Abstract. In this paper we study the Fuchsian Riemann-Hilbert (inverse monodromy) problem corresponding to Frobenius structures on Hurwitz spaces. We find a solution to this Riemann-Hilbert problem in terms of integrals of certain meromorphic differentials over a basis of an appropriate relative homology space over a Riemann surface, study the corresponding monodromy group and compute the monodromy matrices explicitly for various special cases.

Contents

1 Introduction ........................................... 2
2 The Fuchsian Riemann-Hilbert problem in Frobenius manifolds theory .................................................. 7
3 Solution to the Fuchsian system associated to the Hurwitz Frobenius manifolds .................................. 10
   3.1 Preliminaries ............................................. 10
   3.2 Construction of a solution to the Fuchsian system ................................................................. 12
   3.3 Dependence of the solution on the choice of homology basis ..................................................... 16
4 Monodromy group of the Fuchsian system ........................................................................ 19
   4.1 Preliminaries ................................................. 20
   4.2 Monodromy group ........................................... 22
      4.2.1 Spaces of meromorphic functions with simple poles .......................................................... 22
      4.2.2 Spaces of meromorphic functions with poles of higher multiplicity .................................. 24
5 Action of braid group on solution to the Fuchsian system .................................................. 25
   5.1 Braid monodromy group .................................... 25
   5.2 Genus one coverings of degree 2 ......................... 26
   5.3 Action of braid group on monodromy matrices ......................... 30
6 Concluding remarks ........................................ 32

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1 Introduction

The matrix Riemann-Hilbert problems (or inverse monodromy problems) naturally arise in the context of systems of linear differential equations with meromorphic coefficients (we consider here the simplest Fuchsian case, when all the poles of the coefficients in the right-hand side are simple):

\[
\frac{\partial \Phi}{\partial \lambda} = \sum_{j=1}^{N} \frac{A_j}{\lambda - \lambda_j} \Phi,
\]

(1.1)

where \(A_j\) are \(n \times n\) matrices independent of \(\lambda\); \(\Phi(\lambda)\) is \(n \times n\) solution matrix. The solution \(\Phi\) is single-valued on the universal cover of the Riemann sphere with punctures at \(\lambda_1, \ldots, \lambda_N\) and \(\infty\). If one starts at a point \(\lambda_0\) on some sheet of the universal cover, and analytically continues \(\Phi\) along an element \(\gamma\) of the fundamental group of the punctured sphere, one gets a new solution, \(\Phi_\gamma\), of the same system; therefore, \(\Phi_\gamma\) is related to \(\Phi\) by a right multiplier, \(M_\gamma\), which is called the monodromy matrix: \(\Phi_\gamma = \Phi M_\gamma\). The monodromy matrix corresponding to the product \(\gamma_1 \gamma_2\) is given by \(M_{\gamma_1 \gamma_2} = M_{\gamma_2} M_{\gamma_1}\); therefore, in this way one gets the group anti-homomorphism of the fundamental group of Riemann sphere with \(N+1\) punctures to \(GL(n)\). The image of this anti-homomorphism is called the monodromy group of the system (1.1). The Riemann-Hilbert (or inverse monodromy) problem is the problem of finding the matrix-valued function \(\Phi\) (and, therefore, also the coefficients \(A_j\)) knowing the positions of singularities \(\lambda_j\) and the corresponding monodromy matrices.

It is natural to deform the whole picture by changing infinitesimally the positions of singularities \(\lambda_j\) in such a way that the monodromy matrices remain unchanged. Such a deformation (called isomonodromic deformation) implies a set of non-linear differential equations for matrices \(A_j\) as functions of \(\{\lambda_k\}\); these equations are called the Schlesinger equations.

Therefore, the solution of the non-linear Schlesinger equations reduces to solution of a the complex-analytic inverse monodromy problem. Typically, one starts with a set of monodromy matrices, finds a solution \(\Phi\) of the corresponding Riemann-Hilbert problem, and, finally, gets a solution \(\{A_j\}\) of the Schlesinger system. In particular, a class of Riemann-Hilbert problems whose monodromy groups are subgroups of the torus normalizer was solved in [29]; this allowed to find a class of solutions of the Schlesinger system associated to the Hurwitz spaces. Another class of solutions of the Schlesinger system related to the Hurwitz spaces was discussed in [10].

The Riemann-Hilbert problems of some special type and the corresponding Schlesinger system play an important role in the theory of Frobenius manifolds [12,13]. In this context the corresponding
monodromy groups provide a way of classification of Frobenius manifolds; corresponding Schlesinger systems are equivalent to equations for rotation coefficients of the Darboux-Egoroff metric corresponding to the Frobenius manifold.

To each Frobenius manifold one can naturally associate two systems of linear differential equations: a Fuchsian system and a non-Fuchsian one. In the case of the non-Fuchsian system the coefficients have both first and second order poles. These systems are related by a formal Laplace transform. For the class of Frobenius manifolds associated to the Hurwitz spaces (Hurwitz Frobenius manifolds), the non-Fuchsian systems were recently solved in [38] (although many essential elements of this construction were already given by Dubrovin in [12, 13]); the corresponding Stokes and monodromy matrices were also computed in [38]. In principle, one can apply the formal Laplace transform to the solution from [38] and get solutions to the corresponding Fuchsian systems, however, this does not give a satisfactory final result due to a non-trivial superposition of various Laplace transforms.

The goal of this paper is to present a different approach (not involving the Laplace transform of solutions to the dual problem) to constructing solutions to the Fuchsian Riemann-Hilbert problems corresponding to the Hurwitz Frobenius manifolds and study the monodromy group.

The coefficients of the system of Fuchsian linear ODE’s with meromorphic coefficients corresponding to a given Frobenius manifold are written in terms of rotation coefficients $\Gamma_{ij}$ of the Darboux-Egoroff metric on the manifold. In the context of Frobenius manifolds the number of singularities $N$ in (1.1) is the dimension of the Frobenius manifold; $\{\lambda_i\}_{i=1}^{N}$ are the canonical coordinates on the Frobenius manifold. In the case under consideration, the number of singularities $\lambda_j$ coincides with the matrix dimension of the system (1.1).

The residues $A_j$ are given by

$$A_j = -E_j(V + qI),$$

where $E_j = \text{diag}(0, \ldots, 1, \ldots, 0)$ is the diagonal $N \times N$ matrix with 1 on $j$th place; $q \in \mathbb{C}$ is an arbitrary constant. The matrix $V$ is defined as follows:

$$V := [\Gamma, U],$$

where $\Gamma$ is the matrix of rotation coefficients: $(\Gamma)_{jk} := \Gamma_{jk}$ if $j \neq k$ and $(\Gamma)_{jj} := 0$; $U := \text{diag}(\lambda_1, \ldots, \lambda_N)$. Thus each matrix $A_j$ in (1.1) has only one non-trivial row (the $j$th row).

Dubrovin in [12, 13] studies the linear system with $q = 1/2$. In this paper we focus on the case $q = -1/2$; the relationship between systems (1.1), (1.2) with the values of $q$ different by an integer is discussed in Remark 1 below.

In the context of the Fuchsian system (1.1), (1.2), (1.3) the solution of the Schlesinger system is given by the rotation coefficients, which were found earlier in [12, 26]. Moreover, in [27] the corresponding Jimbo-Miwa isomonodromic tau-function was explicitly computed. This tau-function turned out to be an object of fundamental importance: it appears in various contexts from the large $N$ limit of of Hermitian matrix models to the determinants of Laplacians on Riemann surfaces [30] and geometry of Hurwitz spaces [31].

However, a solution to the corresponding Fuchsian Riemann-Hilbert problem (which coincides with the solution of the Fuchsian system (1.1)) was missing so far. It is this gap which we fill in this paper: we solve this Fuchsian system, compute the monodromy matrices and describe the corresponding monodromy group. Thus, the logic of this paper is different from the logic of the paper [29], where the Riemann-Hilbert problem was solved first, and the solution of the corresponding Schlesinger system was found as a corollary.
We also discuss the transformation of the solution $\Phi$ under the action of the braid group on the set of singularities $\lambda_j$, by introducing the notion of the braid monodromy group. In particular, we discuss the action of the braid group on the set of monodromy matrices of the system \([11]\) following the ideas of the work by Dubrovin and Mazzocco \([15]\) where such an action was considered in the context of algebraic solutions of the P-VI equation.

Let us now describe the settings and our results in more details.

The Hurwitz space $\mathcal{H}_{g,d}(k_1, \ldots, k_m)$ is the space of equivalence classes of pairs $\mathcal{X} := (\mathcal{L}, f)$, where $\mathcal{L}$ is a compact Riemann surface of genus $g$, and $f$ is a meromorphic function of degree $d$ on $\mathcal{L}$ with simple critical points and $m$ poles of multiplicities $k_1, \ldots, k_m$ ($k_1 + \cdots + k_m = d$); two pairs $\mathcal{X}_1 := (\mathcal{L}_1, f_1)$ and $\mathcal{X}_2 := (\mathcal{L}_2, f_2)$ are equivalent if there exists a biholomorphic map $h : \mathcal{L}_1 \to \mathcal{L}_2$, such that $f_1 = f_2 \circ h$. Using the function $f$ we can realize the Riemann surface $\mathcal{L}$ as a $d$-sheeted branched covering of the Riemann sphere; the branch points of this covering are given by the critical values of the function $f$. Therefore, the Hurwitz space can be viewed as a space of branched covering of the Riemann sphere with the fixed number of sheets and the fixed branching structure.

The Frobenius structures can be defined on each space $\mathcal{H}_{g,d}(k_1, \ldots, k_m)$. The branch points, which we denote by $\lambda_1, \ldots, \lambda_N$ (the corresponding ramification points on $\mathcal{L}$ are the critical points of the function $f$; they are denoted by $P_1, \ldots, P_N$, i.e., we have $\lambda_i = f(P_i)$), can be used as local coordinates on the Hurwitz space; they also play the role of canonical coordinates on the corresponding Frobenius manifold.

The main tool in our construction is the canonical meromorphic bidifferential $W(P, Q)$ on the Riemann surface $\mathcal{L}$. To define this bidifferential we have to choose some weak marking of the Riemann surface $\mathcal{L}$, i.e., a canonical basis $(a_\alpha, b_\alpha)$ $(\alpha = 1, \ldots, g)$ of the homologies $H_1(\mathcal{L})$ with coefficients in $\mathbb{Z}$.

The bidifferential $W$ is symmetric, has a quadratic pole on the diagonal $P = Q$ with biresidue 1 and is normalised by the condition of vanishing of all periods along cycles $a_\alpha$ with respect to both $P$ and $Q$. Therefore, in fact, $W$ depends only on the choice of a Lagrangian subspace (the $a$-cycles) in $H_1(\mathcal{L})$. Therefore it is natural to introduce the space $\mathcal{H}^a_{g,d}(k_1, \ldots, k_m)$ which is the space of pairs $(\mathcal{X}, \{a\})$, where $\mathcal{X} = (\mathcal{L}, f) \in \mathcal{H}_{g,d}(k_1, \ldots, k_m)$ and $\{a\}$ is a Lagrangian subspace of $H_1(\mathcal{L})$.

The rotation coefficients (the matrix $\Gamma$ in \([13]\)) of the Frobenius structures on the Hurwitz spaces are written in terms of the bidifferential $W$ \([26]\):

$$\Gamma_{jk} = \frac{1}{2} W(P_j, P_k) := \frac{1}{2} \left. dW(P, Q) \right|_{P=P_j, Q=P_k} \left( \frac{W(P, Q)}{df(P) df(Q)} \right). \tag{1.4}$$

Here $x_j(P)$ in a local parameter on $\mathcal{L}$ near the branch point $P_j$ defined by the equations $2x_j dx_j = df$ and $x_j(P_j) = 0$. These two conditions imply that $x_j(P) := \pm \sqrt{f(P) - \lambda_j}$; these local parameters near the ramification points $P_j$ are called distinguished. Different choices of the signs of $x_j(P)$ in \((1.4)\) lead to different sets of rotation coefficients. If one considers $\Gamma_{jk}^2$, the freedom of choosing different signs disappears and one can write the following invariant expression:

$$\Gamma_{jk}^2 = 2 \left. \text{Res} \right|_{P=P_j} \left. \text{Res} \right|_{Q=P_k} \frac{W^2(P, Q)}{df(P) df(Q)}.$$

For a given covering $(\mathcal{L}, f)$ and a Lagrangian subspace $\{a\}$ we therefore get $2^N$ different sets of rotation coefficients. Each set gives rise to a family of $N$ Frobenius manifolds of dimension $N$.

To construct a solution of the Fuchsian linear system \([11]\) we introduce, for any $\lambda \in \mathbb{C}$, the relative homology group $H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda))$ with coefficients in $\mathbb{Z}$. This is the homology group of the
Riemann surface $\mathcal{L}$ punctured at the poles of the function $f$ relative to the set of points (the number of these points equals $d$ unless $\lambda$ coincides with a branch point or $\infty$) on $\mathcal{L}$ where the value of $f$ equals $\lambda$. The dimension of this relative homology space equals $N = 2g + d + m - 2$, where $m$ is the number of poles of the function $f$, i.e., the number of points in the set $f^{-1}(\infty)$.

Our first main result is that for any contour $s \in H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda))$ the vector function with the components

$$\Phi_j^{(s)}(\lambda) := \lambda \int_s W(P, P_j) - \int_s f(P)W(P, P_j),$$

where $j = 1, \ldots, N$, satisfies the linear system (1.1)-(1.3) with $q = -1/2$. In (1.5)

$$W(P, P_j) := \frac{W(P, Q)}{dx_j(Q)} \bigg|_{Q = P_j},$$

and the signs of the distinguished local parameters $x_j(Q) := \sqrt{f(Q) - \lambda_j}$ in (1.6) have to be chosen in the same way as in (1.4). The square, $W^2(P, P_j)$, is defined by the formula

$$W^2(P, P_j) = 2 \text{Res}_{Q = P_j} \left\{ \frac{W^2(P, Q)}{df(Q)} \right\}.$$

Our second main result is that choosing $s$ to run through a basis in $H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda))$, we get the complete set of $2g + d + m - 2$ independent solutions to (1.1); the proof of this independence is a tedious exercise involving analysis of the behaviour of the bidifferential $W(P, Q)$ at the boundary of the Hurwitz space.

Let us choose a neighbourhood $D$ of a point $\lambda_0 \in \mathbb{C}$ which contains no branch points $\lambda_k$.

A set of basis contours in $H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda))$ can be chosen as follows: a canonical basis of $2g$ cycles on $\mathcal{L}$ (this canonical basis does not necessarily coincide with the set of cycles on $\mathcal{L}$ which enter the definition of the bidifferential $W$); small contours around $m - 1$ points which can be arbitrarily chosen from the set $f^{-1}(\infty)$ consisting of $m$ points. The remaining $d - 1$ contours can be taken to connect pairwise the $d$ points from $f^{-1}(\lambda)$; for the linear independence of these contours one has to require connectedness of the graph whose edges are given by these contours and the vertices are the $d$ points from $f^{-1}(\lambda)$. The bases of cycles in the spaces $H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda))$ for any two values of $\lambda \in D$ can be smoothly deformed one into another on the Riemann surface $\mathcal{L}$. In this way we get a non-degenerate matrix-valued function $\Phi(\lambda)$ solving (1.1), which is analytic for $\lambda \in D$.

The function $\Phi = \Phi(\{\lambda_i\}; \lambda)$ depends on (i) the choice of a Lagrangian subspace in $H_1(\mathcal{L})$ generated by the $a$-cycles (which enters the definition of the canonical bidifferential $W$); (ii) the choice of a basis $s_1, \ldots, s_N$ of the relative homologies $H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda_0))$, where $\lambda_0 \in \mathbb{C} \setminus \{\lambda_k\}$ is some base point, and (iii) the choice of the signs of the distinguished local parameters $x_j$ near the ramification points.

If one preserves the integration contours $s_1, \ldots, s_N$, but changes the Lagrangian subspace $\{a\}$ used for normalization of $W$, the new function $\Phi$ turns out to be related to the old one by a Schlesinger transformation (multiplication of $\Phi$ from the left by a rational function of $\lambda$ of a special form), which we find explicitly. Therefore, a change of normalization of $W$ does not influence the monodromy matrices of the function $\Phi$ (i.e. the new and the old functions have the same set of monodromies although they satisfy the linear system (1.1) with different coefficients).

If, on the other hand, we preserve the normalization of $W$ but change the set of the integration contours $s_1, \ldots, s_N$, the new function $\Phi$ can be obtained from the old one by multiplication from the
right with some constant matrix: this corresponds to a linear transformation in the space of solutions of the linear system (1.1).

Finally, if one changes the sign of some of distinguished local parameters \( x_j \): \( x_j \rightarrow \epsilon_j x_j \) with \( \epsilon_j^2 = 1 \), the new function \( \Phi \) differs from the old one by multiplication from the left by the matrix \( \text{diag}(\epsilon_1, \ldots, \epsilon_N) \).

Let us discuss these two monodromy groups in more detail.

A solution to the system (1.1) is non-singlevalued in the complex plane. Upon analytical continuation with respect to \( \lambda \in \mathbb{C} \) along the generators of the fundamental group \( \pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}) \), the function \( \Phi \) is multiplied from the right by monodromy matrices \( M_k \), \( k = 1, \ldots, N, \infty \).

Since the only non-linear dependence on \( \lambda \) of our solution comes from the \( \lambda \)-dependence of the contours of integration, the monodromy matrices describe the transformation of a chosen basis in \( H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda)) \) under the natural action of an element of \( \pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda) \). Therefore all entries of the monodromy matrices are integer numbers.

If a basis in \( H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda)) \) is chosen as described above, the monodromy matrices possess the following structure:

\[
M_k = \begin{pmatrix}
I & S_k \\
0 & \Sigma_k
\end{pmatrix},
\]

(1.7)

where \( \Sigma_k \) are square \( (d-1) \times (d-1) \) matrices; they generate a subgroup of \( GL(d-1, \mathbb{Z}) \) given by the image in \( GL(d-1, \mathbb{Z}) \) of the monodromy group of the covering \( \mathcal{L} \) under a group homomorphism. The unit matrices in the upper diagonal block are of the size \( (2g+m-1) \times (2g+m-1) \); the matrices \( S_k \) of the size \( (2g+m-1) \times (d-1) \) depend on the choice of a basis in \( H_1(\mathcal{L} \setminus f^{-1}(\infty)) \). However, the change of a basis in \( H_1(\mathcal{L} \setminus f^{-1}(\infty)) \) results in a simultaneous conjugation of all monodromy matrices \( M_k \) by the same matrix; thus the monodromy group is in fact independent of the choice of the basis in \( H_1(\mathcal{L} \setminus f^{-1}(\infty)) \).

The monodromy group formed by the matrices (1.7) can be described as a semidirect product of the free group \( \mathbb{Z}^{(2g+d-1) \times (d-1)} \) and the symmetric group \( S_d \), the monodromy group of the covering. This group coincides with the Weyl group of the algebra of formal power series in \( 2g + d - 1 \) variables with coefficients in \( A_{d-1} \).

A Schlesinger system corresponding to a block-diagonal structure of monodromy matrices as in (1.7), was called reducible in [15]: this means that its solution can be expressed in terms of solutions of two Schlesinger systems of lower dimension \((d-1) \times (d-1)\) and \((2g+m-1) \times (2g+m-1)\) in our case).

The second type of monodromy transformation is the transformation of the solution (1.5) under analytical continuation with respect to the arguments \( \lambda_k \) along the generators of the fundamental group of the covering \( \hat{\mathcal{H}}_{g,d} \) of the Hurwitz space \( \mathcal{H}_{g,d} \). This fundamental group is a subgroup
of the plane braid group on $N$ strands acting on the set of local coordinates $\{\lambda_k\}_{k=1}^N$ on the Hurwitz space. Such an analytical continuation also results in the multiplication of the solution $\Phi$ from the right by some monodromy matrices with integer entries. We called the arising group of transformations the braid monodromy group of the solution to the Fuchsian system.

We notice that the action of the braid group on solutions to the Schlesinger system was used to classify the algebraic solutions of the Painlevé-IV equation in [14]. In the context of Knizhnik-Zamolodchikov equations (which can be considered as a special case of equations of isomonodromic deformations [36, 22]), the braid monodromies were studied starting from Drinfeld’s paper [11]; these monodromies play a fundamental role in the theory of quantum groups [23].

The action of the braid group on the coverings with $\mathbb{Z}_d$ symmetry was recently studied in [32].

The central object associated to any Riemann-Hilbert problem and the corresponding equations of isomonodromic deformations (the Schlesinger system) is the isomonodromic Jimbo-Miwa tau-function, a function of $\{\lambda_k\}$. The divisor of zeros of the tau-function consists of whose configurations of poles $\{\lambda_k\}$ where the Riemann-Hilbert problem loses its solvability (see [7]). In the context of the Frobenius manifold structures on Hurwitz spaces, the tau-function determines the $G$-function of the Frobenius manifold, which is the genus one free energy of the corresponding topological field theory. The isomonodromic tau-function associated to the solutions (1.2), (1.3), (1.4) of the Schlesinger system coincides with the so-called Bergman tau-function on the Hurwitz space [28]. The Bergman tau-function plays a key role in the computation of the determinant of the Laplacian in flat metrics on Riemann surfaces [30] and of the genus one free energy in the Hermitian two-matrix models [18]. In [31] it was constructed a line bundle on compactified Hurwitz spaces (spaces of admissible covers proposed by Harris and Mumford) whose holomorphic section is given by the Bergman tau-function; this line bundle is closely related to the Hodge line bundle on the moduli space of Riemann surfaces.

The paper is organized as follows. Section 2 contains a few basic facts about the Fuchsian and non-Fuchsian Riemann-Hilbert problems appearing in the theory of Frobenius manifolds. In Section 3 we construct a solution to the Fuchsian system and discuss its dependence on the choice of normalization for the main building block of the solution, the bidifferential $W$. In Section 4 we describe the monodromy group of the solution. In Section 5 we discuss the braid group action on the constructed solution, i.e., the behaviour of the solution under the analytical continuation along nontrivial loops in the Hurwitz space; we compute the generators of the braid monodromy group for the case of Hurwitz space of two-fold genus one coverings. Technical details of the computation of monodromy matrices and of the proof of the non-degeneracy of our solution are given in the Appendices A and B, respectively.

2 The Fuchsian Riemann-Hilbert problem in Frobenius manifolds theory

For the reader’s convenience and to set up the notations we shall review here the connections between solutions to systems of linear differential equations with meromorphic coefficients, matrix Riemann-Hilbert (inverse monodromy) problems, and Frobenius manifolds.

Consider a matrix linear differential equation (1.1); depending on the context we shall understand $\Phi$ as either a vector solution to this equation, or a square $N \times N$ matrix of linearly independent vector solutions to this equation. Generically, a solution to equation (1.1) has non-trivial monodromy under the analytical continuation around singularities $\{\lambda_i\}$ and around the point $\lambda = \infty$. Let us choose a set
of generators $\gamma_1, \ldots, \gamma_N, \gamma_\infty$ of the fundamental group of the punctured sphere $\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}$ such that each generator $\gamma_j$ encloses only the point $\lambda_j$, the generator $\gamma_\infty$ goes around the point at infinity, and the following relation is fulfilled:

$$\gamma_1 \cdots \gamma_N \gamma_\infty = id.$$  \hfill (2.1)

Suppose that the solution $\Phi$, being analytically continued along $\gamma_j$, gains the right multiplier $M_j$ (which is called the monodromy matrix). Being analytically continued along $\gamma_\infty$, the solution $\Phi$ gains the right multiplier $M_\infty$. As a corollary of relation (2.1) the monodromy matrices satisfy the relation

$$M_\infty M_N \cdots M_1 = I,$$  \hfill (2.2)

i.e., they give an anti-representation of the fundamental group.

At the poles $\lambda_j$ of the coefficients of the system (1.1), the function $\Phi$ has regular singularities (i.e., $\Phi(\lambda)$ grows at these points not faster than some power of $\lambda - \lambda_j$). If the matrices $A_j$ are diagonalizable (this is the only case considered in this paper), the behaviour of $\Phi$ in a neighbourhood of $\lambda_i$ looks as follows:

$$\Phi(\lambda) = G(\lambda)(\lambda - \lambda_j)^T_i C_j,$$  \hfill (2.3)

where $T_i$ is a diagonal matrix, $G(\lambda) = G_j + O(\lambda - \lambda_i)$ is a function holomorphic in a neighbourhood of $\lambda_j$. If some matrix $A_j$ is non-diagonalizable, the asymptotics of $\Phi$ near $\lambda_j$ contains logarithmic terms.

The monodromy matrices can be expressed in terms of $C_j$ and $T_j$ as follows:

$$M_j = C_j^{-1} e^{2\pi i T_j} C_j.$$  \hfill (2.4)

The Riemann-Hilbert (or inverse monodromy) problem is the problem of reconstructing the function $\Phi$ knowing its monodromy matrices $\{M_j\}$ and the positions of singularities $\{\lambda_j\}$. Obviously, a solution to the Riemann-Hilbert problem is not unique: multiplying such a solution from the left with an arbitrary matrix-valued rational function of $\lambda$, we again get a solution to the same Riemann-Hilbert problem. On the other hand, assuming that $\Phi$ has at $\{\lambda_j\}$ regular singularities of the form (2.3) with the given $\{T_j, C_j\}$, and has no other singularities (including zeros of $\det \Phi$), the solution of the Riemann-Hilbert problem is unique.

Let us now impose the isomonodromy condition, i.e., the condition of independence of the monodromy data $\{T_j, C_j\}$ of the positions of singularities $\{\lambda_j\}$. The isomonodromy condition implies a system of differential equations, called the Schlesinger equations, for the residues $A_j$ as functions of $\{\lambda_j\}$. The Schlesinger equations of a special type together with the corresponding Riemann-Hilbert problem play a significant role in the theory of Frobenius manifolds.

