DEGENERATION OF LOG CALABI-YAU PAIRS VIA LOG CANONICAL PLACES

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Abstract. Let \((X, \Delta)\) be a projective log canonical Calabi-Yau pair and \(L\) an ample \(\mathbb{Q}\)-line bundle on \(X\), we show that there is a correspondence between lc places of \((X, \Delta)\) and weakly special test configurations of \((X, \Delta; L)\).

Contents

1. Introduction 1
2. Local correspondence 4
3. Global correspondence 8
4. Generalized global correspondence 11
5. Remark on global correspondence for log Fano pairs 18
References 19

1. INTRODUCTION

We work over \(\mathbb{C}\) throughout.

Test configurations are basic objects in the study of K-stability of Fano varieties. After the work \([\text{BHJ}17]\), people realize that the language via test configurations can be reformulated by a even more basic concept, i.e. valuations. As a special example of valuations, divisorial valuations play important roles in the recent development of algebraic K-stability theory of Fano varieties (e.g. \([\text{Xu}21]\)). For example, any weakly special test configuration with integral central fiber of a \(\mathbb{Q}\)-Fano variety (i.e. a Fano variety with klt singularities) could be induced by an lc place of some complement (see \([\text{BLX}19]\) Appendix or \([\text{CZ}21]\), and via reformulating these test configurations by the corresponding lc places of complements, many tools in birational geometry come to play the powerful role. In the realm of Calabi-Yau varieties, K-stability theory could be translated into singularity theory by \([\text{Oda}13]\) (see also \([\text{BHJ}17]\) Section 9)), in other words, people are mainly interested in Calabi-Yau varieties with log canonical or semi-log canonical singularities. Inspired by this, we aim to reformulate test configurations via valuations (more precisely via divisorial valuations induced by lc places) for Calabi-Yau varieties, and expect to further study the degenerations of Calabi-Yau varieties in the moduli theory.

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We say that a projective log pair $(X, \Delta)$ is a log canonical (resp. semi-log canonical) Calabi-Yau pair (abbreviated by lc CY pair (resp. slc CY pair)) if $(X, \Delta)$ admits log canonical (resp. semi-log canonical) singularities and $K_X + \Delta \sim_{Q} 0$. We aim to establish the following result.

**Theorem 1.1.** Let $(X, \Delta)$ be a projective lc CY pair and $L$ an ample $\mathbb{Q}$-line bundle on $X$, then we have the following result:

1. If $E$ is an lc place of $(X, \Delta)$, then $E$ is dreamy with respect to $L$;
2. there is a correspondence between lc places of $(X, \Delta)$ and non-trivial weakly special test configurations of $(X, \Delta; L)$ with integral central fibers. More precisely, given an lc place $E$ of $(X, \Delta)$ and a positive integer $c \in \mathbb{Z}^+$, there exists a weakly special test configuration with integral central fiber $(X, \Delta_{tc}; \mathcal{L}) \to \mathbb{A}^1$ of $(X, \Delta; L)$ such that $\text{ord}_{\lambda_0}^{X}(E) = c \cdot \text{ord}_{E}$; conversely, given a weakly special test configuration with integral central fiber $(X, \Delta_{tc}; \mathcal{L}) \to \mathbb{A}^1$ of $(X, \Delta; L)$ such that $\text{ord}_{\lambda_0}^{X}(E) = c \cdot \text{ord}_{E}$ for some $c \in \mathbb{Z}^+$, then $E$ is an lc place of $(X, \Delta)$.

We just note here that $E$ is dreamy with respect to $L$ means the finite generation of the following graded ring

$$
\bigoplus_{m \in \mathbb{N}} \bigoplus_{j \in \mathbb{N}} H^0(Y, g^*(mrL) - jE),
$$

where $g : Y \to X$ is a normal projective birational model such that $E$ is a prime divisor on $Y$, and $r$ is a fixed positive integer such that $rL$ is Cartier. We say that the test configuration $(X, \Delta_{tc}; \mathcal{L}) \to \mathbb{A}^1$ of $(X, \Delta; L)$ is weakly special if $(X', \Delta_{tc} + \lambda_0)$ is log canonical (see Definition 2.2). Theorem 1.1 concerns weakly special test configuration with integral central fiber, while in Section 4, we study the case where the central fiber is reduced. The key ingredient to establish the above global correspondence is the following characterization of log canonical blow-ups (see Definition 2.3).

**Theorem 1.2.** (= Theorem 2.4) Let $(X, \Delta)$ be a projective lc CY pair and $L$ an ample $\mathbb{Q}$-line bundle on $X$. Denote by $Z = \text{Spec} \bigoplus_{m \in \mathbb{N}} H^0(X, mrL)$, where $r$ is a positive integer such that $rL$ is Cartier. Let $\Delta_Z$ be the extension of $\Delta$ on $Z$ and $o \in Z$ the cone vertex of $Z$. Suppose $w$ is a $C^*$-equivariant divisorial valuation corresponding to an lc blow-up over $o \in Z$, then $w$ is a quasi-monomial combination of $\nu_0$ and $\text{ord}_{E_{\infty}}^{X}$ with weight $(\lambda, c)$ for some $\lambda \in \mathbb{Z}^+$ and $c \in \mathbb{Q}^+_{> 0}$, where $\nu_0$ is the canonical valuation, $E_{\infty}$ is the natural extension of some prime divisor $E$ over $X$, and $E$ is an lc place of $(X, \Delta)$.

In Section 4, we will generalize Theorem 1.1 for weakly special test configurations with reduced central fibers, and establish a correspondence between weakly special collections (see Definition 4.2) and weakly special semi-test configurations (see Definition 2.2).

**Theorem 1.3.** (= Theorem 4.3) Let $(X, \Delta)$ be a projective lc CY pair and $L$ an ample $\mathbb{Q}$-line bundle on $X$, then there exists a correspondence between weakly special collections of $(X, \Delta)$ and weakly special semi-test configurations of $(X, \Delta; L)$. More precisely, given a weakly special collection $\{c_1 \cdot \text{ord}_{E_1}, \ldots, c_1 \cdot \text{ord}_{E_l}\}$, we can construct a weakly special semi-test configuration $(X, \Delta_{tc}; \mathcal{L})$ satisfying

$$
\text{ord}_{\lambda_0}^{X}(E_i) = c_i \cdot \text{ord}_{E_i}, i = 1, \ldots, l,
$$
where $\lambda_{0,i}$ are all non-trivial components of $X_0$; conversely, given a weakly special semi-test configuration $(X, \Delta; L)$ satisfying
\[
\text{ord}_{\lambda_{0,i}}|_{K(X)} = c_i \cdot \text{ord}_{E_i}, i = 1, \ldots, l,
\]
where $c_i \in \mathbb{Z}^+$ and $X_{0,i}$ are all non-trivial components of $X_0$, then the set $\{c_1 \cdot \text{ord}_{E_1}, \ldots, c_l \cdot \text{ord}_{E_l}\}$ is a weakly special collection.

As a corollary, we see that there are no non-trivial weakly special semi-test configurations of $(X, \Delta; L)$ if $(X, \Delta)$ is a klt Calabi-Yau pair (e.g. [Oda12 Corollary 4.3]).

We emphasize here that our definition of weakly special collection is a little different from that in [BLX19 Appendix A], as they define weakly special collections directly from weakly special semi-test configurations, while our definition automatically contains the information of lc places (see Definition 4.2). Thus the above theorem is also a correspondence between weakly special semi-test configurations and lc places of a given polarized lc CY pair.

**Remark 1.4.** For Theorem 1.1 and Theorem 1.3, we have explained precisely what the correspondences mean in the both statements. We emphasize that such correspondences are not bijective. For example, given a weakly special collection as in Theorem 1.3, we could construct many weakly special semi-test configurations of $(X, \Delta; L)$ satisfying the requirements (see the proof of Theorem 4.3 and Remark 4.6).

**Notation.** For a given normal variety $X$, we say that $\Delta$ is a $\mathbb{Q}$-divisor on $X$ if $\Delta$ can be put as a finite sum $\Delta := \sum a_i \Delta_i$, where $a_i \in \mathbb{Q}$ and $\Delta_i$ are Weil divisors on $X$. We say that $\Delta$ is $\mathbb{Q}$-Cartier if $m\Delta$ is a Cartier divisor for a sufficiently divisible integer $m \in \mathbb{Z}^+$. We say that $L$ is a $\mathbb{Q}$-line bundle on $X$ if $L^\otimes m$ (or put as $mL$) is a line bundle on $X$ for a sufficiently divisible integer $m \in \mathbb{Z}^+$.

We say that $(X, \Delta)$ is a log pair if $X$ is a normal variety and $\Delta$ is an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that $E$ is a prime divisor over $X$ if there is a proper birational morphism from a normal variety $Y$, denoted by $Y \to X$, such that $E$ is a prime divisor on $Y$.

For a given normal variety $X$, we say that $v$ is a divisorial valuation over $X$ if it is of the form $v = c \cdot \text{ord}_E$, where $c \in \mathbb{R}^+$ and $E$ is a prime divisor over $X$. We refer to [BHJ17 Section 1.3] for a basic introduction of valuation.

