Quadratic Lie algebras and quasi-exact solvability of the two-photon Rabi Hamiltonian

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Abstract

It is proved that the two-photon Rabi Hamiltonian is quasi exactly solvable on the basis of the two different quadratic Lie algebras.

1 Introduction

By definition, a quasi-exactly solvable (QES) Hamiltonian $\hat{H}$ has a finite-dimensional matrix representation within a finite-dimensional subspace of the whole infinite-dimensional Hilbert space. As a result, a partial algebraization of the spectrum occurs. Correspondingly, a QES Lie algebra is the Lie algebra of differential operators $\mathfrak{g}(J_1, J_2, ..., J_N)$ that admits finite-dimensional representation in the subspace $\mathcal{R}_N$ of smooth functions (if $\psi \in \mathcal{R}_N$ and $J_n \in \mathfrak{g}$ then $J_n(\psi) \in \mathcal{R}_N$). Then, the QES Hamiltonian belongs to an universal enveloping QES algebra.

In investigation of QES systems, it was implied up to now that such a system has only one underlying QES algebra. In the present paper, we would like to pay attention that a given QES Hamiltonian may have more than one QES algebra. In doing so, different finite-dimensional representations of such QES algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, ...$ cut out different parts of the spectrum of Hamiltonian $\hat{H}$ and/or correspond to different pieces in the space of its parameters. This observation not only deepens our understanding of the mathematical structure of QES systems but is also important for physical applications since it allows to explore the properties of the spectrum in more detail.

In the present paper, we consider the two-photon Rabi Hamiltonian (TPRH). We find explicitly two different QES algebras that ensure its quasi-exact solvability for certain values of parameters. It turns out that both the algebras are quadratic in contrast to the case of linear $sl(2, \mathbb{C})$ algebra typical of one-dimensional systems [2-9]. (Quadratic algebras in the QES context were also discussed in Ref. [10].)
The fact that TPRH is QES was shown in Ref. [11]. The works [11], [12] and the present one deal with different relationships between the parameters of the system and describe, in general, different parts of the spectrum for which different QES Lie algebras are relevant. Such a rich QES structure of TPRH is a separate interesting property of this system.

2 Two-photon Rabi Hamiltonian

Rabi Hamiltonian [13] describes a two-level system (atom) coupled to a single mode of radiation via dipole interaction. It reads

\[ H_R = \frac{\tilde{\omega}_0}{2} \sigma_z + \tilde{\omega} \cdot b^+ b + \tilde{g} \left( (b+b^+)^2 \right) \cdot (\sigma_+ + \sigma_-) \] (1)

where \( b \) and \( b^+ \) are the Bose annihilation and creation operators respectively \([b, b^+] = 1\) and \( \sigma_z, \sigma_\pm = \sigma_x \pm i \cdot \sigma_y \) are the Pauli matrices. It is often considered in the rotating-wave approximation in which the terms proportional to \( b\sigma_- \) and \( b^+\sigma_+ \) are omitted in which case it is sometimes called the Jaynes-Cummings Hamiltonian [14], [15] (in some works, the latter term applies to the full Hamiltonian [14], [16]).

The simplest generalization of such a Hamiltonian taking into account nonlinear optical processes is TPRH:

\[ H = \frac{\tilde{\omega}_0}{2} \sigma_z + \tilde{\omega} \cdot b^+ b + \tilde{g} \left( (b+b^+)^2 \right) \cdot (\sigma_+ + \sigma_-) \] (2)

The Hamiltonian [2] describes the same physical system "two-level atom plus radiation" but emission and absorption of radiation occurs due to a two-photon process instead of a single-photon ones in the ordinary Rabi Hamiltonian [13]. In the present work, we consider the full form of Hamiltonian (2) without using the rotating wave approximation.

