Evolution of superhigh magnetic fields of magnetars

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Abstract In this paper, we consider the effect of Landau levels on the decay of superhigh magnetic fields of magnetars. Applying $^3P_2$ anisotropic neutron super-fluid theory yield a second-order differential equation for a superhigh magnetic field $B$ and its evolutionary timescale $t$. The superhigh magnetic fields may evolve on timescales $\sim (10^6 - 10^7)$ yrs for common magnetars. According to our model, the activity of a magnetar may originate from instability caused by the high electron Fermi energy.

Keywords Magnetar. Superhigh magnetic fields. Electron capture rate

1 Introduction

It is now universally accepted that pulsars are neutron stars (NSs) with extremely strong magnetic fields. The surface magnetic field manifests itself as synchrotron radiation from the pulsar magnetosphere, where the field magnitude for young NSs has been estimated to be $(10^{12} \sim 10^{13})$ G. The internal magnetic field is expected to be even higher (Mihara 1990). The magnetic field is the main energy source of all the persistent and bursting emission observed in anomalous X-ray pulsars(AXPs) and soft gamma-ray repeaters (SGRs) (Duncan & Thompson 1992; Paczynski 1992; Megreghetti 2008). AXPs and SGRs are considered to be Magnetar candidates, namely isolated neutron stars powered by the decay of huge ($B > B_{cr} = 4.414 \times 10^{13}$ G) magnetic fields inferred from the rotation under the so-called ‘standard assumption’: $\alpha - \Omega$ dynamo (Duncan & Thompson 1992; Thompson & Duncan 1995, 1996). Their common properties are: stochastic outbursts (lasting from days to years) during which they emit very short X/$\gamma$-ray bursts; rotational periods in a narrow range $P \sim (6-12)$ s; compared to other isolated neutron stars, large period derivatives of $(10^{-13} \sim 10^{-10})$ s s$^{-1}$; rather soft (at least for AXPs) X-ray spectra that can be fitted by the sum of a black body ($kT \sim 0.5$ keV) and a power law component (power law index $\Gamma \sim 2-4$), and, in some cases, associated with supernova remnants (SNRs) (Duncan & Thompson 1992; Megreghetti 2008).

Up to now (3 October 2010), there are nine SGRs (seven conformed) and twelve AXPs (nine conformed) detected, so statistically investigating their persistent parameters is available. Observationally, all known magnetars are X-ray pulsars with luminosities of $L_x \sim (10^{32} \sim 10^{36})$ erg s$^{-1}$ (Rea et al. 2010), their high luminosities together with the lack of evidence for ac-
cretion from a stellar companion, result in the conclusion that the energy reservoir fueling the magnetar activity is their superhigh magnetic fields \(10^{14} \sim 10^{15} \) G. \cite{Dib2008, ThompsonDuncan1996, Megreocchi2008, Rea2010}. Such a huge energy reservoir \((E \sim 3 \times 10^{37} (B/10^{15} G)^2 (R/10 km)^3 \) erg) is sufficient to power not only the persistent surface X-ray emission but also a steady stream of low-amplitude Alfvén waves into the magnetosphere for \(\sim 10^4 \) yrs \cite{ThompsonDuncan1995, ThompsonDuncan1996}.

In order to provide a detailed view of the actual evolution of a magnetar, we should consider the effective electron capture rate \(\Gamma_{\text{eff}}\) (the effective number of electrons captured by one proton per second) due to the existence of the Landau levels of charged particles \cite{Gao2010a}. However, only the electrons occupying large Landau levels with high energy \((E > Q, Q\) is the threshold of energy) are allowed to participate in the process \(e^- + p \rightarrow n + \nu_e\) \cite{Yakovlev2001}. In other words, whether an electron could be captured depends not only on the electron’s energy \(E_e\) but also on the number of the Landau level it occupies. Based on the assumption of the Landau level effect coefficient \(q\), we define the effective electron capture rate \(\Gamma_{\text{eff}}\) as: \(\Gamma_{\text{eff}} = q \Gamma\) \cite{Gao2010a}. With respect to the electron Fermi energy \(E_F(e)\), our point of view is that in the case of an intense magnetic field, the stronger the magnetic field, the higher the electron Fermi energy released from the magnetic field energy \(E_e\) \cite{Gao2010a}.

The remainder of this paper is organized as follows: in § 2 a review on the evolution of superhigh magnetic fields of magnetars is given; in § 3 we investigate the relation between \(E_F(e)\) and \(B\), and gain a schematic diagram of \(L_X\) as a function of \(B\); in § 4 a second-order differential equation for a superhigh field and its evolutionary timescale is given; discussions and conclusions are presented in § 5; in Appendix A we consider the effect of a superhigh magnetic field on the proton fraction \(Y_p\) and gain a concise formula of \(E_F(e)\), and in Appendix B we give an explanation for why a wrong notion that \(E_F(e)\) decreases with increasing \(B\) is used as yet.

2 The evolution of magnetic fields inside a magnetar

The evolution of magnetic fields of a magnetar is a rather challenging subject, with strong observational clues and involving complex physics. The strong evidences for spontaneous evolution of magnetic fields inside magnetars are mainly that: SGRs and AXPs appear to radiate substantially more power than available from their rotational energy loss; the dipole field inferred from the observed torque is stronger than that of all other NSs, \(\sim 10^{14} \sim 10^{15} \) G, which make it nature to accept the argument of Thompson & Duncan (1992) that these sources are in fact ‘magnetars’, powered by the dissipation of their magnetic energy; and torque changes of both signs, associated with outbursts from these sources, etc. In order to explain the evolution of superhigh magnetic fields inside magnetars, different models have been proposed recently, as partly listed below.

