Dyadic sets, maximal functions and applications on \(ax + b\)-groups

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Abstract Let \(S\) be the Lie group \(\mathbb{R}^n \ltimes \mathbb{R}\), where \(\mathbb{R}\) acts on \(\mathbb{R}^n\) by dilations, endowed with the left-invariant Riemannian symmetric space structure and the right Haar measure \(\rho\), which is a Lie group of exponential growth. Hebisch and Steger in [Math. Z. 245(2003), 37–61] proved that any integrable function on \((S, \rho)\) admits a Calderón–Zygmund decomposition which involves a particular family of sets, called Calderón–Zygmund sets. In this paper, we show the existence of a dyadic grid in the group \(S\), which has nice properties similar to the classical Euclidean dyadic cubes. Using the properties of the dyadic grid, we prove a Fefferman–Stein type inequality, involving the dyadic Hardy–Littlewood maximal function and the dyadic sharp function. As a consequence, we obtain a complex interpolation theorem involving the Hardy space \(\mathcal{H}^1\) and the space BMO introduced in [Collect. Math. 60(2009), 277–295].

Keywords Exponential growth group, Dyadic set, Complex interpolation, Hardy space, BMO

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1 Introduction

Let \(S\) be the Lie group \(\mathbb{R}^n \ltimes \mathbb{R}\) endowed with the following product: for all \((x, t), (x', t') \in S\),

\[(x, t) \cdot (x', t') \equiv (x + e^t x', t + t').\]

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The group $S$ is also called an $ax+b$–group. Clearly, $o = (0,0)$ is the identity of $S$. We endow $S$ with the left-invariant Riemannian metric

$$ds^2 \equiv e^{-2t}(dx_1^2 + \cdots + dx_n^2) + dt^2,$$

and denote by $d$ the corresponding metric. This coincides with the metric on the hyperbolic space $H^{n+1}(\mathbb{R})$. For all $(x,t)$ in $S$, we have

$$\cosh d((x,t),o) = \frac{e^t + e^{-t} + e^{-t}|x|^2}{2}.$$ (1.1)

The group $S$ is nonunimodular. The right and left Haar measures are given by

$$d\rho(x,t) \equiv dx\,dt \quad \text{and} \quad d\lambda(x,t) \equiv e^{-nt}\,dx\,dt.$$ 

Throughout the whole paper, we work on the triple $(S,d,\rho)$, namely, the group $S$ endowed with the left-invariant Riemannian metric $d$ and the right Haar measure $\rho$. For all $(x,t) \in S$ and $r > 0$, we denote by $B((x,t),r)$ the ball centered at $(x,t)$ of radius $r$. In particular, it is well known that the right invariant measure of the ball $B(o,r)$ has the following behavior

$$\rho(B(o,r)) \sim \begin{cases} r^{n+1} & \text{if } r < 1 \\ e^{nr} & \text{if } r \geq 1. \end{cases}$$

Thus, $(S,d,\rho)$ is a space of exponential growth.

Throughout this paper, we denote by $L^p$ the Lebesgue space $L^p(\rho)$ and by $\|\cdot\|_{L^p}$ its quasi-norm, for all $p \in (0,\infty]$. We also denote by $L^{1,\infty}$ the Lorentz space $L^{1,\infty}(\rho)$ and by $\|\cdot\|_{L^{1,\infty}}$ its quasi-norm.

Harmonic analysis on exponential growth groups recently attracts a lot of attention. In particular, many efforts have been made to study the theory of singular integrals on the space $(S,d,\rho)$.

In the remarkable paper [16], Hebisch and Steger developed a new Calderón–Zygmund theory which holds in some spaces of exponential growth, in particular in the space $(S,d,\rho)$. The main idea of [16] is to replace the family of balls which is used in the classical Calderón–Zygmund theory by a suitable family of rectangles which we call Calderón–Zygmund sets (see Section 2 for their definitions). We let $\mathcal{R}$ denote the family of all Calderón–Zygmund sets.

The Hardy–Littlewood maximal function associated with $\mathcal{R}$ is of weak type $(1,1)$ (see [13, 25]). In [16], it was proven that every integrable function on $(S,d,\rho)$ admits a Calderón–Zygmund decomposition involving the family $\mathcal{R}$. As a consequence, a theory for singular integrals holds in this setting. In particular, every integral operator bounded on $L^2$ whose kernel satisfies a suitable integral Hörmander’s condition is of weak type $(1,1)$. Interesting examples of singular integrals in this setting are spectral multipliers and Riesz transforms associated with a distinguished Laplacian $\Delta$ on $S$, which have been studied by numerous authors in, for example, [2, 8, 11, 12, 16, 15, 20, 22, 23].

Vallarino [26] introduced an atomic Hardy space $H^1$ on the group $(S,d,\rho)$, defined by atoms supported in Calderón–Zygmund sets instead of balls, and a corresponding
BMO space, which enjoy some properties of the classical Hardy and BMO spaces (see [7, 10, 24]). More precisely, it was proven that the dual of $H^1$ may be identified with BMO, that singular integrals whose kernel satisfies a suitable integral Hörmander’s condition are bounded from $H^1$ to $L^1$ and from $L^\infty$ to BMO. Moreover, for every $\theta \in (0, 1)$, the real interpolation space $[H^1, L^2]_{\theta, q}$ is equal to $L^q$ if $\frac{1}{q} = 1 - \frac{\theta}{2}$, and $[L^2, \BMO]_{\theta, p}$ is equal to $L^p$ if $\frac{1}{p} = \frac{1-\theta}{2}$. The complex interpolation spaces between $H^1$ and $L^2$ and between $L^2$ and BMO are not identified in [26].

In this paper, we introduce a dyadic grid of Calderón–Zygmund sets on $S$, which we denote by $D$ and which can be considered as the analogue of the family of classical dyadic cubes (see Theorem 3.1 below). Recall that dyadic sets in the context of spaces of homogeneous type were also introduced by Christ [6]; his construction used the doubling condition of the considered measure, so it cannot be adapted to the current setting. In the $ax+b$–groups, the main tools we use to construct such a dyadic grid are some nice splitting properties of the Calderón–Zygmund sets and an effective method to construct a “parent” of a given Calderón–Zygmund set (see Lemma 2.3 below). More precisely, given a Calderón–Zygmund set $R$, we find a bigger Calderón–Zygmund set $M(R)$ which can be split into at most $2^n$ sub-Calderón–Zygmund sets such that one of these subsets is exactly $R$ and each of these subsets has measure comparable to the measure of $R$. To the best of our knowledge, this is the first time that a family of dyadic sets appears in a space of exponential growth. The dyadic grid $D$ turns out to be a useful tool to study the analogue of maximal singular integrals (see [17]) on the space $(S, d, \rho)$, which will be investigated in a forthcoming paper [19].

