Resummation of infrared divergencies in the theory of atomic Bose gases

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We present a general strong-coupling approach for the description of an atomic Bose gas beyond the Bogoliubov approximation, when infrared divergences start to occur that need to be resummed exactly. We consider the determination of several important physical properties of the Bose gas, namely the chemical potential, the contact, the speed of sound, the condensate density, the effective interatomic interaction and the three-body recombination rate. It is shown how the approach can be systematically improved with renormalization-group methods and how it reduces to the Bogoliubov theory in the weak-coupling limit.

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The main challenge of statistical physics is to describe the many-body properties of a system given the underlying few-body physics. Cold atomic gases provide a versatile experimental testbed for these theoretical descriptions by allowing the investigation of the crossover of many-body systems from weak to strong two-body interactions, using magnetic-field-tunable Feshbach resonances [1–3]. In particular, the universal nature of fermionic many-body systems with resonant two-body interactions has been successfully studied experimentally and theoretically [2, 4]. The most remarkable property of such resonant systems, which have an infinite scattering length and are therefore said to be at unitarity, is that at zero temperature there is no other length scale than the average interatomic distance that is set by the particle density n. As a result all thermodynamic quantities, when appropriately scaled, can be expressed in terms of a set of universal numbers. For the case of the Fermi gas at unitarity, one of the most crucial quantities is the chemical potential

$$\mu = (1 + \beta)\epsilon_F,$$  \hspace{1cm} (1)

which is given by an universal constant times the Fermi energy $\epsilon_F = \hbar^2 k_F^2/2m$, where $k_F = (6\pi^2 n/2s + 1)^{1/3}$ is the Fermi momentum and $s = 1/2$ due to the hyperfine degrees of freedom. The universal constant $\beta$ can be interpreted as describing the deviation from the ideal gas result due to interactions and was found to be $\beta \approx -0.63$ experimentally as well as theoretically [4–6].

Recently there has been increasing experimental interest in the strongly interacting Bose gas [7–12]. It is expected on dimensional grounds that the Bose gas at unitarity, if stable, has similar universal properties as that of the unitary Fermi gas. For instance Eq. (1) is expected to hold also but with $s = 0$ and a different value of $\beta$ due to the different statistics of the atoms. In contrast to the unitary Fermi gas, the realization of the unitary Bose gas is complicated by an increased loss of atoms as a consequence of a strong increase in the rate of inelastic three-body recombination processes caused by the absence of the Pauli principle and the existence of Efimov trimers. These three-body processes result in the formation of molecules, which shows that the actual ground state of these gases is a Bose-Einstein condensate of molecules. Nevertheless, it may still be experimentally possible to create the meta-stable state of a Bose-Einstein condensate of atoms at large scattering lengths for a sufficiently long time [12]. We have little to say about this important problem in this paper, and assume from now on that such a meta-stable state can indeed be realized in the laboratorium.

On the theoretical side, the description of the unitary Bose gas has been challenging and recent theoretical results strongly vary [13–23]. The main difficulty with constructing a theory of unitary Bose gases comes from the fact that there is no small parameter in the theory. Variational studies circumvent this by finding the minimum of the thermodynamic potential. However, since we are interested in the meta-stable state, care should be taken to project out the true many-body groundstate. In addition, diagrammatic approaches beyond the Bogoliubov theory are known to be plagued by logarithmic infrared divergences, as was first noted by Gavoret and Nozières [24].

Motivated by these ongoing efforts to study atomic Bose gases with strong interaction effects, the main objective of this paper is to present a general strong-coupling approach to an interacting Bose gas that can be improved systematically, for instance by renormalization-group methods but also by other non-pertubative methods such as the large-N expansion. Our approach is by construction free of the troublesome infrared divergencies by exactly incorporating the phase fluctuations of the Bose-Einstein condensate, which are known to dominate the long-wavelength behavior of the system [24–26]. More precisely, the theory is first renormalized by all other fluctuations using for instance the renormalization group. Then using this improved theory we next include the effects of the phase fluctuations of the Bose-Einstein condensate, which is reminiscent of bosonization for fermions. That the phase fluctuations are exactly incorporated will be confirmed by reconstructing the exact form of the single-particle propagator in the long-wavelength limit as derived by Nepomnyashchii and Nepomnyashchii [27, 28].

The outline of the paper is as follows. In section I
we give a brief overview of Bogoliubov theory and discuss the difficulties in going beyond this theory, such as the appearance of the above-mentioned infrared divergences. Subsequently, in section II, we present our strong-coupling approach which circumvents these difficulties by incorporating the phase fluctuations of the Bose-Einstein condensate exactly. In particular, the theoretical framework is discussed in subsection II A. In section II B it is first discussed how the Bogoliubov theory is reproduced within this general framework when taking the weak-coupling limit. Next, we discuss as a proof of principle also a first non-trivial approximation that goes beyond the Bogoliubov theory and allows us to obtain finite results for several properties of the Bose gas as a function of the coupling constant, i.e., the scattering length. Finally we conclude our paper in section III and discuss various avenues for further improvement.

I. BOGOLIUBOV THEORY AND BEYOND

In this section we illustrate the difficulties in constructing a theory of the strongly interacting Bose gas, which will be of use when presenting our approach in section II. We first briefly review in section IA Bogoliubov theory as a benchmark for our theory. Next, we recognize that the correct low-energy behavior must be exactly incorporated into the theory. To do so requires going beyond Bogoliubov theory at which point we encounter the above-mentioned logarithmic infrared divergences, which are discussed in section IB.

A. Atomic Bose gas

Here we briefly summarize some of the results of Bogoliubov theory, including the first quantum corrections, as a benchmark for our theory. In general, the Bose gas in cold atom experiments is well described by the Euclidean action $S[\phi^*, \phi] = \int d\tau dx L(x, \tau)$ with a point interaction, where the lagrangian density is

$$L(x, \tau) = \phi^*(x, \tau) \left[ \hbar \partial_\tau - \frac{\hbar^2 \nabla^2}{2m} - \mu \right] \phi(x, \tau) + \frac{1}{2} T^{2B} |\phi(x, \tau)|^4. \quad (2)$$

Here $\phi$ is the atomic field, $\mu$ is the chemical potential, $T^{2B} = 4na(B)\hbar^2/m$ is the exact two-body $T$(ransition) matrix at zero energy and momentum, $a(B)$ is the magnetic-field-tunable scattering length, and $m$ is the mass of the atoms.

In mean-field theory, which amounts to expanding the field in terms of the condensate and neglecting the fluctuations around it, the time-independent equation for the atomic condensate is

$$\mu = nT^{2B} = \frac{\hbar^2}{ma^2} (4\pi na^3), \quad (3)$$

where it was used that at this level of approximation the condensate density $n_c$ is equal to the total density $n$. The first quantum correction to the above result was calculated by Lee-Huang-Yang (LHY) using the Bogoliubov theory that also incorporates the gaussian fluctuations around the mean-field solution, and results in [29]

$$\mu = \frac{\hbar^2}{ma^2} (4\pi na^3) \left( 1 + \frac{16}{3\pi} \sqrt{4\pi na^3} \right). \quad (4)$$

The condensate density to this order is given by

$$n_c = n \left( 1 - \frac{4}{3\pi} \sqrt{4\pi na^3} \right). \quad (5)$$

This shows the depletion from the condensate due to the interaction. Higher-order corrections to the chemical potential have been determined [30], however, these depend also on three-body physics and will not be discussed in detail here.

