Well-posedness and stability for semilinear wave-type equations with time delay

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Abstract
In this paper we analyze a semilinear abstract damped wave-type equation with time delay. We assume that the delay feedback coefficient is variable in time and belonging to $L^1_{\text{loc}}([0, +\infty))$. Under suitable assumptions, we show well-posedness and exponential stability for small initial data. Our strategy combines careful energy estimates and continuity arguments. Some examples illustrate the abstract results.

1 Introduction
Let $H$ be an Hilbert space and let $A : \mathcal{D}(A) \subset H \to H$ be a positive self-adjoint operator with dense domain and compact inverse in $H$. Let us consider the following wave-type equation:

\begin{align*}
    u_{tt}(t) + Au(t) + CC^*u_t(t) + k(t)BB^*u_t(t - \tau) &= \nabla \psi(u(t)), \quad t \geq 0, \\
    u(0) &= u_0, \quad u_t(0) = u_1, \\
    B^*u_t(s) &= g(s), \quad s \in [-\tau, 0],
\end{align*}

(1.1)

where $\tau > 0$ is the time delay, the damping coefficients $k(\cdot)$ is a function in $L^1_{\text{loc}}([0, +\infty))$ and $(u_0, u_1, g)$ are the initial data in suitable spaces. Moreover, for given real Hilbert spaces $W_1$ and $W_2$ that will be identified with their dual spaces, $C : W_1 \to H$ and $B : W_2 \to H$ are bounded linear operators with adjoint $C^*$ and $B^*$ respectively. We assume that the damping operator $CC^*$ satisfies a control geometric property (see e.g. [8] or [11, Chapter 5]). Moreover, on the delay feedback coefficient, we assume

\begin{align*}
    \int_{-\tau}^{t} |k(s)| ds &\leq K, \quad \forall \ t \geq 0, \quad (1.2)
\end{align*}

for some $K > 0$.

Furthermore, $\psi : \mathcal{D}(A^\frac{1}{2}) \to \mathbb{R}$ is a functional having Gâteaux derivative $D\psi(u)$ at every $u \in \mathcal{D}(A^\frac{1}{2})$. In the spirit of [2], we assume the following hypotheses:

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(H1) For every \( u \in D(A^{\frac{1}{2}}) \), there exists a constant \( c(u) > 0 \) such that
\[
|D\psi(u)(v)| \leq c(u)||v||_H \quad \forall v \in D(A^{\frac{1}{2}}).
\]
Then, \( \psi \) can be extended to the whole \( H \) and we denote by \( \nabla\psi(u) \) the unique vector representing \( D\psi(u) \) in the Riesz isomorphism, i.e.
\[
\langle \nabla\psi(u), v \rangle_H = D\psi(u)(v), \quad \forall v \in H;
\]
(H2) for all \( r > 0 \) there exists a constant \( L(r) > 0 \) such that
\[
||\nabla\psi(u) - \nabla\psi(v)||_H \leq L(r)||A^{\frac{1}{2}}(u - v)||_H,
\]
for all \( u, v \in D(A^{\frac{1}{2}}) \) satisfying \( ||A^{\frac{1}{2}}u||_H \leq r \) and \( ||A^{\frac{1}{2}}v||_H \leq r \).
(H3) \( \psi(0) = 0, \nabla\psi(0) = 0 \) and there exists a strictly increasing continuous function \( h \) such that
\[
||\nabla\psi(u)||_H \leq h(||A^{\frac{1}{2}}u||_H)||A^{\frac{1}{2}}u||_H, \tag{1.3}
\]
for all \( u \in D(A^{\frac{1}{2}}) \).

