MAXIMAL SUBGROUPS OF FINITE SOLUBLE GROUPS
IN GENERAL POSITION

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Abstract. For a finite group \( G \) we investigate the difference between the maximum size \( \text{MaxDim}(G) \) of an “independent” family of maximal subgroups of \( G \) and maximum size \( m(G) \) of an irredundant sequence of generators of \( G \).

We prove that \( \text{MaxDim}(G) = m(G) \) if the derived subgroup of \( G \) is nilpotent. However \( \text{MaxDim}(G) - m(G) \) can be arbitrarily large: for any odd prime \( p \), we construct a finite soluble group with Fitting length 2 satisfying \( m(G) = 3 \) and \( \text{MaxDim}(G) = p \).

1. Introduction

Let \( G \) be a finite group. A sequence \((g_1, \ldots, g_n)\) of elements of \( G \) is said to be irredundant if \( \langle g_j \mid j \neq i \rangle \) is properly contained in \( \langle g_1, \ldots, g_n \rangle \) for every \( i \in \{1, \ldots, n\} \). Let \( i(G) \) be the maximum size of any irredundant sequence in \( G \) and let \( m(G) \) be the maximum size of any irredundant generating sequence of \( G \) (i.e. an irredundant sequence \((g_1, \ldots, g_n)\) with the property that \( \langle g_1, \ldots, g_n \rangle = G \)). Clearly \( m(G) \leq i(G) = \max \{ m(H) \mid H \leq G \} \). The invariant \( m(G) \) has received some attention (see, e.g., [2], [9], [7], [1], [4], [5]) also because of its role in the efficiency of the product replacement algorithm [6]. In a recent paper Fernando [3] investigates a natural connection between irredundant generating sequences of \( G \) and certain configurations of maximal subgroups of \( G \). A family of subgroups \( H_i \leq G \), indexed by a set \( I \), is said to be in general position if for every \( i \in I \), the intersection \( \cap_{j \neq i} H_j \) properly contains \( \cap_{j \in I} H_j \). Define \( \text{MaxDim}(G) \) as the size of the largest family of maximal subgroups of \( G \) in general position. It can be easily seen that \( m(G) \leq \text{MaxDim}(G) \leq i(G) \) (see, e.g., [3] Proposition 2 and Proposition 3). However the difference \( \text{MaxDim}(G) - m(G) \) can be arbitrarily large: for example if \( G = \text{Alt}(5) \wr C_p \) is the wreath product of the alternating group of degree 5 with a cyclic group of prime order \( p \), then \( \text{MaxDim}(G) \geq 2p \) but \( m(G) \leq 5 \) [3] Proposition 12. On the other hand Fernando proves that \( \text{MaxDim}(G) = m(G) \) if \( G \) is a finite supersoluble group [3] Theorem 25], but gives also an example of a finite soluble group \( G \) with \( m(G) \neq \text{MaxDim}(G) \) [3] Proposition 16].

In this note we collect more information about the difference \( \text{MaxDim}(G) - m(G) \) when \( G \) is a finite soluble group. In this case \( m(G) \) coincides with the number of complemented factors in a chief series of \( G \) (see [3] Theorem 2]). Our first result is that the equality \( \text{MaxDim}(G) = m(G) \) holds for a class of finite soluble groups, properly containing the class of finite supersoluble groups (see, e.g., [8] 7.2.13)].

Theorem 1. If \( G \) is a finite group and the derived subgroup \( G' \) of \( G \) is nilpotent, then \( \text{MaxDim}(G) = m(G) \).
However, already in the class of finite soluble groups with Fitting length equal to 2, examples can be exhibited of groups $G$ for which the difference $\text{MaxDim}(G) - m(G)$ is arbitrarily large.

**Theorem 2.** For any odd prime $p$, there exists a finite group $G$ with Fitting length 2 such that $m(G) = 3$, $\text{MaxDim}(G) = p$ and $i(G) = 2p$.

