Star flow with singularities of different indices

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Abstract

A vector field \( X \) is called a star flow if every periodic orbit of any vector field \( C^1 \)-close to \( X \) is hyperbolic. It is known that a generic a star flow \( X \) on a 3 or 4-dimensional manifold is such that its chain recurrence classes are either hyperbolic, or singular hyperbolic ([MPP] for 3-dimensional manifolds and [CSW] on 4-dimensional manifolds).

We present here a non empty open set of star flows on a 5-dimensional manifold for which two singular points of different indices belong (robustly) to the same chain recurrence class. This prevent the class to be singular hyperbolic.

We show that this chain recurrence class is robustly chain transitive, and since it has two singular points of different indices, from [GWZ] we have that is not robustly transitive. This is then a new example of this behavior. The first and only example of this phenomena is the one in [BCGP].

We prove that this example exhibits a weak form of hyperbolicity (called strong multisingular hyperbolic) which is a particular case of the multisingular hyperbolicity in [BdL] and it is easier to define in this context. This is used to show that the example we built is actually a star flow, since strong multisingular hyperbolicity implies the star property.

1 Introduction

In the beginning of the theory of dynamical systems, the systems that where first consider where the time continuous dynamical systems, that is, the flows. Later on, we began to study discrete time dynamical systems, that is diffeomorphisms. Both theories seem to be very related. Even more, there is an idea shared by many authors that is:

*The dynamics of a non singular vector field (a vector field that is never zero) in dimension \( n \) should look like the one of a diffeomorphism in dimension \( n - 1 \).*

Several results can be translated from one setting to the other. However, this idea of translating results from one setting to the other does not always work quite straightforwardly. One of the possible situations where this happens is when we are dealing with vector fields with singularities (zeros of the vector field).
The coexistence of singularities and regular orbits in indecomposable parts of
the dynamics has lead to the fact that there are several areas in which the theory
for vector fields is somewhat behind the theory for diffeomorphisms.

The aim of this paper will be to build and example showing how singularities
do introduce extra difficulties in the vector field scenario.

A property of a dynamical system is called $C^r$-robust if it holds on a $C^r$-
open set of systems. If one can understand what makes a property be robust, or
to try to detect when a property is robust just by looking at one system, that
makes it possible to understand properties of a hole open set of systems just by
looking at one of them.

In the spirit of understanding when a property is robust, or even more, of
detecting a robust property when one looks at only one system, there are several
results and conjectures that evidence the link between the robust properties
and some structure related to the differential. But even when the study of
flows began earlier than the study of diffeomorphisms, this questions are far less
understood for flows than for diffeomorphisms.

A famous example of this kind of results, is the $C^1$-stability theorem:

Structurally stable systems (systems such that all their dynamical properties
$C^1$-robust) are characterized by uniform hyperbolicity (a strong uniform
structure related to the differential).

This was conjectured by Palis and Smale in [PaSm] (it was conjectured for
the $C^r$ topology but it was only solved for $r = 1$). The sufficient condition was
proven by [R1] and [R2]. The necessary condition was proven by Mañe in 1988
[Ma2] and by Hayashi [H] 1992.

Less informally, given a compact invariant set $K$ of a diffeomorphism $f$ we
say that $f$ is hyperbolic or uniformly hyperbolic on $K$, if

• there is a continuous, invariant splitting of the tangent space, in two
  spaces: $T_xM = E^s_x \oplus E^u_x$

• the vectors are uniformly contracted in $E^s$

• the vectors are uniformly expanded in $E^u$.

We can define an analogous notion for vector fields.

However, hyperbolicity does not describe all systems that hold robust dynamical
properties. In order to understand when a systems has some property
that persist under small perturbation, we aim to find (weaker) structures that
limit the effect of the small perturbations.

The weakest of this defined structures for diffeomorphisms was introduced
by Mañe and Liao and it is called dominated splitting:

**Definition 1.** Let $f: M \to M$ be a diffeomorphism of a Riemannian manifold
$M$ and $K \subset M$ a compact invariant set of $f$. A splitting $T_xM = E(x) \oplus F(x)$,
for $x \in K$, is called dominated if
• \( \dim(E(x)) \) is independent of \( x \in K \) and this dimension is called the \( s \)-index of the splitting;

• it is \( Df \)-invariant: \( E(f(x)) = Df(E(x)) \) and \( F(f(x)) = Df(F(x)) \) for every \( x \in K \);

• there is \( n > 0 \) so that for every \( x \) in \( K \) and every unit vectors \( u \in E(x) \) and \( v \in F(x) \) one has

\[
\|Df^n(u)\| \leq \frac{1}{2}\|Df^n(v)\|.
\]

One denotes \( TM|_K = E \oplus _F \) the dominated splitting.

We will not ask for this properties to hold over all the points in the manifold, but only in some important subsets that hold the most relevant dynamical properties: that is the chain recurrence classes.

• a point \( x \) is chain recurrent if, for any \( \varepsilon > 0 \), there is an \( \varepsilon \)-pseudo orbit from \( x \) to \( x \), that is, a sequence \( x = x_0, x_1, \ldots, x_k = x, k > 0 \) with \( d(x_i, f(x_{i-1})) < \varepsilon \), for \( i \in \{1, \ldots, k\} \). Equivalently \( x \) is chain recurrent if for any attracting region \( U \) (an open set such that \( f(U) \subset U \)), the orbit of \( x \) is either disjoint from \( U \) or contained in it.

• two points \( x, y \) in \( R(f) \) are in the same chain recurrence class if for any \( \varepsilon > 0 \), there are \( \varepsilon \)-pseudo orbits from \( x \) to \( y \) and from \( y \) to \( x \).

It is shown by Conley in [Co] that this chain classes play the role of fundamental pieces of the dynamics, and the rest of the orbits, simply go from one of this pieces to the other.

Other possible subsets to look at when studying the structure of some system can be the maximal invariant sets. We say a set \( \Lambda \) is maximal invariant in \( U \) for a diffeomorphism \( f \) if

\[
\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).
\]

Some other examples of the relation between this structures and the robustness of the dynamical properties are:

1. A long sequence of papers, starting with the work [Ma] of Mañé, and then [DPU, BDP, BDV] and the most complete result in this spirit, in [BB], show that the dominated splittings is the unique obstruction for mixing the Lyapunov exponents of periodic orbits (and therefore not having any dynamical property robust), by \( C^1 \)-small perturbation of the diffeomorphism.

2. One says that a system is star if all periodic orbits are hyperbolic in a robust fashion: every periodic orbit of every \( C^1 \)-close system is hyperbolic. For a diffeomorphism, to be star is equivalent to be hyperbolic (an important step is done in [Ma] and has been completed in [H]).
However, as noted before, vector fields can have singularities (zero of the vector field). This makes the translation of this results to the flow setting, more complicated. In fact, the $C^1$ stability conjecture for flows was proven by Hayashi ([H2]) almost ten years after the result for diffeomorphisms.

The hyperbolic splitting of a singularity and the one over a periodic orbit are different and a priori not compatible.

This is due to the fact that singularities do not have a space of the splitting of the tangent space spanned by the direction of the flow $X(x)$.

We cannot avoid this problem, since the singularities might be accumulated, in a robust way, by regular chain recurrent orbits.

The first example with this behavior has been indicated by Lorenz in [Lo] under numerical evidences. Then [GuWi] constructs a $C^1$-open set of vector fields in a 3-manifold, having a topological transitive attractor containing periodic orbits (that are all hyperbolic) and one singularity. The examples in [GuWi] are known as the geometric Lorenz attractors.

The Lorenz attractor is also an example of a robustly non-hyperbolic star flow, showing that the result in [H] is not true anymore for flows.

In dimension 3 the difficulties introduced by the robust coexistence of singularities and periodic orbits is now almost fully understood. In particular, Morales, Pacifico and Pujals (see [MPP]) defined the notion of singular hyperbolicity, which requires that the chain recurrence classes admit a dominated splitting in two bundles, one being uniformly contracted (resp. expanded) and the other being volume expanding (resp. volume contracting).

They prove that, for a $C^1$-generic star flows on 3-manifolds, every chain recurrence class is singular hyperbolic. In [BaMo] the authors built a star flow on a 3-manifold having a chain recurrence class which is not singular hyperbolic, showing that the previous result cannot be improved.

There are already many results on the hyperbolic structure of the star flows in dimensions larger than 3. The notion of singular hyperbolicity defined by [MPP] in dimension 3 admits a straightforward generalization to higher dimensions:

Each chain recurrence class admits a dominated splitting in two bundles, one being uniformly contracted (resp. expanded) and the other being sectionally area expanding (resp. sectionally area contracting).

If the chain recurrence set of a vector field $X$ can be covered by filtrating sets $U_i$ in which the maximal invariant set $\Lambda_i$ is singular hyperbolic, then $X$ is a star flow. Conversely, [GLW] and [GWZ] prove that this property characterizes the generic star flows on 4-manifolds and for robustly transitive (i.e. having a dense orbit ) singular sets.

In [GSW] the authors prove the singular hyperbolicity of generic star flows in any dimensions assuming an extra property: if two singularities are in the same chain recurrence class then they must have the same $s$-index (dimension of the stable manifold). Indeed, the singular hyperbolicity implies directly this extra property.
If one assumes that the chain recurrence that we are looking at, is robustly transitive then [GWZ] prove that the condition on the index of the singularities is always verified. A priory we do not know whether the condition on the index is still verified after removing the hypothesis of robust transitivity. Are all generic star flows singular hyperbolic?

The following conjecture was formulated in [GWZ]:

**Conjecture 1.** For every star vector field $X$ of a $d$ dimensional manifold $M$, the chain recurrent set $\mathcal{R}(X)$ is singular-hyperbolic and consists of finitely many chain recurrent classes.

One of the difficulties that persisted for understanding this question in higher dimension is that there are very little examples, illustrating what are the possibilities. Let us mention [BLY] which builds a flow having a robustly chain recurrent attractor containing saddles of different indices.

The contents of this work aim to contribute in this direction. We give a negative answer to the first half of Conjecture [1] (note that the second half is still open even for $d = 2$).

We build an example of a star flow in a 5 dimensional manifold, admitting singularities of different indices which belong to the same chain recurrence class, robustly.

**Theorem 1.** Let $M$ be the manifold $S^3 \times \mathbb{RP}^2$. There is a $C^1$-open set $\mathcal{U}$ of $\mathcal{X}^1(M)$ so that every $X \in \mathcal{U}$ is such that there is an open set $U$,

- such that $X \in \mathcal{U}$ is a star flow in $U$.
- the maximal invariant set in $U$ is a chain recurrence class $C$
- $C$ has two singularities $\sigma_1$ and $\sigma_2$ such that the stable manifold of $\sigma_1$ is 3 dimensional and the stable manifold of $\sigma_2$ is 2 dimensional
- these singularities are such that $\sigma_1$ and $\sigma_2$ belong to $\text{Per}(X)$

This example cannot satisfy the singular hyperbolicity used in [GSW].

The definition of singular hyperbolicity forbids a priori the coexistence of singularities of different indexes in the same chain class.

We know by [GWZ] that this example cannot be robustly transitive, but non the less it satisfies a weaker form of transitivity:

**Definition 2.** We say that a chain recurrence class $C$ of a vector field $X$ is **robustly chain transitive** if there is a neighborhood of $C$, $U$ and a $C^1$-neighborhood of $X$, $\mathcal{U}$, such that $C$ is the maximal invariant set in $U$ for $X$, and for every $Y \in \mathcal{U}$, the maximal invariant set in $U$ is a chain recurrence class.

This example is then, an other kind of systems that has a robustly chain transitive chain recurrence class that is not robustly transitive. The first and only example known of this phenomena until now was built in [BCGP].
In order to prove that the example we construct is actually a star flow, we need some tool that allows us to detect the robust hyperbolicity of the periodic orbits without any information of the neighboring vector fields. For this we define a hyperbolic structure that we call strong multisingular hyperbolicity which is a particular case of the multisingular hyperbolicity defined in [BdL] that is a sufficient condition to be a star flow. We then construct our example so that it is strong multisingular hyperbolic, and therefore a star flow.

The example can be done in such a way that it does not admit any dominated splitting of the tangent space for the flow: therefore the hyperbolic structure we will define does not lie on the tangent bundle, but in the normal bundle with the linear Poincaré flow. However the linear Poincaré flow is only defined far from the singularities, and therefore it cannot be used directly for understanding our example.

In [GLW], the authors define the notion of extended linear Poincaré flow defined on some sort of blow-up of the singularities. The notion of strong multisingular hyperbolicity will be expressed as the hyperbolicity of a reparametrization of this extended linear Poincaré flow, over a well chosen extension of the chain recurrence set.

Since we do not have any information of the neighboring vector fields, we will need to extend the linear Poincaré flow to some set that is interesting to us from the dynamical point of view, that varies upper semi continuously with the vector field, but that does not depend on knowing information from the neighborhood of the vector field. Therefore we will need a different notion as the one defined in [GLW]. We will use the notion of central space as defined in [BdL].

2 Basic definitions and preliminaries

2.1 chain recurrence classes and filtrating neighborhoods

The following notions and theorems are due to Conley [Co] and they can be found in several other references (for example [AN]).

- We say that pair of sequences \( \{ x_i \}_{0 \leq i \leq k} \) and \( \{ t_i \}_{0 \leq i \leq k-1} \), \( k \geq 1 \), are an \( \varepsilon \)-pseudo orbit from \( x_0 \) to \( x_k \) for a flow \( \phi \), if for every \( 0 \leq i \leq k-1 \) one has
  \[
  t_i \geq 1 \quad \text{and} \quad d(x_{i+1}, \phi^{t_i}(x_i)) < \varepsilon.
  \]

- A compact invariant set \( \Lambda \) is called chain transitive if for any \( \varepsilon > 0 \), for any \( x, y \in \Lambda \) there is an \( \varepsilon \)-pseudo orbit from \( x \) to \( y \).

- We say that \( x, y \in M \) are chain related if, for every \( \varepsilon > 0 \), there are \( \varepsilon \)-pseudo orbits form \( x \) to \( y \) and from \( y \) to \( x \). This is an equivalence relation.

- We say that \( x \in M \) is chain recurrent if for every \( \varepsilon > 0 \), there is an \( \varepsilon \)-pseudo orbit from \( x \) to \( x \). We call the set of chain recurrent points, the
**Chain recurrent set** and we note it $\mathcal{R}(M)$. The equivalent classes of this equivalence relation are called *chain recurrence classes*.

**Definition 3.**
• An *attracting region* (also called *trapping region* by some authors) is a compact set $U$ so that $\phi^t(U)$ is contained in the interior of $U$ for every $t > 0$. The maximal invariant set in an attracting region is called an *attracting set*. A repelling region is an attracting region for $-X$, and the maximal invariant set is called a repeller.

• A *filtrating region* is the intersection of an attracting region with a repelling region.

• Let $C$ be a chain recurrence class of $M$ for the flow $\phi$. A *filtrating neighborhood* of $C$ is a (compact) neighborhood which is a filtrating region.

The following is a corollary of the fundamental theorem of dynamical systems [Co].

**Corollary 2.** [Co] Let $X$ be a $C^1$-vector field on a compact manifold $M$. Every chain class $C$ of $X$ admits a basis of filtrating neighborhoods, that is, every neighborhood of $C$ contains a filtrating neighborhood of $C$.

