Constraining the evolutionary history of Newton’s constant with gravitational wave observations

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Space-borne gravitational wave detectors, such as the proposed Laser Interferometer Space Antenna, are expected to observe black hole coalescences to high redshift and with large signal-to-noise ratios, rendering their gravitational waves ideal probes of fundamental physics. The promotion of Newton’s constant to a time-function introduces modifications to the binary’s binding energy and the gravitational wave luminosity, leading to corrections in the chirping frequency. Such corrections propagate into the response function and, given a gravitational wave observation, they allow for constraints on the first time-derivative of Newton’s constant at the time of merger. We find that space-borne detectors could indeed place interesting constraints on this quantity as a function of sky position and redshift, providing a constraint map over the entire range of redshifts where binary black hole mergers are expected to occur. A LISA observation of an inspiral event with redshifted masses of $10^4 - 10^5$ solar masses for three years should be able to measure $\dot{G}/G$ at the time of merger to better than $10^{-11}$ yr$^{-1}$.

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I. INTRODUCTION

The principle of equivalence is at the cornerstone of classical physics as evidenced by the opening paragraph of Newton’s Principia [1]. Constructing a relativistic theory of gravity consistent with this principle was one of the key inspirations that led Einstein to general relativity (GR) [2, 3]. The Strong Equivalence Principle (SEP) states that all physics must satisfy: (i) the Weak Equivalence Principle (WEP), (ii) local Lorentz invariance and (iii) local position invariance. The WEP states that test-particle trajectories are independent of their internal structure, while the latter two properties require experimental results to be independent of the 4-velocity and position of the laboratory frame.

Many known alternative theories of gravity violate the SEP [4, 5]. A classic example of which are scalar-tensor theories [6]. These theories modify the Einstein-Hilbert Lagrangian by multiplying the Ricci scalar by a field $\phi$ and adding a kinetic and a potential term (in the Jordan frame). Scalar-tensor theories are well-motivated from a theoretical standpoint, as they seem unavoidable in unification models inspired by particle-physics. In string theory, for example, there are numerous (dilatonic) scalar fields that couple to the scalar curvature, and perhaps also to the matter sector, then leading to additional violations of the WEP [7, 8]. In loop quantum gravity, the Barbero-Immirzi parameter that controls the minimum eigenvalue of the area operator could be promoted to a field, leading to a classical coupling of Einstein gravity with a scalar-field stress-energy tensor [9, 10].

From an observational standpoint, we also possess evidence for interesting physics that could point to a modification of GR on cosmological scales, the prime example of which is the late-time acceleration of the universe. The cause of this acceleration is usually interpreted within the framework of pure GR as “dark energy”, a new form of matter that is or behaves similar to a cosmological constant. This form of matter has only recently (redshifts less than of order unity) become important in governing the evolution of the universe. Another interpretation of the observed acceleration is that it is purely a gravitational phenomenon, though the theory governing the dynamics of spacetime is not GR, but a new or modified theory that has cosmological solutions that mimic a cosmological constant within pure GR. It is therefore of some interest to search for modifications to GR, in particular as a function of redshift.

Often, gravitational theories that violate the SEP predict a varying gravitational constant. In the scalar-tensor theory case, one can effectively think of the scalar as promoting the coupling between gravity and matter to a field-dependent quantity via $G \rightarrow G(\phi)$, thus violating local position invariance as $\phi$ varies. Another interesting example are higher-dimensional, brane-world scenarios, where although physics remains Lorentz invariant in higher dimensions, “gravitational leakage” into the bulk inexorably leads to a time-varying effective 4D gravitational constant [11, 12]. A constraint on the magnitude of this variation can also be interpreted as a constraint on the curvature radius of extra dimensions [13].

Existing experiments searching for varying $G$ can be classified into two groups: (a) those that search for the present or nearly present rate of variation (at red-
shifts close to zero); (b) those that search for secular variations over long time periods (at very large redshifts). The most stringent bounds of the first class come from lunar ranging observations [10] and pulsar timing observations [17, 18], but also include planetary radar-ranging [19] and surface temperature observations of low-redshift millisecond pulsars [21, 22]. The strongest bounds of the second class come from Big Bang Nucleosynthesis (BBN) calculations [22, 23], but also include the evolution of the Sun [24]. The order of magnitude of the strongest constraint from either class is about $G/G \lesssim 10^{-13} \, \text{yr}^{-1}$, but it varies with the experiment.

In this article, we show that gravitational wave (GW) observations of binary compact object coalescence can also provide constraints on the variation of $G$. We concentrate on these coalescences because of their high expected event rate and signal-to-noise ratio (SNR) with the proposed Laser Interferometer Space Antenna (LISA) [27]. We consider both nearly equal-mass, supermassive black hole (SMBH) merger events (with total mass $M \sim 10^6 M_\odot$), as well as extreme-mass-ratio, quasi-circular inspirals of a stellar-mass black hole (BH) into a SMBH. Only BHs are considered here because they lack internal structure that could lead to additional violations of the WEP [3] and modifications of the multipolar structure of the gravitating body. From this standpoint, BH binaries are cleaner probes of fundamental physics.

One interesting difference between previous tests of $G$ and GW-based ones is that we are using black holes, i.e. vacuum solutions, instead of neutron stars. As mentioned earlier, scalar-tensor theories can be interpreted as variable-$G$ theories, but in that case the variable $G$ is really only a variation in the coupling between gravity and matter. The gravitational constant $G$, however, serves more fundamentally as the quantity that defines the relationship between geometry (length) and energy, and such a quantity is not altered in scalar-tensor theories. We here consider the possibility that this fundamental constant varies with time. In this sense, we are looking at more generic variable-$G$ theories, instead of restricting attention to theories that can reduce to a scalar-tensor theory. In Appendix A, we discuss the relation between generic variable-$G$ theories and scalar tensor ones in more detail.

Another interesting difference between previous tests of $G$ and the GW tests proposed here is that the latter constrain $G/G$ at the time and location of the merger event. Therefore, once a sufficient number of events have been observed, a constraint map could be constructed, which would encode bounds on $G/G$ along our past light-cone, i.e. as a function of redshift $z$ and sky position, out to a redshift of order 10 and limited mostly by the likelihood of merger events at that redshift. In contrast, the first class of tests discussed above can only bound $G/G$ near redshift zero, while the BBN calculations only constrain linear changes in $G$ from $z = 0$ to $z \gg 10^3$ and they are degenerate with limits on the number of relativistic species. GW constraint maps would thus be excellent probes of local position invariance and the SEP, complementing previous bounds in a regime where the latter are inaccessible.

Constraint maps are generic and applicable to any parametrizable GR deviation in the waveform, provided one possesses GW observations from sources at sufficiently large distances. The concept of GW tests, of course, is not new (see e.g. [26–39] for a discussion of GW tests of Brans-Dicke and massive graviton theories), but until now constraint maps had not been really considered. Such maps add a new dimension to the nature of GW tests; this is in contrast to most traditional tests that can only search for deviations in our immediate spacetime neighborhood, or search for particular deviations that are by fiat constant over the age of the Universe. In this sense, GW constraint maps can provide invaluable global information on the SEP and local position invariance that is unattainable with Solar System observations.

