Fast Generation of Spatially Embedded Random Networks

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Spatially Embedded Random Networks such as the Waxman random graph have been used in a variety of settings for synthesizing networks. However, little thought has been put into fast generation of these networks. Existing techniques are $O(n^2)$ where $n$ is the number of nodes in the graph. In this paper we present an $O(n + e)$ algorithm, where $e$ is the number of edges.

I. INTRODUCTION

Random graphs are frequently used as the underlying model in fields such as computer networking, biology and physics, but increasingly the datasets we wish to model involve large numbers of elements. This is problematic because common methodologies that investigate the properties of such large graphs involve the generation of random graphs as examples in order to investigate their asymptotic behavior’s and though there are many works on the analysis of random graphs, there are relatively few on how to generate large graphs efficiently.

This paper is concerned with efficient generation of large Spatially Embedded Random Networks (SERN). These arose soon after the Gilbert-Erdős-Rényi (GER) random graph [1, 2] with the random plane network proposed by Gilbert [3]. However, the most cited example is the Waxman random graph [4], with the original paper being cited several thousand times.

The GER random graph links every pair of vertices independently with a fixed probability, whereas the Waxman graph reflects that in real networks longer links are often more costly or difficult to construct, and their existence is therefore less likely. It links nodes $i$ and $j$ with a probability given by a function of the distance $d_{i,j}$ between them. The form chosen by Waxman was the negative exponential

$$p(d_{i,j}) = q e^{-s d_{i,j}},$$

for parameters $q, s \geq 0$. A Waxman random graph is generated by randomly choosing a set of points in a section of the plane (usually the unit square), and then linking these points independently according to their distance.

The idea of a distance-based probability of connection has been generalized in SERNs [5, 6]. In these the metric space in which the points are embedded is generalized, as is the distance function, but the underlying concept is identical. There are many examples of SERNs, including random plane networks, random geometric graphs, spatial networks, range-dependent random graphs, random connection models, random distance graphs, and partially structured random graphs.

Generation of synthetic random graphs is one of the basic requirements for modeling. Synthetic graphs allow consideration questions such as “how will a network behave if it grows?” The model overlaying the graph may be quite complex, such as a routing protocol in computer networking. This means analysis of the graph may not provide much insight whereas the ability to simulate the model may. Ability to generate a suite of such graphs which match some characteristics of real graphs allows us to test not just predictions, but also their sensitivity to the underlying assumptions (for instance the choice of parameter values). The facility to synthesize graphs is also needed in estimation procedures such as Approximate Bayesian Computation (ABC) [7].

At present, the only algorithms available for generating SERNs use the approach of generating the node locations, calculating the distances, and then generating edges using a series of Bernoulli trials, which is $O(n^2)$ in the number of nodes. We call this the naïve algorithm.

However, many realistic graphs are very sparse in the sense that $O(e) < O(n^2)$, the number of edges grows more slowly that the number of potential edges. So for many real-world examples an $O(n^2)$ algorithm is highly inefficient.

Here we develop a fast, efficient method for creating large sparse SERNs. Our method takes $O(n + e)$ computation, which is the best possible computation time for an exact method. We have used the method to generate graphs with up to a billion nodes.

We also demonstrate a multi-threaded implementation that shows that the method parallelizes.

Code implementing the algorithm is available at github.com/lamestllama/conSERN.

II. BACKGROUND

SERNs [5, 6] constitute a large class of useful random-graph models, including Gilbert’s random plane network [3] (also known by other names such as the random geometric graph [8]) and the Waxman random graph [4]. The Waxman graph has been used in many settings from computer networks to biological cell networks, typically to synthesize random networks. We demonstrate our approach with this particular SERN but the reader should keep in mind that our
implementation already caters for the general case.

We are not aware of any general tools to generate wideclasses of SERNs, but there are a number that have been
designed for generating Waxman random graphs [9–13]. None
seriously consider how to generate these graphs quickly.

