Bulk-Bulk Correspondence in Disordered Non-Hermitian Systems

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The consistency between eigenvalues calculated under open and periodic boundary conditions, named as bulk-bulk correspondence (BBC), can be destroyed in systems with non-Hermitian skin effect (NHSE). In spite of the great success of the generalized Brillouin zone (GBZ) theory in clean non-Hermitian systems, the applicability of GBZ theory is questionable when the translational symmetry is broken. Thus, it is of great value to rebuild the BBC for disorder samples, which extends the application of GBZ theory in non-Hermitian systems. Here, we propose a scheme reconstructing BBC, which can be regarded as the solution of an optimization problem. By solving this optimization problem analytically, we reconstruct the BBC and obtain the modified GBZ theory in several prototypical disordered non-Hermitian models. The modified GBZ theory gives a precise description of NHSE, which predicts the intriguing disorder-enhanced and disorder-irrelevant NHSEs.

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Introduction.— In recent years, non-Hermitian systems have been attracting great interests [1–51], among which the non-Hermitian skin effect (NHSE) is one of the most focused ones [48–74]. The presence of NHSE indicates that the eigenvectors exhibit different distributions for open (OBC) and periodic boundary conditions (PBC) [52], where the eigenvectors are localized at specific areas of the sample under OBC. Such unique features also lead to the failure [75–79] of the bulk-bulk correspondences (BBC) due to the inconsistence of the eigenvalues between OBC and PBC in non-Hermitian systems [54–56]. The savior is the establishment of the generalized Brillouin zone (GBZ) theory [52–54], by which the problems of BBC are dealt with appropriately. The correct open-boundary bulk spectra are available under PBC [54–56]. The validity of BBC also provides a reliable way for the characterization of NHSE, thus, a precise description of NHSE is now available [55]. Enlightened by the GBZ theory, a blossom of studies are reported with the combinations of topology and non-Hermiticity [24–51].

Very recently, the study on disorder effect in non-Hermitian systems has also drawn extensive attentions [80–93]. Importantly, the NHSE could still exist in disordered samples [12–17] and leads to the breakdown of BBC for translational-symmetry-broken samples. Although has achieved great successes in clean systems, the applicability of GBZ theory in disordered systems is still not fully understood because this theory is heavily based on the translational symmetry [52–54]. Therefore, the widely adopted GBZ theory may not be directly applicable [96] when disorder is presented [80–95]. The BBC mechanism as well as the quantitative description of the NHSE in disordered samples remain to be investigated.

In this Letter, we propose that the universal scheme reconstructing BBC in disordered non-Hermitian systems can be reduced to an optimization problem in linear algebra. By analytically solving this problem, we demonstrate two key issues about the BBC: (i) A Hamiltonian preserves its determinant after a similarity transformation under OBC; (ii) The transformed Hamiltonian possesses the BBC. As an illustration, we study the disorder effect on the Hatano-Nelson model. We show that the GBZ theory in the clean limit cannot be directly applied to the disordered non-Hermitian samples. However, the correct BBC can be recovered in the manner of the similarity transformation stated above. In this way, we obtain a modified GBZ theory which extends the application of the GBZ theory into disordered systems. Guided by this theory, we predict the emergence of disorder-enhanced and disorder-irrelevant NHSEs.

Model. — We focus on the disordered Hatano-Nelson
model [94] shown in Fig. 1(a) with Hamiltonian:

$$\mathcal{H} = \sum_{i=1}^{N} -t_{i+1}^+ c_{i}^\dagger c_{i+1} - t_{i-1}^- c_{i}^\dagger c_{i}^-.$$  

(1)

We set $t_{i}^+ = (t + \gamma + w_{i}^+)$ and $t_{i}^- = (t - \gamma + w_{i}^-)$. $w_{i}^\pm \in [-\frac{W}{2}, \frac{W}{2}]$ denotes the Anderson disorder [95] with $W$ the disorder strength. $t_{i}^\pm$ is the asymmetric hopping between the $i$-th and the $(i+1)$-th sites. Supposing that the 1st site overlaps with the $(N+1)$-th site, then one has the PBC (OBC) if $t_R = t_N^+ \equiv (t + \gamma + w_N^+)$ and $t_L = t_N^- \equiv (t - \gamma + w_N^-)$ are nonzero (zero).

When $\gamma \neq 0$ and $W = 0$, the NHSE is presented in such a model. The GBZ theory should be adopted [56] to achieve the BBC. For samples with a finite size, the GBZ theory introduces a transformation [52–54]

$$\mathcal{H}(t_i^+, t_i^-) \rightarrow \tilde{\mathcal{H}}(\tilde{t}_i^+, \tilde{t}_i^-),$$  

(2)

where $\tilde{t}_i^+ = \beta_i t_i^+$, and $\tilde{t}_i^- = \beta_i^{-1} t_i^-$. $\beta_i \equiv \beta_{GBZ} = \sqrt{\frac{\tau_i^+}{\tau_i^-}}$ modulates the Bloch wavefunction. However, the derivation of $\beta_{GBZ}$ relies on the translational symmetry [52, 54]. When the disorder effect is included, a well-defined unit cell is no longer presented [54], hence the $\beta_i$ cannot be obtained by the previous version of the GBZ theory.

**Proposed scheme reconstructing BBC.** — We find that BBC in disordered non-Hermitian systems can be reconstructed as follows, where the translational symmetry is dispensable. Concretely, the construction of BBC for samples with NHSE relies on the following two issues: (i) A similarity transformation $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$ should preserve the determinant under OBC as $\det[E - \mathcal{H}_{OBC}] \equiv \det[E - \tilde{\mathcal{H}}_{OBC}]$; (ii) Such a similarity transformation should eliminate the NHSE so the transformed Hamiltonian $\tilde{\mathcal{H}}$ possesses the BBC as $\det[E - \tilde{\mathcal{H}}_{OBC}] \approx \det[E - \tilde{\mathcal{H}}_{PBC}]$. These relations can be summarized in a schematic diagram shown in Fig. 1(b), which can be formulated as

$$\det[E - \mathcal{H}_{OBC}] \equiv \det[E - \tilde{\mathcal{H}}_{OBC}] \approx \det[E - \tilde{\mathcal{H}}_{PBC}].$$  

(3)

Equation (3) leads to the BBC as the reason shown below. Firstly, $E_{OBC}$, the eigenvalue of $\mathcal{H}_{OBC}$ satisfies $\det[E_{OBC} - \mathcal{H}_{OBC}] = 0$. If Eq. (3) holds, then the eigenvalues of $\tilde{\mathcal{H}}_{PBC}$ should be approximately the same as those in $\mathcal{H}_{OBC}$, i.e., $\det[E_{OBC} - \tilde{\mathcal{H}}_{PBC}] \approx 0$. Physically, Eq. (3) defines the criterion to achieve appropriate bulk states for non-Hermitian systems. In this way, the BBC and a detailed description of NHSE for $\mathcal{H}$ are available. Notably, the validity of this criterion is model-independent since it is insensitive to the form of $\mathcal{H}$. Figure 1(b) sketches the key finding of this Letter, where an appropriate transformation is desired to match Eq. (3).

**BBC as an optimization problem.** — Generally, the accurate eigenvalues under the OBC cannot be obtained under the PBC. Nevertheless, the BBC can still be captured by solving an optimization problem related to Eq. (3) as shown below. Taking the disordered Hatano-Nelson model with sample size $N = n$ as an example, the determinant under PBC reads [97, 98]:

$$\det[E - \tilde{\mathcal{H}}_{PBC}; n \times n] = \det[E - \mathcal{H}_{OBC}; n \times n] + f_{PBC},$$  

(4)

with

$$f_{PBC} = (-1)^T t_R t_L \det[E - \mathcal{H}_{OBC}; n - 2 \times n - 2] + (-1)^{n+1} \prod_{i=1}^{n-1} \left[ t_R t_i^+ + \prod_{i=1}^{n-1} t_L t_i^- \right].$$  

(5)

Here, $\tau \equiv \tau(n; 2, \ldots, n - 1; 1)$ is the permutation of $n$-th order [97, 98] [see Supplementary Material (SM), II]. By considering $E_{OBC}$ as one of the open boundary eigenvalues of $\mathcal{H}$ with dimension $(n - 2)$, one has

$$\det[E_{OBC} - \tilde{\mathcal{H}}_{PBC}; n - 2 \times n - 2] = 0.$$  

(6)

When $n \rightarrow \infty$, it is reasonable [97] to show $\det[E_{OBC} - \tilde{\mathcal{H}}_{PBC}; n \times n] \approx 0$, then Eq. (4) can be rewritten as

$$|\det[E_{OBC} - \tilde{\mathcal{H}}_{PBC}]| \approx |\prod_{i=1}^{n-1} t_R t_i^+ + \prod_{i=1}^{n-1} t_L t_i^-|. $$  

(7)

Here $|\cdots|$ stands for modulus. The right hand side of Eq. (7) is nonzero in general, hence the accurate value of $E_{OBC}$ cannot be obtained under PBC. Moreover, in the presence of NHSE, the asymmetric hopping could significantly increase [97] the modulus in Eq. (7), which leads to the breakdown of BBC.

