Coupling synchronization principle of two pairs counter-rotating unbalanced rotors in the different resonant conditions

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Abstract
The present work investigates the coupling synchronization principle and stability in a vibrating system with two pairs counter-rotating unbalanced rotors (also called exciters). Based on Lagrange equations, the dimensionless coupling differential equations of motion of the system are deduced. The synchronization criterion of two pairs exciters stems from the averaging method, it satisfies the fact that the absolute value of dimensionless residual torque difference between arbitrary two driving motors is less than or equal to the maximum of their dimensionless coupling torques. The stability criterion of the synchronous states complies with Routh-Hurwitz principle. The coupling dynamic characteristics of the system are numerically analyzed in detail, including synchronization and stability ability, maximum of the coupling torque and phase relationship, etc. Some simulation results applying the Runge-Kutta algorithm are performed, it is shown that the motion states of the system can be classified into two types: sub-resonant state and super-resonant state. Generally in engineering, the ideal working points should be selected in sub-resonance region, in this case the expended energy can be saved relatively by 1/5–1/3, which is less than that in super-resonance region under the precondition of the same vibration amplitude value.

Keywords
Unbalanced rotors, synchronization, stability, different resonant conditions, energy-saving

Introduction
As a common special phenomenon in nature, synchronization exists widely in human life and production, such as: synchronous satellites, pendulum clock, neuronal networks, coupling oscillators, self-sustained electromechanical devices, and so on.¹-⁴ Researches for synchronization have a long history, as early as 1960s, Dr Blekman and coworkers⁵-⁷ theoretically explained the synchronization problem of two identical exciters by using the method of direct separation of motion firstly. Subsequently, Wen et al.⁸,⁹ applied such theory to engineering and established a branch of vibration utilization engineering, and invented many self-synchronous vibrating machines. Acebrón et al.¹⁰ applied the Kuramoto model to propose a more detailed analyses of the synchronization problems for the cluster oscillators. Perlikowski et al.¹¹ described the relationship between full synchronization of the response oscillators and generalized synchronization of a driving system. Besides, Balthazar et al.¹²,¹³ gave some reviews on self-synchronization of two or four non-ideal exciters by numeric, and a special phenomenon called “Sommerfeld
effect” was revealed. The synchronization problem of flow shop was researched by Bultmann et al.,\textsuperscript{14} which improved the working efficiency of the machines.

Since synchronization of unbalanced rotors (URs) is put forward and developed, which leads to the fact that vibratory synchronization theory has a wide range of applications in engineering, especially in materials screening, transporting and dewatering, including vibrating screens, vibrating feeders, synchronous dryers, conveyors and dewatering screens, etc. Currently, many vibrating screening and transporting equipments are driven by double motors, which take advantage of the relative stable vibration amplitude of the system in super-resonant state. The synchronization theory of two URs driven by hydraulic motors is further involved.\textsuperscript{15} However, motion function of machine is also available by using multiple motor to enhance the driving powers, which is less consideration. It is, therefore, of great significant to study the synchronization problem of multiple URs.

In previous literatures, synchronization theory of two or three exciters rotating in the same (or opposite) directions, in the far super-resonant vibrating system, was further studied,\textsuperscript{16,17} as well as a cylindrical roller with dry friction in the mechanical system,\textsuperscript{18} stability of the synchronous states of multiple exciters in the system with different resonant conditions, however, is less involved. This paper takes a dynamical model with two pairs counter URs for example synchronization and stability of the two pairs URs in sub-resonant and super-resonant states are investigated, some corresponding to coupling dynamic characteristics of the system are provided. By numeric characteristic and simulation analyses, it is shown that under the precondition of the same response amplitude, the expended energy of the system in sub-resonant state can save $1/5$–$1/3$, which is less than that in super-resonant one. Therefore, the driving power of the vibrating system is relatively reduced in sub-resonant state, and the energy-saving can be implemented. To explain the phenomenon better, this paper will further analyze the stable states of the vibrating system with two pairs URs in sub-resonance and super-resonance regions.

The structure of this paper is as follows: firstly, the dynamic model and differential equations of motion of the vibrating system are given. Section “System dynamic model and motion differential equations” is devoted to deriving the synchronization and stability criterions. The coupling dynamic characteristics of the system are numerically discussed, and some simulation results of which are provided next. Finally, conclusions are drawn.

**System dynamic model and motion differential equations**

As shown in Figure 1, a dynamic model is taken as an example to discuss the synchronization of two pairs of URs (or four URs) in a vibrating system. It is made up of a rigid frame, four URs and isolation springs. Four URs are driven by four motors separately, which are asymmetrical. The rotational centers of four URs are denoted by $o_1$, $o_2$, $o_3$, and $o_4$, respectively; their rotational radiuses are set as the same, denoted by $r$; the connectional lines of $o_1$–$o_3$ and $o_2$–$o_4$ pass through the mass center $o$ of the system. Besides, URs 1 and 3, URs 2 and 4, are symmetrically distributed about the mass center. The angles between the connectional lines from the rotating centers of four URs to the mass center of the system and $y$-axis, are $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_4$, respectively. The rigid frame is supported by an elastic foundation, it exhibits three degrees of freedom, i.e. the displacements $x$ and $y$, the rational angle $\psi$. In the generalized coordinate, the kinetic energy $T$, potential energy $V$, and energy dissipation function $D$ are introduced into the Lagrange’s equation. Neglecting the asymmetry of the microscopic inertial coupling caused by the system, the differential equations of motion of the system are established as follows

$$
M\ddot{x} + f_x\dot{x} + k_x x = \sum_{i=1}^{4} m_i r (\dot{\phi_i}^2 \cos\phi_i + \dot{\phi_i} \sin\phi_i)
$$

$$
M\ddot{y} + f_y\dot{y} + k_y y = \sum_{i=1}^{4} m_i r (\dot{\phi_i}^2 \sin\phi_i - \dot{\phi_i} \cos\phi_i)
$$

$$
J\ddot{\psi} + f_{\psi}\dot{\psi} + k_{\psi}\psi = \sum_{i=1}^{4} m_i r (\dot{\phi_i}^2 \sin\phi_i - \dot{\phi_i} \cos\phi_i - \sigma_i \beta_i)
$$

$$
(J_{ei} + m_i r^2)\dddot{\phi_i} + f_{\phi_i}\dot{\phi_i} = T_{ei} - m_i r (\dot{\phi_i} \cos\phi_i - \sigma_i x \sin\phi_i + l_i \ddot{\psi} \cos(\phi_i - \sigma_i \beta_i))
$$

$$
+ l_i \dot{\psi}^2 \sin(\phi_i - \sigma_i \beta_i), \quad i = 1, 2, 3, 4
$$
Synchronization of the vibration system

