Semiparametric tail-index estimation for randomly right-truncated heavy-tailed data

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Abstract

It was shown that when one disposes of a parametric information of the truncation distribution, the semiparametric estimator of the distribution function for truncated data (Wang, 1989) is more efficient than the nonparametric one. On the basis of this estimation method, we derive an estimator for the tail index of Pareto-type distributions that are randomly right-truncated and establish its consistency and asymptotic normality. The finite sample behavior of the proposed estimator is carried out by simulation study. We point out that, in terms of both bias and root of the mean squared error, our estimator performs better than those based on nonparametric estimation methods. An application to a real dataset of induction times of AIDS diseases is given as well.

Keywords: Extreme value index; Product-limit estimator; Semiparametric; Tail-Empirical process; Truncated data.

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1. Introduction

Let \((X_i, Y_i), i = 1, \ldots, N \geq 1\) be a sample from a couple \((X, Y)\) of independent positive random variables (rv’s) defined over a probability space \((\Omega, \mathcal{A}, P)\), with continuous distribution functions (df’s) \(F\) and \(G\) respectively. Suppose that \(X\) is right-truncated by \(Y\), in the sense that \(X_i\) is only observed when \(X_i \leq Y_i\). Thus, let us denote \((X_i, Y_i), i = 1, \ldots, n\) to be the observed data, as copies of a couple of dependent rv’s \((X, Y)\) corresponding to the truncated sample \((X_i, Y_i), i = 1, \ldots, N\), where \(n = n_N\) is a random sequence of discrete rv’s. By the weak law of large numbers, we have

\[
\frac{n/N}{p} := P(X \leq Y) = \int_0^\infty F(w) dG(w), \quad \text{as } N \to \infty, \quad (1.1)
\]

where the notation \(\frac{\cdot}{\cdot}\) stands for the convergence in probability. The constant \(p\) corresponds to the probability of observed sample which is supposed to be non-null, otherwise nothing is observed. The truncation phenomena frequently occurs in medical studies, when one wants to study the length of survival after the start of the disease: if \(Y\) denotes the elapsed time between the onset of the disease and death, and if the follow-up period starts \(X\) units of time after the onset of the disease then, clearly, \(X\) is right-truncated by \(Y\). For concrete examples of truncated data in medical treatments one refers, among others, to Lagakos et al. (1988) and Wang (1989). Truncated data schemes may also occur in many other fields, namely actuarial sciences, astronomy, demography and epidemiology, see for instance the textbook of Lawless (2002).

From Gardes and Stupfler (2015) the marginal df’s \(F^*\) and \(G^*\) corresponding to the joint df of \((X, Y)\) are given by

\[
F^* (x) := p^{-1} \int_0^x \frac{dG(w)}{G(w)} dF(w) \quad \text{and} \quad G^* (x) := p^{-1} \int_0^x \frac{F(w)}{G(w)} dG(w).
\]

By the previous first equation we derive a representation of the underlying df \(F\) as follows:

\[
F(x) = p \int_0^x \frac{dF^*(w)}{G(w)}, \quad (1.2)
\]

which will be for a great interest thereafter. In the sequel, we are dealing with the concept of regular variation. A function \(\varphi\) is said to be regularly varying at infinity with negative index \(-1/\eta\), notation \(\varphi \in \mathcal{R}V (-1/\eta)\), if

\[
\varphi(st)/\varphi(t) \to s^{-1/\eta}, \quad \text{as } t \to \infty, \quad (1.3)
\]
for $s > 0$. This convergence is known as the first-order condition of regular variation and its corresponding uniform convergence is formulated in terms of ”Potter’s inequalities” as follows: for any small $\epsilon > 0$, there exists $t_0 > 0$ such that for any $t \geq t_0$ and $s \geq 1$, we have

\[(1 - \epsilon) s^{-1/\eta - \epsilon} < \varphi(st) / \varphi(t) < (1 + \epsilon) s^{-1/\eta + \epsilon}. \tag{1.4}\]

See for instance Proposition B.1.9 (assertion 5, page 367) in de Haan and Ferreira (2006). The second-order condition (see de Haan and Stadtmüller, 1996) expresses the rate of the convergence (1.3) above. For any $x > 0$, we have

\[\frac{\varphi(tx) / \varphi(t) - x^{-1/\eta}}{A(t)} \to x^{-1/\eta} x^{\tau/\eta - 1} / \tau \eta, \text{ as } t \to \infty, \tag{1.5}\]

where $\tau < 0$ denotes the second-order parameter and $A$ is a function tending to zero and not changing signs near infinity with regularly varying absolute value with positive index $\tau/\eta$. A function $\varphi$ that satisfies assumption (1.5) is denoted $\varphi \in \mathcal{RV}_2(-1/\eta, \tau, A)$. We now have enough material to tackle the main goal of the paper. To begin, let us assume that the tails of both df’s $F$ and $G$ are regularly varying. That is

\[\overline{F} \in \mathcal{RV}(-1/\gamma_1) \text{ and } \overline{G} \in \mathcal{RV}(-1/\gamma_2), \text{ with } \gamma_1, \gamma_2 > 0. \tag{1.6}\]

Under this assumption, Gardes and Stupfler (2015) showed that

\[\overline{F}^\gamma \in \mathcal{RV}(-1/\gamma_1) \text{ and } \overline{G}^\gamma \in \mathcal{RV}(-1/\gamma_2), \tag{1.7}\]

where

\[\gamma := \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}. \tag{1.8}\]

For details on the proof of this statement, on refers to Benchaira et al. (2016a) (Lemma A1). The estimation of the tail index $\gamma_1$ was recently addressed for the first time in Gardes and Stupfler (2015) where the authors used equation (1.8) to propose an estimator to $\gamma_1$ as a ratio of Hill estimators (Hill, 1975) of the tail indices $\gamma$ and $\gamma_2$. These estimators are based on the top order statistics $X_{n-k:n} \leq \ldots \leq X_{n:n}$ and $Y_{n-k:n} \leq \ldots \leq Y_{n:n}$ pertaining to the samples $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ respectively. The sample fraction $k = k_n$ being a sequence of integers such that, $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. The asymptotic normality of the given estimator is established in Benchaira et al. (2015) by considering both the tail dependence and
the second-order conditions of regular variation. By using a Lynden-bell integral, Worms and Worms (2016) proposed the following estimator for the tail index $\gamma_1$:

$$\hat{\gamma}_1^{(W)}(u) := \frac{1}{F_n^{(1)}(u)} \sum_{i=1}^{n} 1(X_i > u) \frac{F_n^{(1)}(X_i)}{C_n(X_i)} \log \frac{X_i}{u},$$

where $u > 0$ is a given deterministic threshold and

$$F_n^{(1)}(x) := \prod_{X_i > x} \left[1 - \frac{1}{nC_n(X_i)}\right],$$

and

$$C_n(x) := \frac{1}{n} \sum_{i=1}^{n} 1(X_i \leq x \leq Y_i),$$

is the well-known nonparametric maximum likelihood estimator introduced in the well-known work Lynden-Bell (1971). Independently, Benchaira et al. (2016a) used a Woodroofe-integral with a random threshold, to derive the following estimator

$$\hat{\gamma}_1^{(BMN)}(u) := \frac{1}{F_n^{(2)}(X_{n-k:n})} \sum_{i=1}^{k} \frac{F_n^{(2)}(X_{n-i+1:n})}{C_n(X_{n-i+1:n})} \log \frac{X_{n-i+1:n}}{X_{n-k:n}},$$

where

$$F_n^{(2)}(x) := \prod_{X_i > x} \exp \left\{ -\frac{1}{nC_n(X_i)} \right\},$$

is the so-called Woodroofe’s nonparametric estimator (Woodroofe, 1985) of df $F$. To improve the performance of $\hat{\gamma}_1^{(BML)}$, Benchaira et al. (2016b) and Haouas et al. (2019) respectively proposed a Kernel-smoothed and a reduced-biais versions of this estimator and establish their consistency and asymptotic normality. It is worth mentioning that the Lynden-Bell integral estimator $\hat{\gamma}_1^{(W)}$ with a random threshold $u = X_{n-k:n}$ becomes

$$\hat{\gamma}_1^{(W)} := \frac{1}{F_n^{(1)}(X_{n-k:n})} \sum_{i=1}^{k} \frac{F_n^{(1)}(X_{n-i+1:n})}{C_n(X_{n-i+1:n})} \log \frac{X_{n-i+1:n}}{X_{n-k:n}}.$$ 

(1.10)

In a simulation study, Haouas et al. (2018) compared this estimator with $\hat{\gamma}_1^{(BMN)}$. They pointed out that both estimators have similar behaviors in terms of biases and mean squared errors.

Recall that the nonparametric Lynden-Bell estimator $F_n^{(1)}$ was constructed on the basis of the fact that $F$ and $G$ are both unknown. In this paper, we are dealing with the situation when $F$ is unknown but $G$ is parametrized by a known model $G_\theta$, $\theta \in \Theta \subset \mathbb{R}^d$, $d \geq 1$ having a density $g_\theta$ with respect to Lebesgue measure. Wang
(1989) considered this assumption and introduced a semiparametric estimator for df $F$ defined by

$$F_n \left( x; \hat{\theta}_n \right) := P_n \left( \hat{\theta} \right) \frac{1}{n} \sum_{i=1}^{n} \frac{1(X_i \leq x)}{G_{\hat{\theta}}(X_i)}, \quad (1.11)$$

where $1/P_n(\hat{\theta}) := n^{-1} \sum_{i=1}^{n} \frac{1}{G_{\hat{\theta}}(X_i)}$ and

$$\hat{\theta} := \arg \max_{\theta \in \Theta} \prod_{i=1}^{n} g_{\theta}(Y_i) / G_{\theta}(X_i), \quad (1.12)$$

denoting the conditional maximum likelihood estimator (CMLE) of $\theta$, which is consistent and asymptotically normal, see for instance Andersen (1970). On the other hand, Wang (1989) showed that $F_n \left( x; \hat{\theta}_n \right)$ is a uniformly consistent estimator over the $x$-axis and established, under suitable regularity assumptions, its asymptotic normality. Both Wang (1989) and Moreira and de Uña-Álvarez (2010) pointed out that the semiparametric estimate has greater efficiency uniformly over the $x$-axis. In the light of a simulation study, the authors suggest that the semiparametric estimate is a better choice when parametric information of the truncation distribution is available. Since the apparition of this estimation method many papers are devoted to the statistical inference with truncation data, see for instance Bilker and Wang (1996), Li et al. (1997), Qin et al. (2001), Shen (2010), Moreira et al. (2014), and Shen and Hsu (2020).

