The equivalences among $p$-capacity, $p$-Laplace-capacities and Hausdorff measure

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Abstract
Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$. In this paper, we study the equivalences among three capacities and Hausdorff measure. First we present the equivalence between $p$-capacity $C_p(K)$ and $p$-Laplace-capacity $C_{\Delta_p}(K)$ relative to $\Omega$ for any compact set $K \subset \Omega$. Secondly we establish the equivalence between $p$-Laplace capacity $C_p(K, \partial \Omega)$ relative to $\partial \Omega$ and Hausdorff measure $H^{N-1}(K)$ for any compact set $K \subset \partial \Omega$.

1. Introduction
Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 2$) and given a compact set $K \subset \Omega$. In this paper, we shall study the relations among three capacities and Hausdorff measure on the boundary.

First, we shall establish the equivalence of two capacities between $C_{\Delta_p}$ and $C_p$, where $\Delta_p$ denotes the so-called $p$-Laplace operator $\Delta_p$ defined by

$$\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u),$$

(1.1)

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N)$.

By $C_p(K)$ we denote a usual $p$-capacity of a compact set $K \subset \Omega$, that is

$$C_p(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p; \varphi \in W^{1,p}_0(\Omega) \cap C_c^0(\Omega), \varphi \geq 1 \text{ in some neighborhood of } K \right\},$$

where by $C_c^0(\Omega)$ we denote the space of all continuous functions having compact supports in $\Omega$. On the other hand the capacity $C_{\Delta_p}$ is defined in Definition 2.3. When $p = 2$, the equation $C_{\Delta_2}(K) = 2C_2(K)$ was established by Brezis and Marcus and Ponce [3]. Here we extend their result to the case where $p \in (1, \infty)$. 

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Theorem 1.1. For every compact set $K \subset \Omega$, we have
\[ C_{\Delta_p}(K) = 2C_p(K). \] (1.2)

Secondly, we shall establish the equivalence of a capacity $C_{\Delta_p}(K, \partial \Omega)$ relate to the boundary and Hausdorff measure $H^{N-1}$, where $C_{\Delta_p}(K, \partial \Omega)$ is defined in Definition 2.4. In the linear case ($p=2$), Brezis and Ponce [1] have proved the result. Here we extend their result to the case where $p \in (1, \infty)$. The result is the following.

Theorem 1.2. For every compact set $K \subset \partial \Omega$, we have
\[ H^{N-1}(K) = C_{\Delta_p}(K, \partial \Omega). \] (1.3)

Remark 1.1. For the sake of simplicity we assumed the smoothness of the domain $\Omega$. By a technical reason we used solutions to certain Neumann boundary value problems involving $p$-Laplacian, hence in Theorem 1.2 we need $C^2$ regularity of the boundary $\partial \Omega$. But in Theorem 1.1, it suffices to assume that $\Omega$ is bounded in $\mathbb{R}^N$.

2. Preliminaries

In this section, we collect some fundamental definitions in the present article together.

It will be convenient to write
\[ C^k_b(\overline{\Omega}) = \{ u \in C^k_b(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \} \quad (k = 0, 1, 2, \cdots). \]

Let $L^p(\Omega)$, $1 \leq p < \infty$, denote the space of Lebesgue measurable functions, defined on $\Omega$, for which
\[ ||f||_{L^p(\Omega)} = \left( \int_{\Omega} |f|^p \right)^{1/p} < \infty. \]

By $L^p_{\text{loc}}(\Omega)$ we mean the space of functions locally integrable with power $p$ in $\Omega$, and by $L^\infty(\Omega)$ we mean the space of essentially bounded Lebesgue measurable functions. As a norm of $f$ in $L^\infty(\Omega)$ we take its essential supremum, i.e.,
\[ ||f||_{L^\infty(\Omega)} = \inf \{ c > 0 : |f(x)| \leq c \text{ for almost all } x \in \Omega \}. \]

Then we define the following Sobolev spaces:

Definition 2.1. $(W^{1,p}(\Omega), W^{1,p}_{\text{loc}}(\Omega) \text{ and } W^{1,p}_{0}(\Omega))$

