Coupling of eigenvalues of complex matrices
at diabolic and exceptional points

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Abstract

The paper presents a general theory of coupling of eigenvalues of complex matrices of
arbitrary dimension depending on real parameters. The cases of weak and strong coupling are
distinguished and their geometric interpretation in two and three-dimensional spaces is given.
General asymptotic formulae for eigenvalue surfaces near diabolic and exceptional points are
presented demonstrating crossing and avoided crossing scenarios. Two physical examples
illustrate effectiveness and accuracy of the presented theory.

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1 Introduction

Behavior of eigenvalues of matrices dependent on parameters is a problem of general interest hav-
ing many important applications in natural and engineering sciences. Probably, Hamilton (1833)
was the first who revealed an interesting physical effect associated with coincident eigenvalues
known as conical refraction, see also Berry et al. (1999). In modern physics, e.g. quantum
mechanics, crystal optics, physical chemistry, acoustics and mechanics, singular points of ma-
trix spectra associated with specific effects attract great interest of researchers since the papers
Von Neumann and Wigner (1929) [Herring (1937)] [Teller (1937)]. These are the points where
matrices possess multiple eigenvalues. In applications the case of double eigenvalues is the most
important. With a change of parameters coupling and decoupling of eigenvalues with crossing
and avoided crossing scenario occur. The crossing of eigenvalue surfaces (energy levels) is con-
nected with the topic of geometrical phase, see Berry and Wilkinson (1984). In recent papers, see
e.g. Heiss (2000) Dembowski et al. (2001) [Berry and Dennis (2003) Dembowski et al. (2003)]
Keck et al. (2003) Korsch and Mossman (2003) [Heiss (2004) Stehmann et al. (2004)], two im-
portant cases are distinguished: the diabolic points (DPs) and the exceptional points (EPs). From
mathematical point of view DP is a point where the eigenvalues coalesce (become double), while
corresponding eigenvectors remain different (linearly independent); and EP is a point where both
eigenvalues and eigenvectors merge forming a Jordan block. Both the DP and EP cases are in-
teresting in applications and were observed in experiments, see e.g. Dembowski et al. (2001)
Dembowski et al. (2003) Stehmann et al. (2004). In early studies only real and Hermitian ma-
trices were considered while modern physical systems require study of complex symmetric and non-
symmetric matrices, see Mondragon and Hernandez (1993) Berry and Dennis (2003) Keck et al. (2003).
Note that most of the cited papers dealt with specific $2 \times 2$ matrices depending on two or three
parameters. Of course, in the vicinity of an EP (and also DP) the $m$-dimensional matrix problem
becomes effectively two-dimensional, but finding the corresponding two-dimensional space for a
general $m$-dimensional matrix family is a nontrivial problem Arnold (1983).
In this paper we present a general theory of coupling of eigenvalues of complex matrices of arbitrary dimension smoothly depending on multiple real parameters. Two essential cases of weak and strong coupling based on a Jordan form of the system matrix are distinguished. These two cases correspond to diabolic and exceptional points, respectively. We derive general formulae describing coupling and decoupling of eigenvalues, crossing and avoided crossing of eigenvalue surfaces. We present typical (generic) pictures showing movement of eigenvalues, the eigenvalue surfaces and their cross-sections. It is emphasized that the presented theory of coupling of eigenvalues of complex matrices gives not only qualitative, but also quantitative results on behavior of eigenvalues based only on the information taken at the singular points. Two examples on propagation of light in a homogeneous non-magnetic crystal possessing natural optical activity (chirality) and dichroism (absorption) in addition to biaxial birefringence illustrate basic ideas and effectiveness of the developed theory.

The presented theory is based on previous research on interaction of eigenvalues of real matrices depending on multiple parameters with mechanical applications. In [Seyranian (1991)], the important notion of weak and strong coupling (interaction) was introduced for the first time. In the papers [Seyranian and Pedersen (1993)] [Seyranian et al. (1994)] [Mailybaev and Seyranian (1999)] [Seyranian and Kliem (2001)] [Seyranian and Mailybaev (2001)] [Kirillov and Seyranian (2002)] [Seyranian and Mailybaev (2003)] [Kirillov (2004)] [Kirillov and Seyranian (2004)], and in the recent book [Seyranian and Mailybaev (2003)] significant mechanical effects related to diabolic and exceptional points were studied. These include transference of instability between eigenvalue branches, bimodal solutions in optimal structures under stability constraints, flutter and divergence instabilities in undamped nonconservative systems, effect of gyroscopic stabiliza
tion, destabilization of a nonconservative system by infinitely small damping, which were described and explained from the point of view of coupling of eigenvalues. An interesting application of the results on eigenvalue coupling to electrical engineering problems is given in [Dobson et al. (2001)].

The paper is organized as follows. In Section 2 we present general results on weak and strong coupling of eigenvalues of complex matrices depending on parameters. These two cases correspond to the study of eigenvalue behavior near diabolic and exceptional points. Section 3 is devoted to crossing and avoided crossing of eigenvalue surfaces near double eigenvalues with one and two eigenvectors. Two physical examples are presented in Section 4, and finally we end up with the conclusion in Section 5.

## 2 Coupling of eigenvalues

Let us consider the eigenvalue problem

\[ A u = \lambda u \]  

for a general \( m \times m \) complex matrix \( A \) smoothly depending on a vector of \( n \) real parameters \( p = (p_1, \ldots, p_n) \). Assume that, at \( p = p_0 \), the eigenvalue coupling occurs, i.e., the matrix \( A_0 = A(p_0) \) has an eigenvalue \( \lambda_0 \) of multiplicity 2 as a root of the characteristic equation \( \det(A_0 - \lambda_0 I) = 0 \); \( I \) is the identity matrix. This double eigenvalue can have one or two linearly independent eigenvectors \( u \), which determine the geometric multiplicity. The eigenvalue problem adjoint to (1) is

\[ A^* v = \eta v, \]

where \( A^* = A^T \) is the adjoint matrix operator (Hermitian transpose), see e.g. [Lancaster (1969)]. The eigenvalues \( \lambda \) and \( \eta \) of problems (1) and (2) are complex conjugate: \( \eta = \bar{\lambda} \).

