Non-Polynomial Quintic Spline for Numerical Solution of Fourth–Order Time Fractional Partial Differential Equations

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Abstract

This paper presents a novel approach for numerical solution of a class of fourth order time fractional partial differential equations (PDE’s). The finite difference formulation has been used for temporal discretization, whereas, the space discretization is achieved by means of non polynomial quintic spline method. The proposed algorithm is proved to be stable and convergent. In order to corroborate this work, some test problems have been considered and the computational outcomes are compared with those found in the exiting literature. It is revealed that the presented scheme is more accurate as compared to current variants on the topic.

Keywords: Non-Polynomial quintic spline, Backward Euler method, Time fractional Partial differential equation, Caputo fractional derivative.

1 Introduction

In the modern era, fractional order differential equations have gained a significant amount of research work due to their wide range of applications in various branches of science and engineering such as Physics, electrical networks, fluid mechanics, control theory, theory of viscoelasticity, neurology and theory of electromagnetic acoustics [1, 2]. Wang [3] introduced the very first approximate solution of nonlinear fractional Korteweg–de Vries (KdV) Burger equation involving space and time fractional derivatives using Adomian Decomposition method. Zurigat at al. [4] examined the approximate solution of fractional order algebraic differential equations using Homotopy analysis method. Turut and Guzel [5] implemented Adomian decomposition method and multivariate Pade approximation method for solving fractional order nonlinear partial differential equations (PDE’s). In [6], Liu and Hou applied the Generalized differential transform method to solve the coupled Burger equation with space and time fractional derivatives. Khan et al. [7] used Adomian decomposition method and Variational iteration method for numerical solution of fourth order time fractional PDE’s with variable coefficients. Later on, Abbass et al. [8] employed a finite difference approach based on third degree trigonometric B-spline functions for approximate solution.

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of one-dimensional wave equation. Javidi and Ahmad [9] developed a computational technique based on Homotopy perturbation method Laplace transform and Stehfests numerical inversion algorithm for solving fourth-order time-fractional PDE’s with variable coefficients. The fractional differential transform method and modified fractional differential transform method were proposed by Kanth and Aruna [10] for series solution to higher dimensional third-order dispersive fractional PDE’s. Pandey and Mishra [11] applied Sumudu transforms and Homotopy analysis approach for solving time-fractional third order dispersive type of PDE’s. The fractional Variational iteration method was put into action by Prakash and Kumar in [12] for series solution to third-order fractional dispersive PDE’s in higher dimensional space.

The spline approximation techniques have been applied extensively for numerical solution of ODE’s and PDE’s. The spline functions have a variety of significant gains over finite difference schemes. These functions provide a continuous differentiable estimation to solution over the whole spatial domain with great accuracy. The straightforward employment of spline functions provides a solid ground for applying them in the context of numerical approximations for initial/boundary problems.

Khan and Aziz [13], solved third order boundary-value problems (BVP’s) using a numerical method based on quintic spline functions. In [14], non polynomial quintic spline method was employed for numerical solution of fourth order two-point BVP’s. Khan and Sultana [15] proposed non-polynomial quintic spline functions for numerical solution of third order BVP’s associated with odd-order obstacle problems. In [16], Srivastava discussed numerical solution of differential equations using polynomial spline functions of different orders. Siddiqi and Arshed [17] brought the fifth degree basis spline collocation functions into use for approximate solution of fourth order time fractional PDE’s. Rashidinia and Mohsenyzadeh [18] used non-polynomial quintic spline technique for one-dimensional heat and wave equations. Recently, in [19], fifth degree spline approximation technique has been utilized for approximate solution of fourth-Order time-fractional PDE’s. In [20], the new fractional order spline functions were considered to obtain the approximate solution for fractional Bagely-Torvik Equation. Arshed [21] employed quintic B-spline collocation scheme for solving fourth order time-fractional super diffusion equation. More recently, parametric quintic spline approach and Grunwald-Letnikov approximation have been proposed in [22] for a distributed order fractional sub-diffusion problem.

In the field of modern science and engineering the fourth-order initial/boundary value problems are of great importance. For example, airplane wings, bridge slabs, floor systems and window glasses are being modeled as plates subject to different types end supports which are successfully described in terms of fourth-order PDE’s [19]. In this work, we consider the following class of the fourth-order time-fractional PDE’s

\[
\frac{\partial^\gamma y}{\partial t^\gamma} + \alpha \frac{\partial^4 y}{\partial x^4} = u(x,t), \quad t \in [0,T], \quad x \in [0,L],
\]

(1.1)

with the following initial and boundary conditions

\[
y(x,0) = v_0(x)\\
y(0,t) = y(L,t) = 0\\
y_{xx}(0,t) = y_{xx}(L,t) = 0
\]

where \(\gamma \in (0,1)\), is the order of fractional time derivative, \(\alpha\) represents the ratio of flexural-rigidity of beam to its mass per unit length, \(y(x,t)\) is the beam transverse displacement, \(u(x,t)\) describes the
dynamic driving force per unit mass and the function \( v_0(x) \) is known to be continuous on \([0, L]\). There are many descriptions to the concept of fractional differentiation but Caputo and Riemann-Liouville have been the most common definitions. Here, we shall use the Caputo’s approach because it is more appropriate for real world problems and it permits initial and boundary conditions in terms of ordinary derivatives. The Caputo’s definition of fractional derivative of order \( \gamma \) is given by

\[
\frac{\partial^\gamma y(x, t)}{\partial t^\gamma} = \begin{cases} \\
\frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial y(x, s)}{\partial s} \frac{ds}{(t-s)^\gamma}, & 0 < \gamma < 1 \\
\frac{\partial y(x, t)}{\partial t}, & \gamma = 1.
\end{cases}
\]

This paper has been composed with the aim to develop a spline collocation method for approximate solution of fourth order time-fractional PDE’s. The backward Euler’s scheme has been utilized for temporal discretization, whereas, non polynomial quintic spline function, comprised of a trigonometric part and a polynomial part, has been used to interpolate the unknown function in spatial direction. The presented technique has also been proved to be stable and convergent.

This work is arranged as follows: In section 2, a brief explanation of quintic spline scheme has been presented and the consistency relations between the values of spline approximation and its derivatives at the nodal points are derived. Section 3 describes the use of \( L^1 \) approximation in time direction to achieve a backward Euler technique. Non-polynomial quintic spline scheme for the spatial discretization has been discussed in section 4. The computational results and discussions are given in section 5.

2 Description of Non Polynomial Quintic Spline Function

Consider \( x_i = ih \), be the mesh points of uniform partition of \([0, L]\) into sub-intervals \([x_i, x_{i-1}]\), where \( h = \frac{L}{n} \) and \( i = 0, 1, 2, \cdots, n \). Let \( y(x) \) be a sufficiently smooth function defined on \([0, L]\). We denote the non polynomial quintic spline approximation to \( y(x) \) by \( S(x) \). Each non polynomial spline segment \( R_i(x) \) has the following form

\[
R_i(x) = a_i \cos(\xi(x-x_i)) + b_i \sin(\xi(x-x_i)) + c_i(x-x_i) + d_i(x-x_i)^2 + e_i(x-x_i)^3 + f_i, i = 0, 1, 2, \cdots, n. \quad (2.1)
\]

where \( a_i, b_i, c_i, d_i, e_i \) and \( f_i \) are the constants and the parameter \( \xi \), the frequency of the trigonometric functions, will be used to enhance the accuracy of the technique. When \( \xi \) approaches to zero, Eq.(2.1) reduces to quintic polynomial spline function in \([a, b]\). The non polynomial quintic spline can be defined as

\[
S(x) = R_i(x), \ \forall \ x \in [x_i, x_{i+1}], \ i = 0, 1, 2, \cdots, n. \quad (2.2)
\]

\[
R_i(x) \in C^4[0, L] \quad (2.3)
\]

First of all, we establish the consistency relations for all the coefficients involved in (2.1) in terms of \( S_i \)'s, \( M_i \)'s and \( F_i \)'s, where

\[
S_i = S(x_i) = R_i(x_i),
\]

\[
M_i = S''(x_i) = R''_i(x_i)
\]

\[
F_i = S^{(4)}(x_i) = R^{(4)}_i(x_i)
\]
The values of coefficients introduced in (2.1) can be calculated as
\[ a_i = \frac{h^4}{\theta^4} F_i, \]
\[ b_i = \frac{h^4}{\theta^3 \sin(\theta)} (F_{i+1} - F_i \cos(\theta)), \]
\[ c_i = \frac{1}{6h} (M_{i+1} - M_i) + \frac{h}{6\theta^2} (F_{i+1} - F_i), \]
\[ d_i = \frac{1}{2} M_i + \frac{h^2}{2\theta^2} F_i, \]
\[ e_i = \frac{1}{h} (S_{i+1} - S_i) + \left( \frac{h^3}{\theta^4} - \frac{h^3}{3\theta^2} \right) F_i - \left( \frac{h^3}{\theta^4} + \frac{h^3}{6\theta^2} \right) F_{i+1} - \frac{h}{6} (M_{i+1} + 2M_i), \]
\[ f_i = S_i - \frac{h^4}{\theta^4} F_i, \]
where \( \theta = \xi h \) and \( i = 0, 1, \cdots, n - 1 \).

