Geodesic Spanners for Points in $\mathbb{R}^3$ amid Axis-parallel Boxes

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Abstract

Let $P$ be a set of $n$ points in $\mathbb{R}^3$ amid a bounded number of obstacles. When obstacles are axis-parallel boxes, we prove that $P$ admits an $8\sqrt{3}$-spanner with $O(n \log^3 n)$ edges with respect to the geodesic distance.

1. Introduction

When designing a network—like a road or a railway network—our main desire is to have a sparse network in which there is a fast connection between any pair of nodes in the network. This leads to the concept of spanners, as defined next.

In an abstract setting, a metric space $\mathcal{M} = (P, d_\mathcal{M})$ is given, where the points of $P$ represent the nodes in the network, and $d_\mathcal{M}$ is a metric on $P$. A $t$-spanner for $\mathcal{M}$, for a given $t > 1$, is an edge-weighted graph $G = (P, E)$ where the weight of each edge $(p, q) \in E$ is equal to $d_\mathcal{M}(p, q)$, and for all pairs $p, q \in P$ we have that $d_G(p, q) \leq t \cdot d_\mathcal{M}(p, q)$, where $d_G(p, q)$ denotes the length of a shortest path (that is, minimum-weight) from $p$ to $q$ in $G$.

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(the distance between $p$ and $q$ in $\mathcal{G}$, for short). Indeed, the distance between any two points in the spanner $\mathcal{G}$ approximates their original distance in the metric space $\mathcal{M}$ up to a factor $t$. The factor $t$ is called the spanning ratio (or dilation, or stretch factor) of $\mathcal{G}$. Now the main question is whether we can construct a sparse graph with the spanning ratio of at most $t$ for the given constant $t$?

**Previous work.** When the metric space $\mathcal{M}$ does not have any additional properties, one can get a $(2^k - 1)$-spanner of size $O(n^{1+1/k})$, for any integer $k > 0$ by the method given in $\mathcal{M}$ and an improvement on its main lemma (Lemma 6 in $\mathcal{M}$) in $\mathcal{M}$. No methods are known to obtain constant spanning ratio with a spanner of size $O(n \text{polylog } n)$ in general metric spaces. However, for several special types of metric spaces better results can be obtained. We next mention some of them.

When $\mathcal{M}$ is the Euclidean metric (i.e. $P$ is a set of $n$ points in $\mathbb{R}^d$ and the Euclidean distance is used), for any fixed $\varepsilon > 0$ one can then obtain a $(1 + \varepsilon)$-spanner with $O(n)$ edges—see the book by Narasimhan and Smid for fundamental results on geometric spanners. This result was generalized to metric spaces of bounded dimension (a metric space $\mathcal{M} = (P, d_{\mathcal{M}})$ has doubling dimension $d$ if any ball of radius $r$ in the space can be covered by $2^d$ balls of radius $r/2$)

Recently, Abam et al. showed that a set $P$ of $n$ points on a polyhedral terrain $\mathcal{T}$ admits a $(2 + \varepsilon)$-spanner with $O(n \log n)$ edges. This improved two recent results that deal with special cases of geodesic spanners on terrains, namely additively weighted spanners and spanners for points in a polygonal domain with some holes.
The problem definition. Let $P$ be a set of $n$ points in $\mathbb{R}^3$ amid a bounded number of disjoint obstacles (even they do not have a common boundary) and each obstacle is an axis-parallel box. Now, consider the metric space $\mathcal{M} = (P, d_M)$ where $d_M(p, q)$ is the geodesic distance of $p$ and $q$, i.e. the length of a shortest path from $p$ to $q$ avoiding obstacles. The goal is to construct for $\mathcal{M}$ a $t$-spanner of size $O(n \text{ polylog } n)$ for some constant $t$. Note that our desire is to have a spanner whose size is independent of the number of obstacles and indeed we will construct a spanner for $\mathcal{M} = (P, d_M)$ defined over $n$ points while we are not allowed to use any Steiner points like the vertices of obstacles.

Our results. When obstacles are convex $\alpha$-fat objects (not necessarily axis-parallel boxes), the geodesic distance of any two points is at most a constant factor (depending on $\alpha$) of their Euclidean distance. Therefore, $\mathcal{M}$ is a metric space of constant doubling dimension and consequently it has $(1 + \varepsilon)$-spanner of size $O(n)$. When obstacles do not have additional properties, it is unknown how to construct a $t$-spanner of size $O(n \text{ polylog } n)$ for some constant $t$. In this paper, we present an $8\sqrt{3}$-spanner of size $O(n \log^3 n)$ when obstacles are axis-parallel boxes.

