Purely Infinite Simple Ultragraph Leavitt Path Algebras

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Abstract. In this article, we give necessary and sufficient conditions under which the Leavitt path algebra $L_K(\mathcal{G})$ of an ultragraph $\mathcal{G}$ over a field $K$ is purely infinite simple and that it is von Neumann regular. Consequently, we obtain that every graded simple ultragraph Leavitt path algebra is either a locally matricial algebra, or a full matrix ring over $K[x, x^{-1}]$, or a purely infinite simple algebra.

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1. Introduction

The study of algebras associated with combinatorial objects has attracted a great deal of attention in the past years. Part of the interest in these algebras arises from the fact that many properties of the combinatorial object translate into algebraic properties of the associated algebras and their applications to symbolic dynamics. There have been interesting examples of algebras associated with combinatorial objects among which we mention, for example, the following ones: graph $C^*$-algebras, Leavitt path algebras, higher rank graph algebras, Kumjian–Pask algebras, ultragraph $C^*$-algebras (we refer the reader to [1] and [4] for a more comprehensive list).

Ultragraphs were defined by Mark Tomforde in [18] as an unifying approach to Exel-Laca and graph $C^*$-algebras. They have proved to be a key ingredient in the study of Morita equivalence of Exel-Laca and graph $C^*$-algebras [16]. Recently, Gonçalves and Royer have established nice connections between ultragraph $C^*$-algebras and the symbolic dynamics of shift spaces over infinite alphabets (see [10] and [13]).

The Leavitt path algebra associated with an ultragraph was defined by Imanfar, Pourabbas and Larki in [15] (along with a study of graded ideal structures and a proof of a Cuntz–Krieger uniqueness type theorem), and by Gonçalves and Royer in [12] in terms of two different definitions. In [9],
de Castro, Gonçalves and van Wyk showed that the resulting algebras are isomorphic. Furthermore, it was shown in [15] that ultragraph Leavitt path algebras provide examples of algebras that cannot be realized as the Leavitt path algebra of a graph; that is, the class of ultragraph Leavitt path algebras is strictly larger than the class of Leavitt path algebras of graphs. This raises the question of which results about Leavitt path algebras of graphs can be generalized to ultragraph Leavitt path algebras, and whether results from the \( C^* \)-algebraic setting can be proved in the algebraic level. Recently, a number of interesting results regarding this question have been obtained. We mention the following. Gonçalves and Royer [14] extended Chen’s construction of irreducible representations of graph Leavitt path algebras to ultragraph Leavitt path algebras (see [8]); Gonçalves and Royer [12] realized ultragraph Leavitt path algebras as partial skew group rings. Using this realization they characterized artinian ultragraph Leavitt path algebras and gave simplicity criteria for these algebras; and de Castro, Gonçalves and van Wyk [9] realized ultragraph Leavitt path algebras as Steinberg algebras, and applied this result to obtain generalized uniqueness theorems for ultragraph Leavitt path algebras.

The current article is a continuation of this direction. In [2, Theorem 11], the authors characterized purely infinite simple Leavitt path algebras in terms of properties of the associated graph. Also, in [4, Proposition 3.1.14] the authors showed that every graded simple Leavitt path algebra of a graph is either a locally matricial algebra, or a full matrix ring over \( K[x, x^{-1}] \), or a purely infinite simple algebra. The authors of [4] call this result the Trichotomy Principle for graded simple Leavitt path algebras of graphs. Motivated by these interesting results, the main goal of this article is to characterize purely infinite simple ultragraph Leavitt path algebras, and establish the Trichotomy Principle for graded simple ultragraph Leavitt path algebras. Our proofs are based on the ones of [2, Theorem 11] and [4, Proposition 3.1.14], respectively. However, we should mention that although ultragraphs benefit from the same kind of intuition available for graphs, very often the techniques involved in generalizing results know for graphs require non-trivial ideas to be developed. Also, sometimes the graph intuition may be misleading and a detailed approach is necessary to deal with ultragraphs.

The article is organized as follows. In Sect. 2, for the reader’s convenience, we provide subsequently necessary notions and facts on ultragraphs and ultragraph Leavitt path algebras. We show that the ultragraph Leavitt path algebras arising from acyclic ultragraphs are precisely the von Neumann regular ultragraph Leavitt path algebras, and in this case they are exactly locally matricial algebras (Theorem 2.9). In Sect. 3, we give necessary and sufficient conditions on an ultragraph \( G \) so that \( L_K(G) \) is purely infinite simple (Theorem 3.4). This result provides the algebraic analog to the corresponding result for the ultragraph \( C^* \)-algebra \( C^*(G) \) given in [19, Theorem 4.4 and Proposition 4.5] and [17, Theorem 6.6], and extends [2, Theorem 11] to ultragraph Leavitt path algebras. Using Theorems 2.9 and 3.4 and [15, Theorem 3.4], we obtain Theorem 3.9, showing that every graded
simple ultragraph Leavitt path algebra is either a locally matricial algebra, or a full matrix ring over \( K[x, x^{-1}] \), or a purely infinite simple algebra.

2. Regularity Conditions for Ultragraph Leavitt Path Algebras

The main goal of this section is to establish the equivalence of the following conditions for an ultragraph \( G \) and a field \( K \) (Theorem 2.9): (1) \( L_K(G) \) is von Neumann regular. (2) \( G \) is acyclic. (3) \( L_K(G) \) is a locally matricial \( K \)-algebra.

We begin this section by recalling some notions and notes of ultragraph theory introduced by Tomforde in [18] and [19].

An ultragraph \( G = (G^0, G^1, r, s) \) consists of a countable set of vertices \( G^0 \), a countable set of edges \( G^1 \), and functions \( s : G^1 \to G^0 \) and \( r : G^1 \to G^0 \). \( \mathcal{P}(G^0) \) denotes the set of all subsets of \( G^0 \).

A vertex \( v \in G^0 \) is called a sink if \( s^{-1}(v) = \emptyset \) and \( v \) is called an infinite emitter if \( |s^{-1}(v)| = \infty \). A singular vertex is a vertex that is either a sink or an infinite emitter. The set of all singular vertices is denoted by \( \text{Sing}(G) \). A vertex \( v \in G^0 \) is called a regular vertex if \( 0 < |s^{-1}(v)| < \infty \).

For an ultragraph \( G = (G^0, G^1, r, s) \) we let \( G^0 \) denote the smallest subset of \( \mathcal{P}(G^0) \) that contains \{v\} for all \( v \in G^0 \), contains \( r(e) \) for all \( e \in G^1 \), and is closed under finite unions and finite intersections. Elements of \( G^0 \) called generalized vertices. The following lemma gives us another description of \( G^0 \).

**Lemma 2.1.** [18, Lemma 2.12] If \( G := (G^0, G^1, r, s) \) is an ultragraph, then \( G^0 = \{ \bigcap_{e \in X_1} r(e) \cup \cdots \cup \bigcap_{e \in X_n} r(e) \cup F \mid X_1, \ldots, X_n \) are finite subsets of \( G^1 \) and \( F \) is a finite subset of \( G^0 \}. \) Furthermore, \( F \) may be chosen to be disjoint from \( \bigcap_{e \in X_1} r(e) \cup \cdots \cup \bigcap_{e \in X_n} r(e) \).

A finite path in an ultragraph \( G \) is either an element of \( G^0 \) or a sequence \( \alpha_1 \alpha_2 \cdots \alpha_n \) of edges with \( s(\alpha_{i+1}) \in r(\alpha_i) \) for all \( 1 \leq i \leq n-1 \) and we say that the path \( \alpha \) has length \( |\alpha| := n \). We consider the elements of \( G^0 \) to be paths of length \( 0 \). We denote by \( G^* \) the set of all finite paths in \( G \). The maps \( r \) and \( s \) extend naturally to \( G^* \). Note that when \( A \in G^0 \) we define \( s(A) = r(A) = A \).

An infinite path in \( G \) is a sequence \( e_1 e_2 \cdots e_n \cdots \) of edges in \( G \) such that \( s(e_{i+1}) \in r(e_i) \) for all \( i \geq 1 \).

If \( G \) is an ultragraph, then a cycle in \( G \) is a path \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_{|\alpha|} \in G^* \) with \( |\alpha| \geq 1 \) and \( s(\alpha) \in r(\alpha) \). An exit for a cycle \( \alpha \) is one of the following:

(1) an edge \( e \in G^1 \) such that there exists an \( i \) for which \( s(e) \in r(\alpha_i) \) but \( e \neq \alpha_{i+1} \).

(2) a sink \( w \) such that \( w \in r(\alpha_i) \) for some \( i \).

In [18] Mark Tomforde introduced the \( C^* \)-algebra of an ultragraph as an unifying approach to Exel-Laca and graph \( C^* \)-algebras. Leavitt path algebras of ultragraphs were introduced in the literature (see, e.g., [12, 15]).

**Definition 2.2.** (cf. [18, Theorem 2.11] and [12, Definition 2.3]) Let \( G \) be an ultragraph and \( K \) a field. The Leavitt path algebra \( L_K(G) \) of \( G \) with coefficients in \( K \) is the \( K \)-algebra generated by the set \( \{s_e, s_e^* \mid e \in G^1\} \cup \{p_A \mid A \in G^0\} \), satisfying the following relations for all \( A, B \in G^0 \) and \( e, f \in G^1 \):
(1) \( p_\emptyset = 0, p_A p_B = p_{A \cap B} \) and \( p_{A \cup B} = p_A + p_B - p_{A \cap B} \);
(2) \( p_{1(e)} s_e = s_e = s_e p_{r(e)} \) and \( p_{r(e)} s_e^* = s_e^* p_{s(e)} \);
(3) \( s_e^* f = \delta_{e,f} p_{r(e)} \);
(4) \( p_v = \sum_{s(\epsilon)=v} s_e s_e^* \) for any regular vertex \( v \),
where \( p_v \) denotes \( p_{\{v\}} \) and \( \delta \) is the Kronecker delta.

It is worth mentioning the following note.

