SMALL EXTENDED FORMULATIONS FOR CYCLIC POLYTOPES

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Abstract. We provide an extended formulation of size $(\log n)\lfloor \frac{d^2}{2} \rfloor$ for the cyclic polytope with $n$ vertices and dimension $d$. First, we use the framework of reflection relations (see Kaibel and Pashkovich, proc. IPCO 2011), to find an extended formulation of size $\log(n)$ for $d = 2$. Then, we use this as base case to construct small-rank nonnegative factorizations of the slack matrices of higher-dimensional cyclic polytopes, by iterated tensor products. Through Yannakakis’s factorization theorem, these factorizations yield small-size extended formulations for cyclic polytopes of dimension $d \geq 3$.

1. Introduction

Extended formulations is a rapidly developing field with connections to, among others, discrete mathematics and theoretical computer science. Two main reasons that make this field interesting are the facts that: (i) small-size extended formulations allow efficient formulations of various optimization problems; (ii) lower bounds on the sizes of extended formulations show fundamental limits to what can be efficiently expressed through linear programs, and more generally conic programs. Here our focus is on linear programming extended formulation, thus the underlying cone is the nonnegative orthant.

In this paper, we provide an extended formulation of size $(\log n)\lfloor \frac{d^2}{2} \rfloor$ for the $d$-dimensional $n$-vertex cyclic polytope with vertices $(i, i^2, \ldots, i^d)$ for $i \in [t_1, t_2] \cap \mathbb{Z}$. The size of this extended formulation is asymptotically smaller than the size of the trivial “vertex” extended formulation, provided $d < (\log n)/(\log \log n)$.

As a possible application, consider the problem of minimizing a degree-$d$ polynomial $p(t)$ over $t \in [t_1, t_2] \cap \mathbb{Z}$, where $d$ is bounded. Our result implies that this can be formulated as a linear program with only a polylogarithmic number of constraints in $n$.

1.1. Polytopes, Extensions and Extended Formulations. Recall that a polytope $P \subseteq \mathbb{R}^d$ is the convex hull of a finite point set $V \subseteq \mathbb{R}^d$. Without loss of generality, we assume that $P$ is full-dimensional, that is, $\dim P = d$. Then $P$ can alternatively be described as the solution set of a system of finitely many linear inequalities, i.e.

$$P = \{ x \in \mathbb{R}^d : Ax \leq b \}$$

for some $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$. The size of the above linear description is the number of inequalities in the system, i.e. the number $m$. (When $P$ is not full-dimensional its linear description may contain linear equations. They are not taken into account in the computation of the size.)
A polytope \( Q \subseteq \mathbb{R}^d \) together with an affine map \( \pi : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is called an \textit{extension} of polytope \( P \subseteq \mathbb{R}^d \) if \( \pi(Q) = P \). The size of extension \( Q \) is defined as the number of facets of \( Q \), that is, the minimum number of inequality constraints in a linear description of \( Q \).

By choosing appropriately the origin and the basis vectors of \( \mathbb{R}^d \), we may assume that the projection \( \pi \) is given by \( \pi(x, y) := x \). In this case, the \textit{extended formulation} determined by the extension is simply a linear description of \( Q \). Extended formulations and extensions are two basically interchangeable concepts.

The \textit{extension complexity} of polytope \( P \) is the minimum size of an extension of \( P \), or equivalently the minimum size of an extended formulation of \( P \). This is denoted by \( xc(P) \). Thus \( xc(P) \) is the minimum number of facets of a polytope that projects to \( P \).