For each $1 \leq p < \infty$, we set
\[ W^{1,p}(\Omega) = \{ f : \Omega \to R : f \in L^p(\Omega), \partial_i f \in L^p(\Omega) \text{ for } i = 1, \ldots, N \}, \] (2.1)
\[ W^{1,p}_{\text{loc}}(\Omega) = \{ f : \Omega \to R : f \in L^p_{\text{loc}}(\Omega), \partial_i f \in L^p_{\text{loc}}(\Omega) \text{ for } i = 1, \ldots, N \}, \] (2.2)
where $\partial_i f$ is taken as a distributional derivative of $f$ for $x_i$ with $i = 1, \ldots, N$. The space $W^{1,p}(\Omega)$ is equipped with the norm
\[ ||u||_{W^{1,p}(\Omega)} = |||\nabla u|||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}. \] (2.3)

By $W^{1,p}_{0}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the norm $|| \cdot ||_{W^{1,p}(\Omega)}$.

Definition 2.2. $(M(\Omega), M_b(\Omega) \text{ and } M_b(\partial \Omega))$

1. By $M(\Omega)$ we denote the space of all Radon measures on $\Omega$. 

2. By $M_b(\Omega)$ we denote the space of all Radon measures $\mu \in M(\Omega)$ having bounded variation on $\Omega$.

3. By $M_b(\partial \Omega)$ we denote the space of all Radon measures $\mu \in M(\partial \Omega)$ having bounded variation on $\partial \Omega$.

**Definition 2.3.** (A $p$-Laplace-capacity relative to $\Omega$)

$$C_{\Delta_p}(K) = \inf \left\{ \int_{\Omega} |\Delta_p \varphi|; \varphi \in W^{1,p}_0(\Omega) \cap C_0^\infty(\Omega), \varphi \geq 1 \text{ in some neighborhood of } K, \Delta_p \varphi \in M_b(\Omega) \right\}.$$

**Definition 2.4.** (A $p$-capacity relative to $\partial \Omega$)

Let $1 < p < \infty$. For each compact set $K \subset \Omega$ we define a $p$-capacity of $K$ relative to $\partial \Omega$ by

$$C_{\Delta_p}(K, \partial \Omega) = \inf \left\{ \int_{\Omega} |\Delta_p \varphi|; \varphi \in C_0^1(\Omega), \Delta_p \varphi \in C(\Omega), -|\nabla \varphi|^{p-2} \partial \varphi \partial n \geq 1 \text{ in some neighborhood of } K, \right\}$$

where $n$ denotes the unit outer normal.

**Definition 2.5.** (Quasicontinuity)

We say that a function $u: \Omega \to \mathbb{R}$ is quasicontinuous if there exists a sequence of open subsets $\omega_n \subset \subset \Omega$ such that $u|_{\Omega \setminus \omega_n}$ is continuous for $n \geq 1$ and $C_p(\omega_n) \to 0$ as $n \to \infty$.

Lastly we recall that any Radon measure $\mu$ can be uniquely decomposed as a sum of two Radon measures on $\Omega$ (see e.g. [2, 4]):

**Definition 2.6.** (Decomposition of Radon measure)

For any $\mu \in M(\Omega)$, we set $\mu = \mu_a + \mu_s$ ($\mu_a$ is the absolutely continuous part and $\mu_s$ is singular part of $\mu$), where

$$\begin{cases} 
\mu_a(A) = 0 & \text{for any Borel measurable set } A \subset \Omega \text{ such that } C_p(A) = 0, \\
|\mu_s|(\Omega \setminus F) = 0 & \text{for some Borel measurable set } F \subset \Omega \text{ such that } C_p(F) = 0.
\end{cases}$$

(2.4)

Here by $C_p(K)$ we denote a $p$-capacity of a Borel set $K$ relative to $\Omega$. We note that $(\mu_a)^\pm = (\mu^\pm)_a$, $(\mu_s)^\pm = (\mu^\pm)_s$, $|\mu_a| = |\mu|_a$ and $|\mu_s| = |\mu|_s$ hold by the definition, where $\nu^- = \max[\nu, 0] = (-\nu)^+$ and $|\nu| = \nu^+ + \nu^-$ for a Radon measure $\nu$ on $\Omega$.

3. Proof of Theorem 1.1

In order to establish Theorem 1.1 we will need a preliminary result.