**Definition 4.** Let $C$ be a chain recurrence class of $M$ for the vector field $X$. We say that $C$ is *robustly chain transitive* if there exist a filtrating neighborhood $U$ of $C$, and $C^1$ neighborhood of $X$ called $\mathcal{U}$ such that for every $Y \in \mathcal{U}$, the maximal invariant set for $Y (C_Y)$ in $U$ is a unique chain class.

**Definition 5.** Let $C$ be a robustly chain transitive class of $M$ for the vector field $X$. We say that $C$ is *robustly transitive* if there is a $C^1$ neighborhood of $X$ called $\mathcal{U}$ such that for every $Y \in \mathcal{U}$, there is an orbit for $Y$ which is dense in $C_Y$.

### 2.2 Linear cocycle

Let $\phi = \{\phi^t\}_{t \in \mathbb{R}}$ be a topological flow on a compact metric space $K$.

Consider as well:

• A $d$ dimensional linear bundle $E$ over $K$ with $\pi : E \to K$

• A continuous map $A_t : (x, t) \in K \times \mathbb{R} \mapsto GL(E_x, E_{\phi^t(x)})$ that satisfies the following *Cocycle relation* : for any $x \in K$ and $t, s \in \mathbb{R}$ one has:

$$A_{t+s}(x) = A_t(\phi^s(x))A_s(x)$$

We define a *linear cocycle over $(K, \phi)$* as the associated morphism $A^t : K \times \mathbb{R} \to K$ defined by

$$A^t(x, v) = (\phi^t(x), A_t(x)v).$$
Note that $A = \{A^t\}_{t \in \mathbb{R}}$ is a flow on the space $E$ which projects on $\phi^t$.

$$
\begin{array}{ccc}
E & \xrightarrow{A^t} & E \\
\downarrow & & \downarrow \\
K & \xrightarrow{\phi^t} & K
\end{array}
$$

If $\Lambda \subset K$ is a $\phi$-invariant subset, then $\pi^{-1}(\Lambda) \subset E$ is $A$-invariant, and we call the restriction of $A$ to $\Lambda$ the restriction of $\{A^t\}$ to $\pi^{-1}(\Lambda)$.

### 2.3 Hyperbolicity and dominated splitting on linear cocycles

**Definition 6.** Let $\phi$ be a topological flow on a compact metric space $M$ and a $\phi$-invariant connected compact subset $\Lambda$. We consider a vector bundle $\pi: E \to \Lambda$ and a linear cocycle $A$ over $(\Lambda, X)$.

We say that $A$ admits a **Dominated splitting over $\Lambda$** if

- there exists a splitting $E = E^1 \oplus \cdots \oplus E^k$ over $\Lambda$ into $k$ subbundles
- The dimension of the sub-bundles is constant, i.e. $\dim(E^i_x) = \dim(E^i_y)$ for all $x, y \in \Lambda$ and $i \in \{1 \ldots k\}$,
- The splitting is invariant, i.e. $A^t(x)(E^i_x) = E^i_{\phi^t(x)}$ for all $i \in \{1 \ldots k\}$,
- There exists a $t > 0$ such that for every $x \in \Lambda$ and any pair of non vanishing vectors $u \in E^i_x$ and $v \in E^j_x, i < j$ one has
  \[ \frac{\|A^t(u)\|}{\|u\|} \leq \frac{1}{2} \frac{\|A^t(v)\|}{\|v\|} \]  \hspace{1cm} (1)

We denote $E^1 \oplus \cdots \oplus E^k$, or $E^1 \oplus_{\prec} \cdots \oplus_{\prec} E^k$ if one wants to emphasis the role of $t$: in that case one says that the splitting is $t$-**dominated**.

A classical result (see for instance [BDV, Appendix B]) asserts that the bundles of a dominated splitting are always continuous. A given cocycle may admit several dominated splittings. However, the dominated splitting is unique if one prescribes the dimensions $\dim(E^i)$.

One says that one of the bundle $E^i$ is **(uniformly) contracting** (resp. **expanding**) if there is $t > 0$ so that for every $x \in \Lambda$ and every non vanishing vector $u \in E^i_x$ one has $\|A^t(u)\|_u < \frac{1}{2}$ (resp. $\|A^t(u)\|_u < \frac{1}{2}$). In both cases one says that $E^i$ is **hyperbolic**.

Notice that if $E^i$ is contracting (resp. expanding) then the same holds for any $E^j, i < j$ (reps. $j < i$).

**Definition 7.** We say that the linear cocycle $A$ is **hyperbolic over $\Lambda$** if there is a dominated splitting $E = E^s \oplus E^u$ over $\Lambda$ into 2 hyperbolic sub-bundles so that $E^s$ is uniformly contracting and $E^u$ is uniformly expanding.

One says that $E^s$ is the **stable bundle**, and $E^u$ is the **unstable bundle**.
The existence of a dominated splitting or of an hyperbolic splitting is a robust property in the following sense.

**Proposition 3.** Let $K$ be a compact metric space, $\pi: E \to K$ be a $d$-dimensional vector bundle, and $A$ be a linear cocycle over $K$. Let $\Lambda$ be a $\phi$-invariant compact set. Assume that the restriction of $A$ to $\Lambda_0$ admits a dominated splitting $E^1 \oplus \cdots \oplus E^k$, for some $t > 0$.

Then there is a compact neighborhood $U$ of $\Lambda_0$ with the following property. Let $\Lambda = \bigcap_{t \in \mathbb{R}} \phi^t(U)$ be the maximal invariant set of $\phi$ in $U$. Then the dominated splitting admits a unique extension as a $2t$-dominated splitting over $\Lambda$. Furthermore if one of the sub-bundle $E^i$ is contracting (or expanding) over $\Lambda_0$ it is still contracting (or expanding) over $\Lambda$.

As a consequence, if $A$ is hyperbolic over $\Lambda_0$ then (up to shrink $U$ if necessary) it is also hyperbolic over $\Lambda$.

As a consequence of Proposition 3 we get:

**Corollary 4.** Let $\pi: E \to M$ be a linear cocycle over a manifold $M$ and let $\phi_n$ be a sequence of flows on $M$ converging to $\phi_0$ as $n \to \infty$. Let $\Lambda_n$ be a sequence of $\phi_n$-invariant compact subsets so that the upper limit of the $\Lambda_n$, as $n \to \infty$, is contained in $\Lambda_0$.

Let $A_n$ be a sequence of linear cocycles over $\phi_n$ defined on the restriction of $E$ to $\Lambda_n$. Assume that $A_0$ tends to $A_n$ as $n \to \infty$.

Assume that $A_0$ admits a dominated splitting $E = E^1 \oplus \cdots \oplus E^k$ over $\Lambda_0$. Then, for any $n$ large enough, $A_n$ admits a dominated splitting with the same number of sub-bundles and the same dimensions of the sub-bundles. Furthermore, if $E^1$ was contracting (or expanding) over $\Lambda_0$ it is still contracting (or expanding, respectively) for $C_n$ over $\Lambda_n$.

### 2.4 Linear Poincaré flow

Let $X$ be a $C^1$ vector field on a compact manifold $M$. We denote by $\phi^t$ the flow of $X$.

**Definition 8.** The normal bundle of $X$ is the vector bundle $N_X$ over $M \setminus \text{Sing}(X)$ defined as follows: the fiber $N_X(x)$ of $x \in M \setminus \text{Sing}(X)$ is the quotient space of $T_xM$ by the line $\mathbb{R}.X(x)$.

Note that, if $M$ is endowed with a Riemannian metric, then $N_X(x)$ is canonically identified with the orthogonal space of $X(x)$:

$$N_X = \{(x,v) \in TM, v \perp X(x)\}$$

Consider $x \in M \setminus \text{Sing}(M)$ and $t \in \mathbb{R}$. Thus $D\phi^t(x): T_xM \to T_{\phi^t(x)}M$ is a linear automorphism mapping $X(x)$ onto $X(\phi^t(x))$. Therefore $D\phi^t(x)$ passes to the quotient as a linear automorphism $\psi^t(x): N_X(x) \to N_{\phi^t(x)}(x)$:
\[ T_x M \xrightarrow{\partial \phi} T_{\phi^t(x)} M \]
\[ N_X(x) \xrightarrow{\psi^t} N_X(\phi^t(x)) \]

where the vertical arrow are the canonical projection of the tangent space to the normal space to the directions of \( X \).

**Definition 9.** We say that a vector field \( X \) is hyperbolic over \( \Lambda \) if there is a dominated splitting \( TM = E^s \oplus \mathbb{R}X(x) \oplus E^u \) over \( \Lambda \) into 2 hyperbolic sub-bundles so that \( E^s \) is uniformly contracting and \( E^u \) is uniformly expanding.

One says that \( E^s \) is the stable bundle, and \( E^u \) is the unstable bundle.

Note that if \( X \) is non singular the linear Poincaré flow is a linear cocycle.

**Proposition 5.** Let \( X \) be a \( C^1 \) vector field on a manifold and \( \Lambda \) be a compact invariant set of \( X \). Assume that \( \Lambda \) does not contained any zero of \( X \). Then \( \Lambda \) is hyperbolic if and only if the linear Poincaré flow over \( \Lambda \) is hyperbolic.

Notice that the notion of dominated splitting for non-singular flows is sometimes better expressed in term of Linear Poincaré flow: for instance, the linear Poincaré flow of a robustly transitive vector field always admits a dominated splitting, when the flow by itself may not admit any dominated splitting. An example of a diffeomorphism with a robustly transitive set having dominated splitting into two bundles, that none of them is contracting or expanding is exhibited in [BV]. The suspension of this diffeomorphism would not have a dominated splitting of the tangent space.

### 2.5 Extended linear Poincaré flow

We are dealing with singular flows and the linear Poincaré flow is not defined on the zero of the vector field \( X \). However we can extend the linear Poincaré flow to a flow, called extended linear Poincaré flow (as defined in [GLW]), and for which the zeros of \( X \) do not play a specific role.

This flow will be a linear co-cycle define on some linear bundle over a manifold, that we define now.

**Definition 10.** Let \( M \) be a manifold of dimension \( d \).

- We call the projective tangent bundle of \( M \), and denote by \( \Pi_p: \mathbb{P}M \to M \), the fiber bundle whose fiber \( \mathbb{P}_x \) is the projective space of the tangent space \( T_x M \): in other word, a point \( L_x \in \mathbb{P}_x \) is a 1-dimensional vector subspace of \( T_x M \).

- We call normal bundle of \( \mathbb{P}M \) and we denote by \( \Pi_N: \mathcal{N}M \to \mathbb{P}M \), the \( d - 1 \)-dimensional vector bundle over \( \mathbb{P}M \) whose fiber \( \mathcal{N}_L \) over \( L \in \mathbb{P}_x M \) is the quotient space \( T_x M/L \).

If we endow \( M \) with riemannian metric, then \( \mathcal{N}_L \) is identified with the orthogonal hyperplane of \( L \) in \( T_x M \).
Let \( X \) be a \( C^r \) vector field on a compact manifold \( M \), and \( \phi^t \) its flow. The natural actions of the derivative of \( \phi^t \) on \( \mathbb{P}M \) and \( \mathcal{N}M \) define \( C^{r-1} \) flows on these manifolds. More precisely, for any \( t \in \mathbb{R} \),

- We denote by \( \phi^t_P : \mathbb{P}M \to \mathbb{P}M \) the flow defined by
  \[
  \phi^t_P(L_x) = D\phi^t(L_x) \in \mathbb{P}_{\phi^t(x)}.
  \]

- We denote by \( \psi^t_N : \mathcal{N}M \to \mathcal{N}M \) the \( C^{r-1} \) diffeomorphism whose restriction to a fiber \( \mathcal{N}_L \), \( L \in \mathbb{P}_x \), is the linear automorphisms onto \( \mathcal{N}_{\phi^t(L)} \) defined as follows: \( D\phi^t(x) \) is a linear automorphism from \( T_xM \) onto \( T_{\phi^t(x)}M \), which maps the line \( T_L \subset T_xM \) onto the line \( T_{\phi^t(L)} \). Therefore it passe to the quotient in the announced linear automorphism.

\[
\begin{array}{cccc}
T_xM & \xrightarrow{D\phi^t} & T_{\phi^t(x)}M \\
\downarrow & & \downarrow \\
\mathcal{N}_L & \xrightarrow{\psi^t_N} & \mathcal{N}_{\phi^t(L)}
\end{array}
\]

Note that \( \phi^t_P \), \( t \in \mathbb{R} \) defines a flow on \( \mathbb{P}M \) which is a co-cycle over \( \phi^t \) whose action on the fibers is by projective maps.

The one-parameter family \( \psi^t_N \) defines a flow on \( \mathcal{N}M \), which is a linear co-cycle over \( \phi^t \). We call \( \psi^t_N \) the extended linear Poincaré flow. We can summarize by the following diagrams:

\[
\begin{array}{cccc}
\mathcal{N}M & \xrightarrow{\psi^t_N} & \mathcal{N}M \\
\downarrow & & \downarrow \\
\mathbb{P}M & \xrightarrow{\phi^t} & \mathbb{P}M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi^t} & M
\end{array}
\]

**Remark 11.** The extended linear Poincaré flow is really an extension of the linear Poincaré flow defined in the previous section; more precisely:

Let \( S_X : M \setminus \text{Sing}(X) \to \mathbb{P}M \) be the section of the projective bundle defined as \( S_X(x) = \langle X(x) \rangle \in \mathbb{P}_x \) generated by \( X(x) \). Then \( N_X(x) = \mathcal{N}_{S_X(x)} \) and the linear automorphisms \( \psi^t_X : N_X(x) \to N_X(\phi^t(x)) \) and \( \psi^t_N : \mathcal{N}_{S_X(x)} \to \mathcal{N}_{S_X(\phi^t(x))} \)

### 2.6 The reparametrized Linear Poincaré flow

We endow the manifold \( M \) with a smooth Riemannian metric \( \| \cdot \| \). We call reparametrizing map to the map \( h : \mathbb{P}M \times \mathbb{R} \to \mathbb{R} \) defined as follows: \( h(L, t) \) is the norm of the derivative of \( \phi^t \) restrict to the line \( L \): in other words \( h(L, t) = \| D\phi^t(u) \| \), where \( u \) is a non vanishing vector in \( L \).

Note that \( h \) satisfies the following cocycle relation:

\[
h(L, t + s) = h(\phi^t_P(L), s) \cdot h(L, t).
\]
Definition 12. We call reparametrized linear Poincaré flow and we denote $\Psi_t$, the linear cocycle defined on the linear bundle $\Pi_N : \mathcal{N} \to \mathbb{P}M$ as follows:

$$\Psi^t(L, u) = h(L, t) \cdot \psi^t_N(L, u)$$

where $u \in \mathcal{N}_L$.

The fact that this formula defines a Linear cocycle, follows directly from the fact that $\psi_N$ is a linear cocycle and $h$ satisfies (2).