It is a nontrivial problem to determine the extension complexity of a given polytope \( P \). In the first place, \( P \) has an infinite number of extensions. Fortunately, it suffices to look at one single matrix for computing \( xc(P) \). The \textit{slack matrix} of polytope \( P \) relative to point set \( V = \{V(1), \ldots, V(n)\} \subseteq \mathbb{R}^d \) with \( P = \text{conv}(V) \) and linear description \( Ax \leq b \) of \( P \) is the matrix \( M \in \mathbb{R}^{n \times m} \) defined as

\[
M(i,j) := b(j) - \langle A(j), V(i) \rangle .
\]

That is, the entry corresponding to point \( V(i) \) and inequality \( \langle A(j), x \rangle \leq b(j) \) equals the slack \( b(j) - \langle A(j), V(i) \rangle \geq 0 \). We say that nonnegative vectors \( \alpha_i \in \mathbb{R}^d_+ \) for \( i \in [n] \) and \( \beta_j \in \mathbb{R}^d_+ \) for \( j \in [m] \) form a rank-\( r \) \textit{nonnegative factorization} of the matrix \( M \) if the equation

\[
M(i,j) = \langle \alpha_i, \beta_j \rangle
\]

holds for all \( i \in [n] \) and \( j \in [m] \). The \textit{nonnegative rank} of \( M \) is the minimum rank of a nonnegative factorization of \( M \), and is denoted \( rk_+(M) \). The connection between nonnegative rank and extension complexity was provided by Yannakakis \cite{Yannakakis}: \( xc(P) = rk_+(M) \), provided \( \text{dim} \, P \geq 1 \). In other words, the minimum size of an extension of a polytope equals the nonnegative rank of any of its slack matrices.

In the present paper, we mainly work with nonnegative factorizations for slack matrices to guarantee the existence of extended formulations of certain size.

2. **Cyclic Polytopes**

In this section, we define cyclic polytopes and list some of their properties. For more detailed information on cyclic polytopes we refer the reader to \cite{Bogomolov,Fiorini}.

For a pair of integer numbers \((t_1, t_2) \in \mathbb{Z}^2\), \( t_1 \leq t_2 \) and \( d \in \mathbb{N} \) let us define the corresponding \textit{cyclic polytope} \( P^d_{t_1, t_2} \subseteq \mathbb{R}^d \) as the convex hull of the points \( V(i) := (i, i^2, \ldots, i^d), i \in [t_1, t_2] \cap \mathbb{Z}, \) i.e.

\[
P^d_{t_1, t_2} := \text{conv}\{(i, i^2, \ldots, i^d) : i \in [t_1, t_2] \cap \mathbb{Z}\}.
\]

We assume \( t_2 - t_1 + 1 > d \), then the cyclic polytope \( P^d_{t_1, t_2} \) has \( t_2 - t_1 + 1 \) vertices and dimension \( d \). Moreover, the polytope \( P^d_{t_1, t_2} \) is simplicial, i.e. every facet contains exactly \( d \) vertices. In its turn, a set of \( d \) vertices \( V(i), i \in S \) where \( S \subseteq [t_1, t_2] \cap \mathbb{Z} \) defines a facet if and only if it satisfies \textit{Gale’s evenness condition}: the cardinality of the set \([\ell_1, \ell_2] \cap S\) is even for all \( \ell_1, \ell_2 \in [t_1, t_2] \cap \mathbb{Z}, \ell_1 < \ell_2 \).

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1. Throughout the paper we use superindices for dimensions, subindices for enumerating indices.

We use parentheses to refer to a row or an element of a matrix, depending on the number of indices enclosed in the parentheses.
proving that there is a nonnegative factorization of the slack matrix for guaranteed by Lemma 1 together with the Yannakakis’ theorem \[7\], in order to

3.2. The case \(\mathbb{P}\) polytope to a hyperplane, and thus we are not able to use reflection relations here.

\(\mathbb{P}\) dinates with an odd index. This does not correspond to a symmetry with respect \(\mathbb{P}\) the relationship between the polytopes \(\mathbb{P}\) Lemma 1.

3.1. \(\mathbb{P}\) polytopes \(\mathbb{P}\) dimensions.

