I discuss the interplay between inflation and microwave background anisotropies, stressing in particular the accuracy with which inflation predictions need to be made, and the importance of inflation as an underlying paradigm for cosmological parameter estimation.

1 Introduction

Because of calculational simplicity, and because it provides a good fit to the observational data, an initial spectrum of adiabatic density perturbations is normally assumed responsible for all the observed structures in the Universe, such as galaxy clusters and microwave background anisotropies. The inflationary cosmology provides a natural explanation for such an initial spectrum, and indeed the causal generation of large-scale adiabatic perturbations requires a period of inflation.

In studying the microwave background anisotropies, we believe we have a tool through which all of the cosmological parameters, such as the Hubble constant $h$ and the density parameter $\Omega_0$, can be probed. This is because the evolution of perturbations in the matter and radiation fields depends on all the cosmological parameters. On the other hand, the microwave background anisotropies are entirely due to linear perturbation theory, and hence entirely dependent on the initial perturbations you have in the first place. The only reason that one can indeed hope to extract cosmological parameters is because one believes that the initial spectrum takes on a simple form, perhaps a power-law, which can be parametrized simply and then those initial parameters thrown into the melting pot along with the cosmological ones and fitted by the data. One of the delights of inflation is that the initial spectrum is indeed typically predicted to take on a simple form, and furthermore one which can readily be calculated to high precision for a given inflationary model.
2 Models of inflation

In common with normal practice, I’ll focus my discussion on the simplest sub-class of inflationary models, where there is a single scalar field \( \phi \) rolling slowly down some potential \( V(\phi) \). The full range of inflationary models currently under discussion mostly consists of models which fit into this simple class, though I’m keen to stress that there are more complicated models on the market, in particular so-called open inflation models which, needless to say, give an open Universe. I’ll come back to some of these complexities later.

The simplest models give rise to a flat spatial geometry, so first let’s think about how they compare to the observational data. In recent years the flood of data supporting a low matter density has become an avalanche, leaving even the most hard-bitten theorist feeling guilty to suggest otherwise. However, interestingly, the two probes which actually constrain the geometry, rather than just the matter density, appear to favour a flat geometry. One is the location of the first peak in the microwave background spectrum, which is not particularly strongly constrained but certainly looks closer to the \( \ell = 220 \) or so of a flat model than the \( \ell = 500 \) or so of a low-density open model. The second, more secure, evidence comes from the recent type Ia supernovae results (Perlmutter et al. and Kim in these proceedings), which, for a matter density of around 0.3 (the favoured value), support the existence of a cosmological constant at just the right density to make the Universe flat.

The type of person who likes inflation is rather prone to disliking the cosmological constant, on the grounds that it represents exactly the kind of fine-tuning which inflation was supposed to liberate us from. On the other hand, inflation says nothing about the present matter content of the Universe; it just provides a flat spatial geometry and the way in which the matter is divided into different classes is, most likely, a property of present-day physics. So the presence or otherwise of a cosmological constant says nothing about whether inflation really occurred. In fact, one can rescue something from this sorry state of affairs, by noting that the favoured region corresponds to a Universe which is accelerating today. Once people become convinced that the Universe is presently accelerating, it will presumably become rather easier to suggest that the Universe might also have accelerated at some point in its distant past, and as it happens the definition of inflation is precisely any epoch during which the Universe experiences accelerated expansion.

3 Perturbations from inflation

During an inflationary epoch, comoving length scales are continually being stretched to scales larger than the Hubble length. As this happens, quantum fluctuations in the fields, which have to exist in accordance with the uncertainty principle, become ‘frozen in’ — unable to evolve on the Hubble timescale as their wavelength is so long — and begin to act like classical perturbations. The amplitude of these perturbations is readily calculable, and has been studied in many papers. See Liddle and Lyth for a review.

Because the Heisenberg uncertainty principle democratically affects everything, there are perturbations not only in the scalar field, which become adiabatic density perturbations, but also in the gravitational field which become gravitational waves. The precise form of both types of perturbation, their amplitude and scale-dependence, will depend on the potential energy \( V(\phi) \), it being the only input into the problem.

If the inflationary expansion is sufficiently rapid, those scales on which observable perturbations are generated all cross outside the Hubble radius during a very narrow interval of time. In that event, physical conditions were clearly pretty much the same when the smallest interesting scales (corresponding to galaxies) and when the largest interesting scales (corresponding to COBE) acquired their perturbations. Hence a first guess is that the density perturbations on
all scales will be the same, the scale-invariant or Harrison–Zel’’dovich spectrum. Within this approximation all inflation models predict exactly the same outcome, so in particular observations could not distinguish between different models.