- NetworkX [14], aSHIIP [13], NEM [11, 12] and GT-ITM [10] all generate the graph using the naïve \( O(n^2) \) algorithm.
- The Matlab Waxman graph generator [15] also executes a vectorized naïve algorithm.
- BRITE [9] has two algorithms, but both appear to make serious deviations from the standard Waxman model in order to generate connected graphs. Also, although one approach is technically \( O(e) \) it uses an rejection sampling approach that can take hours to generate even small networks.

All of the methods that generate true Waxman graphs are \( O(n^2) \) in computation time [16], and the vectorized Matlab algorithm is \( O(n^2) \) in memory as well.

In modern problems, networks of millions of nodes are common, and billion node networks exist. For instance, FaceBook claims (as of July 2015) over a billion active users, who form part of a large graph. As network modeling moves towards encompassing such graphs, the need for synthesizing very large graphs increases.

There are also some subsequent algorithms that require us to generate a large number of random graphs. For instance ABC [7] requires a large number of synthesized graphs, over a wide range of parameters. The requirement means we need an efficient generator.

A. Mathematical formalities

A graph (or network) consists of a set of \( n \) vertices (we shall synonymously refer to them as nodes), which, without loss of generality, we label \( V = \{1, 2, \ldots, n\} \), and edges (or links) \( E \subset V \times V \). We are primarily concerned here with undirected graphs (though much of the work on random graphs is easy to generalize to directed graphs). We say that two nodes \( i \) and \( j \) are adjacent or neighbors if \((i, j) \in E\).

The GER random graph [1, 2], \( G_{n,p} \) of \( n \) vertices is constructed by assigning each edge \((i, j)\) to be in \( E \) independently, with fixed probability \( p \). A SERN generalizes this by making the probability of each edge dependent on the distance between the two nodes.

Formally, we create a SERN by placing \( n \) nodes randomly within some defined region \( R \) of a metric space \( \Omega \) with distance metric \( d(x, y) \). Each pair of nodes is made adjacent independently, with link probability given by a function of distance \( d_{i,j} = d(x_i, x_j) \) between nodes \( i \) and \( j \). For instance we could define a space, with one of the standard distance metrics

- Euclidean: \( d_{i,j} = ||x_i - x_j||_2 \).
- Manhattan: \( d_{i,j} = ||x_i - x_j||_1 \).
- Discrete: \( d_{i,j} = ||x_i - x_j||_0 \).
- Max: \( d_{i,j} = ||x_i - x_j||_\infty \).

and one of the following link probability functions:

- Waxman: \( p_{i,j} = q e^{-s d_{i,j}}, \) where \( s \in [0, \infty), q \in (0, 1], [4][17] \).
- Clipped Waxman: \( p_{i,j} = \min(q e^{-s d_{i,j}}, 1), \) where \( s \in [0, \infty), q \in (0, 1); \)
- Mixed Waxman-threshold: \( p_{i,j} = q e^{-s d_{i,j}} H(r - d_{i,j}), \) where \( s \in [0, \infty), q \in (0, 1], r \in [0, \infty); \)
- Threshold: \( p_{i,j} = q H(r - d_{i,j}), \) where \( q \in (0, 1], r \in [0, \infty), \) (motivated by the random plane network [3]);
- GER: \( p_{i,j} = q, \) where \( q \in (0, 1], [1, 2]; \)
- Power law: \( p_{i,j} = q (1 + \theta_1 d_{i,j})^{-\theta_2}, \) (e.g., range-dependent random graphs) \[18–21\];
- Cauchy: \( p_{i,j} = q (1 + \theta_1 d_{i,j}^2)^{-1} \) \[22\];
- Exponential: \( p_{i,j} = \frac{q e^{-d_{i,j}}}{L - d_{i,j}}, [10]; \)
- Max entropy: \( p_{i,j} = \frac{q e^{-s d_{i,j}}}{1 + q e^{-s d_{i,j}}}; \)

where \( H(r) \) is the Heaviside step function, and \( L \) denotes the longest possible link in the region in question. Note that our parameterizations are sometimes different from those in the literature so that they can be presented consistently.

