Drinfeld isomorphisms from quantum Stokes matrices

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Abstract

In this paper, we show that the quantum Stokes matrices, of certain meromorphic linear system of ordinary differential equations, give rise to a family of Drinfeld isomorphisms from quantum groups to the undeformed universal enveloping algebra of $\mathfrak{gl}(n)$. In particular, we compute explicitly the Drinfeld isomorphisms corresponding to caterpillar points on the parameter space. Our computation unveils a relation between the asymptotics of confluent hypergeometric functions and the Gelfand-Zeitlin subalgebras.

As by products, we show that the Drinfeld isomorphisms at caterpillar points coincide with the Appel-Gautam isomorphisms. Then by going to the semiclassical limit, we place the results in this paper into the context of Poisson geometry. As an application, we prove the conjecture of Appel and Gautam, i.e., their isomorphisms are canonical quantization of the Alekseev-Meinrenken diffeomorphisms arising from the linearization problem in Poisson geometry.

1 Introduction

1.1 Quantum Stokes matrices and Drinfeld isomorphisms

Throughout this introduction, we will take the Lie algebra $U(\mathfrak{gl}(n))$ generated by $\{E_{ij}\}_{1 \leq i < j \leq n}$ subject to the relation $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$. Let us define the $n$ by $n$ matrix $T = (T_{ij})$ with entries valued in $U(\mathfrak{gl}(n))$

$$T_{ij} = E_{ji}, \quad \text{for } 1 \leq i, j \leq n. \quad (1)$$

For any $u$ in the set $\mathfrak{h}_{\text{reg}}(\mathbb{R})$ of $n$ by $n$ real diagonal matrices with distinct eigenvalues, let us consider the linear system of differential equation

$$\frac{dF}{dz} = \left( iu + \frac{\hbar}{2\pi i} T \right) \cdot F, \quad (2)$$

for an $n$ by $n$ matrix function $F(z)$ with entries in $U(\mathfrak{gl}(n))[\hbar]$. Here $u$ is seen as a matrix with scalar entries in $U(\mathfrak{gl}(n))$, and $U(\mathfrak{gl}(n))[\hbar]$ is seen as a topological Hopf algebra over the ring of formal power series $\mathbb{C}[\hbar]$.

Equation (2) has a canonical solution $F_{h_{\pm}}$ in each Stokes sector $\text{Sect}_{\pm} := \{ z \in \mathbb{C} \mid \pm \text{Re}(z) > 0 \}$. These solutions are specified by a prescribed asymptotics at $z = \infty$ in the corresponding sectors. The quantum Stokes matrices $S_{h_{\pm}}(u)$ are then the transition matrices between the solutions. In particular, $S_{h_{+}}(u)$ and $S_{h_{-}}(u)^{-1}$ are $n$ by $n$ matrices $(s_{ij}^{(\pm)})$, with entries $s_{ij}^{(\pm)}$ in $U(\mathfrak{gl}(n))[\hbar]$. See Section 2.2.

Now let $U(R)$ be the Faddeev-Reshetikhin-Takhtajan realization [26] of the quantum group of $\mathfrak{gl}(n)$ with generators $(l_{ij}^{(\pm)})_{i,j=1}^{n}$, see Section 2. It is a Hopf algebra over $\mathbb{C}[\hbar]$, and is a deformation of $U(\mathfrak{gl}(n))$. In [12], Drinfeld pointed out that there exists an isomorphism between the $\mathbb{C}[\hbar]$ algebras $U(R)$ and $U(\mathfrak{gl}(n))[\hbar]$. Such isomorphisms are not canonical, and one explicit isomorphism was constructed by Appel and Gautam in [2] (for $\mathfrak{g} = \mathfrak{sl}(n)$). Our first result states that Stokes matrices produce a natural family of Drinfeld isomorphisms.

**Theorem 1.1.** For any fixed $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$, the map

$$\nu_{\hbar}(u) : U(R) \to U(\mathfrak{gl}(n))[\hbar] ; \quad l_{ij}^{(\pm)} \mapsto s_{ij}^{(\pm)},$$

is an algebra isomorphism.

The second main result in this paper is to find an explicit Drinfeld isomorphism, using the closure of Stokes matrices developed in [34]. In the following, let us first illustrate the idea in the classical case.
1.2 Closure of Stokes matrices

In [34], we have considered the meromorphic linear systems of ordinary differential equations

\[ \frac{dF}{dz} = \left( iu - \frac{1}{2\pi i z} A \right) F, \]

where \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \) the set of real diagonal matrices with distinct eigenvalues, and \( A \in \text{Herm}(n) \) the space of \( n \) by \( n \) Hermitian matrices. Associated to any \((u, A) \in \mathfrak{h}_{\text{reg}} \times \text{Herm}(n)\) and a chosen branch of \( \log(z) \), there are Stokes matrices \( S_{\pm}(u, A) \).

The isomonodromy (also known as monodromy preserving) deformation problem is to find the matrix valued function \( A(u) \) such that the Stokes matrices \( S_{\pm}(u, A(u)) \) are (locally) constant. In particular, it is the case when the matrix valued function \( A(u) : \mathfrak{h}_{\text{reg}}(\mathbb{R}) \to \text{Herm}(n) \) satisfies the isomonodromy deformation equation with respect to the diagonal entries \( \{u_i\}_{i=1, \ldots, n} \) of \( u \)

\[ \frac{\partial A}{\partial u_k} = \frac{1}{2\pi i} [\text{ad}_{u}^{-1} \text{ad}_{E_k} A, A], \quad k = 1, \ldots, n. \]

Here \( E_k \) is the \( n \times n \) diagonal matrix whose \((k, k)\)-entry is 1 and other entries are 0. Note that \( \text{ad}_{E_k} A \) takes values in the space \( \mathfrak{gl}(n)^{\text{ad}} \) of off diagonal matrices and that \( \text{ad}_{u} \) is invertible when restricted to \( \mathfrak{gl}(n)^{\text{ad}} \). See more detailed discussions in e.g., [6, 20].

The isomonodromy deformation equation with respect to the derivation of \( u_j \) is generated by the time \((u_1, \ldots, u_n)\)-dependent quadratic Hamiltonian

\[ H_j := \left(-\frac{1}{\pi i}\right) \sum_{k < j} a_{kj} a_{kj}, \quad \text{for } 1 \leq j \leq n, \]

where \( a_{ij} \)'s are the entry functions on \( \text{Herm}(n) \). These are the classical non-homogeneous Gaudin Hamiltonians, which are the coefficients of (an appropriate version of) the Knizhnik–Zamolodchikov (KZ) equations [21]. It has been known that the prescription of the asymptotic behavior of solutions of KZ equations is controlled by the geometry of the de Concini-Procesi wonderful compactification \( \mathfrak{h}_{\text{reg}}(\mathbb{R}) \) of \( \mathfrak{h}_{\text{reg}}(\mathbb{R}) \). See [8, 9].

The connected components \( U_{\sigma} \) of \( \mathfrak{h}_{\text{reg}}(\mathbb{R}) \) are labelled by the elements \( \sigma \) in the permutation group \( S_n \), i.e.,

\[ U_{\sigma} := \{ u = \text{diag}(u_1, \ldots, u_n) \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \mid u_{\sigma(1)} < \cdots < u_{\sigma(n)} \}, \]

and the closure of \( U_{\sigma} \) in \( \mathfrak{h}_{\text{reg}}(\mathbb{R}) \) will be denoted by \( \overline{U_{\sigma}} \).

Then similar to the theory of KZ equations, associated to a given point \( u_0 \) in the 0-dimension strata of \( \overline{U_{\sigma}} \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), in [34] we introduce a solution \( A(u) \) of the isomonodromy equation (4), with a prescribed regularized asymptotics \( A_0 \in \text{Herm}(n) \) as \( u \to u_0 \) within \( U_{\sigma} \). Since \( A(u) \) gives an isomonodromy deformation, we know that \( S_{\pm}(u, A(u)) \) is locally constant, therefore is constant when \( u \in U_{\sigma} \). Thus it is natural to define Stokes matrices \( S_{\pm}(u_0, A_0) \), associated to \((u_0, A_0) \in \overline{U_{\sigma}} \times \text{Herm}(n) \), such that

\[ S_{\pm}(u_0, A_0) := S_{\pm}(u, A(u)), \quad u \in U_{\sigma}. \]

The interesting thing is that we can describe \( S_{\pm}(u_0, A_0) \) rather explicitly, using the combinatorial description of the point \( u_0 \) as a planar labelled rooted tree \( T(u_0) \) with \( n \) (ordered) leaves and \( n \) inner vertices. See e.g., [8]. In [33], we show that how the system (4) of rank \( n \) are “decoupled” into multiple much simpler and lower rank systems via isomonodromy deformation, according to the branching of the inner vertices of \( T(u_0) \). As a result, we gave a description of the abstract Stokes matrices \( S_{\pm}(u_0, A_0) \) using the Stokes matrices of the simpler and lower rank systems. In particular, when \( u_0 \) is a caterpillar point in \( \overline{U_{\sigma}} \), the Stokes matrices \( S_{\pm}(u_0, A_0) \) have a closed-form expression, given explicitly by the Gelfand-Zeitlin action and angular variables of \( A_0 \in \text{Herm}(n) \).

**Remark 1.2.** Note that a given \( u_0 \) can live in the boundary of \( \overline{U_{\sigma}} \), for different \( \sigma \in S_n \). The right hand side of (6) is in general different for different \( \sigma \in S_n \). Thus the definition of \( S_{\pm}(u_0, A_0) \) will depend on the choice of \( \sigma \), or equivalently the choice of planar embeddings of the labelled tree \( T(u_0) \). In particular, for \( u_0 \) being a caterpillar point, these different choices correspond to the different chosen Gelfand-Zeitlin chains \( u(1) \subset \cdots \subset u(n) \), and will produce the cactus group actions on Gelfand-Zeitlin patterns. The appearance of Gelfand-Zeitlin integrable systems can be understood as follows. If we choose the time \( t(u) = (t, t^2, \ldots, t^n) \), then when \( t \) approaches to \( \infty \), the Hamiltonians \( \{ H_j \} \) will become the Hamiltonians \( H_j = \frac{1}{\pi i} \sum_{k < j} \frac{a_{kj}}{u_k} \) of Gelfand-Zeitlin systems, see [29]. See more discussions on the Gelfand-Zeitlin systems and Stokes matrices at caterpillar points in [34].
The above construction of closure of Stokes matrices can be extended from \( \text{Herm}(n) \) to generic \( \mathfrak{gl}(n) \) case, see Section[3]. In this way, we can think of Stokes matrices \( S(u, A) \) as an \( n \)-parameter \( u = (u_1, \ldots, u_n) \) family deformation of the rather explicit \( S_{\pm}(u_{\text{cat}}, A) \). Stokes matrices has appeared in many places, but the difficult in application is that they don’t have explicit expression (except for the rank 2 case). However, if the problem that one considers has certain isomonodromy property, then one may take the irregular data \( u \) as a caterpillar point to deduce an explicit formula in the problem. Such an example is provided in [34], where using this method we find an explicit formula of the Ginzburg-Weinstein linearization for the unitary group \( U(n) \). In the following, we will provide another example, giving an explicit formula of Drinfeld isomorphism for quantum groups.

### 1.3 Quantum Stokes matrices at caterpillar points

The closure of Stokes matrices in the classical case carries over to the equation (2). In particular, although the coefficients of the linear system are valued in the non-commutative algebra \( U(\mathfrak{gl}(n))[\hbar] \), one can still introduce the Stokes matrices \( S_{\pm}(u_{\text{cat}}, A) \) at a caterpillar point \( u_{\text{cat}} \). Furthermore, the Gelfand-Zelitin subalgebra in \( U(\mathfrak{gl}(n)) \) naturally arises in this procedure, and makes it possible to compute explicitly \( S_{\pm}(u_{\text{cat}}) \). See the book [24] of Molev for the theory of Gelfand-Zelitin subalgebras.

To state the explicit formula, let us introduce the following notations. Define \( T(\zeta) := \zeta \text{Id} - \frac{\hbar}{2}(k-1)T \). For any \( 1 \leq m \leq n \), \((a_1, \ldots, a_m)\) and \((b_1, \ldots, b_m)\) elements of \( \{1, \ldots, n\} \), let us consider the \( m \) by \( m \) quantum minor

\[
\Delta_{a_1, \ldots, a_m}^{b_1, \ldots, b_m}(T(\zeta)) := \sum_{\sigma \in S_m} (-1)^{\sigma} T_{a_{\sigma(1)}, b_{\sigma(1)}}(\zeta_1) \cdots T_{a_{\sigma(m)}, b_{\sigma(m)}}(\zeta_m) \in U(\mathfrak{gl}(n))[\zeta, \hbar],
\]

where \( \zeta_j = \zeta + \hbar(j - 1) \). For any \( 1 \leq k \leq n \), let

\[
M_k(\zeta) := \Delta_{1, \ldots, k}^{1, \ldots, k}(T(\zeta - \frac{\hbar}{2}(k-1)))
\]

be the upper left \( k \) by \( k \) quantum-minor of \( T \). Then the subalgebra, generated in \( U(\mathfrak{gl}(n))[\hbar] \) by the coefficients in all \( M_k(\zeta) \) for \( 1 \leq k \leq n \), is commutative. If we denote by \( \zeta_1^{(k)}, \ldots, \zeta_k^{(k)} \) the roots of \( M_k(\zeta) \) defined in an appropriate splitting extension, then (see Section[4])

**Theorem 1.3.** For any \( 1 \leq k \leq n-1 \), the \((k, k+1)\)-entry of \( S_{\pm}(u_{\text{cat}}) \), as an element in \( U(\mathfrak{gl}(n))[\hbar] \) is given by

\[
s_{k,k+1}^{(\pm)} = -\hbar e^{-\zeta_k^{(k-1)} - \zeta_{k+1}^{(k-1)}} \frac{\hbar}{4} \sum_{j=1}^{k} \left( \prod_{l=1}^{k} \Gamma(1 + \frac{\zeta_j^{(k)} - \zeta_j^{(k-1)}}{2\pi i}) \prod_{l=1}^{k+1} \Gamma(1 + \frac{\zeta_k^{(k)} - \zeta_k^{(k-1)} - \hbar}{2\pi i}) \right) \left( \prod_{l=1, l \neq j}^{k} \zeta_l^{(k)} - \zeta_j^{(k)} \right) E_{k+1, j}.
\]

Similarly, one can get the expression of \( s_{k-1,k}^{-} \). As a consequence of Theorem[1.1] and the isomonodromy property of quantum Stokes matrices, we have

**Corollary 1.4.** The algebra isomorphism arising from the Stokes matrices \( S_{\pm}(u_{\text{cat}}) \) at a caterpillar point is

\[
\nu_{\hbar}(u_{\text{cat}}) : U(R) \to U(\mathfrak{gl}(n))[\hbar] ; \quad l_{k,k+1}^{(\pm)} \mapsto s_{k,k+1}^{(\pm)},
\]

with \( s_{k,k+1}^{(\pm)} \in U(\mathfrak{gl}(n))[\hbar] \) given in[8] (the expression of \( s_{k,k+1}^{(-)} \) is similar).

