Bounded-Degree Cut is Fixed-Parameter Tractable

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Abstract

In the bounded-degree cut problem, we are given a multigraph $G = (V, E)$, two disjoint vertex subsets $A, B \subseteq V$, two functions $u_A, u_B : V \rightarrow \{0, 1, \ldots, |E|\}$ on $V$, and an integer $k \geq 0$. The task is to determine whether there is a minimal $(A, B)$-cut $(V_A, V_B)$ of size at most $k$ such that the degree of each vertex $v \in V_A$ in the induced subgraph $G[V_A]$ is at most $u_A(v)$ and the degree of each vertex $v \in V_B$ in the induced subgraph $G[V_B]$ is at most $u_B(v)$. In this paper, we show that the bounded-degree cut problem is fixed-parameter tractable by giving a $2^{18k}|G|^{O(1)}$-time algorithm. This is the first single exponential FPT algorithm for this problem. The core of the algorithm lies two new lemmas based on important cuts, which give some upper bounds on the number of candidates for vertex subsets in one part of a minimal cut satisfying some properties. These lemmas can be used to design fixed-parameter tractable algorithms for more related problems.

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1 Introduction

A cut of a graph is a partition of the vertices of the graph into two disjoint subsets. Graph cuts play an important role in combinatorial optimization and graph theory. The classical minimum cut problem is well known to be polynomially solvable [17]. Due to the rich application realm of this problem, many variants and extensions have been investigated. Some problems ask to partition the graph into more than two parts to disconnect some vertices such as the $k$-way cut problem (the $k$-cut problem) [19, 21], the multiterminal cut problem [13, 27] and the multicut problem [5, 25]. Some problems are still going to partition the graph into two parts, but with some additional requirements beyond the disconnectivity. One of the most extensively studied additional requirements is the constraint on the numbers of vertices or edges in each of the two parts. For examples, the balanced cut problem [11, 16, 22] and the minimum bisection problem [11, 14, 15] require the numbers of vertices in the two parts of the cut as close as possible. The (balanced) judicious bipartition problem [23] has conditions on the numbers of edges in the two parts. Some other well studied additional
requirements include conditions on the connectivity of the two parts such as the 2-disjoint connected subgraphs problem [12], and conditions on the degree of the two parts, such as the series of bipartition problems with degree constraints [2, 3, 4, 26, 28].

In this paper, we study the bounded-degree cut problem, which belongs to the latter kind of the extensions: to partition a given graph into two parts with some degree constraints on the induced subgraphs of the two parts. We mainly consider the upper bounds of the degree. The problem is defined as follows.

BOUNDDED-DEGREE CUT (with parameter: $k$)

**Instance:** A multigraph $G = (V, E)$, two disjoint nonempty vertex subsets $A, B \subseteq V$, two functions $u_A$ and $u_B$ from $V$ to $\{0, 1, \ldots, |E|\}$ and an integer $k \geq 0$.

**Question:** Does there exist a minimal $(A, B)$-cut $(V_A, V_B)$ such that the number of edges with one endpoint in $V_A$ and one endpoint in $V_B$ is at most $k$, for each vertex $v \in V_A$, the degree of it in the induced graph $G[V_A]$ is at most $u_A(v)$, and for each vertex $v \in V_B$, the degree of it in the induced graph $G[V_B]$ is at most $u_B(v)$?

During the last decade, cut related problems were extensively studied from the viewpoint of parameterized algorithms. The parameterized complexity of many variants and extensions of the minimum cut problem have be developed. In this paper, we will study bounded-degree cut from the viewpoint of parameterized algorithms. The naive brute-force algorithm to enumerate all partitions can solve bounded-degree cut in $2^{|V|} \cdot |G|^{O(1)}$ time. The exponential part of the running time is related to the input size $|V|$. We show that this problem admits a fixed-parameter tractable (FPT) algorithm with parameter $k$, an algorithm with running time $f(k) \cdot |G|^{O(1)}$ for a computable function $f(\cdot)$. Our main result is the first single-exponential FPT algorithm for bounded-degree cut.

**Theorem 1.** BOUNDED-DEGREE CUT admits an FPT algorithm that runs in $2^{18k} \cdot |G|^{O(1)}$ time.

This theorem also implies that bounded-degree cut can be solved in polynomial time for $k = O(\log |G|)$.

1.1 Related work

There are several interesting contributions on finding a cut or partition of a graph with some additional requirements. It is known that the minimum $(s,t)$-cut problem is polynomially solvable. However, the balanced minimum $(s,t)$-cut problem is NP-hard [10], which is to decide whether there is a minimum $(s,t)$-cut such that the number of vertices in each part is at most $0 < \alpha < 1$ times of the total vertex number. We can add some trivial vertices in the input graph to make $\alpha$ always being 0.5. Note that in this problem, the cut is required to be a minimum $(s,t)$-cut. Let $k$ denote the size of a minimum $(s,t)$-cut. By developing a dynamic programming algorithm, Feige and Mahdian [10] showed that the vertex-deletion variant of the balanced minimum $(s,t)$-cut problem is FPT with the parameter $k$. This algorithm also works for the edge-deletion version. Another related problem is the (vertex) minimum bisection problem, which is to find a (vertex) cut of size at most $k$ such that the two parts of the cut have the same number of vertices. Marx’s result in [24] implies that the vertex minimum bisection problem is W[1]-hard with the parameter $k$. Cygan et.al. [11] showed that the edge vertex version of the minimum bisection problem is FPT. The above problems have requirements on the vertex numbers in the two parts of the cut or partition. The judicious bipartition problem requires that the numbers of edges in the two parts are bounded by $k_1$. 
and $k_2$ respectively. Lokshtanov et al. [23] proved that the judicious bipartition problem is FPT with parameter $k_1 + k_2$. In this paper, we consider BOUNDED-DEGREE CUT, which is a cut problem with additional requirements on the upper bound of the degree of each vertex in the two parts, and take the cut size as the parameter.

