Equisingularity, Multiplicity, and Dependence

Steven L Kleiman

Department of Mathematics, Room 2-278 MIT, 77 Mass Ave, Cambridge, MA 02139-4307, USA
E-mail: kleiman@math.mit.edu

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Abstract. This is a report on some recent work by Gaffney, Massey, and the author, characterizing the conditions $A_f$ and $W_f$ for a family of ICIS germs equipped with a function. First we introduce the work informally. Then we review the formal definitions of $A_f$ and $W_f$, and state the theorems that characterize them by the constancy of Milnor numbers. Next we review the definition of the Buchsbaum–Rim multiplicity, and reformulate the theorems by the constancy of certain Buchsbaum–Rim multiplicities. Finally, we review the theory of integral dependence of elements on submodules of free modules, and apply it to prove the reformulated theorems.

1. Introduction

The conditions $A_f$ and $W_f$ are “relative” forms of the Whitney conditions $A$ and $B$ (or $W$). The latter are important as they are manageable algebraic-geometric conditions on the tangents to a variety $X$, yet they imply the local topological triviality of $X$ along a given smooth subvariety $Y$ such that $X - Y$ is smooth. When $X$ carries a map $f$ that has constant rank off $Y$, then $A_f$ and $W_f$ are defined as the corresponding conditions on the tangents to the level sets (or fibers) of $f$. Thom introduced $A_f$ as the primary condition that would ensure the topological triviality of the pair $X, f$ along $Y$.

By definition, $A_f$ and $W_f$ signify this. For simplicity, assume that $f$ is a nonconstant function vanishing on $Y$, and embed $X$ in $\mathbb{C}^n$ so that $Y$ is a linear subspace through the origin 0. Then $A_f$ holds at 0 if $Y$ lies in every hyperplane obtained as a limit of hyperplanes $H$, each tangent to a level hypersurface of $f$ at a point $x$ of $X - Y$, as $x$ approaches 0. A more stringent condition, $W_f$ signifies that the angle between $H$ and $Y$ approaches 0 as fast as $x$ approaches 0.

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The Thom–Mather second isotopy lemma readily yields the following isotopy theorem: if $W_f$ holds at 0, then, after $X$ is replaced by a neighborhood of 0, the pair $X, f$ is topologically right trivial. More precisely, let $X_y$ denote the fiber of a transverse projection to $Y$, and set $f_y := f|X_y$. Then there is a homeomorphism $h$ from the product $X_0 \times Y$ onto $X$ such that $fh$ is equal to $f_0 \times 1_Y$. Thus, if $W_f$ holds at 0, then, for $y \in Y$ near 0, the pairs $X_y, f_y$ are topologically the same, or “topologically equisingular.”

In the present article, we discuss and develop some recent work done by Gaffney, Massey, and the author in [5] and [6] on the algebraic-geometric significance of $A_f$ and $W_f$. Here $X$ is the total space of a complex analytic family of germs of isolated complete-intersection singularities (ICIS germs) $X_y$, parameterized by a smooth variety $Y$.

For convenience, we identify $Y$ with the subvariety of $X$ traced by the central points $0 \in X_y$ as $y$ varies. We assume $f$ vanishes on $Y$. We choose an embedding of $X$ in $\mathbb{C}^a \times \mathbb{C}^b$ such that $Y$ represents the germ of $0 \times \mathbb{C}^b$. We choose an extension of $f$ over a neighborhood of $X$, and we denote the extension by $f$ too.

Needless to say, our results are independent of these choices.

What does $A_f$ mean by itself? Let $Z_y$ be the level hypersurface through 0 of $f_y$. It turns out that $A_f$ is closely related to the vanishing cycles associated to the individual $X_y$ and $Z_y$. In the case where $X$ represents the germ of the whole affine space $\mathbb{C}^a \times \mathbb{C}^b$ and $Y$ is the critical set $\Sigma(f)$, Lê and Saito [18] proved that if the number of vanishing cycles, the Milnor number $\mu(Z_y)$ at 0, is constant in $y$, then $A_f$ holds.

We prove a definitive generalization of this celebrated theorem of Lê and Saito to ICIS germs. (However, Green and Massey, [7] and [21], showed that other information about the vanishing cycles implies $A_f$ for families with generalized isolated singularities.) Namely, we characterize $A_f$ by the constancy in $y$ of the Milnor numbers $\mu(X_y)$ and $\mu(Z_y)$. Notice that these Milnor numbers refer to the ambient topology of $X_y$ and $Z_y$ in $\mathbb{C}^a$, but the isotopy theorem does not.

We characterize $W_f$ similarly, by the constancy of the sequences of Milnor numbers $\mu_i(X_y)$ and $\mu_i(Z_y)$ of the sections of $X_y$ and $Z_y$ by general linear spaces of codimension $i$ for $i = 0, \ldots, a - k$. The appearance of these linear sections is a reflection of the Lipschitz-like nature of the vector fields that we integrate to prove the isotopy theorem. (The corresponding theorem for $W$ was proved for hypersurface germs in part by Teissier and in part by Briançon and Speder, and for general ICIS germs by Gaffney [4, Thm. 1].)

We also characterize $A_f$ and $W_f$ by the constancy of certain Buchsbaum–Rim multiplicities. Say $X$ is defined in $\mathbb{C}^a \times \mathbb{C}^b$ by the vanishing of $f_1, \ldots, f_k$, and form the Jacobian matrix of $f_1, \ldots, f_k, f$ with respect to the first $a$ variables. For each $y$, the $a$ columns of this matrix generate a module over the local ring $O_{X_y, 0}$. This is a submodule $\mathcal{M}_y$ of finite colength of the free module of rank $k+1$. We characterize $A_f$ by the constancy in $y$ of the Buchsbaum–Rim multiplicity $e(\mathcal{M}_y)$. Let $m_y$ be the maximal ideal of $X_y$; its appearance is a more direct
reflection of the Lipschitz-like nature of the vector fields. We characterize $W_f$ by the constancy of the Buchsbaum–Rim multiplicity of the product, $e(m_Y M_y)$.

