On preheating in $\alpha$-attractor models of inflation

Tomasz Krajewski, Krzysztof Turzyński and Michał Wieczorek

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland
E-mail: tomasz.krajewski@fuw.edu.pl, krzysztof.turzynski@fuw.edu.pl, michal.wieczorek@fuw.edu.pl

Abstract. We study (p)reheating in $\alpha$-attractor T-models of inflation, taking into account both scalar fields present in these models: the inflaton and the spectator. The two-field model has a negative field-space curvature which, at the end of inflation, may lead to geometrical destabilization of the spectator for small values of $\alpha \lesssim 10^{-3}$. We perform the instability (Floquet) analysis of the linear dynamics and a fully non-linear lattice computations with our numerical code, which we specifically designed for a class of two-field models with non-canonical kinetic terms. We find that the perturbations of the spectator field are much more unstable than the perturbations of the inflaton field, so the dynamics of the early stages of preheating is dominated by the evolution of the spectator perturbations. As a result, the transition from the inflationary era to radiation domination era is practically instantaneous and much faster than previously found in an effective theory including only the inflaton field.

Keywords: inflation, preheating, lattice simulations, $\alpha$-attractors

ArXiv ePrint: 1801.01786
1 Introduction

The concept of cosmological inflation has brought about a solution to many problems in big bang cosmology and, therefore, inflation has become a natural ingredient of the standard cosmological model (see e.g. [1] for a pedagogical introduction). However, inflation remains a very general theory and its relation with the Standard Model of particle physics is still unclear. In particular, the dynamics of the passage from the inflationary era to radiation-dominated era, called reheating, remains elusive. Although there exist possible scenarios for reheating, well embedded in the general framework of quantum field theory [2, 3], there are typically no observables that could at present distinguish between various possibilities.

While the large scale structures of the Universe are seeded by quantum fluctuations of the inflaton field around 50 e-folds before the end of inflation, the fluctuations created at later times, around the end of inflation, can have very different, but equally important implications. The evolution of these fluctuations in the background of the homogeneous inflaton oscillating around the minimum of its potential can lead to mode amplification and particle production via a mechanism called parametric resonance [3–5]. The precise knowledge of the evolution of the Universe during this stage, called preheating, is very important, since it affects the interpretation of the CMB data. The main factor that affects this interpretation is the
barotropic parameter $w = \langle p \rangle / \langle \rho \rangle$ and the way it evolves from the value around $-1$, typical of the inflationary era, to the value $1/3$, which characterizes radiation-dominated era.

Certain inflationary models come naturally equipped with a mechanism for efficient reheating. A notable example is the class of $\alpha$-attractor models of inflation, called T-models [6], which have recently attracted a lot of interest because of several appealing features. First, they are originally formulated in the context of supergravity, which gives them a solid theoretical motivation. Second, their predictions are naturally consistent with the Planck data [7]. Last but not least, it has recently been realized that in a single-field effective theory of the inflaton field stemming from these models, at the end of inflation, the inflaton experiences self-resonance [8] and its perturbations may become highly unstable; once they dominate the Universe, the barotropic parameter can quickly approach $1/3$ [9, 10]. In this way, the radiation-dominated era begins very soon after the end of inflation, which greatly reduces the theoretical uncertainty customarily attributed to the reheating era [11–18].

Even in the minimal supergravity construction, one should in principle consider both real degrees of freedom present in the scalar part of the chiral multiplet. Recently, a number of authors have studied the multi-field aspects of inflation in $\alpha$-attractor models [19–21] and discussed the predictions for the perturbations relevant for the CMB scales. However, there are also interesting regions of the parameter space in which the scalar field that does not drive inflation (and that we shall from now on call the spectator field) can have important consequences for reheating. The spectator field is typically heavy when the CMB modes leave the Hubble radius, so its presence can be safely neglected for the calculation of the power spectrum of the curvature perturbations, but at the end of inflation the spectator field becomes transiently tachyonic and unstable, so its perturbations may eventually dominate the Universe. This is possible because the noncanonical form of the kinetic part of the Lagrangian gives rise to geometrical destabilization [22]. In the context of inflation, geometrical destabilization may end inflation prematurely [23] or trigger a new phase of inflation [24]. Also the dynamics of reheating can be affected by the instability of the spectator field and it is therefore interesting to study it in detail.

In this letter, we present the analysis of preheating for $\alpha$-attractor T-models of inflation. We demonstrate that the spectator field may indeed become significant after the end of inflation. We first show this semi-analytically, by performing a Floquet analysis for perturbations of both fields. Because the linearized equations of motion for the perturbations become unreliable when the instability kicks in, we also present results of fully nonlinear lattice simulations of preheating in these models. We find that the spectator field may have a very strong impact on the post-inflationary dynamics in these models.

Our letter is organized as follows. In Chapter 2 we briefly present the $\alpha$-attractor T-models and analyze some of their features, which are crucial for an analysis of preheating. Chapter 3 is devoted to the Floquet analysis of the mode amplification during parametric resonance. In Chapter 4, we present the results of lattice simulations. We draw our conclusions in Chapter 5. A detailed description of the numerical procedure and results of additional simulations supplementing our main hypothesis are deferred to Appendices.

Throughout the paper we adopt natural units with $M_P = 1$, unless indicated otherwise.
2 α-attractor T-models of inflation

2.1 Presentation of the model

We will consider α-attractor T-models of inflation characterized by the following superpotential

\[ W_H = \sqrt{\alpha} \mu S \left( \frac{T - 1}{T + 1} \right)^n, \]  

(2.1)

where \( \mu \) is a constant parameter, and by the Kähler potential

\[ K_H = -\frac{3\alpha}{2} \log \left( \frac{(T - \bar{T})^2}{4TT} \right) + S \bar{S} \]  

(2.2)

with parameters \( \alpha > 0 \) and \( n > 0 \). As shown in [25], the superfield \( S \) can be stabilized during and after inflation and we can assume \( S \equiv 0 \). The scalar sector of the model can be then expressed in terms of two real scalar fields \( \phi \) and \( \chi \), which are related to the scalar component of the superfield \( T \) by

\[ \left| \frac{T - 1}{T + 1} \right|^2 = \left( \frac{\cosh(\beta \phi) \cosh(\beta \chi) - 1}{\cosh(\beta \phi) \cosh(\beta \chi) + 1} \right) \]  

(2.3)

where \( \beta \equiv \sqrt{\frac{2}{3\alpha}} \).