2. A near linear-size spanner

Suppose $P = \{p_1, \ldots, p_n\}$ is a set of $n$ points in $\mathbb{R}^3$ and there are some axis-parallel boxes as obstacles. For $p, q \in \mathbb{R}^3$ being outside obstacles, let $\sigma(p, q)$ be the geodesic distance of $p$ and $q$ (i.e. the length of a shortest path from $p$ to $q$ avoiding obstacles). Also, let $\mathcal{B}(p, q)$ be an axis-parallel box whose two opposite corners are $p$ and $q$. For the ease of presentation, from
now on, we assume that any point that we use, is outside or on the boundary of obstacles unless it is explicitly mentioned.

Usually $\sigma(p, q)$ is measured in the Euclidean norm (or the $L_2$ norm). Since we will deal with the axis-parallel boxes as obstacles, it is more convenient that $\sigma(p, q)$ is measured in the Manhattan norm (or the $L_1$ norm)—several works have been devoted to computing the shortest rectilinear geodesic path in the presence of axis-parallel boxes. \cite{9,10,11}. Let $L_i(p, q)$ ($i = 1, 2$) denote the $L_i$ distance of two points $p$ and $q$ in the absence of obstacles. Since for any $p, q \in \mathbb{R}^3$ we have $1/\sqrt{3}L_1(p, q) \leq L_2(p, q) \leq L_1(p, q)$, then we can easily get the following observation.

**Observation 1.** If $G$ is a $t$-spanner for the metric $M = (P, d_M)$ where distances are measured in the $L_1$ norm, then $G$ is a $\sqrt{3}t$-spanner for the metric $M = (P, d_M)$ when distances are measured in the $L_2$ norm.

Therefore, from now on, we can assume that every distance is measured in the $L_1$ norm.

We start with the main tool of our spanner construction that we believe is of independent interest.

**Lemma 1.** For any two points $p, q \in \mathbb{R}^3$ and any point $o$ inside (or on the boundary of) $B(p, q)$, we have $\sigma(p, o) + \sigma(o, q) \leq 4 \cdot \sigma(p, q)$.

**Proof.** Let $g$ be a shortest geodesic path from $p$ to $q$. The length of $g$ of course is $\sigma(p, q)$. We claim (and prove later) that there is a point $r$ on $g$ such that $\sigma(o, r) \leq (3/2)\sigma(p, q)$. By the triangle inequality, we know that $\sigma(p, o) \leq \sigma(p, r) + \sigma(r, o)$ and $\sigma(o, q) \leq \sigma(o, r) + \sigma(r, q)$. Summing up these
two inequalities and using the claim, we have

\[ \sigma(p, o) + \sigma(o, q) \leq 4 \cdot \sigma(p, q) \]

Then, it remains to prove the claim. W.l.o.g. assume that \( x(p) \leq x(q), y(p) \leq y(q) \) and \( z(p) \leq z(q) \) where \( x(.), y(.) \) and \( z(.) \) denote the \( x, y \) and \( z \) coordinates, respectively. We know that \( L_1(p, q) = L_1(p, o) + L_1(o, q) \) — we recall that \( L_1(p, q) \) denotes the \( L_1 \) distance of \( p \) and \( q \) in the absence of obstacles and obviously \( L_1(p, q) \leq \sigma(p, q) \). Therefore, one of \( L_1(p, o) \) and \( L_1(o, q) \) is at most \((1/2)L_1(p, q)\). W.l.o.g. assume \( L_1(o, q) \leq (1/2)L_1(p, q) \).

Now consider a moving point \( s \) that initially sits at \( o \) and starts moving in the positive directions of the \( x, y \) and \( z \) axes. The point \( s \) can reach itself to a point \( w \) on one of the edges of \( B(p, q) \) incident to \( q \) as at each position of \( s \) at most one direction of the three directions (positive \( x, y \) and \( z \) directions) may be blocked by an obstacle. W.l.o.g. assume \( w \) is on the edge being parallel to the \( z \)-axis. The length of the path traveled by \( s \) from \( o \) to \( w \) is \( L_1(o, w) \) as \( s \) just moves in the positive directions. Since \( L_1(o, w) \leq L_1(o, q) \leq (1/2)L_1(p, q) \leq (1/2)\sigma(p, q) \), the shortest geodesic path from \( o \) to \( w \) (i.e. \( \sigma(o, w) \)) is at most \((1/2)\sigma(p, q)\).

