INDECOMPOSABLE REPRESENTATIONS OF LIE SUPERALGEBRAS

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ABSTRACT. In 1960’s I. Gelfand posed a problem: describe indecomposable representations of any simple infinite dimensional Lie algebra $g$ of polynomial vector fields. Here by applying the elementary technique of Gelfand and Ponomarev a toy model of the problem is solved: finite dimensional indecomposable representations of $\text{vect}(0|2)$, the Lie superalgebra of vector fields on the $(0|2)$-dimensional superspace, are described.

Since $\text{vect}(0|2)$ is isomorphic to $\mathfrak{sl}(1|2)$ and $\mathfrak{osp}(2|2)$, their representations are also described. The result is generalized in two directions: for $\mathfrak{sl}(1|n)$ and $\mathfrak{osp}(2|2n)$. Independently and differently J. Germoni described indecomposable representation of the series $\mathfrak{sl}(1|n)$ and several individual Lie superalgebras.

Partial results for other simple Lie superalgebras without Cartan matrix are reviewed. In particular, it is only for $\text{vect}(0|2) \simeq \mathfrak{sl}(1|n)$ and $\mathfrak{sh}(0|4) \simeq \mathfrak{psl}(2|2)$ that the typical irreducible representations can not participate in indecomposable modules; for other simple Lie superalgebras without Cartan matrix (of series $\text{vect}(0|n)$, $\text{svect}(0|n)$, $\tilde{\text{svect}}(0|n)$, $\mathfrak{spe}(n)$ for $n \geq 3$ and $\mathfrak{sh}(0|m)$ for $m \geq 5$) one can construct indecomposable representations with arbitrary composition factors.

Several tame open problems are listed, among them a description of odd parameters, previously ignored.

To the memory of Misha Saveliev together with whom we began to apply the description of representations of $\mathfrak{osp}(N|2)$ to the solution of $N$-extended Leznov-Saveliev equations, cf. [LSS].

INTRODUCTION

This is a version of the paper published in: In: Sissakian A. N. (ed.), Memorial volume dedicated to Misha Saveliev and Igor Luzenko, JINR, Dubna, 2000, 126–131.

Meanwhile Noam Shomron [SH] rediscovered an unpublished A. Shapovalov’s results on modules over $\text{vect}(0|m)$, $\text{svect}(0|m)$, $\mathfrak{pe}(m)$, $\mathfrak{spe}(m)$ and $\mathfrak{h}(0|m)$, $\mathfrak{sh}(0|m)$. Shomron formulated it only for $\text{vect}(0|m)$ but went further in formulation and his proof is impressive.

In what follows the ground field is $\mathbb{C}$. This paper is an attempt to review the subject, in particular, formulate tame open problems.

0.0. Prehistory. The study of irreducible finite dimensional representations of simple finite dimensional Lie algebras over $\mathbb{C}$ is a natural problem. It turns out that such representations are completely reducible and, therefore, it suffices to study irreducible modules. Situation changes when we consider infinite dimensional modules, even if the module is “semi-infinite” in a sense, e.g., possesses a highest or lowest weight vector. Among such representation and their “infinite in both ways” generalizations, Harish-Chandra modules, there is no complete reducibility but the problem of description of all indecomposable modules in these categories seem to be wild. (So we can describe either finite dimensional (hence, irreducible), or infinite dimensional and irreducible, or nothing.)

1991 Mathematics Subject Classification. 17B10 (Primary) 16G20 (Secondary).

Key words and phrases. Lie superalgebra, indecomposable representation.

Financial support of NFR and technical of V. Serganova, A. Shapovalov and P. Grozman, as well as a crucial hint of A. A. Kirillov, are gratefully acknowledged.
The study of invariant differential operators on manifolds is closely related with continuous in a natural topology infinite dimensional representations of simple infinite dimensional Lie algebras of vector fields $\mathcal{L}$. Though the representations themselves are no longer completely reducible, the description of irreducible ones reduces, to an extent, to finite dimensional representations of simple finite dimensional Lie algebras $L_0$, the linear parts of $\mathcal{L}$ (see [3] and an elucidation [BL]). This reduction to finite dimensional representation of simple Lie algebras was, perhaps, a motivation for I. Gelfand to consider the classification problem of indecomposable representations (say, in the class of modules with highest or lowest weight vectors) of such Lie algebras not hopeless (at least, in the particular and exceptional case of $\text{vect}(1) = \text{der} \mathbb{C}[x]$). Observe that this problem is still open.

