NOETHER DECOMPOSITION FOR BIRATIONAL MAPS

by

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Abstract. — Let $\phi$ be a birational map of the complex projective plane. We know that $\phi$ can be written as a composition of automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ and the standard quadratic birational map $\sigma$. This writing, that is non-unique, is minimal if the number $n(\phi)$ of $\sigma$ is as small as possible. We prove that if $\phi$ is of degree $d \geq 2$, then $\left\lceil \frac{\ln d}{\ln 2} \right\rceil \leq n(\phi) \leq 2(2d - 1)$.

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1. Introduction

A rational map $\phi$ of $\mathbb{P}^2_{\mathbb{C}}$ is a map of the following type

$$\phi: \mathbb{P}^2_{\mathbb{C}} \rightarrow \mathbb{P}^2_{\mathbb{C}}, \quad (x : y : z) \rightarrow (\phi_0(x,y,z) : \phi_1(x,y,z) : \phi_2(x,y,z))$$

where the $\phi_i$’s are homogeneous polynomials of the same degree, and without common factor. The degree of $\phi$ is the degree of the $\phi_i$’s. A birational map $\phi$ of $\mathbb{P}^2_{\mathbb{C}}$ is a rational map of $\mathbb{P}^2_{\mathbb{C}}$ for which there exists a rational map $\psi$ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\phi \psi = \psi \phi = \text{id}$.

Examples 1.1. — if $d = 1$ then $\phi$ is a birational map given by linear forms, i.e. $\phi$ is an element of $\text{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \text{PGL}(3; \mathbb{C})$.

– in the case $d = 2$ we have the following examples:

$\sigma: (x : y : z) \rightarrow (yz : xz : xy), \quad \rho: (x : y : z) \rightarrow (xy : z^2 : yz), \quad \tau: (x : y : z) \rightarrow (x^2 : xy : y^2 - xz)$.

As we will see these three maps play an important role in the description of the set of quadratic birational maps of $\mathbb{P}^2_{\mathbb{C}}$.

If $\phi$ denotes a birational map of the complex projective plane, we denote by $\mathcal{O}(\phi)$ the orbit of $\phi$ under the action of $\text{PGL}(3; \mathbb{C}) \times \text{PGL}(3; \mathbb{C})$

$$\mathcal{O}(\phi) = \{A_1 \phi A_2 \mid A_1, A_2 \in \text{PGL}(3; \mathbb{C})\}.$$

Theorem 1.2 ([3]). — Let $\phi$ be a birational map of $\mathbb{P}^2_{\mathbb{C}}$ of degree 2, then $\phi$ belongs to $\mathcal{O}(\sigma) \cup \mathcal{O}(\rho) \cup \mathcal{O}(\tau)$.
Their statement concerning this group is the following:

**Theorem 1.3** ([2]). — The group Bir($\mathbb{P}^2_\mathbb{C}$) is generated by Aut($\mathbb{P}^2_\mathbb{C}$) and $\sigma$.

In other words any birational map $\phi$ of $\mathbb{P}^2_\mathbb{C}$ can be written

$$(A_1)\sigma A_2 \sigma \ldots \sigma A_{n-1}\sigma(A_n)$$

with $A_i$ in PGL($3; \mathbb{C}$). This writing is of course non-unique, for example

$$\sigma = \sigma(2x : y : z/2)\sigma(2x : y : z/2)\sigma.$$

We will say that the writing of $\phi$ is minimal if the number of $\sigma$ in this writing is as small as possible, and we will denote by $n(\phi)$ this number.

**Examples 1.4.** — If $A$ denotes an automorphism of $\mathbb{P}^2_\mathbb{C}$, then $n(A) = 0$.

- One has

$$n(\sigma) = 1, \quad n(p) = 2, \quad n(\tau) = 4.$$

So, according to Theorem 1.3, if $\phi \in$ Bir($\mathbb{P}^2_\mathbb{C}$) is of degree 2, then $n(\phi) \leq 4$.