All examples of which we are aware have non-increasing link-probability functions. We refer to these as distance deterrence functions, and exploit this property in our algorithm.

Many of the properties of SERNs are known. For instance, the average node degree in the Waxman graph is \[23\]

\[
\bar{k} = (n - 1)q\tilde{G}(s), \tag{2}
\]

where \( \tilde{G}(s) \) is the Laplace transform of \( g(t) \), the probability density function between an arbitrary pair of random points (as in the Line-Picking Problem) \[24, 25\].

B. Fast generation of GER graphs

The common method for generation of GER \( G_{n,p} \) is simply to perform \( O(n^2) \) Bernoulli trials, one for each possible edge. Batagelj and Brandes [26] noted that this algorithm is naïve and that a faster implementation was possible.
The best algorithm is \( \Omega(n + e) \), i.e., no SERN generation algorithm can be faster than a factor of the number of nodes and edges generated, because the edges are independent (conditional on node locations).

Batagelj and Brandes [26] noted that if we list the possible edges, then taking sequential trials generates a discretized Poisson Process. If the graph is sparse, then it is much faster to generate the points of this process by taking geometrically distributed jumps. The result is an \( O(n + e) \) algorithm as it generates one edge per jump.

Somewhat surprising, particularly as [26] presents fast methods for generating \( G(n, m) \), we have seen no works that consider generation by using the fact that \( G_{n,p} = G(n, M) \) where \( M \sim \text{Binomial}(\ell, p) \), where \( \ell \) is the number of possible edges. In this case we might generate \( M \) from the binomial distribution (or more efficiently for sparse networks via its Poisson approximation), and then use the resampling technique of [26] which is \( O(m) \) to generate the network. This evidently scales just as the previous algorithm, but the constant time components are different, and so for some parameter values this approach might be faster.

Regardless, our goal here is to exploit some of these ideas to generate (sparse) SERNs, but it is not so simple: we can not just generate a jump process, because all links are not equal, and likewise we can not sample from the possible links. However, in addition to the insight of Batagelj and Brandes we add that the jump process on the edges allows for the edges to be listed in any order. That means we can exploit the geometrical structure of the SERN to list the possible edges in an advantageous order for synthesis.

III. FAST WAXMAN GENERATION

All SERN generators start by generating a set of \( n \) nodes, which takes \( O(n) \) operations. We discuss methods for doing so quickly and efficiently in \( \S \text{V} \), as this requires some implementation tricks. Here we concentrate on the main performance bottleneck, which is generating the edges.

For simplicity, we describe our edge generation algorithms here specifically for the Waxman SERN, though our code generalizes this for all the cases described above. The naive algorithm for generating the edges of an undirected Waxman graph is shown in Algorithm 1.

Our first algorithm – \( q \)-jumping – uses the observation that

\[ p(d_{i,j}) = q \, e^{-sd_{i,j}} \leq q. \] (3)

Thus, there exists a GER \( G_{n,q} \) random graph that is an “upper bound” on the Waxman random graph, in the sense that each Waxman random graph is a subgraph of a GER random graph. We can generate the the GER graph using the jump process described above, and then filter to obtain the Waxman graph as shown in Algorithm 2.

The computational cost of the \( q \)-jumping algorithm can be seen to be \( O(e_1) \) where \( e_1 \) is the number of edges in the \( GER_{n,q} \) graph. We can derive this number of edges in relation to the Waxman graph by noting [23] that

\[ \mathbb{E}[e_1] = n\bar{k}/2, \]
\[ \mathbb{E}[e] = nk\bar{G}(s)/2, \]

where \( \bar{k} \) is average node degree. The algorithm is therefore overall \( O(e) \), but we want not only good order performance, but also efficient algorithms. The efficiency of this approach depends on the ratio of the two expectations, \( \bar{G}(s) \).

