Hall Effect in Noncommutative Coordinates

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Abstract

We consider electrons in uniform external magnetic and electric fields which move on a plane whose coordinates are noncommuting. Spectrum and eigenfunctions of the related Hamiltonian are obtained. We derive the electric current whose expectation value gives the Hall effect in terms of an effective magnetic field. We present a receipt to find the action which can be utilized in path integrals for noncommuting coordinates. In terms of this action we calculate the related Aharonov–Bohm phase and show that it also yields the same effective magnetic field. When magnetic field is strong enough this phase becomes independent of magnetic field. Measurement of it may give some hints on spatial noncommutativity. The noncommutativity parameter $\theta$ can be tuned such that electrons moving in noncommutative coordinates are interpreted as either leading to the fractional quantum Hall effect or composite fermions in the usual coordinates.

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1 Introduction

To clarify the role which noncommutative coordinates may play in physics a better understanding of quantum mechanics in noncommutative spaces would be useful. Obviously, the simplest case is to consider particles moving in two dimensional noncommutative spaces. Actually, there exist some realistic physical systems like electrons in a uniform external magnetic field which are effectively moving in a two dimensional space which is perpendicular to magnetic field. These electrons are investigated in noncommuting coordinates and interesting phenomena like nonextensive statistics and orbital magnetism are resulted. We would like to consider electrons moving in two dimensional noncommutative space when both uniform external magnetic and electric fields are present. In the usual case this system leads to Hall effect. Indeed, we will show that in noncommuting coordinates one obtains Hall effect in terms of an effective magnetic field.

Once noncommutativity is imposed coordinates behave as operators. However, we can bring in noncommutativity by keeping coordinates as commuting but requiring that composition of their functions is given by star product. After canonical quantization is performed we deal with ordinary coordinates but replace ordinary product with star product. This procedure leads to an ordinary quantum mechanics problem in terms of an effective Hamiltonian depending on the noncommutativity parameter $\theta$. As far as operator description of quantum mechanics is concerned this procedure suits well. However, if one deals with path integrals the suitable action should be given in terms of $c$–number phase space variables. One of the possibilities is to find an effective action which leads to the Green functions which are calculated in terms of operators. We will adopt another method: The effective action which we use in path integrals is found by replacing derivatives appearing in Hamiltonian with $c$–number momentum variables. A similar approach is given in and a different one in . We will use the action obtained in this manner in path integrals to calculate the related Aharonov–Bohm phase after generalizing the action obtained in symmetric gauge to embrace the other vector potentials at the first order in $\theta$.

In Section 2 we recall how one can find energy eigenvalues and eigenfunctions of an electron moving on plane in uniform external magnetic and electric fields. This serves as a guide in
Section 3 when we deal with the same system in noncommuting coordinates. In Section 4 we present an approach to derive the electric current in noncommutative coordinates. Then, we calculate its expectation value utilizing eigenfunctions derived in Section 3, yielding the Hall effect in noncommuting coordinates. This can be envisaged as the usual Hall effect in terms of an effective magnetic field. Section 5 is devoted to calculate Aharonov–Bohm phase in noncommuting coordinates after presenting our receipt to obtain the action suitable to be used in path integrals. This phase is used to define an effective magnetic field in terms of commuting coordinates. We observe that effective magnetic fields obtained in Section 4 and in Section 5 are the same. In the last section by tuning the parameter \( \theta \) and utilizing the effective magnetic field we offer two different interpretations of electrons moving in noncommutative space as either leading to the fractional quantum Hall effect \([7]\) or composite fermions \([8]\) in the usual space. Moreover, we propose to measure the Aharonov–Bohm phase for large magnetic fields which may give some hints on the existence of spatial noncommutativity.

