Open subgroups of free topological groups

Jeremy Brazas

May 2, 2014

Abstract

The theory of covering spaces is often used to prove the Nielsen-Schreier theorem, which states that every subgroup of a free group is free. We apply the more general theory of semicovering spaces to obtain analogous subgroup theorems for topological groups: Every open subgroup of a free Graev topological group is a free Graev topological group. An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.

1 Introduction

A well-known application of covering space theory is the Nielsen-Schreier theorem [19], which states that every subgroup of a free group is free [5, 11]. The corresponding situation for topological groups is more complicated since it is not true that every closed subgroup of a free topological group is free topological [7, 8, 9, 13]. The purpose of this paper is to use the theory of semicovering spaces developed in [4] to prove the following theorem.

Theorem 1. Every open subgroup of a free Graev topological group is a free Graev topological group.

Free topological groups are important objects in the general theory of topological groups and have an extensive literature dating back to their introduction by A.A. Markov [15] in the 1940s. Markov [15] defined the free topological group $F_M(X)$ on a space $X$ and Graev [9] later introduced the free topological group $F_G(X, *)$ on a space $X$ with basepoint $* \in X$. While the existence of these groups (for any $X$) follows abstractly from adjoint existence theorems [18], the theory of free topological groups has traditionally required the condition that $X$ be completely regular since the canonical injections $\sigma : X \to F_M(X)$ and $\sigma_* : X \to F_G(X, *)$ are embeddings if and only if $X$ is completely regular. In this paper, we exploit universal properties and adjoint functors (as opposed to working with complicated characterizations of the topology) to avoid placing any restrictions on $X$. For more on the theory of free topological groups, we refer the reader to [1, 20, 21].
Topological versions of the Nielsen-Schreier Theorem have been attempted for free topological groups on Hausdorff $k_\omega$-spaces, i.e. spaces which are the inductive limit of a sequence of compact subspaces. In this case, a subgroup of a free Graev topological group which admits a continuous Schreier traversal is free (Graev) topological. Theorem 1 is an improvement in the sense that it holds for the free topological group on an arbitrary topological space. Another difference between our approach and that in [6] is that we use topologically enriched graphs and categories as opposed to graphs and categories internal to a category of spaces.

Covering theoretic proofs of the algebraic Nielsen-Schreier Theorem typically require an understanding of covering spaces and fundamental group(oid)s of graphs. Our proof of Theorem 1 generalizes this approach by replacing covering theory with the theory of semicoverings, graphs with Top-graphs (i.e. topological graphs with discrete vertex spaces), and the fundamental groupoid (fundamental group) with the fundamental Top-groupoid (topological fundamental group). Our application of the classification of semicoverings relies heavily on the fact that the theory applies to certain non-locally path connected spaces, called locally wep-connected spaces, which are not included in classical covering space theory.

This paper is structured as follows. In Section 2 we recall the basic theory of free topological groups and include a general comparison of the two notions of free topological groups (those in the sense of Graev and those in the sense of Markov). Using Theorem 1 we obtain a structure theorem for open subgroups of free Markov topological groups (Theorem 5): An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.

In Section 3 we extend the usual notion of an algebraic graph by allowing edge spaces to have non-discrete topologies; the resulting objects are called Top-graphs. We then present some universal constructions of topologically enriched categories and groupoids to be used in the computations of Section 4. In Section 4 we show the fundamental Top-groupoid (resp. topological fundamental group) of a Top-graph is a free Top-groupoid (resp. free Graev topological group). Finally, in Section 5 semicovering theory is applied to Top-graphs. Analogous to the fact that a covering of a graph is a graph, we find that a semicovering of a Top-graph is a Top-graph. The paper concludes with a proof of Theorem 1.

2 Free topological groups

Definition 2. Let X be a topological space. The free Markov topological group on X is the unique (up to isomorphism) topological group $F_M(X)$ equipped with a map $\sigma : X \to F_M(X)$ universal in the sense that every map $f : X \to G$ to a topological group G induces a unique, continuous homomorphism $\hat{f} : F_M(X) \to G$ such that $\hat{f}\sigma = f$.

The existence of free Markov topological groups is guaranteed by the Gen-
eral Adjoint Theorem \[18\]. In particular, if \textbf{Top} is the category of topological spaces and \textbf{TopGrp} is the category of topological groups, then \( F_M : \textbf{Top} \to \textbf{TopGrp} \) is left adjoint to the forgetful functor \( \textbf{TopGrp} \to \textbf{Top} \). Algebraically, \( F_M(X) \) is the free group on the underlying set of \( X \) and \( \sigma : X \to F_M(X) \) is the canonical injection of generators.

**Definition 3.** Let \( X \) be a space with basepoint \( * \in X \). The \text{free Graev topological group} on \((X,*)\) is the unique (up to isomorphism) topological group \( F_G(X,*) \) equipped with a map \( \sigma_* : X \to F_G(X,*) \) such that \( \sigma_*(*) \) is the identity element of \( F_G(X,*) \) and universal in the sense that every map \( f : X \to G \) to a topological group \( G \) which takes \(*\) to the identity element of \( G \) induces a unique, continuous homomorphism \( f : F_G(X,*) \to G \) such that \( f_\sigma = f \).

Similar to the unbased case, if \( \textbf{Top} \) is the category of based topological spaces, then \( F_G : \textbf{Top} \to \textbf{TopGrp} \) is left adjoint to the forgetful functor \( \textbf{TopGrp} \to \textbf{Top} \). Here, the basepoint of a topological group is the identity element. Algebraically, \( F_G(X,*) \) is the free group on the set \( X \setminus \{ * \} \), however, it is not necessarily isomorphic to \( F_M(X \setminus \{ * \}) \) as a topological group. On the other hand, \( F_M(X) \) is isomorphic to the free Graev topological group \( F_G(X,*) \) where \( X_* = X \sqcup \{ * \} \) has an isolated basepoint and \( F_G(X,*) \) is isomorphic to the quotient topological group \( F_M(X)/N \) where \( N \) is the conjugate closure of \( \{ * \} \).

Graev showed in \[9, \text{Theorem 2}\] that the isomorphism class of \( F_G(X,*) \) as a topological group does not depend on the choice of basepoint, i.e., given any other point \( *' \in X \) there is an isomorphism \( F_G(X,*) \to F_G(X,*)' \) of topological groups\(^1\). It remains to understand when \( F_G(X,*) \) is isomorphic to the free Markov topological group \( F_M(Y) \) on some space \( Y \). This is answered by Graev in \[9\] in the case that \( X \) is completely regular; we modify Graev’s argument only slightly in order to obtain a more general result.

**Lemma 4.** If \( X \) is the disjoint union \( X = A_1 \sqcup A_2 \) of open sets \( A_i \subset X \) and \( e_i \in A_i \), then \( F_G(X,e_i) \) is isomorphic to the free Markov topological group \( F_M(A_1 \vee A_2) \) on the wedge sum \( A_1 \vee A_2 = X/\{e_1, e_2\} \).

**Proof.** Let \( q : X \to A_1 \vee A_2 \) be the quotient map making the identification \( q(e_1) = z = q(e_2) \). Define a map \( f : X \to F_M(X) \) by \( f(a) = ae_2^{-1} \) for \( a \in A_1 \) (here \( e_2^{-1} \) is the product in \( F_M(X) \)) and \( f(a) = a \) for \( a \in A_2 \). Since \( F_M(q)(e_1) = F_M(q)(e_2) \), the composition \( \psi = F_M(q)f : X \to F_M(A_1 \vee A_2) \) takes \( e_1 \) to the identity of \( F_M(A_1 \vee A_2) \) and induces a continuous homomorphism \( \tilde{\psi} : F_G(X,e_1) \to F_M(A_1 \vee A_2) \). Note that \( \tilde{\psi}(e_2) = z \).

Now consider the map \( g : X \to F_G(X,e_1) \) where \( g(a) = ae_2, \ a \in A_1 \) is the product in \( F_G(X,e_1) \) and \( g(a) = a, \ a \in A_2 \). Since \( e_1 \) is the identity in \( F_G(X,e_1) \), \( g(e_1) = e_1e_2 = e_2 = g(e_2) \). We obtain a continuous map \( \phi : A_1 \vee A_2 \to F_G(X,e_1) \) on the quotient such that \( \phi(z) = e_2 \) and which induces a continuous homomorphism \( \tilde{\phi} : F_M(A_1 \vee A_2) \to F_G(X,e_1) \).

\(^1\)Graev assumes \( X \) is completely regular, however, the argument given also applies to the general case.
A direct check shows that \( \hat{\psi} \hat{\phi} \) is the identity homomorphism of \( F_G(X, e_1) \) and \( \hat{\psi} \psi \hat{\phi} \) is the identity of \( F_M(A_1 \lor A_2) \). In particular, if \( a \in A_1 \setminus \{e_1\} \), then \( \hat{\psi} \hat{\phi}(a) = \hat{\phi}(ae_2) = a \) and \( \psi \hat{\phi}(a) = \psi(e_2) = ae_2 e_2 = a \). The other cases are straightforward and left to the reader. □

**Theorem 5.** For any space \( X \), the following are equivalent.

1. \( X \) is connected.
2. \( F_G(X, \ast) \) is connected.
3. \( F_G(X, \ast) \) is not isomorphic to a free Markov topological group.

**Proof.**

1. \( \Rightarrow \) 2. Suppose \( X \) is connected and let \( C \) be the connected component of the identity in \( F_G(X, \ast) \). Since \( \sigma : X \to F_G(X, \ast) \) is continuous, the generating set \( \sigma(X) \) is a connected subspace of \( F_G(X, \ast) \) containing the identity and is therefore contained in \( C \). In a general topological group, the connected component of the identity element is a subgroup [1, 1.4.26]. Therefore \( C = F_G(X, \ast) \).