We are interested in studying the well-posedness of system (1.1) and in proving an exponential stability estimate for solutions corresponding to small initial data, under a suitable assumption involving the model’s parameters. A linear version of such a model has been first studied in [20] where a wave equation with frictional damping and delay feedback with constant coefficient has been analyzed, proving an exponential decay estimate under a suitable smallness condition on the delay term coefficient. This result has then been extended to linear wave equations with boundary dissipation (see [9]) and with viscoelastic damping (see [3, 10]). We quote also [16, 17, 12] for related stability results for abstract semilinear evolution equations. However, in the nonlinear setting, the results previously obtained require that the damping operator \( CC^* \) contrasts, in the spirit of [15] (cf. also [22]), the delay feedback. Indeed, in [16, 17], where the delay coefficient is constant, i.e. \( k(t) \equiv k \), in order to have a not increasing energy, it is assumed \( |k| < \frac{1}{\mu} \) and
\[
||B^*u||_{W_2} \leq \mu||C^*u||_{W_1}, \quad \forall u \in D(A^{\frac{1}{2}}).
\]
In [12] the coefficient \( k \) is time dependent, as here, but it is assumed that
\[
k(t) = k^1(t) + k^2(t),
\]
with \( k^1 \in L^1([0, +\infty)), k^2 \in L^\infty([0, +\infty)) \), and \( ||k^2||_\infty \) smaller than a suitable constant depending on the damping operator \( CC^* \). Actually, in [12], the model involves a finite number of time delays \( \tau_1, \ldots, \tau_l \). Here, for sake of clarity, we consider only a delay feedback. However, our analysis could be easily extended to more than one delay term.

Stability results in presence of delay feedback have also been obtained for specific models, with \( k \) constant, mainly in the linear setting (see e.g. [1, 5, 7, 9, 13, 21]).

The rest of the paper is organized as follows. In section 2 we rewrite system (1.1) in an abstract form and we give an exponential stability result under an appropriate well-posedness assumption. Then, we show that the well-posedness assumption is satisfied for model (1.1) and so the exponential decay estimate holds for small initial data. Section 3 illustrates some concrete examples for which the abstract theory is applicable.
2 Exponential stability

Before proving the exponential stability of system (1.1), we rewrite it in an abstract way. Let us introduce the Hilbert space
\[ \mathcal{H} = \mathcal{D}(A^{\frac{1}{2}}) \times H, \]
endowed with the inner product
\[ \left\langle \left( \begin{array}{c} u \\ v \end{array} \right), \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) \right\rangle_{\mathcal{H}} := \langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} \tilde{u} \rangle_{H} + \langle v, \tilde{v} \rangle_{H}. \]

If we denote \( v(t) = u_t(t) \) and \( U(t) := (u(t), v(t))^T \), we can rewrite system (1.1) in the following abstract form
\[ U'(t) = AU(t) - k(t) BU(t - \tau) + F(U(t)), \]
\[ U(0) = U_0, \]
\[ BU(t) = f(t), \quad t \in [-\tau, 0], \tag{2.1} \]
where
\[ A = \begin{pmatrix} 0 & 1 \\ -A & -CC^* \end{pmatrix}, \quad BU(t) = \begin{pmatrix} 0 \\ BB^* v(t) \end{pmatrix} \quad \text{and} \quad F(U(t)) = \begin{pmatrix} 0 \\ \nabla \psi(u(t)) \end{pmatrix}. \]

We know that, under controllability assumptions on the damping operator \( CC^* \) (see for instance [8, 11]), \( A \) generates an exponentially stable \( C_0 \)-semigroup \( \{S(t)\}_{t \geq 0} \), namely there exist \( M, \omega > 0 \) such that
\[ ||S(t)||_{\mathcal{L}(\mathcal{H})} \leq Me^{-\omega t}, \quad \forall t \geq 0. \tag{2.2} \]

Moreover, the previous hypotheses (H2)-(H3) on \( \psi \) imply the following properties on \( F \):
\[ \text{(F1) } F(0) = 0; \]
\[ \text{(F2) for any } r > 0 \text{ there exists a constant } L(r) > 0 \text{ such that} \]
\[ ||F(U) - F(V)||_{\mathcal{H}} \leq L(r)||U - V||_{\mathcal{H}}, \]
whenever \( ||U||_{\mathcal{H}}, ||V||_{\mathcal{H}} \leq r. \]

Let us denote
\[ ||B||_{\mathcal{L}(\mathcal{H})} = ||B^*||_{\mathcal{L}(\mathcal{H})} = b. \tag{2.3} \]

Then, by (2.3),
\[ ||B||_{\mathcal{L}(\mathcal{H})} = b^2. \tag{2.4} \]

We will prove well-posedness and exponential stability for system (1.1), for small initial data, under the assumption
\[ Mb^2 e^{\omega \tau} \int_0^t |k(s + \tau)| ds \leq \gamma + \omega' t, \quad \forall t > 0, \tag{2.5} \]
for suitable constants \( \gamma \geq 0 \) and \( \omega' \in [0, \omega) \).

