Dynamics of an infection-age model with staged-progression

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Abstract. We introduce and analyze an age-structured model with staged-progression. The results reveal that the disease-free equilibrium is globally stable if the basic reproductive number $R_0 < 1$, and the disease is uniformly persistent for $R_0 > 1$. Numerical simulations are performed based on hepatitis C, illustrating the effects of infection age on dynamical process.

1. Introduction
Since the first compartmental model was formulated to study epidemiology by Kermack and McKendrick [1], extensive mathematical models have been developed to explore the mechanisms and dynamics of infectious diseases [2-4]. Considering the heterogeneity in infectious progression, epidemic models with various structures, such as multistage structure [5-6] and age structure [7-8], have been introduced and studied. The interest in the role played by variable infectivity in disease transmission has been considerably increased by some epidemics, such as HIV/AIDS and hepatitis B [8-10]. When modeling the dynamic process of these diseases, the infectivity is allowed to depend on infection-age, which represents the time that has passed since the moment of infection. In recent years, global stability of age-structure models is extensively studied [7-9]. Variability of infectiousness in time has been described by staged progression models. For some specific diseases, the infectious progression consists of two stages, i.e., acute and chronic. In this paper, we consider an epidemic model that includes infection age structure in two successive infectious stages.

2. Model formulation
We consider an infection-age model with two infectious states as following

\[
S'(t) = \Lambda - \left( \int_0^\infty \beta_1(a)i(a,t)da + \int_0^\infty \beta_2(a)e(a,t)da \right)S(t) - \mu S(t) + \int_0^\infty (1 - q(a))\gamma(a)i(a,t)da + \int_0^\infty \alpha(a)c(a,t)da,
\]

\[
E'(t) = \left( \int_0^\infty \beta_1(a)i(a,t)da + \int_0^\infty \beta_2(a)e(a,t)da \right)S(t) - (\mu + \varepsilon)E(t),
\]

\[
\frac{\partial i(a,t)}{\partial a} + \frac{\partial i(a,t)}{\partial t} = -(\mu + \gamma(a))i(a,t),
\]

\[
\frac{\partial c(a,t)}{\partial a} + \frac{\partial c(a,t)}{\partial t} = -(\mu + \alpha(a) + \theta(a))c(a,t), \quad i(0,t) = \varepsilon E(t), \quad c(0,t) = \int_0^a q(a)\gamma(a)i(a,t)da,
\]

with $S(0) = S_0$, $E(0) = E_0$, $i(a,0) = i_0(a)$, $c(a,0) = c_0(a)$. The population is decomposed into four classes, including susceptible, exposed, acute and chronic infection. Let $S(t)$ and $E(t)$ denote the number of individuals in susceptible and exposed compartments at time $t$. The two infectious classes are structured by infection age $a$, which is the time that has elapsed since the infection began, and $i(a,t)$ and
c(a,t) denote the density of acute and chronic infected individuals with respect to a. \( a_i \) is the critical infection age when acute individuals progress through it and move to the chronic state.

In model (1), \( \Lambda \) is the constant flux into susceptible class and \( \mu \) is the death rate. \( \epsilon \) is the transfer rate from the exposed to the acute infection. \( \beta(a), \beta_2(a) \) denote the probabilities of infectiousness as the disease progresses within acute and chronic infection respectively. Individuals who have stayed in the acute stage for duration \( a \) can move out at a rate \( \gamma(a) \) to chronic state with probability \( q(a)\gamma(a) \) or back to the susceptible again with probability \( (1-q(a))\gamma(a) \), where \( q(a) \) denotes the rate of infectious population who progress through acute stage and move to chronic stage. The chronic individuals move back to the the susceptible at \( a_0(a) \) and experience an induced death rate \( \theta(a) \).

We assume that the age-dependent parameters in model (1) are non-negative, bounded and integrable in their defined intervals. The other parameters and the initial conditions are supposed to be non-negative and \( i_0(a), c_0(a) \in L^1_1(0, \infty) \). Introduce the following notations