2. \( \Rightarrow \) 3. Every free Markov topological group is disconnected since the canonical map \( X \to \ast \) collapsing \( X \) to a point induces a continuous homomorphism \( F_M(X) \to F_M(\ast) = \mathbb{Z} \) onto the discrete group of integers. Therefore, if \( F_G(X, \ast) \) is connected, \( F_G(X, \ast) \) cannot be isomorphic to a free Markov topological group.

3. \( \Rightarrow \) 1. This follows directly from Lemma [2]. □

Combining Theorems 1 and 5 and the fact that every free Markov topological group is a free Graev topological group, we obtain a structure theorem for open subgroups of free Markov topological groups. This result generalizes that in [6] for free Markov topological groups on Hausdorff \( k_\omega \)-spaces.

**Theorem 6.** An open subgroup of a free Markov topological group is a free Markov topological group if and only if it is disconnected.

### 3 Topologically enriched graphs and categories

The rest of this paper is devoted to a proof of Theorem 6.

#### 3.1 Top-graphs

A **Top-graph** \( \Gamma \) consists of a discrete space of vertices \( \Gamma_0 \), an edge space \( \Gamma \), and continuous structure maps \( \partial_0, \partial_1 : \Gamma \to \Gamma_0 \). For convenience, we sometimes let \( \Gamma \) denote the Top-graph itself. The set of composable edges in \( \Gamma \) is the pullback \( \Gamma \times_\Gamma \Gamma = \{ (e, e') \in \partial_0(e) = \partial_0(e') \} \).

For each pair of vertices \( x, y \in \Gamma_0 \), let \( \Gamma_x = \partial_0^{-1}(x) \), \( \Gamma_y = \partial_1^{-1}(y) \), and \( \Gamma(x, y) = \Gamma_x \cap \Gamma_y \). Since we require the vertex space of a Top-graph to be discrete, the edge space decomposes as the topological sum \( \Gamma = \bigsqcup_{(x, y) \in \Gamma \times \Gamma} \Gamma(x, y) \) over ordered pairs of vertices.
Since it is possible that both \( \Gamma(x, y) \) and \( \Gamma(y, x) \) are non-empty, we are motivated to make the following construction. Let \( \Gamma(x, y)^{-1} \) denote a homeomorphic copy of \( \Gamma(x, y) \) for each pair \((x, y) \in \Gamma_0 \times \Gamma_0\). Here \( e \in \Gamma(x, y) \) corresponds to \( e^{-1} \in \Gamma(x, y)^{-1} \). Define a new Top-graph \( \Gamma^* \) to have vertex spaces \( \Gamma_0 \) and \( \Gamma^*(x, y) = \Gamma(x, y) \sqcup \Gamma(y, x)^{-1} \). In particular, note that \( \Gamma^*(x, x) = \Gamma(x, x) \sqcup \Gamma(x, x)^{-1} \).

A morphism \( f : \Gamma \to \Gamma' \) of Top-graphs consists of a pair of continuous functions \((f_0, f) : (\Gamma_0, \Gamma) \to (\Gamma'_0, \Gamma')\) such that \( \partial'_i \circ f = f_0 \circ \partial_i \) for \( i = 1, 2 \). Such a morphism is said to be quotient if \( f_0 \) and \( f \) are quotient maps of spaces (note \( f_0 \) only needs to be surjective to be quotient). There is also an obvious notion of sub-Top-graph \( S \subseteq \Gamma \). We say such a sub-Top-graph is wide if \( S_0 = \Gamma_0 \). The category of Top-graphs is denoted \( \text{TopGraph} \).

**Definition 7.** The geometric realization of a Top-graph \( \Gamma \) is the topological space

\[
|\Gamma| = \Gamma_0 \sqcup (\Gamma \times [0, 1]) / \sim \quad \text{where } \partial_i(\alpha) \sim (\alpha, i) \text{ for } i = 0, 1.
\]

A Top-graph is connected if \( |\Gamma| \) is path connected, or equivalently, if for each \( x, y \in \Gamma_0 \), there is a sequence of vertices \( x = a_1, a_2, ..., a_n = y \) such that \( \Gamma^*(a_j, a_{j+1}) \neq \emptyset \) for \( j = 1, ..., n - 1 \).

We typically assume Top-graphs are connected.

**Remark 8.** For any \( 0 < r < 1 \), the image of \( \Gamma_0 \times [0, r] \) in the quotient \( |\Gamma| \) is homeomorphic to the cone \( CX \) on \( X \). Similarly, if \( Z = \Gamma(x, y) \sqcup \Gamma(y, x) \), then the image of \( Z \times [0, 1] \) in the \( |\Gamma| \) is the unreduced suspension \( SZ \). Finally, note

\[
|\Gamma| \setminus \Gamma_0 = \bigsqcup_{(x, y)} (\Gamma(x, y) \times (0, 1)).
\]

![Figure 1](image.png)

**Figure 1:** The realization of a Top-graph \( \Gamma \) with four vertices. Here \( \Gamma(z, x) \), \( \Gamma(w, z) \), and \( \Gamma(z, w) \) are all empty.
Remark 9. It is unfortunate that $|\Gamma|$ need not be first countable at its vertices, however, it is possible to change the topology on $|\Gamma|$ without changing homotopy type so that each vertex has a countable neighborhood base. The vertex neighborhood of $x \in \Gamma_0$ of radius $r \in (0, 1)$ is the image of

$$(\Gamma_x \times [0, r)) \cup (\Gamma^x \times (1 - r, 1])$$

and is denoted $B(x, r)$. An edge neighborhood of a point $(e, t) \in \Gamma(x, y) \times (0, 1)$ is the homeomorphic image of a set $U \times (a, b)$ where $U$ is an open neighborhood of $e$ in $\Gamma(x, y)$ and $0 < a < t < b < 1$. The basis consisting of vertex and edge neighborhoods is closed under finite intersection and generates a topology which may be strictly coarser than the quotient topology (but only at vertices). Note that each vertex neighborhood $B(x, r)$ is contractible onto $x$ and the set $\{B(x, 1/n) | n \geq 1\}$ is a countable neighborhood base at $x$.

From now on, we assume $|\Gamma|$ has the coarser topology generated by vertex and edge neighborhoods.

Example 10. If $\Gamma$ is a Top-graph with a single vertex, then $|\Gamma|$ is the generalized wedge of circles $\Sigma(\Gamma_i)$ (where $\Sigma$ denotes reduced suspension) studied in detail in [2]. When $\Gamma_0 = \{x_0, x_1\}$ and the structure maps are the two constant maps $\partial_i : \Gamma \rightarrow \Gamma_0$, $\partial_i(a) = x_i$ (equivalently, $\Gamma = \Gamma(x_0, x_1)$), then $|\Gamma|$ is the unreduced suspension $S\Gamma$. Thus, unlike a discrete graph, a Top-graph may be simply connected but not contractible, e.g. $\Gamma = S^1$.

Path component spaces 11. We end this section with a useful construction on Top-graphs: The path component space of a topological space $X$ is the quotient space $\pi_0(X)$ where each path component is identified to a point. If $\Gamma$ is a Top-graph, then $\pi_0(\Gamma)$ is the Top-graph with vertex space $\Gamma_0$ and $\pi_0(\Gamma)(x, y) = \pi_0(\Gamma(x, y))$. The canonical quotient morphism of Top-graphs $q : \Gamma \rightarrow \pi_0(\Gamma)$ is the identity on vertices and takes an edge $e$ to its path component $[e]$.

Remark 12. Another useful construction is a section to the path component functor $\pi_0 : \textbf{TopGraph} \rightarrow \textbf{TopGraph}$. Given any space $X$, there a (paracompact Hausdorff) space $h(X)$ and a natural homeomorphism $\pi_0(h(X)) \equiv X$ [10]. Thus for any Top-graph $\Gamma$, we define $h(\Gamma)$ to have object space $\Gamma_0$ and $h(\Gamma)(x, y) = h(\Gamma(x, y))$ so that $\pi_0(h(\Gamma)) \equiv \Gamma$.

3.2 Top-categories and qTop-categories

Our use of enriched categories aligns with that in [14]. If a Top-graph $C$ comes equipped with continuous composition map $C \times_{C_0} C \rightarrow C$ making $C$ a category in the usual way, then $C$ is a Top-category (or a category enriched over Top). Since $\text{Ob}(C) = C_0$ is discrete, $C \times_{C_0} C$ decomposes as a topological sum of products $C(x, y) \times C(y, z)$. Thus to specify a Top-category one only need specify the hom-spaces $C(x, y)$ and continuous composition maps $C(x, y) \times C(y, z) \rightarrow C(x, z)$. If composition maps are only continuous in each variable, then $C$ is an sTop-category (the "s" is for "semitopological" as in [11]). A Top-functor
to the following lemma. 

An involution on a small category $C$ is a function $C \to C$ defined by functions $C(x, y) \to C(y, x), f \mapsto f^*$ such that $(f^*)^* = f,$ $(fg)^* = g^* f^*$, and $(id_x)^* = id_x$. A Top-category (resp. a sTop-category) equipped with a continuous involution is a Top-category with continuous involution (resp. a qTop-category). If $G$ is a Top-category (resp. sTop-category) whose underlying category is a groupoid and the involution given by the inversion functions $G(x, y) \to G(y, x)$ is continuous, then $G$ is a Top-groupoid (resp. qTop-groupoid).

A functor $F : C \to D$ of categories with involution preserves involution if $F(f^*) = F(f)^*$. In particular, a qTop-functor $F : C \to D$ of qTop-categories is an involution preserving functor which is continuous on hom-spaces.