A Laplace transform of a PDF obeys certain properties: \( \bar{G}(0) = 1 \), and \( \bar{G}(s) \to 0 \) for large \( s \), so the \( q \)-jumping algorithm will be quite efficient for small \( s \), but less so as \( s \) grows. On the other hand, the main property of the Waxman graph is that for larger \( s \), long links are unlikely. Thus the very nature of these graphs creates geometric structure, we can exploit in their generation.

We do so by breaking the region into \( M^2 \) “buckets” as shown in Figure 1. Given nodes \( i \) and \( j \) in buckets \( I \) and \( J \), respectively, we can put a lower bound \( D_{i,j} \leq d_{i,j} \) on the distance between the nodes, and thus an upper bound on the probability of a link.

Note that as the GER jumping algorithm does not depend on the order of the potential edges, or even that we generated them all at once, we can use this approach to generate

1: Input: \( n, q, s \)
2: \( \mathcal{E} \leftarrow \emptyset \)
3: for \( i = 1 \ldots n \) do
4: \hspace{1em} for \( j = i+1 \ldots n \) do
5: \hspace{2em} calculate \( d_{i,j} \)
6: \hspace{2em} calculate \( p_{i,j} \leftarrow q \exp(-sd_{i,j}) \)
7: \hspace{2em} generate \( r \sim U[0, 1] \)
8: \hspace{2em} if \( r \leq p_{i,j} \) then
9: \hspace{3em} \( \mathcal{E} \leftarrow \mathcal{E} \cup (i, j) \)
10: \hspace{2em} end if
11: end for
12: end for

ALGORITHM 1. The naïve algorithm for generating the edges of an undirected Waxman graph. We refer here to the uniform random variate on the interval \([0, 1] \) as \( U[0, 1] \).

1: Input: \( n, q, s \)
2: \( \mathcal{E} \leftarrow \emptyset \)
3: \( \mathcal{E}_1 \leftarrow G_{n,q} \)
4: for \( (i, j) \in \mathcal{E}_1 \) do
5: \hspace{1em} calculate \( d_{i,j} \)
6: \hspace{1em} calculate \( p'_{i,j} \leftarrow \exp(-sd_{i,j}) \)
7: \hspace{1em} generate \( r \sim U[0, 1] \)
8: \hspace{1em} if \( r \leq p'_{i,j} \) then
9: \hspace{2em} \( \mathcal{E} \leftarrow \mathcal{E} \cup (i, j) \)
10: \hspace{1em} end if
11: end for

ALGORITHM 2. The \( q \)-jumping algorithm for generating the edges of an undirected Waxman graph.
Algorithm 3. The bucket algorithm for generating the edges of an undirected Waxman graph.

1: Input: \( n, q, s, M \)
2: \( E \leftarrow \phi \)
3: for \( I=1..M \) do
4:     for \( J=1..M \) do
5:         \( N_{I,J} \leftarrow \) number of possible node pairs
6:         \( Q_{I,J} \leftarrow q \exp(-sD_{I,J}) \)
7:         \( E_{I,J} \leftarrow G_{N_{I,J},Q_{I,J}} \)
8:         for \((i,j) \in E_{I,J}\) do
9:             calculate \( d_{i,j} \)
10:            calculate \( \mu_{i,j} \leftarrow \exp(-s(d_{i,j} - D_{I,J})) \)
11:            generate \( r \sim U[0,1] \)
12:            if \( r \leq \mu_{i,j} \) then
13:                \( E \leftarrow E \cup (i,j) \)
14:            end if
15:        end for
16:     end for
17: end for

Algorithm 3. The bucket algorithm for generating the edges of an undirected Waxman graph.

the set of edges between any pair of buckets, using the upper bound given above. The resulting algorithm is shown in Algorithm 3.

The algorithm does not yet describe

1. creating the buckets; and
2. data structures for efficiently storing the component edges, and bringing them back together at the end.

These are necessary to create fast code but do not affect the asymptotic performance of the algorithm, which is again \( O(e) \), but faster than the \( q \)-jumping algorithm for large \( s \). We describe these below in \( \S \) V.