Motivated by the features of the semiparametric estimation, we next propose a new estimator for $\gamma_1$ by means of a suitable functional of $F_n \left( x; \hat{\theta}_n \right)$. We start our construction by noting that from Theorem 1.2.2 in de Haan and Ferreira (2006), the first-order condition (1.6) (for $F$) implies that

$$\lim_{t \to \infty} \frac{1}{F(t)} \int_{t}^{\infty} \log \left( x/t \right) dF(x) = \gamma_1. \quad (1.13)$$

In other words, $\gamma_1$ may viewed as a functional $\psi_t (F)$, for a large $t$, where

$$\psi_t (F) := \frac{1}{F(t)} \int_{t}^{\infty} \log \left( x/t \right) dF(x).$$

Replacing $F$ by $F_n \left( ; \hat{\theta}_n \right)$ and letting $t = X_{n-k:n}$ yield

$$\hat{\gamma}_1 = \psi_{X_{n-k:n}} \left( F_n \left( ; \hat{\theta}_n \right) \right) \quad (1.14)$$

$$= \frac{1}{F_n \left( X_{n-k:n}; \hat{\theta}_n \right)} \int_{X_{n-k:n}}^{\infty} \log \left( x/X_{n-k:n} \right) dF_n \left( x; \hat{\theta}_n \right), \quad (1.15)$$
as new estimator for $\gamma_1$. Observe that
\[
\int_t^\infty \log (x/t) \, dF_n \left( x; \hat{\theta}_n \right)
= P_n (\hat{\theta}) \int_{X_{n-k:n}}^\infty \log (x/X_{n-k:n}) \, 1 (x \geq X_{n-k}) \, dF_n \left( x; \hat{\theta}_n \right),
\]
which may be rewritten into
\[
P_n (\hat{\theta}) \frac{1}{n} \sum_{i=1}^{n} \int_{X_{n-k:n}}^\infty \log (x/X_{n-k:n}) \, 1 (x \geq X_{n-k}) \, dF_n \left( x; \hat{\theta}_n \right) \frac{1}{G_{\hat{\theta}} (X_i)}
= P_n (\hat{\theta}) \frac{1}{n} \sum_{i=1}^{k} \frac{\log (X_{n-i+1}/X_{n-k:n})}{G_{\hat{\theta}} (X_{n-i+1:n})}.
\]
On the other hand, $F \left( X_{n-k:n}; \hat{\theta}_n \right)$ equals
\[
P_n (\hat{\theta}) \frac{1}{n} \sum_{i=1}^{n} \frac{1 (X_{i:n} \leq X_{n-k:n})}{G_{\hat{\theta}} (X_{i:n})} = P_n (\hat{\theta}) \frac{1}{n} \sum_{i=1}^{n-k} \frac{1}{G_{\hat{\theta}} (X_{i:n})}.
\]
Hence
\[
F \left( X_{n-k:n}; \hat{\theta}_n \right) = \frac{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{G_{\hat{\theta}} (X_{i:n})} - \frac{1}{n} \sum_{i=1}^{n-k} \frac{1}{G_{\hat{\theta}} (X_{i:n})}}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{G_{\hat{\theta}} (X_{i:n})} - \frac{1}{n} \sum_{i=1}^{n-k} \frac{1}{G_{\hat{\theta}} (X_{i:n})}}
= P_n (\hat{\theta}) \frac{1}{n} \sum_{i=1}^{k} \frac{1}{G_{\hat{\theta}} (X_{n-i+1:n})}.
\]
Thereby, the final form of our new estimator is
\[
\hat{\gamma}_1 = \frac{\sum_{i=1}^{k} \left( G_{\hat{\theta}} (X_{n-i+1:n}) \right)^{-1} \log (X_{n-i+1}/X_{n-k:n})}{\sum_{i=1}^{k} \left( G_{\hat{\theta}} (X_{n-i+1:n}) \right)^{-1}}.
\] (1.16)
The asymptotic behavior of $\hat{\gamma}_1$ will be established by means of the following tail empirical process
\[
D_n \left( x; \hat{\theta}; \gamma_1 \right) := \sqrt{k} \left( \frac{F_n \left( xX_{n-k:n}; \hat{\theta} \right)}{F_n \left( X_{n-k:n}; \hat{\theta} \right)} - x^{-1/\gamma_1} \right), \text{ for } x > 1.
\]
This method was already used to establish the asymptotic behavior of Hill’s estimator for complete data (de Haan and Ferreira, 2006, page 162) that we will adapt to the truncation case. Indeed, an integration by parts of the integral (1.14), yields
\[
\hat{\gamma}_1 = \int_1^\infty x^{-1} \frac{F_n \left( xX_{n-k:n}; \hat{\theta} \right)}{F_n \left( X_{n-k:n}; \hat{\theta} \right)} \, dx,
\]
and therefore
\[ \sqrt{k} (\hat{\gamma}_1 - \gamma_1) = \int_1^\infty x^{-1} D_n \left( x; \hat{\theta}; \gamma_1 \right) \, dx. \] (1.17)

Thus for a suitable weighted weak approximation to \( D_n \left( \cdot ; \hat{\theta} \right) \), we may easily deduce the consistency and asymptotic normality of \( \hat{\gamma}_1 \). This process may also contribute to the goodness-of-fit test to fitting heavy-tailed distributions via, among others, the Kolmogorov-Smirnov and Cramer-Von Mises type statistics
\[ \sup_{x > 1} \left| D_n \left( x; \hat{\theta}, \hat{\gamma}_1 \right) \right| \text{ and } \int_1^\infty D_n^2 \left( x; \hat{\theta}, \hat{\gamma}_1 \right) \, dx^{-1/\hat{\gamma}_1}. \]

More precisely, these statistics are used when testing the null hypothesis \( H_0 : \)"both \( F \) and \( G \) are heavy-tailed" versus the alternative one \( H_1 : \)"at least one of \( F \) and \( G \) is not heavy-tailed", that is \( H_0 : (1.6) \) holds" versus \( H_1 : (1.6) \) does not hold". This problem has been already addressed by Drees et al. (2006) and Koning and Peng (2008) in the case of complete data. The (uniform) weighted weak convergence of \( D_n \left( x; \hat{\theta}, \hat{\gamma}_1 \right) \) and the asymptotic normality of \( \hat{\gamma}_1 \), stated below, will be of great interest to establish the limit distributions of the aforementioned test statistics. This is out of the scope of this paper whose remainder is structured as follows. In Section 2, we present our main results which consist in the consistency and asymptotic normality of estimator \( \hat{\gamma}_1 \). The performance of the proposed estimator is checked by simulation in Section 3. An application to a real dataset composed of induction times of AIDS diseases is given in Section 4. All proofs are gathered in Section 5. The proofs of two useful lemmas are postponed to the Appendix.

## 2. Main results

The regularity assumptions, denoted [A0], concerning the existence, consistency and asymptotic normality of the CLME estimator \( \hat{\theta} \), given in (1.12), are discussed in Andersen (1970). Here we only state additional conditions on df \( G_\theta \) corresponding to Pareto-type models which are required to establish the asymptotic behavior of our newly estimator \( \hat{\gamma}_1 \).

- [A1] For each fixed \( y \), the function \( \theta \to G_\theta (y) \) is continuously differentiable of partial derivatives \( \mathbf{G}_\theta^{(j)} = \partial G_\theta / \partial \theta_j \), \( j = 1, \ldots, d \).
- [A2] \( \overline{\mathbf{G}}_\theta^{(j)} \in \mathcal{RV}(-1/\gamma_2) \).
- [A3] \( y^{-1} \overline{\mathbf{G}}_\theta^{(j)} (y) / \overline{G}_\theta (y) \to 0 \), as \( y \to \infty \), for any \( \epsilon > 0 \).

For commonly Pareto-type models, one may easily checked that there exist some constants \( a_j \geq 0 \), \( c_j \) and \( d_j \), such that \( \overline{\mathbf{G}}_\theta^{(j)} (y) \sim c_j \left( y^{-1/\gamma_2} + d_j \right) \log y \), for all large
Then one may consider that the assumptions $[A1] - [A3]$ are not very restrictive and they may be acceptable in the extreme value theory.

**Theorem 2.1.** Assume that $F \in \mathcal{RV}_2(-1/\gamma_1; \rho_1, A)$ and $G_\theta \in \mathcal{RV}(-1/\gamma_2)$ satisfying the assumptions $[A0] - [A3]$, and suppose that $\gamma_1 < \gamma_2$. Then on the probability space $(\Omega, \mathcal{A}, P)$, there exists a standard Wiener process $\{W(s), 0 \leq s \leq 1\}$ such that, for any small $0 < \varepsilon < 1/2$:

\[
\sup_{x > 1} x^\varepsilon \left| D_n \left( x; \hat{\theta}, \gamma_1 \right) - \Gamma (x; W) - x^{-1/\gamma_1} x^{\rho_1/\gamma_1} - 1/\rho_1 \sqrt{kA} (a_k) \right| \overset{P}{\to} 0,
\]

provided that $\sqrt{kA} (a_k) = O (1)$, where

\[
\Gamma (x; W) := \frac{\gamma}{\gamma_1} x^{-1/\gamma_1} \left\{ x^{1/\gamma} W \left( x^{-1/\gamma} \right) - W (1) \right\}
\]

\[+ \frac{\gamma}{\gamma_1 + \gamma_2} x^{-1/\gamma_1} \int_0^1 s^{-\gamma/\gamma_2 - 1} \left\{ x^{1/\gamma} W \left( x^{-1/\gamma} s \right) - W (s) \right\} ds,
\]

is a centred Gaussian process and $a_k := F_{\ast \ast -} (1 - k/n)$, where

\[F_{\ast \ast -} (s) := \inf \left\{ x : F_{\ast} (x) \geq s \right\}, 0 < s < 1,
\]

denotes the quantile (or the generalized inverse) function pertaining to df $F_{\ast}$.