Remark 2.3. There have been different definitions of Leavitt path algebras of ultragraphs, and the difference of these definitions lies in how the set of generalized vertices are defined. Given an ultragraph \( G \), let \( B \) denote the smallest subset of \( \mathcal{P}(G^0) \) that contains \( \{v\} \) for all \( v \in G^0 \), contains \( r(e) \) for all \( e \in G^1 \), and is closed under relative complements, finite unions and finite intersections. We denote by \( L_K(G_v) \) the Leavitt path algebra associated with \( G \) by allowing \( A, B \in B \) in item (1) of Definition 2.2, that means, \( L_K(G_v) \) is the algebra as defined in [15, Definition 2.1]. However, in [9, Proposition 5.2] the authors showed that \( L_K(G_v) \) and \( L_K(G) \) are isomorphic to each other.

We usually denote \( s_A := p_A \) for \( A \in G^0 \) and \( s_\alpha := s_{e_1} \cdots s_{e_n} \) for \( \alpha = e_1 \cdots e_n \in \mathcal{G}^* \). It is easy to see that the mappings given by \( p_A \mapsto p_A \) for \( A \in G^0 \), and \( s_e \mapsto s_e^* \) for \( e \in G^1 \), produce an involution on the algebra \( L_K(G) \), and for any path \( \alpha = \alpha_1 \cdots \alpha_n \) there exists \( s_\alpha^* := s_{e_n}^* \cdots s_{e_1}^* \). Also, \( L_K(G) \) has the following universal property: if \( A \) is a \( K \)-algebra generated by a family of elements \( \{b_A, c_e, c_e^* | A \in G^0, e \in G^1\} \) satisfying the relations analogous to (1) - (4) in Definition 2.2, then there always exists a \( K \)-algebra homomorphism \( \varphi : L_K(G) \rightarrow A \) given by \( \varphi(p_A) = b_A, \varphi(s_e) = c_e \) and \( \varphi(s_e^*) = c_e^* \).

Furthermore, we recall a few other useful properties as follows.

Lemma 2.4. ([12, Theorem 3.10]) If \( G \) is an ultragraph and \( K \) is a field, then the Leavitt path algebra \( L_K(G) \) has the following properties:

1. All elements of the set \( \{p_A, s_e, s_e^* | A \in G^0 \setminus \{\emptyset\}, e \in G^1\} \) are nonzero.
2. \( L_K(G) \) is of the form \( \text{Span}_K \{s_\alpha p_A s_{\beta}^* | \alpha, \beta \in G^*, A \in G^0 \text{ and } r(\alpha) \cap A \cap r(\beta) \neq \emptyset\} \).

Furthermore, \( L_K(G) \) is a \( Z \)-graded \( K \)-algebra by the grading
\[
L_K(G)_n = \text{Span}_K \{s_\alpha p_A s_{\beta}^* | \alpha, \beta \in G^*, A \in G^0 \text{ and } |\alpha| - |\beta| = n\} \quad (n \in \mathbb{Z}).
\]

Proof. (1) It follows from [12, Theorem 3.10].

(2) It follows from the last paragraph of the proof of [12, Theorem 3.10], thus finishing the proof. \hfill \Box

In light of Lemma 2.4, an element \( x \in L_K(G)_n \) is called a homogeneous element of degree \( n \). Recall that a ring \( R \) is said to have local units if every finite subset of \( R \) is contained in a subring of the form \( eRe \) where \( e = e^2 \in R \). The following lemma shows that every ultragraph Leavitt path algebra is an algebra with local units.

Lemma 2.5. Let \( G \) be an ultragraph and \( K \) a field. Then \( L_K(G) \) is an algebra with local units (specifically, the set of local units of \( L_K(G) \) is given by \( \{p_A | A \in G^0\} \)). Moreover, \( L_K(G) \) is unital if and only if \( G^0 \in G^0 \); in this case the identity element is \( 1 = p_{G^0} \).
Proof. Consider a finite subset \( \{a_i\}_{i=1}^n \) of \( L_K(\mathcal{G}) \) and use Lemma 2.4 (2) to write
\[
a_i = \sum_{s=1}^n k_i^s s p_i^s p_{A_i}^s q_i^s \quad \text{where} \quad k_i^s \in K \setminus \{0\}, A_i^s \in \mathcal{G}_0, \text{and} \ p_s^i, q_i^s \in \mathcal{G}^*.
\]
Let
\[
A := \{s(p_i^i), s(q_i^i) \mid |p_i^i| \geq 1, |q_i^i| \geq 1\} \cup \left( \bigcup_{|p_s^i| = 0} s(p_s^i) \cup s(q_s^i) \right) \subseteq \mathcal{G}_0.
\]
We then have \( A \in \mathcal{G}_0 \) and \( p_A a_i = a_i = a_i p_A \) for all \( i \), and so \( L_K(\mathcal{G}) \) is an algebra with local units.

The remainder follows from [15, Lemma 2.12] and, just for the reader’s convenience, we briefly sketch it here. Namely, assume that \( L_K(\mathcal{G}) \) is unital and write
\[
1_{L_K(\mathcal{G})} = \sum_{i=1}^n k_i s p_i p_{A_i} s_{q_i}^* \quad \text{where} \quad k_i \in K \setminus \{0\}, \text{and} \ p_i, A_i, q_i \in \mathcal{G}^*.
\]
Let
\[
B := \{s(p_i) \mid |p_i| \geq 1\} \cup \left( \bigcup_{|p_i| = 0} s(p_i) \right) \subseteq \mathcal{G}_0.
\]
It is obvious that \( B \in \mathcal{G}_0 \). If \( \mathcal{G}_0 \not\subseteq \mathcal{G}_0 \), then there exists an element \( v \in \mathcal{G}_0 \setminus B \), and
\[
p_v = p_v \cdot 1_{L_K(\mathcal{G})} = p_v \left( \sum_{i=1}^n k_i s p_i p_{A_i} s_{q_i}^* \right) = 0;
\]
but \( p_v \) is well known to be a nonzero element inside the ultragraph Leavitt path algebra, we get a contradiction, which shows that \( \mathcal{G}_0 \subseteq \mathcal{G}_0 \). The converse is obvious, thus finishing the proof. \( \square \)

Lemma 2.6. Let \( \mathcal{G} \) be an ultragraph and \( K \) a field. Then the algebra \( L_K(\mathcal{G}) \) is generated by \( \{s_e, s_e^* \mid e \in \mathcal{G}^1\} \cup \{p_v \mid v \in \text{Sing}(\mathcal{G})\} \).

Proof. Let \( \mathcal{A} \) be the \( K \)-subalgebra of \( L_K(\mathcal{G}) \) generated by \( \{s_e, s_e^* \mid e \in \mathcal{G}^1\} \cup \{p_v \mid v \in \text{Sing}(\mathcal{G})\} \). We claim that \( L_K(\mathcal{G}) \subseteq \mathcal{A} \). To do so, it is sufficient to show that \( p_A \in \mathcal{A} \) for all \( A \in \mathcal{G}_0 \). Take any \( A \in \mathcal{G}_0 \). By Lemma 2.1, there exist finite subsets \( X_1, \cdots, X_n \) of \( \mathcal{G}^1 \) and a finite subset \( F \) of \( \mathcal{G}_0 \) such that
\[
A = \bigcap_{e \in X_1} r(e) \cup \cdots \cup \bigcap_{e \in X_n} r(e) \cup F \text{ and } F \cap \left( \bigcap_{e \in X_1} r(e) \cup \cdots \cup \bigcap_{e \in X_n} r(e) \right) = \emptyset.
\]
Note that \( p_v \in \mathcal{A} \) for every singular vertex \( v \). Also, if \( v \) is a regular vertex, then \( p_v = \sum_{e \in s^{-1}(v)} s_e s_e^* \in \mathcal{A} \). This implies that \( p_F = \sum_{v \in F} p_v \in \mathcal{A} \).

For every \( e \in \mathcal{G}^1 \), we always have that \( p_{r(e)} = s_e^* s_e \in \mathcal{A} \), and so
\[
p_{A_i} = \prod_{e \in X_i} p_{r(e)} \in \mathcal{A}
\]
for all $1 \leq i \leq n$, where $A_i := \bigcap_{e \in X_i} r(e)$. For $n = 2$, by item (1) of Definition 2.2, $p_{A_1 \cup A_2} = p_{A_1} + p_{A_2} - p_{A_1 \cap A_2} = p_{A_1} + p_{A_2} - p_{A_1 p_{A_2}} \in A$. By induction we obtain that

$$p_{\bigcup_{i=1}^{n} A_i} = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} p_{A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n}} \in A.$$  

This implies that $p_A = p_{\bigcup_{i=1}^{n} A_i} + p_F \in A$, and hence $L_K(G) \subseteq A$. The inverse inclusion is obvious, and so $L_K(G) = A$, thus finishing our proof. \hfill \Box

Let $G = (G^0, G^1, r, s)$ be an ultragraph and let $F$ be a finite subset of $G^1 \cup \text{Sing}(G)$. Write $F^0 := F \cap \text{Sing}(G)$ and $F^1 := F \cap G^1 = \{e_1, e_2, \ldots, e_n\}$. Following [15], we construct a finite graph $G_F$ as follows. For each $\omega = (\omega_1, \ldots, \omega_n) \in \{0,1\}^n \setminus \{0^n\}$, we define

$$r(\omega) := \bigcap_{\omega_i = 1} r(e_i) \bigcup \bigcup_{\omega_j = 0} r(e_j) \text{and } R(\omega) := r(\omega) \setminus F^0.$$  

Notice that $r(\omega) \cap r(\nu) = \emptyset$ for distinct $\omega, \nu \in \{0,1\}^n \setminus \{0^n\}$. Let

$$\Gamma_0 := \{\omega \in \{0,1\}^n \setminus \{0^n\} \mid \text{there are vertices } v_1, \ldots, v_m\text{ such that } R(\omega) = \{v_1, \ldots, v_m\}\text{ and } \emptyset \neq s^{-1}(v_i) \subseteq F^1 \text{ for } 1 \leq i \leq m\}$$

and

$$\Gamma_F := \{\omega \in \{0,1\}^n \setminus \{0^n\} \mid R(\omega) \neq \emptyset \text{ and } \omega \notin \Gamma_0\}.$$  

Now we define the finite graph $G_F = (G^0_F, G^1_F, r_F, s_F)$ as follows:

$$G^0_F := F^0 \cup F^1 \cup \Gamma_F,$$

and

$$G^1_F := \{(e, f) \in F^1 \times F^1 \mid s(f) \in r(e)\}$$

$$\cup \{(e, v) \in F^1 \times F^0 \mid v \in r(e)\}$$

$$\cup \{(e, \omega) \in F^1 \times \Gamma_F \mid \omega_i = 1 \text{ when } e = e_i\}$$

with

$$s_F((e, f)) = e \quad s_F((e, v)) = e \quad s_F((e, \omega)) = e$$

$$r_F((e, f)) = f \quad r_F((e, v)) = v \quad r_F((e, \omega)) = \omega.$$  

For clarification, we illustrate the above construction by presenting the following example.