First, we will give an exponential decay result for the abstract model (2.1) under a suitable well-posedness assumption.
Theorem 2.1. Assume (2.5). Moreover, suppose that

(I) there exist \( \rho > 0, \ C_\rho > 0 \), with \( L(C_\rho) < \frac{\omega - \omega'}{M} \) such that if \( U_0 \in \mathcal{H} \) and if \( f \in C([-\tau, 0]; \mathcal{H}) \) satisfy

\[
||U_0||_{\mathcal{H}}^2 + \int_0^\tau |k(s)| \cdot ||f(s - \tau)||_{\mathcal{H}}^2 ds < \rho^2, \tag{2.6}
\]

then the system (2.1) has a unique solution \( U \in C([0, +\infty); \mathcal{H}) \) satisfying \( ||U(t)||_{\mathcal{H}} \leq C_\rho \) for all \( t > 0 \).

Then, for every solution \( U \) of (2.1), with initial data \((U_0, f)\) satisfying (2.6),

\[
||U(t)||_{\mathcal{H}} \leq Me^\gamma \left(||U_0||_{\mathcal{H}} + \int_0^t e^{\omega s}|k(s)| \cdot ||BU(s - \tau)||_{\mathcal{H}} ds \right) + ML(C_\rho)e^{-\omega t}\int_0^t e^{\omega s}||U(s)||_{\mathcal{H}} ds,
\]

where we have used the fact that \( ||F(U(t))||_{\mathcal{H}} \leq L(C_\rho)||U(t)||_{\mathcal{H}} \) for any \( t \geq 0 \). Then, we get

\[
||U(t)||_{\mathcal{H}} \leq Me^{-\omega t}||U_0||_{\mathcal{H}} + M e^{-\omega t} \int_0^t e^{\omega s}|k(s)| \cdot ||f(s - \tau)||_{\mathcal{H}} ds \nonumber
\]

\[
+ Me^{-\omega t} \int_0^t e^{\omega s}t^2|k(s)| \cdot ||U(s - \tau)||_{\mathcal{H}} ds + ML(C_\rho)e^{-\omega t}\int_0^t e^{\omega s}||U(s)||_{\mathcal{H}} ds.
\]

By change of variables \( s - \tau = z \) we infer

\[
M e^\omega ||U(t)||_{\mathcal{H}} \leq M||U_0||_{\mathcal{H}} + M \int_0^t e^{\omega s}|k(s)| \cdot ||f(s - \tau)||_{\mathcal{H}} ds \nonumber
\]

\[
+ M b^2 e^{\omega t} \int_0^t |k(z + \tau)|e^{\omega z}||U(z)||_{\mathcal{H}} dz + ML(C_\rho) \int_0^t e^{\omega s}||U(s)||_{\mathcal{H}} ds.
\]

Now, let us denote \( \tilde{u}(t) := e^{\omega t}||U(t)||_{\mathcal{H}} \). Hence,

\[
\tilde{u}(t) \leq M||U_0||_{\mathcal{H}} + M \int_0^t e^{\omega s}|k(s)| \cdot ||f(s - \tau)||_{\mathcal{H}} ds 
\]

\[
+ \int_0^t \left[M b^2 e^{\omega t}|k(s + \tau)| + ML(C_\rho)\right] \tilde{u}(s) ds.
\]

Then, Gronwall estimate implies

\[
\tilde{u}(t) \leq \left(M||U_0||_{\mathcal{H}} + M \int_0^t e^{\omega s}|k(s)| \cdot ||f(s - \tau)||_{\mathcal{H}} ds\right) e^{Mb^2 e^{\omega t} \int_0^t |k(s + \tau)| ds + ML(C_\rho)t}.
\]

Then, by definition of \( \tilde{u} \) and by assumption (2.5), we get (2.7).
By Gronwall inequality we then obtain (2.8).

Lemma 2.3. Let \( E(t) \) be a solution to (1.1). If \( E(t) \geq \frac{1}{4} ||u(t)||_{H}^2 \) for any \( t \geq 0 \), then

\[
E(t) \leq \tilde{C}(t) E(0),
\]

for any \( t \geq 0 \), where

\[
\tilde{C}(t) := e^{2 \int_0^t \tilde{b}(k(s) + |k(s + \tau)|) ds}.
\]

Proof. Differentiating \( E \) in time yields

\[
\frac{d}{dt} E(t) = (u_t(t), u_{tt}(t))_H + (A^{\frac{1}{2}} u(t), A^{\frac{1}{2}} u_t(t))_H - (\nabla \psi(u(t)), u_t(t))
+ \frac{1}{2} |k(t + \tau)| \cdot ||B^* u_t(t)||_{W_2}^2 - \frac{1}{2} |k(t)| \cdot ||B^* u_t(t - \tau)||_{W_2}^2.
\]

