Modules in which pure submodule is essential in a direct summand

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Abstract

In this paper, we study the class of modules have the property that every pure submodule is essential in a direct summand. These modules are termed as pure extending modules which is a proper generalisation of extending modules. Examples and counterexamples are given. We study some properties of pure extending modules and characterize regular ring, semisimple ring, local ring and PDS ring in terms of pure extending modules.

Keywords: Extending module; Pure Extending module; Pure-Injective module; Regular ring.

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1 Introduction

Utumi[18] observed $C_1$ condition on a ring which is satisfied if the ring is self injective. Later, similarly $C_1$ condition on a module $M$ is defined as, every submodule of a module $M$ is essential in a direct summand of $M$. A module satisfies $C_1$ condition known as $C_1$ module or extending module. As we know that every submodule of a module $M$ need not be pure submodule (Infact no submodule is pure submodule of $Z$ as $Z$ module). Motivated by the above facts, the objective of this paper is to extend the theory of extending modules to
pure extending modules by using the concept of purity. Pure extending modules are those modules in which every pure submodule is essential in a direct summand. So pure extending modules is a proper generalisation of extending modules i.e. the class of pure extending modules is larger than the class of extending modules.

Notion of pure submodules defined by some authors in different aspects which are listed below:

(i) P.M. Cohn \[5\] called a submodule \(P\) of a right \(R\)-module \(M\) to be a pure submodule, if for each left \(R\)-module \(N\), the sequence \(0 \rightarrow P \otimes N \rightarrow M \otimes N\) is exact whenever the sequence \(0 \rightarrow P \rightarrow M\) is exact.

(ii) According to Anderson and Fuller \[2\], a submodule \(P\) of a module \(M\) is said to be pure if \(IP = P \cap IM\) for every ideal \(I\) of \(R\).

(iii) In \[16\], Ribenboim defined a pure submodule \(P\) of a module \(M\) if for every \(r \in R\), \(rP = rM \cap P\). In particular, \(P\) is known as RD (relatively divisible) pure submodule of \(M\).

In above definitions, (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) but converse need not be true, while in case of \(M\) to be flat module all are equivalent. A right \(R\)-module \(M\) is called flat if whenever \(0 \rightarrow N_1 \rightarrow N_2\) is exact for left \(R\)-modules \(N_1\) and \(N_2\) then \(0 \rightarrow M \otimes N_1 \rightarrow M \otimes N_2\) is also exact. A module \(M\) is said to be pure \(C_2\) module if every pure submodule of \(M\) that is isomorphic to a direct summand of \(M\) is itself a direct summand of \(M\) \[14\]. An \(R\)-module \(M\) is said to be pure \(C_3\) module if \(K\) and \(L\) are disjoint direct summands of \(M\) then \(K \oplus L\) is a pure submodule of \(M\) iff \(K \oplus L\) is a direct summand of \(M\) \[13\].

In section 2, after defining the notion of pure extending modules, several examples and counter examples are given to distinguish this class of modules with the various classes of modules. We show that pure extending module is a proper generalisation of extending module by giving counter examples of pure extending module that are not extending module. We provide a sufficient condition under which pure extending module implies extending module. We prove that a direct summand and a pure submodule of pure extending module are pure extending. Let \(P\) and \(Q\) be \(R\)-modules, then a module \(P\) is \(Q\)-pure injective if for every pure submodule \(L\) of \(Q\), \(f \in \text{Hom}_R(L, P)\) can be extended to \(g \in \text{Hom}_R(Q, P)\) such that \(goh = f\) where \(h \in \text{Hom}_R(L,Q)\). Further, \(P\) is said to be a quasi-pure-injective if \(P\) is a \(P\)-pure-injective, while \(P\) is called a pure-injective if it is \(Q\)-pure injective for every \(R\)-module \(Q\) \((9],[11],[20])\). Here, we show that pure quasi injective module implies pure extending module. In general the following chain holds,

\[
\text{Injective} \downarrow \quad \text{Quasi-injective} \quad \downarrow \quad \text{Extending} \\
\text{Pure-injective} \quad \downarrow \quad \text{Pure Quasi-injective} \quad \downarrow \quad \text{Pure extending}
\]

but converse of the above chain need not be true (see \[9\], \[12\],\[20\] and Example 7). Also, we show that direct sum of pure extending modules is pure extending (see Proposition 20). We call a module \(M\), RD-pure extending if every RD-pure submodule of \(M\) is essential in a direct summand of \(M\). As we
have seen above that not every RD-pure submodule is pure submodule. We show that every RD-pure extending module is extending module and converse need not true.

