The uniform invariant approximation property for compact groups

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Abstract

In this short note we give a proof of the refined version of the uniform invariant approximation property for compact (non-commutative) groups following the Bourgain’s approach ([B]).

1 Introduction

We shall use the following notation: $G$ will stand for a compact group with the normalized Haar measure $m$ and the dual object $\Sigma$ (consisting of equivalence classes of continuous irreducible unitary representations), $L^p(G)$ are the usual Banach spaces of $p$-integrable functions with respect to $m$ and $M(G)$ is the convolution algebra of all complex-valued Borel regular measures endowed with the total variation norm. For $f \in L^1(G)$ we write $\hat{f}(\sigma)$,

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σ ∈ Σ for a matrix defined as follows

\[ \hat{f}(\sigma) = \int_G \sigma(x^{-1}) f(x) dm(x) = \int_G U_x^{(\sigma)} f(x) dm(x) \]

where \( \sigma(x) = U_x^{(\sigma)} \) for all \( x \in G \).

For every \( \sigma \in \Sigma \) let \( d_{\sigma} \) denote the dimension (necessarily finite) of the Hilbert space \( H_{\sigma} \) on which \( \sigma \) acts and let \( \zeta_1^{(\sigma)}, \ldots, \zeta_{d_{\sigma}}^{(\sigma)} \) be a fixed orthonormal basis of \( H_{\sigma} \). With \( \sigma \in \Sigma \) and \( j, k \in \{1, \ldots, d_{\sigma}\} \) we associate a coordinate function (coefficient of the representation) defined by the formula:

\[ u_{jk}(x) = \langle U_x^{(\sigma)} \zeta_j^{(\sigma)}, \zeta_k^{(\sigma)} \rangle. \]

For \( y \in G \) we write \( (l_y f)(x) = f(y^{-1} x) \) and \( (r_y f)(x) = f(xy) \) for \( f \in L^1(G) \), \( x \in G \). A linear operator \( T : X \to X \) where \( X = C(G) \) or \( X = L^p(G) \), \( 1 \leq p < \infty \) is called invariant if for every \( y, z \in G \) we have \( r_y l_z T = T r_y l_z \).

Our main reference for harmonic analysis on compact groups is the first chapter of [HR].

The uniform invariant approximation property for a wide class (translation invariant regular Banach spaces) in the terminology from [K], the prototypical examples are \( L^p(G) \) spaces for \( 1 \leq p < \infty \) of function spaces on a compact group \( G \) is equivalent to the following theorem (see [K] for details).

**Theorem 1.** For every \( k > 1 \) there exists a positive sequence \( q_k(r) \) such that for every finite set \( R \subset \Sigma \) there exists a central function \( g \in L^1(G) \) such that:

1. \( \hat{g}(\sigma) = \text{Id}_{\sigma} \) for \( \sigma \in R \),

2. \( \|g\|_1 \leq k \),

3. \( v(\text{supp} \hat{g}) \leq q_k(v(R)) \) where for \( S \subset \Sigma \) we put \( v(S) := \sum_{\sigma \in S} d_{\sigma}^2 \).

The most important question is how \( q_k(v(R)) \) grows with \( v(R) \). It was proved in [BP] that for Abelian groups one can take \( q_k(r) \simeq r^{4r} \), later the estimate was refined (again for commutative groups) by J. Bourgain in [B] to \( q_k(r) \simeq c^{2r} \) where \( c > 0 \) is an absolute constant. For non-Abelian groups it was proved by J. Krawczyk [K] that the estimate given by Bożejko and Pełczyński holds true. In what follows we will prove that the refined estimate by J. Bourgain is correct also for non-commutative groups by extending the proof presented in [W] to this setting. To be more precise our aim is to prove the following theorem.
Theorem 2. Let \( R \subset \Sigma \) be a finite set. Then for every \( \varepsilon > 0 \) there exists a central function \( f \in L^\infty(G) \) such that:

1. \( \hat{f}(\sigma) = 1d_{d_\sigma} \) for \( \sigma \in R \),
2. \( \|f\|_1 \leq 1 + \varepsilon \),
3. \( v(\text{supp}\hat{f}) \leq \left(\frac{c}{\varepsilon}\right)^{2v(R)} \) where \( c > 0 \) is an absolute constant.