This choice is justified by a particularly simple form of the field-space metric in the kinetic term of the scalar Lagrangian:

\[ \mathcal{L} = -\frac{1}{2} \left( \partial_\mu \chi \partial^\mu \chi + e^{2b(\chi)} \partial_\mu \phi \partial^\mu \phi \right) - V(\phi, \chi). \]  

(2.4)

where \( b(\chi) \equiv \log(\cosh(\beta \chi)) \) and the potential of the model reads

\[ V(\phi, \chi) = M^4 \left( \frac{\cosh(\beta \phi) \cosh(\beta \chi) - 1}{\cosh(\beta \phi) \cosh(\beta \chi) + 1} \right)^n \left( \cosh(\beta \chi) \right)^{2/\beta^2}, \]  

(2.5)

with \( M^4 = \alpha \mu^2 \). With such a field-space metric, the reparametrization invariant field-space curvature is negative

\[ \mathbb{R} = -2(b^2 + b'') = -2\beta^2 = -\frac{4}{3\alpha}, \]  

(2.6)

which, in particular, means that one cannot canonically normalize both degrees of freedom.

We shall present numerical results for four benchmark models, characterized in Table 1. These model have different values of the parameters \( n \) and \( \alpha \), which corresponds to different shapes of the inflaton potential near the minimum and different strengths of the geometrical destabilization, respectively. The contour plots of the potentials are shown in Figure 1. Note that the potential (2.5) has a plateau in the entire \((\phi, \chi)\) plane away from the minimum.

2.2 Inflationary trajectory and first-order perturbations

As shown in [6], the model presented in Section 2.1 admits an inflating solution with the inflationary trajectory proceeding along \( \chi = 0 \). The model is therefore effectively described in terms of a canonically normalized inflaton field \( \phi \) with a potential

\[ V(\phi, 0) = M^4 \tanh^{2n} \left( \frac{\beta \phi}{2} \right). \]  

(2.7)
The model is consistent with Planck data for a wide range of its parameters (see e.g. [26]). At this stage, $\phi$ is the inflaton and $\chi$ does not play any role in the evolution of the Universe.

This single-field description may cease to be adequate when the inflaton field accelerates and eventually leaves the slow-roll regime defined as $\epsilon \equiv -\dot{H}/H^2 \ll 1$. A negative value of field space curvature, can cause a ‘geometrical’ destabilization of the perturbations of the field $\chi$ near the end of inflation [22]. Therefore, in order to track the dynamics of the perturbations accurately, both fields should be taken into account starting from a few e-folds before the end of inflation.

Equations of motion for the perturbations in two-field models described by (2.4) can be found e.g. in [27], with no slow-roll approximation or any additional assumptions. The per-

| model | panel in figures | $n$ | $\alpha$ | $M$ | $k_{\text{max}}$ |
|-------|-----------------|-----|----------|-----|---------------|
| 1     | upper left      | 1   | $10^{-3}$| $5.96 \times 10^{-4}$ | $8.88 \times 10^{-5}$ |
| 2     | upper right     | 1   | $10^{-4}$| $3.37 \times 10^{-4}$ | $8.57 \times 10^{-5}$ |
| 3     | lower left      | 1.5 | $10^{-3}$| $5.97 \times 10^{-4}$ | $8.91 \times 10^{-5}$ |
| 4     | lower right     | 1.5 | $10^{-4}$| $3.38 \times 10^{-4}$ | $8.57 \times 10^{-5}$ |

Table 1. Description of the benchmark models used in the simulations. The last column shows the momentum space cutoff defined in Section 4.1.

Figure 1. Contour plots of the two-field potential (2.5) for $n = 1, \alpha = 10^{-3}$ (upper left), $n = 1, \alpha = 10^{-4}$ (upper right), $n = 1.5, \alpha = 10^{-3}$ (lower left), $n = 1.5, \alpha = 10^{-4}$ (lower right). The red line represents the inflationary trajectory with the onset of the numerical simulations (described in Section 4) and the end of inflation marked as green and yellow dots, respectively.
turbed Friedmann-Robertson-Walker metric (in longitudinal gauge, with only scalar degrees of freedom included and constraints taken into account) reads

\[ ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 - 2\Psi)d^2x. \]

From now on, we will assume \( \chi = 0 \). The relevant equations of motion for the background quantities \( H \equiv \dot{a}/a \) and \( \phi(t) \) are

\[ H^2 = \frac{1}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi, 0) \right], \quad \ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi, 0) = 0, \]

where \( V_\phi \) stands for a derivative in the \( \phi \) direction (\( V_\phi \equiv \partial V / \partial \phi \)). Linear perturbations are described in terms of gauge invariant Mukhanov-Sasaki variables:

\[ Q_\phi \equiv \delta\phi + \frac{\dot{\phi}}{H}\Psi \quad \text{and} \quad Q_\chi \equiv \delta\chi + \frac{\ddot{\chi}}{H}\Psi, \]

which obey the following equations of motion

\[ \ddot{Q}_\phi + 3H\dot{Q}_\phi + \left( \frac{k^2}{a^2} + V_\phi \right)Q_\phi = 0, \]
\[ \ddot{Q}_\chi + 3H\dot{Q}_\chi + \left( \frac{k^2}{a^2} + V_{\chi\chi} + \frac{1}{2} \dot{\phi}^2 R \right)Q_\chi = 0. \]

Writing eqs. (2.11) and (2.12), we ignored all contributions suppressed by the Planck scale, as the energy scale of inflation is much smaller. We also used the assumption \( \chi = 0 \) which immediately implies that \( V_\chi(\phi, 0) = 0 \) and \( V_{\phi\chi}(\phi, 0) = 0 \) for \( V \) given by (2.5).

Comparing eq. (2.6) and (2.12) we can see that for large values of \( \beta \), i.e. for small values of \( \alpha \), we can expect the perturbation \( Q_\chi \) to exhibit a tachyonic instability as \( |\dot{\phi}| \) increases towards the end of inflation. We will analyze this instability quantitatively in Chapter 3, using Floquet theory. However, we would like to address first the issue of the initial conditions for perturbations, which will be important for our lattice simulations.