We find that a variability in Newton’s constant introduces modifications to the phase of the GW response function of an inspiraling binary that could be measurable by LISA to an interesting level. For comparable-mass BH inspirals with total redshifted mass $M_\odot = 10^6 M_\odot$, one could constrain $G/G \lesssim 10^{-9} \, \text{yr}^{-1}$ or better to redshift 10 (using an SNR of $10^3$ [40]), while for extreme mass-ratio inspirals with the same total mass, one could place roughly the same constraint out to redshift 0.3 (using an SNR of $10^2$ [41]). Such a bound becomes stronger for intermediate mass BH inspirals, allowing constraints on the order of $10^{-11} \, \text{yr}^{-1}$. Even though these bounds are, at a first glance, weaker than those from existing tests, the latter only constrain secular variation in $G$, or oscillatory changes if we happen to be near a maximum of $G$ today. GW observations are in principle sensitive to a much larger class of variations in $G$, if a sufficient number of events are detected to build a meaningful constraint map.

We also find that there is an equivalence between the vacuum, varying-$G$ model that we consider here and a time-varying chirp mass. Such an equivalence is established via the mapping $(G/G)M \rightarrow M$. In view of this, one can think of the tests presented here as probes of the time-constancy of the BH mass or horizons. Such time-variation could be induced, for example, due to gravitational leakage into the bulk in braneworld models [13]. Therefore, the results presented here are truly a fundamental GW test of the gravitational interaction.

An outline of the remainder of the paper is as follows. Section II describes the parameterization chosen for the functional dependence Newton’s constant. Section III derives how the inspiral is modified in post-Newtonian (PN) theory with a time-varying $G$ to $O(1/c^3)$, where $c$ is the speed of light (this is commonly referred to as an expansion to 1.5PN order). Section IV estimates the accuracy to which $G/G$ could be measured by LISA, performing a simplified Fisher analysis and a time-domain dephasing study, using both frequency-domain PN waveforms and modified EOB, numerical templates. Section V con-
cludes with some discussion of future extensions of this work. In Appendix [A] we further discuss the mapping between scalar-tensor theories and a variable-G theory, while in Appendix [B] we comment on possible degeneracies due to cosmological expansion.

Henceforth, we use the following conventions: we work exclusively in four spacetime dimensions with signature $(-, +, +, +)$ [42]; Greek letters ($\mu, \nu, \ldots$) in index lists range over all spacetime indices; Latin letters ($a, b, \ldots$) in index lists range over all template parameter indices $\lambda^a$; overhead dots stand for partial differentiation with respect to the time coordinate, $\dot{E} := \partial E/\partial t$; the Einstein summation convention is employed unless otherwise specified; finally, we use geometrized units $G = c = 1$ where possible, with $c$ the speed of light in vacuum, but we shall mostly retain factors of Newton’s constant $G$, as this quantity will be promoted to a time-function.

II. PARAMETERIZATION OF THE VARIABILITY OF NEWTON’S CONSTANT

In constructing the GW signal, we do not want to restrict attention to a particular alternative theory. Rather, we will assume that, during inspiral, the modification can be captured by simply promoting $G$ to a function of time in the GR equations of motion: $G \rightarrow G(t)$. This is analogous to promoting $G$ to a time-function in Newton’s second law, a topic much studied in the past [43][46]. The $G$-varying framework we consider here can thus be thought of as an effective theory that could represent any alternative theory that leads to such a modification. In this section we discuss how we parameterize $G(t)$.

At a given merger event, we will simply describe the evolution of the gravitational constant by

$$G(t, x, y, z) \approx G_c + \dot{G}_c (t_c - t), \quad (1)$$

which is a Taylor expansion of $G(t, x, y, z)$ to first order in time $t$ and position $(x, y, z)$ about the coalescence event $(t_c, x_c, y_c, z_c)$, where $G_c = G(t_c, x_c, y_c, z_c)$, and $\dot{G}_c = (\partial G/\partial t)(t_c, x_c, y_c, z_c)$. Such an expansion is only valid provided the spatial variation in $G$ is much smaller than the temporal variation, and the characteristic period of the temporal variation is longer than the observation window, the latter of which is on the order of a few years for LISA.

Previous bounds on $G(t)$ have assumed linear deviation from the present day value. In turn, this allowed BBN calculations to place such stringent constraints on $\dot{G}$, by exploiting the huge “lever arm” provided by integrating over cosmic time. However, any such bound will be largely insensitive to oscillatory components in $G$ that have periods much less that the cosmic observation time. With the parameterization chosen in Eq. (1), such oscillations can be captured, provided their frequency is much smaller than the inverse of the LISA observation time.

Before proceeding, one might wonder how a variable $G$ can lead to modifications to a vacuum spacetime, such as a BH inspiral. Newton’s constant appears explicitly on the right-hand of Einstein’s equations, $G_{\mu\nu} = 8\pi G/c^4 T_{\mu\nu}$, as a coupling constant between geometry, encoded by the Einstein tensor $G_{\mu\nu}$, and matter, given by the stress energy tensor $T_{\mu\nu}$. If $T_{\mu\nu} = 0$, as is the case in vacuum, then one might naively conclude that a variable $G$ would not introduce any modification. In fact, this is essentially the case in scalar-tensor theories (see Appendix [A]), where the no-hair theorem is protected in vacuum by the scalar field’s evolution equation.

Newton’s constant, however, plays a much more fundamental role than merely a coupling constant: it defines the relationship between energy and length. For example, in the vacuum Schwarzschild solution, the relationship between the radius $R$ of the black hole and the rest-mass energy $E$ of the spacetime is $R = 2G(E/c^4)$. Similarly, in a binary BH spacetime, the BHs have introduced an energy scale into the problem that is quantified by a specification of Newton’s constant$^2$. In our PN implementation of a varying $G$, these issues are avoided as $G$ explicitly couples to a point-particle stress-energy tensor (the PN representation of a BH), and thus, it explicitly appears in the equations of motion. Henceforth, we treat the variable-G modifications as induced by some effective theory and we do not concern ourselves with the specific model from which such modifications arise.

III. POST-NEWTONIAN EXPANSION OF THE MODIFIED GW SIGNAL

In the above parameterization of $G(t, x, y, z)$ we have assumed that the spatio-temporal variation of $G$ is sufficiently long to only induce a secular evolution over the duration of the observed inspiral and merger. Thus, the effect on the waveform will build up slowly over many cycles. We can then ignore the relatively short coalescence and ringdown phases of the merger to a good approximation, and concentrate only in the inspiral. Modifications to the inspiral waveform can then be analyzed within PN theory; here we shall carry all expansions out to $O(1/c^3)$ (ie. 1.5PN order).

The time-dependence of Newton’s constant modifies the generation of GWs due to corrections to the balance law during inspiral. Consider then a binary BH system, represented by PN point-particles with binding energy $E = -\eta M c^2 x/2$, where $\eta = m_1 m_2/M^2$ is the symmetric mass ratio, $M = m_1 + m_2$ is the total mass, with $m_1, 2$ the binary’s component masses, and $x = (2\pi GMF/c^3)^{2/3}$.

$^2$ In a full numerical implementation, one could effectively model a time-varying $G$ by introducing a time-varying length-scale, which is most easily done via a time-varying speed of light $c(t)$. One must point out, however, that such modifications are unrelated to the variable speed of light theories of [47][48].
with \( F(t) \) the orbital frequency. The balance law states that the rate of change of the binary’s binding energy \( \dot{E} \) is equal and opposite to the GW luminosity \( \mathcal{L}_{GW} = 32c^5 \eta^2 x^5 / (5G) \), where the latter can be obtained from Isaacson’s stress-energy tensor \([49, 50]\). The promotion of \( G \to G(t) \) in the conservative energy or Hamiltonian is comparable to a modification to Newton’s second law, while this promotion in the GW flux is a modification to the dissipative sector of the theory.

