HOCHSCHILD DIMENSION OF TILTING OBJECTS

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Abstract. We give a new upper bound for the generation time of a tilting object and use it to verify, in some new cases, a conjecture of Orlov on the dimension of the derived category of coherent sheaves on a smooth variety.

1. Introduction

In [Rou08], R. Rouquier introduced a notion of dimension for triangulated categories. Roughly, this is the infimum over all generators of the minimal number of triangles it takes to build the category from a generator.

Under some mild hypotheses on $X$, Rouquier showed that the dimension of $D^b_{coh}(X)$ is finite, bounded below by the dimension of the variety, and, for a smooth variety, bounded above by twice the dimension of the variety [Rou08].

The following conjecture is due to D. Orlov [Orl08]:

Conjecture 1. Let $X$ be a smooth variety. The dimension of $D^b_{coh}(X)$ equals the dimension of $X$.

In [Rou08], Rouquier showed that Conjecture 1 is true for affine varieties, flag varieties (of type A), and quadrics. Recently, Orlov proved that this conjecture is true for curves [Orl08].

In this paper, we will be interested in the case when $X$ is a smooth variety whose derived category of coherent sheaves possesses a tilting object, $T$. We give a new upper bound on the number of cones needed to build all of $D^b_{coh}(X)$ from $T$.

Theorem 1. Let $i_0$ be the largest $i$ for which $\text{Hom}_X(T, T \otimes \omega_X^i)$ is nonzero. The Hochschild dimension of $\text{End}_X(T)$ is equal to $\dim(X) + i_0$. If $i_0$ is zero, then the Hochschild dimension of $\text{End}_X(T)$, the dimension of $D^b_{coh}(X)$, and the dimension of $X$ are all equal.

Applying Theorem 1 to examples of varieties (and stacks) known to possess tilting objects, we are able to enlarge the set of varieties for which Conjecture 1 is true. Below we list a handful of examples.

Corollary 1. Conjecture 1 holds for:

- del Pezzo surfaces with Picard number no more than seven;
- Fano threefolds of types $V_5$ and $V_{22}$;
- toric surfaces with nef anti-canonical divisor;
- toric Deligne-Mumford stacks of dimension no more than two or Picard number no more than two;
- and Hirzebruch surfaces.

The case of Hirzebruch surfaces is of particular interest. Using Theorem 1, we show that it takes three cones for any tilting bundle to generate the derived category. However, there is an essentially surjective functor from a weighted projective stack to the Hirzebruch surface. Pulling back the tilting bundle from the weighted projective stack gives a generator with generation time two.

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2. Preliminaries

In this section, we introduce some of the necessary background and gather the results which will be of importance to us later on. We will denote an arbitrary field by $k$ and a variety will refer to a separated and reduced scheme of finite type over $k$. On a smooth variety, $\omega_X$ represents the canonical bundle on $X$ and $K$ represents the corresponding divisor.

2.1. Dimension of a triangulated category. Let $\mathcal{T}$ be a triangulated category. For a full subcategory, $\mathcal{I}$, of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the full subcategory of $\mathcal{T}$ whose objects are isomorphic to summands of finite direct sums of shifts of objects in $\mathcal{I}$. In other words, $\langle \mathcal{I} \rangle$ is the smallest full subcategory containing $\mathcal{I}$ which is closed under isomorphisms, shifting, and taking finite direct sums and summands. For two full subcategories, $\mathcal{I}_1$ and $\mathcal{I}_2$, we denote by $\mathcal{I}_1 \ast \mathcal{I}_2$ the full subcategory of objects, $B$, such that there is a distinguished triangle, $B_1 \to B \to B_2 \to B_1[1]$, with $B_i \in \mathcal{I}_i$. Set $\mathcal{I}_1 \ast \mathcal{I}_2 := \langle \mathcal{I}_1 \ast \mathcal{I}_2 \rangle$, $\langle \mathcal{I} \rangle_0 := \langle \mathcal{I} \rangle$, and inductively define,

$$\langle \mathcal{I} \rangle_n := \mathcal{I}_n \ast \mathcal{I}_{n-1} \ast \mathcal{I}. \quad \text{Similarly we define,}$$

$$\langle \mathcal{I} \rangle_\infty := \bigcup_{n \geq 0} \langle \mathcal{I} \rangle_n.$$  

The reader is warned that, in the previous literature, $\langle \mathcal{I} \rangle_0 := 0$ and $\langle \mathcal{I} \rangle_1 := \langle \mathcal{I} \rangle$. With our convention, the index equals the number of cones allowed.

We will also require a slight variation which allows for infinite direct sums. Let $\overline{\mathcal{I}}$ denote the smallest full subcategory of $\mathcal{T}$ closed under isomorphisms, shifts, direct summands, and all direct sums.

**Definition 2.1.** Let $E$ be an object of a triangulated category $\mathcal{T}$. If there is an $n$ with $\langle E \rangle_n = T$, we set,

$$\Theta(E) := \min \{ n \geq 0 \mid \langle E \rangle_n = T \}.$$  

Otherwise, we set $\Theta(E) := \infty$. We call $\Theta(E)$ the generation time of $E$. If $\langle E \rangle_\infty$ equals $T$, we say that $E$ is a generator. If $\Theta(E)$ is finite, we say that $E$ is a strong generator. The dimension of $\mathcal{T}$, denoted $\dim \mathcal{T}$, is the minimal generation time amongst strong generators. It is set to $\infty$ if there are no strong generators.

Let $F : \mathcal{T} \to \mathcal{R}$ be an exact functor between triangulated categories. If every object in $\mathcal{R}$ is isomorphic to a direct summand of an object in the image of $F$, we say that $F$ is dense, or has dense image. The following lemmas are good exercises:

**Lemma 2.2.** If $F : \mathcal{T} \to \mathcal{R}$ has dense image, then $\dim \mathcal{T} \leq \dim \mathcal{R}$.

**Lemma 2.3.** If $\mathcal{T}$ is a finite dimensional triangulated category, then any generator is a strong generator.

**Lemma 2.4.** Let $G$ be an object of $\mathcal{T}$. If $B \in \langle G \rangle_n$, then $F(B) \in \langle F(G) \rangle_n$. Moreover, if $F$ commutes with coproducts and $B \in \langle G \rangle_n$, then $F(B) \in \langle F(G) \rangle_n$.

Let $k$ be a field and $A$ be a $k$-algebra. We will consider the following derived categories associated to $A$: $D(\text{Mod-}A)$, the derived category of unbounded complexes of right $A$-modules; $D^b(\text{mod-}A)$, the derived category of bounded complexes of coherent right $A$-modules; and $D_{\text{perf}}(A)$, the perfect derived category of right $A$-modules. Recall that $D_{\text{perf}}(A)$ is the smallest thick triangulated subcategory generated by the free module $A$ in $D(\text{Mod-}A)$, i.e. $D_{\text{perf}}(A) \equiv \langle A \rangle_\infty$.

In algebraic and geometric situations, the dimension of a triangulated category is related to common homological invariants, e.g. the global dimension and the Hochschild dimension of an algebra $A$. For the convenience of the reader, we now recall the definition of the Hochschild dimension of an algebra.

**Definition 2.5.** Let $A$ be a $k$-algebra. The **Hochschild dimension** of $A$, denoted $\text{hd}(A)$, is the projective dimension of $A$ as an $A \otimes_k A^{\text{op}}$-module.

To compress notation, we set $A^e := A \otimes_k A^{\text{op}}$. The categories of left or right $A^e$-modules are equivalent to the category of $A$-bimodules. The vector space $A \otimes_k A$ has many $A^e$-module structures. We shall consider it as a $A^e$-module via the outer bimodule structure, i.e. left multiplication on the first copy of $A$ and right multiplication on the second copy of $A$. With this bimodule structure, $A \otimes_k A$ and $A^e$ are isomorphic as left $A^e$-modules. Similarly, $A$ is always taken to have the natural bimodule structure given by left multiplication.
on the left and right multiplication on the right. If $A$ is a perfect $A^e$-module, the Hochschild dimension of $A$ can be understood as follows:

**Lemma 2.6.** Assume $A$ is a perfect $A^e$-module. The Hochschild dimension of $A$ is equal to the minimal $m$ for which $A \in (A^e)_m$ in $D(A^e\text{-Mod})$.

