Abstract. Let $\mu$ be an $n$-dimensional finite positive measure on $\mathbb{R}^m$. We obtain a $T_1$ condition sufficient for the boundedness of Calderón–Zygmund operators on RBMO$(\mu)$, the regular BMO space of Tolsa.

1. Introduction

Given a positive Radon measure $\mu$ on $\mathbb{R}^m$, Tolsa [6] introduced RBMO$(\mu)$, the regular BMO space with respect to $\mu$. This space is suitable for the non-doubling measures $\mu$ and it has genuine properties of the classical space BMO such as the John–Nirenberg inequality. Moreover, it is proved in [6] that a bounded on $L^2(\mu)$ Calderón–Zygmund operator maps $L^\infty(\mu)$ into RBMO$(\mu)$. Motivated by this result, we consider Calderón–Zygmund operators on RBMO$(\mu)$: we obtain a $T_1$ condition sufficient for the boundedness of Calderón–Zygmund operators on RBMO$(\mu)$; see Section 1.4 for a precise formulation.

1.1. Cubes and $n$-dimensional measures. In what follows, a cube is a closed cube in $\mathbb{R}^m$ with sides parallel to the axes and centered at a point of $\text{supp} \mu$. For a cube $Q$, let $\ell = \ell(Q)$ denote its side-length. Also the notation $Q(x, \ell)$ is used to indicate explicitly the center $x$ of the cube under consideration.

As in [6], we always assume that $\mu$ is an $n$-dimensional measure on $\mathbb{R}^m$ for a real number $0 < n \leq m$. By definition, it means that

$$\mu(Q) \leq C\ell^n(Q) \quad \text{for any cube } Q \subset \mathbb{R}^m, \, \ell(Q) > 0,$$

with a universal constant $C > 0$.

1.2. Calderón–Zygmund operators. Let $d(\cdot, \cdot)$ denote the standard distance between points of $\mathbb{R}^m$. A Calderón–Zygmund kernel associated with an $n$-dimensional measure $\mu$ on $\mathbb{R}^m$ is a measurable function $K(x, y)$ on $\mathbb{R}^m \times \mathbb{R}^m \setminus \{(x, x) : x \in \mathbb{R}^m\}$ satisfying the following conditions:

$$|K(x, y)| \leq Cd^{-n}(x, y),$$

$$|K(x_1, y) - K(x_2, y)| + |K(y, x_1) - K(y, x_2)| \leq C\frac{d(x_1, x_2)}{d^{n+s}(x_1, y)}, \quad 2d(x_1, x_2) \leq d(x_1, y),$$

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and

\[(1.4) \quad \left| \int_{Q(x,R) \setminus Q(x,r)} K(x,y) \, d\mu(x) \right| \leq C, \quad 0 < r < R, \]

where $C > 0$ is a universal constant and $\delta$, $0 < \delta \leq 1$, is a regularity constant specific to the kernel $K$.

**Remark 1.** Restrictions (1.2) and (1.3) are standard. Condition (1.4) is a more special cancellation property.

The Calderón-Zygmund operator associated to the kernel $K(x,y)$ and the measure $\mu$ is defined as

\[T f(x) = \int_{\mathbb{R}^m} K(x,y) f(y) \, d\mu(y)\]

for $x \not\in \text{supp}(f \mu)$. So, in the general setting, one introduces the following truncated operators $T_\varepsilon$:

\[T_\varepsilon f(x) = \int_{\mathbb{R}^m \setminus Q(x,\varepsilon)} K(x,y) f(y) \, d\mu(y).\]

The operator $T$ is said to be bounded on $L^p(\mu)$ if the operators $T_\varepsilon$ are bounded on $L^p(\mu)$ uniformly in $\varepsilon > 0$.

1.3. **Regular BMO space.** In this section, we give an equivalent definition of $\text{RBMO}(\mu)$, the regular BMO space introduced by Tolsa [6].

1.3.1. **Coefficients $K(Q,R)$.** Given two cubes $Q \subset R$ in $\mathbb{R}^m$, put

\[K(Q,R) = 1 + \sum_{j=1}^{N_{Q,R}} \frac{\mu(2^j Q)}{\ell^n(2^j Q)},\]

where $N_{Q,R}$ is the minimal integer $k$ such that $\ell(2^k Q) \geq \ell(R)$. Clearly, $K(Q,R) \geq 1$. On the other hand, $K(Q,R)$ is bounded above by $C \log(\ell(R)/\ell(Q))$ because $\mu$ is $n$-dimensional.

1.3.2. **Doubling cubes.**

**Definition 1.** Let $\alpha > 1$ and $\beta > \alpha^n$. A cube $Q$ is called $(\alpha, \beta)$-doubling if $\mu(\alpha Q) < \beta \mu(Q)$.

Let $\mu$ be a Radon measure on $\mathbb{R}^m$ and $\alpha > 1$. As indicated in [6], it is known that for a sufficiently large $\beta = \beta(\alpha, n)$, for $\mu$-almost all $x \in \mathbb{R}^m$ there is a sequence of $(\alpha, \beta)$-doubling cubes $\{Q_k\}_{k=1}^\infty$ centered at $x$ and with $\ell(Q_k)$ tending to $0$ as $k \to \infty$. Let $\beta_n$ denote two times the infimum of the corresponding constants $\beta(4,n)$.

**Definition 2.** A cube $Q \subset \mathbb{R}^m$ is called doubling if $Q$ is $(4, \beta_n)$-doubling.

**Remark 2.** The original definition of a doubling cube and further results in [6] are given for $\alpha = 2$ and under assumption $1 \leq n \leq m$. Nevertheless, it is known that the results of Tolsa [6] are extendable to larger values of $\alpha$ and for $0 < n \leq m$; see, for example, [4] for a generalization of this theory for a wide class of measures on appropriate metric spaces. So, in what follows, we use the above definition with $\alpha = 4$ and still refer to original results of Tolsa [6].
1.3.3. Definition of RBMO.

