An Overview of Transverse Momentum Dependent Factorization and Evolution

T. C. Rogers

Department of Physics, Old Dominion University, Norfolk, VA 23529, USA
Theory Center, Jefferson Lab, 12000 Jefferson Avenue, Newport News, VA 23606, USA

September 15, 2015 JLAB-THY-15-2133

Abstract. I review TMD factorization and evolution theorems, with an emphasis on the treatment by Collins and originating in the Collins-Soper-Sterman (CSS) formalism. I summarize basic results while attempting to trace their development over that past several decades.

PACS. 12.38.Aw – 12.38.Bx

1 Introduction

I will summarize the basic ideas of the Collins-Soper-Sterman (CSS) approach to TMD factorization [1][2][3][4][5], and the updated version in Ref. [6, Chapt. (10,13,14)], for formulating transverse momentum dependent factorization. In this context, “transverse momentum dependent” (TMD) refers to QCD treatments of inclusive observables with at least one perturbatively hard scale \( Q \), and a separate transverse momentum \( q_T \) that can vary from 0 to order \( Q \). It also refers to related objects like TMD parton distribution functions (PDFs) and TMD fragmentation functions (FFs). Studies of TMD objects have been driven by a diverse set of motivations that include testing perturbative QCD, probing hadronic structure, and providing calculations for general particle and nuclear physics experiments.

A trend in TMD physics is that initial intuition frequently needs to be revised as quantum field theoretical details come into focus. I will build on this observation to motivate greater general interest in TMD physics as an arena for unpacking foundational QCD concepts.

In its complete form, a TMD factorization theorem should apply to a variety of processes and allow for comparisons between them. But for a concrete discussion, I begin by considering Drell-Yan (DY) scattering. The process is shown in cartoon form in Fig. [1] In the center of mass frame, a hadron enters from the left with large “plus” momentum and another enters from the right with large “minus” momentum. An antiquark from one hadron annihilates with a quark from the other, and the resulting virtual photon splits into a lepton-antilepton (\( l^+l^- \)) pair with total four-momentum \( q \). The relevant observable is

\[
\frac{d\sigma}{dq^4dq}\, ,
\]

where \( d\Omega \) is the phase space of the \( l^+l^- \) pair. Dependence on the total transverse momentum \( q_T \) of the \( l^+l^- \) pair.

Fig. 1. A cartoon depiction of Drell-Yan scattering with an antiproton entering from the left and a proton entering from the right. The explosion is the hard part. In (a), the gluon is taken to be part of the proton wavefunction, making this a Type I picture in the language of Ref. [7]. In (b) the gluon is associated with hard perturbative QCD radiation, making this a type II picture. (Or course, there are many more gluons than just the one shown explicitly.)
is of special interest at small values and when possible
correlations to spin dependence are included. Descriptions
of Eq. (1) naturally generalize to the production of other
colorless bosons ($W_{\pm}, Z_{\text{Higgs}}$).

Other processes where TMD factorization applies in-
clude semi-inclusive deep inelastic scattering (SIDIS) and
$t^+t^-$ annihilation into back-to-back jets or hadrons. In
each of these, there are two large forward and backward
directions analogous to the large positive and negative ra-
pidities of the proton and antiproton in Fig. 1 and each
has a hard scale $Q$ and transverse momentum that be-
comes sensitive to intrinsic transverse momentum when
small.

The question of how to describe TMD cross sections
like Eq. (1) in QCD has traditionally been approached
from two rather opposite conceptual starting points, which
Feynman, Field and Fox [7] (FFF) identified early on, call-
ing them “Type I” and “Type II” descriptions. They dif-
ferrented between the two pictures as follows:

**Type I**: A description in terms of colliding bound states
is taken very literally, with the total transverse mo-
m entum of the $t^+t^-$ pair assumed to originate entirely
from the nonperturbative intrinsic motion of partons,
such as that associated with an incoming “wavefunc-
tion.” (See Fig. 1(a).)

**Type II**: The transverse momentum of the $t^+t^-$ pair is
understood to originate from the radiation of gluons
before and after the collision. In a type II picture, it is
assumed that all or most of such radiation is describ-
able using small $\alpha_s$ methods. (See Fig. 1(b).)

Note the similarity between Fig. 1 of this paper and Fig. 6
of FFF. In reference to their Fig. 6, FFF write

“While there has been much speculation about how much
of the dimuon $k_T$ spectra shown in Fig. 7 [DY data
from Ref. SD1011] is due to the wave function
(Type I) and how much is explained by QCD per-
turbation calculations (Type II).”

To some extent, this quote remains true today, though the
situation is in some ways even more interesting after the
advances in QCD theory of the intervening decades.

Although it is a useful starting point for general dis-
cussions of TMD physics, the type I / type II dichotomy
is rather artificial, and a sharp distinction becomes elu-
sive when one tries to formalize it. Figure 6 of FFF al-
ready exhibits some of this difficulty. Compare, for ex-
ample, the relatively large intrinsic transverse momentum of
$\langle k_T \rangle \sim 848$ MeV found by FFF with other expec-
tations suggested around the same period, such as the
$\langle k_T \rangle \sim 300$ MeV proposed in Ref. [12] on the basis of a
parton model. Also, given what is now understood about
nonperturbative evolution, a significant fraction of the
transverse momentum width is likely actually due to non-
perturbative radiation that does not fit into either a Type
I or Type II category in an obvious way.

In type I oriented approaches of the past, one typically
finds discussions of TMD parton models, TMD PDFs, and
effects from nonperturbative wavefunctions. It is an ap-
proach that is used to address problems in hadron struc-
ture, such as the orbital angular momentum composition of
hadrons. See, for example, Ref. [13] from this collection
for a review.

By contrast, in type II oriented approaches one typi-
cally finds discussions of fixed high order calculations
and/or $q_T$-resummation, with nonperturbative transverse
momenta only entering in the form of small corrections.
Applications are to be to high energy physics and very
large hard scales, where nonperturbative effects tend to
get washed out.

One important development of roughly the last decade
is a trend toward a convergence of Type I and type II
oriented approaches into a single TMD formalism. This
will be an organizing theme for this review.

In Sect. 2 I will expand on the motivations for TMD
physics that came mainly from the hadron structure
perspective, and which has been traditionally seen as a more
type-I-oriented perspective, while in Sect. 3 I briefly men-
tion the type-II-oriented perspective. In Sect. 4 I will give
an overview of the development of full QCD approaches
to TMD factorization. In Sect. 5 I will summarize the ba-
sic formulas of TMD factorization as they now stand. In
Sect. 6 I will discuss solutions to the evolution equations,
and in Sect. 7 I will end with concluding remarks.

2 Type I Approaches: TMD functions in
hadronic structure and nonperturbative
physics

For describing Type I physics, the parton model is of-	en generalized to include an intrinsic transverse momen-
tum for parton distribution functions and fragmentation
functions – see, for example, Ref. [14, Eq. (9.13)] and the
surrounding discussion. In the DY case, for example, one
writes

\[
\frac{d\sigma}{dq^2d\hat{\Omega}} = \sum_{j,j'} H_{jj'}
\times \int d^2k_T F_{j/A}(x_A,k_T,S_A)F_{j'/B}(x_B,q_T-k_T,S_B)
+ \text{p.s.c.}
\]  

(2)

The basic structure is analogous to collinear factorization,
but the usual collinear parton distribution functions, with
their dependence on longitudinal momentum fractions $x_A$
and $x_B$, are replaced by TMD PDFs with additional de-
pendence on the intrinsic transverse momenta ($k_{T,A} = k_T$
and $k_{T,B} = q_T - k_T$). $F_{j/A}(x_A,k_T,S_A)$ labels a proba-
bility density for finding a parton of flavor $f$ inside a hadron
of species $H$. The overall hard part is $H_{jj'}$. It is usu-
ally set equal to the zeroth order partonic vertex in a type I
approach. Possible spin dependence is indicated by $S_H$.
The “p.s.c.” means “power-suppressed corrections.” For
processes like DISID and $e^+e^-$ annihilation into back-to-
back hadrons, formulas analogous to Eq. (2) are needed,
but with TMD fragmentation functions.
From here forward, I will refer to TMD parton distribution functions (TMD PDFs) and TMD fragmentation functions (TMD FFs) generically as “TMDs.”

The sensitivity to an intrinsic transverse direction makes TMDs natural objects of interest for spin physics because new nonperturbative correlations become possible relative to the collinear leading twist case. As early as 1978, Cahn used TMD PDFs to describe azimuthal asymmetries in SIDIS [15]. Intrinsic transverse momentum can become correlated with various components of spin and induce asymmetries in the cross section. In 1990, Sivers proposed a now famous TMD mechanism [16] for explaining the asymmetries in the cross section. In 1995/1996, Mulders and Tangerman classified the TMD dependence:

$$F_{q/p}(x,k_T) = f_{q/p}(x)\Theta(k_T),$$

and then requiring

$$\int d^2k_T F_{q/p}(x,k_T) = f_{q/p}(x).$$

The modulating factor $\Theta(k_T)$ is usually taken to be Gaussian and $x$ and $z$ independent. See, for example, Ref. [13 Eq. (9.14)].

While the role of nonperturbative transverse momentum was becoming less of a focus in many of the applications of $q_T$-resummation to high energy physics, efforts like those summarized in the last few paragraphs, particularly when applied to spin physics, focused on intrinsic transverse momentum as a way to study non-trivial aspects of fundamental QCD.