**Proof.** Since $A$ is a perfect $A^e$-module, $A$ lies in $(A^e)_n$ for some $n$. The Ghost Lemma, see Lemma 4.11 of [Rou08], implies that $\text{Ext}_A^n(A, \bullet)$ vanishes on $D(A^e\text{-Mod})$ when $l \geq n+1$. Thus, $\text{hd}(A) \leq n$. If the Hochschild dimension of $A$ is equal to $n$, then $A \in (A^e)_n$. Proposition 2.2.4 of [BV03] implies that $A \in (A^e)_m$. □

**Lemma 2.7.** The generation time of $A$, in $D_{\text{per}}(A)$, is bounded above by the Hochschild dimension of $A$.

**Proof.** The statement is vacuous if $\text{hd}(A) = \infty$ so we assume that $\text{hd}(A)$ is finite. Thus, one has $A \in (A^e)_m$ for some $m$. Taking $M \in D_{\text{per}}(A)$, and applying the exact functor, $M \otimes_A \bullet$, we get

$$M \cong M \otimes_A A \in (M \otimes_k A)_m = (A)_m.$$  

Applying A. Bondal and M. Van den Bergh’s Proposition 2.2.4, we conclude that $M$ lies in $(A)_m$. Thus, the generation time of $A$ is less than $m$. □

We will use the following lemma to compute the Hochschild dimension:

**Lemma 2.8.** Assume that $A$ is a perfect $A^e$-module. The Hochschild dimension of $A$ is the maximal $i$ for which $\text{Ext}_A^n(A, A^e)$ is nonzero.

**Proof.** We have seen that $A$ has finite Hochschild dimension. Take a resolution of $A$ by projective $A^e$-modules:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0.$$  

Let $i_0$ be the maximal $i$ so that $\text{Ext}_A^n(A, A^e)$ is nonzero. It is clear that $i_0$ must be less than or equal to $n$. If $i_0$ is strictly less than $n$, then $\text{Ext}_A^n(A, P)$ is zero for any projective module $P$. Thus, the map $P_n \to P_{n-1}$ must split allowing us to shorten the projective resolution. □

H. Krause and D. Kussin, using a construction due to J. D. Christensen, prove the following (see [Ch98, KK06]):

**Theorem 2.9.** Let $A$ be a right-coherent $k$-algebra and view it as an object of $D^b(\text{mod-}A)$. The generation time of $A$ is the global dimension of $A$.

In a special case of importance to us, we have equality of global and Hochschild dimensions. A proof of the following lemma can be found in [Rou08]:

**Lemma 2.10.** If $A$ is a finite-dimensional algebra over a perfect field $k$, then the Hochschild dimension of $A$ equals the global dimension of $A$.

Consequently, if $A$ is a finite-dimensional algebra over a perfect field, the Hochschild dimension, the global dimension, and the generation time of $A$ are equal.

For a variety (or an algebraic stack), we propose the following definition which is a weaker analogue of the Hochschild dimension:

**Definition 2.11.** Let $X$ be a variety. The **diagonal dimension** of $X$, denoted $\text{dim}_\Delta(X)$, is the minimal $n$ such that the diagonal, $O_{\Delta X}$, is in $(G \boxtimes H)_n$, for some $G \boxtimes H \in D^b_{\text{coh}}(X \times X)$. It is set to $\infty$ if no such $n$ exists.

The diagonal dimension has the following nice properties, the proofs of which, for the most part, are embedded in the next section:

**Lemma 2.12.** Let $X$ be a variety. One has:

1. $\text{dim}_\Delta(X \times Y) \leq \text{dim}_\Delta(X) + \text{dim}_\Delta(Y)$;
2. if $X$ is proper, then $\text{dim}_{D^b_{\text{coh}}}(X) \leq \text{dim}_\Delta(X)$;
3. if $X$ is smooth, then $\text{dim}_\Delta(X) \leq 2\dim X$.

Throughout this paper we obtain upper bounds on $\text{dim}_{D^b_{\text{coh}}}(X)$ by bounding $\text{dim}_\Delta(X)$, but, for the most part, we will simply state this bound either on $\text{dim}_{D^b_{\text{coh}}}(X)$ or on the generation time of the object being considered.
2.2. Dimension for Deligne-Mumford stacks. While stacks are not essential to the main arguments in this paper, they may provide a useful means for proving Conjecture [1] see subsection [2]. It is also natural to generalize Theorem [3] to stacks to obtain a greater class of examples. Consequently, in this subsection, we extend some of the basic results on dimension to smooth and tame Deligne-Mumford stacks with quasi-projective coarse moduli spaces. All stacks are of finite-type over $k$.

Lemma 2.13. Let $\mathcal{X}$ be a tame Deligne-Mumford stack. The dimension of $\mathcal{D}_{\text{coh}}(\mathcal{X})$ is at least the dimension of $\mathcal{X}$.

Proof. There is an open and dense substack, $[U/G]$, of $\mathcal{X}$ with $U = \text{Spec } A$ affine of dimension $n := \text{dim } \mathcal{X}$ and $G$ a finite group acting on $U$ (see [Kre08]). We have an essentially surjective localization map $\mathcal{D}_{\text{qcoh}}(\mathcal{X}) \to D_{\text{qcoh}}([U/G])$. Since we can extend coherent sheaves from $[U/G]$ to $\mathcal{X}$, the induced map $\mathcal{D}_{\text{coh}}(\mathcal{X}) \to \mathcal{D}_{\text{coh}}([U/G])$ is essentially surjective. Consider the forgetful functor $\mathcal{D}_{\text{qcoh}}([U/G]) \to \mathcal{D}_{\text{coh}}(U)$. The adjoint functor can be described algebraically as follows: take the inclusion of $A$ into the skew group algebra, $A \to A * G$, since $G$ is a group, $A * G$ is a free $A$-module of rank $|G|$, the functor, $\bullet \otimes_A (A * G)$, is exact and thus immediately provides a functor $\mathcal{D}_{\text{qcoh}}(U) \to \mathcal{D}_{\text{coh}}([U/G])$. Composing with the forgetful functor, we get $\text{Ind}_{\mathcal{D}_{\text{coh}}(U)}$. Thus, the image of $\mathcal{D}_{\text{qcoh}}([U/G])$ is dense in $\mathcal{D}_{\text{coh}}(U)$. Each of these maps preserves boundedness and coherence. Therefore, the dimension of $\mathcal{D}_{\text{coh}}(\mathcal{X})$ is greater than the dimension of $\mathcal{D}_{\text{coh}}(U)$, which is greater than $n$ by Theorem 7.17 of [Rou08].

To get an upper bound on the dimension of $\mathcal{D}_{\text{coh}}(\mathcal{X})$, in terms of the dimension of $\mathcal{X}$, we further restrict our class of stacks.

Definition 2.14. Let $\mathcal{X}$ be a Deligne-Mumford stack with a coarse moduli space $\pi : \mathcal{X} \to X$. A locally-free coherent sheaf, $\mathcal{E}$, on $\mathcal{X}$ is called a generating sheaf if for any quasi-coherent sheaf, $\mathcal{F}$, on $\mathcal{X}$, the natural morphism

$$\pi^*(\pi_*\text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})) \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F}$$

is surjective.