Another important quantity of the atomic Bose gas is called the contact $C = 31–36$. It is determined by the short-wavelength behavior of the single-particle distribution function, namely $n(k) \simeq C/k^4$. In Bogoliubov theory the contact is given by [33] $C = (4\pi na^2) \left[ 1 + \frac{48}{3\pi} \sqrt{4\pi na^3} \right], \quad (6)$

where also the first quantum correction is shown, consistent with the Lee-Huang-Yang correction of the chemical potential.

Clearly all the above quantities diverge in the unitarity limit $a \to \infty$, which is not surprising since they are expansions in terms of the small parameter $\sqrt{4\pi na^3}$. This is a consequence of the fact that in Bogoliubov theory no many-body corrections on the scattering length have been taken into account, such that the effective interaction cannot become finite at unitarity. Therefore, to be able to describe the Bose gas in the strongly-interacting limit $(na^3 \gg 1)$ the action $S[\phi^*, \phi]$ needs to be properly renormalized as we discuss in much more detail later on.

B. Difficulties beyond Bogoliubov theory

In Bogoliubov theory it thus appears that we cannot reach the strongly interacting regime. Therefore, we now want to go beyond Bogoliubov theory. To correctly describe the low-energy behavior of the Bose-Einstein condensate it appears natural to use Bogoliubov theory to describe the excitations above the condensate. However, it proves difficult to renormalize the action of the atomic Bose gas of Eq. (2) using the Bogoliubov propagator. Here we discuss some of the difficulties we encounter when trying to renormalize the action after using the Bogoliubov substitution. Again we expand the field around the condensate density $n_c$, i.e.,

$$\phi(x, \tau) = \sqrt{n_c} + \phi'(x, \tau), \quad (7)$$
and we obtain the mean-field equation
\[ \mu = n_c \Delta T^{2B}. \]

In Bogoliubov theory only terms quadratic in the fluctuations are taken into account in the lagrangian, giving for the fluctuations the action
\[ \frac{1}{2} \sum_{k,n} \Phi^\dagger(k, \omega_n) \left[-\hbar G^{-1}(k, \omega_n)\right] \Phi(k, \omega_n), \]
with \( \Phi^\dagger(k, \omega_n) = [\phi^\dagger(k, \omega_n), \phi^{\ast\dagger}(-k, -\omega_n)]^T \). The components of the \( 2 \times 2 \) (Nambu space) inverse Green’s function \( G^{-1} \) are
\[ -\hbar G_{11}^{-1}(k, \omega_n) = \imath \hbar \omega_n + \epsilon_k - \mu + \hbar \Sigma_{11}, \]
\[ \hbar \Sigma_{11} = 2n_c \Delta T^{2B}, \quad \hbar \Sigma_{12} = n_c \Delta T^{2B}, \]
with the properties \( G_{11}^{-1}(k, \omega_n) = G_{22}^{-1}(-k, -\omega_n) \) and also \(-\hbar G_{12}^{-1}(k, \omega_n) = -\hbar G_{21}^{-1}(-k, -\omega_n) = \hbar \Sigma_{12} \). Moreover, \( \omega_n \) are the bosonic Matsubara frequencies and \( \epsilon_k = \hbar^2 k^2 / 2m \) is the free atomic dispersion. Taking the inverse of the matrix in Eq. (8) we obtain the \( 2 \times 2 \) Bogoliubov Green’s function, whose components are
\[ -\hbar^{-1} G_{11}(k, \omega_n) = \frac{\imath \hbar \omega_n + \epsilon_k + n_c \Delta T^{2B}}{-(\imath \hbar \omega_n)^2 + (\hbar \omega_k)^2}, \]
\[ -\hbar^{-1} G_{12}(k, \omega_n) = \frac{-n_c \Delta T^{2B}}{-(\imath \hbar \omega_n)^2 + (\hbar \omega_k)^2}, \]
where the mean-field equation was used to eliminate the chemical potential and we defined the dispersion \( \hbar \omega_k \) as
\[ \hbar \omega_k = \sqrt{\epsilon_k (\epsilon_k + 2n_c \Delta T^{2B})}. \]

To go beyond the Bogoliubov approximation, which as we have seen is necessary to describe a strongly interacting Bose gas, we need to compute the corrections to the propagator, or more precisely to the self-energy matrix \( \hbar \Sigma \). Doing so, the one-loop correction gives rise to an infrared logarithmic divergency in the normal and anomalous self-energy as a consequence of the linear mode in the normal and anomalous propagators, as was previously noted in Refs. [24, 37]. This is easily shown by realizing that at low momenta and low frequencies both the normal and anomalous propagator are of the relativistic form \( 1/K^2 \) with the four-vector \( K = (\imath \hbar \omega_n, \sqrt{2n_c \Delta T^{2B} \epsilon_k}) \) and thus first-order corrections to the normal and anomalous self-energies give rise to a logarithmically divergent quantity \( \Delta \Sigma \) proportional to
\[ \int \frac{d^4K'}{K^2 (K' - K)^2} \propto \log \left[ \frac{-(\imath \hbar \omega_n)^2 + 2n_c \Delta T^{2B} \epsilon_k}{\Lambda^2} \right], \]
where \( \Lambda \) is some high-energy cut-off obeying \( K^2 \ll \Lambda^2 \). This logarithmic divergence makes it increasingly difficult to apply a self-consistent diagrammatic renormalization procedure to find the effective interaction and self-energies of the atoms. Nevertheless, it was shown by Nepomnyashchii and Nepomnyashchii that an important consequence of these divergencies is that the exact anomalous self-energy vanishes for zero momentum and energy, i.e., \( \hbar \Sigma_{12}(0, 0) = 0 \) [27, 28]. This indicates another difficulty with the Bogoliubov substitution, since it gives rise to a non-zero anomalous self-energy, as for example in Eq. (9).

In general, when encountering infrared divergencies we need to perform a resummation of an infinite amount of diagrams. Indeed, a resummation of the one-loop diagrams gives in the long-wavelength limit \( \hbar \Sigma_{11} = \mu + \Delta \Sigma^{-1} + \mathcal{O}(\omega, \epsilon_k) \) and \( \hbar \Sigma_{12} = \Delta \Sigma^{-1} + \mathcal{O}(\omega^2, \epsilon_k) \) [28], where \( \Delta \Sigma \) is again the above logarithm. Thus after resummation the anomalous self-energy satisfies the exact relation \( \hbar \Sigma_{12}(0, 0) = 0 \). Also, to obtain a consistent theory of the Bose gas it is necessary to make sure that the theory has a gapless mode at each level of approximation as a consequence of Goldstone’s theorem. This statement is equivalent to demanding that the self-energies satisfy the Hugenholtz-Pines relation \( \hbar \Sigma_{11}(0, 0) - \hbar \Sigma_{12}(0, 0) = \mu \) [38]. The resummed self-energies indeed satisfy this relation, quite simply as \( \hbar \Sigma_{11}(0, 0) = \mu \).

Now, we may think that because we have obtained reasonable self-energies, we are in a position to further investigate the effects of interactions. This turns out to be no simple task, especially since the full self-energies are quite involved. As an example, in order to re-obtain the sound mode in the propagators in the long-wavelength limit it is already necessary to deal with precise cancellations of the logarithms, as was shown by Nepomnyashchii and Nepomnyashchii [28].