Example 2.7. Let $G$ be the ultragraph with $G^0 = \{v_n \mid n \in \mathbb{Z}, n \geq 1\}$, $G^1 = \{e_n \mid n \in \mathbb{Z}, n \geq 1\}$ and $s(e_1) = v_1, s(e_n) = v_{n+1}$ for every $n \geq 2$, and $r(e_1) = \{v_n \mid n \geq 2\}$, $r(e_2) = G^0$, $r(e_3) = \{v_5, v_6, v_7\}$, $r(e_n) = \{v_m \mid m \geq n + 2\}$ for every $n \geq 4$. We then have $\text{Sing}(G) = G^0 \setminus \{v_1\}$.

Let $F := \{e_1, e_2\} \cup \{v_1, v_2\} \subseteq G^1 \cup \text{Sing}(G)$, $F^0 := F \cap \text{Sing}(G) = \{v_1, v_2\}$ and $F^1 := F \cap G^1 = \{e_1, e_2\}$. We have $r(1,0) = r(e_1) \setminus r(e_2) = \emptyset$, $r(0,1) = r(e_2) \setminus r(e_1) = \{v_1\}$ and $r(1,1) = r(e_1) \cap r(e_2) = \{v_n \mid n \geq 2\}$, and so $R(1,0) = r(1,0) \setminus F^0 = \emptyset$, $R(0,1) = r(0,1) \setminus F^0 = \emptyset$ and $R(1,1) =$
for every finite direct sum of full finite dimensional matrix algebras over the field locally matricial $A$ not necessarily-unital transition homomorphisms). In [5, Theorem 1], Abrams

Let theorem extends this result to ultragraph Leavitt path algebras.

and in this case they are exactly locally matricial algebras. The following

acyclic graphs are precisely the von Neumann regular Leavitt path algebras,

\[
\begin{align*}
\text{(e_1,v_2)} & \quad (e_1,(1,1)) \\
\text{(e_2,v_2)} & \quad (e_1,e_2) \quad (e_2,(1,1)) \\
\text{v_2} & \quad (e_2,e_1) \\
\text{e_2} & \quad (1,1) \\
\text{(e_2,v_1)} & \\
\text{v_1}
\end{align*}
\]

As usual, an ultragraph is called acyclic if it has no cycles. The following lemma gives us a criterion for acyclic ultragraphs.

**Lemma 2.8.** An ultragraph $G$ is acyclic if and only if $G_F$ is acyclic for every non-empty finite subset $F$ of $G^1 \cup \text{Sing}(G)$.

**Proof.** $(\Rightarrow)$ Assume that $G$ is an acyclic ultragraph and $F$ is a finite subset of $G^1 \cup \text{Sing}(G)$. We claim that the graph $G_F$ is acyclic. Indeed, suppose $G_F$ is not acyclic, that means, it has a cycle $c = c_1c_2\cdots c_m$, where $c_i \in G^1_F$ for all $i$. By the definition of $G_F$, we must have that $c_i \in \{(e,f) \in F^1 \times F^1 \mid s(f) \in r(e)\}$ for all $i$. So, without loss of generality, we may assume $c_i = (e_i,f_i) \in F^1 \times F^1$ with $s(f_i) \in r(e_i)$. Then, since $c = c_1c_2\cdots c_m$ is a cycle in $G_F$, we have that $e_1f_1e_2f_2\cdots e_mf_m$ is a cycle in $G$, which is a contradiction, and hence $G_F$ is acyclic.

$(\Leftarrow)$ Assume that $G$ has a cycle $\alpha = e_1e_2\cdots e_n$, where $e_i \in G^1$ for all $i$. Let $F := \{e_1,e_2,\ldots,e_n\} \subset G^1$. Since $\alpha$ is a cycle in $G$, $s(e_{i+1}) \in r(e_i)$ for all $1 \leq i \leq n-1$, and $s(e_1) \in r(e_n)$. This implies that $(e_i,e_{i+1}) \in G^1_F$ for all $1 \leq i \leq n-1$ and $(e_n,e_1) \in G^1_F$, and $(e_1,e_2)(e_2,e_3)\cdots(e_n,e_1)$ is a cycle in the graph $G_F$, thus finishing our proof. \hfill $\Box$

A (not necessarily unital) ring $R$ is called von Neumann regular in case for every $r \in R$ there exists $s \in R$ such that $r = rsr$. A matricial $K$-algebra is a finite direct sum of full finite dimensional matrix algebras over the field $K$. A locally matricial $K$-algebra is a direct limit of matricial $K$-algebras (with not necessarily-unital transition homomorphisms). In [5, Theorem 1], Abrams and Rangaswamy showed that the graph Leavitt path algebras arising from acyclic graphs are precisely the von Neumann regular Leavitt path algebras, and in this case they are exactly locally matricial algebras. The following theorem extends this result to ultragraph Leavitt path algebras.

**Theorem 2.9.** Let $G$ be an ultragraph and $K$ a field. Then the following conditions are equivalent:

1. $L_K(G)$ is von Neumann regular;
2. $G$ is acyclic;
(3) $L_K(G)$ is a locally matricial $K$-algebra, that is, $L_K(G)$ is a union of a chain of matricial $K$-subalgebras.

Proof. (1)$\implies$(2). The proof is essentially based on the ideas in the proof of (2)$\implies$(3) in [5, Theorem 1], using Lemma 2.4.

Assume that $L_K(G)$ is von Neumann regular, and there exists a cycle $c$ in $G$; denote $s(c)$ by $v$. We claim that $p_v - s_c p_v \neq 0$. Indeed, suppose that $p_v - s_c p_v = 0$. By Lemma 2.4 (2), we must have $p_v = s_c p_v = 0$. On the other hand, by Lemma 2.4 (1), we always have $p_v \neq 0$, which is a contradiction, thus proving the claim.

Since $L_K(G)$ is von Neumann regular, there exists an element $\beta \in L_K(G)$ such that $p_v - s_c p_v = (p_v - s_c p_v) \beta (p_v - s_c p_v)$. Replacing $\beta$ by $p_v \beta p_v$ if necessary, there is no loss of generality in assuming that $\beta = p_v \beta p_v$. By Lemma 2.4, we may write $\beta$ as a sum of homogeneous elements $\beta = \sum_{i=1}^{n} \beta_i$, where $m, n \in \mathbb{Z}$, $\beta_m \neq 0$, $\beta_n \neq 0$, and $\deg(\beta_i) = i$ for all nonzero $\beta_i$ with $m \leq i \leq n$. Since $\deg(p_v) = 0$, we have $p_v \beta p_v = \beta_i$ for all $i$. Then

$$p_v - s_c p_v = (p_v - s_c p_v) \left( \sum_{i=m}^{n} \beta_i \right) (p_v - s_c p_v).$$

Equating the lowest degree terms on both sides, we obtain that $\beta_m = p_v$. Since $\deg(p_v) = 0$, we must have that $m = 0$ and $\beta_0 = p_v$. Thus, $\beta = \sum_{i=0}^{n} \beta_i$.

Let $\deg(s_c) = s > 0$. By again equating terms of like degree in the displayed equation, we see that $\beta_i = 0$ whenever $i$ is nonzero and not a multiple of $s$, so that

$$\sum_{i=m}^{n} \beta_i = p_v + \sum_{t=1}^{k} \beta_{ts}.\]$$

We then have

$$p_v - s_c p_v = (p_v - s_c p_v) p_v (p_v - s_c p_v) + (p_v - s_c p_v) \left( \sum_{t=1}^{k} \beta_{ts} \right) (p_v - s_c p_v),$$

which shows that $0 = -p_v + \left( s_c p_v \right)^2 + (p_v - s_c p_v) \left( \sum_{t=1}^{k} \beta_{ts} \right) (p_v - s_c p_v)$. By equating the degree $s$ components on both sides we obtain $\beta_s = s_c p_v$.

Similarly, by equating the degree $2s$ components, we obtain $0 = (s_c p_v)^2 - (s_c p_v) \beta_s - \beta_s (s_c p_v) + \beta_{2s}$, so $\beta_{2s} = (s_c p_v)^2$, and continuing in this manner we get $\beta_{ts} = (s_c p_v)^t$ for all $t$. In particular, we conclude that every homogeneous component $\beta_i$ of $\beta$ commutes with $s_c p_v$ in $L_K(G)$. This yields that $(s_c p_v) \beta = \beta (s_c p_v)$. But then the equation $p_v - s_c p_v = (p_v - s_c p_v) \beta (p_v - s_c p_v)$ becomes

$$p_v - s_c p_v = \beta (p_v - s_c p_v)^2.$$ 

But this is not possible, as follows. Let $i$ be maximal with the property that $\beta_i ((p_v - s_c p_v)^2) \neq 0$ (Such $i$ exists, since $\beta_0 = p_v$ has this property.) Then the right-hand side contains terms of degree $2s + i$ (namely, $\beta_i (s_c p_v)^2$), while the maximum degree of terms on the left-hand side is $s$.

(2)$\implies$(3). Since $G^0$ and $G^1$ are countable sets, we may rename the edges of $G^1$ as a sequence $\{e_i\}_{i=1}^{\infty}$ and rename the vertices of $\text{Sing}(G)$ a sequence $\{v_i\}_{i=1}^{\infty}$. For $n \in \mathbb{N}$, we denote by $B_n$ the subalgebra of $L_K(G)$ generated
by \{s_{e_i}, s_{e_i}^*p_{e_i} \mid i = 1, \ldots, n\}. We then have that \(B_n \subseteq B_{n+1}\) for all \(n\), and by Lemma 2.6, \(L_K(\mathcal{G}) = \bigcup_{n=1}^{\infty} B_n\). By repeating approach described in the proof of [15, Lemma 2.13], we have that \(B_n \cong L_K(G_{F_n})\), where \(F_n := \{e_1, \ldots, e_n\} \cup \{v_1, \ldots, v_n\}\). Since \(\mathcal{G}\) is acyclic, by Lemma 2.8, the finite graph \(G_{F_n}\) is acyclic for all \(n\). By [3, Proposition 3.5], \(L_K(G_{F_n})\) is a matricial \(K\)-algebra. Therefore, \(L_K(\mathcal{G})\) is a locally matricial \(K\)-algebra.