Following [EHKV01, OS03, Kre09], we can give a useful construction of a generating sheaf. Assume that $\mathcal{X}$ is isomorphic to a global quotient stack, i.e. $\mathcal{X} \cong [Y/G]$ where $Y$ is a scheme and $G$ is a subgroup of $GL_n$ acting on $Y$. Take a $G$-representation $W$ which has an open subset $U$ where $G$ acts freely. At every geometric point of the vector bundle $[(Y \times W)/G]$, the stabilizer’s action is faithful. Denote the associated locally-free coherent sheaf by $\mathcal{E}$. Then, $\mathcal{E}^\otimes r$ is a generating sheaf for $r$ large.

This explicit construction of a generating sheaf lets us make a useful observation: since all the above constructions respect products, there is a generating sheaf on $\mathcal{X} \times \mathcal{X}$ which is an exterior product. Recall that an exterior product, $\mathcal{F} \boxtimes \mathcal{G}$, of sheaves, $\mathcal{F}$ and $\mathcal{G}$, is called K"unneth-type. We can combine this observation with another from [Kre09]. Assume $[Y/G]$ has a quasi-projective coarse moduli space and let $\mathcal{L}$ be an ample line bundle on it. For any quasi-coherent sheaf, $\mathcal{F}$, on $\mathcal{X}$, there exists an $n_0$ so that the map,

$$\text{Hom}_X(\mathcal{E}^\otimes r \otimes_{\mathcal{O}_X} \pi^*(\mathcal{L}(-n)), \mathcal{F}) \otimes_k (\mathcal{E}^\otimes r \otimes_{\mathcal{O}_X} \pi^*(\mathcal{L}(-n))) \to \mathcal{F}$$

is surjective for $n \geq n_0$. In particular, on $\mathcal{X} \times \mathcal{X}$, we can use $\mathcal{E}^\otimes r \boxtimes \mathcal{E}^\otimes r$ for our generating sheaf and $\mathcal{L} \boxtimes \mathcal{L}$ as the ample line bundle on the coarse moduli space. Thus, for any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X} \times \mathcal{X}$, there is a locally-free K"unneth-type sheaf surjecting onto $\mathcal{F}$. This will be useful in the next lemma.

Lemma 2.15. Let $\mathcal{X}$ be a smooth and tame Deligne-Mumford stack with quasi-projective coarse moduli space. Then the dimension of $\mathcal{D}_{\text{coh}}(\mathcal{X})$ is bounded by twice the dimension of $\mathcal{X}$.

Proof. By Theorem 4.4 of [Kre09], $\mathcal{X}$ is automatically a global quotient stack. Take the structure sheaf of the diagonal, $\mathcal{O}_{\Delta \mathcal{X}}$, and resolve it by finite rank locally-free K"unneth-type sheaves:

$$\cdots \to \mathcal{H}_m \boxtimes \mathcal{G}_m \to \cdots \to \mathcal{H}_0 \boxtimes \mathcal{G}_0 \to \mathcal{O}_{\Delta \mathcal{X}} \to 0.$$
Take a bounded complex of coherent sheaves, \( F \), on \( \mathcal{X} \). Applying \( \Phi_\bullet(F) := R p_{2*}(\bullet \otimes_{\mathcal{O}_{\mathcal{X}}} L p^*_1(F)) \) to the previous statement, we get

\[
F \cong \Phi_{\mathcal{O}_{\Delta \mathcal{X}}}(F) \in \left( \bigoplus_{i=0}^{2n} H^i(\mathcal{X}, \mathcal{H}_i \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}) \otimes_k \mathcal{G}_i \right) \cong \left( \bigoplus_{i=0}^{2n} \mathcal{G}_i \right)_{2n}.
\]

Applying Bondal and Van den Bergh’s Proposition 2.2.4, we see that \( F \) lies in \( \langle \bigoplus_{i=0}^{2n} \mathcal{G}_i \rangle_{2n} \).

2.3. Tilting objects and Serre functors.

**Definition 2.16.** Let \( T \) be a \( k \)-linear triangulated category. An object, \( T \), of \( T \) is called a **tilting object** if the following two conditions hold:

1. \( \text{Hom}_T(T, T[i]) = 0 \) for all \( i \neq 0 \);
2. \( T \) is a generator for \( T \).

Our tilting objects will mainly reside in the bounded derived category of coherent sheaves on a smooth variety, \( X \).

**Proposition 2.17.** Let \( T \) be a tilting object in \( D^b_{\text{coh}}(X) \), where \( X \) is smooth, and set \( A := \text{End}_X(T) \). Then the functors \( R \text{Hom}_X(T, \bullet) \) and \( \bullet \otimes_A T \) define exact equivalences between \( D^b_{\text{coh}}(X) \) and \( D_{\text{perf}}(A) \).

**Proof.** Consider the functor \( \Psi := \bullet \otimes_A T \). We have \( \Psi(A) = A \otimes_A T \cong T \) and this identification induces an isomorphism, \( A \cong \text{Hom}_A(A, A) \cong \text{Hom}_X(\Psi(A), \Psi(A)) \cong \text{Hom}_X(T, T) = A \). Therefore, \( \Psi \) is full and faithful on the object \( A \). Since \( \Psi \) commutes with shifts, taking direct sums, taking direct summands, and sends triangles to triangles, \( \Psi \) is full and faithful on \( (A)_\infty \). Now, as \( \Psi \) is full, faithful, and exact, the essential image of \( \Psi \) is triangulated. Since \( \Psi(A) \cong T \), \( \Psi \) essentially surjects onto the smallest thick triangulated subcategory of \( D^b_{\text{coh}}(X) \) containing \( T \) which by assumption is all of \( D^b_{\text{coh}}(X) \). \( \square \)

**Definition 2.18.** A \( k \)-linear exact autoequivalence, \( S \), of \( T \), is called a **Serre functor** if for any pair of objects, \( X \) and \( Y \) of \( T \), there exists an isomorphism of vector spaces,

\[
\text{Hom}_T(Y, X)^* \cong \text{Hom}_T(X, S(Y)),
\]

which is natural in \( X \) and \( Y \).

A Serre functor, if it exists, is determined uniquely up to natural isomorphism. If \( F : T \rightarrow S \) is an exact equivalence of triangulated categories possessing Serre functors, then \( F \) commutes with those Serre functors [Huy05].

3. Generation time for tilting objects

**Theorem 3.1.** Suppose \( X \) is a smooth variety and \( T \) is a tilting object in \( D^b_{\text{coh}}(X) \). Let \( i_0 \) be the largest \( i \) for which \( \text{Hom}_X(T, T \otimes \omega_X^i[i]) \) is nonzero. The Hochschild dimension of \( \text{End}_X(T) \) is equal to \( \dim(X) + i_0 \).

Consequently, the generation time of \( T \) is bounded above by \( \dim(X) + i_0 \). If \( X \) is proper over a perfect field, then the generation time of \( T \) is equal to \( \dim(X) + i_0 \).