To summarize, after the Bogoliubov substitution we encounter difficulties to go beyond the Bogoliubov approximation because of logarithmic infrared divergencies. To perform self-consistent calculations of the effective interaction and the normal and anomalous self-energies that always satisfy the Hugenholtz-Pines relation and the requirement of a linear mode in the single-particle Green’s function quickly becomes practically unfeasible. In the following we will isolate these troublesome infrared divergences, which will be seen to originate from the phase fluctuations of the Bose-Einstein condensate, and most importantly show how to exactly incorporate these fluctuations in our approach.

II. RENORMALIZED BOSONIZATION

Here the general framework of our strong-coupling approach is presented. Subsequently, after discussing the weak-coupling limit where the Bogoliubov theory is reproduced, we discuss a first non-trivial application of the general framework to obtain several properties of the Bose gas as a function of scattering length, such as the chemical potential, the contact, the speed of sound, the condensate density and the effective interatomic interaction. Lastly, we also discuss the unitarity-limited three-

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As described above, it was used that $\phi''(x, \tau)$ and $n_0(x, \tau)$ or $\theta(x, \tau)$ are orthogonal to each other, i.e., the space-time integral over their products vanish. Then by performing the path integral over the non-phase fluctuations $\phi''$, the phase-fluctuation-dependent part of the action in lowest order in the derivatives is

$$\int d\tau d\mathbf{x} \left\{ n(x, \tau) (i\hbar \partial_\tau) \theta(x, \tau) + \frac{\hbar^2 n(x, \tau)}{2m} (\nabla \theta(x, \tau))^2 \right\}. $$

Here we introduced the total density

$$n(x, \tau) = n_0(x, \tau) + \langle \phi''(x, \tau)\phi''^*(x, \tau) \rangle = n_0(x, \tau) + \langle \phi'(x, \tau)\phi'^*(x, \tau) \rangle. $$

Expanding the latter around its equilibrium value $n(x, \tau) = n + \delta n(x, \tau)$ the gaussian part of the action can be written in momentum space as

$$\frac{1}{2} \sum_{\mathbf{k}, n} \left[ \frac{\delta n(\mathbf{k}, \omega_n)}{\theta(\mathbf{k}, \omega_n)} \right]^\dagger \left( \chi_{nn}(\mathbf{k}) - \frac{\hbar \omega_n}{\hbar \omega_n} 2 n c_k \right) \left[ \frac{\delta n(\mathbf{k}, \omega_n)}{\theta(\mathbf{k}, \omega_n)} \right],$$

where we introduced the exact density-density correlation function $\chi_{nn}(\mathbf{k})$. The phase-fluctuation propagator is thus found to be

$$\langle \theta(\mathbf{k}, \omega_n)\theta^*(\mathbf{k}, \omega_n) \rangle = \frac{1}{\pi} \frac{mc^2}{(\hbar \omega_n)^2 + 2mc^2 c_k},$$

where the speed of sound is $c = \sqrt{n\chi_{nn}(0)/m}$. Note that we have obtained in this manner the exact phase-fluctuation propagator in the long-wavelength limit.

**Propagator of the non-phase fluctuations**

When integrating out the non-phase fluctuations $\phi'$ the phase of the condensate must be considered as non-fluctuating. Therefore, the propagator of the non-phase fluctuations can be determined from the action with a constant phase. Comparing the expansions of the field in Eq. (12) with Eq. (7) we see that the quadratic part of the action $S[n_0, \theta, \phi'^*, \phi']$ with constant phase, for simplicity take $\theta = 0$, is given by the Bogoliubov action of Eq. (8). The usual Bogoliubov propagators, however, contain contributions of the phase fluctuations, which can be identified by their proportionality to $n_0$, since in Bogoliubov theory the phase fluctuations are described by $n_0 \exp [i\theta(x, \tau)] \sqrt{n_0}. \exp [i\theta(x, \tau)] - \sqrt{n_0} \approx i \sqrt{n_0}\theta(x, \tau)$. We thus see that the contributions from the phase fluctuations are

$$\langle \phi'(x, \tau)\phi'^*(x, \tau) \rangle \propto n_0 \langle \theta(x, \tau)\theta(x, \tau) \rangle,$$

$$\langle \phi'(x, \tau)\phi''^*(x, \tau) \rangle \propto -n_0 \langle \theta(x, \tau)\theta(x, \tau) \rangle.$$

**A. Theory**

In view of the problems discussed in the previous section, we now show how to incorporate the phase fluctuations exactly and automatically resum all infrared divergences in the theory. To describe the Bose-Einstein condensed phase, we expand the field as

$$\phi(x, \tau) = \sqrt{n_0(x, \tau)} \exp [i\theta(x, \tau)] + \phi'(x, \tau),$$

where $n_0 = \langle n_0(x, \tau) \rangle$ should now be viewed as the quasi-condensate density [3] and not as the density of atoms in the condensate $n_c$. The latter will be related to $n_0$ by the large-distance behavior of the fluctuations in the phase of the condensate $\theta(x, \tau)$ as we will see shortly. Roughly speaking, the first term of the expansion describes the low-energy modes of the field, as shown in Fig. 1, and includes the phase fluctuations. The fluctuations $\phi'(x, \tau)$ describe the high-energy modes and are defined such that they do not contain phase fluctuations. The non-phase fluctuations $\phi'$ are thus orthogonal to the first term in Eq. (12). By inserting the expansion into Eq. (2), the action $S[n_0, \theta, \phi'^*, \phi']$ is obtained.

To proceed, we first show how to obtain the exact phase-fluctuation propagator and the propagator of non-phase fluctuations from this action. The latter will then be used to renormalize the theory using the renormalization group, after which the exact contributions of the phase fluctuations are re-introduced.

**Propagator of the phase fluctuations**

The action for the phase fluctuations can be found by eliminating the phase dependence of the part of the action involving $\phi'$ through the replacement

$$\phi'(x, \tau) \rightarrow \exp [i\theta(x, \tau)] \phi''(x, \tau).$$

This procedure of extracting the overall phase of the field $\phi$ is reminiscent of bosonization for fermions. The phase-fluctuation-dependent part of the action $S[n_0, \theta, \phi'^*, \phi'']$ reduces to

$$\int d\tau d\mathbf{x} \left\{ \left[ n_0(x, \tau) + |\phi''(x, \tau)|^2 \right] (i\hbar \partial_\tau) \theta(x, \tau) + \frac{\hbar^2}{2m} \left[ n_0(x, \tau) + |\phi''(x, \tau)|^2 \right] (\nabla \theta(x, \tau))^2 \right\}. $$