(3)\(\Rightarrow\)(1). It is well known that every matricial \(K\)-algebra is a von Neumann regular ring, and so is any direct union of such algebras, thus finishing our proof. \(\square\)

3. Purely Infinite Simplicity
The results regarding purely infinite simple ultragraph Leavitt path algebras could be derived by general groupoid results ([7, Theorem 3.2] and [9, Theorem 5.5]). The main goal of this section is both to give a graph-theoretic criterion for purely infinite simple ultragraph Leavitt path algebras (Theorem 3.4) and provide a complete description of graded simple ultragraph Leavitt path algebras (Theorem 3.9).

Recall (see, e.g. [6]) that an idempotent \(e\) in a ring \(R\) is called infinite if \(eR\) is isomorphic as a right \(R\)-module to a proper direct summand of itself. \(R\) is called purely infinite in case every nonzero right ideal of \(R\) contains an infinite idempotent. The following lemma provides us with a useful criterion for purely infinite simple rings with local units.

Lemma 3.1. ([2, Proposition 10]) For any ring with local units \(R\), the following conditions are equivalent:

1. \(R\) is purely infinite simple;
2. \(R\) is not a division ring, and \(R\) has the property that for every pair of nonzero elements \(\alpha, \beta \in R\) there exist elements \(a, b \in R\) such that \(a\alpha b = \beta\).

We will use the above lemma to characterize the purely infinite simplicity of ultragraph Leavitt path algebras. Before doing so, we recall some notions and facts introduced by Tomforde in [19]. Let \(\mathcal{G}\) be an ultragraph. A subset \(H \subseteq \mathcal{G}^0\) is called hereditary if the following conditions are satisfied:

1. whenever \(e\) is an edge with \(\{s(e)\} \in H\), then \(r(e) \in H\);
2. \(A \cup B \in H\) for all \(A, B \in H\);
3. \(A \in H, B \in \mathcal{G}^0\) and \(B \subseteq A\), imply that \(B \in H\).

A subset \(H \subseteq \mathcal{G}^0\) is called saturated if for any \(v \in \mathcal{G}^0\) with \(0 < |s^{-1}(v)| < \infty\), we have that

\[\{r(e) \mid e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq H\text{ implies }\{v\} \in H.\]

Note that \(\emptyset\) and \(\mathcal{G}^0\) are two saturated hereditary subsets of \(\mathcal{G}^0\). We denote by \(\mathcal{H}\) the set of all saturated hereditary subsets of \(\mathcal{G}^0\). For \(H \subseteq \mathcal{G}^0\), we denote by \(\overline{H}\) the smallest saturated hereditary subset of \(\mathcal{G}^0\) containing \(H\). In [19, Lemma 3.12] Tomforde gave a useful description of \(\overline{H}\) as follows.
Lemma 3.2. ([19, Lemma 3.12]) Let \( G := (G^0, G^1, r, s) \) be an ultragraph and let \( H \subseteq G^0 \) be a hereditary subset. Set \( H_0 := H \) and for \( n \in \mathbb{N} \) define

\[
H_{n+1} := \{ A \cup F \mid A \in H_n \text{and} F \text{ is a finite subset of } S_n \}
\]

where \( S_n := \{ w \in G^0 \mid 0 < |s^{-1}(w)| < \infty \text{ and } \{ r(e) \mid s(e) = w \} \subseteq H_n \} \).

Then \( \overline{H} = \bigcup_{i=0}^{\infty} H_i \) and every \( X \in \overline{H} \) has the form \( X = A \cup F \) for some \( A \in H \) and for some finite set \( F \subseteq \bigcup_{i=1}^{\infty} S_i \).

Following [19], if \( G \) is an ultragraph and \( v, w \in G^0 \), we write \( w \geq v \) to mean that there exists a path \( \alpha \in G^* \) with \( s(\alpha) = w \) and \( v \in r(\alpha) \). Also, we write \( G^0 \geq \{ v \} \) to mean that \( w \geq v \) for all \( w \in G^0 \). We say that a vertex \( w \) connects to a cycle \( \alpha = e_1 \cdots e_n \) if there exists a path \( \beta \in G^* \) with \( s(\beta) = w \) and \( s(e_i) \in r(\beta) \) for some \( 1 \leq i \leq n \). Note that if \( w \) is a sink on a cycle (i.e. \( w \in r(e_i) \) for some \( i \)), then \( w \) does not connect to a cycle. We say that a vertex \( w \) connects to an infinite path \( \alpha = e_1 \cdots e_n \cdots \) if there exists an \( i \in \mathbb{N} \) such that \( v \geq s(e_i) \).

If \( v \in G^0 \) and \( A \subseteq G^0 \), then we write \( v \rightarrow A \) to mean that there exist a finite number of paths \( \alpha_1, \ldots, \alpha_n \in G^* \) such that \( s(\alpha_i) = v \) for all \( 1 \leq i \leq n \) and \( A \subseteq \bigcup_{i=1}^{n} r(\alpha_i) \). Note that if \( A = \{ w \} \), then \( v \rightarrow \{ w \} \) if and only if \( v \geq w \).

The following lemma provides us with a criterion for ultragraphs \( G \) with \( H_G = \{ \varnothing, G^0 \} \), its proof is given in [19, Theorem 3.11]; and just for the reader’s convenience, we reproduce it here.

Lemma 3.3. For every ultragraph \( G \), \( H_G = \{ \varnothing, G^0 \} \) if and only if the following conditions are satisfied:

1. Every vertex connects to every infinite path.
2. \( G^0 \geq \{ v \} \) for every singular vertex \( v \in G^0 \).
3. If \( e \in G^1 \) is an edge for which the set \( r(e) \) is infinite, then for every \( w \in G^0 \) there exists a set \( A_w \subseteq r(e) \) for which \( r(e) \setminus A_w \) is finite and \( w \rightarrow A_w \).

Proof. \((\Rightarrow)\) Let \( \alpha = e_1 e_2 \cdots \) be an infinite path and set \( K := \{ w \in G^0 \mid w \not\in s(e_i) \text{ for all } i \} \). Let \( H := \{ A \in G^0 \mid A \subseteq K \} \). We then claim that \( H \) is a saturated hereditary subset of \( G^0 \). Indeed, let \( e \in G^1 \) with \( \{ s(e) \} \in H \) (i.e. \( s(e) \in K \)). If \( r(e) \not\in H \), then \( r(e) \not\subseteq K \), and so there exist \( w \in r(e) \) and \( i \in \mathbb{N} \) such that \( w \geq s(e_i) \). This implies that \( s(e) \geq w \geq s(e_i) \), that means, \( s(e) \not\in K \), a contradiction, and hence \( r(e) \in H \). It is also obvious that \( A \cup B \in H \) for all \( A, B \in H \), and \( B \in H \) for all \( B \subseteq A \in H \). Therefore, \( H \) is a hereditary subset of \( G^0 \).

Let \( v \in G^0 \) with \( 0 < |s^{-1}(v)| \leq \infty \) and \( \{ r(e) \mid e \in G^1 \text{ and } s(e) = v \} \subseteq H \). Assume that \( \{ v \} \not\in H \). Then, there exists \( i \in \mathbb{N} \) such that \( v \geq s(e_i) \), that means, there exists a path \( p = f_1 f_2 \cdots f_m \in G^* \) such that \( s(p) = s(f_1) = v \) and \( s(e_i) \in r(p) \). This implies that \( s(f_2) \geq s(e_i) \), and so \( s(f_2) \not\in K \). On the other hand, since \( r(f_1) \subseteq K \) and \( s(f_2) \in r(f_1) \), we must have that \( s(f_2) \in K \), we get a contradiction, which shows that \( \{ v \} \in H \). So \( H \) is a saturated subset of \( G^0 \), thus proving the claim.
Since \( \{s(e_1)\} \notin \mathcal{H} \), we have that \( \mathcal{H} \neq \mathcal{G}^0 \), and so \( \mathcal{H} = \emptyset \), by our hypothesis that \( \mathcal{H}_G = \{\emptyset, \mathcal{G}^0\} \). So, for every vertex \( v \in \mathcal{G}^0 \) there exists an \( i \in \mathbb{N} \) such that \( v \geq s(e_i) \), showing item (1).

Let \( v \in \mathcal{G}^0 \) be a singular vertex. Fix any vertex \( w \in \mathcal{G}^0 \) and define \( K := \{x \in \mathcal{G}^0 \mid w \geq x\} \). Let \( \mathcal{H} := \{A \in \mathcal{G}^0 \mid A \subseteq K\} \). Similar to the above case, we obtain that \( \mathcal{H} \) is a hereditary subset of \( \mathcal{G}^0 \).

We denote by \( \overline{\mathcal{H}} \) the smallest saturated hereditary subset of \( \mathcal{G}^0 \) containing \( \mathcal{H} \). Since \( \{w\} \in \mathcal{H} \), \( \overline{\mathcal{H}} \neq \emptyset \), and so \( \overline{\mathcal{H}} = \mathcal{G}^0 \). Using the notation of Lemma 3.2 and since \( v \) is a singular vertex, we get that \( v \notin S_i \) for all \( i \).

By Lemma 3.2 and since \( \{v\} \in \overline{\mathcal{H}} \), we immediately get that \( \{v\} \in \mathcal{H} \). This implies that \( v \in K \) and \( w \geq v \). Hence \( \mathcal{G}^0 \geq \{v\} \), proving item (2).

Let \( e \in \mathcal{G}^1 \) be an edge such that \( r(e) \) is an infinite set. Fix \( w \in \mathcal{G}^0 \) and set \( \mathcal{H} := \{A \in \mathcal{G}^0 \mid w \rightarrow A\} \). We claim that \( \mathcal{H} \) is a hereditary subset of \( \mathcal{G}^0 \). Indeed, if \( f \) is an edge with \( \{s(f)\} \in \mathcal{H} \), then \( w \rightarrow \{s(f)\} \), and hence \( w \geq s(f) \). Thus, there exists a path \( \beta \in \mathcal{G}^* \) such that \( s(\beta) = w \) and \( s(f) \in r(\beta) \). We then have that \( w \rightarrow r(f) \) via the path \( \beta \). Thus, \( r(f) \in \mathcal{H} \).