### 1.4 The Appel-Gautam isomorphisms and the Alekseev–Meinrenken diffeomorphisms

In [2], Appel and Gautam constructed two Drinfeld isomorphisms \( \Psi_{AG} \) and \( \Phi_{AG} \), associated to Gamma and hyperbolic functions respectively, see [2] Remark 2.6 (2)]. After identifying \( U(R) \) with the Drinfeld-Jimbo quantum group \( U_{\hbar}(\mathfrak{gl}(n)) \) (see Section[5.2]), the isomorphism in (9) coincides with the expression of the Appel-Gautam isomorphism \( \Psi_{AG} \) in [2] Formula 2.5].
Corollary 1.5. As $u$ being the caterpillar point, the Drinfeld isomorphism $\nu_{\hbar}(u_{\text{cat}})$ coincides with the isomorphism $\Psi_{\text{AG}}$.

Similar to the classical case [34], we can get another Drinfeld isomorphism from $\nu_{\hbar}(u_{\text{cat}})$ by a gauge transformation, see Section 5.3.2. The resulting explicit isomorphism will be given by hyperbolic functions, and coincides with the Appel-Gautam isomorphism $\Phi_{\text{AG}}$ involving hyperbolic functions [2] Remark 2.6 (2)]. Thus we have explained the two types of Appel-Gautam isomorphisms as the Drinfeld isomorphism at a caterpillar point and its gauge transformation. In particular, this viewpoint helps to verify the conjecture in [2]: on the one hand, the semiclassical limit of quantum Stokes matrices are the classical Stokes matrices (see [34]); on the other hand, the (gauge transformation of) Stokes matrices at caterpillar points give rise to the Alekseev–Meinrenken diffeomorphism [34]. Thus as an immediate consequence (see Section 5.4), it verifies the conjecture in [2], i.e.,

Corollary 1.6. The isomorphism $\Phi_{\text{AG}}$ is a canonical quantization of the Alekseev–Meinrenken diffeomorphism.

Here the Alekseev–Meinrenken diffeomorphism [1] is a Poisson isomorphism from the dual of the Lie algebra $u(n)^*$ to the the dual Poisson Lie group $U(n)^*$ (see e.g., [22]). It is a distinguished Ginzburg-Weinstein linearization [16] for the unitary group. See [1] or Section 5.4 for more details.

More interestingly, the results in this paper unveil a relation between Stokes matrices at caterpillar points, or equivalently the asymptotics of confluent hypergeometric functions, and the Gelfand-Zeitlin subalgebras. See Section 5.5. It motivates a more general relation between the Stokes matrices at a generic point $u \in \hat{\mathfrak{h}}_{\text{reg}}(\mathbb{R})$ and the theory of the shift of argument subalgebras parameterized by the same $u \in \hat{\mathfrak{h}}_{\text{reg}}(\mathbb{R})$. We leave it for our future work.

The organization of the paper is as follows. Next section gives the preliminaries of Stokes data of meromorphic linear systems, and proves that quantum Stokes matrices give a family of Drinfeld isomorphisms. Section 2 recalls the definition of Stokes matrices at the caterpillar points on $\hat{\mathfrak{h}}_{\text{reg}}(\mathbb{R})$. Section 3 introduces the quantum Stokes matrices at caterpillar points, and derives their explicit formula using the Gelfand-Zeitlin subalgebras. The last section discusses the relation between quantum dual exponential maps at caterpillar points and Appel-Gautam isomorphisms. It also discusses the semiclassical analog and algebraic characterization of quantum Stokes matrices.

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2 Stokes phenomenon and Drinfeld isomorphisms

In this section, we prove that there is a natural family of Drinfeld isomorphisms for classical Lie algebras, arising from the quantum Stokes matrices. In particular, Section 2.1 recalls the definition of Stokes matrices at the caterpillar points on $\hat{\mathfrak{h}}_{\text{reg}}(\mathbb{R})$. Section 3 introduces the quantum Stokes matrices at caterpillar points, and derives their explicit formula using the Gelfand-Zeitlin subalgebras. The last section discusses the relation between quantum dual exponential maps at caterpillar points and Appel-Gautam isomorphisms. It also discusses the semiclassical analog and algebraic characterization of quantum Stokes matrices.

2.1 Stokes data of meromorphic linear systems

In this subsection, we recall the canonical solutions, Stokes matrices and connection matrices of certain meromorphic linear systems.

2.1.1 Canonical solutions and Stokes matrices

Let $\mathfrak{h}(\mathbb{R})$ (resp. $\mathfrak{h}_{\text{reg}}(\mathbb{R})$) denote the set of $n$ by $n$ diagonal matrices with (resp. distinct) real eigenvalues. Let us consider the meromorphic linear system of differential equations for a $\mathbb{C}^n$-valued function $F(z)$

$$
\frac{dF}{dz} = \left( iu - \frac{1}{2\pi i} A \right) \cdot F,
$$

(10)
where \( u \in \mathfrak{h}(\mathbb{R}) \) and \( A \in \mathfrak{gl}(n) \). The system has an order two pole at \( \infty \) and (if \( A \neq 0 \)) a first order pole at \( 0 \).

**Definition 2.1.** The two Stokes sectors \( \text{Sect}_\pm \) of the system are the right/left half planes \( \text{Sect}_\pm = \{ z \in \mathbb{C} \mid \pm \Re(z) > 0 \} \). The corresponding supersectors are \( \text{Sect}_+ := \mathbb{C} \setminus \mathbb{R}_{\geq 0} \) and \( \text{Sect}_- := \mathbb{C} \setminus \mathbb{R}_{\leq 0} \).

In this paper, let us choose the branch of \( \log(z) \), which is real on the positive real axis, with a cut along the positive imaginary axis \( i\mathbb{R}_{\geq 0} \). The following result can be found in e.g., [3, Chapter 8] or [4, 7, 23] in different generalities.

**Theorem 2.2.** On each \( \text{Sect}_\pm \), there is a unique (therefore canonical) holomorphic fundamental solution \( F_\pm(z) \) of equation (10) such that \( F_\pm \cdot e^{-iu z \frac{\delta(A)}{2i}} \) can be analytically continued to the supersector \( \text{Sect}_\pm \), and satisfies

\[
F_\pm(z) \cdot e^{-iu z \frac{\delta(A)}{2i}} \rightarrow \text{Id}_n \text{ as } z \rightarrow \infty \text{ within } \text{Sect}_\pm.
\]

Here \( \text{Id}_n \) is the rank \( n \) identity matrix, \( \delta(A) \) is the projection of \( A \) to the centralizer of \( u \) in \( \mathfrak{gl}(n) \). In particular, if \( u \) has distinct eigenvalues, \( \delta(A) \) is the diagonal part of \( A \).

**Definition 2.3.** The Stokes matrices of the system (10) (with respect to \( \text{Sect}_+ \) and the chosen branch of \( \log(z) \)) are the elements \( S_{\pm}(A,u) \in \mathfrak{gl}(n) \) determined by

\[
F_+(z) = F_-(z) \cdot e^{-\frac{\delta(A)}{2} z} S_+(A,u), \quad F_-(z) = F_+(z) \cdot S_-(A,u) e^{\frac{\delta(A)}{2} z},
\]

where the first (resp. second) identity is understood to hold in \( \text{Sect}_- \) (resp. \( \text{Sect}_+ \)) after \( F_+ \) (resp. \( F_- \)) has been analytically continued clockwise.

**Remark 2.4.** Our notation of Stokes matrices is different from the one in e.g., [5]. Actually, the matrices \( S_{\pm}(A,u) \) here are \( e^{-\frac{\delta(A)}{2}} S_+(u,-A)^{-1} \) and \( S_-(u,-A)^{-1} e^{\frac{\delta(A)}{2}} \) as the notation of [5].

**Example 2.5.** Given any \( u_1 \neq u_2 \) and generic \( A = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \), following [4] Proposition 8] or Section 3.2.2, the Stokes matrix \( S_- \) of the 2 by 2 system

\[
\frac{dF}{dz} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} F + \frac{1}{2} \begin{pmatrix} t_1 & b_2 \\ b_1 & t_2 \end{pmatrix} F,
\]

is

\[
S_- = \begin{pmatrix} e^{\imath t_1} & 0 \\ -\frac{2\pi i b_1 (u_2-u_1) t_1 e^{\imath t_2}}{1-(1-\lambda_1+t_1)l(1-\lambda_2+t_1)} & e^{\imath t_2} \end{pmatrix},
\]

where \( \lambda_1, \lambda_2 \) are the eigenvalues of \( A \).

### 2.1.2 Connection matrices and monodromy relation

As for the local picture around the simple pole \( z = 0 \), the following fact is well-known (see e.g [30] Chapter 2, Theorem 5]).

**Lemma 2.6.** If the system (10) is non-resonant, i.e., no two eigenvalues of \( \frac{A}{2\pi i} \) differ by a positive integer, then it has a unique holomorphic fundamental solution \( F_0(z) \in \mathfrak{gl}(n,\mathbb{C}) \) on a neighbourhood of \( 0 \) slit along \( i\mathbb{R}_{>0} \) such that \( F_0(z) \cdot z^{\frac{\delta(A)}{2i}} \rightarrow \text{Id}_n \) as \( z \rightarrow 0 \). Here \( \text{Id}_n \) is the identity matrix.

**Definition 2.7.** The connection matrix \( C(A,u) \in \mathfrak{gl}(n) \) of the system (10) (with respect to \( \text{Sect}_+ \)) is determined by the identity \( F_0(z) = F_+(z) \cdot C(u,A) \) in \( \text{Sect}_+ \).

In a global picture, the connection matrix is related to the Stokes matrices by the following monodromy relation, which follows from the fact that a simple negative loop (i.e., in clockwise direction) around \( 0 \) is a simple positive loop around \( \infty \):

\[
C(A,u) e^A C(A,u)^{-1} = S_-(A,u) S_+(A,u).
\] (11)
2.2 Stokes matrices of generalized KZ equations

Let us take the complex Lie algebra \( \mathfrak{gl}(n) \), and the Casimir element \( \Omega := \sum_{1 \leq i, j \leq n} E_{ij} \otimes E_{ji} \). We consider the equation for a \( U(\mathfrak{gl}(n)) \)-valued function \( F(z) \),

\[
\frac{dF}{dz} = \left( iu^{(2)} + \frac{\hbar}{2\pi i} \Omega \right) \cdot F
\]

(12)

where \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), and \( u^{(2)} \) denotes the action of \( u \) on the second \( U(\mathfrak{gl}(n)) \) component. The equation (2) is a linear system with coefficients valued in an infinite dimension non-commutative ring. The Stokes phenomenon of this equation was first studied by Toledano Laredo in [27]. Then motivated by [27], we introduce and study the quantum Stokes matrices in \([28, 31, 33, 35]\). In particular, similar to the classical case, this equation has two canonical solutions \( F_{h \pm}(z) \) with prescribed asymptotics at \( z = \infty \) within \( \text{Sect}_\pm = \{ z \in \mathbb{C} \mid \pm \Re(z) > 0 \} \), see e.g., [33], and

**Definition 2.8.** For any \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), the quantum Stokes matrices \( S_{h \pm}(u) \in U(\mathfrak{gl}(n)) \) are defined by

\[
F_{h +} = F_{h -} \cdot e^{\frac{i\hbar}{2\pi i} \cdot \Omega} S_{h +}(u) \quad \text{and} \quad F_{h -} = F_{h +} \cdot S_{h -}(u) e^{\frac{i\hbar}{2\pi i} \cdot \Omega}
\]

where \( \Omega_0 := \sum_{i=1}^n E_{ii} \otimes E_{ii} \), and the first (resp. second) identity is understood to hold in \( \text{Sect}_- \) (resp. \( \text{Sect}_+ \)) after \( F_{h +}(z, u) \) (resp. \( F_{h -} \)) has been analytically continued clockwise.

The following theorem can be found in [28, 33], and see [31, 35] for a proof in the categorical setting.

**Theorem 2.9.** For any \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), the Stokes matrices \( S_{h \pm}(u) \in U(\mathfrak{gl}(n)) \) satisfy the Yang-Baxter equation. Furthermore, we have \( S_{h +}(u) = S_{h -}(u) S_{h +}^{-1}(u) \).

Let us compute explicitly their evaluation on the standard representation.

**Lemma 2.10.** The evaluation of the quantum Stokes matrix \( S_{h -}^{-1}(u) \) on the standard representation is the \( R \)-matrix \( R \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)[\hbar] \) given by (under a proper basis of \( \mathbb{C}^n \otimes \mathbb{C}^n \))

\[
R = \sum_{i \neq j, i = 1}^n E_{ii} \otimes E_{jj} + e^{\frac{i\pi}{2}} \sum_{i=1}^n E_{ii} \otimes E_{ii} + (e^{\frac{i\pi}{2}} - e^{-\frac{i\pi}{2}}) \sum_{1 \leq j < i \leq n} E_{ij} \otimes E_{ji}.
\]

(13)

**Proof.** Let us first give the \( \mathfrak{gl}_2 \) example. That is to consider the equation \( \frac{dF}{dz} = (iu + \frac{\hbar}{2\pi i} \Omega) F \), with

\[
F = \begin{pmatrix}
  u_1 & 0 & 0 & 0 \\
  0 & u_1 & 0 & 0 \\
  0 & 0 & u_2 & 0 \\
  0 & 0 & 0 & u_2
\end{pmatrix}
\]

and \( \hbar \Omega = \begin{pmatrix}
  h & 0 & 0 & 0 \\
  0 & h & 0 & 0 \\
  0 & 0 & h & 0 \\
  0 & 0 & 0 & h
\end{pmatrix} \).

