A FEM FOR AN OPTIMAL CONTROL PROBLEM OF FRACTIONAL POWERS OF ELLIPTIC OPERATORS

HARBIR ANTIL† AND ENRIQUE OTÁROLA‡

Abstract. We study solution techniques for a linear-quadratic optimal control problem involving fractional powers of symmetric coercive elliptic operators in a bounded domain. These operators can be realized as the Dirichlet-to-Neumann map for a nonuniformly elliptic problem posed on a semi-infinite cylinder in one more spatial dimension. Thus, we consider an equivalent optimal control problem with a nonuniformly elliptic operator as the state equation. The rapid decay of the optimal state suggests a truncation that is suitable for numerical approximation. We discretize the proposed truncated state equation using first degree tensor product finite elements on anisotropic meshes. For the control problem we consider and analyze two approaches: one that is semi-discrete based on the so-called variational approach, where the control is not discretized, and the other one is fully discrete via the discretization of the control by piecewise constant functions. For both approaches, we derive a priori error estimates with respect to the degrees of freedom on anisotropic meshes.

Key words. linear-quadratic optimal control problem, fractional derivatives, fractional diffusion, weighted Sobolev spaces, finite elements, stability, anisotropic estimates.

AMS subject classifications. 35R11, 35J70, 49J20, 49M25, 65N12, 65N30.

1. Introduction. We are interested in the design and analysis of numerical schemes for a linear-quadratic optimal control problem involving fractional powers of elliptic operators. To be concrete, let Ω be an open, connected and bounded domain of \( \mathbb{R}^n \) \((n \geq 1)\), with boundary \( \partial \Omega \). Given \( s \in (0,1)\), and a desired state \( u_d : \Omega \to \mathbb{R} \), we shall be concerned with the following problem:

\[
\min J(u,z) = \frac{1}{2} \|u - u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2} \|z\|^2_{L^2(\Omega)},
\]

subject to the state equation

\[
\begin{cases}
L^s u = z, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and the control constraints

\[a(x') \leq z(x') \leq b(x') \quad \text{a.e } x' \in \Omega,\]

where \( \lambda > 0 \) is the so-called regularization parameter, and \( a \) and \( b \) are real functions in \( L^2(\Omega) \) having the property that \( a(x') \leq b(x') \) for almost every \( x' \in \Omega \). The operator \( L^s \), with \( s \in (0,1) \), denotes the fractional powers of a general second order, symmetric and uniformly elliptic operator \( L \), which is supplemented with homogeneous Dirichlet boundary conditions and is defined by

\[
Lw = -\text{div}_{x'}(A\nabla_{x'}w) + cw,
\]

where \( c \in L^\infty(\Omega) \) with \( c \geq 0 \) almost everywhere, and \( A \in C^{0,1}(\Omega, \text{GL}(n,\mathbb{R})) \) is symmetric and positive definite, with \( \text{GL}(n,\mathbb{R}) \) denoting the group of \( n \times n \) invertible

\*
EO has been supported in part by the NSF grant DMS-1109325.

†Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA.
hantil@gmu.edu

‡Department of Mathematics, University of Maryland, College Park, MD 20742, USA.
kike@math.umd.edu

1
matrices of real numbers. For convenience, we will refer to the optimal control problem defined by (1.1)-(1.3) as the \textit{fractional control problem}; see § 3.1 for a precise definition.

One of the main difficulties in the study of problem (1.2) is that the fractional powers of the operator \( L \) are nonlocal operators; see [11, 30, 34, 38]. A possible approach to overcome this nonlocality property is given by the result of Caffarelli and Silvestre in \( \mathbb{R}^n \) [11] and its extensions to both bounded domains [10, 12] and a general class of elliptic operators [38]. Fractional powers of the operator \( L \) can be realized as an operator that maps a Dirichlet boundary condition to a Neumann condition via an extension problem on the semi-infinite cylinder \( C = \Omega \times (0, \infty) \). This extension leads to the following mixed boundary value problem:

\[
\begin{cases}
L \mathcal{U} - \frac{\alpha}{y} \partial_y \mathcal{U} - \partial_{yy} \mathcal{U} = 0, & \text{in } C, \\
\mathcal{U} = 0, & \text{on } \partial_L C, \\
\frac{\partial \mathcal{U}}{\partial \nu^\alpha} = d_s z, & \text{on } \Omega \times \{0\},
\end{cases}
\]  

(1.5)

where \( \partial_L C = \partial \Omega \times [0, \infty) \) is the lateral boundary of \( C \), and \( d_s \) is a positive normalization constant that depends only on \( s \); see [11, 38] for details. The parameter \( \alpha \) is defined as

\[
\alpha = 1 - 2s \in (-1, 1),
\]  

(1.6)

and the so-called conormal exterior derivative of \( \mathcal{U} \) at \( \Omega \times \{0\} \) is

\[
\frac{\partial \mathcal{U}}{\partial \nu^\alpha} = - \lim_{y \to 0^+} y^\alpha \mathcal{U}_y.
\]  

(1.7)

We will call \( y \) the \textit{extended variable} and the dimension \( n + 1 \) in \( \mathbb{R}^{n+1} \) the \textit{extended dimension} of problem (1.5). The limit in (1.7) must be understood in the distributional sense; see [11, 38]. As noted in [10, 11, 12, 38], we can relate the fractional powers of the operator \( L \) with the Dirichlet-to-Neumann map of problem (1.5): \( d_s \mathcal{L}^s u = \frac{\partial \mathcal{U}}{\partial \nu^\alpha} \) in \( \Omega \). Notice that the differential operator in (1.5) is \( -\text{div} (y^\alpha A \nabla \mathcal{U}) + y^\alpha c \mathcal{U} \) where, for all \((x', y) \in C, A(x', y) = \text{diag}\{A(x')\}, 1\} \in C^{0,1}(C, GL(n + 1, \mathbb{R})) \). Consequently, we can rewrite problem (1.5) as follows:

\[
\begin{cases}
-\text{div} (y^\alpha A \nabla \mathcal{U}) + y^\alpha c \mathcal{U} = 0, & \text{in } C, \\
\mathcal{U} = 0, & \text{on } \partial_L C, \\
\frac{\partial \mathcal{U}}{\partial \nu^\alpha} = d_s z, & \text{on } \Omega \times \{0\}.
\end{cases}
\]  

(1.8)

Before proceeding with the description and analysis of our method, let us give an overview of those advocated in the literature. The study of solution techniques for problems involving fractional diffusion, such that the state equation (1.2), is a relatively new but rapidly growing area of research. We refer to [34, 35] for an overview of the state of the art and restrict the list of papers to those strictly related to our work. On the other hand, numerical strategies for solving a discrete optimal control problem with PDE constraints have been widely studied in the literature; see [24, 25, 27] for an extensive list of references. They are mainly divided in two categories. Both of them rely on an agnostic discretization of the state and adjoint equations, but they differ on whether or not the admissible set of controls is also discretized. The first approach [5, 13, 20, 32, 37] discretizes the admissible control set. The second approach [23] induces a discretization of the optimal control by
A FEM for a control problem for fractional powers of operators

projecting the discrete adjoint state into the admissible control set. Mainly, these studies are concerned with control problems governed by local linear and semilinear elliptic and parabolic PDEs. We note that to the best of the authors’ knowledge, this is the first paper addressing the numerical approximation of a linear-quadratic optimal control problem involving fractional powers of elliptic operators in general domains. We will provide a comprehensive treatment to a linear-quadratic optimal control problem involving evolution equations with fractional diffusion and fractional time derivative in a forthcoming paper.

The main contribution of this work is the study of solution techniques for the fractional control problem (1.1)-(1.3). We overcome the nonlocality of the operator \( \mathcal{L}^s \) by using the localization results of Caffarelli and Silvestre [11]. To be concrete, we consider the following equivalent optimal control problem:

\[
\min J(U, z) := \frac{1}{2} \| U(\cdot, 0) - u_d \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| z \|^2_{L^2(\Omega)},
\]

subject to the state equation (1.8) and the control constraints (1.3). We will refer to the optimal control problem described above as the extended control problem; see §3.2 for a precise definition.

Inspired by [34], we propose the following simple strategy to find the solution of the fractional control problem (1.1)-(1.3): given \( s \in (0, 1) \), \( a, b \in L^2(\Omega) \) satisfying \( a(x') \leq b(x') \) a.e. \( x' \in \Omega \), \( \lambda > 0 \) and a desired state \( u_d : \Omega \to \mathbb{R} \), we solve the equivalent extended control problem, thus obtaining an optimal control \( \bar{z}(x') \) and an optimal state \( \bar{U} : (x', y) \in \mathcal{C} \to \bar{U}(x', y) \in \mathbb{R} \). Setting \( \bar{u} : x' \in \Omega \mapsto \bar{u}(x') = \bar{U}(x', 0) \in \mathbb{R} \), we obtain the optimal pair \((\bar{u}, \bar{z})\) solving the control problem (1.1)-(1.3).

In this paper we propose and analyze two discrete schemes to solve the fractional control problem. Both of them rely on a discretization of the state equation (1.8) and the corresponding adjoint equation, via first degree tensor product finite elements on anisotropic meshes as in [34]. However they differ on whether or not the set of controls is discretized as well. The first approach is semi-discrete and is based on the so-called variational approach [23], in which the set of controls is not discretized. The second approach is fully discrete and discretizes the set of controls by piecewise constant functions [5, 13, 37].

The outline of this paper is as follows. In §2 we introduce some terminology used throughout this work. We recall the definition of the fractional powers of elliptic operators on a bounded domain via spectral theory in §2.2, and in §2.3 we introduce the functional framework that is suitable to analyze problems (1.2) and (1.8). In §3 we define the fractional and extended control problems. For both of them, we derive existence and uniqueness results together with first order sufficient and necessary optimality conditions. Moreover, we prove that both problems are equivalent. The numerical analysis of the fractional control problem begins in §4. Here we introduce a truncation of the state equation (1.8), and propose the truncated control problem. We derive approximation properties of its solution. Section 5 is devoted to the study of discretization techniques to solve the fractional control problem. In §5.1 we review the a priori error analysis developed in [34] for the state equation (1.8): graded meshes in the extended variable allow for a quasi-optimal rate of convergence with respect to degrees of freedom. In §5.2 we study a semi-discrete scheme for the fractional control problem and derive a priori error estimate for both the optimal control and state. Such estimates are quasi-optimal with respect to the degrees of freedom on anisotropic meshes. Finally, in §5.3 we propose a fully-discrete scheme for the control problem.
we study the regularity properties of the optimal control and we derive a priori error estimates for the optimal variables.

2. Notation and preliminaries.

2.1. Notation. Throughout this work $\Omega$ is an open, bounded and connected domain of $\mathbb{R}^n$, $n \geq 1$, with polyhedral boundary $\partial \Omega$. We define the semi-infinite cylinder with base $\Omega$ and its lateral boundary, respectively, by $$C := \Omega \times (0, \infty), \quad \partial_t C := \partial \Omega \times [0, \infty).$$ Given $\gamma > 0$, we define the truncated cylinder with base $\Omega$ by $C_\gamma := \Omega \times (0, \gamma)$. The lateral boundary $\partial_t C_\gamma$ is defined accordingly.

Throughout our discussion we will be dealing with objects defined in $\mathbb{R}^{n+1}$ and it will be convenient to distinguish the extended dimension. A vector $x \in \mathbb{R}^{n+1}$, will be denoted by

$$x = (x^1, \ldots, x^n, x^{n+1}) = (x', x^{n+1}) = (x', y),$$

with $x^i \in \mathbb{R}$ for $i = 1, \ldots, n+1$, $x' \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces, we write $\mathcal{X} \hookrightarrow \mathcal{Y}$ to denote that $\mathcal{X}$ is continuously embedded in $\mathcal{Y}$. We denote by $\mathcal{X}'$ the dual of $\mathcal{X}$ and by $\| \cdot \|_{\mathcal{X}}$ the norm of $\mathcal{X}$. The relation $a \lesssim b$ indicates that $a \leq Cb$, with a constant $C$ that does not depend on $a$ or $b$ nor the discretization parameters. The value of $C$ might change at each occurrence.

2.2. Fractional powers of general second order elliptic operators. Our definition is based on spectral theory. For any $f \in L^2(\Omega)$, the Lax Milgram Lemma provides the existence and uniqueness of $w \in H^1_0(\Omega)$ that solves

$$\mathcal{L} w = f \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega.$$ 

The operator $\mathcal{L}^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ is compact, symmetric and positive, whence its spectrum $\{ \lambda_k^{\frac{1}{2}} \}_{k \in \mathbb{N}}$ is discrete, real, positive and accumulates at zero. Moreover, there exists $\{ \varphi_k \}_{k \in \mathbb{N}} \subset H^1_0(\Omega)$ which is an orthonormal basis of $L^2(\Omega)$ and satisfies

$$\mathcal{L} \varphi_k = \lambda_k \varphi_k \text{ in } \Omega, \quad \varphi_k = 0 \text{ on } \partial \Omega,$$

for all $k \in \mathbb{N}$. Fractional powers of the operator $\mathcal{L}$ can be defined for $w \in C_0^\infty(\Omega)$ by

$$\mathcal{L}^s w := \sum_{k=1}^{\infty} \lambda_k^s w_k \varphi_k,$$

where $w_k = \int_\Omega w \varphi_k$. By density $\mathcal{L}^s$ can be extended to the space

$$H^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k \varphi_k : \sum_{k=1}^{\infty} \lambda_k^s w_k^2 < \infty \right\} = \begin{cases} H^s(\Omega), & s \in (0, \frac{1}{2}), \\ H^{1/2}_0(\Omega), & s = \frac{1}{2}, \\ H^s_0(\Omega), & s \in (\frac{1}{2}, 1). \end{cases}$$

The characterization given by the second equality is shown in [31, Chapter 1]; see [7, § 2] for a discussion. The space $H^{1/2}(\Omega)$ is the so-called Lions-Magenes space, which can be characterized as

$$H^{1/2}(\Omega) = \left\{ w \in H^{\frac{1}{2}}_0(\Omega) : \int_\Omega \frac{w^2(x')}{\text{dist}(x', \partial \Omega)} \, dx' < \infty \right\},$$

see [31, Theorem 11.7] and [39, Chapter 33]. For $s \in (0, 1)$ we denote by $H^{-s}(\Omega)$ the dual space of $H^s(\Omega)$. 