By a strength application of this weak approximation, we establish both consistency and asymptotic normality of our newly estimator $\hat{\gamma}_1$, that we state there in the following Theorem.

**Theorem 2.2.** Under the assumptions of Theorem 2.1, we have

\[
\hat{\gamma}_1 - \gamma_1 = k^{-1/2} \int_1^\infty x^{-1} \Gamma (x; W) \, dx + A (a_k) \int_1^\infty x^{-1/\gamma_1 - 1} x^{\rho_1/\gamma_1} - 1/\rho_1 \sqrt{kA} (a_k) \, dx + o_P (k^{-1/2}),
\]

this implies that $\hat{\gamma}_1 \overset{P}{\to} \gamma_1$. Whenever $\sqrt{kA} (a_k) \to \lambda < \infty$, we get

\[
\sqrt{k} (\hat{\gamma}_1 - \gamma_1) \overset{D}{\to} \mathcal{N} \left( \frac{\lambda}{1 - \rho_1}, \sigma^2 \right),
\]

where $\sigma^2 := \gamma^2 (1 + \gamma_1/\gamma_2) (1 + (\gamma_1/\gamma_2)^2) (1 - \gamma_1/\gamma_2)^3 1 (\gamma_1 < \gamma_2)$, and $1 (\mathcal{A})$ stands for the indicator function pertaining to a set $\mathcal{A}$.
3. Simulation study

In this section we will perform a simulation study in order to compare the finite sample behavior of our newly semiparametric estimator \( \hat{\gamma}_1 \), given in (1.16), with the Woodroffe and the Lynden-Bell integral estimators \( \hat{\gamma}_1^{(BMN)} \) and \( \hat{\gamma}_1^{(W)} \), given respectively in (1.9) and (1.10). The truncation and truncated distributions functions \( F \) and \( G \), will be chosen among the following two models:

- Burr \((\gamma, \delta)\) distribution with right-tail function:
  \[
  H(x) = \left(1 + x^{1/\delta}\right)^{-\delta/\gamma}, \quad x \geq 0, \quad \delta > 0, \quad \gamma > 0;
  \]

- Fréchet \((\gamma)\) distribution with right-tail function:
  \[
  H(x) = 1 - \exp\left(-x^{-1/\gamma}\right), \quad x > 0, \quad \gamma > 0.
  \]

The simulation study be made in fours scenarios following to the choice of the underlying df’s \( F \) and \( G_\theta \):

- \([S1]\) Burr \((\gamma_1, \delta)\) truncated by Burr \((\gamma_2, \delta)\); with \( \theta = (\gamma_2, \delta) \)
- \([S2]\) Fréchet \((\gamma_1)\) truncated by Fréchet \((\gamma_2)\); with \( \theta = \gamma_2 \)
- \([S3]\) Fréchet \((\gamma_1)\) truncated by Burr \((\gamma_2, \delta)\); with \( \theta = (\gamma_2, \delta) \)
- \([S4]\) Burr \((\gamma_1, \delta)\) truncated by Fréchet \((\gamma_2)\); with \( \theta = \gamma_2 \)

To this end, we fix \( \delta = 1/4 \) and choose the values 0.6 and 0.8 for \( \gamma_1 \) and 55% and 90% for the portions of observed truncated data given in (1.1) by

\[
p = \int_0^\infty F(w) \, dG_\theta(w), \quad (3.18)
\]

so that the assumption \( \gamma_1 < \gamma_2 \) stated in Theorem 2.1 be hold. In other terms the values of \( p \) have to be greater than 50%. For each couple \((\gamma_1, p)\), we solve the equation (3.18) to get the pertaining \( \gamma_2 \)-value, which we summarize as follows:

\[
(p, \gamma_1, \gamma_2) = (55\%, 0.6, 1.4), (90\%, 0.6, 5.4), (55\%, 0.8, 1.9), (90\%, 0.8, 7.2). \quad (3.19)
\]

For each scenario, we simulated 1000 random samples of size \( N = 300 \) and compute the root mean squared error (RMSE) and the absolute bias (ABIAS) corresponding to each estimator \( \hat{\gamma}_1 \), \( \hat{\gamma}_1^{(BMN)} \) and \( \hat{\gamma}_1^{(W)} \). The comparison is carry out by plotting the ABIAS and RMSE as functions of the sample fraction \( k \) which is vary from 2 to 120. The end points of this range is chosen so that it contains the optimal number of upper extremes \( k^* \) used in the computation of the tail index estimate. There are many heuristic methods to select the optimal choice of \( k^* \), see for instance Caeiro and Gomes (2015), here we use the algorithm proposed by Reiss and Thomas.
(2007) in page 137, which is incorporated in the R software “Xtremes” package. Note that the computation the CLME of $\theta$ is made by means of the syntax ”maxLik” of the maxLik R software package. The optimal sample fraction $k^*$ is defined, in this procedure, by

$$k^* := \arg \min_{1 < k < n} \frac{1}{k} \sum_{i=1}^{k} i^\theta |\hat{\gamma}(i) - \text{median}\{\hat{\gamma}(1), \ldots, \hat{\gamma}(k)\}|,$$

for suitable constant $0 \leq \theta \leq 1/2$, where $\hat{\gamma}(i)$ corresponds to an estimator of tail index $\gamma$, based on the $i$ upper order statistics, of a Pareto-type model. We observed, in our simulation study, that $\theta = 0.3$ allows better results both in terms of bias and rmse. It is worth mentioning that making $N$ vary did not provide notable findings, therefore we kept the size $N$ be fixed. The finite sample behavior of the above mentioned estimators are illustrated in Figures 3.1-3.8. On the overall, the biases of three estimators are almost equal, however in the case of moderate truncation ($p \approx 50\%$) the RMSE of our newly semiparametric $\hat{\gamma}_1$ is clearly the smaller compared that of $\hat{\gamma}_1^{(BMN)}$ and $\hat{\gamma}_1^{(W)}$. Actually, the moderate truncation situation is the most frequently in real data, while up to our knowledge the strong truncation remains theoretic. In this sense, we may consider that the semiparametric estimator is more efficient than the two other ones. We point out that the two estimators $\hat{\gamma}_1^{(BMN)}$ and $\hat{\gamma}_1^{(W)}$ have almost the same behavior which actually is noticed before by Haouas et al. (2018). The optimal sample fractions $k^*$ of each tail index estimator are given in Tables 1-4.

|        | $k^*$ | $\hat{\gamma}_1$ | $k^*$ | $\hat{\gamma}_1^{(BMN)}$ | $k^*$ | $\hat{\gamma}_1^{(W)}$ |
|--------|-------|-------------------|-------|--------------------------|-------|------------------------|
| S1     | 44    | 0.600             | 41    | 0.599                    | 40    | 0.600                  |
| S2     | 18    | 0.601             | 17    | 0.600                    | 16    | 0.597                  |
| S3     | 21    | 0.601             | 20    | 0.601                    | 19    | 0.599                  |
| S4     | 30    | 0.603             | 27    | 0.600                    | 25    | 0.598                  |

**Table 1.** Optimal sample fraction $k^*$ and the estimated value of each estimator of the tail index $\gamma_1 = 0.6$ based on 1000 samples for the four scenarios with $p = 0.55$.

4. **Real data example**

In this section, we give an application to the AIDS data set, available in the ”DTDA” R package, used before by Lagakos et al. (1988). The data present the infection and
Figure 3.1. Absolute bias (left two panels) and RMSE (right two panels) of $\hat{\gamma}_1$ (black) and $\hat{\gamma}_1^{\text{BMN}}$ (red) and $\hat{\gamma}_1^{\text{W}}$ (blue), corresponding to two situations of scenario $S_1$: ($\gamma_1 = 0.6, p = 55\%$) and ($\gamma_1 = 0.6, p = 90\%$) based on 1000 samples of size 300.

Table 2. Optimal sample fraction $k^*$ and the estimated value of each estimator of the tail index $\gamma_1 = 0.6$ based on 1000 samples for the four scenarios with $p = 0.9$.

|     | $k^*$ | $\hat{\gamma}_1$ | $k^*$ | $\hat{\gamma}_1^{\text{BMN}}$ | $k^*$ | $\hat{\gamma}_1^{\text{W}}$ |
|-----|-------|-------------------|-------|-------------------------------|-------|---------------------|
| $S_1$ | 82    | 0.610             | 82    | 0.611                         | 82    | 0.611               |
| $S_2$ | 37    | 0.640             | 37    | 0.640                         | 37    | 0.640               |
| $S_3$ | 46    | 0.633             | 37    | 0.625                         | 37    | 0.625               |
| $S_4$ | 52    | 0.610             | 52    | 0.610                         | 52    | 0.610               |

induction times for $n = 258$ adults who were infected with HIV virus and developed AIDS by June 30, 1986. The time in years, measured from April 1, 1978, when adults were infected by the virus from a contaminated blood transfusion and the waiting time to development of AIDS measured from the date of infection. We interest here to the estimation of the end-time of the induction of the AIDS virus which corresponds to the high quantile in basis the given observations. The variable of interest here is the time of induction $T$ of the disease duration which elapses between
Figure 3.2. Absolute bias (left two panels) and RMSE (right two panels) of \( \hat{\gamma}_1 \) (black) and \( \hat{\gamma}_1^{(BMN)} \) (red) and \( \hat{\gamma}_1^{(W)} \) (blue), corresponding to two situations of scenario \( S_1 : (\gamma_1 = 0.8, p = 55\%) \) and \( (\gamma_1 = 0.8, p = 90\%) \) based on 1000 samples of size 300.