However, the work discussed in this section so far was mostly done in the context of a TMD parton model or parton-model-like description. The situation becomes very interesting when going beyond a TMD parton model picture. In incorporating perturbative QCD, one expects the TMD functions to acquire scale dependence through renormalization group (RG) dependence, analogously to collinear PDFs. Perturbative QCD predicts a large transverse momentum dependence that is power-like rather than Gaussian. In 1991, Chay, Ellis and Stirling used TMD PDFs to describe azimuthal asymmetries in SIDIS, with a matching to perturbative behavior at large transverse momentum [31]. A more recent discussion of the matching of large and small transverse momentum regions in TMD functions is in Ref. [32].

Much of the work of the full QCD approach began very early, but was not commonly incorporated into type-I nonperturbative physics studies like those discussed above until relatively recently. This will be discussed in more detail in Sect. 4.

3 Type II physics and collinear factorization

In traditions that approach observables like Eq. (1) from a more Type-II-like perspective, one typically starts from calculations of large transverse momentum in perturbative QCD, and attempts to extend the description to smaller $q_T$. One works entirely in collinear factorization so that the only nonperturbative objects that appear are the collinear PDFs and FFs. One example is transverse momentum resummation [33,34], which incorporates large logarithms of $q_T/Q$ to all orders in $\alpha_s$. These approaches typically assume $\Lambda_{QCD} \ll q_T \ll Q$ and ignore intrinsic nonperturbative transverse momentum effects, or at least refrain from accounting for them with detailed QCD considerations. In certain practical circumstances, such as at very high energies and large hard scales, these approaches may be sufficient, because sensitivity to nonperturbative transverse momentum become suppressed in the limit of infinite $Q$, even down to small $q_T$ [35]. The advantage is that calculations can be done without needing nonperturbative input that is often unknown or poorly constrained. However, working from a purely type II approach means that one cannot directly connect results to lower $Q$ measurements where nonperturbative transverse momentum definitely becomes important, and one also abandons the study of nonperturbative transverse structure itself via the extraction of TMD functions. Of course, in a full QCD treat-
ment resummation-like results should emerge naturally in the large $Q$ limit.

For more discussion of $q_T$-resummation, see for example Ref. [11] Chapt. 9.0 and Ref. [36] Chapt. 6 and references therein. See also the cautionary remarks regarding $q_T$-resummation techniques in Ref. [3].

4 TMDs and QCD

Notions of intrinsic transverse momentum and TMD functions appeared early on in considerations of full QCD \[37\,38\]. It is clear from the FFF discussions that a complete QCD formalism, like a TMD factorization formalism, would involve combination of type I and type II physics. In fact, the CSS formalism is a TMD factorization, though the way it was originally presented and subsequently applied possibly discouraged its rapid adoption in areas like spin physics. An early exception to the tendency to neglect TMD evolution in hadron structure phenomenology are the papers of Boer [39,40]. These are the first cases I know of where CSS-style evolution is applied directly to the phenomenology of single-spin asymmetries and azimuthal asymmetries directly identified with TMDs in the type-I sense of Sect. 2. See also Refs. [41,42]. CSS style treatments similar to Refs. [13,14,15,16] for unpolarized SIDIS were extended to the polarized case in Refs. [43,44].

An extra complication with TMD parton model approaches is that definitions that use the naive number density operator contain extra “light-cone” divergences. The light-cone divergences remain if light-like Wilson lines are used to enforce gauge invariance in TMD definitions, even if infrared and ultraviolet divergences are regulated. Unlike the normal infrared divergences, which signal the onset of genuine nonperturbative physical phenomena in the region of soft physics, the light-cone divergences are artifacts of the approximations that separate the cross section into different factors for widely separated regions of rapidity. (They are partly artifacts of ignoring the role of soft gluons.) Light-cone divergences describe gluons with infinite rapidity in the direction opposite that of the parent hadron. For the approximations to be consistent with factorization, therefore, the light-cone divergences need to be regulated and dealt with in some way.

In 1981, Collins and Soper (CS) introduced operator definitions for TMD PDFs and TMD FFs \[1\] Eqs. (2.1) and (4.9), and these definitions remain adequate for many purposes. The TMD FFs from Ref. [1] were used in the derivation of the CS equation in Ref. [2]. See also Eq. (3.8)]. Light-cone divergences were handled by defining TMDs in a non-light-like axial gauge. Early discussions of light-cone divergences and the axial-gauge-method of dealing with them are discussed in the pioneering work of Refs. [37,45]. (These papers also contain useful references for many of the now standard techniques, such as the use of non-light-like axial gauges.) In particular, Ref. [37,45] showed how TMD parton correlation functions acquire dependence on an auxiliary scale $\zeta = (2P \cdot n)^2/(-n^2)$, where $P$ is the proton four-momentum and $n$ is a non-light-like gauge fixing vector $n^2 \neq 0$. The dependence on $\zeta$ gives rise to logarithmic scaling violations in a complete cross section. For the Sudakov form factor, a derivation of the corresponding evolution in perturbative QCD was given in Ref. [49], and for the CS TMD functions in Refs. [12]. An analysis of TMD PDFs in structure functions is given in Ref. [50]. The $\zeta$ scale, with its connection to the gauge fixing vector $n$, can be thought of as a cutoff on light-cone divergences. The CS equation gives the evolution with respect to the direction of the gauge fixing vector $n$ and restores predictive power which would otherwise be lost by having an extra parameter $\zeta$.

The TMDs, defined in a non-light-like axial gauge as in Refs. [1,2], use the same auxiliary scales $\zeta$ as in Ref. [48]. Since they make explicit use of the gauge fixing vector in treating light-cone divergences, the early definitions of the TMDs in Refs. [1,2] were not gauge invariant. Treatments in Ref. [34] did propose gauge invariant definitions for the TMD PDFs, using non-light-like Wilson lines to regulate light-cone divergences, and similar procedures have come to be preferred. However, the CSS formalism for hadron-hadron scattering, as it was presented in Ref. [5], was based on the earlier derivation of TMD factorization for $e^+e^-$ annihilation into back-to-back jets with the non-light-like axial gauge definitions of Ref. [1]. The original definitions of the TMD functions are also modified in Ref. [5] relative to those of Refs. [1,2] such that the overall hard factor is unity and there is no explicit $U(b)$ in the factorization formula. (See the footnote at Eq. (3.3) of Ref. [5] and the discussion that begins with that equation.)

In most derivations of factorization, especially when initial state hadrons are involved, determining a method for dealing with gluons in the “Glauber” region is a major step toward the ultimate factorization, and it is the source of many of the subtleties that affect the TMD definitions. The Glauber region describes gluons whose longitudinal momentum components vanish while the the transverse components remain small (say $\sim Q_{CD}$) but fixed. The approximations that would normally allow one to apply Ward identities and eikonalize soft and collinear gluons fail in the Glauber region. So Glauber gluons threaten to spoil factorization \[52,53\]. Therefore, any derivation of factorization (either collinear or TMD) must show that Glauber region contributions either cancel in a sum over all graphs, or are avoided by contour deformations in the integrals over gluon momenta. Observables that involve collisions between two hadrons are especially challenging because gluon exchanges between parton spectators are “pinched” in the Glauber region, blocking straightforward

---

2 The term “decay function” was used in Refs. [1,2] rather than “fragmentation function.”

3 Early work such as Refs. [45,11,25] did not use the terminology “TMD,” which became common only later.

4 $U(b)$ is related to the well-known soft factor, often called $S(b)$.

5 An analogy can be made between Glauber gluon exchanges at the partonic level and multiple nucleon interactions in a Glauber model of nucleus-nucleus scattering.
contour deformations. For inclusive DY, the Glauber region contributions can be shown to cancel for spectator-spectator interactions.\textsuperscript{6} See also Refs.\textsuperscript{51,54} for more recent pedagogical overviews.\textsuperscript{5}

In the TMD case, many of the steps for showing a cancellation of Glauber region effects, especially in spectator-spectator interactions, carry over from the inclusive collinear case. CSS used this in Ref.\textsuperscript{5} to carry TMD factorization results originally obtained for $\ell^+\ell^-$ annihilation over to the DY case.

The details of how Glauber regions are avoided by contour deformations are closely connected to establishing good TMD definitions.\textsuperscript{58} The corresponding subtleties are associated with many of the nonintuitive aspects of TMD factorization (see the discussion of the Sivers effect below).

The original CS definitions in Refs.\textsuperscript{12}, and subsequent definitions in Ref.\textsuperscript{5} that are based on them, are sufficient for capturing much of the physics needed to set up basic factorization theorems. The CSS presentation is the starting point for successful applications to pion phenomenology, especially in unpolarized Drell-Yan-type processes and $e^+e^-$ annihilation into back-to-back hadrons and jets. It was extended to the SIDIS case in Refs.\textsuperscript{43,44}. The CSS formalism in this or roughly similar forms has been widely applied. See, for example, Refs.\textsuperscript{59,60,61,62,63,64,65}. It is often the most convenient way of formulating cross section calculations for some practical calculations, especially in contexts where there is comparatively little sensitivity to, and/or interest in, the precise details of nonperturbative transverse momentum. It also clearly displays the underlying simplicity of solutions to the TMD evolution equations, and it makes very explicit the matching to collinear correlation functions and collinear factorization in the limits of large $q_T$ and $Q$.