Passing to Lie superalgebras, we are forced to consider their indecomposable representations even in the above problems for several reasons:

1. even finite dimensional representations of simple Lie superalgebras are never completely reducible (with the only exception of $\mathfrak{osp}(1|2n)$; the proof of this fact is similar to that for Lie algebras, cf. [3]);

2. the description of irreducible and continuous in a natural topology infinite dimensional representations of simple infinite dimensional Lie superalgebras of vector fields $\mathcal{L}$ does not reduce to finite dimensional representations of the corresponding finite dimensional Lie superalgebras $L_0$ but even if we confine ourselves to such representations we have to face problem (1). (This was the reason problem (1) was first tackled; G. Shmelev [Sh1], [Sh2] considered problem (1) only in connection with problem (2) in a particular case of the latter: for $L_0 = \mathfrak{osp}(m|2n)$.)

One more reason is provided by the necessity to classify embeddings of $\mathfrak{osp}(N|2)$ into the simple Lie superalgebras; this classification is a part of the explicit solution of vector-valued generalizations of $N$-extended Leznov-Saveliev equations considered, so far, for $N = 1$ only, see [LSS].

0.1. History. Main result. Unless otherwise stated, in what follows $\mathfrak{g}$ is either $\mathfrak{sl}(1|n)$ or $\mathfrak{osp}(2|2n)$. For $\mathfrak{g}$, the description of irreducible finite dimensional representations is complete, cf. [3], [L]. The next step — a description of indecomposable modules — was performed under the assumption of $\mathfrak{h}$-diagonalizability in [Sh1], [Sh2], [L2] who used a key observation A. A. Kirillov made: he related the problem with a result of [ZN].

A draft of this paper was written in 1989; I delayed the publication because I wanted to compute the odd parameters. Regrettably, it is still an open problem. Meanwhile the problem considered in this paper was solved for $\mathfrak{sl}(1|2)$ by Su Yucai [S2] (with minor omissions). He also classified indecomposable infinite dimensional (Harish-Chandra) modules over $\mathfrak{osp}(1|2)$, see [S2].

Here we complete the description of indecomposable finite dimensional representations of $\mathfrak{g}$ in full generality. Our results are based on the deep results of I. Gelfand and Ponomarev [GP]. Not only are they deep, they are obtained by elementary methods and I consider this as an advantage.

Meanwhile J. Germoni independently considered the case $\mathfrak{sl}(1|n)$ by much more sophisticated methods (which enabled him to relate the result with quivers, etc.), cf. [G1], and completely classified indecomposable representation. However, [G2] and its continuation [G3] do not mention either $\mathfrak{osp}(2|2n)$ considered in [L2] or odd parameters discussed here.

Observe that the description of indecomposable representations for $\mathfrak{sl}(1|n)$ and $\mathfrak{osp}(2|2n)$ are obtained by practically identical means, be they either elementary (as here) or more involved (as in [G1]).
0.2. A related problem. For $\mathfrak{g}(m|n)$ or $\mathfrak{s}(m|n)$, Sergeev proved that any finite dimensional representation realizable in the space of tensors $T(V \oplus \Pi(V))$ is completely reducible. There is no complete reducibility for representations realizable in the general tensor algebra $T(V \oplus \Pi(V)) \oplus V^* \oplus \Pi(V^*)$. To distinguish a subalgebra of the tensor algebra inside of which the complete reducibility takes place for $\mathfrak{g} \neq \mathfrak{g}(1)$ is an interesting open problem for various representations $V$, the identity representation of the matrix algebra, the representations of minimal dimension and the adjoint representation are most interesting to consider.

0.3. On tame and wild representations. The problem of description of indecomposable representations of $\Lambda(n)$ for $n > 2$ is a wild one (this inference from $\mathfrak{G}$ is made in $\mathfrak{N}$). Similarly, one deduces that to describe indecomposable representations of the direct sum of several copies of $\Lambda(2)$ is a wild problem.

Germoni proved that the description of indecomposable representations of $\mathfrak{g} = \mathfrak{s}(p|q)$, where $1 < p \leq q$, is a wild problem $\mathfrak{G}$ by cohomology arguments. Roughly speaking, it seem to be related with the possibility to embed the direct sum of two copies of $\mathfrak{s}(1|1)$ into $\mathfrak{g}$ since this reduces to description of indecomposable representations of the direct sum of several copies of $\Lambda(2)$. The arguments are, however, subtler, and it is an open problem to consider the indecomposable representations of simple (and close to them) Lie superalgebras of other types.