With the parameterization of Sec. \( \text{II} \) the balance law can be rewritten as the following, modified differential equation for the orbital frequency:

\[
\dot{F} = \frac{\dot{G}_c}{G_c} F + \frac{48}{5\pi c^5} (G_c M)^{-1} x^{11/2} \eta^{11/5} \\
\times \left\{ 1 - \left( \frac{743}{336} + \frac{11}{4} \eta \right) x + 4\pi x^{3/2} \right\} \\
- \frac{5 \dot{G}_c}{3 G_c} (t - t_c) \left[ 1 - \left( \frac{743}{240} + \frac{77}{20} \eta \right) x + \frac{32}{5} \pi x^{3/2} \right],
\]

(2)

where \( M = \eta^{3/5} M \) is the chirp mass. By construction, this expansion is valid to 1.5 PN order, i.e., for orbital velocities \( v \ll c \) to relative \( \mathcal{O}(v^2) \). This expression can be simplified by using the zeroth-order in \( \dot{G}_c \) relationship between time and frequency:

\[
t_{GR}(F) = t_c - \frac{5}{256} G_c M x_c^{-4} \eta^{-8/5} \left[ 1 + \left( \frac{743}{252} + \frac{11}{3} \eta \right) x_c - \frac{32\pi}{5} x_c^{3/2} \right],
\]

(3)

which then renders Eq. (2) into

\[
\dot{F} = \frac{48}{5\pi c^5} (G_c M)^{-1} x^{11/2} \eta^{11/5} \\
\times \left\{ 1 - \left( \frac{743}{336} + \frac{11}{4} \eta \right) x + 4\pi x^{3/2} \right\} \\
+ \frac{65}{768} G_c M x^{-4} \eta^{-8/5} \left[ 1 - \left( \frac{743}{13104} + \frac{11}{156} \eta \right) x \right].
\]

(4)

The promotion of \( G \) to a time-function with \( M \) constant is similar to the promotion of \( M \) to a time-function with \( G \) constant. One can establish this similarity by noting that the evolution of the GW luminosity function and binding energy is governed by the PN quantity \( x \), which is always given in terms of the product \( G M \). If one were to assume that \( M \) varied but not \( G \), one would obtained the same type of evolution equation for the frequency [Eq. (2)], but with a slightly different prefactor. One can then qualitatively think of a varying \( G \) as a varying \( M \), or equivalently a varying chirp mass.

The differential equation in Eq. (2) can be solved perturbatively in \( G_c M \ll 1 \) up to errors of \( \mathcal{O}(G_c^2 M^2) \):

\[
t(F) = t_c - \frac{5}{256} G_c M \eta^{-8/5} x_c^{-4} \left[ 1 + \left( \frac{743}{252} + \frac{11}{3} \eta \right) x_c - \frac{32\pi}{5} x_c^{3/2} \right] \\
- \frac{65}{1536} \dot{G}_c M \eta^{-8/5} x_c^{-4} - \frac{55}{768} \left( \frac{743}{252} + \frac{11}{3} \eta \right) G_c M \eta^{-8/5} x_c^{-3} + \frac{5\pi}{12} \dot{G}_c M \eta^{-8/5} x_c^{-5/2},
\]

(5)

where \( x_c = x \) is evaluated at \( G(t) = G_c \). One might also expect corrections to quadrupole emission (such as dipolar or monopolar emission) due to a varying Newton’s constant. Such non-quadrupolar emission could only be sourced by energy carried away by the \( G \) field. However, if we expect modified \( G \)-theories to still have well-defined, conserved energies, then the effective energy associated with \( G \) would scale as \( G_c^2 \), which is sub-leading. If this behavior is generic, we can ignore such corrections, and we do so henceforth.

Note that the PN equations are ill-behaved at the time of coalescence \( t_c \). Though we will stop the integration of the equations before this time, one might be concerned about the validity of Taylor expanding Newton’s constant about \( t_c \) as in Eq. (1). This may have been an issue if we had taken into account possible back-reaction effects of the orbital motion onto the local value of \( G \), which we are not, as just discussed. Hence, to this order, it is valid to picture \( G(t) \) as an independent, cosmological field, whose functional behavior is not tied to the orbital motion of the binary.

A modification to the frequency directly affects the GW phase evolution and thus the response function. The Fourier transform of this function in the stationary-phase approximation \([51]\) becomes \( \hat{h}(f) = A(f) \exp(i\Psi) \), with
\[ A(f) = -\frac{\sqrt{30\pi}}{48} \frac{(GcMz)^2}{D_L} Q(\iota, \beta) u^{-7/6} \left\{ 1 - \frac{5}{512} \frac{\dot{G}cM_z}{Gc} u^{-8/3} \left[ 1 + \left( \frac{3715}{6048} + \frac{55}{72} \right) \eta^{-3/5} u^{2/3} + 2\pi \eta^{-3/5} u \right] \right\} \] (6)

\[ \Psi = 2\pi f_c - \phi_c + \frac{3}{128} u^{-5/3} \left\{ 1 + \left( \frac{3715}{756} + \frac{55}{9} \right) \eta^{-2/5} u^{2/3} - 16\pi \eta^{-3/5} u \right\} - \frac{25}{1536} \frac{\dot{G}cM_z}{Gc} u^{-8/3} \left[ 1 + \left( \frac{743}{126} + \frac{22}{3} \right) \eta^{-2/5} u^{2/3} - \frac{64\pi}{5} \eta^{-3/5} u \right] \] (7)

where \( u \equiv \pi GcM_z f \) is a dimensionless parameter with \( f \) the GW frequency, \( M_z = (1 + z)M \) is the redshifted chirp mass, and \( \phi_c \) is a constant phase of coalescence. \( Q(\iota, \beta) \) is some function of the inclination angle \( \iota \) and the polarization angle \( \beta \), which depends on the beam-pattern functions \([51]\). In Eq. (6), we have omitted all the known PN amplitude corrections of GR and we have only explicitly written out the modifications to the amplitude induced by a time-varying \( G \).

The frequency dependence of the \( \dot{G}c \) modification \( (f^{-13/3}) \) is unique within the space of known alternative theories of gravity \([52]\). Notice that the controlling factor of the expansion is changed by the \( \dot{G}c \) terms from \( u^{-5/3} \) to \( u^{-13/3} \), which implies that to obtain corrections proportional to positive powers of the reduced frequency \( u \), one would need to include corrections induced by 4PN terms. Comparing Eq. (7) to the analogous Brans-Dicke modification, we first notice that the \( \dot{G}c \) term couples to \( M \eta^{-2} \) to leading order instead of to a positive power of the symmetric mass ratio \( \eta \), as in Brans-Dicke theory. In the latter, dipolar radiation activates when there are differences in the gravitational and inertial centers of mass (WEP violation), which requires objects with different self-gravitational binding energies (sensitivities) and mass ratios. On the other hand, the \( \dot{G}c \) modification can be present in the absence of matter (i.e. it modifies the point-particle representation of a BH spacetime). In spite of these differences, the leading-order \( \dot{G}c \) modification can be recovered by the simplest form of the recently proposed parameterized post-Einsteinian framework \([52]\), with the leading-order post-Einsteinian parameters

\[ \alpha = \frac{5}{512} \frac{\dot{G}c}{Gc}(GcM_z), \quad a = -\frac{8}{3} \]
\[ \beta = -\frac{25}{65536} \frac{\dot{G}c}{Gc}(GcM_z), \quad b = -\frac{13}{3} \] (8)

**IV. PLACING A BOUND ON \( \dot{G} \)**

With the insight gained from the analytic study of the previous section, we shall now concentrate on producing estimates of the accuracy to which \( \dot{G_c}/G_c \) could be measured with a LISA GW observation. In Sec. [IV.A] we briefly describe the basics of a dephasing accuracy estimate and of a Fisher estimate. In Sec. [IV.B] we then proceed to calculate estimates using the PN expansion found in Sec. [III]. The resultant expressions, computed using only the leading order PN terms, give insight into the scaling behavior of the estimates with system parameters. More accurate estimates can be obtained if one includes higher order PN terms in the phasing. In Sec. [IV.C] we compute the bounds on a \( \dot{G_c}/G_c \) measurement using numerically generated waveforms that employ a certain parameterization of the PN equations of motion and include higher order PN terms. We discuss all the different estimates in Sec. [IV.D].

