ON THE HODGE STRUCTURES OF COMPACT HYPERKÄHLER MANIFOLDS

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Abstract. The purpose of this note is to give an account of a well-known folklore result: the Hodge structure on the second cohomology of a compact hyperkähler manifold uniquely determines Hodge structures on all higher cohomology groups. We discuss the precise statement and its proof, which are somewhat difficult to locate in the literature.

1. Introduction

Compact hyperkähler manifolds have been extensively studied in recent decades. One of the central results of their theory is the global Torelli theorem [V4]. It addresses the problem of reconstructing a hyperkähler manifold from the Hodge structure on its second cohomology group. It is known that in general one can not reconstruct the manifold uniquely, and the global Torelli theorem explains the reasons for this. It gives a description of the moduli space of hyperkähler manifolds as a certain non-Hausdorff covering space of the period domain for the Hodge structures on the second cohomology group, see e.g. the discussion in [H3].

Despite of the fact that it is impossible to reconstruct a hyperkähler manifold from the Hodge structure on $H^2$, one can still ask if it is possible to recover the rational Hodge structures on higher cohomology groups from the Hodge structure on $H^2$. It turns out that in a certain sense this is possible, and such statements have appeared in the literature (e.g. in the preprint version of [LL] or [GHJ, Corollary 24.5]). In this note we prove a more precise version of this result, Theorem 3.6. A more standard version is stated as Corollary 3.7. Let us remark that the proof of Theorem 3.6 does not use the global Torelli theorem.

In section 2 we recall all necessary definitions and results about the structure of the cohomology algebra of hyperkähler manifolds and sketch some of the proofs. In section 3 we discuss sufficient conditions for a complex structure to be of hyperkähler type, Proposition 3.1. In the end we prove the main result, Theorem 3.6.

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2. Cohomology of hyperkähler manifolds

2.1. Topological invariants. Let $X$ be a compact $C^\infty$-manifold, $\dim_{\mathbb{R}}(X) = 2n$. The singular cohomology of $X$ with rational coefficients $H^\bullet(X, \mathbb{Q})$ is a finite-dimensional graded $\mathbb{Q}$-algebra. We will denote by $p_i(X) \in H^{4i}(X, \mathbb{Q})$ the rational Pontryagin classes of $X$.

Definition 2.1. Consider the following groups:

1. $\mathcal{G}(X)$ – automorphisms of the graded algebra $H^\bullet(X, \mathbb{Q})$ that stabilize all $p_i(X)$;
2. $\mathcal{G}^+(X)$ – automorphisms of the graded subalgebra $H^{2\bullet}(X, \mathbb{Q})$ that stabilize all $p_i(X)$.
Define the operator $\theta \in \text{End}(H^\bullet(X, \mathbb{Q}))$ as follows:

$$\theta|_{H^k(X, \mathbb{Q})} = (k - n)\text{Id}.$$  

For an element $h \in H^2(X, \mathbb{Q})$, let $L_h \in \text{End}(H^\bullet(X, \mathbb{Q}))$ denote the operator of cup product with $h$. We will say that $h$ has Lefschetz property, if

$$L_h^k: H^{n-k}(X, \mathbb{Q}) \sim H^{n+k}(X, \mathbb{Q})$$

is an isomorphism for all $k = 0, \ldots, n$. If $h$ has Lefschetz property, then there exists a unique $\Lambda_h \in \text{End}(H^\bullet(X, \mathbb{Q}))$, such that $(\Lambda_h, \theta, L_h)$ is an sl$_2$-triple.

**Definition 2.2.** Let us denote by $\mathfrak{g}_{\text{tot}}(X)$ the minimal Lie subalgebra of $\text{End}(H^\bullet(X, \mathbb{Q}))$ containing $\theta, \Lambda_h$ and $L_h$ for all $h \in H^2(X, \mathbb{Q})$ with the Lefschetz property.

**Remark 2.3.** The groups $\mathcal{G}(X), \mathcal{G}^+(X)$ and the Lie algebra $\mathfrak{g}_{\text{tot}}(X)$ depend only on the homeomorphism type of $X$. For $\mathfrak{g}_{\text{tot}}(X)$ this is clear from the definition, and for $\mathcal{G}(X), \mathcal{G}^+(X)$ it follows from a theorem of Novikov [No].