FIG. 1. Schematic representation of the expansion of the field $\phi$ in terms of the condensate and its phase fluctuations and the non-phase fluctuations $\phi'$, c.f. Eq. (12).
Thus we can remove the phase fluctuations from the Bogoliubov propagators in Eq. (10) by writing

$$\langle \phi'(k, \omega_n) \phi'^* (k, \omega_m) \rangle = \frac{i \hbar \omega_n + \epsilon_k + n_0 T^{2B}}{(\hbar \omega_n)^2 + (\hbar \omega_k)^2} n_0 T^{2B} - \hbar \left( \frac{(\hbar \omega_n)^2 + (\hbar \omega_k)^2}{(\hbar \omega_n)^2 + (\hbar \omega_k)^2} \right),$$

$$\langle \phi'(k, \omega_n) \phi'(k, \omega_m) \rangle = \frac{i n_0 T^{2B}}{(\hbar \omega_n)^2 + (\hbar \omega_k)^2} n_0 T^{2B} + \frac{\hbar}{(\hbar \omega_n)^2 + (\hbar \omega_k)^2} = 0,$$

where the second term on both right-hand sides is the phase-fluctuation propagator with $mc^2 = n_0 T^{2B}$ and the dispersion $\sqrt{2mc^2 \epsilon_k}$ is extended to the full Bogoliubov dispersion $\hbar \omega_k = \sqrt{\epsilon_k (\epsilon_k + 2mc^2)}$. In contrast to the exact phase-fluctuation propagator, the factor $n_c/n$ is not present in the Bogoliubov propagator, which can be attributed to a renormalization not present in Bogoliubov theory. After the subtraction of the phase fluctuations, the propagator of the non-phase fluctuations is given by

$$\tilde{h}^{-1} \langle \phi'(k, \omega_n) \phi'^* (k, \omega_m) \rangle = \frac{i \hbar \omega_n + \epsilon_k}{(\hbar \omega_n)^2 + (\hbar \omega_k)^2},$$

(15)

while the anomalous averages vanish, i.e.,

$$\langle \phi'(k, \omega_n) \phi'(k, \omega_m) \rangle = \langle \phi'^* (k, \omega_n) \phi'^* (k, \omega_n) \rangle = 0.$$

The vanishing of the anomalous averages means that the Green’s function is diagonal in Nambu space and this greatly simplifies the renormalization procedure of the interaction.

**Renormalization due to the non-phase fluctuations**

The accuracy of the action $S [ n_0, \theta, \phi'^*, \phi']$ can be improved systematically by incorporating the $\phi'$ fluctuations into a renormalization of the action. However, due to the fundamental Ward identities associated with the $U(1)$ invariance of the theory, it turns out to be more convenient to carry out this renormalization immediately at the level of $S [ \phi^*, \phi]$, cf. Eq. (2), and then apply the expansion of the field, as in Eq. (12). To be useful for a strong-coupling situation this renormalization should be carried out by a non-perturbative method, such as for instance the large-$N$ expansion or the renormalization group. We here discuss only the latter choice. The exact Wilsonian renormalization-group flow equation for the action $S [ \phi^*, \phi]$ is

$$\frac{dS}{d\Lambda} = \frac{\hbar}{2} \left( \delta^2 \ln \left( \frac{1}{\hbar} \delta^2 S_{\text{int}} + \frac{1}{\hbar} \delta \Phi \delta \Phi^* \right) \right),$$

which is derived in Appendix B. Here $S [ \phi^*, \phi; \Lambda]$ is the effective action obtained by integrating out all non-phase fluctuations above the momentum $\hbar \Lambda$, $G'$ is the matrix propagator of the non-phase fluctuations, $S_{\text{int}}$ is the non-gaussian part of the effective action, the trace is over space, imaginary time and Nambu space $\Phi(k, \omega_n) = [\phi'(k, \omega_n), \phi'^*(-k, -\omega_n)]^T$, and $\delta \Lambda = \delta (k - \Lambda)$. Although there are no small parameters in the theory of unitary Bose gases, the renormalization group can distinguish between the relevance of the various coupling constants based on their scaling dimension under renormalization. As the effective interaction evaluated at zero momentum and zero frequency is expected to be a crucial variable, since it induces a flow of the chemical potential that corresponds to the most relevant operator of the action, let us here restrict our attention to these parameters, allowing us also to give an explicit illustration of the general procedure. The running of the chemical potential and effective interaction $g$ are in general found to be given in terms of the so-called beta functions by

$$\Lambda \frac{d\mu}{d\Lambda} = \beta_\mu (\mu, g), \quad \Lambda \frac{dg}{d\Lambda} = \beta_g (\mu, g).$$

By solving these equations the renormalized action $S [ \phi^*, \phi; \Lambda]$ is found. Then, after inserting the expansion of the field, the renormalized action $S [ n_0, \theta, \phi'^*, \phi'; \Lambda]$ is obtained. This action defines the propagator of the non-phase fluctuations in terms of the effective interaction, which in this case is simply Eq. (15) with the interaction replaced by the effective interaction at zero momentum and zero frequency, namely $mc^2 \equiv n_0 g$. This can thus be seen as a self-consistency condition on the propagator of the non-phase fluctuations, which should be generalized when more running coupling constants are included.

Before we turn to the solution of the above renormalization-group equations, we first show that our approach reproduces the exact propagator in the long-wavelength limit derived by Nepomnyashchii and Nepomnyashchii, as mentioned in the introduction, and that the condensate density and the total density can in general be expressed in terms of the quasi-condensate density and the effective interaction at zero frequency and momentum.

**Exact normal and anomalous propagators**

To reproduce the exact propagator in the long-wavelength limit we take the Fourier transform of the exact one-particle correlation function, which in our theory is given by

$$\langle \phi(x, \tau) \phi^* (0, 0) \rangle = n_0 \left( \exp \left( i (\theta(x, \tau) - \theta(0, 0)) \right) + \langle \phi'(x, \tau) \phi'^* (0, 0) \rangle \right)$$

By expanding the exponential we find that the dominant long-wavelength behavior is due only to the first three terms in the expansion, where the first term is the condensate density and the second term is simply the exact phase-fluctuation propagator in Eq. (14). The third term gives a non-trivial logarithmic term, which results from a
convolution of two phase-fluctuation propagators as was shown in section IB. The expansion is thus

$$\frac{1}{\hbar} \langle \phi(k, \omega_n) \phi^*(k, \omega_n) \rangle \approx n_c \beta V \delta_{k,0} \delta_{n,0} + \frac{m c^2}{(\hbar \omega_n)^2 + 2mc^2\epsilon_k} + \frac{3\sqrt{mc^2}}{32\sqrt{2}\epsilon_F} \frac{n_c}{n} \log \left[ \frac{(\hbar \omega_n)^2 + 2mc^2\epsilon_k}{(8mc^2\epsilon_k)} \right].$$

(16)

To obtain the denominator inside the logarithm using the phase-fluctuation propagator in Eq. (14) with $mc^2 = n_0 g$, an ultra-violet subtraction was needed. This subtraction removes the ultra-violet divergences associated with a point interaction $[3]$, and is a result of the renormalization of the bare coupling to $T^{2B}$, as explained in appendix C. Also, it was used that

$$n_0 \langle \exp \left[ i \left( \theta(x, \tau) - \theta(0,0) \right) \right] \rangle = n_0 \exp \left[ -\frac{1}{2} \langle \left( \theta(x, \tau) - \theta(0,0) \right)^2 \rangle \right] = n_c \exp \left[ \langle \theta(x, \tau) \theta(0,0) \rangle \right],$$

and the condensate density is defined in terms of the off-diagonal long-range order of the one-particle density matrix

$$n_c = \lim_{|x| \to \infty} \langle \phi(x,0) \phi^*(0,0) \rangle \quad \text{(17)}$$

$$= n_0 \exp \left[ -\langle \theta(0,0) \theta(0,0) \rangle \right].$$

In the last line it was used that in the limit of large separation $\langle \theta(x,0) \theta(0,0) \rangle = 0$, as is also shown in Appendix C.