If \( A, B \in \mathcal{H} \), then \( w \rightarrow A \) and \( w \rightarrow B \), and so there exist a finite number of paths \( \alpha_1, \ldots, \alpha_n \in \mathcal{G}^* \) such that \( s(\alpha_i) = w \) for all \( 1 \leq i \leq n \) and \( A \subseteq \bigcup_{i=1}^n r(\alpha_i) \), and there exist a finite number of paths \( \beta_1, \ldots, \beta_m \in \mathcal{G}^* \) such that \( s(\beta_j) = w \) for all \( 1 \leq j \leq m \) and \( B \subseteq \bigcup_{j=1}^m r(\beta_j) \). It is obvious that \( A \cup B \subseteq (\bigcup_{i=1}^n r(\alpha_i)) \cup (\bigcup_{j=1}^m r(\beta_j)) \), and so \( w \rightarrow A \cup B \). Therefore, \( A \cup B \in \mathcal{H} \).

If \( A \in \mathcal{H}, B \in \mathcal{G}^0, \) and \( B \subseteq A \), then we have that \( w \rightarrow A \), and so there exist a finite number of paths \( \alpha_1, \ldots, \alpha_n \in \mathcal{G}^* \) such that \( s(\alpha_i) = w \) for all \( 1 \leq i \leq n \) and \( A \subseteq \bigcup_{i=1}^n r(\alpha_i) \). This implies that \( B \subseteq \bigcup_{i=1}^n r(\alpha_i) \), and hence \( B \in \mathcal{H} \), thus proving the claim.

Since \( \{w\} \in \mathcal{H}, \mathcal{H} \) is nonempty, and hence \( \overline{\mathcal{H}} = \mathcal{G}^0 \). Thus \( r(e) \in \overline{\mathcal{H}} \). By Lemma 3.2, \( r(e) = A_w \cup F \) for some \( A_w \in \mathcal{H} \) and some finite set \( F \subseteq \bigcup_{i=1}^\infty S_i \). Then \( w \rightarrow A_w \) and \( r(e) \setminus A_w \) is finite, showing item (3).

\((\iff)\) Let \( \mathcal{H} \) be a nonempty saturated hereditary subset of \( \mathcal{G}^0 \). We first claim that for every \( w \in \mathcal{G}^0 \) with \( \{w\} \notin \mathcal{H} \), there exists an edge \( e \in \mathcal{G}^1 \) such that \( s(e) = w \) and \( r(e) \) contains a vertex \( w' \) for which \( \{w'\} \notin \mathcal{H} \). Indeed, since \( \{w\} \notin \mathcal{H} \) and \( \mathcal{G}^0 \geq \{v\} \) for every singular vertex \( v, w \) is a regular vertex because otherwise it implies that \( w \) is a singular vertex, so \( \mathcal{G}^0 \geq \{w\} \), but since \( \mathcal{H} \neq \emptyset \), there exists a vertex \( v \in \mathcal{G}^0 \) such that \( \{v\} \in \mathcal{H} \) and \( v \geq w \). Since \( \mathcal{H} \) is hereditary, we must have that \( \{w\} \in \mathcal{H} \), we get a contradiction.

Now, since \( \mathcal{H} \) is saturated, there exists an edge \( e \in \mathcal{G}^1 \) such that \( s(e) = w \) and \( r(e) \notin \mathcal{H} \). If \( r(e) \) is finite, then since \( \mathcal{H} \) is closed under unions, there must exist a vertex \( w' \in r(e) \) such that \( \{w'\} \notin \mathcal{H} \), as desired. Consider the case that \( r(e) \) is infinite. If \( \{v\} \notin \mathcal{H} \) for every vertex \( v \in r(e) \), then the claim is obvious. If there exists a vertex \( x \in \mathcal{G}^0 \) such that \( \{x\} \in \mathcal{H} \), then there exists \( A_x \subseteq r(e) \) such that \( w \rightarrow A_x \) and \( r(e) \setminus A_x \) is a finite set. Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{G}^* \) be paths with \( s(\alpha_i) = x \) and \( A_x \subseteq \bigcup_{i=1}^n r(\alpha_i) \). Since \( \{x\} \in \mathcal{H} \) and \( \mathcal{H} \) is hereditary, \( r(\alpha_i) \in \mathcal{H} \) for all \( i \), and so \( \bigcup_{i=1}^n r(\alpha_i) \in \mathcal{H} \). If \( r(e) \setminus A_x \in \mathcal{H} \), then \( \bigcup_{i=1}^n r(\alpha_i) \cup (r(e) \setminus A_x) \) is an element in \( \mathcal{H} \) containing \( r(e) \). Then, since \( \mathcal{H} \) is hereditary, we immediately get that \( r(e) \in \mathcal{H} \), which
is a contradiction. Therefore, we must have that $r(e) \setminus A_x$ is not in $H$, that means, \( \{w'\} \not\in H \) for some \( w' \in r(e) \setminus A_x \), as desired.

Now suppose that $H \neq G^0$. Then there exists \( \{w_1\} \not\in H \). By the above claim, there exist an edge $e_1$ and a vertex $w_2$ such that $s(e_1) = w_1, w_2 \in r(e_1)$, and \( \{w_2\} \not\in H \). Continuing inductively, we create an infinite path $e_1e_2e_3 \cdots$ with \( \{s(e_i)\} \not\in H \) for all $i$. Since $H \neq \emptyset$, there exists a vertex $v \in G^0$ with \( \{v\} \in H \). By condition (1), we obtain that $v \geq s(e_n)$ for some $n$. Then, since $H$ is hereditary and \( \{v\} \in H \), we must have that \( \{s(e_n)\} \in H \), which is a contradiction. It implies that $H = G^0$, thus finishing the proof. \( \square \)

We now have all the necessary ingredients in hand to prove the first main result of this section, which both characterizes the purely infinite simple Leavitt path algebras in terms of properties of the associated graph and extends Abrams and Aranda-Pino’s result [2, Theorem 11] to ultragraph Leavitt path algebras.

**Theorem 3.4.** Let $G$ be an ultragraph and $K$ a field. Then $L_K(G)$ is purely infinite simple if and only if the following three conditions are satisfied:

1. The only hereditary and saturated subsets of $G^0$ are $\emptyset$ and $G^0$;
2. Every cycle in $G$ has an exit;
3. Every vertex connects to a cycle.

Equivalently, (3) may be replaced by

3’. $G$ contains at least one cycle.

**Proof.** ($\Rightarrow$) Assume that $L_K(G)$ is purely infinite simple. By [12, Theorem 4.7] we have (1) and (2). If $G$ has no cycles, i.e. $G$ is acyclic, then by Theorem 2.9, $L_K(G)$ is a locally matricial $K$-algebra, that is, it is a direct union of matricial $K$-subalgebras. Since every matricial $K$-algebra is finite dimensional, and by [2, Lemma 8], $L_K(G)$ is not purely infinite, which is a contradiction. Hence, $G$ must contain at least one cycle $c$. We then have an infinite path $c^\infty := cc \cdots c \cdots$ in $G$. By Lemma 3.3, every vertex connects to $c^\infty$; equivalently, every vertex connects to $c$, proving (3). Notice that the above argument shows that conditions (3) and (3’) are equivalent in the presence of conditions (1) and (2).

($\Leftarrow$) Assume that (1), (2) and (3) hold. By [12, Theorem 4.7] we immediately obtain that $L_K(G)$ is simple. By Lemmas 2.5 and 3.1, it suffices to show that $L_K(G)$ is not a division ring, and that for every pair of nonzero elements $\alpha, \beta$ in $L_K(G)$ there exist elements $\alpha, \beta$ in $L_K(G)$ such that $\alpha \alpha b = \beta$. Condition (3) implies that there exists a cycle $c = e_1e_2 \cdots e_n$ with $n \geq 1$. By condition (2), $c$ has an exit, that is, there exists $f \in G^1$ such that $s(f) \in r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$ (where $e_{n+1} := e_1$), or $r(e_j)$ contains a sink $v$ for some $j$. If the first case occurs, then we have that $s_j^* s_{e_{i+1}} = 0$. If the second one occurs, then we have that $p v s_{e_j} = p v p s(e_j) s_{e_j} = 0 \cdot s_{e_j} = 0$. This implies that $L_K(G)$ has zero divisors, and thus $L_K(G)$ is not a division ring.

We now apply the Reduction Theorem [11, Theorem 3.2] to find $\bar{\pi}, \bar{b} \in L_K(G)$ such that either $\bar{\pi}\alpha \bar{b} = p_A$ for some nonempty set $A \in G^0$, or $\bar{\pi}\alpha \bar{b} = \sum_{i=1}^m k_i s^i_c$, where $c$ is a cycle without exit. By condition (2), the later cannot
happen, and so we must obtain that \( p_\omega \alpha \beta = p_A \). Take any \( w \in A \). We then have

\[
p_w p_\omega \alpha \beta = p_w p_A = p_w.
\]

By condition (3), \( w \) connects to a cycle \( c' = e'_1 e'_2 \cdots e'_n \), and so there exists a path \( p \in \mathcal{G}^* \) with \( s(p) = w \) and \( s(e'_j) \in r(p) \) for some \( 1 \leq j \leq n \). We note that \( s_p^* p_w s_p = s_p^* s_p = p_{r(p)} \). Let \( v := s(e'_i) \), \( a' := s_p^* \) and \( b' := s_p p_v \). We have that

\[
a' p_w b' = s_p^* p_w s_p p_v = p_{r(p)} p_v = p_v.
\]

By condition (2), \( c' \) has an exit, that is, there exists \( g \in \mathcal{G}^1 \) such that \( s(g) \in r(e'_i) \) and \( g \neq e'_i+1 \) for some \( 1 \leq i \leq n \) (where \( e'_{n+1} := e'_1 \)), or \( r(e'_k) \) contains a sink \( u \) for some \( 1 \leq k \leq n \). Assume that the later occurs. Then, since \( \mathcal{H}_G = \{ \emptyset, \mathcal{G}^0 \} \) and by Lemma 3.3, \( w \) connects to \((c')^\infty \), so \( u \) is not a sink, which is a contradiction. Hence, we are in the first case: there exists \( g \in \mathcal{G}^1 \) such that \( s(g) \in r(e'_i) \) and \( g \neq e'_{i+1} \) for some \( 1 \leq i \leq n \). Take any \( w \in r(g) \). Since \( \mathcal{H}_G = \{ \emptyset, \mathcal{G}^0 \} \) and by Lemma 3.3, \( w \) connects to \((c')^\infty \); equivalently, \( w \) connects to \( c' \), so there exist two distinct cycles \( p, q \) with \( v := s(p) = s(q) \), \( p \) is not a subpath of \( q \), and \( q \) is also not a subpath of \( p \).