However, these early definitions had shortcomings that made them non-ideal for confronting some of the issues discussed in Sect.\textsuperscript{2}. Some of the problems are mainly organizational. One possibly confusing aspect of the presentation in Ref.\textsuperscript{5} is that the central result—that the formalism is, first and foremost, a TMD factorization formalism—appeared only later in the paper, in Eqs. (5.2,5.8). The first equation of the paper, which might appear to a casual reader to be the main result, contains no explicit intrinsic nonperturbative transverse coordinate ($b_T$) dependence. Perhaps as a result, the CSS formalism is now frequently referred to as a resummation. However, it was intended to be more powerful than resummation methods; the CSS formalism was meant to be a true TMD factorization formalism, with a valid pQCD perturbation expansion of the hard part and renormalization group equations for all $b_T$, even in the limit of $b_T \rightarrow \infty$ where $b_T$-dependence is nonperturbative (Collins, private communication).

If the hard scale is extremely large, the effects of perturbative radiation becomes so dominant that all nonperturbative transverse momentum effects are washed out even for $q_T \approx 0$.\textsuperscript{33} The process gradually becomes entirely a type II process. In this sense, questions of whether interesting nonperturbative TMD phenomena like the Sivers effect are washed out at larger $Q$ are correlated to the question about whether $Q$ is large enough to make perturbative radiative corrections negligible. The process one gets in the Drell-Yan process is the opposite sign to the one in SIDIS, not that it vanishes. Thus, what might at first seem like a contradiction between factorization and the way this procedure gets modified to factorize different processes introduces a process dependent sign and connection between TMD definitions, as they had been presented up to that time, and their origins in a factorization derivation. The TMD factorization derivation relies on contour deformations that avoid the Glauber region, and the way this procedure gets modified to factorize different processes introduces a process dependent sign and the associated Wilson line direction.

Another issue with the factorization as organized in Ref.\textsuperscript{5} is that perturbatively calculable process dependence was moved out of the hadronic part of the factorization. In 2002, Brodsky, Hwang, and Schmidt (BHS)\textsuperscript{71} used an explicit model calculation to demonstrate that final state interactions in SIDIS can give a transverse single spin asymmetry at leading power in $Q$. This was presented as a direct conflict with factorization itself; they argued that the effect cannot be associated with parton densities or fragmentation functions. But Collins showed\textsuperscript{72} that in fact the result is consistent with TMD factorization. Instead the BHS calculation effectively demonstrated that a Sivers-like effect is non-vanishing.

Even when nonperturbative $b_T$-dependence is included, the factorization formulas of Ref.\textsuperscript{5} do not directly resemble TMD parton model formulas like Eq.\textsuperscript{1}. The connection between TMD functions (including the many subtleties involved in defining those functions) and the evolved factors used in actual cross sections becomes somewhat indirect. An example of how this can lead to practical consequences can be seen in discussions of the Sivers function. But Collins showed\textsuperscript{72} that in fact the result is consistent with TMD factorization. Instead the BHS calculation effectively demonstrated that a Sivers-like effect is non-vanishing. While the TP-invariance argument\textsuperscript{26} mentioned earlier appeared to show that the Sivers function vanished, there is a loop-hole arising from the Wilson lines in a gauge-invariant definition of TMD functions. When applied to those definitions, TP invariance shows that the Sivers function used in the Drell-Yan process has the opposite sign to the one in SIDIS, not that it vanishes. Thus, what might at first seem like a contradiction between factorization and the direct calculations of Ref.\textsuperscript{71} is largely due to an unclear connection between TMD definitions, as they had been presented up to that time, and their origins in a factorization derivation. The TMD factorization derivation relies on contour deformations that avoid the Glauber region, and the way this procedure gets modified to factorize different processes introduces a process dependent sign and the associated Wilson line direction.

Reference\textsuperscript{3} pg. 12 notes an interesting similarity between Glauber cancelations and the AGK unitarity cancelations in Regge theory.
out by de Florian and Grazzini in Refs. [73,74]. Catani et al. subsequently developed an approach to organizing nonuniversal contributions into a separate hard factor [75,76,77,78,79]. But with well-defined TMD definitions, including a specification of a renormalization scheme, the hard part is uniquely determined automatically from Eq. (2).

I have listed several issues that highlight the importance of considering TMD definitions in detail and examining their origins in a factorization theorem. The discussions of the last few paragraphs also motivate natural conditions for optimally defining TMD PDFs and FFs: a.) They should have definite operator definitions that are separately gauge invariant and account for any instances of nonperturbative process dependent signs. b.) Apart from the process dependent signs, they should be universal and provide a prescription for calculating hard parts. c.) They should combine perturbative and nonperturbative information in a way that allows one to simultaneously extract maximum advantage from the universality of nonperturbative parts and from the perturbative calculability of small coupling parts.

In fact, the details of how to precisely define the TMDs becomes a guiding question for setting up the newer TMD definitions in next section. The universality of the TMD functions needs to be modified from the parton model in at least two respects: 1.) there is process dependence in the sense of dependence on a hard scale Q via evolution (just as in collinear factorization) and 2.) there is dependence on the direction of the Wilson line. The second of these is a novel modification of the more familiar universality concept encountered when dealing with collinear PDFs.

The early 2000s saw growing attention paid to the issue of precisely defining TMD functions, largely due to gradually increasing recognition of the relevance of predictions like the changing Sivers function sign and the inadequacy of light-cone Wilson lines for definitions. See, especially, 5.1 Factorization formulas

Expressions with TMDs are most easily expressed in transverse coordinate space:

\[ F_{j/(p,x,k_{T},S_{A};\mu,\zeta_{PDF})} = \frac{1}{(2\pi)^2} \int d^2b_T \ e^{i b_T \cdot k_{T}} \tilde{F}_{j/p}(x,b_T,S_{A};\mu,\zeta_{PDF}) , \]

\[ D_{H/j}(z,zk_{T},S_{B};\mu,\zeta_{FF}) = \frac{1}{(2\pi)^2} \int d^2b_T \ e^{-i b_T \cdot k_{T}} \tilde{D}_{H/j}(z,b_T,S_{B};\mu,\zeta_{FF}) . \]

Note the convention to write the left side of Eq. (6) as a function of zk_{T} rather than k_{T}. The nonperturbative behavior associated with small q_{T} is associated with large b_{T} behavior in the coordinate space functions.

TMD factorization theorems are best established theoretically for the classic electromagnetic processes of DY, SIDIS and the annihilation of e^{+}e^{-} pairs into a back-to-back hadron pairs. The basic statement of TMD factorization for these three processes is [6]:

\[ \frac{d\sigma}{dq_{T}^{2} \cdots} = \mathcal{H}_{DY}^{DY}(\mu/Q;\alpha_{s}(\mu)) \int d^2b_T \ e^{i q_{T} \cdot b_{T}} \tilde{F}_{\gamma/A}^{[-]}(x_{A},b_T,S_{A};\gamma,\mu) \tilde{F}_{\gamma_B}^{[-]}(x_B,b_T,S_B;Q^2/\gamma,\mu) + Y_{DY} , \]

\[ \frac{d\sigma}{dq_{T}^{2} \cdots} = \mathcal{H}_{SIDIS}^{SIDIS}(\mu/Q;\alpha_{s}(\mu)) \int d^2b_T \ e^{-i q_{T} \cdot b_{T}} \tilde{F}_{\gamma/A}^{[+]}(x_{A},b_T,S_{A};\gamma,\mu) \tilde{D}_{B/j}(z_B,b_T,S_B;Q^2/\gamma,\mu) + Y_{SIDIS} , \]

\[ \frac{d\sigma}{dq_{T}^{2} \cdots} = \mathcal{H}_{JJ}^{e^{+}e^{-}}(\mu/Q;\alpha_{s}(\mu)) \int d^2b_T \ e^{-i q_{T} \cdot b_{T}} \tilde{D}_{\gamma_A/j}(z_A,b_T,S_{A};\gamma,\mu) \tilde{D}_{\gamma_B/j}(z_B,b_T,S_B;Q^2/\gamma,\mu) + Y_{e^{+}e^{-}} . \]

The left side is a cross section differential in at least transverse momentum q_{T}, defined in an appropriate reference frame, and the “\cdots” represents possible dependence on other kinematic variables like rapidities. The first term in each equation has a structure like that of a TMD parton model. For example, use Eq. (5) to write

\[ \int d^2b_T \ e^{i q_{T} \cdot b_{T}} \tilde{F}_{\gamma/A}^{[-]}(x_A,b_T,S_{A};\gamma,\mu) \times \tilde{F}_{\gamma_B}^{[-]}(x_B,b_T,S_B;Q^2/\gamma,\mu) \]

\[ = \int d^2k_{T,1} \int d^2k_{T,2} \ F_{\gamma_A/A}^{[-]}(x_A,k_{T,1},S_{A};\gamma,\mu) \times F_{\gamma_B}^{[-]}(x_B;k_{T,2},S_B;Q^2/\gamma,\mu) \delta^{(2)}(q_{T} - k_{T,1} - k_{T,2}) , \]

References [80,81,82,83,84,85,86,87,88,89,90,91]. Reference [82] contains a useful summary of the issues as they stood approximately a decade ago. These considerations led to the formulation that will be discussed in the next section.
and compare with Eq. (2). In each of Eqs. (2)-(9), there is a convolution of two TMD functions, each being either a TMD PDF (labeled by \( F \)) or a TMD FF (labeled by \( D \)). Capital letters are used here to distinguish TMDs from collinear PDFs and FFs. The first term is called the “W-term.” The W-terms differ from a parton model picture by the presence of an explicit hard factor \( H_{ij}(\mu/Q; \alpha_i(\mu)) \), and by the appearance of evolution scales \( \zeta \) and \( \mu \). The scales are exactly arbitrary, but in applications they should be set to values of order \( Q^2 \) and \( Q \) respectively to enable well-behaved perturbative calculations. The “\([\pm]\)” superscripts represent process dependent Wilson line directions, following the notation of [83]. The \([+]\) means future-pointing and the \([-]\) means past-pointing. The \( S_A \) and \( S_B \) denote possible polarization dependence. From here forward, the power suppressed correction term will be assumed implicit.