Similarly, the exact anomalous propagator is given by

$$\langle \phi(x, \tau) \phi(0,0) \rangle = n_0 \langle \exp \left[ i \left( \theta(x, \tau) - \theta(0,0) \right) \right] \rangle = n_c \exp \left[ -\langle \theta(x, \tau) \theta(0,0) \rangle \right],$$

such that the Fourier transform in the long-wavelength limit only differs from Eq. (16) by a minus sign in front of the second term in the right-hand side. The above expressions are the exact normal and anomalous propagators in the long-wavelength limit, as derived in a different manner in Refs. [27, 28]. In particular, this leads to the counter-intuitive conclusion that the anomalous self-energy vanishes at zero momentum and zero frequency [28].

### Condensate density and total density

The condensate density can be expressed in terms of the quasicondensate density and the effective interaction using Eq. (17) as

$$n_c = n_0 \exp \left[ \frac{3}{4} \left( 2\sqrt{2} - \pi \right) \left( \frac{n_0 g}{\epsilon_F} \right)^{3/2} \right], \quad \text{(18)}$$

In order to determine the condensate density, the quasicondensate density $n_0$ needs to be eliminated in favor of the total density $n = \langle \phi(x, \tau) \phi^*(x, \tau) \rangle$ using

$$n = n_0 + \frac{1}{4} \left( 8\sqrt{2} - 3\pi \right) \left( \frac{n_0 g}{\epsilon_F} \right)^{3/2} n,$$ \quad \text{(19)}$$

where the second term is the contribution from the high-energy fluctuations $n' = \langle \phi'(x, \tau) \phi'^*(x, \tau) \rangle$, see Eq. (15). As required, exactly the same ultra-violet subtraction was used for the high-energy fluctuations as in Eq. (18). To solve these equations only the effective interaction at zero momentum and zero frequency remains to be determined using the renormalization group.

To summarize thus our general approach, the action $S[\phi, \phi]$ of the Bose gas can be systematically renormalized by the non-phase fluctuations $\phi'$ using for instance the renormalization-group flow equation, giving in particular rise to an effective coupling $g$ and a renormalized chemical potential $\mu$. The propagators of the non-phase fluctuations are determined self-consistently after expansion of the field $\phi$. After this renormalization step has been performed, during which no infrared divergencies will occur, the exact propagator of the phase fluctuations can be used to reproduce the exact normal and anomalous propagators in the long-wavelength limit.

### B. Applications

In this section we apply this general framework within the simplest approximation that goes beyond the Bogoliubov theory to obtain several quantities of the Bose gas as a function of scattering length without encountering any infrared divergencies. We use this particular approximation mostly for illustrational purposes of the general procedure and as a proof of principle that in this manner finite results can be obtained even at unitarity. To formulate the most accurate approximation at unitarity is beyond the scope of this paper and is left for future work.

**Bogoliubov theory revisited**

Within Bogoliubov theory the effective interaction is assumed not to be running and we simply have that $g = g(\Lambda = 0) = g(\Lambda = \infty) = T^{2B}(-2n_0 g)$ [3], where the energy dependence of the two-body $T$ matrix is given by

$$T^{2B}(E) = \frac{4\pi a \hbar^2}{m} \frac{1}{1 - a \sqrt{-mE/\hbar^2}}.$$ \quad \text{(20)}$$

Note that in the boundary condition that is used, i.e., $g(\Lambda = \infty) = T^{2B}(-2n_0 g)$, the particular value of the energy argument of the two-body $T$ matrix is such that indeed not only the dominant but also the subdominant ultra-violet term in $\beta_g$ is cancelled as shown explicitly.
in Eq. (23) below. The chemical potential is running, however, with

$$
\beta_\mu = -2g \frac{4\pi \Lambda^3}{(2\pi)^3} \left[ \frac{\epsilon_\Lambda - i\hbar \omega_\Lambda}{2\hbar \omega_\Lambda} + \frac{n_0g}{2\epsilon_\Lambda + 2n_0g} \right],
$$

(21)

where the dispersions are evaluated at \( \Lambda \). Integrating the resulting renormalization-group equation with the boundary condition \( \mu(\Lambda = \infty) = gn_0 \) gives ultimately

$$
\mu = \mu(\Lambda = 0) = (2n' + n_0)T^{2B}(-2n_0g),
$$

(22)

with \( n' = n - n_0 \) determined from Eq. (19). As desired, the latter equation exactly reproduces the chemical potential of the Bogoliubov theory, including the Lee-Huang-Yang correction. Furthermore, Eqs. (18) and (19) also reproduce the condensate depletion of Eq. (5) at weak coupling.

**Effective interaction, (quasi-)condensate density and one-particle density matrix**

To go beyond the Bogoliubov theory, we must now determine the effective interaction \( g \) in a better approximation. Taking only the renormalization of the coupling constant and the chemical potential into account, which we here use to illustrate the general procedure but interestingly enough turns out to be very accurate for the unitary Fermi gas [39], the beta functions are given by

$$
\beta_\mu = -2g \frac{4\pi \Lambda^3}{(2\pi)^3} \left[ \frac{\epsilon_\Lambda}{2\hbar \omega_\Lambda} + \frac{1}{2\epsilon_\Lambda + 2n_0g} \right],
$$

$$
\beta_n = g \frac{4\pi \Lambda^3}{(2\pi)^3} \left[ \frac{u_\Lambda^4 + v_\Lambda^4 - 8u_\Lambda^2 v_\Lambda^2}{2\hbar \omega_\Lambda} - \frac{1}{2\epsilon_\Lambda + 2n_0g} \right],
$$

(23)

where the Bogoliubov dispersion \( \hbar \omega_{k} \) and the coherence factors \( u_\Lambda^2 = v_\Lambda^2 + 1 = (\hbar \omega_k + \epsilon_k)/2\hbar \omega_k \) are evaluated at \( \Lambda \). For a derivation of these expressions, compare with appendix A for the derivation of the ladder sum \( \Xi \) and bubble sum \( \Pi \) contributions. The equation also shows the non-perturbative nature of the renormalization group.

The effective interaction and the condensate density in terms of the total density are found analytically as a function of scattering length by solving Eqs. (18), (19), and (24), and are plotted in Figs. 3 and 4. As can be seen from Fig. 3 the position of the resonance shifts due to many-body effects to negative scattering lengths as a consequence of the screening effects of the bubble sum. In the unitarity limit, \( T^{2B} \to \infty \), the effective interaction and condensate density are in this first approximation given by

$$
\frac{ng}{\epsilon_F} = \frac{2}{3\sqrt{3}} \left( 1 + \lambda \right)^{1/3} \approx 1.09,
$$

$$
\frac{n_c}{n} = \frac{1 + \lambda}{1 + \lambda - \frac{\lambda}{\sqrt{2}(1 + \lambda)}} \approx 0.59,
$$

$$
\frac{n'}{n} = \frac{n - n_0}{n} = \frac{\lambda}{1 + \lambda} \approx 0.31,
$$

(24)

where

$$
\lambda \equiv \frac{n'}{n_0} \approx \frac{1}{3\sqrt{2}} \left( 8\sqrt{2} - 3\pi \right) \approx 0.45.
$$

The depletion from the condensate is given by \( 1 - n_c/n \approx 0.41 \), which clearly differs from the density of particles contributing to the non-phase fluctuating modes \( n' \) by phase-fluctuation contributions.