By means of the dyadic collection $D$, in Section 4 below, we prove a relative distributional inequality involving the dyadic Hardy–Littlewood maximal function and the dyadic sharp maximal function on $S$, which implies a Fefferman–Stein type inequality involving those maximal functions; see Stein’s book [24, Chapter IV, Section 3.6] and Fefferman–Stein’s paper [10] for the analogous inequality in the Euclidean setting. The previous inequality is the main ingredient to prove that the complex interpolation space $(L^2, \BMO)_{\theta, q}$ is equal to $L^\rho$ if $\frac{1}{\rho} = \frac{1-\theta}{2}$ and $(H^1, L^2)_{\theta, q}$ is equal to $L^\rho$ if $\frac{1}{\rho} = 1 - \frac{\theta}{2}$. This implies complex interpolation results for analytic families of operators (see Theorems 5.2 and 5.3 below). In particular, the complex interpolation result for analytic families of operators involving $H^1$ could be interesting and useful to obtain endpoint growth estimates of the solutions to the wave equation associated with the distinguished Laplacian $\Delta$ on $ax+b$–groups, as was pointed out by Müller and Vallarino [21, Remark 6.3].

We remark that the corresponding complex interpolation results for the classical Hardy and BMO spaces were proven by Fefferman and Stein [10]. Recently, an $H^1$–BMO theory was developed by Ionescu [18] for noncompact symmetric spaces of rank 1 and, more generally, by Carbonaro, Mauceri and Meda [5] for metric measure spaces which are non-doubling and satisfy suitable geometric assumptions. In those papers, the authors proved a Fefferman–Stein type inequality for the maximal functions associated with the family of balls of small radius: the main ingredient in their proofs is an isoperimetric property which is satisfied by the spaces studied in [18, 5]. As a consequence, the authors in [18, 5] obtained some complex interpolation results involving a Hardy space defined only by means of atoms supported in small balls and a corresponding BMO space. Notice that
the space \((S,d,\rho)\) which we study here does not satisfy the isoperimetric property \([5, (2.2)]\). Moreover, we consider atoms supported both in “small” and “big” sets. Then we have to use different methods to obtain a suitable Fefferman–Stein inequality and complex interpolation results involving \(H^1\) and \(\text{BMO}\).

Due to the existence of the dyadic collection \(\mathcal{D}\), it makes sense to define a dyadic \(\text{BMO}_D\) space and its predual dyadic Hardy space \(H^1_D\) on \(S\) (see Definitions 4.2 and 4.3 below). Though in Theorem 4.5 below, it is proven that \(H^1_D\) is a proper subspace of \(H^1\), the complex interpolation result given by \(H^1_D\) and \(L^2\) is the same as that given by \(H^1\) and \(L^2\); see Remark 5.1 below.

Finally, we make some conventions on notations. Set \(Z_+ \equiv \{1, 2, \ldots\}\) and \(\mathbb{N} = Z_+ \cup \{0\}\). In the following, \(C\) denotes a \textit{positive finite constant} which may vary from line to line and may depend on parameters according to the context. Constants with subscripts do not change through the whole paper. Given two quantities \(f\) and \(g\), by \(f \lesssim g\), we mean that there exists a positive constant \(C\) such that \(f \leq Cg\). If \(f \lesssim g \lesssim f\), we then write \(f \sim g\). For any bounded linear operator \(T\) from a Banach space \(A\) to a Banach space \(B\), we denote by \(\|T\|_{A \to B}\) its \textit{operator norm}.

2 Preliminaries

We first recall the definition of Calderón–Zygmund sets which appears in [16] and implicitly in [13]. In the sequel, we denote by \(Q\) the \textit{collection of all dyadic cubes in} \(\mathbb{R}^n\).

\textbf{Definition 2.1.} A \textit{Calderón–Zygmund set} is a set \(R \equiv Q \times [t-r, t+r)\), where \(Q \in \mathcal{Q}\) with side length \(L\), \(t \in \mathbb{R}\), \(r > 0\) and

\[
\begin{align*}
e^2 e^t r \leq L < e^8 e^t r & \quad \text{if } r < 1, \\
e^t e^{2r} \leq L < e^t e^{8r} & \quad \text{if } r \geq 1.
\end{align*}
\]

We set \(t_R \equiv t\), \(r_R \equiv r\) and \(x_R \equiv (c_Q, t)\), where \(c_Q\) is the \textit{center} of \(Q\). For a Calderón–Zygmund set \(R\), its \textit{dilated set} is defined as \(R^* \equiv \{x \in S : d(x, R) < r_R\}\). Denote by \(\mathcal{R}\) the family of all Calderón–Zygmund sets on \(S\). For any \(x \in S\), denote by \(\mathcal{R}(x)\) the family of the sets in \(\mathcal{R}\) which contain \(x\).

\textbf{Remark 2.1.} For any set \(R \equiv Q \times [t-r, t+r) \in \mathcal{R}\), we have that

\[
\rho(R) = \int_Q \int_{t-r}^{t+r} ds \, dx = 2r|Q| = 2rL^n,
\]

where \(|Q|\) and \(L\) denote the Lebesgue measure and the side length of \(Q\), respectively.

The following lemma presents some properties of the Calderón–Zygmund sets.

\textbf{Lemma 2.1.} Let all the notation be as in Definition 2.1. Then there exists a constant \(\kappa_0 \in [1, \infty)\) such that for all \(R \in \mathcal{R}\), the following hold:

(i) \(B(x_R, r_R) \subset R \subset B(x_R, \kappa_0 r_R)\);
(ii) \(\rho(R^*) \leq \kappa_0 \rho(R)\);
(iii) every $R \in \mathcal{R}$ can be decomposed into mutually disjoint sets $\{R_i\}_{i=1}^k$, with $k = 2$ or $k = 2^n$, $R_i \in \mathcal{R}$, such that $R = \bigcup_{i=1}^k R_i$ and $\rho(R_i) = \rho(R)/k$ for all $i \in \{1, \cdots, k\}$.

We refer the reader to [16, 26] for the proof of Lemma 2.1. We recall here an idea of the proof of the property (iii) from [16, 26]. Given a Calderón–Zygmund set $R \equiv Q \times [t-r, t+r)$, when the side length $L$ of $Q$ is sufficiently large with respect to $e^t$ and $r$, it suffices to decompose $Q$ into $2^n$ smaller dyadic Euclidean cubes $\{Q_1, \cdots, Q_{2^n}\}$ and define $R_i \equiv Q_i \times [t-r, t+r)$ for $i \in \{1, \cdots, 2^n\}$. Otherwise, it suffices to split up the interval $[t-r, t+r)$ into two disjoint sub-intervals $\{I_1, I_2\}$, which have the same measure, and define $R_i \equiv Q \times I_i$ for $i \in \{1, 2\}$. This construction gives rise to either $2^n$ or 2 smaller Calderón–Zygmund sets satisfying the property (iii) above.

