Topology and Sizes of H II Regions during Cosmic Reionization

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ABSTRACT
We use the results of large-scale simulations of reionization to explore methods for characterizing the topology and sizes of H II regions during reionization. We use four independent methods for characterizing the sizes of ionized regions. Three of them give us a full size distribution: the friends-of-friends (FOF) method, the spherical average method (SPA) and the power spectrum (PS) of the ionized fraction. These latter three methods are complementary: While the FOF method captures the size distribution of the small scale H II regions, which contribute only a small amount to the total ionization fraction, the spherical average method provides a smoothed measure for the average size of the H II regions constituting the main contribution to the ionized fraction, and the power spectrum does the same while retaining more details on the size distribution. Our fourth method for characterizing the sizes of the H II regions is the average size which results if we divide the total volume of the H II regions by their total surface area, (i.e. $3V/A$), computed in terms of the ratio of the corresponding Minkowski functionals of the ionized fraction field. To characterize the topology of the ionized regions, we calculate the evolution of the Euler Characteristic. We find that the evolution of the topology during the first half of reionization is consistent with inside-out reionization of a Gaussian density field. We use these techniques to investigate the dependence of size and topology on some basic source properties, such as the halo mass-to-light ratio, susceptibility of haloes to negative feedback from reionization, and the minimum halo mass for sources to form. We find that suppression of ionizing sources within ionized regions slows the growth of H II regions, and also changes their size distribution. Additionally, the topology of simulations including suppression is more complex, as indicated by the evolution of the Euler characteristic of the ionized regions. We find density and ionized fraction to be correlated on large scales, in agreement with the inside-out picture of reionization.

Key words: H II regions–ISM: bubbles–ISM: galaxies: high-redshift–galaxies:formation–intergalactic medium–cosmology:theory

1 INTRODUCTION
The Cosmic Microwave Background (CMB) discovered in 1965 was evidence that the hot big bang universe cooled and recombined (Penzias & Wilson 1965). That same year, however, the intergalactic medium (IGM) at $z = 2$ was found to be largely devoid of neutral hydrogen atoms, when astronomers failed to detect their Lyman alpha resonant scattering in the spectra of the first quasars discovered with high enough redshift to make the transition visible from the ground (Gunn & Peterson 1965; Oke 1966). This was soon interpreted to mean that, unless the IGM were many orders of magnitude less dense than the average density of a critical universe, most of the hydrogen atoms there must have been reionized sometime between $z = 1000$ and $z = 2$.

Although we have since then learned much more about both the CMB and the HI absorption towards high redshift QSOs, currently it is still those two observables which constrain the epoch of reionization (EoR). The results from the WMAP measurements of the CMB have constrained the optical depth due to electron scattering, $\tau_{\text{es}}$, to $0.088 \pm 0.015$, implying that an instantaneous reionization would have happened at $z = 10.4 \pm 1.2$ (Komatsu et al 2010). The QSO spectra obtained within the Sloan Digital Sky Survey (SDSS) indicate a low, but rapidly rising neutral fraction around redshift 6 (Fan et al 2006; Willott et al 2007). The combination

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of these two measurements suggests that the epoch of reionization extended over several redshift units.

However, a series of measurements are being performed or prepared that are expected to give completely new constraints on this early epoch of galaxy formation. Radio telescopes capable of measuring at low frequencies (GMRT\(^1\), 21CMA\(^2\), LOFAR\(^3\), MWA\(^4\)) should be able to detect the signature of redshifted 21cm radiation from neutral hydrogen during the EoR. These measurements are challenging due to the presence of the strong foreground emission of, mostly, our own Milky Way as well as ionospheric distortions. If successful, these experiments should produce a detection before the 50th anniversary of the discovery of the CMB and the Gunn-Peterson effect.

In preparation for these 21cm observations, many groups have been numerically simulating the reionization process on large scales (e.g. Iliev et al. 2006b, McQuinn et al. 2007, Shin et al. 2008, to name a small selection); see (Trac & Gnedin 2009) for a review on simulations of reionization. Semi-numerical models have also been developed, first by Zahn et al. (2007), later improved by others (Mesinger & Furlanetto 2007, Santos et al. 2008, Alvarez et al. 2009, Choudhury et al. 2009, Mesinger et al. 2010, Alvarez & Abel 2010). What both of these types of calculations give us is the evolution of the ionized fraction of intergalactic hydrogen in the Universe, \( \langle x(r, t) \rangle \).

The simulation results show a great amount of complexity in \( x(r, t) \). As the sources of reionization are likely to be clustered in space, individual H II regions typically contain many sources (e.g., Furlanetto et al. 2004, Iliev et al. 2005, 2006b), and obtain complex shapes in 3D space. Rare sources are more biased than more abundant ones, and it is expected that the level of bias will largely determine the characteristic scale of the reionization process (Iliev et al. 2006b, Furlanetto et al. 2008). Accurate theoretical predictions for the morphology and size of H II regions depend upon an understanding of the abundances and clustering of the ionizing sources themselves, in addition to the underlying inhomogeneous density field. Quantitative analysis of the distribution of ionized material is expected to constrain the ionization fraction field \( x(r, t) \) and its evolution.

The planned observations of the 21cm line of neutral hydrogen are expected to constrain the ionization fraction field \( x(r, t) \) statistically, as the power spectrum of neutral hydrogen fluctuations is the most directly observed quantity via 21–cm radio observations (e.g. Zaldarriaga et al. 2000, Mellema et al. 2006, Harker et al. 2010). Future observations, for example with the SKA (Square Kilometer Array) may have enough sensitivity to actually image the redshifted 21cm signal as a function of frequency and thus reveal the spatial structure of the ionization fraction field \( x(r, t) \).

Ultimately, we are less interested in the function \( x(r, t) \) itself but more in “why it is like it is”, i.e. in the properties of the sources and sinks of reionization. For this, one has to find out how the statistical properties of ionization fraction field depend on different source and sink properties. Simulations of reionization can thus be said not to aim at reproducing the actual \( x(r, t) \), but rather at showing the same statistical behavior as the real epoch of reionization: on the one hand, because only statistical quantities can be measured (as mentioned above), on the other hand because the input of simulations is only statistically comparable to the real conditions.

This paper has two parts. In the first part we investigate the usefulness, in terms of characterizing size distributions of ionized regions, of different kinds of statistics (for example the power spectrum of \( x(r, t) \)) of a simulated ionization fraction field. In the second part we employ these statistics to investigate the effect of different source properties. This is useful to draw conclusions on the sources, once statistical properties of the real \( x(r, t) \) of reionization can be measured.

We focus on the early and intermediate stages of reionization, when the morphology of H II regions is most well defined and the photon mean-free-path is determined by the patchiness of the reionization process itself. At the latest stages of the reionization process, after overlap, fluctuations in the UV background are expected to be sensitive to the small fraction of gas which is left neutral in the form, for example, of Lyman-limit systems (Miralda-Escudé et al. 2000, Gnedin & Fan 2006, Alvarez & Abel 2010, Prochaska et al. 2009). We limit ourselves to analyzing the ionization fraction fields \( x(r, t) \) from the simulations, not on producing the observable quantities. This is the necessary first step before proceeding to evaluate whether different scenarios can be observationally distinguished. The observables will be discussed in a follow-up paper (Iliev et al., in prep.).

In terms of sections, the paper is organized as follows: §2 introduces the simulations included in this study. §3 introduces the analysis methods used to investigate these simulations. In §4 we test the effect of numerical parameters on the statistics of \( x(r, t) \). In §5, we test the effect of source properties on the statistics of \( x(r, t) \). We end with our conclusions in §6.

2 SIMULATIONS

Our simulation methodology has been previously described in detail (Iliev et al. 2006b, Mellema et al. 2006, Iliev et al. 2007). Here, we will briefly summarize the underlying N-body simulations that

\[ \frac{3}{2} \text{Giant Metrewave Telescope, http://gmrt.ncra.tifr.res.in} \]
\[ \text{21 Centimeter Array, http://21cma.bao.ac.cn} \]
\[ \text{Low Frequency Array, http://www.lofar.org} \]
\[ \text{Murchison Widefield Array, http://www.mwatelescope.org} \]
\[ \text{Precision Array to Probe the EoR, http://astro.berkeley.edu/~dbacker/eor} \]

\[ \text{Figure 1. Evolution of the effective efficiency factor } g_{\text{eff}} \text{ for the simulation with imposed photon production history (53Mpc}_\text{uvS}_\text{e09) as a function of time and global ionization fraction.} \]
Table 1. Simulation parameters, volumes derived from this and global (mass averaged) reionization history results for simulations with WMAP5 cosmology parameters. The box sizes can be directly inferred from the simulation names. For all simulations, the mesh consists of $256^3$ cells. The ionization time step for all simulations is $\Delta t_i = 5.75 \times 10^6$ yr. $g_{\gamma}$ is the efficiency parameter as explained in the text; the old efficiency parameter $f_{\gamma}$ is given in brackets; $V_{\min}$ is the comoving volume of the minimum size H II region, ionized by the least efficient source during one time step, assuming the density to be the average density of the universe; $\tau_{es}$ is the electron scattering optical depth calculated for each simulation.

| Simulation  | $V_{cell}/\text{Mpc}^3$ | $V_{min}/\text{Mpc}^3$ | $z_{10\%}$ | $z_{50\%}$ | $z_{70\%}$ | $z_{99\%}$ | $\tau_{es}$ |
|-------------|--------------------------|------------------------|-------------|-------------|-------------|-------------|-------------|
| 53Mpc_g8.7_130S | 0.0088 | 0.1361 | 13.6 | 10.6 | 9.2 | 8.6 | 0.083 |
| 163Mpc_g8.7_130S | 0.2575 | 0.1361 | 13.3 | 10.4 | 8.9 | 8.3 | 0.080 |
| 53Mpc_g8.7_130 | 0.0088 | 0.1361 | 15.8 | 14.6 | 13.6 | 13.0 | 0.13 |
| 53Mpc_g1.7_8.7S | 0.0088 | 0.0136 | 10.1 | 8.5 | 7.3 | 6.7 | 0.058 |
| 53Mpc_g0.4_5.3 | 0.0088 | 0.0084 | 11.7 | 10.3 | 9.7 | 6.7 | 0.078 |
| 53Mpc_g8.7_130S | 0.1627 | >0.1361 | 13.7 | 10.6 | 9.7 | 8.6 | 0.084 |
| 53Mpc_g10.4_0 | 0.0088 | 0.0088 | 10.5 | 9.6 | 8.8 | 8.3 | 0.071 |

Figure 2. Ionization maps (ionization fraction according to colour bar) of all included simulations at 30% global ionized fraction: from top left to bottom right: 53Mpc_g10.4_0, 53Mpc_g8.7_130S, 53Mpc_g8.7_130, 163Mpc_g8.7_130S, 53Mpc_g4.9_130, 53Mpc_g8.7_130S and 53Mpc_g10.4_0. Each panel is for a slice which is one cell thick ($\sim 0.64$ Mpc and $\sim 0.21$ Mpc, respectively for the 163 Mpc and 53 Mpc simulations).
we analyze. On the scales of interest to us here, the intergalactic medium and dark matter followed each other as cosmic structure arose in the ΛCDM universe during this epoch. Even during reionization, since ionization fronts which reionized the IGM moved supersonically, the back reaction of the gas due to mass motions related to pressure forces can be neglected to first approximation (Shapiro & Girard [1987]), and, hence, the radiative transfer can be done as a post-processing of the N-body density field.