2.2. Fractional powers of general second order elliptic operators.
2.3. The Caffarelli-Silvestre extension problem. To exploit the Caffarelli-Silvestre result [11], or its variants [8, 11, 12], we need to deal with the nonuniformly elliptic equation (1.3). To this end, we consider weighted Sobolev spaces with the weight \( |y|^{\alpha} \), \( \alpha \in (-1, 1) \). If \( D \subset \mathbb{R}^{n+1} \), we then define \( L^2(\Omega) \) as the space of all measurable functions \( w \) defined on \( D \) such that
\[
\|w\|_{L^2(\Omega)}^2 = \frac{1}{2} \alpha \int_D |y|^{\alpha} w^2 < \infty.
\]
Similarly we define the weighted Sobolev space
\[
H^1(|y|^{\alpha}, D) = \{ w \in L^2(|y|^{\alpha}, D) : |\nabla w| \in L^2(|y|^{\alpha}, D) \},
\]
where \( \nabla w \) is the distributional gradient of \( w \). We equip \( H^1(|y|^{\alpha}, D) \) with the norm
\[
\|w\|_{H^1(|y|^{\alpha}, D)}^2 = \left( \|w\|_{L^2(|y|^{\alpha}, D)}^2 + \|\nabla w\|_{L^2(|y|^{\alpha}, D)}^2 \right) \frac{1}{2}.
\]  
(2.5)

Since \( \alpha \in (-1, 1) \) we have that \( |y|^{\alpha} \) belongs to the so-called Muckenhoupt class \( A_2(\mathbb{R}^{n+1}) \); see [19, 21, 33, 41]. This, in particular, implies that \( H^1(|y|^{\alpha}, D) \) equipped with the norm (2.5), is a Hilbert space and the set \( C^\infty(D) \cap H^1(|y|^{\alpha}, D) \) is dense in \( H^1(|y|^{\alpha}, D) \) (cf. [11] Proposition 2.1.2, Corollary 2.1.6). We recall now the definition of Muckenhoupt classes; see [33, 41].

Definition 2.1 (Muckenhoupt class \( A_2 \)). Let \( \omega \) be a weight and \( N \geq 1 \). We say \( \omega \in A_2(\mathbb{R}^N) \) if
\[
C_{2,\omega} = \sup_B \left( \int_B \omega \right) \left( \int_B \omega^{-1} \right) < \infty,
\]  
(2.6)
where the supremum is taken over all balls \( B \) in \( \mathbb{R}^N \).

If \( \omega \) belongs to the Muckenhoupt class \( A_2(\mathbb{R}^N) \), we say that \( \omega \) is an \( A_2 \)-weight, and we call the constant \( C_{2,\omega} \) in (2.6) the \( A_2 \)-constant of \( \omega \).

To study the extended control problem, we define the weighted Sobolev space
\[
\tilde{H}^1_L(y^\alpha, \mathcal{C}) := \{ w \in H^1(y^\alpha, \mathcal{C}) : w = 0 \text{ on } \partial_L \mathcal{C} \}.
\]  
(2.7)

As [33, (2.21)] shows, the following weighted Poincaré inequality holds:
\[
\int_{\mathcal{C}} y^\alpha w^2 \leq C_\Omega \int_{\mathcal{C}} y^\alpha |\nabla w|^2, \quad \forall w \in \tilde{H}^1_L(y^\alpha, \mathcal{C}),
\]  
(2.8)
where \( C_\Omega \) denotes a positive constant that depends only on \( \Omega \). Then, the semi-norm on \( \tilde{H}^1_L(y^\alpha, \mathcal{C}) \) is equivalent to the norm (2.5). For \( w \in H^1(y^\alpha, \mathcal{C}) \), we denote by \( \text{tr}_\Omega w \) its trace onto \( \Omega \times \{0\} \), and we recall that the trace operator \( \text{tr}_\Omega \) satisfies (see [11] Proposition 1.8 for \( s = 1/2 \) and [12] Proposition 2.1) for any \( s \in (0, 1) \setminus \{\frac{1}{2}\} \)
\[
\text{tr}_\Omega \tilde{H}^1_L(y^\alpha, \mathcal{C}) = \mathbb{H}^s(\Omega), \quad \|\text{tr}_\Omega w\|_{\mathbb{H}^s(\Omega)} \leq C_{\text{tr}_\Omega} \|w\|_{\tilde{H}^1_L(y^\alpha, \mathcal{C})}.
\]  
(2.9)

Let us now describe the Caffarelli-Silvestre result and its extension to second order operators; [11, 33]. Consider a function \( u \) defined on \( \Omega \). We define the \( \alpha \)-harmonic extension of \( u \) to the cylinder \( \mathcal{C} \), as the function \( \mathcal{U} \) that solves the boundary value problem
\[
\begin{cases}
-\text{div}(y^\alpha \mathbf{A} \nabla \mathcal{U}) + y^\alpha c \mathcal{U} = 0, & \text{in } \mathcal{C}, \\
\mathcal{U} = 0, & \text{on } \partial_L \mathcal{C}, \\
\mathcal{U} = u, & \text{on } \Omega \times \{0\}.
\end{cases}
\]  
(2.10)
Problem (2.10) has a unique solution $\mathcal{U} \in H^1_L(y^\alpha, C)$ whenever $u \in H^s(\Omega)$. We define the Dirichlet-to-Neumann operator $N : H^s(\Omega) \to H^{-s}(\Omega)$

$$u \in H^s(\Omega) \mapsto N(u) = \frac{\partial u}{\partial \nu} \in H^{-s}(\Omega),$$

where $\mathcal{U}$ solves (2.10) and $\frac{\partial u}{\partial \nu}$ is given in (1.7). The fundamental result of [11], see also [12] Lemma 2.2 and [38] Theorem 1.1, is stated below.

**Theorem 2.2** (Caffarelli–Silvestre extension). If $s \in (0, 1)$ and $u \in H^s(\Omega)$, then
d$$d_s(-\Delta)^su = N(u),$$
in the sense of distributions. Here $\alpha = 1 - 2s$ and $d_s$ is given by

$$d_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}, \tag{2.11}$$

where $\Gamma$ denotes the Gamma function.

The relation between the fractional Laplacian and the extension problem is now clear. Given $z \in H^{-s}(\Omega)$, a function $u \in H^s(\Omega)$ solves (2.12) if and only if its $\alpha$-harmonic extension $\mathcal{U} \in H^1_L(y^\alpha, C)$ solves (2.13).

We now present the weak formulation of problem (1.2): seek $\mathcal{U} \in H^1_L(y^\alpha, C)$ such that
d$$a(\mathcal{U}, \phi) = d_s(z, \text{tr}_\Omega \phi)_{H^{-s}(\Omega) \times H^{s}(\Omega)}, \quad \forall \phi \in H^1_L(y^\alpha, C), \tag{2.12}$$

where, for $w, \phi \in H^1_L(y^\alpha, C)$

$$a(w, \phi) := \int_C y^\alpha A(x)\nabla w \cdot \nabla \phi + y^\alpha c(x')w\phi \tag{2.13}$$

and $\langle \cdot, \cdot \rangle_{H^{-s}(\Omega) \times H^{s}(\Omega)}$ denotes the duality pairing between $H^{s}(\Omega)$ and $H^{-s}(\Omega)$, which, as a consequence of (2.9), is well defined for $z \in H^{-s}(\Omega)$ and $\phi \in H^1_L(y^\alpha, C)$.

**Remark 2.3** (equivalent semi-norm). Notice that the regularity assumed for the coefficients $A$ and $c$, together with the weighted Poincaré inequality (2.8), imply that the bilinear form $a$, defined in (2.13), is bounded and coercive in $H^1_L(y^\alpha, C)$. In what follows we shall use repeatedly the fact that $a(w, w)^{1/2}$ is a norm, equivalent to the semi-norm in $H^1_L(y^\alpha, C)$.

Remark (2.3) in conjunction with [12] Proposition 2.1 for $s \in (0, 1) \setminus \{\frac{1}{2}\}$ and [10] Proposition 2.1 for $s = \frac{1}{2}$ provide us the following estimates for problem (2.12):

$$\|\mathcal{U}\|_{H^1_L(C, y^\alpha)} \lesssim \|u\|_{H^s(\Omega)} \lesssim \|z\|_{H^{-s}(\Omega)} \tag{2.14}$$

We conclude with a representation of the solution of problem (2.12) using the eigenpairs $\{\lambda_k, \varphi_k\}$ defined in (2.1). Let the solution to (1.2) be given by $u(x') = \sum_k u_k\varphi_k(x')$. The solution $\mathcal{U}$ of problem (2.12) can then be written as

$$\mathcal{U}(x, t) = \sum_{k=1}^{\infty} u_k\varphi_k(x')\psi_k(y), \tag{2.15}$$

where $\psi_k$ solves

$$\psi''_k + \frac{\alpha}{y}\psi'_k - \lambda_k\psi_k = 0, \quad \psi_k(0) = 1, \quad \lim_{y \to \infty} \psi_k(y) = 0. \tag{2.16}$$
If \( s = \frac{1}{2} \), then clearly \( \psi_s(y) = e^{-\sqrt{\lambda} y} \). For \( s \in (0, 1) \setminus \{ \frac{1}{2} \} \) we have that if \( c_s = \frac{2^{1-s}}{\Gamma(s)} \), then \[ \psi_s(y) = c_s \left( \sqrt{\lambda} y \right)^s K_s(\sqrt{\lambda} y), \]
where \( K_s \) is the modified Bessel function of the second kind \([11\text{ Chapter 9.6}]\).

3. The optimal fractional and extended control problems. In this section, we describe and analyze the fractional and extended control problems. For both of them, we derive existence and uniqueness results together with first order necessary and sufficient optimality conditions. We conclude the section by showing the equivalence between both optimal control problems, which set the stage to propose numerical algorithms to solve the fractional control problem.

3.1. The fractional control problem. We start by recalling the fractional control problem introduced in \([11]\), which reads as follows:

\[
\min J(u, z) := \frac{1}{2} \| u - u_d \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| z \|^2_{L^2(\Omega)},
\]
subject to the state state equation \((1.2)\) and the control constraints \((1.3)\). The set of admissible controls \( Z_{ad} \) is defined by

\[
Z_{ad} := \{ w \in L^2(\Omega) : a(x') \leq w(x') \leq b(x'), \quad \text{a.e. } x' \in \Omega \},
\]
where \( a, b \in L^2(\Omega) \) and satisfy \( a(x') \leq b(x') \) a.e. \( x' \in \Omega \). The function \( u_d \in L^2(\Omega) \) denotes the desired state and \( \lambda > 0 \) the so-called regularization parameter.

In order to study the existence and uniqueness of this problem, we follow \([10\ § 2.5]\) and we introduce the so-called fractional control-to-state operator.

**Definition 3.1** (fractional control-to-state operator). We define the fractional control to state operator \( S : \mathbb{H}^{-s}(\Omega) \to \mathbb{H}^{s}(\Omega) \) such that for a given control \( z \in \mathbb{H}^{-s}(\Omega) \) it associates a unique state \( u(z) \in \mathbb{H}^{s}(\Omega) \) via the state equation \((1.2)\).

The operator \( S \) is a linear and continuous mapping from \( \mathbb{H}^{-s}(\Omega) \) into \( \mathbb{H}^{s}(\Omega) \), as a consequence of \((2.14)\). Moreover, in view of the continuous embedding \( \mathbb{H}^{s}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathbb{H}^{-s}(\Omega) \), we may also consider the operator \( S \) acting from \( L^2(\Omega) \) and with range in \( L^2(\Omega) \). For simplicity, we keep the notation \( S \) for such an operator.

We define the fractional optimal state-control pair as follows.

**Definition 3.2** (fractional optimal state-control pair). A state-control pair \( (\bar{u}(z), \bar{z}) \in \mathbb{H}^{s}(\Omega) \times Z_{ad} \) is called optimal for the fractional control problem, if \( \bar{u}(z) = Sz \) and

\[
J(\bar{u}(z), \bar{z}) \leq J(u(z), z),
\]
for all \( (u(z), z) \in \mathbb{H}^{s}(\Omega) \times Z_{ad} \) such that \( u(z) = Sz \).

Now, given that \( \mathcal{L}^s \) is a self-adjoint operator, it follows that \( S \) is self-adjoint operator as well, and then, we have the following definition for the adjoint state.