\[
\begin{align*}
\pi_i(a) &= e^{-\int_{a}^{a_i} \mu \gamma(s) \, ds}, \quad a \in [0, a_i), \\
\pi_2(a) &= e^{-\int_{a_i}^{\infty} \mu \gamma(s) \, ds}, \quad a \in [a_i, \infty), \\
W_1 &= \int_{0}^{a} \beta(a) \pi_i(a) \, da, \\
W_2 &= \int_{a_i}^{a} \beta_2(a) \pi_2(a) \, da, \\
W_3 &= \int_{a}^{\infty} \alpha(a) \pi_2(a) \, da, \\
W_4 &= \int_{0}^{a} \alpha(a) \pi_i(a) \, da, \\
W_5 &= \int_{a_i}^{a} \alpha(a) \pi_2(a) \, da, \\
\tilde{W}_1(a) &= \int_{0}^{a} e^{-\mu \gamma(s) \alpha(s)} \, ds, \\
\tilde{W}_2(a) &= \int_{a_i}^{a} e^{-\mu \gamma(s) \alpha(s)} \, ds, \\
\tilde{W}_3(a) &= \int_{a}^{\infty} \alpha(a) \pi_2(a) \, da, \\
\tilde{W}_4(a) &= \int_{a}^{\infty} \alpha(a) \pi_2(a) \, da, \\
\tilde{W}_5(a) &= \int_{a}^{\infty} e^{-\mu \gamma(s) \alpha(s)} \, ds.
\end{align*}
\]

For \( \pi_i(a) \) and \( \pi_2(a) \), they denote the survival probabilities. The infected possibility can be expressed as \( W_i \) or \( W_2 \). \( W_3 \) and \( W_5 \) give the total number of hosts that leaving from acute class. \( W_4 \) denotes the infections that recover and become susceptible again. Integrate \( i(a,t) \) and \( c(a,t) \) in (1) along characteristic curves yields

\[
i(a,t) = \begin{cases} 
\int_{0}^{a} \pi_i(a) \, da, & t > a, \\
i_i(a-t), & t \leq a, \\
(\pi_i(a-t) / \pi_i(a), & t = a,
\end{cases} \quad a \in [0, a_i) \tag{2}
\]

\[
c(a,t) = \begin{cases} 
c(a, t-a) \pi_2(a), & t + a > a, \\
c_i(a) - a + t \pi_2(a), & a + t \leq a, \\
c_i(a) \pi_2(a) \pi_2(a), & t + a \leq a, \\
c_i(a-a) \pi_2(a), & t + a > a,
\end{cases} \quad a \in [a_i, \infty) \tag{3}
\]

By (2) and (3), it can be seen that \( i(a,t) \) and \( c(a,t) \) remain non-negative for any non-negative initial value. Further, by classical existence and uniqueness results for functional differential equations, we know that system (1) has a unique solution. For all \( t \geq 0 \) and all non-negative initial values, the solutions are non-negative and all the solutions of (1) are uniformly bounded.

3. Steady states and stability

By the next generation matrix \[11\], the basic reproductive number \( R_0 \) is defined as

\[
R_0 = \frac{\Lambda \epsilon (W_1 + W_2 W_3)}{\mu (\mu + \epsilon)}.
\]

There always exists a disease-free equilibrium \( R_0 = (S_0, 0, 0) \), where \( S_0 = \sqrt{\Lambda \mu} \), and there may exist a positive steady state \( P^* = (S^*, E^*, i^*(a), c^*(a)) \). Noticing that \( (\mu + \epsilon) > \epsilon (W_1 W_4 + W_2) \) is always valid, thus it is obvious that the unique endemic equilibrium \( P^* \) is feasible if and only if \( R_0 > 1 \), and

\[
\begin{align*}
S^* &= \frac{S_0}{R_0}, \\
E^* &= \frac{\Lambda (1 - \frac{1}{R_0})}{(\mu + \epsilon) - \epsilon (W_1 W_4 + W_2)}, \\
i^*(a) &= \epsilon E^* \pi_i(a), \\
c^*(a) &= \epsilon E^* W_5 \pi_2(a).
\end{align*}
\]
Theorem 1. If $R_0 < 1$, the disease-free equilibrium $P_0$ is globally stable, and it is unstable if $R_0 > 1$.

Proof. Assume $S(t) = s(t) + S_0$, $E(t) = e(t)$, $i(a,t) = u_i(a,t)$, $c(a,t) = u_c(a,t)$ and the linear system of (1) at $P_0$ has exponential solutions of $s(t) = e^{\lambda t}$, $e(t) = e^{\lambda t}$, $u_i(a,t) = e^{\lambda t}u_i(a)$, $u_c(a,t) = e^{\lambda t}u_c(a)$. By the boundary conditions, we have $u_i(0,t) = \beta_i(0)e^{\lambda t}$, $u_c(a,t) = \beta_c(a)e^{\lambda t}$, $u_c(a,t) = \beta_c(a)e^{\lambda t} = \int_0^a q(a)\gamma(a)\beta_i(a)\lambda e^{\lambda t}da$.