**Definition 3.** The space $\text{RBMO}(\mu)$ consists of those $f \in L^1_{\text{loc}}(\mu)$ for which there exists a constant $C_\varepsilon > 0$ and a collection of constants $\{f_Q\}$ (one constant for each doubling cube $Q \subset \mathbb{R}^m$) such that

$$\frac{1}{\mu(Q)} \int_Q |f - f_Q| \, d\mu \leq C_\varepsilon$$

and

$$|f_Q - f_R| \leq C_\varepsilon K(Q, R)$$

for all doubling cubes $Q, R, Q \subset R$. Let $\|f\| = \|f\|_E$ denote the infimum of the corresponding constants $C_\varepsilon > 0$.

Standard arguments guarantee that $\|\cdot\|$ is a norm on the space $\text{RBMO}(\mu)$ modulo constants.

1.4. Main theorem. Suppose that a Calderón-Zygmund operator $T$ is bounded on $L^2(\mu)$. Then, as mentioned above, $T$ maps $L^\infty(\mu)$ boundedly into $\text{RBMO}(\mu)$. Moreover, in the classical situation of homogeneous metric spaces and under additional assumption $T1 = 0$, the operator $T$ is known to be bounded on BMO type spaces; see, for example, [5, Ch. 4, Sect. 4]. In the present paper, we obtain a $T1$ condition sufficient for the boundedness of $T$ on $\text{RBMO}(\mu)$.

Given a cube $Q \subset \mathbb{R}^m$, put

$$K(Q) = K(Q, 2^k Q),$$

where $k$ is the smallest positive integer such that $\mu(2^k Q) > \frac{1}{2} \mu(\mathbb{R}^m)$.

**Theorem 1.1.** Let $\mu$ be a finite positive $n$-dimensional measure on $\mathbb{R}^m$. Let $T$ be a Calderón–Zygmund operator bounded on $L^2(\mu)$. Assume that for each doubling cube $Q \subset \mathbb{R}^m$, there exists a constant $b_Q$ such that

$$\frac{1}{\mu(Q)} \int_Q |T1 - b_Q| \, d\mu \leq \frac{C}{K(Q)}$$

for all doubling cubes $Q$

and

$$|b_Q - b_R| \leq C \frac{K(Q, R)}{K(Q)}$$

for all doubling cubes $Q, R, Q \subset R$, where the constant $C > 0$ does not depend on $Q$ and $R$. Then $T$ is bounded on $\text{RBMO}(\mu)$.

Remark 3. We say that $T$ is bounded on $\text{RBMO}(\mu)$ if the operators $T_\varepsilon$, $\varepsilon > 0$, are uniformly bounded on $\text{RBMO}(\mu)$. Also, (1.7) and (1.8) similarly mean that these estimates hold for $T_\varepsilon$ uniformly in $\varepsilon > 0$.

Remark 4. We implicitly assume in Theorem 1.1 that $T1 \in L^\infty(\mu)$. Indeed, this property follows from (1.4); see Lemma 2.5. If $T1$ is a constant, then Theorem 1.1 clearly guarantees that $T$ is bounded on $\text{RBMO}(\mu)$.

Remark 5. Property (1.7) is, in a sense, similar to the oscillation condition used in [1], where a $T1$ theorem for $\text{BMO}_H$ in the Hermite-Calderón-Zygmund setting is obtained.

The final step in the proof of Theorem 1.1 uses $\|\cdot\|$ only; however, the following semi-norm on $\text{RBMO}(\mu)$ is crucial for certain auxiliary results.
**Definition 4.** Let \( f \in L^1_{\text{loc}}(\mu) \). Fix a constant \( \rho > 1 \). Let \( \|f\|_{A,\rho} \) denote the infimum of the constants \( C_A = C_{A,\rho} > 0 \) with the following properties: for each cube \( Q \), there exists \( f_Q \in \mathbb{R} \) such that

\[
\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f(x) - f_Q| \, d\mu(x) \leq C_A,
\]

\[
|f_Q - f_R| \leq C_A K(Q, R)
\]

for any two cubes \( Q \subset R \).

**Remark 6.** The original semi-norm on \( \text{RBMO}(\mu) \) from \([6]\) is different from those introduced in Definitions \([3]\) and \([4]\). Nevertheless, all these semi-norms are equivalent; see Section \([2]\) for further details.

1.5. **Notation.** As usual, the symbol \( C \) denotes an absolute constant whose value can vary from line to line. Notation \( C_A, C_B, \) etc. is used in certain specific situations.

1.6. **Organization of the paper.** Auxiliary results are presented in Section \([2]\). Section \([3]\) is devoted to estimates related to the main technical decomposition of functions from \( \text{RBMO} \); equivalence of Definitions \([3]\) and \([4]\) is essential on this step. The proof of Theorem \([1.1]\) is given in Section \([4]\).

## 2. Auxiliary results

2.1. **Equivalent definitions of \( \text{RBMO} \).** As mentioned in the introduction, Definition \([3]\) is not the original one for \( \text{RBMO}(\mu) \) in \([6]\). In the present section, we show that Definition \([3]\) and the definitions of the regular \( \text{BMO} \) from \([6]\) are equivalent.

Firstly, recall several notions introduced by Tolsa \([6]\). Let \( \rho > 1 \) and \( f \in L^1_{\text{loc}}(\mu) \). Given a cube \( Q \subset \mathbb{R}^n \), let \( \langle f \rangle_Q \) denote the standard \( \mu \)-average of \( f \) over \( Q \), that is,

\[
\langle f \rangle_Q = \frac{1}{\mu(Q)} \int_Q f \, d\mu.
\]

- Let \( \|f\|_{B,\rho} \) denote the infimum of the constants \( C_B = C_{B,\rho} > 0 \) with the following properties:

\[
\frac{1}{\mu(\rho Q)} \int_Q |f - \langle f \rangle_Q| \, d\mu \leq C_B
\]

for any cube \( Q \) (centered at some point of \( \text{supp}(\mu) \)),

\[
|\langle f \rangle_Q - \langle f \rangle_R| \leq C_B K(Q, R)
\]

for any two doubling cubes \( Q \subset R \),

where \( \bar{Q} \) denotes the smallest doubling cube in the sequence \( Q, 4Q, 4^2Q, \ldots \).