We shall now briefly outline the way the equations of the type (1.1) appear in the Frobenius manifold theory. We skip the complete description of the notion of a Frobenius manifold and associated objects, referring the reader to [12, 13]. We recall only that to each Frobenius manifold one can associate a Darboux-Egoroff (i.e., diagonal flat potential) metric. The poles $\lambda_j$, $j = 1, \ldots, N$, of the coefficients in (1.1) coincide with the canonical coordinates on the Frobenius manifold. The following two differential operators are also associated to a Frobenius manifold structure: $e = \sum_{j=1}^N \frac{\partial}{\partial \lambda_j}$, called the unit vector field, and $E = \sum_{j=1}^N \lambda_j \frac{\partial}{\partial \lambda_j}$, called the Euler vector field.

For the Darboux-Egoroff metrics appearing in the theory of Frobenius manifolds the rotation coefficients $\Gamma_{ij}$ satisfy the following system of equations:

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \Gamma_{ik} \Gamma_{jk},$$  \hfill (2.5)
where all \( i, j, k \) are distinct, and

\[
e(\Gamma_{ij}) = 0, \quad E(\Gamma_{ij}) = -\Gamma_{ij}.
\] (2.6)

The non-linear system (2.5), (2.6) is the compatibility condition for the following system of linear differential equations [12, 13]:

\[
d\Phi/d\lambda = - \sum_{j=1}^{N} E_j(V + qI) \lambda - \lambda_j \Phi,
\] (2.7)

\[
d\Phi/d\lambda_j = \left( \frac{E_j(V + qI)}{\lambda - \lambda_j} + [\Gamma, E_j] \right) \Phi,
\] (2.8)

where \( \Phi \) is an \( N \times N \) matrix-valued function of \( \lambda \) and \( \{\lambda_j\} \); \( q \in \mathbb{C} \) is an arbitrary constant; matrices \( V, \Gamma \) and \( E_j \) are defined after (1.1).

The system (2.8) provides the isomonodromy condition for the Fuchsian system (2.7).

In this way, a family (2.7), (2.8) of isomonodromic linear systems and the corresponding Riemann-Hilbert problems are associated to any semisimple Frobenius manifold.

The Fuchsian linear system introduced in the original papers [12, 13] corresponds to the value \( q = 1/2 \). In this paper, we shall study the case \( q = -1/2 \); below (see Remark 1) we discuss the relationship between linear systems (2.7), (2.8) with the values of \( q \) which differ by an integer.

In the sequel we shall use the following convenient alternative formulation of the linear system (2.7), (2.8).

**Proposition 1** A vector \( \Phi := (\varphi_1, \ldots, \varphi_N)^T \) satisfies the linear system (2.7), (2.8) if and only if the following equations are fulfilled

\[
\lambda \frac{\partial \varphi_j}{\partial \lambda} + E(\varphi_j) = -q \varphi_j
\] (2.9)

\[
\frac{\partial \varphi_j}{\partial \lambda} + e(\varphi_j) = 0
\] (2.10)

\[
\frac{\partial \varphi_j}{\partial \lambda_k} = \Gamma_{jk} \varphi_k, \quad j \neq k.
\] (2.11)

**Proof.** Equation (2.7) for the vector \( (\varphi_1, \ldots, \varphi_N)^T \) reads in the components:

\[
\frac{\partial \varphi_j}{\partial \lambda} = - \frac{1}{\lambda - \lambda_j} \left( q \varphi_j + \sum_{k=1, k\neq j}^{N} \Gamma_{kj}(\lambda_k - \lambda_j)\varphi_k \right).
\] (2.12)

Similarly, equation (2.8) for the vector \( (\varphi_1, \ldots, \varphi_N)^T \) is equivalent to

\[
\frac{\partial \varphi_j}{\partial \lambda_k} = \Gamma_{jk} \varphi_k, \quad j \neq k,
\] (2.13)

\[
\frac{\partial \varphi_j}{\partial \lambda_j} = \frac{1}{\lambda - \lambda_j} \left( q \varphi_j + \sum_{k=1, k\neq j}^{N} \Gamma_{kj}(\lambda_k - \lambda_j)\varphi_k \right) - \sum_{k=1, k\neq j}^{N} \Gamma_{kj} \varphi_k.
\]

The latter equation rewrites due to (2.12) as

\[
\frac{\partial \varphi_j}{\partial \lambda_j} = -\frac{\partial \varphi_j}{\partial \lambda} - \sum_{k=1, k\neq j}^{N} \Gamma_{kj} \varphi_k.
\]
which, by virtue of (2.13), coincides with (2.10).

We thus need to show the equivalence of equations (2.9) and (2.12) provided (2.10) and (2.11) hold. Using (2.11), we rewrite (2.12) as follows:

\[
\frac{\partial \varphi_j}{\partial \lambda} = -\frac{1}{\lambda} \lambda_j \left( q \varphi_j + \sum_{k=1, k \neq j}^{N} (\lambda_k - \lambda_j) \frac{\partial \varphi_j}{\partial \lambda} \right).
\]

Adding and subtracting \(\lambda_j \frac{\partial \varphi_j}{\partial \lambda}\) in the right hand side and using the unit and Euler vector field s, we obtain

\[
(\lambda - \lambda_j) \frac{\partial \varphi_j}{\partial \lambda} = -q \varphi_j - e(\varphi_j) + \lambda_j e(\varphi_j) = 0.
\]  

(2.14)

Plugging equation (2.10) into the above relation (2.14), we obtain (2.9).

**Remark 1** Using Proposition 1 we can easily see that the solutions to the linear systems (2.7), (2.8) corresponding to values \(q\) and \(q + 1\) are related by differentiation in \(\lambda\). Namely, let us indicate explicitly the dependence of a solution to the system (2.7), (2.8) on \(q\), i.e., we denote \(\Phi\) by \(\Phi^q\). Then

\[
\Phi^{q+1} = \frac{\partial \Phi^q}{\partial \lambda} \equiv A^q(\lambda) \Phi^q(\lambda),
\]

(2.15)

where \(A^q(\lambda) = -\sum_{i=1}^{N} \frac{E_i(V + qI)}{\lambda - \lambda_i}\) is the matrix of coefficients in (2.7).

In this paper we find a complete system of linearly independent solutions to the system (2.7), (2.8) for the case \(q = -1/2\). Several columns of our solution \(\Phi\) turn out to be independent of \(\lambda\), therefore formula (2.15) cannot be used to generate fundamental solutions to the system with \(q = -1/2 + n\) for integer \(n \geq 1\). However, from our solution for \(q = -1/2\) we can obtain the complete system of solutions for any negative half-integer value of \(q\) (see also Remark 3 below).

**Remark 2** The same system of equations (2.5), (2.6) describes isomonodromic deformations of the non-Fuchsian equation

\[
\frac{d\Psi}{dz} = (U + \frac{1}{z} V) \Psi.
\]

(2.16)

A solution \(\Psi\) to the system (2.16) has an irregular singularity of Poincaré rank 1 at \(z = \infty\), and a regular singularity at the origin. Solutions to the Fuchsian system (2.7) and the non-Fuchsian system (2.16) are related by a formal Laplace transform (see [12], p. 87, (3.149)).

## 3 Solution to the Fuchsian system associated to the Hurwitz Frobenius manifolds

### 3.1 Preliminaries

Let \(\mathcal{L}\) be a Riemann surface of genus \(g\) and \(f\) be a meromorphic function on \(\mathcal{L}\) of degree \(d\). Let us fix the degrees of the poles of \(f\) to be \(k_1, \ldots, k_m\) \((k_1 + \cdots + k_m = d)\), and assume that all finite critical points of the function \(f\) (i.e., zeros of \(df\)) are simple; we denote them by \(P_1, \ldots, P_N\), where, according to the Riemann-Hurwitz formula, \(N = 2g + d + m - 2\). We denote by \(\mathcal{H}_{g,d}(k_1, \ldots, k_m)\) the Hurwitz space, i.e., the space of equivalence classes of pairs (or branched coverings) \(\mathcal{X} := (\mathcal{L}, f)\) (two coverings \(\mathcal{X}_1 := (\mathcal{L}_1, f_1)\) and \(\mathcal{X}_2 := (\mathcal{L}_2, f_2)\) are called equivalent if there exists a biholomorphic isomorphism
h : \mathcal{L}_1 \to \mathcal{L}_2 such that f_1 = f_2 \circ h). The critical values of the function f, i.e., the values \( \lambda_k := f(P_k) \) with \( k = 1, \ldots, N \), can be chosen to be the local coordinates on this Hurwitz space.

The branched covering \( \mathcal{X} = (\mathcal{L}, f) \) is a \( d \)-sheeted covering of \( \mathbb{CP}^1 \) ramified at the points \( P_1, \ldots, P_N \) as well as at those poles of \( f \) whose degrees are higher than 1. The critical values \( \{ \lambda_k \} \) are the finite branch points of the covering \( \mathcal{X} \). In a neighbourhood of the ramification point \( P_j \) we introduce a local parameter \( x_j(P) \) (called “distinguished” \([39]\)) satisfying equations

\[
df = 2x_j dx_j , \quad x_j(P_j) = 0 .
\]

This differential equation has two solutions: \( x_j(P) = \pm \sqrt{f(P) - \lambda_j} \). Therefore, for each \( j \) we have two possible choices (which differ by a sign) of the distinguished local parameter. Altogether we get \( 2^N \) sets of distinguished local parameters.

Let us introduce the canonical meromorphic bidifferential \( W(P,Q) \) on \( \mathcal{L} : P,Q \in \mathcal{L} \). This bidifferential is symmetric; it has a quadratic pole on the diagonal \( P = Q \) with the singular part given by \( dx(P)dx(Q)(x(P) - x(Q))^{-2} \) in any local parameter \( x \), and is normalized by the requirement that all of its \( a \)-periods with respect to some symplectic basis \( (a_\alpha, b_\alpha) \) in \( H_1(\mathcal{L}) \) vanish. Let us also introduce the canonical basis of holomorphic differentials \( w_1, \ldots, w_g \) on \( \mathcal{L} \) normalized by \( f_{a_\alpha} w_\beta = \delta_{\alpha\beta} \), where \( \delta_{\alpha\beta} \) is the Kronecker symbol and \( \alpha, \beta = 1, \ldots, g \). Integrals of these differentials over the cycles \( b_\alpha \) give the Riemann matrix \( \mathbb{B} \) of the surface: \( \mathbb{B}_{\alpha\beta} = \oint_{b_\alpha} w_\beta \).

We are going to use the following Rauch variational formulas, which describe the dependence of \( w_\alpha \), \( W \) and \( \mathbb{B} \) on the branch points \( \{ \lambda_k \} \) (see \([33, 28]\)):

\[
\frac{d}{d\lambda_j} \{ \mathbb{B}_{\alpha\beta} \} = 2\pi i \text{Res}_{Q=P_j} \left\{ \frac{w_\alpha(Q)w_\beta(Q)}{df(Q)} \right\} ; \tag{3.1}
\]

\[
\frac{d}{d\lambda_j} \big|_{f(P)} \{ w_\alpha(P) \} = \text{Res}_{Q=P_j} \left\{ \frac{w_\alpha(Q)W(Q,P_j)}{df(Q)} \right\} ; \tag{3.2}
\]

\[
\frac{d}{d\lambda_j} \big|_{f(P),f(Q)} \{ W(P,Q) \} = \text{Res}_{R=P_j} \left\{ \frac{W(P,R)W(Q,R)}{df(R)} \right\} . \tag{3.3}
\]

Here the derivative with respect to \( \lambda_k \) is taken keeping the projections \( f(P) \) and \( f(Q) \) of the points \( P \) and \( Q \) to \( \mathbb{CP}^1 \) constant.

Notice that the variational formulas for normalized holomorphic differentials \([3.2]\) can be alternatively stated as horizontality of the the column vector \( (w_1(P), \ldots, w_g(P))^T \) with respect to the Gauss-Manin connection on the Hurwitz space

\[
d = \sum_{j=1}^{N} \Theta_j d\lambda_j , \tag{3.4}
\]

where the connection coefficients \( \Theta_j \) are the \( g \times g \) diagonal matrices with

\[
(\Theta_j)_{\alpha\alpha} := \text{Res}_{Q=P_j} \left\{ \frac{w_\alpha(Q)W(P,Q)}{df(Q)w_\alpha(P)} \right\} .
\]

The formulas \([3.1] - [3.3]\) can be alternatively rewritten in the following less invariant form which we are going to use below:

\[
\frac{d}{d\lambda_j} \{ \mathbb{B}_{\alpha\beta} \} = \pi i w_\alpha(P_j)w_\beta(P_j) ; \tag{3.5}
\]
Let us fix some $\lambda \in \mathbb{C}P^1$ which does not coincide with any of $\lambda_j$, i.e., such that its pre-image $f^{-1}(\lambda)$ consists of $d$ different points $\lambda^{(k)}$, $k = 1, \ldots, d$. Let us also enumerate in some way the points of $f^{-1}(\infty)$, which we denote by $\infty^{(s)}$, $s = 1, \ldots, m$ (if some of $\infty^{(s)}$ are ramification points then $m < d$).

Let us introduce the homology group $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$, with coefficients in $\mathbb{Z}$, of the Riemann surface $\mathcal{L}$ punctured at $m$ points $\infty^{(s)}$, $s = 1, \ldots, m$, relative to the set $f^{-1}(\lambda)$ of $d$ points $\lambda^{(k)}$, $k = 1, \ldots, d$. The dimension of $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$ is $2g + d + m - 2$. We notice that this dimension equals $N$, the number of the branch points $\{\lambda_j\}$. The set of basis contours $s_k$, $k = 1, \ldots, 2g + d + m - 2$ in $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$ can be chosen as follows:

$$s_{2\alpha - 1} := a_\alpha, \quad s_{2\alpha} := b_\alpha, \quad \alpha = 1, \ldots, g,$$

where $(a_\alpha, b_\alpha)$ is a canonical basis of cycles in the homology space $H_1(\mathcal{L}, \mathbb{Z})$;

$$s_{2g + s} := l_s, \quad s = 1, \ldots, m - 1,$$

where $l_s$ is the closed contour encircling $\infty^{(s)}$ in the positive direction (in $H_1(\mathcal{L}, \mathbb{Z})$ the contour $l_s$ is trivial);

$$s_{2g + m - 1 + n} := \gamma_{n, n + 1}(\lambda), \quad n = 1, \ldots, d - 1,$$

where

$$\frac{d}{d\lambda_j} f(P) \{w_\alpha(P)\} = \frac{1}{2} w_\alpha(P_j) W(P, P_j); \quad (3.6)$$

$$\frac{d}{d\lambda_j} f(P, f(Q)) \{W(P, Q)\} = \frac{1}{2} W(P, P_j) W(Q, P_j), \quad (3.7)$$

and $x_j(Q)$ is one of the possible sets of distinguished local parameters.

For the squares, $w_\alpha(P_j)^2$ and $W^2(P, P_j)$, we have the following invariant expressions:

$$w_\alpha^2(P_j) = 2 \text{Res} \left|_{P=P_j} \left\{ \frac{w_\alpha^2(P)}{df(P)} \right\} \right. , \quad W^2(P, P_j) = 2 \text{Res} \left|_{Q=P_j} \left\{ \frac{W^2(P, Q)}{df(Q)} \right\} \right. . \quad (3.9)$$

In the next section we are going to solve the linear system (2.7), (2.8), where the rotation coefficients are given by (1.4) [12, 28]: i.e.,

$$\Gamma_{jk} = \frac{1}{2} W(P_j, P_k) := \frac{1}{2} \left. \frac{W(P, Q)}{dx_j(Q)} \right|_{P=P_j, Q=P_k} . \quad (3.10)$$

where \{x_j\} is some set of distinguished local parameters. By changing signs of some of the $x_j$ we get $2^N$ different sets of rotation coefficients. These coefficients satisfy the system (2.5), (2.6) as a simple corollary of the Rauch formulas (3.7).

The squares $\Gamma^2_{jk}$ of rotation coefficients, are defined by the following residue formulas:

$$\Gamma^2_{jk} = 2 \text{Res} \left|_{P=P_j} \right. \text{Res} \left|_{Q=P_k} \left\{ \frac{W^2(P, Q)}{df(Q)} \right\} \right. .$$

### 3.2 Construction of a solution to the Fuchsian system

Let us fix some $\lambda \in \mathbb{C}P^1$ which does not coincide with any of $\lambda_j$, i.e., such that its pre-image $f^{-1}(\lambda)$ consists of $d$ different points $\lambda^{(k)}$, $k = 1, \ldots, d$. Let us also enumerate in some way the points of $f^{-1}(\infty)$, which we denote by $\infty^{(s)}$, $s = 1, \ldots, m$ (if some of $\infty^{(s)}$ are ramification points then $m < d$).

Let us introduce the homology group $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$, with coefficients in $\mathbb{Z}$, of the Riemann surface $\mathcal{L}$ punctured at $m$ points $\infty^{(s)}$, $s = 1, \ldots, m$, relative to the set $f^{-1}(\lambda)$ of $d$ points $\lambda^{(k)}$, $k = 1, \ldots, d$. The dimension of $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$ is $2g + d + m - 2$. We notice that this dimension equals $N$, the number of the branch points $\{\lambda_j\}$. The set of basis contours $s_k$, $k = 1, \ldots, 2g + d + m - 2$ in $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$ can be chosen as follows:

$$s_{2\alpha - 1} := a_\alpha, \quad s_{2\alpha} := b_\alpha, \quad \alpha = 1, \ldots, g,$$

where $(a_\alpha, b_\alpha)$ is a canonical basis of cycles in the homology space $H_1(\mathcal{L}, \mathbb{Z})$;

$$s_{2g + s} := l_s, \quad s = 1, \ldots, m - 1,$$

where $l_s$ is the closed contour encircling $\infty^{(s)}$ in the positive direction (in $H_1(\mathcal{L}, \mathbb{Z})$ the contour $l_s$ is trivial);

$$s_{2g + m - 1 + n} := \gamma_{n, n + 1}(\lambda), \quad n = 1, \ldots, d - 1,$$
where $\gamma_{n,n+1}(\lambda)$ is some contour connecting the points $\lambda^{(n)}$ and $\lambda^{(n+1)}$ from the pre-image $f^{-1}(\lambda)$.

It is sometimes convenient to choose the symplectic basis $(a_\alpha, b_\alpha)$ in $H_1(\mathcal{L})$ which forms a part of the basis in the space of relative homologies $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$ independently of the basis $(a_\alpha, b_\alpha)$ used for normalization of the bidifferential $W$ and the holomorphic 1-forms $w_\alpha$ (see Section 3.1). That’s why we denote these two bases of $H_1(\mathcal{L})$ by different letters.

Let us consider the meromorphic differentials $W(P, P_j)$ on $\mathcal{L}$ (see (3.8)). This is the Abelian differential of the second kind, having a second order pole at $\infty$, generically, the differential $f(P)W(P, P_j)$ does not satisfy any normalization conditions.

Now we are going to construct a solution to the Fuchsian system (2.7) and isomonodromy equations (2.8) in terms of integrals of the differentials $W(P, P_j)$ and $f(P)W(P, P_j)$ over the basis (3.11)-(3.13) in the relative homology space $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$.

Consider some point $\lambda_0 \in \mathbb{C}$ which does not coincide with any of $\lambda_j$. Consider an open simply-connected neighbourhood $D \subset \mathbb{C}$ of $\lambda_0$ such that $f^{-1}(D)$ consists of $d$ connected components.

For all $\lambda \in D$ we choose the basis elements (3.11)-(3.13) of the space $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$ to be obtained by a small smooth deformation from the respective elements of $H_1(\mathcal{L} \setminus f^{-1}(\lambda_0); f^{-1}(\lambda_0))$ (this concerns in fact only the contours $\gamma_{n,n+1}(\lambda)$: we require that for all $\lambda \in D$ these contours differ from $\gamma_{n,n+1}(\lambda_0)$ only by paths connecting the endpoints $[\lambda_0^{(n)}, \lambda^{(n)}]$ and $[\lambda_0^{(n+1)}, \lambda^{(n+1)}]$ within $f^{-1}(D)$).

For any contour $s \in H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$ we introduce the column vector-function $\Phi^{(s)} j$ with values in $\mathbb{C}^N$ whose $j$th component $(j = 1, \ldots, N)$ is given by:

$$\Phi^{(s)} j(\lambda) := \lambda \int_s W(P, P_j) - \int_s f(P)W(P, P_j),$$

where $\lambda \in \mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N\}$.

Let us choose for a moment the canonical basis of cycles $(a_\alpha, b_\alpha)$, with respect to which the meromorphic bidifferential $W$ is normalized (see Section 3.1), to coincide with the canonical basis of cycles $(a_\alpha, b_\alpha)$ from the basis (3.11) in $H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda))$. Then the vectors $\Phi^{(a_\alpha)} j$, $\alpha = 1, \ldots, g$, do not depend on $\lambda$, since $a$-periods of the differentials $W(P, P_j)$ vanish:

$$\Phi^{(a_\alpha)} j(\lambda) = - \oint_{a_\alpha} f(P)W(P, P_j).$$

The vectors $\Phi^{(b_\alpha)} j$, $\alpha = 1, \ldots, g$, are linear in $\lambda$; since $b$-periods of $W$ are given by the holomorphic normalized differentials $\{w_\alpha\}$:

$$\Phi^{(b_\alpha)} j(\lambda) = 2\pi i \lambda w_\alpha(P_j) - \oint_{b_\alpha} f(P)W(P, P_j).$$

The columns corresponding to the contours $l_s$ do not depend on $\lambda$ either, since the differentials $W(P, P_j)$ are non-singular at $\infty^{(s)}$:

$$\Phi^{(l_s)} j(\lambda) = - 2\pi i \text{res}_{\lambda = \infty^{(s)}}[f(P)W(P, P_j)], \quad s = 1, \ldots, m - 1.$$
In particular, if all $\infty^{(s)}$ are not ramification points, i.e., $m = d$, the residues in (3.15) can be easily computed to give

$$\Phi^{(\delta)}_j(\lambda) = -2\pi i W(\infty^{(s)}, P_j) := -2\pi i \frac{W(Q, P_j)}{dz_s(Q)} \bigg|_{Q = \infty}, \quad s = 1, \ldots, d - 1,$$

(3.16)

where $z_s = 1/\lambda$ is the local parameter at $\infty^{(s)}$. The columns $\Phi^{(\gamma_n,n+1)(\lambda)}$ depend on $\lambda$ non-trivially through the dependence of the integration contours $\gamma_n,n+1(\lambda)$ on $\lambda$.

**Theorem 1** For any contour $s \in H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$, the vector function $\Phi^{(s)}(\lambda)$ defined by (3.14), satisfies the linear system (2.7), (2.8) with $q = -1/2$ and $\lambda \in D$.

**Proof.** We shall check that the vector $\Phi^{(s)}(\lambda) = (\Phi_1^{(s)}(\lambda), \ldots, \Phi_N^{(s)}(\lambda))^T$ satisfies the system (2.9), (2.10), (2.11) with $q = -1/2$, which is equivalent to the original system (2.7), (2.8) with the same value of the parameter $q$.

The validity of equations (2.11) is an immediate consequence of (3.10) and the Rauch variational formulas for the bidifferential $W(P, Q)$.

To verify (2.10) we lift the functions $\Phi^{(s)}_j(\lambda), \lambda \in D \subset \mathbb{CP}^1$, (3.14) to the function $\Phi^{(s)}_j(f(P))$ on the Riemann surface $\mathcal{L}$.

The equation (2.10) is an infinitesimal form of the invariance of the function $\Phi^{(s)}(f(P))$ under a simultaneous translation of all $\lambda_j$ and $\lambda = f(P)$ by a constant. Namely, consider a biholomorphic mapping of the Riemann surfaces $\mathcal{L} \to \mathcal{L}^\delta$ which acts in every sheet of $\mathcal{L}$ by sending the point $P$ with the projection $\lambda = f(P)$ to the point $P^\delta$ projecting to $\lambda^\delta := f(P^\delta) = f(P) + \delta$ on the base of the covering. The branch points $\{\lambda_j\}$ are then mapped to $\{\lambda_j + \delta\}$. Due to the invariance of the local parameters $x_j(P) = \sqrt{f(P) - \lambda_j}$ under the mapping and the invariance of the bidifferential $W$ under all biholomorphic mappings of the surfaces, the equality $W(P, P_j) = W^\delta(P^\delta, P_j^\delta)$ holds, where $W^\delta$ is the bidifferential $W$ defined on $\mathcal{L}^\delta$. Therefore, for the function $\Phi^{(s)}_j(f(P))$ we have:

$$\Phi^{(s)}_j(\delta)(f(P^\delta)) := f(P^\delta) \int_{s^\delta} W^\delta(Q, P_j^\delta) - \int_{s^\delta} f(Q)W^\delta(Q, P_j^\delta)$$

$$= (f(P) + \delta) \int_s W(Q, P_j) - \int_s (f(Q) + \delta) W(Q, P_j),$$

where the second equality is obtained by changing the variable of integration $Q \mapsto Q^\delta$ and using the invariance $W(P, P_j) = W^\delta(P^\delta, P_j^\delta)$. Differentiating the above relation with respect to $\delta$ at $\delta = 0$ we get $\partial_\lambda \Phi^{(s)}(\lambda) + e(\Phi^{(s)}_j(\lambda)) = 0$, i.e., the first equation in (2.10).