In section 3, we characterize regular ring, semisimple ring, local ring and PDS ring in terms of pure extending modules. Every pure extending module is flat over regular ring (see Proposition 27). We find the equivalent conditions of pure $C_i$ for $i = 1, 2, 3$ to be projective modules over semisimple ring (see Proposition 28). A module $C$ is called cotorsion if $\text{Ext}_1^R(F, C) = 0$ for any flat module $F$ [7]. We show that flat cotorsion module is pure extending module (see Proposition 31).

Throughout this article, we consider all rings to be associative with unity and all modules are right unital unless otherwise specified. The notations $N \leq M$, $N \leq \oplus M$ and $N \leq e M$ will denote $N$ is a submodule of $M$, $N$ is a direct summand of $M$ and $N$ is an essential submodule of $M$ respectively. A regular ring means to be von Neumann regular ring. $E(M)$ and $PE(M)$ denote the injective hull and pure injective hull of a module $M$ respectively. For undefined terms and notions, please refer to [2].

# 2 Pure Extending Modules

In this section, we study extending modules in terms of purity by taking pure submodules. Now we introduce pure extending modules.

**Definition 1** An $R$-module $M$ is called pure extending (or pure $C_1$), if every pure submodule of $M$ is essential in a direct summand of $M$.

**Example 2**

(i) Since, only pure submodules of $\mathbb{Z}$-module $\mathbb{Z}$ are $\{0\}$ and $\mathbb{Z}$ itself. Therefore $\mathbb{Z}$ as a $\mathbb{Z}$-module is a pure extending.

(ii) Any pure injective module $M$ is pure extending module (since every pure injective module is pure quasi injective and by proposition 16).

(iii) Semisimple modules and injective modules are pure extending.

(iv) Any finitely generated module over a Noetherian ring is pure extending module. Since its pure submodules are just direct summands [Corollary 4.91, [12]]. In particular, every finitely generated $\mathbb{Z}$-module is pure extending.

We observed that every $R$-module (in particular $\mathbb{Z}$-module) need not be pure extending.

**Example 3** Consider a $\mathbb{Z}$-module $M$ such that $M = \Pi_{p \in P}\mathbb{Z}/ < p >$ where $p$ varies through all primes. Let $P = \oplus_{p \in P}\mathbb{Z}/ < p >$ be pure submodule which is not essential in a direct summand.

**Proposition 4** Pure submodule of pure extending module is pure extending.
Proof Let $M$ be a pure extending module and $P$ be a pure submodule of $M$. Consider $K$ be a pure submodule of $P$. Thus from [Proposition 7.2,[9]], $K$ is pure submodule of $M$. So there exists a direct summand $L$ of $M$ such that $K \leq^e L$ which implies $K \leq^e L \cap P$. Since $L \leq^e M$, so $M = L \oplus L'$ for some submodule $L'$ of $M$. Now, $M \cap P = (L \oplus L') \cap P$ which implies $P = (L \cap P) \oplus (L' \cap P)$. Hence $P$ is pure extending.

Remark 5 Submodule of a pure extending module need not be pure extending. Let $N$ be a module which is not a pure extending and $PE(N)$ be pure injective hull of $N$. Then $PE(N)$ be pure injective module which implies $PE(N)$ is pure extending while $N \leq PE(N)$ is not pure extending. While over regular ring, every submodule of a pure extending module is pure extending.

Proposition 6 Direct summand of a pure extending module is pure extending.

Proof Let $N$ be a direct summand such that $M = N \oplus N'$ of a pure extending module $M$ and $P$ be a pure submodule of $N$. Since $P \leq N \leq^e M$, $P$ be a pure submodule of $M$. As $M$ is pure extending module, therefore $P$ is essential in a direct summand of $M$. Since $P \leq N$ and $P \cap N' = 0$ then $P$ is essential in a direct summand of $N$. □

The following examples justifies that the class of pure extending modules is a proper generalisation of extending.

Example 7 Consider $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ as $\mathbb{Z}$ module, where $p$ be any prime. In particular for $p = 2$, $P = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$ module. As its each pure submodule is direct summand [Corollary 4.91,[12]], so $P$ is pure extending whereas $P$ is not extending. In fact, its submodule $\mathbb{Z}(1 + 2\mathbb{Z}, 2 + 8\mathbb{Z})$ is not essential in any direct summand of $P$.

Example 8 Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, then $R_R$ is finitely generated and noetherian module so it is pure extending module whereas it is not extending module [Example 6.2,[4]].

In the following proposition, we give a sufficient condition for the pure extending modules to be extending modules.

Proposition 9 A ring $R$ is regular iff every pure extending right (resp., left ) $R$-module is extending module.

Proof Let $M$ be pure extending module then every pure submodule of $M$ is essential in a direct summand $M$. As $R$ is a von-Neumann regular ring so every submodule of an $R$-module $M$ is pure. Hence $M$ is a extending module. Conversely, every pure extending right (resp., left ) $R$-module is extending module, that implies every submodule of $R$-module $M$ is pure. Hence, $R$ be a von-Neumann regular ring. □
**Proposition 10** Let $M$ be a module which is fully invariant in its pure injective hull, then $M$ is pure extending module.