**Remark 7.** The original definition of \( \bar{Q} \) in \([6]\) uses the sequence \( Q, 2Q, 2^2Q, \ldots \). The present definition of \( \bar{Q} \) is based on \( \alpha = 4 \); related details are given in Definition \([2]\) and Remark \([2]\). See also \([1]\) for similar definitions with \( \alpha > 1 \).

- Let \( \|f\|_{E,\rho} \) denote the infimum of the constants \( C_E = C_{E,\rho} > 0 \) with the following properties: for any cube \( Q \)

\[
\int_Q |f - \langle f \rangle_Q| \, d\mu \leq C_E \mu(\rho Q),
\]

\[
|\langle f \rangle_Q - \langle f \rangle_R| \leq C_E K(Q, R) \left( \frac{\mu(\rho Q)}{\mu(Q)} + \frac{\mu(\rho R)}{\mu(R)} \right)
\]

for any two cubes \( Q \subset R \).
\( \|f\|_D \) denote the infimum of the constants \( C_D > 0 \) with the following properties:

\[
\int_Q |f - \langle f \rangle_Q| \, d\mu \leq C_D \mu(Q)
\]

for any doubling cube \( Q \) and

\[
|\langle f \rangle_Q - \langle f \rangle_R| \leq C_D K(Q, R)
\]

for any two doubling cubes \( Q \subset R \).

The property \( \|f\|_{B, \rho} < \infty \) is used to define the regular BMO space in \([6]\). By \([6\text{, Lemma 2.6}]\), the norms \( \| \cdot \|_{A, \rho} \) are equivalent for different \( \rho > 1 \); by \([6\text{, Lemma 2.8}]\), \( \| \cdot \|_{B, \rho} \) and \( \| \cdot \|_{A, \rho} \) are equivalent. By \([6\text{, Lemma 2.10}]\), \( \| \cdot \|_{C, \rho} \) and \( \| \cdot \|_{D} \) are equivalent to \( \| \cdot \|_{A, \rho} \).

Standard arguments show that Definition 3 and the property \( \|f\|_D < \infty \) define the same space, with equivalent norms. Indeed, if \( \|f\|_D < \infty \), then trivially \( f \in \text{RBMO}(\mu) \) with \( C_E = \|f\|_D \). Now, assume that \( f \in \text{RBMO}(\mu) \). Property (1.5) guarantees that

\[
|\langle f \rangle_Q - f_Q| \leq C_E
\]

for any doubling \( Q \). Hence, (1.5) implies (2.1) with \( C_D = 2C_E \); (1.6) implies (2.2) with \( C_D = 3C_E \). Therefore, \( \| \cdot \|_D \) and \( \| \cdot \|_{A, \rho} \) are equivalent.

In the arguments related to the proof of Theorem 1.1, we will use \( \| \cdot \|_{A, \rho} \). Thus, for further reference, we separately formulate a particular conclusion from the above arguments as the following lemma.

**Lemma 2.1.** Given a constant \( \rho > 1 \), \( \| \cdot \|_{A, \rho} \) is an equivalent norm on the space \( \text{RBMO}(\mu) \) modulo constants.

### 2.2. John–Nirenberg inequality for RBMO.

Tolsa \([6]\) proved the following version of the John–Nirenberg inequality for RBMO.

**Theorem 2.2** (see \([6\text{, Theorem 3.1}]\)). Let \( f \in \text{RBMO}(\mu), \rho > 1 \), and let \( \{b_Q\} \) be a collection of numbers satisfying

\[
\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |f(x) - b_Q| \, d\mu(x) \leq C\|f\|_{B, \rho}
\]

\[
|b_Q - b_R| \leq CK(Q, R)\|f\|_{B, \rho}
\]

for any two cubes \( Q \subset R \), with an absolute constant \( C > 0 \). Then for any cube \( Q \) and any \( \lambda > 0 \), we have

\[
\mu\{x \in Q : |f(x) - b_Q| > \lambda\} \leq C_{\text{JN}} \mu(\rho Q) \exp\left(-\frac{c_{\text{JN}} \lambda}{\|f\|_{B, \rho}}\right)
\]

with \( C_{\text{JN}}, c_{\text{JN}} > 0 \) depending on \( \rho \).

We will need the following corollary related to Definition 4.

**Lemma 2.3.** Let \( 1 < p < \infty \) and \( f \in \text{RBMO}(\mu), \rho > 1 \). Let \( \{f_Q\} \) be such numbers that (1.9) and (1.10) hold with \( C_A = 2\|f\|_{A, \rho} \). Then

\[
\left(\frac{1}{\mu(\rho Q)} \int_Q |f - f_Q|^p \right)^{\frac{1}{p}} \leq C\|f\|
\]

for any cube \( Q \subset \mathbb{R}^m \), where \( C = C(C_{\text{JN}}, c_{\text{JN}}, p, \rho) > 0 \).
Proof. A standard argument is applicable. Indeed, by Theorem 2.2 and Lemma 2.1,

\[
\frac{1}{\mu(pQ)} \int_Q |f - f_Q|^p \, d\mu = \frac{1}{\mu(pQ)} \int_0^\infty p \lambda^{p-1} \mu \{ x : |f(x) - f_Q| > \lambda \} \, d\lambda \\
\leq C_{\text{JN}} \int_0^\infty p \lambda^{p-1} \exp \left( - \frac{C_{\text{JN}} \lambda}{\|f\|_{\mathfrak{A}, \rho}} \right) \, d\lambda \\
\leq C \|f\|_{\mathfrak{A}, \rho}^p \\
\leq C \|f\|_p^p,
\]

as required. \( \square \)

2.3. Properties of RBMO.

Lemma 2.4. Let \( \mu \) be a finite \( n \)-dimensional measure, \( f \in \text{RBMO}(\mu) \), \( \rho > 1 \), and let \( \{ f_Q \} \) be numbers such that (1.9) and (1.10) hold with \( C_{\mathfrak{A}} = 2 \|f\|_{\mathfrak{A}, \rho} < \infty \). Then

\[ |f_Q| \leq C_f K(Q) \quad \text{for all cubes } Q \subset \mathbb{R}^m, \]

where the number \( C_f > 0 \) does not depend on \( Q \).