It is assumed that the average phase of four URs is \( \varphi \), and the phase differences among them, are \( 2\alpha_1 \), \( 2\alpha_2 \), and \( 2\alpha_3 \). Then, we have

\[
\varphi = \frac{1}{4} \sum_{i=1}^{4} \varphi_i, \quad \varphi_1 - \varphi_2 = 2\alpha_1, \quad \varphi_2 - \varphi_3 = 2\alpha_2, \quad \varphi_3 - \varphi_4 = 2\alpha_3 \tag{2}
\]

Arranging equation (2) leads to

\[
\begin{align*}
\varphi_1 &= \varphi + \frac{3}{2} \alpha_1 + \alpha_2 + \frac{1}{2} \alpha_3 = \varphi + \alpha_1, \\
\varphi_2 &= \varphi - \frac{1}{2} \alpha_1 + \alpha_2 + \frac{1}{2} \alpha_3 = \varphi + \alpha_2, \\
\varphi_3 &= \varphi - \frac{1}{2} \alpha_1 - \alpha_2 + \frac{1}{2} \alpha_3 = \varphi + \alpha_3, \\
\varphi_4 &= \varphi - \frac{1}{2} \alpha_1 - \alpha_2 - \frac{3}{2} \alpha_3 = \varphi - \alpha_4
\end{align*}
\tag{3}
\]

When the four motors operate synchronously in the vibrating system with small damping, their synchronous angular velocity is denoted by \( \omega_{m0} \), i.e. \( \dot{\varphi} = \omega_{m0} \), and the responses of steady-state in \( x \)-, \( y \)-, and \( \psi \)-directions, can be expressed in form.

\[
\begin{align*}
\dot{x} &= -\frac{\tau_{mY}}{\mu_x} \sum_{i=1}^{4} \eta_i \cos(\varphi + \nu_i + \gamma_x) \\
\dot{y} &= -\frac{\tau_{mY}}{\mu_y} \sum_{i=1}^{4} \eta_i \sin(\varphi + \nu_i + \gamma_y) \\
\dot{\psi} &= -\frac{\tau_{mY}}{\mu_\psi} \sum_{i=1}^{4} \eta_i \sin(\varphi + \nu_i - \beta_i + \gamma_\psi)
\end{align*}
\tag{4}
\]

Synchronization criterion of the four URs

Differentiating equation (4) to obtain \( \ddot{x}, \dot{y}, \ddot{\psi}, \) and \( \dot{\psi} \), inserting them into the last three formulae of equation (1), and integrating them over \( \varphi = 0 \sim 2\pi \), the average balanced equations of the four URs can be obtained as

\[
T_{ei} - \bar{T}_{Li} = f_i \omega_{m0}, \quad i = 1, 2, 3, 4
\tag{5}
\]
The coefficients $\chi_{fi}$ and $\chi_{ai}$ ($i, j = 1, 2, 3$) in equation (6) are listed in Appendix 1, it should be noted here that $\bar{x}_i$ ($i = 1, 2, 3$) in equation (6) denotes the value of $x_i$ after the averaging. If the four URs can operate synchronously, the balanced equations for electromagnetic torques of four motors are acquired as

$$
T_{01} = T_{010} - f_1 \omega_{m0} = m_0 r^2 \omega_{m0} (\chi_{f1} + \chi_{a1})
$$

$$
T_{02} = T_{020} - f_2 \omega_{m0} = m_0 r^2 \omega_{m0} (\chi_{f2} + \chi_{a2})
$$

$$
T_{03} = T_{030} - f_3 \omega_{m0} = m_0 r^2 \omega_{m0} (\chi_{f3} + \chi_{a3})
$$

$$
T_{04} = T_{040} - f_4 \omega_{m0} = m_0 r^2 \omega_{m0} (\chi_{f4} + \chi_{a4})
$$

The differences of output torques between motors 1 and 2 ($\Delta T_{012}$), 2 and 3 ($\Delta T_{023}$), and 3 and 4 ($\Delta T_{034}$), can be expressed as the following:

$$
\Delta T_{012} = T_{01} - T_{02} = T_{m}[(\eta_1^2 - \eta_2^2)W_{c0} + \eta_1 \eta_2 W_{c12}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c12}) - \eta_2 \eta_3 W_{c123}\cos(2\bar{x}_1 + \theta_{c23}) + \eta_2 \eta_4 W_{c124}\cos(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_1 \eta_3 W_{c13}\cos(2\bar{x}_1 + 2\bar{x}_3 + \theta_{c13}) + 2\eta_1 \eta_2 W_{c12}\sin(2\bar{x}_1 + \theta_{c12}) + \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{c23}) + \eta_1 \eta_4 W_{c14}\sin(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_2 \eta_4 W_{c24}\sin(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24})]
$$

$$
\Delta T_{023} = T_{02} - T_{03} = T_{m}[(\eta_2^2 - \eta_3^2)W_{c0} + \eta_1 \eta_2 W_{c12}\cos(2\bar{x}_1 + \theta_{c12}) - \eta_2 \eta_3 W_{c13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) + \eta_2 \eta_4 W_{c124}\cos(2\bar{x}_1 + 2\bar{x}_3 + \theta_{c24}) - \eta_1 \eta_4 W_{c13}\cos(2\bar{x}_1 + 2\bar{x}_3 + \theta_{c13}) + 2\eta_1 \eta_2 W_{c12}\sin(2\bar{x}_1 + \theta_{c12}) + \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{c23}) + \eta_1 \eta_4 W_{c14}\sin(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_2 \eta_4 W_{c24}\sin(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24})]
$$