The notion of qTop-groupoid is particularly relevant and is studied in Section 4 of [4]. The following Lemma is a useful fact asserting that the category qTopGrpd of qTop-groupoids is a full reflective subcategory of the category qTopCat of qTop-groupoids.

**Lemma 13.** [4 Lemma 4.5] The forgetful functor qTopGrpd $\to$ qTopCat has a left adjoint $\tau : qTopGrpd \to TopGrpd$ which is the identity on the underlying groupoids and functors.

**Free Top-categories 14.** The free Top-category generated by a Top-graph $\Gamma$ is the Top-category $\mathcal{E}(\Gamma)$ with object space $\Gamma_0$ and in which morphisms are finite sequences $e_1 e_2 \ldots e_n$ of composable edges $e_i \in \Gamma$. In particular, the hom-space $\mathcal{E}(\Gamma)(x, y)$ is topologized as the topological sum $\bigoplus \Gamma(x, a_1) \times \Gamma(a_1, a_2) \times \ldots \times \Gamma(a_n, y)$ where the sum ranges over all finite sequences $a_1, \ldots, a_n$ in $\Gamma_0$. In order to obtain a category, we add an isolated identity morphism $\{id_x\}$ to each space $\mathcal{E}(\Gamma)(x, x)$. Note this construction yields a functor $\mathcal{E} : TopGraph \to qTopCat$ left adjoint to the forgetful functor $TopCat \to TopGraph$.

The construction of $\mathcal{E}(\Gamma)$ is easily modified to include a continuous involution. In particular, the free Top-category with (continuous) involution on $\Gamma$ is $\mathcal{E}^+(\Gamma) = \mathcal{E}(\Gamma^+)$, the free Top category on the Top-graph $\Gamma^+$ described in the previous section. Thus a generic (non-identity) morphism of $\mathcal{E}^+(\Gamma)$ may be given by a sequence $e_1^\delta_1 e_2^\delta_2 \ldots e_n^\delta_n$ where $e_i \in \Gamma, \delta_i \in \{\pm 1\}$.

**Remark 15.** The construction of the path component Top-graph $\pi_0(\Gamma)$ also applies to Top-categories. Recall that for spaces $X, Y$, there is a canonical, continuous bijection $\psi : \pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$ which is not necessarily a homeomorphism [2]. Consequently, if $\Gamma$ is a Top-category (with continuous involution), then $\pi_0(\Gamma)$ naturally inherits the structure of a sTop-category (qTop-category) but is not always a Top-category (with continuous involution). While it is possible to avoid this difficulty by restricting to a cartesian closed category of spaces, we remain in the usual topological category in order to prove Theorem 1 in full generality.

The observation on products in the previous remark immediately extends to the following lemma.
Definition 18. A logical group. We now show each vertex group of a free Topological groupoid \( F_\Gamma \) is a monoid with continuous involution on \( C_\text{space} \). \( \Gamma \) vertex \( v \) groupoid \( a \) discrete Topological property. In the case that \( \Gamma \) is a reduced word \( e_1^\delta e_2^\delta \cdots e_n^\delta \in \mathcal{E}^+(\Gamma) \). It is straightforward to verify that this groupoid has the desired universal property. The underlying groupoid of \( F_\Gamma \) is simply the free groupoid generated by the underlying algebraic graph of \( \Gamma \), i.e. \( \text{Ob}(\mathcal{F}(\Gamma)) = \Gamma_0 \) and a morphism is a reduced word \( e_1^\delta e_2^\delta \cdots e_n^\delta \in \mathcal{E}^+(\Gamma) \). See [5, 12] for more on free groupoids. The topological structure of \( \mathcal{F}(\Gamma) \) is characterized as follows: Let \( \mathcal{F}_R(\Gamma) \) be the free groupoid on the underlying algebraic graph of \( \Gamma \) which is the quotient of \( \mathcal{E}^+(\Gamma) \) with respect to the word reduction functor \( R : \mathcal{E}^+(\Gamma) \rightarrow \mathcal{F}_R(\Gamma) \). Note that \( \mathcal{F}_R(\Gamma) \) is a \( \mathfrak{q}\text{-top} \)-groupoid. The free \( \mathfrak{q}\text{-top} \)-groupoid is the \( \tau \)-reflection

\[ \mathcal{F}(\Gamma) = \tau (\mathcal{F}_R(\Gamma)). \]

It is straightforward to verify that this groupoid has the desired universal property. In the case that \( \Gamma \) has a single vertex \( [2], \mathcal{E}^+(\Gamma) \) is the free topological monoid with continuous involution on \( \Gamma \) and \( \mathcal{F}(\Gamma) \) is the free topological group \( F_M(\Gamma) = F_G(\Gamma_+, \ast) \).

3.3 Vertex groups and free topological groups

We now show each vertex group of a free \( \text{Top} \)-groupoid is a free Graev topological group.

Definition 18. A \( \text{Top} \)-graph \( T \) is a tree if \( T \) is discrete and \( |T| \) is contractible. If \( \Gamma \) is a \( \text{Top} \)-graph, a tree \( T \subseteq \Gamma \) is maximal in \( \Gamma \) if \( T_0 = \Gamma_0 \). Even though \( T \) is itself a discrete \( \text{Top} \)-graph, the edge space \( T \) need not be open in \( \Gamma \). A tree groupoid is a groupoid \( \mathcal{G} \) such that each set \( \mathcal{G}(x, y) \) has exactly one element. Note that if \( T \) is a tree, then \( \mathcal{F}(\Gamma) \) is a discrete tree groupoid.

The standard argument that every graph contains a maximal tree is the same for \( \text{Top} \)-graphs.

Lemma 19. Every connected \( \text{Top} \)-graph contains a maximal tree.

Fix a \( \text{Top} \)-graph \( \Gamma \), a maximal tree \( T \subseteq \Gamma \), and a vertex \( v \in \Gamma_0 \). Let \( \mathcal{F}(\Gamma)(v) = \mathcal{F}(\Gamma)(v, v) \) be the vertex topological group at \( v \). Recall that if \( \Gamma \) has a single vertex \( v \), then \( \mathcal{F}(\Gamma) = \mathcal{F}(\Gamma)(v) \cong F_M(\Gamma) \cong F_G(\Gamma_+, \ast) \). Therefore, we restrict to the case when \( \Gamma \) has more than one vertex. In this case, \( T \) has non-empty edge space.

For vertex \( x \in \Gamma_0 \), let \( \gamma_{v,x} \) be the unique element of \( \mathcal{F}(T)(v, x) \). Define a retraction \( r_T : \mathcal{F}(\Gamma) \rightarrow \mathcal{F}(\Gamma)(v) \) of groupoids so that if \( \alpha \in \mathcal{F}(\Gamma)(x, y) \), then \( r_T(\alpha) = \gamma_{v,x} \alpha \gamma_{y,v} \). By definition, if \( i : \mathcal{F}(\Gamma)(v) \rightarrow \mathcal{F}(\Gamma) \) is the inclusion of the
vertex group, then \( r_{\tau}i \) is the identity of \( \mathcal{F}(\Gamma)(v) \). In fact, since composition in \( \mathcal{F}(\Gamma) \) is continuous, we have a \textbf{Top}-functor.

**Lemma 20.** \( r_{\tau} : \mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma)(v) \) is a retraction of \textbf{Top}-groupoids.

Let \( \sigma : \Gamma \to \mathcal{F}(\Gamma) \) be the canonical \textbf{Top}-graph morphism. It is known that the underlying group of \( \mathcal{F}(\Gamma)(v) \) is freely generated by the set \( r_{\tau}\sigma(\Gamma\setminus T) \) (See, for instance, [5, 8.2.3]).

Note that if \( y \in \Gamma \), then \( r_{\tau}\sigma(y) \) is the identity element of the group \( \mathcal{F}(\Gamma)(v) \) if and only if \( y \in T \). Let \( \Gamma/T \) be the quotient of the edge space \( \Gamma \) and choose the basepoint \( * \in \Gamma/T \) to be the the image of \( T \). Thus the function \( r_{\tau}\sigma : \Gamma \to \mathcal{F}(\Gamma)(v) \) induces a continuous injection \( s : \Gamma/T \to \mathcal{F}(\Gamma)(v) \) such that \( sq = r_{\tau}\sigma \) and where \( s(\gamma) \) is the identity element of \( \mathcal{F}(\Gamma) \). Since \( r_{\tau}\sigma(\Gamma\setminus T) \) freely generates \( \mathcal{F}(\Gamma)(v) \), the continuous group homomorphism \( \tilde{s} : F_G(\Gamma/T,*) \to \mathcal{F}(\Gamma)(v) \) induced by \( s \) is an isomorphism of groups.

**Theorem 21.** If \( \Gamma \) has more than one vertex and \( T \subset \Gamma \) is a maximal tree, then the vertex group \( \mathcal{F}(\Gamma)(v) \) is isomorphic to the free Graev topological group \( F_G(\Gamma/T,*) \).

**Proof.** It suffices to show the inverse of \( \tilde{s} : F_G(\Gamma/T,*) \to \mathcal{F}(\Gamma) \) is continuous. Let \( \sigma_* : \Gamma/T \to F_G(\Gamma/T,*) \) be the inclusion of generators and \( q : \Gamma \to \Gamma/T \) be the quotient map. The composition \( g = \sigma_*q : \Gamma \to F_G(\Gamma/T,*) \) may be viewed as a morphism of \textbf{Top}-graphs taking all vertices of \( \Gamma \) to the unique vertex of \( F_G(\Gamma/T,*) \). Since \( F_G(\Gamma/T,*) \) is a \textbf{Top}-groupoid, there is a unique \textbf{Top}-functor \( \hat{g} : \mathcal{F}(\Gamma) \to F_G(\Gamma/T,*) \) such that \( \hat{g}\sigma = g \). If \( i : \mathcal{F}(\Gamma)(v) \to \mathcal{F}(\Gamma) \) is the inclusion of the vertex group, the composition \( \hat{g}i : \mathcal{F}(\Gamma)(v) \to F_G(\Gamma/T,*) \) is continuous. We check that \( \hat{g}i \) is the inverse of \( \tilde{s} \).