Double eigenvalues appear at sets in parameter space, whose codimensions depend on the matrix type and the degeneracy (EP or DP). In Table 1 we list these codimensions based on the results of the singularity theory [Von Neumann and Wigner (1929)] [Arnold (1983)]. In this paper we analyze general (nonsymmetric) complex matrices. The EP degeneracy is the most typical for this type of matrices. In comparison with EP, the DP degeneracy is a rare phenomenon in systems described by general complex matrices. However, some nongeneric situations may be interesting from the
| matrix type         | codimension of DP | codimension of EP |
|---------------------|-------------------|-------------------|
| real symmetric      | 2                 | non-existent      |
| real nonsymmetric   | 3                 | 1                 |
| Hermitian           | 3                 | non-existent      |
| complex symmetric   | 4                 | 2                 |
| complex nonsymmetric| 6                 | 2                 |

Table 1: Codimensions of eigenvalue degeneracies.

physical point of view. As an example, we mention complex non-Hermitian perturbations of symmetric two-parameter real matrices, when the eigenvalue surfaces have coffee-filter singularity, see Mondragon and Hernandez (1993) [Berry and Dennis (2003) Keck et al. (2003). A general theory of this phenomenon will be given in our companion paper Kirillov et al. (2004).

Let us consider a smooth perturbation of parameters in the form \( p = p(\varepsilon) \), where \( p(0) = p_0 \) and \( \varepsilon \) is a small real number. For the perturbed matrix \( A = A(p(\varepsilon)) \), we have

\[
A = A_0 + \varepsilon A_1 + \frac{1}{2} \varepsilon^2 A_2 + o(\varepsilon^2),
\]

\[
A_0 = A(p_0), \quad A_1 = \sum_{i=1}^{n} \frac{\partial A}{\partial p_i} dp_i, \quad A_2 = \sum_{i=1}^{n} \frac{\partial A}{\partial p_i} d^2 p_i + \sum_{i,j=1}^{n} \frac{\partial^2 A}{\partial p_i \partial p_j} dp_i dp_j.
\]

The double eigenvalue \( \lambda_0 \) generally splits into a pair of simple eigenvalues under the perturbation. Asymptotic formulae for these eigenvalues and corresponding eigenvectors contain integer or fractional powers of \( \varepsilon \) [Vishik and Lyusternik (1960)].

### 2.1 Weak coupling of eigenvalues

Let us consider the coupling of eigenvalues in the case of \( \lambda_0 \) with two linearly independent eigenvectors \( u_1 \) and \( u_2 \). This coupling point is known as a diabolic point. Let us denote by \( v_1 \) and \( v_2 \) two eigenvectors of the complex conjugate eigenvalue \( \eta = \lambda \) for the adjoint eigenvalue problem satisfying the normalization conditions

\[
(u_1, v_1) = (u_2, v_2) = 1, \quad (u_1, v_2) = (u_2, v_1) = 0,
\]

where \( (u, v) = \sum_{i=1}^{n} u_i v_i \) denotes the Hermitian inner product. Conditions [4] define the unique vectors \( v_1 \) and \( v_2 \) for given \( u_1 \) and \( u_2 \) [Seyranian and Mailybaev (2003)].

For nonzero small \( \varepsilon \), the two eigenvalues \( \lambda_+ \) and \( \lambda_- \) resulting from the bifurcation of \( \lambda_0 \) and the corresponding eigenvectors \( u_\pm \) are given by

\[
\lambda_\pm = \lambda_0 + \mu_\pm \varepsilon + o(\varepsilon), \quad u_\pm = \alpha_\pm u_1 + \beta_\pm u_2 + o(1).
\]

The coefficients \( \mu_\pm, \alpha_\pm, \) and \( \beta_\pm \) are found from the \( 2 \times 2 \) eigenvalue problem (see e.g. Seyranian and Mailybaev (2003))

\[
\begin{pmatrix}
(A_1 u_1, v_1) & (A_1 u_2, v_1) \\
(A_1 u_1, v_2) & (A_1 u_2, v_2)
\end{pmatrix}
\begin{pmatrix}
\alpha_+ \\
\beta_+
\end{pmatrix}
= \mu_\pm
\begin{pmatrix}
\alpha_+ \\
\beta_+
\end{pmatrix}.
\]

Solving the characteristic equation for (6), we find

\[
\mu_\pm = \frac{(A_1 u_1, v_1) + (A_1 u_2, v_2)}{2} \pm \sqrt{\frac{((A_1 u_1, v_1) - (A_1 u_2, v_2))^2}{4} + (A_1 u_1, v_2)(A_1 u_2, v_1)}.
\]

We note that for Hermitian matrices \( A \) one can take \( v_1 = u_1 \) and \( v_2 = u_2 \) in (6), where the eigenvectors \( u_1 \) and \( u_2 \) are chosen satisfying the conditions \( (u_1, u_1) = (u_2, u_2) = 1 \) and \( (u_1, u_2) = 0 \), and obtain the well-known formula, see Courant and Hilbert (1953).
Figure 1: Eigenvalue coupling: (a) weak, (b) strong.

As the parameter vector passes the coupling point \( p_0 \) along the curve \( p(\varepsilon) \) in parameter space, the eigenvalues \( \lambda_+ \) and \( \lambda_- \) change smoothly and cross each other at \( \lambda_0 \), see Figure 1a. At the same time, the corresponding eigenvectors \( u_+ \) and \( u_- \) remain different (linearly independent) at all values of \( \varepsilon \) including the point \( p_0 \). We call this interaction weak coupling. By means of eigenvectors, the eigenvalues \( \lambda_\pm \) are well distinguished during the weak coupling.

We emphasize that despite the eigenvalues \( \lambda_\pm \) and the eigenvectors \( u_\pm \) depend smoothly on a single parameter \( \varepsilon \), they are nondifferentiable functions of multiple parameters at \( p_0 \) in the sense of Frechét [Schwartz (1967)].

2.2 Strong coupling of eigenvalues

Let us consider coupling of eigenvalues at \( p_0 \) with a double eigenvalue \( \lambda_0 \) possessing a single eigenvector \( u_0 \). This case corresponds to the exceptional point. The second vector of the invariant subspace corresponding to \( \lambda_0 \) is called an associated vector \( u_1 \) (also called a generalized eigenvector [Lancaster (1969)]); it is determined by the equation

\[
A_0^* u_1 = \lambda_0 u_1 + u_0.
\]

An eigenvector \( v_0 \) and an associated vector \( v_1 \) of the matrix \( A^* \) are determined by

\[
A_0^* v_0 = \bar{\lambda}_0 v_0, \quad A_0^* v_1 = \bar{\lambda}_0 v_1 + v_0, \quad (u_1, v_0) = 1, \quad (u_1, v_1) = 0,
\]

where the last two equations are the normalization conditions determining \( v_0 \) and \( v_1 \) uniquely for a given \( u_1 \).