The one-particle density matrix is defined by

$$
n(x) = \langle \phi(x,0)\phi^*(0,0) \rangle = n_c \exp\left[ \langle \theta(x,0)\theta(0,0) \rangle \right] + \langle \phi'(x,0)\phi^{*'}(0,0) \rangle,
$$

where the expressions of the phase-fluctuation and the non-phase fluctuation propagator with the appropriate ultra-violet subtractions can be found in appendix C. The one-particle density matrix is shown for several scattering lengths in Fig. 5, including at unitarity. Clearly the condensate density reduces to the total density in the weak-coupling limit.

**Chemical potential and speed of sound**

The change in the chemical potential follows from integrating Eq. (23) and is given by \( \Delta \mu = 2ng \). According
The chemical potential at unitarity is usually written as \( \mu = (1 + \beta)\epsilon_F \), such that we have for the universal constant \( \beta \simeq 0.42 \). Furthermore, the speed of sound at unitarity is given by

\[
\frac{mc^2}{\epsilon_F} = \frac{n_0g}{\epsilon_F} = \frac{1}{1 + 2\lambda}\frac{\mu}{\epsilon_F} \simeq 0.53\frac{\mu}{\epsilon_F} \simeq 0.75.
\]

The expected value for the speed of sound at unitarity in terms of the chemical potential is \( mc^2 = n(d\mu/dn) = 2\mu/3 \simeq 0.66\mu \), which is close to our result and gives an indication of the accuracy of the simplest first approximation that we have presented here.

In comparison to the literature, our results for the chemical potential differ from the variational studies which find \( \beta \approx -0.2 \) \([15]\) and \( \beta \approx 1.93 \) \([13]\) and the renormalization-group study \( \beta \approx -0.34 \) \([16]\). As mentioned in the introduction, it is not clear that the variational studies are always inside a Hilbert space orthogonal to the true many-body ground state. Also, as correctly presented in these articles, these variational results should not be viewed as upper bounds to \( \beta \), as it is the energy which is determined variationally and not its derivative with respect to the number of atoms. Furthermore, the variational study in Ref. \([15]\) always has an attractive interaction whose normal mean-field contribution is treated in the Hartree-Fock approximation, which presumably explains its negative value of \( \beta \). In contrast, our result uses for both the normal and anomalous contributions an effectively repulsive interaction and as a result \( \beta \) becomes positive.

**Contact**

Another interesting property is called the contact \( C \) and is related to the short-wavelength behavior of the momentum distribution, namely \([32, 40]\)

\[
n(k) \simeq C/k^4.
\]

The value of the contact is determined by the non-phase fluctuations and is found after performing the Matsubara sum over Eq. (15) and expanding for large momenta to be

\[
\frac{C}{k^2} = \left( \frac{n_0g}{2\epsilon_F} \right)^2.
\]

This expression is of the same form as that found in Bogoliubov theory \([35]\) but with the two-body \( T \) matrix replaced by the effective interaction. At unitarity, its value

\[
\frac{mc^2}{\epsilon_F} = \frac{n_0g}{\epsilon_F} = \frac{1}{1 + 2\lambda}\frac{\mu}{\epsilon_F} \simeq 0.53\frac{\mu}{\epsilon_F} \simeq 0.75.
\]
is
\[ \frac{C}{k_F^2} = \frac{1}{3^{4/3}} \frac{1}{(1 + \lambda)^{4/3}} \simeq 0.14 \]

An equivalent definition of the contact is through the average of the interaction term in the action [33, 35, 40]
\[ \frac{C}{k_F^2} = \left( \frac{T^2}{2\epsilon_F} \right)^2 \langle |\phi|^4 \rangle . \tag{26} \]
Assuming that the action is first renormalized, such that the two-body \( T \)-matrix is replaced by the effective interaction \( g \), and that all non-phase fluctuations have been included into the renormalization of the action, i.e., we take \( \langle |\phi|^4 \rangle = n_0^2 \) to avoid double counting, we re-obtain Eq. (25).

Yet another definition of the contact can be given in terms of the derivative of the total energy or the chemical potential with respect to the scattering length, namely
\[ \frac{C}{k_F} = -\frac{4\pi}{\epsilon_F k_F^2} \frac{d(E/V)}{d(1/a)}, \tag{27} \]
where the total energy per volume is obtained from the chemical potential as \( E/V = \int_0^\infty \mu(n',a)dn' \). By neglecting the contribution of the non-phase fluctuations in the chemical potential, i.e., taking \( \mu = n_0 g \), we analytically re-obtain the same value of the contact at unitarity as obtained from Eq. (25) and numerically we re-obtain the same contact as a function of scattering length. If the contact is determined through the derivative of the complete chemical potential, which includes contributions from the non-phase fluctuations, it becomes larger. We expect that this difference is a consequence of a double counting, since the effects of non-phase fluctuations have already been included in the effective interaction and should not be included again through the derivative of the self-energy contribution \( 2n'g \) of the chemical potential.

### Energy-dependent effective interaction and bound state

The center-of-mass energy dependence is most easily investigated by generalizing Eq. (24) for non-zero frequencies, giving
\[ \frac{1}{g(\hbar \omega_n)} = \frac{1}{T_{2B}} - \left[ \Xi(0,\omega_n) + 4\Pi(0,0) \right] . \]
Here the bubble sum contribution \( \Pi \) is not energy dependent, since at this level of approximation it only depends on the relative energy. The frequency-dependent ladder contribution can be found analytically and its integral expression is shown in appendix A. For high energies the expression reduces to the vacuum expression in Eq. (20). The frequency dependence of the effective interaction at unitarity is shown in Fig. 6, where the Kramers-Kronig-like feature in the real and imaginary parts, that is a consequence of the molecular bound state, is clearly visible. This feature shifts to more negative frequencies for decreasing scattering lengths.

### Three-body recombination rate

As mentioned in the introduction, the atomic Bose gas is meta-stable. The primary mechanism for the system to decay to the true ground state of a Bose-Einstein condensate of molecules is by inelastic three-body collisions. In these collisions three particles interact to form a diatomic molecule and a free atom. The molecular binding energy is then released in the form of kinetic energy of the molecule and atom, which results in a loss of atoms from the shallow traps used in cold atomic gas experiments. Here the dependence of the decay rate on the scattering length is investigated using our knowledge of the contact and of how the bound-state energy is shifted away from the original position of the resonance due to many-body effects.
The particle loss is written as
\[ \frac{dn}{dt} = -Ln^3, \]
where \( L \) is the three-body loss rate [17, 41]. The dependence on the scattering length of the loss rate is found by application of Fermi’s golden rule
\[ L \propto |\langle f | V | i \rangle|^2 q_f. \]
Here \(|f\rangle\) and \(|i\rangle\) indicate the final and initial state, respectively, and \(q_f\) is the wavevector of the final state. The final state is the Feshbach bound state [3] and is given by
\[ \langle r | f \rangle = \frac{1}{2\pi a_b} \frac{e^{-r/\lambda} \sqrt{\lambda}}{r}, \]
where we defined the effective scattering length \(a_b\) that without many-body corrections is just equal to \(a(B)\). The wavevector of the final state is given by \(q_f \propto a_b^{-1}\).

The two-body scattering states in the open channel [3] are given by
\[ \lim_{r \to 0} \langle r_1 r_2 r_3 | i \rangle \propto a^3. \]
When no many-body corrections are present, we therefore expect
\[ L^{3B}(a) \propto \left( \frac{1}{\sqrt{a_b}} \right)^2 \frac{l}{a_b} \propto \frac{a^6}{a_b^4} \propto a^4. \]

From Efimov physics it is known that for a shallow bound state
\[ L^{3B}(a) = F(a) \frac{\hbar}{2m} a^4, \] (28)
where \(F(a)\) is a logarithmically periodic function of the scattering length and its maximum value is \(F_{\text{max}} \simeq 67.12\) [41–43]. From now on we neglect the Efimov physics and concentrate on the maximum value \(L^{3B}(a) = F_{\text{max}} \hbar a^3 / 2m\).