We will use the following notations. The $\mathbb{R}$-Lie algebra $\mathfrak{g}_{\text{tot}}(X) \otimes_{\mathbb{Q}} \mathbb{R}$ will be denoted by $\mathfrak{g}_{\text{tot}}(X)_{\mathbb{R}}$. For any $\psi \in \mathcal{G}(X)$ we have $\psi L_h \psi^{-1} = \theta, \psi L_h \psi^{-1} = L_h \psi(h)$ and hence $\psi \Lambda_h \psi^{-1} = \Lambda \psi(h)$ for all $h \in H^2(X, \mathbb{Q})$. This shows that the adjoint action of $\psi$ preserves $\mathfrak{g}_{\text{tot}}(X)$. We will denote by $\text{ad}_{\psi}$ the corresponding endomorphism of $\mathfrak{g}_{\text{tot}}(X)$.

### 2.2. Hyperkähler manifolds

Given a complex structure $I \in \text{End}(TX)$, we will denote by $X_I$ the corresponding complex manifold, and by $\Omega^k_{X_I}$ the sheaves of holomorphic differential forms on $X_I$. The canonical bundle will be denoted by $K_{X_I}$.

**Definition 2.4.** Assume that the manifold $X$ is compact and $\pi_1(X) = 1$. We will say that a complex structure $I$ is of hyperkähler type, if

1. $X_I$ admits a Kähler metric;
2. $H^0(X_I, \Omega^2_{X_I})$ is spanned by a symplectic form.

In this case $X_I$ is called a hyperkähler manifold. We will say that $X$ is of hyperkähler type, if it admits a complex structure of hyperkähler type.

Assume that $I$ is of hyperkähler type, and let $\sigma \in H^0(X_I, \Omega^2_{X_I})$ be a symplectic form. The dimension of any symplectic manifold is even, and we let $2n = \dim \mathbb{C}(X_I)$. The form $\sigma$ defines an isomorphism $T_{X_I} \cong \Omega^1_{X_I}$, which shows that all odd Chern classes of $T_{X_I}$ vanish. The total Todd class of a complex vector bundle with vanishing odd Chern classes can be expressed as a universal polynomial in the Pontryagin classes of the underlying real bundle. Evaluating this polynomial on the Pontryagin classes of $X$ gives an element $\text{td}(X) \in H^{4\bullet}(X, \mathbb{Q})$ that does not depend on the choice of the complex structure $I$. This element is the total Todd class of $X_I$ for any $I$ of hyperkähler type. Since $\text{td}(X) = 1$, there is a unique square root of $\text{td}(X)$ with degree zero term equal to 1, and we denote it by $\sqrt{\text{td}(X)} \in H^{4\bullet}(X, \mathbb{Q})$.

It was shown in [HS], that $\int_{X_I} \sqrt{\text{td}(X)} > 0$. The integral here means evaluation of the degree $4n$ component of $\sqrt{\text{td}(X)}$ on the fundamental class of $X$. The latter is determined by the orientation of $X$ induced by $I$. In particular, this shows that all complex structures of hyperkähler type induce the same orientation on $X$, and that all diffeomorphisms of $X$ are orientation-preserving, since they have to fix all polynomial expressions in Pontryagin classes. From now on we will implicitly assume that we have fixed the orientation of $X$. 

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Definition 2.5. The Beauville-Bogomolov-Fujiki (BBF) form of $X$ is the quadratic form $q \in S^2H^2(X, \mathbb{Q})^*$ given by
\[ q(a) = \int_X a^2 \sqrt{\text{td}(X)} \]
for all $a \in H^2(X, \mathbb{Q})$.

Remark 2.6. Usually the BBF form is defined via the Fujiki relations (2.2) that we recall below. We prefer the above definition to avoid the ambiguity in the choice of the constant in (2.2). The fact that the definition above is equivalent up to a scalar factor to the usual one is due to [Ni], see also the discussion in [H2, section 4].

Let us list a few properties of the BBF form.

1. The form $q$ is non-degenerate of signature $(3, b_2(X) - 3)$. If $\omega \in H^2(X, \mathbb{R})$ is a Kähler class for a complex structure of hyperkähler type, then $q(\omega) > 0$, see [Be, Théorème 5 and p. 773] and [H2, Theorem 4.2].

