Quantum information becomes classical when distributed to many users

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(Dated: January 25, 2022)

Any physical transformation that equally distributes quantum information over a large number $M$ of users can be approximated by a classical broadcasting of measurement outcomes. The accuracy of the approximation is at least of the order $O(M^{-1})$. In particular, quantum cloning of pure and mixed states can be approximated via quantum state estimation. As an example, for optimal qubit cloning with 10 output copies, a single user has error probability $p_{\text{err}} \geq 0.45$ in distinguishing classical from quantum output—a value close to the error probability of the random guess.

PACS numbers: 03.67.Hk, 03.65.Ta

Differently from classical information, which can be perfectly read out and copied, quantum information cannot, since nonorthogonal quantum states can be neither perfectly distinguished [1], nor perfectly copied [2]. Since ideal distribution of quantum information is impossible, one is then interested in the performance limits of optimal distribution, and such interest has focused much attention in the literature to the problem of optimal cloning [3]. Optimal cloning consists in finding the physically realizable cloning transformation with $\rho_{\text{out}}^{(k)} = \text{Tr}_{M-k}[\mathcal{E}(\rho)]$, $k = O(M)$ can survive. The scaling $M^{-1}$ is a general upper bound holding for all physical transformations that equally distribute quantum information among $M$ users, including pure state cloning and mixed state broadcasting. Of course for specific transformations the actual scaling can be even faster.

The mathematical description of a quantum channel that transforms states on the Hilbert space $\mathcal{H}_{\text{in}}$ into states on the Hilbert space $\mathcal{H}_{\text{out}}$ is provided by a completely positive trace-preserving map $\mathcal{E}$. Since here we focus on channels that distribute quantum information to $M$ users, we have $\mathcal{H}_{\text{out}} = \mathcal{H}^\otimes M$, with $\mathcal{H}$ denoting the single user's Hilbert space. Moreover, since we require the information to be equally distributed among all users, for any input state $\rho$ on $\mathcal{H}_{\text{in}}$ the state $\mathcal{E}(\rho)$ must be invariant under permutations of the $M$ output spaces. Invariance under permutations implies that any group of $k$ users will receive the same state

$$\rho_{\text{out}}^{(k)} = \text{Tr}_{M-k}[\mathcal{E}(\rho)] ,$$

(1)

$\mathcal{E}$ is a channel for symmetric distribution of information (SDI-channel, for short). Our goal will be to approximate any SDI-channel $\mathcal{E}$ with a classical channel $\bar{\mathcal{E}}$, corresponding to measure the input and broadcast the measurement outcome, with each user preparing locally the same state accordingly. Such channels have the special form

$$\bar{\mathcal{E}}(\rho) = \sum_i \text{Tr}[P_i \rho] \rho_i^\otimes M ,$$

(2)

where $P_i$ denotes partial trace over $n$ output spaces, no matter which ones. In particular, each single user receives the same state $\rho_{\text{out}}^{(1)} = \text{Tr}_{M-1}[\mathcal{E}(\rho)]$. In the following, we will name a channel with the above properties a channel for symmetric distribution of information (SDI-channel, for short). Our goal will be to approximate any SDI-channel $\mathcal{E}$ with a classical channel $\bar{\mathcal{E}}$, corresponding to measure the input and broadcast the measurement outcome, with each user preparing locally the same state accordingly. Such channels have the special form

$$\bar{\mathcal{E}}(\rho) = \sum_i \text{Tr}[P_i \rho] \rho_i^\otimes M ,$$

(2)
where the operators \( \{ P_i \} \) represent the quantum measurement performed on the input \( (P_i \geq 0, \sum_i P_i = 1 \), and \( \rho_i \) is the state prepared conditionally to the outcome \( i \). The accuracy of the approximation is given by the trace-norm distance \( ||\rho^{(1)}_{\text{out}} - \rho^{(1)}_{\text{out}}||_1 = \text{Tr}|\rho^{(1)}_{\text{out}} - \rho^{(1)}_{\text{out}}|\) between the single user output states. The trace-norm distance governs the distinguishability of states [1], namely the minimum error probability \( p_{\text{err}} \) in distinguishing between two equally probable states \( \rho_1 \) and \( \rho_2 \) is given by