Notice that if $G$ is a soluble group with $m(G) \neq \text{MaxDim}(G)$, then $m(G) \geq 3$. Indeed if $m(G) \leq 2$, then a chief series of $G$ contains at most two complemented factors and it can be easily seen that this implies that $G'$ is nilpotent.

2. **Groups whose derived subgroup is nilpotent**

**Definition 3.** A family of subgroups $H_i \leq G$, indexed by a set $I$, is said to be in general position if for every $i \in I$, the intersection $\cap_{j \neq i} H_j$ properly contains $\cap_{j \in I} H_j$ (equivalently, $H_i$ does not contain $\cap_{j \neq i} H_j$).

Note that the subgroups $\{H_i \mid i \in I\}$ are in general position if and only, whenever $I_1 \neq I_2$ are subsets of $S$, then $\cap_{i \in I_1} H_i \neq \cap_{i \in I_2} H_i$ (see, e.g., Definition 1 in [3]).

**Lemma 4.** Let $\mathbb{F}$ be a field of characteristic $p$. Let $V$ a finite dimension $\mathbb{F}$-vector space, let $H = \langle h \rangle$ where $h \in \mathbb{F}^* \text{ such that } \mathbb{F} = \mathbb{F}_p[h]$ and set $G = V \rtimes H$.

If $M_1, \ldots, M_r$ is a set of maximal subgroups of $G$ supplementing $V$, then

$$M_1 \cap \ldots \cap M_r = W \rtimes K$$

where $W$ is a $\mathbb{F}$-subspace of $V$ and $K$ is either trivial or a conjugate $H^v$ of $H$, for some $v \in V$.

**Proof.** By induction on $r$ we can assume that $T_1 = M_1 \cap \ldots \cap M_{r-1} = W_1 \rtimes K_1$, where $W_1$ is a subspace of $V$ and $K_1 = \{1\}$ or $K_1 = H^v$, $v \in V$. The maximal subgroup $M_r$ is a supplement of $V$, so we can write $M_r = W_2 \rtimes H^w$, where $W_2$ is a subspace of $V$ and $w \in V$. For shortness, set $T_2 = M_r$ and $T = T_1 \cap T_2$. Since $W_1$ and $W_2$ are normal Sylow $p$-subgroups of $T_1$ and $T_2$, respectively, their intersection $W = W_1 \cap W_2$ is a normal Sylow $p$-subgroup of $T$. In the case where $T$ is not a $p$-group, then $T = W \rtimes K$ where $K$ is a non-trivial $p'$-subgroup of $T$. Then $K$ is contained in some conjugates $H^v_1$ and $H^v_2$ of the $p'$-subgroups of $T_1$ and $T_2$, respectively. In particular, there exists $1 \neq y \in K$ such that $y = h_1^v = h_2^v$ for some $h_1, h_2 \in H$. It follows that $1 \neq h_1 = h_2 \in C_H(v_1 - v_2)$. From $C_H(v_1 - v_2) \neq \{1\}$, we deduce that $v_1 = v_2$. Thus we have $T_1 = W_1 \rtimes H^v_1$, $T_2 = W_2 \rtimes H^v_2$ and $T = W \rtimes H^v$. \hfill $\square$

**Corollary 5.** In the hypotheses of Lemma 4, if $M_1, \ldots, M_r$ are in general position, then

1. $r \leq \dim(V) + 1$;
2. if $r = \dim(V)+1$, then, for a suitable permutation of the indices, $\bigcap_{i=1}^{r-1} M_i = H^v$ for some $v \in V$, and $\bigcap_{i=1}^r M_i = \{1\}$.

**Proof.** Let $n = \dim V$. Since the subgroups $M_1, \ldots, M_r$ are in general position, the set of the intersections $T_j = \cap_{i=1}^j M_i$, for $j = 1, \ldots, r$, is a strictly decreasing chain of subgroups. By Lemma 4, $T_i = W_i \rtimes K_i$, where $W_i$ is a $\mathbb{F}$-subspace of $W_{i-1}$ and $K_i$ is either trivial or a conjugate of $H$. Note that $n-1 = \dim W_1 \geq \dim W_2 \geq \dim W_{i+1}$. Moreover, if $\dim W_i = \dim W_{i+1}$ for some index $i$, then $W_i = W_{i+1}$ and, since $T_i \neq T_{i+1}$, we have that
• $K_1, \ldots, K_i$ are non-trivial;
• $K_{i+1} = \cdots = K_r = \{1\}$.

