What really matters in Hilbert-space stochastic processes

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Abstract

The relationship between discontinuous and continuous stochastic processes in Hilbert space is investigated. It is shown that for any continuous process there is a parent discontinuous process, that becomes the continuous one in the proper infinite frequency limit. From the point of view of solving the quantum measurement problem, what really matters is the choice of the set of operators whose value distributions are made sharp. In particular, the key role of position sharpening is emphasized.

1 Introduction

In attempting to overcome the difficulties connected with the quantum measurement problem, two types of stochastic processes in Hilbert space have been considered, the so-called hitting or discontinuous processes and the continuous ones. All proposed processes aim at describing reduction as a physical process. A common feature is non-linearity, which originates from the need of maintaining normalization while reducing the state vector. One finds sometimes the statement that the continuous processes allow to deal with systems which cannot be treated with the hitting processes. The aim of the present contribution is to discuss the relationship between discontinuous and continuous processes. Actually, it will be shown that all continuous processes have a physically equivalent parent discontinuous one, applicable to the same class of physical systems.

In all examples of stochastic process in Hilbert space a set of physical quantities (observables) appears, represented by the corresponding set of selfadjoint operators. The process acts inducing the sharpening of the distribution of values of those quantities around a stochastically chosen centre. What really matters from the physical point of view is the choice of the set of quantities, rather than the details of the sharpening procedure.

Both approaches have their benefits. In the discontinuous processes the meaning of the parameters appearing in their definition is transparent and the physical consequences are easy to grasp, while the continuous processes are undoubtedly more elegant from the mathematical point of view. The continuous processes depend on one parameter less than the corresponding discontinuous ones. But this difference is illusory since, as discussed
below, the effectiveness of the discontinuous process depends on two of its parameters only through their product.

In the next section a simplified heuristic derivation of a generic continuous process as the proper infinite frequency limit of the parent hitting process is given. In the last section the three physically most relevant examples of discontinuous and corresponding continuous processes are presented.

2 Infinite frequency limit

In the present section we show how a generic hitting process becomes a continuous stochastic process in the Hilbert space under a suitable infinite frequency limit.

A hitting process is characterized by a choice of the quantities whose distribution is to be made sharp, by the mean frequency of the Poisson distribution of hitting times, and by the accuracy of the sharpening of the chosen quantities. A probabilistic rule is assumed for the distribution of the hitting centres. The effectiveness of the hitting process actually depends on the products of the mean frequency times the accuracy parameters, so that a given effectiveness can be obtained by increasing the hitting frequency and, at the same time, appropriately decreasing the accuracy parameters. Taking the infinite frequency limit with this prescription, one gets the corresponding continuous process.

The above feature of the effectiveness of a hitting process was first noticed by [1] in a different conceptual context. Later, the same feature was highlighted by [5] and the infinite frequency limit was considered by [4] with reference to the time evolution of the statistical operator. Last, [2] in a particular case and [6] in the general case considered the same limit for the time evolution of the state vector, obtaining the continuous stochastic process.

In what follows for the sake of simplicity we use evenly spaced hitting times instead of random ones, and, at least initially, we ignore the Schrödinger evolution. In the present section the sharpened quantities are assumed, for simplicity, to have a purely discrete spectrum.

Let the set of compatible quantities characterizing the considered discontinuous stochastic process be

\[ \hat{A} \equiv \{ \hat{A}_p; p = 1, 2, \ldots, K \}, \quad [\hat{A}_p, \hat{A}_q] = 0, \quad \hat{A}^\dagger_p = \hat{A}_p, \]  

and the sharpening action be given by the operator

\[ S_i = \left( \frac{\beta}{\pi} \right)^{K/4} \exp(R_i), \quad R_i = -\frac{1}{2} \beta (\hat{A} - a_i)^2. \]

The parameter \( \beta \) rules the accuracy of the sharpening and \( a_i \) is the centre of the hitting \( i \). It is assumed that the hittings occur with frequency \( \mu \).

