A MAZUR–ULAM THEOREM IN NON-ARCHIMEDEAN NORMED SPACES

MOHAMMAD SAL MOSLEHIAN AND GHADIR SADEGHI

Abstract. The classical Mazur–Ulam theorem which states that every surjective isometry between real normed spaces is affine is not valid for non-Archimedean normed spaces. In this paper, we establish a Mazur–Ulam theorem in the non-Archimedean strictly convex normed spaces.

1. Introduction and preliminaries

A non-Archimedean field is a field $K$ equipped with a function (valuation) $|·|$ from $K$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. An example of a non-Archimedean valuation is the mapping $|·|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial.

In 1897, Hensel [3] discovered the $p$-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number $p$. For any nonzero rational number $x$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where $a$ and $b$ are integers not divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the $p$-adic number field; see [15]. During the last three decades $p$-adic numbers have gained the interest of physicists for their research in particular in problems coming from quantum physics, $p$-adic strings and superstrings (cf. [4]).

Let $\mathcal{X}$ be a vector space over a scalar field $K$ with a non-Archimedean valuation $|·|$. A function $\|·\| : \mathcal{X} \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|$ \hspace{1cm} (r \in K, x \in \mathcal{X});

(iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \hspace{1cm} (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, \|·\|)$ is called a non-Archimedean normed space. A non-Archimedean normed space over a valued field $K$ with $|2| = 1$ satisfying $\|\mathcal{X}\| := \{\|x\| : x \in \mathcal{X}\} = \{|r| : r \in K\}$ is called strictly convex if $\|x + y\| = \max\{\|x\|, \|y\|\}$ and $\|x\| = \|y\|$ imply $x = y$. The assumption $|2| = 1$ is necessary in order a space $\mathcal{X}$ to be satisfies the implication. A non-Archimedean normed space is called spherically complete if every collection of closed balls in $\mathcal{X}$ which is totally ordered by inclusion has a non-empty intersection [15]. Every spherically complete space is complete but the converse is not true in general. The notion of spherical completeness is more

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suitable than the notion of completeness in the study of non-Archimedean spaces. Theory of non-Archimedean normed spaces is not trivial, for instance there may not be any unit vector. Although many results in classical normed space theory have a non-Archimedean counterpart, but their proofs are essentially different and require an entirely new kind of intuition, cf. [6, 7].

A valued space is a non-Archimedean normed linear space over a trivially field with characteristic 0. A \( V \)-space \( X \) is a valued space which is complete in its norm topology such that

\[ \|x\| \subset \{0\} \cup \{\rho^n : n \in \mathbb{Z}\} \]

for some real number \( \rho > 1 \) [11].

The theory of isometric mappings had its beginning in the classical paper [5] by S. Mazur and S. Ulam, who proved that every isometry of a real normed vector space onto another real normed vector space is a linear mapping up to translation. The property is not true for normed complex vector spaces (for instance, consider the conjugation on \( \mathbb{C} \)). The hypothesis surjectivity is essential. Without this assumption, J. A. Baker [2] proved that every isometry from a normed real space into a strictly convex normed real space is a linear up to translation. A number of mathematicians have had deal with Mazur–Ulam theorem; see [9, 10] and references therein. Mazur–Ulam Theorem is not valid in the contexts of non-Archimedean normed spaces, in general. As a counterexample take \( \mathbb{R} \) with the trivial non-Archimedean valuation and define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^3 \). Then \( f \) is clearly a surjective isometry and \( f(0) = 0 \), but \( f \) is not linear.

In this paper, by using some ideas of [2], we establish a Mazur–Ulam type theorem in the framework of strictly convex non-Archimedean normed spaces. We also provide an example to show that the assumption of strict convexity is essential.

2. Main results

**Lemma 2.1.** Let \( X \) be a non-Archimedean normed space over a valued filed \( K \) which is strictly convex and let \( x, y \in X \). Then \( \frac{x + y}{2} \) is the unique member of \( X \) which is of distance \( \|x - y\| \) from both \( x \) and \( y \).

