Microarcsecond light bending by Jupiter

M T Crosta1 and F Mignard

Observatoire de la Côte d’Azur, UMR CNRS 6202, Le Mont Gros, BP 4229, 06304 Nice cedex 4, France

E-mail: crosta@obs-nice.fr and mignard@obs-nice.fr

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Abstract
The detectors designed for Gaia, the next ESA space astrometry mission to be launched in 2011, will allow us to repeatedly observe stars very close to Jupiter’s limb. This will open a unique opportunity to test general relativity by performing many Eddington-like experiments through the comparison between the pattern of a starfield observed with or without Jupiter. We have derived the main formulae relevant for the monopole and quadrupole light deflection by an oblate planet and have developed a simulator to investigate the processing of the Gaia astrometric observations in the vicinity of the planet. The results show that such an experiment carried out with the Gaia data will provide a fully independent determination of \( \gamma \) by means of differential astrometric measurements and, more importantly, for the first time will evidence the bending effect due to the quadrupole moment with a 3\( \sigma \) confidence level. Given the accuracy of the experiment for the monopole deflection, this will permit testing of alternative models of light bending by moving masses.

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1. Introduction

Several key questions of modern astrophysics regarding the formation and evolution of the Milky Way will be clarified with the next space astrometry mission Gaia, approved in 2000 as a cornerstone within the European Space Agency science program [1]. The mission is now funded and will enter the C/D phase in 2006 for a launch scheduled in late-2011 and nominal operations over the next 5 years. Thanks to its high precision (20 \( \mu \)arcsecond in angular measurements) and multi-epoch astrometry, Gaia will be able to detect the relative positional change of a star resulting from the tiny curvature of the light ray brought about by the gravitational pull of the Sun, and to a lesser extent by the giant planets.

1 Present address: Observatory of Turin, Strada Osservatorio 20, Pino Torinese (To), Italy.
As far as the light deflection produced by the solar mass is concerned, the science case has included from the outset an investigation of the value of the PPN parameter $\gamma$, considered as a global unknown in the astrometric model. This parameter, equal to one in general relativity (GR), indicates the amount of space-time curvature produced by a unit rest-mass and it is a fundamental part of the so-called parametrized post-Newtonian (PPN) formalism [2], originally developed by Eddington for the famous experiment during the solar eclipse in 1919. The PPN method aims to quantify the violations of the equivalence principle by parametrizing deviations which modify both local physical laws and large-scale gravitational phenomena, including the constancy of the constants. The PPN formalism is valid for a broad class of metric theories and includes GR as a reference case, i.e. as a current standard theory of gravitation. The slow motion and weak field limit allows one to use PPN metric expansion as a function of several parameters, varying from theory to theory, useful to discern gravitational experiments in the solar system. The PPN parameter $\gamma$ is the most important, since all other parameters to all relativistic orders within the alternative theories of gravity converge to their GR value in proportion to $|1 - \gamma|$ [3].

The most general formulation of these theories contains an arbitrary function of a scalar field coupled to the stress–energy tensor in order to merge quantum mechanics with gravity. Recent works on scalar–tensorial theories consistent with various cosmological scenarios suggest the search for discrepancies from unity of $\gamma$ at the levels of $10^{-5}$ to $10^{-7}$; more precisely, Damour and Nordtvedt [4] assume that there exists a cosmological attractor mechanism towards GR quantified from deviation of $\gamma$ from unity within the above level of accuracy depending on the total mass density of the Universe. Then, an experiment purposely designed could reveal the coexistence of the two mentioned fields throughout cosmological evolution up to the present where pure tensor gravity explains the gravitational interaction from small to large scale. In addition, some dynamical models predict scalar fields dominating the current energy density of the universe, which should contribute to the value of $\gamma$ at the level of $10^{-7}$ to $10^{-9}$, thus providing a universe evolution without the need for dark matter [3]. All these issues make precision tests of gravity in space a fascinating challenge for the next decades, as in the past Eddington’s observations in 1919 of star line-of-sight did, confirming the amount of $1.72^\prime\prime$ as a deflection angle predicted by GR. Nowadays, all present experimental tests are compatible with the predictions of GR. In particular, experiments conducted in the solar system have tested all the weak-field predictions of GR theory at better than the $10^{-3}$ level and down to $2 \times 10^{-5}$ level for $\gamma$ as seen from the additional Doppler shift in radio-wave beams connecting the Earth to the Cassini spacecraft when they passed near the Sun [5].

New space projects will go deeper in experimental gravitation. The NASA Gravity Probe B mission [6], whose first results are expected very soon, should directly measure $\gamma$ to better than the $10^{-5}$ level, although this is not the main objective of the experiment and remains subject to efficient instrumental calibrations. An early analysis of Gaia’s capabilities indicates that its measurements should provide a precision of $5 \times 10^{-7}$ for the parameter [7, 8], an improvement of more than two orders of magnitude on the current best estimate mentioned above. This is quite comparable to the expectations of the ASTROD I, LATOR and ASTROD projects [9, 10], which are designed to use optical devices between two spacecrafts and aim to determine $|1 - \gamma|$ at the level of $10^{-7}$, $10^{-8}$ and $10^{-9}$, respectively. For more information on space experiments planned to test relativistic gravity in the next 25 years, we refer the reader to the comparison reported in [11].