$$
\Delta T_{034} = T_{03} - T_{04} = T_{m}[(\eta_3^2 - \eta_4^2)W_{c0} + \eta_2 \eta_3 W_{c23}\cos(2\bar{x}_2 + \theta_{c23}) - \eta_2 \eta_4 W_{c24}\cos(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24}) + \eta_1 \eta_3 W_{c13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_1 \eta_4 W_{c14}\cos(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) + \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{c23}) - \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) + 2\eta_1 \eta_2 W_{c12}\sin(2\bar{x}_1 + \theta_{c12}) + \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{c23}) + \eta_1 \eta_4 W_{c14}\sin(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_2 \eta_4 W_{c24}\sin(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24})]
$$

where $T_{m} = m_0 r^2 \omega_{m0}^2 / 2$ denotes the kinetic energy of the standard UR.
Rearranging equations (8) to (10), yields

\[
\frac{\Delta T_{012}}{T_{tu}} - (\eta_1^2 - \eta_2^2) W_{s0} = \tau_{c12}(\bar{x}_1, \bar{x}_2, \bar{x}_3) \tag{11}
\]

\[
\frac{\Delta T_{023}}{T_{tu}} - (\eta_2^2 - \eta_3^2) W_{s0} = \tau_{c23}(\bar{x}_1, \bar{x}_2, \bar{x}_3) \tag{12}
\]

\[
\frac{\Delta T_{034}}{T_{tu}} - (\eta_3^2 - \eta_4^2) W_{s0} = \tau_{c34}(\bar{x}_1, \bar{x}_2, \bar{x}_3) \tag{13}
\]

with

\[
\tau_{c12}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \eta_1 \eta_2 W_{c13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_2 \eta_3 W_{c23}\cos(2\bar{x}_2 + \theta_{c23}) + \eta_1 \eta_4 W_{c14}\cos(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_2 \eta_4 W_{c24}\cos(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24}) + 2\eta_1 \eta_2 W_{c12}\sin(2\bar{x}_1 + \theta_{c12}) + \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{c23}) + \eta_1 \eta_4 W_{c14}\sin(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_2 \eta_4 W_{c24}\sin(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24})
\]

\[
\tau_{c23}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \eta_1 \eta_2 W_{c12}\cos(2\bar{x}_1 + \theta_{c12}) - \eta_1 \eta_3 W_{c13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) + \eta_2 \eta_4 W_{c24}\cos(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24}) - \eta_3 \eta_4 W_{c34}\cos(2\bar{x}_3 + \theta_{c34}) + 2\eta_1 \eta_2 W_{c12}\sin(2\bar{x}_1 + \theta_{c12}) + \eta_1 \eta_3 W_{c13}\sin(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{c23}) + \eta_1 \eta_4 W_{c14}\sin(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_2 \eta_4 W_{c24}\sin(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24})
\]

\[
\tau_{c34}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \eta_2 \eta_3 W_{c23}\cos(2\bar{x}_2 + \theta_{c23}) - \eta_2 \eta_4 W_{c24}\cos(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24}) + \eta_1 \eta_3 W_{c13}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{c13}) - \eta_1 \eta_4 W_{c14}\cos(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) - \eta_2 \eta_3 W_{c23}\sin(2\bar{x}_2 + \theta_{c23}) + \eta_1 \eta_4 W_{c14}\sin(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14}) + 2\eta_2 \eta_3 W_{c34}\sin(2\bar{x}_3 + \theta_{c34}) + \eta_2 \eta_4 W_{c24}\sin(2\bar{x}_2 + 2\bar{x}_3 + \theta_{c24}) + \eta_1 \eta_4 W_{c14}\sin(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{c14})
\]

where \(\tau_{c12}(\bar{x}_1, \bar{x}_2, \bar{x}_3)\), \(\tau_{c23}(\bar{x}_1, \bar{x}_2, \bar{x}_3)\), and \(\tau_{c34}(\bar{x}_1, \bar{x}_2, \bar{x}_3)\) describe the dimensionless coupling torques between URs 1 and 2, 2 and 3, and 3 and 4, respectively. Since they are the limited functions of \(\bar{x}_1\), \(\bar{x}_2\), and \(\bar{x}_3\), we have

\[
|\tau_{c12}(\bar{x}_1, \bar{x}_2, \bar{x}_3)| \leq \tau_{c12\text{max}} \tag{14}
\]
where and are satisfied, the four URs can implement synchronization. So, the synchronization criterion of the four URs is respectively limited function, i.e.

\[
\frac{\Delta T_{012}}{T_u} - (\eta_1^2 - \eta_2^2) W_{s0} \leq \tau_{c12\text{max}}
\]

\[
\frac{\Delta T_{023}}{T_u} - (\eta_2^2 - \eta_3^2) W_{s0} \leq \tau_{c23\text{max}}
\]

\[
\frac{\Delta T_{034}}{T_u} - (\eta_3^2 - \eta_4^2) W_{s0} \leq \tau_{c34\text{max}}
\]

When the following conditions are satisfied, the four URs can implement synchronization. So, the synchronization criterion of the four URs is that the absolute value of dimensionless residual torque difference between arbitrary two motors, is less than or equal to the maximum of their dimensionless coupling torques.

Adding the four formulae of equation (7), and the mean is carried out, we have

\[
\tau_a(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{1}{4 T_u} \sum_{i=1}^{4} T_{0i}
\]

\[
= \frac{1}{4} \left[ \sum_{i=1}^{4} \eta_i^2 W_{s0} + 2 \eta_1 \eta_2 W_{sc12}\cos(2\bar{x}_1 + \theta_{s12}) + 2 \eta_1 \eta_3 W_{sc12}\cos(2\bar{x}_1 + 2\bar{x}_2 + \theta_{s12}) + 2 \eta_2 \eta_4 W_{sc14}\cos(2\bar{x}_1 + 2\bar{x}_2 + 2\bar{x}_3 + \theta_{s14}) + 2 \eta_2 \eta_3 W_{sc23}\cos(2\bar{x}_2 + \theta_{s23}) + 2 \eta_3 \eta_4 W_{sc23}\cos(2\bar{x}_2 + 2\bar{x}_3 + \theta_{s34}) + 2 \eta_3 \eta_4 W_{sc34}\cos(2\bar{x}_3 + \theta_{s34}) \right]
\]

