How to Measure the Cosmic Curvature

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Abstract

The conventional method to determine the cosmic curvature is to measure the total mass density $\Omega_{\text{tot}}$. Unfortunately the observational $\Omega_{\text{tot}}$ is closely near the critical value 1. The computation of this paper shows that $\Omega_{\text{tot}} \approx 1$ is an inevitable result for the young universe independent of the spatial topology. So the mass density is not a good criterion to determine the cosmic curvature. In this paper, we derive a new criterion based on the galactic distribution with respect to redshift $z$, which only depends on the cosmological principle and geometry. The different type of spatial topology will give different results, then the case can be definitely determined.

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1 Introduction

Whether the space of our universe is open, flat or closed is one of the most fundamental problems in cosmology. The solutions to some other important problems such as the property of dark matter, the evolution of the universe, are closely related to the spatial topology of the universe. Almost all used methods to determine the cosmic curvature are based on computing the total mass density $\Omega_{\text{tot}}$. However, the most authoritative empirical data show that the total mass density $\Omega_{\text{tot}} = 1.02 \pm 0.02 [1]-[4]$, which is closely near the critical density 1, so we can not get a definite solution to this fundamental problem.

Recent years, lots of attention have been attracted to the resolution of this problem [5]-[15]. In [7], by analyzing the large-scale correlation function of red galaxies from the Sloan Digital Sky Survey (SDSS) and combining with the cosmic microwave background (CMB) acoustic scale, we arrive at $\Omega_K = -0.010 \pm 0.009$ under the assumption of $w = -1$. In [8], by measured the large-scale power spectrum $P(k)$ for the luminous red galaxies from SDSS, the authors improve the evidence for spatial flatness, and sharpen the curvature constraint to $\Omega_{\text{tot}} = 1.003 \pm 0.010$.

In [9, 10], using the full dataset of CMB and large scale structure (LSS), and by Markov Chain Monte Carlo global fit, the authors find the best fit value of $|\Omega_K| < 0.015$ for some dark energy models and $|\Omega_K| > 0.06$ are excluded. The $\Lambda$CDM model is consistent with all the data, and a flat universe is preferred.

From the calculation of the above papers, we find that the results are strongly and nonlinearly correlated with the equation of state $w(z)$. In [14], the authors solved $w(z)_F$ and $w(z)_d$, two equations of state for the same model, respectively from the Friedmann equation and the luminosity distance equation. The numerical results show that the two functions $w(z)$ unstably depend on $\Omega_K$ and depend on $z$ towards opposite trends, which lead to great errors for adequately large $z$. Similar conclusion also confirmed in [15], where the authors find bounds on cosmic curvature are less stringent if dark energy density is a free function of cosmic time.

However, in our point of view, $\Omega_{\text{tot}}$ or $\Omega_K$ may be not the best criterion to determine the cosmic curvature. At first, $\Omega_{\text{tot}} \approx 1$ is an inevitable results for the young universe as shown bellow. Secondly, since the Friedmann equation is a dynamic equation of $a(t)$, it is inadequate to analyze this equation as algebraic equation by introducing Hubble’s parameter $H$. Thus looking for a more effective method and criterion to determine the spatial type is necessary.
Hubble once realized to measure the spatial curvature by counting galaxies[16], but the corresponding equation derived is very complicated, which not only depends on \( z \) but also on \( H \) and \( \Omega_{\text{tot}} \). This situation is caused by the calculation, which is too direct to simplify the model, so that the idea was not developed for practical application.

However, the geometrical problem may be more naturally and effectively solved by geometrical method. In this paper we develop the Hubble’s idea by designing a simple reference function of the galactic distribution in flat space, and then compare it with the practical counting data. The result of comparison may be able to give a definite answer to the fundamental question.