Given a normal projective variety $X$, suppose $L$ is an ample line bundle on $X$ and $Z := \text{Spec } \oplus_{m \in \mathbb{N}} H^0(X, mL)$ is the affine cone over $X$ with respect to $L$. Let $o \in Z$ be the cone vertex, then $Z \setminus o$ is a $\mathbb{C}^*$-bundle over $X$. Thus any prime divisor over $X$ admits a natural $\mathbb{C}^*$-equivariant extension over $Z$. More generally, any valuation over $X$ can be naturally extend to a $\mathbb{C}^*$-invariant valuation over $Z$. Suppose $\tilde{Z} \to Z$ is the blow-up of $o \in Z$, we write $v_0$ for the divisorial valuation corresponding to the exceptional divisor which is isomorphic to $X$ ($v_0$ is also called canonical valuation). Let $v$ be a valuation over $X$ and $\tilde{v}$ the valuation over $Z$ via the natural extension, then we put the quasi-monomial combination of $v_0$ and $\tilde{v}$ with weight $(\lambda, c)$ as $\lambda \cdot v_0 + c \cdot \tilde{v}$ for convenience.\footnote{We emphasize here that the valuation $\lambda \cdot v_0 + c \cdot \tilde{v}$ does not necessarily have the property $(\lambda \cdot v_0 + c \cdot \tilde{v})(f) = \lambda \cdot v_0(f) + c \cdot \tilde{v}(f)$ for every $f \in K(Z)^*$, it is just a notation of quasi-monomial combination for convenience.}
For various type of singularities in birational geometry such as lc, klt, dlt, slc, etc., we refer to \cite{KM98,Kol13}.

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2. LOCAL CORRESPONDENCE

In this section, we study weakly special test configuration from a local point of view. We first recall some basic concepts.

**Definition 2.1.** Let \((X, \Delta)\) be a log pair and \(E\) a prime divisor over \(X\), then we define the log discrepancy of \(E\) with respect to \((X, \Delta)\) as follows:

\[
A_{X, \Delta}(E) = \text{ord}_E(K_Y - g^*(K_X + \Delta)) + 1,
\]

where \(g : Y \to X\) is a log resolution such that \(E\) is a prime divisor on \(Y\). If \(A_{X, \Delta}(E) = 0\), we say that \(E\) is an lc place of \((X, \Delta)\).

**Definition 2.2.** Let \((X, \Delta)\) be a projective log pair and \(L\) an ample \(\mathbb{Q}\)-line bundle on \(X\). A semi-test configuration \(\pi : (X, \Delta_{tc}; L) \to \mathbb{A}^1\) is a family over \(\mathbb{A}^1\) consisting of the following data:

1. \(\pi : X \to \mathbb{A}^1\) is a projective flat morphism from a normal variety \(X\), \(\Delta_{tc}\) is an effective \(\mathbb{Q}\)-divisor on \(X\) and \(L\) is a relatively semi-ample \(\mathbb{Q}\)-line bundle on \(X\),
2. the family \(\pi\) admits a \(\mathbb{C}^*\)-action which lifts the natural \(\mathbb{C}^*\)-action on \(\mathbb{A}^1\) such that \((X, \Delta_{tc}; L) \times_{\mathbb{A}^1} \mathbb{C}^*\) is \(\mathbb{C}^*\)-equivariantly isomorphic to \((X, \Delta; L) \times_{\mathbb{A}^1} \mathbb{C}^*\).

We say that the semi-test configuration is a test configuration if \(L\) is relatively ample. We say that the (semi-)test configuration is weakly special if \((X, \Delta, \Delta_{tc})\) is log canonical. In this case, we also say that \((X, \Delta_{tc})\) is a weakly special degeneration of \((X, \Delta)\). For a given weakly special (semi-)test configuration \((X, \Delta_{tc}; L)\), if the central fiber \(X_0\) is integral, we say that it is a weakly special (semi-)test configuration with integral central fiber.

**Definition 2.3.** Let \((X, \Delta)\) be a log pair and \(x \in X\) is a closed point. Let \(E\) be a prime divisor over \(X\), we say that \(E\) is a log canonical blow-up (abbreviated by lc blow-up) over \(x\) if there exists a proper birational morphism \(f : Y \to X\) such that \(Y \setminus E \cong X \setminus x\), \((Y, f_x^{-1}\Delta + E)\) is log canonical, and \(-E\) is relatively ample.

In the rest of this section, we fix the following notation.

Let \((X, \Delta)\) be a projective lc CY pair and \(L\) an ample \(\mathbb{Q}\)-line bundle on \(X\). Consider the affine cone over \(X\) with respect to \(rL\):

\[
Z := \text{Spec} \bigoplus_{m \in \mathbb{N}} H^0(X, mrL),
\]

where \(r\) is a positive integer such that \(rL\) is Cartier. Denote by \(o \in Z\) the cone vertex and \(\Delta_Z\) the extension of \(\Delta\) on \(Z\). We need the following theorem to establish the global correspondence in Section 3.
Theorem 2.4. Suppose $w$ is a $\mathbb{C}^*$-equivariant divisorial valuation corresponding to an lc blow-up over $o \in Z$, then $w$ is a quasi-monomial combination of $v_0$ and $\text{ord}_{E_\infty}$ with weight $(\lambda, c)$ for some $\lambda \in \mathbb{Z}^+, c \in \mathbb{Q}^{\geq 0}$, where $E_\infty$ is the natural extension of some prime divisor $E$ over $X$, and $E$ is an lc place of $(X, \Delta)$.

Proof. If $w$ is obtained by the blow-up of $o \in Z$: $\tilde{Z} \to Z$, then $\lambda = 1, c = 0$. We may assume that $w$ is not obtained via this blow-up. By [BHJ17, Lemma 4.2], we can express $w$ as a quasi-monomial combination of $v_0$ and $r(w)$ with weight $(\lambda, 1)$ for some $\lambda \in \mathbb{Z}^+$,

$$w = \lambda \cdot v_0 + r(w),$$

where $r(w) = w|_{K(X)}$, and $\overline{r(w)}$ is the natural extension of $r(w)$. As $w$ is a divisorial valuation, by [BHJ17, Lemma 4.1], $r(w)$ is also divisorial. Thus one can find some $c \in \mathbb{R}^+$ and a prime divisor $E$ over $X$ such that $r(w) = c \cdot \text{ord}_E$. Denote by $E_\infty$ the natural extension of $E$, then $w$ is a quasi-monomial combination of $v_0$ and $\text{ord}_{E_\infty}$ with weight $(\lambda, c)$,

$$w = \lambda \cdot v_0 + c \cdot \text{ord}_{E_\infty}.$$

As $w$ is a divisorial valuation, the value group of $w$ has rational rank 1, which implies that $c \in \mathbb{Q}$. It remains to show that $E$ is an lc place of $(X, \Delta)$. We denote $f : W \to Z$ to be the lc blow-up which gives rise to $w$, and $G \subset W$ the exceptional divisor. Then we have $w = \text{ord}_G$ and

$$K_W + f_*^{-1} \Delta_Z + G = f^*(K_Z + \Delta_Z) + A_{Z, \Delta_Z}(G)G,$$

where $(W, f_*^{-1} \Delta_Z + G)$ is log canonical. If $A_{Z, \Delta_Z}(G) = 0$, then $G$ is already an lc place of $(Z, \Delta_Z)$. As $v_0$ is an lc place of $(Z, \Delta_Z)$ (see [Kol13, Proposition 3.14]), by [JM12, Proposition 5.1], $r(w)$ is also an lc place of $(Z, \Delta_Z)$, which implies that $E$ is an lc place of $(X, \Delta)$. From now on, we assume $A_{Z, \Delta_Z}(G) > 0$ and derive the contradiction.

Denote by

$$K_G + \Delta_G := (K_W + f_*^{-1} \Delta_Z + G)|_G,$$

then $(G, \Delta_G)$ is semi-log canonical by the sub-adjunction, and

$$-(K_G + \Delta_G) \sim_{\mathbb{Q}} -A_{Z, \Delta_Z}(G)G|_G$$

is ample. Choose a sufficiently divisible $k \in \mathbb{Z}^+$ such that

$$k(K_W + f_*^{-1} \Delta_Z + G), kG$$

are all Cartier. Consider the following exact sequence:

$$0 \to \mathcal{O}_W(-G - k(K_W + f_*^{-1} \Delta_Z + G)) \to \mathcal{O}_W(-k(K_W + f_*^{-1} \Delta_Z + G)) \to \mathcal{O}_G(-k(K_G + \Delta_G)) \to 0.$$  

Note that

$$-G - k(K_W + f_*^{-1} \Delta_Z + G) - (K_W + f_*^{-1} \Delta_Z) = -(k + 1)(K_W + f_*^{-1} \Delta_Z + G)$$

is ample over $Z$, by an lc version of Kawamata-Viehweg vanishing theorem (see [Fuj14, Theorem 1.7]), we have the following surjective map:

$$H^0(W, -k(K_W + f_*^{-1} \Delta_Z + G)) \to H^0(G, -k(K_G + \Delta_G)) \to 0.$$  