For certain particular representation of any superalgebra one can always obtain the final result. For example, P. Grozman recently computed with the help of his package SuperLie, see $\mathfrak{G}$, the decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ for certain particular Lie superalgebras $\mathfrak{g}$. In $\mathfrak{G}$ Germoni considered indecomposable representations of two of the exceptional Lie superalgebras.

0.4. Miscellanea. Several such partial results obtained with P. Grozman constitute explicit decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ for $\mathfrak{g} = \mathfrak{s}(1|1), \mathfrak{s}(1|2), \mathfrak{ps}(2|2)$ and some other algebras. They are prompted by A. Vaintrob who hopes to relate them with the description of invariants of links and tangles à la $\mathfrak{B}$ and are presented elsewhere.

§1. Representations of $\mathfrak{s}(1|1)$

Thanks to the super analog of I. Schur lemma, to consider the irreducible representations of $\mathfrak{s}(1|1)$ is the same as to consider same of the (associative) Clifford superalgebra $C_{\mathfrak{cl}}(2) = U(\mathfrak{s}(1|1))/(E - h)$ for $h \in \mathbb{C}$, where $E$ is the central element of $\mathfrak{s}(1|1)$. General Algebra and Linear Algebra in Superspaces yield the following statement:

1.1. Proposition. The irreducible representations of $C_{\mathfrak{cl}}(2)$ are:

1) for $h \neq 0$: the 1|1-dimensional modules $V^h$ with even highest vector on which $E$ acts as the operator of multiplication by the scalar $h$ and $\Pi(V^h)$;

2) for $h = 0$: the 1|0-dimensional trivial module $1$ and $\Pi(1)$.

Problem Describe representations of $U(\mathfrak{s}(1|1))/(E^n - h)$.

Observe that for $h = 0$ the superalgebra $C_{\mathfrak{cl}}(2)$ turns into the Grassmann superalgebra $\Lambda(2)$ with two generators. In order to study indecomposable representations of $\Lambda(n)$, recall some general results on $\Lambda(n)$-modules.

Modules over Grassmann superalgebras are described in $\mathfrak{B}$ and $\mathfrak{G}$. Let $\Lambda(n)$ be Grassmann superalgebra with $n$ indeterminates, i.e., the free supercommutative superalgebra with $n$ odd indeterminates $\theta_1, \ldots, \theta_n$. Setting $\deg \theta_i = 1$ we define a $\mathbb{Z}$-gradation on $\Lambda(n)$. In this section we only consider left unitary $\mathbb{Z}$-graded modules with a compatible $\mathbb{Z}/2$-grading, i.e., the parity of the elements is their degree modulo 2 and $\Lambda(n)_i \otimes V_j \subset V_{i+j}$ for $i, j \in \mathbb{Z}$. Given a $\Lambda(n)$-module $V = \oplus V_i$ and $r \in \mathbb{Z}$, define $V[r]$ by setting $V[r]_i = V_{r-i}$.
A Λ(n)-module V is called reduced if ΘV = 0 for Θ = θ₁ · · · θₙ (the highest term in Λ(n)).

Lemma . ([BG]) i) Any free Λ(n)-module V is of the form F = ⊕₀Λ(n)[r].
    ii) Any Λ(n)-module V can be represented in the form F ⊕ V^{rd}, where F is free and V^{rd} is reduced.
    iii) Any indecomposable Λ(n)-module is either isomorphic to Λ(n)[r] for some r or is reduced.

Since we are only interested in compatibility of the action with parity, it suffices to consider shifts of grading modulo 2 in which case they coincide with the change of parity functor Π.

1.3. Reduced modules over Λ(2). Let a and b be generators of Λ(2). On a reduced Λ(2)-module V the operators a and b satisfy

\[ a^2 = b^2 = ab = ba = 0 \]

and, therefore, results from [GP] are applicable. Let us recall these results (and adjust them to supercase). We will depict Λ(2)-modules by directed graphs whose nodes stand for subsuperspaces of V; the action of a is depicted by a horizontal arrow, that of b by a vertical one; up to changes of parity both actions are isomorphisms representable in a basis by the identity matrix. The curvy arrow may stand for either a or b and corresponds to the Jordan cell with eigenvalue µ ∈ C.

