Prehomogeneous vector spaces obtained from triangle arrangements

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Abstract

In this paper, we construct a new series of prehomogeneous vector spaces from figures made up of triangles, called triangle arrangements. Our main theorem states that, under suitable assumptions, we are able to construct a prehomogeneous vector space obtained from a triangle arrangement by attaching two triangle arrangements corresponding to prehomogeneous vector spaces at a vertex. We also give examples of prehomogeneous vector spaces obtained from triangle arrangements. Many of them seem to be new.

Introduction

The theory of prehomogeneous vector spaces, constructed by M. Sato [13] (see also Sato–Shintani [14], Kimura [8, Introduction]) enables us to construct zeta functions satisfying functional equations systematically. The key fact is that basic relative invariants satisfy a local functional equation, that is, the Fourier transform of a product of complex powers of basic relative invariants is essentially given by a product of complex powers of some polynomials. It is known that some polynomials which are not basic relative invariants of any prehomogeneous vector space satisfy a functional equation (cf. Faraut–Korányi [6], Kogiso–Sato [9, 10]). Local functional equations are also studied in the fields of algebraic geometry and projective geometry (cf. Etingof–Kazhdan–Polishchuk [3]), and in these fields, they are related homaloidal polynomials which are homogeneous polynomials whose gradient-log maps are bi-rational. Many authors including [1, 2, 4, 5, 7, 11] deal with homaloidal polynomials, and basic relative invariants of regular prehomogeneous vector spaces are recognized as good examples of homaloidal polynomials (cf. [3]). Therefore, finding new concrete examples of regular prehomogeneous vector spaces is important both for the theory of prehomogeneous vector spaces and algebraic geometry.

In this paper, we construct a new series of prehomogeneous vector spaces from figures made up of triangles, called triangle arrangements. Our main theorem, Theorem 4.1, states that, under suitable assumptions, we are able to construct a prehomogeneous vector space obtained from a triangle arrangement by attaching two triangle arrangements corresponding to prehomogeneous vector
spaces at a vertex. We also give examples of prehomogeneous vector spaces obtained from triangle arrangements in Section 5. Combining results in Section 5 with Theorem 4.1, we are able to construct a lot of prehomogeneous vector spaces. Many of them seem to be new.

We organize this paper as follows. Section 1 collects a basic tool that we need later. In particular, a notion of triangle arrangements is introduced. In Section 2, we view triangulation of convex polygons as triangle arrangements and consider which triangulation corresponds to a prehomogeneous vector space. Section 3 is devoted to study a structure of Lie algebras corresponding to triangle arrangements which have no edge sharing. Our main theorem, Theorem 4.1 is stated and proved in Section 4. In Section 5, we give four examples of triangle arrangements which correspond to prehomogeneous vector spaces.

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1 Preliminaries

Let $V = \mathbb{C}^n$. We denote the natural representation of $GL(V) = GL(n, \mathbb{C})$ on $V$ by $\rho$. For a given homogeneous polynomial $p(x)$ on $V$, we introduce a group $G[p] := GL(1) \times G_0[p]$ where

$$G_0[p] := \{ g \in GL(V); p(\rho(g)x) = p(x) \text{ for all } x \in V \}.$$

Then, it is easily verified that $G[p]$ is an algebraic subgroup of $GL(V)$. By definition, we see that $p(x)$ is relatively invariant under the action of $G[p]$. We also use the symbol $\rho$ for the action of $G[p]$ on $V$. In this paper, we work on the following problem.

Problem 1.1. For which homogeneous polynomial $p(x)$ a triplet $(G[p], \rho, V)$ admits a structure of a prehomogeneous vector space?

As in [8], the prehomogeneity is an infinitesimal condition so that we shall describe the condition of a triplet $(G[p], \rho, V)$ being a prehomogeneous vector space in terms of Lie algebra. Let $g_0[p]$ be the Lie algebra corresponding to $G_0[p]$. A bilinear form $\langle \cdot | \cdot \rangle$ on $V$ is defined to be

$$\langle x | y \rangle = \langle x | y \rangle = \sum_{i=1}^{n} x_i y_i \quad (x, y \in V).$$

Then, we have

$$g_0[p] := \{ M \in \mathfrak{gl}(V); \langle dp(M)x | \nabla_x p(x) \rangle = 0 \text{ for all } x \in V \},$$

where $dp$ is a differential of $\rho$. Thus, the Lie algebra $g[p]$ of $G[p]$ is given as

$$g[p] = \mathfrak{gl}(1) \dot{+} g_0[p].$$

Here, the symbol $\dot{+}$ means a direct sum of vector spaces. By [8, Proposition 2.2], we see that the condition of the triplet $(G[p], \rho, V)$ being a prehomogeneous vector space is described by using its Lie algebra $g[p]$ as follows.
**Lemma 1.2** (cf. [8 Proposition 2.2]). The triplet \((\mathfrak{g}[p], dp, V)\) admits a structure of prehomogeneous vector space if and only if linear maps \(A(x) : \mathfrak{g}[p] \to V \ (x \in V)\), defined by \(A(x)M := dp(M)x \ (M \in \mathfrak{g}[p])\) have full generic rank.

In what follows, we concentrate the case of homogeneous polynomials \(p(x)\) of degree three, in particular, those constructed from figures made up of triangles.

Triangle arrangements are figures made up of triangles in such a way that finite triangles are glued at some vertices or some edges. In what follows, the symbol \(T\) denote triangle arrangements. We label number 1, 2, 3, \ldots to vertices of a triangle arrangement \(T\). We assign variable \(x_i\) to vertex \(i\) and, to each triangle with vertices \(i, j, k\) in \(T\), we associate a monomial \(x_ix_jx_k\). Then, we construct a polynomial \(p(x)\) from \(T\) by summing up monomials \(x_ix_jx_k\) with respect to each triangle with vertices \(i, j, k\) in \(T\). We call \(p(x)\) a polynomial with respect to \(T\), or more simply a polynomial of \(T\).

For brevity, we call a polynomial \(p(x)\) is prehomogeneous if the triplet \((\mathfrak{g}[p], dp, V)\) is a prehomogeneous vector space. Moreover, if \(p(x)\) is obtained from a triangle arrangement, then we also say that \(T\) is prehomogeneous. In this case, we often write \(
\mathfrak{g}[T] \) instead of \(\mathfrak{g}[p]\), where \(p(x)\) is a polynomial of \(T\).

**Example 1.3.** The following figures are three examples of triangle arrangements.

If we assign a monomial to each grayed triangles, then the corresponding polynomials are given as follows.

\[
\begin{align*}
p_A(x) &= x_1x_4x_5 + x_2x_5x_6 + x_3x_6x_7 \\
p_B(x) &= x_1x_2x_3 + x_2x_4x_5 + x_3x_5x_6 \\
p_C(x) &= x_1x_5x_6 + x_2x_6x_7 + x_2x_3x_7 + x_3x_4x_8
\end{align*}
\]

Let \(T\) be a triangle arrangement with \(n\) vertices. Set

\[
\mathcal{T} := \{T = \{i, j, k\} \subset [n]; \ a \ triangle \ of \ vertices \ i, j, k \ is \ contained \ in \ T\},
\]

where \([n] := \{1, 2, \ldots, n\}\). We call \(\mathcal{T}\) a hypergraph with respect to \(T\). For each vertex \(i \in [n]\), the set \(\mathcal{T}(i)\) consists of \(T \in \mathcal{T}\) including \(i\), that is,

\[
\mathcal{T}(i) := \{T \in \mathcal{T}; \ i \in T\}.
\]

If a vertex \(i\) satisfies \(|\mathcal{T}(i)| = 1\), then \(i\) is said to be an isolated vertex. If \(\mathcal{T}\) contains two triangles \(T_1, T_2\) such that \(|\mathcal{T}(T_1 \cap T_2)| = 2\), then we say that \(T\) has edge sharing.

For example, \(T_A\) in Example 1.3 have

\[
\mathcal{T} = \{\{1, 4, 5\}, \{2, 5, 6\}, \{3, 6, 7\}\}, \quad \mathcal{T}(5) = \{\{1, 4, 5\}, \{2, 5, 6\}\},
\]

and isolated vertices are \(1, 2, 3, 4, 7\). The triangle arrangement \(T_A\) does not have an edge sharing, whereas \(T_C\) does.
2 Triangulation of convex polygons

In this section, we view triangulation of convex polygons as triangle arrangements and consider those prehomogeneity. Since prehomogeneity is independent of the action of $GL(V)$ on $p(x)$, we first make a reduction of triangulation of polygons in order to decrease cases which we consider. Let us explain this reduction by a concrete example.

The polynomial $p(x)$ associated with Figure 1 (left) is described as

$$p(x) = x_1x_2x_3 + x_1x_3x_4 + x_1x_4x_5 + x_1x_5x_6.$$

If we change variables

$$z_2 = x_2 + x_4, \quad z_i = x_i \quad (i = 1, 3, 4, 5, 6),$$

then $p(x)$ transfers to

$$p(x) = x_1z_3 + x_1z_5(z_4 + z_6) = w_1w_2w_3 + w_1w_5w_6,$$

and we finally get a polynomial consisting of two monomials. Along this reduction, triangles having the vertex 4 disappear. Since the original triangulation have 6 vertex, we should calculate for 5-variable polynomial $p(w)$ on 6-dimensional vector space $\mathbb{C}^6$.

In Figure 4, we exhibit triangle arrangements obtained by reduction of triangulation of $n$-polygons up to $n = 10$. We can see prehomogeneity at the top of each figure. Its proofs are left to Section 5. A black circle • in figures indicates a vertex which does not appear as a vertex of triangles, like the vertex 4 in the above example. For simplicity, triangle arrangements with a black circle are also called just triangle arrangements. Note that, if a triangle arrangement $\mathcal{T}$ is a triangle arrangement $\mathcal{T}'$ with a black circle ($\mathcal{T}'$ does not have a black circle), then the corresponding Lie algebras $\mathfrak{g}[\mathcal{T}]$ and $\mathfrak{g}[\mathcal{T}']$ are related as

$$\mathfrak{g}[\mathcal{T}] = \left\{ M = \begin{pmatrix} M' & 0 \\ M'x & 0 \end{pmatrix} : M' \in \mathfrak{g}[\mathcal{T}'], x \in \mathbb{C}^{\mathcal{T}'}, m \in \mathbb{C} \right\}.$$
| n   | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| (A) | 3  | 4  | 12 | 27 | 82 | 228 | 733 | 2282 | 7528 | 24834 | 83898 | 285357 |
| (B) | 3  | 2  | 7  | 7  | 26 | 37 | 137 | 298 | 993 | 2726 | 8749 | 26446 |
| (C) | 2  | 2  | 4  | 3  | 9  | 7  | 23  | 18  | 61  | 56  | 174 | 186  |

Table 1: The row of (A) indicates numbers of triangulation of \( n \)-polygons up to rotations and reflections, (B) those of reduced triangle arrangements and (C) those of prehomogeneous vector spaces.

In particular, the prehomogeneity of \( T \) is the same as that of \( T' \).

We see from Figure 3 that triangle arrangements obtained by reduction of triangulation of \( n \)-gons may unconnected. We shall show in Proposition 5.2 that they cannot be prehomogeneous unless one of the connected components is a black circle discussed in the previous paragraph.

Table 1 includes numbers of (A) triangulation of \( n \)-polygons under rotations and reflections, (B) those of reduced triangle arrangements and (C) those of prehomogeneous vector spaces.