The sharpening operator for the \( i \)-th hitting \( S_i \) acts on the normalized state vector \( |\psi_t\rangle \) giving the normalized state vector \( |\chi_{t,i}\rangle \) according to

\[ |\psi_{t,i}\rangle = \frac{|\chi_{t,i}\rangle}{||\chi_{t,i}\||}, \quad |\chi_{t,i}\rangle = S_i |\psi_t\rangle. \]

The probability that the hitting takes place around \( a_i \) is

\[ \mathcal{P}(\psi_t | a_i) = ||\chi_{t,i}||^2. \]

When considering \( n \) evenly spaced hittings in \( (t, t + dt] \), so that the time interval between two adjacent hittings is \( \tau = dt/n \), the hitting frequency is given by \( \mu = 1/\tau \). In this case, assumptions (4) and (5) become

\[ |\psi_{t+i\tau}\rangle = \frac{|\chi_{t+i\tau}\rangle}{||\chi_{t+i\tau}||}, \quad |\chi_{t+i\tau}\rangle = S_i |\psi_{t+(i-1)\tau}\rangle. \]
with probability
\[ P(\psi_{t+(i-1)t} \mid a_i) = P(a_1, a_2, \ldots, a_{i-1} \mid a_i) = \|\chi_{t+it}\|^2. \] (6)

The continuous process based on the same quantities \( \dot{A} \equiv \{\dot{A}_p; p = 1, 2, \ldots, K\} \) is ruled by the Itô stochastic differential equation
\[ d|\psi\rangle = \left[ \sqrt{\gamma}(\dot{A} - \langle \dot{A}\rangle_{\psi_t}) \cdot dB - \frac{1}{2} \gamma (\dot{A} - \langle \dot{A}\rangle_{\psi_t})^2 dt \right] |\psi\rangle, \] (7)
where
\[ \langle \dot{A}\rangle_{\psi_t} = \langle \psi_t \mid \dot{A} \mid \psi_t \rangle \] (8)
and
\[ dB \equiv \{dB_p; p = 1, 2, \ldots, K\}, \quad dB = 0, \quad dB_p dB_q = \delta_{pq} dt. \] (9)
The parameter \( \gamma \) sets the effectiveness of the process.

We shall show that taking the infinite frequency limit of the discontinuous process (5) and (6) with the prescription
\[ \beta \mu = \text{constant} = 2\gamma, \] (10)
one gets the continuous process (7). As a consequence it becomes apparent that, for \( t \to \infty \), the continuous process drives the state vector to a common eigenvector of the operators \( \dot{A} \), the probability of a particular eigenvector \( |a_r\rangle \) being \( |\langle a_r \mid \psi_0 \rangle|^2 \), for the state vector \( |\psi_0\rangle \) at a given arbitrary initial time.

In what follows for any set of stochastic variables \( \nu \) we use the notation \( \nu_p \) for the mean value and \( \nu_p \nu_q \) for the variances and covariances.