2 Electron Moving on Plane

An electron moving on the plane \((x, y)\) in the uniform external electric field \(\vec{E} = -\vec{\nabla}\phi\) and the uniform external magnetic field \(B\) which is perpendicular to the plane is described by the Hamiltonian

\[
H = \frac{1}{2m}(\vec{p} + \frac{e}{c}\vec{A})^2 - e\phi. \tag{1}
\]

We neglect the spin, because taking it into account does not affect our results.

Let us adopt the symmetric gauge

\[
\vec{A} = (-\frac{B}{2}y, \frac{B}{2}x). \tag{2}
\]

During the related experiments the electric field \(\vec{E}\) is taken in one of the two possible directions. Thus let the scalar potential be

\[
\phi = -Ex. \tag{3}
\]

Making use of (2) and (3) in (1) leads to the Hamiltonian function

\[
H(\vec{p}, \vec{r}) = \frac{1}{2m} \left[ \left( p_x - \frac{eB}{2c}y \right)^2 + \left( p_y + \frac{eB}{2c}x \right)^2 \right] + eEx. \tag{4}
\]
As usual canonical quantization of this system is achieved by introducing the coordinate and momentum operators \( \hat{r}_i, \hat{p}_i \) satisfying

\[
[\hat{r}_i, \hat{p}_j] = i\hbar \delta_{ij}
\]  

and dealing with the Hamiltonian operator \( \hat{H} \) obtained from (4) as \( \hat{H} = H(\hat{p}, \hat{r}) \).

To discuss the eigenvalue problem

\[ \hat{H} \Psi = E \Psi, \]  

it is convenient to perform the change of variables

\[ \hat{\mathbf{z}} = \hat{x} + i\hat{y}, \quad \hat{p}_z = \frac{1}{2}(\hat{p}_x - i\hat{p}_y). \]

and introduce two sets of creation and annihilation operators:

\[
\begin{align*}
    b^\dagger &= -2i\hat{p}_z + \frac{eB}{2c} \hat{\mathbf{z}} + \lambda, \\
    b &= 2i\hat{p}_z + \frac{eB}{2c} \hat{\mathbf{z}} + \lambda,
\end{align*}
\]

and

\[
\begin{align*}
    d &= 2i\hat{p}_z - \frac{eB}{2c} \hat{\mathbf{z}}, \\
    d^\dagger &= -2i\hat{p}_z - \frac{eB}{2c} \hat{\mathbf{z}},
\end{align*}
\]

where \( \lambda = \frac{mecE}{B} \). These two sets commute with each other and satisfy the commutation relations

\[
[b, b^\dagger] = 2m\hbar \omega, \quad [d^\dagger, d] = 2m\hbar \omega,
\]

where \( \omega = \frac{eB}{mc} \) is the cyclotron frequency. Now, the Hamiltonian \( \hat{H} \) can be written as

\[
\hat{H} = \frac{1}{4m}(b^\dagger b + bb^\dagger) - \frac{\lambda}{2m}(d^\dagger + d) - \frac{\lambda^2}{2m}.
\]

To calculate the eigenvalues \( E \) and the eigenfunctions \( \Psi \) we separate (11) into two mutually commuting parts:

\[ \hat{H} = \hat{H}_{osc} - \hat{T}, \]

where \( \hat{H}_{osc} \) denotes the harmonic oscillator part

\[
\hat{H}_{osc} = \frac{1}{4m}(b^\dagger b + bb^\dagger)
\]
and the part linear in $d$ and $d^\dagger$ is given by
\[
\hat{T} = \frac{\lambda}{2m} (d^\dagger + d) + \frac{\lambda^2}{2m}.
\] (12)