Recall that \( sq = r_{\tau}\sigma \). Therefore

\[
\tilde{s}\hat{g}\sigma = \tilde{s}\hat{g} = \tilde{s}\sigma_*q = sq = r_{\tau}\sigma.
\]

Uniqueness of extensions then gives \( \tilde{s}\hat{g} = r_{\tau} \).

\[
\begin{array}{ccc}
\mathcal{F}(\Gamma) & \xrightarrow{\hat{g}} & F_G(\Gamma/T,*) \\
\updownarrow{s} & \downarrow{r_{\tau}} & \downarrow{i} \\
\mathcal{F}(\Gamma)(v) & \xleftarrow{\tilde{s}} & \Gamma/T
\end{array}
\]

It is now clear that \( \tilde{s}\hat{g}i = r_{\tau}i = id \). Finally, since \( \tilde{s}\hat{g}i\tilde{s} = r_{\tau}i\tilde{s} = \tilde{s} \) and \( \tilde{s} \) is injective, we have \( \hat{g}i\tilde{s} = id \). \( \square \)
4 The fundamental Top-groupoid of a Top-graph

4.1 Path spaces and the fundamental Top-groupoid

For a given space $X$, let $PX$ be the space of paths $[0, 1] \to X$ with the compact-open topology generated by subbasis sets $\{a[C, W] = \{a[a(C) \subseteq W]\} \mid C \subseteq [0, 1]$ is compact and $W \subseteq X$ is open. For a closed subinterval $K \subseteq [0, 1]$, let $L_K : [0, 1] \to K$ be the unique, increasing, linear homeomorphism. If $a : [0, 1] \to X$ is a path, let $a_K = a|_K \circ L_K$ be the restricted path of $a$ to $K$. If $K = \{t\} \subseteq [0, 1]$, take $a_K$ to be the constant path $c_{a(t)}$ at $a(t)$. The concatenation $a = a_1 \cdot a_2 \cdot \ldots \cdot a_n$ of paths $a_j$ such that $a_{j+1}(0) = a_j(1)$ is given by letting $a[\frac{t_j}{n}, \frac{t_{j+1}}{n}] = a_j$.

Consider a basic open neighborhood $U = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ of a path $a$ and any closed interval $K \subseteq [0, 1]$. Then $U_K = \bigcap_{\beta \in C, j \in \emptyset} \langle L_K^{-1}(K \cap C_j), U_j \rangle$ is an open neighborhood of $a_K$. If $K = \{t\}$ is a singleton, let $U_K = \langle [0, 1], \bigcap_{t \in C}, U_j \rangle$. On the other hand, if $\beta_K = a$, then $U^K = \bigcap_{j=1}^n \langle L_K(C_j), U_j \rangle$ is an open neighborhood of $\beta$. If $K = \{t\}$ so that $\beta = c_{\alpha(t)}$, let $U^K = \bigcap_{j=1}^n \langle \{t\}, U_j \rangle$.

Lemma 22. Let $U = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ be an open neighborhood in $PX$ such that $\bigcup_{j=1}^n C_j = [0, 1]$. Then

1. For any closed interval $K \subseteq [0, 1]$, $(U^K)_K = U \subseteq (U_K)^K$,
2. If $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = 1$, then $U = \bigcap_{j=1}^n \langle U_{\{t_{j-1}, t_j\}} \rangle_{\{0, 1, \ldots, n\}}$.

In the case that $X$ is a Top-graph $[\Gamma]$, recall that vertex neighborhoods $B(x, r)$ and edge neighborhoods $U \times (a, b)$ (as in Remark [9]) form a basis $B_T$ for the topology of $[\Gamma]$ which is closed under finite intersection. Thus sets of the form $\bigcap_{j=1}^n \langle \{\frac{t_j-1}{n}, \frac{t_j}{n}\}, U_j \rangle$, where $\{U_j\} \subseteq B_T$, give basis generating the topology of $P[\Gamma]$, which is convenient for our purposes. An open set of this form is said to be standard.

The path space $P[\Gamma]$ can be used to construct a Top-graph $\mathcal{P}_T$ of paths:

- The object space of $\mathcal{P}_T$ is the vertex set $\Gamma_0$ and $\mathcal{P}_T(x, y)$ is the subspace of $P[\Gamma]$ consisting of paths from $x$ to $y$. Though concatenation of paths gives a continuous operation $\cdot : \mathcal{P}_T(x, y) \times \mathcal{P}_T(y, z) \to \mathcal{P}_T(x, z)$, $\mathcal{P}_T$ is not a Top-category because concatenation is not associative. One can obtain a Top-category by replacing paths with Moore paths, however, to remain consistent with [4], we refrain from doing so.

We now recall the topologically enriched version of the usual fundamental groupoid used in [4] as applied to the case of Top-graphs.

Definition 23. The fundamental $q$Top-groupoid of a Top-graph $\Gamma$ is the $q$Top-groupoid $\pi^{q\text{Top}}(\Gamma, \Gamma_0) = \pi_0(\mathcal{P}_T)$ whose object space is $\Gamma_0$ and $\pi^{q\text{Top}}(\Gamma, \Gamma_0)(x, y)$ is the path component space $\pi_0(\mathcal{P}_T(x, y))$. The canonical quotient morphism is denoted $\pi : \mathcal{P}_T \to \pi^{q\text{Top}}(\Gamma, \Gamma_0)$. The fundamental Top-groupoid of $\Gamma$ is the $\tau$-reflection $\pi^\tau(\Gamma, \Gamma_0) = \tau(\pi^{q\text{Top}}(\Gamma, \Gamma_0))$.

The underlying groupoid of $\pi^{q\text{Top}}(\Gamma, \Gamma_0)$ and $\pi^\tau(\Gamma, \Gamma_0)$ is the familiar fundamental groupoid $\pi([\Gamma], \Gamma_0)$ with set of basepoints $[\Gamma]$ [5].
4.2 \( \pi^1(\Gamma, \Gamma_0) \) is a free Top-groupoid

**Definition 24.** A path \( \alpha : [0,1] \to |\Gamma| \) is an edge path if \( \alpha^{-1}(\Gamma_0) = \{0,1\} \). An edge path \( \alpha \) is trivial if it is a null-homotopic loop. Equivalently, an edge path is non-trivial if and only if the endpoints are distinct or if it traverses a generalized wedge of circles \( \Sigma(\Gamma(x),x) \subset |\Gamma| \) at a vertex \( x \). Let \( \delta_\Gamma \) be the wide sub-\( \text{Top} \)-graph of \( \mathcal{P}_\Gamma \) consisting of non-trivial edge paths.

The following Lemma is a straightforward application of the existence of Lebesgue numbers.

**Lemma 25.** If \( \mathcal{V} \) is an open neighborhood of an edge path \( \alpha \) in \( \delta_\Gamma(x,y) \), then there is a standard neighborhood \( \mathcal{A} = \bigcap_{n=1}^\infty \left\{ \left[ \frac{1}{n}, \frac{1}{n} \right], U \right\} \), such that \( \mathcal{A} \cap \delta_\Gamma(x,y) \subseteq \mathcal{V} \) and such that \( U_1, U_2, \ldots, U_{n-1} \) are edge neighborhoods.

Note for each edge \( e \in \Gamma(x,y) \), there is a canonical non-trivial edge path \( \alpha_x : [0,1] \to |\Gamma| \) where \( \alpha_x(t) \) is the image of \( (e,t) \in \Gamma(x,y) \times [0,1] \) in \( |\Gamma| \).

**Lemma 26.** There is a canonical embedding \( \Gamma^+ \to \delta_\Gamma \) of \( \text{Top} \)-graphs which induces an isomorphism \( \pi_0(\mathcal{C}(\Gamma^+)) = \pi_0(\mathcal{C}(\delta_\Gamma)) \to \pi_0(\mathcal{C}(\delta_\Gamma)) \) of \( \text{qTop} \)-categories.

**Proof.** The embedding is the identity on objects and \( e \mapsto \alpha_e \) for \( e \in \Gamma(x,y) \) and \( e^{-1} \mapsto \alpha_{e^{-1}} \) for \( e^{-1} \in \Gamma(y,x)^{-1} \). Here \( \beta(t) = \beta(1-t) \) denotes the reverse path of \( \beta \).

Define a \( \text{Top} \)-graph morphism \( \text{eval} : \delta_\Gamma \to \Gamma^+ \) as follows: It is the identity on vertices. If \( \alpha \in \delta_\Gamma(x,y) \), then \( \alpha(1/2) \) is the image of a point \( (g,s) \in \Gamma(x,y) \sqcup \Gamma(y,x) \times (0,1) \) in \( |\Gamma| \). Take \( \text{eval}(\alpha) = g \). It is straightforward to check that \( \text{eval} \) induces the inverse \( \text{qTop} \)-functor \( \pi_0(\mathcal{C}(\delta_\Gamma)) \to \pi_0(\mathcal{C}(\Gamma^+)) \).

