Hamiltonian Analysis of Gauged $CP^1$ Model, with or without Hopf term, and fractional spin

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Abstract: Recently it has been shown by Cho and Kimm that the gauged $CP^1$ model, obtained by gauging the global $SU(2)$ group of $CP^1$ model and adding a corresponding Chern-Simons term, has got its own soliton. These solitons are somewhat distinct from those of pure $CP^1$ model, as they cannot always be characterised by $\pi_2(CP^1) = \mathbb{Z}$. In this paper, we first carry out the Hamiltonian analysis of this gauged $CP^1$ model. Then we couple the Hopf term, associated to these solitons and again carry out its Hamiltonian analysis. The symplectic structures, along with the structures of the constraints, of these two models (with or without Hopf term) are found to be essentially the same. The model with Hopf term, is then shown to have fractional spin, which however depends not only on the soliton number $N$ but also on the nonabelian charge.

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1. Introduction

Recently there has been an upsurge of interest in the study of physics of 2 + 1-dimensional systems. Particularly because of the strange nature of the Poincare group ISO(2, 1) in 2 + 1-dimension, in contrast to ISO(3, 1), there arises possibilities of nontrivial configuration space \( \mathcal{Q} \) and the associated fractional spin and statistics. These possibilities can be realised in practice by adding topological terms like the Chern-Simons(CS) or Hopf term in the model\[1,2\]. Fractional spin and Galilean/Poincare’ symmetry in these various models have been exhibited in detail in the literature, where both path integral\[2,3,4\] and the canonical analysis\[2,5-9\] have been performed.

The CS term (abelian) is a local expression (\( \sim \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \)) involving a “photon-less” gauge field \( a_\mu[5] \). This gauge field is basically introduced to mimic, in the manner of Aharanov-Bohm, the phase acquired by the system in traversing a nontrivial loop in the configuration space\[2\]. On the other hand, the Hopf term is usually constructed by writing the conserved current \( j^\mu (\partial_\mu j^\mu = 0) \) of a model as a curl of a ‘fictitious’ gauge field \( a_\mu \):

\[
j^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda
\]

and then contracting \( j^\mu \) with \( a_\mu \) to get the Hopf term \( H \) as,

\[
H \sim \int d^3x j^\mu a_\mu
\]

Written entirely in terms of \( a_\mu \) (using (1.1)), this Hopf term has also the appearance of CS term. However there is a subtle difference. In the case of CS term, the gauge field \( a_\mu \) should be counted as an independent variable in the configuration space\[7-9\]. This is despite the fact that the gauge field is “photon-less”. On the other hand, the \( a_\mu \) in (1.1) is really a ‘fictitious’ gauge field (as we have mentioned above) and is not an independent variable in the configuration space. It has to be, rather, determined by inverting (1.1), by making use of a suitable gauge fixing condition. Once done that, the Hopf term (1.2) represents a non-local current-current interaction. It should however be mentioned that this distinction, in terminology, has not always been maintained in the literature (see for example \[4\]).

This inherent non-locality arising in the very construction of the Hopf term, can however be avoided in certain cases by enlarging the phase space. For example, consider the \( O(3) \) non-linear sigma model(NLSM). The model has solitons\[10\] and an associated topological current. The Hopf term here is again non-local. On the other hand, one can consider \( CP^1 \) model, which is known to be equivalent to NLSM\[10,11\]. This is a \( U(1) \) gauge theory, having an enlarged phase space. Here the Hopf term becomes local\[11,12\]. In this case, the Hopf term has a geometrical interpretation and provides a representation of \( \pi_1(\mathcal{Q}) \). Note that the configuration space for NLSM is basically given as the space \( \mathcal{Q} = \text{Map}(S^2, S^2) \), so that \( \pi_1(\mathcal{Q}) = \pi_3(S^2) = Z \). However it should be mentioned that this trick of enlarging the phase space and writing a local expression of the Hopf term may not work all the time, as we shall see later in this paper.
That the Hopf term can impart fractional spin, was demonstrated initially by Wilczek and Zee[3], in the context of the NLSM, using path integral technique. It has been found to depend on the soliton number. This result was later corroborated by Bowick et al.[13], using canonical quantization. On the other hand, the fractional spin obtained in the models involving abelian(nonabelian) CS term, have been found to depend on the total abelian (nonabelian) charge of the system.

This is an important observation, considering the fact that NLSM has become almost ubiquitous in physics, appearing in various circumstances where the original \(O(3)\) symmetry is broken spontaneously. For example, in particle physics, the model is considered a prototype of QCD, as the model is asymptotically free in \((1 + 1)\) dimension. On the other hand, in condensed matter physics, this model can describe antiferromagnetic spin chain in its relativistic version[14]. And in its nonrelativistic version, it can describe a Heisenberg ferromagnetic system in the long wavelength limit, i.e. the Landau-Lifshitz(LL) model[15,16]. Besides, the Hopf term can arise naturally in this NLSM, when one quantises a \(U(1)\) degree of freedom hidden in the configuration space \(Q\), as has been shown recently by Kobayashi et. al.[17]. Further, it has been shown recently in [12], that the Hopf term can alter the spin algebra of the LL model drastically.