Finally, the equation (2.9) with $q = -1/2$ can be verified by considering the transformation of the function $\Phi^{(s)}(f(P))$ under the biholomorphic mapping of the Riemann surfaces $\mathcal{L} \to \mathcal{L}^\epsilon$ which maps the point $P$ with the projection $f(P)$ to the point $P^\epsilon$ belonging to the same sheet and projecting to $f(P^\epsilon) = (1 + \epsilon)f(P)$ on the base. The local parameters $x_j(P)$ get multiplied by $\sqrt{1 + \epsilon}$ and the bidifferential $W$ stays invariant, i.e., $W(P, Q) = W^\epsilon(P^\epsilon, Q^\epsilon)$. Thus for the differential $W(Q, P_j)$ we have $W^\epsilon(Q^\epsilon, P_j^\epsilon) = W(Q, P_j)/\sqrt{1 + \epsilon}$, see (3.8). Therefore, for the function $\Phi^{(s)}_j(f(P))$ (3.14) we have:

$$\Phi^{(s)}_j(\epsilon)(f(P^\epsilon)) := f(P^\epsilon) \int_{s^\epsilon} W^\epsilon(Q, P_j^\epsilon) - \int_{s^\epsilon} f(Q)W^\epsilon(Q, P_j^\epsilon)$$

$$\quad = \sqrt{1 + \epsilon} \left[ f(P) \int_s W(Q, P_j) - \int_s f(Q) W(Q, P_j) \right].$$
where the second equality is obtained by changing the variable of integration \( Q \mapsto Q^\epsilon \) and using the relation
\[
W^\epsilon(Q^\epsilon, P^j_\epsilon) = W(Q, P_j)/\sqrt{1 + \epsilon}.
\]
This implies for the function \( \Phi_j^{(s)}(\lambda(P)) \):
\[
(\Phi_j^{(s)}(f(P^\epsilon))) = \sqrt{1 + \epsilon} \Phi_j^{(s)}(f(P)).
\]
Differentiating this relation with respect to \( \epsilon \) at \( \epsilon = 0 \) we get
\[
\lambda \partial_\lambda \Phi_j^{(s)}(\lambda) + E(\Phi_j^{(s)}(\lambda)) = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \Phi_j^{(s)}(\lambda^\epsilon) = \frac{1}{2} \Phi_j^{(s)}(\lambda).
\]

**Theorem 2** The matrix \( \Phi(\lambda) \) (3.17) gives a complete set of linearly independent solutions to the Fuchsian linear system (2.7) for \( \lambda \in D \) with \( q = -1/2 \). The matrix \( \Phi(\lambda) \) also satisfies the isomonodromy deformation equations (2.8).

**Proof.** The matrix \( \Phi \) satisfies equations (2.7) and (2.8) since each of its columns satisfies these equations. The proof of linear independence of its columns is rather tedious. We postpone it to Appendix B which is entirely devoted to this proof. \( \square \)

**Remark 3** The solution (3.14) can be formally rewritten in the following form:
\[
\Phi^s(\lambda) = \int_s W(R, P_i), \tag{3.18}
\]
where \( s \) is again one of the integration contours (2.7) - (3.13) and we assume that the closed contours start and end at one of the points from the set \( f^{-1}(\lambda) \) (not necessarily the same for all contours). This can be achieved by deformation of contours. This solution satisfies our linear system with \( q = -1/2 \). Similarly to the proof of (3.18) (Theorem 1) one can prove that a solution for the system (2.7), (2.8) with \( q = -3/2 \) can be written in the form
\[
\Phi^s(\lambda) = \int_s W(R, P_i). \tag{3.19}
\]
Adding one more integration, we get a solution to the system (2.7), (2.8) with \( q = -5/2 \):
\[
\Phi^s(\lambda) = \int_s W(R, P_i), \tag{3.20}
\]
and so on.

Let us perform integration by parts in (3.19) and (3.20). Then the solutions take, respectively, the form:
\[
\Phi^s(\lambda) = \frac{1}{2} \lambda^2 \int_s W(P, P_i) - \lambda \int_s f(P)W(P, P_i) + \frac{1}{2} \int_s f^2(P)W(P, P_i). \tag{3.21}
\]
\[ \hat{\Phi}(\lambda) = \frac{1}{6} \lambda^3 \int_s W(P, P_t) - \frac{1}{2} \lambda^2 \int_s f(P) W(P, P_t) + \frac{1}{2} \lambda \int_s f^2(P) W(P, P_t) - \frac{1}{6} \int_s f^3(P) W(P, P_t). \] (3.22)

A straightforward differentiation of (3.22) with respect to \( L \) gives (3.21), and differentiation of (3.21) gives (3.14), in agreement with (2.15).

Similarly, we get solutions to systems (2.7), (2.8) for any negative half-integer value of \( q \). However, for \( q = 1/2, 3/2 \ldots \) we don’t get a complete system of solutions of (2.7), (2.8) since some of columns of (3.14) don’t depend on \( \lambda \) and turn into zero vectors after differentiation.

### 3.3 Dependence of the solution on the choice of homology basis

In this section we discuss the dependence of the solution \( \Phi \) (3.14), (3.17) on the choice of a Lagrangian subspace \( \{ a \} \) generated by the \( a \)-cycles \( a_1, \ldots, a_g \) in \( H_1(\mathcal{L}) \) with respect to which \( W(P, Q) \) is normalized, on the choice of the integration contours \( s_1, \ldots, s_N \) and on the choice of the signs of distinguished local parameters \( x_j \).

Let us denote by \( a \) and \( b \) the column vectors of basis cycles: \( a := (a_1, \ldots, a_g)^T \) and \( b := (b_1, \ldots, b_g)^T \). Consider a new symplectic basis, \((\hat{a}, \hat{b})\), in \( H_1(\mathcal{L}) \) which is related to the old one by a symplectic transformation:

\[
\begin{pmatrix}
\hat{b} \\
\hat{a}
\end{pmatrix} = \begin{pmatrix} A & B \\
C & D \end{pmatrix} \begin{pmatrix} a \\
b \end{pmatrix}.
\] (3.23)

Then the canonical bidifferentials \( \hat{W} \) and \( W \) corresponding to the bases \((\hat{a}, \hat{b})\) and \((a, b)\), respectively, are related by (see [21], p.10):

\[
\hat{W}(P, Q) = W(P, Q) - 2\pi i w^T(P)(CB + D)^{-1}Cw(Q),
\] (3.24)

where \( w \) is the vector of holomorphic differentials, \( w := (w_1, \ldots, w_g)^T \), normalized by \( \int a_\alpha w_\beta = \delta_{\alpha\beta} \), and \( B \) is the matrix of \( b \)-periods: \( B_{\alpha\beta} := \int_{a_\alpha} w_\beta \).

Let us denote by \( s \) the row vector whose components are given by the contours \( s_1, \ldots, s_N \). For another basis \( \hat{s} = (\hat{s}_1, \ldots, \hat{s}_N) \) in \( H_1(\mathcal{L}\setminus f^{-1}(\infty), f^{-1}(\lambda)) \) we have \( \hat{s} = sR \), where \( R \) is a non-degenerate \( N \times N \) matrix with integer entries.

Then we can form a matrix-function \( \hat{\Phi}(\lambda) \) defined by the formulas (3.14), (3.17) with the bidifferential \( W \) replaced by the transformed bidifferential \( \hat{W} \), with integration contours \( \{ \hat{s}_k \} \in H_1(\mathcal{L}\setminus f^{-1}(\infty), f^{-1}(\lambda)) \), and a new set of distinguished local parameters \( \hat{x}_j = \epsilon_j x_j \) with \( \epsilon_j^2 = 1 \). The function \( \hat{\Phi}(\lambda) \) solves the system (2.7), (2.8) with the matrix \( V \) built from the rotation coefficients given by the deformed bidifferential:

\[
\Gamma_{ij} = \frac{\hat{W}(P, Q)}{d\hat{x}_i(P) d\hat{x}_j(Q)} \bigg|_{P=P_i, Q=P_j}.
\]

**Theorem 3** The matrix-functions \( \hat{\Phi} \) and \( \Phi \) are related as follows:

\[
\hat{\Phi}(\lambda) = Y(1 - T(\lambda)) \Phi(\lambda) R,
\] (3.25)

where \( 1 \) denotes the \( N \times N \) identity matrix; \( T \) is a symmetric matrix with the entries:

\[
(T)_{ij} = \pi i (\lambda_j - \lambda) \sum_{\alpha, \beta = 1}^g [(CB + D)^{-1}C]_{\alpha\beta} w_\alpha(P_i) w_\beta(P_j),
\] (3.26)
where \( w_\alpha(P_j) := (w_\alpha(P)/dx_j(P))|_{P=P_j}; \) the constant matrices \( C \) and \( D \) are blocks of the symplectic transformation \( \Phi \) between the two canonical homology bases; \( R \) is the transformation matrix between the sets of new and old integration contours: \( s = sR; \) the diagonal matrix \( Y \) is formed by the factors \( \epsilon_j \), i.e., \( Y := \text{diag}(\epsilon_1, \ldots, \epsilon_N) \).

**Proof.** It is sufficient to check the statement of the theorem in three cases:

1. The symplectic matrix in \( \Phi \) is the unit matrix, all \( \epsilon_j = 1 \) (i.e. \( Y = 1 \)), while the transformation matrix \( R \) between the bases \( s \) and \( \hat{s} \) in \( H_1(L \setminus f^{-1}(\infty), f^{-1}(\lambda)) \) is non-trivial. Then \( W = \hat{W} \) and the only difference between \( \Phi \) and \( \hat{\Phi} \) is the choice of the integration contours; therefore, \( \hat{\Phi} = \Phi R \).

2. Matrix \( R \) is the unit matrix (i.e. the contours of integration \( s_j \) remain unchanged), all \( \epsilon_j = 1 \) while the symplectic transformation matrix in \( \Phi \) is non-trivial.

In this case the formula \( \text{(3.25)} \) with \( Y = R = 1 \) can be proved by a direct computation as follows. Relation \( \text{(3.25)} \) is equivalent to

\[
\lambda \int_{s_k} \hat{W}(P, P_1) - \int_{s_k} f(P)\hat{W}(P, P_1) = \sum_{j=1}^N (1 - T)_{ij} \left( \lambda \int_{s_k} W(P, P_j) - \int_{s_k} f(P)W(P, P_j) \right),
\]

for any \( i = 1, \ldots, N \) and \( k = 1, \ldots, N \). Using the definition \( \text{(3.26)} \) of the matrix \( T \) and the Rauch variational formula \( \text{(3.6)} \) for the holomorphic differentials \( w_\alpha \), we obtain:

\[
\sum_{j=1}^N (1 - T)_{ij} \int_{s_k} f(P)W(P, P_j) = \int_{s_k} f(P)W(P, P_i)
\]

\[
- 2\pi i \sum_{\alpha, \beta = 1}^g [(C\alpha + D)^{-1}]_{\alpha\beta} w_\alpha(P) \left[ E \left( \int_{s_k} f(P)w_\beta(P) \right) - \lambda e \left( \int_{s_k} f(P)w_\beta(P) \right) \right], \tag{3.28}
\]

where \( E = \sum_{j=1}^N \lambda_j \partial_{\lambda_j} \) is the Euler vector field and \( e = \sum_{j=1}^N \partial_{\lambda_j} \) is the unit vector field on the Frobenius manifold. We compute the action of these fields on our integrals using the invariance of the holomorphic differentials \( w_\alpha \) with respect to the biholomorphic mappings of Riemann surfaces \( L \rightarrow L^\epsilon \) and \( L \rightarrow L^\delta \) from the proof of Theorem [1].

\[
E \left( \int_{s_k} f(P)w_\beta(P) \right) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \int_{s_k} f(P)w_\beta(P) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \int_{s_k} f(P^\epsilon)w_\beta(P^\epsilon)
\]

\[
= \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \int_{s_k} f(P)(1 + \epsilon)w_\beta(P) = \int_{s_k} f(P)w_\beta(P). \tag{3.29}
\]

\[
e \left( \int_{s_k} f(P)w_\beta(P) \right) = \frac{d}{d\delta} \bigg|_{\delta=0} \int_{s_k} f(P)w_\beta(P) = \frac{d}{d\delta} \bigg|_{\delta=0} \int_{s_k} (f(P) + \delta)w_\beta(P) = \int_{s_k} w_\beta(P). \tag{3.30}
\]

To obtain the second equalities in the above lines we used the invariance \( w_\beta(P^\epsilon) = w_\beta(P) \) and \( w_\beta(P^\delta) = w_\beta(P) \) of the normalized holomorphic differentials under the biholomorphic mappings.
Similarly, for the first summand in the right hand side of (3.27) we get:

\[
\sum_{j=1}^{N} (1 - T)_{ij} \lambda \int_{s_k} W(P, P_j) = \lambda \int_{s_k} W(P, P_i) \\
- 2\pi i \lambda \sum_{\alpha, \beta=1}^{g} [(C \mathbb{B} + D)^{-1} C]_{\alpha\beta} w_{\alpha}(P_i) \left[ E \left( \int_{s_k} w_{\beta}(P) \right) - \lambda e \left( \int_{s_k} w_{\beta}(P) \right) \right] = \lambda \int_{s_k} W(P, P_i).
\]

(3.31)

Where the action of the fields \( E \) and \( e \) are computed similarly to the above:

\[
E \left( \int_{s_k} w_{\beta}(P) \right) = 0 \\
e \left( \int_{s_k} w_{\beta}(P) \right) = 0.
\]

Thus, plugging relations (3.28), (3.29), (3.30) and (3.31) into (3.27) and using the expression (3.24) for the transformed bidifferential \( W \), we get (3.27).

3. The integration contours \( s_k \), as well as \( W(P, Q) \), remain unchanged, but some of distinguished local parameters change sign, i.e. \( Y \neq 1 \). Then the differentials \( W(P, P_j) \) change to \( \epsilon_j W(P, P_j) \), which implies the transformation \( \Phi \rightarrow Y \Phi \) of the matrix \( \Phi \).

\[\square\]

Note that we can rewrite the transformation (3.25) in the form \( \hat{\Phi}(\lambda) = Y \left( 1 + T_1 - \lambda T_2 \right) \Phi(\lambda) R \), where the matrices \( T_1 \) and \( T_2 \) do not depend on \( \lambda \).

Therefore, in the case \( R = 1 \) (3.25) is nothing but a special type of the Schlesiger transformation (multiplication from the left by a rational function); this transformation does not change the monodromy matrices of \( \Phi \). In the case of a non-trivial matrix \( R \) the monodromy matrices of \( \hat{\Phi} \) are obtained from monodromy matrices of \( \Phi \) via the conjugation by \( R^{-1} \).

Let us formulate the following technical lemma:

**Lemma 1** The matrix \( 1 - T \) from Theorem 3 is non-degenerate. Its inverse is given by \( 1 + T \).

**Proof.** The statement of the lemma follows from the relation \( T^2 = 0 \), which holds due to the following identity:

\[
\sum_{j=1}^{N} (\lambda_j - \lambda) w_{\alpha}(P_j) w_{\beta}(P_j) = 0 \quad \text{for any} \quad \alpha, \beta = 1, \ldots, g.
\]

(3.32)

To prove (3.32) we notice that by virtue of the Rauch variational formulas (3.3) for the Riemann matrix, the left hand side of (3.32) is a multiple of the quantity \( E(\mathbb{B}_{\alpha\beta}) - \lambda e(\mathbb{B}_{\alpha\beta}) \). The constancy of the Riemann matrix \( \mathbb{B} \) along the Euler and the unit vector fields, \( E(\mathbb{B}_{\alpha\beta}) = 0 \) and \( e(\mathbb{B}_{\alpha\beta}) = 0 \), is proved as in (3.31) choosing the contour of integration to be \( s_k = b_{\beta} \).

This lemma implies the following corollary of Theorem 3, which will be used in the proof of the completeness of the constructed set of solutions to the Fuchsian system (2.7), (2.8):

**Corollary 1** Assume that the matrix \( \Phi \) is a fundamental matrix of solutions to the system (2.7), (2.8) for some choice of symplectic basis \( (a_\alpha, b_\alpha) \) in \( H_1(L) \) and a basis \( \{s_j\} \) in \( H_1(L \setminus f^{-1}(\infty), f^{-1}(\lambda)) \). Then the matrix \( \hat{\Phi} \) corresponding to any other choice of the bases in these homology spaces is also a fundamental matrix of solutions. In other words, the non-degeneracy of \( \Phi \) for some \( \lambda \), implies the non-degeneracy of \( \hat{\Phi} \).
through the dependence on $\lambda$ from [37] to Frobenius structures with rotation coefficients $\Gamma$ in [37], where the corresponding deformations of Frobenius structures were built - the Frobenius $2$-degeneracy of the matrix $C$ can be characterized as a unique symmetric bidifferential with a second order pole at the diagonal $P = Q$ with biresidue $1$, normalized by the conditions:

$$\sum_{\alpha=1}^{g} C_{\beta\alpha} \oint_{b_\alpha} \tilde{W}^C(P,Q) + \sum_{\alpha=1}^{g} D_{\beta\alpha} \oint_{a_\alpha} \tilde{W}^C(P,Q) = 0,$$

the integration being done with respect to either of the arguments. (Notice that due to the non-degeneracy of the matrix $C + D$, the vanishing of the above combinations of periods of a holomorphic differential $v$, namely, $\sum_{\alpha=1}^{g} C_{\beta\alpha} v + \sum_{\alpha=1}^{g} D_{\beta\alpha} v = 0$ for all $\beta = 1, \ldots, g$ implies $v = 0$.)

The variational formulas for $\tilde{W}^C$ have the same form as the Rauch variational formulas (3.7) for the $W$. The deformed bidifferential $\tilde{W}^C$ is also invariant with respect to biholomorphic transformations of the Riemann surface.

Thus the matrix $\hat{\Phi}^C(\lambda)$ given by (3.17), (3.14), (3.11)-(3.13) with the $W$ replaced by its deformation $\tilde{W}^C$ solves the system (2.7), (2.8) with $q = -1/2$ and the matrix $V$ built from the entries $V_{ij} = \tilde{W}^C(P_i, P_j)(\lambda_i - \lambda_j)/2$. The deformed system is related to the original one by the Schlesinger transformation of the form (3.25), (3.26) with $Y = R = 1$ and the matrices $C$ and $D$ having complex-valued entries.

If the matrix $C$ is invertible, the definition (3.33) yields the bidifferential $W_q(P,Q) = W(P,Q) - 2\pi i w^T(P)(C^2 + D)^{-1} Cw(Q)$, where $q = C^{-1}D$. This is the deformation of the bidifferential $W$ considered in [37], where the corresponding deformations of Frobenius structures were built - the Frobenius structures with rotation coefficients $\Gamma_{ij} = W_q(P_i, P_j)/2$. Apparently, one can generalize the deformations from [37] to Frobenius structures with rotation coefficients $\Gamma_{ij} = \tilde{W}^C(P_i, P_j)/2$. (Here the values of $\tilde{W}^C$ and $W_q$ at the points $\{P_i\}$ are defined similarly to (3.10).)

4 Monodromy group of the Fuchsian system

In this section we study the transformations of the solution $\Phi$ (3.17) under analytical continuation with respect to $\lambda$ along the paths from $\pi_1(\mathcal{C} \setminus \{\lambda_1, \ldots, \lambda_N, \lambda_0\})$. Since $\Phi(\lambda)$ depends non-linearly on $\lambda$ only through the dependence on $\lambda$ of the integration contours $\gamma_{n,n+1}(\lambda)$, the monodromy transformation in question is given by the corresponding tranformation of the integration contours in the space $H_1(\mathcal{C} \setminus f^{-1}(\infty), f^{-1}(\lambda))$. 

Remark 4 Note that while the transformation (3.23) of the homology basis is done by a symplectic matrix with integer entries, we can construct a bidifferential $\tilde{W}^C$ as in (3.24) with $C$ and $D$ being the corresponding blocks of a symplectic matrix with complex entries. Such a bidifferential $\tilde{W}^C$ gives also a “deformation” of the original bidifferential $W$.

Namely, let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{C})$$

and assume the matrix $C + D$ to be non-degenerate. Then the bidifferential $\tilde{W}^C(P,Q), P,Q \in \mathcal{L}$, given by

$$\tilde{W}^C(P,Q) = W(P,Q) - 2\pi i w^T(P)(C + D)^{-1} Cw(Q), \quad (3.33)$$

can be characterized as a unique symmetric bidifferential with a second order pole at the diagonal $P = Q$ with biresidue $1$, normalized by the conditions:

$$\sum_{\alpha=1}^{g} C_{\beta\alpha} \oint_{b_\alpha} \tilde{W}^C(P,Q) + \sum_{\alpha=1}^{g} D_{\beta\alpha} \oint_{a_\alpha} \tilde{W}^C(P,Q) = 0,$$
4.1 Preliminaries

For any set of \(d\) points \(Q_1, \ldots, Q_d\) on a Riemann surface \(\mathcal{L}\) one can introduce the surface braid group \(B_d(\mathcal{L}, \{Q_j\}_{j=1}^d)\) (see \[5\]; if \(\mathcal{L}\) is the complex plane, the surface braid group coincides with the Artin braid group).

For a description of the monodromy group of the Fuchsian system \(2.7\) we introduce the surface braid group \(B_d(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda_0))\). The corresponding strands end at \(d\) points from \(f^{-1}(\lambda_0)\), i.e., at \(\lambda_0^{(1)}, \ldots, \lambda_0^{(d)}\).

The lift \(f^{-1}(\gamma)\) of a path \(\gamma \in \pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda_0)\) from \(\mathbb{CP}^1\) to \(\mathcal{L} \setminus f^{-1}(\infty)\) consists of \(d\) non-intersecting (other than at the end points) paths on \(\mathcal{L}\) which start and end in the set \(\{\lambda_0^{(1)}, \ldots, \lambda_0^{(d)}\}\). Therefore, \(f^{-1}(\gamma)\) naturally defines an element of the group \(B_d(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda_0))\) (see review \[34\]). We denote this map which takes a loop \(\gamma\) to the corresponding surface braid by \(f^{-1}\). Obviously, for any two elements \(\gamma\) and \(\tilde{\gamma}\) of \(\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda_0)\) the element of the surface braid group corresponding to \(\gamma \circ \tilde{\gamma}\) coincides with that corresponding to the product \(f^{-1}(\gamma) \circ f^{-1}(\tilde{\gamma})\). Therefore, we get the following

**Proposition 2** The map \(f^{-1}\) from \(\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda_0)\) to \(B_d(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda_0))\) defined above is a group homomorphism.

There exists also a standard homomorphism from the surface braid group \(B_d(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda_0))\) to the symmetric group \(S_d\) acting on the set of \(d\) points \(\lambda_0^{(1)}, \ldots, \lambda_0^{(d)}\). The superposition of this homomorphism with the homomorphism \(f^{-1}\) from Proposition 2 gives the standard group homomorphism \(h\) from \(\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda_0)\) to the symmetric group \(S_d\); the image of \(\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda_0)\) under the homomorphism \(h\) is called the monodromy group of the covering.

Now, for any Riemann surface \(\mathcal{L}\) and a set of \(d\) points \(\{Q_n \in \mathcal{L}\}_{n=1}^d\) one can define a natural action of the surface braid group \(B_d(\mathcal{L}, \{Q_n\}_{n=1}^d)\) on the relative homology group \(H_1(\mathcal{L}, \{Q_n\}_{n=1}^d)\). Namely, on the space of absolute homologies \(H_1(\mathcal{L})\) (which is a linear subspace of \(H_1(\mathcal{L}, \{Q_n\}_{n=1}^d)\)) the group \(B_d(\mathcal{L}, \{Q_n\}_{n=1}^d)\) acts identically. On an element of \(H_1(\mathcal{L}, \{Q_n\}_{n=1}^d)\) represented by an oriented contour \(\gamma_{mn}\) which starts at the point \(Q_m\) and ends at \(Q_n\) an element \(G \in B_d(\mathcal{L}, \{Q_n\}_{n=1}^d)\) acts in the following way. The element \(G\) induces a permutation \((i_1, \ldots, i_d)\) of points \(Q_1, \ldots, Q_d\) and is defined by \(d\) oriented paths \(\{l_n\}\) on \(\mathcal{L}\); the path \(l_n\) goes from \(Q_n\) to \(Q_{i_n}\). The natural action of \(G \in B_d(\mathcal{L}, \{Q_n\}_{n=1}^d)\) on a contour \(\gamma_{mn}\) is defined by

\[
\gamma_{mn} \rightarrow \gamma_{mn} - l_m + l_n =: \gamma_{i_m i_n}.
\]

In this way to each \(G \in B_d(\mathcal{L}, \{Q_n\}_{n=1}^d)\) a linear automorphism of \(H_1(\mathcal{L}, \{Q_n\}_{n=1}^d)\) is assigned.

**Proposition 3** This map from \(B_d(\mathcal{L}, \{Q_n\}_{n=1}^d)\) to the group of linear automorphisms of \(H_1(\mathcal{L}, \{Q_n\}_{n=1}^d)\) is a group homomorphism.

The proof is geometrically obvious: it is easy to see that the action of the product of two elements of \(B_d(\mathcal{L}, \{Q_n\}_{n=1}^d)\) on \(H_1(\mathcal{L}, \{Q_n\}_{n=1}^d)\) corresponds to the superposition of the automorphisms corresponding to each of these elements.

Let us now denote by \(R\) the homomorphism from the surface braid group \(B_d(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda_0))\) to the group of linear automorphisms of the vector space \(H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda_0))\).

The superposition \(T := R \circ f^{-1}\) defines a group homomorphism from \(\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda_0)\) to \(\text{Aut}[H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda_0))]\).

