Quantum Theory of Cavity-Assisted Sideband Cooling of Mechanical Motion

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Recent years have seen significant advances in fabricating, measuring, and controlling mechanical systems on the micro- and nanometer scale 1, 2, 3, 4, 5, 6. These advances have opened the possibility of observing quantum effects in mechanical devices. A number of experiments have been proposed, including measuring their coherence and entangling them with other systems 7, 8, 9, 10, 11, 12, generating nonclassical states 13, 14, and making quantum-limited measurements 15, 16. Many of these proposals require the mechanical system to be cooled to its ground state, i.e. to temperatures below 20 mK even for 1 GHz resonators. This is difficult or impossible using bulk refrigeration, and a more promising means is to make use of non-equilibrium cooling techniques, analogous to the laser-cooling schemes developed for trapped ions and neutral atoms 17.

Such schemes fall into two categories. In the first, cooling is achieved via an active feedback loop which is used to cancel the cantilever’s thermal motion 18, 19, 20. This approach (including its quantum limits 21) has been considered elsewhere. Here we will focus on the second category: passive, non-feedback-based cooling. In this approach the cantilever displacement is coupled parametrically to a driven resonator (or two-level system). When the frequency of the drive applied to the resonator is chosen appropriately, this parametric coupling ensures that the cantilever is preferentially driven to lower energy states. The lowest temperature to which this process can cool the cantilever is determined by the resonator’s quantum fluctuations (photon shot-noise), which drive the cantilever randomly and compete with the cooling.

This balance has been considered theoretically for a few specific realizations of the cooling system: a Cooper-pair box 22, the superconducting single-electron transistor 23, 24, quantum dots 25, and ions 26. In this paper we present a quantum mechanical description of the self-cooling of a cantilever coupled to an optical cavity via radiation pressure. This system is of particular interest because of its relative simplicity - the cantilever is cooled by a single mode of the electromagnetic field. It is also of interest because this system has already been realized experimentally, and recently produced very promising results on self-cooling using both photothermal forces and radiation pressure forces 27, 28, 29, 30, 31, 32, 33, 34, 35. Cooling rates and steady state temperatures for such a scheme (and of related systems like a driven LC circuit 35) have been derived only (semi-)classically so far. Most notably, Braginsky and Vyatchanin 36 considered the optomechanical damping within a semiclassical theory. Here, we use the quantum noise approach to find simple and transparent, fully quantum-mechanical expressions, starting directly from the spectrum of the force fluctuations. Our results are valid both for the good-cavity regime (resolved mechanical sidebands) and the bad-cavity regime (unresolved sidebands). Earlier (semi-) classical calculations 27, 32, 35 are reproduced in the appropriate limits. We show in particular that it should be possible to cool the cantilever to its quantum mechanical ground state by choosing the cantilever resonance frequency much larger than the cavity ringdown rate, a regime that has not been considered so far. Cooling the cantilever to its ground state will enable the realization of many of the aforementioned tasks, as well as investigations into the optomechanical instability 37, 38, 39 in the quantum regime. The results are complemented by an exact solution of the coupled equations of motion to account for the “strong cooling” limit, where the cooling rate exceeds the cavity ringdown rate.

Consider a mechanical degree of freedom $\hat{x}$ coupled parametrically with strength $A$ to the cavity oscillator

$$\hat{H} = \hbar (\omega_R - A \hat{x}) \left( \hat{a}^{\dagger} \hat{a} - \langle \hat{a}^{\dagger} \hat{a} \rangle \right) + \hat{H}_M + \hat{H}_{\text{drive}} + \hat{H}_\kappa + \hat{H}_\Gamma \quad (1)$$
where \( \omega_R \) is the cavity resonance frequency at the equilibrium cantilever position \( x = 0 \) in the presence of the mean radiation pressure, \( H_M = h\omega_M \hat{c}^\dagger \hat{c} \) is the mechanical oscillator, \( \hat{H}_{\text{drive}} \) is the optical drive, \( \hat{H}_\kappa \) represents the cavity damping, and \( \hat{H}_T \) represents the mechanical damping. For a cavity of length \( L \), the change in cavity length with \( x \) gives a radiation pressure coupling \( A = +\omega_R/L \).