\[
p_{\text{err}} = \frac{1}{4} - \frac{1}{4}||\rho_1 - \rho_2||_1,
\]
and for small distances it approaches the random guess value \( p_{\text{err}} = 1/2 \). In our case, a small distance \( ||\rho^{(1)}_{\text{out}} - \rho^{(1)}_{\text{out}}||_1 \) means that a single user has little chance of distinguishing between the outputs of the two channels \( E \) and \( \tilde{E} \) by any measurement on his local state. In addition, to discuss the multipartite entanglement in the state \( \rho^{(k)}_{\text{out}} \), we will consider the distance \( ||\rho^{(k)}_{\text{out}} - \rho^{(k)}_{\text{out}}||_1 \). Since the state \( \rho^{(k)}_{\text{out}} \) coming from \( \tilde{E} \) in Eq.(2) is separable, a small distance means that any group of \( k \) users has a little chance of detecting entanglement.

The key idea of this letter is to get the approximation of SDI-channels exploiting the invariance of their output states under permutations. In fact, permutationally invariant states have been thoroughly studied in the research about quantum de Finetti theorem [9], where the goal is to approximate any such state \( \rho \) on \( \mathcal{H}^{\otimes M} \) with a mixture of identically prepared states \( \tilde{\rho} = \sum_i p_i \rho_i^{\otimes M} \). In particular, as we will see in the following, the recent techniques of Ref. [10] provide a very useful tool to prove our results. For simplicity, we will first start by considering the special case of SDI-channel with output states in the totally symmetric subspace \( \mathcal{H}^{\otimes M}_+ \subset \mathcal{H}^{\otimes M} \), which is the case, for example, of the optimal cloning of pure states. Then, all results will be extended to the general case of arbitrary SDI-channels.

In order to approximate channels we use the following finite version of quantum de Finetti theorem, which is proved with the same techniques of Ref.[10], with a slight improvement of the bound given therein [11]:

**Lemma 1** For any state \( \rho \) on \( \mathcal{H}^{\otimes M}_+ \subset \mathcal{H}^{\otimes M} \), consider the separable state

\[
\tilde{\rho} = \int d\psi \, p(\psi) \, |\psi\rangle \langle \psi|^{\otimes M},
\]
where the probability distribution \( p(\psi) \) is given by

\[
p(\psi) = \text{Tr}(\Pi^i \rho), \quad \Pi^i = d^\dagger M \langle \psi| |\psi|^{\otimes M},
\]

where \( d \psi \) denotes the normalized Haar measure over the pure states \( |\psi\rangle \in \mathcal{H} \), and \( d^\dagger M = \dim(\mathcal{H}^{\otimes M}) \). Then, one has

\[
||\rho^{(k)} - \tilde{\rho}^{(k)}||_1 \leq 4s_{M,k} \sqrt{\frac{d^M - k}{d^M}},
\]
where \( \rho^{(k)} \) denotes the reduced state \( \rho^{(k)} = \text{Tr}_{M-k}[\rho] \).

**Proof.** The identity in the totally symmetric subspace \( \mathcal{H}^{\otimes n}_+ \subset \mathcal{H}^{\otimes n} \) can be written as

\[
\mathbb{1}^+ n = d^+ n \int d\psi \, P_n(\psi),
\]

where \( P_n(\psi) = |\psi\rangle \langle \psi|^{\otimes n} \). Using Eq.(7) with \( n = M - k \), we can write \( \rho^{(k)} = d^k M - k \int d\psi \, \rho_k(\psi), \) where \( \rho_k(\psi) = \text{Tr}_{M-k}[\rho |\psi\rangle \langle \psi|^{\otimes M-k}] \). On the other hand, the reduced state \( \tilde{\rho}^{(k)} \) can be written as \( \tilde{\rho}^{(k)} = d^k M \int d\psi \, \rho_k(\psi) \rho_k(\psi) P_k(\psi) \). Then, the difference between \( \rho^{(k)} \) and \( \tilde{\rho}^{(k)} \), denoted by \( \Delta(k) \), is given by

\[
\Delta(k) = d^k M \int d\psi \left[ \rho_k(\psi) - \frac{d^k M}{d^k M - k} \rho_k(\psi) \rho_k(\psi) P_k(\psi) \right].
\]