2.7 Maximal invariant set and lifted maximal invariant set

Let $X$ be a vector field on a manifold $M$ and $U \subset M$ be a compact subset. The maximal invariant set $\Lambda = \Lambda_U$ of $X$ in $U$ is the intersection

$$\Lambda_U = \bigcap_{y \in RR} \phi^t(U).$$

We say that a compact $X$-invariant set $K$ is locally maximal if there exist a compact neighborhood $U$ of $K$ so that $K = \Lambda_U$.

Definition 13. We call lifted maximal invariant set in $U$, and we denote by $\Lambda_{P, U} \subset \mathbb{P}M$ (or simply $\Lambda_P$ if one may omit the dependence on $U$), to the closure of the set of lines $\langle X(x) \rangle$ for regular points $x \in \Lambda_U$:

$$\Lambda_{P, U} = S_X(\Lambda_U \setminus \text{Sing}X) \subset \mathbb{P}M,$$

where $S_X : M \setminus \text{Sing}X \to \mathbb{P}M$ is the section defined by $X$.

The next remark is very useful for getting robust properties:

Remark 14. Let $U$ be a compact subset of $M$.

- The maximal invariant set $\Lambda_{X, U}$ depends upper semi-continuously on $X$: if $Y$ is close to $X$ then $\Lambda_{Y, U}$ is contained in an arbitrarily small neighborhood of $\Lambda_{X, U}$.

- if $X$ has no zero in $U$ (that is $\text{Sing}(X) \cap U = \emptyset$), then the lifted maximal invariant set $\Lambda_{U, P}(Y)$ depends upper-semi continuously on the vector field $Y$ in a small neighborhood of $X$: for $Y$ close to $X$ the intersection $\text{Sing}(Y) \cap U$ is empty and $\Lambda_{U, P}(Y)$ is just the image $S_Y(\Lambda_{Y, U})$, and $S_Y$ depends continuously on $Y$. Therefore, the semi-continuity of $\Lambda_{U, P}(Y)$ is a straightforward consequence of the one of $\Lambda_{Y, U}$.

One difficulty that we need to deal with when one considers the lifted maximal invariant set is that it no longer depends semi-continuously on the flow, when there are singularities in $U$. 

12
2.8 Strong stable, strong unstable and center spaces associated to a hyperbolic singularity.

Let $X$ be a vector field and $\sigma \in \text{Sing}(X)$ be a hyperbolic singular point of $X$. Let $\lambda_1^s \ldots \lambda_2^s < 0 < \lambda_1^u < \lambda_2^u \ldots \lambda_l^u$ be the Lyapunov exponents of $\phi_t$ at $\sigma$ and let $E_{k}^s \oplus \cdots \oplus E_{2}^s \oplus E_{1}^s \oplus E_{1}^u \oplus E_{2}^u \cdots \oplus E_{l}^u$ be the corresponding (finest) dominated splitting over $\sigma$.

A subspace $F$ of $T_M$ is called a center subspace if it is of one of the possible form below:

- Either $F = E_{k}^s \oplus \cdots \oplus E_{2}^s \oplus E_{1}^s$
- Or $F = E_{1}^s \oplus E_{2}^s \cdots \oplus E_{l}^s$
- Or else $F = E_{i}^s \oplus \cdots \oplus E_{i+1}^s \oplus E_{i}^s$ for some $i$.

A subspace of $T_M$ is called a strong stable space, and we denote it $E_{i}^{ss}(\sigma)$, is there in $i \in \{1, \ldots, k\}$ such that:

$$E_{i}^{ss}(\sigma) = E_{k}^s \oplus \cdots \oplus E_{j+1}^s \oplus E_{i}^s$$

A classical result from hyperbolic dynamics asserts that for any $i$ there is a unique infectively immersed manifold $W_{i}^{ss}(\sigma)$, called a strong stable manifold tangent at $E_{i}^{ss}(\sigma)$ and invariant by the flow of $X$.

We define analogously the strong unstable spaces $E_{j}^{uu}(\sigma)$ and the strong unstable manifolds $W_{j}^{uu}(\sigma)$ for $j = 1, \ldots, l$.

We can also define the strong stable and unstable manifolds in an analogue way, for regular points $x$ in an invariant set $\Lambda$.

3 Controlling Stable and unstable Manifolds

3.1 Pliss lemma and controlling invariant manifolds near singularities.

We now present some results that allow us a better control of the size of the invariant manifolds near singularities. We need for this the definition of $(\eta, T, E)^*$ contracting orbit arcs.

Definition 15. Given $\phi_t$ a flow induced by $X \in \mathcal{X}^1(M)$, $\Lambda$ a compact invariant set of $\phi_t$, and $E \subset N_{\Lambda \setminus \text{Sing}(X)}$ an invariant bundle of the linear Poincaré flow $\psi_t$. For $C > 0$, $\eta > 0$ and $T > 0$, a periodic point $p \in \Lambda \setminus \text{Sing}(X)$ of period $\tau(p)$ is called $(\eta, T, E)$ contracting at the period (w.r.t $\psi_t$) if there exist $m \in \mathbb{N}$ and a partition of times $0 = t_0 < t_1 < \cdots < t_n = m\tau(p)$ such that, $t_{j+1} - t_j \geq T$ for all $j \leq n - 1$,

$$\prod_{i=0}^{n-1} \left\| \psi_{t_{i+1} - t_i}^{*} \chi_{E(\phi_{t_i}(x))} \right\| \leq C e^{-\eta}.$$

Similarly $p \in \Lambda \setminus \text{Sing}(X)$ is called $(\eta, T, F)$ expanding at the period if it is $(\eta, T, F)$ contracting at the period for $-X$. 

13
Note that in a star flow, periodic orbits are uniformly contracting at the period.

**Definition 16.** Given \( \phi_t \) a flow induced by \( X \in C^1(M) \), \( \Lambda \) a compact invariant set of \( \phi_t \), and \( E \subset N_{\Lambda \setminus Sing(X)} \) an invariant bundle of the linear Poincaré flow \( \psi_t \).

For \( C > 0 \), \( \eta > 0 \) and \( T > 0 \), \( x \in \Lambda \setminus Sing(X) \) is called \((\eta, T, E)\) contracting if for any partition of times \( 0 = t_0 < t_1 < \cdots < t_n < \cdots \) such that, \( t_{n+1} - t_n \geq T \) for all \( n \in \mathbb{N} \) and \( t_n \to \infty \) when \( n \to \infty \), que have that

\[
\prod_{i=0}^{n-1} \left\| \psi_{t_{i+1} - t_i}^* | E(\phi_{t_i}(x)) \right\| \leq Ce^{-n\eta}.
\]

Similarly \( x \in \Lambda \setminus Sing(X) \) is called \((\eta, T, F)^*\) expanding if it is \((\eta, T, F)\) contracting for \(-X\).

To find the \((\eta, T, E)\) contracting orbit arcs, one needs the classical result due to V.Pliss:

**Lemma 6.** \([PL]\) (Pliss lemma) Given a number \( A \). Let \( \{a_1, \cdots, a_n\} \) be a sequence of numbers which are bounded from above by \( A \). Assume that there exists a number \( \xi < A \) such that \( \sum_{i=1}^{n} a_i \geq n \cdot \xi \), then for any \( \xi' < \xi \), there exist \( l \) integers \( 1 \leq t_1 < \cdots < t_l \leq n \) such that

\[
\frac{1}{t_j - k} \sum_{i=k+1}^{t_j} a_i \geq \xi', \text{for any } j = 1, \cdots, l \text{ and any integer } k = 0, \cdots t_j - 1.
\]

Moreover, one has the estimate \( \frac{l}{n} \geq \frac{\xi - \xi'}{A - \xi'} \).

For a star flow, and using the Pliss lemma above one can find orbits arcs that are at the same time \((\eta, T, E)\) contracting and \((\eta, T, F)^*\) expanding in the periodic orbits with big enough period, (where \( E \) and \( F \) are the invariant sub-bundles of the normal space from the hyperbolicity of the periodic orbit).

**Lemma 7.** Let \( \phi_t \) be a flow induced by a \( C^1 \) vector field. Let \( p \) be a periodic point such that there exists an \( \psi_t \) invariant splitting \( N_{O(p)} = E \oplus F \), where \( \psi_t \) is the linear Poincaré flow for \( \phi_t \). Assume, in addition, that \( p \) is \((\eta, T, E)\) contracting at the period and \((\eta, T, F)^*\) expanding at the period. then there exist points \( x \in O(p) \), such that \( x \) is both, \((\eta, T, E)\) contracting and \((\eta, T, F)^*\) expanding.

We call the point \( x \) as \((\eta', T)\) bi-pliss point or bi-pliss point for simplicity.

**Definition 17.** Let \( M \) be a compact Riemannian manifold with a metric \( d \).

Let \( A \) be a sub-manifold of \( M \) of dimension \( i \). We say that \( A \subset M \) has inner diameter bigger than \( k \) at \( x \) if the ball of center \( x \) and radius \( k \) for \( M \) intersects \( A \) in a \( i \)-dimensional ball of center \( x \) and radius \( k \) for the restriction of \( d \) to \( A \).

**Theorem 8.** \([LT]\) Let \( X \in C^1(M) \) and \( \Lambda \) be a compact invariant set of \( \phi_t \) associated to \( X \). Given \( \eta > 0, T > 0 \) assume that \( \| \Lambda - Sing(X) \| = E \oplus F \) is an \((\eta, T)\)-dominated splitting with respect to the linear Poincaré flow. Then there is \( \delta > 0 \) such that if \( x \) is \((\eta, T, E)^*\) contracting, then the inner diameter of the stable manifold of \( x \) at \( x \), is bigger than \( \delta\|X(x)\| \).
3.2 Generic properties

We say that a vector field is Kupka-Smale if the following two properties hold:

- All periodic and singular orbits are hyperbolic.
- The intersections of stable and unstable manifolds of closed hyperbolic orbits are transversal.

A famous theorem by Kupka and Smale show that this conditions are generic.

Lemma 9 (Connecting lemma). [BC]. Given $\phi_t$ induced by a Kupka-Smale vector field $X \in X^1(M)$. For any $C^1$ neighborhood $U$ of $X$ and $x, y \in M$, if $y$ is attainable from $x$, then there exists $Y \in U$ and $t > 0$ such that $\phi^Y_t(x) = y$. Moreover, for every $k \geq 1$, let \( \{x_{i,k}, t_{i,k}\}_{i=0}^{n_k} \) be a $(1/k, T)$-pseudo orbit from $x$ to $y$ and denote by

$$\Delta_k = \bigcup_{i=0}^{n_k-1} \phi_{[0,t_{i,k}]}(x_{i,k}).$$

Let $\Delta$ be the upper Hausdorff limit of $\Delta_k$. Then for any neighborhood $U$ of $\Delta$, there exists $Y \in U$ with $Y = X$ on $M - U$ and $t > 0$ such that $\phi^Y_t(x) = y$.

Remark 18. From the proof of connecting lemma for pseudo orbit, one can obtain the following stronger statement: for any neighborhood $U$ of $\Delta$, and for any finitely many (hyperbolic) critical elements $c_i$, $i = 1, \ldots, j$, there exists a neighborhood $V_i$ of $c_i$ and $Y \in U$ with $Y = X$ on $(M - U) \cup (\bigcup_{i=1}^j V_i)$ and $t > 0$ such that $\phi^Y_t(x) = y$.

The following result is a consequence of Connecting Lemma for pseudo-orbits made by S. Crovisier and C. Bonatti in [BC].

Theorem 10. [C] There exists a $G_{\mathrm{approx}} \subset X^1(M)$ a generic set such that for every $X \in G_{\mathrm{approx}}$ and for every $C$ a chain recurrence class there exists a sequence of periodic orbits $\gamma_n$ which converges to $C$ in the Hausdorff topology.

4 multisingular hyperbolicity

4.1 The lifted maximal invariant set and the singular points

The aim of this section is to find a bigger set than the lifted maximal invariant set $\Lambda_{\phi, U}$ in which to look for hyperbolic properties but that varies upper semi-continuous properties. We do this by adding some subset of the projective over the singular points, as in [BdL]. All the proofs of the following lemmas and propositions can be find there.

Let $U$ be a compact region, and $X$ be a vector field and $\sigma$ be a hyperbolic singularity of $X$, contained in the interior of $U$.

We define the escaping stable space of $\sigma$ in $U$ $E_{\sigma, U}^{es}$ as the biggest strong stable space $E_j^{es}(\sigma)$ such that the invariant manifold tangent to it (that we call
escaping strong stable manifold) \( W^{ss}_{j}(\sigma) \) is such that all orbits in it, escape \( U \). That is,

\[
W^{ss}_{j}(\sigma) \text{ is such that } \Lambda_{X,U} \cap W^{ss}_{j}(\sigma) = \{ \sigma \}.
\]

We define the escaping unstable space of \( \sigma \) in \( U \) and the escaping strong unstable manifolds analogously.

We define the central space of \( \sigma \) in \( U \) and we denote \( E^{c}_{\sigma,U} \) the space such that

\[
T_{\sigma}M = E^{ss}_{\sigma,U} \oplus E^{c}_{\sigma,U} \oplus E^{uu}_{\sigma,U}.
\]

We denote by \( \mathbb{P}^{i}_{\sigma,U} \) the projective space of \( E^{i}((\sigma,U) \) where \( i = \{ ss, uu, c \} \).

**Lemma 11.** Let \( U \) be a compact region and \( X \) a vector field whose singular points are hyperbolic and contained in the interior of \( U \). Then, for any \( \sigma \in \text{Sing}(X) \cap U \), one has:

\[
\Lambda_{X,U} \cap \mathbb{P}^{ss}_{\sigma,U} = \Lambda_{X,U} \cap \mathbb{P}^{uu}_{\sigma,U} = \emptyset.
\]

As a consequence we get the following characterization of the central space of \( \sigma \) in \( U \):

**Lemma 12.** The central space \( E^{c}_{\sigma,U} \) is the smallest center space containing \( \Lambda_{X,U} \cap \mathbb{P}^{ss}_{\sigma,U} \).

We are now able to define the subset of \( \mathbb{P}M \) which extends the lifted maximal invariant set and which has the upper-semicontinuity properties.

**Definition 19.** Let \( U \) be a compact region and \( X \) a vector field whose singular points are hyperbolic, and disjoint from the boundary \( \partial U \). Then the set

\[
B(X,U) = \Lambda_{X,U} \cup \bigcup_{\sigma \in \text{Sing}(X) \cap U} \mathbb{P}^{c}_{\sigma,U} \subset \mathbb{P}M
\]

is called the extended maximal invariant set of \( X \) in \( U \)

**Proposition 13.** Let \( U \) be a compact region and \( X \) a vector field whose singular points are hyperbolic, and disjoint from the boundary \( \partial U \).

Then the extended maximal invariant set \( B(X,U) \) of \( X \) in \( U \) is a compact subset of \( \subset \mathbb{P}M \).

Furthermore, there is a \( C^{1} \)-neighborhood \( \mathcal{U} \) of \( X \) for which the map \( Y \to B(Y,U) \) depends upper semi-continuously on \( Y \in \mathcal{U} \).