Opening the port used to supply the classical drive to the cavity also admits vacuum noise. Writing the cavity field in terms of its classical and quantum parts \( \hat{a}(t) = e^{-\omega_c t}[\hat{a} + \hat{d}(t)] \) yields the photon number autocorrelation function

\[
\hat{S}_{nn}(t) = \langle \hat{a}^\dagger(t) \hat{a}(t) \hat{a}^\dagger(0) \hat{a}(0) \rangle - \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle^2 = \bar{n} e^{\Delta t - \frac{\kappa}{2} t},
\]

where \( \Delta = \omega_L - \omega_R \) is the laser-cavity detuning and \( \bar{n} \) is the mean photon number. Because the number fluctuations are the result of interference between the classical drive amplitude and the vacuum fluctuation amplitude, they decay at the amplitude decay rate \( \kappa/2 \), not the energy decay rate \( \kappa \) as is sometimes naively assumed.

![FIG. 1: (Color online) (a) Noise spectrum of the photon number in a driven cavity as a function of frequency when the cavity drive frequency is detuned from the cavity resonance by \( \Delta = +5\kappa \) (leading to heating: dashed line) and \( \Delta = -5\kappa \) (cooling: solid line). (b) Effective noise temperature \( T_{\text{eff}} \) as a function of the detuning from the cavity resonance, see Eq. (3), for a cantilever frequency matching the optomechanical damping mechanism. Note however that it is the asymmetry in the noise which leads to cooling or heating. The noise of the radiation pressure force \( \hat{F} = h\bar{n} \hat{a} \) at positive frequency, \( S_{FF}(+\omega_M) \), corresponds to the ability of the cavity to absorb a quantum of energy from the cantilever, while noise at negative frequency \( S_{FF}(-\omega_M) \) corresponds to the ability of the cavity to emit a quantum of energy into the cantilever. Assuming \( \omega_M \gg \Gamma_M \), the net optical damping rate of the cantilever from Fermi’s Golden Rule is

\[
\Gamma_{\text{opt}} = \frac{1}{\hbar^2} [S_{FF}(\omega_M) - S_{FF}(-\omega_M)] x_{ZPF}^2
\]

where \( x_{ZPF} \) is the cantilever zero point position uncertainty, and \( S_{FF} = \hbar^2 A^2 S_{nn} \).

As can be seen from Eq. (3) and in Fig. (1b), for positive detuning, the noise peaks at negative \( \omega \) meaning that the noise tends to heat the degree of freedom \( \hat{x} \). For negative detuning the noise peaks at positive \( \omega \) corresponding to the cavity absorbing energy from \( \hat{x} \). Essentially, the interaction with \( \hat{x} \) (three wave mixing) tries to Raman scatter the drive photons into the high density of states at the cavity frequency. If this is uphill in energy, then \( \hat{x} \) is cooled. A somewhat similar mechanism involving an oscillator with non-linear damping was considered by Dykman [10].

The radiation pressure noise is completely non-equilibrium but can be assigned a unique effective temperature assuming that the mechanical oscillator is perfectly harmonic with sufficiently weak optical and mechanical damping, \( \Gamma_M + \Gamma_{\text{opt}} \ll \kappa \), so that it responds to the noise only at frequencies \( \pm \omega_M \). In the absence of mechanical damping, the mean number of mechanical quanta \( \bar{n}_M \) present in the steady state of the cantilever is given by the detailed balance expression (for \( \Delta < 0 \))

\[
\frac{\bar{n}_M + 1}{\bar{n}_M} = \frac{S_{FF}(+\omega_M)}{S_{FF}(-\omega_M)} = \exp \left( \frac{\hbar \omega_M}{T_{\text{eff}}} \right).
\]