Thus our theorems provide necessary and sufficient conditions for $A_f$ (resp., for $W_f$) to hold. These conditions require the constancy in $y$ of certain numerical invariants of the individual pairs $X_y, f_y$. Therefore, when these invariants are constant, then, in whatever way the individual pairs are glued together to form a pair $X, f$, necessarily $A_f$ (resp., $W_f$) holds. Conversely, if one pair $X, f$ exists for which $A_f$ (resp., $W_f$) holds, then these invariants of $X_y, f_y$ have the same values for all $y \in Y$ near 0. Thus these numerical invariants may be considered as indicators of “$A_f$-equisingularity” (resp., “$W_f$-equisingularity”).

The invariants depend only on the germs of $X_y$ and $f_y$ at 0. Yet, of course, there is a significant difference between these germs and the larger representatives themselves. Assume that 0 is the only critical point of the germ of $f_y$. Still, the representative $f_y$ may have an additional critical point no matter how close $y$ is to 0. However, $f_y$ has no additional critical point if our numerical invariants are constant. Furthermore, if there is no such critical point, then $A_f$ holds. Conversely, if $A_f$ holds, then, for all $y$ close to 0, each additional critical point of $f_y$ is also a critical point of $f$.

Our treatments of $A_f$ and $W_f$ run in parallel. Not only are the statements similar, but, to a fair extent, the proofs are similar. First, we prove that the two numerical characterizations of $A_f$ (resp., of $W_f$) are equivalent by proving that the constancy of the Milnor numbers is equivalent to the constancy of the Buchsbaum–Rim multiplicities. This proof involves the theorem of Lê and Greuel, which re-expresses the Milnor numbers algebraically, and some theorems of Buchsbaum and Rim, which re-express the multiplicity $e(M_y)$ as a length (resp., and a more recent theorem, which re-expresses $e(m_Y M_y)$ as a linear combination of the polar multiplicities of $X_y$).

We prove the characterization of $A_f$ (resp., of $W_f$) by Buchsbaum–Rim multiplicities using the theory of integral dependence of elements on submodules of free modules. Let $M$ be the module, over the local ring of $X$ at 0, generated by the $a$ columns of the Jacobian matrix above. Let $g_j$ be the column vector of partial derivatives of $f_1, \ldots, f_k, f$ with respect to the $j$th coordinate variable on $C^b$. Finally, let $m_Y$ be the ideal of $Y$ in $X$.

We characterize $A_f$ (resp., $W_f$) by the integral dependence of $g_1, \ldots, g_b$ on $M$ (resp., on $m_Y M$). We prove this characterization of $A_f$ using a remarkable result, Lemma (5.7) in [6], concerning the geometry of the relative conormal variety. Technically, this result is the new ingredient, which permits perfecting the treatment of $A_f$ in [5]. We prove the characterization of $W_f$ via a direct computation with analytic inequalities.

The restrictions $g_j|X_y$ are always integrally dependent on $M_y$ (resp., on $m_Y M_y$) for all $y$ in a dense Zariski open subset of $Y$, as $A_f$ (resp., $W_f$) holds for all $y$ in such a subset by a celebrated theorem proved by Hironaka (resp., by Henry, Merle, and Sabbah). It follows that, if the Buchsbaum–Rim multiplicity
$e(M_y)$ (resp., $e(m_y M_y)$) is constant, then the $g_j$ are dependent on $M$ (resp., on $m_y M$) by virtue of the “principle of specializations of integral dependence.”

This principle is the main algebraic result, Theorem (1.8), in [5]. For ideals, it was discovered, named, and proved by Teissier [24, 3.2, p. 330] and [25, App. I]. To prove it for modules, we follow Teissier’s approach, but first we must generalize certain basic results in the theory of multiplicity for ideals, including upper semicontinuity, Rees’s characterization of integral dependence, and Böger’s generalization of it (Teissier rediscovered the latter in the case at hand). Upper semicontinuity of the Buchsbaum–Rim multiplicity is proved in Proposition (1.1) of [5]. The generalizations of Rees’s theorem and of Böger’s theorem are proved in Theorems (6.7a)(iii) and (10.9) in [14]. Moreover, Böger’s theorem is given a new proof in [16]; see its subsections (1.4) and (1.7). This proof is simpler, shorter, and more direct. It was inspired by Lemma (5.7) in [6].

The present article is meant to give a feeling for the nature and the spirit of the work in [5] and its extension in [6]. The latter article also surveys a lot of other work in equisingularity theory, and discusses its historical development. This is a masterful report, and may be highly recommended.

The present discussion is, of course, limited, and so achieves greater focus. Some results in [5] are not discussed here, and some are discussed in restricted generality. Some proofs are abridged, and some are omitted. Also, the general theory is recalled in the special case at hand. However, some noteworthy results were not explicitly discussed before, and they are stated and proved here.

In Section 2, we review the formal definitions of $A_f$ and $W_f$. We discuss isotopy, and state the two theorems, which characterize $A_f$ and $W_f$ by the constancy of Milnor numbers. In Section 3, we review the formal definition of the Buchsbaum–Rim multiplicity. Then we reformulate the theorems in terms of the constancy of Buchsbaum–Rim multiplicities, and we prove that the two formulations are equivalent. In Section 4, we review the theory of integral dependence of elements on submodules of free modules, and we characterize $A_f$ and $W_f$ in terms of integral dependence. Finally, we apply this theory to prove the reformulated theorems.