### 3 Structure of \( g[p] \) without edge sharing

Let \( T \) be a triangle arrangement with \( n \) vertices and \( p(x) \) the corresponding polynomial. Suppose that \( T \) has no edge sharing. In this section, we investigate a structure of \( g[p] \), which will be needed to prove our main theorem. By [1], it is enough to calculate \( \langle dp(M)x|\nabla x p(x)\rangle = 0 \). By definition, \( p(x) \) can be described by using the hyper graph \( \mathcal{T} \) associated with \( T \) as

\[
p(x) = \sum_{\{i,j,k\} \in \mathcal{T}} x_i x_j x_k.
\]

Thus, we have

\[
\langle dp(M)x|\nabla x p(x)\rangle = \sum_{i=1}^{n} \sum_{a=1}^{n} M_{ia} x_a \cdot \sum_{\{i,j,k\} \in \mathcal{T}(i)} x_j x_k
\]

\[
= \sum_{i=1}^{n} \sum_{a=1}^{n} \sum_{\{i,j,k\} \in \mathcal{T}(i)} M_{ia} x_a x_j x_k.
\]

We first exhibit a calculation of a Lie algebra of a concrete polynomial.

**Example 3.1.** Let \( p(x) \) be a homogeneous polynomial whose hyper graph is given as \( \mathcal{T} = \{\{1,2,3\},\{1,4,5\}\} \), that is,

\[
p(x) = x_1 x_2 x_3 + x_1 x_4 x_5.
\]

In this case, we have

\[
\langle dp(M)x|\nabla x p(x)\rangle
\]

\[
= M_{11} x_1 (x_2 x_3 + x_4 x_5) + M_{12} x_2 (x_2 x_3 + x_4 x_5) + M_{13} x_3 (x_2 x_3 + x_4 x_5)
\]

\[
+ M_{14} x_4 (x_2 x_3 + x_4 x_5) + M_{15} x_5 (x_2 x_3 + x_4 x_5)
\]

\[
+ M_{21} x_1 \cdot x_1 x_3 + M_{22} x_2 \cdot x_1 x_3 + M_{23} x_3 \cdot x_1 x_3 + M_{24} x_4 \cdot x_1 x_3 + M_{25} x_5 \cdot x_1 x_3
\]

\[
+ M_{31} x_1 \cdot x_1 x_2 + M_{32} x_2 \cdot x_1 x_2 + M_{33} x_3 \cdot x_1 x_2 + M_{34} x_4 \cdot x_1 x_2 + M_{35} x_5 \cdot x_1 x_2
\]

\[
+ M_{41} x_1 \cdot x_1 x_5 + M_{42} x_2 \cdot x_1 x_5 + M_{43} x_3 \cdot x_1 x_5 + M_{44} x_4 \cdot x_1 x_5 + M_{45} x_5 \cdot x_1 x_5
\]

\[
+ M_{51} x_1 \cdot x_1 x_4 + M_{52} x_2 \cdot x_1 x_4 + M_{53} x_3 \cdot x_1 x_4 + M_{54} x_4 \cdot x_1 x_4 + M_{55} x_5 \cdot x_1 x_4
\]
and hence
\[\langle dp(M)x \mid \nabla_x p(x) \rangle = (M_{11} + M_{22} + M_{33})x_1x_2x_3 + (M_{11} + M_{44} + M_{55})x_1x_4x_5 + (M_{24} + M_{53})x_1x_3x_4 + (M_{25} + M_{43})x_1x_3x_5 + (M_{34} + M_{52})x_1x_2x_4 + (M_{35} + M_{42})x_1x_2x_5 + M_{12}x_1^2x_3 + M_{12}x_2x_4x_5 + M_{13}x_2x_3^2 + M_{13}x_3x_4x_5 + M_{14}x_2x_3x_4 + M_{14}x_3^2x_5 + M_{15}x_3x_5 + M_{15}x_4x_5^2 + M_{21}x_1^2x_3 + M_{23}x_1x_3^2 + M_{31}x_1^2x_2 + M_{32}x_1x_2^2 + M_{41}x_1^3x_5 + M_{45}x_1x_5^2 + M_{51}x_1^2x_4 + M_{54}x_4x_1x_5^2.\]

Thus, solving the equation \(\langle dp(M)x \mid \nabla_x p(x) \rangle = 0\), we need have all coefficients of each monomial must be equal to zero, and hence \(g[p]\) consists of matrices of the form
\[t \cdot I_5 + \begin{pmatrix} M_{11} & 0 & 0 & 0 & 0 \\ 0 & M_{22} & 0 & M_{24} & M_{25} \\ 0 & 0 & M_{33} & M_{34} & M_{35} \\ 0 & -M_{35} & -M_{25} & M_{44} & 0 \\ 0 & -M_{34} & -M_{24} & 0 & M_{55} \end{pmatrix}, \quad M_{11} + M_{22} + M_{43} = 0, \quad M_{11} + M_{44} + M_{55} = 0\]

where \(t \in \mathbb{C}\) and \(M_{ij} \in \mathbb{C}\). Here, \(I_5\) is the identity matrix of size 5.

As in Example 3.1, it is important to find out how many times each monomial \(x_ia_jx_k\) appears. In what follows, we investigate it for \(T\) without edge sharing in detail enough to prove our main theorem.

Let \(T\) be a triangle arrangement with \(n\) vertices. Suppose that \(T\) has no edge sharing.

Fix a vertex \(i\). Then, a monomial \(x_ia_jx_k\) appears for each triangle \(\{i,j,k\} \in T(i)\) and for each \(a \in I_n\). Let us find out from which vertex the monomial \(x_ia_jx_k\) appears.

At first, there are no duplicate in terms which are given from the vertex \(i\). In fact, let us suppose that the monomial \(x_ia_jx_k\) appears from triangles in \(T(i)\) other than \(\{i,j,k\}\). Then, at least one of \(\{i,a,j\}\) and \(\{i,a,k\}\) is included in \(T(i)\) because two of \(x_a, x_j\) and \(x_k\) come from partial derivative of \(p(x)\). In this case, however, \(T\) has edge sharing, which leads to a contradiction.

Next, assume that the monomial \(x_ia_jx_k\) comes from a vertex \(l \in I_n\) such that \(l \neq i\). Then, we see that at least one of \(\{a,j,l\} \in T\) and \(\{a,k,l\} \in T\) need satisfy by the same reason to the case \(i\). In this situation, let us consider the positions of \(i, a\) and \(l\) in the graph consisting of edges of all triangles in \(T\). Let us denote by \(d_{\text{graph}}(i,a)\) the graph distance. If \(d_{\text{graph}}(i,a) \geq 3\), then there are no vertex \(l\) satisfies \(\{a,j,l\} \in T\) or \(\{a,k,l\} \in T\) and hence the monomial \(x_ia_jx_k\) never appear from vertices other than \(i\). This means that \(M_{ia} = 0\) whenever \(d_{\text{graph}}(i,a) \geq 3\), or equivalently, \(M_{ia} \neq 0\) occurs only if \(d_{\text{graph}}(i,a) \leq 2\).

(0) The case \(d_{\text{graph}}(i,a) = 0\), that is, \(a = i\). In this case, we have by setting \(a = i\) in \(\ref{eq:1}\)

\[
\sum_{i=1}^{n} M_{ii}x_i \sum_{\{i,j,k\} \in T(i)} x_jx_k = \sum_{\{i,j,k\} \in T} (M_{ii} + M_{jj} + M_{kk})x_i x_j x_k,
\]

which leads to the following conditions.

\[M_{ii} + M_{jj} + M_{kk} = 0 \text{ if } \{i,j,k\} \in T.\]
(1) The case $d_{\text{graph}}(i,a) = 1$. Assume that $\{i,j,k\}, \{i,s,t\} \in \mathcal{T}(i)$ with $\{j,k\} \neq \{s,t\}$. Suppose $a = j$. Then, we have $x_a x_j x_k = x_j^2 x_k$. If the monomial $x_j^2 x_k$ arises from the other vertex $l$, then the triangle $\{j,k,l\}$ must included in $\mathcal{T}$, but then $\mathcal{T}$ has edge sharing at the edge $jk$. It creates a contradiction. Next, we suppose that $a = s$. Although the monomial $x_a x_j x_k = x_s x_j x_k$ has no information, we know that $\frac{\partial}{\partial x_j} p(x)$ have a monomial $x_s x_t$ so that $x_a x_s x_t = x_s^2 x_t$ appears in $\mathcal{B}$. Then, by the same reason on the case $a = j$, we face a contradiction and hence we conclude that, if $\mathcal{T}$ has no edge sharing, then $M_{ia} = 0$ whenever $d_{\text{graph}}(i,a) = 1$.

(2) The case $d_{\text{graph}}(i,a) = 2$. Let $\{j,a,b\} \in \mathcal{T}$, that is, the vertex $a$ is linked to $j$. The assumption that there are no edge sharing implies $k \not\in \{a,b\}$.

(2-i) At first, suppose that the vertex $i$ is an isolated vertex. In this case, the partial derivative $\frac{\partial}{\partial x_j} p(x)$ is exactly a monomial $x_j x_k$ so that $x_a \frac{\partial}{\partial x_j} p(x) = x_a x_j x_k$, and its factor $x_a x_j$ appears as a monomial in $\frac{\partial}{\partial x_j} p(x)$. Hence, if $\sharp \mathcal{T}(b) = 1$, then we obtain

$$x_a x_j x_k = x_a \frac{\partial}{\partial x_j} p(x) = x_k \frac{\partial}{\partial x_j} p(x)$$

and this monomial never arises from the other vertices so that we obtain a condition

$$M_{ia} + M_{bk} = 0. \quad (3)$$

Assume that $\sharp \mathcal{T}(b) \geq 2$. If there are no $T \in \mathcal{T}(b)$ such that $k \in T$, then we need have $M_{bk} = 0$ and hence we also have $M_{ia} = 0$. For the case that there exists $T \in \mathcal{T}(b)$ such that $k \in T$, we do not discuss in detail and only give one remark as follows. If we set $T = \{k,b,c\}$, then $c \not\in \{i,j,k,a,b\}$ because there are no edge sharing and hence the following three triangles $\{i,j,k\}, \{j,a,b\}$ and $\{k,b,c\}$ form a ring of triangles $\mathcal{T}_B$ as in Example 1.3.

(2-ii) Next, suppose that the vertex $i$ is not an isolated vertex. Let $\{i,j,k\}, \{i,s,t\} \in \mathcal{T}(i)$ with $\{j,k\} \cap \{s,t\} = \emptyset$. In this case, polynomial $x_a \frac{\partial}{\partial x_j} p(x)$ includes two monomials $x_a x_j x_k$ and $x_a x_s x_t$. If the vertex $a$ is not linked to $s$ and $t$, then the monomials $x_a x_s$ and $x_a x_t$ cannot appear when we differentiate $p(x)$ so that we obtain $M_{ia} = 0$. For the case that the vertex $a$ is linked to $s$ or $t$, then we do not discuss in detail and only give one remark as follows. If we set $\{a,c,t\} \in \mathcal{T}$, then $c \not\in \{a,b,i,j,s,t\}$ because there are no edge sharing ($c = k$ may occur), so that $\mathcal{T}$ must include at least one of the following triangle arrangements.