Notice that the integrand on the r.h.s. has the form \( A - BAB \), with \( A(\psi) = \rho_k(\psi) \) and \( B(\psi) = \sqrt{d^k M/d^k M - k} P_k(\psi) \). Using the relation

\[
A - BAB = A(\mathbb{1} - B) + (\mathbb{1} - B)A(\mathbb{1} - B)
\]
we obtain

\[
\Delta(k) = d^k M - k (C + C^t - D),
\]

where

\[
C = \int d\psi \, A(\psi) [\mathbb{1} - B(\psi)],
\]
\[
D = \int d\psi \, [\mathbb{1} - B(\psi)] A(\psi) [\mathbb{1} - B(\psi)].
\]

The operator \( C \) is easily calculated using the relation

\[
\int d\psi \, \rho_k(\psi) P_k(\psi) = \int d\psi \, \text{Tr}_{M-k}[\rho \rho_{M}(\psi)]
\]

\[
= \frac{\text{Tr}_{M-k}[\rho]}{d^k M} = \frac{\rho^{(k)}}{d^k M},
\]

which follows from Eq. (7) with \( n = M \). In this way we obtain \( C = s_{M,k}/d^k M - k \rho^{(k)} \). Since \( C \) is nonnegative, we have \( ||C||_1 = \text{Tr}[C] = s_{M,k}/d^k M - k \). Moreover, due to definition (11) also \( D \) is nonnegative, then we have \( ||D||_1 = \text{Tr}[D] = \text{Tr}[C + C^t] \), as follows by taking the trace on both sides of Eq.(9). Thus, the norm of \( D \) is

\[
||D||_1 = 2 ||C||_1.
\]

Finally, taking the norm on both sides of Eq. (9), and using triangular inequality we get

\[
||\Delta(k)||_1 \leq 4d^k M - k ||C||_1 = 4s_{M,k},
\]

that is bound (6). Since the dimension of the totally symmetric subspace \( \mathcal{H}^{\otimes n}_+ \) is given by \( d^k n = (d+n-1) \), \( d = \dim(\mathcal{H}) \), for \( M \gg kd \) the ratio \( d^k M - k / d^k M \) tends to \( 1 - k(d-1) / M \). Therefore, Lemma 1 yields

\[
||\rho^{(k)} - \tilde{\rho}^{(k)}||_1 \leq \frac{2(d-1)k}{M}, \quad M \gg kd.
\]
and therefore less distinguishable—as obtained that $\tilde{T}_k[\rho]$ since they are obtained by applying a completely positive identity-preserving, namely $E$, on $H$ (12). Applying Lemma 1 to the output state $\rho_{out} = E(\rho)$, we get $\rho_{out} = \int d\psi Tr[\rho_{out} E(\rho)] |\psi\rangle \langle \psi|^M$. Since $Tr[\rho_{out}] = Tr[\tilde{E}(\rho)]$, we immediately obtain that $\rho_{out} = \tilde{E}(\rho)$, with $\tilde{E}$ as in Eq. (13). The operators $\rho_{out}$ represent a quantum measurement on $H_{in}$, since they are obtained by applying a completely positive identity-preserving map to $\Pi_\psi$, which is a measurement on $H_{out}$. Finally, the bound (14) then follows from Eq. (12).