2 Main result

We need to recall first a few facts from the theory of Banach spaces. We start with II.E.13 from [W].

Proposition 3. For every \( n \)-dimensional (complex) Banach space \( X \) and for every \( \delta > 0 \) there exists \( N \leq (1+\delta)^{2n} \) and an embedding \( u : X \to l_1^N \) with \( (1-\delta)\|x\| \leq \|u(x)\| \leq \|x\| \).

The next is III.E.14 from [W].

Proposition 4. For any \( \delta > 0 \) and every Banach space \( X \), every subspace \( Y \subset X \) and every finite rank operator \( T : Y \to C(K) \) there exists an operator \( \tilde{T} : X \to C(K) \) such that \( \tilde{T}|_Y = T \) and \( \|\tilde{T}\| \leq (1+\delta)\|T\| \).

Definition 5. An operator \( T : X \to Y \) is absolutely summing, if there exists a constant \( C < \infty \) such that for all finite sequences \( (x_j)_{j=1}^n \subset X \) we have

\[
\sum_{j=1}^n \|Tx_j\| \leq C \sup \left\{ \sum_{j=1}^n |x^*(x_j)| : x^* \in X^*, \|x^*\| \leq 1 \right\}.
\]

We define the absolutely summing norm of an operator \( T \) by

\[
\pi_1(T) = \inf \{C : \text{the above holds for all } (x_j)_{j=1}^n \subset X, \ n = 1, 2, \ldots\}.
\]

The collection of all absolutely summing operators forms an operator ideal (for a precise definition see [W]). In particular, every finite rank operator is absolutely summing and \( \pi_1(BTA) \leq \|B\|\pi_1(T)\|A\| \) for bounded operators \( A, B \) and \( T \) whenever the composition makes sense.

Definition 6. Let \( G \) be a compact group. A measure \( \mu \in M(G) \) is called central if \( \mu * \nu = \nu * \mu \) for every \( \nu \in M(G) \), i.e. \( \mu \) is in the center of the convolutive algebra \( M(G) \).
The next theorem gives equivalent conditions for centrality (see Theorem 28.48 in [HR]).

**Theorem 7.** Let $G$ be a compact group. The following properties of a measure $\mu \in M(G)$ are equivalent:

1. $\mu$ is central,
2. $\mu * u^{(\sigma)}_{jk} = u^{(\sigma)}_{jk} * \mu$ for some set of coordinate functions $\{u^{(\sigma)}_{jk}\}$ and every $\sigma \in \Sigma$,
3. $\hat{\mu}(\sigma) = \alpha(\mu, \sigma) I_{d_\sigma}$ for all $\sigma \in \Sigma$ where $\alpha(\mu, \sigma) \in \mathbb{C}$.

Now we have a non-commutative analogue of III.F.12 from [W]

**Proposition 8.** Let $G$ be a compact group and let $T : C(G) \to C(G)$ be a bounded linear invariant operator which is absolutely summable. Then there exists a central $h \in L^\infty(G)$ such that $T f = f * h$ for $f \in C(G)$. Moreover $\pi_1(T) = \|h\|_\infty$.

**Proof.** By Theorem 1.2 in [BE] there exist $\mu, \nu \in M(G)$ such that $T f = f * \mu = \nu * f$. Taking into account that the adjoint $T^* : M(G) \to M(G)$ is given by the very similar formula to $T$ and inserting $\delta_e$ into the definition of $T^*$ we obtain $\mu = \nu$. It follows now from Theorem 7 that $\mu$ is a central measure (as the coordinate functions are continuous). The rest of the proof is the same as the argument for justifying III.F.12 in [W].

We shall also use the basic Peter-Weyl theorem (see 27.40 and 28.43 in [HR]).