1.2 Our methods

The main idea of the algorithm is to construct from a given instance in a graph $G$ a set of at most $2^{O(k)}$ new “easy” instances on the same graph with a special structure such that (i) the feasibility of each easy instances can be tested in $|G|^{O(1)}$ time; and (ii) the original instance is feasible if and only if at least one of the easy instances is feasible. Constructing such easy instances and testing the feasibility of all these give an FPT algorithm for the original instance. The idea of converting a general instance to a set of “easy” instances has been used to design parameterized algorithms for several hard and important problems [20, 21, 9, 11]. The construction of easy instances is the most important step. Some of the crucial techniques used in our construction is based on the concept of important cut (or important separator) introduced by Marx [24]. Important cuts and separators play an important role in designing FPT algorithms for cut problems. The fixed-parameter tractability of the (directed) multiterminal cut problem [6, 10], the multicut problem [25], the directed feedback vertex set problem [7, 8] and many other important problems were proved by using important cuts and separators together with some other techniques. We will apply important cuts in a nontrivial way to obtain some general lemmas for bounded sets related to cuts. These are crucial for us to design FPT algorithms.

The framework of our algorithm is as follows. For a given instance $(G, A, B, u_A, u_B, k)$ with a feasible $(A, B)$-cut $(V_1, V_2)$, we try to guess some subsets $V_1' \subseteq V_1 \setminus A$ and $V_2' \subseteq V_2 \setminus B$ so that the new instance $(G, A^* = A \cup V_1', B^* = B \cup V_2', u_A, u_B, k)$ remains feasible and is an “easy” instance in the sense that the feasibility can be tested in $n^{O(1)}$ time. We call a vertex $v$ in $G$ $A$-unsatisfied (resp., $B$-unsatisfied) if its degree in $G$ is greater than $u_A(v)$ (resp., $u_B(v)$), and call an $A$- or $B$-unsatisfied vertex unsatisfied. We first guess whether each unsatisfied vertex belongs to $V_1$ or $V_2$. Although the number of unsatisfied vertices may not be bounded by a function of $k$, the set $Z_{A1}$ of $A$-unsatisfied vertices in $V_1$ can contain at most $k$ vertices, because each vertex in $Z_{A1}$ must be adjacent to a vertex in $V_2$ to satisfy the degree constraint. Symmetrically the set $Z_{B2}$ of $B$-unsatisfied vertices in $V_2$ can contain at most $k$ vertices. By applying the result on important cuts and our new lemmas, we construct at most $2^{O(k)}$ pairs $(X_1, X_2)$ of vertex subsets one of which is equal to $(Z_{A1}, Z_{B2})$. For the set $(X_1, X_2) = (Z_{A1}, Z_{B2})$ and the set $Z_{A2}$ (resp., $Z_{B1}$) of $A$-unsatisfied vertices in $V_2$ (resp., $B$-unsatisfied vertices in $V_1$), we see that the new instance $(G, A' = A \cup Z_{A1} \cup Z_{B1}, B' = B \cup Z_{A1} \cup Z_{B2}, u_A, u_B, k)$ remains feasible. However, this instance may not be “easy” yet in our sense, because whether the degree constraint on a vertex in $Z_{A1}$ or $Z_{B2}$ holds or not depends on a choice of an $(A', B')$-cut in the new instance. We next guess whether each neighbor of a vertex in $Z_{A1} \cup Z_{B2}$ belongs to $V_1$ or $V_2$. We see that the set $W_{B1}$ of neighbors of $Z_{B2}$ belonging to $V_1$ can contain at most $k$ vertices, because the number of such neighbors is bounded from above by the cut-size of $(V_1, V_2)$. Symmetrically the set $W_{A2}$ of neighbors of $Z_{A1}$ belonging to $V_2$ contains at most $k$ vertices. By applying the result on important cuts and our new lemmas again, we construct at most $2^{O(k)}$ pairs $(Y_1, Y_2)$ of vertex subsets one of which is equal to $(W_{B1}, W_{A2})$. Let $W_{B2}$ (resp., $W_{A1}$) denote of neighbors of $Z_{B2}$ belonging to $V_2$ (resp., of $Z_{A1}$ belonging to $V_1$). Then for the right choice $(X_1, X_2) = (Z_{A1}, Z_{B2})$ and $(Y_1, Y_2) = (W_{B1}, W_{A2})$, the resulting instance $(G, A^* = A \cup Z_{A1} \cup Z_{B1} \cup W_{A1} \cup W_{B1}, B^* = B \cup Z_{A1} \cup Z_{B2} \cup W_{A2} \cup W_{B2}, u_A, u_B, k)$ remains
feasible and is an easy instance where the degree constraint on a vertex in \( Z_{A1} \) or \( Z_{B2} \) holds or not does not depend on a choice of an \((A^*, B^*)\)-cut in the new instance.

