Common Fixed Points for Pairs of Mappings with Variable Contractive Parameters

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1. Introduction and Preliminaries

The metric fixed point theory has a vast literature; the Banach-Caccioppoli contraction principle is one of the most outstanding results in this theory. Since its appearance, several generalizations of this result have appeared in the literature. In 1976, Jungck [1] generalized this principle by considering two commuting mappings and proved a common fixed point theorem for these mappings. Afterwards, the commutative property of the mappings assumed by Jungck has been relaxed by introducing “weak” alternative notions as weak commutativity, (non)compatibility, R-weak commutativity, and weak compatibility, among others, which allowed extending several well-known common fixed point theorems for Lipschitz type of mapping pairs.

Here, we are going to establish existence and uniqueness results of common fixed point for a pair of contractive type of mappings whose contractive parameters are nonconstants and its contractive inequality is controlled by a positive function satisfying a stability condition at 0 (see (11)). As a particular case, our results are valid if we control the mentioned inequality by using the well-known altering distance functions [2]. To attain our goals, we will assume that the mappings under consideration are weakly compatible, which is a minimal noncommuting notion for contractive type of mapping pairs. Also, alternatively, we are going to assume that the pair of mappings satisfy some strong conditions like E. A. property and CLRₜ property.

In order to establish our results the following notions will be needed. A pair of self-mappings (S, T) on a metric space (M, d) is said to be compatible [3] if and only if \[ \lim_{n \to \infty} d(TSx_n, STx_n) = 0, \] whenever \( (x_n) \subset M \) is such that

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t, \quad (1) \]

for some \( t \in M \). A pair of mappings \( (S, T) \) is said to be noncompatible if there exits at least one sequence \( (x_n) \subset M \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in M \), but \( \lim_{n \to \infty} d(STx_n, TSx_n) \) is either nonzero or nonexistent. A pair of self-mappings \( (S, T) \) is said to satisfy the property (E.A.) [4] if there exists a sequence \( (x_n) \subset M \) such that

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t, \quad (2) \]

for some \( t \in M \).

A pair of self-mappings \( (S, T) \) is said to satisfy the common limit in the range of \( T \) property (in short CLRₜ) [5] if there exists a sequence \( (x_n) \subset M \) such that

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = Tt, \quad (3) \]

for some \( t \in M \). It may be observed that the CLRₜ property avoids the requirement of the condition of closeness of the ranges of the involved mappings.
Finally, a pair of mappings \( \{x, y\} \) is called \( \text{orbitally continuous} \) if \( z = \lim_{n \to \infty} s^n x \) implies \( s = \lim_{n \to \infty} s^n y \) [6].

If \( S \) and \( T \) are self-maps of a metric space \((M, d)\) and if \( \{x_n\} \) is a sequence in \((M, d)\) such that \( Sx_n = Tx_{n+1}, n = 0, 1, \ldots \), then the set \( O(x, S) = \{Sx_n, n = 0, 1, \ldots \} \) is called the \( \text{orbit of S at x} \) and \( S \) is called \( \text{orbitally continuous} \) if \( \lim_{n \to \infty} s^n x \) exists in \((M, d)\) such that \( \lim_{n \to \infty} s^n x = s \) always holds.

A point \( z \in M \) is called a \( \text{coincidence point} \) (CP) of \((S, T)\) if \( Sx = Tx \). If \( Sx = Tx \) and \( \text{ort} \geq 1 \), then the set \( \{Sx, Tx\} \) is called the \( \text{orbit} \) of \( S \) and \( T \). A point \( x \) is called a \( \text{coincidence point} \) (CP) of \((S, T)\) if \( Sx = Tx \).