2. Equisingularity

Let $X$ be a complex-analytic germ at 0 in $\mathbb{C}^a \times \mathbb{C}^b$. Say that, on a (Euclidean) neighborhood of 0, we have

$$X : f_1, \ldots, f_k = 0,$$

where each $f_i$ is an analytic function $f_i(x, y)$ of the two sets of variables,

$$x = (x_1, \ldots, x_a) \text{ and } y = (y_1, \ldots, y_b).$$

Assume that $X$ is a reduced complete intersection of codimension $k$ with $k < a$. 
For fixed $y$, let $X_y \subset \mathbb{C}^a$ denote the locus of $x$ such that $(x, y) \in X$; so $X_y$ is the locus of zeros of the $f_i(x, y)$ as $x$ varies. Assume that, if $X_y$ is nonempty, then $0 \in X_y$. Let $Y$ be the locus of $y$ with $(0, y) \in X$, assume that $Y$ contains a neighborhood of 0 in $\mathbb{C}^b$, and identify $Y$ with $0 \times Y$. View $Y$ as the parameter space and $X$ as the total space of the family of $X_y$. Assume that the $X_y$ represent germs at 0 of isolated complete-intersection singularities (ICIS germs) of codimension $k$.

Let $f$ be a nonconstant analytic function on $X$. Set

$$f_y := f|_{X_y} \text{ and } Z := X \cap f^{-1}0 \text{ and } Z_y := X_y \cap f^{-1}0.$$ 

Assume that $f$ vanishes on $Y$, so $Y \subset Z$.

If $x \in X$ is a simple point of the level hypersurface $f^{-1}fx$, then $x$ must be a simple point of $X$. Let $\Sigma(f)$ denote the critical set, the union of the singular sets of the various level hypersurfaces. Then, in other words, $\Sigma(f)$ contains the singular set of $X$.

In turn, $\Sigma(f)$ is contained in the union of the critical sets of the various restrictions $f_y$. Denote this union by $\Sigma_Y(f)$. Note that $\Sigma_Y(f)$ consists of all the singular points of all the level hypersurfaces of all the restrictions $f_y$ for all the $y$ in $Y$. Replacing $X$ by a smaller representative of the same germ, we may also assume that every component of $\Sigma_Y(f)$ contains 0. Similarly, if we assume that $Z_0$ represents a germ with an isolated singularity at 0, then we may also assume that the projection $\Sigma_Y(f) \to Y$ is finite. Note that we work only with the reduced structures on the sets $\Sigma(f)$ and $\Sigma_Y(f)$.

The Thom condition $A_f$ can be formulated succinctly using the relative conormal variety $C(X, f)$ and the absolute conormal variety $C(Y)$. These varieties are defined as follows. Both are closures in $X \times \mathbb{P}^{a+b-1}$, or more correctly, in the restriction to $X$ of the projectivized cotangent bundle of $\mathbb{C}^a \times \mathbb{C}^b$. The former closure is that of the set of pairs $(x, H)$ such that $x$ is a point in $X - \Sigma(f)$ and $H$ is a hyperplane tangent at $x$ to the level surface $f^{-1}fx$. The latter closure is that of the set of pairs $(x, H)$ such that $x \in Y \subset H$. In these terms, $A_f$ is said to be satisfied by the pair $(X - \Sigma(f), Y)$ at 0 if the fiber of $C(X, f)$ over $0 \in X$ lies in $C(Y)$.

Suppose the germs of $\Sigma(f)$ and $Y$ at 0 are equal. If $A_f$ holds at 0, then $A_f$ holds at every $y$ in a neighborhood $U$ of 0 in $Y$. This statement is not obvious from the definition, but follows from Proposition (4.2)(2) below.

Hence, $C(Y)$ contains the preimage of $U$ in $C(X, f)$, which will be denoted by $C(X, f)|U$. Since the intersection $C(X, f) \cap C(Y)$ always projects into $Y$, we conclude that $A_f$ holds at 0 if and only if, after we replace $X$ by a smaller representative of the same germ, we obtain the set-theoretic equation,

$$C(X, f) \cap C(Y) = C(X, f)|Y.$$

In other words, $A_f$ holds at 0 if and only if, along the fiber of $C(X, f)$ over $0 \in X$, the ideal of the intersection $C(X, f) \cap C(Y)$ has the same radical as the ideal of the preimage $C(X, f)|Y$; compare with the Remarque on p. 550 in [19].
The condition $A_f$ can also be expressed analytically in terms of the “angular distance” $\text{dist}(Y, T_x f^{-1} f x)$ from $Y$ to the tangent space at $x$ to the level hypersurface. Namely, $A_f$ holds at $0$ if and only if this distance approaches $0$ as $x$ approaches $0$ along any analytic path $\phi: (C, 0) \to (X, 0)$ such that $\phi(u)$ lies in $X - \Sigma(f)$ for $u \neq 0$. Now, this distance approaches $0$ if and only if the inequality,
\[
\text{dist}(Y, T_x f^{-1} f x) \leq c \cdot \text{dist}(x, Y)^e,
\]
holds where the constant $c$ and the exponent $e$ depend on the path $\phi$.

The Whitney condition relative to $f$, denote $W_f$, is defined by requiring the preceding analytic inequality to hold for some constant $c$ independent of $\phi$ and for $e = 1$. This condition generalizes Teissier’s condition of ‘$c$-equisingularity’ (see [19, top, p. 550]). It reduces to Whitney’s Condition B, in the equivalent form of Verdier’s condition $W$ [27, Sect. 1], when $f$ is constant (and so vanishes).

