Credible equilibrium

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Note: This is a preliminary draft to ask for feedback and references from friends and colleagues. Comments and literature suggestions are welcome.
10th June, 2022

Abstract

Credible equilibrium is a solution concept that imposes a stronger credibility notion than subgame perfect equilibrium. A credible equilibrium is a refinement of subgame perfect equilibrium such that if a threat in a subgame \( g \) is "credible," then it must also be credible in every subgame \( g' \) that is "equivalent" to \( g \). I show that (i) a credible equilibrium exists in multi-stage games, and (ii) if every stage game has a unique Nash equilibrium, then the credible equilibrium is unique even in infinite horizon multi-stage games. Moreover, in perfect information games, credible equilibrium is equivalent to subgame perfect equilibrium.

JEL: C70, D81

Keywords: subgame perfect equilibrium, noncooperative games, Nash equilibrium

1 Introduction and definition

Let \( G = (G^1, G^2, ..., G^T) \) be a multi-stage game where at time \( t \) stage game \( G^t = (S^t_i, u^t_i)_{i \in N} \) is played. Let \( N \) denote the finite set of players, \( u^t_i \) the von Neumann-Morgenstern (vNM) utility function of player \( i \) at \( t \), and \( s^t_i \in S^t_i \) a mixed strategy in \( G^t \). It may be that \( T = \infty \). A game \( G \) can be denoted by \( (\delta, (S^t_i, u^t_i)_{i \in N}) \), where \( \delta \in [0, 1) \) is the common discount factor, \( u_i \) the discounted vNM utility of \( i \), \( s_i \in S^t_i \) a mixed strategy in \( G^t \), and \( s \in S \) a mixed strategy profile.

A game \( g = (G^k, G^{k+1}, ..., G^T) \), where \( k \geq 1 \), is called a subgame of \( G \). Two subgames \( g = (G^k, G^{k+1}, ..., G^T) \) and \( g' = (G^{k'}, G^{k'+1}, ..., G^T) \) are called equivalent if \( k = k' \). Let \( (s|g) \) denote \( s \) restricted to \( g \). A strategy profile \( s \) is called a Nash equilibrium (Nash, 1950) if for every \( i \), \( s_i \in \arg \max_{s_i' \in S_i} u_i(s_i', s_{-i}) \). If for every subgame \( g \) \( (s|g) \) is a Nash equilibrium, then \( s \) is called a subgame perfect equilibrium (SPNE) (Selten, 1965).

Definition 1. A strategy profile \( s^* \) in a game \( G \) is called a credible equilibrium if (i) for every subgame \( g \) of \( G \), \( (s^*|g) \) is a Nash equilibrium, and (ii) for every equivalent subgames \( g \) and \( g' \) of \( G \) if \( (s^*_i|g) \neq (s^*_i|g') \), then \( u_i(s_i|g) = u_i(s_i|g') \).

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1If \( T \) is finite, then \( \delta \) may be equal to 1.
Credible equilibrium imposes a stronger credibility notion than an SPNE. The intuition is that the threat of a punishment should be credible with respect to not only what happens in the punishment subgame but also what happens in the no-punishment subgame whenever those two subgames are equivalent. A player’s threat should not be considered credible if her punishment strategy strictly decreases her own payoff in the subgame relative to her payoff in an equivalent subgame in the no-punishment case. Thus, any threat (or promise) must be “credible.” In other words, a credible equilibrium is a refinement of SPNE such that if for a player it prescribes a different strategy to two equivalent subgames, then the player’s utility must be the same in these subgames.