Besides the determination of $\gamma$ based on the global astrometry capabilities of Gaia, the mission will also allow us to carry out dedicated small field experiments from the observations performed close to the surface of a giant planet like Jupiter or Saturn. The light bending
due to their gravitational field will be detectable in the visible spectrum, and we show in this paper that even the quadrupole field of Jupiter will be evidenced in such an experiment. It is therefore possible to carry out the same kind of eclipse experiment made by Dyson, Eddington and Davidson in 1920, but now by comparing stellar positions in the immediate vicinity of the planets. Not only will this experiment be a further confirmation of a GR prediction, but it will also help understand the very difficult question of light bending by a moving source and foster discussions about the propagation of gravitation, although for the latter Gaia seems a bit short in accuracy to contribute significantly to a settlement of the debate.

Using Jupiter to test GR, Kopeikin and Fomalont claimed they have determined the speed of gravity through the VLBI astrometric measurement of Shapiro time delay of light from the quasar J0842+1835, as its image passed within 3.7 arcmin of the planet (about 10 Jupiter radii, a far cry from what Gaia can achieve) [12]. According to the post-Minkowskian formalism assumed [13], the gravitational field due to a moving body propagates with finite speed; consequently, it influences a photon with some retardation. The speed of gravity enters as an extra tiny velocity-dependent correction to the Shapiro time delay formula of the order of 4.8 ps. At present there is no general consensus on the results, especially upon the method to extend GR to the case where the velocity of gravity differs from the velocity of light [14, 15]. In particular, given the observed limits on $\gamma$, the Fomalont–Kopeikin measurements are not accurate enough yet to determine the speed of gravity [14]. Some authors interpret the result within the Lorentz transformation properties of the weak gravitational field and consider it as a violation not only of the Lorentz invariance but also of the Galilean one [16, 17]. Paper [18] settles the significance of the Jupiter experiment as a simply standard aberration of light which propagates across an active medium with an effective index of refraction induced by the gravitational field of a lens in motion: the tiny corrections to the Shapiro time delay should be included into the fitting of microlensing events. However, even if there is still a controversy about the question of whether the results depend on the speed of gravity [19] or on the aberration of light [16, 18, 20], there is a general agreement that this measurement is a remarkable precise VLBI detection of a post-Newtonian effect due to a planet.

The possibility of applying methods of relativistic astrometry to test GR has prompted several groups to model highly accurate angular observations in the relativistic framework. New formalisms have been proposed ([13, 21–25] and reference therein) to tackle the relativistic problem of the light path reconstruction, which, at least at the $\mu$as accuracy, implies taking into account all the contributions not only due to the bulk mass of the solar system bodies, but also to their quadrupole moment and to the time-dependent gravitational field generated by their motions (see table 1).

In particular, two papers [26] and [27] have gone through the post-Newtonian treatment of the light propagation in the case of an isolated axisymmetric body, in order to take into account the influence of the multipole moments. The former is based on a mathematical solution of geodesic equation whereby mathematical expressions for the light deflection in any order of multipole perturbations are obtained. In the second paper, this contribution stems from the prior computation of the time transfer function between two points located at a finite distance. However, neither approach has provided hints on how to apply the intricate formulae to a practical case like the one faced in the Gaia modelling.

Having in mind the capabilities of Gaia to observe stellar sources very close to Jupiter’s limb (at a fraction of the planet radius), we have evaluated the light deflection produced by an oblate planet on grazing photons coming from distant stars. This study is a part of a wider project called GAia Relativistic EXperiment (GAREX), which aims to investigate all

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2 VLBI is the acronym for very long baseline interferometry.
Table 1. Light deflection amounts at 1 μarcsec due to the planets for a photon crossing the solar system: \( pN \) is for post-Newtonian order and \( Q \) for the quadrupolar moment. The values are computed for grazing rays in the Gaia observing geometry; figures in the last column give the angular distances between the perturbing body and the star at which the effect is still 1 μarcsecond, with Gaia at the Sun–Earth L2. Where two values are reported they refer, respectively, to \( pN \) and quadrupole effect.

| Planet   | \( \delta\Phi_{pN} \)  | \( \delta\Phi_Q \)  | \( \delta\Phi_{max} \)  |
|----------|-------------------------|----------------------|--------------------------|
| Sun      | 1.75 ± 1 μas (180°)    |                      |                          |
| Mercury  | 83 – 9                 |                      |                          |
| Venus    | 493 – 4.5°             |                      |                          |
| Earth    | 574 0.6 178°           |                      |                          |
| Moon     | 26 – 9°                |                      |                          |
| Mars     | 116 0.2 25°            |                      |                          |
| Jupiter  | 16280 239 90°/3°       |                      |                          |
| Saturn   | 5772 94 18°/51°        |                      |                          |
| Uranus   | 2081 25 72°/6°"       |                      |                          |
| Neptune | 2535 9 51°/3"          |                      |                          |
| Pluto    | 7 – 8°"                |                      |                          |

possibilities to test GR with Gaia measurements. In the solar system, this mission will mainly carry out the following:

- light deflection experiments, divided into (i) global astrometry, in particular highly accurate determinations of \( \gamma \) by observing the change in star position at different angular distances to the Sun; (ii) small field experiments, with the examination of the light propagation by means of differential measurements of stellar positions near the planets;

- perihelion precession effect, related to the determinations of PPN parameter \( \beta \) from the orbit fitting of several thousands of minor bodies [28].

In this paper we concentrate on the case of light bending by Jupiter at the microarcsecond level and its detection by the small field astrometric observations with Gaia. In [29] we have already explored the favourable circumstances for Gaia in 5 years of continuous observations, from approximately 2012 to 2018, to detect the quadrupole light bending effect predicted by GR, but never observed. For Jupiter the magnitude of the monopole deflection for a grazing ray is \( \sim 16 \) milliarcsecond, to which a component from the quadrupole moment is superimposed with an amplitude of \( \sim 240 \) microarcsecond, as shown in table 1. This secondary deflection has a very specific pattern as a function of (i) the position of the star with respect to the oblate deflector and (ii) the orientation of its spin axis. In section 2 we derive the relevant formulae to express this light bending effect using the PPN formalism at large and small angles from the planet. In section 3 we describe all the steps needed to simulate the small field experiment when observing the stellar background in the vicinity of Jupiter. Finally, in section 4, we discuss the data processing of a set of Monte Carlo runs and the results in the determination of the PPN parameter \( \gamma \) and the quadrupole effect from Jupiter.