For any \( m \geq 1 \) let \( c_m \) denote the path \( p^{m-1} q \), where \( p^0 := p_v \). We then have that \( s_{c_m}^* s_{c_m} = \delta_{m,n} p_{r(q)} \) for every \( m, n \geq 1 \), where \( \delta \) is the Kronecker delta, and so \( s_{c_m}^* s_{c_n} p_v = \delta_{m,n} p_{r(q)} p_v = \delta_{m,n} p_v \) for every \( m, n \geq 1 \) (since \( v \in r(q) \)).

Since \( L_K(\mathcal{G}) \) is simple and \( p_v \neq 0 \), there exist \( \{ a_i, b_i \in L_K(\mathcal{G}) \mid 1 \leq i \leq t \} \) such that \( \beta = \sum_{i=1}^t a_i p_v b_i \). Let \( a_\beta := \sum_{i=1}^t a_i s_{c_i}^* \) and \( b_\beta := \sum_{j=1}^t s_{c_j} b_j \). We then obtain that

\[
a_\beta p_v b_\beta = \sum_{i=1}^t a_i s_{c_i}^* p_v (\sum_{j=1}^t s_{c_j} b_j) = \sum_{i=1}^t a_i p_v b_j = \beta.
\]

Finally, letting \( a = a_\beta a' p_w \alpha \) and \( b = b' b_\beta \), we immediately get that \( aab = \beta \), as desired, thus finishing the proof. \( \square \)

It is worth mentioning the following remark.

**Remark 3.5.** It was shown in [7, Theorem 3.2] that the Steinberg algebra \( A_K(\mathfrak{G}) \) of an ample Hausdorff groupoid \( \mathfrak{G} \) over a field \( K \) is purely infinite simple if and only if \( A_K(\mathfrak{G}) \) is simple, and the characteristic function \( 1_U \) is an infinite idempotent for every compact open \( U \) of the unit space \( \mathfrak{G}^{(0)} \) of \( \mathfrak{G} \). Also, in [9, Theorem 5.5] the authors proved that the Leavitt path algebra \( L_K(\mathcal{G}) \) of an ultragraph \( \mathcal{G} \) over a field \( K \) is isomorphic to the Steinberg algebra \( A_K(\Gamma(T, \sigma)) \), where \( \Gamma(T, \sigma) \) is the groupoid associated with \( \mathcal{G} \), which was defined in [9, Subsection 3.1]. Thus, the Leavitt path algebra \( L_K(\mathcal{G}) \) is purely infinite simple if and only if the following three conditions are satisfied: (1) the only hereditary and saturated subsets of \( \mathcal{G}^0 \) are \( \emptyset \) and \( \mathcal{G}^0 \); (2) every cycle in \( \mathcal{G} \) has an exit; (3) the characteristic function \( 1_U \) is an infinite idempotent for every compact open \( U \) of the unit space \( \Gamma(T, \sigma)^{(0)} \) of \( \Gamma(T, \sigma) \).
For an ultragraph \( \mathcal{G} \) and subset \( \mathcal{H} \subseteq \mathcal{G} \), we denote by \( I(\mathcal{H}) \) the ideal of \( L_K(\mathcal{G}) \) generated by the idempotents \( \{p_A \mid A \in \mathcal{H}\} \). If \( \mathcal{H} = \{v\} \) with \( v \in \mathcal{G}^0 \), then we denote \( I(\mathcal{H}) \) by \( I(v) \). The following lemma provides us with a complete description of the ideal \( I(\mathcal{H}) \) for every hereditary subset \( \mathcal{H} \).

**Lemma 3.6.** Let \( \mathcal{G} \) be an ultragraph, \( K \) a field, and \( \mathcal{H} \) a hereditary subset of \( \mathcal{G}^0 \). Then
\[
I(\mathcal{H}) = I(\mathcal{H}) = \text{Span}_K \{s_{\alpha \beta}^* s_{\nu}^* \mid \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{H}, r(\alpha) \cap A \cap r(\beta) \neq \emptyset\}.
\]

**Proof.** We first claim that
\[
I(\mathcal{H}) = \text{Span}_K \{s_{\alpha \beta}^* s_{\nu}^* \mid \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{H}, r(\alpha) \cap A \cap r(\beta) \neq \emptyset\}.
\]
Indeed, we denote by \( J \) the right-hand side of the above equality. We note that for every \( \alpha, \beta, \mu, \nu \in \mathcal{G}^* \backslash \mathcal{G}^0 \) and every \( A, B \in \mathcal{G}^0 \), we have
\[
(s_{\alpha \beta}^* s_{\nu}^*)(s_{\mu \beta}^* s_{\nu}^*) = \begin{cases} 
  s_{\mu \nu}^* s_{\beta}^* & \text{if } \mu = \beta', s(\mu') \in A \cap r(\alpha) \text{ and } |\mu'| \geq 1, \\
  s_{\alpha \beta}^* s_{\nu}^* & \text{if } \mu = \beta, \\
  s_{\alpha \beta}^* s_{\nu}^* & \text{if } \beta = \mu \beta', s(\beta') \in B \cap r(\nu) \text{ and } |\beta'| \geq 1, \\
 0 & \text{otherwise.}
\end{cases}
\]
By this note and Lemma 2.4 (2), we immediately get that \( J \) is an ideal of \( L_K(\mathcal{G}) \). Moreover, we have \( p_A = pAPAPA \in J \) for all \( A \in \mathcal{G}^0 \), i.e. \( J \) is an ideal of \( L_K(\mathcal{G}) \) containing \( p_A \) for all \( A \in \mathcal{G}^0 \), and so \( I(\mathcal{H}) \subseteq J \). It is obvious that \( J \subseteq I(\mathcal{H}) \), and hence \( J = I(\mathcal{H}) \), thus proving the claim.

It is obvious that \( I(\mathcal{H}) \subseteq I(\mathcal{H}) \). To show the inverse inclusions, it is sufficient to prove that
\[ p_B \in I(\mathcal{H}) \quad \text{for all} \quad B \in \mathcal{H}. \]
We first note that since \( \mathcal{H} \) is a hereditary subset of \( \mathcal{G}^0 \) and by Lemma 3.2, we have that
\[ \mathcal{H} = \bigcup_{i=0}^{\infty} \mathcal{H}_i, \]
where \( \mathcal{H}_0 := \mathcal{H} \) and
\[
\mathcal{H}_{n+1} := \{A \cup F \mid A \in \mathcal{H}_n \text{ and } \text{F is a finite subset of } S_n\}, \\
S_n := \{w \in \mathcal{G}^0 \mid 0 < |s^{-1}(w)| < \infty \text{and}\{r(e) \mid s(e) = w\} \subseteq \mathcal{H}_n\}.
\]
By induction on \( n \), we claim that \( p_B \in I(\mathcal{H}) \) for all \( B \in \mathcal{H}_n \). Indeed, if \( n = 0 \), then \( B \in \mathcal{H} \), and so \( p_B \in I(\mathcal{H}) \). Now we proceed inductively. For \( n > 1 \), we have that \( B = A \cup F \) for some \( A \in \mathcal{H}_{n-1} \) and for some finite subset \( F \subseteq S_{n-1} \). As is shown in the proof of [19, Lemma 3.12], \( \mathcal{H}_i \) is hereditary for all \( i \in \mathbb{N} \). By this fact and the induction hypothesis, we get that \( p_A, p_{(A \cup F)} \in I(\mathcal{H}) \), and \( p_{r(e)} \in I(\mathcal{H}) \) for all \( e \in \mathcal{G}^1 \) with \( r(e) \in \mathcal{H}_{n-1} \).

Write \( F = \{w_1, w_2, \ldots, w_k\} \subseteq S_{n-1} \). Then, for \( 1 \leq i \leq k \), \( w_i \) is a regular vertex and \( r(e) \in \mathcal{H}_{n-1} \) for all \( e \in s^{-1}(w_i) \). This implies that
\[
p_{w_i} = \sum_{e \in s^{-1}(w_i)} s^* e s^* = \sum_{e \in s^{-1}(w_i)} s e p_{r(e)} s^* \in I(\mathcal{H}),
\]
so \( p_F = \sum_{i=1}^{k} p_{w_i} \in I(\mathcal{H}) \) and \( p_B = p_{\Lambda \cap F} = p_A + p_F - p_{\Lambda \cap F} \in I(\mathcal{H}) \), showing the claim. Using this claim and the above note, we immediately get that \( p_B \in I(\mathcal{H}) \) for all \( B \in \mathcal{H} \), and so \( I(\mathcal{H}) = I(\mathcal{H}) \), thus finishing the proof. \( \square \)

Let \( \mathcal{G} \) be an ultragraph and \( c = e_1 e_2 \cdots e_n \) a cycle without exits in \( \mathcal{G} \). We then have that \( s(e_i) \neq s(e_j) \) for all \( 1 \leq i \neq j \leq n \), and \( r(e_i) = \{ s(e_{i+1}) \} \) for all \( 1 \leq i \leq n \) (where \( e_{n+1} := e_1 \)). We denote by \( c^0 \) the set of all subsets of \( \{ s(e_i) \mid 1 \leq i \leq n \} \). It is obvious that \( c^0 \) is a hereditary subset of \( \mathcal{G}^0 \). We say that a path \( p \) contains \( c \) if \( p = qc^n \) for some \( q \in \mathcal{G}^* \) and \( n \geq 1 \). We say that a path \( p \) ends at a vertex \( v \) if \( r(p) = \{ v \} \). The following lemma provides us with a complete description of the ideal \( I(c^0) \).