**A. Accuracy Estimates**

One criterion that can be used to estimate the accuracy to which \( G_c \) can be measured is a dephasing measure. This measure requires that one compute the GW phase difference between a waveform with \( G_c = G_0 \) and \( G_c = 0 \). When this phase difference is equal to the inverse of the SNR \( 1/\rho \), then one estimates that a \( G_c \) correction of magnitude \( G_0 \) or larger could be detected. Such a measure follows from the analysis of \([53, 54]\).

A more popular measure for determining the accuracy to which parameters could be estimated given a GW observation is via a Fisher analysis. Consider then the Fourier transform of the response function \( \tilde{h}(f, \lambda^a) \) as a frequency-series that depends on certain parameters \( \lambda^a \) that characterize the system. In our case, these parameters define a 6-dimensional space, as we have neglected spin, that is spanned by \( \lambda^a = (A_0, f_c, \phi_c, M, \eta, \dot{G}_c) \), where \( A_0 \) is the first term of Eq. (6) without the \( f^{-7/6} \) factor. In reality, there are additional (so-called extrinsic) parameters that also enter the response function and characterize the motion of the detector around the Sun through time-dependent beam-pattern function. One of the approximations we shall make here is to average over all angles so that the detector’s proper motion can be ignored.

In the Fisher scheme, the accuracy to which a template parameter \( \lambda^a \) can be measured is given by \( \Delta \lambda^a = \sqrt{\Sigma_{nn}} \), where \( \Sigma_{nn} \) is the \( (n, n) \) element of the covariance matrix (no summation over \( n \) implied), defined as the inverse of the Fisher matrix

\[ \Gamma_{ab} = 4 \Re \int_{f_{lo}}^{f_{hi}} \frac{df}{S_n(f)} \frac{\partial \tilde{h}}{\partial \lambda^a} \frac{\partial \tilde{h}^*}{\partial \lambda^b}, \] (9)
with the asterisk denoting complex conjugation. The quantity $S_n(f)$ is the spectral noise density, which describes the noise in the observation. We shall here employ the $S_n$ of Eq. (6), which includes white-dwarf confusion noise. The upper limit of integration $f_{hi}$ is chosen to be the minimum of the innermost stable circular orbit (ISCO) frequency $(6^{3/2} \pi M_\odot)^{-1}$ and 1 Hz, while for the lower limit of integration $f_{lo}$ we choose the maximum of the frequency 1 year prior to reaching the ISCO and $10^{-5}$ Hz (see eg. [32] for further details).

Another approximation we make here is to ignore correlations between $G_c$ and other parameters. With this approximation at hand, the Fisher matrix becomes diagonal and its inverse tells us how well parameter estimation is observed, we should be able to place a constraint on the constraint, but we relegate such considerations to future work. In our case, the Fisher analysis will reveal how much degeneracies deteriorate the constraint, and its inverse tells us how well parameter $\lambda^n$ can be measured: $\Delta \lambda^n \sim (\Gamma_{nn})^{-1/2}$. In our case, if no $\dot{G}_c$ modification is observed, we should be able to place a constraint of approximately $\Delta \dot{G}_c \equiv \dot{G}_c/G_c \lesssim (\Gamma_{\dot{G}_c G_c})^{-1/2}$.

The estimate obtained from $\sqrt{\Sigma_{nn}}$, with the considerations described above, is what we refer to as a simplified Fisher analysis in this paper.

Correlations generically tend to weaken the accuracy to which parameters can be measured. In our case, the inclusion of the mass ratio seems to deteriorate the correlation, but it is unclear whether this is the case. This is because such effects appear at high PN order, multiplied by a high power of the GW frequency and become important in the late inspiral, while the $\dot{G}_c$ corrections modify the leading-order Newtonian term, multiplied by a different power of the GW frequency, and dominate during the very early inspiral. Only a full Fisher analysis will reveal how much degeneracies deteriorate the constraint, but we relegate such considerations to future work.

### B. Stationary Phase, 1.5 PN Estimates

Let us now apply these measures to the frequency-domain PN waveform found in Sec. (III). We begin with the dephasing measure and make use of the fact that the GW frequency to leading PN order and zeroth-order in $G_c$ is given by

$$f = \frac{5^{3/8}}{8\pi} M_z^{5/8} T^{-3/8} + \mathcal{O}(\dot{G}_c),$$

where $T$ is the integration time. Inserting this into Eq. (7), retaining only the leading-order $G_c$ term and requiring that $\Psi(\dot{G}_c) - \Psi(\dot{G}_c = 0) = 1/\rho$ leads to the bound

$$\Delta \dot{G}_c \lesssim \frac{8^{3/8}}{5^{3/8}} \frac{1}{\rho} \eta^{3/8} M_z^{5/8} T^{-13/8},$$

where $M_z = M(1 + z)$ is the redshifted total mass. In turn, this bound leads to the estimates

$$\left( \Delta \dot{G}_c \right)_{\text{EMRI}} \lesssim \frac{3 \times 10^{-8}}{\text{yr}} \left( \frac{100}{\rho} \right) \left( \frac{\eta}{10^{-5}} \right)^{3/8} \times \left( \frac{M_z}{10^6 M_\odot} \right)^{5/8} \left( \frac{T}{1\text{yr}} \right)^{-13/8},$$

$$\left( \Delta \dot{G}_c \right)_{\text{SMBH}} \lesssim \frac{1.5 \times 10^{-7}}{\text{yr}} \left( \frac{1000}{\rho} \right) \left( \frac{\eta}{1/4} \right)^{3/8} \times \left( \frac{M_z}{10^6 M_\odot} \right)^{5/8} \left( \frac{T}{1\text{yr}} \right)^{-13/8}.$$

Clearly, the longer we observe the signal and the higher its SNR, the stronger the bound we can place, eg. a 3-year observation or an order of magnitude decrease in the total mass gains us roughly an order of magnitude on the constraint for each.