2.1 N-body simulations

As a basis for our radiative transfer calculations, we begin with the time-dependent density field extracted from N-body simulations of structure formation. We use the CUBEPM code which was developed from the PMFAST code (Merz et al. 2005), see Iliev et al. (2008) for a short description of the CUBEPM code. It uses particle-particle interactions at sub grid distances and a particle-mesh method for larger distances. Here, we use the results of two simulations, performed with CUBEPM, one for a volume of 163 Mpc on a side, another with 53 Mpc. The former has 30723 particles and mesh size of 61443 cells, while the latter has 10243 particles and 20483 cells, which imply particle masses of 5.5 × 106 M⊙ and 5.1 × 106 M⊙ respectively. These parameters guarantee a minimum resolved halo mass of 106 M⊙ which is approximately the minimum mass of halos able to cool by atomic hydrogen cooling. The cosmological parameters used were for a flat ΛCDM universe with Ωm = 0.27, Ωb = 0.044, h = 0.7, n = 0.96, and σ8 = 0.8, based on the five year WMAP results (Komatsu et al. 2009).

2.2 Radiative transfer runs

Table 1 gives an overview of the seven different radiative transfer runs that we analyze in this paper. These are a sub-set of a larger suite of simulations, to be presented in a follow-up paper (Iliev et al., in prep.). This sub-set was chosen as the minimum one needed to illustrate the points we want to make in this work. All the radiative transfer simulations were performed using the C2-Ray method (Mellama et al. 2006) on a uniform rectilinear grid containing 2563 grid cells. The density is assigned to the mesh by smoothing the density from the underlying N-body simulation using an SPH kernel function: each N-body particle is assigned a compact, spherical smoothing kernel whose width is adjusted so as to encompass its 32 nearest-neighbors. Particle mass is then assigned to the cells of our radiative transfer grid by integrating each kernel function over the volume of each cell it overlaps. To convert what is a the IGM dark matter density into a baryon density, we assume that in the IGM the gas distribution follows the dark matter. This is valid on the scales of the radiative transfer cells (0.2 or 0.6 comoving Mpc) as at the mean density of the IGM they are much larger than the local Jeans length.

There are physical effects below our resolution limit which influence reionization. These are small scale density variations (‘clumping’) and the presence of unresolved absorbers (such as mini-halos and the structures of unknown origin which are observed as Lyman Limit Systems at lower redshifts). All of these will slow down the reionization process as they increase the number of photons absorbed. In the simulations presented here we do not consider these effects, but see e.g. Ciardi et al. (2006); McQuinn et al. (2007); Alvarez & Abel (2010); Crociani et al. (2010) for studies about the effect of different types of unresolved absorbers and e.g. Iliev et al. (2006b); McQuinn et al. (2007); Iliev et al. (2008) for comparative studies of the effect of clumping.

Simulations are labeled with the parameter gγ, which is an efficiency factor for the ionizing photon production of halos per source halo baryon per unit time. Each halo of mass M is assigned a luminosity

\[ \dot{N}_γ = gγ \frac{M\Omega_b}{10^9 m_Ω m_p}, \]

where \( \dot{N}_γ \) is the number of ionizing photons emitted per Myr, M is the halo mass, and \( m_p \) is the proton mass. Halos are assigned different luminosities according to whether their mass is above (“large sources”) or below (“small sources”) 108 M⊙ (but above 107 M⊙). For example, 53Mpcg8.7,130S indicates that large sources have an efficiency \( gγ = 8.7 \), while small sources have an efficiency \( gγ = 130 \), and the symbol “S” means that the small sources are suppressed in regions where the ionization fraction is higher than 10%.

In previous simulations performed with C2-Ray, the source efficiencies were characterized by \( fγ \), the number of ionizing photons released per source halo baryon per star-forming episode (i.e. per simulation time-step for updating the source halo catalogue from the N-body results). The relation between \( fγ \) and \( gγ \) is given by

\[ gγ = fγ \left( \frac{10^{4} \text{Myr}}{\Delta t} \right) \]

where \( \Delta t \) is the time between two snapshots from the N-body simulation. For example Iliev et al. (2008) considered a simulation called f250. In the new naming scheme this would be called g112. The reason for switching to a new naming scheme is that the previous scheme hid the dependence on the size of the time step \( \Delta t \), since \( fγ \) ionizing photons were released over a time \( \Delta t \) per baryon for all the baryons inside source halos when that step began. This made it more difficult to compare simulations involving different time-steps, since the results depend on BOTH \( fγ \) AND \( \Delta t \), while the instantaneous luminosities of source halos depend only upon their ratio \( fγ / \Delta t \), not \( fγ \) alone.

The suite of simulations presented in Table 1 allows us to see how the morphology and characteristic scales of reionization depend upon various important numerical and physical parameters which are not yet well understood. The simulation 53Mpcg8.7,130S is our standard case for this paper. We refer to this as the fiducial simulation. It produces an electron scattering optical depth consistent with the 1–σ range allowed by the seven year WMAP results, \( \tau_{\text{ss}} = 0.088 \pm 0.015 \) (Komatsu et al. 2010). To test the effect of weaker sources, and thus more extended reionization we also present 53Mpcg1.7,8.7, which ends considerably later and has an optical depth consistent with the seven year WMAP results when considering the 2–σ range (and assuming a gaussian error distribution) for \( \tau_{\text{ss}} \). These two simulations are used to introduce the different analysis methods in section 3.

One of the physical effects which may be present during reionization and which we study in this paper, is source suppression due to Jeans mass filtering, in which ionizing radiation from sources hosted by halos with a mass below some threshold is suppressed when the halos are located within ionized regions (e.g. Shapiro et al. 1994). This concept was introduced in our simulation models in Iliev et al. (2007). By comparing, for example 53Mpcg8.7,130S to 53Mpcg8.7,130 (the latter without source suppression), it is possible to isolate the effects due solely to source suppression. However, the simulation with no suppression will end at a much higher redshift and therefore the halo populations are not suppressed. The small halos and mini-halos have an optical depth consistent with the seven year WMAP results, \( \tau_{\text{ss}} = 0.088 \pm 0.015 \) (Komatsu et al. 2010). To test the effect of weaker sources, and thus more extended reionization we also present 53Mpcg1.7,8.7, which ends considerably later and has an optical depth consistent with the seven year WMAP results when considering the 2–σ range (and assuming a gaussian error distribution) for \( \tau_{\text{ss}} \). These two simulations are used to introduce the different analysis methods in section 3.

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comparable at corresponding stages of reionization (e.g., at $z_{50\%}$). Hence, we also include the simulation $53\text{Mpc}_g^{0.4,5.3}$ which does not have suppression, but which due to the weaker source luminosities, ends approximately at the same time as our fiducial simulation $53\text{Mpc}_g^{8.7,130S}$. This way the different reionization stages (except the earliest ones) occur at similar times, and thus these two simulations have similar halo populations at the different stages of reionization.

For the 54 Mpc simulation volume at $z \sim 13.6$, there are roughly 330 cells containing source halos more massive than $10^9 M_\odot$ (this corresponds to roughly $2 \times 10^{-3} \%$ of all cells). Additionally, there are about 35 000 cells ($2.5 \times 10^{-1}\%$) containing low mass source halos between $10^6 M_\odot$ and $10^7 M_\odot$. However, for the fiducial simulation for example, roughly 88\% of these low mass source halos are suppressed. For the large simulation volume, these numbers (also at roughly 10\% global ionization fraction, i.e. $z \sim 13.2$) are: 12 000 cells containing massive halos ($7.8 \times 10^{-2}\%$ of all cells) and 750 000 cells ($4.8\%$ of all cells) containing low mass halos, of which roughly 86\% are suppressed. At overlap, the small (large) simulation volume has about 17 000 (440 000) cells containing massive source halos, which corresponds to 0.1\% and 2.8\% of all cells, respectively. The number for low mass halos are 280 000 (3 600 000) cells, or 1.8\% and 13\% of all cells, almost all are completely suppressed.

Throughout this study, we will make comparisons like indicated above for the case of source suppression, in order to see how physical effects manifest themselves and to find which quantitative measurements best discriminate among different reionization scenarios. Instead of comparing the simulations at equal redshifts, we do the comparison at equal mass averaged (global) ionization fraction, $\langle x \rangle$. Table 1 lists the redshifts at which the global ionization fraction (i.e. mass-weighted average, unless otherwise stated) for each simulation are $\langle x \rangle \sim 0.1, 0.3, 0.5, 0.7, 0.99$. The epoch at which the H II regions globally "overlap", $z_{ov}$, will, by convention, be taken here to be the redshift at which $\langle x \rangle = 0.99$, although the value of $z_{ov}$ which results is not very sensitive to this particular choice as long as $\langle x \rangle$ is close to unity.

Beside the above mentioned simulations, there are three more simulations included in this study: the simulation labeled $53\text{Mpc}_g^{uvS_1}$ has the same (imposed) global photon production history as our fiducial simulation, but only halos more massive than $10^8 M_\odot$ are allowed to host luminous sources. This results in a $g_7$ that is time dependent. Its evolution is plotted in Fig. 3 as a function of time and mass averaged ionization fraction $\langle x \rangle$.

As a second simulation with only high mass sources, we include simulation $53\text{Mpc}_g^{10.4,0}$. Unlike $53\text{Mpc}_g^{uvS_1}$ it has a constant mass to light ratio which is chosen so that reionization ends roughly at the same time as in our fiducial simulation. This yields later $z_{10\%} - z_{70\%}$ and thus a lower value of $\tau_{ov}$. Note that the efficiency of sources in high mass halos had to be boosted only by a factor 1.2. This means that in our fiducial simulation, sources in low mass halos only contribute about 17\% to the ionizing photon budget.

Simulation $53\text{Mpc}_g^{8.7,130S}$ has the same physical parameters and the same mass resolution and, hence, halo mass range as our fiducial simulation, but the simulation box volume is about 30 times bigger. Therefore, it is capable of catching structure on larger scales. On the other hand, the resolution in the radiative transfer simulation is worse than in the small box simulation since the number of cells in both simulations is 256 per side. This simulation is included to check for cosmic-variance effects and to test the effect of resolution on our investigation methods.

Snapshots of the simulations at the 30\% global (mass averaged) ionized fraction are shown in Figure 2. The slices are to the same comoving physical scale to make it more easy to see the morphological and topological differences between the models which we will discuss in detail below.

