Bondi-Sachs energy-momentum and the energy of gravitational radiation

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Abstract

We construct the gravitational energy-momentum of the Bondi-Sachs space-time, in the framework of the teleparallel equivalent of general relativity (TEGR). The Bondi-Sachs line element describes gravitational radiation in the asymptotic region of the space-time, and is determined by the mass aspect and by two functions, $c$ and $d$, that yield the news functions, which are interpreted as the radiating degrees of freedom of the gravitational field. The standard expression for the Bondi-Sachs energy-momentum is constructed in terms of the mass aspect only. The expression that we obtain in the context of the TEGR is given by the standard expression, which represents the gravitational energy of the source, plus a new term that is determined by the two functions $c$ and $d$. We interpret this new term as the energy of gravitational radiation.

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1 Introduction

One of the most interesting consequences of general relativity is the description of gravitational radiation generated by isolated astrophysical configurations, that lose energy in the form of radiation. The space-time around these configurations is not strictly asymptotically flat because although the metric tensor components fall off in the expected way as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(1/r)$ in the asymptotic limit $r \to \infty$, the time derivative of $h_{\mu\nu}$ is of order $1/r$ at spacelike infinity in Cartesian coordinates. The first significant work on this issue is due to Bondi and collaborators [1], who established the coordinates and notation that are currently employed in the analysis of gravitational radiation. The line element obtained by Bondi and collaborators is constructed out of the functions $M(u, \theta)$ and $c(u, \theta)$, where $u$ is the retarded time $(u = t - r)$, and $r$ and $\theta$ are spherical coordinates. These functions do not depend on the spherical coordinate $\phi$ because the line element is axially symmetric. The function $M$ is called the mass aspect, and the time derivative $\partial_0 c = \partial c/\partial u$ is identified as the first news function.

Bondi’s line element was subsequently generalised by Sachs [2], who abandoned the axial symmetry and obtained the most general metric tensor that describes gravitational radiation at spacelike and null infinities. In the work by Sachs there also appears the function $d$, that yields the second news function $\partial_0 d$. The two news functions are interpreted as the radiating degrees of freedom of the gravitational field. The metric tensor obtained by Sachs yields what is presently known as the Bondi-Sachs space-time. The mathematical expression of the Bondi-Sachs metric tensor is rather intricate. There are very good review articles that clarify the several aspects of the subject, and also explain the emergence of the Bondi-Sachs energy-momentum vector [3] [4] [5] [6] [7] [8].

The Bondi-Sachs energy-momentum is constructed out of the mass aspect $M(u, \theta, \phi)$ only, i.e., it does not depend on the functions $c(u, \theta, \phi)$ and $d(u, \theta, \phi)$ (see Eq. (4.4) of Ref. [6]). To some extent, it is intriguing that the total energy and momentum do not depend on the these functions. Recently, the total Bondi-Sachs energy-momentum has been compared to the ADM expression [9] of the total gravitational energy-momentum at spacelike infinity of the Bondi-Sachs space-time. The metric tensor of the latter has been rewritten in the ordinary $(t, r, \theta, \phi)$ coordinates by parametrizing the spacelike hypersurfaces by the standard time $t$. It has been found [10] [11] [12] that the resulting expression for the total ADM energy-momentum depends
on the mass aspect, as expected, but also depends on the functions $c$ and $d$. Although the total ADM energy-momentum is strictly constructed for asymptotically flat space-times, the deviation of the resulting expression from the standard Bondi-Sachs energy-momentum and its dependence on the functions $c$ and $d$ - is a very interesting result.

In this article we obtain the expression of the total gravitational energy-momentum of the Bondi-Sachs space-time in the context of the teleparallel equivalent of general relativity (TEGR) [13]. The latter is an alternative geometrical formulation of general relativity based on the tetrad field. The TEGR provides a natural geometrical setting for consistent definitions of energy, momentum and angular momentum of the gravitational field. The definitions are given by surface integrals, they satisfy conservation equations, the algebra of the Poincaré group, and arise from well defined densities [14]. These definitions are possible because of the structure of the field equations and the covariance of the tetrad fields under global SO(3,1) transformations. The expressions for the gravitational energy-momentum and angular momentum are covariant under global SO(3,1) transformations. In special relativity the energy-momentum and angular momentum of localised material systems are frame dependent, and so they are in the TEGR, since the presence of gravitational fields (in the Newtonian approximation, for instance) do not modify this situation.

We find that the total energy-momentum of the Bondi-Sachs space-time is given by the standard integral of the mass aspect, plus a new term that contains the time derivative of the functions $c$ and $d$. This new term is interpreted as the energy of gravitational radiation. Attempts have been made in the past to arrive at such quantity, but failed because there was no guarantee that the integrals would be convergent [3]. The expression that we obtain in the present geometrical framework is finite. We test our expression by using a simple expression for the news function suggested long time ago in the literature.

In section II we briefly present the geometrical framework of the TEGR, the definition of the energy-momentum 4-vector, and the conservation equations. The Bondi-Sachs line element is presented in section III. In this section we display the asymptotic expansion of the metric tensor components and also construct the set of tetrad fields that will be used in the calculations. Section IV contains all relevant expressions and steps that yield the gravitational energy-momentum of the Bondi-Sachs space-time. The application to the news function suggested by Papapetrou [20], Halliday and Janis [21] and
Hobill [22] is given in section V. Finally in section VI we present the final remarks.

**Notation:** space-time indices $\mu, \nu, \ldots$ and SO(3,1) (Lorentz) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad fields are represented by $e^a_\mu$, and the torsion tensor by $T_{a\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu$. The flat space-time metric tensor raises and lowers tetrad indices, and is fixed by $\eta_{ab} = e^a_\mu e^b_\nu g^{\mu\nu} = (-1, +1, +1, +1)$. The frame components are given by the inverse tetrads $\{e^a_{\mu}\}$. The determinant of the tetrad field is written as $e = \det(e^a_{\mu})$.