The above theorem proves that for large $M$ the quantum information distributed to a single user can be efficiently replaced by the classical information about the measurement outcome $\psi$. In fact, the single user output states of the channels $E$ and $\tilde{E}$ become closer and closer—and therefore less distinguishable—as $M$ increases. For large $M$, the error probability in distinguishing between $\rho_{out}^{(1)}$ and $\rho_{out}^{(1)}$ has to satisfy the bound

$$p_{err} \geq \frac{1}{2} - \frac{d-1}{2M},$$

namely it approaches $1/2$ at rate $M^{-1}$. For example, for qubits Eq. (15) gives already with $M = 10$ an error probability $p_{err} \geq 0.45$, quite close to the error probability of a purely random guess. More generally, the bound (14) implies that for any group of $k$ users there is almost no entanglement in the state $\rho_{out}^{(k)}$, since it is close to a completely separable state. As the number of users grows, multipartite entanglement vanishes at any finite order: only $k$-partite entanglement with $k = O(M)$ can survive.

Applying our approximation theorem to the particular case of pure state cloning, we obtain a complete proof of its asymptotic equivalence with state estimation. In fact, taking $E$ as an optimal pure state cloning, the channel $\tilde{E}$ yields an approximation of $E$ based on state estimation (the measurement outcomes of $P_\psi$ are in one to one correspondence with the pure states on $H$). On one hand, when applied to a pure state $|\phi\rangle$, the optimal cloning gives fidelity $F_{clon} = \langle \phi|\rho_{out}^{(1)}|\phi\rangle$. On the other hand, since the measurement $P_\psi$ gives a possible estimation strategy, the fidelity of the optimal estimation $F_{est}$ cannot be smaller than $\langle \phi|\rho_{out}^{(1)}|\phi\rangle$. Therefore, the difference between the two fidelities can be bounded as

$$0 \leq F_{clon} - F_{est} \leq \|\rho_{out}^{(1)} - P_{out}^{(1)}\| \leq 2(d-1)/M, \quad M \gg d,$$

namely it approaches zero at rate $1/M$. A part from a constant, this is the optimal rate one can obtain in a general fashion holding for any kind of pure state cloning. In fact, $1/M$ is the exact rate in the case of universal cloning, where $F_{clon} - F_{est} = 1/(M + d) + o(1)$ (see [12] for the single-cloning fidelity). In addition, from Eq. (16) it immediately follows that any quantum cloning map for large numbers $N$ of input copies is approximated by state estimation, since for cloning one has $M > N$, and $M$ is necessarily large. In this way we proved the asymptotic equivalence between cloning and state estimation for any kind of cloning (see also the following Theorem 2 for the general case of $H_{out} \neq H_{out}^{\otimes 2}$), for either large $N$ or large $M$ (see open problems in Ref. [6]). We emphasize that the $M = \infty$ result of Ref. [7] cannot be used to prove the large $N$ asymptotics.

All results obtained for SDI-channels with output in the totally symmetric subspace can be easily extended to arbitrary SDI-channels, exploiting the fact that any permutationally invariant state can be purified to a totally symmetric one [10]:

**Lemma 2** Any permutationally invariant state $\rho$ on $H_{out}^{\otimes M}$ can be purified to a state $|\Phi\rangle \in K_{out}^{\otimes M} \subset K_{out}^{\otimes 2}$.

Once the state $\rho$ has been purified, we can apply Lemma 1 to the state $|\Phi\rangle$, thus approximating its reduced states. The reduced states of $\rho$ are then obtained by taking the partial trace over the ancillae used in the purification. This implies the following:

**Lemma 3** For any permutationally invariant state $\rho$ on $H_{out}^{\otimes M}$, purified to $|\Phi\rangle \in K_{out}^{\otimes M}, K = H_{out}^{\otimes 2}$, consider the separable state

$$\tilde{\rho} = \int d\Psi p(\Psi) \rho(\Psi)^{\otimes M}$$

where $d\Psi$ is the normalized Haar measure over the pure states $|\Psi\rangle \in K, \rho(\Psi)$ is the reduced state $\rho(\Psi) = Tr_K[|\Psi\rangle \langle \Psi|]$, and $p(\Psi)$ is the probability distribution given by $p(\Psi) = Tr[\Pi_\psi|\Psi\rangle \langle \Psi|]$, with $\Pi_\psi =$
$$D_M^+ \left| \Psi \right\rangle \left\langle \Psi \right\rangle^\otimes M, \quad D_M^+ = \dim(K^\otimes M).$$ Then, one has

$$\left| \rho_{k,A} - \hat{\rho}_{k,A} \right|_1 \leq 4 S_{M,k}, \quad S_{M,k} = 1 - \sqrt{\frac{D_{M-k}}{D_M}}. \quad (18)$$

