Leonard pairs having LB-TD form

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Abstract
Fix an algebraically closed field $F$ and an integer $d \geq 3$. Let $\text{Mat}_{d+1}(F)$ denote the $F$-algebra consisting of the $(d+1) \times (d+1)$ matrices that have all entries in $F$. We consider a pair of diagonalizable matrices $A, A^*$ in $\text{Mat}_{d+1}(F)$, each acts in an irreducible tridiagonal fashion on an eigenbasis for the other one. Such a pair is called a Leonard pair in $\text{Mat}_{d+1}(F)$. For a Leonard pair $A, A^*$ there is a nonzero scalar $q$ that is used to describe the eigenvalues of $A$ and $A^*$. In the present paper we find all Leonard pairs $A, A^*$ in $\text{Mat}_{d+1}(F)$ such that $A$ is lower bidiagonal with subdiagonal entries all 1 and $A^*$ is irreducible tridiagonal, under the assumption that $q$ is not a root of unity. This gives a partial solution of a problem given by Paul Terwilliger.

1 Introduction
Throughout the paper $F$ denotes an algebraically closed field. All scalars will be taken from $F$. Fix an integer $d \geq 0$ and a vector space $V$ over $F$ with dimension $d+1$. Let $F^{d+1}$ denote the $F$-vector space consisting of the column vectors of length $d+1$ and $\text{Mat}_{d+1}(F)$ denote the $F$-algebra consisting of the $(d+1) \times (d+1)$ matrices. The algebra $\text{Mat}_{d+1}(F)$ acts on $F^{d+1}$ by left multiplication.

We begin by recalling the notion of a Leonard pair. We use the following terms. A square matrix is said to be tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

**Definition 1.1** [7, Definition 1.1] By a Leonard pair on $V$ we mean an ordered pair of linear transformations $A: V \to V$ and $A^*: V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

By a Leonard pair in $\text{Mat}_{d+1}(F)$ we mean an ordered pair $A, A^*$ in $\text{Mat}_{d+1}(F)$ that acts on $F^{d+1}$ as a Leonard pair.

**Note 1.2** According to a common notational convention, $A^*$ denotes the conjugate transpose of $A$. We are not using this convention. In a Leonard pair $A, A^*$ the matrices $A$ and $A^*$ are arbitrary subject to the conditions (i) and (ii) above.

We refer the reader to [5][11] for background on Leonard pairs.
A square matrix is said to be lower bidiagonal whenever each nonzero entry lies on either the diagonal or the subdiagonal. Paul Terwilliger gave the following problem:

**Problem 1.3** [10] Problem 36.14 Find all Leonard pairs $A, A^*$ in $\text{Mat}_{d+1}(\mathbb{F})$ that satisfy the following conditions: (i) $A$ is lower bidiagonal with subdiagonal entries all 1; (ii) $A^*$ is irreducible tridiagonal.

The above problem is related to “Leonard triples” [13], “adjacent Leonard pairs” [2], and “q-tetrahedron algebras” [4]. In the present paper we give a partial solution of Problem 1.3 We use the following terms:

**Definition 1.4** An ordered pair of matrices $A, A^*$ in $\text{Mat}_{d+1}(\mathbb{F})$ is said to be $LB$-$TD$ whenever $A$ is lower bidiagonal with subdiagonal entries all 1 and $A^*$ is irreducible tridiagonal.

**Definition 1.5** A Leonard pair $A, A^*$ on $V$ is said to have $LB$-$TD$ form whenever there exists a basis for $V$ with respect to which the matrices representing $A, A^*$ form an $LB$-$TD$ pair in $\text{Mat}_{d+1}(\mathbb{F})$.

**Note 1.6** Let $A, A^*$ be a Leonard pair on $V$. For scalars $\alpha, \alpha^*$, the pair $A + \alpha I, A^* + \alpha^* I$ is also a Leonard pair on $V$, which is called a translation of $A, A^*$. Here $I$ denotes the identity. If $A, A^*$ has $LB$-$TD$ form, then any translation of $A, A^*$ has $LB$-$TD$ form.

Below we display a family of $LB$-$TD$ Leonard pairs in $\text{Mat}_{d+1}(\mathbb{F})$. Consider the following $LB$-$TD$ pair in $\text{Mat}_{d+1}(\mathbb{F})$: 

$$A = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} x_0 & y_1 & \cdots & \cdots & \cdots & 0 \\ z_1 & x_1 & y_2 & \cdots & \cdots & \vdots \\ z_2 & z_1 & x_2 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & z_d & x_d \end{pmatrix}. \quad (1)$$

**Proposition 1.7** Fix a nonzero scalar $q$ that is not a root of unity. Let $\alpha, \alpha^*, a, a', b, b'$, $c$ be scalars with $c \neq 0$. Define scalars $\{\theta_i\}_{i=0}^d$, $\{x_i\}_{i=0}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ by

$$\theta_i = \alpha + aq^{2i-d} + a'q^{d-2i}, \quad (2)$$

$$x_i = \alpha^* + (b + b')q^{d-2i} + a'cq^{d-2i}(q^{d+1} + q^{-d-1} - q^{-2i-1} - q^{-d-2i+1}), \quad (3)$$

$$y_i = (q^i - q^{-i})(q^{d+i+1} - q^{-d-i})((b - a'cq^{d-2i+1})(b' - a'cq^{d-2i+1})c^{-1}, \quad (4)$$

$$z_i = -cq^{d-2i+1}. \quad (5)$$

Then the matrices $A, A^*$ form a $LB$-$TD$ Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$ if and only if the scalars $a, a', b, b', c$ satisfy the following inequalities:

$$a \notin \{a'q^{2d-2}, a'q^{2d-4}, \ldots, a'q^{2-d}\}, \quad (6)$$

$$b \notin \{b'q^{2d-2}, b'q^{2d-4}, \ldots, b'q^{2-d}\}, \quad (7)$$

$$bc^{-1}, b'c^{-1} \notin \{aq^{d-1}, aq^{d-3}, \ldots, aq^{1-d} \} \cup \{a'q^{d-1}, a'q^{d-3}, \ldots, a'q^{1-d}\}. \quad (8)$$

To state our further results, we recall some materials concerning Leonard pairs. Consider a Leonard pair $A, A^*$ on $V$. We first recall some facts concerning the eigenvalues of
A, A*. By [7, Lemma 1.3] each of A, A* has mutually distinct $d + 1$ eigenvalues. Let $\{\theta_i\}_{i=0}^d$ be an ordering of the eigenvalues of A. For $0 \leq i \leq d$ pick an eigenvector $v_i \in V$ of A associated with $\theta_i$. Then the ordering $\{\theta_i\}_{i=0}^d$ is said to be standard whenever the basis $\{v_i\}_{i=0}^d$ satisfies Definition 1.1(ii). A standard ordering of $A^*$ is similarly defined. For a standard ordering $\{\theta_i\}_{i=0}^d$ of the eigenvalues of A, the ordering $\{\theta_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result applies to $A^*$. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be a standard ordering of the eigenvalues of A (resp. $A^*$). By [7, Theorem 1.9] the expressions
\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}
\]
are equal and independent of $i$ for $2 \leq i \leq d - 1$. Next we recall the notion of a parameter array of A, A*.

**Lemma 1.8** [7, Theorem 3.2] For a Leonard pair A, $A^*$ on $V$ and a standard ordering $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) of the eigenvalues of A (resp. $A^*$), there exists a basis $\{u_i\}_{i=0}^d$ for V and there exist scalars $\{\phi_i\}_{i=1}^d$ such that the matrices representing A, $A^*$ with respect to $\{u_i\}_{i=0}^d$ are
\[
A: \begin{pmatrix} \theta_0 & \theta_1 & \theta_2 & \cdots & \theta_d \\ 1 & \theta_1 & \theta_2 & \cdots & \theta_d \\ \theta_1 & 1 & \theta_2 & \cdots & \theta_d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \theta_1 & \theta_2 & \cdots & 1 \end{pmatrix}, \quad A^*: \begin{pmatrix} \theta_0^* & \phi_1 & \phi_2 & \cdots & \phi_d \\ \theta_1^* & \theta_2^* & \theta_3^* & \cdots & \theta_d^* \\ \phi_1 & \theta_2^* & \theta_3^* & \cdots & \theta_d^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \phi_1 & \phi_2 & \cdots & \theta_2^* \end{pmatrix}.
\]

The sequence $\{\phi_i\}_{i=0}^d$ is uniquely determined by the ordering $\{(\theta_i)_{i=0}^d, (\theta_i^*)_{i=0}^d\}$. Moreover $\phi_i \neq 0$ for $1 \leq i \leq d$.