Suppose \( S \) and \( T \) are a contractive type pair of self-mappings of a metric space \((M, d)\) having a common fixed point, say \( z \); then \( z = Sx = Tz \). If \( Sx = Tz \) and \( \text{ort} \geq 1 \), then the set \( \{Sx, Tx\} \) is called the \( \text{orbit} \) of \( S \) and \( T \). A point \( z \) is called a \( \text{coincidence point} \) (CP) of \((S, T)\) if \( Sx = Tx \).

Remark 1. We would like to show that weak compatibility is a necessary, hence minimal, condition for the existence of common fixed points of contractive type mapping pairs. Suppose \( S \) and \( T \) are a contractive type pair of self-mappings of a metric space \((M, d)\) having a common fixed point, say \( z \); then \( z = Sx = Tz \). If \( Sx = Tz \) and \( \text{ort} \geq 1 \), then the set \( \{Sx, Tx\} \) is called the \( \text{orbit} \) of \( S \) and \( T \).

2. The Class of Pairs of Mappings with Nonconstant Contractive Parameters

In order to introduce the class of mappings which will be the focus of study of this paper, as in [10], we are going to use the functions \( \alpha, \beta, \gamma : [0, 1) \to [0, 1) \) which satisfy that \( \alpha(t) + \beta(t) + \gamma(t) < 1 \), for all \( t \in [0, 1) \), and

\[
\lim_{s \to 0} \alpha(s) + \beta(s) - \gamma(s) < 1, \quad \forall t > 0.
\]

Now, we introduce the following class of pair of contraction type of mappings.

**Definition 3.** Let \((M, d)\) be a metric space and let \( S, T : M \to M \) be mappings. The pair \((S, T)\) is called a \( \psi-(\alpha, \beta, \gamma)\)-contraction pair if for all \( x, y \in M \)

\[
\psi(d(Sx, Sy)) \leq \alpha(d(Tx, Ty)) \psi(d(Tx, Ty)) + \beta(d(Tx, Ty)) \psi(d(Sx, Tx)) + \gamma(d(Tx, Ty)) \psi(d(Sy, Ty)),
\]

where \( \alpha, \beta, \gamma : [0, 1) \to [0, 1) \) are functions satisfying the conditions (9) and \( \psi : [0, 1) \to [0, 1) \) is a continuous function satisfying that

\[
\psi(t_n) \to 0 \text{ implies that } t_n \to 0.
\]

**Proposition 4.** Let \( S \) and \( T \) be two self-maps on a metric space \((M, d)\). Let one assume that the pair \((S, T)\) is a \( \psi-(\alpha, \beta, \gamma)\)-contraction pair. If \( S \) and \( T \) have a POC in \( M \) then it is unique.

**Proof.** Let \( z \in M \) be a POC of the pair \((S, T)\). Then there exists \( x \in M \) such that \( Sx = Tx = z \). Suppose that, for some \( v \in M \), \( Sv = Tv = w \) with \( z \neq w \). Then, by (10) we have

\[
\psi(d(z, w)) = \psi(d(Sx, Sv)) \leq \alpha(d(Tx, Tv)) \psi(d(Tx, Tv)) + \beta(d(Tx, Tv)) \psi(d(Sx, Tx)) + \gamma(d(Tx, Tv)) \psi(d(Sy, Tv)).
\]

It follows that

\[
\psi(d(z, w)) \leq \alpha(d(z, w)) \psi(d(z, w)) + \beta(d(z, w)) \psi(d(z, w)) + \gamma(d(z, w)) \psi(w, w).
\]
Thus, we get
\[ \psi(d(z, w)) \leq \alpha(d(z, w)) \psi(d(z, w)) < \psi(d(z, w)) \] (14)
which is a contradiction. Therefore, \( z = w. \)

Proposition 5. Let \((M, d)\) be a metric space and let \( S, T : M \rightarrow M \) be mappings with \( S(M) \subset T(M). \) If the pair \((S, T)\) is a \( \psi(\alpha, \beta, \gamma)\)-contraction pair, then for any \( x_0 \in M, \) the sequence \((y_n)\) defined by
\[ y_n = Sx_n = Tn+1, \quad n = 0, 1, \ldots \] (15)
satisfies
1. \( \lim_{n \to \infty} d(y_n, y_{n+1}) = 0; \)
2. \( (y_n) \subset M \) is a Cauchy sequence in \( M. \)