**Type I modules** V(p + qε; dir): are determined by their dimension p + qε, where q = p or q = p ± 1, and the direction (dir = in or dir = out) indicating the submodule (each node is 1-dimensional). Here is the diagram representing V(p + qε; dir) for (p, q) = (3, 2) and (2, 3), respectively:

![Diagram for Type I modules](image)

**Type II modules** V(p; m + nε; dir; µ) are generalizations of Type I modules by means of the operator depicted by a curvy arrow (here depicted by the composition of arrows of two solid lines) from the space represented by the most left and upper node to the most right and low one or the other way round.

![Diagram for Type II modules](image)

Modules V(p; m + nε; dir; µ) are determined by three nonnegative integers: the superdimension m + nε of the subsuperspace corresponding to each node and the total number of nodes, p; in either of the two slanted lines the graph “lies” on and the directions dir of the first arrow (since we only consider graded modules, the odd operators a and b must change parity; hence, the direction of the last noncurvy arrow must coincide with that of the first one), and the parameter µ corresponds to the eigenvalue of the Jordan cell represented by the curvy arrow (not depicted).

The following result is an easy corollary of [GP], pp. 59–60 (compare with [ZN], where the description of type II modules contains an omission).
Lemma. Let $V$ be a reduced indecomposable $\Lambda(2)$-module. Up to the change of parity, $V$ is one of the above modules $V(p + q\varepsilon; \text{dir})$ of type I or modules $V(p; m + n\varepsilon; \text{dir}; \mu)$ of type II.

1.4. Indecomposable representations of $Cl_h(2)$ for $h \neq 0$. Let $X_\pm$ be the generators of $\mathfrak{sl}(1|1)$. Denote by $V^h(n)$ for $n > 0$ the $\mathfrak{sl}(1|1)$-module induced from the $n$-dimensional representation $\rho_n$ of the uppertriangular subalgebra $b = \text{Span}(E, X_+)$:

$$\rho_n(E) = J_n(h) \text{ for the Jordan cell } J_n(h), \quad \rho_n(X_+) = 0.$$ 

Clearly, $V^h(1) = V^h$ and on $V^h(n)$ we have:

$$\rho_n(X_-) = \begin{pmatrix} 0 & 0 \\ 1_n & 0 \end{pmatrix}, \quad \rho_n(E) = \begin{pmatrix} J_n(h) & 0 \\ 0 & J_n(h) \end{pmatrix}, \quad \rho_n(X_+) = \begin{pmatrix} 0 & J_n(h) \\ 0 & 0 \end{pmatrix}.$$

Lemma. Up to the change of parity, all the indecomposable representations of $Cl_h(2)$ for $h \neq 0$ are realized in the modules $V^h(n)$.

Proof. A representation from Lemma is, clearly, indecomposable. Such representatins realize the only way to glue two copies of $V^h$: indeed, $\text{dim} \, H^1(\mathfrak{sl}(1|1); \text{End}(V^h)) = 1$. Since $\text{dim} \, H^1(\mathfrak{sl}(1|1); V^h \otimes V^h) = 0$ for $h \neq \tilde{h}$, it is impossible to glue modules $V^h$ for different $h$'s. Similar argument applies to $V^h(n)$.

Now, let us collect our results.

1.5. Theorem. Indecomposable finite dimensional representations of $\mathfrak{sl}(1|1)$ are realized, up to the change of parity, in the modules

1. $V^h(n)$ of dimension $n + n\varepsilon$;
2. $V(p + q\varepsilon; \text{dir})$ of type I; its dimension is equal to $p + q\varepsilon$ with $q = p$ or $p \pm 1$;
3. $V(p; m + n\varepsilon; \text{dir}; \mu)$ of type II; its dimension is equal to $p(m + n)(1 + \varepsilon)$;
4. $\mathfrak{gl}(1|1)$; this is a free $\Lambda(2)$-module.

§2. IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{g} \not\cong \mathfrak{sl}(1|1)$

The Lie superalgebras $\mathfrak{g}$ of series $\mathfrak{sl}$ and $\mathfrak{osp}(2|2n)$ have a $\mathbb{Z}$-grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Hence, (BL)), any irreducible $\mathfrak{g}$-module $V^h$ with highest weight $h$ and even highest vector is the quotient of the induced module $I(V^h) = \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} V^h$ for an irreducible $\mathfrak{g}_0$-module $V^h$ with highest weight $h$.