We find $\det[E_{OBC} - \tilde{\mathcal{H}}_{PBC}]$ gets closer to zero by inserting the transformation $\tilde{t}_i^\pm = \beta_i^{\pm 1} t_i^\pm$ adopted in Eq. (2). Importantly, such a transformation preserves the determinant under OBC as $\det[E - \mathcal{H}_{OBC}(\tilde{t}_i^+, \tilde{t}_i^-)] \equiv \det[E - \mathcal{H}_{OBC}(t_i^+, t_i^-)] = f(t_i^+, t_i^-)$ since $\tilde{t}_i^\pm = t_i^\pm / \beta_i$ and $f(t_i^+, t_i^-)$ is a polynomial only depending on $t_i^+$ and $E$ [see SM. II [97]]. Thus, the first issue for obtaining Eq. (3) is satisfied, and Eq. (4) can be rewritten as:

$$\tilde{F} \equiv |\det[E_{OBC} - \tilde{\mathcal{H}}_{PBC}]|.$$  

(8)

where $\tilde{F} \equiv |\det[E_{OBC} - \tilde{\mathcal{H}}_{PBC}]|$. For a suitable $\beta_i$, $\tilde{F}$ reaches its minimum so the eigenvalues of $\mathcal{H}$ under PBC ($\tilde{E}_{PBC}$) fit better with $E_{OBC}$. This is equivalent to an optimization problem minimizing $|\tilde{E}_{PBC} - E_{OBC}|$.

Specifically, minimizing $\tilde{F}$ requires $\beta_i \equiv \sqrt{\tilde{t}_i^- / \tilde{t}_i^+}$ [see SM. III [97]]. When disorder is absent, it reduces $\beta_i \equiv \sqrt{(t - \gamma)/(t + \gamma)}$, which is the same as the GBZ theory. For the Hermitian case without NHSE, Eq. (7) is already minimized since $\tilde{t}_i^+ = t_i^+$, which ensures that $\det[E -$
\( \mathcal{H}_{OBC} \approx \text{det}[E - \mathcal{H}_{OBC}] \) holds. Similarly, \( \beta_i \equiv \sqrt{t_i^-/t_i^+} \) leads to \( \tilde{t}_i^+ = \tilde{t}_i^- = \sqrt{t_i^+/t_i^-} \) so that \( \mathcal{H} \) has no asymmetric hopping. Thus, \( \tilde{\mathcal{H}} \) should have no NHSE and the BBC is rebuilt as \( \text{det}[E - \tilde{\mathcal{H}}_{OBC}] \approx \text{det}[E - \tilde{\mathcal{H}}_{PBC}] \).

In short, for obtaining Eq. (3), one needs to manage a similarity transformation \( t_i^+ = \beta_i^\pm t_i^\pm \) which minimizes \( \tilde{F} \). The \( \beta_i \) is reduced to \( \sqrt{(t - \gamma)/\gamma} \) in the clean limit by the GBZ theory \cite{101}. Due to the universality of Eq. (3) and its related results, one is able to study the BBC in disordered non-Hermitian systems.

Failure of \( \beta_{GBZ} \) in disordered samples.— To demonstrate the advantage of the proposed scheme reconstructing BBC, we clarify that the NHSE is still presented for disordered samples, which is supported by the numerical results shown in Figs. 2(a) and (b). Here the eigenvalues of \( \mathcal{H}_{PBC} \) form a closed loop, which implies the presence of the NHSE \cite{9}. Furthermore, \( \mathcal{H}_{PBC} \) and \( \mathcal{H}_{OBC} \) show different eigenvalues, indicating that the BBC is destroyed.

Although no longer in the clean limit, for the sake of comparison, we can still try to utilize the GBZ theory with \( \beta_{GBZ} = \sqrt{(t - \gamma)/\gamma} \) to estimate the eigenvalues in disordered samples. Based on Figs. 2(a)-(c), one notices that \( \beta_{GBZ} = \sqrt{(t - \gamma)/\gamma} \) gives the wrong (correct) open boundary spectrum under strong (weak) disorder. Since \( \beta_{GBZ} \) preserves \( \text{det}[E - \mathcal{H}_{OBC}] = \text{det}[E - \mathcal{H}_{OBC}] \), the failure of GBZ theory is attributed to the presence of NHSE for \( \beta_i = \beta_{GBZ} \) [see the closed loops in Figs. 2(b) and (c)], which gives rise to \( \text{det}[E - \mathcal{H}_{PBC}] \neq \text{det}[E - \mathcal{H}_{OBC}] \), i.e. violating Eq. (3).

Even though the GBZ theory \cite{52, 54} gives correct results under weak disorder [see Fig. 2(a)], strong disorder could significantly alter the NHSE. Therefore, for disordered samples, \( \beta_{GBZ} \) can not minimize \( \tilde{F} \) [see SM. VI \cite{97}] and the GBZ theory is no longer directly applicable.

**Modified GBZ theory.—** Now we adopt the proposed scheme to rebuild the BBC for disordered samples. Although the transformation \( \beta_i = \sqrt{t_i^-/t_i^+} \) minimizes \( \tilde{F} \) \cite{97}, \( \beta_i \) differs from sample to sample when disorder is present. Now, we demonstrate that for a fixed disorder strength \( W \), the \( \beta_i \) in the modified GBZ theory can be replaced by a universal transformation parameter \( \langle \beta \rangle \). Since \( \langle \beta \rangle \) always preserves \( \text{det}[E - \mathcal{H}_{OBC}] = \text{det}[E - \mathcal{H}_{OBC}] \), one only needs to ensure that \( \langle \beta \rangle \) preserves \( \text{det}[E - \mathcal{H}_{PBC}] \approx \text{det}[E - \mathcal{H}_{OBC}] \). Based on Eq. (8), we find that the minimum \cite{97} of \( |\langle \beta \rangle|^{N_{tR}} \prod_i t_i^+ + |\langle \beta \rangle|^{-N_{tL}} \prod_i t_i^- \) ensures that \( \text{det}[E - \mathcal{H}_{PBC}] \approx \text{det}[E - \mathcal{H}_{OBC}] \). Thus, one has

\begin{align}
(\langle \beta \rangle)^{2N} \equiv \prod_{i=1}^{N-1} t_{iL}^{-1} t_{iR} = \prod_{i=1}^{N-1} (t - \gamma + w_i^-) \prod_{i=1}^{N-1} (t + \gamma + w_i^+) \approx 1.
\end{align}

After some algebra including a self-average \cite{97}, \( \langle \beta \rangle \) for the modified GBZ theory with Anderson disorder reads:

\begin{align}
\langle \beta \rangle = \left[ \frac{(t - \gamma + W/2)^{t-g-W} + (t + \gamma - W/2)^{t+g-W}}{(t - \gamma - W/2)^{t-g-W} + (t + \gamma + W/2)^{t+g+W}} \right]^{1/2}. \quad (10)
\end{align}

\( \langle \beta \rangle \) is a universal transformation parameter for different realizations when sample size \( N \) is large enough.

The analytical results show that \( \langle \beta \rangle \) could be complex as \( \langle \beta \rangle = |\langle \beta \rangle| e^{i\theta} \) [see Fig. 2(c)]. However, the modulus is more important for experimental observations. In Fig. 2(d), we plot the evolution of \( |\langle \beta \rangle| \) versus \( \gamma \) and \( W \), and some specific features can be identified from both Eq. (10) and the figure. In case of weak disorder, one has

\( \lim_{W \to \infty} |\langle \beta \rangle| \sim 1 \) implies the absence of NHSE with \( \mathcal{H} \sim \mathcal{H} \). Since \( \beta \) should eliminate the NHSE of \( \mathcal{H} \) to achieve the correct results, \( |\langle \beta \rangle| \) can be considered as the strength of NHSE \cite{52, 54}. Consequently, Fig. 2(d)
The proposed theory deepens our understanding of the Hermitian Hamiltonian with disorder, we predict the presence of disorder-enhanced and disorder-irrelevant NHSEs. As a demonstration, we consider the following two disorder mechanisms.

**Conclusion and Discussion.** — In summary, we studied the BBC in the non-Hermitian system $\mathcal{H}$ with the combination of the disorder effect and the NHSE. We find that the GBZ theory is not directly applicable for disordered systems, and the BBC for samples without translational symmetry requires: (i) A transformation $\mathcal{H} \to \tilde{\mathcal{H}}$ that preserves the determinant under OBC as $\det[\mathcal{E} - \mathcal{H}_{\text{OBC}}] = \det[\mathcal{E} - \tilde{\mathcal{H}}_{\text{OBC}}]$; (ii) The transformed Hamiltonian possesses the BBC as $\det[\mathcal{E} - \tilde{\mathcal{H}}_{\text{OBC}}] \approx \det[\mathcal{E} - \tilde{\mathcal{H}}_{\text{PBC}}]$. Based on these two requirements, we constructed the BBC in disordered non-Hermitian systems and obtained a modified GBZ theory. The modified transformation parameter is the order parameter characterizing the NHSE, and the disorder effect on NHSE is uncovered. The validity of our theory can be confirmed by detecting the proposed disorder-enhanced and disorder-irrelevant NHSEs.