This NLSM has global \(O(3)\) symmetry. Recently Nardelli[18] has shown that if this \(O(3)\) group is gauged by adding an \(SO(3)\) CS term, then the resulting model also has got its own soliton. This work was later extended by Cho and Kimm[19] for the general \(CP^N\) model, where one has to gauge the global \(SU(N+1)\) group and add a corresponding CS term. These solitons are somewhat distinct from those of pure \(CP^N\) model, in the sense that these are not always characterised by the second homotopy group \((\pi_2(CP^N) = \mathbb{Z})\) of the manifold, unlike the pure \(CP^1\) model[10].

The purpose of the paper is to investigate \((N = 1)\ case), whether a Hopf term ,associated with this new soliton number, can be added to the model to obtain fractional spin. The question is all the more important, as the model has already got an nonabelian \(SU(2)\) CS term, needed for the very existence of these new type of solitons. And, as we have mentioned earlier, the CS term, is likely to play its own role in imparting fractional spin to the model. We find that the fractional angular momentum is given in terms of both the soliton number and the nonabelian charge.

To that end, we organise the paper as follows. In section 2, we carry out the Hamiltonian analysis of the gauged \(CP^1\) model. The Hopf term is introduced in section 3 and again the Hamiltonian analysis of the resulting model is performed. In section 4, we compute the fractional spin of the model. Finally we conclude in section 5.

2. Hamiltonian Analysis of Gauged \(CP^1\) model

We are going to carry out the Hamiltonian analysis of the gauged \(CP^1\) model, as introduced
by Cho and Kimm[19]. The model is given by,
\[
\mathcal{L} = (D_{\mu}Z)^\dagger (D^{\mu}Z) + \theta e^{\mu\nu\lambda} [A^{a}_{\mu}\partial_{\nu}A^{a}_{\lambda} + \frac{g}{3} e^{abc}A^{b}_{\mu}A^{c}_{\nu}A^{a}_{\lambda}] - \lambda(Z^\dagger Z - 1) \tag{2.1}
\]
where \(Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\) is an \(SU(2)\) doublet satisfying
\[
Z^\dagger Z = 1 \tag{2.2}
\]
and enforced by the Lagrange multiplier \(\lambda\) in (2.1). The covariant derivative operator \(D_{\mu}\) is given as
\[
D_{\mu} = \partial_{\mu} - ia_{\mu} - igA^{a}_{\mu}T^{a} \tag{2.3}
\]
with \(T^{a} = \frac{1}{2}\sigma^{a}\) (\(\sigma^{a}\), the Pauli matrices), representing the \(SU(2)\) generators and \(g\) a coupling constant. \(\theta\) represents the CS parameter. And finally \(a_{\mu}\) and \(A^{a}_{\mu}\) represent the \(U(1)\) and \(SU(2)\) gauge fields respectively. Note that there is no dynamical CS term for the\( a_{\mu}\) field.

The canonically conjugate momenta variables corresponding to the configuration space variables \((a_{\mu}, A^{a}_{\mu}, z_{\alpha}, z^{\star}_{\alpha})\) are given as,
\[
\pi^\mu = \frac{\delta L}{\delta \dot{a}_{\mu}} = 0 \\
\Pi^{ia} = \frac{\delta L}{\delta \dot{A}^{a}_{i}} = \theta e^{ij} A^{a}_{j}, \Pi^{0a} = \frac{\delta L}{\delta \dot{A}^{a}_{0}} = 0 \\
\pi_{\alpha} = \frac{\delta L}{\delta \dot{z}_{\alpha}} = (D_{0}Z)^{\ast}_{\alpha}, \pi^{\ast}_{\alpha} = \frac{\delta L}{\delta \dot{z}^{\ast}_{\alpha}} = (D_{0}Z)_{\alpha} \tag{2.4}
\]
where \(L = \int d^{2}x \mathcal{L}\) is the Lagrangian.

The Legendre transformed Hamiltonian
\[
\mathcal{H} = \pi^\mu \dot{a}_{\mu} + \Pi^{ia} \dot{A}^{a}_{i} + \pi_{\alpha} \dot{z}_{\alpha} + \pi^{\ast}_{\alpha} \dot{z}^{\ast}_{\alpha} - \mathcal{L} \tag{2.5a}
\]
when expressed in terms of the phase space variables (2.4), gives
\[
\mathcal{H} = \pi^{\ast}_{\alpha} \pi_{\alpha} + ia_{0}(\pi_{\alpha} z_{\alpha} - \pi^{\ast}_{\alpha} z^{\ast}_{\alpha}) + \frac{1}{2} i g A^{a}_{0}[\pi_{\alpha}(\sigma^{a} Z)_{\alpha} - \pi^{\ast}_{\alpha}(Z^\dagger \sigma^{a})_{\alpha}] - 2\theta A^{a}_{0} B^{a} + (D_{i}Z)^{\dagger}(D_{i}Z) + \lambda(Z^\dagger Z - 1) \tag{2.5b}
\]
where
\[
B^{a} \equiv F^{a}_{12} = (\partial_{1} A^{a}_{2} - \partial_{2} A^{a}_{1} + \frac{g}{3} e^{abc} A^{b}_{1} A^{c}_{2}) \tag{2.5c}
\]
is the non-abelian \(SU(2)\) magnetic field.
Clearly the fields $a_0, A^a_0$ and $\lambda$ play the role of Lagrange multipliers, which enforce the following constraints,

