Exquisitely accurate energies for the general quartic oscillator

Pavel Okun, Kieron Burke

Department of Chemistry, University of California, Irvine, CA 92697, USA and
Departments of Physics and Astronomy and of Chemistry

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Recent advances in the asymptotic analysis of energy levels of potentials produce relative errors in eigenvalue sums of order $10^{-33}$, but few non-trivial potentials have been solved numerically to such accuracy. We solve the general quartic potential (arbitrary linear combination of $x^4$ and $x^2$) beyond this level of accuracy using a basis of several hundred oscillator states. We list the lowest 20 eigenvalues for 9 such potentials. We confirm the known asymptotic expansion for the levels of the pure quartic oscillator, and extract the next 2 terms in the asymptotic expansion. We give analytic formulas for expansion in up to 3 even basis states. We confirm the virial theorem for the various energy components to similar accuracy. The sextic oscillator levels are also given. These benchmark results should be useful for extreme tests of approximations in several areas of chemical physics and beyond.

1. INTRODUCTION

Since the early days of quantum mechanics, potentials with analytic solutions have played a crucial role in providing both insight into more complex problems, and benchmarks for more general quantum solution methods [1, 2]. The quartic oscillator is iconic in being a simple potential without a built-in length scale which does not have a simple analytic solution [3–7]. The general quartic oscillator (adding both quadratic and linear terms) is not scale-invariant, and has been studied in many different contexts in physics [8–10]. In particular, the Mexican hat shape of symmetric double wells is a paradigm of simple symmetry breaking [8, 11].

In chemical physics, the double well provides important tests of theories of tunneling in quantum nuclear dynamics of liquids [8, 11, 12]. In particle physics, it is a prototype of symmetry breaking, such as occurs in simple field theories [13, 14]. In mathematical physics, it is a simple case to test and explore asymptotic approximations [15]. Asymptotic analysis, especially hyperasymptotics, can yield exquisitely accurate approximations [16–19]. In the past, many developments and tests of these methods have been applied to scale invariant potentials [20–22], but the general quartic oscillator provides opportunities to look at more complex cases.

Recent work on one-dimensional potentials [20–22] has established a deep explicit connection between the gradient expansion of density functional theory and asymptotic expansions in powers of $\hbar$ [23]. In one case fractional errors were below the picoyocto range, i.e., of order $10^{-33}$ [22]. To further develop and test methods in this area, there is a need for benchmark calculations of this level of accuracy for non-trivial potentials. This exceeds even quadruple precision on standard computers, rendering standard numerical algorithms, even pushed to their convergence limits, difficult to apply. There is also a new area of application: The breaking of symmetry is a simple prototype of a bond breaking, in which electrons localize in two separate wells [24]. Such bond breaking is very difficult to model with standard semilocal density functionals, and their failure has been traced back to the change in asymptotic expansions in going from one well to two [21].

FIG. 1. Ground-state densities and potentials. The potentials are shown with dashed lines and the densities with solid lines. Legend: $\lambda = 0$ (blue), $\lambda = \lambda_c$ (magenta), $\lambda = 4$ (orange).

In order to generate such benchmarks and as a simple example, we consider the generalized quartic oscillator potential:

$$v_\lambda(x) = \frac{x^4}{4} - \lambda \frac{x^2}{2},$$

where $\lambda$ is a real number, either positive or negative. For $\lambda = 0$, this is a pure quartic oscillator, which has been the subject of many investigations. In this paper we will present the pure quartic oscillator energies for more states and to more digits than previously computed in Refs. [3–5]. We also numerically examine the WKB series for the quartic oscillator closely following Bender & Orszag’s book [3]. Previous investigations of the WKB approximation of the quartic oscillator can be found in Refs. [9, 25]. We examine the variation of the energy with $\lambda$ and the effect of a linear term as in Ref. [9]. Our exact energies can be used as inputs to test the semiclassical analysis of Ref. [26]. The exact solution of the quartic oscillator was studied in Refs. [6, 7]. For $\lambda < 0$, the minimum is always at $x = 0$, with vibrational frequency $\sqrt{|\lambda|}$. For $\lambda > 0$, the most interesting case, two distinct wells appear, with minima at $\pm \sqrt{\lambda}$, and
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2. METHOD

