MEAN-FIELD BACKWARD-FORWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND NONZERO SUM STOCHASTIC DIFFERENTIAL GAMES

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ABSTRACT. We study a general class of fully coupled backward-forward stochastic differential equations of mean-field type (MF-BFSDE). We derive existence and uniqueness results for such a system under weak monotonicity assumptions and without the non-degeneracy condition on the forward equation. This is achieved by suggesting an implicit approximation scheme that is shown to converge to the solution of the system of MF-BFSDE. We apply these results to derive an explicit form of open-loop Nash equilibrium strategies for nonzero sum mean-field linear-quadratic stochastic differential games with random coefficients. These strategies are valid for any time horizon of the game.

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1. INTRODUCTION

We study the solvability of the following backward-forward stochastic differential equation of mean-field type (MF-BFSDE): for every \( t \leq T \),

\[
\begin{aligned}
    X_t &= x + \int_0^t f(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s, Y_s)}) \, ds + \int_0^t \sigma(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s, Y_s)}) \, dW_s, \\
    Y_t &= g(X_T, \mathbb{P}_{X_T}) - \int_t^T h(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s, Y_s)}) \, ds - \int_t^T Z_s \, dW_s,
\end{aligned}
\]

where \( W := (W_t)_{t \leq T} \) is a standard Brownian motion on \( \mathbb{R}^m \) defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), \( \mathbb{P}_{(X_t, Y_t)} \) is the \( t \)-marginal distribution of \( (X_t, Y_t) \) and \( f, h, \sigma \) and \( g \) are Lipschitz continuous functions with appropriate dimensions.

This class of MF-BSDEs appears in the analysis of optimal control problems (the stochastic maximum principle) and nonzero-sum games related to nonlinear stochastic dynamical systems of McKean-Vlasov type (see e.g. [AD11][BDL11][BLM15][CD13][CD15][DH18], the list of related papers being far longer). It is an extension of the standard BFSDEs studied in several papers including [Ant93][Ham98][HY00][HP95][MPY94][MPQ14][PW99][HP95].

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Under Lipschitz continuity and monotonicity conditions on the coefficients we derive existence and uniqueness results for the system (1.1). Compared with e.g. [CD13], we do not require non-degeneracy of the diffusion coefficient of the forward process. We further allow it to depend on $Z$.

The monotonicity condition appears first in the paper by Hu and Peng [HP95] in order to remedy the assumption related to the length $T$ of the horizon $[0, T]$ when dealing with the existence and uniqueness of the solution to the standard backward-forward SDE (equation (1.1) when the coefficients $f$, $g$, $h$ and $\sigma$ do not depend on $\nu$). See also [Ant93] for more details. Subsequent papers on the solvability of standard BFSDEs where the monotonicity condition is substantially weakened include [Ham98, PW99].

As mentioned above, when the data $f$, $g$, $h$ and $\sigma$ do not depend on $\nu$, the monotonicity condition is sufficient to obtain existence and uniqueness of a solution to the standard BFSDE. Therefore, an important issue is, beside the monotonicity condition, what kind of assumptions should be further imposed on the data (especially w.r.t. $\nu$) in order to obtain existence and uniqueness of a solution to the mean-field BFSDE (1.1). We show that if the Lipschitz constants of $f$, $g$, $h$ and $\sigma$ w.r.t. $\nu$ are small enough then (1.1) has a unique solution. When $\sigma$ does not depend on $\nu$, we give a refinement of that result, under a relaxed monotonicity condition. This feature on $\sigma$ appears in the study of some linear-quadratic nonzero-sum differential games, which we consider in the second part of this paper.

In the second part of the paper we deal with the linear-quadratic nonzero-sum differential game. The coefficients are stochastic processes and not necessarily deterministic. By using the stochastic maximum principle for optimal stochastic control problems obtained in [AD11], we reduce the problem of existence of Nash equilibrium point (NEP for short) of the game to the solution of an associated MF-BFSDE of the type considered in the first part. Then, we provide conditions on the data of the game under which this latter MF-BFSDE has a unique solution and, consequently, the game has a NEP for any horizon $T$ whose explicit expression is also given. To the best of our knowledge, this result seems new. Finally, there are some (rather few) other papers on linear-quadratic nonzero-sum differential games including [DaH18, MP19], whose frameworks are however different from ours. Actually, in [DaH18], the main tool is the square completion technique, and in [MP19], the method is based on the resolution of the associated Riccati equation.

The paper is organized as follows. In Section 2, we formulate the problem and present our main results about existence and uniqueness of solutions to two classes of MF-BFSDEs under two different sets of monotonicity conditions, (H1) and (H1'). In Section 3, we derive necessary and sufficient conditions for the existence of open loop Nash equilibrium strategies for an $n$-players nonzero sum mean-field linear-quadratic SDEs with random coefficients. Moreover, we give an explicit form of those strategies. Finally, we give a counterexample to show that a Nash equilibrium may not exist when the monotonicity condition on the coefficients is not satisfied.

2. MEAN-FIELD BACKWARD-FORWARD STOCHASTIC DIFFERENTIAL EQUATIONS

Before we describe the framework defining the system of backward-forward SDEs, we introduce the Wasserstein distance between two probability measures. Denote by $M_2(\mathbb{R}^k)$ the set of probability measures on $\mathbb{R}^k$ with finite moments of order 2. For $\mu_1, \mu_2 \in M_2(\mathbb{R}^k)$, the 2-Wasserstein distance is defined by the formula

$$d(\mu_1, \mu_2) := \inf \left\{ \left( \int_{\mathbb{R}^k \times \mathbb{R}^k} |x - y|^2 F(dx, dy) \right)^{1/2} ; F(\cdot, \mathbb{R}^k) = \mu_1, F(\mathbb{R}^k, \cdot) = \mu_2 \right\}$$

(2.1)

i.e., the infimum is taken over $F \in M_2(\mathbb{R}^k \times \mathbb{R}^k)$ with marginals $\mu_1$ and $\mu_2$. It has also the following formulation in terms of a coupling between two square-integrable random variables $\xi$.
and $\xi'$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$d(\mu, \nu) = \inf \left\{ \left( \mathbb{E} \left[ |\xi - \xi'|^2 \right] \right)^{1/2}, \text{law}(\xi) = \mu_1, \text{law}(\xi') = \mu_2 \right\}, \tag{2.2}$$

from which is derived the following inequality involving the Wasserstein metric between the laws of the square integrable random variables $\xi, \tilde{\xi}$ and their $L^2$-distance:

$$d^2(\mathbb{P}_\xi, \mathbb{P}_{\tilde{\xi}}) \leq \mathbb{E}[|\xi - \tilde{\xi}|^2], \tag{2.3}$$

where $\mathbb{P}_\xi := \text{law}(\xi)$ and $\mathbb{P}_{\tilde{\xi}} := \text{law}(\tilde{\xi})$.

Next let $(W_t)_{0 \leq t \leq T}$ denote a standard $m$-dimensional Brownian motion, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose natural filtration is $(\mathcal{F}_t^0)_{0 \leq t \leq T}$, where $\mathcal{F}_s^0 = \sigma(W_u, s \leq t)$ and we denote by $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ its completion with the $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\mathcal{P}$ be the $\sigma$-algebra of $\mathbb{F}$-progressively measurable sets on $[0, T] \times \Omega$. Set $\mathbb{R}^{m+m+m \times m} := \mathbb{R}^m \times \mathbb{R}^m \times L(\mathbb{R}^m, \mathbb{R}^m)$ and let $\mathcal{M}^{2k}$ denote the space of $\mathcal{P}$-measurable and $\mathbb{R}^k$-valued processes which belong to $L^2([0, T] \times \Omega, dt \otimes d\mathbb{P})$. Next, we introduce the following spaces:

(i) $\mathcal{S}^{2m}$ is the space of continuous $\mathcal{P}$-measurable $\mathbb{R}^m$-valued processes $\xi := (\xi_t)_{t \leq T}$ such that $\mathbb{E}[\sup_{0 \leq t \leq T} |\xi_t|^2] < \infty$;

(ii) $\mathcal{H}^{2m \times m}$ is the space of $\mathcal{P}$-measurable $\mathbb{R}^{m \times m}$-valued processes $\theta := (\theta_t)_{t \leq T}$ such that $\mathbb{E}[\int_0^T |\theta_t|^2] < \infty$.

For $x, y \in \mathbb{R}^m$, $x \cdot y$ denotes the scalar product and for any $A, B \in L(\mathbb{R}^m, \mathbb{R}^d)$, $[A, B] = \sum_{j=1}^d A^j B^j$, $A^j, B^j$ being the $j$th columns of $A$ and $B$, respectively. Furthermore, for $u = (x, y, z) \in \mathbb{R}^{m+m+m \times m}$, we set $\|u\|^2 := |x|^2 + |y|^2 + |z|^2$, where $|z|^2 = \text{trace}(zz^\top)$; $(\cdot)^\top$ is the transpose operation.

We make the following assumptions.

(1) $g$ is a function defined on $\Omega \times \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)$ and valued in $\mathbb{R}^m$ such that,

(a) for any $(x, \mu) \in \mathbb{R}^m \times \mathcal{M}_2(\mathbb{R}^m)$, $g(x, \mu)$ is $\mathcal{F}^0_t$-measurable and square-integrable;

(b) $g$ is Lipschitz in $(x, \mu)$ uniformly in $\omega \in \Omega$, i.e. there exists positive constants $C^x_g$ and $C^\mu_g$ such that, for any $x, x' \in \mathbb{R}^m$ and any $\nu, \nu' \in \mathcal{M}_2(\mathbb{R}^m)$,

$$|g(x, \mu) - g(x', \nu')| \leq C^x_g |x - x'| + C^\mu_g d(\mu, \nu'), \quad \mathbb{P}\text{-a.s.} \tag{2.4}$$

(2) $f, h$ and $\sigma$ are functions defined on $[0, T] \times \Omega \times \mathbb{R}^{m+m+m \times m} \times \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R}^m)$, valued respectively in $\mathbb{R}^m, \mathbb{R}^m$ and $L(\mathbb{R}^m; \mathbb{R}^m)$ and satisfy

(a) For any $\nu \in \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R}^m)$, $u = (x, y, z) \in \mathbb{R}^{m+m+m \times m}$, the processes $(f(t, u, \nu))_{0 \leq t \leq T}$, $(h(t, u, \nu))_{0 \leq t \leq T}$ and $(\sigma(t, u, \nu))_{0 \leq t \leq T}$ belong respectively to $\mathcal{M}^{2m}_t, \mathcal{M}^{2m}_t$ and $\mathcal{M}^{2m \times m}_t$.