V. Kac gave a description [K] of conditions under which the module $I(V^h)$ is irreducible, if $\text{dim} \, V^h < \infty$. A weight $h$ is typical if $(h + \rho, \varphi_i) \neq 0$ for the Killing form $(\cdot, \cdot)$ on $\mathfrak{g}$ and every root $\varphi_i$ of the odd space $\mathfrak{g}_{-1}$. (Here $\rho$ is a half sum of positive even roots and negative odd ones.)

For atypical weights the modules $I(h)$ constitute infinite in both directions acyclic complexes, first described in [BL] and [L]. Recall this result.

2.1. The complex of integral and differential forms and its generalizations. For $\mathfrak{sl}(1|2) \cong \mathfrak{osp}(2|2) \cong \mathfrak{vect}(0|2)$ the interpretation of the modules in the realization by vector fields is the most graphic one. Every irreducible finite dimensional $\mathfrak{vect}(0|2)$-module is of the following form (up to $\Pi$). Let $(a, b)$ be the weights with respect to $\xi_1 \partial_1$ and $\xi_2 \partial_2$, respectively. Then $I(V^{(a, b)})$ is irreducible if $(a, b) \neq (0, -n)$ or $(n + 1, 1)$. In the latter case, set $\Sigma_n = I(V^{(0, -n)})$ and $\Omega^n = I(V^{(n + 1, 1)})$. These spaces are called the superspaces of integral $(-n)$-forms and differential $n$-forms, respectively. They constitute an acyclic complex

$$\cdots \xrightarrow{d} \Sigma_n \xrightarrow{d} \cdots \xrightarrow{d} \Sigma_{-1} \xrightarrow{d} \Sigma_0 \xrightarrow{f} \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} \cdots$$
Denote by $i(k)$ the kernel of the outgoing arrow at the $k$-th place of the above complex. This is an irreducible module and two modules with neighboring numbers are glued in one indecomposable one, more exactly,

$$\Sigma_{-n} \simeq i(-n + 1) \longrightarrow \Pi(i(-n)) \text{ and } \Omega^n \simeq \Sigma_{-n} \simeq i(n) \longrightarrow \Pi(i(n + 1)).$$

These arrows can be organized in graphs similar to the cases $g = \mathfrak{sl}(1|1)$. Let us number the nodes in graphs corresponding to modules of type I downwards, starting with an integer $k$. Let the $n$-th node denote the module $i(n)$ or $\Pi(i(n))$, and denote the corresponding module by $V(p + q \varepsilon, \text{dir}; k)$. The dimension of the module obtained is equal to $2p(k + p - 1)(1 + \varepsilon)$ if $p = q$ and to $N + (N - 1)\varepsilon$, where $N = k(2p + 1) + 2p^2$, for $p = q - 1$, to $N + (N + 1)\varepsilon$, where $N = k(2p + 1) + 2(p^2 - p + 1)$, for $p = q + 1$. The following module $S\bar{q}(k)$ of dimension $(4k - 2)(1 + \varepsilon)$ is an analog of the representation of $\mathfrak{sl}(1|1)$ in $\mathfrak{gl}(1|1)$, i.e., the free $\Lambda(2)$-module of rank 1:

$$i(k) \quad \rightarrow \quad \Pi(i(k + 1)) \downarrow \quad \Pi(i(k - 1)) \quad \rightarrow \quad i(k)$$

**Lemma.** (Sh1) For $g \not\cong \mathfrak{sl}(1|1)$ we have $\text{Ext}^1(I(\varphi), I(\chi)) = 0$;

$$\text{Ext}^1(i(\varphi), i(\chi)) = \begin{cases} \mathbb{C} & \text{if, up to } \Pi, (i(\varphi), i(\chi)) = (I(\varphi), I(\chi)) \\ 0 & \text{otherwise.} \end{cases}$$

This lemma shows that indecomposable modules over $g \not\cong \mathfrak{sl}(1|1)$ are simpler than those over $\mathfrak{sl}(1|1)$.

§3. SUPERVARIETIES OF REPRESENTATIONS

So far we have described points of the *variety* that parametrizes $g$-modules.