Proof. Without loss of generality, we assume that \( \mu(\mathbb{R}^m) = 1 \). Fix a cube \( Q_1 \) such that \( \ell(Q_1) \geq 1 \) and \( \mu(Q_1) > 0 \). Put

\[ C_1(f) = |f_{Q_1}|. \]

Firstly, let \( Q_2 \) be a cube such that \( \mu(Q_2) \geq \frac{1}{2} \). Select a cube \( Q_3 \) such that \( Q_3 \supset Q_1 \cup Q_2 \). By the choice of \( f_{Q_j} \), \( j = 1, 2, 3 \),

\[
|f_{Q_2}| \leq |f_{Q_2} - f_{Q_1}| + |f_{Q_1} - f_{Q_3}| + |f_{Q_3}| \\
\leq 2 K(Q_2, Q_3) \|f\|_{\mathfrak{A}, \rho} + 2 K(Q_1, Q_3) \|f\|_{\mathfrak{A}, \rho} + C_1(f).
\]

Since \( \mu \) is \( n \)-dimensional, we have \( \ell(Q_2) \geq \kappa > 0 \). Hence,

\[
K(Q_m, Q_3) \leq 1 + \sum_{j \geq 1} \frac{\mu(2^j Q_m)}{\ell^n(2^j Q_m)} \leq 1 + \sum_{j \geq 1} \frac{1}{\ell^n(2^j Q_m)} \\
\leq C
\]

for \( m = 1, 2 \). Therefore,

\[
|f_{Q_2}| \leq C \|f\|_{\mathfrak{A}, \rho} + C_1(f) := C_0(f).
\]

Now, consider a cube \( Q \subset \mathbb{R}^m \) such that \( \mu(Q) < \frac{1}{4} \). Let \( k \) be the smallest positive integer such that \( \mu(2^k Q) \geq \frac{1}{2} \). Since the cube \( 2^k Q \) has the properties of \( Q_2 \), we obtain

\[
|f_Q| \leq |f_Q - f_{2^k Q}| + |f_{2^k Q}| \\
\leq 2 K(Q, 2^k Q) \|f\|_{\mathfrak{A}, \rho} + C_0(f) \\
\leq C_f K(Q),
\]

as required. \( \square \)
2.4. Estimates of \( T_1 \).

**Lemma 2.5.** Let \( \mu \) be a finite positive measure on \( \mathbb{R}^m \). Let \( T \) be a Calderón–Zygmund operator. Then \( T_1 \in L^{\infty}(\mu) \).

**Proof.** Since \( \mu \) is a finite measure, we have
\[
|T_1(x)| \leq \int_{\mathbb{R}^m \setminus Q(x,2)} |K(x, y)| \, d\mu(y) + \int_{Q(x,2) \setminus Q(x,\varepsilon)} K(x, y) \, d\mu(y) \leq C\mu(\mathbb{R}^m) + C \leq C
\]
by (1.2) and (1.4). \( \square \)

**Lemma 2.6.** Let the assumptions of Theorem 1.1(ii) hold. Then
\[
\frac{1}{\mu(Q)} \int_Q |T_1 - \langle T_1 \rangle_Q| \, d\mu \leq C \frac{K(Q, \rho)}{K(\rho)} \quad \text{for any doubling cube } Q
\]
and
\[
|\langle T_1 \rangle_Q - \langle T_1 \rangle_R| \leq C \frac{K(Q, \rho)}{K(\rho)} \quad \text{for any two doubling cubes } Q \subset R,
\]
where the constant \( C > 0 \) does not depend on \( Q \) and \( R \).

**Proof.** Put \( h = T_1 \). Firstly, let \( Q \subset \mathbb{R}^m \) be a doubling cube. The following standard arguments guarantee that (1.7) implies (2.3):
\[
\int_Q |h - \langle h \rangle_Q| \, d\mu \leq \int_Q |h - b_Q| \, d\mu + \int_Q |b_Q - \frac{1}{\mu(Q)} \int_Q h \, d\mu| \, d\mu \leq C\mu(Q) + \frac{1}{\mu(Q)} \int_Q \int_Q |h - b_Q| \, d\mu \, d\mu \leq C\mu(Q)
\]
by (1.7).

