Simulation of Rare Events in Stochastic Systems

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Abstract. The paper presents algorithms for simulation of rare events in stochastic systems based on the theory of large deviations. Here, this approach is used in conjunction with the tools of optimal control theory to estimate the probability that some observed states in a stochastic system will exceed a given threshold by some upcoming time instant. Algorithms for obtaining controlled extremal trajectory (A-profile) of the system, along which the transition to a rare event (threshold) occurs most likely under the influence of disturbances that minimize the action functional, are presented. It is also shown how this minimization can be efficiently performed using numerical-analytical methods of optimal control for linear and nonlinear systems. These results are illustrated by an example for a precipitation-measured monsoon intraseasonal oscillation (MISO) described by a low-order nonlinear stochastic model.

1. Introduction

In reality, all dynamical systems and processes are affected by random perturbations, which can often be eliminated by methods of stabilization and robust control at finite time intervals. At the same time, even a low noise can radically affect the dynamics, moving it from a stable state to an unstable one, for large or infinite time intervals. Rare events, such as deviations from nominal stable states or transitions between metastable states, play an important functional role in applied problems of analysis and control. Therefore, it is important to monitor these events in real time with the issuance of probabilistic estimates of their occurrence.

To determine these events numerically, we can use the controlled motion of the system, which makes rare events more likely [1, 2]. This approach is used in conjunction with the tools of the large deviations theory (LDT) and optimal control to estimate the probability that some states observed in a dynamical system will exceed a given threshold after some time (a rare event will occur). The system corresponding to such a controlled movement to a rare undesirable event is usually called the path system [3, 4]. The connection between a weakly perturbed dynamical system and the corresponding optimal control problem at a finite time interval for a system of paths means that the trajectories of this system of paths with optimal control are of interest as large deviations of the perturbed dynamical system under consideration. Thus, the system of paths with optimal control makes it possible to assess the probability of large deviations, determining the profile of the development of a critical situation (A-profile) [3].

In the linear case, this approach is particularly effective, since it allows us to obtain an analytical solution to the corresponding optimal control problem, which is especially important for real-time systems. For example, [4] shows a solution in the open loop form, and [5, 6] shows some results on the corresponding solution in the feedback form. In the nonlinear case, everything is much more
complicated, and therefore the solution is sought within the framework of simplified schemes, where some form of approximation is performed. One group of such approximate methods is methods using State-dependent coefficients (SDC) [7-10].

The well-known method of the state-dependent Riccati equation (SDRE) is quite simple and effective, especially for problems on an infinite time interval [7, 8]. This method considers the original nonlinear optimal control problem pointwise in the form of the linear-quadratic controller (LQR) problem. As a result, a set of LQR problems is solved sequentially at each time point, into which the entire time domain is sampled. However, the non-stationarity of control for problems at a finite time interval leads to the following problem: an exact solution to the problem is impossible without knowing the future states of the system, which are unknown at the current time. In [11], an approximate algorithm is proposed that uses the hypothesis of a weak change in the state vector (the hypothesis of "frozen" coefficients).

In this paper, we consider the development of numerical methods for the analysis of large deviations of dynamical systems for rare events control based on the solution of the corresponding optimal control problems. The peculiarity of the formulation of the optimal control problem as a finite time optimal control without a penalty on the system state in the action functional allows us to obtain a numerical-analytical solution of the problem, which is especially important for real-time implementation.

This approach to analyzing large deviations becomes a natural tool for obtaining quantitative information about rare events. Its numerical applicability in this paper is shown by the example of a large-scale monsoon intraseasonal oscillation (MISO) as measured by precipitation using a low-order nonlinear stochastic model. The works of many researchers are devoted to solving the problems of modeling, real-time monitoring and forecast the modes of the MISO. For example, [12] shows an ensemble prediction system or the extended range prediction of MISO. In [13-15] several indices have been proposed for the real-time monitoring and extended-range forecast of the MISO, also an index based on the nonlinear Laplacian spectral analysis [16, 17]. In [19] a prediction framework for the large-scale MISO precipitation was developed. In this paper we propose to use the LDT to simulate the extreme events for the MISO model.