The effect of \( n \) hitting processes in the time interval \( (t, t+dt] \) is described by
\[ |\chi_{t+dt}⟩ = \left( \frac{\beta}{\pi} \right)^{nK/4} \exp(R_n) \ldots \exp(R_2) \exp(R_1) |\psi_t⟩, \quad |\psi_{t+dt}⟩ = \frac{|\chi_{t+dt}⟩}{||\chi_{t+dt}||}. \] (11)
By using the properties of the exponential function, the final non-normalized state vector is then given by
\[ |\chi_{t+dt}⟩ = \left( \frac{\beta}{\pi} \right)^{nK/4} \exp \left( -\frac{1}{2} \beta \sum_{i=1}^{n} a_i^2 \right) \exp \left\{ \sum_{i=1}^{n} \left[ -\frac{1}{2} \beta \left( \dot{A}^2 - 2 \dot{A} \cdot a_i \right) \right] \right\} |\psi_t⟩ \]
\[ = F \exp \left\{ -\frac{1}{2} \beta \left[ n \dot{A}^2 - 2 n \dot{A} \cdot \sum_{i=1}^{n} a_i \right] \right\} |\psi_t⟩ \]
\[ = F \exp \left\{ -\frac{1}{2} \beta n \left[ \dot{A}^2 - 2 \dot{A} \cdot \frac{1}{n} \sum_{i=1}^{n} (a_i - (\langle a_i \rangle_{\psi_t} - 2 \dot{A} \cdot \langle \dot{A}\rangle_{\psi_t}) \right] \right\} |\psi_t⟩, \] (12)
with joint probability
\[ P(a_1, \ldots, a_n) = ||\chi_{t+dt}||^2. \] (13)
Actually, in the limit \( \beta \to 0 \) the joint probability factorizes as
\[ P(a_1, \ldots, a_n) = P(a_n) \cdots P(a_1) \] (14)
where
\[ P(a_i) = \left( \frac{\beta}{\pi} \right)^{K/2} |\psi_t⟩ \exp \left( -\beta (\dot{A} - a_i)^2 \right) |\psi_t⟩ \] (15)
as illustrated in the following two-hitting example. The joint probability for two hittings is given by
\[ P(a_1, a_2) = P(a_1 | a_2) P(a_1), \] (16)
where \( P(a_1 | a_2) \) is the conditional probability of \( a_2 \) given \( a_1 \). In turn,
\[ \mathcal{P}(a_1 | a_2) = \| \chi_{t+2r} \|^2 \]

\[ = \frac{(\beta)}{(\pi)}^{K/2} \langle \psi_{t+r} | \exp \left( -\beta (\hat{A} - a_2)^2 \right) | \psi_{t+r} \rangle \]

\[ = \frac{(\beta)}{(\pi)}^{K/2} \left[ \frac{(\beta)}{(\pi)}^{-K/2} \langle \psi_{t} | \exp \left( -\beta (\hat{A} - a_1)^2 \right) | \psi_{t} \rangle^{-1} \right] \]

\[ \langle \chi_{t+r} | \exp \left( -\beta (\hat{A} - a_2)^2 \right) | \chi_{t+r} \rangle \]

\[ = \frac{(\beta)}{(\pi)}^{K/2} \left[ \frac{(\beta)}{(\pi)}^{-K/2} \langle \psi_{t} | \exp \left( -\beta (\hat{A} - a_1)^2 \right) | \psi_{t} \rangle^{-1} \right] \]

\[ \left( \frac{(\beta)}{(\pi)} \right)^{K/2} \langle \psi_{t} | \exp \left( -\beta (\hat{A} - a_2)^2 \exp \left( -\beta (\hat{A} - a_1)^2 \right) | \psi_{t} \rangle \right. \]

\[ \left. \rightarrow_{\beta \to 0} \left( \frac{(\beta)}{(\pi)} \right)^{K/2} \langle \psi_{t} | \exp \left( -\beta (\hat{A} - a_2)^2 \right) | \psi_{t} \rangle = \mathcal{P}(a_2) \right. \]

(17)

We now compute the statistical properties of the variables \( a_i \). The average value is given by

\[ \overline{a}_i \rightarrow_{\beta \to 0} \int da_i a_i \mathcal{P}(a_i) = \int da a \mathcal{P}(a) \]

\[ = \int da a \left( \frac{(\beta)}{(\pi)} \right)^{K/2} \langle \psi_{t} | \exp \left( -\beta (\hat{A} - a)^2 \right) | \psi_{t} \rangle. \]

(18)

By inserting the expansion of the identity in terms of the common eigenvectors of the operators \( A \), satisfying \( \hat{A}^p | \alpha_k \rangle = \alpha_k p | \alpha_k \rangle \), one finds

\[ \overline{a}_i \rightarrow_{\beta \to 0} = \sum_k \int da a \left( \frac{(\beta)}{(\pi)} \right)^{K/2} \langle \psi_{t} | \exp \left( -\beta (\alpha_k - a)^2 \right) | \alpha_k \rangle \langle \alpha_k | \psi_{t} \rangle \]

\[ = \sum_k | \langle \alpha_k | \psi_{t} \rangle |^2 \int da a \left( \frac{(\beta)}{(\pi)} \right)^{K/2} \exp \left( -\beta (\alpha_k - a)^2 \right) \]

\[ = \sum_k \mathcal{P}_{\psi_k}(\alpha_k) \alpha_k = \langle \hat{A} \rangle_{\psi_t}. \]