Table 3. Optimal sample fraction \( k^* \) and the estimated value of each estimator of the tail index \( \gamma_1 = 0.8 \) based on 1000 samples for the four scenarios with \( p = 0.55 \).

| Scenario | \( k^* \) | \( \hat{\gamma}_1 \) | \( k^* \) | \( \hat{\gamma}_1^{(BMN)} \) | \( k^* \) | \( \hat{\gamma}_1^{(W)} \) |
|----------|----------|----------------|----------|----------------|----------|----------------|
| S1       | 59       | 0.799         | 57       | 0.800          | 54       | 0.799          |
| S2       | 21       | 0.803         | 21       | 0.803          | 20       | 0.799          |
| S3       | 24       | 0.802         | 22       | 0.798          | 22       | 0.801          |
| S4       | 51       | 0.799         | 52       | 0.800          | 50       | 0.801          |

the date of infection \( M \) and the date \( M + T \) of the declaration of the disease. The sample \( (T_1, M_1), \ldots, (T_n, M_n) \) are taken between two fixed dates: "0" and "8", i.e. between April 1, 1978, and June 30, 1986. The initial date "0" denotes an infection occurring in the three months: from April 1, 1978, to June 30, 1978. Let us assume that \( M \) and \( T \) are the observed rv’s, corresponding to the underlying rv’s \( M \) and \( T \), given by the truncation scheme \( 0 \leq M + T \leq 8 \), which in turn may be rewritten
Figure 3.3. Absolute bias (left two panels) and RMSE (right two panels) of $\hat{\gamma}_1$ (black) and $\hat{\gamma}_1^{(BMN)}$ (red) and $\hat{\gamma}_1^{(W)}$ (blue), corresponding to two situations of scenario $S_2$: ($\gamma_1 = 0.6, p = 55\%$) and ($\gamma_1 = 0.6, p = 90\%$) based on 1000 samples of size 300.

Table 4. Optimal sample fraction $k^*$ and the estimated value of each estimator of the tail index $\gamma_1 = 0.8$ based on 1000 samples for the four scenarios with $p = 0.9$.

|   | $k^*$ | $\hat{\gamma}_1$ | $k^*$ | $\hat{\gamma}_1^{(BMN)}$ | $k^*$ | $\hat{\gamma}_1^{(W)}$ |
|---|------|---------------|------|----------------|------|----------------|
| S1 | 90   | 0.804         | 90   | 0.806           | 90   | 0.807          |
| S2 | 34   | 0.845         | 34   | 0.846           | 34   | 0.846          |
| S3 | 40   | 0.831         | 40   | 0.831           | 40   | 0.831          |
| S4 | 71   | 0.814         | 71   | 0.814           | 71   | 0.815          |

into

$$0 \leq M \leq S,$$

(4.20)

where $S := 8 - T$. To work within the framework of the present paper, let us make the following transformations:

$$X := \frac{1}{S + \epsilon} \quad \text{and} \quad Y := \frac{1}{M + \epsilon},$$

(4.21)
Figure 3.4. Absolute bias (left two panels) and RMSE (right two panels) of \( \hat{\gamma}_1 \) (black) and \( \hat{\gamma}_1^{(BMN)} \) (red) and \( \hat{\gamma}_1^{(W)} \) (blue), corresponding to two situations of scenario \( S_2 \): \( (\gamma_1 = 0.8, p = 55\%) \) and \( (\gamma_1 = 0.8, p = 90\%) \) based on 1000 samples of size 300.

where \( \epsilon = 0.05 \) so that the two denominators be non-null. Thus, in view of (4.20), we have \( X \leq Y \), which means that \( X \) is randomly right-truncated by \( Y \). Thereby, for the given sample \( (T_1, M_1), ..., (T_n, M_n) \), from \( (T, M) \), the previous transformations produce us a new ones \( (X_1, Y_1), ..., (X_n, Y_n) \) from \( (X, Y) \).

Let us now denote by \( F \) and \( G \) the df’s of the underling rv’s \( X \) and \( Y \) corresponding to the truncated rv’s \( X \) and \( Y \), respectively. By using parametric likelihood methods, Lui et al. (1986) fit both df’s of \( M \) and \( S \) by the two-parameter Weibull model, this implies that the df’s of \( F \) and \( G \) by may be fitted by two-parameter Fréchet model, namely \( H_{(a,r)} (x) = \exp (-a^r x^{-r}) \), \( x > 0, a > 0, r > 0 \), hence both \( F \) and \( G \) are heavy-tailed. The estimated parameters corresponding to the fitting of df \( G \) are \( a_0 = 0.004 \) and \( r_0 = 2.1 \), see also Lagakos et al. (1988) page 520. Thus on may consider that df \( G \) is known and equals \( G_{\theta} = H_{(a_0,r_0)} \), where \( \theta = (a_0, r_0) \). By using the Thomas and Reiss algorithm, given above, we compute the optimal sample fraction \( k^* \) corresponds to the tail index estimator \( \hat{\gamma}_1 \) of df \( F \) is \( \gamma_1 \). We find

\[
k^* = 19, \ X_{n-k:n} = 0.356 \text{ and } \hat{\gamma}_1 = 0.917.
\]
Figure 3.5. Absolute bias (left two panels) and RMSE (right two panels) of $\hat{\gamma}_1$ (black) and $\hat{\gamma}_1^{\text{(MBN)}}$ (red) and $\hat{\gamma}_1^{\text{(W)}}$ (blue), corresponding to two situations of scenario $S_3$: ($\gamma_1 = 0.6, p = 55\%$) and ($\gamma_1 = 0.6, p = 90\%$) based on 1000 samples of size 300.

The well-known Weissman estimator (Weissman, 1978) of the high quantile, $q_v := F^{-1}(1 - v_n)$, corresponding to the underlying df $F$ is given by

$$\hat{q}_v := X_{n-k:n} \left( \frac{v}{F_n(X_{n-k:n})} \right)^{-\hat{\gamma}_1},$$

where $v = 1/(2n)$, and $F_n$ is the semiparametric estimator of df $F$ of $X$ given in (1.11). From the values (4.22), we get $\hat{q}_v = 0.061$. Let us now compute the high quantile of $T$ based on the original data, $T_1, ..., T_n$. Recall that $P(X \geq q_v) = v$ and $X = 1/(8 - T + \epsilon)$, this implies that $P(T \geq 1/q_v - 8 + \epsilon) = v$, this means that $1/q_v - 8 + \epsilon$ is the high quantile of $T$, which corresponds to the end-time $t_{\text{end}}$ that we want to estimate. Thereby $\hat{t}_{\text{end}} = 1/\hat{q}_v - 8 + 10^{-2} = 1/0.061 - 8 + 10^{-2} = 8.40$, the value the end time of induction of AIDS is: 8 years, 4 months and 24 days.
Figure 3.6. Absolute bias (left two panels) and RMSE (right two panels) of $\hat{\gamma}_1$ (black) and $\hat{\gamma}_1^{(\text{BMN})}$ (red) and $\hat{\gamma}_1^{(W)}$ (blue), corresponding to two situations of scenario $S_3$: ($\gamma_1 = 0.8, p = 55\%$) and ($\gamma_1 = 0.8, p = 90\%$) based on 1000 samples of size 300.

5. Proof of Theorems

5.1. Proof of Theorem 2.1. Let us first notice that the semiparametric estimator of df $F$ given in (1.12) may be rewritten into

$$F_n(x; \tilde{\theta}_n) = P_n(\tilde{\theta}) \int_0^x \frac{dF_n^*(w)}{G_{\tilde{\theta}}(w)},$$

and $1/P_n(\tilde{\theta}) = \int_0^\infty dF_n^*(w)/G_{\tilde{\theta}}(w)$, where $F_n^*(w) := n^{-1} \sum_{i=1}^n 1(X_i \leq w)$ denotes the usual empirical df pertaining to the observed sample $X_1, \ldots, X_n$. It is worth mentioning that by using the strong law of large numbers $P_n(\tilde{\theta}) \to P(\theta)$ (almost surely) as $n \to \infty$, where $P(\theta) = 1/\int_0^\infty dF^*(w)/G_\theta(w)$ (see e.g. Lemma 3.2 in Wang (1989)). On the other hand by using the first equation in (1.2), we deduce that $p = 1/\int_0^\infty dF^*(w)/G(w)$, it follows that $p \equiv P(\theta)$ because we already assumed that $G \equiv G_\theta$. Next we use the distribution tail

$$\overline{F}(x; \theta) = P(\theta) \int_x^\infty \frac{dF^*(w)}{G_\theta(w)},$$

(5.24)
Figure 3.7. Absolute bias (left two panels) and RMSE (right two panels) of $\hat{\gamma}_1$ (black) and $\hat{\gamma}_1^{(BMN)}$ (red) and $\hat{\gamma}_1^{(W)}$ (blue), corresponding to two situations of scenario $S_4$: ($\gamma_1 = 0.6, p = 55\%$) and ($\gamma_1 = 0.6, p = 90\%$) based on 1000 samples of size 300.

and its empirical counterpart

$$\overline{F}_n(x; \hat{\theta}) = P_n(\hat{\theta}) \int_{x}^{\infty} \frac{dF^*_n(w)}{G_{\hat{\theta}}(w)}.$$  

