A singular stochastic differential equation driven
by fractional Brownian motion∗

Yaozhong Hu†, David Nualart‡, Xiaoming Song
Department of Mathematics
University of Kansas
Lawrence, Kansas, 66045 USA

Abstract
In this paper we study a singular stochastic differential equation driven by an additive fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.
Under some assumptions on the drift, we show that there is a unique solution, which has moments of all orders. We also apply the techniques of Malliavin calculus to prove that the solution has an absolutely continuous law at any time $t > 0$.

1 Introduction

The aim of this paper is to study a stochastic differential equation, driven by an additive fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$, assuming that the drift $f(t,x)$ has a singularity at $x = 0$ of the form $x^{-\alpha}$, where $\alpha > \frac{1}{1-H} - 1$.

The study of this type of equations is partially motivated by the equation satisfied by the $d$-dimensional fractional Bessel process $R_t = |B^H_t|$, $d \geq 2$ (see Guerra and Nualart [7], and Hu and Nualart [8]):

$$R_t = Y_t + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds,$$

where the process $Y_t$ is equal to a divergence integral, $Y_t = \int_0^t \sum_{i=1}^d \frac{B^H_{t,i}}{R_s} \delta B^{H,i}_s$.

The process $Y$ is not a one-dimensional fractional Brownian motion (see Eisenbaum and Tudor [5] and Hu and Nualart [8] for some results in this direction), although it shares with the fBm similar properties of scaling and $\frac{1}{H}$-variation.

Notice that here the initial condition is zero.

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We are considering the case where the initial condition \( x_0 \) is strictly positive. Using arguments based on fractional calculus inspired by the estimates obtained by Hu and Nualart in [9], we will show that there exist a unique global solution which has moments of all orders, and even negative moments, in the particular case \( f(t, x) = Kx^{-1} \), if \( t \) is small enough. We will also show that the solution has an absolutely continuous law with respect to the Lebesgue measure, using the techniques of Malliavin calculus for the fractional Brownian motion. As an application we obtain the existence of a unique solution with moments of all orders for a fractional version of the CIR model in mathematical finance ([3]), which is a singular stochastic differential equation driven by fractional Brownian motion with the diffusion coefficient being \( \sqrt{x} \).

The paper is organized as follows. In the first section we will consider the case of a deterministic differential equation driven by a H"older continuous function, and with singular drift. The case of the fractional Brownian motion is developed in Section 3.

2 Singular equations driven by rough paths

For any \( s \leq t \), \( C([s,t]) \) denotes the Banach space of continuous functions equipped with the supremum norm, and \( C^\beta([s,t]) \) denotes the space of H"older continuous functions of order \( \beta \) on \([s,t] \). For any \( x \in C([s,t]) \) we put
\[
\|x\|_{s,t,\infty} = \sup\{|x(r)|, s \leq r \leq t\},
\]
and if \( x \in C^\beta([s,t]) \) we put
\[
\|x\|_{s,t,\beta} = \sup\{|x(u) - x(v)| / |u - v|^{\beta}, s \leq u,v \leq t\}.
\]

Fix \( \beta \in (1/2, 1) \). Suppose that \( \varphi : \mathbb{R}_+ \to \mathbb{R} \) is a function such that \( \varphi(0) = 0 \), and \( \varphi \in C^\beta([0,T]) \) for all \( T > 0 \). Consider the following deterministic differential equation driven by the rough path \( \varphi \)
\[
x_t = x_0 + \int_0^t f(s, x_s)ds + \varphi(t), \quad (2.1)
\]
where \( x_0 > 0 \) is a constant. We are going to impose the following assumptions on the coefficient \( f \):

(i) \( f : [0, \infty) \times (0, \infty) \to [0, \infty) \) is a nonnegative, continuous function which has a continuous partial derivative with respect to \( x \) such that \( \partial_x f(t, x) \leq 0 \) for all \( t > 0, x > 0 \).

(ii) There exists \( x_1 > 0 \) and \( \alpha \geq 1/\beta \) such that \( f(t, x) \geq g(t)x^{-\alpha} \), for all \( t \geq 0 \) and \( x \in (0, x_1) \), where \( g(t) \) is a nonnegative continuous function with \( g(t) > 0 \) for all \( t > 0 \).
(iii) \( f(t, x) \leq h(t) \left( 1 + \frac{1}{x} \right) \) for all \( t \geq 0 \) and \( x > 0 \), where \( h(t) \) is a certain nonnegative locally bounded function.

**Theorem 2.1** Under the assumptions (i)-(ii), there exists a unique solution \( x_t \) to equation (2.1) such that \( x_t > 0 \) on \([0, \infty)\).

