MINIMAL MODEL THEOREM FOR TORIC DIVISORS

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ABSTRACT. Minimal model conjecture for a proper variety $X$ is that if $\kappa(X) \geq 0$, then $X$ has a minimal model with the abundance and if $\kappa = -\infty$, then $X$ is birationally equivalent to a variety $Y$ which has a fibration $Y \to Z$ with $-K_Y$ relatively ample. In this paper, we prove this conjecture for a $\Delta$-regular divisor on a proper toric variety by means of successive contractions of extremal rays and flips of ambient toric variety. Furthermore, for such a divisor $X$ with $\kappa(X) \geq 0$ we construct a projective minimal model with the abundance in a different way; by means of "puffing up" of the polytope, which gives an algorithm of a construction of a minimal model.

0. Introduction

Let $k$ be an algebraically closed field of arbitrary characteristic. Varieties in this paper are all defined over $k$. Let $X$ be a proper algebraic variety. A proper algebraic variety $Y$ is called a minimal model of $X$, if (1) $Y$ is birationally equivalent to $X$, (2) $Y$ has at worst terminal singularities and (3) the canonical divisor $K_Y$ is nef. A minimal model $Y$ is said to have the abundance if the linear system $|mK_Y|$ is basepoint free for sufficiently large $m$. The minimal model conjecture states: an arbitrary proper variety with $\kappa \geq 0$ has a minimal model with the abundance and an arbitrary proper variety with $\kappa = -\infty$ has a birationally equivalent model $Y$ with at worst terminal singularities and a fibration $Y \to Z$ to a lower dimensional variety with $-K_Y$ relatively ample.

The conjecture holds true for 2-dimensional case which is known as a classical result. For 3-dimensional case the conjecture for $k = \mathbb{C}$ is proved by Mori [4] and Kawamata [3], while it is not yet proved for higher dimensional case. As a special case of higher dimension, Batyrev [1] proved, among other results, the existence of a minimal model for a $\Delta$-regular anti-canonical divisor of a Gorenstein Fano toric variety $T_N(\Delta)$. In this paper we prove the minimal model conjecture for every $\Delta$-regular divisor $X$ on a toric variety of arbitrary dimension by means of successive contractions of extremal rays and flips. Furthermore for such a divisor with $\kappa \geq 0$, we construct a projective minimal model with the abundance in a different way; by means of "puffing up" of the polytope corresponding to the adjoint divisor. By this method one can concretely construct a projective minimal model. As a corollary,
for a field \( k \) of characteristic 0, the minimal model conjecture holds for a general member of a basepoint free linear system on a proper toric variety over \( k \). The half of this work was done during the author’s stay at the Johns Hopkins University on April 1996. She expresses her gratitude to Professors Shokurov and Kawamata who made her stay possible. She is also grateful to the Johns Hopkins University for their hospitality. She would like to thank Professor Reid who gave useful suggestions and Professor Batyrev who called her attention to this problem and pointed out an error of the first draft of this paper.

1. The minimal model theorem for toric divisors

Definition 1.1. ([1]) A divisor \( X \) of a toric variety \( T_N(\Delta) \) defined by a fan \( \Delta \) is called \( \Delta \)-regular, if for every \( \tau \in \Delta \) the intersection \( X \cap \text{orb}(\tau) \) is either a smooth divisor of \( \text{orb}(\tau) \) or empty.

Definition 1.2. Let \( V \) and \( V' \) are toric varieties defined by fans \( \Delta \) and \( \Delta' \) respectively and \( f : V' \to V \) a toric birational map: i.e. \( \Delta' \) is obtained by successive subdivisions and converse of subdivisions from \( \Delta \). Let \( T \) be the maximal orbit in \( V \). If an irreducible divisor \( X \) on \( V \) satisfies \( X \cap T \neq \phi \), the divisor \( X' = f^{-1}(X \cap T) \) on \( V' \) is called the proper transform of \( X \) on \( V' \).

Definition 1.3. Let \( X \) a divisor on a normal variety \( V \) such that \( K_V + X \) is a \( \mathbb{Q} \)-Cartier divisor and \( f : V' \to V \) a birational morphism. Let \( X' \) be the proper transform of \( X \). If
\[
K_{V'} + X' = f^*(K_V + X) + \sum_i a_i E_i,
\]
where \( E_i \)'s are the exceptional divisors of \( f \), then \( a_i \) is called the discrepancy of \( K_V + X \) at \( E_i \).

Definition 1.4. Let \( V \) be a toric variety defined by a simplicial fan \( \Delta \) and \( X \) an irreducible divisor on \( V \). The divisor \( K_V + X \) is called terminal, if the following hold:

(1) there exists a morphism \( f : V' = T_N(\Delta') \to V \) corresponding to a non-singular subdivision \( \Delta' \) of \( \Delta \) (\( \Delta' \neq \Delta \)) such that the proper transform \( X' \) of \( X \) on \( V' \) is \( \Delta' \)-regular, in particular \( X \cap T \neq \phi \) for the maximal orbit \( T \) in \( V \), and

(2) for every such morphism as in (1) the discrepancy of \( K_V + X \) at every exceptional divisor on \( V' \) is positive.

Lemma 1.5. If \( V = T_N(\Delta) \) is non-singular and an irreducible divisor \( X \) on \( V \) is \( \Delta \)-regular, then \( K_V + X \) is terminal

Proof. For every non-singular subdivision \( \Delta' \) of \( \Delta \), where \( \Delta' \neq \Delta \), the proper transform \( X' \) of \( X \) by the corresponding morphism \( f : V' = T_N(\Delta') \to V \) is \( \Delta' \)-regular by 3.2.1 of [1]. Since \( X' = f^*X \) and \( K_{V'} = f^*K_V + \sum_i a_i E_i \), where \( a_i > 0 \) for every exceptional divisor \( E_i \) on \( V' \), it follows that the discrepancy of \( K_V + X \) at each \( E_i \) is positive. \( \Box \)
Proposition 1.6. Let $V$ be a toric variety defined by a simplicial fan $\Delta$ and $X$ an irreducible divisor on $V$. Then the divisor $K_V + X$ is terminal if and only if the following hold:

(i) there exists a morphism $f : V' = T_N(\Delta') \to V$ corresponding to a non-singular subdivision $\Delta'$ of $\Delta$ ($\Delta' \neq \Delta$) such that the proper transform $X'$ of $X$ on $V'$ is $\Delta'$-regular.