In the twisting magnetic field model, the magnetic field is supposed to be dominated by a toroidal component larger than the external dipole. The persistent quiescent X-ray emission of a magnetars could be brought out by the twisting of the external magnetic field caused by the motions of the star interior. The twisting crustal motions generally drive currents outside a magnetar, and generate X-rays \cite{Thompson2000}. Furthermore, a discussion about the globally-twisted magnetosphere was proposed by Thompson et al (2002). However, up to now, a quantitative calculation of X-ray luminosity in the twisting magnetic field model is still up in the air.

In the thermal evolution model, the main points of view include: The field could decay directly as a result of the non-zero resistivity of the matter Ohmic decay or ambipolar diffusion, or indirectly as a result of Hall drift producing a cascade of the field to high wave number components, which decay quickly through Ohmic decay \cite{GoldreichReisenegger1992, Pons2007, RheinhardtGeppert2002}; magnetic field decay can be a main source of internal heating; the enhanced thermal conductivity in the strongly magnetized envelope contributes to raise the surface temperature \cite{HevlHernquist1997, HevlKulkarni1998}, etc.

In currently existing magnetar models, the evolution timescale of superhigh magnetic fields of magnetars is estimated just according to conservation of energy, rather than depending on an accurate equation of \(B\) and its evolution timescale \(t\). For example, a strong dependence of neutron star surface temperature on the dipolar component of the magnetic field \(B_d\) for stars with \(B_d > 10^{13} \) G reveals that the thermal evolution is almost fully controlled by the amount of magnetic field energy the star has stored in its crust by the time the star has reached an age of \(\sim 10^8\) yrs (earlier, for magnetars) \cite{Pons2007}. One more example, while the field decay is governed by Ohmic dissipation and Hall drift, the most interrelated process inside a magnetar is ambipolar diffusion, which has a characteristic timescale

\[
t_{\text{amb}} \sim 10^4 \times (B_{\text{core}}/10^{15} G)^{-2} \text{yrs},
\]
where $B_{\text{core}}$ denotes the core magnetic field (Thompson & Duncan 1993). In this article, unlike the previous magnetar model, we calculate the decay timescale of superhigh magnetic fields of magnetars by employing the Landau level effect coefficient $g$ and the effective electron capture rate $\Gamma_{\text{eff}}$ (Gao et al. 2010a). For further details, see § 4.

With respect to the origin of superhigh magnetic fields of magnetars, a more popular hypothesis is that magnetars are formed from rapidly rotating proto-neutron stars, differential rotation and convection would result in an efficient $\alpha - \Omega$ dynamo (Duncan & Thompson 1992, 1996; Thompson & Duncan 1993; Vink & Kuiper 2006). The dynamo responsible for the superhigh magnetic field generation requires that magnetars be born with ultrashort initial spin periods $P_i \sim 1 \text{ ms}$, the associated free energy is $E_{\text{dir}} \sim 10^{52} (P_i/1\text{ms})^{-2} \text{ erg}$, and the magnetic field is as strong as $\sim 3 \times 10^{17} (P_i/1\text{ms})^{-1} \text{ G}$ (Thompson & Duncan 1993, 1996). However, there is as yet no mechanism to explain such a high efficiency of energy transformation. When investigating solar flare using the $\alpha - \Omega$ model, apart from the difficulty in explaining such a high energy transformation efficiency, there are also many other problems to be settled at present. Note, up to now, there is no observational indication on the existence of fields $B \geq 10^{16} \text{ G}$ in the interior of a NS. Furthermore, the explosion energies of these supernova remnants associated with AXPs and SGRs (AXP 1E 1841-045, AXP 1E 2259+586, and SGR 0526-26), are near to the canonical supernova explosion energies of $\sim 10^{51} \text{ erg}$, suggesting $P_i \geq 5 \text{ ms}$ (Vink & Kuiper 2006). Enough has been said to show that $\alpha - \Omega$ dynamo is a mere assumption, lacking observational support.

Although the origin of superhigh magnetic fields of magnetars, as mentioned above, is uncertain, a possible mechanism that superhigh magnetic fields of magnetars originate from the induced magnetic fields at a moderate lower temperature, has been proposed (Peng & Tong 2007, 2009). It is universally supposed that $^3P_2$ anisotropic neutron superfluid could exist in the interior of a NS, where the mean thermal kinetic energy of nucleons is far less than nucleon interaction energy (in the order of $\sim 1 \text{ MeV}$) due to strong nucleon-nucleon attractive interaction (the mean distance of nucleons is $\sim 1 \text{ fm}$). It is this strong nucleon-nucleon interaction of attraction that results in the formations of proton Cooper pairs and $^3P_2$ anisotropic neutron Cooper pairs. The nucleonic $^3P_2$ pairing gaps in NSs have been investigated by Elgaroy et al (1996) in detail. Their important results may be briefly summarized as follows:

The $^3P_2$ neutron pair energy gap $\Delta(^3P_2)$ appears in the region $1.32 < k_F (\text{fm}^{-1}) < 2.34$, where $\Delta(^3P_2)$ is an increasing function of the density for the region $1.32 < k_F (\text{fm}^{-1}) < 1.8$ and it is a rapidly decreasing function of the density for the region $2.1 < k_F (\text{fm}^{-1}) < 2.34$, where $k_F$ is the Fermi wave number of the neutrons.

(2). The maximum of the $^3P_2$ neutron pair energy gap $\Delta_{\text{max}}(^3P_2)$ is about $0.048 \text{ MeV}$ at $k_F = 1.96 \text{ fm}^{-1}$.

(3). $\Delta(^3P_2)$ is almost a constant about the maximum with error less than 3% in the rather wide range $1.8 < k_F (\text{fm}^{-1}) < 2.1$, corresponding to the density region $3.3 \times 10^{14} < \rho (\text{g cm}^{-3}) < 5.2 \times 10^{14}$.