The requirements in Eq. (3) to achieve BBC should be suitable for all the non-Hermitian systems. In SM [97], similar analyses for the Su-Schrieffer-Heeger model [52] can be regarded as the phase diagram of NHSE, where $|\langle \beta \rangle|$ is analytically drawn from Eq. (10). The NHSE is presented when $|\langle \beta \rangle|$ deviates from one [102] that the more $|\langle \beta \rangle|$ deviates from one, the stronger the NHSE is. The direction of NHSE is determined by the sign of $|\langle \beta \rangle| - 1$. More importantly, by adopting $\langle \beta \rangle$, as shown in Figs. 2(a)-(c), the results for the modified GBZ theory under PBC (red dots) overlap perfectly with the eigenvalues for $\mathcal{H}_{\text{OBC}}$ (blue circles) [104], verifying our theory. Such verification is independent of the parameters [97]. Furthermore, the advantage of the modified GBZ theory can be exhibited by comparing $|\beta_{\text{GBZ}}|$ and $|\langle \beta \rangle|$. Taking $\gamma = 1.4$ as an example [see Fig. 2(e)], $\beta_{\text{GBZ}}$ in the GBZ overlaps with $|\langle \beta \rangle|$ in the modified GBZ under weak disorder, where the GBZ theory still holds [see Fig. 2(a)]. For strong disorder, the modified GBZ captures the disorder-dependent NHSE, while $|\langle \beta \rangle|$ fails. The disorder dependence of NHSE can be identified in Figs. 2(a)-(c), where the eigenvalues of $\mathcal{H}_{\text{PBC}}$ tend to lose their closed loop characteristics. It agrees with Fig. 2(e), where $|\langle \beta \rangle| \to 1$ for strong disorder. Experimentally, the variation of NHSE can be detected by measuring the average density $\langle \rho \rangle = \sum_{i\in\mathbb{U}} |\langle \psi_i |^2|$ where $\mathcal{H}_{\text{OBC}} \psi_i = E_i \psi_i$ [66] and $\langle \cdot \cdot \cdot \rangle$ is the ensemble average. As plotted in Fig. 2(f), the NHSE becomes weaker with the increase of the disorder strength. These numerical results are consistent with the analytical results of $|\langle \beta \rangle|$ shown in Fig. 2(e).

** Disorder-enhanced and disorder-irrelevant NHSEs. ** — The proposed theory deepens our understanding of BBC in disordered non-Hermitian systems. As a demonstration, we predict the presence of disorder-enhanced and disorder-irrelevant NHSEs. Specifically, we start from a Hermitian Hamiltonian with $t^+_i = t^-_i = t$ and $t = 1$ in Eq. (1). Such a model has no NHSE with $\beta_{\text{GBZ}} \equiv 1$. Two disorder schemes are considered as shown in Fig. 3(a), where $t^+_i = t + w_i$ and $t^-_i = t$ in the left panel, while $t^+_i = te^{iw_i}$ and $t^-_i = t$ in the right panel. The $w_i$ is uniformly distributed as $w_i \in [-\frac{W}{2}, \frac{W}{2}]$.

At first glance, the above two models look the same, as the locally asymmetric hopping term is exerted. However, these two disorder mechanisms exhibit distinct behaviors based on the criterion of BBC. Figure 3(b) shows the variation of NHSE. For $W < 2$, $|\langle \beta \rangle|$ decreases by increasing $W$, which shows the existence of disorder-enhanced NHSE. In moderate disorder region, the direction of NHSE reverses with $|\langle \beta \rangle| - 1$ changing its sign. For even larger $W$, the disorder-enhanced NHSE is more obvious. In Fig. 3(d), the plot of $\langle \rho \rangle$ is consistent with the variation of $|\langle \beta \rangle|$, where the disorder-enhanced NHSE is identified. The proposed disorder-enhanced NHSE could be realized and detected in artificial systems [49, 93]. Finally, effects of other types of disorder can also be evaluated with the help of the modified GBZ theory. More details are given in SM. V [97].
are presented. It follows the modified GBZ theory here is the same as that in the Hatano-Nelson model. Previously, the study on disorder effect in topological systems inevitably suffers from the finite-size effect due to the introducing of the OBC [87, 96]. The reconstruction of the BBC paves a possible way to overcome such a problem. Besides, several numerical evidences have indicated that the GBZ theory holds for disordered higher-dimensional systems [80, 86]. We leave the applications of our theory for samples with long-range hopping and higher dimensions for further investigations [105]. Our work unveils the mechanism of BBC in disordered non-Hermitian systems and extends the universality of the GBZ theory.

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See Supplementary Materials at [URL will be inserted by publisher] for more details which includes: (i) the deviations of the determinants and the proofs; (ii) the modified GBZ theory for different samples; (iii) disorder-dependent of eigenvalues; (iv) additional numerical results with Ref. [17, 52, 54, 94, 98, 99].

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Since H and ˜H have the same open boundary spectrum, they should have the same topological features and phase diagrams. This explains why GBZ theory can capture the bulk-boundary correspondences in non-Hermitian systems.

For GBZ theory, the first issue is naturally ensured. The second issue is available by ensuring the extended wavefunction distributions of ˜H_{OBC} [52]. It suggests ˜H has no NHESE, consisting with our theory.

The modified transformation parameter (β) looks similar to the disorder-induced renormalization of topological mass in topological Anderson insulator [103].

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The PBC still holds for the modified GBZ theory since the GBZ theory with β_i ≡ β → β_{GBZ} gives the wrong eigenvalues.

Zhi-Qiang Zhang, and Hua Jiang, in preparation.
Supplementary Materials for “Bulk-Bulk Correspondence in Disordered Non-Hermitian Systems”

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I. Introduction for the Supplementary Materials

In this Supplementary Materials, we present more details of the bulk-bulk correspondence (BBC) proposed in the main text. Specifically, the BBC is defined as: the consistency between the eigenvalues calculated under open and periodic boundary conditions. In Section II, the derivation and the formulas of the determinants under open (OBC) and periodic boundary conditions (PBC) are given. In Section III, we show additional details of the universal criterion of BBC in disordered non-Hermitian systems. In Section IV, the details of the modified generalized Brillouin zone (GBZ) theory for different disordered models are exhibited. Finally, Sections VI and VII give additional numerical results.

References

I. INTRODUCTION FOR THE SUPPLEMENTARY MATERIALS

In this Supplementary Materials, we present more details of the bulk-bulk correspondence (BBC) proposed in the main text. Specifically, the BBC is defined as: the consistency between the eigenvalues calculated under open and periodic boundary conditions. In Section II, the derivation and the formulas of the determinants under open (OBC) and periodic boundary conditions (PBC) are given. In Section III, we show additional details of the universal criterion of BBC in disordered non-Hermitian systems. In Section IV, the details of the modified generalized Brillouin zone (GBZ) theory for different disordered models are exhibited. Finally, Sections VI and VII give additional numerical results.
II. ANALYTICAL RESULTS FOR SAMPLES WITH SIZE $n$

It is known that the eigenvalues calculated under open and periodic boundary conditions are different when samples are in small sizes. To achieve reliable $BBC$, large sample size with $n \to \infty$ should be considered. In this section, we present the analytical results for samples in which $n \to \infty$ while the translational symmetry is absent. The main results are independent of $n$ when $n \to \infty$. For simplicity, we set $n = 4k$ as an even number throughout the deviations only.

A. Determinant under open boundary condition: $\det[E - \mathcal{H}_{OBC}; n \times n]$

We focus on the disordered Hatano-Nelson model [S1] with Hamiltonian:

$$\mathcal{H} = \sum_{i=1}^{n} -t_i^+ c_i^+ c_{i+1} - t_i^- c_{i+1}^+ c_i.$$  \hfill (S1)

The determinant $\det[E - \mathcal{H}_{OBC}; n \times n]$ under OBC has the form of:

$$\det[E - \mathcal{H}_{OBC}; n \times n] = \begin{vmatrix}
    E & t_1^+ & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
    t_1^- & E & t_2^+ & 0 & 0 & 0 & 0 & \cdots & 0 \\
    0 & t_2^- & E & t_3^+ & 0 & 0 & 0 & \cdots & 0 \\
    & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & t_n^- & E & t_{n-1}^+ & 0 & \cdots & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & t_{n-2}^- & E & t_{n-1}^+ \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{vmatrix}. \hfill (S2)$$

Generally, the recurrence relation

$$\det[E - \mathcal{H}_{OBC}; n \times n] = \det[E - \mathcal{H}_{OBC}; n-1 \times n-1]E - t_1^+ \Delta_x,$$ \hfill (S3)

can be adopted to evaluate the determinant, where the $\Delta_x$ reads

$$\Delta_x = \begin{vmatrix}
    t_1^- & t_2^+ & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
    0 & E & t_3^+ & 0 & 0 & 0 & 0 & \cdots & 0 \\
    0 & t_3^- & E & t_4^+ & 0 & 0 & 0 & \cdots & 0 \\
    & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & t_n^- & E & t_{n-1}^+ & 0 & \cdots & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & t_{n-2}^- & E & t_{n-1}^+ \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{vmatrix}. \hfill (S4)$$

