Riemann–Hilbert–Birkhoff inverse problem for semisimple flat $F$-manifolds and convergence of oriented associativity potentials

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Funding information
HIM; FCT, Grant/Award Numbers: UIDB/00208/2020, PTDC/MAT-PUR/30234/2017, 2021.01521.CEECIND, 2022.03702.PTDC (GENIDE)

Abstract
In this paper, we address the problem of classification of quasi-homogeneous formal power series providing solutions of the oriented associativity equations. Such a classification is performed by introducing a system of monodromy local moduli on the space of formal germs of homogeneous semisimple flat $F$-manifolds. This system of local moduli is well defined on the complement of the strictly doubly resonant locus, namely, a locus of formal germs of flat $F$-manifolds manifesting both coalescences of canonical coordinates at the origin, and resonances of their conformal dimensions. It is shown how the solutions of the oriented associativity equations can be reconstructed from the knowledge of the monodromy local moduli via a Riemann–Hilbert–Birkhoff boundary value problem. Furthermore, standing on results of B. Malgrange and C. Sabbah, it is proved that any formal homogeneous semisimple flat $F$-manifold, which is not strictly doubly resonant, is actually convergent. Our semisimplicity criterion for convergence is also reformulated in terms of solutions of Losev–Manin commutativity equations, growth estimates of correlators of $F$-cohomological field theories, and solutions of open Witten–Dijkgraaf–Verlinde–Verlinde equations.

MSC 2020
53D45, 34M50 (primary), 34M40, 34M55 (secondary)
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1 | INTRODUCTION

Oriented associativity equations. In this paper, we address both the problem of classification and the convergence issues of formal solutions, in the ring of formal power series with complex coefficients, of the oriented associativity equations \([66, 67, 74]\). These consist of the overdetermined system of nonlinear partial differential equations, in \(n\) functions \(F^1(t), \ldots, F^n(t)\) depending on \(n\) variables \(t = (t^1, \ldots, t^n)\), given by

\[
\sum_{\alpha} \frac{\partial F^\alpha}{\partial t^\beta} \frac{\partial F^\mu}{\partial t^\gamma} \frac{\partial F^\nu}{\partial t^\varepsilon} = \sum_{\mu} \frac{\partial F^\alpha}{\partial t^\beta} \frac{\partial F^\mu}{\partial t^\gamma} \frac{\partial F^\nu}{\partial t^\varepsilon}, \quad \alpha, \beta, \gamma, \varepsilon = 1, \ldots, n,
\]

\[
\sum_{\mu} A^\mu \frac{\partial F^\alpha}{\partial t^\mu} \frac{\partial F^\beta}{\partial t^\varepsilon} = \delta^\alpha_\beta, \quad A^\mu \in \mathbb{C}, \quad \alpha, \beta = 1, \ldots, n.
\]

The oriented associativity equations are a natural generalization of Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) associativity equations \([34, 87]\). Their solutions \((F^1, \ldots, F^n)\) reflect the rich geometry of \(F\)-manifolds with a compatible flat structure, for short flat \(F\)-manifolds \([74]\).

Flat \(F\)-manifolds. In the early 1990s, B. Dubrovin introduced Frobenius manifolds as geometrical materialization of solutions of WDVV equations \([35–38]\). This notion turns up in many areas of mathematics: for example, Frobenius manifolds play a key role in mirror symmetry, singularity theory, quantum cohomology, integrable systems, and symplectic geometry.

It was soon realized, however, that weaker (i.e., with relaxed axioms) variants of the Frobenius structure are of interest per se. The core notion is that of \(F\)-manifolds, introduced by C. Hertling and Y.I. Manin in \([47]\). Such a notion not only strictly includes the Frobenius structures, but it also greatly broadens the scope of the examples and applications. Examples of \(F\)-manifolds, indeed, arise not only in singularity theory \([46]\), but also in quantum \(K\)-theory \([59]\), differential-graded deformation theory \([75, 76]\), and even information geometry \([21]\).

Flat \(F\)-manifolds — introduced by Y.I. Manin \([74]\) — are an intermediate notion, weaker than Frobenius, but stronger than \(F\)-manifolds:

\[
\text{Frobenius manifolds} \subset \text{flat \(F\)-manifolds} \subset \text{\(F\)-manifolds}.
\]

Flat \(F\)-manifolds are equipped with the minimum amount of structures to share some of the deeper properties of Frobenius manifolds, including Dubrovin’s deformed connection, Dubrovin’s almost duality, and also operadic descriptions. See \([2, 4, 63, 73, 74]\). These structures are also studied in \([41]\), where they are called Dubrovin manifolds.

A flat \(F\)-manifold (in the analytic category) is a complex manifold \(M\) whose tangent spaces are equipped with an associative, commutative, and unital algebra structure — analytically depending in the point — whose product \(\circ\) is compatible with a given flat connection \(\nabla\). This means that each element of the pencil \((\nabla^z)_{z \in \mathbb{C}}\), defined by \(\nabla^z X = \nabla_X Y + zX \circ Y\), is required to be flat and torsionless.

The compatibility of \((\circ, \nabla)\) implies a potentiality condition. In \(V\)-flat local coordinates \(t = (t^1, \ldots, t^n)\) on \(M\), with \(n = \dim \mathbb{C} M\), the product \(\circ\) descends from a vector potential: there exists \(F = (F^1, \ldots, F^n)\) such that

\[
\frac{\partial}{\partial t^\beta} \circ \frac{\partial}{\partial t^\gamma} = \sum_{\alpha} \frac{\partial^2 F^\alpha}{\partial t^\beta \partial t^\gamma} \frac{\partial}{\partial t^\alpha}, \quad \beta, \gamma = 1, \ldots, n.
\]
The associativity of $\circ$ is equivalent to the oriented associativity equations for $F$. Vice versa, starting from a solution $F$ of the oriented associativity equations, we can define a flat $F$-structure via Equation (1.1). If the starting solution $F$ is a tuple of formal power series in $k[[t]]$ (with $k$ a $\mathbb{Q}$-algebra), the resulting flat $F$-structure is said to be formal over $k$. It can be seen as a flat $F$-structure on the formal spectrum $\text{Spf } k[[t]]$.

**Homogeneity, semisimplicity, and double resonance.** In this paper, we consider only quasi-homogenous solutions $F$ of the oriented associativity equations, that is, satisfying a further condition of the form

$$\sum_{\alpha} [(1 - q_{\alpha})t^{\alpha} + r^{\alpha}] \frac{\partial F^\beta}{\partial t^{\alpha}} = (2 - q_{\beta})F^\beta(t) + \text{linear terms in } t \text{ and constant terms},$$

for suitable complex numbers $q_{\alpha}, r^{\alpha} \in \mathbb{C}$. The resulting flat $F$-manifold is said to be homogeneous, or conformal. The vector field $E = \sum_{\alpha} [(1 - q_{\alpha})t^{\alpha} + r^{\alpha}] \frac{\partial}{\partial t^{\alpha}}$ is then an Euler vector field, that is, it satisfies the conditions $\nabla \nabla E = 0$ and $L_E(\circ) = \circ$. We say that $p \in M$ is tame if the spectrum of the operator $(E \circ)_p \in \text{End}(T_pM)$ is simple; otherwise, we say that $p$ is coalescing.

An analytic flat $F$-manifold is said to be semisimple if there exists an open dense subset of points $p$ whose corresponding algebra $(T_pM, \circ_p)$ is without nilpotent elements. This is equivalent to the existence of idempotent vectors $\pi_1, \ldots, \pi_n \in T_pM$: $\pi_i \circ \pi_j = \pi_i \delta_{ij}$ for $i, j = 1, \ldots, n$. If a manifold is both homogenous and semisimple, the eigenvalues $u^1, \ldots, u^n$ of the tensor $(E \circ)_p$ can be chosen as local holomorphic coordinates in a neighborhood of any semisimple point $p \in M$. Tame points are necessarily semisimple, whereas coalescing points may or may not be semisimple. In the (tame/coalescing) semisimple case, the vector fields $\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}$ are the idempotent vectors.

With each homogeneous semisimple (analytic/formal) flat $F$-manifold, we can associate a tuple $(\delta_1, \ldots, \delta_n)$ of numerical invariants called conformal dimensions. Fix a semisimple point $p \in M$, and introduce the operator

$$\mu^0_p \in \text{End}(T_pM), \quad \mu^0_p \left( \frac{\partial}{\partial t^{\alpha}} \right) = q_{\alpha} \frac{\partial}{\partial t^{\alpha}}, \quad \alpha = 1, \ldots, n.$$  

The conformal dimensions can be defined as the numbers $\delta_1, \ldots, \delta_n \in \mathbb{C}$ satisfying

$$\mu^0_p(\pi_i) \circ \pi_i = \delta_i \pi_i, \quad i = 1, \ldots, n.$$  

They are defined up to ordering, and they actually do not depend on the chosen semisimple point $p \in M$. The flat $F$-manifold is be said to be conformally resonant if $\delta_i - \delta_j \in \mathbb{Z} \setminus \{0\}$ for some $i, j$. The conformal dimensions of a Frobenius manifold are all equal, with common value $d^2$ (the number $d \in \mathbb{C}$ is the charge of the Frobenius structure). In particular, Frobenius manifolds are never conformally resonant.

In the formal case, all the conditions on points introduced above (tameness, coalescence, and semisimplicity) are intended to be referred to the origin $t = 0$, the only geometric point of the formal spectrum $\text{Spf } k[[t]]$.

A (germ of) pointed flat $F$-manifold $(M, p)$ is said to be doubly resonant if $M$ is conformally resonant, and $p$ is coalescing. We say that $(M, p)$ is strictly doubly resonant if we
have\footnote{Any ordering of the eigenvalues $u^1, \ldots, u^n$ induces an ordering of the idempotent vectors $\pi_i \equiv \delta_{u^i}$, and consequently of the conformal dimensions $\delta_i$.}

\[ u^i(p) = u^j(p), \quad \delta_i - \delta_j \in \mathbb{Z} \setminus \{0\}, \quad \text{for some } i, j \in \{1, \ldots, n\}, \ i \neq j. \]

**Results.** One of the main aspects of Dubrovin's analytic theory of Frobenius manifolds is their isomonodromic approach. Under the quasi-homogeneity assumption of the WDVV potential, the semisimple part of a Frobenius manifold can be locally identified with the space of isomonodromic deformation parameters of ordinary differential equations on $\mathbb{P}^1$ with rational coefficients, see [37, 38].

In this paper, we extend to the case of homogenous semisimple flat $F$-manifolds both Dubrovin's analytical theory as well as its refinement developed in [27, 29]. The key ingredient is a family $\left(\hat{\nabla}^{\lambda}\right)_{\lambda \in \mathbb{C}}$ of flat extended deformed connections on $\pi^*TM$, with $\pi: M \times \mathbb{C}^* \to \mathbb{C}^*$, whose restrictions to $M \times \{z\}$ equal $\nabla^z = \nabla + z(\sigma)$. These families of flat connections $\hat{\nabla}^{\lambda}$ on homogeneous flat $F$-manifolds were first introduced by Y.I. Manin [74].

**Remark 1.1.** In [79, Chapter VII, §1], the notion of Saito structure without metric is introduced. These structures are defined in terms of several data on a complex manifold $M$. Among them, there is a flat meromorphic connection $\hat{\nabla}$ on the bundle $\pi^*TM$ on $M \times \mathbb{P}^1$. It turns out that the notion of Saito structure without metric is equivalent to the notion of homogeneous flat $F$-manifold, and that $\hat{\nabla}$ is one of the connections $\hat{\nabla}^{\lambda}$ above. See also [53], in which it is shown that the space of isomonodromic deformation parameters of extended Okubo systems can be equipped with Saito structures without metrics.

For any germ $(M, p)$, semisimple and not strictly doubly resonant, we introduce a tuple of numerical data, the **monodromy data** of the flat $F$-manifold. These data split into two pieces: a pair $(\mu^1, R)$ of matrices called “monodromy data at $z = 0$,” and a 4-tuple $(S_1, S_2, \Lambda, C)$ of matrices called “monodromy data at $z = \infty$.” Precise definitions are given in Section 4. In the case of Frobenius manifolds, all these data are subjected to several constraints: the final amount of data coincides with the 4-tuple $(\mu, R, S, C)$ of monodromy data introduced by Dubrovin in [37, 38]. If $M$ is analytic, the monodromy data define **local invariants** of $M$: if $p_1, p_2 \in M$ are sufficiently close, the data of the germs $(M, p_1), (M, p_2)$ are equal.

**Theorem 1.2** (Cf. Theorems 6.15 and 6.23). Any homogeneous semisimple (analytic/formal) pointed germ of flat $F$-manifold, which is not strictly doubly resonant, is uniquely determined by its monodromy data. In particular, the vector potential $F$ can be explicitly reconstructed from the monodromy data via a Riemann–Hilbert–Birkhoff (RHB) boundary value problem.

We show that the totality of local isomorphism classes of germs of $n$-dimensional flat $F$-manifolds can be parametrized by points of a “stratified” space, whose generic stratum has dimension $n^2 - n$. The monodromy data provide a system of local coordinates. The Frobenius structures correspond to a locus of generic dimension $\frac{1}{2}(n^2 - n)$. See Theorem 6.27.

The reconstruction procedure of the flat $F$-structure is based on a crucial property of a joint system of “generalized” Darboux–Egoroff equations [62]: solutions $\Gamma(u)$ of this system of nonlinear partial differential equations are uniquely determined by its initial value $\Gamma_o$ at one point $u_o \in \mathbb{C}^n$ (with possibly $u^i_o = u^j_o$ for $i \neq j$), provided that there is no conformal resonance. See Lemma 6.24.
We underline that both the initial values \((\mathbf{u}_0, \Gamma_0)\) and the monodromy data provide a system of local coordinates on the space of germs of flat \(F\)-manifolds. The reconstruction procedure of \(F\) in terms of the initial values, however, is generally impossible, the dependance being typically transcendental (e.g., for \(n = 3\), the general Darboux–Egoroff system reduces to the full-parameters family of Painlevé equations PVI, see [62]). This makes the monodromy data “preferable” as a system of coordinates for the classification of flat \(F\)-structures.

There is a further advantage in choosing the monodromy data as a system of local moduli. Indeed, they make possible the study of convergence issues.

**Theorem 1.3** (Cf. Theorem 6.25). Let \(F \in \mathbb{C}[[t]]^m\) be a quasi-homogeneous solution of the oriented associativity equations. If \(F\) defines a semisimple formal flat \(F\)-manifold, which is not strictly doubly resonant, then \(F\) is a tuple of convergent functions.

For the proof, we invoke results of B. Malgrange [69–71] and C. Sabbah [80] on the solvability of families of RHB problems. More precisely, we use their equivalent formulations given in [24]. Notice that Theorem 1.3 generalizes [24, Theorem 5.1].

**Cohomological field theories.** Frobenius structures are intrinsically correlated to the notion of cohomological field theories. The geometry of Frobenius structures reflects properties of the cohomology rings \(H^*(\overline{M}_{0,n}, \mathbb{C})\), with \(n \geq 3\), and they can indeed be defined in terms of \(H^*(\overline{M}_{0,n}, \mathbb{C})\)-valued polylinear maps. See [58, 72, 77].

At the level of flat \(F\)-manifolds, such a construction has been generalized in two different (a posteriori) equivalent ways. The first one is due to A. Losev and Y.I. Manin [66, 67], the second one to A. Buryak and P. Rossi [20]. See also [5–7].

In [66], a new compactification \(\overline{L}_n\) of \(\overline{M}_{0,n}\) is introduced. The boundary strata represent isomorphism classes of stable \(n\)-pointed chains of projective lines, in which the marked points do not play a symmetric role. In [67], a notion of genus 0 extended modular operad, and \(\mathcal{L}\)-algebras over it are studied. Moreover, it is shown that the differential equations satisfied by generating functions of correlators of \(\mathcal{L}\)-algebras lead to two differential geometric pictures: the first one is the study of pencils of flat connections, based on the commutativity equations, the second one is the study of flat \(F\)-manifolds, based on the oriented associativity equations. These two pictures are actually locally equivalent, if a further amount of data — a primitive element — is given. See also the constructions in [64, 65].

Notice that the compactifications \(\overline{L}_n\) and \(\overline{M}_{0,n}\), and their higher genus analogs, both arise in the more general construction of [45] as compactified moduli spaces of weighted pointed stable curves, for two different choices of the weights. See also [73].

In [20], A. Buryak and P. Rossi introduced the notion of \(F\)-cohomological field theories (\(F\)-CohFT) as a generalization of both cohomological field theories [58, 72], and partial cohomological field theories [61]. An \(F\)-CohFT is defined by the datum of a family of \(H^*(\overline{M}_{0,n}, \mathbb{C})\)-valued polylinear maps on a tensor product \(V^n \otimes V^\otimes n\) (\(V\) is an arbitrary \(\mathbb{C}\)-vector space), which satisfy some natural \(\mathfrak{S}_n\)-equivariance and gluing properties. Given an \(F\)-CohFT, its genus zero sector (or tree level) defines a formal flat \(F\)-structure on \(V\).