Transverse momentum dependence can be factored into separate TMD functions only when \( q_T \ll Q \). For very large \( q_T \), a correction is needed, though it is calculable in pure collinear factorization. The correction is indicated by the last term in each equation, called the “Y-term.”

It is probably best to apply the term “TMD factorization theorem” to the complete collection of formulas in Eqs. (2)-(9), including the set of universalities properties of large distance parts and the Y-terms, rather than to any one equation alone.

5.2 Definitions

Each TMD function is defined in terms of quark and gluon field operators. I will use SIDIS as a reference process for setting up the definitions, with the directions of the incoming proton and outgoing hadron defining the large “+” and “−” directions respectively. Define space-like directions by the vectors

\[
\begin{align*}
 n_A(y_A) &= (1, -e^{-2y_A}, 0, t), \\
 n_B(y_B) &= (-e^{2y_B}, 1, 0, t).
\end{align*}
\]

These approach light-like plus (minus) vectors when \( y_A(y_B) \) approach \(+(-)\infty\).

To build up TMD definitions, one first needs to define a soft factor. Define the Wilson line from \( x \) to \( \infty \) along \( n \) in terms of a bare coupling and the bare gluon field operator:

\[
W(\infty, x; n) = P \exp \left[ -ig_0 \int_0^\infty ds \cdot A^\alpha_0(x + sn)t^\alpha \right].
\]

This Wilson line is in the color triplet representation, with \( t_a \) being the SU(3) generators in the fundamental representation. \( P \) is a path-ordering operator. Following [6], Eq. (13.39)], the soft factor is the vacuum expectation value of a Wilson loop:

\[
\begin{align*}
\tilde{S}_{(0)}(b_T; y_A, y_B) &= \frac{1}{N_C} \langle 0 | W(b_T/2, \infty; n_B(y_B))_{\alpha a}^\dagger W(b_T/2, \infty; n_A(y_A))_{\beta b} W(-b_T/2, \infty; n_B(y_B))_\beta W(-b_T/2, \infty; n_A(y_A))_{\alpha a} | 0 \rangle_{\text{No S.I.}}.
\end{align*}
\]

The “No S.I.” means Wilson line self-interactions are temporarily excluded, in addition to interactions with transverse Wilson lines at light-cone infinity. The Greek letters are color triplet indices. An analogous soft factor can be defined in an octet representation. The soft factor is designed mainly to describe QCD radiation of gluons that are both nearly on-shell and at central rapidities.

For the TMDs, one would like functions that are reminiscent of number the densities with light-like Wilson lines, but these suffer from light-cone divergences as previously discussed. To regulate them while maintaining gauge invariance, one starts by defining TMDs with non-light-like Wilson lines. These are called the “unsubtracted” definitions. For quark TMD PDFs and FFs, respectively, they are

\[
\begin{align*}
&\tilde{S}_{(0)}^Q(b_T; y_A, y_B), \\
&\tilde{S}_{(0)}^G(b_T; y_A, y_B).
\end{align*}
\]
\[ \tilde{F}_{f/P}^{\text{unsub},[+]}(x, b_T, b_T; n_B(y_B)) = \text{Tr}_{C,D} \int \frac{dw}{2\pi} e^{-iwP^+w^-} \times \]
\[ \times \langle P, S|\tilde{\psi}_f \left( \frac{w}{2} \right) W \left( \frac{w}{2}, \infty; n_B(y_B) \right) \gamma^+ W \left( -\frac{w}{2}, \infty; n_B(y_B) \right) \psi_f \left( -\frac{w}{2} \right) |P, S\rangle \langle S, 0 \rangle, \]
\[ \tilde{D}_{H/f}^{\text{unsub}}(z, b_T, S; n_A(y_A)) = \sum_x \frac{1}{i2N_{C,f}} \text{Tr}_{C,D} \int \frac{dw}{2\pi} e^{ik^+w^-} \times \]
\[ \times \langle 0|\gamma^- W \left( \frac{w}{2}, \infty; n_A(y_A) \right) \psi_f \left( \frac{w}{2} \right) |H, S, X\rangle \langle H, S, X|\tilde{\psi}_f \left( -\frac{w}{2} \right) W \left( -\frac{w}{2}, \infty; n_A(y_A) \right) \langle 0 \rangle \rangle \langle S, 0 \rangle. \]

(See [3] Eqs. (13.108,13.41) and associated discussions.) The definitions here do not use bare fields because the renormalization factors will be included in the final unsubtracted definition. The Tr_{C,D} denotes traces over color and Dirac indices. The definition for \( \tilde{F}_{f/P}^{\text{unsub},[-]}(x, b_T, S; n_B(y_B)) \) is exactly the same as in Eq. (14) but with the main Wilson lines past pointing rather than future pointing.

The final definitions to be used in Eqs. (7-9) are defined with the soft factors included, ordinary renormalization, and in the limits of infinite rapidities for the main Wilson lines:

\[ \tilde{F}_{f/P}^{[+]}(x, b_T; \mu, \zeta_{PDF}) = \lim_{y_A \rightarrow +\infty, y_B \rightarrow -\infty} \tilde{F}_{f/P}^{\text{unsub},[+]}(x, b_T; n_B(y_B)) \times \]
\[ \times \frac{\tilde{S}_{(0)}(b_T; y_A, y_B)}{S_{(0)}(b_T; y_A, y_B)S_{(0)}(b_T; y_A, y_B)} \times \]
\[ \times \text{UV ren}, \]
\[ \tilde{D}_{H/f}(z, b_T; \mu, \zeta_{FF}) = \lim_{y_A \rightarrow +\infty, y_B \rightarrow -\infty} \tilde{D}_{H/f}^{\text{unsub}}(z, b_T; n_A(y_A)) \times \]
\[ \times \frac{\tilde{S}_{(0)}(b_T; y_A, y_B)}{S_{(0)}(b_T; y_A, y_B)S_{(0)}(b_T; y_A, y_B)} \times \]
\[ \times \text{UV ren}. \]

(See [3] Eqs. (13.106,13.42) and associated discussions.) The soft rapidity \( y_s \) now regulates light-cone divergences. The factor “UV ren” is an instruction to apply UV renormalization and remove the UV regulator after the limits of \( y_A(y_B) \rightarrow +(-)\infty \) have been taken. In Eqs. (16-17), Wilson line self-energies and interactions with transverse Wilson lines at light-cone infinity can be included now because they cancel between the soft factors and the main Wilson line in the unsubtracted TMDs. Thus the “No S.I.” has been removed.

The sensitivity to \( y_s \) in each TMD is contained in the auxiliary parameters \( \zeta_{PDF} \) and \( \zeta_{FF} \):

\[ \zeta_{PDF} = x^2 M_P^2 e^{2(y_P - y_s)}, \]
\[ \zeta_{FF} = M_P^2 \frac{Y}{2} e^{2(y_s - Y)}. \]

So,

\[ \zeta_{PDF} \zeta_{FF} = Q^4. \]

The \( \zeta \)-parameters carry the memory of the need to regulate Wilson lines to define separate TMD functions.

For the Drell-Yan and \( e^+ e^- \) processes in Eqs. (7-9), the TMD PDFs (FFs) for hadrons moving with large minus (plus) momentum have the same definitions but with the plus and minus directions reversed, and corresponding replacements of \( n_A(n_B) \) with \( n_B(n_A) \).

The notational complexity in Eqs. (16-17) maybe disguises an important simplicity in these definitions. Divergences are removed by multiplying unsubtracted TMDs by factors with relatively simple and universal properties. This is closely analogous to ordinary ultraviolet (UV) renormalization, where bare operators are renormalized by multiplying with renormalization factors. It is useful to define a notation that emphasizes this analogy. First write

\[ Z_{CS}(y_s) \equiv \frac{\tilde{S}_{(0)}(b_T; +\infty, y_s)}{\tilde{S}_{(0)}(b_T; -\infty, y_s)\tilde{S}_{(0)}(b_T; +\infty, -\infty)}. \]
\[ Z_{CS}(y_s) \equiv \frac{\tilde{S}_{(0)}(b_T; y_s, -\infty)}{\tilde{S}_{(0)}(b_T; y_s, +\infty)\tilde{S}_{(0)}(b_T; +\infty, -\infty)}. \]

The infinite plus and minus rapidities in the arguments on the right side should be taken to mean that one applies the limits of infinity plus and minus rapidities in combination with whatever \( Z_{CS}(y_s) \), \( Z_{CS}(y_s) \) multiply on their left.
Then Eqs. \([16]-[17]\) are

\[
\tilde{F}_{j/P}^{[\pm]}(x, b_T; \mu, \zeta) = \frac{\Gamma_{j/P}^{\text{annub}}[\pm](x, b_T; n_B(-\infty))Z_{\text{CS}}(y_s)Z_{\text{PDF}}(\mu)Z_2(\mu)}{\mu},
\]

\[
(23)
\]

where the products are defined to include first the limits of \(y_A(y_B) \to +(-\infty)\), and then the application of UV renormalization with removal of UV regulators. The factors \(Z_{\text{PDF,FF}}(\mu)\) are the UV renormalization factors for the PDF and FF. The factor \(Z_2(\mu)\) is the ordinary UV field strength renormalization factor.