With reference to Lemma 1.8 we refer to $\{\phi_i\}_{i=1}^d$ as the first split sequence of A, $A^*$ associated with the ordering $\{(\theta_i)_{i=0}^d, (\theta_i^*)_{i=0}^d\}$. By the second split sequence of A, $A^*$ associated with the ordering $\{(\theta_i)_{i=0}^d, (\theta_i^*)_{i=0}^d\}$ we mean the first split sequence of A, $A^*$ associate with the ordering $\{(\theta_{d-i})_{i=0}^d, (\theta_i^*)_{i=0}^d\}$. By a parameter array of A, $A^*$ we mean the sequence
\[
\{(\theta_i)_{i=0}^d, (\theta_i^*)_{i=0}^d, (\phi_i)_{i=1}^d, (\phi_i^*)_{i=1}^d\},
\]
where $\{\theta_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A, $\{\theta_i^*\}_{i=0}^d$ is a standard ordering of the eigenvalues of $A^*$, and $\{\phi_i\}_{i=0}^d$ (resp. $\phi_i^*\}_{i=0}^d$) is the first split sequence (resp. second split sequence) of A, $A^*$ associated with the ordering $\{(\theta_i)_{i=0}^d, (\theta_i^*)_{i=0}^d\}$.

For the Leonard pair given in Propositions 1.7, the corresponding parameter array is as follows:
Proposition 1.9. With reference to Proposition 1.7, assume $A, A^*$ is an LB-TD Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$. Define scalars $\{\theta_i^*\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ by

$$\theta_i^* = \alpha^* + bq^{2i-d} + b'q^{d-2i}, \quad \varphi_i = (q^i-q^{-i})(q^{d-i+1}-q^{i-d-1})(b - a'cq^{d-2i+1})(b' - acq^{2i-d-1})c^{-1}, \quad \phi_i = (q^i-q^{-i})(q^{d-i+1}-q^{i-d-1})(b - acq^{d-2i+1})(b' - a'cq^{2i-d-1})c^{-1}.$$  

Then (11) is a parameter array of $A, A^*$.

For the rest of this section, we assume $d \geq 3$. Let $A, A^*$ be a Leonard pair on $V$, and let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be a standard ordering of the eigenvalues of $A$ (resp. $A^*$). Let $\beta$ be one less the common value of (9). We call $\beta$ the fundamental parameter of $A, A^*$. Let $q$ be a nonzero scalar such that $\beta = q^2 + q^{-2}$. We call $q$ a quantum parameter of $A, A^*$. We now give a solution of Problem 1.3 for the case that $q$ is not a root of unity:

Theorem 1.10. Consider sequences of scalars $\{\theta_i\}_{i=0}^d$, $\{x_i\}_{i=0}^d$, $\{y_i\}_{i=1}^d$, $\{z_i\}_{i=1}^d$ such that $y_iz_i \neq 0$ for $1 \leq i \leq d$, and consider the matrices $A, A^*$ in (11). Assume $A, A^*$ is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$ with quantum parameter $q$ that is not a root of unity. Then, after replacing $q$ with $q^{-1}$ if necessary, there exist scalars $\alpha$, $\alpha^*$, $\alpha'$, $\beta$, $\beta'$, $c$ with $c \neq 0$ that satisfy (2)–(8).

Let $A, A^*$ be a Leonard pair on $V$ with parameter array (11). Our next result gives a necessary and sufficient condition on the parameter array for that $A, A^*$ has LB-TD form. To state this result, we use the following notation. Let $q$ be a quantum parameter of $A, A^*$, and assume $q$ is not a root of unity. By [7, Lemma 9.2] there exist scalars $\alpha$, $\alpha^*$, $\alpha'$, $\beta$, $\beta'$ such that

$$\theta_i = \alpha + aq^{2i-d} + a'q^{d-2i} \quad (0 \leq i \leq d),$$

$$\theta_i^* = \alpha^* + bq^{2i-d} + b'q^{d-2i} \quad (0 \leq i \leq d).$$

By [6, Lemma 13.1] there exists a scalar $\xi$ such that

$$\varphi_i = (q^i-q^{-i})(q^{d-i+1}-q^{i-d-1})(\xi + abq^{2i-d-1} + a'b'q^{d-2i+1}) \quad (1 \leq i \leq d),$$

$$\phi_i = (q^i-q^{-i})(q^{d-i+1}-q^{i-d-1})(\xi + a'bq^{2i-d-1} + ab'q^{d-2i+1}) \quad (1 \leq i \leq d).$$

Theorem 1.11. With the above notation, the following (i) and (ii) are equivalent:

(i) $A, A^*$ has LB-TD form.

(ii) At least two of $\alpha \alpha'$, $\beta \beta'$, $\xi$ are nonzero.

This paper is organized as follows. In Sections 2 and 3 we recall some materials concerning Leonard pairs. In Section 4 we prove Propositions 1.7 and 1.9. In Section 5 we recall the Askey-Wilson relations. In Section 6 we use Askey-Wilson relations to obtain the entries of $A^*$ in (1). In Section 8 we prove Theorem 1.11. In Section 9 we prove Theorem 1.10. Leonard pairs have been classified in [10, Section 35]: there are 7 types for the case that $q$ is not a root of unity. In Section 10 we explain about which types of Leonard pairs have LB-TD form.
2 Recurrent sequences

In this section we recall the notion of a recurrent sequence. We also mention some lemmas for later use. Assume $d \geq 3$ and consider a sequence \( \{\theta_i\}_{i=0}^d \) consisting of mutually distinct scalars. We say \( \{\theta_i\}_{i=0}^d \) is recurrent whenever the expression
\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}
\]
(19)
is independent of \( i \) for \( 2 \leq i \leq d - 1 \). For a scalar \( \beta \), we say \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent whenever
\[
\theta_{i-2} - (\beta + 1)\theta_{i-1} + (\beta + 1)\theta_i - \theta_{i+1} = 0 \quad (2 \leq i \leq d - 1).
\]
Observe that \( \{\theta_i\}_{i=0}^d \) is recurrent if and only if it is \( \beta \)-recurrent for some \( \beta \). In this case, the value of (19) is equal to \( \beta + 1 \).

Lemma 2.1 [7, Lemmas 8.4, 8.5] Assume \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent for some scalar \( \beta \). Then the following hold:

(i) There exists a scalar \( \gamma \) such that
\[
\gamma = \theta_{i-1} - \beta \theta_i + \theta_{i+1} \quad (1 \leq i \leq d - 1).
\]
(20)

(ii) Let \( \gamma \) be from (i). Then there exists a scalar \( \varrho \) such that
\[
\varrho = \theta_{i-1}^2 - \beta \theta_{i-1} \theta_i + \theta_i^2 - \gamma (\theta_{i-1} + \theta_i) \quad (1 \leq i \leq d).
\]
(21)

Lemma 2.2 [7, Lemma 8.4] Assume \( \{\theta_i\}_{i=0}^d \) satisfies (20) for some scalars \( \beta \) and \( \gamma \). Then \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent.