Secondly, for any doubling cube \( R \supset Q \), we have
\[
|b_R - \langle h \rangle_R| \leq \frac{1}{\mu(R)} \int_R |h - b_R| \, d\mu \leq C \frac{K(Q, \rho)}{K(\rho)} \leq C \frac{K(Q, \rho)}{K(\rho)}
\]
by (1.7). Clearly, \( K(R) \leq C K(Q) \) for \( Q \subset R \). Therefore,
\[
|\langle h \rangle_Q - \langle h \rangle_R| \leq |\langle h \rangle_Q - b_Q| + |b_Q - b_R| + |b_R - \langle h \rangle_R| \leq C \frac{K(Q, \rho)}{K(\rho)}
\]
by (1.8). The proof of the lemma is finished. \( \square \)

3. Main construction

Let \( T \) be a Calderón-Zygmund operator bounded on \( L^2(\mu) \). Let \( f \in \text{RBMO}(\mu) \), \( \rho = 2 \). Using Definition 4 for each cube \( Q \subset \mathbb{R}^m \), select a number \( f_{2Q} \) such that (1.5) and (1.6) hold with \( C_{\alpha, \rho} = 2\|f\|_{\text{a}, \rho} \) and with \( 2Q \) in the place of \( Q \). In particular, the assumptions of Lemma 2.3 are satisfied. Also, to explain further arguments and estimates, it is worth mentioning that \( 2Q \) is not necessarily doubling even if \( Q \) is a doubling cube.
In the present section, we give estimates related to the following functions:

\[ f_1 = f_{1,Q} = f_{2Q}, \]
\[ f_2 = f_{2,Q} = (f - f_{2Q})\chi_{2Q}, \]
\[ f_3 = f_{3,Q} = (f - f_{2Q})\chi_{\mathbb{R}^m \setminus 2Q}. \]

Observe that

\[ f = f_1 + f_2 + f_3. \]

This decomposition ascends to \([3]\); see also \([2]\).

Let \(b_{2,Q} = 0\) and \(b_{3,Q} = \mu(Q)\)

\[ \int_Q T f_{3,Q}(y) \, d\mu(y). \]

In the following lemma, we assume that \(Q\) is a doubling cube.

**Lemma 3.1.** There exists a constant \(C > 0\) such that

\[ \frac{1}{\mu(Q)} \int_Q |T f_k - b_{k,Q}| \, d\mu \leq C \|f\|, \quad k = 2, 3, \]

for any doubling cube \(Q\).

**Proof.** Put

\[ I_k = \frac{1}{\mu(Q)} \int_Q |T f_k - b_{k,Q}| \, d\mu, \quad k = 2, 3. \]

For \(k = 2\), we have

\[ I_2 \leq \left( \frac{1}{\mu(Q)} \int_{2Q} |f_{2Q}|^2 \, d\mu \right)^{\frac{1}{2}}. \]

by H"older’s inequality. Since \(T\) is bounded on \(L^2(\mu)\), the definition of \(f_2\) guarantees that

\[ I_2 \leq C \left( \frac{1}{\mu(Q)} \int_{2Q} |f - f_{2Q}|^2 \, d\mu \right)^{\frac{1}{2}}. \]

Using the doubling property of \(Q\) and Lemma \([2,3]\) with \(\rho = p = 2\) and \(2Q\) in the place of \(Q\), we obtain

\[ I_2 \leq C \left( \frac{\mu(4Q)}{\mu(Q)} \right)^{\frac{1}{2}} \left( \frac{1}{\mu(4Q)} \int_{2Q} |f - f_{2Q}|^2 \, d\mu \right)^{\frac{1}{2}}, \]

as required for \(k = 2\).

For \(k = 3\), we have

\[ I_3 = \frac{1}{\mu(Q)} \int_Q \frac{1}{\mu(Q)} \int_{2Q} |T f_3(x) - T f_3(y)| \, d\mu(y) \, d\mu(x) \]

\[ \leq \frac{1}{\mu(Q)} \frac{1}{\mu(Q)} \int_Q \int_{\mathbb{R}^m \setminus 2Q} |K(x,u) - K(y,u)||f(u) - f_{2Q}| \, d\mu(u) \, d\mu(y) \, d\mu(x). \]

For \(x, y \in Q\) and \(u \in \mathbb{R}^m \setminus 2Q\), the defining property of \(K(\cdot, \cdot)\) guarantees that

\[ |K(x,u) - K(y,u)| \leq C \frac{d^q(x,y)}{d^{n+q}(x,u)}. \]
Therefore,
\[
I_3 \leq \frac{C}{\mu(Q)} \int_Q \int_{\mathbb{R}^n \setminus 2Q} \frac{\ell}{d^{n+\delta}(x,u)} |f(u) - f_{2Q}| \, d\mu(u) \, d\mu(x)
\]
\[
\leq \frac{C}{\mu(Q)} \int_Q \sum_{k=1}^\infty \frac{\ell}{d^{n+\delta}(x,u)} \int_{2^{k+1}Q \setminus 2^kQ} |f(u) - f_{2Q}| \, d\mu(u)
\]
\[
\leq C \sum_{k=1}^\infty \frac{\ell}{(2^{k-1}\ell)^{n+\delta}} \int_{2^{k+1}Q \setminus 2^kQ} |f(u) - f_{2Q}| \, d\mu(u)
\]
\[
\leq C \sum_{k=1}^\infty \frac{\ell}{(2^k\ell)^{n+\delta}} \left( \int_{2^{k+1}Q} |f(u) - f_{2^{k+1}Q}| \, d\mu(u) + \mu(2^{k+1}Q)\mu(2^kQ) \right)
\]
by the triangle inequality. The choice of the constants $f_{2Q}$ and $f_{2^{k+1}Q}$ guarantees that
\[
I_3 \leq C \sum_{k=1}^\infty \frac{\ell}{(2^k\ell)^{n+\delta}} (\mu(2^{k+2}Q) + \mu(2^{k+1}Q)\mu(2^kQ)) \|f\|_{\alpha,2}.
\]
Since $\mu$ is $n$-dimensional, we have
\[
I_3 \leq C\|f\|_{\alpha,2} \sum_{k=1}^\infty \frac{\ell}{(2^k\ell)^{n+\delta}} \mu(2^{k+2}Q)
\]
\[
\leq C\|f\|_{\alpha,2} \sum_{k=1}^\infty \frac{\mu(2^{k+1}Q)}{2^k \ell}
\]
\[
\leq C\|f\|_{\alpha,2} \left( 1 + \sum_{k=1}^\infty q^{-k} \sum_{j=2}^{k+1} \frac{\mu(2^jQ)}{(2^j)^n} \right)
\]
\[
\leq C\|f\|
\]
by Lemma 2.1. The proof of the lemma is finished. \qed

In the following lemma, the cubes under consideration are not assumed to be doubling. However, in the proof of Theorem 1.1 we apply this lemma in the doubling setting.