**Definition 3.3** (fractional adjoint state). Given a control \( z \in \mathbb{H}^{-s}(\Omega) \), we define the fractional adjoint state \( p(z) \in \mathbb{H}^{s}(\Omega) \) as \( p(z) := S(u - u_d) \).

We now present the following result about existence and uniqueness of the optimal control together with the first order necessary and sufficient optimality condition of the fractional control problem.

**Theorem 3.4** (existence, uniqueness and optimality conditions). The fractional control problem \([11\ § 3.3]\) has a unique optimal solution \( (\bar{u}, \bar{z}) \in \mathbb{H}^{s}(\Omega) \times Z_{ad} \). The
The optimality conditions \( \bar{u} = Sz \in H^s(\Omega) \), \( \bar{p} = S(\bar{u} - u_d) \in H^s(\Omega) \), and
\[
\bar{z} \in Z_{ad}, \quad (\lambda \bar{z} + \bar{p} , z - \bar{z})_{L^2(\Omega)} \geq 0 \quad \forall z \in Z_{ad}
\] (3.3)
hold. These conditions are necessary and sufficient.

Proof. We start by noticing that using the control-to-state operator \( S \), the fractional control problem reduces to the following quadratic optimization problem:
\[
\min_{z \in Z_{ad}} f(z) := \frac{1}{2} \| Sz - u_d \|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \| z \|_{L^2(\Omega)}^2.
\]
Since \( \lambda > 0 \), it is immediate that the functional \( f \) is strictly convex. Moreover, the set \( Z_{ad} \) is nonempty, closed, bounded and convex in \( L^2(\Omega) \). Then, invoking an infimizing sequence argument, followed by the well-posedness of the state equation, we derive the existence of an optimal control \( \bar{z} \in Z_{ad} \); see \([10, \text{Theorem 2.14}]\). The uniqueness of \( \bar{z} \) is a consequence of the strict convexity of \( f \). The first order optimality condition (KKT) is a direct consequence of \([10, \text{Theorem 2.22}]\). \( \square \)

3.2. The extended control problem. In order to overcome the nonlocality feature in the fractional control problem, which is due to the presence of the operator \( L^* \), we introduce an equivalent problem: the extended control problem. The main advantage of the latter, which was already motivated in \( \S 1 \) is its local nature. We will rigorously prove the equivalence between the fractional and extended control problems at the end of this section.

We start by defining the extended control problem as follows:
\[
\min J(\mathcal{W}, q) := \frac{1}{2} \text{tr}_\Omega \mathcal{W} - u_d \|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \| q \|_{L^2(\Omega)}^2,
\] (3.4)
subject to the state equation
\[
a(\mathcal{W}, \phi) = d_s(q, \text{tr}_\Omega \phi)_{H^{-s}(\Omega) \times H^s(\Omega)}, \quad \forall \phi \in \dot{H}^1_L(y^s, C).
\] (3.5)
and the control constraints
\[
q \in Z_{ad}.
\] (3.6)

**Definition 3.5 (extended control-to-state operator).** We define the control to state operator \( G : H^{-s}(\Omega) \to H^s(\Omega) \) such that for a given control \( q \in H^{-s}(\Omega) \), the operator \( G \) associates a unique state \( \text{tr}_\Omega q \in H^s(\Omega) \) via the state equation (3.3).

As a consequence of Theorem 2.2, if \( u(q) \in H^s(\Omega) \) denotes the solution to (1.2) with \( q \in H^{-s}(\Omega) \) as a datum, and \( \mathcal{W}(q) \) solves (3.5), then
\[
\text{tr}_\Omega \mathcal{W}(q) = u(q).
\]
Consequently, the action of the operators \( S \) and \( G \) coincide. On the other hand, the operator \( G \) is well defined, linear and continuous; see \([10, \text{Lemma 2.6}]\) for \( s = 1/2 \) and \([12, \text{Proposition 2.1}]\) for any \( s \in (0, 1) \setminus \{ \frac{1}{2} \} \). Moreover, since \( H^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega) \), we may also consider the control-to-state operator acting from \( L^2(\Omega) \) with range in \( L^2(\Omega) \). For simplicity, we keep the notation \( G \) for such an operator.

With this notation we define the extended optimal state-control pair as follows.

**Definition 3.6 (extended optimal state-control pair).** A state-control pair \( (\mathcal{W}(q), q) \in \dot{H}^1_L(y^s, C) \times Z_{ad} \) is called optimal for the control problem (3.4)-(3.6), if \( \mathcal{W}(q) = Gq \) and
\[
J(\mathcal{W}(q), q) \leq J(\mathcal{W}(q), q),
\]
for all \((ɛ, q) ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C}) \times \mathcal{Z}_{ad}\) such that \(ɛ(q) = \mathbf{G}q\).

We define the adjoint operator of \(\mathbf{G}\) as follows.

**Definition 3.7** (extended adjoint operator). The adjoint operator \(\mathbf{G}^*: L^2(\Omega) \rightarrow L^2(\Omega)\) is given by \(\mathbf{G}^*q = d_s \mathbf{tr}_\Omega \mathcal{V}, \) where \(\mathcal{V} ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C})\) solves

\[
a(\mathcal{V}, \phi) = (q, \mathbf{tr}_\Omega \phi)_{L^2(\Omega)}, \quad \forall \phi ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C}). \tag{3.7}
\]

Since the above definition of the adjoint operator \(\mathbf{G}^*\) is not easy to understand intuitively, we now consider a formal Lagrangian approach (see [40 §2.10]). This will give us an alternative and simple approach to define the adjoint state. To do so, we introduce the Lagrangian function \(\mathbf{L}\) defined by

\[
L : \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C}) \times \mathcal{Z}_{ad} × \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C}) → \mathbb{R}
\]

\[
(ɛ, q, \mathcal{P}) → J(ɛ, q) - \int_\mathcal{C} y^\alpha (\mathbf{A} \nabla \mathcal{P} + c \mathcal{P} \mathcal{P}) + d_s \int_\Omega q \mathbf{tr}_\Omega \mathcal{P}.
\]

A formal computation via integration by parts shows that

\[
L = J(ɛ, q) - \int_\mathcal{C} (-\text{div}(y^\alpha \mathbf{A} \nabla \mathcal{P}) + cy^\alpha \mathcal{P}) \mathcal{U} - \int_\Omega \left( \frac{\partial \mathcal{P}}{\partial \nu^\alpha} \mathbf{tr}_\Omega \mathcal{U} - d_s q \mathbf{tr}_\Omega \mathcal{P} \right),
\]

where we have used \((1.7)\). Recalling the Lagrange principle we expect the pair \((\mathcal{U}, \bar{q})\) together with the Lagrange multiplier \(\mathcal{P}\) to satisfy the optimality conditions associated with the problem

\[
\min \mathcal{L}(\mathcal{U}, q, \mathcal{P}), \quad \mathcal{U} ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C}), \quad q ∈ \mathcal{Z}_{ad}.
\]

Since \(\mathcal{U}\) is now unconstrained in \(\hat{H}^1(\mathbf{y}^\alpha, \mathcal{C})\), we have \(D_\mathcal{U} \mathcal{L}(\mathcal{U}, \bar{\mathcal{Z}}, \mathcal{P})h = 0\) for all \(h ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C})\), which reads

\[
\int_\Omega (\mathbf{tr}_\Omega \mathcal{U} - u_d) \mathbf{tr}_\Omega h - \int_\mathcal{C} (-\text{div}(y^\alpha \mathbf{A} \nabla \mathcal{P}) + y^\alpha c \mathcal{P}) h - \int_\Omega \frac{\partial \mathcal{P}}{\partial \nu^\alpha} \mathbf{tr}_\Omega h = 0, \tag{3.8}
\]

for all \(h ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C})\). By a density result (see [40 Proposition 2.1.2]), we consider a function \(h ∈ C_0^\infty(\mathcal{C})\) in \((3.8)\) to arrive at

\[
\int_\mathcal{C} (-\text{div}(y^\alpha \mathbf{A} \nabla \mathcal{P}) + y^\alpha c \mathcal{P}) h = 0 \quad ⇒ \quad -\text{div}(y^\alpha \mathbf{A} \nabla \mathcal{P}) + y^\alpha c \mathcal{P} = 0 \quad \text{in } \mathcal{C}.
\]

Next, we consider a smooth function \(h\) such that \(h ≠ 0\) on \(\Omega × \{0\}\). For all such \(h\) evidently \((3.8)\) implies

\[
\int_\Omega \left( \mathbf{tr}_\Omega \mathcal{U} - u_d - \frac{\partial \mathcal{P}}{\partial \nu^\alpha} \right) \mathbf{tr}_\Omega h = 0 \quad ⇒ \quad \frac{\partial \mathcal{P}}{\partial \nu^\alpha} = \mathbf{tr}_\Omega \mathcal{U} - u_d \text{ on } \Omega × \{0\}.
\]

Thus, as a consequence of the formal analysis developed above, we define the extended optimal adjoint state as follows.

**Definition 3.8** (extended optimal adjoint state). We define the extended optimal adjoint state \(\hat{\mathcal{P}}(\bar{q}) ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C})\), associated with the optimal state \(\mathcal{U}(\bar{q}) ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C})\), to be the solution of

\[
a(\hat{\mathcal{P}}(\bar{q}), \phi) = (\mathbf{tr}_\Omega \mathcal{U}(\bar{q}) - u_d, \mathbf{tr}_\Omega \phi)_{L^2(\Omega)}, \quad \forall \phi ∈ \hat{H}^1(\mathbf{y}^\alpha, \mathcal{C}). \tag{3.9}
\]
As a consequence of Definitions 3.5, 3.7 and 3.8, the following important relation holds true: \( \text{tr}_\Omega \tilde{\mathcal{P}}(\tilde{q}) = \frac{1}{d_s} \text{G}^* (\text{G}q - u_d) \).

We also remark that the formal analysis developed above yields the following variational inequality

\[
(D_\Omega L(\tilde{W}, \tilde{q}, \tilde{\mathcal{P}}), q - \tilde{q})_{L^2(\Omega)} \geq 0 \quad \forall q \in \mathbb{Z}_{ad},
\]

because \( q \) is constrained in the set \( \mathbb{Z}_{ad} \). Equivalently,

\[
\int_\Omega (d_s \text{tr}_\Omega \tilde{\mathcal{P}} + \lambda \tilde{q})(q - \tilde{q}) \geq 0 \quad \forall q \in \mathbb{Z}_{ad},
\] \hspace{1cm} (3.10)

We then conclude the existence and uniqueness of the extended control problem, together with the first order necessary and sufficient optimality condition.

**Theorem 3.9** (existence, uniqueness and optimality system). *The extended control problem (3.4), (3.6) has a unique optimal solution \( (\tilde{W}, \tilde{q}) \in \tilde{H}_1^1(\mathbb{R}^n, \mathcal{C}) \times \mathbb{Z}_{ad} \). The optimality system

\[
\begin{cases}
\tilde{W} = \tilde{W}(\tilde{q}) \in \tilde{H}_1^1(\mathbb{R}^n, \mathcal{C}) \text{ solution of (3.5),} \\
\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(\tilde{q}) \in \tilde{H}_1^1(\mathbb{R}^n, \mathcal{C}) \text{ solution of (3.8),} \\
\tilde{q} \in \mathbb{Z}_{ad}, \quad (d_s \text{tr}_\Omega \tilde{\mathcal{P}} + \lambda \tilde{q}, q - \tilde{q})_{L^2(\Omega)} \geq 0 \quad \forall q \in \mathbb{Z}_{ad},
\end{cases}
\] \hspace{1cm} (3.11)

hold. These conditions are necessary and sufficient.

**Proof.** We write the extended control problem as a quadratic optimization problem:

\[
\min_{q \in \mathbb{Z}_{ad}} g(q) := \frac{1}{2} \| \text{G}q - u_d \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| q \|^2_{L^2(\Omega)}.
\]

Then, the rest of the proof follows the same arguments employed in the proof of Theorem 3.4. For brevity, we leave the details to the reader. \( \square \)

We conclude this section with the following important Lemma which shows the equivalence between the fractional and extended control problems.

**Theorem 3.10** (equivalence of the fractional and extended control problems). *If \( (\tilde{u}(\tilde{z}), \tilde{z}) \in \tilde{H}_1^s(\Omega) \times \mathbb{Z}_{ad} \) and \( (\tilde{W}(\tilde{q}), \tilde{q}) \in \tilde{H}_1^1(\mathbb{R}^n, \mathcal{C}) \) denote the optimal solution to the fractional and extended control problems respectively, then

\[
\tilde{z} = \tilde{q} \quad \text{and} \quad \text{tr}_\Omega \tilde{W} = \tilde{u}.
\]

**Proof.** We proceed with a contradiction argument: let us assume \( \tilde{q} \neq \tilde{z} \). In view of Theorem 2.2, we have that \( \text{tr}_\Omega \tilde{W}(\tilde{q}) = \tilde{u}(\tilde{q}) \) and \( \text{tr}_\Omega \tilde{W}(\tilde{z}) = \tilde{u}(\tilde{z}) \). This, together with the fact that \( \tilde{W}(\tilde{q}) \), \( \tilde{q} \) is optimal for the extended control problem, yields

\[
J(\tilde{u}(\tilde{q}), \tilde{q}) = \mathcal{J}(\tilde{W}(\tilde{q}), \tilde{q}) \leq \mathcal{J}(\tilde{W}(\tilde{z}), \tilde{z}) = J(\tilde{u}(\tilde{z}), \tilde{z}),
\]

which is a contradiction with the fact that \( \tilde{u}(\tilde{z}), \tilde{z} \in \tilde{H}_1^s(\Omega) \times \mathbb{Z}_{ad} \) is the unique optimal pair solving the fractional control problem (1.1)-(1.3). Consequently, \( \tilde{q} = \tilde{z} \) and then \( \text{tr}_\Omega \tilde{W} = \tilde{u} \). \( \square \)
4. A truncated optimal control problem. Since $\mathcal{C}$ is unbounded, the state equation (3.5) cannot be directly approximated with finite element-like techniques. The first step towards the discretization is to truncate the domain $\mathcal{C}$. In this Section we will recall that the optimal state $\bar{U}$ of the optimal control in Lemma 4.8.