IV. RESULTS

We test the performance of the algorithms described above using a C implementation, for which we provide stand-alone code, library functions, and R and Matlab bindings. We use the last to provide a mechanism to time generation through Matlab’s \texttt{tic()}/\texttt{toc()} function, which provide a wall-clock time estimate which we can compare against existing Matlab code. We test timing by performing 100 generations and taking the shortest times for each on a Ubuntu 12.10 Linux box running on an Intel i7 X990 CPU with 6 cores running at 3.47 GHz, with Matlab (R2013a), and gcc 4.7.2. In each case we generate a network with fixed average node degree \( \bar{k} = 1 \), i.e., a sparse graph with \( O(e) = O(n) \).

Figure 2 shows the results for a small \( s \) value, over a range of network sizes \( n \), and for two bucket grid sizes \( M = 1 \) and 10. The dashed blue curve shows results for a vectorized Matlab implementation as a benchmark. The dashed red curve is the naïve algorithm, which shows clear \( O(n^2) \) performance, with roughly a two times speed up in comparison to the Matlab implementation. NB: Matlab has the capability to use multiple threads to speed up vectorized computations, whereas this C-version uses a single thread, hence the C-code speedup is not as great as might be expected. The Matlab implementation uses \( O(n^2) \) memory, so we do not try to perform any very large tests.

The solid curves show the bucket-based algorithm for two bucket grid sizes \((M = 1 \text{ and } 10)\). Note that when \( M = 1 \) the bucket algorithm is equivalent to the \( q \)-jumping algorithm.

We can see for both values of \( M \) that the performance for large \( n \) is \( O(e) \), and that the bucket grid size \( M \) has negligible impact for large \( n \). For small \( n \) we can see the overhead (which is \( O(M^2) \)) in the initial bucket generation procedure.

Figure 3 shows the performance for large \( s \), and although we see the same broad features as in the previous figure, we now also see the benefit of the bucket. A larger number of buckets improves the algorithm for larger \( s \), though there is a diminishing return as \( M \) increases.

We consider the effect of bucket dimension more carefully in Figure 4, which shows the performance for fixed \( n \) over
a range of $s$ values, for different bucket dimensions. Most obviously, any fixed number of buckets has a “sweet spot” where it best balances the initial overhead of bucket creation with the performance drop-off as $s$ increases. However, a relatively small number of buckets (around $M = 20$) provides good performance over a very wide range of parameters (note the log scales).

The final results shown in Figure 5 show the multi-threading performance in comparison to the ideal parallelized performance. The figure shows that the parallelization works, but that the multi-thread implementation has significant overhead in bringing the edges back together. If one were aiming to calculate statistical properties of the graph that did not require it to be stored as a whole (for instance, average node degrees or link distances), then one could construct the information required, in parallel, without this overhead, and thus attain the ideal performance.

V. IMPLEMENTATION DETAILS

Our implementation is based on a shared C library which can be built using the supplied makefiles on Linux or Xcode project file for OSX, using only standard C libraries already present.

The makefiles in the package will create the following:

- a command line based application that outputs a GraphML file [27]; and
- a shared library, suitable to for linking to a program written in any language that can support the C call by reference calling convention.

The shared library is linkable with any other software capable of using the C call by reference calling convention: we provide as examples

- Matlab MEX bindings,
- Rcpp based R bindings,

High-level tools such as as Matlab and R do their own garbage collection, so data passed back to them must be allocated using specific functions that allow access by the callers’ garbage collection routines. Rather than conditionally compile the routines on a per application basis they can take pointers to functions for allocating and reallocating memory and will use these (if supplied) to create any memory that will be returned to the caller.

The implementation was developed under OSX and Linux concurrently and the thread model chosen was the POSIX pthread library as it was well supported on both.