2. For every $k = 1, \ldots, n$ there exists a non-zero constant $C_{X,k} \in \mathbb{Q}$, such that for all $a \in H^2(X, \mathbb{Q})$
\[ \int_X a^{2k} \sqrt{\text{td}(X)} = C_{X,k} q(a)^k. \]
This follows from [H2, Theorem 4.2] and the inequality $\int_X \sqrt{\text{td}(X)} > 0$ from [HS]. In particular, for $k = n$ we get the Fujiki relation:
\[ \int_X a^{2n} = C_{X,n} q(a)^n. \]

3. For all $a, b \in H^2(X, \mathbb{C})$ we have:
\[ \int_X a^{2n-1}b = C_{X,n} q(a)^{n-1} q(a, b). \]
This relation follows from (2.2) by substituting $a + tb$ in place of $a$ and comparing the coefficients of the obtained polynomials in $t$.

4. For all $a, b \in H^2(X, \mathbb{C})$ such that $q(a, b) = 0$, we have:
\[ (2n - 1) \int_X a^{2n-2}b^2 = C_{X,n} q(a)^{n-1} q(b). \]
This follows from (2.2) or from [Be, Théorème 5c].

5. Let $\mathcal{I} \subset S^*H^2(X, \mathbb{C})$ denote the ideal generated by $a^{n+1}$ for all $a \in H^2(X, \mathbb{C})$ with $q(a) = 0$. Then according to [B1, Theorem 2.4 and Lemma 2.2] the multiplication in cohomology induces an embedding
\[ S^*H^2(X, \mathbb{C})/\mathcal{I} \hookrightarrow H^*(X, \mathbb{C}). \]

2.3. The Lie algebra action. We assume that $X_I$ is a hyperkähler manifold. It follows from Calabi’s conjecture proven by Yau, that in this case $X$ admits two other complex structures $J, K$ and a Riemannian metric $g$, such that $K = IJ = -JI$ and $g$ is Kähler with respect to $I, J$ and $K$, see e.g. [Be] or [GHJ].

We will use the following notation: $\omega_I, \omega_J$ and $\omega_K$ will denote the Kähler forms, $L_I, L_J$ and $L_K$ the corresponding Lefschetz operators, $\Lambda_I, \Lambda_J$ and $\Lambda_K$ the dual Lefschetz operators. The complex structures can be extended as derivations to act on the differential $k$-forms on $X$ for all $k$. The corresponding operators will be denoted by $W_I, W_J$ and $W_K$. For instance, $W_I$ acts on the differential forms of $I$-type $(p, q)$ as multiplication by $i(p - q)$. In the proposition below, $\Lambda^*T^*X$ denotes the graded vector bundle of real differential forms on $X$, and $\text{End}(\Lambda^*T^*X)$ denotes the algebra of its endomorphisms.
Proposition 2.7. The Lie subalgebra of $\text{End}(\Lambda^\bullet T^\ast X)$ generated by the operators $L_I$, $L_J$, $L_K$, $\Lambda_I$, $\Lambda_J$ and $\Lambda_K$ is isomorphic to $\mathfrak{so}(4,1)$. We have the following commutator identities:

$$[\Lambda_I, L_J] = W_K; \quad [\Lambda_J, L_K] = W_I; \quad [\Lambda_K, L_I] = W_J;$$

$$[\Lambda_I, \Lambda_J] = [\Lambda_J, \Lambda_K] = [\Lambda_K, \Lambda_I] = 0.$$

Proof. The proof of this statement can be found in [V3, Theorem 8.1], see also references therein. We sketch an alternative proof, based on the theory of $k$-symplectic structures from [KSV].

It clearly suffices to prove the commutator identities pointwise, so we are reduced to the following linear-algebraic problem. Consider $M = \mathbb{H}^n$ as a left $\mathbb{H}$-module with the standard metric $g(x, y) = \sum_{k=1}^{n} x_k \bar{y}_k$, where the bar denotes quaternionic conjugation. We have the operators of multiplication by imaginary quaternions $I, J, K \in \text{End}(M)$ and the corresponding two-forms $\omega_I, \omega_J, \omega_K \in \Lambda^2 M^\ast$. We need to prove the commutator identities for the Lefschetz operators and their duals in $\text{End}(\Lambda^\bullet M^\ast)$.