The rest of the paper is structured as follows. Section 2 provides a brief description of the LDT method for rare events analysis and provides a statement of the problem. The solution of the problem for linear and nonlinear systems by the SDRE methods is given in Sections 3. In section 4, numerical example is given for using the developed methods for simulation rare events for MISO model. Discussion of the results and conclusions are given in sections 5.

2. Large deviations theory and statement of the problem

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A closed nonlinear dynamical system is considered:

\[ \frac{dx}{dt} = f(x), \quad x(t_0) = x_0, \]  

perturbed by the addition of a small noise

\[ \frac{dx}{dt} = f(x) + \epsilon \alpha(x) \dot{w}, \quad x(t_0) = x_0, \]  

where \( x \in \mathbb{R}^n, \dot{w} \in \mathbb{R}^r \) is a white noise vector, \( \epsilon > 0 \) is a small parameter, \( f(x), \alpha(x) \) are smooth matrix functions satisfying the growth conditions, as in [19, chapter 2], and \( f(0) = 0, \forall t \in \mathbb{R} \).

At \( \epsilon \to 0 \) with a probability tending to 1 the trajectories of the perturbed system converge to the trajectories of the unperturbed system at any finite time interval (the exact meaning is in [19], chapter 2, theorem 1.2). In this case, the trajectories of the system (2), which are far from the trajectories of (1), are of interest. These are events which probabilities are close to zero, but it is possible to distinguish those that are overwhelmingly more likely than others. The estimation of the probabilistic
quantities associated with these rare events, which are called large deviations, involves an optimal problem formulation that is closely related to the dynamics of the perturbed system.

The difficulty of moving along a given path \( \bar{\varphi} \) is measured by the cost of control actions \( \nu \in \mathbb{R}^r \) necessary to control the state of the system when moving along this path:

\[
d\varphi/dt = f(\varphi) + \sigma(\varphi)\nu, \quad \varphi(t_0) = x_o,
\]

where the cost of control \( \nu \) is measured by functional

\[
S_{\varphi,t_f}(\nu) = \frac{1}{2} \int_{t_0}^{t_f} \nu^T \nu dt.
\]

This relationship between the controlled system of paths (3), (4) and the perturbed system (2) means that the available trajectories of (3), (4) are of interest as large deviations of the perturbed system (2) (i.e., those to which the system (3) can be directed using the control \( \nu \) corresponding to the constraints (4)). Thus, the global properties of the system (2) are described using a system of paths (3), with a particular focus on the possible deviations of the system state (2) from attractor in the direction of the boundaries \( \partial_D \) of the operating region \( D \subset O_\chi \) (\( D \) is - open set, \( O_\chi \) is a basin of attraction, \( \chi \) is a stable equilibrium state (attractor) of an unperturbed system (1)).

We denote the following family for paths \( F = \{ \varphi \in C_{a,t_f}(\mathbb{R}^n) : \varphi_{t_0} \in D, \varphi_{t_f} \in \mathbb{R}^n \setminus D \} \). For the set \( D \) and systems (2), (3) the equality is true [19]:

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \ln P\{x_{t_f} \in \mathbb{R}^n \setminus D\} = -\min_{\varphi \in F} S_{\varphi,t_f}(\varphi,\nu),
\]

where the functional \( S_{\varphi,t_f}(\varphi,\nu) \) is defined in accordance with (4) on the solutions of the system (3), for which the boundary condition for reaching the critical state (rare event) at the time \( t_f \) is \( y(t_f) = C\varphi(t_f) \in \partial_K \), where \( C \) is a full rank matrix.