The noise temperature \( T_{\text{eff}} \) will become negative when the noise leads to heating (see Fig. 1). From Eq. (3) we obtain

\[
\bar{n}_M = -\frac{(\omega_M + \Delta)^2 + (\kappa/2)^2}{4\omega_M \Delta}.
\]

For the special case of detuning \( \Delta = -\omega_M \), we have the simple limit

\[
\bar{n}_M^O = \left( \frac{\kappa}{4\omega_M} \right)^2
\]

which shows that in the limit \( \omega_M \gg \kappa \), the quantum ground state can be approached provided that the optical damping dominates over the mechanical damping. For the case \( \Delta = -\omega_M \), we obtain from Eq. (1):

\[
\Gamma_{\text{opt}} = 4 \left( \frac{x_{ZPF}}{L} \right)^2 \omega_M^2 \bar{n} \kappa \left( \frac{1}{1 + (\kappa/4\omega_M)^2} \right)^2
\]

If the mechanical damping \( \Gamma_M \) is not negligible compared to \( \Gamma_{\text{opt}} \), then a rate equation yields the full expression for
the mean number of mechanical quanta in steady state (Fig. 4).
\[ \bar{n}_M = \frac{\Gamma_{\text{opt}} \bar{n}_M^O + \Gamma_M \bar{n}_M^T}{\Gamma_{\text{opt}} + \Gamma_M} \] (9)

where \( \bar{n}_M^T \) is the equilibrium mechanical mode occupation number determined by the mechanical bath temperature. In the limit of large mode occupation we can replace the occupation numbers by the corresponding temperatures and we see that this expression reduces to the expected classical expression for the final effective temperature.

We emphasize once more that using a large value of the detuning \(|\Delta| = \omega_M \gg \kappa\) offers the advantage of being able to reach an arbitrarily small minimum phonon number by optomechanical cooling, see Eq. (7). The only price to pay is to increase the input intensity, in order to keep constant the number of photons inside the cavity, \( \bar{n} = \bar{n}_{\text{max}}/(1 + (2\Delta/\kappa)^2) \), which determines the cooling rate. Here \( \bar{n}_{\text{max}} \) is the photon number at resonance, proportional to the input power. (Note however that \( \bar{n} \) is limited by our weak coupling assumption \( \Gamma_{\text{opt}} + \Gamma_M \ll \kappa \).)

In the good cavity limit, the optimum cooling condition \( \kappa = \omega_M \) requires detuning the drive laser by more than a cavity linewidth. Although this decreases the cooling efficiency in terms of the input power, it increases the efficiency in terms of the circulating power. This is advantageous in experiments where the limiting factor is heating of the cantilever by residual absorption from the circulating power.

Apart from the cooling or heating of the cantilever, there is the well-known optomechanical frequency shift of the mechanical mode ("optical spring effect") (see e.g. [34, 36]), which can also be expressed in terms of the force spectrum, using second-order perturbation theory:

\[ \delta \omega_M = \frac{\hbar^2}{\kappa^2} \int \frac{d\omega}{2\pi} S_F(\omega) \left[ \frac{1}{\omega_M - \omega} - \frac{1}{\omega_M + \omega} \right]. \] (10)

We emphasize that Eqs. (1), (5), (9), and (10) can be used to obtain the optical spring effect, the cooling rate, and the cooling limited phonon number for arbitrary force noise spectra. Such more complicated spectra might, for example, result from the contribution of more than one mode, or relate to a model different from the one discussed here, e.g. the coupling of the cantilever to some electrical circuit [35]. A related discussion in the context of cooling by a single-electron transistor has been provided in Ref. [23].