(ii) for one such morphism as in (i) the discrepancy of $K_V + X$ at every exceptional divisor on $V'$ is positive.

Proof. Let $f : V' = T_N(\Delta') \to V$ be the morphism satisfying the condition (i) and (ii) and $g : V'' \to V$ be another morphism satisfying (i). Take a nonsingular toric variety $\tilde{V}$ which dominates both $V'$ and $V''$. Then by 1.5, $K_{V'} + X'$ is terminal. Therefore the discrepancy of $K_V + X$ at every exceptional divisor on $\tilde{V}$ is positive which yields the positivity of it at every exceptional divisor on $V''$. \qed

Lemma 1.7. Let $V$ be a toric variety defined by a simplicial fan $\Delta$ and $X$ an irreducible divisor on $V$. If the divisor $K_V + X$ is terminal, then $V$ has at worst terminal singularities.

Proof. This follows from the fact that a discrepancy of $K_V$ is greater than or equal to that of $K_V + X$. \qed

Here we summerize the results of Reid ([7]) which are used in this section.

Proposition 1.8. ([7]) Let $V$ be the toric variety defined by a proper simplicial fan $\Delta$.

(i) $NE(V) = \sum_{i=1}^r \mathbb{R}_{\geq 0}[\ell_i]$, where $\ell_i$'s are 1-dimensional strata on $V$. Here each $\mathbb{R}_{\geq 0}[\ell_i]$ is called an extremal ray.

(ii) For every extremal ray $R$ there exist a toric morphism $\varphi_R : V \to V'$ which is an elementary contraction in the sense of Mori theory: $\varphi_R \mathcal{O}_V = \mathcal{O}_{V'}$ and $\varphi_R C = pt$ if and only if $[C] \in R$. Let $A \subset V$ and $B \subset V'$ be the loci on which $\varphi_R$ is not an isomorphism, then $\varphi_R|_A : A \to B$ is a flat morphism and all of whose fibers are weighted projective spaces of the common dimension.

(iii) If $\varphi_R : V \to V' = T_N(\Delta')$ is birational and not isomorphic in codimension one, then the exceptional set of $\varphi_R$ is an irreducible divisor and $\Delta'$ is proper simplicial. Here this $\varphi_R$ is called a divisorial contraction.

(iv) If $\varphi_R : V \to V' = T_N(\Delta')$ is isomorphic in codimension one, then there exists the following commutative diagram:

\begin{equation*}
\begin{array}{c}
\tilde{V} \\
\psi \downarrow \\
V \\
\varphi_R \downarrow \\
V' \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{c}
V_1 = T_N(\Delta_1) \\
\psi_1 \downarrow \\
V_1 \\
\end{array}
\end{equation*}

Here $\phi_R$ is a morphism.
such that $\Delta_1$ is proper simplicial, $\Delta_1(1) = \Delta(1)$, all morphisms are elementary contractions of extremal rays, $\psi$ and $\psi_1$ are birational morphisms with the exceptional divisor $D$, $\varphi_R$ and $\varphi_1$ are birational morphisms with the exceptional sets $\psi(D)$ and $\psi_1(D)$ respectively, and identifying $N_1(V)$ and $N_1(V_1)$, $-R$ is an extremal ray in $NE(V_1)$ and $\varphi_1 = \varphi_{-R}$. Here the birational map $\varphi_1^{-1} \circ \varphi_R : V \to V_1$ is called a flip.

**Lemma 1.9.** Let $V$ be a toric variety defined by a proper simplicial fan $\Delta$ and $X$ an irreducible divisor such that $K_V + X$ is terminal. Let $R$ be an extremal ray such that $(K_V + X)R < 0$. Then the following hold:

(i) if $\varphi_R : V \to V'$ is a divisorial contraction, then $K_{V'} + X'$ is terminal, where $X'$ is the proper transform of $X$ on $V'$;

(ii) let $\varphi_R : V \to V'$ be isomorphic in codimension one; for the diagram

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\psi} & V_1 = T_N(\Delta_1) \\
V & \downarrow \psi_1 & \\
\varphi_R & \downarrow & \varphi_1 \\
& V' & \\
\end{array}
\]

of (iv), 1.8, let $X_1$ be the proper transform of $X$ on $V_1$, $D$ the exceptional divisor of $\psi$ and $\psi_1$, $\alpha$ the discrepancy of $K_V + X$ at $D$ and $\alpha'$ the discrepancy of $K_{V_1} + X_1$ at $D$; then $\alpha < \alpha'$ and $K_{V_1} + X_1$ is terminal.

**Proof.** For the proof of (i), first one should remark that $V'$ is $\mathbb{Q}$-factorial, because $\Delta'$ is simplicial. Let $E$ be the exceptional divisor for $\varphi_R$.

**Claim 1.10.** $ER < 0$.

For the proof of the claim, take an irreducible divisor $H$ on $V'$ such that $H \supset \varphi_R(E)$. Then $\varphi^*H = [H] + aE$ with $a > 0$, where $[H]$ is the proper transform of $H$ on $V$. Since $(\varphi_R^*H)R = 0$ and $[H]R > 0$, it follows that $aER < 0$ which completes the proof of the claim.

Denote $K_V + X$ by $\varphi_R(K_{V'} + X') + bE$, then $b > 0$. In fact, by $(K_V + X)R < 0$, $\varphi_R^*(K_{V'} + X')R = 0$ and $ER < 0$, it follows that $b > 0$. Let $\overline{\Delta}$ be a non-singular subdivision of $\Delta$ such that the proper transform $\overline{X}$ of $X$ on $\overline{V} = T_N(\overline{\Delta})$ is $\overline{\Delta}$-regular. Since $K_V + X$ is terminal, the discrepancy of $K_V + X$ at every exceptional divisor for $\overline{V} \to V$ is positive. By this, and $b > 0$, it follows that the discrepancy of $K_{V'} + X'$ at every exceptional divisor for $\overline{V} \to V'$ is positive. For the proof of (ii), take a curve $\ell$ on $\tilde{V}$ such that $\psi_1(\ell) = pt$ and $\psi(\ell) \neq pt$. This is possible, because if a curve contracted by both $\psi$ and $\psi_1$ exists, then the extremal rays corresponding to $\psi$ and $\psi_1$ coincide which implies $V \simeq V_1$ and $\varphi_R = \varphi_1$ a contradiction to $\varphi_1 = \varphi_{-R}$ in (iv) of 1.8. For this $\ell$, one can prove that $D\ell < 0$ in the same way as in the claim above. Now as $\psi_1(\ell)$ is
contracted to a point by \( \varphi_R, [\psi_*(\ell)] \in R \), therefore \( \psi^*(K_V + X)\ell = (K_V + X)\psi_*(\ell) < 0 \). By intersecting \( \ell \) with \( K_V + X = \psi^*(K_V + X) + \alpha D \), we obtain

\[
(K_V + X)\ell < \alpha D\ell.
\]