Let us now perform a simplified Fisher analysis with the waveform of Sec. (III). To obtain less complicated expressions where the scaling with system parameters is apparent, we employ the so-called restricted PN approximation, where one neglects all amplitude corrections to the waveform. We find that the accuracy to which $G_c$ can be measured is approximately given by

$$\Delta \dot{G}_c \lesssim \frac{65536}{25} \frac{1}{\rho} \frac{\eta^2}{M_z} (\pi M_z f_0)^{13/3} \frac{1}{\sqrt{J_{33}}}$$

where again we have retained only the leading-order $\dot{G}_c$ correction to the GW phase in Eq. (7). In Eq. (13) we have defined the reduced moment $J_p = I_p / I_7$ and the moment of the distribution $58$

$$I_p \equiv \int_{x_{lo}}^{x_{hi}} dx \frac{x^{-p/3}}{S_n(x f_0)},$$

with the dimensionless variable $x = f / f_0$ and the arbitrary normalization constant $f_0 = 10^{-4}$ Hz. Equation (13) can be used to derive the following scaling rules:

$$\left( \Delta \dot{G}_c \right)_{\text{EMRI}} \lesssim \frac{4 \times 10^{-8}}{\text{yr}} \left( \frac{100}{\rho} \right) \left( \frac{\eta}{10^{-5}} \right)^2 \times \left( \frac{M_z}{10^6 M_\odot} \right)^{10/3},$$

$$\left( \Delta \dot{G}_c \right)_{\text{SMBH}} \lesssim \frac{2 \times 10^{-5}}{\text{yr}} \left( \frac{1000}{\rho} \right) \left( \frac{\eta}{1/4} \right)^2 \times \left( \frac{M_z}{10^6 M_\odot} \right)^{10/3}.$$

The scaling of this bound with total mass and mass ratio is different to that of Eq. (12), because we have dropped in Eq. (15) the $J_{33}$ dependence, which itself depends on observation time, total mass and mass ratio.

It is somewhat surprising that the Fisher and dephasing estimates are quite consistent for EMRIs, though
much less so for SMBH mergers. One reason for this is that we have neglected sub-leading PN corrections in the frequency-domain waveform of Eq. 7, which one would expect to have a much larger effect for equal mass systems. To check this, we can compute the sub-leading PN corrections to \( \Gamma_{G_c} \), which leads to the bound

\[
\Delta G_c \lesssim \frac{65536}{25} \frac{1}{\rho} \frac{M_z}{M} (\pi M_{c} f_0)^{13/3} \frac{1}{\sqrt{J_{33}}} |1 - \frac{743}{126} \left( \frac{22}{3} \eta \right) (f_0 M \pi)^{2/3} J_{31} | J_{33} + \frac{64\pi}{5} (\pi M_{c} f_0) J_{30}. \tag{16}
\]

For a system with the parameters \( M_z = 10^6 M_\odot, \eta = 1/4 \) and \( \rho = 1000 \), Eq. (16) gives \( \Delta G_c \lesssim 8 \times 10^{-7} \text{ yr}^{-1} \), which is more than an order of magnitude stronger than that obtained in Eq. (15). This result gives us a sense of the importance of the inclusion of higher-order corrections to the GW phase and it indicates that the SMBH bound in Eq. (15) may be unreliable (note that due to the \( \eta \) dependence in (16) the EMRI estimate is largely unaffected). With this in mind, the dephasing and simplified Fisher measures suggest that roughly the same order of magnitude constraint on \( G_c \) from either EMRIs or SMBHs could be obtained by LISA.

C. Effective-One-Body Estimates

In the previous subsection we have seen that the inclusion of higher-order corrections can influence the estimated accuracy to which \( G_c \) can be measured. For this reason, in this subsection we calculate a more accurate GW phase, obtained via numerical methods through the so-called effective-one-body (EOB) formalism \([56–69]\). Essentially, one numerically evolves the bodies’ trajectories by solving the Hamilton-Jacobi equations for a certain resummed, high-order PN Hamiltonian. The waveform is then constructed via another resummed PN expression that depends on the evolved trajectories. We shall not here review the EOB framework, but instead we follow the prescription described in detail in \([70–72]\). In particular, we employ the uncalibrated 5.5PN order EOB model, including \( \eta \)-deformations of \([73, 74]\), although we find our results are rather insensitive to the specific EOB model chosen.

The \( G_c \) modification is straightforward to implement in the EOB scheme. The conservative dynamics (the Hamiltonian) depend on reduced coordinates and conjugate momenta (denoted by overhead hats), which are normalized to the total mass and mass ratio \( \mu = \eta M \) respectively. The \( G_c \) corrections to the Hamiltonian can then be modeled by rescaling certain reduced quantities by \( G(t) \) (see eg. Sec. III of \([72]\))^3: \( t \to \hat{t} MG(t), r \to \hat{r} MG(t), \phi \to \hat{\phi} M\mu G(t) \). The dissipative dynamics (the radiation-reaction force) can be modeled by a certain Pade resummation of the energy flux, which itself is a function of the orbital velocity. The \( G_c \) corrections to the radiation-reaction force can then be modeled by reinstanting the factors of \( G \) and employing Kepler’s third law, which states \( v = [G(t)M\omega]^{1/3} \), i.e. the \( \phi \)-component of this force is rescaled via \( F_\phi(v) \to F_\phi [G(t)]^{1/2} \) (see eg. Sec. IIIb in \([57]\)). In this way, the Hamiltonian and the dissipative force become explicit functions of time, leading to modified Hamilton-Jacobi equations with new terms proportional to \( G_c \), whose numerical solution allows us to compute \( G_c \)-modified EOB waveforms.

We begin first by applying the dephasing measure to these EOB waveforms. Figure 4 plots the difference in the GW phase as a function of time in units of months for different values of \( G_c \). Clearly, as \( G_c \) is chosen to be larger, the difference in GW phase also increases. With the criteria described above, these evolutions suggest LISA would be able to detect a \( G_c \sim 10^{-9} \text{ yr}^{-1} \) when observing extreme-mass ratio inspirals (EMRIs) with \( \rho \sim 250 \) or SMBH mergers with \( \rho \sim 3300 \). As expected from Eq. (7), EMRI systems are preferred over SMBH/SMBH coalescences due to the extra factors of \( \eta \) multiplying the \( G_c \) correction (or intuitively, due to the slower inspiral rate, an EMRI spends many more cycles in the strong-field regime prior to merger, enhancing the secular deviations induced by \( G_c \)).

Let us now apply a numerical version of the simplified Fisher analysis to the modified EOB waveforms. Via Parseval’s theorem, we can convert a frequency-domain integration into a time-domain integration. The Fisher matrix, however, contains the spectral noise density in the denominator, which makes this conversion non-trivial. We shall here treat this density as an averaged quantity that we evaluate at the time \( t(f) \). With this in mind, we can rewrite the \( (G_c, G_c) \) element of the \( \rho^2 \)-normalized Fisher matrix as

\[
\hat{\Gamma}_{G_c G_c} = \frac{4}{\rho^2} \int_0^{1 \text{yr}} dt \frac{S_n[f(t)]}{S_n} \left| \frac{\partial h}{\partial G_c} \right|^2, \tag{17}
\]

and the SNR is simply

\[
\rho^2 = 4 \int_0^{1 \text{yr}} dt \frac{S_n[f(t)]}{S_n} |h|^2. \tag{18}
\]

The pseudo-averaging of the spectral noise density is similar to what is done in the stationary phase approximation, where the generalized Fourier integral that defines the Fourier transform is evaluated at the stationary

---

3 Since we study quasi-circular orbits, we can neglect the \( \theta \) evolution. The Hamilton-Jacobi equations do not acquire overall multiplicative \( G_c \) corrections, as this can be cast completely in terms of dimensionless quantities. The only \( G_c \) corrections arise via the rescaling of the coordinates and momenta inside the effective Hamiltonian.
point\(^4\). The accuracy to which \(\dot{G}_c\) can be measured is then simply \(\Delta \dot{G}_c \lesssim \rho^{-1}(\hat{\Gamma}_{\dot{G}_c \dot{G}_c})^{-1/2}\).