We may assume that $k$ is sufficiently large such that $-k(K_G + \Delta_G)$ is very ample, thus one can choose a general element $A_G$ in the linear system $|-k(K_G + \Delta_G)|$ such that $(G, \Delta_G + A_G)$
is still semi-log canonical. Via the above surjective map, one can extend $A_G$ to a divisor $A$ on $W$ such that $A|_G = A_G$. By the inversion of adjunction, the pair $(W, f_s^{-1}(Z) + G + A)$ is log canonical around $G$. Denote by $\Theta := f_s(1/kA)$, then it is not hard to see the following:

$$K_W + f_s^{-1}(Z) + G + \frac{1}{k}A = f^*(K_Z + Z + \Theta).$$

By the choice of $k$, we see that $k\Theta$ is a Cartier divisor on $Z$, then we may write $k\Theta = \text{div}(h)$ for some $h \in O_{Z,o}$. Note here that $(Z, Z + \Theta)$ is log canonical around $o \in Z$ and $v_0$ is an lc place of $(Z, Z + \Theta)$. Denote by $\Theta' := \frac{1}{k} \cdot \text{div}(\text{in}(h))$, where $\text{in}(h)$ is the initial part of $h$, then by \cite{dFEM10} Theorem 3.1, the pair $(Z, Z + \Theta')$ is also log canonical around $o \in Z$. Since $\Theta'$ is $\mathbb{C}^*$-equivariant, it can be obtained as the extension of a divisor $D$ on $X$, or in other words, the pair $(Z, Z + \Theta')$ is the cone over $(X, \Delta + D)$ with respect to $rL$. By our construction of $\Theta'$, it is not hard to see that $\Theta'$ is an effective divisor which is not zero, and $K_Z + Z + \Theta'$ is $\mathbb{Q}$-Cartier. By \cite{Kol13} Proposition 3.14(4), we have

$$L \sim_{\mathbb{Q}} a \cdot (K_X + \Delta + D)$$

for some $a \in \mathbb{Q}^+$. This is a contradiction, since the cone over a stable pair can never be log canonical (see \cite{Kol13} Lemma 3.1). The contradiction implies that $A_{Z,\Delta}(G) = 0$, and thus $E$ is an lc place of $(X, \Delta)$.

Remark 2.5. This remark is pointed out by one of the Referees. In the above proof, by assuming $A_{Z,\Delta}(G) > 0$, we obtain a log canonical pair $(Z, Z + \Theta)$ such that $v_0$ is an lc place. As $v_0$ is already the lc place of $(Z, Z)$ and $\Theta$ passes through the cone vertex via construction, thus $v_0$ cannot be an lc place of $(Z, Z + \Theta)$. This already deduces a contradiction which implies that $A_{Z,\Delta}(G) = 0$ and we do not actually need the argument on initial degeneration. We thank the referee for this simplified argument.

We next establish the following local correspondence, which is similar to the local correspondence for log Fano cone singularities (see \cite{Xu21} Proposition 4.25 or \cite{LWX21} Lemma 2.21).

Theorem 2.6. There is a correspondence between non-trivial weakly special test configurations with integral central fibers of $(X, \Delta; L)$ and non-trivial $\mathbb{C}^*$-equivariant lc blow-up\footnote{This means that the blow-up is not obtained via canonical blow-up of $o \in Z$ which gives rise to $v_0$.} over $o \in Z$. More precisely, given a weakly special test configuration with integral central fiber $(X, \Delta_c; L) \rightarrow \mathbb{A}^1$ of $(X, \Delta; L)$ such that $\text{ord}_{X_0}|_{K(X)} = a \cdot \text{ord}_E$ for some $a \in \mathbb{Z}^+$, then one could obtain a $\mathbb{C}^*$-equivariant lc blowup of the form $\lambda \cdot v_0 + a \cdot \text{ord}_{E_\infty}$ for some $\lambda \in \mathbb{Z}^+$; conversely, given a non-trivial $\mathbb{C}^*$-equivariant lc blow-up of the form $\lambda \cdot v_0 + c \cdot \text{ord}_{E_\infty}$ for some $\lambda \in \mathbb{Z}^+$ and $c \in \mathbb{Q}^+$, then for any given $a \in \mathbb{Z}^+$ one could construct a weakly special test configuration with integral central fiber $(X, \Delta_{c}\mathbb{C}; L)$ such that $\text{ord}_{X_0}|_{K(X)} = a \cdot \text{ord}_E$.

Proof. Given a non-trivial $\mathbb{C}^*$-equivariant lc blow-up over $o \in Z$, denoted by $f : W \rightarrow Z$, and let $G$ be the exceptional divisor. By the proof of Theorem 2.4 there is a prime divisor $E$ over $X$ such that then $\text{ord}_G$ is a quasi-monomial combination of $v_0$ and $\text{ord}_{E_\infty}$ with weight $(\lambda, c)$ for some $\lambda \in \mathbb{Z}^+$ and $c \in \mathbb{Q}^+$, where $E_\infty$ is the natural extension of $E$ and $E$ is an lc place of $(X, \Delta)$. The proofs of Lemma 3.1 and Proposition 3.3 give the way to construct the...
weakly special test configuration with integral central fiber $(X, \Delta_{tc}; L)$ of $(X, \Delta; L)$ satisfying \( \text{ord}_{X_0}\mid_{K(X)} = c' \cdot \text{ord}_E \) for any \( c' \in \mathbb{Z}^+ \). It remains to consider the converse direction.

Suppose we are given a weakly special test configuration with integral central fiber \((X, \Delta_{tc}; L)\) of \((X, \Delta; L)\) such that
\[
v_{X_0} := \text{ord}_{X_0}\mid_{K(X)} = a \cdot \text{ord}_E
\]
for some prime divisor \( E \) over \( X \) and \( a \in \mathbb{Z}^+ \), we aim to create a \( \mathbb{C}^*\)-equivariant lc blow-up associated to \( E \). Denote by
\[
R_m := H^0(X, mrL) \quad \text{and} \quad F_j^{v_{X_0}} R_m := \{ s \in R_m \mid v_{X_0}(s) \geq j \}.
\]
By \cite{BHJ17}, we have the following reformulation:
\[
X = \text{Proj} \bigoplus_{m \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} F_j^{v_{X_0}} R_m \cdot t^{-j} \quad \text{and} \quad L = \frac{1}{r} O_X(1).
\]
Let us use the following notation for convenience:
\[
R := \bigoplus_{m \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} F_j^{v_{X_0}} R_m \cdot t^{-j}.
\]
It is clear that \( R \) is a finitely generated \( \mathbb{C}[t] \)-algebra with two gradings, \( m \) and \( j \). Thus \( R \) admits a \( \mathbb{C}^* \times \mathbb{C}^* \)-action, where the relative cone structure corresponds to the action by the co-weight \((1, 0)\), and \( \mathbb{C}^* \)-action from the test configuration corresponds to the co-weight \((0, 1)\). Denote by
\[
w_1 := v_0 + a \cdot \text{ord}_{E_\infty}, \quad a_{m,p} := \{ s \in R_m \mid w_1(s) \geq p \}, \quad a_p := \{ s \in R \mid w_1(s) \geq p \},
\]
then
\[
a_p = \bigoplus_{m \in \mathbb{N}} a_{m,p} \quad \text{and} \quad a_{m,p} = F_j^{v_{X_0}} R_m.
\]
Thus we have the following
\[
\bigoplus_{p \in \mathbb{Z}} a_p \cdot t^{-p} = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{m \in \mathbb{Z}} F_j^{v_{X_0}} R_m \cdot t^{-p} \cong \bigoplus_{m \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} F_j^{v_{X_0}} R_m \cdot t^{-j}.
\]
Therefore, \( \bigoplus_{p \in \mathbb{Z}} a_p \) is finitely generated and the central fiber of the test configuration can be expressed as
\[
\mathcal{X}_0 := \text{Proj} \bigoplus_{m \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} F_j^{v_{X_0}} R_m / F_j^{v_{X_0}} R_m \cong \text{Proj} \bigoplus_{p \in \mathbb{Z}} a_p / a_{p+1}.
\]
Take a weighted blow-up of \( o \in Z \) with respect to the filtration induced by \( w_1 \), denoted by \( \mu_1 : W_1 \to Z \). More concretely,
\[
W_1 = \text{Proj}_Z \bigoplus_{p \in \mathbb{Z}} a_p = \text{Proj}_Z \bigoplus_{m \in \mathbb{Z}} \bigoplus_{p \in \mathbb{Z}} a_{m,p}.
\]
Let \( E_1 \) be the exceptional divisor of \( \mu_1 \), then by \cite{BHJ17} Lemma 1.13, \( E_1 = \text{Proj} \bigoplus_{p \in \mathbb{Z}} a_{pl} / a_{pl+1} \)
for a sufficiently divisible positive integer \( l \), and \(-E_1\) is ample over \( Z \). Consider the following degeneration of \( Z \):

\[
Z := \text{Spec} \bigoplus_{p \in \mathbb{Z}} a_p \cdot t^{-p} \to \mathbb{A}^1
\]

with the central fiber

\[
Z_0 := \text{Spec} \bigoplus_{p \in \mathbb{Z}} a_p/a_{p+1}, \quad (2.2)
\]

and denote by \( \Delta_Z \) the natural extension of \( \Delta_Z \), then \((Z, \Delta_Z + Z_0)\) is log canonical since \((X, \Delta + X_0)\) is log canonical. By sub-adjunction we see that \((Z_0, \Delta_{Z_0})\) is slc, where \( \Delta_{Z_0} := \Delta_Z|_{Z_0} \). Write

\[
K_{E_1} + \Delta_{E_1} := (K_W + \mu_1^{-1} \Delta_Z + E_1)|_{E_1},
\]

then by (2.1) and (2.2) we see that \((Z_0, \Delta_{Z_0})\) is log canonical since \((Z_0, \Delta_Z)\) is slc. By the inversion of adjunction, \( \mu_1 : (W_1, E_1) \to Z \) is a \( \mathbb{C}^* \)-equivariant lc blow-up and the proof is finished. It is worth to say one more sentence. Recall that \( \text{ord}_{E_1} v_0 \)

\[E \text{ is an lc place of } (X, \Delta).\]

3. Global Correspondence

In this section, we prove Theorem 1.1. We begin with the following dreamy property.