When the scattering length becomes large, many-body effects become important. The scattering state is then renormalized by the wavefunction renormalization factor \(\sqrt{Z(a)}\), which leads to the renormalized initial state
\[ \langle r_1 r_2 r_3 | i \rangle \propto (\sqrt{Z} a)^3 = \left( \frac{C^{3/2}}{(4\pi n)^{3/2}} \right), \]
where it was used that the wavefunction renormalization factor can be related to the contact by \(C = Z(4\pi n a_n)^2\) [17]. The effective scattering length \(a_b\) is given in terms of the bound-state energy \(E_b(a) = -\hbar^2 / ma_b^2\). The many-body loss rate can then be expressed in terms of the contact and the bound-state energy, namely
\[ L^{MB}(a) = -\frac{F_{\text{max}}}{2m} \left( \frac{C^{3/2}}{\sqrt{a_b}} \right)^2 \frac{1}{a_b} \]
\[ = -F_{\text{max}} C^{3/2} E_b(a) / (4\pi n)^{3/2} \]
In dimensionless form the many-body recombination rate is
\[ L^{MB} / (\hbar \epsilon) = -F_{\text{max}} \frac{\pi^2}{2} \left( \frac{3}{4} \right)^3 \left( \frac{C}{k_F^2} \right)^{3/2} E_b / \epsilon, \]
Here the last line is found using Eq. (25) for the contact. The many-body recombination rate as a function of scattering length is shown in Fig. 7.

At unitarity, where the bound-state energy is \(E_b \simeq -2.39\hbar \pi a \simeq -1.80\hbar \epsilon_F\), see Fig. 6, this gives for the universal recombination rate
\[ L^{MB} / (\hbar \epsilon) \simeq \frac{\pi^2}{2} \left( \frac{2.39 F_{\text{max}}}{2} \right) \frac{3/2}{3/3} \left( 1 + \lambda \right)^{3/2} \simeq 0.61. \]

The dependence of \(L^{MB}\) indicates that the many-body loss rate saturates at unitarity to a finite value. A similar saturation of the loss rate was seen experimentally in non-degenerate Bose gases at unitarity in Ref. [11], where the saturation is determined by the temperature. When the temperature becomes small the many-body loss rate is eventually set only by the density and this crossover is determined by a universal function of \(k_B T / \epsilon_F\) [12].

**III. DISCUSSION AND CONCLUSIONS**

Due to the fact that we have for illustrative purposes made the simplifying assumption of having only two running coupling constants, all quantities in this article have been determined analytically as a function of scattering length, which allows us to compare to the known weak-coupling results, of which some are shown in section IA. Furthermore, we have taken the full-renormalized value of the effective interaction inside the Bogoliubov dispersions of the renormalization-group flow equations, in Eq. (23). Therefore, it would be interesting to see the effect of a full numerical solution of the coupled renormalization-group-flow equations, which is also of interest for a study of the stability of the present results. The latter is also true for the study of the effects of various other coupling constants. Two important effects immediately come to mind. Due to the presence of a Feshbach bound state, the energy dependence of the effective interaction may play an important role. In addition, an important feature of the Bose gases near a Feshbach resonance is Efimov physics, which can also be studied by renormalization-group methods [44, 45]. By including the running of the appropriate three-body coupling constants in the renormalization-group equation it may be
possible to investigate how much of the Efimov physics survives in a many-body setting when also medium effects are playing an important role. Another useful direction is to obtain the renormalized thermodynamic potential of the theory. This will allow for the determination of all quantities using thermodynamic relations.

In summary, we have constructed a general self-consistent approach to describe strongly interacting Bose gases as a function of scattering length, which is free of infrared divergencies, can be improved systematically by renormalization-group methods or other non-pertubative methods, and reduces to the Bogoliubov theory for small scattering lengths. The generalization of the theory to non-zero temperature is straightforward, see appendix A. Furthermore, we expect that the approach can be applied to other systems with a broken continuous symmetry, where similar infrared divergencies occur as a consequence of the presence of Goldstone modes. We hope that our results stimulate further experimental developments toward unitarity-limited Bose gases in the near future.

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Appendix A: Ladder and bubble-sum contributions

In this section the ladder and bubble-sum contributions to the effective interaction are derived. The full energy-momentum and temperature-dependent ladder contribution is

\[
\Xi(k,\omega_n) = \frac{1}{\hbar^2 V} \sum_{k',\omega_n'} G'(k'_+,\omega_n')G'(k'_-,-\omega_n') \tag{A1}
\]

\[
= \frac{1}{V} \sum_{k'} \left\{ \frac{u_{k_+}^2 u_{k_-}^2}{i\hbar\omega_n - \hbar\omega_{k_+} - \hbar\omega_{k_-}} \left[ 1 + N(\hbar\omega_{k_+}) \right] \left[ 1 + N(\hbar\omega_{k_-}) \right] - N(\hbar\omega_{k_+})N(\hbar\omega_{k_-}) \right\} 
\]

\[
- \frac{v_{k_+}^2 v_{k_-}^2}{i\hbar\omega_n - \hbar\omega_{k_+} + \hbar\omega_{k_-}} \left[ 1 + N(\hbar\omega_{k_+}) \right] \left[ 1 + N(\hbar\omega_{k_-}) \right] - N(\hbar\omega_{k_+})N(\hbar\omega_{k_-}) \right\} 
\]

\[
+ \frac{v_{k_+}^2 v_{k_-}^2}{i\hbar\omega_n + \hbar\omega_{k_+} + \hbar\omega_{k_-}} \left[ 1 + N(\hbar\omega_{k_+}) \right] \left[ 1 + N(\hbar\omega_{k_-}) \right] - N(\hbar\omega_{k_+})N(\hbar\omega_{k_-}) \right\} ,
\]

where we defined \( k'_\pm = k/2 \pm k' \), \( n'_\pm = n/2 \pm n' \), \( (\hbar\omega_k)^2 = \epsilon_k(\epsilon_k + 2mc^2) \) and the coherence factors \( u_{k_\pm}^2 = v_{k_\pm}^2 = 1 = (\hbar\omega_k + \epsilon_k)/2\hbar\omega_k \). The bubble diagram is given by

\[
\Xi(k,\omega_n) = \frac{1}{V} \sum_{k'} \left\{ \frac{u_{k_+}^2 u_{k_-}^2}{i\hbar\omega_n - \hbar\omega_{k_+} - \hbar\omega_{k_-}} \right\} \tag{A2}
\]

\[
- \frac{v_{k_+}^2 v_{k_-}^2}{i\hbar\omega_n + \hbar\omega_{k_+} + \hbar\omega_{k_-}} \right\} .
\]

In cold atomic gases the momentum dependence of these quantities is of little importance. The momentum-independent ladder and bubble sum contributions can then be integrated analytically, however, due to the size of the expressions they are not shown here. Evaluating the expressions also at zero frequency we obtain

\[
\Xi(0,0) = \frac{3}{4\sqrt{2\pi}^2} \sqrt{\frac{n\omega g^2 k_F^3}{\epsilon_F}} ,
\]

\[
\Pi(0,0) = -\frac{1}{4\sqrt{2\pi}^2} \sqrt{\frac{n\omega g^2 k_F^3}{\epsilon_F}} ,
\]
For the non-interacting case, where \( h\omega_k = \epsilon_k \) and \( u_k^2 = u_k^2 + 1 = 1 \), we have that the bubble sum contribution vanishes and that at zero momentum the ladder contribution becomes

\[
\Xi(0, z) = \frac{1}{V} \sum_{k'} \left( \frac{1}{z - 2\epsilon_{k'}} + \frac{1}{2\epsilon_{k'}} \right) = \frac{1}{8\sqrt{2\pi}} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{-z}, \quad (A3)
\]

where \( z = ih\omega_n \) and an ultra-violet subtraction was needed as a consequence of the point interaction and in agreement with Eq. (20).