$$G_1(x) \equiv i(\pi_\alpha(x)z_\alpha(x) - \pi^*_\alpha(x)z^*_\alpha(x)) \approx 0 \quad (2.6)$$

$$G^a_2(x) \equiv \frac{ig}{2}[\pi_\alpha(x)(\sigma^a Z(x))_\alpha - \pi^*_\alpha(x)(Z^\dagger(x)\sigma^a)_\alpha] - 2\theta B^a(x) \approx 0 \quad (2.7)$$

$$\chi_1(x) \equiv Z^\dagger(x)Z(x) - 1 \approx 0. \quad (2.8)$$

Apart from all these constraints, we have yet another primary constraint,

$$\chi_2(x) \equiv (\pi_\alpha(x)z_\alpha(x) + \pi^*_\alpha(x)z^*_\alpha(x)) \approx 0 \quad (2.9)$$

Also the preservation of the primary constraint $\pi^i(x) \equiv 0 \quad (2.4)$ yield the following secondary constraint,

$$2iZ^\dagger\partial_j Z + 2a_j + gA^a_0M^a \approx 0 \quad (2.10)$$

Here

$$M^a = Z^\dagger\sigma^a Z \quad (2.11)$$

is a unit 3-vector, obtained from the $CP^1$ variables using the Hopf map. We are left with a pair of primary constraints from the CS gauge field sector in (2.4),

$$\xi^{ia} \equiv \Pi^{ia} - \theta \epsilon^{ij} A^a_j \approx 0 \quad (2.12)$$

This pair of constraints can be implemented strongly by the bracket,

$$\{A^a_i(x), A^b_j(y)\} = \frac{1}{2\theta} \epsilon^{ij} \delta^{ab} \delta(x - y) \quad (2.13)$$

obtained either by using Dirac method[20] or by the symplectic technique of Faddeev-Jackiw[21].

Also note that the constraint $\pi^i \approx 0 \quad (2.4)$ is conjugate to the constraint (2.10) and can again be strongly implemented by the Dirac bracket (DB),

$$\{\pi^i(x), a_j(y)\} = 0 \quad (2.14)$$

With this the ‘weak’ equality in (2.10) is actually rendered into a strong equality and the field $a_i$ ceases to be an independent degree of freedom.

Finally the constraints $\chi_1 \quad (2.8)$ and $\chi_2 \quad (2.9)$ are conjugate to each other and are implemented strongly by the following DBs,

$$\{z_\alpha(x), z_\beta(y)\} = \{z_\alpha(x), z^*_\beta(y)\} = 0$$

$$\{z_\alpha(x), \pi_\beta(y)\} = (\delta_{\alpha\beta} - \frac{1}{2}z_\alpha z^*_\beta)\delta(x - y)$$
Precisely the same set of brackets (2.15) are obtained in the case of $CP^1$ model also[22]. We are thus left with the constraints (2.6) and (2.7) and are expected to be the Gauss constraints generating $U(1)$ and $SU(2)$ gauge transformations respectively. The fact that this is indeed true will be exhibited by explicit computations. But before we proceed further, let us note that the constraints (2.8),(2.9) and (2.10) hold strongly now. In view of this, the constraint $G_1$ (2.6) can be simplified as,

$$G_1(x) = 2i\pi_\alpha(x)z_\alpha(x) \approx 0$$

At this stage, one can substitute $\pi_\alpha = (D_0Z)^\dagger_\alpha$ from (2.4) and solve for $a_0$ to get,

$$a_0 = -iZ^\dagger\partial_0Z - \frac{1}{2}gA_\mu^aM^a$$

Clearly this is not a constraint equation, as it involves time derivative. It is nevertheless convenient to club it with the expression of $a_i$, obtained from (2.10) and write covariantly as,

$$a_\mu = -iZ^\dagger\partial_\mu Z - \frac{1}{2}gA_\mu^aM^a$$

Here the first term ($-iZ^\dagger\partial_\mu Z$) is the pullback, onto the spacetime, of the $U(1)$ connection on the $CP^1$ manifold[16]. The second term on the other hand has nothing to do with $CP^1$ connection and arises from the presence of the CS gauge field $A_\mu^a$.

It is now quite trivial to show that $G_1(x)$ (2.16) generates $U(1)$ gauge transformation on the $Z$ fields

$$\delta Z(x) = \int d^2y f(y)\{Z(x), G_1(y)\} = if(x)Z(x)$$

but leaves the CS gauge field $A_\mu^a$ unaffected

$$\delta A_\mu^a(x) = \int d^2y f(y)\{A_\mu^a(x), G_1(y)\} = 0$$

Consequently $M^a = Z^\dagger\sigma^a Z$ (2.11) remains invariant under this transformation and hence $a_\mu$ (2.18) undergoes the usual gauge transformation

$$\delta a_\mu(x) = \int d^2y f(y)\{a_\mu(x), G_1(y)\} = \partial_\mu f(x)$$
Here in the equations (2.19-2.21) we have taken \( f(x) \) to be an arbitrary differentiable functions with compact support.