$\lambda_1$ denotes the first eigenvalue of the operator $\mathcal{L}$.

**Proof.** The estimate (4.1) follows directly from [34, Proposition 3.1] in conjunction with [36, Proposition 4.1].

As a consequence of Proposition 4.1, given a control $r \in \mathbb{H}^{-s}(\Omega)$, we can consider the following truncated state equation

$$\begin{cases}
-\text{div}(y^\alpha A \nabla v) + y^\alpha cv = 0, & \text{in } \mathcal{C}_Y, \\
v = 0, & \text{on } \partial\mathcal{L}\mathcal{C}_Y \cup \Omega \times \{Y\},
\end{cases}$$

(4.2)

for $Y$ sufficiently large. In order to obtain a weak formulation of (4.2) and formulate an appropriate optimal control problem, we define

$$\tilde{\mathcal{H}}^1_L(y^\alpha, \mathcal{C}_Y) := \{ w \in H^1(y^\alpha, \mathcal{C}_Y) : w = 0 \text{ on } \partial\mathcal{L}\mathcal{C}_Y \cup \Omega \times \{Y\} \},$$

and

$$a_Y(w, \phi) := \int_{\mathcal{C}_Y} y^\alpha A(x) \nabla w \cdot \nabla \phi + y^\alpha c(x') w \phi, \quad w, \phi \in \tilde{\mathcal{H}}^1_L(y^\alpha, \mathcal{C}_Y).$$

(4.3)

We are now in position to define the truncated control problem as follows:

$$\min \mathcal{J}(v, r) := \frac{1}{2} \| \mathbf{v} \|_{L^2(\Omega)}^2 + \frac{\lambda_1}{2} \| r \|_{L^2(\Omega)}^2,$$

(4.4)

subject to the truncated state equation

$$a_Y(v, \phi) = d_s (r, \text{tr}_\Omega \phi)_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)}, \quad \forall \phi \in \tilde{\mathcal{H}}^1_L(y^\alpha, \mathcal{C}_Y),$$

(4.5)

and the control constraints

$$r \in \mathbb{Z}_{ad}.$$

(4.6)

Before continuing with the analysis of the truncated control problem, we state the following exponential estimates.

**Proposition 4.2** (exponential convergence). If $\mathcal{V}(r) \in \tilde{\mathcal{H}}^1_L(y^\alpha, \mathcal{C})$ solves problem (3.5) with $q = r$ and $v(r) \in \tilde{\mathcal{H}}^1_L(y^\alpha, \mathcal{C}_Y)$ solves (4.5) then, for any $Y \geq 1$, we have

$$\| \nabla (\mathcal{V}(r) - v(r)) \|_{L^2(y^\alpha, \mathcal{C})} \lesssim e^{-\sqrt{\lambda_1}Y/4} \| r \|_{\mathbb{H}^{-s}(\Omega)},$$

(4.7)
and
\[
\| \text{tr}_Ω(\mathcal{H}(r) - v(r)) \|_{H^s(Ω)} \lesssim e^{-\sqrt{\lambda_1} γ/4} \| r \|_{H^{-s}(Ω)},
\]
where \( λ_1 \) denotes the first eigenvalue of the operator \( \mathcal{L} \).

**Proof.** The estimate (4.7) follows from [34, Theorem 3.5] and Remark 2.8. On the other hand, the trace estimate (2.9) in conjunction with (4.7) yields (4.8). □

Now, we introduce the truncated control-to-state operator as follows.

**Definition 4.3** (truncated control-to-state operator). We define the truncated control-to-state operator \( \mathcal{H} : H^{-s}(Ω) → H^{s}(Ω) \) such that for a given control \( r ∈ H^{-s}(Ω) \) it associates a unique state \( \text{tr}_Ω v(r) ∈ H^{s}(Ω) \) via (4.5).

The operator above is well defined, linear and continuous; see [10, Lemma 2.6] for \( s = 1/2 \) and [12, Proposition 2.1] for any \( s ∈ (0,1) \setminus \{ 1/2 \} \). We may also consider \( \mathcal{H} \) acting on \( L^2(Ω) \) and with range in \( L^2(Ω) \), and for simplicity we keep the same notation for this operator.

We define the truncated optimal state-control pair as follows.

**Definition 4.4** (truncated optimal state-control pair). A state-control pair \( (\bar{v}(r), r) ∈ \bar{H}_L^1(y^σ, C_γ) × Z_{ad} \) is called optimal for the truncated control problem, if \( \bar{v}(r) = \mathcal{H} r \) and
\[
\mathcal{J}(\bar{v}(r), r) \leq \mathcal{J}(v(r), r),
\]
for all \( (v(r), r) ∈ H^s(Ω) × Z_{ad} \) such that \( v(r) = \mathcal{H} r \).

We now proceed to define the adjoint operator of \( \mathcal{H} \) and the optimal adjoint state.

**Definition 4.5** (truncated adjoint operator). The adjoint operator of \( \mathcal{H} \), i.e., \( \mathcal{H}^*: L^2(Ω) → L^2(Ω) \) is given by \( \mathcal{H}^* r = d_α \text{tr}_Ω w \), where \( w ∈ \bar{H}_L^1(y^σ, C_γ) \) solves
\[
a_{γ}(w, φ) = (r, \text{tr}_Ω φ)_{L^2(Ω)}, \quad \forall φ ∈ \bar{H}_L^1(y^σ, C_γ).
\]

**Definition 4.6** (truncated optimal adjoint state). We define the optimal adjoint state \( \bar{p}(r) ∈ \bar{H}_L^1(y^σ, C_γ) \) associated with the optimal state \( \bar{v}(r) \) to be the solution of
\[
a_{γ}(\bar{p}(r), φ) = (\text{tr}_Ω \bar{v}(r) - u_d, \text{tr}_Ω φ)_{L^2(Ω)}, \quad \forall φ ∈ \bar{H}_L^1(y^σ, C_γ).
\]

In the same spirit of §3.2 and as a consequence of Definitions 4.3, 4.5 and 4.6, we have the relation \( d_α \text{tr}_Ω \bar{p} = \mathcal{H}^*(\mathcal{H} r - u_d) \). Moreover, the same arguments developed in §3.1 and §3.2 allow us to derive the following result.

**Theorem 4.7** (existence, uniqueness and optimality system). The truncated control problem (4.3)-(4.6) has a unique optimal solution \( (\bar{v}, \bar{r}) ∈ \bar{H}_L^1(y^σ, C_γ) × Z_{ad} \). The optimality system
\[
\begin{cases}
\bar{v} = \bar{v}(r) ∈ \bar{H}_L^1(y^σ, C_γ) \text{ solution of (4.5),} \\
\bar{p} = \bar{p}(r) ∈ \bar{H}_L^1(y^σ, C_γ) \text{ solution of (4.10),} \\
\bar{r} ∈ Z_{ad}, \quad (d_α \text{tr}_Ω \bar{p} + λ_ś \bar{r} - \bar{r})_{L^2(Ω)} ≥ 0 \quad \forall r ∈ Z_{ad},
\end{cases}
\]
hold. These conditions are necessary and sufficient.

The next result shows the approximation properties of the optimal pair \( (\bar{v}(r), \bar{r}) \) solving the truncated control problem (4.3)-(4.6).

**Lemma 4.8** (exponential convergence). For every \( γ ≥ 1 \), we have
\[
\| \bar{r} - \bar{z} \|_{L^2(Ω)} ≤ e^{-γ \sqrt{\lambda_1}/4} \left( \| \bar{r} \|_{L^2(Ω)} + \| u_d \|_{L^2(Ω)} \right),
\]
We start by setting \( q = \tilde{r} \in \mathbb{Z}_{ad} \) and \( r = \tilde{z} \in \mathbb{Z}_{ad} \) in the variational inequalities of the systems \((3.11)\) and \((4.11)\) respectively. Adding the obtained results, we derive

\[
\lambda \| r - z \|^2_{L^2(\Omega)} \leq (d_s \text{tr}_\Omega (\mathcal{P} - \tilde{P} r), r - \tilde{z})_{L^2(\Omega)},
\]

where we have used the result of Theorem \(3.10\). Since \( q = \tilde{z} \). Now, we add and subtract \( \text{tr}_\Omega (\mathcal{P}(\tilde{r})) \) in the inequality above to get

\[
\lambda \| r - z \|^2_{L^2(\Omega)} \leq (d_s \text{tr}_\Omega (\mathcal{P} - \mathcal{P}(\tilde{r})), r - \tilde{z})_{L^2(\Omega)} + (d_s \text{tr}_\Omega (\mathcal{P}(\tilde{r}) - \tilde{P} r), r - \tilde{z})_{L^2(\Omega)} = I + II.
\]

We now proceed to estimate II in 3 steps. Let \( \psi = \mathcal{P}(\tilde{r}) - \tilde{P} r - \mathcal{P}(\tilde{r})(\cdot, \gamma') \) be a modification of \( \mathcal{P}(\tilde{r}) - \tilde{P} r \) with vanishing trace at \( y = \gamma' \). We observe that \( \psi \) satisfies

\[
a_{\gamma'}(\psi, \phi) = a(\mathcal{P}(\tilde{r}), \phi) - a_{\gamma'}(\tilde{P} r, \phi) - a_{\gamma'}(\mathcal{P}(\tilde{r})(\cdot, \gamma'), \phi), \quad \forall \phi \in \mathcal{H}_1^1(\gamma', C_r),
\]

by extending \( \phi \) by zero to \( C \). Since \( \tilde{P} r \in \mathcal{H}_1(\gamma', C_r) \) solves problem \((4.10)\) and \( \mathcal{P}(\tilde{r}) \in \mathcal{H}_1^1(\gamma', C_r) \) solves problem \((3.11)\) with \( q \) replaced by \( \tilde{r} \), the expression above becomes

\[
a_{\gamma'}(\psi, \phi) = (\text{tr}_\Omega (\mathcal{P}(\tilde{r}) - \tilde{P} r), \phi)_{L^2(\Omega)} - a_{\gamma'}(\mathcal{P}(\tilde{r})(\cdot, \gamma'), \phi), \quad (4.14)
\]

for all \( \phi \in \mathcal{H}_1^1(\gamma', C_r) \).

An estimate for \( \psi \) in the \( \mathcal{H}_1^1(\gamma', C_r) \) semi-norm, follows easily from Remark \(2.3\) provided we can estimate the right-hand side of the expression \((4.14)\). We start estimating the second term in \((4.14)\) by exploiting the expression

\[
\mathcal{P}(\tilde{r}) = \sum_{k=1}^{\infty} b_k \varphi_k(x') \psi_b(y),
\]

where \( \text{tr}_\Omega (\mathcal{P}(\tilde{r})) = \sum_{k=1}^{\infty} b_k \varphi_k(x') \) and the functions \( \varphi_k(x') \) and \( \psi_b(y) \) are defined by \((3.1)\) and \((2.1)\) respectively; see \((2.2)\). In fact, we have the following estimate

\[
|a_{\gamma'}(\mathcal{P}(\tilde{r})(\cdot, \gamma'), \phi)| \lesssim \| \nabla \mathcal{P}(\tilde{r})(\cdot, \gamma') \|_{L^2(\gamma'), \gamma'} \| \nabla \phi \|_{L^2(\gamma'), \gamma'}
\]

\[
\| \nabla \mathcal{P}(\tilde{r})(\cdot, \gamma') \|_{L^2(\gamma'), \gamma'} \lesssim \gamma^{1+\alpha} \sum_{k=1}^{\infty} b_k^2 \lambda_k \psi_b^2(\gamma').
\]

Now, since \( |\psi_b(y)| \lesssim (\sqrt{\lambda_k} y)^s e^{-\sqrt{\lambda_k} y} \) for \( y \geq 1 \), we easily see that

\[
\| \nabla \mathcal{P}(\tilde{r})(\cdot, \gamma') \|_{L^2(\gamma'), \gamma'} \lesssim \gamma^{2(1-s)} \sum_{k=1}^{\infty} \lambda_k b_k^2 (\sqrt{\lambda_k} y)^{2s} e^{-2\sqrt{\lambda_k} y}
\]

\[
\lesssim e^{-\sqrt{\lambda_k} y} \sum_{k=1}^{\infty} \lambda_k b_k^2 = e^{-\sqrt{\lambda_k} y} \| \text{tr}_\Omega (\mathcal{P}(\tilde{r})) \|_{\mathcal{H}_1^1(\Omega)}^2.
\]

\[
\| \text{tr}_\Omega (\mathcal{P}(\tilde{r}) - \tilde{P} r) \|_{L^2(\Omega)} \lesssim e^{-\sqrt{\lambda_k} y} \| \text{tr}_\Omega (\mathcal{P}(\tilde{r})) \|_{\mathcal{H}_1^1(\Omega)}^2.
\]
We set \( \phi = \psi \in \dot{H}^1_1(y^\alpha, C_\gamma) \). Then equality (4.14), in conjunction with Remark 2.3 and the estimate above, yields

\[
\| \nabla \psi \|_{L^2(\Omega)}^2 \lesssim \| \text{tr}_\Omega(\mathcal{W}(\bar{r} - \bar{v}(\bar{r})) \|_{L^2(\Omega)} \| \text{tr}_\Omega \psi \|_{L^2(\Omega)} + \epsilon^{-\sqrt{\lambda}/2} \| \text{tr}_\Omega \mathcal{P}(\bar{r}) \|_{H^1(\Omega)} \| \nabla \psi \|_{L^2(\Omega)}),
\]

which, by employing the trace estimate (2.9), the exponential estimate (4.8) and \( \| \text{tr}_\Omega \mathcal{P}(\bar{r}) \|_{H^1(\Omega)} \lesssim \| \bar{r} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \), allows us to derive

\[
\| \nabla \psi \|_{L^2(\Omega)} \lesssim \epsilon^{-\sqrt{\lambda}/4}(\| \bar{r} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)}).
\]

5 By recalling that \( \psi = \mathcal{P}(\bar{r}) - \bar{p} - \mathcal{P}(\bar{r})(\cdot, \gamma) \), the estimate above implies

\[
\| \mathcal{P}(\bar{r}) - \bar{p} - \mathcal{P}(\bar{r})(\cdot, \gamma) \|_{L^2(\Omega)} \lesssim \epsilon^{-\sqrt{\lambda}/4}(\| \bar{r} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)}),
\]

which, together with (1) and (2), concludes the proof of (4.12).