Now, using the first and third derivative continuity conditions at the knots, i.e. \( R_i^{(r)}(x_i) = R_i^{(r)}(x_{i+1}) \), for \( \tau = 1, 3 \), we can derive the following important relations

\[ M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} (S_{i-1} - 2S_i + S_{i+1}) + \frac{6h^2}{\theta^2} \left( \frac{1}{\theta^4} \right) - \frac{1}{\theta^4} - \frac{1}{6} (F_{i+1} + F_{i-1}) \]
\[ + \frac{6h^2}{\theta^2} \left( \frac{2}{\theta^4} \right) - \frac{2 \cos(\theta)}{\theta \sin(\theta)} - \frac{4}{6} F_i \]
\[ (2.4) \]

and

\[ M_{i-1} - 2M_i + M_{i+1} = \frac{h^2}{\theta(\sin(\theta))} - \frac{1}{\theta^2} (F_{i+1} + F_{i-1}) + 2h^2 \left( \frac{1}{\theta^4} - \frac{\cos(\theta)}{\theta \sin(\theta)} \right) F_i \]
\[ (2.5) \]

Solving (2.4) and (2.5), we get

\[ M_i = \frac{1}{h^2} (S_{i-1} - 2S_i + S_{i+1}) + h^2 \left( \frac{1}{\theta^4 \sin(\theta)} - \frac{1}{6\theta^2 \sin(\theta)} \right) \left( F_{i+1} + F_{i-1} \right) \]
\[ + h^2 \left( \frac{2}{\theta^4} - \frac{2 \cos(\theta)}{\theta^3 \sin(\theta)} + \frac{2 \cos(\theta)}{6 \theta^2 \sin(\theta)} \right) \left( \frac{1}{\theta^4} \right) F_i \]
\[ (2.6) \]

Using (2.6) (2.0), we get the following consistency relation involving \( F_i \) and \( S_i \) for \( i = 2, 3, \cdots, n - 2 \).

\[ S_{i+2} - 4S_{i+1} + 6S_i - 4S_{i-1} + S_{i-2} = h^4 (\alpha_1 F_{i-2} + \beta_1 F_{i-1} + \gamma_1 F_i + \beta_1 F_{i+1} + \alpha_1 F_{i+2}) \]
\[ (2.7) \]

where

\[ \alpha_1 = \left( \frac{1}{\theta^4} + \frac{1}{6\theta \sin(\theta)} - \frac{1}{\theta^2 \sin(\theta)} \right), \]
\[ \beta_1 = \left( \frac{2}{\theta^4} + \frac{2 \cos(\theta)}{\theta^2 \sin(\theta)} - \frac{2 - \cos(\theta)}{3 \theta \sin(\theta)} - \frac{4}{\theta^4} \right) \]
\[ \gamma_1 = \left( \frac{1 - 4 \cos(\theta)}{3 \theta \sin(\theta)} - \frac{2 + 4 \cos(\theta)}{\theta^2 \sin(\theta)} + \frac{6}{\theta^4} \right) \]

The relation (2.7) provides \((n - 3)\) linear equations with \((n - 1)\) unknowns \( S_i, i = 1(1)n - 1 \). Hence, we require two more equations for direct calculation of \( S_i \), one at each end of the range of integration, which can be formulated as setting \( i = 1, 2 \) in (2.4) we have

\[ M_0 + 4M_1 + M_2 = \frac{6}{h^2} (S_0 - 2S_1 + S_2) + \tilde{\lambda} (F_0 + F_2) + \tilde{\mu} F_1 \]
\[ (2.8) \]
and 
\[ M_1 + 4M_2 + M_3 = \frac{6}{h^2}(S_1 - 2S_2 + S_3) + \tilde{\lambda}(F_1 + F_3) + \tilde{\mu}F_2 \] 

(2.9)

Similarly, for \( i = 1, 2 \) the expression (2.5) returns the following two equations

\[ M_0 - 2M_1 + M_2 = \tilde{\lambda}(F_0 + F_2) + \tilde{\mu}F_1 \] 

(2.10)

and

\[ M_1 - 2M_2 + M_3 = \tilde{\lambda}(F_1 + F_3) + \tilde{\mu}F_2. \] 

(2.11)

where

\[ \tilde{\lambda} = \frac{6h^2}{\theta^2} \left( \frac{1}{\theta \sin \theta} - \frac{1}{\theta^2} - \frac{1}{6} \right), \quad \tilde{\mu} = \frac{6h^2}{\theta^2} \left( \frac{2}{\theta^2} - \frac{2 \cos(\theta)}{\theta \sin(\theta)} - \frac{4}{6} \right). \]

From (2.8) and (2.10), we have

\[ M_1 = \frac{1}{h^2}(S_0 - 2S_1 + S_2) + \frac{\tilde{\lambda} - \lambda}{6}(F_0 + F_2) + \frac{\tilde{\mu} - \mu}{6}F_1 \] 

(2.12)

Similarly, subtracting (2.11) from (2.9), we get

\[ M_2 = \frac{1}{h^2}(S_1 - 2S_2 + S_3) + \frac{\tilde{\lambda} - \lambda}{6}(F_1 + F_3) + \frac{\tilde{\mu} - \mu}{6}F_2 \] 

(2.13)

Now, the first end condition is obtained by substituting (2.12), (2.13) into (2.8) for \( i = 1 \).

\[ -2S_0 + 5S_1 - 4S_2 + S_3 = -h^2M_0 + h^4(\omega_0F_0 + \omega_1F_1 + \omega_2F_2 + \omega_3F_3) \] 

(2.14)

Similarly, the second end condition for \( i = n \), is given by

\[ S_{n-3} - 4S_{n-2} + 5S_{n-1} - 2S_n = -h^2M_n + h^4(\omega_3F_{n-3} + \omega_2F_{n-2} + \omega_1F_{n-1} + \omega_0F_n) \] 

(2.15)

where

\[ \omega_0 = \left( \frac{2}{\theta^3 \sin(\theta)} - \frac{2}{\theta^4} + \frac{4}{6\theta \sin(\theta)} - \frac{1}{\theta^2} \right), \quad \omega_1 = \frac{1 - 8 \cos(\theta)}{6\theta \sin(\theta)} - \frac{1 + 4 \cos(\theta)}{\theta^3 \sin(\theta)} + \frac{5}{\theta^4}, \quad \omega_2 = \left( \frac{2 + 2 \cos(\theta)}{\theta^3 \sin(\theta)} + \frac{2 - \cos(\theta)}{3\theta \sin(\theta)} - \frac{4}{\theta^4} \right), \quad \omega_3 = \frac{1}{6\theta \sin(\theta)} - \frac{1}{\theta^3 \sin(\theta)} + \frac{1}{\theta^4}. \]

**Lemma 2.1.** The local truncation error \( t_{1,i} = 1(1)n - 1 \) associated with the Eqs (2.7), (2.14) and (2.15)
is given by

\[
t_i = \begin{cases} 
(11/12 - \omega_0 - \omega_1 - \omega_2 - \omega_3)h^4y_i^{(4)} + (1/12 + \omega_0 - \omega_2 - 2\omega_3)h^5y_i^{(5)} \\
+ (1/90 - \frac{1}{3}\omega_0 - \frac{1}{2}\omega_2 - 2\omega_3)h^6y_i^{(6)} + (\frac{1}{60} + \frac{1}{6}\omega_0 - \frac{5}{6}\omega_2 - \frac{4}{3}\omega_3)h^7y_i^{(7)} \\
+ (\frac{17}{225} - \frac{1}{9}\omega_0 - \frac{1}{2}\omega_2 - \frac{2}{3}\omega_3)h^8y_i^{(8)} + O(h^9), & i = 1 \\
(1 - 2\alpha_1 - 2\beta_1 - \gamma_1)h^4y_i^{(4)} + (\frac{1}{6} - 4\alpha_1 - \beta_1)h^6y_i^{(6)} \\
+ (\frac{1}{150} - \frac{1}{3}\alpha_1 - \frac{1}{12}\beta_1)h^8y_i^{(8)} + (\frac{17}{30240} - \frac{1}{45}\alpha_1 - \frac{1}{360}\beta_1)h^{10}y_i^{(10)} + O(h^{11}), & i = 2(1)n - 2 \\
(1/12 - \omega_0 - \omega_1 - \omega_2 - \omega_3)h_i^4 + (\frac{1}{12} + \omega_0 - \omega_2 - 2\omega_3)h^5y_i^{(5)} \\
+ (1/90 - \frac{1}{3}\omega_0 - \frac{1}{2}\omega_2 - 2\omega_3)h^6y_i^{(6)} + (\frac{1}{60} + \frac{1}{6}\omega_0 - \frac{5}{6}\omega_2 - \frac{4}{3}\omega_3)h^7y_i^{(7)} \\
+ (\frac{17}{225} - \frac{1}{9}\omega_0 - \frac{1}{2}\omega_2 - \frac{2}{3}\omega_3)h^8y_i^{(8)} + O(h^9), & i = n - 1 
\end{cases}
\]