Let $U \subset \mathbb{H}$ be the three-dimensional subspace of imaginary quaternions with the quadratic form $\rho(a) = \text{Re}(a^2)$. The Clifford algebra $\mathcal{C} = \text{Cl}(U, \rho)$ is by construction endowed with a natural morphism $\mathcal{C} \to \mathbb{H}$, making $M$ a left $\mathbb{C}$-module. The metric $g$ is a $\mathbb{C}$-invariant symmetric bilinear form in the sense of [KSV, Definition 3.1]. We define a map $\eta: U \to \Lambda^2 M^\ast$ by sending $a \in U$ to the form $\omega_a$, such that $\omega_a(x, y) = g(ax, y)$. The complexification of the image of $\eta$ is a 3-symplectic structure on $M_{\mathbb{C}}$ in the sense of [KSV, Definition 1.1]. It is clear that the image of $\eta$ is the linear span of $\omega_I, \omega_J$ and $\omega_K$. The statement now follows from [KSV, Theorem 3.10 and Lemma 3.12].

It is known that the Lefschetz operators and their duals commute with the Laplacian of the Riemannian metric $g$. Hence all the operators from the above proposition act on the cohomology of $X$, and we obtain an embedding of Lie algebras $\mathfrak{so}(4,1) \hookrightarrow \mathfrak{g}_{\text{tot}}(X)_{\mathbb{R}}$. This embedding depends on the choice of the complex structure $I$ and the hyperkähler metric $g$. Using local deformation theory of complex structures on $X$, we can obtain enough $\mathfrak{so}(4,1)$-subalgebras in $\mathfrak{g}_{\text{tot}}(X)_{\mathbb{R}}$ to conclude that all dual Lefschetz operators on $X$ pairwise commute. This observation leads to the description of $\mathfrak{g}_{\text{tot}}(X)$ that we give below.

Definition 2.8. Let us denote by $V$ the $\mathbb{Q}$-vector space $H^2(X, \mathbb{Q})$. Define the graded $\mathbb{Q}$-vector space $	ilde{V} = \langle e_0 \rangle + V \oplus \langle e_4 \rangle$ with $e_k$ of degree $k$ and $V$ in degree 2. Define the quadratic form $q \in S^2 V^\ast$, such that $q|_V = q$, and $\langle e_0, e_4 \rangle$ is a hyperbolic plane orthogonal to $V$ with $q(e_0) = q(e_4) = 0$ and $q(e_0, e_4) = 1$.

The graded Lie algebra $\mathfrak{so}(\tilde{V}, \tilde{q})$ has components of degrees $-2$, 0 and 2. The semisimple part of $\mathfrak{so}(\tilde{V}, \tilde{q})$ is isomorphic to $\mathfrak{so}(V, q)$, and we have the following isomorphisms of $\mathfrak{so}(V, q)$-modules: $\mathfrak{so}(\tilde{V}, \tilde{q})^{-2} \simeq \mathfrak{so}(\tilde{V}, \tilde{q})^2 \simeq V$, see e.g. [KSV, section 3.4].

Proposition 2.9. There exists an isomorphism of graded Lie algebras $\mathfrak{g}_{\text{tot}}(X) \simeq \mathfrak{so}(\tilde{V}, \tilde{q})$. The subalgebra $\mathfrak{so}(V, q) \subset \mathfrak{g}_{\text{tot}}(X)$ acts on $H^\bullet(X, \mathbb{Q})$ by derivations.

Proof. It is proven in [LL, Proposition 4.5] that $\mathfrak{g}_{\text{tot}}(X)_{\mathbb{R}} \simeq \mathfrak{so}(\tilde{V}_R, \tilde{q})$. To deduce the corresponding statement over $\mathbb{Q}$, note that under the embedding $\mathfrak{g}_{\text{tot}}(X)_{\mathbb{R}} \subset \text{End}(H^\bullet(X, \mathbb{R}))$ the components $\mathfrak{so}(\tilde{V}_R, \tilde{q})^2$ and $\mathfrak{so}(\tilde{V}_R, \tilde{q})^{-2}$ are mapped to the subspaces of Lefschetz operators, respectively dual Lefschetz operators. These embeddings are defined over $\mathbb{Q}$, since for $x \in H^2(X, \mathbb{Q})$ with $q(x) \neq 0$ both $L_x$ and $\Lambda_x$ are defined over $\mathbb{Q}$. Since $\mathfrak{so}(\tilde{V}, \tilde{q})$ is generated by the components of degree $\pm 2$, this proves the first statement of the proposition. The second statement follows directly from [LL, Proposition 4.5].
It follows from Proposition 2.9 that there exists a representation of \( \text{Spin}(V, q) \) in the group of algebra automorphisms of \( H^*(X, \mathbb{Q}) \). Recall Definition 2.1 of the groups \( \mathcal{G}(X) \) and \( \mathcal{G}^+(X) \), and observe that there exists a natural homomorphism \( \mathcal{G}(X) \to \mathcal{G}^+(X) \).