Proceeding similarly, one can show that the constraints \( G^a_2(x) \) (2.7) generates \( SU(2) \) gauge transformation,

\[
\delta Z(x) \equiv \int d^2y f^a(y)\{Z(x), G^a_2(y)\} = igf^a(x)(T^a Z(x)) \tag{2.22a}
\]

\[
\delta A^a_i(x) \equiv \int d^2y f^b(y)\{A^a_i(x), G^b_2(y)\} = \partial_i f^a(x) - g\epsilon^{abc}f^b(x)A^c_i(x) \tag{2.22b}
\]

Using these one can also show that,

\[
\delta F^a_{ij} = -g\epsilon^{abc}f^bF^c_{ij} \tag{2.23a}
\]

\[
\delta M^a = -g\epsilon^{abc}f^bM^c \tag{2.23b}
\]

but \( M^aF^a_{ij} \) is an \( SU(2) \) scalar as

\[
\delta(M^aF^a_{ij}) = 0 \tag{2.24}
\]

just as \( A^a_\mu(x) \) remains unaffected by this \( G_2^a \),

\[
\delta a_\mu(x) = \int d^2y f^a(y)\{a_\mu(x), G^a_2(y)\} = 0 \tag{2.25}
\]

It also follows from (2.23a) and (2.18) that \( a_\mu \) remains unaffected by this \( G_1^a \)

\[
G_1(x) = 2i(D_0 Z)^\dagger Z \approx 0 \tag{2.26}
\]

\[
\{G_1(x), G_2^a(y)\} = 0 \tag{2.26}
\]

It also follows after a straightforward algebra that \( G_2^a \)'s satisfy an algebra isomorphic to \( SU(2) \) Lie algebra and thus vanishes on the constraint surface,

\[
\{G^a_2(x), G^b_2(y)\} = 2\epsilon^{abc}G_2^c(x)\delta(x - y) \approx 0 \tag{2.27}
\]

Finally note that (2.18) really corresponds to the Euler-Lagrange’s equation for the \( a_\mu \) field. The corresponding equations for \( Z \) and \( A^a_\lambda \) are given by,

\[
D_\mu D^\mu Z + \lambda Z = 0 \tag{2.28}
\]

\[
\theta\epsilon^{\mu\nu\lambda}F^a_{\mu\nu} = ig[(D^\mu Z)^\dagger T^a Z - Z^\dagger T^a (D^\mu Z)] \tag{2.29}
\]
3. Introducing the Hopf term

In order to introduce the Hopf term, it will be convenient to provide a very brief review of some of the essential features of these new solitons. For this we essentially follow [19]. The symmetric expression for the energy-momentum (EM) tensor, as obtained by functionally differentiating the action $S = \int d^3x L$ with respect to the metric, is given by

$$T_{\mu\nu} = (D_\mu Z)\dagger(D_\nu Z) + (D_\nu Z)\dagger(D_\mu Z) - g_{\mu\nu}(D_\rho Z)\dagger(D_\rho Z) \quad (3.1)$$

The energy functional

$$E = \int d^2x T_{00} = \int d^2x[2(D_0 Z)\dagger(D_0 Z) - (D_\mu Z)\dagger(D^\mu Z)] \quad (3.2)$$

can be expressed alternatively as,

$$E = \int d^2x(|D_0 Z|^2 + |(D_1 \pm iD_2)Z|^2) \pm 2\pi N \quad (3.3a)$$

where

$$N = \frac{1}{2\pi i} \int d^2x \epsilon^{ij}(D_i Z)\dagger(D_j Z) \quad (3.3b)$$

is the soliton charge.

It immediately follows that the energy functional satisfy the following inequality,

$$E \geq 2\pi|N| \quad (3.4)$$

The corresponding saturation conditions are,

$$|D_0 Z|^2 = |(D_1 \pm iD_2)Z|^2 = 0 \quad (3.5)$$

For static configuration ($\dot{Z} = 0$), this yields

$$A_0^a = kM^a \quad (3.6)$$

where $k$ is an arbitrary constant.

Again assuming the static case, one can easily show that $\mu = 0$ component of the Euler-Lagrange equation (2.29) implies that the $SU(2)$ magnetic field $B^a$ vanishes,

$$B^a = 0 \quad (3.7)$$

where use of (3.6) has been made. This in turn implies that $A_i^a$ is a pure gauge, so that one can write without loss of generality

$$A_i^a = 0 \quad (3.8)$$
In this gauge, the soliton charge \( N \) (3.3b) reduces to the standard \( CP^1 \) soliton charge,
\[
N = \frac{1}{2\pi i} \int d^2x \epsilon^{ij} (D_i Z)^\dagger (D_j Z) 
\]
where
\[
D_i = D_i|_{A^a_i=0} = \partial_i - (Z^\dagger \partial_i Z) 
\]
is the covariant derivative operator for the standard \( CP^1 \) model. Thus in this gauge (3.8), the “soliton charge” is essentially characterised by \( \pi_2(CP^1) = Z \). Nonetheless, it is possible to make “large” topology changing gauge transformation, where \( A^a_i \) is no longer zero and one has to make use of (3.3b), rather than (3.9a), to compute the solitonic charge. Of course this will yield the same value for the charge, but the various solitonic sectors will not be characterised by \( \pi_2(CP^1) \) anymore.