Here, \( \tau_a(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) is referred to as the average dimensionless loading torque of the four motors and it is a limited function, i.e.

\[
\left| \tau_a(\bar{x}_1, \bar{x}_2, \bar{x}_3) \right| \leq \tau_{\text{amax}}
\]

and \( \tau_{\text{amax}} \) is the maximum of average dimensionless loading torques of the four motors. The coefficients of synchronization ability between URs 1 and 2, 2 and 3, 3 and 4, and 4 and 1, are defined as the following, respectively

\[
\zeta_{12} = \frac{\tau_{c12\text{max}}}{\tau_{\text{amax}}}, \quad \zeta_{23} = \frac{\tau_{c23\text{max}}}{\tau_{\text{amax}}}, \quad \zeta_{34} = \frac{\tau_{c34\text{max}}}{\tau_{\text{amax}}}, \quad \zeta_{41} = \frac{\tau_{c41\text{max}}}{\tau_{\text{amax}}}
\]

where

\[
\left| \tau_{c41}(\bar{x}_1, \bar{x}_2, \bar{x}_3) \right| \leq \tau_{c41\text{max}}
\]

\[
\tau_{c41}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = -\left[ \tau_{c12}(\bar{x}_1, \bar{x}_2, \bar{x}_3) + \tau_{c23}(\bar{x}_1, \bar{x}_2, \bar{x}_3) + \tau_{c34}(\bar{x}_1, \bar{x}_2, \bar{x}_3) \right]
\]
The synchronization ability coefficient $\zeta_{ij}$ describes the ratio between coupling torque and the average loading torque of the system. The larger the synchronization ability coefficient $\zeta_{ij}$, the greater the amount of system coupling, and the easier to implement synchronization of the system.

**Stability criterion of the synchronous states**

According to equation (3), we can deduce $v_1 = 3/2\bar{x}_1 + \bar{x}_2 + 1/2\bar{x}_3$, $v_2 = -1/2\bar{x}_1 + \bar{x}_2 + 1/2\bar{x}_3$, $v_3 = -1/2\bar{x}_1 - \bar{x}_2 + 1/2\bar{x}_3$ and $v_4 = -1/2\bar{x}_1 - \bar{x}_2 - 3/2\bar{x}_3$. Supposing that $\Delta\nu_i = \nu_i - \nu_0$, linearizing $\Delta\nu_i$ around $\bar{x}_{10}$, $\bar{x}_{20}$ and $\bar{x}_{30}$, we have

\[
\begin{align*}
(\Delta\nu_1)' &= -\sum_{i=1}^{3} \left( \frac{\partial f_{ai}}{\partial \bar{x}_i} \right)_0 (\Delta x_i) \\
(\Delta\nu_2)' &= -\sum_{i=1}^{3} \left( \frac{\partial f_{ai}}{\partial \bar{x}_i} \right)_0 (\Delta x_i) \\
(\Delta\nu_3)' &= -\sum_{i=1}^{3} \left( \frac{\partial f_{ai}}{\partial \bar{x}_i} \right)_0 (\Delta x_i) \\
(\Delta\nu_4)' &= -\sum_{i=1}^{3} \left( \frac{\partial f_{ai}}{\partial \bar{x}_i} \right)_0 (\Delta x_i)
\end{align*}
\]

where

$$\bar{x}_1 = \bar{x}_{10}, \quad \bar{x}_2 = \bar{x}_{20}, \quad \bar{x}_3 = \bar{x}_{30}, \quad \Delta x_i = \bar{x}_i - \bar{x}_0 \quad (i = 1, 2, 3)$$

Arranging equation (23), it can be arranged as

\[
\Delta \dot{x} = D\Delta \bar{x}
\]

with $D = (d_{ij})_{3 \times 3}$,

\[
\begin{align*}
d_{11} &= \left(-\frac{\omega_{m0}}{2}\right) \left[ \eta_1 \eta_2 \left( \frac{1}{k_{11}} + \frac{1}{k_{22}} \right) W_{cc12} \cos(2\bar{x}_{10} + \theta_{c12}) \\
&+ \eta_1 \eta_3 W_{cc13} \cos(2\bar{x}_{10} + 2\bar{x}_{20} + \theta_{c13})/k_{11} \\
&+ \eta_1 \eta_4 W_{cc14} \cos(2\bar{x}_{10} + 2\bar{x}_{20} + 2\bar{x}_{30} + \theta_{c14})/k_{11} \right] \\
\end{align*}
\]

\[
\begin{align*}
d_{12} &= \left(\frac{\omega_{m0}}{2}\right) \left[ \eta_2 \eta_3 W_{cc23} \cos(2\bar{x}_{20} + \theta_{c23})/k_{22} \\
&+ \eta_2 \eta_4 W_{cc24} \cos(2\bar{x}_{20} + 2\bar{x}_{30} + \theta_{c24})/k_{22} \\
&- \eta_1 \eta_3 W_{cc13} \cos(2\bar{x}_{10} + 2\bar{x}_{20} + \theta_{c13})/k_{11} \\
&- \eta_1 \eta_4 W_{cc14} \cos(2\bar{x}_{10} + 2\bar{x}_{20} + 2\bar{x}_{30} + \theta_{c14})/k_{11} \right] \\
\end{align*}
\]

\[
\begin{align*}
d_{13} &= \left(\frac{\omega_{m0}}{2}\right) \left[ \eta_2 \eta_4 W_{cc24} \cos(2\bar{x}_{20} + 2\bar{x}_{30} + \theta_{c24})/k_{22} \\
&- \eta_1 \eta_3 W_{cc13} \cos(2\bar{x}_{10} + 2\bar{x}_{20} + \theta_{c13})/k_{11} \\
&- \eta_1 \eta_4 W_{cc14} \cos(2\bar{x}_{10} + 2\bar{x}_{20} + 2\bar{x}_{30} + \theta_{c14})/k_{11} \right] \\
\end{align*}
\]