Based on the analysis above, the critical temperature of the $^3P_2$ neutron Cooper pairs can be evaluated as follows: $T_{\text{cm}} = \Delta_{\text{max}}(^3P_2)/2k_F \approx 2.78 \times 10^{8}\text{K}$, the maximum of internal temperature $T$ (rather than the maximum of core temperature) cannot exceed $T_{\text{cm}}$ according to our model (Peng & Luc 2006, Peng & Tong 2009). A NS is often approximately considered as a system of magnetic dipoles because of the existence of $^3P_2$ anisotropic neutron superfluid in its interior. In the presence of a background magnetic field, the dipoles have a tendency to be aligned along the direction of this field. If temperature falls below a critical temperature, a phase transition from paramagnetism to ferromagnetism could occur, which provides a possible explanation for superhigh magnetic fields of magnetars. The maximum induced magnetic field is estimated to be $(3.0 \sim 4.0) \times 10^{15} \text{ G}$ (Peng & Luc 2006).

In the case of field-free, for the direct Urca reactions to take place, there exists the following inequality among the Fermi momenta of the proton ($P_p$), the electron ($K_F$) and the neutron ($Q_F$): $P_p + K_F \geq Q_F$ required by momentum conservation near the Fermi surface. Together with the charge neutrality condition, the above inequality brings about the threshold for proton concentration $Y_p = n_p/(n_p + n_n) \geq 1/3 \sim 0.33$, this means that, in the field-free case, direct Urca reactions are strongly suppressed in the degenerate $n - p - e$ system (Baiko & Yakovlev 1999; Lai & Shapiro 1991; Leinson & Pérez 1997; Yakovlev et al 2001). However, when in a superhigh magnetic field $B \gg B_{\text{cr}}$, things could be quite different. A superhigh magnetic field can generate a noticeable magnetic broadening of the direct Urca process threshold, as well as of the thresholds of other reactions in the interior of a magnetar (Yakovlev et al, 2001). If the magnetic field is large enough for protons and electrons to be confined to the lowest landau levels ($n = 0, 1$), the field-free threshold condition on proton concentration no longer holds, and direct Urca reactions are open for an arbitrary proton concentration (Leinson & Pérez, 1997). Strong magnetic field can alter matter compositions and increase
phase pace for protons which leads to a branch of the Fermi surface (Lai & Shapiro 1991). By strongly modifying the phase space of protons and electrons, ultrastrong magnetic fields (∼ 10^{20} G) can cause a substar version, as a result, the system is con proton-rich matter with distinctively higher Fermi energy of the outgoing neutrons will be transformed into thermal energy and then transformed into the radiation energy with X-ray and soft γ-ray (Peng & Tong 2009). The \( ^3P_2 \) Cooper pairs will be destroyed quickly by these high-energy outgoing neutrons. Thus the anisotropic superfluid and the superhigh magnetic fields induced by the \( ^3P_2 \) Cooper pairs will disappear.

3 The electron Fermi energy and direct Urca process

This section is composed of two subsections. For each subsection we present different methods and considerations.

3.1 The Fermi energy of electrons in superhigh magnetic fields

As shown in Fig.1, in the presence of an extremely strong magnetic field, the electrons are situated in disparate energy states in order one by one from the lowest energy state up to the Fermi energy (the highest energy) with the highest momentum \( p_F(z) \) along the magnetic field, according to the Pauli exclusion principle.

In the interior of a magnetar, different forms of intense magnetic field could exist simultaneously and a high \( E_F(e) \) could be generated by the release of magnetic field energy. When \( B \gg B_{cr} \), the Landau column becomes a very long and very narrow cylinder along the magnetic field, the electron Fermi energy is determined by

\[
\frac{3\pi}{B^2} \left( \frac{m_e c}{\hbar} \right)^3 (\gamma_e)^4 \int_0^1 \left( 1 - \frac{1}{\gamma_e} - \chi^2 \right) \frac{d\chi}{2B}\chi^2 = N_A \rho Y_e,
\]

where \( B^* \), \( \chi \) and \( \gamma_e \) are three non-dimensional variables, which are defined as \( B^* = B/B_{cr}, \chi = (\rho \gamma_e)/(E_F/m_{e\gamma c}) = p_e c/E_F \) and \( \gamma_e = E_F/m_e c^2 \), respectively; \( 1/\gamma_e \) is the modification factor; \( N_A = 6.02 \times 10^{23} \) is the Avogadro constant; \( Y_e = Y_p = Z/A \), here \( Y_c, Y_p, Z \) and \( A \) are the electron fraction, the proton fraction, the nucleon number and the nucleon number of a given nucleus, respectively (Gao et al. 2010). From Eq.(2), we gain the relations of \( E_F(e) \) vs. \( B \) and \( E_F(e) \) vs. \( \rho \), which are shown in Fig. 2.

Seeing from Fig.2, it’s obvious that \( E_F(e) \) increases with the increase in \( B \) when \( \rho \) and \( Y_e \) are given; \( E_F(e) \) also increases with the increasing \( \rho \) when \( B \) and \( Y_e \) are given. The high Fermi energy of electrons could be from the release of the magnetic energy. A possible interpretation of high \( E_F(e) \) is given as follows: an envelope of the Landau circles with maximum quantum number \( n_{max} \) will approximately form a sphere, i.e. Fermi sphere. For a given electron number density with a highly degenerate state in the interior of a NS, the stronger the magnetic field, the larger the maximum of \( p_z \), is, hence the lower the number of states in the \( x-y \) plane according to the Pauli exclusion principle. In other words, \( n_{max} \) and the number of electrons in the \( x-y \) plane decrease with the increase of \( B \), the radius of the Fermi sphere \( p_F \) is expanded which implies that the electron Fermi energy \( E_F(e) \) also increases. The higher the Fermi energy \( E_F(e) \), the more obvious the ‘expansion’ of the Fermi sphere is, however, the majority of the momentum space in the Fermi sphere is empty for not being occupied by electrons. However, one incorrect notion that \( E_F(e) \) decreases with increasing \( B \) in an intense field \( (B \gg B_{cr}) \) has been universally adopted.
The electron capture rates can be calculated for reactions, but at non-zero temperature \((kT \ll E_{F})\), we assume that at zero-temperature the NS is \(\beta\)-stable, and further details are presented in Appendix A.