Bifurcation of \( \lambda_0 \) into two eigenvalues \( \lambda_\pm \) and the corresponding eigenvectors \( u_\pm \) are described by (see e.g. [Seyranian and Mailybaev (2003)])

\[
\lambda_\pm = \lambda_0 \pm \sqrt{\mu_1 \varepsilon} + \mu_2 \varepsilon + o(\varepsilon),
\]

\[
u_\pm = u_0 \pm u_1 \sqrt{\mu_1 \varepsilon} + (\mu_1 u_0 + \mu_2 u_1 - G^{-1} A_1 u_0) \varepsilon + o(\varepsilon),
\]

where \( G = A_0 - \lambda_0 I + u_1 v_1^* \). The coefficients \( \mu_1 \) and \( \mu_2 \) are

\[
\mu_1 = (A_1 u_0, v_0), \quad \mu_2 = ((A_1 u_0, v_1) + (A_1 u_1, v_0))/2.
\]

With a change of \( \varepsilon \) from negative to positive values, the two eigenvalues \( \lambda_\pm \) approach, collide with infinite speed (derivative with respect to \( \varepsilon \) tends to infinity) at \( \lambda_0 \), and diverge in the perpendicular direction, see Figure 1b. The eigenvectors interact too. At \( \varepsilon = 0 \), they merge to \( u_0 \) up to a scalar complex factor. At nonzero \( \varepsilon \), the eigenvectors \( u_\pm \) differ from \( u_0 \) by the leading...
term \( \pm u_1 \sqrt{\mu_1 \varepsilon} \). This term takes the purely imaginary factor \( i \) as \( \varepsilon \) changes the sign, for example altering from negative to positive values.

We call such a coupling of eigenvalues as strong. An exciting feature of the strong coupling is that the two eigenvalues cannot be distinguished after the interaction. Indeed, there is no natural rule telling how the eigenvalues before coupling correspond to those after the coupling.

3 Crossing of eigenvalue surfaces

3.1 Double eigenvalue with single eigenvector

Let, at the point \( p_0 \), the spectrum of the complex matrix family \( \Lambda(p) \) contain a double complex eigenvalue \( \lambda_0 \) with an eigenvector \( u_0 \) and an associated vector \( u_1 \). The splitting of the double eigenvalue with a change of the parameters is governed by equations (10) and (11). Introducing the real \( n \)-dimensional vectors \( f, g, h, r \) with the components

\[
\begin{align*}
    f_s &= \text{Re} \left( \frac{\partial A}{\partial p_s} u_0, v_0 \right), \\
    g_s &= \text{Im} \left( \frac{\partial A}{\partial p_s} u_0, v_0 \right), \\
    h_s &= \text{Re} \left( \left( \frac{\partial A}{\partial p_s} u_0, v_1 \right) + \left( \frac{\partial A}{\partial p_s} u_1, v_0 \right) \right), \\
    r_s &= \text{Im} \left( \left( \frac{\partial A}{\partial p_s} u_0, v_1 \right) + \left( \frac{\partial A}{\partial p_s} u_1, v_0 \right) \right),
\end{align*}
\]

and neglecting higher order terms, we obtain from (10) an asymptotic formula

\[
\text{Re}\Delta \lambda + i\text{Im}\Delta \lambda = \pm \sqrt{\langle f, \Delta p \rangle + i\langle g, \Delta p \rangle} + \frac{1}{2}(\langle h, \Delta p \rangle + i\langle r, \Delta p \rangle),
\]

where \( \Delta \lambda = \lambda_\pm - \lambda_0 \), \( \Delta p = p - p_0 \), and angular brackets denote inner product of real vectors:

\[
\langle a, b \rangle = \sum_{s=1}^{n} a_s b_s.
\]

From equation (14) it is clear that the eigenvalue remains double in the first approximation if the two following equations are satisfied

\[
\begin{align*}
    \langle f, \Delta p \rangle &= 0, \\
    \langle g, \Delta p \rangle &= 0.
\end{align*}
\]

This means that the double complex eigenvalue with the Jordan chain of length 2 has codimension 2. Thus, double complex eigenvalues occur at isolated points of the plane of two parameters, and in the three-parameter space the double eigenvalues form a curve [Arnold (1983)]. Equations (15) define a tangent line to this curve at the point \( p_0 \).

Taking square of (14), where the terms linear with respect to the increment of parameters are neglected, and separating real and imaginary parts, we derive the equations

\[
\begin{align*}
    (\text{Re}\Delta \lambda)^2 - (\text{Im}\Delta \lambda)^2 &= \langle f, \Delta p \rangle, \\
    2\text{Re}\Delta \lambda \text{Im}\Delta \lambda &= \langle g, \Delta p \rangle.
\end{align*}
\]

Let us assume that \( f_1^2 + g_1^2 \neq 0 \), which is the nondegeneracy condition for the complex eigenvalue \( \lambda_0 \). Isolating the increment \( \Delta p_1 \) in one of the equations (16) and substituting it into the other one we get

\[
g_1(\text{Re}\Delta \lambda)^2 - 2f_1\text{Re}\Delta \lambda \text{Im}\Delta \lambda - g_1(\text{Im}\Delta \lambda)^2 = \gamma,
\]

where \( \gamma \) is a small real constant

\[
\gamma = \sum_{s=2}^{n} (f_s g_1 - f_1 g_s) \Delta p_s.
\]

Equation (17) describes hyperbolic trajectories of the eigenvalues \( \lambda_\pm \) in the complex plane when only \( \Delta p_1 \) is changed and the increments \( \Delta p_2, \ldots, \Delta p_n \) are fixed. Of course, any component of the vector \( \Delta p \) can be chosen instead of \( \Delta p_1 \).
Let us study movement of eigenvalues in the complex plane in more detail. If $\Delta p_j = 0$, $j = 2, \ldots, n$, or if they are nonzero but satisfy the equality $\gamma = 0$, then equation (17) yields two perpendicular lines which for $g_1 \neq 0$ are described by the expression

$$g_1 \text{Re}(\lambda - \lambda_0) - \left( f_1 \pm \sqrt{f_1^2 + g_1^2} \right) \text{Im}(\lambda - \lambda_0) = 0. \quad (19)$$

These lines intersect at the point $\lambda_0$ of the complex plane. Due to variation of the parameter $p_1$ two eigenvalues $\lambda_{\pm}$ approach along one of the lines (19), merge to $\lambda_0$ at $\Delta p_1 = 0$, and then diverge along the other line (19), perpendicular to the line of approach; see Figure 2, where the arrows show motion of eigenvalues with a monotonous change of $p_1$. Recall that the eigenvalues that born after the coupling cannot be identified with the eigenvalues before the coupling.