\[
\begin{align*}
d_{21} &= \left(-\frac{\omega_{m0}}{2}\right) \left[ \eta_1 \eta_2 W_{cc12} \cos(2\bar{x}_{10} + \theta_{c12})/k_{22} \\
&- \eta_1 \eta_3 W_{cc13} \cos(2\bar{x}_{10} + 2\bar{x}_{20} + \theta_{c13})/k_{33} \right]
\end{align*}
\]
\[d_{22} = \left(-\frac{\omega_{m0}}{2}\right) \left[\eta_2 \eta_3 \left(\frac{1}{k_{22}} + \frac{1}{k_{33}}\right) W_{ee23} \cos(2\varphi_{20} + \theta_{23})
+ \eta_2 \eta_4 W_{ee24} \cos(2\varphi_{20} + 2\varphi_{30} + \theta_{24})/k_{22} + \eta_1 \eta_3 W_{ee13} \cos(2\varphi_{10} + 2\varphi_{20} + \theta_{13})/k_{33}\right]\]

\[d_{23} = \left(-\frac{\omega_{m0}}{2}\right) \left[\eta_3 \eta_4 W_{ee34} \cos(2\varphi_{30} + \theta_{34})/k_{33} - \eta_2 \eta_4 W_{ee24} \cos(2\varphi_{20} + 2\varphi_{30} + \theta_{24})/k_{22}\right]\]

\[d_{31} = \left(-\frac{\omega_{m0}}{2}\right) \left[\eta_1 \eta_3 W_{ee13} \cos(2\varphi_{10} + 2\varphi_{20} + \theta_{13})/k_{33} - \eta_1 \eta_4 W_{ee14} \cos(2\varphi_{10} + 2\varphi_{20} + 2\varphi_{30} + \theta_{14})/k_{44}\right]\]

\[d_{32} = \left(-\frac{\omega_{m0}}{2}\right) \left[\eta_2 \eta_3 W_{ee23} \cos(2\varphi_{20} + \theta_{23})/k_{33} + \eta_1 \eta_3 W_{ee13} \cos(2\varphi_{10} + 2\varphi_{20} + \theta_{13})/k_{33}
- \eta_1 \eta_4 W_{ee14} \cos(2\varphi_{10} + 2\varphi_{20} + 2\varphi_{30} + \theta_{14})/k_{44} - \eta_2 \eta_4 W_{ee24} \cos(2\varphi_{20} + 2\varphi_{30} + \theta_{24})/k_{44}\right]\]

\[d_{33} = \left(-\frac{\omega_{m0}}{2}\right) \left[\eta_3 \eta_4 \left(\frac{1}{k_{33}} + \frac{1}{k_{44}}\right) W_{ee34} \cos(2\varphi_{30} + \theta_{23})
+ \eta_1 \eta_4 W_{ee14} \cos(2\varphi_{10} + 2\varphi_{20} + 2\varphi_{30} + \theta_{14})/k_{44}
+ \eta_2 \eta_4 W_{ee24} \cos(2\varphi_{20} + 2\varphi_{30} + \theta_{24})/k_{44}\right]\]

Assuming that the eigenvalue of matrix \(D\) is \(\lambda\), solving the eigenvalue equation \(\det(D - \lambda I) = 0\), we can obtain the characteristic equation as

\[\lambda^3 - (d_{11} + d_{22} + d_{33})\lambda^2 + (d_{11}d_{33} + d_{22}d_{33} + d_{11}d_{22} - d_{13}d_{31} - d_{23}d_{32} - d_{12}d_{21})\lambda
- (d_{11}d_{22}d_{33} + d_{12}d_{23}d_{31} + d_{13}d_{21}d_{32} - d_{13}d_{23}d_{31} - d_{11}d_{23}d_{32} - d_{12}d_{21}d_{33}) = 0\]

(25)

According to Routh-Hurwitz criterion,\(^{10}\) only when \(\lambda\) has negative real parts, are the solutions of equation (25) stable, i.e. the following conditions

\[H_1 = -d_{11} - d_{22} - d_{33} > 0\]
\[H_2 = d_{11}d_{33} + d_{22}d_{33} + d_{11}d_{22} - d_{13}d_{31} - d_{23}d_{32} - d_{12}d_{21} > 0\]
\[H_3 = -(d_{11}d_{22}d_{33} + d_{12}d_{23}d_{31} + d_{13}d_{21}d_{32} - d_{13}d_{23}d_{31} - d_{11}d_{23}d_{32} - d_{12}d_{21}d_{33}) > 0\]

should be satisfied, i.e.

\[H_1 > 0, \ H_2 > 0, \ H_3 > 0\]

(27)

Here, \(H_1\), \(H_2\), and \(H_3\) are defined as the stability ability coefficients (SAC) of the system, and equation (27) is the stability criterion of the synchronous states, in other words, the vibrating system is stable when equation (27) is satisfied.

**Numerical qualitative characteristic analyses about the coupling dynamics of the system**

In this section, some numerical qualitative discussions are given to validate the above theoretical results. Specific parameters are set as follows: \(m = 1440\ \text{kg}\), \(m_0 = 10\ \text{kg}\), \(r = 0.15\ \text{m}\), \(I_\psi = 1750\ \text{kg} \cdot \text{m}^2\), \(\zeta_{\text{imp}} = 0.07\), \(f_s = 7.6\ \text{kN} \cdot \text{s/m}\), \(k_s = k_y = 40915\ \text{kN/m}\), \(k_\psi = 29385\ \text{kN/m}\). The types of the four motor are the same (three-phase squirrel cage, Model VB-1082-W, 380 V, 50 Hz, 4-pole, \(A\)-connection, 0.75 kW, rotated speed 980 r/min), and their parameters include: rotor resistance \(R_r = 3.40\ \Omega\), stator resistance \(R_s = 3.35\ \Omega\), rotor
inductance $L_r = 170$ mH, stator inductance $L_s = 170$ mH, mutual inductance $L_m = 164$ mH, $f_1 = f_2 = 0.05$. It should be noted that, the selection principle of $b_i$ tends to practical engineering. According to the above parameters, the natural frequencies of the system can be obtained: $\omega_{nx} = \omega_{ny} = 166.3$ rad/s, $\omega_{n\beta} = 129.6$ rad/s.