There is also a characterization of $W_f$ in terms of the conormal varieties $C(X, f)$ and $C(Y)$, which strengthens the corresponding characterization of $A_f$ in an interesting way. Indeed, Lê and Teissier proved (a more general version of) the following result in their Proposition 1.3.8 on p. 550 of [19] (see Proposition (6.1) in [5] for another treatment of the general case): $W_f$ holds at $0$ if and only if, along the fiber of $C(X, f)$ over $0 \in X$, the ideal of the intersection $C(X, f) \cap C(Y)$ is integral over the ideal of the preimage $C(X, f)|Y$.

The condition $W_f$ implies that the pair $X, f$ is topologically right trivial over $Y$. Indeed, Thom, Mather, and Teissier, Verdier, Gaffney and others introduced and developed the requisite methods to prove this triviality by integrating vector fields. Namely, it is possible to take the constant tangent vector field to $Y$ and carefully lift it to $X$ so that the lift is Lipschitz-like (Fr. rugueux), hence integrable, and is tangent to the level hypersurfaces of $f$, so that the integral gives an appropriate continuous flow on $X$.

Thus we obtain the following isotopy theorem: If the critical set $\Sigma(f)$ is equal to $Y$ and if the pair $(X - Y, Y)$ satisfies $W_f$ at $0$, then, after $X$ is replaced by a neighborhood of $0$, there is a homeomorphism $h: X_0 \times Y \to X$ such that $fh = f_0 \times 1_Y$ and $h$ is $C^\infty$ off $Y$.

Let $X^i$ be the section of $X$ by a general linear space of codimension $i$ containing $Y$, and set $f^i := f|X^i$; so $X^0 = X$ and $f^0 = f$. If $W_f$ holds, then $W_{f^i}$ holds for $i = 0, \ldots, a - k - 1$ because the requisite analytic inequality follows from that for $W_f$. Hence the pair $X^i, f^i$ is also topologically trivial by the isotopy theorem.

The next theorem characterizes $W_f$ in terms of the Milnor numbers $\mu_i(X_y)$ and $\mu_i(Z_y)$ of the sections of $X_y$ and $Z_y$ by a general linear space through $0$ of codimension $i$ for $i = 0, \ldots, a - k$. By convention, $\mu_{a-k}(X_y)$ and $\mu_{a-k-1}(Z_y)$ are the ordinary multiplicities at $0$ diminished by $1$, and $\mu_{a-k}(Z_y) = 1$. However, for $\mu_i(Z_y)$ to be defined, $Z_y$ too must have an isolated singularity at $0$. For $y$ near $0$, it does if the germ of $\Sigma(f)$ and $Y$ at $0$ are equal, and if $W_f$, or simply $A_f$, holds; indeed, then the germ of $\Sigma(f)$ and $\Sigma_y(f)$ at $0$ are equal by Lemma (4.3) below.
**Theorem (2.1)** The following three conditions are equivalent:

(i) the germs of $\Sigma(f)$ and $Y$ at 0 are equal, and the pair $(X - \Sigma(f), Y)$ satisfies $W_f$ at 0;

(ii) for the $y$ in a neighborhood of 0 in $Y$, the level hypersurface $Z_y$ has an isolated singularity at 0, and the sequences of Milnor numbers, $\{\mu_i(X_y)\}$ and $\{\mu_i(Z_y)\}$, are constant in $y$;

(ii') for the $y$ in a neighborhood of 0 in $Y$, the level hypersurface $Z_y$ has an isolated singularity at 0, and the sequence of sums of Milnor numbers, $\{\mu_i(X_y) + \mu_i(Z_y)\}$, is constant in $y$.

This theorem is part of Theorem (6.4) in [5]. The latter also asserts that (i) and (ii) are equivalent to the following condition: the germs of $\Sigma_Y(f)$ and $Y$ at 0 are equal, and both pairs $(X - Y, Y)$ and $(Z - Y, Y)$ satisfy the absolute Whitney condition $W$ at 0.

This additional condition is implied by (i). Indeed, the two germs are equal by Lemma (4.3) below because $W_f$ implies $A_f$. Furthermore, (i) implies the requisite analytic inequalities. Indeed, $T_x f^{-1} f x \subset T_x X$ and, if $x \in Z$, then $T_x f^{-1} f x = T_x Z$.

The additional condition implies (ii); indeed, this implication is virtually the assertion of Théorème (10.1) on p. 223 of [22]. Trivially, (ii) implies (ii').

Finally, (ii') implies (i), but our proof is more involved, and runs as follows: in Theorem (3.1) below, we replace (ii') with an equivalent condition involving a Buchsbaum–multiplicity, and then at the end of Section 4, we prove that latter implies (i). On the other hand, there is a direct proof that the additional condition implies (i); indeed, Briançon, Maisonobe and Merle gave such a proof in [1, Thm. 4.3.2, p. 543] in a more general setting using a different approach.

The next theorem characterizes $A_f$ in terms of Milnor numbers.

**Theorem (2.2)** The following four conditions are equivalent:

(i) the germs of $\Sigma(f)$ and $Y$ at 0 are equal, and the pair $(X - \Sigma(f), Y)$ satisfies $A_f$ at 0;

(ii) the germs of $\Sigma_Y(f)$ and $Y$ at 0 are equal;

(iii) for the $y$ in a neighborhood of 0 in $Y$, the level hypersurface $Z_y$ has an isolated singularity at 0, and the Milnor numbers, $\mu(X_y)$ and $\mu(Z_y)$, are constant in $y$;

(iii') for the $y$ in a neighborhood of 0 in $Y$, the level hypersurface $Z_y$ has an isolated singularity at 0, and the sum of Milnor numbers, $\mu(X_y) + \mu(Z_y)$, is constant in $y$.