If $\gamma \neq 0$, then equation (17) defines a hyperbola in the complex plane. Indeed, for $g_1 \neq 0$ it is transformed to the equation of hyperbola

$$(g_1 \text{Re}(\lambda - \lambda_0) - f_1 \text{Im}(\lambda - \lambda_0))^2 - (\text{Im}(\lambda - \lambda_0))^2 (f_1^2 + g_1^2) = \gamma g_1 \quad (20)$$

with the asymptotes described by equation (19). As $\Delta p_1$ changes monotonously, two eigenvalues $\lambda_{\pm}$ moving each along its own branch of hyperbola come closer, turn and diverge; see Figure 2, c. Note that for a small $\gamma$ the eigenvalues $\lambda_{\pm}$ come arbitrarily close to each other without coupling that means avoided crossing. When $\gamma$ changes the sign, the quadrants containing hyperbola branches are changed to the adjacent.

Expressing $\text{Im}\Delta\lambda$ from the second of equations (16), substituting it into the first equation and then isolating $\text{Re}\Delta\lambda$, we find

$$\text{Re}\lambda_{\pm} = \lambda_0 + \frac{1}{2} \langle h, \Delta p \rangle \pm \sqrt{\frac{1}{2} \left( \langle f, \Delta p \rangle + \sqrt{\langle f, \Delta p \rangle^2 + \langle g, \Delta p \rangle^2} \right)}.$$

Similar transformation yields

$$\text{Im}\lambda_{\pm} = \lambda_0 + \frac{1}{2} \langle r, \Delta p \rangle \pm \sqrt{\frac{1}{2} \left( - \langle f, \Delta p \rangle + \sqrt{\langle f, \Delta p \rangle^2 + \langle g, \Delta p \rangle^2} \right)}.$$

Equations (21) and (22) describe behavior of real and imaginary parts of eigenvalues $\lambda_{\pm}$ with a change of the parameter. On the other hand they define hypersurfaces in the spaces $(p_1, p_2, \ldots, p_n, \text{Re}\lambda)$ and $(p_1, p_2, \ldots, p_n, \text{Im}\lambda)$. The sheets $\text{Re}\lambda_+(p)$ and $\text{Re}\lambda_-(p)$ of the eigenvalue hypersurface (21) are connected at the points of the set

$$\text{Re}\Delta\lambda = \frac{1}{2} \langle h, \Delta p \rangle, \quad \langle g, \Delta p \rangle = 0, \quad \langle f, \Delta p \rangle \leq 0, \quad (23)$$

where the real parts of the eigenvalues $\lambda_{\pm}$ coincide: $\text{Re}\lambda_- = \text{Re}\lambda_+$. Similarly, the set

$$\text{Im}\Delta\lambda = \frac{1}{2} \langle r, \Delta p \rangle, \quad \langle g, \Delta p \rangle = 0, \quad \langle f, \Delta p \rangle \geq 0, \quad (24)$$

glues the sheets $\text{Im}\lambda_+(p)$ and $\text{Im}\lambda_-(p)$ of the eigenvalue hypersurface (22).

To study the geometry of the eigenvalue hypersurfaces we look at their two-dimensional cross-sections. Consider for example the functions $\text{Re}\lambda(p_1)$ and $\text{Im}\lambda(p_1)$ at fixed values of the other parameters $p_2, p_3, \ldots, p_n$. When the increments $\Delta p_s = 0$, $s = 2, 3, \ldots, n$, both the real and imaginary parts of the eigenvalues $\lambda_{\pm}$ cross at $p_1 = p_1^n$, see Figure 2. The crossings are described by the double cusps defined by the equations following from (21) and (22) as

$$\text{Re}\Delta\lambda = \pm \sqrt{\frac{f_1 \pm \sqrt{f_1^2 + g_1^2}}{2} \Delta p_1 + \frac{h_1}{2} \Delta p_1}, \quad \text{Im}\Delta\lambda = \pm \sqrt{-\frac{f_1 \pm \sqrt{f_1^2 + g_1^2}}{2} \Delta p_1 + \frac{r_1}{2} \Delta p_1}, \quad (25)$$

where the signs of the square roots are chosen with a change of the sign, the quadrants containing hyperbola branches are changed to the adjacent.
Figure 2: Crossing and avoided crossing of eigenvalues.
For the fixed $\Delta p_s \neq 0$, $s = 2, 3, \ldots, n$, either real parts of the eigenvalues $\lambda_\pm$ cross due to variation of $p_1$ while the imaginary parts avoid crossing or vice-versa, as shown in Figure 2a,c. Note that these two cases correspond to level crossing and width repulsion or vice-versa studied in Heiss (2000). The crossings, which occur at $p_1^* = p_1^0 - \sum_{s=2}^{n} (g_s / g_1) \Delta p_s$ and

$$
\text{Re} \lambda = \text{Re} \lambda_0 - \frac{1}{2g_1} \sum_{s=2}^{n} (h_1 g_s - g_1 h_s) \Delta p_s, \quad \text{Im} \lambda = \text{Im} \lambda_0 - \frac{1}{2g_1} \sum_{s=2}^{n} (r_1 g_s - g_1 r_s) \Delta p_s,
$$

are described by the equations (21) and (22). In the vicinity of the crossing points the tangents of two intersecting curves are

$$
\text{Re} \lambda = \text{Re} \lambda_0 + \left( \frac{h_1}{2} \pm \frac{g_1}{2} \sqrt{\frac{g_1}{\gamma}} \right) (p_1^0 - p_1^*),
$$

$$
\text{Im} \lambda = \text{Im} \lambda_0 + \left( \frac{r_1}{2} \pm \frac{g_1}{2} \sqrt{\frac{-g_1}{\gamma}} \right) (p_1^0 - p_1^*),
$$

where the coefficient $\gamma$ is defined by equation (18). Lines (27) and (28) tend to the vertical position as $\gamma \to 0$ and coincide at $\gamma = 0$. The avoided crossings are governed by the equations (21) and (22).

If the vector of parameters consists of only two components $\mathbf{p} = (p_1, p_2)$, then in the vicinity of the point $\mathbf{p}_0$, corresponding to the double eigenvalue $\lambda_0$, the eigenvalue surfaces (21) and (22) have the form of the well-known Whitney umbrella; see Figure 3. The sheets of the eigensurfaces are connected along the rays (23) and (24). We emphasize that these rays are inclined with respect to the plane of the parameters $p_1, p_2$. The cross-sections of the eigensurfaces by the planes orthogonal to the axis $p_2$, described by the equations (25)–(28), are shown in Figure 2.