### 2.3 Initial conditions for perturbations

One usually chooses the Bunch-Davies initial conditions for perturbations \( Q_\phi \) and \( Q_\chi \), appropriate for quantum fields in time-dependent, de Sitter background. This procedure is well known for fields with trivial field-space metric and is readily generalized for non-trivial cases. For two-field models with kinetic terms as in eq. (2.4), the prescription for initial conditions of the perturbations can be found e.g. in [27]. This prescription becomes very simple, if one assumes that \( \chi = 0 \) and neglects the contributions suppressed by the Planck scale. Defining \( u_\phi \equiv aQ_\phi \) and \( u_\chi \equiv aQ_\chi \), we can write the second-order action for the model as [28]:

\[ S_2 = \frac{1}{2} \int d\tau d^3k \left[ (u'_\phi)^2 + (u'_\chi)^2 - k^2u_\phi^2 - k^2u_\chi^2 \right]
+ a^2 \left( 2H^2 - V_\phi \right) u_\phi^2 + a^2 \left( 2H^2 - \left( V_{\chi\chi} + \frac{1}{2} \dot{\phi}^2 R \right) \right) u_\chi^2. \]

This action is a sum of two parts: one that depends only on \( u_\phi \) and one that depends only on \( u_\chi \). Each of these parts has the form of the action for a harmonic oscillator with time-dependent mass. Therefore, a usual single-field quantization procedure (see e.g. [29]) can be
performed. In so-called adiabatic approximation, it provides the following initial conditions for perturbations:

$$u_\phi(k, \tau_0) = \frac{1}{\sqrt{2\omega_{\phi,k}}} \exp^{-i\omega_{\phi,k}\tau_0}, \quad u_\phi'(k, \tau_0) = -i\frac{\sqrt{\omega_{\phi,k}}}{2} \exp^{-i\omega_{\phi,k}\tau_0} \tag{2.14}$$

and

$$u_\chi(k, \tau_0) = \frac{1}{\sqrt{2\omega_{\chi,k}}} \exp^{-i\omega_{\chi,k}\tau_0}, \quad u_\chi'(k, \tau_0) = -i\frac{\sqrt{\omega_{\chi,k}}}{2} \exp^{-i\omega_{\chi,k}\tau_0}, \tag{2.15}$$

where

$$\omega_{\phi,k}^2 \equiv k^2 + a^2(V_{\phi\phi} + 2H^2) \quad \text{and} \quad \omega_{\chi,k}^2 \equiv k^2 + a^2(V_{\chi\chi} + 2H^2 + \frac{1}{2}\dot{\phi}^2 R). \tag{2.16}$$

The formulae for energy density of perturbations per mode have the standard form

$$E_\phi(k, \tau) = \frac{1}{2} \left( |u_\phi'(k, \tau)|^2 + \omega_{\phi,k}^2 |u_\phi(k, \tau)|^2 \right) \tag{2.17}$$

and

$$E_\chi(k, \tau) = \frac{1}{2} \left( |u_\chi'(k, \tau)|^2 + \omega_{\chi,k}^2 |u_\chi(k, \tau)|^2 \right). \tag{2.18}$$

We shall use expressions (2.14) and (2.15) to set Gaussian initial conditions for perturbations in our lattice simulations.

### 3 Floquet analysis of perturbations.

From equations (2.17) and (2.18) it is clear that the amplitudes of fluctuations $u_\phi$ and $u_\chi$ (and hence $Q_\phi$ and $Q_\chi$) play a crucial role in the expression for energy density of perturbations. If these perturbations are unstable and their amplitudes grow sufficiently large, they can eventually dominate the energy density of the Universe, thereby affecting its equation of state. On one hand, this can be treated as unavoidable uncertainty related to the reheating era. On the other hand, within each model the evolution of the perturbations can be in principle tracked numerically and the history of the Universe between the end of inflation and the onset of the radiation-dominated era can be reconstructed.

In this Section we shall discuss the growth of amplitudes of the perturbations semi-analytically, using a Floquet analysis along the lines of Ref. [30], deferring the full numerical analysis to Section 4. We write eqs. (2.11) and (2.12) as two sets of two first order equations

$$\begin{pmatrix}
\dot{Q}_{\phi,k} \\
\Pi_{\phi,k}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-\left(\frac{k^2}{a^2} + V_{\phi\phi}\right) & -3H
\end{pmatrix}
\begin{pmatrix}
Q_{\phi,k} \\
\Pi_{\phi,k}
\end{pmatrix} \tag{3.1}
$$

and

$$\begin{pmatrix}
\dot{Q}_{\chi,k} \\
\Pi_{\chi,k}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-\left(\frac{k^2}{a^2} + V_{\chi\chi} + \frac{1}{2}\dot{\phi}^2 R\right) & -3H
\end{pmatrix}
\begin{pmatrix}
Q_{\chi,k} \\
\Pi_{\chi,k}
\end{pmatrix}. \tag{3.2}
$$

After inflation the homogeneous field $\phi(t)$ begins to oscillate around the minimum of the potential. Because the timescale of these oscillation is typically much smaller that the timescale
of the expansion of the Universe, $\phi(t)$ can be to a good approximation treated as a periodic function for at least a few efolds. This implies that the matrices in eqs. (3.1) and (3.2) are also periodic functions of time, as their time dependence comes primarily from their dependence on $\phi(t)$. Therefore, by the Floquet Theorem, the fundamental matrices $O_{\phi,\chi}(t, t_0)$ of solutions of the eqs. (3.1) and (3.2) can be written as

$$O_{\phi,\chi}(t, t_0) = P_{\phi,\chi}(t, t_0) \exp \left[ (t - t_0) \Lambda_{\phi,\chi}(t_0) \right],$$

(3.3)

where $P_{\phi,\chi}(t, t_0)$ are periodic matrices with the same period as matrices in eqs. (3.1) and (3.2), satisfying $P_{\phi,\chi}(t_0, t_0) = I$. Matrices $\Lambda_{\phi,\chi}(t_0)$ are constant (but $k$-dependent) matrices, whose eigenvalues $\mu_{\phi,\chi}$ are called Floquet exponents. A positive real part of a Floquet exponent indicates that the amplitude of the corresponding mode grows exponentially. Therefore, a calculation of Floquet exponents for a range of modes and for different values of the amplitude of oscillating $\phi(t)$ may reveal which of these modes are unstable and, if both modes are unstable, which one grows faster.

3.1 Floquet exponents for inflaton and spectator perturbations for $\alpha$-attractor T-models

In Figures 2-4, we present the Floquet exponents of inflaton and spectator perturbations for parameters $n = \frac{3}{2}$ and $\alpha = 10^{-2}$, $10^{-3}$ and $10^{-4}$. We computed the Floquet exponents for a range of values of the amplitude $\bar{\phi}$ of the oscillating background field $\phi(t)$, because if we follow the evolution of the Universe for many efolds, the Hubble friction leads to a slow decrease of this amplitude, which is described by the relation $\bar{\phi} \propto a^{-3/(n+1)}$. Similarly, in the expanding Universe the effective wave number decreases and satisfies $k_{\text{eff}} = k/a$. Therefore, an initial condition at the end of inflation corresponds to a particular value of the amplitude of the homogeneous inflaton field, which in the plots is indicated with a red line. As the Universe expands, a given mode corresponds to a certain path on the $(k_{\text{eff}}, \bar{\phi})$ plane. Therefore, to describe the growth of the particular modes during their evolution, we need to compute Floquet exponents for different amplitudes and wave numbers. In Figures 2-4, the $(k_{\text{eff}}, \bar{\phi})$-paths for a few different modes are drawn as white curves.