3.2 Direct Urca process in superhigh magnetic fields

We assume that at zero-temperature the NS is \(\beta\)-stable, but at non-zero temperature \((kT \ll E_{F}(i), i = n, p, e, k = 1.38 \times 10^{-16} \text{ erg K}^{-1}\) is the Boltzmann constant\), reactions \(e^{-} + p \rightarrow n + \nu_{e}\) and \(n \rightarrow e^{-} + p + \nu_{e}^{-}\) proceed near the Fermi energies \(E_{F}(i)\) of the participating particles. The electron capture rates can be calculated as follows:

\[
N = N_{\text{eff}} = \frac{2\pi G_{F}^{2} C_{F}^{2} (1 + 3a^{2})}{(2\pi^{2} \hbar^{3} c^{3})^2} I
\]

\[
I = \int_{0}^{E_{F}(e)} \left( E_{e}^2 - m_{e}^{2} c^{4} \right)^{3/2} E_{e}^{1/2} dE_{e}
\]

\[
(E_{e} - 60)^{2} \left( \frac{1}{e^{E_{e} - E_{F}(e)} + 1} - \frac{1}{e^{E_{e} + E_{F}(e)} + 1} \right) + 1
\]

where \(Q = E_{F}(n) - E_{F}(p) \approx 60 \text{ MeV}, \Lambda \approx 0.018 (\text{MeV})^{-5} \text{s}^{-1}, E_{F}(e) = 40(B/B_{c})^{1/2} \text{ MeV}\) (c.f. Gao et al 2010a and Appendix A of this paper), and other terms appearing in Eqs.(3-4) have already been defined in Chapter 18 of Shapiro & Teukolsky [1983].

This paper, for the purpose of convenient calculation, we select \(\rho = \rho_{0}\). The range of \(B\) is assumed to be \((2.2346 \times 10^{14} \sim 3.0 \times 10^{15}) \text{ G}\) corresponding to \(E_{F}(e) \approx (60 \sim 114.85) \text{ MeV}\), where \(2.2346 \times 10^{14} \text{ G}\) is the minimum of \(B\) denoted as \(B_{f}\). When \(B\) drops below \(B_{f}\), the direct Urca process is quenched everywhere in the magnetar interior. Combining the relation

\[
\langle E_{\nu} \rangle = \int_{Q}^{E_{F}(e)} S(E_{e} - Q)^{3} E_{e}^{1/2} dE_{e}/I,
\]

with the relation

\[
\langle E_{n} \rangle = E_{F}(e) - \langle E_{\nu} \rangle - 1.29 \text{ MeV},
\]

gives the mean kinetic energy of neutrons. The average X-ray luminosity \(L_{X}\) can be expressed as follows:

\[
L_{X} = \Gamma_{\text{eff}} n_{p} V (3P_{2}) \langle E_{n} \rangle = q \Gamma \rho_{p} V (3P_{2}) \langle E_{n} \rangle,
\]

where \(q\) is the Landau level effect coefficient, \(V (3P_{2})\) denotes the volume of \(3P_{2}\) anisotropic neutron superfluid (\(V (3P_{2}) = (\frac{4}{3} \pi R_{0}^{3}, R_{0} = R/10^{5} \text{ cm}\) and \(n_{p} = n_{e} = 9.6 \times 10^{35} \text{ cm}^{-3}\). For a magnetar with initial magnetic field \(B_{0} = 3.0 \times 10^{15} \text{ G}\), if its initial X-ray luminosity \(L_{X0}\) is assumed to be \((1.0 \sim 9.0) \times 10^{36} \text{ erg s}^{-1}\), then \(q\) is estimated to be \((0.216 \sim 1.94) \times 10^{-18}\) and the initial effective capture rate is \((0.221 \sim 1.98) \times 10^{-11} \text{ s}^{-1}\). We can also construct a schematic diagram of the average X-ray luminosity as a function of magnetic field for the process of electron capture, which is shown as in Fig.4.

As discussed above, the decay of magnetic fields is the ultimate energy source of all the persistent and bursting emission observed in AXPs and SGRs. In Table 1, the related persistent parameters of the confirmed magnetars with observed X-ray fluxes are listed according to observations performed in the last two decades.

From Table 1, we see weak correlation between \(T_{BB}\) and \(B\). Magnetars are massive cooling neutron stars, according to neutron star cooling theory (modified Urca reactions), the typical magnetar internal temperature is about \(3 \times 10^{8} \text{ K}\), its surface temperature is lower than its internal temperature by two orders of magnitude (Yakovlev et al [2001]), which is consistent with the surface temperature data in Table 1. The assumption that \(3P_{2}\) anisotropic neutron superfluid could exist in the interior of a NS is supported by the data of column 3 in Table 1.
Canuto & Chiu 1971; Dorofeev et al 1985; Lai & Shapiro (Baiko & Yakovlev 1999; Bandyopadhyay et al 1998; high magnetic fields have been studied since late 1960’s original data by using 1 KeV SGR 1806-20. The data of column 3 are gained from the erg s$^{-1}$ -notes: from Thompson & Duncan 1996. The sign a de -notes: from Thompson & Duncan 1996. The sign a de -notes: from Thompson & Duncan 1996. The sign a de