6 Extending \( \bar{v}(\bar{r}) \in \dot{H}^1_1(y^\alpha, C_\gamma) \) by zero to \( C \), we have that \( \mathcal{W}(\bar{z}) - \bar{v}(\bar{r}) \in \dot{H}^1_1(y^\alpha, C) \) solves

\[
a(\mathcal{W}(\bar{z}) - \bar{v}(\bar{r}), \phi) = (\bar{z} - \bar{r}, \text{tr}_\Omega \phi)_{L^2(\Omega)}, \quad \forall \phi \in \dot{H}^1_1(y^\alpha, C).
\]

The well-posedness of the problem above, in conjunction with Remark 2.3, implies

\[
\| \nabla(\mathcal{W}(\bar{z}) - \bar{v}(\bar{r})) \|_{L^2(\Omega)} \lesssim \| \bar{z} - \bar{r} \|_{L^2(\Omega)} \lesssim \epsilon^{-\sqrt{\lambda}/4}(\| \bar{r} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)}),
\]

which yields (4.13) and concludes the proof.

5. A priori error estimates. In this Section, we propose and analyze two simple numerical strategies to solve the fractional control problem (1.1)-(1.3): a semi-discrete scheme based on the so-called variational approach introduced by Hinze in [23] and a fully-discrete scheme which discretizes both the state and control spaces. Before proceeding with the analysis of our method, it is instructive to review the a priori error analysis for the numerical approximation of the state equation (5.5) developed in [34]. In an effort to make this contribution self-contained, such results are briefly presented in the following Subsection.

5.1. A finite element method for the state equation. In order to study the finite element discretization of problem (4.5), we must first understand the regularity of the solution \( \mathcal{W} \). This is because an error estimate for \( \psi \), solution of (4.5), depends on the regularity of \( \mathcal{W} \); see [34] §4.1. We recall that [34] Theorem 2.7 reveals that the second order regularity of \( \mathcal{W} \) is significantly worse in the extended direction, since it requires a stronger weight \( y^\beta \):

\[
\| \mathcal{L} \mathcal{W} \|_{L^2(\Omega)} + \| \partial_\gamma \nabla x^\gamma \mathcal{W} \|_{L^2(\Omega)} \lesssim \| z \|_{H^{1-\epsilon}(\Omega)},
\]

\[
\| \mathcal{W}_{y^\beta} \|_{L^2(\Omega)} \lesssim \| z \|_{L^2(\Omega)},
\]

where \( \beta > 2\alpha + 1 \). However, estimate (5.1) requires \( z \in H^{1-\epsilon}(\Omega) \), which might be too strong an assumption since it does not allow for meaningful duality arguments. For this reason, we also present an improvement over (5.1), in which the regularity of \( z \) has been weakened. This is done at the expense of strengthening the weight from \( y^\alpha \)
to \( y^\beta \) as in [5.2], which is already needed to control the term \( \mathcal{U}_{yy} \). This improved regularity result of [35, Theorem 2.7] reads:

\[
\| \mathcal{U} \|_{H^2(y^\beta, \mathcal{C})} \lesssim \| z \|_{L^2(\Omega)},
\]

where \( \beta > 2\alpha + 1 \). Concerning the domain \( \Omega \), in the analysis that follows, we will tacitly assume the regularity result

\[
\| w \|_{H^2(\Omega)} \lesssim \| \mathcal{L}w \|_{L^2(\Omega)}, \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega),
\]

which is valid if the domain \( \Omega \) is convex [22].

The regularity estimates (5.1)-(5.3) suggest that \textit{graded meshes} in the extended variable \( y \) play a fundamental role. In fact, since, \( \mathcal{U}_{yy} \approx y^{-\alpha-1} \) as \( y \approx 0 \), which follows from the behavior of the Bessel functions (see [34, (2.35)]), anisotropy in the extended variable is fundamental to recover quasi-optimality; see [34, § 5]. This in turn motivates the construction of a mesh over \( \mathcal{C}_Y \) as follows. We first consider a graded partition \( \mathcal{I}_Y \) of the interval \([0, Y]\) with mesh points

\[
y_k = \left( \frac{k}{M} \right) \gamma, \quad k = 0, \ldots, M, \tag{5.4}
\]

where \( \gamma > 3/(1 - \alpha) = 3/2s > 1 \). To avoid technical difficulties we have assumed that the boundary of \( \Omega \) is polygonal. The difficulties inherent to curved boundaries could be handled, for instance, with the methods of [4]. We then consider \( \mathcal{T}_{\Omega} = \{ K \} \) to be a conforming and shape regular mesh of \( \Omega \), where \( K \subset \mathbb{R}^n \) is an element that is isoparametrically equivalent either to the unit cube \([0,1]^n\) or the unit simplex in \( \mathbb{R}^n \). The collection of these triangulations \( \mathcal{T}_{\Omega} \) is denoted by \( \mathcal{T}_{\Omega} \). We construct the mesh \( \mathcal{T}_\mathcal{Y} \) as the tensor product triangulation of \( \mathcal{T}_{\Omega} \) and \( \mathcal{I}_Y \). In order to obtain a global regularity assumption for \( \mathcal{T}_\mathcal{Y} \), we assume that there is a constant \( \sigma_\mathcal{Y} \) such that if \( T_1 = K_1 \times I_1 \) and \( T_2 = K_2 \times I_2 \in \mathcal{T}_\mathcal{Y} \) have nonempty intersection, then

\[
\frac{h_{I_1}}{h_{I_2}} \leq \sigma_\mathcal{Y}, \tag{5.5}
\]

where \( h_I = |I| \). It is well known that this weak regularity condition allows for anisotropy in the extended variable [17, 34]. The set of all triangulations of \( \mathcal{C}_Y \) that are obtained with this procedure and satisfy these conditions is denoted by \( \mathcal{T} \).

Remark 5.1 (\( s \)-independent mesh grading). We note that the term \( \gamma = \gamma(s) \), which defines the graded mesh \( \mathcal{I}_Y \) by (5.4), deteriorates as \( s \) becomes small because \( \gamma > 3/2s \). However, a modified mesh grading in the \( y \)-direction has been proposed in [15, §7.3], which does not change the ratio of the degrees of freedom in \( \Omega \) and the extended dimension by more than a constant and provides a uniform bound with respect to \( s \in (0, 1) \).

For \( \mathcal{T}_\mathcal{Y} \in \mathcal{T} \), we define the finite element space

\[
\mathcal{V}(\mathcal{T}_\mathcal{Y}) = \{ W \in C^0(\overline{\mathcal{C}_Y}) : W|_T \in \mathcal{P}_1(K) \otimes \mathcal{P}_1(I) \forall T \in \mathcal{T}_\mathcal{Y}, \ W|_{\Gamma_D} = 0 \}, \tag{5.6}
\]

where \( \Gamma_D = \partial_{\mathcal{L}} \mathcal{C}_Y \cup \Omega \times \{ Y \} \) is called the Dirichlet boundary; the space \( \mathcal{P}_1(K) \) is \( \mathcal{P}_1(K) \) when the base \( K \) of an element \( T = K \times I \) is simplicial, and \( \mathcal{Q}_1(K) \) when it is an \( n \)-rectangle.

The Galerkin approximation of (1.15) is given by the unique function \( V_{\mathcal{Y}} \in \mathcal{V}(\mathcal{T}_\mathcal{Y}) \) such that

\[
\int_{\mathcal{C}_Y} y^\alpha \nabla V_{\mathcal{Y}} \cdot \nabla W = d_s(r, \text{tr}_\Omega W)_{H^{-s}(\Omega) \times H^s(\Omega)}, \quad \forall W \in \mathcal{V}(\mathcal{T}_\mathcal{Y}). \tag{5.7}
\]
Existence and uniqueness of \( V_{\mathcal{F}} \) immediately follows from \( \mathcal{V}(\mathcal{F}) \subset H^1(y^\alpha, \mathcal{F}) \) and the Lax-Milgram lemma.

We define the space \( U(\mathcal{F}_\Omega) = \text{tr}_\Omega \mathcal{V}(\mathcal{F}_\Omega) \), which is nothing more than a \( P_1 \) finite element space over the mesh \( \mathcal{F}_\Omega \). The finite element approximation of \( u \in H^s(\Omega) \), solution of (1.2) with \( r \) as a datum, is then given by

\[
U_{\mathcal{F}_\Omega} = \text{tr}_\Omega V_{\mathcal{F}} \in U(\mathcal{F}_\Omega). \tag{5.8}
\]

It is trivial to obtain a best approximation result \( \text{à la Cea} \) for problem \( \text{5.7} \). This best approximation result reduces the numerical analysis of problem \( \text{5.7} \) to a question in approximation theory, which in turn can be answered with the study of piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces; see \( \text{[34, Theorem 4.6–4.8, and [36]} \, \text{for details. However, since } \mathcal{W}_{yy} \approx y^{-\alpha-1} \text{ as } y \approx 0, \text{we realize that } \mathcal{W} \notin H^2(y^\alpha, \mathcal{F}) \text{ and the second estimate is not meaningful for } j = n + 1. \text{In view of estimate (5.2)} \), it is necessary to measure the regularity of \( \mathcal{W}_{yy} \) with a stronger weight and thus compensate with a graded mesh in the extended dimension. This makes anisotropic estimates essential.

Notice that \( \# \mathcal{F}_\Omega = M \# \mathcal{F}_0 \), and that \( \# \mathcal{F}_\Omega \approx M^n \) implies \( \# \mathcal{F}_\Omega \approx M^{n+1} \). Finally, if \( \mathcal{F}_\Omega \) is shape regular and quasi-uniform, we have \( h_{\mathcal{F}_\Omega} \approx \left( \# \mathcal{F}_\Omega \right)^{-1/n} \). All these considerations allow us to obtain the following result, which follows from \( \text{[34, Theorem 5.4 and [34 Corollary 7.11]} \) in conjunction with \( \text{[35, Proposition 4.7].}

**Theorem 5.2 (a priori error estimate).** Let \( \mathcal{F}_\mathcal{T} \subseteq \mathcal{T} \) be a tensor product grid, which is quasi-uniform in \( \Omega \) and graded in the extended variable so that \( \text{[5.3]} \) holds. If \( r \in L^2(\Omega), u \) denotes the solution of (1.2) with \( r \) as a datum, \( v \) solves problem (1.5), \( V_{\mathcal{F}_\mathcal{T}} \in \mathcal{V}(\mathcal{F}_\mathcal{T}) \) is the Galerkin approximation defined by \( \text{[5.7]} \), and \( U_{\mathcal{F}_\mathcal{T}} \in U(\mathcal{F}_\mathcal{T}) \) is the approximation defined by \( \text{[5.8]} \), then we have

\[
\| \text{tr}_\Omega v - U_{\mathcal{F}_\mathcal{T}} \|_{L^2(\Omega)} \leq C 2^{s \left( \# \mathcal{F}_\mathcal{T} \right)^{-2/(n+1)} \| r \|_{L^2(\Omega)}, \tag{5.9}
\]

and

\[
\| u - U_{\mathcal{F}_\mathcal{T}} \|_{L^2(\Omega)} \leq C 2^{s \left( \# \mathcal{F}_\mathcal{T} \right)^{-2/(n+1)} \| r \|_{L^2(\Omega)}. \tag{5.10}
\]

where \( C \approx \log(\# \mathcal{F}_\mathcal{T}). \)