**Proposition 2.10.** The action of the group \( \text{Spin}(V, q) \) on the algebra \( H^*(X, \mathbb{Q}) \) obtained from Proposition 2.9 is induced by a homomorphism \( \alpha : \text{Spin}(V, q) \to \mathcal{G}(X) \). The action of \( \text{Spin}(V, q) \) on \( H^2(X, \mathbb{Q}) \) factors through \( \text{SO}(V, q) \):

\[
\begin{array}{ccc}
\text{Spin}(V, q) & \xrightarrow{\alpha} & \mathcal{G}(X) \\
\downarrow & & \downarrow \\
\text{SO}(V, q) & \xrightarrow{\alpha^+} & \mathcal{G}^+(X)
\end{array}
\]

*Proof.* Since the Pontryagin classes of \( X \) are of Hodge type \((p, p)\) for all complex structures admitting a Kähler metric, one deduces that \( \text{Spin}(V, q) \) fixes all the Pontryagin classes, see [LL, Proposition 4.8]. This gives a homomorphism \( \alpha \). It was shown in [V3, Corollary 8.2] that the composition of \( \alpha \) and the homomorphism \( \mathcal{G}(X) \to \mathcal{G}^+(X) \) factors through \( \text{SO}(V, q) \). \( \square \)

**Proposition 2.11.** For an element \( \psi \in \mathcal{G}(X) \) let \( \varphi = \psi^2 \) denote its degree two component acting on \( V = H^2(X, \mathbb{Q}) \). Then \( \varphi \in \Omega(V, q) \). For any \( x \in \mathfrak{s}\mathfrak{o}(V, q) \subset \mathfrak{g}_{\text{tot}}(X)^0 \), we have \( \text{ad}_x(x) = \text{ad}_\varphi(x) \).

*Proof.* The first statement follows from Definition 2.5, because \( \mathcal{G}(X) \) fixes all the Pontryagin classes, and \( H^{4n}(X, \mathbb{Q}) \) is spanned by a polynomial in the Pontryagin classes (see the paragraph before Definition 2.5).

For the second statement, consider the composition of the inclusion \( \mathfrak{s}\mathfrak{o}(V, q) \subset \mathfrak{g}_{\text{tot}}(X) \) and the homomorphism \( \mathfrak{g}_{\text{tot}}(X) \to \text{End}(V) \) obtained by restricting the action of \( \mathfrak{g}_{\text{tot}}(X) \) to \( H^2(X, \mathbb{Q}) \). This composition equals the canonical embedding \( \mathfrak{s}\mathfrak{o}(V, q) \subset \text{End}(V) \) (see e.g. [LL, Claim 1 on p. 392]), so the adjoint action of \( \psi \) on \( \mathfrak{s}\mathfrak{o}(V, q) \) is determined by the action of its degree two component, which is \( \varphi \). \( \square \)

### 3. Hodge structures on the cohomology of hyperkähler manifolds

#### 3.1. Complex structures of hyperkähler type

If \( I_1 \) and \( I_2 \) are two complex structures on \( X \), and \( I_1 \) is of hyperkähler type, it is not a priori clear that \( I_2 \) is also of hyperkähler type. Two conditions are necessary for this: \( I_2 \) should admit a Kähler metric, and the canonical bundle of \( X_{I_2} \) should be trivial. The following lemma shows that these conditions are also sufficient under a technical assumption on \( b_2(X) \).

**Proposition 3.1.** Assume that \( X \) is of hyperkähler type with \( b_2(X) \geq 5 \). Let \( I \) be an arbitrary complex structure on \( X \). The following conditions are equivalent:

1. \( I \) is of hyperkähler type;
2. \( I \) admits a Kähler metric and \( c_1(K_{X_I}) = 0 \).