**Proof** It is easy to see that there exists a continuous local solution \( x_t \) to equation (2.1) on some interval \([0, T)\), where \( T = \inf \{ t > 0 : x_t = 0 \} \). Then it suffices to show that \( T = \infty \). Suppose that \( T < \infty \). Then, the solution \( x_t \rightarrow 0 \) as \( t \uparrow T \). Since \( \varphi \in C^\beta(\[0, T\]) \), there exists a constant \( C > 0 \) such that \( |\varphi(t) - \varphi(s)| \leq C|t - s|^{\beta} \), for all \( s, t \in [0, T] \). Since \( x_t \) satisfies the equation (2.1), for all \( t \in [0, T] \) we have

\[
0 = x_T = x_t + \int_t^T f(s, x_s)ds + \varphi(T) - \varphi(t).
\]

Since \( f(s, x_s) \) is positive, for all \( t \in [0, T] \) we have

\[
x_t \leq x_t + \int_t^T f(s, x_s)ds = \varphi(t) - \varphi(T) \leq C(T - t)^\beta.
\]

From the assumption (ii), there exist \( t_0 \in (0, T) \) and a constant \( K > 0 \), such that \( g(t) \geq K \) and \( x_s \in (0, x_1) \) for all \( t \in [t_0, T) \). Then, for all \( t \in [t_0, T) \) we have

\[
f(t, x_t) \geq \frac{g(t)}{x_t^\alpha} \geq \frac{K}{x_t^\alpha} \geq \frac{K}{C^\alpha (T - t)^{\alpha\beta}}.
\]

Consequently, for all \( t \in [t_0, T) \) we obtain

\[
\frac{K(T - t)^{1 - \alpha\beta}}{C^\alpha (1 - \alpha\beta)} = \int_t^T \frac{K}{C^\alpha (T - s)^{\alpha\beta}}ds \leq \int_t^T f(s, x_s)ds \leq C(T - t)^\beta,
\]

which is a contradiction because \( 1 - \alpha\beta - \beta < 0 \) and \( t \) can be arbitrarily close to \( T \). Therefore, \( T = \infty \). This proves the existence of the solution for all \( t \).

Now we show the uniqueness. If \( x_{1,t} \) and \( x_{2,t} \) are two positive solutions to equation (2.1), then

\[
x_{1,t} - x_{2,t} = \int_0^t [f(s, x_{1,s}) - f(s, x_{2,s})]ds.
\]

Because \( \partial_x f(t, x) \leq 0 \) for all \( t > 0, x > 0 \), we deduce

\[
(x_{1,t} - x_{2,t})^2 = 2 \int_0^t (x_{1,s} - x_{2,s})[f(s, x_{1,s}) - f(s, x_{2,s})]ds \leq 0.
\]

So \( x_{1,t} = x_{2,t} \).

Thus we conclude that there exists a unique solution \( x_t \) to the equation (2.1) such that \( x_t > 0 \) on \([0, \infty)\).
Theorem 2.3  Let the assumptions (i)-(iii) be satisfied. If equation (2.1), then for any \( \gamma > 0 \), formula (see Zähle [17]) yields

\[
\phi(x) = \text{constant}.
\]

From Assumption (iii), we have

\[
\phi(x) = \text{constant}.
\]

The second integral in (2.3) is a Riemann-Stieltjes integral (see Young [16]).

Remark 2.2  From the continuity of \( x_t \) and \( f(t,x) \) and the Hölder continuity of \( \phi(t) \), we obtain that for any \( T > 0 \), \( x \in C^2([0,T]) \).

The next result provides an estimate on the supremum norm of the solution in terms of the Hölder norm of the driving function \( \phi \).

Theorem 2.3  Let the assumptions (i)-(iii) be satisfied. If \( x_t \) is the solution to equation (2.1), then for any \( \gamma > 2 \), and for any \( T > 0 \),

\[
\|x\|_{0,T,\infty} \leq C_{1,\gamma,\beta,T} \left( |x_0| + 1 \right) \exp \left\{ C_{2,\gamma,\beta,T} \left( 1 + \|\phi\|_{0,T,\beta} \right) \right\},
\]

where \( C_{1,\gamma,\beta,T} \) and \( C_{2,\gamma,\beta,T} \) are constants depending on \( \beta, \gamma, \|h\|_{0,T,\infty} \) and \( T \).

Proof  Fix a time interval \([0,T]\). Let \( y_t = x_t^\gamma \). Then the chain rule applied to \( x_t^\gamma \) yields

\[
y_t = x_0^\gamma + \gamma \int_0^t f(s,y_s) y_s^{\frac{1}{\gamma} - \frac{1}{\beta}} ds + \gamma \int_0^t y_s^{\frac{1}{\beta} - \frac{1}{\gamma}} d\phi(s).
\]

The second integral in (2.3) is a Riemann-Stieltjes integral (see Young [16]).

From Assumption (iii), we have

\[
\|y_t - y_s\| = \gamma \left| \int_s^t f(u,y_u) y_u^{\frac{1}{\beta} - \frac{1}{\gamma}} du + \int_s^t y_u^{\frac{1}{\beta} - \frac{1}{\gamma}} d\phi(u) \right|
\]

\[
\quad \leq K_T \gamma \int_s^t \left[ \frac{1}{y_u^{\frac{1}{\beta} - \frac{1}{\gamma}}} + y_u^{\frac{1}{\gamma} - \frac{1}{\beta}} \right] du + \gamma \int_s^t y_u^{\frac{1}{\beta} - \frac{1}{\gamma}} d\phi(u),
\]

where \( K_T = \sup_{t \in [0,T]} h(t) \). Since \( \gamma > 2 \), we have

\[
\int_s^t y_u^{\frac{1}{\gamma} - \frac{1}{\beta}} du \leq \left( \|y\|_{s,T,\infty} + \|y\|_{1,T,\infty} \right) (t - s).
\]