2. The light deflection produced by an axisymmetric planet

Most stellar sources, planets and observers have a small velocity compared to the velocity of light and a weak inter-body gravitational field exists inside bound systems. Moreover,

\[ \text{PPN parameter } \beta \text{ measures the nonlinearity in the superposition of gravitational fields.} \]
if we consider isolated distribution of matter, the geometry can be assumed Minkowskian, asymptotically flat away \[30\]. Let us consider, then, a set of PPN coordinates \(\{x^\alpha\}\) (\(\alpha = 0, 1, 2, 3\)) and a solar system locally perturbed by isolated stationary axisymmetric masses. To the post-Newtonian (pN) accuracy, the spatial part of the geodesic equation of a light ray coming from a distant star can be easily transformed into \[2\]:

\[
\frac{d^2 x^i}{d\xi^0} = U_{,i} \left( 1 + \gamma \delta_{jk} \frac{dx^j}{d\xi^0} \frac{dx^k}{d\xi^0} \right) - 2(1 + \gamma) \frac{dx^i}{d\xi^0} \left( \delta_{jk} \frac{dx^j}{d\xi^0} U_{,k} \right),
\]

(1)

where \(U = M/\sqrt{(\delta_{ij} x^i x^j)}\) (\(i, j, k = 1, 2, 3\)) is the Newtonian potential (in geometrized units) and \(U_{,i}\) is its partial derivative with respect to the spatial coordinates. A solution of the above equation can be obtained by tracing a Newtonian zero order straight-line \(x^i_N\) plus all the relativistic deviations \(x^i_D\), i.e.

\[
x^i = x^i_N + x^i_D.
\]

(2)

Then, by decomposing \(x^i_D\) into two components, parallel \((x^i_D)^\parallel\) and perpendicular \((x^i_D)^\perp\) to \(x^i_N\), equation (1) becomes

\[
\frac{d^2 x^i_D}{d\xi^0} = (1 + \gamma)(U_{,i} - t^i (\delta_{jk} t^j U_{,k})) = (1 + \gamma) \nabla^\perp U,
\]

(3)

where \(t = dx^i/d\xi\) is the unit tangent vector on the unperturbed light path in the direction of the observer. Here the symbol \(\nabla^\perp\) indicates the derivative perpendicular to the light ray, assumed to pass outside the matter distribution. Placing the origin of coordinates \(x^i_0\) at the centre of an axisymmetric planet, i.e. \(z\) considered as its axis of maximum moment of inertia, located between the star and the observer, the positional vector of the photon with respect to the principal axes centred at the planet can be expressed as

\[
r^i = x^i - x^i_0 = t^i \ell - n^i b,
\]

(4)

where \(\ell\) represents the length of the photon path and \(n\) the radial direction perpendicular to the unperturbed ray pointing towards the centre of gravity, along the impact parameter \(b\).

The light deflection is given by the deviation of the tangent vector \(t\) from the straight line along the photon’s path as

\[
\Delta \Phi \equiv \int \frac{dt}{d\ell} d\ell
\]

(5)

which leads with (3) to

\[
\Delta \Phi = (1 + \gamma) \int \nabla^\perp U \, dl.
\]

(6)

By taking the gravitational potential up to the quadrupole term one obtains

\[
\nabla^\perp U = \left[ -\frac{b^2}{r^2} \left( -\frac{M}{r^2} + \frac{3M}{r^4} J_2 R^2 \cos^2 \theta - \frac{1}{2} \right) + \left( -\frac{3M}{r^4} J_2 R^2 \cos \theta \right) (z \cdot n) \right] n
\]

(7)

\[+ \left( -\frac{3M}{r^4} J_2 R^2 \cos \theta \right) (z \cdot m)m,
\]

where \(r\) and \(\theta\) are respectively the radial distance and the co-latitude (measured from the planet’s north pole) of a field point on the light trajectory. \(J_2\) is the dimensionless coefficient of the second zonal harmonic, \(R\) is the radius and \(M\) is the mass of the planet, and, finally, \(m\) represents the orthoradial component (see figures 1 and 2).

The length \(\ell\) of the photon path can be scaled by the impact parameter into a dimensionless parameter as follows:

\[
d\ell = b \, d\lambda,
\]

(8)
Figure 1. The geometry of light deflection due to a planet (P): the spin axis of the planet z is out of the plane; t represents the unit tangent vector from a distant star (S) to the observer (O) on the unperturbed light trajectory; u is the unit direction from O to P along their distance a; finally, χ is the angle $SO\ Pi$, and b is the impact parameter.