It reduces to the computation of the 2 by 2 system

\[
\frac{dF}{dz} = \begin{pmatrix}
  u_1 & 0 & 0 & 0 \\
  0 & u_2 & 0 & 0
\end{pmatrix} F + \frac{1}{z} \begin{pmatrix}
  0 & 0 & \frac{h}{2\pi i} & 0 \\
  0 & 0 & 0 & \frac{h}{2\pi i}
\end{pmatrix} F.
\]

Applying the formula for 2 by 2 systems in Example 2.5 the (lower triangular) Stokes matrix is expressed by the Gamma functions of eigenvalues \( \pm \frac{h}{2\pi i} \) of the residue matrix. Using the Euler’s reflection formula \( \Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \) for \( z \notin \mathbb{Z} \) to rewrite the Gamma function by hyperbolic function, we get the Stokes matrix \( S_{h -}^{-1}(u) = \begin{pmatrix}
  e^{\frac{h}{2\pi i}} & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & e^{\frac{h}{2\pi i}} & 1 \\
  0 & 0 & 0 & e^{\frac{h}{2\pi i}}
\end{pmatrix} \).

Thus the Stokes matrix of the 4 by 4 system is

\[
S_{h -}(u) = \begin{pmatrix}
  e^{\frac{h}{2\pi i}} & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & e^{\frac{h}{2\pi i}} & 1 \\
  0 & 0 & 0 & e^{\frac{h}{2\pi i}}
\end{pmatrix} \cdot
\]

Note that it doesn’t depend on \( u \), and coincides with the \( R \)-matrix \( R \) defined in [13] for \( \mathfrak{gl}_2 \).

For general \( n \), the evaluation of [12] on \( \mathbb{C}^n \otimes \mathbb{C}^n \) is a system of rank \( n^2 \). Let \( \{ v_i \}_{1 \leq i \leq n} \) be the standard basis of \( \mathbb{C}^n \), and let us take a basis \( \{ v_i \otimes v_j \} \) of \( \mathbb{C}^n \otimes \mathbb{C}^n \) (with an order remained to be fixed). Since \( \Omega \) is a permutation operator, i.e., \( \Omega(v_i \otimes v_j) = v_j \otimes v_i \), the linear system of rank \( n^2 \) can be composed to multiple rank 2 and rank 1 systems. Since the Stokes matrices of rank 2 systems have closed form, the lemma follows by a direct computation. The only thing one should be careful about is, in order to make the evaluation of \( S_{h -}^{-1}(u) \) on the standard representation have the desired form, the chosen basis \( \{ v_i \otimes v_j \} \) should be compatible with the orders of \( u_1, \ldots, u_n \).
2.3 Faddeev-Reshetikhin-Takhtajan presentation of the quantum group

Let us recall the RLL formalism [26] of the quantized universal enveloping algebra of \( \mathfrak{gl}(n) \), by means of solutions of the Yang-Baxter equation. Let us take the R-matrix \( R \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)[\hbar] \) given in (13).

**Definition 2.11.** The quantized universal enveloping algebra \( U(R) \) is an algebra generated by elements \( t_{ij}^{(+)} \), \( t_{ij}^{(-)} \), \( 1 \leq i \leq j \leq n \). Let \( L_{\pm} = (t_{ij}^{(+)})_{i,j=1}^{n} \in \text{End}(\mathbb{C}^n) \otimes U(R) \), with \( t_{ij}^{(+)} = t_{ij}^{(-)} = 0 \) for \( 1 \leq j < i \leq n \). Then the defining relations are given in matrix form

\[
R_{12}L_{\pm}^{23} = L_{\pm}^{23}R_{12}, \tag{14}
\]

\[
R_{12}L_{\mp}^{23} = L_{\pm}^{23}R_{12}, \tag{15}
\]

and

\[
i_{ii}^{(+)}i_{ii}^{(-)} = i_{ii}^{(-)}i_{ii}^{(+)} = 1, \quad \text{for } i = 1, \ldots, n.
\]

The algebra \( U(R) \) has a Hopf algebra structure with coproduct and counit given by \( \Delta(L_{\pm}) = L_{\pm}^{12}L_{\pm}^{13} \) and \( \varepsilon(L_{\pm}) = 1 \).

2.4 Drinfeld isomorphisms from quantum Stokes matrices

The equation (2) in the introduction

\[
\frac{dF}{dz} = \left( u + \frac{\hbar}{2\pi i} T \right) \cdot F,
\]

can be obtained from equation (12) by evaluating the standard representation \( \mathbb{C}^n \) on the first component. Thus for any \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), the evaluation of \( S_{h^{+}} \) and \( S_{h^{-}}^{-1} \) on the first component by the standard representation coincide with the Stokes matrices \( S_{h^{+}}(u; hT) \) and \( S_{h^{-}}(u; hT)^{-1} \in \text{End}(\mathbb{C}^n) \otimes U(\mathfrak{gl}(n))[\hbar] \) of equation (2) respectively. Now we are ready to prove

**Theorem 2.12.** For any \( u \in \mathfrak{h}_{\text{reg}}(\mathbb{R}) \), the map

\[ \nu_{\hbar}(u) : U(R) \rightarrow U(\mathfrak{gl}(n))[\hbar] ; \quad t_{ij}^{(\pm)} \mapsto s_{ij}^{(\pm)}, \]

is an algebra isomorphism. Here recall that \( t_{ij}^{(\pm)} \) are the entries of the \( L_{\pm} \) operator, and \( s_{ij}^{(\pm)} \) are the entries of the \( S_{h^{+}}(u; hT) \) and \( S_{h^{-}}(u; hT)^{-1} \) respectively.

**Proof.** First by Lemma 2.10 the evaluation of quantum Stokes matrix \( S_{h^{-}}(u)^{-1} \) on \( \mathbb{C}^n \otimes \mathbb{C}^n \) is the R-matrix in (13). If we set \( S_{+} := S_{h^{+}}(u; hT) \) and \( S_{-} := S_{h^{-}}(u; hT)^{-1} \) as the \( n \) by \( n \) matrices with entries \( s_{ij}^{(\pm)} \in U(\mathfrak{gl}(n))[\hbar] \), then it follows from Theorem 2.9, i.e., the Yang-Baxter equation

\[
S_{h^{+}}^{12}S_{h^{-}}^{13}S_{h^{+}}^{23} = S_{h^{-}}^{23}S_{h^{-}}^{13}S_{h^{+}}^{12} \in U(\mathfrak{gl}(n))[\hbar]^{\otimes 3},
\]

and the relation \( S_{h^{+}} = S_{h^{+}}^{21} \) that (when evaluates on \( \mathbb{C}^n \otimes \mathbb{C}^n \))

\[
R_{12}S_{h^{+}}^{23} = S_{h^{-}}^{23}R_{12}, \tag{16}
\]

\[
R_{12}S_{h^{-}}^{23} = S_{h^{-}}^{23}R_{12}, \tag{17}
\]

Comparing the above relation with the defining relation (14)-(15) of the (algebra structure of) \( U(R) \), we have that the entries \( s_{ij}^{(\pm)} \in U(\mathfrak{gl}(n))[\hbar] \) satisfy the defining relation of \( U(R) \).

**Remark 2.13.** Given the RLL formulation of quantum groups for other classical Lie algebras (see [26]), to generalize the result in this section to other classical Lie algebras, we only need to compute the evaluation \( R \) of quantum Stokes matrices, of kKZ equations for other types, on natural representation (an analog of Lemma 2.10 for other types). Besides, the (quantum) Stokes matrices are compatible with involutions on Lie algebras. See [35] and [35] for some particular classical and quantum cases respectively. In particular, we showed that the quantum Stokes matrices of cyclotomic KZ equations give rise to universal \( K \)-matrices for quantum symmetric pairs of type AI [35]. Thus the Drinfeld isomorphism \( \nu_{\hbar}(u) \) in Theorem 2.12 restricts to an algebra isomorphism for the classical and quantum symmetric pairs of type AI. It is interesting to generalize the results in this section to quantum groups of classical Lie algebras and to quantum symmetric pairs.
2.5 Explicit Drinfeld isomorphisms

We have constructed a family of Drinfeld isomorphisms, parameterized by the space $\mathfrak{h}_{\text{reg}}(\mathbb{R})$, between quantum and classical $U(\mathfrak{gl}(n))$. However, this construction uses the Stokes phenomenon at essential singularities of certain functions in complex analysis, and thus is transcendental and hard to work with. The interesting thing is that this family has an extension from $\mathfrak{h}_{\text{reg}}(\mathbb{R})$ to the compactification $\hat{\mathfrak{h}}_{\text{reg}}(\mathbb{R})$, and the Drinfeld isomorphisms corresponding to caterpillar points on $\hat{\mathfrak{h}}_{\text{reg}}(\mathbb{R})$ have explicit expression and already capture many aspects of representation theory of quantum groups. In the following, the explicit Drinfeld isomorphisms will be given.

3 Stokes matrices at caterpillar points

This section is an exposition of some related results in [34]. In particular, Section 3.1 recalls the definition of Stokes matrices $S_{\pm}(u_{\text{cat}})$ at a caterpillar point $u_{\text{cat}}$ on $\mathfrak{h}_{\text{reg}}(\mathbb{R})$. Then Section 3.2 recalls the explicit expression of $S_{\pm}(u_{\text{cat}})$.

3.1 Stokes matrices at the caterpillar point $u_{\text{cat}}$

In this subsection, we will recall the definition of Stokes matrices at a caterpillar point $u_{\text{cat}}$ on $\mathfrak{h}_{\text{reg}}(\mathbb{R})$ (seen as a planar tree with the chosen embedding in Figure 1). We refer the reader to [34] for more details.

![Figure 1: A caterpillar point $u_{\text{cat}}$ with a chosen planar embedding](image)

Given any matrix $A \in \mathfrak{gl}(n)$, we denote by $A^{(k)} \subset \mathfrak{gl}(k)$ its upper left $k$ by $k$ principal submatrix, and denote by $\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_k^{(k)}$ the eigenvalues of $A^{(k)}$. We denote by $\mathfrak{gl}_0(n) \subset \mathfrak{gl}(n)$ the subspace consisting of all $n$ by $n$ matrices $A$ such that for any $1 \leq k \leq n$, $A^{(k)}$ is diagonalizable, and such that any two of the numbers $\{\lambda_j^{(k)}\}_{1 \leq j \leq n}$ do not differ by an integer.

Given any $A \in \mathfrak{gl}_0(n)$ and any $1 \leq k \leq n$, we consider the rank $k$ system

$$
\frac{dF}{dz} = \left(iE_k - \frac{1}{2\pi i} \frac{A^{(k)}}{z}\right)F,
$$

where $E_k = (0, \ldots, 0, 1) \in \mathfrak{gl}(k)$, and $A^{(k)} \in \mathfrak{gl}(k)$. Let $S_{\pm}^{(k)}(A^{(k)}) \in \text{GL}(k)$ denote the two Stokes matrices. See the Appendix for an introduction to Stokes matrices. Then by Definition 2.3 and the asymptotics of $F_{\pm}$, we have (here to derive the second formula, the change in choice of $\log(z)$ is accounted for)

$$
F_{+}(z)F_{-}^{-1}(z) = e^{iE_k z} e^{-\frac{\delta(A^{(k)})}{2\pi i}} S_{\pm}^{(k)} e^{\frac{\delta(A^{(k)})}{2\pi i}} e^{-iE_k z} \sim \text{Id}_k,
$$

as $z \to \infty$ in the third quadrant,

$$
F_{-}(z)F_{+}^{-1}(z) = e^{iE_k z} e^{-\frac{\delta(A^{(k)})}{2\pi i}} S_{\pm}^{(k)} e^{\frac{\delta(A^{(k)})}{2\pi i}} e^{-iE_k z} \sim \text{Id}_k,
$$

as $z \to \infty$ in the first quadrant,

here recall that $\delta(A^{(k)})$ takes the projection of $A^{(k)}$ to the centralizer of $E_k$ in $\mathfrak{gl}(n)$. It follows that the Stokes matrices take the form

$$
S_{-}^{(k)}(A^{(k)}) = \begin{pmatrix} e^{A^{(k-1)}(k)} & 0 \\ b^{(k)} \end{pmatrix}, \quad S_{+}^{(k)}(A^{(k)}) = \begin{pmatrix} e^{A^{(k-1)}(k)} & b^{(k)} \\ 0 & * \end{pmatrix},
$$

(19)
where \( b^{(k)} \) is a column vector. Furthermore, if we denote by \( C^{(k)}(A^{(k)}) \) the connection matrix, then the monodromy relation (11) gives rise to

\[
C^{(k)}(A^{(k)})e^{A^{(k)}} C^{(k)}(A^{(k)})^{-1} = S_+^{(k)}(A^{(k)}) S_-^{(k)}(A^{(k)}),
\]

where \( \delta(A^{(k)}) = \begin{pmatrix} A^{(k-1)} & 0 \\ 0 & * \end{pmatrix} \) for \( A^{(k-1)} \) the upper left \( k - 1 \)-th submatrix of \( A^{(k)} \).

**Remark 3.1.** The chosen the minus sign in (18) ensures the monodromy relation (20), in order to be compatible with the chosen Gelfand-Zeitlin chain (\( gl(k) \subset gl(k + 1) \) as the upper left corner), see [34] for more details.

Now for each \( 1 \leq k \leq n \), let us introduce the connection map (denoted by the same letter by abuse of notation)

\[
C^{(k)} : gl_0(n) \to GL(k) \subset GL(n); \ A \mapsto C^{(k)}(A^{(k)}),
\]

where \( GL(k) \subset GL(n) \) denotes the obvious inclusion of \( GL(k) \) as the upper left corner of \( GL(n) \), extended by 1’s along the diagonal. In the following, \( C^{(k)}(A^{(k)}) \) will denote the invertible \( k \times k \) matrix, and \( C^{(k)}(A) \) will denote its image in \( GL(n) \). Since \( GL(k - 1) \) is in the centralizer of the irregular term \( iE_k \) of the equation (18), we have

**Lemma 3.2.** Each connection map \( C^{(k)} \) is \( GL(k - 1) \)-equivariant, that is \( C^{(k)}(gAg^{-1}) = gC^{(k)}(A)g^{-1} \) for any \( g \in GL(k - 1) \subset GL(k) \).

**Definition 3.3.** The connection map \( C_n := C^{(1)} \cdots C^{(n-1)}C^{(n)} : gl_0(n) \to GL(n) \), associated to the caterpillar point, is the pointwise ordered multiplication of all \( C^{(k)} \)'s. That is

\[
C_n(A) := C^{(1)}(A) \cdots C^{(n-1)}(A)C^{(n)}(A) \in GL(n), \quad \text{for any} \ A \in gl_0(n).
\]

**Definition 3.4.** For any \( A \in gl_0(n) \), the Stokes matrices \( S_{\pm}(u_{cat}, A) \) at the caterpillar point \( u_{cat} \), are respectively the upper and lower triangular matrices determined by the identity (Gauss decomposition)

\[
C_n(A)e^{A} C_n(A)^{-1} = S_-(u_{cat}, A) S_+(u_{cat}, A),
\]

and with the diagonal part \([S_+(u_{cat}, A)] = [S_-(u_{cat}, A)] = e^{\frac{A}{2\pi i}}\).