Substituting $u_i(a,t)$ and $u_c(a,t)$ into their equations and integrating about age, we have $u_i(z,t) = \beta_i(0)\hat{W}_i(z)$.

Solving $e(t)$ and substituting $\hat{W}_i(a)$, $\hat{W}_c(a)$ lead to the characteristic equation

$$1 = \frac{\lambda s_0 e^{\lambda t} \hat{W}_i(z) + \hat{W}_c(z)\hat{W}_i(z)}{z + \mu + \epsilon}, \quad (4)$$

where $z$ denotes an eigenvalue. If $z=0$, it is easy to see that (4) becomes $10 = 10$.

Denoting $f^*$ as $\lim_{t \to \infty} f(t)$. By (5) we have $c^*(a) = e^{E^*}W_i$, and then $E^* = 0$ when $R_0 < 1$, which implies that $\lim_{t \to \infty} i(t(a)) = 0$ and $\lim_{t \to \infty} c(a(t)) = 0$. Therefore, $\lim_{t \to \infty} S(t) = S_0$ follows.

Theorem 2. When $R_0 > 1$, the endemic equilibrium $P^*$ is locally stable.

Proof. Take $S(t) = s(t) + S^*$, $E(t) = e(t) + E^*$, $i(a,t) = u_i(a,t) + i^*_i(a)$, $c(a,t) = u_c(a,t) + i^*_c(a)$, and $s(t) = e^{\lambda t}S^*$, $e(t) = e^{\lambda t}E^*$, $u_i(a,t) = e^{\lambda t}\hat{W}_i(a)$, $u_c(a,t) = e^{\lambda t}\hat{W}_c(a)$. Substituting $i(a,t)$ and $c(a,t)$ into their equations leads to

$$i^*(a) = e^{\lambda t}\hat{W}_i(a), \quad \hat{W}_i(a) = e^{\lambda t}\int_0^a (z + \mu + \epsilon)\hat{W}_i(z)dz, \quad (5)$$

where $z$ denotes an eigenvalue. If $z=0$, it is easy to see that (4) becomes $10 = 10$.

Denoting $f^*$ as $\lim_{t \to \infty} f(t)$. By (5) we have $c^*(a) = e^{E^*}W_i$, and then $E^* = 0$ when $R_0 < 1$, which implies that $\lim_{t \to \infty} i(t(a)) = 0$ and $\lim_{t \to \infty} c(a(t)) = 0$. Therefore, $\lim_{t \to \infty} S(t) = S_0$ follows.
which is a contradiction, so (6) has eigenvalues only with negative real part.

**Remark.** When \( q(a) = 0, \alpha(a) = 0 \), global stability of \( P' \) can be proved by using a Lyapunov function as \( \mathcal{L}(t) = F(S(t), S'(a)) + F(E(t), E'(a)) + \int_a^b \varphi_1(a)F(i(a,t), i'(a,t))\,da + \int_a^b \varphi_2(a)F(c(a,t), c'(a,t))\,da \), with

\[
F(x, x') = x - x' - x' \ln \frac{x}{x'}, \quad \varphi_1(a) = \int_a^b [\beta_1(s)S'(a) + \varphi_2(a, i(s))\gamma(s)]\frac{\pi_1(s)}{\pi_1(a)}\,ds, \quad \varphi_2(a) = \int_a^b \beta_2(s)x'\frac{\pi_2(s)}{\pi_2(a)}\,ds.
\]

4. Uniform persistence

By reformulating system (1) as an abstract Cauchy problem, we establish the uniform persistence for (1) when \( R_0 > 1 \). Denote \( u = (S, E, i, c) \in \bar{\mathcal{X}} = R^2 \times L^1((0, +\infty), R^2) \) and introduce the notations

\[
\mathcal{X}_1 = R_1^2 \times L^1((0, +\infty), R^2), \quad \mathcal{X}_2 = \bar{\mathcal{X}} \times R^2, \quad \mathcal{X}_3 = \bar{\mathcal{X}} \times R^2, \quad \mathcal{X}_4 = \bar{\mathcal{X}} \times [0 \times \emptyset], \quad \mathcal{X}_5 = \mathcal{X}_4 \cap \mathcal{X}_3.
\]