**Numerical qualitative discussions on synchronization**

The synchronization ability of the four URs is reflected by the synchronization ability coefficients $\zeta_{ij}$ ($i = 12, 23, 34, 13, 24, 41$). The larger the synchronization ability coefficient, the better the synchronization of the system. By changing the masses of URs, the synchronization ability of the four URs is given. As illustrated in Figure 2(a), the synchronization ability coefficient curves of the four identical URs are obtained. Here, $\zeta_{ij}$ represents the synchronization ability coefficient between exciters $i$ and $j$. It is easy to see that the synchronization ability coefficients among the four URs decrease with the increasing frequency $\omega_m$ in sub-resonant region, and reach the peak values in near-resonant region, while in super-resonance region they increase with the increasing of $\omega_m$. Adjusting the mass of UR4, and fixing the masses of URs 1, 2, and 3, we obtain synchronization ability coefficient curves for different URs. They have similar trends in Figure 2(a), and bring about a fluctuation at $\omega_{n\beta}$. As shown in the enlarged views of (a) and (b), it can be seen the fact of $\zeta_{13} = \zeta_{24} > \zeta_{34} = \zeta_{41} > \zeta_{12} = \zeta_{23}$, no matter whether four URs are identical.

**Numerical qualitative discussions on maximum of coupling torques**

The maximum of coupling torques among the four URs is obtained according to the dimensionless coupling torques, which are shown in Figure 3. Here, $\tau_{cij\text{max}}$ is the maximum of coupling torques between UR $i$ and $j$.

Figure 3(a) shows the curves of maximum of coupling torques in the case of the identical four URs. With the increasing of $\omega_m$, whether in sub-resonance or super-resonance region, the dimensionless maximums of coupling torques are in the vicinity of zero, except for points A and B, which are the resonant points, and in this case they reflect a greater peak fluctuation. Similarly, the other maximums of coupling torques among different URs are shown in Figure 3(b), where their variation tendencies are the same as Figure 3(a), and there exists a greater fluctuation at two resonant points. Simultaneously, it can be found that $\tau_{c23\text{max}}$ and $\tau_{c41\text{max}}$ are less than the others at the resonant points, whether the four URs parameters are completely identical or not.

**Numerical qualitative discussions on stability**

The system parameters are inserted into $H_1$, $H_2$, and $H_3$, the SAC of the system are obtained as shown in Figure 4. Figure 4(a) illustrates the SAC curves of the system with four identical URs. It can be seen that the SAC are all greater than zero in both sub-resonant and super-resonant regions. Relatively speaking, the SAC is relatively
greater in super-resonant region. Figure 4(b) shows the SAC curves of the system under the condition of four non-identical URs, the overall trend is consistent with Figure 4(a). It is worth noticing that the stability coefficient is significantly increased in super-resonant region.

Numerical qualitative discussions on phase relationship in the steady-state

For the present dynamical model, the phase relationship among the four URs, is an important index to analyze the synchronization and stability and evaluate the machine’s function.

As shown in Figure 5(a), the phase relationships among the four URs with the same parameter at the steady-state are obtained. Here, $2\alpha_{12}$ represents the phase difference between URs 1 and 2, similarly, $2\alpha_{23}$ denotes that between URs 2 and 3, and $2\alpha_{34}$ is referred to that between URs 3 and 4.

When the system is stable, the phase differences are all near to zero in sub-resonant region, i.e. $2\alpha_{12} = -10$, $2\alpha_{23} = 0$, $2\alpha_{34} = 10$; while the phase differences are in the neighborhood of Pi in super-resonant region.
As shown in Figure 5(b), the stable phase differences among the four non-identical URs are similar with that among four identical ones. In contrast, the excessive values of phase differences appear in near-resonant region. However, there is no excessive value of phase differences when four URs are identical.

**Computer simulations**

Runge-Kutta algorithm is applied to equation (1) for comparing numerical qualitative characteristic results quantitatively. The parameters of the system and motors are listed in section “Numerical qualitative characteristic analyses about the coupling dynamics of the system.” In the simulation process, the parameters of the four URs are completely identical ($g_1 = g_2 = g_3 = g_4 = 1$), the displacement in $x$-, $y$-, and $\psi$-direction by selecting two groups of parameters in sub-resonant and super-resonant regions, respectively.

**Simulations in sub-resonant region**

As shown in Figure 6, here, $k_x = k_y = 80000 \text{ kN/m}$, $k_\psi = 60000 \text{ kN/m}$, $z_x = 0.44$, the four URs operate in sub-resonant state, their synchronous rotational velocities are stabilized in the range of 815–895 r/min. A disturbance ($\pi/6$) is added to the motor 2 at 10 s, and interrupted at 35 s. After the disturbance, the rotational velocities appear some little fluctuations, but soon after it returns to the previous state, which indicates that the interference and power outage have no significant impact on speed of motors, the system is stable.

The curves of phase differences between neighboring exciters are shown in Figure 6(b) to (e). It can be found that the phase differences are stabilized around zero before disturbance. After disturbance, the phase differences have no change, which manifests that the system has a relative strong stability. But, the phase difference associated with motor 2 causes a transition after a power outage. The above results are consistent with the above numeric qualitative analyses.

In Figure 6(f) to (h), the displacements in $x$-, $y$-, and $\psi$-directions are given. Before 10 s, the displacements are 0 mm in $x$- and $\psi$-directions, while the range of displacement is –18.2–18.2 mm in $y$-direction. It can be noted that the system only embodies the vibrations in $y$-direction. After 10 s, a disturbance is given to the motor 2, three displacements still do not change. While the motor 2 is interrupted at 35 s, the displacement in $y$-direction is reduced to –15.8–15.8 mm. At the same time, the motion type of the rigid frame in $x$- and $\psi$-direction is always non-vibration.

**Simulations in super-resonant region**

Here, the parameters are set as: $k_x = k_y = 2000 \text{ kN/m}$, $k_\psi = 2500 \text{ kN/m}$, and $z_x = 2.8$, the simulation results in super-resonant state for $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 1$, are shown in Figure 7. Similarly, the same disturbance and power outage are added to the motor 2.
Figure 6. Simulation results in sub-resonant region for $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 1$: (a) rotational velocities of four motors; (b) phase difference between exciters 1 and 2; (c) phase difference between exciters 2 and 3; (d) phase difference between exciters 3 and 4; (e) phase difference between exciters 4 and 1; (f) displacement in $x$-directions; (g) displacement in $y$-directions; (h) displacement in $\psi$-direction.