The question is: if $\phi \in$ Bir($\mathbb{P}^2_\mathbb{C}$) is of degree $d$, can we bound $n(\phi)$?

**Theorem A.** — Let $\phi$ be a birational map of $\mathbb{P}^2_\mathbb{C}$ of degree $d \geq 2$ then

$$\left\lfloor \frac{\ln d}{\ln 2} \right\rfloor \leq n(\phi) \leq 2(2d - 1).$$

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2. A first bound

We will first of all give a lower but "immediate" bound. Let $\phi$ be an element of Bir($\mathbb{P}^2_\mathbb{C}$); it is given by

$$(x : y : z) \mapsto (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z))$$

for some homogeneous polynomials of the degree $d$, and without common factor. The linear system of $\phi$ is the preimage of the linear system of lines of $\mathbb{P}^2_\mathbb{C}$ and is denoted $\Lambda_\phi$. The degree of the curves of $\Lambda_\phi$ is obviously $d$.

If $\phi$ has some points where $\phi$ is not defined, we choose one, that we denote $p_1 \in \mathbb{P}^2_\mathbb{C}$; let $\pi_1 : S_1 \to \mathbb{P}^2_\mathbb{C}$ be its blow-up. The map $\psi_1 = \phi \pi_1 : S_1 \to \mathbb{P}^2_\mathbb{C}$ is a birational one. If $\psi_1$ has at least one base point, we again choose one that we denote $p_2 \in S_2$ and $\pi_2 : S_2 \to S_1$ its blow-up. Again the map $\psi_2 = \phi \pi_2 : S_2 \to \mathbb{P}^2_\mathbb{C}$ is a birational one. We continue the process until $\psi_n$ becomes a morphism. Let us justify the existence of such a $n$. The linear system $\Lambda_\phi$ consists of curves of degree $d$ all passing through the $p_i$'s with multiplicity $m_i$. Recall that a blow-up $\pi : Y \to X$ of a point $p \in X$ induces the map $\pi^* : \text{Pic}(X) \to \text{Pic}(Y)$, which sends a curve $C \subset X$ into $\pi^{-1}(C)$. Furthermore if $C \subset X$ is an irreducible curve, the strict transform of $C$, denoted $\tilde{C}$, is obtained by taking the closure of $\pi^{-1}(C \setminus \{p\})$. In $\text{Pic}(Y)$ we have

$$\pi^*(C) = \tilde{C} + m_p(C)E$$
where $E = \pi^{-1}(p)$. Applying this $n$ times the members of $\Lambda_{\psi_n}$ are equivalent to

$$dL - \sum_{i=1}^{n} m_i E_i$$

so these curves have self-intersection $d^2 - \sum_{i=1}^{n} m_i^2$ that has to be non-negative; therefore the number $n$ exists.

Denote by $E_i \subset S_i$ the $(-1)$-curve $\pi_i^{-1}(p_i)$ and by

$$E_i = (\pi_{i+1} \ldots \pi_n)^* E_i \in \text{Pic}(S_n).$$

The points $p_i$ are called base points of $\phi$, some of them belong to $\mathbb{P}^2_C$ (these are the common zeros of the $\phi_i$ often called the indeterminacy points of $\phi$) some of them don’t; we say that these last one are infinitely near $\mathbb{P}^2_C$. We denote by $\text{Base}\phi$ the set of base points of $\phi$.

By construction the map $\psi_n$ is a birational morphism, let us denote it $\xi$. In fact any birational morphism between smooth projective surfaces is a sequence of blow-ups so

$$\xi = \xi_q \ldots \xi_1$$

where $\xi_q$ is the blow-up of a point $q_i \in S_i'$ with $S_0' = \mathbb{P}^2_C$ and $S_q' = S_n$ (it follows from the computations of the rank of the Picard group that $q = n$).