In particular there exists at most one index $i$ such that $\dim W_i = \dim W_{i+1}$. As $\dim W_1 = n - 1$, it follows that we can have at most $n + 1$ subgroups $T_i$, hence $r \leq n + 1$.

In the case where $r = n + 1$, we actually have that $\dim W_i = \dim W_{i+1}$ for at least one, and precisely one, index $i$. This implies that $W_i = W_{i+1}$ and, setting $J = \{1, \ldots, n + 1\} \setminus \{i + 1\}$ and $T = \cap_{i \in J} M_i$, we get that $W_{n+1}$ coincides with the Sylow $p$-subgroup of $T$. Since $\dim W_{n+1} = 0$ and $T \neq 1$ we deduce that $T = H^v$, for some $v \in V$. Finally, $T \cap M_{i+1} = \{1\}$. □

A proof of the following lemma is implicitly contained in Section 1 of [3], but, for the sake of completeness, we sketch a direct proof here.

Lemma 6. Let $H$ be an abelian finite group. The size of a set of subgroups in general position is at most $m(H)$.

Proof. The proof is by induction on the order of $H$. Let $\Omega = \{A_1, \ldots, A_r\}$ be a set of subgroups of $H$ in general position. Without loss of generality we can assume that $\cap_{i=1}^{r} A_i = \{1\}$. If $m = m(H)$, then $H$ decomposes as a direct product of $m$ cyclic groups of prime-power order. Let $B$ be one of these factors, and let $X$ be the unique minimal normal subgroup of $B$. Since $\cap_{i=1}^{r} A_i = \{1\}$, there exists at least an integer $i$ such that $X$ is not contained in $A_i$. It follows that $A_i \cap B = \{1\}$, hence $A_i \cong A_i B / B \leq H / B$ and

$$m(A_i) \leq m(H / B) = m - 1.$$ 

Now, the set of subgroups of $A_i$

$$\Omega^* = \{A_j \cap A_i \mid j \neq i, \ 1 \leq j \leq r\}$$

is in general position, hence, by inductive hypothesis, $|\Omega^*| = r - 1 \leq m(A_i)$. Therefore $r \leq m$. □

Proof of Theorem 1. Since

$$m(G) = m(G / \Frat(G)) \quad \text{and} \quad \MaxDim(G) = \MaxDim(G / \Frat(G)),$$

without loss of generality we can assume that $\Frat(G) = 1$. In this case the Fitting subgroup $F$ of $G$ is a direct product of minimal normal subgroups of $G$, it is abelian and complemented. Let $H$ be a complement of $F$ in $G$; note that, being $G'$ nilpotent by assumption, $H$ is abelian. We can write $F$ as a product of $H$-irreducible modules

$$F = V_1^{n_1} \times \cdots \times V_r^{n_r}$$

where $V_1, \ldots, V_r$ are irreducible $H$-modules, pairwise not $H$-isomorphic.

By [4] Theorem 2] $m(G)$ coincides with the number of complemented factors in a chief series of $G$, hence

$$m(G) = \sum_{i=1}^{r} n_i + m(H).$$

Let $\mathcal{M}$ be a family of maximal subgroups of $G$ in general position. Let $M_{0,1}, \ldots, M_{0,v_0}$ the elements of $\mathcal{M}$ containing $F$. We can write

$$M_{0,i} = F \rtimes Y_i$$
where \( Y_i \) is a maximal subgroup of \( H \). Note that \( Y_1, \ldots, Y_{\nu_0} \) are maximal subgroups of \( H \) in general position, hence, by Lemma 3, \( \nu_0 \leq \text{MaxDim}(H) \leq m(H) \).