The aim is to create very large graphs so the use of memory is important. We avoid the use of data structures which fragment memory and thus cause the processor to try to cache memory from both ends of the available address space simultaneously. Using linear data structures maximizes the effectiveness of the cache.
We achieve very close to lower bound on the amount of memory needed to fully specify a SERN. Other than the lookup table \( Q \) and a small buffer for each bucket, we store only the \((x, y)\) coordinates of the nodes and \((\text{from}, \text{to})\) pairs of node identifiers that define the links. We do not store labels for each node as they have an implicit ordering.

Our code has an upper limit of \(2^{20}\) on the number of nodes \( N \), so links \((i, j)\) can be represented as two 32 bit integers. Coordinates \((x_i, y_i)\) and optionally the distances for each link \(d_{i,j}\) are represented as 32 bit floats. So the total storage requirements are

- \((x_i, y_i)\) for \(i = 1, \ldots, N\) takes \(2 \times 4 \times n\) bytes;
- \(E = \{(i, j)\}\) takes \(2 \times 4 \times e\) bytes; and
- optionally \(d_{i,j}\) for \((i, j) \in E\) takes \(1 \times 4 \times e\) bytes.

Total memory usage is \(8(n+e)\) to store the graph, or \(8n+12e\) bytes if we include distances. In the largest example we have considered, with \(n = 10^9\) and \(E[k] = 3\), the memory usage was around 20 gigabytes (measured in powers of ten).

The implementation has two phases:

- Node generation within bucket data structures given the restrictions of the geometry used to define the region; and
- Link generation following algorithm described above.

The following sections give an overview of the techniques used in each phase.

A. Node generation

The running time of our algorithm is dominated by the time spent creating links, but most of the code is devoted to setting the preconditions for the algorithm to work efficiently. We need to make the discovery of the connection buckets and the nodes they contain an \(O(1)\) operation. Our implementation does this by creating an array of bucket data structures, each of which record a node count and an offset to the start of its data in the arrays of \(x\) and \(y\) values associated with the nodes. Whilst this arrangement is straightforward to produce naively, particularly when the SERN is embedded in a square region, our implementation allows for parallel execution and three types of region in which to embed the SERN:

- A rectangular region, generalizing the square region initially investigated by Waxman [4];
- An elliptical region allowing investigation of SERN’s where there is no corner effects; and
- A user defined polygon allowing real world boundary data to be used.

The technique is to find a rectangular area that covers the defined region, and then divide this into \(M_1 \times M_2\) square buckets, where we denote \(M = \max(M_1, M_2)\).

We then calculate the area \(A_{i,j}\) of the intersection of the defined region with bucket \((i, j)\).

- For non-square rectangular regions we simply compare the dimensions of the defined region with the dimensions of the area covered by the \(M_1 \times M_2\) square buckets,
- For elliptical regions if all the points defining a bucket lie wholly inside or outside the region then \(A_{i,j}\) is the area of the bucket or zero respectively. If the bucket intersects the boundary of the region then the method described by Groves [28] is used to calculate the intersection area.
- For a region defined using a polygon if the bucket is wholly inside or outside the defined region the \(A_{i,j}\) are as in the case of an elliptical region. If the bucket intersects the boundary of the region then the Sutherland-Hodgman [29] algorithm is used to calculate the intersection area.

At present, these precalculations are done per call, but it is clear that if more than one graph is to be generated on the same region, these could be precalculated once.

The probability of a node being in a particular bucket is \(P_{i,j} = A_{i,j}/A\). We calculate the number of nodes in each bucket in advance using a multinomial distribution \(\text{Mult}(n, P_{i,j})\) to allow memory allocation to be performed once. That makes node creation and allocation to buckets embarrassingly parallel. Also, all nodes can be stored in a single contiguous memory block with a separate pointer to the start of each bucket, rather than separate memory for each bucket.

The algorithm for generating multinomials is simply the conditioned repeated use of the algorithm for generating binomial random variates due to Knuth [30].

The resultant output is an array of \(M_1 \times M_2\) bucket data structures containing a count of the number of nodes within them and offsets into the larger array containing the \(x\) and \(y\) coordinates of the nodes. The set of nodes are not mixed or overlapped between buckets within this array, meaning the link creation algorithm can operate on them blindly.