The probability in (5) can be estimated by solving the optimal control problem (the Lagrange-Pontryagin problem): on the solutions of the path system (3), minimize the action functional (4) under the boundary condition \( y(t_f) = C\varphi(t_f) \in \partial_K \).

3. Solution of the Lagrange-Pontryagin problem

3.1. Linear system case

Let's represent the path system (3) in a linear state-space form (where \( f(\varphi) = A\varphi, \sigma(\varphi) = B \)):

\[
d\varphi/dt = A\varphi + B\nu, \quad \varphi(t_0) = \varphi_o,
\]

and formulate the problem of minimizing the criterion (4) under the conditions of constraints (6).

The solution to problem (7), (4), (6) in the form of feedback can be obtained by analogy with [20] with a slight modification that takes into account the matrix \( C \):

\[
\nu = -B^T W^T C^T M^{-1}(C W \varphi - y(t_f)),
\]

\[
dW/dt = -WA, \quad W(t_f) = I,
\]

\[
dM/dt = -CWB^T W^T C^T, \quad M(t_f) = 0.
\]

For a given LQR problem, the matrix \( W(t) \) and \( M(t) \) can be represented analytically [6]:

\[
W(t) = e^{A(t_f-t)}, \quad M(t) = C(W(t)DW^T(t) - D)C^T
\]
where the matrix $D$ is a solution to the Lyapunov equation
\[
AD + DA^T - BB^T = 0. 
\] (11)

The minimum value of the functional (4) (the normalized action functional):
\[
S_{t_0,t_f} = (C \Phi - y(t_f))^T M^{-1} (C \Phi - y(t_f)) \bigg|_{t_0}^{t_f}.
\] (12)

Applying (7) to (6), we obtain the $\Phi(t), \ t \in [t_0,t_f]: \Phi(t) = e^{A_{\varepsilon}t} \Phi + \int_{t_0}^{t} e^{A_{\varepsilon}(t-t')} G y(t_f) dt'$.

The solution of this LQR problem provides analytical expressions for control in the form of feedback, for the profile of the development of a critical event (A-profile) and for the minimum value of the quality criterion, and therefore for assessing the probability. The condition for the existence of this solution is the controllability of the pair $(A, B)$ and the Hurwitz property of the matrix $A$.

Let us briefly describe an algorithm for estimating the probability of a rare event on the basis of the described solution of the LQR problem.

**Algorithm 1 (Linear case)**

Step 1: Solve equation (11) for $D$.
Step 2: Calculate $W(t)$ and $M(t)$ using (10).
Step 3: Calculate the control $\nu$ using (7).
Step 4: Apply the control $\nu$ to the path system to define $\Phi(t)$.
Step 5: Calculate the $S_{t_0,t_f}$ through (12) and get a probability measure: $\ln P^* \{x \in R^n / D \} = -e^{-2S_{t_0,t_f}}$.
Step 6: If, the state vector $x(t)$ of the system (2) is located in $\varepsilon$-vicinity of the vector $\Phi(t)$ of the path system, then in the case of $P^* > \rho$, where $\rho$ is a given threshold value of the probability of occurrence of a rare event, we get that the rare event probably will occur.

3.2. **Nonlinear system case. SDRE technique**

In the nonlinear case, we consider the system of paths (3), which we represent in the SDC form:
\[
d\Phi/dt = A(\Phi)\Phi + B(\Phi)\nu, \quad \Phi(t_0) = \Phi_0, \quad f(\Phi) = A(\Phi)\Phi, \ \sigma(\Phi) = B(\Phi).
\] (13)

Solution of the problem (13), (4), (5) in the feedback form, obtained using the SDRE technique and the method of "frozen" coefficients, has the form of equations (7), (8), (9), in which (following the hypothesis of "frozen" coefficients [11] for each $\Phi(t) \ \forall \Phi(t), \ t \in [t_0,t_f]$) we denote $A = A(\Phi), B = B(\Phi)$.