In what follows we will use the representation of the Pauli matrices

\[ \sigma_x = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \sigma_y = \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right), \quad \sigma_z = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right). \]

Then, the spectral problem \( H \psi = \tilde{E} \psi \) for Hamiltonian (2) can be represented in the form

\[ \begin{pmatrix} L_+ & \omega_0 \\ \omega_0 & L_- \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \tilde{E} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \] (3)

where \( L_\pm = b^+ b \pm g/2 \cdot (b^+ b)^2, \omega_0 = \tilde{\omega}_0/\tilde{\omega}, g = 4 \cdot \tilde{g}/\tilde{\omega}, E = \tilde{E}/\tilde{\omega}. \)

Eliminating the function \( \varphi_2 \) from the system of equations (3) we arrive at the fourth-order equation for the component \( \varphi_1 \) of the wave function:

\[ L_1 \varphi_1 = 0 \]

\[ L_1 = (L_-L_+ - E (L_+ + L_-) + E^2 - \omega_0^2) \] (4)
We find the conditions under which the operator $L_1$ \((4)\) is QES and, hence, so is Hamiltonian \((2)\). We would like to draw attention to the following subtlety. We are dealing with the two-component wave function. Usually, while considering matrix QES Hamiltonians \([17]-[20]\) it was assumed that all components of the wave function belong to the invariant subspace. Now, this is, generally speaking, not the case. We require that, say, the component $\varphi_1$ obey QES equation. However, the second component $\varphi_2 = 1/\omega_0 \cdot (E - L_+) \varphi_1$ does not, in general, belong to the invariant subspace. (We may interchange the role of $\varphi_1$ and $\varphi_2$ and require that $\varphi_2$ obey the QES equation.)

Now we will show that the operator $L_1$ \((4)\) can be represented in terms of generators of two QES algebras and, therefore, the operator $L_1$ itself possesses the same invariant subspaces as the aforementioned generators. The explicit description of corresponding generators and their invariant subspaces is given in Appendix A. The fact that the generators under consideration are quasi-exactly solvable is by no means obvious and was established within the approach called QES-extension. It was shown there that the differential operators $J_2^\pm$ \((26)\), $J_3^\pm$ \((29)\) possess invariant subspaces $\mathbb{R}^N_3$, $\mathbb{R}^3_3$ which are not polynomials and the generators cannot be obtained as a polynomial deformation of the linear algebra $sl(2, R)$ \([10]\). It turned out that they can be expressed in terms of hypergeometric functions \([12]\). I use here these results from the previous work \([12]\) with the notations borrowed from that paper. It is also interesting that the corresponding algebras are quadratic in terms of aforementioned operators as it is seen from the commutation relations \((28)\), \((31)\).

Let us use the coordinate representation

$$b = \frac{\partial_x + x}{\sqrt{2}}, \quad b^+ = \frac{-\partial_x + x}{\sqrt{2}}$$

(5)

in which $2 \cdot L_4 = (\pm g - 1) \cdot \partial_x^2 + (1 \pm g) \cdot x^2 - 1$. We want to express $L_1$ \((4)\) in terms of operators $J_2^-, J_2^+$ or $J_3^-, J_3^+$. As these operators have the second order in derivatives while $L_1$ is of the fourth order, we are seeking for the suitable quadratic combinations. Using also the gauge freedom, we rely on the relation

$$e^{\beta(x)} \cdot L_1 \cdot e^{-\beta(x)} = P \left( J_2^-, J_2^+ \right) |_{\eta(x)}$$

(6)

(similarly for $J_3^-, J_3^+$) where $\beta(x)$ is a gauge function, $P (z, y) = c_{zz} \cdot z^2 + c_{yy} \cdot y^2 + c_{yz} \cdot yz + c_{zy} \cdot zy + c_z \cdot z + c_y \cdot y + c_0$ — is the second order polynomial, $\eta (x)$ is the function that defines the change of variable. Making a simple choice $\beta (x) = \text{const} \cdot x^2$, $\eta (x) = \text{const} \cdot x^2$ we obtain the system of equations for the coefficients of the polynomial $P (z, y)$. Below we give several solutions which we managed to obtain.

**1) The subspace $\mathbb{R}^2_N$:**

$$e^{e^x z^2} L_1 e^{-e^x z^2} = \left| 4g^2 \cdot (J_2^2)^2 + 4g \cdot S_2 - 2g^2 \cdot J_2^+ 4J_2^+ - \omega_0^2 - a_0 \right|_{t = \xi \cdot x^2}$$

(7)

where $a_0 = (3 + 4N) \cdot (4N (g^2 - 1) - 3 + 5g^2) / 4$, $\alpha = -3/4, s = -1/2$ (the parameters of subspace $\mathbb{R}^2_N$)
c = \frac{1}{2} \sqrt{\frac{1+g}{1-g}} \quad \text{(the parameter of gauge transformation)} \quad (9)