Proof. Applying Lemma 1 to \( \tau = \left| \Phi \right\rangle \left\langle \Phi \right\rangle \), we get the state \( \hat{\tau} = \int d \Psi p(\Psi) \left| \Psi \right\rangle \left\langle \Psi \right\rangle^\otimes M \). The state \( \hat{\rho} \) is then obtained by tracing out the ancillae used in the purification, namely it is given by Eq. (17). Since partial traces can only decrease the distance, the bound (18) immediately follows from the bound (6).

It is then immediate to obtain the following:

Theorem 2 Any SDI-channel \( \mathcal{E} \) can be approximated by a classical channel

$$\tilde{\mathcal{E}}(\rho) = \int d \Psi \text{Tr}[P_{\Psi} \rho] \left| \Psi \right\rangle \left\langle \Psi \right\rangle^\otimes M, \quad (19)$$

where \( P_{\Psi} \) is a quantum measurement, namely \( P_{\Psi} \succeq 0 \) and \( \int d \Psi P_{\Psi} = \mathbb{1}_{\text{in}} \). For large \( M \), the accuracy of the approximation is

$$\| \rho_{(k)_{\text{out}}} - \rho_{(k)_{\text{out}}} \|_{1} \leq \frac{2(d^2 - 1)k}{M}, \quad M \gg kd^2. \quad (20)$$

This theorem extends Theorem 1 and all its consequences to the case of arbitrary SDI-channels. In particular, it proves that asymptotically the optimal cloning of mixed state can be efficiently simulated via mixed states estimation. The results of the measurement \( P_{\Psi} \) are indeed in correspondence with pure states on \( \mathcal{H} \otimes \mathcal{H} \), and, therefore, with mixed states on \( \mathcal{H} \). Accordingly, the knowledge of the classical result \( \Psi \) is enough to reproduce efficiently the output of the optimal cloning machine.

Notice the dependence on the dimension of the single user’s Hilbert space in both Theorems 1 and 2: increasing \( d \) makes the bounds (14) and (20) looser, leaving more room to cloning/broadcasting of genuine quantum nature. Rather surprisingly, instead, the efficiency of our approximations does not depend on the dimension of the full input Hilbert space, e. g. it doesn’t depend on the number \( N \) of the input copies of a broadcasting channel. No matter how large is the physical system carrying the input information, if there are many users at the output there is no advantage of quantum over classical information processing. Accordingly, our results can be applied to channels from \( \mathcal{H}^\otimes N \) to \( \mathcal{H}^\otimes M \), even with \( M < N \). As long as \( M \gg kd^2 \) any such channel can be efficiently replaced by a classical one. In particular, this argument holds also for the purification of quantum information [13, 14]: if \( M \) is enough large, any strategy for quantum purification can be approximated by a classical measure-and-prepare scheme. Only for small \( M \) one can have a really quantum purification.

In conclusion, we have considered the general class of quantum channels that equally distribute information among \( M \) users, showing that for large \( M \) any such channel can be efficiently approximated by a classical one, where the input system is measured and the measurement outcome is broadcast, and each user prepares locally the same state accordingly. The approximating channel can be regarded as the concatenation of a quantum-to-classical channel (the measurement), followed by a classical-to-quantum channel (the local preparation). Actually, the latter channel is needed only for the sake of comparison with the original quantum transformation to be approximated, since, due to the data processing inequality, this additional stage can only decrease the amount of information contained in the classical probability distribution of measurement outcomes. Therefore, asymptotically, there is no broadcasting of quantum information, but just an announcement of the classical information extracted by a measurement. In synthesis, we cannot distribute more information than what we are able to read out.

Acknowledgments.— This work has been founded by Ministero Italiano dell’Università e della Ricerca (MIUR) through PRIN 2005.