(b) $f, h$ and $\sigma$ are Lipschitz in $(x, y, z, \omega)$ uniformly in $(t, \omega) \in [0, T] \times \Omega$, i.e. for $\varphi = f, h, \sigma$, there exist positive constants $C^x_\varphi$ and $C^\nu_\varphi$ such that for any $t \in [0, T], u = (x, y, z), u' = (x', y', z') \in \mathbb{R}^{m+m+m \times m}, \nu, \nu' \in \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R}^m)$

$$|\varphi(t, u, \nu) - \varphi(t, u', \nu')| \leq C^x_\varphi \|u - u'\| + C^\nu_\varphi d(\nu, \nu'), \quad \mathbb{P}\text{-a.s.} \tag{2.5}$$

Hereafter, we will use $C^u := \max(C^x_f, C^x_h, C^x_\sigma)$ and $C^\nu := \max(C^x_f, C^x_h, C^x_\sigma)$ as common Lipschitz constants of $f, h$ and $\sigma$ w.r.t. $u$ and $\nu$, respectively.
A solution to the backward-forward stochastic differential equation associated with \((f, \sigma, h, g)\) is a triple of processes \((X,Y,Z) := (X_t, Y_t, Z_t)_{t \leq T}\) which is \(\mathbb{R}^{m+m+1\times m}\)-valued such that
\[
\begin{align*}
X_t &= x + \int_0^t f(s, X_s, Y_s, Z_s) ds + \int_0^t \sigma(s, X_s, Y_s, Z_s) dW_s, \quad t \leq T; \\
Y_t &= g(X_T, Y_T) - \int_t^T h(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \leq T.
\end{align*}
\] (2.6)

Next, for \(t \in [0, T]\), \(v \in \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R}^m)\), \(u = (x,y,z)\) and \(u' = (x',y',z')\) in \(\mathbb{R}^{m+m+1\times m}\), we define the function \(A\) by
\[
A(t,u,u',v) := (f(s,x,y,z,v) - f(s,x',y',z',v)) \cdot (y - y') + (h(s,x,y,z,v) - h(s,x',y',z',v)) \cdot (x - x') + [\sigma(s,x,y,z,v) - \sigma(s,x',y',z',v)] \cdot (z - z').
\] (2.7)

We consider the following assumption.
\[
(A) \begin{cases}
(i) \text{ there exists } k > 0, \text{ s.t. for all } t \in [0, T], \forall v \in \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R}^m), u, u' \in \mathbb{R}^{m+m+1\times m}, A(t,u,u',v) \leq -k(|x - x'|^2 + |y - y'|^2 + |z - z'|^2), & \text{P-a.s.} \\
(ii) \text{ there exists } k' > 0, \text{ s.t. for all } v \in \mathcal{M}_2(\mathbb{R}^m \times \mathbb{R}^m), x, x' \in \mathbb{R}^m, (g(x,v) - g(x',v)) \cdot (x - x') \geq k'|x - x'|^2, & \text{P-a.s.}
\end{cases}
\] (H1)

Remark 2.1. In the case when \(A\) and \(g\) do not depend on \(v\), Assumption (H1) appear first in a paper by Hu-Peng [HP95] to study the existence and uniqueness of the solution of the backward-forward SDE (1.1) in the framework where the coefficients do not depend on \(v\). This assumption is then weakened in several papers including [Ham98], [PW99].

In the next section we prove existence and uniqueness of the solution of system (2.6) of backward-forward SDEs under the assumptions (H1).

### 2.1. Existence and uniqueness results under (H1).

**Theorem 2.2** (Existence and Uniqueness of a solution). Let assumption (H1) hold. If the constant \(C_1^n, C_2^n\) satisfy the inequality
\[
C_1^n C_2^n < \min\{ (\sqrt{3} - 1)k', \frac{\sqrt{3}}{3}k \}
\] (2.8)

then there exists a unique process \(U^n = (X^n, Y^n, Z^n)\) which solves the system (2.6) of Backward-Forward SDE of mean-field type.

**Proof.** (i) **Existence of a solution:** Let \(\delta > 0\) and consider the sequence \(U^n = (X^n, Y^n, Z^n)_{n \geq 0}\) of processes defined recursively as follows: \((X^0, Y^0, Z^0) = (0,0,0)\) and, for \(n \geq 0, U^{n+1}\) satisfies, for every \(0 \leq t \leq T\),
\[
\begin{align*}
U^{n+1} &= (X^{n+1}, Y^{n+1}, Z^{n+1}) \in S^{2m} \times S^{2m} \times H^{2m \times m}; \\
X^{n+1}_t &= x + \int_0^t \left\{ f(s, U^{n+1}_s, v^n_s) - \delta(Y^{n+1}_s - Y^n_s) \right\} ds \\
&\quad + \int_0^t \left\{ \sigma(s, U^{n+1}_s, v^n_s) - \delta(Z^{n+1}_s - Z^n_s) \right\} dW_s, \\
Y^{n+1}_t &= g(X^{n+1}_T, Y^{n+1}_T) - \int_t^T h(s, U^{n+1}_s, v^n_s) ds - \int_t^T Z^{n+1}_s dW_s,
\end{align*}
\] (2.9)

where \(v^n_s := \mathbb{P}(X^n_s, Y^n_s)\) and \(\mu^n_T := \mathbb{P}(X^n_T)\). By Theorem 1.2 in [Ham98] (see also [HP95], pp.282 or [PW99], pp.833), the system (2.9) admits a unique solution. First we will show that \((U^n)_{n \geq 0}\) is a
Cauchy sequence in $\mathcal{M}^{2,m+m+m\times m}$ and $(X^n_T)_{n \geq 0}$ is a Cauchy sequence in $L^2(d\mathbb{P})$. For $n \geq 1$, $t \in [0,T]$, set

$$\hat{X}^{n+1}_t := X^{n+1}_t - X^n_t, \quad \hat{Y}^{n+1}_t := Y^{n+1}_t - Y^n_t, \quad \hat{Z}^{n+1}_t := Z^{n+1}_t - Z^n_t$$

(2.10)

and for $\varphi = f, h, \sigma$,

$$\hat{\varphi}^{n+1}(t) := \varphi(t, U^{n+1}_t, \nu^n_t) - \varphi(t, U^n_t, \nu^n_t), \quad \overline{\varphi}^{n}(t) := \varphi(t, U^n_t, \nu^n_t) - \varphi(t, U^n_t, \nu^n_t).$$

(2.11)

Applying Itô’s formula, we obtain

$$\hat{X}^{n+1}_T \cdot \hat{Y}^{n+1}_T - \hat{X}^{n+1}_0 \cdot \hat{Y}^{n+1}_0 = \int_0^T \hat{Y}^{n+1}_s \cdot \{ \hat{\varphi}^{n+1}(s) - \delta(\hat{Z}^{n+1}_s - \hat{Z}^n_s) \} ds$$

$$+ \int_0^T \hat{X}^{n+1}_s \cdot [\hat{\varphi}^{n+1}(s) - \delta(\hat{Z}^{n+1}_s - \hat{Z}^n_s)] dW_s$$

$$+ \int_0^T \hat{Y}^{n+1}_s \cdot \hat{h}^{n+1}(s) ds + \int_0^T \hat{X}^{n+1}_s \cdot \hat{Z}^{n+1}_s dW_s$$

$$+ \int_0^T [\hat{\varphi}^{n+1}(s) - \delta(\hat{Z}^{n+1}_s - \hat{Z}^n_s)] ds. \quad (2.12)$$

Furthermore, using standard estimates of BSDEs and the Burkholder-Davis-Gundy inequality, it is easy to see that the stochastic integrals in (2.12) are true martingales. We may take expectation to obtain

$$\mathbb{E}[\hat{X}^{n+1}_T \cdot \{ g(\hat{X}^{n+1}_T, \mu^n_T) - g(\hat{X}^{n}_T, \mu^n_{T-}) \}] + \delta \mathbb{E} \left[ \int_0^T (|\hat{Y}^{n+1}_{s}|^2 + \|\hat{Z}^{n+1}_s\|^2) ds \right]$$

$$= \delta \mathbb{E} \left[ \int_0^T (\hat{Y}^{n+1}_s \cdot \hat{Y}^n_s + |\hat{Z}^{n+1}_s - \hat{Z}^n_s|) ds \right]$$

$$+ \mathbb{E} \left[ \int_0^T (\hat{X}^{n+1}_s \cdot \hat{h}^{n+1}(s) + \hat{Y}^n_s \cdot f^{n+1}(s) + [\hat{\varphi}^{n+1}(s), \hat{Z}^{n+1}_s]) ds \right]. \quad (2.13)$$

Using the Lipschitz continuity of $g$, Young’s inequality, (2.3) and (H1(iii)), we have, for any $\varepsilon > 0$,

$$\mathbb{E}[\hat{X}^{n+1}_T \cdot (g(\hat{X}^{n+1}_T, \mu^n_T) - g(\hat{X}^{n}_T, \mu^n_{T-}))] = \mathbb{E}[\hat{X}^{n+1}_T \cdot (g(\hat{X}^{n+1}_T, \mu^n_T) - g(\hat{X}^{n}_T, \mu^n_T))] + \mathbb{E}[\hat{X}^{n+1}_T \cdot (g(\hat{X}^{n}_T, \mu^n_T) - g(\hat{X}^{n}_T, \mu^n_{T-}))]$$

$$\geq k' \mathbb{E}[|\hat{X}^{n+1}_T|^2] - C'_S \mathbb{E}[|\hat{X}^{n+1}_T|] d(\mu^n_T, \mu^n_{T-})$$

$$\geq k' \mathbb{E}[|\hat{X}^{n+1}_T|^2] - \frac{C'_E}{2} \mathbb{E}[|\hat{X}^{n+1}_T|^2] - \frac{C'_E}{2\varepsilon} d^2(\mu^n_T, \mu^n_{T-})$$