If \( M \) is a maximal subgroup supplementing \( F \), then \( M \) contains the subgroup \( U_i = \prod_{j \not= i} V_j^{n_j} \) for some index \( i \). In particular \( M = (U_i \times W_i) \times H^v \) for some \( v \in V_i^{n_i} \) and some hyperplane \( W_i \) of \( V_i^{n_i} \).

Assume, by contradiction, that for example \( v_i \in V_i^{n_i} \). Our next task is to prove that \( U_i \) generated by a primitive element. In particular we can apply Corollary 5 to the group \( V_i^{n_i} \rtimes H_i \). Let \( M_{i,1}, \ldots, M_{i,\nu_i} \) the maximal subgroups in \( M \) containing \( U_i \); say

\[
M_{i,l} = (U_i \times W_{i,l}) \times H^v_{i,l},
\]

where \( v_{i,l} \in V_i^{n_i} \). Note that the subgroups \( \overline{M}_{i,l} = W_{i,l} \times H^v_{i,l} \), for \( l \in \{1, \ldots, \nu_i\} \), are maximal subgroups of \( V_i^{n_i} \rtimes H_i \) in general position, hence, by Corollary 5

\[
\nu_i \leq n_i + 1.
\]

If \( \nu_i \leq n_i \) for every \( i \neq 0 \), then

\[
|\mathcal{M}| = \sum_{i=1}^{r} \nu_i + \nu_0 \leq \sum_{i=1}^{r} n_i + m(H) = m(G),
\]

and the result follows.

Otherwise let \( J \) be the set of the integers \( i \in \{1, \ldots, r\} \) such that \( \nu_i = n_i + 1 \). By Corollary 5 we can assume that, for some \( v_i \in V_i^{n_i} \),

\[
\bigcap_{l=1}^{n_i} M_{i,l} = U_i \rtimes H^{v_i},
\]

\[
\bigcap_{l=1}^{n_i+1} M_{i,l} = U_i \rtimes C_i.
\]

Recall that the \( M_{0,j} = F \rtimes Y_j \), for \( j = 1, \ldots, \nu_0 \), are the elements of \( \mathcal{M} \) containing \( F \). Our next task is to prove that

\[
\Omega = \{C_i \mid i \in J\} \cup \{Y_j \mid j = 1, \ldots, \nu_0\}
\]

is a set of subgroups of \( H \) in general position.

Assume, by contradiction, that for example \( C_1 \geq (\cap_{i \not= 1} C_i) \cap (\cap_{j=1}^{\nu_0} Y_j) \); then

\[
M_{1, n_1+1} \geq U_1 \rtimes C_1 \geq (\cap_{l=1}^{n_1} M_{1,l}) \cap (\cap_{i \not= 1} (\cap_{l=1}^{n_i+1} M_{i,l})) \cap (\cap_{j=1}^{\nu_0} M_{0,j})
\]

against the fact that \( \mathcal{M} \) is in general position. Similarly, if \( Y_1 \geq (\cap_{i \in J} C_i) \cap (\cap_{j \not= 1} Y_j) \), then

\[
M_{0,1} = F \rtimes Y_1 \geq (\cap_{i \in J} (\cap_{l=1}^{n_i+1} M_{i,l})) \cap (\cap_{j \not= 1} M_{0,j}),
\]

a contradiction.

Now we can apply Lemma 5 to get that \( |\Omega| \leq m(H) \). Therefore we conclude that

\[
|\mathcal{M}| = \sum_{i=1}^{r} \nu_i + \nu_0 \leq \sum_{i=1}^{r} n_i + |J| + \nu_0 = \sum_{i=1}^{r} n_i + |\Omega| \leq \sum_{i=1}^{r} n_i + m(H) = m(G),
\]

and the proof is complete.
3. Finite soluble groups with \( m(G) = 3 \) and \( \text{MaxDim}(G) \geq p \)

