BIDIRECTED GRAPHS I: SIGNED GENERAL KOTZIG-LOVÁSZ
DECOMPOSITION

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ABSTRACT. This paper is the first from a series of papers that establish a common analogue of the strong component and basilica decompositions for bidirected graphs. A bidirected graph is a graph in which a sign + or − is assigned to each end of each edge, and therefore is a common generalization of digraphs and signed graphs. Unlike digraphs, the reachabilities between vertices by directed trails and paths are not equal in general bidirected graphs. In this paper, we set up an analogue of the strong connectivity theory for bidirected graphs regarding directed trails, motivated by factor theory. We define the new concepts of circular connectivity and circular components as generalizations of the strong connectivity and strong components. In our main theorem, we characterize the inner structure of each circular component; we define a certain binary relation between vertices in terms of the circular connectivity and prove that this relation is an equivalence relation. The nontrivial aspect of this structure arises from directed trails starting and ending with the same sign, and is therefore characteristic to bidirected graphs that are not digraphs. This structure can be considered as an analogue of the general Kotzig-Lovász decomposition, a known canonical decomposition in 1-factor theory. From our main theorem, we also obtain a new result in b-factor theory, namely, a b-factor analogue of the general Kotzig-Lovász decomposition.

1. INTRODUCTION

1.1. Background.

1.1.1. General Kotzig-Lovász decomposition for 1-Factors. The Kotzig-Lovász decomposition [7, 10, 15–19, 21] is a canonical decomposition in 1-matching theory [19]. Canonical decompositions of graphs are fundamental tools in 1-matching theory and have the distinction of being uniquely determined for each given graph. The classical Kotzig-Lovász decomposition [15–19, 21] is a decomposition for 1-factor connected graphs, a special class of graphs with 1-factors, and is known for producing many celebrated results, such as the two ear theorem and the tight cut lemma [19, 21]. A general graph with 1-factors can be considered as being built up by joining multiple 1-factor connected graphs with edges under a certain rule; each of these component graphs is called a 1-factor component. Comparably recently, the classical Kotzig-Lovász decomposition was generalized for arbitrary graphs with 1-factors [7, 10]; we call this the general Kotzig-Lovász decomposition or sometimes just the Kotzig-Lovász decomposition. The general Kotzig-Lovász decomposition is by itself a canonical decomposition and is also a piece of another more comprehensive canonical decomposition, the basilica decomposition [7, 9, 10]. As such, the general Kotzig-Lovász decomposition has produced new results in 1-matching.

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theory, such as a characterization of maximal barriers in general graphs \cite{8,11} and new proofs of Lovász’s cathedral theorem for saturated graphs \cite{12} and the tight cut lemma \cite{13}.

1.1.2. Bidirected Graphs. Bidirected graphs \cite{21} are a common generalization of digraphs and signed graphs. A bidirected graph is a graph in which a sign + or − is assigned to each end of each edge. A digraph is a special bidirected graph in which the ends of an edge have distinct signs. A signed graph is a graph in which a single sign is assigned to each edge, and can be considered as a bidirected graph in which the ends of each edge have the same sign. The concept of bidirected graphs is first proposed by Edmonds and Johnson \cite{3,21} to provide a general framework that integrates matchings and network flows. Various problems, such as capacitated nonsimple b-matchings, in which 1-matchings and simple b-matchings are included, capacitated b-edge covers, and minimum cost flows, are given a single unified formulation as an optimization problem over bidirected graphs. Bidirected graphs have also gained attention in the studies of nowhere-zero integral flows \cite{1,4,22} and totally unimodular matrices \cite{2,20}. Needless to say, numerous studies exist regarding digraphs and signed graphs; see, e.g., Schrijver \cite{21} or Zaslavsky \cite{23}.

1.2. Our Aim. We can naturally define a bidirected counterpart of directed paths and trails in digraphs: A ditrail in a bidirected graph is a trail such that, for each vertex term \( v \), the signs of \( v \) assigned with respect to \( e_1 \) and \( e_2 \) are distinct, where \( e_1 \) and \( e_2 \) are the edge terms immediately before and after \( v \), respectively. A dipath in a bidirected graph can be defined as a ditrail in which no vertex is contained twice or more.