To begin let us decompose $k^{-1/2}D_n(x; \hat{\theta})$ for $x > 1$ into the sum of

$$M_{n1}(x) := x^{-1/\gamma_1} \frac{\overline{F}_n(xX_{n-k:n}; \hat{\theta}) - \overline{F}_n(xX_{n-k:n}; \theta)}{\overline{F}(xX_{n-k:n}; \theta)},$$

$$M_{n2}(x) := x^{-1/\gamma_1} \frac{\overline{F}_n(xX_{n-k:n}; \theta) - \overline{F}(xX_{n-k:n}; \theta)}{\overline{F}(xX_{n-k:n}; \theta)},$$

$$M_{n3}(x) := \frac{\overline{F}(xX_{n-k:n}; \hat{\theta}) \overline{F}_n(X_{n-k:n}; \theta) - \overline{F}(xX_{n-k:n}; \theta)}{\overline{F}(X_{n-k:n}; \theta)},$$

$$M_{n4}(x) := \frac{\overline{F}(xX_{n-k:n}; \theta) \overline{F}_n(X_{n-k:n}; \hat{\theta}) - \overline{F}(xX_{n-k:n}; \theta)}{\overline{F}(xX_{n-k:n}; \hat{\theta})} - x^{-1/\gamma_1}$$

and

$$M_{n5}(x) := \frac{\overline{F}(xX_{n-k:n}; \theta)}{\overline{F}(X_{n-k:n}; \theta)} - x^{-1/\gamma_1}.$$
Our goal is to provide a weighted weak approximation to the tail empirical process $D_n \left( x; \theta; \gamma_1 \right)$. To begin, let $\xi_i := F^\ast (X_i), i = 1, ..., n$ be a sequence of independent and identically rv’s. Recall that both df’s $F$ and $G_\theta$ are assumed to be continuous, this implies that $F^\ast$ is continuous as well, therefore $P (\xi_i \leq u) = u$, this means that $(\xi_i)_{i=1,n}$ are uniformly distributed on $(0, 1)$. Let us now define the corresponding uniform tail empirical process

$$\alpha_n (s) := \sqrt{k} (U_n (s) - s), \text{ for } 0 \leq s \leq 1,$$

(5.25)

where

$$U_n (s) := k^{-1} \sum_{i=1}^{n} 1 (\xi_i < ks/n)$$

(5.26)

denotes the tail empirical df pertaining to the sample $(\xi_i)_{i=1,n}$. In view of Proposition 3.1 of Einmahl et al. (2006), there exists a Wiener process $W$ such that for every $0 \leq \epsilon < 1/2$,

$$\sup_{0 \leq s < 1} s^{-\epsilon} |\alpha_n (s) - W (s)| \overset{P}{\to} 0, \text{ as } n \to \infty.$$

(5.27)
Let us fix a sufficiently small $0 < \epsilon < 1/2$. We will successively show that, under the first-order of regular variation conditions \((1.6)\), uniformly on $x \geq 1$, for all large $n$:

\[
\sqrt{k} M_{n2} (x) = \frac{\gamma}{\gamma_1} x^{1/\gamma_2} W (t^{-1/\gamma}) + \frac{\gamma}{\gamma_1} \int_{x^{1/\gamma_2}}^{\infty} W (t^{-\gamma/2}) dt + o_p \left( x^{\frac{1}{2} \left( \frac{1}{\gamma_2} - \frac{1}{\gamma} \right) + \epsilon} \right), \tag{5.28}
\]

and

\[
\sqrt{k} M_{n3} (x) = -x^{-1/\gamma_1} \left( \frac{\gamma}{\gamma_1} W (1) + \frac{\gamma}{\gamma_1} \int_{1}^{\infty} W (t^{-\gamma/2}) dt \right) + o_p \left( x^{-1/\gamma_1 + \epsilon} \right). \tag{5.29}
\]

while

\[
\sqrt{k} M_{n1} (x) = o_p \left( x^{-1/\gamma_1 + \epsilon} \right), \quad \sqrt{k} M_{n4} (x) = o_p \left( x^{\frac{1}{2} \left( \frac{1}{\gamma_2} - \frac{1}{\gamma} \right) + \epsilon} \right), \tag{5.30}
\]

and

\[
\sqrt{k} M_{n5} (x) = x^{-1/\gamma_1} x^{\rho_1/\gamma_1} \frac{1}{\rho_1 \gamma_1} \sqrt{k} A (a_k) + o_p \left( x^{-1/\gamma_1} \right). \tag{5.31}
\]

Throughout the proof, without loss of generality, we assume that $a \epsilon \equiv \epsilon$, for any constant $a > 0$. We point out that all the rest terms of the previous approximations are negligible in probability, uniformly on $x > 1$. Let us begin by the term $M_{n1} (x)$ which may be made into

\[
\frac{x^{-1/\gamma_1}}{F (x X_{n-k:n}; \theta)} P_n \left( \tilde{\theta} \right) \left( \int_x^{\infty} \frac{dF_n^* (X_{n-k:n}w)}{G_{\tilde{\theta}} (X_{n-k:n}w)} - \int_x^{\infty} \frac{dF_n^* (X_{n-k:n}w)}{G_{\hat{\theta}} (X_{n-k:n}w)} \right)
= \frac{x^{-1/\gamma_1}}{F (x X_{n-k:n}; \theta)} P_n \left( \tilde{\theta} \right) \int_x^{\infty} \left( \frac{1}{G_{\tilde{\theta}} (X_{n-k:n}w)} - \frac{1}{G_{\hat{\theta}} (X_{n-k:n}w)} \right) dF_n^* (X_{n-k:n}w).
\]

By applying the mean value theorem (for several variables) to function $\theta \rightarrow 1/G_{\theta} (\cdot)$, yields

\[
\frac{1}{G_{\tilde{\theta}} (z)} - \frac{1}{G_{\hat{\theta}} (z)} = \sum_{i=1}^{d} (\tilde{\theta}_i - \hat{\theta}_i) \frac{G_{\tilde{\theta}}^{(i)} (z)}{G_{\hat{\theta}}^{(i)} (z)}, \quad \text{for any } z > 1,
\]

where $\tilde{\theta}$ is such that $\tilde{\theta}_i$ is between $\theta_i$ and $\hat{\theta}_i$, for $i = 1, ..., d$, therefore

\[
M_{n1} (x) = \frac{x^{-1/\gamma_1}}{F (x X_{n-k:n}; \theta)} P_n \left( \hat{\theta} \right) \sum_{i=1}^{d} (\tilde{\theta}_i - \hat{\theta}_i) \int_x^{\infty} \frac{G_{\tilde{\theta}}^{(i)} (X_{n-k:n}w)}{G_{\hat{\theta}}^{(i)} (X_{n-k:n}w)} dF_n^* (X_{n-k:n}w).
\]

Recall that by assumptions \((1.6)\) and [42] both $G_{\theta}$ and $G_{\theta}^{(i)}$ are regularly varying with the same index $(-1/\gamma_2)$ and on the other hand, $X_{n-k:n} \xrightarrow{P} \infty$ and $w > 1$, then $X_{n-k:n}w \xrightarrow{P} \infty$. Then by applying Pooter’s inequalities \((1.4)\), yields

\[
\frac{G_{\tilde{\theta}} (X_{n-k:n}w)}{G_{\hat{\theta}} (X_{n-k:n})} = (1 + o_P (1)) w^{-1/\gamma_2 + \epsilon} = \frac{G_{\tilde{\theta}}^{(i)} (X_{n-k:n}w)}{G_{\hat{\theta}}^{(i)} (X_{n-k:n})},
\]
it follows that
\[ M_{n1}(x) = (1 + o_P(1)) P_n \left( \theta - \frac{x^{1/\gamma_1}}{G_{\hat{\theta}}(X_{n-k:n}) F(x,X_{n-k:n};\theta)} \right) \times \sum_{i=1}^{d} \frac{G_{\hat{\theta}}^{(i)}(X_{n-k:n})}{G_{\hat{\theta}}(X_{n-k:n})} | \hat{\theta}_i - \theta_i | \int_x^\infty w^{1/\gamma_2} dF^*_n(X_{n-k:n:w}). \]

For some regularity assumptions, Andersen (1970) stated that \( \sqrt{n}(\hat{\theta} - \theta) \) is asymptotically a centred multivariate normal rv, which implies that \( \hat{\theta}_i - \theta_i = O_P(n^{-1/2}) \) thus \( \hat{\theta} \xrightarrow{P} \theta \). On the other hand, by the law of large numbers \( P_n(\theta) \xrightarrow{P} P(\theta) \) as \( n \to \infty \), then we may readily show that \( P_n(\hat{\theta}) \xrightarrow{P} P(\theta) \) as \( n \to \infty \) as well. Note since \( \hat{\theta} \) is consistent estimator for \( \theta \) then \( \hat{\theta} \) it is, then by using the fact that \( X_{n-k:n} \xrightarrow{P} \infty \), and the two assumptions \([A1]\) and \([A3]\) together, we show readily that
\[ (X_{n-k:n})^{-1} \frac{G_{\hat{\theta}}^{(i)}(X_{n-k:n})}{G_{\hat{\theta}}(X_{n-k:n})} \xrightarrow{P} 0, \quad \text{as } n \to \infty, \]
and \( \frac{G_{\hat{\theta}}(X_{n-k:n})}{G_{\hat{\theta}}(X_{n-k:n})} \xrightarrow{P} 1 \). In view of Lemma A1 in Benchaira et al. (2016a), we infer that \( X_{n-k:n} = (1 + o_P(1)) (k/n)^{-\gamma} \), thus
\[ M_{n1}(x) = (k/n)^{-\gamma} o_P(n^{-1/2}) \tilde{M}_{n1}(x), \]
where
\[ \tilde{M}_{n1}(x) := \frac{x^{1/\gamma_1} P(\theta)}{G_{\hat{\theta}}(X_{n-k:n}) F(x,X_{n-k:n};\theta)} \int_x^\infty w^{1/\gamma_2} dF^*_n(X_{n-k:n:w}). \]

Making use of representation (5.24), we write
\[ \tilde{M}_{n1}(x) = x^{-1/\gamma_1} \left( \int_x^\infty \frac{G_{\hat{\theta}}(X_{n-k:n})}{G_{\hat{\theta}}(X_{n-k:n:w})} \frac{dF^*_n(X_{n-k:n:w})}{F^*(X_{n-k:n})} \right)^{-1} \times \left( \int_x^\infty w^{1/\gamma_2} dF^*_n(X_{n-k:n:w}) \right). \]