Note that the rays (23), (24) and the point $\mathbf{p}_0$ are well-known in crystal optics as the branch cuts and the singular axis, respectively Berry and Dennis (2003). We emphasize that the branch cut is a general phenomenon: it always appears near the EP degeneracy. In general, branch cuts may be infinite or end up at another EP. The second scenario is always the case when the complex matrix $\mathbf{A}(\mathbf{p})$ is a small perturbation of a family of real symmetric matrices Kirillov et al. (2004).

Consider the movement of the eigenvalues in the complex plane near the point $\mathbf{p}_0$ due to cyclic variation of the parameters $p_1$ and $p_2$ of the form $\Delta p_1 = a + r \cos \varphi$ and $\Delta p_2 = b + r \sin \varphi$, where $a$, $b$, and $r$ are small parameters of the same order. From equations (10) we derive

$$(g_1 \text{Re} \Delta \lambda^2 - 2f_1 \text{Re} \Delta \lambda \text{Im} \Delta \lambda - g_1 \text{Im} \Delta \lambda^2 - b(f_2 g_1 - f_1 g_2))^2 +$$
Figure 4: Movement of eigenvalues due to cyclic evolution of the parameters.

\[ (g_2 \text{Re} \Delta \lambda^2 - 2 f_2 \text{Re} \Delta \lambda \text{Im} \lambda - g_2 \text{Im} \Delta \lambda^2 - a(f_1 g_2 - g_1 f_2))^2 = (f_2 g_1 - f_1 g_2)^2 r^2. \]  

Movement of eigenvalues on the complex plane governed by equation (29) is shown in Figure 4. If the contour encircles the point \( p_0 \), then the eigenvalues move along the curve (29) around the double eigenvalue \( \lambda_0 \) in the complex plane, see Figure 4c. Indeed, in this case \( a^2 + b^2 < r^2 \) and the loop (29) crosses the lines \( \text{Re} \lambda = \text{Re} \lambda_0 \) and \( \text{Im} \lambda = \text{Im} \lambda_0 \) at the four points given by the equations

\[
(\text{Im} \Delta \lambda)^2 = \left( f_2 g_1 - f_1 g_2 \right) \left( g_2 a - g_1 b \pm \sqrt{(g_2 a - g_1 b)^2 + (r^2 - a^2 - b^2)(g_1^2 + g_2^2)} \right) / (g_1^2 + g_2^2),
\]

and

\[
(\text{Re} \Delta \lambda)^2 = \left( f_2 g_1 - f_1 g_2 \right) \left( g_1 b - g_2 a \pm \sqrt{(g_1 b - g_2 a)^2 + (r^2 - a^2 - b^2)(g_1^2 + g_2^2)} \right) / (g_1^2 + g_2^2),
\]

respectively. When \( a^2 + b^2 = r^2 \) the loop overlaps at the double eigenvalue and its form depends on the sign of the quantity \( \sigma = (f_2 g_1 - f_1 g_2)(g_1 b - g_2 a) \). If \( \sigma < 0 \) the eigenvalues cross the line \( \text{Re} \lambda = \text{Re} \lambda_0 \) (Figure 4b), otherwise they cross the line \( \text{Im} \lambda = \text{Im} \lambda_0 \) (Figure 4d). Eigenvalues strongly couple at the point \( \lambda_0 \) in the complex plane. For \( a^2 + b^2 > r^2 \) the circuit in the parameter plane does not contain the point \( p_0 \) and the eigenvalues move along the two different closed paths (“kidneys”, [Arnold (1989)]) in the complex plane, see Figure 4a, e. Each eigenvalue crosses the line \( \text{Re} \lambda = \text{Re} \lambda_0 \) twice for \( \sigma < 0 \) (Figure 4a), and for \( \sigma > 0 \) they cross the line \( \text{Im} \lambda = \text{Im} \lambda_0 \) (Figure 4d).

Note that the ”kidneys” in the complex plane were observed in [Korsch and Mossman (2003)] for the specific problem of Stark resonances for a double \( \delta \) quantum well.

### 3.2 Double eigenvalue with two eigenvectors

Let \( \lambda_0 \) be a double eigenvalue of the matrix \( A_0 = A(p_0) \) with two eigenvectors \( u_1 \) and \( u_2 \). Under perturbation of parameters \( p = p_0 + \Delta p \), the bifurcation of \( \lambda_0 \) into two simple eigenvalues \( \lambda_+ \) and \( \lambda_- \) occurs. Using (5) and (7), we obtain the asymptotic formula for \( \lambda_\pm \) under multiparameter perturbation as

\[
\lambda_\pm = \lambda_0 \pm \frac{\langle d_{11} + d_{22}, \Delta p \rangle}{2} \pm \sqrt{\langle d_{11} - d_{22}, \Delta p \rangle^2 + 4 \langle d_{12}, \Delta p \rangle \langle d_{21}, \Delta p \rangle},
\]

where \( d_{ij} = (d_{ij}^1, \ldots, d_{ij}^n) \) is a complex vector with the components

\[
d_{ij}^k = \left( \frac{\partial A}{\partial p_k} u_i, v_j \right).
\]
and \((d_{ij}, \Delta p) = \langle \text{Re } d_{ij}, \Delta p \rangle + i \langle \text{Im } d_{ij}, \Delta p \rangle\). In the same way as we derived formulae (21) and (22), we obtain from (32) the expressions for real and imaginary parts of \(\lambda_\pm\) in the form

\[
\text{Re } \lambda_\pm = \text{Re } \lambda_0 + \text{Re } \langle d_{11} + d_{22}, \Delta p \rangle/2 \pm \sqrt{(|c| + \text{Re } c)/2},
\]

(34)

\[
\text{Im } \lambda_\pm = \text{Im } \lambda_0 + \text{Im } \langle d_{11} + d_{22}, \Delta p \rangle/2 \pm \sqrt{(|c| - \text{Re } c)/2},
\]

(35)

where

\[
c = \langle d_{11} - d_{22}, \Delta p \rangle^2/4 + \langle d_{12}, \Delta p \rangle \langle d_{21}, \Delta p \rangle.
\]

(36)

Considering the situation when \(\lambda_0\) remains double under perturbation of parameters, i.e. \(\lambda_+ = \lambda_-\), we obtain the two independent equations

\[
\text{Re } c = 0, \quad \text{Im } c = 0.
\]

(37)

By using (5)–(7), one can show that the perturbed double eigenvalue \(\lambda_+ = \lambda_-\) possesses a single eigenvector \(u_+ = u_-\), i.e., the weak coupling becomes strong due to perturbation, see [Seyranian and Mailybaev (2003)].