We should however note that general bidirected graphs have the following two features that digraphs do not possess, which make the structure of bidirected graphs rich and complicated. First, general bidirected graphs have four types of dipaths or ditrails, whereas digraphs have only two types. That is, in digraphs, any dipath or ditrail clearly starts with − and ends with +, or vice versa. In contrast, in bidirected graphs, there can be dipaths and ditrails that start and end with − and − or + and +, in addition to those with + and − or − and +. Second, in bidirected graphs, even if two vertices are connected by a ditrail, it does not necessarily follow that these vertices are connected by a dipath. In digraphs, if there is a directed trail from a vertex \( u \) to a vertex \( v \), then clearly there is a directed path from \( u \) to \( v \); however, this property fails for general bidirected graphs.

Thus, in this paper, we initiate a bidirected analogue of the strong connectivity theory with respect to ditrails. Our motivation for considering ditrails comes from b-factor theory \cite{19,21}. Given a graph \( G \) and a mapping \( b \) from the vertex set onto the set of integers, a set of edges \( F \) is a \( b \)-factor if the number of edges adjacent to each vertex \( v \) is \( b(v) \). When discussing \( b \)-factors, we are often required to detect “alternating trails,” that is, a trail in a graph along which edges in \( F \) and not in \( F \) show up alternately, where \( F \) is a given \( b \)-factor. This task can be considered as a task of detecting ditrails in the signed graph generated from \( G \) by assigning − and + to each edge in \( F \) and not in \( F \), respectively. Because signed graphs are special bidirected graphs, the theory for ditrails in bidirected graphs has implications for \( b \)-factor theory.

1.3. Our Results in This Paper.
1.3.1. Main Theorem for Bidirected Graphs. We introduce the new concepts of circular connectivity and circular components of bidirected graphs as generalizations of the strong connectivity and strong components of digraphs. Just as a digraph is made up of its strong components and the edges joining them, a bidirected graph is made up of its circular component and the edges joining them.

In our main theorem, we obtain the inner structure of each circular component in a bidirected graph. This structure is stated by a certain equivalence relation over the vertex set; we first define a certain binary relation, $\leftrightarrow$, between vertices considering whether two vertices are connected by a ditrail that starts and ends with the same sign, and then prove that this relation is an equivalence relation, the quotient set of which, in fact, provides the inner structure of each circular component. The nontrivial aspect of this structure is characteristic to nondigraphic bidirected graphs. In bidirected graphs, the vertex set of each circular component can consist of any number of equivalence classes. In contrast, for digraphs, this structure is trivial in that each equivalence class coincides with the vertex set of a strong component. In our subsequent papers, we show that these equivalence classes can be considered as the fundamental units for considering the ditrail reachability between vertices.

1.3.2. Consequence for $b$-Factor Theory. This result for bidirected graphs contains an analogue of the general Kotzig-Lovász decomposition for $b$-factors. The original general Kotzig-Lovász decomposition for 1-factors is the quotient set of a certain equivalence relation $\sim$ defined over the vertex set. We define a $b$-matching analogue of this equivalence relation, $\sim_b$, and prove that $\sim_b$ is also an equivalence relation in the following way: Given a graph $G$ and a $b$-factor $M$, create a bidirected graph $G^M$ from $G$ by assigning the sign $-$ to every end of every edge in $M$ and the sign $+$ to every end of every edge not in $M$. It is easily observed that this relation $\sim_b$ of $G$ coincides with the relation $\leftrightarrow$ of $G^M$, and thus our main result for bidirected graphs immediately proves the claim. The counterpart of 1-factor components for $b$-matchings is the concept known as $b$-flexible components [14]. The $b$-flexible components of $G$ correspond to the circular components of the auxiliary bidirected graph $G^M$, and accordingly, this $b$-matching analogue of the general Kotzig-Lovász decomposition in fact provides the inner structure of each $b$-flexible component.