Our Schrödinger equation is (in units where \( \hbar = m = 1 \))

\[
-\frac{1}{2} \frac{d^2 \psi}{dx^2} + v(x)\psi(x) = \epsilon \psi(x),
\]

so all energies are in Hartrees, all distances in Bohr radii. We expand the eigenfunctions in a basis of harmonic oscillator states, where \( \omega \) can be freely chosen. The Hamiltonian is pentadiagonal, with only a few non-zero matrix elements no more than 2 double-steps off the diagonal. The nonzero matrix elements of the Hamiltonian in the harmonic basis are \( \mathcal{H}_{n,n+2k} = \hbar \kappa \sqrt{\text{nk}} / 16 \omega^2 \) where \( h_2 = 1 \) and

\[
\begin{align*}
h_0 &= 4\omega(\omega^2 - \lambda)(2n + 1) + 3(2n^2 + 2n + 1), \\
h_1 &= 2[2n + 3 - 2\omega(\lambda + \omega^2)],
\end{align*}
\]

and we use the shorthand

\[
\alpha_p = \prod_{m=1}^{p} (\alpha + m), \quad \alpha_0 = 1.
\]

We closely follow Ref. 5 and use the Eigensystem function in Mathematica to diagonalize this matrix for various values of \( \lambda \) and choices of \( \omega \) [27]. We denote by \( N_B \) the number of basis functions included in the calculation (both odd and even, since we did not take advantage of parity). Our default choice of \( [\omega/N_B] \) is \( [2/200] \) but we use \( [2/400] \) as a baseline for ‘exact’ energies, and report errors relative to those values.

A special case is \( \epsilon = 0 \) for the ground state (magenta in Fig. 1). This happens at \( \lambda = \lambda_c \) which we found using a golden section search to be \( 1.3982585455298953502585947187218312604396, \) at which the ground state energy is \(-3.955 \times 10^{-41}\).

3. RESULTS

In this section, we report many different results that may be of interest to different communities under different circumstances. In each case, we also provide a minimal analysis.

3.1. Energetics for different potentials

Here, we simply survey the behavior of the energies and eigenfunctions for various values of \( \lambda \). Our focus is primarily on positive values of \( \lambda \), which produce the Mexican hat double-well potential.

| \( \lambda \) | \( \lambda = -1 \) | \( \lambda = 0 \) | \( \lambda = 2 \) | \( \lambda = 4 \) |
|---|---|---|---|---|
| 0 | 0.62092703 | 0.42080497 | -0.29952137 | -2.66144807 |
| 1 | 2.02596616 | 1.50790124 | 0.04637108 | -2.65173172 |
| 2 | 3.69845032 | 2.95879483 | 1.22797281 | -0.51029304 |
| 3 | 5.55757714 | 4.62122032 | 2.45984143 | -0.18079943 |
| 4 | 7.18323497 | 6.45350993 | 3.93826197 | 1.16951434 |
| 5 | 9.03984811 | 8.42453888 | 5.58129195 | 2.36490189 |
| 6 | 11.96454362 | 10.52783077 | 7.36888889 | 3.83579483 |
| 7 | 14.32326520 | 12.7383694 | 9.28322263 | 5.4430452 |
| 8 | 16.77645279 | 15.04975293 | 11.31134968 | 7.18323497 |
| 9 | 19.31695430 | 17.45393416 | 13.44312537 | 9.03984811 |

TABLE I. The energies at various values of \( \lambda \). See Table S1 for more values of \( \lambda \), more states, and more digits.

Our first results are the energetics of the first several eigenstates of the generalized quartic oscillator. These values are given to 8 digits in Table I for four values of \( \lambda \). In Table S1 in the supplementary information, we give 40 digits for 9 values of \( \lambda \) for the first 20 eigenvalues. Here \( \lambda = 0 \) corresponds to the pure quartic oscillator. As \( \lambda \) grows, the eigenvalues inside the double well come in pairs, with ever smaller splitting.

FIG. 2. First three eigenfunctions (orange, red, magenta) with potentials (blue) at various values of \( \lambda \).

We also show the first three stationary states and potentials at various values of \( \lambda \) in Fig. 2. As \( \lambda \) grows, the
ground-state wavefunction develops a minimum at the origin, and the first excited state almost matches it in the bulk of the minimum. By \( \lambda = 8 \), the wavefunctions are almost indistinguishable, except for their sign.

![Graph showing behavior of coefficients](image)

In Fig. 3 we show the overlap \( c_{m} \) of the ground-state wavefunction for the pure quartic oscillator (blue) and double-well potential (red, \( \lambda = 8 \)) in the basis \([2/200]\). In the lower panel stars and open circles denote \( c_{2m} \) of opposite signs. See Table S2 for more digits.

In this section, we examine both the zero point energy and the tunneling between the symmetric wells that occur for positive \( \lambda \). As mentioned before, the vibrational frequency is \( \sqrt{|\lambda|} \) for negative \( \lambda \), and \( \sqrt{2|\lambda|} \) for positive \( \lambda \). Fig. 5 shows the exact zero-point energy and it’s harmonic approximation, which becomes accurate as \( |\lambda| \) grows.