The linear system $\Lambda_\phi$ of $\phi$ corresponds to the strict pull-back by $\xi$ of the system $O_{\mathbb{P}^2}(1)$. Let $L$ be a general line, which does not pass through the $p_i$; its pull-back $\xi^{-1}(L)$ corresponds to a smooth curve on $S_n$ which has self-intersection 1 and genus 0. By adjunction formula one gets

$$(\xi^{-1}(L))^2 = 1, \quad \xi^{-1}(L) \cdot K_{S_n} = -3.$$ 

Since the members of $\Lambda_\xi$ are equivalent to $dL - \sum_{i=1}^{n} m_i E_i$ and since $K_{S_n} = -3L + \sum_{i=1}^{n} E_i$ one has

$$\left\{ \begin{array}{l}
  d^2 - \sum_{i=1}^{n} m_i^2 = 1 \\
  3d - \sum_{i=1}^{n} m_i = 3
\end{array} \right. \quad (2.1)$$
Remark that not all solutions of (2.1) correspond to the base points of a birational map of degree $d$. Nevertheless the solution $m_0 = d - 1$, $m_1 = \ldots = m_{2d - 2} = 1$ is realized, for example by (see [4])

$$f_d = (x^{d-1} + y^d ; y^{d-1} : z^d);$$

more precisely $f_d$ has one indeterminacy point with multiplicity $d - 1$ and $2d - 2$ base points infinitely near each of them having multiplicity 1. Applying the previous construction one gets that to write it. Then

two sequences of

Lemma 2.1. — Let $\phi$ be a birational map of $\mathbb{P}_C^2$ of degree $d \geq 2$, then $\phi$ has at most $2d - 1$ base points.

Proof. — Let us denote by $p_1, \ldots, p_n$ the base points of $\phi$ and by $m_i$ the multiplicity of $p_i$.

The inequality $d \geq m_i + 1$ holds for any $i$. In fact if $p_i$ belongs to $\mathbb{P}_C^2$, the pencil of lines passing through $p_i$ has to intersect positively the linear system so $d \geq m_i + 1$. If $p_j$ is infinitely near to a point $p_i$ we have $m_j \leq m_i$ so $d \geq m_i + m_j + 1$.

Let us order the $p_i$'s such that $m_1 \geq m_2 \geq \ldots \geq m_n$; the following inequality holds

$$m_1 + m_2 + m_3 \geq d.$$ \hspace{1cm} (2.2)

Indeed, from (2.1) we get

$$\sum_{i=1}^{n} m_i^2 - m_3 = d^2 - 1 - 3(d - 1)m_3$$

that gives

$$(d - 1)(m_1 + m_2 + m_3 - (d + 1)) = (m_1 - m_3)(d - (1 + m_1)) + (m_2 - m_3)(d - (1 + m_2)) + \sum_{i=4}^{n} m_i(m_i - m_3)$$

Of course $\sum_{i=4}^{n} m_i(m_i - m_3) \geq 0$ and as we just see $d \geq m_i + 1$ for any $i$; hence

$$m_1 + m_2 + m_3 - (d + 1) \geq 0.$$ \hfill \Box

We have established the following statement.

Proposition 2.2. — Let $\phi$ be a birational map of degree $d \geq 2$; then

$$n(\phi) \leq 4(2d - 1).$$

3. A bound for birational maps of $\mathbb{P}_C^2$ coming from polynomial automorphisms of $\mathbb{C}^2$

This section is based on [6]; in this article the author gives a geometric proof of JUNG’s Theorem:

Theorem 3.1 ([5]). — The group of polynomial automorphisms of $\mathbb{C}^2$ denoted $\text{Aut}(\mathbb{C}^2)$ has a structure of amalgamated product:

$$\text{Aut}(\mathbb{C}^2) = A \ast_S E$$

where

$$A = \{(x, y) \mapsto (a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2) | a_1 b_2 - a_2 b_1 \neq 0\},$$

$$E = \{(\alpha x + P(y), \beta y + \gamma) | \alpha, \beta, \gamma \in \mathbb{C}, \alpha \beta \neq 0, P \in \mathbb{C}[y]\}$$

and $S = A \cap E$. 
Before giving the sketch of the proof, let us recall what are Hirzebruch surfaces, a Hirzebruch surface, denoted by $\mathbb{F}_n$, is a ruled surface over the projective line defined by

$$\mathbb{F}_n = \mathbb{P}^1 \langle \mathcal{O} \rangle \oplus \mathcal{O}(-n) \quad \forall n \geq 2.$$ 

The surface $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up at a point. Hirzebruch surfaces for $n > 0$ have a special rational curve $s_\infty(\mathbb{F}_n)$ on them: $\mathbb{F}_n$ is the projective bundle of $\mathcal{O}(-n)$ and the curve $s_\infty(\mathbb{F}_n)$ is the zero section. This curve has self-intersection $-n$, and this is the only irreducible curve with negative self-intersection. The only irreducible curves with zero self-intersection are the fibers $f_\infty(\mathbb{F}_n)$ of $\mathbb{F}_n$.

We say that a birational map $\phi : S \rightarrow \mathbb{P}^2_\mathbb{C}$ from a surface $S$ to $\mathbb{P}^2_\mathbb{C}$ comes from a polynomial automorphism of $\mathbb{C}^2$ if

1. $S = \mathbb{C}^2 \cup D$ where $D$ is a union of irreducible curves called divisor at infinity;
2. $\mathbb{P}^2_\mathbb{C} = \mathbb{C}^2 \cup L$ where $L$ is a line;
3. $\phi$ induces an isomorphism between $S \setminus D$ and $\mathbb{P}^2_\mathbb{C} \setminus L$.

This situation implies strong constraints on the base points of $\phi$:

**Lemma 3.2 ([6], Lemma 9).** Let $\phi : S \rightarrow \mathbb{P}^2_\mathbb{C}$ be a birational map from a surface $S$ to $\mathbb{P}^2_\mathbb{C}$ that comes from a polynomial automorphism of $\mathbb{C}^2$. Then

1. $\phi$ has only one base point in $\mathbb{P}^2_\mathbb{C}$ on the divisor at infinity of $S$;
2. $\phi$ has base points $p_1, \ldots, p_n$, with $n \geq 1$, such that
   - $p_1$ is the indeterminacy point in $\mathbb{P}^2_\mathbb{C}$;
   - for any $i = 2, \ldots, n$ the point $p_i$ belongs to the exceptional divisor obtained by blowing up $p_{i-1}$;
3. any of the irreducible curves contained in the divisor at infinity of $S$ is contracted onto a point by $\phi$;
4. the first curve contracted by $p_2$ is the strict transform of a curve contained in the divisor at infinity of $S$;
5. in particular if $S = \mathbb{P}^2_\mathbb{C}$ then the first curve contracted by $p_2$ is the strict transform of the line at infinity.

Let us explain the strategy used by LAMY to prove JUNG’s Theorem. Let $\phi$ be a birational map of $\mathbb{P}^2_\mathbb{C}$ coming from a polynomial automorphism of $\mathbb{C}^2$ of degree $n$.

The first step is the blow up the only indeterminacy point of $\phi$, one thus gets the following diagram:

$$\begin{array}{c}
\mathbb{P}^2_\mathbb{C} \\
\phi \downarrow \downarrow \phi_1 \\
\mathbb{P}^1_1 \\
\end{array}$$

where $\phi_1^{-1}$ is the blow up to $(1 : 0 : 0)$, and $\#\text{Base} \phi_1 = \#\text{Base} \phi - 1$. According to Lemma 4.1 the only indeterminacy point of $\phi_1$ is $f_\infty(\mathbb{F}_1) \cap s_\infty(\mathbb{F}_1)$.