Table I also lists values for the smallest possible H II region $V_{min}$ which could be formed during a single radiative transfer time step $\Delta t_i$, if the surrounding IGM has the average density of the Universe and recombinations can be neglected. This number is likely to be an overestimate as recombinations and density peaks will reduce it. However, it is a useful number to compare the resolution of various radiative transfer simulations with. The number of emitted ionizing photons from the smallest halos of mass $M_{min}$ is given by

$$N_{min} = \dot{N}_i \Delta t_i = \frac{g_7 M_{min} \Omega_b}{10 \mu m_\odot \Omega_b \rho_{crit} 10^{10} \text{Myr}} \Delta t_i,$$

which then gives a minimum volume of

$$V_{min} = \frac{N_{min}}{n_H} \approx \frac{g_7 M_{min} \Omega_b}{10 \mu m_\odot \rho_{crit} 10^{10} \text{Myr}} \Delta t_i \approx 0.0016 \text{Mpc}^3 \left(\frac{M_{min}}{10^8 M_\odot}\right) g_7,$$

where we have used the radiative transfer time step $\Delta t_i = 5.75 \times 10^6$ yr and $n_H = \rho_{crit}/\mu m_\odot$ is the hydrogen number density. $V_{min}$ can be compared to the cell-sizes of the simulations which are also listed in Table I.

3 Introducing the Analysis Methods

In this section we introduce our analysis methods by means of two simulations that differ only in the mass-to-light ratio of the halos, $53\text{Mpc}_g^{8.7,130S}$ and $53\text{Mpc}_g^{1.7,8.7S}$. Whenever we refer to "the simulations" in this section, we mean these two. The focus in this section is on the ability of our analysis methods to discriminate between the two simulations. The results of all methods together can be seen as a characterization of the morphology of the ionization fraction.

3.1 Size distribution

One of the most basic measures of reionization is the size distribution of H II regions. However, as will become clear below, "size of an H II region" is a quantity which can be defined in different ways. Under the assumption that most of the volume is either highly-ionized or highly-neutral, H II regions can be considered to be topologically connected volumes of space. We previously used a friends-of-friends (FOF) method [Iliev et al. 2006b] to identify such regions, using the condition $x > 0.5$ for a cell to be considered ionized. In contrast to this measure of the volume of connected ionized space, [Zahn et al. (2007)] used a different method, introduced as "the bubble probability distribution". For reasons we explain below, we refer to this method as the "spherical average" method. We now describe these two methods in more detail.

3.1.1 Friends-of-friends method

Our first method for identifying the size distribution of H II regions relies on a literal definition of "H II region": a connected region in which hydrogen is mostly ionized. For grid data, the obvious way to identify such a connected region is to use a "friends-of-friends"
(FOF) approach, in which two neighbouring cells are considered friends if they both fulfill the same condition. Cells are grouped into distinct regions according to whether they are linked together in an extended network of mutual friends. The algorithm we use to group cells together is the equivalence class method, described in Press et al. (1992). Unless otherwise specified, we use $x > 0.5$ for a cell to be considered ionized, and $x \leq 0.5$ for a cell to be considered neutral, so that every point in the simulation box is either in an H I or an H II region. Our method was first described in (Iliev et al., 2006). In contrast to all other size measures presented below, the FOF does not care about how contorted an H II region is. Therefore, the sizes of H II regions found by the FOF method and the sizes of H II regions found by other methods which will be introduced below, give complementary information about the morphology of the ionization fraction field.

The FOF method has been used extensively for halo finding in cosmological N-body simulations (Davis et al., 1985). Our implementation is more straightforward, since each cell has only 6 direct neighbours, the identities of which are known in advance, as opposed to particle data, in which it is necessary to perform costly searches to identify the groups. Another significant difference between the two methods is the role played by free parameters. In the halo finding FOF method, the free parameter is the linking length, which is the distance within which two particles are considered to be friends. In the region finding method, the free parameter is the threshold, $x_{th}$, for a cell to be considered ionized or neutral.

As seen in Fig. 3, we test the effect of varying $x_{th}$ (using our fiducial simulation 53Mpc_g8.7_130S at 50% global ionization fraction), where we used three different values for $x_{th} = 0.1, 0.5$ and 0.9. The thin dot-dashed black line shows at every volume $V$, how much a single region with this volume would contribute. This means that each bin cannot be filled less (i.e. or the bin must be empty) than to the point where this line crosses the lower limit of each volume bin.

As can be inferred from Table 1, the minimum volume ionized by a source in a single radiative transfer time step for our fiducial simulation is about an order of magnitude bigger than the cell size. This means that at the end of the radiative transfer time step at which the source turned on, the cell hosting the halo will be completely ionized and surrounded by partly ionized cells. If regions are connected through such partly-ionized border cells, it strongly depends on $x_{th}$ if the regions count as two disconnected or as one connected region. In every analysis method that depends on a threshold value this effect is bigger if the partly ionized borders of ionization regions are comparable to the regions’ size. Nevertheless, as can be seen in Fig. 3, the qualitative picture remains unchanged, with a few few-cell regions, a substantial contribution of regions of intermediate size, and the main contribution coming from a single large region comparable in size to the simulation box. The absence of single-cell regions for $x_{th} = 0.1$ can be explained by looking at the minimum number of photons from a single source released during one time step. Sufficient photons are produced to ionize the cells surrounding the source cell more than 10%, even if the density of the cells is nine times the average density of the universe, see Eq. (4).

Below, we convert the volume bins into equivalent radius bins, $R_{equi,FOF} = [3/(4\pi V)]^{1/3}$ which corresponds to the radius $R_{equi,FOF}$ of a spherical region with the same volume. We want to stress that this does not mean that the H II regions are spherical. We convert to an equivalent radius only to allow a more direct comparison to the spherical average method and the power spectrum which are described in the following sections. We normalize to the total volume and not to ionized volume, to allow for a more direct comparison to the power spectrum which is normalized in the same way.

In order to show the time-evolution of the FOF size distribution of ionized regions for a simulation in a single plot, we choose a fixed threshold value ($x_{th} = 0.5$) and color code the contribution $V dp/dV$. The color coding makes it possible to show a histogram (like Fig. 3) in a single line or column. Each individual column of Fig. 4 (left panel) is a histogram as Fig. 3 at a different global ionization fraction. Fig. 4 thus shows the evolution of the size distribution with global ionization fraction. The box- and the cell size are marked with $>$ on the ordinate. By construction the FOF method will not result in sizes larger than the former and smaller than the latter. As we will see later, H II regions start to merge very early on in the course of reionization which results in shapes far from spherical and a complex topology already at global ionization fractions $\langle x \rangle \sim 0.2$. The concept of distinct H II regions and their sizes quickly becomes meaningless. Therefore, all size distribution estimates are only shown up to a global ionization fraction $\langle x \rangle \sim 0.6$.

Three things catch the eye when analyzing the size evolution in Fig. 4:

1. Already at $\langle x \rangle \sim 0.15$, the distribution for both simulations is not continuous, but shows a gap. Most of the ionized volume is contained within one region of a size falling into a size-bin which is separated from the rest. This is an inherent property of the FOF method, where regions are grouped together as soon as they touch and local H II region percolation occurs quite early in the evolution. If the H II regions reach a certain size, which depends on the clustering of the sources and on their efficiency, they will percolate and form bigger H II regions. As those smaller H II regions grow and merge into the larger one, both their numbers and the fraction of the ionized volume that they occupy decrease. A doubling of the volume (merging of two bubbles with the same size) thus cor-

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**Figure 3.** Effect of varying the threshold $x_{th}$ for the FOF method for the 53Mpc_g8.7_130S simulation at $z = 9.7$, when the global ionized fraction $\langle x \rangle$ is about 50 percent. The results for three different threshold limits are shown, as indicated in the figure. The thin dot-dashed black line shows at every volume $V$, how much a single region with this volume would contribute. Indicated by $^*$ on the abscissa are the cell- and box- volume.
responds to a jump over one effective radius bin. The largest H II region grows through mergers with smaller H II regions, as well as due to sources within the region, and approaches the box-size by \( \langle x \rangle \sim 0.6 \).

2. The distribution of the smaller scales (i.e. everything except the one big region) is much flatter for the simulation 53Mpc.g1.7.8.7S. This simulation has more single- or few-cell-sized regions than our fiducial simulation 53Mpc.g8.7.130S.

3. While the size-bin which contributes most to the global ionization fraction is increasing with \( \langle x \rangle \), the shape of the distribution of the rest does not change much. However, its total contribution to the global ionization fraction decreases with \( \langle x \rangle \): At the same rate that small H II regions grow bigger and merge into even bigger H II regions, new small ones are "born" but contribute in the course of reionization less and less to the global ionized-fraction.

The FOF-method applied to an ionization field in a finite simulation box can only sample the true underlying size distribution function (as it would be in an infinite simulation box) up to a lower limit. This is indicated by the thin dot-dashed black line in Fig. This lower limit depends on the size of the simulation box. The gap mentioned above will be smaller the better the sampling, that is the computational volume, averaging the ionization fraction inside these spheres and finding the largest such sphere for which the average ionization fraction is greater than a certain threshold. This lower limit depends on the size of the simulation box. The gap mentioned above will be smaller the better the sampling, that is the computational volume, averaging the ionization fraction inside these spheres and finding the largest such sphere for which the average ionization fraction is greater than a certain threshold.

3.1.2 Spherical Average method

The spherical average method (SPA) was described by Zahn et al. (2007); it can be easily used for comparisons with analytical models. It is based on constructing spheres around every cell in the computational volume, averaging the ionization fraction inside these spheres and finding the largest such sphere for which the average ionization fraction is greater than a certain threshold \( x_{th} \). We chose \( x_{th} = 0.9 \). Because of this, we call it the spherical average method (SPA). It yields a smoother distribution of H II region sizes than the one obtained by the FOF method. It does not measure the size of a connected ionized space but instead it is a measure of the scales of spherical bubbles which would cover the ionized space.

Motivated by the analysis given in Appendix A we multiply the radius found by the spherical average method by a factor of 4, \( s = 4 \times R \). We call this the scale of the spherical average method. In Fig. we plot the spherical average distribution in the same way as the FOF distribution in Fig. \( R_{dP/dR} \) is normalized to the whole volume. We define \( R_{\text{max}}(\langle x \rangle) \) to be the position of the maximum of \( R_{dP/dR} \) at every \( \langle x \rangle \). We see that \( R_{\text{max}} \) is increasing with \( \langle x \rangle \) which means an increase in the average bubble size with global ionization fraction, as expected. Although initially smaller, it can be seen that the average scale of simulation 53Mpc.g1.7.8.7S is growing faster with respect to \( \langle x \rangle \) than the average scale of simulation 53Mpc.g8.7.130S. From Fig. it can be further seen that the distribution of the model with the smaller efficiencies shows initially (at \( \langle x \rangle \lesssim 0.1 \)) a wider distribution of bubble sizes with a peak at smaller scales: The smallest H II regions are smaller than in the fiducial model, but the contribution from small regions to total ionized fraction is smaller. Therefore the bigger H II regions have to grow to a bigger size in the 53Mpc.g1.7.8.7 model to reach an ionization fraction of 10%. At higher average ionization fractions, the peak of the distribution is slightly shifted towards bigger scales with respect to the 53Mpc.g8.7.130S model.