Notice that (iii) implies (iii') trivially. The converse holds because $\mu(X_y)$ and $\mu(Z_y)$ are each upper semicontinuous by [20, bot. p. 126]. We prove the rest of the theorem in two steps too: first, we replace (iii') with an equivalent condition involving a Buchsbaum–Rim multiplicity, obtaining Theorem (3.2); then at the end of Section 4, we prove Theorem (3.2).
In the case where $X$ represents the germ of $\mathbb{C}^a \times \mathbb{C}^b$, Lê and Saito [18] proved that (iii) implies (i). They used Morse theory, but Teissier reproved their theorem almost immediately using more algebraic-geometric methods. Teissier’s work served as a model for most of the work in this report.

Combining the preceding two theorems, we obtain the following corollary, which asserts the equivalence of $W_f$, $W_{f_1}$, and $A_{f_1}$.

**Corollary (2.3)** If $\Sigma(f) = Y$, then the following conditions are equivalent:

(i) the pair $(X - Y, Y)$ satisfies $W_f$ at 0;
(ii) the pair $(X^i - Y, Y)$ satisfies $W_{f_i}$ at 0 for every $i$;
(iii) the pair $(X^i - Y, Y)$ satisfies $A_{f_i}$ at 0 for every $i$.

### 3. Multiplicity

In this section, we reformulate Theorems (2.1) and (2.2). Instead of Milnor numbers, we use certain Buchsbaum–Rim multiplicities. Thus we obtain Theorems (3.1) and (3.2). In the next section, we discuss the proofs of these reformulated theorems.

The Buchsbaum–Rim multiplicity was introduced by Buchsbaum and Rim [2] in 1963, and the theory has been developed more recently by Kirby and Rees [13], by Henry and Merle [10], and by Thorup and the author [14] and [15]. For our purposes here, it suffices to work over the local ring $O$ of a complex-analytic germ, say one of dimension $d$. Let $E$ be a free $O$-module, and $M$ a submodule of finite colength; that is, the vector-space dimension $\dim \mathbb{C}(E/M)$ is finite.

The Buchsbaum–Rim multiplicity generalizes the ordinary multiplicity. In the case where $E$ is the ring $O$ and where $M$ is an ideal $I$, Samuel defined the multiplicity $e(I)$ in 1951 to be the rectified leading coefficient $e$ of the Hilbert–Samuel polynomial,

$$\dim \mathbb{C}(E/I^n) = e n^d/d! + \cdots$$

for $n \gg 0$.

Buchsbaum and Rim considered the case in which $E$ is free of arbitrary (finite) rank $r$. They generalized Samuel’s definition essentially as follows. Form the symmetric algebra $O[E]$; it is just the polynomial algebra in $r$ variables with coefficients in $O$. Form the Rees algebra $O[M]$; it is just the subalgebra generated by $M$ placed in degree 1. Both these algebras are graded; denote their $n$th graded pieces by $O[E]_n$ and $O[M]_n$. For example, if $E = O$ and $M = I$, then $O[E]_n = O$ and $O[M]_n = I^n$.

Buchsbaum and Rim formed the quotient of these two graded pieces, and proved that its dimension is eventually given by a polynomial in $n$ of degree $d + r - 1$; thus,

$$\dim \mathbb{C}(O[E]/O[M])_n = e n^{d+r-1}/(d + r - 1)! + \cdots$$

for $n \gg 0$.

Then they defined the multiplicity $e(M)$ to be $e$. 
To use the Buchsbaum–Rim multiplicity, return to the setup described at the beginning of Section 2. Fix an extension of $f$ over a neighborhood of $X$ in $\mathbb{C}^a \times \mathbb{C}^b$ on which $f_1, \ldots, f_k$ are defined, and abusing notation, denote the extension too by $f$. Form the Jacobian matrix with respect to the first set of variables:

$$
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_a} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \ldots & \frac{\partial f_k}{\partial x_a} \\
\frac{\partial f}{\partial x_1} & \ldots & \frac{\partial f}{\partial x_a}
\end{bmatrix}
$$

Its columns generate a module over the affine ring $\mathcal{O}_X$,

$$JM_x(f_1, \ldots, f_k; f) \subset \mathcal{E} := \mathcal{O}^{k+1}_X,$$

called the Jacobian module of $f_1, \ldots, f_k; f$ with respect to the $x$-variables.

Given $y \in Y$, form the image $\mathcal{M}_y$ of $JM_x(f_1, \ldots, f_k; f)$ in the free module $\mathcal{E}_y$ of rank $k + 1$ over the local ring $\mathcal{O}_{X_y,0}$; so

$$\mathcal{M}_y := JM_x(f_1, \ldots, f_k; f)|_{X_y} \subset \mathcal{E}_y.$$ 

In addition, denote the maximal ideal of $\mathcal{O}_{X_y,0}$ by $\mathfrak{m}_y$.

Suppose for a moment that each $Z_y$ has an isolated singularity at 0. Since $\Sigma_Y(f) = \text{Supp}(\mathcal{E}/JM_x(f_1, \ldots, f_k; f))$, the module $\mathcal{M}_y$ has finite colength. Hence the Buchsbaum–Rim multiplicities $e(\mathcal{M}_y)$ and $e(\mathfrak{m}_y, \mathcal{M}_y)$ are defined.

We can now reformulate Theorems (2.1) and (2.2); we obtain Theorems (3.1) and (3.2). After stating them, we discuss the proof that the two formulations are equivalent.

**Theorem (3.1)** The following two conditions are equivalent:

(i) the germs of $\Sigma(f)$ and $Y$ at 0 are equal, and the pair $(X - \Sigma(f), Y)$ satisfies $W_f$ at 0;

(ii) for the $y$ in a neighborhood of 0 in $Y$, the level hypersurface $Z_y$ has an isolated singularity at 0, and the multiplicity $e(\mathfrak{m}_y, \mathcal{M}_y)$ is constant in $y$.