Hence, $\Delta_x = t_1^- \det[E - \mathcal{H}_{OBC}; n-2 \times n-2]$ and then one has

$$\det[E - \mathcal{H}_{OBC}; n \times n] = \det[E - \mathcal{H}_{OBC}; n-1 \times n-1]E - t_1^+ t_1^- \det[E - \mathcal{H}_{OBC}; n-2 \times n-2].$$ \hfill (S5)

Similarly we have:

$$\det[E - \mathcal{H}_{OBC}; n-1 \times n-1] = \det[E - \mathcal{H}_{OBC}; n-2 \times n-2]E - t_1^+ t_2^- \det[E - \mathcal{H}_{OBC}; n-3 \times n-3],$$ \hfill (S6)

$$\det[E - \mathcal{H}_{OBC}; n-2 \times n-2] = \det[E - \mathcal{H}_{OBC}; n-3 \times n-3]E - t_3^+ t_4^- \det[E - \mathcal{H}_{OBC}; n-4 \times n-4].$$

In such a way, the determinant can be simply marked as:

$$\det[E - \mathcal{H}_{OBC}; n \times n] = \sum_{m=0}^{n/2} (-1)^m \left( \sum_{\text{pairs}} \prod_{i \in [1,n-1]} t_i^+ t_i^- \right) E^{n-2m} = f(t_i^+ t_i^-, E). \hfill (S7)$$
m pair means that one has to select m pair of $t^+_i t^-_i$ with $i \equiv [i_1, i_2, \cdots, i_x, \cdots, i_y, \cdots, i_m] \in [1, n - 1]$. For simplicity, we mark it as $m \equiv i$ hereafter. The summation $\sum_i$ suggests that one has to include all the possible permutation of $i \equiv [i_1, i_2, \cdots, i_x, \cdots, i_y, \cdots, i_m]$ for fixed $m$, where the restriction $|i_x - i_y| \geq 2$ for $\forall i_x, i_y \in [1, n - 1]$ should be considered at the same time. In specific, we give the detailed formula as follow. Eq. (S7) can be rewritten as:

$$
\det[\mathcal{E} - \mathcal{H}_{OBC}; n \times n] = E^n - \sum_{i_1=1}^{n-1} T_{(n-i_1)} E^{n-2} + \sum_{i_1=1}^{n-3} T_{(n-i_1)} \sum_{i_2=i_1+2}^{n-1} T_{(n-i_2)} E^{n-4} - \sum_{i_1=1}^{n-5} \sum_{i_2=i_1+2}^{n-3} \sum_{i_3=i_2+2}^{n-1} \cdots \sum_{i_n=n-2}^{1} T_{(n-i_n)} E^0
$$

(S8)

with the mark $T_i = t^+_i t^-_i$. Actually, the detailed expression of $\det[\mathcal{E} - \mathcal{H}_{OBC}; n \times n]$ is unimportant. Instead, the important thing is that such a determinant only contains $T_i = t^+_i t^-_i$ and $E^2$ terms for OBC, while the isolated $t^+_i$ or $t^-_i$ term is absent. Such a fact is very important to achieve a specific transformation preserving $\det[\mathcal{E} - \mathcal{H}_{OBC}; n \times n]$. One only needs to ensure that $t^+_i t^-_i$ remains unchanged for $\forall i \in [1, n - 1]$. For the sake of convenience, in the following derivation, the corresponding determinant is expressed by the simplified formula as Eq. (S7).

B. Determinant under periodic boundary condition: $\det[\mathcal{E} - \mathcal{H}_{PBC}; n \times n]$

The matrix form of Hamiltonian in the main text under PBC is given:

$$
E - \mathcal{H}_{PBC}; n \times n = \begin{pmatrix}
E & t^+_1 & 0 & 0 & 0 & 0 & 0 & \cdots & t_L \\
t^-_1 & E & t^+_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & t^-_2 & E & t^+_3 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & t^-_3 & E & t^+_4 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & t^-_4 & E & t^+_5 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & t^-_5 & E & t^+_6 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & t^-_6 & E & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
t_R & 0 & 0 & 0 & 0 & 0 & t^-_{n-2} & E & t^+_{n-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t^-_{n-1} & E
\end{pmatrix}
$$

(S9)

with $t^+_n = t_R \equiv (t + \gamma + w^+_n)$ and $t^-_n = t_L \equiv (t - \gamma + w^-_n)$. The determinant can be rewritten as:

$$
\det[\mathcal{E} - \mathcal{H}_{PBC}; n \times n] = \det[\mathcal{E} - \mathcal{H}_{OBC}; n \times n] + f(t_L, t_R),
$$

(S10)

with [S4]

$$
f(t_L, t_R; E) = (-1)^{\tau[n; 2, \cdots, n-1]} t_L t_R \det[\mathcal{E} - \mathcal{H}_{OBC}; n-2 \times n-2] +
(-1)^{\tau[n; 1, \cdots, n-1]} t_L t^+_1 t^-_2 \cdots t^-_m \cdots t^-_{n-1} +
(-1)^{\tau[2, \cdots, n]} t_R t^+_1 t^+_2 \cdots t^+_m \cdots t^+_{n-1}.
$$

(S11)

The permutation of order $n$ is in the form of:

$$
(-1)^{\tau[n; 2, \cdots, n-1]} = (-1)^{|(n-1)+(n-2)|} = -1;
(-1)^{\tau[n; 1, \cdots, n-1]} = (-1)^{n-1} = -1;
(-1)^{\tau[2, \cdots, n]} = (-1)^{n-1} = -1;
$$

(S12)
with \( n = 4k \). Thus,

\[
f(t_L, t_R; E) = -t_LT_R \det[\mathcal{E} - \mathcal{H}_{\text{OBC}}|_{n-2 \times n-2}] + \\
- t_LT_1^{-} t_2^{-} \cdots t_m^{-} t_{n-1}^{-} \\
- t_RT_1^{+} t_2^{+} \cdots t_m^{+} t_{n-1}^{+}.
\]  

(S13)

Based on Eq. (S7), one has

\[
\det[\mathcal{E} - \mathcal{H}_{\text{OBC}}|_{n-2 \times n-2}] = \sum_{m=0}^{n/2-1} (-1)^m \prod_{i \in [2, n-2]} t_i^+ t_i^- |E^{(n-2)-2m}.
\]  

(S14)

Then, Eq. (S11) can be written as

\[
f(t_L, t_R; E) = -t_LT_R \left\{ \sum_{m=0}^{n/2-1} (-1)^m \prod_{i \in [2, n-2]} t_i^+ t_i^- |E^{(n-2)-2m} \right\} \\
- t_LT_1^{-} t_2^{-} \cdots t_m^{-} t_{n-1}^{-} \\
- t_RT_1^{+} t_2^{+} \cdots t_m^{+} t_{n-1}^{+}.
\]  

(S15)

Based on these results, Eq. (S10) is given as follows:

\[
\det[\mathcal{E} - \mathcal{H}_{\text{BC}}|_{n \times n}] = \det[\mathcal{E} - \mathcal{H}_{\text{OBC}}|_{n \times n}] + f(t_L, t_R) \\
= \sum_{m=0}^{n/2} (-1)^m \prod_{i \in [1, n-1]} t_i^+ t_i^- |E^{n-2m} \right\} + t_LT_R \left\{ \sum_{m=0}^{n/2-1} (-1)^m \prod_{i \in [2, n-2]} t_i^+ t_i^- |E^{(n-2)-2m} \right\} \\
- [t_LT_1^{-} t_2^{-} \cdots t_m^{-} t_{n-1}^{-}] - [t_RT_1^{+} t_2^{+} \cdots t_m^{+} t_{n-1}^{+}].
\]  

(S16)

III. HOW TO REBUILD THE BBC

A. Derivations of the requirements in the main text

In this section, we demonstrate how to rebuild the BBC in disordered non-Hermitian systems with nearest-neighbor hopping. For a disordered \( \mathcal{H} \), the determinant under PBC reads:

\[
\det[\mathcal{E} - \mathcal{H}_{\text{BC}}|_{n \times n}] = \det[\mathcal{E} - \mathcal{H}_{\text{OBC}}|_{n \times n}] - t_LT_R \det[\mathcal{E} - \mathcal{H}_{\text{OBC}}|_{n-2 \times n-2}] \\
- [t_LT_1^{-} t_2^{-} \cdots t_m^{-} t_{n-1}^{-}] - [t_RT_1^{+} t_2^{+} \cdots t_m^{+} t_{n-1}^{+}].
\]  

(S17)

For a specific eigenvalue \( E_{\text{OBC}} \) belonging to the open boundary spectrum, one should have

\[
\det[E_{\text{OBC}} - \mathcal{H}_{\text{OBC}}|_{n-2 \times n-2}] = 0.
\]  

(S18)

If \( n \to \infty \), it is reasonable to demonstrate that

\[
\det[E_{\text{OBC}} - \mathcal{H}_{\text{OBC}}|_{n \times n}] \approx 0.
\]  

(S19)