The torsion tensor $T_{a\mu\nu}$ is sometimes related to the object of anholonomity $\Omega^\lambda_{\mu\nu}$ via $\Omega^\lambda_{\mu\nu} = e^a_{\lambda} T^a_{\mu\nu}$. It is important to note that we assume that the space-time geometry is defined by the tetrad fields only, and thus the only possible non-trivial definition for the torsion tensor is given by $T^a_{\mu\nu}$. This tensor is related to the antisymmetric part of the Weitzenböck connection $\Gamma^\lambda_{\mu\nu} = e^{a\lambda} \partial_\mu e^a_{\nu}$, which determines the Weitzenböck space-time and the distant parallelism of vector fields.

## 2 A brief review of the TEGR

The teleparallel equivalent of general relativity is a theory for the gravitational field based on the tetrad field. The dynamics of the gravitational field in the TEGR is exactly the same as in the standard metric formulation of general relativity. The tetrad fields have 16 independent components, and the extra six components, compared to the ten components of the metric tensor, allow the establishment of additional geometric structures. In the geometrical framework determined by the tetrad fields, one may dispose of the concepts of both the Riemannian and Weitzenböck geometries. The equivalence of the TEGR with Einstein’s general relativity is established by means of an identity between the scalar curvature $R(e)$, constructed out of the tetrad fields, and a combination of quadratic terms of the torsion tensor,

$$eR(e) \equiv -e\left(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^a_{\mu}T^a_{\mu}\right) + 2\partial_\mu(eT^\mu).$$  \hspace{1cm} (1)

The formulation of Einstein’s general relativity in the context of the teleparallel geometry is discussed in several references, see [13, 14, 15, 16, 17, 18]. The Lagrangian density of the TEGR is given by the combination of the quadratic terms on the right hand side of Eq. (1),
\[ L = -k e \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) - L_M \]
\[ \equiv -k e \Sigma^{abc} T_{abc} - L_M, \]  
\[ (2) \]
where \( k = c^3 / 16 \pi G \), \( T_a = T^b_{ba} \), \( T_{abc} = e_b \mu e_c \nu T_{a \mu \nu} \), and
\[ \Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c). \]  
\[ (3) \]
\( L_M \) represents the Lagrangian density of the matter fields. The field equations derived from (2) are equivalent to Einstein’s equations. They read [13]

\[ e_{a \lambda} e_{b \mu} \partial_\nu (e \Sigma^{b \lambda \nu}) - e (\Sigma^{b \mu} e T_{a \nu \mu} - \frac{1}{4} e_{a \mu} T_{b c d} \Sigma^{b c d}) = \frac{1}{4 k} e T_{a \mu}, \]  
\[ (4) \]
where \( \delta L_M / \delta e_{a \mu} = e T_{a \mu} \). It is possible to show that the left hand side of the equation above may be rewritten as \( \frac{1}{4} e \left[ R_{a \mu} (e) - \frac{1}{2} e_{a \mu} R(e) \right] \), which proves the equivalence of the present formulation with the standard metric theory.

Equation (4) may be rewritten in a simplified form as

\[ \partial_\nu (e \Sigma^{a \lambda \nu}) = \frac{1}{4 k} e e^a \mu (t^{a \lambda \mu} + T^{a \lambda \mu}), \]  
\[ (5) \]
where \( T^{a \lambda \mu} = e_a \lambda T^{a \mu} \), and \( t^{a \lambda \mu} \) is defined by

\[ t^{a \lambda \mu} = k (4 \Sigma^{b c \lambda} T_{b c \mu} - g^{\lambda \mu} \Sigma^{b c d} T_{b c d}). \]  
\[ (6) \]
In view of the antisymmetry property \( \Sigma^{a \mu \nu} = -\Sigma^{a \nu \mu} \), it follows that

\[ \partial_\lambda \left[ e e^a \mu (t^{a \lambda \mu} + T^{a \lambda \mu}) \right] = 0. \]  
\[ (7) \]
The equation above yields the continuity (or balance) equation,

\[ \frac{d}{dt} \int_V d^3 x e e^a \mu (t^{0 \mu} + T^{0 \mu}) = - \oint_S dS_j \left[ e e^a \mu (t^{j \mu} + T^{j \mu}) \right]. \]  
\[ (8) \]
We identify \( t^{a \lambda \mu} \) as the gravitational energy-momentum tensor [16], and

\[ P^a = \int_V d^3 x e e^a \mu (t^{0 \mu} + T^{0 \mu}), \]  
\[ (9) \]
as the total energy-momentum contained within a volume \( V \) of the three-dimensional space. In view of (5), Eq. (9) may be written as
\[ P^a = - \int_V d^3 x \partial_j \Pi^{aj}, \tag{10} \]

where \( \Pi^{aj} = -4k \epsilon \Sigma^{a0j} \), which is the momentum canonically conjugated to \( e_{a0j} \). The quantity \( \partial_j \Pi^{aj} \) is a well-defined space-time scalar density, that transforms as a vector under the global SO(3,1) group. The expression above may be transformed into a surface integral, and is the definition for the gravitational energy-momentum discussed in Refs. \[14, 16, 19\], obtained in the framework of the Hamiltonian vacuum field equations. Note that Eq. (8) is a true energy-momentum conservation equation.

The emergence of a non-trivial total divergence is a feature of theories with torsion. The integration of this total divergence yields a surface integral. If we consider the \( a = (0) \) component of Eq. (10) in spherical coordinates and a spacelike surface \( S \) determined by \( r = \) constant, we have

\[ P^{(0)} = \frac{E}{c} = - \oint_S dS_j \Pi^{(0)j} = 4k \oint_S d\theta d\phi \Sigma^{(0)01}. \tag{11} \]

Adopting asymptotic boundary conditions for the tetrad fields, we find \[19\] that in the limit \( S \to \infty \) the resulting expression is precisely the surface integral at spacelike infinity that defines the ADM energy \[9\]. This fact is an indication that Eq. (9) does indeed represent the gravitational energy-momentum vector. But note that there is no restriction regarding the applicability of definition (10). The latter may be applied to arbitrary space-times, with arbitrary boundary conditions.