Since \( \alpha > \frac{1}{\beta} - 1 \), we have \( \alpha > \alpha \beta > 1 - \beta \). Thus \( 1 - \alpha < \beta \). From Remark 1.1, we know that \( y_t \in C^2([0,T]) \), for any \( T > 0 \). A fractional integration by parts formula (see Zähle [17]) yields

\[
\int_s^t y_u^{\frac{1}{\gamma} - \frac{1}{\beta}} d\phi(u) = (-1)^{-\alpha} \int_s^t D_s^\alpha y_u^{\frac{1}{\beta} - \frac{1}{\gamma}} D_t^\alpha \phi_t(t) du,
\]

where \( \phi_{t-}(u) = \phi(u) - \phi(t) \), and \( D_{s+}^\alpha \) and \( D_{t-}^\alpha \) denote the left and right-sided fractional derivatives of orders \( \alpha \) and \( 1 - \alpha \), respectively (see [13]), defined by

\[
D_{s+}^\alpha y_u^{\frac{1}{\gamma} - \frac{1}{\beta}} = \frac{1}{\Gamma(1-\alpha)} \left( \frac{1}{y_u^{\frac{1}{\beta} - \frac{1}{\gamma}}} + \alpha \int_s^u \frac{1}{y_r^{\frac{1}{\beta} - \frac{1}{\gamma}}} - \frac{1}{y_r^{\frac{1}{\beta} - \frac{1}{\gamma}}} \frac{1}{(u - r)^{\alpha + 1}} dr \right),
\]

and

\[
D_{t-}^\alpha \phi_t(t) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left( \frac{\phi(u) - \phi(t)}{(t - u)^{1-\alpha}} + (1 - \alpha) \int_u^t \frac{\phi(u) - \phi(r)}{(r - u)^{2-\alpha}} dr \right).
\]
From (2.7), and using the H"older continuity of \( y \) we obtain
\[
|D_x^\alpha y | \leq C \left( \|y\|_{s,t,\infty}^{1-\frac{x}{\gamma}} \right)^{1-\frac{\alpha}{\gamma}} + \int_s^t \frac{|y_u - y_s|^{1-\frac{\alpha}{\gamma}}}{(u-r)^{\alpha+1}} \, dr
\]
\[
\leq C \left( \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} \right)^{1-\frac{\alpha}{\gamma}} + \int_s^t \frac{|y_u - y_s|^{1-\frac{\alpha}{\gamma}}}{(u-r)^{\alpha+1}} \, dr
\]
\[
\leq C \left( \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} \right)^{1-\frac{\alpha}{\gamma}} + \|y\|_{s,t,\beta} \int_s^t (u-r)^{\beta(1-\frac{\alpha}{\gamma})-\alpha-1} \, dr
\]
where and in what follows, \( C \) denotes a generic constant depending on \( \alpha, \beta \) and \( T \). On the other hand, from (2.8) we have
\[
|\int_s^t y_u^{1-\frac{\alpha}{\gamma}} \, d\varphi(u)| \leq C\|\varphi\|_{0,T,\beta} (t-u)^{\alpha+\beta-1}. \tag{2.10}
\]
Substituting (2.9) and (2.10) into (2.6) yields
\[
\left| \int_s^t y_u^{1-\frac{\alpha}{\gamma}} \, d\varphi(u) \right| \leq C \int_s^t \left( \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} + \|y\|_{s,t,\beta}^{1-\frac{\alpha}{\gamma}} (u-s)^{\beta(1-\frac{\alpha}{\gamma})-\alpha} \right)
\times \|\varphi\|_{0,T,\beta} (t-u)^{\alpha+\beta-1} \, du
\]
\[
\leq C \|\varphi\|_{0,T,\beta}
\times \left( \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} (t-s)^{\beta} + \|y\|_{s,t,\beta}^{1-\frac{\alpha}{\gamma}} (t-s)^{\beta(2-\frac{\alpha}{\gamma})} \right). \tag{2.11}
\]
Substituting (2.11) and (2.5) into (2.4) we obtain
\[
|y_t - y_s| \leq K_T \gamma \left[ \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} + \|y\|_{s,t,\beta}^{1-\frac{\alpha}{\gamma}} \right] (t-s) + C\gamma \|\varphi\|_{0,T,\beta}
\times \left( \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} (t-s)^{\beta} + \|y\|_{s,t,\beta}^{1-\frac{\alpha}{\gamma}} (t-s)^{\beta(2-\frac{\alpha}{\gamma})} \right).
\]
Consequently, using the estimate \( x^{1-\frac{\alpha}{\gamma}} \leq 1 + x \) for all \( x > 0 \), we obtain
\[
\|y\|_{s,t,\beta} \leq K_T \gamma \left[ \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} + \|y\|_{s,t,\beta}^{1-\frac{\alpha}{\gamma}} \right] (t-s)^{1-\beta} + C\gamma \|\varphi\|_{0,T,\beta}
\times \left( \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} + (1 + \|y\|_{s,t,\beta}) (t-s)^{\beta(1-\frac{\alpha}{\gamma})} \right),
\]
which implies
\[
\left[ 1 - C\gamma \|\varphi\|_{0,T,\beta} (t-s)^{\beta(1-\frac{\alpha}{\gamma})} \right] \|y\|_{s,t,\beta} \leq K_T \gamma \left[ \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} + \|y\|_{s,t,\beta}^{1-\frac{\alpha}{\gamma}} \right]
\times (t-s)^{1-\beta} + C\gamma \|\varphi\|_{0,T,\beta} \left( \|y\|_{s,t,\infty}^{1-\frac{\alpha}{\gamma}} + (t-s)^{\beta(1-\frac{\alpha}{\gamma})} \right).
\]