**Remark 5.3 (Computational complexity).** The cost of solving the discrete problem \( \text{[5.7]} \) is related to \( \# \mathcal{F}_\mathcal{T} \), and not to \( \# \mathcal{F}_\mathcal{T} \), but the resulting system is sparse. The structure of \( \text{[5.7]} \) is so that fast multilevel solvers can be designed with complexity proportional to \( \# \mathcal{F}_\mathcal{T} (\log(\# \mathcal{F}_\mathcal{T}))^{1/(n+1)} \); see \( \text{[15, On the other hand, we comment that a discretization of the intrinsic integral formulation of the fractional Laplacian \text{{[9, 11]}} would result in a dense matrix. Moreover, it involves the development of accurate quadrature formulas for singular integrands; see \text{{[26]}} to observe these complications even in a one-dimensional setting.}} \)
5.2. The variational approach: a semi-discrete scheme. In this Subsection we consider the variational approach introduced and analyzed by Hinze in [23], which only discretizes the state space; the control space $Z_{ad}$ is not discretized. It guarantees conformity since the continuous and discrete admissible sets coincide and induces a discretization of the optimal control by projecting the discrete adjoint state into the admissible control set. Following [23], we consider the following semi-discretized optimal control problem:

$$\min J(V, g) := \frac{1}{2} \| \text{tr}_\Omega V - u_d \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| g \|^2_{L^2(\Omega)},$$  \hfill (5.11)

subject to the discrete state equation

$$a_{\gamma'}(V, W) = d_s(g, \text{tr}_\Omega W)_{H^{-s}(\Omega) \times H^s(\Omega)}, \quad \forall W \in \mathcal{V}(\mathcal{F}_Y),$$  \hfill (5.12)

and the control constraints

$$g \in Z_{ad}.\quad \hfill (5.13)$$

We denote by $(\bar{V}, \bar{g}) \in \mathcal{V}(\mathcal{F}_Y) \times Z_{ad}$ the optimal pair solving the control problem (5.11)-(5.13). Then, by defining

$$\bar{U} := \text{tr}_\Omega \bar{V},$$  \hfill (5.14)

we obtain an approximation $(\bar{U}, \bar{g}) \in \mathcal{U}(\mathcal{F}_Y) \times Z_{ad}$ of the optimal pair $(\bar{u}, \bar{z}) \in H^s(\Omega) \times Z_{ad}$ solving the fractional control problem (1.1)-(1.3).

**Remark 5.4 (locality).** The main advantage of the semi-discrete optimal problem (5.11)-(5.13) is its local nature, thereby mimicking that of problem (3.4)-(3.6).

In order to study the control problem defined above, we define the control-to-state operator $H^T_Y : Z_{ad} \rightarrow \mathcal{U}(\mathcal{F}_Y)$, which given a control $g \in Z_{ad}$ associates a unique discrete state $\text{tr}_\Omega V(g) = H^T_Y g$ solving problem (5.12). This operator is linear and continuous as a consequence of the Lax-Milgram Lemma.

We define the optimal adjoint state $\bar{P} = \bar{P}(\bar{g}) \in \mathcal{V}(\mathcal{F}_Y)$ as the solution of

$$a_{\gamma'}(\bar{P}, W) = (\text{tr}_\Omega \bar{V} - u_d, \text{tr}_\Omega W)_{L^2(\Omega)}, \quad \forall W \in \mathcal{V}(\mathcal{F}_Y).$$  \hfill (5.15)

We also define the adjoint operator $H^*_\mathcal{F}_Y : L^2(\Omega) \rightarrow \mathcal{U}(\mathcal{F}_Y)$ such that $H^*_\mathcal{F}_Y g = d_s \text{tr}_\Omega R$, where $R$ solves the discrete problem

$$a_{\gamma'}(R, W) = (g, \text{tr}_\Omega W)_{L^2(\Omega)}, \quad \forall W \in \mathcal{V}(\mathcal{F}_Y).$$  \hfill (5.16)

The following result states the existence and uniqueness of the optimal control together with the first order necessary and sufficient optimality condition for problem (5.11)-(5.13).

**Theorem 5.5 (existence, uniqueness and optimality system).** The semi-discrete control problem (5.11)-(5.13) has a unique optimal solution $(\bar{V}, \bar{g}) \in \mathcal{V}(\mathcal{F}_Y) \times Z_{ad}$. The optimality system

$$\begin{cases} \bar{V} = \bar{V}(\bar{g}) \in \mathcal{V}(\mathcal{F}_Y) & \text{solution of (5.12)}, \\
\bar{P} = \bar{P}(\bar{g}) \in \mathcal{V}(\mathcal{F}_Y) & \text{solution of (5.15)}, \\
\bar{g} \in Z_{ad}, & (d_s \text{tr}_\Omega \bar{P} + \lambda g - \bar{g})_{L^2(\Omega)} \geq 0 \quad \forall g \in Z_{ad}, \end{cases} \quad (5.17)$$

hold. These conditions are necessary and sufficient.
\[ \| (\mathbf{H} - \mathbf{H}_\mathcal{P}) r \|_{L^2(\Omega)} \lesssim \gamma^{2s}(\#\mathcal{P})^{-2/(n+1)} \| r \|_{L^2(\Omega)}. \] 

(5.18)

Moreover, as a consequence of Definition 5.1 and the definition of the discrete adjoint operator via problem (5.16), we also have

\[ \| (\mathbf{H}^* - \mathbf{H}_\mathcal{P}^*) r \|_{L^2(\Omega)} \lesssim \gamma^{2s}(\#\mathcal{P})^{-2/(n+1)} \| r \|_{L^2(\Omega)}. \] 

(5.19)

We now present an a priori error estimate for the semi-discrete scheme. The proof is similar to the one introduced by Hinz in [23] and it is based in the error estimates (5.18) and (5.19). However, we recall the arguments to verify that they are still valid in the anisotropic framework of [18], which is summarized in §5.1.

**Theorem 5.6** (Variational approach: error estimate). If \((\bar{v}, \bar{r})\) denotes the optimal pair solving \((4.1) - (4.6)\) and \((\bar{V}(\bar{g}), \bar{g})\) solves the semi-discrete control problem \((5.11) - (5.17)\), then we have the following error estimates

\[ \| \bar{r} - \bar{g} \|_{L^2(\Omega)} \lesssim \gamma^{2s}(\#\mathcal{P})^{-2/(n+1)} \left( \| \bar{r} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right), \] 

(5.20)

and

\[ \| \text{tr}_\Omega \bar{v} - \text{tr}_\Omega \bar{V} \|_{L^2(\Omega)} \lesssim \gamma^{2s}(\#\mathcal{P})^{-2/(n+1)} \left( \| \bar{r} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right), \] 

(5.21)

where \(\gamma \approx |\log(\#\mathcal{P})|\).

**Proof.** We start by setting \(r = \bar{g} \in Z_{ad}\) and \(g = \bar{r} \in Z_{ad}\) in the variational inequalities of systems (4.11) and (5.17) respectively. Adding the obtained results, we arrive at

\[ \lambda\| \bar{r} - \bar{g} \|^2_{L^2(\Omega)} \leq (d_s \text{tr}_\Omega \bar{p} - \bar{P}, \bar{g} - \bar{r})_{L^2(\Omega)}. \]

We now proceed to use the relations \(d_s \text{tr}_\Omega \bar{p} = d_s \text{tr}_\Omega \bar{p}(\bar{r}) = \mathbf{H}^*(\mathbf{H} \bar{r} - u_d)\) and \(d_s \text{tr}_\Omega \bar{P} = d_s \text{tr}_\Omega \bar{P}(\bar{g}) = \mathbf{H}_\mathcal{P}^*(\mathbf{H}_\mathcal{P} \bar{g} - u_d)\), to rewrite the inequality above as

\[ \lambda\| \bar{r} - \bar{g} \|^2_{L^2(\Omega)} \leq \mathbf{H}^* \mathbf{H} \bar{r} - \mathbf{H}_\mathcal{P}^* \mathbf{H}_\mathcal{P} \bar{g} - \mathbf{H}_\mathcal{P}^* (\mathbf{H}^* - \mathbf{H}^*) u_d, \bar{g} - \bar{r} \|_{L^2(\Omega)} \]

which, by adding and subtracting the term \(\mathbf{H}_\mathcal{P}^* \mathbf{H}_\mathcal{P} \bar{r}\), yields

\[ \lambda\| \bar{r} - \bar{g} \|^2_{L^2(\Omega)} \leq ((\mathbf{H}\mathbf{H} - \mathbf{H}_\mathcal{P}\mathbf{H}_\mathcal{P}) \bar{r} - (\mathbf{H}_\mathcal{P}\mathbf{H}_\mathcal{P} - \mathbf{H}^*) u_d, \bar{g} - \bar{r})_{L^2(\Omega)} - \| \mathbf{H}_\mathcal{P} (\bar{r} - \bar{g}) \|^2_{L^2(\Omega)}. \]

We now add and subtract the term \(\mathbf{H}^* \mathbf{H}_\mathcal{P} \bar{r}\) to arrive at

\[ \lambda\| \bar{r} - \bar{g} \|^2_{L^2(\Omega)} \leq ((\mathbf{H}\mathbf{H} - \mathbf{H}_\mathcal{P}\mathbf{H}_\mathcal{P}) \bar{r} + (\mathbf{H}^* - \mathbf{H}_\mathcal{P}^* \mathbf{H}_\mathcal{P}) \mathbf{H}_\mathcal{P} \bar{r} - (\mathbf{H}_\mathcal{P}^* \mathbf{H}_\mathcal{P} - \mathbf{H}^*) u_d, \bar{g} - \bar{r})_{L^2(\Omega)} \]

\[ \leq \left( \| \mathbf{H}^* \|_{L^2(\Omega)} \| \mathbf{H} - \mathbf{H}_\mathcal{P} \| L^2(\Omega) + \| (\mathbf{H}^* - \mathbf{H}_\mathcal{P}^* \mathbf{H}_\mathcal{P}) \mathbf{H}_\mathcal{P} \bar{r} \|_{L^2(\Omega)} + \| (\mathbf{H}^* - \mathbf{H}_\mathcal{P}^* \mathbf{H}_\mathcal{P}) u_d \|_{L^2(\Omega)} \right) \| \bar{r} - \bar{g} \|_{L^2(\Omega)}. \] 

(5.22)
We use (5.18) and (5.19), together with the continuity of \( H \) and \( H^* \), to derive
\[
\| \bar{\mathbf{r}} - \bar{\mathbf{g}} \|_{L^2(\Omega)} \lesssim \gamma^{2s} (\# \mathcal{T}_p)^{-2/(n+1)} \left( \| \bar{\mathbf{r}} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right).
\]
Finally, the estimate (5.21) follows easily. In fact, since \( \bar{v} = \bar{v}(\bar{r}) \) and \( \operatorname{tr}_\Omega \vec{V} = \operatorname{tr}_\Omega \vec{V}(\bar{g}) \), we conclude that
\[
\| \operatorname{tr}_\Omega \bar{v} - \operatorname{tr}_\Omega \vec{V} \|_{L^2(\Omega)} = \| H \bar{r} - H_{\mathcal{T}_p} \bar{g} \|_{L^2(\Omega)} \leq \| H(\bar{r} - \bar{g}) \|_{L^2(\Omega)} + \| (H - H_{\mathcal{T}_p}) \bar{g} \|_{L^2(\Omega)}
\]
which, via estimates (5.18) and (5.20), yields (5.21) and concludes the proof. \( \square \)

**Remark 5.7** (variational approach: advantages and disadvantages). The key advantage of the so-called variational approach is obtaining an optimal quadratic rate of convergence [20, Theorem 2.4] for the control. In our case, it allows us to derive the quasi-optimal estimate (5.20), with respect to the degrees of freedom. However, from an implementation perspective this projection may lead to a control which is not discrete in the current mesh and thus requires an independent mesh.

**Remark 5.8** (anisotropic meshes). Examining the proof of Theorem 5.6, we realize that the critical step, where the anisotropy of the mesh \( \mathcal{T}_p \) is needed, is (5.22). The use of quasi-uniform meshes would give a suboptimal estimate in terms of order; see [35, Proposition 4.7]. We note that the analysis developed in [23] allows the use anisotropic meshes through the estimates (5.18) and (5.19). This fact can be easily observed in [22], and has also been exploited to address control problems on nonconvex domains; see [2, § 6], [3, § 3] and [4, § 4].

We now present the following two consequences of Theorem 5.6.