Under this result for $b$-factors, our result for bidirected graphs can be considered as a generalization of the general Kotzig-Lovász decomposition. Hence, we name this structure described by $\leftrightarrow$ the general Kotzig-Lovász decomposition for bidirected graphs.

Considering the relationship between the original Kotzig-Lovász decomposition and 1-factor theory, we can expect from this new result for $b$-factors further new consequences in $b$-factor theory. For example, the $b$-factor analogues of the two ear theorem and tight cut lemma and their further consequences might be derived.

1.4. Further Consequences. This paper is in fact the first from a series of papers that establish the circular connectivity theory of bidirected graphs [5,6]. Just as the strong component decomposition for digraphs tells us how an entire graph is structured from its strong components, how an entire bidirected graph is made up of its circular components will be revealed in our subsequent papers. As is also the case for digraphs, this entire structure can be stated in terms of a partial order.
between circular components. However, the structure will be again much richer and more complicated for bidirected graphs.

Here, our general Kotzig-Lovász decomposition for bidirected graphs, the inner structure of each circular component, will turn out to be related to the entire graph structure.

This whole theory of circular connectivity provides a bidirected analogue of the basilica decomposition mentioned in Section 1.1.1. The original basilica decomposition for 1-factors is a canonical decomposition applicable for any graph with 1-factors, and consists of three main concepts regarding 1-factor components: the general Kotzig-Lovász decomposition, which provides the inner structure of each 1-factor component; the basilica order, which is a partial order between 1-factor components; and the relationship between the two. In our circular connectivity theory, analogues of these concepts are provided regarding circular components.

This theory can be considered as a common generalization of the strong component decomposition and basilica decomposition. As is also the case in this paper, this theory derives a new result for $b$-factors, that is, the $b$-factor analogue of the basilica decomposition.

The general Kotzig-Lovász decomposition for bidirected graphs can be computed in polynomial time. The algorithm will be introduced in another paper by us [6].

1.5. Organization of Paper. The remainder of this paper is organized as follows. In Section 2 we explain basic notation and definitions used in this paper. In Section 3 we make some remarks about the disparities between digraphs and general bidirected graphs regarding dipaths and ditrails. In Section 4 we introduce the concept of circular connectivity of bidirected graphs, and show that this is a generalization of the strong connectivity of digraphs. In Section 5 we prove our main result, the analogue of general Kotzig-Lovász decomposition for bidirected graphs. In Section 6 we show that the result in Section 5 easily derives a $b$-factor analogue of a known result in 1-factor theory, i.e., the general Kotzig-Lovász decomposition.

2. Notation

2.1. Graphs. For basic notation, we mostly follow Schrijver [21]. In the following, we list exceptions or nonstandard definitions that are used. Let $G$ be an (undirected) graph. The vertex set and edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. Let $X \subseteq V(G)$. The set of edges joining $X$ and $V(G) \setminus X$ is denoted by $\delta_G(X)$. The subgraph of $G$ induced by $X$ is denoted by $G[X]$. The graph $G[V(G) \setminus X]$ is denoted by $G \setminus X$. As usual, we often denote a singleton $\{x\}$ by $x$.

Let $u, v \in V(G)$. A walk from $u$ to $v$ is a sequence $(s_1, \ldots, s_k)$, where $k$ is an odd number with $k \geq 1$, such that

(i) for each odd $i \in \{1, \ldots, k\}$, $s_i$ is a vertex of $G$, for which $s_1 = u$ and $s_k = v$, and

(ii) for each even $i \in \{1, \ldots, k\}$, $s_i$ is an edge of $G$ that connects $s_{i-1}$ and $s_{i+1}$.