Assume \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent, and take a nonzero scalar \( q \) such that \( \beta = q^2 + q^{-2} \). Assume \( q \) is not a root of unity. By [7, Lemma 9.2] there exist scalars \( \alpha, a, a' \) such that
\[
\theta_i = \alpha + a q^{2i-d} + a' q^{d-2i} \quad (0 \leq i \leq d).
\]
(22)

Lemma 2.3 Let the scalars \( \gamma, \varrho \) be from Lemma 2.1. Then
\[
\gamma = -\alpha(q - q^{-1})^2,
\]
(23)
\[
\varrho = \alpha^2(q - q^{-1})^2 - aa'(q^2 - q^{-2})^2.
\]
(24)

Proof. Routine verification. \( \square \)
Lemma 2.4 Let the scalars $\gamma, \varrho$ be from Lemma 2.1 and let $\{\tilde{\theta}_i\}_{i=0}^d$ be a reordering of $\{\theta_i\}_{i=0}^d$ that satisfies both
\[
\gamma = \tilde{\theta}_{i-1} - \beta \tilde{\theta}_i + \tilde{\theta}_{i+1} \quad (1 \leq i \leq d - 1),
\]
\[
\varrho = \tilde{\theta}_0^2 - \beta \tilde{\theta}_0 \tilde{\theta}_1 + \tilde{\theta}_1^2 - \gamma(\tilde{\theta}_0 + \tilde{\theta}_1).
\]
Then the sequence $\{\tilde{\theta}_i\}_{i=0}^d$ coincides with either $\{\theta_i\}_{i=0}^d$ or $\{\theta_{d-i}\}_{i=0}^d$.

Proof. Note that the sequence $\{\tilde{\theta}_i\}_{i=0}^d$ is $\beta$-recurrent by (25) and Lemma 2.2. By this and [7, Lemma 9.2] there exist scalars $\tilde{\alpha}, \tilde{\alpha}'$ such that
\[
\tilde{\theta}_i = \tilde{\alpha} + \tilde{\alpha}' q^{2i-d} + \tilde{\alpha}' q^{d-2i} \quad (0 \leq i \leq d).
\]
By (23) and (25) for $i = 1$ one finds $\tilde{\alpha} = \alpha$. By this and (23), (24), (26), one finds $a\tilde{\alpha}' = a\tilde{\alpha}'$. Using these comments and the assumption that $\{\tilde{\theta}_i\}_{i=0}^d$ is a permutation of $\{\theta_i\}_{i=0}^d$, one routinely finds that either (i) $\tilde{\alpha} = \alpha$, $\tilde{\alpha}' = \alpha'$ or (ii) $\tilde{\alpha} = \alpha'$, $\tilde{\alpha}' = \alpha$. The result follows.

\[\square\]

3 Parameter arrays

In this section we recall some materials concerning Leonard pairs.

We first recall the notion of an isomorphism of Leonard pairs. Consider a vector space $V'$ over $\mathbb{F}$ that has dimension $d + 1$. For a Leonard pairs $A, A^*$ on $V$ and a Leonard pair $B, B^*$ on $V'$, by an isomorphism of Leonard pairs from $A, A^*$ to $B, B^*$ we mean a linear bijection $\sigma : V \to V'$ such that both $\sigma A = B \sigma$ and $\sigma A^* = B^* \sigma$. We say that the Leonard pairs $A, A^*$ and $B, B^*$ are isomorphic whenever there exists an isomorphism of Leonard pairs from $A, A^*$ to $B, B^*$.

Next we recall some facts concerning a parameter array of a Leonard pair.

Definition 3.1 By a parameter array over $\mathbb{F}$ we mean a sequence (11) consisting of scalars in $\mathbb{F}$ that satisfy (i)–(v) below:

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \leq i, j \leq d)$.

(ii) $\varphi_i \neq 0$, $\phi_i \neq 0$ $(1 \leq i \leq d)$.

(iii) $\varphi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*) (\theta_{i-1} - \theta_d)$ $(1 \leq i \leq d)$.

(iv) $\phi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*) (\theta_{d-i+1} - \theta_0)$ $(1 \leq i \leq d)$.

(v) The expressions
\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}
\]
are equal and independent of $i$ for $2 \leq i \leq d - 1$. 

6
Lemma 3.2 \[7\] Theorem 1.9] Consider sequences of scalars \(\{\theta_i\}_{i=0}^d\), \(\{\varphi_i\}_{i=0}^d\), \(\{\phi_i\}_{i=1}^d\). Let \(A : V \to V\) and \(A^* : V \to V\) be linear transformations that are represented as in (10) with respect to some basis for \(V\). Then the following (i) and (ii) are equivalent:

(i) The pair \(A, A^*\) is a Leonard pair with parameter array (11).

(ii) The sequence (11) is a parameter array over \(\mathbb{F}\).

Suppose (i) and (ii) hold above. Then \(A, A^*\) is unique up to isomorphism of Leonard pairs.

Lemma 3.3 Let \(A, A^*\) be a Leonard pair on \(V\) with parameter array (11). Assume \(d \geq 3\), and let \(q\) be a quantum parameter of \(A, A^*\). Let \(a, a^*, a', b, b', \xi\) be scalars that satisfy (15)–(18). Assume at least two of \(aa', bb', \xi\) are nonzero. Then there exists a nonzero scalar \(c\) such that \(\xi = -aa'c - bb'c^{-1}\). Moreover, this scalar satisfies

\[
\begin{align*}
\varphi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{-d-i})(b - acq^{d-2i+1})(b' - acq^{2i-d-1})c^{-1} \\
\phi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{-d-i})(b - acq^{d-2i+1})(b' - acq^{2i-d-1})c^{-1}
\end{align*}
\]

\((1 \leq i \leq d)\). (29) (30)

Proof. Such a scalar \(c\) exists since \(\mathbb{F}\) is algebraically closed. To get (29) and (30), set \(\xi = -aa'c - bb'c^{-1}\) in (17) and (18). \(\square\)

4 Proof of Propositions [1.7] and [1.9]

Proof of Propositions [1.7] and [1.9]. Fix a nonzero scalar \(q\) that is not a root of unity. Let \(a, a^*, a', b, b', c\) be scalars with \(c \neq 0\). Define scalars \(\{\theta_i\}_{i=0}^d\), \(\{x_i\}_{i=0}^d\), \(\{y_i\}_{i=1}^d\), \(\{z_i\}_{i=1}^d\) by (2)–(3), and define scalars \(\{\theta_i^d\}_{i=0}^d\), \(\{\varphi_i^d\}_{i=1}^d\), \(\{\phi_i^d\}_{i=1}^d\) by (12)–(14). One checks that the conditions Definition 3.1(i), (ii) are satisfied if and only if (6)–(8) hold. In this case, the conditions Definition 3.1(iii)–(v) are satisfied. Therefore (11) is a parameter array over \(\mathbb{F}\) if and only if (5)–(8) hold. Let the matrices \(A, A^*\) be from (11). For \(0 \leq r \leq d\) define \(u_r \in \mathbb{F}^{d+1}\) that has \(i\)th entry

\[
(u_r)_i = (-1)^{r+i}c^{d-r}q^{(d+r-i)(d-r+i-1)/2} \prod_{h=0}^{d+r-i-1} (q^{d-h} - q^{h-d}) \prod_{h=0}^{r-1} (b' - acq^{2h-d+1}) \prod_{h=0}^{d-i-1} (b' - acq^{2h-d+1})
\]

for \(0 \leq i \leq d\). One routine verifies that

\[
\begin{align*}
Au_r &= \theta_r u_r + u_{r+1} \\
&= (0 \leq r \leq d - 1),
\end{align*}
\]

\[
\begin{align*}
Au_d &= \theta_d u_d, \\
&= (0 \leq r \leq d - 1),
\end{align*}
\]

Therefore the matrices representing \(A, A^*\) with respect to \(\{u_r\}_{r=0}^d\) are as in (10). By these comments and Lemma 3.2, \(A, A^*\) is a Leonard pair with parameter array (11) if and only if (5)–(8) hold. In this case, observe that \(y_i z_i \neq 0\) for \(1 \leq i \leq d\). The results follow. \(\square\)
5 The Askey-Wilson relations

For the rest of the paper we assume \( d \geq 3 \). In this section we recall the Askey-Wilson relations for a Leonard pair. Consider a Leonard pair \( A, A^* \) on \( V \) with parameter array \((11)\) and fundamental parameter \( \beta \). Note that \( \beta \) is well-defined by our assumption \( d \geq 3 \).