Lemma 3.1. Let \((X, \Delta)\) be a projective lc CY pair and \( L \) an ample \( \mathbb{Q} \)-line bundle on \( X \). If \( E \) is an lc place of \((X, \Delta)\), then \( E \) is dreamy with respect to \( L \).

Proof. Choose a divisible positive integer \( r \) such that \( rL \) is Cartier. Let \( Z \) be the affine cone over \( X \) with respect to \( rL \), i.e.,

\[
Z = \text{Spec} \bigoplus_{m \in \mathbb{N}} H^0(X, mrL),
\]

and denote by \( o \in Z \) the cone vertex. Let \( \Delta_Z \) be the extension of \( \Delta \) on \( Z \). Consider the projection morphism \( p : Z \setminus o \to X \), we denote \( E_\infty \) to be the prime divisor over \( Z \setminus o \) via pulling back \( E \) over \( X \). Let \( \mu : \tilde{Z} \to Z \) be the blow-up of the vertex and \( X_0 \) the exceptional divisor. Write \( v_0 := \text{ord}_{X_0} \) for the canonical valuation. It is clear that \( E_\infty \) is an lc place of the pair \((Z, \Delta_Z)\), and by [Kol13] Proposition 3.14, \( v_0 \) is also an lc place of \((Z, \Delta_Z)\).

Let \( w_k := k \cdot v_0 + \text{ord}_{E_\infty} \) be the quasi-monomial valuation with weight \((k, 1)\) along \( v_0 \) and \( \text{ord}_{E_\infty} \) for some positive integer \( k \). By [JM12] Proposition 5.1, \( w_k \) is an lc place of \((Z, \Delta_Z)\) whose center is exactly the cone vertex \( o \in Z \). By [CZ21] Lemma 5.1, there is an extraction morphism \( f : W \to Z \) which has a unique exceptional divisor corresponding to \( w_k \), denoted by \( E_k \subset W \), such that \(-E_k\) is ample over \( Z \), and we have the following:

\[
K_W + f_*^{-1} \Delta_Z + E_k = f^*(K_Z + \Delta_Z).
\]

Consider the following exact sequence for \( pl \in \mathbb{N} \), where \( l \in \mathbb{N} \) and \( p \) is a fixed divisible positive integer such that \( pE_k \) is Cartier:

\[
0 \to \mathcal{O}_W(-(pl + 1)E_k) \to \mathcal{O}_W(-plE_k) \to \mathcal{O}_{E_k}(-plE_k) \to 0.
\]
Write $a_l := f_* O_W(-lE_k)$, then
$$a_{pl}/a_{pl+1} \cong H^0(E_k, -plE_k|E_k)$$
for $l \in \mathbb{N}$, since $-E_k$ is $f$-ample and $R^l f_* O_W(-(pl+1)E_k) = 0$ by an lc version of Kawamata-Viehweg vanishing (see [Fuj14, Theorem 1.7]). Thus the graded algebra $\bigoplus_{l \in \mathbb{N}} a_{pl}/a_{pl+1}$ is finitely generated, hence so is $\bigoplus_{l \in \mathbb{N}} a_l/a_{l+1}$. Therefore, the graded algebra $\bigoplus_{l \in \mathbb{N}} a_l$ is finitely generated.

Denote by
$$R_m := H^0(X, mrL) \quad \text{and} \quad R := \bigoplus_{m \in \mathbb{N}} R_m.$$ 
We naturally extend $\bigoplus_{l \in \mathbb{N}} a_l$ and $R$ to graded algebras indexed by $\mathbb{Z}$ via defining $R_m = 0$ for $m < 0$ and $a_l = O_Z$ for $l < 0$. We use the notation
$$\mathcal{F}_{ord_E}^j R_m := \{s \in R_m \mid ord_E(s) \geq j\} \quad \text{and} \quad a_{m,l} := \{s \in R_m \mid km + ord_E(s) \geq l\}.$$
It is clear that $a_l = \bigoplus_{m \in \mathbb{Z}} a_{m,l}$, and $s \in \mathcal{F}_{ord_E}^j R_m$ if and only if $s \in a_{m,km+j}$. Then we have
$$\bigoplus_{m \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} \mathcal{F}_{ord_E}^j R_m \cong \bigoplus_{m \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} a_{m,km+j} \cong \bigoplus_{m \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} a_{m,j} \cong \bigoplus_{l \in \mathbb{Z}} a_l,$$
which are all finitely generated graded algebras. Note that
$$\mathcal{F}_{ord_E}^j R_m \cong H^0(Y, g^*(mrL) - jE),$$
where $g : Y \to X$ is a projective normal birational model such that $E$ is a prime divisor on $Y$ (see [CZ21, Lemma 5.1]). Then $\bigoplus_{m \in \mathbb{N}} \bigoplus_{j \in \mathbb{N}} \mathcal{F}_{ord_E}^j R_m$ being finitely generated implies that $E$ is dreamy with respect to $L$. The proof is finished. \hfill \Box

**Remark 3.2.** The idea of the above proof is the same as that of [CZ21, Theorem 1.5], and [CZ21, Theorem 1.5] is just a direct corollary of Lemma 3.1.

Next we show that an lc place induces a weakly special degeneration.

**Proposition 3.3.** Let $(X, \Delta)$ be a projective lc CY pair and $L$ an ample $\mathbb{Q}$-line bundle on $X$. Suppose $E$ is an lc place of $(X, \Delta)$, then for any given $c \in \mathbb{Z}^+$, there exists a non-trivial weakly special test configuration with integral central fiber $(\mathcal{X}, \Delta_{tc}; L)$ of $(X, \Delta; L)$ such that $ord_{KX}|_{K(X)} = c \cdot ord_E$.

**Proof.** By [CZ21, Lemma 5.1], there exists an extraction $g : Y \to X$ which only extracts $E$, and we have
$$K_Y + g_*^{-1}\Delta + E = g^*(K_X + \Delta).$$
If $E$ is a prime divisor on $X$, then we may just take $g = id$. We only deal with the case where $E$ is exceptional in the remaining proof, as one can prove similarly for the case where $E$ is a divisor on $X$. According to Lemma 3.1, we know that the following $\mathbb{N}^2$-graded ring
$$\bigoplus_{k \in \mathbb{N}} \bigoplus_{j \in \mathbb{N}} H^0(Y, g^*(krL) - jE)$$
is finitely generated for some $r \in \mathbb{Z}^+$ such that $rL$ is Cartier. Let
$$R_k := H^0(X, krL) \quad \text{and} \quad \mathcal{F}_{ord_E}^j R_k := \{s \in R_k \mid c \cdot ord_E(s) \geq j\}.$$
We construct the following degeneration family over $\mathbb{A}^1$:

$$\mathcal{X} := \text{Proj} \bigoplus_{k \in \mathbb{N}} \bigoplus_{j \in \mathbb{Z}} \left( F^j_{c \cdot \text{ord}_E} R_k \right) t^{-j} \to \mathbb{A}^1.$$  

By the same proof as that of [Fuj17, Lemma 3.8], we know that $(\mathcal{X}, \Delta; \mathcal{O}_X(1))$ is a test configuration of $(X, \Delta; rL)$ whose central fiber $X_0$ is integral, where $\Delta_X$ is the extension of $\Delta$ on $\mathcal{X}$. Then it holds that $\text{ord}_{X_0} |_{K(X)} = c \cdot \text{ord}_E$. We next show that $(\mathcal{X}, \Delta_X + \lambda_0)$ is log canonical, which will imply that $(\mathcal{X}, \Delta_X; \mathcal{O}_X(1))$ is a weakly special test configuration of $(X, \Delta; L)$ induced by $E$.

