The $h$-vectors of the edge rings of a special family of graphs

Akihiro Higashitani and Nayana Shibu Deepthi

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Suita, Osaka, Japan

**ABSTRACT**

The $h$-vectors of homogeneous rings are one of the most important invariants that often reflect ring-theoretic properties. On the other hand, there are a few examples of edge rings of graphs whose $h$-vectors are explicitly computed. In this paper, we compute the $h$-vector of a special family of graphs, by using the technique of initial ideals and the associated simplicial complex.

**ARTICLE HISTORY**

Received 16 May 2023
Revised 20 June 2023
Communicated by Scott Chapman

**KEYWORDS**
Almost Gorenstein; edge rings; initial complex; Gröbner basis; $h$-vector; toric ideals

**2020 MATHEMATICS SUBJECT CLASSIFICATION**
Primary: 13P10; Secondary: 05E40

**1. Introduction**

Throughout the paper, we assume all graphs to be connected and have no loops and multiple edges.

Many researchers have conducted intensive studies on the edge rings and toric ideals of graphs. Especially, Ohsugi and Hibi initiated the investigation and have been developing the theory of edge rings. For instance, the characterization for the edge rings to be normal was given in [10]. Note that, almost at the same time, the same result was obtained by Simis-Vasconcelos-Villarreal independently in [12]. For the introduction to the edge rings and toric ideals of graphs, we refer the reader to [4, Section 5] and [15, Section 10].

One of the most important invariants of homogeneous rings is the Hilbert series. In fact, the Gorensteinness of homogeneous normal Cohen–Macaulay domains is characterized by the symmetry of their $h$-vectors [13]. Moreover, there are many other results claiming that the $h$-vectors of homogeneous (or semi-standard graded) normal Cohen–Macaulay rings (or domains) have some connection with their ring-theoretic properties (see, e.g., [1, 5, 7, 16], and so on). On the other hand, the $h$-vector of an edge ring is scarcely known. As far as we know, the $h$-vectors (or their counterparts) of the edge rings of the following graphs have been computed:

- Let $K_{m,n}$ denote the complete bipartite graph with $m + n$ vertices. Then

$$h([K_{m,n}]; t) = \sum_{i=0}^{\min\{m,n\}} \binom{m−1}{i} \binom{n−1}{i} t^i.$$ 

- Let $K_m$ denote the complete graph with $m$ vertices. Then

$$h([K_m]; t) = 1 + \frac{m(m−3)}{2} t + \sum_{i=2}^{\lfloor \frac{m}{2}\rfloor} \binom{m}{2i} t^i.$$ 

**CONTACT** Nayana Shibu Deepthi nayanasd@ist.osaka-u.ac.jp Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565-0871, Japan.

© 2023 Taylor & Francis Group, LLC
• The Hilbert functions of the edge rings of complete multipartite graphs were computed in [11, Theorem 2.6].
• The Hilbert series of the edge rings of bipartite graphs are described using their interior polynomials ([8]).

(On the notations used above, see Section 2.) Regarding the results on $K_{m,n}$ and $K_m$, see [14], or [15, Section 10]. Note that if the edge rings of the graphs are normal, then the Hilbert functions (resp. $h$-vectors) of the edge rings agree with the “Ehrhart polynomials (resp. $h^*$-vectors) of the edge polytopes arising from the graphs”. The goal of this paper is to explicitly compute the $h$-vector for some particular class of graphs in order to suggest more examples.

Let $n \geq 2$ be an integer. We introduce a connected non-bipartite graph $G_n$ as shown in Figure 1. Clearly, $G_n$ satisfies the odd cycle condition (see Section 2).

Figure 1. The graph $G_n$.  

Our main result of this paper is the following:

**Theorem 1.1.** The $h$-polynomial of $\mathcal{k}[G_n]$ is as follows:

$$h(\mathcal{k}[G_n]; t) = \binom{n}{0} + \binom{n}{1} t + \binom{n}{2} t^2 + \cdots + \binom{n}{n} t^n = (1 + t)^n - t.$$  

Moreover, $\mathcal{k}[G_n]$ is almost Gorenstein but not Gorenstein if $n \geq 3$.