3 The Bondi-Sachs space-time and the tetrad fields

The Bondi-Sachs metric tensor describes gravitational radiation at null and spatial infinities, and in both asymptotic limits the the metric tensor \( g_{\mu\nu} \) approaches the flat space-time metric tensor \( \eta_{\mu\nu} \), but time derivatives of \( g_{\mu\nu} \) fall off as \( 1/r \). As a consequence, gravitational waves may in principle be detected at spatial or null infinities. The line element of the Bondi-Sachs space-time in spherical \((u, r, \theta, \phi)\) coordinates, where \( u = t - r \) is the retarded time, is constructed out of the functions \( M(u, \theta, \phi) \), \( c(u, \theta, \phi) \) and \( d(u, \theta, \phi) \). It is given by
\[ ds^2 = g_{00} \, du^2 + g_{22} \, d\theta^2 + g_{33} \, d\phi^2 + 2g_{01} \, du \, dr + 2g_{02} \, du \, d\theta + 2g_{03} \, du \, d\phi + 2g_{23} \, d\theta \, d\phi , \]  

(12)

where

\[ g_{00} = \frac{V}{r} e^{2\beta} - r^2 (e^{2\gamma} U^2 \cosh 2\delta + e^{-2\gamma} W^2 \cosh 2\delta + 2UW \sinh 2\delta) , \]
\[ g_{01} = -e^{2\beta} , \]
\[ g_{02} = -r^2 (e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta) , \]
\[ g_{03} = -r^2 \sin \theta (e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta) , \]
\[ g_{22} = r^2 e^{2\gamma} \cosh 2\delta , \]
\[ g_{33} = r^2 e^{-2\gamma} \cosh 2\delta \sin^2 \theta , \]
\[ g_{23} = r^2 \sinh 2\delta \sin \theta . \]  

(13)

We adopt the usual convention \((u, r, \theta, \phi) = (x^0, x^1, x^2, x^3)\). The functions \(\beta, \gamma, \delta, U\) and \(W\) in the equations above are given only in asymptotic form, in powers of \(1/r\).

In this article we will present the asymptotic expansions of all field quantities that are effectively needed in the calculations, i.e., we will dispense with the powers of \(1/r\) of the field quantities that do not contribute to the calculations. Thus, the asymptotic form of the functions above are

\[ V \approx -r + 2M , \]
\[ \beta \approx -\frac{c^2 + d^2}{4r^2} , \]
\[ \gamma \approx \frac{c}{r} , \]
\[ \delta \approx \frac{d}{r} , \]
\[ U \approx -\frac{l(u, \theta, \phi)}{r^2} , \]
\[ W \approx -\frac{\bar{l}(u, \theta, \phi)}{r^2} , \]  

(14)

where

\[ l = \partial_2 c + 2c \cot \theta + \partial_3 d \csc \theta , \]

\[ \bar{l} = \partial_2 \bar{c} + 2\bar{c} \cot \theta + \partial_3 d \csc \theta . \]
\[
\bar{\ell} = \partial_2 \bar{d} + 2d \cot \theta - \partial_3 c \csc \theta .
\]

In the limit \( r \to \infty \), the asymptotic form of the functions above yield

\[
\begin{align*}
g_{00} & \simeq -1 + \frac{2M}{r}, \\
g_{01} & \simeq -1 + \frac{c^2 + d^2}{2r^2}, \\
g_{02} & \simeq l + \frac{1}{r}(2cl + 2d \bar{l} - p), \\
g_{03} & \simeq l \sin \theta + \frac{1}{r}(-2cl + 2dl - \bar{p}) \sin \theta, \\
g_{22} & \simeq r^2 + 2cr + 2(c^2 + d^2), \\
g_{33} & \simeq [r^2 - 2cr + 2(c^2 + d^2)] \sin^2 \theta, \\
g_{23} & \simeq 2dr \sin \theta + \frac{4d^3}{3r^2} \sin \theta.
\end{align*}
\] (15)

The functions \( p \) and \( \bar{p} \) are defined in Refs. [10,11]. They depend on functions that are not defined above, but since they will not contribute to the final expressions, we will not present their definitions here.

The expressions of the contravariant components of the metric tensor are calculated by means of the standard procedure out of Eqs. (13) (not out of Eqs. (15)). The inverse components are given by \( g^{\mu \nu} = (-1)^{\mu+\nu} (1/g) M_{\mu \nu} \), where \( g = -g_{01}^2 (g_{22} g_{33} - g_{23}^2) \) is the determinant of the metric tensor, and \( M_{\mu \nu} \) is the co-factor of the \( \mu \nu \) component (we have taken into account all necessary powers of \( 1/r \) of the functions given in Eq. (14)). We find

\[
\begin{align*}
g_{00} &= g_{02} = g_{03} = 0, \\
g_{01} &\simeq -1 - \frac{c^2 + d^2}{2r^2}, \\
g_{11} &\simeq 1 - \frac{2M}{r}, \\
g_{12} &\simeq \frac{l}{r^2}, \\
g_{13} &\simeq \frac{\bar{l} \sin \theta}{r^2}, \\
g_{22} &\simeq \frac{1}{r^2}.
\end{align*}
\]
\[ g^{33} \approx \frac{1}{r^2 \sin^2 \theta}, \]
\[ g^{23} \approx -\frac{2d}{r^3 \sin^2 \theta}. \]  

(16)