Proof. To prove (1), let \( x_0 \in M \) be an arbitrary point. Since \( S(M) \subset T(M), \) then there exists \( x_1 \in M \) such that \( Sx_0 = Tx_1. \) By continuing this process inductively we obtain a sequence \((x_n)\) in \( M \) such that
\[ y_n = Sx_n = Tn+1. \] (16)

Now, we have
\[ \psi(d(Tn+1, Tn+2)) = \psi(d(Sx_n, Sx_{n+1})) \leq \alpha(d(Tn, Tn+1)) \psi(d(Tn, Tn+1)) + \beta(d(Tn, Tn+1)) \psi(d(Sx_n, Tn+1)) + \gamma(d(Tn, Tn+1)) \psi(d(Sx_{n+1}, Tn+1)). \] (17)

It follows that
\[ \psi(d(Tn+1, Tn+2)) \leq \alpha(d(Tn, Tn+1)) \psi(d(Tn, Tn+1)) + \beta(d(Tn, Tn+1)) \psi(d(Tn, Tn+1)) + \gamma(d(Tn, Tn+1)) \psi(d(Tn, Tn+1)). \] (18)

Therefore, we obtain
\[ \psi(d(Tn+1, Tn+2)) \leq \frac{\alpha(d(Tn, Tn+1)) + \beta(d(Tn, Tn+1))}{1 - \gamma(d(Tn, Tn+1))} \psi(d(Tn, Tn+1)). \] (19)
from which, together with (9), we conclude that
\[ \psi(d(Tn+1, Tn+2)) < \psi(d(Tn, Tn+1)). \] (20)

Thus, \((\psi(d(Tn, Tn+1)))_n\) is a nondecreasing sequence, bounded below by zero, and so converges to \( a \geq 0. \) Now, if \( a > 0 \) then by taking \( \lim sup \) from both sides of the above inequality we have a contradiction. Thus, \( a = 0. \) Now, from condition (11) we conclude that \( d(Tn, Tn+1) \to 0. \)

To prove (2), we are going to suppose that \((y_n) \subset T(M)\) is not a Cauchy sequence. Then, from Lemma 2 there exist \( \epsilon > 0 \) and sequences \((m(k))\) and \((n(k))\) with \( m(k) > n(k) > k \) such that
\[ \lim_{k \to \infty} d(Tn(k), Tn(k)) = \epsilon, \]
\[ \lim_{k \to \infty} d(Tm(k), Tm(k)) = \epsilon. \] (21)

In this way we have
\[ 0 < \psi(\epsilon) = \lim_{k \to \infty} \psi(d(Tn(k), Tn(k))) \]
\[ = \lim_{k \to \infty} \psi(d(Sm(k), Sm(k))) \]
\[ \leq \lim_{k \to \infty} \psi(d(Tm(k), Tn(k))) \]
\[ \leq \lim_{s \to \epsilon} \psi(d(Tm(k), Tm(k))) \]
\[ + \lim_{s \to \epsilon} \psi(d(Sm(k), Tm(k))) \]
\[ + \lim_{s \to \epsilon} \psi(d(Tm(k), Sm(k))) \]
\[ < \psi(\epsilon), \] (22)
which is a contradiction; hence, \((Tn)\) is a Cauchy sequence in \( M. \)

3. On the Existence and Uniqueness of Common Fixed Points

In this section we prove our main results concerning the existence and uniqueness of common fixed points for a \( \psi(\alpha, \beta, \gamma)\)-contraction pair of mappings without continuity requirement.