The second step is based on the following statement:

**Lemma 3.3.** Let $k \geq 1$, and let $\psi : \mathbb{F}_k \rightarrow \mathbb{P}^2_\mathbb{C}$ be a birational map that comes from a polynomial automorphism of $\mathbb{C}^2$. Assume that the unique proper indeterminacy point of $\psi$ is the point $p = s_\infty(\mathbb{F}_k) \cap f_\infty(\mathbb{F}_k)$. Let
us consider the following commutative diagram

\[
\begin{array}{c}
\mathbb{P}^2 \xrightarrow{\psi'} \mathbb{P}^2 \\
\mathbb{P}^2 \xrightarrow{\psi} \mathbb{P}^2 \\
\end{array}
\]

where \( \psi \) is the blow up of the point \( p \) composed with the contraction of the strict transform of \( f_\infty(\mathbb{F}_k) \). Then the birational map \( \psi' = \psi \phi^{-1} \) satisfies the following properties:

1. \( \# \text{Base}(\psi') = \# \text{Base}(\psi) - 1 \);
2. the indeterminacy point of \( \psi' \) is on \( f_\infty(\mathbb{F}_k) \).

After the first step we are under the assumptions of Lemma 3.3 with \( k = 1 \). We get a map \( \psi' : \mathbb{F}_2 \rightarrow \mathbb{F}_2 \) with a unique indeterminacy point; this point lies on \( f_\infty(\mathbb{F}_2) \). Repeating this process as soon as we satisfy assumptions of Lemma 3.3 one gets the following diagram

\[
\begin{array}{c}
\mathbb{P}^2 \xrightarrow{\phi_2} \mathbb{P}^2 \\
\mathbb{P}^2 \xrightarrow{\phi_1} \mathbb{P}^2 \\
\end{array}
\]

where \( \phi_2 \) is obtained by applying \( n - 1 \) times Lemma 3.3. Furthermore one has

\[ \# \text{Base}(\phi_2) = \# \text{Base}(\phi_1) - n + 1 \]

and the indeterminacy point of \( \phi_2 \) is on \( f_\infty(\mathbb{F}_n) \) but not on \( s_\infty(\mathbb{F}_n) \).

The third step relies on the following result:

**Lemma 3.4.** — Let \( k \geq 2 \), and let \( \psi : \mathbb{F}_k \rightarrow \mathbb{P}^2 \) be a birational map that comes from a polynomial automorphism of \( \mathbb{C}^2 \). Assume that the unique indeterminacy point \( p \) of \( \psi \) lies on \( f_\infty(\mathbb{F}_k) \) but not on \( s_\infty(\mathbb{F}_k) \). Let us consider the following diagram

\[
\begin{array}{c}
\mathbb{P}^2 \xrightarrow{\phi_2} \mathbb{P}^2 \\
\mathbb{P}^2 \xrightarrow{\phi_1} \mathbb{P}^2 \\
\end{array}
\]

where \( \phi_2 \) is the blow up of \( p \) composed with the contraction of the strict transform of \( f_\infty(\mathbb{F}_n) \). Then the map \( \psi' \) satisfies the two following properties

1. \( \# \text{Base}(\psi') = \# \text{Base}(\psi) - 1 \);
2. the proper indeterminacy point of \( \psi' \) lies on \( f_\infty(\mathbb{F}_{k-1}) \) but not on \( s_\infty(\mathbb{F}_{k-1}) \).

After the second step the assumptions of Lemma 3.4 are satisfied. Moreover as soon as \( k \geq 3 \), the map \( \psi' \) given by Lemma 3.4 still satisfies the assumption of this lemma. So after applying \( n - 1 \) times Lemma 3.4 we get

\[
\begin{array}{c}
\mathbb{P}^2 \xrightarrow{\phi_3} \mathbb{P}^2 \\
\mathbb{P}^2 \xrightarrow{\phi_2} \mathbb{P}^2 \\
\end{array}
\]
with \#Baseφ3 = #Baseφ2 − n + 1, and the only proper indeterminacy point of φ3 lies on \( f_\infty(\mathbb{P}_1) \) but not on \( s_\infty(\mathbb{P}_1) \). According to Lemma 4.1 and ZARISKI’s Theorem we get