The Hardy–Littlewood maximal function associated to the family $\mathcal{R}$ is defined as follows.

**Definition 2.2.** For any locally integrable function $f$ on $S$, the Hardy–Littlewood maximal function $Mf$ is defined by

$$Mf(x) \equiv \sup_{R \in \mathcal{R}(x)} \frac{1}{\rho(R)} \int_R |f| \, d\rho \quad \forall \ x \in S. \quad (2.1)$$

The maximal operator $M$ has the following boundedness properties [13, 16, 25].

**Proposition 2.2.** The Hardy–Littlewood maximal operator $M$ is bounded from $L^1$ to $L^{1, \infty}$, and also bounded on $L^p$ for all $p \in (1, \infty)$.

By Proposition 2.2, Lemma 2.1 and a stopping-time argument, Hebisch and Steger [16] showed that any integrable function $f$ on $S$ at any level $\alpha > 0$ has a Calderón–Zygmund decomposition $f = g + \sum b_i$, where $|g|$ is a function almost everywhere bounded by $\kappa_0 \alpha$ and functions $\{b_i\}$, have vanishing integral and are supported in sets of the family $\mathcal{R}$. This was proven to be a very useful tool in establishing the boundedness of some multipliers and singular integrals in [16], and the theory of the Hardy space $H^1$ on $S$ in [26].

Lemma 2.1(iii) states that given a Calderón–Zygmund set, one can split it up into a finite number of disjoint subsets which are still in $\mathcal{R}$. We shall now study how, starting from a given Calderón–Zygmund set $R$, one can obtain a bigger set containing it which is still in $\mathcal{R}$ and whose measure is comparable to the measure of $R$.

**Definition 2.3.** For any $R \in \mathcal{R}$, $M(R) \in \mathcal{R}$ is called a parent of $R$, if

(i) $M(R)$ can be decomposed into 2 or $2^n$ mutually disjointed sub-Calderón–Zygmund sets, and one of these sets is $R$;

(ii) $3p(R)/2 \leq \rho(M(R)) \leq \max\{3, 2^n\} \rho(R)$.

For any $R \in \mathcal{R}$, a parent of $R$ always exists, but it may not be unique. The following lemma gives three different kinds of extensions for sets $R \equiv Q \times [t-r, t+r) \in \mathcal{R}$ when $r \geq 1$. Precisely, if $Q$ has small side length, then we find a parent of $R$ by extending $R$ “horizontally”; if $Q$ has large side length, then we find a parent of $R$ by extending $R$ either “vertically up” or “vertically down”.

Lemma 2.3. Suppose that \( R \equiv Q \times [t-r, t+r) \in \mathcal{R} \), where \( t \in \mathbb{R} \), \( r \equiv r_R \geq 1 \) and \( Q \subset \mathbb{R}^n \) is a dyadic cube with side length \( L \) satisfying \( e^t e^{2r} \leq L < e^t e^{8r} \). Then the following hold:

(i) If \( e^t e^{2r} \leq L < e^t e^{8r}/2 \), then \( R_1 \equiv Q' \times [t-r, t+r) \) is a parent of \( R \), where \( Q' \subset \mathbb{R}^n \) is the unique dyadic cube with side length \( 2L \) that contains \( Q \). Moreover, \( \rho(R_1) = 2^n \rho(R) \).

(ii) If \( e^t e^{8r}/2 \leq L < e^t e^{8r} \), then \( R_2 \equiv Q \times [t-r, t+3r) \) is a parent of \( R \). Moreover, the set \( R' \equiv Q \times [t+r, t+3r) \) belongs to the family \( \mathcal{R} \), \( R_2 = R \cup R' \) and

\[
\rho(R) = \rho(R') = \rho(R_2)/2.
\]

The set \( R_2 \) is also a parent of \( R' \).

(iii) If \( e^t e^{8r}/2 \leq L < e^t e^{8r} \), then \( R_3 \equiv Q \times [t-5r, t+r) \) is a parent of \( R \). Moreover, the set \( R'' \equiv Q \times [t-5r, t-r) \) belongs to the family \( \mathcal{R} \), \( R_3 = R \cup R'' \) and

\[
\rho(R) = \rho(R'')/2 = \rho(R_3)/3.
\]

The set \( R_3 \) is also a parent of \( R'' \).

Proof. We first prove (i). Since \( 2e^t e^{2r} \leq 2L < e^t e^{8r} \), we have that \( R_1 \in \mathcal{R} \). Obviously \( Q \times [t-r, t+r) \) can be decomposed into \( 2^n \) sub-Calderón–Zygmund sets and one of these sets is \( R \). By Remark 2.1, we have \( \rho(R_1) = 2^n \rho(R) \). Thus, (i) holds.

To show (ii), notice that \( r_{R_2} = 2r \) and \( t_{R_2} = t+r \). Since \( r \geq 1 \) and \( e^t e^{8r}/2 \leq L < e^t e^{8r} \), we have that \( e^{t+r} e^{4r} \leq L < e^{t+r} e^{16r} \), which implies that \( R_2 \in \mathcal{R} \). If we set \( R' \equiv Q \times [t+r, t+3r) \),

then \( t_{R'} = t+2r \) and \( r_{R'} = r \). Since \( r \geq 1 \) and \( e^t e^{8r}/2 \leq L < e^t e^{8r} \), we obtain \( e^{t+2r} e^r \leq L < e^{t+2r} e^{8r} \), and hence \( R' \in \mathcal{R} \). By Remark 2.1, \( \rho(R) = \rho(R') = \rho(R_2)/2 \). Thus, (ii) holds.

Finally, we show (iii). Observe that \( t_{R_3} = t-2r \) and \( r_{R_3} = 3r \). Since \( r \geq 1 \) and \( e^t e^{8r}/2 \leq L < e^t e^{8r} \), we have that \( e^{t-2r} e^{6r} \leq L < e^{t-2r} e^{24r} \) and hence \( R_3 \in \mathcal{R} \). Set \( R'' \equiv Q \times [t-5r, t-r) \). Notice that \( t_{R''} = t-3r \) and \( r_{R''} = 2r \). Again by \( r \geq 1 \) and \( e^t e^{8r}/2 \leq L < e^t e^{8r} \), we obtain \( e^{t-3r} e^{4r} \leq L < e^{t-3r} e^{16r} \) and hence \( R'' \in \mathcal{R} \). It is easy to see that \( R_3 = R \cup R'' \), \( \rho(R'') = 2\rho(R) \) and \( \rho(R_3) = 3\rho(R) \). Therefore, we obtain (iii), which completes the proof. \( \square \)

We conclude this section by recalling the definition of the Hardy space \( H^1 \) and its dual space \( \text{BMO} \) (see [26]).

Definition 2.4. An \( H^1 \)-atom is a function \( a \) in \( L^1 \) such that

(i) \( a \) is supported in a set \( R \in \mathcal{R} \);

(ii) \( \|a\|_{L^\infty} \leq [\rho(R)]^{-1} \);

(iii) \( \int_R a \, dp = 0 \).