To make things explicit, consider a typical solitonic configuration:
\[
Z = \frac{1}{\sqrt{r^2 + \lambda^2}} \left( \begin{array}{c} re^{-i\Phi} \\ \lambda \end{array} \right) 
\]
\[
A^a_i = 0 
\]
where \((r, \Phi)\) represents the polar coordinates in the two-dimensional plane and \( \lambda \) is the size of the soliton. The corresponding unit vector \( M^a(2.11) \) takes the form,
\[
M^1 = sin\Theta cos\Phi = \frac{2r\lambda}{r^2 + \lambda^2} cos\Phi \\
M^2 = sin\Theta sin\Phi = \frac{2r\lambda}{r^2 + \lambda^2} sin\Phi \\
M^3 = cos\Theta = \frac{r^2 - \lambda^2}{r^2 + \lambda^2} 
\]
We therefore have for the time component of the gauge field \( A^a_0 = kM^a \) (3.6).

At this stage, one can make a “large” topology changing gauge transformation,
\[
Z \rightarrow Z' = UZ = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) 
\]
where,
\[
U = \frac{1}{\sqrt{r^2 + \lambda^2}} \left( \begin{array}{cc} \lambda & -re^{-i\Phi} \\ re^{i\Phi} & \lambda \end{array} \right) \in SU(2) 
\]
so that \( A^a_0 \) undergoes the transformation,
\[
A^a_0 \rightarrow A'^a_0 = -k \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) 
\]
The spatial components on the other hand, undergoes the transformation,

\[ A_i \rightarrow A_i' = UA_iU^{-1} + \frac{i}{g} U\partial_i U^{-1} = -\frac{i}{g} (\partial_i U) U^{-1} \]

which on further simplification yields the following form for the connection one-form in the cartesian coordinate system,

\[ A^1 = \frac{2\lambda}{g(r^2 + \lambda^2)} dy \]
\[ A^2 = -\frac{2\lambda}{g(r^2 + \lambda^2)} dx \]  \hspace{1cm} (3.12d)
\[ A^3 = -\frac{2}{g(r^2 + \lambda^2)} (xdy - ydx) \]

Now it is a matter of straightforward exercise to calculate the soliton charge ‘\(N\)’ in either of these gauges (3.10) and (3.12). For example, in the gauge (3.10), this can be computed by using (3.9a) to get,

\[ N = \frac{1}{2\pi} \int d(-iZ^\dagger dZ) = -1 \]  \hspace{1cm} (3.13)

On the other hand, the same soliton charge can also be computed in the gauge (3.12), but where the use of (3.3b), rather than (3.9a), has to be made. Note that the topological density \( j^0 \) (\( N \equiv \int d^2 x j^0 \)) can be written as,

\[ j^0 = \frac{1}{2\pi i} \epsilon^{ij} (D_i Z)^\dagger (D_j Z) = \tilde{j}^0 + \frac{g}{4\pi} \epsilon^{ij} A_i^a (\partial_j M^a + \frac{g}{2} \epsilon^{abc} A_j^b M^c) \]  \hspace{1cm} (3.14a)

where,

\[ \tilde{j}^0 = \frac{\epsilon^{ij}}{2\pi i} (D_i Z)^\dagger (D_j Z) \]  \hspace{1cm} (3.14b)

is the expression of topological density in the gauge (3.10). But in the gauge (3.12), this \( \tilde{j}^0 \) vanishes, and one can rewrite \( N \) completely in terms of the CS gauge field as,

\[ N = \frac{g^2}{8\pi k} \int d^2 x \epsilon^{ij} \epsilon^{abc} A_i^a A_j^b A_c^c \]  \hspace{1cm} (3.15)

The corresponding \( Z \) field configuration being trivial (\( Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)), the soliton number \( N \) cannot be captured by \( \pi_2(CP^1) \).

Since ‘\(N\)’ is a conserved soliton charge, with an associated topological density \( j^0(3.14a) \), one can regard \( j^0 \) to be the time-component of a conserved topological 3(= 2 + 1)-current

\[ j^\mu = \frac{1}{2\pi i} \epsilon^{\mu\nu\lambda} (D_\nu Z)^\dagger (D_\lambda Z) \]  \hspace{1cm} (3.16)
This can therefore be expressed as the curl of a ‘fictitious’ $U(1)$ gauge field $A_\lambda$:

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$  \hspace{1cm} (3.17)

Unlike the case of pure $CP^1$ model[11,12], this equation cannot be solved trivially for $A_\lambda$ in a gauge independent manner[2,16]. We therefore find it convenient to follow Bowick et.al.[13], to solve (3.17) for $A_\lambda$ in the radiation gauge ($\partial_\lambda A_\lambda = 0$), where one can prove the following identity,

$$\int d^3 x j_0 A_0 = -\int d^3 x j_\mu A_\mu$$ \hspace{1cm} (3.18)

so that the Hopf action

$$S_{Hopf} = \Theta \int d^3 x j^\mu A_\mu,$$ \hspace{1cm} (3.19)

($\Theta$ being the Hopf parameter and should not be confused with the spherical angles introduced in (3.11)) simplifies to the following non-local term

$$S_{Hopf} = -2\Theta \int d^3 x j_\mu A_\mu.$$ \hspace{1cm} (3.20)

Adding this term to the original model (2.1), we get the following model,

$$\mathcal{L} = (D_\mu Z)\dagger (D^\mu Z) + \theta \epsilon^{\mu\nu\lambda}[A^a_\mu \partial_\nu A^a_\lambda + \frac{g}{3} \epsilon^{abc} A^a_\mu A^b_\nu A^c_\lambda]$$

$$+ \frac{\Theta}{\pi i} \epsilon^{ij} A_i [(D_j Z)\dagger (D_0 Z) - (D_0 Z)\dagger (D_j Z)] - \lambda (Z\dagger Z - 1)$$ \hspace{1cm} (3.21)

In the rest of this section, we shall be primarily concerned with the Hamiltonian analysis of this model. As the Hopf term is linear in time derivative of the $Z$ variable, the analysis is expected to undergo only minor modification. Indeed we shall verify this by explicit computations.