Put the morphism $g$ into the trivial family $G : Y^{\lambda_1} \to X^{\lambda_1}$, then we have

$$K_{Y^{\lambda_1}} + G^{-1}_s \Delta_{\lambda_1} + E_{\lambda_1} + Y_0 = G^* (K_{X^{\lambda_1}} + \Delta_{\lambda_1} + X_0),$$

where $X_0$ (resp. $Y_0$) is the central fiber of the family $X^{\lambda_1} \to \mathbb{A}^1$ (resp. $Y^{\lambda_1} \to \mathbb{A}^1$). Let $v$ be the divisorial valuation over $X^{\lambda_1}$ with weight $(c, 1)$ along divisors $E^{\lambda_1}$ and $Y_0$, then

$$A_{X^{\lambda_1}, \Delta_{\lambda_1} + X_0} (v) = A_{Y^{\lambda_1}, G^{-1}_s \Delta_{\lambda_1} + E_{\lambda_1} + Y_0} (v) = 0.$$

By [CZ21, Lemma 5.1], one can extract $v$ to a divisor on a birational model $h : Y \to X^{\lambda_1}$ as follows:

$$K_Y + h^{-1}_s \Delta_{\lambda_1} + h^{-1}_s X_0 + E = h^* (K_{X^{\lambda_1}} + \Delta_{\lambda_1} + X_0),$$

where $v = \text{ord}_E$. We note here that the pair $(Y, h^{-1}_s \Delta_{\lambda_1} + h^{-1}_s X_0 + E)$ is log canonical and log Calabi-Yau over $\mathbb{A}^1$.

Now we could compare the two pairs,

$$(Y, h^{-1}_s \Delta_{\lambda_1} + h^{-1}_s X_0 + E) \quad \text{and} \quad (\mathcal{X}, \Delta_X + \lambda_0).$$

Since $\mathcal{X}$ and $\lambda_0$ induce the same divisorial valuation on $K(\mathcal{X}) \cong K(X \times \mathbb{A}^1) \cong K(\mathcal{X})$, we have the following birational contraction map:

$$(Y, h^{-1}_s \Delta_{\lambda_1} + h^{-1}_s X_0 + E) \dashrightarrow (\mathcal{X}, \Delta_X + \lambda_0).$$

As both two sides are log Calabi-Yau over $\mathbb{A}^1$, they are crepant, i.e., for any common log resolution $p : W \to Y$ and $q : W \to \mathcal{X}$ one has

$$p^* (K_Y + h^{-1}_s \Delta_{\lambda_1} + h^{-1}_s X_0 + E) = q^* (K_{\mathcal{X}} + \Delta_X + \lambda_0).$$

It follows that $(\mathcal{X}, \Delta_X + \lambda_0)$ is log canonical, and this completes the proof.

We turn to the converse direction.

**Proposition 3.4.** Let $(X, \Delta)$ be a projective lc CY pair and $L$ an ample $\mathbb{Q}$-line bundle on $X$. Given a non-trivial weakly special test configuration with integral central fiber $(\mathcal{X}, \Delta_{\text{lc}}; L)$ of $(X, \Delta; L)$ such that $\text{ord}_{\lambda_0} |_{K(X)} = c \cdot \text{ord}_E$ for some prime divisor $E$ over $X$ and $c \in \mathbb{Z}^+$, then $E$ is an lc place of $(X, \Delta)$.

**Proof.** By the proof of Theorem 2.6 there is a $\mathbb{C}^*$-equivariant lc blow-up over $a \in \mathbb{Z}$, denoted by $\mu_1 : W_1 \to Z$, such that

$$\text{ord}_{E_1} = c \cdot \text{ord}_{E_{\infty}} + v_0,$$

where $E_1$ is the exceptional divisor of $\mu_1$, and $E_{\infty}$ is the natural extension of $E$. By Theorem 2.4 we see that $E$ is an lc place of $(X, \Delta)$. \qed
4. Generalized Global Correspondence

In this section, we will generalize Theorem 1.1 for weakly special test configurations without assuming the irreducibility of central fibers.

Let \((X, \Delta)\) be a projective lc CY pair and \(L\) an ample \(\mathbb{Q}\)-line bundle on \(X\). Given a weakly special semi-test configuration \((\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})\) of \((X, \Delta; L)\), and denote by

\[ v_{\chi_0,i} := \text{ord}_{\chi_0,i} K(X) = c_i \cdot \text{ord}_{E_i}, \]

where \(\chi_0,i, i = 1, ..., k\), are all non-trivial components\(^3\) in the central fiber \(\chi_0\), \(c_i \in \mathbb{Z}^+\), and \(E_i\) are prime divisors over \(X\). We have the following characterization of \(E_i\).

**Theorem 4.1.** Notation as above. Then \(E_i\) is an lc place of \((X, \Delta)\) for every \(i\).

**Proof.** Let

\[ h : (\mathcal{Y}, \Delta_\mathcal{Y} + \mathcal{Y}_0) \rightarrow (\mathcal{X}, \Delta_{\text{tc}} + \mathcal{X}_0) \]

be a \(\mathbb{C}^*\)-equivariant \(\mathbb{Q}\)-factorial dlt modification such that

\[ K_\mathcal{Y} + \Delta_\mathcal{Y} + \mathcal{Y}_0 = h^*(K_\mathcal{X} + \Delta_{\text{tc}} + \mathcal{X}_0). \]

Let \((\mathcal{Y}_t, \Delta_{\mathcal{Y}_t})\) be the fiber over \(0 \neq t \in \mathbb{A}^1\), then we see that

\[ \phi : (\mathcal{Y}_t, \Delta_{\mathcal{Y}_t}) \rightarrow (X, \Delta) \]

is a crepant lc model of \((X, \Delta)\), and \((\mathcal{Y}_t, \Delta_\mathcal{Y}; h^*\mathcal{L})\) is a semi-test configuration of \((\mathcal{Y}_t, \Delta_{\mathcal{Y}_t}; L')\), where \(L' = \phi^*L\). In particular, \((\mathcal{Y}_t, \Delta_{\mathcal{Y}_t})\) is an lc CY pair. Denote \(\mathcal{Y}_0, i\) to be the strict transform of \(\chi_0, i\) and we still have \(\text{ord}_{\mathcal{Y}_0, i} K(\mathcal{Y}_t) = c_i \cdot \text{ord}_{E_i}\), where \(E_i\) can also be viewed as a prime divisor over \(\mathcal{Y}_t\). It suffices to show that \(E_i\) is an lc place of \((\mathcal{Y}_t, \Delta_{\mathcal{Y}_t})\).

Consider the pair \((\mathcal{Y}, \Delta_\mathcal{Y} + \mathcal{Y}_0, i)\), it is clear to see the following:

\[ K_\mathcal{Y} + \Delta_\mathcal{Y} + \mathcal{Y}_0, i \sim_{\mathbb{Q}, \mathbb{A}^1} -(\mathcal{Y}_0 - \mathcal{Y}_0, i). \]

By [Bir12, Theorem 1.1] or [HX13, Theorem 1.6], one may run a \(\mathbb{C}^*\)-equivariant MMP on \(K_\mathcal{Y} + \Delta_\mathcal{Y} + \mathcal{Y}_0, i\) over \(\mathbb{A}^1\), which terminates with a minimal model, denoted by

\[ (\mathcal{Y}, \Delta_\mathcal{Y} + \mathcal{Y}_0, i) \rightarrow (\mathcal{Y}', \Delta_\mathcal{Y}' + \mathcal{Y}_0', i), \]

where \(\Delta_\mathcal{Y}'\) (resp. \(\mathcal{Y}_0', i\)) is the push-forward of \(\Delta_\mathcal{Y}\) (resp. \(\mathcal{Y}_0, i\)). Thus \(-(\mathcal{Y}_0' - \mathcal{Y}_0, i)\) is nef over \(\mathbb{A}^1\), or equivalently, \(\mathcal{Y}_0', i\) is nef over \(\mathbb{A}^1\). By the Zariski lemma (e.g. [LX14, Section 2.4]), \(\mathcal{Y}_0', i = \mathcal{Y}_0'\). It is not hard to see that the following two pairs

\[ (\mathcal{Y}, \Delta_\mathcal{Y} + \mathcal{Y}_0) \quad \text{and} \quad (\mathcal{Y}', \Delta_\mathcal{Y}' + \mathcal{Y}_0', i) \]

are isomorphic over \(\mathbb{A}^1 \setminus 0\) and crepant to each other, thus \((\mathcal{Y}', \Delta_\mathcal{Y}')\) is a weakly special degeneration of \((\mathcal{Y}_t, \Delta_{\mathcal{Y}_t})\) in the sense of Definition 2.2. Put \(A := \mathcal{O}(1)|_{\mathcal{Y}_t}\), then \((\mathcal{Y}', \Delta_\mathcal{Y}'; \mathcal{O}_{\mathcal{Y}'}(1))\)

\[^3\text{Here we say that a component in } \chi_0 \text{ is non-trivial if it is not the strict transform of } X \times 0 \text{ with respect to the birational map } X' \rightarrow X \times \mathbb{A}^1, \text{ see [BHJ17, Definition 4.4].}\]
is a weakly special test configuration with integral central fiber of \((Y_t, \Delta_{Y_t}; A)\). Note that \(\text{ord}_{Y_t}^c\) and \(\text{ord}_{Y'_t}\) are the same divisorial valuation over the function field

\[ K(X) = K(Y) = K(Y') = K(X)(t), \]

thus \(E_i\) is an lc place of \((Y_t, \Delta_{Y_t}); A\) by Theorem 2.4 and Theorem 2.6. The proof is finished. \(\square\)

We still denote \((Z, \Delta_Z)\) to be the affine cone over the lc CY pair \((X, \Delta)\) with respect to \(rL\), and \(o \in Z\) is the cone vertex.