For a discussion of the Floquet exponents, it is convenient to introduce explicitly the reduced Planck mass $M_P$ to keep track of mass dimensions of different quantities. Values

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Floquet exponents for the inflaton (left panel) and the spectator (right panel) perturbations with $n = 3/2$ and $\alpha = 10^{-2}$.}
\end{figure}
Figure 3. Floquet exponents for the inflaton (left panel) and the spectator (right panel) perturbations with \( n = 3/2 \) and \( \alpha = 10^{-3} \).

of Floquet exponents are then given in units \( M^2/M_P \). These units are natural for Floquet exponents for two reasons. First, the values of Floquet exponents are \( M \)-independent in such units. Second, the Hubble rate is of order of \( M^2/M_P \) at the end of inflation. Therefore, values of Floquet exponents describe naturally the rate of the exponential growth of the amplitude of a given mode: with \( \tilde{\mu}_k \equiv \mu_k M^2/M_P \), the amplitude grows roughly by \( \sim e^{\tilde{\mu}_k} \) during one Hubble time.

In our computation of Floquet exponents we have taken into account the expansion of the Universe. We used the full equation of motion (2.9) for the background quantities and eqs. (3.1) and (3.2) for the perturbations. This, in particular, implies that \( \phi(t) \) is an almost periodic function modulated by a slowly decreasing envelope. We calculated the Floquet exponents in a standard way, comparing the values of \( Q_{\tilde{\phi},k} \) and \( \Pi_{\tilde{\phi},k} \) at two subsequent maxima of \( \phi(t) \); the same calculation was applied for \( Q_{\tilde{\chi},k} \) and \( \Pi_{\tilde{\chi},k} \). This shifts the obtained values of Floquet exponents by \(-\frac{3}{2}H\) with respect to the computation with no expansion included. Taking this into account, the Floquet exponents for the inflaton shown in Figures 2–4 are consistent with the results presented in [9, 10].

For all cases, the Floquet exponents for the spectator are much larger than those for the inflaton. This is consistent with the observation that there is a ‘geometrical’ tachyonic instability that sets in for the spectator field shortly before the end of inflation [22]. We conclude that it is fairly plausible that small-scale spectator perturbations may be primarily responsible for the dynamics of reheating, as they seem to grow much faster than the small-scale inflaton perturbations. However, we have so far used a linear theory for the perturbations. In this way, we may follow the early stage of the instability, but as the perturbations grow – and interact – beyond the linear level, we need to study the evolution of the perturbations resorting to fully non-linear lattice simulations.

4 Lattice Simulations

4.1 Description of simulations

There are many computer codes written to simulate preheating after inflation on the lattice (see e.g [30] for a recent review). However, most of them can be applied only to models with canonical kinetic term in the Lagrangian (e.g. [31–33]). One exception is GABE [34],
designed to make simulations for models with non-canonical kinetic terms. However, this code is based on the Runge-Kutta method, which is not symplectic and turned out unsuitable for our purposes. Using a non-symplectic method significantly decreases accuracy of the long-time simulations and spoils energy conservation. Fortunately, the $\alpha$-attractor T-models field space metric (2.4) is very particular in the sense that one can construct the explicit symplectic method in this case. Its description is provided in Appendix A. We performed the simulations using a code written by ourselves which is based on this method. At this point, our goal was to perform an exploratory analysis to catch a glimpse of the behavior of the spectator perturbations. Therefore, we used cubic lattices of a quite modest size: throughout the paper, we report on results obtained with $N_{\text{lattice}} = 128$, unless indicated otherwise. We also checked that results of our computations are stable against reducing the lattice size to $N_{\text{lattice}} = 64$. We performed the simulations for $\alpha = 10^{-3}$ and $\alpha = 10^{-4}$, where we expected strong instability of the spectator, based on the results of the Floquet analysis presented in Section 3.1. We used the momentum space cutoff $k_{\text{max}} = 250 M^2 / M_P$ and $k_{\text{max}} = 750 M^2 / M_P$ for $\alpha = 10^{-3}$ and $\alpha = 10^{-4}$ respectively, which we found to be a good trade-off between granularity in Floquet instability regions and factoring in higher frequency modes. The implied lattice spacing is $h = \sqrt{3} \pi / k_{\text{max}}$.

In our model the definition of the barotropic parameter $w$ can be written in terms of fields as

$$w \equiv \frac{\langle \rho \rangle}{\langle \rho \rangle} = \left\langle \frac{e^{2 \delta(x) \phi^2 + \chi^2}}{2} - \frac{(e^{2 \delta(x) (\nabla \phi)^2} + (\nabla \chi)^2)}{6a^2} - V(\phi, \chi) \right\rangle \left\langle \frac{(e^{2 \delta(x) \phi^2 + \chi^2})}{2} + \frac{(e^{2 \delta(x) (\nabla \phi)^2} + (\nabla \chi)^2)}{2a^2} + V(\phi, \chi) \right\rangle,$$

where the brackets $\langle \rangle$ denote the (optional) time average over a few oscillations of the homogeneous inflaton field $\phi(t)$. If the potential for the inflaton behaves as $V \sim |\phi|^{2n}$ around the minimum and the Universe is dominated by the homogeneous inflaton field $\phi$, the barotropic parameter satisfies $w = \frac{n-1}{n+1}$ [1], which for our benchmark models would give $w = 0.2$ ($w = 0$) for $n = 1.5$ ($n = 1$). However, it has been shown in Ref. [9] that in the single-field $\alpha$-attractor T-model, the growth of the inflaton perturbations due to self-resonance [8] can be so large that for $n = 1.5$ these perturbations dominate the Universe which expands according to the equation of state with $w = 1/3$, i.e. it evolves as a radiation dominated (for $n = 1$ oscillons are
formed and the effective equation of state with \( w = 0 \) does not change). Since we found that the growth of the spectator perturbations estimated from the Floquet analysis is faster that the growth of the inflaton perturbations, we may hypothesize that spectator perturbations are the main factor that makes the Universe approach the radiation-like state with \( w = 1/3 \). Such a hypothesis can only be tested by means of numerical simulations.

4.2 Results

Lattice simulations of preheating for a single field \( \alpha \)-attractor T-models (i.e. without the perturbations of \( \chi \)) have been already performed by the authors of Ref. [9]. As one of the tests of our code, we repeated these simulations and our results are in agreement with those shown in [9]. We show in Figure 5 the evolution of barotropic parameter \( w \) calculated within the four benchmark models described in Section 2.1.