| Name          | $P$       | $T_{BB}$ | $B$  | $L_X$ |
|---------------|-----------|----------|------|-------|
| SGR1900+14    | 7.783     | 4.98     | 6.42 | 1.8-2.8 |
| SGR0526-66    | 6.5       | 6.14     | 7.32 | 2.1   |
| CXOU.0100     | 1.88(8)   | 4.41     | 3.94 | 0.78  |
| 4U 0142+61    | 0.196     | 4.60     | 1.32 | >0.53 |
| 1E 1841-045   | 4.1551    | 5.10     | 7.08 | 2.2   |
| 1E 2259+586   | 0.048     | *        | 0.59 | 0.18  |
| 1RXS J1708    | 1.945     | 5.29     | 4.68 | 1.9   |
| SGR1900+14    | 54.9      | 7.54     | 21   | (50$^6$) |
| 4U 0142+61    | 0.196     | 4.60     | 1.32 | >0.53 |
| 1E 1841-045   | 4.1551    | 5.10     | 7.08 | 2.2   |
| 1E 2259+586   | 0.048     | *        | 0.59 | 0.18  |
| 1RXS J1708    | 1.945     | 5.29     | 4.68 | 1.9   |
| CXOU.0100     | 1.88(8)   | 4.41     | 3.94 | 0.78  |
| 4U 0142+61    | 0.196     | 4.60     | 1.32 | >0.53 |
| 1E 1841-045   | 4.1551    | 5.10     | 7.08 | 2.2   |
| 1E 2259+586   | 0.048     | *        | 0.59 | 0.18  |
| 1RXS J1708    | 1.945     | 5.29     | 4.68 | 1.9   |

Note: The units of $P$, $T_{BB}$, $B$ and $L_X$ are $10^{-11}$ s$^{-1}$, $10^6$ K, $10^{14}$ G and $10^{35}$ erg s$^{-1}$, respectively, where $T_{BB}$ is the temperature of a magnetar. The sign a de -notes: from Thompson & Duncan 1996. The sign a de -notes: from Thompson & Duncan 1996. The sign a de -notes: from Thompson & Duncan 1996. The sign a de

The inverse $\beta$-decay and related reactions in super-high magnetic fields have been studied since late 1960’s (Baiko & Yakovlev 1999; Bandyopadhyay et al 1998; Canuto & Chiu 1997; Dorofeev et al 1985; Lai & Shapiro 1998; Lattimer et al 1991). The conventional wisdom is that the neutrino emissivity $Q_\nu \propto T_9^6 (kT_9 \sim 0.086$ MeV) in the direct Urca process, and the ‘standard magnitude’ of $Q_\nu$ is $10^{27} T_9^6$ erg cm$^{-3}$ s$^{-1}$ $\sim 10^{25} T_9^6$ erg s$^{-1}$, where $T_9$ is the temperature in units of $10^9$ K and it is assumed that the super-high magnetic field permeates the entire volume ($R_{star} \sim 10^6$ cm) of the magnetar interior. With respect to the ‘standard magnitude’ of $Q_\nu$, the main assumptions include: $E_i \sim E_F(i)$ ($i$=e, p, n, $\nu$); $F_F(e)$ decreases with increasing $B$; $E_F(p) \sim E_F(n) \sim E_F(\nu) \sim kT_9$. From the expression $Q_\nu \propto T_9^6$, it is easy to imagine that the neutrino flux comes from the thermal energy in the magnetar interior, rather than from the free energy of the super-high magnetic field. Why should we say this? Results of our analysis appear below: In the interior of a NS, the majority of fermions are situated in the bottom of Fermi sea (Landau & Lifshitz 1963), their energies are far less than their Fermi energies, only in this case can we write $E_i \sim kT$; however, in the vicinity of Fermi surface, there are but few fermions with energies $E_i \sim E_F(i)$. We must bear in mind that the minimum of $E_n$ of the outgoing neutrons in the process of electron capture is no less than the neutron Fermi energy ($E_F(n) \geq 60$ MeV (Shapiro & Teukolsky 1983)), otherwise the outgoing neutrons can not escape from the surface of Fermi sea, thus the inverse $\beta$-decay will not occur. In the interior of a magnetar where the direct Urca reaction takes place, neutrons can be sorted into two kinds according to their energies: low- energy neutrons with thermal kinetic energy $\sim kT$ and high- energy outgoing neutrons with energy $\sim E_F(n)$, for neutrinos, the same are true. Although the energies of outgoing neutrons and outgoing neutrinos are directly from the energies of electrons participating in direct Urca reaction, $L_X$ and $Q_\nu$ are ultimately determined by $B$, rather than by $T$, the reason is that magnetic field energy is the main source of emissions, whereas $T$ is only equivalent to background temperature, therefore both $L_X$ and $Q_\nu$ are just weak functions of $T$. With regard to the calculations of X-ray luminosity and neutrino luminosity in super-high magnetic fields, for further details, refer to Gao et al (2010a). The magnetic field distribution of a magnetar, though attracting general attentions of researchers, still lacks a definite conclusion now. If the direct Urca reaction occurs in the crust of a magnetar, the assumption of $T \sim 10^9$ K is contradictory to the observational values of $T_{BB}$ in Table 1. If the direct Urca reaction occurs in the interior of a magnetar where electrons are relativistic, neutrons and protons are non-relativistic, the so-called ‘standard magnitude’ of $Q_\nu$ will be higher than $10^{36}$ erg s$^{-1}$ at least according to our model (see § 3.2). By analyzing, we can make a comparison between the observed values ($Q_\nu$
and $L_X$) and the conventional values ($Q_\nu$ and $L_X$): For a magnetar, a typical observed soft X-ray luminosity is about $3 \times 10^{35}$ erg s$^{-1}$ (Thompson et al. 2002), the amount of total electromagnetic radiation is about one thirteenth of that of the neutrino emissivity due to the fact that photons are partly absorbed by star internal matter, (Heyl & Kulkarni 1998; Thompson & Duncan 1996; Thompson et al. 2002), then we obtain the neutrino emissivity $\sim 10^{37}$ erg s$^{-1}$; if all emissions of a magnetar are from the modified Urca reaction (the magnitude of which is about 6-orders lower than that of the direct Urca reaction), then the calculated value of $Q_\nu$, is $\sim 10^{50}$ erg s$^{-1}$ or above, which is far larger than the observed values of $Q_\nu(\sim 10^{37}$ erg s$^{-1}$), and the calculated value of $L_X$ in the modified Urca process is about $10^{48}$ erg s$^{-1}$ or above, accordingly, which is far higher than the observed value of $L_X$, where we assume that the energies of all particles participating in the process of electron capture are calculated according to our model. Our model will be a challenge for the conventional way of calculating neutrino emissivity, if our analysis is reasonable.