Vertices and edges from a walk may not be distinct. A trail is a walk whose edges are all distinct. Let $W$ be a walk $(s_1, \ldots, s_k)$. We say that $W$ is closed if $s_1 = s_k$. For vertices $s_i$ and $s_j$ from $W$ with $i \leq j$, $s_iWs_j$ denote the walk $(s_i, \ldots, s_j)$. The reverse walk of $W$ is the sequence $(s_k, \ldots, s_1)$ and is denoted by $W^{-1}$. Let $U$ be another walk $(s_k, \ldots, s_l)$, where $k \leq l$. Then, $W + U$ denotes the walk $(s_1, \ldots, s_k, \ldots, s_l)$, namely, the concatenation of $W$ and $U$. 
2.2. Bidirected Graphs. A bidirected graph is a graph in which each end of each edge is assigned a sign $+$ or $-$. The precise definition is as follows: A bidirected graph is a graph $G$ endowed with two mappings $\partial_+$ and $\partial_-$ over $E(G)$ such that, for each $e \in E(G)$ with ends $u$ and $v$,

(i) $\partial_+(e)$ and $\partial_-(e)$ are subsets of $\{u, v\}$ one of which can be empty,
(ii) $\partial_+(e) \cup \partial_-(e) = \{u, v\}$, and
(iii) if $u \neq v$ then $\partial_+(e) \cap \partial_-(e) = \emptyset$.

We say that $a \in \{+, -\}$ is a sign of $u \in V(G)$ over $e \in E(G)$ if $u \in \partial_a(e)$ holds. Bidirected graphs are a generalization of (usual) digraphs. A digraph is a bidirected graph in which each end of each edge is assigned a sign $+$ or $-$. The precise definition is as follows: A digraph is a graph $G$ endowed with two mappings $\partial_+$ and $\partial_-$ over $E(G)$ such that, for each $e \in E(G)$ with ends $u$ and $v$,

(i) $\partial_+(e)$ and $\partial_-(e)$ are subsets of $\{u, v\}$ one of which can be empty,
(ii) $\partial_+(e) \cup \partial_-(e) = \{u, v\}$, and
(iii) if $u \neq v$ then $\partial_+(e) \cap \partial_-(e) = \emptyset$.

A dipath is a closed $(\alpha, \beta)$-ditrail, where $\alpha, \beta \in \{+, -\}$, if $k > 1$ holds, and $W$ is a ditrail in which the sign of $s_1$ over $s_2$ is $\alpha$, whereas the sign of $s_k$ over $s_{k-1}$ is $\beta$. If $k = 1$, we also define $W$ as an $(\alpha, \beta)$-ditrail for any $\alpha, \beta \in \{+, -\}$ with $\alpha \neq \beta$. Note that if $W$ is an $(\alpha, \beta)$-ditrail, then $W^{-1}$ is a $(\beta, \alpha)$-ditrail. We call $W$ a cyclic ditrail if $W$ is a closed $(\alpha, -\alpha)$-ditrail for some $\alpha \in \{+, -\}$. We call a ditrail a dipath if no vertex is contained more than once.

3. Digraphs Versus Bidirected Graphs

In this section, we make some remarks comparing digraphs and nondigraphic bidirected graphs regarding ditrails or dipaths. Note the following observations for digraphic bidirected graphs.

Observation 3.1. Let $G$ be a digraphic bidirected graph, and let $\alpha, \beta \in \{+, -\}$. Then, for any $s, t \in V(G)$, there is an $(\alpha, \beta)$-ditrail from $s$ to $t$ if and only if there is an $(\alpha, \beta)$-dipath from $s$ to $t$.

That is, in considering the strong connectivity of digraphs, we are not required to distinguish between ditrails and dipaths. However, for general bidirected graphs, the connectivities by ditrails and by dipaths are distinct; clearly, two vertices connected by $(\alpha, \beta)$-ditrails do not necessarily imply that they are connected by $(\alpha, \beta)$-dipaths. In this paper, we study the connectivity of bidirected graphs by ditrails. As we will see in Section 4, this study has implications for factor theory.

Observation 3.2. In a digraphic bidirected graph, any ditrail is a $(-, +)$- or $(+, -)$-ditrail.

That is, in the strong connectivity theory of digraphs, we need only consider $(-, +)$-ditrails between two vertices. In contrast, in general digraphs, there are also $(-, -)$- and $(+, +)$-ditrails. This variety is peculiar to nondigraphic bidirected graphs and allows for a new structure that we introduce in the following sections.