If perturbations of the inflaton become so important in single-field simulations, it is quite reasonable to expect, that the more strongly amplified perturbations of the spectator may be the main force that drives the Universe towards a radiation-like state. Indeed, in two-field simulations, the growth of spectator perturbations for the values of parameters that gave effective reheating in the single-field case is so strong that our variable-step simulations typically stall within an efold after the end of inflation, while our fixed step simulations report unacceptably large errors. However, before that happens our determination of the barotropic parameter \( w \) is physically relevant and we are allowed to conclude that tachyonic instability of the spectator causes a very fast growth of the barotropic parameter just after the end of inflation, with \( w \) approaching 1/3 in all benchmark models except for \( n = 1, \alpha = 10^{-3} \).
Figure 6. The power spectrum of the spectator perturbations for $n = 1, \alpha = 10^{-3}$ (upper left), $n = 1, \alpha = 10^{-4}$ (upper right), $n = 1.5, \alpha = 10^{-3}$ (lower left), $n = 1.5, \alpha = 10^{-4}$ (lower right) for different number of e-folds $N$ after the end of inflation.

benchmark model$^1$. In this sense, in these models reheating may be completed much faster than it follows from single-field simulations. We therefore find that the presence of the spectator leads to practically immediate reheating both for $n = 1$ and $n = 1.5$.

One of the sources of numerical difficulties that we ran into involves multiple evaluations of the hyperbolic sines and cosines with arguments larger than one, necessary for calculating numerical values of the potential (2.5) for different field configurations. As we are mostly interested in the evolution of the coupled two-field system near the minimum at $\phi = \chi = 0$, we repeated our simulations for a simplified version of the potential, containing the Taylor expansion of the hyperbolic cosines, coinciding with (2.5) in the region of our interest, but easier to handle numerically. Details of this analysis are presented in Appendix B.2; our simplification postpones the onset of numerical instabilities and more strongly suggests that the Universe enters the radiation-domination-like regime $w \approx 1/3$ in a fraction of an efold for all four benchmark models considered.

The growth of spectator perturbations can be investigated further with Fourier analysis. In Figure 6, we show the plots of the power spectrum of the spectator field for different moments after the end of inflation for our four benchmark models. The strong growth of modes which fall into Floquet instability bounds and subsequent rescattering resulting in a ‘ringing’ pattern are clearly visible. After initial growth these modes backreact and cause the growth of modes with larger wavenumber $k$. The obtained initial growth of small-$k$ modes is consistent with the approximation obtained from the linear Floquet analysis $^2$. The fact that

$^1$However, results of calculations on a smaller lattice $N_{\text{lattice}} = 64$ presented in Appendix B.1 strongly suggest that $w$ approaches $1/3$ already at $N = 1.2$ efolds after inflation in this benchmark model

$^2$For a back-of-the-envelope estimate, we can consider as an example values of the spectator power spectrum for $n = 1.5, \alpha = 10^{-4}$ and $k_{\text{eff}} = 60 M^2/M_P$ at two different moments: $N = 0.01$, $N = 0.09$ efolds after inflation. Then from the Floquet analysis, we obtain:

$$\frac{\left| \chi_{N=0.01} \right|^2}{\left| \chi_{N=0.01} \right|^2} \approx \left( \exp \left( \frac{\mu_{\text{eff}}}{H} \Delta N \right) \right)^2 \approx \exp \left( 25 \frac{M^2}{M_P} \sqrt{3} \frac{M_P}{M^2} \cdot 0.08 \cdot 2 \right) \approx 10^3 \quad (4.2)$$

This value is in agreement with results shown on the right panel in Figure 6.
the growth of these modes is faster for $\alpha = 10^{-4}$ than for $\alpha = 10^{-3}$ can be easily understood, since the smaller $\alpha$ is, the stronger tachyonic instability the spectator exhibits.

A gradual growth of higher frequency modes cannot be predicted by the Floquet theory, since it is a purely nonlinear effect. However, this effect plays a very important role in reheating, since it leads to the fragmentation of fields. On the timescales considered here, this crucially depends on the evolution of the field $\chi$ that drives the instability; as the two fields $\phi$ and $\chi$ are tightly coupled through the non-canonical kinetic term, higher frequency modes of $\phi$ quickly follow those of $\chi$, which can be seen comparing Figures 6 and 7.

For completeness, we also show in Figure 8, the evolution of different components of the energy density for our four benchmark models. Except for $n = 1, \alpha = 10^{-3}$, for which the simulations end prematurely, we find that kinetic and gradient energies of the fields start domination very quickly. That the contributions coming from the perturbations of $\phi$ dominates over those of $\chi$ follows from the factor of $\cosh^2(\beta \chi)$ multiplying the kinetic term of $\phi$.

In the benchmark models studied in this paper, a strong instability of the spectator perturbations results from a sufficiently small value of $\alpha$. It is therefore interesting to check what are the values of $\alpha$ for which this instability leads to immediate reheating, i.e. $w \to 1/3$. To this end, we performed a series of simulations on smaller lattices with $N_{\text{lattice}} = 64$ in which we kept $n = 1.5$ fixed and we varied $\alpha$. We found that for $\alpha \gtrsim 2 \times 10^{-3}$ there is no immediate reheating and the barotropic parameter $w$ approaches 0.2, as in the single-field case (before self-resonance).

### 5 Conclusions and outlook

In this paper we examined the issue of parametric resonance preheating for $\alpha$-attractor $T$-models of inflation. We confirm the fact, that for small values of parameter $\alpha$ in these models, the parametric resonance can be very effective. For $\alpha \lesssim 10^{-3}$, it can lead to the fragmentation of homogeneous inflaton and to complete reheating soon after the end of
Figure 8. The evolution of the components of the energy density for $n = 1, \alpha = 10^{-3}$ (upper left), $n = 1, \alpha = 10^{-4}$ (upper right), $n = 1.5, \alpha = 10^{-3}$ (lower left), $n = 1.5, \alpha = 10^{-4}$ (lower right). $N$ is the number of e-folds after the end of inflation.

inflation. This conclusion is true for both one-field and two-field models, even if the inflaton potential is approximately quadratic around the minimum.

We have also shown that during preheating the spectator field may play an important role, driving the exponential growth of the energy density and pressure perturbations. Results of our numerical simulations indicate that the tachyonic instability of the spectator is the main factor driving the parametric resonance. For very small values of the parameter $\alpha$, the growth of perturbations is so strong that it is very hard to tract them numerically and obtained overdensities are so huge that they can possibly lead to primordial black holes formation, which may lead to lower bounds on $\alpha$. Albeit very interesting, these issues lie beyond the scope of the present note and we shall address them in future analyses. In future, we also plan to refine our analysis by employing larger lattice sizes and to use more powerful computational facilities.