**Lemma 3.2.** There exists a constant $C > 0$ such that
\[
|b_{k,Q} - b_{k,R}| \leq C\|f\|K(Q,R), \quad k = 2, 3.
\]
for any two cubes $Q \subset R$.

**Proof.** For $k = 2$, the required estimate is trivial; so assume that $k = 3$. Given cubes $Q \subset R$, we have
\[
|b_{3,Q} - b_{3,R}| \leq |b_{3,Q} - b_{3,Q_0}| + |b_{3,Q_0} - b_{3,R}| = I + J,
\]
where $Q_0 = 2^kQ$ and $k$ is the minimal integer such that $2^kQ \supset R$. 
Observe that
\[
I = \left| \frac{1}{\mu(Q_0)} \int_{Q_0} \frac{1}{\mu(Q)} \int_Q T f_{3,Q}(y) \, d\mu(y) \, d\mu(z) \right| - \left| \frac{1}{\mu(Q)} \int_Q \frac{1}{\mu(Q_0)} \int_{Q_0} T f_{3,Q_0}(z) \, d\mu(z) \, d\mu(y) \right|
\]

\[
= \left| \frac{1}{\mu(Q)} \frac{1}{\mu(Q_0)} \int_Q \int_{Q_0} T f_3(y) - T f_3(z) \, d\mu(z) \, d\mu(y) \right|.
\]

By the definitions of \( f_{3,Q} \) and \( f_{3,Q_0} \),
\[
T f_{3,Q}(y) - T f_{3,Q_0}(z) = \int_{\mathbb{R}^m \setminus 2Q} K(y, \cdot)(f - f_{2Q}) \, d\mu - \int_{\mathbb{R}^m \setminus 2Q_0} K(z, \cdot)(f - f_{2Q_0}) \, d\mu
\]
\[
= \int_{2Q \setminus 2Q} K(y, \cdot)(f - f_{2Q}) \, d\mu + \int_{\mathbb{R}^m \setminus 2Q_0} K(y, \cdot)(f_{2Q_0} - f_{2Q}) \, d\mu
\]
\[
+ \int_{\mathbb{R}^m \setminus 2Q_0} (K(y, \cdot) - K(z, \cdot))(f - f_{2Q_0}) \, d\mu
\]
\[
= D + E + F.
\]

We split \( D \) into dyadic sums as follows:
\[
D = \sum_{j=1}^{k} \int_{2^{j+1}Q \setminus 2^jQ} K(y, \cdot)(f - f_{2Q}) \, d\mu
\]
\[
= \sum_{j=1}^{k} \int_{2^{j+1}Q \setminus 2^jQ} K(y, \cdot)(f - f_{2^{j+1}Q}) \, d\mu
\]
\[
+ \sum_{j=1}^{k} (f_{2^{j+1}Q} - f_{2Q}) \int_{2^{j+1}Q \setminus 2^jQ} K(y, \cdot) \, d\mu
\]
\[
= D_1 + D_2.
\]

To estimate \( D_1 \), we apply (1.2) and obtain
\[
|K(y, x)| \leq \frac{C}{d^n(x, y)} \leq \frac{C}{(\ell(Q)2^{j+2})^n} \quad \text{for} \quad y \in Q, \ x \in 2^{j+1}Q \setminus 2^jQ.
\]

Hence,
\[
|D_1| \leq C \sum_{j=1}^{k} \frac{\mu(2^{j+2}Q)}{(\ell(Q)2^{j+2})^n} \frac{1}{\mu(2^{j+2}Q)} \int_{2^{j+1}Q \setminus 2^jQ} |f - f_{2^{j+1}Q}| \, d\mu
\]
(3.1)
\[
\leq C K(Q, Q_0) \| f \|_{\mathbf{A}, 2}
\]
\[
\leq C K(Q, Q_0) \| f \|
\]

by Lemma 2.1. Below we repeatedly use the equivalence of \( \| \cdot \|_{\mathbf{A}, 2} \) and \( \| \cdot \| \) without explicit reference to Lemma 2.1.
Next, applying summation by parts, we obtain
\[
D_2 = (f_{2^k+1} - f_{2^k}) \int_{2^k+1Q\setminus 2Q} K(y, \cdot) \, d\mu \\
- \sum_{j=1}^{k-1} (f_{2^{j+2}Q} - f_{2^{j+1}Q}) \int_{2^{j+1}Q\setminus 2Q} K(y, \cdot) \, d\mu.
\]

By the cancellation property (1.4),
\[
\bigg| \int_{2^{j+1}Q\setminus 2Q} K(y, \cdot) \, d\mu \bigg| \leq C, \quad j = 1, 2, \ldots, k.
\]

Thus, the choice of the numbers \(f_{2^{j+1}Q}, j = 0, 1, \ldots, k\), guarantees that
\[
|D_2| \leq CK(2Q, 2Q_0)\|f\|_{\alpha, 2} + C \sum_{j=1}^{k-1} K(2^{j+1}Q, 2^{j+2}Q)\|f\|_{\alpha, 2}
\]
\[
\leq CK(Q, Q_0)\|f\|_{\alpha, 2} + C K(2Q, 2Q_0)\|f\|_{\alpha, 2}.
\]

(3.2)

To estimate \(E\), we also use the cancellation property (1.4) and we obtain
\[
|E| = \bigg| (f_{2Q} - f_{2Q_0}) \int_{R^m\setminus 2Q_0} K(y, \cdot) \, d\mu \bigg|
\]
\[
\leq C|f_{2Q} - f_{2Q_0}|
\]
\[
\leq CK(Q, Q_0)\|f\|_{\alpha, 2}
\]
\[
\leq CK(Q, Q_0)\|f\|.
\]

(3.3)

Now, consider \(F\). We have
\[
|K(y, x) - K(z, x)| \leq C \frac{\ell^{\delta}(Q_0)}{d^{n+\delta}(x, Q_0)}
\]
for \(x \in R^m \setminus 2Q_0\).