**Definition 4.2.** We say that a finite set of divisorial valuations over \(X\), denoted by

\[ \{c_1 \cdot \text{ord}_{E_1}, \ldots, c_l \cdot \text{ord}_{E_l}\}, \]

is a weakly special collection of \((X, \Delta)\) if \(c_i \in \mathbb{Z}^+\) and \(E_i\) is an lc place of \((X, \Delta)\) for every \(i\).

Note here that our definition of weakly special collection is a little different from that in [BLX19 Appendix A], since all \(w_i\) are automatically induced by lc places of \((X, \Delta)\). However, we will later see that these definitions are essentially equivalent. We aim to establish the following generalized global correspondence.

**Theorem 4.3.** Let \((X, \Delta)\) be a projective lc CY pair and \(L\) an ample \(\mathbb{Q}\)-line bundle on \(X\), then there exists a correspondence between weakly special collections of \((X, \Delta)\) and weakly special semi-test configurations of \((X, \Delta; L)\). More precisely, given a weakly special collection \(\{c_1 \cdot \text{ord}_{E_1}, \ldots, c_l \cdot \text{ord}_{E_l}\}\), we can construct a weakly special semi-test configuration \((\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})\) satisfying

\[ \text{ord}_{\mathcal{X}_{0,i}} |_{K(X)} = c_i \cdot \text{ord}_{E_i}, i = 1, \ldots, l, \]

where \(\mathcal{X}_{0,i}\) are all non-trivial components of \(\mathcal{X}_0\); conversely, given a weakly special semi-test configuration \((\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})\) satisfying

\[ \text{ord}_{\mathcal{X}_{0,i}} |_{K(X)} = c_i \cdot \text{ord}_{E_i}, i = 1, \ldots, l, \]

where \(c_i \in \mathbb{Z}^+\) and \(\mathcal{X}_{0,i}\) are all non-trivial components of \(\mathcal{X}_0\), then the set \(\{c_1 \cdot \text{ord}_{E_1}, \ldots, c_l \cdot \text{ord}_{E_l}\}\) is a weakly special collection.

**Proof.** Given a weakly special semi-test configuration of the form in the theorem, denoted by \((\mathcal{X}, \Delta_{\text{tc}}; \mathcal{L})\), by Theorem 1.1 every \(E_i\) is an lc place of \((X, \Delta)\). Thus \(\{c_i \cdot \text{ord}_{E_i}\}\) is a weakly special collection of \((X, \Delta)\) induced by the given weakly special test configuration.

Now we consider the converse direction. Given a weakly special collection

\[ \{c_1 \cdot \text{ord}_{E_1}, \ldots, c_l \cdot \text{ord}_{E_l}\}, \]

we aim to construct a weakly special semi-test configuration as in the theorem. Let \(G_i(\lambda_i)\) be the prime divisor over \(Z\) such that

\[ \text{ord}_{G_i(\lambda_i)} := \lambda_i \cdot v_0 + c_i \cdot \text{ord}_{E_i, \infty}, \]

where \(\lambda_i \in \mathbb{Z}^+\). Then all \(G_i(\lambda_i)\) are lc places of \((Z, \Delta_Z)\) centered at \(o \in Z\). We have the following claim:

**Claim 4.4.** There exists a birational model \(\mu : W \to Z\) which extracts part of \(G_i(\lambda_i)\), say \(\{G_{j_1}(\lambda_{j_1}), \ldots, G_{j_{1'}}(\lambda_{j_{1'}})\}\), such that \(-\sum_{s=1}^{1'} G_{j_s}(\lambda_{j_s})\) is \(\mu\)-ample. Here \(\{j_1, ..., j_{1'}\}\) is a non-repeating subset of \(\{1, ..., l\}\).
Proof of the Claim. Let $\gamma : Y \to Z$ be a $\mathbb{C}^*$-equivariant $\mathbb{Q}$-factorial dlt modification such that all $G_i(\lambda_i)$ appear as prime divisors on $Y$. We write
\[ K_Y + \gamma^{-1}_* \Delta_Z + \sum_{i=1}^{l} G_i(\lambda_i) + C = \gamma^*(K_Z + \Delta_Z), \]
where $C$ is the reduced sum of lc places other than $\sum_{i=1}^{l} G_i(\lambda_i)$. Choose a small $0 < \epsilon \ll 1$ and consider the pair
\[ (Y, \gamma^{-1}_* \Delta_Z + (1 - \epsilon) \cdot \sum_{i=1}^{l} G_i(\lambda_i) + C), \]
then we have
\[ K_Y + \gamma^{-1}_* \Delta_Z + (1 - \epsilon) \cdot \sum_{i=1}^{l} G_i(\lambda_i) + C \sim_{\gamma, \mathbb{Q}} \epsilon \sum_{i=1}^{l} G_i(\lambda_i). \]
By applying [Bir12, Theorem 1.1] or [HX13, Theorem 1.6] and arguing by the same way as [CZ21, Lemma 5.1], one can run a $\mathbb{C}^*$-equivariant MMP over $Z$ on
\[ K_Y + \gamma^{-1}_* \Delta_Z + (1 - \epsilon) \cdot \sum_{i=1}^{l} G_i(\lambda_i) + C \]
to get a good minimal model over $Z$, denoted by
\[ Y \dasharrow W'/Z, \]
and the ample model $\mu : W \to Z$ is exactly what we want. Consider the birational contraction $Y \dasharrow W$, by the same argument of [CZ21, Lemma 5.1], we know that the components in $\sum_{i=1}^{l} G_i(\lambda_i)$ cannot be all contracted (but some may be contracted), and this is why we say that only part of $G_i(\lambda_i)$ are extracted in the statement of the Claim. For convenience, we just assume $\{j_1, \ldots, j_{l'}\} = \{1, \ldots, l'\}$ for some $l' < l$, and still use $\sum_{i=1}^{l'} G_i(\lambda_i)$ to denote the push-forward of $\sum_{i=1}^{l} G_i(\lambda_i)$ via $Y \dasharrow W$. Thus $\sum_{i=1}^{l'} G_i(\lambda_i)$ is $\mu$-ample. □

We first construct a weakly special test configuration $(\mathcal{X}, \Delta_{\mathcal{T}}; \mathcal{L})$ satisfying
\[ \text{ord}_{X_0,i}|_{K(\mathcal{X})} = c_i \cdot \text{ord}_{E_i}, \ i = 1, \ldots, l', \]
where $X_{0,i}, i = 1, \ldots, l'$, are all non-trivial components of $X_0$. Recall that
\[ Z = \text{Spec } R = \text{Spec } \bigoplus_{m \in \mathbb{N}} R_m = \text{Spec } \bigoplus_{m \in \mathbb{N}} H^0(X, mrL), \]
we first introduce the following ideals for $i = 1, \ldots, l'$:
\[ a^{(i)}_{m,p} := \{ s \in R | \text{ord}_{G_i(\lambda_i)}(s) = c_i \cdot \text{ord}_{E_i}(s) + \lambda_i \cdot v_0(s) \geq p \}, \]
\[ a^{(i)}_p := \{ s \in R | \text{ord}_{G_i(\lambda_i)}(s) = c_i \cdot \text{ord}_{E_i}(s) + \lambda_i \cdot v_0(s) \geq p \}, \]
\[ I_{m,p} := \cap_i a^{(i)}_{m,p} \quad \text{and} \quad I_p := \cap_i a^{(i)}_p. \]
Note that $-\sum_{i=1}^{l'} G_i(\lambda_i)$ is ample over $Z$ (see Claim 4.4), thus by [BHJ17, Lemma 1.8 and Theorem 1.10], we have the following formulation of $W$:

$$W = \text{Proj}_Z \bigoplus_{p \in Z} I_p = \text{Proj}_Z \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in Z} I_{m,p}.$$ 

Note that the ideal sequence $I_\bullet$ induces a degeneration of $Z$ over $\mathbb{A}^1$ via

$$Z := \text{Spec} \bigoplus_{p \in Z} I_p : t^{-p} = \text{Spec} \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in Z} I_{m,p} : t^{-p} \to \mathbb{A}^1,$$

where the central fiber is given by

$$Z_0 := \text{Spec} \bigoplus_{p \in \mathbb{N}} I_p / I_{p+1} = \text{Spec} \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{N}} I_{m,p} / I_{m,p+1}.$$ 

Let $\Delta_{Z_0}$ be the restriction of $\Delta_Z$ on $Z_0$, where $\Delta_Z$ is the extension of $\Delta_Z$ on $Z$. Then we know that $(Z, \Delta_Z, \xi; \eta)$ is a test configuration of $(Z, \Delta_Z, \xi)$ in the sense of [LWX21, Definition 2.14], where $\xi$ (resp. $\eta$) is the vector field on $Z$ (resp. $Z$) induced by the grading $m$ (resp. $p$). We have the following claim:

**Claim 4.5.** Notation as above, $(Z, \Delta_Z, \xi; \eta)$ is a weakly special test configuration of $(Z, \Delta_Z, \xi)$ in the sense of [LWX21, Definition 2.14], more precisely, $(Z_0, \Delta_{Z_0})$ is semi-log canonical and it is an orbifold cone over an slc CY pair.

