A quaternionic generalisation of the Riccati differential equation

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Abstract
A quaternionic partial differential equation is shown to be a generalisation of the traditional Riccati equation and its relationship with the Schrödinger equation is established. Various approaches to the problem of finding particular solutions to this equation are explored, and the generalisations of two theorems of Euler on the Riccati equation, which correspond to this partial differential equation, are stated and proved.

1 Introduction
The Riccati equation
\[ \partial u = pu^2 + qu + r, \]
where \( p, q \) and \( r \) are functions, has received a great deal of attention since a particular version was first studied by Count Riccati in 1724, owing to both its peculiar properties and the wide range of applications in which it appears. For a survey of the history and classical results on this equation, see for example [13], [5] and [12]. This equation can be reduced to its canonical form [4],
\[ \partial y + y^2 = -v, \quad (1) \]
and this is the form that we will consider.
One of the reasons for which the Riccati equation has so many applications is that it is related to the general second order homogeneous differential equation. In particular, the one-dimensional Schrödinger equation

\[-\partial^2 u - vu = 0\]  

(2)

where \( v \) is a function, is related to the (1) by the easily inverted substitution \( y = \frac{\partial u}{u} \). This substitution, which as its most spectacular application reduces Burger’s equation to the standard one-dimensional heat equation, is the basis of the well-developed theory of logarithmic derivatives for the integration of nonlinear differential equations \([1]\). A generalisation of this substitution will be used in this work.

A second relation between the one-dimensional Schrödinger equation and the Riccati equation is as follows. The one-dimensional Schrödinger operator can be factorised in the form

\[-\partial^2 - v(x) = -(\partial + y(x))(\partial - y(x))\]

if and only if (1) holds.

Among the peculiar properties of the Riccati equation stand out two theorems of Euler, dating from 1760. The first of these \([13]\) states that if a particular solution \( y_0 \) of the Riccati equation is known, the substitution \( y = y_0 + z \) reduces (1) to a Bernoulli equation which in turn is reduced by the substitution \( z = \frac{1}{u} \) to a first order linear equation. Thus given a particular solution of the Riccati equation, the general solution can be found in two integrations. The second of these theorems \([12]\) states that given two particular solutions \( y_0, y_1 \) of the Riccati equation, the general solution can be found in the form

\[y = k y_0 \exp(\int y_0 - y_1) - y_1 k \exp(\int y_0 - y_1) - 1\]  

(3)

where \( k \) is a constant. That is, given two particular solutions of (1), the general solution can be found in one integration.

Other interesting properties are those discovered by Picard and Weyr \([13]\). The first is that given a third particular solution \( y_3 \), the general solution can be found without integrating. That is, an explicit combination of three particular solutions gives the general solution. The second is that given a fourth particular solutions \( y_4 \), the cross ratio

\[\frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_3 - y_2)}\]

is a constant.

This article is concerned with a quaternionic generalisation of the Riccati equation and versions of the above-mentioned theorems of Euler corresponding to this generalisation. Some necessary notation will be introduced in Section ???. In Section ?? we propose the quaternionic generalisation of the Riccati equation,
which is shown to be a good generalisation for various reasons, including that it is related to the three-dimensional Schrödinger equation

\[ \triangle u + vu = 0 \quad (4) \]

(where \( \triangle \) is the three-dimensional laplacian) in the same ways as the Riccati equation is related to (2). In Section 4, we turn our attention to cases in which particular solutions of (8) can be found, some of which differ considerably from the one-dimensional case. Finally, in Section 5, generalisations of Euler’s theorems will be stated and proved.

2 Preliminaries

The complex numbers and complex quaternions are denoted by \( \mathbb{C} \), \( \mathbb{H}(\mathbb{C}) \) respectively. The latter consists of elements of the form

\[ a = \sum_{k=0}^{3} a_k i_k \]

where the \( a_k \in \mathbb{C} \) and the base units \( i_k \) satisfy the following rules of multiplication

\[
\begin{align*}
i_0^2 &= i_0 = -i_k^2, & i_0i_k &= i_k, & k = 1, 2, 3, \\
i_1i_2 &= -i_2i_1 = i_3, & i_2i_3 &= -i_3i_2 = i_1, & i_3i_1 &= -i_1i_3 = i_2.
\end{align*}
\]

The complex unit \( i \) commutes with the \( i_k \). Frequently it is useful to consider a quaternion \( a \) as being the sum of a scalar and a vector part, denoted respectively

\[ a_0 := \text{Sc}(a), \quad \vec{a} := \text{Vec}(a) = \sum_{k=1}^{3} a_k i_k. \]

Conjugation is defined as follows,

\[ \overline{a} := a_0 - \vec{a} \]

and the modulus is

\[ |a|^2 = a \cdot \overline{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \]

so that in particular \( \overline{a}^2 = -|\overline{a}|^2 \).

