SPECTRAL SHAPE PRESERVING APPROXIMATION

VLADIMIR S. CHELYSHKOV*

Abstract. We introduce an algorithm of joint approximation of a function and its first derivative by alternative orthogonal polynomials on the interval \([0, 1]\). The algorithm exhibits properties of shape preserving approximation for the function. A weak formulation of approximation is presented. An example on shape preserving extrapolation is given. The weak form is reduced to approximation on a discrete set of abscissas.

Key words. Orthogonal polynomials, Gaussian abscissas, shape preserving approximation, weak formulation, polynomial reproduction

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1. Introduction. Theory of shape preserving approximation by polynomials has been developing intensively, and we refer here to survey [9] and monograph [8] for most recent assembled results. In these publications, among topics discussed, much attention was paid to degree of shape preserving approximation with various constrains, and a number of Bernstein-type operators were described (see also paper [10]).

Bernstein polynomials [1] approximate uniformly any continuous function and provide shape preserving approximation. Bernstein operator is linear and monotone. Approximation by the polynomials admits differentiations, and convergence of the polynomial expansion even for smooth functions is slow.

In this paper, a distinct, less general approach for shape preserving approximation is developed. We use joint approximation of a function and its first derivative by alternative orthogonal polynomials [4] (see also [6]) to formally construct an operator for shape preserving approximation of the function. Evidently, the operator is not valid to represent the Weierstrass approximation theorem, it has \(n\)-th degree polynomial reproduction property and exhibits faster convergence for smooth functions.

A few numerical examples on approximation are given, but rigorous mathematical arguments qualifying our statements on shape preserving approximation and on the rate of convergence are not presented in this paper.

The paper is self-contained for numerical implementation of the algorithm.

2. A-kind system. We use a system of alternative orthogonal polynomials

\[
A_n(x) = \{A_{nk}(x)\}_{k=n}^{0}
\]  

(2.1)

for constructing the approximation algorithm. The system has nice properties, which are similar to properties of the classical orthogonal polynomials [4]. Some properties are shown below.

The polynomials \(A_{nk}(x)\) obey the orthogonality relations

\[
\int_0^1 \frac{1}{x} A_{nk}(x) A_{nl}(x) dx = \frac{\delta_{kl}}{k+l}, \quad k = n, n-1, \ldots, 0, \quad l = n, n-1, \ldots, 1,
\]  

(2.2)

but the polynomial \(A_{n0}(x)\) is not normalizable with the given weight. Thus, (2.1) is a marginal system that contains a singular term \(A_{n0}(x)\). From properties of the system it follows that \(A_{n0}(x)\) are shifted to the interval \([0, 1]\) Legendre polynomials.

*University of Pikeville, 147 Sycamore St., Pikeville, KY 41501 (VolodymyrChelyshkov@upike.edu).
The polynomials can be calculated by the three-term recurrence relation
\[ A_{n,n}(x) = x^n, \quad A_{n,n-1}(x) = (2n-1)x^{n-1} - 2nx^n, \]
\[ (2k+1)(n+k)(n-k+1)A_{n,k-1}(x) \]
\[ = 2k[(2k-1)(2k+1)x^{-1} - 2(n^2 + k^2 + n)]A_{nk}(x) \]
\[ - (2k-1)(n-k)(n+k+1)A_{n,k+1}(x). \]

Also,
\[ A'_{nk}(x) = k \frac{A_{nk}(x)}{x} + 2 \sum_{l=k+1}^{n}(-1)^{l-k}l \frac{A_{nl}(x)}{x}, \quad k = 1, ..., n, \] (2.3)

and
\[ A_{nk}(1) = (-1)^{n-k}. \] (2.4)

The system \( A_n(x) \) generates an alternative Gauss-type quadrature with the given weight function. The quadrature is exact for \( x^m, 1 \leq m \leq 2n \) on the interval \([0, 1] \). This results in the second (discrete) orthogonality property
\[ \sum_{j=1}^{n} w_j x_j A_{nk}(x_j) A_{nl}(x_j) = \frac{\delta_{kl}}{k+l}, \quad k, l = n, n-1, ..., 1 \] (2.5)

where \( w_j \) and \( x_j \) are the weights and abscissas of the shifted Legendre-Gauss quadrature.

**Fig. 2.1. Basis functions: \( n = 5, k = 1 - 5 \).**

We form an almost orthogonalized polynomial basis
\[ A_n(x) = \{1, A_{nk}(x)\}_{k=0}^{n-1} \] (2.6)
in vector space \( \Pi_n \) of polynomials of degree less or equal than \( n \) and introduce a set
\[ (A_n \cup A_{n0}) (x). \] (2.7)

for function approximation on the interval \([0, 1] \).