In the following, for the consistence of notation, we will denote the matrices \( S_{\pm}(u_{cat}, A) \) of rank \( n \) by \( S_{n\pm}(A) \).

### 3.2 Evaluation of Stokes and connection matrices at caterpillar points

In this subsection, we will recall the explicit formula of the relative Stokes matrices.

#### 3.2.1 Diagonalization

For any \( A \in gl_0(n) \), to compute the Stokes matrix \( S(A) \) at the caterpillar point, we need to consider for each \( 1 \leq k \leq n - 1 \), the system of rank \( k + 1 \) taking the form

\[
\frac{dF}{dz} = \left( iE_{k+1} - \frac{1}{2\pi i} \frac{A^{(k+1)}}{z} \right) \cdot F
\]

Its connection map \( C^{(k+1)} : gl_0(n) \to GL(n) \) is \( GL(k - 1) \)-equivariant, that is \( C^{(k+1)}(gAg^{-1}) = gC^{(k+1)}(A)g^{-1} \) for any \( g \in GL(k) \subset GL(k + 1) \). Thus to simplify the computation, we can first diagonalize the upper left \( k \)-th submatrix \( A^{(k)} \) of \( A \).

Denoted by \( \lambda_1^{(k)}, \lambda_2^{(k)}, ..., \lambda_k^{(k)} \) the eigenvalues of \( A^{(k)} \), and \( \text{adj}(A^{(k)} - \lambda) \) the adjugate of \( A^{(k)} - \lambda \), i.e, the transpose of the cofactor matrix of \( A^{(k)} - \lambda \). Since for any \( i = 1, ..., k \), we have

\[
\text{adj}(A^{(k)} - \lambda_i^{(k)}) \cdot (A^{(k)} - \lambda_i^{(k)}) = \det(A^{(k)} - \lambda_i^{(k)}) = 0,
\]

9
thus the last row of \( \text{adj} \left( A^{(k)} - \lambda_i^{(k)} \right) \) is an eigenvector of \( A^{(k)} \) associated to the eigenvalue \( \lambda_i^{(k)} \). That is if we denote by \( \mathcal{P}_{ij}^{(k)} \) the product of \((-1)^{k-j}\) and the \( k - 1 \) by \( k - 1 \) minor \( \Delta_{1,\ldots,k-1}^{(k)} \left( A^{(k)} - \lambda_i^{(k)} \cdot \text{Id}_k \right) \) (here \( j \) means that the index \( j \) is omitted), then the row vector

\[
v_i = \frac{1}{\prod_{t=1, t \neq i}^{k} (\lambda_i^{(k)} - \lambda_t^{(k)})} (\mathcal{P}_{i1}^{(k)}, \ldots, \mathcal{P}_{ik}^{(k)})
\]

satisfies \( v_i \cdot A^{(k)} = \lambda_i^{(k)} v_i \). Here the scale normalizer is introduced to simplify the computation in the following sections. Therefore, the \( k \) by \( k \) matrix \( \mathcal{P}^{(k)}(A) = (\mathcal{P}_{ij}^{(k)}) \) diagonalizes \( A^{(k)} \), and \( \mathcal{P}^{(k)}(A) \in \text{GL}(k) \subset \text{GL}(n) \) is such that \( A_k := \mathcal{P}^{(k)}(A) \mathcal{P}^{(k)}(A)^{-1} \) takes the form

\[
A_k = \begin{pmatrix}
\lambda_1^{(k)} & a_1^{(k)} & \cdots \\
\vdots & \ddots & \vdots \\
b_1^{(k)} & \cdots & \lambda_{k+1}^{(k)}
\end{pmatrix}.
\] (25)

**Definition 3.5.** We define the function \( a_j^{(k)}(A) \) of \( A \in \mathfrak{gl}_0(n) \) as the \((i, k+1)\) entry of \( A_k \) for any \( 1 \leq j \leq k \leq n - 1 \). Note that we have \( a_i^{(k)} = \sum_{j=1}^{k} \mathcal{P}_{ij}^{(k)} A_{j,k+1} \).

Let \( L^{(k+1)}(A) \in \text{GL}(k+1) \subset \text{GL}(n) \) be the matrix given by

\[
L_{ij}^{(k+1)}(A) := \frac{a_i^{(k)}(A)}{\lambda_j^{(k+1)} - \lambda_i^{(k)}}, \quad \text{for } i \neq k + 1, \quad j = 1, \ldots, k + 1;
\] (26)

\[
L_{k+1,j}^{(k+1)}(A) := 1, \quad \text{for } j = 1, \ldots, k + 1.
\] (27)

One checks that the matrix \( L^{(k+1)}(A)^{-1} \) diagonalizes the \((k+1)\)-th principal submatrix \( A_k^{(k+1)} \) of \( A_k \), and furthermore we have the identity (which can be seen as a recursive definition of) \( \mathcal{P}^{(k)}(A) \) for all \( k = 1, \ldots, n \),

\[
\mathcal{P}^{(k+1)}(A) = L^{(k+1)}(A)^{-1} \mathcal{P}^{(k)}(A) \in \text{GL}(n).
\]

**Definition 3.6.** For any integer \( 1 \leq k \leq n \), we define the normalized connection map

\[
\tilde{C}^{(k)} : \mathfrak{gl}_0(n) \to \text{GL}(n); \quad A \mapsto C^{(k)} \left( \mathcal{P}^{(k-1)}(A) \mathcal{P}^{(k-1)}(A)^{-1} \right) \cdot \mathcal{P}^{(k-1)}(A) \mathcal{P}^{(k)}(A)^{-1}.
\] (28)

That is \( \tilde{C}^{(k)}(A) = C^{(k)}(A_{d_k-1}) L^{(k)}(A) \), for any \( A \in \mathfrak{gl}_0(n) \). Here \( C^{(k)} : \mathfrak{gl}_0(n) \to \text{GL}(n) \) is the connection map given in Section 3.1.

**Lemma 3.7.** Let us define \( \tilde{C}_n = \tilde{C}^{(1)} \tilde{C}^{(2)} \cdots \tilde{C}^{(n)} : \mathfrak{gl}_0(n) \to \text{GL}(n) \) as the ordered pointwise multiplication, then

\[
\tilde{C}_n(A) e^{A_n} \tilde{C}_n(A)^{-1} = C_n(A) e^{A_n} C_n(A)^{-1}, \quad \forall A \in \mathfrak{gl}_0(n).
\]

Here recall that \( A_n = \text{diag}(\lambda_1^{(n)}, ..., \lambda_n^{(n)}) \).

**Proof.** It follows from the Definition 3.3 of \( C_n(A) \) and the Definition (28) of \( \tilde{C}^{(k)} \) that \( \tilde{C}^{(k)}(A) = C_n(A) \cdot P(n)(A) \). Then the lemma follows from the identity \( P(n)(A)e^{A_n} P(n)(A)^{-1} = e^A \). ■
### 3.2.2 Explicit evaluation

For any $A \in \mathfrak{gl}_n(n)$, the Stokes matrix $S^{(k+1)}_+(A^{(k+1)}_k)$ and normalized connection matrix $\tilde{C}^{(k+1)}_+(A^{(k+1)}_k)$ of the $k+1$ by $k+1$ system, $k = 1, \ldots, n-1$,

$$\frac{dF}{dz} = \left( iE_{k+1} - \frac{1}{2\pi i} A^{(k+1)}_k \right) \cdot F, \quad (29)$$

are described by the following proposition. Here $A^{(k+1)}_k$ takes the upper left submatrix of $A_k$ given in (25). It can be derived directly from the result of Balser [3], see Remark [3.10]. However, for the use of next section, we give a proof showing how the explicit formula can be derived from the known asymptotics of generalized confluent hypergeometric functions.

**Proposition 3.8.** (1) The $j$-th entry of the last column $b^{(k+1)}_j$ (above the diagonal) of $S^{(k+1)}_+(A^{(k+1)}_k)$ is

$$b^{(k+1)}_j = \frac{e^{\lambda^{(k)}_j}}{4} \prod_{i=1}^{k+1} \Gamma(1 + \frac{\lambda^{(k)}_i - \lambda^{(k)}_j}{2\pi i}) \cdot a^{(k)}_j, \quad j = 1, \ldots, k. \quad (2)$$

The entries of the matrix $\tilde{C}^{(k+1)}(A^{(k+1)}_k)$ are given by

$$\tilde{C}^{(k+1)}_{ij} = -\frac{e^{\lambda^{(k)}_i - \lambda^{(k)}_j}}{4} \prod_{v=1}^{k} \Gamma(1 + \frac{\lambda^{(k)}_v - \lambda^{(k)}_j}{2\pi i}) \prod_{v=1, v \neq i}^{k+1} \Gamma(1 + \frac{\lambda^{(k)}_v - \lambda^{(k+1)}_j}{2\pi i}) \cdot a^{(k)}_i, \quad \text{for } 1 \leq j \leq k+1, 1 \leq i \leq k, \text{ and}$$

$$\tilde{C}^{(k+1)}_{k+1,j} = \frac{e^{\lambda^{(k+1)}_i - \lambda^{(k+1)}_i}}{4} \prod_{v=1}^{k} \Gamma(1 + \frac{\lambda^{(k)}_v - \lambda^{(k+1)}_j}{2\pi i}) \prod_{v=1, v \neq i}^{k+1} \Gamma(1 + \frac{\lambda^{(k)}_v - \lambda^{(k+1)}_j}{2\pi i}) \cdot a^{(k)}_i, \quad \text{for } 1 \leq j \leq k+1. \quad (3)$$

**Proof.** Under certain generic additional assumptions one can explicitly compute a Floquet solution of the equation (29), using the generalized confluent hypergeometric functions. Recall that they are the functions, for any $m \geq 1, \alpha_j \in \mathbb{C}, \beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, 1 \leq j \leq m$, are

$$kF_k(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_m)_n}{(\beta_1)_n \cdots (\beta_m)_n} \cdot z^n, \quad (30)$$

where $(\alpha)_0 = 1, (\alpha)_n = \alpha \cdots (\alpha + n - 1), n \geq 1.$

The $k+1$ by $k+1$ matrix $L^{(k+1)}$ defined by (26)-(27) diagonalize $A^{(k+1)}_k$, i.e., $L^{(k+1)}A^{(k+1)}_k L^{(k+1)}^{-1} = A_k^{(k+1)} = \text{diag}(\lambda^{(k)}_1, \ldots, \lambda^{(k)}_{k+1})$.

**Lemma 3.9.** The equation (29) has a fundamental solution $F(z)$ taking the form

$$F(z) = Y \cdot H(z) \cdot z^{-\frac{1}{2\pi i} A^{(k+1)}_k}, \quad (31)$$

where $Y = \text{diag}(a_1^{(k)}, \ldots, a_k^{(k)}, 1)$ and $H(z)$ is the $k+1$ by $k+1$ matrix $H(z)$ given by

$$H(z)_{ij} = \frac{1}{\lambda^{(k+1)}_j - \lambda^{(k)}_i} \cdot kF_k(\alpha_{ij,1}, \ldots, \alpha_{ij,k}, \beta_{ij,1}, \ldots, \beta_{ij,k+1; i}; iz), \quad 1 \leq i \leq k, 1 \leq j \leq k+1,$$

$$H(z)_{k+1,j} = kF_k(\alpha_{k+1,j+1,1}, \ldots, \alpha_{k+1,j+1,k}, \beta_{k+1,j+1,1}, \ldots, \beta_{k+1,j+1,k+1; i}; iz), \quad i = k+1, 1 \leq j \leq k+1, \quad (32)$$

with the variables $\{\alpha_{ij,l}\}$ and $\{\beta_{ij,l}\}$ given by

$$\alpha_{ij,l} = \frac{1}{2\pi i} (\lambda^{(k)}_l - \lambda^{(k+1)}_j), \quad 1 \leq l \leq k, 1 \leq j \leq k+1,$$

$$\alpha_{k+1,j,l} = \frac{1}{2\pi i} (\lambda^{(k)}_l - \lambda^{(k+1)}_{j+1}), \quad 1 \leq j \leq k+1,$$

$$\alpha_{ij,l} = \frac{1}{2\pi i} (\lambda^{(k)}_l - \lambda^{(k+1)}_j), \quad l \neq i, 1 \leq l \leq k, 1 \leq i, j \leq k+1,$$

$$\beta_{ij,l} = \frac{1}{2\pi i} (\lambda^{(k)}_{j+1} - \lambda^{(k+1)}_j), \quad l \neq j, 1 \leq l \leq k+1, 1 \leq i, j \leq k+1.
Proof. The first $k$ rows of the matrix equation (29) follows from the definition of the functions $H(z)_{ij}$ and the special arguments $\alpha_{ij,l}$ and $\beta_{ij,l}$, that the functions $H(z)_{ij}$ satisfy

$$z \frac{dH_{ij}}{dz} = \frac{\lambda_j^{(k+1)} - \lambda_i^{(k)}}{2\pi i} H_{ij} - \frac{1}{2\pi i} H_{k+1,j}, \quad \text{for} \ 1 \leq i \leq k.$$  

(32)

For the rest of the equation, one just needs the identity $a_i^{(k)} b_i^{(k)} = -\prod_{j=1}^{k+1} (\lambda_j^{(k+1)} - \lambda_i^{(k)}) / \prod_{j=1}^{k+1} (\lambda_j^{(k)} - \lambda_i^{(k)})$, for any $1 \leq i \leq k$, which follows from the character polynomial of $A_k$.