The proof is given in the next subsection. Thus, for systems with non-Hermitian skin effect (NHSE), one has:

\[
\det[E_{\text{OBC}} - \mathcal{H}_{\text{BC}}|_{n \times n}] \approx \det[E_{\text{OBC}} - \mathcal{H}_{\text{OBC}}|_{n \times n}] - t_LT_R \det[E_{\text{OBC}} - \mathcal{H}_{\text{OBC}}|_{n-2 \times n-2}] \\
- [t_LT_1^{-} t_2^{-} \cdots t_m^{-} t_{n-1}^{-}] - [t_RT_1^{+} t_2^{+} \cdots t_m^{+} t_{n-1}^{+}] \\
\approx 0 - [t_LT_1^{-} t_2^{-} \cdots t_m^{-} t_{n-1}^{-}] - [t_RT_1^{+} t_2^{+} \cdots t_m^{+} t_{n-1}^{+}].
\]  

(S20)

Suppose such an eigenvalue also belongs to the spectrum under the PBC, one should have

\[
\det[E_{\text{OBC}} - \mathcal{H}_{\text{BC}}|_{n \times n}] \approx - [t_LT_1^{-} t_2^{-} \cdots t_m^{-} t_{n-1}^{-}] - [t_RT_1^{+} t_2^{+} \cdots t_m^{+} t_{n-1}^{+}] \approx 0.
\]  

(S21)
However, it is not the case for $\mathcal{H}$. Nevertheless, by introducing a transformation $H \to \tilde{H}$, $E_{\text{OBC}}$ could be approximately the eigenvalues of $\mathcal{H}$ under $P\text{B}C$. Now one can see what is the requirement to obtain $\det[E_{\text{OBC}} - \tilde{\mathcal{H}}_{\text{PBC}}; n \times n] \to 0$. Generally, there are two key issues:

(i) One has to ensure that the transformation preserves the following equation: $\det[E_{\text{OBC}} - \tilde{\mathcal{H}}_{\text{PBC}}; n-2 \times n-2] = 0$ and $\det[E_{\text{OBC}} - \tilde{\mathcal{H}}_{\text{PBC}}; n \times n] = \det[E_{\text{OBC}} - \tilde{\mathcal{H}}_{\text{PBC}}; n \times n]$.

(ii) One has to minimize $|t_L t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-| + |t_R t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-|$ to ensure the accuracy of $\det[E_{\text{OBC}} - \tilde{\mathcal{H}}_{\text{PBC}}; n \times n] \to 0$.

Thus, $t_i^+ t_i^-$ should be fixed to preserve the first requirement, for instance $t_i^+ \to (\beta)t_i^+$ and $t_i^- \to (\beta)^{-1}t_i^-$. As for the second key issue, one has to minimize $|f_x|$ with:

$$f_x = (\beta)^n [t_L t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-] + (\beta)^{-n}[t_L t_i^- t_2^- \cdots t_m^- \cdot t_{n-1}^-] = xT^+ + x^{-1}T^-,$$

and $x = (\beta)^n$. With the variation of $x$, the minimum is available when $\frac{df_x}{dx} = 0$. The validity of our results is confirmed numerically, see Sec. VI(A) for details. Then, one has $T^+ - T^- x^{-2} = 0$ with

$$x^2 = (\beta)^2 = \frac{t_L t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-}{t_R t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-}.$$

If $t^+ t^-$ is invariant when $\beta$ is in the form shown in Eq. (S23), then $\det[E_{\text{OBC}} - \tilde{\mathcal{H}}_{\text{PBC}}; n \times n] \to 0$ holds to the maximum extent. Thus, all of $\tilde{\mathcal{H}}_{\text{PBC}}$, $\tilde{\mathcal{H}}_{\text{OBC}}$ and $\mathcal{H}_{\text{OBC}}$ tend to have the same eigenvalue $E_{\text{OBC}}$.

In short, one has

$$f_x = (\beta)^n [t_L t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-] + (\beta)^{-n}[t_L t_i^- t_2^- \cdots t_m^- \cdot t_{n-1}^-] = 2\sqrt{|t_L t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-||t_R t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-|}.$$

Such a result is also consistent with the transformation $\beta_i \equiv \sqrt{t_i^- / t_i^+}$ with $\tilde{t}_i^+ = \beta_i t_i^+$ and $\tilde{t}_i^- = \beta_i^{-1}t_i^-$, where

$$f_x = \prod_i \beta_i[t_L t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-] + \prod_i \beta_i^{-1}[t_L t_i^- t_2^- \cdots t_m^- \cdot t_{n-1}^-] = 2\sqrt{|t_L t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-||t_R t_i^+ t_2^+ \cdots t_m^+ \cdot t_{n-1}^-|}.$$

B. Equation (S19) is an appropriate approximation

In this subsection, we prove that if $\det[E_{\text{OBC}} - \mathcal{H}_{\text{OBC}}; n-2 \times n-2] = 0$ and $n \to \infty$, then

$$\det[E_{\text{OBC}} - \mathcal{H}_{\text{OBC}}; n \times n] \approx 0.$$

is an appropriate approximation.

Since $\mathcal{H}_{\text{OBC}}(t_i^+, t_i^-)$ has the same eigenvalues with $\tilde{\mathcal{H}}_{\text{OBC}}(\beta_i t_i^+, \beta_i^{-1}t_i^-)$, we pay our attention to $\tilde{\mathcal{H}}_{\text{OBC}}(\beta_i t_i^+, \beta_i^{-1}t_i^-)$, in which $\beta_i t_i^+ = \beta_i^{-1}t_i^- = \sqrt{t_i^+ t_i^-}$ is a constant. $\tilde{\mathcal{H}}_{\text{OBC}}(\beta_i t_i^+, \beta_i^{-1}t_i^-)$ satisfies the eigenvalue equation:

$$\tilde{\mathcal{H}}_{\text{OBC}, n-2 \times n-2} V = E_{\text{OBC}}.$$

Noticing that $\tilde{\mathcal{H}}_{\text{OBC}, n-2 \times n-2}$ has no asymmetric hopping, therefore the NHSE is absent based on our theory. The above equation can be rewritten as $V^{-1} \tilde{\mathcal{H}}_{\text{OBC}, n-2 \times n-2} V = E_{\text{OBC}}$. Consider the following transformation:

$$U^{-1} \tilde{\mathcal{H}}_{\text{OBC}, n \times n} U = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & V^{-1}
\end{pmatrix} \begin{pmatrix}
(\beta_1 t_1^+ & 0 & 0 \\
0 & \beta_2^{-1} t_1^- & 0 \\
0 & 0 & \beta_2^{-1} V^{-1} T^-
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & V
\end{pmatrix}$$

(S28)
where $U$ satisfying $U^{-1}U = I$ does not change the eigenvalues. Furthermore, one has $T^{\pm} \equiv [t_2^\pm, 0, \cdots, 0]_{n-2}$, which leads to the expression of $(\beta_2 T^+ V)_k = \beta_2 t_2^+ V_{1,k}$.

Generally, the extended states are equally distributed in the lattice, which means that the eigenvector $V$ satisfies $V_{1,k} \propto \frac{1}{n-2}$. For a localized state, instead, one has $V_{1,k} \sim 0$ since the eigenvalues do not tend to concentrate at the first site for a sample with size $(n - 2)$. Based on Gershgorin circle theorem [S5] and $\beta t_i^+ \equiv \text{constant}$, the eigenvalues of $\tilde{H}_{OBC,n-2\times n-2} \in \text{E}_{OBC}$ are related to the eigenvalues of $\tilde{H}_{OBC,n\times n} \in \text{E}_{OBC,n}$ as

$$\lim_{n \to \infty} |E_{k,n} - E_{k,n-2}| < \lim_{n \to \infty} |\beta t_2^+ V_{1,k}| \propto \lim_{n \to \infty} \left|\frac{\beta t_2^+}{n - 2}\right| \approx 0.$$  (S29)

It suggests that $E_{k,n} \approx E_{k,n-2} \in \text{E}_{OBC}$ is also approximately the eigenvalue of $\tilde{H}_{OBC,n\times n}$.

Thus, if $\det[E_{OBC} - \tilde{H}_{OBC}; n-2 \times n-2] = 0$, one has $\det[E_{OBC} - \tilde{H}_{OBC}; n \times n] \approx 0$. Noticing that $\det[E - \tilde{H}_{OBC}] = \det[E - \tilde{H}_{OBC}]$, one can prove that if $\det[E_{OBC} - \tilde{H}_{OBC}; n-2 \times n-2] = 0$, then $\det[E_{OBC} - \tilde{H}_{OBC}; n \times n] \approx 0$. Although $\tilde{H}_{OBC,n\times n}$ has two additional eigenvalues compared with $H_{OBC,n-2 \times n-2}$, such an approximation can still be regarded as valid.

IV. UNIFIED TRANSFORMATION FOR DIRTY SAMPLES

We present the renormalized $\beta$ for the modified GBZ theory in this section. Based on previous results, one has:

$$\langle \beta \rangle^{2n} = \frac{t_L t_1^- t_2^- \cdots t_m^- t_{n-1}^-}{t_R t_1^+ t_2^+ \cdots t_m^+ t_{n-1}^+}. \tag{S30}$$

A. Clean samples: revisit GBZ theory

For clean samples one has:

$$t_m^- \equiv t_1^- = t - \gamma; \quad t_m^+ \equiv t_1^+ = t + \gamma. \tag{S31}$$

Thus, one has $\beta^{2n} = \left(\frac{t_1^+}{t_1^-}\right)^n$ and $\beta = \sqrt{\frac{t_1^+}{t_1^-}} = \sqrt{\frac{t + \gamma}{t - \gamma}}$, which is the same as those in the GBZ theory.