4 Attaching two triangle arrangements at a vertex

In this section, we present our main theorem stating that we are able to construct a prehomogeneous triangle arrangement by attaching two prehomogeneous triangle arrangements at a vertex.
**Theorem 4.1.** Let $T_{\nu}$ ($\nu = 1, 2$) be two prehomogeneous triangle arrangements with no edge sharing and let $(g_{\nu}, d\rho_{\nu}, V_{\nu})$ be the corresponding prehomogeneous vector spaces. Suppose that there exist vertices $0^{(\nu)}$ of $T_{\nu}$ and subalgebras $h_{\nu}$ of $g_{\nu}$ ($\nu = 1, 2$) such that

1. triplets $(h_{\nu}, d\rho_{\nu}|_{h_{\nu}}, V_{\nu})$ are prehomogeneous vector spaces,
2. variables $x_{\nu(\nu)}$ corresponding to the vertices $0^{(\nu)}$ are relatively invariant under the actions of $h_{\nu},$
3. for each $\nu = 1, 2$, there exists at least one triangle $\{0^{(\nu)}, a^{(\nu)}, \tilde{a}^{(\nu)}\} \in T(0^{(\nu)})$ such that $a^{(\nu)}$ is an isolated vertex and the variable $x_{\nu(\nu)}$ is relatively invariant under the action of $h_{\nu}.$

Then, the triangle arrangement $T$ obtained by attaching two triangle arrangements $T_{\nu}$ at vertices $0^{(\nu)}$ is prehomogeneous.

**Example 4.2.** Let $T_1 = T_A$ and $T_2 = T_D$ as in Examples 3.3 and 3.1. Let $T$ be the triangle arrangement obtained by attaching $T_{\nu}$ ($\nu = 1, 2$) at vertices $0^{(1)} = 5$ in $T_1$ and $0^{(2)} = 2$ in $T_2$. Then, $T$ is drawn as follows.

In the figure of $T$, we attach the symbol prime ′ to vertices coming from $T_2$ in order to distinguish those from $T_1$. Although the polynomial $x_2$ is not a relative invariant with respect to the Lie algebra $g_2 = g_D$, a subalgebra $h_2 = \{M \in g_2; M_{24} = M_{25} = 0\}$ acts on $V_2 = \mathbb{C}^5$ prehomogeneously and $x_2$ is a relatively invariant polynomial with respect to $h_2$ so that we can apply Theorem 4.1. Thus, we see that $T$ is also prehomogeneous.

**Proof of Theorem 4.1** Set $\dim V_{\nu} = n_{\nu} + 1$. Then, vertices of $T_{\nu}$ are $0^{(\nu)}, 1^{(\nu)}, \ldots, n^{(\nu)}_{\nu}.$ For a basis of $V_{\nu}$, we choose the standard basis $e_{0^{(\nu)}}, e_{1^{(\nu)}}, \ldots, e_{n^{(\nu)}_{\nu}}.$ Since a polynomial $x_{\nu^{(\nu)}}$ is relatively invariant for each $\nu = 1, 2$, a general element $M^{(\nu)}$ of $h_{\nu}$ is described as

$$M^{(\nu)} = \begin{pmatrix} m^{(\nu)} \\ h^{(\nu)} \end{pmatrix}, \quad m^{(\nu)} \in \mathbb{C}, \; h^{(\nu)} \in \mathbb{C}^{n_{\nu}}, \; \tilde{M}^{(\nu)} \in \text{Mat}(n_{\nu}, \mathbb{C}).$$

Note that $m^{(\nu)}, h^{(\nu)}$ and $\tilde{M}^{(\nu)}$ may be depend on each other.

Let $T$ be the triangle arrangement which is obtained by attaching $T_{\nu}$ ($\nu = 1, 2$) at the points $0^{(\nu)}.$ The corresponding vector space $V$ has dim $V = n_1 + n_2 + 1$. Vertices of $T$ are labelled as

$0 = 0^{(1)} = 0^{(2)}, \; i = i^{(1)} \; (i = 1, \ldots, n_1), \; n_1 + j = j^{(2)} \; (j = 1, \ldots, n_2).$

Put $V := \{0, 1, \ldots, n_1 + n_2\}$. We denote by $g$ the Lie algebra corresponding to $T$. 

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The proof is separated into two parts, one is determining a structure of \( g \), and the other is proving prehomogeneity.

(i) We first investigate a structure of \( g \). Since \( T \) obviously has no edge sharing, for an element \( M = (M_{ij}) \in g \), a condition \( M_{ij} \) occur only if \( d_{\text{graph}}(i,j) \) for \( i,j \in V \). If \( i,j \) are included in one \( T_\nu \), then it relates to \( g_\nu \) and thus we do not consider again. Hence, it is enough to consider the case that one of \( i,j \) is included in \( T_1 \) and the other one is in \( T_2 \). Since \( T_1 \) and \( T_2 \) are joined at one point, such a pair \( i,j \) must satisfy \( d_{\text{graph}}(0,i) = d_{\text{graph}}(0,j) = 1 \). In what follows, we use a symbol \( i \) for vertex in \( T_1 \) and a symbol \( a \) (instead of \( j \)) for vertex in \( T_2 \) in order to distinguish \( T_\nu \) easily by symbols. Triangles in \( T(0) \) are written like \( \{0,a,\bar{a}\} \), that is, we use bar symbol \( \bar{a} \) for the remaining vertex.

The current situation is included in a situation discussed in the previous section by setting \( j = 0 \). Since \( T_1 \) and \( T_2 \) are joined at one point 0, rings consisting of triangles, which are excluded in the discussion of the previous section, never appear. Thus, (3) is the only non-trivial relation which implies for each pair of \( \{0^{(1)}, i, \bar{i}\} \in T_1(0^{(1)}) \) and \( \{0^{(2)}, a, \bar{a}\} \in T_2(0^{(2)}) \), we have

\[
M_{ia} + M_{ai} = 0,
\]

and \( M_{ia} = 0 \) or \( M_{ai} = 0 \) otherwise. By discussion (2) in the previous section, an element in (3) do not vanish if and only if \( i \) is an isolated vertex in \( T_1 \) and \( a \) is an isolated vertex in \( T_2 \). The assumption (3) ensures existence of such vertices \( i, a \), that is, there exists at least one \( \{0^{(1)}, i, \bar{i}\} \in T_1(0^{(1)}) \) such that \( i \) is an isolated vertex and a polynomial \( x_i \) is relatively invariant with respect to \( h_1 \), and similarly there exists at least one \( \{0^{(2)}, a, \bar{a}\} \in T_2(0^{(2)}) \) such that \( a \) is an isolated vertex and a polynomial \( x_a \) is relatively invariant with respect to \( h_2 \).

Therefore, we have confirmed that \( g \) includes a subalgebra \( h \) consisting of matrices of the form

\[
M = \begin{pmatrix}
m & 0 & 0 \\
h^{(1)} & M^{(1)} & Z_1 \\
h^{(2)} & Z_2 & M^{(2)}
\end{pmatrix},
\]

where \( m = m^{(1)} = m^{(2)} \in \mathbb{C} \), \( h^{(\nu)} \in \mathbb{C}^{n_\nu} \) and \( \widetilde{M}^{(\nu)} \in \text{Mat}(n_\nu, \mathbb{C}) \) are as in (4). For matrices \( Z_1 = (M_{ia})_{1 \leq i \leq n_1, 1 \leq a \leq n_2} \) and \( Z_2 = (M_{ia})_{1 \leq i \leq n_2, 1 \leq a \leq n_1} \), elements \( M_{ia} \) and \( M_{ai} \) are zeros except for the case

(a) \( \{0^{(1)}, i, \bar{i}\} \in T_1(0^{(1)}) \) and \( \{0^{(2)}, a, \bar{a}\} \in T_2(0^{(2)}) \),

(b) the vertex \( i \) is an isolated vertex in \( T_1 \),

(c) a polynomial \( x_a \) is relatively invariant with respect to \( h_2 \),

and in this case we have

\[
M_{ai} = -M_{ia}.
\]

(ii) Next we investigate the prehomogeneity of \( T \). To do so, we consider basic relative invariants with respect to \( h \). We set

\[
p_0(x) = p_0(x_0, x^{(1)}, x^{(2)}) = p^{(1)}(x_0, x^{(1)}) + p^{(2)}(x_0, x^{(2)}),
\]

\[
p_1(x) = p_1(x_0, x^{(1)}, x^{(2)}) = x_0.
\]

It is obvious that \( p_0(x) \) and \( p_1(x) \) are relatively invariant polynomials. For \( \nu = 1, 2 \), let us denote by \( p^{(\nu)}(x_0^{(\nu)}, x^{(\nu)}) \) and \( q^{(\nu)}_j(x_0^{(\nu)}, x^{(\nu)}) \) \((j = 0, 1, \ldots, k_\nu)\) the basic relative invariants with respect to \( h_\nu \). Here, \( p^{(\nu)}(x_0^{(\nu)}, x^{(\nu)}) \) is the
polynomial corresponding to triangle arrangement $T_\nu$ and $q_j^{(\nu)}(x_0^{(\nu)}, x^{(\nu)})$ are the other ones. We set $q_0^{(\nu)}(x_0^{(\nu)}, x^{(\nu)}) = x_0^{(\nu)}$.

Among polynomials $q_j^{(\nu)}(x_0^{(\nu)}, x^{(\nu)})$ ($j = 1, \ldots, k_0$), we pick ones such that $\partial_j q_0^{(\nu)}(x_0^{(\nu)}, x^{(\nu)}) = 0$ for any isolated vertex $j$ in triangles in $T_\nu(0^{(\nu)})$, and rename them as $p_2(x), \ldots, p_k(x)$.

We shall show that the basic relative invariants with respect to $h$ are exactly $p_j(x)$ ($j = 0, 1, \ldots, k$), and the prehomogeneity is proved in the same time.

Let $H = \exp h$ be a connected and simply connected Lie group of $h$. Let us take a reference point $x_* \in V$ such that $p_j(x_*) \neq 0$ for all $j = 0, 1, \ldots, k$. What we want to prove is to show that any regular element $x \in V$, that is, $p_j(x) \neq 0$ for any $j = 0, 1, \ldots, k$ can be moved to $x_*$ by the action of $H$. To do so, we decompose $h$ into 4 spaces as a vector space. At first, we put

$$h' = \{ M \in h; \langle dp(M)x | \nabla_x p_0(x) \rangle = 0 \text{ for all } x \in V \}$$

and

$$h'' = \{ M' \in h'; dp(M')x_0 = 0 \} \subset h'.'$$

Then, $h''$ is a subalgebra of $h'$ so that there exists an $A \in h'$ such that

$$h' = \mathbb{C} A + h''.$$

Note that we can take $A$ as a diagonal matrix. Then, an element $M$ as in (6) can be decomposed into

$$M = t I + m'A + \begin{pmatrix} 0 & 0 & 0 \\ 0 & M'_1 & 0 \\ 0 & 0 & M'_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Z_1 \\ 0 & Z_2 & 0 \end{pmatrix}. \tag{8}$$

Here, $I$ is the identity matrix of suitable size. Set

$$M' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & M'_1 & 0 \\ 0 & 0 & M'_2 \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Z_1 \\ 0 & Z_2 & 0 \end{pmatrix}.$$

Then, we have $M', Z \in h''$. By definition, the polynomial $p_0(x)$ is invariant under the actions of $A$, $M'$ and $Z$, and the polynomial $p_1(x)$ is invariant under the actions of $M'$ and $Z$. This means that we can match a value of $p_0(x)$ to $p_0(x_*)$ by the action $\exp(t I)$, and then a value of $p_1(x)$ to $p_1(x_*)$ by the action of $\exp(m'A)$.