**Proof.** There is nothing to prove if \( x = y \). Let \( x \neq y \). The point \( \frac{x + y}{2} \) is of distance \( \|x - y\| \) from both \( x \) and \( y \), since

\[
\|x - \frac{x + y}{2}\| = \left\| \frac{x - y}{2} \right\| = \|x - y\|;
\]

\[
\|y - \frac{x + y}{2}\| = \left\| \frac{x - y}{2} \right\| = \|x - y\|.
\]

Assume that \( z, t \in X \) with

\[
\|x - z\| = \|x - t\| = \|y - z\| = \|y - t\| = \|x - y\|.
\]

Then

\[ (2.1) \quad \|x - \frac{z + t}{2}\| \leq \max\{\|\frac{x - z}{2}\|, \|\frac{x - t}{2}\|\} = \|x - y\|. \]

Similarly

\[ (2.2) \quad \|y - \frac{z + t}{2}\| \leq \|x - y\|. \]
If both of inequalities (2.1) and (2.2) were strict we would have
\[ \|x - y\| \leq \max\{\|x - \frac{z + t}{2}\|, \|y - \frac{z + t}{2}\|\} < \|x - y\|, \]
a contradiction. So at least one of the equalities holds in (2.1). Without lose of generality assume that equality holds in (2.1). Then \( \|x - \frac{z + t}{2}\| = \max\{\|\frac{x - z}{2}\|, \|\frac{y - z}{2}\|\} \). By strictly convexity we obtain \( \frac{z}{2} = \frac{t}{2} \) that is \( z = t \). \( \square \)

**Theorem 2.2.** Suppose that \( \mathcal{X} \) and \( \mathcal{Y} \) are non-Archimedean normed spaces and \( \mathcal{Y} \) is strictly convex. If \( f: \mathcal{X} \to \mathcal{Y} \) is an isometry, then \( f - f(0) \) is additive.

**Proof.** Let \( g(x) = f(x) - f(0) \). Then \( g \) is an isometry and \( g(0) = 0 \).

\[ \|g\left(\frac{x + y}{2}\right) - g(x)\| = \|\frac{x + y}{2} - x\| = \|x - y\| = \|g(x) - g(y)\| \quad (x, y \in \mathcal{X}) \]

and similarly
\[ \|g\left(\frac{x + y}{2}\right) - g(y)\| = \|g(x) - g(y)\| \quad (x, y \in \mathcal{X}) \]

It follows from Lemma 2.1 that
\[ g\left(\frac{x + y}{2}\right) = \frac{g(x) + g(y)}{2} \]

Hence \( g = f - f(0) \) is additive. \( \square \)

**Remark 2.3.** If \( \mathcal{K} = \mathbb{Q}_p \) with the valuation \( |.|_p \), where \( p \neq 2 \) then \( f - f(0) \) in Theorem 2.2 is a linear mapping.

**Example 2.4.** Let \( \mathcal{Y} \) be a \( V \)-space over a trivially valued field \( \mathcal{K} \) such that \( 1 \in \|\mathcal{Y}\| \) i.e there exists \( y_0 \in \mathcal{Y} \) with \( \|y_0\| = 1 \). Clearly \( |2| = 1 \) and \( \mathcal{Y} \) is not strictly convex. Now define the mapping \( f: \mathcal{K} \to \mathcal{Y} \) by \( f(\alpha) = \alpha^2 y_0 \). Then \( f \) is an isometry and \( f(0) = 0 \) but \( f \) is not additive. Therefore the assumption that \( \mathcal{Y} \) is strictly convex cannot be omitted in Theorem 2.2.

It is well know that \( \mathbb{Q}_2 \) is spherically complete (see [12]) and \( |2|_2 \neq 1 \). Let \( f: \mathbb{Q}_2 \to \mathbb{Q}_2 \) be defined by
\[ f(x) = \begin{cases} 1/x & x = \frac{a}{2^b} \text{ for some } a \neq 0, b \text{ with } (a, b) = 1 \\ x & \text{otherwise} \end{cases} \]

Then \( f \) is an isometry (note that in a non-Archimedean field \( |a - b| = \max\{|a|, |b|\} \) for all \( a, b \) with \( |a| \neq |b| \)), \( f(0) = 0 \) but \( f \) is not an additive mapping. Regarding this fact and using some ideas of [13], we have the next result which seems to be interesting on its own right.