Once again by using the routine manipulations of Potter’s inequalities, we show that the first quantity between two brackets is
\[ (1 + o_P(1)) \int_x^\infty w^{1/\gamma_2+\epsilon/2} \frac{dF^*_n(X_{n-k:n:w})}{F^*(X_{n-k:n:w})}. \]
By using an integration by parts to the previous integral yields
\[ w^{1/\gamma_2+\epsilon/2} \frac{F^*_n(X_{n-k:n})}{F^*(X_{n-k:n})} + (1/\gamma_2 + \epsilon/2) \int_x^\infty w^{1/\gamma_2+\epsilon/2-1} \frac{F^*_n(X_{n-k:n:x})}{F^*(X_{n-k:n:x})} dw. \]
Recall that from (1.7) we have \( \bar{F}^* \in \mathcal{R}Y_{(-1/\gamma)} \), then
\[
\frac{\bar{F}^* (X_{n-k:n} w)}{\bar{F}^* (X_{n-k:n})} = (1 + o_P (1)) w^{-1/\gamma + \epsilon/2},
\]
uniformly on \( w > 1 \). Therefore the previous quantity reduces into
\[
(1 + o_P (1)) \left( 1 + \frac{1/\gamma_2 + \epsilon/2}{-1/\gamma_1 + \epsilon} \right) x^{-1/\gamma_1 + \epsilon}.
\]
Thereby the first expression between two brackets in (5.32) equals \( O_P \left( x^{1/\gamma - \epsilon} \right) \). Let us consider the second factor in (5.32). By similar arguments as used for the first factor, we show that
\[
x^{1/\gamma_2 + \epsilon/2} \frac{\bar{F}^* (X_{n-k:n} x)}{\bar{F}^* (X_{n-k:n})} + (1/\gamma_2 + \epsilon/2) \int_x^{\infty} w^{1/\gamma_2 + \epsilon/2} \frac{\bar{F}^* (X_{n-k:n} x)}{\bar{F}^* (X_{n-k:n})} dw,
\]
multiplied by \( (1 + o_P (1)) \), uniformly on \( x > 1 \). From Lemma 7.1, we have
\[
\frac{\bar{F}^* (X_{n-k:n} w)}{\bar{F}^* (X_{n-k:n})} = O_P \left( w^{-1/\gamma + \epsilon/2} \right),
\]
which implies that the previous expression equals \( O_P \left( x^{-1/\gamma + \epsilon} \right) \), thus \( \tilde{M}_{n1} (x) = O_P \left( w^{-1/\gamma + \epsilon} \right) \) and therefore
\[
\sqrt{k} M_{n1} (x) = (k/n)^{1/2 - \epsilon_1} O_P \left( w^{-1/\gamma + \epsilon} \right).
\]
By assumption \( k/n \to 0 \), it follows that \( \sqrt{k} M_{n1} (x) = o_P \left( x^{-1/\gamma + \epsilon} \right) \) which meets the result of (5.32). Let now consider the second term \( M_{n2} (x) \) which may be rewritten into
\[
- x^{-1/\gamma} \int_x^{\infty} \frac{\bar{F} (X_{n-k:n}; \theta) / \bar{G}_\theta (X_{n-k:n})}{\bar{F} (x, X_{n-k:n}; \theta) / \bar{F} (X_{n-k:n}; \theta)} \frac{\bar{F}^* (X_{n-k:n} w)}{\bar{F}^* (X_{n-k:n})} \frac{d}{k/n} \frac{\bar{F}^* (X_{n-k:n} w) - \bar{F}^* (X_{n-k:n})}{\bar{F}^* (X_{n-k:n}) - \bar{F} (x, X_{n-k:n}; \theta)}.
\]
In view of Potter’s inequalities, it is clear that
\[
\frac{\bar{F} (X_{n-k:n}; \theta)}{\bar{F} (X_{n-k:n}) / \bar{G}_\theta (X_{n-k:n})} \overset{P}{\to} \gamma_1 / \gamma \quad P \theta
\]
and
\[
\frac{\bar{F} (X_{n-k:n}; \theta)}{\bar{F} (x, X_{n-k:n}; \theta)} \overset{P}{\to} x^{1/\gamma}.
\]
Note that \( \bar{F}^* (X_{n-k:n}) \overset{d}{=} \xi_{k+1:n} \) and Smirnov’s lemma (see, e.g., Lemma 2.2.3 in de Haan and Ferier, 2006) implies that \( \xi_{k+1:n} \overset{P}{\to} 1 \), hence \( \bar{F}^* (X_{n-k:n}) = 1 + o_P (1) \). Therefore
\[
M_{n2} (x) = - (1 + o_P (1)) \frac{\gamma}{\gamma_1} \int_x^{\infty} \frac{\bar{G}_\theta (X_{n-k:n})}{\bar{G}_\theta (X_{n-k:n})} \frac{d}{k/n} \frac{\bar{F}^* (X_{n-k:n} w) - \bar{F}^* (X_{n-k:n})}{\bar{F}^* (X_{n-k:n}) - \bar{F} (x, X_{n-k:n}; \theta)}.
\]
On the other hand, by using an integration by parts yields
\[
M_{n2}^{(2)}(x) = (1 + o_P(1)) \frac{\gamma_1}{\gamma} \left( M_{n2}^{(1)}(x) + M_{n2}^{(2)}(x) \right),
\]
where
\[
M_{n2}^{(1)}(x) := \int_x^{\infty} \frac{F_n^t(X_{n-k_n}; t; \theta)}{k/n} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} \, dt
\]
and
\[
M_{n2}^{(2)}(x) := \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} \frac{F_n^t(X_{n-k_n}; t; \theta)}{k/n} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)}.
\]
By using the change of variables \( t = \gamma_0(X_{n-k_n}) \int \gamma_0(X_{n-k_n}; w) \), it is easy to verify that
\[
M_{n2}^{(1)}(x) = \int_x^{\infty} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} \frac{n}{k} \left( F_n^t(\vartheta_n(t; \theta)) - F_n^t(\vartheta_n(t; \theta)) \right) \, dt,
\]
where \( \vartheta_n(t; \theta) := \frac{n}{k} F_n^t(\gamma_0(X_{n-k_n}) t^{-1}) \). Observe that
\[
M_{n2}^{(1)}(x) = \int_x^{\infty} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} \left( \gamma_n(t; \theta) - \vartheta_n(t; \theta) \right) \, dt,
\]
where \( \gamma_n(t; \theta) \) being the tail empirical df given in (5.26), thereby
\[
\sqrt{k} M_{n2}^{(1)}(x) = \int_x^{\infty} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} \alpha_n(\vartheta_n(t; \theta)) \, dt,
\]
where \( \alpha_n \) being the tail empirical process defined in (5.25). Let us decompose the previous integral into
\[
\int_x^{\infty} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} \left( \alpha_n(\vartheta_n(t; \theta)) - W(\vartheta_n(t; \theta)) \right) \, dt + \int_x^{\infty} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} W(\vartheta_n(t; \theta)) \, dt
\]
\[= S_n + R_n.\]
By applying weak approximation (5.27) we get
\[
S_n = o_P(1) \int_x^{\infty} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} (\vartheta_n(t; \theta))^{1/2-\epsilon} \, dt.
\]
Observe that \( F_n^t(\gamma_0(X_{n-k_n}) t^{-1}) = F_n^t(X_{n-k_n}) \), thereby
\[
\vartheta_n(t; \theta) = \frac{n}{k} F_n^t(X_{n-k_n}) \frac{F_n^t(\gamma_0(X_{n-k_n}) t^{-1})}{F_n^t(\gamma_0(X_{n-k_n}) t^{-1})}.
\]
It is easy to check that \( F_n^t(\gamma_0(X_{n-k_n}) t^{-1}) \in RV(\gamma_2/\gamma) \), then once again by means of Pooter’s inequality, we show that \( \vartheta_n(t; \theta) = (1 + o_P(1)) t^{-\gamma_2/\gamma+\epsilon} \), therefore
\[
S_n = o_P(1) \int_x^{\infty} \frac{\gamma_0(X_{n-k_n})}{\gamma_0(X_{n-k_n}; t)} (t^{-\gamma_2/\gamma+\epsilon})^{1/2-\epsilon} \, dt.
\]
By using an elementary integration we get
\[ S_n = o_p \left( 1 \right) \left( \frac{G_{\theta} (X_{n-k:n})}{G_{\theta} (X_{n-k:n} \gamma)} \right)^{-\gamma \epsilon / \gamma + \epsilon} = o_p \left( x^{-\frac{1}{2} - \frac{1}{2\gamma} + \epsilon} \right). \]

By replacing \( \gamma \) by its expression given in (1.8), we end up with
\[ S_n = o_p \left( x^{\frac{1}{2} \left( \frac{1}{\gamma} - \frac{1}{\gamma} \right) + \epsilon} \right). \]

The term \( R_n \) may be decomposed into
\[ \int_{x^{1/2}} x^{1/2} W (\vartheta_n (t; \theta)) \, dt + \int_{x^{1/2}} \infty W (\vartheta_n (t; \theta)) \, dt = R_{n1} + R_{n2}. \]

It is clear that
\[ |R_{n1}| < \left\{ \frac{\sup_{t > x^{1/2}} \frac{W (\vartheta_n (t; \theta))}{(\vartheta_n (t; \theta))}}{\sup_{0 < s < x^{1/2}} |W (s)|} \right\} \int_{x^{1/2}} x^{1/2} (\vartheta_n (t; \theta))^\epsilon \, dt. \]

It is ready to check, by using the change of variables \( \vartheta_n (t; \theta) = s \), that the previous first factor between the curly brackets equals
\[ \sup_{0 < s < \frac{x^{1/2}}{n} F (X_{n-k:n}; \theta)} |W (s)| < \sup_{0 < s < \frac{x^{1/2}}{n} F (X_{n-k:n}; \theta)} s^\epsilon. \]