2 Analysis and method

2.1 Flatness problem of the space

In mean sense, the universe is highly homogeneous and isotropic, and the metric is described by Friedmann-Robertson-Walker (FRW) metric. The corresponding line element is given by

\[
ds^2 = d\tau^2 - a^2(\tau) \left( \frac{dr^2}{1 - K r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right),
\]

where \( K = 1, 0 \) and \(-1\) correspond to the closed, flat and open universe respectively. Some literatures intended to replace the constant \( K \) by some continuous function \( K(r) \) or \( K(\tau) \). In fact, such replacement violate the symmetry of the spacetime. The cosmic radius of curvature is determined by \( a(t) \) rather than by \( K \). The spatial topology of the homogeneous and isotropic spacetime has only the three cases[17]-[20], and the metric must be equivalent to (2.1). In this paper, we adopt the conformal coordinate system, because in this form \( a(t) \) can be expressed as Taylor series near \( a \to 0 \). The corresponding metric becomes

\[
g_{\mu\nu} = a^2(t) \text{diag}[1, -1, -f^2(r), -f^2(r) \sin^2 \theta],
\]

where \( d\tau = adt \), and

\[
f = \begin{cases} 
\sin r & \text{if } K = 1, \\
r & \text{if } K = 0, \\
\sinh r & \text{if } K = -1.
\end{cases}
\]

The critical density \( \rho_c \) and the Hubble’s parameter \( H \) are defined by

\[
\rho_c = \frac{3}{8\pi G} H^2 \sim 8 \times 10^{-30} (\text{g/cm}^3), \quad H = \frac{a'(t)}{a^2}.
\]
By Friedmann equation[16], we have

$$\Omega_K \equiv K \left(\frac{a}{a'}\right)^2 = \Omega_m + \Omega_\Lambda - 1 = \Omega_{\text{tot}} - 1,$$

(2.5)

where $\Omega_m = \frac{\rho_c}{\rho}$, $\Omega_\Lambda = \frac{\Lambda}{3H^2}$. Theoretically, by (2.5) we have the following judgment: $\Omega_{\text{tot}} > 1$, $\Omega_{\text{tot}} = 1$ and $\Omega_{\text{tot}} < 1$ correspond to the closed, flat and open universe respectively. The following analysis shows that, this judgement is not sharp for a young universe.

The original equation of (2.5) is the dynamic equation

$$a'^2 = \frac{1}{3}\Lambda a^4 + \frac{8\pi G}{3}\rho a^4 - Ka^2.$$  

(2.6)

For most matter models[21], such as nonlinear spinors[22, 23, 24] and Casimir effect, when $a \to 0$, we have the main part of density as

$$\rho = \rho_0 \left(\frac{1}{a^3} - \frac{\sigma}{a^4}\right),$$

(2.7)

where $\rho_0$ is a constant, and $|\sigma| \ll a$ is a slowly varying function of $a$, which can be treated as a constant for the following calculation. For homogeneous scalar field models such as quintessence and phantom field $\phi$ [25], we can also get the corresponding $\rho(a)$ via transforming $\frac{d\phi}{dt} = \frac{d\phi}{da}a'$ and then solving the differential equation. However, we need not to solve the specific model for the present purpose, because we only need the assumption that the cosmic equation of state $\rho(a)$ exists and has singularity not worse than (2.7). There are a lot of works on constructing complicated equation of state $w(z)$ and fitting the empirical data. It seems unnecessary, because we can hardly find out the effective $w(z)$ [14].

We take (2.7) with constant $\sigma$ as model to show the properties of $a(t)$ and $\Omega_K$ for small $t$. We take $R = \frac{4\pi G}{3}\rho_0 = 1$ as length unit, which is the mean scale factor. Substituting (2.7) into (2.6) we get

$$a'^2 = 2(a - \sigma) - Ka^2 + \lambda a^4,$$

(2.8)

where $\lambda = \frac{1}{3}\Lambda R^2$ is dimensionless constant. Choosing suitable starting time $t_0 = 0$, for small $t$ the solution of (2.8) can be expressed by

$$a(t) = a_0 + \frac{1}{2}(1 - Ka_0 + 2\lambda a_0^3)t^2 +$$

$$\left(\frac{-K}{24} + \frac{\lambda\sigma + K^2}{24}\right)a_0 - \frac{3\lambda}{4}a_0^2 + \frac{\lambda K}{6}a_0^3 \right) t^4 + O(t^6),$$

(2.9)
The limit deviation of mass density

\[ F(t) = |\Omega_K| < F \]

Figure 1: The Maximum Deviation of Mass Density Caused by Space Type

where \( 0 \leq |a_0| \sim |\sigma| \ll 1 \) is the root of \( 2(a - \sigma) - Ka^2 + \lambda a^4 = 0 \) near zero. For the Big Bang model we have \( a_0 = 0 \). (2.9) shows how the parameters \( K, \sigma, \lambda \) and \( a_0 \) influence the scale factor.