Suppose that $\Delta$ satisfies
\[
\Delta \leq \left( \frac{1}{2C\gamma \|\varphi\|_{0,T,\beta}} \right)^{\frac{1}{\gamma - 1}}. \tag{2.12}
\]
Then for all $s, t \in [0, T]$, $s \leq t$, such that $t - s \leq \Delta$, we have
\[
\|y\|_{s,t,\beta} \leq 2K_T\gamma \left( \|y\|_{s,t,\infty}^{1 - \frac{1}{\gamma}} + \|y\|_{s,t,\infty}^{1 - \frac{1}{2\gamma}} \right) (t - s)^{1 - \beta} + 2C\gamma\|\varphi\|_{0,T,\beta}\|y\|_{s,t,\infty}^{1 - \frac{1}{2\gamma}} + 1,
\]
and this implies
\[
\|y\|_{s,t,\infty} \leq \|y\|_{s,t,\beta}(t - s)^\beta
\leq |y_s| + 2K_T\gamma \left( \|y\|_{s,t,\infty}^{1 - \frac{1}{\gamma}} + \|y\|_{s,t,\infty}^{1 - \frac{1}{2\gamma}} \right) (t - s)
+ 2C\gamma\|\varphi\|_{0,T,\beta}\|y\|_{s,t,\infty}^{1 - \frac{1}{2\gamma}} (t - s)^\beta + (t - s)^\beta.
\]
Using again the inequality $x^\alpha \leq 1 + x$ for all $x > 0$ and $\alpha \in (0, 1)$, we have
\[
\|y\|_{s,t,\infty} \leq |y_s| + 4K_T\gamma (1 + \|y\|_{s,t,\infty}) (t - s)
+ 2C\gamma\|\varphi\|_{0,T,\beta} (1 + \|y\|_{s,t,\infty}) (t - s)^\beta + (t - s)^\beta,
\]
which can be written as
\[
\|y\|_{s,t,\infty} (1 - 2C\gamma\|\varphi\|_{0,T,\beta}(t - s)^\beta - 4K_T\gamma(t - s))
\leq |y_s| + 4K_T\gamma(t - s) + 2(t - s)^\beta. \tag{2.13}
\]
Now we choose $\Delta$ such that
\[
\Delta = \left( \frac{1}{2C\gamma\|\varphi\|_{0,T,\beta}} \right)^{\frac{1}{\gamma - 1}} \land \left( \frac{1}{16K_T\gamma} \right) \land \left( \frac{1}{8C\gamma\|\varphi\|_{0,T,\beta}} \right)^{\frac{1}{\beta}}. \tag{2.14}
\]
Then, for all $s, t \in [0, T]$, $s < t$, such that $t - s \leq \Delta$, the inequality (2.13) implies
\[
\|y\|_{s,t,\infty} \leq 2|y_s| + C_{\gamma,\beta,T}, \tag{2.15}
\]
where $C_{\gamma,\beta,T} = 8K_T\gamma T + 4T^\beta$. Take $n = \left[ \frac{T}{\Delta} \right] + 1$ (where $[a]$ denotes the largest integer bounded by $a$). Divide the interval $[0, T]$ into $n$ subintervals. Applying the inequality (2.15) for $s = 0$ and $t = \Delta$, we have for all $t \in [0, \Delta]$,
\[
\|y\|_{0,t,\infty} \leq 2|y_0| + C_{\gamma,\beta,T}. \tag{2.16}
\]
Applying the inequality (2.16) on the intervals $[\Delta, 2\Delta], \ldots, [(n - 2)\Delta, (n - 1)\Delta], [(n - 1)\Delta, T]$ recursively, we obtain
\[
\|y\|_{0,T,\infty} \leq 2^n|y_0| + 2^{n-1}C_{\gamma,\beta,T} + \cdots + C_{\gamma,\beta,T}
\leq 2^\left( \left[ \frac{T}{\Delta} \right] + 1 \right) (|y_0| + C_{\gamma,\beta,T})
\leq 2^T (2C\gamma\|\varphi\|_{0,T,\beta})^{\frac{1}{\gamma - 1}} \cdot \gamma T \left( 16K_T\gamma \right)^{\frac{1}{\beta}} (8C\gamma\|\varphi\|_{0,T,\beta})^{\frac{1}{\beta}} + 1 (|y_0| + C_{\gamma,\beta,T}).
\]
Therefore, we obtain
\[ \|x\|_{0,T,\infty} \leq C_{1,\gamma,\beta,T}(|x_0| + 1) \exp \left\{ C_{2,\gamma,\beta,T} \left( 1 + \|\phi\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}} \right) \right\}, \]
which concludes the proof of the theorem. ■

3 Singular equations driven by fBm

Let \((B^H_t, t \geq 0)\) be a fractional Brownian motion with Hurst parameter \(H \in (1/2, 1)\), defined in a complete probability space \((\Omega, \mathcal{F}, P)\). Namely, \((B^H_t, t \geq 0)\) is a mean zero Gaussian process with covariance
\[ \mathbb{E}(B^H_t B^H_s) = R_H(s, t) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right). \quad (3.1) \]

We are interested in the following singular stochastic differential equation
\[ X_t = x_0 + \int_0^t f(s, X_s) ds + B^H_t, \quad (3.2) \]
where \(x_0 > 0\), and the function \(f(s, x)\) has a singularity at \(x = 0\) and satisfies the assumptions (i) to (iii). As an immediate consequence of Theorem 2.3 we have the following result.