4. Circularly Connected Components of Bidirected Graph

We now introduce the new concept of circular connectivity in bidirected graphs and prove that this is a generalization of the strong connectivity in digraphs. In this section, unless stated otherwise, let $G$ be a bidirected graph with respect to ditrails.
**Definition 4.1.** We say that an edge $e \in E(G)$ is *circular* if there is a cyclic ditrail that contains $e$. We say that vertices $u$ and $v$ are *circularly connected* if there is a path between $u$ and $v$ whose edges are all circular. A bidirected graph is *circularly connected* if every two vertices are circularly connected. A circularly connected component or *circular component* of $G$ is a maximal circularly connected subgraph of $G$.

An alternative way to define a circular component of $G$ is as follows: Let $F \subseteq E(G)$ be the set of circular edges of $G$. A circular component is a subgraph of the form $G[V(C)]$, where $C$ is a connected component of the subgraph of $G$ determined by $F$. A bidirected graph consists of its circular components, which are disjoint, and edges joining distinct circular components, which are not circular.

The circular connectivity of bidirected graphs is a generalization of the strong connectivity of digraphs. In a digraph, two vertices $u$ and $v$ are *strongly connected* if there are $(-, +)$-dipaths from $u$ to $v$ and from $v$ to $u$. A maximal strongly connected subgraph is called a *strongly connected component* or *strong component*. The next statement is a basic fact.

**Fact 4.2** (see, e.g., Schrijver [21]). No two distinct strong components share vertices. Accordingly, a digraph is made up of its strong components, which are mutually disjoint, and the edges joining them.

Under Observation 3.1 two vertices $u$ and $v$ are strongly connected if and only if there are $(-, +)$-ditrails from $u$ to $v$ and from $v$ to $u$. Accordingly, an edge $uv$ is circular if and only if $u$ and $v$ are strongly connected.

This further implies the following statement.

**Observation 4.3.** Let $G$ be a digraphic bidirected graph. Let $u, v \in V(G)$. Then, $u$ and $v$ are strongly connected if and only if $u$ and $v$ are circularly connected.

That is, the circular connectivity of bidirected graphs is a generalization of the strong connectivity of digraphs, and circular components are a generalization of strong components.

5. **General Kotzig-Lovász Decomposition for Bidirected Graphs**

In this section, we prove the main result of this paper: the analogue of the general Kotzig-Lovász decomposition for bidirected graphs. In this section, unless stated otherwise, let $G$ be a bidirected graph and let $\alpha \in \{+, -\}$. In the following, we define a binary relation $\overset{\alpha}{\leftrightarrow}$ and then prove in Theorem 5.4 that this relation is an equivalence relation, the quotient set of which in fact provides the inner structure of each circular component.

**Definition 5.1.** Define a binary relation $\overset{\alpha}{\leftrightarrow}$ over $V(G)$ as follows: For $u, v \in V(G)$, we let $u \overset{\alpha}{\leftrightarrow} v$ if $u$ and $v$ are identical or if $u$ and $v$ are circularly connected and there is no $(\alpha, \alpha)$-ditrail between $u$ and $v$.

We prove in the following that $\overset{\alpha}{\leftrightarrow}$ is an equivalence relation.