**Proof.** Write \( A \) as shorthand for \( \text{End}_X(T) \) and set \( T^\vee = R \text{Hom}_{\mathcal{O}_X}(T, \mathcal{O}_X) \). Then \( A^\vee \) is isomorphic to \( \text{End}_{\mathcal{X} \times \mathcal{X}}(T \boxtimes T^\vee) \). By Proposition 2.17, this yields an equivalence of categories between \( D^b_{\text{coh}}(X \times X) \) and \( D_{\text{perf}}(A^\vee) \) under which \( \mathcal{O}_\Delta \) corresponds to \( A \) with its natural bimodule structure. Consider the object \( T \boxtimes T^\vee \) in \( D^b_{\text{coh}}(X \times X) \). From \( T \boxtimes T^\vee \), we can get all Künneth-type sheaves. Furthermore, since \( X \) is smooth, it possesses an ample family of line bundles. We can resolve any bounded complex of coherent sheaves on \( X \times X \) using exterior products of line bundles in this ample family. Hence \( T \boxtimes T^\vee \) is a generator. Since \( D^b_{\text{coh}}(X \times X) \) has finite dimension, Lemma 2.3 says that \( T \boxtimes T^\vee \) is a strong generator. Consequently, \( A \) must lie in \( (A^\vee)_d \) for some \( d \). We have isomorphisms:

\[
\text{Hom}_{A^\vee}(A, A^\vee[i]) \cong \text{Hom}_{X \times X}(\mathcal{O}_\Delta, T \boxtimes T^\vee[i])
\cong \text{Hom}_X(\mathcal{O}_X, \Delta^!(T \boxtimes T^\vee)[i])
\cong \text{Hom}_X(T, T \otimes \omega_X^i[i - \dim(X)]).
\]

Applying Lemma 2.8 yields the upper bound. If \( X \) is proper over a perfect field, Theorem 2.10 and Lemma 2.10 imply that the generation time is equal to \( \dim(X) + i_0 \). \( \square \)
Corollary 3.2. Let $X$ be a smooth variety and $T$ be a tilting object in $D^b_{coh}(X)$. If $\text{Hom}_X(T, T \otimes \omega_X^{-i}[j])$ is zero for $i$ positive, then the generation time of $T$ is equal to the dimension of $X$ and Conjecture 1 holds for $X$.

Corollary 3.3. Let $X$ be a smooth variety and $T$ a tilting sheaf in $D^b_{coh}(X)$. The generation time of $T$ is bounded above by $2 \dim X$.

The same proof works for our class of stacks.

Theorem 3.4. Let $\mathcal{X}$ be a smooth and tame Deligne-Mumford stack with a projective coarse moduli space. Suppose that $T$ is a tilting object in $D^b_{coh}(\mathcal{X})$ and $i_0$ is the largest $i$ for which $\text{Hom}_X(T, T \otimes \omega_X^{-i}[j])$ is nonzero. The Hochschild dimension of $A := \text{End}_X(T)$ is equal to $\dim(X) + i_0$. Consequently, the generation time of $T$ is bounded above by $\dim \mathcal{X} + i_0$. If $\mathcal{X}$ is proper and $k$ is perfect, then the generation time of $T$ is equal to $\dim \mathcal{X} + i_0$.

Proof. The proof of Proposition 3.4 tells us that we have enough objects of Künneth-type so $T \boxtimes T^\vee$ generates $D^b_{coh}(\mathcal{X} \times \mathcal{X})$. It also implies that $D^b_{coh}(\mathcal{X})$ is compactly generated. By general results of Neeman [Nee96], $\Delta$, has a right adjoint, $\Delta^!$. As $\mathcal{X}$ is smooth, $\Delta$, takes bounded complexes of locally-free sheaves to bounded complexes of locally-free sheaves. The projection formula, plus some formal nonsense, tells us that $\Delta^!(\bullet) \cong \Delta^!(\bullet) \otimes_{\mathcal{O}_X} \omega_X^{[-\dim \mathcal{X}]}$. Therefore, we can proceed as in the proof of Theorem 3.1.

\hspace{1cm} \square

We also have a statement for more general triangulated categories. See [Kel94] for the definition of an algebraic triangulated category.

Proposition 3.5. Let $T$ be a $k$-linear algebraic triangulated category with finite dimensional morphism spaces. Assume that $T$ possesses a tilting object, $T$, and that $A := \text{End}_T(T)$ lies in $D_{\text{perf}}(A^e)$. Let $S$ be the Serre functor for $T$ and $j_0$ be the largest $j$ for which $\text{Hom}_T(T, S^{-j}(T)[j])$ is nonzero. The Hochschild dimension of $\text{End}_X(T)$ is equal to $j_0$.

Proof. From our assumption that $T$ is algebraic, there is an exact functor $R\text{Hom}_T(T, -) : T \to D_{\text{perf}}(A)$. The same argument as in Proposition 3.4 shows that $R\text{Hom}_T(T, -)$ is an equivalence. Since $T$ is has finite-dimensional morphism spaces, $A$ is a finite-dimensional algebra. By Theorem 7.26 of [Rou08], $A$ has finite global dimension and $D^b(\text{mod-A})$ is equivalent to $D_{\text{perf}}(A)$. By Proposition 20.5.5 of [Gin05], $D^b(\text{mod-A})$ has a Serre functor and its inverse is $\bullet \otimes_A R\text{Hom}_{A^e}(A, A^e)^e$. By naturality, $\text{Hom}_T(T, S^{-j}(T)[j]) \cong \text{Hom}_{A^e}(A, R\text{Hom}_{A^e}(A, A^e)^e)[j] \cong \text{Ext}^j_{A^e}(A, A^e)$.

Applying Lemma 3.5 again, we get the result.

Remark 3.6. We can relax the need for finite dimensional morphism spaces in Proposition 3.5 by using relative Serre functors, [Gin06].

Returning to the case of varieties, we have the following simple but useful observations:

Lemma 3.7. Let $X$ be a smooth variety of dimension $n$ such that the anti-canonical divisor is effective. Any tilting bundle, $T$, has generation time at most $2n - 1$.

Proof. Let $Y$ be a subscheme representing the anti-canonical class. Consider the exact sequence,

$$ 0 \to T \otimes T^\vee \to T \otimes T^\vee \otimes \omega_X^Y \to T \otimes T^\vee \otimes \omega_X^Y \otimes O_Y \to 0. $$

One knows that $T \otimes T^\vee$ has no higher cohomology by assumption and $T \otimes T^\vee \otimes \omega_X^Y \otimes O_Y$ has no cohomology in degree $n$ since it is supported in degree $n - 1$. Hence, $T \otimes T^\vee \otimes \omega_X^Y$ does not have cohomology in degree $n$.

\hspace{1cm} \square

Lemma 3.8. Let $X$ be a smooth proper variety over of dimension $n$ over a perfect field. Suppose that for some $i$, $H^i(X, \omega_X^Y)$ is nonzero. Then any tilting bundle $T$ (or more generally any tilting object which contains a vector bundle as a summand) has generation time at least $\dim(X) + i$.

Proof. By assumption $O_X$ is a summand of $T \otimes T^\vee$. Hence, $H^i(X, \omega_X^Y)$ is a summand of $\text{Hom}_X(T, T \otimes \omega_X^Y[i])$.

\hspace{1cm} \square
For the remainder of this section and the paper, we use our new bound on generation time of tilting objects to investigate Conjecture 1 in some examples. In the following subsections, we will assume that our base field has characteristic zero to both achieve sharper statements and assure all stacks encountered are tame. We leave the reader to formulate the appropriate statements when $k$ has characteristic $p$.

3.1. **Rational surfaces.** The following lemma is a useful computational aid:

**Lemma 3.9.** Let $X$ be a smooth proper surface such that the anti-canonical divisor is effective and the corresponding linear system contains a smooth connected curve, $C$. Let $D$ be a divisor satisfying: $H^i(X,\mathcal{O}(D)) = 0$ for $i > 0$. The line bundle $\mathcal{O}(D - K)$ has no higher cohomology if and only if $(K - D)|_C$ is non-trivial and $(K - D) \cdot K \geq 0$. If $T$ is a tilting object that is a direct sum of line bundles, then $T$ has generation time two if and only if $(K - D)|_C$ is non-trivial and $(K - D) \cdot K \geq 0$ for every summand, $\mathcal{O}(D)$, of $T \otimes T^\vee$.

**Proof.** Consider the following exact sequence,

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D - K) \to \mathcal{O}_C(D - K).$$

As $H^i(X,\mathcal{O}(D)) = 0$ for $i > 0$, one has $H^i(X,\mathcal{O}(D - K)) \cong H^i(C,\mathcal{O}_C(D - K))$ for $i > 0$. Since $C$ is a smooth curve of genus one, $H^i(C,\mathcal{O}_C(D - K)) = 0$ for $i > 0$ if and only if $(K - D)|_C$ is non-trivial and $(K - D) \cdot K \geq 0$.