![Schematic diagram of $L_X$ vs. $B$. Triangles and circles mark the values of variables corresponding to $q = 1.94 \times 10^{-18}$ and $2.16 \times 10^{-19}$, respectively. The ranges of $B$ is assumed to be $(2.24 \times 10^{14} \sim 3.0 \times 10^{15}$ Gauss) considering that, when $B \leq B_t$, the direct Urca process ceases, while the modified Urca process still occurs, from which weaker X-ray and weaker neutrino flux are produced.](image)

4 The calculations of reaction time

When the electron Fermi energy is much greater than the threshold energy of inverse $\beta$-decay, the electron capture reaction will occur immediately. Thus the $^3P_2$ Cooper pairs will be destroyed quickly by the outgoing high-energy neutrons, and the anisotropic superfluid and the superhigh induced magnetic field will disappear. Employing Eqs.(3-4) can allow us to write a differential equation

$$
\frac{d\Gamma}{dt} = \Lambda S(E_F(e) - Q)^2 E_F(e)(E_F^2(e) - m_e^2 c^4)\frac{1}{4} B^{\frac{13}{2}} \frac{dB}{dt},
$$

where the approximation $S \approx 1$ is used. Using the binomial expansion theorem, the term $(E_F^2(e) - m_e^2 c^4)\frac{1}{4}$ can be expanded as:

$$
(E_F^2(e) - m_e^2 c^4)\frac{1}{4} = E_F(e) \times (1 - m_e^2 c^4 / 2 E_F^2(e)) - m_e^2 c^4 / 8 E_F^2(e) + \cdots \approx 40(B / B_{cr})^{\frac{13}{2}}
$$

$$
\times (1 - 542B^{-\frac{13}{2}} - 146932B^{-1} + \cdots).
$$

Since $542B^{-\frac{13}{2}} \sim 10^{-5}$ and $146932B^{-1} \sim 10^{-10}$, reserving the first two terms in the bracket of the expansion above gives

$$
\frac{d\Gamma}{dt} = 16000\Lambda(1600B^{\frac{13}{2}} B_{cr}^{\frac{13}{2}} - 4800B_{cr}^{-1}) + 3600B^{\frac{13}{2}} B_{cr}^{\frac{13}{2}})(1 - 542B^{-\frac{13}{2}}) \frac{dB}{dt}.
$$

A neutron star can be treated as a system of magnetic dipoles $\mu = BR^3/2$, where $\mu$, $B$, and $R$ are the polar magnetic moment, the polar magnetic field strength and the radius of the neutron star, respectively (Shapiro & Teukolsky 1983). If one outgoing neutron in the process of $e^- + p \rightarrow n + \nu_e$ is assumed to destroy one $^3P_2$ Cooper pair, then the decay rate of the superhigh magnetic field is

$$
\frac{dB}{dt} = 2 \frac{d}{dt} \left( -q \Gamma 2 \mu_n n_e V(^3 P_2) \right) = -\frac{4\mu_n n_e V(^3 P_2)q\Gamma}{R^3},
$$

where $\mu_n = 0.966 \times 10^{-23}$ erg G$^{-1}$ is the absolute value of the neutron abnormal magnetic moment. Combining Eq.(10) with Eq.(11) and eliminating $\Gamma$, we get a
second-order differential equation

\[ \frac{d^2 B}{dt^2} + \frac{64000 \Delta n_0 n_\beta V^3 P_2 q}{R^3} (1600 B^\frac{4}{3} B_C^\frac{1}{3} - 4800 \ B_C^{-1} + 3600 B^{-\frac{4}{3}} B_C^{-\frac{1}{3}}) \frac{dB}{dt} = 0, \quad (12) \]

In order to obtain the total electron capture time \( t \), this second-order differential equation can be treated as follows: firstly, decrease the order of Eq.(12); secondly, apply the initial reaction condition \( B = 2.2346 \times 10^{14} \) G, \( dB/dt = 0 \) to determine the constant of integration \( \sim 1619.4 \); thirdly, integrating over \( B \) gives the expression for \( t \); and lastly, using the integral transform \( B^\frac{4}{3} \rightarrow x \) and \( dB \rightarrow 4x^3 dx \) gives the integral equation

\[
\begin{align*}
 t &= \frac{1}{q} \int_{B_C^{25}}^{B_1^{25}} (0.502516 x^5 - 4857.23 x^4 + 1.25198 \\
 & \quad \times 10^7 x^3 - 453.939 x^2 + 5.26524 \times 10^6 x^2 - 2.03578 \\
 & \quad \times 10^8 x - 6.71187 \times 10^{18} )^{-1} 4x^3 dx,
\end{align*}
\]  

(13)

Simplifying Eq.(13) further, we get

\[
\frac{t}{q} = \frac{1}{B_1^{25}} \int_{B_C^{25}}^{B_1^{25}} (0.502516 x^5 - 4857.23 x^4 + 1.25198 \\
\times 10^7 x^3 - 453.939 x^2 + 5.26524 \times 10^6 x^2 - 2.03578 \\
\times 10^8 x - 6.71187 \times 10^{18} )^{-1} 4x^3 dx,
\]  

(14)

where \( B_1 \) is the value of \( B_1 \), and the mean value of \( q \approx 1.94 \times 10^{-18} \). Solving Eq.(14) gives \( t \approx 2.36 \times 10^{14} \) s \( = 7.48 \times 10^6 \) yrs when \( B_2 = 3 \times 10^{15} \) G and \( B_1 = 2.2346 \times 10^{14} \) G; similarly, if \( B_2 = 3.0 \times 10^{15} \) G and \( B_1 = 3.7 \times 10^{14} \) G, then \( t \approx 2.66 \times 10^{14} \) s \( \approx 7.1 \times 10^6 \) yrs corresponding to \( L_X \sim (9.0 \times 10^{36} - 1.0 \times 10^{34}) \) erg s\(^{-1}\); if \( B_2 = 3.7 \times 10^{14} \) G and \( B_1 = 2.25 \times 10^{14} \) G, then \( t \approx 3.7 \times 10^5 \) yrs corresponding to \( L_X \sim (1.0 \times 10^{34} - 1.75 \times 10^{28}) \) erg s\(^{-1}\).