Definition 2.5. The Hardy space \( H^1 \) is the space of all functions \( g \) in \( L^1 \) which can be written as \( g = \sum_j \lambda_j a_j \), where \( \{a_j\}_j \) are \( H^1 \)-atoms and \( \{\lambda_j\}_j \) are complex numbers such that \( \sum_j |\lambda_j| < \infty \). Denote by \( \|g\|_{H^1} \) the infimum of \( \sum_j |\lambda_j| \) over such decompositions.
In the sequel, for any locally integrable function \( f \) and any set \( R \in \mathcal{R} \), we denote by \( f_R \) the average of \( f \) on \( R \), namely, \( \frac{1}{\rho(R)} \int_R f \, dp \).

**Definition 2.6.** For any locally integrable function \( f \), its sharp maximal function is defined by

\[
\| f^\sharp \|_{BMO} \equiv \left( \frac{1}{R} \int_R |f - f_R| \, dp \right)_{\text{ess}} \quad \forall x \in S.
\]

The space \( BMO \) is the space of all locally integrable functions \( f \) such that \( f^\sharp \in L^\infty \). The space \( BMO \) is the quotient of \( \ell^1 \) module constant functions. It is a Banach space endowed with the norm \( \| f \|_{BMO} \equiv \| f^\sharp \|_{L^\infty} \).

The space \( BMO \) is identified with the dual of \( H^1 \); see [26, Theorem 3.4]. More precisely, for any \( f \) in \( BMO \), the functional \( \ell \) defined by \( \ell(g) \equiv \int fg \, dp \) for any finite linear combination \( g \) of atoms extends to a bounded functional on \( H^1 \) whose norm is no more than \( C \| f \|_{BMO} \). On the other hand, for any bounded linear functional \( \ell \) on \( H^1 \), there exists a function \( f^\ell \) in \( BMO \) such that \( \| f^\ell \|_{BMO} \leq C \| \ell \|_{(H^1)} \), and \( \ell(g) = \int f^\ell g \, dp \) for any finite linear combination \( g \) of atoms.

## 3 A dyadic grid on \((S, d, \rho)\)

The main purpose of this section is to introduce a dyadic grid of Calderón–Zygmund sets on \((S, d, \rho)\), which can be considered as an analogue of Euclidean dyadic cubes (see [24, p.149] or [14, p.384]). The key tools to construct such a grid are Lemmas 2.1 and 2.3.

**Theorem 3.1.** There exists a collection \( \{D_j\}_{j \in \mathbb{Z}} \) of partitions of \( S \) such that each \( D_j \) consists of pairwise disjoint Calderón–Zygmund sets, and

(i) for any \( j \in \mathbb{Z} \), \( S = \bigcup_{R \in D_j} R \);
(ii) if \( \ell \leq k \), \( R \in D_\ell \) and \( R' \in D_k \), then either \( R \subset R' \) or \( R \cap R' = \emptyset \);
(iii) for any \( j \in \mathbb{Z} \) and \( R \in D_j \), there exists a unique \( R' \in D_{j+1} \) such that \( R \subset R' \) and \( \rho(R') \leq 2^n \rho(R) \);
(iv) for any \( j \in \mathbb{Z} \), every \( R \in D_j \) can be decomposed into mutually disjoint sets \( \{R_1\}_{i=1}^k \subset D_{j-1} \), with \( k = 2^m \) or \( k = 2^n \), such that \( R = \bigcup_{i=1}^k R_i \) and \( \frac{\rho(R)}{2^m} \leq \rho(R_i) \leq \frac{\rho(R)}{2^n} \) for all \( i \in \{1, \ldots, k\} \);
(v) for any \( x \in S \) and for any \( j \in \mathbb{Z} \), let \( R^*_j \) be the unique set in \( D_j \) which contains \( x \), then \( \lim_{j \to -\infty} \rho(R^*_j) = 0 \) and \( \lim_{j \to +\infty} \rho(R^*_j) = \infty \).

**Proof.** We write \( S = \Omega_1 \cup \Omega_2 \), where \( \Omega_1 = \mathbb{R}^n \times [0, \infty) \) and \( \Omega_2 = \mathbb{R}^n \times (-\infty, 0) \), and construct a sequence \( \{D^1_j\}_{j \in \mathbb{N}} \) of partitions of \( \Omega_1 \) as well as a sequence \( \{D^2_j\}_{j \in \mathbb{N}} \) of partitions of \( \Omega_2 \), respectively.

Let us first construct the desired partitions \( \{D^1_j\}_{j \in \mathbb{N}} \) of \( \Omega_1 \) by the following four steps.

**Step 1.** Choose a Calderón–Zygmund set \( R_0 \equiv Q_0 \times [t_0 - r_0, t_0 + r_0] \), where \( t_0 = r_0 \geq 1 \) and \( Q_0 = [0, t_0] \in \mathcal{Q} \), with \( e^{t_0} e^{2r_0} \leq t_0 < e^{t_0} e^{8r_0} \). To find a parent of \( R_0 \), we consider the following two cases, separately.
Case 1: $e^{t_0} e^{8r_0}/2 \leq \ell_0 < e^{t_0} e^{8r_0}$. In this case, by Lemma 2.3(ii),

$$R_1 \equiv Q_0 \times [t_0 - r_0, t_0 + 3r_0]$$

is a parent of $R_0$ and $\rho(R_1) = 2\rho(R_0)$.

Case 2: $e^{t_0} e^{2r_0} \leq \ell_0 < e^{t_0} e^{8r_0}/2$. By Lemma 2.3(i), $R_1 \equiv Q_1 \times [t_0 - r_0, t_0 + r_0)$ is a parent of $R_0$, where $Q_1 \subset \mathbb{R}^n$ is the unique dyadic cube with side length $2\ell_0$ that contains $Q_0$, namely, $Q_1 = [0, 2\ell_0)^n$.

We then proceed as above to obtain a parent of $R_1$, which is denoted by $R_2$. By repeating this process, we obtain a sequence of Calderón-Zygmund sets, $\{R_j\}_{j \in \mathbb{N}}$, such that each $R_{j+1}$ is a parent of $R_j$.

Without loss of generality, for any $j \in \mathbb{N}$, we may set $R_j \equiv Q_j \times [t_j - r_j, t_j + r_j)$, where $r_{j+1} \geq r_j \geq 1$, $t_j = r_j$ and $Q_j = [0, \ell_j)^n \in Q$, with $e^{t_j} e^{2r_j} \leq \ell_j < e^{t_j} e^{8r_j}$.