The remaining part of the paper is organized as follows. Section 2 reviews basic notations on graphs and cuts and properties on minimum cuts and important cuts. Section 3 introduces some technical lemmas, which will be building blocks of our algorithm. We believe that these lemmas can be used to design FPT algorithms for more problems. Section 4 defines “easy” instances and proves the polynomial-time solvability. Based on the results in Section 3, Section 5 describes how to generate candidate set pairs \((X_1, X_2)\) for the pair \((Z_{A1}, Z_{B2})\) and \((Y_1, Y_2)\) for the pair \((W_{B1}, W_{A2})\). Finally Section 6 makes some concluding remarks.

## 2 Preliminaries

In this paper, a graph \( G = (V, E) \) stands for an undirected multigraph with a vertex set \( V \) and an edge set \( E \). We will use \( n \) and \( m \) to denote the sizes of \( V \) and \( E \), respectively. Let \( X \) be a subset of \( V \). We use \( G - X \) to denote the graph obtained from \( G \) by removing vertices in \( X \) together with all edges incident to vertices in \( X \). Let \( G[X] \) denote the graph induced by \( X \), i.e., \( G[X] = G - (V \setminus X) \). Let \( N_G(v) \) denote the set of neighbors of a vertex \( v \) in \( G \), and let \( N_G(v; X) = N_G(v) \cap X \). Let \( N_G(X) \) denote the set of neighbors \( u \in V \setminus X \) of a vertex \( v \in X \), i.e., \( N_G(X) = \bigcup_{v \in X} N_G(v; V \setminus X) \). For two disjoint vertex subsets \( X \) and \( Y \), the set of edges with one endpoint in \( X \) and one endpoint in \( X \) is denoted by \( E_G(X, Y) \), and \( E_G(X, V \setminus X) \) may be simply written as \( E_G(X) \). Define \( \deg_G(v) = |E_G(\{v\})| \) and \( \deg_G(v; X) = |E_G(\{v\}, X \setminus \{v\})| \).

**Definition 2.** \(((S,T)\)-cuts) For two disjoint vertex subsets \( S \) and \( T \), an ordered pair \((V_1, V_2 = V \setminus V_1)\) is called a \((S,T)\)-cut if \( S \subseteq V_1 \) and \( T \subseteq V_2 \), and its cut-size is defined to be \(|E_G(V_1)|\).

**Definition 3.** (minimal \((S,T)\)-cuts, minimum \((S,T)\)-cuts and MM \((S,T)\)-cuts) An \((S,T)\)-cut \((V_1, V_2)\) is minimal if \( E_G(V_1) \) does not contain \( E_G(V'_1) \) or \( E_G(V'_2) \) as a subset for any \( S \subseteq V'_1 \subseteq V_1 \) and \( T \subseteq V'_2 \subseteq V_2 \). An \((S,T)\)-cut \((V_1, V_2)\) is minimum if its cut-size \(|E_G(V_1, V_2)|\) is minimum over all \((S,T)\)-cuts. An \((S,T)\)-cut \((V_1, V_2)\) is called an MM \((S,T)\)-cut if it is a minimum \((S,T)\)-cut such that \( |V_1| \) is maximum over all minimum \((S,T)\)-cuts.

**Lemma 4.** \([17, 13]\) For two disjoint vertex subsets \( S, T \subseteq V \), an MM \((S,T)\)-cut is unique and it can be found in \( O(\min\{n^{2/3}, m^{1/2}\}) \) time.

**Definition 5.** (important cuts) A minimal \((S,T)\)-cut \((X, V \setminus X)\) is called an important \((S,T)\)-cut if there is no \((S,T)\)-cut \((X', V \setminus X')\) such that \( X' \supseteq X \) and \( |E_G(X')| \leq |E_G(X)| \).

The following result is known \([6, 25]\).

**Lemma 6.** Let \( S, T \subseteq V \) be non-empty subsets in a graph \( G = (V, E) \).

(i) For any subset \( X \) with \( S \subseteq X \subseteq V \setminus T \), the MM \((X,T)\)-cut is an important \((S,T)\)-cut;

(ii) There are at most \( 4^k \) important \((S,T)\)-cuts of size at most \( k \) and one can list all of them in \( 4^k (n + m)^{O(1)} \) time.

## 3 Candidate Sets

We introduce the next technical lemmas, which will be used to build blocks of our algorithm. These lemmas are crucial for us to design FPT algorithms.
Lemma 7. Let $A, B, C \subseteq V$ be non-empty subsets in a graph $G = (V, E)$ and $k$ and $\ell$ be nonnegative integers. Then one can find in $2^{3(k+\ell)(n+m)^{O(1)}}$ time a family $X$ of at most $2^{3(k+\ell)}$ subsets of $C$ with a property that $C \cap V_1 \in X$ for any minimal $(A, B)$-cut $(V_1, V_2)$ with size at most $k$ such that $|C \cap V_1| \leq \ell$.