By using (1.1), we obtain

\[
\frac{d}{dt} E(t) = - ||C^* u_t(t)||_{W_2}^2 - k(t)(B^* u_t(t), B^* u_t(t - \tau))_{W_2}
+ \frac{1}{2} |k(t + \tau)| \cdot ||B^* u_t(t)||_{W_2} - \frac{1}{2} |k(t)| \cdot ||B^* u_t(t - \tau)||_{W_2}.
\]

We use Young inequality in the second term in the previous equality and we get

\[
\frac{d}{dt} E(t) \leq \frac{1}{2} (|k(t)| + |k(t + \tau)|) ||B^* u_t(t)||_{W_2}^2
\leq 2b^2 (|k(t)| + |k(t + \tau)|) \frac{1}{4} ||u_t(t)||_{H}^2,
\]

where for the last inequality we used (2.3). Since we have assumed \( E(t) \geq \frac{1}{4} ||u(t)||_{H}^2 \) for any \( t \geq 0 \), we obtain

\[
\frac{d}{dt} E(t) \leq 2b^2 (|k(t)| + |k(t + \tau)|) E(t).
\]

By Gronwall inequality we then obtain (2.8). \( \blacksquare \)

In order to prove the well-posedness assumption (I) for system (1.1), we need the following two lemmas.

Lemma 2.3. Let us consider the system (2.1) with initial data \( U_0 \in \mathcal{H} \) and \( f \in C([\tau, 0]; \mathcal{H}) \). Then, there exists a unique continuous local solution \( U(\cdot) \) defined on a time interval \([0, \delta]\), with \( \delta \leq \tau \).
Proof. Since \( t \in [0, \tau] \), we can rewrite the abstract system (2.1) as an undelayed problem:

\[
\begin{align*}
U'(t) &= AU(t) - k(t)f(t - \tau) + F(U(t)), \quad t \in (0, \tau), \\
U(0) &= U_0.
\end{align*}
\]

Then, we can apply the classical theory of nonlinear semigroups (see e.g. [19]) obtaining the existence of a unique solution on a set \([0, \delta]\), with \( \delta \leq \tau \).

**Lemma 2.4.** Let \( U(\cdot) \) be a non-null solution to (2.1) defined on the interval \([0, \delta]\), with \( \delta \leq \tau \). Let \( h \) be the strictly increasing function appearing in (1.3). Then,

1. if \( h(||A^{\frac{1}{2}}u_0(0)||_{H}) < \frac{1}{2} \), then \( E(0) > 0 \);
2. if \( h(||A^{\frac{1}{2}}u_0(0)||_{H}) < \frac{1}{2} \) and \( h \left( 2\bar{C}(\tau)E^{\frac{1}{2}}(0) \right) < \frac{1}{2} \), with \( \bar{C}(\cdot) \) defined as in (2.9), then

\[
E(t) > \frac{1}{4}||u(t)||_{H}^2 + \frac{1}{4}||A^\frac{1}{2}u(t)||_{H}^2 + \frac{1}{4} \int_{t-\tau}^{t} |k(s + \tau)| \cdot ||B^* u_t(s)||_{W_2}^2 ds \quad (2.10)
\]

for all \( t \in [0, \delta] \). In particular,

\[
E(t) > \frac{1}{4}||U(t)||_{H}^2, \quad \text{for all} \quad t \in [0, \delta]. \quad (2.11)
\]

**Proof.** First, from assumption (H3) on \( \psi \) we can infer that

\[
|\psi(u)| \leq \int_{0}^{1} |(\nabla \psi(su), u)| ds \leq ||A^{\frac{1}{2}}u||_{H}^2 \int_{0}^{1} h(s)||A^{\frac{1}{2}}u||_{H} ds \leq \frac{1}{2} h(||A^{\frac{1}{2}}u||_{H})||A^{\frac{1}{2}}u||_{H}^2. \quad (2.12)
\]

Hence, under the assumption \( h(||A^\frac{1}{2}u_0(0)||_{H}) < \frac{1}{2} \), we have that

\[
E(0) = \frac{1}{2}||u_1||_{H}^2 + \frac{1}{2}||A^\frac{1}{2}u_0(0)||_{H}^2 - \psi(u_0(0)) + \frac{1}{2} \int_{-\tau}^{0} |k(s + \tau)| \cdot ||B^* u_t(s)||_{W_2}^2 ds \leq \frac{1}{2} h(||A^\frac{1}{2}u||_{H})||A^\frac{1}{2}u||_{H}^2. \quad (2.13)
\]

Note that, if the right-hand side of (2.13) is zero, then \( U(\cdot) \) is the null solution. Therefore, we have proven claim 1.