B. Dirty samples: modified GBZ theory and renormalized transformation parameter

For a dirty sample considered, one has

$$\langle \beta \rangle^{2n} = \frac{t_L t_1^- t_2^- \cdots t_m^- t_{n-1}^-}{t_R t_1^+ t_2^+ \cdots t_m^+ t_{n-1}^+}. \tag{S32}$$

Therefore, the renormalized $\langle \beta \rangle$ for a specific disordered sample considered in the main text $[t_m^+ = t + \gamma + w_m^+ \text{ and } t_m^- = t - \gamma + w_m^-]$ can be marked as:

$$\langle \beta \rangle^{2n} = \frac{\prod_{m=1}^{n}(t - \gamma + w_m^-)}{\prod_{m=1}^{n}(t + \gamma + w_m^+)} \tag{S33}$$

For $w_m^+ \in [-\frac{W}{2}, \frac{W}{2}]$, the renormalized $\beta$ can be obtained as follows [S6]:

$$\ln(\langle \beta \rangle) = \lim_{n \to \infty} \frac{1}{2n} \left[ \sum_{m=1}^{n} \ln(t - \gamma + w_m^-) - \sum_{m=1}^{n} \ln(t + \gamma + w_m^+) \right]$$

$$\quad = \int_{-\frac{W}{2}}^{\frac{W}{2}} \ln(t - \gamma + x) \frac{dx}{2W} - \int_{-\frac{W}{2}}^{\frac{W}{2}} \ln(t + \gamma + x) \frac{dx}{2W}. \tag{S34}$$
For simplicity, if \( t + \gamma + x > 0, \forall x \in [-\frac{W}{2}, \frac{W}{2}] \), then one has [considering \( \int \ln(a + bx)dx = (x + \frac{a}{b})[\ln(a + bx) - 1] \)]

\[
\ln(\langle \beta \rangle) = \frac{1}{2W} \left\{ \int_{-W/2}^{W/2} \ln[t - \gamma + x] \, dx - \int_{-W/2}^{W/2} \ln[t + \gamma + x] \, dx \right\} = \frac{1}{2W} \left\{ \int_{-W/2}^{W/2} \ln[t - \gamma + x] \, dx - \int_{-W/2}^{W/2} \ln[t + \gamma + x] \, dx \right\}
\]

\[
= \frac{1}{2W} \left\{ (t - \gamma + x)[\ln(t - \gamma + x) - 1] \right\} - \int_{-W/2}^{W/2} \left\{ \int_{-W/2}^{W/2} \ln[t + \gamma + x] \, dx \right\}
\]

\[
= \frac{1}{2W} \left\{ (t - \gamma + x)[\ln(t + \gamma + x) - 1] \right\} - \int_{-W/2}^{W/2} \left\{ \int_{-W/2}^{W/2} \ln[t + \gamma + x] \, dx \right\}
\]

\[
= \frac{1}{2W} \left\{ (t - \gamma + x)[\ln(t + \gamma + x) - 1] - (t - \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2} - 1] \right\}
\]

\[
- (t + \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) - 1] + (t + \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) - 1] \right\}
\]

\[
= \frac{1}{2W} \left\{ (t - \gamma + x)[\ln(t + \gamma + x) - 1] - (t - \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) - 1] \right\}
\]

\[
- (t + \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) + (t + \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) - 1] \right\}
\]

Finally, we capture the second key point: the analytical results for the renormalized \( \beta \) versus disorder strength \( W \) and \( \gamma \)

\[
\langle \beta \rangle = \exp \left\{ \frac{1}{2W} \left\{ \ln(t - \gamma + \frac{W}{2})[\ln(t - \gamma + \frac{W}{2}) - 1] \right\} - \ln(t + \gamma + \frac{W}{2})[\ln(t + \gamma + \frac{W}{2}) - 1] \right\}
\]

\[
- \ln(t + \gamma + \frac{W}{2})[\ln(t + \gamma + \frac{W}{2}) - 1] + \ln(t + \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) - 1] \right\}
\]

\[
\exp \left\{ \frac{1}{2W} \left\{ (t - \gamma + x)[\ln(t + \gamma + x) - 1] - (t - \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) - 1] \right\}
\]

\[
- (t + \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) + (t + \gamma - \frac{W}{2})[\ln(t + \gamma - \frac{W}{2}) - 1] \right\}
\]

\[
\right\}
\]

The average of \( \beta \) is similar to the self-average in disordered mesoscopic samples. It is not an average on different samples. Thus, \( \langle \beta \rangle \) is actually a well-defined and unified transformation parameter when \( n \) is large enough.

Importantly, we still have to prove that Eq. (S36) is well-defined over the entire parameter region of \( (W, \gamma) \), especially when

\[
t + \gamma + x \leq 0. \quad (S37)
\]

The key point is to solve the integral \( \int_{-\epsilon}^{a} \ln(x)dx \) when \( a > 0 \) and \( b < 0 \). Since \( \ln[x]|_{x=0} \) is not well-defined and has no lower boundary, the improper integral instead the usual integral has to be adopted. Nevertheless, these two processes give the same result, which verifies the validity of Eq. (S36) over the entire parameter region. The proof is given as follows as the integral \( \int_{-\epsilon}^{a} \ln(x)dx \) can be rewritten as:

\[
\int_{-\epsilon}^{a} \ln(x)dx = \lim_{\epsilon \to 0} \left\{ \int_{-\epsilon}^{0} \ln(x)dx + \int_{0}^{b} \ln(x)dx \right\} = \lim_{\epsilon \to 0} \left\{ [x \ln(x) - x]|_{-\epsilon}^{0} + \int_{0}^{b} \ln(x)dx \right\}
\]

\[
= \lim_{\epsilon \to 0} (x \ln(x) - x)|_{\epsilon}^{0} + \int_{0}^{b} \left\{ \ln(-\epsilon) + \epsilon \right\}
\]

\[
= \lim_{\epsilon \to 0} (x \ln(x) - x)|_{\epsilon}^{0} + \int_{0}^{b} \left\{ \ln(-\epsilon) + \epsilon \right\}
\]

\[
= \lim_{\epsilon \to 0} (x \ln(x) - x)|_{\epsilon}^{0} + \int_{0}^{b} \left\{ \ln(-\epsilon) + \epsilon \right\}
\]

The relation \( \lim_{x \to 0} [\pm x \ln(\pm x)] = 0 \) has been inserted in the final step. Thus, Eq. (S36) is well-defined over the entire parameter space, except for the points where the denominator equals to zero. Fortunately, these points are strongly limited. Thus, it is appropriate to claim that Eq. (S36) is well-defined over the entire parameter region.

C. Modified GBZ theory for models in Fig. 3 in the main text

For model \( H_{L} = \sum_{i} -t_{i}^{+} c_{i+1}^{\dagger} - (t + w_{i}) c_{i+1}^{\dagger} c_{i}, \) one has \( t_{i}^{+} = t \) and \( t_{i}^{-} = t + w_{i} \) with \( w_{i} \in [-\frac{W}{2}, \frac{W}{2}] \). We set \( t = 1 \). Based on

\[
\langle \beta \rangle^{2n}_{L} = \frac{\{t_{L}^{+} t_{L}^{-} \cdots t_{m}^{+} t_{m}^{-} \cdots t_{n-1}^{+} \}}{\{t_{R}^{+} t_{R}^{-} \cdots t_{m-1}^{+} t_{m-1}^{-} \cdots t_{n-1}^{+} \}}, \quad (S39)
\]
one has

$$\langle \beta \rangle_{2n}^{L} = \left[ \frac{t_{L} l_{1}^{-} t_{2}^{-} \cdots t_{m}^{-} \cdots t_{n}^{-}}{t_{R} l_{1}^{+} t_{2}^{+} \cdots t_{m}^{+} \cdots t_{n}^{+}} \right] = \frac{\prod_{m=1}^{n} (t + w_{m})}{t^{n}} = \prod_{m=1}^{n} (1 + w_{m}).$$ \hspace{1cm} (S40)

Thus,

$$\ln(\langle \beta \rangle_{L}) = \lim_{n \to \infty} \frac{1}{2n} \left[ \sum_{m=1}^{n} \ln(1 + w_{m}) \right] = \int_{-W/2}^{W/2} \ln(1 + x) \frac{dx}{2W}$$

\[= \frac{1}{2W} (1 + x) \left[ \ln(1 + x) - 1 \right] W^{1/2} - W^{1/2} \]

\[= \ln \left[ \left( 1 + \frac{W}{2} \right)^{1/2} e^{-W} \right] \left( 1 - \frac{W}{2} \right)^{1/2} \right] \frac{1}{2W}. \hspace{1cm} (S41)\]

Finally, one has \(\langle \beta \rangle_{L} = \left[ \frac{(1 + \frac{W}{2})^{1/2} e^{-W}}{(1 - \frac{W}{2})^{1/2}} \right] \frac{1}{2W}.\]

**FIG. S1:** (Color online). (a) and (b) \(\text{Re}[E] \) versus \(\text{Im}[E] \) for different disorder strengths. The Hamiltonian reads \(\mathcal{H}_{R} = \sum_{i} -t_{i}^{+} c_{i+1}^{\dagger} - t_{i}^{\dagger} c_{i}^{\dagger} c_{i+1} \) with \(w_{i} \in [-\frac{W}{2}, \frac{W}{2}] \). \(W \) is the disorder strength. \(H_{\text{OBC}} \) and \(H_{\text{PBC}} \) correspond to the initial Hamiltonian under open and periodic boundary conditions, respectively. (c) \(\langle \rho \rangle \) for different disorder strengths.