Acknowledgements

T.K. is supported by grants No. DEC-2012/04/A/ST2/00099 and 2016/23/N/ST2/0311 from the National Science Centre (Poland). M.W. and K.T. are supported by grant No. 2014/14/E/ST9/00152 from the National Science Centre (Poland).

A The symplectic numerical method to simulate preheating in the $\alpha$-attractor T-model of inflation

The method presented here is a modification of the method presented in description of PyCOOL [31] – the lattice code for simulating preheating for scalar fields inflationary models.
PyCOOL can be used only for models with canonical kinetic term in their Lagrangian. Therefore, for $\alpha$-attractor T-models, we wrote a code based on our modified method.

A.1 Action and Hamiltonian

Our goal is to solve numerically equations which come from the action

$$ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \mathcal{R} - \frac{1}{2} e^{2b(\chi)} \left( \partial_{\mu} \phi \right) \left( \partial^{\mu} \phi \right) - \frac{1}{2} \left( \partial_{\mu} \chi \right) \left( \partial^{\mu} \chi \right) - V(\phi, \chi) \right] $$  \hspace{1cm} (A.1)

We assume here that the spacetime is spatially homogeneous, isotropic and flat, i.e. we have

$$ ds^2 = a^2(-d\tau^2 + dx^2), $$  \hspace{1cm} (A.2)

which implies

$$ \sqrt{-g} = a^4 \quad \text{and} \quad \mathcal{R} = 6 \frac{a''}{a'^3}, $$  \hspace{1cm} (A.3)

where the prime denotes the derivative with respect to the conformal time $\tau$. After discretization in space the action (A.1) can be written as:

$$ S = (dx)^3 \int L d\tau = $$

$$ = (dx)^3 \int \left[ -3a'^2 V_L + \sum_{x} a^2 \left( e^{2b(\chi_x)} \left( \frac{\phi'_x}{2} \right)^2 - \frac{G(\phi_x, \chi_x)}{(dx)^2} \right) + \left( \chi'_x \right)^2 - \frac{G(\chi_x, \chi_x)}{(dx)^2} \right] d\tau, $$  \hspace{1cm} (A.4)

where $(dx)^3 V_L$ equals the volume of the periodic lattice and

$$ G(Y, \bar{x}) = \frac{1}{2} \sum_{x_1-1}^{x_1+1} \sum_{x_2-1}^{x_2+1} \sum_{x_3-1}^{x_3+1} c_{d(\alpha)} (Y_\alpha - Y_0)^2 $$  \hspace{1cm} (A.5)

is the second order discretization of the squared spatial gradient operator

$$ \left( \nabla Y \right)^2(\bar{x}) \simeq G(Y, \bar{x}) $$  \hspace{1cm} (A.6)

with $c_1 = 1$ and $c_0 = -6$ (see [33]).

After the Legendre transformation we obtain the following Hamiltonian density:

$$ \mathcal{H} = -\frac{p_0^2}{12 V_L} + \sum_{\bar{x}} a^2 \left( \frac{\pi_{\phi, \bar{x}}^2}{2a^2} + \frac{\pi_{\chi, \bar{x}}^2}{2a^2} + e^{2b(\chi_x)} \frac{G(\phi_x, \chi_x)}{2(dx)^2 a^2} + \frac{G(\chi_x, \chi_x)}{2(dx)^2 a^2} + V(\phi_x, \chi_x) \right), $$  \hspace{1cm} (A.7)

where canonical momenta are defined by the formulae

$$ p_a = \frac{\partial L}{\partial a'}, \quad \pi_{\phi, \bar{x}} = \frac{\partial L}{\partial \phi'_x} = a^2 e^{2b(\chi_x)} \phi'_x \quad \text{and} \quad \pi_{\chi, \bar{x}} = \frac{\partial L}{\partial \chi'_x} = a^2 \chi'_x. $$  \hspace{1cm} (A.8)
A.2 Discretization scheme

The time upgrade scheme is based on the fact that we can divide this Hamiltonian into four parts

\[ H = H_1 + H_2 + H_3 + H_4 \]  

in such the way that none of the parts depends on both the field and its canonical momentum. An example of such division is

\[ H_1 = -\frac{p_a^2}{12V_L} \]  
\[ H_2 = \sum_{\vec{x}} a^4 \left( \frac{\pi_{\phi,\vec{x}}^2}{2a^2} \right) \]  
\[ H_3 = \sum_{\vec{x}} a^4 \left( \frac{\pi_{\chi,\vec{x}}^2}{2a^2} \right) \]  
\[ H_4 = \sum_{\vec{x}} a^4 \left( e^{2b(\chi,\vec{x})} \frac{G(\phi,\vec{x})}{2(dx)^2a^2} + \frac{G(\chi,\vec{x})}{2(dx)^2a^2} + V(\phi,\chi,\vec{x}) \right) \]

To solve the Hamiltonian system numerically, we define for time step \( h = \delta \tau \) the transformations

\[ \Phi_1(h) : (a,p_a,\phi,\vec{x},\pi_{\phi,\vec{x}},\chi,\vec{x},\pi_{\chi,\vec{x}}) \to \left( a + \frac{\partial H_1}{\partial p_a} h, p_a, \phi,\vec{x},\pi_{\phi,\vec{x}},\chi,\vec{x},\pi_{\chi,\vec{x}} \right) \]  
\[ \Phi_2(h) : (a,p_a,\phi,\vec{x},\pi_{\phi,\vec{x}},\chi,\vec{x},\pi_{\chi,\vec{x}}) \to \left( a, p_a - \frac{\partial H_2}{\partial a} h, \phi,\vec{x} + \frac{\partial H_2}{\partial \pi_{\phi,\vec{x}}} h, \pi_{\phi,\vec{x}},\chi,\vec{x},\pi_{\chi,\vec{x}} - \frac{\partial H_2}{\partial \chi} h \right) \]  
\[ \Phi_3(h) : (a,p_a,\phi,\vec{x},\pi_{\phi,\vec{x}},\chi,\vec{x},\pi_{\chi,\vec{x}}) \to \left( a, p_a - \frac{\partial H_3}{\partial a} h, \phi,\vec{x},\pi_{\phi,\vec{x}},\chi,\vec{x} + \frac{\partial H_3}{\partial \pi_{\chi,\vec{x}}} h, \pi_{\chi,\vec{x}} \right) \]  

and

\[ \Phi_4(h) : (a,p_a,\phi,\vec{x},\pi_{\phi,\vec{x}},\chi,\vec{x},\pi_{\chi,\vec{x}}) \to \left( a, p_a - \frac{\partial H_4}{\partial a} h, \phi,\vec{x},\pi_{\phi,\vec{x}},\chi,\vec{x} - \frac{\partial H_4}{\partial \phi,\vec{x}} h, \pi_{\chi,\vec{x}} \right) \]