By Lemma \((2.1)\) there exist scalars \( \gamma, \gamma^*, \varrho, \varrho^* \) such that
\[
\gamma = \theta_{i-1} - \beta \theta_i + \theta_{i+1} \quad (1 \leq i \leq d-1), \tag{31}
\]
\[
\gamma^* = \theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* \quad (1 \leq i \leq d-1), \tag{32}
\]
\[
\varrho = \theta_{i-1}^2 - \beta \theta_{i-1} \theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \quad (1 \leq i \leq d), \tag{33}
\]
\[
\varrho^* = \theta_{i-1}^{*2} - \beta \theta_{i-1}^* \theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \quad (1 \leq i \leq d). \tag{34}
\]

Lemma 5.1 \((11)\) Theorem 1.5] \( \) There exist scalars \( \omega, \eta, \eta^* \) such that both
\[
A^2 A^* - \beta AA^* A + A^* A^2 - \gamma(AA^* + A^* A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I, \quad (35)
\]
\[
A^{*2} A - \beta A^* AA^* + AA^{*2} - \gamma^*(AA^* + AA^*) - \varrho^* A = \gamma A^{*2} + \omega A^* + \eta^* I. \quad (36)
\]

The scalars \( \omega, \eta, \eta^* \) are uniquely determined by \( A, A^* \).

The relations \((35)\) and \((36)\) are known as the Askey-Wilson relations. Below we describe the scalars \( \omega, \eta, \eta^* \). Define scalars \( \{a_i\}_{i=0}^d \) and \( \{a_i^*\}_{i=0}^d \) by
\[
a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_i} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}} \quad (1 \leq i \leq d-1),
\]
\[
a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1}, \quad a_d = \theta_d + \frac{\varphi_d}{\theta_d^* - \theta_{d-1}},
\]
\[
a_i^* = \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} \quad (1 \leq i \leq d-1),
\]
\[
a_0^* = \theta_0^* + \frac{\varphi_1}{\theta_0 - \theta_1}, \quad a_d^* = \theta_d^* + \frac{\varphi_d}{\theta_d - \theta_{d-1}}.
\]

For notational convenience, define \( \theta_{-1}, \theta_{d+1} \) (resp. \( \theta_{-1}^*, \theta_{d+1}^* \)) so that \((31)\) (resp. \((32)\)) holds for \( i = 0 \) and \( i = d \). Let the scalars \( \omega, \eta, \eta^* \) be from Lemma 5.1.

Lemma 5.2 \((11)\) Theorem 5.3] \( \) With the above notation,
\[
\omega = a_i^*(\theta_i - \theta_{i+1}) + a_{i-1}^*(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_{i-1} + \theta_i) \quad (1 \leq i \leq d), \tag{37}
\]
\[
\eta = a_i^*(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^* \theta_i^2 - \omega \theta_i \quad (0 \leq i \leq d), \tag{38}
\]
\[
\eta^* = a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - \gamma \theta_i^{*2} - \omega \theta_i^* \quad (0 \leq i \leq d). \tag{39}
\]

Let \( q \) be a quantum parameter of \( A, A^* \), and assume \( q \) is not a root of unity. Let \( a, a^*, a', b, b', \xi \) be scalars that satisfy \((15)-(18)\).
Lemma 5.3 With the above notation, assume $\alpha = 0$ and $\alpha^* = 0$. Then

$$
\begin{align*}
\gamma &= 0, & \gamma^* &= 0, \\
\varrho &= -aa'(q^2 - q^{-2})^2, \\
\varrho^* &= -bb'(q^2 - q^{-2})^2, \\
\omega &= (q - q^{-1})^2 ((q^{d+1} + q^{-d-1})\xi - (a + a')(b + b')), \\
\eta &= -(q - q^{-1})(q^2 - q^{-2}) ((a + a')\xi - ad'(b + b')(q^{d+1} + q^{-d-1})), \\
\eta^* &= -(q - q^{-1})(q^2 - q^{-2}) ((b + b')\xi - bb'(a + a')(q^{d+1} + q^{-d-1})).
\end{align*}
$$

Proof. The lines (30)–(32) follows from Lemma 2.3. The lines (33)–(35) are routinely verified.

\[\square\]

6 Evaluating the Askey-Wilson relations

Let $\{\theta_i\}_{i=0}^d$, $\{x_i\}_{i=0}^d$, $\{y_i\}_{i=0}^d$, $\{z_i\}_{i=0}^d$ be scalars such that $y_i z_i \neq 0$ for $1 \leq i \leq d$. Consider the matrices $A, A^*$ from (1), and assume $A, A^*$ is a Leonard pair in $\text{Mat}_{d+1}(\mathbb{F})$ with fundamental parameter $\beta$ and quantum parameter $q$ that is not a root of unity. In this section we evaluate the Askey-Wilson relations to obtain some relations between the entries of $A$ and $A^*$. For each matrix in $\text{Mat}_{d+1}(\mathbb{F})$ we index the rows and columns by $0, 1, \ldots, d$.

Lemma 6.1 With the above notation, $\{\theta_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of $A$.

Proof. Clearly $\{\theta_i\}_{i=0}^d$ is an ordering of the eigenvalues of $A$. Compute the $(i-1, i+1)$-entry of (36) for $1 \leq i \leq d - 1$ to find

$$y_i y_{i+1}(\theta_{i-1} - \beta \theta_i + \theta_{i+1}) = y_i y_{i+1} \gamma.$$ 

So $\gamma = \theta_{i-1} - \beta \theta_i + \theta_{i+1}$ for $1 \leq i \leq d - 1$. Compute the $(0, 1)$-entry of (36) to find

$$y_0(\theta_0^2 - \beta \theta_0 \theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1)) = y_0 \varrho.$$

So $\varrho = \theta_0^2 - \beta \theta_0 \theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1)$. By these comments and Lemma 2.4 we find the result. \[\square\]

Let $\{\theta_i^*\}_{i=0}^d$ be a standard ordering of the eigenvalues of $A^*$. Let $\alpha$, $a$, $a'$ (resp. $\alpha^*$, $b$, $b'$) be scalars that satisfy (15) (resp. (16)). We assume $\alpha = 0$ and $\alpha^* = 0$. Let the scalars $\gamma$, $\gamma^*$, $\varrho$, $\varrho^*$ be from (31)–(34). Let $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) be the first split sequence (resp. second split sequence) of $A, A^*$ associated with the ordering $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$. Let $\xi$ be a scalar that satisfies (17) and (18). Let the scalars $\omega$, $\eta$, $\eta^*$ be from Lemma 5.1. Note that the scalars $\gamma$, $\gamma^*$, $\varrho$, $\varrho^*$, $\omega$, $\eta$, $\eta^*$ are written as in (36)–(45).
Lemma 6.2 With the above notation, after replacing $q$ with $q^{-1}$ if necessary,

$$z_i = z_1 q^{2-2i} \quad (1 \leq i \leq d). \quad (46)$$

Proof. Compute the $(i+1, i-2)$-entry of (35) to find

$$z_{i-1} - \beta z_i + z_{i+1} = 0 \quad (2 \leq i \leq d-1). \quad (47)$$

By (47) for $i = 2$

$$z_3 = \beta z_2 - z_1. \quad (48)$$

Compute the $(3, 0)$-entry of (36) to find

$$z_1 z_2 - \beta z_1 z_3 + z_2 z_3 = 0.$$

In this equation, eliminate $z_3$ using (48), and simplify the result using $\beta = q^2 + q^{-2}$ to find

$$(z_2 - q^2 z_1)(z_2 - q^{-2} z_1) = 0.$$

So either $z_2 = z_1 q^2$ or $z_2 = z_1 q^{-2}$. After replacing $q$ with $q^{-1}$ if necessary, we may assume $z_2 = z_1 q^{-2}$. Now the result follows from this and (47). $\square$

For the rest of this section, we choose $q$ that satisfies (46).