We now compare with commonly employed simpler models [27, 30, 35], where one postulates the cavity light intensity to approach its position-dependent equilibrium value in an exponential fashion, \( dn/dt = \gamma (\bar{n}(x) - n) \), with \( \bar{n}(x) \) denoting the equilibrium photon number as a function of position. Linearizing this equation and solving for the motion of \( x \) yields an effective optomechanical damping rate [27, 30, 35]

\[ \Gamma'_{\text{opt}} = \frac{\hbar A^2}{m \gamma} \frac{1}{1 + (\omega_M/\gamma)^2} \frac{\partial \bar{n}}{\partial \Delta}. \] (11)

In general, this is not equal to the correct quantum-mechanical result (1), if we use a fixed \( \gamma \propto \kappa \), and it would predict an optimum damping rate for \( \gamma \) comparable to \( \omega_M \). For damping by radiation pressure, the full analysis shows that we must instead employ a detuning-dependent effective decay rate, \( \gamma \equiv ((\dot{\theta})^2 + \Delta^2)/\kappa \). Then we recover Eq. (1) in the limit \( \omega_M \ll \kappa \). We note that Eq. (11) is always valid when applied to bolometric forces, where one describes the time-lag of the bolometric force due to a finite heat relaxation rate \( \gamma \ll \kappa \), and where a quantum theory would be different from the one presented here due to the dissipative nature of the force.

The quantum result (11) for \( \Gamma_{\text{opt}} \) can be reproduced by a classical theory based on linearizing the equation of motion for the complex light amplitude \( a \) itself [32, 36], of the form \( da/dt = i(\Delta + A x)a - \frac{\gamma}{2}(a - \bar{a}) \). Of course...
it is still not possible to recover the correct formula \((9)\) for the steady-state phonon number within such a purely classical theory, unless it is extended to include the zero-point fluctuations.

In order to have direct access to the mechanical and optical fluctuation spectra, we now derive an exact solution of the linearized Heisenberg equations of motion for \(\ddot{d}\) and \(\dot{c}\), where we use the input-output formalism \((11)\) to take into account the damping and the fluctuations, both for the optical and the mechanical degree of freedom (see also \((31)\)):

\[
\begin{align*}
\ddot{d} &= i\Delta \dot{d} - \frac{\kappa}{2} \dot{d} - \sqrt{\kappa} \dot{d} + i\alpha (\dot{c} + \dot{\dot{c}}) \\
\dot{c} &= -i \omega_M \dot{c} - \frac{\Gamma_M}{2} \dot{c} - \sqrt{\Gamma_M} \dot{c} + i (\alpha^* \dot{d} + \alpha \ddot{d}). (13)
\end{align*}
\]

Here the effective light amplitude has been expressed in terms of a frequency, \(\alpha = \tilde{\alpha} (\omega_R + 2 \sqrt{\Gamma_R} / L)\) with \(|\tilde{\alpha}|^2 = \bar{n} \).

The solution yields the cantilever spectrum \(S_{cc}(\omega) = \int dt e^{i\omega t} \langle \dot{c}^\dagger(t) \dot{c} \rangle\):

\[
S_{cc}(\omega) = \frac{\Gamma_M \sigma_{th}(\omega) + i |\tilde{\alpha}|^2 \sigma_{opt}(\omega)}{|\mathcal{N}(\omega)|^2}, (14)
\]

where

\[
\begin{align*}
\sigma_{th}(\omega) &= (\bar{n}_{\text{M}} + 1) |\Sigma(\omega)|^2 + \bar{n}_{\text{M}} \chi_M^{-1}(\omega) + i \Sigma(\omega) \mathcal{F}(15) \\
\sigma_{opt}(\omega) &= \kappa^2 |\chi_R(\omega)|^2 |\chi_M^{-1}(\omega)|^2 (16) \\
|\mathcal{N}(\omega)|^2 &= \chi_M^{-1}(\omega) \chi_M^{-1*}(-\omega) + 2 \omega_M |\Sigma(\omega)| (17).
\end{align*}
\]

We introduced the response functions of mirror and optical resonator, \(\chi_M(\omega) = \chi_{\text{opt}}(\omega) = (\frac{\delta_{\text{M}}}{\delta_{\text{opt}}} - i(\omega - \omega_M))^{-1} \) and \(\chi_R(\omega) = (\frac{\delta_{\text{M}}}{\delta_{\text{opt}}} - i(\omega + \Delta))^{-1} \), and we defined the optomechanical “self-energy” \(\Sigma(\omega) = -i |\tilde{\alpha}|^2 (\chi_R(\omega) - \chi_R^*(-\omega))\).