**Proposition 5.2.** Let $G$ be a circularly connected bidirected graph, let $\alpha \in \{+, -\}$, and let $s \in V(G)$. Then, for any $t \in V(G)$, there exists $\beta \in \{+, -\}$ such that $G$ has an $(\alpha, \beta)$-ditrail from $s$ to $t$. 
Proof. Define $S \subseteq V(G)$ as follows: Let $x \in S$ if there exists $\beta \in \{+,-\}$ such that $G$ has an $(\alpha, \beta)$-ditrail from $s$ to $x$. We prove $S = V(G)$ in the following. Suppose, to the contrary, $S \subsetneq V(G)$. Because $G$ is circularly connected, there is a circular edge $uv \in \delta_G(S)$, where $u \in S$ and $v \in V(G) \setminus S$. Because $u \in S$ holds, there is an $(\alpha, \beta)$-ditrail $P$ from $s$ to $u$, where $\beta$ is either $+$ or $-$. Because $v \not\in S$ holds, the vertex $v$ is not contained in $P$, and therefore $P + (u, uv, v)$ is a trail from $s$ to $v$. This further implies that the sign of $u$ over the edge $uv$ is $\beta$. Let $\gamma$ be the sign of $v$ over $uv$. Because $uv$ is circular, there is a $(-\beta, -\gamma)$-ditrail $Q$ from $u$ and $v$. If $P$ and $Q$ do not share an edge, then $P + Q$ is an $(\alpha, -\gamma)$-ditrail from $s$ to $v$, which implies $v \in S$; this is a contradiction. Hence, consider now the case in which $P$ and $Q$ share edges. Trace $P$ from $s$, and let $wz$ be the first encountered edge shared by $E(Q)$. Without loss of generality, let $w$, $wz$, and $z$ appear in this order over $P$. First, assume that $w$, $wz$, and $z$ also appear in this order over $Q$. Then, $sPz + zQv$ is an $(\alpha, -\gamma)$-ditrail from $s$ to $v$, that is, $v \in S$ holds, which is a contradiction. Assume now that $z$, $zw$, and $w$ appear in this order over $Q$. Then, $sPz + zQ^{-1}u$ is an $(\alpha, -\beta)$-ditrail from $s$ to $u$. Therefore, $sPz + zQ^{-1}u + (u, uv, v)$ is an $(\alpha, \gamma)$-ditrail from $s$ to $v$, which is again a contradiction. This completes the proof. \[ \]

Remark 5.3. The value of $\beta$ in Proposition 5.2 is not exclusive. That is, for two vertices $s$ and $t$, there may be both $(\alpha, +)$- and $(\alpha, -)$-ditrails from $s$ to $t$.

Theorem 5.4. Let $G$ be a bidirected graph, and let $\alpha \in \{+, -\}$. Then, $\overset{\alpha}{\leftrightarrow}$ is an equivalence relation over $V(G)$.

Proof. Reflexivity and symmetry are obvious from the definition. Let $u, v, w \in V(G)$ be vertices with $u \overset{\alpha}{\leftrightarrow} v$ and $v \overset{\alpha}{\leftrightarrow} w$. We prove $u \overset{\alpha}{\leftrightarrow} w$ in the following. If any two from $u, v, w$ are identical, the statement obviously holds. Hence, assume that these are mutually distinct. It is obvious that $u, v, w$ are mutually circularly connected. Suppose $u \overset{\alpha}{\leftrightarrow} w$ does not hold, and let $P$ be an $(\alpha, \alpha)$-ditrail from $u$ to $w$. From Proposition 5.2 there is an $(\alpha, -\alpha)$-ditrail $Q$ from $v$ to $u$. Trace $Q$ from $v$, and let $x$ be the first encountered vertex in $P$. Then, either $vQx + xPuv$ or $vQx + xP^{-1}u$ is an $(\alpha, \alpha)$-ditrail from $v$ to $w$ or $u$, respectively. This contradicts either $u \overset{\alpha}{\leftrightarrow} v$ or $v \overset{\alpha}{\leftrightarrow} w$. Therefore, $u \overset{\alpha}{\leftrightarrow} w$ is proved. \[ \]

For each $\alpha \in \{+, -\}$, we denote as $\mathcal{P}^\alpha(G)$ the family of equivalence classes of $\overset{\alpha}{\leftrightarrow}$, and call this family the general Kotzig-Lovász decomposition or simply the Kotzig-Lovász decomposition of the bidirected graph $G$ regarding the sign $\alpha$.

Let $H$ be a circular component of $G$. From the definition of the equivalence relation, the family $\{S \in \mathcal{P}^\alpha(G) : S \subseteq V(H)\}$ forms a partition of $V(H)$. Therefore, this decomposition of a bidirected graph can be considered as providing the inner structure of each circular component. If $G$ is digraphic, then $\{S \in \mathcal{P}^\alpha(G) : S \subseteq V(H)\}$ coincides with $\{V(H)\}$. The nontrivial aspects of the general Kotzig-Lovász decomposition are characteristic to bidirected graphs that are not digraphs.