Note that in terms of scalars and vectors, the quaternionic product can be written

\[ ab = (a_0 + \vec{a})(b_0 + \overline{b}) = a_0b_0 + a_0\overline{b} + b_0\vec{a} - \left( \vec{a} \cdot \overline{b} \right) + \left[ \vec{a} \times \overline{b} \right] \]
where \((a, b)\) is the standard inner product and \([a \times b]\) the standard cross product in \(\mathbb{R}^3\). In particular

\[
\{ \overrightarrow{a}, \overrightarrow{b} \} = -2 \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle
\]  

(5)

where \(\{a, b\} = ab + ba\) is the standard anticommutator.

In what follows functions \(g : \Omega \to \mathbb{H}(\mathbb{C})\) will be considered, where \(\Omega\) is some domain in \(\mathbb{R}^3\). The Moisil-Theodoresco operator \(D\) is defined on differentiable functions \(g\) as follows:

\[
Dg = \sum_{k=1}^{3} i_k \partial_k g
\]

where \(\partial_k = \frac{\partial}{\partial x_k}\). Due to properties of the quaternionic product, this can be written

\[
Dg = -\text{div} \overrightarrow{g} + \text{grad} g_0 + \text{rot} \overrightarrow{g}.
\]

Thus it follows that

\[
\text{Sc}(D \overrightarrow{g}) = -\text{div} \overrightarrow{g},
\]
\[
\text{Vec}(D \overrightarrow{g}) = \text{rot} \overrightarrow{g},
\]

and for scalar functions \(u\)

\[
Du = \text{grad} u.
\]

The theorem of Leibnitz for this operator is the following: given a differentiable scalar function \(u\) and a differentiable quaternionic function \(g\),

\[
D(ug) = D(u)g + uD(g).
\]

(6)

The logarithmic derivative (in the sense of Marchenko) of a scalar function \(u\) such that \(u \neq 0\) in \(\Omega\), is defined as

\[
\partial u = u^{-1} Du.
\]

The function \(\partial u\) is a vector. The derivative is logarithmic in the following sense: given two scalar functions \(u_1, u_2\) that do not vanish in \(\Omega\), formula (6) implies that

\[
\partial(u_1 u_2) = \partial u_1 + \partial u_2.
\]
3 A three-dimensional generalisation of the Riccati equation

The substitution of the Jackiw-Nohl-Rebbi-'t Hooft ansatz \[8\] in the self-duality equation can be written \[7\] in the following quaternionic form

\[ \partial_t g + Dg + |g|^2 = 0 \]  

where the subscript \( t \) denotes differentiation with respect to time. This equation has obvious formal similarities with equation (1). In \[9\] the relation of (7) to the F"uter operator

\[ \partial_t + D \]

was shown. In particular, it was shown that for any \( f \in \ker(\partial_t + D) \), the function

\[ 2(\text{grad } f_0 - \text{div } \vec{f}) \]

is a solution of (8), that is, a class of instantons was obtained. In what follows we will concentrate on the case of time independent, purely vectorial quaternionic functions, but the nonhomogeneous equation will be considered.

The following result generalises to three dimensions the relation, mentioned in the introduction, between the one-dimensional Schrödinger operator and the Riccati differential equation via the logarithmic derivative.

**Proposition 1** \( \varphi \) is a solution of (4) if and only if \( \vec{f} := \partial \varphi \) is a solution of

\[ D \vec{f} + \vec{f}^2 = v \]

**Proof.** Suppose that there exists a function \( \varphi \) such that \( \vec{f} = \partial \varphi \). Applying (6) gives

\[ D \vec{f} = \frac{1}{\varphi^2} (\nabla \varphi, \nabla \varphi) - \frac{1}{\varphi} \Delta \varphi, \]

\[ \vec{f}^2 = -\frac{1}{\varphi^2} (\nabla \varphi, \nabla \varphi), \]

so that \( -\frac{1}{\varphi^2} \Delta \varphi = v \), or equivalently \( \Delta \varphi + v \varphi = 0 \). Conversely, given a solution \( \varphi \) of (6), \( \vec{f} = \partial \varphi \) is a solution of (8). \( \blacksquare \)

In \[2\] and \[3\] it was shown that the three-dimensional Schrödinger operator can be factorised in the following way

\[ -\Delta - vI = (D + M \vec{f})(D - M \vec{f}) \]
where $I$ is the identity operator and $M^T q := q \overrightarrow{f}$, if and only if equation (8) holds.