3.2. Satisfaction of virial theorem

The virial theorem \([28]\) is a useful check on the accuracy of eigenstates in a basis. It is particularly simple here, as the potential is a sum of two powers of \( x \). For \( v_{\lambda}(x) \), the virial theorem requires, for any eigensolution

\[
\left\langle p^{2} \right\rangle + \lambda \left\langle x^{2} \right\rangle = \left\langle x^{4} \right\rangle,
\]

with nonzero matrix elements

\[
\bar{x}_{0}^{2} = \bar{p}_{0}^{2} = 2n + 1, \quad \bar{x}_{1}^{2} = -\bar{p}_{1}^{2} = \sqrt{n_{2}},
\]

\[
\bar{x}_{k}^{2} = 3(2n^{2} + 2n + 1), \quad \bar{x}_{k}^{4} = 2\sqrt{n_{2}}(2n + 3), \quad \bar{x}_{k}^{8} = \sqrt{n_{4}},
\]

where \( \bar{x}_{k} = x_{n,n+2k}\sqrt{2/\omega} \) and \( \bar{p}_{k} = p_{n,n+2k}\sqrt{2/\omega} \). In particular, at \( \lambda_{c} \), we find the simple formula:

\[
\frac{\left\langle p^{2} \right\rangle}{\left\langle x^{2} \right\rangle} = \frac{\lambda_{c}}{3}.
\]

In Table II we show how closely our solutions satisfy Eq. 5. This confirms that with \([2/200]\) we have a very good approximation to the exact ground states. Eq. 7 is satisfied to 39 decimal places.

![Graph showing error in ground state density](image)

**TABLE II.** Expectation values and their virial sum for different wells with \([2/200]\). See Table S3 for more digits.

3.3. Tunneling between wells

In this section, we examine both the zero point energy and the tunneling between the symmetric wells that occur for positive \( \lambda \). As mentioned before, the vibrational frequency is \( \sqrt{|\lambda|} \) for negative \( \lambda \), and \( \sqrt{2|\lambda|} \) for positive \( \lambda \). Fig. 5 shows the exact zero-point energy and it’s harmonic approximation, which becomes accurate as \( |\lambda| \) grows.
3.3 **Tunneling between wells**

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0.434910378705254929874279292555363392774
1.6483110636571093650578372048997389227058
2.2347092671416130103185131119292879955987
3.567413742993262212079377944120214255830389
4.8245988596347452041423953299512400416730
5.11.1345828009733275940909285184369395669
6.14.289882708235237836486899279259398944806
7.17.02352219545620790967806369145795203975459
8.21.3511174819949061642485892730226003053565
9.25.2217126857306792812484143857544098122878
10.29.30205391825152643146857417753838413530
11.33.5818407424144144759439497544089122878
12.38.65216382472115780049203687472674910548
13.42.7502160161321923679887360117322196371241762
14.47.5340594514494442611381955694530788687525146
15.52.53252145969659249911539322573825024702200
16.57.69501952928699138098650728087269497456
17.63.0164936093670075261990621857703983520713
18.68.49235342797181295476164180074890075654
19.74.11827728288342368118468014333735452468298

**TABLE III.** First twenty energies of the sextic oscillator calculated with $[2/800]$. The energies are accurate to all 41 digits shown.

\[ \sqrt{2(\epsilon - v_\lambda(x))} \] over the classically forbidden region between the two wells

\[
\phi = 2 \int_0^{x_1} dx \sqrt{2[v_\lambda(x) + \lambda^2/4 - \omega_0/2]}, \tag{10}
\]

where \(x_1 = \sqrt{\lambda - 2^{3/4}\lambda^{1/4}}\) is the inner turning point and \(-\lambda^2/4 + \omega_0/2\) is the harmonic approximation to the ground state energy. For the approximation to be meaningful, the inner turning point must be positive, so that \(\lambda > 2\). The appendix shows how to find the asymptotic behavior of the splitting for large \(\lambda\):

\[
\Delta \epsilon = \frac{2^{11/4}}{\pi} \sqrt{\gamma} \lambda^{5/4} \exp \left( -\frac{2\lambda^{3/2}}{3} \right). \tag{11}
\]

Fig. 6 shows just how accurate this approximation is. We have confirmed this expansion numerically.