If we have $q_j^{(\nu)}(x_0^{(\nu)}, x^{(\nu)}) \neq 0$ for any $j = 0, 1, \ldots, k$ and for $\nu = 1, 2$, then the prehomogeneity of $h_0$, imply that of $h$, and hence of $g$. The problem here is that there are some $q_j^{(\nu)}(x_0^{(\nu)}, x^{(\nu)})$ which are not included in $p_j(x)$ ($j = 0, 1, \ldots, k$). Such $q_j^{(\nu)}(x_0^{(\nu)}, x^{(\nu)})$ satisfy $\partial_j q_j^{(\nu)}(x_0^{(\nu)}, x^{(\nu)}) \neq 0$ for some isolated vertex $i$ in a triangle in $T_\nu(0^{(\nu)})$. By (7), we can take $Z = E_{i0} - E_{00}$, where $E_{st}$ is the matrix unit of size $\dim V$ having one at the position $(s,t)$ and zeros elsewhere. Since $\exp Z = I + Z$, polynomials $q_j^{(\nu)}(x_0^{(\nu)}, x^{(\nu)})$ which are not included in $p_j(x)$ can be take non-zero values by applying actions of $\exp Z$. Therefore, we can conclude that any regular element $x \in V$ can be moved to the reference point $x_* \in V$ by the action of $H$. Namely, we have proved that $H$ acts transitively on the set of regular elements and hence $(H, \rho, V)$ is a prehomogeneous vector space and so is $(G, \rho, V)$. \hfill \Box
Remark 4.3. The condition (3) in Theorem 4.1 is necessary. We shall confirm this by the following example.

Both triangle arrangements $T_\nu (\nu = 1, 2)$ above are prehomogeneous, and variables corresponding to the vertices in black circles in each figures are relatively invariant polynomials, but there are no triangles in $T_1(3)$ including isolated points. In this case, the triangle arrangement $T$ obtained by attaching $T_\nu (\nu = 1, 2)$ at the vertices of black circles is not prehomogeneous. We note that if we choose a vertex from one of $\{6, 8, 10, 12, 14, 16\}$ for the attaching point in $T_1$, then the condition (3) is satisfied so that the triangle arrangement obtained by attaching this point is prehomogeneous.

5 Examples

In this section, we give some series of prehomogeneous triangle arrangements. We also exhibit triangle arrangements which does not correspond to prehomogeneous vector spaces.

Theorem 5.1. The following triangle arrangements are prehomogeneous.

1. daisy cases: triangle arrangements constructed by attaching $n$ triangles at one vertex ($n \geq 2$),
2. chain cases: triangle arrangements constructed by arraying $n$ triangles in a row ($n \geq 2$),
3. circular cases: triangle arrangements constructed by arraying $n$ triangles circularly ($n \geq 3$),
4. edge gluing cases: triangle arrangements constructed by gluing $n$ triangle arrangements $T_B$ (as in Example 1.3) edges 24 and 36 ($n \geq 2$).

Examples of triangle arrangements in Theorem 5.1 are given in Figure 2. In particular, Theorem 5.1 (4) tells us that the condition of edge-sharing is not a necessary condition for prehomogeneity.

We shall prove this theorem in the following subsections by giving detailed structures of Lie algebras. We note here that, in this section, the dual vector space $V^*$ be identified with $V$ through $\langle \cdot | \cdot \rangle$. Before going to proofs, we give an example of triangle arrangements which are not prehomogeneous.

Proposition 5.2. Triangle arrangements, which are not connected as graphs, are not prehomogeneous.

Proof. If a triangle arrangement $T$ is not connected, then there exist $T_1$ and $T_2$ such that $T = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Let $p(x)$ and $q(y)$ be the corresponding polynomials associated with $T_1$ and $T_2$, respectively. Then, it is obvious that the polynomial corresponding to $T$ is $P(x, y) = p(x) + q(y)$. Then, since there is
Figure 2: Examples of triangle arrangements of Theorem 5.1

no overlapping at variables of \( p \) and \( q \), the corresponding Lie algebra \( \mathfrak{g}[P] \) can be described as

\[
\mathfrak{g}[P] = \left\{ tI + \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} : t \in \mathbb{C}, M_1 \in \mathfrak{g}_0[p], M_2 \in \mathfrak{g}_0[q] \right\},
\]

where \( \mathfrak{g}[p], \mathfrak{g}[q] \) are the Lie algebras corresponding to \( p \) and \( q \), respectively. Thus, it is easily verified that if one of \( \mathfrak{g}[p] \) or \( \mathfrak{g}[q] \) is not prehomogeneous, then so is not \( \mathfrak{g}[P] \). Therefore, we can assume that both of them are prehomogeneous, and we shall prove the assertion for this case for a more general situation. To do so, we recall the definition of homaloidal polynomials. A homogeneous polynomial \( p(x) \) of degree \( d \) is said to be homaloidal if the polar map \( \varphi_p(x) := \text{grad } \log p(x) \) is birational. Then, there exists a homogeneous polynomial \( \varphi^*(x) \), called the dual polynomial of \( \varphi(x) \), such that

\[
\varphi^*(x) = \varphi(x) - 1.
\]

For a relatively invariant polynomial \( \varphi(x) \) of a prehomogeneous vector space, it is known that the polar map \( \varphi(x) \) is homaloidal or a zero map. Thus, it is enough to prove the following claim.

Claim 5.3. Let \( d \geq 3 \). For two homaloidal polynomials \( p(x) = p(x_1, \ldots, x_n) \) and \( q(y) = q(y_1, \ldots, y_m) \) of degree \( d \), we set \( P(x,y) := p(x) + q(y) \). Then, \( P(x,y) \) cannot be a homaloidal polynomial.

Proof. Let \( p_*(x) \) and \( q_*(y) \) be the dual polynomials of \( p(x) \) and \( q(y) \), respectively, that is, we have

\[
p_*(\nabla_x p(x)) = p(x)^{d-1}, \quad q_*(\nabla_y q(y)) = q(y)^{d-1}.
\]

Put

\[
P_*(x,y) := (p_*(x) \frac{x}{x})^{d-1} + q_*(y) \frac{y}{y}
\]

Then, we have

\[
P_*(\nabla_x P(x,y)) = P_*(\nabla_x p(x), \nabla_y q(y))
\]

\[
= \left\{ \left\{ p_*(\nabla_x p(x)) \right\} \frac{x}{x} + \left\{ q_*(\nabla_y q(y)) \right\} \frac{y}{y} \right\}^{d-1}
\]

\[
= \left\{ p(x) + q(y) \right\}^{d-1} = P(x,y)^{d-1}.
\]

Since \( P_*(x,y) \) is not a rational function, we see that the polar map \( \varphi_P(x,y) \) cannot be birational so that \( P(x,y) \) cannot be a homaloidal polynomial.
5.1 Daisy cases

Let $T$ be a triangle arrangement which is constructed by attaching $n$ triangles at one vertex for $n \geq 2$ (see (1) of Figure 2). In this case, the number of vertices are $2n+1$, that is, $\dim V = 2n+1$. The $i$-th triangle consists of vertices $(i, i+1, 2n+1)$. The corresponding polynomial $p(x)$ is

$$p(x) = (x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n})x_{2n+1}.$$ 

Obviously, $p(x)$ is obtained as a product of polynomials of degree one and two, which do not share variables, so that the corresponding triplet $(\mathfrak{g}[p], d\rho, V)$ is a prehomogeneous vector space. Let $J$ be a $2n \times 2n$ matrix defined by

$$J = \text{diag}(J', \ldots, J'), \quad J' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

**Lemma 5.4.** The prehomogeneous vector space $(\mathfrak{g}[p], d\rho, V)$ is regular and reductive. A general element $M$ of $\mathfrak{g}[p]$ is of the form

$$M = \begin{pmatrix} tI_{2n} + M' & 0 \\ 0 & M_{2n+1} \end{pmatrix}, \quad (t, M_{2n+1} \in \mathbb{C}, \ M' \in \mathfrak{so}(J)),$$

and hence one has $\dim \mathfrak{g}[p] = 2n^2 - n + 2$. The basic relative invariants are given as

$$p_0(x) = p(x), \quad p_1(x) = x_{2n+1} \quad (x \in V).$$

**Proof.** It is enough to mention that $\mathfrak{so}(J)$ is a Lie algebra with respect to the following bilinear form

$$x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} = \frac{1}{2} \langle x' | Jx' \rangle \quad (x' = (x_1, x_2, \ldots, x_{2n}),$$

and we have $p(x) = \frac{1}{2} \langle x' | Jx' \rangle \cdot x_{2n+1}$. \hfill \Box

5.2 Chain cases

Let $T$ be a triangle arrangement which is constructed by arraying $n$ triangles in a row with $n \geq 3$ (see (2) in Figure 2). In this case, the number of vertices are $2n+1$, that is, $\dim V = 2n + 1$. The $i$-th triangle consists of vertices $(i, n+i, n+i+1)$. The corresponding polynomial $p(x)$ is given as

$$p(x) = \sum_{i=1}^{n} x_ix_{n+i}x_{n+i+1} \quad (x \in V).$$

Let $E^{(m)}_{ij}$ (resp. $E'_{ij}$) be a matrix unit of size $m \times m$ (resp. $n \times (n+1)$) having one on the position $(i, j)$ and zeros elsewhere.

**Lemma 5.5.** A general element $M$ of $\mathfrak{g}[p]$ is of the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$
where
\[
\begin{align*}
M_{11} &= bE_{21}^{(n)} + cE_{n-1,n}^{(n)} + \sum_{i=1}^{n}(t - t_i - t_{i+1})E_{ii}^{(n)}, \\
M_{12} &= \sum_{i=1}^{n-1}a_i(E_{i+1,i}^{(n)} - E_{i,i+2}^{(n)}), \\
M_{22} &= -bE_{13}^{(n+1)} - cE_{n+1,n+1}^{(n+1)} + \sum_{i=1}^{n+1}t_iE_{ii}^{(n+1)}.
\end{align*}
\]

\textbf{Proof.} Since the triangle arrangement $T$ does not have edge sharing, we can apply the discussion in Section 3. Let $j = 1, \ldots, n$. Then, the vertex $j$ is isolated, and the equation (3) can be written as
\[
M_{j,n} + M_{j+2,n} + M_{j+1,n+j} = 0 \quad (j = 1, \ldots, n).
\]
Moreover, the vertices $i = n + 1$ and $i = 2n + 1$ are also isolated so that the equation (3) again implies
\[
M_{n+1,n+3} + M_{2,n+1} = 0, \quad M_{2n+1,2n-1} + M_{n-1,n} = 0.
\]
The other terms are all zeros so that the proof is now completed.

This lemma yields that $\mathfrak{g}[p]$ is a solvable Lie algebra of dim $\mathfrak{g}[p] = 2n + 3$.

\textbf{Lemma 5.6.} The triplet $(\mathfrak{g}[p], d\rho, V)$ is a regular prehomogeneous vector space for all $n \geq 3$. Its basic relative invariants $p_i(x)$ ($i = 1, \ldots, n$) are given as
\[
p_1(x) = p(x), \quad p_i(x) = x_{n+i} \quad (i = 2, \ldots, n).
\]

\textbf{Proof.} We use Lemma 1.2. To make discussion simple, we consider a subalgebra $\mathfrak{h}$ of $\mathfrak{g}[p]$ defined by
\[
\mathfrak{h} := \{M \in \mathfrak{g}[p]; b = c = 0\},
\]
where we use an expression of $M \in \mathfrak{g}[p]$ as in the above lemma. For $x \in V$, let $A(x): \mathfrak{g}[p] \rightarrow V$ be a linear map defined by $A(x)M := d\rho(M)x$ ($M \in \mathfrak{h}$). Then, since $\dim \mathfrak{h} = \dim V$, we see that $A(x)$ is a square matrix and by the above lemma

Thus, we can calculate its determinant by using cofactor expansion at the first
column as

$$\det A(x) = x_{n+1} \cdots x_{2n+1} \det \begin{pmatrix} x_1 & -x_{n+3} & 0 & \cdots & 0 \\ x_2 & x_{n+1} & -x_{n+4} & \cdots & \\ \vdots & 0 & x_{n+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -x_{2n+1} \\ x_n & 0 & \cdots & 0 & x_{2n-1} \end{pmatrix}$$

$$= x_{n+1} \cdots x_{2n+1} \sum_{i=1}^{n} x_i x_{n+i} x_{n+i+1} \cdots x_{2n-1}$$

$$= x_{n+1} x_{n+2} (x_{n+3} \cdots x_{2n-1})^2 x_{2n} x_{2n+1} \cdot p(x).$$

This shows that $A(x)$ has generic full rank and hence the triplet $(g[p], dp, V)$ is a prehomogeneous vector space. This also shows that polynomials $x_{n+i}$ ($i = 1, \ldots, n+1$) are basic relative invariants with respect to $h$. Among them, two polynomials $x_{n+1}$ and $x_{2n+1}$ are not relatively invariant because $g[p]$ has $b, c$ parts, but the other ones are relatively invariant with respect to $g[p]$.