**Theorem 3.1** Under the assumptions (i)-(iii), there is a unique pathwise solution \(X = (X_t, t \geq 0)\) to Equation (3.2), such that \(X_t > 0\) almost surely on \([0, \infty)\) and for any \(T > 0\), \(\|X\|_{0,T,\infty} \in L^p(\Omega)\), for all \(p > 0\).

**Proof** Fix \(\beta \in \left(\frac{1}{2}, H\right)\) and \(T > 0\). Applying Theorem 2.3 we obtain that there is a unique pathwise solution \(X = (X_t, t \geq 0)\) to Equation (3.2), such that \(X_t > 0\) almost surely on \([0, \infty)\) and
\[ \|X\|_{0,T,\infty} \leq C_{1,\gamma,\beta,T}(|x_0| + 1) \exp \left\{ C_{2,\gamma,\beta,T} \left( 1 + \|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}} \right) \right\}. \quad (3.3) \]

If we choose \(\gamma\) such that \(\gamma > \frac{2\beta}{2\beta - 1}\), then \(\frac{\gamma}{\beta(\gamma-1)} < 2\), and by Fernique’s theorem (see [9]), we obtain
\[ \mathbb{E}(e^{C\|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}}}) < \infty, \quad (3.4) \]
for all \(C > 0\), which implies that \(\mathbb{E}(\|X\|_{0,T,\infty}^p) < \infty\) for all \(p > 1\). ■

Theorem 3.1 implies the existence of a unique solution to the following stochastic differential equation with non Lipschitz diffusion coefficient:
\[ Y_t = y_0 + \int_0^t f(s, Y_s) ds + \int_0^t \sqrt{Y_s} dB^H_s, \quad (3.5) \]
where \(y_0\) is a positive constant and \(f\) is a nonnegative continuous function satisfying the following conditions:
(a) There exists $x_1 > 0$ such that $f(t,x) \geq g(t)$ for all $t > 0$ and $x \in (0,x_1)$, where $g$ is a continuous function such that $g(t) > 0$ if $t > 0$.

(b) $f(t,x) \geq x \partial_x f(t,x)$ for all $t > 0$ and $x > 0$.

(c) $f(t,x) \leq h(t)(x+1)$ for all $t \geq 0$ and $x > 0$, where $h$ is a nonnegative locally bounded function.

The term $\sqrt{Y_s}$ appears in a fractional version of the CIR process in financial mathematics (see [3]) and cannot be treated directly by the approaches in Lyons [10], Nualart and Răşcanu [12], since function $g(x) = \sqrt{x}$ does not satisfy the usual Lipschitz conditions commonly imposed. We make the change of variables $X_t = 2\sqrt{Y_t}$. Then, from the chain rule for the Young integral, it follows that a positive stochastic process $Y = (Y_t, t \geq 0)$ satisfies (3.5) if and only if $X_t$ satisfies the following equation:

$$X_t = 2\sqrt{y_0} + \int_0^t 2f(s,X_s) ds + B_H^t.$$  

(3.6)

Let $f_1(t,x) = 2f(t,x)x^{-1}$. Then $f_1(t,x)$ satisfies all assumptions (i)-(iii), and hence from Theorem 3.1 we know that there exists a unique positive solution $X_t$ to equation (3.6) with all positive moments. So $Y_t = X_t^2/4$ is the unique positive solution to Equation (3.5), and it has finite moments of all orders.

The next result states the scaling property of the solution to Equation (3.2), when the coefficient $f(s,x)$ satisfies some homogeneity condition on the variable $x$.

**Proposition 3.2 (Scaling Property)** We denote by $Eq(x_0,f)$ the equation (3.2). Suppose that $x_0 > 0$, and $f(t,x)$ satisfies assumptions (i)-(iii), and $f(t,x)$ is homogeneous, that is, $f(st,xy) = s^m y^n f(t,x)$ for some constants $m, n$. Then, the process $(a^H X_{a,t}, t \geq 0)$ has the same law as the solution to the Equation $Eq(a^H x_0, a^H - nH - m^{-1} f)$.

**Proof** For each $a > 0$, we know that $\{a^{-H} B_{at}^H, t \geq 0\}$ is a fractional Brownian motion. We denote $X_{a,t}$ the solution to the following equation:

$$X_{a,t} = x_0 + \int_0^t f(s,X_{a,s}) ds + a^{-H} B_{at}^H.$$ 

So $(X_t, t \geq 0)$ (the solution to $Eq(x_0, f)$) has the same distribution as $(X_{a,t}, t \geq 0)$. Then

$$a^H X_{a,\cdot} = a^H x_0 + \int_0^\cdot a^H f(s,X_{a,s}) ds + B_t^H$$

$$= a^H x_0 + \int_0^t a^H - nH f(r,a^H X_{a,\cdot}) dr + B_t^H,$$

which implies the result. □

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As an example, we can consider the function \( f(t, x) = s^\gamma x^{-\alpha} \), where \( \alpha > \frac{1}{H} - 1 \), and \( \gamma > 0 \). Then, if \((X_t, t \geq 0)\) is the solution to Equation

\[
X_t = x_0 + \int_0^t s^\gamma X_s^{-\alpha} ds + B^H_t
\]

where \( B^H_t \) is the fractional Brownian motion with Hurst parameter \( H \), then \((a^H X_t, t \geq 0)\) has the same law as the solution to the Equation

\[
X_t = a^H x_0 + a^H - \alpha H - \gamma - 1 \int_0^t s^\gamma X_s^{-\alpha} ds + B^H_t.
\]