Figure 7. Simulation results in super-resonant region for $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 1$: (a) rotational velocities of four motors; (b) phase difference between exciters 1 and 2; (c) phase difference between exciters 2 and 3; (d) phase difference between exciters 3 and 4; (e) phase difference between exciter 4 and 1; (f) displacement in $x$-directions; (g) displacement in $y$-directions; (h) displacement in $\psi$-direction.
Figure 7. Simulation results in super-resonant region for $n_1 = n_2 = n_3 = n_4 = 1$: (a) rotational velocities of four motors; (b) phase difference between exciters 1 and 2; (c) phase difference between exciter 2 and 3; (d) phase difference between exciters 3 and 4; (e) phase difference between exciter 4 and 1; (f) displacement in $x$-directions; (g) displacement in $y$-directions; (h) displacement in $\psi$-direction.
Figure 7(a) shows the rotational velocities of the four motors in the steady state, they are all stabilized at about 983 r/min. After giving the disturbance, the rotational velocities return to the previous states rapidly. Likewise, power outages also have no effect on motor rotational velocities, which indicates that the system has a strong stability.

As illustrated in Figure 7(b) to (e), the phase differences between neighboring URs are all near $\pi$ in the steady state, i.e. $2\pi_{12} = 2\pi_{23} = 2\pi_{34} = 2\pi_{41} = \pi$. When the system is given a disturbance, there appear some fluctuations, however, they finally return to $\pi$ soon after. Similarly, the phase differences generate a transition process after a power outage. The above results indicate that the phase differences of the system approach $\pi$ in super-resonant state and have a relative strong stability.

The displacement curves in $x$, $y$, and $\psi$-direction, are shown in Figure 7(f) to (h), respectively. It is easy to found that the displacements are zero in three directions, no matter how much the disturbance is.

Conclusions

Through the theoretical and numerical analyses, the following conclusions are stressed.

Based on the averaging method, the synchronization criterion of the system is derived. The synchronization ability coefficients among four URs decreases with the increasing of $\omega_{n0}$ in sub-resonant state, while increases in super-resonant state. In addition, it is easy to find that the synchronization ability is approximately considered independent of the masses of URs.

In light of the SAC, they are all greater than zero in both sub-resonant and super-resonant regions when the four URs are identical, and the overall trends are the same as that of the four non-identical URs. It is worth noting that the SAC are significantly increased in near-resonant region.

The results of simulation analyses show that the phase differences are stabilized in the vicinity of zero in sub-resonant state, and the exciting forces are positively superimposed so that the amplitude is relative large and stable. According to our engineering experiences, to achieve the same amplitude, the exciting forces required in sub-resonant state is only 1/5–1/3 of that in super-resonant one, therefore, the energy is saved in this case. These results can be seen as the theoretical foundation for the engineering designs.

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References

1. Senator M. Synchronization of two coupled escapement-driven pendulum clocks. J Sound Vib 2006; 291: 566–603.
2. Orange S and Verdiere N. Nonlinear synchronization on connected undirected networks. Nonlinear Dyn 2014; 76: 47–55.
3. Tanaka HA, Lichtenberg AJ and Oishi S. Self-synchronization of coupled oscillators with hysteretic responses. Phys D 1997; 100: 279–300.
4. Yamapi R and Woafo P. Dynamics and synchronization of coupled self-sustained electromechanical devices. J Sound Vib 2005; 285: 1151–1170.
5. Blekhman II. Synchronization in science and technology. New York: ASME Press, 1988.
6. Blekhman II, Fradkov AL, Tomchina OP, et al. Self-synchronization and controlled synchronization: general definition and example design. Math Comput Simul 2002; 58: 367–384.
7. Blekhman II and Sorokin VS. On the separation of fast and slow motions in mechanical systems with high-frequency modulation of the dissipation coefficient. J Sound Vib 2010; 329: 4936–4949.
Appendix

\textbf{Notation}

\begin{itemize}
  \item $f_i$ damping coefficient of the axes of induction motor $i$, $i = 1, 2, 3, 4$
  \item $f_\phi$ damping constant of the vibrating system in $\phi$-direction, $f_\phi = \left( \frac{f_x I_x^2 + f_y I_y^2}{2} \right)$
  \item $f_x, f_y$ damping constants of the vibrating system in $x$- and $y$-direction
  \item $J_m$ inertia moment of the rigid frame about its mass center
  \item $J$ inertia moment of the vibrating system about its mass center, $J = J_m + (m_1 + m_2 + m_3 + m_4) l_0^2$
  \item $m_i$ mass of the exciter $i$, $i = 1, 2, 3, 4$
  \item $m$ mass of the rigid frame
  \item $m_0$ mass of the standard exciter
  \item $M$ mass of the vibrating system with a rigid frame and four exciters, $M = m + m_1 + m_2 + m_3 + m_4$
  \item $r_m$ mass ratio between the standard exciter and the vibrating system, $r_m = m_0 / M$
  \item $k_\psi$ stiffness of the vibrating system in $\psi$-direction.
  \item $k_x, k_y$ stiffness of the vibrating system in $x$- and $y$-direction
  \item $u_i$ $u_i = 1 - \omega_{ni}^2 / \omega_{n0}^2$, $i = x, y, \psi$
  \item $\beta_i$ angle between the line from the rotary center of the exciter $i$ to the mass center of the vibrating system and $y$-axis
  \item $\gamma_i$ difference between $\pi$ and the phase angle of the vibrating system in $i$-direction, $i = x$.
  \item $\eta_i$ mass ratio between the exciter $i$ and the standard exciter, $\eta_i = m_i / m_0$
  \item $\zeta_{ni}$ critical damping ratio of the vibrating system in $i$-direction, $i = x, y, \psi$
  \item $\zeta_{ji}$ non-dimensional coupling stiffness coefficient for angular velocity between the induction motors $i$ and $j$, $i \neq j$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$
  \item $\chi_{ji}$ non-dimensional moment of coupling inertia between the exciters $i$ and $j$, $i \neq j$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$
  \item $\omega_{n0}$ average angular velocity of the four exciters when the vibrating system operates in the steady-state
  \item $\omega_{ni}$ natural frequency of the vibrating system in $i$-direction, $i = x, y, \psi$
\end{itemize}
Appendix I