**Theorem (3.2)** The following three conditions are equivalent:

(i) the germs of $\Sigma(f)$ and $Y$ at 0 are equal, and the pair $(X - \Sigma(f), Y)$ satisfies $A_f$ at 0;

(ii) the germs of $\Sigma_Y(f)$ and $Y$ at 0 are equal;

(iii) for the $y$ in a neighborhood of 0 in $Y$, the level hypersurface $Z_y$ has an isolated singularity at 0, and the multiplicity $e(\mathcal{M}_y)$ is constant in $y$.

The new theorems are equivalent to the old because of the following lemma. Indeed, the summands in (3.3.2) are upper semicontinuous; so they are constant if and only if $e(\mathfrak{m}_y, \mathcal{M}_y)$ is. For future use, note that therefore the lemma yields this: if $e(\mathfrak{m}_y, \mathcal{M}_y)$ is constant, then so is $e(\mathcal{M}_y)$. 
Lemma (3.3) For each \( y \in Y \), we have these two equations:

\[
e(M_y) = \mu(X_y) + \mu(Z_y); \quad (3.3.1)
\]

\[
e(m_y M_y) = \sum_{i=0}^{a-k} \binom{a-1}{i} (\mu_i(X_y) + \mu_i(Z_y)). \quad (3.3.2)
\]

Moreover, each sum \( \mu_i(X_y) + \mu_i(Z_y) \) is upper semicontinuous in \( y \).

To prove Equation (3.3.1), note that \( X_y \) is a complete intersection of dimension \( a-k \) and that \( M_y \) is a submodule of \( O_{X_y,0}^{k+1} \) generated by \( a \) elements, namely, the columns of the Jacobian matrix. Denote the ideal of maximal minors of this matrix by \( \mathcal{J} \). Then some theorems of Buchsbaum and Rim [2, 2.4, 4.3, 4.5] yield

\[
e(M_y) = \dim C(O_{X_y,0}/\mathcal{J}O_{X_y,0}). \quad (3.3.3)
\]

The right side is equal to the sum \( \mu(X_y) + \mu(Z_y) \) by the theorem of Lê [17, Thm. 3.7.1, p. 130] and Greuel [8, Kor. 5.5, p. 263]. Thus Equation (3.3.1) holds.

To prove Equation (3.3.2), for each \( i \), let \( P_i \) be a general linear space of codimension \( i \) in \( C^a \), and let \( \Pi^i \) be the \( i \)-dimensional “relative polar subscheme” of \( f_y \) with \( P_i \) as pole. By definition, \( \Pi^i \) is the closure in \( X_y \) of the locus of simple points \( x \) of the level hypersurface surface \( X_y \cap f^{-1} f x \) such that there exists a tangent hyperplane at \( x \) that contains \( P_i \). Algebraically, \( \Pi^i \) is cut out of \( X_y \) by the maximal minors of the Jacobian matrix of the map \( C^a \to C^{k+1} \times C^i \) with components \( f_1, \ldots, f_k, f \) and \( p_i \), where \( p_i: C^a \to C^i \) is the linear map with kernel \( P_i \). Hence \( \Pi^i \) is Cohen–Macaulay as \( X_y \) is.

From the polar multiplicity formula [14, Thm. (9.8)(i)] (compare with [10, 4.2.7] and [4, §3]), it follows that

\[
e(m_y M_y) = \sum_{i=0}^{a-k} \binom{a-1}{i} m(\Pi^i),
\]

where \( m(\Pi^i) \) is the ordinary multiplicity at 0 of \( \Pi^i \). In particular, \( m(\Pi^{a-k}) \) is simply the multiplicity of \( X_y \) at 0; so it is equal to \( \mu_{a-k}(X_y) + 1 \). For any \( i \), since \( \Pi^i \) is Cohen–Macaulay,

\[
m(\Pi^i) = \dim C(O_{\Pi^i,0}/\mathcal{I}_i), \quad (3.3.4)
\]

where \( \mathcal{I}_i \) is the ideal of any linear space \( L_i \) of codimension \( i \) in \( C^a \) that is transverse to \( \Pi^i \). It follows that \( m(\Pi^i) \) is upper semicontinuous.

Remarkably, although \( \Pi^i \) is defined using \( P_i \), nevertheless \( P_i \) is transverse to \( \Pi^i \) simply because \( P_i \) is general. This important result was proved by Teissier in [26, (4.1.8), p. 569] as a consequence of his general idealistic Bertini theorem. The result was also proved, at about the same time, by Henry and Merle [9, Cor. 2, p. 195]. The result is reproved in Lemma (6.2) of [5] in a new way, using the theory of the \( W_f \) condition in the spirit of the current work. By this transversality result, we may take \( L_i \) to be \( P_i \). Then, for \( i < a-k \), the right
side of Equation (3.3.4) is equal to $\mu_i(X_y) + \mu_i(Z_y)$ by the theorem of Lê and Greuel. The asserted formula follows immediately, and the proof is complete.

4. Dependence

In this section, we discuss the proofs of Theorems (3.1) and (3.2), and thereby complete the proofs of Theorems (2.1) and (2.2). Our main technical tool is the theory of integral dependence of elements on modules, which generalizes the older theory for ideals. We begin by reviewing this theory.

Let $O$ be the local ring of a complex-analytic germ $(X, 0)$. Let $E$ be a free $O$-module, $M$ a submodule, and $g \in E$ an element. By definition, $g$ is integrally dependent on $M$ if, when $g$ is viewed as an element of degree 1 in the symmetric algebra $O[E]$, then $g$ is integrally dependent on the Rees algebra $O[M]$ in the usual sense; namely, $g$ satisfies an equation of integral dependence,

$$g^n + r_1 g^{n-1} + \cdots + r_n = 0,$$

where $n \geq 1$ and $r_i \in O[M]$, both depending on $g$. Of course, each $r_i$ may be replaced by its homogeneous piece of degree $i$ if desired.