Limits associated with factors of \(Z\) should be taken in order from left to right; the order of factors in Eqs. \((23)-(24)\) is important. The limits of infinite Wilson line rapidities needs to be applied before UV regulators are removed (i.e., before \(\epsilon \to 0\) in dimensional regularization). See \([6\text{, Sect. }10.8.2]\) for a detailed discussion of the non-commuting limits.

Note that all the dependence on \(y_s\) (or, equivalently, \(\zeta\)) is in \(Z_{\text{CS}}(y_s)\). The UV factors and rapidity renormalization factors are independent of the nature of the target or measured hadrons. The overall cross section is independent of \(\mu\) and \(y_s\), but separate factors acquire \(\mu\) and \(y_s\) dependence from \(Z\)-factors. In this sense, \(Z_{\text{CS}}(y_s)\) is very much like a generalization of a standard renormalization factor.

### 5.3 Evolution

The ordinary renormalization plus rapidity evolution in Eqs. \((23)-(24)\) gives a system of evolution equations. (See, for example, \([6\text{, Eqs. }13.47,13.49,13.50]\).) The rapidity evolution equation for the TMD PDF is

\[
\frac{\partial \ln \tilde{F}_{j/P}^{[\pm]}(x, b_T; \mu, \zeta)}{\partial \ln \sqrt{\xi}} = K(b_T; \mu).
\]

\[
(25)
\]

The right side is the CS evolution kernel \(K(b_T; \mu)\), which is calculable in perturbation theory at small \(b_T\) and using \(\sim 1/b_T\) as a hard scale. It obeys its own RG equation:

\[
\frac{dK(b_T; \mu)}{d \ln \mu} = -\gamma_K(\alpha_s(\mu)).
\]

\[
(26)
\]

The RG equation for the TMD PDF is

\[
\frac{d \ln \tilde{F}_{j/P}^{[\pm]}(x, b_T; \mu, \zeta)}{d \ln \mu} = \gamma_{j,PDF}(\alpha_s(\mu); 1) - \frac{1}{2} \gamma_K(\alpha_s(\mu)) \ln \frac{\zeta}{\mu^2}.
\]

\[
(27)
\]

\(\gamma_K\) and \(\gamma_{j,PDF}\) are anomalous dimensions for \(K(b_T; \mu)\) and the TMD PDF respectively.\(^7\) At small \(b_T\), \(1/b_T\) becomes a hard scale and the individual TMD PDFs can be expanded in a perturbative series in terms of collinear PDFs using an operator product expansion (OPE):

\[
\tilde{F}_{j/H}(x, b_T; \zeta, \mu) = \sum_k \int_{x-}^{x+} \frac{d\xi}{\xi} \tilde{C}_{j/k}(x/\xi, b_T; \zeta, \mu, \alpha_s(\mu)) \times f_k/H(\zeta; \mu) + O([mbT]^0) .
\]

\[
(28)
\]

The \(C_{j/k}\) are Wilson coefficients, and \(p > 0\). Equation \((28)\) is for an unpolarized TMD PDF so I have dropped the \([\pm]\) for the Wilson line direction. TMD PDFs like the Sivers function need to be expanded in terms of twist three hard coefficients in the small \(b_T\) limit. A set of equations analogous to \((23)-(24)\) holds for TMD FFs.

The right side of Eq. \((25)\) is perturbatively calculable if \(1/b_T\) is much larger than \(O(A_{QCD})\) and \(\mu\) is fixed to \(\sim 1/b_T\). The anomalous dimensions \(\gamma_K(\alpha_s(\mu))\) and \(\gamma_{j,PDF}(\alpha_s(\mu); 1)\) are perturbatively calculable as long as \(\mu\) is much larger than \(O(A_{QCD})\).

One striking difference between TMD evolution and collinear evolution is that Eq. \((25)\) implies that, in the limit of large \(b_T\), the evolution itself becomes nonperturbative. Predictive power is maintained because the \(K(b_T; \mu)\) has strong universality, meaning it is independent not only of the process, but also the species of hadrons involved or any polarizations involved. It is even the same \(K(b_T; \mu)\) for TMDs and FFs. It follows from the universality of the \(Z_{CS}(y_s)\) from the previous section. Testing the strong universality of nonperturbative evolution is an important part of TMD phenomenology.

### 6 Solutions

The TMD evolution equations only involve products of factors in transverse coordinate space, making solutions simple to write. In preparation for writing the solutions, one needs several definitions associated with the organization of perturbative and nonperturbative parts.

For small \(b_T\), the right sides of Eqs. \((25)\) and \((28)\) can be calculated entirely in collinear perturbation theory with \(1/b_T\) acting as the hard scale. The only nonperturbative inputs then are the collinear PDFs and FFs. To define what one means by “large” and “small” \(b_T\), one must define a cutoff scale \(b_{\text{max}}\). Above \(b_{\text{max}}\), one allows for nonperturbative \(b_T\)-dependence, while below \(b_{\text{max}}\) one relies on collinear perturbation theory. A standard procedure is to define a \(b_s(b_T)\) such that

\[
b_s(b_T) = \begin{cases} b_T & b_T \ll b_{\text{max}} \\ b_{\text{max}} & b_T \gg b_{\text{max}} \end{cases}.
\]

\[
(29)
\]

\(^7\) A function that is basically equivalent to \(K\) is called \(-2D\) in \([22]\) and \(-F_{qq}\) in \([23]\).
A smooth transition function is typically used. One of the 
most common is \[ b_\epsilon(b_T) = \sqrt{\frac{b_T^2}{1 + b_T^2/b_{\text{max}}^2}}. \] (30)

One ultimately evolves to a scale \[ \mu_\epsilon = C_1/b_\epsilon. \] (31)

The perturbative and nonperturbative \( b_T \)-dependence in \( \tilde{K}(b_T; \mu) \) is separated by defining

\[ -g_K(b_T; b_{\text{max}}) = -\tilde{K}(b_T; \mu_0) + \tilde{K}(b_T; \mu_0). \] (32)
which by construction vanishes like a power at small \( b_T \). In Eq. (32), the scale \( \mu_0 \) is arbitrary.

Solving Eqs. (23)-(24) (and the equivalent equations for the FFs) to obtain the TMDs at arbitrary scales \( \zeta \) and \( \mu \) gives:

\[
\tilde{F}_{f/P}(x, b_T; \mu, \zeta) = \sum_j \int_0^x \frac{d\xi}{\xi} C_{f/j}^{PDF}(x/\xi, b_{\text{max}}, \mu_0, \alpha_s(\mu_0)) f_{f/P}(\xi, \mu_0)
\times \exp \left\{ \ln \sqrt{\frac{\zeta}{\mu_0}} \tilde{K}(b_T; \mu_0) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[ \gamma_{f/P}(\alpha_s(\mu'); 1) - \ln \sqrt{\frac{\zeta}{\mu'}} \gamma_K(\alpha_s(\mu')) \right] \right\}
\times \exp \left\{ -g_{f/P}(x, b_T; b_{\text{max}}; Q_0) - g_K(b_T; b_{\text{max}}; Q_0) \ln \sqrt{\frac{\zeta}{Q_0}} \right\},
\] (33)

and

\[
\tilde{D}_{H/f}(z, b_T; \mu, \zeta) = \sum_j \int_0^z \frac{d\xi}{\xi^3} C_{f/j}^{FF}(z/\xi, b_{\text{max}}, \mu_0, \alpha_s(\mu_0)) d_{H/j}(\xi, \mu_0)
\times \exp \left\{ \ln \sqrt{\frac{\zeta}{\mu_0}} \tilde{K}(b_T; \mu_0) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[ \gamma_{f/FF}(\alpha_s(\mu'); 1) - \ln \sqrt{\frac{\zeta}{\mu'}} \gamma_K(\alpha_s(\mu')) \right] \right\}
\times \exp \left\{ -g_{H/f}(z, b_T; b_{\text{max}}; Q_0) - g_K(b_T; b_{\text{max}}; Q_0) \ln \sqrt{\frac{\zeta}{Q_0}} \right\}.
\] (34)

In practice, one usually sets \( \mu = \sqrt{\zeta} = Q \) to enable perturbative calculations.

I have organized the solutions here into three factors, labeled “AA,” “BB,” and “CC,” following the method of Ref. [95] to highlight the different components of an evolved TMD and connect type I and type II pictures.