Observe that $R_{j+1}$ is obtained by extending $R_j$ either “vertically up” (see Case 1) or “horizontally” (see Case 2). Notice that the definition of Calderón–Zygmund sets implies that we cannot always extend $R_j$ “horizontally” to obtain its parent $R_{j+1}$; in other words, for some $j$, to obtain $R_{j+1}$, we have to extend $R_j$ “vertically up”. Thus, $\lim_{j \to \infty} (t_j + r_j) = \infty$. This, combined with the fact that $t_j = r_j$, implies that

$$\Omega_1 = \bigcup_{j \in \mathbb{N}} (\mathbb{R}^n \times [t_j - r_j, t_j + r_j))$$

(3.1)

**Step 2.** For any $j \in \mathbb{N}$ and $R_j$ as constructed in **Step 1**, we set

$$\mathcal{N}_j \equiv \{Q \times [t_j - r_j, t_j + r_j) : Q \in Q, \ell(Q) = \ell(Q_j)\}.$$  

(3.2)

Then $\mathcal{N}_j \subset \mathcal{R}$ and we put all sets of $\mathcal{N}_j$ into $\mathcal{D}^1_j$. If $R_{j+1}$ is obtained by extending $R_j$ “vertically up” as in Case 1 of **Step 1**, then we set

$$\widetilde{\mathcal{N}}_j \equiv \{Q \times [t_j + r_j, t_j + 3r_j) : Q \in Q, \ell(Q) = \ell(Q_j)\}.$$ 

(3.3)

By Lemma 2.3(ii), $\widetilde{\mathcal{N}}_j \subset \mathcal{R}$. If $R_{j+1}$ is obtained by extending $R_j$ “horizontally” as in Case 2 of **Step 1**, then we set $\widetilde{\mathcal{N}}_j = \emptyset$. We also put all sets of $\widetilde{\mathcal{N}}_j$ into $\mathcal{D}^1_j$.

We claim that for any fixed $j \in \mathbb{N}$,

$$\Omega_1 = \bigcup_{R \in \mathcal{N}_j \cup (\cup_{k=j}^{\infty} \widetilde{\mathcal{N}}_k)} R.$$ 

(3.4)

Indeed,

$$\mathbb{R}^n \times [0, t_j + 3r_j) = \bigcup_{R \in \mathcal{N}_j \cup \widetilde{\mathcal{N}}_j} R.$$ 

(3.5)

Rewrite the sequence $\{\mathcal{N}_k : k > j, \mathcal{N}_k \neq \emptyset\}$ as $\{\mathcal{N}_{\ell_k}\}_{k=1}^{\infty}$, where

$$j + 1 \leq \ell_1 < \ell_2 < \cdots < \ell_k < \ldots$$
We have that
\[ t_j + 3r_j = t_{\ell_1} + r_{\ell_1} \quad \text{and} \quad t_{\ell_k-1} + 3r_{\ell_k-1} = t_{\ell_k} + r_{\ell_k} \quad \forall \, k \geq 1. \]  
(3.6)
Since
\[ \mathbb{R}^n \times [t_{\ell_k} + r_{\ell_k}, t_{\ell_k} + 3r_{\ell_k}) = \bigcup_{R \in \tilde{N}_{\ell_k}} R \]
and \( \lim_{k \to \infty} (t_{\ell_k} + 3r_{\ell_k}) = \infty \), by (3.6), we obtain that
\[ \mathbb{R}^n \times [t_j + 3r_j, \infty) = \bigcup_{k \geq 1} \bigcup_{\ell \geq j+1} R = \bigcup_{\ell \geq j+1} \bigcup_{R \in \tilde{N}_\ell} R. \]  
(3.7)

The claim (3.4) follows by (3.5) and (3.7).

**Step 3.** Now fix \( j \in \mathbb{N} \) and take \( \ell \geq j + 1 \) such that \( \tilde{N}_\ell \neq \emptyset \). For any \( R \in \tilde{N}_\ell \), by Lemma 2.1(iii), there exist mutually disjoint sets \( \{ R_i \}_{i=1}^k \subset \mathcal{R} \) with \( k = 2 \) or \( k = 2^n \) such that \( R = \bigcup_{i=1}^k R_i \), and \( \rho(R)/2^n \leq \rho(R_i) \leq \rho(R)/2 \) for all \( i \in \{1, \ldots, k\} \). Denote by \( \tilde{N}_\ell^1 \) the collection of all such small Calderón–Zygmund sets \( R_i \) obtained by running \( \mathcal{R} \) over all elements in \( \tilde{N}_\ell \). Observe that sets in \( \tilde{N}_\ell^1 \) are mutually disjoint. Next, we apply Lemma 2.1(iii) to every \( R \in \tilde{N}_\ell^1 \) and argue as above; we then obtain a collection of smaller Calderón–Zygmund sets, which is denoted by \( \tilde{N}_\ell^2 \). By repeating the above procedure \( i \) times, we obtain a collection of Calderón–Zygmund sets which we denote by \( \tilde{N}_\ell^i \). In particular, we put the collection \( \tilde{N}_\ell^{\ell-j} \) obtained after \( \ell - j \) steps into \( D_j^1 \).

Thus, for any \( j \in \mathbb{N} \), we define
\[ D_j^1 = \mathcal{N}_j \bigcup \tilde{N}_j \bigcup \left( \bigcup_{\ell \geq j+1} \tilde{N}_\ell^{\ell-j} \right). \]  
(3.8)

By construction, the sets in \( D_j^1 \) are mutually disjoint. Moreover, since for all \( j \geq 0 \) and \( \ell \geq j + 1 \),
\[ \bigcup_{R \in \tilde{N}_\ell^{\ell-j}} R = \bigcup_{R \in \tilde{N}_\ell} R, \]
from the formula (3.4), we deduce that \( \Omega_1 = \bigcup_{R \in D_j^1} R \). This shows that \( D_j^1 \) satisfies the property (i).

**Step 4.** For any \( 0 \leq \ell \leq k, \, R \in D_j^1 \) and \( R' \in D_k^1 \), by (3.8), (3.2), (3.3) and the construction above, it is easy to verify that either \( R \subset R' \) or \( R \cap R' = \emptyset \), namely, the property (ii) is satisfied.

Let \( R \) be in \( D_j^1 \) for some \( j \in \mathbb{N} \). If \( R \) is in \( \mathcal{N}_j \cup \tilde{N}_j \) and if \( R_{j+1} \) is obtained by extending \( R_j \) “horizontally”, then there exists one parent of \( R \) in \( D_{j+1} \) whose measure is \( 2^n \rho(R) \). If \( R \) is in \( \mathcal{N}_j \cup \tilde{N}_j \) and if \( R_{j+1} \) is obtained by extending \( R_j \) “vertically up”, then there exists one parent of \( R \) in \( D_{j+1} \) whose measure is \( 2 \rho(R) \). If \( R \) is in \( \tilde{N}_\ell^{\ell-j} \) for some \( \ell \geq j + 1 \), then it has a parent in \( \tilde{N}_\ell^{\ell-j-1} \subset D_{j+1} \) whose measure is either \( 2 \rho(R) \) or \( 2^n \rho(R) \). Thus, the property (iii) is satisfied.
So far, we have proven that there exists a sequence $\{D_j^1\}_{j \in \mathbb{N}}$ of partitions of $\Omega_1$ whose elements satisfy the properties (i)--(iii).