\[
\begin{array}{c}
\mathbb{P}_1 \xrightarrow{\phi_4} \mathbb{P}_1 \xrightarrow{\phi_4} \mathbb{P}_1 \\
\phi_4
\end{array}
\]

where \( \phi_4 \) is the blow up of some point \( q \) whose exceptional divisor is \( s_\infty(\mathbb{P}_1) \). Since \( \phi_4 \) is defined up to isomorphism one can assume that \( q = (1 : 0 : 0) \). Furthermore #Baseφ3 = #Baseφ4.

**Conclusion:** finally

\[
\begin{array}{c}
\mathbb{P}_1 \xrightarrow{\phi_4 \phi_2 \phi_1} \mathbb{P}_1 \xrightarrow{\phi_4} \mathbb{P}_1 \\
\phi = \phi_4 \phi_2 \phi_1
\end{array}
\]

where \( \phi = \phi_4 \phi_2 \phi_1 \) is an element of \( \text{Aut}(\mathbb{C}^2) \) that preserves the pencil of lines through \( (1 : 0 : 0) \), i.e. \( \phi \in E \), and #Baseφ4 = #Baseφ − 2n + 1.

Hence a birational map \( \phi \) of \( \mathbb{P}_2^2 \) of degree \( d \) that comes from a polynomial automorphism of \( \mathbb{C}^2 \) can be written as follows

\[
\phi = \phi \psi
\]

where \( \psi \) is an affine automorphism, \( \phi \) is a sequence of \( 2d − 1 \) blow-ups. Since a blow-up can be written with \( 2 \sigma \), the map \( \phi \) can be written \( 2(2d − 1) \sigma \).

**Remark 3.5.** — For \( d = 2 \), we need \( 4 \sigma \): the map \( \tau = (x^2 : xy : y^2 − xz) \) can be written \( \ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_2 \sigma \ell_4 \) with

\[
\begin{align*}
\ell_1 &= (y − x : 2y − x : z − y + x), & \ell_3 &= (−y : x + z − 3y : x), \\
\ell_2 &= (x + z : x : y), & \ell_4 &= (y − x : z − 2x : 2x − y),
\end{align*}
\]

and our bound gives \( 6 \sigma \).

For \( d = 3 \), we need \( 8 \sigma \): the map \( \psi = (x^2 + y^3 : y^2 z : z^3) \) can be written \( \ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_4 \sigma \ell_6 \sigma \ell_2 \sigma \ell_7 \) with

\[
\begin{align*}
\ell_1 &= (z − y : y : y − x), & \ell_2 &= (y + z : z : x), & \ell_3 &= (−z : −y : x − y), \\
\ell_4 &= (x + z : x : y), & \ell_5 &= (−y : x − 3y + z : x), & \ell_6 &= (−x : −y − z : x + y), \\
\ell_7 &= (x + y : z − y : y),
\end{align*}
\]

and our bound gives \( 10 \sigma \).

### 4. Proof of NOETHER Theorem and consequences

**4.1. NOETHER Theorem (I).** — An element of Bir(\( \mathbb{P}_2^2 \)) which preserves a pencil of rational curves is classically called JONQUIÈRES transformation. If \( \phi \) is a JONQUIÈRES map of degree \( d \), it has a base point of multiplicity \( d − 1 \) and \( 2d − 2 \) base points of multiplicity 1.