The harmonic oscillator eigenvalue equation $\hat{H}_{\text{osc}} \Phi_n = E_{\text{osc}}^n \Phi_n$ is easily solved:
\[
\Phi_n = \frac{1}{\sqrt{(2m\hbar\omega)^n n!}} (b^\dagger)^n |0>,
\]
\[
E_{\text{osc}}^n = \hbar \omega \left(2n + 1\right), \quad n = 0, 1, 2...
\] (13)
leading to a discrete spectrum. However, the eigenvalue equation $\hat{T} \phi = E \phi$ can be analyzed in terms of the eigenvalues of the operators $\hat{r}_i$ denoted by $r_i$ as
\[
\phi_\alpha = e^{i(\alpha y + \frac{\hbar \omega}{2m} xy)},
\]
\[
E_\alpha = \frac{\hbar \omega}{m} \alpha + \frac{\lambda^2}{2m}, \quad \alpha \in \mathbb{R},
\] (14)
yielding a continuous spectrum labeled by $\alpha$.

Therefore, the eigenfunctions and the energy spectrum of the Hamiltonian $\hat{H}$ are
\[
\Psi_{(n,\alpha)} = \Phi_n \otimes \phi_\alpha \equiv |n, \alpha>,
\]
\[
E_{(n,\alpha)} = \hbar \omega \left(2n + 1\right) - \frac{\hbar \lambda}{m} \alpha - \frac{\lambda^2}{2m}, \quad n = 0, 1, 2..., \quad \alpha \in \mathbb{R},
\] (15)
where $\otimes$ denotes the direct product.

## 3 Electron Moving on Noncommutative Plane

Let the coordinates of the plane be noncommuting:
\[
[x, y] = i\theta.
\] (16)
The parameter $\theta$ is a real constant. Noncommutativity can be imposed by treating the coordinates as commuting but requiring that composition of their functions is given in terms of the star product
\[
\star \equiv \exp \frac{i\theta}{2} \left( \partial_x \partial_y - \partial_y \partial_x \right).
\] (17)
Now, we deal with the commutative coordinates $x$ and $y$ but replace the ordinary products with the star product (17). For example, instead of the commutator (16) one defines
\[
x \star y - y \star x = i\theta.
\] (18)
We would like to study the Hamiltonian (4) in terms of the noncommutative coordinates (16). First we quantize this system by establishing the commutation relations (5). Then, the noncommutativity of the coordinates is taken into account by defining a new operator as

\[ \hat{H} \Psi(\vec{r}) \equiv \hat{H}_{nc} \Psi(\vec{r}). \]  

(19)

This definition yields the Hamiltonian operator\[ \hat{H}_{nc} = \frac{1}{2m} \left[ \left( (1 - \kappa)\hat{p}_x - \frac{eB}{2c} \hat{y} \right)^2 + \left( (1 - \kappa)\hat{p}_y + \frac{eB}{2c} \hat{x} \right)^2 \right] + eE(\hat{x} - \frac{\theta}{2\hbar}\hat{p}_y), \] when the coordinate representation of momentum \[ \hat{p}_i = -i\hbar \partial_i \] is used and \[ \kappa = \frac{e\theta B}{4\hbar c}. \]

The eigenvalue problem

\[ \hat{H}_{nc} \Psi^{nc} = E^{nc}\Psi^{nc} \]  

(21)

is as in ordinary quantum mechanics in spite of the fact that electron moves on noncommutative plane. The solutions of this problem can be worked out in a manner similar to the one used in the previous section although here there exists a term linear in momentum which was not present in \( \hat{H} \). Then, let us introduce two sets of operators

\[ \tilde{b}^\dagger = -2i\hat{p}_x + \frac{eB}{2c} \hat{z} + \lambda_-, \]

\[ \tilde{b} = 2i\hat{p}_x + \frac{eB}{2c} \hat{z} + \lambda_-, \]  

(22)

and

\[ \tilde{d} = 2i\hat{p}_x - \frac{eB}{2c} \hat{z}, \]

\[ \tilde{d}^\dagger = -2i\hat{p}_x - \frac{eB}{2c} \hat{z}, \]  