The notion of almost Gorenstein homogeneous rings was introduced by Goto-Takahashi-Taniguchi in [3], as a new class of graded rings between Cohen–Macaulay rings and Gorenstein rings. After this work, almost Gorenstein homogeneous rings have been studied further, e.g., in [5, 9]. On almost Gorensteinness of edge rings, known examples of almost Gorenstein non-Gorenstein edge rings are presumably rare. However, almost Gorenstein edge rings arising from complete multipartite graphs were completely characterized in [6]. According to [6, Examples 1.5–1.7], we know that $K_{2,m}$, $K_{1,1,m}$, $K_{1,m,m}$ with $m \geq 3$ and $K_{1,1,m,m}$ with $m \geq 2$ give almost Gorenstein but not Gorenstein edge rings, where $K_{a_1,\ldots,a_r}$ denotes the complete $r$-partite graph. Theorem 1.1 gives a new family of graphs whose edge rings are almost Gorenstein but not Gorenstein.

A brief structure of this paper is as follows. In Section 2, we recall some fundamental notions on homogeneous rings, and prepare some materials on the edge rings and toric ideals of graphs for the computation of the $h$-polynomial. In Section 3, we compute the $h$-polynomial of $\mathcal{k}[G_n]$ by using a certain initial complex of the toric ideal of $G_n$. In Section 4, we prove the almost Gorensteinness of $\mathcal{k}[G_n]$. On the non-Gorensteinness of $\mathcal{k}[G_n]$ for $n \geq 3$, see Remark 2.1.

**2. Preliminaries**

First, we recall some fundamental materials like, notions on commutative algebra, the edge rings and toric ideals of graphs.
2.1. Homogeneous rings

In this subsection, we collect some notions on homogeneous rings used in this paper. We refer the readers to [2, 15] for more detailed information on homogeneous rings.

Let R be a Cohen–Macaulay homogeneous ring of dimension d over a field \( k \). Then the Hilbert series of R looks as follows:

\[
\sum_{i \geq 0} \dim_k R_i t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1 - t)^d},
\]

where \( R_i \) denotes the homogeneous part of \( R \) of degree \( i \), \( \dim_k \) denotes the dimension as a \( k \)-vector space, and we assume that \( h_s \neq 0 \). We call the polynomial \( h_0 + h_1 t + \cdots + h_s t^s \) appearing in the numerator as the \( h\)-polynomial of \( R \), denoted by \( h(R; t) \), and the sequence of the coefficients \( (h_0, h_1, \ldots, h_s) \) as the \( h\)-vector of \( R \), denoted by \( h(R) \).

**Remark 2.1.** It follows from [13] that, if \( R \) is a normal Cohen–Macaulay homogeneous domain, then \( R \) is Gorenstein if and only if \( h(R) \) is symmetric, i.e., \( h_i = h_{s-i} \) for \( i = 0, 1, \ldots, s \). Hence, once we get the \( h\)-vector of \( k[\mathcal{G}_n] \), as described in Theorem 1.1, we obtain that \( k[\mathcal{G}_n] \) is Gorenstein if and only if \( n \leq 2 \).

We denote the Cohen–Macaulay type of \( R \) by \( r(R) \). It is well known that \( r(R) \) coincides with the number of elements in the minimal system of generators of the canonical module of \( R \).

2.2. Edge rings

In this subsection, we recall the definitions of edge rings and toric ideals of graphs, and some fundamental theorems regarding them.

First of all, let us recall the definition of edge rings. Let \( G \) be a graph on the vertex set \( [d] \) with edge set \( E(G) = \{e_1, \ldots, e_m\} \). We define a polynomial ring \( R = \k[t_1, \ldots, t_d] \) in \( d \) variables and another one \( S = \k[x_1, \ldots, x_m] \) in \( m \) variables, where \( \k \) is a field. Let \( \pi : S \to R \) be the ring homomorphism defined by \( \pi(x_i) = t^e_i \) for \( i = 1, \ldots, m \), where \( t^e_i := t_{v_1} t_{v_2} \) for any edge \( e = \{v_1, v_2\} \in E(G) \). The image \( \text{Im}(\pi) \), which is a subalgebra of \( R \), is called the edge ring of \( G \), and the kernel \( \text{Ker}(\pi) \), an ideal of \( S \), is called the \( \text{toric ideal} \) of \( G \). We denote the edge ring of \( G \) by \( k[G] \) and the toric ideal of \( G \) by \( I_G \). Clearly, we have the ring isomorphism \( k[G] \cong S/I_G \). It is known that \( k[G] \) is \( d\)-dimensional if \( G \) is not bipartite (cf. [10, Proposition 1.3]).