In Section 7, we review all these cohomological field theoretical approaches to flat \(F\)-manifolds and their equivalences, and we rephrase our semisimplicity criterion of convergence in terms of solutions of Losev–Manin commutativity equations and growth estimates of correlators of \(F\)-CohFT’s. Furthermore, in Appendix B, we prove\(^\dagger\) that any formal flat \(F\)-manifold over \(\mathbb{C}\)

\(^\dagger\)The author is not aware of a proof of this fact in the literature.
descend from a unique $F$-CohFT in the sense of P. Rossi and A. Buryak. This is in complete analogy with the Frobenius manifolds case, see [72].

**Structure of the paper.** In Section 2, we present some preliminary material and basic properties of flat $F$-manifolds, in both formal and analytic categories. We recall definitions of homogeneity, semisimplicity of flat $F$-manifolds. We also introduce the notion of local isomorphisms, pointed germs, and irreducibility of flat $F$-manifolds.

In Section 3, we first describe how to reconstruct oriented associativity potentials from deformed coordinates on a flat $F$-manifold. We then introduce a family of flat extended deformed connections $\hat{\nabla}^\lambda$, and we develop the analytical theory of its flat (co)section. We study the differential system of $\hat{\nabla}^\lambda$-flatness in three different frames: the flat frame, the idempotent frame, and a normalized idempotent frame (the normalizing factors are the so-called Lamé coefficients). We also introduce Darboux–Tsarev and Darboux–Egoroff systems of equations.

In Section 4, after introducing the notion of spectrum of a flat $F$-manifolds, we define the monodromy data of a homogeneous semisimple flat $F$-manifold. We also describe their mutual constraints. In Section 5, we then clarify the dependence of the monodromy data on all the choices of normalizations involved in their definition. Different choices of normalizations affect the numerical values of the data via the action of suitable groups. We show that the analytic continuation of the flat $F$-structure is described by a braid group action on the tuple of monodromy data.

Section 6 contains the main results of the paper. After introducing the notion of admissible data and the related RHB boundary value problem, we recall some results of B. Malgrange and C. Sabbah as formulated in [24]. We show that germs of flat $F$-structures can be constructed starting from solutions of RHB problems. Moreover, we show that any germ, which is not strictly doubly resonant, is of such a form: it can be reconstructed from its monodromy data. It is also proved that any formal germ of homogenous semisimple flat $F$-manifold (over $\mathbb{C}$), which is not strictly doubly resonant, is actually convergent.

In Section 7, we review equivalent approaches for defining flat $F$-manifolds. We recall the notions of Losev–Manin commutativity equations, Losev–Manin cohomological field theories (LM-CohFT), and $F$-cohomological field theories ($F$-CohFT) in the sense of A. Buryak and P. Rossi. We discuss the equivalence of these notions. Furthermore, we also discuss relations with the open WDVV (OWDVV) equations. Our semisimplicity criterion is reformulated in terms of growth estimates of correlators of LM-CohFT and $F$-CohFT, and convergence of solutions of OWDVV equations.

In Appendix A, we provide a proof for the following characterization of irreducibility of flat $F$-structures in terms of Euler vector fields: a flat $F$-manifold is irreducible if and only if any two arbitrary Euler vector fields differ by a scalar multiple of the unit vector field.

In Appendix B, we prove that any formal flat $F$-manifold descends from a unique tree-level $F$-CohFT.

## 2 | FLAT $F$-MANIFOLDS

### 2.1 | Analytic flat $F$-manifolds

Let $M$ be a complex analytic manifold with dimension $\text{dim}_\mathbb{C} M = n$. Denote by $TM$, $T^*M$ the holomorphic tangent and cotangent bundles, and by $\mathcal{T}_M$, $\Omega^1_M$ their sheaves of sections. If $E \to M$ is a holomorphic bundle with sheaf of sections $\mathcal{E}$, we denote by $\Gamma(E) = \Gamma(M, \mathcal{E})$ the space of global
sections. By usual abuse of notations, we will write $X \in \mathcal{E}$ for $X \in \Gamma(U, \mathcal{E})$ for some (or arbitrary) open set $U \subseteq M$.

Let $(M, \nabla, c, e)$ be the datum of

1. a connection $\nabla : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M$ on $TM$;
2. a section $c \in \Gamma(TM \otimes \bigodot^2 T^*M)$;
3. a vector field $e \in \Gamma(TM)$ such that
   a) $c(\cdot, e, \cdot) = c(\cdot, \cdot, e) \in \Gamma(\text{End} TM)$ is the identity morphism,
   b) $\nabla e = 0$.

Denote by $X \circ Y := c(\cdot, X, Y)$ the commutative product defined by the tensor $c$, and introduce the one-parameter family of connections $(\nabla^z)_{z \in \mathbb{C}}$ defined by $\nabla^z X Y := \nabla X Y + z X \circ Y$ for $X, Y \in \mathcal{T}_M$.

**Definition 2.1.** We say that $(M, \nabla, c, e)$ is a flat $F$-manifold if the connection $\nabla^z$ is flat and torsionless for any $z \in \mathbb{C}$.

Let $t = (t^1, \ldots, t^n)$ be a system of $\nabla$-flat local coordinates on $M$. Set $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$, for $\alpha = 1, \ldots, n$, and define $c^{\alpha}_\beta := c(dt^\alpha, \partial_\beta, \partial_\gamma)$. The flatness and torsionless of $\nabla^z$ is equivalent to the associativity of $\circ$, the symmetry of $\partial_\alpha c^{\delta}_{\beta \gamma}$ in $(\alpha, \beta, \gamma)$, and the flatness of $\nabla$. Hence, there locally exist analytic functions $F = (F^1, \ldots, F^n) \in \mathcal{O}_M^n$ such that

$$c^{\alpha}_\beta = \frac{\partial^2 F^\alpha}{\partial t^\beta \partial t^\gamma}, \quad \alpha, \beta, \gamma = 1, \ldots, n.$$  

In what follows the Einstein summation rule is used over repeated Greek indices. Let $A^\alpha \in \mathbb{C}$ be constants such that $e = A^\alpha \partial_\alpha$. From the associativity of $\circ$ and the properties of $e$, we have

$$A^\mu \frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\beta} = \delta^\alpha_\beta, \quad \alpha, \beta = 1, \ldots, n,$$  

(2.1)

$$\frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\beta} \frac{\partial^2 F^\mu}{\partial t^\gamma \partial t^\delta} = \frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\gamma} \frac{\partial^2 F^\mu}{\partial t^\beta \partial t^\delta}, \quad \alpha, \beta, \gamma, \delta = 1, \ldots, n.$$  

(2.2)

Equations (2.1) and (2.2) are called oriented associativity equations, and $F$ is the oriented associativity potential of the flat $F$-structure.

A flat $F$-manifold is said to be homogeneous if there it is equipped with an Euler vector field, that is, a vector field $E \in \Gamma(TM)$ such that

$$\nabla E = 0, \quad \mathcal{L}_E c = c.$$  

**Lemma 2.2.** We have $[e, E] = e$.

**Proof.** The condition $\mathcal{L}_E c = c$ is equivalent to $[E, Y \circ Z] - [E, Y] \circ Z - [E, Z] \circ Y = Y \circ Z$, for $Y, Z \in \mathcal{T}_M$. If $Y = Z = e$, the identity follows.  

□
Homogeneous flat \( F \)-manifold are also called Saito structures without metric. [79, Ch. VII]. We assume that the \((1,1)\)-tensor \( \nabla E \) is diagonalizable, and in diagonal form in the \( t \)-coordinates:

\[
E = \sum_{\alpha=1}^{n} ((1 - q_\alpha) t_\alpha + r_\alpha) \frac{\partial}{\partial t_\alpha}, \quad q_\alpha, r_\alpha \in \mathbb{C}.
\] (2.3)

The condition \( \mathfrak{L}_E c = c \) is thus equivalent to

\[
E_\mu \frac{\partial F_\alpha}{\partial t_\mu} = (2 - q_\alpha) F_\alpha + A_\beta t_\beta + B_\alpha, \quad A_\beta, B_\alpha \in \mathbb{C}.
\] (2.4)

**Definition 2.3.** A flat \( F \)-manifold \((M, \nabla, c, e)\) is a Frobenius manifold if there exist a symmetric nondegenerate \( \mathcal{O}_M \)-bilinear form \( \eta \in \Gamma(\bigwedge^2 T^* M) \), called metric, such that

\[
\nabla \eta = 0, \quad \text{and} \quad \eta(X \circ Y, Z) = \eta(X, Y \circ Z), \quad X, Y, Z \in \mathfrak{T}_M.
\] (2.5)

In such a case, \( \nabla \) is the Levi–Civita connection of \( \eta \). A vector field \( E \in \Gamma(TM) \) is Euler if it satisfies the conditions

\[
\mathfrak{L}_E c = c, \quad \mathfrak{L}_E \eta = (2 - d) \eta,
\] (2.6)

where the number \( d \in \mathbb{C} \) is the conformal dimension (or charge) of the Frobenius manifold.

**Remark 2.4.** An Euler vector field \( E \) for a Frobenius manifold is automatically an Euler vector field for the underlying flat \( F \)-manifold. The condition \( \nabla \nabla E = 0 \) is indeed implied by the conformal Killing condition (2.6), and the flatness of \( \nabla \).

**Remark 2.5.** In the case a flat \( F \)-manifold is actually Frobenius, the oriented associativity potentials \( F = (F^1, \ldots, F^n) \), solutions of (2.1) and (2.2), can be shown to locally descend from a single WDVV potential \( F(t) \), that is, a solution of the system of equations

\[
A_\mu \frac{\partial^5 F}{\partial t_\mu \partial t_\alpha \partial t_\beta \partial t_\gamma \partial t_\delta} = \eta_{\alpha\beta} = \text{const.}, \quad \eta = (\eta_{\alpha\beta})_{\alpha,\beta}, \quad \eta^{-1} = (\eta^{\alpha\beta})_{\alpha,\beta} \quad \alpha, \beta = 1, \ldots, n,
\]

\[
\frac{\partial^3 F}{\partial t_\alpha \partial t_\beta \partial t_\mu} \eta^{\gamma\nu} \frac{\partial^3 F}{\partial t_\gamma \partial t_\nu \partial t_\delta} = \frac{\partial^3 F}{\partial t_\alpha \partial t_\beta \partial t_\mu} \eta^{\gamma\nu} \frac{\partial^3 F}{\partial t_\gamma \partial t_\nu \partial t_\delta}, \quad \alpha, \beta, \gamma, \delta = 1, \ldots, n.
\]

The potentials \( F^\alpha \)'s are the components of the \( \eta \)-gradient of \( F \), that is, \( F^\alpha(t) = \eta^{\alpha\beta} \partial_\beta F(t) \).

## 2.2  Formal flat \( F \)-manifolds

Let

- \( k \) be a commutative \( \mathbb{Q} \)-algebra,
- \( H \) be a free \( k \)-module of finite rank,
- \( K := k[[H^*]] \) be the completed symmetric algebra of \( H^* := \text{Hom}_k(H, k) \).
Fix a basis \((\Delta_1, \ldots, \Delta_n)\) of \(H\), and denote by \(t = (t^1, \ldots, t^n)\) the dual coordinates. The algebra \(K\) is then identified with the algebra of formal power series \(k[[t]]\). Denote by \(\text{Der}_k(K)\) the \(K\)-module of \(k\)-linear derivations of \(K\). Put \(\partial_\alpha = \frac{\partial}{\partial t^\alpha} : K \to K\). The module \(\text{Der}_k(K)\) is a free \(K\)-module with basis \((\partial_1, \ldots, \partial_n)\).

Elements of \(H_K := K \otimes_k H\) will be identified with derivations on \(K\), by \(\Delta_\alpha \mapsto \partial_\alpha\).

**Definition 2.6.** A formal flat \(F\)-manifold structure on \(H\) is given by an \(n\)-tuple \(\Phi = (\Phi_1, \ldots, \Phi_n) \in K^n\), satisfying the oriented associativity Equations (2.1) and (2.2), where \(A^\mu \in k\).

Define the \(K\)-linear multiplication \(\circ\) on \(H_K\) by

\[
\Delta_\alpha \circ \Delta_\beta := c^\gamma_{\alpha\beta} \Delta_\gamma, \quad c^\gamma_{\alpha\beta} := \frac{\partial^2 \Phi^\gamma}{\partial t^\alpha \partial t^\beta}, \quad \alpha, \beta = 1, \ldots, n.
\]

The oriented associativity equations imply that \(\circ\) is associative, and that \(e := A^\mu \Delta_\mu\) is the unit of the algebra \((H_K, \circ)\). A vector \(E \in H_K\) is an Euler vector if it is if the form (2.3), and the pair \((E, \Phi)\) satisfies Equations (2.4).

Let \(\text{Diff}_1(H_K, H_K)\) denote the set of \(\mathcal{D} \in \text{Hom}_k(H_K, H_K)\) such that

\[
ab \mathcal{D}(p) - b \mathcal{D}(ap) - a \mathcal{D}(bp) + \mathcal{D}(abp) = 0, \quad a, b \in K, \quad p \in H_K.
\]

Both \(\text{Der}_k(K)\) and \(\text{Diff}_1(H_K, H_K)\) are naturally equipped with a \(K\)-module structure. A (formal) connection on \(H_K\) is defined by a \(K\)-linear morphism \(\nabla : \text{Der}_k(K) \to \text{Diff}_1(H_K, H_K)\), \(u \mapsto \nabla_u\) satisfying the Leibniz rule

\[
\nabla_u(a p) = u(a)p + a \nabla_u p, \quad a \in K, \quad p \in H_K.
\]

The torsion and curvature of \(\nabla\) are the \(K\)-bilinear morphisms \(T, R : \text{Der}_k(K) \times \text{Der}_k(K) \to \text{Hom}_K(H_K, H_K)\) defined by

\[
T(u, v) := \nabla_u v - \nabla_v u - [u, v], \quad u, v \in \text{Der}_k(K) \cong H_K,
\]

\[
R(u, v) := [\nabla_u, \nabla_v] - \nabla_{[u, v]}, \quad u, v \in \text{Der}_k(K).
\]

We can thus introduce the one-parameter family \((\nabla^z)_{z \in k}\) of (formal) connection given by \(\nabla^z_{\alpha\beta} := z \partial_\alpha \circ \partial_\beta\). The formal connection \(\nabla^z\) is flat and torsionless for any \(z \in k\).

**Remark 2.7.** If \(F := (F^1, \ldots, F^n)\) is a (formal/analytic) solution of the oriented associativity Equations (2.2), then also \(\bar{F} := (\lambda_1 F^1, \ldots, \lambda_n F^n)\) is a solution for any \((\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n\). If the original flat \(F\)-manifold has a unit \(e := A^\mu \partial_\mu\), then the rescaled flat \(F\)-manifold structure has unit \(e' = \frac{1}{\lambda_\mu} A^\mu \partial_\mu\).

**Remark 2.8.** If \(E\) is an Euler vector field for a given flat \(F\)-manifold structure, then also \(E - \lambda e\) is an Euler vector field, for \(\lambda \in \mathbb{C}\). Under two further assumptions of semisimplicity and irreducibility, one can prove that any Euler vector field is of this form. See Theorem 2.20.
2.3 Local isomorphisms and pointed germs

Let \((M_i, \nabla_i, c_i, e_i)\), with \(i = 1, 2\), be two analytic flat \(F\)-manifolds. A biholomorphism \(\varphi : M_1 \to M_2\) is an isomorphism of flat \(F\)-manifolds if

1. \(d\varphi(\ker \nabla_1) = \ker(\varphi^* \nabla_2)\), where \(d\varphi : T_{M_1} \to \varphi^* T_{M_2}\) is the differential of \(\varphi\), and \(\varphi^* \nabla_2\) is the pulled-back connection on \(\varphi^* T_{M_2}\).
2. For each \(p \in M_1\), the map \(d\varphi_p : T_p M_1 \to T_{\varphi(p)} M_2\) is an isomorphism of unital algebras.

**Lemma 2.9.** Let \(M_1\) and \(M_2\) be two isomorphic flat \(F\)-manifolds. Given two systems of local flat coordinates, \(t\) on \(M_1\) and \(\tilde{t}\) on \(M_2\), the corresponding local potentials \(F_1(t)\) and \(F_2(\tilde{t})\) are related by

\[
F_2^\alpha(\tilde{t}) = G_\lambda^\alpha F_1^\lambda(t) + \text{linear terms in } t, \quad \alpha = 1, \ldots, n,
\]

\[
\tilde{t} = \varphi(t) = Gt + c,
\]

where \(c \in \mathbb{C}^n\) is a constant vector, and the matrix \(G \in GL(n, \mathbb{C})\) satisfies the relation \(GA_1 = A_2\), the column vectors \(A_i = (A_1^i, \ldots, A_n^i)^T \in \mathbb{C}^n\), for \(i = 1, 2\), being such that

\[
e_1 = \sum_{\alpha=1}^{n} A_1^\alpha \frac{\partial}{\partial t^\alpha}, \quad e_2 = \sum_{\alpha=1}^{n} A_2^\alpha \frac{\partial}{\partial \tilde{t}^\alpha}.
\]

**Proof.** If \(\tilde{t} = \varphi(t)\) defines the isomorphism in the chosen local flat coordinates, then the Jacobian matrix \((\frac{\partial \varphi^\alpha}{\partial t^\beta})_{\alpha,\beta=1}^n\) must be constant. Hence, \(\tilde{t} = Gt + c\), for some \(G \in GL(n, \mathbb{C})\) and \(c \in \mathbb{C}^n\).