For an example of an SPNE that is not a credible equilibrium consider the twice repeated game below. Notice that CC in the first stage can be supported as an SPNE by playing EE on-path and playing DD off-path. However, this is not a credible equilibrium because by playing D in the second stage a player punishes not only their opponent but also themself. Therefore, such a punishment is not credible relative to the no-punishment case. Since no matter what happens in the first period, the second period subgame will be the same and players receive strictly less payoff in the off-path subgame than the on-path subgame, this SPNE is not a credible equilibrium.

\[
\begin{array}{ccc}
C & D & E \\
C & 4,4 & 0,0 & 0,5 \\
D & 0,0 & 1,1 & 0,0 \\
E & 5,0 & 0,0 & 3,3 \\
\end{array}
\]

**Theorem 1 (Existence).** Every multi-stage game \( G \) has a credible equilibrium. 

**Proof.** Every \( G \) has an SPNE because for every \( t \) \( G^t \) has a Nash equilibrium. Moreover, any two equivalent subgames must have the same set of SPNEs. Thus, we can construct an SPNE \( s^* \) such that \( (s^*|g) = (s^*|g') \) in every equivalent subgames \( g \) and \( g' \) of \( G \). It implies that \( s^* \) is a credible equilibrium. \( \square \)

Next, I provide a sufficient condition for the uniqueness of credible equilibrium.

**Proposition 1 (Uniqueness).** Let \( G \) be a multi-stage game. Suppose that for every \( t \) \( G^t \) has a unique Nash equilibrium. Then, there is a unique credible equilibrium \( s \) of \( G \) where for every \( t \) \( s^t \) is the unique Nash equilibrium \( G^t \).

**Proof.** It is straightforward to see that if for every \( t \) \( s^t \) is the unique Nash equilibrium \( G^t \), then \( s \) is a credible equilibrium. I next show that \( s \) is the unique credible equilibrium. To reach a contradiction suppose that there is another credible equilibrium \( s' \neq s \) such that for some \( t \), \( s'^t \) is not the Nash equilibrium of \( G^t \). Then, there must be a player \( i \) who has a unilateral profitable deviation from \( s'^t \) in \( G^t \). This deviation must also be profitable for \( i \) in \( G \) because \( i \)‘s payoff is the same in \( (G^{t+1}, G^{t+2}, ..., G^T) \) whether \( i \) deviates in \( G^t \) or not by the uniqueness of Nash equilibria in the stage games and by definition of credible equilibrium. Thus, \( s' \) cannot be a credible equilibrium, which contradicts our supposition. \( \square \)
Figure 1 illustrates a two-stage game in which a non-Nash outcome of a stage game can be supported as part of a credible equilibrium. Assume that $\delta = 1$ and consider the following strategy profile. Row player: Play B in the first stage (left), and if BB were played in the first stage, then play B in the second stage (right); otherwise play A in the second stage. Column player: Always play B in both stages. Notice that this profile is a credible equilibrium because Row player’s threat in the second stage is credible—i.e., punishing the opponent does not hurt Row relative to the no-punishment case.

$$
\begin{array}{cc}
A & B \\
0 & 0, 0 & 1, 1 \\
1, 3 & 4, 2 & 1, 3
\end{array}
$$

Figure 1: A credible equilibrium that supports a non-Nash outcome.

A simple example of an SPNE that is not a credible equilibrium is a cooperative SPNE in the infinitely repeated prisoners’ dilemma (PD). As is well-known cooperation can be supported as an SPNE by grim-trigger strategies off-path (for some $\delta$). But irrespective of what happens in the first period, the subgame from the second period onwards will be exactly the same. In the off-path, by playing a grim-trigger strategy a player punishes themself forever to punish their opponent. Thus, in the context of this paper such a punishment is not credible. As a result, the only credible equilibrium in the infinitely repeated PD is D,D in every period regardless of the discount factor.

Credible equilibrium can be defined in general extensive form games in an analogous way. Two subgames $g$ and $g'$ are called equivalent if their extensive forms are the same up to a relabeling of strategies, histories, and nodes. Then, a credible equilibrium is an SPNE with the condition that if for a player it prescribes a different strategy to two equivalent subgames, then the player’s utility must be the same in these subgames.

Proposition 2 (Perfect information equivalence). In perfect information games, a strategy profile is a credible equilibrium if and only if it is an SPNE.

The proof is based on the simple observation that in every subgame in a perfect information game all the best responses of the players lead to the same payoffs. Therefore, condition (ii) of credible equilibrium is automatically satisfied in perfect information games.

References

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