(23)

where \( \hat{p}_z = \gamma \hat{p}_z; \gamma = 1 - \kappa \). The real parameter \( \lambda_- \) will be fixed later. The sets of operators \( (\tilde{b}, \tilde{b}^\dagger) \) and \( (\tilde{d}, \tilde{d}^\dagger) \) commute with each other. Moreover, they satisfy the commutation relations

\[ [\tilde{b}, \tilde{b}^\dagger] = 2m\hbar\bar{\omega}, \quad [\tilde{d}^\dagger, \tilde{d}] = 2m\hbar\bar{\omega}, \]  

(24)

where \( \bar{\omega} = \gamma \omega \). The Hamiltonian \( \hat{H}_{nc} \) can be written as

\[ \hat{H}_{nc} = \frac{1}{4m} (\tilde{b}^\dagger \tilde{b} + \tilde{b} \tilde{b}^\dagger) - \frac{\lambda_+}{2m} (\tilde{d}^\dagger + \tilde{d}) - \frac{\lambda_-^2}{2m}, \]  

(25)

where the parameters \( \lambda_{\pm} \) are fixed to be

\[ \lambda_{\pm} = \lambda \pm \frac{emE\theta}{4\gamma\hbar} \]  

(26)
We take into account only the values of the noncommutativity parameter $\theta \neq 4\hbar/c/eB$. Otherwise, $\lambda_\pm$ diverge. We will give a brief discussion of this fact in the last section.

Observing the similarity between the Hamiltonians (10) and (25) the solutions of the eigenvalue problem (21) can be read from (15) as

$$
\Psi^{nc}_{(n,\alpha,\theta)} \equiv |n, \alpha, \theta > = \frac{1}{\sqrt{(2\pi\hbar c\omega)^n n!}} e^{i(ny + \frac{m\omega}{2\hbar}xy)}(\hat{b}^\dagger)^n |0>,
$$

$$
E^{nc}_{(n,\alpha,\theta)} = \frac{\hbar\omega}{2}(2n + 1) - \frac{\hbar\gamma\lambda_\pm}{m}\alpha - \frac{m\lambda_\pm^2}{2} n = 0, 1, 2..., \quad \alpha \in \mathbb{R}.
$$

We would like to emphasize that the results of the previous section are recovered if the noncommutativity parameter $\theta$ is switched off.

4 Hall Conductivity on Noncommutative Plane

We would like to find conductivity resulting from the Hamiltonian $\hat{H}_{nc}$. The first step in this direction is to define the related current. Although the identification of derivatives with momentum operators $\partial_i = i\hat{p}_i$ is only valid in coordinate representation, we will use this definition in defining the current operator $\hat{J}$ on noncommutative plane as

$$
\hat{J} = \frac{ie\rho}{\hbar}[\hat{H}_{nc}, \hat{r}] = \frac{e\gamma\rho}{m}(\gamma\hat{p} + \frac{e}{c}\hat{A} + \hat{a}),
$$

where $\hat{a} = (0, -\frac{meE\theta}{2\hbar c})$ and $\rho$ denotes electron density.

Now, the expectation value of the current operator $<\hat{J}>$ can be calculated with respect to the eigenstates $|n, \alpha, \theta >$ [27] leading to

$$
<\hat{J}_x > = 0,
$$

$$
<\hat{J}_y > = -\gamma\left(\frac{\rho e\gamma}{B}\right)E.
$$

Therefore, the Hall conductivity on noncommutative plane, denoted by $\sigma^{nc}_H$, is

$$
\sigma^{nc}_H = -\gamma\left(\frac{\rho e\gamma}{B}\right).
$$

Recall that in the ordinary case the Hall conductivity $\sigma_H$ and the filling factor $\nu$ are given as

$$
\sigma_H = \frac{e^2}{h}\nu, \quad \nu = \frac{\Phi_0\rho}{B},
$$

(31)
where $\Phi_0 = \hbar c/e$. Comparison of (30) with (10) suggests that one can interpret the noncommutative case as a theory of Hall effect on commuting plane with an effective magnetic field

$$B_{eff} = \frac{B}{1 - \frac{e\theta B}{4hc}}. \quad (32)$$

Moreover, the effective filling factor

$$\nu_{eff} = \frac{\Phi_0 \rho}{B(1 - \frac{e\theta B}{4hc})}, \quad (33)$$

can also be defined.