There are two useful criteria for integral dependence. The first is a form of the valuative criterion, but it is also known in the trade as the “curve criterion.”

(Curve criterion) For $g \in E$ to be integrally dependent on $M$ it is necessary that, for every map germ $\phi: (C, 0) \to (X, 0)$, the pullback $\phi^* g$ lie in the pullback $\phi^* M$, viewed in the free $O_{C,0}$-module $\phi^* E$, or put more concisely,

$$\phi^* g \in \phi^* M \subset \phi^* E;$$

conversely, it is sufficient that this condition obtain for every nonconstant $\phi$ whose image meets any given dense Zariski open subset of $X$.

The second criterion is an analytic inequality, which re-expresses integral dependence in terms of speeds of vanishing.

(Analytic criterion) For $g \in E$ to be integrally dependent on $M$ it is necessary that, for any finite set of generators $g_i$ of $M$, there exist a Euclidean neighborhood $U$ of 0 in $X$ and a constant $c$ such that $|g(x)| \leq c \max |g_i(x)|$ for any $x \in U$; conversely, it is sufficient that this condition obtain for some finite set of generators $g_i$ of $M$.

Indeed, the condition in the “curve criterion” is equivalent to the condition in the “analytic criterion” by Proposition 1.11 on p. 306 of [3]. In fact, on p. 303 in [3], Gaffney takes the former as the defining condition of integral dependence. On p. 305, he proves that this definition is equivalent to Rees’s definition [23, p. 435]. Finally, Theorem 1.5 in [23, p. 437] yields the curve criterion.

In fact, neither [3] nor [23] treat the present version of the curve criterion, with its weaker sufficiency condition. However, it is not difficult to extend that work: if $\phi^* g \notin \phi^* M$, then $\phi$ can be tweaked, preserving this relation, so that its image does meet the given open set (see the proof of Prop. 1.7 on p. 304 in [3]).
From now on, work in the setup described at the beginning of Section 2. The next lemma is the main algebraic result, Theorem (1.8), in [5]. Let $E, M,$ and $g$ be as above, but assume that they arise from a free module, a submodule, and an element defined over the affine ring of $X$. In addition, for each $y \in Y$, form the restriction $E_y$ of the free module, the image $M_y \subset E_y$ of the submodule, and the image $g_y \in E_y$ of the element. Finally, assume that $M_y$ has finite colength.

Lemma (4.1) (Principle of specialization of integral dependence) Assume that the Buchsbaum–Rim multiplicity $e(M_y)$ is constant in $y$. Then $g$ is integrally dependent on $M$ if $g_y$ is integrally dependent on $M_y$ for all $y$ in a dense Zariski open subset of $Y$.

Of course, if $g$ is integrally dependent on $M$, then $g_y$ is integrally dependent on $M_y$ for all $y$ in a neighborhood of 0, but the latter condition is strictly weaker than the former. Indeed, a simple example is given in Example (1.3) of [5]. Thus the conclusion of the lemma is stronger than it might seem at first.

To prove Theorems (3.1) and (3.2), take $M$ to be the Jacobian module,

$$M := JM_x(f_1, \ldots, f_k; f) \subset E := \mathcal{O}_X^{k+1},$$

the column space of the Jacobian matrix; see Section 3. For convenience, let $M$ also denote the induced module over the local ring $\mathcal{O}_{X,0}$. For $1 \leq j \leq b$, let $g_j$ be the column vector,

$$g_j := \begin{bmatrix} \partial f_1 / \partial y_j \\ \vdots \\ \partial f_k / \partial y_j \\ \partial f / \partial y_j \end{bmatrix}.$$

Finally, denote the ideal of $Y$ in $X$ by $\mathfrak{m}_Y$; so $\mathfrak{m}_Y := (x_1, \ldots, x_a) \mathcal{O}_X$.

The next result characterizes $A_f$ (resp., $W_f$) by the integral dependence of the $g_j$ on the submodule $M$ (resp., $\mathfrak{m}_Y M$) of $E$.

Proposition (4.2) Assume that $\Sigma(f) = Y$.

(1) Then $(X - Y, Y)$ satisfies $W_f$ at 0 if and only if the columns $g_1, \ldots, g_b$ are all integrally dependent on $\mathfrak{m}_Y M$.

(2) Then $(X - Y, Y)$ satisfies $A_f$ at 0 if and only if the columns $g_1, \ldots, g_b$ are all integrally dependent on $M$.

Indeed, (1) is part of Proposition (6.1) in [5]. The characterization is proved by developing the analytic inequalities involved in the definition of $W_f$ until the condition in the analytic criterion is met.

Part (2) is not explicitly stated in either [5] or [6], but may be derived as follows. First note the formula (Gaffney, priv. comm., 1990),

$$C(X, f) := \text{Projan}(\mathcal{O}_X[M, g_1, \ldots, g_b]).$$

Indeed, both sides are closed subvarieties of $X \times \mathbb{P}^{a+b-1}$. Both are equal, over $X - \Sigma(f)$, to the set of pairs $(x, H)$ such that $H$ is a hyperplane tangent at $x$
to the level surface \( f^{-1}fx \). The left side is, by definition, the closure of this set. The right side is also the closure of this open subset of itself, because its algebra is, by construction, a subalgebra of the symmetric algebra \( \mathcal{O}[E] \).

Let \( V \) be an arbitrary component of the preimage of \( Y \) in \( C(X, f) \). Since \( \Sigma(f) = Y \), the dimension of \( V \) is \( a + b - 1 \) by Gaffney and Massey’s Lemma (5.7) in [6]; see also Theorem 4.2 in [21] and the corollary in [16, (1.2)].