Now consider two moving points \( s_1 \) and \( s_2 \) initially sitting at \( q \) and \( w \), respectively. Point \( s_1 \) moves on \( g \), the shortest geodesic path from \( q \) to \( p \), and both \( s_1 \) and \( s_2 \) follow the rules described below in the given order:

(i) If \( s_2 \) is free to move in the positive \( z \) direction, \( s_2 \) moves in the positive \( z \) direction and \( s_1 \) stays unmoved.

(ii) If \( s_1 \) moves in the \( z \) direction, \( s_2 \) stays unmoved.
(iii) Otherwise, $s_2$ follow $s_1$ in the $x$ or $y$ directions (i.e. both keep the same $x$ and $y$ coordinates during their motions).

Step (iii) is done when $s_2$ is blocked in the positive $z$ direction, so $s_2$ is free to move in the $x$ and $y$ directions and can follow $s_1$ without any obstacle blocking it. At the initial time $z(q) = z(s_1) \geq z(s_2) = z(w)$. Since $s_2$ moves in the positive $z$ direction, and $s_1$ at some time reaches a point whose $z$-coordinate is $z(w)$ (note that $s_1$ moves on $g$ and finally it must reach $p$ whose coordinate is at most $z(w)$), and $s_1$ and $s_2$ keep their $x$ and $y$ coordinates the same, then at some time $s_1$ and $s_2$ must collide at a point $r$ on $g$.

The length of the path traveled by $s_2$ from $w$ to $r$ is at most the length of $g$ (i.e. $\sigma(p, q)$). The reason is that $s_2$ behaves like $s_1$ in the $x$ and $y$ directions and the $z$ distance traveled by $s_2$ is at most the $z$ distance traveled by any moving point from $p$ to $q$ on $g$ (note that $s_2$ only moves in the positive $z$ direction and starts its journey from $w$ whose $z$ coordinate is at least $z(p)$).

Now consider the paths described above from $o$ to $w$ and then from $w$ to $r$. The lengths of these paths are at most $(1/2)\sigma(p, q)$ and $\sigma(p, q)$, respectively. Therefore, $\sigma(o, r) \leq (3/2)\sigma(p, q)$ as the claim says.

Before we explain our $t$-spanner construction, we introduce our final tool used in our construction, namely the cone-separated pair decomposition. Let $\mathcal{C}$ be the set of 4 cones constructed by the $xy$, $xz$ and $yz$ planes and being above the $xy$-plane. For a cone $\mu \in \mathcal{C}$ and any point $p \in \mathbb{R}^3$, let $\mu(p)$ denote the translated copy of $\mu$ whose apex coincide with $p$. Also let $\bar{\mu}(p)$ be the reflection of $\mu(p)$ about $p$. For a cone $\mu \in \mathcal{C}$ and a set $P$ of $n$ points in $\mathbb{R}^3$, the cone-separated pair decomposition is defined as follows:
Definition 1. A cone-separated pair decomposition, or CSPD for short, for $P$ with respect to $\mu$ is a collection $\Psi_\mu := \{(A_1, B_1), \ldots, (A_m, B_m)\}$ of pairs of subsets from $P$ such that

(i) For every two points $p, q \in P$ with $q \in \mu(p)$, there is a unique pair $(A_i, B_i) \in \Psi_\mu$ such that $p \in A_i$ and $q \in B_i$.

(ii) For any pair $(A_i, B_i) \in \Psi_\mu$ and every two points $p \in A_i$ and $q \in B_i$, we have $q \in \mu(p)$ and, hence, $p \in \bar{\mu}(q)$.

Abam and de Berg showed that a CSPD $\Psi_\mu := \{(A_1, B_1), \ldots, (A_m, B_m)\}$ can be constructed with the property that $\sum_{i=1}^m |A_i| + |B_i| = O(n \log^3 n)$.

Spanner construction. Next we show how to compute a spanner $G = (P, E)$ for the metric space $\mathcal{M} = (P, d_M)$ for the set $P$ of $n$ points in $\mathbb{R}^3$ amid axis-parallel boxes as obstacles.