Proof. We have to find local truncation error \( t_i, i = 1, 2, ..., n - 1 \) for the present scheme. First of all, we write Eqs. (2.7), (2.14), and (2.15) as

\[
t_1 = -2y_0 + 5y_1 - 4y_2 + y_3 + h^2M_0 - h^4(\omega_0y_0^{(4)} + \omega_1y_1^{(4)} + \omega_2y_2^{(4)} + \omega_3y_3^{(4)}), \\
t_i = y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} - h^4(\alpha_1y_{i-2}^{(4)} + \beta_1y_{i-1}^{(4)} + \gamma_1y_i^{(4)} + \beta_1y_{i+1}^{(4)} + \alpha_1y_{i+2}^{(4)}), \\
t_{n-1} = y_{n-3} - 4y_{n-2} + 6y_{n-1} - 4y_n + h^2M_n + h^4(\omega_3y_{n-3}^{(4)} + \omega_2y_{n-2}^{(4)} + \omega_1y_{n-1}^{(4)} + \omega_0y_n^{(4)})
\]

The expressions for \( t_i, i = 1, 2, ..., n - 1 \) can be obtained by expanding the terms \( y_0, y_1, y_2, y_3, y_3 \) etc about the points \( x_i, i = 1, 2, ..., n - 1 \), using Taylor series respectively.

Equating the coefficients of \( y_i^{(\tau)} \) for \( \tau = 4, 5, 6, 7 \), we get

\[
\alpha_1 = -\frac{1}{720}, \beta_1 = \frac{1}{180}, \gamma_1 = \frac{1}{720}, \omega_0 = \frac{1}{3}, \omega_1 = \frac{1}{12}, \omega_2 = -\frac{7}{36}, \text{ and } \omega_3 = \frac{1}{360}
\]

The local truncation error given in Eq. (2.16) takes the following form

\[
t_i = \begin{cases} 
-\frac{241}{60480}h^8y_i^{(8)} + O(h^9), & i = 1 \\
\frac{69}{30240}h^{10}y_i^{(10)} + O(h^{11}), & i = 2(1)n - 2 \\
-\frac{241}{60480}h^8y_i^{(8)} + O(h^9), & i = n - 1 
\end{cases}
\]

3 Temporal Discretization

In order to discretize the time fractional derivative, backward Euler scheme is employed. We consider, \( t_p = p\Delta t \) for \( p = 0(1)K \) with \( \Delta t = \frac{T}{K} \) as the step size in time direction. The computation of Caputo time-fractional derivative at \( t = t_{p+1} \) can be made as

\[
\int_0^{t_{p+1}} \frac{\partial y(x, w)}{\partial w} (t_{p+1} - w)^{-\gamma} dw = \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} \frac{\partial y(x, w)}{\partial w} (t_{p+1} - w)^{-\gamma} dw
\]
Caputo fractional derivative gives the following relation.

\[ G_{\gamma}^\beta \]

where,

\[ \text{Using (3.1), the above equation can be written as} \]

Then, Eq. (3.1) can be written as

\[ \text{Moreover, the coefficients} \ b_j \ \text{involved in (3.1) have the following properties} \]

Where \( b_j = (j + 1)^{1 - \gamma} - j^{1 - \gamma} \) and \( y = (t_{p+1} - w) \). The above equation along with the definition of Caputo fractional derivative gives the following relation.

\[ \frac{\partial^\gamma y(x, t_{p+1})}{\partial t^\gamma} = \frac{1}{\Gamma(2 - \gamma)} \sum_{j=0}^{p} b_j \frac{y(x, t_{p-j+1}) - y(x, t_{p-j})}{\Delta t^{\gamma}} + \frac{p+1}{\Delta t} \]  \hspace{1cm} (3.1)

Now, we define a semi-discrete fractional differential operator \( G_t^\gamma \) as

\[ G_t^\gamma y(x, t_{p+1}) = \frac{1}{\Gamma(2 - \gamma)} \sum_{j=0}^{p} b_j \frac{y(x, t_{p-j+1}) - y(x, t_{p-j})}{\Delta t^{\gamma}} \]

Then, Eq. (3.1) can be written as

\[ \frac{\partial^\gamma y(x, t_{p+1})}{\partial t^\gamma} = G_t^\gamma y(x, t_{p+1}) + \frac{p+1}{\Delta t} \]  \hspace{1cm} (3.2)

Here, \( p^{1+1}_{\Delta t} \) denotes the truncation error between \( \frac{\partial^\gamma y(x, t_{p+1})}{\partial t^\gamma} \) and \( G_t^\gamma y(x, t_{p+1}) \). Let \( G_t^\gamma y(x, t_{p+1}) \) be the approximation of Caputo time-fractional derivative at \( t = t_{p+1} \), then Eq. (3.1) can be expressed as

\[ G_t^\gamma y(x, t_{p+1}) + c \frac{\partial^4}{\partial x^4} y(x, t_{p+1}) = u(x, t_{p+1}) \]  \hspace{1cm} (3.3)

Using (3.1), the above equation can be written as

\[ y^{p+1}(x) + \beta \alpha y^{p+1}_{xxx} = (b_0 - b_1) y^p(x) + \sum_{j=1}^{p-1} (b_j - b_{j+1}) y^{p-j}(x) + b_{p+1} y^0(x) + \beta u^{p+1}(x), \]

where, \( \beta = \Gamma(2 - \gamma) \Delta t^{\gamma} \) and \( y^{p+1}(x) = y(x, t_{p+1}) \) with the initial and boundary conditions as follow

\[ y^0 = v_0(x), \quad x \in [0, L]. \]

Moreover, the coefficients \( b_j \) involved in (3.1) have the following properties

- \( b_j \)s are non-negative for \( j = 0, 1, \cdots, p \).
\[ b_0 > b_1 > b_2 > b_3 > \cdots > b_p, \quad b_p \to 0 \text{ as } p \to \infty \]
\[ \sum_{j=0}^{p} (b_j - b_{j+1}) + b_{p+1} = (b_0 - b_1) + \sum_{j=1}^{p-1} (b_j - b_{j+1}) + b_p = 1 \]
The truncation error in (3.2) is bounded, i.e.
\[ |t_{p+1}^m| \leq c \Delta t^{2-\gamma} \]  

where the constant \( c \) is dependant on \( y \). To apply this scheme, we need the values \( y^0 \) and \( y^1 \).

For \( p = 0 \), (3.4) takes the following form
\[ y^1(x) + \beta \alpha y^1_{xxx} = v^0(x) + \beta u^1(x) \]  

(3.6)

For \( p = 1 \), (3.4) becomes
\[ y^2(x) + \beta \alpha y^2_{xxx} = (b_0 - b_1)y^1(x) + b_1y^0(x) + \beta u^2(x) \]

Now (3.4) and (3.6) with initial and boundary conditions formulate a complete set of semi-discrete problem for (1.1).

The error term \( l_{p+1}^m \) can also be defined as [23]
\[ l_{p+1}^m = \beta \left( \frac{\partial^\gamma}{\partial t^\gamma} y(x, t_{p+1}) - G^\gamma_i y(x, t_{p+1}) \right). \]  

(3.7)

From Eqs. (3.2) and (3.5), the error term can be expressed as
\[ |l_{p+1}^m| = \Gamma(2-\gamma) \Delta t^\gamma |l_{p+1}^m| \leq c_9 \Delta t^{2} \]  

(3.8)

Now, we define some functional spaces and their standard norms as
\[ H^2(\eta) = \{ g \in L^2(\eta), g_x, g_{xx} \in L^2(\eta) \} \]
\[ H_0^2(\eta) = \{ g \in H^2(\eta), g|_{\partial \eta} = 0, g_x|_{\partial \eta} = 0 \} \]
\[ H^n(\eta) = \{ g \in L^2(\eta), g^{(r)}_{x}, \forall r \leq n \} \]

where \( L^2(\eta) \) denotes the space of all measurable functions whose square is Lebesgue integrable in \( \eta \). The inner product and norm in \( L^2(\eta) \) are given by
\[ \langle f, g \rangle = \int_{\eta} fg dx, \quad \|g\|_0 = \langle g, g \rangle^\frac{1}{2} \]

The inner product and norm in \( S^2(\eta) \) are given by
\[ \langle f, g \rangle_2 = \langle f, g \rangle + \langle f_x, g_x \rangle + \langle f_{xx}, g_{xx} \rangle, \quad \|g\|_2 = \langle g, g \rangle^\frac{1}{2} \]
Also, the norm \( \| \cdot \| \) in \( H^\eta(\eta) \) is defined in the following way

\[
\| g \|_n = \left( \sum_{r=0}^n \| g(r) \|_0^2 \right)^{\frac{1}{2}}
\]

It is also preferred to define \( \| \cdot \|_2 \)

\[
\| g \|_2 = (\| g \|_0^2 + \beta \| g(2) \|_0^2)^{\frac{1}{2}}
\] (3.9)

Now, for the stability and convergence analysis, we are to find \( y_{p+1} \in H^0_0(\eta) \) such that for all \( g \in H^0_0(\eta) \), Eqs. (3.4) and (3.6) give the following two relations

\[
< y_{p+1}, g > + \beta \alpha < y_{p+1}^{xxx}, g > = (1 - b_1) < y^0, g > + \sum_{j=1}^{p-1} (b_j - b_{j+1}) < y^{p-j}, g >
\]
\[+ b_p < y^0, g > + \beta < u^{p+1}, g >, \] (3.10)

and

\[
< y^1, g > + \beta \alpha < y^{1xxx}, g > = < y^0, g > + \beta < u^1, g >
\] (3.11)

The theorem given below describes the unconditional stability of the semi–discrete problem.