*Proof.* Since the top exterior power of a symplectic form trivializes the canonical bundle, the implication \((1) \Rightarrow (2)\) is obvious. Let us prove the converse.

According to the decomposition theorem of Bogomolov [B2],

\[ X_I \cong Y \times Z_1 \times \ldots \times Z_m, \]

where \( Y \) is a Calabi-Yau manifold with \( h^{2,0}(Y) = 0 \) and \( Z_i \) are hyperkähler manifolds in the sense of Definition 2.4. Let \( \dim \mathbb{C}(X_I) = 2n \) and \( \dim \mathbb{C}(Z_i) = 2n_i \).

It follows from (2.5) that the multiplication map \( S^n H^2(X, \mathbb{C}) \to H^{2n}(X, \mathbb{C}) \) is injective. Assume that \( n_i < n \) for some \( i \). Let \( \pi : X_I \to Z_i \) be the projection and \( \sigma \in H^0(Z_i, \Omega^2_{Z_i}) \) be the symplectic form. Then \( (\pi^* \sigma)^n = \pi^*(\sigma^n) = 0 \), which is a contradiction. We conclude that \( m \leq 1 \), and if \( m = 1 \), then \( X_I \cong Z_1 \).
It remains to exclude the case \( m = 0 \), i.e. \( X_I \simeq Y \). Assuming that this is the case, let \( \omega \in H^2(X, \mathbb{R}) \) be a Kähler class for \( I \) and \( H^2_\omega(X, \mathbb{R}) = \{ a \in H^2(X, \mathbb{R}) \mid \int_X \omega^{n-1} a = 0 \} \). It follows from (2.2) that \( q(\omega) \neq 0 \). The equation (2.3) shows that \( H^2_\omega(X, \mathbb{R}) \) is the \( q \)-orthogonal complement of \( \omega \). Since we assume that \( h^{2,0}(X_I) = 0 \), the Hodge-Riemann bilinear relations and the formula (2.4) with \( a = \omega \) imply that \( q \) is sign-definite on \( H^2_\omega(X, \mathbb{R}) \). Since the signature of \( q \) is \( (3, b_2(X) - 3) \), this contradicts our assumptions on \( b_2(X) \). This completes the proof. \( \square \)

3.2. Hodge structures. As before, we will denote by \( q \) the BBF form on \( V = H^2(X, \mathbb{Q}) \), see Definition 2.5. Let \( I_1, I_2 \) be complex structures of hyperkähler type on \( X \). Then \( H^2(X_{I_1}, \mathbb{Q}) \) and \( H^2(X_{I_2}, \mathbb{Q}) \) are rational Hodge structures having the same underlying vector space \( V \).

Definition 3.2. A rational Hodge isometry between \( H^2(X_{I_1}, \mathbb{Q}) \) and \( H^2(X_{I_2}, \mathbb{Q}) \) is an element \( \varphi \in O(V, q) \), such that \( \varphi(H^{p,q}(X_{I_1})) = H^{p,q}(X_{I_2}) \) for all \( p + q = 2 \).

Definition 3.3. Define the following subgroups of \( O(V, q) \):

1. \( J^+ \) is the image of the homomorphism \( G^+(X) \rightarrow O(V, q) \) (see Proposition 2.11);
2. \( J \subset J^+ \) is the image of the composition \( G(X) \rightarrow G^+(X) \rightarrow O(V, q) \).

We are interested in Hodge isometries that are contained either in \( J \) or in \( J^+ \). Let us give some sufficient conditions for an isometry of \( (V, q) \) to be contained in one of these groups. Recall that there exists a group homomorphism

\[ SN: SO(V, q) \rightarrow \mathbb{Q}^\times / (\mathbb{Q}^\times)^2, \]

called the spinor norm, and that

\[ \text{Ker}(SN) = \text{Im}(\text{Spin}(V, q) \rightarrow SO(V, q)), \]

see e.g. [Kn, Abschnitt 8].