Theorem 6. Let \( S \) and \( T \) be self-maps on a complete metric space \((M, d)\) such that
\[ (i) S(M) \subset T(M); \]
\[ (ii) the pair \((S, T)\) is a \( \psi(\alpha, \beta, \gamma)\)-contraction pair. \]
Then,

1. the pair \((S, T)\) has a unique POC;
2. if \(S\) and \(T\) are orbitally continuous and if the pair \((S, T)\) is compatible, then \(S\) and \(T\) have a unique common fixed point.

Proof. Let \(y_n = Sx_n = Tx_{n+1}, n = 0, 1, \ldots, \) be the Cauchy sequence defined in Proposition 5 which, as was proved, satisfies that \((y_n) = (Tx_{n+1}) \subset T(M).\) Since \((M, d)\) is complete, there exists a point \(z\) in \((M, d)\) such that

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_{n+1} = z.
\]

Compatibility and orbital continuity of \(S\) and \(T\) imply \(Sz = Tz;\) hence \(z\) is a POC of \(S\) and \(T\). From Proposition 4 we conclude that \(z\) is the unique POC.

On the other hand, since the pair \((S, T)\) is compatible and compatible mappings commute at their coincidence point, \(SSz = TSz = TSz = TTz.\) Using (ii), we get

\[
\psi \left( d \left( Sz, SSz \right) \right) \leq \alpha \left( d \left( Sz, TSz \right) \right) \psi \left( d \left( Sz, TZz \right) \right) + \beta \left( d \left( Sz, TSz \right) \right) \psi \left( d \left( SSz, Tz \right) \right) + \gamma \left( d \left( Sz, TSz \right) \right) \psi \left( d \left( SSz, Tz \right) \right)
\]

(24)

\[
= \alpha \left( d \left( Tz, TSz \right) \right) \psi \left( d \left( Sz, SSz \right) \right);
\]

that is, \(Sz = SSz.\) Hence, \(Sz = SSz = TSSz,\) and \(Sz\) is a common fixed point of the pair \((S, T).\) The uniqueness of the common fixed point follows easily.

**Theorem 7.** Let \(S\) and \(T\) be self-mappings on a metric space \((M, d)\) such that

1. \(S(M) \subset T(M);\)
2. \(T(M) \subset M\) is a complete subspace of \(M;\)
3. the pair \((S, T)\) is a \(\psi(\alpha, \beta, \gamma)-\)contraction pair.

Then,

1. the pair \((S, T)\) has a unique POC;
2. if the pair \((S, T)\) is nontrivially weakly compatible, then \(S\) and \(T\) have a unique common fixed point.

Proof. Let \(y_n = Sx_n = Tx_{n+1}, n = 0, 1, \ldots, \) be the Cauchy sequence defined in Proposition 5 which, as was proved, satisfies that \((y_n) = (Tx_{n+1}) \subset T(M).\) Since \(T(M) \subset M\) is a complete subspace of \(M\), then there exists \(z \in T(M)\) such that

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_{n+1} = z.
\]

(25)

and thus we can find \(u \in M\) such that \(Tu = z.\) Now, we are going to assume that \(Su \neq z.\) Then,

\[
\psi \left( d \left( Sx_{n+1}, Su \right) \right) \leq \alpha \left( d \left( Tx_{n+1}, Tu \right) \right) \psi \left( d \left( Tx_{n+1}, Tu \right) \right) + \beta \left( d \left( Tx_{n+1}, Tu \right) \right) \psi \left( d \left( Sx_{n+1}, Tx_{n+1} \right) \right) + \gamma \left( d \left( Tx_{n+1}, Tu \right) \right) \psi \left( Su, Tu \right).
\]

(26)

Letting \(n \to \infty,\) we obtain

\[
\psi \left( d \left( Su, z \right) \right) \leq \lim_{n \to \infty} \alpha \left( d \left( Tx_{n+1}, Tu \right) \right) \psi \left( d \left( z, Tu \right) \right) + \lim_{n \to \infty} \beta \left( d \left( Tx_{n+1}, Tu \right) \right) \psi \left( d \left( z, z \right) \right) + \lim_{n \to \infty} \gamma \left( d \left( Tx_{n+1}, Tu \right) \right) \psi \left( d \left( Su, Tu \right) \right) < \psi \left( d \left( Su, z \right) \right)
\]

(27)

which is a contradiction; therefore \(Su = z;\) hence \(z\) is a POC of \(S\) and \(T.\) The theorem now follows easily using arguments similar to those which have been used in the corresponding part of Theorem 6.