Given an edge \( e = \{i, j\} \in E(G) \), let \( \rho(e) = e_i + e_j \), where \( e_1, \ldots, e_d \in \R^d \) are the unit vectors of \( \R^d \).

Let \( A_G = \{\rho(e) : e \in E(G)\} \). For \( A \subset \R^d \), let \( A A_G = \left\{ \sum_{e \in E(G)} a_e \rho(e) : a_e \in A \right\} \). In this paper, we only consider the cases where \( A \) is \( \Q_{\geq 0} \) or \( \Z \) or \( \Z_{\geq 0} \). We can regard \( k[G] \) as a monoid algebra of an affine monoid \( \Z_{\geq 0} A_G \). In particular, the structure of the cone \( \Q_{\geq 0} A_G \) plays a crucial role in the study of \( k[G] \).

Next, let us describe the cone \( \Q_{\geq 0} A_G \) in terms of linear inequalities. For the description, we need to introduce some more notions on graphs.

- For a subset \( W \subset V(G) \), let \( G \setminus W \) be the subgraph on \( V(G) \setminus W \) with the edge set \( \{e \in E(G) : e \subset V(G) \setminus W\} \). If \( W = \{w\} \), then we write \( G \setminus w \) instead of \( G \setminus \{w\} \).
- For \( v \in V(G) \), let \( N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\} \), and for any subset \( W \subset V(G) \), let \( N_G(W) = \bigcup_{w \in W} N_G(w) \).
- A non-empty subset \( T \subset V(G) \) is called an independent set if \( \{v, w\} \notin E(G) \) for any \( v, w \in T \).
- We call a vertex \( v \) of \( G \) regular if each connected component of \( G \setminus v \) contains an odd cycle.
- We say that an independent set \( T \) of \( V(G) \) is a fundamental set if
  - the bipartite graph on the vertex set \( T \cup N_G(T) \) with the edge set \( \{\{v, w\} : v \in T, w \in N_G(T)\} \cap E(G) \) is connected, and
Theorem 2.3. Fundamental properties of $\mathbb{k}[G]$.

Now, let us focus on our graph $G_n$. We identify the edges of $G_n$ with the variables of the polynomial ring $S$, as depicted in Figure 1. Namely, we consider

$$S = \mathbb{k}[x_1, y_1, z_1, \ldots, x_n, y_n, z_n]$$

and we regard $I_{G_n}$ as an ideal of $S$.

For $G_n$, we see that, from [4, Lemma 5.11], every primitive even closed walk consists of two 3-cycles with exactly one common vertex, and is given by

$$(x_i, z_i, y_i, x_j, z_j, y_j); \quad 1 \leq i < j \leq n.$$  

Hence, the toric ideal $I_{G_n}$ is generated by the binomials:

$$x_iz_jy_j - z_ix_jy_j; \quad 1 \leq i < j \leq n. \quad (3)$$

Let $\preceq_{\text{lex}}$ be the graded lexicographic order on $S$ induced by the ordering of the variables

$$x_1 \prec_{\text{lex}} y_1 \prec_{\text{lex}} z_1 \prec_{\text{lex}} \cdots \prec_{\text{lex}} x_n \prec_{\text{lex}} y_n \prec_{\text{lex}} z_n. \quad (4)$$

For the fundamental materials on initial ideals and Gröbner basis, consult, e.g., [4, Section 1].
Lemma 2.4. The binomials in (3) form a Gröbner basis of $I_{\mathcal{G}_n}$ with respect to the monomial order $<_{\text{lex}}$.

Proof. The result follows from the straightforward application of Buchberger’s criterion to the set of generators (3) of $I_{\mathcal{G}_n}$. (For Buchberger’s criterion, see, e.g., [4, Theorem 1.29].)

Let $f = x_iy_iz_j - z_ix_zy_j$ and $g = x_py_qz_q - z_px_yq$ be two generators. If $i \neq p$ and $j \neq q$, then the leading terms of $f$ and $g$ are relatively prime and thus the S-polynomial $S(f, g)$ will reduce to 0 by [4, Lemma 1.27].