3.4 **Sextic oscillator**

In this section, we apply exactly the same technology to finding the energies of the sextic oscillator:

\[ v(x) = \frac{x^6}{6}. \tag{12} \]

The quartic and sextic oscillators both belong to the class of potentials whose exact solutions are given by Heun’s special function [29]. In our harmonic basis, the nonzero Hamiltonian matrix elements are \( \hat{H}_{n,n+2k} = \sqrt{n} k h_{k}^{(6)} / 48 \omega^4 \) where

\[
\epsilon_{\pm} = \frac{\omega_0}{2} \pm \frac{\omega_0}{2} e^{-\phi}, \tag{8}
\]

for the lowest two levels, where \(\omega_0\) is the vibrational frequency, and \(\phi\) is the decay rate for tunneling, evaluated on the ground-state energy. The splitting is

\[
\Delta \epsilon = \frac{\omega_0}{\pi} e^{-\phi(\lambda)}, \tag{9}
\]

and \(\omega_0 = \sqrt{2\lambda}\) in the harmonic approximation. Here \(\phi\) is the integral of the absolute value of the momentum \(p(x) = \frac{d\epsilon}{dx}\).
\[ h_3^{(6)} = 1 \text{ and } h_0^{(6)} = (2n + 1)[10n(n + 1) + 3 \left( 4\omega^4 + 5 \right)], \]
\[ h_1^{(6)} = 3[5n(n + 3) - 4\omega^4 + 15], \]
\[ h_2^{(6)} = 3(2n + 5), \]

i.e., they go one more step away from the diagonal. The energies of the first twenty sextic oscillator states are given in Table III.

### 3.5. Analytic results for a few states

![Graph](image-url)

**FIG. 7.** The errors of the analytic expressions for the approximate ground state with 1, 2, and 3 basis functions and \( \lambda = \lambda_c. \)

It can often be useful to find an approximate solution using just a few basis functions, instead of hundreds. Here we give analytic formulas for the lowest lying even energies as functions of \( \omega \) and \( \lambda \) when only 1, 2, and 3 even oscillator states are used. These expressions can be useful for quick estimates of low-lying eigenvalues. The approximate ground-state energy with one even basis function is

\[ \epsilon_0 = \frac{3}{16\omega^2} + \frac{\omega}{4} - \frac{\lambda}{4\omega}, \quad (N_B = 1). \]  

(14)

The approximate ground- and second-excited states with two even basis functions are:

\[ \epsilon_{\pm} = \frac{3(\omega^2 - \lambda)}{4\omega} + \frac{21 \pm 2\sqrt{D}}{16\omega^2}, \quad (N_B = 3), \]

\[ D = 8\omega[3\omega(\lambda^2 + \omega^4 + 2\omega) - 2\lambda(\omega^3 + 6)] + 99. \]  

(15)

With three even basis functions the first three approximate even state energies are \( (n = 0, 2, 4): \)

\[ \epsilon_n = \frac{1}{48\omega^2} \left[ 15(11 - 4\lambda\omega + 4\omega^3) - (-1)^n \cdot 28\sqrt{6D} \cos \left( \frac{\phi}{3} + \frac{(n + 1)\pi}{6} \right) \right], \quad (N_B = 5), \]

\[ D = 15[\omega^2 (\lambda^2 + \omega^4 + \omega) - 7\lambda\omega + 13] - 2\lambda\omega^4, \]

\[ \sin \phi = \frac{9B}{8\sqrt{6DD}}, \]

\[ B = 20\lambda\omega \left( \lambda \omega [51 - 4\omega(\lambda + \omega^2)] + 2(2\omega^6 + 7(\omega^3 - 15)) \right) \]

\[ + 4\omega^6 (20\omega^3 - 57) + 5575. \]  

(16)

At \( \lambda_c \) (Fig. 7), the least error in the ground state energy is \( 5.467 \times 10^{-2} \) at \( \omega = 0.7595 \) with Eq. 14, \( 4.320 \times 10^{-3} \) at \( \omega = 1.383 \) with Eq. 15, and \( 4.563 \times 10^{-4} \) at \( \omega = 1.854 \) with Eq. 16.

### 3.6. Error dependence on \( \omega \)

![Graph](image-url)

**FIG. 8.** The errors of the ground state and 19th excited state (i.e. 10th odd state) as a function of \( \omega \). See Table S5 for more digits.

In this paper we have usually set the basis set angular frequency \( \omega \) to 2. Now we analyze what happens to the
error of the ground and a highly excited state of the pure quartic oscillator as \( \omega \) is varied. The error as a function of \( \omega \) for a fixed number of basis states is complicated and has several local minima, as we found in the previous section. Nevertheless there is a clear trend for the pure quartic oscillator as seen in Fig. 8: the error tends to level off to a very low value as \( \omega \) increases, though it must increase if \( \omega \) becomes too large. The error for the 19th excited state is very low value as \( \omega \) oscillator as seen in Fig. 8: the error tends to level off to a

Nevertheless there is a clear trend for the pure quartic os-
several local minima, as we found in the previous section.