$$\geq (k' - \frac{C'_E}{2}) \mathbb{E}[|\hat{X}^{n+1}_T|^2] - \frac{C'_E}{2\varepsilon} \mathbb{E}[|\hat{X}^{n}_T|^2]. \quad (2.14)$$
Again, by the Lipschitz continuity of \( f, h, \sigma, \) Young's inequality, (2.3) and (H1(i)), we also have, for every \( 0 \leq t \leq T \) and any \( \alpha > 0 \),
\[
\hat{X}_t^{n+1} \cdot \hat{h}^{n+1}(t) + \hat{Y}_t^{n+1} \cdot \hat{f}^{n+1}(t) + \langle \hat{\sigma}^{n+1}(t), \hat{Z}_t^{n+1} \rangle
\]
\[
= \mathcal{A}(t, U_t^{n+1}, U_t^n, v_t^n) + \hat{X}_t^{n+1} \hat{h}^{n}(t) + \hat{Y}_t^{n+1} \cdot \hat{f}^{n}(t) + \langle \hat{\sigma}^{n}(t), \hat{Z}_t^{n+1} \rangle.
\]
\[
\leq -k \left\{ |\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2 + |\hat{Z}_t^{n+1}|^2 \right\} + C' \|v_t^n, v_t^{n-1}\| \left( |\hat{X}_t^{n+1}| + |\hat{Y}_t^{n+1}| + |\hat{Z}_t^{n+1}| \right)
\]
\[
\leq -k \left\{ |\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2 + |\hat{Z}_t^{n+1}|^2 \right\} + C'E \left( |\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2 \right) + \frac{3\alpha}{2} \|v_t^n, v_t^{n-1}\|^2.
\]
Now since \( d^2(v_t^n, v_t^{n-1}) \leq \mathbb{E}[|\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2] \), we have
\[
\mathbb{E} \left[ f_0^T \left( \hat{X}_t^{n+1} \cdot \hat{h}^{n+1}(s) + \hat{Y}_t^{n+1} \cdot \hat{f}^{n+1}(s) + \langle \hat{\sigma}^{n+1}(s), \hat{Z}_t^{n+1} \rangle \right) ds \right]
\]
\[
\leq \mathbb{E} \left[ f_0^T \left( \frac{C'E}{2\alpha} - k \right) \left\{ |\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2 + |\hat{Z}_t^{n+1}|^2 \right\} + \frac{3\alpha}{2} \mathbb{E}[|\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2] \right] ds.
\]
On the other hand, in view of Young’s inequality, we also have, for any \( \rho > 0 \),
\[
\mathbb{E} \left[ f_0^T (\hat{Y}_t^{n+1} + \hat{Z}_t^{n+1}) ds \right] \leq \frac{1}{2} \mathbb{E} \left[ f_0^T \left( \rho |\hat{Y}_t^{n+1}|^2 + \rho |\hat{Z}_t^{n+1}|^2 \right) \right] + \frac{1}{\rho} |\hat{Y}_t^{n+1}|^2 + \frac{1}{\rho} |\hat{Z}_t^{n+1}|^2 \right].
\]
Applying now (2.14), (2.15) and (2.16) to (2.13), yields
\[
(k' - \frac{C'E}{2\alpha}) \mathbb{E}[|\hat{X}_t^{n+1}|^2] - \frac{C'E}{2\alpha} \mathbb{E}[|\hat{X}_t^{n+1}|^2] + \delta \mathbb{E} \left[ f_0^T (|\hat{Y}_t^{n+1}|^2 + |\hat{Z}_t^{n+1}|^2) \right]
\]
\[
- \frac{C'E}{2\alpha} \mathbb{E}[|\hat{Y}_t^{n+1}|^2] + \delta \mathbb{E} \left[ f_0^T \left( \frac{\delta}{2\alpha} |\hat{Y}_t^{n+1}|^2 + \frac{\delta}{\alpha} \hat{Z}_t^{n+1} \right) \right]
\]
\[
\leq \delta \mathbb{E} \left[ f_0^T \left( \frac{\delta}{2\alpha} |\hat{Y}_t^{n+1}|^2 + \frac{\delta}{\alpha} \hat{Z}_t^{n+1} \right) \right] + \frac{3\alpha}{2} \mathbb{E} \left[ f_0^T \left( |\hat{Y}_t^{n+1}|^2 + \hat{Y}_t^{n+1} \right) \right].
\]
Rearranging terms, we obtain
\[
(k' - \frac{C'E}{2\alpha}) \mathbb{E}[|\hat{X}_t^{n+1}|^2] + \mathbb{E} \left[ f_0^T (k - \frac{C'E}{2\alpha}) |\hat{X}_t^{n+1}|^2 ds \right] + \frac{C'E}{2\alpha} \mathbb{E}[|\hat{X}_t^{n+1}|^2] + \delta \mathbb{E} \left[ f_0^T \left( |\hat{Y}_t^{n+1}|^2 + |\hat{Z}_t^{n+1}|^2 \right) \right]
\]
\[
\leq \frac{C'E}{2\alpha} \mathbb{E}[|\hat{X}_t^{n+1}|^2] + \mathbb{E} \left[ f_0^T \left( \frac{3\alpha C'E}{2} |\hat{X}_t^{n+1}|^2 + \frac{\delta}{2\alpha} \hat{Y}_t^{n+1} |\hat{Y}_t^{n+1}|^2 + \frac{\delta}{\alpha} |\hat{Z}_t^{n+1}|^2 \right) \right].
\]
By setting
\[
\lambda(\epsilon, \delta, \alpha, \rho) := \min \left\{ k' - \frac{C'E}{2\alpha}, k - \frac{C'E}{2\alpha}, \delta(1 - \frac{\delta}{\alpha}) + k - \frac{C'E}{2\alpha} \right\},
\]
\[
\theta(\epsilon, \delta, \alpha, \rho) := \max \left\{ \frac{C'E}{2\alpha}, \frac{\delta}{2\alpha} + \frac{3\alpha C'E}{2} \right\},
\]
we obtain
\[
\mathbb{E}[|\hat{X}_t^{n+1}|^2] + \mathbb{E} \left[ f_0^T \left( |\hat{Y}_t^{n+1}|^2 \right) \right] \leq \frac{\delta}{\pi} \left( \mathbb{E}[|\hat{X}_t^{n+1}|^2] + \mathbb{E} \left[ f_0^T \left( |\hat{Y}_t^{n+1}|^2 \right) \right] \right).
\]
Now, if there exist \( \alpha, \epsilon, \delta \) and \( \rho \) so that
\[
\lambda(\epsilon, \delta, \alpha, \rho) > \theta(\epsilon, \delta, \alpha, \rho)
\]
then the inequality (2.17) becomes a contraction, which implies that \((X^n_s)_{n \geq 0}\) is a Cauchy sequence in \(L^2(\Omega, \mathbb{P})\) and \((X^n)_{n \geq 0}, (Y^n)_{n \geq 0}\) and \((Z^n)_{n \geq 0}\) are Cauchy sequences in \(L^2([0, T] \times \Omega, dt \otimes d\mathbb{P})\). Therefore going back to (2.9), using Itô’s formula and, by now standard calculations, we obtain
\[
\mathbb{E} \sup_{s \leq T} (|X^n_s - X^n_s|^2 + |Y^n_s - Y^n_s|^2) \to 0 \text{ as } n, m \to \infty.
\]
Consequently, there exist \(\mathcal{F}\)-adapted continuous processes \(X\) and \(Y\) and an \(\mathcal{F}\)-progressively measurable process \(Z\) such that
\[
\mathbb{E} \sup_{s \leq T} (|X_s|^2 + |Y_s|^2) + \int_0^T \|Z_s\|^2 ds \to 0 \text{ as } n \to \infty.
\]
Moreover,
\[
\mathbb{E} \sup_{s \leq T} (|X_s|^2 + |Y_s|^2) + \int_0^T \|Z_s\|^2 ds < \infty.
\]
Finally, taking the limits in equation (2.9) we obtain that \((X, Y, Z)\) is a solution of MF-BFSDE (2.6).

Next, we are going to show that such \(\alpha, \epsilon, \delta\) and \(\rho\) exist when the condition (2.8) is satisfied. In fact, to make the contraction meaningful, we assume \(k' - \frac{C'_v}{2\alpha}, k - \frac{C'_v}{2\alpha}\) and \(1 - \frac{\rho}{2}\) are positive. It is easily shown that \((1 - \frac{\rho}{2}) \leq \frac{1}{3}\) and the terms of this inequality are equal if and only if \(\rho = 1\). So let us take \(\rho = 1\) and set
\[
\theta^*(\epsilon, \alpha) = \lim_{\delta \to 0} \theta(\epsilon, \delta, \alpha, 1) = \max \left\{ \frac{C'_v}{2\epsilon}, \frac{3\alpha C'_v}{2} \right\}
\]
and
\[
\lambda^*(\epsilon, \alpha) = \lim_{\delta \to 0} \lambda(\epsilon, \delta, \alpha, 1) = \min \left\{ k' - \frac{C'_v}{2}, k - \frac{C'_v}{2\alpha} \right\}.
\]
Now if, for some \(\epsilon, \alpha\), we have \(\lambda^*(\epsilon, \alpha) > \theta^*(\epsilon, \alpha)\), then there exists \(\delta\) small enough such (2.18) is satisfied with those \(\epsilon, \alpha, \delta\) and \(\rho = 1\). Finally, in order to have \(\lambda^*(\epsilon, \alpha) > \theta^*(\epsilon, \alpha)\), it is equal to have the following inequalities:
\[
\begin{align*}
k' - &\frac{C'_v \epsilon}{2} > \frac{C'_v}{2\epsilon} \\
k &\frac{C'_v}{2\alpha} > \frac{C'_v}{2\epsilon} \\
k &\frac{C'_v}{2} > \frac{3\alpha C'_v}{2} \\
k' - &\frac{C'_v \epsilon}{2} > \frac{3\alpha C'_v}{2} .
\end{align*}
\]
For these inequalities, noticing that \(\frac{C'_v \epsilon}{2} + \frac{C'_v}{2\alpha} + \frac{3\alpha C'_v}{2}\) reach their minimum when \(\epsilon = 1\) and \(\alpha = \frac{\sqrt{3}}{\sqrt{2}}\), respectively. To give a sufficient condition on \(C'_v, C'_v\), we choose \(\alpha = \frac{\sqrt{3}}{\sqrt{2}}\), \(\epsilon = 1\), and set \(\gamma_1, \gamma_2 > 0\) to be the coefficients satisfying \(C'_v, C'_v < \min \{\gamma_1 k, \gamma_2 k'\}\).
If \(\gamma_1 k \leq \gamma_2 k'\), then (2.19) holds if the following system of inequalities hold.
\[
\begin{align*}
k' &> C'_v \\
k &> \sqrt{3} C'_v \\
k &\frac{\sqrt{3} \gamma_1 k}{2} > \frac{\gamma_1 k}{2} \\
k' - &\frac{\gamma_1 k}{2} > \frac{\sqrt{3} \gamma_1 k}{2} .
\end{align*}
\]
From the third inequality, we obtain \( \gamma_1 < \sqrt{3} - 1 \). For the forth inequality in (2.20), it is enough to show
\[
\frac{\gamma_1 k}{\gamma_2} - \frac{\gamma_2 k}{2} > \frac{\sqrt{3} \gamma_1 k}{2},
\]
which means \( \gamma_2 < \sqrt{3} - 1 \). It is easily checked that we obtain the same result under the other condition \( \gamma_1 k > \gamma_2 k' \). Finally, compared with \( C' \sqrt{3} k \), we obtain the sufficient condition \( C' \sqrt{3} k < \min \left\{ (\sqrt{3} - 1)k', \frac{\sqrt{3} \gamma_1 k}{2}, C \varepsilon \right\} \) for which \( \gamma(e, \delta, \alpha, \rho) < \theta(e, \delta, \alpha, \rho) \) when \( \rho = \varepsilon = 1, \alpha = \frac{\sqrt{3}}{2} \) and \( \delta > 0 \) is small enough.

(ii) Uniqueness of the solution: Let \( U' = (X', Y', Z') \) be another solution to (2.6). Set
\[
\Gamma_T := \mathbb{E} \left[ \int_0^T \{(f(s, U'_s, v'_s) - f(s, U_s, v_s)) \cdot (Y'_s - Y_s) \\
+ (h(s, U'_s, v'_s) - h(s, U_s, v_s)) \cdot (X'_s - X_s) \\
+ \sigma(s, U'_s, v'_s) - \sigma(s, U_s, v_s), Z'_s - Z_s \} \, ds \right].
\]
(2.21)

Applying Itô’s formula to the product \( (X'_T - X_T) \cdot (Y'_T - Y_T) \) and taking expectation, we obtain
\[
\mathbb{E}[(X'_T - X_T) \cdot (Y'_T - Y_T)] = \Gamma_T.
\]
(2.22)

In view of (H1), (2.3) and the Lipschitz continuity of \( f, h, \sigma \) and \( g \), and the Cauchy-Schwarz inequality we have
\[
\Gamma_T = \mathbb{E}[(X'_T - X_T) \cdot (Y'_T - Y_T)] \\
\geq k' \mathbb{E}[|X'_T - X_T|^2] - C' \mathbb{E}[|X'_T - X_T|] \mathbb{E}[|X'_T - X_T|^2]^{\frac{1}{2}} \\
\geq (k' - C' \nu) \mathbb{E}[|X'_T - X_T|^2].
\]
Therefore,
\[
\Gamma_T \geq (k' - C' \nu) \mathbb{E}[|X'_T - X_T|^2].
\]
(2.23)

On the other hand, we have
\[
\Gamma_T \leq \mathbb{E} \left[ \int_0^T \{A(s, U_s, U'_s, v'_s) + (C' |X'_s - X_s| + C' |Y'_s - Y_s| + C' |Z'_s - Z_s|) \} \, ds \right].
\]
But
\[
d(v_s, v'_s) \leq \sqrt{\mathbb{E}[|X'_s - X_s|^2 + |Y'_s - Y_s|^2]}
\]
Therefore, by the use of Young’s inequality three times we obtain
\[
\Gamma_T \leq \mathbb{E} \left[ \int_0^T \left\{ -k(|X'_s - X_s|^2 + |Y'_s - Y_s|^2 + |Z'_s - Z_s|^2) + \frac{C'}{2} (3\alpha + \frac{1}{\alpha}) |X'_s - X_s|^2 + \frac{C'}{2} (3\alpha + \frac{1}{\alpha}) |Y'_s - Y_s|^2 + \frac{C'}{2\alpha} |Z'_s - Z_s|^2 \right\} \, ds \right].
\]
(2.24)