Now we turn to the construction of the tetrad fields. The inverse tetrads \( e_\alpha^\mu \) determine the frame adapted to a particular class of observers in space-time. Let the curve \( x^\mu(\tau) \) represent the timelike worldline \( C \) of an observer in space-time, where \( \tau \) is the proper time of the observer. The velocity of the observer along \( C \) is given by \( u^\mu = dx^\mu/d\tau \). A frame adapted to this observer is constructed by identifying the timelike component of the frame \( e_{(0)}^\mu \) with the velocity \( u^\mu \) of the observer: \( e_{(0)}^\mu = u^\mu(\tau) \). The three other components of the frame, \( e_{(i)}^\mu \), are orthogonal to \( e_{(0)}^\mu \), and may be oriented in the three-dimensional space according to the symmetry of the physical system. A static observer in space-time is defined by the condition \( u^\mu = (u^0, 0, 0, 0) \). Thus, a frame adapted to a static observer in space-time must satisfy the conditions \( e_{(0)}^i(t, x^k) = (0, 0, 0) \). It is easy to verify, by means of a coordinate transformation, that in terms of the retarded time \( u \) we also have \( e_{(0)}^i(u, r, \theta, \phi) = (0, 0, 0) \). The Bondi-Sachs space-time is not axially symmetric, and therefore there are no distinguished directions at spacelike infinity. Since we will evaluate surface integrals at spacelike infinity, i.e., we will be interested only in total quantities (we will also integrate over a surface \( S \) determined by \( r = \text{constant} \), for \( r \) finite but sufficiently large), any set of tetrad fields that satisfy the asymptotic expansion \( e_{\alpha}^\mu \approx \eta_{\alpha\mu} + (1/2)h_{\alpha\mu}(1/r) \) in Cartesian coordinates when \( r \to \infty \), and that satisfy the conditions \( e_{(0)}^i(u, r, \theta, \phi) = 0 \), will serve our purposes. For such a frame, \( e_{(1)}^\mu \), \( e_{(2)}^\mu \) and \( e_{(3)}^\mu \) will define the usual unit frame vectors in the \( x \), \( y \) and \( z \) directions, respectively, in the limit \( r \to \infty \), provided \( e_{\alpha}^\mu \) is constructed in Cartesian coordinates. If we restrict the Bondi-Sachs metric to the Bondi metric tensor (by making \( d = 0 = \bar{l} \)), then the latter is axially symmetric and \( e_{(3)}^\mu \) will define the unit vector in the \( z \) direction at spacelike infinity.

It is not straightforward to construct a simple set of tetrad fields that yields Eq. (12), and that satisfy the conditions \( e_{(0)}^i = 0 \). Note that \( e_{(0)}^i = 0 \) implies \( e_{(i)}^0 = 0 \). One such set of tetrad fields that satisfy these requirements, and acquires the asymptotic form \( e_{\alpha\mu} \approx \eta_{\alpha\mu} + (1/2)h_{\alpha\mu} \) at spacelike infinity, is given in \( (u, r, \theta, \phi) \) coordinates by

\[ e_{(0)\mu} = (-A, -E, -F, -G), \]
\[ e_{(1)\mu} = \left( 0, B_1 \sin \theta \cos \phi + B_2 \cos \theta \cos \phi - B_3 \sin \theta \sin \phi, \
C_1 r \cos \theta \cos \phi + C_2 \sin \theta \sin \phi, 
-D r \sin \theta \sin \phi \right), \]
\[ e_{(2)\mu} = \left( 0, B_1 \sin \theta \sin \phi + B_2 \cos \theta \sin \phi + B_3 \sin \theta \cos \phi, 
C_1 r \cos \theta \sin \phi - C_2 \sin \theta \cos \phi, 
D r \sin \theta \cos \phi \right), \]
\[ e_{(3)\mu} = \left( 0, B_1 \cos \theta - B_2 \sin \theta, -C_1 r \cos \theta, 0 \right). \]

The quantities \( A, B_1, B_2, B_3, C_1, C_2, D, E, F, G \) are determined by requiring that \( e_{a\mu} \) yields the metric tensor components (15) according to \( e_{a\mu} e_{b\nu} \eta^{ab} = g_{\mu\nu} \). The determination of exact form of these quantities in terms of the metric tensor components (13) is very complicated and useless for our purposes. We will need the components of the torsion tensor only in the asymptotic limit \( r \to \infty \). These quantities must satisfy the following equations,

\[
-A^2 = g_{00} \\
-AE = g_{01} \\
-AF = g_{02} \\
-AG = g_{03} \\
-E^2 + B_1^2 + B_2^2 + B_3^2 \sin^2 \theta = g_{11} = 0 \\
-EF + B_2(C_1 r) + B_3 C_2 \sin^2 \theta = g_{12} = 0 \\
-EG + B_3(Dr) \sin^2 \theta = g_{13} = 0 \\
-F^2 + (C_1 r)^2 + C_2^2 \sin^2 \theta = g_{22} \\
-G^2 + (Dr)^2 \sin^2 \theta = g_{33} \\
-FG + C_2(Dr) \sin^2 \theta = g_{23}.
\]

As we mentioned earlier, in the asymptotic expansion of the field quantities we will display the terms only up to the power of \( 1/r \) that is actually needed in the calculations, taking care that we do not neglect any relevant term up to \( (1/r)^2 \). Terms of order \( (1/r)^n \), with \( n \geq 3 \), do not contribute to the final, total expressions. We find

\[ A \simeq 1 - \frac{M}{r}, \]
\[ E \simeq 1 + \frac{M}{r}, \]
\[ F \simeq -l - \frac{1}{r}(2cl + 2d\bar{l} + Ml - p), \]
\[ G \simeq -\sin \theta \left[ \bar{l} + \frac{1}{r}(-2c\bar{l} + 2dl + M\bar{l} - \bar{p}) \right], \]
\[ B_1 \simeq 1 + \frac{M}{r}, \]
\[ B_2 \simeq -\frac{l}{r} - \frac{1}{r^2}(2Ml + cl - p), \]
\[ B_3 \simeq -\frac{1}{\sin \theta} \left[ \frac{l}{r} + \frac{1}{r^2}(2M\bar{l} - cl + 2dl - \bar{p}) \right], \]
\[ C_1 \simeq 1 + \frac{c}{r} + \frac{1}{r^2} \left[ \frac{l^2}{2} + c^2 - d^2 \right], \]
\[ C_2 \simeq \frac{1}{\sin \theta} \left[ 2d + \frac{1}{r} (\bar{l} + 2cd) \right], \]
\[ D \simeq 1 - \frac{c}{r} + \frac{1}{r^2} \left( \frac{l^2}{2} + c^2 + d^2 \right). \]  

The expressions above completely fix the set of tetrad fields given by Eq. (17).