Whether this approximation is adequate depends on the data set under consideration. Throughout the nineteen eighties, it was advertised as the prediction of inflation because existing data, basically the galaxy correlation function out to around $10h^{-1}$ Mpc, covered only a narrow range of scales and hence said little about the scale-dependence. Interestingly, the COBE data set (seen in Figure 1) taken in isolation can also be discussed in this approximation. It is consistent with the scale-invariant spectrum and only weakly constrains scale-dependence of the spectrum, insufficiently to rule out any inflation model on its own.

Fortunately, current data sets, where COBE is combined with short-scale information such as the galaxy power spectrum and the galaxy cluster abundance, are already of high enough quality that the scale-invariant assumption is inadequate, and one must be more precise in assessing the inflationary predictions. The next level of sophistication approximates the density perturbations and gravitational waves as power-laws; each needs an amplitude and a spectral index making four numbers in all, and these can readily be computed in any given inflation model. In fact there is a degeneracy between density perturbations and gravitational waves, the so-called consistency equation, which reduces this to three, but in practice it seems optimistic to believe that more than one piece of information about the gravitational waves can be observationally extracted, and one normally just considers a quantity $r$ which measures the fractional contribution of gravitational waves to the COBE signal.

Within the power-law approximation different inflation models give different predictions for the spectra. This represents a reduction in the predictability of inflation as a paradigm, but on the other hand means that more information is potentially available to observation, yielding extra information on the inflation mechanism and hence, perhaps, on very high energy physics.

Although it is often suggested otherwise, the realization that inflation doesn’t give a perfect scale-invariant spectrum is a huge success for the inflationary cosmology. It means that the lowest-order approximation fits the data well enough that one has to worry about corrections to it, and that the data has reached a quality whereby one can attempt to measure the size of these corrections. This is progress of the same sort as occurs in particle physics when one realizes that a tree-level calculation is no longer good enough and one has to worry about the one-loop corrections.
Table 1: Estimated uncertainties on parameters expected from the Planck satellite, assuming the Standard Cold Dark Matter model is correct. Successive columns introduce more freedom into the description of the initial parameters. The upper block contains the cosmological parameters and the lower one the inflationary parameters. Here $\Omega_b$ is the baryon density, $h$ the Hubble parameter, $\Omega_\Lambda$ a possible cosmological constant and $\tau$ the optical depth to the last-scattering surface. $\Omega_{\text{cdm}}$ is fixed by the assumption of spatial flatness so the second row estimates the uncertainty in $h$.

| Parameter | Planck with polarization |
|-----------|--------------------------|
| $\delta \Omega_b h^2 / \Omega_b h^2$ | 0.007 0.009 0.01 |
| $\delta \Omega_{\text{cdm}} h^2 / h^2$ | 0.02 0.02 0.02 |
| $\delta \Omega_\Lambda h^2 / h^2$ | 0.04 0.05 0.05 |
| $\tau$ | 0.0006 0.0006 0.0006 |
| $\delta n$ | 0.004 0.04 0.14 |
| $\delta r$ | 0.04 0.05 0.05 |
| $dn/d\ln k$ | $-$ 0.006 0.04 |
| $d^2 n / d(\ln k)^2$ | $-$ $-$ 0.005 |

4 Scale-dependence of the spectral index

We're always being told how vastly superior the upcoming microwave background observations will be in comparison to the data we presently have, so given that present observations already require corrections to the Harrison–Zel’dovich spectrum, might we have to worry about correcting the corrections? This requires one to consider scale dependence of the spectral indices. The optimal strategy appears to be to expand the log of the spectra as Taylor series in $\ln k$, i.e.

$$\ln \delta_H(k) = \ln \delta_H(k_*) + (n_* - 1) \ln k / k_* + \frac{1}{2} \left. \frac{dn}{d \ln k} \right|_* \ln^2 k / k_* + \cdots. \quad (1)$$

Further details can be found in Lidsey et al. The expansion scale $k_*$ is arbitrary but presumably best chosen in the centre of the data. This expansion is to be truncated as soon as an adequate fit to the data is obtained. The first term corresponds to the Harrison–Zel’dovich spectrum, and the first two taken together to the power-law approximation. To these, a general inflationary model adds a sequence of derivatives of the spectral index, evaluated at the scale $k_*$. In a given model, these are readily calculable, in the same way $n$ itself is. Typically one finds that only the first two terms are significant, but there are models where higher terms are important too.