Making use of the basis \( A_n(x) \) and orthogonality (2.2), or discrete orthogonality (2.5) on the abscissas \( x_j \) of \( A_{n0}(x) \), one can approximate a function by, correspondingly, minimization error in a space of quadratically integrable functions with interpolation at \( x = 0 \), or interpolation at \( n + 1 \) points \( \{0, x_j\}_{j=1}^{n} \).
3. Spectral approximation

We initially suppose that \( f(x) \in C^1([0, 1]) \) and expand the function \( f(x) \) in sums using the basis \( A_n(x) \) as follows

\[
f(x) = f(0) + f_0(x), \quad f_0(x) = \lim_{n \to \infty} f_n(x),
\]

where

\[
f_n(x) = \sum_{k=1}^{n} a_{nk} A_{nk}(x), \quad a_{nk} \equiv a_{nk}(f_0).
\]

To find coefficients \( a_{nk} \) we consider finite projection of

\[
f_0'(x) = \lim_{n \to \infty} f_n'(x) = \lim_{n \to \infty} \sum_{k=1}^{n} a_{nk} A_{nk}'(x)
\]
on \( \{A_{nl}(x)\}_{l=1}^{n} \). Making use of (3.1), (2.3) and (2.2) we get a system of linear equations

\[
\sum_{k=1}^{n} T_{lk} a_{nk} = b_{nl},
\]

where

\[
T_{lk} = \begin{cases} 
0, & l < k, \\
1/2, & l = k, \\
(-1)^{l-k}, & l > k
\end{cases}
\]

and

\[
b_{nl} = \int_0^1 f_0'(x) A_{nl}(x) dx, \quad b_{nl} \equiv b_{nl}(f_0').
\]

Solving equations (3.2) for \( a_{nk} \), substituting the solution to (3.1), and applying simple transformations we obtain

\[
f_n(x) = 2 \sum_{k=1}^{n} b_{nk} B_{nk}(x),
\]

where

\[
B_{nk}(x) = A_{nk}(x) + 2 \sum_{l=k+1}^{n} A_{nl}(x).
\]

Additionally, from (3.4) it follows

\[
B_{n0}(x) = 1.
\]

It is worthy of notice

\[
B_{nk}'(x) = k \frac{A_{nk}(x)}{x}, \quad k = 1, 2, \ldots, n,
\]

\[\footnote{We identify the meaning of the term “spectral approximation” with that one given in [3], page 31.}
and we find that (3.4) and (3.5) form integral co-basis \( B_n(x) = \{1, 2B_{nk}(x)\}_{k=1}^n \) in \( \Pi_n \) with
\[
B_{nk}(x) = k \int_0^x \frac{A_{nk}(t)}{t} \, dt, \quad k = 1, 2, \ldots, n.
\]

Essentially, basis \( B_n(x) \) is not orthogonal, but the system \( B'_n(x) \) is orthogonal
\[
\int_0^1 xB'_{nk}(x)B'_{nl}(x) \, dx = \frac{k+l}{4} \delta_{kl}, \quad k, l = 1, 2, \ldots, n,
\]
and the systems \( A_n(x) \) and \( B'_n(x) \) are mutually orthogonal
\[
\int_0^1 A_{nk}(x)B'_{nl}(x) \, dx = \frac{\delta_{kl}}{2}, \quad k, l = 0, 1, 2, \ldots, n.
\]

![Fig. 3.1. \( B_{nk}(x), n = 5, k = 0 - 5. \)](image)

From (3.6) and (2.2) it follows that (3.8) is the result of expansion of \( f'(x) \) in \( A_{nk}(x)/x \), and
\[
f(x) = f(0) + 2 \lim_{n \to \infty} \sum_{k=1}^n b_{nk}B_{nk}(x), \quad (3.7)
\]
\[
f'(x) = 2 \lim_{n \to \infty} \sum_{k=1}^n b_{nk}B'_{nk}(x). \quad (3.8)
\]
is joint approximation of the function \( f(x) \) and its first derivative by the basis \( A_n(x) \) and the integral co-basis \( B_n(x) \).

Let us consider three examples on function approximation on the interval \([0, 1]\) by expansion (3.7).