### 3.1 Absolute continuity of the law of the solution

In this subsection we will apply the Malliavin calculus to the solution to Equation (3.2) in order to study the absolute continuity of the law of the solution at a fixed time \( t > 0 \). We will first make some preliminaries on the Malliavin calculus for the fractional Brownian motion, and we refer to Decreusefond and Üstünel [4], Nualart [11] and Saussereau and Nualart [14] for a more complete treatment of this topic.

Fix a time interval \([0, T]\). Denote by \( E \) the set of real valued step functions on \([0, T]\) and let \( H \) be the Hilbert space defined as the closure of \( E \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_H \) given in (3.1). We know that

\[
R^H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H} \, dudr = \int_0^{t\wedge s} K^H(t,r)K^H(s,r) \, dr,
\]

where \( K^H(t,s) = c_H s^{\frac{3}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{3}{2}} du \mathbf{1}_{\{s < t\}} \) with \( c_H = \sqrt{\frac{H(2H-1)}{B(2-2H,H-\frac{3}{2})}} \) and \( B \) denotes the Beta function, and \( \alpha_H = H(2H-1) \). In general, for any \( \varphi, \psi \in E \) we have

\[
\langle \varphi, \psi \rangle_H = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u \, dudr.
\]

The mapping \( \mathbf{1}_{[0,t]} \mapsto B^H_t \) can be extended to an isometry between \( H \) and the Gaussian space \( H_1 \) spanned by \( B^H \). We denote this isometry by \( \varphi \mapsto B^H(\varphi) \).

We consider the operator \( K^*_H : E \to L^2(0,T) \) defined by

\[
(K^*_H \varphi)(s) = \int_s^T \varphi(t) \frac{\partial K^H}{\partial t}(t,s) \, dt.
\]

(3.7)

Notice that \( (K^*_H(\mathbf{1}_{[0,t]}))(s) = K^H(t,s) \mathbf{1}_{[0,t]}(s) \). For any \( \varphi, \psi \in E \) we have

\[
\langle \varphi, \psi \rangle_H = \langle K^*_H \varphi, K^*_H \psi \rangle_{L^2(0,T)} = \mathbb{E}(B^H(\varphi)B^H(\psi)),
\]

(3.8)
and $K_H^*$ provides an isometry between the Hilbert space $\mathcal{H}$ and a closed subspace of $L^2([0,T])$. We denote $K_H : L^2([0,T]) \rightarrow \mathcal{H} := K_H(L^2([0,T]))$ the operator defined by $(K_H h)(t) := \int_0^t K_H(t,s)h(s)ds$. The space $\mathcal{H}$ is the fractional version of the Cameron-Martin space. Finally, we denote by $R_H = K_H \circ K_H^* : \mathcal{H} \rightarrow \mathcal{H}$ the operator defined by $R_H \varphi = \int_0^T K_H(\cdot,s)(K_H^* \varphi)(s)ds$. We remark that for any $\varphi \in \mathcal{H}$, $R_H \varphi$ is Hölder continuous of order $H$.

If we assume that $\Omega$ is the canonical probability space $C_0([0,T])$, equipped with the Borel $\sigma$-field and the probability $P$ is the law of the fBm. Then, the injection $R_H : \mathcal{H} \rightarrow \Omega$ embeds densely into $\Omega$ and $(\Omega, \mathcal{H}, P)$ is an abstract Wiener space in the sense of Gross. In the sequel we will make this assumption on the underlying probability space.

Let $S$ be the space of smooth and cylindrical random variables of the form

$$F = f(B^H(\varphi_1), \ldots, B^H(\varphi_n)), \quad (3.9)$$

where $f \in C_0^\infty(\mathbb{R}^n)$ ($f$ and all its partial derivatives are bounded). For a random variable $F$ of the form $(3.9)$ we define its Malliavin derivative as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \ldots, B^H(\varphi_n))\varphi_i.$$ 

We denote by $D^{1,2}$ the Sobolev space defined as the completion of the class $S$, with respect to the norm

$$\|F\|_{1,2} = \left[\mathbb{E}(F^2) + \mathbb{E}\left(\|DF\|^2_{\mathcal{H}}\right)\right]^{1/2}.$$ 

The basic criterion for the existence of densities (see Bouleau and Hirsch [1]), says that if $F \in D^{1,2}$, and $\|DF\|_{\mathcal{H}} > 0$ almost surely, then the law of $F$ has a density with respect to the Lebesgue measure on the real line. Using this criterion we can show the following result.