Let $\mathcal{B}_t$ be any blow-up of $\mathbb{P}^2$ at any finite set of points and $\pi : \mathcal{B}_t \to \mathbb{P}^2$ be the projection (this is a slight abuse of notation as $\mathcal{B}_t$ depends on the set and not just the number of points). Consider the following vector bundles:

$$T_1 := \mathcal{O} \oplus \mathcal{O}(H) \oplus \mathcal{O}(2H) \oplus \mathcal{O}(E_1) \oplus \cdots \oplus \mathcal{O}(E_t),$$

$$T_2 := \mathcal{O} \oplus \mathcal{O}(H) \oplus \mathcal{O}(2H) \oplus \mathcal{O}_{E_1} \oplus \cdots \oplus \mathcal{O}_{E_t},$$

where $\mathcal{O}(H)$ is the pullback of the hyperplane bundle and $E_1, \ldots, E_t$ are the exceptional divisors.

**Proposition 3.10.** If $t \leq 2$ or $t = 3$ and the points are not collinear, then the generation time of $T_1$ is two, whereas if $t > 3$ or $t = 3$ and the points are collinear, then the generation time of $T_1$ is three. The generation time of $T_2$ is 3 for all $\mathcal{B}_t$. Moreover, any tilting bundle on $\mathcal{B}_t$ for $t > 10$ has generation time at least three.

**Proof.** We leave the proof that $T_1$ and $T_2$ are tilting as an exercise for the interested reader, see [KO94].

Any line bundle summand of $T_1 \otimes T_1^\vee \otimes \omega_{\mathcal{B}_t}^\vee$ can be expressed as $\mathcal{O}(nH + \sum b_i E_i)$ with $n \geq 1$. As $-nH - \sum b_i E_i$ is not effective for $n \geq 1$, $T_1$ must have generation time at most three.

Consider the cohomology of $\mathcal{O}(H - E_1 - \cdots - E_t)$. The self-intersection of this divisor is $-t + 1$. The intersection with the canonical divisor is $t - 3$. Thus, by Riemann-Roch, $\chi(\mathcal{O}(H - E_1 - \cdots - E_t))$ is negative and $\text{Ext}_{\mathcal{B}_t}(\mathcal{O}(2H), \mathcal{O}(3H - E_1 - \cdots - E_t))$ is nonzero unless $t \leq 3$. Hence $T_1$ has generation time three when $t > 3$. In the case, $t = 3$, the Euler characteristic of $\mathcal{O}(H - E_1 - E_2 - E_3)$ is zero and $\mathcal{O}(H - E_1 - E_2 - E_3)$ has a section if and only if the points are collinear. Hence, $T_1$ has generation time three when the points are collinear.

Now for $t \leq 3$, write $T = L_1 \oplus \cdots \oplus L_{t+3}$ and let $\mathcal{O}(D_{ij}) = L_i \otimes L_j$. Then $(D_{ij} - K) \cdot K \geq 0$ with equality if and only if $D_{ij} = -2H$ and $t = 3$. We already saw that $\mathcal{O}(H - E_1 - E_2 - E_3)$ has no higher cohomology when the points are not collinear. By Lemma 3.9, $T_1$ has generation time two when $t \leq 2$ or $t = 3$ and the points are not collinear.

Now we consider $T_2$. Some of the Ext-groups we need to compute were covered in the argument for $T_1$. The new ones are $\text{Ext}_{\mathcal{B}_t}^2(\mathcal{O}_{E_i} \otimes \omega_{\mathcal{B}_t} \otimes \mathcal{O}(mH))$, $\text{Ext}_{\mathcal{B}_t}^1(\mathcal{O}(mH), \mathcal{O}_{E_i} \otimes \omega_{\mathcal{B}_t}^\vee)$, and $\text{Ext}_{\mathcal{B}_t}^1(\mathcal{O}_{E_i} \otimes \omega_{\mathcal{B}_t}^\vee \otimes \mathcal{O}(mH))$. The cohomology group $\text{Ext}_{\mathcal{B}_t}^2(\mathcal{O}_{E_i} \otimes \omega_{\mathcal{B}_t} \otimes \mathcal{O}(mH))$ is isomorphic to $\text{Ext}_{\mathcal{B}_t}^2(\mathcal{O}_{E_i}, \mathcal{O}(-E_i))$, which is nonzero for $i = 1$. Thus, the generation time of $T_2$ is at least three for any $t$. Apply $\text{Hom}_{\mathcal{B}_t}(-, \mathcal{O}(-E_i))$ to the short exact sequence

$$0 \to \mathcal{O}(-E_i) \to \mathcal{O} \to \mathcal{O}_{E_i} \to 0.$$ 

Since $\mathcal{O}$ and $\mathcal{O}(-E_i)$ have no higher cohomology, $\text{Ext}_{\mathcal{B}_t}^2(\mathcal{O}_{E_i}, \mathcal{O}(-E_i))$ is zero. In addition, the cohomology group $\text{Ext}_{\mathcal{B}_t}^1(\mathcal{O}(mH), \mathcal{O}_{E_i} \otimes \omega_{\mathcal{B}_t}^\vee)$ is isomorphic to $\text{Ext}_{\mathcal{B}_t}^1(\mathcal{O}, \mathcal{O}_{E_i}(1))$ which is nonzero for positive $i$. One also has an isomorphism between $\text{Ext}_{\mathcal{B}_t}^1(\mathcal{O}_{E_i} \otimes \mathcal{O}(\omega_{\mathcal{B}_t}^\vee))$ and $\text{Ext}_{\mathcal{B}_t}^1(\mathcal{O}_{E_i}(-1), \mathcal{O}_{E_i})$. Apply $\text{Hom}_{\mathcal{B}_t}(-, \mathcal{O}_{E_i})$ to the short exact sequence,

$$0 \to \mathcal{O} \to \mathcal{O}(E_i) \to \mathcal{O}_{E_i}(-1) \to 0.$$
As \( \mathcal{O}_{E_t} \) and \( \mathcal{O}_{E_t}(1) \) have no higher cohomology, \( \text{Ext}^2_{\mathcal{E}}(\mathcal{O}_{E_t}(-1), \mathcal{O}_{E_t}) \) is zero. Thus, by Theorem 3.1, the generation time of \( T_2 \) is three for all \( t \).

For the final statement, note that the Euler characteristic of the anti-canonical divisor is \( 10 - t \). Thus, for \( t > 10 \), \( \omega_{X_t} \) has nontrivial cohomology in degree one. Applying Lemma 3.8, we see that the generation time must be at least three. \( \square \)

**Remark 3.11.** From \([KO94]\), these two exceptional collections are related by mutation. Thus, generation time is not invariant under mutation.

In \([HP08]\), L. Hille and M. Perling systematically studied the question of when rational surfaces admit full strong exceptional collections consisting of line bundles. We recall one of their definitions:

**Definition 3.12.** Let \( E_0, \ldots, E_n \) be an exceptional collection on a smooth variety, \( X \). We say that the collection is **strongly cyclic** if \( E_s, \ldots, E_n, E_0 \otimes \omega_X^s, \ldots, E_{s-1} \otimes \omega_X^s \) is a strong exceptional collection for any \( s \). Equivalently, one requires that

\[ \text{Ext}^l_X(E_j, E_i \otimes \omega_X^s) = 0 \text{ for } l > 0 \text{ and } i < j. \]

One of the main theorems of \([HP08]\) is the following:

**Theorem 3.13.** Let \( X \) be a smooth proper rational surface. If \( X \) possesses a full strongly cyclic exceptional collection consisting of line bundles, then \( \text{rk} \, \text{Pic}(X) \leq 7 \). If \( X \) is a del Pezzo surface with \( \text{rk} \, \text{Pic}(X) \leq 7 \), then \( X \) admits a strongly cyclic exceptional collection consisting of line bundles.