For \( n = 3 \) we have
\[
\ln(1 + x) \approx (342 - 492\ln 2)x - (645 - 930\ln 2)x^2 + \left(\frac{1040}{3} - 500\ln 2\right)x^3,
\]
\[
1 - \sin(\pi x) \approx 1 + \frac{12}{\pi^2}((17\pi^2 - 180)x - (35\pi^2 - 360)x^2 + (20\pi^2 - 200)x^3),
\]

\[^2\text{At this point, the algorithm is suggestive of a spectral method for solving an initial value problem.}\]
for \( n = 5 \)

\[
\sqrt{x} \approx \frac{2}{11} (15x - 35x^2 + 56x^3 - 45x^4 + 14x^5).
\]

The examples represent low degree approximation of a monotonic function, of an even convex function by a polynomial of odd degree, as well as approximation of a monotonic function that is not differentiable at the left end of the interval \([0, 1]\). Graphs of all the three expose shape preserving approximation.

One may state that the expansions admit one time differentiation as \( n \to \infty \), and the derivatives of the functions and their approximations have points of intercepts. For, say, \( n = 3 \) and \( f(x) = 1 - \sin(\pi x) \) the points are \( x_1 = 0.213063, x_2 = 0.585763, \) and \( x_3 = 0.907986 \). Since \( x_1, x_2 \) and \( x_3 \) cannot be determined unless the expansion is obtained, distribution of the nodes in the interval \((0, 1)\) depends on shape of the derivative at large.

In this paper, we use the term “concealed interpolation” for approximation that results in interpolation of a function in \( O(n) \) non-preassign nodes as \( n \to \infty \).

4. Weak formulation, composition and asymmetry. Joint approximation \((3.7) – (3.8)\) can be expressed in terms of \( f(x) \) as follows.

We introduce the operator

\[
\Omega_n(f; x) := f(0) + 2 \sum_{k=1}^{n} \int_{0}^{x} \frac{A_{nk}(s)}{s} ds \int_{0}^{1} f'_0(t)A_{nk}(t) dt
\]

that corresponds to approximation \((3.7)\). Integrating second integral in \((4.1)\) by parts and applying \((2.3)\) and \((2.4)\) we find

\[
\int_{0}^{1} f'_0(t)A_{nk}(t) dt = (-1)^{n-k} f_0(1) - c_{nk}/2 - \sum_{l=k+1}^{n} (-1)^{l-k} c_{nl}, \tag{4.2}
\]

where

\[
c_{nk} = 2k \int_{0}^{1} \frac{1}{t} f_0(t)A_{nk}(t) dt, \quad k = 1, \ldots, n. \tag{4.3}
\]

Making use of \((4.2)\) we explicate operator \((4.1)\) in the weak form

\[
\Omega_n(f; x) := f(0) + \sum_{k=1}^{n} a_{nk}A_{nk}(x)
\]

with

\[
a_{nk} = (-1)^{n-1} \cdot 2f_0(1) + \sum_{l=1}^{n} S_{kl} c_{nl}, \tag{4.4}
\]

and

\[
S_{kl} = \begin{cases} 
-1, & k = l \quad \text{odd}, \\
3, & k = l \quad \text{even}, \\
(-1)^l \cdot 2, & k \neq l.
\end{cases} \tag{4.5}
\]

That is,

\[
\Omega_n(f; x) := f(0) + 2 \sum_{k=1}^{n} A_{nk}(x) \left( (-1)^{n-1} f_0(1) + \sum_{l=1}^{n} S_{kl} \int_{0}^{1} \frac{1}{t} f_0(t)A_{nl}(t) dt \right). \tag{4.6}
\]
Also, following property of orthogonality (2.2), we find that evaluation of coefficients \( c_{nk} \) results in approximation of \( f_0(x) \) by

\[
\varphi_n(x) := \sum_{k=1}^{n} c_{nk} A_{nk}(x)
\]  

(4.7)
in weighted “conditional” \( L^2[0, 1] \), and (4.7) yields the operator

\[
\hat{\Omega}_n(f; x) := f(0) + 2 \sum_{k=1}^{n} A_{nk}(x) k \int_0^1 \frac{1}{t} f_0(t) A_{nk}(t) dt.
\]  

(4.8)

We can state now that mapping coefficients \( c_{nk} \) to \( a_{nk} \) by linear transformation (4.4), (4.5) forms a composition of approximations \( \hat{\Omega}_n(f; x) \) and \( \Omega_n(f; x) \).

Let us consider an example on the composition. For \( n = 4 \) operator (4.8) results in expansion

\[
\sin(\pi x) \approx \frac{6(3\pi^2-28)}{\pi^4} A_{41}(x) + \frac{6}{\pi^4} A_{42}(x) + \frac{6(\pi^2-20)}{\pi^4} A_{43}(x) + \frac{8(\pi^2-6)}{\pi^4} A_{44}(x),
\]  

(4.9)

and we observe that coefficients in (4.9) represent eigenvector of matrix (4.5) with eigenvalue 1. In general, this pattern of approximation can be described as follows.