Here the left hand side is \( \psi_1^*(K_{V_1} + X_1)\ell + \alpha' D\ell \), and \( \psi_1^*(K_{V_1} + X_1)\ell = 0 \) because of the definition of \( \ell \). This proves that \( \alpha < \alpha' \). To prove the last statement, take a non-singular subdivision \( \tilde{\Delta} \) of \( \Delta \) such that the proper transform \( \tilde{X} \) of \( X \) is \( \tilde{\Delta} \)-regular. Let \( \lambda : \tilde{V} = T_N(\tilde{\Delta}) \rightarrow \tilde{V} \) be the corresponding morphism. Then \( K_{\tilde{V}} + \tilde{X} = \lambda^*\psi^*(K_V + X) + \sum_i \beta_i E_i \), where \( \beta_i > 0 \) for every exceptional divisor \( E_i \), because \( K_V + X \) is terminal. Now by substituting \( \psi^*(K_V + X) = \psi_1^*(K_{V_1} + X_1) + (\alpha' - \alpha)D \) into the equality above, the discrepancy of \( K_{V_1} + X_1 \) at every exceptional divisor on \( \tilde{V} \) turns out to be positive. \( \square \)

**Theorem 1.11.** Let \( V \) be a toric variety defined by a proper simplicial fan \( \Delta \) and \( X \) an irreducible divisor on \( V \) such that \( K_V + X \) is terminal. Then there exists a sequence of birational toric maps:

\[
V = V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{r-1}} V_r
\]

where

(i) each \( \varphi_i \) is either a divisorial contraction or a flip, in particular \( V_i \) is defined by a proper simplicial fan;

(ii) for the proper transform \( X_i \) of \( X \) on \( V_i \) \( (i = 1, \ldots, r) \), \( K_{V_i} + X_i \) is terminal;

(iii) either that \( K_{V_r} + X_r \) is nef or that there exists an extremal ray \( R \) on \( V_r \) such that \( (K_{V_r} + X_r)R < 0 \) and the elementary contraction \( \varphi_R : V_r \rightarrow Z \) is a fibration to a lower dimensional variety \( Z \).

**Proof.** If \( K_V + X \) is nef, then the statement is obvious. If \( K_V + X \) is not nef, then there is an extremal ray \( R \) such that \( (K_V + X)R < 0 \). Take the elementary contraction \( \varphi_R : V \rightarrow V' \). If \( \dim V' < \dim V \), then the statement holds. So assume that \( \varphi_R \) is birational. If \( \varphi_R \) is divisorial, then define \( \varphi_1 := \varphi_R : V \rightarrow V' =: V_2 \). If \( \varphi_R \) is not divisorial, then let \( \varphi_1 : V \rightarrow V_2 \) be the flip. Then in both cases, \( K_{V_2} + X_2 \) is terminal by Lemma 1.9. Now if \( K_{V_2} + X_2 \) is nef, then the proof is completed. If it is not nef, make the same procedure as above. By the successive procedure, one obtains a sequence of divisorial contractions and flips:

\[
V = V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{r-1}} V_r \cdots
\]

It is sufficient to prove that the sequence terminates at finite stage. Let us assume that there exists such a sequence of infinite length. Since the divisorial contraction makes the Picard number strictly less, the number of divisorial contractions in the sequence is finite. So we may assume that there is \( m_0 \in \mathbb{N} \) such that \( \varphi_m \)'s are all flips for \( m \geq m_0 \). By (iv) of 1.8 the set of one dimensional cones of the fan defining
Let \( m < m' \) such that \( \varphi_{m'-1} \circ \cdots \circ \varphi_m : V_m \to V_{m'} \) is identity. For each flip \( \varphi_j \) \((j = m, \ldots, m' - 1)\), take the dominating variety \( V'_j \) as in (iv) of 1.8:

\[
\begin{array}{c}
\psi_j \downarrow \psi'_j \\
V_j \quad \quad \quad V_{j+1}
\end{array}
\]

Let \( D_j \) be the exceptional divisor of \( \psi_j \) and \( \psi_{j+1} \). Then take a proper toric variety \( \tilde{V} = T_N(\tilde{\Delta}) \) which dominates all \( V'_j, j = m, \ldots, m' - 1 \) and on which the proper transform \( \tilde{X} \) of \( X_j \)'s is \( \tilde{\Delta} \)-regular. This is possible, because \( K_{V_j} + X_j \)'s are terminal.

Here one should note that the set of exceptional divisors on \( \tilde{V} \) for all morphisms \( \tilde{V} \to V_j \) \((j = m, \ldots, m' - 1)\) are common. For every \( j = m, \ldots, m' - 1 \), the discrepancy \( \alpha \) of \( K_{V_j} + X_j \) at \( D_j \) is less than the discrepancy \( \alpha' \) of \( K_{V_{j+1}} + X_{j+1} \) at \( D_j \) by 1.9. By this fact, for every exceptional divisor \( E \) on \( \tilde{V} \), the discrepancy \( \alpha_E \) of \( K_{V_j} + X_j \) at \( E \) and the discrepancy \( \alpha'_E \) of \( K_{V_{j+1}} + X_{j+1} \) at \( E \) satisfy \( \alpha_E < \alpha'_E \) and for at least one exceptional divisor \( E, \alpha_E < \alpha'_E \). Therefore comparing \( K_{V_m} + X_m \) and \( K_{V_{m'}} + X_{m'} \), there exists an exceptional divisor on \( \tilde{V} \) at which the discrepancy of \( K_{V_m} + X_m \) is less than that of \( K_{V_{m'}} + X_{m'} \), which is the contradiction to that \( V_m \to V_{m'} \) is the identity. \( \square \)

To apply the theorem above to the minimal model problem for a toric divisor, one needs the following lemma.