3.7. Quartic potential as perturbation

Consider the case where \( \lambda \) is large and negative, and treat the quartic potential as a perturbation. The zeroth, first, and second order contributions to the energies are

\[
\begin{align*}
\epsilon_n^{(0)} &= \left( n + \frac{1}{2} \right) \sqrt{\lambda}, \\
\epsilon_n^{(1)} &= \frac{3(2n^2 + 2n + 1)}{16|\lambda|}, \\
\epsilon_n^{(2)} &= -\frac{(1 + 2n)|17n(n + 1) + 21|}{128|\lambda|^{5/2}}.
\end{align*}
\]

Fig. 9 shows the resulting error in the ground state energy.

3.8. Asymmetric wells

We now examine the effect of breaking the symmetry of \( v_\lambda(x) \) by adding a linear term

\[
v_{\lambda,\alpha}(x) = x^4 - \lambda x^2 + \alpha x.
\]

We only examine the case \( \lambda = 4 \). In Table IV, we show both the energies for the case \( \alpha = 0.1 \) and their difference from the unperturbed case \( \alpha = 0 \). As one side of the well is depressed and the other elevated, for the low-lying states, the differences alternate in sign. As one goes further up the well, eventually all states are lower than their symmetric counterparts.

In Fig. 10 we show how the ground state density varies as \( \alpha \) is increased. Even a very small value of \( \alpha \) causes substantial asymmetry in the ground-state density, with almost all the weight in the lower well when \( \alpha = 0.1 \).

3.9. Asymptotic analysis of pure quartic

The asymptotic solution of the pure and generalized quartic oscillator has been studied many times before [3, 9, 10, 25]. We analyze only the pure quartic oscillator and closely follow Bender & Orszag [3]. The WKB series for a pure quartic oscillator with potential \( v(x) = x^4/4 \) yields the implicit formula

\[
\sum_{m=0}^{\infty} A_{2m}(4e^{3/2})^{1/2-m} = \left( n + \frac{1}{2} \right) \pi,
\]

TABLE IV. The first twenty energies when \( \lambda = 4 \) and \( \alpha = 0.1 \) calculated with \([3/200]\). The difference from \( \alpha = 0 \) is reported. All energies are accurate to the 41 digits given.

| \( n \) | \( \text{Energy} \) | \( \text{difference} \) |
|---|---|---|
| 0 | -2.841788149705852588888844101277927051 | -0.1900336276 |
| 1 | -2.4715425318023322653808989492564422611375 | 0.1785774606 |
| 2 | -0.55873119537087765308220775703245718616 | -0.0413089972 |
| 3 | -0.138474055714196891853704272418272968978 | 0.042053773 |
| 4 | 1.64170069238701178393006368992169773 | -0.005440297 |
| 5 | 2.36735322904390725934415582236798864 | 0.001343161 |
| 6 | 3.835681404437941741468079899019165511556 | -0.0001317882 |
| 7 | 5.44302758496120716436759670624192068853 | 0.000227263 |
| 8 | 7.183200460005102453964222654704137331752 | -0.000345135 |
| 9 | 9.039793505571445602108810644675535316811 | -0.000546026 |
| 10 | 11.0024485729203955435367821153968384860653 | -0.000706268 |
| 11 | 13.06271604872508975571968767010426920279753 | -0.000980253 |
| 12 | 15.21369451947288710175932607522153655338 | -0.000983813 |
| 13 | 17.459854772748716577855945106099214542188 | -0.000985162 |
| 14 | 19.76542796690516123491940192713302188667 | -0.000965616 |
| 15 | 22.156922727743638689214205215068733955026 | -0.000987886 |
| 16 | 24.6202945181054068163027588935212543080114 | -0.000101585 |
| 17 | 27.15219883634313049678476189026302920896 | -0.000108965 |
| 18 | 29.749647511110868811782340924631425560723 | -0.000101638 |
| 19 | 32.40985266600747257538662259578163091406390 | -0.000107975 |

FIG. 9. The error in the ground state energy from zeroth (blue) and first order (red) perturbation theory. See Table S6 for accurate numbers.