Let \( f_0(x) \) be an even/odd function, and \( n \) is an even/odd number. Then

\[
\hat{\varphi}_n(x) := \hat{\Omega}_n(f_0; x)
\]
is symmetric concealed interpolation of \( f_0(x) \) and

\[
\Omega_n(\hat{\varphi}_n; x) \equiv \hat{\varphi}_n(x).
\]  

(4.10)

We find that identity (4.10) is reminiscent of our primary choice of approximation of \( f(x) \) in the form

\[
f(x) \equiv f(0) + \int_0^x f'(x) dx, \quad f(x) \in C^1[0, 1],
\]

and the algorithm does not provide symmetric shape preserving approximation for symmetric \( f_0(x) \).

Remarkably, \( \{A_{nk}(x)\}_{k=1}^{n} \) is the system of asymmetric functions, and one can change the given pattern of approximation by distorting its symmetry, that is, by choosing odd/even degree of approximation \( n \) for even/odd function \( f_0(x) \). Then \( \hat{\Omega}_n(f_0; x) \) represents asymmetric concealed interpolation, and \( \Omega_n(f_0; x) \) stands for asymmetric shape preserving approximation. Hereinafter we consider this option for \( n \) as a component of the algorithm. Accordingly, no special choice of \( n \) is required for an asymmetric \( f_0(x) \), for any asymmetric function can be represented as a sum of even and odd function.

With the pattern selected, we find that the composition of approximations consists of two different types of approximation by the same system of functions.

It was mentioned above that expansion of \( f(x) = \sqrt{x} \) by operator \( \Omega_n(f; x) \) is one time differentiable, and it is easy to verify that expansion of the same function by \( \hat{\Omega}_n(f; x) \) is not.

We infer that operator (4.6) may be applied for approximation of functions from a wider class, say, \( f(x) \in C \cap BV([0, 1]) \), as well as for approximation of the derivative \( f'(x) \) where it exists on \([0, 1] \).

\[\text{3Below we suppose that the second integral in (4.1) is Henstock-Kurzweil integral.}\]
5. An example on shape preserving extrapolation. In this section we numerically compare the two types of approximation of a particular function of $C^\infty([0,1])$ class. We choose

$$f(x) = \sin\left(\frac{\pi}{2} x\right), \quad f_1(x) = \hat{\Omega}_9(f;x), \quad f_2(x) = \Omega_9(f;x), \quad x \in [0,1].$$

Numerical results for $n = 9$ are presented in Fig. 7.1. The function $f_1(x)$ (yellow curve) represents $L_2$-approximation with concealed nodes of interpolation in the interval $(0, 1)$ and the node at $x = 0$, whereas shape preserving approximation $f_2(x)$ (green curve) intersects $f(x)$ only at $x = 0$. The two curves are visually very close to the graph of $\sin\left(\frac{\pi}{2} x\right)$ (red curve) in a much longer interval, say, $[-1, 2]$. One may also observe that shape preserving approximation provides a “better” extrapolation, for it is in a compliance with the $f(x)$ if $x \in [-1.5, -1] \cup [2, 2.5]$. We see that shape preserving approximation may lead to a better result near turning points outside of the original interval.

The observation is more of general interest, rather than of computational implementation, since calculation of orthogonal polynomials outside of the interval of orthogonality meets difficulty related to a number of significant figures required.

6. Discretization and pseudo-basis representation. Let us consider a discrete weak formulation of joint approximation. Lagrange interpolation in the Gaussian nodes of a function from the above selected class converges uniformly [11], and we employ discrete orthogonality property (2.5) to interpolate the $f_0(x)$ by the polynomial

$$\psi_n(x) := \sum_{k=1}^n d_{nk} \mathcal{A}_{nk}(x) \quad (6.1)$$

with

$$d_{nk} = 2k \sum_{i=1}^n \frac{w_j}{x_j} f_0(x_j) \mathcal{A}_{nk}(x_j).$$

Following (6.1) we introduce the operator

$$\hat{\mathcal{W}}_n(f; x) := f(0) + 2 \sum_{j=1}^n f_0(x_j) \mathcal{Q}_{nj}(x), \quad (6.2)$$

$$\mathcal{Q}_{nj}(x) := \frac{w_j}{x_j} \sum_{k=1}^n k \mathcal{A}_{nk}(x_j) \mathcal{A}_{nk}(x)$$
that interpolates $f(x)$ in $n$ Gaussian nodes and in $x = 0$.