**Theorem 9** (Peter-Weyl). Let $G$ be a compact group. The set of functions $d_\sigma^1 u^{(\sigma)}_{jk}$ is an orthonormal basis for $L^2(G)$. Thus for $f \in L^2(G)$ we have

$$f = \sum_{\sigma \in \Sigma} d_\sigma \sum_{j,k=1}^{d_\sigma} \langle \hat{f}(\sigma) \zeta_k^{(\sigma)}, \zeta_j^{(\sigma)} \rangle u^{(\sigma)}_{jk} \text{ the series converging in } L^2(G).$$

Moreover, $\|f\|^2 = \|\hat{f}\|^2 = \sum_{\sigma \in \Sigma} d_\sigma \|\hat{f}(\sigma)\|_{HS}^2$ where $\|\hat{f}(\sigma)\|_{HS} = \sqrt{\text{tr}(\hat{f}(\sigma)\hat{f}(\sigma)^*)}$ is the Hilbert-Schmidt norm of a matrix $\hat{f}(\sigma)$.

**Lemma 10.** Let $G$ be a compact group and let $\sigma \in \Sigma$ and $f \in L^1(G)$. Then the following holds true:
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1. For every \( y, z \in G \) and \( j, k \in \{1, \ldots, d_\sigma\} \) we have
   \[
   l_y r_z u^{(\sigma)}_{jk} \in \text{lin} \left\{ u^{(\sigma)}_{jk} : j, k \in \{1, \ldots, d_\sigma\} \right\}.
   \]

2. If \( f \cdot u^{(\sigma)}_{jk} = u^{(\sigma)}_{jk} \) for every \( j, k \in \{1, \ldots, d_\sigma\} \) then \( \tilde{f}(\sigma) = I_{d_\sigma} \).

Proof. We have
   \[
   (l_y r_z u^{(\sigma)}_{jk})(x) = u^{(\sigma)}_{jk}(y^{-1}xz) = \langle U^{(\sigma)}_{y^{-1}xz} \zeta_k^{(\sigma)}, \zeta_j^{(\sigma)} \rangle =
   \langle U^{(\sigma)}_{x} U^{(\sigma)}_{z} \zeta_k^{(\sigma)}, \zeta_j^{(\sigma)} \rangle = \langle U^{(\sigma)}_{x} \zeta_k^{(\sigma)}, U^{(\sigma)}_{y} \zeta_j^{(\sigma)} \rangle.
   \]
Writing \( U^{(\sigma)}_{z} \zeta_k^{(\sigma)} = \sum_{i=1}^{d_\sigma} c_i \zeta_k^{(\sigma)} \) and \( U^{(\sigma)}_{y} \zeta_j^{(\sigma)} = \sum_{i=1}^{d_\sigma} c'_i \zeta_j^{(\sigma)} \) for some complex coefficients \( c_i \) and \( c'_i \) we obtain the assertion of the first part of the lemma.

In order to prove the second part let us observe that \( \overline{u}^{(\sigma)}_{jk}(\sigma) = e_{jk} \) (matrix unit in \( M_{d_\sigma}(\mathbb{C}) \)). Hence \( \tilde{f}(\sigma)e_{jk} = e_{jk} \) for every \( j, k \in \{1, \ldots, d_\sigma\} \) which implies the desired conclusion.

After these preparations we are ready to prove Theorem 2.

By Proposition 8 and the second part of Lemma 10 the assertion of the theorem is equivalent to the existence of a certain linear bounded invariant operator \( T : C(G) \rightarrow C(G) \). Let us fix a number \( \delta \) satisfying \( 0 < \delta < 1 \). By Proposition 8 there exists a positive integer \( N < \left( \frac{1+\delta}{\delta} \right)^{2v(R)} \) (observe that \( v(R) = \dim R \)) and an embedding \( u : R \rightarrow l_N^\infty \) with \( \|u\| \cdot \|u^{-1}\| \leq \frac{1}{1-\delta} \).

Applying the Hahn-Banach theorem coordinatewise we get \( \tilde{u} : C(G) \rightarrow l_N^\infty \) - the extension of \( u \) with \( \|u\| = \|\tilde{u}\| \). In addition, let \( v : l_N^\infty \rightarrow C(G) \) be an extension of \( u^{-1} \) with \( \|v\| \leq (1+\delta)\|u^{-1}\| \) (such extension is possible by Proposition 4). Put \( T_1 := v\tilde{u} : C(G) \rightarrow C(G) \). Then, obviously \( T_1|_R = Id_R \) and using the ideal property of absolutely summing operators (see the comment following Definition 5) and an elementary calculation \( \pi_1(id : l_N^\infty \rightarrow l_N^\infty) = N \) we get
   \[
   \pi_1(T_1) \leq \|v\| \cdot \|\tilde{u}\| \pi_1(id : l_N^\infty \rightarrow l_N^\infty) \leq \|v\| \cdot \|\tilde{u}\| \cdot N \leq N \frac{1+\delta}{1-\delta}.
   \]