Suppose $i = p$. Then
\[
S(f, g) = \frac{\text{lcm}(\text{in}_{<_{\text{lex}}}(f), \text{in}_{<_{\text{lex}}}(g))}{\text{in}_{<_{\text{lex}}}(f)} f - \frac{\text{lcm}(\text{in}_{<_{\text{lex}}}(f), \text{in}_{<_{\text{lex}}}(g))}{\text{in}_{<_{\text{lex}}}(g)} g
\]
\[= z_qf - z_qg
\]
\[= z_q(x_iy_iz_j - z_ix_zy_j) - z_j(x_py_qz_q - z_px_yq)
\]
\[= z_i(x_iy_qz_j - z_qx_iy_j).
\]

Note that, up to sign, $x_qy_qz_j - z_qx_iy_j$ is a generator of $I_{\mathcal{G}_n}$ and therefore $S(f, g)$ will reduce to 0. The $j = q$ case is similar.

Corollary 2.5. The initial ideal in $<_{\text{lex}} (I_{\mathcal{G}_n})$ of $I_{\mathcal{G}_n}$ with respect to the monomial order $<_{\text{lex}}$ is generated by the squarefree monomials
\[x_iy_iz_j; \quad 1 \leq i < j \leq n. \tag{5}
\]

By this corollary, since the given monomial ideal is squarefree, we can associate a simplicial complex whose Stanley-Reisner ideal coincides with the initial ideal generated by (5).

3. Computation of the $h$-polynomial of $k[\mathcal{G}_n]$

Let $\Delta_n$ be the simplicial complex whose Stanley-Reisner ideal coincides with the initial ideal of the toric ideal $I_{\mathcal{G}_n}$ with respect to $<_{\text{lex}}$. (For the introduction to Stanley-Reisner theory, consult, e.g., [2, Section 5].) Let $\mathcal{F}(\Delta_n)$ be the set of all facets of $\Delta_n$. By definition, any facet of our simplicial complex $\Delta_n$ can be expressed as the maximal set that does not contain the triplet $\{x_i, y_i, z_i\}; \quad 1 \leq i < j \leq n$. Since $x_n, y_n$ and $z_1$ will be contained in all the facets, with out loss of generality, we write the facets without indicating these elements. Therefore, any $F \in \mathcal{F}(\Delta_n)$ can be expressed as:

\[F = \bigcup_{1 \leq i \leq n-1} \{x_i\} \cup \bigcup_{1 \leq j \leq n-1} \{y_j\} \cup \bigcup_{2 \leq k \leq n} \{z_k\},
\]

which is maximal and does not contain the triplet $\{x_i, y_i, z_i\}; \quad 1 \leq i < j \leq n$. Let us try to get a more concrete representation for the facets in $\mathcal{F}(\Delta_n)$. Consider any $F \in \mathcal{F}(\Delta_n)$.

Case 1:
Let $i \in I \cap J$. This implies $z_{i+1}, \ldots, z_n \notin F$, since $F$ does not contain the triplet $\{x_i, y_i, z_i\}; \quad 1 \leq i < j \leq n$.

Case 2:
Let us consider $i \in I$. If there exists some $k$ with $i < k \leq n$ such that $z_k \in F$, then we have $y_i \notin F$, that is, $i \notin I \setminus J$.

If $z_k \notin F$ for all $k$ with $i < k \leq n$, then by the maximality of $F$, we have $y_i \in F$ and $i \in I \cap J$.

Case 3:
Let us consider $j \in J$. If there exists some $k$ with $j < k \leq n$ such that $z_k \in F$, then we have $x_j \notin F$, that is, $j \notin J \setminus I$. 

If $z_k \notin F$ for all $k$ with $j < k \leq n$, then by the maximality of the set $F$, we have $x_j \in F$ and $j \in I \cap J$.

According to our observations from the three situations above, any facet in $\mathcal{F}(\Delta_n)$ is of the form:

$$\{w_1, \ldots, w_{j-1}\} \cup \{x_j, y_j, \ldots, x_{n-1}, y_{n-1}\} \cup \{z_2, \ldots, z_j\}, \tag{6}$$

where $w_i \in \{x_i, y_i\}$ and $j = 1, \ldots, n$.

**Remark 3.1** (Mac-Mullen characterization of $h$-vectors). Here, we recall a well-known method to compute the $h$-vector of a shellable simplicial complex. (For example, see [2, Corollary 5.1.14].)