It may be questioned whether there are sufficient anisotropic \( ^3P_2 \) superfluid Cooper pairs able to be destroyed to significantly decrease the induced magnetic field in the magnetar interior. Allow us to make following calculations: a neutron star’s mass is \( \sim 1.4 \) \( M_\odot \), \( M_\odot \approx 2.0 \times 10^{33} \) g and the mass fraction of \( ^3P_2 \) anisotropic neutron superfluid in the mass of neutron star is more than 0.1. Therefore, \( N_{\text{Cooper}} \approx 0.1 \times 1.4 \ M_\odot / m(^3P_2) = 0.14 \times 1.4 / 2n_\text{m} = 2.8 \times 10^{32} / (2 \times 1.67 \times 10^{-24}) \approx 8.38 \times 10^{55} \). Meanwhile \( \Delta N_{\text{Cooper}} \), the total number of \( ^3P_2 \) Cooper pairs destroyed by outgoing neutrons is determined by \( \Delta N_{\text{Cooper}} = n_p V(^3P_2) \Gamma_{\text{tot}}, \) where \( \Gamma_{\text{tot}} \), the average total electron number captured by one proton, can be expressed as

\[
\Gamma_{\text{tot}} = \int q \Gamma dt.
\]  

(15)

Using Eq.(15), we find

\[
\Gamma_{\text{tot}} = \int_{B_2}^{B_1} \frac{R^3}{4 n_p n_\beta V^3 P_2} dB \approx 17926,
\]  

(16)

where \( B_2 = 3.0 \times 10^{15} \) G and \( B_1 = 2.2346 \times 10^{14} \) G. Thus \( \Delta N_{\text{Cooper}} \) can be evaluated to be \( 1.2 \times 10^{55} \). Be note, in the interior of a NS, the processes of electron capture and \( \beta \)-decay exist at the same time required by electric neutrality, the depleted protons (electrons) are recycled for many times, so the variable \( \Delta Y_e \) could be very small. From evaluations above, it is obvious that \( \Delta N_{\text{Cooper}} < N_{\text{Cooper}}, \) which demonstrates that our results are reasonable.

5 Discussions and Conclusions

In this paper, we review briefly the evolution of magnetic fields of magnetars in currently existing magnetar models, investigate the relation between magnetar soft X-ray Luminosity \( L_X \) and magnetic field strength \( B \), and calculate magnetar magnetic field evolution timescale \( t \). From the analysis and the calculations above, the main conclusions are as follows:

1. Superhigh magnetic fields genuinely produce a remarkable magnetic broadening of the direct Urca process threshold. The \( ^3P_2 \) Cooper pairs will be destroyed quickly by the high-energy outgoing neutrons via the process of electron capture, so the induced magnetic field will disappear.

2. The kinetic energy of the outgoing neutrons will be first transformed into thermal energy and then into radiation energy as X-rays and soft \( \gamma \)-rays. The observed X-ray luminosity \( L_X \) is related to \( \langle E_n \rangle \) and \( \Gamma_{\text{eff}}, \) but is ultimately determined by \( B \).

3. The relationship between \( B \) and \( t \) can be expressed as a second-order differential equation. If \( L_{X0} \) is assumed to be \( (1 < 9) \times 10^{36} \) erg s\(^{-1}\), then \( q \) is \( (0.216 \sim 1.94) \times 10^{-18} \) and \( t \) is \((6.7 \sim 0.748) \times 10^7 \) yrs, respectively. The origin of the magnetic fields in magnetars shows that magnetars could exist, but that they would be unstable due to the high electron Fermi energy.

Finally, we are hopeful that our calculations can soon be combined with astrophysical studies of magnetars and new observations, to provide a deeper understanding of the nature of the superhigh magnetic fields in magnetars.
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Appendix

A  The effect of a superhigh magnetic field on $Y_p$

As we know, in the case of field-free, for reactions $e^- + p \rightarrow n + \nu_e$ and $n \rightarrow p + e^- + \nu_e$ to take place, there exists the following inequality among the Fermi momenta of the proton ($P_F$), the electron ($K_F$) and the neutron ($Q_F$): $P_F + K_F \geq Q_F$ required by momentum conservation near the Fermi surface, $Y_p$ is the proton fraction, defined as the mean proton number per baryon. Together with the charge neutrality condition, the above inequality brings about

$$\rho_{n} \geq 1/9,$$

where the solution of Eq.(B1) is used (also c.f. Page 460 of Quantum Mechanics (Landau & Lifshitz 1965)). In the case of field-free, the value of $\rho_{n}$ is expected to be considerably enhanced, where $B^{p}_{cr}$ is the quantum critical magnetic field of protons ($\sim 1.48 \times 10^{20}$ G); by strongly modifying the phase spaces of protons and electrons, magnetic fields of such magnitude ($\sim 10^{20}$ G) can cause a substantial proton-rich matter with distinctly softer EOS, compared to the field-free case. Though magnetic fields of such magnitude inside NSs are unauthentic, and are not consistent with our model, their calculations are useful in supporting our following assumption: when $B \sim 10^{14} - 15$ G, the value of $Y_p$ may be enhanced, and could be slightly higher than the mean value of $Y_p$ inside a NS ($\sim 0.05$). Based on this assumption, we can gain a concise expression

$$E_{F}(e) = 39.9(B/B_{cr})^{1/2}(\frac{\rho_{e}}{\rho_{e}^{0.005}})^{1/2} \text{MeV} \leq 0.08 \text{MeV}$$

by solving equation(19) of Gao Z. F et al (2010b).