\xi = \frac{g}{\sqrt{1-g^2}} \quad \text{(the parameter of change of variable)} \quad (10)

\[ E = -\frac{1}{2} + 2 (N+1) \cdot \sqrt{1-g^2} \quad \text{(the energy of the Hamiltonian \,(3).)} \quad (11) \]

II) The subspace \( \mathbb{R}_N^3 \):

\[ e^{c-x^2} L_1 e^{-c-x^2} = \left| 4g^2 \cdot (J_3^-)^2 - 4g \cdot S_3 - 2g^2 \cdot J_3^- + 4J_3^+ - \omega_0^2 - a_0 \right|_{t=\xi \cdot x^2} \quad (12) \]

where \( a_0 = (1+2N) \cdot (2N(g^2-1)-1+3g^2) /4, \) \( \alpha = \frac{3}{4} - \frac{N}{2}, \) \( s = 1/2 \) \( \text{(the parameters of subspaces \( \mathbb{R}_N^3 \)}) \) \( (13) \)

\[ e = \frac{1}{2} \sqrt{\frac{1-g}{1+g}} \quad \text{(the parameter of gauge transformation)} \quad (14) \]

\[ \xi = \frac{-g}{\sqrt{1-g^2}} \quad \text{(the parameter of change of variable)} \quad (15) \]

\[ E = -\frac{1}{2} + (N+1) \cdot \sqrt{1-g^2} \quad \text{(the energy of the Hamiltonian \,(3).)} \quad (16) \]

The expressions \( (7), (12) \) demonstrate explicitly that Hamiltonian \( (2) \) is indeed QES for corresponding values of parameters. Using the asymptotic formulas for the hypergeometric function \( [21] \), one can check that for the values of parameters \( (9), (10), (14), (15) \) the functions belong to the space \( L_2 (-\infty, +\infty). \) For example, for the subspace \( \mathbb{R}_N^3 \) for \( x \to \infty \) we have \( \varphi_1 \sim \exp \left( - (c - \xi) x^2 \right) \) \( [21] \) where \( c - \xi > 0, \) \( 0 < g < 1 \) \( (9). \)

Now, the matrix representation of the operator \( L_1 \) \( (1) \) follows from those of \( J_\pm_2 \) \( (26) \) and \( J_\pm_3 \) \( (29). \) Below we list explicit formulas for small values of \( N. \) For convenience, we also include in the subspaces \( \mathbb{R}_N^2, \mathbb{R}_N^3, \) the gauge factor \( e^{-c x^2} \) from eqs. \( (7), (12). \)

I) The subspace \( \mathbb{R}_0^0 \) \( (N = 0): \)

\[ \mathbb{R}_0^0 = \text{span} \left\{ e^{-c x^2} \, _1F_1 \left[ -\frac{3}{4}; -\frac{1}{2}; \xi x^2 \right], \ e^{-c x^2} \, _1F_1 \left[ \frac{1}{4}; \frac{1}{2}; \xi x^2 \right] \right\} \quad (17) \]

where \( c \) and \( \xi \) are the parameters \( (9), (10). \) Finite-dimensional representations of the operators \( J_\pm_2 \) take the form:

\[ J_2^+ \to \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad J_2^- \to \frac{1}{4} \begin{pmatrix} -3 & 6 \\ 0 & 1 \end{pmatrix} \quad (18) \]
whence, taking into account eq. (7) we obtain the matrix representation for the operator $L_1$ on the subspace $\mathbb{R}_2^3$:

$$\begin{pmatrix}
3g + 9/4 - \omega_0^2 & -3g(2g + 1) \\
2(g + 1) & 1/4 - 3g - 4g^2 - \omega_0^2
\end{pmatrix}$$  \hspace{1cm} (19)

The spectral problem (4) possesses a non-zero solution if the determinant of the matrix (19) is equal to zero. The only real solution $g(\omega_0)$ of the corresponding equation that obeys the condition $0 < g < 1$ has the form

$$g(\omega_0) = \frac{\sqrt{(4\omega_0^2 - 9)(1 - 4\omega_0^2)}}{8\omega_0}, \quad \frac{1}{2} < \omega_0 < \frac{3}{2}. \hspace{1cm} (20)$$

The components of the wave function corresponding to the solution (20) for the particular values of parameters are listed in Appendix B.