FIG. 10. The ground state density with \( \lambda = 4 \) and various values of \( \alpha \): (blue), 0.01 (magenta), 0.005 (orange), 0.01 (cyan).
with the known $A_{2n}$ reported in Table V and in Ref. [3]. One can invert this implicit expression to an explicit formula for each level:

$$
\epsilon_n = 2^{-1/3} \sum_{m=0}^{\infty} B_{2m} \left( n + \frac{1}{2} \right)^{4/3-2m}.
$$  

We give the known $A_{2n}$ and $B_{2n}$ coefficients numerically in Table VI to twenty decimal places. The analytic forms of the $B_{2n}$ coefficients are given by

$$
B_{2n} = (-1)^{[n/2]} \frac{\pi^{2-n} \beta_n}{18^{1/3} \Gamma(1/4)^{8/3}},
$$

where the $\beta_n$ are polynomials of order $[n/2]$ in $\gamma$:

$$
\beta_n = C_n \sum_{k=0}^{[n/2]} a_k \gamma^k,
$$

where $\gamma = \Gamma(1/4)^{8/4}$. This allows the 7 known $\beta_{2n}$ to be given by the constants in Table VII.

We can use our highly accurate energies to extract higher order coefficients. We define the deviation from the $2m$-th order WKB approximation as

$$
\Delta \epsilon^{(2m)}(n) = \epsilon_n - \epsilon^{(2m-2)}_{WKB,n},
$$

which, according to Eq. 20, has the asymptotic form

$$
\Delta \epsilon^{(2m)}(n) = B_{2m}X_n^{2/3-m} + B_{2m+2}X_n^{2/3-2m-1} + ..., \quad (24)
$$

where

$$
X_n = \left( n + \frac{1}{2} \right)^{-2},
$$

yielding

$$
\Delta \epsilon^{(2m)}(n) = B_{2m} + B_{2m+2}X_n + B_{2m+4}X_n^2 + ... \quad (26)
$$

Thus by calculating accurate energies, multiplying them by $X_n^{-2/3+m}$, and fitting to a parabola, we confirm the WKB coefficients up to twelfth order and find the next two coefficients numerically, as shown in Fig. 11. Our most accurate approximations to $B_{14}$ and $B_{16}$ were calculated using $[3/1400]$ to be $B_{14} = -0.34434940$ and $B_{16} = 3.5192337$, which are accurate to the number of digits shown.

4. CONCLUSIONS

We have used Blinder's method to extract many quantities from the general quartic oscillator to many digits [5]. We have considered many distinct limits and scenarios where these benchmark results might be useful. We have covered energetics of eigenstates, the virial theorem, tunneling between wells, the sextic oscillator, analytic forms in a few basis functions, error dependence on choice of $\omega$, perturbation theory in the quadratic term, asymmetric wells, and asymptotic analysis of WKB results for the pure quartic case. In all cases, we have provided preliminary analysis and compared with the exact results. We expect future studies in specific areas to perform more thorough analysis in the cases considered here, and in others we have not anticipated.

5. DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.
6. ACKNOWLEDGMENTS

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Appendix A: Derivation of asymptotic splitting formula

We now explain how to derive Eq. 11, the asymptotic approximation to \( \Delta \epsilon = \epsilon_1 - \epsilon_0 \) in the limit \( \lambda \to \infty \).

We introduce the shorthand \( \eta = (2/\lambda)^{3/4} \) so \( \lambda \to \infty \Rightarrow \eta \to 0_+ \). In terms of \( \eta \) Eq. 10 of the main text becomes

\[
\phi(\eta) = \frac{4}{\eta^2} \int_0^{\sqrt{1-\eta}} dx \sqrt{(1-x^2)^2 - \eta^2}.
\]  

(A1)

In the limit \( \eta \to 0_+ \),

\[
\phi^{(0)}(\eta) = \frac{4}{\eta^2} \int_0^1 dx (1 - x^2) = \frac{(2\lambda)^{3/2}}{3}.
\]  

(A2)

We evaluate Eq. A1:

\[
\phi(\eta) = \frac{8}{3\eta^2} \sqrt{1 + \eta F(\eta)},
\]  

(A3)

where \( F(\eta) = E(y) - \eta K(y), \ y = (1-\eta)/(1+\eta) \) and

\[
K(x) = \int_0^{\pi/2} \frac{d\theta}{f(x,\theta)}, \ E(x) = \int_0^{\pi/2} d\theta f(x,\theta),
\]  

(A4)

with \( f(x,\theta) = \sqrt{1-x \sin^2 \theta} \) [30]. The following expansion will prove useful shortly:

\[
F(\eta) = 1 - \frac{\eta}{2} + \frac{3}{16} \eta^2 (1 - 6\ln 2 + 2\ln \eta) + O(\eta^3),
\]  