(19)

Similarly, for the variances one gets

\[ \overline{a^2_{ip}} \rightarrow_{\beta \to 0} \int da_i a^2_{ip} \mathcal{P}(a_i) - \overline{a^2_{ip}} = \int da a^2_p \mathcal{P}(a) - \overline{a^2_p} \]

\[ = \int da_p a^2_p \left( \frac{(\beta)}{(\pi)} \right)^{1/2} \langle \psi_{t} | \exp \left( -\beta (\hat{A}_p - a_p)^2 \right) | \psi_{t} \rangle - \overline{a^2_p} \]

\[ = \sum_k \int da_p a^2_p \left( \frac{(\beta)}{(\pi)} \right)^{1/2} \langle \psi_{t} | \exp \left( -\beta (\alpha_{kp} - a_p)^2 \right) | \alpha_k \rangle \langle \alpha_k | \psi_{t} \rangle - \overline{a^2_p}. \]

(20)

By properly shifting the integration variable one then finds

\[ \overline{a^2_{ip}} \rightarrow_{\beta \to 0} \sum_k | \langle \alpha_k | \psi_{t} \rangle |^2 \int db (b + \alpha_{kp})^2 \left( \frac{(\beta)}{(\pi)} \right)^{1/2} \exp \left( -\beta b^2 \right) - \overline{a^2_p} \]

\[ = \sum_k \mathcal{P}_{\psi_k}(\alpha_k) \left( \frac{1}{2\beta} + \alpha^2_{kp} \right) - \overline{a^2_p} \]

\[ = \frac{1}{2\beta} + \langle \hat{A}^2 \rangle_{\psi_t} - \langle \hat{A}^2 \rangle_{\psi_t} \rightarrow_{\beta \to 0} \frac{1}{2\beta}. \]

(21)
Last, for the covariances, without the need of shifting the integration variables one can write
\[
\overline{aa_{ip}a_{iq}} \xrightarrow{\beta \to 0} \int \overline{da_i a_{ip} a_{iq}} \mathcal{P}(a_i) - \overline{a_{ip} a_{iq}} = \int \overline{da a_{ip} a_{iq}} \mathcal{P}(a) - \overline{a_{ip} a_{iq}}
\]
\[
= \sum_k \mathcal{P}_\psi(\alpha_k) \alpha_{kp} \alpha_{kq} - \overline{\alpha_{p} \alpha_{q}} = \langle \hat{A}_p \hat{A}_q \rangle_\psi - \langle \hat{A}_p \rangle_\psi \langle \hat{A}_q \rangle_\psi.
\] (22)

Let us define the set of stochastic variables
\[
dB = \sqrt{\frac{2 \beta}{\mu}} \sum_{i=1}^{n} (a_i - \langle \hat{A} \rangle_\psi).
\] (23)

Taking the limit \( \beta \to 0 \) according to the prescription (10), \( \mu \) and \( n \) go to infinity in the same way, the conditions of the central limit theorem are satisfied so that the variables \( dB \) are Gaussian with the properties
\[
\overline{dB} = 0,
\]
\[
\overline{dB^2} = \frac{2 \beta}{\mu} n = n = n\tau = dt,
\]
\[
\overline{dB_p dB_q} = \frac{2 \beta}{\mu} n (\langle \hat{A}_p \hat{A}_q \rangle_\psi - \langle \hat{A}_p \rangle_\psi \langle \hat{A}_q \rangle_\psi) \xrightarrow{\beta \to 0} 0.
\] (24)