Similarly, for model \(\mathcal{H}_{R} = \sum_{i} -t_{i}^{+} c_{i+1}^{\dagger} - t_{i}^{\dagger} w_{i} c_{i}^{\dagger} c_{i+1} \), one has \(t_{i}^{+} = t \) and \(t_{i}^{-} = t e^{w_{i}} \) with \(w_{i} \in [-\frac{W}{2}, \frac{W}{2}] \). Thus,

$$\langle \beta \rangle_{2n}^{R} = \left[ \frac{t_{L} l_{1}^{-} t_{2}^{-} \cdots t_{m}^{-} \cdots t_{n}^{-}}{t_{R} l_{1}^{+} t_{2}^{+} \cdots t_{m}^{+} \cdots t_{n}^{+}} \right] = \frac{\prod_{m=1}^{n} t e^{w_{m}}}{t^{n}} = \prod_{m=1}^{n} e^{w_{m}}. \hspace{1cm} (S42)$$

One has:

$$\ln(\langle \beta \rangle_{R}) = \lim_{n \to \infty} \frac{1}{2n} \left[ \sum_{m=1}^{n} \ln(e^{w_{m}}) \right] = \frac{1}{2n} \left[ \sum_{m=1}^{n} w_{m} \right]$$

\[= \frac{1}{2W} \int_{-W/2}^{W/2} x \, dx \hspace{1cm} (S43)\]

Thus, \(\langle \beta \rangle_{R} = 1 \) for different disorder strengths and the NHSE is absent. As shown in Figs. S1(a) and (b), eigenvalues calculated under \(PBC \) show the absence of the close loop characteristics. Furthermore, the eigenvalues calculated under \(OBC \) and \(PBC \) spectrums fit perfectly. The density distributions \(\langle \rho \rangle = \sum_{i \in \text{all}} |\psi_{i}|^{2} \) in Fig. S1(c) also verifies the fact that the NHSE feature is absent. The deviation comes from the finite size effect. These results are consistent with our analysis.
D. Modified GBZ theory for Su-Schrieffer-Heeger model

In this subsection, the Su-Schrieffer-Heeger model \([S2]\) is considered

\[
\mathcal{H}_{SSH} = \sum_i -\{t_0c_i^\dagger c_{i+1} + h.c.\} - [(t + \gamma + w_{i+1}^+)c_{i+1}^\dagger c_{i+2} + (t - \gamma + w_{i+1}^-)c_{i+2}^\dagger c_{i+1}] .
\]  

(S44)

with \(i \in \{1,3,5, \cdots, 2m - 1 \cdots \}\). We mark the hopping term as follows

\[
t_{i+1}^+ = (t + \gamma + w_{i+1}^+) ,
\]

\[
t_{i+1}^- = (t - \gamma + w_{i+1}^-) ,
\]

\[
t_i^\pm = t_0 .
\]

(S45)

We introduce the transformation \(t_i^\pm \equiv t_0 \rightarrow t_0, t_{i+1}^+ \rightarrow \beta t_{i+1}^+\) and \(t_{i+1}^- \rightarrow \beta^{-1} t_{i+1}^-\) for the clean samples. One has:

\[
\beta^n = \left[ t_{L} t_{L}^\dagger t_{2} t_{2}^\dagger \cdots t_{i} t_{i}^\dagger \cdots t_{n} t_{n}^\dagger \right] .
\]

(S46)

Thus, \(\beta = \sqrt{\frac{t_0}{\overline{t_0}} = \sqrt{\frac{t_{L} t_{L}^\dagger}{t_{2} t_{2}^\dagger}} \cdot \frac{t_{L} t_{L}^\dagger}{t_{2} t_{2}^\dagger}}\). Such a result is consistent with the previous studies \([S2, S3]\).

When disorder is introduced, the transformation follows a unified parameter \(\langle \beta \rangle\) that: \(t_{i+1}^+ \rightarrow \langle \beta \rangle t_{i+1}^+\) and \(t_{i+1}^- \rightarrow \langle \beta \rangle^{-1} t_{i+1}^-\) with \(i \in \{1,3,5, \cdots \}\). Then the elimination of the asymmetric hopping of \(\hat{H}\) requires:

\[
\langle \beta \rangle^{n/2}[t_{L} t_{L}^\dagger t_{2} t_{2}^\dagger \cdots t_{i} t_{i}^\dagger \cdots t_{n} t_{n}^\dagger] = (\langle \beta \rangle)^{-n/2}[t_{L} t_{L}^\dagger t_{2} t_{2}^\dagger \cdots t_{i} t_{i}^\dagger \cdots t_{n} t_{n}^\dagger] ,
\]

(S47)

which is equivalent to

\[
(\langle \beta \rangle)^n = \frac{t_{L} t_{L}^\dagger t_{2} t_{2}^\dagger \cdots t_{i} t_{i}^\dagger \cdots t_{n} t_{n}^\dagger}{t_{L} t_{L}^\dagger t_{2} t_{2}^\dagger \cdots t_{i} t_{i}^\dagger \cdots t_{n} t_{n}^\dagger} = \prod_{m=1}^{n/2}(t - \gamma + w_m^-) = \prod_{m=1}^{n/2}(t + \gamma + w_m^+) .
\]

(S48)

Thus, the \(\langle \beta \rangle\) here is the same as that in the Hatano-Nelson model.

E. Eigenfunctions correlated with \(\langle \beta \rangle\)

\(\mathcal{H}\) and \(\hat{H}\) are the original and the transformed Hamiltonian, respectively. Here, \(\hat{H} = S^{-1}\mathcal{H} S\) and \(S = \text{diag}[\langle \beta \rangle, \langle \beta \rangle^2, \langle \beta \rangle^3, \cdots, \langle \beta \rangle^n]\). They satisfy the eigen-equations as:

\[
\mathcal{H}\psi_i = E_i\psi_i; \hat{H}\tilde{\psi}_i = E_i\tilde{\psi}_i .
\]

(S49)

Thus, one has \(\tilde{\psi}_i = S\psi_i\). The detectable quantity is the absolute value of the eigenvectors \(|\tilde{\psi}_i|\) and \(|\psi_i|\) in which the phase of \(\langle \beta \rangle\) is irrelevant.

V. DISORDER-DEPENDENCE OF THE BULK-BULK CORRESPONDENCE AND THE EIGENVALUES

Based on the universal criterion of \(BBC\), one is able to qualitatively illuminate the universality of the disorder-dependent of \(\beta_{GBZ}\) and the eigenvalues for different types of disorder. For the nearest hopping model, (i) If the disorder preserves \(t_i^+ t_i^-\), then the open boundary spectrum \(E_{OBC}\) is invariant since \(\det[E - \mathcal{H}_{OBC}] = f(t_i^+ t_i^- , E)\); (ii) If the disorder preserves \(\prod_i t_i^- / t_i^+\), then the GBZ theory and \(\langle \beta \rangle = \beta_{GBZ}\) hold since \(\langle \beta \rangle = \beta_{GBZ} \propto \prod_i \frac{t_i^-}{t_i^+}\); (iii) If the disorder preserves \(\prod_i t_i^+ t_i^-\) and \(\prod_i t_i^\pm\), then \(\det[E - \mathcal{H}_{PBC}]\) as well as \(E_{PBC}\) are invariant. If disorder does not preserve all these three classes of quantities, then nothing remains invariant. For illustration, several types of disorder are considered, and the corresponding results are summarized in TABLE. I. More details are exhibited in Sec. VI (C).
TABLE I: The influence of disorder on Hamiltonian \( \mathcal{H}_1 = -\sum_i [(t + \gamma)e^{w_i^+}c_i^{\dagger}c_{i+1} + (t - \gamma)e^{w_i^-}c_i^{\dagger}c_i + \text{H.c.}] \). One has \( t_i^+ = (t + \gamma)e^{w_i^+} \) and \( t_i^- = (t - \gamma)e^{w_i^-} \) for \( \mathcal{H}_1 \). \( w_i^\pm \) is random numbers. \( [w_i^\pm] \) stands for the sets of all possible values. \( \langle \beta \rangle \) is the transformation coefficient of the modified GBZ theory. \( E_{\text{PBC/OBC}} \) is the eigenvalues calculated under different boundary conditions. \( \text{Fixed/changed} \) suggests the value is unchanged/changed compared with disordered cases. \( \text{Depends} \) means that it depends.

| disorder type | \( w_i^\pm \in \) | \( \langle \beta \rangle \) | \( \mathcal{E}_{\text{PBC}} \) | \( \mathcal{E}_{\text{OBC}} \) | example |
|--------------|-----------------|-----------------|-----------------|-----------------|--------|
| \( w_i^{\gamma} = -w_i^{\gamma} \) | any | depends | depends | fixed | Fig. S4(c)/(d) |
| \( w_i^{\gamma} = -w_i^{\gamma} \) | \( [-\frac{W}{2}, \frac{W}{2}] \) | fixed | fixed | fixed | Fig. S4(c) |
| \( [w_i^\gamma] = [w_i^\gamma] \) | any | fixed | changed | changed | Fig. S4(d) |

VI. NUMERICAL RESULTS FOR DISORDERED HATANO-NELSON MODEL

In this section, we give some numerical results to support our analytical discussions.