4 The gravitational energy-momentum

4.1 Gravitational energy

The gravitational energy contained within a two-dimensional spacelike surface \( S \) (defined by \( r = \text{constant} \)) in an arbitrary space-time, in spherical coordinates, is obtained by evaluating the quantity \( \Sigma^{(0)01} \), as indicated in Eq. (11). The simplification of this quantity is crucial to arrive at the final result. Taking into account Eq. (3), and the fact that \( g^{00} = g^{02} = g^{03} = 0 \), we find

\[
\Sigma^{(0)01} = e^{(0)}_{\mu} \Sigma_{\mu 01} \\
= e^{(0)}_{0} \Sigma_{001} + e^{(0)}_{1} \Sigma_{101} + e^{(0)}_{2} \Sigma_{201} + e^{(0)}_{3} \Sigma_{301} \\
= A \left[ \frac{1}{4} \left( T^{001} + T^{001} - T^{100} \right) + \frac{1}{2} g^{01} T^{0} \right]
\]

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\[ + E \left[ \frac{1}{4} (T^{101} + T^{011} - T^{110}) + \frac{1}{2} \left( g^{11} T^0 - g^{01} T^1 \right) \right] \]
\[ + F \left[ \frac{1}{4} (T^{201} + T^{021} - T^{120}) + \frac{1}{2} g^{21} T^0 \right] \]
\[ + G \left[ \frac{1}{4} (T^{301} + T^{031} - T^{130}) + \frac{1}{2} g^{31} T^0 \right] . \]  
\text{(20)}

where \( T^\mu = T^\alpha \alpha^\mu \).

In Eq. (11), \( \Sigma^{(0)01} \) is multiplied by the determinant \( e \), whose asymptotic expression is \( e \simeq r^2 \sin \theta \). To obtain a non-vanishing value for the gravitational energy when the surface \( S \) of integration approaches the limit \( r \to \infty \), we must select the terms of \( \Sigma^{(0)01} \) that are of the order \( 1/r^2 \). Terms of the order \( 1/r^n \), with \( n \geq 3 \), do not contribute to the final expression.

By expanding Eq. (20) in terms of the covariant torsion tensor \( T^\alpha_{\mu \nu} \), we observe that all terms will be of the type \( g^{\mu \alpha} g^{\beta \gamma} T^\alpha_{\beta \gamma} \). In view of Eq. (16), we see that all products of the type \( g^{\mu \alpha} g^{\beta \gamma} g^{12} T^\alpha_{2 \beta \gamma} \) that arise in the expansion of Eq. (20) are at least of the order \( 1/r^2 \). Products of the order \( 1/r^n \), with \( n \geq 3 \), yield a vanishing contribution in the limit \( r \to \infty \). Therefore we keep only the products that are of the order \( 1/r^2 \), and consider the contributions from \( T^\alpha_{\mu \nu} \) whose values at spacelike infinity are of the order \( 1/r^0 \). It is important to mention that there does not arise any divergent term (of order \( O(r) \) or higher) in the expansion of \( e \Sigma^{(0)01} \). The final expression turns out to be finite. After long calculations, we conclude that the non-vanishing value of \( \Sigma^{(0)01} \) is simplified to

\[
\Sigma^{(0)01} = A \Sigma^{001} + E \Sigma^{101} + F \Sigma^{201} + G \Sigma^{301} \\
= -\frac{1}{2} A \left( g^{01} g^{01} g^{22} T_{212} + g^{01} g^{01} g^{33} T_{313} \right) \\
+ \frac{1}{2} E \left( g^{01} g^{01} g^{22} T_{202} + g^{01} g^{01} g^{33} T_{303} \right) \\
+ \frac{1}{4} F g^{01} g^{01} g^{22} (T_{012} - T_{201} - T_{102}) \\
+ \frac{1}{4} g^{01} g^{01} g^{33} (T_{013} - T_{301} - T_{103}). \]  
\text{(21)}

Inspection of the right hand side of the equation above indicates that we need to calculate only 10 components of \( T^\alpha_{\mu \nu} \). Dispensing with terms of order \( (1/r)^2 \) and higher orders, these components are given asymptotically by
\[ T_{201} \approx -\partial_0 l + \frac{1}{r} \left[ -c\partial_0 l - \partial_0 (cl + 2ML) - 2d\partial_0 \bar{l} + \partial_0 p \right], \]

\[ T_{301} \approx - (\partial_0 \bar{l}) \sin \theta + \frac{\sin \theta}{r} \left[ \bar{l} \partial_0 M + c\partial_0 \bar{l} - \partial_0 (2Ml - cl + 2dl) + \partial_0 \bar{p} \right], \]

\[ T_{102} \approx \partial_0 l + \frac{1}{r} \left[ -\partial_2 M + M\partial_0 l + \partial_0 (2cl + 2d\bar{l} + Ml) - l\partial_0 c - 2\bar{l}\partial_0 d \right], \]

\[ T_{202} \approx -l\partial_0 l + r\partial_0 c + c\partial_0 c + 4d\partial_0 d, \]

\[ T_{103} \approx (\partial_0) \sin \theta \]

\[ + \frac{\sin \theta}{r} \left[ -\partial_3 M + \bar{l}\partial_0 c + M\partial_0 \bar{l} + \partial_0 (M\bar{l} - 2c\bar{l} + 2dl) - \partial_0 \bar{p} \right], \]

\[ T_{303} \approx - (\bar{l}\partial_0 \bar{l} + r\partial_0 c - c\partial_0 c) \sin^2 \theta, \]

\[ T_{012} \approx \frac{\partial_2 M}{r}, \]

\[ T_{212} \approx \partial_2 l - M, \]

\[ T_{013} \approx \frac{\partial_3 M}{r}, \]

\[ T_{313} \approx (\partial_3 \bar{l}) \sin \theta - M \sin^2 \theta + l \sin \theta \cos \theta. \]  