**Theorem 1.** The discrete problem is unconditionally stable in such a way that \( \forall \Delta t > 0 \), it holds

\[
\| y^{p+1} \|_2 \leq (\| y^0 \|_0 + \beta \sum_{j=1}^{p+1} \| u^j \|_0), \quad p = 0, 1, 2, \cdots, K - 1
\] (3.12)

where \( \| \cdot \|_2 \) is discussed in Eq. 3.9.

**Proof.** In order to prove this result, mathematical induction is used. For \( p = 0 \) and \( g = y^1 \), Eq. (3.11) takes the following form

\[
< y^1, y^1 > + \beta \alpha < y^{1xxx}, y^1 > = < y^0, y^1 > + \beta < u^1, y^1 >
\]

Integrating by parts, the above result can be written as

\[
< y^1, y^1 > + \beta \alpha < y^{1xxx}, y^1 > = < y^0, y^1 > + \beta < u^1, y^1 >
\] (3.13)

Due to the boundary conditions on \( g \), all the boundary related contributions are disappeared. From Schwarz inequality and the inequality \( \| g \|_0 \leq \| g \|_2 \), Eq. 3.13 becomes

\[
\| y^1 \|_2^2 \leq \| y^0 \|_0 \| y^1 \|_0 + \beta \| u^1 \|_0 \| y^1 \|_0
\]
\[\leq \| y^0 \|_0 \| y^1 \|_2 + \beta \| u^1 \|_0 \| y^1 \|_2
\]
\[\| y^1 \|_2 \leq (\| y^0 \|_0 + \beta \| u^1 \|_0)
\]

Suppose that the result is true for \( g = y^j \) i.e.

\[
\| y^j \|_2 \leq (\| y^0 \|_0 + \beta \sum_{i=1}^j \| u^i \|_0), \quad j = 2, 3, \cdots, p.
\] (3.14)
Let \( g = y^{p+1} \) in Eq. \((3.10)\)

\[
< y^{p+1}, y^{p+1} > + \beta \alpha < y_{xxxx}^{p+1}, y_{xxxx}^{p+1} > = (1 - b_1) < y^p, y^{p+1} > + \sum_{j=1}^{p-1} (b_j - b_{j+1}) < y^{p-j}, y^{p+1} > \\
+ b_p < y^0, y^{p+1} > + \beta < u^{p+1}, y^{p+1} > \tag{3.15}
\]

Integrating by parts, we get

\[
< y^{p+1}, y^{p+1} > + \beta \alpha < y_{xx}^{p+1}, y_{xx}^{p+1} > = (1 - b_1) < y^p, y^{p+1} > + \sum_{j=1}^{p-1} (b_j - b_{j+1}) < y^{p-j}, y^{p+1} > \\
+ b_p < y^0, y^{p+1} > + \beta < u^{p+1}, y^{p+1} > \tag{3.16}
\]

Again due to the boundary conditions on \( g \) all the boundary related contributions are disappeared. From Schwarz inequality and the inequality \( \|g\|_0 \leq \|g\|_2 \), the above expression changes to

\[
\|y^{p+1}\|_2^2 \leq (1 - b_1)\|y^p\|_0\|y^{p+1}\|_0 + \sum_{j=1}^{p-1} (b_j - b_{j+1})\|y^{p-j}\|_0\|y^{p+1}\|_0 \\
+ b_p\|y^0\|_0\|y^{p+1}\|_0 + \beta\|u^{p+1}\|_0\|y^{p+1}\|_0,
\]

or

\[
\|y^{p+1}\|_2^2 \leq (1 - b_1)\|y^p\|_0\|y^{p+1}\|_2 + \sum_{j=1}^{p-1} (b_j - b_{j+1})\|y^{p-j}\|_0\|y^{p+1}\|_2 \\
+ b_p\|y^0\|_0\|y^{p+1}\|_2 + \beta\|u^{p+1}\|_0\|y^{p+1}\|_2,
\]

or

\[
\|y^{p+1}\|_2 \leq (1 - b_1)\|y^p\|_0 + \sum_{j=1}^{p-1} (b_j - b_{j+1})\|y^{p-j}\|_0 + b_p\|y^0\|_0 + \beta\|u^{p+1}\|_0
\]

Using \((3.14)\), the above relation takes the following form

\[
\|y^{p+1}\|_2 \leq \left(\|y^p\|_0 + \beta \sum_{j=1}^{p} \|u^j\|_0\right) (1 - b_1) + \sum_{j=1}^{p-1} (b_j - b_{j+1}) + b_p \|y^0\|_0 + \beta\|u^{p+1}\|_0
\]

Using the properties of \( b_j \), we can write

\[
\|y^{p+1}\|_2 \leq \left(\|y^p\|_0 + \beta \sum_{j=1}^{p} \|u^j\|_0\right)
\]

\[ \square \]

**Lemma 3.1.** Let \( \{y^p\}_{p=0}^K \) be the time discrete solution to Eqs. \((3.10) - (3.14)\) and \( y \) be the exact solution of \((1.1)\), then

\[
\|y(t_p) - y^p\|_2 \leq c_0 b_{p-1} \Delta t^2, \quad p = 1, 2, \ldots, K. \tag{3.17}
\]
Proof. Consider $e^p = y(x, t_p) - y^p(x)$, for $p = 1$, the error equation takes the following form by combining Eqs. (1.1), (3.11) and (3.9)

$$< e^1, g > + \beta \alpha < e^1_{xx}, g_{xx} > = < e^0, g > + < l^1, g >, \quad \forall g \in S_0^2(\eta).$$

Let $g = e^1$ and $e^0 = 0$ gives the following relation

$$\| e^1 \|_2 \leq \| l^1 \|_0$$

(3.18)

Eq. (3.18) along with (3.18), gives

$$\| y(t_1) - y^1 \|_2 \leq c_y b_0^{-1} \Delta t^2.$$  

(3.19)

For $p = 1$, Eq. (3.17) is satisfied.

Next, suppose that (3.17) is true for $p = 1, 2, 3, \cdots, r$. i.e.

$$\| y(t_p) - y^p \|_2 \leq c_y b_p^{-1} \Delta t^2$$  

(3.20)

Using (1.1), (3.9), (3.10) and for $p = r + 1$, the error equation is obtained as

$$< e^{r+1}, g > + \beta \alpha < e^{r+1}_{xx}, g_{xx} > = (1 - b_1) < e^p, g > + \sum_{j=1}^{p-1} (b_j - b_{j+1}) < e^{p-j}, g > + b_p < e^0, g > + < l^{r+1}, g >.$$  

(3.21)

Now, using the induction assumption and taking $g = e^{r+1}$ along with the relation $\frac{b_{j-1}}{b_{j+1}} < 1$ for all positive integer $j$, Eq. (3.21) can be written as

$$\| e^{r+1} \|_2 \leq c_y b_p^{-1} \Delta t^2.$$  

Hence, proved. \qed

Also, from the definition of $b_p$, the following useful equation can be formulated

$$\lim_{p \to \infty} b_p^{-1} = \lim_{p \to \infty} \frac{p^{-\gamma}}{p^{1-\gamma} - (p-1)^{1-\gamma}}$$  

$$= \lim_{p \to \infty} \frac{p^{-1}}{1 - (1 - \frac{1}{p})^{1-\gamma}}$$  

$$= \frac{1}{1 - \gamma}$$

The function $\psi(z)$ is defined as $\psi(z) = \frac{z^{1-\gamma}}{z^{1-\gamma} - (z-1)^{1-\gamma}}$, as $\psi(z) \geq 0 \forall z > 1$, the function $\psi(z)$ is increasing on $z$. This indicates that as $1 < p \to \infty$, $\frac{b_p^{-1}}{p}$ increasingly approaches to $\frac{1}{1 - \gamma}$.
Since, for \( p = 1 \), \( p^\gamma b_{p-1}^{-1} = 1 \). Therefore, it can be written in the following form
\[
p^{-\gamma} b_{p-1}^{-1} \leq \frac{1}{1 - \gamma}, \quad p = 1, 2, \cdots, K.
\]
Therefore, \( \forall \ p \) such that \( p\Delta t \leq T \),
\[
\|y(t_p) - y^p\|_2 \leq c_y b_{p-1}^{-1} \Delta t^2
\]
\[
= c_y p^{-\gamma} b_{p-1}^{-1} p^{-\gamma} \Delta t^{2-\gamma} + \gamma
\]
\[
\leq c_y \frac{1}{1 - \gamma} (p\Delta t)^\gamma (\Delta t)^{2-\gamma}
\]
\[
\leq c_y T^\gamma \Delta t^{2-\gamma}
\]
The above discussion can be summed up in following theorem.