Then the first order symplectic numerical method (see [35]) has the form

\[ \tilde{\Phi}(h) = \Phi_4(h) \circ \Phi_3(h) \circ \Phi_2(h) \circ \Phi_1(h). \]

With that and its adjoint (again, see [35]), we can create the second order symplectic method for this system, namely

\[ \Phi(h) \equiv \tilde{\Phi}(h/2) \circ \tilde{\Phi}(h/2) = \Phi_1(h/2) \circ \Phi_2(h/2) \circ \Phi_3(h/2) \circ \Phi_4(h) \circ \Phi_3(h/2) \circ \Phi_2(h/2) \circ \Phi_1(h/2). \]  

It can be written as the following set of explicit upgrades:

\[ a_{n+1/2} = a_n + \frac{h}{2} \frac{\partial H_1}{\partial p_a}(p_{a,n}) \]  
\[ \tilde{p}_{a,n+1/2} = p_{a,n} - \frac{h}{2} \frac{\partial H_2}{\partial a}(a_{n+1/2},\pi_{\phi,n},\chi_n) \]
Table 2. Description of the benchmark models used in the simulations presented in Appendix B.2. The last column shows the momentum space cutoff defined in Section 4.1.

| model | panel in figures | n     | \(\alpha\)      | \(M\)           | \(k_{\text{max}}\) |
|-------|-----------------|-------|-----------------|-----------------|-------------------|
| 1     | upper left      | 1     | \(10^{-3}\)     | \(1.72 \times 10^{-4}\) | \(7.40 \times 10^{-4}\) |
| 2     | upper right     | 1     | \(10^{-4}\)     | \(1.29 \times 10^{-5}\) | \(1.25 \times 10^{-3}\) |
| 3     | lower left      | 1.5   | \(10^{-3}\)     | \(1.81 \times 10^{-5}\) | \(8.19 \times 10^{-4}\) |
| 4     | lower right     | 1.5   | \(10^{-4}\)     | \(1.37 \times 10^{-3}\) | \(1.41 \times 10^{-4}\) |

Note that this procedure needs only one array of numbers for every variable and for its associated canonical momenta. These variables are modified consecutively.

B Supplementary results

In order to corroborate our claim that in \(\alpha\)-attractor models it is the spectator field which is mainly responsible for self-resonance and almost immediate reheating, we compare the results presented in the main text with a number of alternative simulations run on smaller lattices with \(N_{\text{lattice}} = 64\). All the plots presented herein support our main hypothesis.

B.1 Results for \(N_{\text{lattice}} = 64\)

The plots presented here show the results obtained according to the procedure described in the main text but on a smaller lattice with \(N_{\text{lattice}} = 64\). In Figures 9, 10, 11 and 12, we show, respectively, the evolution of the barotropic parameter \(w\), the power spectra of the fields \(\chi\) and \(\phi\), and the contributions to the energy density of the Universe. The results are consistent with those presented in Section 4.2, but the simulation for \(n = 1, \alpha = 10^{-3}\) was run longer before being stalled by instabilities and the regime \(w \to 1/3\) is visible.

B.2 Results for \(N_{\text{lattice}} = 64\) and simplified potential

In our simulations, multiple evaluations of the hyperbolic sines and cosines with arguments larger than one is necessary for calculating numerical values of the potential (2.5) for different field configurations. As we are mostly interested in the evolution of the coupled two-field system near the minimum at \(\phi = \chi = 0\), we repeated our simulations for a simplified version of the potential, containing the Taylor expansion of the hyperbolic cosines, coinciding with
Figure 9. The evolution of the barotropic parameter $w$ (not averaged) for $n = 1$, $\alpha = 10^{-3}$ (upper left), $n = 1$, $\alpha = 10^{-4}$ (upper right), $n = 1.5$, $\alpha = 10^{-3}$ (lower left), $n = 1.5$, $\alpha = 10^{-4}$ (lower right). $N$ is the number of e-folds after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$.

Figure 10. The power spectrum of the spectator perturbations for $n = 1$, $\alpha = 10^{-3}$ (upper left), $n = 1$, $\alpha = 10^{-4}$ (upper right), $n = 1.5$, $\alpha = 10^{-3}$ (lower left), $n = 1.5$, $\alpha = 10^{-4}$ (lower right) for different number of e-folds $N$ after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$.

(2.5) in the region of our interest, but easier to handle numerically. To this end, we used the
The power spectrum of the inflaton perturbations for $n = 1$, $\alpha = 10^{-3}$ (upper left), $n = 1$, $\alpha = 10^{-4}$ (upper right), $n = 1.5$, $\alpha = 10^{-3}$ (lower left), $n = 1.5$, $\alpha = 10^{-4}$ (lower right) for different number of e-folds $N$ after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$.

The evolution of the components of the energy density for $n = 1$, $\alpha = 10^{-3}$ (upper left), $n = 1$, $\alpha = 10^{-4}$ (upper right), $n = 1.5$, $\alpha = 10^{-3}$ (lower left), $n = 1.5$, $\alpha = 10^{-4}$ (lower right). $N$ is the number of e-folds after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$.

potential

$$V = M^4 \left(1 + \frac{(\beta \psi)^2}{2}\right)^2 \beta^2 \left(1 + \frac{(\beta \phi)^2}{2}\right) \left(1 + \frac{(\beta \psi)^2}{2}\right)^n \left(1 + \frac{(\beta \phi)^2}{2}\right) + 1 \right)^n$$

(B.1)
Similarly as before, we perform simulations for four benchmark models whose parameters are shown in Table 2. Contour plots of the potential (B.1) for the benchmark models are shown in Figure 13; they reveal a presence of the plateau. A different energy scale, compared to Table 1 and Figure 1, results from adjusting the scale $M$ to obtain the correct normalization of the power spectrum at scales visible in the CMB data.

The simulations were run on a smaller lattice with $N_{\text{lattice}} = 64$. In Figures 14, 15, 16 and 17, we show, respectively, the evolution of the barotropic parameter $w$, the power spectra of the fields $\chi$ and $\phi$, and the contributions to the energy density of the Universe. The results are consistent with those presented in Section 4.2, but the simulation for $n = 1$, $\alpha = 10^{-3}$ was run longer before being stalled by instabilities and the regime $w \rightarrow 1/3$ is visible.