To obtain the desired partitions $\{D_j^2\}_{j \in \mathbb{N}}$ on $\Omega_2$, we apply (i) and (iii) of Lemma 2.3 and proceed as for $\Omega_1$: the details are left to the reader.

We define $D_j = D_j^1 \cup D_j^2$ for all $j \geq 0$. We now construct the partitions $D_j$ for $j < 0$.

By applying Lemma 2.1(iii) to each $R \in D_0$, we find mutually disjoint sets $\{R_i\}_{i=1}^k$, with $k = 2$ or $k = 2^n$, such that $R_i \in \mathcal{R}$, $R = \bigcup_{i=1}^k R_i$, and $\rho(R)/2^n \leq \rho(R_i) \leq \rho(R)/2$ for all $i \in \{1, \ldots, k\}$. Then we define $D_{-1}$ to be the collection of all such small Calderón-Zygmund sets $R_i$ obtained by running $R$ over all elements in $D_0$. Clearly $D_{-1}$ is still a partition of $S$. Again, applying Lemma 2.1(iii) to each element of $D_{-1}$ and using a similar splitting argument to this, we obtain a collection of smaller Calderón–Zygmund sets, which is defined to be $D_{-2}$. By repeating this process, we obtain a collection $\{D_j\}_{j < 0}$, where each $D_j$ is a partition of $S$. By the construction of $\{D_j\}_{j < 0}$ and by Lemma 2.1(iii), it is easy to check that the sets in $\{D_j\}_{j < 0}$ satisfy the properties (i)--(iii).

It remains to prove the properties (iv) and (v). For a set $R \in D_j$, with $j \leq 0$, the property (iv) is easily deduced from Lemma 2.1(iii). Take now a set $R \in D_j^1$ for some $j > 0$. If $R$ is in $N_j$, then it has either $2^n$ disjoint subsets in $N_j-1$ or 2 disjoint subsets in $N_j-1 \cup \tilde{N}_j$. If $R$ is in $N_j$, then it has either $2^n$ or 2 disjoint subsets in $N_j^2 \subset D_j_{-1}$. Finally, if $R$ is in $\tilde{N}_j^2$ for some $\ell \geq j + 1$, then it has either $2^n$ or 2 subsets in $\tilde{N}_j^2 \subset D_j$. In all the previous cases, $R$ satisfies the property (iv). The case when $R$ is in $D_j^2$ for some $j > 0$ is similar and omitted.

As far as the property (v) is concerned, given a point $x$ in $S$, for any $j \in \mathbb{Z}$, let $R_j^x$ be the set in $D_j$ which contains $x$. By the construction and the property (iv), for any $j \in \mathbb{Z}$, there exists a set $R_{j+1}^x \subset D_{j+1}$ which is a parent of $R_j^x$, so that

$$\rho(R_{j+1}^x) \geq \frac{3}{2} \rho(R_j^x) \geq \left(\frac{3}{2}\right)^j \rho(R_0^x);$$

this shows that $\lim_{j \to -\infty} \rho(R_j^x) = \infty$. For any $j < 0$, we have that

$$\rho(R_j^x) \leq \frac{2}{3} \rho(R_{j+1}^x) \leq \left(\frac{2}{3}\right)^j \rho(R_0^x);$$

this shows that $\lim_{j \to -\infty} \rho(R_j^x) = 0$ and concludes the proof of the theorem.

\[\square\]

Remark 3.1. (i) It should be pointed out that a sequence $\{D_j\}_{j \in \mathbb{Z}}$ satisfying Properties (i)--(v) of Theorem 3.1 is not unique.

(ii) For any given $j \in \mathbb{Z}$, the measures of any two elements in $D_j$ may not be comparable. This is an essential difference between the collection of Euclidean dyadic cubes and of dyadic sets in spaces of homogeneous type [6] and the dyadic sets which we introduced above.

We now choose one collection $D \equiv \{D_j\}_{j \in \mathbb{Z}}$ of dyadic sets in $S$ constructed as in Theorem 3.1. In the sequel, $D$ always denotes this collection.


## 4 Dyadic maximal functions

By using the collection $\mathcal{D}$ introduced above, we define the corresponding Hardy–Littlewood dyadic maximal function and dyadic sharp maximal function as follows.

**Definition 4.1.** For any locally integrable function $f$ on $(S, d, \rho)$, the Hardy–Littlewood dyadic maximal function $M_D f$ is defined by

$$
M_D f(x) \equiv \sup_{R \in \mathcal{R}(x), R \in \mathcal{D}} \frac{1}{\rho(R)} \int_R |f| \, d\rho \quad \forall x \in S, \tag{4.1}
$$

and the dyadic sharp maximal function $f_D^\#$ by

$$
f_D^\#(x) \equiv \sup_{R \in \mathcal{R}(x), R \in \mathcal{D}} \frac{1}{\rho(R)} \int_R |f - f_R| \, d\rho \quad \forall x \in S. \tag{4.2}
$$

Recall that $f_R \equiv \frac{1}{\rho(R)} \int_R f \, d\rho$.

It is easy to see that for all locally integrable functions $f$ and almost every $x \in S$, $f(x) \leq M_D f(x) \leq M f(x)$ and $f_D^\#(x) \leq f^\#(x)$. This combined with Proposition 2.2 implies the following conclusion.

**Corollary 4.1.** The operator $M_D$ is bounded from $L^1$ to $L^{1, \infty}$, and also bounded on $L^p$ for all $p \in (1, \infty]$.

**Remark 4.1.**

(i) It is obvious that $M_D f(x) \leq M f(x)$ for any locally integrable function $f$ at any point $x \in S$. However, there exist functions $f$ such that $M f$ and $M_D f$ are not pointwise equivalent. To see this, we take a set $R \equiv Q \times [0, 2r)$ in $\mathcal{D}$ such that $Q = [0, 2^{\ell_0})^n$ for some $\ell_0 \in \mathbb{Z}$. Then, for all points $(y_1, \ldots, y_n, s) \in S$ such that $y_1 < 0$ for all $j \in \{1, \ldots, n\}$ and $s < 0$, we have $M_D (\chi_R)(y, s) = 0$ and $M(\chi_R)(y, s) > 0$. So there does not exist a positive constant $C$ such that $M(\chi_R) \leq C M_D(\chi_R)$.