**Theorem 2.** Let \( y \) be the analytical exact solution to \( (4.1) \) and \( \{y^p\}_{p=0}^K \) be the time discrete solution to Eq.\( (3.10) \) and Eq.\( (3.11) \) subject to the initial condition \( y^0 = v_0(x) \), \( x \in [0, L] \), then the following holds
\[
\|y(t_p) - y^p\|_2 \leq c_y T^\gamma \Delta t^{2-\gamma}, \quad p = 1, 2, 3, \cdots, K. \tag{3.22}
\]

### 4 Discretization in Space

Let \( (x_i, t_p) \) be the grid points which uniformly discretize the region \([0, L] \times [0, T]\) with \( x_i = ih \), \( t_p = p\Delta t \), \( T = K\Delta t \), where, \( i = 0(1)n \) and \( p = 0(1)K \). The parameters \( h, \Delta t \) are the grid sizes in the space and time directions respectively. The space discretization of Eq.\( (4.4) \) using non polynomial quintic spline is formulated as
\[
S_i^{p+1} + \beta \alpha F^{p+1} = (1 - b_1)S_i^p + \sum_{j=1}^{p-1} (b_j - b_{j+1}) S_i^{p-j} + b_p v_i + \beta u_i^{p+1}. \tag{4.1}
\]
The operator \( \Phi \) is defined as
\[
\Phi S_j = \alpha_1 S_{j-2} + \beta_1 S_{j-1} + \gamma_1 S_j + \beta_1 S_{j+1} + \alpha_1 S_{j+2}. \tag{4.2}
\]
Now, Eq.\( (4.2) \) takes the following form
\[
\Phi F_i = \frac{1}{h^4} (S_{i-2} - 4S_{i-1} + 6S_i - 4S_{i+1} + S_{i+2}). \tag{4.3}
\]
Applying the operator \( \Phi \) on Eq.\( (4.1) \), we get the following result
\[
\alpha_1 S_{i-2}^{p+1} + \beta_1 S_{i-1}^{p+1} + \gamma_1 S_i^{p+1} + \beta_1 S_{i+1}^{p+1} + \alpha_1 S_{i+2}^{p+1} + \frac{\beta \alpha}{h^4} (S_{i-2}^{p+1} - 4S_{i-1}^{p+1} + 6S_i^{p+1} - 4S_{i+1}^{p+1} + S_{i+2}^{p+1})
\]
\[
= (1 - b_1)(\alpha_1 S_{i-2}^p + \beta_1 S_{i-1}^p + \gamma_1 S_i^p + \beta_1 S_{i+1}^p + \alpha_1 S_{i+2}^p) + \sum_{j=1}^{p-1} (b_j - b_{j+1})(\alpha_1 S_{i-2}^{p-j} + \beta_1 S_{i-1}^{p-j} + \gamma_1 S_i^{p-j} + \beta_1 S_{i+1}^{p-j} + \alpha_1 S_{i+2}^{p-j})
\]
\[
+ \beta (\alpha_1 u_{i-2}^{p+1} + \beta_1 u_{i-1}^{p+1} + \gamma_1 u_i^{p+1} + \beta_1 u_{i+1}^{p+1} + \alpha_1 u_{i+2}^{p+1}), \quad p = 1, 2, 3, \cdots, K - 1. \tag{4.4}
\]
The proposed algorithm is a five point scheme. In order to implement it, the numerical values of

\[(\alpha_1 + \frac{\beta \alpha}{h^4})S_{i-2}^{p+1} + (\beta_1 - 4 \frac{\beta \alpha}{h^4})S_{i-1}^{p+1} + (\gamma_1 + 6 \frac{\beta \alpha}{h^4})S_i^{p+1} + (\beta_1 - 4 \frac{\beta \alpha}{h^4})S_{i+1}^{p+1} + (\alpha_1 + \frac{\beta \alpha}{h^4})S_{i+2}^{p+1} = Q_i, \quad i = 2, 3, \cdots, n - 2, \quad p = 1, 2, \cdots, K - 1. \tag{4.5}\]

where

\[Q_i = (1 - b_1)(\alpha_1 S_{i-2}^p + \beta_1 S_{i-1}^p + \gamma_1 S_i^p + \beta_1 S_{i+1}^p + \alpha_1 S_{i+2}^p) + \sum_{j=1}^{p-1} (b_j - b_{j+1})(\alpha_1 S_{i-2-j}^p + \beta_1 S_{i-1-j}^p + \gamma_1 S_i^{p-j} + \beta_1 S_{i+1-j}^{p-j} + \alpha_1 S_{i+2-j}^{p-j}) + b_p(\alpha_1 v_{i-2} + \beta_1 v_{i-1} + \gamma_1 v_i + \beta_1 v_{i+1} + \alpha_1 v_{i+2}) + \beta(\alpha_1 u_{i-2}^p + \beta_1 u_{i-1}^p + \gamma_1 u_i^p + \beta_1 u_{i+1}^p + \alpha_1 u_{i+2}^p) \tag{4.6}\]

System (4.6) provides \((n - 3)\) equations involving \(S_i^{p+1}, i = 1, 2, \cdots, n - 1\). Therefore, we further need two equations for complete solution of \(S_i^{p+1}\). The required two end conditions can be derived using simply supported boundary conditions as

\[(\omega_0 - 2 \frac{\beta \alpha}{h^4})S_0^{p+1} + (\omega_1 + 5 \frac{\beta \alpha}{h^4})S_1^{p+1} + (\omega_2 - 4 \frac{\beta \alpha}{h^4})S_2^{p+1} + (\omega_3 + \frac{\beta \alpha}{h^4})S_3^{p+1} = (1 - b_1)(\omega_0 S_0^p + \omega_1 S_1^p + \omega_2 S_2^p + \omega_3 S_3^p) + \sum_{j=1}^{p-1} (b_j - b_{j+1})(\omega_0 S_{j-2}^p + \omega_1 S_{j-1}^p + \omega_2 S_j^p + \omega_3 S_{j+1}^p) + b_p(\omega_0 v_0 + \omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3) + \beta(\omega_0 u_0^p + \omega_1 u_1^p + \omega_2 u_2^p + \omega_3 u_3^p) \tag{4.7}\]

Similarly

\[(\omega_3 + \frac{\beta \alpha}{h^4})S_n^{p+1} + (\omega_2 - 4 \frac{\beta \alpha}{h^4})S_{n-1}^{p+1} + (\omega_1 + 5 \frac{\beta \alpha}{h^4})S_{n-2}^{p+1} + (\omega_0 - 2 \frac{\beta \alpha}{h^4})S_{n-3}^{p+1} = (1 - b_1)(\omega_3 S_n^p + \omega_2 S_{n-1}^p + \omega_1 S_{n-2}^p + \omega_0 S_{n-3}^p) + \sum_{j=1}^{p-1} (b_j - b_{j+1})(\omega_3 S_{n-2-j}^p + \omega_2 S_{n-1-j}^p + \omega_1 S_{n-j}^p + \omega_0 S_{n+1-j}^p) + b_p(\omega_3 v_{n-3} + \omega_2 v_{n-2} + \omega_1 v_{n-1} + \omega_0 v_n) + \beta(\omega_3 u_{n-3}^p + \omega_2 u_{n-2}^p + \omega_1 u_{n-1}^p + \omega_0 u_n^p) \tag{4.8}\]