**II** The subspace $\mathbb{R}_2^3$ ($N = 2$):

$$\mathbb{R}_2^3 = \text{span} \left\{ e^{-cx^2} \begin{pmatrix} 1 & 1/2 & \xi x^2 \end{pmatrix}, e^{-cx^2} \begin{pmatrix} 3/4 & 1/2 & \xi x^2 \end{pmatrix} \right\} \hspace{1cm} (21)$$

where $c$ and $\xi$ are the parameters (13), (15). Taking into account (30), (12) we find the finite-dimensional representations of operators $J^\pm_3$:

$$J^+_3 \rightarrow \frac{1}{4} \begin{pmatrix} -2 & 2 & 0 \\
1 & 2 & -3 \\
0 & 10 & -10
\end{pmatrix}, \hspace{1cm} J^-_3 \rightarrow \frac{1}{4} \begin{pmatrix} -1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 7
\end{pmatrix} \hspace{1cm} (22)$$

whence, with (12) taken into account, we also obtain the matrix representation for the operator $L_1$ on the subspace $\mathbb{R}_2^3$:

$$\begin{pmatrix}
17/4 - \omega_0^2 - 8g^2 & 2(g + 1) \\
1 - g & 33/4 - \omega_0^2 - 8g^2 \\
0 & 10(1 - g)
\end{pmatrix} \hspace{1cm} (23)$$

The spectral problem (4) has a non-zero solution if the determinant of the matrix (23) is equal to zero. We find two solutions $g_{\pm}(\omega_0)$ satisfying the condition $0 < g_{\pm}(\omega_0) < 1$:

$$g_{\pm}(\omega_0) = \frac{1}{8} \sqrt{(5 + 2\omega_0)(2\omega_0 \pm 3)(1 \pm 2\omega_0)}/\omega_0, \quad \frac{1}{2} < \omega_0 < 2 \pm \frac{1}{2}. \hspace{1cm} (24)$$

The components of the wave function corresponding to the solution $g_{+}(\omega_0)$ are listed in Appendix B for particular values of parameters.

Equations (7), (12) obtained above give evidence that Hamiltonian (2) is quasi-exactly solvable on the basis on two quadratic Lie algebras (see Appendix A). Each of them cuts out some set of eigenvalues from the spectrum of the total Hamiltonian. Thus, different parts of the spectrum are described by different Lie algebras.
3 Summary and outlook

It is shown that the two-photon Rabi Hamiltonian (2) is quasi-exactly solvable on the basis of two different quadratic Lie algebras. The results of the present paper demonstrate that different parts of the spectrum correspond to different Lie algebras. It is especially interesting that the functions which realize invariant subspaces are expressed in terms of special functions (cf. also [12]) whereas application of the linear Lie algebra leads to subspaces spanned on polynomials [11].

The results obtained provoke also further questions which we outline here briefly. First of all, it concerns the comparison of the present article with previous results. Both in Ref. [11] and in our paper QES solutions are obtained for the energies

\[ E = -\frac{1}{2} + M \cdot \sqrt{1 - g^2} \]  

(25)

where \( M \) is a half-integer in [11] and an integer in our paper. In [11], these energies were claimed to correspond to level-crossing of the TPRH but not to all of them. It was conjectured there that integer \( M \) might correspond to the remaining crossings. Now, in view of the present results, it would be especially interesting to elucidate the relationship between level-crossings and different types of quasi-exact solvability.

In the previous paper [12] we used another representation - the Fock-Bargmann one instead of the coordinate one in the present work. Correspondingly, the operators \( L_1 \) and its analogue \( L \) from [12] do not coincide, so that the conditions of their quasi-exact solvability impose different restrictions on the parameters of Hamiltonian. It turned out that in the first case it reduces to \( g^* = \frac{2}{\sqrt{6}} \) [12] independently of \( N \) whereas in our case \( g \) is the function of \( \omega_0 \), the concrete form of this function being different for different \( N \) (see eq. (20) for \( N = 0 \) and (24) for \( N = 2 \)).

It is of separate interest the issue of classification of states. In particular, it concerns the question as to which multiplet the ground state belongs. If the corresponding relevant QES algebra is known, the problem of minimization of the functional \( \langle \Psi, \hat{H} \Psi \rangle \) reduces from the whole Hilbert space to its finite-dimensional part that simplifies greatly the problem and can be useful for applications.

4 Acknowledgments

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5 Appendix A.