From Lemma 3.2 in Einmahl et al. (2006) \( \sup_{0 < s < 1} s^{-\delta} |W (s)| = O_p (1) \), for any \( 0 < \delta < 1/2, \) then since \( nF_{\theta} (x_{n-k:n}; \theta) / k \to P \), as \( n \to \infty \), we infer that
\[ \sup_{0 < s < \frac{x^{1/2}}{n} F (X_{n-k:n}; \theta)} s^{-\epsilon} |W (s)| = O_p (1). \]

for all large \( n \). On the other hand, we already pointed out above that
\[ \vartheta_n (t; \theta) = (1 + o_p (1)) t^{-\gamma \epsilon / \gamma + \epsilon}, \]

which implies that the second factor is equal to
\[ O_p (1) \int_{x^{1/2}} x^{1/2} (t^{-\gamma \epsilon / \gamma + \epsilon})^\epsilon \, dt = O_p (1) \int_{x^{1/2}} x^{1/2} t^{-\epsilon \gamma / \gamma + \epsilon} \, dt, \]

which after integration yields
\[ O_p (1) \left\{ \left( \frac{G_{\theta} (X_{n-k:n})}{G_{\theta} (X_{n-k:n} \gamma)} \right)^{-\epsilon \gamma / \gamma + \epsilon + 1} - \left( x^{-1 / \gamma} \right)^{-\epsilon \gamma / \gamma + \epsilon + 1} \right\}. \]

Recall that from formula (1.8) we have \( \gamma \gamma \theta > 1 \), then by using the mean value theorem and Pooter’s inequalities, we get \( R_{n1} = o_p (x^{-\epsilon}). \) The second term \( R_{n2} \) may be decomposed into
\[ R_{n2} = \int_{x^{1/2}} \infty (W (\vartheta_n (t; \theta)) - W (t^{-\gamma \epsilon / \gamma})) \, dt + \int_{x^{1/2}} \infty W (t^{-\gamma \epsilon / \gamma}) \, dt. \]
From Proposition B.1.10 in de Haan and Ferreira (2006), with high probability,
\[ c_n (t; \theta) := \left| \vartheta_n (t; \theta) - t^{-\gamma_2/\gamma} \right| \leq \epsilon t^{-\gamma_2/\gamma - \epsilon}, \quad \text{as } n \to \infty, \tag{5.33} \]
this means that \( \sup_{x \geq 1} \sup_{t > x^{1/\gamma_2}} c_n (t; \theta) \to 0, \) as \( n \to \infty. \) This implies by using Levy’s modulus of continuity of the Wiener process (see, e.g., Theorem 1.1.1 in Csörgő and Révész, 1981), that
\[ \left| W \left( \vartheta_n (t; \theta) \right) - W \left( t^{-\gamma_2/\gamma} \right) \right| \leq 2 \sqrt{c_n (t; \theta) \log \left( 1/c_n (t; \theta) \right)}, \]
with high probability. By using the fact that \( \log s < \epsilon s^{-\epsilon}, \) for \( s \downarrow 0 \) together with inequality (5.33), we show that
\[ \left| W \left( \vartheta_n (t; \theta) \right) - W \left( t^{-\gamma_2/\gamma} \right) \right| < 2 \epsilon t^{-\left( \gamma_2/\gamma - \epsilon \right)/2}, \]
uniformly on \( t > x^{1/\gamma_2}, \) it follows that
\[ \left| \int_{x^{1/\gamma_2}}^{\infty} \left( W \left( \vartheta_n (t; \theta) \right) - W \left( t^{-\gamma_2/\gamma} \right) \right) dt \right| = o_{\mathbb{P}} \left( 1 \right) \left| \int_{x^{1/\gamma_2}}^{\infty} t^{-\left( \gamma_2/\gamma - \epsilon \right)/2} dt \right|. \]
Recall that the assumption \( \gamma_1 < \gamma_2 \) together with the equation \( 1/\gamma = 1/\gamma_1 + 1/\gamma_2, \) imply that \( \gamma_2 / (2 \gamma) > 1, \) thus \( - (\gamma_2 / \gamma - \epsilon) / 2 + 1 < 0, \) therefore \( \left| \int_{x^{1/\gamma_2}}^{\infty} t^{-\left( \gamma_2/\gamma - \epsilon \right)/2} dt \right| = o_{\mathbb{P}} \left( x^{-1/\gamma_1 - \epsilon} \right) \). Then we showed that
\[ R_{n1} = o_{\mathbb{P}} \left( x^{-\epsilon} \right) \quad \text{and} \quad R_{n2} = \int_{x^{1/\gamma_2}}^{\infty} W \left( t^{-\gamma_2/\gamma} \right) dt + o_{\mathbb{P}} \left( x^{-1/\gamma_1 - \epsilon} \right), \]
hence
\[ \sqrt{k} M_{n2}^{(1)} (x) = R_n + S_n = \int_{x^{1/\gamma_2}}^{\infty} W \left( t^{-\gamma_2/\gamma} \right) dt + o_{\mathbb{P}} \left( x^{-1/\gamma_1 - \epsilon} \right) + o_{\mathbb{P}} \left( x^{1/2 \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right) + \epsilon} \right). \]
It is clear that
\[ \left( - \frac{1}{\gamma_1} - \epsilon \right) - \left( \frac{1}{2} \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right) + \epsilon \right) = - \frac{\gamma_1 + \gamma_2 + 4 \epsilon \gamma_1 \gamma_2}{2 \gamma_1 \gamma_2} < 0, \]
then
\[ \sqrt{k} M_{n2}^{(1)} = \int_{x^{1/\gamma_2}}^{\infty} W \left( t^{-\gamma_2/\gamma} \right) dt + o_{\mathbb{P}} \left( x^{1/2 \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right) + \epsilon} \right). \]
By using similar arguments we end up with
\[ \sqrt{k} M_{n2}^{(2)} (x) = x^{1/\gamma_2} W \left( t^{-1/\gamma} \right) \quad \text{and} \quad \sqrt{k} M_{n2}^{(2)} (x) = x^{1/\gamma_1} \int_{x^{1/\gamma_2}}^{\infty} W \left( t^{-\gamma_2/\gamma} \right) dt + o_{\mathbb{P}} \left( x^{1/2 \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right) + \epsilon} \right), \]
therefore we omit further details.
\[ \sqrt{k} M_{n2} = \frac{\gamma}{\gamma_1} x^{1/\gamma_2} W \left( t^{-1/\gamma} \right) + \frac{\gamma}{\gamma_1} \int_{x^{1/\gamma_2}}^{\infty} W \left( t^{-\gamma_2/\gamma} \right) dt + o_{\mathbb{P}} \left( x^{1/2 \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right) + \epsilon} \right). \]
Let us now focus of the term $M_{n3}$. By letting $x = 1$ in the previous weak approximation we infer that
\[
\sqrt{k} \frac{\mathbf{F}_n(x_{n-k:n}; \theta) - \mathbf{F}(x_{n-k:n}; \theta)}{\mathbf{F}(x_{n-k:n}; \theta)} = \frac{\gamma}{\gamma_1} W(1) + \frac{\gamma}{\gamma_1} \int_1^\infty W\left(t^{-\gamma/\gamma_1}\right) dt + o_P(1),
\]
(5.34)
which implies that
\[
\sqrt{k} \frac{\mathbf{F}_n(x_{n-k:n}; \theta) - \mathbf{F}(x_{n-k:n}; \theta)}{\mathbf{F}(x_{n-k:n}; \theta)} = O_P(1).
\]
In other terms, we have
\[
\mathbf{F}_n(x_{n-k:n}; \theta) = 1 + O_P(k^{-1/2}).
\]
(5.35)
The regular variation of $\mathbf{F}(\cdot; \theta)$ and (5.35) together imply that
\[
\frac{\mathbf{F}(x_{n-k:n}; \theta)}{\mathbf{F}_n(x_{n-k:n}; \theta)} = x^{-1/\gamma_1} + o_P(x^{-1/\gamma_1+\epsilon}).
\]
(5.36)
By combining the results (5.34) and (5.36) we get
\[
\sqrt{k} M_{n3}(x) = -x^{-1/\gamma_2}\left(\frac{\gamma}{\gamma_1} W(1) + \frac{\gamma}{\gamma_1} \int_1^\infty W\left(t^{-\gamma/\gamma_1}\right) dt + o_P(x^{-1/\gamma_1+\epsilon})\right).
\]
For the forth term we write
\[
\sqrt{k} M_{n4}(x) = \left(\frac{\mathbf{F}(x_{n-k:n}; \theta)}{\mathbf{F}_n(x_{n-k:n}; \theta)} - x^{-1/\gamma_1}\right) \left(\sqrt{k} \frac{\mathbf{F}_n(x_{n-k:n}; \theta) - \mathbf{F}(x_{n-k:n}; \theta)}{\mathbf{F}(x_{n-k:n}; \theta)}\right).
\]
From (5.36) the first factor of the previous equation equals $o_P(x^{-1/\gamma_1+\epsilon})$. On the other hand by using the change of variables $s = t^{-\gamma/\gamma_2}$, yields
\[
\int_{x^{1/\gamma_2}}^\infty W\left(t^{-\gamma/\gamma_2}\right) dt = \frac{\gamma}{\gamma_2} \int_0^{x^{-1/\gamma}} s^{-\gamma/\gamma_2-1} W(s) ds.
\]
Since $\sup_{0<s<1} s^{-1/\gamma+\epsilon} |W(s)| = O_P(1)$, then we easily show that
\[
\int_{x^{1/\gamma_2}}^\infty W\left(t^{-\gamma/\gamma_2}\right) dt = O_P\left(x^{\frac{1}{2}\left(\frac{\gamma}{\gamma_2} - \frac{1}{\gamma_1}\right)+\epsilon}\right),
\]
it follows that $\sqrt{k} M_{n2} = O_P\left(x^{\frac{1}{2}\left(\frac{\gamma}{\gamma_2} - \frac{1}{\gamma_1}\right)+\epsilon}\right)$ as well. Therefore
\[
\sqrt{k} \frac{\mathbf{F}_n(x_{n-k:n}; \theta) - \mathbf{F}(x_{n-k:n}; \theta)}{\mathbf{F}(x_{n-k:n}; \theta)} = x^{1/\gamma_1} O_P\left(x^{\frac{1}{2}\left(\frac{\gamma}{\gamma_2} - \frac{1}{\gamma_1}\right)+\epsilon}\right) = O_P\left(x^{\frac{1}{2\gamma}+\epsilon}\right).
\]
Hence we have
\[
\sqrt{k} M_{n4}(x) = o_P\left(x^{-1/\gamma_1+\epsilon}\right) = O_P\left(x^{\frac{1}{2\gamma}+\epsilon}\right) = o_P\left(x^{\frac{1}{2}\left(\frac{\gamma}{\gamma_2} - \frac{1}{\gamma_1}\right)+\epsilon}\right).
\]
By assumption $\mathbf{F}$ satisfies the second order condition of regular variation function (1.5), this means that for

$$
\lim_{t \to \infty} \frac{\mathbf{F}(tx) / \mathbf{F}(t) - x^{-1/\gamma_1}}{\mathbf{A}(t)} = x^{-1/\gamma_1} \frac{x^{\rho_1/\gamma_1} - 1}{\rho_1 \gamma_1},
$$