For any \( u = (u_1, u_2, u_3, u_4)^T \in \mathcal{X}_5 \), define operator \( A \) and operator \( F \) as

\[
Av = \begin{pmatrix}
-\mu u_1 \\
-(\mu + \varepsilon) u_2 \\
-\left(\frac{\partial}{\partial a} + \mu + \rho(a)\right) u_3 \\
-\left(\frac{\partial}{\partial a} + \mu + \theta(a)\right) u_4 \\
-u_1(0) \\
u_2(0)
\end{pmatrix}, \quad F(v) = \begin{pmatrix}
\Lambda - \sum \int_0^a \beta_1(u_1(a,t))\,da + \int_0^a \beta_2(u_1(a,t))\,da \\
0 \\
0 \\
0 \\
\int_0^a q(a)\gamma(a)u_3(a,t)\,da + \int_0^a c(a)\,da
\end{pmatrix}
\]

with \( \Sigma = \int_0^a \beta_1(u_1(a,t))\,da + \int_0^a \beta_2(u_1(a,t))\,da + \int_0^a (1-q(a))\gamma(a)u_3(a,t)\,da + \int_0^a c(a)\,da \). Then we deal with system (1) as the abstract Cauchy problem

\[
\frac{dv(t)}{dt} = Av + F(v), \quad v(0) = v_0 \in \mathcal{X}_5,
\]

for any \( t > 0 \). By Theorems in [12-14], there exists a unique solution semiflow \( U(t) : \mathcal{X}_5 \to \mathcal{X}_5 \) defined by (7). Furthermore, \( U(t) \) is compact for any \( t > 0 \) and has a compact global attractor in \( \mathcal{X}_5 \). Let \( \mathcal{M} = \{(i, c) \in \mathcal{L}^1((0, +\infty), R^2) \} \), \( \mathcal{M} = \bar{\mathcal{X}} \times \{0 \times \emptyset \} \cap \mathcal{M} \) is positively invariant under the semiflow \( \{U(t)\}_{t \geq 0} \). Moreover, \( U(t) \xrightarrow{t \to \infty} \mathcal{X} = (\bar{\mathcal{X}}, 0, 0, 0, 0, 0) \) for each \( x \in \partial \mathcal{M} \) as \( t \) is large enough. By the results in [7], we obtain the following theorem.

**Theorem 3.** If \( R_0 > 1 \), the solution semiflow \( \{U(t)\}_{t \geq 0} \) is uniformly persistent with respect to \( (\mathcal{M}, \partial \mathcal{M}) \), i.e., there exists \( \eta > 0 \) such that the solution of system (1) with initial value \( (S_0, E_0, i_0, c_0, 0, 0) \in \mathcal{M} \) satisfies \( \lim_{t \to \infty} E(t) = \eta, \lim_{t \to \infty} \|i(t)\| > \eta, \lim_{t \to \infty} \|c(t)\| > \eta, \lim_{t \to \infty} \|E(t)\| > \eta \). 

**Proof.** We will study the behavior of the solutions starting in \( \mathcal{M} \) in some neighborhood of \( \mathcal{X} \). Then we verify the case that for any \( \delta > 0 \) and \( x = (S_0, E_0, i_0, c_0, 0, 0) \in \mathcal{M} \), there exists \( t_0 \geq 0 \) such that \( \|\mathcal{X} - U(t_0)x\| < \delta \), i.e., \( W^i(\mathcal{X}) \cap \mathcal{M} = \emptyset \). By contradiction, supposing that for any \( t \geq 0 \) and each integer \( n \geq 0 \), we can find

\[
x_n = (S^n_0, E^n_0, i^n_0, c^n_0, 0, 0) \in \mathcal{M} : \|\mathcal{X} - y\| \leq \frac{1}{n + 1},
\]

satisfying \( \|\mathcal{X} - U(t)x_n\| \leq \frac{1}{n + 1} \). Let \( E^n(t) = (S^n(t), E^n(t), i^n(t), a^n(t), c^n(t), 0, 0) \), then \( S^n(t) - S_n(t) \leq \frac{1}{n + 1} \). Consider

\[
E^n(t) = \left(\int_0^a \beta_1(u_1(a,t))\,da + \int_0^a \beta_2(u_1(a,t))\,da\right)S^n(t) - (\mu + \varepsilon)E^n(t),
\]