We now calculate on the dual prehomogeneous vector space. Let us consider a linear map $A^*(x): h \to V (x \in V)$ defined by $A^*(x)M := dp(M)x$ ($M \in h$). Then, $A^*(x)$ is a matrix of the form

$$A^*(x) = \begin{pmatrix} x_1 & -x_1 & -x_2 & \cdots & \cdots & -x_{n-1} \\ x_2 & x_2 & x_3 & \cdots & \cdots & -x_2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\ x_{n-1} & x_{n-2} & x_{n-3} & \cdots & \cdots & x_2 \\ x_n & 0 & \cdots & \ddots & \ddots & x_n \\ 0 & x_{n+1} & x_{n+2} & \ddots & \cdots & -x_1 \end{pmatrix}$$

Its determinant can be calculated as if $n = 2k$ is even then

$$\det A^*(x) = x_1 \cdots x_n \times \det \begin{pmatrix} x_{n+1} & 0 & x_2 & \cdots & \cdots & x_n \\ 0 & x_{n+2} & 0 & \cdots & \cdots & 0 \\ x_{n+3} & 0 & -x_1 & \cdots & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & x_{2n} & -x_{n-2} & 0 & \cdots & x_1 \\ x_{2n+1} & 0 & \cdots & \ddots & \cdots & -x_{n-1} \end{pmatrix}$$

and if $n = 2k + 1$ is odd, then

$$\det A^*(x) = x_1 \cdots x_n \times \det \begin{pmatrix} x_{n+1} & 0 & x_2 & \cdots & \cdots & x_n \\ 0 & x_{n+2} & 0 & \cdots & \cdots & 0 \\ x_{n+3} & 0 & -x_1 & \cdots & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & x_{2n} & -x_{n-2} & 0 & \cdots & x_1 \\ x_{2n+1} & 0 & \cdots & \ddots & \cdots & -x_{n-1} \end{pmatrix}$$
Thus, the dual prehomogeneous vector space has basic relative invariants \( q_i(x) \) \((i = 0, 1, \ldots, n - 1)\) defined by

\[
q_i(x) = x_i \quad (i = 2, \ldots, n - 1).
\]
The other ones \( q_0(x) \) and \( q_1(x) \) are defined according to \( n \) is odd or even. If \( n \) is even, then

\[
q_0(x) = \begin{vmatrix}
x_{n+1} & x_2 & x_3 & \cdots & x_n \\
x_{n+3} & -x_1 & x_4 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
x_{2n+1} & 0 & -x_{n-3} & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & -x_{n-1}
\end{vmatrix},
q_1(x) = \begin{vmatrix}
x_{n+2} & x_3 & x_4 & \cdots & x_n \\
x_{n+4} & -x_2 & x_5 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
x_{2n+1} & 0 & -x_{n-4} & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & -x_{n-2}
\end{vmatrix}
\]

and if \( n \) is odd, then

\[
q_0(x) = \begin{vmatrix}
x_{n+1} & x_2 & x_3 & \cdots & x_n \\
x_{n+3} & -x_1 & x_4 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
x_{2n+1} & 0 & -x_{n-4} & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & -x_{n-1}
\end{vmatrix},
q_1(x) = \begin{vmatrix}
x_{n+2} & x_3 & x_4 & \cdots & x_n \\
x_{n+4} & -x_2 & x_5 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
x_{2n+1} & 0 & -x_{n-3} & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & -x_{n-1}
\end{vmatrix}
\]

**Remark 5.7.** Let \( n = 3 \). In this case, we have

\[
p(x) = p_1(x) = x_1x_4x_5 + x_2x_5x_6 + x_3x_6x_7 \quad (x \in V),
\]

and the other basic relative invariants are \( p_2(x) = x_5 \) and \( p_3(x) = x_6 \). Let us change variables as follows.

\[
a_{12} = \frac{x_3 - x_7}{2}, \quad b_{12} = \frac{x_3 + x_7}{2\sqrt{-1}}, \quad a_{13} = \frac{x_1 - x_4}{2}, \quad b_{13} = \frac{x_1 + x_4}{2\sqrt{-1}}
\]

Then, the polynomials \( p_i(x) \) \((i = 1, 2, 3)\) are transferred to

\[
p_1(z) = z_{11}z_{22}z_{33} - z_{22}(a_{13}^2 + b_{13}^2) - z_{33}(a_{12}^2 + b_{12}^2), \quad p_2(z) = z_{22}, \quad p_3(z) = z_{33}.
\]

These three polynomials \( p_i(z) \) \((i = 1, 2, 3)\) are exactly all the basic relative invariants of a homogeneous open convex cones \( \Omega \) defined by

\[
\Omega := \left\{ Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12} & z_{22} & 0 \\ z_{13} & 0 & z_{33} \end{pmatrix} \in \text{Herm}(3, \mathbb{C}); \det Z > 0, z_{22}, z_{33} > 0 \right\}.
\]

This cone \( \Omega \) is a typical example of non-symmetric homogeneous open convex cone, which can be viewed as a generalization of the so-called Vinberg cone. Therefore, the corresponding prehomogeneous vector space associated with \( p(x) \) is linearly isomorphic to the prehomogeneous vector space obtained from the cone \( \Omega \).

Note that the excluded case \( n = 2 \) is calculated in Example 3.1.
5.3 Circular cases

Let $\mathcal{T}$ be a triangle arrangement which is constructed by arranging $n$ triangles circularly with $n \geq 5$ (see (3) in Figure 2). In this case, the number of vertices are $2n$, that is, $\dim V = 2n$. Let $\varphi_n(k) := k \mod n \in \{1, \ldots, n\}$. The $i$-th triangle consists of vertices $\{i, n + i, n + \varphi_n(i + 1)\}$. Then, the corresponding polynomial $p(x)$ is described as

$$p(x) = \sum_{i=1}^{n} x_{i} x_{n+i} x_{n+\varphi_n(i+1)}.$$

Let $E_{ij}$ be a matrix unit having 1 on the position $(i, j)$ and zeros elsewhere.

**Lemma 5.8.** A general element $M$ in $\mathfrak{g}[p]$ is of the form

$$M = \left( \begin{array}{cc} \text{diag}(t_0 - t_i - t_{\varphi_n(i+1)}) & X \\ 0 & \text{diag}(t_i) \end{array} \right)$$

where $X$ is an $n \times n$ matrix defined by

$$X = \sum_{i=1}^{n} X_{i+1,i} (E_{\varphi_n(i+1),i} - E_{i,\varphi_n(i+2)}).$$

**Proof.** Since the triangle arrangement $\mathcal{T}$ does not have edge sharing, we can apply discussion in Section 3. Let $j = 1, \ldots, n$. Then, the vertex $j$ is isolated, and the equation (3) can be written as

$$M_{j,n+\varphi_n(j+2)} + M_{\varphi_n(j+1),n+j} = 0 \quad (j = 1, \ldots, n).$$

The other terms are all zeros so that the proof is now completed.

This lemma implies that $\mathfrak{g}[p]$ is a solvable Lie algebra with $\dim \mathfrak{g}[p] = 2n + 1$.

**Lemma 5.9.** Let $n \geq 5$. The dual triplet $(\mathfrak{g}[p], d\rho^*, V)$ is also a prehomogeneous vector space if and only if $n$ is an odd number. If $n$ is odd, then its basic relative invariants are given as

$$q_0(x) = \sum_{i=1}^{n} x_{i} x_{n+i} \prod_{j=0}^{k} x_{\varphi_n(i+2j)}, \quad q_i(x) = x_i \quad (i = 1, \ldots, n).$$

Here, indices of $x$ in the product symbol run through $1, \ldots, n$ modulo $n$.

**Proof.** We use Lemma 1.2. For simplicity, we set $y_i := x_{n+i}$ ($i = 1, \ldots, n$) with indices being taken in $1, \ldots, n$ by modulo $n$. For $x \in V$, let $A(x) : \mathfrak{g}[p] \to V$ be a linear map defined by $A(x)M := d\rho(M)x$ ($M \in \mathfrak{g}[p]$). Then, since $\dim \mathfrak{g}[p] = \dim V + 1$, we see that $A(x)$ is a matrix of size $\dim V \times (\dim V + 1)$. By the above lemma, we have

$$A(x) = \begin{pmatrix} x_1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & x_{n-1} & 0 & 0 \\ 0 & \cdots & 0 & x_0 & 0 \\ 0 & \cdots & 0 & 0 & x_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & x_{n-1} \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

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where

\[
B = \begin{pmatrix}
  y_n & -y_3 & 0 & \cdots & \cdots & 0 \\
  0 & y_1 & -y_4 & \ddots & \cdots & \vdots \\
  \vdots & 0 & y_2 & \ddots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 & y_{n-2} & -y_1 \\
- y_2 & 0 & \cdots & \cdots & 0 & y_{n-1}
\end{pmatrix}.
\]

Let \( A'(x) \) be a square matrix obtained by removing the second column from \( A(x) \). Set

\[
B' = \begin{pmatrix}
  x_1 & -y_3 & 0 & \cdots & \cdots & 0 \\
  x_2 & y_1 & -y_4 & \ddots & \cdots & \vdots \\
  \vdots & 0 & y_2 & \ddots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  x_{n-1} & 0 & \cdots & 0 & y_{n-2} & -y_1 \\
  x_n & 0 & \cdots & \cdots & 0 & y_{n-1}
\end{pmatrix}.
\]

Then, we have

\[
\det A'(x) = y_1 \cdots y_n \times \det B'.
\]

By taking cofactor expansion on \( B' \) at the first column, the \( i \)-th row element is given as

\[
(-1)^{i+1} x_i \times \det \begin{pmatrix}
  -y_3 & 0 & \cdots & 0 \\
  y_1 & -y_4 & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & y_{i-2} & -y_{i+1}
\end{pmatrix}
\]

\[
\times \det \begin{pmatrix}
  y_i & -y_{i+3} & 0 \\
  0 & y_{i+1} & \cdots \\
  \vdots & \ddots & \ddots \\
  0 & \cdots & 0 & y_{n-1}
\end{pmatrix}
\]

\[
= (-1)^{i+1} x_i \times (-1)^{i-1} (y_3 y_4 \cdots y_{i+1}) \times (y_{i+1} y_{i+2} \cdots y_{n-1})
\]

\[
= x_i y_i y_{i+1} \times y_{i+2} \cdots y_{n-1}.
\]

Here, we ignore the first det when \( i = 1 \), and ignore the last det when \( i = n \), and recall that indices run through \( 1, \ldots, n \) modulo \( n \). This implies that

\[
\det A'(x) = y_1 y_2 y_3^2 y_4^2 \cdots y_{n-1}^2 y_n \sum_{i=1}^{n} x_i y_i y_{i+1} = y_1 y_2 y_3^2 y_4^2 \cdots y_{n-1}^2 y_n \times p(x),
\]

which shows that the triplet \((\mathfrak{g}, p, d\rho, V)\) is a prehomogeneous vector space, and it is easily verified that polynomials \( y_i = x_{n+i} \ (i = 1, \ldots, n) \) are basic relative invariants.