To begin with, note that the only change in the form of canonically conjugate momenta variables takes place in the variables $\tilde{\pi}_\alpha$ and its complex conjugates, counterpart of $\pi_\alpha$ and $\pi_\alpha^*$ (2.4)-the momenta variables for the model (2.1). They are now given as,

$$\tilde{\pi}_\alpha = (D_0 Z)_\alpha^* + \frac{\Theta}{\pi i} \epsilon^{ij} A_i (D_j Z)_\alpha^* = \pi_\alpha + \frac{\Theta}{\pi i} \epsilon^{ij} A_i (D_j Z)_\alpha^*$$

$$\tilde{\pi}_\alpha^* = (D_0 Z)_\alpha - \frac{\Theta}{\pi i} \epsilon^{ij} A_i (D_j Z)_\alpha = \pi_\alpha^* - \frac{\Theta}{\pi i} \epsilon^{ij} A_i (D_j Z)_\alpha$$ \hspace{1cm} (3.22)

Rest of the momenta variables undergo no change from that of (2.4).

The Legendre transformed Hamiltonian $\tilde{\mathcal{H}}$ can be calculated in a straightforward manner to get,

$$\tilde{\mathcal{H}} = \mathcal{H} + \frac{g\Theta}{2\pi} A^a_0 \epsilon^{ij} A_i (D_j M)^a$$ \hspace{1cm} (3.23)

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where $H$ is just the expression of the Legendre transformed Hamiltonian density (2.5) corresponding to the model (2.1) and $(D_j M)^a$ is given by,

$$D_j M^a = \partial_j M^a + g_{abc} A_j^b M^c$$

as can be easily obtained by using the Hopf map (2.11) and the fact that the covariant derivative operator $D_{\mu}$ boils down, using (2.3) and (2.18) to,

$$D_{\mu} Z = \partial_{\mu} Z - (Z^\dagger \partial_{\mu} Z) Z + \frac{ig}{2} A_{\mu}^a (M^a - \sigma^a) Z$$

Clearly the structure of all the constraints remain the same, except the $SU(2)$ Gauss constraint. This is clearly given as,

$$\tilde{G}^a_2 = G^a_2 + \frac{g\Theta}{2\pi} \epsilon^{ij} A_i (D_j M)^a$$

where $G^a_2$ is given in (2.7). But we have to rewrite this in terms of $\tilde{\pi}_\alpha$ and $\tilde{\pi}_\alpha^*$. Once we do this, we find that that the $SU(2)$ Gauss constraint $G^a_2$ (2.7) for the model (2.1) is now given by,

$$G^a_2 = ig\left( [\tilde{\pi}_\alpha (T^a Z)_{\alpha} - \tilde{\pi}_\alpha^* (Z^\dagger T^a)_{\alpha}] + \frac{i\Theta}{2\pi} \epsilon^{ij} A_i (D_j M)^a \right) - 2\theta B^a \approx 0$$

Substituting this in (3.26), $\tilde{G}^a_2$ is found to have the same form as that of $G^a_2$ (2.7) with the replacement $\pi_\alpha \rightarrow \tilde{\pi}_\alpha$ and $\pi_\alpha^* \rightarrow \tilde{\pi}_\alpha^*$,

$$\tilde{G}^a_2 = ig(\tilde{\pi}_\alpha (T^a Z)_{\alpha} - \tilde{\pi}_\alpha^* (Z^\dagger T^a)_{\alpha}) - 2\theta B^a \approx 0$$

The other $U(1)$ Gauss constraint $\tilde{G}_1$ (2.6) can also be seen to take the same form, with identical replacement,

$$\tilde{G}_1(x) = i(\tilde{\pi}_\alpha (x) z_\alpha (x) - \tilde{\pi}_\alpha^* z_\alpha^* (x)) \approx 0$$

, where use of the identity $Z^\dagger D_{\mu} Z = 0$ has been made.

Finally note that the constraint (2.9) also preserves its form, i.e.

$$\chi_2(x) = \tilde{\pi}_\alpha (x) z_\alpha (x) + c.c$$

and is again conjugate to $\chi_1$ (2.8). Thus these pair of constraints can be implemented strongly by using the DB (2.15), again taken with the replacement $\pi_\alpha \rightarrow \tilde{\pi}_\alpha$ and $\pi_\alpha^* \rightarrow \tilde{\pi}_\alpha^*$. On the other hand, the pair of second class constraints (2.12) are implemented strongly by the brackets (2.13) in this case also. These set of DB furnishes us with the symplectic structure of the model (3.21).