Proposition 3.4. We have the following inclusions:

1. \( \text{SO}(V, q) \subset J^+ ; \)
2. \( \text{Ker}(SN) \subset J. \)

Proof. Both inclusions easily follow from the definitions and Proposition 2.10. \( \square \)

Remark 3.5. The inclusions in Proposition 3.4 are in general strict. For example, any diffeomorphism \( \Phi: X \rightarrow X \) induces an isometry \( \varphi = \Phi^* \) of \( (V, q) \), and \( \varphi \in J \). But such \( \varphi \) does not in general preserve the orientation on \( V \), so does not always lie in \( \text{SO}(V, q) \).

Since \( \text{SO}(V, q) \) is of index two in \( O(V, q) \), it is enough to produce one element of \( J \) that does not preserve the orientation on \( V \) to prove that \( J^+ = O(V, q) \). For all known examples of compact hyperkähler manifolds one can do that, because their monodromy group (see [Ma, Definition 1.1]) contains reflections along the classes of prime exceptional divisors (see [Ma, Definition 5.1]). For varieties of K3\[n\] type, generalized Kummer type and O’Grady’s 10-dimensional example, see Theorem 9.1 and two paragraphs after Conjecture 10.6 in [Ma]. For O’Grady’s 6-dimensional example the existence of a prime exceptional divisor follows from [Na].

We can now state the main result.

Theorem 3.6. Let \( I_1 \) and \( I_2 \) be two complex structures of hyperkähler type on a compact simply-connected manifold \( X \) with \( \dim \mathbb{R}(X) = 4n \). Assume that there exists a rational Hodge isometry

\[ \varphi: H^2(X_{I_1}, \mathbb{Q}) \overset{\sim}{\rightarrow} H^2(X_{I_2}, \mathbb{Q}). \]
If \( \varphi \in \mathcal{J} \), then there exists an isomorphism of rational Hodge structures

\[ \psi: H^\ast(X_1, \mathbb{Q}) \cong H^\ast(X_2, \mathbb{Q}) \]

that extends \( \varphi \), respects the grading and the algebra structure;

\( \varphi \in \mathcal{J}^+ \), then there exists an isomorphism of rational Hodge structures

\[ \psi: H^{2\ast}(X_1, \mathbb{Q}) \cong H^{2\ast}(X_2, \mathbb{Q}) \]

that extends \( \varphi \), respects the grading and the algebra structure.

Proof. Assume that \( \varphi \in \mathcal{J} \), and let \( \psi \in \mathcal{G}(X) \) be a preimage of \( \varphi \), see Definition 3.3. The action of \( \psi \) respects the algebra structure and the grading by the definition of \( \mathcal{G}(X) \). It remains to check that \( \psi \) is a morphism of Hodge structures. The complex structures \( I_1 \) and \( I_2 \) can be completed to a pair of hyperkähler structures \( I_1, J_1, K_1 \) and \( I_2, J_2, K_2 \), see section 2.3. Consider the operators \( W_{I_1} \) and \( W_{I_2} \) from Proposition 2.7. The action of these operators on differential forms descends to cohomology, so let us denote by \( w_1 \) and \( w_2 \) the corresponding endomorphisms of \( H^\ast(X, \mathbb{C}) \). The endomorphisms \( w_1 \) and \( w_2 \) are the Weil operators that induce the Hodge decomposition on the cohomology. It follows from Proposition 2.7, that \( w_1, w_2 \in \mathfrak{so}(V, q) \subset \mathfrak{g}_{tot}(X)^0 \).

By our assumptions, \( \varphi \) is a morphism of Hodge structures. In terms of the Weil operators, this means \( \text{ad}_\varphi (w_1) = w_2 \). By Proposition 2.11 the adjoint action of \( \psi \) on \( \mathfrak{so}(V, q) \) is determined by the action of its degree two component, which equals \( \varphi \). Hence we have \( \text{ad}_\psi (w_1) = w_2 \). This shows that the components \( \psi_k \) in every degree \( k \) are morphisms of Hodge structures. This proves the first part of the theorem. The proof of the second part is analogous.

Corollary 3.7. Let \( I_1 \) and \( I_2 \) be two complex structures of hyperkähler type on a compact simply-connected manifold \( X \). Assume that \( I_1 \) and \( I_2 \) define the same Hodge structure on \( H^2(X, \mathbb{Q}) \). Then they define the same Hodge structure on \( H^k(X, \mathbb{Q}) \) for all \( k \).

Proof. Apply the previous theorem to \( \varphi = \text{Id} \), and note that in its proof we can choose \( \psi = \text{Id} \).

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