**Lemma 3.7.** Let \( \mathcal{G} \) be an ultragraph, \( K \) a field, and \( c \) a cycle without exits. Let \( v := s(c) \), and let \( \Lambda_v \) be the (possibly infinite) set of all finite paths in \( \mathcal{G} \) which end at \( v \), and which do not contain \( c \). Then

\[
I(\overline{c^0}) = I(c^0) = I(v) \cong M_{\Lambda_v}(K[x, x^{-1}]),
\]

where \( \overline{c^0} \) is the smallest saturated hereditary subset of \( \mathcal{G}^0 \) containing \( c^0 \).

**Proof.** We first claim that

\[
I(v) = \text{Span}_K \{ s_{\alpha}s_{\beta}^* \mid \alpha, \beta \in \mathcal{G}^*, r(\alpha) = \{ v \} = r(\beta) \}.
\]

Indeed, we denote by \( J \) the right-hand side of the above equality. By repeating the method described in the proof of Lemma 3.6, we immediately get that \( J \) is an ideal of \( L_K(\mathcal{G}) \). Moreover, we have \( p_v = p vp_v \in J \), i.e. \( J \) is an ideal of \( L_K(\mathcal{G}) \) containing \( p_v \), and so \( I(v) \subseteq J \). For \( \alpha, \beta \in \mathcal{G}^* \) with \( r(\alpha) = \{ v \} = r(\beta) \), we obtain that \( s_{\alpha}s_{\beta}^* = s_{\alpha}p_v s_{\beta}^* \in I(v) \), and hence \( J \subseteq I(v) \), thus proving the claim.

It is obvious that \( I(v) \subseteq I(c^0) \). To show the inverse inclusions, we write \( c = e_1 e_2 \cdots e_n \) \((n \geq 1)\) and \( v_i := s(e_{i+1}) \) for all \( 1 \leq i \leq n - 1 \) (note that \( v = s(c) = s(e_1) \)). For \( 1 \leq i \leq n - 1 \), there exists a path \( p := e_1 \cdots e_i \) with \( s(p) = v \) and \( r(p) = \{ v_i \} \), and so \( p_v \circ s_{\alpha} \circ s_{\beta}^* = s_{\alpha}p_v s_{\beta}^* \in I(v) \). Take any a non-empty set \( A \in c^0 \). We then have that \( A \) is a subset of \( \{ v, v_i \mid 1 \leq i \leq n - 1 \} \), and \( p_A = \sum_{w \in A} p_w \in I(v) \). This implies that \( I(c^0) \subseteq I(v) \), and so \( I(v) = I(c^0) \). By Lemma 3.6, \( I(c^0) = I(\overline{c^0}) \), and so \( I(v) = I(c^0) = I(\overline{c^0}) \).

Consider the family

\[
\mathcal{B} := \{ s_{\alpha}s_{\beta}^k \mid \alpha, \beta \in \Lambda, k \in \mathbb{Z} \},
\]

where as usual \( s_0^0 \) denotes \( p_v \) and \( c^k \) denotes \( (s_{\alpha})^{-k} \) for \( k < 0 \).

We note that since \( c \) is a cycle without exits, we have \( s_c s_c^* = p_v \). Also, for \( \alpha \in \mathcal{G}^* \) with \( r(\alpha) = \{ v \} \), it may be written in the form: \( \alpha = pc^m \) for some \( p \in \Lambda_v \) and for some \( m \in \mathbb{N} \) (where \( c^0 := v \)). Using this note, we immediately obtain that for all \( p, q \in \mathcal{G}^* \) with \( r(p) = \{ v \} = r(q) \), \( sp^*_s = s_{\alpha}s_{\beta}^k \) for some \( \alpha, \beta \in \Lambda_v \) and for some \( k \in \mathbb{Z} \). This implies that \( \mathcal{B} \) generates \( I(v) \).

We next claim that \( \mathcal{B} \) is a \( K \)-linearly independent set. Consider the equation

\[
\sum_{i=1}^{n} k_is_{\alpha_i}s_{\alpha_i}^m_i \cdot s_{\beta_i}^* = 0,
\]

\((*)\)
where \( k_i, m_i \in \mathbb{N} \) and \( \alpha_i, \beta_i \in \Lambda_v \). By induction on \( n \) we prove that \( k_i = 0 \) for all \( 1 \leq i \leq n \). If \( n = 1 \), then we have \( k_1 s_{\alpha_1} m_i s_{\beta_1} = 0 \), and so \( k_1 p_v = k_1 (s_c^m) s_{\alpha_1} m_i s_{\beta_1} s_{\beta_1} = 0 \). By Lemma 2.4 (1), \( p_v \neq 0 \), and hence \( k_1 = 0 \).

Now we proceed inductively. For \( n > 1 \), if \( s(\alpha_1) \neq s(\alpha_j) \) for some \( 1 \leq j \leq n \), then we have

\[
\sum_{i=2}^{n} k_i p_s(\alpha_j) s_{\alpha_i} m_i s_{\beta_i} = p_s(\alpha_j) (\sum_{i=1}^{n} k_i s_{\alpha_i} m_i s_{\beta_i}) = 0. 
\]

Using equations (*) and (**), and the induction hypothesis, we immediately get that \( k_i = 0 \) for all \( i \). If \( s(\beta_1) \neq s(\beta_j) \) for some \( 1 \leq j \leq n \), then we have

\[
\sum_{i=2}^{n} k_i s_{\alpha_i} m_i s_{\beta_i} p_s(\beta_j) = (\sum_{i=1}^{n} k_i s_{\alpha_i} m_i s_{\beta_i}) p_s(\beta_j) = 0. 
\]

Using equations (*) and (**), and the induction hypothesis, we obtain that \( k_i = 0 \) for all \( i \).

Consider the case that \( s(\alpha_1) = s(\alpha_i) \) and \( s(\beta_1) = s(\beta_i) \) for all \( i \). Then, since \( r(\alpha_i) = \{v\} = r(\beta_i) \) for all \( i \), we have that

\[
s_{\alpha_i} m_i = \delta_{\alpha_1, \alpha_i} p_r(\alpha_i) = \delta_{\alpha_1, \alpha_i} p_v \quad \text{and} \quad s_{\beta_i} m_i = \delta_{\beta_1, \beta_i} p_r(\beta_i) = \delta_{\beta_1, \beta_i} p_v
\]

for all \( i \) (where \( \delta \) is the Kronecker delta), and

\[
0 = s_{\alpha_1} m_i s_{\beta_1} = \sum_{j=1}^{r} k_{ij} s_{c_i} s_{\beta_i}
\]

where \( 1 \leq r \leq n \), \( \{i_j \mid 1 \leq j \leq r\} \subseteq \{1, 2, \ldots, n\} \) and \( \alpha_1 = \alpha_{i_j}, \beta_1 = \beta_{i_j} \) for all \( 1 \leq j \leq r \), and \( m_{i_j} \)'s are distinct. Now the grading in \( L_K(G) \) (see Lemma 2.4 (2)) shows that \( k_{ij} = 0 \) for all \( 1 \leq j \leq r \). From this observation and equation (*), and by the induction hypothesis, we obtain that \( k_i = 0 \) for all \( i \), showing the claim. Therefore \( B \) is a \( K \)-basis for \( I(v) \).

We define \( \phi : I(v) \rightarrow M_{\alpha} (K [x, x^{-1}]) \) by setting \( \phi(s_{\alpha} s_{c} s_{\beta}^*) = x^k E_{\alpha, \beta} \) for each \( s_{\alpha} s_{c} s_{\beta}^* \in B \), where \( x^k E_{\alpha, \beta} \) denotes the element of \( M_{\alpha} (K [x, x^{-1}]) \) which is \( x^k \) in the \((\alpha, \beta)\) entry, and zero otherwise. Then we easily check that \( \phi \) is a \( K \)-algebra isomorphism, thus finishing the proof. \( \square \)

The following lemma provides us with a criterion for ultragraph Leavitt path algebras to be graded simple.

**Lemma 3.8.** Let \( G = (G^0, G^1, r, s) \) be an ultragraph and \( K \) a field. Then \( L_K(G) \) is graded simple if and only if \( \mathcal{H}_G = \{0, G^0\} \).

**Proof.** (\( \Longleftrightarrow \)) Assume that \( \mathcal{H}_G = \{0, G^0\} \) and \( I \) is a nonzero graded ideal of \( L_K(G) \). Take any \( \alpha \in I \setminus \{0\} \). By the Reduction Theorem [11, Theorem 3.2], there exist \( a, b \in L_K(G) \) such that either \( aab = p_a \) for some nonempty set \( A \in G^0 \), or \( aab = \sum_{i=1}^{m} k_i s_c^i \neq 0 \), where \( k_i \in K \) and \( c \) is a cycle without exit in \( G \). If the first case occurs, then we have \( p_a \in I \). If the second case occurs, then we have \( \sum_{i=1}^{m} k_i s_c^i = aab \in I \). Since \( I \) is a graded ideal, \( k_i s_c^i \in I \) for all \( 1 \leq i \leq m \). Since \( aab \neq 0 \), there exists an element \( 1 \leq j \leq m \) such that \( k_j \neq 0 \), and so \( p_{r(c)} = k_j^{-1} (s_c^j)^* (k_j s_c^j) \in I \). In any case, we obtain that there exists a nonempty set \( B \in G^0 \) such that \( p_B \in I \). Let \( \mathcal{H} := \{A \in G^0 \mid p_A \in I\} \).
We then have $B \in \mathcal{H}$. By repeating the method described in the proof of [19, Lemma 3.4], we immediately get that $\mathcal{H}$ is a saturated hereditary subset of $\mathcal{G}^0$, and so $\mathcal{H} = \mathcal{G}^0$. From this observation, and by Lemma 2.5, we have $I = L_K(\mathcal{G})$, showing that $L_K(\mathcal{G})$ is graded simple.

($\implies$) It is borrowed from [19, Lemma 3.7]. Assume that $L_K(\mathcal{G})$ is graded simple, and $\mathcal{H}$ is a nonempty saturated hereditary subset of $\mathcal{G}^0$. Let $K := \{w \in \mathcal{G}^0 \mid \{w\} \in \mathcal{H}\}$ and $S := \mathcal{G}^0 \setminus K$. We define an ultragraph $\mathcal{F} = (F^0, \mathcal{F}^1, r_F, s_F)$ as follows: $F^0 := S$, $\mathcal{F}^1 := \{e \in \mathcal{G}^1 \mid r(e) \cap S \neq \emptyset\}$, and $s_F := s(e)$, $r_F := r(e) \cap S$ for all $e \in \mathcal{F}^1$. We note that if $e \in \mathcal{F}^1$, then $r(e) \cap S \neq \emptyset$, so $r(e) \notin \mathcal{H}$. Since $\mathcal{H}$ is hereditary, $\{s(e)\} \notin \mathcal{H}$ and $s(e) \in S$. Therefore, $\mathcal{F}$ is well defined.