Now, combine (2.23) and (2.24) to obtain
\[
(k' - C' \nu) \mathbb{E}[|X'_T - X_T|^2] + k \mathbb{E} \left[ \int_0^T \{|X'_s - X_s|^2 + |Y'_s - Y_s|^2 + |Z'_s - Z_s|^2 \} \, ds \right]
\leq \mathbb{E} \left[ \int_0^T \left\{ \frac{C'}{2} (3\alpha + \frac{1}{\alpha}) |X'_s - X_s|^2 + \frac{C'}{2} (3\alpha + \frac{1}{\alpha}) |Y'_s - Y_s|^2 + \frac{C'}{2\alpha} |Z'_s - Z_s|^2 \right\} \, ds \right].
\]
or
\[
0 \leq (k' - C'_8)E[|X_T^i - X_T^j|^2] \leq E\left[ \int_0^T \left\{ \frac{C'}{2}(3\alpha + 1) - k \right\} |X_s^i - X_s^j|^2 + \frac{C'}{2}(3\alpha + 1) - k \right] |Y_s^i - Y_s^j|^2 + \frac{C'}{2\alpha} - k \right\} |Z_s^i - Z_s^j|^2 \right] ds.
\]
Noticing now that \( C'_8, C' \leq \min\{ (\sqrt{3} - 1)k', \frac{\sqrt{7}}{2}k \}, \) with \( \alpha = \frac{\sqrt{7}}{2}, \) all the coefficients of the right hand side of the above inequality are negative, which implies that, \( \mathbb{P} \)-a.s. for all \( 0 \leq s \leq T, \) \( X_s^i = X_s, Y_s^i = Y_s, \) and \( Z_s^i = Z_s, ds \otimes d\mathbb{P} \)-a.e. Thus the solution of (2.6) is unique. \( \square \)

2.2. Existence and uniqueness results when \( \sigma \) does not depend on the mean-field term.

Assuming \( \sigma \) does not depend on \( \mathbb{P}_{(X_t,Y_t)} \) i.e. the MF-BFSDE (2.6) becomes: \( \forall t \leq T, \)
\[
\begin{align*}
X_t, Y_t &\in S^{2,m} \text{ and } Z_t \in \mathcal{H}^{2,m \times m}; \\
X_t &= x + \int_0^t f(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s,Y_s)}) ds + \int_0^t \sigma(s, X_s, Y_s, Z_s) dW_s, \quad t \leq T, \\
Y_t &= g(X_t, \mathbb{P}_{X_t}) - \int_t^T h(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s,Y_s)}) ds - \int_t^T Z_s dW_s, \quad t \leq T.
\end{align*}
\]
In this framework, the condition (H1) can be relaxed to the following assumption.

(\( H1' \))
\[
(i) \text{there exists } k > 0, \text{ s.t. for all } t \in [0, T], v \in \mathbb{M}_2(\mathbb{R}^{m \times m}), u, u' \in \mathbb{R}^{m+m \times m}, \text{ and } v \cdot (x-x') \in \mathbb{R}^m,
\]
\[
\mathcal{A}(t, u, u', v) \leq -k(|x-x'|^2 + |y-y'|^2), \quad \mathbb{P}\text{-a.s.}
\]

(ii) there exists \( k' > 0, \) s.t. for all \( v \in \mathbb{M}_2(\mathbb{R}^{m \times m}), x, x' \in \mathbb{R}^m, \) \( (g(x,v) - g(x',v)) \cdot (x-x') \geq k'|x-x'|^2, \quad \mathbb{P}\text{-a.s.} \)

Following similar steps as in the proof of Theorem 2.2, we have the following:

Theorem 2.3 (Existence and Uniqueness of a solution). Let Assumption (\( H1' \)) hold. If the constants \( C'_8, C', k, k' \) satisfy the inequalities
\[
C'_8, C' < \min\{ 2(\sqrt{2} - 1)k', \frac{\sqrt{7}}{2}k \},
\]
then there exists a unique process \( U = (X, Y, Z) \) which belongs to \( S^{2,m} \times S^{2,m} \times \mathcal{H}^{2,m \times m} \) and which solves the MF-BFSDE (2.25).

Proof: (i) Existence of a solution: Let \( \delta > 0 \) and consider the sequence \( U^n = (X^n, Y^n, Z^n)_{n \geq 0} \) of processes defined recursively as follows: \( (X^n, Y^n, Z^n) = (0, 0, 0) \) and, for \( n \geq 0, U^{n+1} \) satisfies, for every \( 0 \leq t \leq T, \)
\[
\begin{cases}
U^{n+1} = (X^{n+1}, Y^{n+1}, Z^{n+1}) \in S^{2,m} \times S^{2,m} \times \mathcal{H}^{2,m \times m}; \\
X^{n+1}_t = x + \int_0^t \{ f(s, U^n_s, v^n_s) - \delta(Y^n_s - Y^n_s) \} ds \\
+ \int_0^t \sigma(s, U^n_s) dW_s, \\
Y^{n+1}_t = g(X^{n+1}_t, \mu^n_t) - \int_t^T h(s, U^n_s, v^n_s) ds - \int_t^T Z^n_s dW_s,
\end{cases}
\]
where \( v^n_t := \mathbb{P}_{(X^n_t,Y^n_t)} \) and \( \mu^n_t := \mathbb{P}_{X^n_t} \). By Theorem 1.2 in [Ham98] (or [PW99], pp.833), the system

(2.27)

admits a unique solution. We will show that \( (U^n)_{n \geq 0} \) is a Cauchy sequence in \( \mathcal{M}^{2,m+m \times m} \) and \( (X^n_T)_{n \geq 0} \) is a Cauchy sequence in \( L^2(d\mathbb{P}) \). For \( n \geq 1, t \in [0, T], \) recall the processes \( \hat{X}^{n+1}, \) \( \hat{Y}^{n+1}, \) \( Z^{n+1}, \) \( \hat{\varphi}^{n+1} \) and \( \overline{\varphi}^{n} \) defined respectively in (2.10) and (2.11).
Applying Itô's formula, we obtain
\[
\begin{align*}
X_T^{n+1} \cdot \hat{Y}_T^{n+1} - X_0^{n+1} \cdot \hat{Y}_0^{n+1} &= \int_0^T \hat{\phi}_s^{n+1} \cdot \{ \hat{\phi}_s^{n+1} - \delta(\hat{Y}_s^{n+1} - \hat{Y}_s^n) \} \, ds \\
&+ \int_0^T \hat{\phi}_s^{n+1} \cdot \hat{\delta}_s^{n+1} \, dW_s \\
&+ \int_0^T \hat{X}_s^{n+1} \cdot \hat{f}_s^{n+1} \, ds + \int_0^T \hat{X}_s^{n+1} \cdot \hat{Z}_s^{n+1} \, dW_s \\
&+ \int_0^T \hat{\sigma}_s^{n+1} \, ds.
\end{align*}
\] (2.28)

Similarly as above, we take expectation to obtain
\[
\begin{align*}
\mathbb{E}[X_T^{n+1} \cdot (g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1}))] &= \mathbb{E}[X_T^{n+1} \cdot (g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^n))] \\
&+ \mathbb{E}[X_T^{n+1} \cdot (g(X_T^n, \mu_T^n) - g(X_T^n, \mu_T^{n-1}))] \\
&\geq k' \mathbb{E}[\|X_T^{n+1}\|^2] - C_\nu^e \mathbb{E}[\|X_T^{n+1}\|] \, d(\mu_T^n, \mu_T^{n-1}) \\
&\geq k' \mathbb{E}[\|X_T^{n+1}\|^2] - C_\nu^e \mathbb{E}[\|X_T^{n+1}\|^2] - \frac{C_\nu}{\alpha} \, d^2(\mu_T^n, \mu_T^{n-1}) \\
&\geq (k' - \frac{C_\nu}{\alpha}) \mathbb{E}[\|X_T^{n+1}\|^2] - \frac{C_\nu}{\alpha} \mathbb{E}[\|X_T^{n+1}\|^2].
\end{align*}
\] (2.30)

Using the Lipschitz continuity of \( g \), Young’s inequality, (2.3) and (H1’(iii)), we have, for any \( \varepsilon > 0 \),
\[
\mathbb{E}[\hat{X}_T^{n+1} \cdot (g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1}))] = \mathbb{E}[\hat{X}_T^{n+1} \cdot (g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^n))] \\
+ \mathbb{E}[\hat{X}_T^{n+1} \cdot (g(X_T^n, \mu_T^n) - g(X_T^n, \mu_T^{n-1}))] \\
\geq k' \mathbb{E}[\|\hat{X}_T^{n+1}\|^2] - C_\nu^e \mathbb{E}[\|\hat{X}_T^{n+1}\|] \, d(\mu_T^n, \mu_T^{n-1}) \\
\geq k' \mathbb{E}[\|\hat{X}_T^{n+1}\|^2] - C_\nu^e \mathbb{E}[\|\hat{X}_T^{n+1}\|^2] - \frac{C_\nu}{\alpha} \, d^2(\mu_T^n, \mu_T^{n-1}) \\
\geq (k' - \frac{C_\nu}{\alpha}) \mathbb{E}[\|\hat{X}_T^{n+1}\|^2] - \frac{C_\nu}{\alpha} \mathbb{E}[\|\hat{X}_T^{n+1}\|^2].
\] (2.31)

Again, by the Lipschitz continuity of \( f, h, \sigma \), Young’s inequality, (2.3) and (H1’(ii)), we also have, for every 0 \( \leq t \leq T \) and any \( \alpha > 0 \),
\[
\hat{X}_t^{n+1} \cdot \hat{h}_t^{n+1}(t) + \hat{Y}_t^{n+1} \cdot \hat{f}_t^{n+1}(t) + [\hat{\sigma}_t^{n+1}(t), \hat{Z}_t^{n+1}] \\
= A(t, U_t^{n+1}, U_t^n, v_t^n) + \hat{X}_t^{n+1} \cdot \hat{h}_t^{n}(t) + \hat{Y}_t^{n+1} \cdot \hat{f}_t^{n}(t) \\
\leq -k \left\{ |\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2 \right\} + |\hat{X}_t^{n+1}||\hat{h}_t^{n}(t)| + |\hat{Y}_t^{n+1}||\hat{f}_t^{n}(t)| \\
\leq -k \left\{ |\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2 \right\} + C^v \, d(v_t^n, v_t^{n-1}) \left( |\hat{X}_t^{n+1}| + |\hat{Y}_t^{n+1}| \right) \\
\leq -k \left\{ |\hat{X}_t^{n+1}|^2 + |\hat{Y}_t^{n+1}|^2 \right\} + C^v |\hat{X}_t^{n+1}|^2 + \frac{C^v}{\alpha^2} |\hat{Y}_t^{n+1}|^2 + aC^v \cdot d^2(v_t^n, v_t^{n-1}).
\]

Since \( d^2(v_t^n, v_t^{n-1}) \leq \mathbb{E}[|\hat{X}_t^n|^2 + |\hat{Y}_t^n|^2] \), then
\[
\mathbb{E} \left[ \int_0^T \left( \hat{X}_s^{n+1} \cdot \hat{h}_s^{n+1}(s) + \hat{Y}_s^{n+1} \cdot \hat{f}_s^{n+1}(s) + [\hat{\sigma}_s^{n+1}(s), \hat{Z}_s^{n+1}] \right) \, ds \right] \\
\leq \mathbb{E} \int_0^T \left( \left( \frac{C^v}{\alpha^2} - k \right) (|\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2) + aC^v \mathbb{E}[|\hat{X}_s^n|^2 + |\hat{Y}_s^n|^2] \right) \, ds.
\]
Furthermore, we have, for any \( \rho > 0 \),
\[
\mathbb{E} \left[ \int_0^T \dot{Y}^n_{t+1} \cdot \dot{Y}^n_s \, ds \right] \leq \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \rho \left| \dot{Y}^n_{t+1} \right|^2 + \frac{1}{\rho} \left| \dot{Y}^n_s \right|^2 \right) \, ds \right].
\] (2.32)