The asymptotics expansion of $k F_k$ via gamma functions are (See [25, Page 411]),

$$k F_k(\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k; z) = \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\prod_{i=1}^{k} \Gamma(\beta_i)} \sum_{m=1}^{k} \Gamma(\alpha_m) \prod_{i \neq m}^{k} \Gamma(\alpha_i - \alpha_m) \prod_{i=1}^{k} \Gamma(\beta_i - \alpha_m)(z^{-\alpha_m} + O(1/z)) + e^{z} \sum_{l=1}^{k} (\alpha_l - \beta_l)(1 + O(1/z)).$$  

(33)

(34)

where upper or lower signs are chosen according as $z$ lies in the upper or lower half-plane. Using this, one can get explicitly the asymptotics of $F_0(z)$ as $z \to 0$ in the two different Stokes sectors, and particularly its comparison with the unique formal solution $\hat{F}(z) = (I + O(z)) e^{iz} He^{k+1} z^{-\frac{1}{2}} A^{(k+1)}$. That is

$$F(z) \sim \hat{F}(z) \cdot Y U_+, \quad \text{as} \ z \to 0 \ \text{in} \ \hat{H}_+,$$

(35)

$$F(z) \sim \hat{F}(z) \cdot Y U_-, \quad \text{as} \ z \to 0 \ \text{in} \ \hat{H}_-.$$  

(36)

where $Y = \text{diag}(a_1^{(k)}, ..., a_k^{(k)}, 1)$, and $U_{\pm}$ are explicit invertible matrices with entries given by Gamma functions. By the uniqueness of fundamental solutions in $\hat{H}_\pm$, one get $F_0(z) = F_{\pm} \cdot Y U_{\pm}$ in the two corresponding sectors. Then by the definition of Stokes matrices and the chosen branch of $\log(z)$, the Stokes matrices are given by

$$S_+(A^{(k+1)}_k) = Y U_+^{-1} Y^{-1}, \quad S_-(A^{(k+1)}_k) = Y U_- e^{A^{(k+1)}_k} U_+^{-1} Y^{-1}. $$  

(37)

The explicit computation of $U_{\pm}$ using the asymptotics of $k F_k$ is straight-forward, and may be omitted here. For example, the explicit computation of 2 by 2 cases in this approach can be found in [4, Proposition 8].

Remark 3.10. The proof in [3] didn’t make use of any known results on the global behaviour of the functions $k F_k$. In particular, the expression of the central connection factors $\Omega_0$ of the meromorphic linear system $dF = (E_{k+1} + \frac{1}{z}) F$ are given in [3, Formula 7.3-7.4]. Following [3, Formula 6.3] (where our $C^{(k+1)}(A_k)$ is denoted by $\Omega_0$ there), the connection factor $\Omega_0$ is related to the connection matrix by

$$\Omega_0 = C^{(k+1)}(A_k) \cdot \tilde{L}_0,$$

where (see the definition in [3, Formula 5.2-5.3]) the matrix $\tilde{L}_0 = L^{(k+1)} \cdot D$ with $D$ given by

$$D_{jj} = \frac{\prod_{i=1}^{k} (\lambda_j^{(k+1)} - \lambda_i^{(k)})}{\prod_{i=1}^{k} (1 + \lambda_j^{(k+1)} - \lambda_i^{(k+1)})}, \quad 1 \leq j \leq k,$$

$$D_{k+1,k+1} = \frac{1}{\prod_{i=1}^{k} (1 + \lambda_j^{(k+1)} - \lambda_i^{(k+1)})}.$$  

Thus the normalized connection matrix $C^{(k+1)}(A_k)L^{(k+1)}$, what we are computing, differs from the connection factor $\Omega_0$ by $D$. One checks that multiplying the formula [3, Formula 7.3-7.4] for $\Omega_0$ by $D$ gives rise to the formula in Proposition 3.8 (provided that the replacement of the matrix $A$ by $-A$ and the irregular term $E_{k+1}$ by $iE_{k+1}$ are accounted for).

Remark 3.11. Recall that our previous paper [34] studied the unitary case, i.e., for $A \in \text{Herm}(n)$, where we need to introduce a normalizer to make the diagonalization matrix $L^{(k+1)}$ unitary, that ensures the connection matrices to be unitary.
3.2.3 Entries of Stokes matrices

Recall that the Stokes matrix $S_{n+}(A)$ at the caterpillar point $u_{\text{cat}}$ is upper-triangular. For any $1 \leq k \leq n-1$, let us denote by $b_{k+1}(A)$ the column vector consisting of the first $k$ elements of the $k+1$-th column of $S_{n+}(A)$. They encode the part of the Stokes matrix above the diagonal, and can be computed recursively as follows.

First for each $1 \leq k \leq n-1$, we define the $k$-th normalized connection map $\tilde{C}_k : \mathfrak{gl}_n(k) \to \text{GL}(k)$ as the multiplication $\tilde{C}(1) \ldots \tilde{C}(k)$. Then for any $A \in \mathfrak{gl}_n(k)$, we define upper and lower $k$ by $k$ matrices $S_{k\pm}(A)$ via

$$
\tilde{C}_k(A)e^{\lambda_k \tilde{C}_k(A)} = S_{k-}(A)S_{k+}(A),
$$

with the diagonal part $[S_{k+}(A)] = [S_{k-}(A)] = e^{\frac{\lambda_k}{2}}$. Thus $S_{k\pm}(A)$ are just the Stokes matrices of rank $k$ at the caterpillar point, and only depend on the $k$-th principal submatrix $A^{(k)}$ of $A$.

**Lemma 3.12.** The column vector $b_{k+1}(A)$ is given by

$$
b_{k+1}(A) = S_{k+}(A_{d_{k-1}}) \tilde{C}_k(A^{(k)})e^{-\frac{\lambda_k}{2}} b^{(k+1)}(A^{(k+1)}),
$$

where $b^{(k+1)}(A^{(k+1)})$ takes the first $k$ entries of the $k+1$-th column of the Stokes matrix $S_{n+}^{(k+1)}(A^{(k+1)})$ of equation (29).

**Proof.** It follows from the monodromy relation that the upper left $k+1$ submatrix of $S_{n+}(A) = S(u_{\text{cat}}, A)$ coincides with $S_{k+1}(A^{(k+1)})$. So it is enough to prove the case $k = n - 1$. Using the monodromy relation

$$
\tilde{C}^{(n)}(A)e^{A_n \tilde{C}^{(n)}(A)} = S^{(n)}_{n-1}(A_{n-1})S^{(n)}_{n+}(A_{n-1}),
$$

and the expression

$$
S^{(n)}_{n-1}(A_{n-1}) = \begin{pmatrix} e^{\frac{1}{2}A_{n-1}^{(n-1)}} & * \\ b_{n-1}^{(n)}(A_{n-1}) & * \end{pmatrix}, \quad S^{(n)}_{n+}(A_{n-1}) = \begin{pmatrix} e^{\frac{1}{2}A_{n-1}^{(n-1)}} & b_{n}^{(n)}(A_{n-1}) \\ 0 & * \end{pmatrix},
$$

we get

$$
S_{n-}(A)S_{n+}(A) = \tilde{C}_{n-1}(A)\tilde{C}^{(n)}(A)e^{A_n \tilde{C}^{(n)}(A)} - 1 \tilde{C}_{n-1}(A)^{-1} = \left( \begin{array}{cc} \tilde{C}_{n-1}(A^{(n-1)}) & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} e^{\frac{1}{2}A_{n-1}^{(n-1)}} & 0 \\ b_{n-1}^{(n)} & * \end{array} \right) \left( \begin{array}{cc} e^{\frac{1}{2}A_{n-1}^{(n-1)}} & b_{n}^{(n)} \\ 0 & * \end{array} \right) \left( \begin{array}{cc} \tilde{C}_{n-1}(A^{(n-1)})^{-1} & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} \tilde{C}_{n-1}(A^{(n-1)})e^{A_{n-1}^{(n-1)}} \tilde{C}_{n-1}(A^{(n-1)})^{-1} & \tilde{C}_{n-1}(A^{(n-1)})e^{A_{n-1}^{(n-1)}} b_{n}^{(n)} \\ b_{n}^{(n)} e^{\frac{1}{2}A_{n-1}^{(n-1)}} \tilde{C}_{n-1}(A^{(n-1)})^{-1} & * \end{array} \right) = \left( \begin{array}{cc} S_{n-1,} & \tilde{C}_{n-1}^{-1}e^{-\frac{1}{2}A_{n-1}^{(n-1)}} b_{n}^{(n)} \\ b_{n}^{(n)} e^{\frac{1}{2}A_{n-1}^{(n-1)}} \tilde{C}_{n-1}^{-1} & * \end{array} \right).
$$

Here in the last equality, we use again the monodromy relation

$$
\tilde{C}_{n-1}(A^{(n-1)})e^{A_{n-1}^{(n-1)}} \tilde{C}_{n-1}(A^{(n-1)})^{-1} = S_{n-1,} - (A_{n-2}^{(n-1)}) S_{n-1,} + (A_{n-2}^{(n-1)}).
$$

Then the proof is finished for $k = n - 1$. ■

**Corollary 3.13.** The $k$-th entry of the column vector $b_{k+1}(A)$ is given by

$$
b_{k+1}(A) = \frac{1}{\prod_{j=1}^{k} \Gamma(1 + \frac{\lambda_j}{2})} \prod_{l=1}^{k} \Gamma(1 + \frac{\lambda_k - \lambda_l}{2}) \prod_{l=1}^{k} \Gamma(1 + \frac{\lambda_l - \lambda_k}{2}) \left( \sum_{J} \frac{(-1)^{k-J} \prod_{i=1}^{k-J} (A^{(k)} - \lambda_i) d_i}{\prod_{l=1, i \neq j} \lambda^j_k - \lambda^j_k} A_{J,K+1} \right)
$$

where $J$ is a subset of $\{1, \ldots, k\}$ with $|J| = k$.
Thus the elements \( \{ \eta_i^{(k)} \}_{i=1}^{k} \) generate the center of \( U(\mathfrak{gl}(k)) \), and the following proposition is standard, see e.g., [24] 4.7.

**Proposition 4.1.** The elements \( \{ \eta_i^{(k)} \}_{1 \leq i \leq k \leq n} \) form a maximal commutative subalgebra of \( U(\mathfrak{gl}(n)) \), i.e., the Gelfand-Zeitlin subalgebra. Furthermore, the roots \( \zeta_1^{(k)}, \ldots, \zeta_k^{(k)} \) of \( M_k(\zeta) \) are distinct.

**Remark 4.2.** We give a remark on the notation: our matrix \( T = (E_{ij}) \) is transpose to the one \( T = (E_{ij}) \) used in [2]. However, the quantum minor given in [7] is the row-determinant, i.e., the row expansion of the determinant starting from the first row, while the quantum-minor in [2] is the column-determinant (taking into account the same Capelli correction). Therefore the quantum minors \( M_k(\zeta) \) (thus the roots \( \zeta_1^{(k)} \)) are same as the ones denoted by the same symbol in [2].

Any highest weight representation \( L(\lambda) \) is equipped with the Gelfand-Zeitlin basis, on which the elements in the Gelfand-Zeitlin subalgebra, particularly \( \{ \zeta_i^{(k)} \}_{1 \leq i \leq k \leq n} \), act diagonally. Elements in such a basis (associated to the highest weight \( \lambda = (\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}) \)) are parameterized by the collection of numbers \( \{ \lambda_{ij}^{(i)} \}_{1 \leq i \leq j \leq n} \) satisfying the interlacing relation \( \lambda_{ij}^{(i)} - \lambda_{ij}^{(i-1)} \in \mathbb{Z}_+ \), \( \lambda_{ij}^{(i-1)} - \lambda_{ij}^{(i)} \in \mathbb{Z}_+ \). See e.g., [24] Section 2 for more details.
4.2 Equations in representation spaces

Instead of working in the formal setting, in the following we will switch to a categorical setting. Let $W$ be a finite dimension vector space with a representation $\rho : \mathfrak{gl}(n) \to \text{End}(W)$. Denote by the same letter $T = (T_{ij}) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(W)$ the $n$ by $n$ matrix with entries in $\text{End}(W)$ given by $T_{ij} = \rho(E_{ji})$. For any $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$ and real number $h$, let us consider the linear system (a categorical version of (2))

$$\frac{dF}{dz} = \left( iu + \frac{h}{2\pi i} T \right) \cdot F,$$

for a function $F(z) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(W)$.

The system has rank $n \times \text{dim}(W)$, and has an irregular term with repeated eigenvalues $u_i$. It has Stokes matrices $S_\pm(u, hT) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(W)$. One can study the isomonodromy deformation of such a system with respect to $u$, while keeping the multiplicity of each eigenvalue $u_i$, see [35] and see e.g., [2] for a more general setting. Similarly the closure of Stokes matrices in [34] can be generalized to this setting. In particular, Stokes matrices $S_\pm(u_{\text{cat}}, hT)$ at the caterpillar point $u_{\text{cat}}$ can be defined. However, to derive the explicit formula of $S_\pm(u_{\text{cat}}, hT)$ in the following subsections, we will treat the system as a system of rank $n$ with coefficients in $\text{End}(W)$. Although the entries are not commutative, the computation in Section 3.2 is valid. The only difference is that we need to replace the variables $\lambda^{(i)}_j$ and $a^{(i)}_j$ with their quantum analog.

4.3 Diagonalization in non-commutative cases

Recall that associated to a representation $\rho : \mathfrak{gl}(n) \to \text{End}(W)$, we have defined the matrix $T = (\rho(E_{ji}))$. In this subsection, we show the diagonalization in stages of the matrix $T$ with entries in the non-commutative ring, as a quantum analog of the diagonalization in Section 3.2.1. It is a consequence of algebraic expression manipulation using the structure of commutators $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$.

For any real number $h \neq 0$, set $T(\zeta) = (\text{Id} - hT)$. As we have seen, the subalgebra, generated in $\text{End}(W)$ by the coefficients in all $M_k(\zeta) = \Delta^{1 \ldots k}_{1 \ldots k\ldots k}(T(\zeta - \frac{h}{2}(k - 1)))$ for $1 \leq k \leq n$, is commutative, and the roots $\zeta^{(k)}_1, ..., \zeta^{(k)}_k$ of $M_k(\zeta)$ act diagonally on the Gelfand-Zeitlin basis in $W$.