Assume that \( g_1, \ldots, g_b \) are integrally dependent on \( M \). Then, after \( X \) is replaced by a neighborhood of 0, the inclusion of \( \mathcal{O}_X[M] \) into \( \mathcal{O}_X[M, g_1, \ldots, g_b] \) induces a finite map,

\[
\gamma: C(X, f) \to X \times \mathbb{P}^{a-1},
\]

since \( M \) is generated by \( a \) elements. Hence \( \dim \gamma(V) = a + b - 1 \). Since \( \gamma(V) \) is contained in \( Y \times \mathbb{P}^{a-1} \), therefore these two sets are equal. Hence \( V \) maps onto \( Y \times \mathbb{P}^{a-1} \), so \( V \) is equal to \( \gamma(V) \).

The converse is a special case of (1) of the following lemma, and so the proof of the proposition is complete.

Both parts of the lemma are used below to complete the proofs of Theorems (3.1) and (3.2). Moreover, both are interesting in their own right.

**Lemma (4.3)**

(1) If \((X - \Sigma(f), Y)\) satisfies \( A_f \) at 0, then each \( g_j \) is integrally dependent on \( M \).

(2) If each \( g_j \) is integrally dependent on \( M \), then the germs of \( \Sigma(f) \) and \( \Sigma_Y(f) \) at 0 are equal.

To prove (2), suppose that the germ of \( \Sigma_Y(f) \) is strictly larger than that of \( \Sigma(f) \). Then there is a path \( \phi: (C, 0) \to (X, 0) \) whose image lies in the former set, but outside the latter. Now,

\[
\Sigma_Y(f) = \text{Supp}(E/M) \quad \text{and} \quad \Sigma(f) = \text{Supp}(E/(M, g_1, \ldots, g_b)).
\]

Hence, for some \( j \), the pullback \( \phi^*g_j \) does not lie in the pullback \( \phi^*M \). So this \( g_j \) is not integrally dependent on \( M \) by the curve criterion. Thus (2) holds.

To prove (1), take a \( \phi: (C, 0) \to (X, 0) \) whose image lies outside \( \Sigma_Y(f) \), so outside \( \Sigma(f) \). The gradients of \( f_1, \ldots, f_k, f \) define hyperplanes tangent to the level hypersurfaces of \( f \). Each hyperplane must approach along \( \phi \) a hyperplane that contains \( Y \) since \( A_f \) holds. Consider the last \( b \) components of each gradient. Each such component must, therefore, vanish at 0 along \( \phi \) to order higher than the order of one, or more, of the first \( a \) components. Denote the maximal ideal of \( (C, 0) \) by \( m \). Then, therefore,

\[
\phi^*g_1, \ldots, \phi^*g_b \in mM \subset M.
\]

So, by the curve criterion, \( g_j \) is integrally dependent on \( M \). Thus (1) holds.
To prove Theorem (3.2), let $\mathcal{J}$ be the ideal of maximal minors of the Jacobian matrix in Section 3. Then

$$
\Sigma_Y(f) = \text{Supp}(E/\mathcal{M}) = \text{Supp}(\mathcal{O}_X/\mathcal{J}).
$$

Now, if either (ii) or (iii) holds, then each $Z_y$ has an isolated singularity; hence, we may assume that $\Sigma_Y(f)$ is finite over $Y$. Consequently, $\mathcal{O}_X/\mathcal{J}$ is determinantal, so a Cohen–Macaulay ring by Eagon’s theorem. Therefore, $\mathcal{O}_X/\mathcal{J}$ is a Cohen–Macaulay $\mathcal{O}_Y$-module, and so, by the Auslander–Buchsbaum formula, free. Hence the following number is constant in $y$:

$$
e'(y) := \dim_C(\mathcal{O}_X/\mathcal{J})(y).
$$

For each $y$, this number $e'(y)$ is a sum of positive numbers, one for each point $z$ in $\Sigma_Y(f)$ lying over $y$, and the number corresponding to $z = 0$ is equal to $e(M_y)$ by Equation (3.3.3). Hence, we have

$$
e(M_0) = e'(0) = e'(y) \geq e(M_y),
$$

with equality at the end for every $y$ if and only if $\Sigma_Y(f) = Y$. Therefore, (ii) and (iii) are equivalent.

Assume (i). Then (ii) follows directly from Lemma (4.3).

Conversely, assume (ii). Then, since $\Sigma_Y(f)$ and $\Sigma(f)$ and $Y$ are nested, all three represent the same germ. Furthermore, (iii) holds; so $e(M_y)$ is constant. Now, to prove that $A_f$ holds, we use Proposition (4.2)(2). We have to show that the $g_j$ are integrally dependent on $\mathcal{M}$. However, they are so by the principle of specialization of integral dependence. Indeed, $(X - \Sigma(f), Y)$ satisfies $A_f$ at every $y$ in a dense Zariski open subset $U$ of $Y$ by Hironaka’s Theorem 2 on p. 247 in [12]. Hence, for $y \in U$, the restrictions $g_j|X_y$ are integrally dependent on $M_y$ by Proposition (4.2)(2) again. Thus Theorem (3.2) is proved.

Finally, consider Theorem (3.1). We have left to prove that (ii) implies (i). So assume (ii); that is, $e(m_y,M_y)$ is constant. Then $e(M_y)$ is constant too; we noted this consequence before stating Lemma (3.3). Hence, by Theorem (3.2), the germs of $\Sigma(f)$ and $Y$ at 0 are equal. To prove that $W_f$ holds, we use Proposition (4.2)(1) and the principle of specialization of integral dependence much as we just did for $A_f$; we need only replace Hironaka’s theorem by Henry, Merle, and Sabbah’s Théorème 5.1 on p. 255 of [11]. (They attribute this result to Navarro, who didn’t publish it.) Thus Theorem (3.1) is proved.

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