Since concatenation \( (\alpha, \beta) \mapsto \alpha \cdot \beta \) of paths is continuous, the inclusion \( \delta_\Gamma \to \mathcal{P}_\Gamma \) gives rise to a \( \text{Top} \)-graph morphism \( \mathcal{C}(\delta_\Gamma) \to \mathcal{P}_\Gamma \), \( \alpha_1 \alpha_2 \ldots \alpha_n \to \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n \) on the free \( \text{Top} \)-category. Application of the path component space functor gives a \( \text{qTop} \)-functor \( \phi : \pi_0(\mathcal{C}(\delta_\Gamma)) \to \pi_0(\mathcal{P}_\Gamma) = \pi^{qTop}(\Gamma, \Gamma_0) \) to the fundamental \( \text{qTop} \)-groupoid.

**Lemma 27.** The \( \text{qTop} \)-functor \( \phi : \pi_0(\mathcal{C}(\delta_\Gamma)) \to \pi^{qTop}(\Gamma, \Gamma_0) \) is quotient.

**Proof.** Consider the following factorization of \( \pi \):

\[
\begin{array}{ccc}
\mathcal{C}(\delta_\Gamma) & \xrightarrow{\phi} & \mathcal{P}_\Gamma \\
\downarrow{\pi} & & \downarrow{\pi} \\
\pi_0(\mathcal{C}(\delta_\Gamma)) & \xrightarrow{\phi} & \pi^{qTop}(\Gamma, \Gamma_0)
\end{array}
\]

where \( \phi \) is the decomposition morphism defined as follows. If \( \alpha \in \mathcal{P}_\Gamma(x,y) \), there are finitely many intervals \([a_1, b_1], \ldots, [a_n, b_n] \subset [0,1] \) (ordered with respect to the ordering of \([0,1]\)) such that \( \alpha_i = \alpha_{[a_i, b_i]} \) is a non-trivial edge path. Now
\(D(a)\) is defined as the word \(\alpha_1 \alpha_2 \cdots \alpha_n\) in \(E(G)\). If no restriction of \(\alpha\) is a non-trivial edge path, then \(\alpha\) is a null-homotopic loop based at some vertex \(x \in \Gamma_0\) and so we take \(D(\alpha)\) to be the identity \(id_x\). Since \(\alpha\) and \(\alpha_1 \cdot \alpha_2 \cdots \cdot \alpha_n\) are homotopic paths, the diagram commutes. The decomposition morphism is a direct generalization of the decomposition function in [2, pp. 793]; it is important to note that \(D\) is only a morphism of underlying algebraic graphs since it is not continuous on edge spaces.

For convenience, rename the sets \([0, a_1], [b_1, a_2], \ldots, [b_{n-1}, a_n], [b_n, 1]\) (some of which may be singletons) as \(K_1, K_2, \ldots, K_n, K_{n+1}\). Note that each restriction \(\alpha_{K_i}\) is a trivial loop based at a vertex \(x_i\) which may be singleton as described above. In particular, \(x = x_1\) and \(x_n = y\).

Given vertices \(x, y \in \Gamma_0\), suppose \(U \subseteq \pi_1^{\phi^q}(\Gamma, \Gamma_0)(x, y)\) such that \(\phi^q(U)\) is open in \(\pi_0(E(G))(x, y)\). Thus \(q^{-1}(\phi^q(U))\) is open in \(E(G)(x, y)\). Since \(\pi\) is quotient, it suffices to show \(\pi^{-1}(U) = D^{-1}(q^{-1}(\phi^q(U)))\) is open in \(\mathcal{P}_x(x, y)\). Suppose \(x \in \pi^{-1}(U)\) and note \(D(\alpha)\) lies in the open neighborhood \(q^{-1}(\phi^q(U))\). If \(D(\alpha) = id_x\) for some \(x\), then the image of \(\alpha\) lies in the contractible vertex neighborhood \(B(x, 1)\). The neighborhood \(\{\beta \in \mathcal{P}_x(x, y) | \text{Im}(\beta) \subseteq B(x, 1)\}\) of \(\alpha\) contains only null-homotopic loops and is therefore contained in \(\pi^{-1}(U)\).

On the other hand, suppose \(D(\alpha) = \alpha_1 \cdots \alpha_n \subseteq q^{-1}(\phi^q(U))\) with decomposition as describe above. In particular, \(\alpha_i \in E(G)(x_i, x_{i+1})\). We construct a neighborhood of \(\alpha\) contained in \(\pi^{-1}(U)\). Recall that \(B_i = \{0, 1\}, B(x_i, r_i)\) is a neighborhood of \(\alpha_{K_i}\), and thus \(B_i^{K_i}\) is a neighborhood of \(\alpha\) for each \(i\). Since \(E(G)(x_i)\) is a \textit{Top}-category, there are open neighborhoods \(V_i\) of \(\alpha_i \in E(G)(x_i, x_{i+1})\) such that the product \(V_1 V_2 \cdots V_n\) is contained in \(q^{-1}(\phi^q(U))\). By Lemma 25 there is a standard neighborhood \(\mathcal{A} = \bigcap_{i=1}^n \left\{ \left[ \frac{1}{n_i}, \frac{1}{n_i} \right], U_i \right\}, n_i > 2\) of \(\alpha_i\) in \(\mathcal{P}\) such that \(U_i = V_i \times E(G)(x_i, x_{i+1}) \subseteq V_i\) and such that \(U_i, U_{i+1}, U_{n+1}\) are vertex neighborhoods and \(U_{2i}, U_{2i+1}\) are edge neighborhoods. In particular, choose the \(\mathcal{A}\) so that \(U_i \subseteq B(x_{i+1}, r_{i+1})\).

Now \(\mathcal{W} = \bigcap_{i=1}^n \mathcal{A}[a_i, b_i] \cap \bigcap_{i=1}^{n+1} B_i^{K_i} \cap \mathcal{P}_x(x, y)\) is an open neighborhood of \(\alpha\) in \(\mathcal{P}_x(x, y)\).

Suppose \(\beta \in \mathcal{W}\). We clearly have \(\beta(K_{i+1}) \subseteq B(x_{i+1}, r_{i+1})\), however if \(x_i = x_{i+2}\), it is possible that \(\beta_{[a_i, b_{i+1}]}\) does not hit the vertex \(x_{i+1}\). To deal with this possibility, we replace “small” portions of \(\beta\). For each \(i = 2, \ldots, n\), let \(s_i = L_{[a_i, b_{i+1}]} \left( 1 - \frac{1}{n_i} \right)\) and \(t_i = L_{[a_i, b_{i}]} \left( \frac{1}{n_i} \right)\) so that \(K_i = [b_{i-1}, a_i] \subset [s_i, t_i]\). Now define a path \(\gamma\) to equal the path \(\beta\) with the following exceptions: replace the portion of \(\beta\) from \(s_i\) to \(b_{i-1}\) with the canonical arc from \(\beta(s_i)\) to \(x_i\), take \(\gamma\) to be the constant at \(x_i\) on \([b_{i-1}, a_i]\), and replace the portion of \(\beta\) from \(a_i\) to \(t_i\) with the canonical arc from \(x_i\) to \(\beta(t_i)\). Since \(\gamma\) is given by changing \(\beta\) only in contractible neighborhoods, \(\gamma\) and \(\beta\) are homotopic paths, i.e. \(\pi(\beta) = \pi(\gamma)\). Moreover, \(\gamma_i = \gamma_{[a_i, b_i]}\) is an edge path for each \(i\) contained in \(U_i\). Thus \(D(\gamma) = \gamma_1 \gamma_2 \cdots \gamma_n \in U_1 U_2 \cdots U_n \subseteq V_1 V_2 \cdots V_n \subseteq q^{-1}(\phi^q(U))\).
Finally, we see that
\[ \pi(\beta) = \pi(\gamma) = \phi(q(\mathcal{P}(\gamma))) \in U \]
giving the inclusion \( \mathcal{W} \subseteq \pi^{-1}(U) \).

**Theorem 28.** The fundamental \( \text{Top} \)-groupoid \( \pi^1(\Gamma, \Gamma_0) \) is naturally isomorphic to the free \( \text{Top} \)-groupoid \( \mathcal{F}(\pi_0(\Gamma)) \).

**Proof.** The embedding \( \Gamma \to \mathcal{P}_1 \) given by \( e \mapsto \alpha_e \) induces a \( \text{Top} \)-graph morphism \( \pi_0(\Gamma) \to \pi_0(\mathcal{P}_1) = \pi^{\text{top}}(\Gamma, \Gamma_0) \). Additionally, the identity functor \( \pi^{\text{top}}(\Gamma, \Gamma_0) \to \pi^1(\Gamma, \Gamma_0) \) is a morphism of \( \text{qTop} \)-groupoids. The composition \( \sigma : \pi_0(\Gamma) \to \pi^1(\Gamma, \Gamma_0) \) of these two morphisms is a morphism of \( \text{Top} \)-graphs which induces a morphism \( \delta : \mathcal{F}(\pi_0(\Gamma)) \to \pi^1(\Gamma, \Gamma_0) \) of \( \text{Top} \)-groupoids. A straightforward generalization of \([3, 3.14]\) to \( \text{Top} \)-graphs with more than one vertex gives that \( \delta \) is an isomorphism of the underlying groupoids. Therefore, it suffices to check the inverse \( \delta^{-1} : \pi^1(\Gamma, \Gamma_0) \to \mathcal{F}(\pi_0(\Gamma)) \) is a \( \text{Top} \)-functor.