(A5)

as \( \eta \to 0_+ \) [31]. Inserting Eq. A5 into Eq. A3 and expanding around \( \eta = 0 \), yields

\[
\phi^{(2)}(\lambda) = \frac{(2\lambda)^{3/2}}{3} - \frac{3}{4} \ln \lambda - \frac{1}{4} (2 + 9 \ln 2).
\]  

(A6)

The above equation combined with Eq. 9 leads to the final result, Eq. 11 of the main text.
Appendix B: Supplemental Info
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| $\alpha$ | $\omega$ | $\lambda$ |
|---------|---------|---------|
| 0.6929/0.3292/0.8297 | 0.9680/0.8635/1.2820 | 0.9882/0.9882/1.1420 |
| 0.2692/0.4516/0.6972 | 0.3297/0.5162/0.8320 | 0.2384/0.3682/0.5082 |
| 3.6845/0.4310/0.6830 | 0.3247/0.5106/0.7367 | 2.1052/0.3435/0.4868 |
| 5.5357/0.7185/0.6090 | 0.4335/0.5180/0.6627 | 5.0535/0.7362/0.6930 |
| 7.5684/0.2352/0.5999 | 0.2435/0.3352/0.4630 | 5.0535/0.7362/0.6930 |
| 9.7357/0.4787/0.6145 | 0.3455/0.4798/0.5479 | 7.0535/0.8721/0.7967 |
| 11.9465/0.4626/0.7036 | 0.4375/0.5262/0.6756 | 15.5734/0.7675/0.8382 |
| 14.3235/0.2915/0.7016 | 0.2405/0.3735/0.4947 | 4.5734/0.4516/0.5782 |
| 17.7562/0.4727/0.7749 | 0.4316/0.5284/0.6936 | 15.1475/0.8721/0.7967 |
| 20.1965/0.5086/0.8316 | 0.5376/0.6342/0.7829 | 10.2936/0.5245/0.6824 |
| 22.8603/0.5296/0.8553 | 0.5325/0.6273/0.7782 | 7.5295/0.4925/0.6662 |
| 25.1615/0.5176/0.7763 | 0.6324/0.7155/0.8315 | 10.9326/0.4525/0.6824 |
| 27.4073/0.5321/0.7967 | 0.7324/0.8035/0.8724 | 11.9214/0.4927/0.6824 |
| 29.6431/0.5432/0.8165 | 0.8323/0.8914/0.9123 | 14.1572/0.5872/0.6824 |

**TABLE S1.** Exact energies given to 40 digits.
TABLE S2. The expansion coefficients of the ground state for several values of \( \lambda \) calculated with \( [2/200] \). The normalization error is defined as \( \text{Norm. Error} = 1 - \sum_{n=0}^{\infty} c_n^2 \). All entries are accurate for all digits shown.

TABLE S3. Testing the virial theorem on the ground state in the basis [2/200] for various \( \lambda \).
TABLE S4. Zero point energy (\(\epsilon_0\)) for the general quartic oscillator.

| \(\lambda\) | \(\epsilon_0\) | \(\Delta\) |
|---|---|---|
| 0.1 | 0.9093118844888522.0226103899099989.0242 |
| 0.2 | 0.1905028260.2268484721397359091110.1224 |
| 0.3 | 0.143147487.61554950119094991235.6250 |
| 0.4 | 0.536786667297313998169216052434509.100 |
| 0.5 | 1.29671129031102422108960865480212.111 |
| 0.6 | 0.5721199106171318941548475425511060561.010 |
| 0.7 | 0.59688938848168725490349959990670.177 |
| 0.8 | 0.1144316147331961327512113359084762.100 |
| 0.9 | 0.107602797175747750864723755487369.074 |
| 1.0 | 0.133659492524652848671972190203225.228 |
| 1.1 | 0.0131770576757440238022893425103686.198 |
| 1.2 | 0.058285277456939903829662389941533.162 |
| 1.3 | 0.0041338039583659099186326238948060.100 |
| 1.4 | 0.0057453248479381490323764690325877.000 |
| 1.5 | 0.0051592675245651920766529340913246.060 |
| 1.6 | 0.0048149312403995026550978462513465.000 |
| 1.7 | 0.0040363548125997498157209380333940.000 |
| 1.8 | 0.0033614643441992633920538693131389.000 |
| 1.9 | 0.0027469768375513763718873293535135.000 |
| 2.0 | 0.0022006869272693228303855014401539.000 |

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The calculations made with \(\lambda = 2/200\).