**Theorem 3.3** Suppose that $f$ satisfies the assumptions (i)-(iii). Let $X_t$ be the solution to Equation $(3.2)$. Then for any $t \geq 0$, $X_t \in D^{1,2}$. Furthermore, for any $t > 0$ the law of $X_t$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

**Proof** Fix a time interval $[0,T]$, and let $\beta \in \left(\frac{1}{2}, H\right)$. We want to compute the directional derivative $\langle DX_t, \varphi \rangle_{\mathcal{H}}$, for some $\varphi \in \mathcal{H}$. The function $h = R_H \varphi$ belongs to $C^\beta([0,T])$ and $h_0 = 0$. Taking into account the embedding given by $R_H : \mathcal{H} \rightarrow \Omega$ mentioned before, we will have

$$\langle DX_t, \varphi \rangle_{\mathcal{H}} = \frac{dX_t^\varphi}{d\epsilon}{\big|}_{\epsilon=0}, \quad (3.10)$$

where $X_t^\varphi$ is the solution to the following equation

$$X_t^\varphi = x_0 + \int_0^t f(s, X_s^\varphi)ds + B_t^H + ch_t, \quad (3.11)$$

\[10\]
\( t \in [0, T] \), where \( \epsilon \in [0, 1] \).

From Theorem 3.1 it follows that there is a constant \( C \) independent of \( \epsilon \) such that
\[
E \left( \sup_{0 \leq t \leq T} |X^\epsilon_t|^p \right) \leq C < \infty ,
\]
for all \( p \geq 1 \). From equations (3.2) and (3.11), we deduce
\[
X^\epsilon_t - X_t = \int_0^t (f(s, X^\epsilon_s) - f(s, X_s))ds + \epsilon h_t. \tag{3.12}
\]
By using Taylor expansion, the equation (3.12) becomes:
\[
X^\epsilon_t - X_t = \int_0^t \Theta_s(X^\epsilon_s - X_s)ds + \epsilon h_t, \tag{3.13}
\]
where \( \Theta_s = \partial_x f(s, X_s + \theta_s(X^\epsilon_s - X_s)) \) for some \( \theta_s \) between 0 and 1. By using (3.7) the solution to equation (3.13) is given by
\[
X^\epsilon_t - X_t = \epsilon \int_0^t \left( \int_s^t \exp \left( \int_r^s \partial_s f(r, X_r)dr \right) \right) \partial_s f(s, u)dsdu.
\]
Using (3.7) and (3.3) we can write
\[
X^\epsilon_t - X_t = \epsilon \int_0^t \left( \int_s^t \exp \left( \int_r^s \partial_s f(r, X_r)dr \right) \right) \partial_s f(s, u)dsdu.
\]
Since \( \partial_x f(t, x) \) is continuous and \( \partial_x f(t, x) \leq 0 \) for all \( t > 0 \) and \( x > 0 \), we have \( \exp \left( \int_0^t \partial_s f(r, X_r)dr \right) \leq 1 \). As a consequence,
\[
\lim_{\epsilon \to 0} \frac{X^\epsilon_t - X_t}{\epsilon} = \alpha H \int_0^t \int_0^s \varphi(s) \exp \left( \int_u^s \partial_x f(r, X_r)dr \right) |s - u|^{2H-2}duds
\]
where the limit holds almost surely and in \( L^2(\Omega) \). Then, taking into account (3.10), by the results of Sugita [15], we have \( X_t \in D^{1,2} \), and
\[
DX_t = \exp \left( \int_t^s \partial_x f(r, X_r)dr \right) 1_{[0, t]} . \tag{3.14}
\]
Finally,

\[ \|DF\|_{H}^{2} = \alpha_{H} \int_{0}^{t} \int_{0}^{t} \exp \left( \int_{s}^{t} \partial_{x} f(r, X_{r}) dr \right) \times \exp \left( \int_{u}^{t} \partial_{x} f(r, X_{r}) dr \right) |s - u|^{2H - 2} du ds > 0. \]

In the next proposition we will show the existence of negative moments for
the function \( f(t, x) = \frac{K}{x} \), where \( K > 0 \) is a constant. The proof is based again
on the techniques of Malliavin calculus.

**Proposition 3.4** Let \( (X_{t}, t \geq 0) \) be the solution to the equation

\[ X_{t} = x_{0} + \int_{0}^{t} \frac{K}{X_{s}} ds + B_{t}^{H}. \quad (3.15) \]

Then, for all \( p \geq 1 \) and \( t \leq \left( \frac{K}{(p+1)H} \right)^{\frac{1}{2H - 1}} \), we have \( E(X_{t}^{-p}) \leq x_{0}^{-p} \).

**Proof** Obviously the function \( f(t, x) = \frac{K}{x} \) satisfies all the conditions (i)-(iii).
From Equation (3.14), we have

\[ D_{s} X_{t} = - \int_{0}^{t} \frac{K}{X_{r}} D_{s} X_{r} dr + 1_{[0, t]}(s). \]

So,

\[ D_{s} X_{t} = \exp \{- \int_{0}^{t} \frac{K}{X_{r}^{2}} dr \} 1_{[0, t]}(s). \]

For any fixed \( p \geq 1 \), we construct the family of functions \( \varphi_{\epsilon}(x) = \frac{1}{(\epsilon + x)^{p}}, \)
\( x > 0 \). Then \( \varphi_{\epsilon} \uparrow x^{-p} \), as \( \epsilon \downarrow 0 \). For each \( \epsilon > 0 \), \( \varphi_{\epsilon} \) is a bounded continuously
differentiable function and all its derivatives are bounded.