Therefore,
\[
|F| \leq CF^{\delta}(Q_0) \int_{R^m\setminus 2Q_0} \frac{|f - f_{2Q_0}|}{d^{n+\delta}(x, Q_0)} \, d\mu(x)
\]
\[
\leq C \sum_{j=1}^{\infty} \frac{\ell^{\delta}(Q_0)}{(2^{j+2}\ell(Q_0))^{n+\delta}} \int_{2^{j+1}Q_0\setminus 2Q_0} |f - f_{2Q_0}| \, d\mu
\]
\[
\leq C \sum_{j=1}^{\infty} \frac{\ell^{\delta}(Q_0)}{(2^{j+2}\ell(Q_0))^{n+\delta}} \left( \int_{2^{j+1}Q_0\setminus 2Q_0} |f - f_{2^{j+1}Q_0}| \, d\mu + |f_{2^{j+1}Q_0} - f_{2Q_0}| \int_{2^{j+1}Q_0\setminus 2Q_0} \, d\mu \right) = F_1 + F_2.
\]
Firstly,

\[
F_1 \leq \sum_{j=1}^{\infty} \frac{\epsilon^j(Q_0) \mu(2^{j+2}Q_0)}{(2^{j+2} \ell(Q_0))^{n+\delta}} \|f\|_{\mathcal{A},2} \leq C \sum_{j=1}^{\infty} \frac{\|f\|_{\mathcal{A},2}}{(2^j)^\delta} \leq C\|f\|.
\]  
(3.4)

Secondly,

\[
F_2 \leq \sum_{j=1}^{\infty} \frac{\epsilon^j(Q_0) \mu(2^{j+2}Q_0)}{(2^{j+2} \ell(Q_0))^{n+\delta}} \|f\|_{\mathcal{A},2} K(2Q_0, 2^jQ_0) \leq C \sum_{j=1}^{\infty} \frac{\|f\|_{\mathcal{A},2} K(2Q_0, 2^jQ_0)}{(2^j)^\delta}.
\]

Since \(\mu\) is \(n\)-dimensional, we have \(K(2Q_0, 2^jQ_0) \leq C j\) with a universal constant \(C > 0\). Thus,

\[
F_2 \leq C\|f\| \sum_{j=1}^{\infty} \frac{j}{(2^j)^\delta} \leq C\|f\|.
\]  
(3.5)

Combining (3.1, 3.5) and integrating with respect to \(z\) and \(y\), we obtain the required estimate for \(I = |b_{3,Q} - b_{3,Q_0}|\). Therefore, it remains to estimate \(J\).

We have

\[
J = |b_{3,Q_0} - b_{3,R}| = \left| \frac{1}{\mu(R)} \int_{R_m} \frac{1}{\mu(Q_0)} \int_{Q_0} Tf_{3,Q_0}(y) d\mu(y) d\mu(w) - \frac{1}{\mu(Q_0)} \int_{Q_0} \frac{1}{\mu(R)} \int_{R_m} Tf_{3,R}(w) d\mu(w) d\mu(y) \right|
\]

\[
= \left| \frac{1}{\mu(Q_0)} \mu(R) \int_{Q_0} \int_{R_m} (Tf_{3,Q_0}(y) - Tf_{3,R}(w)) d\mu(w) d\mu(y) \right|.
\]

By the definitions of \(f_{3,Q_0}\) and \(f_{3,R}\),

\[
Tf_{3,Q_0}(z) - Tf_{3,R}(w) = \int_{R_m \setminus 2Q_0} K(z, \cdot) (f - f_{2Q_0}) d\mu - \int_{R_m \setminus 2R} K(w, \cdot) (f - f_{2R}) d\mu
\]

\[
= - \int_{2Q_0 \setminus 2R} K(w, \cdot) (f - f_{2R}) d\mu + (f_{2Q_0} - f_{2R}) \int_{R_m \setminus 2Q_0} K(w, \cdot) d\mu
\]

\[
+ \int_{R_m \setminus 2Q_0} (K(z, \cdot) - K(w, \cdot)) (f - f_{2Q_0}) d\mu
\]

\[
= \mathcal{D} + \mathcal{E} + \mathcal{F}.
\]

Firstly, \(\mathcal{D}\) is estimated similarly to \(D\) and even simpler:

\[
\mathcal{D} = - \int_{2Q_0 \setminus 2R} K(w, \cdot) (f - f_{2Q_0}) d\mu + (f_{2R} - f_{2Q_0}) \int_{2Q_0 \setminus 2R} K(w, \cdot) d\mu
\]

\[
= \mathcal{D}_1 + \mathcal{D}_2.
\]
Since $\ell(R)$ and $\ell(Q_0)$ are comparable and $\mu$ is $n$-dimensional, we have

\begin{align}
|D_1| &\leq \int_{2Q_0 \setminus 2R} \frac{C}{d^n(R, 2Q_0 \setminus 2R)} |f - f_{2Q_0}| \, d\mu \\
&\leq \frac{C}{\ell^n(4R)} \frac{\mu(4Q_0)}{\mu(4Q_0)} \int_{2Q_0} |f - f_{2Q_0}| \, d\mu \\
&\leq C \frac{\mu(4Q_0)}{\ell^n(4R)} \|f\|_{A, 2} \\
&\leq C \|f\|.
\end{align}