Let $F_1, \ldots, F_t$ be the facets of a pure simplicial complex $\Delta$ of dimension $d - 1$. Let $\langle F_1, \ldots, F_m \rangle$ be the unique smallest simplicial complex which contains all $F_i$, $1 \leq i \leq m$. The ordering of the facets is said to be a *shelling* if it satisfies that $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a non-empty set of maximal proper faces of $F_i$ for all $2 \leq i \leq t$. We say that a pure simplicial complex is *shellable* if it has a shelling. Throughout our further study, we may refer the subcomplex $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ as the *intersection subcomplex* corresponding to the $i$th shelling step. Let $r_i$ be the number of maximal proper faces of $F_i$ in $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ for $2 \leq i \leq t$, and let $r_1 = 0$. Then the $h$-vector of $\Delta$, $h(\Delta) = (h_0, \ldots, h_d)$, is obtained by $h_i = |\{j: r_j = i\}|$.

Now, we consider an ordering of the facets $F^0_n, \ldots, F^n_{t_n}$ of $\Delta_n$. From the structure of each facet as shown in (6), we have $t_n = \sum_{i=0}^{n-1} 2^i$ and $|F^0_i| = 2n - 2$ for all $1 \leq i \leq t_n$. Let us consider each facet as a $(2n - 2)$-tuple of $x_1, y_1, x_2, y_2, z_2, \ldots, x_{n-1}, y_{n-1}, z_{n-1}, z_n$. Lexicographic order $<_L$ in $\mathcal{F}(\Delta_n)$ is defined by

$$(a_1, \ldots, a_{2n-2}) <_L (b_1, \ldots, b_{2n-2})$$

if and only if either $a_i \neq b_i$ for some $i$ and $a_i <_lex b_i$ with respect to (4). Now, we consider an ordering of the facets $F^0_n, \ldots, F^n_{t_n}$ of $\Delta_n$ such that they are arranged in lexicographically increasing order of their corresponding $(2n - 2)$-tuple.

Let $r^n_i$ be the number of maximal proper faces of $F^n_i$ that generates the $i$th intersection subcomplex for $2 \leq i \leq t_n$. We define $\delta_n = \{r^n_2, \ldots, r^n_{t_n}\}$ with $n \geq 2$ as a multi-set.

**Lemma 3.2.** For each $n \geq 2$, we have

$$\delta_{n+1} = \{1, \delta_n, 2, \delta_n + 1\},$$

where $\delta_n + 1 = \{\alpha + 1: \alpha \in \delta_n\}$.

By induction on $n$, and using Lemma 3.2, we can obtain the $h$-vector of $\Delta_n$. Our goal is to show that

$$h(\Delta_n) = \left(\begin{array}{c}n \\ 0\end{array}\right), \left(\begin{array}{c}n \\ 1\end{array}\right), \left(\begin{array}{c}n \\ 2\end{array}\right), \ldots, \left(\begin{array}{c}n \\ n\end{array}\right)$$

for any $n \geq 2$.

For $n = 2$, since $\mathcal{F}(\Delta_2) = \{\{x_1, y_1\}, \{x_1, z_2\}, \{y_1, z_2\}\}$, we obtain the $h$-vector as $(1, 1, 1)$ which is equal to our formula $\left(\begin{array}{c}2 \\ 0\end{array}\right), \left(\begin{array}{c}2 \\ 1\end{array}\right), \left(\begin{array}{c}2 \\ 2\end{array}\right)$.

By the hypothesis of induction, assume that our formula holds for an arbitrary $n$. Therefore, the $h$-vector associated with $\Delta_n$ is $\left(\begin{array}{c}n \\ 0\end{array}\right), \left(\begin{array}{c}n \\ 1\end{array}\right) - 1, \left(\begin{array}{c}n \\ 2\end{array}\right), \ldots, \left(\begin{array}{c}n \\ n\end{array}\right)$. Let $h(\Delta_{n+1}) = (h_0^{n+1}, h_1^{n+1}, \ldots, h_{n+1}^{n+1})$. By Lemma 3.2, we see the following:

$$h_0^{n+1} = 1 = \left(\begin{array}{c}n + 1 \\ 0\end{array}\right),$$

$$h_1^{n+1} = h_1^n + 1 + \left(\begin{array}{c}n \\ 1\end{array}\right) - 1 = \left(\begin{array}{c}n + 1 \\ 1\end{array}\right) - 1,$$
Moreover, when we look at the first \( \frac{t_n+1}{2} \) facets, we observe that each of the facets differs from the preceding one by just an element. Thus with these observations, we claim that \( F_1, \ldots, F_n \) is a shelling of \( \Delta_n \).