Note that this inner structure of circular components is determined in the context of the entire bidirected graph; recall that the definition of $\overset{\alpha}{\leftrightarrow}$ is given considering the entire $G$. It is easily confirmed that the family $\{S \in \mathcal{P}^\alpha(G) : S \subseteq V(H)\}$ is not equal to $\mathcal{P}^\alpha(H)$ in general but is a refinement of $\mathcal{P}^\alpha(H)$.

6. Consequences for $b$-Factor Theory
6.1. Definitions regarding $b$-Factors. Let $G$ be a graph, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. A set of edges $M \subseteq E(G)$ is a $b$-matching if $|\delta_G(v) \cap M| \leq b(v)$ holds for each $v \in V(G)$. A $b$-matching is maximum if it has the maximum number of edges. A $b$-matching $M$ is perfect if $|\delta_G(v) \cap M| = b(v)$ holds for each $v \in V(G)$. A perfect $b$-matching is also known as a $b$-factor. A $b$-factor is a maximum $b$-matching, however the converse does not necessarily hold. We say that $G$ is $b$-factorizable if it has a $b$-factor. We denote $b$ by $1$ if $b(v)$ is $1$ for every $v \in V(G)$. Thus, 1-matchings and 1-factors are the most fundamental concepts in matching theory, also known as matchings or perfect matchings.

Now, let $G$ be $b$-factorizable. An edge $e \in E(G)$ is $b$-allowed if $G$ has a $b$-factor that contains $e$; otherwise, we say that $e$ is $b$-forbidden.

A $b$-allowed edge is $b$-flexible if $G$ also has a $b$-factor that does not contain $e$; otherwise, we say that $e$ is $b$-essential.

For a subgraph $H$, $b|_H$ denotes the mapping $V(H) \to \mathbb{Z}_{\geq 0}$ such that $b|_H(v) := b(v) - k$ for each $v \in V(H)$, where $k$ denotes the number of $b$-essential edges that connect $v$ and $V(G) \setminus V(H)$. For twoappings $b_1, b_2 : V(G) \to \mathbb{Z}_{\geq 0}$, $b_1 + b_2$ and $b_1 - b_2$ denote the mappings $V(G) \to \mathbb{Z}$ such that $(b_1 + b_2)(v) = b_1(v) + b_2(v)$ and $(b_1 - b_2)(v) = b_1(v) - b_2(v)$ for each $v \in V(G)$. As usual, we utilize the associativity of these operations over mappings. Given a vertex $u \in V(G)$, $\chi_u$ denotes the mapping $V(G) \to \mathbb{Z}_{\geq 0}$ such that $\chi_u = 1$ and $\chi_v = 0$ for each $v \in V(G) \setminus \{u\}$.

Vertices $u, v \in V(G)$ are $b$-flexibly connected (resp. $b$-factor connected) if there is a path between $u$ and $v$ in which every edge is $b$-flexible (resp. $b$-allowed). A subgraph $H$ of $G$ is $b$-flexibly connected (resp. $b$-factor connected) if any two vertices in $H$ are $b$-flexibly connected (resp. $b$-factor connected) in $G$. A maximal $b$-flexibly connected subgraph is called a $b$-flexibly connected component or $b$-flexible component (resp. a $b$-factor connected component or $b$-factor component) of $G$. The graph $G$ consists of its $b$-flexible components, which are disjoint, and edges joining distinct $b$-flexible components, which are $b$-forbidden or $b$-essential. A $b$-factor component consists of some $b$-flexible components and edges joining them.

A set $M \subseteq E(G)$ is a $b$-factor if and only if it is the union of the set of $b$-essential edges and a set $M^* \subseteq E(G)$ of the form $M^* = \bigcup \{M_C : C \in \mathcal{G}(G, b)\}$, where $M_C$ is a $b|_C$-factor of $C \in \mathcal{G}(G, b)$. The $b$-flexible components form the most fine-grained set of subgraphs that satisfies this property. Hence, $b$-flexible components can be considered as the fundamental units for $b$-factors.