Thus the two relations between the Riccati equation (1) and the one-dimensional Schrödinger equation (3) mentioned in the introduction have natural counterparts relating (8) and the three-dimensional Schrödinger equation (4). These relationships suggest that (8) can be considered a good generalisation of (1). It should also be noted that if $\overrightarrow{f} = f_k(x_k)i_k$, then (8) is reduced to

$$\partial_k f_k + f_k^2 = -v.$$ 

That is, a one-dimensional solution of (8) is a solution of (1). Equation (8) will be referred to as the Riccati PDE.

Note that the scalar and vector components of equation (8) are respectively

$$-\text{div} \overrightarrow{f} + \overrightarrow{f}^2 = v,$$

$$\text{rot} \overrightarrow{f} = 0.$$ 

The second equation implies that for a simply-connected domain $\Omega$, there exists a scalar function $\phi$ such that $\overrightarrow{f} = \text{grad} \phi$. Substituting this in the first equation gives

$$\Delta \phi + \langle \nabla \phi, \nabla \phi \rangle = -v. \quad (9)$$

This equivalence of the Riccati PDE with a scalar elliptic partial differential equation will be used frequently in what follows.

It should also be noted that if the function $v$ in (8) is zero, the substitution $\overrightarrow{f} = \partial \phi$ reduces the equation to

$$\Delta \phi = 0.$$ 

Thus the homogeneous Riccati equation can be solved explicitly, and its solutions are of the form $\partial \phi$, where $\phi \in \ker \Delta$. This generalises the highly restricted class of solutions of the homogeneous equation (1), which are precisely functions of the form

$$\frac{1}{x + c},$$

$c$ constant.

4 Particular solutions of the equation

In the next section it will be shown that given one solution of the Riccati PDE it can be linearised. Thus in this section we discuss some possibilities for
obtaining particular solutions of (8). First we note that if the function \(v\) is of the form

\[ v(x) = v_1(x_1) + v_2(x_2) + v_3(x_3), \]

then assuming that \(\tilde{f} = f_1(x_1)i_1 + f_2(x_2)i_2 + f_3(x_3)i_3\), equation (8) reduces to the system of Riccati ordinary differential equations

\[ \partial_k f_k + f_k^2 = -v_k, \quad k = 1, 2, 3. \]

Thus in this case, a particular solution of (8) can be found if and only if each of the above equations can be solved. Obviously, if any of the \(v_k\)'s are zero, this task is greatly simplified. This situation corresponds to the Schrödinger equation (4) with potential \(v\) of the form given above, in which case the variables can be separated.

The existence of a large class of solutions of the homogeneous equation, as described in Section 3, motivates the following procedure, which reduces (8) to various scalar differential equations. Substituting the sum of two functions \(\tilde{f}_1, \tilde{f}_2 \in C^1(\Omega)\) into equation (8) gives the following possible decomposition:

\[
\begin{align*}
D\tilde{f}_1 + \tilde{f}_1^2 &= v_1, \\
D\tilde{f}_2 + \tilde{f}_2^2 &= v_2, \\
\{\tilde{f}_1, \tilde{f}_2\} &= v_3,
\end{align*}
\]

\[ v = v_1 + v_2 + v_3. \]

In particular, if \(\tilde{f}_1, \tilde{f}_2\) are solutions of the homogeneous equation, this is reduced to

\[ \{\tilde{f}_1, \tilde{f}_2\} = -2\langle \hat{\partial}\varphi_1, \hat{\partial}\varphi_2 \rangle = v, \tag{10} \]

where \(\varphi_1, \varphi_2\) are harmonic functions.

If \(\varphi_1 = x_1, \tilde{f}_1 = \frac{i_1}{x_1}\), this becomes

\[ \partial_1 \varphi_2 = -\frac{1}{2}vx_1 \varphi_2 \]

which has solution

\[ \varphi_2 = A(x_2, x_3) \exp(-\frac{1}{2} \int vx_1 dx_1) \]

where \(A(x_2, x_3)\) is an arbitrary function. Thus if \(\varphi_2 \in \ker \Delta\) and \(\varphi_2\) satisfies the above equation, the sum

\[ \frac{i_1}{x_1} + \hat{\partial} \varphi_2 \]

is a particular solution of the Riccati PDE.
If instead of choosing $\varphi_1$ as above $\varphi_1 = \varphi_2$ is substituted in (10), the eikonal equation

$$2(\dot{\varphi}^2) = -v$$

(11)

results. Thus for a scalar function $\varphi \in \ker \triangle_3$, which is also a solution of the above eikonal equation, $\vec{f} = 2\dot{\varphi}$ is a solution of $\triangle_3$.