Proof. Let $\text{Cut}(A, B, C, k, \ell; G)$ denote the set of minimal $(A, B)$-cuts $(V_1, V_2)$ in $G$ with size at most $k$ such that $|C \cap V_1| \leq \ell$. Construct a multigraph $H_b$ from $G$ by choosing a vertex $b \in B$ and adding a new edge between $b$ and each vertex $u \in C$, and let $\text{ICut}(A, B, k + \ell; H_b)$ denote the set of important $(A, B)$-cuts in $H_b$ of size at most $k + \ell$. By Lemma 5(ii), $|\text{ICut}(A, B, k + \ell; H_b)| \leq 4^{k+\ell}$ holds, and $\text{ICut}(A, B, k + \ell; H_b)$ can be found in time $4^{k+\ell}(n + m)^{O(1)}$.

For any minimal $(A, B)$-cut $(V_1, V_2) \in \text{Cut}(A, B, C, k, \ell; G)$, we see by Lemma 6(i) that the MM $(A \cup (C \cap V_1), B)$-cut $(S, T)$ in $H_b$ is an important $(A, B)$-cut in $H_b$ of size at most $k + |C \cap V_1| \leq k + \ell$, where $C \cap V_1 \subseteq N_{H_b}(b) \cap S$ holds. Construct the family $X$ of subsets $X \subseteq N_{H_b}(b) \cap S$ with $|X| \leq \ell$ for each $(A, B)$-cut $(S, T) \in \text{ICut}(A, B, k + \ell; H_b)$.

Then $X$ contains the set $C \cap V_1$ for each $(A, B)$-cut $(V_1, V_2) \in \text{Cut}(A, B, C, k, \ell; G)$.

For each important $(A, B)$-cut $(S, T) \in \text{ICut}(A, B, k + \ell; H_b)$, the family $X$ contains at most

$$\sum_{i=0}^{\ell} \binom{|N_{H_b}(b) \cap S|}{i} \leq \sum_{i=0}^{\ell} \binom{k + \ell}{i} < 2^{k+\ell}$$

subsets $X$. Since $|\text{ICut}(A, B, k + \ell; H_b)| \leq 4^{k+\ell}$, it holds that $|X| \leq 4^{k+\ell} \cdot 2^{k+\ell} = 2^{3(k+\ell)}$ and the family $X$ can be constructed in $4^{k+\ell}(n + m)^{O(1)} + 2^{3(k+\ell)(n + m)^{O(1)}}$ time. This proves the lemma.

Lemma 8. Let $A, B, B' \subseteq V$ be non-empty subsets in a graph $G = (V, E)$, where $B' \subseteq B$, and $k$ be a nonnegative integer. Then one can find in $2^{3k(n + m)^{O(1)}}$ time a family $Y$ of at most $2^{2k}$ subsets of $N_G(B')$ with a property that $N_G(B') \cap V_1 \in Y$ for any minimal $(A, B)$-cut $(V_1, V_2)$ with size at most $k$.

Proof. Let $\text{Cut}(A, B, k)$ denote the set of minimal $(A, B)$-cuts in $G$ with size at most $k$. Let $\text{ICut}(A, B, k)$ denote the set of important $(A, B)$-cuts in $G$ with size at most $k$. By Lemma 5(ii), $|\text{ICut}(A, B, k)| \leq 4^k$ holds, and $\text{ICut}(A, B, k)$ can be found in $4^k n^{O(1)}$ time.

For any minimal $(A, B)$-cut $(V_1, V_2) \in \text{Cut}(A, B, k)$, we see by Lemma 6(i) that the MM $(A \cup (N_G(B') \cap V_1), B)$-cut $(S, T)$ is an important $(A, B)$-cut of size at most $k$, where $N_G(B') \cap V_1 \subseteq N_G(B') \cap S$ holds. Construct the family $Y$ of subsets $Y \subseteq N_G(B') \cap S$ for each $(A, B)$-cut $(S, T) \in \text{ICut}(A, B, k)$.

Then $Y$ contains the set $N_G(B') \cap V_1$ for each $(A, B)$-cut $(V_1, V_2) \in \text{Cut}(A, B, k)$.