The proposed algorithm is a five point scheme. In order to implement it, the numerical values of

\[S^2 = [S_1^2, S_2^2, S_3^2, \cdots, S_{n-1}^2]^T\]

and \(S^1 = [S_1^1, S_2^1, S_3^1, \cdots, S_{n-1}^1]^T\) are needed. To calculate the values of \(S^2\), it is required to find \(S^1\). Solving Eq. (4.3) and using the non polynomial quintic spline technique, value of \(S^1\) can be found as:

\[(\alpha_1 + \frac{\beta \alpha}{h^4})S_{i-2}^1 + (\beta_1 - 4 \frac{\beta \alpha}{h^4})S_{i-1}^1 + (\gamma_1 + 6 \frac{\beta \alpha}{h^4})S_i^1 + (\beta_1 - 4 \frac{\beta \alpha}{h^4})S_{i+1}^1 + (\alpha_1 + \frac{\beta \alpha}{h^4})S_{i+2}^1 = J_i, \quad i = 2, 3, \cdots, n - 2. \tag{4.9}\]

where

\[J_i = (\alpha_1 v_{i-2} + \beta_1 v_{i-1} + \gamma_1 v_i + \beta_1 v_{i+1} + \alpha_1 v_{i+2}) + \beta(\alpha_1 u_{i-2}^p + \beta_1 u_{i-1}^p + \gamma_1 u_i^p + \beta_1 u_{i+1}^p + \alpha_1 u_{i+2}^p)\]
The system (4.10) consists of \((n - 3)\) equations involving \(S_i^1, i = 1, 2, \cdots, n - 1\). Hence, to get a unique solution to this system, two additional end equations can be obtained from simply supported boundary conditions in the following way

\[
\begin{align*}
(\omega_0 - 2 \frac{\beta \alpha}{h^4})S_0^1 + (\omega_1 + \frac{\beta \alpha}{h^4})S_1^1 + (\omega_2 - 4 \frac{\beta \alpha}{h^4})S_2^1 + (\omega_3 + \frac{\beta \alpha}{h^4})S_3^1 &= (\omega_0 v_0 + \omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3) + \beta(\omega_0 u_0^1 + \omega_1 u_1^1 + \omega_2 u_2^1 + \omega_3 u_3^1) \\
(\omega_0 + \frac{\beta \alpha}{h^4})S_{n-3}^1 + (\omega_1 - 4 \frac{\beta \alpha}{h^4})S_{n-2}^1 + (\omega_2 + \frac{\beta \alpha}{h^4})S_{n-1}^1 + (\omega_3 - 2 \frac{\beta \alpha}{h^4})S_n^1 &= (\omega_3 v_{n-3} + \omega_2 v_{n-2} + \omega_1 v_{n-1} + \omega_0 v_n) + \beta(\omega_3 u_{n-3}^1 + \omega_2 u_{n-2}^1 + \omega_1 u_{n-1}^1 + \omega_0 u_n^1)
\end{align*}
\]

(4.10)

Suppose \(v = [v_1, v_2, \cdots, v_{n-1}]^T\), \(u = [u_1, u_2, \cdots, u_{n-1}]^T\), \(\tilde{v} = [v_0, 0, \cdots, 0, v_n]^T\) and \(\tilde{u} = [u_0, 0, \cdots, 0, u_n]^T\) are column vectors with dimension \((n - 1)\). The system in (4.9) - (4.11) can be expressed as

\[
A S_i^1 = (v + \beta u) + C(\tilde{v} + \beta \tilde{u})
\]

where \(A, B\) and \(C\) are square matrices of order \((n - 1)\), such that

\[
A = 
\begin{pmatrix}
\omega_1 + \frac{5 \beta \alpha}{h^4} & \omega_2 - 4 \frac{\beta \alpha}{h^4} & \omega_3 + \frac{\beta \alpha}{h^4} & 0 & 0 & 0 & \cdots & 0 \\
\beta_1 - 4 \frac{\beta \alpha}{h^4} & \gamma_1 + 6 \frac{\beta \alpha}{h^4} & \beta_1 - 4 \frac{\beta \alpha}{h^4} & \alpha_1 + \frac{\beta \alpha}{h^4} & 0 & 0 & \cdots & 0 \\
\alpha_1 + \frac{\beta \alpha}{h^4} & \beta_1 - 4 \frac{\beta \alpha}{h^4} & \gamma_1 + 6 \frac{\beta \alpha}{h^4} & \beta_1 - 4 \frac{\beta \alpha}{h^4} & \alpha_1 + \frac{\beta \alpha}{h^4} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \alpha_1 + \frac{\beta \alpha}{h^4} & \beta_1 - 4 \frac{\beta \alpha}{h^4} & \gamma_1 + 6 \frac{\beta \alpha}{h^4} & 0 & \beta_1 - 4 \frac{\beta \alpha}{h^4} \\
0 & \cdots & 0 & 0 & \alpha_1 + \frac{\beta \alpha}{h^4} & \beta_1 - 4 \frac{\beta \alpha}{h^4} & \gamma_1 + 6 \frac{\beta \alpha}{h^4} & \beta_1 - 4 \frac{\beta \alpha}{h^4} \\
0 & \cdots & 0 & 0 & 0 & \omega_3 + \frac{\beta \alpha}{h^4} & \omega_2 - 4 \frac{\beta \alpha}{h^4} & \omega_1 + 5 \frac{\beta \alpha}{h^4}
\end{pmatrix}
\]

\[
B = 
\begin{pmatrix}
\omega_1 & \omega_2 & \omega_3 & 0 & 0 & 0 & \cdots & 0 \\
\alpha_1 & \beta_1 & \gamma_1 & \beta_1 & \alpha_1 & 0 & \cdots & 0 \\
\alpha_1 & \beta_1 & \gamma_1 & \beta_1 & \alpha_1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \alpha_1 & \beta_1 & \gamma_1 & \beta_1 & \alpha_1 \\
0 & \cdots & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 & \beta_1 \\
0 & \cdots & 0 & 0 & 0 & \omega_3 & \omega_2 & \omega_1
\end{pmatrix}
\]

\[
C = 
\begin{pmatrix}
\omega_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

4.1 Calculation of Truncation Error

The Eq.(4.11) can be written in the following form

\[
\begin{align*}
h^4(\alpha_1 S_{i-2}^p + \beta_1 S_{i-2}^p + \gamma_1 S_{i-1}^p + \beta_1 S_{i-1}^p + \alpha_1 S_i^p + \beta_1 S_i^p + \alpha_1 S_{i+1}^p + \beta_1 S_{i+1}^p + \alpha_1 S_{i+2}^p + \beta_1 S_{i+2}^p) + \beta \alpha (S_{i-2}^p - 4 S_{i-1}^p + 6 S_i^p - 4 S_{i+1}^p + S_{i+2}^p) \\
+ S_{i+2}^{p+1} = h^4(1 - b_1)(\alpha_1 S_{i-2}^p + \beta_1 S_{i-1}^p + \gamma_1 S_i^p + \beta_1 S_i^p + \alpha_1 S_{i+1}^p + \beta_1 S_{i+1}^p + \alpha_1 S_{i+2}^p + \beta_1 S_{i+2}^p) + \sum_{j=1}^{p-1} h^4(b_j - b_{j+1}) \\
+ (\alpha_1 S_{i-2}^{p-j} + \beta_1 S_{i-1}^{p-j} + \gamma_1 S_i^{p-j} + \beta_1 S_i^{p-j} + \alpha_1 S_{i+1}^{p-j} + \beta_1 S_{i+1}^{p-j} + \alpha_1 S_{i+2}^{p-j} + \beta_1 S_{i+2}^{p-j}) + h^4 b_p (\alpha_1 v_{i-2} + \beta_1 v_{i-1} + \gamma_1 v_i \\
+ \beta_1 v_{i+1} + \alpha_1 v_{i+2}) + h^4 \beta (\alpha_1 u_{i-2}^p + \beta_1 u_{i-1}^p + \gamma_1 u_i^p + \beta_1 u_{i+1}^p + \alpha_1 u_{i+2}^p)
\end{align*}
\]

\[
p = 1, 2, 3, \cdots, K - 1
\]

(4.12)
or

\[ h^4 ((a_1 S^{p+1}_{i-2} + \beta_1 S^{p+1}_{i-1} + \gamma_1 S^{p+1}_i + \beta_1 S^{p+1}_{i+1} + \alpha_1 S^{p+1}_{i+2}) + \beta \alpha (S^{p+1}_{i-2} - 4S^{p+1}_{i-1} + 6S^{p+1}_i - 4S^{p+1}_{i+2}) + S^{p+1}_{i+2}) = h^4 (1 - b_1) (a_1 S^p_{i-1} + \gamma_1 S^p_i + \beta_1 S^p_{i+1} + \alpha_1 S^p_{i+2}) + \sum_{j=1}^{p-1} h^4 (b_j - b_{j+1}) (a_1 S^{p-j}_{i-2} + \gamma_1 S^{p-j}_i + \beta_1 S^{p-j}_{i+1} + \alpha_1 S^{p-j}_{i+2}) + h^4 b_1 (a_1 u_{i-2} - \gamma_1 v_i - \beta_1 v_{i+1} + \beta_1 u_{i+2}) + h^4 \beta (\alpha_1 u_{i-1} + \beta_1 u_{i+1} + \gamma_1 v_i + \alpha_1 v_{i+2} + \gamma_1 u_{i+1} + \beta_1 u_{i+2}), \quad p = 1, 2, 3, \ldots, K - 1 \]