Inserting the definition (23) into equation (12) one gets
\[
|\chi_{t+dt} = F \exp \left\{ -\gamma (\hat{A}^2 - 2 \hat{A} \cdot \langle \hat{A} \rangle_\psi) dt + \sqrt{\gamma} \hat{A} \cdot dB \right\} |\psi_t \rangle.
\] (25)

By expanding the exponential and using the rules of Itô calculus one eventually obtains
\[
|\chi_{t+dt} = F \left[ 1 - \frac{1}{2} \gamma (\hat{A}^2 - 4 \hat{A} \cdot \langle \hat{A} \rangle_\psi) dt + \sqrt{\gamma} \hat{A} \cdot dB \right] |\psi_t \rangle,
\]
\[
||\chi_{t+dt}||^{-1} = F^{-1} \left[ 1 - \frac{1}{2} \gamma (\hat{A}^2 dt - \sqrt{\gamma} \langle \hat{A} \rangle_\psi \cdot dB \right],
\]
so that
\[
d|\psi_t \rangle = |\psi_{t+dt} \rangle - |\psi_t \rangle = ||\phi_{t+dt}||^{-1} |\phi_{t+dt} \rangle - |\psi_t \rangle
\]
\[
= \left[ \sqrt{\gamma} (\hat{A} - \langle \hat{A} \rangle_\psi) \cdot dB - \frac{1}{2} \gamma (\hat{A} - \langle \hat{A} \rangle_\psi)^2 dt \right] |\psi_t \rangle.
\] (26)

By assuming that both the Schrödinger evolution and the stochastic process are there, and taking into account that the two terms in the stochastic differential equation (26) are of the order \( \sqrt{dt} \) and \( dt \), respectively, one can write on the whole
\[
d|\psi_t \rangle = \left[ -\frac{i}{\hbar} \hat{H} dt + \sqrt{\gamma} (\hat{A} - \langle \hat{A} \rangle_\psi) \cdot dB - \frac{1}{2} \gamma (\hat{A} - \langle \hat{A} \rangle_\psi)^2 dt \right] |\psi_t \rangle.
\] (27)

This is the form of the evolution equation normally assumed for continuous stochastic processes in Hilbert space, corresponding to eq. (7) with the addition of the term describing the Schrödinger dynamics.

The above argument is worked out with reference to a case in which the quantity label runs over a finite numerable set. There are relevant situations in which the quantity label runs over a measurable continuous set. Two such cases will be examined in the following section, together with all the necessary changes.
3 Three relevant implementations

In the present Section we present three physically most relevant implementations of discontinuous stochastic processes and the corresponding continuous evolution equations.

As discussed in Section 2, both the discontinuous and the equivalent continuous processes are characterized by the choice of the sharpened quantities \( \hat{A}_p, p \in \{1, \ldots, K\} \).

The discontinuous process is further specified by a sharpening frequency \( \mu \) and a sharpening accuracy \( \beta \). The probability distribution of the hitting centres \( a_{p,i} \) for the \( i \)-th hitting is assumed to be

\[
\mathcal{P}(\psi_t | a_i) = \left( \frac{\beta}{\pi} \right)^{K/2} \exp \left[ -\beta \sum_{p=1}^{K} (\hat{A}_p - a_{p,i})^2 \right] |\psi_t\rangle.
\] (28)

The continuous process is ruled by equation (27) specified by the strength parameter \( \gamma \) and by the properties of the Gaussian random variables

\[
dB = 0,
\]
\[
dB^2_p = dt,
\]
\[
dB_p dB_q = 0.
\] (29)

For equivalence of the two processes, the parameter \( \gamma \) must be given by \( \gamma = \beta \mu / 2 \).

Distinguishable particles

For \( N \) distinguishable particles the sharpened quantities are the three-dimensional positions \( \hat{x}_l, l \in \{1, \ldots, N\} \).