**Proposition 4.3.** (a). For any $2 \leq k \leq n$, the $k$ by $k$ matrix $P^{(k)}_{ij} = (P^{(k)}_{ij})$ with entries in $\text{End}(W)$,

$$P^{(k)}_{ij} := \frac{(-1)^{k-j}}{\prod_{l=1, l\neq i}^{k}(\zeta^{(k)}_l - \zeta^{(k)}_i)} \Delta^{1 \ldots j \ldots k\ldots k}_{1 \ldots i \ldots k\ldots k}(T(\zeta - \frac{h}{2}(k - 3))),$$

$$P^{(k)}_{ii} := 1, \quad i > k, \quad P^{(k)}_{ij} := 0, \quad \text{otherwise},$$

diagonalizes the $k$-th principal submatrix $hT^{(k)}$ of $hT$, and is such that $hT^{(k)} := P^{(k)} \cdot hT \cdot P^{(k)}^{-1}$ takes the form

$$hT^{(k)} = \begin{pmatrix}
\zeta^{(k)}_1 - \frac{h}{2}(k - 1) & a^{(k)}_1 & \cdots \\
\vdots & \ddots & \cdots \\
0 & \cdots & \zeta^{(k)}_k - \frac{h}{2}(k - 1) & a^{(k)}_k \\
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.$$

(b). Furthermore, for any $1 \leq k \leq n - 1$, and $1 \leq i, j \leq k$, we have the commutators

$$[\zeta^{(k)}_i, \zeta^{(k)}_j] = \delta_{ij} h \alpha^{(k)}_j.$$

**Proof.** The Laplace expansion, of the quantum minor $M_k(\zeta)$ as a weighted sum of $k$ quantum minors of size $k - 1$, shows that for any tuples $(a_1, ..., a_k)$ and $(b_1, ..., b_k)$,

$$\Delta^{a_1 \ldots a_k}_{b_1 \ldots b_k}(T)(\zeta) = \sum_{j=1}^{k} (-1)^{k-j} \Delta^{a_1 \ldots a_{j-1}, a_j \ldots a_k}_{b_1 \ldots b_{j-1}, b_j \ldots b_k}(T)(\zeta + h) \cdot T_{a_jb_k}(\zeta).$$

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Set \((b_1, ..., b_k) = (1, ..., k)\) and \((a_1, ..., a_{k-1}) = (1, ..., k-1)\). As \(a_k = k\), we have

\[
\Delta^{1, ..., k}_{i_1, ..., i_k}[(T)(\zeta)] = \sum_{j=1}^{k} (-1)^{k-j} \Delta^{1, ..., j-1, j-k}_{i_1, ..., i_{k-1}}[(T)(\zeta + h)] \cdot T_{jk}(\zeta).
\]

Then for each \(i = 1, ..., k\), plugging in \(\zeta^{(k)}_i - \frac{h}{2}(k-1)\), we get the identity

\[
0 = \Delta^{1, ..., k}_{i_1, ..., i_k}[(T)(\zeta^{(k)}_i - \frac{h}{2}(k-1))] = \sum_{j=1}^{k} (-1)^{k-j} \Delta^{1, ..., j-1, j-k}_{i_1, ..., i_{k-1}}[(T)(\zeta^{(k)}_i - \frac{h}{2}(k-3))] \cdot T_{jk}\left(\zeta^{(k)}_i - \frac{h}{2}(k-1)\right).
\]

Similarly, taking \(a_k\) as other values eventually implies that for each \(i\), the row vector

\[
v_i = \frac{1}{\prod_{j=1, j \neq i}^{k} (\zeta^{(k)}_j - \zeta^{(k)}_i)} \left(P^{(k)}_{i1}, ..., P^{(k)}_{ik}\right)
\]

satisfies \(v_i \cdot T^{(k)} = \left(\zeta^{(k)}_i - \frac{h}{2}(k-1)\right)v_i\). Here we denote by \(T^{(k)}\) the upper left \(k\)-th principal submatrix of \(T\). It verifies \((a)\).

To prove \((b)\), without loss of generality, let us assume \(W\) is a highest representation \(L(\lambda)\) of \(\mathfrak{gl}(k+1)\). The restriction of \(L(\lambda)\) to the subalgebra \(\mathfrak{gl}_k\) is isomorphic to the direct sum of irreducible representations \(L(\mu)\), summed over the highest weights \(\mu\) of \(\mathfrak{gl}(k)\) satisfying the interlacing relation with the given \(\lambda\). Denote by \(L(\lambda)^+\) the subspace of \(\mathfrak{gl}_k\) highest vectors in \(L(\lambda)\). Given a \(\mathfrak{gl}_k\) weight \(\mu = (\mu_1, ..., \mu_k)\), denote by

\[
L(\lambda)^+_{\mu} = \{ \eta \in L(\lambda)^+ \mid E_{ii}\eta = \mu_i\eta, i = 1, ..., k \}
\]

the corresponding weight subspace in \(L(\lambda)^+\).

First note that the element \(a_{i}^{(k)}\) for \(i = 1, ..., k\) is just the \(k\) by \(k\) quantum minors

\[
\tau(\zeta_i^{(k)}) = \Delta^{1, ..., k}_{i_1, ..., i_{k-1}, k+1}(T(\zeta_i^{(k)} - h(k-3)/2)).
\]

**Lemma 4.4.** (1) For any \(\eta \in L(\lambda)^+_{\mu}\), we have

\[
\tau(\zeta_i^{(k)})\eta \in L(\lambda)^+_{\mu - \delta_i^{(k)}}, \quad i = 1, ..., k,
\]

where \(\mu - \delta_i^{(k)}\) is obtained from \(\mu\) by replacing \(\mu_i\) with \(\mu_i - 1\).

(2) For any indices \(a = 1, ..., k\), \(b = 1, ..., k-1\), we have

\[
[E_{ab}, \tau(\zeta_i^{(k)})] = 0, \quad \forall \ i = 1, ..., k.
\]

**Proof.** It follows from [24] Proposition 2.18 and [24] 2.37. Note that the \(T\) in this paper is transpose to the element \(E\) appearing in [24] Proposition 2.18. Thus up to the algebra isomorphism \(\theta : U(\mathfrak{gl}(n)) \to U(\mathfrak{gl}(n)) : E_{ij} \mapsto -E_{ji}\), for any \(i, j = 1, ..., n\), the properties (1) and (2) follow from the properties of \(E\) given in [24] Proposition 2.18 and [24] 2.37 respectively.

Now since for any \(\eta \in L(\lambda)^+_{\mu}\), we have

\[
\zeta_i^{(k)}\eta = h(\mu_i + (k-1)/2) \cdot \eta, \quad \forall \ i = 1, ..., k,
\]

thus \((43)\) implies that

\[
\tau(\zeta_j^{(k)})\zeta_i^{(k)}\eta - \zeta_i^{(k)}\tau(\zeta_j^{(k)})\eta = h(\mu_i + (k-1)/2) \cdot \tau(\zeta_j^{(k)})\eta - h(\mu_j - \delta_{ij} + (k-1)/2) \cdot \tau(\zeta_j^{(k)})\eta = \delta_{ij}h\tau(\zeta_j^{(k)})\eta,
\]

which verifies \((b)\) for the \(\mathfrak{gl}(k)\) highest vectors.

However, if we take \(b \subset \mathfrak{gl}(k)\) as the set of lower triangular matrices, then \(L(\mu)\) is generated by the action of \(U(b)\) on the highest vector \(\eta\). Since any element \(b\) in \(U(b)\) commute with \(\lambda_i^{(k)}\), and by \((44)\) commute with \(\tau(\zeta_i^{(k)})\) as well, we get

\[
\tau(\zeta_j^{(k)})\zeta_i^{(k)}b\eta - \zeta_i^{(k)}\tau(\zeta_j^{(k)})b\eta = b(\tau(\zeta_j^{(k)})\zeta_i^{(k)}\eta - \zeta_i^{(k)}\tau(\zeta_j^{(k)})\eta) = \delta_{ij}h\tau(\zeta_j^{(k)})b\eta.
\]
with the variables \( \{a_{ij}^{(k)}\}_{1 \leq i \leq k} \) and \( \{\beta_{ij,l}^{(k)}\}_{1 \leq i \leq k, 1 \leq l \leq k} \).}

Then \( L^{(k+1)-1} \cdot hT_k \cdot L^{(k+1)} = hT_{k+1} \), and \( P^{(k)} = L^{(k+1)}P^{(k+1)} \). In particular, the upper left \( k + 1 \)-th principal submatrix of \( L^{(k+1)-1}hT_kL^{(k+1)} = hT_{k+1} \) is diag\( (\zeta_1^{(k+1)} - \frac{hk}{2}, \ldots, \zeta_{k+1}^{(k+1)} - \frac{hk}{2}) \).

### 4.4 Confluent hypergeometric functions in representation spaces

Recall that we let \( W \) be a finite dimension representation of \( gl(n) \), and for any \( 1 \leq k \leq n - 1 \) we think of \( T_k^{(k+1)} \) as a \( k + 1 \) by \( k + 1 \) matrix with entries in \( \text{End}(W) \). Let us consider a linear system for a function \( F(z) \in \text{End}(C^{k+1}) \otimes \text{End}(W) \), taking the form

\[
\frac{dF}{dz} = \left( iE_{k+1} + \frac{h}{2\pi i} T_k^{(k+1)} \right) F.
\]

(47)

Here \( E_{k+1} \in \text{End}(C^{k+1}) \otimes \text{End}(W) \) is the matrix whose all entries are zero but the \((k+1,k+1)\)-entry, which equals to \( 1 \in \text{End}(W) \).

A solution of this system is described as follows, just like Lemma 3.9. First note that \( \{\zeta_i^{(k)}\}_{1 \leq i \leq k} \) are commutative elements in \( \text{End}(W) \), thus we can define the following confluent hypergeometric functions valued in \( \text{End}(W) \),

\[
H^W(z)_{ij} = \frac{1}{\zeta_i^{(k+1)} - \zeta_i^{(k)} - \frac{h}{2}}, \quad kF_k(\alpha_{ij,1}, \ldots, \alpha_{ij,k}, \beta_{ij,1}, \ldots, \beta_{ij,k+1}; iz), \quad 1 \leq i \leq k, 1 \leq j \leq k + 1,
\]

\[
H^W(z)_{k+1,j} = kF_k(\alpha_{k+1,j,1}, \ldots, \alpha_{k+1,j,k}, \beta_{k+1,j,1}, \ldots, \beta_{k+1,j,k+1}; iz), \quad i = k + 1, 1 \leq j \leq k + 1,
\]

with the variables \( \{\alpha_{ij,l}\} \) and \( \{\beta_{ij,l}\} \) as in Lemma 3.9 for the classical case, except replacing \( \lambda_i^{(k)} \) and \( \lambda_i^{(k+1)} \) with \(-\left(\zeta_i^{(k)} - \frac{h(k-1)}{2}\right)\) and \(-\left(\zeta_i^{(k+1)} - \frac{hk}{2}\right)\) respectively. That is

\[
\alpha_{ij,l} = \frac{1}{2\pi i} \left( \zeta_j^{(k)} - \zeta_i^{(k)} - \frac{h}{2} \right), \quad 1 \leq i \leq k, 1 \leq j \leq k + 1,
\]

\[
\alpha_{(k+1),j,l} = \frac{1}{2\pi i} \left( \zeta_j^{(k+1)} - \zeta_i^{(k)} - \frac{h}{2} \right), \quad 1 \leq j \leq k + 1,
\]

\[
\alpha_{ij,l} = \frac{1}{2\pi i} \left( \zeta_j^{(k+1)} - \zeta_i^{(k+1)} - \frac{h}{2} \right), \quad l \neq i, 1 \leq l \leq k, 1 \leq i, j \leq k + 1,
\]

\[
\alpha_{ij,l} = \frac{1}{2\pi i} \left( \zeta_j^{(k+1)} - \zeta_i^{(k+1)} - \frac{h}{2} \right), \quad l \neq j, 1 \leq l \leq k + 1, 1 \leq i, j \leq k + 1.
\]

**Proposition 4.5.** The equation (47) has a fundamental solution \( F^W(z) \in \text{End}(C^{k+1}) \otimes \text{End}(W) \) taking the form

\[
F^W(z) = Y \cdot H^W(z) \cdot z^{T_{k+1}^{(k+1)}},
\]

(48)

where \( Y = \text{diag}(a_1^{(k)}, \ldots, a_k^{(k)}, 1) \) and \( H^W(z) = \left( H^W_{ij}(z) \right)_{i,j=1}^{k+1} \).

**Proof.** We only need to verify the identities

\[
\frac{d}{dz}(F^W(z)_{ij}) = \frac{1}{2\pi i} \left( \zeta_j^{(k)} - \frac{h}{2}(k-1) \right) F^W(z)_{ij} + \frac{1}{2\pi i} a_{ij}^{(k)} F^W(z)_{k+1,j}, \quad \text{for } 1 \leq i \leq k,
\]

(49)

\[
\frac{d}{dz}(F^W(z)_{k+1,j}) = iz F^W(z)_{k+1,j} + \frac{1}{2\pi i} \sum_{i=1}^{k} b_i^{(k)} F^W(z)_{ij} + \frac{1}{2\pi i} \zeta_j^{(k)} F^W(z)_{k+1,j},
\]

(50)
for the functions
\[ F^W(z)_{ij} = a_{i}^{(k)} \cdot H_{ij}^W(z) \cdot z^{-\frac{h}{2}(\zeta_i^{(k+1)} + \frac{h}{2})}, \text{ for } 1 \leq i \leq k, \]
\[ F^W(z)_{k+1,j} = H_{k+1,j}^W(z) \cdot z^{-\frac{h}{2}(\zeta_i^{(k+1)} + \frac{h}{2})}. \]

To compare the left and right hand sides of (49), we need to keep the entries \(a_i^{(k)} \) in the front. That is to pull the term \(a_i^{(k)} \) in \((\zeta_i^{(k)} - \frac{h}{2}(k-1))\) \(F^W(z)_{ij} = (\zeta_i^{(k)} - \frac{h}{2}(k-1)) \cdot a_{i}^{(k)} H_{ij}^W(z) z^{-\frac{h}{2}(\zeta_i^{(k+1)} + \frac{h}{2})}\) in front, using the commutator relation
\[(\zeta_i^{(k)} - \frac{h}{2}(k-1)) \cdot a_{i}^{(k)} = a_{i}^{(k)} \cdot (\zeta_i^{(k)} - \frac{h}{2}(k+1)).\]

Then the identity (49) follows from the equation (32) satisfied by the function \(H_{ij}^W\) as in the classical case.