As we have already noted, much of the noticeable difference in the FOF curves comes from a very small fraction of the volume, and a log scale is required to see such differences in the FOF plots. The spherical average distributions are much smoother, and therefore offer a less detailed, more global picture of the spatial structure of the ionized regions. However, the spherical average shows much more clearly the difference in size of the large scale H II regions between the two simulations.

Another method which is similar to the SPA is a method used in Mesinger & Furlanetto (2007). We examine this method in Appendix B. We find that it yields qualitatively the same results as the SPA and PS but its use in simulations with a continuous distribution of ionization fractions (i.e. not a binary field) creates complications.

3.1.3 3VA method

As another estimate of the scale of bubbles, one could use the ratio of the total volume of the H II regions and their total surface area:

\[
3 \times \frac{\sum_{\text{H II regions}} \text{Volume}}{\sum_{\text{H II regions}} \text{Area}}.
\]

The volume \( V \) and the surface area \( A \) were calculated from the zeroth and first minkowski functionals, respectively: \( V = V_3 \) and \( A = 6 \times V_1 \). For a distribution of disconnected spherical bubbles, \( 3V/A \) is the surface weighted average radius. For three dimensional bodies, one could say that \( 3V/A \) is proportional to the sur-
over a multiple of inverse dominant structures are 2-dimensional, i.e. disc-like, then face weighted average of the smallest scale of each object. If the dominant structure is 1-dimensional, i.e. bar-like, the dimensionless power per logarithmic interval in wavenumber, \( \langle P \rangle \equiv k^3 P(k)/(2\pi^2) \), for a single spherical top-hat bubble of radius \( s \) would be located at \( k_{max} \approx 2.46/s \), which is why we use \( 2.46/k_{max} \) to characterize the typical radius of regions. Indeed, comparison of the maxima of the spherical average model and the peaks of the power spectra shows that they are approximately related by \( 2.5/k_{max} \sim 4R_{max} \) as can be seen by comparing Fig. 5 and Fig. 6. It should be noted that both, the spherical average and the power spectrum do not show a pronounced peak at all global ionization fractions, but are instead rather flat at higher global ionization fractions. This is in agreement with one of the trends that \( \text{Zahn et al. 2010} \) identify for their tested reionization models.

To better show this behaviour of the PS at higher global ionization fractions, we show the power spectrum of the fiducial simulation as a function of \( k \) for four different global ionization fractions as a line-plot in the left panel of Fig. 7. However, the other trend they identify, the shift towards smaller \( k \) with increasing lower ionization fraction is less apparent for our fiducial simulation. Instead, the form of the power spectrum of the fiducial simulation suggests that at global ionization fraction greater than \( \langle x \rangle \approx 0.4 \) there are two main populations of bubbles: One at sizes where the PS initially peaks, around 0.8 Mpc and the other with sizes around 6 Mpc, which is of the size of clusters of galaxies, i.e. the typical clustering scale of the source halos at this epoch, not to be confused with the length scale which encompasses the mass of the higher-mass, virialized galaxy clusters familiar at lower redshift. The first scale results from suppression of sources after initial turn on of a source in a low-mass halo. Growth of the H II region is completely halted until a high-mass halo forms in that region. A less stringent suppression criterion would slow down the growth but probably not halt it entirely. The second scale results from merging of bubbles emerging from galaxies in the same cluster. These details in the size distribution are washed out in the spherical average distribution. However, it should be noted that a scale of 6 Mpc is a substantial fraction of a 53 Mpc box and therefore it is questionable if the sampling at this scale is high enough. At global ionization fractions larger than \( \langle x \rangle \sim 0.4 \), it can be seen that there is considerable power on scales comparable to the box size.

For the 53Mpc_\text{g1.7.8.7S} simulation it can be seen that there is more power on smaller scales but also that the power spectrum is flatter than the one of the fiducial model as there is no distinct peak below global ionization fractions of 20%. Further it can be seen that the slope of isochromatic lines is greater for this model than for the fiducial one. This means that the sizes of H II regions in the 53Mpc_\text{g1.7.8.7S} are growing faster with respect to global ionization fraction. This was also seen in the spherical average results. The absence of the peak at smaller scales that is present in the
fiducial model, is due to the fact that already at a global ionization fraction of roughly 10\%, the contribution from sources hosted by massive halos is about the same as the contribution from sources in low-mass halos, while this is true at about 70\% global ionized fraction for our fiducial simulation. Therefore, the relative contribution from H II regions produced by sources in low mass halos in isolated cells is smaller. H II regions produced by sources in massive halos will grow continuously, explaining the flat distribution below several Mpc.

The right panel of Fig. 7 shows the cross-correlation coefficient of ionized fraction and density field, $r_{\delta x}(k) \equiv \frac{\Delta^2 \delta_x(k)}{[\Delta^2 \delta(k) \Delta^2 x(k)]^{1/2}}$ at two different $\langle x \rangle \sim 0.05$ and 0.5 for both simulations plotted against $s = 2.46/k$. When $r_{\delta x} = -1$, the ionized fraction and density field are perfectly (anti-)correlated, while $r_{\delta x} = 0$ implies they are uncorrelated. As seen from the figure, the ionized fraction and density fields are nearly perfectly correlated on large scales, $s \gtrsim 8$ Mpc. It can be seen that the scale $s$ at which the correlation starts to decrease, is increasing with global ionization fraction. This is due to the fact that while the H II regions grow, they also start ionizing the voids. At very low global ionization fractions, represented here by $\langle x \rangle \sim 0.05$, the correlation coefficient for simulation 53Mpc.g1.7.8.75 is greater than the one for the fiducial simulation (especially at smaller $s$). This is expected because the ionizing radiation of the sources in the simulation with lower efficiencies can less easily "break out" of high density regions. Additionally, less efficient sources trace the high density regions better since clustered low mass sources are less suppressed: individually they form smaller H II regions and therefore do not suppress each other. At later stages of reionization this difference disappears: The simulation with low source efficiencies reaches the same global ionization fraction as the fiducial simulation at much later times when massive sources are more common. Those massive sources form bigger HII regions which also grow into the voids.

### 3.3 Topology of Reionization

Minkowski functionals have been used extensively in cosmology to characterize the topology of large scale structure \cite{Gott1986, Mecke1994, Schmalzing1996} and also the non-Gaussianity of the cosmic microwave background \cite{Komatsu2008}. Recent work has focused on using Minkowski functionals as a way to characterize the morphological structure of reionization \cite{Gleiser2006, Lee2008}.

Both works focused on the topology of the H I density field. They showed the Euler Characteristic (or genus, respectively) as a function of neutral density for several different times (i.e. global ionization fractions) and concentrated on the increasing deviations from the typical curve of a gaussian random field. Here, we will take a complementary approach based upon the topology of the ionization fraction field, rather than the fluctuating neutral density field. Unlike the neutral density field, which takes values spread continuously over a very wide range, the ionized fraction field ranges only between 0 and 1, and, ideally, there would essential be only two values possible to assign to any given point in space, either "neutral" values close to zero or "ionized" values close to unity. In that ideal case, the Euler Characteristic for the ionized fraction field would only be a function of time (or of the evolving globally-averaged ionized fraction) and be largely independent of the choice of ionization fraction threshold.

We follow the definition and notation of \cite{Schmalzing1996} and \cite{Schmalzing1997}. Consider a scalar function $f(x)$ defined at each point $x \in \mathbb{R}^3$. The set $F_{th}$ of all points $x$ for which $f(x) > f_{th}$ defines bodies in three dimensional space. The zeroth Minkowski functional, $V_0(f_{th})$, is simply the volume of those bodies:

$$ V_0(f_{th}) = \int_V \Theta(f_{th} - f(x)) d^3x, \quad (7) $$

where $\Theta$ is the Heaviside step function. The next three Minkowski functionals are defined as surface integrals over the boundary of the bodies:

$$ V_1(f_{th}) = \frac{1}{6} \int_{\partial F_{th}} (x) d^2 A, \quad (8) $$

$$ V_2(f_{th}) = \frac{1}{6\pi} \int_{\partial F_{th}} (x) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) d^2 A, \quad (9) $$

$$ V_3(f_{th}) = \frac{1}{4\pi} \int_{\partial F_{th}} (x) \frac{1}{R_1 R_2} d^2 A, \quad (10) $$

where $R_1$ and $R_2$ are the principal radii of curvature along the surface $\partial F_{th}$. The first Minkowski functional is proportional to the integrated surface area. This and the zeroth Minkowski functional were used when calculating the size estimator $\mathcal{V}/A$. The Minkowski functional $V_3$, which is proportional to the integral of the Gaussian curvature over the surface, is also known as the Euler characteristic, and is equal to:

$\#\text{parts} - \#\text{tunnels} + \#\text{cavities}$

Applied to the ionization fraction field, two disconnected ionized cells would count as two parts, a ring-like ionized region constitutes a tunnel and one part and a neutral cell completely surrounded by ionized cells is a cavity. Table 1 in \cite{Schmalzing1996} gives an overview of different notations for the Minkowski functionals which only differ in constant factors. The better known quantity genus, $g$ (number of complete cuts one can make through the object without dividing it into disconnected parts), is related to the Euler characteristic by the simple relation $g = 1 - V_3$.\footnote{This is true if one considers the Euler Characteristic of the volume defined by the set of points. Note that others consider the Euler characteristic of the surface of the set of points defining the volume. In this case, the Euler Characteristic $\chi$ is a factor of two greater, resulting in the relation to genus: $\chi = 2(1 - g)$. This is consistent with the relation $\chi(\partial A) =}$

![Figure 7](image-url)
one part and one tunnel. A sphere, on the other hand, has \( V_3 = 1 \), since it has one part and no tunnels, and positive total curvature.

We oversample the ionization fraction fields before calculating the Euler characteristic. We do this to minimize critical connections of HI and H II regions. A critical connection is for example an ionized cell which is connected via an edge to another ionized cell in an otherwise neutral neighbourhood. Appendix C explains in more detail the problems that are involved. Important for the following analysis is to note that oversampling reduces ambiguities concerning the connectivity at a given threshold value but introduces higher dependences of \( V_3 \) on the threshold value.