Let \( \phi \) be a birational map of \( \mathbb{P}_2^2 \) of degree \( d \). Let \( p_1, \ldots, p_n \) denote the base points of \( \phi \) and \( m_i \) the multiplicity of \( p_i \). Assume that the \( p_i \) are ordered such that \( m_1 \geq m_2 \geq \ldots \geq m_n \). Set \( f_\phi = \frac{d − m_1}{2} \) and let \( h_\phi \) be
the number of $p_{i}$, with $i \neq 1$, such that $m_{i} > j_{0}$. The integer $2j_{0}$ measures the complexity of $\Lambda_{q}$. Remark that $j_{0} \geq \frac{1}{2}$ with equality if and only if $\phi$ is a JONQUIÈRES transformation.

Lemma 4.1 (1). — If $\phi$ is a birational map of $\mathbb{P}_{\mathbb{C}}^{2}$ of degree $d \geq 2$, the integer $h_{q}$ satisfies the following properties:

1. $h_{q} \geq 2$;
2. if $h_{q} \geq 3$, then $\sum_{i=1}^{h} m_{i} > d$;
3. if $h_{q} \geq 3$ and if the points $p_{1}, \ldots, p_{h_{q}}$ are in $\mathbb{P}_{\mathbb{C}}^{2}$, then they are not all aligned.

Proof. — To see that $h_{q} \geq 2$, it is sufficient to prove that the following inequality holds: $\sum_{i=1}^{h} (m_{i} - j_{0}) > m_{0} - j_{0}$. By definition of $h_{q}$ we have

$$\sum_{i=1}^{h_{q}} m_{i}(m_{i} - j_{0}) \geq \sum_{i=1}^{n} m_{i}(m_{i} - j_{0}).$$

But

$$\sum_{i=1}^{n} m_{i}(m_{i} - j_{0}) = \sum_{i=1}^{n} m_{i}^{2} - j_{0} \sum_{i=1}^{n} m_{i} = (d - 1)(d - 3j_{0} + 1) = d(d - 3j_{0}) + 3j_{0} - 1 = d(m_{1} - j_{0}) + 3j_{0} - 1$$

i.e. $\sum_{i=1}^{n} m_{i}(m_{i} - j_{0}) > 2 j(m_{1} - j_{0})$. But for any $i$ the integer $m_{i}$ is smaller than $2j_{0}$ hence $\sum_{i=1}^{h} m_{i} - j_{0} > m_{1} - j_{0}$.

From $\sum_{i=1}^{h_{q}} m_{i} - j_{0} > m_{1} - j_{0}$ one gets $\sum_{i=1}^{h_{q}} m_{i} > hj_{0} + m_{1} - j_{0}$. But $h_{q}j_{0} + m_{1} - j_{0} = d + j(h_{q} - 3)$ therefore

$$\sum_{i=1}^{h_{q}} m_{i} > d + j_{0}(h_{q} - 3).$$

As a consequence $\sum_{i=1}^{h_{q}} m_{i} > d$ as soon as $h_{q} \geq 3$. \hfill $\square$

Let $q$ be a quadratic birational map whose indeterminacy points are $p_{1}, A$ and $B$, we also say that $q$ is a quadratic birational map centered at $p_{1}, A$ and $B$. Set $\phi' = \phi q$ and $d' = \deg \phi'$. The idea is the following: choose $A$ and $B$ such that $(j_{0}, h_{q}) > (j_{0}', h_{q}')$ for the lexicographic order. After a finite number of such steps, one obtains an automorphism of $\mathbb{P}_{\mathbb{C}}^{2}$.

Let us first assume that $p_{1}$ is not the point of largest multiplicity of $\phi'$. If $\Lambda$ and $\Lambda'$ are two linear systems of $\mathbb{P}_{\mathbb{C}}^{2}$, the free intersection of $\Lambda$ and $\Lambda'$ is a non-negative integer, which counts the number of free points, that is points which are not base points of $\Lambda$, $\Lambda'$ in the intersection of a general member of $\Lambda$ and a general member of $\Lambda'$. 
The free intersection of a generic line through $p_1$ and $\phi^*L$, where $L$ denotes a pencil of lines of $\mathbb{P}_C^2$, is $d - m_1 = 2j_0$. If $p_1$ is a base point of multiplicity $m_1^*$ for $\phi'$ we have $d' - m_1^* = d - m_1 = 2j_0$. If $P$ denotes the base point of largest multiplicity $m_P$ for $\phi'$ then