The next theorem states that, essentially, the image of \(\pi_1(\mathbb{CP}^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\}, \lambda_0)\) in \(\text{Aut}[H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda_0))]\) under \(T\) coincides with the monodromy group of the Fuchsian system \((2.16)\).
Theorem 4 Consider a standard system of generators γ₁, ..., γₙ, γₙ⁺ in the fundamental group \( \pi₁(\mathbb{CP}^1 \setminus \{λ₁, ..., λₙ, ∞\}, λ₀) \) based at \( λ₀ \). Let a solution \( Φ(λ) \) to the Fuchsian system (2.7) in the neighbourhood \( D \) of a base point \( λ₀ \) be given by (3.14), (3.17), where the basis \{σₗ\} in the relative homology group \( H₁(\mathcal{L} \setminus f⁻¹(∞); f⁻¹(λ)) \) is given by (3.11), (3.12) and (3.13). Let the automorphisms \( T(γₖ) \in \text{Aut}[H₁(\mathcal{L} \setminus f⁻¹(∞); f⁻¹(λ₀))] \) (where the homomorphism \( T \) is defined before the theorem) be defined in the basis \{σₗ\} by the matrices \( Fₖ \). Then the solution \( Φ(λ) \) transforms under the analytical continuation along the path \( γₖ \) as follows: \( Φ \rightarrow Φ Mₖ \), where the monodromy matrices \( Mₖ \) are related to the matrices \( Fₖ \) by:

\[
Mₖ = (Fₖ)ᵀ, \quad k = 1, ..., N, ∞.
\]

Proof. To prove the theorem one has to remember that the neighbourhood \( D \) of \( λ₀ \) was chosen such that for \( λ \in D \) the contours \( σₗ(λ) \) can be obtained by a smooth deformation from the contours \( σₗ(λ₀) \). Then the statement of the theorem is just a corollary of the definition of the function \( Φ \) (3.14), (3.17) in terms of integrals of certain meromorphic differentials over the contours \( σₗ(λ) \), as well as of the definitions of monodromy matrices and the homomorphism \( T \).

The transposition in the relation (4.2) between the matrices \( Mₖ \) and \( Fₖ \) appears since the cycles \( σₗ \) label the columns of matrix \( Φ \). Thus the map from \( π₁(\mathbb{CP}^1 \setminus \{λ₁, ..., λₙ, ∞\}, λ₀) \) to \( GL(N, \mathbb{C}) \) given by the monodromy map is an anti-homomorphism (i.e., the monodromy matrices multiply in the order opposite to the order of multiplication of the corresponding paths in \( π₁(\mathbb{CP}^1 \setminus \{λ₁, ..., λₙ, ∞\}, λ₀) \), see (2.1), (2.2).

In our situation, when all finite branch points are simple and the covering is connected, the monodromy group of the covering \( \mathcal{L} \) (i.e., the image of \( π₁(\mathbb{CP}^1 \setminus \{λ₁, ..., λₙ, ∞\}, λ₀) \) in \( S_d \) under the homomorphism \( h \)) coincides with the whole symmetric group \( S_d \). Let us denote the permutations corresponding to the loops \( γₖ \) by \( σₗ \), i.e., \( σₗ = h(γₖ) \), \( k = 1, ..., N, ∞ \). The permutations satisfy the relation

\[
σ₁σ₂...σₙσ∞ = id.
\]

One can make the following statement about the structure of the monodromy matrices:

Theorem 5 The monodromy matrices of the function \( Φ \) defined by (3.14), (3.17) have the following block structure:

\[
Mₖ = \begin{pmatrix}
I & Sₖ \\
0 & Σₖ
\end{pmatrix},
\]

where \( I \) is the \((2g+m−1) \times (2g+m−1)\) identity matrix; \( 0 \) is the \((d−1) \times (2g+m−1)\) matrix with zero entries; \( Sₖ \) and \( Σₖ \) are matrices with integer entries of size \((2g+m−1) \times (d−1)\) and \((d−1) \times (d−1)\), respectively. Moreover, the matrix \( Σₖ \) depends only on the element \( σₗ \) of the monodromy group of the covering.

Proof. The diagonal unit block of the size \((2g+m−1) \times (2g+m−1)\) and the zero matrix in the left lower corner of \( Mₖ \) appear since the first \( 2g+m−1 \) columns of the matrix \( Φ \) are either linear functions of \( λ \) or constant with respect to \( λ \); these \( 2g+m−1 \) columns remain thus invariant under the analytical continuation of \( Φ \) along any \( γₖ \) (this can also be seen from the fact that the contours \( σₗ \), \( k = 1, ..., 2g+m−1 \), are independent of \( λ \) and, therefore, do not change under \( T(γₖ) \)). The matrices \( Sₖ \) and \( Σₖ \) define the transformation of the contours \( γₙ,n+1(λ₀) \), \( n = 1, ..., d−1 \), under the homomorphism \( T(γₖ) \). The contour \( γₙ,n+1(λ₀) \) gets mapped under such a transformation to some contour connecting the points \( λ₀^{(i_n)} \) and \( λ₀^{(i_{n+1})} \) (where \( (i₁, ..., i_d) \in S_d \) is an element \( h(γₖ) \) of the monodromy group of the covering \( \mathcal{L} \) corresponding to \( γₖ \)). This contour can be expressed in \( H₁(\mathcal{L} \setminus f⁻¹(∞); f⁻¹(λ₀)) \) as
a linear combination of the contours \( \{ \gamma_{n,n+1}(\lambda_0) \}_{n=1}^{d-1} \), basis \( a \)- and \( b \)-cycles, and the cycles around \( \infty^{(s)} \). The coefficients in front of \( \{ \gamma_{n,n+1}(\lambda_0) \} \) are given by the matrix \( \Sigma_k \); clearly, they depend only on the permutation \( h(\gamma_k) \); thus the matrices \( \Sigma_k \) are entirely determined by the monodromy group of the covering \( L \). The matrices \( S_k \), which determine the coefficients in front of the \( a \)- and \( b \)-cycles, and the cycles around \( \infty^{(s)} \), depend also on the choice of a canonical basis of cycles in \( H_1(L) \).

It is thus easy to see that under a change of the basis \((a_\alpha, b_\alpha, l_s)\) in \( H_1(L \setminus f^{-1}(\infty)) \) the matrices \( \Sigma_k \) do not change; the matrices \( S_k \) transform in an obvious way given by the next proposition. We notice also that the matrices \( \Sigma_k \) satisfy the relation
\[
\Sigma_\infty \Sigma_N \ldots \Sigma_1 = \text{id}.
\]

**Proposition 4** Let a \((2g + m - 1) \times (2g + m - 1)\) matrix \( Q \) define a transformation between a basis \((a_\alpha, b_\alpha, l_s)\) in \( H_1(L \setminus f^{-1}(\infty)) \) and a new basis \((\tilde{a}_\alpha, \tilde{b}_\alpha, \tilde{l}_s)\), i.e.,
\[
\begin{pmatrix}
a_\alpha \\
b_\alpha \\
l_s
\end{pmatrix}
= Q
\begin{pmatrix}
\tilde{a}_\alpha \\
\tilde{b}_\alpha \\
\tilde{l}_s
\end{pmatrix}.
\] (4.4)

Then the new monodromy matrices (the monodromy matrices of the solution \( \Phi \) given by the integrals \((3.14)\) over the new basis of contours) have the form \((4.3)\) with the same matrices \( \Sigma_k \) and new matrices \( \tilde{S}_k \) given by:
\[
\tilde{S}_k = Q^T S_k Q.
\] (4.5)

The proof is an immediate corollary of the definition of the matrices \( S_k \); it is also easy to observe that the simultaneous transformation \((4.5)\) of all matrices \( S_k \) preserves the relation \((2.2)\) between the monodromy matrices. Indeed, the transformed monodromy matrices \( \tilde{M}_k \) \((4.3), (4.5)\) are related to the matrices \( M_k \) by a simultaneous conjugation:
\[
\tilde{M}_k = \begin{pmatrix} Q^T & 0 \\ 0 & I \end{pmatrix} M_k \begin{pmatrix} (Q^T)^{-1} & 0 \\ 0 & I \end{pmatrix}^{-1};
\]
the corresponding solutions of the Fuchsian system are related by
\[
\tilde{\Phi} = \Phi \begin{pmatrix} (Q^T)^{-1} & 0 \\ 0 & I \end{pmatrix}^{-1}.
\] (4.6)

**Remark 5** We would like to stress that in Proposition 4 we only consider the dependence of \( \Phi \) on the change of some of the integration contours \( s_k \) in \((3.14)\); the canonical basis of cycles \((a_\alpha, b_\alpha)\) used in the definition of the bidifferential \( W \) (see Section 3.3) is assumed to remain the same. The dependence of \( \Phi \) on the choice of a basis \((a_\alpha, b_\alpha)\) (i.e., on the normalization of \( W \)) was discussed in Section 3.3.

### 4.2 Monodromy group

#### 4.2.1 Spaces of meromorphic functions with simple poles

Here we describe the group \( \mathcal{M} \) generated by the monodromy matrices computed in Appendix A.1 as a semidirect product of the free group \( \mathbb{Z}^{(2g + d - 1) \times (d - 1)} \), where \( d \) is the degree of the covering, and the symmetric group \( S_d \), the monodromy group of the covering.
Consider a Hurwitz space of coverings represented by Figure 1. Let the group \( \tilde{M} \) be generated by the monodromy matrices \( M_1, M_{2g+3}, M_{2g+5}, \ldots, M_{N-1} \), i.e.,

\[
\tilde{M} := \langle M_1, \{ M_{2g+2n-1} \}_{n=2}^{d-1} \rangle.
\]

These generators have the form (see Appendix A, (A.5) and (A.9)):

\[
M_i = \begin{pmatrix} I & 0 \\ 0 & \Sigma_i \end{pmatrix}, \quad \text{for } i = 1 \text{ or odd } i \geq 2g + 3, \tag{4.7}
\]

where the matrix \( \Sigma_i \) corresponds to the element \( \sigma_i = h(\gamma_i) \) of the monodromy group of the covering.

As is easy to see, for the coverings from Figure 1, \( \tilde{M} \) is isomorphic to the monodromy group of the covering, i.e., to the symmetric group \( S_d \).

Denote by \( \hat{M} \) the following group:

\[
\hat{M} := \langle \{ M_1 M_k \}_{k=2}^{2g+2} ; \{ M_{2g+2n-1} M_{2g+2n} \}_{n=2}^{d-1} \rangle, \tag{4.8}
\]

and consider its normal closure \( \widehat{\hat{M}}^{\mathcal{M}} \) in \( \mathcal{M} \). Then the monodromy group \( \mathcal{M} \) is represented as a semidirect product \( \mathcal{M} = \widehat{\hat{M}}^{\mathcal{M}} \rtimes \tilde{M} \).

**Theorem 6** The normal closure \( \widehat{\hat{M}}^{\mathcal{M}} \) of the group \( \hat{M} \) (4.8) in the monodromy group \( \mathcal{M} = \langle \{ M_k \}_{k=1}^{N} \rangle \) is isomorphic to the free group \( \mathbb{Z}^{(2g+d-1)(d-1)} \). Here \( d \) is the degree and \( g \) is the genus of the covering; \( N \) is the number of simple finite branch points.

**Proof.** The matrices generating the group \( \hat{M} \) (4.8) have the form (see Appendix A):

\[
M_1 M_k = \begin{pmatrix} I & S_k \\ 0 & I \end{pmatrix}, \quad M_{2g+2n-1} M_{2g+2n} = \begin{pmatrix} I & S_{2g+2n} \\ 0 & I \end{pmatrix}, \tag{4.9}
\]

where again \( k = 2, \ldots, 2g+2; \ n = 2, \ldots, d-1 \) and \( S_l \) is the block above the diagonal in the monodromy matrix \( M_l \), see (4.3). We recall that the second diagonal block in \( M_l M_k \) (the block \( \Sigma \) in (4.3)) depends only on the permutation \( h(\gamma_k \gamma_l) \). The permutations \( h(\gamma_k \gamma_l) \) and \( h(\gamma_k \gamma_l) \) are trivial for the coverings from Figure 1, thus the corresponding diagonal blocks are trivial in (4.9).

From (4.9) and (4.7) we get elements of \( \widehat{\hat{M}}^{\mathcal{M}} \) in the form:

\[
M = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}. \tag{4.10}
\]
with some matrix $S$. We shall now show that $S$ can be any matrix with integer entries.

Consider an element of the normal closure $\hat{\mathcal{M}}^M$, obtained from matrices (4.9) by conjugation with one of the generators (4.7) of the group $\hat{\mathcal{M}}$. We get

$$M_i \begin{pmatrix} I & S_k \\ 0 & I \end{pmatrix} M^{-1}_i = \begin{pmatrix} I & S_k \Sigma_i \\ 0 & I \end{pmatrix},$$

where $k$ runs through the set $\{2, 3, \ldots, 2g + 2\} \cup \{2g + 2n\}^{d-1}$.

Now we note that the last $d - 1$ rows in the block $S_\infty$ above the diagonal in the monodromy matrix $M_\infty$ (A.16) form an $(d - 1) \times (d - 1)$ matrix which coincides with the Cartan matrix for $A_{d-1}$. Therefore, each row of the matrix $M_\infty$ is a coordinate vector of a root of $A_{d-1}$ with respect to a basis of weight vectors $\{v_i\}$ defined by

$$\langle v_i, r_j \rangle = \delta_{ij},$$

where $\{r_i\}$ are the root vectors and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. The Weyl group for $A_{d-1}$ is the symmetric group $S_d$, thus the orbit of one of the root vectors of $A_{d-1}$ under the action of $S_d$ contains all the roots of $A_{d-1}$.

Furthermore, as can be seen from (A.7), (A.8), (A.11), and (A.16), the only non-zero row in each of the blocks $S_k$ and $S_{2g+2n}$ in the matrices (4.9) is the respective row from $M_\infty$, i.e., a row of a Cartan matrix for $A_{d-1}$.

Therefore, the matrices (4.11) for $i = 1$ and odd $i \geq 2g + 3$ and for $k = 2, \ldots, 2g + 2$ and $k = 2g + 2n$ with $n = 2, \ldots, d - 1$ have above the diagonal the products $S_k \Sigma$, where matrices $\Sigma$ represent all generators of the group $S_d$ and the nonzero rows in $S_k$ run through all the root vectors of $A_{d-1}$. Thus, for each of the above $k$, the only nonzero row in the product $S_k \Sigma_i$, the $k$th row, runs through all the root vectors of $A_{d-1}$, i.e., through an integer basis in $\mathbb{R}^{d-1}$. Multiplying matrices of the form (4.11), we get matrices of the form (4.10) with all possible integer blocks $S$ of the size $(2g + d - 1) \times (d - 1)$ above the diagonal, which implies that the normal closure $\hat{\mathcal{M}}^M$ contains the free group $\mathbb{Z}^{(2g+d-1)(d-1)}$. Since from Section 4.1 we know that entries of the blocks $S$ in (4.10) are always integer numbers, we arrive at the isomorphism $\hat{\mathcal{M}}^M \simeq \mathbb{Z}^{(2g+d-1)(d-1)}$. \(\Box\)

To summarize, we repeat that in the case of Hurwitz spaces of coverings shown in Fig. 1, Theorem 6 implies the isomorphism between the monodromy group $\mathcal{M}$ of the solution $\Phi(\lambda)$ (3.14) to the Fuchsian system (2.7), (2.8) and a semidirect product of the free abelian group $\mathbb{Z}^{(2g+d-1)(d-1)}$ and the symmetric group $S_d$, i.e., $\mathcal{M} \simeq \mathbb{Z}^{(2g+d-1)(d-1)} \rtimes S_d$.

### 4.2.2 Spaces of meromorphic functions with poles of higher multiplicity

We consider a covering with ramification over the point at infinity as a limit case of the coverings from Fig. 1 when some of the points $P_{2g+2n}$ with $n \geq 1$ tend to the point at infinity without crossing any branch cuts on the covering.

As discussed in Appendix A.2, the monodromy matrices corresponding to the ramification points that don’t merge in the limit are obtained from the matrices for simple coverings (see Appendix A.1) by deleting a trivial row and the corresponding trivial column. The monodromy matrices corresponding to the ramification points that are sent to infinity disappear in the limit.

Therefore, the reasoning from the proof of Theorem 6 remains valid in the limit: the nonzero rows of the blocks $S_k$ above the diagonal remain unchanged and coincide with rows of the Cartan matrix for $A_{d-1}$. Since sending a ramification point to infinity results in deleting one of the first $2g + d - 1$
rows and one of the first \(2g + d - 1\) columns from the monodromy matrices, the dimension of the blocks \(S_k\) in the limit is \((2g + m - 1) \times (d - 1)\), where \(m\) is the number of points projecting to \(\lambda = \infty\) on the base of the covering arising in the limit.

Thus we get a similar result for the spaces of coverings ramified over the point at infinity: the corresponding monodromy group is isomorphic to the semidirect product \(\mathcal{M} \simeq \mathbb{Z}^{(2g + m - 1)\times (m - 1)} \rtimes S_d\).

For the space of polynomials of degree \(N\) we have \(g = 0\) and \(m = 1\); then the monodromy group of the Fuchsian system \([2.7]\) coincides with the symmetric group \(S_d\) - the Weyl group of \(A_{d-1}\) (as well as with the monodromy group of the covering \(X\)).

If \(g = 0\) and \(m = 2\) (this is the space of rational functions of degree \(d\) with one simple pole and one pole of degree \(d - 1\)), then \(2g + m - 1 = 1\) and the monodromy group of the Fuchsian system coincides with the Weyl group \(\mathbb{Z} \rtimes S_d\) of the affine Lie algebra \(A_{d-1}\), i.e. the algebra of formal power series of one variable with coefficients from \(A_{d-1}\).

For arbitrary \(g\) and \(m\) the monodromy group is the Weyl group of the Lie algebra of formal power series in \(2g + m - 1\) variables with coefficients from \(A_{d-1}\).

## 5 Action of braid group on solution to the Fuchsian system

### 5.1 Braid monodromy group

The braid group \(B_N\) on \(N\) strands (on the plane) naturally acts on the set \(\{\lambda_k\}_{k=1}^N\) and thus on our Hurwitz space. To each covering \(X = (\mathcal{L}, f) \in \mathcal{H}_{g,d}\) one can naturally associate a subgroup \(B_N(X)\) of \(B_N\) such that any element \(\sigma \in B_N(X)\) transforms the covering \(X\) into a covering \(X'\) which is holomorphically equivalent to \(X\) (i.e., \(B_N(X)\) is the fundamental group of \(\mathcal{H}_{g,d}\) with the base at \(X\)). In particular, for \(d = 2\), when the covering \(X\) is hyperelliptic, the subgroup \(B_N(X)\) coincides with the whole braid group \(B_N\). An equivalence between \(X\) and \(X'\) is defined by an element of \(S_d\); therefore, in the case when the automorphism group of \(X\) is trivial, we get a group homomorphism from \(B_N(X)\) to \(S_d\); the image of this homomorphism we call the braid monodromy group of the covering \(X\). The action of the braid group on coverings with \(\mathbb{Z}_d\) symmetry (all branch points have in this case multiplicity \(d - 1\)) was recently studied in [32].

In the case when the covering \(X\) admits no automorphisms (this is obviously the case when all branch points are simple and distinct with the exception of hyperelliptic coverings), each element \(\sigma \in B_N(X)\) naturally induces some \(Sp(2g, \mathbb{Z})\) transformation on homologies \(H_1(\mathcal{L}, \mathbb{Z})\); in this way one gets a group homomorphism

\[ h : B_N(X) \to Sp(2g, \mathbb{Z}) \, . \]

The image \(\Gamma(X)\) of \(B_N(X)\) under the homomorphism \(h\) is a subgroup of \(Sp(2g, \mathbb{Z})\). Some partial results about the subgroup \(\Gamma(X)\) were obtained (in the simplest case of hyperelliptic coverings) in [1] [33]. In particular, it was proved in [1] that \(\Gamma(X)\) coincides with the whole group \(Sp(2g, \mathbb{Z})\) for hyperelliptic coverings with 3, 4 and 6 branch points, and only in these cases. In [33] it was shown that the image of the subgroup of pure braids of \(B_N\) in \(Sp(2g, \mathbb{Z})\) under the homomorphism \(h\) coincides with the principal congruence subgroup \(\Gamma(2)\).

Let us fix some canonical basis of cycles \(\{a_{\alpha}, b_{\alpha}\}\) on \(\mathcal{L}\). In this section we shall identify the symplectic basis \((a_{\alpha}, b_{\alpha})\) of \(H_1(\mathcal{L})\) with respect to which \(W\) is normalized with the symplectic basis \((a_{\alpha}, b_{\alpha})\) which forms a part of the set of the integration contours \(s_j\):

\[ \{a_{\alpha}, b_{\alpha}\} = \{a_{\alpha}, b_{\alpha}\} \, . \]
Denote by \( B_{N}^{(a)}(\mathcal{X}) \) the subgroup of \( B_{N}(\mathcal{X}) \) whose elements preserve the \( g \)-dimensional subspace spanned by the set of \( a \)-cycles. The image of this subgroup in \( Sp(2g, \mathbb{Z}) \) is a subgroup \( \Gamma(\mathcal{X}, \{a\}) \) consisting of matrices \( S \in Sp(2g, \mathbb{Z}) \) \( (3.23) \) with \( C = 0 \):

\[
\left( \begin{array}{c} \hat{b} \\ \hat{a} \end{array} \right) = S \left( \begin{array}{c} b \\ a \end{array} \right), \quad S = \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right). \tag{5.1}
\]

As before, we consider the space \( \mathcal{H}_{g,d}^{(a)} \) which is the space of equivalence classes of pairs \((\mathcal{X}, \{a\})\), where \( \mathcal{X} = (\mathcal{L}, f) \) is a covering of genus \( g \), and degree \( d \) with simple branch points, and \( \{a\} \) is a choice of a subspace of dimension \( g \) in \( H_{1}(\mathcal{X}, \mathbb{Z}) \) spanned by \( a \)-cycles. The subgroup \( B_{N}^{(a)}(\mathcal{X}) \) coincides with the fundamental group of the space \( \mathcal{H}_{g,d}^{(a)} \) with the base point given by the pair \((\mathcal{X}, \{a\})\):

\[
B_{N}^{(a)}(\mathcal{X}) = \pi_{1}(\mathcal{H}_{g,d}^{(a)}, (\mathcal{X}, \{a\})).
\]

The role of the subgroup \( B_{N}^{(a)}(\mathcal{X}) \) in our context is the following: this subgroup consists of braids which not only map the covering \( \mathcal{X} \) to a holomorphically equivalent covering, but also preserve the canonical bidifferential \( W \). This follows from the normalization of \( W(P, Q): \int_{a_{\alpha}} W(\cdot, Q) = 0 \) for all \( \alpha = 1, \ldots, g \).

Therefore, any transformation \( \sigma \in B_{N}^{(a)}(\mathcal{X}) \) preserves the coefficients \( (3.10) \) of the Fuchsian linear system \( (2.7), (2.8) \). However, the solution \( \Phi \) of the system \( (2.7), (2.8) \) may transform under the action of any braid \( \sigma \in B_{N}^{(a)}(\mathcal{X}) \) to a new solution \( \Phi^{\sigma} \) of the same system, which differs from \( \Phi \) by a right monodromy factor \( M^{\sigma}(\{a\}) \) independent of \( \lambda \) and \( \{\lambda_{j}\} \) (but dependent on the choice of the subspace spanned by \( a \)-cycles):

\[
\Phi^{\sigma} = \Phi M^{\sigma}, \quad \sigma \in B_{N}^{(a)}(\mathcal{X}). \tag{5.2}
\]

One therefore obtains a monodromy representation of the fundamental group \( B_{N}^{(a)}(\mathcal{X}) \) of the space \( \mathcal{H}_{g,d}^{(a)} \) in \( GL(L, \mathbb{C}) \).

The corresponding group, which we call the braid monodromy group of the Fuchsian system, will be denoted by \( M^{(a)} \) (the index \( \{a\} \) indicates that this group may depend on the choice of the subspace of \( a \)-cycles). This group is of course different from the monodromy group \( M \) of the Fuchsian system discussed above in Section 4.

It seems rather hard to study the groups \( M^{(a)} \) explicitly for general coverings: even description of the subgroup of the braid group preserving a given covering seems to be not known in general. In the case of hyperelliptic coverings every braid from \( B_{N} \) preserves the covering; however there remains the problem of describing the subgroup of the braid group which preserves the chosen subspace spanned by \( a \)-cycles.

Here we restrict ourselves to the simplest case of the space \( \mathcal{H}_{1,2}(1, 1) \) which consists of two-sheeted coverings of genus 1 with four finite branch points \( \lambda_{1}, \ldots, \lambda_{4} \) and give an explicit description of the corresponding group \( M^{(a)} \). In particular, we shall show that in this case the groups \( M^{(a)} \) corresponding to different choices of the \( a \)-cycle are isomorphic to each other.

### 5.2 Genus one coverings of degree 2

The braid group \( B_{4} \) on 4 strands has three standard generators: \( \sigma_{1} \) (interchanging \( \lambda_{1} \) and \( \lambda_{2} \)), \( \sigma_{2} \) (interchanging \( \lambda_{2} \) and \( \lambda_{3} \)) and \( \sigma_{3} \) (interchanging \( \lambda_{3} \) and \( \lambda_{4} \)), see Figure [2] These generators satisfy the standard relations \( \sigma_{1}\sigma_{2}\sigma_{1} = \sigma_{2}\sigma_{1}\sigma_{2}, \sigma_{2}\sigma_{3}\sigma_{2} = \sigma_{3}\sigma_{2}\sigma_{3} \) and \( \sigma_{1}\sigma_{3} = \sigma_{3}\sigma_{1} \).
Note that although every element of \( B_4 \) preserves the covering \( \mathcal{X} \), there are two ways of identifying the initial covering and the transformed one due to the existence of the nontrivial automorphism of \( \mathcal{X} \), which interchanges the sheets. However, in our case we have an additional marking of the sheets of \( \mathcal{X} \), namely, the choice of the contour \( l_1 \) encircling the point \( \infty^{(1)} \) on the first sheet of the covering. We therefore have a natural choice of identification of two coverings - we identify the sheets in a way that the contour \( l_1 \) stays on the first sheet.

We thus get a group homomorphism (which we denoted by \( h \)) from \( B_4 \) to \( Sp(2, \mathbb{Z}) \).