Since the ordering pattern repeats after the \( \left( \frac{t_n+1}{2} \right) \)th stage, for \( 1 \leq i \leq \frac{t_n-1}{2} \), we see that \( r_n^{t_{i+1}} \) maximal proper faces will always be contained in the intersection subcomplex for each \( k \)th shelling step, \( \frac{t_n+3}{2} \leq k \leq t_n \), and the intersection subcomplex also contains the maximal face \( F_{i+1}^n \setminus \{x_1\} \). Therefore, we have

\[
r_{\frac{t_n+1}{2}+i}^n = 1 + r_{i+1}^n \quad \text{for any } 1 \leq i \leq \frac{t_n-1}{2}.
\]

Hence, we can express \( \delta_n \) as

\[
\delta_n = \{1, r_3^n, \ldots, r_{\frac{t_n+1}{2}}^n, 2, r_5^n + 1, \ldots, r_{\frac{t_n+1}{2}}^n + 1\}.
\]

The set \( \mathcal{F}(\Delta_n) \) consists of (6). Namely, each facet \( F_k^n \), \( 1 \leq k \leq t_n \), can be denoted as:

\[
\bigcup_{i=1}^{j-1} \{w_i\} \cup \bigcup_{i=j}^{n-1} \{x_i, y_i\} \cup \bigcup_{i=2}^j \{z_i\}, \quad j = 1, \ldots, n.
\]

For each \( 1 \leq j \leq n \), we have \( 2^{j-1} \) number of facets corresponding to it. We can show that there exists a one-to-one correspondence between the facets \( F_k^n \) and \( F_{k+1}^{n+1} \) for all \( n \geq 2 \) and \( 2 \leq k \leq t_n \). The one-to-one correspondence is given by

\[
\phi: \left\{ F_k^n: 2 \leq k \leq t_n \right\} \to \left\{ F_{k+1}^{n+1}: 3 \leq k' \leq \frac{t_{n+1}+1}{2} \right\}
\]

\[
\phi(F_k^n) = F_{k+1}^{n+1}, \quad 2 \leq k \leq t_n,
\]
In this section, we are in the process of obtaining a theoretical proof to state that \( k[\mathcal{G}_n] \) is almost Gorenstein for all \( n \geq 2 \).

We first recall the necessary and sufficient condition for a homogeneous ring to be almost Gorenstein from [5].

### 4. On the almost Gorensteinness of \( k[\mathcal{G}_n] \)

In this section, we are in the process of obtaining a theoretical proof to state that \( k[\mathcal{G}_n] \) is almost Gorenstein for all \( n \geq 2 \).

We first recall the necessary and sufficient condition for a homogeneous ring to be almost Gorenstein from [5].
Proposition 4.1. [5, Corollary 2.7] Let \( R \) be a Cohen–Macaulay homogeneous ring of dimension \( d \) over a field \( k \) and let \( h(R) = (h_0, h_1, \ldots, h_s) \) be its \( h \)-vector. Then \( R \) is almost Gorenstein if and only if the following equality holds:

\[
r(R) - 1 = \sum_{j=0}^{s-1} ((h_s + \cdots + h_{s-j}) - (h_0 + \cdots + h_j)) =: \tilde{e}(R).
\]

Note that \( r(R) - 1 \leq \tilde{e}(R) \) is always satisfied, so almost Gorensteinness is equivalent to the inequality

\[
r(R) \geq \tilde{e}(R) + 1.
\]

Let us assume that \( R \) is a domain. Then there is an ideal \( I_R \) which is isomorphic to a canonical module of \( R \) as an \( R \)-module. We know that the number of elements of a minimal system of generators of \( I_R \) coincides with \( r(R) \).

Let us consider \( R = \mathbb{k}[G_n] \). Note that \( G_n \) satisfies the odd cycle condition, so \( \mathbb{k}[G_n] \) is normal. By the first part of Theorem 1.1, we can compute \( \tilde{e}(R) \) as follows:

\[
\sum_{j=0}^{n-1} \left\{ \left( \left( \begin{array}{c} n \\ j \end{array} \right) + \cdots + \left( \begin{array}{c} n \\ n-j \end{array} \right) \right) - \left( \left( \begin{array}{c} n \\ 0 \end{array} \right) + \left( \begin{array}{c} n \\ 1 \end{array} \right) - 1 \right) + \cdots + \left( \begin{array}{c} n \\ j \end{array} \right) \right\}
= \sum_{j=1}^{n-2} 1 = n - 2.
\]