To consider the supervariety is not difficult: its *points* are the same as those of the underlying variety. The tangent space to the supervariety of parameters at its point corresponding to an indecomposable module $V$ is, as follows from the cohomology theory (cf., e.g., [F]), isomorphic to $H^1(g; \text{End}(V))$.

**Theorem.** (Sh1) $H^1(g; \text{End}(V))|_1 = 0$ for $g \not\cong \mathfrak{sl}(1|1)$ and any indecomposable module $V$ or if $g \cong \mathfrak{sl}(1|1)$ and $V \cong V_h(n)$.

Proof is straightforward, with the help of the Cazimir element.

For $V \not\cong V_h(n)$ there are odd parameters. Indeed, consider the simplest cases: $V \cong 1$ or $\Pi(1)$. In either case, $\text{End}(V) \cong 1$ and, since $H^1(\mathfrak{sl}(1|1)) \cong \mathbb{C}[\xi, \eta]/(\xi \eta)$, where $\xi$ and $\eta$ are odd 1-cocycles, see [FL], $\dim H^1(\mathfrak{sl}(1|1))|_1 = 2$. There are no obstructions to globalization of these deformations.

Computation of $H^1(g; \text{End}(V))$ for modules more complicated than the trivial one seems at the moment to be too difficult to handle by bare hands or even bare computers.

§4. ON INDECOMPOSABLE REPRESENTATIONS OF VECTORIAL LIE SUPERALGEBRAS

Consider any of the simple finite dimensional vectorial Lie superalgebras $L$ except $\mathfrak{svect}(0|n)$ or $\mathfrak{sp}(n)$ in their standard $\mathbb{Z}$-grading: $L = \bigoplus_{-1}^{k} L_i$. Set

$$L_{\geq} = \bigoplus_{i \geq 0} L_i; \quad L_{>0} = \bigoplus_{i > 0} L_i; \quad L_{\leq} = \bigoplus_{i < 0} L_i; \quad L_{\leq0} = \bigoplus_{i \leq 0} L_i.$$

Let $V$ be an irreducible $L_0$-module. We will consider two types of $L$-modules:

1) let $L_{>0}V = 0$ and set $I(V) = U(L) \otimes_{U(L_{\geq})} V$. 

2) let $L_\leq V = 0$ and set $\mathcal{I}(V) = U(\mathcal{L}) \otimes U(L_\leq V)$.

The Lie superalgebras with Cartan matrix and with a compatible grading are of the form

$$\mathcal{L} = L_{-1} \oplus L_0 \oplus L_1,$$

where $L_1 \approx L_{-1}$. Therefore, for them $I(V) \simeq \mathcal{I}(V^*)$.

Contrarywise for the vectorial Lie superalgebras the difference between modules $I(V)$ and $\mathcal{I}(V)$ is crucial. As is easy to see with the help of the Poincaré-Birkhoff-Witt theorem, $I(V) \simeq \Lambda(L_\leq) \otimes V$, as spaces. For vectorial Lie superalgebras distinct from $\text{svect}(0|2)$, $\text{svect}(0|3) \simeq \text{spe}(3)$ and $\text{sh}(0|4)$ the space $\mathcal{I}(V)$ is of infinite dimension.

For $\text{svect}(0|3)$ and $\text{spe}(n)$, though $\mathcal{I}(V)$ is of finite dimension, still $I(V) \not\simeq \mathcal{I}(V^*)$. However, thanks to the existence of complete description of typical $\text{spe}(n)$-modules, see \cite{Shapovalov} one may hope for a complete description of indecomposable $\text{spe}(n)$-modules, at least, for $n = 3$. The limitations for this hope are set by Theorem 4.2.

4.1. Theorem . (Shapovalov, 1985) Typical irreducible representations of $\text{vect}(0|2)$ and $\text{sh}(0|4)$ are direct summands.

4.2. Theorem . (Shapovalov, 1985) Indecomposable representations of $\text{vect}(0|n)$ for $n > 2$, $\text{spe}(n)$, $\text{svect}(0|n)$, $\text{svect}(0|n)$ for $n > 3$ and $\text{sh}(0|n)$ for $n > 4$ may include any irreducible module as the composition factor.

No proof of these theorems was ever published. Recent result of Shomron (given with proof) \cite{Shomron} covers the $\text{vect}(0|n)$ case and indicates how to tackle the other cases.