In order to prove the second statement, we argue by contradiction. Let us denote

\[
r := \sup \{ s \in [0, \delta) : (2.10) \text{ holds} \forall t \in [0, s) \}.
\]

We suppose by contradiction that \( r < \delta \). Then, by continuity, we have

\[
E(r) = \frac{1}{4}||u_t(r)||_{H}^2 + \frac{1}{4}||A^\frac{1}{2}u(r)||_{H}^2 + \frac{1}{4} \int_{r-\tau}^{r} |k(s + \tau)| \cdot ||B^* u_t(s)||_{W_2}^2 ds. \quad (2.14)
\]

Now, since from (2.14)

\[
\frac{1}{4}||A^\frac{1}{2}u(r)||_{H}^2 \leq E(r),
\]
we can infer, by using Proposition 2.2, that
\[
h(||A^{\frac{1}{2}}u(r)||_H) \leq h \left(2E^{\frac{1}{2}}(r)\right) < h \left(2\bar{C}^{\frac{1}{2}}(\tau)E^{\frac{1}{2}}(0)\right) < \frac{1}{2}.
\] (2.15)

Hence, we have that
\[
E(r) = \frac{1}{2}||u_t(r)||_H^2 + \frac{1}{2}||A^{\frac{1}{2}}u(r)||_H^2 - \psi(u(r)) + \frac{1}{2} \int_{r-\tau}^r |k(s + \tau)| \cdot ||B^*u_t(s)||_{W^2_2}^2 ds
\]
\[
> \frac{1}{4}||u_t(r)||_H^2 + \frac{1}{4}||A^{\frac{1}{2}}u(r)||_H^2 + \frac{1}{4} \int_{r-\tau}^r |k(s + \tau)| \cdot ||B^*u_t(s)||_{W^2_2}^2 ds,
\]
where in the last estimate we used (2.12) and (2.15). This contradicts the maximality of \(r\).
Hence, \(r = \delta\) and this concludes the proof of the lemma.

We are now ready to prove the well-posedness assumption for system (1.1).

**Theorem 2.5.** If hypothesis (2.5) is satisfied, then problem (2.1), with initial data \(U_0 \in H\) and \(f \in C([-\tau, 0]; H)\), satisfies the well-posedness assumption (I). Hence, for solutions of (2.1) corresponding to sufficiently small initial data the exponential decay estimate (2.7) holds true.

**Proof.** Let us fix \(N \in \mathbb{N}\) such that
\[
C_N := 2M^2e^{2\gamma}\left(1 + Ke^{\omega\tau}b^2\right)(1 + e^{2\omega K})e^{-(\omega - \omega')N\tau} \leq 1.
\] (2.16)
Moreover let \(\rho > 0\) be such that
\[
\rho \leq \frac{1}{2C(N\tau)}h^{-1}\left(\frac{1}{2}\right),
\]
and assume
\[
||u_1||_H^2 + ||A^{\frac{1}{2}}u_0(0)||_H^2 + \int_{-\tau}^0 |k(s + \tau)| \cdot ||g(s)||_{W^2_2}^2 ds \leq \rho^2.
\] (2.17)
We observe that this is equivalent to (by considering the abstract formulation (2.1))
\[
||U_0||_H^2 + \int_0^\tau |k(s)| \cdot ||f(s - \tau)||_H^2 ds \leq \rho^2.
\]
First of all, from Lemma 2.3 we know that there exists a solution \(u\) to (1.1) on the time interval \([0, \delta]\) with \(\delta \leq \tau\). Now, we have that
\[
h(||A^{\frac{1}{2}}u_0(0)||_H) < h(\rho) \leq h \left(h \left(\frac{1}{2C^{\frac{1}{2}}(N\tau)}h^{-1}\left(\frac{1}{2}\right)\right)\right) < \frac{1}{2},
\]
where we have used the fact that \(\bar{C}(N\tau) > 1\). Hence, by Lemma 2.4 \(E(0) > 0\). Moreover, from (2.12) we get
\[
E(0) \leq \frac{1}{2}||u_1||_H^2 + \frac{3}{4}||A^{\frac{1}{2}}u_0(0)||_H^2 + \frac{1}{2} \int_{-\tau}^0 |k(s + \tau)| \cdot ||g(s)||_{W^2_2}^2 ds \leq \rho^2,
\]
which gives us
\[
    h \left( 2\bar{C}^{\frac{1}{2}}(N\tau)E^{\frac{1}{2}}(0) \right) < h \left( 2\bar{C}^{\frac{1}{2}}(N\tau)\rho \right) < h \left( h^{-1} \left( \frac{1}{2} \right) \right) = \frac{1}{2}.
\]