(22)

No term of order $1/r$ in the expressions above will contribute to $P^{(0)}$. However, they will contribute to the momenta $P^{(i)}$. The substitution of Eqs. (16) and (22) into (21) yields

\[ \Sigma^{(0)01} = \frac{1}{r^2} \left[ M - \frac{1}{2} \left( \partial_2 l + \frac{\partial_3 \bar{l}}{\sin \theta} + l \frac{\cos \theta}{\sin \theta} \right) \right. \]

\[ - \frac{1}{2} l\partial_0 l - \frac{1}{2} \bar{l}\partial_0 \bar{l} + c\partial_0 c + 2d\partial_0 d \]. \]

(23)

Assuming $\bar{l}(\phi) = \bar{l}(\phi + 2\pi)$ and $l \sin \theta = 0$ for $\theta = 0$ and $\theta = \pi$, it is easy to see that, under integration, the second term in the expression above vanishes,

\[ \int_0^{2\pi} \int_0^\pi d\theta d\phi \sin \theta \left( \partial_2 l + \frac{\partial_3 \bar{l}}{\sin \theta} + l \frac{\cos \theta}{\sin \theta} \right) = 0. \]

(24)

Finally, substitution of Eq. (23) and $e = r^2 \sin \theta$ into definition (11), in the limit $r \to \infty$, results in

\[ P^{(0)} = 4k \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left[ M + \partial_0 F \right], \]

(25)
where

\[ F = -\frac{1}{4}(l^2 + \bar{l}^2) + \frac{1}{2}c^2 + d^2. \]  \hfill (26)

The integral of \( M(u, \theta, \phi) \) yields the mass aspect, but the total value of the gravitational energy depends also on the functions \( c(u, \theta, \phi) \) and \( d(u, \theta, \phi) \), as one would a priori expect. The new term \( \partial_0 F \) generalises the standard Bondi-Sachs energy.

For finite values of the ordinary time \( t \), the limit \( r \to \infty \) corresponds to \( u \to -\infty \), and therefore in Eq. (25) we have \( P(0)(u = -\infty) \). However, this is not the general case. We may consider large but finite values of the radial coordinate \( r \) (\( r \gg M \) and \( r \gg \partial_0 F \)) such that Eq. (23) is verified. In this case, \( P(0) \) depends on arbitrary and finite values of \( u \).

In equations (22), we observe that \( T_{202} \) and \( T_{303} \) depend linearly on the radial coordinate \( r \). This dependence could, in principle, lead to a divergent expression for \( P(0) \). However, the two contributions cancel each other, and the final expression turns out to be finite.

In order to check the consistency of the result above, we calculated \( P(0) \) out of a slightly different set of tetrad fields that satisfies the same requirements that led to Eq. (17), namely, that the frame is adapted to a static observer and that the set of tetrad fields has the asymptotic form \( e_{a\mu} \simeq \eta_{a\mu} + (1/2)h_{a\mu} \) in the limit \( r \to \infty \). We considered \( e_{a\mu} \) given by

\[
\begin{align*}
  e_{(0)\mu} &= (-A, -E, -F, -G), \\
  e_{(1)\mu} &= (0, B_1 \sin \theta \cos \phi + B_2 \cos \theta \cos \phi - B_3 \sin \theta \sin \phi, \\
&\hspace{1cm} Cr \cos \theta \cos \phi, \\
&\hspace{1cm} -D_1 r \sin \theta \sin \phi + D_2 \cos \theta \cos \phi), \\
  e_{(2)\mu} &= (0, B_1 \sin \theta \sin \phi + B_2 \cos \theta \sin \phi + B_3 \sin \theta \cos \phi, \\
&\hspace{1cm} Cr \cos \theta \sin \phi, \\
&\hspace{1cm} D_1 r \sin \theta \cos \phi + D_2 \cos \theta \sin \phi), \\
  e_{(3)\mu} &= (0, B_1 \cos \theta - B_2 \sin \theta, -C_1 r \cos \theta, -D_2 \sin \theta). \\
\end{align*}
\hfill (27)
\]

The values of \( A, B, \cdots \) in the equation above are of course different from Eq. (19), but eventually we obtain \( P(0) \) exactly as given by Eq. (25). The latter is, therefore, the gravitational energy that a static observer would measure.
at large distances from the source determined by $M(u, \theta, \phi)$. Assuming the speed of light $c = 1$ as well as $G = 1$, the quantity

$$E_{rad} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta (\partial_0 F),$$

may be interpreted as the energy of the gravitational radiation.

### 4.2 Gravitational momenta

The evaluation of the gravitational momenta $P^{(i)}$ requires the calculation of $\Sigma^{(i)01} = e^{(i)}_\mu \Sigma^{\mu01}$. The expressions of $\Sigma^{\mu01}$ were already obtained in the evaluation of $P^{(0)}$. These are the quantities that arise on the right hand side of Eq. (21). In the course of the calculations we find that the radial dependence of the tetrad components $e^{(i)2}$ and $e^{(i)3}$ in Eq. (17) requires the values of some components of $T_{\alpha\mu\nu}$ of order $1/r$, which are already presented in Eq. (22). As we mentioned earlier, the contributions from the functions $p$ and $\bar{p}$ cancel out in the calculations. After a number of simplifications and cancellations, we arrive at

$$P^{(1)} = 4k \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[ \sin^2 \theta \cos \phi (\partial_0 F) 
+ \frac{1}{4} \left( \sin \theta \cos \theta \cos \phi (2\partial_2 M + l\partial_0 M) 
- \sin \phi (2\partial_3 M + \bar{l} \sin \theta \partial_0 M) \right) \right],$$

$$P^{(2)} = 4k \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[ \sin^2 \theta \sin \phi (\partial_0 F) 
+ \frac{1}{4} \left( \sin \theta \cos \theta \sin \phi (2\partial_2 M + l\partial_0 M) 
+ \cos \phi (2\partial_3 M + \bar{l} \sin \theta \partial_0 M) \right) \right],$$

$$P^{(3)} = 4k \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[ \sin \theta \cos \phi (\partial_0 F) 
- \frac{1}{4} \sin^2 \theta (2\partial_2 M + l\partial_0 M) \right].$$