(5.37)

for any $x > 0$, where $\rho_1 < 0$ is the second-order parameter and $\mathbf{A}$ is $\mathcal{RV}(\rho_1/\gamma_1)$. The uniform inequality corresponding to (5.37) says: there exist $t_0 > 0$, such that for any $t > t_0$, we have

$$
\left| \frac{\mathbf{F}(tx) / \mathbf{F}(t) - x^{-1/\gamma_1}}{\mathbf{A}(t)} - x^{-1/\gamma_1} \frac{x^{\rho_1/\gamma_1} - 1}{\rho_1 \gamma_1} \right| < \epsilon x^{-1/\gamma_1 + \rho_1/\gamma_1 + \epsilon},
$$

see for instance assertion (2.3.23) of Theorem 2.3.9 in de Haan and Ferreira (2006). It is easy to check that the later inequality implies that

$$
\sqrt{k} \mathbf{M}_{n5}(x) = \sqrt{k} \left( \frac{\mathbf{F}(x X_{n-k:n}; \theta)}{\mathbf{F}(X_{n-k:n}; \theta)} - x^{-1/\gamma_1} \right)
$$

$$
= x^{-1/\gamma_1} \frac{x^{\rho_1/\gamma_1} - 1}{\rho_1 \gamma_1} \sqrt{k} \mathbf{A}(X_{n-k:n}) + o_P \left( x^{-1/\gamma_1} \frac{x^{\rho_1/\gamma_1} - 1}{\rho_1 \gamma_1} \right) \sqrt{k} \mathbf{A}(X_{n-k:n})
$$

Recall that $a_k := F_{\mathcal{RV}}^*(1 - k/n)$ and notice that $X_{n-k:n}/a_k \xrightarrow{P} 1$ as $n \to \infty$, then in view of the regular variation of $\mathbf{A}$ we infer that $\mathbf{A}(X_{n-k:n}) = (1 + o_P(1)) \mathbf{A}(a_k)$. On the other hand, by assumption $\sqrt{k} \mathbf{A}(a_k)$ is asymptotically bounded, therefore

$$
\sqrt{k} \mathbf{M}_{n5}(x) = x^{-1/\gamma_1} \frac{x^{\rho_1/\gamma_1} - 1}{\rho_1 \gamma_1} \sqrt{k} \mathbf{A}(a_k) + o_P \left( x^{-1/\gamma_1} \right).
$$

To summarize, as this stage we showed that

$$
\mathbf{D}_n \left( x; \hat{\theta} \right) = \frac{\gamma}{\gamma_1} x^{1/\gamma_2} W \left( t^{-1/\gamma_1} \right) + \frac{\gamma}{\gamma_1} \int_{x^{1/\gamma_2}}^{\infty} W \left( t^{-\gamma_2/\gamma_1} \right) dt
$$

$$
- x^{-1/\gamma_2} \left( \frac{\gamma}{\gamma_1} W(1) + \frac{\gamma}{\gamma_1} \int_1^{\infty} W \left( t^{-\gamma_2/\gamma_1} \right) dt \right)
$$

$$
+ x^{-1/\gamma_1} \frac{x^{\rho_1/\gamma_1} - 1}{\rho_1 \gamma_1} \sqrt{k} \mathbf{A}(a_k) + \zeta(x),
$$

where $\zeta(x) := o_P \left( x^{-1/\gamma_1 + \epsilon} \right) + o_P \left( x^{-1/\gamma_1} \right) + o_P \left( x^{\frac{1}{2} \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right) + \epsilon} \right)$. By using a change of variables, we show that sum of the first three terms equals the Gaussian process $\Gamma(x; W)$ stated in Theorem 2.1. Recall that $\gamma_1 < \gamma_2$ and

$$
\frac{1}{2} \left( \frac{1}{\gamma_2} - \frac{1}{\gamma_1} \right) + \epsilon < 0,
$$
then it is easy to verify that \( \zeta(x) = o_P \left( x^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{n} \right) + \epsilon} \right) \). It follows that

\[
x^\epsilon \left\{ D_n \left( x; \hat{\theta} \right) - \Gamma(x; W) - x^{-1/\gamma_1} x^{\rho_1/\gamma_1} - 1 \sqrt{k A} (a_k) \right\} = o_P \left( x^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{n} \right) + 2\epsilon} \right) = o_P \left( 1 \right),
\]

uniformly on \( x > 1 \), therefore

\[
\sup_{x > 1} x^\epsilon \left| D_n \left( x; \hat{\theta} \right) - \Gamma(x; W) - x^{-1/\gamma_1} x^{\rho_1/\gamma_1} - 1 \sqrt{k A} (a_k) \right| = o_P \left( 1 \right),
\]

for any small \( 0 < \epsilon < 1/2 \), which completes the proof of Theorem 2.1.

5.2. Proof of Theorem 2.2. From the representation (1.17) we write

\[ \hat{\gamma}_1 - \gamma_1 = T_{n1} + T_{n2} + T_{n3}, \]

where

\[
T_{n1} := k^{-1/2} \int_1^\infty x^{-1} \left\{ D_n \left( x; \hat{\theta}; \gamma_1 \right) - \Gamma(x; W) - x^{-1/\gamma_1} x^{\rho_1/\gamma_1} - 1 \sqrt{k A} (a_k) \right\} dx
\]

\[
T_{n2} := k^{-1/2} \int_1^\infty x^{-1} \Gamma(x; W) dx
\]

and

\[
T_{n3} := -A (a_k) \int_1^\infty x^{-1/\gamma_1 - 1} x^{\rho_1/\gamma_1} - 1 dx.
\]

By using 2.1 yields \( T_{n1} = o_P \left( k^{-1/2} \right) \int_1^\infty x^{-1+\epsilon} dx = o_P \left( k^{-1/2} \right) \). Since \( E \left| W(s) \right| \leq s^{1/2} \), then it is easy to show that \( \int_1^\infty x^{-1} \Gamma(x; W) dx = O_P \left( 1 \right) \), it follows that \( T_{n2} = O_P \left( k^{-1/2} \right) \). By using an elementary integration, we get \( T_{n3} := \frac{A(a_k)}{1 - \rho_1} \). Since both \( k^{-1/2} \) and \( A(a_k) \) tend to zero as \( n \to \infty \), then \( T_{n3} = o \left( 1 \right) \), it follows that \( \hat{\gamma}_1 \xrightarrow{P} \gamma_1 \), which gives the first result of Theorem. To establish the asymptotic normality

\[
\sqrt{k} \left( \hat{\gamma}_1 - \gamma_1 \right) = \sqrt{k} T_{n1} + \sqrt{k} T_{n2} + \sqrt{k} T_{n3},
\]

where

\[
\sqrt{k} T_{n1} = o_P \left( 1 \right), \quad \sqrt{k} T_{n2} = \int_1^\infty x^{-1} \Gamma(x; W) dx
\]

and

\[
\sqrt{k} T_{n2} = \frac{\sqrt{k A (a_k)}}{1 - \rho_1}.
\]

Note that \( \Gamma(x; W) \) is a centred Gaussian process and by using the assumption \( \sqrt{k A (a_k)} \to \lambda < \infty \), we end up with

\[
\sqrt{k} \left( \hat{\gamma}_1 - \gamma_1 \right) \xrightarrow{D} \mathcal{N} \left( \frac{\lambda}{1 - \rho_1}; E \left[ \int_1^\infty x^{-1} \Gamma(x; W) dx \right]^2 \right).
\]
By using elementary calculation we show that $E \left[ \int_1^\infty x^{-1} \Gamma (x; W) dx \right]^2 = \sigma^2$, that we omit the details.

6. Conclusion

In basis on a semiparametric estimator of the underlying distribution function, we proposed a new estimation method to the tail index of Pareto-type distributions for randomly right-truncated data. Compared with the existing ones, this estimator behaves well both in terms of bias and rmse. A useful weak approximation of the corresponding tail empirical process allowed us to establish both the consistency and asymptotic normality of the proposed estimator.

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7. Appendix

**Lemma 7.1.** For any small $\epsilon > 0$, we have

$$
\frac{\hat{F}_n(X_{n-k:n}w)}{F(X_{n-k:n})} = O_P\left(w^{-1/\gamma+\epsilon/2}\right), \text{ uniformly on } w \geq 1.
$$

**Proof.** Let $V_n(t) := n^{-1} \sum_{i=1}^{n} 1(\xi_i \leq t)$ be the uniform empirical df pertaining to the sample $\xi_i := F^*(X_i), i = 1, ..., n$, of iid uniform$(0, 1)$ rv’s. It is clear that, for an arbitrary $x$, we have $V_n\left(F^*(x)\right) = F_n(x)$ almost surely. From Assertion 7 in *Shorack and Wellner (1986)* (page 415), $V_n(t)/t = O_P\left(1\right)$ uniformly on $1/n \leq t \leq 1$. 

this implies that
\[
\frac{\bar{F}_n(X_{n-k:n})}{\bar{F}(X_{n-k:n})} = O_p(1), \text{ uniformly on } w \geq 1. \quad (7.38)
\]

On the other hand, by applying Potter’s inequalities (1.4) to \(\bar{F}^v\), we get
\[
\frac{\bar{F}^v(X_{n-k:n})}{\bar{F}(X_{n-k:n})} = O_p\left(w^{-1/\gamma+\varepsilon/2}\right), \text{ uniformly on } w \geq 1. \quad (7.39)
\]

Combining the two statements (7.38) and (7.39) gives the desired results. \(\square\)