Moreover, by definition of isomorphism, we have

\[
\frac{\partial^2 \varphi^\mu(t)}{\partial t^\lambda \partial t^\alpha \partial \tilde{t}^\beta} \bigg|_{t=\varphi(t)} = \frac{\partial^2 F_2^\nu(t)}{\partial \tilde{t}^\mu \partial \tilde{t}^\nu} = \frac{\partial^2 (F_2^\nu \circ \varphi)(t)}{\partial \tilde{t}^\mu \partial \tilde{t}^\nu}.
\]

The claim follows by double integration. \(\square\)

**Remark 2.10.** The group of transformations \(t \mapsto \tilde{t} = Gt + c\) of Lemma 2.9 has dimension \(n^2\). Notice that here we are not requiring \(VE\) to be diagonal in both systems of coordinates. That further constraint decreases the dimension of the group of admissible transformations. If we want both \(\frac{\partial}{\partial t^\alpha}\) and \(\frac{\partial}{\partial \tilde{t}^\alpha}\), with \(\alpha = 1, \ldots, n\), to be eigenvectors of \(VE\), we need to impose a constraint on \(G\): namely, the eigenspaces of \(VE\) must be invariant under the linear transformation \(\frac{\partial}{\partial t^\alpha} \mapsto \sum_\beta (G^{-1})^\beta_\alpha \frac{\partial}{\partial \tilde{t}^\beta}\). In particular, if \(VE\) has simple spectrum (i.e., the parameters \(q_1, \ldots, q_n\) are pairwise distinct), the matrix \(G\) should be diagonal. In total, we would have a group of transformations of dimension \(2n\). If the spectrum of \(VE\) is not simple, then the dimension of possible transformations increases.

A pointed flat \(F\)-manifold is a pair \((M, p)\), where \(M\) is a flat \(F\)-manifold, and \(p \in M\) is a fixed base point. Isomorphisms between pointed flat \(F\)-manifold will always be assumed to be base point preserving. Given \((M, p)\), we will always consider flat coordinates \(t\) vanishing at \(p\).
Two pointed flat $F$-manifolds $(M_1, p_1)$, $(M_2, p_2)$ are locally isomorphic if there exist open neighborhoods $\Omega_1 \subseteq M_1$ of $p_1$, and $\Omega_2 \subseteq M_2$ of $p_2$, respectively, with isomorphic induced flat $F$-structures, that is, $(\Omega_1, p_1) \cong (\Omega_2, p_2)$.

A pointed germ is a (local isomorphism) equivalence class of pointed flat $F$-manifolds.

Any analytic pointed flat $F$-manifold $(M, p)$ induces a formal flat $F$-manifold $(H, \Phi)$ over $k = \mathbb{C}$. Choose flat coordinates $t$ vanishing at $p$, and set $H := T_p M$. Let $\mathcal{O}_{M, p}$ be the local ring of germs at $p$, and $\mathfrak{m}$ be its maximal ideal. The formal potential $\Phi^\alpha$ is given by the image of $F^\alpha$ in the completion $\hat{\mathcal{O}}_{M, p} := \lim \rightarrow_{\mathfrak{m} / \mathfrak{m}^l}$ of the local ring $\mathcal{O}_{M, p}$; this means that $\Phi^\alpha$ is defined by the Taylor series expansion of $F^\alpha$ at $p$ in coordinates $t$. Moreover, the formal flat $F$-structure $(H, \Phi)$ is also equipped with a flat unit $e\big|_p$. If $M$ has Euler vector field $E$, then $(H, \Phi)$ has Euler vector field $E\big|_p$. We will say that the formal flat $F$-structure constructed in this way, starting from an analytic one, is convergent.

Vice versa, let us assume that $(H, \Phi)$ is a formal flat $F$-structure over $k = \mathbb{C}$ (with Euler vector field). If the common domain of convergence $\Omega \subseteq H$ of the power series $\Phi^\alpha \in k[[t]]$ is nonempty, it is easily seen that $\Omega$ is equipped with an analytic flat $F$-structure (with Euler vector field).

2.4 | Semisimple flat $F$-manifolds

Let $(M, \nabla, c, e)$ be an analytic flat $F$-manifold. A point $p \in M$ is called semisimple if the algebra $(T_p M, \circ_p)$ is without nilpotent elements. This is equivalent to

- the existence of idempotent vectors $\pi_1, \ldots, \pi_n \in T_p M$, that is, such that $\pi_i \circ_p \pi_j = \pi_i \delta_{ij}$,
- the existence of $v \in T_p M$ such that $v \circ_p : T_p M \to T_p M$ has simple spectrum.

Semisimplicity is an open property: if $p \in M$ is semisimple, then there exists an open neighborhood $V$ of $p$, such that any point of $V$ is semisimple. Moreover, if $V$ is small enough, we have well-defined local idempotent holomorphic vector fields $\pi_1, \ldots, \pi_n \in \Gamma(V, \mathcal{T}_M)$. See, for example, [46, Ch. II] for a detailed discussion.

Let $(H, \Phi)$ be a formal flat $F$-manifold. Denote by $\circ_0$ the product on $H$ defined by structure constants $c^\alpha_{\beta\gamma}(0) := \delta^2_{\alpha\beta} \Phi|_{t=0}$. We will say that $(H, \Phi)$ is

- semisimple at the origin if we have an isomorphism of $k$-algebras $(H, \circ_0) \cong k^n$,
- formally semisimple if we have an isomorphism of $K$-algebras $(H, \circ) \cong K^n$.

Formal semisimplicity is thus equivalent to the existence of vectors $\pi_i \in H_K$ such that $\pi_i \circ \pi_j = \pi_i \delta_{ij}$.

**Lemma 2.11.** A formal flat $F$-manifold is formally semisimple if and only if it is semisimple at the origin.

**Proof.** The proof of [24, Lemma 4.2] works verbatim. \[ \square \]

**Remark 2.12.** In both formal and analytic cases, we have $e = \sum_{i=1}^n \pi_i$. 

Proposition 2.13. For both formal and analytic semisimple flat $F$-manifolds, the idempotent vectors $\pi_1, \ldots, \pi_n$ are pairwise commuting, that is, $[\pi_i, \pi_j] = 0$. Hence, there exist local coordinates $u = (u^1, \ldots, u^n)$ such that $\pi_i = \frac{\partial}{\partial u^i}$ for $i = 1, \ldots, n$. The local coordinates $u$ will be called canonical.

In the formal case, the functions $u^i$ are formal functions, that is, elements of $K$. Canonical coordinates $u$ are defined up to permutations and shifts by constants. We set $\tilde{\pi}_i := \frac{\partial}{\partial u^i}$ for $i = 1, \ldots, n$. If an Euler vector field is given, then the shift freedom can be actually frozen.

Proposition 2.14. A vector field $E \in \Gamma(TM)$ satisfies $\mathcal{L}_E c = c$ if and only if in canonical coordinates, it has the form $E = \sum_j (u^j + c^j) \partial_j$. Up to shifts of canonical coordinates $u$, we have $E = \sum_j u^j \partial_j$.

Hence, the eigenvalues of the tensor $(E \circ) \in \Gamma(\text{End}TM)$ may and will be chosen as local canonical coordinates.

Definition 2.15. A point $p \in M$ will be called

- tame if the operator $E \circ_p : T_p M \to T_p M$ has simple spectrum
- coalescing, otherwise.

If a point is tame, then it is necessarily semisimple, and with pairwise distinct canonical coordinates.

The same definition can be adapted to the formal case, relatively at the origin $t = 0$, by looking at the spectrum of $E \circ_0 : H \to H$.

At coalescing points $p$, we have $u(p) \in \Delta$ where $\Delta \subseteq \mathbb{C}^n$ denotes the big diagonal

\[ \Delta := \bigcup_{i \neq j} \{ u_i = u_j \}. \]

For a given flat $F$-manifold $M$, we define an algebraic symmetry of $M$ to be a diffeomorphism $\varphi : M \to M$ whose differential $d\varphi_p : T_p M \to T_{\varphi(p)} M$ is an isomorphism of unital algebras for any $p \in M$. Denote by $\text{AlgSym}(M)$ the group of algebraic symmetries of $M$.

Remark 2.16. The group $\text{AlgSym}(M)$ is never empty: if the $\nabla$-flat coordinates $t$ are normalized so that $\frac{\partial}{\partial t^1} = e$, any translation $t^1 \mapsto t^1 + c$, with $c \in \mathbb{C}$, defines an algebraic symmetry. Moreover, notice that an algebraic symmetry $\varphi$ is an isomorphism of flat $F$-structures if and only if $\varphi$ preserves the $\nabla$-affine structure, namely, $d\varphi(\text{ker } \nabla) = \text{ker}(\varphi^* \nabla)$.

For semisimple flat $F$-manifolds, Proposition 2.13 allows to compute the connected component $\text{AlgSym}(M) \circ_0$ of the identity.

Proposition 2.17. If $M$ is a semisimple flat $F$-manifold, then the connected component $\text{AlgSym}(M) \circ_0$ of the identity is a commutative $n$-dimensional Lie group. Moreover, it acts locally transitively on $M$.

Proof. The Lie algebra of $\text{AlgSym}(M)$ can be identified with the Lie algebra of vector fields $X \in \Gamma(TM)$ on $M$ such that $\mathcal{L}_X c = 0$. This is equivalent to the condition

\[ [X, Y \circ Z] - [X, Y] \circ Z - [X, Z] \circ Y = 0, \quad Y, Z \in \mathcal{F}_M. \]
In local canonical coordinates $u$, set $X = \sum X^i \delta_i$ and take $Y = Z = \delta_j$. We have $\delta_j X^k = 0$ for all $j, k$. Hence, locally, $X$ is a constant linear combination of the idempotent vector fields $\delta_j$. In local canonical coordinates, the flow of $X$ reads as shifts $u^i \mapsto u^i + c^i$. □

Remark 2.18. On a semisimple flat $F$-manifold, we have two different affine structures: the first one is defined by $\nabla$, whereas the second one is defined by the atlas of canonical coordinates. As the proof of Proposition 2.17 shows, algebraic symmetries of a semisimple flat $F$-manifold preserve the second affine structure. In general, the two affine structures are not compatible: this only happens when the canonical coordinates $u$ are $\nabla$-flat. In such a case, the oriented associativity potentials take the form $F^i(u) = \frac{1}{2} u^2_i$, for $i = 1, \ldots, n$. The resulting flat $F$-manifold is trivial: it is isomorphic to $\mathbb{C}^n$ equipped with the product $\mathbb{C}$-algebra structure $(\mathbb{C}, +, \cdot , 1)^{\times n}$, seen as a flat $F$-manifold.

Remark 2.19. In [2], a notion of biflat $F$-manifold is introduced. This consists of the datum of two flat $F$-manifolds structures $(\nabla, \circ, e)$ and $(\nabla^*, \ast, E)$, on the same manifold $M$, satisfying the following compatibility conditions:

1. $E$ is $\circ$-invertible on $M$,
2. $\nabla^*_E(\circ) = \circ$,
3. $X \ast Y = E^{-1} \circ X \circ Y$, for all $X, Y \in \mathcal{T}_M$,
4. $(d_{\nabla} - d_{\nabla^*})(X \circ) = 0$, for all $X \in \mathcal{T}_M$ (here $d_{\nabla}$ is the exterior covariant derivative).

In [4], in the tame semisimple case (pairwise distinct canonical coordinates), it is proved that a bi-flat $F$ structure is actually equivalent to the datum of a homogeneous flat $F$-manifold with invertible Euler vector field $E$. We also refer the reader to [55], where the identification of biflat $F$-structures and homogeneous flat $F$-structures (in loc. cit. they are called Saito structures) is established even beyond the semisimple case. Furthermore, in [55], Dubrovin’s almost duality formalism for Frobenius manifolds is extended to homogeneous flat $F$-manifolds.

### 2.5 Irreducible flat $F$-manifolds

If $M_1, M_2$ are two flat $F$-manifolds, their product $M_1 \times M_2$ is naturally equipped with a flat $F$-structure, called the sum $M_1 \oplus M_2$. If $M_1, M_2$ are homogenous, then also $M_1 \oplus M_2$ is homogeneous.

We say that a flat $F$-manifold $M$ is irreducible if no pointed germ $(M, p)$ is locally isomorphic to a pointed sum $(M_1 \oplus M_2, p')$.

In the semisimple homogeneous case, we have the following characterization of irreducibility.

**Theorem 2.20.** Let $M$ be a formal/analytic semisimple and homogeneous flat $F$-manifold. The following conditions are equivalent:

1. $M$ is irreducible;
2. if $E_1, E_2 \in \Gamma(TM)$ are two Euler vector fields, then $E_2 = E_1 - \lambda e$ for some $\lambda \in \mathbb{C}$.

The proof of this result can be found in Appendix A.
Remark 2.21. Theorem 2.20 underlines how much selective is the condition $\nabla \nabla E = 0$. In the category of $F$-manifolds (not necessarily flat), Euler vector fields are defined† by the condition $\mathcal{L}_E c = c$ only. For semisimple $F$-manifolds, given an Euler vector field $E$, all other Euler fields are of the form $E + \sum_{i=1}^n C \pi_i$. See [46, Ex. 2.12(ii)].

3 | EXTENDED DEFORMED CONNECTIONS

3.1 | $\nabla^z$-flat coordinates and oriented associativity potentials

In both the analytic and the formal context (over $k$), we can look for $\nabla^z$-flat coordinates of the flat $F$-structure, that is, functions $\tilde{t}^z(t, z)$ such that $\nabla^z d\tilde{t}^z = 0$. Assume they are of the form

$$\tilde{t}^z(t, z) := \sum_{p=0}^{\infty} h^z_p(t) z^p \in k[t, z], \quad h^z_0(t) = t^\alpha, \quad \alpha = 1, \ldots, n.$$

Theorem 3.1. The functions $h^z_p$ satisfy the recursive equations

$$h^z_0(t) = t^\alpha, \quad \partial_\gamma \partial_\beta h^z_p = c^\lambda_\gamma^\beta h^z_{p+1}, \quad p \in \mathbb{N}.$$

Proof. The $\nabla^z$-flatness equations for a one-form $\xi = \xi_\alpha dt^\alpha$ are

$$\partial_\gamma \xi_\beta = z c^\lambda_\gamma^\beta \xi_\lambda.$$

Corollary 3.2. The functions $h^z_1(t)$ equal the oriented associativity potentials $F^z(t)$ up to linear terms.

Proof. We have $\partial_\gamma \partial_\beta h^z_1 = c^\alpha_\beta^\gamma$.

3.2 | Family of extended deformed connections

Following [74, Section 3], we introduce a one-parameter family $(\widehat{\nabla}^\lambda)_\lambda$ of flat connections, which “rigidify” the family $(\nabla^z)_z$. See also [10, Section 4.3] and [5, Section 1.4].

Analytic case. Let $(M, \nabla, c, e, E)$ be a homogenous flat $F$-manifold. Introduce the (1,1)-tensors $U, \mu^\lambda \in \Gamma(\text{End}(TM))$, with $\lambda \in \mathbb{C}$, by the formulae

$$U(X) = E \circ X, \quad \mu^\lambda(X) := (1 - \lambda)X - \nabla_X E, \quad X \in \mathcal{F}_M.$$

By Equation (2.3), in $t$-coordinates, we have $\mu^\lambda = \text{diag}(q_1 - \lambda, \ldots, q_n - \lambda)$.

Denote by $\pi : M \times \mathbb{C}^* \to M$ the canonical projection on the first factor. If $\mathcal{F}_M$ denotes the tangent sheaf of $M$, then $\pi^* \mathcal{F}_M$ is the sheaf of sections of $\pi^* TM$, and $\pi^{-1} \mathcal{F}_M$ is the sheaf of sections of $\pi^* TM$ constant along the fibers of $\pi$. All the tensors $c, e, E, U, \mu$ can be lifted to the pullback bundle $\pi^* TM$, and we denote these lifts with the same symbols. Consequently, also the connection $\nabla$ can be uniquely lifted on $\pi^* TM$ in such a way that $\nabla_{\frac{\partial}{\partial z}} Y = 0$ for $Y \in \pi^{-1} \mathcal{F}_M$.

†A more general notion of Euler vector field of weight $d \in \mathbb{C}$ is discussed in [46, 72]: these are vector fields $E$ such that $\mathcal{L}_E c = d \cdot c$. If $d \neq 0$, one can always rescale $E$ in order to be of weight 1.
The extended deformed connection $\hat{\nabla}^\lambda$, with $\lambda \in \mathbb{C}$, is the connection on $\pi^*TM$ defined by the formulae

$$\hat{\nabla}^\lambda_{\frac{\partial}{\partial t}} Y = \nabla_{\frac{\partial}{\partial t}} Y + z \frac{\partial}{\partial t} \circ Y, \quad \hat{\nabla}^\lambda_{\frac{\partial}{\partial z}} Y = \nabla_{\frac{\partial}{\partial z}} Y + \mathcal{U}(Y) - \frac{1}{z} \mu^\lambda(Y),$$

(3.1)

where $Y \in \pi^*\mathcal{T}M$.