**Lemma 3.1.** Let $K \subset \Omega$ be a compact set. Given any $\varepsilon > 0$, there exists $\psi \in W^{1,p}_0(\Omega) \cap C^\infty(\Omega)$ such that $0 \leq \psi \leq 1$ in $\Omega$, $\psi = 1$ in some neighborhood of $K$, and

$$\int_{\Omega} |\Delta_p \psi| \leq 2C_p(K) + \varepsilon. \quad (3.1)$$
Proof of Lemma 3.1:
Firstly we assume that $K = \omega$ for a smooth open set $\omega$. Let $u$ denote the capacitary potential of $K$. More precisely, let $u \in W^{1,p}_0(\Omega)$ be such that $u = 1$ on $K$ and
$$\int_{\Omega} |\nabla u|^p \, dx = C_p(K).$$
Note that $u$ is super $p$-harmonic in $\Omega$ and $p$-harmonic in $\Omega \setminus K$. In particular, $0 \leq u \leq 1$. Since $\text{supp} \Delta_p u \subset [u = 1]$ and $u$ is of class $C^{1,\sigma}$ with some $\sigma \in (0,1)$ near the boundary as a $p$-harmonic function, we have
$$||\Delta_p u||_{M_\delta} = -\int_{\Omega} \Delta_p u = \int_{\Omega} |\nabla u|^p = C_p(K).$$
Given $\delta > 0$ small, set
$$v = \frac{(u - \delta)^+}{1 - \delta}.$$
Since $v$ has compact support in $\Omega$, we have
$$\int_{\Omega} \Delta_p v = 0 \quad (3.2)$$
Since $v \in W^{1,p}_0(\Omega)$, it is to see that $\Delta_p v$ is absolutely continuous with respect to $p$-capacity $C_p$. Moreover we have
$$\text{supp} \Delta_p v \subset [v = 1] \cup [v = 0]. \quad (3.3)$$
By Kato’s inequalities for $p$-Laplace operator $\Delta_p$ in [5] (Theorem 1.1 and Corollary 1.1), we can easily deduce the following inequalities in the sense of distribution.
$$\Delta_p v \geq 0 \text{ in } [v = 0] \text{ and } \Delta_p v \leq 0 \text{ in } [v = 1]. \quad (3.4)$$
For the sake of selfcontainedness we prove (3.4) in Appendix. It then follows from (3.2) - (3.4) that
$$||\Delta_p v||_{M_\delta} = 2 \int_{[v = 1]} |\Delta_p v|. \quad (3.5)$$
Since $\Delta_p v = \frac{1}{1 - \delta} \Delta_p u$ in $[v = 1]$, we conclude that
$$||\Delta_p v||_{M_\delta} \leq \frac{2}{1 - \delta} ||\Delta_p u||_{M_\delta}. \quad (3.5)$$
Since $K$ is smooth, we have
$$||\Delta_p v||_{M_\delta} \leq \frac{2}{1 - \delta} ||\Delta_p u||_{M_\delta} = \frac{2}{1 - \delta} C_p(K).$$
Choosing $\delta > 0$ so that
$$\frac{\delta}{1 - \delta} C_p(K) < \frac{\varepsilon}{4},$$
we have
$$2(1 + \frac{\delta}{1 - \delta})C_p(K) \leq 2C_p(K) + \varepsilon.$$ Then we have (3.1).
Secondly we prove (3.1) when \( K \) is a compact set. For any \( \varepsilon > 0 \), let \( \omega \subset\subset \Omega \) be open and smooth such that \( K \subset \omega \) and
\[
C_p(K) \leq C_p(\omega) \leq C_p(K) + \frac{\varepsilon}{4}.
\] (3.6)

Let \( u \) denote the capacitary potential of \( \omega \). More precisely, let \( u \in W^{1,p}_0(\Omega) \) be such that \( u = 1 \) on \( \omega \) and
\[
\int_{\Omega} |\nabla u|^p dx = C_p(\omega).
\]

Note that \( u \) is super \( p \)-harmonic in \( \Omega \) and \( p \)-harmonic in \( \Omega \setminus \omega \) and
\[
||\Delta_p u||_{M_b} = -\int_{\Omega} \Delta_p u = \int_{\Omega} |\nabla u|^p = C_p(\omega).
\]