Let us choose a canonical basis of cycles \((a, b)\) on the covering \( \mathcal{X} \) as shown in Fig.3. Then the action of the generators \( \sigma_1 \) and \( \sigma_3 \) on \( \mathcal{X} \) is a Dehn half-twist with respect to the cycles \( a \) and \( -a \), respectively; for \( \sigma_2 \) it is a Dehn half-twist along the \( b \)-cycle.

The Picard-Lefschetz formulas (see for example [16], Th. 24.3) give the following transformations of a contour \( l \in H_1(L) \) under such an action of \( \sigma_i \):

\[
\sigma_1 : \ l \mapsto l + (l \circ a)a; \quad \sigma_2 : \ l \mapsto l + (l \circ b)b; \quad \sigma_3 : \ l \mapsto l + (l \circ a)a,
\]

where \((l \circ \gamma)\) stands for the intersection index of two contours \( l \) and \( \gamma \).

Thus we get the images of \( \sigma_i \) in \( Sp(2, \mathbb{Z}) \) (acting on the column \((b, a)^T\)):

\[
\begin{align*}
\mathcal{A} := h(\sigma_1) &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, & \mathcal{B} := h(\sigma_2) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & h(\sigma_3) &= h(\sigma_1). \\
\end{align*}
\]

The transformations \((5.4)\) can also be obtained by an appropriate deformation of the elliptic curve in Figure 3. In Fig.4 we draw the deformation of the \( a \)- and \( b \)-cycles induced by the action of \( \sigma_1 \) on the set of branch points of \( \mathcal{X} \).

From \((5.4)\) we see that \( h(\sigma_1) \) and \( h(\sigma_2) \) span the whole group \( Sp(2, \mathbb{Z}) \) since

\[
\begin{align*}
h(\sigma_2\sigma_1\sigma_2) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & h(\sigma_1^{-1}) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]
Figure 4: Action of generator $\sigma_1$ on basis cycles.

are the standard generators of $\text{Sp}(2, \mathbb{Z})$.

We are now in a position to formulate the following

**Lemma 2** The group $B_4^{(a)}(X)$ corresponding to the choice of the basis cycles shown in Figure 3 is the subgroup of the braid group $B_4$ generated by the elements $\sigma_1$, $\sigma_3$, $\sigma_2^2\sigma_1\sigma_2$ and $\sigma_2^2\sigma_3\sigma_2$.

**Proof.** We need to identify the preimage under $h$ of the subgroup $U$ of upper triangular matrices in $\text{Sp}_2(\mathbb{Z})$, i.e., the subgroup generated by the matrices

$$h(\sigma_1^{-1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad h(\sigma_2^2\sigma_1\sigma_2^2) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}. \quad (5.6)$$

Let us denote the second braid from (5.6) by $\theta := \sigma_2^2\sigma_1\sigma_2^2$. Then we have

$$U = \langle h(\sigma_1), h(\theta) \rangle. \quad (5.7)$$

Due to the equality $h(\sigma_3) = h(\sigma_1)$ (see (5.4)), we can factorize the braid group $B_4$ over the relation $\sigma_1 = \sigma_3$ when we study the image of the braid group under $h$. In other words, for our purposes, we need to consider the braid group $B_3$.

Let us denote by $\tilde{h} : B_3 \to \text{Sp}_2(\mathbb{Z})$ the restriction of the homomorphism $h$ to $B_3$. We claim that

$$\tilde{h}^{-1}(U) = \langle \sigma_1, \theta \rangle. \quad (5.8)$$

To prove (5.8) we show that the kernel of $\tilde{h}$ is a subgroup of the group $\langle \sigma_1, \theta \rangle$. Indeed, consider the composition $F \circ \tilde{h} : B_3 \to \text{PSL}(2, \mathbb{Z})$, where $F : \text{Sp}(2, \mathbb{Z}) \to \text{PSL}(2, \mathbb{Z})$ is the factorization over plus or minus the identity matrix. Let us first show that

$$\text{Ker}(F \circ \tilde{h}) = \langle (\sigma_2\sigma_1\sigma_2)^2 \rangle = Z(B_3), \quad (5.9)$$

where the last equality, the fact that the center $Z(B_3)$ of the braid group is generated by $(\sigma_2\sigma_1\sigma_2)^2$, was obtained in [24].

From (5.5) we get the inclusion $\langle (\sigma_2\sigma_1\sigma_2)^2 \rangle < \text{Ker}(F \circ \tilde{h})$.

Now, since $B_3$ is a central extension of $\text{PSL}_2(\mathbb{Z})$, we have the isomorphism $\text{PSL}_2(\mathbb{Z}) \simeq B_3/N$, where $N$ is a subgroup of the center $Z(B_3)$ of the braid group. On the other hand, groups $\text{PSL}_2(\mathbb{Z})$ and $B_3/\text{Ker}(F \circ \tilde{h})$ are isomorphic. Therefore, $\text{Ker}(F \circ \tilde{h})$ is isomorphic to a subgroup $N$ of $Z(B_3)$ and thus, due to the above inclusion $Z(B_3) < \text{Ker}(F \circ \tilde{h})$, equality (5.9) holds.

It is easy to verify that $\tilde{h}(\langle (\sigma_2\sigma_1\sigma_2)^2 \rangle) = -I$; thus

$$\text{Ker}(\tilde{h}) = \langle (\sigma_2\sigma_1\sigma_2)^2 \rangle. \quad (5.10)$$
We thus have the inclusion $\text{Ker}(\hat{h}) < \langle \sigma_1, \theta \rangle$, which follows from (5.10) and the relation $(\sigma_2 \sigma_1 \sigma_2)^4 = (\sigma_1 \theta)^2$. This inclusion and the fact that every upper triangular matrix can be obtained (see (5.7)) as an image under $\hat{h}$ of some braid from the group $\langle \sigma_1, \theta \rangle$ implies that every braid mapped to an upper triangular matrix by $\hat{h}$ belongs to the group $\langle \sigma_1, \theta \rangle$, i.e., (5.8) holds. Now using the relation $h^{-1}(U) = h^{-1}(U) / \{ \sigma_1 = \sigma_3 \}$, we complete the proof.

\[ \square \]

**Remark 6** If we start with a choice of an $a$-cycle different from the one shown in Figure 8, the fundamental group $\pi_1 \left( \mathcal{H}_{1,2}^{(a)}, \mathcal{L}, f, \{ \tilde{a} \} \right)$ will be a different subgroup $B_4^{(a)}$ of the braid group $B_4$. However, by virtue of the result in [1], for any two cycles $a$ and $\tilde{a}$ there exists a braid $\sigma \in B_4$ such that the transformation $h(\sigma)$ in the homology group $H_1(\mathcal{L}, \mathbb{Z})$ takes $a$ to $\tilde{a}$. Then it is straightforward to see that the two subgroups of $B_4$ are related by conjugation, i.e., $B_4^{(a)} = \sigma B_4^{(a)} \sigma^{-1}$.

Now we compute the braid monodromy matrices for the solution $\Phi = (\Phi^{\gamma_1}, \Phi^{\gamma_2}, \Phi^{\gamma_3}, \Phi^{\gamma_4})$ (5.17), i.e., the transformations of $\Phi$ induced by the braids from the group $B_4^{(a)} := B_4^{(a)}(\mathcal{L}, f)$. Note that only the contours of integration change in the solution $\Phi(\lambda)$ under these transformations, since the bidifferential $W(P, Q)$ remains invariant.

Let us choose the contour $\gamma_1, \gamma_2$ as in Appendix A.1, see Figure 6 below. In other words, for the standard basis $\{ \gamma_k \}_{k=1}^{4}$ in the fundamental group of the punctured sphere $\pi_1(\mathbb{CP}^1 \setminus \{ \lambda_1, \ldots, \lambda_4, \infty \}, \lambda_0)$ with some base point $\lambda_0$, we consider lift$(\gamma_1)$, the lift of $\gamma_1$ to the first sheet of the covering. Then we take $\gamma_1, \gamma_2$ to be a deformation of lift$(\gamma_1)$ which takes the end points $\gamma_0^{(1,2)}$ to $\gamma^{(1,2)}$, respectively.

From the action of the generators $\sigma_i$ of the braid group $B_4$ on the loops $\{ \gamma_k \}_{k=1}^{4}$ in $\mathbb{CP}^1$,

\[ \sigma_i(\gamma_i) = \gamma_{i+1}, \quad \sigma_i(\gamma_{i+1}) = \gamma_{i+1}^{-1} \gamma_i \gamma_{i+1}, \]

we get the action of the generators of $B_4^{(a)}$ on the contour $\gamma_1, \gamma_2$. Namely, using the invariance of the points $\gamma_0^{(1)}$ and $\gamma^{(1)}$ under the action of the braids on the covering, we have

\[ \begin{align*}
\sigma_1 : \gamma_1, \gamma_2 & \mapsto \text{lift}^{(1)}(\gamma_2); \\
\sigma_2 : \gamma_1, \gamma_2 & \mapsto \text{lift}^{(1)}(\gamma_3^{-1} \gamma_2 \gamma_3); \\
\sigma_3 : \gamma_1, \gamma_2 & \mapsto \gamma_1, \gamma_2.
\end{align*} \]

(5.12)

Drawing the contours on the covering, one sees that

\[ \text{lift}^{(1)}(\gamma_2) = \gamma_1, \gamma_2 - a + l_1 \quad \text{and} \quad \text{lift}^{(1)}(\gamma_3^{-1} \gamma_2 \gamma_3) = \gamma_1, \gamma_2 - a + l_1 - 2b. \]

(5.13)

Note that the action induced by the braids from $B_4^{(a)}$ on the $a$- and $b$-cycles is only partially described by the matrices $h(\sigma_i)$ (5.4), since these matrices give the transformation of the contours in the homology space $H_1(\mathcal{L}, \mathbb{Z})$. Now we need to find the transformation of the contours in the relative homology space $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$.

To compute this transformation we proceed as follows. The $a$- and $b$-cycles can be written in the form:

\[ a = \text{lift}^{(1)}(\gamma_4 \gamma_3), \quad b = \text{lift}^{(1)}(\gamma_2 \gamma_3). \]

(5.14)

Note that lift$(\gamma_4 \gamma_3)$ is a closed contour encircling the ramification point $P_i$ and is thus trivial in $H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda))$. Therefore, we have, for example: $b = \text{lift}^{(1)}(\gamma_2 \gamma_3) = \text{lift}^{(1)}(\gamma_2^{-1} \gamma_3) = \text{lift}^{(1)}(\gamma_2^{-1} \gamma_3^{-1})$.

Using this observation, (5.11) and the invariance of the point $\gamma_0^{(1)}$ under the transformations, we get:

\[ \begin{align*}
\sigma_1 : & \quad a \mapsto a; \\
\sigma_2 : & \quad a \mapsto a + b; \\
\sigma_3 : & \quad a \mapsto a; \\
\sigma_1 : & \quad b \mapsto b - a + l_1; \\
\sigma_2 : & \quad b \mapsto b; \\
\sigma_3 : & \quad b \mapsto b - a.
\end{align*} \]

(5.15)
Let us prove, for example, that $\sigma_1$ induces the claimed transformation on the $b$-cycle: from \((5.14)\) we get that the contour $b$ is mapped to lift$^1(\sigma_1(\gamma_2\gamma_3)) = \text{lift}^1(\gamma_2^{-1}\gamma_1\gamma_2\gamma_3) = \text{lift}^1(\gamma_4\gamma_3) + b = -a + l_1 + b$, where the last equality is obtained by noting that lift$^1(\gamma_2\gamma_1)$ is the contour encircling counterclockwise the points $P_1$ and $P_2$ on the second sheet of the covering.

Now for the action of the remaining generators of $B_4^{(a)}$ we find:

\[
\begin{align*}
\sigma_2^2\sigma_1\sigma^2 : & \ a \mapsto -a + 2l_1; \quad \sigma_2^2\sigma_3\sigma^2 : \ a \mapsto -a; \\
\sigma_2^2\sigma_1\sigma^2 : & \ b \mapsto -a - b + l_1; \quad \sigma_2^2\sigma_3\sigma^2 : \ b \mapsto -a - b.
\end{align*}
\]

The contour $l_1$ stays invariant under our transformations.

We thus proved the following

**Theorem 7** Consider the Hurwitz space $\mathcal{H}_{g,d}^{(a)}(1,1)$ of two-fold elliptic coverings with the choice of the canonical homology basis as in Fig. 3. Then the braid monodromy group $M(\{a\})$ of the corresponding solution $\Phi = (\Phi^{\gamma_1}, \Phi^{\gamma_2}, \Phi^{a}, \Phi^{b})$ \((3.17)\) to the Fuchsian system \((2.7), (2.8)\) is generated by the following monodromy matrices:

\[
M_{\sigma_1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 \\
-1 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}; \quad M_{\sigma_2^2\sigma_1\sigma^2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 \\
-1 & 0 & -1 & -1 \\
-2 & 0 & 0 & -1
\end{pmatrix};
\]

\[
M_{\sigma_3} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}; \quad M_{\sigma_2\sigma_3\sigma_2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & -1
\end{pmatrix};
\]

these matrices define the group homomorphism from the subgroup $B_4^{(a)} = \langle \sigma_1, \sigma_3, \sigma_2^2\sigma_1\sigma_2^2, \sigma_2^3\sigma_3\sigma_2^2 \rangle$ of the braid group $B_4$ to $GL(4, \mathbb{Z})$.

### 5.3 Action of braid group on monodromy matrices

Here we discuss the action of the braid group on the set of monodromies of our solution $\Phi$ to system \((2.7), (2.8)\).

The action of the braid group on the sets of monodromy matrices was used in \([13, 3]\) to study the algebraic solutions of the Painlevé VI equation. Finiteness (up to a simultaneous conjugation) of the orbit of the action of the braid group on the set of monodromies of a given Painlevé VI equation is a necessary condition for algebraicity of the corresponding solution.

Although an analogous result seems to be not explicitly formulated for the Schlesinger systems of an arbitrary dimension with an arbitrary number of singularities, it is instructive to see how the braid group acts on the set of monodromies of the function $\Phi$.

Let us briefly recall how $B_N$ acts on monodromies $\{M_j\}$ of an arbitrary Fuchsian system of the form \((1.1)\). Choose the standard set of generators of $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\})$ satisfying relation \((2.1)\); then the monodromy matrices satisfy \((2.2)\).

The action of the generator $\sigma_k \in B_N$ on the generators $\gamma_j$ of $\pi_1(\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_N, \infty\})$ looks as in \((5.11)\): $\sigma_k(\gamma_k) = \gamma_{k+1}$; $\sigma_k(\gamma_{k+1}) = \gamma_k^{-1}\gamma_k\gamma_{k+1}$; $\sigma_k(\gamma_j) = \gamma_j$ for $j \neq k, k + 1$. 

30
Therefore, under the action of \( \sigma_k \), the set of matrices \( \{M_j\} \) transforms into the set \( \{M_j^{\sigma_k}\} \) where

\[
M_k^{\sigma_k} = M_k^{-1} M_{k+1} M_k, \quad M_{k+1}^{\sigma_k} = M_k, \quad M_j^{\sigma_k} = M_j, \quad j \neq k, k+1. \tag{5.16}
\]

Recall that the subgroup of \( B_N \) consisting of the braids which map the covering \( \mathcal{X} \) to an equivalent one was denoted by \( B_N(\mathcal{X}) \). The index of \( B_N(\mathcal{X}) \) in \( B_N \) is finite; it is given by the number of inequivalent coverings with simple branch points of given degree and given ramification at \( \infty \) - the Hurwitz number, which is denoted by \( h_{g,d}(k_1, \ldots, k_m) \).

The following theorem is a simple corollary of Theorem \( 3 \):

**Theorem 8** Let \( \sigma \in B_N(\mathcal{X}) \). Suppose the action of \( \sigma \) on the chosen basis \( \mathbf{s} = (s_1, \ldots, s_N) \) in \( H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda)) \) is defined by a matrix \( R_\sigma \), i.e., \( \sigma : \mathbf{s} \mapsto sR_\sigma \). Then \( \sigma \) acts on monodromy matrices of solution \( (3.14) \) to the system \( (2.7), (2.8) \) by simultaneous conjugation with the matrix \( R_\sigma \):

\[
M_k^{\sigma} = (R_\sigma)^{-1}M_k R_\sigma. \tag{5.17}
\]

**Proof.** An action of \( \sigma \) may result in a transformation of a basis \( \{a, b\} \) in \( H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda)) \) to normalize \( W \) and the basis \( \{s_k\} \) in \( H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda)) \). It may also change signs of some of distinguished local parameters. A transformation of \( \{a, b\} \) results in a change of normalization of \( W \); according to Theorem \( 3 \), the induced action on \( \Phi \) is given by a Schlesinger transformation with the matrix \( \mathbf{T} - \mathbf{T} \) \( (3.26) \), and, therefore, does not change the monodromy matrices \( M_k \). A change of signs of some of \( x_j \) corresponds to multiplication of \( \Phi \) from the left by a constant matrix \( Y \), which does not change the monodromy matrices either.

A transformation of the basis \( \{s_k\} \) leads to the right multiplication of the solution with the matrix \( R_\sigma \) \( (3.25) \), which implies the transformation \( (5.17) \) of monodromy matrices.

\( \Box \)

Let us introduce the equivalence relation \( \sim \) on the space of the sets of monodromy matrices; two sets \( \{M_k\} \) and \( \tilde{M}_k \) are called equivalent if there exists a matrix \( J \) such that \( \tilde{M}_k = JM_kJ^{-1} \) for all \( k \).

Theorem \( 8 \) implies the following immediate corollary:

**Corollary 2** Consider a covering \( \mathcal{X} \in H_{g,d}(k_1, \ldots, k_m) \), choose some Lagrangian subspace \( \{a\} \) of \( a \)-cycles and a basis \( \{s_j\} \) in \( H_1(\mathcal{L} \setminus f^{-1}(\infty), f^{-1}(\lambda)) \); define function \( \Phi \) by \( (3.14) \) and denote the corresponding monodromy matrices by \( \{M_1, \ldots, M_N\} \). The braid group \( B_N \) acts on the set \( \{M_1, \ldots, M_N\} \) according to \( (5.16) \). Then the number of inequivalent (modulo equivalence relation \( \sim \)) sets of \( N \) matrices obtained by the action of \( B_N \) on the set \( \{M_1, \ldots, M_N\} \) equals the Hurwitz number \( h_{g,d}(k_1, \ldots, k_m) \).

**Proof.** Recall that the Hurwitz number equals the number of inequivalent simple \( d \)-sheeted coverings whose branching at \( \infty \) has the type \( (k_1, \ldots, k_m) \). According to Theorem \( 8 \), if \( \sigma \in B_N(\mathcal{X}) \), then the initial set of monodromy matrices is equivalent to the set of monodromies obtained by the action of an element \( \sigma \) of the braid group.

Conversely, consider two coverings, \( \mathcal{X} \) and \( \tilde{\mathcal{X}} \) with the same sets of branch points, denote corresponding solutions of the Fuchsian system by \( \Phi \) and \( \tilde{\Phi} \) and the sets of monodromy matrices by \( \{M_k\} \) and \( \{\tilde{M}_k\} \), respectively. Assume that the sets \( \{M_k\} \) and \( \{\tilde{M}_k\} \) are equivalent i.e. there exists a matrix \( T \) such that \( \tilde{M}_k = TM_kT^{-1} \). Since matrices

\[
M_k = \begin{pmatrix} 1 & S_k \\ 0 & \Sigma_k \end{pmatrix}, \quad \tilde{M}_k = \begin{pmatrix} 1 & \tilde{S}_k \\ 0 & \tilde{\Sigma}_k \end{pmatrix}
\]

31
have upper block-triangular structure, the matrix $T$ must have the same upper block-diagonal structure. Namely, denote the lower off-diagonal block of the matrix $T$ by $T_0$. Then relation $\hat{\Sigma}_k = TM_kT^{-1}$ implies that $T_0 = \hat{\Sigma}_kT_0$ for all $k$, i.e. each non-vanishing column of $T_0$ is an eigenvector with the eigenvalue 1 of all matrices $\Sigma_k$ simultaneously. Using the explicit form (A.6), (A.10) of the matrices $\Sigma_k$ one easily sees that this is impossible i.e. $T_0 = 0$.

Denoting the lower diagonal block of the matrix $T$ by $T_1$ we see that $\hat{\Sigma}_k = T_1\Sigma_kT_1^{-1}$ for all $k$. On the other hand, the group generated by $\{\Sigma_k\}$ is isomorphic to the monodromy group of the corresponding covering. Since the groups generated by $\{\Sigma_k\}$ and by $\{\hat{\Sigma}_k\}$ are isomorphic, corresponding monodormy groups of the coverings $\hat{\mathcal{X}}$ and $\mathcal{X}$ are isomorphic, too, and these two coverings are equivalent.

$\square$

6 Concluding remarks

Present work poses a number of interesting questions.

- According to the general idea of the work by Dubrovin and Mazzocco [14], finiteness of the orbit of the action of the braid group on the set of monodromy matrices of a Fuchsian system is the necessary condition for the algebraicity of the corresponding solution to the Schlesinger system (see also [3]). For the solutions to the Fuchsian system coconstructed here, these numbers are finite and equal to the Hurwitz numbers $h_{g,d}(k_1, \ldots , k_m)$. Therefore, it seems natural to expect that all solutions to the Fuchsian linear system discussed in this paper, as well as the corresponding solutions to the Schlesinger system, are algebraic, which would reflect the algebraic nature of the Hurwitz spaces. To find a complete proof of this fact would be an interesting problem.

- In another work [15] by Dubrovin and Mazzocco the idea of reducibility of Schlesinger systems was developed: a solution to a Schlesinger system is called reducible if it can be expressed in terms of solutions to Schlesinger systems with smaller number of singularities or lower matrix dimension. In particular, it was proved that if all monodromy matrices have the same block-triangular structure, than the solution is reducible. Since all monodromy matrices of the $N \times N$ Fuchisan systems considered here have the structure of this type, their solutions should be expressible in terms of solutions of lower-dimensional $(d-1) \times (d-1)$ Schlesinger system with the same number of singularities ($N = d - 1$ only in the case of the space of polynomials of degree $d$); the monodromy matrices of this $(d-1) \times (d-1)$-dimensional system are supposed to coincide with matrices $\Sigma_i$ from (4.3).

It is natural to ask how does the solution to the corresponding $(d-1) \times (d-1)$ Riemann-Hilbert problem look like and what is the corresponding Jimbo-Miwa tau-function. Is it different from the Bergman tau-function?

- In this paper we solve the Fuchsian systems corresponding to the value $q = -1/2$ from the one-parametric family of Fuchsian systems arrising from the Frobenius structures on Hurwitz spaces, while the system used by Dubrovin in [12] has $q = 1/2$. In principle, one could get a solution to Dubrovin’s system by a simple differentiation of the solution $\Phi$ to our system; however, since generically some of the columns of our matrix $\Phi$ are constants, in this way one does not get a complete set of solutions to the system with $q = 1/2$. Therefore, there arises a problem of finding the missing set of vector functions satisfying Dubrovin’s system.
Two solutions to the Fuchsian system are equivalent up to a multiplication with constant factors from both sides if the corresponding coverings are equivalent as elements of the Hurwitz space $\mathcal{H}_{g,d}^{(a)}(k_1, \ldots, k_m)$. The fundamental group of this space is a subgroup of the braid group which preserves the covering together with the Lagrangian subspace spanned by the $a$-cycles in the homologies. In this paper we described this subgroup in the simplest case of two-sheeted coverings of genus 1. An extension of this result to hyperelliptic and more general coverings is an interesting problem.

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A Explicit form of monodromy matrices

Here we compute the explicit form of monodromy matrices of our solution of the the Fuchsian system (1.1).

A.1 Spaces of meromorphic functions with simple poles

Consider the Hurwitz space \( H_{g,d}(1, \ldots, 1) \) of functions with \( d \) simple poles and simple critical points on a Riemann surface of genus \( g \). The branched covering \( L \) corresponding to such a function has \( N = 2g + 2d - 2 \) finite branch points \( \lambda_j \) and no branching over \( \lambda = \infty \); the covering \( L \) can be defined by a choice of \( N \) generators of the fundamental group \( \pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N, \lambda_0\}) \) of the base of the covering and a set of elements of the symmetric group \( S_d \) assigned to these generators. For an explicit computation of monodromy matrices of the solution (3.14), (3.17) to the Fuchsian system (2.7), (2.8) it is useful to represent the branched covering \( L \) in a standard form. For that purpose we make use of Clebsch’s result ([8], see [17] for the modern exposition) stating that one can always choose generators \( \{\gamma_j\} \) of \( \pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N, \lambda_0\}) \) satisfying (2.1) in such a way that the loop \( \gamma_j \) encircles only the point \( \lambda_j \) and the set of the corresponding elements \( \sigma_k \in S_d \) of the monodromy group of the covering has the form:

\[
\sigma_1, \ldots, \sigma_N = (1, 2), (1, 2), \ldots, (1, 2), (2, 3), (2, 3), (3, 4), (3, 4), \ldots, (d-1, d), (d-1, d) ,
\]

where the first transposition \((1, 2)\) occurs \( 2g + 2 \) times at the beginning and the other transpositions \((j, j+1), j \geq 2\), each occur twice, in order. Such a covering can be visualized as a hyperelliptic Riemann surface of genus \( g \) with \( d-2 \) Riemann spheres attached to it, see the Hurwitz diagram from Figure 1.

Assume the canonical homology basis to be chosen on the hyperelliptic part of the Riemann surface in the standard way, i.e., the cycle \( a_\alpha \) encircles the ramification points \( P_{2\alpha+1}, P_{2\alpha+2} \) on the second sheet, and the cycle \( b_\alpha \) goes around the points \( P_2 \) and \( P_{2\alpha+1} \), see Figure 5. Assume also that the branch cuts are chosen to connect the points \( P_{2k-1} \) and \( P_{2k} \) for \( k = 1, \ldots, g + d - 1 \).