By the chain rule we obtain,

\[
\begin{align*}
\varphi_{\epsilon}(X_{t}) &= \varphi_{\epsilon}(x_{0}) + \int_{0}^{t} \varphi'_{\epsilon}(X_{s}) \frac{K}{X_{s}} ds + \int_{0}^{t} \varphi''_{\epsilon}(X_{s}) dB_{s}^{H} \\
&= \varphi_{\epsilon}(x_{0}) - p \int_{0}^{t} \frac{K}{X_{s}(\epsilon + X_{s})^{p+1}} ds \\
&- p \int_{0}^{t} \frac{1}{(\epsilon + X_{s})^{p+1}} dB_{s}^{H} \quad (3.16)
\end{align*}
\]

Then, Proposition 5.3.2 in [11] implies that

\[
\int_{0}^{t} \frac{1}{(\epsilon + X_{s})^{p+1}} dB_{s}^{H} = \delta \left( \frac{1}{(\epsilon + X_{s})^{p+1}} 1_{[0, t]}(s) \right) - (p + 1) \alpha_{H} \times \int_{0}^{t} \int_{0}^{t} \frac{D_{r} X_{s}}{(\epsilon + X_{r})^{p+2}} |s - r|^{2H - 2} dr ds, \quad (3.17)
\]
where $\delta$ is the divergence operator with respect to fractional Brownian motion. Substituting (3.17) into (3.16) yields

$$\varphi_\epsilon(X_t) \leq \varphi_\epsilon(x_0) - p \int_0^t \frac{K}{(\epsilon + X_s)^{p+2}} ds - p\delta \left( \frac{1}{(\epsilon + X_s)^{p+1}} \right) 1_{[0,t]}(s)$$

$$= \varphi_\epsilon(x_0) - p \int_0^t \frac{K - (p + 1)Ht^{2H - 1}}{(\epsilon + X_s)^{p+2}} ds - p\delta \left( \frac{1}{(\epsilon + X_s)^{p+1}} \right) 1_{[0,t]}(s).$$

Fix some $t$, such that $K - (p + 1)Ht^{2H - 1} \geq 0$, that is $t \leq \left( \frac{K}{(p+1)H} \right)^{\frac{1}{2H-1}}$. Taking expectation on above inequality, we have

$$\mathbb{E}(\varphi_\epsilon(X_t)) \leq \varphi_\epsilon(x_0) \leq x_0^{-p}.$$

Let $\epsilon$ tends to 0. Applying monotone convergence theorem, for any fixed $p \geq 1$, we obtain

$$\mathbb{E}(X_t^{-p}) \leq x_0^{-p}.$$

References

[1] Bouleau, N.; Hirsch, F. *Dirichlet forms and analysis on Wiener space*. de Gruyter in Mathematics, 14. Walter de Gruyter & Co., Berlin, 1991.

[2] Cherny, A. S.; Engelbert, H.-J. *Singular stochastic differential equations*. Lecture Notes in Mathematics 1858. Springer-Verlag, Berlin, 2005.

[3] Cox, J. C.; Ingersoll Jr., J. E.; Ross, S.A. A theory of term structure of interest rates. *Econometrica* 53 (1985), 385–407.

[4] Decreusefond, L.; "Ust"unel; A.S. Stochastic Analysis of the Fractional Brownian Motion. *Potential Analysis* 10 (1998), 177–214.

[5] Eisenbaum, N.; Tudor, C. A. On squared fractional Brownian motions. In: Séminaire de Probabilités XXXVIII, *Lecture Notes in Math.* 1857 2005, 282–289.

[6] Fernique, X. Regularité des trajectoires des fonctions aléatoires gaussiennes. In: École d’Été de Probabilités de Saint-Flour, IV-1974. *Lecture Notes in Math.* 480 (1975), 1–96.

[7] Guerra, J. M. E.; Nualart, D. The $1/H$-variation of the divergence integral with respect to the fractional Brownian motion for $H > 1/2$ and fractional Bessel processes. *Stochastic Process. Appl.* 115 (2005), no. 1, 91–115.
[8] Hu, Y.; Nualart, D. Some processes associated with fractional Bessel processes. *J. Theoret. Probab.* 18 (2005), no. 2, 377–397.

[9] Hu, Y.; Nualart, D. Differential equations driven by Hölder continuous functions of order greater than 1/2. In *The Abel Symposium on Stochastic Analysis*, 399-423. Springer, 2007.

[10] Lyons, T. Differential equations driven by rough signals (I): An extension of an inequality of L.C. Young. *Mathematical Research Letters* 1 (1994), 451–464.

[11] Nualart, D. *The Malliavin calculus and related topics.* Second edition. Springer, 2006.

[12] Nualart, D; Răşcanu, A. Differential equations driven by fractional Brownian motion. *Collect. Math.* 53 (2002), no. 1, 55–81.

[13] Samko S. G.; Kilbas A. A.; Marichev O. I. *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach, 1993.

[14] Saussereau, B.; Nualart, D. Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. Preprint.

[15] Sugita, H. On a characterization of the Sobolev spaces over an abstract Wiener space. *J. Math. Kyoto Univ.* 25 (1985), no. 4, 717–725.

[16] Young, L. C. An inequality of the Hölder type connected with Stieltjes integration. *Acta Math.* 67 (1936), 251–282.

[17] Zähle, M. Integration with respect to fractal functions and stochastic calculus, I. *Prob. Theory Related Fields* 111 (1998), 333–374.