We compute the relevant element of the Fisher matrix numerically. The partial derivatives in Eq. (17) are approximated via the rule \(\partial h / \partial \dot{G}_c \sim [h(\dot{G}_0) - h(0)] / \dot{G}_0\) for some very small \(\dot{G}_0\). We have calculated the Fisher estimate for different values of \(\dot{G}_0 \in \{10^{-10}, 10^{-7}\} \text{ yr}^{-1}\) and found the constraint to be robust to such changes. The waveforms employed are aligned in an initial time window, in the low-frequency domain: we have time and phase-shifted them so that they agree to at least one part in \(10^5\) during the first day of evolution. Such a calibration amounts to a matched filtering maximization of the SNR in white noise over the time and phase of coalescence \(^6\) (see \(71\) for more details on this procedure).

Using this numerical Fisher prescription and the modified EOB code, we find that for an equal-mass, SMBH inspiral with \(M_z = 10^6 \text{ M}_\odot\), \(\dot{G}_c\) could be measured to \(\Delta \dot{G}_c \lesssim 3.5 \times 10^{-9} \text{ yr}^{-1}(1000 / \rho)\). Similarly, for an extreme-mass ratio inspiral with the same total mass and mass ratio \(\eta = 10^{-5}\), \(\dot{G}_c\) could be measured to \(\Delta \dot{G}_c \lesssim 4.5 \times 10^{-9} \text{ yr}^{-1}(1000 / \rho)\). These estimates are in rough agreement with the graphical dephasing measure presented in Sec. [IV.C] as well as with the frequency-domain simplified Fisher analysis of Sec. [IV.B] keeping in mind the various approximations employed.

\(^4\) Ref. \([13]\) has verified that this time-domain implementation of a Fisher analysis produces very similar results to the standard frequency-domain implementation, when the Fourier transform of the waveform in the latter is treated in a stationary-phase approximation.

\(^6\) Ref. \([69]\) has verified that this time-domain implementation of a Fisher analysis produces very similar results to the standard frequency-domain implementation, when the Fourier transform of the waveform in the latter is treated in a stationary-phase approximation.

### D. Comparing Estimates

We have found that the dephasing and simplified Fisher measures are rather consistent among themselves. Differences, however, arise when considering a Taylor-expanded, stationary-phase approximated, 1.5PN order waveform versus a modified EOB wave. These similarities and differences can be better appreciated in Table I. At some level, we expect the EOB waveforms to lead to a better bound, as they include many more \(\dot{G}_c\) corrections than the analytic 1.5PN model. As already found in Sec. [IV.B] these higher-order PN terms have more of an impact on comparable mass systems than on EMRIs.

Overall then, the numerical estimates suggest a typical GW inspiral could be used to constrain \(G_c / G_e\) to a level of \(\sim 10^{-9} \text{ yr}^{-1}\). The best bounds come from intermediate mass BH inspirals, e.g., for an inspiral with redshifted masses \((10^4 : 10^6) \text{ M}_\odot\), one can constrain \(G_c / G_e \lesssim 10^{-11} \text{ yr}^{-1}\), assuming an SNR of \(10^5\) and an observation time of 3 years. Such a constraint may even become stronger once spin and eccentricity are included, as then one can possibly obtain more cycles in the sen-
since the typical, observable solar mass merger events of these events, together with the shorter integration LIGO/VIRGO/GEO. However, the lower expected SNR GW observations of solar mass BH mergers with constituents in $\dot{\gamma}$.

Nevertheless, as explained in the Introduction, a sufficiently large number of observations will allow for the construction of a map of the constraints on $\dot{\gamma}$ as a function of redshift and sky position, a task that is impossible with Solar System observations. As such, GW tests constitute a new, complementary class of $\dot{\gamma}$ tests, allowing for a unique opportunity to build a constraint map during the cosmological epoch of structure formation.

V. CONCLUSIONS

The promotion of Newton’s constant to a time-dependent quantity introduces modifications in the GW frequency and phase of an inspiraling binary. We have analytically computed the $\dot{\gamma}$ modifications to the response function in the stationary phase approximation to 1.5PN order. We have also numerically solved a $\dot{\gamma}$-modified version of the EOB equations of motion to estimate the magnitude of the $\dot{\gamma}$ corrections to higher PN order. We have used both a dephasing and a Fisher measure to obtain order of magnitude estimates of the possible constraints that could be placed with LISA given a GW observation.

An observation of one merger event could be used to constrain deviations from $\dot{\gamma} = \text{const.}$ at the spacetime location of merger. A single GW observation of an equal-mass inspiral event with total redshifted mass of $10^9 M_\odot$ for 3 years should be able to constrain $\dot{\gamma} c / \dot{\gamma} c$ to $10^{-11} \text{yr}^{-1}$.

Given a sufficient number of GW observations from events at different sky locations and redshifts, one could in principle construct a constraint map of $\dot{\gamma}$ as a function of sky angles and redshift. Conversely, if $\dot{\gamma}$ is in truth not constant, and deviations are measurable, GW observations could allow for a reconstruction of $\dot{\gamma}(t, x, y, z)$ along our past lightcone. Solar system bounds constrain $\dot{\gamma}$ at redshift zero, while cosmological BBN calculations bound a secular drift in $\dot{\gamma}$ from redshift 0 to $z \gg 10^4$. With these restrictions, existing bounds are a couple of orders of magnitude stronger than putative GW observations; however, GWs will allow for entirely new constraints on $\dot{\gamma}$ at intermediate redshifts, and thus as more events are observed throughout the universe observations will become increasingly sensitive to oscillatory components in $\dot{\gamma}$ (with periods greater than of order a year).

Certainly, the same techniques described here could be used to place constraints on $\dot{\gamma}$ from ground-based GW observations of solar mass BH mergers with LIGO/VIRGO/GEO. However, the lower expected SNR of these events, together with the shorter integration time, imply the bounds would be much weaker than existing bounds or putative LISA bounds. Furthermore, since the typical, observable solar mass merger events will occur at relatively low redshifts, ground-based detectors will also not be able to offer a measurement that is complimentary to the already strong solar system constraints.

Future work could concentrate on performing a more detailed data analysis of the accuracy to which $\dot{\gamma} c$ can be measured, in particular taking into account possible degeneracies with other GR binary merger parameters. We have estimated that such correlations will deteriorate the accuracy to which $\dot{\gamma}$ can be constrained by roughly an order of magnitude, when degeneracies with the chirp mass and symmetric mass ratio are included, a result in qualitative agreement with [32]. The inclusion of additional physical effects, such as spin, eccentricity or higher-order PN harmonics, however, might break some of these degeneracies and restore some of the strength in the constraint [37][38]. The spin corrections, for example, first enter in the phase at 1.5 PN order via a spin-orbit interaction, and as such are only dominant close to merger, while the $\dot{\gamma}$ modification also affects the early inspiral. Even if these degeneracies deteriorate the bound dramatically, however, the GW constraint remains unique in its sensitivity to non-linear behavior in $\dot{\gamma}$, if a sufficient number of detections are made to measure $\dot{\gamma}$ at different locations in the universe.