As computed in [22, 23], we have the following conclusions. (I) Since \( \lambda \) has effects only on the cosmic scale, we have \( |\lambda| < 1 \). (II) The present comoving time \( t_0 \sim 18^\circ = 0.314(\text{rad}) \). In this paper, we are only concerned with the behavior of the universe after the galaxies formed, so we have the estimation for maximum of \( |\Omega_K| \) as follows

\[ |\Omega_K| \leq \frac{a(t)^2}{a'(t)^2} \left( \frac{\varepsilon}{t} + \frac{t}{2} \right)^2 \equiv F(t), \quad \varepsilon \equiv \frac{a_0}{1 - Ka_0 + 2\lambda a_0^3} \rightarrow 0. \] (2.10)

The typical values are displayed in Fig.(1), which visually tells us \( \Omega_K \) is not a good criterion to determine the cosmic curvature. If \( a_0 \leq 0 \), then \( \Omega_K(t) = 0 \) has solution \( t_{\text{singular}} \geq 0 \), which inevitably leads to the so called “fine-tuning mass density”, thus the dynamic process can not be understood with static equation.

### 2.2 Galactic distribution and reference function

Now we design the new method to determine the cosmic curvature \( K \), which only depends on the redshift distribution of galaxies at a few selected point \( z_k \). The cosmo-
logical redshift

\[ z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{a(t)}{a(t_0)} - 1, \]  

(2.11)

where \( t = t_0 + r \) is the present time, and \( t_0 \) is the time when the photon was emitted, \( r \) is the radial coordinate of the light source. Denote the number of all galaxies in a ball with radius coordinate \( r \) by \( N(r) \), we consider \( N(r) \) in the domain \( [r - \frac{1}{2} \Delta r, r + \frac{1}{2} \Delta r] \), or correspondingly in \( [z - \frac{1}{2} \delta z, z + \frac{1}{2} \delta z] \). The increment of redshift is given by

\[ \delta z \equiv \Delta a(t) \left| \frac{a(t)}{a(t_0)} \right| \Delta t = (1 + z) \left| \frac{a'(t)}{a(t_0)} \right| \Delta r, \]  

(2.12)

where we used \( \Delta t = \Delta r \) for photons in the metric (2.2). Then the galaxies in the domain \( [r - \frac{1}{2} \Delta r, r + \frac{1}{2} \Delta r] \) can be counted by

\[ \delta N = 4\pi \rho_g f^2(r) \Delta r = 4\pi \rho_g f^2 \left| \frac{a(t_0)}{a(t_0)} \right| \frac{\delta z}{1 + z}, \]  

(2.13)

where \( \rho_g \) is the number density of galaxies in comoving volume \( dV = f^2 \sin \theta dr d\theta d\phi \), which is a constant independent of \( t \). By (2.13), we get the following rigorous equation

\[ (1 + z) \frac{dN}{dz} \left| \frac{a'(t_0)}{a(t_0)} \right| = 4\pi \rho_g f^2 r. \]  

(2.14)

By (2.9) and (2.11), we can solve \( r = r(t, z) \), then get the following function

\[ \phi \equiv \left| \frac{a'(t_0)}{a(t_0)} \right|. \]  

(2.15)

Generally speaking, \( \phi = \phi(t, z) \) is a complicated function. However for a small redshift, by the empirical Hubble’s law, we have

\[ \phi = \left| \frac{a'(t_0)}{a(t_0)} \right| D_r = H(t_0)D_r \equiv z, \]  

(2.16)

where \( D_r = a(t_0)r \) is the distance corresponding to radial coordinate \( r \) at time \( t_0 \). For flat space, the number of galaxies in a ball with radius \( r \) is given by

\[ N(r) = \frac{4\pi}{3} \rho_g r^3 = \frac{4\pi}{3} \rho_g f^2 r. \]  

(2.17)

Substituting (2.16) and (2.17) into (2.14), we get the distributive equation for flat space

\[ \frac{dN}{Ndz} = \frac{3}{z(1 + z)}. \]  

(2.18)

The solution of (2.18) is given by

\[ N(z) \equiv N_1 \left( \frac{z}{1 + z} \right)^3. \]  

(2.19)
By the precision of Hubble’s law, we learn (2.19) is a good approximation for galaxy count at least within the range $z < 1$.