Figure 2. Light deflection by a planet: tangent plane on the sky. The position of the star is displaced both in the radial ($-n$) and orthoradial m directions. The spin axis of the planet z (not shown here) does not lie in this plane in general.

so the radial distance becomes

$$r = b(1 + \lambda^2)_{1/2}.$$ (9)

Each integral entering (6) must be computed with $\lambda$ running positively in the same direction as the photon from $\lambda = -\infty$ to $\lambda = 1/\tan \chi$, with $\chi$ standing for the angular separation between the directions star/observer and observer/planet (figure 1). At the closest approach on the unperturbed ray one has $\lambda = 0$. The explicit expressions of the integrals are given in appendix A. After some algebra, the light deflection vector is split into two components, the first one along n and the second one along m, both including the monopole and the quadrupole contribution of the planet in the function of the angular separation $\chi$:

$$\Delta \Phi = \Delta \Phi_1 n + \Delta \Phi_2 m,$$ (10)

where, precisely,

$$\Delta \Phi_1 = (1 + \gamma) \frac{2M}{b} \left\{ (1 + \cos \chi) + J_2 \frac{R^2}{b^2} \left[ \left( 1 + \cos \chi + \frac{1}{2} \cos \chi \sin^2 \chi \right) 
- 2 \left( 1 + \cos \chi + \frac{1}{2} \cos \chi \sin^2 \chi + \frac{3}{4} \cos \chi \sin^4 \chi \right) (n \cdot z)^2 + (\sin^3 \chi - 3 \sin^5 \chi) \right] 
\times (n \cdot z)(t \cdot z) - \left( 1 + \cos \chi + \frac{1}{2} \cos \chi \sin^2 \chi - \frac{3}{2} \cos \chi \sin^4 \chi \right) (t \cdot z)^2 \right\}.$$ (11)
and
\[
\Delta \Phi_2 = \frac{(1 + \gamma)M J_2 R^2}{b^3} \left[ 2 \left( 1 + \cos \chi + \frac{1}{2} \cos \chi \sin^2 \chi \right) (\mathbf{n} \cdot \mathbf{z})(\mathbf{m} \cdot \mathbf{z}) + \sin^3 \chi (\mathbf{m} \cdot \mathbf{z})(\mathbf{t} \cdot \mathbf{z}) \right].
\]

The first term in the radial component (that along \( \mathbf{n} \)) is the classical monopole deflection, widely used in astronomy. All the other terms factored by \( J_2 \) come from the quadrupole of the planet. One can specialize these expressions to near-grazing rays as, unlike with the solar deflection, the effect is too small to be observed at a large angle from the planet. Hence when \( \chi \ll 1 \) this leads to the more convenient and accurate enough formulae,
\[
\Delta \Phi_1 = \frac{2(1 + \gamma)M}{b} \left[ 1 + J_2 \frac{R^2}{b^2} (1 - 2(\mathbf{n} \cdot \mathbf{z})^2 - (\mathbf{t} \cdot \mathbf{z})^2) \right]
\]
\[
\Delta \Phi_2 = \frac{4(1 + \gamma)M J_2 R^2}{b^3} (\mathbf{m} \cdot \mathbf{z})(\mathbf{n} \cdot \mathbf{z}),
\]

which are similar to a derivation obtained in [31] in the case of the Sun.

The deflection vector depends on the orientation of the spin axis of the planet and, moreover, on the direction of the star with respect to the planet as seen in the terms proportional to \( (\mathbf{n} \cdot \mathbf{z}) \) and \( (\mathbf{t} \cdot \mathbf{z}) \). In general the quadrupole deflection vector depends both on the impact parameter and on the direction of the light source with respect to the spin axis projected on the plane of the sky. However, for a given impact parameter, its modulus (10) depends only on the star’s direction with respect to the planet’s reference frame. This property is established in appendix B and has been used as an additional check on the computer implementation of the simulation (see section 3.2). One should note that the above formulation has left aside and neglected refinements in the modelling like the motion of Jupiter, the retarded effect on the light propagation and multipolar terms of higher orders in the potential expansion. We recall that the theoretical treatment of the light deflection adopted to obtain (13)–(14) is sufficient to our purpose, namely investigating in the simplest static case and considering only the monopole and quadrupole terms, if the Gaia measurements are sensitive or not to the quadrupole light bending effect.

Considering \( \gamma = 1 \) is equivalent to saying that equations (13)–(14) constitute the reference modelling in the standard GR. Therefore, in the same way as one introduces \( \gamma \) in the monopole deflection to characterize departures from GR, we multiply \( J_2 \) by a dimensionless parameter \( \epsilon \) whose value is 0 if no effect is seen (say this may correspond to a lack of coupling between gravity and electromagnetism seen in the multipole moments) and 1 for the standard GR. Hereafter we call this parameter the quadrupole efficiency factor (QEF) and the goal is to see whether it is significantly determined from the Gaia observations. Although \( \epsilon \) is not a PPN parameter, it represents the first level of a post-Newtonian test whose goal is to determine whether the effect is detected, i.e. if one can say that \( \epsilon \) is equal to unity, with a certain uncertainty. Thus, for the processing of the experiment of light deflection near the planet, we adopt the following formulation,
\[
\Delta \Phi_1 = \frac{2(1 + \gamma)M}{b} \left[ 1 + J_2 \frac{R^2}{b^2} (1 - 2(\mathbf{n} \cdot \mathbf{z})^2 - (\mathbf{t} \cdot \mathbf{z})^2) \right]
\]
\[
\Delta \Phi_2 = \frac{4M(1 + \gamma)\epsilon J_2 R^2}{b^3} (\mathbf{m} \cdot \mathbf{z})(\mathbf{n} \cdot \mathbf{z}),
\]

including the two unknowns \( \gamma \) and \( \epsilon \), which are equal to 1 in the simulation.
3. Small field experiments with Jupiter

The one GAREX experiment we consider in this paper is the light deflection in the vicinity of Jupiter. The goal is to measure in a fully independent way the two parameters $\gamma$ and $\epsilon$ expressed in light deflection formulae (15) and (16), together with their associated errors, knowing the astrometric accuracy that will be achieved by Gaia. As $J_2$ is well known from space probe tracking with a precision better than $10^{-5}$, we will here consider it as a constant (as we do for the mass or $G$), and infer instead an estimate of the QEF factor $\epsilon$. The goal is not to measure the flattening but only its effect on the light path.