The expressions above may be simplified by making integrations by parts in the angular variables. The expressions are further simplified by introducing the three-dimensional vectors.
\[ \hat{r}^i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \]
\[ \hat{\theta}^i = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \]
\[ \hat{\phi}^i = (-\sin \phi, \cos \phi, 0). \]  
(30)

We finally obtain
\[ P^{(i)} = 4k \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left[ (M + \partial_0 F)\hat{r}^i + \frac{1}{4}(l\partial_0 M)\hat{\theta}^i + \frac{1}{4}(\bar{l}\partial_0 M)\hat{\phi}^i \right]. \]  
(31)

The expression above and \( P^{(0)} \) given by Eq. (25) constitute the gravitational energy-momentum of the Bondi-Sachs space-time in the realm of the TEGR. One important observation is the following. It is well known that the time derivative \( \partial_0 M \) may be written, in view of the field equations, in terms of the time derivatives of the functions \( c \) and \( d \) [2,5] according to
\[ \partial_0 M = -[(\partial_0 c)^2 + (\partial_0 d)^2] + \frac{1}{2} \partial_0 \left( \partial_2 l + l \cot \theta + \frac{\partial_3 \bar{l}}{\sin \theta} \right). \]  
(32)

If we evaluate all integrals in \( P^a \) in the limit \( r \to \infty \), we are actually taking the limit \( u \to -\infty \). If we further assume that the news functions satisfy the initial conditions
\[ \partial_0 c(-\infty) = 0, \quad \partial_0 d(-\infty) = 0, \]  
(33)
then the total energy-momentum \( P^a = (P^{(0)}, P^{(i)}) \) given by (25) and (31) reduces to the well known expression for the Bondi-Sachs energy-momentum. These initial conditions might be physically reasonable, but there is no justification for assuming them. We asserted above that we can evaluate \( (P^{(0)}, P^{(i)}) \) over a surface \( S \) of integration sufficiently large from the source, such that Eq. (23) is verified. This fact allows us to dispense with the initial conditions given by Eq. (33).
5 News function in an axially symmetric space-time

In this section we will discuss an application of Eq. (28) by requiring the space-time geometry to be axially symmetric. The restriction to such symmetry is achieved by enforcing $d = 0 = \bar{l}$. The latter equations imply that the components of the space-time metric tensor do not depend on the variable $\phi$, ensuring the axial symmetry of the configuration. This simplification allows the use a general form of the news function suggested by Papapetrou [20], Halliday and Janis [21] and Hobill [22]. It is very difficult to obtain the exact form of the news function $c(u, \theta)$ because the latter must be consistent with the mass coefficient $M(u, \theta)$ such that Eq. (32) is verified, and the verification of the latter equation is non-trivial.

Adopting the usual convention $x \equiv \cos \theta$, the suggested form of the news function is written as [22]

$$c(u, x) = (1 - x^2) \sum_{n=2} a_n(u) \frac{d^2}{dx^2} P_n(x),$$

where $P_n(x)$ are the Legendre polynomials and $a_n(u)$ are general functions of the retarded time $u$, that reduce to constants when the source is not radiating. Of course, it is assumed that the summation in the expression above (as well as all summations in this section) converges. Hobill [22] specified a form of the time dependence of $a(u)$ and proceeded to obtain a particular expression for $c(u, x)$. Below we will consider the more general form given by Eq. (34).

The function $F$ given by Eq. (26) may be rewritten as

$$F = -\frac{(\partial c\theta)^2}{4} - \cot \theta c(\partial_c c) - \frac{3}{2} c^2 \cot^2 \theta + \frac{c^2}{2 \sin^2 \theta}.$$  (35)

With the help of identities involving the Legendre polynomials, we write $c(u, x)$ in the form

$$c(u, \theta) = \sum_{n=2} a_n(u) \left[ 2x \frac{d}{dx} P_n(x) - n(n+1)P_n(x) \right].$$

Substitution of Eq. (35) into definition (28) yields

$$E_{rad} = \frac{1}{8} \partial_\theta \int_{-1}^{1} \left[ -(1 - x^2)(\partial_x c)^2 + 4x c \partial_x c + 2c^2(1 - 3x^2) \right] dx.$$  (37)
The equation above may be further simplified by taking into account Eq. (36). We arrive at

\[ E_{\text{rad}} = \frac{1}{8} \partial_0 \left[ -\int_{-1}^{1} (1 - x^2) (\partial_x c)^2 dx - 6 \sum_{n,m} a_n a_m \int_{-1}^{1} x^2 P_n^2 P_m^2 dx \right], \quad (38) \]

where \( P_n^m \) are the associated Legendre polynomials. Finally, using

\[ \partial_x c = \sum_{n=2} a_n \left[ \frac{P_n^3}{(1 - x^2)^{1/2}} - \frac{2xP_n^2}{(1 - x^2)} \right], \]

we obtain

\[
E_{\text{rad}} = -\frac{1}{8} \partial_0 \sum_n a_n^2 (n + 2)! \frac{(16n^4 + 28n^3 + 20n^2 + 11n + 105)}{(n - 2)!(2n + 1)(2n - 1)(2n + 3)} \\
+ \frac{1}{8} \partial_0 \sum_{n,m} a_n a_m (n + 2)! \frac{[1 + (-1)^{n+m}]}{(n - 2)!} \\
- \frac{3}{2} \partial_0 \sum_n a_n a_{n+2} (n + 4)! \frac{1}{(n - 2)!(2n + 1)(2n + 3)(2n + 5)}.
\]

The Bondi mass \( m(u) \) is defined as the usual integral over the mass aspect \( M(u, \theta) \),

\[ m(u) = \frac{1}{2} \int_{-1}^{1} dx \ M(u, x). \quad (40) \]

It has been demonstrated \([20, 21, 22]\) that if one requires the total variation of the Bondi mass, due to gravitational radiation, to be equal to the total variation of the mass aspect, then the summation in Eqs. (38) and (39) cannot have a finite number of terms, i.e., the summation cannot be truncated.