The discontinuous process (5) is defined by the localization frequency \( \lambda_l \) for particle \( l \) and by the localization accuracy \( \alpha \). The probability distribution of localization centres \( \mathbf{x}_l,i \) for the \( i \)-th hitting on particle \( l \) is

\[
\mathcal{P}(\psi_t | \mathbf{x}_{l,i}) = \left( \frac{\beta}{\pi} \right)^{3/2} \exp \left[ -\alpha (\hat{x}_l - \mathbf{x}_{l,i})^2 \right] |\psi_t\rangle.
\] (30)

The corresponding continuous process is ruled by the stochastic differential equation

\[
d|\psi_t\rangle = \left[ -\frac{i}{\hbar} \hat{H} dt + \sum_{l=1}^{N} \sqrt{\gamma_l} (\mathbf{x}_l - \langle \hat{x}_l | \psi_t \rangle) \cdot dB_l - \frac{1}{2} \sum_{l=1}^{N} \gamma_l (\mathbf{x}_l - \langle \hat{x}_l | \psi_t \rangle)^2 dt \right] |\psi_t\rangle,
\] (31)

where the stochastic variables \( dB_l \) are \( N \) independent three-dimensional Gaussian variables whose statistical properties are described in Eqs. (29). For equivalence, the strength parameters \( \gamma_l \) must be given by \( \gamma_l = \alpha \lambda_l / 2 \).

Identical particles

In this case the localization effect is obtained by sharpening the particle density \( \hat{N}(\mathbf{x}) \) around each point \( \mathbf{x} \) in physical space. The particle densities can be defined in the second quantization language as

\[
\hat{N}(\mathbf{x}) = \left( \frac{\alpha}{2\pi} \right)^{3/2} \sum_s \int d\mathbf{x}' \exp \left( -\frac{1}{2} \alpha (\mathbf{x}' - \mathbf{x})^2 \right) a_{\mathbf{x},s} a_{\mathbf{x}',s}^\dagger,
\] (32)

\( a_{\mathbf{x},s}^\dagger \) and \( a_{\mathbf{x},s} \) being the creation and annihilation operators of a particle at point \( \mathbf{x} \) with spin component \( s \). The smooth volume used to define the particle density has linear dimensions of the order of \( 1/\sqrt{\alpha} \).

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1 J.S. Bell, private communication, 1987.
For the discontinuous process the sharpening frequency and the sharpening accuracy of the density $\hat{N}(x)$ are $\mu$ and $\beta$, respectively. It is to be noted that, because of the nature of the domain of the quantity label $x$, the “centre” of the sharpening for the $i$-th hitting is now a number density profile $n_i(x)$ and its probability density (in the functional space of number density profiles) is given by

$$P[n_i] = |C|^2 \langle \psi_t \rangle \exp \left[ -\beta \int dx \left( \hat{N}(x) - n_i(x) \right)^2 \right] |\psi_t\rangle. \quad (33)$$

The coefficient $C$ is given by the normalization condition

$$\int P[n] P[n_i] = 1. \quad (34)$$

The corresponding continuous process is ruled by the equation (35)

$$d|\psi_t\rangle = \left[ -\frac{i}{\hbar}\hat{H}dt + \sqrt{\gamma} \int dx \left( \hat{N}(x) - \langle \hat{N}(x) \rangle_{\psi_t} \right) dB(x) - \frac{1}{2\gamma} \int dx \left( \hat{N}(x) - \langle \hat{N}(x) \rangle_{\psi_t} \right)^2 dt \right] |\psi_t\rangle, \quad (35)$$

where the Gaussian random variables $dB(x)$ have the properties

$$dB(x) = 0, \quad dB(x)dB(x') = \delta(x - x')dt. \quad (36)$$

For equivalence, the strength parameter $\gamma$ must be given by $\gamma = \beta \mu/2$.

**Several kinds of identical particles**

In the case of several kinds of identical particles, the most established formulation sharpens the mass density around each point in physical space by using a universal stochastic field $dB(x)$. The particle density operators $\hat{N}(x)$ are then replaced, both in the discontinuous and the continuous processes, by the mass densities $\hat{M}(x)$ where

$$\hat{M}(x) = \sum_k m_k \hat{N}_k(x), \quad (37)$$

$m_k$ being the mass of the particle of kind $k$.