Another interesting avenue to pursue would be to investigate how the bounds on a varying $\dot{\gamma}$ could be strengthened if in addition to the GW signal an electromagnetic (EM) counterpart was observed. When dealing with pure vacuum spacetimes, such an EM signature is not likely, but recent work on numerical simulations of mergers embedded in circumbinary disks suggest that EM signals might be emitted [74][75]. Given a coincident EM/GW detection, one would then be able to determine the position of the source [80] and its redshift [81][82], while given a cosmology, in the form of a distance-redshift relationship, one would then be able to independently estimate $\dot{\gamma}$ at the source. The tandem GW/EM observation would thus simultaneously constraint $\dot{\gamma}$ and $\dot{\gamma}$ at the merger event.

One last path to explore is the possible degeneracies between the $\dot{\gamma} c$ effect and the GR time-dependence of redshift $z$, induced by the expansion of the Universe. Both effects lead to a modification to the GW phase that is proportional to the same power of the GW frequency, making these effects exactly degenerate. We performed a preliminary exploration of such a degeneracy in Appendix [E] where we found that when cosmological information is employed, the $z$ effect can be subtracted off to still allow for bounds on $\dot{\gamma}$ at the levels discussed in this paper.

Note added after submission: Shortly after this paper was submitted, one [34] appeared in preprint considering the possibility of constraining the size of string-theory inspired extra-dimensions, which induce a time-dependence in the BH mass. As we discussed earlier, such an effect is equivalent to a varying $\dot{\gamma}$, with the mapping $\dot{\gamma} c / \dot{\gamma} c \rightarrow M / M$. Because the extra-dimension effect
is strongest for the smallest BHs, we note that EMRIs are the best systems to explore this effect with GWs. A direct GW detection could place a bound on the size of the extra-dimension of $\ell \gtrsim 1$ micrometers, assuming a constraint on $M/M$ on the order of $10^{-9}$ yr$^{-1}$. Such a constraint can be improved by one order of magnitude if the system is observed for more than one year.

VI. ACKNOWLEDGEMENTS

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Appendix A: Generic Variable-G Theories and Scalar-Tensor Models

In this appendix, we discuss the relation between the generic variable-G theories we consider in this paper and one particular scalar-tensor theory: Brans-Dicke [32]. In Brans-Dicke theory the action is given by

$$S = \int d^4 x \sqrt{-g} \left( \frac{\phi}{16\pi G} R - \frac{\omega_{BD}}{\phi} \partial_\mu \phi \partial^\mu \phi \right) + S_{\text{mat}}, \quad (A1)$$

where $g$ is the determinant of the metric, $R$ is the Ricci scalar, $\phi$ is a scalar field, $\omega_{BD}$ is a coupling constant, $\text{commas stand for partial differentiation and } S_{\text{mat}}$ stands for additional matter degrees of freedom. This action leads to modified field equations, as well as an evolution equation for the scalar field:

$$\Box_g \phi = \frac{8\pi}{3 + 2\omega_{BD}} \left( T - 2 \phi \frac{\partial T}{\partial \phi} \right), \quad (A2)$$

where $\Box_g$ is the D’Alembertian operator of curved spacetime and $T$ is the trace of the matter stress-energy tensor.

Brans-Dicke theory is often interpreted as a variable-$G$ theory by defining a new Brans-Dicke gravitational parameter via $G_{BD} = G \phi^{-1}$. Thus, if $\phi$ varies in spacetime, so does the effective coupling parameter $G_{BD}$. However, note that Newton’s constant $G$ is still exactly that: a constant. What is changing is the coupling between other forms of matter and gravity, as mediated by $\phi$.

In order to illustrate this point, consider the original weak-field solution found by Brans and Dicke describing the geometry and scalar field outside a stationary spherically symmetric body of mass $M$ [80]:

$$\phi = \phi_0 + \frac{2M}{r} G (3 + 2\omega_{BD})^{-1},$$
$$g_{00} = -1 + \frac{2M G}{r \phi_0} [1 + (3 + 2\omega_{BD})^{-1}],$$
$$g_{ii} = 1 + \frac{2M G}{r \phi_0} [1 - (3 + 2\omega_{BD})^{-1}], \quad i = 1, 2, 3$$
$$g_{ij} = 0, \quad i \neq j \quad (A3)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. From the above, the identification of an effective $G$ via the action gives

$$G_{BD} = \frac{G}{\phi_0 + 2GM r^{-1} (3 + 2\omega_{BD})^{-1}}. \quad (A4)$$

Clearly, $G_{BD}$ varies with distance from the source and it asymptotes to $G_{BD} \sim G \phi_0^{-1}$ in the limit $M/r \ll 1$. However, from the $(0,0)$-component of the metric, the effective Newtonian potential is $-G_{eff} M/r$, with

$$G_{eff} = \frac{G}{\phi_0 \left[ 1 + (3 + 2\omega_{BD})^{-1} \right]} \quad (A5)$$

We notice that this effective potential is the same as that of general relativity, but with a constant rescaling of $G \to G_{eff}$. Note further that, in the spatial components of the metric, a different constant rescaling of $G$ would bring those components to a form analogous to Schwarzschild. One of the consequences of this, as originally pointed out by Brans and Dicke, is that to this order in the weak-field approximation circular orbits are completely unaffected compared to GR predictions in the solar system, although light deflection is for example modified.

The example above shows that Brans-Dicke theory is not fundamentally a variable $G$ theory: absorbing $\phi$ into an effective variable-$G$ coupling at the level of the action is a formal device and care must be taken in how this is interpreted. More significantly, Brans-Dicke theory does not predict any GR deviations in vacuum. In particular, binary BH mergers, as considered in this paper, are completely unaffected compared to GR predictions in the solar system, although light deflection is for example modified.

We can elucidate this further by taking an effective field theory point of view. Consider the GR action, linearized in the perturbations $h_{\mu \nu}$ about Minkowski spacetime $\eta_{\mu \nu}$, i.e. $g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}$:

$$S = \int d^4 x \left[ (h^{\sigma \tau}_{\mu \nu} - \eta^{\sigma \tau} \Box h_{\mu \nu}) + 16\pi G_{\text{mat}} \mathcal{L}_{\text{mat}} + 16\pi G G_{\nu} \mathcal{L}_{\text{GW}} + \alpha^{-1} \left( \partial_{\mu} G_{\nu} \right) \left( \partial^\mu G_{\nu} \right) \right], \quad (A6)$$
where $\Box_g$ is the D’Alembertian operator of flat space, $\mathcal{L}_{\text{matt}}$ is some matter Lagrangian density and $\mathcal{L}_{\text{GW}}$ is the second-order, effective Lagrangian density associated with GWs. We have included two coupling constants here: $G_{\text{matt}}$ associated with the coupling between gravity and matter; and $G_g$ associated with the self-gravitational coupling. In GR, both of these coupling constants are identical, but in Brans-Dicke $G_{\text{matt}} = G/\phi$, whereas $G_g$ is still $G$. Essentially, in Sec. III of this paper we have promoted $G_g$ to a function of spacetime, and this cannot be mimicked by Brans-Dicke or similar scalar-tensor theories.

The last term in Eq. (A6) gives dynamics to the gravitational constant, which leads to the equation of motion

$$\Box_g G_g = \alpha \mathcal{L}_{\text{GW}}, \quad (A7)$$

where $\alpha$ is a coupling constant. Notice that this coupling is different from that which arises in scalar-tensor theories, as in the latter the scalar field couples to the trace of the stress-energy tensor, instead of to GW nonlinearities.