However, for the flat space $K = 0$, we can derive more accurate $N(z)$ for large range $z$ as follows. By (2.9), we always have solution to high accuracy

$$a(t) = \frac{1}{2} t^2, \quad \text{for } 5^\circ \leq t \leq 60^\circ,$$

(2.20)

$t \geq 5^\circ$ is because only the evolution after galaxies formed is concerned here, then we can omit the influence of $a_0$. The condition $t \leq 60^\circ \sim 1(\text{rad})$ is derived by omitting the higher order term in the series (2.9).

Substituting (2.20) into (2.11), we get

$$z = \frac{(t_0 + r)^2 - t_0^2}{t_0^2} = \frac{r}{t_0} \left(2 + \frac{r}{t_0}\right),$$

(2.21)

or equivalently

$$\frac{r}{t_0} = \sqrt{1 + z} - 1 = \frac{z}{1 + \sqrt{1 + z}},$$

(2.22)

so we have

$$\varphi = \frac{a'(t_0)}{a(t_0)} r = \frac{2r}{t_0} = \frac{2z}{1 + \sqrt{1 + z}}.$$  

(2.23)

Substituting (2.23) and (2.17) into (2.14), we get the distributive equation for flat space to high accuracy

$$\frac{dN}{Ndz} = \frac{3(1 + \sqrt{1 + z})}{2z(1 + z)}.$$  

(2.24)

The solution of (2.24) is given by

$$N(z) = N_0 \left(\frac{z}{1 + z + \sqrt{1 + z}}\right)^3.$$  

(2.25)

Define the reference function by

$$n(z) \equiv \ln \left(\frac{N'(z)}{N'(z_0)}\right) = 2 \ln(\sqrt{1 + z} - 1) - \frac{5}{2} \ln(1 + z) - n_0,$$

(2.26)

where $z_0$ is the first redsheft point measured, from which we count galaxies,

$$n_0 = 2 \ln(\sqrt{1 + z_0} - 1) - \frac{5}{2} \ln(1 + z_0)$$  

(2.27)

is a constant. The function $n(z)$ is the criterion which we are looking for, because for all $z > z_0$, we definitely have

$$n(z)_{\text{open}} > n(z)_{\text{flat}} = 2 \ln(\sqrt{1 + z} - 1) - \frac{5}{2} \ln(1 + z) - n_0 > n(z)_{\text{closed}}.$$  

(2.28)
The trend curves of \( n(z) \)

\[
n(z) = \ln \left[ \frac{N'(z)}{N'(z_0)} \right]
\]

Figure 2: The Distributive Functions of Counting Galaxies

due to the direct influence of \( f(r) \) on \( N(z) \). The trend curves of \( n(z) \) are displayed in Fig.(2). The practical counting data \( n(z) \) should also be a smooth function of \( z \) due to the law of large number. By this method, we will compare the trends of two smooth functions rather than two data, so there should be less ambiguous consequence caused by experiment and model errors.

### 3 Discussion and conclusion

From the above computation and analysis, we can get the following results.

(C1). From Fig(1) we learn that the present mass energy \( \Omega_K \) corresponding to the spatial type is generally small in value for a young universe, which is not sharp to give a definite answer for the spatial type.

(C2). The space type can be determined by counting the galactic distribution with respect to the redshift \( z \), which is a smooth function of \( z \) and can be sharply compared with the theoretical reference function (2.26).

(C3). The cosmic equation of state \( \rho(a) \) is a useful concept for cosmology, because \( \rho \) is a superposable intensity with clear physical meanings, and it can be easily derived for concrete matter models via action principle, which seems to be more effective than \( w(z) \).
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