Applying now the three inequalities (2.30), (2.31) and (2.32) to (2.13), yields
\[
(k' - \frac{C\nu}{2\delta}) \mathbb{E}[|\dot{X}^{n+1}_T|^2] - \frac{C\nu}{2\alpha} \mathbb{E}[|\dot{X}^n_T|^2] + \delta \mathbb{E} \left[ \int_0^T \left| \dot{Y}^{n+1}_s \right|^2 \, ds \right]
\]
\[
\leq \mathbb{E} \left[ \int_0^T \left\{ \left( \frac{C\nu}{2\alpha} - k \right) \cdot |\dot{X}^{n+1}_t|^2 + \left( \frac{C\nu}{2\alpha} - k \right) \cdot |\dot{Y}^{n+1}_t|^2 + \alpha C\nu \mathbb{E}[|\dot{X}^n_t|^2 + |\dot{Y}^n_t|^2] \right\} \, ds \right]
\]
\[
\leq \delta \mathbb{E} \left[ \int_0^T \left( \frac{\nu}{2\delta} |\dot{Y}^{n+1}_s| + \frac{1}{\rho} |\dot{Y}^n_s|^2 \right) \, ds \right].
\]

Rearranging terms, we obtain
\[
(k' - \frac{C\nu}{2\delta}) \mathbb{E}[|\dot{X}^{n+1}_T|^2] + \mathbb{E} \left[ \int_0^T (k - \frac{C\nu}{2\alpha}) |\dot{X}^{n+1}_t|^2 \, ds \right] + \mathbb{E} \left[ \int_0^T (\nu(1 - \frac{\nu}{\delta}) + k - \frac{C\nu}{2\alpha}) |\dot{Y}^{n+1}_t|^2 \, ds \right]
\]
\[
\leq \frac{C\nu}{2\delta} \mathbb{E}[|\dot{X}^n_T|^2] + \mathbb{E} \left[ \int_0^T \left( \alpha C\nu |\dot{X}^n_t|^2 + \left( \frac{\nu}{2\delta} + \alpha C\nu \right) |\dot{Y}^n_t|^2 \right) \, ds \right].
\]

Taking \( \rho = 1 \), we obtain
\[
(k' - \frac{C\nu}{2\delta}) \mathbb{E}[|\dot{X}^{n+1}_T|^2] + \mathbb{E} \left[ \int_0^T (k - \frac{C\nu}{2\alpha}) |\dot{X}^{n+1}_t|^2 \, ds \right] + \mathbb{E} \left[ \int_0^T \left( \frac{1}{2} \delta + k - \frac{C\nu}{2\alpha} \right) |\dot{Y}^{n+1}_t|^2 \, ds \right]
\]
\[
\leq \frac{C\nu}{2\delta} \mathbb{E}[|\dot{X}^n_T|^2] + \mathbb{E} \left[ \int_0^T \left( \alpha C\nu |\dot{X}^n_t|^2 + \left( \frac{1}{2} \delta + \alpha C\nu \right) |\dot{Y}^n_t|^2 \right) \, ds \right].
\]

Let us set
\[
\lambda(\epsilon, \delta, \alpha) := \min \left\{ k' - \frac{C\nu}{2}, \frac{1}{2} \delta + k - \frac{C\nu}{2\alpha} \right\},
\]
\[
\theta(\epsilon, \delta, \alpha) := \max \left\{ \frac{C\nu}{2\delta}, \frac{1}{2} \delta + \alpha C\nu \right\}. \tag{2.33}
\]

Then, it holds that
\[
\mathbb{E}[|\dot{X}^{n+1}_T|^2] + \mathbb{E} \left[ \int_0^T |\dot{X}^{n+1}_t|^2 + |\dot{Y}^{n+1}_t|^2 \, ds \right] \leq \frac{\lambda}{\theta} \mathbb{E} \left[ |\dot{X}^n_T|^2 \right] + \mathbb{E} \left[ \int_0^T |\dot{X}^{n+1}_t|^2 + |\dot{Y}^{n+1}_t|^2 \, ds \right]. \tag{2.34}
\]

Now, if there exist \( \alpha, \epsilon, \delta \) so that \( \theta < \lambda \), the inequality (2.34) becomes a contraction. Thus, \( (X^n_T)_{n \geq 0} \) is a Cauchy sequence in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) and \( (X^n)_{n \geq 0} \) and \( (Y^n)_{n \geq 0} \) are Cauchy sequences in \( L^2([0, T] \times \Omega, dt \otimes d\mathbb{P}) \).

To make the contraction meaningful, we assume \( k' - \frac{C\nu}{2} \) and \( k - \frac{C\nu}{2\alpha} \) are positive. Next, similarly to Theorem 2.2 since \( \delta > 0 \) can be chosen small enough, we only need to solve the following system of inequalities (which stem from the limits of \( \lambda \) and \( \theta \) as \( \delta \to 0 \)):
\[
\begin{align*}
k' - \frac{C\nu}{2} &> \frac{C\nu}{2\delta} \\
k - \frac{C\nu}{2\alpha} &> \frac{C\nu}{2\delta} \\
k - \frac{C\nu}{2\alpha} &> \alpha C\nu \\
k' - \frac{C\nu}{2} &> \alpha C\nu.
\end{align*}
\] (2.35)
As in the proof of Theorem 2.2 for those inequalities, we choose \( \alpha = \frac{\sqrt{3}}{2}, \varepsilon = 1 \) and set \( \gamma_3, \gamma_4 > 0 \) to be the coefficients satisfying the \( C^v_{\gamma_3}, C^v < \min \{ \gamma_3 k, \gamma_4 k \} \). Assuming \( \gamma_3 k \leq \gamma_4 k' \), (2.35) holds if the following system of inequalities holds:

\[
\begin{align*}
  k' &> C^v_{\gamma_3} \\
  k &> \sqrt{2} C^v \\
  k - \frac{\sqrt{2} \gamma_3 k}{2} &> \frac{\gamma_3 k}{2} \\
  k' - \gamma_3 k &> \frac{\sqrt{2} \gamma_3 k}{2}.
\end{align*}
\] (2.36)

From the third inequality, we obtain \( \gamma_3 < 2(\sqrt{2} - 1) \). For the forth inequality in (2.36) to be satisfied, it is enough to have

\[
\frac{\gamma_3 k}{2} - \frac{\gamma_3 k}{2} > \frac{\sqrt{2} \gamma_3 k}{2}
\]

which means \( \gamma_4 < 2(\sqrt{2} - 1) \). The result under the other condition \( \gamma_3 k > \gamma_2 k' \) turns out to be the same, as it can be easily checked. Finally, compared with \( C^v < \frac{\sqrt{2}}{2} k \), we obtain a sufficient condition \( C^v_{\gamma_3}, C^v < \min \{ 2(\sqrt{2} - 1) k', \frac{\sqrt{2}}{2} k \} \) satisfying \( \gamma(\varepsilon, \delta, \alpha, \rho) < \theta(\varepsilon, \delta, \alpha, \rho) \), when \( \varepsilon = 1, \alpha = \frac{\sqrt{2}}{2} \) and \( \delta > 0 \) is small enough.

We now show that \( (Z^n)_{n \geq 0} \) is also a Cauchy sequence in \( L^2([0, T] \times \Omega, dt \otimes d\mathbb{P}) \).

For \( n, m \geq 0 \), we have

\[
d(Y^n_t - Y^m_t) = (h(t, U^n_t, v^n_t - 1) - h(t, U^m_t, v^m_t - 1))dt + (Z^n_t - Z^m_t)dW_t.
\]

By applying the Itô formula and then taking expectation, we obtain

\[
\mathbb{E}[|Y^n_t - Y^m_t|^2] - |Y^n_t - Y^m_t|^2 = \mathbb{E}[\int_t^T 2|Y^n_s - Y^m_s| |h(s, U^n_s, v^n_s - 1) - h(s, U^m_s, v^m_s - 1)| + |Z^n_s - Z^m_s|^2 | ds].
\]

In view of the Lipschitz condition on the coefficients and Young’s inequality, we have, for any \( \beta > 0 \),

\[
\mathbb{E} \int_t^T |Z^n_s - Z^m_s|^2 ds \leq \mathbb{E}[|Y^n_t - Y^m_t|^2] + \mathbb{E}[\int_t^T 2C^n|Y^n_s - Y^m_s| (|X^n_s - X^m_s| + |Y^n_s - Y^m_s| + \|Z^n_s - Z^m_s\|) + \|Z^n_s - Z^m_s\|^2 + 2\beta(C^n)^2 |Y^n_s - Y^m_s|^2 + C^n (|Y^n_s - Y^m_s|^2 + |X^n_s - X^m_s|^2 + |Y^n_s - Y^m_s|^2 + |X^n_s - X^m_s|^2 + |Y^n_s - Y^m_s|^2) | ds].
\]

Let \( \beta = 1 \) and \( t = 0 \). We then have

\[
\frac{1}{2} \mathbb{E} \int_0^T |Z^n_t - Z^m_t|^2 ds \leq \mathbb{E}[|Y^n_t - Y^m_t|^2] + \mathbb{E}[\int_0^T 2C^n|Y^n_s - Y^m_s| (|X^n_s - X^m_s| + |Y^n_s - Y^m_s| + 2(C^n)^2 |Y^n_s - Y^m_s|^2 + C^n (|Y^n_s - Y^m_s|^2 + |X^n_s - X^m_s|^2 + |Y^n_s - Y^m_s|^2) | ds].
\] (2.37)
Since $(X^n_T)_{n \geq 0}$ is a Cauchy sequence in $L^2(\Omega, \mathbb{P})$ and $(X^n)_{n \geq 0}$ and $(Y^n)_{n \geq 0}$ are Cauchy sequences in $L^2([0, T] \times \Omega, dt \otimes d\mathbb{P})$, $(Z^n)_{n \geq 0}$ is also a Cauchy sequence in $L^2([0, T] \times \Omega, dt \otimes d\mathbb{P})$. Therefore, by standard calculations, we also have

$$
\mathbb{E}[\sup_{s \leq T} (|X^n_s - X^m_s|^2 + |Y^n_s - Y^m_s|^2)] \to 0 \text{ as } n, m \to \infty.
$$

Consequently, there exist $\mathbb{F}$-adapted continuous processes $X$ and $Y$ and an $\mathbb{F}$-progressively measurable process $Z$ such that

$$
\mathbb{E}[\sup_{s \leq T} (|X_s|^2 + |Y_s|^2) + \int_0^T \|Z_s - Z_s\|^2 ds] \to 0 \text{ as } n \to \infty.
$$

Moreover,

$$
\mathbb{E}[\sup_{s \leq T} (|X_s|^2 + |Y_s|^2) + \int_0^T \|Z_s\|^2 ds] < \infty.
$$

By taking the limit with respect to $n$ in equation (2.27) we obtain that $(X, Y, Z)$ is a solution of MF-BFSDE (2.25).

(ii) Uniqueness of the solution: Let $U' = (X', Y', Z')$ be another solution to (2.25) and set

$$
\Gamma_T := \mathbb{E}\left[\int_0^T \{(f(s, U'_s, \nu'_s) - f(s, U_s, \nu_s)) \cdot (Y'_s - Y_s) + (h(s, U'_s, \nu'_s) - h(s, U_s, \nu_s)) \cdot (X'_s - X_s) + \nu(s, U'_s, Z'_s - Z_s) ds\}\right].
$$

Applying Itô’s formula to the product $(X'_s - X_T) \cdot (Y'_s - Y_T)$ and taking expectation, we obtain

$$
\mathbb{E}[(X'_s - X_T) \cdot (Y'_s - Y_T)] = \Gamma_T.
$$

In view of (H1'), (2.3), the Lipschitz continuity of $f, h, \sigma$ and $g$, and the Cauchy-Schwarz inequality, we have

$$
\Gamma_T = \mathbb{E}[(X'_s - X_T) \cdot (Y'_s - Y_T)] \geq k'\mathbb{E}[(X'_s - X_T)^2] - C^v_s\mathbb{E}[(X'_s - X_T)]d(\mu'_s, \mu_T) \geq k'\mathbb{E}[(X'_s - X_T)^2] - C^v_s\mathbb{E}[(X'_s - X_T)]\mathbb{E}[(X'_s - X_T)^2]^{\frac{1}{2}} \geq (k' - C^v_s)\mathbb{E}[(X'_s - X_T)^2].
$$

Therefore,

$$
\Gamma_T \geq (k' - C^v_s)\mathbb{E}[(X'_s - X_T)^2].
$$

On the other hand, we have

$$
\Gamma_T \leq \mathbb{E}\left[\int_0^T \{\mathcal{A}(s, U_s, U'_s, \nu_s) + (C^v|X'_s - X_s| + C^v|Y'_s - Y_s|)d(\nu_s, \nu'_s)\} ds\right].
$$

But,

$$
d(\nu_s, \nu'_s) \leq \sqrt{\mathbb{E}[(X'_s - X_s)^2 + (Y'_s - Y_s)^2]}.
$$

Therefore, by the use of Young’s inequality three times we obtain

$$
\Gamma_T \leq \mathbb{E}\int_0^T \left(-k(|X'_s - X_s|^2 + |Y'_s - Y_s|^2) + \frac{C^v}{2}(2a + \frac{1}{a})(|X'_s - X_s|^2 + |Y'_s - Y_s|^2)\right) ds. \tag{2.41}
$$
Remark 2.4.