Lemma 6.3 With the above notation, for $1 \leq i \leq d-1$

$$q^{-3} x_{i-1} - (q + q^{-1}) x_i + q^3 x_{i+1} = 0. \quad (49)$$

Proof. Compute the $(i+1, i-1)$-entry of (36) to find

$$z_i z_{i+1} (\theta_{i-1} - \beta \theta_i + \theta_{i+1}) + x_i (z_i + z_{i+1}) + x_{i-1} (z_i - \beta z_{i+1}) + x_{i+1} (z_{i+1} - \beta z_i) = 0.$$

In this equation, the first term is zero by (31). Simplify the remaining terms using (46) to find

$$(50)$$

Lemma 6.4 With the above notation, for $1 \leq i \leq d-1$

$$x_{i-1} - \beta x_i + x_{i+1} = a' q^{d-2i-1}(q^2 - q^{-2})(q^3 - q^{-3}) z_i. \quad (50)$$

Proof. Compute $(i+1, i-1)$-entry of (35) to find

$$x_{i-1} - \beta x_i + x_{i+1} + \theta_{i-1} (z_{i+1} - \beta z_i) + \theta_i (z_i + z_{i+1}) + \theta_{i+1} (z_i - \beta z_{i+1}) = 0.$$

In this line, eliminate $z_{i+1}$ using (46), and simplify the result using (15) to find (50). $\square$

For notational convenience, define $y_0 = 0$ and $y_{d+1} = 0$. 

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Lemma 6.5  With the above notation, for $1 \leq i \leq d$

\[ y_{i-1} - \beta y_i + y_{i+1} = (\theta_{i-2} - \theta_{i-1})x_{i-1} + (\theta_{i+1} - \theta_i)x_i + \omega. \]  

(51)

Proof.  Compute the $(i, i-1)$-entry of (35) to find

\[ z_i(\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \theta) + y_{i-1} - \beta y_i + y_{i+1} = (\beta\theta_{i-1} - \theta_{i-1} - \theta_i)x_{i-1} + (\beta\theta_i - \theta_{i-1} - \theta_i)x_i + \omega. \]

Simplify this using (31) and (33) to get (51).

Lemma 6.6  With the above notation, for $1 \leq i \leq d$

\[ x_{i-1}^2 - \beta x_{i-1}x_i + x_i^2 - \theta^* = (q + q^{-1})(q^{-1}y_{i-1} - (q + q^{-1})y_i + qy_{i+1})z_i. \]  

(52)

Proof.  Compute the $(i, i-1)$-entry of (36), and simplify the result using (46) to find that the left-hand side of (52) is equal to $z_i$ times

\[ q^{-2}y_{i-1} - 2y_i + q^2y_{i+1} + (\beta\theta_{i-1} - \theta_{i-1} - \theta_i)x_{i-1} + (\beta\theta_i - \theta_{i-1} - \theta_i)x_i + \omega. \]

Using (31) one finds that the above expression is equal to

\[ q^{-2}y_{i-1} - 2y_i + q^2y_{i+1} + (\beta\theta_{i-2} - \theta_{i-2} - \theta_i)x_{i-1} + (\beta\theta_{i+1} - \theta_i)x_i + \omega. \]

Using Lemma 6.5 one finds that $z_i$ times the above expression is equal to the right-hand side of (52).

Lemma 6.7  With the above notation,

\[ (\theta_0 - \beta\theta_0 + \theta_1)y_1 = ((\beta - 2)\theta_0^2 + \theta)x_0 + \omega\theta_0 + \eta, \]  

(53)

\[ (\theta_{d-1} + \theta_d - \beta\theta_d)y_d = ((\beta - 2)\theta_d^2 + \theta)x_d + \omega\theta_d + \eta. \]  

(54)

Proof.  Compute the $(0, 0)$-entry and the $(d, d)$-entry of (35).

Lemma 6.8  With the above notation,

\[ ((1 - \beta)x_0 + x_1 + (\theta_0 - \theta_2)z_1)y_1 = (\beta - 2)\theta_0 x_0^2 + \omega x_0 + \theta^* \theta_0 + \eta^*. \]  

(55)

Proof.  Compute the $(0, 0)$-entry of (36) to find

\[ (2 - \beta)\theta_0 x_0^2 + (1 - \beta)x_0 y_1 + x_1 y_1 + (2\theta_0 - \beta\theta_1)y_1 z_1 - \omega x_0 - \theta^* \theta_0 - \eta^* = 0. \]

By (31) $2\theta_0 - \beta\theta_1 = \theta_0 - \theta_2$. By these comments we find (55).

\[ \Box \]
7 Obtaining the entries of \( A^* \)

Let \( \{\theta_i\}_{i=0}^d, \{x_i\}_{i=0}^d, \{y_i\}_{i=1}^d, \{z_i\}_{i=1}^d \) be scalars such that \( y_i z_i \neq 0 \) for \( 1 \leq i \leq d \). Consider the matrices \( A, A^* \) from (1), and assume \( A, A^* \) is a Leonard pair in \( \text{Mat}_{d+1}(\mathbb{F}) \). In this section we obtain the entries of \( A^* \).

By Lemma 6.1 \( \{\theta_i\}_{i=0}^d \) is a standard ordering of the eigenvalues of \( A \). Let \( \{\theta_i^*\}_{i=0}^d \) be a standard ordering of the eigenvalues of \( A^* \). Let \( \beta \) (resp. \( q \)) be the fundamental parameter (resp. quantum parameter) of \( A, A^* \), and assume \( q \) is not a root of unity. Let \( \alpha, a, a' \) (resp. \( \alpha^*, b, b' \)) be scalars that satisfy (15) (resp. (16)). We assume \( \alpha = 0 \) and \( \alpha^* = 0 \). Let \( \{\phi_i\}_{i=1}^d \) (resp. \( \{\phi_i^*\}_{i=1}^d \)) be the first split sequence (resp. second split sequence) of \( A, A^* \) associated with the ordering \( \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d \). Let \( \xi \) be a scalar that satisfies (17) and (18). Let \( \gamma, \gamma^*, q, \eta \) be from (31)–(34), and the scalars \( \omega, \eta, \eta^* \) be from Lemma 5.1. Note that these scalars are written as in Lemma 5.3.

**Lemma 7.1** With the above notation,

\[
x_i = q^{-2i}x_0 - a'z_1q^{d-3i+1}(q + q^{-1})(q^i - q^{-i}) \quad (0 \leq i \leq d).
\]  

**Proof.** Routinely obtained from (50) for \( i = 1 \) and (59) for \( i = 1, 2, \ldots, d - 1 \). \( \Box \)

**Lemma 7.2** With the above notation, for \( 1 \leq i \leq d \), \( y_i \) is equal to \( q^{-i}(q^i - q^{-i}) \) times

\[
\begin{align*}
& a^2 q^{2d-i+3}(q^{i-1} - q^{-1}) z_1 + aa' q(q + q^{-1}) z_1 + bb' q^{-2}(q^i + q^{-i}) z_1^{-1} \\
& - q^{-i}(aq^{i-d-1} + a' q^{d-i+1}) x_0 - (q^{d+1} + q^{-d-1}) \xi + (a + a')(b + b').
\end{align*}
\]

**Proof.** Obtained from (52) for \( i = 1 \) and (53) for \( i = 1, 2, \ldots, d - 1 \) using Lemma 7.1. \( \Box \)

We first consider the case that \( a \neq a' q^{2d+2} \) and \( a' \neq a q^{2d+2} \).