Using again that $\ell(R)$ and $\ell(Q_0)$ are comparable and $\mu$ is $n$-dimensional, we obtain

\begin{align}
|D_2| &\leq |f_{2Q_0} - f_{2R}| \frac{\mu(2Q_0)}{d^n(R, 2Q_0 \setminus 2R)} \\
&\leq C |f_{2Q_0} - f_{2R}| \frac{\mu(2Q_0)}{\ell^n(R)} \\
&\leq C |f_{2Q_0} - f_{2R}| \\
&\leq CK(2R, 2Q_0) \|f\|_{A, 2} \\
&\leq C \|f\|.
\end{align}

To estimate $E$, we use the cancellation property (1.4) and obtain

\begin{align}
|E| = \left| f_{2R} - f_{2Q_0} \right| \int_{R^m \setminus 2Q_0} K(w, \cdot) \, d\mu \\
&\leq C |f_{2R} - f_{2Q_0}| \\
&\leq CK(2R, 2Q_0) \|f\|_{A, 2} \\
&\leq C \|f\|.
\end{align}

Now, we estimate $F$. Property (1.3) guarantees that

\begin{align}
|F| &\leq \int_{R^m \setminus 2Q_0} \frac{d^\delta(z, w)}{d^n + s(z, Q_0)} |f(x) - f_{2Q_0}| \, d\mu(x).
\end{align}

We have

\begin{align}
d^\delta(z, w) &\leq \ell^\delta(Q_0).
\end{align}

Hence, using dyadic decompositions and summation by parts, we repeat the arguments applied to estimate $|F|$ and we obtain the following analog of (3.4) and (3.5):

\begin{align}
|F| &\leq C \|f\|.
\end{align}

Now, combining estimates (3.6) and (3.9), we conclude that

\begin{align}
J = |b_{3, Q} - b_{3, R}| \leq CK(Q, R) \|f\|
\end{align}

for any two cubes $Q \subset R$. The proof of Lemma 3.2 is finished. \qed
4. Proof of Theorem 1.1

Given an $f \in \text{RBMO}(\mu)$, we have to prove that $Tf \in \text{RBMO}(\mu)$. Put $g = Tf$.

In this section, for every doubling cube $Q \subset \mathbb{R}^m$, we find a constant $g_Q \in \mathbb{R}$ such that

$$\frac{1}{\mu(Q)} \int_Q |g - g_Q| \, d\mu \leq C$$

and

$$|g_Q - g_R| \leq CK(Q, R)$$

for any two doubling cubes $Q \subset R$, where $C = C_f > 0$.

So, given an $f \in \text{RBMO}(\mu)$ and a doubling cube $Q \subset \mathbb{R}^m$, we apply the construction described in Section 3 and we obtain

$$f = f_1, Q + f_2, Q + f_3, Q = f_1 + f_2 + f_3.$$ 

Also, we have constants $b_{2, Q}$ and $b_{3, Q}$. By Lemma 2.6 properties (2.3) and (2.4) hold for the constants $\langle T1 \rangle_Q$ and $\langle T1 \rangle_R$. Put

$$g_Q = f_{2Q} \langle T1 \rangle_Q + b_{2, Q} + b_{3, Q}.$$ 

4.1. Oscillation condition (4.1). We have

$$\frac{1}{\mu(Q)} \int_Q |g - g_Q| \, d\mu \leq \frac{1}{\mu(Q)} \left( \int_Q |Tf_1 - f_{2Q} \langle T1 \rangle_Q| \, d\mu + \int_Q |Tf_2 - b_{2, Q}| \, d\mu + \int_Q |Tf_3 - b_{3, Q}| \, d\mu \right) \leq \frac{1}{\mu(Q)} \int_Q |Tf_1 - f_{2Q} \langle T1 \rangle_Q| \, d\mu + C\|f\|$$

by Lemma 3.4. Recall that $Tf_1 = Tf_{2Q} = f_{2Q}T1$. Therefore, applying (2.3) and Lemma 2.4 we obtain

$$\frac{1}{\mu(Q)} \int_Q |Tf_1 - f_{2Q} \langle T1 \rangle_Q| \, d\mu = \frac{|f_{2Q}|}{\mu(Q)} \int_Q |T1 - \langle T1 \rangle_Q| \, d\mu \leq \frac{C_fK(2Q)}{K(Q)} \leq C_f.$$ 

Hence, the oscillation condition (4.1) is proved.

4.2. K-condition (4.2). Let $Q \subset R$ be doubling cubes. Combining the triangle inequality and Lemma 3.2 we obtain

$$|g_Q - g_R| \leq |f_{2Q} \langle T1 \rangle_Q - f_{2R} \langle T1 \rangle_R| + |b_{3, Q} - b_{3, R}| \leq |f_{2Q} \langle T1 \rangle_Q - f_{2R} \langle T1 \rangle_R| + C\|f\|K(Q, R).$$

Next, the choice of the constants $f_{2Q}$ and $f_{2R}$ guarantees that

$$|f_{2Q} - f_{2R}| \leq C\|f\|_{\mathcal{A}_2}K(2Q, 2R) \leq C\|f\|K(Q, R).$$

Also, we have $T1 \in L^\infty(\mu)$ by Lemma 2.3 hence $|\langle T1 \rangle_Q| \leq C$. Therefore,

$$|f_{2Q} \langle T1 \rangle_Q - f_{2R} \langle T1 \rangle_R| \leq |\langle T1 \rangle_Q||f_{2Q} - f_{2R}| + |f_{2R}||\langle T1 \rangle_Q - \langle T1 \rangle_R| \leq C\|f\|K(Q, R) + |f_{2R}||\langle T1 \rangle_Q - \langle T1 \rangle_R|.$$
Applying Lemma 2.6 and Lemma 2.7, we obtain
\[ |f_{2R}| |(T1)_{Q} - (T1)_{R}| \leq C_{f} K(2R) \frac{K(Q, R)}{K(R)} \leq C_{f} K(Q, R). \]

Combining the above estimates, we conclude that (4.2) holds. This ends the proof of the theorem.

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