Note that to calculate the matrices $W(t)$ and $M(t)$, we can use the analytical relations (10), (11), which are calculated at the current values of the state vector from the current time $t$ to the final $t_f$ time at each step. The condition for the existence of this solution is the pointwise controllability of the pair $(A(\Phi), B(\Phi))$ and the Hurwitz property of the matrix $A(\Phi) \ \forall \Phi(t), t \in [t_0,t_f]$.

**Algorithm 2 (Nonlinear case, SDRE technique)**

At each time-step repeat:
Step 1: Measure $\phi$ and evaluate $A(\phi)$ and $B(\phi)$. Fix $A = A(\phi), B = B(\phi)$ and solve the LQR problem using algorithm 1.
Step 2: If, the state vector $x(t)$ of the system (2) is located in $\varepsilon$-vicinity of the vector $\Phi(t)$ of the path system, then in the case of $P^* > \rho$, where $\rho$ is a given threshold value of the probability of occurrence of a rare event, we get that the rare event probably will occur.

The action functional can also be determined on the basis of (12), but it should be understood that this will be a suboptimal solution, which will not give an accurate probability estimate. It only allows us to qualitatively judge the occurrence of a rare event (critical situation).
4. Example
To describe the temporal intermittency and the randomness in the oscillation frequency of the MISO indices, the following low-order stochastic model is used [19]. Here the two components, MISO1 and MISO2, are denoted by \( u_1 \) and \( u_2 \) respectively:

\[
\begin{align*}
\frac{du_1}{dt} &= -d_u u_1 + \gamma (\nu + \nu_f) u_1 - (a + \omega_u) u_2 + \sigma_u \dot{W}_{u_1}, \\
\frac{du_2}{dt} &= -d_u u_2 + \gamma (\nu + \nu_f) u_2 - (a + \omega_u) u_1 + \sigma_u \dot{W}_{u_2}, \\
\frac{d\nu}{dt} &= -d_\nu \nu - \gamma (u_1^2 + u_2^2) + \sigma_\nu \dot{W}_\nu, \\
\frac{d\omega_u}{dt} &= -d_{\omega_u} \omega_u + \sigma_{\omega_u} \dot{W}_{\omega_u},
\end{align*}
\]

where \( \nu_f(t) = f_0 + f_1 \sin(\omega_f t + \phi) \), \( d_u = 0.8, d_\nu = 0.6, d_{\omega_u} = 0.5, \sigma_u = 0.5, \sigma_\nu = 0.5, \sigma_{\omega_u} = 0.5, \gamma = 0.3, a = 4.1, f_0 = 1.0, f_1 = 4.7, \omega_f = 2\pi/12, \phi = -2 \). In addition to the two observed MISO variables, the other two variables \( \nu \) and \( \omega_u \) are unobserved, which represent the stochastic damping and stochastic phase, respectively.

We will consider the problem of the A-profile simulation for the MISO1 index. Based on the features of the system behavior, we will take the critical value (threshold) of the index \( u_1 = 3.5 \), which is reached by the moment \( t = 6 \). To build the A-profile, we use algorithm 2. The simulation results are shown in Figure 1, where the A-profile is shown by a red line. Here, the ensemble of 100 implementations of the system (1) that satisfy the specified conditions is shown in gray color, the average implementation is shown in black. It can be seen from the simulation that the average implementation and the A-profile are quite close to each other. This allows us to use the algorithm 2 for approximate calculation of A-profile as a tool for predicting the system's output to the critical index value.

![Figure 1. Simulation result.](image_url)

5. Conclusion
In this study, an algorithms for approximate calculation of A-profiles for linear and nonlinear stochastic systems are proposed. The advantage of the proposed algorithms is the use of analytical expressions to calculate the control at each moment of time, which allows us to perform A-profile simulation online in direct time.
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Acknowledgments
This reported study was funded by the Russian Science Foundation (Project No. 21-11-00202).