Since \( \bar{C}(N\tau) \geq \bar{C}(\tau) \), then
\[
    h \left( 2\bar{C}^{\frac{1}{2}}(\tau)E^{\frac{1}{2}}(0) \right) \leq h \left( 2\bar{C}^{\frac{1}{2}}(N\tau)E^{\frac{1}{2}}(0) \right) < \frac{1}{2},
\]

Hence, we can apply again Lemma \( \text{[2.4]} \) and we can infer that \( \text{(2.11)} \) is satisfied for any \( t \in [0, \delta) \). Then, we can use Proposition \( \text{[2.2]} \) getting
\[
    0 < \frac{1}{4} ||u(t)||_H^2 + \frac{1}{4} ||A^{\frac{1}{2}}u(t)||_H^2 + \frac{1}{4} \int_{t-\tau}^t |k(s+\tau)| \cdot ||B^*u_t(s)||_{W_2}^2 ds < E(t) \leq \bar{C}(\tau)E(0),
\]

for any \( t \in [0, \delta) \). Then, we can extend the solution in \( t = \delta \) and on the entire interval \([0, \tau]\). Now, from \( \text{(2.19)} \) and \( \text{(2.18)} \), we obtain
\[
    h(||A^{\frac{1}{2}}u(\tau)||_H) \leq h(2E(\tau)) \leq h \left( 2\bar{C}^{\frac{1}{2}}(\tau)E^{\frac{1}{2}}(0) \right) < \frac{1}{2}.
\]

Therefore, there exists \( \delta' \in [0, \tau] \) such that
\[
    h(||A^{\frac{1}{2}}u(t)||_H) < \frac{1}{2}, \quad \forall \ t \in [\tau, \tau + \delta').
\]

So, arguing analogously to the proof of Lemma \( \text{[2.4]} \) one can obtain
\[
    E(t) > \frac{1}{4} ||u(t)||_H^2 + \frac{1}{4} ||A^{\frac{1}{2}}u(t)||_H^2 + \frac{1}{4} \int_{t-\tau}^t |k(s+\tau)| \cdot ||B^*u_t(s)||_{W_2}^2 ds,
\]

for all \( t \in [\tau, \tau + \delta') \). Then,
\[
    E(t) > \frac{1}{4} ||u(t)||_H^2, \quad \forall \ t \in [\tau, \tau + \delta').
\]

Hence, we can apply once again Proposition \( \text{[2.2]} \) which yields
\[
    0 < \frac{1}{4} ||u(t)||_H^2 + \frac{1}{4} ||A^{\frac{1}{2}}u(t)||_H^2 + \frac{1}{4} \int_{t-\tau}^t |k(s+\tau)| \cdot ||B^*u_t(s)||_{W_2}^2 ds < E(t) \leq \bar{C}(2\tau)E(0),
\]