The perturbed double eigenvalue has two eigenvectors only when the matrix in the left-hand side of (6) is proportional to the identity matrix. This yields the equations

\[
\langle d_{11}, \Delta p \rangle = \langle d_{22}, \Delta p \rangle, \quad \langle d_{12}, \Delta p \rangle = \langle d_{21}, \Delta p \rangle = 0,
\]

(38)

Conditions (38) imply (37) and represent six independent equations taken for real and imaginary parts. Thus, weak coupling of eigenvalues is a phenomenon of codimension 6, which generically occurs at isolated points in 6-parameter space, see [Arnold (1983), Mondragon and Hernandez (1993)].

First, let us study behavior of the eigenvalues \(\lambda_+\) and \(\lambda_-\) depending on one parameter, say \(p_1\), when the other parameters \(p_2, \ldots, p_n\) are fixed in the neighborhood of the coupling point \(\lambda_+(p_0) = \lambda_-(p_0) = \lambda_0\). In case \(\Delta p_2 = \cdots = \Delta p_n = 0\), expression (32) yields

\[
\lambda_\pm = \lambda_0 + \left(d_{11}^2 + d_{22}^2ight)/2 \pm \sqrt{(d_{11}^2 - d_{22}^2)^2/4 + d_{12}^2 d_{21}^2} \Delta p_1.
\]

(39)

The two eigenvalues couple when \(\Delta p_1 = 0\) with the double eigenvalue \(\lambda_0\), see Figure 5a. As we showed in Section 2, the eigenvalues \(\lambda_+\) and \(\lambda_-\) behave as smooth functions at the coupling point; they possess different eigenvectors, which are smooth functions of \(\Delta p_1\) too.

If the perturbations \(\Delta p_2, \ldots, \Delta p_n\) are nonzero, the avoided crossing of the eigenvalues \(\lambda_\pm\) with a change of \(p_1\) is a typical scenario. We can distinguish different cases by checking intersections of real and imaginary parts of \(\lambda_+\) and \(\lambda_-\). By using (34), we find that \(\text{Re } \lambda_+ = \text{Re } \lambda_-\) if

\[
\text{Im } c = 0, \quad \text{Re } c < 0.
\]

(40)

Analogously, from (35) it follows that \(\text{Im } \lambda_+ = \text{Im } \lambda_-\) if

\[
\text{Im } c = 0, \quad \text{Re } c > 0.
\]

(41)

Let us write expression (36) in the form

\[
c = c_0 + c_1 \Delta p_1 + c_2 (\Delta p_1)^2,
\]

(42)

where

\[
c_0 = \sum_{k,l=2}^{n} \left[(d_{11}^k - d_{22}^k)(d_{11}^l - d_{22}^l)/4 + d_{12}^k d_{21}^l\right] \Delta p_k \Delta p_l,
\]

\[
c_1 = \sum_{k=2}^{n} \left[(d_{11}^k - d_{22}^k)(d_{11}^k - d_{22}^k)/2 + (d_{12}^k d_{21}^k + d_{12}^k d_{21}^k)d_{12}^k d_{21}^k\right] \Delta p_k,
\]

\[
c_2 = (d_{11}^2 - d_{22}^2)^2/4 + d_{12}^2 d_{21}.
\]

(43)
Figure 5: Weak coupling of eigenvalues and avoided crossing.
If the discriminant $D = (\text{Im } c_1)^2 - 4 \text{Im } c_0 \text{Im } c_2 > 0$, the equation $\text{Im } c = 0$ yields two solutions

$$
\Delta p^0_1 = \frac{-\text{Im } c_1 - \sqrt{D}}{2 \text{Im } c_2}, \quad \Delta p^0_2 = \frac{-\text{Im } c_1 + \sqrt{D}}{2 \text{Im } c_2}.
$$

(44)

There are no real solutions if $D < 0$, and the single solution corresponds to the degenerate case $D = 0$. At the points $p^0_1 = p^0_1 + \Delta p^0_1$ and $p^0_2 = p^0_2 + \Delta p^0_2$ the values of $c$ are real, and we denote them by $c_a$ and $c_b$, respectively. According to (40) and (41), the sign of $c_{a,b}$ determines whether the real or imaginary parts of $\lambda_\pm$ coincide at $p^0_{a,b}$.

In the nondegenerate case $D \neq 0$, there are four types of avoided crossing shown in Figure 5b–e. The first case corresponds to $D < 0$ when both real and imaginary parts of the eigenvalues $\lambda_\pm$ are separate at all $p_1$, see Figure 5a. In other cases $D > 0$, so that there are two separate points $p^0_1$ and $p^0_2$. For the second case we have $c_a > 0$ and $c_b < 0$, when both real and imaginary parts of $\lambda_\pm$ have a single intersection, see Figure 5d. The equivalent situation when $c_a < 0$ and $c_b > 0$ is obtained by interchanging the points $p^0_1$ and $p^0_2$ in Figure 5. The third case is represented by $c_{a,b} < 0$, when the real parts of $\lambda_\pm$ have two intersections and $\text{Im } \lambda_\pm$ do not intersect, see Figure 5e. Finally, if $c_{a,b} > 0$, when the real parts of $\lambda_\pm$ do not intersect and $\text{Im } \lambda_\pm$ intersect at both $p^0_1$ and $p^0_2$, see Figure 5f. The last column in Figure 5 shows behavior of the eigenvalues $\lambda_\pm$ on the complex plane. In each of the cases b–e, the trajectories of eigenvalues on the complex plane may intersect and/or self-intersect, which can be studied by using expression (32). Note that intersections of the eigenvalue trajectories on the complex plane do not imply eigenvalue coupling since the eigenvalues $\lambda_\pm$ and $\lambda_-$ pass the intersection point at different values of $p_1$. The small loops of the eigenvalue trajectories on the complex plane, shown in Figure 5b–e, shrink as the perturbations of the parameters $\Delta p_2, \Delta p_3, \ldots, \Delta p_n$ tend to zero. Finally, we mention that the case of Figure 5f is the only avoided crossing scenario when the eigenvalues follow the initial directions on the complex plane after interaction. In the other three cases (b, d, and e) the eigenvalues interchange their directions due the interaction.

Let us consider a system depending on two parameters $p_1$ and $p_2$ with the weak coupling of eigenvalues at $p_1 = p^0_1$ and $p_2 = p^0_2$. The double eigenvalue $\lambda_0$ bifurcates into a pair $\lambda_\pm$ under perturbation of the parameters $\Delta p_1$ and $\Delta p_2$. Conditions (46) and (47) determine the values of parameters, at which the real and imaginary parts of $\lambda_\pm$ coincide.