In turn, the polytope \(\mathbb{P}\) is an image of the polytope \(\mathbb{P}\) \(\mathbb{P}\) \(\mathbb{P}\) \(\mathbb{P}\)\(\mathbb{P}\) is an image of the polytope \(\mathbb{P}\)\(\mathbb{P}\) where \(\mathbb{P}\) \(\mathbb{P}\) \(\mathbb{P}\) \(\mathbb{P}\)\(\mathbb{P}\) - 0 (a map defined by the sign change of the first coordinate). This fact allows us from the two dimensional case, going to even dimensions and finally considering odd dimensions.

\[ M^d_{t_1,t_2}(i,S) := \prod_{j \in S} |j - i|, \]

where \(i \in [t_1,t_2] \cap \mathbb{Z}\) and \(S \subseteq [t_1,t_2] \cap \mathbb{Z}\), \(|S| = d\) satisfy Gale’s evenness condition.

It is not hard to see that for two pairs of integers \((t_1,t_2)\) and \((k_1,k_2)\) the polytopes \(\mathbb{P}^{d}_{t_1,t_2}\) and \(\mathbb{P}^{d}_{k_1,k_2}\) are affinely isomorphic if and only if the equation \(t_2 - t_1 = k_2 - k_1\) holds. Indeed, if \(t_2 - t_1\) is not equal to \(k_2 - k_1\) the polytopes \(\mathbb{P}^{d}_{t_1,t_2}\) and \(\mathbb{P}^{d}_{k_1,k_2}\) have different number of vertices, and thus can not be affinely isomorphic. On the other hand, if \(t_2 - t_1\) equals \(k_2 - k_1\) then they have the same slack matrix because translating indices preserves the difference \(j - i\) in (1). Concretely, the following affine map defines an isomorphism between \(\mathbb{P}^{d}_{t_1,t_2}\) and \(\mathbb{P}^{d}_{k_1,k_2}\): \((x_1,\ldots,x_d) \mapsto (y_1,\ldots,y_d)\), where

\[ y_i := (k_1 - t_1)^i + \sum_{j=1}^{i} \binom{i}{j} (k_1 - t_1)^{i-j} x_j. \]

3. Extended Formulation

In this section, we construct extended formulations for cyclic polytopes: starting from the two dimensional case, going to even dimensions and finally considering odd dimensions.

3.1. The case \(d = 2\). Due to (2), we can affinely transform the polytope \(\mathbb{P}^{d}_{t_1,t_2}\) for a pair \((t_1,t_2) \in \mathbb{Z}^2\) into the polytope \(\mathbb{P}^{d}_{k_1,k_2}\) for \(k \in \mathbb{Z}\) whenever \(t_2 - t_1\) equals \(2k + 1\). In turn, the polytope \(\mathbb{P}^{d}_{-k,k}\), \(k \in \mathbb{Z}\) can be represented as the convex hull of two polytopes \(\mathbb{P}^{d}_{-k,0}\) and \(\mathbb{P}^{d}_{0,k}\), i.e.

\[ \mathbb{P}^{d}_{-k,k} = \text{conv}(\mathbb{P}^{d}_{-k,0},\mathbb{P}^{d}_{0,k}). \]

The above relation holds for every dimension, but there is a particularly nice relationship between \(\mathbb{P}^{d}_{-k,0}\) and \(\mathbb{P}^{d}_{0,k}\) in the two dimensional case: the polytope \(\mathbb{P}^{2}_{-k,0}\) is an image of the polytope \(\mathbb{P}^{2}_{0,k}\) under the reflection with respect to the hyperplane \(x_1 = 0\) (a map defined by the sign change of the first coordinate). This fact allows us to use reflection relations (see Theorem 2 in [6]) to state that every size-\(f\) extended formulation of the polytope \(\mathbb{P}^{2}_{0,k}\) leads to a size-\((f + 2)\) extended formulation of the polytope \(\mathbb{P}^{2}_{k,k}\). Due to (2), we are able to apply this argumentation iteratively, and thus to prove the next lemma.

Lemma 1. For the polytope \(\mathbb{P}^{2}_{t_1,t_2}\) there is an extension of size \(O(\log(t_2 - t_1))\).