**Lemma 7.3** With the above notation, assume \( a \neq a' q^{2d+2} \). Then

\[
\xi = q^d(aa' q z_1 + bb' q^{-1} z_1^{-1}) - aqx_0 + a(b + b') q^{d+1}.
\]  

**Proof.** In (53), eliminate \( y_1 \) using Lemma 7.2 and simplify the result to find

\[
q^{-d-1}(q - q^{-1})(q^2 - q^{-2})(aq^{-d-1} - a'q^{d+1}) \\
\times (\xi - q^d(aa' q z_1 + bb' q^{-1} z_1^{-1}) + aqx_0 - a(b + b') q^{d+1}) = 0.
\]

By this and \( aq^{-d-1} \neq a'q^{d+1} \) we find (57). \( \Box \)
Lemma 7.4 With the above notation, assume \( a \neq a' q^{2d+2} \) and \( a' \neq a q^{2d+2} \). Then
\[
x_0 = a'(q^d - q^{-d})z_1 + (b + b')q^d,
\]
\[
\xi = aa'q^{1-d}z_1 + bb'q^{d-1}z_1^{-1}.
\]

Proof. In (57), eliminate \( x_d \) and \( y_d \) using Lemmas 7.1 and 7.2. Then eliminate \( \xi \) using (57). Simplify the result using (15) and Lemma 5.3 to find
\[
x_0 - a'(q^d - q^{-d})z_1 - (b + b')q^d = 0.
\]
We have \( a' \neq a q^{2d+2} \); otherwise \( \theta_{d-1} = \theta_d \). By our assumption \( a' \neq a q^{2d+2} \). By these comments we get (58). Line (59) follows from (57) and (58).

Lemma 7.5 With the above notation, assume \( a \neq a' q^{2d+2} \) and \( a' \neq a q^{2d+2} \). Then
\[
x_i = (b + b')q^{d-2i} - a'q^{1-2i}(q^{i+1} + q^{d-i} - q^{d-2i-1})z_1 \quad (0 \leq i \leq d),
\]
\[
y_i = q^{d-i}(q^i - q^{-i})(q^{d-i+1} - q^{-d+i})z_1 \quad (0 \leq i \leq d),
\]
\[
z_i = q^{2-2i}z_1 \quad (1 \leq i \leq d).
\]

Proof. The values of \( x_i \) and \( y_i \) are obtained from Lemmas 7.1, 7.2 and (58). The value of \( z_i \) is given in Lemma 6.2.

Next consider the case \( a = a' q^{2d+2} \). Note that \( a \neq 0 \) in this case; otherwise \( \theta_i = 0 \) for \( 0 \leq i \leq d \).

Lemma 7.6 With the above notation, assume \( a = a' q^{2d+2} \). Then
\[
x_0 = aq^{-d-2}(1 + q^2 - q^{-2d})z_1 + a^{-1}bb'q^{3d}z_1^{-1} - a^{-1}q^{2d+1}z_1^{-1} + (b + b')q^d.
\]

Proof. In (51) for \( i = d \), eliminate \( y_{d-1}, y_d, x_{d-1}, x_d \) using Lemmas 7.1 and 7.2. Then simplify the result to find that
\[
-ax_0 + a^2q^{-d-2}(1 + q^2 - q^{-2d})z_1 + bb'q^{3d}z_1^{-1} - q^{2d+1}z_1^{-1} + (b + b')q^d
\]
is zero. The result follows.

Lemma 7.7 With the above notation, assume \( a = a' q^{2d+2} \). Then at least one of the following (i)–(iii) holds:
(i) \( z_1 = -a^{-1}bq^{2d} \).
(ii) \( z_1 = -a^{-1}b'q^{2d} \).
(iii) \( \xi = a^2q^{-3d-1}z_1 + bb'q^{d-1}z_1^{-1} \).

Proof. In (55), eliminate \( x_1 \) and \( y_1 \) using Lemmas 7.1 and 7.2. Then eliminate \( x_0 \) using Lemma 7.3 to find that \( z_1^{-1}q^{-d-1}(q - q^{-1})(q^2 - q^{-2}) \) times
\[
(b + az_1q^{-2d})(b' + az_1q^{-2d})(aq^{-2d}z_1 + a^{-1}bb'q^{2d}z_1^{-1} - a^{-1}q^{d+1}z_1^{-1}
\]
is zero. The result follows.
Lemma 7.8 With reference to Lemma 7.7, assume (i) holds. Then
\[ x_i = -a^{-1}q^{2d-2i+1}\xi + bq^{-2i-1}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1}) \quad (0 \leq i \leq d), \]
\[ y_i = q^{d-2i-1}(q^i - q^{i-1})(q^{d-i+1} - q^{d-i-1})(q^{d+1}\xi + abq^{-2i} + ab'q^{2i}) \quad (1 \leq i \leq d), \]
\[ z_i = -a^{-1}bq^{2d-2i+2} \quad (1 \leq i \leq d). \]

Proof. The values of \( x_i \) and \( y_i \) are obtained from Lemmas 7.1, 7.2, 7.6. The value of \( z_i \) is given in Lemma 6.2.

Lemma 7.9 With reference to Lemma 7.7, assume (ii) holds. Then
\[ x_i = -a^{-1}q^{2d-2i+1}\xi + b'q^{-2i-1}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1}) \quad (0 \leq i \leq d), \]
\[ y_i = q^{d-2i-1}(q^i - q^{i-1})(q^{d-i+1} - q^{d-i-1})(q^{d+1}\xi + abq^{2i} + ab'q^{-2i}) \quad (1 \leq i \leq d), \]
\[ z_i = -a^{-1}b'q^{2d-2i+2} \quad (1 \leq i \leq d). \]

Proof. Similar to the proof of Lemma 7.8.

Lemma 7.10 With reference to Lemma 7.7, assume (iii) holds. Then
\[ x_i = (b + b')q^{d-2i} - aq^{-2d-2i-1}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1})z_1 \quad (0 \leq i \leq d), \]
\[ y_i = q^{d-1}(q^i - q^{i-1})(q^{-d-1} - q^{d-i+1})(b + aq^{-2d-2i}z_1)(b' + aq^{-2d-2i}z_1)z_1^{-1} \quad (1 \leq i \leq d), \]
\[ z_i = q^{2-2i}z_1 \quad (1 \leq i \leq d). \]

Proof. Similar to the proof of Lemma 7.8.

Next consider the case \( a' = aq^{2d+2} \). The following lemmas can be shown in a similar way as Lemmas 7.6, 7.10.

Lemma 7.11 With the above notation, assume \( a' = aq^{2d+2} \). Then
\[ x_0 = aq^{d+2}z_1 + a^{-1}bb'q^{d-2}z_1^{-1} - a^{-1}q^{d-1}\xi + (b + b')q^d. \]

Lemma 7.12 With the above notation, assume \( a' = aq^{2d+2} \). Then at least one of the following (i)–(iii) holds:
(i) \( z_1 = -a^{-1}bq^{-2} \).
(ii) \( z_1 = -a^{-1}b'q^{-2} \).
(iii) \( \xi = a^2q^{d+3}z_1 + bb'q^{d-1}z_1^{-1} \).

Lemma 7.13 With reference to Lemma 7.12, assume (i) holds. Then
\[ x_i = -a^{-1}q^{-2i-1}\xi + bq^{2d-2i+1}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1}) \quad (0 \leq i \leq d), \]
\[ y_i = q^{3d-2i-3}(q^i - q^{i-1})(q^{d-i+1} - q^{d-i-1})(q^{d-1}\xi + abq^{2d-2i+2} + ab'q^{2i-2d-2}) \quad (1 \leq i \leq d), \]
\[ z_i = -a^{-1}bq^{-2i} \quad (1 \leq i \leq d). \]
Lemma 7.14 With reference to Lemma 7.12, assume (ii) holds. Then
\[
x_i = -a^{-1}q^{-2i-1}\xi + b'q^{2d-2i+1}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1}) \quad (0 \leq i \leq d),
\]
\[
y_i = q^{3d-2i+3}(q^{i} - q^{-i})(q^{i+1} - q^{-d-1})(q^{d-1}\xi + abq^{2i-2d-2} + ab'q^{2d-2i+2}) \quad (1 \leq i \leq d),
\]
\[
z_i = -a^{-1}b'q^{-2i} \quad (1 \leq i \leq d).
\]

Lemma 7.15 With reference to Lemma 7.12, assume (iii) holds. Then
\[
x_i = (b + b')q^{d-2i} - aq^{2d-2i+3}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1})z_1 \quad (0 \leq i \leq d),
\]
\[
y_i = q^{d-1}(q^{i} - q^{-i})(q^{i-d-1} - q^{-d-1})(q^{d-1}\xi + abq^{2d-2i+4}z_1)(b' + aq^{2d-2i+4}z_1)z_1^{-1} \quad (1 \leq i \leq d),
\]
\[
z_i = q^{2-2i}z_1 \quad (1 \leq i \leq d).
\]

8 Proof of Theorem 1.11

Let $A, A^*$ be a Leonard pair on $V$ with parameter array (11). Let $q$ be a quantum parameter of $A, A^*$, and assume $q$ is not a root of unity. Let $\alpha, \alpha^*$, $a$, $a'$, $b$, $b'$, $\xi$ be scalars that satisfy (15)–(18).