Note that the size of $N_G(B') \cap S$ is at most the size of the cut $(S, T)$. For each important $(A, B)$-cut $(S, T) \in \text{ICut}(A, B, k)$, the family $Y$ contains at most

$$2^{|N_G(B') \cap S|} \leq 2^k$$

subsets $Y$. Since $|\text{ICut}(A, B, k)| \leq 4^k$, it holds that $|Y| \leq 4^k \cdot 2^k = 2^{2k}$ and the family $Y$ can be constructed in $4^k(n + m)^{O(1)} + 2^{2k(n + m)^{O(1)}}$ time. This proves the lemma.
### 4 Restriction to an Easy Case

Recall that an instance of bounded-degree cut is defined by a tuple \((G = (V, E), A, B, u_A, u_B, k)\) such that \(G\) is a multigraph, \(A, B \subseteq V\) are two disjoint vertex subsets, \(u_A\) and \(u_B\) are two functions from \(V\) to \(\{0, 1, \ldots, |E|\}\), and \(k\) is a nonnegative integer. We will use \(I = (G = (V, E), A, B)\) to denote an instance of the problem, where \(u_A, u_B\) and \(k\) are omitted since they remain unchanged throughout our argument. We call a minimal \((A, B)\)-cut \((V_A, V_B = V \setminus V_A)\) feasible to an instance \(I\) if

- \(|E_G(V_A)| \leq k;\)
- \(\deg_G(v; V_A) \leq u_A(v)\) for all vertices \(v \in V_A;\) and
- \(\deg_G(v; V_B) \leq u_B(v)\) for all vertices \(v \in V_B,\)

where the last two conditions are also called the degree constraint. A feasible \((A, B)\)-cut in \(I\) is called a solution to \(I\), and an instance \(I\) is called feasible if it admits a solution.

**Bounded-degree cut** is to decide whether a given instance \(I\) is feasible or not. We observe the next.

#### Lemma 9.

For an instance \(I = (G = (V, E), A, B)\) and disjoint nonempty subsets \(X, Y \subseteq V, \) let \(I_{X,Y}\) denote the instance \((G, A \cup X, B \cup Y).\)

1. If \(I\) is infeasible, then \(I_{X,Y}\) is infeasible for any \(X, Y \subseteq V;\)
2. If \(I\) is feasible and \(X \subseteq V_A\) and \(Y \subseteq V_B\) hold for a feasible \((A, B)\)-cut \((V_A, V_B)\) to \(I,\) then \(I_{X,Y}\) admits a feasible \((A \cup X, B \cup Y)\)-cut, which is also feasible to \(I.\)

**Proof.** (i) Assume to the contrary that \(I_{X,Y}\) is feasible and \((V_A, V_B)\) is a feasible \((A \cup X, B \cup Y)\)-cut. Then it holds that \(A \subseteq V_A\) and \(B \subseteq V_B\) and the cut \((V_A, V_B)\) satisfies the conditions in the definition of feasible cuts. Thus, \((V_A, V_B)\) is also a feasible \((A, B)\)-cut, a contradiction to the fact that \(I\) is infeasible.

(ii) First of all, it is clear that \((V_A, V_B)\) is still a feasible \((A \cup X, B \cup Y)\)-cut. Then \(I_{X,Y}\) admits feasible \((A \cup X, B \cup Y)\)-cuts. Let \((V_1, V_2)\) be an arbitrary feasible \((A \cup X, B \cup Y)\)-cut. The above proof for (i) shows that \((V_1, V_2)\) is also a feasible \((A, B)\)-cut.

Let \(I = (G, A, B)\) be an instance. We use \(Z_A\) and \(Z_B\) to denote the sets of \(A\)-unsatisfied vertices and \(B\)-unsatisfied vertices, respectively, i.e.,

\[
Z_A \triangleq \{v \in V \mid \deg_G(v) > u_A(v)\} \quad \text{and} \quad Z_B \triangleq \{v \in V \mid \deg_G(v) > u_B(v)\},
\]

where \(Z_A \cap Z_B\) may not be empty. We call \(I\) an easy instance if it holds that

1. \(Z_A \cup Z_B \subseteq A \cup B,\)
2. \(N_G(Z_A \cap A) \subseteq A \cup B,\) and
3. \(N_G(Z_B \cap B) \subseteq A \cup B.\)

#### Lemma 10.

The feasibility of an easy instance of bounded-degree cut can be tested in \((n + m)^{O(1)}\) time.

**Proof.** Let \(I = (G, A, B)\) be an easy instance. Note that the degree constraint to each vertex in \(V \setminus (A \cup B)\) is satisfied for any \((A, B)\)-cut in \(I.\) First we test in \(n^{O(1)}\) time whether there is a vertex \(v \in Z_A \cap A\) with \(\deg_G(v; A) > u_A(v)\) (resp., \(v \in Z_B \cap B\) with \(\deg_G(v; B) > u_B(v)\) or not. If so, then clearly \(I\) admits no solution. Assume that no such vertices exist in \(I.\) Then each \(v \in Z_A\) satisfies \(v \notin A\) or \(v \in Z_A \cap A\). In the former, it holds that \(v \in B\) since \(Z_A \subseteq A \cup B,\) where we do not need to consider the degree constraint by \(u_A(v).\) In the latter, it holds that \(N_G(v) \subseteq A \cup B\) since \(N_G(Z_A \cap A) \subseteq A \cup B,\) where \(\deg_G(v; V_1) = \deg_G(v; A) \leq u_A(v)\) for any \((A, B)\)-cut \((V_1, V_2)\) in \(I,\) satisfying the degree constraints.
constraint by $u_A(v)$. Analogously no vertex in $Z_B$ violates the degree constraint by $u_B(v)$ for any choice of $(A, B)$-cuts $(V_1, V_2)$ in $I$.

Now $I$ admits a solution if and only if it has an $(A, B)$-cut with size at most $k$, which can be checked in $(n + m)^O(1)$ time by Lemma 4. This proves the lemma.