**Lemma 1.12.** (Lemma 2.7, [2]) Let \( Y \subset Z \) be an irreducible Weil divisor on a variety \( Z \). Assume that \( Z \) admits at worst \( \mathbb{Q} \)-factorial log-terminal singularities. Let \( \varphi : \tilde{Y} \to Y \) be a resolution of singularities on \( Y \). Assume \( K_{\tilde{Y}} = \varphi^*((K_Z + Y)|_Y) + \sum_i m_i E_i \) with \( m_i > -1 \) for all \( i \), where \( E_i \)'s are the exceptional divisors of \( \varphi \).

Then \( Y \) is normal, and \( Y \) has at worst log-terminal singularities.

In particular, if \( m_i > 0 \) for all \( i \), then \( Y \) has at worst terminal singularities.

**Corollary 1.13.** Let \( V \) be a toric variety defined by a proper fan \( \Delta \) and \( X \) a \( \Delta \)-regular divisor on \( V \). If \( \kappa(X) \geq 0 \), then \( X \) has a minimal model with the abundance. If \( \kappa(X) = -\infty \), then \( X \) is birationally equivalent to a proper variety \( Y \) with at worst terminal singularities and a fibration \( \varphi : Y \to Z \) to a lower dimensional variety \( Z \) with \(-K_Y \) relatively ample.

**Proof.** Let \( V_1 \) be the toric variety defined by a non-singular subdivision \( \Delta_1 \) of \( \Delta \) and \( X_1 \) be the proper transform of \( X \) on \( V_1 \). Then \( X_1 \) is \( \Delta_1 \)-regular and therefore \( K_{V_1} + X_1 \) is terminal by 1.5. Then one obtains a sequence:

\[
V_1 \to V_2 \to \cdots \to V_r
\]
as in Theorem 1.11. One can prove that for each \( j = 1, \ldots, r \), \( X_j \) has at worst terminal singularities. In fact, take a morphism \( \varphi : \tilde{V} \to V_j \) corresponding to a non-singular subdivision \( \Delta \) of the fan \( \Delta_j \) of \( V_j \) such that the proper transform \( \tilde{X} \) of \( X \) is \( \Delta \)-regular. Then, as \( K_{V_j} + X_j \) is terminal, it follows that

\[
(K_{\tilde{V}} + \tilde{X})|_{\tilde{X}} = \varphi^*((K_{V_j} + X_j)|_{X_j}) + \sum a_i E_i|_{\tilde{X}} \quad (a_i > 0 \text{ for all } i).
\]

Here the left hand side is the canonical divisor \( K_{\tilde{X}} \) of a non-singular variety \( \tilde{X} \).

Therefore by Lemma 1.12, one sees that \( X_j \) has at worst terminal singularities. By (iii) of 1.11 there are two cases for \( V_r \).

**Case 1.** \( K_{V_r} + X_r \) is nef.

Then the linear system \( |m(K_{V_r} + X_r)| \) is basepoint free for some \( m \in \mathbb{N} \). This is proved by a slight modification of the proof of Toric Nakai Criterion (2.18, [6]). Therefore \( |mK_{X_r}| \) is basepoint free, which implies that \( X_r \) is a minimal model with the abundance. In this case, \( \kappa(X) = \kappa(X_r) \geq 0 \).

**Case 2.** There exists an extremal ray \( R \) on \( V_r \) such that \((K_{V_r} + X_r)R < 0\) and the elementary contraction \( \varphi_R : V_r \to Z \) is a fibration to a lower dimensional variety \( Z \).

Under this situation, first consider the case:

**Subcase.** \( \dim X_r > \dim \varphi_R(X_r) \).

Let \( F \) be a fiber of \( \varphi_R \). Then by (ii) of 1.8, \( F \) is a weighted projective space and \((K_{V_r} + X_r)C < 0\) for every curve \( C \) in \( F \), which implies that \(-(K_{V_r} + X_r)\) is relatively ample over \( Z \). Hence \(-K_{X_r}\) is relatively ample over \( \varphi_R(X_r) \). This yields that \( \kappa(X) = \kappa(X_r) = -\infty \), and \( \varphi_R|_{X_r} : X_r \to \varphi_R(X_r) \) is a desired fibration.

**Subcase.** \( \dim X_r = \dim \varphi_R(X_r) \).

In this case \( \dim Z = \dim V_r - 1 \) and every fiber \( \ell \) of \( \varphi_R : V_r \to Z \) is \( \mathbb{P}^1 \) by (ii) of 1.8. Therefore \( K_{V_r}\ell = -2 \). On the other hand, because \( \varphi|_{X_r} \) is generically finite, \( X_r\ell > 0 \). Here, since \( V_r \) has at worst terminal singularities by 1.7, the singular locus has codimension greater than 2 and therefore the divisor \( X_r \) is a Cartier divisor along a general fiber \( \ell \), which yields that \( X_r\ell \) is an integer. By \((K_{V_r} + X_r)\ell < 0\), it follows \( X_r\ell = 1 \). It implies that \( \varphi_R|_{X_r} : X_r \to Z \) is a birational morphism, therefore \( X_r \) is rational. So \( X \) and \( X_r \) are birationally equivalent to \( \mathbb{P}^n \) which has ample anti-canonical divisor and of course \( \kappa(X) = -\infty \).

\[ \square \]

**Corollary 1.14.** Let the ground field \( k \) be of characteristic zero. Let \( V \) be a proper toric variety, \(|L|\) a linear system without a basepoint and \( X \) a general member of \(|L|\). Then the statements of Corollary 1.13 hold for \( X \).

**Proof.** By the Bertini’s Theorem, \( X \) is \( \Delta \)-regular. \[ \square \]
Corollary 1.15. Let \( V \) be a toric variety defined by a proper fan \( \Delta \) and \( X \) a \( \Delta \)-regular divisor on \( V \). Assume \( \kappa(X) \geq 0 \). Then there exists a non-singular subdivision \( \tilde{\Delta} \) of \( \Delta \) such that \( \tilde{V} = T_N(\tilde{\Delta}) \) and the proper transform \( \tilde{X} \) of \( X \) on \( \tilde{V} \) satisfy the following:

\[
\kappa(\tilde{V}, K_{\tilde{V}} + \tilde{X}) \geq 0.
\]

Proof. Use the notation of the proof of 1.13. Take a nonsingular subdivision \( \tilde{\Delta} \) of both \( \Delta \) and \( \Delta_r \), which is the fan of \( V_r \). Then the proper transform \( \tilde{X} \) of \( X \) on \( \tilde{V} = T_N(\tilde{\Delta}) \) is \( \tilde{\Delta} \)-regular. Since \( K_{V_r} + X_r \) is terminal and \( |m(K_{V_r} + X_r)| \) is basepoint free for some \( m \in \mathbb{N} \),

\[
0 \neq \Gamma(V_r, m(K_{V_r} + X_r)) \subset \Gamma(\tilde{V}, m(K_{\tilde{V}} + \tilde{X})).
\]

\( \Box \)

2. Divisors and Polytopes

2.1. Here we summarize the basic notion of an invariant divisor of a toric variety and the corresponding polytope which will be used in the next section. In this paper, a polytope in an \( \mathbb{R} \)-vector space means the intersection of finite number of half-spaces \( \{m \mid f_i(m) \geq a_i\} \) for linear functions \( f_i \).