Making use of the interpolant $\psi_n(x)$ for evaluating integrals in (4.6) we finally obtain the operator of discrete approximation of $f(x)$

$$W_n(f; x) := f(0) + 2 \sum_{k=1}^{n} A_{nk}(x) \left( (-1)^{n-1} f_0(1) + \sum_{l=1}^{n} S_{kl} l \sum_{j=1}^{n} \frac{w_j}{x_j} f_0(x_j) A_{nl}(x_j) \right). \quad (6.3)$$

Again, linear transformation (4.4), being applied to $d_{nk}$, maps interpolation (6.2) to shape preserving approximation (6.3), the operators form a composition of two discrete approximations by the same system of polynomials, and the nodes of interpolation are abscissas for the joint approximation.

Formalism (4.6), (6.1), (6.3) represents exact projection that unites weak formulation and collocation method; it was introduced as compound spectral formalism in [5] for solving a problem on linear theory of hydrodynamic stability in ordinary differential equations.

Operator (6.3) can be expressed in similar to (6.2) form as

$$W_n(f; x) = f(0) P_{n0}(x) + \sum_{j=1}^{n} f_0(x_j) P_{nj}(x) + f_0(1) P_{n,n+1}(x), \quad (6.4)$$

$$P_{n0}(x) = 1, \quad P_{n,n+1}(x) = 2(-1)^{n-1} \sum_{k=1}^{n} A_{nk}(x),$$

$$P_{nj}(x) = 2 \frac{w_j}{x_j} \sum_{k=1}^{n} A_{nk}(x) \sum_{l=1}^{n} S_{kl} A_{nl}(x_j).$$

where $\{P_{nj}(x)\}_{j=0}^{n+1}$ is a pseudo-basis, and $P_{n,n+1}(x)$ is considered as a linear combination of $P_{nj}(x), 0 \leq j \leq n$. Approximation by (6.4) possesses $n$-th degree polynomial reproduction property; it interpolates $f(x)$ only at $x_0 = 0$, and approximation of the first derivative of $f(x)$ is a concealed interpolation.

The polynomials $P_{nj}(x)$ $j = 0, 1, ..., n$ are not orthogonal. They have no zeros in the interval $[0, 1]$, increase in absolute value for $j > 0$ as $n$ increases, and alternate in sign with respect to $j$ in $[0, 1]$. That is, representation of $W_n(f; x)$ in form of (6.4) may result in numerical instability for greater $n$, and the algorithm of discrete approximation should be implemented in its original form (6.3).

7. Convergence. Obviously, approximation of $f_0(x)$ by $\varphi_n(x)$ may not be distinguished as a separate part of the algorithm; furthermore, interpolation of $f_0(x)$ by $\psi_n(x)$ is not the necessary step of the discrete variant of the algorithm. Indeed, coefficients $d_{nk}$ are the result of evaluation of $c_{nk}$ by the $n$-th order alternative Gauss quadrature in (4.3). So, both continuous and discrete variants of the algorithm may be considered without introducing the compositions.

At this point we recognize the algorithm as a whole entity and assume that the most general class of functions for approximation of $f(x)$ might be defined in the following way.

Conjecture. Let $f(x)$ be a continuous almost everywhere differentiable function in the interval $[0, 1]$. Then approximation of $f(x)$ by the operator $\Omega_n(f; x)$ in weak formulation (4.6) and its discrete analog $W_n(f; x)$ converges uniformly in $[0, 1]$. 

V. S. CHELYSHKOV
8. Conclusion. Presented algorithm of joint approximation does not provide interpolation of a function at the right end of the interval, but example on extrapolation of a smooth function shows that low degree approximation may be good enough in the closed interval $[0, 1]$ without interpolation at the endpoint. Thus, the algorithm may originate polynomial curves for solving problems of isogeometric analysis [2].

Formally, the algorithm can be reconstructed for shape preserving approximation on a half-line, and it can be done in two different ways.

Firstly, the system of orthogonal exponential functions $E_{nk}(t) := A_{nk}(e^{-t})$, $n > 0$, described in [4, 6] can be employed for reformulation of the algorithm. One can find that the rearranged version of the algorithm provides approximation for functions $f(t) \in C^1[0, \infty)$, $f(t) \sim e^{-\alpha t}$ as $t \to \infty$, and $\alpha > 0$.

Secondly, reformulation can be performed by choosing introduced in [7] system of orthogonal rational functions $R_{nk}(t) := A_{nk}(t^{-1})$. Then, this version of the algorithm can be employed for approximation of functions $f(t) \in C^1[1, \infty)$ and $f(t) \sim t^{-\alpha}$ as $t \to \infty$.

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