3.1. Ephemerides

First, we have investigated what are the favourable circumstances to perform experiments using Jupiter during the mission lifetime [29]. This depends on the number of times Jupiter will cross one of the astrometric fields during the mission and on the stellar density in its immediate surroundings during these observations. At present, the details of the former can only be known statistically, as the precise initial conditions of the Gaia scanning law remain unknown. On the other hand the sky background can be assessed with more certainty as it depends only on the motion of Jupiter between 2012 and 2018, and not very accurately on when the observations take place, provided they are reasonably scattered during the observing period. The ephemerides of Jupiter have been computed in galactic coordinates between these two epochs for an observer orbiting the Sun–Earth L2 in the same way as Gaia will do. The observability conditions are shown in figure 3 with the visibility periods when the angular distance to the Sun makes the observations feasible with Gaia (no observations can be made around conjunctions and oppositions). Galactic coordinates are used just because the primary dependence of the stellar density is on the galactic latitude.

We have chosen the date 2012 for the start of the simulations, since Gaia will take at least 6 months to be operational at the Lagrangian point L2 of the Earth–Moon–Sun system. Jupiter will cross the galactic plane in mid-2013 and, consequently, we expect favourable observations of the planet in front of a dense stellar background as shown in figure 4. The plot
Figure 4. Mean number of stars per square degree down to the magnitude limit $V = 20$ in the field of Jupiter during the Gaia operations. The dashed lines represent the period when Jupiter crosses the galactic plane and the central one the epoch when the maximum number of stars is observed.

gives the average number of stars per square degree around Jupiter from 2012 to 2018 as seen from Gaia.

3.2. Simulation of the experiment

The principle of the simulated experiment is to generate a series of pairs of observations over the mission length of 5 years, so that in each pair Jupiter is, or is not, at the centre of the stellar field. In both cases astrometric observations similar to that expected from Gaia are computer generated, with the proper noise function of the star brightness. The number of stars simulated in each field is determined by the galactic latitude of Jupiter at the observation time and a simple latitude dependent model of the stellar density. Then the two star patterns are compared and the differences in along-scan positions (the only direction of accurate measurements with Gaia) are fitted to the model with the two unknowns $\gamma$ and $\epsilon$. This is an idealized version of the Gaia procedure as the two fields (with or without Jupiter) will not be observed exactly at the same time, leading to correction for the relative orientations of the fields and for the proper motion of the stars. But these are technical details that will be carefully handled during the data processing and have no impact on the feasibility study demonstrated in this paper. The numerical parameters used in the simulation are listed in table 2.

In order to use equations (15)–(16) in the simulation, we have fixed the origin of the coordinate system at Jupiter’s centre and assumed that it coincides with the centre of the astrometric focal plane. Given the galactic coordinates $(l, b)$ of Jupiter we extracted the relevant stellar density in the magnitude range $V = 12.0–20.0$ from a realistic galactic model based on actual star counts fitted to convenient mathematical expressions as a function of the galactic latitude and longitude. The magnitude range is determined by the rejection of saturated stars (brighter than $V = 12.0$) on the bright side and by keeping stars observable with Gaia with a reasonable astrometric accuracy on the faint side. One must keep in mind that the number of close approaches between Jupiter and stars brighter than $V \simeq 12$ will be a rare event, albeit very important in this context, but not liable to statistical treatment. It must
Table 2. Main parameters used for the simulation.

| Parameter                        | Numerical value |
|----------------------------------|-----------------|
| Number of observations          | 90              |
| Area of the field of view       | 0.6 deg × 0.6 deg |
| Magnitude range                  | 12.0 to 20.0    |
| \( R_J \)                        | \( 7.13 \times 10^7 \) m |
| \( M = GM_J/c^2 \)              | 1.41 m          |
| Jupiter \( J_2 \)               | 0.014736        |
| \( \gamma \)                    | 1               |
| \( \epsilon \)                  | 1               |
| \( \sigma_{\text{pos along the scan at } V = 15} \) | 100 \( \mu \) as |
| \( \sigma_{\text{pos across the scan at } V = 15} \) | 300 \( \mu \) as |

be handled on a case-by-case basis with a real sky and this will be done later when the final parameters of the scanning law are known. After having computed the number of stars per interval of 0.5 magnitude inside the field of view at each observational epoch, we generated the observed stars in the field of view. For the fields with Jupiter we have computed the proper stellar directions by taking into account the full deflection due to the monopole and the quadrupole. Then for any pair and any epoch we have added a Gaussian noise in both coordinates. The standard deviation of the noise is determined by the magnitude of the star and the coordinates (along- or across-scan) with [1]:

\[
\sigma_{\text{als}} = a \times 10^{0.2(V - 15)}
\]

for stars with magnitude \( V > 12.5 \). The coefficient \( a \) is the scaling factor giving the single observation astrometric accuracy for a star of 15 magnitude. For stars brighter than 12.5, \( \sigma_{\text{als}} \) is constant and chosen to be 30 \( \mu \) as for the measurements along the scan and 100 \( \mu \) as across.

We have used the property expressed in (B.3) to cross check the contribution of the quadrupole term in the simulation. Other checks were also performed in different reference frames that helped improve the reliability of the simulator. Figures 5 and 6 illustrate several important features of the simulated data.

The plots in figure 5 show the absolute values of the monopole and of the quadrupole deflections, obtained with the simulator for different epochs and impact parameters. Figure 6 is an example of the observed field corresponding to mid 2013 when Jupiter crosses the Galactic plane. Even if this date represents a favourable observation, the relatively limited number of stars close to Jupiter does not lead to an obvious signature of the stellar displacements produced, in particular, by the quadrupole effect. For this reason we have also plotted the cumulative effect in figure 7 by combining all the epochs. As expected one sees clearly the radial nature of the monopole deflection and the more complex pattern of the quadrupole effect.