Consider the following commutative diagram. The upper horizontal functors are the \( \text{qTop} \)-isomorphism from Lemma \([25]\) and the canonical \( \text{qTop} \)-functor \( \psi : \pi_0(\mathcal{E}_0^\pm(\Gamma)) \to \mathcal{E}_0^\pm(\pi_0(\Gamma)) \) from Lemma \([16]\). The vertical functor \( R \) is the quotient \( \text{qTop} \)-functor given by word reduction (See Remark \([17]\) and \( \phi \) is the quotient \( \text{qTop} \) functor of Lemma \([27]\)).

\[
\begin{array}{ccc}
\pi_0(\mathcal{E}_0^\pm(\Gamma)) & \xrightarrow{\psi} & \mathcal{E}_0^\pm(\pi_0(\Gamma)) \\
\phi \downarrow & & \downarrow R \\
\pi^{\text{top}}_1(\Gamma, \Gamma_0) & \xrightarrow{\delta^{-1}} & \mathcal{F}_R(\pi_0(\Gamma))
\end{array}
\]

Since the top composition is a \( \text{qTop} \)-functor and \( \phi \) is quotient, \( \delta^{-1} : \pi^{\text{top}}_1(\Gamma, \Gamma_0) \to \mathcal{F}_R(\pi_0(\Gamma)) \) is continuous on hom-spaces (by the universal property of quotient spaces) and is therefore a \( \text{qTop} \)-functor. Applying the \( \tau \)-reflection gives that
\[
\delta^{-1} : \pi^1(\Gamma, \Gamma_0) = \tau(\pi^{\text{top}}_1(\Gamma, \Gamma_0)) \to \tau(\mathcal{F}_R(\pi_0(\Gamma))) = \mathcal{F}(\pi_0(\Gamma))
\]
is a \( \text{Top} \)-functor. \( \square \)

The topological group \( \pi^1(\Gamma, \Gamma_0)(v) \) at a vertex \( v \in \Gamma \) is, by definition, the topological fundamental group \( \pi_1^1([\Gamma], v) \) of \([3, 4]\). In light of Section \( 3.3 \) we have the following Corollary.

**Corollary 29.** The topological fundamental group \( \pi_1^1([\Gamma], v) \) of a \( \text{Top} \)-graph \( \Gamma \) is a free Graev topological group. In particular, if \( \Gamma \) has more than one vertex and \( \Gamma \subset \pi_0(\Gamma) \) is a maximal tree, then \( \pi_1^1([\Gamma], v) \cong F_0(\pi_0(\Gamma)/T, *) \). If \( \Gamma \) has a single vertex, then \( \pi_1^1([\Gamma], v) \cong F_0(\pi_0(\Gamma), *) \cong F_M(\pi_0(\Gamma)) \).

**Corollary 30.** Every free \( \text{Top} \)-groupoid is the fundamental \( \text{Top} \)-groupoid of a \( \text{Top} \)-graph.

**Proof.** According to Remark \([12]\) a given \( \text{Top} \)-graph \( \Gamma \) is isomorphic to \( \pi_0(h(\Gamma)) \) for some \( \text{Top} \)-graph \( h(\Gamma) \). By Theorem \([28]\) \( \pi^1(h(\Gamma), h(\Gamma)_0) \cong \mathcal{F}(\pi_0(h(\Gamma))) \cong \mathcal{F}(\Gamma) \). \( \square \)
5 Semicoverings and a proof of Theorem 1

We recall the theory of semicovering spaces and apply it to Top-graphs. Our definitions and notation are those used in [4]. Given a space $X$ and point $x \in X$, let $(\mathcal{P} X)_x$ and $(\Delta X)_x$ be spaces of paths and homotopies (rel. endpoints) of paths starting at $x$ respectively with the compact-open topology. Let $\mathcal{P} X(x, x')$ be the subspace of $(\mathcal{P} X)_x$ consisting of paths ending at $x'$. In particular, $\mathcal{P} X(x, x) = \Omega(X, x)$ is the space of based loops.

**Definition 31.** A semicovering is a local homeomorphism $p : Y \to X$ such that for each $y \in Y$, the induced maps $\mathcal{P} p : (\mathcal{P} Y)_y \to (\mathcal{P} X)_{p(y)}$ and $\Phi p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ are homeomorphisms. The space $Y$ is a semicovering (space) of $X$. If $\alpha$ is a path or homotopy of paths starting at $p(y)$, then $\bar{\alpha}_y$ denotes the unique lift of $\alpha$ starting at $y \in Y$.

Every covering (in the classical sense) is a semicovering, however, if $X$ does not have a simply connected cover (e.g. the Hawaiian earring), a semicovering of $X$ need not be a covering. Of particular importance to the current paper is the fact that a semicovering $p : Y \to X$ induces an open covering morphism $\pi^1 p : \pi^1 Y \to \pi^1 X$. In particular, for each $y_1, y_2 \in Y$, $p(y_1) = x_1$, the induced map $p_* : \pi^1 Y(y_1, y_2) \to \pi^1 X(x_1, x_2)$ is an open embedding of spaces (of topological groups when $y_1 = y_2$). Just as in classical covering theory, if $\beta$ is a loop based at $p(y)$, then $\tilde{\beta}_y$ is a loop based at $y$ if and only if $[\beta]$ lies in the image of the embedding $p_*$. Thus, since $\pi : \mathcal{P} X(x, x') \to \pi^1 X(x_1, x_2)$ is continuous, $\{[\beta] | \tilde{\beta}_y(1) = y_2 \}$ is an open subspace of $\mathcal{P} X(x_1, x_2)$.

Since Top-graphs can fail to be locally path connected, we cannot apply classical covering theory to study them. We use the fact that semicovering theory applies to certain non-locally path connected spaces called locally wep-connected spaces [4, Definition 6.4].

**Definition 32.** Let $X$ be a space.

1. A path $\alpha : [0, 1] \to X$ is well-targeted if for every open neighborhood $\mathcal{U}$ of $\alpha$ in $(\mathcal{P} X)_{\alpha(0)}$ there is an open neighborhood $V_1$ of $\alpha(1)$ such that for each $b \in V_1$, there is a path $\beta \in \mathcal{U}$ with $\beta(1) = y$.

2. A path $\alpha : [0, 1] \to X$ is locally well-targeted if for every open neighborhood $\mathcal{U}$ of $\alpha$ in $(\mathcal{P} X)_{\alpha(0)}$ there is an open neighborhood $V_1$ of $\alpha(1)$ such that for each $b \in V_1$, there is a well-targeted path $\beta \in \mathcal{U}$ with $\beta(1) = y$.

See also the closely related definition of (locally) well-ended path in [4]. A space $X$ is locally wep-connected if for every pair of points $x, y \in X$, there is a locally well-targeted path from $x$ to $y$.

Every locally path connected space is locally wep-connected. There are also many spaces which are locally wep-connected but not locally path connected. For example, if $\Gamma$ is a Top-graph with a single vertex, the generalized wedge of circles $[\Gamma] = \Sigma(\Gamma_+)$ is locally wep-connected [4, Proposition 6.7] but is only locally path connected if $\Gamma$ is locally path connected. The following Lemma generalizes this special case to arbitrary Top-graphs; the proof is nearly identical.
Lemma 33. Every Top-graph is locally wep-connected.

Proof. Since \(|\Gamma|\) is locally path connected at each vertex, it suffices to find a locally well-targeted path from a vertex to each point \(z \in |\Gamma| \setminus \Gamma_0\). Suppose \(z\) is the image of \((e, t)\) in \(\Gamma(x, y) \times (0, 1)\). Any path \(\alpha : [0, 1] \to |\Gamma|\) such that \(\alpha(0) = x, \alpha(1) = z\), and having image on the edge \([e] \times (0, 1) \subset |\Gamma|\) is locally well-targeted. The argument that \(\alpha\) is locally well-targeted is identical to that in [4, Proposition 6.7]. □

Since Top-graphs are locally well-connected, we may apply semicovering theory to obtain the last ingredient for our proof of Theorem 1.

Lemma 34. [4, Corollary 7.20] If \(\Gamma\) is a Top-graph, \(x \in \Gamma_0\) is a vertex, and \(\mathcal{H}\) is an open subgroup of the topological fundamental group \(\pi_1^*(|\Gamma|, x)\), then there is a semicovering \(p : Y \to |\Gamma|\), \(p(y) = x\) such that the induced homomorphism \(p_* : \pi_1^*(Y, y) \to \pi_1^*|\Gamma|, x)\) is a topological embedding onto \(\mathcal{H}\).

5.1 A semicovering of a Top-graph is a Top-graph

The following result generalizes the fact that a covering (in the classical sense) of a graph is a graph and provides the last ingredient for a proof of Theorem 1.

Theorem 35. A semicovering of a Top-graph is a Top-graph.

Proof. It suffices to assume the semicovering and Top-graph in question are connected. Let \(p : Y \to |\Gamma|\) be a connected semicovering of Top-graph \(\Gamma\). We find a Top-graph \(\Gamma\) such that \(|\Gamma| \cong Y\). Since \(\Gamma_0\) is a discrete subspace of \(|\Gamma|\) and \(p\) is a local homeomorphism, \(p^{-1}(\Gamma_0)\) is a discrete subspace of \(Y\). Define the vertex space \(\Gamma_0 = p^{-1}(\Gamma_0)\). For \(y_1, y_2 \in \Gamma_0\) such that \(x_i = p(y_i)\) define

\[ \Gamma(y_1, y_2) = \{ e \in \Gamma(x_1, x_2) | (\alpha_e)_{y_1}(1) = y_2 \} \]

with the subspace topology of \(\Gamma(x_1, x_2)\).