Now let us investigate the dual prehomogeneous vector space. Let us consider a linear map \( A^*(x): \mathfrak{h} \to V \ (x \in V) \) defined by \( A^*(x)M := d\rho(M)x \ (M \in \mathfrak{h}) \).
Then, $A^*(x)$ is a matrix of size $2n \times (2n + 1)$ given as

$$
A^*(x) = \begin{pmatrix}
  x_1 & -x_1 & -x_1 \\
  x_2 & -x_2 & -x_2 \\
  \vdots & \ddots & \ddots \\
  x_{n-1} & -x_{n-1} & -x_{n-1} \\
  x_n & -x_n & \cdots \\
  0 & y_1 & \cdots \\
  0 & y_2 & \cdots \\
  \vdots & \ddots & \ddots \\
  0 & \cdots & y_n \\
\end{pmatrix}
$$

Let us consider a rank of the following matrix.

$$
B(x) := \begin{pmatrix}
  x_1 & -x_1 & -x_1 \\
  x_2 & -x_2 & -x_2 \\
  \vdots & \ddots & \ddots \\
  x_{n-1} & -x_{n-1} & -x_{n-1} \\
  x_n & -x_n & \cdots \\
\end{pmatrix}
$$

If $n$ is even, then sum of columns 2, 4, \ldots, $n$ is equal to $(-(x_1, \ldots, x_n)$, and so

\[\text{is that of columns 3, 5, \ldots, } n + 1.\]

This means that the rank of $B(x)$ must be smaller than or equal to $n - 1$, which implies rank $A^*(x) \leq 2n - 1$. Since the size of $A^*(x)$ is $2n$, $A^*(x)$ cannot be full rank. Thus, if $n \geq 5$ is even, then the dual triplet $(g[p], dp^*, V)$ cannot be a prehomogeneous vector space.

Next, we assume that $n$ is odd. We shall calculate a determinant of a matrix obtained by removing the last column from $A^*(x)$.

$$
A'(x) = \begin{pmatrix}
  x_1 & -x_1 & -x_1 \\
  x_2 & -x_2 & -x_2 \\
  \vdots & \ddots & \ddots \\
  x_{n-1} & -x_{n-1} & -x_{n-1} \\
  x_n & -x_n & \cdots \\
  0 & y_1 & \cdots \\
  0 & y_2 & \cdots \\
  \vdots & \ddots & \ddots \\
  0 & \cdots & y_n \\
\end{pmatrix}
$$

At first, we add the columns 2, 3, \ldots, $n$ and twice of the column 1 to the column
\[ n + 1 \] to obtain
\[
\det A' (x) = \begin{vmatrix}
  x_1 & -x_1 & -x_1 & \cdots & 0 \\
  x_2 & -x_2 & \ddots & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  x_{n-1} & \ddots & \ddots & -x_{n-2} & 0 \\
  x_n & -x_n & 0 & \cdots & 0
\end{vmatrix}
\]

Thus, we see that \( \det A'(x) \) can be factorized as a product of
\[
A_1 = \det \begin{vmatrix}
  x_1 & -x_1 & -x_1 & \cdots \\
  x_2 & -x_2 & \ddots & \cdots \\
  \vdots & \ddots & \ddots & \ddots \\
  x_{n-1} & \ddots & \ddots & -x_{n-2} \\
  x_n & -x_n & 0 & \cdots 
\end{vmatrix}
\]

and
\[
A_2 = \det \begin{vmatrix}
  y_1 & x_2 & 0 & \cdots & 0 \\
  y_2 & 0 & x_3 & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  y_{n-1} & 0 & \ddots & \ddots & x_n \\
  y_n & x_n & 0 & \cdots & 0
\end{vmatrix}
\]

Let us calculate \( A_1 \) and \( A_2 \). Set
\[
B_n = \det \begin{vmatrix}
  1 & 1 & 0 \\
  1 & 1 & -1 \\
  \vdots & \ddots & \ddots \\
  1 & -1 & -1 \\
  1 & 0 & -1
\end{vmatrix}
\]

Then, \( A_1 \) can be expressed, up to signature, as
\[
A_1 = x_1 \cdots x_n \cdot \det \begin{vmatrix}
  1 & -1 & -1 \\
  1 & -1 & \ddots \\
  \vdots & \ddots & \ddots \\
  1 & -1 & -1 \\
  1 & -1 & 0
\end{vmatrix} = (\text{sgn}) x_1 \cdots x_n \cdot B_n.
\]

By a cofactor expansion along the last row, we see that \( B_n \) can be calculated as
\[
B_n = (-1)^n + (-1)^{n-1} + (-1) \cdot B_{n-1} = 1 - B_{n-1}
\]
\[
= 1 - ((-1)^{n-1} + (-1)^{n-2} + (-1) \cdot B_{n-2})
\]
\[
= B_{n-2} = \cdots = B_3 = 1,
\]

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whence
\[ A_1 = (\text{sgn}) \, x_1 \cdots x_n. \]

Next we consider \( A_2 \). Let \( n = 2k + 1 \). By changing an order of columns as \( 1, 2, 4, \ldots, 2k \) and then \( 3, 5, \ldots, 2k + 1 \) and that of rows as \( 1, 3, \ldots, 2k + 1 \) and then \( 2, 4, \ldots, 2k \) to obtain

\[
A_2 = \begin{vmatrix}
  y_1 & x_2 & -x_1 & x_4 \\
  y_3 & -x_1 & x_4 & \\
  \vdots & \ddots & \ddots & \\
  y_{2k-1} & -x_{2k-3} & x_{2k} & -x_{2k-1} \\
  y_{2k+1} & x_3 & -x_2 & x_5 \\
  \vdots & \ddots & \ddots & \\
  y_2 & x_3 & -x_2 & x_5 \\
  y_4 & x_3 & -x_2 & x_5 \\
  \vdots & \ddots & \ddots & \\
  y_{2k} & x_3 & -x_2 & x_5
\end{vmatrix}.
\]

By using a cofactor expansion along the first column, we obtain

\[
A_2 = \sum_{i=1}^{k+1} (-1)^i y_2 x_{2i-1} \prod_{j=1}^{i-1} x_{2j} \prod_{j=1}^{k} x_{2j+1} + \sum_{i=1}^{k} (-1)^{k+i} y_2 x_{2i} \prod_{j=1}^{k} x_{2j-1} \prod_{j=i}^{i-1} (-x_{2j}) \prod_{j=1}^{2k+1} x_{2j+1} = (-1)^k x_3 x_5 \cdots x_{2k-1} q(x, y),
\]

where \( q \) is a homogeneous polynomial of degree \( k + 2 \) defined by

\[
q(x, y) := \sum_{i=1}^{k+1} y_2 x_{2i-1} x_{2i+1} \cdots x_{2k+1} x_2 x_4 \cdots x_{2i-2} + \sum_{i=1}^{k} y_2 x_{2i} x_{2i+2} x_{2k} x_1 x_3 \cdots x_{2i-1}.
\]

Since \( y_i = x_{n+i} \), we see that

\[
q(x, y) = \sum_{i=1}^{n} x_{n+i} \prod_{j=0}^{k} x_{\varphi_0(i+2j)} = q_0(x)
\]
and thus a generic rank of \( A^* (x) \) is full so that the dual triplet \((g[p], d\rho^*, V)\) is a prehomogeneous vector space. By structure of \( g[p] \), it is easily verified that polynomials \( q_i(x) = x_i \) \((i = 1, \ldots, n)\) are relatively invariant under the action of \( g[p] \). The proof is now completed. \( \square \)

**Remark 5.10.** Let \( n = 2k + 1 \) be an odd number. Then, \( q_0(x) \) is a homogeneous polynomial of degree \( k + 2 \) \(= \frac{n+3}{2} \). This is an example that, for a given arbitrary integer \( N \), we can construct a relative invariant of degree \( N \) of a prehomogeneous vector space.

In what follows, we deal with cases \( n = 3, 4 \). Both cases correspond to prehomogeneous vector spaces, and general elements \( M \) of \( g[p] \) are given for the
case \( n = 3 \) as
\[
M = \begin{pmatrix}
  t_0 - t_1 & 0 & 0 & X_1 & 0 & 0 \\
  0 & t_0 - t_2 & 0 & 0 & X_2 - X_1 & 0 \\
  0 & 0 & t_0 - t_3 & 0 & 0 & -X_2 \\
  0 & 0 & 0 & t_1 & 0 & 0 \\
  0 & 0 & 0 & 0 & t_2 & 0 \\
  0 & 0 & 0 & 0 & 0 & t_3 \\
\end{pmatrix},
\]
and for the case \( n = 4 \) as
\[
M = \begin{pmatrix}
  t_0 - t_1 - t_2 & Y_{12} & Y_{13} & Y_{14} & 0 & 0 & -X_{21} & 0 & 0 & 0 & 0 & X_{14} \\
  Y_{12} & t_0 - t_2 - t_3 & 0 & 0 & 0 & 0 & Y_{23} & 0 & 0 & 0 & 0 & 0 \\
  Y_{13} & 0 & t_0 - t_3 - t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  Y_{14} & 0 & 0 & t_0 - t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Note that both cases are excluded from detailed discussion in Section 3. In particular, the matrices of case \( n = 4 \) has the same (1,2) block of the case \( n \geq 5 \), but it has additional variables \( Y_{ij} \) in diagonal blocks. Moreover, for the case \( n = 4 \), the prehomogeneous vector space \((\mathfrak{g}[p], d\rho, V)\) is not regular, but its dual \((\mathfrak{g}[p], d\rho^*, V)\) is also a prehomogeneous vector space with a unique basic relative invariant
\[
q_0(x) = x_1x_3 - x_2x_4 \quad (x \in V).
\]
We are also able to confirm that \((\mathfrak{g}[p], d\rho, V)\) is not regular by checking that not all variables appear in \(q_0(x)\).

### 5.4 Edge gluing cases

Let \( T \) be a triangle arrangement constructed by gluing \( n \) triangle arrangements \( T_B \) at edges 24 and 36. Note that \( \dim V = 4n + 2 \). To each vertex, we attach variables \( x_i, y_i (i = 0, 1, \ldots, n) \) and \( z_j, w_j (j = 1, \ldots, n) \) as in Figure 3. Then, the corresponding polynomial \( p(x) \) is described as
\[
p(x) = \sum_{i=1}^{n} \left( x_i x_{i+1} z_i + x_i y_{i+1} w_i + x_{i+1} y_i w_i \right) \quad (x = (x, y, z, w) \in V).
\]

The order of basis of \( V \) is \( x, w, y \) and \( z \). Let \( E_{ij}^{(m)} \) (resp. \( E_{ij}^{(n)} \)) be a matrix unit of size \( m \times m \) (resp. \( n \times (n + 1) \)) having one on the position \((i, j)\) and zeros elsewhere.

**Lemma 5.11.** A general element \( M \) in \( \mathfrak{g}[p] \) is of the form
\[
M = \begin{pmatrix}
  M^{xx} & 0 & 0 & 0 \\
  M^{wx} & M^{ww} & 0 & 0 \\
  M^{yz} & 0 & M^{yy} & 0 \\
  M^{zz} & M^{zw} & M^{zy} & M^{zz} \\
\end{pmatrix}.
\]
where $M^{ab}$ ($a, b = x, y, z, w$) are given as

\[
\begin{align*}
M^{xx} &= \text{diag}(t_i)_{i=1}^{n+1}, \\
M^{ww} &= uI_n, \\
M^{yy} &= \text{diag}(t - u - t_i)_{i=1}^n, \\
M^{zz} &= \text{diag}(t - t_i - t_{i+1})_{i=1}^n,
\end{align*}
\]

\[
M^{wx} = \sum_{i=1}^{n} d_i (E'_{i,i+1} - E'_{i+1,i+1}), \\
M^{zy} = \sum_{i=1}^{n} d_i (E'_{i,i+1} - E'_{i+1,i}), \\
M^{yx} = \sum_{i=1}^{n} b_i E_{i,i+1} + c_i E_{i+1,i} + (b_i + c_i)E_{i,i+1} + c_i E_{i+1,i+1}, \\
M^{zw} = -\sum_{i=1}^{n} b_i E_{i,i} + (b_i + c_i)E_{i+1,i} + c_i E_{i,i+1} + \sum_{i=1}^{n} a_i (E'_{i,i+2} - E'_{i+1,i+1}).
\]

We shall give a proof of this lemma at the end of this subsection. This lemma shows that $g[p]$ is a solvable Lie algebra and $\dim g[p] = 5n + 1$.