APPENDIX

A.1. Linear algebra in superspaces. Generalities. A superspace is a $\mathbb{Z}/2$-graded space; for a superspace $V = V_0 \oplus V_1$ denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $(\Pi(V))_i = V_{i+1}$. The superdimension of $V$ is $\dim V = p + q\varepsilon$, where $\varepsilon^2 = 1$ and $p = \dim V_0$, $q = \dim V_1$. (Usually $\dim V$ is expressed as a pair $(p, q)$ or $p|q$; this obscures the fact that $\dim V \otimes W = \dim V \cdot \dim W$ which becomes manifest with the use of $\varepsilon$.)

A superspace structure in $V$ induces the superspace structure in the space $\text{End}(V)$. A basis of a superspace is always a basis consisting of homogeneous vectors; let $\text{Par} = (p_1, \ldots, p_{\dim V})$ be an ordered collection of their parities. We call $\text{Par}$ the format of the basis of $V$. A square supermatrix of format $(p)$ is a $\dim V \times \dim V$ matrix whose $i$th row and $i$th column are of the same parity $p_i$. The matrix unit $E_{ij}$ is supposed to be of parity $p_i + p_j$ and the bracket of supermatrices (of the same format) is defined via Sign Rule:

\begin{itemize}
  \item if something of parity $p$ moves past something of parity $q$ the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.
\end{itemize}

Examples of application of Sign Rule: setting $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$ we get the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra and the Lie superalgebra (that in addition to superskew-commutativity satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule). The derivation of a superalgebra $A$ is a linear map $D : A \to A$ such that satisfies the Leibniz rule (and Sign rule)

$$D(ab) = D(a)b + (-1)^{p(a)p(b)} aD(b).$$

In particular, let $A = \mathbb{K}[x]$ be the free supercommutative polynomial superalgebra in $x = (x_1, \ldots, x_n)$, where the superstructure is determined by the parities of the indeterminates: $p(x_i) = p_i$. Partial derivatives are defined (with the help of Leibniz and Sign Rules) by the formulas $\frac{\partial a}{\partial x_j} = \delta_{i,j}$. 


Clearly, the collection \( \text{det} A \) of all superdifferentiations of \( A \) is a Lie superalgebra \( \text{vect}(m|n) \) on \( m \) even and \( n \) odd indeterminates whose elements are of the form

\[
D = \sum f_i(x) \frac{\partial}{\partial x_i}.
\]

The divergence of such \( D \) is defined to be

\[
\text{div} D = \sum (-1)^p f_i \frac{\partial f_i}{\partial x_i}.
\]

Set \( \text{svect}(m|n) = \{ D \in \text{vect}(m|n) \mid \text{div} D = 0 \} \) and \( \tilde{\text{svect}}(m|n) = \{ D \in \text{vect}(m|n) \mid \text{div}(1 + t\xi_1 \ldots \xi_n)D = 0 \} \), where \( p(t) \equiv n \mod 2 \).

Define the superalgebra of differential forms as the supercommutative superalgebra of polynomials in the \( x_i \) and \( dx_i \) with \( p(dx_i) = p(x_i) + 1 \). Define the exterior differential \( d \) by the formula \( d(f) = \sum dx_i \frac{\partial f}{\partial x_i} \) and extend \( d \) to forms of higher degrees (in \( dx \)) via Leibniz and Sign Rules. Define the Lie derivative \( L_D \) along \( D \in \text{vect}(m|n) \) by the formula \( L_D d(f) = (-1)^p(D)d(D(f)) \) extended to higher forms via Leibniz and Sign Rules.

Set \( \omega = \sum (dx_i)^2 \) and \( b(0|n) = \{ D \in \text{vect}(0|n) \mid L_D(\omega) = 0 \} \).

The general linear Lie superalgebra of all supermatrices of size \( Par \) is denoted by \( \mathfrak{gl}(Par) \); usually, \( \mathfrak{gl}(\bar{0}, \ldots, \bar{0}, \bar{1}, \ldots, \bar{1}) \) is abbreviated to \( \mathfrak{gl}(\dim V_0|\dim V_1) \). Usually, \( Par \) is of the form \( (\bar{0}, \ldots, \bar{0}, 1, \ldots, 1) \). Such a format is called standard. In this paper we can do without nonstandard formats but they are vital in the study of systems of simple roots that the reader might be interested in. Any matrix from \( \mathfrak{gl}(Par) \) can be expressed as the sum of its even and odd parts; in the standard format this is the block expression:

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p \left( \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right) = 0, \quad p \left( \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right) = 1.
\]

The supertrace is the map \( \mathfrak{gl}(Par) \rightarrow \mathbb{C}, (A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii} \). Since \( \text{str}[x, y] = 0 \), the space of supertraceless matrices constitutes the special linear Lie superalgebra \( \mathfrak{sl}(Par) \).