To place nodes on non-square regions we generate candidates on a square that covers the region and then accept or reject based on their membership of the region. The rejection rate can be controlled via the bucket size.

The only other prerequisite for the operation of the fast link creation algorithm is the generation of a lookup table \(Q\) (Algorithm 3 step 6) to enable filtering based on the distances between the buckets. \(Q\) is implemented as an array where \(Q[n, m]\) represents the probability of a link of the minimum length between buckets that are \(n\) buckets apart
in the $x$ axis and $m$ buckets apart in the $y$ axis. The discussion of distance and link probability calculation is dealt with hereunder as it also applies to the link generation phase.

In early development it was apparent that the C stdlib library random number generating functions have a number of limitations in this setting. The srand function is not re-entrant nor is it thread safe because it stores state internally thus can be immediately disqualified. The rand_r function and drand48_r functions allow the caller to provide storage space for the state but here we will be calling the function $2N$ times to create the coordinates for $N$ nodes and the overhead of a function call becomes substantial. Based on the work of Marsaglia [31] we implemented a multiply with carry (MWC) random number generator with separate state for each thread and because it is not a library function we were able to use the C99 inline calling convention to eliminate the overhead of a function call. This random number generator is the source for creating the random variates drawn from geometric and binomial distributions used in all sections of the implementation.

B. Link generation

The general algorithm for fast link generation has been covered in §III so here we will use the notation of Algorithm 3 to discuss implementation details.

The implementation combines steps 7 and 8 of the algorithm by looping from 0 to $N + 1$ and in skips of length $2N$ times to create the coordinates for $N$ nodes and the overhead of a function call becomes substantial. Based on the work of Marsaglia [31] we implemented a multiply with carry (MWC) random number generator with separate state for each thread and because it is not a library function we were able to use the C99 inline calling convention to eliminate the overhead of a function call. This random number generator is the source for creating the random variates drawn from geometric and binomial distributions used in all sections of the implementation.

VI. SERNs in general

As noted, the current code allows generation on rectangles, ellipses and arbitrary polygons on $\mathbb{R}^2$.

Currently our standard wrappers implement the four metrics and nine link probability functions described in §II A, but the routines for both bucket and link generation have been parameterized to accept pointers to functions that implement distance and link probabilities, so creating new models involves only five or six lines of code.

Taking all the combinations of space, metric and probability functions we get a total of nearly 100 different models implemented in the default code, with the ability to add new models simply by passing function handles, so our code can generate a very large class of SERNs.

The major limitation at present is that the points lie in $\mathbb{R}^2$, but the bucket algorithm extends to $\mathbb{R}^n$ and to surfaces other than the plane, such as the sphere and cylinder. It is our intention to add these as possibilities, but this requires additional complexity in the generation of the buckets and management of memory. Moreover, at present we have no clear application for SERNs in high dimensions.
VII. CONCLUSION AND FUTURE WORK

This paper describes an algorithm to perform fast $O(n+e)$ generation of SERNs. The results from the implementation described show that the performance is several orders of magnitude faster than competing code for large graphs. However, there are still many improvements that could be made, and are the topic of current work:

- Better threading by predicting the amount of links that will be generated and allocating memory in advance with sufficient leeway to allow for the variance in the number of links so that we can pick an acceptable failure rate. The algorithm could then run at whatever bandwidth the memory is capable of for the vast majority of the time and would only have to restart occasionally due to insufficient memory being allocated.

- The current implementation of polygon shapes can be optimized by recursively finding the intersection of buckets with the defined region. Rather than compare the whole region with each bucket we can compare an arbitrary small portion of the region with each bucket.

- Portions of the code are suitable for moving to a graphics processing unit (GPU) which would speed execution of these sections but reduce portability.

- The current implementation is multi-threaded, but the algorithm is suitable for a multi-processor implementation, for instance using map-reduce.

- We aim to extend the approach to higher-dimensional spaces, and non-Euclidean manifolds such as the surface of a sphere.

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