Define a map \(h : |\Gamma| \to Y\) as follows: The restriction of \(h\) to \(\Gamma_0\) is the identity. The map \(h_{y_1, y_2} : \Gamma(y_1, y_2) \times [0, 1] \to Y\) given by \(h_{y_1, y_2}(e, t) = (\alpha_e)_{y_1}(1)\) is continuous since \(\Gamma(x_1, x_2) \to \mathcal{P}_{|\Gamma|}(x_1, x_2)\), \(e \mapsto \alpha_e\) is continuous, \(\mathcal{P}p : (\mathcal{P}Y)_{y_1} \to (\mathcal{P}X)_{x_1}\) is a homeomorphism, and evaluation \(\mathcal{P}_Y \times [0, 1] \to Y, (\beta, t) \mapsto \beta(t)\) is continuous. The maps \(h_{y_1, y_2}\) induce the function \(h\) on the image of \(\Gamma(y_1, y_2) \times [0, 1]\) in \(\Gamma\). It follows from Lemma 36 that \(h\) is a bijection, Lemma 37 that \(h\) is continuous, and Lemma 38 that \(h\) is an open map. Therefore \(h\) is a homeomorphism. □

To prove the following Lemmas, we make some observations about the edge spaces \(\Gamma(y_1, y_2)\). Let \(A = \{ e \in \mathcal{P}_{|\Gamma|}(x_1, x_2) | e \in \Gamma(x_1, x_2) \}\) and recall \(B = \{ \beta \in \mathcal{P}_{|\Gamma|}(x_1, x_2) | \beta_{y_1}(1) = y_2 \}\) is open in \(\mathcal{P}_{|\Gamma|}(x_1, x_2)\). It is now clear that \(A \cap B\) is open in \(A\) and is the image of \(\Gamma(y_1, y_2)\) under the homeomorphism \(\Gamma(x_1, x_2) \cong A, e \mapsto \alpha_e\). Therefore, \(\Gamma(y_1, y_2)\) is an open subspace of \(\Gamma(x_1, x_2)\). It follows that whenever \(p(y_1) = x_1\),
• \( \widetilde{\Gamma}_{y_1} = \Gamma_{x_1} \) and

• \( \Gamma(x_1, x_2) \) decomposes as the topological sum \( \bigsqcup_{p(y_0)=x_2} \Gamma(y_1, y_2) \).

**Lemma 36.** \( h : \widetilde{\Gamma} \to Y \) is a bijection.

*Proof.* First, we show \( h \) is surjective. It suffices to consider a point \( y \in Y \setminus \Gamma_0 \). Note \( p(y) \) is the image of a pair \((e, t) \in \Gamma(x_1, x_2) \times (0, 1) \) in \( \Gamma \). Fix \( x_0 \in \Gamma_0 \) and \( y_0 \in p^{-1}(x_0) \) and let \( \beta \) be a path from \( y_0 \) to \( y \) in \( Y \). Since \( \Gamma \) is a connected **top**-graph, there is a sequence of vertices \( x_0, x_1, \ldots, x_{n-1}, x_n \), edges \( e_i \in \Gamma(x_{i-1}, x_i) \), and \( \delta_i \in \{ \pm 1 \} \) such that \( p\beta \) is homotopic (rel. endpoints) to the concatenation

\[
\alpha = \alpha_{\delta_1} \circ \cdots \circ \alpha_{\delta_n} \circ (\alpha_r)_{[0,1]}.
\]

In particular, \( \tilde{\alpha}_{y_0}(1) = y \). Let \( y_1 \) be the endpoint of the lift of \( \alpha_{\delta_1} \circ \cdots \circ \alpha_{\delta_n} \) starting at \( y_0 \) and \( y_2 = \tilde{\alpha}_e(1) \). By our choice of \( \alpha \), \( p(y_1) = x_1 \). Then the lift of \( (\alpha_r)_{[0,1]} \) starting at \( y_1 \) ends at \( h(e, t) = (\tilde{\alpha}_r)_{y_0}(t) = y \).

For injectivity, suppose \( z, z' \in \widetilde{\Gamma} \). If one of \( z \) or \( z' \) is a vertex and \( h(z) = h(z') \), then \( z, z' \in \Gamma_0 \) and it follows that \( z = z' \). Therefore it suffices to check that \( h \) is injective on \( \Gamma \setminus \Gamma_0 \). Suppose \( z \) is the image of \( (e, t) \in \Gamma(y_1, y_2) \times (0, 1) \) under \( h \). If \( h(z) = (\tilde{\alpha}_e)_{y_1}(t) = (\tilde{\alpha}_f)_{y_2}(u) = h(z') \), then \( \alpha_e(t) = \alpha_f(u) \) in \( \Gamma \), however, this only occurs if \( e = f \) and \( t = u \) and thus \( z = z' \). \( \square \)

**Lemma 37.** \( h : \widetilde{\Gamma} \to Y \) is continuous.

*Proof.* Since the maps \( h_{y_0, y} \) are continuous, \( h \) is continuous if \( \Gamma \) is given the quotient topology. However, we using the coarser topology on \( \Gamma \) (which may differ from the quotient topology at vertices). Thus it remains to check that \( h \) is continuous at each vertex of \( \Gamma \). Suppose \( y_0 \in \Gamma_0 \) and \( V \) is an open neighborhood of \( h(y_0) = y_0 \) in \( Y \) such that \( p \) maps \( V \) homeomorphically onto a vertex neighborhood \( B(x_0, r) \) of \( p(y_0) = x_0 \). Since \( h \) is a bijection (Lemma 36).

\[
h(B(y_0, r)) = \{ (\alpha_r)_{y_0}(t) \in Y | 0 < t < r, e \in \Gamma_{y_0} \} \cup \{ (\alpha_{y_0})_{y_0}(t) \in Y | 0 < t < r, f \in \Gamma_{y_0} \}.
\]

But \( p \) maps \( V \) homeomorphically onto the path connected set \( B(x_0, r) \). Therefore, the lifts of all paths in \( B(x_0, r) \) starting at \( y_0 \) have image in \( V \). It follows that \( h(B(y_0, r)) = V \). \( \square \)

**Lemma 38.** \( h : \widetilde{\Gamma} \to Y \) is an open map.

*Proof.* Since \( \Gamma \) is locally wep-connected (Lemma 33), \( Y \) is locally wep connected \( \mathbb{H} \) Corollary 6.12. Thus for each \( y \in Y \), evaluation \( ev_1 : (P\Gamma)_y \to Y \), \( ev_1(\beta) = \beta(1) \) is quotient \( \mathbb{I} \) Proposition 6.2. Additionally, if \( p(y) = x \), \( Pp : (P\Gamma)_y \to (P\Gamma)_x \) has a continuous inverse \( L : (P\Gamma)_x \to (P\Gamma)_y \). To show \( h \) is open, we use the fact that the composition \( ev_1 L : (P\Gamma)_x \to Y \), \( \beta \mapsto \beta_y(1) \) is quotient whenever \( p(y) = x \).
Fix a vertex \( y_0 \in \overline{\Gamma}_0 \) and \( p(y_0) = x_0 \). Suppose \( \bar{B}(y_0, r) \), \( 0 < r < 1 \) is a vertex neighborhood of \( y_0 \in \overline{\Gamma}_0 \). As in the previous lemma, if
\[
V = \{ (\alpha \gamma)_{y_0} (t) \in Y | 0 \leq t < r, \alpha \in \Gamma_{y_0} \} \cup \{ (\tilde{\alpha}_e)_{y_0} (t) \in Y | 0 \leq t < r, f \in \Gamma_{y_0} \},
\]
then \( h(\bar{B}(y_0, r)) = V \). Again, we use the fact that \( p(V) = B(x_0, r) \) and if \( \gamma : [0, 1] \to B(x_0, r) \) is the canonical arc from \( x_0 \) to a given point \( z \in B(x_0, r) \), then \( \gamma_{y_0} \) has image in \( V \). We claim that \( V \) is open in \( Y \).

Since \( ev_1 L \) is quotient, it suffices to show \( L^{-1}(ev_1^{-1}(V)) \) is open in \( (\mathcal{P}\Gamma)_{x_0} \). If \( \beta \in L^{-1}(ev_1^{-1}(V)) \), then \( \tilde{\beta}_{y_0}(1) \in V \) and thus \( \beta(1) \in B(x_0, r) \). Let \( \gamma \) be the canonical arc from \( x_0 \) to \( \beta(1) \) in \( B(x_0, r) \) and recall \( \text{Im}(\gamma_{y_0}) \subset V \). Thus \( \beta \cdot \gamma_{y_0} \) is a loop based at \( y_0 \) and \( [\beta \cdot \gamma] \) lies in the open subgroup \( p_*(\pi_1(Y, y_0)) \) of \( \pi_1(\Gamma, x_0) \). Since \( \pi : \Omega(X, x_0) \to \pi_1(X, x_0) \) is continuous, there is a basic open neighborhood \( U = \bigcap_{j=1}^n \left( \left[ \frac{i}{n} \right], \frac{i+1}{n} \right) \cup U_j \) of \( \beta \cdot \gamma \) in \( \mathcal{P}\Gamma \) such that \( U \cap \Omega(X, x_0) \subseteq \pi^{-1}(p_*(\pi_1(Y, y_0))) \).

Since \( U(x_0, r) \) is contractible, we may assume 1. \( n \) is even, 2. \( U_1 = B(x_0, r) \), and 3. \( U_k = B(x_0, r) \) for \( k \geq n/2 \). Now \( V = U_{[1/2]} \cap (\mathcal{P}\Gamma)_{x_0} \) is an open neighborhood of \( \beta \) in \( (\mathcal{P}\Gamma)_{x_0} \) which we claim is a subset of \( L^{-1}(ev_1^{-1}(V)) \). Suppose \( \beta' \in V \). Then \( \tilde{\beta}'(1) \in U_{n/2} = B(x_0, r) \) and, if \( \gamma' \) is the canonical arc from \( x_0 \) to \( \tilde{\beta}'(1) \), then \( \beta \cdot \gamma' \in \mathcal{U} \cap \Omega(X, x_0) \subseteq \pi^{-1}(p_*(\pi_1(Y, y_0))) \). Thus \( \beta \cdot \gamma_{y_0}(1) = y_0 \). Since \( \gamma_{y_0} \) has image in \( V \),
\[
ev_1 L(\beta) = \tilde{\beta}_{y_0}(1) = \gamma_{y_0}(1) \in V.
\]
It follows that \( L^{-1}(ev_1^{-1}(V)) \) is open in \( (\mathcal{P}\Gamma)_{x_0} \).