Proposition 8.1 With the above notation, assume at least two of $aa'$, $bb'$, $\xi$ are nonzero. Then $A, A^*$ is isomorphic to the Leonard pair given in Proposition 1.7 for some nonzero scalar $c$.

Proof. Without loss of generality, we assume $\alpha = 0$ and $\alpha^* = 0$. By Lemma 3.3 there exists a nonzero scalar $c$ that satisfies (29) and (30). Using conditions (i) and (ii) in Definition 3.4, one checks that the scalars $a, a', b, b', c$ satisfy the inequalities (3–8). By Proposition 1.9 the Leonard pair from Proposition 1.7 has the same parameter array as $A, A^*$. By this and Lemma 3.2 $A, A^*$ is isomorphic to the Leonard pair given in Proposition 1.7.

Lemma 8.2 With the above notation, assume $A, A^*$ has LB-TD form. Then at least one of $aa'$, $bb'$ is nonzero.

Proof. Without loss of generality, we may assume $A, A^*$ is an LB-TD pair in Mat$_{d+1}(\mathbb{F})$. In view of Lemma 6.11 we may assume that $A$ is the matrix from (1) after replacing $\{\theta_i\}_{i=0}^d$ with $\{\theta_{d-i}\}_{i=0}^d$ if necessary. Write $A^*$ as in (11), and note that $y_iz_i \neq 0$ for $1 \leq i \leq d$. In view of Note 1.6 we may assume $\alpha = 0$ and $\alpha^* = 0$. We show that at least one of $aa'$, $bb'$ is nonzero. By way of contradiction, we assume $aa' = 0$ and $bb' = 0$. Note that $a' \neq a'q^{2d+2}$ and $a' \neq aq^{2d+2}$; otherwise both $a = 0$ and $a' = 0$ by $aa' = 0$. Let $\{\phi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) be the first split sequence (resp. second split sequence) of $A, A^*$ associated with the ordering $\{\theta_i\}_{i=0}^d, \{\theta_i\}_{i=0}^d$. Let $\xi$ be a scalar that satisfies (17) and (18). By (59) $\xi = 0$. By this and (17), (18)
\[
\varphi_1 = (q - q^{-1})(q^{d} - q^{-d})(abq^{1-d} + a'b'q^{d-1}),
\]
\[
\phi_1 = (q - q^{-1})(q^{d} - q^{-d})(a'bq^{1-d} + ab'q^{d-1}).
\]
Note that $\varphi_1 \neq 0$ and $\phi_1 \neq 0$ by Lemma 3.2(ii). So
\[ abq^{1-d} + a'b'q^{d-1} \neq 0, \quad a'bk^{1-d} + ab'q^{d-1} \neq 0. \]
Therefore, if $a = 0$ then both $b' \neq 0$ and $b \neq 0$, and if $a' = 0$ then both $b \neq 0$ and $b' \neq 0$. This contradicts $bb' = 0$. The result follows.

**Lemma 8.3** With the above notation, assume $A, A^*$ has LB-TD form. Assume either $a = a'q^{2d+2}$ or $a' = aq^{2d+2}$. Then at least one of $bb', \xi$ is nonzero.

**Proof.** Without loss of generality, we may assume $A, A^*$ is an LB-TD pair in $\text{Mat}_{d+1}(F)$. In view of Lemma 5.1 we may assume that $A$ is the matrix from (1) after replacing $\{\theta_i\}_{i=0}^d$ with $\{\theta_{d-i}\}_{i=0}^d$ if necessary. Write $A^*$ as in (1), and note that $y_i z_i \neq 0$ for $1 \leq i \leq d$. In view of Note 1.6 we may assume $\alpha = 0$ and $\alpha^* = 0$. First consider the case $a = a'q^{2d+2}$. Note that $aa' \neq 0$; otherwise $\theta_0 = \theta_1$. By Lemma 7.7 at least one of (i)–(iii) holds. First assume (iii) holds. Then at least one of $bb', \xi$ is nonzero; otherwise $a_z = 0$. Next assume (i) holds. By way of contradiction, assume both $bb' = 0$ and $\xi = 0$. We must have $b' = 0$ since $b \neq 0$ by $0 \neq z_1 = -a^{-1}bk^{2d}$. By these comments and Lemma 7.8
\[
x_i = bq^{2i-1}(q^{d+1} + q^{-d-1} - q^{d-2i-1} - q^{d-2i+1}) \quad (0 \leq i \leq d),
y_i = abq^{-d-4i-1}(q^i - q^{-i})(q^{d-i+1} - q^{-d-i}) \quad (1 \leq i \leq d),
z_i = -a^{-1}bk^{2d-2i+2} \quad (1 \leq i \leq d).
\]
We claim that $\det A^* = 0$. To see the claim, for $0 \leq k \leq d$ we define $(k+1) \times (k+1)$ matrix $M_k$ that consists of the rows $0, 1, \ldots, k$ and columns $0, 1, \ldots, k$ of $A^*$. So
\[
M_0 = (x_0), \quad M_1 = \begin{pmatrix} x_0 & y_1 \\ z_1 & x_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} x_0 & y_1 & 0 \\ z_1 & x_1 & y_2 \\ 0 & z_2 & x_2 \end{pmatrix}, \quad \ldots \quad M_d = A^*.
\]
Using induction on $k = 0, 1, \ldots, d$ one routinely finds that
\[
\det M_k = b^{k+1}q^{-(k+1)(d+k+2)} \prod_{\ell=0}^k (1 - q^{2d-2\ell}) \quad (0 \leq k \leq d).
\]
Thus $\det M_d = 0$ and the claim is proved. By elementary linear algebra, $\det A^* = \theta_0^* \theta_1^* \cdots \theta_d^*$. So $A^* \neq 0$ since $\theta_i^* = bq^{2i-d}$ for $0 \leq i \leq d$. This contradicts the claim. We have shown that at least one of $\xi, bb'$ is nonzero for the case of (i). Next assume (ii) holds in Lemma 7.7. We can show the assertion in a similar way as above. We have shown the assertion for the case of $a = a'q^{2d+2}$. The proof is similar for the case of $a' = aq^{2d+2}$.

**Proof of Theorem 1.11** (i)⇒(ii): By Lemma 8.2 at least one of $aa', bb'$ is nonzero. First assume $a \neq a'q^{2d+2}$ and $a' \neq aq^{2d+2}$. By Lemma 7.4 $\xi = aa'q^{1-d}z_1 + bb'q^{d-1}z_1^{-1}$. If one of $aa', bb'$ is zero, then $\xi \neq 0$. Thus at least two of $aa', bb', \xi$ are nonzero. Next assume $a = a'q^{2d+2}$ or $a' = aq^{2d+2}$. In this case $aa' \neq 0$; otherwise $\theta_0 = \theta_1$. Moreover, at least one of $bb', \xi$ is nonzero by Lemma 8.3. Thus at least two of $aa', bb', \xi$ are nonzero.