In a similar way, we also have (3.5), then we have
\[
||\Delta_p u||_{M_b} \leq \frac{2}{1-\delta} ||\Delta_p u||_{M_b} = 2(1 + \frac{\delta}{1-\delta})C_p(\omega)
\]
Choosing \( \delta > 0 \) so that
\[
\frac{\delta}{1-\delta}C_p(K) < \frac{\varepsilon}{4},
\]
we have
\[
||\Delta_p u||_{M_b} \leq 2C_p(\omega) + \frac{\varepsilon}{2},
\]
and
\[
||\Delta_p u||_{M_b} \leq 2C_p(K) + \varepsilon.
\]

Thus we have the desired estimate (3.1). \( \square \)

**Proof of Theorem 1.2:**

In view of Lemma 3.1, it suffices to show that
\[
C_p(K) \leq \frac{1}{2} C_{\Delta_p}(K).
\] (3.7)

Let \( \phi \in W^{1,p}_0(\Omega) \cap C^0(\Omega) \) satisfy \( 0 \leq \phi \leq 1 \), \( \Delta_p \phi \in M_b(\Omega) \) and \( \phi = 1 \) in some neighborhood of \( K \). Then we have
\[
C_p(K) \leq \int_{\Omega} |\nabla \phi|^p = \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \nabla \phi = -\int \phi \Delta_p \phi.
\]

Note that \( \phi \) has a compact support and \( 0 \leq \phi \leq 1 \). Hence \( \int \Delta_p \phi = 0 \) and we have
\[
C_p(K) \leq -\int (\phi - \frac{1}{2}) \Delta_p \phi \leq \frac{1}{2} \int |\Delta_p \phi|.
\]

Since \( \phi \) is arbitrary, we have
\[
C_p(K) \leq \frac{1}{2} C_{\Delta_p}(K).
\] \( \square \)

**4. Proof of Theorem 1.2**

In order to prove Theorem 1.2, we prepare two lemmas which are similar to those in Brezis and Ponce [1] provided that \( p = 2 \). They are proved on a basis of PDE theory.
Lemma 4.1. Let $K \subset \partial \Omega$ be a compact set. Given $\varepsilon > 0$, there exists $\psi \in C^1_0(\Omega)$ such that $\psi \geq 0$ in $\Omega$, $\Delta_p \psi \in C(\Omega)$, $-|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} \geq 1$ in some neighborhood of $K$ and
\[
\int_{\Omega} |\Delta_p \psi| \leq C_{\Delta_p}(K, \partial \Omega) + \varepsilon. \tag{4.1}
\]

Proof of Lemma 4.1:
Given $\varepsilon > 0$, let $\xi \in C^1_0(\Omega)$ be such that $\xi \geq 0$ in $\Omega$, $\Delta_p \xi \in C(\Omega)$, $-|\nabla \xi|^{p-2} \frac{\partial \xi}{\partial n} \geq 1$ in some neighborhood of $K$ and
\[
\int_{\Omega} |\Delta_p \xi| \leq C_{\Delta_p}(K, \partial \Omega) + \varepsilon. \tag{4.2}
\]
Let $\psi$ be the unique solution of
\[
\begin{cases}
-\Delta_p \psi = |\Delta_p \xi| & \text{in } \Omega, \\
\psi = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By a standard theory we see $\psi \in C^{1, \sigma}(\Omega)$ with some $\sigma \in (0, 1)$ and by the maximum principle, $\xi \leq \psi$ in $\Omega$. Since $\xi = \psi = 0$ on $\partial \Omega$, we have
\[-\frac{\partial \xi}{\partial n} \leq -\frac{\partial \psi}{\partial n} \quad \text{on } \partial \Omega.\]
Since $|\nabla \psi| = |\frac{\partial \psi}{\partial n}|$ on $\partial \Omega$, we have
\[-|\nabla \xi|^{p-2} \frac{\partial \xi}{\partial n} \leq -|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} \quad \text{on } \partial \Omega. \tag{4.3}\]
Therefore we see that
\[-|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} \geq 1 \quad \text{in some neighborhood of } K. \tag{4.4}\]
Hence we have
\[
\int_{\Omega} |\Delta_p \psi| = \int_{\Omega} |\Delta_p \xi| \leq C_{\Delta_p}(K, \partial \Omega) + \varepsilon. \tag{4.5}\]
□