where we used the fact that \( \delta' \leq \tau \) and the monotonicity of \( \bar{C}(\cdot) \). Then, as before, we can extend the solution to the interval \([0, 2\tau]\). Iterating this procedure, one can extend the solution to the whole time interval \([0, N\tau]\) with \( N \) satisfying \( \text{(2.16)} \). Now for \( t = N\tau \) we have that
\[
    h(||A^{\frac{1}{2}}u(N\tau)||_H) \leq h(2E^{\frac{1}{2}}(N\tau)) \leq h(2\bar{C}^{\frac{1}{2}}(N\tau)E^{\frac{1}{2}}(0)) \leq h(2\bar{C}^{\frac{1}{2}}(N\tau)\rho) < \frac{1}{2}.
\]
Moreover,
\[ \frac{1}{4} \|U(t)\|_{\mathcal{H}}^2 \leq E(t) \leq \bar{C}(N\tau_{\text{min}})E(0) < \bar{C}(N\tau)\rho^2, \quad \forall t \in [0, N\tau], \]
and so
\[ \|U(t)\|_{\mathcal{H}} \leq C_\rho := 2\bar{C}_\frac{1}{2}(N\tau)\rho, \]
for any \( t \in [0, N\tau] \). Now, eventually choosing smaller values of \( \rho \), we suppose that \( \rho \) is such that \( L(C_\rho) < \omega - \omega' \). Therefore, assumption (I) is satisfied along the interval \([0, N\tau]\). Hence, Theorem 2.1 gives us the following estimate:
\[ \|U(t)\|_{\mathcal{H}} \leq M e^{\gamma} \left( \|U_0\|_{\mathcal{H}} + \int_0^T e^{\omega s} |k(s)| \cdot \|f(s - \tau)\|_{\mathcal{H}} ds \right) e^{-\frac{\omega - \omega'}{2} t} \]
(2.20)
for any \( t \in [0, N\tau] \). By using assumption (1.2) and Hölder inequality we get
\[
\int_0^T |k(s)| e^{\omega s} \|f(s - \tau)\|_{\mathcal{H}} ds \leq e^{\omega T} \left( \int_0^T |k(s)| ds \right)^{\frac{1}{2}} \left( \int_0^T |k(s)| \cdot \|f(s - \tau)\|_{\mathcal{H}}^2 ds \right)^{\frac{1}{2}} 
\leq e^{\omega T} K \frac{1}{2} \rho.
\]
Therefore, from (2.20) we get
\[ \|U(N\tau)\|_{\mathcal{H}} \leq 2M^2 \rho^2 e^{2\gamma} (1 + e^{2\omega T} K) e^{-(\omega - \omega')N\tau}. \]
Moreover, if \( s \in [N\tau, N\tau + \tau] \), then \( s - \tau \in [N\tau - \tau, N\tau] \). Then,
\[
\|U(N\tau)\|_{\mathcal{H}} + \int_{N\tau}^{N\tau + \tau} e^{\omega(s - N\tau)} |k(s)| \cdot \|B^* u(t)(s - \tau)\|_{W^2}^2 ds \leq C N \rho^2 \leq \rho^2,
\]
where we have used (2.16). Hence, we can infer that
\[ \|U(N\tau)\|_{\mathcal{H}} + \int_{N\tau}^{N\tau + \tau} |k(s)| \cdot \|B^* u(t)(s - \tau)\|_{W^2}^2 ds \leq \rho^2. \]
We can proceed by applying a similar argument shown before on the interval \([N\tau, 2N\tau]\), obtaining a solution on the interval \([0, 2N\tau]\). Iterating the process, we find a unique global solution to (2.1) satisfying the well-posedness assumption (I). Hence, the theorem is proved.

3 Examples

In this section, we give some applications of previous abstract well-posedness and stability results. We will show that the following systems can be rewritten in the abstract form (2.1) and so, under suitable assumptions, global existence and stability decay estimates hold for small initial data.
### 3.1 Wave equation with damping and source term

Let $\Omega$ be an open subset of $\mathbb{R}^d$, with boundary $\Gamma$ of class $C^2$ and let $\mathcal{O} \subset \Omega$ be an open subset which satisfies the geometrical control property in [8]. For instance, denoting by $m$ the standard multiplier $m(x) = x - x_0$, $x_0 \in \mathbb{R}^d$, as in [13], $\mathcal{O}$ can be the intersection of $\Omega$ with an open neighborhood of the set

$$\Gamma_0 = \{ x \in \Gamma : m(x) \cdot \nu(x) > 0 \}.$$  

Moreover, let $\tilde{\mathcal{O}} \subset \Omega$ be another open subset. Denoting by $\chi_D$ the characteristic function of a set $D$, we consider the following wave equation

$$u_{tt}(x, t) - \Delta u(x, t) + a\chi_D(x)u_t(x, t) + k(t)\chi_D(x)u_t(x, t - \tau) = u(x, t)|u(x, t)|^\beta, \quad (x, t) \in \Omega \times (0, +\infty),$$