It is convenient also to consider a basis of contours in the space \( H_1(L \setminus f^{-1}(\infty); f^{-1}(\lambda)) \) different from the basis (3.11)-(3.13). Namely, assume the contour \( \gamma_{1,2}(\lambda) \) (see (3.13)) to go around the point \( P_1 \) when passing from the first sheet to the second, i.e, \( \gamma_{1,2} = \text{lift}^{(1)}(\gamma_1) \), the lift of the generator
\[ \gamma_1 \in \pi_1(C \setminus \{\lambda_1, \ldots, \lambda_N\}, \lambda_0). \] Analogously, for \( n = 2, \ldots, d-1 \) we assume \( \gamma_{n,n+1}(\lambda) = \text{lift}^{(n)}(\gamma_{2g+2n-1}), \) i.e., \( \gamma_{n,n+1}(\lambda) \) passes from \( n \)th to \( (n+1) \)st sheet of the covering by going around the point \( P_{2g+2n-1} \).

Recall from Section 4 that there is a homomorphism \( T : \pi_1(C \setminus \{\lambda_1, \ldots, \lambda_N\}, \lambda) \to \text{Aut} \{H_1(L \setminus f^{-1}(\infty); f^{-1}(\lambda))\} \). Let us denote the images of the generators \( \gamma_k \) of the fundamental group under \( T \) by \( T_{\lambda_k} \).

Then \( T_{\lambda_1}[\gamma_{n,n+1}] \) is the transformation of the contour \( \gamma_{n,n+1}(\lambda) \) as \( \lambda \) goes around \( \lambda_j \) on the base of the covering. Let us take the basis in \( H_1(L \setminus f^{-1}(\infty); f^{-1}(\lambda)) \) formed by the following \( 2g + 2d - 2 \) paths on the surface \( L \):

\[
\begin{align*}
S_j & := \frac{1}{2} (\gamma_{1,2} + T_{\lambda_j}[\gamma_{1,2}]), & \text{for } j = 2, \ldots, 2g + 2; & \quad (A.2) \\
S_{2g+2n} & := \frac{1}{2} (\gamma_{n,n+1} + T_{\lambda_{2g+2n}}[\gamma_{n,n+1}])), & \text{for } n = 2, \ldots, d - 1; & \quad (A.3) \\
\gamma_{n,n+1}(\lambda), & \text{for } n = 1, \ldots, d - 1. & \quad (A.4)
\end{align*}
\]

The factor of \( 1/2 \) is introduced for computational convenience in what follows. Note that the paths \( 2S_k \) are closed contours on the surface.

In this section we compute monodromy matrices for the fundamental matrix-solution to our Fuchsian system (2.7), (2.8), associated to the Hurwitz space \( H_g,d(1, \ldots, 1) \), whose columns are the vectorsolutions (3.14) with integration contours \( d \) given by the above basis \( (A.2) - (A.4) \) in this order.

The monodromy matrices of the solution \( \Phi(\lambda) \) with integration paths \( (A.2) - (A.4) \) in the given order still have the structure \( (4,3) \), where the number \( m \) of pre-images of the point at infinity equals \( d \), the degree of the covering. The monodromy matrices are determined by the transformations of the contours \( (A.2) - (A.4) \) in the relative homology group \( H_1(L \setminus f^{-1}(\infty); f^{-1}(\lambda)) \) which occur as the point \( \lambda \) describes the loops \( \gamma_k \) on the base of the covering (i.e., under the automorphisms \( T_{\lambda_k} \)). The first \( 2g + d - 1 \) columns of the matrix \( \Phi \) remain unchanged under these transformations.

We now look at the transformations of the last \( d - 1 \) columns of the matrix \( \Phi(\lambda) \) given by the integrals (3.14) over the contours \( \gamma_{n,n+1}(\lambda) \) with \( n = 1, \ldots, d - 1 \) and find the corresponding \( S_k \) and \( \Sigma_k \) (see (4,3)) for \( k = 1, \ldots, 2g + 2d - 2, \infty \).

**Monodromy matrix \( M_1 \).**

When \( \lambda \) travels along the loop \( \gamma_1 \) on the base, the contour \( \gamma_{1,2} \) transforms to \( -\gamma_{1,2} \), as shown in Figure 6. Note that the sum of the two contours in Figure 6 is the closed contour encircling the point \( P_1 \); this contour is trivial in the space \( H_1(L \setminus f^{-1}(\infty); f^{-1}(\lambda)) \).

![Figure 6: The transformation of the contour \( \gamma_{1,2} \) corresponding to the monodromy matrix \( M_1 \).](image)

As is easy to see, the contour \( \gamma_{2,3} \) becomes the sum \( \gamma_{2,3} + \gamma_{1,2} \) under the automorphism \( T_{\lambda_1} \). The other contours \( \gamma_{n,n+1} \) with \( n > 2 \) do not change. Thus, the monodromy matrix \( M_1 \) has the form

\[
M_1 = \begin{pmatrix} I_{2g+d-1} & 0 \\ 0 & \Sigma_1 \end{pmatrix},
\]

(A.5)
where $\Sigma_1$ is the following $d-1 \times d-1$ matrix

$$
\Sigma_1 = \begin{pmatrix}
-1 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}.
$$

(A.6)

**Monodromy matrix $M_2$.**

The image of the contour $\gamma_{1,2}$ under $T_{\lambda_2} \in \text{Aut}[H_1(L \setminus f^{-1}(\infty); f^{-1}(\lambda))]$ is shown in Figure 7. Let

![Figure 7](image)

Figure 7: The transformation of the contour $\gamma_{1,2}$ corresponding to monodromy around $\lambda_2$.

us denote the closed contour encircling the branch cut $[P_1, P_2]$ counter-clockwise on the first sheet, the closed contour from Figure 7 by $A_{12}$. The sum of the non-closed contour in the right hand side in Figure 7 and $\gamma_{1,2}$ (the contour $-\gamma_{1,2}$ from Figure 6 with inverse orientation) is a closed contour encircling clockwise the branch cut $[P_1, P_2]$ on the second sheet, i.e., again the contour $A_{12}$. By (A.2) we have $T_{\lambda_2}[\gamma_{1,2}] = 2S_2 - \gamma_{1,2}$. Thus, we get $S_2 = A_{12}$, see Figure 8.

![Figure 8](image)

Figure 8: The contour $S_2$

The transformation of the contour $\gamma_{2,3}$ under $T_{\lambda_2}$ is shown in Figure 9. From the figure we see that the sum of the contour $T_{\lambda_2}[\gamma_{2,3}]$ and the the non-closed contour from the right hand side of Figure 7 gives $\gamma_{2,3}$. In other words, $T_{\lambda_2}[\gamma_{2,3}] = \gamma_{2,3} + \gamma_{1,2} - S_2$.

The paths $\gamma_{n,n+1}$ with $n > 2$ remain unchanged when $\lambda$ goes around $\lambda_2$. Thus, the monodromy matrix $M_2$ has the form:

$$
M_2 = \begin{pmatrix}
I_{2g+d-1} \quad S_2 \\
0 \quad \Sigma_1
\end{pmatrix},
$$

(A.7)
where

\[
S_2 = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 
\end{pmatrix}
\]

Similarly, we find other monodromy matrices corresponding to the ramification points on the hyper-elliptic part of the surface.

**Monodromy matrix** \( M_{\lambda_{2k+1}} \) for \( 1 \leq k \leq g \).

The contour \( \gamma_{1,2} \) as its end points go counterclockwise around the point \( \lambda_{2k+1} \), \( 1 \leq k \leq g \), results in the contour shown in Figure 10 for \( k = 1, \ldots, g + 1 \). As before, the paths on the first sheet are drawn with dash line and solid line corresponds to the second sheet. The sum of the original contour \( \gamma_{1,2} \) and the non-closed component in the right hand side in Figure 10 gives a closed contour equivalent to \( 2b_k \).

The transformation of the contour \( \gamma_{2,3} \) under \( T_{\lambda_{2k+1}} \in \text{Aut}[H_1(\mathcal{L} \setminus f^{-1}(\infty) ; f^{-1}(\lambda))] \) is shown in Figure 12. Note that after subtracting \( \gamma_{1,2} \) and \( \gamma_{2,3} \) from the contour in the right hand side of Figure 12, we get the contour equivalent to \( -S_{2k+1} \) (see Figure 11). Therefore, \( T_{\lambda_{2k+1}}[\gamma_{2,3}] = \gamma_{1,2} + \gamma_{2,3} - S_{2k+1} \).

The remaining paths \( \gamma_{n,n+1} \) with \( n > 2 \) do not change under \( T_{\lambda_{2k+1}} \).

**Monodromy matrix** \( M_{\lambda_{2k+2}} \), \( 1 \leq k \leq g \).

Analogously, the image \( T_{\lambda_{2k+2}}[\gamma_{1,2}] \) of \( \gamma_{1,2} \) is shown in Figure 13. As is easy to see, the first closed contour from the right hand side in Figure 13 is equivalent to the sum of the other two closed contours. On the other hand, from (A.2), we have \( T_{\lambda_{2k+2}}[\gamma_{1,2}] = 2S_{2k+2} - \gamma_{1,2} \). Thus, the basis contour \( S_{2k+2} \) has the form given by Figure 14.

Similarly, from Figure 15 we see that \( T_{\lambda_{2k+2}}[\gamma_{2,3}] = \gamma_{1,2} + \gamma_{2,3} - S_{2k+2} \).

The contours \( \gamma_{n,n+1} \) with \( n > 2 \) do not change under the automorphism \( T_{\lambda_{2k+2}} \).

Hence, we obtain the monodromy matrices \( M_n \) in the form:

\[
M_n = \begin{pmatrix}
I_{2g + d - 1} & S_n \\
0 & \Sigma_1
\end{pmatrix}, \quad 2 \leq n \leq 2g + 2.
\]

37
where

\[
S_n = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 \\
2 & -1 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

Here, \( \Sigma_1 \) is the \((d-1) \times (d-1)\) matrix given by (A.6), and the nontrivial row in the block above the diagonal is the the \((n-1)\)st one, i.e., the row corresponding to the integration contour \( S_n \).

\[
S_{2k+1} = P_2 \cdots P_{2k} + b_k
\]

Figure 10: The transformation of the contour \( \gamma_{1,2} \) corresponding to monodromy around \( \lambda_{2k+1}, k > 1 \).

Figure 11: The basis contour \( S_{2k+1} \).
Monodromy matrix $M_{\lambda_{2g+2k-1}}$, $2 \leq k \leq d - 1$.

The automorphism $T_{\lambda_{2g+2k-1}}$ transforms the three contours $\gamma_{k-1,k}(\lambda)$, $\gamma_{k,k+1}(\lambda)$, $\gamma_{k+1,k+2}(\lambda)$ - those passing through the $k$th and $(k+1)$st sheets. The contour $\gamma_{k,k+1}$ transforms to $-\gamma_{k,k+1}$ similarly to Figure 8. The contours $\gamma_{k-1,k}$ and $\gamma_{k+1,k+2}$ transform to $\gamma_{k-1,k} + \gamma_{k,k+1}$ and $\gamma_{k,k+1} + \gamma_{k+1,k+2}$, respectively.

The monodromy matrix thus has the form:

$$M_{2g+2k-1} = \begin{pmatrix} I_{2g+d-1} & \Sigma_{2g+2k-1} \\ 0 & \Sigma_{2g+2k-1} \end{pmatrix}, \quad 2 \leq k \leq d - 1. \quad (A.9)$$

Here, $\Sigma_{2g+2k-1}$ is the $(d - 1) \times (d - 1)$ matrix, corresponding to the permutation of the sheets of the covering, associated to the branch point $\lambda_{2g+2k-1}$ - the matrix is given by

$$\Sigma_{2g+2k-1} = \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 1 & -1 & 1 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \quad 2 \leq k \leq d - 1, \quad (A.10)$$

where the nontrivial $3 \times 3$ diagonal block is formed by the rows from $(k-1)$st to $(k+1)$st and the respective columns.

Monodromy matrix $M_{\lambda_{2g+2k}}$, $2 \leq k \leq d - 1$.

The same columns of the matrix $\Phi(\lambda)$ transform, when $\lambda$ describes the loop $\gamma_{2g+2k}$, $k \geq 2$, on the base of the covering.

The transformation of $\gamma_{k,k+1}$ under $T_{\lambda_{2g+2k}}$ is analogous to that in Figure 7, where the ramification points are $P_{2k-1}$ and $P_{2k}$ instead of $P_1$ and $P_2$, respectively. Since from A.3 we have $T_{\lambda_{2g+2k}}[\gamma_{k,k+1}] = 2S_{2g+2k} - \gamma_{k,k+1}$, we see, similarly to Figure 8 that the basis contour $S_{2g+2k}$ is equivalent to the closed contour encircling the branch cut $[P_{2g+2k-1}, P_{2g+2k}]$ on the $k$th sheet, see Figure 16.
Figure 13: The transformation of the contour $\gamma_{1,2}$ corresponding to monodromy around $\lambda_{2k+2}$. The parts of the contours lying on the first sheet are drawn with dash line, the ones on the second sheet with solid line.

The transformations of $\gamma_{k-1,k}$ and $\gamma_{k+1,k+2}$ under $T_{\lambda_{2g+2k}}$ are shown in Figures 17 and 18 respectively.

The sum of the contour in the right hand side of Figure 17 and the contour $-\gamma_{k,k+1}$ is equivalent to $\gamma_{k-1,k}$ plus the contour encircling the branch cut $[P_{2g+2k-1}, P_{2g+2k}]$ clockwise on the $k$th sheet (we use the triviality of the contour encircling one ramification point). Therefore, we get

$$T_{\lambda_{2g+2k}}[\gamma_{k-1,k}] = \gamma_{k-1,k} + \gamma_{k,k+1} - S_{2g+2k}.$$

Analogously, adding $-\gamma_{k,k+1}$ to the contour in the right hand side of Figure 18 we get $\gamma_{k+1,k+2}$ plus a closed contour around the branch cut $[P_{2g+2k-1}, P_{2g+2k}]$ oriented clockwise on the $k$th sheet. Thus the contour $\gamma_{k+1,k+2}$ transforms to $\gamma_{k+1,k+2} + \gamma_{k,k+1} - S_{2g+2k}$ as $\lambda$ describes the loop $\gamma_{2g+2k}$ around $\lambda_{2g+2k}$.

The monodromy matrix thus has the form:

$$M_{2g+2k} = \begin{pmatrix} I_{2g+d-1} & \Sigma_{2g+2k-1} \end{pmatrix}, \quad 2 \leq k \leq d - 1. \quad (A.11)$$

where

$$S_{2g+2k} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & -1 & 2 & -1 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The nontrivial row in the block above the diagonal is the $(2g + k)$th one, i.e., the row corresponding to the integration contour $S_{2g+2k}$; the nonzero columns being the $(k-1)st$, $kth$ and $(k+1)st$. The
\[ S_{2k+2} = \sum_{1 \leq k \leq g} b_k \]

Figure 14: The basis contour \( S_{2k+2} \)

\((d - 1) \times (d - 1)\) matrix \( \Sigma_{2g+2k-1} \) is given by \((A.10)\).

**Monodromy matrix** \( M_\infty \).

As \( \lambda \) goes counterclockwise around the point at infinity, each contour \( \gamma_{k,k+1} \) transforms to \( \gamma_{k,k+1} - l_k + l_{k+1} \), where the contour \( l_s \ ((A.12)) \) is a closed contour around the point \( \infty^{(s)} \). We need to express these contours in terms of our basis \((A.2)-(A.4)\). As is easy to see from Figures 3, 11, 14 and 16, the following relations hold in \( H_1(\mathcal{L} \backslash f^{-1}(\infty); f^{-1}(\lambda)) \):

\[
\begin{align*}
l_1 &= -S_{2g+2} + b_g = \sum_{n=1}^{g} (S_{2n+1} - S_{2n}) - S_{2g+2}, \\
l_2 &= -S_{2g+1} + S_{2g+2} - b_g = -\sum_{n=1}^{g} (S_{2n+1} - S_{2n}) + S_{2g+2} - S_{2g+4}, \\
l_k &= S_{2g+2k-2} - S_{2g+2k}, \quad 2 \leq k \leq d - 1, \\
l_d &= S_{2g+2d-2}.\end{align*}
\]

Thus, the monodromy matrix has the form:

\[
M_\infty = \begin{pmatrix} I_{2g+d-1} & S_\infty \\ 0 & I_{d-1} \end{pmatrix},
\]

where

\[
S_\infty = \begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 2 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
Figure 15: The transformation of the contour $\gamma_{2,3}$ corresponding to monodromy around $\lambda_{2k+2}$. The correspondence between style of the lines and sheets of the covering is as in Figure 12.

One can check that the monodromy matrices (A.5), (A.7), (A.8), (A.9), (A.11), (A.16) satisfy the relation $M_{\infty}M_\text{dots}M_1 = I$, see (2.2).

Note that all the above monodromy matrices except for $M_\infty$ are rank one perturbations of the identity matrix, according to the general theory [13].

Note also that the lower $(d-1) \times (d-1)$ block of the matrix $S_\infty$ is equal to the Cartan matrix of the group $A_{d-1}$.

A.2 Spaces of meromorphic functions with poles of higher multiplicity

Consider the coverings with branching over the point at infinity as the limits of simple coverings when some of the branch points tend to infinity. In the covering represented by the Hurwitz diagram in

\[ S_{2g+2k} = \]

Figure 16: The basis contour $S_{2g+2k}$. 

42
Figure 17: The transformation of the contour $\gamma_{k-1,k}$ corresponding to the monodromy around $\lambda_{2g+2k}$.

Figure 1 let one of the points $P_{2g+2k}$, $1 \leq k \leq d-1$, tend to infinity without leaving the sheets it belongs to, i.e., without crossing any branch cuts. The dimension of the Hurwitz space is thus reduced by one. Then in the space $H_1(\mathcal{L} \setminus \tilde{f}^{-1}(\infty) ; \tilde{f}^{-1}(\lambda))$ which corresponds to the covering $\tilde{f} : \mathcal{L} \rightarrow \mathbb{CP}^1$ arising in the limit, we take the basis consisting of the contours (A.2)-(A.3) less the contour $S_{2g+2k}$ corresponding to the ramification point taken to infinity. Then the matrix $\Phi(\lambda)$ (3.14) with integration contours given by this basis in the space $H_1(\mathcal{L} \setminus \tilde{f}^{-1}(\infty) ; \tilde{f}^{-1}(\lambda))$, solves the Fuchsian problem (2.7), (2.8) associated to the Hurwitz space of the coverings $\tilde{f} : \mathcal{L} \rightarrow \mathbb{CP}^1$ arising in the limit considered.

As we have seen in the previous section, a basis contour $S_n$ only plays a role in the corresponding monodromy transformation $T_{\lambda_n}$. Therefore, in the limit $P_{2g+2k} \rightarrow \infty$ with $1 \leq k \leq d-1$ the monodromies of the solution $\Phi(\lambda)$ around the remaining finite branch points are obtained from the respective monodromy matrices computed in Section A.1 by deleting the $(2g+k)$th row and column, which are trivial and which correspond to the disappearing in the limit integration contour $S_{2g+2k}$. In other words, the monodromy matrices $M_n$, $n \neq \infty$, still have the form (A.5) - (A.11) computed in Section A.1. The monodromy matrix $M_\infty$ can be obtained from relation (2.2).

By repeating the procedure for some of the remaining ramification points $P_{2g+2k}$ one can arrive at a covering with any number (not exceeding $d$) of points projecting to $\lambda = \infty$ on the base.

### A.3 Space of polynomials

Here we consider a partial case of the Hurwitz spaces from Section A.2 the Hurwitz space $\mathcal{H}_{0,d}(d)$ with a degenerate ramification over $\lambda = \infty$ where all $d$ sheets are glued together, represented by the Hurwitz diagram from Figure 19. This space can be regarded as a space of polynomial functions on $\mathbb{CP}^1$.

As before, we assume the generator $\gamma_k$ of the fundamental group $\pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_{d-1}, \lambda_0\})$ to encircle only one branch point, namely $\lambda_k$. 

43
The $k$th column of the solution $\Phi$ corresponding to this Hurwitz space is given by the integral (3.14) over the contour $\gamma_{k,k+1}$ (3.13) for $k = 1, \ldots, d - 1$ (the contour $\gamma_{k,k+1}$ is again defined as the lift of the loop $\gamma_k$ to the $k$th sheet, it starts on the $k$th and ends on the $(k + 1)$st sheet). Then, as $\lambda$ describes the loop $\gamma_k$ on the base of the covering, the contours $\gamma_{1,2}, \ldots, \gamma_{d-1,d}$ change as follows: $\gamma_{k-1,k}$ becomes $\gamma_{k-1,k} + \gamma_{k,k+1}$; the contour $\gamma_{k,k+1}$ turns into its negative $-\gamma_{k,k+1}$ (as in Figure 6), and $\gamma_{k+1,k+2}$ becomes $\gamma_{k,k+1} + \gamma_{k+1,k+2}$. Here we assume that if one of the indexes becomes 0 or $d$, the contour equals 0.

All other contours $\gamma_{j,j+1}$ for $j \neq k-1,k,k+1$ remain unchanged.

Figure 19: A Hurwitz diagram for the space $\mathcal{H}_{0,d}(d)$. 
Thus the monodromy matrices have the following form:

\[
M_k = \begin{pmatrix}
I_{k-2} & 0 & 0 \\
0 & M & 0 \\
0 & 0 & I_{d-k-2}
\end{pmatrix}, \quad 1 < k < d - 1,
\]

where the block \( M \) is

\[
M = \begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Note that the matrices \( M_k \) coincide with the blocks \( \Sigma_k \) from (4.3) in this case. The monodromies at \( \lambda_1 \) and \( \lambda_{d-1} \) are given by

\[
M_1 = \begin{pmatrix}
-1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_{d-3}
\end{pmatrix}, \quad M_{d-1} = \begin{pmatrix}
I_{d-3} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

To compute the monodromy at \( \lambda = \infty \) we note that since the covering surface is of genus zero and since the preimage \( f^{-1}(\infty) \) consists of just one point, all closed contours on the covering are trivial in the relative homology space \( H_1(L \setminus f^{-1}(\infty)) \). Therefore, the non-closed contours from this space can be characterized by their end points, i.e., any contour connecting points from \( f^{-1}(\lambda) \) on the \( k \)th and \((k + 1)\)th sheet is equivalent to \( \gamma_{k,k+1} \) up to orientation. Then it is easy to see that the monodromy matrix corresponding to the loop \( \gamma_\infty \) based at \( \lambda_0 \) and going around \( \lambda = \infty \) counterclockwise has the form:

\[
M_\infty = \begin{pmatrix}
0 & 0 & \ldots & 0 & -1 \\
1 & 0 & \ldots & 0 & -1 \\
0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -1
\end{pmatrix}.
\]

**B Completeness of the set of solutions to the Fuchsian system**

Here we are going to prove the completeness of the set of solutions to the system (2.7), (2.8) given by formula (3.14) with the integration contours (3.11) - (3.13) forming a basis in \( H_1(L \setminus f^{-1}(\infty); \pi^{-1}(\lambda)) \).

The whole section will be devoted to the proof of the following theorem:

**Theorem 9** The determinant of the matrix function \( \Phi \) defined by (3.14), (3.17) is given by:

\[
\det \Phi = C \prod_{j=1}^{N} (\lambda - \lambda_j)^{1/2},
\]

where \( C \neq 0 \) is a constant independent of \( \lambda \) and \( \{\lambda_j\} \).

**Proof.** Since the function \( \Phi \) satisfies linear system (2.7) with \( q = -1/2 \), we have:

\[
\frac{d}{d\lambda} \log \det \Phi = \text{tr} \left\{ -\sum_{j=1}^{N} E_j (V - \frac{1}{2} I) \frac{1}{\lambda - \lambda_j} \right\} = \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j},
\]

45
where we used the relation \( \text{tr} V = 0 \). Analogously, from (2.8) we get
\[
\frac{d}{d\lambda_j} \log \det \Phi = -\frac{1}{\lambda_j - \lambda_j}.
\]
Therefore, \( \det \Phi \) has the form (B.1) with some constant \( C \). What remains to check is that \( C \) is not equal to 0, i.e., the columns of the matrix \( \Phi(\lambda) \) form a complete set of linearly independent solutions to (2.7), (2.8).

For simplicity, we restrict ourselves to the space of coverings with no branching at infinity, i.e., \( m = d \). According to the Riemann-Hurwitz formula we have in this case \( N = 2g + 2d - 2 \).

Let us choose generators of the fundamental group \( \pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_N\}, \lambda_0) \) in such a way that the corresponding generators of the monodromy group of the covering are given by (A.1).

The branch cuts can then be chosen to connect the branch points \( P_{2k+1} \) and \( P_{2k+2} \), \( k = 0, \ldots, g + d - 1 \). The branch cuts \( [P_1, P_2], \ldots, [P_{2g+1}, P_{2g+2}] \) connect the sheets number \( d - 1 \) and \( d \); the branch cut \( [P_{2g+3}, P_{2g+4}] \) connects sheets number \( d - 1 \) and \( d - 2 \) etc; the branch cut \( [P_{N-1}, P_N] \) connects sheets number 2 and 1. In this way we realize the branch covering \( \mathcal{X} \) as a hyperelliptic Riemann surface of genus \( g \) with \( d - 2 \) Riemann spheres attached to it.

Due to Corollary 11 and relations (4.4), (4.6), the completeness of the set of our solutions to the system (2.7), (2.8) depends neither on the choice of a symplectic basis \( (a_\alpha, b_\alpha) \) used in the normalization of the bidifferential \( W \), nor on the choice of a symplectic basis \( (a_\alpha, b_\alpha) \) in (3.11) used as integration contours in (3.14). Therefore, we shall verify the completeness choosing these two bases to our convenience. First, we choose them to coincide: \( (a_\alpha, b_\alpha) = (a_\alpha, b_\alpha) \). Second, we choose these contours to lie on the “hyperelliptic part” of the covering as shown in Figure 5: the cycle \( a_\alpha \) encircles the ramification points \( P_{2a+1}, P_{2a+2} \) on the \( d \)th sheet, and the cycle \( b_\alpha \) goes around the points \( P_2 \) and \( P_{2a+1} \).