**Proof of the Claim.** Denote by $G := \sum_{i=1}^{l'} G_i(\lambda_i)$, then $-G$ is ample over $Z$ as we have mentioned. Choose a divisible $k \in \mathbb{N}$ such that $-kG$ is Cartier. By [BHJ17, Lemma 1.8 and Theorem 1.10], we have the following characterization for any $p \in Z$:

$$I_p = \mu_k \ast O_W(-pG).$$

Consider the $\mu_k$-action on $(Z, \Delta_Z)$, where $\mu_k$ is the multiplicative group of $k$-th roots of unity. Let $(Z', \Delta_{Z'}) := (Z, \Delta_Z) / \mu_k$, then we have

$$Z' := \text{Spec} \bigoplus_{p \in Z} I_{kp} : t^{-p} \to \mathbb{A}^1,$$

and the quotient map $\sigma : Z \to Z'$ is a lifting of the map $\mathbb{A}^1 \to \mathbb{A}^1, t \mapsto t^k$. It is clear that $\sigma$ is étale away from the central fiber. Recall that

$$Z_0 := \text{Spec} \bigoplus_{p \in \mathbb{N}} I_p / I_{p+1},$$

thus

$$Z_0 / \mu_k = \text{Spec} \bigoplus_{p \in \mathbb{N}} I_{kp} / I_{kp+1} \text{ and } \text{Supp } Z_0 / \mu_k = \text{Supp } Z'_0.$$ 

Let $\text{red}(Z_0)$ (resp. $\text{red}(Z'_0)$) be the reduced part of $Z_0$ (resp. $Z'_0$), then clearly we have

$$K_Z + \Delta_Z + \text{red}(Z_0) = \sigma^*(K_{Z'} + \Delta_{Z'} + \text{red}(Z'_0)).$$

Hence it suffices to show the following three points to complete the proof:

1. the pair $(Z', \Delta_{Z'} + \text{red}(Z'_0))$ is log canonical;
2. $Z_0 = \text{red}(Z_0)$, i.e., $Z_0$ is reduced;
(3) \((K_Z + \Delta_Z + \text{red}(Z'_0))|_{\text{red}(Z'_0)}\) is an affine cone over an slc CY pair.

The first two points together imply that \((Z, \Delta_Z + Z_0)\) is log canonical (e.g. [KM98 Proposition 5.20]), and the third point implies that \((Z_0, \Delta_{Z_0})\) is an orbifold cone over some slc CY pair.

We first show that \(Z_0/\mu_k = C_a(G, O_G(-kG))\), where \(C_a(G, O_G(-kG))\) is the affine cone over \(G\) with respect to \(-kG|_G\). Consider the following exact sequence

\[
0 \to O_W(-(kp + 1)G) \to O_W(-kpG) \to O_G(-kpG) \to 0.
\]

By Claim 4.4 we have

\[
K_W + \mu^{-1}_s \Delta_Z + G = \mu^*(K_Z + \Delta_Z),
\]

where both two sides are lc CY pairs. Thus by an lc version of Kawamata-Viehweg vanishing theorem (see [Fuj14 Theorem 1.7]), we see

\[
R^1 \mu_* O_W(-(kp + 1)G) = 0
\]

for any \(p \in \mathbb{N}\). Therefore, we have

\[
I_{kp}/I_{kp+1} \cong H^0(G, -kpG).
\]

and hence

\[
\text{red}(Z'_0) = Z_0/\mu_k = \text{Spec} \bigoplus_{p \in \mathbb{N}} I_{kp}/I_{kp+1} = \text{Spec} \bigoplus_{p \in \mathbb{N}} H^0(G, -kpG).
\]

Denote by

\[
K_G + \Delta_G := (K_W + \mu^{-1}_s \Delta_Z + G)|_G;
\]

which by adjunction is an slc CY pair, thus the affine cone \((\text{red}(Z'_0), \Delta_{\text{red}(Z'_0)})\) over \((G, \Delta_G)\) with respect to \(-kG|_G\) is slc, where

\[
K_{\text{red}(Z'_0)} + \Delta_{\text{red}(Z'_0)} := (K_Z + \Delta_Z + \text{red}(Z'_0))|_{\text{red}(Z'_0)}.
\]

This proves the third point. Apply the inversion of adjunction we conclude the first point. It remains to show the second point. Recall the following characterization:

\[
\bigoplus_{p \in \mathbb{N}} I_p/I_{p+1} = \bigoplus_{p \in \mathbb{N}} \mu_* O_W(-pG)/\mu_* O_W(-(p + 1)G).
\]

Suppose \(0 \neq f \in I_p/I_{p+1}\), then there must exist some \(i\) such that \(\text{ord}_{G_i}(f) = p\), thus \(\text{ord}_{G_i}(f^b) = bp\) for any \(b \in \mathbb{Z}^+\). This implies that \(0 \neq f^b \in I_{bp}/I_{bp+1}\) for any \(b \in \mathbb{Z}^+\) and hence \(Z_0\) is reduced. The proof of the Claim is finished. \(\square\)

Define the following degeneration over \(\mathbb{A}^1:\)

\[
\mathcal{X} := \text{Proj} \bigoplus_{p \in \mathbb{Z}} I_p \cdot t^{-p} = \text{Proj} \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} I_{m,p} \cdot t^{-p} \to \mathbb{A}^1.
\]

Then the central fiber can be formulated as follows:

\[
\mathcal{X}_0 = \text{Proj} \bigoplus_{p \in \mathbb{P}} I_p/I_{p+1} = \text{Proj} \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{N}} I_{m,p}/I_{m,p+1}.
\]

It is clear that \((\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1))\) is a test configuration of \((X, rL)\), where \(\mathcal{O}_{\mathcal{X}}(1)\) is the relatively ample line bundle on \(\mathcal{X}\) under the grading \(m\). Denote \(\Delta_{tc}\) the extension of \(\Delta\) on \(\mathcal{X}\), then we aim to show that \((\mathcal{X}, \Delta_{tc}; \frac{1}{r} \mathcal{O}_{\mathcal{X}}(1))\) is a weakly special test configuration of \((X, \Delta; L)\).
By our construction, \((Z_0, \Delta_{Z_0})\) is the affine cone over \((X_0, \Delta_{tc,0})\) with respect to \(\mathcal{O}_{X_0}(1)\), where \(\mathcal{O}_{X_0}(1) := \mathcal{O}_X(1)|_{X_0}\). By Claim 4.5 we see that \((X_0, \Delta_{tc,0})\) is semi-log canonical, and thus \((X, \Delta_{tc} + \lambda_0)\) is log canonical by inversion of adjunction. It is not hard to see from the construction that the components in the central fiber exactly produce the divisorial valuations \(\{c_1 \cdot \text{ord}_{E_1}, ..., c_l \cdot \text{ord}_{E_l}\}\), which is a part of the given weakly special collection.

The final step is to construct a weakly special semi-test configuration \((X', \Delta'_{tc}; L')\) satisfying

\[
\text{ord}_{X'_{0,i}}|_{K(X)} = c_i \cdot \text{ord}_{E_i}, i = 1, ..., l,
\]

where \(X'_{0,i}, i = 1, ..., l\), are all non-trivial components of \(X'\).

Take a dlt modification \(Y \to X\) which exactly extracts all \(E_i, i = 1, ..., l\), (by the similar argument as [CZ21] Lemma 5.1 or Claim 4.4). Then we consider the product space \(Y \times \mathbb{A}^1 \to X \times \mathbb{A}^1\).

Let us write \(X_0, Y_0\) for the central fibers and still denote \(E_i, \infty\) to be the extension of \(E_i\) on \(Y \times \mathbb{A}^1\). Then we get a collection of divisorial valuations

\[
\{\lambda_j \cdot \text{ord}_{Y_0} + c_j \cdot \text{ord}_{E_i, \infty} \mid j = 1, ..., l\}
\]

over \(X \times \mathbb{A}^1\). Let \(E_j\) be a prime divisor over \(X \times \mathbb{A}^1\) such that

\[
\text{ord}_{E_j} := \lambda_j \cdot \text{ord}_{Y_0} + c_j \cdot \text{ord}_{E_i, \infty},
\]

we see that all \(E_j\) are log canonical places of \((X \times \mathbb{A}^1, \Delta \times \mathbb{A}^1 + X_0)\) centered in \(X_0\). It is not hard to see that \((X \times \mathbb{A}^1, \Delta \times \mathbb{A}^1 + X_0)\) is crepant to the weakly special test configuration \((X, \Delta_{tc} + \lambda_0)\) which we have constructed above, thus \(\{\text{ord}_{E_j'} \mid l' < j \leq l\}\) are all lc places of \((X, \Delta_{tc} + \lambda_0)\). Apply the similar argument as Claim 4.4, one can find a \(\mathbb{C}^*\)-equivariant extraction \(\sigma : X' \to X\) which exactly extracts \(\{\text{ord}_{E_j'} \mid l' < j \leq l\}\). Let \(\Delta'_{tc}\) be the extension of \(\Delta\) on \(X'\), then \((X', \Delta'_{tc}; \sigma* L)\) is the weakly special semi-test configuration as we want. \(\square\)