**Formal case.** Let $k$ be a commutative $\mathbb{Q}$-algebra and $(H, \Phi, E)$ a formal homogeneous flat $F$-manifold over $k$. Denote by $k[[z]]$ the $k$-algebra of formal Laurent series in an auxiliary indeterminate $z$. Set $K[[z]] := k[[t]] [[z]]$ and $H_K[[z]] := H \otimes_k K[[z]]$.

In what follows we assume that the $K$-linear operator $\nabla^0 E : \text{Der}_k(K) \cong H_K \rightarrow H_K$ is (diagonalizable and) in diagonal form in the basis $(\Delta_1, \ldots, \Delta_n)$. Define the $K$-linear operators $\mathcal{U}, \mu^\lambda$, with $\lambda \in k$, by the formulae

$$\mathcal{U} : H_K \rightarrow H_K, \quad X \mapsto E \circ X,$$

$$\mu^\lambda : \text{Der}_k(K) \cong H_K \rightarrow H_K, \quad X \mapsto (1 - \lambda)^{-1} \nabla_X E.$$

All the tensors $\circ, \mathcal{U}, \mu^\lambda$ can be $K((z))$-linearly extended to $H_K((z))$. We will denote such an extension by the same symbols.

Denote by $\text{Diff}_1(H_K[[z]], H_K[[z]])$ the set of morphisms $\mathcal{D} \in \text{Hom}_k(H_K[[z]], H_K[[z]])$ such that

$$ab \mathcal{D}(p) - b \mathcal{D}(ap) - a \mathcal{D}(bp) + \mathcal{D}(abp) = 0, \quad a, b \in K[[z]], \quad p \in H_K[[z]].$$

Both $\text{Der}_k(K[[z]])$ and $\text{Diff}_1(H_K[[z]], H_K[[z]])$ are naturally equipped with an $K((z))$-module structure.

The extended deformed connection $\hat{\nabla}^\lambda : \text{Der}_k(K[[z]]) \rightarrow \text{Diff}_1(H_K[[z]], H_K[[z]])$ is the $K((z))$-linear operator defined by the formulae

$$\hat{\nabla}^\lambda_{\frac{\partial}{\partial t}} X = \nabla_{\frac{\partial}{\partial t}} X, \quad \hat{\nabla}^\lambda_{\frac{\partial}{\partial z}} X = \frac{\partial}{\partial z} X + \mathcal{U}(Y) - \frac{1}{z} \mu^\lambda(X),$$

where $Y \in H_K[[z]]$.

In both the analytic and formal pictures, the following result holds.

**Theorem 3.3.** The connection $\hat{\nabla}^\lambda$ is flat for any $\lambda \in \mathbb{C}$ (resp. $\lambda \in k$).

**Proof.** The flatness of $\hat{\nabla}^\lambda$ is equivalent to the following conditions: $\partial_{x^\alpha} c^\delta_{\beta \gamma}$ is completely symmetric in $(\alpha, \beta, \gamma)$, the product $\circ$ is associative, $\nabla \nabla E = 0$, and $\mathcal{U} c = c$. This can be easily checked by a straightforward computation. \hfill \Box

**Remark 3.4.** For $\lambda = \frac{1}{2}d$, the connection $\hat{\nabla}^\lambda$ equals the extended deformed connection $\hat{\nabla}$ as defined by Dubrovin [36–38]. In that case, the tensor $\mathcal{U}$ (resp. $\mu = \mu^\lambda$) is $\eta$-self-adjoint (resp. $\eta$-anti-self-adjoint). It follows that if $\zeta_1, \zeta_2 \in \pi^*\mathcal{T}M$ are two $\hat{\nabla}$-flat vector fields, then the pairings $\langle \zeta_1, \zeta_2 \rangle_\pm := \eta(\zeta_1(t, e^{\pm \pi \sqrt{-1} z}), \zeta_2(t, z))$ do not depend on $(t, z)$. See also [27, Section 2].
Remark 3.5. In both the analytic and formal cases, we have $\mu^\lambda(e) = -\lambda e$. This follows from the torsionless of $V = V^0$, the $V$-flatness of $e$, and Lemma 2.2. Without loss of generality, we can then assume that the flat coordinate $t^1$ is such that $\frac{\partial}{\partial t^1} = e$ (analytic case), and that $\Delta_0 = e$ (formal case). In such a case, the parameter $q_1$ in (2.3) satisfies $q_1 = 0$.

### 3.3 $\hat{\nabla}^\lambda$-flat covectors

In both the analytic and formal pictures, the extended connections $\hat{\nabla}^\lambda$ induce connections on the whole tensor algebra of $\pi^*TM$ (resp. $H_K((z))$). So, for example, let $\xi$ denote a $\hat{\nabla}^\lambda$-flat section of the bundle $\pi^*(T^*M)$. In the coframe $(dt^\alpha)_{\alpha=1}^n$, the equation $\hat{\nabla}^\lambda \xi = 0$ can be written, in more convenient matrix notations, as the joint system of differential equations

$$\frac{\partial \xi}{\partial t^\alpha} = z C^\lambda_{\alpha} \xi, \quad \frac{\partial \xi}{\partial z} = \left(U - \frac{1}{z} \mu^\lambda \right)^T \xi, \quad (3.2)$$

where $\xi = (\xi_1, ..., \xi_n)^T$ is a column vector of components w.r.t. $(dt^\alpha)$, and

$$(C^\alpha_{\beta})_{\gamma}^\lambda := c^\gamma_{\alpha \beta}, \quad (U^\beta)^\alpha_{\gamma} = E^\gamma c^\beta_{\alpha \gamma}, \quad (\mu^\lambda)^\beta_{\alpha} = (q_\alpha - \lambda) \delta_{\alpha \beta}.$$ 

We will refer to the second of Equations (3.2) as the $\partial_z$-equation of the flat $F$-manifold.

### 3.4 Matrices $\bar{\Psi}, \bar{V}_i, \bar{V}^\lambda, \bar{\Gamma}$

Assume that $(M, \nabla, c, e, E)$ is a semisimple homogeneous flat $F$-manifold, and introduce the Jacobian matrix $\bar{\Psi}$ by

$$\bar{\Psi}^i_{\alpha} := \frac{\partial u^i}{\partial t^\alpha}, \quad i, \alpha = 1, ..., n.$$ 

Remark 3.6. If the flat coordinate $t^1$ is chosen so that $\frac{\partial}{\partial t^1} = e$, then we have $\bar{\Psi}_1^i = 1$ for any $i = 1, ..., n$.

In canonical coordinates $u$, under the gauge transformation $\bar{x} = (\bar{\Psi}^{-1})^T \xi$, the system (3.2) becomes

$$\frac{\partial \bar{x}}{\partial u^i} = (z E_i - \bar{V}_i)^T \bar{x}, \quad \frac{\partial \bar{x}}{\partial z} = \left(U - \frac{1}{z} \bar{V}^\lambda \right)^T \bar{x}, \quad (3.3)$$

where

$$(E_i)^{jk} = \delta_{ij} \delta_{ik}, \quad U := \text{diag}(u^1, ..., u^n), \quad \bar{V}_i := \partial_i \bar{\Psi} \cdot \bar{\Psi}^{-1}, \quad \bar{V}^\lambda := \bar{\Psi} \cdot \mu^\lambda \cdot \bar{\Psi}^{-1}.$$
For any matrix $A \in M_n(\mathbb{C})$, we denote by $A'$ and $A''$ the diagonal and off-diagonal part of $A$, respectively. So, we have the decomposition:

$$A = A' + A'', \quad (A')_{ij} = 0, \quad (A'')_{ij} = 0, \quad i, j = 1, \ldots, n, \quad i \neq j.$$

**Proposition 3.7.** The following facts hold true for both formal and analytic semisimple homogeneous flat $F$-manifolds.

1. There exists an off-diagonal matrix $\tilde{\Gamma}$ such that

   $$\tilde{V}_i = \tilde{V}'_i + [\tilde{\Gamma}, E_i], \quad i = 1, \ldots, n,$$

   $$\tilde{V}^\lambda = (\tilde{V}^\lambda)' + [\tilde{\Gamma}, U].$$

   In particular, $\tilde{\Gamma}_i^j = -(\tilde{V}_i)^j_i$ for $i \neq j$.

2. We have

   $$[E_i, \tilde{V}^\lambda] = [U, \tilde{V}_i], \quad \partial_i \tilde{V}^\lambda = [\tilde{V}_i, \tilde{V}^\lambda].$$

3. The diagonal entries of the matrix $\tilde{V}^\lambda$ are constant w.r.t. $u$.

4. We have $\partial_i \tilde{V}'_j = \partial_j \tilde{V}'_i$.

**Proof.** The compatibility condition $\partial_i \partial_j = \partial_j \partial_i$ implies the constraints

$$\partial_i \tilde{V}_j - \partial_j \tilde{V}_i = [\tilde{V}_i, \tilde{V}_j], \quad [E_i, E_j] = 0, \quad [E_i, \tilde{V}_j] = [\tilde{V}_j, E_i].$$

Identities (3.7) are trivially satisfied, by definition of the matrices $E_i$ and $\tilde{V}_i$. From (3.8), we deduce

$$(\delta_{jh} - \delta_{jk})(\tilde{V}_i)^h_k = (\delta_{ih} - \delta_{ik})(\tilde{V}_j)^h_k \implies (\tilde{V}_i)^h_j = (\delta_{ij} - \delta_{ik})(\tilde{V}_j)^h_k = [\tilde{\Gamma}, E_i]^h_j,$$

where $\tilde{\Gamma} = (\tilde{\Gamma}_i^j)_{i,j,k}$ is defined by $\tilde{\Gamma}_i^j := -(\tilde{V}_i)^j_i$. This proves (3.4). The compatibility condition $\partial_i \partial_z = \partial_z \partial_i$ implies the constraints

$$[E_i, U] = 0, \quad [E_i, \tilde{V}^\lambda] = [U, \tilde{V}_i], \quad \partial_i \tilde{V}^\lambda = [\tilde{V}_i, \tilde{V}^\lambda].$$

Identity (3.9) is trivially satisfied. Identity (3.5) follows from (3.4) and the first of (3.10). The constancy of $(\tilde{V}^\lambda)'$ follows from the second identity of (3.10), and Equations (3.4) and (3.5). Finally, from the first identity of (3.7), we have $\partial_i \tilde{V}'_j - \partial_j \tilde{V}'_i = [\tilde{V}_i, \tilde{V}_j]' = 0$, by (3.4), (3.5). \[\square\]

### 3.5 Darboux–Tsarev equations, and conformal dimensions

Let us introduce the Christoffel symbols $K^h_{ij}$ by $\nabla_{\partial_i} \partial_j = \sum_h K^h_{ij} \partial_h$. 

Lemma 3.8. We have $K^h_{ij} = - (\mathcal{V}_i)^h_j$.

Proof. The claim follows from the following computation:

$$\nabla_{\delta_j} \partial_j = \nabla_{\delta_j} [(\Psi^{-1})_j^\alpha \partial_\alpha] = [\delta_j (\Psi^{-1})_j^\alpha] \partial_\alpha = - \sum_c (\Psi^{-1})_c^\alpha \partial_c \Psi^\alpha (\Psi^{-1})^\delta_\beta \partial_\delta = - \sum_c (\mathcal{V}_C)_j^\alpha \partial_\alpha .$$

□

Proposition 3.9. The following identities hold true:

$$K^q_{ij} = 0, \quad i, j, h \text{ distinct}, \quad (3.11)$$

$$K^i_{ij} = K^i_{ji} = - K^i_{jj} = \tilde{\Gamma}_i^j, \quad i \neq j, \quad (3.12)$$

$$K^i_{ii} = - \sum_h \tilde{\Gamma}_i^h. \quad (3.13)$$

Moreover, the functions $\tilde{\Gamma}_j^i$ satisfy the Darboux–Tsarev equations

$$\partial_k \tilde{\Gamma}_j^i = - \tilde{\Gamma}_j^i \partial_k + \tilde{\Gamma}_j^j \tilde{\Gamma}_k^i k \neq j, \quad i, j \text{ distinct}, \quad (3.14)$$

$$\sum_k \partial_k \tilde{\Gamma}_j^i = 0, \quad i \neq j. \quad (3.15)$$

Proof. In canonical coordinates, we have $c_{jk}^i = \delta_j^i \delta_k^i$. Consequently, $(\nabla \varepsilon c)^i_{jk} = \sum_p \delta_j^p \delta_k^p K^i_{jp} - \delta_k^i \delta_j^j - \delta_j^i K^i_{jk}$. We have $K^h_{ij} = K^h_{ji}$ because $\nabla$ is torsion free. Hence, from the symmetry $(\nabla \varepsilon c)^i_{jk} = (\nabla_j c)^i_{jk}$, one obtains (3.11), and the first two equalities of (3.12). The equality $K^i_{ij} = \tilde{\Gamma}_j^i$ follows from Lemma 3.8. Equation (3.13) follows from the condition $\nabla \varepsilon = 0$. By flatness of $\nabla$, the components $R^i_{jkl}$ of the Riemann tensor equal zero. By definition, we have $R^i_{jkl} = \partial_k K^i_{jl} - \partial_l K^i_{jk} + \sum_p K^i_{kp} K^p_{jl} - \sum_p K^i_{lp} K^p_{jk}$. Darboux–Tsarev equation (3.14) is equivalent to $R^i_{jkl} = 0$. From the identity $\nabla_{\delta_j} \partial_\alpha = \sum_l (\delta_l \Psi^l_\alpha) \partial_j + \sum_l \Psi^l_\alpha \nabla_{\delta_j} \partial_j$, summing over $i$, we obtain

$$0 = \nabla \varepsilon \delta_\alpha = \sum_l \nabla_{\delta_j} \partial_\alpha = \sum_j \left( \sum_l \delta_l \Psi^l_\alpha \right) \partial_j + \sum \Psi^l_\alpha \nabla \varepsilon \partial_j , \quad (3.16)$$

$$\implies \sum_l \delta_l \Psi = 0. \quad (3.17)$$

We have

$$\partial_k \tilde{\Gamma}_j^i = - \partial_k^2 \tilde{\Psi}^i_\alpha (\Psi^{-1})_j^\alpha + \partial_l \Psi^l_\alpha \sum_h (\Psi^{-1})_h^\alpha \partial_k \Psi^h_\gamma (\Psi^{-1})^\gamma_j.$$
Corollary 3.10. For $i = 1, \ldots, n$, the matrix $\tilde{V}_i$ has the following structure:

$$
\tilde{V}_i = \begin{pmatrix}
-\tilde{\Gamma}_1^i & -\tilde{\Gamma}_2^i & \cdots & -\tilde{\Gamma}_{i-1}^i \\
-\tilde{\Gamma}_1^i & -\tilde{\Gamma}_2^i & \cdots & -\tilde{\Gamma}_{i-1}^i \\
\vdots & \vdots & \ddots & \vdots \\
-\tilde{\Gamma}_1^i & -\tilde{\Gamma}_2^i & \cdots & \tilde{\Gamma}_n^i
\end{pmatrix}.
$$

Proposition 3.11. We have $\tilde{V}^\lambda = -\lambda \cdot 1 + \sum_i u_i \tilde{V}_i$.

Proof. For any $i$, we have

$$
\nabla_{\partial_i} E = \nabla_{\partial_i} \sum_j u^i \partial_j = \partial_i + \sum_j u^i \nabla_{\partial_i} \partial_j = \partial_i + \sum_{j,h} u^i K^h_{ij} \partial_h = \partial_i + \sum_{j,h} u^i K^h_{ij} \partial_h.
$$

By Lemma 3.8, one concludes.

Corollary 3.12. The following identities hold true:

$$
\sum_j u^i \partial_j \tilde{V}_i = -\tilde{V}_i, \quad (3.18)
$$

$$
\sum_j u^i \partial_j \tilde{\Gamma} = -\tilde{\Gamma}, \quad (3.19)
$$

$$
(u^i - u^j) \partial_i \tilde{\Gamma}_j^i = \sum_{\epsilon \neq i,j} (u^i - u^\epsilon) \{-\tilde{\Gamma}_j^i \tilde{\Gamma}_j^i + \tilde{\Gamma}_j^i \tilde{\Gamma}_j^\epsilon + \tilde{\Gamma}_j^\epsilon \tilde{\Gamma}_j^\epsilon \} - \tilde{\Gamma}_j^i, \quad (3.20)
$$

$$
(u^i - u^j) \partial_j \tilde{\Gamma}_i^j = \sum_{\epsilon \neq i,j} (u^i - u^\epsilon) \{-\tilde{\Gamma}_i^j \tilde{\Gamma}_i^j + \tilde{\Gamma}_i^j \tilde{\Gamma}_i^\epsilon + \tilde{\Gamma}_i^\epsilon \tilde{\Gamma}_i^\epsilon \} - \tilde{\Gamma}_i^j. \quad (3.21)
$$

Proof. Equation (3.18) follows from the second equation of (3.6) and the first equation of (3.7). Equation (3.19) is easily deduced. Equations (3.20) and (3.21) follow from (3.14), (3.15), and (3.19).

Remark 3.13. In this section, we started from a given semisimple flat $F$-manifold and we obtained a solution $\tilde{\Gamma}_j^i$ of the Darboux–Tsarev equations. This is also proved in [3]. In loc. cit., it is proved that the opposite construction works as well: the datum of

- a solution $\tilde{\Gamma}_j^i$ of (3.14), (3.15),
- the connection $\nabla$ with Christoffel symbols $K^i_{jk}$ given by (3.11), (3.12), and (3.13),
- the structure constants $c^i_{jk} = \delta^i_j \delta^j_k$,
- the vector field $e := \sum_i \partial_i$,

locally defines a (tame) semisimple flat $F$-manifold structure on $\mathbb{C}^n \setminus \Delta$. 