**Remark 9.** Notice that in Theorem 7 we cannot replace nontrivially weakly compatible mappings by OWC. In fact, under the \(\psi(\alpha, \beta, \gamma)-\)contraction pair of Theorem 6, assumption of OWC and the existence of a unique common fixed point are equivalent conditions. To see this, first suppose that \(A\) and \(S\) satisfy the \(\psi \left( \alpha, \beta, \gamma \right)-\)contraction condition of Theorem 6 above. If \(A\) and \(S\) have a common fixed point, say \(z,\) then \(Az = Sz, \sqrt{A} ASz = SAz = z,\) and \(A\) and \(S\) are, therefore, OWC mappings. On the other hand, if \(A\) and \(S\) are OWC mappings such that \(Ax = Sx\) and \(ASx = AAx = SSx\) for some \(x\) then, using \(\psi \left( \alpha, \beta, \gamma \right)-\)contraction condition, we get

\[
\psi \left( d \left( Sx, SSx \right) \right) \leq \alpha \left( d \left( Tx, TSx \right) \right) \psi \left( d \left( Tx, TSx \right) \right) + \beta \left( d \left( Tx, TSx \right) \right) \psi \left( d \left( Sx, Tx \right) \right) + \gamma \left( d \left( Tx, TSx \right) \right) \psi \left( d \left( SSx, TSx \right) \right)
\]

(28)

\[
= \alpha \left( d \left( Tx, TSx \right) \right) \psi \left( d \left( Sx, SSx \right) \right).
\]

That is, \(Sx = SSx.\) Since \(\psi \left( \alpha, \beta, \gamma \right)-\)contraction condition excludes the existence of two coincidence points \(x, y\) for \(A\) and \(S,\) we get \(Sx = SSx(= ASx).\) This means that \(Sx = Ax\) is a common fixed point of \(A\) and \(S.\) Therefore, one should be really careful before using OWC under any contractive conditions (see, also, [11]).
Theorem 6. Let \((M, d)\) be a metric space and \(S, T : M \to M\) nontrivially weakly compatible mappings satisfying the property \((E.A.)\). Let one suppose that the pair \((S, T)\) is a \(\psi-(\alpha, \beta, \gamma)\)-contraction pair. If \(T(M) \subset M\) is closed, then \(S\) and \(T\) have a unique common fixed point.

**Proof.** Since the pair \((S, T)\) satisfies the property \((E.A.)\), then there exists a sequence \((x_n)\) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z
\]
for some \(z \in M\). Since \(T(M)\) is closed, then \(z \in T(M)\) and \(z = Tu\) for some \(u \in M\). In the proof of Theorem 7, we can prove that \(z = Tu = Su\) and that \(z\) is the unique POC of \(S\) and \(T\). The rest of the proof follows from Theorem 6.

Remark 11. Since two noncompatible self-mappings on a metric space \((M, d)\) satisfy the property \((E.A.)\), then the conclusion of Theorem 10 remains valid if we consider \(S\) and \(T\) noncompatible self-mappings.

In the next theorem we drop closeness of the range of mapping and replace property \((E.A.)\) with \(CLR_T\) property.