A chain recurrence class admits a basis of filtrating neighborhood. That is, for any chain recurrence class we can find a sequence of neighborhoods ordered by inclusion \( U_{n+1} \subset U_{n} \), such that \( C = \bigcap U_{n} \). We define

\[
\widehat{\Lambda}(C) = \bigcap_{n} \Lambda(X, \widehat{U}_{n}) \text{ and } B(C) = \bigcap_{n} B(X, U_{n}).
\]

These two sets are independent of the choice of the sequence \( U_{n} \).
4.2 Strong multisingular hyperbolicity

We are now ready to define the notion of hyperbolicity we will use in this paper. It is expressed in term of the reparametrized linear Poincaré flow defined in Section 2.6.

**Definition 20.** Let \( U \subset M \) be a compact region and \( X \) a \( C^1 \)-vector field on \( M \) and \( C \) a chain recurrence class of \( X \). We say that \( X \) is strong multisingular hyperbolic in \( C \) if \( X \) has hyperbolic singularities in \( U \) and if the restriction of the reparametrized linear Poincaré flow \( \|D\phi^t(u)\|_\psi t \) to \( B(C) \) is a uniformly hyperbolic linear cocycle over \( \phi P \).

For \( L \in B(C) \) we denote,
\[
N_L = N^s(L) \oplus N^u(L)
\]
the stable and unstable spaces of the reparametrized linear Poincaré flow.

We call \( \dim c N^s(L) \) the \( s \)-index of multisingular hyperbolicity of \( X \).

Note that the reparametrized linear Poincaré flow \( \Psi^t \) is one of the possible reparametrizing cocycles considered in [BdL] and therefore if \( \Psi^t \) is hyperbolic then \( X \) is multisingular hyperbolic according to the definition in [BdL]. For simplicity we work with this stronger definition since our aim is just to make an example. Being Multisingular Hyperbolic is a robust property and the proof of that is also in [BdL].

**Lemma 14.** If \( X \) is multisingular hyperbolic in \( U \) then every periodic orbit \( \gamma \subset \Lambda_{X,U} \) is hyperbolic of \( s \)-index equal to the index of multisingular hyperbolicity.

**Proof.** Since \( \gamma \) is a closed orbit, let \( T \) be its period. We define \( \Gamma = \{ L_x \in B(X, U) \text{ such that } x \in \gamma \} \).

We recall that we define \( h(L, T) \) as \( h(L, T) = \|D\phi^T(u)\|_u \) where \( u \) is a non-vanishing vector in \( L \). Since \( x \) is a regular orbit then \( X(x) \) is a non-vanishing vector in \( L_x \).

Since for every \( n \in \mathbb{Z} \) we have that
\[
h(L, nT) = \frac{\|D\phi^{nT}(X(x))\|}{\|X(x)\|} = 1,
\]
then \( h(L, t) \) is bounded restricted to \( \Gamma \) from above and its uniformly away from 0 since \( \gamma \) is a compact invariant set with no singularities. In other words there exist 2 constants \( m_1 \) and \( m_2 \) such that
\[
0 < m_1 < h(L, t) < m_2 \quad \text{ for every } L \in \Gamma \text{ and } t \in \mathbb{R}
\]

The hyperbolicity of \( \Psi^t \) in \( B(X, U) \) gives us that for every \( L \in B(X, U) \) there is an invariant and continuous splitting of
\[
N_L = N^s_L \oplus N^u_L
\]
such that there exist 2 constants $C > 0$ and $\lambda > 0$ such that
\[
\| \Psi^t(L, v) \| = \| \psi_N^t(L, v) \| h(L, t) < Ce^{-t\lambda} \| v \| \quad v \in \mathcal{N}^s
\]
\[
\| \Psi^{-t}(L, v) \| = \| \psi_N^{-t}(L, v) \| h(L, -t) < Ce^{-t\lambda} \| v \| \quad v \in \mathcal{N}^u
\]

As a consequence of $h(L, t)$ being bounded we have that
\[
\| \psi_N^t(L, v) \| m_1 < \| \psi_N^t(L, v) \| h(L, t) < Ce^{-t\lambda} \| v \| \quad v \in \mathcal{N}^s
\]
\[
\| \psi_N^{-t}(L, v) \| m_1 < \| \psi_N^{-t}(L, v) \| h(L, -t) < Ce^{-t\lambda} \| v \| \quad v \in \mathcal{N}^u.
\]

Therefore taking $C' = \frac{C}{m_1}$
\[
\| \psi_N^t(L, v) \| < C'e^{-t\lambda} \| v \| \quad v \in \mathcal{N}^s
\]
\[
\| \psi_N^{-t}(L, v) \| < C'e^{-t\lambda} \| v \| \quad v \in \mathcal{N}^u.
\]

Therefore the extended Poincaré flow $\psi_N^t$ is hyperbolic of index $s$ with constants $C'$ and $\lambda$, restricted to $\Gamma$.

But since $\gamma$ is a periodic orbit, and therefore a compact invariant set without singularities, this implies that the linear Poincaré flow $\psi^t$ is hyperbolic of index $s$ restricted to $\gamma$. By [5] we conclude that $\gamma$ hyperbolic of index $s$ for the tangent flow.

\[\square\]

Corollary 15. If $X$ is multisingular hyperbolic in $U$ then $X$ is a star flow in $U$.

Proof. Since $X$ is multisingular hyperbolic then in [?] it is proven that there is an $n_0$ such that if $Y_n \to X$ with $n \to +\infty$ in the $C^1$-topology then $Y_n$ is multisingular hyperbolic for every $n > n_0$. But now lemma [14] gives us that every periodic orbit $\gamma \subset \Lambda_{Y_n, U}$ is hyperbolic of $s$-index equal to the index of the multisingular hyperbolicity of $Y_n$ which is the same as the $s$-index equal to the index of the multisingular hyperbolicity of $X$ giving us that $X$ is a star flow. \[\square\]

4.3 Extension of hyperbolicity along an orbit

Let us consider now a linear cocycle a linear cocycle $A$ over $(\Lambda, X)$, a hyperbolic set $\Lambda$ for the cocycle $A$ and an orbit $y$ such that the $\alpha$-limit of $y$, and the $\omega$-limit of $y$ are in $\Lambda$.

The splitting $E_{\alpha(y)} = E_{\alpha(y)}^s \oplus E_{\alpha(y)}^u$ will have an unstable cone field and a stable cone field that are strictly invariant on a neighborhood of $\Lambda$. Then the next lemma shows we can extend the hyperbolic structure of our cocycle to $\Lambda \cup o(y)$.

Lemma 16. Let $\Lambda$ be a hyperbolic, maximal invariant set in $U$, for $A$, and $E_\Lambda = E^s \oplus E^u$. Suppose as well that

- The $\alpha$-limit of $y$, $\alpha(y)$ is in $\Lambda$. Since $\Lambda$ is hyperbolic then $E_{\alpha(y)} = E_{\alpha(y)}^s \oplus E_{\alpha(y)}^u$
• The $\omega$-limit of $y$ is in $\Lambda$. Since $\Lambda$ is hyperbolic then $E_{\omega(y)} = E^s_{\omega(y)} \oplus E^u_{\omega(y)}$

• there exists a compact neighborhood $U'$ such that $\Lambda \cup o(y)$ is a maximal invariant set in $U'$.

Then there exist a unique hyperbolic splitting along the orbit of $y$ $E^u_y \oplus E^s_y = E_y$ such that

• $\dim(E^u_y) = \dim(E^s_y)$

• $\dim(E^s_y) = \dim(E^s_\Lambda)$

• The set $\Lambda \cup o(y)$ is hyperbolic with that splitting.

Proof. For that, the space $E^u_{\omega(y)}$. It extends by continuity in a small neighborhood $u_\omega$ of the $\omega$-limit of $y$ and around it it extends an invariant unstable cone field along the piece of orbit of $y$ that stays inside $u_\omega$. Then $E^u_y$ is exactly the set of vectors which do not enter in this unstable cone for large positive iterates We define $\dim(E^u_y)$ analogously. By construction the dimensions must match. and the continuity comes form the fact that the unstable and stable cone fields along the orbit of $y$ coincide with the cone fields given by the hyperbolicity of $\Lambda$ around the piece of orbit of $y$ that never leaves $u_\omega$ for the future.

We want to show that under similar assumptions the hyperbolicity of the reparametrized linear Poincaré flow of some maximal invariant set, extends to this set and one extra orbit with $\alpha$ and $\omega$ limits in this set. We now show that the extended maximal invariant set of $\Lambda \cup o(y)$ is the one of $\Lambda$ and only one extra orbit. Then we will prove that the transversal intersection of the center stable and unstable spaces given by 16 which will give us the desired result.

Proposition 17. Suppose that $\Lambda$ is a multisingular hyperbolic, maximal invariant set in $U$. Suppose as well that

• $y$ is such that the $\alpha$ and $\omega$ limits of $y$, $\alpha(y)$ and $\omega(y)$ are in $\Lambda$.

• there exists a compact neighborhood $U'$ such that $\Lambda \cup o(y)$ is a maximal invariant set in $U'$,

• The orbit of $y$ does not intersect any escaping stable or unstable manifold of any singularity in $\Lambda$

Then the extended maximal invariant set $\Lambda_{\pi}(X, U')$ is $\Lambda_{\pi}(X, U) \cup O(L)$ where $L = S_X(y)$ and $O(L)$ is the orbit of $L$ by $\phi^t_{\pi}$.

Proof. The set $S_X(\Lambda U' \setminus \text{Sing}(X))$ gives only one point of $\mathbb{P}M$ for every regular point in the maximal invariant set of $U'$. Therefore

$$S_X(\Lambda U' \setminus \text{Sing}(X)) = S_X(\Lambda U \setminus \text{Sing}(X)) \cup O(L).$$
The hypothesis above, that state that the orbit of \( y \) is away from the escaping stable and unstable manifolds of the singularity, and the fact that the \( \alpha \) and \( \omega \) limits of \( y \) are in \( \Lambda \)

\[ S_X(\Lambda_U \setminus \text{Sing}(X)) \cup \overline{O(L)} \subset S_X(\Lambda_U \setminus \text{Sing}(X)) \cup O(L). \]

Therefore

\[
\begin{align*}
S_X(\Lambda_{U'} \setminus \text{Sing}(X)) & = S_X(\Lambda_U \setminus \text{Sing}(X)) \cup \overline{O(L)} \\
& = S_X(\Lambda_U \setminus \text{Sing}(X)) \cup O(L) \\
& = \Lambda_p(X, U) \cup O(L).
\end{align*}
\]

\[ \square \]

**Corollary 18.** Suppose that \( \Lambda \) is a multisingular hyperbolic, maximal invariant set in \( U \), for \( X \). Suppose as well that there is a point \( y \) such that:

- the \( \alpha \) and \( \omega \) limits of \( y \), \( \alpha(y) \) and \( \omega(y) \) are in \( \Lambda \).
- there exists a compact neighborhood \( U' \) such that \( \Lambda \cup o(y) \) is a maximal invariant set in \( U' \),
- The orbit of \( y \) does not intersect any escaping stable or unstable manifold of any singularity in \( \Lambda \)
- The stable and unstable spaces along the orbit of \( S_X(y) \) given by Lemma \[\text{[17]}\] intersect transversally.

Then \( \Lambda \cup o(y) \) is multisingular hyperbolic.

Let us consider the set of chain recurrent points in a maximal invariant set \( \Lambda \cap \mathcal{R} \) and suppose that this set is maximal invariant in a smaller neighborhood \( U' \), i.e.

\[ \bigcap \phi^t(U') = \Lambda \cap \mathcal{R}. \]

Applying the same argument to a set of orbits in the hypothesis of proposition \[\text{[17]}\] we get that if the non chain recurrent orbits in a maximal invariant set do not intersect the escaping spaces of the singularities, then

\[ B(X, U') \cup S(\Lambda \cap \mathcal{R}^c) = B(X, U). \]

As a consequence:

**Corollary 19.** Let \( \Lambda \) be the maximal invariant set in \( U \). We consider the set of the chain recurrent orbits \( \Lambda \cap \mathcal{R} \) and the set of the non chain recurrent orbits \( \Lambda \cap \mathcal{R}^c \). We lift the chain recurrent orbits \( S(\Lambda \cap \mathcal{R}) \). If

- The set of chain recurrent orbits in the extended maximal invariant set \( B(X, U) \cap S(\Lambda \cap \mathcal{R}) \) is hyperbolic for the reparametrized linear Poincaré flow with the same index for all connected components).
• Every non chain recurrent orbit \( y \in \Lambda \) does not intersect any escaping stable of unstable manifold of any singularity \( \Lambda \).

• The stable and unstable spaces along the lifted non chain recurrent orbit \( S_X(y) \) given by Lemma 16 intersect transversally.

Then \( \Lambda \) is multisingular hyperbolic.

5 A strong multisingular hyperbolic set in \( \mathbb{R}^3 \)

This section will be dedicated to the building a set in \( S^3 \) containing 2 singularities of different indexes that will be strong multisingular hyperbolic. However this set will not be recurrent.

**Definition 21.** We say that a hyperbolic singularity is strong Lorenz like if its tangent space splits into 3 invariant spaces. If the stable index is 2 then the Lyapunov exponents satisfy:

\[
\lambda_{sa}^s < \lambda_{sa}^u < 0 < -\lambda_{sa}^u < \lambda_{sa}^{ss}.\]

If the unstable index is 2 then:

\[
-\lambda_{ur}^{uu} < \lambda_{ur}^s < -\lambda_{ur}^u < 0 < \lambda_{ur}^u < \lambda_{ur}^{uu}.\]

Recall that this section is dedicated to prove:

**Theorem.** There exists an open set of vector fields \( U \subset X^1(S^3) \) such that every \( X \in U \) has the following properties.

• There is a filtrating region \( U = U_a \cap U_r \).

• \( \Lambda \) is the maximal invariant set of a filtrating region \( U \) i.e.

\[
\Lambda = \bigcap_{t \in \mathbb{R}} \phi^t(U)
\]

where \( \phi \) is the flow of \( X \).

• All singularities contained in \( \Lambda \) are strong Lorenz like.

• The set \( \Lambda \) contains a singularity \( \sigma_a \) that is accumulated by periodic orbits and that has a stable separatrix escaping \( U_a \).

• \( \Lambda \) contains a singularity \( \sigma_r \) that is accumulated by periodic orbits and that has an unstable separatrix escaping \( U_r \).

• There is an orbit \( \sigma(y) \) in \( \Lambda \) such that the \( \alpha \)-limit of \( y \) is in the chain-recurrent class of \( \sigma_r \) (that we call \( L_r \)) and the \( \omega \)-limit of \( y \) is in the chain-recurrent class of \( \sigma_a \) (that we call \( L_a \)).

• The set \( \Lambda \) is multisingular hyperbolic.
5.1 The Lorenz attractor and the stable foliation

In this subsection we will shortly comment on the construction of a geometric Lorenz attractor, done in [GuWi].

5.1.1 Guckenheimer Williams, geometric model

We consider a Flow in $\mathbb{R}^3$ as in [GuWi], having a transitive singular attractor, that we call $L_a$. This set has the following properties:

- It has a singularity in the origin with three different real Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3$, with the following relation:
  
  $$-\lambda_2 > \lambda_1 > -\lambda_3 > 0,$$

  (We call the singularities with this relation between the Lyapunov exponents as strong Lorenz like, and it implies the Lorenz like condition).

- The corresponding invariant spaces of this Lyapunov exponents are the cartesian axis, where the strong stable space is in the direction of $z$ and the unstable in the direction of $y$.