We define
   \[
   T_2 = \int_{G \times G} l_{y^{-1}z^{-1}} T_1 r_z l_y dm(z) dm(y).
   \]
The operator \( T_2 \) is invariant and by the first part of Lemma 10 we have \( T_2|_R = Id \). Moreover,
   \[
   \|T_2\| \leq \|T_1\| \leq \frac{1+\delta}{1-\delta};
   \]
   \[
   \pi_1(T_2) \leq \pi_1(T_1) \leq N \frac{1+\delta}{1-\delta}.
   \]
From Proposition 8 (actually, we use the version of Proposition 8 for functions which is explicitly stated as Theorem 28.49 in [HR]) we infer that $T_2$ is a convolution with a central $h \in L^\infty(G)$ satisfying

$$
\hat{h}(\sigma) = Id_{d_\sigma} \text{ for } \sigma \in R,
$$

$$
\|h\|_1 \leq \frac{1+\delta}{1-\delta},
$$

$$
\|h\|_\infty \leq N \frac{1+\delta}{1-\delta}.
$$

Last two inequalities give $\|h\|_2 \leq \frac{1+\delta}{1-\delta} \sqrt{N}$. Let us define $g = h \ast h \ast h$. Then $g$ is also central and by Theorem 7 we have $\tilde{g}(\sigma) = \alpha(g, \sigma)Id_{d_\sigma} = \alpha^3(h, \sigma)Id_{d_\sigma}$ for every $\sigma \in \Sigma$. Applying the Peter-Weyl theorem to $g$ we have

$$
g = \sum_{\sigma \in \Sigma} d_\sigma \alpha(g, \sigma) \sum_{j=1}^{d_\sigma} u_{jj}^{(\sigma)}.
$$

Put

$$
f = \sum_{\sigma \in \Sigma, |\alpha(g,\sigma)| > N^{-4}} d_\sigma \alpha(g, \sigma) \sum_{j=1}^{d_\sigma} u_{jj}^{(\sigma)}.
$$

Then, using the equality $\|h\|_2 = \|\hat{h}\|_{HS}$, we obtain

$$
\|f\|_1 \leq \|g\|_1 + \|g - f\|_1 \leq \|h\|_1^3 + \sum_{\sigma \in \Sigma, |\alpha(g,\sigma)| \leq N^{-4}} d_\sigma^2 |\alpha(g, \sigma)| \leq
$$

$$
\leq \left( \frac{1+\delta}{1-\delta} \right)^3 + N^{-\frac{4}{3}} \sum_{\sigma \in \Sigma} d_\sigma^2 |\alpha(g, \sigma)|^3 =
$$

$$
= \left( \frac{1+\delta}{1-\delta} \right)^3 + N^{-\frac{4}{3}} \|h\|_2^3 \leq \left( \frac{1+\delta}{1-\delta} \right)^3 + \left( \frac{1+\delta}{1-\delta} \right)^2 N^{-\frac{1}{3}}.
$$

Finally, with the aid of $L^2$ theory again, we get

$$
v \left( \{ \sigma \in \Sigma : \tilde{f}(\sigma) \neq 0 \} \right) \leq
$$

$$
\leq \sum_{\sigma \in \Sigma, |\alpha(h,\sigma)| > N^{-\frac{4}{3}}} d_\sigma^2 \leq \sum_{\sigma \in \Sigma, |\alpha(h,\sigma)| > N^{-\frac{4}{3}}} |\alpha(h, \sigma)|^2 d_\sigma^2 \leq
$$

$$
\leq N^{\frac{8}{3}} \|h\|_2^3 \leq N^{\frac{14}{3}} \left( \frac{1+\delta}{1-\delta} \right)^2 \leq N^4 \left( \frac{1+\delta}{1-\delta} \right)^2.
$$

Choosing correct $\delta$ to $\varepsilon$ finishes the proof (the exact dependence is difficult to calculate but asymptotically $\varepsilon \simeq \delta^4$).
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