A similar property can be stated for $b$-factor components; that is, a set $M \subseteq E(G)$ is a $b$-factor if and only if it is a union of a $b|_C$-factor, where $C$ is taken over every $b$-factor component. In 1-factor theory, 1-factor components have been used as fundamental units of a graph. For example, the Dulmage-Mendelsohn decomposition theory, a classical canonical decomposition for bipartite graphs, relates to a poset over 1-factor components. More examples can be found in the basilica decomposition theory and the polyhedral studies of 1-factors.

6.2. General Kotzig-Lovász Decomposition for $b$-Factors. We now show that Theorem 6.3 implies the analogy of general Kotzig-Lovász decomposition for $b$-factors. In this section, unless stated otherwise, let $G$ be a $b$-factorizable graph, where $b : V(G) \to \mathbb{Z}_{\geq 0}$. We define a binary relation $\sim_b$ that is uniquely determined for $G$ and $b$. This relation is proved to be an equivalence relation by generating an auxiliary bidirected graph $G^M$ from $G$ and some $b$-factor $M$ and then applying Theorem 6.3 for $G^M$. 

Definition 6.1. We define binary relations \( \sim_b \) and \( \not\sim_b \) over \( V(G) \) as follows: For \( u, v \in V(G) \), let \( u \sim_b v \) (resp. \( u \not\sim_b v \)) if \( u \) and \( v \) are identical or if \( u \) and \( v \) are \( b \)-flexibly connected and \( G \) does not have a \( b - \chi_u - \chi_v \)-factor (resp. \( b + \chi_u + \chi_v \)-factor).

In the following, we show that \( \sim_b \) and \( \not\sim_b \) are equivalence relations.

Definition 6.2. Let \( M \subseteq E(G) \). We denote by \( G^M \) the bidirected graph obtained by endowing mappings \( \partial_+ \) and \( \partial_- \) as follows: for each \( e \in E(G) \) with ends \( u, v \in V(G) \), let \( \partial_+(e) = \emptyset \) and \( \partial_-(e) = \{u, v\} \) if \( e \) is an edge from \( M \); otherwise, let \( \partial_+(e) = \{u, v\} \) and \( \partial_-(e) = \emptyset \).

The next lemma is easily confirmed from classical observations regarding \( b \)-matchings.

Lemma 6.3. Let \( G \) be a \( b \)-factorizable graph, where \( b : V(G) \to \mathbb{Z}_{\geq 0} \), and let \( M \) be a \( b \)-factor of \( G \). Then, for each \( u, v \in V(G) \), \( u \sim_b v \) holds in \( G \) if and only if \( u \not\rightarrow v \) holds in \( G^M \).

Under Lemma 6.3, Theorem 5.4 thus derives the next theorem.

Theorem 6.4. Let \( G \) be a \( b \)-factorizable graph, where \( b : V(G) \to \mathbb{Z}_{\geq 0} \). Then, \( \sim_b \) and \( \not\sim_b \) are equivalence relations over \( V(G) \).

For each \( \alpha \in \{+, -\} \), we denote as \( \mathcal{P}^\alpha(G, b) \) the family of equivalence classes of \( \sim_b \). We call \( \mathcal{P}^+(G, b) \) and \( \mathcal{P}^-(G, b) \) the general Kotzig-Lovász decompositions or simply the Kotzig-Lovász decompositions of the \( b \)-factorizable graph \( G \) by addition and subtraction, respectively.

For each \( b \)-flexible component \( H \), the family \( \{S \in \mathcal{P}^\alpha(G, b) : S \subseteq V(H)\} \) is a partition of \( V(H) \), and thus \( \mathcal{P}^\alpha(G, b) \) can be regarded as providing the inner canonical structure of each \( b \)-flexible component. If \( b = 1 \) and \( \alpha = - \), then this family provides the nontrivial aspect of the known general Kotzig-Lovász decomposition for 1-factorizable graphs.

As is also the case in bidirected graphs, this inner structure \( \{S \in \mathcal{P}^\alpha(G, b) : S \subseteq V(H)\} \) is determined in the context of the entire graph \( G \) and is a refinement of \( \mathcal{P}^\alpha(H, b | H) \).

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