Expanding equation (4.13) with Taylor series in terms of \( S(x_i, t_p) \) and its spatial derivatives, the truncation error is obtained as

\[
T_i = (\mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) S^{p+1}_i + \beta \alpha (h^4 D_x^4 + \frac{1}{6} h^6 D_x^6 + \frac{1}{80} h^8 D_x^8 + \cdots) S^{p+1}_i - (\sum_{j=1}^{p-1} (b_j - b_{j+1}) \mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) S^{p-j}_i - b_1 (\mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) v_i - \beta (\mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) u_{i+1}
\]

(4.13)

Where

\[
\mu_1 = 2\alpha_1 + 2\beta_1 + \gamma_1, \quad \mu_2 = 4\alpha_1 + \beta_1 \quad \text{and} \quad \mu_3 = \frac{4}{3} \alpha_1 + \beta_1.
\]

\[
T_i = (\mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) S^{p+1}_i + \beta \alpha (h^4 D_x^4 + \frac{1}{6} h^6 D_x^6 + \frac{1}{80} h^8 D_x^8 + \cdots) S^{p+1}_i - (\sum_{j=1}^{p-1} (b_j - b_{j+1}) \mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) S^{p-j}_i - b_1 (\mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) v_i - \beta (\mu_1 h^4 + \mu_2 h^6 D_x^2 + \mu_3 h^8 D_x^4 + \cdots) u_{i+1}
\]

From above discussion and Theorem 2, It is concluded that the scheme is of \( O(h^4 + \Delta t^{4-\gamma}) \)

## 5 Numerical Results

In this section, we consider three test problems to check the validity and efficiency of the proposed numerical scheme. The approximate results are compared with quintic spline collocation method (QnSM) used in [19]. All the computations are executed in Mathematica 9.0. The accuracy of presented technique is tested by error norms \( L_\infty, L_2 \) and order of convergence \( (\chi) \), which are calculated as

\[
L_\infty = \max |y_i - Y_i|, \quad L_2 = \sqrt{\frac{\sum_{i=0}^{n} |y_i - Y_i|^2}{\sum_{i=0}^{n} |y_i|^2}}
\]

\[
\chi = \frac{1}{\log(2)} \left[ \log \frac{L_\infty(n)}{L_\infty(2n)} \right]
\]

where \( y_i, Y_i \) represent the exact and approximate solution at \( i^{th} \) knot respectively.

**Problem 1.** Consider the fourth order time-fractional PDE

\[
\frac{\partial^\gamma y}{\partial t^\gamma} + \alpha \frac{\partial^4 y}{\partial x^4} = u(x, t), \quad 0 \leq x \leq 1, \quad 0 < t \leq T
\]
with initial condition

\[ y(x, 0) = \sin(\pi x) \]

and the boundary conditions

\[ y(0, t) = y(1, t) = 0 \]
\[ y_{xx}(0, t) = y_{xx}(1, t) = 0 \]

The exact solution is \( y(x, t) = \sin(\pi x)e^{\gamma t} \). The computational error norms \( L_{\infty} \) and \( L_2 \) corresponding to different values of \( \gamma \) are listed in Table 1 when \( \alpha = 0.01 \) and \( n = 100 \). It is obvious that our proposed computational approach produces more accurate results with \( \Delta t = 0.01 \) as compared to QnSM used in [19] with \( \Delta t = 0.000001 \). A comparison of \( L_{\infty} \), \( L_2 \) and order of convergence \( \chi \) with QnSM [19] at \( t = 1 \) corresponding to \( \gamma = 0.5 \) and \( \Delta t = h \) is reported in Table 2. It is observed that the order of convergence in numerical results exhibits a good agreement with the theoretical estimation. In Figure 1, three-dimensional visuals of exact and approximate solutions are displayed for \( n = 100, \Delta t = 0.01 \) and \( \gamma = 0.5 \). The absolute numerical error at \( t = 1 \) corresponding to \( n = 100, \Delta t = 0.01 \) and \( \gamma = 0.5 \) is portrayed in Figure 2.

**Table 1: Comparison of absolute error for Problem 1 when \( n = 100 \)**

| \( \gamma \) | Method in [19] | Proposed method |
|--------------|-----------------|-----------------|
| \( \Delta t = 0.000001, t = 0.0001 \) | \( L_{\infty} \) | \( L_2 \) | \( L_{\infty} \) | \( L_2 \) |
| 0.25         | 1.2346 \times 10^{-5} | 8.7299 \times 10^{-7} | 6.4023 \times 10^{-8} | 1.6489 \times 10^{-8} |
| 0.50         | 1.7841 \times 10^{-6} | 1.2616 \times 10^{-7} | 5.6896 \times 10^{-8} | 1.7455 \times 10^{-8} |
| 0.75         | 5.1222 \times 10^{-7} | 6.6219 \times 10^{-8} | 4.0157 \times 10^{-9} | 1.3770 \times 10^{-9} |
| 1.00         | 9.2130 \times 10^{-7} | 6.5146 \times 10^{-8} | 9.0571 \times 10^{-9} | 3.3451 \times 10^{-9} |

(a) Exact solution  
(b) Approximate solution

Figure 1: Exact and approximate solution for Problem 1 when \( n = 100, \Delta t = 0.01 \) and \( \gamma = 0.25 \)

**Table 2: Computational error norms and order of convergence for Problem 1 when \( \Delta t = h \)**

| \( n \) | Method in [19] | Proposed method |
|-------|----------------|-----------------|
| \( \Delta t = 0.000001, t = 0.0001 \) | \( L_{\infty} \) | \( \chi \) | \( L_2 \) | \( \chi \) | \( L_{\infty} \) | \( \chi \) | \( L_2 \) | \( \chi \) |
| 10    | \( 1.3279 \times 10^{-2} \) | \( 2.0095 \times 10^{-3} \) | \( 5.3290 \times 10^{-4} \) | \( 1.5741 \times 10^{-4} \) | \( 2.9936 \times 10^{-5} \) | \( 4.1539 \) | \( 8.2199 \times 10^{-4} \) | \( 4.2592 \) |
| 20    | \( 4.4730 \times 10^{-3} \) | \( 1.5700 \) | \( 8.0009 \times 10^{-4} \) | \( 1.3918 \) | \( 1.5100 \times 10^{-6} \) | \( 4.3092 \) | \( 4.5177 \times 10^{-8} \) | \( 4.1873 \) |
| 40    | \( 1.5282 \times 10^{-3} \) | \( 1.5496 \) | \( 3.2081 \times 10^{-5} \) | \( 1.3183 \) | \( 7.5135 \times 10^{-8} \) | \( 4.3289 \) | \( 2.7099 \times 10^{-8} \) | \( 4.0573 \) |
| 80    | \( 5.2715 \times 10^{-4} \) | \( 1.6357 \) | \( 9.9469 \times 10^{-5} \) | \( 1.6892 \) | \( 4.2651 \times 10^{-9} \) | \( 4.1388 \) | \( 1.1677 \times 10^{-9} \) | \( 4.0136 \) |
| 160   | \( 5.2715 \times 10^{-4} \) | \( 1.6357 \) | \( 9.9469 \times 10^{-5} \) | \( 1.6892 \) | \( 4.2651 \times 10^{-9} \) | \( 4.1388 \) | \( 1.1677 \times 10^{-9} \) | \( 4.0136 \) |
Problem 2. Consider the following fourth order time-fractional PDE \[ \frac{\partial^\gamma y}{\partial t^\gamma} + 0.05 \frac{\partial^4 y}{\partial x^4} = u(x,t), \quad 0 \leq x \leq 1, \quad 0 < t \leq T \]

with initial condition
\[ y(x,0) = 0 \]

and the boundary conditions
\[ y(0,t) = y(1,t) = 0 \]
\[ y_{xx}(0,t) = y_{xx}(1,t) = 0 \]

The close form solution is \( y(x,t) = t \sin(\pi x) \). The computational error norms \( L_\infty \) and \( L_2 \) for \( n = 100 \) and different choices of \( \gamma \) are presented in Table 3. It can be observed that our presented approach yields more accurate results with \( \Delta t = 0.01 \) as compared to QuSM employed in [19] with \( \Delta t = 0.000001 \). Table 3 presents the error norms \( L_\infty \), \( L_2 \) and the corresponding order of convergence at \( t = 1 \) for \( \Delta t = \text{hand} \gamma = 0.5 \). Figure 3 shows three dimensional plots of exact and approximate solutions for \( n = 100 \), \( \Delta t = 0.01 \) and \( \gamma = 0.5 \). The absolute numerical error at \( t = 1 \) corresponding to \( n = 100 \), \( \Delta t = 0.01 \) and \( \gamma = 0.25 \) is portrayed in Figure 4.