2.2. Let \( M \) be the free abelian group \( \mathbb{Z}^n \) \((n \geq 3)\) and \( N \) be the dual \( Hom_{\mathbb{Z}}(M, \mathbb{Z}) \). We denote \( M \otimes_{\mathbb{Z}} \mathbb{R} \) and \( N \otimes_{\mathbb{Z}} \mathbb{R} \) by \( M_\mathbb{R} \) and \( N_\mathbb{R} \), respectively. Define \( M_\mathbb{Q} \) and \( N_\mathbb{Q} \) in the same way. Then one has the canonical pairing \((, ,): N \times M \to \mathbb{Z}\), which can be canonically extended to \((, ,): N_\mathbb{R} \times M_\mathbb{R} \to \mathbb{R}\). For a fan \( \Delta \) in \( N_\mathbb{R} \), we construct the toric variety \( T_N(\Delta) \). The fan \( \Delta \) is always assumed to be proper, i.e. the support \( |\Delta| = N_\mathbb{R} \). Denote by \( \Delta(k) \) the set of \( k \)-dimensional cones in \( \Delta \). Denote by \( \Delta[1] \) the set of primitive vectors \( q = (q_1, \ldots, q_r) \in N \) whose rays \( \mathbb{R}_{\geq 0}q \) belong to \( \Delta(1) \). For \( q \in \Delta[1] \), denote by \( D_q \) the corresponding divisor which is denoted by \( orb \mathbb{R}_{\geq 0}q \) in [5]. Denote by \( U_\sigma \) the invariant affine open subset which contains \( orb \sigma \) as the unique closed orbit.

Definition 2.3. For \( p \in N_\mathbb{R} \) and a subset \( K \subset M_\mathbb{R} \), define

\[
p(K) := \inf_{m \in K} (p, m)
\]

Definition 2.4. Let \( \Delta \) be a proper fan in \( N_\mathbb{R} \). A continuous function \( h: N_\mathbb{R} \to \mathbb{R} \) is called a \( \Delta \)-support function, if

1. \( h|_\sigma \) is \( \mathbb{R} \)-linear for every cone \( \sigma \in \Delta \) and
2. \( \sigma \) is \( \mathbb{Q} \)-valued on \( N_\mathbb{Q} \).

A \( \Delta \)-support function \( h \) is called integral if

1. \( \sigma \) is \( \mathbb{Z} \)-valued on \( N \).
**Proposition 2.5.** For a \(\Delta\)-support function \(h\), define \(D_h = -\sum_{p \in \Delta[1]} h(p)D_p\). Then the correspondence \(h \mapsto D_h\) gives a bijective map:
\[
\{\text{\(\Delta\)-support functions}\} \simeq \{\text{invariant \(\mathbb{Q}\)-Cartier divisors on } T_N(\Delta)\}\).
\]
Here \(D_h\) is a Cartier divisor, if and only if \(h\) is integral.

**Definition 2.6.** For a \(\Delta\)-support function \(h\), define
\[
\square_h := \{m \in M_\mathbb{R} | (p, m) \geq h(p), \forall p \in N_\mathbb{R}\},
\]
and call it the polytope associated with \(h\) or with \(D_h\). Actually it is a polytope by 2.10 and compact since the fan \(\Delta\) is proper.

**Proposition 2.7.** (see [6]) For an integral \(\Delta\)-support function \(h\), the following are equivalent:
(i) the linear system \(|D_h|\) is basepoint free;
(ii) \(h\) is upper convex; i.e. for arbitrary \(n, n' \in N_\mathbb{R}\), \(h(n) + h(n') \leq h(n + n')\);
(iii) \(\square_h\) is the convex hull of \(\{h_\sigma | \sigma \in \Delta(n)\}\), where \(h_\sigma\) is a point of \(M\) which gives the linear function \(h|_\sigma\) for \(\sigma \in \Delta(n)\).

**Proposition 2.8.** (see [6]) For a \(\Delta\)-support function \(h\), the following are equivalent:
(i) the \(\mathbb{Q}\)-Cartier divisor \(D_h\) is ample;
(ii) \(h\) is strictly upper convex; i.e. \(h\) is upper convex and \(h(n) + h(n') < h(n + n')\), if there is no cone \(\sigma\) such that \(n, n' \in \sigma\);
(iii) \(\square_h\) is of dimension \(n\) and the correspondence \(\sigma \mapsto h_\sigma\) gives the bijective map \(\Delta(n) \simeq \{\text{the vertices of } \square_h\}\), where \(h_\sigma\) is a point of \(M_\mathbb{Q}\) which gives the linear function \(h|_\sigma\) for \(\sigma \in \Delta(n)\).

Now we show simple lemmas which are used in the next section.

**Lemma 2.9.** Let \(h\) be a \(\Delta\)-support function. If \(h_\sigma \in \square_h\) for every \(\sigma \in \Delta(n)\), then \(h(p) = (p, m)\) for every \(p \in N_\mathbb{R}\), and the polytope \(\square_h\) is the convex hull of the set \(\{h_\sigma\}\).