5 The Aharonov–Bohm Effect

We would like to calculate the Aharonov–Bohm effect on noncommutative plane by examining the action appearing in the related path integral. When we deal with quantum mechanics in the usual spaces it is the related classical action. However, it is not clear what should be the definition of action appropriate for path integrals when noncommutativity is taken into account. Because, we define Hamiltonian operators in terms of the receipt used in (19) where, we identify $\partial_i \equiv (i/\hbar)\hat{p}_i$. We propose to define the path integral in noncommutative space as

$$Z = \int d^2p \ d^2r \ e^{\frac{\hbar}{i} \int dt [\vec{p} \cdot \dot{\vec{r}} - H_{eff}(\vec{r}, \vec{p})]}, \quad (34)$$

where $(\vec{r}, \vec{p})$ define the commuting phase space and $H_{eff}(\vec{r}, \vec{p})$ will be obtained from the related Hamiltonian operator in noncommutative space by replacing the operators $\hat{p}$, $\hat{r}$ with c–number variables $\vec{p}$, $\vec{r}$.

Let us deal with the Hamiltonian operator on noncommutative plane in the constant external electric field $\vec{E} = (E_x, E_y)$ and the constant magnetic field $B$ in the symmetric gauge (2):

$$\hat{H}'_{nc} = \frac{1}{2m} \left[ \left( \gamma \hat{p}_x - \frac{eB}{2c} \hat{y} \right)^2 + \left( \gamma \hat{p}_y + \frac{eB}{2c} \hat{x} \right)^2 \right] + eE_x(\hat{x} - \frac{\theta}{2\hbar} \hat{y}) + eE_y(\hat{y} + \frac{\theta}{2\hbar} \hat{x}). \quad (35)$$

Although, the Hamiltonian operator

$$\hat{H}_\theta \equiv \frac{\gamma^2}{2m} \hat{\rho}^2 + \frac{e^2}{2mc^2} \vec{A}^2 + \frac{e\gamma}{2mc} (\hat{\rho} \cdot \vec{A} + \vec{A} \cdot \hat{\rho}) + \hat{\rho} \cdot \vec{K} + e\vec{E} \cdot \vec{r}, \quad (36)$$

where $\vec{K} = \frac{e\theta}{2\hbar}(-E_y, E_x)$, is equivalent to (33) only when the vector potential $\vec{A}$ is as given in (2), we assume that at least at the first order in $\theta$ it is valid for any gauge potential.
The c–number effective Hamiltonian corresponding to (36) is

\[ H_{\text{eff}} = \frac{\gamma^2}{2m} \vec{p}^2 + \frac{e^2}{2mc^2} \vec{A}^2 + \vec{p} \cdot \left( \frac{e\gamma}{mc} \vec{A} + \vec{K} \right) + e \vec{E} \cdot \vec{r}. \] (37)

Thus, the partition function can be written as

\[ Z_{nc} = N \int d^2 p \ d^2 r \ e^{\frac{i}{\hbar} \int_{t_1}^{t_2} dt \left[ \frac{m}{2\gamma^2} \dot{\vec{r}}^2 - \frac{m}{\gamma} \dot{\vec{r}} \left( \frac{e\gamma}{mc} \vec{A} + \vec{K} \right) + \frac{e}{c\gamma} \vec{A} \cdot \vec{K} \right]} , \] (38)

where \( N \) is a normalization constant. Now, we can integrate over the momenta \( \vec{p} \) to obtain

\[ Z_{nc} = \frac{2m}{\gamma^2} N \int d^2 r \ e^{\frac{i}{\hbar} \int_{t_1}^{t_2} dt \left[ \frac{m}{2\gamma^2} \dot{\vec{r}}^2 - \frac{e}{mc} \vec{E} \cdot \vec{r} \right]} , \] (39)