**Proposition 2.5.** Suppose that \( \mathcal{X} \) and \( \mathcal{Y} \) are non-Archimedean normed spaces on a non-Archimedean field \( \mathcal{K} \) with \( |k| \neq 1 \) for some \( k \in \mathbb{N} \). Assume that \( \mathcal{X} \) or \( \mathcal{Y} \) is spherically complete. If \( f: \mathcal{X} \to \mathcal{Y} \) is a surjective isometry, then for each \( u \in \mathcal{X} \) there exists a unique \( v \in \mathcal{X} \) such that \( f(u) + f(v) = f\left(\frac{u + v}{2}\right) \).

**Proof.** We first show that both spaces are spherically complete. Let \( \mathcal{X} \) be a spherically complete and \( \mathcal{B} \) is a collection of closed balls in \( \mathcal{Y} \) which is totally ordered by inclusion. Since \( f \) is a surjective isometry, \( f^{-1}(\mathcal{B}) \) is a collection of closed balls in \( \mathcal{X} \) which is totally ordered by inclusion. Thus \( \cap f^{-1}(\mathcal{B}) \neq \emptyset \). Therefore we have \( \cap \mathcal{B} \neq \emptyset \). Similarly one can prove that if \( \mathcal{Y} \) is spherically complete then so is \( \mathcal{X} \).
Let \( u \in X \). The mapping \( \varphi : X \to X \) defined by \( \varphi(x) := kx - u \) is a contractive mapping since
\[
\|\varphi(x) - \varphi(y)\| = \|kx - ky\| = |k|\|x - y\| < \|x - y\|.
\]
Define the isometry mapping \( \psi : Y \to Y \) by \( \psi(y) := f(u) + y \).
The mapping \( h := \varphi f - \psi f \) is a contractive mapping on \( X \), since
\[
\|h(x) - h(y)\| = \|\varphi f^{-1} \psi f(x) - (\varphi f^{-1} \psi f)(y)\|
\leq \|(f^{-1} \psi f)(x) - (f^{-1} \psi f)(y)\|
= \|(\psi f)(x) - (\psi f)(y)\|
= \|f(x) - f(y)\|
\]
(2.3)
By [8, Theorem 1] \( h \) has a unique fixed point \( v \). Then we have
\[
(\varphi f^{-1} \psi f)(v) = h(v) = v = \varphi \left( \frac{u + v}{k} \right)
\]
Therefore \( \psi(f(v)) = f \left( \frac{u + v}{k} \right) \), since \( \varphi \) is one to one. Hence \( f(u) + f(v) = f \left( \frac{u + v}{k} \right) \).

Finally, suppose that \( E \) and \( F \) are metric spaces and \( f : E \to F \) is a mapping. A real number \( r > 0 \) is called a conservative distance for \( f \) if \( d(x,y) = r \) implies \( d(f(x),f(y)) = r \). In 1970, A. D. Aleksandrov [1] posed the following problem: “Under what conditions is a mapping preserving a distance \( r \) an isometry?” This problem is not easy to solve even in the case where \( E \) and \( F \) are normed spaces. A number of mathematicians have discussed Aleksandrov problem under certain additional conditions, see [9, 10]. In the spirit of the Mazur–Ulam theorem, we pose the following problem:

**Problem 2.6 (Aleksandrov Problem in non-Archimedean spaces).** Let \( X \) and \( Y \) be non-Archimedean normed spaces. Under what conditions is a mapping preserving a distance \( r \) an isometry?

Another related subject is study of the stability of isometries (see [13]) in the framework of non-Archimedean normed spaces as follows.

**Problem 2.7 (Stability of isometries in non-Archimedean spaces).** Let \( X \) and \( Y \) be non-Archimedean normed spaces and \( f : X \to Y \) be a mapping satisfying
\[
\|f(x) - f(y)\| - \|x - y\| \leq \varepsilon
\]
for some \( \varepsilon \) and for all \( x, y \in X \). Under what assumptions are there a constant \( \kappa \) and an isometry \( T : X \to Y \) such that \( \|f(x) - T(x)\| \leq \kappa \varepsilon \) for all \( x \in X \)?

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