$$u(x, 0) = u_0(x), \quad x \in \Omega,$n

$$u_t(x, s) = u_1(x, s) \quad (x, s) \in \Omega \times [-\tau, 0],$$

where $a$ is a positive constant, $\tau > 0$ is the time delay, $\beta > 0$ and the delayed damping coefficient $k(\cdot) : [0, +\infty) \to (0, +\infty)$ is a $L^1_{loc}([0, +\infty))$ function satisfying (1.2). System (3.1) falls in the form (1.1) with $A = -\Delta$ with dense domain $\mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Setting $v(t) = u_t(t)$ and $U(t) = (u(t), v(t))^T$ for any $t \geq 0$, we can rewrite system (3.1) in the abstract form (2.1), with $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega),$

$$\mathcal{A} = \begin{pmatrix} 0 & Id \\ \Delta & -a\chi_D \end{pmatrix}$$

and $\mathcal{B}$ and $F$ defined as

$$\mathcal{B} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -\chi_D v \end{pmatrix}, \quad F(U(t)) = \begin{pmatrix} 0 \\ u(t)|u(t)|^\beta \end{pmatrix}, \quad \forall \ t \geq 0.$$

We know that $\mathcal{A}$ generates an exponentially stable $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ (see e.g. [11]), namely there exist $\omega, M > 0$ such that

$$||S(t)||_{L(\mathcal{H})} \leq Me^{-\omega t}, \quad \forall \ t \geq 0.$$  

Moreover, we consider the following functional:

$$\psi(u) = \frac{1}{\beta + 2} \int_\Omega |u(x)|^{\beta + 2}dx,$$

for any $u \in H^1_0(\Omega)$. For any $\beta \in \left(0, \frac{4}{\pi^2}\right)$, $\psi$ is well-defined by Sobolev’s embedding theorem. Furthermore, $\psi$ is Gâteaux differentiable for any $u \in H^1_0(\Omega)$ with Gâteaux derivative given by

$$D\psi(u)(v) = \int_\Omega |u(x)|^{\beta}u(x)v(x)dx, \quad \forall \ v \in H^1_0(\Omega).$$

As in [2], it is possible to show that $\psi$ satisfies hypotheses (H1), (H2) and (H3), provided that $\beta \in \left(0, \frac{2}{\pi^2}\right]$. We define the following energy functional for any $t \geq 0$:

$$E(t) := \frac{1}{2} \int_\Omega |u_1(x, t)|^2dx + \frac{1}{2} \int_\Omega |\nabla u(x, t)|^2dx - \psi(u(x, t)) + \frac{1}{2} \int_{t-\tau}^t \int_{\tilde{\mathcal{O}}} |k(s + \tau)| \cdot |u_t(x, s)|^2dxds.$$  

Then, Theorem [2.1] can be applied to system (3.1), under the assumption (2.5), obtaining well-posedness and stability results for small initial data.
Remark 3.1. We want to emphasize that our result holds true for every pair of subsets $(${\mathcal O}, \bar{\mathcal O}$) of $\Omega$. The only condition that we require is that $\mathcal O$ satisfies a geometric control property. On the contrary, in [16, 17, 12], in the nonlinear setting, it is required $\bar{\mathcal O} \subseteq \mathcal O$. Same remark applies to the following example.

3.2 Plate equation with damping and source term

Let $\Omega \subset \mathbb{R}^d$ be an open subset with boundary $\Gamma$ of class $C^2$ and let $\mathcal O \subset \Omega$ be an open subset which satisfies the geometrical control property in [5]. Let $\bar{\mathcal O}$ be another subset of $\Omega$. We consider the following plate equation:

$$
\begin{align*}
\ddot{u}(x,t) + \Delta^2 u(x,t) + a\chi_{\mathcal O}(x)u_t(x,t) + k(t)\chi_{\bar{\mathcal O}}(x)u_t(x,t - \tau) \\
= u(x,t)|u(x,t)|^\beta, \quad (x,t) \in \Omega \times (0, +\infty), \\
u(x,0) = u_0(x), \quad x \in \Omega, \\
u_t(x,s) = u_1(x,s) \quad (x,s) \in \Omega \times [-\tau, 0],
\end{align*}
$$

(3.2)

with $a,k(\cdot),\tau$ as before and $\beta > 0$. As for the previous case, one can rewrite system (3.2) in the abstract form (2.1). Now, since the nonlinear term satisfies hypotheses (H1)-(H2)-(H3) for $(d-4)\beta \leq 4$ (cf. e.g. [14]), then we can apply Theorem 2.1 for $\beta$ in this range. Therefore, under assumption (2.5), for the model (3.2) we have well-posedness and stability results for small initial data.

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