The most interesting feature of Eq. (39) is the dependence of \( E_{\text{rad}} \) on a quadratic combination of the coefficients \( a_n(u) \). It might be possible to impose conditions on the coefficients \( a_n(u) \) such that the summation simplifies considerably, but we have not attempted to carry out any simplification, which would require an analysis of the field equations.
6 Conclusions

The Bondi-Sachs space-time is determined by three functions, \( M, c \) and \( d \), which depend on \((u, \theta, \phi)\). The standard expression of the gravitational energy of the Bondi-Sachs space-time is the integral of the mass aspect \( M(u, \theta, \phi) \) in the angular variables, and restricting to the Bondi space-time, it is identified with \( m(u) \) given by Eq. (40). It is not clear why both expressions do not depend on the functions \( c \) and \( d \). The expression that we obtained in the realm of the TEGR, Eq. (25), does depend on \( M, c \) and \( d \). We also obtained contributions of the functions \( c \) and \( d \) to the total gravitational momenta, as given by Eq. (31). Altogether, Eqs. (25) and (31) constitute the gravitational energy-momentum of the Bondi-Sachs space-time in the framework of the TEGR.

The total energy-momentum of the Bondi-Sachs space-time, evaluated by means of the ADM definitions in \((t, r, \theta, \phi)\) coordinates, does depend on the functions \( c \) and \( d \) [12]. However, the ADM definitions are strictly valid only for asymptotically flat space-times, which is not the case of the Bondi-Sachs space-time.

In view of the structure of Eq. (25), we may identify the term that depends only on \( c \) and \( d \), Eq. (28), as the gravitational energy of radiation \( E_{rad} \), since the functions \( c \) and \( d \) are non-vanishing at spacelike infinity and the news functions propagate over the whole three-dimensional space. The expression of \( E_{rad} \) is clearly finite.

By requiring \( d = 0 = \bar{l} \), we restrict the Bondi-Sachs space-time to the Bondi space-time, which is endowed with axial symmetry. In this space-time, a suggested form for the news function \( c(u, \theta) \) is given by Eq. (34). The functions \( c \) and \( d \) yield the news functions, which are assumed as the radiating degrees of freedom of the gravitational field. From this point of view, it is interesting that the energy of gravitational radiation (28) results in a quadratic combination of the coefficients \( a_n \). These coefficients would be naturally amenable to quantization.

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References

[1] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, Proc. R. Soc. London A269, 21 (1962).

[2] R. K. Sachs, Proc. R. Soc. London A270, 103 (1962).

[3] A. Trautman, “Conservation Laws in General Relativity”, in “Gravitation: an Introduction to Current Research”, page 169, edited by L. Witten (Wiley, New York, 1962).

[4] F. A. E. Pirani, “Gravitational Radiation”, in “Gravitation: an Introduction to Current Research”, page 199, edited by L. Witten (Wiley, New York, 1962).

[5] R. K. Sachs, “Gravitational Radiation”, in “Relativity, Groups and Topology”, page 522, lectures delivered at the Les Houches Summer School 1963, edited by C. de Witt and B. de Witt (Gordon and Breach, New York, 1963).

[6] J. N. Goldberg, “Invariant Transformations, Conservation Laws and Energy-Momentum”, in “General Relativity and Gravitation”, page 469, edited by A. Held (Plenum, New York, 1980).

[7] J. N. Goldberg, Phys. Rev. 131, 1367 (1963).

[8] P. T. Chruściel, J. Jezierski and M. A. H. MacCallum, Phys. Rev. D 58, 4001 (1998).

[9] R. Arnowitt, S. Deser and C. W. Misner, “The Dynamics of General Relativity”, in “Gravitation: an Introduction to Current Research”, page 199, edited by L. Witten (Wiley, New York, 1962).

[10] Xiao Zhang, Adv. Theor. Math. Phys. 10, 261 (2006), arXiv:gr-qc/0511036.

[11] Wen-Ling Huang, Shing Tung Yau and Xiao Zhang, Rend. Lincei Mat. Appl. 17, 335 (2006), arXiv:math/0604155.

[12] Wen-Ling Huang and Xiao Zhang, Proc. of the 4th International Congress of Chinese Mathematicians, Hangzou 2007, Vol. III, 367-379 (Higher Education Press, 2008), arXiv:gr-qc/0511037.
[13] J. W. Maluf, J. Math. Phys. 35, 335 (1994).

[14] J. W. Maluf, Ann. Phys. (Berlin) 525, 339 (2013).

[15] F. W. Hehl, in “Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergravity”, Erice, 1979, edited by P. Bergmann and V. de Sabbata (Plenum, New York, 1980).

[16] J. W. Maluf, Ann. Phys. (Berlin) 14, 723 (2005).

[17] F. Gronwald, Int. J. Mod. Phys. D 6 (1997) 263.

[18] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne’eman, Phys. Rep. 258, 1 (1995).

[19] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toribio and K. H. Castello-Branco, Phys. Rev. D 65, 124001 (2002).

[20] A. Papapetrou, Ann. Inst. H, Poincaré 11, 57 (1969).

[21] W. Halliday and A. Janis, J. Math. Phys. 11, 578 (1970).

[22] D. W. Hobill, J. Math. Phys. 25, 3527 (1984).