**Proof.** By the definition of \(\square_h\), \(h(p) \leq (p, m)\) for all \(m \in \square_h\). Therefore \(h(p) \leq p(\square_h)\). Let \(\sigma\) be the cone in \(\Delta(n)\) such that \(p \in \sigma\), then \(h(p) = (p, h_\sigma) \geq p(\square_h)\), since \(h_\sigma \in \square_h\). For the second assertion, assume a vertex \(m \in \square_h\) does not belong to the convex hull of \(\{h_\sigma\}\). Then there exists \(p \in N_\mathbb{R}\) such that \((p, m) < (p, h_\sigma)\) for every \(\sigma \in \Delta(n)\), where the left hand side is greater than or equal to \(h(p)\) by the definition of \(\square_h\). This is a contradiction, because for \(\sigma \in \Delta(n)\) such that \(p \in \sigma\), \(h(p) = (p, h_\sigma)\). \(\square\)
Lemma 2.10. Denote an invariant divisor $D_h = \sum_{p \in \Delta[1]} m_p D_p$. Then
\[ \square_h = \bigcap_{p \in \Delta[1]} \{ m \in M_\mathbb{R}(\mathfrak{p}, m) \geq -m_p \}. \]

Proof. By 2.5, $m_p = -h(\mathfrak{p})$, then the inclusion $\square_h \subset \bigcap_{p \in \Delta[1]} \{ m \in M_\mathbb{R}(\mathfrak{p}, m) \geq -m_p \}$ is obvious. Take an element $m$ from the right hand side. For an arbitrary $\mathfrak{p} \in N_\mathbb{R}$, take $\sigma \in \Delta(n)$ such that $\mathfrak{p} \in \sigma$. Let $\sigma$ be spanned by $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$ $(\mathfrak{p}_i \in \Delta[1])$, then $\mathfrak{p} = \sum a_i \mathfrak{p}_i$ with $a_i \geq 0$. One obtains that $(\mathfrak{p}, m) = \sum a_i(\mathfrak{p}_i, m) \geq \sum a_i(\mathfrak{p}_i) = \sum a_i(\mathfrak{p}_i, h_\sigma) = h(\mathfrak{p})$, which shows that $m$ belongs to $\square_h$. \[ \square \]

Definition 2.11. Let $\square$ be a polytope in $M_\mathbb{R}$ defined by $\bigcap_{i=1}^s H_i$, where $H_i = \{ m \in M_\mathbb{R}(\mathfrak{p}_i, m) \geq a_i \}$. We say that $H_i$ contributes to $\square$, if $\square \cap \{ m \in M_\mathbb{R}(\mathfrak{p}_i, m) = a_i \} \neq \phi$. And we say that $H_i$ contributes properly to $\square$, if $\bigcap_{j \neq i} H_j \neq \square$.

Definition 2.12. Let $\square$ be an $n$-dimensional compact polytope in $M_\mathbb{R}$. Define the dual fan $\Gamma_{\square}$ of $\square$ as follows: $\Gamma_{\square} = \{ \gamma^* \}$, where $\gamma$ is a face of $\square$ and $\gamma^* := \{ m \in N_\mathbb{R} |$ the function $u_{\square}$ attains the minimal value at all points of $\gamma \}$. Then $\Gamma_{\square}$ turns out to be a proper fan.

2.13. If $\Delta$ is the dual fan of the polytope $\square_h$ corresponding to a $\Delta$-support function $h$, then by 2.8 $D_h$ is ample, therefore the variety $T_X(\Delta)$ turns out to be a projective variety.

3. The construction of a minimal model

3.1. In this section we concretely construct a projective minimal model with the abundance for a $\Delta$-regular toric divisor $X$ with $\kappa(X) \geq 0$ by means of a polytope of the adjoint divisor. Let $V$ be a toric variety defined by a proper fan $\Delta$ and $X$ a $\Delta$-regular divisor with $\kappa(X) \geq 0$. To construct a minimal model of $X$ we may assume that $V$ is non-singular and $\kappa(V, K_V + X) \geq 0$, by Corollary 1.15.

3.2. The construction Let $h$ be a $\Delta$-support function such that $K_X(\Delta) + X \sim D_h$. Then, by $\kappa(T_X(\Delta), K_T(\Delta) + X) \geq 0$, it follows that $\square_h \neq \phi$. Let $\Delta[1] = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_s \}$ and $H_i = \{ m \in M_\mathbb{R}(\mathfrak{p}_i, m) \geq h(\mathfrak{p}_i) \}$. Then by 2.10 $\square_h = \bigcap_{i=1}^s H_i$. Assume that $H_1, \ldots, H_r (r \leq s)$ are all that contribute to $\square_h$. For $\epsilon_i > 0$ $i = 1, \ldots, r$, define $H_{i, \epsilon_i} := \{ m \in M_\mathbb{R}(\mathfrak{p}_i, m) \geq h(\mathfrak{p}_i) - \epsilon_i \}, \partial H_{i, \epsilon_i} := \{ m \in M_\mathbb{R}(\mathfrak{p}_i, m) = h(\mathfrak{p}_i) - \epsilon_i \}$ and $\square(\epsilon) := \bigcap_{i=1}^r H_{i, \epsilon_i}$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$. Here one should note that the polytope $\square_h$ may not be of the maximal dimension. By "puffing up" this, one get a polytope $\square(\epsilon)$ of the maximal dimension. The subset $Z = \{ \epsilon \in \mathbb{R}_{>0}^r | \partial H_{i, \epsilon_i} \text{ is not of normal crossings} \}$ is Zariski closed and the complement $\mathbb{R}_{>0}^r \backslash Z$ is divided into finite number of chambers. Take a chamber $W$ such that:

(3.2.1) $0 \in W$.
Now we are going to prove that $X(\Sigma)$ satisfies desired conditions for a minimal model. First note that $\Sigma[1] = T\bigcup h$ is a projective, because an invariant $Q$-Cartier divisor $\sum p_i \in \Sigma[1] (h(p_i) - e_i)D_{p_i}$, with all $e_i$ rational and $\epsilon \in W$ is ample since $\Sigma$ is the dual fan of the corresponding polytope to this divisor (2.13). Hence the projectivity of $X(\Sigma)$ follows automatically.

Claim 3.4. The divisor $K_{T_N(\Sigma)} + X(\Sigma)$ is linearly equivalent to an invariant divisor $-\sum_{i=1}^r h(p_i) D_{p_i}$. Let $k$ be the $\Sigma$-support function corresponding to this divisor, then $h(p_i) = k(p_i)$ for $i = 1, \ldots, r$ and $\Box_h = \Box_k$.

Proof. The first assertion follows from that the divisor $K_{T_N(\Sigma)} + X(\Sigma)$ is the proper transform of $K_{T_N(\Delta)} + X(\Sigma) \sim -\sum_{i=1}^r h(p_i) D_{p_i}$. The second assertion is obvious and the last assertion follows from 2.10 and the fact that $H_1, \ldots, H_r$ are all that contribute to $\Box_h$. □

Claim 3.5. For all $\sigma \in \Sigma(n)$, it follows that $k_\sigma \in \Box_k$.