The AA factor is a fixed order calculation in collinear perturbation theory of the small \( b_T \) dependence. The BB factor is a perturbative evolution factor for relating scales \( \mu_0 \) and \( \mu_0' \) to general \( \mu \) and \( \zeta \). The CC factor in the last line includes all nonperturbative transverse coordinate dependence. Note the \( g_K(b_T; b_{\text{max}}) \) from Eq. (32). The \( g_{f/P}(x, b_T; b_{\text{max}}) \) and \( g_{H/f}(z, b_T; b_{\text{max}}) \) parameterize the transition from the OPE calculation at fixed scale in Eq. (25) to the region where nonperturbative \( b_T \) dependence is included. I have highlighted the roles of perturbative and nonperturbative behavior by making functions that are to be calculated entirely in fixed order perturbation theory red while those that include nonperturbative behavior are in blue. Note in particular that \( g_K(b_T; b_{\text{max}}) \) has no scale dependence, no subscript for \( f \), \( P \) or \( H \), no \( x \) or \( z \) dependence, and is the same function in both Eq. (33) and Eq. (34). This emphasizes its strong universality, discussed earlier. In Eq. (33)-(34), I have included \( Q_0 \) as an argument of \( g_K(b_T; b_{\text{max}}; Q_0) \), \( g_{H/f}(z, b_T; b_{\text{max}}; Q_0) \), and \( g_{f/P}(x, b_T; b_{\text{max}}; Q_0) \) to emphasize that the exact partitioning of non-perturbative \( b_T \)-dependence between the evolution and the intrinsic parts depends on this reference scale.

\footnotetext[8]{I use double letters here to distinguish from other common uses of “A,” “B,” and “C.”}
The \( g_{f/p}(x, b_T; b_{\text{max}}) \), \( g_{H/f}(z, b_T; b_{\text{max}}) \) and \( g_{K}(b_T; b_{\text{max}}) \) functions all show their explicit dependence on \( b_{\text{max}} \). Since it is an arbitrary cutoff, the overall cross section is exactly independent of \( b_{\text{max}} \). If \( b_{\text{max}} \) is varied, changes in perturbatively calculated parts should be compensated by changes in \( g_{f/p}(x, b_T; b_{\text{max}}) \), \( g_{H/f}(z, b_T; b_{\text{max}}) \) and \( g_{K}(b_T; b_{\text{max}}) \). This was exploited in Ref. \( [95] \) to improve the transition between the dominantly perturbative and dominantly nonperturbative regions of \( b_T \)-dependence.

The solutions in Eqs. \((33)-(34)\) are written in a way that maximizes the amount of perturbative input, but there are other ways of writing solutions that might be preferred, depending on the context. This was discussed recently in Ref. \( [95] \). For example, to match to an exactly parton-model-like picture for some initial scale \( Q_0 \), Eq. \((24)\) may be preferred. If one wishes to organize evolution with respect to center of mass energy \( \sqrt{s} \) in Drell-Yan, rather than \( Q \), then \( [95] \) Eq. \((18)\) might be preferred.

Equations \((33)-(34)\) are written here for unpolarized and azimuthally symmetric functions. There are similar formulas for other TMDs like the Sivers function \([97,98]\).

### 7 Comments

I will end with some general remarks about the outlook of TMD applications and work that I believe is still needed.

One important refinement needed for TMD evolution is to include flavor number transitions in the full TMD evolution analogous to what is done for collinear PDFs in the ACOT \([99,100]\) formalism. Another complication is with the implementation of \( Y \)-term corrections. Achieving a smooth matching between the \( W \)-term and the \( Y \)-term is more complicated in practice than a straightforward implementation of the definitions in the factorization derivation \([101]\). More progress in this area is likely possible with further refinements in the details of implementations.

Improvements in nonperturbative theory in treating the TMDs at low \( Q \) and low \( q_T \) will be important for phenomenology since the factorization theorems alone provide few detailed constraints on the these three dimensional objects. Specific methods will be discussed in other articles in this collection. Since TMDs describe inclusive processes, including the radiation of soft hadrons, efforts to constrain them will benefit from more detailed pictures of hadronization and fragmentation. An interesting example of fragmentation theory applications to TMDs is the use of the string model to describe the Collins mechanism in Ref. \([102]\). My perspective is that accounting for the constraints of factorization is critical for guiding the formulation of general pictures of the underlying physics.

There are by now many other formulations of TMD factorization (or frameworks closely analogous to TMD factorization), and unfortunately it is not possible to discuss any one of them in detail here. An especially active approach in recent years is soft-collinear effective theory (SCET). There are at least three versions of TMD factorization that start from the perspective of SCET \([103,104,105,106]\). Another approach to TMD factorization is that of Ji, Ma, and Yuan \([11]\).

It is likely that insight can be gained by determining if and how different formulations of TMD factorization have meaningful differences, or whether they are actually equivalent formulations with different notation and/or conventions for intermediate steps. In Ref. \([105]\), a particular version of SCET was shown to be equivalent to the TMD factorization approach described in Sects. \([16,17)\) of this article, at least to one-loop order. Other issues to consider are the small and large \( x \) limits, higher twist corrections, and the relationship to exclusive scattering. I refer the reader to other recent reviews such as \([106,107]\) for discussions of some of these topics and relevant references.

Detailed theoretical considerations indicate that TMD factorization should break down in some processes where more familiar parton model intuition might suggest that it applies \([108,109,110,111,112,113]\). The mechanisms for TMD factorization breaking have the potential to produce interesting physical effects themselves, though more work is needed to determine how to calculate them. For high energies, effects associated with TMD factorization breaking are likely calculable in perturbation theory in the form of higher order large logarithms and resummation techniques.

Finally, TMD factorization is also expected to breakdown in certain kinematical limits. For example, when target and hadron masses are important, or when the distribution of remnant masses are considered in detail, the approximations that give TMD factorization no longer suffice and corrections are needed. In such cases, it might be necessary to formulate other forms of factorization. For example, one might need something more like the fully unintegrated factorization advocated in Refs. \([114,115,116,117,118]\), but probably more closely analogous to TMD factorization as it is formulated in Sect. \([16)\) of this review.

I acknowledge many useful conversations with J. Collins who also provided suggestions on the text. I also thank C. Aidala, M. Diehl, and R. Fatemi for helpful comments on the text. This work was supported by the DOE contract No. DE-AC05-06OR23177, under which Jefferson Science Associates, LLC operates Jefferson Lab.

### References

1. J. C. Collins and D. E. Soper. Parton distribution and decay functions. *Nucl. Phys.* B194:445–492, 1982.
2. J. C. Collins and D. E. Soper. Back-to-back jets in QCD. *Nucl. Phys.* B193:381–443, 1981. Erratum: *B213*, 545 (1983).
3. John C. Collins, Davison E. Soper, and George Sterman. Does the Drell-Yan cross-section factorize? *Phys. Lett.*, B109:388, 1982.
4. John C. Collins, Davison E. Soper, and George F. Sterman. Factorization for One Loop Corrections in the Drell-Yan Process. *Nucl.Phys.*, B223:381, 1983.
5. J. C. Collins, D. E. Soper, and G. Sterman. Transverse momentum distribution in Drell-Yan pair and W and Z boson production. *Nucl. Phys.* B250:199–224, 1985.
6. J. C. Collins. *Foundations of Perturbative QCD*. Cambridge University Press, Cambridge, 2011.
7. R.P. Feynman, R.D. Field, and G.C. Fox. A Quantum Chromodynamic Approach for the Large Transverse Momentum Production of Particles and Jets. *Phys. Rev.*, D18:3320, 1978.

8. D. C. Hom et al. Observation of High Mass Dilepton Pairs in Hadron Collisions at 400-GeV. *Phys. Rev. Lett.*, 36:1236, 1976.

9. D. C. Hom et al. Production of High Mass Muon Pairs in Hadron Collisions at 400-GeV. *Phys. Rev. Lett.*, 37:1374–1377, 1976.

10. S. W. Herb et al. Observation of a Dimuon Resonance at 9.5-GeV in 400-GeV Proton-Nucleus Collisions. *Phys. Rev. Lett.*, 39:252–255, 1977.

11. Walter R. Innes et al. Observation of structure in the $T$ region. *Phys. Rev. Lett.*, 39:1240, 1977. [Erratum: *Phys. Rev. Lett.* 39, 1640(1977)].

12. C. S. Lam and Tung-Mow Yan. Transverse Momentum Distribution of Partons in QCD. *Phys. Lett.*, B71:173, 1977.

13. Keh-Fei Liu and Cedric Lorce. The Parton Orbital Angular Momentum: Status and Prospects. 2015.

14. R. K. Ellis, W. J. Stirling, and B. R. Webber. *QCD and Collider Physics*. Cambridge University Press, Cambridge, 1996.

15. Robert N. Cahn. Azimuthal Dependence in Leptoproduction: A Simple Parton Model Calculation. *Phys. Lett.*, B78:269, 1978.

16. D. W. Sivers. Single spin production asymmetries from the hard scattering of point-like constituents. *Phys. Rev. D41*:83–90, 1990.

17. R.D. Klem, J.E. Bowers, H.W. Courant, H. Kagan, M.L. Marshak, et al. Measurement of Asymmetries of Inclusive Pion Production in Proton Proton Interactions at 6-GeV/c and 11.8-GeV/c. *Phys. Rev. Lett.*, 36:929–931, 1976.

18. W.H. Dragoset, J.B. Roberts, J.E. Bowers, H.W. Courant, H. Kagan, et al. Asymmetries in Inclusive Proton-Nucleon Scattering at 11.75-GeV/c. *Phys. Rev.*, D18:3939–3954, 1978.