References

[1] V. F Mukhanov *Physical foundations of cosmology*, Cambridge University Press 2005
[2] J. H. Traschen and R. H. Brandenberger, Phys. Rev. D 42 (1990) 2491.
[3] L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. 73 (1994) 3195 [hep-th/9405187].
Figure 14. The evolution of the barotropic parameter $w$ (not averaged) for $n = 1$, $\alpha = 10^{-3}$ (upper left), $n = 1$, $\alpha = 10^{-4}$ (upper right), $n = 1.5$, $\alpha = 10^{-3}$ (lower left), $n = 1.5$, $\alpha = 10^{-4}$ (lower right). $N$ is the number of e-folds after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$ and with a simplified form of the potential (B.1).

Figure 15. The power spectrum of the spectator perturbations for $n = 1$, $\alpha = 10^{-3}$ (upper left), $n = 1$, $\alpha = 10^{-4}$ (upper right), $n = 1.5$, $\alpha = 10^{-3}$ (lower left), $n = 1.5$, $\alpha = 10^{-4}$ (lower right) for different number of e-folds $N$ after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$ and with a simplified form of the potential (B.1).

[4] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, Phys. Rev. D 51 (1995) 5438 [hep-ph/9407247].

[5] L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. D 56 (1997) 3258 [hep-ph/9704452].
Figure 16. The power spectrum of the inflaton perturbations for $n = 1, \alpha = 10^{-3}$ (upper left), $n = 1, \alpha = 10^{-4}$ (upper right), $n = 1.5, \alpha = 10^{-3}$ (lower left), $n = 1.5, \alpha = 10^{-4}$ (lower right) for different number of e-folds $N$ after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$ and with a simplified form of the potential (B.1).

Figure 17. The evolution of the components of the energy density for $n = 1, \alpha = 10^{-3}$ (upper left), $n = 1, \alpha = 10^{-4}$ (upper right), $n = 1.5, \alpha = 10^{-3}$ (lower left), $n = 1.5, \alpha = 10^{-4}$ (lower right). $N$ is the number of e-folds after the end of inflation. These results were obtained on a lattice with $N_{\text{lattice}} = 64$ and with a simplified form of the potential (B.1).

[6] J. J. M. Carrasco, R. Kallosh and A. Linde, Phys. Rev. D 92 (2015) no.6, 063519 [arXiv:1506.00936 [hep-th]].

[7] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594 (2016) A13 [arXiv:1502.01589 [astro-ph.CO]].
[8] M. A. Amin, R. Easther, H. Finkel, R. Flauger and M. P. Hertzberg, Phys. Rev. Lett. 108 (2012) 241302 [arXiv:1106.3335 [astro-ph.CO]].
[9] K. D. Lozanov and M. A. Amin, Phys. Rev. Lett. 119 (2017) no.6, 061301 [arXiv:1608.01213 [astro-ph.CO]].
[10] K. D. Lozanov and M. A. Amin, arXiv:1710.06851 [astro-ph.CO].
[11] A. R. Liddle and S. M. Leach, Phys. Rev. D 68 (2003) 103503 [astro-ph/0305263].
[12] P. Creminelli, R. Easther, J. Pritchard and A. Loeb, JCAP 1102 (2011) 021 [arXiv:1007.3748 [astro-ph.CO]].
[13] P. Creminelli, D. López Nacir, M. Simonović, G. Trevisan and M. Zaldarriaga, Phys. Rev. Lett. 112 (2014) no.24, 241303 [arXiv:1404.1065 [astro-ph.CO]].
[14] L. Dai, M. Kamionkowski and J. Wang, Phys. Rev. Lett. 113 (2014) 041302 [arXiv:1404.6704 [astro-ph.CO]].
[15] J. Martin, C. Ringeval and V. Vennin, Phys. Rev. Lett. 114 (2015) no.8, 081303 [arXiv:1410.7958 [astro-ph.CO]].
[16] J. L. Cook, E. Dimastrogiovanni, D. A. Easson and L. M. Krauss, JCAP 1504 (2015) 047 [arXiv:1502.04673 [astro-ph.CO]].
[17] Y. Ueno and K. Yamamoto, Phys. Rev. D 93 (2016) no.8, 083524 [arXiv:1602.07427 [astro-ph.CO]].
[18] M. Eshaghi, M. Zarei, N. Riazi and A. Kiasatpour, Phys. Rev. D 93 (2016) no.12, 123517 [arXiv:1602.07914 [astro-ph.CO]].
[19] A. Achúcarro, R. Kallosh, A. Linde, D. G. Wang and Y. Welling, JCAP 1804 (2018) no.04, 028 [arXiv:1711.09478 [hep-th]].
[20] A. Linde, D. G. Wang, Y. Welling, Y. Yamada and A. Achúcarro, arXiv:1803.09911 [hep-th].
[21] P. Christodoulidis, D. Roest and E. I. Sfakianakis, arXiv:1803.09841 [hep-th].
[22] S. Renaux-Petel and K. Turzyński, Phys. Rev. Lett. 117 (2016) no.14, 141301 [arXiv:1510.01281 [astro-ph.CO]].
[23] S. Renaux-Petel, K. Turzyński and V. Vennin, JCAP 1711 (2017) no.11, 006 [arXiv:1706.01835 [astro-ph.CO]].
[24] S. García-Saenz, S. Renaux-Petel and J. Ronayne, arXiv:1804.11279 [astro-ph.CO].
[25] J. J. M. Carrasco, R. Kallosh, A. Linde and D. Roest, Phys. Rev. D 92 (2015) no.4, 041301 [arXiv:1504.05557 [hep-th]].
[26] R. Kallosh, A. Linde and D. Roest, JHEP 1311 (2013) 198 [arXiv:1311.0472 [hep-th]].
[27] Z. Lalak, D. Langlois, S. Pokorski and K. Turzyński, JCAP 0707 (2007) 014 [arXiv:0704.0212 [hep-th]].
[28] D. Langlois and S. Renaux-Petel, JCAP 0804 (2008) 017 [arXiv:0801.1085 [hep-th]].
[29] V. F. Mukhanov, S. Winitzki Quantum effects in gravity, Cambridge University Press 2007
[30] M. A. Amin, M. P. Hertzberg, D. I. Kaiser and J. Karouby, Int. J. Mod. Phys. D 24 (2014) 1530003 [arXiv:1410.3808 [hep-ph]].
[31] J. Sainio, JCAP 1204 (2012) 038 [arXiv:1201.5029 [astro-ph.IM]].
[32] G. N. Felder and I. Tkachev, Comput. Phys. Commun. 178 (2008) 929 doi:10.1016/j.cpc.2008.02.009 [hep-ph/0011159].
[33] A. V. Frolov, JCAP 0811 (2008) 009 [arXiv:0809.4904 [hep-ph]].
[34] “http://cosmo.kenyon.edu/gabe.html”
[35] E. Hairer, Lecture notes on Geometric Numerical Integration: “http://www.unige.ch/hairer/poly_geoint/week2.pdf”