Conformal dimensions. By Propositions 3.7(3) and 3.11, there exist complex numbers \( \delta_1, ..., \delta_n \in \mathbb{C} \) such that

\[
(\tilde{V}^\lambda)' = -\lambda \cdot 1 + \text{diag}(\delta_1, ..., \delta_n).
\]

**Definition 3.14.** The numbers \( \delta_1, ..., \delta_n \) are called conformal dimensions of the (formal or analytic) semisimple flat \( F \)-manifold. We will say that a (formal/analytic) semisimple flat \( F \)-manifold is conformally resonant if \( \delta_i - \delta_j \in \mathbb{Z} \setminus 0 \) for some \( i, j \).

**Remark 3.15.** We have \( \delta_i = \sum_k u^k (\tilde{V}_k)^i = \sum_{k \neq i} (u^i - u^k) \tilde{\Gamma}^i_k \).

**Remark 3.16.** In the case of Frobenius manifolds, all conformal dimensions equal \( \frac{1}{2} d \), where \( d \) is the conformal dimension of Equation (2.6). In particular, a Frobenius manifold is never conformally resonant.

### 3.6 Lamé coefficients, matrices \( \Psi, V_i, V^\lambda, \Gamma, \) and Darboux–Egoroff equations

For any \( j = 1, ..., n \), define the one form

\[
\omega_j(u) := -\sum_{i=1}^n (\tilde{V}_i(u))^j du^i.
\]

**Proposition 3.17.** The one forms \( \omega_j \) are closed. There locally exist functions \( H_j(u) \) such that

\[
d \log H_j = \omega_j, \quad j = 1, ..., n.
\]

**Proof.** It follows from point (4) of Proposition 3.7. \( \square \)

The functions \( H_j \) are called Lamé coefficients, and they are defined up to scalar rescaling \( H_j \mapsto \lambda_j H_j, \lambda_j \in \mathbb{C}^* \).

Arrange the Lamé coefficients in the diagonal matrix \( H := \text{diag}(H_1, ..., H_n) \), and define the matrices

\[
\Psi := H \tilde{\Psi}, \quad V_i := H \tilde{V}_i H^{-1} + \partial_i H \cdot H^{-1}, \quad V^\lambda := H \tilde{V}^\lambda H^{-1}, \quad \Gamma := H \tilde{\Gamma} H^{-1}.
\]

**Proposition 3.18.** The functions \( H_1, ..., H_n \) satisfy the following system:

\[
\partial_j H_i = \Gamma^i_j H_j, \quad i \neq j, \quad \partial_i H_i = -\sum_{k \neq i} \Gamma^i_k H_k, \quad \sum_j u^j \partial_j H_i = -\delta_i H_i.
\]

**Proof.** It easily follows from the definitions and identities of the previous section. \( \square \)

**Remark 3.19.** If the \( V \)-flat coordinate \( t^1 \) is such that \( \frac{3}{\partial t^1} = e \), as in Remark 3.5, then we have \( \Psi^i_1 = H_i \) for any \( i = 1, ..., n \). In other words, the Lamé coefficients can be read from the first column of the
Ψ-matrix. Moreover, we have

\[ V^\lambda H = -\lambda H, \quad H = (H_1, ..., H_n)^T. \] (3.22)

Under the gauge transformation \( x = (H^{-1})^T \bar{x} \), the system (3.3) becomes

\[ \frac{\partial x}{\partial u^i} = (zE_i - V_i)^T x, \quad \frac{\partial x}{\partial z} = \left( U - \frac{1}{z} V^\lambda \right)^T x. \] (3.23)

**Proposition 3.20.** The following identities hold true:

\[ V_i = [\Gamma, E_i], \quad i = 1, ..., n \] (3.24)

\[ V^\lambda = (V^\lambda)' + [\Gamma, U], \quad (V^\lambda)' = \text{diag}(\delta_1 - \lambda, ..., \delta_n - \lambda), \] (3.25)

\[ \frac{\partial}{\partial z} \Psi \cdot \Psi^{-1} = V_i, \quad [E_i, V^\lambda] = [U, V_i], \quad \frac{\partial}{\partial z} V^\lambda = [V_i, V^\lambda]. \] (3.26)

**Proof.** It easily follows from the definitions and identities of the previous section. \( \square \)

**Proposition 3.21.** The matrix \( \Gamma \) satisfies the Darboux–Egoroff equations

\[ \frac{\partial_k \Gamma^i_j}{\partial_k} = \Gamma^i_k \Gamma^k_j, \quad i, j, k \text{ distinct}, \] (3.27)

\[ \sum_k \partial_k \Gamma^i_j = 0, \quad i \neq j, \] (3.28)

\[ \sum_k u^k \partial_k \Gamma^i_j = (\delta_j - \delta_i - 1) \Gamma^i_j, \quad i \neq j, \] (3.29)

\[ (u^j - u^l) \partial_j \Gamma^l_j = \sum_k \partial_k \Gamma^i_j = (u^k - u^l)(\Gamma^i_k \Gamma^k_j - (\delta_j - \delta_i - 1) \Gamma^i_j), \] (3.30)

\[ (u^i - u^l) \partial_j \Gamma^l_j = \sum_k \partial_k \Gamma^i_j = (u^k - u^l)(\Gamma^i_k \Gamma^k_j - (\delta_j - \delta_i - 1) \Gamma^i_j). \] (3.31)

**Proof.** It easily follows from the definitions, the Darboux–Tsarev system for \( \bar{\Gamma} \), and the homogeneity identities (3.19) of the previous section. \( \square \)

**Remark 3.22.** For \( n = 3 \), the Darboux–Egoroff joint system of Equations (3.27), (3.28), and (3.29) is equivalent to the full family of Painlevé equations PVI. See remarkable formulas of [62, Theorem 4.1].

**Remark 3.23.** In the case of Frobenius manifolds, there is a canonical choice for the Lamé coefficients: \( H_i = \eta(\partial_i, \partial_i)^{\frac{1}{2}} \) for \( i = 1, ..., n \). The resulting coefficients \( \Gamma^i_j \) are the rotation coefficients of the metric \( \eta \). They satisfy the further symmetry condition \( \Gamma^i_j = \Gamma^j_i \).

**Remark 3.24.** In the light of Remark 3.23, given a semisimple flat \( F \)-manifold with a fixed choice of the Lamé coefficients \( H_i \), we can define a metric by \( \eta := \sum_{i=1}^n H_i^2 du^i \). Such a metric clearly is compatible with the product, in the sense that the second of Equations (2.5) is satisfied. The flatness of \( \eta \) is the obstruction for the flat \( F \)-manifold to be actually Frobenius. For a more invariant description of the metric \( \eta \), see [5, Prop. 1.8].

In [5], a further notion of semisimple Riemannian \( F \)-manifold is introduced. In loc. cit., it is also proved the local equivalence of semisimple flat \( F \)-manifolds and semisimple Riemannian \( F \)-manifolds. Notice that the notion of semisimple Riemannian \( F \)-manifolds given in [5] relaxes the axioms of analog structures introduced in [32, 63]. See also the recent preprint [7].
From a given homogeneous semisimple flat $F$-manifold, we obtained a joint system of Equations (3.23). In the analytic case, such a joint system defines an isomonodromic system, because of integrability equations (3.26).

Vice versa, one can start from such an isomonodromic system to construct the homogeneous semisimple flat $F$-manifold structure. This is exactly the point of view of the definition of Saito structures without metric given in [79, Ch. VII, §1.a].

Notice that one can actually work with a companion Fuchsian system, obtained via a Fourier–Laplace transform. This is the point of view of [53], in which Saito structures without metric are constructed on the space of isomonodromic deformation parameters for extended Okubo systems.

In both [53, 79], coalescences of the parameters of deformations (the entries of $U = \text{diag}(u^1, \ldots, u^n)$ of Equation (3.23)) are not taken into account. In our Equation (3.23), on the contrary, we allow coalescences, provided that the matrix $\Psi$ is not singular, that is, provided that the geometric point of the flat $F$-structure is semisimple. In the recent paper [43], D. Guzzetti extended the results of [9, 42] to the case of Fuchsian systems with confluent singularities. Furthermore, in [43], a notion of isomonodromic Laplace transform is introduced: with such an analytic tool, the study of the correspondence

Monodromy data of an irregular system $\leftrightarrow$ Monodromy data of a Fuchsian system, originally developed in [9], has been extended to the isomonodromic case (possibly with coalescences/confluences). This also gives a new proof of the results of [26, 30].

The point of view of the current paper differs from the perspective of [53, 79], via a Riemann–Hilbert correspondence. In Section 6, we will show a one-to-one correspondence between (local isomorphism classes of) homogeneous semisimple flat $F$-structures and solvable RHB problems.

### 4 MONODROMY MODULI OF ADMISSIBLE GERMS OF SEMISIMPLE FLAT $F$-MANIFOLDS

#### 4.1 $\mu$-nilpotent operators and $\mu$-parabolic group

Let $(V, \mu)$ be the datum of a $n$-dimensional complex vector space, and a diagonalizable operator $\mu : V \to V$. Denote by $\text{spec}(\mu) = (\mu_1, \ldots, \mu_n)$ the spectrum of $\mu$, and by $V_{\mu_\alpha}$ the eigenspace corresponding to the eigenvalue $\mu_\alpha$.

We say that $A \in \text{End}(V)$ is $\mu$-nilpotent if

$$AV_{\mu_\alpha} \subseteq \bigoplus_{m \geq 1} V_{\mu_\alpha+m} \quad \text{for all } \mu_\alpha \in \text{spec}(\mu).$$

In particular, such an operator is nilpotent in the usual sense. Denote by $\epsilon(\mu)$ the set of all $\mu$-nilpotent operators. It is easy to see that the set $\epsilon(\mu)$ is a Lie algebra w.r.t. the commutator $[\cdot, \cdot]$ in $\text{End}(V)$. We can decompose a $\mu$-nilpotent operator $A$ in components $A_k$, $k \geq 1$, such that

$$A_k V_{\mu_\alpha} \subseteq V_{\mu_\alpha+k} \quad \text{for any } \mu_\alpha \in \text{spec}(\mu),$$

This will be explained in details the next section.
so that the following identities hold:

\[ z^\mu A z^{-\mu} = A_1 z + A_2 z^2 + A_3 z^3 + \ldots, \quad [\mu, A_k] = kA_k \quad \text{for } k = 1, 2, 3, \ldots. \]

**Lemma 4.1.** Let \((V, \mu)\) as above, and let us fix a basis \((v_i)_{i=1}^n\) of eigenvectors of \(\mu\).

1. The operator \(A \in \text{End}(V)\) is \(\mu\)-nilpotent if and only if its associate matrix w.r.t. the basis \((v_i)_{i=1}^n\) satisfies the condition \((A)^2 = 0\) unless \(\mu_\alpha - \mu_\beta \in \mathbb{N}^*\).
2. If \(A \in \text{End}(V)\) is a \(\mu\)-nilpotent operator, then the matrices associated with its components \((A_k)_{k \geq 1}\) w.r.t. the basis \((v_i)_{i=1}^n\) satisfy the condition \((A_k)^2 = 0\) unless \(\mu_\alpha - \mu_\beta = k\), with \(k \in \mathbb{N}^*\).

Define the \(\mu\)-parabolic group to be the Lie group \(\mathcal{G}(\mu)\) of operator \(G \in GL(V)\) such that \(G = \mathbf{1} + A\), with \(A \in \mathfrak{c}(\mu)\). We have the canonical identification of Lie algebras \(T_1 \mathcal{C}(\mu) = \mathfrak{c}(\mu)\), and the canonical adjoint action \(\text{Ad} : \mathcal{C}(\mu) \to \text{Aut}(\mathfrak{c}(\mu))\) defined by

\[ \text{Ad}_G(A) := GAG^{-1}, \quad G \in \mathcal{C}(\mu), \quad A \in \mathfrak{c}(\mu). \]

**Remark 4.2.** Consider the space \(V^* := \text{Hom}_\mathbb{C}(V, \mathbb{C})\). Each \(f \in \text{End}_\mathbb{C}(V)\) induces a dual map \(f^* \in \text{End}_\mathbb{C}(V^*)\), defined by \(f^*(w) := w \circ f\), where \(w \in V^*\). This defines an anti-isomorphism of Lie algebras

\[ (-)^* : \text{End}_\mathbb{C}(V) \to \text{End}_\mathbb{C}(V^*), \quad [f_1, f_2]^* = -[f_1^*, f_2^*]. \]

The image of \(\mathfrak{c}(\mu)\) coincides with \(\mathfrak{c}(-\mu^*)\).

### 4.2 Spectrum of a flat F-manifold

Consider an analytic pointed flat F-manifold \((M, p)\). For any \(\lambda \in \mathbb{C}\), we have a pair \((T_pM, \mu^\lambda_p)\) satisfying all the assumption of Section 4.1. We can consequently introduce the Lie algebra \(\mathfrak{c}(\mu^\lambda_p)\), and the Lie group \(\mathcal{C}(\mu^\lambda_p)\).

Let \((V_1, \mu_1), (V_2, \mu_2)\) be two pairs as in Section 4.1. We say that they are **equivalent pairs** if there exist \(\kappa \in \mathbb{C}\) and a \(\mathbb{C}\)-vector space isomorphism \(f : V_1 \to V_2\) such that \(f \circ \mu_1 = (\mu_2 - \kappa \cdot \text{id}_{V_2}) \circ f\).

Given \(\lambda_1, \lambda_2 \in \mathbb{C}\), the pairs \((T_pM, \mu^\lambda_1_p)\) and \((T_pM, \mu^\lambda_2_p)\) are equivalent. Moreover, given \(p_1, p_2 \in M\), the two pairs attached to the germs \((M, p_1)\) and \((M, p_2)\) are equivalent: using the connection \(\nabla\), for any path \(\gamma : [0, 1] \to M\) with \(\gamma(0) = p_1\) and \(\gamma(1) = p_2\), the parallel transport along \(\gamma\) provides an isomorphism realizing the equivalence of the pairs at \(p_1\) and \(p_2\).

As a result, with any homogeneous flat F-manifold \((M, \nabla, c, e, E)\) (not necessarily semisimple), we can canonically associate an equivalence class \([\mathcal{C}(\mu)]\) of pairs as above, which will be called the **spectrum** of \(M\).

Fix a system of flat coordinates \(t = (t^1, \ldots, t^n)\) diagonalizing \(\mu^\lambda = \text{diag}(q_1 - \lambda, \ldots, q_n - \lambda)\). We can thus introduce the \(\lambda\)-independent matrix Lie algebras

\[
\mathfrak{c}(\mu) := \left\{ R \in \mathfrak{gl}(n, \mathbb{C}) : R_{\alpha\beta} = 0 \text{ unless } q_\alpha - q_\beta \in \mathbb{Z}_{>0} \right\},
\]

\[
\mathfrak{c}(-\mu^*) := \left\{ R \in \mathfrak{gl}(n, \mathbb{C}) : R_{\alpha\beta} = 0 \text{ unless } q_\alpha - q_\beta \in \mathbb{Z}_{<0} \right\},
\]
which are canonically anti-isomorphic via transposition, see Remark 4.2. We also denote by $C(\mu)$ and $C(-\mu^*)$ the corresponding parabolic Lie groups.

### 4.3 Solutions in Level $t$ normal forms and monodromy data at $z = 0$

We now introduce some formal invariant of the given analytic flat $F$-manifold, by studying Level $t$ normal forms of solutions of the joint system of differential Equations (3.2).

**Theorem 4.3.**

1. There exist $n \times n$-matrix valued functions $(G_p(t))_{p \geq 1}$, analytic in $t$, and a $t$-independent matrix $R \in \mathfrak{e}(-\mu^*)$, such that the matrix

$$
\Xi(t, z) = G(t, z)z^{-\mu}z^R, \quad G(t, z) = 1 + \sum_{p=1}^{\infty} G_p(t)z^p,
$$

is a (formal) solution of the joint system (3.2).

2. The series $G(t, z)$ converges to an analytic function in $(t, z)$. The matrix $\Xi(t_0, z)$ is a fundamental system of solutions of the $\partial_z$-equation of (3.2) for any fixed $t_0$.