B  A wrong conclusion on the Fermi energy of electrons in superhigh magnetic fields

It is universally accepted that $E_{F}(e)$ decreases with increase in $B$ in the case of superhigh magnetic fields. The reasons for this are as follows: when in the presence of a uniform external magnetic field along the $z$-axis, solving the non-relativistic Schrödinger Equation for electrons gives electron energy level

$$E_{e}(p_{z}, B, n, \sigma) = \frac{p_{z}^{2}c^{2}}{2m_{e}} + (2n + 1 + \sigma \hbar \omega_{B}},$$

where $\hbar \omega_{B} = 2\mu_{e}B$, $\omega_{B} = eB/m_{e}c$ is the well-known non-relativistic electron cyclotron frequency, (c.f Page 460 of Quantum Mechanics (Landau & Lifshitz 1963)). In the interval $[p_{z}, p_{z} + dp_{z}]$ along the magnetic field, for a non-relativistic electron gas, the possible microstate numbers are calculated by $N_{ph}(p_{z}) = \frac{eB \ dp_{z}}{4\pi \hbar^{2}}$. Therefore, we obtain

$$N_{pha} = \int_{0}^{p_{F}} N_{pha}(p_{z})dp_{z} = \frac{eB \ E_{F}(e)}{4\pi \hbar^{2}} \frac{c^{2}}{c^{2}},$$

where the solution of Eq.(B1) is used (also c.f. Page 460 of Quantum Mechanics (Landau & Lifshitz 1963)). In the light of the Pauli exclusion principle, the electron number density should be equal to the its microstate density, one can gain the expression

$$N_{pha} = n_{e} = \frac{eB \ E_{F}(e)}{4\pi \hbar^{2}} \frac{c^{2}}{c^{2}} = N_{A} \rho Y_{e}.$$

From Eq.(B3), it is easy to see $E_{F}(e) \propto B^{-1}$ when $n_{e}$ is given. We then ask why such a phenomenon exists. After careful analysis, we find that the solution of the non-relativistic electron cyclotron motion equation $\hbar \omega_{B}$ is wrongly (or unsuitably) applied to calculate energy state density in a relativistic degenerate electron gas. It’s interesting to note that, in the Page 12 of Canuto & Chiu (1971), in order to evaluate the degeneracy of the $n$-th Landau level
ωn, they firstly introduce the cylindrical coordinates \((p_\perp, \phi)\) where \(\phi = \arctan \frac{p_x}{p_y}\), then gain an approximate relation

\[
\omega_n = (2\pi\hbar) \int_0^{2\pi} d\phi \int_{A<p_\perp^2<B} p_\perp dp_\perp = 2\pi(2\pi\hbar)^{-1}(B - A) = \frac{1}{2\pi}(\hbar/m_e c)^{-2} \cdot (B/B_{cr}), \tag{B4}
\]

where \(A = m^2 c^2 \frac{B_{cr}}{2n} 2n\) and \(B = m^2 c^2 \frac{B}{m_e 2(n+1)}\) \cite{Canuto & Chiu 1971}. In the paper, authors stated clearly that this relation is hold only when \(B = 0\), in other words, this relation is just an approximation in the case of weak magnetic field \((B \ll B_{cr})\). Surprisingly, this relation has been misused for nearly 40 years since then. Even in some textbooks on statistical physics, the statistical weight is calculated unanimously using the expression

\[
\frac{1}{\hbar^2} \int dp_x dp_y = \frac{1}{\hbar^2} \pi p_\perp^2 |_{n}^{n+1} = \frac{4\pi m_e \mu_e B}{\hbar^2}. \tag{B5}
\]

This expression will also cause the wrong deduction: \(E_F(e) \propto B^{-1}\), which is exactly the same as that from Eq.(B3). The sources of the wrong deduction lie in: no consideration of the conditions of Eq.(B4); the assumption that the torus located between the \(n\)-th Landau level and the \((n+1)\)-th Landau level in momentum space is ascribed to the \((n+1)\)-th Landau level. For the latter, the electron energy (or momentum) will change continuously in the direction perpendicular to the magnetic field, which is contradictory to the quantization of energy (or momentum) in the presence of superhigh magnetic field. Actually, electrons are relativistic and degenerate in the interior of a NS, so that Eq.(B1) and Eq.(B2) are no longer applicable, and need to be modified. Considering this, we replace Eq.(B1) and Eq.(B2) by Eq.(B6) and Eq.(B7), respectively:

\[
E^2_e(p_z, B, n, \sigma) = n^2_e c^4 + p_z^2 c^2 + (2n + 1 + \sigma)2m_e c^2 \mu_e B, \tag{B6}
\]

\[
N_{pha} = \frac{2\pi}{\hbar^3} \int dp_z \sum_{n=0}^{n_{m}(p_z, \sigma, B^*)} \sum_{g_n} \int \delta\left(\frac{p_\perp}{m_e c} - [(2n + 1 + \sigma)B^*]^{1/2}\right) p_\perp dp_\perp, \tag{B7}
\]

where \(\delta\left(\frac{p_\perp}{m_e c} - [(2n + 1 + \sigma)B^*]^{1/2}\right)\) is the Dirac \(\delta\)-function. Failing to do so would cause us to reach the wrong conclusion: \(E_F(e)\) decreases with the increase in \(B\) when in an intense field. (Cited partly from Gao Z. F et al 2010b.)