Our proof of the non-vanishing of the constant \( C \) will be inductive: first we check that \( C \neq 0 \) for any covering with \( d = 2 \) (i.e., a hyperelliptic covering) of any genus. Second, we check that \( C \) remains non-vanishing when we attach any number of Riemann spheres to the 2-sheeted covering keeping the genus of the covering unchanged.

### B.1 Example: two sheets, two branch points

In this section we discuss the simplest case of rational functions \( f \) of degree two with simple poles, whose equivalence classes form the Hurwitz space \( \mathcal{H}_{0,2}(1,1) \). Up to a Möbius transformation in the \( \gamma \)-plane, any degree two rational function with critical values \( \lambda_1 \) and \( \lambda_2 \) is equivalent to the function
\[
f(\gamma) = \frac{\lambda_1 - \lambda_2}{4} \left( \gamma + \frac{1}{\gamma} \right) + \frac{\lambda_1 + \lambda_2}{2}.
\]

The function \( f(\gamma) \) (B.2) defines a two-sheeted genus zero branched covering \( \mathcal{X} \) of the Riemann sphere with two branch points \( \lambda_1 \) and \( \lambda_2 \); this covering is the Riemann surface of the function \( \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} \). For simplicity, in this section we identify the ramification points \( P_{1,2} \) with the corresponding branch points \( \lambda_{1,2} \).

The uniformisation map, i.e., the map from the covering \( \mathcal{X} \) to the Riemann sphere, is given by the function
\[
h(\lambda) = \frac{2}{\lambda_1 - \lambda_2} \left\{ \lambda - \frac{\lambda_1 + \lambda_2}{2} + \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} \right\}.
\]

46
the value of \( \lambda \) together with the sign of the square root \( \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} \) determines the point \( P \in \mathcal{X} \). The functions \( f \) \((B.2)\) and \( h \) \((B.3)\) are related by \( f \circ h(\lambda) = \lambda \). In terms of the function \( h \) the bidifferential \( W \) has the form:

\[
W(\lambda, \mu) = \frac{dh(\lambda) \, dh(\mu)}{(h(\lambda) - h(\mu))^2} .
\]

(B.4)

The relative homology group \( H_1(\mathcal{L} \setminus f^{-1}(\infty); f^{-1}(\lambda)) \) is in this case two-dimensional; a basis in this group can be chosen to consist of a closed contour \( s_1 := l_1 \) around \( \infty^{(1)} \) \((3.12)\), and a contour \( s_2 := \gamma_{1,2}(\lambda) \) \((3.13)\) connecting in some way the points \( \lambda^{(1)} \) and \( \lambda^{(2)} \); we shall choose \( \gamma_{1,2}(\lambda) \) to consist of two segments: the first segment lies on the first sheet and connects the points \( \lambda_1 \) and \( \lambda_2 \) with the branch point \( \lambda_1 \); the second interval lies on the second sheet and connects the points \( \lambda_1 \) and \( \lambda_2 \).

If one of the arguments of the \( W \) coincides with a branch point (see \((3.8)\)), we get from \((B.3)\), \((B.4)\) and \((3.10)\):

\[
W(\lambda, \lambda_1) = \frac{\sqrt{\lambda_1 - \lambda_2}}{2} \frac{d\lambda}{(\lambda - \lambda_1)^{3/2}(\lambda - \lambda_2)^{1/2}} ;
\]

(B.5)

\[
W(\lambda, \lambda_2) = \frac{\sqrt{\lambda_2 - \lambda_1}}{2} \frac{d\lambda}{(\lambda - \lambda_2)^{3/2}(\lambda - \lambda_1)^{1/2}}
\]

The choice of the sign in these formulas corresponds to the choice of the signs of distinguished local parameters near \( \lambda_1 \) and \( \lambda_2 \) i.e. the parameters \( \epsilon_1 \) and \( \epsilon_2 \).

Therefore, according to \((3.16)\), for the first column of the matrix \( \Phi \) we get:

\[
\phi_1^{(s_1)} = 2\pi i \, W(\infty^{(1)}, \lambda_1) = -2\pi i \, \frac{\sqrt{\lambda_1 - \lambda_2}}{2},
\]

(B.6)

\[
\phi_2^{(s_1)} = 2\pi i \, W(\infty^{(1)}, \lambda_2) = -2\pi i \, \frac{\sqrt{\lambda_2 - \lambda_1}}{2}.
\]

(B.7)

Integration over the contour \( \gamma_{12}(\lambda) \) gives the following expressions for the second column of the matrix \( \Phi \):

\[
\phi_1^{(s_2)} = -\frac{2}{\sqrt{\lambda_1 - \lambda_2}} \left\{ \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} + \frac{1}{2}(\lambda_1 - \lambda_2) \log h(\lambda) \right\},
\]

(B.8)

\[
\phi_2^{(s_2)} = -\frac{2}{\sqrt{\lambda_2 - \lambda_1}} \left\{ \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} + \frac{1}{2}(\lambda_2 - \lambda_1) \log h(\lambda) \right\}.
\]

(B.9)

Computing the determinant of the matrix function \( \Phi \) \((B.6) - (B.9)\), we get

\[
\det \Phi = \pm 8\pi \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}.
\]

B.2 Completeness for the case of two simple poles \((d = 2)\)

We start by proving a few auxiliary facts related to degeneration of hyperelliptic Riemann surfaces. Consider a hyperelliptic Riemann surface \( \mathcal{X}_g \) defined by the equation

\[
\nu^2 = \Pi_{2g+2}(\lambda) := \prod_{k=1}^{2g+2} (\lambda - \lambda_k).
\]

We are going to study the behaviour of the bidifferential \( W \) under the degeneration of one of the branch cuts: we put \( \lambda_0 := \lambda_{2g+1} \) and consider the limit \( \lambda_{2g+2} \to \lambda_0 \).

47
As a result of the degeneration of the covering $X_g$ there arises the hyperelliptic Riemann surface $X_{g-1}$ of genus $g - 1$ defined by the equation

$$\nu^2 = \Pi_{2g}(\lambda) := \prod_{k=1}^{2g} (\lambda - \lambda_k). \quad (B.10)$$

Due to the choice of a canonical basis of cycles $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ on $X_g$ as shown in Figure 5, the cycles $\{a_\alpha, b_\alpha\}_{\alpha=1}^{g-1}$ in the limit $\lambda_{2g+2} \to \lambda_{2g+1}$ provide a canonical basis of cycles on $X_{g-1}$.

Let us denote by $W_g(P, Q)$ the canonical meromorphic bidifferential $W$ on the covering $X_g$ of genus $g$. Consider the behaviour of $W_g(P, Q)$ in the limit $\lambda_{2g+2} \to \lambda_{2g+1} \equiv \lambda_0$. According to the general theory (see [20]), no second or higher order pole of $W_g$ at $P_0$ arises under such degeneration. Since all $a$-periods of $W_g(P, Q)$ with respect to both of its arguments vanish, and in the limit the $a_0$ period becomes the residue at $P_0$, the first order pole at $P_0$ also does not arise in the limit, and the bidifferential $W_g(P, Q)$ does not gain any singularity at $P_0$ on $X_{g-1}$. At all other points, the singularity structure of $W_g(P, Q)$ under the degeneration coincides with that of $W_{g-1}(P, Q)$. Therefore, if $f(P)$ and $f(Q)$ remain independent of $\lambda_{2g+2}$ and lie outside of a fixed neighbourhood of $\lambda_0$, we have as $\lambda_{2g+2} \to \lambda_0$:

$$W_g(P, Q) = W_{g-1}(P, Q) + o(1). \quad (B.11)$$

The analysis becomes more subtle if one of the arguments of $W$ coincides with $P_{2g+1}$ or $P_{2g+2}$:

**Lemma 3** Let $f(P)$ lie outside of a fixed neighbourhood of $\lambda_0 := \lambda_{2g+1}$ and be independent of $\lambda_{2g+2}$. Then

$$W_g(P, P_{2g+2}) = \frac{\sqrt{\lambda_{2g+2} - \lambda_0}}{2} \{W_{g-1}(P, P_0) - W_{g-1}(P, P_0^*) + o(1)\}, \quad (B.12)$$

$$W_g(P, P_{2g+1}) = \frac{\sqrt{\lambda_0 - \lambda_{2g+2}}}{2} \{W_{g-1}(P, P_0) - W_{g-1}(P, P_0^*) + o(1)\}, \quad (B.13)$$

as $\lambda_{2g+2} \to \lambda_0$, where $P_0$ and $P_0^*$ are the points on the 1st and 2nd sheets of $\mathcal{L}_{g-1}$, respectively, projecting to $\lambda_0$ on the $\lambda$-plane.

**Proof.** The proof of this lemma can be obtained analogously to ([20], p.51, 52) using the Rauch variational formulas. Consider, for example, (B.12). In the hyperelliptic case considered here, the asymptotics (B.12) can alternatively be derived from an explicit formula for $W_g(P, P_{2g+2})$. Namely, the differential $W_g(P, P_{2g+2})$ can be written as follows:

$$W_g(P, P_{2g+2}) = W^0(P) - \sum_{\alpha=1}^{g} \left\{ \oint_{a_\alpha} W^0 \right\} w_\alpha(P), \quad (B.14)$$

where

$$W^0(P) := \frac{1}{\lambda - \lambda_{2g+2}} \sqrt{\Pi_{2g}(\lambda_{2g+2})} \sqrt{\lambda_{2g+2} - \lambda_0} d\lambda \quad (B.15)$$

(with $\lambda = f(P)$) is a non-normalized meromorphic differential having the same singular part as $W_g(P, P_{2g+2})$; a linear combination of holomorphic differentials in (B.14) ensures the vanishing of all $a$-periods of the right hand side.

In the limit $\lambda_{2g+2} \to \lambda_0$ we have

$$\frac{W^0(P)}{\sqrt{\lambda_{2g+2} - \lambda_0}} \to \frac{d\lambda}{2(\lambda - \lambda_0)^2} \frac{\sqrt{\Pi_{2g}(\lambda_0)}}{\sqrt{\Pi_{2g}(\lambda)}}. \quad (B.16)$$
The holomorphic terms in (B.14) guarantee the vanishing of all periods of the differential \((\lambda_{2g+2} - \lambda_0)^{-1/2}W_g(P, P_{2g+2})\), as well as the vanishing of the residues at \(P_0\) and \(P_0^*\) of this differential in the limit considered. The coefficient in front of \((\lambda - \lambda_0)^{-2}\) in the expansion at \(P_0\) and \(P_0^*\) of the differential in the limit coincides with that in (B.16); therefore, taking into account the normalization condition \(\oint_{a_k} W = 0, \ k = 1, \ldots, g\), we arrive at (B.12).

Below we use also the following

**Lemma 4** In the limit \(\lambda_{2g+2} \rightarrow \lambda_{2g+1} := \lambda_0\), the following asymptotics hold true:

\[
\oint_{a_g} f(P)W_g(P, P_{2g+2}) = \pi i (\lambda_{2g+2} - \lambda_0)^{1/2}(1 + o(1)), \tag{B.17}
\]

\[
2\pi i w_g(P_{2g+2}) = (\lambda_{2g+2} - \lambda_0)^{-1/2}(2 + o(1)), \tag{B.18}
\]

\[
\oint_{b_g} f(P)W_g(P, P_{2g+2}) = (\lambda_{2g+2} - \lambda_0)^{-1/2}(2\lambda_0 + o(1)). \tag{B.19}
\]

and

\[
\oint_{a_g} f(P)W_g(P, P_{2g+1}) = \pi i (\lambda_0 - \lambda_{2g+2})^{1/2}(1 + o(1)), \tag{B.20}
\]

\[
2\pi i w_g(P_{2g+1}) = (\lambda_0 - \lambda_{2g+2})^{-1/2}(2 + o(1)), \tag{B.21}
\]

\[
\oint_{b_g} f(P)W_g(P, P_{2g+1}) = (\lambda_0 - \lambda_{2g+2})^{-1/2}(2\lambda_0 + o(1)). \tag{B.22}
\]

**Proof.** We prove only the set of formulas involving \(P_{2g+2}\). To prove (B.17) we make use of the asymptotics (B.12), which implies as \(\lambda_{2g+2} \rightarrow \lambda_0\)

\[
\frac{1}{\pi i (\lambda_{2g+2} - \lambda_0)^{1/2}} \oint_{a_g} f(P)W_g(P, P_{2g+2}) \rightarrow \text{res}_{P=P_0} \{ f(P) W_{g-1}(P, P_0) \} = 1 ,
\]

which yields (B.17).

To prove (B.18), let us write the differential \(w_g\) in the form:

\[
w_g(P) = \frac{1}{2\pi i} \frac{d\lambda}{\sqrt{(\lambda - \lambda_0)(\lambda - \lambda_{2g+2})}} \frac{Q_{g-1}(\lambda)}{\Pi_{2g}(\lambda)} \lambda = f(P), \tag{B.20}
\]

where \(Q_{g-1}(\lambda)\) is a polynomial of degree \(g - 1\) with coefficients depending on \(\{\lambda_k\}\). In the limit \(\lambda_{2g+2} \rightarrow \lambda_0\), the differential \(w_g\) becomes the normalized Abelian differential of the third kind with poles at \(P_0\) and \(P_0^*\) and residues \(+1\) and \(-1\), respectively (this follows from the normalization \(\oint_{a_g} w_{\alpha} = \delta_{\alpha g}\)). Therefore, if we first take the limit \(\lambda_{2g+2} \rightarrow \lambda_0\), and then put \(\lambda = \lambda_0\), we get \(Q_{g-1}(\lambda_0) = \sqrt{\Pi_{2g}(\lambda_0)}\). Since from (B.20) we have

\[
w_g(P_{2g+2}) = \frac{1}{\pi i} \frac{1}{\sqrt{\lambda_{2g+2} - 2g+1}} \frac{Q_{g-1}(\lambda_{2g+2})}{\sqrt{\Pi_{2g}(\lambda_{2g+2})}} ,
\]

in the limit \(\lambda_{2g+2} \rightarrow \lambda_0\) we arrive at (B.18).

The asymptotics (B.19) can be deduced from (B.18) and (B.17) by noticing that the integral \(\oint_{b_g} (f(P) - \lambda_0)W(P, P_{2g+2})\) remains finite in the limit \(\lambda_{2g+2} \rightarrow \lambda_0\). One should also use the relation

\[
2\pi i w_g(P_{2g+2}) = \oint_{b_g} W(P, P_{2g+2}).
\]

Now we are in a position to prove the following
Proposition 5 The constant $C$ in (B.7) is non-vanishing for $d = 2$, i.e., for all hyperelliptic coverings of genus $g$ (with no branching at $\infty$).

Proof. For $d = 2$ the number of ramification points is $N = 2g + 2$. We prove the proposition by induction over the genus of the covering. That is we reduce the computation of the determinant of the $(2g + 2) \times (2g + 2)$-dimensional matrix $\Phi_g$ to the computation of the determinant of the $2g \times 2g$-dimensional matrix $\Phi_{g-1}$ arising from $\Phi_g$ in the limit $\lambda_{2g+2} \to \lambda_{2g+1} \equiv \lambda_0$. The base case of the induction, the determinant of the $2 \times 2$ matrix-solution $\Phi_0$ corresponding to the genus zero two-fold coverings with two ramification points, was computed in Section B.1 $\det \Phi_0 = \pm 8\pi \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}$.

Consider the $2g \times 2g$ matrix obtained from $\Phi_g$ by crossing out two columns and two rows. The resulting matrix, due to (B.11), tends in the limit $\lambda_{2g+2} \to \lambda_{2g+1}$ to a solution $\Phi_{g-1}$ given by (3.14) to the Riemann-Hilbert problem associated to the hyperelliptic curve (B.10) of genus $g - 1$. According to the assumption of our induction, $\det \Phi_{g-1}(\lambda) \neq 0$ for $\lambda \in \C \setminus \{\lambda_1, \ldots, \lambda_N\}$.

Due to Lemma 3, $W_g(P, P_{2g+2})$ and $W_g(P, P_{2g+1})$ tend to 0 as $\lambda_{2g+2} \to \lambda_{2g+1} \equiv \lambda_0$ if $f(P)$ is independent of $\lambda_{2g+1}$ and $\lambda_{2g+2}$. Therefore, the entries of the deleted $(2g + 1)$st and $(2g + 2)$nd rows of the matrix $\Phi_g$ not belonging to the deleted columns tend to 0 as $\lambda_{2g+2} \to \lambda_{2g+1}$.

Thus in our limit, $\det \Phi_g$ tends to the product of $\det \Phi_{g-1}$ and the determinant of the $2 \times 2$ block at the crossing of the deleted rows and columns:

$$\det \Phi_g \rightarrow \det A \det \Phi_{g-1},$$

where

$$A = \lim_{\lambda_{2g+2} \to \lambda_0} \left( \begin{array}{cc} \frac{f_{a_g}}{f_{a_g}} f(P) W(P, P_{2g+1}) & 2\pi i w_1(P_{2g+1}) \lambda - \frac{f_{b_g}}{f_{b_g}} f(P) W(P, P_{2g+1}) \\ \frac{f_{a_g}}{f_{a_g}} f(P) W(P, P_{2g+2}) & 2\pi i w_1(P_{2g+2}) \lambda - \frac{f_{b_g}}{f_{b_g}} f(P) W(P, P_{2g+2}) \end{array} \right).$$

Using Lemma 4, we find the behaviour of $\det A$ in the limit:

$$\det \left( \begin{array}{cc} \pi i(\lambda_{2g+1} - \lambda_{2g+2})^{1/2} & 2(\lambda_{2g+1} - \lambda_{2g+2})^{-1/2}(\lambda - \lambda_0) \\ \pi i(\lambda_{2g+2} - \lambda_{2g+1})^{1/2} & 2(\lambda_{2g+2} - \lambda_{2g+1})^{-1/2}(\lambda - \lambda_0) \end{array} \right) = \pm 4\pi (\lambda - \lambda_0).$$

The corresponding constants in (B.1) are thus related by $C_g = \pm 4\pi C_{g-1}$ and $C_g \neq 0$ if $C_{g-1} \neq 0$. $\square$

B.3 Completeness for an arbitrary number of simple poles

We are going to prove the completeness of our set of solutions for the Hurwitz diagram shown in Fig.1, i.e., the permutations associated to branch points are given by (A.1). Then the completeness for an arbitrary set of elementary permutations follows from the well-known fact that the space $H_{g,d}(1, \ldots, 1)$ is connected and independence of the constant $C$ from (B.1) on $\lambda$ and $\{\lambda_j\}$.

For the coverings defined by (A.1) we shall perform an induction over the number of sheets without changing the genus of the covering $\mathcal{X}$ (in this section we denote it by $\mathcal{X}_d$); on each step we detach one sheet by a degeneration of one branch cut. Put $P_0 := P_{N-1}$ (and $\lambda_0 := \lambda_{N-1}$) and take the limit $P_N \to P_0$. In this limit the first sheet of $\mathcal{X}_d$ detaches and the $d$-sheeted covering splits into a $(d - 1)$-sheeted covering $\mathcal{X}_{d-1}$ of the same genus with the ramification points $\{P_k\}_{k=1}^{N-2}$, and the Riemann sphere, which we denote by $\mathcal{X}_1$. Denote the bidifferential $W$ on $\mathcal{X}_d$ by $W_d$, on $\mathcal{X}_{d-1}$ by $W_{d-1}$ and on
$\mathcal{X}_1$ by $W_1$ (note that $W_1(\lambda, \mu) = (\lambda - \mu)^{-2}d\lambda d\mu$). The points in the set $f^{-1}(\lambda_0)$ on the covering we denote by $\lambda_0^{(k)}$ (the upper index indicates the sheet number).

Let us prove a few auxiliary facts about this type of degeneration. First, we determine the behaviour of the bidifferential $W_d(P, Q)$ in our limit. Assuming that $f(P)$ and $f(Q)$ are independent of $\lambda_N$ and $\lambda_{N-1}$ we have the following obvious asymptotics (see [20]):

\[ W_d(P, Q) \to W_{d-1}(P, Q), \quad P, Q \in \mathcal{X}_{d-1}; \tag{B.21} \]

\[ W_d(P, Q) \to W_1(P, Q) \equiv \frac{d\mu(P) d\mu(Q)}{(\mu(P) - \mu(Q))^2}, \quad P, Q \in \mathcal{X}_1, \]

where $\mu$ is a coordinate on the Riemann sphere $\mathcal{X}_1$; and

\[ W_d(P, Q) \to 0, \quad P \in \mathcal{X}_{d-1}, \quad Q \in \mathcal{L}_1. \]

The next lemma is less trivial.

**Lemma 5** There are the following asymptotic expansions as $P_N \to P_{N-1} = P_0$:

\[ W_d(P, P_0) = \frac{\sqrt{\lambda_0 - \lambda_N}}{2} \{ W_{d-1}(P, \lambda_0^{(2)}) + O(\lambda_0 - \lambda_N) \}, \quad P \in \mathcal{X}_{d-1}, \tag{B.22} \]

and

\[ W_{d-1}(P, \lambda_0^{(2)}) := \left. \frac{W_{d-1}(P, Q)}{df_0(Q)} \right|_{Q = \lambda_0^{(2)}}, \]

where $f_0$ is the meromorphic function on $\mathcal{L}_{d-1}$ which defines the covering $\mathcal{X}_{g-1}$;

\[ W_d(P, P_0) = \frac{\sqrt{\lambda_0 - \lambda_N}}{2} \{ W_1(P, \lambda_0^{(1)}) + O(\lambda_0 - \lambda_N) \}, \quad P \in \mathcal{X}_1; \tag{B.23} \]

and

\[ W_1(P, \lambda_0^{(1)}) := \frac{d\mu(P)}{(\mu(P) - \lambda_0)^2}, \]

$\mu$ being the coordinate on the Riemann sphere $\mathcal{X}_1$.

**Proof.** Following [20], Chapter 3, consider a domain $\Omega \subset \mathcal{L}_d$, which contains the segment $[P_0, P_N]$ on both 1st and 2nd sheets, and can be conformally mapped to an annulus by the map

\[ h(\lambda) = \frac{1}{\lambda_0 - \lambda_N} \left\{ \lambda - \frac{\lambda_0 + \lambda_N}{2} + \sqrt{(\lambda - \lambda_0)(\lambda - \lambda_N)} \right\}; \]

the union of two banks of the branch cut $[P_0, P_N]$ is mapped by the function $h(\lambda)$ to the unit circle. The Laurent series for $W_d(P, P_0)$ in the coordinate $h(\lambda)$ in a neighbourhood of the unit circle can be written as follows in terms of the coordinate $\lambda$ within the domain $\Omega$ [20]:

\[ W_d(P, P_0) = \frac{1}{\sqrt{(\lambda - \lambda_0)(\lambda - \lambda_N)}} \sum_{k=-1}^{\infty} a_k(\tau)(\lambda - \lambda_0)^k d\lambda + \sum_{k=0}^{\infty} b_k(\tau)(\lambda - \lambda_0)^k d\lambda, \tag{B.24} \]

where $\lambda = f(P)$; $\tau = \sqrt{\lambda_N - \lambda_0}$; coefficients $a_k(\tau)$ and $b_k(\tau)$ are holomorphic at $\tau = 0$. The first sum in (B.24) starts from $k = -1$ since $W_d(P, P_0)$ has a quadratic pole at $P_0$. Since the singular part of
$W(P, P_0)$ at $P = P_0$ has the form $(\lambda - \lambda_0)^{-1}d\sqrt{\lambda - \lambda_0}$, we have $a_{-1}(\tau) = \sqrt{\lambda_0 - \lambda_N}/2$. The term in the second sum in (B.24) corresponding to $k = -1$ is absent since the residue of $W_d(P, P_0)$ at $P = P_0$ equals zero.

Therefore, the differential

$$\lim_{\lambda_N \to \lambda_0} \frac{2}{\sqrt{\lambda_0 - \lambda_N}} W_d(P, P_0), \quad P \in \Omega,$$

has a singular part of the form

$$\frac{d\lambda}{(\lambda - \lambda_0)^2}, \quad \lambda = f(P),$$

in neighbourhoods of $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$. The term containing the first order pole must vanish since the integral of (B.25) over the (homologous to zero) contour on $\mathcal{X}_d$ encircling the branch cut $[P_0, P_N]$ is zero; thus the residues of (B.25) at $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$ vanish.

The differential (B.25) does not have any other singularities on either $\mathcal{X}_{d-1}$ or $\mathcal{X}_1$; this differential has all vanishing $a$-periods on $\mathcal{X}_{d-1}$. Therefore, we arrive at (B.22), (B.23). □

Lemma 6 There are the following asymptotic expansions as $\lambda_N \to \lambda_{N-1} = \lambda_0$:

$$\sqrt{\lambda_0 - \lambda_N} \int_P^Q W_d(R, P_0) = 2 + O(\lambda_N - \lambda_0);$$

(B.26)

$$\sqrt{\lambda_0 - \lambda_N} \int_P^Q f(R) W_d(R, P_0) = 2\lambda_0 + O(\lambda_N - \lambda_0),$$

(B.27)

where $P \in \mathcal{X}_1$, $Q \in \mathcal{X}_{d-1}$; $f(P)$ and $f(Q)$ are assumed to be independent of $\lambda_N$.