**Proof.** Consider $n$ functions $\tilde{t}^\alpha(t, z) = \sum_{p=0}^{\infty} h^\alpha_p(t)z^p$ such that $\nabla_z d\tilde{t}^\alpha = 0$, and $\tilde{t}^\alpha(t, 0) = t^\alpha$. This translates in the following recursive equations for the coefficients $h^\alpha_p$:

$$
h^\alpha_0(t) = t^\alpha, \quad \partial_\gamma \partial_\beta h^\alpha_{p+1} = c_\gamma^\epsilon \partial_\epsilon h^\alpha_p, \quad p \geq 0.
$$

Introduce the Jacobian matrix $J(t, z) := (J_\alpha^\beta)_{\alpha, \beta}$, with $J_\alpha^\beta(t, z) := \frac{\partial \tilde{t}^\alpha}{\partial t^\beta}$. Under the gauge transformation $\xi = J^T \tilde{\xi}$, the joint system (3.2) becomes

$$
\frac{\partial \tilde{\xi}}{\partial t^\alpha} = \left( zJC_\alpha J^{-1} - \frac{\partial J}{\partial t^\alpha} J^{-1} \right)^T \tilde{\xi} = 0,
$$

$$
\frac{\partial \tilde{\xi}}{\partial z} = \left( J \left( U - \frac{1}{z^{\mu}} \right) J^{-1} - \frac{\partial J}{\partial z} J^{-1} \right)^T \tilde{\xi} = \left( -\frac{1}{z}(\mu^C)^T + U_1^T + zU_2^T + z^2U_3^T + ... \right) \tilde{\xi},
$$

for suitable matrices $U_k$. From the compatibility $\partial_\alpha \partial_z = \partial_z \partial_\alpha$, it follows that the matrices $U_k$ are $t$-independent. Up to a further gauge transformation $\tilde{\xi} \mapsto G(z) \tilde{\xi}$, of the form $G(z) = 1 + \sum_{k=1}^{\infty} G_k z^k$, the differential Equation (4.1) can be put in a normal form

$$
\frac{\partial \tilde{\xi}}{\partial z} = \left( -\frac{1}{z} \mu^C + R_1 + zR_2 + z^2R_3 + ... \right),
$$

$$(R_k)_{\beta}^\alpha \neq 0 \text{ only if } \mu^C_\alpha - \mu^C_\beta = -k, \quad k \geq 1.$$
Indeed, from the recursion relations
\[ R_n = U_n^T + nG_n - [G_n, \mu^\lambda] + \sum_{k=1}^{n-1} (G_{n-k} U_k^T - R_k G_{n-k}), \]
we determine the entries \((R_n)_{\alpha \beta}\) for \((\mu^\lambda)_{\alpha} - (\mu^\lambda)_{\beta} = -n\), and \((G_n)_{\alpha \beta}\) for \((\mu^\lambda)_{\alpha} - (\mu^\lambda)_{\beta} \neq -n\). We set \((G_n)_{\alpha \beta} = 0\) for \((\mu^\lambda)_{\alpha} - (\mu^\lambda)_{\beta} = -n\). See also [40]. A fundamental system of solutions of (4.2) is given by \(\bar{\xi}(z) = z^{-\mu^\lambda} z^R\), where \(R = \sum_k R_k\). The proof of the convergence of the series \(G(t, z)\) is standard, the reader can consult, for example, [83, 86].

**Corollary 4.4.** The monodromy matrix \(M_0\), defined by \(\Xi(t, e^{2\pi \sqrt{-1} z}) = \Xi(t, z) M_0\), is independent of \(t\). We have \(M_0 := \exp(-2\pi \sqrt{-1} \mu^\lambda) \exp(2\pi \sqrt{-1} R)\).

Solutions \(\Xi(t, z)\) of the form above will be said to be in Level t normal form.

**Theorem 4.5.** Assume that
\[ \Xi(t, z) = G(t, z) z^{-\mu^\lambda} z^R, \quad G(t, z) = 1 + \sum_{p=1}^{\infty} G_p(t) z^p, \quad R \in c(-\mu^*) \]
\[ \bar{\Xi}(t, z) = \bar{G}(t, z) z^{-\mu^\lambda} z^{-\bar{R}}, \quad \bar{G}(t, z) = 1 + \sum_{p=1}^{\infty} \bar{G}_p(t) z^p, \quad \bar{R} \in c(-\mu^*), \]
are two solutions of the joint system (3.2), in Level t normal form. Then, there exists a unique \(C \in C(-\mu^*)\) such that \(\bar{C} - 1 R C\), the function \(p_C(z) := z^{-\mu^\lambda} z^R C z^{-\bar{R}} z^{\mu^\lambda}\) is polynomial in \(z\), and \(\bar{G}(t, z) = p_C(z) G(t, z)\).

**Proof.** By assumption, there exist an unique invertible matrix \(C \in M_n(\mathbb{C})\) such that \(\bar{\Xi} = \Xi C\). This implies that
\[ G^{-1} \bar{G} = z^{-\mu^\lambda} z^R C z^{-\bar{R}} z^{\mu^\lambda}. \]
We deduce that the r.h.s. is a series in \(z\) of the form \(z^{-\mu^\lambda} z^R C z^{-\bar{R}} z^{\mu^\lambda} = 1 + H_1 z + H_2 z^2 + \ldots\). Actually, this sum is finite (i.e., a polynomial in \(z\)) because both \(R\) and \(\bar{R}\) are nilpotent. We can also rewrite this identity as follows:
\[ z^R C z^{-\bar{R}} = z^{\mu^\lambda} (1 + H_1 z + H_2 z^2 + \ldots) z^{-\mu^\lambda}. \] (4.3)
The l.h.s. is a polynomial in log \(z\), and the r.h.s. contains only powers of \(z\). Hence, both sides should actually be independent of \(z\). The \((\alpha, \beta)\)-entry of the r.h.s. equals
\[ \delta_{\alpha \beta} + H_1 \delta_{\alpha \beta} z^{(\mu^\lambda)_{\alpha} - (\mu^\lambda)_{\beta} + 1} + H_2 \delta_{\alpha \beta} z^{(\mu^\lambda)_{\alpha} - (\mu^\lambda)_{\beta} + 2} + \ldots, \]
which is \(z\)-independent if and only if \((H_k)_{\alpha \beta} = 0\) for \((\mu^\lambda)_{\alpha} - (\mu^\lambda)_{\beta} \neq -k\). Set \(z = 1\) in (4.3): we have \(C = 1 + \sum_k H_k\). This shows that \(C \in C(-\mu^*)\).
We have just shown that both sides of (4.3) are $z$-independent and they equal $C$. The l.h.s. of Equation (4.3) can also be written as $Cz^{C^{-1}RC}z^{-\tilde{R}}$. Thus, $Cz^{C^{-1}RC}z^{-\tilde{R}} = C$. It follows that $\tilde{R} = C^{-1}RC$.

**Definition 4.6.** We call monodromy data at $z = 0$ of the flat $F$-manifold the datum $(\lambda, \mu^1, [R])$, where $[R]$ is the adjoint orbit, in the Lie algebra $c(-\mu^*)$, of the exponents of solutions of (3.2) in Levelt normal form.

**Remark 4.7.** The notion of spectrum of a flat $F$-manifold generalizes the corresponding notion for Frobenius manifolds given in [27, 38]. In the Frobenius manifolds case, the group $C(\mu)$ is replaced by its subgroup $G(\eta, \mu)$ called $(\eta, \mu)$-parabolic orthogonal group: this is due to the fact that solutions in Levelt normal form satisfy a further $\eta$-orthogonality requirement described in Remark 3.4. Notice that in the Frobenius case, we have $-\mu^* = \eta \mu \eta^{-1}$. See [27, Section 2.1].

### 4.4 Admissible germs and monodromy data at $z = \infty$

Let $(M, \nabla, c, e, E)$ be an analytic semisimple homogenous flat $F$-manifold. Under semisimplicity assumption, the joint system of differential Equations (3.2) is gauge equivalent to the joint system (3.23). By studying this system, we are going to introduce another set of invariants of pointed germs of the flat $F$-manifold.

**Semisimple, doubly resonant, and admissible germs.** An analytic pointed germ $(M, p)$ will be called

- (tame/coalescing) **semisimple** if the base point $p$ is (tame/coalescing) semisimple,
- **doubly resonant** if $p$ is coalescing and $M$ is conformally resonant,
- **strictly doubly resonant** if, for any arbitrarily fixed ordering $u_o = (u_{o1}, \ldots, u_{on}) \in \mathbb{C}^n$ of the eigenvalues of $U'(p)$, we have
  
  $$u_i' = u_j^j, \quad \delta_i - \delta_j \in \mathbb{Z} \setminus \{0\}, \quad \text{for some } i, j \in \{1, \ldots, n\}, i \neq j,$$

- **admissible** if it is semisimple but not strictly doubly resonant.

In this section, we will consider admissible pointed germs $(M, p)$.

**Remark 4.8.** According to Definition 2.15, the specification “tame/coalescing” depends on the choice of the Euler vector field. In case $M$ is irreducible, it does not depend on such a choice. This follows from Theorem 2.20.

**Formal solutions.** Let $(M, p)$ be an admissible germ. Fix an ordering $u_o = (u_{o1}, \ldots, u_{on}) \in \mathbb{C}^n$ of the spectrum of the operator $U'(p) : T_pM \to T_pM$. Consider the $\partial z$-equation of the joint system (3.23) specialized at $u = u_o$.

**Theorem 4.9.** There exist unique $n \times n$-matrices $(\hat{A}_k)_{k \geq 1}$ such that the matrix

$$\hat{X}_{for}(z) = \left(1 + \sum_{k=1}^{\infty} \frac{\hat{A}_k}{z^k}\right) z^\Lambda e^{zU_o}, \quad \Lambda := \lambda \cdot 1 - \text{diag}(\delta_1, \ldots, \delta_n), \quad U_o = \text{diag}(u_{o1}, \ldots, u_{on}),$$
is a formal solution of the $\partial_z$-equation of (3.23), specialized at $u = u_o$. Moreover,\footnote{Recall the notations introduced in Section 3.4: for any matrix $A \in M_n(\mathbb{C})$, we have the diagonal/off-diagonal decomposition $A = A' + A''$.} we have $\hat{A}_1'' = \Gamma(u_o)^T$.

**Proof.** The matrix $X_{\text{for}}(z)$ is a solution of the $\partial_z$-equation of (3.23) if and only if we have

$$
(1 - k)\hat{A}_{k-1} + \hat{A}_{k-1}A = [U_o, \hat{A}_k] - V(u_o)^T\hat{A}_{k-1}, \quad k \geq 1, \quad \hat{A}_0 := I.
$$

We can compute recursively the matrices $\hat{A}_k$. Let us start with $\hat{A}_1$.

- For $(i, j)$, with $i \neq j$, and so that $u_i^o \neq u_j^o$: from (4.4) specialized at $k = 1$, we deduce

$$
(u_i^o - u_j^o)(\hat{A}_1)_{ij} = V_i^j = (u_i^o - u_j^o)\Gamma(u_o)_{ij} \implies (\hat{A}_1)_{ij} = \Gamma(u_o)_{ij}.
$$

- For $(i, j)$, with $i \neq j$, and so that $u_i^o = u_j^o$: from (4.4) specialized at $k = 2$, we deduce

$$
(\hat{A}_1)_{ij} = \frac{1}{1 - \delta_i + \delta_j} \sum_{\ell \neq i} V(u_o)^\ell_i (\hat{A}_1)^\ell_j = \frac{1}{1 - \delta_i + \delta_j} \sum_{\ell \neq i, j} (u_i^o - u_o^\ell)\Gamma(u_o)^\ell_i \Gamma(u_o)_{ij} = \Gamma(u_o)_{ij}.
$$

In the last equality, we used identity (3.30) specialized at $u = u_o$.

- For the diagonal entries: from (4.4) specialized at $k = 2$, we deduce

$$
(\hat{A}_1)_{ii} = \sum_{\ell \neq i} V(u_o)^\ell_i (\hat{A}_1)^\ell_i = \sum_{\ell \neq i, j} (u_i^o - u_o^\ell)\Gamma(u_o)^\ell_i \Gamma(u_o)_{ij}.
$$

This completes the computation of $\hat{A}_1$, and also proves that $\hat{A}_1'' = \Gamma(u_o)^T$.

Assume now to have computed $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_{h-1}$. The matrix $\hat{A}_h$ can be computed by repeating the same procedure. Namely, from (4.4), with $k = h$, one can compute the entries $(\hat{A}_h)_{ij}$ for $i \neq j$ such that $u_i^o \neq u_j^o$. From (4.4), with $k = h + 1$, one can compute the remaining entries of $\hat{A}_h$. \qed

**Theorem 4.10.** Let $\Omega \subseteq \mathbb{C}^n$ be a simply connected open neighborhood of $u_o$. If $\Omega$ is sufficiently small, then:

1. For any $u \in \Omega$ there exist unique $n \times n$-matrices $(A_k(u))_{k \geq 1}$ such that the matrix

$$
X_{\text{for}}(u, z) = \left(1 + \sum_{k=1}^{\infty} \frac{A_k(u)}{z^k}\right)z^U e^{zU}
$$

is a formal solution of the $\partial_z$-equation of (3.23). Moreover, we have $A_1(u)'' = \Gamma(u)^T$.

2. We have $A_k(u_o) = \hat{A}_k$ for $k \geq 1$, and $X_{\text{for}}(u_o, z) = \hat{X}(z)$.

**Proof.** Point (1) can be proved following the same computations as for Theorem 4.9. Point (2) follows by uniqueness. \qed
Remark 4.11. From the computations above, it is clear that the coefficients $A_k$ are holomorphic at point $u \in \Omega$ such that $u^i \neq u^j$ for $i \neq j$. Below we will prove that the coefficients $A_k$ are actually holomorphic on the whole $\Omega$.

The identity $X_{\text{for}}(u, ze^{2\pi \sqrt{-1}}) = X_{\text{for}}(u, z)e^{2\pi \sqrt{-1}\Lambda}$ justifies the following terminology.

**Definition 4.12.** The matrix $\Lambda = \text{diag}(\lambda - \delta_1, \ldots, \lambda - \delta_n)$ is called *formal monodromy* matrix.

**Admissible directions** $\tau$. Let $q \in M$ be an arbitrary point, and fix an ordering $u(q) := (u^1(q), \ldots, u^n(q)) \in \mathbb{C}^n$ of the eigenvalues of the operator $U(q): T_q M \to T_q M$. Denote by $\text{Arg}(z) \in [-\pi, \pi]$ the principal branch of the argument of the complex number $z$. Set

$$\delta(q) := \left\{ \text{Arg}\left(-\sqrt{-1}(u^i(q) - u^j(q)) + 2\pi k \right): k \in \mathbb{Z}, i, j \text{ are s.t. } u^i(q) \neq u^j(q) \right\}.$$ 

Any element $\tau \in \mathbb{R} \setminus \delta(q)$ will be called an *admissible direction* at $q$.

**Remark 4.13.** The notion of admissibility only depends on the set $\{u^1(q), \ldots, u^n(q)\}$.

**Asymptotic solutions.** Though the formal series defining $X_{\text{for}}$ are typically divergent, $X_{\text{for}}$ contains asymptotical information about genuine analytic solutions of the $\partial_z$-equation of (3.23).

Let $(M, p)$ be an admissible germ, $\Omega$ as in Theorem 4.10, and $\tau$ an admissible direction at $p$. Consider a sufficiently small simply connected open neighborhood $\widetilde{\Omega} \subseteq M$ of $p$ such that:

1. $\widetilde{\Omega} \subseteq M_{ss}$, that is, any point $q \in \widetilde{\Omega}$ is semisimple,
2. A coherent choice of ordering $u$ of eigenvalues of $U$ is fixed on $\widetilde{\Omega}$, so that $u: \widetilde{\Omega} \to \mathbb{C}^n$ defines a local system of canonical coordinates, with $u(p) = u_o$,
3. $u(\widetilde{\Omega}) \subseteq \Omega$,
4. $\tau$ is admissible at any $q \in \widetilde{\Omega}$.

**Theorem 4.14.** If $\widetilde{\Omega}$ is as above, then the following facts hold.

1. For any $q \in \widetilde{\Omega}$, there exist three fundamental systems of solutions $X_1, X_2, X_3$ of the $\partial_z$-equation of (3.23) specialized at $u = u(q)$, uniquely determined by the asymptotics

$$X_i(u, z) \sim X_{\text{for}}(u, z), \quad |z| \to +\infty, \quad \tau - (3 - h)\pi < \text{arg} z < \tau + (h - 2)\pi, \quad h = 1, 2, 3. \quad (4.6)$$

2. The functions $X_1$ are holomorphic w.r.t. $u \in u(\widetilde{\Omega})$, and the asymptotics (4.6) holds true uniformly in $u$.

3. The solutions $X_1(u, z)$ are solutions of the joint system of differential Equations (3.23).

4. The solutions $X_1$ and $X_3$ satisfy the identity $X_3(u, ze^{2\pi \sqrt{-1}}) = X_1(u, z)e^{2\pi \sqrt{-1}\Lambda}$, for $z \in \mathbb{C}^n$. 