In the left panel of Fig. 8 we show the evolution of the Euler characteristic \( V_3 \) of the ionized fraction as a function of the threshold \( x_{th} \) for our (oversampled) fiducial simulation. We choose to normalize \( V_3 \) by dividing by the box size, to have an easier comparison when dealing with different box sizes. Between threshold values \( x_{th} = 0.2 \) and \( 0.6 \), the evolution of \( V_3 \) is largely independent on the actual choice of \( x_{th} \). \( V_3 \) rises to a maximum value at a mean ionization fraction of about 5%, after which \( V_3 \) decreases and gets negative before 20% global ionization fraction is reached. It rises again after the ionization fraction passed 50% but never reaches positive values again.

This behaviour can be qualitatively understood by considering inside-out reionization in an approximately Gaussian density field. \( V_3 \) for a Gaussian random field (or any monotone and steady function of it) as a function of threshold value, as plotted for example in figure 1 in Schmalzing & Buchert (1997), shows in the second half a rise to positive values and decreases again. In Fig. 9 we show \( V_3 \) of the density field from the 53 Mpc box simulation at redshift \( z \approx 26.1 \) as a function of density-threshold value \( \rho \).

\[ \chi(A)(1 + (-1)^{d-1}), \] where \( d \) is the dimension, \( A \) a \( d \)-dimensional body and \( \partial A \) its \( d-1 \)-dimensional surface, see equation (18) in Mecke et al. (1994).

\( V_3 \) of the original density field shows an asymmetry between isolated regions and isolated cavities, for details see Appendix E. A Gaussian smoothing with \( \sigma \sim 3 \) cells would be necessary to account for this. This removes all small scale structure, therefore \( V_3 \) is substantially reduced. It also removes extreme over- and under- densities which is why the \( V_3 \)-curve gets narrower. Using sub-grid sampling enhances cells with intermediate densities and therefore changes the distribution away from gaussian distribution.

Therefore we see deviations from the expected curve for a gaussian random field, which has \[ \frac{V_3^{min}}{V_3^{max}} = 1/2 \exp(-3/2) \approx 0.0078, \] as can be calculated for example with equation (14) in Schmalzing & Buchert (1997).
dependent on the threshold value. Since the extrema of $V_3$ of different simulations can vary quite a bit, we choose to plot the function $f(V_3) = \text{Re}(\sqrt{1 + \text{Im}(V_3)})$ instead of just $V_3$. In the interval $[V_3^{\text{max}}, V_3^{\text{min}}] \rightarrow [f(V_3^{\text{max}}), f(V_3^{\text{min}})]$, $f$ is bijective (i.e. the function has an inverse function), therefore we continue to refer to it as $V_3$.

4 BOX-SIZE AND RESOLUTION

In this section, we investigate the effect of simulation volume size on the simulation and the effect of resolution on our analysis methods. We compare our fiducial simulation to a simulation with the same source properties but in a bigger volume, 163Mpc, 8.7, 130S. Before we do so, we analyze a smoothed version of the data of our fiducial simulation to test the effect of resolution on our analysis methods. We replace the ionization fraction data in each cell with the average over a three cell width centered on the cell in question, i.e. an average over 27 cells. We will refer to this as three-cell smoothing. This results in a resolution similar to the one in the 163 Mpc simulation. It should be kept in mind that smoothing over three cells does not remove all structure smaller than three cells.

In Sect. 3 we introduced three measures of size distribution and one estimate for the average bubble size. In Fig. 10 we plot all four measures as a function of global ionization fraction for the smoothed version of 53Mpc, 8.7, 130S. We also show curves of the peaks of the spherical average and the power spectrum. As a reference, these same curves (including the $3V/A$ estimator) for the fiducial simulation are included as gray lines. We first concentrate on the $3V/A$ estimate. As can be seen in the middle panel of Fig. 10 (comparing the two solid lines), above a global ionization fraction $\langle x \rangle \sim 0.1$, the smoothed version yields larger values for $3V/A$. However, the difference is never greater than 20%.

The effect on the spherical average distribution (same panel) is mainly a reduction of contribution from scales below 0.5 Mpc. Additionally, the peak of the spherical average (compare dot-dashed lines in the same panel) is slightly shifted towards larger scales for the smoothed data.

A somewhat contrary effect can be seen in the FOF size distribution: The contribution from scales below $s \sim 0.3$ Mpc is enhanced in the smoothed data. This can be explained as follows. If a larger ionized structure is elongated (i.e. no structure in 2 dimensions) and very inhomogeneous in its ionization fraction, then 3-dimensional smoothing would break up the structure in smaller parts.

In the right panel of Fig. 10 we show the power spectrum of the three-cell smoothed data, its maximum curve and the maxima from the fiducial simulation without smoothing. It can be seen, similar to the spherical average distribution, that power on scales below $s \sim 0.5$ is removed. Also, it can be seen, that at higher global ionization fraction, the distinct peak at scales around 0.8 Mpc diminishes while the peak at $s \sim 6$ Mpc is as pronounced as in the unsmoothed data.

Since smoothing reduces the small scales and therefore reduces the critical (vertex/edge) H II/H I region-connections, over-sampling the smoothed data does not change $V_3$ more than 10%. It can be seen that the form of the evolution of $V_3$ for the fiducial simulation stays roughly the same even for an eleven-cell smoothing of he data. However, since the smaller scales are smoothed out, the total amplitude of $V_3$ is reduced.

Equipped with an idea about which effects can be due to the changed resolution, we now turn to the larger volume simulation. In Fig. 12 we plot all 4 size measures for simulation 163Mpc, 8.7, 130S. We concentrate first on the $3V/A$ size estimate, see the middle panel of that figure (black solid line) and compare it to the $3V/A$ estimate of the fiducial simulation (gray solid line): except for very low global ionization fractions $\langle x \rangle \leq 0.08$ where the estimate for the fiducial simulation is of the order of the cell size of the 163 Mpc simulation volume, the two curves almost coincide up to a global ionization fraction $\langle x \rangle \sim 0.3$, after which the scale in the 163 Mpc simulation grows faster.

The SPA distribution shows a similar behaviour as $3V/A$: below ionization fractions $\langle z \rangle \sim 0.08$, the SPA peaks of the fiducial simulation are at scales comparable to the cell size of the 163 Mpc simulation while the peaks of 163 Mpc simulation are slightly larger (compare the dot-dashed lines in the same panel). The fiducial simulation shows also a wider distribution with contributions from smaller as well as from larger scales, see left panel in Fig. 5. Between $\langle x \rangle \sim 0.08 - 0.3$ the evolution of SPA distribution of the two simulation is very similar. At larger global ionization fractions, the scale of the SPA peak is growing faster in the 163 Mpc simulation.

The power spectrum also shows that, below global ionization fractions $\langle z \rangle \sim 0.3$, despite the different resolution, the simulations with different simulation volumes agree well. At global ionization fractions $\langle x \rangle \geq 0.4$ there is considerable power on scales that are not captured in the 53 Mpc simulation which might explain the shift found by the other size distribution estimates. The power spectrum suggests that the 163 Mpc simulation captures the most relevant scales involved in reionization (that is H II region sizes up to roughly $\langle x \rangle \sim 0.5$, H I region sizes above this) since there is little power on scales above $s \sim 30$ Mpc. A noticeable difference between the simulations is the lack of the peak at scales around 0.8 Mpc in the large simulation volume. As found earlier by the smoothing test, this may be a resolution effect. The peak at scales $s \sim 6$ Mpc is not as clear in the 163 Mpc volume as in the 53 Mpc volume. After $\langle x \rangle \sim 0.5$ it shifts to larger scales.

The FOF size distribution (left panel of Fig. 12 and left panel of Fig. 4), of the two simulations look on first sight very different.

Figure 11. Evolution of $V_3$ for smoothed versions (degree of smoothing as indicated in the figure by line thickness) of the fiducial simulation at threshold value $x_{th} = 0.5$. Note the similarity of the behavior of the curves for different smoothing lengths.
It can be easily understood why: the smallest scale of H II regions in the small simulation volume is smaller than that in the large simulation volume. Looking at the cell-volume limits which are indicated on the abscissa, it can be seen that the additional population of small scale H II regions present in the 53 Mpc simulation is below the cell-size of the 163 Mpc simulation and therefore below its resolution limit. All those H II regions are partly ionized in the large simulation volume. Therefore, some of them which are more ionized than $x_{th} = 0.5$ appear as additional population in the FOF distribution of the large simulation volume at scales of its cell size. At 10% global ionization fraction, it can be seen that the 53 Mpc simulation has slightly larger large-scales than the 163 Mpc simulation, so $s \sim 4$Mpc and $s \sim 5$Mpc, respectively. Partly ionized cells that are ionized below the threshold value in the larger simulation volume and therefore do not count as belonging to the H II region cannot account completely for this difference in size. This slight mismatch in size between large and small simulation volume might be an artifact from our implementation of suppression: since we suppress all sources inside a cell that has a mass averaged ionization fraction $\bar{x}_m$ larger than 10%, the volume in which sources are suppressed can be overestimated in simulations with larger cell sizes.

While the size distribution in the small simulation volume shows a gap after $\langle x \rangle \sim 0.1$, the gap emerges first at $\langle x \rangle \sim 0.25$ in the large simulation volume. This is because a single region in any of the bins that are empty in the small simulation volume but populated in the large one, would already exceed the contribution it makes in the small volume. This sampling effect was already mentioned in the previous section.

At 30% global ionization fraction, the ionized volume of the single large connected region in the 163 Mpc simulation is approximately 10% of the total simulation volume which is larger than the size of the 53 Mpc volume. Similarly, the volume of the largest connected ionized region in the 53 Mpc simulation is also about 10% of the total volume. The fact that the largest connected region is a constant fraction of the simulation volume already at 30% ionized fraction suggests that this region pervades the whole simulation volume. This statement is strengthened by the Euler Characteristic of these simulations: $V_3$ is already highly negative at $\langle x \rangle \sim 0.3$ (for the 163 Mpc simulation this is only true for lower threshold values), see left panel of Fig. [13]. It should be noted that $V_3$ of the 163 Mpc simulation shows very similar evolution to $V_3$ of the 53 Mpc simulation (for a low threshold value for the 163 Mpc volume). The fact that $V_3$ in the large simulation volume is highly dependent on
threshold value, shows that the resolution is not sufficient to use $V_3$ as a reliable analysis tool: the ambiguity as to whether regions are connected or not is too high, as can be seen comparing the curves for different threshold values in the left panel of Fig. 13. However, it can be seen that the effect of including “lost” photons, is much smaller in the large simulation volume: relatively fewer photons travel distances longer than a box distance in the large simulation volume.

The cross correlation between ionized fraction and density at low global ionization fractions is almost identical for both simulation volumes as the thin lines in the right panel of Fig. 13 show. It differs more at higher global ionization fractions where it shifts towards larger scales for the 163 Mpc simulation, probably indicating that the scales dominating the ionization field at that global ionization fraction exceed the size of the 53 Mpc simulation.