\[
\frac{\partial i^n(a,t)}{\partial a} + \frac{\partial i^n(a,t)}{\partial t} = -\left(\mu + \rho(a)\right)i^n(a,t), \quad \frac{\partial c^n(a,t)}{\partial a} + \frac{\partial c^n(a,t)}{\partial t} = -\left(\mu + \alpha(a) + \theta(a)\right)c^n(a,t).
\]
\[ \dot{i}^*(0, t) = \partial E^*(t), \quad c^*(a, t) = \int_0^a q(a)\gamma(a)\Phi(a, t)\partial a, \quad E^*(0) = E_0^*, \quad \dot{i}^*(., 0) = i_0^*, \quad c^*(., 0) = c_0^* \quad (i_0^*, c_0^*) \in \hat{M}. \]

By the comparison principle, we deduce that
\[ E^*(t) \geq \tilde{E}^*(t), \quad \dot{i}^*(., t) \geq \tilde{i}^*(., t), \quad c^*(., t) \geq \tilde{c}^*(., t), \]
where \((\tilde{E}^*(t), \tilde{i}^*(., t), \tilde{c}^*(., t))\) is a solution of the following system
\[ \begin{align*}
\dot{\tilde{E}}^*(t) &= \int_0^a \beta_1(a)\tilde{E}^*(a, t)\partial a + \int_0^a \beta_2(a)\tilde{c}^*(a, t)\partial a \left(\tilde{S}_0 - \frac{1}{n+1}\right) - (\mu + \varepsilon)\tilde{E}^*(t), \\
\dot{\tilde{i}}^*(a, t) &= -\left(\mu + \gamma(a)\right)\tilde{i}^*(a, t), \\
\dot{\tilde{c}}^*(a, t) &= -\left(\mu + \alpha(a) + \theta(a)\right)\tilde{c}^*(a, t),
\end{align*} \]
\[ \tilde{i}^*(0, t) = \varepsilon \tilde{E}^*(t), \quad \tilde{c}^*(a, t) = \int_0^a q(a)\gamma(a)\tilde{E}^*(a, t)\partial a, \quad \tilde{E}^*(0) = E_0^*, \quad \tilde{i}^*(., 0) = i_0^*, \quad \tilde{c}^*(., 0) = c_0^* \quad (i_0^*, c_0^*) \in \hat{M}. \]

We deduce that for all \(n \geq 0 \) large enough, the dominant eigenvalue of system (9) satisfies
\[ \frac{\dot{G}(z_n^*)}{G(z_n^*)} = \frac{W(z_n) + W(z_n)W(z_n)}{z_n + \mu + \varepsilon} = 1, \quad (10) \]

Since \( G(z_n) \) has the similar property as \( G(z) \), it follows that (10) has a unique real root \( z_n^* > 0 \) for all \( n \) large enough. Denote \( w(t) = (\tilde{E}^*(t), \tilde{i}^*(., t), \tilde{c}^*(., t)) \) and \( \Delta(z_n) = 1 - G(z_n) \). Define \( \frac{dw(t)}{dt} = Bw(t) \) with
\[ Bw(t) = \begin{pmatrix}
-\left(\mu + \gamma(a)\right)\tilde{i}^*(t) \\
-\frac{d\tilde{E}^*(t)}{da} - (\mu + \gamma(a))\tilde{i}^*(t) \\
-\frac{d\tilde{c}^*(t)}{da} - (\mu + \alpha(a) + \theta(a))\tilde{c}^*(t)
\end{pmatrix}, \]
\[ \tilde{E}^*(0) = \left(\tilde{S}_0 - \frac{1}{n+1}\right)\int_0^a \beta_1(a)\tilde{i}^*(a, t)\partial a + \int_0^a \beta_2(a)\tilde{c}^*(a, t)\partial a, \quad \tilde{i}^*(0) = \varepsilon \tilde{E}^*(t), \quad \tilde{c}^*(0) = \int_0^a q(a)\gamma(a)\tilde{E}^*(a, t)\partial a. \]

Then the operator \( B \) can generate a strongly continuous semigroup \( \{\tilde{U}(t)\}_{t \geq 0} \). For \( x_n \in \hat{M} \), let \( \Pi_n \) be the projector on the eigenspace associated to \( z_n^* \). By the results in [7, 15], we have
\[ \Pi_n \tilde{U}(t) \begin{pmatrix} E_0^* \\ i_0^* \\ c_0^* \end{pmatrix} = e^{zt} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}. \]