To compare the left and right hand sides of (49), like in the classical case, one needs to use the identity
\[ a_i^{(k)} b_i^{(k)} = -\prod_{i \neq j} (\zeta_i^{(k)} - \zeta_j^{(k+1)} + \frac{h}{2}) \cdot h a_i^{(k)}, \text{ for any } 1 \leq i \leq k. \]

The analog of Proposition 3.8 is as follows. The Stokes matrix \(S_+(T_k^{(k+1)})\) and the renormalized connection matrix \(\tilde{C}^{(k+1)}(T) := C^{(k+1)}(T_k^{(k+1)}) \cdot L^{(k)}\) of the linear system (47) is described by

**Proposition 4.6.** (1) The \(j\)-th entry of the last column \(b^{(k+1)}\) (above the diagonal part) of \(S_+^{(k+1)}(T_k^{(k+1)})\) is
\[ b_i^{(k+1)} = -\frac{1}{2} (\zeta_i^{(k)} + \zeta_i^{(k+1)} + \frac{h}{2}) \prod_{k=1, l \neq j} \Gamma(1 + \frac{\zeta_i^{(k)} - \zeta_j^{(k+1)} + h}{2 \pi i}) \cdot h a_j^{(k)}. \]

(2) The entries of the matrix \(\tilde{C}^{(k+1)}(T)\) are given by
\[ \tilde{C}_{ij}^{(k+1)} = \frac{e^{\frac{1}{2}(\zeta_i^{(k-1)} - \zeta_j^{(k+1)} - \frac{h}{2})} \prod_{k=1}^{k-1} \Gamma(1 + \frac{\zeta_i^{(k)} - \zeta_j^{(k+1)} + h}{2 \pi i}) \prod_{k=1}^{k+1} \Gamma(1 + \frac{\zeta_i^{(k)} - \zeta_j^{(k+1)} + h}{2 \pi i}) \cdot h a_i^{(k)}, \]
for \(1 \leq j \leq k+1, 1 \leq i \leq k,\) and
\[ \tilde{C}_{k+1,j}^{(k+1)} = \frac{e^{\frac{1}{2}(\zeta_{k+1,j}^{(k+1)} - \zeta_{k+1,j}^{(k+1)} - \frac{h}{2})} \prod_{k=1}^{k} \Gamma(1 + \frac{\zeta_{k+1,j}^{(k+1)} - \zeta_{j}^{(k+1)} + h}{2 \pi i}) \cdot \prod_{k=1}^{k+1} \Gamma(1 + \frac{\zeta_{k+1,j}^{(k+1)} - \zeta_{j}^{(k+1)} + h}{2 \pi i}) \cdot \prod_{k=1}^{k+1} \Gamma(1 + \frac{\zeta_{k+1,j}^{(k+1)} - \zeta_{j}^{(k+1)} + h}{2 \pi i}), \text{ for } 1 \leq j \leq k+1. \]

**Proof.** It follows from Proposition 4.5 and the asymptotics of the functions \(k F_k(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k; z)\): since the arguments \((\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)\) involved are commutative, thus the ordinary asymptotics (33) are valid, and can be used to get the Stokes matrices. In the non-commutative case, the only different (thus the thing we should be careful about) is that the elements \(a_i^{(k)}\) are not commutative with other elements appearing in the solution \(F^W(z)\). However, the appearance of \(a_i^{(k)}\) in \(F^W(z)\) is rather simple, thus the computation is same as Proposition 3.8 since all \(a_i^{(k)}\) terms are kept in front, we have
\[ F^W(z) \sim \hat{F}(z) \cdot Y \cdot U_+, \text{ as } z \to 0 \text{ in } \hat{H}_+, \]
\[ F^W(z) \sim \hat{F}(z) \cdot Y \cdot U_-, \text{ as } z \to 0 \text{ in } \hat{H}_-, \]
where \(\hat{F}(z) = (\text{Id} + O(z)) e^{i z E_k(z)} z^{-\frac{1}{2}(T^{(k+1)})}\) is the unique formal solution, and \(U_+\) are the same matrices as in (33)–(36), with entries given by Gamma functions of same forms provided replacing \(\lambda_i^{(k)}\) and \(\lambda_i^{(k+1)}\) by \(-\left(\lambda_i^{(k)} - \frac{h(k-1)}{2}\right)\) and \(-\left(\lambda_i^{(k+1)} - \frac{h k}{2}\right)\) respectively. Then the Stokes matrices are given by
\[ S_+^{(k+1)}(T_k^{(k+1)}) = Y U_+^{-1} Y^{-1}, \quad S_-^{(k+1)}(T_k^{(k+1)}) = Y U_-^{-1} Y^{-1}. \]
Since all \(\{s_{j}^{(k)}\}_{j=1, \ldots, k}\) and \(\{s_{j}^{(k+1)}\}_{j=1, \ldots, k+1}\) commute, the matrix \(U_- U_+\) takes the same form as the classical case, provided replacing \(\lambda_i^{(k)}\) and \(\lambda_i^{(k+1)}\) by \(-\left(\zeta_i^{(k)} - \frac{h(k-1)}{2}\right)\) and \(-\left(\zeta_i^{(k+1)} - \frac{h k}{2}\right)\) respectively.
In the end, to get the expression of \( b^{(k+1)}_j \) in the proposition, we use \( (b) \) in Proposition 42 to commute \( a^{(k)}_j \) in \( Y \) with the arguments \( \zeta_j^{(k)} \) in \( U_- U_+^{-1} \), which brings the extra factor \( \hbar \) in the expression \( \prod_{l=1, l \neq j}^{k} \Gamma(1 + \frac{\zeta_{l}^{(k)} - \zeta_{j}^{(k)} + \hbar}{2\pi i}) \). Similarly, one can get the expression of the quantum connection matrix. ■

4.4.1 Entries of quantum Stokes matrices above the diagonal

In the last subsection, we have computed the quantum connection and Stokes matrices for each step \( k, k = 1, \ldots, n \), with respect to a representation \( W \). Then Definitions 3.3 and 3.4 carry over to the quantum case, and define the quantum Stokes matrices \( S_{\pm}(u_{\text{cat}}, \hbar T) \) at caterpillar points. In particular, \( S_{+}(u_{\text{cat}}, \hbar T) \) is an \( n \) by \( n \) upper-triangular matrix with entries in \( \text{End}(W) \). For any \( 1 \leq k \leq n - 1 \), its element \( s_{k,k+1}^{(+)} \) at the \( k \)-th row and \( k + 1 \)-th column can be computed as in Section 3.2.3 by replacing \( A \) with \( \hbar T \). As in the proof of Corollary 3.13, the computation of \( s_{k,k+1}^{(+)} \) only involves commutative elements in \( \text{End}(W) \). Thus it immediately gives a proof of Theorem 4.3 in a categorical setting.

5 Quantum dual exponential maps and Alekseev-Meinrenken diffeomorphisms

In this section, we discuss the relation between quantum Stokes matrices at caterpillar points with the Appel-Gautam isomorphisms, and quantization of Alekseev-Meinrenken diffeomorphisms. Section 5.1 recalls the relation between quantum and classical Stokes matrices and dual exponential maps. Section 5.2 recalls the two realization of quantum groups. Section 5.3 shows that up to the two realization of quantum groups, quantum dual exponential maps at caterpillar points coincide with the Appel-Gautam isomorphisms. Then Section 5.4 unveils the relation with quantization of Alekseev-Meinrenken diffeomorphisms. In the end, Section 5.5 discusses briefly the algebraic characterization of quantum Stokes matrices.

5.1 Quantum dual exponential maps

By the quantum duality principle of Drinfeld and Gavarini [11, 15], the Hopf algebras \( U(R) \) and \( U(\mathfrak{gl}(n))[[\hbar]] \) contain quantized formal series Hopf subalgebras \( U(R)' \) and \( U(\mathfrak{gl}(n))[[\hbar]]' \) respectively. Then for any \( u \in \hbar_{\text{reg}}(\mathbb{R}) \), the map

\[
\nu_{\hbar}(u) : U(R) \to U(\mathfrak{gl}(n))[[\hbar]] ; \quad \iota_{ij}^{(\pm)} \mapsto \delta_{ij}^{(\pm)},
\]

restricts to an isomorphism from \( U(R)' \) to \( U(\mathfrak{gl}(n))[[\hbar]]' \). Furthermore, the semiclassical limit of the restricted \( \nu_{\hbar}(u) \) recovers (the dual of) Boalch’s dual exponential map

\[
\nu(u) : \mathfrak{gl}(n)^* \to G^* ; \quad A \mapsto \left( S_{-}(u, A)^{-1}, S_{+}(u, A) \right),
\]

where \( S_{\pm}(u, A) \) are the Stokes matrices of the meromorphic linear system

\[
\frac{dF}{dz} = \left( iu - \frac{A}{z} \right) F.
\]

As pointed out by Boalch [5], the classical Stokes matrices \( S_{\pm}(u) \) can be seen as classical L-operator (with entries in \( S(\mathfrak{gl}(n)) \)). In this paper, we have seen that the quantum Stokes matrices \( S_{\hbar,\pm}(u) \) are actually quantum L-operators. In particular, as \( \hbar \to 0 \), the RLL formulation gives the classical r-matrix formulation of the Poisson brackets on \( G^* \), and then the dual exponential map \( \nu(u) \) is Poisson follows from the classical RLL formalism.

In this way, the maps \( \nu_{\hbar}(u) \) can be seen as quantum dual exponential maps parameterized by \( \hbar_{\text{reg}}(\mathbb{R}) \), and \( \nu_{\hbar}(u_{\text{cat}}) \) is their extension to the caterpillar point on \( \hbar_{\text{reg}}(\mathbb{R}) \). In the following sections, we will focus on the story at caterpillar points.

5.2 Two realization of quantum groups

The Drinfeld-Jimbo quantum group \( U_{\hbar}(\mathfrak{gl}(n)) \) is a unital associative algebra over \( \mathbb{C}[[\hbar]] \) with generators \( q^\pm h_j, e_i, f_i \), \( 1 \leq j \leq n, 1 \leq i \leq n \) and relations (where we set \( q = e^{\hbar/2} \));
for each $1 \leq i \leq n, 1 \leq j \leq n - 1$,

$$q^h_i q^{-h_i} = q^{-h_i} q^h_i = 1, \quad q^h_i e_j q^{-h_i} = q^h_i q^{-h_i} = q^h_i q^{-h_i} f_j.$$ 

for each $1 \leq i, j \leq n - 1$,

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i-h_{i+1}} - q^{-h_i+h_{i+1}}}{q - q^{-1}};$$

for $|i - j| = 1$,

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0,$$

$$f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0,$$

and for $|i - j| \neq 1$, $[e_i, e_j] = 0 = [f_i, f_j]$.

The algebra $U_h(\mathfrak{gl}(n))$ is a Hopf algebra with coproduct and counit given, respectively, by

$$\Delta(q^{\pm h_i}) = q^{\pm h_i} \otimes q^{\pm h_i},$$

$$\Delta(e_i) = e_i \otimes q^{h_i-h_{i+1}} + 1 \otimes e_i,$$

$$\Delta(f_i) = f_i \otimes 1 + q^{-h_i+h_{i+1}} \otimes f_i,$$

and $\varepsilon(h_i) = \varepsilon(e_i) = \varepsilon(f_i) = 0$. Note that as $h \to 0$, $U_h(\mathfrak{gl}(n))$ becomes the universal enveloping algebra $U(\mathfrak{gl}(n))$. Following [26], the two different realization of quantum groups are isomorphic. See also [10] for a proof.

**Proposition 5.1.** There is an isomorphism $I : U_h(\mathfrak{gl}(n)) \cong U(R)$, where for any $i = 1, \ldots, n$,

$$l_i^{(+)} = q^{-h_i}, \quad l_i^{(+)}_{i+1} = -(q - q^{-1})q^{-h_i}e_i,$$

$$l_i^{(-)} = (q - q^{-1})f_i q^{-h_i}.$$

### 5.3 The Appel-Gautam isomorphisms

#### 5.3.1 Isomorphisms associated to gamma functions

To compare the Drinfeld isomorphism $\nu_h(\mathfrak{ucal})$ with the Appel-Gautam isomorphism $\Psi_{AG}$ in [2, Formula 2.5], we introduce

- $U_h(\mathfrak{sl}(n))$ the subalgebra generated by $\{e_i, f_i, q^{h_i-h_{i+1}}\}$;
- the algebra isomorphism $\theta : U(\mathfrak{gl}(n)) \to U(\mathfrak{gl}(n)) : E_{ij} \mapsto -E_{ji}$, for any $i, j = 1, \ldots, n$.

Then after identifying $U_h(\mathfrak{sl}(n))$ with the corresponding subalgebra of $U(R)$ by

$$l_i^{(+)} = q^{-h_i}, \quad l_i^{(+)}_{i+1} = -(q - q^{-1})e_i e^{-h(h_i+h_{i+1})/4}, \quad l_i^{(-)}_{i+1,i} = (q - q^{-1})f_i e^{h(h_i+h_{i+1})/4},$$

we see that the composition of $\nu_h(\mathfrak{ucal})$ given in Corollary [1,4] with $\theta$ coincides with the (complex conjugate of) Appel-Gautam isomorphism $\Psi_{AG} : U_h(\mathfrak{sl}(n)) \to U(\mathfrak{sl}(n))[h]$ in [2, Formula 2.5]

$$\Psi_{AG}(\epsilon_k) = \frac{h}{q - q^{-1}} \sum_{j=1}^{k} \left( \prod_{l=1}^{j} \Gamma(1 + \frac{\zeta^{(k)}(l)}{2\pi i}) \prod_{l=j+1}^{k} \Gamma(1 + \frac{\zeta^{(k)}(j) - \zeta^{(k)}(l)}{2\pi i}) \prod_{l=1}^{j} (1 + \frac{\zeta^{(k)}(l) - \zeta^{(k)}(j)}{2\pi i}) \prod_{l=1}^{j+1} (1 + \frac{\zeta^{(k)}(j) - \zeta^{(k)}(l)}{2\pi i}) \right) \cdot \tilde{a}^{(k)}_j \quad (51)$$

$$\Psi_{AG}(f_k) = \frac{h}{q - q^{-1}} \sum_{j=1}^{k} \left( \prod_{l=1}^{j} \Gamma(1 - \frac{\zeta^{(k)}(l)}{2\pi i}) \prod_{l=j+1}^{k} \Gamma(1 - \frac{\zeta^{(k)}(j) - \zeta^{(k)}(l)}{2\pi i}) \prod_{l=1}^{j} (1 - \frac{\zeta^{(k)}(l) - \zeta^{(k)}(j)}{2\pi i}) \prod_{l=1}^{j+1} (1 - \frac{\zeta^{(k)}(j) - \zeta^{(k)}(l)}{2\pi i}) \right) \cdot \tilde{b}^{(k)}_j. \quad (52)$$
where

\[
\tilde{a}_j^{(k)} = \sum_{j=1}^{k} \frac{(-1)^{k-j} \Delta_{1,\ldots,k-1}^{j} \left( \zeta_i^{(k)} - \frac{h}{2} \right)}{\prod_{l=1,l\neq j}^{k}(\zeta_l^{(k)} - \zeta_i^{(k)})} E_{k+1,j}
\]

\[
\tilde{b}_j^{(k)} = \sum_{j=1}^{k} \frac{(-1)^{k-j} \Delta_{1,\ldots,k-1}^{j} \left( \zeta_i^{(k)} - \frac{h}{2} \right)}{\prod_{l=1,l\neq j}^{k}(\zeta_l^{(k)} - \zeta_i^{(k)})} F_{j,k+1}.
\]

Here the complex conjugate takes each term of gamma function \(\Gamma(1 + \frac{x}{2\pi})\) to \(\Gamma(1 - \frac{x}{2\pi})\). That is (up to the complex conjugate)

**Corollary 5.2.** We have the identity

\[
\Psi_{AG} = \theta \circ \eta(h(u_{can})) \circ I : U_{h}(\mathfrak{sl}(n)) \to U(\mathfrak{g}(n))[h].
\]

**Remark 5.3.** The reason why we need the extra \(\theta\) is that the \(T\) in this paper is transpose to the one given in \([2]\). It also explains the minus sign in the differential equation: if instead of the natural representation, we take its dual representation, then the evaluation of gKZ equation (12) will become \(\frac{dF}{dz} = (iu - \frac{h}{2\pi i})F\), where \(T' = (T_{ij}')\) is the \(n\) by \(n\) matrix with entries \(T_{ij}' = E_{ij} \in U(\mathfrak{g}(n))\).