**Lie superalgebras that preserve bilinear forms: two types.** To the linear map \( F \) of superspaces there corresponds the dual map \( F^* \) between the dual superspaces; if \( A \) is the supermatrix corresponding to \( F \) in a basis of the format \( Par \), then to \( F^* \) the supertransposed matrix \( A^{*t} \) corresponds:

\[
(A^{*t})_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}.
\]

The supermatrices \( X \in \mathfrak{gl}(Par) \) such that

\[
X^{*t}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(Par)
\]

constitute the Lie superalgebra \( \text{mut}(B) \) that preserves the bilinear form on \( V \) with matrix \( B \).

Recall that the supersymmetry of the homogeneous form \( \omega \) means that its matrix \( B \) satisfies the condition \( B^u = B \), where \( B^u = \begin{pmatrix} R^t & (-1)^p(B^t)T^t \\ (-1)^p(B)S^t & -U^t \end{pmatrix} \) for the matrix \( B = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \).

Similarly, skew-supersymmetry of \( B \) means that \( B^u = -B \). Thus, we see that the upsetting of bilinear forms \( u : \text{Bil}(V, W) \rightarrow \text{Bil}(W, V) \), which for the spaces \( V = W \) is expressed on matrices in terms of the transposition, is a new operation.

Most popular canonical forms of the nondegenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, \( B_{ev} \) or \( B'_{ev} \):

\[
B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},
\]
or
\[ B'_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \ldots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}. \]

The usual notation for \( \text{aut}(B_{ev}(m|2n)) \) is \( \mathfrak{osp}(m|2n) \) or \( \mathfrak{osp}^{sy}(m|2n) \). (Observe that the passage from \( V \) to \( \Pi(V) \) sends the supersymmetric forms to superskew-symmetric ones, preserved by the “symplectico-orthogonal” Lie superalgebra \( \mathfrak{sp}'\mathfrak{o}(2n|m) \) or \( \mathfrak{osp}^{sk}(m|2n) \) which is isomorphic to \( \mathfrak{osp}^{sy}(m|2n) \) but has a different matrix realization. We never use notation \( \mathfrak{sp}'\mathfrak{o}(2n|m) \) in order not to confuse with the special Poisson superalgebra.)

In the standard format the matrix realizations of these algebras are:
\[
\mathfrak{osp}(m|2n) = \begin{cases} 
\begin{pmatrix} E & Y & X^t \\
X & A & B \\
-Y^t & C & -A^t \end{pmatrix} 
\end{cases}; \\
\mathfrak{osp}^{sk}(m|2n) = \begin{cases} 
\begin{pmatrix} A & B & X \\
C & -A^t & Y^t \\
Y & -X^t & E \end{pmatrix} 
\end{cases},
\]
where \( \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \ E \in \mathfrak{o}(m) \) and \( ^t \) is the usual transposition.

A nondegenerate supersymmetric odd bilinear form \( B_{odd}(n|n) \) can be reduced to the canonical form whose matrix in the standard format is \( J_{2n} \). A canonical form of the superskew odd nondegenerate form in the standard format is \( \Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \). The usual notation for \( \text{aut}(B_{odd}(\text{Par})) \) is \( \mathfrak{pe}(\text{Par}) \). The passage from \( V \) to \( \Pi(V) \) sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism \( \mathfrak{pe}^{sy}(\text{Par}) \cong \mathfrak{pe}^{sk}(\text{Par}) \). This Lie superalgebra is called, as A. Weil suggested, \textit{periplectic}. In the standard format these superalgebras are isomorphic. The usual notation for \( \mathfrak{pe}^{sy}(n) \) is \( \{X \in \mathfrak{pe}(n) : \text{str}X = 0 \} \). Observe that though the Lie superalgebras \( \mathfrak{osp}^{sy}(m|2n) \) and \( \mathfrak{pe}^{sk}(2n|m) \), as well as \( \mathfrak{pe}^{sy}(n) \) and \( \mathfrak{pe}^{sk}(n) \), are isomorphic, the difference between them is sometimes crucial.

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