4. Measurement of the deflection

4.1. Processing the observations for \( \gamma \) and \( J_2 \)

The observed quantities used to fit the relativistic effect in the light deflection are the along-scan positional differences of the stars referred to a common origin, namely the centre of gravity of Jupiter. More precisely, we have estimated the measurements \( (\Delta \Phi_{\text{als}}) \) along scan
between two transits: the first one with Jupiter ($\Phi_{+J}$) and the second one without Jupiter ($\Phi_{-J}$), i.e.,
\[ \Delta \Phi_{als} = \Phi_{+J} - \Phi_{-J}. \] (18)

As stated by (18), the method rests only on the comparison between small fields taken at two distinct epochs (each point on the sky will be mapped at least three times over six months) and it is largely independent of the attitude reconstruction of Gaia. This makes it possible, in estimating the final accuracy, to consider only random errors and not the systematic attitude errors shared by all the sources at the two epochs.
The set of \( n \) observations can be considered as a system of \( n \) equations where the unknowns are the parameters \( \gamma \) and \( \epsilon \). The condition matrix is calculated by evaluating the partial derivatives with respect to \( \gamma \) and \( \epsilon \) in (18). The large number of stellar measurements, about 160,000 for 90 Jupiter observations scattered over the 5 year mission, is solved for the two unknowns with a weighted least-squares procedure using singular value decomposition. The off-diagonal terms of the final covariance matrix are not significant, indicating a very weak correlation between the two fitted parameters. Finally anomalous observations are filtered out with an iterative Student ratio test on the residuals.

The estimate of the accuracy with which Gaia could determine \( \gamma \) and \( \epsilon \) from small field astrometry has been estimated by running this numerical simulation more than 100 times with random drawings stochastically independent in each experiment. The distribution of the 150 values of \( \gamma \) and \( \epsilon \) show that there is no bias in the determination as seen in figure 8. As for the precision, it is available as a formal error from the residuals to the fit combined with the normal matrix. We have in practice used an estimate based on the scatter as a more robust way to assess the accuracy than the formal error of the diagonal terms of the covariance matrix. However we have carefully checked that both approaches were in complete agreement, since in the real world we will only have one experiment and a single formal error to deal with. Figure 9, which shows the distribution of the normalized residuals to the formal errors, clearly indicates that results are unbiased and the formal errors are quite consistent with the accuracy of this experiment.

Mean and scatter have been evaluated by using the usual point estimates valid for \( n_{\text{run}} \gg 1 \):

\[
\langle \gamma \rangle = \frac{1}{n_{\text{run}}} \sum_{i=1}^{n_{\text{run}}} \gamma(i) \tag{19}
\]

\[
\langle \epsilon \rangle = \frac{1}{n_{\text{run}}} \sum_{i=1}^{n_{\text{run}}} \epsilon(i) \tag{20}
\]
Figure 8. The distribution of the values of $\gamma$ (left-hand side) and $\epsilon$ (right-hand side) over 150 Monte Carlo simulations. The bars are the formal standard deviations in each experiment.

Figure 9. The histograms of the values of $(\gamma - 1)/\sigma_\gamma$ (left-hand side) and $(\epsilon - 1)/\sigma_\epsilon$ (right-hand side) over 150 Monte Carlo simulations. The distributions are comparable to a reference Gaussian with zero mean and unit dispersion and, as expected, it is not affected by bias.

$$\sigma_\gamma = \sqrt{\frac{1}{n_{\text{run}}} \sum_{i=1}^{n_{\text{run}}} \gamma(i)^2 - \langle \gamma \rangle^2}$$  \hspace{1cm} (21)

$$\sigma_\epsilon = \sqrt{\frac{1}{n_{\text{run}}} \sum_{i=1}^{n_{\text{run}}} \epsilon(i)^2 - \langle \epsilon \rangle^2}.$$  \hspace{1cm} (22)
### Table 3. Results for the mean value and scatter of $\gamma$ and $\epsilon$ in Monte Carlo experiments of different lengths.

| Runs | $\langle \gamma \rangle$ | $\sigma_\gamma$ | $\langle \epsilon \rangle$ | $\sigma_\epsilon$ |
|------|----------------|--------------|----------------|--------------|
| 50   | 0.9910 | $1.0295 \times 10^{-3}$ | 1.012 | 0.335 |
| 100  | 1.0000 | $1.1473 \times 10^{-3}$ | 0.992 | 0.358 |
| 150  | 1.0000 | $1.1456 \times 10^{-3}$ | 0.986 | 0.361 |

#### 4.2. Results

The main results are listed in table 3 with the average and scatter on $\gamma$ and $\epsilon$ for three independent Monte Carlo runs.

It is found with all the assumptions described in the previous sections that $\gamma$ can be determined from the monopole light deflection by Jupiter with an uncertainty of $1.1 \times 10^{-3}$. This is three times better than the optical determination achieved by Hipparcos, the previous astrometric ESA mission, with stellar astrometry at large angles from the Sun [32] and 5 million observations of stars. The interest rests not only on the accuracy achieved (the global astrometry should reach four orders of magnitude better with solar light bending), but on the fact that (i) it can be done with a planet, (ii) it is a prerequisite to the detection of the quadrupole effect in the residuals.