3.2. The case \(d = 2q\). Unfortunately, in the three dimensional case and higher the relationship between the polytopes \(\mathbb{P}^{d}_{-k,0}\) and \(\mathbb{P}^{d}_{0,k}\) is more complicated: the polytope \(\mathbb{P}^{d}_{-k,0}\) is an image of the polytope \(\mathbb{P}^{d}_{0,k}\) under the sign change of all coordinates with an odd index. This does not correspond to a symmetry with respect to a hyperplane, and thus we are not able to use reflection relations here.

However, we may use the nonnegative factorization for the slack matrix of \(\mathbb{P}^{2}_{k,k}\) guaranteed by Lemma 11 together with the Yannakakis’ theorem [7], in order to prove that there is a nonnegative factorization of the slack matrix for \(\mathbb{P}^{2}_{k,k}\) of
certain size. Let us consider even dimensions first, i.e. assume \( d \) to be equal to \( 2q \), \( q \in \mathbb{N} \).

For two matrices \( A \) and \( B \) of the same size, we define the elementwise or Hadamard product \( A \odot B \) by the following equation \( (A \odot B)(i,j) := A(i,j)B(i,j) \). We will need the following folklore result.

**Lemma 2.** For all matrices \( A, B \) with the same number of rows and columns:

\[
\text{rk}_+(A \odot B) \leq \text{rk}_+(A) \text{rk}_+(B)
\]

*Proof.* We may assume that both matrices \( A, B \) are nonnegative. If matrices \( A, B \) admit nonnegative factorizations given by vectors \( \alpha_i \in \mathbb{R}^r_+ \), \( \beta_j \in \mathbb{R}^r_+ \) and vectors \( \gamma_i \in \mathbb{R}^s_+ \), \( \zeta_j \in \mathbb{R}^s_+ \) respectively, then the matrix \( A \odot B \) has a size-\( rs \) nonnegative factorization defined by the vectors \( \alpha_i \otimes \gamma_i \in \mathbb{R}^{rxs} \) and \( \beta_j \otimes \zeta_j \in \mathbb{R}^{rxs} \). Here, for every two vectors \( \mu \in \mathbb{R}^r \) and \( \tau \in \mathbb{R}^s \) the vector \( \mu \otimes \tau \) lies in \( \mathbb{R}^{rs} \) and is defined as \( (\mu \otimes \tau)(i,j) := \mu(i)\tau(j) \). \( \square \)

**Lemma 3.** The polytope \( P_{t_1,t_2}^{2q} \) has an extension of size \( O((\log(t_2 - t_1))^q) \).

*Proof.* We construct \( q \) matrices \( M_1, \ldots, M_q \) of suitable dimension such that the elementwise product \( M_1 \odot \cdots \odot M_q \) equals the slack matrix \( M_{t_1,t_2}^{2q} \). In order to do that note that every set \( S \subseteq [t_1, t_2] \cap \mathbb{Z}, |S| = 2q \) satisfying Gale’s evenness condition can be partitioned into \( q \) pairs \( S_1, \ldots, S_{2q} \), where each pair is equal to \( \{t_1, t_2\} \) or consists of two consecutive integers from \([t_1, t_2]\). Since every set \( S_r, 1 \leq r \leq q \) also satisfies Gale’s evenness condition and consists of two elements, there is a column index \( S_r \) such that

\[
M_{t_1,t_2}^{2q}(i, S_r) = \prod_{j \in S_r} |j - i|.
\]

Now, define the entries of the matrices \( M_1, \ldots, M_q \) in the column indexed by the set \( S \) as \( M_r(i, S) := M_{t_1,t_2}^{2q}(i, S_r) \). Notice that

\[
(M_1 \odot \cdots \odot M_q)(i, S) = \prod_{r=1}^{q} \prod_{j \in S_r} |j - i| = \prod_{j \in S} |j - i| = M_{t_1,t_2}^{2q}(i, S).
\]