**Remark 4.6.** In the above proof, given a weakly special collection \(\{c_1 \cdot \text{ord}_{E_1}, ..., c_l \cdot \text{ord}_{E_l}\}\) and a vector of positive integers \(\vec{\lambda} := (\lambda_1, \lambda_l) \in (\mathbb{Z}^+)^{\oplus l}\), we define the corresponding prime divisors

\[
\{G_1(\lambda_1), ..., G_l(\lambda_l)\}
\]

over \(s \in \mathbb{Z}\). Then, we use the sequence of ideals associated to

\[
G := \sum_{i=1}^{l'} G_i(\lambda_i)
\]

to construct the following weakly special test configuration

\[
\mathcal{X}(\vec{\lambda}) := \text{Proj} \bigoplus_{p \in \mathbb{Z}} I_p \cdot t^{-p} = \text{Proj} \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} I_{m,p} \cdot t^{-p} \to \mathbb{A}^1
\]

with the polarization \(\mathcal{O}_{\mathcal{X}(\vec{\lambda})}(1)\), which is the relatively ample line bundle on \(\mathcal{X}(\vec{\lambda})\) under the grading \(m\). Finally, we construct a weakly special semi-test configuration via a \(\mathbb{C}^*\)-equivariant extraction

\[
\sigma : \mathcal{X}(\vec{\lambda})' \to \mathcal{X}(\vec{\lambda}),
\]
which extracts the rest components in the central fiber associated to the given weakly special collection. This means that we could construct various weakly special semi-test configurations of \((X, \Delta; \mathcal{L})\), denoted by
\[
\{(X(\tilde{\lambda})', \Delta_{X(\tilde{\lambda})'}; \frac{1}{r} \sigma^* O_{X(\tilde{\lambda})}(1))\}_{\tilde{\lambda} \in (\mathbb{Z}^+)^{\oplus l}},
\]
from the given weakly special collection as we vary \(\tilde{\lambda} \in (\mathbb{Z}^+)^{\oplus l}\), where \(\Delta_{X(\tilde{\lambda})'}\) is the extension of \(\Delta\) on \(X(\tilde{\lambda})'\). By Remark 4.7 we will see that all these
\[
\{(X(\tilde{\lambda})', \Delta_{X(\tilde{\lambda})'}; \frac{1}{r} \sigma^* O_{X(\tilde{\lambda})}(1))\}_{\tilde{\lambda} \in (\mathbb{Z}^+)^{\oplus l}}
\]
are crepant to each other. Let \(X(\tilde{\lambda}_0, i)\) be the component of \(X(\tilde{\lambda})_0\) which corresponds to \(c_i \cdot \text{ord}_{E_i}\), and still write \(c_i \cdot \text{ord}_{E_i, \infty}\) for the valuation over \(X(\tilde{\lambda})'\) induced by \(c_i \cdot \text{ord}_{E_i}\). By the construction, we see
\[
\text{ord}_{X(\tilde{\lambda}_0, i)} = \lambda_i \cdot \text{ord}_t + c_i \cdot \text{ord}_{E_i, \infty},
\]
where \(t\) is the parameter of \(\mathbb{A}^1\). We also note here that, the data of the vector of positive integers somewhat gives the information on the polarization of the corresponding weakly special semi-test configuration. For example, if \(\tilde{\lambda} := (\lambda_1, ..., \lambda_l)\) and \(\tilde{\lambda}' := (\lambda'_1, ..., \lambda'_l)\) are two vectors satisfying that \(\lambda_i - \lambda'_i = a\) is constant for \(1 \leq i \leq l\), then we could make sure that the two corresponding weakly special semi-test configurations share the same ambient space, and the two polarizations are different by a multiple of the central fiber (the multiple is \(a\)).

**Remark 4.7.** In this remark, we give an alternative way to construct weakly special degeneration from a weakly special collection, following the spirit of Proposition 3.3. Given a weakly special collection \(\{c_1 \cdot \text{ord}_{E_1}, ..., c_l \cdot \text{ord}_{E_l}\}\) and a vector \(\tilde{\lambda} := (\lambda_1, ..., \lambda_l) \in (\mathbb{Z}^+)^{\oplus l}\), we first take a dlt modification \(Y \to X\) which exactly extracts all \(E_i\) by the similar argument as [CZ21, Lemma 5.1] or Claim 4.4. Then we consider the product space
\[
Y \times \mathbb{A}^1 \to X \times \mathbb{A}^1.
\]
Let us write \(X_0, Y_0\) for the central fibers and still denote \(E_i, \infty\) to be the extension of \(E_i\) on \(Y \times \mathbb{A}^1\). Thus we get a collection of divisorial valuations
\[
\{\lambda_j \cdot \text{ord}_{Y_0} + c_j \cdot \text{ord}_{E_j, \infty} \mid j = 1, ..., l\}
\]
over \(X \times \mathbb{A}^1\). Let \(E_j\) be a prime divisor over \(X \times \mathbb{A}^1\) such that
\[
\text{ord}_{E_j} := \lambda_j \cdot \text{ord}_{Y_0} + c_j \cdot \text{ord}_{E_j, \infty},
\]
we see that all \(E_j\) are log canonical places of \((X \times \mathbb{A}^1, E \times \mathbb{A}^1 + X_0)\) centered in \(X_0\). Apply the similar argument as Claim 4.4 one can find a \(\mathbb{C}^*\)-equivariant birational model of \(X \times \mathbb{A}^1\) which exactly extracts all \(E_j\), denoted by
\[
\mathcal{Y}(\tilde{\lambda}) \to X \times \mathbb{A}^1.
\]
\[\text{This means that we do not care about the information on the polarization.}\]
The central fiber of $\mathcal{Y}(\tilde{\lambda}) \to \mathbb{A}^1$ clearly contains the trivial component, i.e., the strict transform of $X_0$. However, one can contract this trivial component via an MMP sequence as in the proof of Theorem 4.1, denoted by

$$\mathcal{Y}(\tilde{\lambda}) \dasharrow \mathcal{X}(\tilde{\lambda})'.$$

If we denote $\Delta_{\mathcal{Y}(\tilde{\lambda})}$ (resp. $\Delta_{\mathcal{X}(\tilde{\lambda})'}$) to be the extension of $\Delta$ on $\mathcal{Y}(\tilde{\lambda})$ (resp. $\mathcal{X}(\tilde{\lambda})'$), then it is not hard to see that the following three pairs are crepant to each other:

(1) $(\mathcal{Y}(\tilde{\lambda}), \Delta_{\mathcal{Y}(\tilde{\lambda})})$, $(\mathcal{X}(\tilde{\lambda})', \Delta_{\mathcal{X}(\tilde{\lambda})'})$, $(\mathcal{X} \times \mathbb{A}^1, \Delta \times \mathbb{A}^1)$,

and $(\mathcal{X}(\tilde{\lambda})', \Delta_{\mathcal{X}(\tilde{\lambda})'})$ here is exactly what we get in Remark 4.6 up to isomorphism in codimension one.

5. Remark on global correspondence for log Fano pairs

In the work [CZ21], for a given log canonical log Fano pair, we establish the correspondence between lc places of complements and weakly special test configurations with integral central fibers (see [CZ21, Theorem 1.2]). The method used there heavily depends on the computation of log canonical slope (see [CZ21, Theorem 3.1 and Theorem 6.2]), which follows the spirit of the proof of [Xu21, Theorem 4.10]. However, this method seems hard to apply to weakly special test configurations with reduced central fibers and to establish the correspondence as Theorem 1.3 and [BLX19, Theorem A.2]. In this section, we give a remark on this topic.

Recall that a projective log pair $(X, B)$ is called log Fano if $-K_X - B$ is ample. A log canonical log Fano pair means a log Fano pair with log canonical singularities.

**Definition 5.1.** Let $(X, B)$ be an lc log Fano pair and $L := -K_X - B$. A test configuration (resp. semi-test configuration) $(\mathcal{X}, B; L) \to \mathbb{A}^1$ of $(X, B; L)$ is weakly special if $(X, B + X_0)$ is log canonical and $L \sim_{\mathbb{Q}} -K_X - B$ is relatively ample (resp. semiample).

Let $(X, B)$ be an lc log Fano pair and $L := -K_X - B$. Choose a positive integer $r$ such that $rL$ is Cartier. Let $(Z, B_Z)$ be the affine cone over $(X, B)$ with respect to $rL$, where $B_Z$ is the extension of $B$ on $Z$. Denote by $o \in Z$ the cone vertex, and we still write $v_0$ for the canonical valuation obtained by the blow-up of $o \in Z$.

**Definition 5.2.** Notation as above. A finite set of divisorial valuations over $X$, denoted by

$$\{c_1 \cdot \text{ord}_{E_1}, \ldots, c_l \cdot \text{ord}_{E_l}\},$$

is called a weakly special collection of $(X, B)$ if the following two conditions are satisfied:

1. for each $i$, $c_i \in \mathbb{Z}^+$ and $E_i$ is a prime divisor over $X$;
2. there exists some effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X - B$ such that $(X, B + D)$ is log canonical and all $E_i$ are lc places of $(X, B + D)$.

We also note that the definition of weakly special collection here is different from that in [BLX19, Appendix A], as we already include the information of lc places. It is natural to ask the following question:
Question 5.3. Suppose that \((X, B)\) is an lc log Fano pair. Is there a correspondence between weakly special semi-test configurations of \((X, B; -K_X - B)\) and weakly special collections of \((X, B)\)?

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