- It is a robustly transitive attractor.

- For the attractor $L_a$, we can consider a region of attraction $U_a$ such that the boundary of this neighborhood is a bi-torus.

- The singularity is accumulated by periodic orbits in a persistent way,

- The strong stable spaces of the points in $U_a$ are well define and parallel to the $y$ direction.

  Additionally the expansion rate is bounded form below by $\sqrt{2}$ and from above by $2$. This is a consequence of the way the example is constructed. So additionally we ask that the strong contraction rate is bigger that $4$ and smaller than $5$.

5.1.2 Attracting region:

Since we aim to construct an example in $S^3$, it will be more convenient to work with an attracting region $U_a$ with is a ball.

Let us consider two saddle singularities the holes of the toral trapping region from the above construction. This singularities will have $1$ dimensional stable space and a $2$ dimensional unstable space with complex Lyapunov exponents. The unstable spaces will cut the toral trapping region and the stable spaces are parallel to the $y$ direction.

Then we can find an attracting region $U_a$ such that the maximal invariant set contained on it is $L^a$ and the $2$ singularities, and the boundary of $U_a$ is $S^2$. For a more detail description we refer the reader to Guckenheimer Williams’s work [GuWi].
We choose one of these 2 singularities, that we call $p$. We call $C_a \subset U_a$, to a subset having a smooth boundary homeomorphic to $D \times [0, \varepsilon]$ with axis one side of the stable manifold of $p$. We ask that there exist $[\delta, \rho] \subset [0, \varepsilon]$ such that $C_a$ and $D \times [0, \varepsilon]$ coincide exactly at $S^1 \times [\delta, \rho]$.

In addition to this we ask that this cylinder $C_a$ cuts the boundary of $U_a$ and does not contain $p$.

Now we consider $U_a \setminus C_a$. The boundary of this new attracting region is such that the strong stable manifolds of the points in $L_a$, cut the boundary of the cylinder $D \times [\delta, \rho]$ parallel to the $y$ direction, that is also parallel to the stable manifold of $P$. We can consider a function $h : U_a \setminus C_a \to U_a$ such that

- $h$ is the identity except on a small neighborhood of the boundary (that doesn’t intersect any recurrent orbit),
- $h$ is a diffeomorphism.
- The image of restriction of $h$ to $S^1 \times [\delta, \rho]$ is an annulus such that any line parallel to the axis goes to a radius. We call this annulus $A_a$
- Consider $C_a \setminus D \times [\delta, \rho]$.

One of the connected components has a point of intersection of the stable manifold of $P$. We call the image of this component under $h$, $D_a$.

Finally we get an attracting region $U_a$ such that :

- The boundary of $U_a$ is $S^2$
- There is an annulus $A_a$ in $S^2$ such that the strong stable manifolds of $L_a$ intersect $A_a$ along a radial foliation
- The annulus $A_a$ bounds a disc $D_a$ containing the intersection of the stable manifold of $p$ and not of the other extra singularity.
5.2 A plug

The goal of this section is to prove the following theorem:

**Theorem 20.** There exist a vector field \( \chi \) such that its flow \( \phi_\chi \) defined in \( S^3 \) has the following properties:

There is a region \( S^2 \times [0, 1] \subset S^3 \) such that

- The vector field \( \chi \) is entering at \( S^2 \times 0 \) and points out at \( S^2 \times 1 \)
- The vector field \( \chi \) is such that the chain recurrent set consists of 2 sources singularities, \( p_1 \) and \( p'_1 \), 2 sinks singularities, \( p_2 \) and \( p'_2 \), and 2 periodic saddles, \( p_3 \) and \( p'_3 \).
- The intersection of the invariant manifolds of the saddles, with the boundary of \( S^2 \times [0, 1] \), are disjoint circles that we name as follows:
  - \( W^s(p_3) \cap S^2 \times [0, 1] = c_0 \) in \( S^2 \times 0 \)
  - \( W^s(p'_3) \cap S^2 \times [0, 1] = c'_0 \) in \( S^2 \times 0 \)
  - \( W^u(p_3) \cap S^2 \times [0, 1] = c_1 \) in \( S^2 \times 1 \)
  - \( W^u(p'_3) \cap S^2 \times [0, 1] = c'_1 \) in \( S^2 \times 1 \).
- The circle \( c_0 \) bounds a disc not containing \( c'_0 \), that we call \( D_0 \). The circle \( c'_0 \) bounds a disc containing \( c_0 \), that we call \( D'_0 \). And they both bound an annulus called \( A_0 \). Analogously we define \( D_1 \), \( D'_1 \) and \( A_1 \).
- The orbit \( O(x) \) of a point \( x \) in \( S^2 \times \{0\} \), crosses \( S^2 \times \{1\} \) if and only if \( x \in A_0 \) and \( O(x) \cap S^2 \times \{1\} \in A_1 \).
- There is a well defined crossing map \( P : A_0 \to A_1 \). Consider the radial foliation \( V_0 \) in \( A_0 \). Then the image of a radial foliation under \( P \) intersect transversally a radial foliation in \( A_1 \) and it extends to a foliation in \( \tilde{A} \cup c_1 \cup c'_1 \).

The complement of \( S^2 \times [0, 1] \) in \( S^3 \) are 2 balls, one in the basin of attraction of a source \( r \) (that has \( S^2 \times 0 \) in the boundary), and the other in the basin of attraction of a sink \( a \).

Most of the ideas here presented are similar than the ones in [BBY], the vector field that we aim to define is a plug in the sense of this article, and we refer the reader to this article to see a more careful presentation on how to glue plugs and what you can construct with them. We construct here a plug according to the specific needs of our example.

We consider the set \( K = \{ (x, y) \mid |y| \leq 1 \text{ and } 0 \leq x \leq 1 \} \), and in this set, a flow \( \phi_0 \) of a vector field \( Y_0 \) in \( \mathbb{R}^2 \) with the following properties:

- The vector field \( Y_0 \) is Morse-Smale with a source \( p_1 = (0, 1/2) \), a sink \( p_2 = (0, -1/2) \) and a saddle \( p_3 = (1/2, 0) \).
- The flow is linear in a neighborhood of the interval \( \{ (0, y) \mid -1 \leq y \leq 1 \} \)
The saddle $p_3$ is such that a branch of the unstable manifold intersects the basin of the sink, the other intersects a corner of $K$.

The saddle $p_3$ is such that a branch of the stable manifold intersects the basin of the source, the other intersects a corner of $K$.

We take an orbit $q = (1, 1 - \epsilon)$ for some positive and small $\epsilon$, that flows near the stable and unstable branch of the saddle that do not intersect the basins of the sink and the source. We consider another point of the orbit of $q$ that we call $q'$ with $x$ coordinate 1. We call $K'$ to the "square" delimited by

- the segments $\{(0, y) \mid y \leq 1\}$,
- the segment $\{(x, 1) \mid 0 \leq x \leq 1\}$,
- the segments $\{(1, y) \mid 1 - \epsilon \leq y \leq 1\}$,
- the vertical segment that joins $q'$ with $(1, -1)$,
- the orbit segment joining $q$ and $q'$,
- the segment $\{(x, -1) \mid 0 \leq x \leq 1\}$. 

25
We define

\[ C = \{ (x, y) \mid 0 \leq x \leq 2 \quad y \mid y \mid \leq 1 \} \, . \]

There is a diffeomorphism \( d : K' \to C \) that:

\begin{itemize}
  \item fixes the segment \( \{ (x, 1) \mid 0 \leq x \leq 1 \} \),
  \item takes the segment \( \{ (1, y) \mid 1 - \varepsilon < y \leq 1 - \varepsilon \} \), to the segment \( \{ (x, 1) \mid 1 \leq x \leq 2 \} \)
  \item takes the point \( q \) to \( (2, 1) \)
  \item fixes the segment \( \{ (x, -1) \mid 0 \leq x \leq 1 \} \),
  \item takes the segment \( \{ (1, y) \mid -1 \leq y \mid y \mid < -1 + \varepsilon \} \), to the segment \( \{ (x, 1) \mid 0 \leq x \leq 2 \} \).
  \item takes the point \( q' \) to \( (2, -1) \)
\end{itemize}

And so that \( d \) is the identity out of a small neighborhood of this boundary components which doesn’t include any of the singular points.

We call \( Y_1 \) to the vector field tangent to the flow \( \phi_1 \) obtained from \( d(\phi_0(d^{-1}(x))) \).

Now let \( (a, 0) \) be the new coordinates of the saddle \( p_3 \).

We define a \( C^\infty \) function \( f : \mathbb{R} \to \mathbb{R} \) such that:

\begin{itemize}
  \item \( f(0) = 0 \)
  \item \( f(a) = 1 \)
  \item \( f \) is decreasing \( (a, 2) \)
  \item \( f'(x) \neq 0 \) in \( [a, 2] \)
  \item \( f(2) = 0 \).
\end{itemize}

Now we consider \( C \times S^1 \) and in \( S^1 \) we take the vector field

\[ Y_2(\theta) = f(x) \, d\theta , \]

where \( x \in [0, 2] \). We get the vector field \( \chi^+ = (Y_1, Y_2) \) in the product. We can also consider

\[ Y_3(\theta) = -f(x) \, d\theta , \]
in another copy of $C \times S^1$. We get the vector field $\chi^- = (Y_1, Y_3)$ in the product. We call $p'_1$, $p'_2$ and $p'_3$ to the source, the sink and the saddle for $\chi^-$. We re write $C \times S^1$ as $D^2 \times [-1, 1]$ so that the vector field is entering at $D^2 \times \{1\}$, points out at $D^2 \times \{-1\}$ and is tangent to $(\partial(D^2)) \times [-1, 1]$. We paste 2 copies of $D^2 \times [-1, 1]$ along $(\partial(D^2)) \times [-1, 1]$. In one copy we have $\chi^+$ and in the other we have $\chi^-$. Since the vector fields are equal in $(\partial(D^2)) \times [-1, 1]$, both are $C^\infty$ even restricted to the boundary and no orbit crosses $(\partial(D^2)) \times [-1, 1]$, we can define a gluing map such that the resulting vector field $\chi$ defined in $\mathbb{R}^2 \times [-1, 1]$ is smooth.

The next lemma is to check all the conditions of theorem 21, except for the transversality condition, that we will check in the next subsection.

For our convenience, the intersections of the stable and unstable manifolds of the saddle periodic orbits are named as follows.

- $W^s(p_3) \cap \mathbb{R}^2 \times [-1, 1] = c_0$ in $\mathbb{R}^2 \times -1$,
- $W^s(p'_3) \cap \mathbb{R}^2 \times [-1, 1] = c'_0$ in $\mathbb{R}^2 \times -1$,
- $W^u(p_3) \cap \mathbb{R}^2 \times [-1, 1] = c_1$ in $\mathbb{R}^2 \times 1$,
- $W^u(p'_3) \cap \mathbb{R}^2 \times [-1, 1] = c'_1$ in $\mathbb{R}^2 \times 1$.
- The intersection of the invariant manifolds of the saddles with the boundary of $\mathbb{R}^2 \times [-1, 1]$ are the disjoint circles, $c_0$, $c'_0$, $c_1$, $c'_1$. 27
• The circle $c_0$ bounds a disc not containing $c'_0$, that we call $D_0$. The circle $c'_0$ bounds a disc not containing $c_0$, that we call $D'_0$. And they both bound an annulus called $A_0$. Analogously we define $D_1$, $D'_1$ and $A_1$.

To complete the flow to $S^3$ we add a sink $a$ and a source $r$ in the remaining space.

For doing this we can define a dynamic with a sink and a source in an open region of $S^3$ with boundaries $\mathbb{R}^2 \times -1$ and $\mathbb{R}^2 \times 1$. We define this so that the vector field is entering at $\mathbb{R}^2 \times 1$, and this boundary is in the basin of attraction of $a$, and vector field is pointing out at $\mathbb{R}^2 \times -1$. We take a neighborhood of the boundaries and compose the flow with an isotopy so that the vector field at the boundaries is now normal and orthogonal to the boundaries. We do the same in a neighborhood of the boundaries for $\chi$. Then we identify the boundaries. (this has been done in more details in several other works, see for instance [?])

This way, the orbits behave in one of the following ways (see figure 5.2):

1. They go from from $r$ to $a$.
2. They go to the periodic orbit $p'_3$ for the future or the past.
3. They enter the plug at $D'_0$ or for the past at $D'_1$.

Figure 1: The vector field $\chi$ in $S^3$.

Lemma 21. The vector field $\chi$ defined in $\mathbb{R}^2 \times [-1,1]$ has the following properties:

• The vector field $\chi$ is such that the chain recurrent set consists of 2 sources $p_1$ and $p'_1$, 2 sinks $p_2$ and $p'_2$, and 2 periodic saddles $p_3$ and $p'_3$.

• The orbit $O(x)$ of a point $x$ in a point in $\mathbb{R}^2 \times -1$ crosses $\mathbb{R}^2 \times 1$ if and only if $x \in A_0$ and $O(x) \cap \mathbb{R}^2 \times 1 \in A_1$
Proof. The flow $\phi_{Y_1}$ is such that the only recurrent points are the sinks or sources. This was not altered by the diffeomorphism $d$ and by rotating it. By construction, there are no orbits crossing from one copy of $\mathbb{D}^2 \times [-1,1]$ to the other. Also the intersection of the critical elements was transverse for $\phi_{Y_1}$ and this was also preserved.

The second item comes from the fact that all orbits of the points in the segment \{ $(1, y) \mid 1 - \epsilon < y < 1$ \}, cross the vertical segment that joints $q'$ with $(1, -1)$ for the flow $\phi_{Y_0}$.

Then $d$ takes this segments to \{ $(x, 1) \mid 1 < x \leq 2$ \} and \{ $(x, -1) \mid 1 < x \leq 2$ \}. Rotating this segments we obtain one half of $A_0$ and $A_1$, the other two halves are obtained after gluing $\chi^+$ with $\chi^-$. Since the same is valid for $\chi^-$, then we get that all orbits that cut $A_0$, also cut $A_1$.

5.3 A radial foliation and the image of the crossing map $P$

The aim of this subsection is to prove the last part of theorem 21.

Since every orbit of the points in $A_0$ cuts $A_1$ at some moment, we define the first return map $P : A_0 \to A_1$. We take polar coordinates in $A_0$ and $A_1$ (that is, we take coordinates in $S^1 \times (0,1)$). We write the diffeomorphism $P$ in this coordinates as

$$P(\theta, r) = (P_\theta(\theta, r), P_r(\theta, r))$$

![Figure 2: The lift of the map $P : A_0 \to A_1$ to $\mathbb{R} \times (0,1)$](image)

**Lemma 22.** Let $P : A_0 \to A_1$ be the first return map from $A_0$ to $A_1$ defined by the vector field $\chi$. If $DP(0, V_r) = (w, z)$ then $w \neq 0$ for all $V_r \in (0,1)$. As a consequence, the image of a radius under $P$ cuts transversally any radius at $A_1$. 