Because of being a constant \( K^2 \) term is irrelevant. In terms of a new normalization constant \( N' \) we can write

\[ Z_{nc} = N' \int d^2 r \ e^{\frac{i}{\hbar} \int_{t_1}^{t_2} dt \left[ \frac{m}{2\gamma^2} \dot{\vec{r}}^2 - \frac{e}{mc} \vec{E} \cdot \vec{r} \right]} , \] (40)

where we defined

\[ S_0 = \int_{t_1}^{t_2} dt \left[ \frac{m}{2\gamma^2} \dot{\vec{r}}^2 - e \vec{E} \cdot \vec{r} + \frac{e}{c\gamma} \vec{A} \cdot \vec{K} \right] . \] (41)

The last term in the exponent of (40) can be written as

\[ i\delta = -\frac{im}{\gamma^2 \hbar} \int_{\vec{r}(t_1)}^{\vec{r}(t_2)} d\vec{r} \cdot \left( \frac{e\gamma}{mc} \vec{A} + \vec{K} \right) . \] (42)

To investigate the Aharonov–Bohm effect for noncommuting coordinates, let \( \vec{A} = \vec{\nabla} f(\vec{r}) \). Then

\[ \delta = -\frac{m}{\gamma^2 \hbar} \int_{\vec{r}(t_1)}^{\vec{r}(t_2)} d\vec{r} \cdot \left( \frac{e\gamma}{mc} \vec{A} + \vec{K} \right) \] (43)

depends only on the points \( \vec{r}(t_1), \vec{r}(t_2) \) which are kept fixed in path integrals. Therefore, it is a phase factor. i.e. propagation with the action \( S_0 \) is changed up to the phase factor (43).

The Aharonov–Bohm effect can now be calculated as the integral of the phase factor (43) along a loop enclosing a magnetic flux. \( \vec{K} \) is a constant vector so that, it does not contribute to the Aharonov–Bohm phase:

\[ \oint d\vec{r} \cdot \vec{K} = 0. \] (44)

As in the ordinary case the unique contribution is due to the gauge potential

\[ \Phi^{nc}_{AB} = -2\pi \frac{BS}{\gamma \Phi_0} , \] (45)
where $S$ denotes the surface enclosed. Obviously, when $\theta = 0$ the usual Aharonov–Bohm phase results

$$\Phi_{AB} = -2\pi \frac{BS}{\Phi_0}. \tag{46}$$

Thus we can envisage the noncommutative case as a theory in commuting coordinates with an effective magnetic field

$$B_{eff} = \frac{B}{1 - \frac{e\theta B}{4\hbar c}}, \tag{47}$$

which is the one obtained previously (32).

6 Discussions

Electrons moving on a noncommutative plane when uniform external magnetic and electric fields are present can be envisaged as the usual motion of electrons experiencing an effective magnetic field (32). This is one of our main results. It followed by considering either Hall effect or Aharonov–Bohm phase in noncommuting coordinates. By tuning the value of the noncommutativity parameter $\theta$ we can offer two different interpretations of this fact:

The fractional quantum Hall effect is one of the most interesting features of low dimensional systems [7]. For electrons moving on a plane in a magnetic field which is perpendicular to the plane and a uniform external electric field which is in the plane, the observed Hall conductivity is

$$\sigma_H = f \frac{e^2}{h},$$

where $f = 1/3, 2/3, 1/5, \cdots$, denoting the fractional quantized values of the filling factor $\nu$. We would like to interpret this phenomena, which is known as the fractional quantum Hall effect, in terms of the Hall effect on noncommutative plane. More precisely we identify the effective filling factor (33) with the observed value $f$ by fixing the value of $\theta$ to be $\theta_H$:

$$\nu_{eff}|_{\theta=\theta_H} = f. \tag{48}$$

In fact, this can be solved as

$$\theta_H = \frac{2\Phi_0}{\pi B} \left(1 - f \frac{B}{\Phi_0 \rho}\right). \tag{49}$$