Proof. Let $\{e^{(m)}_i\}_m$ be a series of rational points in $W$ which converge to 0. Let $k^{(m)}$ be the $\Sigma$-support function corresponding to a $Q$-Cartier divisor $\sum_{i=1}^r (-h(p_i) + e^{(m)}_i)D_{p_i}$. Then by 2.10 it follows that $\Box_{k^{(m)}} = \Box(e^{(m)})$, and therefore by 2.13 the divisor is ample. Replacing $\{e^{(m)}_i\}_m$ by suitable subsequence, one can assume there exists $\lim_{m \to \infty} k^{(m)}_\sigma$ for every $\sigma \in \Sigma(n)$. Indeed, replacing by suitable subsequence, one may assume that $e^{(m)}_i \geq e^{(m+1)}_i$ for every $i$, then $\Box_{e^{(m)}_i} = \Box_{e^{(m+1)}_i} = \cdots$; therefore for every $\sigma \in \Sigma(n)$ and $m$ it follows that $k^{(m)}_\sigma \in \Box(e^{(m)})$ which is compact; so $\{k^{(m)}_\sigma\}$ have an accumulating point. Let $k'_\sigma := \lim_{m \to \infty} k^{(m)}_\sigma$, then $k'_\sigma \in \Box_k$, because the ampleness of $D_{k^{(m)}_\sigma}$ yields $k^{(m)}_\sigma \in \Box(e^{(m)})$. The collection $\{k'_\sigma\}_{\sigma \in \Sigma(n)}$ defines a function $k'$ on $N_R$. In fact, for every $m$, $k^{(m)}_\sigma = k^{(m)}_\tau$ as a function on $\sigma \cap \tau$, which yields that $k'_\sigma = k'_\tau$ as a function on $\sigma \cap \tau$. Now one obtains that $k' = k$. This is proved as follows: for every $p_i \in \Sigma[1]$ take $\sigma \in \Sigma(n)$ such that $p_i \in \sigma$; $k'(p_i) = (p_i, k'_\sigma) = \lim_{m \to \infty} (p_i, k^{(m)}_\sigma) = \lim_{m \to \infty} (h(p_i) - e^{(m)}_i) = h(p_i) = k(p_i)$, since $k^{(m)}_\sigma$ is on the hyperplane $(p_i, m) = h(p_i) - e^{(m)}_i$. Hence it follows that $k = k'$ and therefore $k_\sigma = k'_\sigma$ for every $\sigma \in \Sigma(n)$, which shows that $k_\sigma \in \Box_k$. □

Now by 2.9 and 2.7 the linear system $|mD_k| = |m(K_{T_N(\Sigma)} + X(\Sigma))|$ has no basepoint for such $m$ that $mD_k$ is a Cartier divisor.
3.6. Let $\tilde{\Sigma}$ be a non-singular subdivision of $\Sigma$ and $\Delta$. Let

$$\psi \rightarrow T_N(\Delta)$$

$$\varphi \rightarrow T_N(\Sigma)$$

be the corresponding morphisms and $X(\tilde{\Sigma})$ the proper transform of $X$ in $T_N(\tilde{\Sigma})$. Since $X(\tilde{\Sigma})$ is $\tilde{\Sigma}$-regular by [1], it is non-singular and $\varphi|_{X(\tilde{\Sigma})}$ is birational.

**Claim 3.7.** It follows that

$$K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) = \varphi^*(K_{T_N(\Sigma)} + X(\Sigma)) + \sum_{p \in \Sigma[1] \setminus \Delta[1]} m_p D_p,$$

where $m_p > 0$ for $p$ such that $D_p \cap X(\tilde{\Sigma}) \neq \emptyset$.

**Proof.** Denote

$$K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) = \psi^*(K_{T_N(\Delta)} + X) + \sum_{p \in \Sigma[1] \setminus \Delta[1]} \alpha_p D_p,$$

then $\alpha_p > 0$ for $p$ such that $D_p \cap X(\tilde{\Sigma}) \neq \emptyset$, since $X$ is non-singular. Putting $\alpha_p = 0$ for $p \in \Delta[1]$, one obtains that $K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) \sim \sum_{p \in \Sigma[1] \setminus \Delta[1]} (-h(p) + \alpha_p) D_p$, as $K_{T_N(\Delta)} + X \sim D_h$. On the other hand,

$$K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) = \varphi^*(K_{T_N(\Sigma)} + X(\Sigma)) + \sum_{p \in \Sigma[1] \setminus \Sigma[1]} m_p D_p.$$

Putting $m_p = 0$ for $p \in \Sigma[1]$, one obtains that $K_{T_N(\tilde{\Sigma})} + X(\tilde{\Sigma}) \sim \sum_{p \in \Sigma[1]} (-k(p) + m_p) D_p$, as $K_{T_N(\Sigma)} + X(\Sigma) \sim D_k$.

Therefore $\sum_{p \in \Sigma[1]} (-h(p) + \alpha_p) D_p \sim \sum_{p \in \Sigma[1]} (-k(p) + m_p) D_p$. As $h(p) = k(p)$ and $\alpha_p = m_p = 0$ for $p \in \Sigma[1]$, one obtains that

$$\sum_{p \in \Sigma[1] \setminus \Sigma[1]} ((-h(p) + \alpha_p) - (-k(p) + m_p)) D_p \sim 0.$$

Here $D_p$ ($p \in \tilde{\Sigma}[1] \setminus \Sigma[1]$) are all exceptional for $\varphi$. Then the divisor above is not only linearly equivalent to 0 but also equal to 0. Therefore $(-h(p) + \alpha_p) - (-k(p) + m_p) = 0$ for every $p \in \tilde{\Sigma}[1] \setminus \Sigma[1]$, where $k(p) = p(\square_h)$ by $k_\sigma \in \square_k = \square_h$ and 2.9. Now consider the divisor $D_p$ such that $D_p \cap X(\tilde{\Sigma}) \neq 0$. For $p \in \tilde{\Sigma}[1] \setminus \Delta[1]$, $m_p = p(\square_h) - h(p) + \alpha_p \geq \alpha_p > 0$. For $p \in \Delta[1] \setminus \Sigma[1]$, it follows that $m_p = p(\square_h) - h(p) > 0$, because $\{m| (p, m) \geq h(p)\}$ does not contribute to $\square_h$ by the definition of $\Sigma$ (c.f 3.2). This completes the proof. □
3.8. Since $T_N(\Sigma)$ has at worst quotient singularities, one can apply Lemma 1.12 to our situation and obtain that $X(\Sigma)$ is normal and has at worst terminal singularities. And the linear system of $mK_{X(\Sigma)} = m(K_{T_N(\Sigma)} + X(\Sigma))|_{X(\Sigma)}$ ($m \gg 0$) has no basepoint, because $|m(K_{T_N(\Sigma)} + X(\Sigma))|$ is basepoint free as is noted after the proof of 3.5. This completes the proof of that $X(\Sigma)$ is a projective minimal model with the abundance.