29
Proof. Let us consider new coordinates in the annulus $A_0$ and $A_1$. We consider the change of coordinates $g_0 : S^1 \times (0, 1) \to \mathbb{R}^2 \times \{-1\}$ and $g_0 : \mathbb{R}^2 \times \{-1\} \to S^1 \times (0, 1)$. We can extend the polar coordinates to the closure of $A_0$ and in this case $c_0 = S^1 \times \{0\}$. We define this change of coordinates so that the set

$$[O(q) \times S^1] \cap \mathbb{R}^2 \times \{-1\}$$

has now coordinates $(1/2, \theta)$.

Let us first consider the lift of $A_0$, $\hat{A}_0$ which is a strip $\mathbb{R} \times (0, 1)$, we also take the lift of $A_1$, to $\hat{A}_1$, and the lift of $P$, $\hat{P}$. We orient this lifts by considering the rotation in the sense of $p_3$ as positive.

Let us take a point $x = (\theta, r_x)$ in $A_0$ with the new coordinates. The time that it takes for $x$ to reach $A_1$ is $T_x$.

Suppose that $r < 1/2$ Recall that the vector field $Y_2$ is defined as

$$Y_2(\theta) = f(g(r)) d\theta,$$

and therefore

$$\frac{\partial P_\theta(\theta, r)}{\partial r} = T_r f(g(r))' g(r)'$$

and therefore non vanishing. Suppose that $r > 1/2$, then the vector field $Y_3$ is defined as

$$Y_3(\theta) = -f(g(r)) d\theta,$$

and therefore

$$\frac{\partial P_\theta(\theta, r)}{\partial r} = T_r f(g(r))' g(r)'$$

and therefore non vanishing.

At $r = 1/2$ since the lateral derivatives are not 0 and the function is smooth then

$$\frac{\partial P_\theta(\theta, r)}{\partial r} \neq 0.$$

5.4 Gluing the pieces: defining a flow on $S^3$

Let us consider the vector field $\chi$ defined above, we remove the 2 balls in the complement $S^2 \times [0,1]$. We glue instead a ball which is the attracting region of a Lorenz attractor $L_a$ and 2 singularities (the one from subsection 5.1.2, called $U_a$, instead of the ball that has $S^2 \times \{1\}$ on the boundary.

Recall that from subsection 5.1.2 we have that

- The boundary of $U_a$ is $S^2$
- There is an annulus $A_a$ in $S^2$ such that the strong stable manifolds of $L_a$ intersect $A_a$ along a radial foliation
Figure 3: The ball \( U_a \).

- The annulus \( A_a \) bounds a disc \( D_a \) containing the intersection of the stable manifold of \( p \) and does not intersect the stable manifold of the other extra singularity.

Consider the attractor \( a \) in \( S^3 \) for \( \chi \). There is a neighborhood \( U_a' \) of \( a \) in its basin of attraction such that the boundary is diffeomorphic to \( S^2 \) and the boundary contains \( \mathbb{R}^2 \times \{ 1 \} \). We remove this neighborhood and we glue the boundary of \( U_a \) to the boundary of \( U_a' \) so that

- \( A_a \) is mapped to an annulus containing \( A_1 \), a radial foliation of \( A_a \) is send to cut \( A_1 \) in a radial foliation.
- \( D_a \) is mapped to the interior of \( D_1 \).

Figure 4: Gluing \( U_a \) to \( S^2 \times \{ 1 \} \)

We consider a repelling region defined as the one from subsection 5.1.2, called \( U_r \), but with the reverse time. The maximal invariant set in this ball is a Lorenz repeller \( L_r \) and 2 other singularities. We glue this ball instead of the ball that has \( \mathbb{R}^2 \times \{ -1 \} \) in its boundary in an analogous way as we did with \( U_a \).
Note that by doing this process we do not create any new recurrent orbits. We call the resulting vector field in $S^3$, $X$.

### 5.5 The Filtrating neighborhood

Let us consider $X$ from subsection 5.4.

If we remove some small neighborhoods inside the basin of the 2 sources $p_1$ and $p_1'$, we get a repelling region $V_r$.

If we remove some small neighborhoods inside the basin of the 2 sinks $p_2$ and $p_2'$, we get an attracting region $V_a$.

The resulting open set $U = V_a \cap V_r$ is a filtrating neighborhood. We call the maximal invariant set in it $\Lambda$.

**Lemma 23.** For the vector field $X$ the maximal invariant set $\Lambda \subset U$ is strong multisingular hyperbolic.

**Proof.** The Lorenz attractor is singular hyperbolic, i.e.

$$T_x S^3 = E^{ss} \oplus E^{cu} \quad \text{for all } x \in L_a$$

(see [MPP]). The strong stable space of $L_a$ is escaping, and therefore the center space is $E^{cu}$. The singularities in $L_a$ are strong Lorenz like, and in fact, the expansion rate can never be bigger that 2 while the contraction rate is always bigger that 4. As a consequence

$$\Psi^t(L, u) = h(L, t) \cdot \psi^t_{X_a}(L, u)$$

still contracts $N^s(L)$ since the biggest possible expansion rate for $h(L, t)$ is smaller than 2. Since $E^{cu}$ expands volume, that means that

$$\Psi^t(L, u) = h(L, t) \cdot \psi^t_{X_u}(L, u)$$

expands $N^u(L)$.

The periodic orbits are also strong multisingular hyperbolic since $h(L, t)$ does not expand or contract exponentially along a periodic orbit.

We need to check the the strong multisingular hyperbolicity in the wondering orbits that go from $A_0$ to $A_1$. For this, lemma 19 tells us we need to check that the stable and unstable spaces that extend along this orbits, intersect transversely. This is a consequence of lemma 22 and the fact that the stable foliation of $L_a$ intersects $A_1$ radially, and the unstable foliation of $L_r$ intersects $A_0$ radially.
6 A multisingular hyperbolic set in $M^5$

The aim of this section is to find a chain recurrent set that is multisingular hyperbolic with 2 singularities of different indexes. For this, the strategy will be to multiply the vector field $X$ in $S^3$ from section 5 times $\mathbb{RP}^2$ with a simple vector field. Then modify the resulting set to obtain new recurrence.

The following lemma will be proven in the next section.

Lemma 24. There exist a vector field $Y$ in $\mathbb{RP}^2$ with the following properties:

- $Y$ is a $C^\infty$ vector field
- It has 3 singularities: a saddle singularity $s$, a source $\alpha$ and a sink $\omega$. It is linear in a neighborhood of the singularities.
- The contracting and expanding Lyapunov exponents of the saddle are equal in absolute value ($\lambda_{sss} = -\lambda_{uuu}$), and $\lambda_{uuu} > 6$.
- One of the stable branches of $s$ (that is an orbit) has its $\alpha$-limit in $\alpha$.
- One of the unstable branches of $s$ (that is an orbit) has its $\omega$-limit in $\omega$.
- The other two branches form an orbit with $\alpha$-limit and $\omega$-limit in $s$ and we call this orbit $\gamma$.
- There is a transverse section to $\gamma$ and to the flow, that we call $\mathcal{T} = [1, a] \times [1, a]$. $\mathcal{T} \cap \gamma = 0 \times a$ and the flow of $Y$, $\phi^Y(x,y,t)$ is such that:
  - If $s = (x,y)$ is such that $x > 0$, then $\phi^Y(s,t)$ does not cross $T$ for any $t > 0$ and has $\omega$-limit in $\omega$. And for $t < 0$ there exists only one $t_s < 0$ such that $\phi^Y(s,t_s) = s' \in T$ with $s' = (x',y')$, $x' < 0$ and the $\alpha$-limit of $s$ is $\alpha$.
  - If $s = (x,y)$ and $x < 0$, then $\phi^Y(s,t)$ does not cross $T$ for any $t < 0$ and has $\alpha$-limit in $\alpha$. And for $t > 0$ there exists only one $t_s > 0$ such that $\phi^Y(s,t_s) = s' \in T$ with $s' = (x',y')$, $x' > 0$ and the $\omega$-limit of $s$ is $\omega$.

6.1 The vector field in $M^5$

We start by considering the vector field $Z_{id} = (X,Y)$ in the manifold $M^5 = S^3 \times \mathbb{RP}^2$ and it’s flow $\phi_{id}$. Let us define the section

$$\sum = S^3 \times T$$

which is transverse to $Z_{id}$, and a flow-box $\sum \times [-1,0]$.

Proposition 25. Let $H : \sum \rightarrow \sum$ be a $C^\infty$ diffeomorphism isotopic to identity and that is the identity on the boundary. There exist a $C^1$ vector field $Z_H$ such that $Z_H = Z_{id}$ in the complement of the flow-box $\sum \times [-1,0]$, and in the flow-box $(H(z),0) = Z_H((z,-1),1)$.

33
Proof. Since $H$ is isotopic to the identity we have that there exist a diffeomorphism $F : \Sigma \times [-1,0] \to \Sigma$ such that $F(\Sigma, -1) = \text{id}$ and $F(\Sigma, 0) = H$. We also have that there exist $F' : \Sigma \times [-1,0] \to \Sigma$ such that $F'(\Sigma, -1) = H^{-1}$ and $F'(\Sigma, 0) = \text{id}$. Let us define the flow $\phi_H$ as follows:

- $\phi_H(y,t) = \phi_{\text{id}}(y,t)$ for every $t$ such that $\phi_H(y,t) \notin \Sigma \times [-1,0]$.
- If $t_0$ is such that $\phi_H(y,t_0) \in \Sigma \times \{-1\}$ then
  $$\phi_H(y,t) = F(\phi_{\text{id}}(y,t_0), s),$$
  for every $s = t - 1 - t_0$ such that $-1 \leq s \leq 0$.
- If $t_1$ is such that $\phi_H(y,t_1) \in \Sigma \times \{0\}$ then
  $$\phi_H(y,t) = F'(\phi_{\text{id}}(y,t_1), s),$$
  for every $s = t - t_1$ such that $-1 \leq s \leq 0$.

Now we define the vector field $Z_H$ by taking at any point, the derivative (on $t$) of $\phi_H(y,t)$ and since $\phi_H(y,t)$ is sufficiently smooth, then so is $Z_H$.

6.1.1 A filtrating region for $Z_H$

We recall that $U$ is a filtrating region defined in Section 5. We define now the filtrating region in $M^5$ that is interesting to us: We consider a repelling region $u_\alpha \subset \mathbb{R}P^2$ of $\alpha$ for $Y$, such that $\alpha$ is the maximal invariant set in $u_\alpha$. Similarly, consider a trapping region $u_\omega \subset \mathbb{R}P^2$ We take the respective repelling and trapping regions of this singularities in $M^5$. We define the repelling region $U_\alpha = S^3 \times u_\alpha$ and the trapping region $U_\omega = S^3 \times u_\omega$. We define as well $U_0 = M^5 / \{U_\alpha \cup U_\omega\}$.

$$V = U_0 \cap (U \times \mathbb{R}P^2).$$

Let us consider the maximal invariant set for $Z_{\text{id}}$ in $V$ that we call $\Lambda_{\text{id}}$.

Proposition 26. The maximal invariant set $\Lambda_{\text{id}}$ in $V$ (for $Z_{\text{id}}$) intersects $\Sigma$. For any $H$ as above, any orbit in the maximal invariant set $\Lambda_H \in V$ (for $Z_H$) either crosses $\Sigma$ or is contained in $S^3 \times \{s\}$.

Proof. Let us consider the saddle singularity in $Y$ that we called $s$. By construction, there is a unique orbit of $Y$, formed by a branch of the stable and unstable manifold of $s$, that crosses $T$. Since the contraction and expansion rates in $Y$ are stronger than in $X$, then the points in $S^3 \times \{s\}$ have a connection between the strong stable and unstable manifolds and the orbits in this connections cross $\Sigma$.
If the orbit $\gamma_y$ of a point $y = \{(x,l)\}$ never crosses $\sum$ then

$$Z_{id} |_{\gamma_y} = Z_M |_{\gamma_y}.$$  

Let us see that $\Lambda_{id}$ is contained in $S^3 \times \{s\}$ or it crosses $\sum$.

We take $u_0 = \mathbb{RP}^2 / u_\alpha \cup u_\omega$. Then the maximal invariant set in $u_0$ for $Y$ is the saddle $s$ and the saddle connection (the orbit that contains one unstable branch and one stable branch of $s$). All other points have their $\alpha$ and $\omega$-limits in the singularities $\alpha$ and $\omega$ (see the properties of $Y$ in (24)). So if there is a point $y \in \gamma_y$ such that $y \notin S^3 \times \{s\}$ and $\gamma_y \cap \sum = \emptyset$ then the orbit of $l$ by $Y$ has $\alpha$ or $\omega$-limits in the singularities $\alpha$ and $\omega$. This implies that $y$ has $\alpha$ and $\omega$-limits in $U_\alpha \cup U_\omega$. Therefore $\gamma_y \notin \Lambda_{id}$.

We recall that there are 2 saddles singularities in $S^3$, $\sigma_a$ and $\sigma_r$. By construction of the Lorenz attractor (see [GuWi]) there is a small linear neighborhood around the singularity, in which we can consider the coordinates $(x,y,z)$ to correspond to the strong unstable, weak stable and stable spaces. The singularity is approached by orbits of $L_a$ only in one semi space that corresponds to the points with positive $y$ value. We say then that $\sigma_a$ has an escaping separatrix $W_{cs}^-$ which is the half stable manifold that escapes from a neighborhood of $L_a$. Note that $W_{cs}^-$ (that is an orbit) in the basin of attraction of is a source. Therefore there is an open neighborhood of $W_{cs}^-$ that we call $D_a$ that is a repelling region. In the same way there is an escaping separatrix $W_{cu}^+$ for the singularity $\sigma_r$ in $L_r$. We consider a small neighborhood

$$u_a = \{(x,y,z)\}$$  

such that $-\delta < x < \delta$ $-\delta < z < \delta$ $-\delta < y < 0$

choosing $\delta$ so that $u_a \subset D_a$ is in the linearized neighborhood of $\sigma_a$.

Analogously we define $u_r$ for $\sigma_r$. Note that here the stable and unstable manifolds refer to the dynamics of $X$. We define now the corresponding repelling and trapping regions in $M^5$. That is $V_i = \mathbb{RP}^2 \times D_i$ for $i = \{r,a\}$. 

Figure 5: The ball $u_a$. 

$$u_a = \{(x,y,z)\}$$  

such that $-\delta < x < \delta$ $-\delta < z < \delta$ $-\delta < y < 0$

choosing $\delta$ so that $u_a \subset D_a$ is in the linearized neighborhood of $\sigma_a$. 

Analogously we define $u_r$ for $\sigma_r$. Note that here the stable and unstable manifolds refer to the dynamics of $X$. We define now the corresponding repelling and trapping regions in $M^5$. That is $V_i = \mathbb{RP}^2 \times D_i$ for $i = \{r,a\}$. 

35
6.2 The chain recurrent set with different singularities

We are going to start with a maximal invariant set $\lambda_{id}$ for a flow $Z_{id}$ which is a skew product, and alter some cross section of it by a diffeomorphism $H$ so that the result is a flow with a multisingular chain recurrent set in $M^5$. For that we now need to choose some more properties on the diffeomorphism $H$ from proposition $26$. The following lemma will be proven in section $8$.