Proof. The proof is similar to the proof of the previous lemma. Consider (B.26). The integral of $W_d(R, S)$ with respect to $R$ between the points $P$ and $Q$ is an abelian differential of the third kind in $S$ with simple poles at $S = P$ and $S = Q$ and residues $-1$ and $1$, respectively. We denote this differential by $W_d^{P,Q}(S) := \int_P^Q W_d(\cdot, S)$. Since the sum of the residues of the differential $W_1(S) := \lim_{\lambda_N \to \lambda_0} W_d^{P,Q}(S)$ on $\mathcal{X}_1$ must vanish, we conclude that $W_1(S)$ has two simple poles on $\mathcal{X}_1$; the pole with the residue $-1$ at $S = P$, inherited from $W_d^{P,Q}(S)$, and a new pole at $\lambda_0^{(1)}$, arising as a result of the degeneration, with the residue $+1$ (the absence of higher order terms of $W_1(S)$ at $\lambda_0^{(1)}$ follows from the expansion (B.24) for $W(P, P_0)$). Similarly, on $\mathcal{X}_{d-1}$, the differential $W_d^{P,Q}(S)$ tends to the normalized abelian differential of the third kind with simple poles at $S = \lambda_0^{(2)}$ and $S = Q$ and residues $-1$ and $+1$, respectively.

Let us now write down an analog of the expansion (B.24) for $W_d^{P,Q}(S)$, when $S \in \Omega$:

$$W_d^{P,Q}(S) = \frac{1}{\sqrt{\lambda - \lambda_0}(\lambda - \lambda_N)} \sum_{k=0}^{\infty} c_k(\tau)(\lambda - \lambda_0)^k d\lambda + \sum_{k=0}^{\infty} d_k(\tau)(\lambda - \lambda_0)^k d\lambda,$$

(B.28)

where $\lambda = f(S)$; as before, $\tau := \sqrt{\lambda_0 - \lambda_N}$; the coefficients $c_k(\tau)$ and $d_k(\tau)$ are holomorphic at $\tau = 0$. Both sums in (B.28) start from $k = 0$ since the differential $W_d^{P,Q}(S)$ is holomorphic at $S = P_0 \equiv P_{N-1}$ and $S = P_N$. Since in our limit the differential $W_d^{P,Q}(S)$ gains simple poles at $S = \lambda_0^{(2)}$ and $S = \lambda_0^{(1)}$ with residues $-1$ and $+1$, respectively, we conclude that $c_0 = 1 + o(\tau)$ as $\tau \to 0$. Now, taking $S = P_0$,
and evaluating $W_d^P Q$ at $P_0$ with respect to the local parameter $\sqrt{\lambda - \lambda_0}$ similarly to (3.8), we arrive at (3.26).

The asymptotics (3.27) easily follows from (3.26) since the integral $\int_P^Q (f(R) - \lambda_0) W_d(R, P_0)$ behaves as $o(1)$ in our limit. □

We note that all the asymptotics computed in the above lemmas are symmetric under the interchange of $\lambda_N$ and $\lambda_{N-1}$.

By the assumption of the induction, the constant $C$, we denote it by $C_{d-1}$, in relation (3.11) corresponding to the branch covering $X_{d-1}$ is non-vanishing. One needs to prove the non-vanishing of the constant $C_d$ corresponding to the covering $X_d$.

Denote the function $\Phi$ (3.14) corresponding to the $d$-sheeted covering $X_d$ by $\Phi_d$, and the function $\Phi$ corresponding to the $(d - 1)$-sheeted covering $X_{d-1}$ by $\Phi_{d-1}$. The columns of $\Phi_d$ given by the integrals over the contours $l_1$ encircling $\infty^{(1)}$, and the contour $\gamma_{1, 2}(\lambda)$ have, according to (3.14) and (3.16), the form:

$$\Phi_k^{(\gamma_{1, 2}(\lambda))} = -\int_{\lambda^{(1)}}^{\lambda^{(2)}} f(P) W(P, P_k) + \lambda \int_{\lambda^{(1)}}^{\lambda^{(2)}} W(P, P_k),$$

and

$$\Phi_k^{(l_1)} = -2\pi i W(\infty^{(1)}, P_k).$$

The contours $l_1$ and $\gamma_{1, 2}(\lambda)$ are absent from the set of integration contours determining $\Phi_{d-1}$. The rows corresponding to $P_{N-1}$ and $P_N$ are also missing in $\Phi_{d-1}$. The $2 \times 2$ block on the intersection of these rows and columns in the matrix $\Phi_d$ looks as follows:

$$\mathbf{B} = \begin{pmatrix} -\int_{\lambda^{(1)}}^{\lambda^{(2)}} f(P) W(P, P_{N-1}) + \lambda \int_{\lambda^{(1)}}^{\lambda^{(2)}} W(P, P_{N-1}) & -2\pi i W(P_{N-1}, \infty^{(1)}) \\ -\int_{\lambda^{(1)}}^{\lambda^{(2)}} f(P) W(P, P_N) + \lambda \int_{\lambda^{(1)}}^{\lambda^{(2)}} W(P, P_N) & -2\pi i W(P_N, \infty^{(1)}) \end{pmatrix}.$$ 

According to (3.21), the $(2N-2) \times (2N-2)$ minor in the matrix $\Phi_d$ obtained by deleting these two rows and two columns tends to $\Phi_{d-1}$ in our limit. Since all other entries of the two rows of $\Phi_d$ corresponding to $P_{N-1}$ and $P_N$, tend to 0 as $P_N \to P_0 = P_{N-1}$, we see that in this limit $\det \Phi_d \to \det \mathbf{B} \det \Phi_{d-1}$.

Now, due to Lemmas 5 and 6 in this limit

$$\det \mathbf{B} \to \begin{pmatrix} -2 & -\lambda - \lambda_0 \\ -\lambda - \lambda_0 & \sqrt{\lambda_N - \lambda_{N-1}}^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_N - \lambda_{N-1}} & \sqrt{\lambda_{N-1} - \lambda_N} \\ \sqrt{\lambda_{N-1} - \lambda_N} & \lambda - \lambda_0 \end{pmatrix} (\lambda - \lambda_0) = \pm 2i (\lambda - \lambda_0),$$

where $\lambda_0 = f(P_0)$; therefore, $C_d = \pm 2i C_{d-1}$, i.e., $C_{d-1} \neq 0$ implies $C_d \neq 0$. □

**B.4 Completeness for an arbitrary number of poles of arbitrary multiplicities**

Here we prove that the set of solutions (3.17), (3.14) is complete for any Hurwitz space $H_{g, d}(k_1, \ldots, k_m)$. The proof will again be inductive. Namely, we are going to consider an element of the space $H_{g, d}(k_1, \ldots, k_m)$ as a limit of elements of the space $H_{g, d}(1, \ldots, 1)$. Speaking in terms of meromorphic functions, we consider the limit when some poles of the meromorphic function merge (and unavoidably they also merge with some critical points due to the Riemann-Hurwitz formula).

This is a partial case of the idea proposed by Harris and Mumford [25] who introduced the so-called spaces of admissible coverings which provide a natural compactification of Hurwitz spaces. The boundary components of the Hurwitz space are obtained by merging a subset of simple branch points to get a branch point of an arbitrary branching type.
If our Riemann surface is represented as a branched covering, this procedure is equivalent to sending some of the finite branch points to infinity and considering the covering obtained in the limit.

For simplicity consider the space $H_{g,d}(k+1,1,\ldots,1)$ (the number of 1’s is equal to $d-k-1$; we denote the multiple infinity by $\infty^{(1)}$). Any branched covering $X_{k+1}$ from this space can be obtained as a limit of branched coverings $X_k$ from the space $H_{g,d}(k,1,\ldots,1)$ (where the number of 1’s is equal to $d-k$, and the multiple infinity is also denoted by $\infty^{(1)}$) when a finite branch point (say, $\lambda_1$) on $X_k$ tends to $\infty$. The non-branched point over $\lambda=\infty$ which merges with $\infty^{(1)}$ in this limit will be denoted by $\infty^{(2)}$. The point $\lambda_1$ should be chosen such that the monodromy group of the covering $X_k$ turns in the limit into the monodromy group of the covering $X_{k+1}$. This requirement implies that the product of permutations corresponding to $\lambda=\lambda_1$ and $\lambda=\infty$ on $X_k$ coincides with the permutation corresponding to $\lambda=\infty$ on $X_{k+1}$: $\sigma^{(k)}_{\lambda_1} \sigma^{(k)}_{\infty} = \sigma^{(k+1)}_{\infty}$.

We show this limit schematically in Fig.20. Let us emphasize that in this limit the Riemann surface does not degenerate, i.e., no double point is formed; thus the analysis of the behavior of all ingredients of the solution to the Fuchsian system becomes less complicated.

Let us now consider the behavior of all objects entering the solution associated to $X_k$ in the limit $\lambda_1 \to \infty$.

To study this limit one needs to introduce a local coordinate on $X_k$ in a neighbourhood containing three points of interest: $P_1$, branched infinity $\infty^{(1)}$, and the simple infinity $\infty^{(2)}$. Such local parameter should remain regular in the limit $\lambda_1 \to \infty$. Clearly, neither the standard local coordinate $\sqrt{f(P)-\lambda_1}$ in a neighbourhood of $P_1$ nor the standard local coordinates $f(P)^{-1/k}$ in a neighbourhood of $\infty^{(1)}$ are suitable for studying this limit.

\[ f(P) = \left( \frac{\gamma^{k+1}}{k + 1 - b \gamma^{k}} \right)^{-1}, \]  

(B.29)

![Figure 20](image-url)  

Figure 20: Uniformization of a neighbourhood containing points $\infty^{(1)}$, $\infty^{(2)}$ and $P_1$ by a domain in the $\gamma$-plane. The limit $\lambda_1 \to \infty$ corresponds to $b \to 0$; in this limit the points $\infty^{(1)}$, $\infty^{(2)}$ and $P_1$ merge together.

Instead we consider on $X_k$ the local coordinate $\gamma(P)$ (see Fig.20) related to the function $f$ as follows:
for some $b \in \mathbb{C}$. In a neighbourhood of the points $P_1$, $\infty^{(1)}$ and $\infty^{(2)}$ we introduce a local holomorphic parameter $\gamma(P)$ in such a way that the family of coverings is locally (in a neighbourhood of $P_1$, $\infty^{(1)}$, $\infty^{(2)}$) given by $h(f(P), \gamma(P)) = 0$ with $h(f, \gamma) = f^{k+1} + (1 + k)b\gamma^k - b$ and some suitably chosen $b \in \mathbb{C}$. The ramification of the curves $h(f, \gamma) = 0$ projected to the $\lambda$-sphere occurs at the points where $h'(f, \gamma) = f^{k-1} + (\gamma - b) = 0$, i.e.,

$$\frac{1}{f^2} \frac{df}{d\gamma} = -\gamma^{k-1}(\gamma - b).$$

Thus the function $f(\gamma)$ \textbf{(B.29)} has two critical points: the point of multiplicity 1 at $\gamma = b$ and the point of multiplicity $k - 1$ at $\gamma = 0$. The critical value at $\gamma = b$ is assumed to coincide with $\lambda_1$ (i.e. the $\gamma$-coordinate of $P_1$ equals $b$), which implies

$$\lambda_1 = -\frac{k(k + 1)}{b^{k+1}}. \quad \text{(B.30)}$$

The critical value at $\gamma = 0$ is $\infty$ (thus $\gamma = 0$ is the $\gamma$-coordinate of $\infty^{(1)}$).

The point $\infty^{(2)}$ corresponds to the simple pole of the function \textbf{(B.29)}, which is given by

$$\gamma = \frac{b^{k+1}}{k}. \quad \text{(B.31)}$$

In the limit $\lambda_1 \to \infty$ (i.e., $b \to 0$) the points $\infty^{(1)}$, $\infty^{(2)}$ and $P_1$ merge. The local parameter $\gamma$ tends (up to a constant factor) to the standard local parameter $f(P)^{-1/(k+1)}$ on the Riemann surface $\mathcal{X}_{k+1}$:

$$\gamma \to (k + 1)^{1/(k+1)} f(P)^{-1/(k+1)} \quad \text{(B.32)}$$

as $b \to 0$.

To study the limit $\lambda_1 \to \infty$ of our solution of the Fuchsian system we need to study the asymptotics of $W(P, P_1)$, $W(P, \infty^{(2)})$ and $W(P_1, \infty^{(2)})$ in the limit. For that purpose one needs to relate the values of the bidifferential $W$ calculated in the standard local parameters at $\infty^{(1)}$, $\infty^{(2)}$ and $P_1$ to the values of $W$ calculated in the local parameter $\gamma$.

As $\gamma \to b$ (i.e., $P \to \lambda_1$) we have the following link between the local parameter $\gamma - b$ and the distinguished local parameter $\sqrt{f(\gamma) - \lambda_1}$:

$$\sqrt{f(\gamma) - \lambda_1} = \left( -\frac{1}{2} \right)^{1/2} \frac{k(k + 1)}{b^{(k+3)/2}} (\gamma - b)(1 + O(\gamma - b)). \quad \text{(B.33)}$$

The Jacobian between $\gamma$ and the local parameter $1/f$ at the point $\infty^{(2)}$ has the form:

$$\left. \frac{d(1/f)}{d\gamma} \right|_{\infty^{(2)}} = \gamma^{k-1}(\gamma - b) \quad \bigg|_{\gamma = b^{(k+1)/k}} = \frac{(k + 1)^{k-1}}{k^k} b^k. \quad \text{(B.34)}$$

The Jacobian between $\gamma$ and the local parameter $f(P)^{1/k}$ at $\infty^{(1)}$ we are not going to use.

**Remark 7** Suppose we have a one-parametric family of Riemann surfaces $\mathcal{L}^t$ such that in the limit $t \to 0$ the Riemann surface remains non-degenerate. Suppose, moreover, that the coordinate maps $\varphi^t_1$ defining the Riemann surface $\mathcal{L}^t$ holomorphically depend on $t$ and in the limit $t \to 0$ give the coordinate maps of the Riemann surface $\mathcal{L} := \mathcal{L}^{t=0}$. Suppose also that on each Riemann surface $\mathcal{L}^t$ we choose two points $P^t$ and $Q^t$ such that the family of points $P^t$ converges to $P \in \mathcal{L}$ and the family
of points $Q^t$ converges to $Q \in \mathcal{L}$. This convergence is understood in the following sense: the images of the points $P^t$ under a corresponding coordinate map $\varphi^t_1$ converge to the image of the point $P$ under the coordinate map $\varphi_1 := \varphi^t_{t=0}$, i.e., $\varphi^t_1(P^t) \to \varphi_1(P)$ as $t \to 0$. The same applies to the family $Q^t$ with corresponding coordinate map (denoted by $\varphi_2$ below).

Under these assumptions we have the convergency of the bidifferentials: $W^t(P^t, Q^t) \to W(P, Q)$; this convergency is understood in the following sense:

$$
\frac{W^t(P^t, Q^t)}{d\varphi_1^t(P^t)d\varphi_2^t(Q^t)} \to \frac{W(P, Q)}{d\varphi_1(P)d\varphi_2(Q)}.
$$

(B.35)

In our context the role of the parameter $t$ is played by the parameter $b$ related to $\lambda_1$ by (B.30). The coordinate map on $\mathcal{X}_k$ in a neighborhood containing all three points $\infty^{(1)}$, $\infty^{(2)}$ and $P_1$ is given by $\gamma(\lambda)$ which in the limit $b \to 0$ turns into the local parameter (B.32) around the point $\infty^{(1)}$ on $\mathcal{X}_{k+1}$.

As $b \to 0$, the bidifferential $W_k(P, Q)$ on $\mathcal{X}_k$ tends (in the sense described above) to the bidifferential $W_{k+1}(P, Q)$ on $\mathcal{X}_{k+1}$.

Taking this remark into account, we formulate

**Lemma 7** The following asymptotics hold as $\lambda_1 \to \infty$:

$$
W_k(P, P_1) = c_1 \lambda_1^{k+3/2} W_{k+1}(P, \infty^{(1)})(1 + o(1)) ; \tag{B.36}
$$

$$
W_k(P, \infty^{(2)}) = c_2 \lambda_1^{k+1} W_{k+1}(P, \infty^{(1)})(1 + o(1)) , \tag{B.37}
$$

and

$$
W_k(\infty^{(2)}, P_1) = c_3 \lambda_1^{1/2}(1 + o(1)) \tag{B.38}
$$

where $P \in \mathcal{X}_k$ is a point such that $\lambda := f(P)$ is independent of $\lambda_1$; $c_1$, $c_2$ and $c_3$ are constants; the point $\infty^{(1)}$ on $\mathcal{X}_{k+1}$ is an infinite point with branching of order $k$ which appears as a result of merging of the points $\infty^{(1)}$ and $\infty^{(2)}$ on $\mathcal{X}_k$.

**Proof.** Consider (B.36). We have

$$
W_k(P, P_1) := \frac{W_k(P, Q)}{d\sqrt{f(Q)} - \lambda_1} \bigg|_{Q=P_1} = \frac{W_k(P, Q)}{d\gamma(Q)} \bigg|_{Q=P_1} \frac{d\gamma(Q)}{d\sqrt{f(Q)} - \lambda_1} \bigg|_{Q=P_1} . \tag{B.39}
$$

From (B.33) we get:

$$
d\gamma(P) \bigg|_{P=P_1} = \left( -\frac{1}{2} \right)^{-1/2} \frac{b^{k+3/2} k(3k+1)}{k(k+1)} . \tag{B.40}
$$

To express this formula in terms of $\lambda_1$ we use (B.30), which gives

$$
b = \left\{ -\frac{k(k+1)}{\lambda_1} \right\}^{1/(k+1)} . \tag{B.41}
$$

To find the asymptotics of the first multiplier in the right hand side of (B.39) we recall that the coordinate $\gamma$ behaves in a regular way (see (B.32)) in the limit $\lambda_1 \to \infty$; therefore (see discussion before the lemma)

$$
\frac{W_k(P, Q)}{d\gamma(Q)} \bigg|_{Q=P_1} \to (k+1)^{1/(k+1)} \frac{W_{k+1}(P, Q)}{d(f(Q))^{1/(k+1)}} \bigg|_{Q=\infty^{(1)}} = (k+1)^{1/(k+1)} W_{k+1}(P, \infty^{(1)}). \tag{B.42}
$$

56
Combining (B.40) with (B.42) we come to (B.36), where the constant \(c_1\) is given by
\[
c_1 = 2^{1/2}(-1)^{\frac{k+1}{k+1}}(k+1)^{-\frac{k+3}{2(k+1)}}.
\]

Consider (B.37). We have
\[
W_k(P, \infty (2)) := \left. \frac{W_k(P, Q)}{d(1/f(Q))} \right|_{Q=\infty (2)} = \left. \frac{d\gamma(Q)}{d\gamma(P)} \right|_{Q=\infty (2)} \left( \frac{d(1/f(Q))}{d(1/f(P))} \right)_{Q=\infty (2)}.
\]
The first multiplier, in analogy to (B.42), behaves in the limit \(b \to 0\) as follows:
\[
\left. \frac{W_k(P, Q)}{d\gamma(Q)} \right|_{Q=\infty (2)} \to (k+1)^{1/(k+1)} W_{k+1}(P, \infty (1)). \tag{B.43}
\]

The second multiplier can be easily computed since from (B.29) and (B.31)
\[
\left. \frac{d(1/f)}{d\gamma} \right|_{P=\infty (2)} = \gamma^{k-1}(\gamma - b) \bigg|_{\gamma=b(k+1)/k} = \left( \frac{k+1}{k} \right)^{k-1} b^k, \tag{B.44}
\]
which, taking into account the link (B.30) between \(b\) and \(\lambda_1\), leads to (B.37) with
\[
c_2 = (-1)^{\frac{k}{k+1}}(k+1)^{-k} \left( \frac{k^2}{k+1} \right).
\]

Consider (B.38). In the left hand side we have
\[
W_k(\infty (2), P_1) := \left. \frac{W(P, Q)}{d(1/f(P))d\sqrt{f(Q)} - \lambda_1} \right|_{P=\infty (2), Q=P_1}
\]
\[
= \left. \frac{W_k(P, Q)}{d\gamma(P) d\gamma(Q)} \right|_{P=P_1, Q=\infty (2)} \left\{ \frac{d\gamma}{d(1/f)} \bigg|_{\gamma=b(k+1)/k} \right\} \left\{ \frac{d\gamma}{d\sqrt{f} - \lambda_1} \bigg|_{\gamma=b} \right\}. \tag{B.45}
\]
Furthermore,
\[
\left. \frac{W_k(P, Q)}{d\gamma(P) d\gamma(Q)} \right|_{P=P_1, Q=\infty (2)} = \left. \frac{W_k(\gamma, \mu)}{d\gamma d\mu} \right|_{\gamma=b, \mu=b(k+1)/k} = \frac{1}{(b(k+1)/k - b)^2} + O(1) = \frac{k^2}{b^2} + O(1) \tag{B.46}
\]
as \(b \to 0\).

Substituting (B.46), (B.44) and (B.40) into (B.45) we come to (B.38) with
\[
c_3 = \sqrt{2} \left( \frac{k}{k+1} \right)^{k+1/2}. \tag{B.47}
\]

Now we are in a position to prove that the determinant of the solution (3.17), (3.14) corresponding to the covering \(X_k\) (we shall denote this solution by \(\Phi_k\)) does not vanish identically. The proof is by induction. The base case of the induction is proved in Section (B.3): the determinant of the solution to the Fuchsian system which corresponds to a branch covering with no branching at infinity does not identically vanish. It remains to prove the following

Lemma 8 Suppose that \(\det \Phi_k \neq 0\). Then \(\det \Phi_{k+1} \neq 0\).
Proof. Since \( \det \Phi_k \neq 0 \) then

\[
\det \Phi_k (\lambda) = C \prod_{n=1}^{N} (\lambda - \lambda_n)^{1/2},
\]

where \( C \neq 0 \) is a constant. Let us now relate \( \det \Phi_k (\lambda) \) to \( \det \Phi_{k+1} (\lambda) \).

Recall that the closed contours around the points \( \infty^{(1)} \) and \( \infty^{(2)} \) on \( L_k \) are denoted by \( l_1 \) and \( l_2 \), respectively. The contour \( l \equiv l_1 + l_2 \) in the limit \( \lambda_1 \to \infty \) encircles the point \( \infty^{(1)} \) (the infinite branch point of order \( k \)). The solution \( \Phi_k (\lambda) \) is built using the basis in the group \( H_1 (L \setminus f^{-1}(\infty) : f^{-1}(\lambda)) \) defined by (3.11), (3.12), (3.13). Let us consider another solution of the same Riemann-Hilbert problem, which we denote by \( \tilde{\Phi}_k (\lambda) \). The solution \( \tilde{\Phi}_k (\lambda) \) is obtained from \( \Phi_k (\lambda) \) by adding the column corresponding to the integration contour \( l_2 \) to the column of \( \Phi_k (\lambda) \) corresponding to the contour \( l_1 \). Moreover, the column corresponding to the contour \( l_2 \) is kept unchanged and is made the first column in the matrix \( \tilde{\Phi}_k (\lambda) \). Obviously,

\[
(\lambda - \lambda_1)^{-1/2} \det \tilde{\Phi}_k (\lambda) = \tilde{C} \prod_{n=2}^{N} (\lambda - \lambda_n)^{1/2},
\]

where the constant \( \tilde{C} = \pm C \).

In the limit \( \lambda_1 \to \infty \) the \( (N-1) \times (N-1) \) minor of the matrix \( \tilde{\Phi}_k (\lambda) \) not containing the first column and the first row tends to the matrix \( \Phi_{k+1} (\lambda) \). The \((11)\)-entry of the matrix \( \Phi_{k+1} (\lambda) \) behaves as \( c_3 \lambda_1^{1/2} \), according to (B.38). All other entries of the first row of \( \Phi_k (\lambda) \) behave according to (B.36) as \( \text{const} \times \lambda_1^{-k+1/2} \) (where the \( \lambda_1 \)-independent constant is different for different entries). On the other hand, all entries of the first column of \( \tilde{\Phi}_k (\lambda) \) behave according to (B.37) as \( \text{const} \times \lambda_1^{k+1/2} \), where again the \( \lambda_1 \)-independent constants are different for the different entries.

Let us divide now the first column of the matrix \( \tilde{\Phi}_k \) by \( \lambda_1^{1/2} \). Then we have, as \( \lambda_1 \to \infty \),

\[
\det \tilde{\Phi}_k (\lambda) = \lambda_1^{1/2} \det \begin{pmatrix} c_3 + o(1) & c_2 \lambda_1^{-k+3/2} \frac{1}{2} \left( 1 + o(1) \right) \\ c_1 \lambda_1^{k+1} \left( 1 + o(1) \right) & \Phi_{k+1} (1 + o(1)) \end{pmatrix},
\]

where \( c_1 \) is a column-vector of length \( N-1 \) constant with respect to \( \lambda_1 \); \( c_2 \) is a row-vector of length \( N-1 \) constant with respect to \( \lambda_1 \). Since

\[
\frac{k+3}{2(k+2)} + \frac{1}{2} = \frac{k+2}{k+1} > \frac{k}{k+1},
\]

we have in the limit \( \lambda_1 \to \infty \) that

\[
(\lambda - \lambda_1)^{-1/2} \det \tilde{\Phi}_k (\lambda) \to c_3 \det \Phi_{k+1} (\lambda),
\]

where \( c_3 \) is a non-vanishing constant independent of \( \lambda \) and all \( \lambda_n \) given by (B.47). Due to (B.48) we get

\[
\det \Phi_{k+1} (\lambda) = C_0 \prod_{n=2}^{N} (\lambda - \lambda_n)^{1/2},
\]

where \( C_0 \) is another non-vanishing constant. \( \square \)

This finishes the proof of completeness of the set of solutions (3.14) for an arbitrary covering with one branched infinity. The proof of completeness for an arbitrary branching at infinity (i.e., for the case of an arbitrary Hurwitz space \( H_{g,d} (k_1, \ldots, k_m) \)) is an obvious generalization of the above proof for \( H_{g,d} (k, 1, \ldots, 1) \).
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