### Appendix B: Cosmological Degeneracies

The modification to the GW response function, derived in Eq. (7), is not unique. As mentioned in the Introduction, this effect can be also obtained in a theory where $G$ is not varying, but $\mathcal{M}$ is time-dependent. Both of these are wave generation effects, where the evolution equation for the GW frequency is modified due to the new time dependence. In addition to this, a similar modification arises if the background cosmology is time-dependent, i.e. in an expanding universe. Although such a modification is a wave propagation one, it leads to the same frequency modification in the frequency-domain GW phase to leading order in the stationary-phase approximation. In this appendix we show how this comes about and consider its implications in the constraint of $\mathcal{G}_c$.

Consider an expanding cosmology, where the metric upon which GWs propagate, is described by an unperturbed Friedmann-Robertson-Walker line element:

$$ds^2 = -dt^2 + a^2(t)dx^i dx^j \delta_{ij}, \quad (B1)$$

where $a(t)$ is the scale factor and $t$ is cosmic time. The scale factor allows us to define the Hubble parameter in the usual way: $H(t) \equiv \dot{a}/a$, where the overhead dot stands for partial differentiation with respect to cosmic time. Using the standard definition for the redshift, $z \equiv a_0/a - 1$, where $a_0 = a(t_0)$ corresponds to the present value of the scale factor, we can derive the evolution equation

$$\frac{dz}{dt} = (1 + z)^2 H_0 - (1 + z) H(z), \quad (B2)$$

The constant $H_0$ is the value of the Hubble parameter $H(z = 0)$ today, $H_0 = (70.5 \pm 1.3) \text{ km s}^{-1} \text{ Mpc}^{-1}$ [87]. Notice that this expression differs from that derived by McVittie [88] $[dz/dt_0 = (1 + z) H_0 - H(z)]$, because here we differentiate redshift with respect to cosmic time at the source, instead of at the observer.

The Hubble parameter depends on the Friedmann equations, and neglecting any $G(t)$ dependence, one can write it as

$$H(z) = H_0 \sqrt{\Omega_{m,0} (1 + z)^3 + \Omega_c + \ldots}, \quad (B3)$$

where $\Omega_{m,0}$ and $\Omega_c$ stand for the present-day matter density and the cosmological constant respectively, while the $\ldots$ represent other terms that are negligible in the regime $z < 10$. In a matter-dominated era, the regime of most relevance for LISA, we can approximate the Hubble parameter as $H(z) \approx H_0 (1 + z)^{3/2} \Omega_{m,0}^{1/2}$, thus leading to

$$\frac{dz}{dt} = H_0 \left[ (1 + z)^2 - (1 + z)^{5/2} \Omega_{m,0}^{1/2} \right]. \quad (B4)$$

Notice that this effect not only affects gravitational wave observations, but also any electromagnetic one that involves a sufficiently long, coherent time-integration.

Let us now connect $\dot{z}$ to the GW response function. As already pointed out, the response function, as measured on Earth, is sensitive to the redshifted chirp mass $\mathcal{M}_z = (1 + z) \mathcal{M}$. Though since the redshift changes with time, which to leading order we can expand about the time of coalescence as

$$z(t) \approx z_c + \frac{dz}{dt} \bigg|_{z = z_c} (t - t_c), \quad (B5)$$

$$\approx z_c + H_0 \left[ (1 + z_c)^2 - (1 + z_c)^{5/2} \Omega_{m,0}^{1/2} \right] (t - t_c),$$

one sees that $\mathcal{M}_z$ becomes time-dependent. Such an effect contributes to any cosmological observable that depends on redshift, as first pointed out by McVittie [88].

The calculation of the GW observable is not yet complete as one must relate the time dependence ($t - t_c$) in Eq. (B3) to GW frequency. As pointed out in Sec. III the Fourier transform of the GW response function in the stationary phase approximation is given by Eqs. (6)-(7), except that now redshift has become a function of time in the inspral frame. One can rewrite this function of time as a function of GW frequency, through Eq. (10) with the substitutions $F \rightarrow f/2$ and $\mathcal{M} \rightarrow \mathcal{M}(1 + z_c)$, which appropriately redshifts all physical lengths into the observer frame.

Concentrating on the dominant term in the GW phase, and defining $\Psi_{1.o.}$ to be the leading order term of Eq. (7) and $\delta \Psi_{z}$ to be the new contribution to Eq. (7), such that $\Psi = \Psi_{1.o.}(1 + \delta \Psi_{z})$, we find

$$\delta \Psi_{z} = \frac{25}{768} z_c (1 + z_c)^{-1} G_c \mathcal{M}_{z_c} u^{-8/3}, \quad (B6)$$

where now $u \equiv \pi G_c \mathcal{M}_{z_c} f$ and $\mathcal{M}_{z_c} \equiv \mathcal{M}(1 + z_c)$. The new relative contribution, $\delta \Psi_{z}$, depends on $f^{-8/3}$, or
equivalently, $u^{-8/3}$, just as the $\dot{G}_c$ term in Eq. (7), and thus the two effects are degenerate.

This raises the question of whether the $\dot{z}$ effect could overwhelm the $\dot{G}_c$ effect, rendering the bounds derived in this paper unattainable. To answer this question, we will use the results of the PN dephasing/Fisher analysis of the $\dot{G}_c$ effect in Sec. IV D. Using the measured values of the cosmological parameters obtained in the five-year WMAP analysis, $H_0 = 70.5 \pm 1.1 $ km s$^{-1}$ Mpc$^{-1}$, $\Omega_{m,0} = 0.26$, and defining the ratio $r \equiv \dot{\Psi}_z/\dot{\Psi}_{\dot{G}_c}$ of the $\delta \dot{\Psi}_z$ correction induced by the $\dot{z}$ term to the $\delta \dot{\Psi}_{\dot{G}_c}$ correction induced by the $\dot{G}_c$ term (the first term in the second line of Eq. (7), we find that $r \approx 0.008 \left( 10^{-8} \text{ yr}^{-1}/\dot{G}_c \right)$ at $z_c = 1$, growing to $r \approx 0.02 \left( 10^{-8} \text{ yr}^{-1}/\dot{G}_c \right)$ at $z_c = 5$ for equal-mass binaries. Since this ratio is less than one, it implies the cosmological effect is not significant, and does not need to be accounted for.

One might wonder though why we did not use the more sensitive bounds for $G_c$ obtained from the EOBB analysis in the above. The reason is that the $\dot{z}$ effect is purely a propagation effect, and would not be enhanced by inclusion of higher-order PN terms. This is in contrast to the effect of $G_c$, which changes the local dynamics of the binary, and for near-equal mass systems this is strongly reflected in the higher order PN terms. Nevertheless, even if we did assume the EOBB $G_c$ bounds carried over verbatim to $\dot{z}$, one can still subtract the $\dot{z}$ effect using cosmological data prior to testing for $G_c$. How much this will reduce the $\dot{z}$ effect depends on how accurately the cosmology is known. Assuming a relative error of $10\%$ in $H_0$ and $\Omega_{m,0}$ (consistent with how accurately these numbers are known today), the ratio is reduced to $r \approx 0.2 \left( 10^{-10} \text{ yr}^{-1}/\dot{G}_c \right)$ at $z = 1$ and $r \approx 0.5 \left( 10^{-10} \text{ yr}^{-1}/\dot{G}_c \right)$ at $z = 5$. From new and anticipated observations, provided for example by Planck [90], it is not unreasonable to assume by the anticipated time of LISA the relative error in the cosmology would be an order of magnitude less, further reducing $r$ by an order of magnitude. This is then well below our most optimistic bounds for measuring $G_c$.

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