3.9. Pursuing elementary contractions and flips is like groping for a minimal model in the dark. The reason why the discussion of this section goes well without contractions nor flips is because in toric geometry every exceptional divisor is visible as a vector in the space $N$. Then one can prepare so that every discrepancy of adjoint divisor is positive (cf. 3.7), which makes the singularities terminal. In the discussion, one puffed up the polytope of the adjoint divisor and took its dual fan $\Sigma$. This implies that in $T_N(\Sigma)$ the adjoint divisor is the limit of a sequence of ample divisors (cf. 3.5), which makes the adjoint divisor nef; or equivalently semi-ample.

4. Examples

In this section the base field $k$ is always assumed to be of characteristic zero. Let $M$ be $\mathbb{Z}^3$ and $N$ be its dual.

Example 4.1. Let $p_i$ ($i = 1, \ldots, 6$) and $q_j$ ($j = 1, \ldots, 8$) be points in $N$ as follows: $p_1 = (1, 0, 0)$, $p_2 = (-1, 0, 0)$, $p_3 = (0, 1, 0)$, $p_4 = (0, -1, 0)$, $p_5 = (0, 0, 1)$, $p_6 = (0, 0, -1)$, $q_1 = (1, 1, 1)$, $q_2 = (-1, -1, -1)$, $q_3 = (1, 1, -1)$, $q_4 = (-1, -1, 1)$, $q_5 = (1, -1, 1)$, $q_6 = (-1, 1, -1)$, $q_7 = (-1, 1, 1)$, $q_8 = (1, -1, -1)$. Let them generate one-dimensional cones $\mathbb{R}_{\geq 0}p_i$, $\mathbb{R}_{\geq 0}q_j$ and construct a fan $\Delta$ with these cones as in Figure 1. Here note that Figure 1 is the picture of the fan which is cut by a hypersphere with the center the origin and unfolded onto the plane. This fan is the dual fan of the polytope of Figure 2 and it is easy to check that it is non-singular. Let $X$ be a general member of a base-point-free linear system $|\sum_{i=1}^{6} D_{p_i} + 2 \sum_{j=1}^{8} D_{q_j}|$. Let $h$ be the $\Delta$-support function such that $K_{T_N(\Delta)} + X \sim D_h$. Then the polytope $\square_h$ is one point $\cap_{i=1}^{6}\{m|(p_i, m) \geq 0\}$ and the half spaces contributing to this polytope are $\{m|(p_i, m) \geq 0\}$, $i = 1, \ldots, 6$, because $K_{T_N(\Delta)} + X \sim \sum D_{q_j}$. Therefore, for a sufficiently small general $\epsilon$, the polytope $\square(\epsilon) = \cap_{i=1}^{6}\{m \in \mathbb{R}|(p_i, m) \geq -\epsilon_i\}$ is a hexahedron whose picture is as Figure 3. The dual fan $\Sigma$ of $\square(\epsilon)$ (Figure 4) gives a minimal model $X(\Sigma)$ of $X$. Since $K_{T_N(\Sigma)} + X(\Sigma) \sim 0$, it follows that $\kappa(X) = 0$.

Example 4.2. Let $p_i$ and $q_j$ be as in 4.1 and $\Delta$ be the fan with the cones generated by these vectors as Figure 5. This fan is the dual fan of the polytope of Figure 6 and it is easy to check that it is non-singular. Let $X$ be a general member of a base-point-free linear system $|2D_{p_1} + 2D_{p_2} + \sum_{i=3}^{6} D_{p_i} + 3 \sum_{j=1}^{8} D_{q_j}|$. Let $h$ be the $\Delta$-support function such that $K_{T_N(\Delta)} + X \sim D_h$. Then the polytope $\square_h$ is a segment $\cap_{i=1}^{6}\{m|(p_i, m) \geq -1\} \cap \cap_{i=3}^{6}\{m|(p_i, m) \geq 0\}$ and the half spaces contributing to this polytope are
\{m|(p_i, m) \geq -1\} \ (i = 1, 2) \ and \ \{m|(p_i, m) \geq 0\} \ (i = 3, \ldots, 6), \ because \ K_{T_N(\Delta)} + X \sim D_{p_1} + D_{p_2} + 2\sum D_{q_j}. \ Therefore, for a sufficiently small general \(\epsilon\), the polytope \\
\square(\epsilon) = (\intersection_{i=1}^2\{m \in M_{\mathbb{R}}|(p_i, m) \geq -1 - \epsilon_i\}) \cap (\intersection_{i=3}^6\{m \in M_{\mathbb{R}}|(p_i, m) \geq -\epsilon_i\}) \ is \ a \ hexahedron \ whose \ picture \ is \ as \ Figure \ 7. \ The \ dual \ fan \ \Sigma \ of \ \square(\epsilon) \ (Figure \ 4) \ gives \ a \ minimal \ model \ X(\Sigma) \ of \ X. \ Since \ \square_h \ is \ of \ one \ dimension, \ \dim \Gamma(T_N(\Sigma), m(K_{T_N(\Sigma)} + X(\Sigma))) \ grows \ in \ order \ 1, \ and \ therefore \ \dim \Phi|m_{K_{T_N(\Sigma)} + X(\Sigma)}|(T_N(\Sigma)) = 1. \ This \ shows \ that \ \dim \Phi|m_{K_{T_N(\Sigma)} + X(\Sigma)}|(X(\Sigma)) \leq 1. \ As \ the \ dual \ fan \ of \ the \ polytope \ of \ X(\Sigma) \sim 2D_{p_1} + 2D_{p_2} + \sum_{i=3}^6 D_{p_i}, \ is \ \Sigma, \ X(\Sigma) \ is \ ample \ by \ 2.13. \ Hence \ X(\Sigma) \ intersects \ all \ fibers \ of \ \Phi|m_{K_{T_N(\Sigma)} + X(\Sigma)}|(T_N(\Sigma)), \ which \ shows \ that \ \kappa(X) = 1.

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