After some calculation, we obtain
\[ \phi_1 = \left(\tilde{S}_0 - \frac{1}{n+1}\right)\int_0^a q(a)\gamma(a)\int_0^a e^{-\int_0^E(z_0 + \mu + \gamma(t))dt} i_0^* \partial a, \]
\[ + \int_0^a \beta_1(a)\int_0^a e^{-\int_0^E(z_0 + \mu + \gamma(t))dt} i_0^* \partial a \]
\[ = \varepsilon \phi_1 + W(z_n^*) + \int_0^a q(a)\gamma(a)\int_0^a e^{-\int_0^E(z_0 + \mu + \gamma(t))dt} i_0^* \partial a. \]

It follows that \( \lim_{t \to \infty} E^*(t) = +\infty \), \( \lim_{t \to \infty} \|i^*(., t)\| = +\infty \), \( \lim_{t \to \infty} \|c^*(., t)\| = +\infty \), which contradicts with (8). By Theorem 4.2 in [16], it is deduced that the semiflow \( \{U(t)\}_{t \geq 0} \) is uniformly persistent with respect to \( (M, \partial M) \), and the result follows.

5. Numerical simulations
Based on the transmission of hepatitis C, which is well-suited to model (1), we conduct simulations by the method of Runge-Kutta with four order. Set \([0, 0.5]\) and \([0.5, 70]\) to be the two infectious periods respectively, which is obvious that the acute period is much shorter than that of the chronic stage, thus the age-dependent parameters in the first stage is assumed to be constant. Since the acute infection
generally lasts about 3 months on average, we take $\gamma(a) = 4$. Assume the initial conditions to be $S(0) = 0.8$, $E(0) = 0.001$, $i_0(a) = 1.05e^{-0.4a}$, $c_0(a) = e^{-1.5a}$ and $\Lambda = 1$, $\mu = 0.02$, $e = 1$, $\alpha(a) = 0.25q(a)$

\[
\theta(a) = 1/(1 + \exp(1 - 0.17(a - 0.5) + 0.0007(a - 0.5)^2)). \]

Suppose $\beta_1(a) \geq \beta_2(a)$ due to the fact that the infectivity in acute stage is generally greater than that of chronic. We use the forms of transmission functions $\beta_1(a), \beta_2(a)$ and function $q(a)$ as

\[
\beta_1(a) = \begin{cases} 0.1774, & a < 0.5y, \\ 0, & a \geq 0.5y \end{cases}, \quad \beta_2(a) = \begin{cases} 0, & a < 0.5y, \\ 0.1774e^{-0.05(a-0.5)}, & a \geq 0.5y \end{cases}, \quad q(a) = \begin{cases} 0.885, & a < 0.5y, \\ e^{-0.645a^{0.5}}, & a \leq 0.5y. \end{cases}
\]

With these values, we have $R_0 > 1$ and the disease becomes endemic. Since the global dynamics when $R_0 < 1$ are proved completely in Theorem 1, we only focus on the numerical simulations when $R_0 > 1$, which are presented in Figure 1 and Figure 2. As shown in Figure 1, the endemic equilibrium is globally asymptotically stable when $R_0 > 1$ although we are only capable of proving the uniform persistence and the global stability of a special case. Furthermore, the effect of infection age on the two infectious states are illustrated in Figure 2. The variations of $i(a,t)$ and $c(a,t)$ with respect to infection age $a$ are simulated in Figure 2, showing the relations of the two distributions with $a$.

![Graphs](image)

**Figure 1.** When $R_0 > 1$, the solution of model (1) converges asymptotically to $P^*$.  

6
6. Discussions
In this paper, we seek to study the mathematical behaviors of the system analytically, including the stable analysis of the equilibria and uniform persistence of the disease. Our theoretical results reveal that the disease free equilibrium is globally asymptotically stable provided that $R_0 < 1$. Conversely, the endemic equilibrium is locally stable and the system is uniformly persistent when $R_0 > 1$. Although complete global stability of the endemic equilibrium is absent in the analysis, we perform numerical simulations with the chosen parameters, which extends the theoretical result by showing the global stability. On the other hand, it is possible that the endemic equilibrium can destabilize through a Hopf bifurcation for some forms of age-dependent parameters, which we will focus on in the future. Furthermore, the influence of age on distributions of infected individuals has been revealed by the curves and surfaces in Figure 2, reflecting the transmission heterogeneity of individuals in two infectious stages. Therefore, incorporation of infection age seems to be an important consideration in some disease modelling.

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