**Corollary 5.9** (fractional control problem: error estimate). Let \((\vec{V}, \bar{g}) \in \mathbb{V}(\mathcal{T}_p) \times Z_{ad}\) solve the semi-discrete problem (5.11)-(5.13) and \( \bar{U} \in U(\mathcal{T}_p) \) be defined as in (5.14). Then, we have
\[
\| \bar{u} - \bar{U} \|_{L^2(\Omega)} \lesssim \left| \log(\# \mathcal{T}_p) \right|^{2s} \left( \# \mathcal{T}_p \right)^{-2/(n+1)} \left( \| \bar{\mathbf{r}} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right) \tag{5.23}
\]
and
\[
\| \bar{z} - \bar{g} \|_{L^2(\Omega)} \lesssim \left| \log(\# \mathcal{T}_p) \right|^{2s} \left( \# \mathcal{T}_p \right)^{-2/(n+1)} \left( \| \bar{\mathbf{r}} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right) \tag{5.24}
\]

**Proof.** We recall that \((\vec{u}, \bar{z}) \in H^1_0(\Omega, \mathcal{C}) \times Z_{ad}\) and \((\bar{v}, \bar{r}) \in \dot{H}^1_0(\Omega, \mathcal{C}_p) \times Z_{ad}\) solve problems (3.4)-(3.6) and (4.4)-(4.6) respectively. Then, triangle inequality in conjunction with Lemma 4.8 and Theorem 5.6 yields
\[
\| \bar{z} - \bar{g} \|_{L^2(\Omega)} \leq \| \bar{z} - \bar{r} \|_{L^2(\Omega)} + \| \bar{r} - \bar{g} \|_{L^2(\Omega)} 
\]
\[
\lesssim \left( e^{-\sqrt{n+1}\gamma/4} + \gamma^{2s} \left( \# \mathcal{T}_p \right)^{-2/(n+1)} \right) \left( \| \bar{\mathbf{r}} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right)
\]
\[
\lesssim \left| \log(\# \mathcal{T}_p) \right|^{2s} \left( \# \mathcal{T}_p \right)^{-2/(n+1)} \left( \| \bar{\mathbf{r}} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right)
\]
where in the last inequality we have used \( \gamma \approx \log(\#(\mathcal{T}_p)) \); see [24, Remark 5.5] for details. This gives the desired estimate (5.24). In order to derive (5.23), a similar argument shows that
\[
\| \bar{u} - \bar{U} \|_{L^2(\Omega)} \leq \| \bar{u} - \operatorname{tr}_\Omega \bar{v} \|_{L^2(\Omega)} + \| \operatorname{tr}_\Omega \bar{v} - \bar{U} \|_{L^2(\Omega)}
\]
\[
\lesssim \left| \log(\# \mathcal{T}_p) \right|^{2s} \left( \# \mathcal{T}_p \right)^{-2/(n+1)} \left( \| \bar{\mathbf{r}} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)} \right),
\]
where we have used (4.13) and (5.21). This concludes the proof. \( \square \)
5.3. A fully discrete scheme. The goal of this section is to introduce and analyze a fully discrete scheme to solve the fractional control problem (1.1)-(1.3).

We start by considering the following discretization of the truncated control problem (4.4)-(4.6):

$$\min J(V, Z) := \frac{1}{2} \| \text{tr}_\Omega V - u_d \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| Z \|^2_{L^2(\Omega)},$$

subject to the discrete state equation

$$a_s(V, W) = d_s(Z, \text{tr}_\Omega W), \quad \forall W \in \mathcal{V}(\mathcal{F}),$$

and the discrete control constraints

$$Z \in \mathcal{Z}_{ad}(\mathcal{F}_1),$$

where the discrete space $\mathcal{V}(\mathcal{F})$ is defined by (5.6) and

$$\mathcal{Z}_{ad}(\mathcal{F}_1) := \mathcal{Z}_{ad} \cap \{ Z \in C^0(\overline{\Omega}) : Z|_K \in \mathcal{P}_0(K) \quad \forall K \in \mathcal{F}_1 \},$$

i.e., the admissible set of controls is discretized by piecewise constants functions. We denote by $(\bar{V}, \bar{Z}) \in \mathcal{V}(\mathcal{F}) \times \mathcal{Z}_{ad}(\mathcal{F}_1)$ the optimal pair solving the discrete control problem defined above. Then, by defining

$$\bar{U} := \text{tr}_\Omega \bar{V},$$

we obtain an approximation $(\bar{U}, \bar{Z}) \in U(\mathcal{F}_1) \times \mathcal{Z}_{ad}(\mathcal{F}_1)$ of the optimal pair $(\bar{u}, \bar{z}) \in H^1(\Omega) \times \mathcal{Z}_{ad}$ solving the fractional control problem (1.1)-(1.3).

**Remark 5.10** (locality). The main advantage of the fully-discrete optimal problem (5.25)-(5.27) is its local nature, thereby mimicking that of problem (3.4)-(3.6).

We define the discrete control-to-state operator $H_{\mathcal{F}} : \mathcal{Z}_{ad}(\mathcal{F}_1) \rightarrow U(\mathcal{F})$, which given a discrete control $Z \in \mathcal{Z}_{ad}(\mathcal{F}_1)$ associates a unique discrete state $\text{tr}_\Omega V(Z) = H_{\mathcal{F}} Z$ solving problem (5.26).

We define the optimal adjoint state $\bar{P} = \bar{P}(\bar{Z}) \in \mathcal{V}(\mathcal{F})$ as the solution of

$$a_s(\bar{P}, W) = (\text{tr}_\Omega \bar{V} - u_d, \text{tr}_\Omega W)_{L^2(\Omega)}, \quad \forall W \in \mathcal{V}(\mathcal{F}),$$

and the adjoint operator of $H_{\mathcal{F}}$ i.e., $H_{\mathcal{F}}^* : L^2(\Omega) \rightarrow L^2(\Omega)$ such that $H_{\mathcal{F}}^* Z = d_s \text{tr}_\Omega R$, where $R$ solves the discrete problem

$$a_s(R, W) = (Z, \text{tr}_\Omega W)_{L^2(\Omega)}, \quad \forall W \in \mathcal{V}(\mathcal{F}).$$

**Theorem 5.11** (existence, uniqueness and optimality system). The fully-discrete control problem (5.25)-(5.27) has a unique optimal solution $(\bar{V}, \bar{Z}) \in \mathcal{V}(\mathcal{F}) \times \mathcal{Z}_{ad}(\mathcal{F}_1)$.

The optimality system

$$\begin{cases}
\bar{V} = \bar{V}(\bar{Z}) \in \mathcal{V}(\mathcal{F}) \text{ solution of (5.26)}, \\
\bar{P} = \bar{P}(\bar{Z}) \in \mathcal{V}(\mathcal{F}) \text{ solution of (5.29)}, \\
\bar{Z} \in \mathcal{Z}_{ad}(\mathcal{F}_1), \quad (d_s \text{tr}_\Omega \bar{P} + \lambda \bar{Z}, Z - \bar{Z})_{L^2(\Omega)} \geq 0 \quad \forall Z \in \mathcal{Z}_{ad}(\mathcal{F}_1),
\end{cases}$$

hold. These conditions are necessary and sufficient.

**Proof.** The proof follows the same arguments employed in the proof of Theorem 3.9. For brevity, we skip the details. \[\square\]
Before analyzing an a priori error estimate for the scheme (5.25)-(5.27) it is fundamental to discuss the regularity properties of the optimal control \( \bar{r} \) solving the truncated control problem (4.4)-(4.6).

**Lemma 5.12** (regularity of the control). Let \( \bar{r} \in Z_{\text{ad}} \) be the optimal control of problem (4.4)-(4.6), and \( a, b \in H^1(\Omega) \), then \( \bar{r} \in H^s(\Omega) \).

**Proof.** Let \( u \in H^s(\Omega) \) be the solution of

\[
\begin{cases}
L^s u = \bar{r}, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Clearly \( u \in H^s(\Omega) \) and consequently, if \( \bar{v} = \bar{v}(\bar{r}) \) and \( \bar{p} = \bar{p}(\bar{r}) \) solve (16) and (18) respectively, Theorem 2.2 implies

\[
\text{tr}_\Omega \bar{v}(\bar{r}) \in H^s(\Omega), \quad \text{tr}_\Omega \bar{p}(\bar{r}) \in H^s(\Omega).
\]

We now resort to the well known projection formula

\[
\bar{r}(x') = \text{proj}_{[a,b]} \left( -\frac{1}{\lambda} \text{tr}_\Omega \bar{p}(x') \right),
\]

which is equivalent to the variational inequality of (1.12) (Section 2.8) for details), and where \( \text{proj}_{[a,b]}(v) := \min \{b, \max \{a, v\}\} \). Next, we define the operator

\[
A : H^1(\Omega) \to H^1(\Omega)
\]

such that \( Aw = \max \{w, 0\} \). This operator is bounded, i.e., \( ||Aw||_{H^1(\Omega)} \lesssim ||w||_{H^1(\Omega)} \) for all \( w \in H^1(\Omega) \); see [28, Theorem A.1]. Moreover, it is not difficult to see that given \( w_1, w_2 \in L^2(\Omega) \), we have that \( ||Aw_1 - Aw_2||_{L^2(\Omega)} \lesssim ||w_1 - w_2||_{L^2(\Omega)} \). In fact, since \( Aw_1(x') \) \( Aw_2(x') = 0 \) on \( \Omega \setminus \Omega_0 \), where \( \Omega_0 := \{x' \in \Omega : w_1(x') > 0 \text{ and } w_2(x') > 0\} \), we have

\[
||Aw_1 - Aw_2||^2_{L^2(\Omega)} = \int_{w_1 > 0} |w_1|^2 + \int_{w_2 > 0} |w_2|^2 - 2 \int_{\Omega} w_1w_2 + 2 \int_{\Omega} w_1w_2 \\
\leq \int_{w_1 > 0} |w_1|^2 + \int_{w_2 > 0} |w_2|^2 - 2 \int_{\Omega} w_1w_2 \leq ||w_1 - w_2||^2_{L^2(\Omega)},
\]

where we have used the Cauchy-Schwarz inequality applied to \( \int_{\Omega} w_1w_2 \). We finally apply an interpolation argument based on [28, Lemma 28.1] to conclude \( ||Ap||_{H^s(\Omega)} \lesssim ||p||_{H^s(\Omega)} \), which in view of the fact that \( \bar{r} = \text{proj}_{[a,b]} \left( -\frac{1}{\lambda} \bar{p} \right) = \min \{b, \max \{a, -\frac{1}{\lambda} \bar{p} \}\} \) immediately implies that \( \bar{r} \in H^s(\Omega) \).

We now recall the definition of the standard \( L^2 \)-orthogonal projection operator \( \mathfrak{P} : L^2(\Omega) \to \mathbb{P}_0(\mathcal{T}_h) \), which is defined by

\[
(r - \mathfrak{P}r, Z) = 0 \quad \forall Z \in \mathbb{P}_0(\mathcal{T}_h);
\]

see [10, 18]. The space \( \mathbb{P}_0(\mathcal{T}_h) \) denote the space of piecewise constants over the mesh \( \mathcal{T}_h \). We also recall the following properties of such operator:

\[
\forall r \in L^2(\Omega), \quad ||\mathfrak{P}r||_{L^2(\Omega)} \lesssim ||r||_{L^2(\Omega)},
\]

and

\[
\forall r \in H^1(\Omega), \quad ||r - \mathfrak{P}r||_{L^2(\Omega)} \lesssim h \mathfrak{T}_h ||r||_{H^1(\Omega)},
\]
where $h_{\mathcal{T}_h}$ denotes the mesh-size of $\mathcal{T}_h$; see [13] Lemma 1.131 and Proposition 1.134. Moreover, given $r \in L^2(\Omega)$, we have that $\mathfrak{f}r|_K = \frac{1}{|K|} \int_K r$, which follows directly from (5.28). If we assume $a(x') = a$ and $b(x') = b$ in $\Omega$ in the definition of the admissible set $Z_{ad}$, we have that $\mathfrak{F}r \in Z_{ad}(\mathcal{T}_h)$, and then $\mathfrak{F} : L^2(\Omega) \to Z_{ad}(\mathcal{T}_h)$ is well defined.

We also write the following approximation estimates, which follows from the property (5.33), the estimate (5.35) and an interpolation argument:

$$\forall r \in H^s(\Omega), \quad \|r - \mathfrak{F}r\|_{L^2(\Omega)} \lesssim h_{\mathcal{T}_h}^s \|r\|_{H^s(\Omega)}. \quad (5.36)$$

We now present the following a priori error estimate for the scheme (5.25)-(5.27).