### 5.3.2 Isomorphisms associated to hyperbolic functions

The coefficients \(f_j^{(i)}\) and \(g_j^{(i)}\) before \(\tilde{a}_j^{(k)}\) and \(\tilde{b}_j^{(k)}\) in (51)–(52) are not real. The Appel-Gautam isomorphism \(\Phi_{AG}\) associated to hyperbolic functions \([2]\) Remark 2.6 (2)] replaces \(f_j^{(i)}\) and \(g_j^{(i)}\) with their "norm" respectively. That is

\[
\Psi_{AG}(\epsilon_k) = \frac{h}{q - q^{-1}} |f_j^{(i)}| \cdot \tilde{a}_j^{(k)},
\]

\[
\Psi_{AG}(f_k) = \frac{h}{q - q^{-1}} |g_j^{(i)}| \cdot \tilde{b}_j^{(k)},
\]

where \(|f_j^{(i)}|\) and \(|g_j^{(i)}|\) are given respectively by \(f_j^{(i)}\) and \(g_j^{(i)}\) with each term of gamma function \(\Gamma(1 + \frac{x}{2\pi})\) replaced by \(\left(\frac{1}{\pi} \sinh(\frac{x}{2})\right)^\frac{1}{2}\). Therefore, the isomorphisms \(\Phi_{AG}\) and \(\Psi_{AG}\) differs by the automorphism \(\mathcal{X}_h\) of \(U(\mathfrak{sl}(n))[h]\),

\[
\mathcal{X}_h(\zeta_j^{(i)}) = \zeta_j^{(i)}, \quad \mathcal{X}_h(\tilde{a}_j^{(k)}) = \frac{f_j^{(i)}}{|f_j^{(i)}|} \cdot \tilde{a}_j^{(k)}, \quad \mathcal{X}_h(\tilde{b}_j^{(k)}) = \frac{g_j^{(i)}}{|g_j^{(i)}|} \cdot \tilde{b}_j^{(k)}.
\]

That is

\[
\Phi_{AG} = \Psi_{AG} \circ \mathcal{X}_h.
\]

### 5.4 Darboux coordinates and the Alekseev-Meinrenken diffeomorphisms

In this subsection, let us turn to the semiclassical limit, and place the results in this paper into the context of Poisson geometry. To this end, let us consider the Lie algebra \(u(n)\) of the unitary group \(U(n)\), consisting of skew-Hermitian matrices, and identify \(\text{Herm}(n) \cong u(n)^*\) via the pairing \(\langle A, \zeta \rangle = 2\text{Im}(\text{tr} A \zeta)\). Thus \(\text{Herm}(n)\) inherits a Poisson structure from the canonical linear (Kostant-Kirillov-Souriau) Poisson structure on \(u(n)^*\). Furthermore, the unitary group \(U(n)\) carries a standard structure as a Poisson Lie group (see e.g. \([22]\)). The dual Poisson Lie group \(U(n)^*\), which is the group of complex upper triangular matrices with strictly positive diagonal entries, is identified with \(\text{Herm}^+(n)\), by taking the upper triangular matrix \(X \in U(n)^*\) to the positive Hermitian matrix \((X^*X)^{1/2} \in \text{Herm}^+(n)\). The Ginzburg-Weinstein linearization theorem \([16]\) states that the dual Poisson Lie group \(U(n)^* \cong \text{Herm}^+(n)\) is Poisson isomorphic to the dual of the Lie algebra \(u(n)^* \cong \text{Herm}(n)\).

The Poisson manifolds \(u(n)^*\) and \(U(n)^*\) carry extra structures: Guillemin-Sternberg \([17]\) introduced the Gelfand-Zeitlin integrable system on \(u(n)^*\); Flaschka-Ratiu \([14]\) described a multiplicative Gelfand-Zeitlin
Theorem 5.5. For any $a_j \in \mathbb{R}$, the dual exponential maps

$$\nu(u) : u(n)^* \cong \text{Herm}(n) \rightarrow U(n)^* \cong \text{Herm}^+(n) ; A \mapsto S_-(A, u) S_+(A, u),$$

associating the Stokes matrices of $\{a_j^{(i)}\}_{1 \leq j \leq n}$ and angle coordinates $\{\phi_j^{(i)} := \text{Arg}(a_j^{(i)})\}_{1 \leq j \leq n}$ (recall the functions $a_j^{(i)}$ are given in Definition 3.5). Under these coordinates, the Poisson bracket on the open dense subset of $u(n)^* \cong \text{Herm}(n)$ takes the Darboux form

$$\{\lambda_j^{(i)}, \lambda_l^{(k)}\} = 0, \quad \{\lambda_j^{(i)}, \phi_l^{(k)}\} = \delta_{ik} \delta_{jl}, \quad \{a_j^{(i)}, a_k^{(l)}\} = 0.$$

Similarly, on certain open dense subset $U(n)^*_0$ of $U(n)^*$, the functions $\{\log(\lambda_j^{(i)})\}_{1 \leq j \leq n}$ and $\{\phi_j^{(i)}\}$ define a multiplicative Gelfand-Zeitlin action-angle coordinate system, under which the Poisson bracket of $U(n)^* \cong \text{Herm}^+(n)$ takes the Darboux form. See [1] or [34] for more details. Then in [1] Alekseev and Meinrenken proved that

**Theorem 5.4.** The map $\Gamma_{AM}$ from $u(n)^*_0 \rightarrow U(n)^*_0$ intertwining the two Gelfand-Zeitlin action-angle variables on the open dense subset, extends to a diffeomorphism from $u(n)^*$ to $U(n)^*$.

Such a distinguished Ginzburg-Weinstein linearization $\Gamma_{AM}$ is called the Alekseev-Meinrenken diffeomorphism. Another canonical family of Ginzburg-Weinstein linearization is given by Boalch’s dual exponential maps. In particular, Boalch proved that

**Theorem 5.5.** [5] For any $u \in \mathfrak{h}_{\text{reg}}(\mathbb{R})$, the dual exponential maps

$$\nu(u) : u(n)^* \cong \text{Herm}(n) \rightarrow U(n)^* \cong \text{Herm}^+(n) ; A \mapsto S_-(A, u) S_+(A, u),$$

The closure of Stokes matrices in [34] extends the family of dual exponential maps $\nu(u)$ from $\mathfrak{h}_{\text{reg}}(\mathbb{R})$ to $\mathfrak{h}_{\text{reg}}(\mathbb{R})$, in an isomonodromy way. In particular, we get a dual exponential map at a caterpillar point, which coincides with the diffeomorphism $\Gamma_{AM}$ up to an explicit gauge transformation.

**Theorem 5.6.** [32] [34] The dual exponential map $\nu(u_{\text{cat}}) : u(n)^* \cong \text{Herm}(n) \rightarrow U(n)^* \cong \text{Herm}^+(n)$ at the caterpillar point is a Ginzburg-Weinstein linearization. Furthermore, there exists an explicit Poisson automorphism $\mathcal{X}$ of $u(n)^*$ such that

$$\Gamma_{AM} = \nu(u_{\text{cat}}) \circ \mathcal{X}.$$

Explicit expression of $\mathcal{X}$ can be found in [34]. In particular, under the Gelfand-Zeitlin action-angle variables $(\lambda_j^{(i)}, a_j^{(i)})$ on $u(n)^*_0$, the automorphism $\mathcal{X}$ is given by

$$\mathcal{X}(\lambda_j^{(i)}) = \eta_j^{(i)}, \quad \mathcal{X}(a_j^{(i)}) = \text{scl}(f_j^{(i)}) \cdot a_j^{(i)}, \quad \mathcal{X}(b_j^{(i)}) = \text{scl}(g_j^{(i)}) \cdot b_j^{(i)},$$

where $\text{scl}(\cdot)$ takes the semiclasscal limit of elements in the quantized formal series Hopf subalgebra $U(\mathfrak{gl}(n))[\hbar]$. Therefore, for the quantum analog of various Ginzburg-Weinstein linearization, we have the quantum Stokes matrices at caterpillar points, i.e., the Drinfeld isomorphism $\nu_\hbar(u_{\text{cat}})$, and can introduce another isomorphism as the composition of automorphism $\mathcal{X}_\hbar$ with $\nu_\hbar(u_{\text{cat}})$. Furthermore, we have the commutative diagram

$$\begin{array}{ccc}
\nu_\hbar(u) & \xrightarrow{\text{isomonodromy}} & \nu_\hbar(u_{\text{cat}}) \\
\hbar = 0 & \downarrow & \hbar = 0 \\
\nu_\hbar(u) & \text{isomonodromy} & \nu(u_{\text{cat}}) \\
\hbar = 0 & \downarrow & \hbar = 0 \\
\nu_\hbar(u) & \xrightarrow{\text{isomonodromy}} & \nu(u_{\text{cat}}) \circ \mathcal{X} \\
\hbar = 0 & \downarrow & \hbar = 0 \\
\nu_\hbar(u) & \xrightarrow{\text{isomonodromy}} & \nu(u_{\text{cat}}) \circ \mathcal{X}.
\end{array}$$

Following Theorem 5.6, we get

**Corollary 5.7.** The restriction to $u(n)^*$ of the semiclassical limit of the Drinfeld isomorphism $\nu_\hbar(u_{\text{cat}}) \circ \mathcal{X}_\hbar$ coincides with the Alekseev-Meinrenken diffeomorphism $\Gamma_{AM} = \nu(u_{\text{cat}}) \circ \mathcal{X}$. Since under the isomorphism of two realization of quantum groups, $\nu_\hbar(u_{\text{cat}}) \circ \mathcal{X}_\hbar$ coincides with the Appel-Gautum isomorphism $\Phi_{AG}$, therefore the above corollary verifies the conjecture in [24], i.e., $\Phi_{AG}$ is a quantization of the diffeomorphism $\Gamma_{AM}$. 

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5.5 Algebraic characterization of quantum Stokes matrices

In this paper, we use the theory of essential singularities of confluent hypergeometric functions \(_F_k\) to construct the distinguished Drinfeld isomorphism/quantum Alekseev-Meinrenken map \(\nu_h(u_{\text{cat}}) \circ \chi_h\). However, it is interesting to derive it directly in a purely algebraic way. Recall that in the classical setting, \(\Gamma_{AM}\) is the unique diffeomorphism intertwining the Gelfand-Zeitlin action-angle coordinates on \(u(n)^*\) and \(U(n)^*\). While in the quantum setting, the "Darboux coordinates", quantum analog of Gelfand-Zeitlin action-angle variables, on (certain extension of) \(U(gl(n))\) are \(\{s_{ij}\}_{1 \leq i \leq n}\) and the (logarithm of) \(\{a_{ij}\}_{1 \leq i \leq n-1}\) given in Section 4.3. It is expected that there exists a similar "Darboux coordinates", quantum analog of Gelfand-Zeitlin action-angle variables on \(U(n)^*\), on \(U_h(gl(n))\), such that the quantum Alekseev-Meinrenken map \(\nu_h(u_{\text{cat}}) \circ \chi_h\) is the unique isomorphism from \(U_h(gl(n))\) to \(U(gl(n))[\hbar]\) intertwining the two "Darboux coordinates". In this way, the quantum Stokes matrices at caterpillar points are characterized by the Gelfand-Zeitlin subalgebras of \(U(gl(n))\) and \(U_h(gl(n))\).

**Remark 5.8.** As remarked in [34] for the classical setting, the fact that \(\nu_h(u_{\text{cat}}) \circ \chi_h\) is an algebra isomorphism is equivalent to the identity for the function \(s(x) = \sinh(x)\):

\[
s\left(\sum_{j=1}^{n} x_j\right) = \frac{n}{\prod_{k=1, k \neq j}^{n} s(x_k + \alpha_k - \alpha_j)}
\]

for any \(x_i\) and \(\alpha_j\) such that \(s(\alpha_k - \alpha_j) \neq 0\). Note that the above identity is true for any function \(s(x)\) that satisfies

\[
s(x + y) = \frac{s(x)s(y + \beta - \alpha)}{s(\beta - \alpha)} + \frac{s(y)s(x + \alpha - \beta)}{s(\alpha - \beta)}
\]

for any \(x, y, \alpha, \beta\) s.t. \(s(\alpha - \beta) \neq 0\).

Motivated by the above observation for the caterpillar point, we expect that various properties of the quantum Stokes matrices at a generic point \(u \in \mathfrak{b}_{\text{reg}}(R)\) are related to the theory of the shift of argument subalgebras \(\mathcal{A}(u) \subset U(gl(n))\) parameterized by the same \(u \in \mathfrak{b}_{\text{reg}}(R)\). Here, just as (quantum) Stokes matrices \(S(u)\) can be seen as a family of deformation of the Stokes matrices \(S(u_{\text{cat}})\) at caterpillar points (along isomonodromy deformation), the algebras \(\mathcal{A}(u)\) can be seen as a family of deformation of the Gelfand-Zeitlin subalgebras. See e.g., [13] for more details on the shift of argument subalgebras. The semiclassical analog would be that the dual exponential map \(\nu(u)\) intertwine the classical shift of argument subalgebras (also known as Mishchenko–Fomenko subalgebras) on \(u(n)^*\) and its multiplicative analog on \(U(n)^*\). For \(u\) being a caterpillar point, it has been verified in [32, 34]. We will postpone the study along this direction to our future work.

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