A. Plots of Eq. (S22)

The plots of \( \bar{F} \equiv \langle f(\beta) \rangle = |\beta^n [t_R t_1^+ \cdots t_m^+ \cdots t_{n-1}^+] + \beta^{-n} [t_L t_1^- \cdots t_m^- \cdots t_{n-1}^-]| \) in Eq. (S22) for different disorder strengths \( W \) are given in Fig. S2. One can see that the minimum of \( \bar{F} \) are consistent with our analytical results \( \langle \beta \rangle \). For weak disorder strength, one has \( |\beta_{\text{GBZ}}| \approx |\langle \beta \rangle| \), which is located at the minimum of \( \bar{F} \). However, when disorder is strong enough, \( |\beta_{\text{GBZ}}| \neq |\langle \beta \rangle| \).

![Fig. S2](Color online). The plots of \( \bar{F} \equiv \langle f(\beta) \rangle = |\beta^n [t_R t_1^+ \cdots t_m^+ \cdots t_{n-1}^+] + \beta^{-n} [t_L t_1^- \cdots t_m^- \cdots t_{n-1}^-]| \) in Eq. (S22) for different disorder strengths \( W \) are given in Fig. S2. One can see that the minimum of \( \bar{F} \) are consistent with our analytical results \( \langle \beta \rangle \). For weak disorder strength, one has \( |\beta_{\text{GBZ}}| \approx |\langle \beta \rangle| \), which is located at the minimum of \( \bar{F} \). However, when disorder is strong enough, \( |\beta_{\text{GBZ}}| \neq |\langle \beta \rangle| \).

B. Stability of different transformations

Taking the Hatano-Nelson model as an example, the Hamiltonian reads [S1]:

\[
\mathcal{H} = \sum_{i=1}^{N} [-t_i^+ c_i^{\dagger} c_{i+1} - t_i^- c_i^{\dagger} c_i].
\]

with \( t_i^+ = (t + \gamma + w_i^+) \) and \( t_i^- = (t - \gamma + w_i^-) \). The Andoher disorder is introduced as \( w_i^\pm \in [-\frac{W}{2}, \frac{W}{2}] \). The stability of the transformation \( t_i^+ \rightarrow \beta_{\text{GBZ}} t_i^+ \) and \( t_i^- \rightarrow \beta_{\text{GBZ}}^{-1} t_i^- \), in which \( \beta_{\text{GBZ}} = \sqrt{\frac{t_i^+}{t_i^-}} \). The modified GBZ is the same with the previous GBZ except that \( \beta_{\text{GBZ}} \) is replaced by \( \langle \beta \rangle \). \( H_{\text{OBC}} \) and \( H_{\text{PBC}} \) correspond to results calculated by using the original Hamiltonian.
Other parameters are \( t = 1 \) and \( N = 100 \). (a-5), (b-5) and (c-5) show the \( \beta \) versus disorder strength. GBZ corresponds to the cases with \( \beta = \beta_{GBZ} = \sqrt{1 - \frac{\gamma}{\gamma + 2t^2}} \), and \( t^+ \to \beta t^+_i, t^- \to \beta^{-1} t^-_i \). Modified GBZ is the same with GBZ except that \( \beta_{GBZ} \) is replaced by \( \langle \beta \rangle \). \( H_{OBC} \) and \( H_{PBC} \) correspond to the initial Hamiltonian under OBC and PBC, respectively. \( \tilde{H}_{PBC} \) corresponds to the Hamiltonian with \( \tilde{t}^+_i = \tilde{t}^-_i = \sqrt{t^+_i t^-_i} \) under the PBC.

under OBC and PBC, respectively. \( \tilde{H}_{PBC} \) corresponds to the effective Hamiltonian with \( \tilde{t}^+_i = \tilde{t}^-_i = \sqrt{t^+_i t^-_i} \) under the PBC.

As one can see, results for the modified GBZ and \( \tilde{H}_{PBC} \) overlap with \( H \) under the OBC for different parameters. In addition, one should notice that the results for GBZ with \( \beta_{GBZ} = \sqrt{1 - \frac{\gamma}{\gamma + 2t^2}} \) is the same with the modified GBZ as well as the \( H_{OBC} \) for weak disorder since \( |\beta_{GBZ}| \approx |\langle \beta \rangle| \) as shown in Figs. S3(a-5),(b-5) and (c-5). Nevertheless, GBZ is totally wrong when disorder is strong enough.

Taking \( \gamma = 0.5 \) as an example, GBZ give the correct result as shown in Fig. S3(a-1), since \( \langle \beta \rangle \approx \beta_{GBZ} \approx 0.577 \) [see Fig. S3(a-5)]. By increasing \( W \), the results of GBZ deviate from the open boundary results, as shown in Fig. S3 (a-2) since \( \langle \beta \rangle \) deviates from \( \beta_{GBZ} \approx 0.577 \). By further increasing \( W \), the result of GBZ overlap with the open boundary results again [see Fig. S3 (a-3)] since \( |\langle \beta \rangle| \) approaches to 0.577 again. For stronger disorder strength, the GBZ transformation \( \beta_{GBZ} \) can not give correct results since \( |\langle \beta \rangle| \) tends to approach one.

These numerical results are consistent with our discussions and analytical results.

**C. Hopping disorder with exponential forms**

We check the stability of the results shown in TABLE. 1. The Hamiltonian reads:

\[
\mathcal{H} = \sum_{i=1}^{n} -t^+_i c^+_i c_{i+1} - t^-_i c^+_i c_i.
\]  

(S51)

where the hopping terms are set as \( t^+_i = (t + \gamma)e^{w^+_i} \) and \( t^-_i = (t - \gamma)e^{w^-_i} \). The three different kinds of disorder calculated are:

(i). For \( w^+_i = -w^-_i \), then \( t^+_i t^-_i \) is an invariant. Based on Eq. (S7), the open boundary spectrum remains unchanged no matter which kind of disorder \( w^+_i \) is applied [see Figs. S4 (a), (c) and (d)].
FIG. S4: (Color online). Re$[E]$ versus Im$[E]$. The disorder strength $W$ has been marked in the figure. Other parameters are $t = 1$, $\gamma = 0.5$ and $n = 100$. (a) without disorder. (b) $w_i^+ = w_i^-$; (c) $w_i^- = -w_i^+$ with $w_i^+ \in [-\frac{W}{2}, \frac{W}{2}]$. (d) $w_i^- = -w_i^+$ with $w_i^+ \in [0, W]$. $W$ is the disorder strength.

(ii). If $w_i^+ = -w_i^-$ and $w_i^+ \in [-\frac{W}{2}, \frac{W}{2}]$, then det$[E - H_{PBC}]$ and $\beta_{GBZ}$ are invariant based on Eqs. (S16) and (S23). Thus, the periodic boundary spectrum also remains unchanged, no matter how large $W$ is [as shown in Fig. S4 (a) and (c)]. Generally, $\beta_{GBZ}$ is invariant when the sample size $n$ is large enough with $\prod_{i=1}^n e^{w_i^+} \sim 1$. Furthermore, det$[E - H_{PBC}]$ remains almost unchanged since $\prod_{i=1}^n t_i^+ \sim (t + \gamma)^n$, $\prod_{i=1}^n t_i^- \sim (t - \gamma)^n$ and $t_i^+ t_i^- = t^2 - \gamma^2$ holds.

(iii). If $w_i^+ = w_i^-$, then $\beta_{GBZ}$ is invariant based on Eq. (S23). As shown in Fig. S4(b), the results of GBZ still overlap with the open boundary cases when $W \neq 0$.

VII. NUMERICAL RESULTS FOR SU-SCHRIEFFER-HEEGER MODEL

FIG. S5: (Color online). Re$[E]$ versus Im$[E]$. The disorder strength $W$ has been marked in the figure. Other parameters are $t = 1$, $\gamma = 0.5$ and $N = 50$. (a)-(d) $t_0 = 0.2$; (e)-(h) $t_0 = 0.6$.

We pay attention to the model in Eq. (S44). In Eq. (S48), we have pointed out that the Su-Schrieffer-Heeger model has the same modified GBZ theory and the transformation parameter $\langle \beta \rangle$ as those in the Hatano-Nelson model. $\langle \beta \rangle$ is only applied to the translational symmetry. $t_0$ is fixed at its initial value. The modified GBZ with transformation parameter $\langle \beta \rangle$ is the same as those in Eq. (S36).

As clearly shown in Fig. S5, the modified GBZ theory overlaps perfectly with the bulk spectrum under OBC. Notably, the zero eigenvalues are only available for the open boundary cases, which shows the periodic boundary features for other cases. Thus, our result is applicable for the Su-Schrieffer-Heeger model. These results strongly support our theory.
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