**Theorem 5.13** (fully discrete scheme: error estimate). If $(\bar{v}(\bar{r}), \bar{r})$ denotes the optimal pair solving (4.4) and $(\bar{V}(\bar{Z}), \bar{Z})$ solves the fully-discrete control problem (5.25), (5.26), then we have the following error estimates

$$\|\bar{r} - \bar{Z}\|_{L^2(\Omega)} \lesssim \gamma^{2s}(\# \mathcal{T}_h)^{-s/(n+1)} \left(\|\bar{r}\|_{H^s(\Omega)} + \|\bar{Z}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}\right) \quad (5.37)$$

and

$$\|\text{tr}_\Omega \bar{v} - \bar{U}\|_{H^s(\Omega)} \lesssim \gamma^{2s}(\# \mathcal{T}_h)^{-s/(n+1)} \left(\|\bar{r}\|_{H^s(\Omega)} + \|\bar{Z}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}\right). \quad (5.38)$$

**Proof.** We proceed in five steps.

1. Since $Z_{ad}(\mathcal{T}_h) \subset Z_{ad}$, we can consider $r = \bar{Z}$ in the variational inequality of (4.11) to arrive at

$$(d_s \text{tr}_\Omega \bar{p} + \lambda \bar{r}, \bar{Z} - \bar{r})_{L^2(\Omega)} \geq 0.$$  

On the other hand, setting $Z = \mathfrak{F}r \in Z_{ad}(\mathcal{T}_h)$ in the variational inequality of (5.31), and adding and substracting $\bar{r}$, we derive

$$(d_s \text{tr}_\Omega \bar{P} + \lambda \bar{Z}, \mathfrak{F}r - \bar{r})_{L^2(\Omega)} + (d_s \text{tr}_\Omega \bar{P} + \lambda \bar{Z}, \bar{r} - \bar{Z})_{L^2(\Omega)} \geq 0,$$

where $\mathfrak{F}$ is $L^2$-orthogonal projection operator defined by (5.36). Consequently, adding the derived expressions we have

$$(d_s \text{tr}_\Omega (\bar{p} - \bar{P}) + \lambda (\bar{r} - \bar{Z}), \bar{Z} - \bar{r})_{L^2(\Omega)} + (d_s \text{tr}_\Omega \bar{P} + \lambda \bar{Z}, \mathfrak{F}r - \bar{r})_{L^2(\Omega)} \geq 0,$$

and then

$$\lambda \|\bar{r} - \bar{Z}\|_{L^2(\Omega)} \lesssim (d_s \text{tr}_\Omega (\bar{p} - \bar{P}), \bar{Z} - \bar{r})_{L^2(\Omega)} + (d_s \text{tr}_\Omega \bar{P} + \lambda \bar{Z}, \mathfrak{F}r - \bar{r})_{L^2(\Omega)} = I + II.$$  

2. The estimate for the term $I$ follows immediately as a consequence of the arguments employed in the proof of Theorem 5.6

$$|I| \lesssim \gamma^{2s}(\# \mathcal{T}_h)^{-2/(n+1)} \left(\|\bar{r}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}\right) \|r - \bar{Z}\|_{L^2(\Omega)}.$$  

3. We now proceed to estimate the term $II$ following [14] Theorem 1.2 and exploiting the properties of the $L^2$-orthogonal projection operator. We write

$$II = (d_s \text{tr}_\Omega \bar{P} + \lambda \bar{Z}, \mathfrak{F}r - \bar{r})_{L^2(\Omega)} = (d_s \text{tr}_\Omega \bar{P}, \mathfrak{F}r - \bar{r})_{L^2(\Omega)} = d_s (\text{tr}_\Omega \bar{P} - \mathfrak{F} \text{tr}_\Omega \bar{P}, \mathfrak{F}r - \bar{r})_{L^2(\Omega)},$$
The estimate above derived for II together with the bound for I derived in [2], yield the desired estimate (5.37). Finally, the estimate (5.38) follows easily. In fact, since we have that \( \bar{\Psi} \), as in (5.28)

where we have used (4.13) and (5.38). This concludes the proof.

The argument shows

where in the last inequality we have used \( Y \). This is a consequence of the regularity properties of the optimal control \( \bar{u} \), optimal in terms of regularity but it is suboptimal in terms of the degrees of freedom.

Lemma 5.12: \( \bar{\Psi} \), imply

problems (3.4)-(3.6) and (4.4)-(4.6) respectively. Then, Lemma 4.8 and Theorem 5.13 (5.3), yields (5.38) and concludes the proof.

We now present the following two consequences of Theorem 5.13.

Corollary 5.14 (fractional control problem: error estimate). Let \((\bar{V}, \bar{Z}) \in V(\mathcal{F}_0) \times Z_{ad}\) solve the fully-discrete problem (5.25) and (5.27) and \( \bar{U} \in U(\mathcal{F}_0) \) be defined as in (5.28). Then, we have

\[
\|\bar{u} - \bar{U}\|_{H^s(\Omega)} \lesssim (\log(\#(\mathcal{F}_0)))^{2s} (\|\bar{z}\|_{H^s(\Omega)} + \|\bar{Z}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}) ,
\]

and

\[
\|\bar{z} - \bar{Z}\|_{L^2(\Omega)} \lesssim (\log(\#(\mathcal{F}_0)))^{2s} (\|\bar{r}\|_{H^s(\Omega)} + \|\bar{Z}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}) ,
\]

where \( \mathcal{Y} \approx \log(\#(\mathcal{F}_0)) \).

Proof. We recall that \((\bar{u}, \bar{z}) \in H^1_L(y^o, \mathcal{C}) \times Z_{ad}\) and \((\bar{r}, \bar{t}) \in H^1_L(y^o, \mathcal{C}) \times Z_{ad}\) solve problems (3.4)-(3.6) and (4.4)-(4.6) respectively. Then, Lemma 4.8 and Theorem 5.13 imply

\[
\|\bar{z} - \bar{Z}\|_{L^2(\Omega)} \leq \|z - \bar{r}\|_{L^2(\Omega)} + \|\bar{r} - \bar{Z}\|_{L^2(\Omega)}
\]

\[
\lesssim \left(e^{-\sqrt{\mathcal{Y}/4}} + (\#(\mathcal{F}_0))^{-s/(n+1)} \right) \left(\|\bar{r}\|_{H^s(\Omega)} + \|\bar{Z}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}\right)
\]

\[
\lesssim (\log(\#(\mathcal{F}_0)))^{2s} (\|\bar{r}\|_{H^s(\Omega)} + \|\bar{Z}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}) ,
\]

where in the last inequality we have used \( \mathcal{Y} \approx \log(\#(\mathcal{F}_0)) \); see [34] Remark 5.5 for details. This gives the desired estimate (5.40). In order to derive (5.39), a similar argument shows

\[
\|\bar{u} - \bar{U}\|_{H^s(\Omega)} \leq \|u - \bar{u}\|_{H^s(\Omega)} + \|\bar{r}\|_{H^s(\Omega)} + \|u_d\|_{L^2(\Omega)}
\]

\[
\lesssim (\log(\#(\mathcal{F}_0)))^{2s} (\|\bar{r}\|_{H^s(\Omega)} + \|\bar{Z}\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}) ,
\]

where we have used (5.13) and (5.38). This concludes the proof.

Remark 5.15 (suboptimal and quasi-optimal estimates). The estimate (5.39) is optimal in terms of regularity but it is suboptimal in terms of the degrees of freedom. This is a consequence of the regularity properties of the optimal control \( \bar{r} \), studied in Lemma 5.12. If we assume \( \bar{r} \in H^1(\Omega) \cap H^s(\Omega) \), the same analysis employed
in the proof of Theorem 5.13 together with the estimate (5.35), it would allow us to derive the following quasi-optimal error estimate in terms of degrees of freedom

\[ \| \bar{z} - \bar{Z} \|_{L^2(\Omega)} \lesssim |\log(#\mathcal{T}_r)|^{2s} (#\mathcal{T}_r)^{\frac{1}{2}} (\| r \|_{L^1(\Omega)} + \| \bar{Z} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)}) . \]  

(5.41)

The following error estimate is also valid when \( \bar{r} \in H^1(\Omega) \cap \mathbb{H}^s(\Omega) \):

\[ \| \bar{u} - \bar{U} \|_{H^1(\Omega)} \lesssim |\log(#\mathcal{T}_r)|^{2s} (#\mathcal{T}_r)^{\frac{1}{2}} (\| r \|_{H^1(\Omega)} + \| \bar{Z} \|_{L^2(\Omega)} + \| u_d \|_{L^2(\Omega)}) . \]  

(5.42)

**Remark 5.16** (quasi-optimal estimate for the optimal state). Exploiting the results of [52] would make it possible to derive a quasi-optimal \( L^2 \)-estimate in terms of the degrees of freedom for the optimal state. Such a result is valid under the following assumptions:

- \( \bar{r} \in W^{2,p}(\Omega) \) for a certain \( p > 2 \).
- \( \bar{r} \in C^{0,1}(\bar{\Omega}) \).
- \( \{ \{ K \in \mathcal{T}_r : \| r \|_{C^{0,1}(\bar{\Omega})} \} \} \lesssim h_{\mathcal{T}_0} \), where \( h_{\mathcal{T}_0} \) denotes the mesh-size of the mesh \( \mathcal{T}_0 \). However, such assumptions are valid only in some special situations, which would restrict the applicability of our results.

**REFERENCES**

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of National Bureau of Standards Applied Mathematics Series. 1964.

[2] T. Apel, Johannes P., and A. Rösch. Finite element error estimates for Neumann boundary control problems on graded meshes. *Comput. Optim. Appl.*, 52(1):3–28, 2012.

[3] T. Apel, A. Rösch, and D. Sirch. \( L^\infty \)-error estimates on graded meshes with application to optimal control. *SIAM J. Control Optim.*, 48(3):1771–1796, 2009.

[4] T. Apel, A. Rösch, and G. Winkler. Optimal control in non-convex domains: a priori discretization error estimates. *Calcolo*, 44(3):137–158, 2007.

[5] N. Arada, E. Casas, and F. Tröltzsch. Error estimates for the numerical approximation of a semilinear elliptic control problem. *Comput. Optim. Appl.*, 23(2):201–229, 2002.

[6] C. Bernardi. Optimal finite-element interpolation on curved domains. *SIAM J. Numer. Anal.*, 26(5):1212–1240, 1989.

[7] M Bonforte, Y Sire, and J.L. Vázquez. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. arXiv:1404.6195, 2014.

[8] C. Brändle, E. Colorado, A. de Pablo, and U. Sánchez. A concave–convex elliptic problem involving the fractional laplacian. *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, 143:39–71, 2013.

[9] X. Cabré and Y. Sire. Nonlinear equations for fractional Laplacians ii: Existence, uniqueness and qualitative properties of solutions. arXiv:1111.0796, 2011.

[10] X. Cabré and J. Tan. Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.*, 224(5):2052–2093, 2010.

[11] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Part. Diff. Eqs.*, 32(7-9):1245–1260, 2007.

[12] A. Capella, J. Dávila, L. Dupaigne, and Y. Sire. Regularity of radial extremal solutions for some non-local semilinear equations. *Comm. Part. Diff. Eqs.*, 36(8):1353–1384, 2011.

[13] E. Casas, M. Mateos, and F. Tröltzsch. Error estimates for the numerical approximation of boundary semilinear elliptic control problems. *Comput. Optim. Appl.*, 31(2):193–219, 2005.

[14] E. Casas and F. Tröltzsch. Error estimates for linear-quadratic elliptic control problems. In *Analysis and optimization of differential systems (Constanta, 2002)*, pages 89–100. Klumer Acad. Publ., Boston, MA, 2003.

[15] L. Chen, R.H. Nochetto, E. Otárola, and A.J. Salgado. Multilevel methods for nonuniformly elliptic operators. arXiv:1403.4278, 2014.

[16] P.G. Ciarlet. *The finite element method for elliptic problems*, volume 40 of Classics in Applied Mathematics. SIAM, Philadelphia, PA, 2002.
[17] R.G. Durán and A.L. Lombardi. Error estimates on anisotropic $Q_1$ elements for functions in weighted Sobolev spaces. *Math. Comp.*, 74(252):1679–1706 (electronic), 2005.

[18] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[19] E. B. Fabes, C.E. Kenig, and R.P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Part. Diff. Eqs.*, 7(1):77–116, 1982.

[20] P. Gamallo and E. Hernández. Error estimates for the approximation of a class of optimal control systems governed by linear PDEs. *Numer. Funct. Anal. Optim.*, 30(5-6):523–547, 2009.

[21] V. Gol’dshtein and A. Ukhlov. Weighted Sobolev spaces and embedding theorems. *Trans. Amer. Math. Soc.*, 361(7):3829–3850, 2009.

[22] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[23] M. Hinze. A variational discretization concept in control constrained optimization: the linear-quadratic case. *Comput. Optim. Appl.*, 30(1):45–61, 2005.

[24] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE constraints*, volume 23 of *Mathematical Modelling: Theory and Applications*. Springer, New York, 2009.

[25] M. Hinze and F. Tröltzsch. Discrete concepts versus error analysis in PDE-constrained optimization. *GAMM-Mitt.*, 33(2):148–162, 2010.

[26] Y. Huang and A. Oberman. Numerical methods for the fractional laplacian part i: a finite difference-quadrature approach. [arXiv:1311.7691] 2013.

[27] K. Ito and K. Kunisch. *Lagrange multiplier approach to variational problems and applications*, volume 15 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[28] G. Kinderlehrer, D.and Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[29] A. Kufner and B. Opic. How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carolin.*, 25(3):537–554, 1984.

[30] N.S. Landkof. *Foundations of modern potential theory*. Springer-Verlag, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[31] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*. Springer-Verlag, New York, 1972.

[32] C. Meyer and A. Rösch. Superconvergence properties of optimal control problems. *SIAM J. Control Optim.*, 43(3):970–985, March 2004.

[33] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[34] R.H. Nochetto, E. Otárola, and A.J. Salgado. A PDE approach to fractional diffusion in general domains: a priori error analysis. *Found. Comput. Math.* accepted for publication.

[35] R.H. Nochetto, E. Otárola, and A.J. Salgado. A PDE approach to space-time fractional parabolic problems. [arXiv:1404.0068] 2014.

[36] R.H. Nochetto, E. Otárola, and A.J. Salgado. Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications. [arXiv:1402.1916v1] 2014.

[37] A. Rösch. Error estimates for linear-quadratic control problems with control constraints. *Optim. Methods Softw.*, 21(1):121–134, 2006.

[38] P.R. Stinga and J.L. Torrea. Extension problem and Harnack’s inequality for some fractional operators. *Comm. Part. Diff. Eqs.*, 35(11):2092–2122, 2010.

[39] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Berlin, 2007.

[40] F. Tröltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications*. Graduate Studies in Mathematics. American Mathematical Society, 2010.

[41] B.O. Turesson. *Nonlinear potential theory and weighted Sobolev spaces*, volume 1736 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.