4. Angular momentum

In this section, we are going to find the fractional spin imparted by the Hopf term. As was done for the models involving the CS[7-9] and Hopf[13] term, the fractional spin was essentially revealed by computing the difference $(J^s - J^N)$ between the expression of angular momentum
$J^s$, obtained from the symmetric expression of the EM tensor $T_{\mu\nu}$ ($\sim \frac{\delta S}{\delta g^{\mu\nu}}$) and the one $J^N$, obtained by using Noether’s prescription. It is $J^s$, which is taken to be the physical angular momentum. This is because it is gauge invariant by construction, in contrast to $J^N$, which turn out to be gauge invariant only on the Gauss constraint surface and that too usually under those gauge transformations, which tend to identity asymptotically[8,9].

To that end, let us consider the generator of linear momentum. This is obtained by integrating the (0i) component of the EM tensor (3.1), which undergoes no modification as the metric independent topological (Hopf) term (3.20) is added to the original Lagrangian (2.1) to get the model (3.21).

$$P^s_i = \int d^2x T^s_{0i} = \int d^2x [(D_0 Z)^\dagger (D_i Z) + (D_i Z)^\dagger (D_0 Z)]$$ (4.1)

Expressing this in terms of phase-space variables (3.22), one gets

$$P^s_i = \int d^2x [\tilde{\pi}_\alpha (D_i Z)_{\alpha} + \tilde{\pi}^*_{\alpha} (D_i Z)^*_{\alpha} + 2\Theta A_i(x) j^0(x)]$$ (4.2)

This can now be re-expressed as,

$$P^s_i = \int d^2x [\tilde{\pi}_\alpha \partial_i z_\alpha + \tilde{\pi}^*_\alpha \partial_i z^*_\alpha - 2\theta A^a_i B^a + 2\Theta A_i j^0 - a_i \tilde{G}_1 - A^a_i \tilde{G}_2]$$ (4.3)

However this cannot be identified as an expression of linear momentum, as this fails to generate appropriate translation,

$$\{Z(x), P^s_i \} \approx D_i Z$$ (4.4)

in contrast to the corresponding expression of linear momentum

$$P^N_k = \int d^2x T^N_{0k} = \int d^2x [\tilde{\pi}_\alpha \partial_k z_\alpha + \tilde{\pi}^*_\alpha \partial_k z^*_\alpha - \theta \epsilon^{ij} A^a_i \partial_k A^a_j]$$ (4.5)

obtained through Noether’s prescription, as this generates appropriate translation by construction,

$$\{Z(x), P^N_k \} = \partial_k Z(x)$$

$$\{A^a_i(x), P^N_k \} = \partial_k A^a_i$$ (4.6)

The adjective “appropriate” in this context means that the bracket $\{\Phi(x), G\}$ is just equal to the Lie derivative ($L_{V_G} (\Phi(x))$) of a generic field $\Phi(x)$ with respect to the vector field $V_G$, associated to the symmetry generator $G$. We have not, of course, displayed any indices here. The field $\Phi$ may be a scalar, spinor, vector or tensor field in general. In this case, it can correspond either to the scalar field $Z(x)$ or the vector field $A^a_i(x)$. And $G$ can be, for example, the momentum($P_i$) or angular momentum ($J$) operator generating translation and spatial rotation respectively. The
associated vector fields $V_G$ are thus given as $\partial_i$ and $\partial_\phi$ respectively ($\phi$ being the angle variable in the polar coordinate system in 2-dimensional plane).

Coming back to the translational generator $P_i^a$ (4.3), we observe that the EM tensor (3.1) is not unique by itself. One has the freedom to modify it to $\tilde{T}_{\mu\nu}$ by a linear combination of first class constraint(s), here $\tilde{G}_1$ (3.28) and $\tilde{G}^a_2$ (3.27) with arbitrary tensor valued coefficients $u_{\mu\nu}$ and $v^a_{\mu\nu}$:

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + u_{\mu\nu}\tilde{G}_1 + v^a_{\mu\nu}\tilde{G}^a_2$$

Choosing,

$$u_{0i} = a_i$$
$$v^a_{0i} = A^a_i$$

one can easily see that the corresponding modified expression of momentum

$$\tilde{P}_i = \int d^2x\tilde{t}_{0i} = \int d^2x[\tilde{\pi}_\alpha \partial_i z_\alpha + \tilde{\pi}^*_\alpha \partial_i z^*_\alpha - 2\theta A^a_i B^a + 2\Theta A^a_i j_0]$$

generate appropriate translation,

$$\{Z(x), \tilde{P}_i\} = \partial_i Z(x)$$
$$\{A^a_i(x), \tilde{P}_i\} = \partial_i A^a_i(x)$$

just like $P^N_i$ (4.4).