Lemma 4.2. Let $K \subset \partial \Omega$ be a compact set. Given $\varepsilon > 0$, there exists $\psi \in C^1_0(\Omega)$ such that $0 \leq \psi$ in $\Omega$, $\Delta_p \psi \in C(\Omega)$, $-|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} \geq 1$ in some neighborhood of $K$ and
\[
\int_{\Omega} |\Delta_p \psi| \leq \mathcal{H}^{N-1}(K) + \varepsilon. \tag{4.6}\]

Proof of Lemma 4.2:
Let $\delta > 0$ be such that
\[
\mathcal{H}^{N-1}(N_{\delta}(K) \cap \partial \Omega) \leq \mathcal{H}^{N-1}(K) + \varepsilon, \tag{4.7}\]
where $N_{\delta}(K)$ is an open set of $\mathbb{R}^N$ consisting of all points with distance from $K$ being less than $\delta$.
We fix $\xi \in C^\infty(\partial \Omega)$ such that $\xi = 1$ in $N_{\frac{\delta}{2}} \cap \partial \Omega$, $\xi = 0$ in $\partial \Omega \setminus N_{\frac{\delta}{2}}(K)$ and $0 \leq \xi \leq 1$ on $\partial \Omega$. Let $f \in C^\infty_c(\Omega)$ satisfy $f \geq 0$ and $\int_{\Omega} f = \int_{\partial \Omega} \xi$. Now let $\psi$ be one of solutions the following Neumann problem:
\[
\begin{cases}
-\Delta_p \psi = f & \text{in } \Omega, \\
-|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} = \xi & \text{on } \partial \Omega. \tag{4.8}
\end{cases}
\]
Again we see that $\psi \in C^{1,\sigma}(\Omega)$ with some $\sigma \in (0, 1)$. Replacing $\psi$ by $\psi - \min_{\Omega} \psi$, we may assume that $\psi \geq 0$ in $\Omega$. Then

$$||\Delta_p \psi||_{M_b} = \int_{\Omega} |\Delta_p \psi| = - \int_{\Omega} \Delta_p \psi$$

$$= - \int_{\Omega} |\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} = \int_{\partial \Omega} \xi$$

$$\leq \mathcal{H}^{N-1}(N_\delta(K) \cap \partial \Omega) \leq \mathcal{H}^{N-1}(K) + \varepsilon.$$

Thus, we have our result.

Now we present Proof of Theorem 1.2:

Given $\varepsilon > 0$, let $\psi \in C^1_0(\Omega)$ be the function given by Lemma 4.1. Since $\psi \geq 0$ in $\Omega$, we have $-|\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} \geq 0$ on $\partial \Omega$. Hence integrating by parts and using (4.1) we have

$$\mathcal{H}^{N-1}(K) \leq - \int_{\partial \Omega} |\nabla \psi|^{p-2} \frac{\partial \psi}{\partial n} \leq - \int_{\partial \Omega} \Delta_p \psi \leq \int_{\partial \Omega} |\Delta_p \psi| \leq C_{\Delta_p}(K, \partial \Omega) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that

$$\mathcal{H}^{N-1}(K) \leq C_{\Delta_p}(K, \partial \Omega).$$

From Lemma 4.2 we have the reverse inequality,

$$C_{\Delta_p}(K, \partial \Omega) \leq \mathcal{H}^{N-1}(K).$$

5. Appendix (Proof of the inequalities in (3.4))

We begin with recalling the admissible class in $W^{1,p}_\text{loc}(\Omega)$ and improved Kato’s inequalities; Theorem 1.2 in [5] (See also [6]).