Here we give explicit formulas for two QES quadratic Lie algebras used in the text [12].
I) Finite dimensional function subspace $\mathcal{R}_N^2 = \text{span}\{f_n^+, \ldots, f_N^+, f_n^- \ldots, f_N^-\}$ $(N = 0, 1, 2, \ldots)$, dim $(\mathcal{R}_N^2) = 2(N + 1)$, formed by functions $f_n^+ = t^n \cdot 1 F_1\left[\alpha; t\right], f_n^- = t^n \cdot 1 F_1\left[\alpha + 1; t\right]$ $(n = 0, 1, \ldots, N, -1, N)$ \cite{21}, is invariant for the operators $J_2^\pm$:

\[
J_2^+ = t \frac{d^2}{dx^2} + (1 + s - t) \frac{d}{dt} \\
J_2^- = t^2 \frac{d^2}{dx^2} + (s - 2N - t) \cdot t \frac{d}{dt} + t(N - \alpha)
\]  \hspace{1cm} (26)

The operators $J_2^\pm$ act on the functions $f_n^\pm(x) \in \mathcal{R}_N^2$ as follows:

\[
J_2^+ \left(\frac{f_n^+}{f_n^-}\right) = \left(\frac{n(A_n - 1 + s) \cdot f_n^+ - B_n \cdot f_{n+1}^+ + 2\alpha B_n \cdot s \cdot f_{n+1}^-}{(s - n)(1 - A_n) \cdot f_n^- + B_n \cdot f_{n+1}^- + s \cdot (2B_n - 1) \cdot f_n^-}\right)
\]

\[
J_2^- \left(\frac{f_n^+}{f_n^-}\right) = \left(\frac{(\alpha - n) \cdot f_n^+ + n(n + s) \cdot f_{n+1}^- + \alpha(1 + 2n) \cdot s \cdot f_{n+1}^-}{(\alpha + n + 1) \cdot f_n^+ + n(n - s) \cdot f_{n+1}^- + 2ns \cdot f_{n+1}^-}\right)
\]  \hspace{1cm} (27)

where $A_n = n - 2N, B_n = n - N$. The commutation relations of the operators $J_2^\pm$ are:

\[
[J_2^+, J_2^-] = S_2 \\
[J_2^+, S_2] = 4 \cdot J_2^+ J_2^- - 2 \cdot S_2 - c_5^+ J_2^+ - c_5^- J_2^- - c_7^+
\]

\[
[J_2^-, S_2] = -2 \cdot (J_2^+)^2 - J_2^- + c_5^+ J_2^- - (\alpha - N)(s + 1)
\]  \hspace{1cm} (28)

where $c_5^+ = 2(1 + \alpha + s), c_5^- = (2N - s)(s - 2 - 2N), c_7^+ = (N - \alpha)(2 + 2N - s) \cdot (s + 1)$.

II) Finite dimensional function subspace $\mathcal{R}_N^3 = \text{span}\{f_0, f_1, \ldots, f_N\} \ (N = 0, 1, 2, \ldots), \dim (\mathcal{R}_N^3) = N + 1$, formed by functions $f_n = 1 F_1\left[\alpha + n; t\right]$ $(n = 0, 1, \ldots, N, -1, N)$ \cite{21}, is invariant for the operators $J_3^\pm$:

\[
J_3^- = t \frac{d^2}{dx^2} + (s - t) \frac{d}{dt} \\
J_3^+ = t^3 \frac{d^2}{dx^2} + (s - N - t) \cdot t \frac{d}{dt} - \alpha
\]  \hspace{1cm} (29)

The operators $J_3^\pm$ act on the functions $f_n(x) \in \mathcal{R}_N^3$ as follows:

\[
J_3^- (f_n) = (n + \alpha) \cdot f_n
\]

\[
J_3^+ (f_n) = (s \cdot n + C_n) \cdot f_n + \alpha_n \cdot B_n \cdot f_{n+1} + n(\alpha_n - s) \cdot f_{n-1}
\]  \hspace{1cm} (30)

where $\alpha_n = n + \alpha, C_n = \alpha_n \cdot (N - 2n), B_n = n - N$. The commutation relations of the operators $J_3^\pm$ are:

\[
[J_3^+, J_3^-] = S_3 \\
[J_3^+, S_3] = 4 \cdot J_3^+ J_3^- - 2 \cdot S_3 - c_5^+ J_3^+ + c_5^- J_3^- + c_7^+
\]

\[
[J_3^-, S_3] = -2 \cdot (J_3^+)^2 - J_3^- + c_5^+ J_3^- - s \cdot \alpha
\]  \hspace{1cm} (31)

where $c_5^+ = s + 2\alpha + N, c_5^- = (N + 2 - s) \cdot (N - s), c_7^+ = s\alpha \cdot (N + 2 - s)$. 