(ii)⇒(i): Follows from Proposition 8.1.
9 Proof of Theorem 1.10

Proof of Theorem 1.10. Consider sequences of scalars \( \{ \theta_i \}_{i=0}^d, \{ y_i \}_{i=1}^d, \{ z_i \}_{i=1}^d \) such that \( y_i z_i \neq 0 \) for \( 1 \leq i \leq d \), and consider the matrices \( A, A^* \) from (1). Assume \( A, A^* \) is an LB-TD Leonard pair in \( \text{Mat}_{d+1}(F) \) with quantum parameter \( q \) that is not a root of unity. By Lemma 6.1 \( \{ \theta_i \}_{i=0}^d \) is a standard ordering of the eigenvalues of \( A \). Let \( \{ \theta_i^* \}_{i=0}^d \) be a standard ordering of the eigenvalues of \( A^* \). Let \( \beta \) (resp. \( q \)) be the fundamental parameter (resp. quantum parameter) of \( A, A^* \), and assume \( q \) is not a root of unity. Let \( \alpha, a, a^* \) (resp. \( \alpha^*, b, b^* \)) be scalars that satisfy (15) (resp. (16)). We may assume \( \alpha = 0 \) and \( \alpha^* = 0 \).

Let \( \{ \varphi_i \}_{i=1}^d \) (resp. \( \{ \phi_i \}_{i=1}^d \)) be the first split sequence (resp. second split sequence) of \( A, A^* \) associated with the ordering \( \{ \theta_i \}_{i=0}^d, \{ \theta_i^* \}_{i=0}^d \). Let \( \xi \) be a scalar that satisfies (17) and (18). Note that at least two of \( aa^*, bb^*, \xi \) are nonzero by Theorem 1.11 and so Lemma 3.3 applies.

First consider the case that \( a \neq a'q^{2d+2} \) and \( a' \neq aq^{2d+2} \). By Lemma 7.7 the scalars \( \{ x_i \}_{i=0}^d, \{ y_i \}_{i=1}^d, \{ z_i \}_{i=1}^d \) are written as in Lemma 7.9. Setting \( z_1 = -cq^{d-1} \) in these expressions, we obtain (3)–(5).

Next consider the case \( a = a'q^{2d+2} \). By Lemma 7.7 at least one of (i)–(iii) holds in that lemma. First assume (i) holds. By Lemma 7.13 the scalars \( \{ x_i \}_{i=0}^d, \{ y_i \}_{i=1}^d, \{ z_i \}_{i=1}^d \) are written as in Lemma 7.8. Setting \( \xi = -aa'c - bb'c^{-1} \) in these expressions, and replacing \( b, b', c \) with \( (acq^{d-1}, a^{-1}bb'c^{-1}q^{d-1}, a^{-1}bq^{d-1}) \), we obtain (3)–(5). Next assume (ii) holds. Observe the expressions in Lemma 7.9 are obtained from the expressions in Lemma 7.8 by exchanging \( b \) and \( b' \). Now proceed as above after exchanging \( b \) and \( b' \). Next assume (iii) holds. By Lemma 7.10 the scalars \( \{ x_i \}_{i=0}^d, \{ y_i \}_{i=1}^d, \{ z_i \}_{i=1}^d \) are written as in Lemma 7.10. Setting \( z_1 = -cq^{d-1} \) in these expressions, we obtain (3)–(5).

Next consider the case \( a' = aq^{2d+2} \). By Lemma 7.12 at least one of (i)–(iii) holds in that lemma. First assume Lemma 7.12(i) holds. By Lemma 7.13 the scalars \( \{ x_i \}_{i=0}^d, \{ y_i \}_{i=1}^d, \{ z_i \}_{i=1}^d \) are written as in Lemma 7.13. Setting \( \xi = -aa'c - bb'c^{-1} \) in these expressions, and replacing \( b, b', c \) with \( (acq^{d+1}, a^{-1}bb'c^{-1}q^{-d-1}, a^{-1}bq^{-d-1}) \), we obtain (3)–(5). Next assume Lemma 7.12(ii) holds. Observe the expressions in Lemma 7.14 are obtained from the expressions in Lemma 7.13. Now proceed as above after exchanging \( b \) and \( b' \). Next assume Lemma 7.12(iii) holds. By Lemma 7.15 the scalars \( \{ x_i \}_{i=0}^d, \{ y_i \}_{i=1}^d, \{ z_i \}_{i=1}^d \) are written as in Lemma 7.15. Setting \( z_1 = -cq^{d-1} \) in these expressions, we obtain (3)–(5).

We have shown that there exist scalars \( \alpha, \alpha^*, a, a', b, b', c \) with \( c \neq 0 \) that satisfy (3)–(5). By the construction, (2) holds. By Proposition 1.7 these scalars satisfy the inequalities (6)–(8).

\[
(\{ \theta_i \}_{i=0}^d, \{ \theta_i^* \}_{i=0}^d, \{ \varphi_i \}_{i=1}^d, \{ \phi_i \}_{i=1}^d)
\]

10 Types of Leonard pairs

Let \( A, A^* \) be a Leonard pair on \( V \) with quantum parameter \( q \) that is not a root of unity. Let

\[
(\{ \theta_i \}_{i=0}^d, \{ \theta_i^* \}_{i=0}^d, \{ \varphi_i \}_{i=1}^d, \{ \phi_i \}_{i=1}^d)
\]
be a parameter array of $A, A^*$. By \[7, \text{Lemma 9.2}\] there exist scalars $\alpha, \alpha^*, a, a', b, b'$ such that

$$
\theta_i = \alpha + aq^{2i-d} + a'q^{d-2i} \quad (0 \leq i \leq d),
$$

$$
\theta_i^* = \alpha^* + bq^{2i-d} + b'q^{d-2i} \quad (0 \leq i \leq d).
$$

By \[6, \text{Lemma 13.1}\] there exists a scalar $\xi$ such that

$$
\varphi_i = (q^i - q^{-1})(q^{i-d-1} - q^{d-i+1})(\xi + abq^{2i-d-1} + a'b'q^{d-2i+1}) \quad (1 \leq i \leq d),
$$

$$
\phi_i = (q^i - q^{-1})(q^{i-d-1} - q^{d-i+1})(\xi + a'qb^{2i-d-1} + ab'q^{d-2i+1}) \quad (1 \leq i \leq d).
$$

By \[10, \text{Section 35}\], according to the values of $a, a', b, b', \xi$, the Leonard pair $A, A^*$ has one of the following types:

| $a$ | $a'$ | $b$ | $b'$ | $\xi$ | Name          |
|-----|------|-----|------|------|---------------|
| $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | any | $q$-Racah      |
| $0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | $q$-Hahn      |
| $\neq 0$ | $0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | dual $q$-Hahn |
| $\neq 0$ | $\neq 0$ | $\neq 0$ | $0$ | $\neq 0$ | quantum $q$-Krawtchouk |
| $0$ | $\neq 0$ | $\neq 0$ | $0$ | $\neq 0$ | $q$-Krawtchouk |
| $\neq 0$ | $0$ | $\neq 0$ | $\neq 0$ | $\neq 0$ | affine $q$-Krawtchouk |
| $\neq 0$ | $\neq 0$ | $0$ | $\neq 0$ | $\neq 0$ | dual $q$-Krawtchouk |
| $\neq 0$ | $\neq 0$ | $0$ | $\neq 0$ | $\neq 0$ | dual $q$-Krawtchouk |

By Theorem [11] $A, A^*$ has LB-TD form if and only if at least two of $aa', bb', \xi$ are nonzero. Therefore we obtain:

**Corollary 10.1** Let $A, A^*$ be a Leonard pair on $V$ with quantum parameter $q$ that is not a root of unity. Then the following (i) and (ii) are equivalent:

(i) $A, A^*$ has LB-TD form.

(ii) $A, A^*$ has one of the types: $q$-Racah, $q$-Hahn, dual $q$-Hahn.

11 Acknowledgments

The author thanks the referee for many insightful comments that lead to great improvements in the paper.
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Keywords. Leonard pair, tridiagonal pair, Askey-Wilson relation, orthogonal polynomial

2010 Mathematics Subject Classification. 05E35, 05E30, 33C45, 33D45