The continuous process (35) is ruled by Equation (35), with $\hat{N}(x)$ replaced by $\hat{M}(x)$.

### 4 Final considerations

Some final comments are in order. From the discussion of Sections 2 and 3 it is apparent that the discontinuous processes bear the same generality as the continuous ones as far as their applicability to physical systems is concerned. In particular, contrary to what has been sometimes stated in the literature, discontinuous processes can be formulated for systems of identical particles or of several kinds of identical particles. To deal with such physical systems resorting to continuous formulations is not necessary.

As explicitly shown in Section 2 discontinuous processes give rise, in a proper infinite frequency limit, to corresponding continuous ones, thus showing the physical equivalence of the two formulations for sufficiently high hitting frequencies. Stated differently, for any continuous process there is a discrete process which turns out to induce a dynamics as near as wanted to the corresponding continuous one and which becomes identical to it when the infinite frequency limit is taken.

One could ask what really means “sufficiently high” frequencies. From the purely formal point of view, the equivalence of discontinuous and continuous processes requires that, in the time interval $[t, t + dt]$ one has a large enough number of hittings, so that
the central limit theorem can be applied. Having said that, the effectiveness of the 
hitting process depends on the product $\beta \mu$, so that for fixed effectiveness one can still 
maintain a finite frequency, provided that a sufficient number of hittings occur on the 
time scale relevant to the solution of the measurement process. For the sake of simplicity, 
in Section 2 it has been assumed that the hittings occur at evenly spaced times; in order 
to preserve time translation invariance (5) one should use random times with a certain 
mean frequency. Then the mean frequency has to be sufficiently large to guarantee that 
reduction takes place in the time interval of interest. There remains, however, a small 
probability that no reduction takes place. The same thing happens in the continuous 
process that leads certainly to a common eigenstate of the considered quantities only 
when $t \to \infty$.

As a last comment we stress that the continuous processes, on one hand, are undoubt-
edly mathematically more elegant, while, on the other hand, the discontinuous processes 
show immediately the physical effect of reduction, so that, taken for granted the infinite 
frequency limit, they show the reduction properties of the continuous ones too.

What really matters is the choice of the quantities induced to have a sharp distribution. 
We think it is important to stress the role of positions as the quantities that allow the 
strengthening of the process in going from microscopic to macroscopic degrees of freedom. 
In the case of distinguishable particles the variables undergoing the process are directly 
the positions of individual particles. In the case of identical particles or several kinds of 
identical particles the variables undergoing the process are the number or mass densities 
around the running point in physical space, that play the role of positions respecting 
the identity of particles. The final effect is again to make definite the position in space 
of macroscopic objects, thus providing a viable and conceptually simple solution to the 
measurement problem.

References

[1] A. Barchielli, L. Lanz, and G. M. Prosperi. A model for macroscopic description and 
continuous observations in quantum mechanics. *Nuovo Cimento*, 72B:79, 1982.

[2] L. Diosi. Continuous quantum measurement and Itô formalism. *Phys. Lett.*, A129:419–
423, 1988.

[3] G. C. Ghirardi, R. Grassi, and F. Benatti. Describing the macroscopic world - Closing 
the circle within the dynamical reduction program. *Found. Phys.*, 25:5–38, 1995.

[4] G. C. Ghirardi, P. M. Pearle, and A. Rimini. Markov processes in Hilbert space 
and continuous spontaneous localization of systems of identical particles. *Phys. Rev.*, 
A42:78–79, 1990.

[5] G. C. Ghirardi, A. Rimini, and T. Weber. Unified dynamics for microscopic and 
macroscopic systems. *Phys. Rev.*, D34:470, 1986.

[6] O. Nicrosini and A. Rimini. On the relationship between continuous and discontinuous 
stochastic processes in Hilbert space. *Found. Phys.*, 20:1317–1327, 1990.

[7] P. M. Pearle. Combining stochastic dynamical state vector reduction with spontaneous 
localization. *Phys. Rev.*, A39:2277–2289, 1989.