Lemma 27. There exist a $C^\infty$ diffeomorphism isotopic to identity, $H: \Sigma \to \Sigma$, where $\sum = S^3 \times T$, that is the identity on the boundary. We take coordinates for $T$ in $[-1, 1]$ and in this coordinates,

$$H(x, l) = (r_l(x), \theta_x(l))$$

where $r_l : \sum \to S^3$, $\theta_x : \sum \to T$. We can construct such a function having the following properties:

- $H$ is the identity outside of $V \cap \sum$ and in $V_a$ and $V_r$.
- The map $r_l(x)$ is the identity for $l = 1$ or $l = -1$, or if $x \in u_a \cup u_r$.
- Consider a compact ball $B_r \subset S^3$ that intersects the maximal invariant set $\Lambda \subset U$ only in a point $z' \in W^s_r(\sigma_r)$. Analogously consider a compact ball $B_a$ that intersects the maximal invariant set $\Lambda \subset U$ only in a point $z \in W^s_a(\sigma_a)$.

The image of $r_l(B_a) = B_r$, and $r_l(z) = z'$ for all $l \in [-1/2, 1/2]$.

- The balls $B_a$ and $B_r$ can be taken so that there exist $K_Y > t_0$ such that $\phi^t_X(B_r) \subset (u_r)$ and $\phi^t_X(B_a) \subset (u_a)$ for all $t > t_0$. Recall that $K_Y + 1$ is the minimum of the times that it takes for a point in $T_1$ to return to $T$ for $Y$ and $K_Y > 0$.

- If $l \in [-1/2, 1/2]$ then $\theta_x(l) = l$.
- If $l \in [0, 1/2]$ and $x \notin B_a$ then $\theta_x(l) > 0$.
- If $l \in [0, 1/2]$ and $x \in B_a$ then $-\epsilon < \theta_x(l) \leq \epsilon$.
- The only point $l$ such that $H(z, l) = (z', 0)$, is $l = 0$.

Proposition 28. We consider $H$ as in $27$, then the orbits in the maximal invariant set $\Lambda_H$ are contained in $\Lambda \times \{s\}$ or cross the flow box $\sum \times [-1, 0]$ in

$$B_a \times [0, 1/2] \times \{0\}.$$ 

Proof. Suppose that $\gamma$ is an orbit in $\Lambda_H$ that doesn’t cross $\sum$. From Proposition $26$ these orbits of $\Lambda_H$ are in $S^3 \times \{s\}$. Let $y$ be a point of $\gamma$ of coordinates $(x, p) \in S^3 \times \mathbb{R}P^2$. If $x$ is not in $\Lambda$ (for $X$) then, the alpha or the omega limit of $x$ must be in $U_c$. Therefore, for a $t$ large enough,

$$\phi^t_H(x, l) \notin V = U_0 \cap U.$$
Figure 6: The function $\theta_x(l)$. Note that the stripped area depicts orbits with their $\alpha$ limits in $U_\alpha$ and the painted area depicts orbits with their $\omega$ limits in $U_\omega$.

Then if $\gamma$ doesn’t intersect $\Sigma$, it must be in $\Lambda \times \{s\}$.

Let us suppose now that $\gamma$ intersects $\Sigma$. Let $y$ be a point in $\gamma \cap \Sigma \times [-1,0]$ such that $y \in \Sigma \times \{-1\}$. We write $y$ as $(z, -1)$ and $z$ as $z = (x, l) \in \Sigma$.

1. If $l > 1/2$, or if $x \notin B_\alpha$ with $l > 0$, then $H(x, l) = (r_1(x), \theta(l))$ with $\theta(l) > 0$ and then $\phi^t_H(y) = (r_1(x), \theta(l)) \times \{0\}$. Since outside of the flow-box $Z_{id} = Z_H$ now we can look at $Z_{id}$. From the properties of $Y$ ($24$) we have that the future orbit of $\theta(l) > 0$, does not cross $T$ and the $\omega$-limit is $\omega$. Then the orbit for $\phi^t_H$ is in $U_\omega$ for a large enough $t$. Then $\gamma$ is not in $\Lambda_H$.

2. If $l < 0$, since $y$ goes outside of the flow-box for the past (where $Z_{id} = Z_H$) now we can look at $Z_{id}$. From the properties of $Y$ ($24$) we have that the orbit of $l < 0$ does not cross $T$ for the past and the $\alpha$-limit is $\alpha$. Then $\gamma$ does not cross again the flow-box for the past. The orbit for $\phi^t_H$ is in $U_\alpha$ for a negatively large enough $t$ and $\gamma$ is not in $\Lambda_H$.

3. If $x$ is not in $B_\alpha$ and $l = 0$ then $\phi^1_H(y) = ((H(x), \theta_x(l)), 0)$ and $\theta_x(l) > 0$. Then, as before, we have that the orbit of $\theta_x(l)$ for $Y$ does not cross $T$ for the future and the $\omega$-limit for $Y$ is $\omega$. Then $\gamma$ does not cross again the flow-box for the future and $\gamma$ is not in $\Lambda_H$.

Then the only other case in which $\gamma \in \Lambda_H$ is if $\gamma$ crosses the flow box $\Sigma \times [-1,0]$ in $B_\alpha \times [0,1/2] \times \{-1\}$. \qed
Figure 7: The ball $B_a$

**Proposition 29.** There is a unique orbit $\gamma$ in $\Lambda_H$ that crosses $\sum$, that orbit is the orbit of $(z,0) \times \{-1\} \in \sum$.

**Proof.** Let $\gamma$ be an orbit in $\Lambda_H$. From proposition (28), we already know that if an orbit of $\Lambda_H$ crosses $\sum$ then it crosses at a point $y = (x,l,-1) \in B_a \times [0,1/2] \times \{-1\}$.

If $\theta_x(l) < -\epsilon < 0$ recall that the properties of $H$ (Lemma 27) give us that then $l < 0$.

Suppose now that $l \geq 0$ and that $\theta_x(l) > 0$. As in our previous proposition this implies that $\phi_{tH}^t(y) \notin U$ for $t$ large enough.

If $l \geq 0$ and $-\epsilon < \theta_x(l) \leq 0$ then $x \in B_a$. Suppose that $x \neq z$. Then

$$\phi_{tH}^t(y) \notin \sum \times [-1,0]$$

for all $K_Y > t > 0$, and therefore $\phi_{tH}^{t+1}(y) = \phi_{id}^t(\phi_{H}^1(y))$ for all $K_Y > t > 0$. Let us consider $t_0$ as in the properties of $H$ (Lemma 27). Recall that $t_0$ is such that $\phi_{X}(B_r) \subset (v_r)$ and $\phi_{H}^{t}(B_a) \subset (v_a)$ for all $K_Y > t > t_0$. We call

$$\phi_{id}^{t_0}(\phi_{H}^1(y)) = (x_1,z_1) \in S^3 \times \mathbb{RP}^2.$$

Since $x_1 \in u_r$ and is not $x$, then $x_1$ is in the attracting region of a sink $p_r$ of $X$ (see subsection 5.4). Now, for all $t > t_0$ even the ones bigger than $K_Y$, if we call $s = t_0 + 1 - t$, we have that

$$\phi_{H}^s(x_1,z_1) = (\phi_{H}^s(x_1,z_1),\phi_{H}^s(x_1,z_1))$$
and since every time for the future that this orbit crosses the flow-box $\sum \times [-1,0]$, the function $r_l$ is the identity, then

$$\phi^s_H(x_1,z_1) = (\phi^s_X(x_1),\phi^s_H(x_1,z_1)).$$

Since $\phi^s_X(x_1) \notin U$ for $t > t_0$ big enough, then $\phi^s_H(y)$ is eventually not in $V$ for some $t > t_0$. Then $\gamma$ is not in $\Lambda_H$ as wanted.

If $l \geq 0$ and $-\epsilon < \theta_x(l) < 0$ but $x = z$. Let $t_y$ be a time in which the orbit returns to the flow-box. That is $t_y$ is such that

$$\phi^{t_y}_H(y) = (x_1,l_1,-1) \in \sum \times \{-1\}.$$  

Recall from the properties of $H$ (Lemma 27) that $t_y \geq K_Y > t_0$ with $t_0$ such that $\phi^s_X(B_r) \subset (u_r)$ and $\phi^u_H(B_a) \subset (u_a)$ for all $t > t_0$. Since $-\epsilon < \theta_x(l) \leq 0$ and after returning to $\sum \times \{-1\}$ the orientation was reversed, then $l_1$ is positive.

Since now $x_1$ is not in $B_a$, then $\theta_{x_1}(l_1) > 0$. So, now for any $t > 0$, we have that

$$\phi^{t+ts+1}_{t_y}(y) = \phi^{t}_{id}(\phi^{t+s+1}_{t_y}(y)).$$

This implies that the orbit of $y$ never cuts the flow box again, and therefore, for a big enough $t$, $\phi^{t}_{t_y}(y)$ is in $U_\omega$. As a consequence $\gamma$ is not in $\Lambda_H$ as wanted.

The only case left is $x = z$ and $\theta_x(l) = 0$. The last property of $H$ (Lemma 27) tells us that $l = 0$, so the objective now is to prove that the orbit of $y = (z,0)$ never leaves

$$V = U_0 \times U \times \mathbb{R}^2.$$ 

But $(z,0)$ is in the stable manifold of $\sigma_r$ and in the unstable manifold of $\sigma_a$, and then then the orbit of $y$ is in $\Lambda_H$.

\[\Box\]

### 6.3 Multisingular hyperbolicity

Until now we have constructed a vector field having a chain recurrent class such that

- Two singularities of different indexes one in $L_a$ and the other in $L_r$.
- All the periodic orbits have the same index and the singularities are in the closure of the periodic orbits.
- There are periodic orbits in $L_a$ such that their stable manifolds intersect the unstable manifolds of periodic orbits in $L_r$.
- There is only one orbit in the class with the $\alpha$-limit in $L_a$ and the $\omega$-limit in $L_r$.

The goal now is to show that we can choose a diffeomorphism $H$ so that this vector field would be strong multisingular hyperbolic. After that we will perturb this vector field to an other that will still be strong multisingular hyperbolic,
but having a homoclinic connection between periodic orbits in \( L_a \) and periodic orbits in \( L_r \). This will finish the proof of theorem \( \frac{1}{1} \).

In the following section we will prove not only that there exist a diffeomorphism \( H \) the properties defined in Lemma \( \frac{27}{27} \) but also that this function can be constructed with the following additional property, we require that the image of

\[
S^3 \times \{0\} \times \{-1\} \in \Sigma \times \{-1\}
\]

under \( H \) cuts transversally

\[
S^3 \times \{0\} \times \{0\} \in \Sigma \times \{0\}.
\]

This last property guaranties that the set \( \Lambda_H \) will be strong multisingular hyperbolic.

To show that \( \Lambda_H \) is strong multisingular hyperbolic we need to check that we are in the hypothesis of lemma \( \frac{13}{13} \). Since we have already shown the other hypothesis the following lemma implies strong multisingular hyperbolicity.

**Lemma 30.** Let \( y \in \Sigma \times \{-1\} \) be such that \( \Lambda_H = \Lambda \cup O(y) \). There exist a diffeomorphism \( H \) such that

- The stable and unstable spaces along the orbit of \( S_X(y) \) intersect transversally,
- The orbit of \( y \) does not intersect the escaping spaces of the singularities for \( Z_H \).

then \( \Lambda_H \) is multisingular hyperbolic.

**Proof.** Consider the points in \( S^3 \times \mathbb{R}^2 \), \( a = (z, s) \) and \( b = (z', s) \). Let us take \( y = (z, 0, -1) \in \Sigma \times \{-1\} \). The orbit of \( y \) is in the strong unstable manifold of \( a \), (since unstable manifold of \( s \) intersects \( T \) at 0 for \( X \)). Analogously \( y \) is in the strong stable manifold of \( b \) since \( \phi_H^1(y) = (z', 0, 0) \). Observe that \( a \) and \( b \) are regular orbits and \( z \in W^u(\sigma_a) \) and \( z' \in W^s(\sigma_r) \) for \( X \). therefore \( \gamma \) does not intersect the escaping spaces of the singularities for \( Z_H \). From Proposition \( \frac{17}{17} \) this implies that the center space of the singularities of \( \Lambda_H \) and \( \Lambda \) are the same.

From lemma \( \frac{16}{16} \) we have that there exists an unstable space (for the reparametrized linear Poincaré flow ) at \( a \) that we call \( E_y^u \). We choose a metric so that the normal space at \( y \) is tangent to \( \Sigma \times \{-1\} \). We take a vector \( v \in E_y^u \) at \( y \). This vector is tangent to

\[
S^3 \times \{0\} \times \{-1\} \in \Sigma \times \{-1\}
\]

at \( y \). Let us recall that we have assumed at the beginning of the subsection that the image of

\[
S^3 \times \{0\} \times \{-1\} \in \Sigma \times -1
\]

under \( H \) cuts transversally

\[
S^3 \times \{0\} \times \{0\} \in \Sigma \times \{0\}.
\]

40
Then the image of $v$ under the differential of $H$ (and of $\phi^1_H$) is transverse to $S^3 \times 0 \times O(y)$ at $\phi^1_H(y)$, and then so is the image of $v$ under $\Psi^1(v)$, since the direction of the flow is not tangent to $T \times \{0\}$. On the other hand lemma [16] also gives us a stable space $E_y^s$ at $\phi^1_H(y)$ that is tangent to $S^3 \times \{0\} \times \{0\}$ at $\phi^1_H(y)$. Then the stable and unstable spaces of the reparametrized linear Poincaré flow are transversal. Then we are in the hypothesis of [18] and this completes the proof.

With this last lemma we know that the maximal invariant set $\Lambda_H$ is multi-singular hyperbolic. But this is not enough, since a small perturbation of $Z_H$ could brake the connection between $L_a$ and $L_r$ and have $\sigma_a$ and $\sigma_r$ in different chain classes. We need now to show that the right perturbation of $Z_H$ will generate the intersection of the stable and unstable manifolds of periodic orbits in $L_a$ and $L_r$. Since $\Lambda_H$ is multisingular hyperbolic for $Z_H$, so will it be for this new vector field and now the singularities will be robustly in the same chain recurrence class.

The following lemma implies Theorem [1].

---

**Figure 8:** A perturbation of $Z_H$, in particular of $r_l(x)$.

**Lemma 31.** There is an arbitrarily small perturbation of $Z_H$, that we call $Z_H$, and a $C^1$ neighborhood of $Z_H$, called $V$ so that any vector field $Z \in V$ has a maximal invariant set $\Lambda_Z$ that is multisingular hyperbolic and there is a chain class $C \in \Lambda_Z$ that has two singularities of different index accumulated by periodic orbits.

**Proof.** We will make a small perturbation of $H$ and this will result in a small perturbation of $Z_H$. Let us recall that we can write $H$ as

$$H(x,l) = (r_l(x), \theta_x(l))$$
where \( r_l : \sum \to S^3, \theta_x : \sum \to T. \)

We will only perturb \( r_l(x) \) to \( r'_l(x) \) so that \( \circ B_a \cap L_a = b_a \) is a small ball (relative to \( L_a \)), and the same for \( b_r \). We can also ask that \( r'_l(b_a) \cap b_r \). This can be done with an arbitrarily small \( C^r \) perturbation, so that the resulting vector field \( Z_{H_\varepsilon} \) is still \( C^1 \) and multisingular hyperbolic.