7
The operators $J_s^\pm (\alpha, s, N)$ (26) can be transformed into the operators $J_s^\pm (\alpha', s', N')$ (28) by means of substitution of the parameters $s \rightarrow s' - 1$, $\alpha \rightarrow \alpha' + (N' - 1)/2$, $N \rightarrow (N' - 1)/2$

$$J_s^\pm (\alpha' + (N' - 1)/2, s' - 1, (N' - 1)/2) = J_s^\pm (\alpha', s', N').$$

(32)

This substitution has meaning for odd $N'$ only. It is also worth noting that the subspace $\mathbb{R}_N^3$ is the subset of the subspace $\mathbb{R}_N^2$ ( $\mathbb{R}_N^3 \subset \mathbb{R}_N^2$ ). This claim follows from the properties of the hypergeometric functions [21].

6 Appendix B.

I) The subspace $\mathbb{R}_0^3$ ($N = 0$):

According to eq. (11), we need the eigenvector corresponding to the zero eigenvalue of the transposed matrix (19) where we should take into account (20). Then, we obtain for the wave function:

$$\Psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} p_{11} \cdot h_1 (x) + p_{12} \cdot h_2 (x) \\ p_{21} \cdot h_1 (x) + p_{22} \cdot h_2 (x) \end{pmatrix}$$

(33)

where $h_1 (x) = e^{-cx^2} F_1 \left[ -3/4, -1/2; \xi x^2 \right]$, $h_2 (x) = e^{-cx^2} F_1 \left[ 1/4, 1/2; \xi x^2 \right]$, $c = (\sqrt{18+1})/2 (\sqrt{18}-1)$, $\xi = \sqrt{15}$, $p_{11} = 250 + 68 \sqrt{15}$, $p_{12} = -235 - 60 \sqrt{15}$, $p_{21} = 70 + 28 \sqrt{15}$, $p_{22} = 35 - (120 + 20 \sqrt{15}) x^2$. The second component $\varphi_2$ was found from relation (3):

$$\varphi_2 = \frac{1}{\omega_0} (E - L_+) \varphi_1$$

(34)

The solution (33) (not normalized) corresponds to the energy value $E = 5/4$.

II) The subspace $\mathbb{R}_N^3$ ($N = 2$)

Proceeding along the same lines as in I) and taking into account (28) we obtain the wave function corresponding to the solution $g_+ (\omega_0)$ at $\omega_0 = 1$.

$$\Psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} p_{11} \cdot h_1 (x) + p_{12} \cdot h_2 (x) \\ p_{21} \cdot h_1 (x) + p_{22} \cdot h_2 (x) \end{pmatrix}$$

(35)

where $h_1 (x) = e^{-cx^2} F_1 \left[ 3/4, 3/2; \xi x^2 \right]$, $h_2 (x) = e^{-cx^2} F_1 \left[ -1/4, 1/2; \xi x^2 \right]$, $c = (\sqrt{8-3\sqrt{5}})/(2 \sqrt{8+3\sqrt{5}})$, $\xi = -3 \sqrt{8 \sqrt{10}}$, $p_{11} = 322 \left( (7110 + 3184 \sqrt{5}) x^2 - \sqrt{19} (371 + 170 \sqrt{5}) \right)$, $p_{12} = -\frac{1}{4} \left( \sqrt{19} (6368 + 2844 \sqrt{5}) x^2 - (1235 + 418 \sqrt{5}) \right)$, $p_{21} = \frac{57}{4} \left( 79 + 32 \sqrt{5} \right) x^2$, $p_{22} (x) = \frac{49}{4} \left( 76 x^2 - \sqrt{19} (70 - 31 \sqrt{5}) \right)$. The second component $\varphi_2$ was obtained from relation (34). Solution (35) (not normalized) corresponds to the energy value $E = 3 \sqrt{19}/8 - 1/2$. 

8
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