1. For each of 4 cones $\mu \in \mathcal{C}$, we construct a CSPD $\Psi_\mu$.

2. For each pair $(A, B) \in \Psi_\mu$, let $o$ be a point such that $\mu(o)$ and $\bar{\mu}(o)$ contain all points of $B$ and all points of $A$, respectively. The point $o$ may be inside an obstacle $O$. Let $o_{x+}$ and $o_{x-}$ be the extreme points on $O$ (recall that $O$ is an axis-parallel box) respectively in the positive $x$ axis and the negative $x$ axis such that $y(o) = y(o_{x+}) = y(o_{x-})$ and $z(o) = z(o_{x+}) = z(o_{x-})$. We define $o_{y+}, o_{y-}, o_{z+}$ and $o_{z-}$ in a similar way for the $y$ and $z$ axes. For each point in the set $\{o_{x+}, o_{x-}, o_{y+}, o_{y-}, o_{z+}, o_{z-}\}$, for instance $o_{x+}$, we find a point $p \in A \cup B$ whose geodesic distance to $o_{x+}$ (i.e. $\sigma(p, o_{x+})$) is minimum. For each $q \in A \cup B$, we add the edge $(p, q)$ to our spanner $G$. 

Lemma 2. The construction above gives an 8-spanner with respect to the geodesic distance. Moreover, the spanner has $O(n \log^3 n)$ edges.

Proof. The number of edges we add to the spanner for each pair $(A, B)$ of a CSPD is at most $6(|A| + |B|)$ (6 comes from $|\{o_{x+}, o_{x-}, o_{y+}, o_{y-}, o_{z+}, o_{z-}\}|$). Since $|\mathcal{C}| = 4$ and for each $\mu \in \mathcal{C}$, $\sum_{(A, B) \in \Psi_{\mu}} |A| + |B| = O(n \log^3 n)$, therefore the total number of edges we have in our spanner is $O(n \log^3 n)$.

Let $p, q$ be two arbitrary points in $P$. There is $\mu \in \mathcal{C}$ and a pair $(A, B) \in \Psi_{\mu}$ such that $p \in A$ and $q \in B$. Consider the point $o$ and the obstacle $O$ and the set $\{o_{x+}, o_{x-}, o_{y+}, o_{y-}, o_{z+}, o_{z-}\}$ as defined in the construction.

We first prove that at least one point in the set $\{o_{x+}, o_{x-}, o_{y+}, o_{y-}, o_{z+}, o_{z-}\}$ is inside $B(p, q)$. W.l.o.g. assume $x(p) \leq x(q)$, $y(p) \leq y(q)$ and $z(p) \leq z(q)$. It is obvious that $o$ is inside $B(p, q)$ (notice that $q \in \mu(o)$ and $p \in \mu(o)$). If $o_{x+}, o_{x-} \notin B(p, q)$, then $x(o_{x+}) > x(q)$ and $x(o_{x-}) < x(p)$. Similar inequalities hold for the $y$ and $z$ axes if $o_{y+}, o_{y-}, o_{z+}, o_{z-} \notin B(p, q)$. All these together imply both $p$ and $q$ must be inside $O$ which is a contradiction.

W.l.o.g. assume $o_{x+} \in B(p, q)$. Let $r \in A \cup B$ be the point such that $\sigma(r, o_{x+})$ is minimum among all points in $A \cup B$. We know that edges $(p, r)$ and $(q, r)$ exist in our spanner. Therefore,

$$d_G(p, q) \leq \sigma(p, r) + \sigma(r, q)$$

$$\leq (\sigma(p, o_{x+}) + \sigma(o_{x+}, r)) + (\sigma(r, o_{x+}) + \sigma(o_{x+}, q))$$

$$= (\sigma(p, o_{x+}) + \sigma(o_{x+}, q)) + 2\sigma(o_{x+}, r)$$

$$\leq 2(\sigma(p, o_{x+}) + \sigma(o_{x+}, q))$$

$$\leq 8 \cdot \sigma(p, q)$$

The last inequality is obtained from Lemma 1.
Remark. We focused on proving the existence of the spanner and the construction time of the spanner was not our desire. But it is easy to see the spanner can be computed in a polynomial time based on $n$ and $m$ where $m$ is the number of obstacles—see\ref{2} and\ref{2} for how to compute CSPDs and the shortest $L_1$ geodesic paths in a polynomial time, respectively.

Putting all this together, we get our main theorem.

**Theorem 1.** Suppose $P$ is a set of $n$ point in $\mathbb{R}^3$ amid a bounded number of obstacles, and obstacles are axis-parallel boxes. $P$ admits an $8\sqrt{3}$-spanner with $O(n \log^3 n)$ edges with respect to the geodesic distance where distances are measured in the $L_2$ norm.

3. Conclusion

We have shown that any set of $n$ points in $\mathbb{R}^3$ amid axis-parallel boxes as obstacles admits a geodesic spanner of spanning ratio $8\sqrt{3}$ and with $O(n \log^3 n)$ edges. This is the first geodesic spanner for points in $\mathbb{R}^3$ amid obstacles. We leave designing a spanner with fewer edges and smaller spanning ratio as an open problem for future research.

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