**Proof.** Let us temporarily assume that $q$ is tame, that is, $u^i(q) \neq u^j(q)$ for $i \neq j$. For the proof of points (1) and (2), see, for example, [8, 86]. Fix $h \in \{1, 2, 3\}$, and set $W_i(u, z) := \partial_z X_h(u, z) - (zE_i - V_i)^T X_h(u, z)$ for $i = 1, \ldots, n$ and $h = 1, 2, 3$. A simple computation, invoking the identities (3.26), shows that $W_i(u, z)$ is a solution of the $\partial_z$-equation of (3.23). Hence, there exist a matrix $C(u)$ such that $W_i(u, z) = X_h(u, z)C(u)$. Denote by $F(u, z) = 1 + z^{-1}A_1(u) + O(z^{-2})$ the formal
power series in \((4.5)\). For \(|z| \to +\infty\) in the sector
\[
V_{\tau,h} := \{ z \in \mathbb{C}^\times : \tau - (3 - h)\pi < \arg z < \tau + (h - 2)\pi \},
\]
the function \(W_i(u,z)\) has asymptotics
\[
W_i(u,z) \sim \delta_iF(u,z)z^\Lambda e^{zU} + F(u,z)z^\Lambda e^{zU} + V^T_i F(u,z)z^\Lambda e^{zU} = (\delta_iF(u,z) + zF(u,z)E_i - zE_iF(u,z) + V^T_i F(u,z))z^\Lambda e^{zU}.
\]
But we also have \(W_i(u,z) \sim F(u,z)z^\Lambda e^{zU} C(u)\). As a consequence, we deduce
\[
z^\Lambda e^{zU} C(u) e^{-zU} z^{-\Lambda} = \text{formal power series in } \frac{1}{z}. \tag{4.7}
\]
For \(j \neq k\), the sector \(V_{\tau,h}\) contains rays of points \(z\) along which \(\Re(z(u^j - u^k)) > 0\). Hence, necessarily, we deduce that the \((j,k)\)-entry of \(C(u)\) vanishes, otherwise we would have a divergence on the l.h.s. of (4.7). So, the matrix \(C(u)\) is diagonal, and
\[
C(u) = z^\Lambda e^{zU} C(u) e^{-zU} z^{-\Lambda}
= F(u,z)^{-1}(\delta_iF(u,z) + zF(u,z)E_i - zE_iF(u,z) + V^T_i F(u,z))
= z(E_i - E_i) + (A_1E_i - E_iA_1 + V^T_i) + O\left(\frac{1}{z}\right) = O\left(\frac{1}{z}\right),
\]
where we used the identity \(V^T_i = [E_i, \Gamma^T] = [E_i, A_1]\). Hence, \(C(u) = 0\). This proves point (3) in the case \(q\) is tame.

The coefficients \(A_k\) of Theorem 4.10 are holomorphic at \(u\) such that \(u^i \neq u^j\). Moreover, from the computations above, we deduce that
\[
[A_{k+1}, E_i] = [A_1, E_i]A_k - \delta_iA_k, \quad k \geq 1. \tag{4.8}
\]
This formula recursively determines the off-diagonal matrix \(A''_{k+1}\) in terms of \(A_1, \ldots, A_k\). On the other hand, the diagonal entries of \(A_{k+1}\) can be computed as in Theorem 4.10, so that
\[
(k + 1)(A_{k+1})_{i}^{i} = \sum_{\ell \neq i} V_{\ell}^i (A_{k+1})_{i}^{\ell} = \sum_{\ell \neq i} (u^i - u^\ell)\Gamma^i_{\ell}(A_{k+1})_{i}^{\ell} \tag{4.9}
\]
Since \(A''_{k} = \Gamma^T\) is holomorphic also at coalescing points \(u\), an inductive argument shows that all the matrices \(A_k\) are holomorphic at coalescing point, by using formulae (4.8) and (4.9). See also [26, Prop. 19.3]. The system (3.23) is a completely integrable Pfaffian system with holomorphic coefficients on \(u(\bar{\Omega})\): the solutions \(X_i(z,u)\) can be \(u\)-analytically continued as single-valued holomorphic functions on \(u(\bar{\Omega})\), see [26, Cor. 19.1]. The assumptions of [26, Th. 14.1] are thus satisfied, and (1)–(3) hold true also at coalescing points. Finally, notice that the two functions \(X_1(u,z)e^{2\pi \sqrt{-1} \Lambda}\) and \(X_3(u, ze^{2\pi \sqrt{-1}})\) have the same asymptotics on the sector \(\tau - 2\pi < \arg z < \tau - \pi\). By uniqueness, it follows point (4).

\(\square\)

Remark 4.15. For any \(h = 1, 2, 3\), the precise meaning of the uniform asymptotic relation (4.6) is the following: for any compact \(K \subseteq u(\bar{\Omega})\), for any \(\ell \in \mathbb{N}\), and for any unbounded closed
subsector \( \overline{V} \) of \( \mathcal{V}_{\tau,h} := \{ z \in \mathbb{C}^2 : \tau - (3 - h)\pi < \arg z < \tau + (h - 2)\pi \} \), there exists a constant \( C_{h,K,\epsilon,\overline{V}} \in \mathbb{R}_{>0} \) such that

\[
z \in \overline{V} \setminus \{0\} \implies \sup_{u \in K} \left\| X_h(u, z)e^{-zU}z^{-\Lambda} - \left( 1 + \sum_{m=1}^{\ell-1} A_m(u) \right) \right\| < \frac{C_{h,K,\epsilon,\overline{V}}}{|z|^\ell}.
\]

**Stokes and central connection matrices.** Let \((M, p)\) be an admissible germ, and \( \tau \) an admissible direction at \( p \). Let \( \Xi(t, z) \) be a solution in Level \( t \) form of the joint system (3.2), and \( X_h(u, z) \), with \( h = 1, 2, 3 \), be the solutions of the joint system (3.23) as in Theorem 4.14. Let \( t_0 = t(p) \) and \( u_o = u(p) \) the values of the flat and canonical coordinates at \( p \), respectively.

We define the *Stokes matrices* \( \hat{S}_1, \hat{S}_2 \) at \( p \) to be the matrices defined by

\[
X_2(u_o, z) = X_1(u_o, z) \hat{S}_1, \quad X_3(u_o, z) = X_2(u_o, z) \hat{S}_2.
\]

We define the *central connection matrix* \( \hat{C} \) at \( p \) to be the matrix defined by

\[
X_2(u_o, z) = (\Psi(u_o)^{-1})^T \cdot \Xi(t_0, z) \cdot \hat{C}.
\]

**Proposition 4.16.** We have

1. the matrices \( \hat{S}_1, \hat{S}_2, \hat{C} \) are invertible, with \( \det \hat{S}_1 = \det \hat{S}_2 = 1 \),
2. \( (\hat{S}_1)_{ii} = (\hat{S}_2)_{ii} = 1 \),
3. if \( i \neq j \) then \( (\hat{S}_1^{-1})_{ij} = 0 \) if \( \Re(e^{\sqrt{-1}(\tau-\pi)}(u_o^i - u_o^j)) > 0 \),
4. if \( i \neq j \) then \( (\hat{S}_2)_{ij} = 0 \) if \( \Re(e^{\sqrt{-1}\tau}(u_o^i - u_o^j)) > 0 \),
5. we have

\[
\hat{S}_1^{-1}e^{2\pi \sqrt{-1}\Lambda}S_2^{-1} = \hat{C}^{-1}e^{-2\pi \sqrt{-1}\mu}e^{2\pi \sqrt{-1}\kappa} \hat{C}.
\]

**Proof.** The proof of points (1)–(4) is standard, see [86]. Point (5) follows from point (4) of Theorem 4.14.

**Proposition 4.17** [26, 30]. If \( p \) is a coalescing point, define the partition \( \{1, \ldots, n\} = \bigsqcup_{r \in J} I_r \) such that for any \( r \in J \), we have \( \{i, j\} \subseteq I_r \) if and only if \( u_o^i = u_o^j \). We then have the further vanishing condition

\[
(S_1)_{ij} = (S_1)_{ji} = (S_2)_{ij} = (S_2)_{ji} = 0 \quad \text{if} \quad i, j \in I_r \text{ for some } r \in J.
\]

For a D-modules theoretical proof of Proposition 4.17, see the recent preprint [81].

**Definition 4.18.** We call *monodromy data at* \( z = \infty \) of the admissible germ \((M, p)\), computed w.r.t. the admissible direction \( \tau \), the 4-tuple of matrices \((\hat{S}_1, \hat{S}_2, \Lambda, \hat{C})\).

**Remark 4.19.** In the case of Frobenius manifolds, with the standard choice \( \lambda = \frac{d}{2} \), we have \( \Lambda = 0 \) and \( \hat{S}_1^{-1} = \hat{S}_2^T \). This follow from the (anti-)self-adjointness properties of \( U^r \) and \( \mu \). For detailed proofs, see [27, Th. 2.42]. In the notations of *loc. cit.*, we have \( \hat{S}_1 = S \) and \( \hat{S}_2 = S_1^{-1} \). Moreover, the
Stokes matrices are uniquely determined by the metric, the central connection matrix, and the monodromy data at $z = 0$:

$$S = C^{-1}e^{-\pi \sqrt{-1}R}e^{-\pi \sqrt{-1}\mu}\eta^{-1}(C^{-1})^T, \quad S_- = S^T = C^{-1}e^{\pi \sqrt{-1}R}e^{\pi \sqrt{-1}\mu}\eta^{-1}(C^{-1})^T. \quad (4.13)$$

This is a direct consequence of the symmetries of the joint system (3.2), see Remark 3.4.

In the next paragraph, we show that the monodromy data at $z = \infty$ define local invariants of the germ, that is, that they are locally constant w.r.t. small perturbations of both the point $p$ and the admissible direction $\tau$.

**Isomonodromicity Property.** Let $\Omega$ be an open neighborhood of $p$ as above. By Theorems 4.3 and 4.14, if we let vary the point $q$ in $\Omega$, we have well defined solutions $\Xi(t(q), z), X_i(u(q), z), i = 1, 2, 3$, of the joint systems (3.2) and (3.23), respectively. We can thus introduce the Stokes and central connection matrices $(S_1, S_2, C)$ as functions of $q \in \Omega$ by the formulae

$$X_2(u(q), z) = X_1(u(q), z)S_1(u(q)) \quad (4.14)$$

$$X_3(u(q), z) = X_2(u(q), z)S_2(u(q)), \quad (4.15)$$

$$X_2(u(q), z) = (\Psi(u(q))^{-1})^T \cdot \Xi(t(q), z) \cdot C(u(q)). \quad (4.16)$$

**Theorem 4.20.** The functions $S_1, S_2, C$ are constant on $\Omega$. In particular, we have $S_1(q) = \tilde{S}_1, S_2(q) = \tilde{S}_2, C(q) = \tilde{C}$, for all $q \in \Omega$.

**Proof.** Let us prove the statement for $S_1$. We have

$$\partial_i S_1(u) = \hat{\partial}_i \left[ X_1(u(q), z)^{-1}X_2(u(q), z) \right]$$

$$= -X_1(u(q), z)^{-1} \cdot \partial_i X_1(u(q), z) \cdot X_1(u(q), z)^{-1} \cdot X_2(u(q), z)$$

$$+ X_1(u(q), z)^{-1} \partial_i X_2(u(q), z)$$

$$= -X_1(u(q), z)^{-1} \cdot (zE_i - V_i)^T \cdot X_2(u(q), z)$$

$$+ X_1(u(q), z)^{-1} \cdot (zE_i - V_i)^T \cdot X_2(u(q), z) = 0.$$
\[ X'_h(u(p), z) \sim X_{\text{for}}(u(p), z), \quad |z| \to +\infty, \quad z \in \mathcal{V}_{\tau', h}, \quad h = 1, 2, 3, \]

by Theorem 4.14, see also Remark 4.15. We prove that \( X'_h = X_h \) for all \( h = 1, 2, 3 \).

Let \( K_h \) be the connection matrix s.t. \( X'_h = X_h K_h \). We have

\[ z^\Lambda e^z U K_h e^{-z U} z^{\Lambda} \sim 1, \quad |z| \to +\infty, \quad z \in \mathcal{V}_{\tau, h} \cap \mathcal{V}_{\tau', h}. \]

By taking the \((j, k)\)-entry, for any \( \ell \in \mathbb{N} \), we have

\[ (K_h)_{jk} e^{z(u^h - u^k(p))} z^{\delta_k - \delta_j} = \delta_{jk} + O(|z|^{-\ell}), \quad |z| \to +\infty, \quad z \in \mathcal{V}_{\tau, h} \cap \mathcal{V}_{\tau', h}. \]

Assume \( j \neq k \). If \( u^h(p) = u^k(p) \), then necessarily \((K_h)_{jk} = 0\). If \( u^h(p) \neq u^k(p) \), notice that in \( \mathcal{V}_{\tau, h} \cap \mathcal{V}_{\tau', h} \), there are rays along which \( \text{Re}(z(u^h(p) - u^k(p))) \) is negative, and also rays along which it is positive. So, we necessarily have \((K_h)_{jk} = 0\). This proves that \( K_h \) is diagonal. It follows that \( K_h = 1 \) for \( h = 1, 2, 3 \). \( \square \)

### 4.5 Monodromy data for a formal admissible germ

In Sections 4.2–4.4, the flat \( F \)-manifold structure on \( M \) is assumed to be analytic. The notion of admissible germs and of their monodromy data can, however, be extended to the formal case.

Let \((H, \Phi)\) be a semisimple formal \( F \)-manifold over \( \mathbb{C} \), with Euler field \( E \). Associated with it, we have two joint systems of differential equations (3.2) and (3.23) whose coefficients are matrix-valued formal power series in the coordinates \( t \) and \( u \), respectively.

We will say that \((H, \Phi)\) is

- **doubly resonant**, if the origin is coalescing and the formal flat \( F \)-manifold is conformally resonant,
- **strictly doubly resonant**, if we have

\[ u^i_o = u^j_o, \quad \delta_i - \delta_j \in \mathbb{Z} \setminus \{0\}, \quad \text{for some } i, j \in \{1, \ldots, n\}, i \neq j, \]

- **admissible** if it is semisimple but not strictly doubly resonant.

The \( \partial_z \)-equations of the joint systems (3.2) and (3.23) can be specialized at \( t = 0 \) and \( u = u_o \), respectively. For these specialized systems of equations, we can introduce a triple \((\lambda, \mu^\lambda, [R])\) of monodromy data at \( z = 0 \), and a 4-tuple \((\tilde{S}_1, \tilde{S}_2, \Lambda, \tilde{C})\) of monodromy data at \( z = \infty \), exactly as in the case of an analytic germ \((M, p)\).

The system \((\lambda, \mu^\lambda, [R], \tilde{S}_1, \tilde{S}_2, \Lambda, \tilde{C})\) will be referred to as the **monodromy data** of the formal structure \((H, \Phi)\). \textit{A priori}, Theorem 4.20 cannot be adapted to this formal picture, but Theorem 4.21 still holds true, and its proof works verbatim.

In Section 6.4, we will prove that an admissible formal germ is actually convergent: it defines an analytic flat \( F \)-manifold, so that all the results of Sections 4.2–4.4 apply.
5  |  NORMALIZATIONS AND ANALYTIC CONTINUATION

5.1  |  Choices of normalizations

The monodromy data of an admissible germ \((M, p)\) are defined up to several noncanonical choices:

1. the choice of \(\lambda \in \mathbb{C}\),
2. the choice of a base point in the universal cover \(\widehat{\mathbb{C}}\),
3. the choice of the solution \(\Xi\) in Levelt normal form,
4. the choice of Lamé coefficients \((H_1, \ldots, H_n)\),
5. the choice of ordering of canonical coordinates \((u^1(p), \ldots, u^n(p))\),
6. the choice of an admissible direction \(\tau \in \mathbb{R} \setminus \mathcal{S}(p)\).

Different choices of normalizations affect the numerical values of the monodromy data. These transformations of the data can be described by actions of corresponding suitable groups:

1. the group \(\mathbb{C}\),
2. the deck transformation group \(\text{Deck}(\widehat{\mathbb{C}}) \cong \mathbb{Z}\),
3. the group \(\mathcal{C}(-\mu^*)\),
4. the torus \((\mathbb{C}^*)^n\),
5. the symmetric group \(\mathfrak{S}_n\),
6. the braid group \(\mathfrak{B}_n\).

We first describe actions (1)–(5), and postpone the description of action (6) to the next sections.

- **Action of \(\mathbb{C}\)**: the transformation \(\lambda \mapsto \lambda'\) implies the following transformations of the monodromy data by translations

  \[ \mu^\lambda \mapsto \mu^\lambda' = \mu^\lambda + (\lambda - \lambda') \mathbf{1}, \quad \Lambda \mapsto \Lambda - (\lambda - \lambda') \mathbf{1}. \]

  For irreducible flat \(F\)-manifolds, the choice of \(\lambda\) is equivalent to the choice of an Euler vector field, see Theorem 2.20.

- **Action of \(\text{Deck}(\widehat{\mathbb{C}}) \cong \mathbb{Z}\)**: a different choice of the base point in \(\widehat{\mathbb{C}}\) is equivalent to the choice of a different determination of the logarithm (i.e., of the argument \(\arg z\)). In particular, by changing \(\log z \mapsto \log z + 2\pi k \sqrt{-1}\) with \(k \in \mathbb{Z}\), we have the transformations

  \[ S_1 \mapsto \exp(-2\pi k \sqrt{-1}\Lambda) S_1 \exp(2\pi k \sqrt{-1}\Lambda), \quad S_2 \mapsto \exp(-2\pi k \sqrt{-1}\Lambda) S_2 \exp(2\pi k \sqrt{-1}\Lambda), \]

  \[ C \mapsto M_0^{-k} C \exp(2\pi k \sqrt{-1}\Lambda), \quad M_0 := \exp(-2\pi \sqrt{-1}\mu^* e^{2\pi \sqrt{-1} R}, \quad k \in \mathbb{Z}. \]

- **Action of \(\mathcal{C}(-\mu^*)\)**: for \(A \in \mathcal{C}(-\mu^*)\), the change of solutions \(\Xi \mapsto \Xi A\) implies the transformation of the central connection matrix

  \[ C \mapsto A^{-1} C. \]
• Action of $(\mathbb{C}^\ast)^n$: for $(h_1, \ldots, h_n) \in (\mathbb{C}^\ast)^n$, consider the transformation $(H_1, \ldots, H_n) \mapsto (H_1 h_1, \ldots, H_n h_n)$. The monodromy data transform as follows:

$$S_1 \mapsto h^{-1} S_1 h, \quad S_2 \mapsto h^{-1} S_2 h, \quad C \mapsto C h,$$

where $h := \text{diag}(h_1, \ldots, h_n)$.