**Lemma 5.12.** Let $n \geq 2$. Then, the triplet $(g[p], d\rho, V)$ is a regular prehomogeneous vector space. Its basic relative invariants are described as

\[
p_i(x) = x_i \quad (i = 0, 1, \ldots, n), \quad p_{n+1}(x) = w_1 + \cdots + w_n, \quad p_{n+2}(x) = p(x).
\]

**Proof of Lemma 5.12** We shall calculate directly by using [1]. The gradients of $p(x)$ can be calculated as

\[
\nabla_x p(x) = (x_1 z_1 + y_0 w_1, x_{i-1} z_i + y_i w_i + x_{i+1} z_{i+1} + y_{i+1} w_{i+1}, w_n y_n + x_{n-1} z_n)
\]

\[
\nabla_y p(x) = (x_0 w_1, x_i w_i + x_{i+1} w_{i+1}, x_n w_n)
\]

\[
\nabla_z p(x) = (x_i^{-1} x_i)
\]

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\[ \nabla_w p(x) = (x_{i-1}y_{i-1} + x_iy_i) \]

Thus, \( \langle M \mid \nabla_w p(x) \rangle \) can be expanded as

\[
\langle M \mid \nabla_w p(x) \rangle = \sum M_{ij}^{zb} b_j x_{i-1} x_i + \sum M_{ij}^{wb} b_j (x_{i-1} y_{i-1} + x_i y_i) + \sum M_{ij}^{zb} b_j (x_{i-1} z_{i-1} + y_{i-1} w_{i-1} + x_{i+1} z_{i+1} + y_{i+1} w_{i+1}) + \sum M_{ij}^{wb} b_j (x_i w_{i-1} + x_{i+1} w_{i+1}).
\]

Here, the summation symbol \( \sum \) runs over all possible \( i, j \) and \( b = x, y, z, w \).

Among \( M_{ij}^{ab} \) \( (a, b = x, y, z, w) \), candidates such that \( M_{ij}^{ab} \neq 0 \) are as follows:

\[
M_{i-1,i}^{xz}, \quad M_{i,i+1}^{yx}, \quad M_{i-1,i}^{yz}, \quad M_{i,i+1}^{zx}, \quad M_{i,i+1}^{zy} \]

Here, we set \( s = 0, 1, \ldots, n, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n - 1. \)

We shall calculate for each monomial.

1. \( (M_{i-1,i}^{xz} + M_{i-2,i}^{xz}) x_{i-1} x_i z_{i-1} \quad (i = 2, \ldots, n) \)
   
   Since \( M_{i-1,i}^{xz} = 0 \) we have \( M_{i-1,i}^{xz} = 0. \)

2. \( (M_{i,i+1}^{xz} + M_{i+1,i}^{xz}) x_{i-1} x_i z_{i+1} \quad (i = 1, \ldots, n - 1) \)
   
   Since \( M_{i,i+1}^{xz} = 0, \) we have \( M_{i,i+1}^{xz} = 0. \)

3. \( (M_{i-2,i}^{xz} + M_{i-1,i}^{xz}) x_{i-2} x_{i-1} x_i \quad (i = 2, \ldots, n) \)
   
   We have \( M_{i-2,i}^{xz} + M_{i-1,i}^{xz} = 0. \)

4. \( (M_{i-1,i}^{zw} + M_{i-1,i}^{zy}) x_{i-1} x_i w_{i-1} \quad (i = 2, \ldots, n) \)
   
   We have \( M_{i-1,i}^{zw} + M_{i-1,i}^{zy} = 0. \)

5. \( (M_{i,i+1}^{zw} + M_{i+1,i}^{zw}) x_{i-1} x_i w_{i+1} \quad (i = 1, \ldots, n) \)
   
   We have \( M_{i,i+1}^{zw} + M_{i+1,i}^{zw} = 0. \)

6. \( (M_{i-1,i}^{ww} + M_{i-1,i}^{ww}) x_{i-1} x_i w_{i-1} \quad (i = 1, \ldots, n - 1) \)
   
   We have \( M_{i-1,i}^{ww} + M_{i-1,i}^{ww} = 0. \)

7. \( (M_{i-1,i}^{ww} + M_{i+1,i}^{ww}) x_{i-1} x_i y_{i-1} \quad (i = 1, \ldots, n) \)
   
   Since \( M_{i-1,i}^{ww} = 0, \) we have \( M_{i+1,i}^{ww} = 0. \)

8. \( (M_{i,i+1}^{ww} + M_{i+1,i}^{ww}) x_{i-1} x_i y_{i} \quad (i = 1, \ldots, n) \)
   
   Since \( M_{i+1,i}^{ww} = 0, \) we have \( M_{i,i+1}^{ww} = 0. \)
9. \((M_{i-1,i}^{zz} + M_{i-1,i}^{zz})x_{i-1}z_{i-1}z_{i} (i = 2, \ldots, n)\)
   We have \(M_{i-1,i}^{zz} = M_{i-1,i}^{zz} = 0\)

10. \((M_{i,i}^{zw} + M_{i,i}^{zw})x_{i-1}z_{i-1}w_{i-1} (i = 2, \ldots, n)\)
    Since \(M_{i-1,i}^{zw} = 0\), we have \(M_{i-1,i}^{zw} = 0\)

11. \((M_{i,i}^{zw} + M_{i-1,i}^{yz})x_{i-1}z_{i}w_{i} (i = 1, \ldots, n)\)
    Since \(M_{i,i}^{zw} = 0\), we have \(M_{i-1,i}^{yz} = 0\)

12. \((M_{i,i}^{zw} + M_{i-1,i}^{yz} + M_{i,i}^{wz})x_{i-1}y_{i-1}z_{i} (i = 1, \ldots, n)\)
    Since \(M_{i-1,i}^{yz} = 0\), we have \(M_{i,i}^{wz} = 0\)

13. \((M_{i-1,i}^{zw} + M_{i-1,i}^{yz})x_{i}z_{i}w_{i} (i = 1, \ldots, n)\)
    Since \(M_{i-1,i}^{zw} = 0\), we have \(M_{i-1,i}^{yz} = 0\)

14. \((M_{i-1,i+1}^{zw} + M_{i-1,i+1}^{yz})x_{i}z_{i}w_{i+1} (i = 1, \ldots, n-1)\)
    Since \(M_{i-1,i+1}^{zw} = 0\), we have \(M_{i-1,i+1}^{yz} = 0\)

15. \((M_{i+1,i}^{zw} + M_{i+1,i}^{yz} + M_{i+1,i}^{wz})x_{i}y_{i}z_{i} (i = 1, \ldots, n)\)
    Since \(M_{i+1,i}^{zw} = 0\), we have \(M_{i+1,i}^{yz} + M_{i+1,i}^{wz} = 0\)

16. \((M_{i,i-1}^{yw} + M_{i-1,i}^{yw})x_{i-1}w_{i-1}w_{i} (i = 2, \ldots, n)\)
    We have \(M_{i,i-1}^{yw} = M_{i-1,i}^{yw} = 0\)

17. \((M_{i-1,i}^{yw} + M_{i-1,i}^{xz})x_{i-1}y_{i}w_{i} (i = 1, \ldots, n)\)
    We have \(M_{i-1,i}^{yw} = M_{i-1,i}^{xz} = 0\)

18. \((M_{i-1,i}^{yw} + M_{i-1,i}^{xy})y_{i-1}w_{i-1}w_{i} (i = 2, \ldots, n)\)
    We have \(M_{i-1,i}^{xy} = M_{i-1,i}^{yw} = 0\)

19. \((M_{i,i-1}^{wy} + M_{i-1,i}^{wy})y_{i-1}y_{i}w_{i} (i = 1, \ldots, n)\)
    We have \(M_{i-1,i}^{wy} = 0\) and \(M_{i,i-1}^{wy} = M_{i-1,i}^{yw} = 0\)

20. \((M_{i+1,i}^{wx} + M_{i-1,i}^{wx})x_{i-1}y_{i} (i = 1, \ldots, n+1)\)
    We have \(M_{i-1,i}^{wx} = M_{i,i+1}^{wx} = 0\) and \(M_{i-1,i+1}^{wx} = M_{i,i}^{wx} = 0\)

Diagonal entries

21. \((M_{i,i}^{xx} + M_{i,i}^{zz} + M_{i,i}^{zz})x_{i}x_{i}z_{i} (i = 1, \ldots, n)\)
    We have \(M_{i,i}^{xx} = M_{i,i}^{yy} = M_{i,i}^{zz} = t\)

22. \((M_{i,i}^{yy} + M_{i,i}^{zz} + M_{i,i}^{yy} + M_{i,i}^{zz})x_{i-1}y_{i-1}w_{i} (i = 1, \ldots, n)\)
    Since \(M_{i-1,i}^{yy} = 0\), we have \(M_{i-1,i}^{yy} + M_{i,i}^{xx} + M_{i,i}^{yy} = t\)

23. \((M_{i,i}^{yy} + M_{i,i}^{yy} + M_{i,i}^{yy} + M_{i,i}^{xx})x_{i}y_{i}w_{i} (i = 1, \ldots, n)\)
    Since \(M_{i,i}^{xx} = 0\), we have \(M_{i,i}^{yy} + M_{i,i}^{xx} = t\)
Summing up the above discussion, we see that 3 implies
\[ M_{i,i-2}^{zx} + M_{i-1,i}^{zx} = 0 \quad (i = 2, \ldots, n) \]
\[(\dim = n - 1), 4, 5, 6 \text{ imply } \]
\[ \begin{cases} M_{i,i-1}^{zw} + M_{i-1,i}^{yz} = 0 & (i = 2, \ldots, n) \\ M_{i,i}^{zw} + M_{i-1,i}^{xy} = 0 & (i = 1, \ldots, n) \\ M_{i,i+1}^{zw} + M_{i,i-1}^{xy} = 0 & (i = 1, \ldots, n - 1) \end{cases} \]
\[(\dim = 2n), 7, 8, 20 \text{ tell us that } \]
\[ \begin{cases} M_{i,i-1}^{zy} + M_{i-1,i}^{wx} = 0 & (i = 1, \ldots, n) \\ M_{i,i}^{zy} + M_{i-1,i}^{wx} = 0 & (i = 1, \ldots, n) \\ M_{i,i-1}^{zy} + M_{i,i-1}^{xy} = 0 & (i = 2, \ldots, n - 1) \end{cases} \]
\[(\dim = n - 1). \text{ Moreover, 12, 15, 19 yield that } \]
\[ \begin{cases} M_{i,i-1}^{xy} + M_{i-1,i}^{zw} = 0 & (i = 1, \ldots, n) \\ M_{i,i}^{xy} + M_{i-1,i}^{zw} = 0 & (i = 1, \ldots, n) \\ M_{i,i-1}^{xy} + M_{i,i-1}^{zw} = 0 & (i = 2, \ldots, n - 1) \end{cases} \]
\[(\dim = 0). \text{ Diagonal entries are } \]
\[ \begin{cases} M_{i,i-1}^{xx} + M_{i,i-1}^{yy} = t & (i = 1, \ldots, n) \\ M_{i-1,i-1}^{xx} + M_{i-1,i-1}^{yy} = t & (i = 1, \ldots, n) \\ M_{i,i}^{xx} + M_{i,i}^{yy} = t & (i = 1, \ldots, n) \end{cases} \]
\[(\dim = n + 3). \text{ The second and third lines shows that } M_{i,i}^{ww} \text{ does not depend on } i. \text{ Therefore, let us introduce new variables } \]
\[ a_i = M_{i,i+1}^{zx}, \quad d_i = M_{i,i}^{wx} \quad (i = 1, \ldots, n - 1), \quad b_j = M_{j+1,j}^{yx}, \quad c_j = M_{j,j+1}^{yx} \quad (j = 0, 1, \ldots, n - 1) \]
and
\[ t, \quad M_i^z = M_i^{zx} \quad (i = 0, 1, \ldots, n), \quad M^y = g_{00}^{yy} \]
as a basis. Then we have
\[ M_{i,i}^{ww} = t - M_0^z - y \quad (i = 1, \ldots, n), \quad M_{j,j}^{yy} = t - M_j^z - (t - g_0^x - y) = M_0^z - M_j^z + y \quad (j = 0, 1, \ldots, n). \]