Therefore, when $\theta$ is fixed to be $\theta_H$ one can envisage the Hall effect on noncommutative plane as the usual fractional quantum Hall effect.
**Composite fermions** are new kind of particles appeared in condensed matter physics to provide an explanation of the behavior of electrons moving on plane when a strong magnetic field $B$ is present [8]. Electrons possessing $2p; p = 1, 2, \cdots$, flux quanta (vortices) can be thought of being composite fermions. One of the most important features of them is they feel effectively the magnetic field

$$B^* = B - 2p\Phi_0\rho,$$

where $\rho$ is the electron density. To interpret electrons moving on noncommutative space in the magnetic field $B$ as the usual composite fermions we should tune $\theta$ such that

$$B_{eff}|_{\theta=\theta_c} = B^*,$$

where $B_{eff}$ is given in (32). We solve this to obtain

$$\theta_c = \frac{2\Phi_0}{\pi B} \left[ 1 - (1 - 2p\rho \frac{\Phi_0}{B})^{-1} \right],$$

which in the limit of strong magnetic field leads to

$$\theta_c \approx 4p\frac{\rho}{\pi} \left( \frac{\Phi_0}{B} \right)^2.$$ 

Thus, composite fermions can be envisaged as electrons moving in noncommutative plane in the magnetic field $B$ and the electric field $E$ when we fix $\theta = \theta_c$.

The Aharonov–Bohm phase $\Phi_{AB}^{nc}$ possesses a very interesting limit. Let us deal with $\theta \neq 0$ and the magnetic field satisfying the condition

$$B \gg \frac{4ch}{e\theta}.$$ 

For these values of magnetic field the Aharonov–Bohm phase $\Phi_{AB}^{nc}$ (45) becomes

$$\Phi_{AB}^{nc} \approx \frac{4chS}{e\theta\Phi_0},$$

which is independent of $B$. For $E^2 = 0$ the action $S_0$ reads

$$\tilde{S}_0 \equiv S_0|_{\vec{E}=0} = \frac{m}{2\gamma^2} \int dt \vec{r}^2,$$

which depends on the magnetic field as $B^{-2}$ when (54) is satisfied. If the phase $\Phi_{AB}^{nc}$ can be measured for the particle propagating with the action $\tilde{S}_0$ when $B$ satisfies (54) and observed
that it is independent of \( B \) after a certain value of \( B \), it may be an evidence for spatial noncommutativity. Obviously, this conclusion is valid only for small values of \( \theta \). Because we assume it when we write the action (36). In [3] Aharonov–Bohm effect in noncommutative coordinates was studied in terms of a field theoretical approach where an experiment to detect spatial noncomutativity was proposed.

Critical value of \( \theta \) defined as

\[ \theta^* = \frac{4\hbar c}{eB} \]  

is avoided through this work. At this value of \( \theta \) all of our analysis fail, because \( \gamma = 1 - eB\theta/4\hbar c \) cannot be inverted, thus \( \lambda_\pm \) (26) are not well defined. Indeed, when \( \theta = \theta^* \) the Hamiltonian in noncommutative coordinates (20) becomes

\[ \hat{H}_{nc}(\theta = \theta^*) = \frac{m\omega^2}{8}(\hat{x}^2 + \hat{y}^2) + eE(\hat{x} - \frac{\theta^*}{2\hbar}\hat{y}). \]

i.e. the terms quadratic in momenta disappear, however a term linear in momenta survives. Obviously, this system should be studied separately.

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