**Example 2** Let $\varphi$ be the fundamental solution of the laplacian $\triangle$,

$$\varphi = \frac{1}{4\pi|x|}.$$

This function is harmonic and positive in any domain $\Omega$ which does not include the origin. Furthermore it satisfies equation (11) with

$$v = \frac{1}{|x|^2}.$$

Thus

$$\vec{f} = \frac{-2\varphi}{|x|^2}$$

is a solution of $\triangle_3$.

5 Generalisations of Euler’s theorems on the Riccati equation

We now state and prove the generalisations to the Riccati PDE of Euler’s theorems on the Riccati equation that were mentioned in the introduction. The first of these theorems states that given a particular solution of the Riccati differential equation, the equation can be linearised.

**Proposition 3** (Generalisation of Euler’s first theorem) Let $\vec{h} = \text{grad} \xi$ be an arbitrary particular solution of $\triangle_3$. Then

$$\vec{f} = \vec{g} + \vec{h}$$

(12)

is also a solution of $\triangle_3$, where $\vec{g} = \dot{\Psi}$ and $\Psi$ is a solution of the equation

$$\triangle \Psi + 2 \langle \nabla \xi, \nabla \Psi \rangle = 0,$$

(13)

or equivalently of

$$\text{div}(e^{2\xi} \nabla \Psi) = 0.$$
Proof. Substituting (13) in (8) gives
\[ D \overrightarrow{y} + \{ \overrightarrow{h}, \overrightarrow{y} \} + \overrightarrow{y}^2 = 0 \]
or alternatively, using (5)
\[ D \overrightarrow{y} - 2 \overrightarrow{h} + \overrightarrow{y}^2 = 0. \tag{15} \]
Note that, as in (8), the vector part of (15) is \( \text{rot} \overrightarrow{y} = 0 \), so that
\[ \overrightarrow{y} = \text{grad} \Phi \]
for some function \( \Phi \). If \( \Psi = e^\Phi \), this is equivalent to
\[ \overrightarrow{y} = \hat{\partial} \Psi. \]
Equation (15), written in terms of \( \Psi \), is
\[ -\frac{1}{\Psi^2} (\nabla \Psi)^2 - \frac{1}{\Psi} \Delta \Psi - \frac{2}{\Psi} \langle \nabla \xi, \nabla \Psi \rangle + \frac{1}{\Psi^2} (\nabla \Psi)^2 = 0, \]
so that (15) is equivalent to
\[ \Delta \Psi + 2 \langle \nabla \xi, \nabla \Psi \rangle = 0. \]
Noting that
\[ \text{div}(e^{2\xi} \nabla \Psi) = 2e^{2\xi} \langle \nabla \xi, \nabla \Psi \rangle + e^{2\xi} \Delta \Psi = e^{2\xi}(\Delta \Psi + 2 \langle \nabla \xi, \nabla \Psi \rangle), \]
this equation can be rewritten in the form
\[ \text{div}(e^{2\xi} \nabla \Psi) = 0. \]

Equation (13) is the well-known transport equation, which appears for example in the Ray Method of approximations of solutions to the wave equation, coupled with the eikonal equation. Equation (14) appears in various applications, for example in electrostatics, where \( e^{2\xi} \) is the dielectric permeability and \( \Psi \) is the electric field potential, and as the continuity equation of hydromechanics in the case of a steady flow, where \( e^{2\xi} \) is the density of the medium.

Remark 4 From (14) we have that
\[ e^{2\xi} \nabla \Psi = \text{rot} \overrightarrow{s} \]
for some vector-valued function \( \overrightarrow{s} \), or
\[ \nabla \Psi = e^{-2\xi} \text{rot} \overrightarrow{s}, \]
where \( \overrightarrow{s} \) must satisfy the condition
\[ \text{rot}(e^{-2\xi} \text{rot} \overrightarrow{s}) = 0 \]
as \( e^{-2\xi} \text{rot} \overrightarrow{s} \) must be the gradient of some function.
As mentioned in the introduction, given two particular solutions $y_1$, $y_2$ of the Riccati ordinary differential equation the general solution can be found in one integration. A natural question is whether this property extends to the Riccati PDE. The form (3) of the general solution found in this case, suggests a similar substitution in (8). This line of reasoning gives the following result.