Hence by Proposition 4.1, it is enough to show that \( r(R) \geq n - 1 \). Furthermore, \( r(R) \) is equal to the number of elements of a minimal system of generators of \( I_R \), which is the relative interior of \( \mathbb{Q}_{\geq 0} \mathbb{A} G_n \cap \mathbb{Z} \mathbb{A} G_n \) (see \([2, \text{Theorem 6.3.5}]\)). Let \( C = \mathbb{Q}_{\geq 0} \mathbb{A} G_n \subset \mathbb{R}^{2n+1} \). In what follows, it suffices to show that we need at least \((n - 1)\) elements as a minimal system of generators for the relative interior of the cone \( C \).

Let us denote the vertices of \( G_n \) as follows:

\[
V(G_n) = \{ u_i^{(1)}, u_i^{(2)} : i = 1, \ldots, n \} \cup \{ w \} \quad \text{and let}
\]

\[
x_i = \{ w, u_i^{(1)} \}, \quad y_i = \{ w, u_i^{(2)} \} \quad \text{and } z_i = \{ u_i^{(1)}, u_i^{(2)} \} \quad \text{for } i = 1, \ldots, n.
\]

We use the following notation for each entry of \( \mathbb{R}^{2n+1} \):

\[
\mathbb{R}^{2n+1} = \{ c_{1,i} \mathbf{e}_{1,i} + \cdots + c_{1,n} \mathbf{e}_{1,n} + c_{2,1} \mathbf{e}_{2,1} + \cdots + c_{2,n} \mathbf{e}_{2,n} + c' \mathbf{e}' : c_{1,i}, c_{2,i}, c' \in \mathbb{R} \},
\]

where \( \mathbf{e}_{1,i}, \mathbf{e}_{2,i}, \mathbf{e}' \) are the unit vectors of \( \mathbb{R}^{2n+1} \), each \( \mathbf{e}_{1,i} \) (resp. \( \mathbf{e}_{2,i} \)) corresponds to \( u_i^{(1)} \) (resp. \( u_i^{(2)} \)) and \( \mathbf{e}' \) corresponds to \( w \).

For \( j = 1, \ldots, n - 1 \), let

\[
\alpha_j := \sum_{i=1}^{n} (\mathbf{e}_{1,i} + \mathbf{e}_{2,i}) + 2j\mathbf{e}'.
\]

In what follows, we verify that \( \alpha_j \in C^\circ \cap \mathbb{Z}^{2n+1} \), where \( C^\circ \) denotes the relative interior of \( C \), and they should be included in a minimal system of generators of \( I_R \).

The first step: We check that \( \alpha_j \in C^\circ \). Here, we see the following:

- Each of \( u_i^{(1)} \) s and \( u_i^{(2)} \) s is a regular vertex of \( G_n \), while \( w \) is not.
- A subset \( T \) of \( V(G_n) \) is fundamental if and only if \( T = \{ w \} \) or \( T = \{ u_1, \ldots, u_n \} \), where \( u_i \in \{ u_i^{(1)}, u_i^{(2)} \} \) for each \( i \).
Hence, it follows from (1) that \( \sum_{i=1}^{n} (c_{1,i} \mathbf{e}_{1,i} + c_{2,i} \mathbf{e}_{2,i}) + c' e' \in \mathbb{R}^{2n+1} \), belongs to \( C \) if and only if the following inequalities are satisfied:

\[
\begin{align*}
c_{1,i} \geq 0 & \quad \text{and} \quad c_{2,i} \geq 0 \quad \text{for any} \ i = 1, \ldots, n, \\
\sum_{i=1}^{n} (c_{1,i} + c_{2,i}) & \geq c', \\
\sum_{i \in U} c_{1,i} + \sum_{i \in [n] \setminus U} c_{i,2} & \geq \sum_{i \in [n] \setminus U} c_{1,i} + \sum_{i \in U} c_{i,2} \quad \text{for any} \ U \subset [n].
\end{align*}
\]  

(7)

It is straightforward to check that \( \alpha_j \) satisfies these inequalities with strict inequalities for each \( j \). This implies that \( \alpha_j \in C^n \).

The second step: We prove that \( \alpha_j \) cannot be written as a sum of an element in \( C^o \cap \mathbb{Z}^{2n+1} \) and an element in \( C \cap \mathbb{Z}^{2n+1} \setminus \{0\} \).