*Proof* Proof follows from [lemma 3.1,[11]]. □

**Proposition 11** Let $R$ and $S$ be rings for which there is a Morita equivalence $F : \text{Mod-}R \to \text{Mod-}S$ and let $A \in \text{Mod-}R$. Then $A$ is pure extending iff $F(A)$ is pure extending.

*Proof* It follows from the morita invariant property of pure submodule. □

**Definition 12** A submodule $K$ of a module $M$ is said to be pure essential in $M$ if $K$ is pure in $M$ and for any non zero submodule $N$ of $M$ either $K \cap N \neq 0$ or $(K \oplus N)/N$ is not pure in $M/N$ [11].

**Proposition 13** If every submodule of a module $M$ is pure essential in $M$, then $M$ is pure extending.

*Proof* Let $N$ be a pure submodule of $M$. Since every submodule is pure essential in $M$, $N$ is essential in $M$. Hence $M$ is pure extending module. □

Converse of the above statement need not true in general. Now, we provide the sufficient condition when it holds true.

**Proposition 14** If $M$ is a pure simple pure extending module then every submodule of $M$ is pure essential submodule.

*Proof* If $M$ be a pure simple module then it has no non proper non trivial pure submodule. Hence every submodule of $M$ is pure essential. □

**Corollary 15** If $M$ is an indecomposable pure extending module then every submodule of $M$ is pure essential submodule.

**Proposition 16** In any quasi injective module $M$, Every pure submodule of $M$ is essential in a direct summand of $M$. 

Proof Let $N$ be a pure submodule of $M$ and write $PE(M) = PE(N) \oplus L$. The pure quasi injectivity of $M$ implies $M \cap PE(M) = M \cap PE(N) \oplus M \cap L$ and $N \leq_e M \cap PE(N)$ it implies pure extending.

Proposition 17 Let $M$ be a $\mathbb{Z}$-module. If $M$ satisfies any one of the following conditions, then $M$ is pure extending module.

(i) $M$ is finitely generated.

(ii) $M$ is divisible.

Proof (i) Since every pure submodule of a finite generated module over a noetherian ring is a direct summand (by Corollary 4.91,[12]). Hence, $M$ is pure extending.

(ii) Let $M$ be a divisible module, this implies that it is an extending $\mathbb{Z}$ module. Hence $M$ is pure extending module.

Proposition 18 Every pure split module is pure extending.

Proof Let $M$ be a pure split $R$-module then every pure submodule of $M$ is direct summand. This implies $M$ is a pure extending module.

Proposition 19 Let $M$ be a module and $M = M_1 \oplus M_2$ be direct sum decomposition . If $N$ be a pure submodule of $M$ then $N = N_1 \oplus N_2$, where $N_i$ is pure submodule of $M_i$ for $i = 1, 2$.

Proof Let $N$ be a pure submodule of $M$ such that $N = N_1 + N_2$. $N_1 \cap N_2 \leq N_1 \leq M_1, N_1 \cap N_2 \leq N_2 \leq M_2$ , which implies $N_1 \cap N_2 \leq M_1 \cap M_2$, so $N_1 \cap N_2 = 0$. Hence, $N = N_1 \oplus N_2$. Since $N_i \leq e \cap N$ then $N_i$ is pure submodule of $N$. $N_i$ is pure in $N$ and $N$ is pure in $M_i$ then $N_i$ is pure in $M_i$ for $i = 1, 2$.

In the next proposition, we show when direct sum of pure extending modules is pure extending module.

Proposition 20 Let $M = \bigoplus_{i \in \Lambda} M_i$, where $\Lambda$ be an arbitrary index set. $M$ is pure extending iff for each $i \in \Lambda$, $M_i$ is pure extending module.

Proof Let $M$ be a pure extending module. So, by proposition 6 $M_i$ is pure extending module for each $i \in \Lambda$.

Conversely, Let $N$ be a pure submodule of $M$. So by proposition (19), $N = \bigoplus_{i \in \Lambda} (N_i)$ such that every $N_i$ is a pure submodule of $M_i$ for each $i \in \Lambda$. Since $M_i$ is pure extending, there exists $X_i \leq e \cap M_i$ such that $N_i \leq e \cap X_i$. Therefore $\bigoplus_{i \in \Lambda} (N_i) \leq e \bigoplus_{i \in \Lambda} X_i \leq e \cap M$. Hence $M$ is pure extending module.
We say an $R$-Module $M$ is $RD$-pure extending if every $RD$-pure submodule is essential in a direct summand of $M$.