Finally, it is straightforward to verify that the matrices \( M_1, \ldots, M_q \) are obtained from \( M_{t_1,t_2}^{2q} \) by duplicating, deleting and reordering columns, and thus the nonnegative rank of every of these matrices is bounded from above by the nonnegative rank of the matrix \( M_{t_1,t_2}^{2q} \). Hence, by Lemmas 2 and 3 the slack matrix \( M_{t_1,t_2}^{2q} \) admits a nonnegative factorization of size \( O((\log(t_2 - t_1))^q) \). \( \square \)

3.3. The case \( d = 2q + 1 \).

**Lemma 4.** We have \( xc(P_{t_1,t_2}^{2q+1}) \leq xc(P_{t_1+1,t_2}^{2q}) + xc(P_{t_1,t_2-1}^{2q}) = 2 xc(P_{t_1,t_2-1}^{2q}) \).

*Proof.* Every set \( S \subseteq [t_1, t_2] \cap \mathbb{Z}, |S| = 2q + 1 \) satisfying Gale’s evenness condition has one of the following forms:

(1) \( t_1 \in S \) and the set \( S \setminus \{t_1\} \) defines a facet of the polytope \( P_{t_1+1,t_2}^{2q} \)

(2) \( t_2 \in S \) and the set \( S \setminus \{t_2\} \) defines a facet of the polytope \( P_{t_1,t_2-1}^{2q} \).

The columns indexed by the sets \( S \) satisfying the condition (1) form a matrix, which is equal to the matrix \( M_{t_1+1,t_2}^{2q} \), where the row \( M_{t_1+1,t_2}^{2q}(i) \), \( i \in [t_1 + 1, t_2] \cap \mathbb{Z} \) is scaled by the positive scalar \( i - t_1 \), plus an appended zero row corresponding to the index \( i = t_1 \). Thus, the nonnegative rank of the submatrix of \( M_{t_1,t_2}^{2q+1} \) indexed by
sets $S$ satisfying the condition \((1)\) and the nonnegative rank of $M_{t_1,t_2}^{2q}$ are equal. Analogously, estimate the nonnegative rank of the submatrix of $M_{t_1,t_2}$ indexed by sets $S$ satisfying the condition \((2)\).

Finally, Lemmas \(3\) and \(4\) together lead to our theorem.

**Theorem 5.** The polytope $P_{t_1,t_2}^d$ has an extension of size $O((\log(t_2 - t_1))^{(d/2)})$.

## 4. Concluding Remarks

We remark that Lemma \(1\) crucially uses the fact that $P_{t_1,t_2}^2$ is the convex hull of the points $(i,i^2)$ for $n$ consecutive integers $i \in [t_1,t_2] \cap \mathbb{Z}$. Actually, \(4\) prove a $\Omega(\sqrt{n}/\log n)$ lower bound on the worst-case extension complexity of a 2-dimensional cyclic polytope of the form $P = \text{conv}\{(i,i^2) : i \in X\}$ where $X \subseteq [1,2n] \cap \mathbb{Z}$ and $|X| = n$.

Finally, there seems to be currently no lower bound on the extension complexity of $P_{t_1,t_2}^d$ that would match the upper bound given by Theorem 5. For instance, it follows from \(3\) that the best lower bound that only relies on the combinatorial structure is $O(d^2 \log n)$, where $n$ is the number of vertices of $P_{t_1,t_2}^d$.

**Appendix A. Explicit Factorizations**

Here, we provide an explicit nonnegative factorization of the slack matrix $M_{t_1,t_2}^2$ of size $2\lfloor \log(t_2 - t_1) \rfloor + 1$. This factorization is similar to the nonnegative factorization of the slack matrix for a regular polygon in \(4\).