**Theorem 12.** Let \((M, d)\) be a metric space and \(S, T : M \to M\) satisfying the \(CLR_T\) property. Let us suppose that the pair \((S, T)\) is a \(\psi-(\alpha, \beta, \gamma)\)-contraction pair. If the pair \((S, T)\) is nontrivially weakly compatible, then \(S\) and \(T\) have a unique common fixed point.

**Proof.** Since the pair \((S, T)\) satisfies the \(CLR_T\) property, then there exists \(z \in M\) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = Tz
\]
for some \(z \in M\).

The rest of the proof of the theorem follows easily.

### 4. Controlling by Altering Distance Functions

In 1976, Delbosco [12] initiated the study of fixed point for contractive conditions using altering distance functions; however, his study was limited to some power functions only. Subsequently, his result was extended by Skof [13] and Khan et al. [2] in 1977 and 1984, respectively. Since then, it has been used to solve several problems in the metric fixed point theory (see, e.g., [14–21]).

**Definition 13.** A function \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) is called an altering distance function if the following properties are satisfied:

\((\Psi_1)\) \(\psi(t) = 0\) if and only if \(t = 0\);
\((\Psi_2)\) \(\psi\) is monotonically nondecreasing;
\((\Psi_3)\) \(\psi\) is continuous.

By \(\Psi\) we are going to denote the set of all the altering distance functions.

Since every nondecreasing map \(\psi\) satisfies (11) (but the converse is not true), then all the previous results are valid, in particular, if we replace functions satisfying the condition (11) for altering distance functions.

On the other hand, in 2002, Branciari [22] extended the Banach-Caccioppoli theorem by using some Lebesgue integrable functions. Since then, several well-known fixed point criteria for contractive type of mappings have been generalized in this way. See, for example, [5, 10, 16, 17, 23–27] and a lot of references therein. In 2009, Jachymski [28] showed that most contractive conditions of integral type given by many authors are mere consequences of classical known ones (see [28] and references therein).

By \(\Phi\) is denoted the set of all mappings \(\phi : \mathbb{R}_+ \to \mathbb{R}_+\) satisfying the following conditions:

\((\Phi_1)\) \(\phi\) is a Lebesgue integrable mapping which is summable on each compact subset of \(\mathbb{R}_+\);
\((\Phi_2)\) \(\phi\) is nonnegative;
\((\Phi_3)\) for each \(\varepsilon > 0\), \(\int_0^\varepsilon \phi(t) dt > 0\).

A relation between these two classes of functions \(\Psi\) and \(\Phi\) is given in the following result ([29, 30]).

**Lemma 14.** For each \(\varphi \in \Phi\), the function \(\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+\) defined by \(\psi_0(s) = \int_0^s \varphi(t) dt, s \in \mathbb{R}_+\), is such that \(\psi_0 \in \Psi\).

In this way, additionally to the class of \(\psi-(\alpha, \beta, \gamma)\)-contraction pairs, we can consider a type of pair of mappings satisfying the following inequality contraction of integral type:

\[
\int_0^{\psi(d(Sx,Sy))} \varphi(t) dt \\
\leq \alpha(d(Tx,Ty)) \int_0^{\psi(d(Tx,Ty))} \varphi(t) dt + \beta(d(Tx,Ty)) \\
\times \int_0^{\psi(d(Sx,Tx))} \varphi(t) dt + \gamma(d(Tx,Ty)) \\
\times \int_0^{\psi(d(Sy,Ty))} \varphi(t) dt
\]

for all \(x, y \in M\), where \(\psi \in \Psi, \varphi \in \Phi, \alpha, \beta, \gamma\) are functions satisfying (9). Since this class can be rewritten as

\[
\psi_0 \left( \psi \left( d(Sx,Sy) \right) \right) \leq \alpha \left( d(Tx,Ty) \right) \psi_0 \left( \psi \left( d(Tx,Ty) \right) \right) \\
+ \beta \left( d(Tx,Ty) \right) \psi_0 \left( d(Sx,Tx) \right) \\
+ \gamma \left( d(Tx,Ty) \right) \psi_0 \left( d(Sy,Ty) \right)
\]

for all \(x, y \in M\), where \(\psi_0 \in \Psi\) is the function defined in Lemma 14, then we have that all the conclusions given for \(\psi-(\alpha, \beta, \gamma)\)-contraction pairs are automatically valid for pair of mappings satisfying the inequality contraction (31).
5. Conclusions and Examples