**Lemma 21** Every $RD$-pure extending module is pure extending.

*Proof* Let $P$ be a pure submodule of $M$. Since every pure submodule is $RD$-pure and $M$ is $RD$-pure extending module. Therefore $P$ is essential in a direct summand of $M$. Hence the module $M$ is pure extending. □

**Example 22** Let $M$ be an $R$ module such that every pure submodule is essential in a direct summand it implies $M$ is pure extending. In [page no.-158][12], there is an example which justify that every $RD$-pure submodule need not be pure submodule. So it may be possible that there exist $RD$-pure submodule of $M$ which are not essential in a direct summand of $M$. Hence $M$ need not be $RD$-pure extending.

**Proposition 23** 1. Direct summand of $RD$-pure extending module is $RD$-pure extending.  
2. $RD$-pure submodule of $RD$-pure extending module is $RD$-pure extending.

*Proof* The proof is similar to proposition 4 and 6. □

**Proposition 24** A flat $R$-module $M$ is pure extending iff $M$ is $RD$-pure extending.

*Proof* The proof follows from lemma 21 and [Corollary 11.21,[8]]. □

**Corollary 25** A free (projective) $R$-module $M$ is pure extending iff $M$ is $RD$-pure extending.

**Corollary 26** A faithful multiplicative $R$-module $M$ is pure extending iff $M$ is $RD$-pure extending.

*Proof* The proof follows by the fact that every faithful multiplicative module is flat. □

### 3 Characterization of rings using Pure extending modules

In the next proposition we characterize the regular ring.
Proposition 27 For a ring $R$, the following conditions are equivalent:
1. $R$ is a regular ring.
2. Every pure extending $R$-module is flat.

Proof (1) $\Rightarrow$ (2) It is clear from [Theorem 4.21,[12]]
(2) $\Rightarrow$ (1) Let $M$ be an right $R$-module and $PE(M)$ be a pure injective hull of $M$. Then $0 \to M \to PE(M) \to PE(M)/M \to 0$ is a pure exact sequence. From hypothesis, $PE(M)$ is a flat module so by [Corollary 4.86,[12]], $PE(M)/M$ is flat. Therefore $M$ is a flat which implies $R$ is regular ring [Corollary 4.21,[12]]. \hfill \square

In the next proposition we characterize the semisimple ring.

Proposition 28 For a ring $R$, the following conditions are equivalent:
1. $R$ is a right semisimple ring.
2. Every pure $C_3$ $R$-module is projective.
3. Every pure $C_2$ $R$-module is projective.
4. Every quasi pure injective $R$-module is projective.
5. Every pure injective $R$-module is projective.
6. Every pure extending $R$-module is projective.

Proof (1) $\Rightarrow$ (2) Let $R$ be a right semisimple ring then every $R$-module $M$ is projective [Proposition 20.7,[20]]. So (2) holds.
(2) $\Rightarrow$ (3) Every pure $C_2$ module is pure $C_3$, so (3) holds.
(3) $\Rightarrow$ (4) Every quasi pure Injective right $R$- module is pure $C_2$, so (4) holds.
(4) $\Rightarrow$ (5) Every pure injective right $R$-module is quasi pure injective, so (5) holds.
(5) $\Rightarrow$ (1) Every pure injective right $R$-module is projective it implies every injective is projective. Therefore (1) holds by [Proposition 20.7,[20]]
(1) $\Leftrightarrow$ (4) It follows from [Proposition 6,[14]].
(1) $\Rightarrow$ (6) Let $R$ be a semisimple ring it implies every $R$-module is injective (in particular pure extending) and projective [Proposition 20.7,[20]], so (6) holds.
(6) $\Rightarrow$ (1) By hypothesis, every pure extending module is projective it implies injective is projective. Hence, $R$ is a right semisimple ring [20, Proposition 20.7]. \hfill \square

Proposition 29 Let $R$ be a local ring then the module $R_R$ is pure extending.

Proof By [Theorem 3,[10]], every local ring is pure simple it implies $R_R$ and 0 are its only pure submodules. Hence, $R_R$ is pure extending module. \hfill \square

A ring $R$ is called left PDS if pure submodule of left $R$-module are direct summand and ring $R$ is said to be a PDS if it is both left and right PDS [9].

Proposition 30 For a PDS ring $R$, every $R$-module is pure extending.
**Proof** Let $M$ be an $R$-module. By hypothesis, $R$ is PDS ring which implies every pure submodule is essential in a direct summand. Hence $M$ be pure extending module.

**Proposition 31** Every flat cotorsion module is pure extending.

**Proof** Let $M$ be a flat and cotorsion module then by [Proposition 3.2,[13]], $M$ is quasi pure injective. Therefore $M$ is a pure extending module.

**Corollary 32** If $R$ is a right cotorsion ring then $R_R$ is pure extending $R$-module.

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