In this section we will assume that \( p \) and \( q \) are two primes and that \( p \) divides \( q - 1 \). Let \( \mathbb{F} \) be the field with \( q \) elements and let \( C = \langle c \rangle \) be the subgroup of order \( p \) of the multiplicative group of \( \mathbb{F} \). Let \( V = \mathbb{F}^p \) be a \( p \)-dimensional vector space over \( \mathbb{F} \) and let \( \sigma = (1, 2, \ldots, p) \in \text{Sym}(p) \). The wreath group \( H = C \wr \langle \sigma \rangle \) has an irreducible action on \( V \) defined as follows: if \( v = (f_1, \ldots, f_p) \in V \) and \( h = (c_1, \ldots, c_p) \sigma \in H \), then \( v^h = (f_{1\sigma^{-1}c_1\sigma^{-1}}, \ldots, f_{p\sigma^{-1}c_p\sigma^{-1}}) \). We will concentrate our attention on the semidirect product

\[
G_{q,p} = V \rtimes H.
\]

**Proposition 7.** \( m(G_{q,p}) = 3 \).

**Proof.** Since \( V \) is a complemented chief factor of \( G_{q,p} \), by [4] Theorem 2 we have \( m(G) = 1 + m(H) = 1 + m(H / \text{Frat}(H)) = 1 + m(C_p \times C_p) = 3 \). \( \square \)

**Proposition 8.** \( i(G_{q,p}) = 2p \).

**Proof.** Let \( B \cong C^p \) be the base subgroup of \( H \) and consider \( K = V \rtimes B \cong (\mathbb{F} \rtimes C)^p \). A composition series of \( K \) has length 2 and all its factors are indeed complemented chief factors, so \( m(K) = 2p \). Now by definition \( i(G_{q,p}) = \max \{ m(X) \mid X \leq G_{q,p} \} \geq m(K) = 2p \). On the other hand, \( m(G) = 3 \) and, if \( X < G_{q,p} \), then \( |X| \) divides \( pq \) and the composition length of \( X \) is at most 2, so \( m(X) \leq 2p \). Therefore \( i(G_{q,p}) \leq 2p \), and consequently \( i(G_{q,p}) = m(K) = 2p \). \( \square \)

**Lemma 9.** \( \text{MaxDim}(G_{q,p}) \geq p \).

**Proof.** Let \( e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_p = (0, 0, \ldots, 1) \in V \) and let \( h_1 = (c, 1, \ldots, 1), h_2 = (1, c, \ldots, 1), \ldots, h_p = (1, 1, \ldots, c) \in C^p \leq H \). For any \( 1 \leq i, j \leq p \), we have

\[
h_i^{e_j} = h_i \text{ if } i \neq j, \quad h_i^{e_i} = ((1/c - 1)e_i)h_i.
\]

But then, for each \( i \in \{1, \ldots, p\} \), we have

\[
h_i \in \cap_{j \neq i} H^{e_j}, \quad h_i \notin H^{e_i},
\]

hence \( H^{e_1}, \ldots, H^{e_p} \) is a family of maximal subgroups of \( G_{q,p} \) in general position. \( \square \)

In order to compute the precise value of \( \text{MaxDim}(G_{q,p}) \), the following lemma is useful.

**Lemma 10.** Let \( v_1 = (x_1, \ldots, x_p) \) and \( v_2 = (y_1, \ldots, y_p) \) be two different elements of \( V = \mathbb{F}^p \) and let \( \Delta(v_1, v_2) = \{ i \in \{1, \ldots, p\} \mid x_i = y_i \} \). Then

- if \( |\Delta(v_1, v_2)| = 0 \), then \( |H^{v_1} \cap H^{v_2}| \leq p \);
- if \( |\Delta(v_1, v_2)| = u \neq 0 \), then \( |H^{v_1} \cap H^{v_2}| = p^u \).