(i) Conditions (2.28) and (2.26) in Theorems 2.2 and 2.3 are only sufficient conditions. Whether or not they are necessary does not seem an easy task. However, as it is well known, the existence of a solution for the MF-BFSDEs (2.6) and (2.25) depends on several parameters including the length of the time horizon and the initial value $x$ of the forward SDE (see Example 2.25 below).

(ii) Conditions (2.28) and (2.26) can be improved if we consider $C_{f}, C_{o}, C_{h}$ instead of $C'$.

3. THE NONZERO-SUM MEAN FIELD GAME: THE OPEN-LOOP FRAMEWORK

In this section $W = (W_{t})_{t \leq T}$ is a one-dimension Brownian motion. For $i = 1, \ldots, m$, let $\mathcal{U}_{i} := \mathcal{M}^{2,m_{i}}$, be the set of open-loop admissible controls for the player $i$. The set $\mathcal{U} := \Pi_{i=1,m} \mathcal{U}_{i}$, is called of open-loop admissible strategies for the players. In the sequel, a stochastic process $\bar{\rho} = (\bar{\rho}_{i}(\omega))_{i \leq T}$ with values in $\mathbb{R}^{t_{1} \times t_{2}}$ is called bounded if

$$
\|\rho\| := \sup_{(t, \omega) \in [0, T] \times \Omega} ||\rho_{i}(\omega)|| < \infty. 
$$

Next, for $u = (u^{i})_{1 \leq i \leq m} \in \mathcal{U}$, let $X^{u} := (X_{t}^{u})_{0 \leq t \leq T}$ be the $\mathbb{R}^{n}$-valued process solution of the following standard SDE of mean-field or McKean-Vlasov type.

$$
X_{t}^{u} = x + \int_{0}^{t} \{A_{s}X_{s}^{u} + \sum_{k=1, m_{i}} C_{s}^{k}u_{s}^{k} + D_{s}\mathbb{E}[X_{s}^{u}] + \beta_{s}\} ds + \int_{0}^{t} \{\sigma_{s}X_{s}^{u} + \alpha_{s}\} dW_{s},
$$

where,

(i) $A = (A_{i})_{0 \leq i \leq T}, D = (D_{i})_{0 \leq i \leq T}, \beta = (\beta_{i})_{0 \leq i \leq T}, \alpha = (\alpha_{i})_{0 \leq i \leq T}$ and $C_{s}^{k} = (C_{s}^{k})_{0 \leq i \leq T}$ are bounded and adapted stochastic processes with values respectively in $\mathbb{R}^{n \times n}, \mathbb{R}^{n \times n}, \mathbb{R}^{n}$, $\mathbb{R}^{n}$ and $\mathbb{R}^{n \times m_{k}}, k = 1, \ldots, m$.

(ii) $\sigma = (\sigma_{i})_{0 \leq i \leq T}$ is an adapted process with values in $\mathbb{R}^{n \times n}$.

Next, to $u = (u^{i})_{1 \leq i \leq m} \in \mathcal{U}$, we associate $m$ payoffs $J_{i}(u), i = 1, \ldots, m$, of the form

$$
J_{i}(u) := \frac{1}{2} \{\mathbb{E}[X_{T}^{u}\mathbb{E}[X_{T}^{u}]] + \mathbb{E}[(X_{T}^{u})^{T}]R^{i}\mathbb{E}[X_{T}^{u}] \\
+ \mathbb{E}[(\int_{0}^{T} \{(X_{s}^{u})^{T}M_{s}^{i}X_{s}^{u} + u_{s}^{T}N_{s}^{i}u_{s} + \mathbb{E}[X_{s}^{u}]^{T}G_{s}\mathbb{E}[X_{s}^{u}]) ds\}]},
$$

where, for any $i = 1, \ldots, m$,

(a) $M^{i} = (M_{s}^{i})_{0 \leq i \leq T}$ are bounded adapted symmetric non-negative matrices with values in $\mathbb{R}^{n \times n},$

(b) $N^{i} = (N_{s}^{i})_{0 \leq i \leq T}$ are bounded adapted symmetric positive matrices with values $\mathbb{R}^{m_{i} \times m_{i}}$.

Moreover, their inverses $(N^{i})^{-1} := ((N_{s}^{i})^{-1})_{0 \leq i \leq T}$ are also bounded,
The meaning of the previous inequalities is that if the player makes the decision to deviate unilaterally from \( u_i^* \), then she is penalized since her cost is at least larger than the cost of using \( u_i^* \). If \( m = 2 \) and \( f_1 + f_2 = 0 \), the game is called of zero-sum type and a NEP \((u_1^*, u_2^*)\) satisfies

\[
J_1(u_1^*, v_2) \leq J_1(u_1^*, u_2^*) \leq J_1(v_1, u_2^*)
\]

for all \( v_1 \in U^1 \) and \( v_2 \in U^2 \).

For the sake of simplicity, in this section we only consider the case where the Brownian motion is one-dimensional. Extension to the multi-dimensional case is straightforward.

For \( i = 1, \ldots, m \), let us denote by \( H_i \) the Hamiltonian associated with the \( i \)-th player which is defined by

\[
H_i(t, \omega, x, u_1, \ldots, u_{m}, \zeta, p_i, q_i) := p_i^\top (A_i(\omega)x + \sum_{k=1}^m C_i^k(\omega)u_k + D_i\zeta + \beta_i)
\]

\[
+ \frac{1}{2}(x^\top M_i(\omega)x + u_i^\top N_i(\omega)u_i + \zeta^\top \Gamma_i^\top \zeta + (\sigma_i^\top x + \alpha_i)q_i),
\]

where \( u_i^i \in \mathbb{R}^{m_i}, z^i \in \mathbb{R}^n \) and \( \zeta \in \mathbb{R}^n \) (\( \zeta \) is the variable which stands for the expectation).

For \( i = 1, \ldots, m \), let \( \tilde{u}^i \) be the functions defined by

\[
\tilde{u}^i(t, \omega, p^i) := -(N_i^i)^{-1}(C_i^i)^\top p_i, \quad 0 \leq t \leq T.
\]

The measurable functions \( \tilde{u}^i \), \( i = 1, \ldots, m \), satisfy for all \( i = 1, \ldots, m \) and all \( u^i \in \mathbb{R}^{m_i} \)

\[
H_i(t, \omega, x, (\tilde{u}^i(t, \omega, p^i))_{1 \leq i \leq m}, \zeta, \tilde{p}_i, q_i)
\]

\[
\leq H_i(t, \omega, x, \bar{u}^i(t, \omega, p^i), \ldots, \bar{u}^{i-1}(t, \omega, p^{i-1}), u, \bar{u}^{i+1}(t, \omega, p^{i+1}), \ldots, \bar{u}^m(t, \omega, z^m), \zeta, \tilde{p}_i, q_i).
\]

The following proposition is a first step toward the proof of existence of a NEP for the game.

**Proposition 3.1.** Let the \( \mathcal{P} \)-measurable processes \((X, (p^1, q^1), \ldots, (p^m, q^m))\) be such that \( X, p^i, i = 1, \ldots, m \) belong to \( \mathcal{S}^{2,n} \) and \( q^i, i = 1, \ldots, m \) belong to \( \mathcal{H}^{2,n} \). Then, they solve the following MF-BFSDE, for all \( 0 \leq t \leq T \),

\[
\begin{cases}
X_t = x + \int_0^t \{ A_s X_s + \sum_{k=1}^m C_i^k(s, p_i^k) + D_i \mathbb{E}[X_s] + \beta_i \} ds
+ \int_0^t \{ \sigma_s X_s^\top + \alpha_s \} dW_s;

p_i^i = (Q^i X_T + R^i \mathbb{E}[X_T]) + \int_t^T \{ A_s^\top p_i^s + M_i^s X_s + \mathbb{E}[D_s^\top p_i^s]
+ \Gamma_i^s \mathbb{E}[X_s] + \sigma_s^\top q_i^s \} ds - \int_t^T q_i^s dW_s, \quad i = 1, \ldots, m.
\end{cases}
\]

If and only if the admissible collective control \( \bar{u} := ((\bar{u}^i)_{1 \leq i \leq m})_{0 \leq t \leq T} \) is a Nash equilibrium point for the mean-field nonzero-sum linear quadratic stochastic differential game.
In the BFSDE (3.5), $X$ is the optimal trajectory and $(p_i, q_i)_{1 \leq i \leq m}$ are the associated adjoint processes (ADIT, CKQ5, Ben82).

Proof. (i) The condition is sufficient. The fact that $\bar{u}$ is an open-loop strategy for the players is an immediate consequence of the boundedness of $C^\iota$, $(N_i^\iota)^{-1}$ and the fact that $(p_i^\iota)_{0 \leq i \leq T}$ belongs to $M^{2,\iota}$ for any $i = 1, \ldots, m$. Next, we will show the inequality (3.3) for $i = 1$. The other cases can be treated in the same manner. Consider $u_1 = (u_1(s))_{0 \leq s \leq T} \in \mathcal{U}^1$, $\bar{u} = (u_1, \bar{u}_2, \ldots, \bar{u}_m)$. We should show that $J_1(\bar{u}) \leq J_1(\bar{u})$.

Indeed,

$$J_1(\bar{u}) - J_1(\bar{u}) = J_1(u_1, \bar{u}_2, \ldots, \bar{u}_m) - J_1(\bar{u}) =$$

$$\frac{1}{2} \left\{ \mathbb{E}[(X^\iota_T - X^\iota_T)^\top Q^1 X^\iota_T] + \mathbb{E}[(X^\iota_T - X^\iota_T)^\top R^1 \mathbb{E}[X^\iota_T]] - \mathbb{E}[(X^\iota_T)^\top Q^1 X^\iota_T + \mathbb{E}[(X^\iota_T)^\top R^1 \mathbb{E}[X^\iota_T]]
+ \frac{1}{2} \mathbb{E} \int_0^T \left\{ (X^\iota_s - X^\iota_s)^\top M^1_s X^\iota_s + u_1(s)^\top N^\iota_1 u_1(s) - X^\iota_s^\top M^1_s X^\iota_s - \bar{u}_1(s)^\top N^\iota_1 \bar{u}_1(s)
+ \mathbb{E}[X^\iota_s^\top \Gamma^1_s \mathbb{E}[X^\iota_s] - \mathbb{E}[X_s]^\top \Gamma^1_s \mathbb{E}[X_s])ds \right\}. \tag{3.6}$$

But, for any symmetric non-negative matrix $\Sigma$ (i.e. $\nu^\top \Sigma \nu \geq 0$, $\forall \nu \in \mathbb{R}^k$), we have

$$\theta_1^\top \Sigma \theta_1 - \theta_2^\top \Sigma \theta_2 = (\theta_1 - \theta_2)^\top \Sigma (\theta_1 - \theta_2) + 2(\theta_1 - \theta_2)^\top \Sigma \theta_2 \geq 2(\theta_1 - \theta_2)^\top \Sigma \theta_2.$$

Therefore,

$$J_1(\bar{u}) - J_1(\bar{u}) \geq \mathbb{E}\left\{ (X^\iota_T - X^\iota_T)^\top Q^1 X^\iota_T \right\} + \mathbb{E}\left\{ \int_0^T \left\{ (X^\iota_s - X^\iota_s)^\top M^1_s X^\iota_s + (u_1(s) - \bar{u}_1(s))^\top N^\iota_1 \bar{u}_1(s)
+ \mathbb{E}[X^\iota_s^\top \Sigma \mathbb{E}[X^\iota_s] - \mathbb{E}[X_s]^\top \Sigma \mathbb{E}[X_s])ds \right\}. \tag{3.6}$$

since the matrices $Q^1, R^1, M^1_s, N^\iota_1$ and $\Gamma^1_s$ are symmetric non-negative.