19. J. Antille, L. Dick, L. Madansky, D. Perret-Gallix, M. Werlen, et al. spin dependence of the inclusive reaction $p p \rightarrow p + X$ at 24-GeV/c for high $p(t)$ at 20-GeV/c in the central region. *Phys. Lett.*, B94:523, 1980.

20. V.D. Apokin, Yu.I. Arestov, O.V. Astafev, N.I. Belikov, B.V. Chukiko, et al. Observation of significant spin effects in hard collisions at 40-GeV/c. *Phys. Lett.*, B243:461–464, 1990.

21. S. Saroff, B.R. Baller, G.C. Blazey, H. Courant, Kenneth J. Heller, et al. Single spin asymmetry in inclusive reactions: polarized $p p$ goes to $p + p$, $p$ and $p$ at high $p(t)$ at 13.3-GeV/c and 18.5-GeV/c. *Phys. Rev. Lett.*, 64:995, 1990.

22. D.L. Adams et al. Comparison of spin asymmetries and cross-sections in $p p$ production by 200-GeV polarized anti-protons and protons. *Phys. Lett.*, B261:201–206, 1991.

23. D. L. Adams et al. Analyzing power in inclusive $p + p$ and $p + p$ production at high $x(F)$ with a 200-GeV polarized proton beam. *Phys. Lett.*, B264:462–466, 1991.

24. Christine A. Aidala, Steven D. Bass, Delia Hasch, and Gerhard K. Mallot. The Spin Structure of the Nucleon. *Rev. Mod. Phys.*, 85:655–691, 2013.

25. Jian-Ping Chen. QCD evolution and TMD/Spin experiments. *Int. J. Mod. Phys. Conf. Ser.*, 20:45–52, 2012.

26. J. C. Collins. Fragmentation of transversely polarized quarks probed in transverse momentum distributions. *Nucl. Phys.*, B396:161–182, 1993.

27. R.D. Tangerman and P.J. Mulders. Intrinsic transverse momentum and the polarized Drell-Yan process. *Phys. Rev.*, D51:3357–3372, 1995.

28. P. J. Mulders and R. D. Tangerman. The complete tree-level result up to order $1/Q$ for polarized deep-inelastic leptoproduction. *Nucl. Phys.*, B461:237–237, 1996.

29. Daniel Boer and P.J. Mulders. Time reversal odd distribution functions in leptoproduction. *Phys. Rev.*, D57:5780–5786, 1998.

30. Mauro Anselmino, Mariaelena Boglione, and Francesco Murgia. Phenomenology of single spin asymmetries in $p + p \rightarrow p + X$. *Phys. Rev.*, D60:054027, 1999.

31. June-gone Chay, Stephen D. Ellis, and W. James Stirling. Azimuthal asymmetry in lepton - photon scattering at high-energies. *Phys. Rev.*, D45:46–54, 1992.

32. Alessandro Bacchetta, Daniel Boer, Markus Diehl, and Piet J. Mulders. Matches and mismatches in the descriptions of semi-inclusive processes at low and high transverse momentum. *HEP*, 08:023, 2008.

33. Yuri L. Dokshitzer, Dmitri Diakonov, and S. I. Troian. Hard Processes in Quantum Chromodynamics. *Phys. Rept.*, 58:269–395, 1980.

34. Yuri L. Dokshitzer, Dmitri Diakonov, and S. I. Trojan. Hard Semiinclusive Processes in QCD. *Phys. Lett.*, B78:290, 1978.

35. G. Parisi and R. Petronzio. Small transverse momentum distributions in hard processes. *Nucl.Phys.*, B154:427, 1979.

36. W. Greiner, S. Schramm, and E. Stein. *Quantum chromodynamics*, 2002.

37. D. E. Soper. Partons and their transverse momenta in QCD. *Phys. Rev. Lett.*, 43:1847–1851, 1979.

38. John C. Collins. Intrinsic transverse momentum. 1. non-gauge theories. *Phys. Rev.*, D21:2962, 1980.

39. Daniel Boer. Sudakov suppression in azimuthal spin asymmetries. *Nucl. Phys.*, B603:195–217, 2001.

40. Daniel Boer. Angular dependences in inclusive two-hadron production at BELE. *Nucl. Phys.*, B806:23–67, 2009.

41. Xiang-Dong Ji, Jian-Ping Ma, and Feng Yuan. QCD factorization for semi-inclusive deep-inelastic scattering at low transverse momentum. *Phys. Rev.*, D71:034005, 2005.

42. P. Nadolsky, D. R. Stump, and C. P. Yuan. QCD factorization for spin-dependent cross sections in DIS and Drell-Yan processes at low transverse momentum. *Phys. Lett.*, B597:299–308, 2004.

43. Ruibin Meng, Fredrick I. Olness, and Davison E. Soper. Semi-inclusive deeply inelastic scattering at electron-proton colliders. *Nucl. Phys.*, B371:79–110, 1996.

44. Ruibin Meng, Fredrick I. Olness, and Davison E. Soper. Semi-inclusive deeply inelastic scattering at small $q_T$. *Phys. Rev.*, D54:1919–1935, 1996.

45. P. Nadolsky, D. R. Stump, and C. P. Yuan. Semi-inclusive hadron production at HERA: The effect of QCD gluon resummation. *Phys. Rev.*, D61:014003, 1999.

46. P. M. Nadolsky, D. R. Stump, and C. P. Yuan. Phenomenology of multiple parton radiation in semi-inclusive deep-inelastic scattering. *Phys. Rev.*, D64:114011, 2001.
Yuji Koike, Junji Nagashima, and Werner Vogelsang. Resummation for polarized semi-inclusive deep-inelastic scattering at small transverse momentum. *Nucl. Phys.*, B774:59–79, 2006.

John P. Ralston and Davison E. Soper. Drell-Yan model at measured $Q_T$: Asymptotic smallness of one loop corrections. *Nucl. Phys.*, B172:445, 1980.

J. C. Collins. Algorithm to compute corrections to the Sudakov form-factor. *Phys. Rev.*, D22:1478–1489, 1980.

John P. Ralston and Davison E. Soper. Production of dimuons from high-energy polarized proton-proton collisions. *Nucl. Phys.*, B152:109–124, 1979.

John Collins. CSS equation, etc, follow from structure of TMD factorization. 2012.

G. T. Bodwin, S. J. Brodsky, and G. P. Lepage. Initial state interactions and the Drell-Yan process. *Phys. Rev. Lett.*, 47:1799–1803, 1981.

J. C. Collins, D. E. Soper, and G. Sterman. Factorization for short distance hadron-hadron scattering. *Nucl. Phys.*, B261:104–142, 1985.

W. W. Lindsay, D. A. Ross, and Christopher T. Sachrajda. On the Cancellation of Long Distance, Nonfactorizing Contributions to the Drell-Yan Cross-section. *Nucl. Phys.*, B214:61, 1983.

Geoffrey T. Bodwin. Factorization of the Drell-Yan Cross-Section in Perturbation Theory. *Phys. Rev.*, D31:2616, 1985. [Erratum: Phys. Rev.D34.3932(1986)].

John C. Collins, Davison E. Soper, and George F. Sterman. Factorization is not violated. *Phys. Lett.*, B438:184–192, 1998.

V. A. Abramovsky, V. N. Gribov, and O. V. Kancheli. Character of Inclusive Spectra and Fluctuations Produced in Inelastic Processes by Multi - Pomeron Exchange. *Yad. Fiz.*, 18:595–616, 1973. [Sov. J. Nucl. Phys.18.308(1974)].

John C. Collins and Andreas Metz. Universality of soft and collinear factors in hard-scattering factorization. *Phys. Rev. Lett.*, 93:252001, 2004.

G.A. Ladinsky and C.P. Yuan. The nonperturbative regime in QCD resummation for gauge boson production at hadron colliders. *Phys. Rev.*, D50:R4239–R4243, 1994.

Csaba Baláz, Jian-Wei Qiu, and C.P. Yuan. Effects of QCD resummation on distributions of leptons from the decay of electroweak vector bosons. *Phys. Lett.*, B355:548–554, 1995.

C. Baláz and C. P. Yuan. Soft gluon effects on lepton pairs at hadron colliders. *Phys. Rev.*, D56:5558–5583, 1997.

F. Landry, R. Brock, G. Ladinsky, and C. P. Yuan. New fits for the non-perturbative parameters in the CSS resummation formalism. *Phys. Rev.*, D63:013004, 2001.

Jian-Wei Qiu and Xiao-Fei Zhang. Role of the nonperturbative input in QCD resummed Drell-Yan $Q_T$ distributions. *Phys. Rev.*, D63:114011, 2001.

Jian-Wei Qiu and Xiao-Fei Zhang. Role of nonperturbative input in QCD resummed heavy boson $Q_T$ distribution. 2002.

George I. Fai, Jian-Wei Qiu, and Xiao-Fei Zhang. Full transverse momentum spectra of low mass Drell-Yan pairs at LHC energies. *Phys. Lett.*, B676:243–250, 2003.

F. Landry, R. Brock, P. M. Nadolsky, and C.-P. Yuan. Tevatron Run-1 Z boson data and Collins-Soper-Sterman resummation formalism. *Phys. Rev.*, D67:073016, 2003.

Anton V. Konychev and Pavel M. Nadolsky. Universality of the Collins-Soper-Sterman nonperturbative function in gauge boson production. *Phys. Lett.*, B633:710–714, 2006.

Pavel M. Nadolsky. Theory of $W$ and $Z$ boson production. *AIP Conf.Proc.*, 753:158–170, 2005.