**Proof.** Clearly \( |H^{v_1} \cap H^{v_2}| = |H \cap H^{v_2 - v_1}| = |C_H(v_2 - v_1)| \). If \( \Delta(v_1, v_2) = \emptyset \), then \( C_H(v_2 - v_1) \cap C^p = \{1\} \), hence \( |C_H(v_2 - v_1)| \leq p \). If \( |\Delta(v_1, v_2)| = u \neq 0 \), then

\[
C_H(v_2 - v_1) = \{c_1, \ldots, c_p \} \in C^p \mid c_i = 1 \text{ if } i \notin \Delta(v_1, v_2) \cong C^u
\]

has order \( p^u \). \( \square \)

**Proposition 11.** If \( p \neq 2 \), then \( \text{MaxDim}(G_{q,p}) = p \).

**Proof.** By Lemma 9 it suffices to prove that \( \text{MaxDim}(G_{q,p}) \leq p \). Assume that \( \mathcal{M} \) is a family of maximal subgroups of \( G = G_{q,p} \) in general position and let \( t = |\mathcal{M}| \). Let \( M \in \mathcal{M} \). One of the following two possibilities occurs:
(1) $M$ is a complement of $V$ in $G$: hence $M = H^v$ for some $v \in V$.

(2) $M$ contains $V$: hence $M = V \rtimes X$ for some maximal subgroup $X$ of $H$.

If $M_1$ and $M_2$ are two different maximal subgroups of type (2), then $M_1 \cap M_2 = V \rtimes \text{Frat}(X)$ is contained in any other maximal subgroup of type (2). Hence $\mathcal{M}$ cannot contain more than 2 maximal subgroups of type (2). Now we prove the following claim: if $\mathcal{M}$ contains at least three different complements of $V$ in $G$, then $t \leq p$. In order to prove this claim, assume, by contradiction that $t > p$. This implies in particular that in the intersection $X$ of any two subgroups of $\mathcal{M}$, the subgroup lattice $L(X)$ must contain a chain of length at least $p - 1$.

Assume that $H^{v_1}, H^{v_2}, H^{v_3}$ are different maximal subgroups in $\mathcal{M}$. It is not restrictive to assume $v_1 = (0, \ldots, 0)$. Let $v_2 = (x_1, \ldots, x_p)$ and $v_3 = (y_1, \ldots, y_p)$. For $i \in \{2, 3\}$, it must $|H \cap H^{v_i}| \geq p^{p-1}$, hence, by Lemma 10, $|\Delta(0, v_2)| = |\Delta(0, v_3)| = p - 1$, i.e. there exists $i_1 \neq i_2$ such that $x_{i_1} \neq 0$, $x_{j_1} = 0$ if $j \neq i_1$, $y_{i_2} \neq 0$, $y_{j_2} = 0$ if $j \neq i_2$. But then $|\Delta(v_2, v_3)| = p - 2$, hence $|H^{v_2} \cap H^{v_3}| = p^{p-2}$, a contradiction. We have so proved that either $t \leq p$ or $\mathcal{M}$ contains at most 2 maximal subgroups of type (1) and at most 2 maximal subgroups of type (2), and consequently $t \leq 4$. It remains to exclude the possibility that $t = 4$ and $p = 3$. By the previous considerations it is not restrictive to assume $\mathcal{M} = \{H, H^v, V \rtimes X_1, V \rtimes X_2\}$ where $X_1$ and $X_2$ are maximal subgroups of $H$ and $|\Delta(0, v)| = 2$. In particular we would have $H \cap H^v \leq C^3$: this excludes $C^3 \in \{X_1, X_2\}$ but then $X_1 \cap C^3 = X_2 \cap C^3 = \text{Frat} H = \{(e_1, e_2, e_3) | e_1e_2e_3 = 1\}$, hence $H \cap H^v \cap X_1 = H \cap H^v \cap X_2$, a contradiction.

Proposition 12. $\text{MaxDim}(G_{q, 2}) = 3$.

Proof. By Lemma 7, $\text{MaxDim}(G_{q, 2}) \geq m(G_{q, 2}) = 3$. Assume now, by contradiction, that $M_1, M_2, M_3, M_4$ is a family of maximal subgroups of $G_{q, 2}$. As in the proof of the previous proposition, at least two of these maximal subgroups, say $M_1$ and $M_2$, are complements of $V$ in $G_{q, 2}$. But then, by Lemma 10, $|M_1 \cap M_2| \leq 2$, hence $M_1 \cap M_2 \cap M_3 = 1$, a contradiction. □

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