An element \( M \in g[p] \) can be described as a block matrix form of
\[
M = \begin{pmatrix}
M^{xx} & 0 & 0 & 0 \\
M^{wx} & M^{ww} & 0 & 0 \\
M^{zx} & 0 & M^{yy} & 0 \\
M^{zz} & M^{zw} & M^{zw} & M^{zz}
\end{pmatrix}.
\]
where

\[
M^{yx} = \begin{pmatrix}
0 & M_{01}^{yx} & 0 & \cdots & 0 \\
M_{10}^{yx} & 0 & M_{12}^{yx} & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & M_{n-1,n-2}^{yx} & 0 & M_{n-1,n}^{yx} \\
0 & \cdots & 0 & M_{n,n-1}^{yx} & 0
\end{pmatrix} \in \text{Mat}(n+1; \mathbb{C})
\]

\[
= \begin{pmatrix}
0 & b_0 & 0 & \cdots & 0 \\
c_0 & 0 & b_1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & c_{n-2} & 0 & b_{n-1} \\
0 & \cdots & 0 & c_{n-1} & 0
\end{pmatrix}
\]

\[
M^{zx} = \begin{pmatrix}
0 & 0 & M_{12}^{zx} & 0 & \cdots & 0 \\
M_{20}^{zx} & 0 & 0 & \ddots & \ddots & \vdots \\
0 & M_{21}^{zx} & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & M_{n,n-2}^{zx} & 0 & 0 \\
0 & \cdots & 0 & 0 & M_{n,n-1}^{zx}
\end{pmatrix} \in \text{Mat}(n \times (n+1); \mathbb{C})
\]

\[
= \begin{pmatrix}
0 & 0 & M_{12}^{zx} & 0 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & M_{n,n-2}^{zx} & 0 & 0 \\
0 & \cdots & 0 & 0 & M_{n,n-1}^{zx}
\end{pmatrix}
\]

\[
M^{wx} = \begin{pmatrix}
M_{10}^{wx} & M_{11}^{wx} & 0 & \cdots & 0 \\
0 & M_{21}^{wx} & M_{22}^{wx} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & M_{n,n-1}^{wx} & M_{n,n}^{wx} \\
0 & d_1 & 0 & \cdots & 0 \\
0 & -d_1 & d_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -d_{n-2} & d_{n-1} \\
0 & \cdots & 0 & 0 & -d_{n-1}
\end{pmatrix} \in \text{Mat}(n \times (n+1); \mathbb{C})
\]
\[ M^{zy} = \begin{pmatrix}
    M_{10}^{zy} & M_{11}^{zy} & 0 & \cdots & 0 \\
    0 & M_{21}^{zy} & M_{22}^{zy} & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & M_{n,n-1}^{zy} & M_{nn}^{zy}
  \end{pmatrix} \in \text{Mat}(n \times (n+1); \mathbb{C}) \\
= \begin{pmatrix}
    -d_1 & 0 & \cdots & 0 & 0 \\
    0 & -d_2 & d_1 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & -d_{n-1} & d_{n-2} \\
    0 & \cdots & 0 & 0 & d_{n-1}
  \end{pmatrix}
\]

\[ M^{zw} = \begin{pmatrix}
    M_{11}^{zw} & M_{12}^{zw} & 0 & \cdots & 0 \\
    M_{21}^{zw} & M_{22}^{zw} & M_{23}^{zw} & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & M_{n-1,n-2}^{zw} & M_{n-1,n-1}^{zw} & M_{nn}^{zw}
  \end{pmatrix} \in \text{Mat}(n \times n; \mathbb{C}) \\
= \begin{pmatrix}
    -b_0 - c_0 & -c_0 & 0 & \cdots & 0 \\
    -b_1 & -b_1 - c_1 & 0 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & -b_{n-1} & -b_{n-2} - c_{n-2} & -c_{n-2} \\
    0 & \cdots & 0 & -b_{n-1} & -b_{n-1} - c_{n-1}
  \end{pmatrix}
\]

This proves the assertion.

**Proof of Lemma 5.12.** Let \( A(x)M := d\rho(M)x \). We set an order of \( g[p] \) by \( M_0^x, M_1^x, \ldots, M_n^x, M^y, t \), and then \( d_i, b_i, c_i, a_i \). Recall that the order of \( V \) is taken as \( x, w, y, z \). Then, \( A(x) \) can be described as a block matrix form as

\[
A(x) = \begin{pmatrix}
    D_1(x) & 0 & 0 & 0 & 0 \\
    D_2(w) & X_1 & 0 & 0 & 0 \\
    D_3(y) & 0 & X_2 & X_3 & 0 \\
    D_4(z) & Y_1 & W_1 & W_2 & X_4
  \end{pmatrix},
\]

where

\[
D_1(x) = \begin{pmatrix}
    x_0 & 0 & 0 & 0 \\
    \vdots & \ddots & 0 & 0 \\
    0 & x_n & 0 & 0
  \end{pmatrix}
\]

\[
D_2(w) = \begin{pmatrix}
    -w_1 & 0 & \cdots & 0 & -w_1 & w_1 \\
    \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
    -w_n & 0 & \cdots & 0 & -w_n & w_n
  \end{pmatrix}
\]

\[
D_3(y) = \begin{pmatrix}
    0 & 0 & \cdots & 0 & y_0 & 0 \\
    y_1 & -y_1 & 0 & y_1 & 0 & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    y_n & 0 & -y_n & y_n & 0
  \end{pmatrix}
\]

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Removing columns corresponding to $c_0, c_1, \ldots, c_{n-2}$ from $A(x)$, we obtain a square matrix $B(x)$ of size $4n + 2$ as

$$B(x) = \begin{pmatrix}
D_1'(x) & 0 & 0 & 0 & 0 & 0 & n + 1 \\
D_2'(w) & -w & w & X_1 & 0 & 0 & n \\
D_3'(y) & 0 & y & 0 & D_5(x) & 0 & n \\
D_4'(z) & 0 & z & Y_1 & W' & X_2 & n + 1 \\
\end{pmatrix}$$

where $D_i'$ are matrices obtained by removing the last two columns from $D_i$, and $D_5(x), W'$ are matrices defined by

$$D_5(x) = \text{diag}(x_1, x_2, \ldots, x_n, x_{n-1})$$

$$W' = \begin{pmatrix}
-w_1 \\
-w_1 - w_2 \\
\vdots \\
-w_{n-1} - w_n - w_n \\
\end{pmatrix}$$
Let us calculate \( \det B(x) \).

\[
\det B(x) = \det D'_1(x) \det \begin{pmatrix} -w & w & x_1 & 0 & 0 \\ 0 & y & 0 & D_5(x) & 0 \\ 0 & z & Y_1 & W' & X_4 \end{pmatrix} = (\text{sgn}) \det D'_1(x) \det \begin{pmatrix} w & x_1 & 0 & 0 \\ 0 & y & D_5(x) & 0 \\ 0 & z & Y_1 & W' & X_4 \end{pmatrix} = (\text{sgn}) \det D'_1(x) \det \begin{pmatrix} w & x_1 \\ z & W' & X_4 \end{pmatrix}.
\]

Then, we are able to continue a calculation on \( \det \begin{pmatrix} w & x_1 \\ z & W' & X_4 \end{pmatrix} \) as follows. Since a signature of its determinant does not affect to a result we want to prove, we omit to calculate signatures and write just (sgn) instead.

\[
\det \begin{pmatrix} w & x_1 \\ z & W' & X_4 \end{pmatrix} = (\text{sgn}) x_1 \cdots x_{n-1} (w_1 + \cdots + w_n).
\]

Recall a formula \( \det \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \det A \det (D - BA^{-1}C) \) when \( \det A \neq 0 \) for block matrices. Using this formula, we can proceed a calculation as follows.

\[
\det \begin{pmatrix} y & D_5(x) \\ z & W' & X_4 \end{pmatrix} = (\text{sgn}) \det \begin{pmatrix} y & D_5(x) \\ W' & z & X_4 \end{pmatrix} = (\text{sgn}) \det D_5(x) \det \left( (z'X_4 - W'D_5(x)^{-1}(y \ 0)) \right) = (\text{sgn}) \det D_5(x) \det (z'|X_4),
\]

where we set

\[
z' = \begin{pmatrix} z_1 + w_1 \frac{y n}{x_1} \\ z_2 + (w_1 + w_2) \frac{y n}{x_2} \\ \vdots \\ z_{n-1} + (w_{n-2} + w_{n-1}) \frac{y n}{x_{n-1}} \\ z_n + (w_{n-1} + w_n) \frac{y n}{x_n} + w_n \end{pmatrix}.
\]

Thus, we obtain

\[
\det(X_2|z') = \det \begin{pmatrix} z'_1 & x_2 & \cdots & x_3 \\ z'_2 & -x_0 & x_1 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z'_n & -x_{n-3} & \cdots & -x_{n-2} \\ z'_n & \end{pmatrix} = \sum_{i=1}^{n} (-1)^{i+1} z'_i \prod_{j=1}^{n-2} (-x_j) \prod_{k=2}^{i} x_k = x_2 \cdots x_{n-2} \sum_{i=1}^{n} x_{i-1}z'_i \\
= x_2 \cdots x_{n-2} \left( x_0 x_1 z_1 + x_0 w_1 y_0 + \sum_{i=1}^{n} (x_{i-1} x_i z_i + x_{i-1} w_{i-1} y_{i-1} + x_{i-1} w_i y_{i-1}) + x_n w_n y_n \right) = x_2 \cdots x_{n-2} p(x)
\]
Summing up the above calculation, we have obtained
\[
\det B(x) = (\text{sgn}) x_0 x_1^3 (x_2 \cdots x_{n-1})^4 x_n^2 (w_1 + \cdots + w_n) p(x).
\]
This shows that a general rank of \( A(x) \) is equal to \( 4n + 2 \), which implies that the triplet \((\mathfrak{g}[p], \rho, V)\) is a prehomogeneous vector space. By a structure of \( \mathfrak{g}[p] \), it is easily verified that polynomials
\[
x_0, x_1, \ldots, x_n \quad \text{and} \quad w_1 + \cdots + w_n.
\]
which are irreducible factors of \( \det B(x) \), are relatively invariant under the action of \( \mathfrak{g}[p] \).

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Figure 4: A table of triangle arrangements obtained by reduction of triangulation of \( n \)-polygon up to \( n \leq 10 \).