Note that since \( b_a \) and \( b_r \) are open, then there is a small neighborhood of \( r'_l(x) \) and therefore a small neighborhood of \( Z_{H_\varepsilon} \), \( \mathcal{V} \) so that the image of \( b_a \) intersects \( b_r \) for all vector fields in the neighborhood.

Now from the fact that periodic orbits are dense in the sets \( L_a \) and \( L_r \), and the fact that \( Z_{H_\varepsilon} \) is star, we get that we can choose a small perturbation by \( \delta \) so that the unstable manifold of some periodic \( p \) orbit in \( L_a \) intersects transversally the stable manifold of a periodic orbit \( q \) in \( L_r \). Recall that the periodic orbits all have the same index.

Also by the connecting lemma we can get by another small enough perturbation, that the stable manifold of \( p \) intersects the stable manifold of \( q \). This homoclinic intersection is roust.

\[ \square \]

7 Construction of the vector field \( Y \) in \( \mathbb{RP}^2 \)

7.1 A vector field with a saddle connection in a Möbius strip

Let us start by defining some simple linear flow in \( \mathbb{R}^2 \). We take a linear vector field \( Y(x, y) = (\lambda_{sss}x, \lambda_{uuu}y) \) defined in \([-2, 2] \times [-2, 2]\). We ask that \( \lambda_{uuu} = -\lambda_{sss} \) and we also ask that \( \lambda_{uuu} > 6 \).

We consider a close curve \( C \) formed by the union of following curves:

- We consider the orbit of a point \((-a, 2)\). This orbit cuts the vertical line \((-2, y)\) in a point \((-2, a')\). The segment of orbit from \((-2, a')\) to \((-a, 2)\) is our first curve \( C_1 \).
- We consider the orbit of a point \((a, 2)\). This orbit cuts the vertical line \((2, y)\) in a point \((2, c)\). The segment of orbit from \((a, 2)\) to \((2, c)\) is \( C_2 \).
- We consider the segment \( \{-2\} \times [a', -a'] \) as our second curve \( C_3 \).
- We take the orbit of \((-2, -a')\) and we call the point where it cuts the horizontal line \( l \) in a point \((-b, -2)\). The segment of orbit from \((-2, -a')\) to \((-b, -2)\) is our third curve \( C_4 \).
- We consider the segment \( \{2\} \times [-c, c] \) as our second curve \( C_5 \).
- We consider the orbit of a point \((2, -c)\). This orbit cuts the horizontal line \((x, -2)\) in a point \((b', -2)\). The segment of orbit from \((2, -c)\) to \((b', -2)\) is \( C_6 \).
- The segment \([b', -b] \times \{-2\}\) our forth curve \( C_7 \).
The segment \([-a,a] \times \{2\}\) our last curve \(C_8\).

There is a diffeomorphism \(d : C_8 \rightarrow C_5\) defined as follows:

\[
d(x) = -\frac{c(x)}{a}.
\]

Now we glue \(C_8\) and \(C_5\) along \(d\). There is a connected component in the complement of \(C\) that contains \((0,0)\). We call the closure of this connected component \(D\). The manifold \(D\) (with boundary \(C\)) obtained from this gluing is a 2-dimensional non-orientable manifold with a connected boundary, therefore it is a Möbius strip.

Note that since the \(d : C_8 \rightarrow C_5\) is such that 0 is pasted to 0, then there is a branch of the stable manifold of \((0,0)\) and a branch of the unstable manifold of \((0,0)\) that intersect. That is, there is an orbit \(\gamma\) such that

\[
\gamma \subset W^s(0,0) \cap W^u(0,0).
\]

We say then that \((0,0)\) has a saddle connection.

### 7.2 Completing the vector field to \(\mathbb{RP}^2\)

Let us consider a linear vector field in \(\mathbb{R}^2\) with a sink \(\omega\), and let us take a neighborhood \(u_\omega\) in its basin of attraction. We choose a curve in the boundary, it will be pointing inwards. We can take \(C_3\), and since the vector field \(Y\) is pointing outwards, we can paste them.

Note that the remaining unstable branch has its \(\omega\)-limit in \(\omega\).

We call the new vector field \(Y\) and what remains of the boundary of \(u_\omega\), we now call it \(C_3'\).
Analogously we attach a neighborhood \( u_\alpha \), containing a source \( \alpha \) and glue it through the segment \( C_7 \). We call the subset of boundary of \( u_\alpha \), that was not glued to \( D, C'_7 \).

Note that the remaining stable branches of \((0,0)\) (that is an orbit) has its \( \alpha \)-limit in \( \alpha \).

We call \( D' \) to the region formed by \( D \) with \( u_\alpha \) and \( u_\omega \) attached. Since \( D' \) is a Möbius strip, then the complement in \( \mathbb{RP}^2 \) is a disc \( R \) having a boundary formed by 4 disjoint curves tangent to the flow \((C_1, C_2, C_6, C')\), one curve transverse to the flow and entering \( D' C'_7 \), and one curve transverse to the flow and exiting \( D' C'_3 \). Therefore we can define the flow in the complement of \( D' \) in the trivial way by sending the points in \( C'_3 \) to \( C'_7 \).

Now we prove Lemma 24

**Proof.**

- Since the original maps are linear, the resulting map after the gluing is also \( C^\infty \).

- The contracting and expanding Lyapunov values of \( Y \) can be taken to be as strong as required.

- As noted above, one branch of each stable and unstable manifold form a saddle connection \( \gamma \) while the others come or go to the sink and source.

- The segment, \( T_0 = C_8 \) is a transverse section to \( \gamma \) by construction and is such that:
  - If \( s > 0 \) \( \phi^Y(s,t) \) never touches \( T_0 \) for any \( t > 0 \) and has \( \omega \)-limit in \( \omega \).
    And for \( t < 0 \) there exists only one \( t_s < 0 \) such that \( \phi^Y(s,t_s) = s' \in T_0 \) with \( s' < 0 \). and the \( \alpha \)-limit of \( s \) is \( \alpha \).
  - If \( s < 0 \) \( \phi^Y(s,t) \) never touches \( T_0 \) for any \( t < 0 \) and has \( \alpha \)-limit in \( \alpha \).
    And for \( t > 0 \) there exists only one \( t_s > 0 \) such that \( \phi^Y(s,t_s) = s' \in T_0 \) with \( s' > 0 \) and the \( \omega \)-limit of \( s \) is \( \omega \).

As a consequence of the fact that that \( C_8 \) was glued to \( C_5 \) reverting orientation.

\[ \square \]

8 Construction of the diffeomorphism \( H \)

In this section we prove the following lemma from the previous section:

**Lemma.** [27] There exist a \( C^\infty \) diffeomorphism isotopic to identity, \( H : \Sigma \to \Sigma \) that is the identity on the boundary. We consider \( \Sigma = S^3 \times T \) and we take coordinates for \( T \) in \([-1,1] \),

\[ H(x,l) = (r_1(x), \theta(x,l)) \]

where \( r_1 : \Sigma \to S^3, \theta : \Sigma \to T \). We can construct such a function having the following properties:
• $H$ is the identity in $V_a$ and $V_r$.
• The map $r_t(x)$ is the identity for $l = 1$ or $l = -1$, or if $x \in u_a \cup u_r$.
• There are two compact balls $B_r \subset S^3$ and $B_a \subset S^3$ such that $r_t(B_a) = B_r$. Moreover $B_r$ intersects the maximal invariant set $\Lambda \subset U$ only in a point $z' \in W^u_\chi(\sigma_r)$, The ball $B_a$ intersects the maximal invariant set $\Lambda \subset U$ only in a point $z \in W^u_\chi(\sigma_a)$, and $r_t(z) = z'$ for all $l \in [-1/2, 1/2]$.
• There exits $K_Y > t_0$ such that $\phi^t_\chi(B_r) \subset (u_r)$ and $\phi^t_H(B_a) \subset (u_a)$ for all $t > t_0$. Recall that $K_Y + 1$ is the minimum of the times that it takes for a Point in $T_1$ to return to $T$ for $Y$ and $K_Y > 0$.
• If $l \in [-1/2, 1/2]^c$ then $\theta_l(l) = l$.
• If $l \in [0, 1/2]$ and $x \notin B_a$ $\theta_l(l) > 0$.
• If $l \geq 0$ and $x \in B_a$ then $-\epsilon \leq \theta_l(l) \leq \epsilon$.
• The only point $l$ such that $H(z, l) = (z', 0)$, is $l = 0$.
• The image of

$$S^3 \times \{0\} \times \{-1\} \in \Sigma \times -1$$

under $H$ cuts transversally

$$S^3 \times \{0\} \times \{0\} \in \Sigma \times \{0\}.$$  

Proof. Let us consider a closed neighborhood of $D_a \cup D_r$ that we call $C$. Since $B_a$ and $B_r$ are subsets of $S^3$, they are isotopic to each other. Moreover, we can choose $C$ so that they are isotopic to each other in $S^3 \setminus C$, since $D_a \cup D_r$ does not disconnect $S^3$. Therefore there is a function $r': S^3 \setminus C^0 \times [0, 1/2] \to S^3$ such that

$$r'(x, 0) = id(x) \text{ and } r(B_a, 1/2) = B_r.$$  

We can choose $r'$ so that it is the identity in the boundary of $C$ for all $t \in [0, 1]$ and such that $r'(x, 1/2) = z'$. We can extend now this function to $S^3$ by asking that $r' | D_a \cup D_r = Id$. Now $r : S^3 \times [-1, 1] \to S^3$ is defined by

$$r(x, l) = \begin{cases} 
    r'(x, l + 1), & \text{if } l \leq -1/2 \\
    r'(x, 1/2), & \text{if } -1/2 < l \leq 1/2 \\
    r'(x, l - 1), & \text{if } l > 1/2.
\end{cases} \quad (3)$$

Now we need to construct $\theta : \Sigma \to [-1, 1]$.

We consider a $C^\infty$ bump function $h : [-1, 1] \to [-1, 1],$

• if $l \in [-1/2, 1/2]^c$ then $h(l) = 0$,
• if $l \in [-1/2, 1/2]$ then $0 < h(l) \leq \frac{\epsilon}{2}$,
• if $l = 0$ then $h(l) = \frac{\epsilon}{4},$

45
We can also assume that \( h \) is sufficiently differentiable. Let \( B_A \) be an arbitrarily small neighborhood of \( B_a \). We consider now a second bump function \( g : S^3 \to [-1, 1] \):

- If \( x \in B_A \) then \( g(x) = 0 \),
- If \( x \in B_a \) then \( \epsilon \leq g(x) \leq \frac{\epsilon}{2} \),
- \( g(z) = -\frac{\epsilon}{4} \) and \( \frac{g(x)}{\partial v} |_{(z)} \neq 0 \) for any given \( v \) direction in \( S^3 \).

We define then \( \theta_x \) as follows:

\[
\theta_x(l) = \text{id}(l) + h(l) + g(x).
\]

Note that the image of the vectors tangent to the coordinates in \( S^3 \), under the differential of \( H \), have a non vanishing component in the direction of \( T \). This is our desired function.

9 Robust chain transitivity

The aim of this section is to prove the only remaining part of 1, that is that the chain class constructed in the previous sections is robustly chain class. We restate this as follows:

**Proposition 32.** Let \( U' \subset X^1(M) \) be the open set in Theorem 1 and \( C_Y \) the chain recurrence class defined in Theorem 1. Then there exist a neighborhood \( U \subset U' \subset X^1(M) \) (and a filtrating set \( U \subset M \) such that for any \( Y \subset U \) \( U \cap R(Y) = C_Y \).

This shows us that the example presented in this chapter is an example of a robustly chain transitive set, that is not robustly transitive. Until now the only other example of this sort, is for diffeomorphisms in a 3 manifold and was presented by [BCGP].

Reasoning by contradiction, if the proposition was not true, for some \( Y \) arbitrarily close to \( Z \) from the previous sections we could find a sequence of chain recurrence classes \( C_n \) converging to \( C \). Since only finitely many of this classes can be aperiodic and only finitely many of this classes can be singular, then the following property is equivalent to the previous one.

**Proposition 33.** Let \( Y \) be a vector field arbitrarily close to \( Z \in U \) also from Theorem 1, and let \( C_Y \) be the chain recurrence class defined in Theorem 1. Suppose that there periodic orbits \( \gamma_n \), for \( Y \), converging to \( C_Y \). Then there exist \( n_0 \) such that for every \( n > n_0 \) we have that \( \gamma_n \subset C_Y \).

**Proof.** Let \( S \subset C_Y \) the be the accumulation points of the sequence \( \gamma_n \). Let us suppose that \( S \) is connected. We can reason in the same way for every connected
component. We can define for $Y$ the continuations of $L_a$, $L_r$ and the periodic orbits $q$ and $q'$. Then

- $S \subset L_a$
- or $S \subset L_r$,
- or $S$ is a periodic orbit,
- or it intersects both $L_r$ and $L_a$.

This is because $S$ is a recurrent set and if we restrict the vector field to $C_Y \setminus (L_a \cup L_r)$ the only recurrent points are a finite number of periodic orbits.

There is a neighborhood of $L_a$ in which the dynamics is normally hyperbolic. The set $L_a$ is robustly transitive for $X$ and therefore it is also robustly transitive for $Z$ and also for $Y$. This is also true for $L_r$.

If $S$ is a hyperbolic periodic orbit in $C$ then, $S$ has a 3 dimensional stable manifold and a 3 dimensional unstable manifold of a given size $\delta$ that Therefore for $n_0$ such that the distance of $\gamma_{n_0}$ to $S$ is less than $\delta$, then $\gamma_n \subset C_Y$ for all $n > n_0$.

If $S$ intersects both $L_r$ and $L_a$, we take a sub sequence of $\gamma_n$ such that there exist points $p_n \in \gamma_n$ and $q_n \in \gamma_n$ such that $p_n \rightarrow p \in L_a$ and $q_n \rightarrow q \in L_r$. The other sub sequences can be treated as before.

Let us first observe that for all the singularities, the strongest stable space and the strongest unstable space are escaping. This means that for any periodic orbit $\gamma$ in the set of periodic orbits in $C_Y$, the angle between the direction of the flow and these strong stable and unstable directions is bounded away from 0. That is, there is a dominated splitting

$$ T_\gamma M = E^{ss} \oplus E^c \oplus E^{uu} $$

where all directions accumulated by flow directions are in $E^c$ that is 3 dimensional, and tangent to $S^3$. $E^{ss}$ is uniformly contracting and one dimensional, and $E^{uu}$ is uniformly expanding and one dimensional. Note that this splitting is over the closure of the set of the periodic orbits in $C_Y$ and not in all $C_Y$.

Then as a consequence there are well defined one dimensional strong stable and unstable manifolds, of uniform size $\delta$ in any point $p_n$ of $\gamma_n$, for an $n$ big enough. The set $L_a$ has a 4 dimensional stable manifold. The points $p_n$ have a one dimensional strong unstable manifold of size $\delta$, so if we take a segment of orbit around each $p_n$ and their strong unstable manifolds, that intersects the stable manifold of $L_a$ transversally.

The same but inverting the role of the stable and unstable manifolds can be said about $L_r$ and the points $q_n$. Therefore then $\gamma_n \subset C$ for all $n > n_0$. \qed
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