We will show that the right-hand side of (3.6) is zero. Indeed, since $p^\iota_1 = Q^1 X^\iota_T + R^1 \mathbb{E}[X^\iota_T]$ and $(p^\iota_1, q^\iota)$ is a solution of a backward SDE of mean-field type, then by Itô's formula we have

$$(X^\iota_T - X^\iota_T)^\top p^\iota_1 = \int_0^T \left\{ -(X^\iota_s - X^\iota_s)^\top \{ A^\iota_s p^\iota_1 + M^1_s X^\iota_s + \mathbb{E}[D_s^\iota p^\iota_1] + \Gamma^1_s \mathbb{E}[X_s] + \sigma^\iota_s q^\iota_2 \} \right\}
+ (X^\iota_s - X^\iota_s)^\top A^\iota_s p^\iota_1 + (u^1(s) - \bar{u}_1(s))^\top (C^\iota_1)^\top p^\iota_1 + \mathbb{E}[X^\iota_s - X^\iota_s]^\top D^\iota_s p^\iota_1 ds \tag{3.7}$$

$$+ \int_0^T (X^\iota_s - X^\iota_s)^\top \sigma^\iota_s q^\iota_2 ds + \int_0^T (X^\iota_s - X^\iota_s)^\top \sigma^\iota_s q^\iota_2 dW^s + \int_0^t (X^\iota_s - X^\iota_s)^\top q^\iota_1 dW^s,$$

since, for any $0 \leq t \leq T$,

$$X^\iota_t - X^\iota_t = \int_0^t \left\{ A_t (X^\iota_s - X^\iota_s) + C^\iota_s (u^1(s) - \bar{u}_1(s)) + D_t \mathbb{E}[X^\iota_s - X^\iota_s] \right\} ds + \int_0^t \sigma_t (X^\iota_s - X^\iota_s) dW^s.$$

Simplifying terms in (3.7) and taking expectation, noting that the stochastic integrals are martingales, we obtain

$$\mathbb{E}[(X^\iota_T - X^\iota_T)^\top p^\iota_1] = \mathbb{E}\{ \left( X^\iota_T - X^\iota_T \right)^\top Q^1 X^\iota_T \} + \mathbb{E}\{ \left( X^\iota_T - X^\iota_T \right)^\top R^1 \mathbb{E}[X^\iota_T] \}$$

$$\mathbb{E}\left\{ \int_0^T \left\{ -(X^\iota_s - X^\iota_s)^\top \{ M^1_s X^\iota_s + \Gamma^1_s \mathbb{E}[X_s] \} + (u^1(s) - \bar{u}_1(s))^\top (C^\iota_1)^\top p^\iota_1 \right\} ds \right\}. \tag{3.8}$$

Finally, insert the right-hand side of (3.6) in (3.8) and take into account that

$$(C^\iota_1)^\top p^\iota_1 + N^\iota_1 \bar{u}_1(s) = 0$$

(see the definition of $\bar{u}_1$ given by (3.4)) to obtain that

$$J_1(u_1, \bar{u}_2, \ldots, \bar{u}_m) - J_1(\bar{u}) \geq 0.$$
(ii) The condition is necessary. Suppose the game has a Nash equilibrium point \( \bar{u} := (\bar{u}^i)_{1 \leq i \leq m} = (\bar{u}^i(t, \omega, p^i))_{0 \leq t \leq T, 1 \leq i \leq m} \) and denote by \( \hat{X} \) its associated optimal trajectory. Then obviously \( \hat{X} \) belongs to \( S^{2^m} \). Next, for \( i = 1, \ldots, m \), let \( (p^i, q^i) \) be the solution of the following backward SDE:

\[
\begin{align*}
\frac{d}{dt}H_i(t, \omega, \hat{X}_t, \bar{u}_t, \ldots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \ldots, u_m, E[\hat{X}_t], p_i, q_i)(u_i - \bar{u}_i) & \geq 0 \quad \mathbb{P}\text{-a.s.} \\
\frac{d}{dt}H_i(t, \omega, \hat{X}_t, \bar{u}_t, \ldots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \ldots, u_m, E[\hat{X}_t], p_i, q_i)(u_i - \bar{u}_i) & \geq 0 \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]

The solution of (3.9) exists, by the results in [BLP09]. Next, by the maximum principle (see [AD11], Theorem 3.1), for any \( u := (u^i)_{1 \leq i \leq m} \in U \), we have

\[
\frac{d}{dt}H_i(t, \omega, \hat{X}_t, \bar{u}_t, \ldots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \ldots, u_m, E[\hat{X}_t], p_i, q_i)(u_i - \bar{u}_i) \geq 0 \quad \mathbb{P}\text{-a.s.}
\]

for all \( t \in [0, T], i = 1, \ldots, m \). That is, for all \( i = 1, \ldots, m \),

\[
((C^i)^\top p^i + N^i u^i)(u_i - \bar{u}_i) \geq 0.
\]

Since \( u_i \in \mathbb{R}^{m_i} \) is arbitrary, we obtain

\[
\bar{u}^i(t, \omega, p^i) := -(N^i)^{-1}(C^i)^\top p^i, \quad 0 \leq t \leq T.
\]

Inserting that value of \( \bar{u}^i \) into (3.9), we have that \( (\hat{X}, (p^1, q^1), \ldots, (p^m, q^m)) \) satisfies MF-BFSDE (3.5).

Next, we are going to provide conditions on the data of the differential game in such a way that a NEP exists. So let us consider the following assumptions:

\[(H2)\]

(i) For any \( i = 1, \ldots, m \), the matrices \( C^i \) and \( N^i \) are time independent. We set \( K^i := C^i(N^i)^{-1}(C^i)^\top \).

(ii) There exist constants \( \eta_1 > 0, \eta_2 > 0 \) such that for any \( x \in \mathbb{R}^n \),

\[
x^\top(\sum_{i=1}^m K^i Q^i)x \geq \eta_1 |x|^2 \text{ and } x^\top(\sum_{i=1}^m K^i M^i)x \geq \eta_2 |x|^2,
\]

for any \( 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \).

(iii) For any \( i = 1, \ldots, m \), \( K^i A^i = A^i K^i, K^i D^i = D^i K^i \) and \( K^i \sigma^i = \sigma^i K^i \),

for any \( 0 \leq t \leq T, \mathbb{P}\text{-a.s.} \).

Note that in the case when \( n = 1 \), those assumptions are rather easy to check.

Let us now consider the following MF-BFSDE.

\[
\begin{align*}
X_t &= x + \int_0^t \{ A_s X_s - \bar{Y}_s + D_s E[X_s] + \beta_s \} ds + \int_0^t \{ \sigma_s X_s u + \alpha_s \} dW_s, \\
\bar{Y}_t &= (\sum_{i=1}^m K^i Q^i) X_T + (\sum_{i=1}^m K^i R^i) E[X_T] - \int_t^T \{ -A^i \bar{Y}_s - (\sum_{k=1}^m K^i M^i) X_s - E[D^i \bar{Y}_s] - \sigma^i \bar{Z}_s \} ds - \int_t^T \bar{Z}_s dW_s.
\end{align*}
\]

Note that if \( (X, (p^1, q^1), \ldots, (p^m, q^m)) \) is a solution of (3.5), then, under (H2), the process \( (X, Y = \sum_{i=1}^m K^i p^i, Z = \sum_{i=1}^m K^i q^i) \) is a solution of the BFSDE (3.12). This is exactly the origin of (3.12).
The functions \( f, g, h \) and \( \sigma \), introduced in Section 1, and related to the BFSDEs (3.12) are defined in (H2).

(a) \( f(t, x, y, z, v) = A_t x - y + D_t \int_{\mathbb{R}^r} x v(dx, dy) + \beta_t \);

(b) \( \sigma(t, x, y, z, v) = \sigma(t, x, z) \); 

(c) \( g(x, \mu) = (\sum_{k=1}^{m} K^i Q^j) x + (\sum_{k=1}^{m} K^i R^j) \int_{\mathbb{R}^r} x v(dx) \).

(d) For any \( t, x, y, z, v \), if \((\xi_1, \xi_2)\) is a random vector on \((\Omega, \mathcal{F}, \mathbb{P})\) whose law is \( v \), then

\[
\xi = -A_t y - (\sum_{k=1}^{m} K^i M^j) x - \mathbb{E}[D_t \xi_2] - \sigma^T z.
\]

To proceed, let us show that \( f \) is uniformly Lipschitz w.r.t. \( v \). Let \( \xi = (\xi_1, \xi_2) \) be a random vector whose distribution is \( v \in \mathcal{M}_2(\mathbb{R}^n) \). We have,

\[
f(t, x, y, z, v) = A_t x - y + D_t \mathbb{E}[\xi_1] + \beta_t.
\]

Next, let \( \nu' \in \mathcal{M}_2(\mathbb{R}^n) \) be given and let \( \xi' = (\xi'_1, \xi'_2) \) be a pair of random variables defined on the same probability as \((\xi_1, \xi_2)\) whose law is \( \nu' \). Therefore,

\[
|f(t, x, y, z, v) - f(t, x, y, z, \nu')| \leq \|D\| \mathbb{E}[|\xi_1 - \xi'_1|^2] \leq \|D\| \mathbb{E}[|\xi - \xi'|^2].
\]

Since \( \xi \) and \( \xi' \) are arbitrary, it holds that

\[
|f(t, x, y, z, v) - f(t, x, y, z, \nu')| \leq \|D\| \inf_{\xi, \xi'} \mathbb{E}[|\xi - \xi'|^2] = \|D\| d(v, \nu'). \tag{3.13}
\]

Finally, linearity implies that \( f \) satisfies (2.5). Similar estimates can be used for \( h \) and \( g \) to show that they satisfy (2.5) and (2.4), respectively.

The operator \( A \) defined in (2.7) reads

\[
A(t, u, u', v) = -|y - y'|^2 - (\sum_{k=1}^{m} K^i M^j) x - x'^2. \tag{3.14}
\]

Therefore, under (H2), \( A \) and \( g \) satisfy Assumption (H1') with \( k = min\{1, \eta_2\}, k' = \eta_1 \) (\( \eta_1 \) and \( \eta_2 \) are defined in (H2)).