Let us write expression (40) in the form

$$
c = c_{11}(\Delta p_1)^2 + c_{12}\Delta p_1 \Delta p_2 + c_{22}(\Delta p_2)^2,
$$

(45)

where

$$
c_{11} = (d_{11}^2 - d_{22}^2)/4 + d_{11}^2d_{21}, \quad c_{22} = (d_{12}^2 - d_{22}^2)/4 + d_{12}^2d_{21},
$$

$$
c_{12} = (d_{11}^2 - d_{22}^2)(d_{21}^2 - d_{22}^2)/2 + d_{11}^2d_{21}^2 + d_{12}^2d_{21}.
$$

(46)

If the discriminant $D' = (\text{Im } c_{11})^2 - 4 \text{Im } c_{12} \text{Im } c_{22} > 0$, the equation $\text{Im } c = 0$ yields the two crossing lines

$$
l_a : 2 \text{Im } c_{11} \Delta p_1 + (\text{Im } c_{12} + \sqrt{D'}) \Delta p_2 = 0,
$$

$$
l_b : 2 \text{Im } c_{11} \Delta p_1 + (\text{Im } c_{12} - \sqrt{D'}) \Delta p_2 = 0.
$$

(47)

There are no real solutions if $D' < 0$, and the lines $l_a$ and $l_b$ coincide in the degenerate case $D' = 0$. On the lines $l_{a,b}$ the values of $c$ are real numbers of the same sign; we denote $\gamma_a = \text{sign } c$ for the line $l_a$, and $\gamma_b = \text{sign } c$ for the line $l_b$. According to (40) and (41), the real or imaginary parts of $\lambda_\pm$ coincide at $l_{a,b}$ for negative or positive $\gamma_{a,b}$, respectively.

One can distinguish four types of the graphs for $\text{Re } \lambda_\pm(p_1, p_2)$ and $\text{Im } \lambda_\pm(p_1, p_2)$ shown in Figure 6. In nondegenerate case $D' \neq 0$, the eigenvalues $\lambda_+ \pm$ and $\lambda_- \pm$ are different for all parameter values except the initial point $p_{1,2} = p^0_{1,2}$. If $D' < 0$, the eigenvalue surfaces are cones with non-elliptic cross-section, see Figure 6a. Other three types correspond to the case $D' > 0$. If $\gamma_a < 0$ and $\gamma_b > 0$ then there is an intersection of the real parts along the line $l_a$, an intersection of the imaginary parts along the line $l_b$ (in case $\gamma_a > 0$ and $\gamma_b < 0$ the lines $l_a$ and $l_b$ are interchanged),
If \( \gamma_a < 0 \) and \( \gamma_b < 0 \) then the real parts intersect along the both lines \( l_a \) and \( l_b \), forming a "cluster of shells", while there is no intersections for the imaginary parts, see Figure 6a. Finally, if \( \gamma_a > 0 \) and \( \gamma_b > 0 \) then there is no intersections for the real parts, while the imaginary parts intersect along the both lines \( l_a \) and \( l_b \), see Figure 6b.

4 Example

As a physical example, we consider propagation of light in a homogeneous non-magnetic crystal in the general case when the crystal possesses natural optical activity (chirality) and dichroism (absorption) in addition to biaxial birefringence, see Berry and Dennis (2003) for the general formulation. The optical properties of the crystal are characterized by the inverse dielectric tensor \( \eta \). The vectors of electric field \( E \) and displacement \( D \) are related as Landau et al. (1984)

\[
E = \eta D.
\]

The tensor \( \eta \) is described by a non-Hermitian complex matrix. The electric field \( E \) and magnetic field \( H \) in the crystal are determined by Maxwell’s equations Landau et al. (1984)

\[
\text{rot} E = -\frac{1}{c} \frac{\partial H}{\partial t}, \quad \text{rot} H = \frac{1}{c} \frac{\partial D}{\partial t},
\]

where \( t \) is time and \( c \) is the speed of light in vacuum.

A monochromatic plane wave of frequency \( \omega \) that propagates in a direction specified by a real unit vector \( s = (s_1, s_2, s_3) \) has the form

\[
D(r, t) = D(s) \exp \left(\frac{n(s)}{c} s^T r - t\right), \quad H(r, t) = H(s) \exp \left(\frac{n(s)}{c} s^T r - t\right),
\]

where \( n(s) \) is a refractive index, and \( r = (x_1, x_2, x_3) \) is the real vector of spatial coordinates. Substituting the wave into Maxwell’s equations, we find

\[
H = n[s, \eta D], \quad D = -n[s, H],
\]

where square brackets indicate cross product of vectors Landau et al. (1984). With the vector \( H \) determined by the first equation of Landau et al. (1984), the second equation of Landau et al. (1984) yields Berry and Dennis (2003)

\[
- [s, [s, \eta D(s)]] = \eta D(s) - s(s^T \eta D(s)) = \frac{1}{n^2(s)} D(s).
\]

Multiplying equation by the vector \( s^T \) from the left we find that for plane waves the vector \( D \) is always orthogonal to the direction \( s \), i.e., \( s^T D(s) = 0 \).

Since the quantity \( s^T \eta D(s) \) is a scalar, we can write in the form of an eigenvalue problem for the complex non-Hermitian matrix \( A(s) \) dependent on the vector of parameters \( s = (s_1, s_2, s_3) \):

\[
A u = \lambda u, \quad A(s) = (I - ss^T)\eta(s),
\]

where \( \lambda = n^{-2}, \ u = D, \) and \( I \) is the identity matrix. Multiplying the matrix \( A \) by the vector \( s \) from the left we conclude that \( s^T A = 0 \), i.e., the vector \( s \) is the left eigenvector with the eigenvalue \( \lambda = 0 \). Zero eigenvalue always exists, because \( \det(I - ss^T) \equiv 0 \), if \( ||s|| = 1 \).

The matrix \( A(s) \) defined by equation \( A(s) \) is a product of the matrix \( I - ss^T \) and the inverse dielectric tensor \( \eta(s) \). The symmetric part of \( \eta \) constitutes the anisotropy tensor describing the birefringence of the crystal. It is represented by the complex symmetric matrix \( U \), which is independent of the vector of parameters \( s \). The antisymmetric part of \( \eta \) is determined by the
Figure 6: Eigenvalue surfaces near a point of weak coupling.
optical activity vector $g(s) = (g_1, g_2, g_3)$, describing the chirality (optical activity) of the crystal. It is represented by the skew-symmetric matrix

$$G = i \begin{pmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{pmatrix}. \quad (54)$$

The vector $g$ is given by the expression $g(s) = \gamma s$, where $\gamma$ is the optical activity tensor represented by a symmetric complex matrix. Thus, the matrix $G(s)$ depends linearly on the parameters $s_1$, $s_2$, $s_3$.