We will construct from a given instance at most $2^{18k}$ easy instances so that where the original instance is feasible if and only if at least one of the easy instances is feasible.

5 Constructing Easy Instances

For a minimal $(A, B)$-cut $\pi = (V_1, V_2)$ (not necessarily feasible) in a given instance $I = (G = (V, E), A, B)$, we define the following notation on vertex subsets:

- $Z_{A1} \triangleq Z_A \cap V_1$ and $Z_{B1} \triangleq Z_B \cap V_1$, $i = 1, 2$;
- $W_A \triangleq N_G(Z_{A1})$ and $W_B \triangleq N_G(Z_{B1})$; $W_{A1} \triangleq W_A \cap V_1$, and $W_{B1} \triangleq W_B \cap V_1$, $i = 1, 2$;
- $A_\pi \triangleq A \cup Z_{A1} \cup Z_{B1} \cup W_{A1} \cup W_{B1}$ and $B_\pi \triangleq B \cup Z_{A2} \cup Z_{B2} \cup W_{A2} \cup W_{B2}$.

See in Fig. 1 for an illustration on these subsets. Observe that the resulting instance $(G, A_\pi, B_\pi)$ is an easy instance. By Lemma 9, the $(A, B)$-cut $\pi = (V_1, V_2)$ is feasible if and only if the corresponding instance $(G, A_\pi, B_\pi)$ is feasible.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a) An $(A, B)$-cut $\pi = (V_A, V_B)$ to $I$ and the partitions $\{Z_{A1}, Z_{A2}\}$ of $Z_A$ and $\{Z_{B1}, Z_{B2}\}$ of $Z_B$ by $\pi$, where possibly $Z_A \cap Z_B \neq \emptyset$. (b) The partitions $\{W_{A1}, W_{B1}\}$ of $W_A = N_G(Z_{A1})$ and $\{W_{A2}, W_{B2}\}$ of $W_A = N_G(Z_{A1})$ by $\pi$, where possibly $W_A \cap W_B \neq \emptyset$.}
\end{figure}

5.1 Partitioning Unsatisfied Vertices

For a minimal $(A, B)$-cut $(V_1, V_2)$ to an instance $I$, let $Z_{A1}$ and $Z_{B2}$ be the subsets defined in the above. We observe that if the cut is feasible, then

$$|Z_{A1}|, |Z_{B2}| \leq k$$

since each vertex in $Z_{A1} \cup Z_{B2}$ has at least one incident edge included in $E_G(V_1, V_2)$ so that the degree constraint on the vertex holds.

By applying Lemma 2 to $(A, B, C = Z_A, k, \ell = k)$, we can construct in $2^{6k}(n + m)^O(1)$ time a family $\mathcal{X}_1$ of at most $2^{6k}$ subsets of $Z_A$ such that $\mathcal{X}_1$ contains the set $Z_{A1}$ defined
to each feasible \((A, B)\)-cut \((V_1, V_2)\) in the instance \(I = (G, A, B)\). Symmetrically it takes \(2^{6k(n + m)}\) time to find a family \(X_2\) of at most \(2^{6k}\) subsets of \(Z_B\) such that \(X_2\) contains the set \(Z_{B2}\) defined to each feasible \((A, B)\)-cut \((V_1, V_2)\) in the instance \(I = (G, A, B)\). Then the set \(X_{1,2}\) of all \((X_1, X_2)\) of disjoint sets \(X_i \in X_i\), \(i = 1, 2\) contains the pair \((Z_{A1}, Z_{B2})\) defined to each feasible \((A, B)\)-cut \((V_1, V_2)\) in \(I\). By noting that \(|X_{1,2}| \leq 2^{6k}2^{6k} = 2^{12k}\), we obtain the next.

**Lemma 11.** Given an instance \(I = (G, A, B)\), one can construct in \(2^{12k(n + m)}\) time at most \(2^{12k}\) new instances \(I' = (G, A', B')\) with \(Z_A \cup Z_B \subseteq A' \cup B'\), one of which is equal to \((G, A \cup Z_{A1} \cup Z_{B1}, B \cup Z_{A2} \cup Z_{B2})\) for each feasible \((A, B)\)-cut \((V_1, V_2)\) to \(I\).

### 5.2 Partitioning Neighbors of Unsatisfied Vertices

For a minimal \((A, B)\)-cut \((V_1, V_2)\) to an instance \(I\), let \(W_{A2}\) and \(W_{B1}\) be the subsets defined in the above. We observe that if the cut is feasible, then

\[
|W_{B1}|, |W_{A2}| \leq k
\]

since each of \(|N_G(Z_{B2}) \cap V_1|\) and \(|N_G(Z_{A1}) \cap V_2|\) is at most \(|E_G(V_1, V_2)| \leq k\) to the feasible \((A, B)\)-cut \((V_1, V_2)\).