The quantum noise results given previously are valid in the weak-coupling limit \(\Gamma_M, \Gamma_{\text{opt}} \ll \kappa\). Then, the optomechanical damping and the “optical spring” frequency shift can be read off the self-energy as \(\text{Im}\Sigma(\omega_M) = -\Gamma_{\text{opt}} / 2\) and \(\text{Re}\Sigma(\omega_M) = \delta \omega_M\), coinciding with the expressions given above (Eqs. \((4)\) and \((11)\)). The steady state average phonon number \(\bar{n}_{\text{M}} = \int \frac{d\omega}{2\pi} S_{cc}(\omega)\) reproduces Eq. \((9)\). The optical output spectrum (of \(\bar{d}_{\text{out}} = \bar{d}_{\text{in}} + \sqrt{\kappa} \bar{d}\)) displays a Stokes peak at \(\omega = -\omega_M\) and an anti-Stokes peak at \(\omega = +\omega_M\), with weights given by \(\Gamma_{\text{opt}} (\bar{n}_{\text{M}} + 1) \bar{n}_{\text{M}}^2\) and \(\Gamma_{\text{opt}} \bar{n}_{\text{M}} (\bar{n}_{\text{M}} + 1)\), respectively, which are the rates of processes leading to heating/cooling of the cantilever by red-/blue-shifting a reflected photon.

The exact solution allows us also to discuss the regime of strong cooling. In the limit \(\Gamma_{\text{opt}} \approx 4 |\tilde{\alpha}|^2 / \kappa \gg \Gamma_M\), it adds a term \(\bar{n}_{\text{M}}^T \Gamma_M / \kappa + 2 \bar{n}_{\text{M}}^T \Gamma_{\text{opt}} / \kappa\) to Eq. \((9)\) for \(\bar{n}_{\text{M}}\) (assuming \(\kappa, \alpha \ll \omega_M\) and \(\Gamma_M \ll \kappa\)). Then a minimal phonon number \(\bar{n}_{\text{M}}^\text{min} \geq \bar{n}_{\text{M}}^T \Gamma_M / \kappa\) is found as a function of \(\Gamma_{\text{opt}}\) at \((\Gamma_{\text{opt}} / \kappa)^2 = (\Gamma_M / \kappa)(\bar{n}_{\text{M}}^T \Gamma_{\text{opt}} / \kappa)^2 / 2\). For \(\Gamma_{\text{opt}} / \kappa > 1 / 2\), the mirror resonance peak splits into a pair of peaks at \(-\omega_M \pm \alpha\), getting hybridized with the driven cavity mode (see arrow in inset of Fig. \(2\)). At even larger photon number or smaller \(\omega_M\), when \(\alpha \approx \omega_M / 2\), the static bistability \((32)\) precludes reaching the desired detuning \(\Delta = -\omega_M\) (see inset of Fig. \(2\)). Thus, we note that the far-detuned regime \(\omega_M \gg \kappa\) has the additional strong advantage of avoiding the bistability, which already interferes with cooling in some current schemes \((33)\).

We have obtained the full quantum theory of cavity assisted sideband cooling of a cantilever, based on the quantum noise approach applied to the fluctuations of the radiation pressure. In the previously unexplored regime of a detuning much larger than the cavity linewidth, the cantilever may be cooled to arbitrarily small phonon numbers. The theory analyzed here may form the basis for the interpretation of future optomechanical experiments that will venture into the quantum regime of mechanical motion.

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