The final result in $\epsilon$ confirms that the gravitational effect due to $J_2$ is detectable with Gaia (albeit marginally), with typically $\epsilon = 1 \pm 0.35$, that is to say a 3-$\sigma$ detection. The effect of $J_4$ is negligible since it is much smaller than the random noise on most stars. Using $J_4 \sim -0.0006 \approx 4J_2/100$ this results for a grazing ray in a deflection of 10 $\mu$as. In addition the effect of $J_4$ decreases as $(R_J/r)^5$ instead of $(R_J/r)^3$ and very few stars close to the planet limb would contribute to the signal.

#### 4.3. Best strategy for the actual experiment

The above results concern a statistical analysis over all the epochs of observations during 5 years, it is however useful to see whether some controlled parameters are more favourable than others to observe the light bending effect and if one can restrict the risk of systematic errors. First of all we have considered the evolution of the errors on $\gamma$ and $\epsilon$ with the limiting magnitude, for various impact parameters and for all the epochs. As we include fainter stars, they get more numerous but each with a smaller weight due to the degradation of the astrometric accuracy. It was not obvious what effect would overcome the other: could many faint stars contribute significantly or not? The risk being to increase significantly the chance of systematic deviation for almost no improvement in the solution.

The results are plotted in figure 10, where the precision is shown when only stars brighter than $V$ are used in the fit. Each curve corresponds to a limitation in the size of the field of view around Jupiter. It is clear (and not surprising) that using the full Gaia field provides the best results, in particular for $\gamma$, just because the monopole deflection decreases more slowly with increasing impact parameter than the quadrupole’s. However going further than $30R_J$ brings very marginal improvement and this can be used to determine what a small field experiment means in practice. Conversely below $10R_J$ the number of stars decreases very quickly and at $5R_J$ we usually have no star brighter than $V = 17.5$. As for the magnitude it seems that one could also reject stars (in the case of calibration problems for example) fainter than $V = 16$ without much loss, as the final precision on $\gamma$ and $\epsilon$ does not improve very much when they are included. That is a very nice feature when comes the time to decide on whether faint stars...
are useful or not. The graphs show also a degradation in precision when only the brightest stars are considered, probably because the number of stars becomes too small to be significant and formula (17) inserts a constant error in this case. Then another interesting experiment has been run to test what would be the result if one selected only the few epochs during which we have the maximum number of stars in the background field, that is to say around 2013 when Jupiter is close to the Galactic plane. To this purpose, figure 11 shows again the evolution of the errors with magnitude, when all observations at every epoch are included (full circles) or only during the crossing of the Galactic plane in 2013–2014 (open circles with only 20 epochs instead of 90 in the nominal experiment). Not surprisingly the errors are larger when considering only the best epochs, but not dramatically larger, showing that a strategy may be
Table 4. Precisions computed after 150 Monte Carlo runs selecting special fields in 2013 and increasing artificially the number of stars \((n^*)\) (\(V = 12\)).

| \(n^*\) | \(\sigma_{\gamma}\) | \(\sigma_{\epsilon}\) |
| --- | --- | --- |
| \(b = 7R_f\) | | |
| 1 | \(7.41 \times 10^{-3}\) | 14.54 |
| 5 | \(3.37 \times 10^{-3}\) | 2.73 |
| 13 | \(1.62 \times 10^{-3}\) | 0.69 |
| 20 | \(1.39 \times 10^{-3}\) | 0.34 |
| 27 | \(1.06 \times 10^{-3}\) | 0.32 |
| 34 | \(9.31 \times 10^{-4}\) | 0.27 |
| \(b = 15R_f\) | | |
| 7 | \(4.48 \times 10^{-3}\) | 9.09 |
| 23 | \(2.12 \times 10^{-3}\) | 1.51 |
| 39 | \(1.70 \times 10^{-3}\) | 0.94 |
| 63 | \(1.31 \times 10^{-3}\) | 0.62 |
| 79 | \(1.26 \times 10^{-3}\) | 0.51 |
| 119 | \(9.16 \times 10^{-4}\) | 0.30 |

Table 5. Precisions computed after 150 Monte Carlo runs selecting special fields in 2013 and increasing artificially the number of stars \((n^*)\) (\(V = 13\)).

| \(n^*\) | \(\sigma_{\gamma}\) | \(\sigma_{\epsilon}\) |
| --- | --- | --- |
| \(b = 7R_f\) | | |
| 9 | \(2.49 \times 10^{-3}\) | 1.51 |
| 29 | \(1.16 \times 10^{-3}\) | 0.37 |
| 49 | \(9.53 \times 10^{-4}\) | 0.26 |
| 66 | \(7.50 \times 10^{-4}\) | 0.20 |
| 91 | \(6.50 \times 10^{-4}\) | 0.15 |
| \(b = 15R_f\) | | |
| 44 | \(1.88 \times 10^{-3}\) | 1.28 |
| 67 | \(1.56 \times 10^{-3}\) | 0.62 |
| 81 | \(1.43 \times 10^{-3}\) | 0.58 |
| 97 | \(1.17 \times 10^{-3}\) | 0.41 |
| 127 | \(1.09 \times 10^{-3}\) | 0.35 |

applied to perform valuable and dedicated experiments during the first 2 years of the mission, to validate the concept and get already significant results.

Finally one should also look at the passage of Jupiter in the middle of a concentrated open cluster with stars brighter than \(v = 13\). There are many such clusters on the ecliptic in the direction of Sagittarius, that Jupiter will cross in 2019. This is after the nominal mission completion, but well within the possible extended mission permitted by the consumable. This kind of experiment would obviously add in the science case to support this extension of the operations. To this aim we have run the simulator with bright stars and a surface density comparable to that found in these open clusters. Results for the determination of \(\gamma\) and \(\epsilon\) are given in tables 4 and 5 with all the stars of 12 or 13 magnitude, respectively. Two realistic concentrations have been considered around Jupiter corresponding to fields of 3 and 6 arcmin around the planet. One sees that a single experiment can do almost as well as the 5 year mission, provided the number of bright stars in the cluster is large enough. Note that we have excluded \textit{a priori} all the stars with a magnitude range between 14 and 16, which statistically
contribute significantly to the precision. At this point this is just an indication showing that a
detail investigation on specific clusters is worth doing.