(ii) It is obvious that $f_D^\#(x) \leq f^\#(x)$ for any locally integrable function $f$ at any point $x \in S$. The same counterexample as in (i) shows that the sharp maximal function and the dyadic sharp maximal function may be not pointwise equivalent. Indeed, if we take the set $R \equiv Q \times [0, 2r)$ as above, then for all points $(y_1, \ldots, y_n, s) \in S$ such that $y_j < 0$ for all $j \in \{1, \ldots, n\}$ and $s < 0$, we have $(\chi_R)^\#(y, s) = 0$ and $(\chi_R)^\#(y, s) > 0$. So there does not exist a positive constant $C$ such that $(\chi_R)^\# \leq C (\chi_R)^\#_D$.

We now state a covering lemma for the level sets of $M_D$, which is proven in a standard way as follows; see also [24, Lemma 1, p.150].

**Lemma 4.2.** Let $f$ be a locally integrable function and $\alpha$ a positive constant such that $\Omega_{\alpha} \equiv \{x \in S : M_D f(x) > \alpha\}$ has finite measure. Then $\Omega_{\alpha}$ is a disjoint union of dyadic sets, $\{R_j\}_j$, with $\alpha < \frac{1}{\rho(R_j)} \int_{R_j} |f| \, d\rho \leq 2^\alpha \alpha$ for all $j$. 
Proof. Since the measure of $\Omega_\alpha$ is finite, for each $x \in \Omega_\alpha$ there exists a maximal dyadic set $R_x \in D$ which contains $x$ such that $\alpha < \frac{1}{\rho(R_x)} \int_{R_x} |f| d\rho$. Any two of these maximal dyadic sets are disjoint. Indeed, by Theorem 3.1, given two points $x, y \in \Omega_\alpha$, either $R_x \cap R_y = \emptyset$ or one is contained in the other; by maximality, this implies that $R_x = R_y$. We denote by $\{R_j\}_j$ this collection of dyadic maximal sets. Then it is clear that $\Omega_\alpha = \bigcup_j R_j$. Moreover, for any $j$, since $R_j$ is maximal, there exists a dyadic set $\tilde{R}_j \in D$ which is a parent of $R_j$ and $\frac{1}{\rho(\tilde{R}_j)} \int_{\tilde{R}_j} |f| d\rho \leq \alpha$. Thus,

$$
\frac{1}{\rho(R_j)} \int_{R_j} |f| d\rho \leq 2^n \frac{1}{\rho(\tilde{R}_j)} \int_{\tilde{R}_j} |f| d\rho \leq 2^n \alpha.
$$

This finishes the proof. \qed

As a consequence of the previous covering lemma, following closely the proof of the inequality [24, (22), p.153], we obtain the following relative distributional inequality. We omit the details.

**Proposition 4.3.** There exists a positive constant $K$ such that for any locally integrable function $f$, and for any positive $c$ and $b$ with $b < 1$,

$$
\rho(\{x \in S : \mathcal{M}_D f(x) > \alpha, f_D^2(x) \leq ca \}) \leq K \frac{c}{1 - b} \rho(\{x \in S : \mathcal{M}_D f(x) > ba \}) \quad (4.3)
$$

for all $\alpha > 0$. The constant $K$ only depends on $n$ and on the norm $\|\mathcal{M}_D\|_{L^1 \rightarrow L^{1,\infty}}$.

By the relative distributional inequality (4.3) and arguing as in [24, Corollary 1, p. 154], we obtain the following Fefferman–Stein type inequality. We also omit the details.

**Corollary 4.4.** Let $p \in (0, \infty)$. There exists a positive constant $A_p$ such that for any locally integrable function $f$ such that $f_D^2$ belongs to $L^p$ and $\mathcal{M}_D f \in L^{p_0}$ with $p_0 \leq p$, then $f$ is in $L^p$ and

$$
\|\mathcal{M}_D f\|_{L^p} \leq A_p \|f_D^2\|_{L^p}.
$$

**Remark 4.2.** Recall that for any locally integrable function $f$, $|f| \leq \mathcal{M}_D f$ and $f_D^2 \leq f^2$. Thus, from Corollary 4.4, we deduce that if $p \in (0, \infty)$, $f^2$ belongs to $L^p$ and $f$ belongs to some $L^{p_0}$ with $p_0 \in (0, p]$, then $f$ is in $L^p$ and

$$
\|f\|_{L^p} \leq A_p \|f^2\|_{L^p}, \quad (4.4)
$$

where $A_p$ is the constant which appears in Corollary 4.4. This generalizes the classical Fefferman–Stein inequality [24, Theorem 2, p.148] to the current setting.

We shall now introduce a dyadic Hardy space and a dyadic BMO space.

**Definition 4.2.** The **dyadic Hardy space** $H^1_D$ is defined to be the space of all functions $g$ in $L^1$ which can be written as $g = \sum_j \lambda_j a_j$, where $\{a_j\}_j$ are $H^1$-atoms supported in dyadic sets and $\{\lambda_j\}_j$ are complex numbers such that $\sum_j |\lambda_j| < \infty$. Denote by $\|g\|_{H^1_D}$ the infimum of $\sum_j |\lambda_j|$ over all such decompositions.
**Definition 4.3.** The space $\text{BMO}_D$ is the space of all locally integrable functions $f$ such that $f_D^2 \in L^\infty$. The space $\text{BMO}_D$ is the quotient of $\text{BMO}_D$ module constant functions. It is a Banach space endowed with the norm $\|f\|_{\text{BMO}_D} \equiv \|f_D^2\|_{L^\infty}$.

It is easy to follow the proof in [25, Theorem 3.4] to show that the dual of $H^1_D$ is identified with $\text{BMO}_D$. We omit the details.

Obviously, $H^1_D \subset H^1$ and $\|g\|_{H^1} \leq \|g\|_{H^1_D}$ for all $g$ in $H^1_D$. It is natural to ask whether the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^1_D}$ are equivalent. The analog problem in the classical setting was studied by Abu-Shammala and Torchinsky [1]. By following the ideas in [1], we obtain the following result.

**Theorem 4.5.** The norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^1_D}$ are not equivalent.

**Proof.** We give the details of the proof in the case when $n = 1$. By the construction of $D$ in Theorem 3.1, there exists $[0, 2^{j-1}) \times [0, 2^{j}) \in D_{k_0}$ for some $k_0 \in \mathbb{Z}$, $\ell_0 \in \mathbb{Z}$ and $r_0 > 0$ such that $R_0 = [0, 2^{j-1}) \times [0, 2^{j}) \in D_{k_0}$ and $E_0 = [2^{j-1}, 2 \cdot 2^{j}) \times [0, 2^{j}) \in D_{k_0}$. Generally, for any $j < 0$, there exist $R_j = [2^{j-1}, 2^{j}) \times [0, 2^{j}) \in D_{k_0}$ and $E_j = [2^{j-1}, 2^{j}) \times [0, 2^{j}) \in D_{k_0}$ such that $R_j \cup E_j \in D_{k_{j+1}}$, where both $\{k_j\}_{j<0}$ and $\{\ell_j\}_{j<0}$ are strictly decreasing sequences which tend to $-\infty$ as $j \to -\infty$, and each $I_j$ is an interval contained in $[0, \infty)$. Notice that for all $j \in \mathbb{N}$, $\rho(R_j) = \rho(E_j) = 2r_j2^{j}$ for some $r_j > 0$. Set $a_j \equiv \frac{1}{2r_j2^{j}}(\chi_{R_j} - \chi_{E_j})$. Obviously, each $a_j$ is an $H^1$-atom and $\|a_j\|_{H^1} \leq 1$.