So finally the corresponding expression of angular momentum can be written as,

$$J^s = \int d^2x\epsilon^{ij}x_i T_{0j} = \int d^2x\epsilon^{ij}x_i[\tilde{\pi}_\alpha \partial_j z_\alpha + \tilde{\pi}^*_\alpha \partial_j z^*_\alpha - 2\theta A^a_j B^a + 2\Theta A^a_j j_0]$$

$$J^N = \int d^2x[\epsilon^{ij}x_i(\tilde{\pi}_\alpha \partial_j z_\alpha + \tilde{\pi}^*_\alpha \partial_j z^*_\alpha - \theta \epsilon^{kl} A^a_k \partial_j A^a_l) - \theta A^a_j A^a_j]$$

Just like the case the case of linear momentum, here too one can show that both $J^s$ and $J^N$ generate appropriate spatial rotation,

$$\{Z(x), J^s\} = \{Z(x), J^N\} = \epsilon^{ij}x_i \partial_j Z(x)$$
$$\{A^a_i(x), J^s\} = \{A^a_i(x), J^N\} = \epsilon^{ij}x_i \partial_j A^a_i(x) + \epsilon_{ki} A^a_i(x)$$

(again the adjective “appropriate” has been used in the sense, mentioned above.) However, they are not identical and the difference $J_f \equiv (J^s - J^N)$ is given as,

$$J_f = \theta \int d^2x \partial_i[x_j A^{a_j} A^{a_i} - x^i A^a_j A^a_j] + 2\Theta \int d^2x \epsilon^{ij} x_i A^a_j j_0$$

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The first $\theta$-dependent boundary term has occurred earlier in [8,9], where some of its properties were studied in detail. For example, it was noted that this term is gauge invariant under only those gauge transformation which tends to identity asymptotically[9]. To evaluate it in a rotationally symmetric configuration therefore, one can make use of the radiation gauge ($\partial_i A_i^a = 0$) condition. To this end, let us rewrite the Gauss constraint (3.27) as,

$$ j_0^a \approx 2\theta B^a $$

(4.15a)

with $j_0^a = ig(\tilde{\pi}_a(T^a Z)_a - \tilde{\pi}_a^*(Z^i T^a)_{\alpha})$ being the time component of the conserved $3(= 2 + 1)$-current

$$ j^{a\mu} = ig[(D^{\mu} Z)^{\dagger} T^a Z - Z^{\dagger} T^a (D^{\mu} Z)]$$

$$ + \frac{\Theta g}{\pi} \epsilon^{ij} A_i[(Z^i T^a (D_j Z) + (D_j Z)^{\dagger} T^a Z)\delta_0^{\mu} - (Z^{\dagger} T^a (D_0 Z) + (D_0 Z)^{\dagger} T^a Z)\delta_j^{\mu}] $$

(4.15b)

arising from the global $SU(2)$ symmetry of the model (3.21). Note that we have expressed this in terms of the phase space variables.

On integrating (4.15a), one gets

$$ Q^a \approx 2\theta \Phi^a $$

(4.16)

where $Q^a(\equiv \int d^2 x j_0^a)$ and $\Phi^a(\equiv \int d^2 x B^a)$ represent the $SU(2)$ charge and magnetic flux respectively. Proceeding as in [8], we can write the following configuration of the $SU(2)$ gauge field,

$$ A_i^a = -\frac{Q^a}{2\pi \theta} \epsilon^{ij} \frac{x_j}{r^2} $$

(4.16)

in the radiation gauge ($\partial_i A_i^a = 0$). Using this, one can easily show that the first $\theta$-dependent term in (4.14) yields $\frac{Q^a Q^a}{2\pi \theta}$ and the second $\Theta$-dependent term yields, following [13], $\Theta N^2$. So finally, we have from (4.14),

$$ J_f = \frac{Q^a Q^a}{2\pi \theta} + \Theta N^2 $$

(4.17)

We thus see that the classical expression of fractional angular momentum (4.17) contains two terms. One depends on the soliton number $N$ and the other on the nonabelian charge $Q^a$. The former is just as in the model, where Hopf term is coupled to NLSM [3,13]. On the other hand, the latter is a typical nonabelian expression, as in [8].

5. Conclusions

In this paper, we have carried out the classical Hamiltonian analysis of the gauged $CP^1$ model of Cho and Kimm[19]. As was shown in [19], the model has got its own solitons, the very existence of which depends crucially on the presence of $SU(2)$ CS term. These solitons are somewhat more general than that of NLSM[10]. We use the adjective “general” to indicate that these solitons can be characterised by $\pi_2(CP^1) = Z$ only for the gauge $A_i^a = 0$ (Note that $A_i^a$ is a pure gauge). One can make large topology changing gauge transformation and thereby
making $A_i^a \neq 0$, without changing the soliton number. We then constructed the Hopf term associated to these solitons and again carried out the Hamiltonian analysis of the model (3.21), obtained by adding Hopf term to (2.1), to find that the symplectic structure and the structure of the constraints undergo essentially no modification, despite the fact that the form of the momenta variables conjugate to $z_\alpha$ and $z_\alpha^*$ undergo changes. We then calculated the fractional angular momentum by computing the difference between $J^s$ and $J^N$, the expressions of angular momenta obtained from the symmetric expression of energy-momentum tensor and the one obtained through Noether’s prescription respectively. We find that this fractional angular momentum consists of two pieces, one is given in terms of the soliton number and the other is given in terms of the nonabelian ($SU(2)$) charge. In absence of the Hopf term ($\Theta = 0$) (i.e. for the model (2.1)), only this latter term will contribute. Again as in [8], this term can be shown to consist of two pieces, one which involves a direct product in the isospin space and characterises a typical nonabelian feature, while the other contains the abelian charge defined in a nonabelian theory.
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