$$2j_0 = d' - m_P < d' - m_1^* = 2j_0.$$ 

In other words if there exist $A$ and $B$ two points in $\mathbb{P}_C^2$ such that after composed $\phi$ with a quadratic birational map centered at $A$, $B$ and $p_1$ the point $p_1$ is not of largest multiplicity then $j_0 < j_0'$. Suppose now that $p_1$ is the point of largest multiplicity of $\phi'$. The point $p_1$ is the point of largest multiplicity of $\phi'$. As we just see, then $j_0 = j_0'$. One of the following holds:

a) there are two points of indeterminacy $p_2$ and $p_3$ with multiplicity $m_2 > j_0$ and $m_3 > j_0$;

b) there is at most one indeterminacy point with multiplicity $> j_0$ and no base point infinitely near $p_1$;

c) there is at most one indeterminacy point with multiplicity $> j_0$ and at least one base point infinitely near $p_1$.

Let us consider all these cases.

a) Let us consider the quadratic birational map $q$ centered at $p_1$, $p_2$ and $p_3$ and set $\phi' = \phi q$. The multiplicity $m_2^*$ of $p_2$ for $\phi'$ is equal to the number of free points of an element of $\phi^*L$ and the line through $p_1$ and $p_3$.

By Bezout one has

$$d = m_1 + m_3 + m_2^*.$$ 

As $d - m_1 = m_3 + m_2^*$ and $d - m_1 = 2j_0$ one has $j_0 > m_2^*$. Similarly $j_0 > m_3$. Thus $h_0 = h_0 - 2$.

b) Assume that the base points $p_1$, $p_2$ and $p_3$ of $\phi$ satisfy the following conditions:

- $p_1$ is of largest multiplicity,
- $p_2$ belongs to $\mathbb{P}_C^2$,
- $p_3$ is infinitely near $p_2$.

Let us choose a point $P$ in $\mathbb{P}_C^2$ such that the line through $P$ and $p_1$ (resp. $m$ and $p_2$) does not contain base point of $\phi$ distinct from $p_1$ (resp. $p_2$). Let us compose $\phi$ with the quadratic birational map centered at $p_1$, $p_2$ and $P$. The point $P$ (resp. $p_1$) becomes a base point of multiplicity $< j_0$ (resp. of multiplicity $2j_0$) and $p_2$ an indeterminacy point of multiplicity $m_2 > j_0$. Hence $h$ is constant and there is one more indeterminacy point with multiplicity $> j_0$. Iterating this process we can assume that all the base points of multiplicity $> j_0$ are in $\mathbb{P}_C^2$. Since $h_0 \geq 2$ we thus are in case a).

c) Let $A$ and $B$ be two generic points of $\mathbb{P}_C^2$. After having composed $\phi$ with the quadratic birational map centered at $A$, $B$ and $p_1$, the integer $h$ has increased by 2 as $A$ and $B$ are of multiplicities $2j_0$; in particular $h_0 \geq 4$. Nevertheless there is no more base point infinitely near $p_1$. According to b) we can assume that $p_1, \ldots, p_h$ are in $\mathbb{P}_C^2$. These points are not all aligned (Lemma 4.1); hence we can apply a) at least two times and $h_0$ decreases by 2. Finally $h_0$ has decreased by 2.

After repeating a) a finite number of times, we obtain a map $\psi$ such that $h_\psi < 2$ then from Lemma 4.1 we get either $\psi \in \text{Aut}(\mathbb{P}_C^2)$ or $j_0 < j_0'$.

### 4.2. Proof of Theorem [A] — The configuration that needs the most $\sigma$ is the configuration of birational maps coming from polynomial automorphisms of $\mathbb{C}^2$ (see [4.1]).

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