5. Conclusion

The objective of this paper has been to study a new way to determine the PPN parameter $\gamma$
by exploiting a method of differential positional measurements around Jupiter and to assess
the detection of the light bending from the quadrupole moment of Jupiter. Thanks to the high
astrometric accuracy to be achieved by the ESA astrometry mission Gaia and the repeated
observations over 5 years this will be feasible as the new design of Gaia’s optical instrument
allows us to process stellar observations very close to the surface of giant bodies. We
have shown that the deviation to GR with the monopole deflection can be assessed to $10^{-3}$
with Jupiter only, a result in the visible better than the optical accuracy already achieved
by Hipparcos with the stars and the solar light bending. With the same observations, the
quadrupole light deflection will be detectable for the first time with a 3-$\sigma$ confidence level.

Although we have designed an ideal and simplified experiment it includes realistic
observations of Jupiter feasible with Gaia, taking into account the astrometric accuracy with
the star’s magnitude deduced from the current error budget analysis. Whereas the Gaia concept
and its design rest entirely on the global astrometry we have given for the first time a realistic
figure on the true strength of Gaia to also carry out relativity testing with small field astrometry
and propose some strategies to carry out this experiment in the best conditions. When the
initial conditions of the Gaia scanning law are known, the same principles will be applied using
a real stellar distribution. The method will be also extended to the observations of Saturn,
at least for the monopole effect. Finally, refinement of the simulations and the experimental
strategies, by including the time-dependent gravitational field and related retarded effects,
could open new way for testing the accurate modelling of the deflection by a moving body (it
is stationary in our Jovian experiments) [33, 34].

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and collaboration.

Appendix A

This appendix gives the fundamental integrals used to obtain expressions (11) and (12) for the
deflection vector $\Delta \Phi$ after having scaled the photon length by the impact parameter:

$$\int \frac{d\lambda}{(1 + \lambda^2)^{3/2}} = \frac{\lambda}{(1 + \lambda^2)^{1/2}}$$

$$\int \frac{d\lambda}{(1 + \lambda^2)^{5/2}} = \frac{\lambda/3}{(1 + \lambda^2)^{3/2}} + \frac{2\lambda/3}{(1 + \lambda^2)^{1/2}}$$

$$\int \frac{d\lambda}{(1 + \lambda^2)^{7/2}} = \frac{\lambda/5}{(1 + \lambda^2)^{3/2}} + \frac{4\lambda/15}{(1 + \lambda^2)^{1/2}} + \frac{8\lambda/15}{(1 + \lambda^2)^{1/2}}$$

$$\int \frac{\lambda d\lambda}{(1 + \lambda^2)^{5/2}} = -\frac{1/3}{(1 + \lambda^2)^{3/2}}$$
\[
\int \frac{\lambda \, d\lambda}{(1 + \lambda^2)^{3/2}} = -\frac{1/5}{(1 + \lambda^2)^{3/2}}
\]
\[
\int \frac{\lambda^2 \, d\lambda}{(1 + \lambda^2)^{3/2}} = -\frac{\lambda/5}{(1 + \lambda^2)^{3/2}} + \frac{\lambda/15}{(1 + \lambda^2)^{3/2}} + \frac{2\lambda/15}{(1 + \lambda^2)^{3/2}}.
\]

Appendix B

Let us consider the angle \( \varphi, \vartheta \) of the Jupiter axis with respect to the orthonormal directions \( \mathbf{n} \) and \( \mathbf{t} \), respectively. From the left-hand side of equation (13) we get

\[
1 - 2(\mathbf{n} \cdot \mathbf{z})^2 - 2(\mathbf{t} \cdot \mathbf{z})^2 = -\sin^2 \vartheta \cdot \cos 2\varphi,
\]  

(B.1)

and from that of equation (14)

\[
2(\mathbf{n} \cdot \mathbf{z})(\mathbf{m} \cdot \mathbf{z}) = \sin^2 \vartheta \sin 2\varphi.
\]  

(B.2)

Then, the modulo of the quadrupole deflection term, let us say \( \Delta \Phi_{\text{quad}} \), in equation (10) will depend only on \( \sin^2 \vartheta \), namely on the angle between the \( \mathbf{z} \) axis of Jupiter and the direction from the star to the observer

\[
|\Delta \Phi_{\text{quad}}| = \frac{2(1 + \gamma)MJR^2}{b^3} \sin^2 \vartheta.
\]  

(B.3)

Clearly, when the \( \mathbf{z} \) lies on the \( \{\mathbf{n}, \mathbf{m}\} \) plane, i.e. \( \mathbf{z} \) perpendicular to the line of sight, the quadrupole term will give its maximum contribution.

For a star in direction \( \alpha \) with respect to the orthonormal axis \( \{\mathbf{m}, \mathbf{n}, \mathbf{t}\} \) the modulo has a fixed value depending only on the star’s direction of arrival with respect to the above frame centred on Jupiter, namely,

\[
|\Delta \Phi_{\text{quad}}| = |\Delta \Phi_{\text{quad}}| \cos 2\alpha
\]  

(B.4)

\[
|\Delta \Phi_{\text{quad}}| = |\Delta \Phi_{\text{quad}}| \sin 2\alpha.
\]  

(B.5)

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