5 PHYSICAL PARAMETERS

Our efficiency parameter $g$, is a product of the efficiency of star formation, production of ionizing photons per stellar atom (related to the initial stellar mass function) and the escape fraction of the photons from the galactic halo into the intergalactic medium. All these quantities are not very well constrained at present. Also the efficiency of suppression due to Jeans-mass filtering can be different from the simple on-off function as implemented in our simulations with suppression (c.f. McQuinn et al. 2007, Mesinger & Dijkstra 2008, Okamoto et al. 2008). To study the effect of our simplified suppression model, we consider in this section some extreme scenarios and use the methods described above to investigate the effect on the scales and the topology of the emerging H II regions.

5.1 Minimum mass of halos hosting sources with escaping ionizing radiation

In this subsection we investigate how a change in the source population affects the simulation. In simulations 53Mpc$_{uvS}$g8.7 and 53Mpc$_{g10.4}$ only halos more massive than $10^9M_\odot$ host sources that emit ionizing radiation into the IGM. The former simulation is constructed such that the number of released ionizing photons in every time step (after the formation of the first massive halos) is the same as for our fiducial simulation. The sum of all photons that were released in the fiducial simulation by low mass halos up to the time at which the first massive halo is forming, is emitted additionally in the first time step after which the first massive source has formed. Since the number of forming halos increases exponentially, the fraction of additionally released photons in this first time step is only about half of the total released photons at that time step. As pointed out in section 3 the resulting source efficiency is variable with time, shown in Fig. 11. This also means that the minimum number of photons released by one source during one time step is decreasing with increasing $(x)$. Therefore, also the minimum size for H II regions decreases to $(x) \sim 0.25$. This can be seen most clearly in the FOF size distribution (Fig. 14 left) and in the power spectrum (Fig. 14 right). To avoid this effect we performed simulation 53Mpc$_{g10.4}$ which has a different ionization history from our fiducial simulation, but the source efficiency of the high mass sources is chosen such that overlap occurs at roughly the same time, as can be seen in table 1.

The FOF size distribution shows that individual H II regions grow larger before merging with the largest H II region in both, the 53Mpc$_{uvS}$g8.7 and 53Mpc$_{g10.4}$ simulations than in the fiducial one; the gap in the distribution is smaller. This is due to the greater average distance between high mass sources. The space in between the large H II regions is neutral, without any ionized spots. Therefore, each individual H II region can grow bigger before merging.

Below global ionization fractions $(x) \sim 0.2$, the evolution of sizes in the 53Mpc$_{uvS}$g8.7 simulation is dominated by the first H II regions emerging around the highly efficient first sources. At higher global ionization fractions the size-evolution is very similar to simulation 53Mpc$_{g10.4}$, Therefore, we concentrate in the following on the 53Mpc$_{g10.4}$ simulation.

Compared to our fiducial simulation, the 3V/A estimate of simulation 53Mpc$_{g10.4}$ (middle panel of Fig. 15) suggests an average bubble scale about a factor 3 greater at all global ionization fractions we consider here. Also the spherical average distribution clearly shows this shift to larger scales. This is best visible when comparing the peak-scales (dot-dashed curves in the same panel). The power spectrum (Fig. 15 right panel), reveals that it is only a shift to bigger scales below global ionization fractions of about 20%. The H II regions that form first seem to be larger than the ones in the fiducial simulation. Later, it is rather a lack of small scales; notably the peak at scales $s \sim 0.8$ Mpc is absent. This is not surprising as we identified the peak to be due to the suppression of sources in low mass halos. The suppression is responsible for halting the growth of the H II regions formed by these sources completely. The peak at scales $s \sim 6$ Mpc is still there. However, there is more power on scales that are not captured by the 53 Mpc simulation volume.

The Euler Characteristic for simulation 53Mpc$_{g10.4}$, see light gray lines in the left panel of Fig. 18 is in total much flatter than the Euler Characteristic for the fiducial simulation. Since we saw in the FOF distribution and the power spectrum that there is not much contribution from scales that are hardly resolved, this is very unlikely to be due to unresolved scales. We therefore conclude that the bigger H II regions around rare sources result in a less complex topology with fewer tunnels and cavities. We saw above that feeding back diffuse photons into the volume, affects $V_3$ at high global ionization fractions. The fraction of those photons for simulation 53Mpc$_{g10.4}$ is already 0.2 at 80% global ionization fraction. Therefore, the evolution of $V_3$ beyond $(x) \sim 0.8$ might be dominated by the effect of these photons, as described in section 3.

It can be seen that $V_3$ for the low threshold value is at some points...
considerably different from the value at $x_{th} = 0.5$. However, since we plot the square root of $V_3$, the differences at values close to 0 are amplified. The actual difference is never greater than 20% of the maximum value.

5.2 Source suppression vs. low efficiency

To test the effect of source suppression in regions where the IGM is ionized, we compare our fiducial simulation (with instantaneous complete suppression of sources in low mass halos in ionized regions) to two simulations without suppression: 53Mpc_g8.7_130 which has the same source efficiencies but which ends much earlier due to the many more released photons and 53Mpc_g0.4_5.3 which has substantially lower source efficiencies to end at roughly the same time as the fiducial simulation.

The comparison of the FOF size distributions between model 53Mpc_g8.7_130 and 53Mpc_g0.4_5.3 (see left panels in Fig. 16 and Fig. 17) shows that the model with the lower source efficiencies shows more very small H II regions than the fiducial simulation, similar to 53Mpc_g1.7_8.7S. The simulation with the same source efficiencies as the fiducial simulation but without suppression shows less small H II regions than the fiducial simulation, because each individual source forming is active longer and so continuously grows its H II region. Also, clustered sources in the same or neighboring cells support the growth of their joined H II region.

The spherical average distribution, see middle panels of Fig. 16 and Fig. 17 for the higher and lower efficiency simulations without suppression, respectively, show a very similar evolution. The rate at which the average size grows seems to be the same, but the scale is shifted towards larger scales for the simulation 53Mpc_g8.7_130 by about a factor 1.5. Also, the $3V/A$ size estimates suggests an almost constant shift to larger scales.

The power spectra, see right panels of the same figures show more power on small scales (below $s \sim 0.5$) and less power on large scales (above $s \sim 6$) for 53Mpc_g0.4_5.3 than for simulation 53Mpc_g8.7_130. However, up to global ionization fraction $\langle x \rangle \sim 0.5$, the peak at scales around $s \sim 6$ Mpc is present in both simulations. The peak at scales $s \sim 0.8$ Mpc is absent in both simulations.

It should be noted that the size distributions found by all three methods as well as the $3V/A$ size estimator of the 53Mpc_g8.7_130 simulation are very similar to the ones from 53Mpc_g10.4_0. However, there is a very significant shift in time between these two simulations. The reason for their similarity if compared at equal $\langle x \rangle$ might be that most halos which first reach masses above $10^8 \, M_\odot$, reach masses above $10^9 \, M_\odot$ at accordingly lower redshifts. Therefore, the same halos hosting sources which turn on as sources in low mass halos in the former simulation, turn on as sources in more massive halos at a later time. This means that if suppression of
sources is not important, the sources "shaping" reionization are the ones in halos with the lowest mass that can form luminous sources.

The Euler Characteristic (left panel in Fig. 18) shows that $V_3$ never becomes negative for both, simulation 53Mpc$_g$0.4$_{5.3}$ (dark gray lines) and simulation 53Mpc$_g$8.7$_{130}$ (black lines): There are many more disconnected H II regions than neutral tunnels through the ionized regions. For a lower threshold value, $x_{th} = 0.3$, $V_3$ goes mildly negative. The high dependence of simulation 53Mpc$_g$8.7$_{130}$ on threshold value is somewhat surprising since the smallest H II regions are larger than for the fiducial simulation. However, the total number of H II and H I regions is small (only a few hundred compared to a few thousand for the fiducial simulation), therefore a few critical connections are enough to introduce a strong dependence of $V_3$ on threshold value. For the 53Mpc$_g$0.4$_{5.3}$ simulation it can further be seen that the maximum $V_3$ is about a factor of four larger (at low threshold value which means most probably overestimating the connections of regions) than for the fiducial simulation. There are more disconnected H II regions due to their smaller minimum sizes. Simulation 53Mpc$_g$8.7$_{130}$ reaches the same global ionization fraction as simulation 53Mpc$_g$0.4$_{5.3}$ at a much higher redshift, at a time where fewer halos have formed. The maximum $V_3$ is smaller for the 53Mpc$_g$8.7$_{130}$ simulation because the individual H II regions grow bigger and merge earlier in terms of global ionization fraction. However, the big differences between $V_3$ at different threshold values show that these simulations do not have sufficient resolution, therefore the ambiguity of connections is too high.

Fig. 18 (right panel) shows that the correlation between ionized fraction and overdensity at small global ionization fractions, here represented by $\langle x \rangle \sim 0.05$, is flatter for simulations without suppression, or simulations without any suppressible sources, like simulation 53Mpc$_g$10.4$_{0}$: on large scales, the correlation is lower than for simulations with suppression like our fiducial simulation;
on small scales, relatively larger. This behaviour can be interpreted together with the results from the Euler Characteristic in the following way: The lack of suppression leads to earlier break out of the ionizing radiation into the low density regions between low mass sources which reduces the number of neutral tunnels and lowers the cross-correlation on larger scales. At the same time, not suppressing sources in partly ionized regions leads to complete ionization of that region which increases the cross-correlation on smaller scales.

At higher global ionization fractions, here represented by \((x) \sim 0.5\), these differences disappear because almost all sources in low mass halos are suppressed in the simulations with suppression and the reionization process is dominated by the sources in high mass halos.

McQuinn et al. (2007) tested the effect of suppression for the case of equally efficient high and low-mass sources. However, their simulations do not resolve halos below \(10^8 \, M_\odot\). They use an analytic prescription to include unresolved halos above the H I atomic cooling mass. Among other things they found that even their most drastic (instantaneous) suppression model (complete suppression of sources in low mass halos) yields an ionization field with similar morphology to a simulation without suppression (their simulations F3 and S1, respectively). Additionally they tested a simulation with higher efficiencies for low mass sources but without suppression (their model S2). To realize this they chose a mass dependent source efficiency factor. Very roughly, this translates into \((10^8 / 10^9)^{-2/3} \approx 5\) times more efficient low \((10^9 > M/M_\odot \geq 10^8)\) than high \((M/M_\odot \geq 10^9)\) mass sources in terms of our step-function efficiency assignment method (our fiducial model has 15 more efficient sources in low mass halos). They found that the effect of boosting the efficiency of low mass sources on ionization field morphology is rather small. From these two tests (F-set vs. S1 and S1 vs. S2) they concluded that suppression, even in the case of high efficient low mass sources, does not affect the morphology of the ionization field. We can confirm that the ionization field morphology does not change much if the efficiency of low mass sources is boosted, as long as the total photon output of the least luminous sources that form the smallest HII regions is held roughly constant (compare for example 53 Mpc \(\times 10^4.0\) and 53 Mpc \(\times 8.7.130\). Fig. 15 and Fig. 16 respectively). However, comparing our fiducial simulation (with suppression) to simulations 53 Mpc \(\times 8.7.130\) and 53 Mpc \(\times 0.4.5.3\) (both without suppression), we find large differences in the topology and in the average size distributions of ionized regions, as discussed above. This is most probably due to the early break out of H II regions formed around sources in low-mass halos in the case of no suppression and high-low mass halo source efficiency.