Next, the Lipschitz constants of \( f, g, h \) and \( \sigma \) w.r.t. \( x, y, z \) are (see (3.1))

\[
C_f^x = \|A\|, \ C_f^y = 1, \ C_f^z = 0, \ C_f^\sigma = \|A\|, \ C_f^\nu = \|\sum_{k=1}^{m} K^i M^j\|, \ C_f^\nu = \|\sigma\|, \tag{3.15}
\]

\[
C_g^x = \|\sigma\|, \ C_g^y = C_g^z = 0 \text{ and } C_g^\nu = \|\sum_{k=1}^{m} K^i Q^j\|.
\]

On the other hand, as for \( f \) in (3.13),

\[
C_f^\nu = \|D\|, \ C_f^\nu = \|D\|, \ C_f^\nu = 0 \text{ and } C_f^\nu = \|\sum_{k=1}^{m} K^i R^j\|. \tag{3.17}
\]

We have the following

**Proposition 3.2.** Assume that (H2) holds and

\[
(i) \quad \|\sum_{k=1}^{m} K^i R^j\| < \min\{2(\sqrt{2} - 1)\eta_1, \sqrt{2}, \sqrt{2} \eta_2\}; \tag{3.18}
\]

\[
(ii) \quad \|D\| < \min\{2(\sqrt{2} - 1)\eta_1, \sqrt{2}, \sqrt{2} \eta_2\}.
\]
Then, there exist \( P \)-measurable processes \((X_i, (p^1, q^1), \ldots, (p^m, q^m))\) such that \( X \) and \( p^i, i = 1, \ldots, m \) belong to \( S^{2,n}_r \) and \( q^i, i = 1, \ldots, m \) belong to \( \mathcal{H}^{2,n} \) which solve the Backward-Forward stochastic differential equation of mean-field type (3.5).

**Proof.** Recall the BFSDE (3.12) is

\[
\begin{align*}
X_t &= x + \int_0^t \{ A_s X_s - \Delta Y_s + D_s \mathbb{E}[\Delta X_s] + \beta_s \} ds + \int_0^t \{ \sigma_s X_s + \alpha_s \} dW_s, \\
\bar{Y}_t &= (\sum_{k=1}^m K^i Q^i) X_T + (\sum_{k=1}^m K^i R^i) \mathbb{E}[X_T] - \int_t^T \{ -A^i_s \bar{Y}_s - (\sum_{k=1}^m K^i M^i_k) X_s - \mathbb{E} [D^i_s \bar{Y}_s] - \sigma^i_s \Delta Z_s \} ds - \int_t^T \Delta Z_s dW_s.
\end{align*}
\]

When (H2) holds,\n
\[
A(t, u, u', v) = -|y - y'|^2 - (\sum_{k=1}^m K^i M^i_k) x - x' |^2
\]

which means that \( k = \min \{ 1, \eta_2 \} \). For any \( x, x' \in \mathbb{R}^n, v \in \mathcal{M}_2(\mathbb{R}^n \times \mathbb{R}^n) \)

\[
g(x, v) - g(x', v) \cdot (x - x') = x^T (\sum_{i=1}^m K^i Q^i) x \geq \eta_1 |x|^2
\]

which means that \( k' = \eta_1 \). Now, under conditions (3.2) we can apply Theorem 2.3 to deduce the existence of \( P \)-measurable processes \((X, \bar{Y}, \bar{Z})\) which solve the MF-BFSDE (3.19).

We will now prove that when (ii) is satisfied, this solution is unique without using Theorem 2.3. This is due to the fact that in this specific case, uniqueness is obtained in an easy way without strong conditions on the Lipschitz constants of \( f, h, g \) and \( \sigma \) as it is the case in Theorem 2.3

Assume there is another solution \((X', \bar{Y}', \bar{Z}')\) of (3.19) and set

\[
\Delta X = X - X', \ \Delta Y = \bar{Y} - \bar{Y}' \ \text{and} \ \Delta Z = \bar{Z} - \bar{Z}'.
\]

We have, for every \( 0 \leq t \leq T, \)

\[
\begin{align*}
\Delta X_t &= \int_0^t \{ A_s \Delta X_s - \Delta Y_s + D_s \mathbb{E}[\Delta X_s] \} ds + \int_0^t \sigma_s \Delta X_s dW_s \\
\Delta Y_t &= (\sum_{k=1}^m K^i Q^i) \Delta X_T + (\sum_{k=1}^m K^i R^i) \mathbb{E}[\Delta X_T] - \int_t^T \{ -A^i_s \bar{Y}_s - (\sum_{k=1}^m K^i M^i_k) X_s - \mathbb{E} [D^i_s \bar{Y}_s] - \sigma^i_s \Delta Z_s \} ds - \int_t^T \Delta Z_s dW_s.
\end{align*}
\]

Next, applying Itô’s formula to \( \Delta X^T \Delta Y \) and taking expectation we obtain

\[
\mathbb{E}[\Delta X^T_T \Delta Y_T] = \mathbb{E} \{ \Delta X^T_T (\sum_{k=1}^m K^i Q^i) \Delta X_T + \mathbb{E}[\Delta X_T]^T (\sum_{k=1}^m K^i R^i) \mathbb{E}[\Delta X_T] \}
\]

\[
= \mathbb{E} \left[ \int_0^T \{ -|\Delta Y_s|^2 - \Delta X^T_s (\sum_{k=1}^m K^i M^i_k) \Delta X_s \} ds \right].
\]

(3.22)

This gives

\[
\mathbb{E}[\eta_1 \Delta X^T_T \Delta X_T] - C^\delta \mathbb{E}[\Delta X^T_T] \mathbb{E}[\Delta X_T] \leq \mathbb{E} \left[ \int_0^T \{ -|\Delta Y_s|^2 - \Delta X^T_s (\sum_{k=1}^m K^i M^i_k) \Delta X_s \} ds \right].
\]

Thus, since \( C^\delta = ||(\sum_{k=1}^m K^i R^i)|| < \eta_1 \), by continuity of the processes we obtain\n
\[
P\text{-a.s., \ } \forall t \leq T, \ \ X_t = X'_t \ \text{and} \ \ Y_t = Y'_t
\]

and finally \( Z'_t = Z_t, dt \otimes dP\text{-a.e.} \) Thus, the solution of (3.19) is unique.
Next, by the results of [BLP09], for $i = 1, \ldots, m$, there exists $(p^i, q^i) \in S^{2,m} \times \mathcal{H}^{2,m}$ solution of the following standard BSDE: $\mathbb{P}$-a.s., $\forall t \leq T$,

$$p^i_t = Q^i X_T + R^i \mathbb{E}[X_T] + \int_t^T \{ A^i_s p^i_s + M^i_s X_s + \mathbb{E}[D^i_s p^i_s] + \sigma^i_s q^i_s \} ds - \int_t^T q^i_s dW_s.$$ 

Therefore, the process $(X, Y = \sum_{i=1}^m K^i p^i, Z = \sum_{i=1}^m K^i q^i)$ is a solution of (3.19). As the solution of this latter is unique, it holds that $\bar{Y} = \sum_{i=1}^m K^i p^i$ and $\bar{Z} = \sum_{i=1}^m K^i q^i$. Replace now $\bar{Y}$ (resp. $\bar{Z}$) with $\sum_{i=1}^m K^i p^i$ (resp. $\sum_{i=1}^m K^i q^i$) in (3.12) to obtain that $(X, (p^i, q^i)_{1 \leq i \leq m})$ satisfy the FBSDE (3.5). The proof is complete. □

As an immediate consequence of Propositions 3.1 and 3.2, we give the main result of this section.

**Theorem 3.3.** Assume that (H2) holds and the following conditions are satisfied:

$$(i) \| (\sum_{k=1}^m K^k R^k) \| < \min\{ 2(\sqrt{2} - 1) \eta_1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \eta_2 \};$$

$$(ii) \| D \| < \min\{ 2(\sqrt{2} - 1) \eta_1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \eta_2 \}.$$ 

Then, the collective strategy $\bar{u} = \left( - (N^i)^{-1} (C^i) \top p_i(t) \right)_{1 \leq i \leq m}$, where $(X, (p^i, q^i)_{1 \leq i \leq m})$ is the solution of FBSDE (3.5), is a Nash equilibrium point for the mean-field LQ differential game.

**Example 3.4** (Nonexistence of a Nash Equilibrium Point of specific game problem without condition (3.23)).

We give an example to illustrate that when (3.23) is not satisfied, the game may not have an equilibrium point. The idea is inspired by the conclusion shown in Section 6 of [Eis82] and Example (4.b) in [Ham99].

Consider following game problem:

$$dX_t = \left\{ X_t - \mathbb{E}[X_t] + u(t) \begin{bmatrix} 1 \\ -2 \end{bmatrix} + v(t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} dt + dW_t, \ t \leq T; X_0 = (1, 2) \top. \tag{3.24}$$

Let $J_1$ and $J_2$ be the cost functionals defined by:

$$J_1(u, v) = \frac{1}{2} \mathbb{E} \left[ \int_0^T (u(t))^2 dt + (X_T^2) \right] \text{ and } J_2(u, v) = \frac{1}{2} \mathbb{E} \left[ \int_0^T (v(t))^2 dt + (X_T^2) \right]$$

where $u, v$ are $\mathbb{R}$-valued and $\mathcal{F}_t$ adapted process. Here, the associated $D = -1$ and then $\| D \| = 1$, which does not satisfy (3.23)-(iii). The Mean-Field FBSDE associated with the game is, for every $t \leq T$,

$$\begin{align*}
X_t &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \int_0^t \{ X_s - \mathbb{E}[X_s] + \begin{bmatrix} 1 \\ -2 \end{bmatrix} [-1 \ 2] p^1_s + \begin{bmatrix} 2 \\ -1 \end{bmatrix} [2 \ -1] p^2_s \} ds + W_t; \\
p^1_t &= \begin{bmatrix} X^1_t \\ 0 \end{bmatrix} + \int_t^T \{ p^1_s - \mathbb{E}[p^1_s] \} ds - \int_t^T q^1_s dW_s; \\
p^2_t &= \begin{bmatrix} 0 \\ X^2_T \end{bmatrix} + \int_t^T \{ p^2_s - \mathbb{E}[p^2_s] \} ds - \int_t^T q^2_s dW_s. \tag{3.25}
\end{align*}$$
Next, set $Y = (E(X_t))_{t \leq T}$, $\mathcal{P}^i = (E[p^i(t)])_{t \leq T}$, $i = 1,2$. Taking expectation in (3.25), we obtain, for every $t \leq T$,

$$
\begin{align*}
Y_t &= \left[ \frac{1}{2} \right] + \int_0^t \left\{ \left[ \begin{array}{c}
1 \\
-2
\end{array} \right] \left[ -1 \right] \left[ \begin{array}{c}
1 \\
2
\end{array} \right] \mathcal{P}^1_s + \left[ \begin{array}{c}
-2 \\
1
\end{array} \right] \left[ \begin{array}{c}
-1 \left[ \begin{array}{c}
2 \\
1
\end{array} \right] \mathcal{P}^2_s
\end{array} \right] ds,
\end{align*}
$$

which is a deterministic system. With the previous system is associated the following deterministic nonzero-sum game

$$
\begin{align*}
dY_t &= \left\{ \mathcal{P}(t) \left[ \begin{array}{c}
1 \\
-2
\end{array} \right] + \mathcal{P}(t) \left[ \begin{array}{c}
-2 \\
1
\end{array} \right] \right\} dt, \ t \leq T,
Y_0 &= \left[ \begin{array}{c}
1 \\
2
\end{array} \right],
\end{align*}
$$

and the cost functionals are given by

$$
\begin{align*}
J_1(\mathcal{P}, \mathcal{P}) &= \frac{1}{2} \left( \int_0^T (\mathcal{P}(t))^2 dt + (Y^1_t)^2 \right) \text{ and } J_2(\mathcal{P}, \mathcal{P}) = \frac{1}{2} \left( \int_0^T (\mathcal{P}(t))^2 dt + (Y^2_t)^2 \right).
\end{align*}
$$

The problem (3.27)-(3.28) is a deterministic nonzero-sum game. Noting that if the game problem (3.24) has a Nash equilibrium point, by Proposition 3.1, the MF-BFSDE (3.25) has a solution. Hence, obviously the FBODE (3.26) has a solution, which means that the deterministic game problem (3.24)-(3.28) has a Nash equilibrium point. However, when we choose $T = 1$, following the conclusion in [Eis82], the game (3.27)-(3.28) does not have a Nash equilibrium point and then the equation (3.26) does not have a solution. Therefore, the MF-BFSDE (3.25) does not have a solution for $T = 1$, from which we deduce that the game (3.24) does not have a Nash equilibrium point.

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