In general, then, one might want to fit the microwave anisotropy data not just for the amplitudes and $n$, but also one or possibly more derivatives of $n$. From an inflationary point of view, this looks like a good thing, because we are saying that there is an extra piece of information available in the microwave anisotropies which we can extract from the data. However, there is a downside, which is that the extra piece of information has been stolen at the expense of all the other parameters; if we say we need to make a fit including one or more extra parameters, then the expected uncertainties on all cosmological parameters will be increased.

We have examined the extent to which the uncertainties are likely to increase, using the Fisher information matrix technique. We consider the standard Cold Dark Matter model for illustration; the actual numbers aren’t very important, what is interesting is the trend as extra parameters describing the initial conditions are added. The results are shown in Table 1.

We assume an experimental configuration of the Planck satellite with polarized detectors. The first column shows the results when the power-law approximation is assumed, and the successive columns each introduce an additional derivative of $n$. First the good news — the cos-
mological parameters take only a very minor hit as extra initial condition freedom is introduced. This leads to the encouraging conclusion that the modelling of the initial perturbation spectra may not have much of an influence on the satellites’ abilities to constrain our cosmology.

The bad news is largely concentrated into the determination of the inflationary parameters, and in particular the measurement of $n$ itself, whose uncertainty is greatly increased. Note that unless a power-law is assumed, this increase in uncertainty applies even if the values of the derivatives are zero to within the observational uncertainties. Nevertheless, the loss in accuracy on $n$ may well be overcompensated by the gain in information on higher derivatives. Indeed, one might expect that, as sneaking in an extra inflationary parameter is a way of transferring a small part of the information content in the microwave background away from the cosmological parameters and into the inflationary ones.

5 Inflationary expectations

As to whether this scale-dependence is likely to show up in practice, we have no better guide than the current theoretical prejudice, which says

- Most slow-roll inflation models do not give a significant scale-dependence, even by the standards of Planck.

- “Designer” models of inflation, for example the broken scale-invariance models, do give a large effect, but not one which is adequately treated within the perturbative framework I’ve outlined. Such models must be confronted with observation on a model-by-model basis.

- Hybrid inflation models can give an observable effect. Partly this is due to the so-called $\eta$-problem; inflation requires that two slow-roll parameters $\epsilon$ and $\eta$ be less than one, but on the other hand supergravity models generically predict $\eta = 1 + \text{‘something’}$. Since the ‘something’ is unlikely to be extremely good at cancelling the 1, such models may well not respect slow-roll very well and this enhances the chance of getting detectable scale-dependence. The best-motivated models at the moment are those of Stewart.

6 Conclusions

Inflation as a paradigm is both eminently and imminently testable by upcoming microwave background observations. For example, the prediction of a peak structure is extremely generic and quite specific to the situation where perturbations begin their evolution on scales much larger than the Hubble radius, and details such as the peak spacing promises a very strong test. Something as simple as an observed spectrum without multiple peaks appears sufficient to rule out inflation (see e.g. Barrow and Liddle). If inflation passes these tests, then detailed fitting to the observations promises startlingly high quality information about the inflationary mechanism.

However, the main purpose of my presentation is to provide a reminder of the important role inflation has in underpinning the microwave background endeavour. I stressed at the start that we can only get highly quality constraints from the present radiation power spectrum if we have a simple form, preferably motivated by theory, for the initial perturbations. Since the observations aim to be accurate at the percent level, the input information needs this accuracy too, and inflationary theory is now in a position where predictions at this level of accuracy can be made for all known models.

This can be contrasted with the situation for topological defect models, where it has proven much harder to make accurate theoretical predictions. Less accurate theoretical predictions will
naturally lead to much more poorly determined cosmological parameters. [In fact, Pen (these proceedings) has also argued that in a defect model the observed spectrum is less sensitive to the cosmological parameters, implying poorer parameter estimation even if the theoretical calculations can after all be made more accurate.]

If all goes well with the observations, and inflation proves to be right, we can indeed look forward to the tiny error bars one hears about so often. If inflation is not correct, the results will still be spectacular but yet after the hype the constraints may seem disappointing.

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