**Appendix B: Renormalization group**

Here we derive the renormalization-group flow equation. Starting from the action of a homogeneous Bose gas as shown in Eq. (2), we take the Fourier transform of the fields

\[
\phi(\mathbf{x}, \tau) = \frac{1}{\sqrt{h\beta V}} \sum_n \sum_{k < \Lambda} \phi_{k,n} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_n \tau)} \quad (B1)
\]

and split up the field in terms of low-momentum \( \phi_\langle \) and high-momentum modes \( \phi_\rangle \) as

\[
\phi(\mathbf{x}, \tau) = \phi_\langle(\mathbf{x}, \tau) + \phi_\rangle(\mathbf{x}, \tau), \quad (B2)
\]

where the low-momentum and high-momentum modes are defined as

\[
\phi_\langle(\mathbf{x}, \tau) = \frac{1}{\sqrt{h\beta V}} \sum_n \sum_{k < \Lambda} \phi_{k,n} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_n \tau)},
\]

\[
\phi_\rangle(\mathbf{x}, \tau) = \frac{1}{\sqrt{h\beta V}} \sum_n \sum_{\Lambda < k < \Lambda + \Delta \Lambda} \phi_{k,n} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_n \tau)}.
\]

Rewriting the partition function leads to

\[
Z = \int D\phi^* D\phi \exp \left\{ -\hbar^{-1} \left( S_0 [\phi^*, \phi] + S_{\text{int}} [\phi^* , \phi] \right) \right\}
\]

\[
= \int D\phi^*_\langle D\phi_\langle \exp \left\{ -\hbar^{-1} \left( S_0 [\phi^*_\langle, \phi_\langle] + S_{\text{int}} [\phi^*_\langle, \phi_\langle] \right) \right\}
\]

\[
\times \left[ \int D\phi^*_\rangle D\phi_\rangle \exp \left\{ -\hbar^{-1} S_0 [\phi^*_\rangle, \phi_\rangle] \right\}
\]

\[
\times \exp \left\{ -\hbar^{-1} \left( S_{\text{int}} [\phi^*, \phi] - S_{\text{int}} [\phi^*_\langle, \phi_\langle] \right) \right\} \right],
\]

where the gaussian part of the action is denoted by \( S_0 [\phi^*, \phi] \) and the non-gaussian part by \( S_{\text{int}} [\phi^*, \phi] \). Expanding up to second order in the high-momentum fields gives

\[
-\hbar^{-1} S_0 [\phi^*_\rangle, \phi_\rangle] - \hbar^{-1} \left( S_{\text{int}} [\phi^*, \phi] - S_{\text{int}} [\phi^*_\langle, \phi_\langle] \right)
\]

\[
= -\frac{1}{2} \text{Tr} \left[ \Phi^\dagger \left( -G_0^{-1} + \frac{1}{\hbar} \frac{\delta S_{\text{int}}}{\delta \Phi^\dagger} \right) [\phi^*_\rangle, \phi^*_\rangle] \right] \Phi_\rangle \right].
\]

Here the trace is over momentum, frequency and Nambu space \( \Phi(\mathbf{k}, \omega_n) = [\phi(\mathbf{k}, \omega_n), \phi^*(-\mathbf{k}, -\omega_n)]^T \). By integrating out the high-momentum fields, we obtain the effective action for the low-momentum fields

\[
-\hbar^{-1} S [\phi^*_\langle, \phi_\langle] = -\hbar^{-1} S_0 [\phi^*_\langle, \phi_\langle] - \hbar^{-1} S_{\text{int}} [\phi^*_\langle, \phi_\langle] + \frac{1}{2} \text{Tr} \ln \left[ -G_0^{-1} + \frac{1}{\hbar} \frac{\delta S_{\text{int}}}{\delta \Phi^\dagger} [\phi^*_\langle, \phi^*_\langle] \right] .
\]

Thus the change in the action after integrating out the high-momentum modes is given by, using that the trace is over an infinitesimal momentum interval \( \Lambda < k < \Lambda + \Delta \Lambda \),

\[
dS = \frac{\hbar}{2} \text{Tr} \delta \ln \left[ -G_0^{-1} + \frac{1}{\hbar} \frac{\delta S_{\text{int}}}{\delta \Phi^\dagger} [\phi^*_\langle, \phi^*_\langle] \right] d\Lambda. \quad (B3)
\]

**Appendix C: Ultra-violet subtractions**

To calculate the condensate density and total density an ultra-violet subtraction is necessary, see Eqs. (17) and (19). This subtraction is a consequence of the renormalization of the bare coupling to the two-body \( T \) matrix \( T^{2B}(-2mc^2) \) [3]. The phase-fluctuation and non-phase-fluctuation propagator in real space are written as

\[
\langle \theta(\mathbf{k}, \omega_n) \theta(\mathbf{k}', \omega_n) \rangle = -\frac{2mc^2}{(h\omega_n)^2 + (\epsilon_k + mc^2)^2}, \quad (C1)
\]

\[
\langle \phi'(\mathbf{k}, \omega_n) \phi'(\mathbf{k}', \omega_n) \rangle + \frac{mc^2}{(h\omega_n)^2 + (\epsilon_k + mc^2)^2},
\]

where \( mc^2 = n_0 g \) and the propagators are defined in Eqs. (14) and (15). This implies that the equal-time correlation function \( \langle \phi(\mathbf{x}, 0) \phi(\mathbf{x}, 0) \rangle \) with ultra-violet subtraction at zero temperature is given by

\[
\frac{1}{V} \left( \frac{1}{mc^2} \right) \sum_k \left( \frac{1}{2h\omega_k} - \frac{1}{2(\epsilon_k + mc^2)} \right) \cos(\mathbf{k} \cdot \mathbf{x}).
\]

In the long-range limit \( (|\mathbf{x}| \to \infty) \) we have that this expression vanishes, which is used to define the condensate density in Eq. (17). Whereas the equal-time correlation function of the non-phase fluctuations \( \langle \phi'(\mathbf{x}, 0) \phi'(\mathbf{x}, 0) \rangle \) with the ultra-violet subtraction at zero-temperature is

\[
\frac{1}{V} \sum_k \left( \frac{\epsilon_k - h\omega_k}{2h\omega_k} + \frac{mc^2}{2(\epsilon_k + mc^2)} \right) \cos(\mathbf{k} \cdot \mathbf{x}).
\]

The contribution to the total density due to non-phase-fluctuations follows from evaluating this expression at equal position \( n' = \langle \phi'(\mathbf{x}, 0) \phi'(\mathbf{x}, 0) \rangle \).
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