Assume that $L_K(\mathcal{F})$ is generated by the set $\{q_A, t_e, t_e^* \mid A \in \mathcal{F}^0, e \in \mathcal{F}^1\}$ satisfying the relations analogous to (1)–(4) in Definition 2.2. Indeed, the same argument as in the proof of [19, Lemma 3.7] shows the relations (1) and (2). For (3), if $e \in \mathcal{F}^1$, then $c_e^*c_e = t_e^*t_e = q_{r(e)} = q_{r(e) \cap S} = b_{r(e)}$. If $e \notin \mathcal{F}^1$, then $r(e) \cap S = \emptyset$, and so $b_{r(e)} = q_\emptyset = 0 = c_e^*c_e$. Therefore, we obtain that $c_e^*c_e = b_{r(e)}$ for all $e \in \mathcal{G}^1$, as desired.

For (4), let $v$ be a regular vertex in $\mathcal{G}$. If $v \notin S$, then $\{v\} \in \mathcal{H}$ and $r(e) \in \mathcal{H}$ for all $e \in s^{-1}(v)$ (since $\mathcal{H}$ is hereditary), so $r(e) \cap S = \emptyset$ for all $e \in s^{-1}(v)$ and

$$\sum_{e \in s^{-1}(v)} c_e c_e^* = 0 = q_\emptyset = q_{\{v\} \cap S} = b_v.$$  

If $v \in S$, then since $s_F^{-1}(v) \subseteq s^{-1}(v)$ we have that $|s_F^{-1}(v)| < \infty$. We claim that $r(e) \cap S \neq \emptyset$ for some $e \in s^{-1}(v)$. Indeed, assume that $r(e) \cap S = \emptyset$ for all $e \in s^{-1}(v)$. Since $L_K(\mathcal{G})$ is graded simple and $\mathcal{H} \neq \emptyset$, $I(\mathcal{H}) = L_K(\mathcal{G})$, so

$$L_K(\mathcal{G}) = \text{Span}_K \{s_\alpha \alpha A_s^\beta s^*_\beta \mid \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{H}, r(\alpha) \cap A \cap r(\beta) \neq \emptyset\},$$

by Lemma 3.6. Then, for each $e \in s^{-1}(v)$, there exist $k_i \in K$, $\alpha_i, \beta_i \in \mathcal{G}^*$, and $A_i \in \mathcal{H}$ for all $1 \leq i \leq n$ such that $p_{r(e)} = \sum_{i=1}^n k_i s_{\alpha_i} \alpha_i A_i s^*_{\beta_i}$. Furthermore, since $p_{r(e)} = p_{r(e)} p_{r(e)} = \sum_{i=1}^n k_i p_{r(e)} s_{\alpha_i} \alpha_i A_i s^*_{\beta_i}$, we may assume that $s(\alpha_i) \in r(e)$ when $|\alpha_i| \geq 1$ and $s(\alpha_i) \subseteq r(e)$ when $|\alpha_i| = 0$. Let $B := (\bigcup_{1 \leq i \leq n} |\alpha_i| = 0 (s(\alpha_i) \cap A_i)) \cup \{s(\alpha_j) \mid 1 \leq j \leq n, |\alpha_j| \geq 1\}$. We then...
have $B \in \mathcal{G}^0$, $B \subseteq r(e)$, and
\[ p_r(e) - p_B = (p_r(e) - p_B)p_r(e) = (p_r(e) - p_B)\sum_{i=1}^{n} k_is_{\alpha_i}p_{A_i}s_{\beta_i}^* = 0. \]
If $r(e) \neq B$, there exists an element $w \in r(e)$ and $w \notin B$, so $p_v = p_v p_r(e) = pvp_B = 0$; but $p_v$ is well-known to be a nonzero element inside the ultragraph Leavitt path algebra, we get a contradiction, which shows that $r(e) = B$. Since $r(e) \cap S = \emptyset$, so $\{w\} \in \mathcal{H}$ for all $w \in r(e)$, so $\{s(\alpha_j)\} \in \mathcal{H}$ for every path $\alpha_j$ with $|\alpha_j| \geq 1$. For any path $\alpha_i$ with $|\alpha_i| = 0$, since $\mathcal{H}$ is hereditary and $A_i \in \mathcal{H}$, $s(\alpha_i) \cap A_i \in \mathcal{H}$. From these observations we immediately get that $B \in \mathcal{H}$, i.e., $r(e) \in \mathcal{H}$. Therefore, we obtain that $r(e) \in \mathcal{H}$ for all $e \in s^{-1}(v)$. Since $\mathcal{H}$ is saturated, $\{v\} \in \mathcal{H}$, and so $v \notin S$, which is a contradiction, which shows the claim. This implies that $|s_{\mathcal{F}}^{-1}(v)| > 0$, i.e., $v$ is a regular vertex in $\mathcal{F}$. Thus, $\sum_{e \in s^{-1}(v)} c_ee^* = \sum_{e \in s_{\mathcal{F}}^{-1}(v)} c_ee^* + \sum_{e \in s^{-1}(v) \setminus s_{\mathcal{F}}^{-1}(v)} c_ee^* = \sum_{e \in s_{\mathcal{F}}^{-1}(v)} c_ee^* + 0 = q_v = q_{\{v\} \cap S} = b_v$, showing relation (4).

Suppose that $L_K(\mathcal{G})$ is generated by the set $\{p_A,s_e,s_e^* \mid A \in \mathcal{G}^0,e \in \mathcal{G}^1\}$ satisfying the relations analogous to (1)–(4) in Definition 2.2. By the universal property of $L_K(\mathcal{G})$, there is a unique $K$-algebra homomorphism $\phi : L_K(\mathcal{G}) \rightarrow L_K(\mathcal{F})$ such that $\phi(p_A) = b_A$, $\phi(s_e) = c_e$ and $\phi(s_e^*) = c_e^*$ for all $A \in \mathcal{G}^0$ and $e \in \mathcal{G}^1$. It is obvious that $\phi$ is a $\mathbb{Z}$-graded homomorphism, and so $\ker(\phi)$ is a graded ideal of $L_K(\mathcal{G})$. Since $p_v \in \ker(\phi)$ for all $v \notin S$ and $\mathcal{H} \neq \emptyset$, $\ker(\phi) \neq 0$. Since $L_K(\mathcal{G})$ is graded simple, $\ker(\phi) = L_K(\mathcal{G})$. This shows that $S = \emptyset$ and $K = \mathcal{G}^0$. Since $\mathcal{H}$ is a saturated hereditary subset, and by Lemma 2.1, we immediately obtain that $\mathcal{H} = \mathcal{G}^0$, thus finishing the proof.

We now have all the tools necessary to generalize [4, Theorem 3.1.14] which the authors of [4] call the Trichotomy Principle for graded simple Leavitt path algebras of graphs. We prove this principle for graded simple ultragraph Leavitt path algebras.

**Theorem 3.9.** Let $\mathcal{G}$ be an ultragraph and $K$ a field. If $L_K(\mathcal{G})$ is graded simple, then exactly one of the following occurs:

1. $L_K(\mathcal{G})$ is locally matricial, or
2. $L_K(\mathcal{G}) \cong M_{\Lambda}(K[x,x^{-1}])$ for some set $\Lambda$, or
3. $L_K(\mathcal{G})$ is purely infinite simple.

**Proof.** By Lemma 3.8, the graded simplicity of $L_K(\mathcal{G})$ is equivalent to that $\mathcal{H}_G = \{\emptyset, \mathcal{G}^0\}$. The three possibilities given in the statement correspond precisely to whether: (1) $\mathcal{G}$ contains no cycles; resp., (2) contains exactly one cycle; resp., (3) contains at least two cycles.

If $\mathcal{G}$ contains no cycle then (1) follows from Theorem 2.9. Consider the case that $\mathcal{G}$ contains at least two cycles. Let $c_1 = e_1 \cdots e_n$ and $c_2 = f_1 \cdots f_m$ be two distinct cycles in $\mathcal{G}$. We then have two infinite paths $(c_1)^\infty = c_1c_1 \cdots c_1 \cdots$ and $(c_2)^\infty = c_2c_2 \cdots c_2 \cdots$. Applying Lemma 3.3 (1), we immediately get that every vertex $v \in \mathcal{G}^0$ connects to both $(c_1)^\infty$ and $(c_2)^\infty$; equivalently, $v$ connects to both $c_1$ and $c_2$. Consequently, every cycle in $\mathcal{G}$ has an exit. Then, by Theorem 3.4, we have that $L_K(\mathcal{G})$ is purely infinite simple.
Now suppose that $G$ contains exactly one cycle $c = \alpha_1\alpha_2 \cdots \alpha_n$. If $c$ has exits then there exists $f \in G^1$ such that $s(f) = r(e_i)$ and $f \neq e_{i+1}$ for some $1 \leq i \leq n$ (where $e_{n+1} := e_1$), or $r(e_j)$ contains a sink $w$ for some $j$. If the first case occurs, then by Lemma 3.3 (1), every vertex in $r(f)$ connects to $c$, and so $G$ has at least two cycles, which is a contradiction. If the second one occurs, then by Lemma 3.3 (1), $w$ connects to the infinite path $c^\infty := c \cdots c \cdots$, and so $w$ is not a sink, we get a contradiction. Therefore, $c$ is a cycle without exits. Now, by Lemma 3.7, $I(c^0) \cong \Lambda(K[x, x^{-1}])$, where $\Lambda$ is the set of all finite paths in $G$ which end at $v := s(c)$, and which do not contain $c$. Since $c^0$ is a non-empty saturated hereditary subset of $G^0$ and $H_G = \{\emptyset, G^0\}$, we must have that $c^0 = G^0$, and so $I(c^0) = L_K(G)$ by Lemma 2.5. This implies that $L_K(G) \cong \Lambda(K[x, x^{-1}])$, thus finishing our proof.

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