**Corollary 3.14.** Let \( X \) be a smooth proper rational surface possessing a strong exceptional collection consisting of line bundles with generation time two, then \( \text{rk} \, \text{Pic}(X) \leq 7 \).

Hille and Perling give explicit strongly exceptional collections for any del Pezzo surface with Picard rank at most seven. For a Picard rank seven del Pezzo, we have

\[
\mathcal{O}, \mathcal{O}(E_2), \mathcal{O}(E_1), \mathcal{O}(H - E_3 - E_4), \mathcal{O}(H - E_5), \mathcal{O}(H - E_1),
\mathcal{O}(2H - E_3 - E_4 - E_5 - E_6), \mathcal{O}(2H - E_3 - E_4 - E_5), \mathcal{O}(2H - E_3 - E_4 - E_5).
\]

Let \( T_3 \) be the sum of the above line bundles.

**Proposition 3.15.** \( T_3 \) has generation time two.

**Proof.** By Bertini’s theorem, there exists a smooth curve representing \( -K \) so we can apply Lemma 3.9. After adding \( -K \), the intersection of the differences, of the line bundles comprising \( T_3 \), with \( -K \) is positive except for \( \mathcal{O}(H - E_1 - E_2 - E_3) \) and \( \mathcal{O}(H - E_1 - E_2 - E_6) \). These restrict to the trivial bundle on an anti-canonical curve of genus one if and only if they have sections. However, our points are in general position so neither bundle has a section. \( \square \)

**Corollary 3.16.** Conjecture \([1]\) holds for del Pezzo surfaces with \( \text{rk} \, \text{Pic}(X) \leq 7 \).

**Proof.** The above Proposition implies that Conjecture \([1]\) holds for blow-ups of \( \mathbb{P}^2 \) at six points in general position. Any other del Pezzo surface with \( \text{rk} \, \text{Pic}(X) \leq 7 \) can be obtained as a blow-down. Suppose \( X \to Y \) is a blow-down. Since \( \mathbf{R} \pi_* \mathcal{O}_X \cong \mathcal{O}_Y \), the projection formula yields: \( \mathbf{R} \pi_* \circ \mathbf{L} \pi^* (B) \cong B \), for any \( B \in \text{D}^b_{\text{coh}}(Y) \). In particular, \( \mathbf{R} \pi_* \) is a dense functor so we may apply Lemma 2.2. \( \square \)

### 3.2. Pullback tilting objects.

**Proposition 3.17.** Suppose \( X \) is a smooth Calabi-Yau variety possessing a tilting object, \( T \). Then the generation time of \( T \) is equal to the dimension of \( X \). In particular, Conjecture \([1]\) holds for \( X \).

**Proof.** This follows immediately from Theorem 3.1. \( \square \)

**Definition 3.18.** Let \( X \) be a smooth variety and \( \pi : \text{Tot}(\omega_X) \to X \) be the projection. We say that tilting bundle, \( T \), (or an exceptional collection) is **pullback** if \( \pi^* T \) is tilting.

A tilting object \( T \) is pullback if and only if,

\[ \text{Hom}_X(T, T \otimes \omega_X^p[l]) = 0 \text{ for } l \neq 0 \text{ and } p \leq 0. \]
Notice that, if $T$ is pullback, then it satisfies the conditions of Theorem 3.1 with $i_0 = 0$. Thus, the generation time of $T$ equals the dimension of $X$. Also, notice that the total space of $\omega_X$ is Calabi-Yau, hence by Proposition 3.14, the generation time of $\pi^* T$ equals $\dim(X) + 1$. The following type of pullback bundle makes quite a few appearances in the literature (see [BP93, Bri05] for instance):

**Definition 3.19.** Let $X$ be smooth variety such that the Grothendieck group, $K_0(X)$, is finitely generated of rank $\dim X + 1$. A full strong pullback exceptional collection on such an $X$ is called a simple (also geometric) exceptional collection.

**Theorem 3.20.** The following varieties possess simple exceptional collections: projective spaces, odd-dimensional quadrics, and Fano threefolds of types $V_5$ and $V_{22}$.

The proof of this theorem is due to A. Beilinson [Be78], M. Kapranov [Kap86], Orlov [Or91], and A. Kuznetsov [Kuz96]. Applying Theorem 3.1 we get the following:

**Corollary 3.21.** Conjecture [φ] is true for for any variety possessing a simple exceptional collection, in particular for projective spaces, odd-dimensional quadrics, and Fano threefolds of types $V_5$ and $V_{22}$.

**Lemma 3.22.** Let $X$ be a smooth rational surface such that the anti-canonical divisor is effective and the corresponding linear system contains a smooth connected curve. If $T$ is a tilting object in $D^b_{\text{coh}}(X)$ with generation time two which is a sum of line bundles, then $T$ is pullback.

**Proof.** Let $T = L_1 \oplus \cdots \oplus L_2$ and $D_{ij} := L_i \otimes L_j^\vee$. By Corollary 3.14, $K^2 \geq 3$. By Lemma 3.19, $(K - D_{ij}) \cdot K \geq 0$ for all $i, j$. Therefore $(nK - D_{ij}) \cdot K \geq 3(n - 1)$ for all $i, j$. Applying Lemma 3.9 one obtains, $H^k(D_{ij} - nK) = 0$ for $k > 0$, $n \geq 2$ and all $i, j$.

**Corollary 3.23.** The tilting objects of generation time two in section 3.1 are pullback.

**Remark 3.24.** The condition

$$\Ext^l_X(E_i, E_j \otimes \omega_X) = 0 \quad \text{for all } i, j \text{ and } l > 0$$

for an exceptional collection $E_0, \ldots, E_n$ can be viewed as first order approximation to being pullback. This condition, as noted previously, is stronger than being cyclic. However, all cyclic exceptional collections in this paper are, in fact, pullback. It would be interesting to ascertain the precise relationship between the notion of cyclic, the above condition, and the notion of pullback.

When $X$ is Fano, and $T$ is a pullback tilting bundle, then Proposition 7.2 of [VdB04] states that $\End(\pi^* T)$ is a noncommutative crepant resolution of the anti-canonical ring. Hence all of the tilting bundles of generation time two produce noncommutative crepant resolutions. As it turns out, any noncommutative crepant resolution $A$ of a affine Gorenstein variety $S$ will have global dimension equal to the dimension of $S$ (see [SV06] Theorem 2.2). In this situation, once again, Theorem 3.1 is quickly verified.

### 3.3. Toric varieties

Smooth toric varieties are conjecturally a fecund ground for tilting bundles. A. King’s conjecture states that any smooth Fano toric variety possesses a full strong exceptional collection. In dimension two, the conjecture is true. In fact, a stronger statement is true thanks to further work of Hille and Perling in [HP08].

**Theorem 3.25.** Let $X$ be a smooth, proper toric surface. The variety $X$ possesses a strongly cyclic exceptional collection of line bundles if and only if the anti-canonical divisor is nef.

Consequently, if the anti-canonical divisor on a toric surface is not nef, we cannot have a strong exceptional collection of line bundles with generation time two. When the anti-canonical divisor is nef, Hille and Perling produce explicit strong cyclic exceptional collections.