By applying Lemma [5] to \((A \cup Z_{A1} \cup Z_{B1}, B \cup Z_{A2} \cup Z_{B2}, B' = Z_{B2}, k)\), we can construct in \(2^{12k(n + m)}\) time a family \(Y_1\) of at most \(2^{12k}\) subsets of \(N_G(Z_{B2})\) such that \(Y_1\) contains the set \(W_{B1} = N_G(Z_{B2}) \cap V_1\) defined to each feasible \((A, B)\)-cut \((V_1, V_2)\) in the instance \(I = (G, A, B)\). Symmetrically it takes \(2^{12k(n + m)}\) time to find a family \(Y_2\) of at most \(2^{12k}\) subsets of \(N_G(Z_{A1})\) such that \(Y_2\) contains the set \(W_{A2} = N_G(Z_{A1}) \cap V_2\) defined to each feasible \((A, B)\)-cut \((V_1, V_2)\) in \(I\). Then the set \(Y_{1,2}\) of all pairs \((Y_1, Y_2)\) of disjoint sets \(Y_i \in Y_i\), \(i = 1, 2\) contains the pair \((W_{B1}, W_{A2})\) defined to each feasible \((A, B)\)-cut \((V_1, V_2)\) in the instance \(I = (G, A, B)\). By noting that \(|Y_{1,2}| \leq 2^{6k}\), we obtain the next.

**Lemma 12.** Given an instance \(I = (G, A, B)\) and the subsets \(Z_{A1}\) and \(Z_{B2}\) defined to a feasible \((A, B)\)-cut \((V_1, V_2)\) in \(I\), one can construct in \(2^{6k(n + m)}\) time at most \(2^{6k}\) new easy instances \(I' = (G, A', B')\), one of which is equal to \((G, A_x, B_x)\) defined to the feasible \((A, B)\)-cut \(\pi = (V_1, V_2)\).

By Lemmas 11 and 12 we obtain the next.

**Lemma 13.** Given an instance \(I = (G, A, B)\), one can construct in \(2^{18k(n + m)}\) time at most \(2^{18k}\) new easy instances \(I' = (G, A', B')\), one of which is equal to \((G, A_x, B_x)\) for each feasible \((A, B)\)-cut \(\pi = (V_1, V_2)\) to \(I\).

This and Lemma 10 imply Theorem 4.

### 6 Concluding Remarks

Cut and partition problems are important problems that have been extensively studied from the viewpoint of FPT algorithms. In this paper, we study a cut problem with additional constraints on the vertex degree of the two parts of the cut and design the first FPT algorithm for this problem. To obtain the FPT algorithm, we develop two new lemmas that are based on important cuts. Important cuts show some properties of bounded-size cuts, while the new lemmas further reveal some properties of vertex subsets of one part of a bounded-size cut. We believe these lemmas can be used to design FPT algorithms for more problems.
In bounded-degree cut, we are going to check whether there is a minimal \((A, B)\)-cut satisfying both the degree constraint and size constraint of most \(k\). We also consider the bounded-degree bipartition problem, which is to check whether there is \((A, B)\)-cut of size at most \(k\) satisfying the degree constraint, without the requirement of being minimal. Note that some \((A, B)\)-cuts of size at most \(k\) satisfying the degree constraint may not be minimal. This kind of cuts are not solutions to bounded-degree cut, but are solutions to bounded-degree bipartition. To solve bounded-degree bipartition, we need some techniques more, which will be introduced in our further work.

References

1. Sanjeev Arora, Satish Rao, and Umesh V. Vazirani. Expander flows, geometric embeddings and graph partitioning. In László Babai, editor, Proceedings of the 36th Annual ACM Symposium on Theory of Computing, Chicago, IL, USA, June 13-16, 2004, pages 222–231. ACM, 2004. doi:10.1145/1007352.1007355

2. Jørgen Bang-Jensen and Stéphane Bessy. Degree-constrained 2-partitions of graphs. CoRR, abs/1801.06216, 2018. arXiv:1801.06216

3. Cristina Bazgan, Zsolt Tuza, and Daniel Vanderpooten. Degree-constrained decompositions of graphs: Bounded treewidth and planarity. Theor. Comput. Sci., 355(3):389–395, 2006. doi:10.1016/j.tcs.2006.01.024

4. Cristina Bazgan, Zsolt Tuza, and Daniel Vanderpooten. Efficient algorithms for decomposing graphs under degree constraints. Discrete Applied Mathematics, 155(8):979–988, 2007. doi:10.1016/j.dam.2006.10.005

5. Gruia Călinescu, Cristina G. Fernandes, and Bruce A. Reed. Multicuts in unweighted graphs and digraphs with bounded degree and bounded tree-width. J. Algorithms, 48(2):333–359, 2003. doi:10.1016/S0196-6774(03)00073-7

6. Jianer Chen, Yang Liu, and Songjian Lu. An improved parameterized algorithm for the minimum node multiway cut problem. Algorithmica, 55(1):1–13, 2009. doi:10.1007/s00453-007-9130-6

7. Jianer Chen, Yang Liu, Songjian Lu, Barry O’Sullivan, and Igor Razgon. A fixed-parameter algorithm for the directed feedback vertex set problem. J. ACM, 55(5):21:1–21:19, 2008. doi:10.1145/1411509.1411511