Notice that due to the minor restrictions on the functions involved in the definition of the class of $\psi(\alpha, \beta, \gamma)$-contraction pairs and the minimal commutative requirements of the mappings, our results extend several common fixed point theorems for classes of well-known contractive type of mappings, including various classes of contractive mappings with inequalities controlled by altering distance functions as well as contractive mappings of the integral type. Even more, the mappings $T$ and $S$ considered here are not necessarily continuous, so in this way our results are more general compared with other results in this line of research.

Next, we are going to show some examples in support of our results.

**Example 15.** Let $M = [0, 1]$ be equipped with the Euclidean metric. We consider the following mappings: $S, T: M \to M$ defined by $Sx = x/16$ and $Tx = x/2$ for all $x \in M$. Let $\alpha, \beta, \gamma: \mathbb{R}_+ \to [0, 1]$ defined by $\alpha(t) = \beta(t) = 1/4$ and $\gamma(t) = 1/8$ for all $x \in \mathbb{R}_+$ and $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula $\psi(t) = t^2$, $t \in \mathbb{R}_+$.

Notice that $\psi \in \mathcal{P}$ and the functions $\alpha, \beta, \gamma$ satisfy the conditions (9); also note that $S(M) \subset T(M)$ and $T(M)$ is a complete subspace of $M$. Moreover, it is not difficult to show that the pair $(S, T)$ is a $\psi(\alpha, \beta, \gamma)$-contraction pair. Besides, $C(S, T) = \{0\}$ and $ST0 = TS0 = 0$, which mean that $(S, T)$ is nontrivially weakly compatible. Then, Theorem 7 guarantees that $\omega = 0$ is the unique common fixed point of $S$ and $T$.

**Example 16.** As in the example before, $M = [0, 1]$ with the usual metric. We define the self-maps $S, T$ on $M$ by

$$
Sx = \begin{cases} 
0, & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{1}{16}, & \text{if } \frac{1}{2} < x \leq 1
\end{cases} \quad (33)
$$

and $Tx = x/2$ for all $x \in M$. Let $\alpha, \beta, \gamma: \mathbb{R}_+ \to [0, 1]$ defined as follows:

$$
\alpha(t) = \frac{1}{8}, \quad \beta(t) = \gamma(t) = \frac{1}{4}, \quad \forall t \in \mathbb{R}_+. \quad (34)
$$

Let $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\psi(t) = t^2$, $t \in \mathbb{R}_+$. Then, the pair $(S, T)$ is a $\psi(\alpha, \beta, \gamma)$-contraction pair satisfying the hypotheses of Theorem 6; thus $\omega = 0$ is the unique POC and moreover the unique common fixed point of $S$ and $T$.

**Example 17.** Let $M = [1/2, 1]$ with the usual metric on $\mathbb{R}$. In this case we consider the mappings $S, T: M \to M$ defined by

$$
Sx = \begin{cases} 
\frac{1}{2}, & \text{if } \frac{1}{2} \leq x < \frac{2}{3} \\
1 - \frac{1}{x}, & \text{if } \frac{2}{3} \leq x \leq 1
\end{cases} \quad (35)
$$

and $Tx$ defined as follows:

$$
Tx = \begin{cases} 
1, & \text{if } \frac{1}{2} \leq x < \frac{2}{3} \\
x, & \text{if } \frac{2}{3} \leq x \leq 1
\end{cases}
$$

Let $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula $\psi(t) = t^2$, $t \in \mathbb{R}_+$.

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