We will not check that each of the exceptional collections produced by Hille and Perling have generation time two; we leave this as an exercise to the interested reader. We are mainly interested in Conjecture [φ] so we content ourselves with a slightly weaker statement:

**Proposition 3.26.** Conjecture [φ] holds for smooth and proper toric surfaces with nef anti-canonical divisor.

**Proof.** We discuss the Picard rank seven toric surface with nef anti-canonical divisor. All others are blow downs of this except two of the Picard rank six cases. The proof for these two cases follows along the same lines. The fan for the toric surface with Picard rank seven is:
We view this fan as an iterated blow-up of $\mathbb{P}^2$ and have labeled the one dimensional cones accordingly. First, we blow up the three torus invariant points of $\mathbb{P}^2$, then we blow up a single point of each of the three exceptional divisors in a cyclic manner. Precisely, the point on the first exceptional curve corresponds to the tangent direction pointing toward the first, the point on the third exceptional curve corresponds to the tangent direction pointing toward the second, and the point on the second exceptional curve corresponds to the tangent direction pointing toward the first. Here, we have used $E_1, E_2,$ and $E_3$ to denote the pullbacks of the exceptional divisors of the first round of blow-ups and $E_4, E_5,$ and $E_6$ to denote the infinitesimal blow-ups. The exceptional collection we wish to consider is

$$\mathcal{O}, \mathcal{O}(E_4), \mathcal{O}(E_2), \mathcal{O}(H - E_3 - E_5), \mathcal{O}(H - E_3), \mathcal{O}(H - E_5),$$

$$\mathcal{O}(2H - E_1 - E_3 - E_5 - E_6), \mathcal{O}(2H - E_1 - E_3 - E_5), \mathcal{O}(2H - E_3 - E_5 - E_6).$$

One can check that there is a smooth and connected divisor in the anti-canonical class. Thus, we can apply Lemma 3.9. After adding $-K$, all the differences of these line bundles have positive intersection with the anti-canonical divisor except $\mathcal{O}(H - E_1 - E_2 - E_4)$ and $\mathcal{O}(H - E_2 - E_4 - E_6)$, which give zero. The restriction of one of these divisors to an anti-canonical curve of genus one is trivial if and only if it has a section. Examining the configuration of the blow-ups on $\mathbb{P}^2$, we see that neither has sections.

\[ \square \]

**Remark 3.27.** For $m \geq 3$, the Hirzebruch surfaces $\mathbb{F}_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$, have non-nef anti-canonical divisor. By the previous corollary, they cannot possess a full strong exceptional collection of line bundles with generation time two. We will further see that, if $m \geq 4$, then it is also impossible for any tilting bundle to have generation time two. However, Conjecture still holds for $\mathbb{F}_m$, for any $m$, see Proposition 3.33.

For a larger class of examples, we move to toric stacks. Motivated by King’s conjecture, L. Borisov and Z. Hua construct full strong exceptional collections of line bundles for all toric Fano Deligne-Mumford stacks of Picard number at most two or dimension at most two in [BH09]. We now prove that the corresponding tilting bundles are pullback.

**Proposition 3.28.** Suppose that $\mathcal{X}$ is a toric Fano Deligne-Mumford stack of Picard number at most two or dimension at most two. Then, there exists a pullback tilting bundle (which is a sum of line bundles). In particular, Conjecture still holds for $\mathcal{X}$.

**Proof.** The setup is as follows: $S$ is a finite set of line bundles and $T := \bigoplus_{\mathcal{L} \in S} \mathcal{L}$. The terminology, notation, and results cited below can be found in [BH09]. We have $\omega_{\mathbb{F}_S}^\vee = \mathcal{O}(E_1 + \ldots + E_n)$.

**Case 1:** $T$ is the generator appearing in Borisov and Hua’s Proposition 5.1. For any two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in S$, we have $\deg(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee) > \deg(K)$. Hence $\deg(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee \otimes \omega_{\mathcal{X}}^{-n}) = \deg(\mathcal{L}_2 \otimes \mathcal{L}_1^\vee) + n \cdot \deg(-K) > \deg(K)$. Hence, it is acyclic by their Proposition 4.5.

**Case 2:** $T$ is the generator appearing in their Theorem 5.11. By their Proposition 5.7, there are three forbidden cones corresponding to the subsets $0, I_+$ and $I_-$ of $\{1, \ldots, n\}$. For any two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in S$
let $\mathcal{L} = \mathcal{L}_2 \otimes \mathcal{L}'_2$. Since $\mathcal{L}$ is not in the forbidden cone corresponding to the empty set neither is $\mathcal{L} \otimes \omega_{\mathbb{P}^n}^{-n}$. Furthermore $|\alpha(\mathcal{L} \otimes \omega_{\mathbb{P}^n}^{-n})| = |\alpha(\mathcal{L})| \leq \frac{1}{2} \sum_{i \in I_+} \alpha_i$. Hence, as in the proof of their Proposition 5.8, $\mathcal{L}$ does not lie in the forbidden cones $I_+$ and $I_-$. 

**Case 3:** $T$ is the generator appearing in their Theorem 7.3. For any two line bundles $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{S}$ let $\mathcal{L} = \mathcal{L}_2 \otimes \mathcal{L}'_2$. Suppose $\mathcal{L} \cong \mathcal{O}(\sum_{i=1}^n x_i E_i)$. As in the proof of their Proposition 7.2, $\sum r_i x_i > -1$. Hence, $\mathcal{L} \otimes \omega_{\mathbb{P}^n}^{-n} \cong \mathcal{O}(\sum_{i=1}^n (x_i + n) E_i)$ and $\sum r_i (x_i + n) = \sum r_i x_i + n \sum r_i = \sum r_i x_i + n > -1$. Therefore $\mathcal{L} \otimes \omega_{\mathbb{P}^n}^{-n}$ is not in the forbidden cone corresponding to the empty set. Now let $\pi : \text{Pic}_{\mathbb{P}^n} \to \hat{\text{Pic}}_{\mathbb{P}^n}$ be the projection and further let $S$ be the set of all other forbidden cones. As in the proof of their Proposition 7.2, $\pi(L) \notin \pi(S)$. Since $\pi(L \otimes \omega_{\mathbb{P}^n}^{-n}) = \pi(L)$, one has $\pi(L \otimes \omega_{\mathbb{P}^n}^{-n}) = \pi(L) \notin \pi(S)$. Hence $L \notin S$. 

**3.4. Weighted projective spaces and projective bundles.** Let $X_{m,n} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-m))$ for $m \geq 0$. Let $\pi : X_{m,n} \to \mathbb{P}^n$ be the projection and $H$ the pullback of the hyperplane section to $X_{m,n}$. Let $S$ denote the class of the zero section of the total space of $\mathcal{O}_{\mathbb{P}^n}$ under the natural inclusion $\mathcal{O}_{\mathbb{P}^n} \to X_{m,n}$, i.e. the relative twisting bundle $\mathcal{O}_{X_{m,n}}(1) = \mathcal{O}(S)$. Consider the object, 

$T := \mathcal{O} \oplus \mathcal{O}(H) \oplus \cdots \oplus \mathcal{O}(nH) \oplus \mathcal{O}(S + mH) \oplus \mathcal{O}(S + (m + 1)H) \oplus \cdots \oplus \mathcal{O}(S + (m + n)H)$.

**Proposition 3.29.** The object $T$ is a tilting generator. If $m < n + 2$, then the generation time of $T$ is $n + 1$, and, if $m \geq n + 2$, then the generation time of $T$ is $2n + 1$. Furthermore, when $m \geq 2n + 2$, any tilting bundle on $X_{m,n}$ has generation time equal to $2n + 1$.

**Proof.** From a more general result of Orlov, [Orl92], $T$ is a generator. One can check that the indecomposable summands of $T$ comprise a strong exceptional collection by using the computations below.

First we check that the canonical bundle on $X_{m,n}$ is $\mathcal{O}(-2S - (n + 1 + m)H)$. The Picard group of $X_{m,n}$ is isomorphic to $\mathbb{Z}^2$ with a basis $S$ and $H$ so the canonical divisor is $aS + bH$ for some $a$ and $b$. The divisor $H$ is isomorphic to $X_{m,n-1}$. Restricting $S$ to $H$ gives $S$ and restricting $H$ gives $H$ (allowing for the abuse of notation). Applying adjunction, we have $\mathcal{O}(aS + (b + 1)H) \cong \omega_{X_{m,n-1}}$. Recall that the canonical bundle of the Hirzebruch surface, $\mathbb{F}_n$, is $\mathcal{O}(-2S - (2 + m)H)$. Proceeding by induction, we get $a = -2$ and $b = -n - m - 1$.