6 CONCLUSIONS

We have explored different methods for characterizing the scales and topology of complex ionization fraction fields produced by simulations of cosmic reionization. For characterizing the length scales or sizes of HII regions, we used three methods that give a distribution of scales: the FOF method, the spherical average method and the power spectrum of the ionization fraction field. In addition we proposed a single valued measure of average size of HII region, given by the ratio of the volume to the surface of all regions. For characterizing the topology we employed the Euler Characteristic or third Minkowski functional, \(V_3\) of the ionization fraction field.

The nature of the size distribution of H II regions can be viewed to be a matter of definition. Applying a literal definition leads to the FOF approach, in which the HII regions are considered to be connected regions of space. Because the topology of reionization can be quite complex, as seen from the Euler characteristic, this definition does not lend itself easily to analytical modeling, and connecting the FOF size distribution to other scale estimators, such as the power spectrum, is by no means trivial. For the FOF method, what is lost in its complexity is gained in the detailed description of the reionization process that it provides. Although most of the volume is already at quite low global ionization fractions contained in one large connected region, there is a wealth of information contained in the number and sizes of the smaller bubbles, which only occupy a small fraction of the volume. As we found in section 3 the FOF size distribution is affected by resolution in terms of cell size and one has to take this into account when interpreting the results and comparing simulations with different resolution. In principle, the FOF method yields the maximum size for isolated ionized regions, before they start to merge into the largest one. However, in practice, this scale is dependent on the sampling of the distribution function and therefore on the size of the simulation volume.

The spherical average method, on the other hand, gives distributions which are much smoother and can more easily be connected to analytical models (Zahn et al. 2007). The results are more sensitive to the large scale HII regions which constitute the main contribution to the global ionization fraction. Due to its averaging nature, it washes out the details. Consider, for example, a collection of two types of spheres with scales \(s_1 = a\) and \(s_2 = b > a\); the spherical average method will only reveal the two distinct scales if \(a/b \leq 1/3\). Also, the spherical average alone tends to substantially underestimate the sizes of HII regions, as shown in Fig. A1 by our toy model. For using it as a size estimator on par with other methods, the spherical average scales should be multiplied by a factor 4.

When the universe is mostly neutral, the peak of the ionization power spectrum is related to the size distribution of ionized regions. For a single top-hat sphere, the first peak in the power spectrum is related to its radius by \(k_{\text{max}} = 2.46/r\) and we choose to use \(2.46/k\) as a size estimator when comparing to the results of other methods. The power spectrum produces size distributions roughly comparable to those from the spherical average method. However, the advantage of the power spectrum over the spherical average method is that it does not wash out the details of the distribution. As mentioned earlier, of all size estimators discussed in this paper, the power spectrum is the one most related to upcoming observations as one component in the expansion of the 21 cm power spectrum (e.g. Furlanetto et al. 2006).

The ratio of the zeroth and first Minkowski functionals can be used to define a mean radius of HII regions, the \(3V/A\) estimate. This estimate is a surface weighted average. It generally gives results consistent with the maximum of the spherical average, and when it falls below that there is a large fraction of small bubbles (which dominate the surface).

The Euler characteristic \(V_3\) of the ionization fraction field offers a rich description of the evolution of the topology of reionization. Taking our fiducial simulation as an example, in the early stages of reionization the value of \(V_3\) is positive as the topology is dominated by a large number of isolated HII regions. However, already beyond global ionization fractions of roughly 20\%, \(V_3\) becomes highly negative, indicating a complex topology of connected HII regions with tunnels. This is consistent with the distribution from the FOF method, which shows that already at \((x) \sim 0.3\) the main contribution to ionized fraction comes from only one large connected region which pervades most of the simulation volume.
As the ionization fronts around stellar sources are quite thin, ideally \( V_3 \) should not depend much on the chosen threshold value, and the most interesting aspect is the change of vertex fields, where the variation of \( V_3 \) with threshold value is used to characterize the field at a given time. However, in practice the resolution of the ionization field may not be sufficient to achieve sharp fronts, and partially ionized cells will occur. The result is different values of \( V_3 \) for different threshold values and even different evolutions of \( V_3 \) at different threshold values. More seriously is the interaction of this effect with the definition of connectivity/adjacency used when calculating the Euler Characteristic. We proposed a new test for establishing how robust the derived values of \( V_3 \) are to a change of adjacency. In this one compares the answer with the value obtained for a field and threshold value which have their sign inverted. Using this we showed that sub-sampling or smoothing is generally required to obtain consistent results, and that for fields with a large fraction of partially ionized cells, it can be difficult to get consistent results.

We subsequently applied the size estimators and Euler Characteristic to study differences and similarities between different reionization simulations. Comparing identical simulations in two different volume sizes, 163 Mpc and 53 Mpc, shows that below global ionization fractions of 30% the average scales of H II regions are roughly the same for both simulations. Beyond that the size distributions in the larger volume start to contain scales beyond those available in the smaller one. Another manifestation of this is that the largest connected region found by the FOF-method for both simulation volume sizes is 10% of the each of the simulation volumes already at 30% global ionization fraction. Therefore, this largest H II region is about a factor 30 larger in the large simulation volume: A region of this size does not fit into the 53 Mpc simulation volume.

Even earlier there are differences between the two FOF distributions showing an absence of H II regions with volumes of a few hundred Mpc\(^3\) in the small box. This is because relatively isolated overdensity regions (surrounded by larger voids) are missing in the small box due to numerical variance. Since the contribution from those scales is very small, the effect on the average scales is negligible. In general, simple size estimates may be biased due to missing scales (e.g. intermediate scales missing in the small box due to the smaller sampling volume, small scales missing in big box due to resolution). If one is not interested in very small scales, the lower resolution in the big box appears not to be a problem. The fact that the position of the peak of the power-spectrum at all times (global ionization fractions) is well below the box-size scale for the 163 Mpc box, indicates that no larger simulation boxes are needed to follow the evolution of the peak position.

Comparing simulations with and without suppression (either with only high mass sources, or with sources in low mass halos not suppressed in ionized regions) shows that the ones without suppression typically have a much less complex ionization fraction field topology and a more steady growth of average H II region sizes. We thus find significant differences between the cases with and without suppression. For the simulations with sources only in high mass halos, this can be explained as follows: the individual H II regions can grow much bigger before merging, due to the larger average distances between high-mass sources. For the simulation without suppression and high efficiency sources a similar argument holds, but the whole process takes place at much earlier redshifts. For the simulation without suppression and with low efficiency sources the explanation is again similar, but now applied to smaller scales instead of at earlier times. This similarity between the cases without suppression is due to the statistical nature of the density field. If suppression is not important, the sources in halos with the lowest mass still capable of forming sources are the ones shaping reionization.

By imposing an external photon-budget on an independently evolved source population, one can in principle separate the effect of the source population from that of the reionization history. However, by necessity this implies an evolution of the source efficiencies, which in the case we studied seriously affected the evolution sizes of HII regions. Up to a global ionization fraction \( (x) \sim 0.2 \) the size distribution is dominated by the size of H II regions generated by the first generation of sources. In the later stages of reionization, the morphology of the ionization field is very similar to a simulation with a fixed source efficiency for high mass sources.

As outlined in the introduction, we concentrated on the analysis of the ionization fraction fields and its evolution. Applying the various analysis methods to the future observations of the redshifted 21cm signal is in principle possible, but requires sufficient sensitivity to image the signal at different frequencies. The first generation of telescopes is not expected to be able to do this, but the planned Square Kilometer Array (SKA) should be. Compared to the simulation results, the observations will have the additional complications of noise and limited spatial resolution (below even that of our 163 Mpc simulation). As we have shown, resolution effects should be treated with care, especially for the Euler Characteristic, and noise peaks can obviously also bias the topology determination. Still, characterizing the morphology of HII regions in the data will be important as they trace the mass and thus the emerging cosmic web. We leave the application of the various size/scale and topology estimates to mock observational data to a future paper.

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REFERENCES

Alvarez M. A., Abel T., 2010, ArXiv e-prints (arXiv:1003.6132)
Alvarez M. A., Busha M., Abel T., Wechsler R. H., 2009, ApJ, 703, L167
Choudhury T. R., Haehnelt M. G., Regan J., 2009, MNRAS, 394, 960
Ciardi B., Scannapieco E., Stoehr F., Ferrara A., Iliev I. T., Shapiro P. R., 2006, MNRAS, 366, 689

http://www.skatelescope.org
APPENDIX A: SPHERICAL AVERAGE SIZE DISTRIBUTION OF A LOG NORMAL DISTRIBUTION

The spherical average method was described by [Zahn et al.] [2007]. In this technique, each cell in the computational volume is considered to be in an ionized region if a sphere centered on that cell has a mean ionized fraction greater than a given threshold, usually \( x_{th} = 0.9 \). The size of the H II region to which it belongs is taken to be the largest such sphere for which the condition is met.

In the following, we assume that gas is either fully ionized or fully neutral, and that all ionized bubbles are non-overlapping spheres with a volume-weighted distribution \( dP/dR \), so that \( P(R + dR) - P(R) \) is the fraction of the ionized volume that lies within bubbles with radii between \( R \) and \( R + dR \). What bubble distribution, \( dP_{sm}/dR \), would be obtained by using the spherical average method? To simplify further, we will take the threshold for the spherical average, \( x_{th} \), to be arbitrarily close to unity, so that a point is considered to be within an ionized sphere of a given radius only if all the matter in that sphere is ionized. For a single ionized sphere of radius \( r \), \( dP_{sm}(R)/dR \) is the surface of a sphere with radius \( r - R \), normalised by the integral of the surface of spheres with radii from 0 to \( r \); \( dP_{sm}(R)/dR = 3(r - R)^2/r^3 \). If we define this as \( W(r, R) \), then

\[
\frac{dP_{sm}}{dR} = \int_{R}^{\infty} drW(r, R) \frac{dP}{dr},
\]

is the bubble size distribution obtained by the spherical average method for the real distribution \( dP/dR \). The lower limit of the integral is \( R \) because only spheres which are larger than \( R \) can contribute to the spherical average bubble distribution at \( R \) (because \( x_{th} \to 1 \); the largest ionized sphere that can be drawn around any given point is always smaller than or equal in radius to the actual ionized sphere in which it lies.