In the present formulation, the problem was studied analytically and numerically in Berry and Dennis (2003). Below we present two specific numerical examples in case of a non-diagonal matrix $\gamma$, for which the structure of singularities was not fully investigated. Unlike Berry and Dennis (2003), where the reduction to two dimensions was carried out, we work with the three-dimensional form of problem (53). Our intention here is to give guidelines for using our theory by means of the relatively simple $3 \times 3$ matrix family, keeping in mind that the main area of applications would be higher dimensional problems.

As a first example, we choose the inverse dielectric tensor in the form

$$\eta = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + i \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -s_1 & 0 \\ s_1 & 0 & -s_3 \\ 0 & s_3 & 0 \end{pmatrix}, \quad (55)$$

where $s_3 = \sqrt{1 - s_1^2 - s_2^2}$. The crystal defined by (55) is dichroic and optically active with the non-diagonal matrix $\gamma$. When $s_1 = 0$ and $s_2 = 0$ the spectrum of the matrix $A$ consists of the double eigenvalue $\lambda_0 = 2$ and the simple zero eigenvalue. The double eigenvalue possesses the eigenvector $u_0$ and associated vector $u_1$:

$$u_0 = \begin{pmatrix} i \\ -1 \\ 0 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (56)$$

The eigenvector $v_0$ and associated vector $v_1$ corresponding to the double eigenvalue $\lambda_0 = 2$ of the adjoint matrix $A^*$ are

$$v_0 = \begin{pmatrix} i \\ 1 \\ 1 + i/2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} i \\ 0 \\ 1/2 - i/4 \end{pmatrix}. \quad (57)$$

The vectors $u_0$, $u_1$ and $v_0$, $v_1$ satisfy the normalization and orthogonality conditions (9). Calculating the derivatives of the matrix $A(s_1, s_2)$ at the point $s_0 = (0, 0, 1)$ we obtain

$$\frac{\partial A}{\partial s_1} = \begin{pmatrix} -2i & -2i & -2 \\ i & 0 & 0 \\ -3 & -i & -2i \end{pmatrix}, \quad \frac{\partial A}{\partial s_2} = \begin{pmatrix} 0 & 0 & 0 \\ -2i & -i & -2 \\ -i & -1 & i \end{pmatrix}. \quad (58)$$

Substitution of the derivatives (58) together with the vectors given by equations (56) and (57) into the formulae (12) and (13) yields the vectors $f$, $g$ and $h$, $r$ as

$$f = (0, 4), \quad g = (-4, 0), \quad h = (0, 0), \quad r = (-4, 0). \quad (59)$$

With the vectors (59) we find from (21) and (22) the approximations of the eigensurfaces $\text{Re}\lambda(s_1, s_2)$ and $\text{Im}\lambda(s_1, s_2)$ in the vicinity of the point $s_0 = (0, 0, 1)$:

$$\text{Re}\lambda_{\pm} = 2 \pm \sqrt{2s_2 + 2\sqrt{s_1^2 + s_2^2}}, \quad \text{Im}\lambda_{\pm} = -2s_1 \pm \sqrt{-2s_2 + 2\sqrt{s_1^2 + s_2^2}}. \quad (60)$$
Calculation of the exact solution of the characteristic equation for the matrix $A$ with the inverse dielectric tensor $\eta$ defined by equation (55) shows a good agreement of the approximations (60) with the numerical solution, see Figure 7. One can see that the both surfaces of real and imaginary parts have a Whitney umbrella singularity at the coupling point; the surfaces self-intersect along different rays, which together constitute a straight line when projected on parameter plane.

As a second numerical example, let us consider the inverse dielectric tensor as

$$\eta = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} + i \begin{pmatrix} 5 & 0 & 4 \\ 0 & 5 & 2 \\ 4 & 2 & 0 \end{pmatrix} + 4i \begin{pmatrix} 0 & -s_1 - is_2 & is_3 \\ s_1 + is_2 & 0 & -s_3 \\ -is_3 & s_3 & 0 \end{pmatrix}. \tag{61}$$

At $s = (0, 0, 1)$, the matrix $A$ has the double eigenvalue $\lambda_0 = 1 + 5i$ with two eigenvectors and the simple zero eigenvalue. The eigenvectors $u_1, u_2$ of $\lambda_0$ and the eigenvectors $v_1, v_2$ of $\overline{\lambda}_0$ for the adjoint matrix $A^*$ are

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{3-4i}{1-3i} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{2i}{1-3i} \end{pmatrix}. \tag{62}$$

These vectors satisfy normalization conditions (4). Taking derivatives of the matrix $A$ with respect to parameters $s_1$ and $s_2$, where $s_3 = \sqrt{1 - s_1^2 - s_2^2}$, and using formula (33), we obtain

$$d_{11} = (-2 - 8i, 0), \quad d_{12} = (6i, -9 - 4i), \quad d_{21} = (-10i, 7 - 4i), \quad d_{22} = (0, -4i). \tag{63}$$

Using (63) in formulae (34)–(36), we find approximations for real and imaginary parts of two nonzero eigenvalues $\lambda_{\pm}$ near the point $s = (0, 0, 1)$ as

$$\text{Re} \lambda_{\pm} = 1 - s_1 \pm \sqrt{(|c| + \text{Re}c)/2}, \quad \text{Im} \lambda_{\pm} = 5 - 4s_1 - 2s_2 \mp \sqrt{(|c| - \text{Re}c)/2}, \tag{64}$$

where $c = (45 + 8i)s_1^2 + 128is_1s_2 + (-83 + 8i)s_2^2$.

Approximations of eigenvalue surfaces (64) and the exact solutions are presented in Figure 8. The eigenvalue surfaces have intersections both in $(s_1, s_2, \text{Re} \lambda)$ and $(s_1, s_2, \text{Im} \lambda)$ spaces. These intersections are represented by two different lines $l_a$ and $l_b$ in parameter space, see Figure 6.

## 5 Conclusion

A general theory of coupling of eigenvalues of complex matrices smoothly depending on multiple real parameters has been presented. Diabolic and exceptional points have been mathematically
described and general formulae for coupling of eigenvalues at these points have been derived. This theory gives a clear and complete picture of crossing and avoided crossing of eigenvalues with a change of parameters. It has a very broad field of applications since any physical system contains parameters. It is important that the presented theory of coupling gives not only qualitative, but also quantitative results on eigenvalue surfaces based only on the information at the diabolic and exceptional points. This information includes eigenvalues, eigenvectors and associated vectors with derivatives of the system matrix taken at the singular points. We emphasize that the developed methods provide a firm basis for analysis of spectrum singularities of matrix operators.

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