Definition 5.1. (Admissible class in $W^{1,p}_\text{loc}(\Omega)$) Let $1 < p < \infty$ and $p^* = \max[1, p-1]$. A function $u \in W^{1,p}_\text{loc}(\Omega)$ is said to be admissible if $\Delta_p u \in M(\Omega)$ and there exists a sequence $\{u_n\}_{n=1}^\infty \subset W^{1,p}_\text{loc}(\Omega) \cap L^{\infty}(\Omega)$ such that:

1. $u_n \to u$ a.e. in $\Omega$ and $u_n \to u$ in $W^{1,p^'}(\Omega)$ as $n \to \infty$.
2. $\Delta_p u_n \in L^1_{\text{loc}}(\Omega)$ ($n = 1, 2, \cdots$) and

$$\sup_n |\Delta_p u_n|_\omega = \sup \int_\omega |\Delta_p u_n| < \infty \quad \text{for every } \omega \subset \subset \Omega. \quad (5.1)$$

For the admissible class we showed the following fundamental properties in [6] (See also [5]).

Proposition 5.1. Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p-1]$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$.

1. Assume that a function $u \in W^{1,p^'}(\Omega)$ is admissible. Then $u^+ = \max[u, 0]$ and $u^- = \max[-u, 0]$ are also admissible.
2. Assume that \( p = 2 \). Then a function \( u \in W^{1,1}_{\text{loc}}(\Omega) \) is admissible if \( \Delta u \in M(\Omega) \).

3. A function \( u \in W^{1,p}_{0}(\Omega) \) is admissible if \( \Delta_p u \in M(\Omega) \).

Under these preparation we have proved in [5] the following:

**Theorem 5.1.** Let \( N \geq 1 \), \( 1 < p < \infty \), \( p' = \max[1,p-1] \) and \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). Let \( \Phi \) be a \( C^1 \) convex function in \( \mathbb{R} \) such that \( \Phi' \geq 0 \) in \( \mathbb{R} \) and \( \Phi' \in L^\infty(\mathbb{R}) \).

When \( p = 2 \), assume that \( u \in L^1_{\text{loc}}(\Omega) \) and \( \Delta u \) is a Radon measure on \( \Omega \).

When \( p \neq 2 \), assume that \( u \in W^{1,p'}_{\text{loc}}(\Omega) \) and \( u \) is admissible in the sense of Definition 5.1.

Then we have

\[
\Delta_p \Phi(u) \geq \Phi'(u)^{p-1}(\Delta_p u)_a - ||\Phi'||^{p-1}_{L^\infty(\mathbb{R})}(\Delta_p u)_s \quad \text{in} \; D'(\Omega). \tag{5.2}
\]

Let \( \{\Phi_n\} \) be a sequence of \( C^1 \) convex functions in \( \mathbb{R} \) such that \( \Phi_n(t) = t \) if \( t \geq \frac{1}{n} \), \( \Phi(t) = \frac{1}{2n} \) if \( t < 0 \), \( 0 \leq \Phi'_n \leq 1 \) in \( \mathbb{R} \). Then it follows from Theorem 5.1 that

\[
\Delta_p \Phi_n(u) \geq \Phi'_n(u)^{p-1}(\Delta_p u)_a - (\Delta_p u)_s \quad \text{in} \; D'(\Omega). \tag{5.3}
\]

By taking a limit as \( n \to \infty \) we have

\[
\Delta_p (u^+) \geq \chi_{\{u>0\}}(\Delta_p u)_a - (\Delta_p u)_s \quad \text{in} \; D'(\Omega). \tag{5.4}
\]

Taking the absolutely continuous part of both sides,

\[
\Delta_p (u^+)_a \geq \chi_{\{u>0\}}(\Delta_p u)_a \quad \text{in} \; D'(\Omega). \tag{5.5}
\]

In particular if \( u \geq 0 \) a.e. in \( \Omega \), then we conclude that

\[
(\Delta_p u)_a|_{\{u=0\}} \geq 0. \tag{5.6}
\]

Now we are in a position to prove (3.4).

**Proof of (3.4)**

Since \( 0 \leq v \in W^{1,p}_{0}(\Omega) \), It follows from Proposition 5.1 that \( v \) is admissible, and it is easy to see that \( \Delta_p v \) is absolutely continuous with respect to \( p \)-capacity so that the concentrated part \( (\Delta_p v)_s = 0 \). Therefore it is clear from (5.6) that the first inequality of (3.4) holds on \( [v=0] \). The second one also follows if we consider \( 1-v \) instead. \( \square \)

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