\[ \chi_{\alpha} = \omega m_0 [\eta_1^2 W_{s0} + \eta_1 \eta_2 W_{sc12} \cos(2x_1 + \theta_{s12}) \\
+ \eta_1 \eta_3 W_{sc13} \cos(2x_1 + 2x_2 + \theta_{s13}) \\
+ \eta_1 \eta_4 W_{sc14} \cos(2x_1 + 2x_2 + 2x_3 + \theta_{s14})]/2 \]  

\[ (38) \]

\[ \chi_{\alpha_1} = \omega m_0 [\eta_1 \eta_2 W_{cc12} \sin(2x_1 + \theta_{c12}) \\
+ \eta_1 \eta_3 W_{cc13} \sin(2x_1 + 2x_2 + \theta_{c13}) \\
+ \eta_1 \eta_4 W_{cc14} \sin(2x_1 + 2x_2 + 2x_3 + \theta_{c14})]/2 \]  

\[ (39) \]

\[ \chi_{\alpha_2} = \omega m_0 [\eta_1 \eta_2 W_{cc12} \sin(2x_1 + \theta_{c12}) \\
+ \eta_2 \eta_3 W_{cc23} \sin(2x_2 + \theta_{c23}) \\
+ \eta_2 \eta_4 W_{cc24} \sin(2x_2 + 2x_3 + \theta_{c24})]/2 \]  

\[ (40) \]

\[ \chi_{\alpha_3} = \omega m_0 [-\eta_1 \eta_3 W_{cc13} \sin(2x_1 + x_2 + \theta_{c13}) \\
- \eta_2 \eta_3 W_{cc23} \sin(2x_2 + \theta_{c23}) \\
+ \eta_3 \eta_4 W_{cc34} \sin(2x_3 + \theta_{c34})]/2 \]  

\[ (41) \]

\[ \chi_{\alpha_4} = \omega m_0 [\eta_1 \eta_4 W_{cc14} \cos(2x_1 + 2x_2 + 2x_3 + \theta_{c14}) \\
+ \eta_2 \eta_4 W_{cc24} \cos(2x_2 + 2x_3 + \theta_{c24}) \\
+ \eta_3 \eta_4 W_{cc34} \cos(2x_3 + \theta_{c34})]/2 \]  

\[ (42) \]

\[ W_{s0} = r_m \left( \frac{r_0^2 \sin(\gamma_\psi) + \sin(\gamma_s) + \sin(\gamma_y)}{\mu_\psi} \right) \]  

\[ (36) \]

\[ W_{c0} = r_m \left( \frac{r_0^2 \cos(\gamma_\psi) + \cos(\gamma_s) + \cos(\gamma_y)}{\mu_\psi} \right) \]  

\[ (37) \]

\[ W_{scij} = r_m a_{scij} + b_{scij}, \quad i = 1, 2, 3; \quad i < j \leq 4. \]  

\[ (38) \]

\[ W_{ccij} = r_m a_{ccij} + b_{ccij}, \quad i = 1, 2, 3; \quad i < j \leq 4. \]  

\[ (39) \]
\[ a_{scij} = \begin{cases} 
\frac{r_i^2}{u_\phi} \cos(\beta_i + \beta_j) \sin \gamma_\phi - \frac{1}{u_x} \sin \gamma_x + \frac{1}{u_y} \sin \gamma_y, & i = 1, j = 2, 3; \ i = 2, 3, j = 4; \\
\frac{r_i^2}{u_\phi} \cos(\beta_i - \beta_j) \sin \gamma_\phi + \frac{1}{u_x} \sin \gamma_x + \frac{1}{u_y} \sin \gamma_y, & i = 2, j = 3; \ i = 1, j = 4. 
\end{cases} \tag{40} \]

\[ a_{ccij} = \begin{cases} 
- \frac{r_i^2 \cos \gamma_\phi}{\mu_\phi} \cos(\beta_i + \beta_j) + \frac{\cos \gamma_x}{\mu_x} - \frac{\cos \gamma_y}{\mu_y}, & i = 1, j = 2, 3; \ i = 2, 3, j = 4; \\
- \frac{r_i^2 \cos \gamma_\phi}{\mu_\phi} \cos(\beta_i - \beta_j) - \frac{\cos \gamma_x}{\mu_x} + \frac{\cos \gamma_y}{\mu_y}, & i = 2, j = 3; \ i = 1, j = 4. 
\end{cases} \tag{41} \]

\[ b_{scij} = \begin{cases} 
\frac{-r_i^2}{u_\phi} \sin(\beta_i + \beta_j) \sin \gamma_\phi, & i = 1, j = 2, 3; \\
\frac{r_i^2}{u_\phi} \sin(\beta_i + \beta_j) \sin \gamma_\phi, & i = 2, 3, j = 4; \\
\frac{-r_i^2}{u_\phi} \sin(\beta_i - \beta_j) \sin \gamma_\phi, & i = 1, j = 4; \\
\frac{r_i^2}{u_\phi} \sin(\beta_i - \beta_j) \sin \gamma_\phi, & i = 2, j = 3. 
\end{cases} \tag{42} \]

\[ b_{ccij} = \begin{cases} 
\frac{-r_i^2 \cos \gamma_\phi}{\mu_\phi} \sin(\beta_i + \beta_j), & i = 1, j = 2, 3; \\
\frac{r_i^2 \cos \gamma_\phi}{\mu_\phi} \sin(\beta_i + \beta_j), & i = 2, 3, j = 4; \\
\frac{-r_i^2 \cos \gamma_\phi}{\mu_\phi} \sin(\beta_i - \beta_j), & i = 1, j = 4; \\
\frac{r_i^2 \cos \gamma_\phi}{\mu_\phi} \sin(\beta_i - \beta_j), & i = 2, j = 3. 
\end{cases} \tag{43} \]

\[ \theta_{ij} = \begin{cases} \arctan(-b_{scij}/a_{scij}), & a_{scij} \geq 0; \\
\pi + \arctan(-b_{scij}/a_{scij}), & a_{scij} < 0; \\
\ i = 1, 2, 3; \ i < j \leq 4. 
\end{cases} \tag{44} \]