The space $\text{Ext}^i_{X_{m,n}}(T, \mathcal{O}(\mathcal{O}(aS + bH)))$ is a direct sum of the cohomology groups $H^i(X_{m,n}, \mathcal{O}(aS + bH))$ where either $a = 1$ and $1 \leq b \leq 2n + 1$, $a = 2$ and $m + 1 \leq b \leq 2n + m + 1$, or $a = 3$ and $2m + 1 \leq b \leq 2m + 2n + 1$. Since $\pi_*$ has no higher direct images when applied to these line bundles, 

$H^i(X_{m,n}, \mathcal{O}(aS + bH)) \cong H^i(\mathbb{P}^n, \pi_* \mathcal{O}(aS + bH))$

$\cong H^i(\mathbb{P}^n, \text{Sym}^n(\mathcal{O} \oplus \mathcal{O}(-m)) \otimes \mathcal{O}(b))$

$\cong \bigoplus_{j=0}^a H^i(\mathbb{P}^n, \mathcal{O}(-jm + b))$.

We will first get nonzero cohomology when either $j = a = -3, b = 2m + 1$ or $j = a = -2, b = m + 1$, and $-am + b \leq -n - 1$, i.e. when $m \geq n + 2$. After we pass this threshold, we will have a nonzero Ext-group of degree $n$. From our calculation, we see there are no nontrivial Ext-groups of degree $n + 1$. So, if $m \leq n + 1$, $T$ has generation time $n + 1$, and, if $m \geq n + 2$, $T$ has generation time $2n + 1$.

In addition, 

$H^i(X_{m,n}, \mathcal{O}(2S + (n + 1 + m)H)) \cong H^i(\mathbb{P}^n, \mathcal{O}(n + 1 + m)) \oplus H^i(\mathbb{P}^n, \mathcal{O}(n + 1)) \oplus H^i(\mathbb{P}^n, \mathcal{O}(n + 1 - m))$.

Since we have a nonzero section of the anti-canonical bundle for any $m$, the generation time must be at most $2n + 1$ by Lemma 3.7. When $m \geq 2n + 2$, we get nonzero cohomology in degree $n$. If $T$ is a tilting object in $\mathbb{D}_{\text{coh}}(X_{m,n})$ with $\mathcal{O}$ a summand of $T \otimes T^\vee$, $T$ must have generation time at least $2n + 1$ by Lemma 3.8. 

Despite the above proposition, the dimension of $\mathbb{D}_{\text{coh}}(X_{m,n})$ is $n + 1$. The dimension is achieved by a generator which is not tilting. Let us denote stacky weighted projective space by $\mathbb{P}(a_0, \ldots, a_n)$. The category of coherent sheaves on this space is described in [AKO08]. The following lemma is inspired by [AKO08]:

**Lemma 3.30.** For $m > n$, $\mathbb{D}_{\text{coh}}(X_{m,n})$ is an admissible subcategory of $\mathbb{D}_{\text{coh}}(\mathbb{P}(1, \ldots, 1, m))$. 


Proof. \( \mathbb{P}(1, \ldots, 1, m) \) has as a strong full exceptional collection consisting of the line bundles \( \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(m+n) \). The following quiver (with relations expressing commutativity) describes the endomorphism algebra of the collection in the case of \( \mathbb{P}(1, 1, 4) \):

![Quiver diagram](image)

The degrees of \( x_0 \) and \( x_1 \) are one and the degree of \( x_2 \) is four. Let \( m > n \). Consider the strong exceptional collection formed by the line bundles \( \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n), \mathcal{O}(m), \mathcal{O}(m+1), \ldots, \mathcal{O}(m+n) \). The quiver associated to this exceptional collection is exactly the quiver for the exceptional collection on \( \mathbb{P}(1, 1, 4) \) given above. In the case of \( \mathbb{P}(1, 1, 4) \), we take \( \mathcal{O}, \mathcal{O}(1), \mathcal{O}(4), \mathcal{O}(5) \) and get the following quiver:

![Quiver diagram](image)

which is the quiver (with relations) for \( \mathbb{F}_4 \).

Let \( E = \sum_{i=0}^{n} \mathcal{O}(i) \oplus \mathcal{O}(m+i) \). Set \( A := \text{Hom}_{\mathbb{P}(1, \ldots, 1, m)}(E, E) \).

\[
\text{RHom}_{\mathbb{P}(1, \ldots, 1, m)}(E, -) : D^b_{\text{coh}}(\mathbb{P}(1, \ldots, 1, m)) \rightarrow D^b_{\text{coh}}(\mathbb{P}(m,n))
\]

is an exact and essentially surjective functor. The left adjoint to \( \text{RHom}_{\mathbb{P}(1, \ldots, 1, m)}(E, -) \) is \( - \otimes_A E \). Furthermore, \( - \otimes_A E \) is full and faithful. Thus, the smallest triangulated category closed under direct summands and containing \( E \) is equivalent to \( D^b_{\text{coh}}(\mathbb{P}(m,n)) \). Since both categories possess Serre functors, \( - \otimes_A E \) also possesses a left adjoint, and \( D^b_{\text{coh}}(\mathbb{P}(m,n)) \) is an admissible subcategory of \( D^b_{\text{coh}}(\mathbb{P}(1, \ldots, 1, m)) \).

\[\square\]

**Lemma 3.31.** The dimension of \( D^b_{\text{coh}}(\mathbb{P}(a_0, \ldots, a_n)) \) is \( n \).

**Proof.** \( \mathbb{P}(a_0, \ldots, a_n) \) is a toric Deligne-Mumford stack of Picard rank one so Propositions 3.28 applies. \[\square\]

**Remark 3.32.** The lemma above can also be realized in two other ways. Firstly, as a more direct application of Theorem 3.4. The relevant computations of cohomology can be found in Theorem 8.1 of [AZ94], see also the discussion in Section 2 of [AKO08]. Secondly, let \( \mu_r \) denote the group of \( r \)th roots of unity and consider the diagonal action of \( G := \mu_{a_0} \times \cdots \times \mu_{a_n} \) on \( \mathbb{P}^n \). One verifies that the terms of the Beilinson resolution have a natural \( \Delta G \)-equivariant structure such that the morphisms are \( \Delta G \) invariant, see [Kaw04]. Hence the category of \( G \)-equivariant sheaves on \( \mathbb{P}^n \), which is equivalent to \( D^b_{\text{coh}}(\mathbb{P}(a_0, \ldots, a_n)) \), has an \( n \)-step generator.

**Proposition 3.33.** Conjecture [4] holds for \( X_{m,n} \).

**Proof.** As noted in the proof of Lemma 3.30, \( \text{RHom}_{\mathbb{P}(1, \ldots, 1, m)}(E, -) \) is essentially surjective. Hence, by Lemma 3.24 the dimension of \( D^b_{\text{coh}}(X_{m,n}) \) is bounded above by the dimension of \( D^b_{\text{coh}}(\mathbb{P}(1, \ldots, 1, m)) \), which is \( n + 1 \) by Lemma 3.31. \[\square\]
Remark 3.34. If one considers noncommutative deformations of weighted projective space \( \mathbb{P}_\theta(a_0, \ldots, a_n) \) as in [AKO08], one can obtain the same upper bound, \( \dim D^b_{\text{coh}}(\mathbb{P}_\theta(a_0, \ldots, a_n)) \leq n + 1 \), using their Proposition 2.7. Similarly, for the corresponding noncommutative deformations of \( X_{m,n} \), we have \( \dim D^b_{\text{coh}}(X_{\theta,m,n}) \leq m + n + 1 \).

However, as these spaces are noncommutative, a good lower bound is unknown. Recent progress on lower bounds for dimension may be useful, see [BOO08, BIKO09, Opp09].

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