Take the function $\phi(x, t) \equiv \chi_{(2^{j-1}, 2^{j})}(x) \log(x - 2^{j}) \equiv h(x)$ for all $(x, t) \in S$. An easy calculation gives that

$$\|\phi\|_{\text{BMO}_D} \leq \sup_{I \text{ is a dyadic interval}} \frac{1}{|I|} \int_I h(x) - \frac{1}{|I|} \int_I h(y) \, dy \, dx < \infty.$$ 

We then have

$$\|a_j\|_{H^1_D} = \sup_{\psi \in \text{BMO}_D} \frac{1}{\|\psi\|_{\text{BMO}_D}} \left| \int_S a_j \psi \, d\rho \right|$$

$$\geq \frac{1}{\|\phi\|_{\text{BMO}_D}} \left| \int_S a_j \phi \, d\rho \right|$$

$$= \frac{1}{2\|\phi\|_{\text{BMO}_D}} \left| \int_{2^{j-1}}^{2^{j}} \log(x - 2^{j}) \, dx \right| = \frac{|1 - \log 2^j|}{2\|\phi\|_{\text{BMO}_D}} \sim |\ell_j|.$$ 

So there exists no positive constant such that $\|a_j\|_{H^1_D} \leq C\|a_j\|_{H^1}$ for all $j < 0$. 

Notice that all the arguments of [26, Section 5] can be adapted to the dyadic spaces $H^1_D$ and $\text{BMO}_D$ such that all results therein also hold for $H^1_D$ and $\text{BMO}_D$. In particular, one can prove that, though $H^1_D$ is a proper subspace of $H^1$, the real interpolation space $[H^1_D, L^2]_{\theta, q}$ is equal to $L^q$, if $\theta \in (0, 1)$ and $\frac{1}{q} = 1 - \frac{\theta}{2}$. 

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Dyadic sets and maximal functions on $ax + b$–groups
5 Complex interpolation

We now formulate an interpolation theorem involving $H^1$ and BMO. In the following, when $A$ and $B$ are Banach spaces and $\theta$ is in $(0, 1)$, we denote by $(A, B)_{[\theta]}$ the complex interpolation space between $A$ and $B$ with parameter $\theta$, obtained via Calderón’s complex interpolation method (see [4, 3]).

**Theorem 5.1.** Suppose that $\theta$ is in $(0, 1)$. Then the following hold:

(i) if $\frac{1}{p_0} = \frac{1-\theta}{2}$, then $(L^2, \text{BMO})_{[\theta]} = L^{p_0};$

(ii) if $\frac{1}{q_0} = 1 - \frac{\theta}{2}$, then $(H^1, L^2)_{[\theta]} = L^{q_0}.$

**Proof.** The proof of (i) is an easy adaptation of the proof of [10, p.156, Corollary 2] and of [5, Theorem 7.4]. We omit the details.

The proof of (ii) follows from Theorem 5.1(i) and [9, Theorem 1]. Alternatively, we may follow [24, p. 175, Theorem 4]. We leave the details to the reader.

A consequence of the previous theorem is the following.

**Theorem 5.2.** Denote by $\Sigma$ the closed strip $\{ s \in \mathbb{C} : \Re s \in [0, 1] \}$. Suppose that $\{ T_s \}_{s \in \Sigma}$ is a family of uniformly bounded operators on $L^2$ such that the map $s \to \int_{\Sigma} T_s(f)g \, d\rho$ is continuous on $\Sigma$ and analytic in the interior of $\Sigma$, whenever $f, g \in L^2$. Moreover, assume that there exists a positive constant $A$ such that

$$\|T_t f\|_{L^2} \leq A \|f\|_{L^2} \quad \forall \, f \in L^2, \, \forall \, t \in \mathbb{R},$$

and

$$\|T_{1+it} f\|_{\text{BMO}} \leq A \|f\|_{\infty} \quad \forall \, f \in L^2 \cap L^\infty, \, \forall \, t \in \mathbb{R}.$$  

Then for every $\theta \in (0, 1)$, the operator $T_\theta$ is bounded on $L^{p_0}$, with $\frac{1}{p_0} = \frac{1-\theta}{2}$ and

$$\|T_\theta f\|_{L^{p_0}} \leq A_\theta \|f\|_{L^{p_0}} \quad \forall \, f \in L^2 \cap L^{p_0}.$$  

Here $A_\theta$ depends only on $A$ and $\theta$.

**Proof.** This follows from Theorem 5.1(i) and [9, Theorem 1]. Alternatively, we may follow the proof of [24, p. 175, Theorem 4]. We leave the details to the reader.

**Theorem 5.3.** Denote by $\Sigma$ the closed strip $\{ s \in \mathbb{C} : \Re s \in [0, 1] \}$. Suppose that $\{ T_s \}_{s \in \Sigma}$ is a family of uniformly bounded operators on $L^2$ such that the map $s \to \int_{\Sigma} T_s(f)g \, d\rho$ is continuous on $\Sigma$ and analytic in the interior of $\Sigma$, whenever $f, g \in L^2$. Moreover, assume that there exists a positive constant $A$ such that

$$\|T_t f\|_{L^1} \leq A \|f\|_{H^1} \quad \forall \, f \in L^2 \cap H^1, \, \forall \, t \in \mathbb{R},$$
and
\[ \|T_{1+t}f\|_{L^2} \leq A \|f\|_{L^2} \quad \forall \ f \in L^2, \forall \ t \in \mathbb{R}. \]

Then for every \( \theta \in (0, 1) \), the operator \( T_\theta \) is bounded on \( L^{q_\theta} \), with \( \frac{1}{q_\theta} = 1 - \frac{\theta}{2} \) and
\[ \|T_\theta f\|_{L^{q_\theta}} \leq A_\theta \|f\|_{L^{q_\theta}} \quad \forall \ f \in L^2 \cap L^{q_\theta}. \]

Here \( A_\theta \) depends only on \( A \) and \( \theta \).

**Proof.** This follows from Theorem 5.1(ii) and [9, Theorem 1]. We omit the details. \( \square \)

**Remark 5.1.** It is easy to see that Theorems 5.1, 5.2 and 5.3 still hold if \( H^1 \) and BMO are replaced by \( H^1_D \) and BMO\(_D\), respectively. We leave the details to the reader.

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