Define $q := \lfloor \log(t_2 - t_1) \rfloor$. Additionally, define $q$ numbers $a_1, \ldots, a_q$ by the following recursive procedure (start the procedure with $j := 1$ and $k := t_2 - t_1$):

- define $a_j := \frac{k}{2}$
- update $k := \lfloor (k+1)/2 \rfloor$ and
- update $j := j + 1$ (stop the procedure if $j > q$)

Now let us assign a nonnegative vector $\alpha_i \in \mathbb{R}_+^{2\lfloor \log(t_2 - t_1) \rfloor}$ to every number $i \in [t_1,t_2] \cap \mathbb{Z}$ by the following procedure (start the procedure with $j := 1$ and $x := i$):

- update $r := x - a_j$
- if $r \geq 0$ update $x := x - 2r$
- if $r \geq 0$ define $\alpha_i(j) := 2r$ and $\alpha_i(j+1) := 0$
- if $r < 0$ define $\alpha_i(j) := 0$ and $\alpha_i(j+1) := -2r$
- update $j := j + 2$ (stop the procedure if $j > 2q$).

and similarly for every set $S := \{t,t+1\}$ with $t \in [t_1,t_2 - 1] \cap \mathbb{Z}$ define a nonnegative vector $\beta_i \in \mathbb{R}_+^{2\lfloor \log(t_2 - t_1) \rfloor}$ by the following procedure (start the procedure with $j := 1$ and $y := t + 1$):

- update $s := y - a_j - 1/2$
- if $s \geq 0$ update $y := y - 2s$
- if $s \geq 0$ define $\beta_i(j) := 0$ and $\beta_i(j+1) := 2s$
- if $s < 0$ define $\beta_i(j) := -2s$ and $\beta_i(j+1) := 0$
- update $j := j + 2$ (stop the procedure if $j > 2q$).

It is not hard to verify that the constructed vectors describe a nonnegative factorization of the slack matrix $M_{t_1,t_2}$ with one column deleted (namely, the column
corresponding to the set $S = \{t_1, t_2\}$, i.e. for every $t \in [t_1, t_2 - 1] \cap \mathbb{Z}$ and every $i \in [t_1, t_2] \cap \mathbb{Z}$

$$
(i - t)(i - t - 1) = \langle \alpha_i, \beta_j \rangle.
$$

An extension of this nonnegative factorization to a factorization of the complete matrix $M^2_{t_1, t_2}$ is straightforward.

**Appendix B. Factorization Examples**

Here, we provide an explicit nonnegative factorization of the matrix $M^2_{1, 8}$ that corresponds to the extended formulation of Lemma 1 and later reappears in the proof of Lemma 3.

Let $n = 8$ and $t_2 - t_1 = n - 1$, then

$$
M^2_{t_1, t_2} = M^2_{1, 8} = \begin{pmatrix}
0 & 2 & 6 & 12 & 20 & 30 & 42 & 0 \\
0 & 0 & 2 & 6 & 12 & 20 & 30 & 6 \\
2 & 0 & 0 & 2 & 6 & 12 & 20 & 10 \\
6 & 2 & 0 & 0 & 2 & 6 & 12 & 12 \\
12 & 6 & 2 & 0 & 0 & 2 & 6 & 12 \\
20 & 12 & 6 & 2 & 0 & 0 & 2 & 10 \\
30 & 20 & 12 & 6 & 2 & 0 & 0 & 6 \\
42 & 30 & 20 & 12 & 6 & 2 & 0 & 0 \\
\end{pmatrix}.
$$

The nonnegative factorization of $M^2_{1, 8}$ implicit in the proof of Lemma 3 is as follows:

$$
M^2_{1, 8} = \begin{pmatrix}
0 & 7 & 2 & 0 & 4 & 0 & 2 & 0 \\
0 & 5 & 0 & 2 & 0 & 0 & 6 \\
0 & 3 & 0 & 0 & 2 & 0 & 10 \\
0 & 1 & 2 & 0 & 0 & 0 & 12 \\
1 & 0 & 2 & 0 & 0 & 0 & 12 \\
3 & 0 & 0 & 0 & 2 & 0 & 10 \\
5 & 0 & 0 & 2 & 0 & 0 & 6 \\
7 & 0 & 0 & 4 & 0 & 2 & 0 \\
\end{pmatrix} \cdot \begin{pmatrix}
6 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 4 & 6 & 0 \\
3 & 1 & 0 & 0 & 0 & 1 & 3 & 0 \\
0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
$$

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