Suppose that \( \alpha_j = \alpha' + \beta \) for some \( \alpha' \in C^o \cap \mathbb{Z}^{2n+1} \) and \( \beta \in C \cap \mathbb{Z}^{2n+1} \setminus \{0\} \). Let

\[
\begin{align*}
\alpha' &= \sum_{i=1}^{n} a_{1,i} \mathbf{e}_{1,i} + \sum_{i=1}^{n} a_{2,i} \mathbf{e}_{2,i} + a' e' \\
\beta &= \sum_{i=1}^{n} b_{1,i} \mathbf{e}_{1,i} + \sum_{i=1}^{n} b_{2,i} \mathbf{e}_{2,i} + b e'.
\end{align*}
\]

Then we see that \( a'_{1,i} \geq 1 \) and \( a'_{2,i} \geq 1 \) for \( 1 \leq i \leq n \) (see (7)). Hence, \( b_{1,i} \leq 0 \) and \( b_{2,i} \leq 0 \). On the other hand, \( b_{1,i} \geq 0 \) and \( b_{2,i} \geq 0 \) should be also satisfied, so we obtain that \( \beta = b e' \). Since \( \beta \neq 0 \), by the second inequality of (7), we have \( b < 0 \), a contradiction to the third inequality.

The third step: By the first and second steps, we see that \( \alpha_1, \ldots, \alpha_{n-1} \) are required for a minimal system of generators of \( I_R \). This shows that \( r(R) \geq n - 1 \), as required.

Funding

The first named author is partially supported by JSPS Grant-in-Aid for Scientists Research (B) 18H01134 and Scientists Research (C) 20K03513.

References

[1] Borzi, A., D’Ali, A. (2021). Graded algebras with cyclotomic Hilbert series. J. Pure Appl. Algebra 225:106764, 9 pp.
[2] Bruns, W., Herzog, J. (1998). Cohen-Macaulay Rings, revised ed. Cambridge: Cambridge University Press.
[3] Goto, S., Takahashi, R., Taniguchi, N. (2015). Almost Gorenstein rings – towards a theory of higher dimension. J. Pure Appl. Algebra 219:2666–2712.
[4] Herzog, J., Hibi, T., Ohsugi, H. (2018). Binomial Ideals. Graduate Texts in Mathematics, 279. Cham: Springer.
[5] Higashitani, A. (2016). Almost Gorenstein homogeneous rings and their \( h \)-vectors. J. Algebra 456:190–206.
[6] Higashitani, A., Matsushita, K. (2022). Levelness versus almost Gorensteinness of edge rings of complete multipartite graphs. Commun. Algebra 50(6):2637–2652.
[7] Higashitani, A., Yanagawa, K. (2018). Non-level semi-standard graded Cohen–Macaulay domain with \( h \)-vector \((h_0, h_1, h_2)\). J. Pure Appl. Algebra 222:191–201.
[8] Kálmán, T., Postnikov, A. (2017). Root polytopes, Root polytopes, Tutte polynomials, and a duality theorem for bipartite graphs. Proc. London Math. Soc. 114(3):561–588.
[9] Matsuoka, N., Murai, S. (2016). Uniformly Cohen–Macaulay simplicial complexes. J. Algebra 455:14–31.
[10] Ohsugi, H., Hibi, T. (1998). Normal polytopes arising from finite graphs. J. Algebra 207:409–426.
[11] Ohsugi, H., Hibi, T. (2000). Compressed polytopes, initial ideals and complete multipartite graphs. Illinois J. Math. 44(2):391–406.
[12] Simis, A., Vasconcelos, W.V., Villarreal, R. H. (1998). The integral closure of subrings associated to graphs. J. Algebra 199:281–289.
[13] Stanley, R. P. (1978). Hilbert functions of graded algebras. Adv. Math. 28:57–83.
[14] Villarreal, R. H. (1996). Normality of subrings generated by square free monomials. J. Pure Appl. Algebra 113:91–106.
[15] Villarreal, R. H. (2015). Monomial Algebras. Monographs and Research Notes in Mathematics. Boca Raton, FL: CRC Press.
[16] Yanagawa, K. (1995). Castelnuovo’s Lemma and \( h \)-vectors of Cohen–Macaulay homogeneous domains. J. Pure Appl. Algebra 105:107–116.