**Proposition 5 (Generalisation of Euler’s second theorem)** Let $\vec{h}_1 = \text{grad} \xi_1$ and $\vec{h}_2 = \text{grad} \xi_2$ be two particular solutions of (8). Then there exists a scalar function $\phi$ such that

$$\vec{f} = \nabla \phi = \frac{\vec{h}_1 w - \vec{h}_2}{w - 1}$$

is also a solution of (8), where $w = Ae^{\xi_1 - \xi_2}$, $A \in \mathbb{C}$.

**Proof.** Equation (8) is equivalent to

$$\Delta \phi + \langle \nabla \phi, \nabla \phi \rangle = -v$$

(16)

where $\vec{f} = \text{grad} \phi$. The scalar functions $\xi_1$ and $\xi_2$ are two solutions of (16). Substituting the expression

$$\frac{\nabla \xi_1 w - \nabla \xi_2}{w - 1}$$

(17)

for $\nabla \phi$ into this equation, where $w$ is a scalar function, gives

$$\Delta \phi = \text{div} \left( \frac{\nabla \xi_1 w - \nabla \xi_2}{w - 1} \right)$$

$$= (w - 1)(w \Delta \xi_1 + \nabla w \cdot \nabla \xi_1 - \Delta \xi_2) - \nabla w \cdot (w \nabla \xi_1 - \nabla \xi_2)$$

and

$$\langle \nabla \phi, \nabla \phi \rangle = \frac{(w \nabla \xi_1)^2 - 2w \nabla \xi_1 \cdot \nabla \xi_2 + (\nabla \xi_2)^2}{(w - 1)^2}.$$ 

Simplifying and using the fact that $\xi_1$ and $\xi_2$ are solutions of (16), the equation is reduced to

$$\nabla \log w = \frac{\nabla w}{w} = \nabla (\xi_1 - \xi_2),$$

so that

$$w = Ae^{\xi_1 - \xi_2}.$$
for an arbitrary constant $A \in \mathbb{C}$. It remains to show that the expression \(17\) is the gradient of some scalar function $\varphi$. This is the case if the rotational of \(17\) disappears. This is shown as follows, where the identities $\text{rot} \nabla \varphi = 0$, $\vec{f} \times \vec{f} = 0$ are used.

\[
\text{rot} \left( \frac{\nabla \xi_1 w - \nabla \xi_2}{w - 1} \right) = \text{rot} \left( \frac{\nabla \xi_1 w}{w - 1} \right) - \text{rot} \left( \frac{\nabla \xi_2}{w - 1} \right)
\]

\[
= \text{grad} \frac{w}{w - 1} \times \nabla \xi_1 + \frac{w}{w - 1} \text{rot} \nabla \xi_1
\]

\[
- \text{grad} \frac{1}{w - 1} \times \nabla \xi_2 - \frac{1}{w - 1} \text{rot} \nabla \xi_2
\]

\[
= \text{grad} \frac{w}{w - 1} \times \nabla \xi_1 - \text{grad} \frac{1}{w - 1} \times \nabla \xi_2
\]

\[
= \frac{(w - 1) \nabla w - w \nabla w}{(w - 1)^2} \times \nabla \xi_1 + \frac{\nabla w}{(w - 1)^2} \times \nabla \xi_2
\]

\[
= \frac{\nabla w}{(w - 1)^2} \times \nabla (\xi_2 - \xi_1)
\]

\[
= -\frac{Ae^{\xi_1 - \xi_2}}{(w - 1)^2} \nabla (\xi_2 - \xi_1) \times \nabla (\xi_2 - \xi_1)
\]

\[
= 0.
\]

It must be noted that the new solution gained is not necessarily the general solution, but a larger class of solutions. The following example illustrates this.

**Example 6** Consider \(8\) with $v = -1$, $D \vec{f} + \vec{f}^2 = -1$.

Two solutions of this equation are $h_1 = i_1 = \text{grad} x_1$, $h_2 = i_2 = \text{grad} x_2$. Applying the above result gives the class of solutions

\[
\frac{i_1 Ae^{x_1 - x_2} - i_2}{Ae^{x_1 - x_2} - 1}, \quad A \in \mathbb{C},
\]

however a third solution $h_3 = i_3$ is not included in the above expression as a special case.

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