All calculations were made with \(\epsilon_0 = \epsilon - \epsilon_0\).
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| $\omega$ | $N_B = 40$ | \(\Delta E_\omega \times 10^4\) | $N_B = 50$ | \(\Delta E_\omega \times 10^4\) | $N_B = 60$ | \(\Delta E_\omega \times 10^4\) | $N_B = 40$ | \(\Delta E_\omega \times 10^2\) | $N_B = 50$ | \(\Delta E_\omega \times 10^2\) | $N_B = 60$ | \(\Delta E_\omega \times 10^2\) |
|-------|-----------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-----------------|
| 1.0   | 2.8316898104 | 7.6749053454 | 1.1671394475 | 9.744457267 | 8.352481902 | 1.2928179048 |
| 1.1   | 1.4303248620 | 6.6348048355 | 1.6786069551 | 4.586104072 | 5.9258296585 | 4.6630523064 |
| 1.2   | 4.1469925551 | 9.1180293837 | 3.6208736723 | 1.5887415607 | 1.9298196092 | 1.660275198 |
| 1.3   | 2.8742739086 | 5.2200569899 | 1.1126424115 | 2.939256233 | 7.620793910 | 7.342354374 |
| 1.4   | 4.2356801829 | 2.1693125997 | 1.9273476827 | 3.3821922647 | 3.7361785356 | 4.2502444003 |
| 1.5   | 3.4039712760 | 7.5543012000 | 1.0429317851 | 3.6273522076 | 2.180691116 | 9.177443632 |
| 1.6   | 2.8212055885 | 7.6197300492 | 1.7298724883 | 1.457859253 | 5.406856329 | 5.1285380535 |
| 1.7   | 1.2974200122 | 1.3167755851 | 1.3929202224 | 1.3615884048 | 2.689092717 | 1.9853288804 |
| 1.8   | 1.1092066597 | 3.340636315 | 2.5156704460 | 8.337723720 | 3.140201708 | 5.0423268707 |
| 1.9   | 3.1199728559 | 2.6229394160 | 2.647725394 | 5.726547288 | 3.953895868 | 1.654595686 |
| 2.0   | 5.2175389623 | 1.763399014 | 5.7037160055 | 7.4920148760 | 6.875822484 | 1.746230443 |
| 2.1   | 4.1048712013 | 6.5059904005 | 7.137451548 | 2.854736030 | 5.625999631 | 6.034684022 |
| 2.2   | 3.4600545600 | 1.1202439689 | 1.9648391489 | 3.0107456778 | 2.534945848 | 6.879284453 |
| 2.3   | 3.1908633139 | 2.1178041626 | 2.7285441719 | 5.279666518 | 1.665006440 | 4.883589413 |
| 2.4   | 1.6416121890 | 8.5785121669 | 1.0118763143 | 1.7617643954 | 8.2250771896 | 3.227812312 |
| 2.5   | 1.2090347328 | 7.9703013537 | 1.4704284879 | 2.584595433 | 2.015104600 | 5.171252408 |

TABLE S5. The error of the ground and 15th excited states calculated in bases composed of various values of $\omega$ and 40, 50, and 60 harmonic oscillator states.

| $\lambda$ | $\epsilon_0$ | $(\epsilon_0)\wedge(0)$ | $\epsilon_0 - (\epsilon_0)\wedge(0)$ | $(\epsilon_0)\wedge(1)$ | $\epsilon_0 - (\epsilon_0)\wedge(1)$ |
|---------|-------------|-----------------|-----------------|-----------------|-----------------|
| -1      | 0.6209270298 | 0.5000000000 | 0.1209270298 | 0.1875000000 | -0.0666579702 |
| -2      | 0.7825872553 | 0.7071067812 | 0.0754805041 | 0.0937500000 | -0.0182694959 |
| -3      | 0.9206648830 | 0.8660254038 | 0.0564394792 | 0.0625000000 | -0.007865208 |
| -4      | 1.0426978264 | 1.0000000000 | 0.0426978264 | 0.0468750000 | -0.0014771736 |
| -5      | 1.1530169674 | 1.1180393887 | 0.0349829786 | 0.0375000000 | -0.0025170214 |
| -6      | 1.2543454649 | 1.2247448714 | 0.0296005935 | 0.0312500000 | -0.0016494065 |
| -7      | 1.3485135506 | 1.3285765455 | 0.0256378951 | 0.0267587143 | -0.0011478192 |
| -8      | 1.4368153589 | 1.4121356264 | 0.0226017966 | 0.0234750000 | -0.0008357034 |
| -9      | 1.5202030526 | 1.5000000000 | 0.0202030526 | 0.0208333333 | -0.0006302807 |
| -10     | 1.5993998650 | 1.5811388301 | 0.0182610350 | 0.0187500000 | -0.0004889650 |

TABLE S6. Comparing perturbation theory with exact results. All values are accurate to the number of digits shown.