Shown in Fig. [A1] are \( rdP/dr \) and the corresponding \( RdP_{sm}/dR \) for three log-normal distributions of bubble sizes

\[
\frac{dP}{d\ln r} = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[ -\frac{(\ln(r) - \ln(r^*)^2}{2\sigma^2} \right],
\]

with different \( \sigma \). As can be seen from the figure, the spherical average tends to change the true bubble distribution in two ways.
First, it smooths the actual bubble distribution with the kernel function $w(r, R, \sigma)$. Second, it lowers the value of the mean bubble radius, $R_{av} = \int R dP/dR$. In our simple toy model, the mean bubble size obtained by the spherical average method is always $1/4$ of the actual one. Our toy model is admittedly crude, most notably in the assumption of a threshold $x_{th} = 1$ and spherical H II regions. A lower value of $x_{th}$ would allow small pockets of neutral gas to be attributed to large ionized regions. In fact, for the case where $x = 1$ and $x = 0$ in ionized and neutral regions, respectively, lower values of $x_{th}$ lead to an overestimate of the volume which is ionized, leading to a violation of the normalization condition,

$$\int_0^\infty R dP/dR = x_v.$$  \hfill (A3)

In most cases this overestimate is not very large. A lower value of $x_{th}$ would also yield larger H II regions, but this effect has been shown to be rather modest \cite{Zahn2007}. The assumption of spherical symmetry is a conservative one, however, in the sense that it provides a lower limit to how much the spherical average method underestimates the “true” H II regions sizes. This is because the spherical average method is sensitive to the smallest dimension of the region in which it lies: the radius is larger than the smallest dimension, the part of the sphere lying in that direction would lie outside the region, and the average ionized fraction would no longer be above the threshold for the region to be considered ionized.

**APPENDIX B: ADDITIONAL SIZE MEASURE**

Mesinger & Furlanetto (2007) used another technique to characterize sizes of ionized (and neutral) regions in their binary ionization fields: They chose a large number of random points; for each point they check if it is ionized (neutral); from each ionized (neutral) point they chose a random direction and measure the distance from this point to the nearest ionized-neutral (neutral-ionized) transition boundary along that line of sight. In the following, we use a method similar to the one in \cite{Mesinger2007}. However, instead of choosing a random direction, we, for simplicity, only choose between the 3 principle axes in one of the two possible orientations. The differences in size distribution obtained with this method only matters for very large and rare H II regions. Small regions should be abundant enough that the orientation should not play a dominant role. Since we have to deal with continuous ionization fields, we have to introduce two parameters: Above which ionization fraction is a random point ionized? What is the limit for the transition boundary along the line of sight? In the following, we consider a point as ionized if its ionization fraction is above $x_{th}$; we count each point along the line of sight ionized as long as its ionization fraction is above $x_{lim}$. In Fig[31] we plot the size distribution curves obtained with this method for the two simulations we use in Section[3]: 53 $\text{M}_{\odot}c_{g}0.7_{\gamma}130S$ (our fiducial simulation, solid lines) and 53 $\text{M}_{\odot}c_{g}1.7_{\gamma}8.7S$ (dashed lines) at two different global ionization fractions, ($x$) $\sim 0.1$ (thin lines) and ($x$) $\sim 0.4$ (thick lines). Qualitatively, we find the same result as with the other size measures: Initially the size distribution peaks earlier for 53 $\text{M}_{\odot}c_{g}1.7_{\gamma}8.7S$; at higher ($x$), the typical size seems to be larger in this simulation than for the fiducial simulation.

With respect to the peak position of the (uncorrected) SPA, especially at low global ionization fraction, the peak of this size distribution is shifted towards larger scales. This is consistent with the findings in appendix A and the fact that the size distribution of a single sphere, found with this method, would peak at the radius of the sphere; therefore, this method yields in theory better results than the SPA. However, the disadvantage is the dependence on two parameters if the field of investigation is not a binary field. We tested the effect of these parameters (compare different colours in Fig[31]) and find large variations for different parameter combinations.

**APPENDIX C: ON THE CALCULATION OF THE EULER CHARACTERISTIC**

The estimation of $V_3$ of a three dimensional field sampled on a finite set of grid points is dependent on the chosen adjacency-pair, if the structure is of the same order as the cell sizes of the grid. An adjacency pair is for example $(26,6)$ which means that cells above the threshold (foreground cells) have 26 neighbours and cells below the threshold (background cells) have 6 neighbours. Because of this dependence on the choice of adjacency pair, we choose to oversample the data to avoid differences in the treatment of isolated/connected H II and isolated/connected H I regions. This introduces a stronger dependency on the choice of the threshold value.

For calculating the Euler Characteristic $V_3$ we use part of a program developed by T. Buchert and J. Schmalzing @Schmalzing1997. The algorithm we use counts the vertices ($V$), edges ($E$), faces ($F$) and lattice cells ($C$) of the foreground cells and calculates $V_3$ according to equation (10) in @Schmalzing1997:

$$V_3 = V - E + F - C$$ \hfill (C1)

In terms of adjacencies, this is equivalent to assigning 26 neighbor cells to any foreground cell and 6 neighbor cells to any background cell. This adjacency pair ensures the 3 dimensional Jordan curve theorem (which basically means that an edge-connection of foreground cells cannot at the same time be a connection for background cells). Ohser et al. @Ohser2002 showed that this adjacency pair
is complementary which means that \( V_3 \) of the background is the same as that of the foreground. If the structure in the data-cube has contributions smaller or of the same size than is sampled by the grid-cells, structure of lower dimensions than 3 can arise. This can cause inconsistencies in the approximation of \( V_3 \) of the set sampled at the grid-points resulting in a violation of the complementarity. This is for example visible in the \((V_2, \delta_{th})\) plot in Fig. 2. The first peak is mainly due to disconnected under-dense regions which are below the density threshold value \( \delta_{th} \). If two of those regions are connected via an edge or a vertex, they count as two disconnected “cavities” since those cells are background cells and have only 6 neighbours. The second peak is somewhat smaller than the first one (although theoretically the peaks should be equal). This peak is mostly due to disconnected over-dense regions that are above the threshold value \( \delta_{th} \) and therefore have 26 neighbours each. Over-densities that are connected via an edge or a vertex count as one connected region, therefore their contribution to \( V_3 \) is smaller.

Osher et al. (2002) showed that the bias of the approximation depends on the choice of adjacencies. Inverting the data cube of ionization fraction, so that at every grid point, \( x' = -x \), computing \( V_3(x'_{th}) = -V_3(x_{th}) \), and comparing \( V_3(x'_{th}) \) to \( V_3(x_{th}) \), is equivalent to changing the adjacencies from \((26/6)\) to \((6/26)\).

For example, consider a 3\(^3\) cube \( C_1 \), with \( C_1(1,1,2) = C_1(2,2,1) = 1 \) and 0 everywhere else and a cube \( C_2 = -C_1 \). Then, \( V_3(C_1, x_{th}) \neq V_3(C_2, -x_{th}) \); \( V_3(C_1, x_{th} = l) = 14 - 23 + 12 - 2 = 1 \), where \( l \in (0, 1) \) (round brackets denoting open intervals) while \( V_3(C_2, x_{th} = l) = (\theta - \xi)(\zeta - (\varphi - 2)) = 2 \), where \( \theta, \xi, \zeta \) and \( \varphi \) are the numbers of vertices, edges, faces and lattice cells of the cube \( C(1 : 3, 1 : 3, 1 : 3) = 0 \), the sum of which is 0 since periodicity is assumed.

Therefore, we use \( V_3(-x, -x_{th}) \) as a check for how good the approximation of \( V_3 \) is. To minimize the asymmetry due to the chosen adjacency, we oversample the data. For the example of the cubes \( C_1 \) and \( C_2 \) from above, this means that \( V_3(C_1, x_{th} = l) = V_3(C_2, x_{th} = -l) = 1 \) for \( l \in (0, 0.5) \) and \( V_3(C_1, x_{th} = l) = V_3(C_2, x_{th} = -l) = 2 \) for \( l \in [0.5, 1) \). Obviously this results in a higher dependency on the chosen threshold value, as this simple example demonstrates, see also Fig. C.1 for a two-dimensional example.

As a second example, we present here the Euler Characteristic of the fiducial simulation and simulation 53Mpc.g1.7.8.7S to demonstrate the effect of oversampling and the choice of threshold value. As explained above, calculating \( V_3(-x_{th}) \) of the inverted field (the data field multiplied by \(-1\), in the following indicated by a “-“), is equivalent to changing foreground to background cells and vice versa. This is the same as changing the adjacencies. Therefore, comparing \( V_3(+x_{th}) \) to \( V_3(-, -x_{th}) \) is a test for the dependence on adjacencies.

Fig. C.2 demonstrates the dependence on the chosen adjacency pair when analyzing the data-fields without smoothing or oversampling (compare \( V_3(+, x_{th} = 0.5) \) to \( V_3(-, x_{th} = -0.5) \) for both simulations) and the lower dependence on the chosen adjacencies after oversampling the data (compare \( V_3^{os}(+, x_{th} = 0.5) \) to \( V_3^{os}(-, x_{th} = -0.5) \)). It also demonstrates the relatively low dependence on threshold value \( x_{th} \) for the fiducial simulation and the high dependence on \( x_{th} \) for 53Mpc.g1.7.8.7S (compare \( V_3^{os}(+, x_{th} = 0.5) \) to \( V_3^{os}(+, x_{th} = 0.3) \)).

Others (e.g. Gleser et al. 2006) chose to smooth the data with a gaussian kernel to remove lower-dimensional parts. This has the disadvantage that some of the small scale structures are suppressed.

Figure C.1. Two-dimensional example to demonstrate the effect of oversampling the data: original structure (left panel) compared to the result from oversampling (right panel). While the original structure counts as one connected region independent of threshold value \( (x_{th} \in (0, 1)) \), the structure in the right panel counts as two disconnected regions if \( |x_{th}| > 0.5 \), and as one connected region otherwise. If the entries would be inverted (multiplied by \(-1\)), the structure in the left panel would count as two disconnected cavities while the structure in the right panel would have the same dependence on the threshold value as before: the cavities count as disconnected if \( |x_{th}| > 0.5 \), and as one connected cavity otherwise.
Figure C2. Evolution of Euler Characteristic $V_3$ for the original field of the fiducial simulation (lower gray thick line) and simulation 53Mpc g1.7–8.7S (lower black thick line) at threshold value $x_{th} = 0.5$. The upper thick lines are the corresponding Euler Characteristic of the inversed fields at threshold value $x_{th} = −0.5$. The thin lines correspond to the oversampled fields of the fiducial simulation (lower gray thin line) and simulation 53Mpc g1.7–8.7S (lower black thin line) at threshold value $x_{th} = 0.5$. The upper thin lines are the corresponding Euler Characteristic of the inversed oversampled fields at threshold value $x_{th} = −0.5$. The gray (fiducial simulation) and black (53Mpc g1.7–8.7S) lines with crosses indicate the Euler Characteristic of the oversampled fields at threshold value $x_{th} = 0.3$. Note the good agreement for $V_{3}^{os}(x_{th} = 0.5)$ and $V_{3}^{os}(-x_{th} = −0.5)$ and the low dependence on threshold value for the fiducial simulation.