Let \( y_1, y_2 \in \overline{\Gamma}_0, p(y_i) = x_i \), and suppose \( U \times (a, b) \subseteq \overline{\Gamma}(y_1, y_2) \times (0, 1) \) is an edge neighborhood in \( \Gamma \). Note that
\[
W = h(U \times (a, b)) = \{ (\alpha \gamma)_{y_1} (t) \in Y | (e, t) \in \overline{\Gamma}(y_1, y_2) \times (0, 1) \}.
\]
We show \( L^{-1}(ev_1^{-1}(W)) \) is open in \( (\mathcal{P}\mathfrak{X})_{x_1} \).

Recall \( \overline{\Gamma}(y_1, y_2) \) is defined to be an open subspace of \( \Gamma(y_1, x_2) \). Therefore \( ph \) maps \( U \times (a, b) \) homeomorphically onto the corresponding edge neighborhood \( U \times (a, b) \subseteq \Gamma \). If \( \beta \in L^{-1}(ev_1^{-1}(W)) \), then \( \tilde{\beta}_{y_1}(1) = (\alpha \gamma)_{y_1} (t) \in W \) for some \( (e, t) \in U \times (a, b) \). Let \( \gamma = (\alpha \gamma)_{[0, 1]} \). Since \( \beta \cdot \gamma_{y_1} \) is a loop based at \( y_1 \), we have, as in the vertex neighborhood case, that \( \beta \cdot \gamma \) lies in the open neighborhood \( \pi^{-1}(p_*(\pi_1(\Gamma, x_1))) \subseteq \Omega(\Gamma, x_1) \). Take a basic open neighborhood \( U = \bigcap_{j=1}^n \left( \left[ \frac{i}{n} \right], \frac{i+1}{n} \right) \cup U_j \) of \( \beta \cdot \gamma \) in \( \mathcal{P}\Gamma \) such that \( U \cap \Omega(\Gamma, x_1) \subseteq \pi^{-1}(p_*(\pi_1(\Gamma, x_1))) \). In particular, we may assume 1. \( n \) is even, 2. \( U_1 = U_{n/2} = B(x_1, r) \) is a vertex neighborhood, 3. when \( n/2 \leq k < n \), \( U_k \) is an edge neighborhood of the form \( A \times (r_{n/2}, s_{n/2}) \) for an open set \( A \subseteq U \), and 4. \( (r_{n/2}, s_{n/2}) = (r_{n/2}+1, s_{n/2}+1) \subseteq (a, b) \). Now \( V = U_{[1/2]} \cap (\mathcal{P}\Gamma)_{x_1} \) is an open neighborhood of \( \beta \) in \( (\mathcal{P}\Gamma)_{x_1} \). Note that \( \beta(1) \in A \times (r_{n/2}, s_{n/2}) \).

We check that \( V \subseteq L^{-1}(ev_1^{-1}(W)) \). If \( \beta' \in V \), then \( \tilde{\beta}'(1) \in A \times (r_{n/2}, s_{n/2}) \subseteq U \times (a, b) \subseteq \Gamma \). If \( \beta'(1) \) is the image of \( (f, u) \in A \times (r_{n/2}, s_{n/2}) \subseteq \Gamma(x_1, x_2) \times (0, 1) \) in \( \Gamma \), then \( f \gamma(u) = \beta'(1) \). Choose any \( \epsilon > 0 \) such that \( \frac{n-1}{n} < \epsilon < 1 \). Define a path \( \gamma' \) so that
• $\gamma'(v) = (\alpha_f)|_{[0,1]}(v)$ for $v \in [0, e]$.
• $(\gamma'|_{[0,1]})$ is the canonical arc from $z = (\alpha_f)|_{[0,1]}(e)$ to $\alpha_f(u)$ in $A \times (r_{n/2}, s_{n/2})$.

Notice that $\gamma'$ is constructed so that 1. $\beta'(1) = \gamma'(1)$, 2. $\beta' \cdot \gamma'$ is a loop in $U \subseteq \pi^{-1}(p,(\pi_1([\Gamma], x_1)))$ based at $x_1$, and 3. $\gamma'$ is homotopic (rel. endpoints) to $(\alpha_f)|_{[0,1]}$ (consequently $\gamma'' y_1(1) = (\alpha_f y_1(u))$. Since $[\beta' \cdot \gamma'] \in p, (\pi_1([\Gamma], x_1))$, we have
$$ev_1 L(\beta) = (\beta') y_1(1) = \gamma'' y_1(1) = (\alpha_f y_1(u)) \in W.$$  
Thus $\mathcal{V} \subseteq L^{-1}(ev_1^{-1}(W))$. 

5.2 A proof of Theorem [1]

We conclude with a proof of Theorem [1]

Proof. Suppose $X$ is a space with basepoint $\ast \in X$ and $H$ is an open subgroup of the free topological group $F_G(X, \ast)$. Let $h(X)$ be a space such that $\pi_0(h(X)) = X$ (see Remark [12]) and $\Gamma$ be the Top graph with $\Gamma_0 = \{a, b\}$ (i.e. two vertices), $\Gamma(a, b) = h(X)$, and $\Gamma(b, a) = \emptyset$. Note that the edge space of $\pi_0(\Gamma)$ is precisely $X$. By Theorem [28], $\pi^1(\Gamma, \Gamma_0)$ is isomorphic to the free Top groupoid $\mathcal{F}(\pi_0(\Gamma))$. A tree $T \subseteq \pi_0(\Gamma)$ is given by taking $T_0 = \{a, b\}$ with edge space $T = \{\ast\}$. Note $\pi_0(\Gamma)/T \cong X$ as based spaces. Theorem [21] gives the middle isomorphism in
$$\pi^1(T, \Gamma_0)(a) = \pi^1(\Gamma, \Gamma_0)(a) \cong F_G(\pi_0(\Gamma)/T, \ast) \cong F_G(X, \ast).$$

By Lemma [54] there is a semicovering $p : Y \to \pi^1(T, \Gamma_0)(a)$ such that the induced homomorphism $\pi^1(Y, y) \to \pi^1(T, \Gamma_0)(a) \cong F_G(X, \ast)$ is a topological embedding onto $H$. By Theorem [35], the semicovering space $Y$ is a Top graph. Finally, Corollary [29] applies to $Y$ to give that $\pi^1(Y, y) \cong H$ is a free Graev topological group.

References

[1] Arhangel’skii, Tkachenko, Topological Groups and Related Structures. Atlantis Studies in Mathematics, 2008.

[2] J. Brazas, The topological fundamental group and free topological groups, Topology Appl. 158 (2011) 779–802.

[3] J. Brazas, The fundamental group as a topological group, Preprint, arXiv:1009.3972v5, 2012.

[4] J. Brazas, Semicoverings: a generalization of covering space theory, Homology Homotopy Appl. 14 no. 1 (2012), 33–63.

[5] R. Brown, Topology and Groupoids, Booksurge PLC, 2006.
[6] R. Brown, J.P.L. Hardy, *Subgroups of free topological groups and free topological products of topological groups*, J. London Math. Soc. (2) 10 (1975), 431–440.

[7] R. Brown, *Some non-projective subgroups of free topological groups*, Proc. Amer. Math. Soc. 52 (1975) 443–440.

[8] F. Clarke, *The commutator subgroup of a free topological group need not be projective*, Proc. Amer. Math. Soc. 57 no. 2 (1976) 354–356.

[9] Graev, M.I. *Free topological groups*. Amer. Math. Soc. Transl. 8 (1962) 305-365.

[10] D. Harris, *Every space is a path component space*, Pacific J. Math. 91 (1980) 95–104.

[11] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[12] P. Higgins, *Notes on categories and groupoids*, Vol. 32, Van Nostrand Reinhold Co. London, 1971, also: Reprints in Theory Appl. Categories No. 7 (2005).

[13] D.C. Hunt, S.A. Morris, *Free subgroups of free topological groups*, Proc. Second Internat. Conf. Theory of Groups, Canberra, Lecture Notes in Mathematics 372 (Springer, Berlin, 1974), 377–387.

[14] G. Kelly, *Basic concepts of enriched category theory*, Vol. 64 of London Math. Soc. Lec. Notes Series, Cambridge University Press, 1982, also: Reprints in Theory Appl. Categories 10 (2005).

[15] Markov, A.A. *On free topological groups*. Izv. Akad. Nauk. SSSR Ser. Mat. 9 (1945) 3-64 (in Russian); English Transl.: Amer. Math. Soc. Transl. 30 (1950) 11-88; Reprint: Amer. Math. Soc. Transl. 8 (1) (1962) 195-272.

[16] P. Nickolas, *A Schreier theorem for free topological groups*, Bulletin of the Australian Math. Soc. 13 (1975) 121–127.

[17] P. Nickolas, *Subgroups of the free topological group on [0,1]*, J. London Math. Soc. (2) 12 (1976) 199–205.

[18] H. Porst, *On the existence and structure of free topological groups*, Category Theory at Work (1991) 165–176.

[19] O. Schreier, *Die Untergruppen der freien Gruppen*, Abh. Mat Sem. Univ. Hamburg 3 (1927) 167-169.

[20] O.V. Sipacheva, *The Topology of Free Topological Groups*. J. Math. Sci. Vol. 131, No. 4, (2005), 5765-5838.

[21] B.V.S. Thomas, *Free topological groups*. General Topology and its Appl. 4 (1974) 51–72.