(a) Exact solution

(b) Approximate solution

Figure 3: Exact and approximate solution for Problem 2 when \( n = 100 \), \( \Delta t = 0.01 \) and \( \gamma = 0.5 \)

Table 3: Comparison of absolute errors for problem 2 when \( n = 100 \)

| Method in [19] | Proposed method |
|----------------|-----------------|
| \( \Delta t = 0.000001 \), \( t = 0.0001 \) | \( \Delta t = 0.01 \), \( t = 1 \) |
| \( L_\infty \) | \( L_\infty \) | \( L_2 \) | \( L_2 \) |
|----------------|-----------------|
| 0.25 \( 9.6400 \times 10^{-7} \) | 0.25 \( 9.5143 \times 10^{-9} \) | 6.8165 \( 10^{-8} \) | 6.8165 \( 10^{-8} \) |
| 0.50 \( 9.9135 \times 10^{-7} \) | 0.50 \( 8.9541 \times 10^{-9} \) | 7.0099 \( 10^{-8} \) | 7.0099 \( 10^{-8} \) |
| 0.75 \( 9.9997 \times 10^{-7} \) | 0.75 \( 9.7525 \times 10^{-9} \) | 7.0709 \( 10^{-8} \) | 7.0709 \( 10^{-8} \) |
| 1.00 \( 1.0000 \times 10^{-6} \) | 1.00 \( 9.8911 \times 10^{-9} \) | 7.0711 \( 10^{-8} \) | 7.0711 \( 10^{-8} \) |
Figure 4: Absolute error for Problem 2 when \( n = 100, \Delta t = 0.01 \) and \( \gamma = 0.5 \)

Table 5: Comparison of absolute errors for problem 3 when \( n = 40 \)

| Method in \([19]\) | Proposed method |
|-------------------|-----------------|
| \( \Delta t = 0.000001, t = 0.0001 \) | \( \Delta t = 0.01, t = 1 \) |
| \( L_\infty \) | \( L_2 \) | \( L_\infty \) | \( L_2 \) |
| 0.25 | -- | -- | 8.7834 \( \times 10^{-8} \) | 7.1567 \( \times 10^{-8} \) |
| 0.50 | 2.1423 \( \times 10^{-4} \) | 2.3952 \( \times 10^{-5} \) | 3.5666 \( \times 10^{-8} \) | 1.5945 \( \times 10^{-8} \) |
| 0.75 | -- | -- | 5.5943 \( \times 10^{-8} \) | 3.4214 \( \times 10^{-8} \) |
| 1.00 | 2.6524 \( \times 10^{-5} \) | 2.9654 \( \times 10^{-6} \) | 1.4199 \( \times 10^{-8} \) | 7.1879 \( \times 10^{-9} \) |

Problem 3. Consider the fourth order time–fractional PDE \([19]\)

\[
\frac{\partial^\gamma y}{\partial t^\gamma} + 0.05 \frac{\partial^4 y}{\partial x^4} = u(x,t), \quad 0 \leq x \leq 1, \quad 0 < t \leq T
\]

with initial condition

\[
y(x,0) = \sin \pi x,
\]

and the boundary conditions

\[
y(0,t) = y(1,t) = 0 \quad y_{xx}(0,t) = y_{xx}(1,t) = 0,
\]

The analytical exact solution is \( y(x,t) = (t + 1) \sin \pi x \). Table 5 shows a comparison of computational error norms with QuSM \([19]\) corresponding to different selections of \( \gamma \). It is found that our computational outcomes are better than QuSM \([19]\). In Figure 5, three dimensional visuals of exact and approximate solutions are displayed for \( n = 40, \gamma = 0.5 \) and \( \Delta t = 0.01 \). The absolute numerical error at \( t = 1 \) corresponding to \( n = 40, \Delta t = 0.01 \) and \( \gamma = 0.5 \) is portrayed in Figure 6. It is obvious that approximate solution is highly consistent with the analytical exact solution, which proves the effectiveness of proposed scheme.

(a) Exact solution  \hspace{1cm} (b) Approximate solution

Figure 5: Exact and approximate solution for Problem 3 when \( \gamma = 0.5, n = 40 \) and \( \Delta t = 0.01 \)

Figure 6: Absolute error for Problem 3 when \( n = 40, \Delta t = 0.01 \) and \( \gamma = 0.5 \)

6 Conclusion

In this work, non polynomial quintic spline collocation method has been employed for approximate solution of fourth order time–fractional partial differential equations. The backward Euler’s method has
been used for temporal discretization, whereas, non polynomial quintic spline function composed of a trigonometric part and a polynomial part has been employed to interpolate the unknown function in spatial direction. The proposed numerical algorithm is proved to be convergent and unconditionally stable. The numerical outcomes are found to be more accurate as compared to QnSM [19].

References

[1] Igor Podlubny. Fractional differential equations, volume 198: An introduction to fractional derivatives, fractional differential equations, to methods of their...(mathematics in science and engineering). 1998.

[2] Kenneth S Miller and Bertram Ross. An introduction to the fractional calculus and fractional differential equations. 1993.

[3] Qi Wang. Numerical solutions for fractional kdv–burgers equation by adomian decomposition method. Applied Mathematics and Computation, 182(2):1048–1055, 2006.

[4] Mohammad Zurigat, Shaher Momani, and Ahmad Alawneh. Analytical approximate solutions of systems of fractional algebraic–differential equations by homotopy analysis method. Computers & Mathematics with Applications, 59(3):1227–1235, 2010.

[5] Veyis Turut and Nuran Güzel. Multivariate pade approximation for solving nonlinear partial differential equations of fractional order. In Abstract and Applied Analysis, volume 2013. Hindawi, 2013.

[6] Jingcun Liu and Guolin Hou. Numerical solutions of the space-and time-fractional coupled burgers equations by generalized differential transform method. Applied Mathematics and Computation, 217(16):7001–7008, 2011.

[7] Najeeb Alam Khan, Nasir-Uddin Khan, Muhammad Ayaz, Amir Mahmood, and Noor Fatima. Numerical study of time-fractional fourth-order differential equations with variable coefficients. Journal of King Saud University-Science, 23(1):91–98, 2011.

[8] Muhammad Abbas, Ahmad Abd Majid, Ahmad Izani Md Ismail, and Abdur Rashid. The application of cubic trigonometric b-spline to the numerical solution of the hyperbolic problems. Applied Mathematics and Computation, 239:74–88, 2014.

[9] M Javidi and Bashir Ahmad. Numerical solution of fourth-order time-fractional partial differential equations with variable coefficients. Journal of Applied Analysis and Computation, 5(1):52–63, 2015.

[10] ASV Ravi Kanth and K Aruna. Solution of fractional third-order dispersive partial differential equations. Egyptian Journal of Basic and Applied Sciences, 2(3):190–199, 2015.

[11] Rishi Kumar Pandey and Hradyesh Kumar Mishra. Homotopy analysis sumudu transform method for timefractional third order dispersive partial differential equation. Advances in Computational Mathematics, 43(2):365–383, 2017.

[12] Amit Prakash and Manoj Kumar. Numerical method for fractional dispersive partial differential equations. Communications in Numerical Analysis, 1:1–18, 2017.
[13] Arshad Khan and Tariq Aziz. The numerical solution of third-order boundary-value problems using quintic splines. *Applied Mathematics and Computation*, 137(2-3):253–260, 2003.

[14] MA Ramadan, IF Lashien, and WK Zahra. Quintic nonpolynomial spline solutions for fourth order two-point boundary value problem. *Communications in Nonlinear Science and Numerical Simulation*, 14(4):1105–1114, 2009.

[15] Arshad Khan and Talat Sultana. Non-polynomial quintic spline solution for the system of third order boundary-value problems. *Numerical Algorithms*, 59(4):541–559, 2012.

[16] Pankaj Kumar Srivastava. Study of differential equations with their polynomial and nonpolynomial spline based approximation. *Acta Technica Corviniensis-Bulletin of Engineering*, 7(3):139, 2014.

[17] Shahid S Siddiqi and Saima Arshed. Numerical solution of time-fractional fourth-order partial differential equations. *International Journal of Computer Mathematics*, 92(7):1496–1518, 2015.

[18] Jalil Rashidinia and Mohamadreza Mohsenyzade. Numerical solution of one-dimensional heat and wave equation by non-polynomial quintic spline. *International Journal of Mathematical Modelling & Computations*, 5(4):291–305, 2015.

[19] Hira Tariq and Ghazala Akram. Quintic spline technique for time fractional fourth-order partial differential equation. *Numerical Methods for Partial Differential Equations*, 33(2):445–466, 2017.

[20] Faraidun K Hamasalh and Pshtiwan O Muhammad. Generalized quartic fractional spline interpolation with applications. *Int. J. Open Problems Compt. Math*, 8(1):67–80, 2015.

[21] Saima Arshed. Quintic b-spline method for time-fractional superdiffusion fourth-order differential equation. *Mathematical Sciences*, 11(1):17–26, 2017.

[22] Patricia J. Y. Li, Xuhao; Wong. An efficient nonpolynomial spline method for distributed order fractional subdiffusion equations. *Mathematical Methods in the Applied Sciences*, 04 2018.

[23] Yumin Lin and Chuanju Xu. Finite difference/spectral approximations for the time-fractional diffusion equation. *Journal of Computational Physics*, 225(2):1533–1552, 2007.