8. Rajesh Hemant Chitnis, Marek Cygan, Mohammad Taghi Hajiaghayi, and Dániel Marx. Directed subset feedback vertex set is fixed-parameter tractable. ACM Trans. Algorithms, 11(4):28:1–28:28, 2015. doi:10.1145/2700209

9. Rajesh Hemant Chitnis, Marek Cygan, MohammadTaghi Hajiaghayi, Marcin Pilipczuk, and Michal Pilipczuk. Designing FPT algorithms for cut problems using randomized contractions. In 53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012, pages 460–469. IEEE Computer Society, 2012. doi:10.1109/FOCS.2012.29

10. Rajesh Hemant Chitnis, MohammadTaghi Hajiaghayi, and Dániel Marx. Fixed-parameter tractability of directed multway cut parameterized by the size of the cutset. SIAM J. Comput., 42(4):1674–1696, 2013. doi:10.1137/12086217X

11. Marek Cygan, Daniel Lokshtanov, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. Minimum bisection is fixed parameter tractable. In David B. Shmoys, editor, Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 - June 03, 2014, pages 323–332. ACM, 2014. doi:10.1145/2591796.2591852

12. Marek Cygan, Marcin Pilipczuk, Michal Pilipczuk, and Jakub Omfry Wojtaszczyk. Solving the 2-disjoint connected subgraphs problem faster than 2 n. Algorithmica, 70(2):195–207, 2014. doi:10.1007/s00453-013-9796-x
XX:10 Bounded-Degree Cut is FPT

13 Elias Dahlhaus, David S. Johnson, Christos H. Papadimitriou, Paul D. Seymour, and Mihalis Yannakakis. The complexity of multiterminal cuts. *SIAM J. Comput.*, 23(4):864–894, 1994. doi:10.1137/0097539792225297

14 Uriel Feige and Robert Krauthgamer. A polylogarithmic approximation of the minimum bisection. *SIAM J. Comput.*, 31(4):1090–1118, 2002. doi:10.1137/S0097539701387660

15 Uriel Feige, Robert Krauthgamer, and Kobbi Nissim. Approximating the minimum bisection size (extended abstract). In F. Frances Yao and Eugene M. Luks, editors, *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing*, May 21-23, 2000, Portland, OR, USA, pages 530–536. ACM, 2000. doi:10.1145/335305.335370

16 Uriel Feige and Mohammad Mahdian. Finding small balanced separators. In Jon M. Kleinberg, editor, *Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, Seattle, WA, USA, May 21-23, 2006, pages 375–384. ACM, 2006. doi:10.1145/1132516.1132573

17 L.R. Ford and Delbert R. Fulkerson. *Flows in networks*. Princeton U. Press, Princeton, NJ, 1962.

18 Lance Fortnow and Salil P. Vadhan, editors. *Proceedings of the 43rd ACM Symposium on Theory of Computing*, STOC 2011, San Jose, CA, USA, 6-8 June 2011. ACM, 2011.

19 Olivier Goldschmidt and Dorit S. Hochbaum. A polynomial algorithm for the k-cut problem for fixed k. *Math. Oper. Res.*, 19(1):24–37, 1994. doi:10.1287/moor.19.1.24

20 Martin Grohe, Ken-ichi Kawarabayashi, Dániel Marx, and Paul Wollan. Finding topological subgraphs is fixed-parameter tractable. In Fortnow and Vadhan [18], pages 479–488. doi:10.1145/1993636.1993700

21 Ken-ichi Kawarabayashi and Mikkel Thorup. The minimum k-way cut of bounded size is fixed-parameter tractable. In Rafail Ostrovsky, editor, *IEEE 52nd Annual Symposium on Foundations of Computer Science, FOCS 2011*, Palm Springs, CA, USA, October 22-25, 2011, pages 160–169. IEEE Computer Society, 2011. doi:10.1109/FOSCS.2011.53

22 Frank Thomson Leighton and Satish Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *J. ACM*, 46(6):787–832, 1999. doi:10.1145/331524.331526

23 Dániel Marx. Parameterized graph separation problems. *Theor. Comput. Sci.*, 351(3):394–406, 2006. doi:10.1016/j.tcs.2005.10.007

24 Dániel Marx. Parameterized graph separation problems. *Theor. Comput. Sci.*, 351(3):394–406, 2006. doi:10.1016/j.tcs.2005.10.007

25 Dániel Marx and Igor Razgon. Fixed-parameter tractability of multicut parameterized by the size of the cutset. In Fortnow and Vadhan [18], pages 469–478. doi:10.1145/1993636.1993699

26 Michael Stiebitz. Decomposing graphs under degree constraints. *Journal of Graph Theory*, 23(3):321–324, 1996. doi:10.1002/(SICI)1097-0118(199611)23:3<321::AID-JGT12>3.0.CO;2-H

27 Mingyu Xiao. Simple and improved parameterized algorithms for multiterminal cuts. *Theory Comput. Syst.*, 46(4):723–736, 2010. doi:10.1007/s00224-009-9215-5

28 Mingyu Xiao and Hiroshi Nagamochi. Complexity and kernels for bipartition into degree-bound bounded induced graphs. *Theor. Comput. Sci.*, 659:72–82, 2017. doi:10.1016/j.tcs.2016.11.011