Marco Guzzi, Pavel M. Nadolsky, and Bowen Wang. Nonperturbative contributions to a resummed leptonic angular distribution in inclusive neutral vector boson production. *Phys. Rev.*, D90:014030, 2014.

Rafael Lopes de Sa. *Measurements of the W Boson Mass with the D0 Detector*. PhD thesis, Stony Brook University, 2013.

S. J. Brodsky, D.-S. Hwang, and I. Schmidt. Final-state interactions and single-spin asymmetries in semi-inclusive deep inelastic scattering. *Phys. Lett.*, B530:99–107, 2002.

J. C. Collins. Leading-twist single-transverse-spin asymmetries: Drell-Yan and deep-inelastic scattering. *Phys. Lett.*, B536:43–48, 2002.

Daniel de Florian and Massimiliano Grazzini. Next-to-next-to-leading logarithmic corrections at small transverse momentum in hadronic collisions. *Phys. Rev. Lett.*, 85:4678–4681, 2000.

Daniel de Florian and Massimiliano Grazzini. The Structure of large logarithmic corrections at small transverse momentum in hadronic collisions. *Nucl. Phys.*, B616:247–285, 2001.

Stefano Catani, Daniel de Florian, and Massimiliano Grazzini. Universality of nonleading logarithmic contributions in transverse momentum distributions. *Nucl. Phys.*, B596:299–312, 2001.

Stefano Catani, Leandro Cieri, Giancarlo Ferrera, Daniel de Florian, and Massimiliano Grazzini. Vector boson production at hadron colliders: a fully exclusive QCD calculation at NNLO. *Phys. Rev. Lett.*, 103:082001, 2009.

Stefano Catani, Leandro Cieri, Daniel de Florian, Giancarlo Ferrera, and Massimiliano Grazzini. Vector boson production at hadron colliders: hard-collinear coefficients at the NNLO. *Eur. Phys. J.*, C72:2195, 2012.

Giuseppe Bozzi, Stefano Catani, Giancarlo Ferrera, Daniel de Florian, and Massimiliano Grazzini. Production of Drell-Yan lepton pairs in hadron collisions: Transverse-momentum resummation at next-to-next-to-leading logarithmic accuracy. *Phys. Lett.*, B696:207–213, 2011.

Stefano Catani, Leandro Cieri, Daniel de Florian, Giancarlo Ferrera, and Massimiliano Grazzini. Universality of transverse-momentum resummation and hard factors at the NNLO. *Nucl. Phys.*, B881:414–443, 2014.

J. C. Collins and F. Hautmann. Infrared divergences and non-lightlike eikonal lines in Sudakov processes. *Phys. Lett.*, B472:129–134, 2000.

A. V. Belitsky, X. Ji, and F. Yuan. Final state interactions and gauge invariant parton distributions. *Nucl. Phys.*, B656:165–198, 2003.

J. C. Collins. What exactly is a parton density? *Acta Phys. Polon.*, B34:3103–3120, 2003.

Daniel Boer, P. J. Mulders, and F. Pijlman. Universality of T-odd effects in single spin and azimuthal asymmetries. *Nucl. Phys.*, B667:201–241, 2003.

C. J. Bomhof, P. J. Mulders, and F. Pijlman. Gauge link structure in quark-quark correlators in hard processes. *Phys. Lett.*, B596:277–286, 2004.
85. C. J. Bomhof, P. J. Mulders, and F. Pijlman. The construction of gauge-links in arbitrary hard processes. *Eur. Phys. J.*, C47:147–162, 2006.
86. I.O. Cherednikov and N.G. Stefanis. New results on gauge-invariant TMD PDFs in QCD. 2008.
87. I.O. Cherednikov and N.G. Stefanis. Renormalization, Wilson lines, and transverse-momentum dependent parton distribution functions. *Phys. Rev.*, D77:094001, 2008.
88. I.O. Cherednikov and N.G. Stefanis. Wilson lines and transverse-momentum dependent parton distribution functions: A renormalization-group analysis. *Nucl. Phys.*, B802:146–179, 2008.
89. F. Hautmann. Endpoint singularities in unintegrated parton distributions. *Phys. Lett.*, B655:26–31, 2007.
90. John Collins. Rapidity divergences and valid definitions of parton densities. *PoS*, LC2008:028, 2008.
91. F. Hautmann. Unintegrated parton distributions and applications to jet physics. *Acta Phys.Polon.*, B40:2139–2163, 2009.
92. Miguel G. Echevarria, Ahmad Idilbi, and Ignazio Scimemi. Factorization theorem for Drell-Yan at low $qT$ and transverse momentum distributions on-the-light-cone. *JHEP*, 1207:002, 2012.
93. Thomas Becher and Matthias Neubert. Drell-Yan production at small $qT$, transverse parton distributions and the collinear anomaly. *Eur. Phys. J.*, C71:1665, 2011.
94. John C. Collins and Davison E. Soper. Back-to-back jets: Fourier transform from $b$ to $kT$. *Nucl. Phys.*, B197:446–476, 1982.
95. S. Mert Aybat and Ted C. Rogers. TMD parton distribution and fragmentation functions with QCD evolution. *Phys. Rev.*, D83:114042, 2011.
96. John Collins and Ted Rogers. Understanding the large-distance behavior of transverse-momentum-dependent parton densities and the Collins-Soper evolution kernel. *Phys. Rev.*, D91(7):074020, 2015.
97. S. Mert Aybat, John C. Collins, Jian-Wei Qiu, and Ted C. Rogers. The QCD evolution of the Sivers function. *Phys. Rev.*, D85:034043, 2012.
98. Alessandro Bacchetta and Alexei Prokudin. Evolution of the helicity and transversity transverse-momentum-dependent parton distributions. *Nucl. Phys.*, B875:536–551, 2013.
99. M. A. G. Aivazis, John C. Collins, Fredrick I. Olness, and Wu-Ki Tung. Leptoproduction of heavy quarks. 2. A unified QCD formulation of charged and neutral current processes from fixed target to collider energies. *Phys. Rev.*, D50:3102–3118, 1994.
100. John C. Collins. Hard-scattering factorization with heavy quarks: A general treatment. *Phys. Rev.*, D58:094002, 1998.
101. M. Boglione, J. O. Gonzalez Hernandez, S. Melis, and A. Prokudin. A study on the interplay between perturbative QCD and CSS/TMD formalism in SIDIS processes. *JHEP*, 02:095, 2015.
102. X. Artru, J. Czyzewski, and H. Yabuki. Single spin asymmetry in inclusive pion production, Collins effect and the string model. *Z. Phys.*, C73:527–534, 1997.
103. Sonny Mantry and Frank Petriello. Factorization and resummation of Higgs boson differential distributions in soft-collinear effective theory. *Phys. Rev.*, D81:093007, 2010.
104. Sonny Mantry and Frank Petriello. Transverse momentum distributions in the non-perturbative region. *Phys. Rev.*, D84:014030, 2011.
105. John C. Collins and Ted C. Rogers. Equality of two definitions for transverse momentum dependent parton distribution functions. *Phys. Rev.*, D87(3):034018, 2013.
106. R. Angeles-Martinez et al. Transverse momentum dependent (TMD) parton distribution functions: status and prospects. 2015.
107. Kai-bao Chen, Shu-yi Wei, and Zuo-tang Liang. Three Dimensional Imaging of the Nucleon and Semi-Inclusive High Energy Reactions. 2015.
108. John Collins and Jian-Wei Qiu. $k_T$ factorization is violated in production of high-transverse-momentum particles in hadron-hadron collisions. *Phys. Rev.*, D75:114014, 2007.
109. John Collins. 2-soft-gluon exchange and factorization breaking. 2007.
110. Ted C. Rogers and Piet J. Mulders. No generalized transverse momentum dependent factorization in hadroproduction of high transverse momentum hadrons. *Phys. Rev.*, D81:094006, 2010.
111. Stefano Catani, Daniel de Florian, and German Rodrigo. Space-like (versus time-like) collinear limits in QCD: Is factorization violated? *JHEP*, 1207:026, 2012.
112. Jeffrey R. Forshaw, Michael H. Seymour, and Andrzej Siodmok. On the Breaking of Collinear Factorization in QCD. *JHEP*, 11:066, 2012.
113. Ted C. Rogers. Extra spin asymmetries from the breakdown of transverse-momentum-dependent factorization in hadron-hadron collisions. *Phys.Rev.*, D88(1):014002, 2013.
114. Christian W. Bauer and Frank J. Tackmann. Gaining analytic control of parton showers. *Phys. Rev.*, D76:114017, 2007.
115. J. C. Collins, T. C. Rogers, and A. M. Staśto. Fully unintegrated parton correlation functions and factorization in lowest order hard scattering. *Phys. Rev.*, D77:085009, 2008.
116. Ted C. Rogers. Next-to-Leading Order Hard Scattering Using Fully Unintegrated Parton Distribution Functions. *Phys. Rev.*, D78:074018, 2008.
117. S. Jadach and M. Skrzypek. QCD evolution in the fully unintegrated form. *Acta Phys. Polon.*, B40:2071–2096, 2009.
118. Anbar Jain, Massimiliano Procura, and Wouter J. Waalewijn. Fully-Unintegrated Parton Distribution and Fragmentation Functions at Perturbative $k_T$. *JHEP*, 04:132, 2012.