• Action of $\mathfrak{S}_n$: for $\sigma \in \mathfrak{S}_n$, consider the permutation of canonical coordinates $(u^1, \ldots, u^n) \mapsto (u^{\sigma(1)}, \ldots, u^{\sigma(n)})$. The monodromy data transform as follows:

$$S_1 \mapsto P S_1 P^{-1}, \quad S_2 \mapsto P S_2 P^{-1}, \quad C \mapsto C P^{-1}, \quad \Lambda \mapsto P \Lambda P^{-1},$$

where $P = (P_{ij})_{i,j}, \quad P_{ij} := \delta_{\sigma(i)j}$.

Remark 5.1. In the above discussion, we have fixed once for all a system of local flat coordinates $t$ in which $\mu$ is diagonal. Different choices of systems of local coordinates as described in Remark 2.10 affect the monodromy data. For example, if we have the transformation

$$t \mapsto G t + c, \quad G \in \text{GL}(n, \mathbb{C}), \quad G \text{ diagonal, } \quad c \in \mathbb{C}^n,$$

then the monodromy data change as follows:

$$R \mapsto G^{-1} R G, \quad C \mapsto G^{-1} C.$$

This is not the most general form of the admissible transformations: if $\mu$ has not simple spectrum, there are also transformations with $G$ not diagonal.

5.2 Triangular and lexicographical orders

If an admissible direction $\tau$ at $p$ is fixed, we will say that the canonical coordinates $(u^i(p))_{i=1}^n$ at $p$ are in triangular order w.r.t. the admissible direction $\tau$ if the Stokes matrix $S_1$ is upper triangular, and $S_2$ is lower triangular.

On the one hand, in general, triangular orders at $p$ are not unique. This happens for example if $p$ is a semisimple coalescing point. In such a case, we have $(S_1)_{ij} = (S_1)_{ji} = 0$ if $u_i = u_j$ with $i \neq j$, by Proposition 4.17. If $S_1$ is upper triangular, then so is $P S_1 P^{-1}$ for $P$ corresponding to the transposition $i \leftrightarrow j$. Similarly, the lower triangular structure of $S_2$ is preserved.

On the other hand, we always have a distinguished triangular order, called lexicographical w.r.t. $\tau$. Introduce the following rays in the complex plane

$$L_j := \{u^j(p) + \rho e^{\sqrt{-1} \left( \frac{\pi}{2} - \tau \right)} : \rho \in \mathbb{R}_+ \}, \quad j = 1, \ldots, n.$$

The ray $L_j$ originates from the point $u^j(p)$, and it is oriented from $u^j(p)$ to $\infty$.

The canonical coordinates $(u^1(p), \ldots, u^n(p))$ are in lexicographical order if $L_j$ is to the left of $L_k$ (w.r.t. the orientation above), for any $1 \leq j < k \leq n$ such that $u^j(p) \neq u^k(p)$.

The lexicographical order is the unique triangular order at $p$ if the number of nonzero entries of $S_1$ or $S_2$ is maximal, that is, \[ \frac{n(n-1)}{2} \]
5.3  Braid group action on matrices

Denote by $U_n$ and $L_n$ the groups of unipotent upper and lower triangular $n \times n$-matrices, and by $\mathfrak{t}$ the Lie algebra of diagonal $n \times n$-matrices.

The (abstract) Artin braid group $B_n$ with $n$-strings is the group with $n - 1$ generators $\beta_1, \ldots, \beta_{n-1}$ satisfying the relations

$$\beta_i \beta_j = \beta_j \beta_i, \quad \text{if } |i - j| > 1, \quad \beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}.$$

(5.1)

Given $g = (g_1, g_2, g_3) \in U_n \times L_n \times \mathfrak{t}$, define $3(n - 1)$ block-diagonal matrices $B_1^{(i)}(g), B_2^{(i)}(g), B_3^{(i)}(g)$, with $i = 1, \ldots, n - 1$, as follows:

$$B_1^{(i)}(g) := 1_{i-1} \oplus [(g_1)_{i,i+1} 1] \oplus 1_{n-i-1},$$

$$B_2^{(i)}(g) := 1_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & (g_2)_{i+1,i} \end{bmatrix} \oplus 1_{n-i-1},$$

$$B_3^{(i)}(g) := 1_{i-1} \oplus \begin{bmatrix} h \cdot (g_1)_{i,i+1} & 1 \\ 1 & 0 \end{bmatrix} \oplus 1_{n-i-1},$$

(5.2)

where

$$h := e^{2\pi \sqrt{-1}(g_3)_{i+1,i} - (g_3)_{i,i}}.$$

For any $\beta_i \in B_n$, define the triple $g^{\beta_i} \in U_n \times L_n \times \mathfrak{t}$ by

$$g^{\beta_i} := \left( B_1^{(i)}(g)^{-1} g_1 B_2^{(i)}(g), \ B_2^{(i)}(g)^{-1} g_2 B_3^{(i)}(g), \ P_i g_3 P_i \right),$$

(5.3)

where $P_i$ is the permutation matrix $i \leftrightarrow i + 1$.

**Lemma 5.2.** The braid group $B_n$ acts on $U_n \times L_n \times \mathfrak{t}$ by mapping $(\beta_i, g) \mapsto g^{\beta_i}$ for $i = 1, \ldots, n - 1$.

**Proof.** By a direct computation, one checks that $g^{\beta} = \text{id}$ for any relator $\beta$ in (5.1). \qed

**Example.** Let $n = 3$, and

$$g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$
We have
\[
\begin{pmatrix}
1 & \alpha & c \\
0 & 1 & b - ac \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\alpha e^{2\sqrt{1/(d_2-d_1)\pi}} & 1 & 0 \\
\alpha \beta e^{2\sqrt{1/(d_2-d_1)\pi}} + \gamma & \beta & 1
\end{pmatrix}
\begin{pmatrix}
d_2 & 0 & 0 \\
0 & d_1 & 0 \\
0 & 0 & d_3
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & b & a + b\gamma \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\beta - \alpha\gamma & 1 & 0 \\
\alpha & e^{2\sqrt{1/(d_3-d_2)\pi}} & 1
\end{pmatrix}
\begin{pmatrix}
d_1 & 0 & 0 \\
0 & d_3 & 0 \\
0 & 0 & d_2
\end{pmatrix}.
\]

If $\beta = (\beta_1 \beta_2)^3$, the triple $g^\beta = (g'_1, g'_2, g'_3)$ equals
\[
g'_1 = \begin{pmatrix}
1 & a e^{2\sqrt{1/(d_2-d_1)\pi}} \\
0 & 1 \\
0 & 0 & 1
\end{pmatrix} = e^{-2\pi \sqrt{-1} g_1} g_1 e^{2\pi \sqrt{-1} g_3},
\]
\[
g'_2 = \begin{pmatrix}
1 & 0 & 0 \\
e^{2\sqrt{1/(d_1-d_2)\pi}} \alpha & 1 & 0 \\
e^{2\sqrt{1/(d_1-d_3)\pi}} \beta & e^{2\sqrt{1/(d_2-d_3)\pi}} \gamma & 1
\end{pmatrix} = e^{-2\pi \sqrt{-1} g_2} g_2 e^{2\pi \sqrt{-1} g_3},
\]
\[
g'_3 = \begin{pmatrix}
d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & d_3
\end{pmatrix} = g_3.
\]

5.4 | Braid mutations of monodromy data

Let $M$ be an analytic homogeneous semisimple flat $F$-manifold, and denote by $M'$ the open set of tame semisimple points $p \in M$, that is, at which the spectrum of the operator $U'(p) : T_pM \to T_pM$ is simple.

Consider the following two different settings:

(I) Assume $g : [0, 1] \to M'$ to be a continuous path such that $g([0, 1])$ is contained in a simply connected open set, on which a coherent choice of normalizations (1)-(5) can be done.

Assume also that
- $\tau$ is admissible at both $g(0)$ and $g(1)$,
- there exists $\tilde{t} \in [0, 1]$ such that $\tau$ is not admissible at $g(\tilde{t})$.

(II) Assume $p \in M'$ is a semisimple point, and fix some choice of normalizations (1)–(5). Let $\tau_0, \tau_1 \in \mathbb{R} \setminus \delta(p)$, and $\tau : [0, 1] \to M$ to be a continuous map such that
- $\tau(0) = \tau_0$ and $\tau(1) = \tau_1$,
- there exists $\tilde{t} \in [0, 1]$ such that $\tau(\tilde{t})$ is not admissible at $p$.

In both cases (I) and (II), for each $t \in \{0, 1\}$, we can introduce a set $\mathcal{M}_t$ of monodromy data.

Problem. In both settings (I) and (II), how to describe the transformation $\mathcal{M}_0 \to \mathcal{M}_1$?
The matrices $\mu, R$ will not depend on $t$, due to the results of Section 4.3. Hence, we need to describe how the matrices $(S_1, S_2, C, \Lambda)$ will transform. In this section, we prove that this is described by an action of the braid group, which on the triple $(S_1, S_2, \Lambda)$ reduces to (5.3).

**Remark 5.3.** Pictures (I) and (II) are “dual” to each other. In (I), we have a fixed $\tau \in \mathbb{R}$ and a variable set $\mathcal{S}(g(t))$ of nonadmissible directions such that $\tau \in \mathcal{S}(g(0)) \cap \mathcal{S}(g(1))$. In (II), we have a fixed set $\mathcal{S}(p) \subseteq \mathbb{R}$ of nonadmissible directions and a continuous map $\tau : [0, 1] \to \mathbb{R}$ with $\tau(0), \tau(1) \in \mathbb{R} \setminus \mathcal{S}(p)$. In both cases, we have to face a wall-crossing phenomenon: the fixed (resp. variable) point $\tau$ is not admissible for some values of the time parameter.

Given $u \in \mathbb{C}^n$ introduce a family of Stokes rays in the universal cover $\hat{\mathbb{C}}^*$: for any pair $(i, j)$ such that $u_i \neq u_j$ set

$$\tau_{ij}(u) := \frac{3\pi}{2} - \text{Arg}(u_i - u_j), \quad R_{ij}^{(k)}(u) := \{z \in \hat{\mathbb{C}}^* : \arg z = \tau_{ij}(u) + 2\pi k\}, \quad k \in \mathbb{Z}.$$  

Also, for any $\tau \in \mathbb{R}$, introduce the admissible ray

$$\ell_{\tau} := \{z \in \hat{\mathbb{C}}^* : \arg z = \tau\}.$$  

Both Stokes and admissible rays are equipped with the natural orientation, from 0 to $\infty$. Any continuous transformations of $u$ and $\tau$ induce continuous rotations of the Stokes and admissible rays. In the case of settings (I) and (II), the oriented ray crosses some of the Stokes rays during the transformation. We will call elementary any such transformation of rays, along which $\ell_{\tau}$ crosses one Stokes ray $R_{ij}^{(k)}$ only.

Let us focus on the picture (I). Fix $u_o \in \mathbb{C}^n \setminus \Delta$ with components in $\tau$-lexicographical order. Consider a continuous map $b_i : [0, 1] \to (\mathbb{C}^n \setminus \Delta)$, with $i = 1, \ldots, n - 1$, such that:

1. $b_i(0) = u_o$,
2. $b_i(t)^h = b_i(0)^h$ for all $h \neq i, i + 1$,
3. $b_i(t)^i$ counterclockwise rotates by one half turn w.r.t. $b_i(t)^{i+1}$ in the plane $\mathbb{C}$,
4. $b_i(0)^i = b_i(1)^{i+1}$ and $b_i(1)^i = b_i(0)^{i+1}$.

The map $b_i$ can be seen as a loop on $\text{Conf}_n(\mathbb{C}) := (\mathbb{C}^n \setminus \Delta) / \mathfrak{S}_n$, the configuration space of $n$ pairwise distinct points in $\mathbb{C}$. The space $\text{Conf}_n(\mathbb{C})$ is aspherical (i.e., $\pi_k(\text{Conf}_n(\mathbb{C})) = 0$ for $k \geq 2$), and its fundamental group is isomorphic to the braid group $B_n$, see [56]. Consider the homotopy classes $[b_i]$ in $\pi_1(\text{Conf}(\mathbb{C}^n), \{u_i(0)\}) \cong B_n$. It is easily seen that

$$[b_i] * [b_j] = [b_j] * [b_i], \quad |i - j| > 1, \quad [b_i] * [b_{i+1}] * [b_i] = [b_{i+1}] * [b_i] * [b_{i+1}],$$

where $*$ denotes the concatenation of loops. We identify $[b_i]$ with the elementary braid $\beta_i$.

In the case of picture (II), any of the maps $b_i$’s can be seen as a map with target $M$, by working in a local chart with canonical coordinates in $\tau$-lexicographical order. It is an elementary transformation: one of the Stokes rays $R_{ij}^{(k)}$ clockwise crosses the ray $\ell_{\tau}$.

In summary, elementary transformations of type (I) can be identified with elements of $B_n$. Dually, by exchanging orientations (counter-clockwise $\leftrightarrow$ clockwise), we can identify $\beta_i$ with the type (II) transformation defined by a counterclockwise rotation of $\ell_{\tau}$ across one of the Stokes rays $R_{ij}^{(k)}$.

Let $(S_1, S_2, \Lambda, C)$ be the 4-tuple of Stokes, formal monodromy, and central connection matrices computed.
• w.r.t. the point \( g(0) \), in case (I);
• w.r.t. the line \( \tau(0) \), in case (II);

In both cases (I) and (II), the monodromy data are always computed w.r.t. the lexicographical order of canonical coordinates, so that \((S_1, S_2, \Lambda) \in U_n \times L_n \times t\).

**Theorem 5.4.** Along the elementary transformation \( \beta_i \), with \( i = 1, \ldots, n - 1 \), the monodromy data transform as follows:

\[
(S_1, S_2, \Lambda) \mapsto (S_1, S_2, \Lambda) \beta_i, \quad C \mapsto CB^{-1},
\]

where

\[
B = B_2^{(i)}(S_1, S_2, \Lambda) = 1_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & (S_2)_{i+1,i} \end{bmatrix} \oplus 1_{n-i-1}.
\]

Cf. Equations (5.2) and (5.3).

**Proof.** Whatever is the case under consideration, (I) or (II), let us consider the initial “frozen” configuration of Stokes and admissible rays, \( R^{(h)}_{ij} \) and \( \ell_{x} \).

Label the Stokes rays as follows: let \( R^{(1)} \) be the first Stokes ray on the left of \( \ell_{x} \), \( R^{(0)} \) the first Stokes ray on the right of \( \ell_{x} \), and extend the numeration \( R^{(k)} \), with \( k \in \mathbb{Z} \), so that the label \( k \) increases in counterclockwise order.

Let \( m \) be the number of Stokes rays in any sector of \( \hat{\mathbb{C}}^n \) defined by

\[
(2j - 1)\pi < |\arg z - \tau| < 2\pi j, \quad j \in \mathbb{Z}.
\]

The number \( m \) also equals the number of Stokes rays in the sectors

\[
2\pi j < |\arg z - \tau| < (2j + 1)\pi, \quad j \in \mathbb{Z}.
\]

For generic initial points \( u_o \in \mathbb{C}^n \setminus \Delta \), we have \( m = \frac{n(n-1)}{2} \), but some Stokes rays may coincide.†

Define \( \Pi^{(k)} \), with \( k \in \mathbb{Z} \), to be the sector in \( \hat{\mathbb{C}}^n \) from the ray \( R^{(k-1)}_{ij} \) to the ray \( R^{(m+k)}_{ij} \). For each \( k \in \mathbb{Z} \), there exists a unique solution \( X^{(k)}(u_o, z) \) of the \( \partial_z \)-equation of (3.23), specialized at \( u = u_o \), such that

\[
X^{(k)}(u_o, z) \sim X_{101}(u_o, z), \quad |z| \to +\infty, \quad z \in \Pi^{(k)}.
\]

Introduce invertible matrices \( K_k \), called **Stokes factors**, such that

\[
X^{(k+1)}(u_o, z) = X^{(k)}(u_o, z) K_k, \quad k \in \mathbb{Z}.
\]

The matrices \( K_k \) have the following structure: all diagonal entries are 1, and the entry \((K_k)_{ab}\) is nonzero only if \( R^{(m+k)}_{ab} \) is one of the rays \( R^{(h)}_{ab} \) with \( h \in \mathbb{Z} \), see [8].

† This happens, for example, if there are three indices \((i, j, k)\) such that \( u_i^o, u_j^o, u_k^o \in \mathbb{C} \) are collinear, or there are four indices \((i, j, k, l)\) such that \( u_i^o, u_j^o \) and \( u_k^o, u_l^o \) define two parallel lines.
Recall that the Stokes matrices $S_1, S_2$ are defined in terms of solutions $X_1, X_2, X_3$, see Equation (4.10). We have

$$X_1(u_o, z) \equiv X^{(1-m)}(u_o, z), \quad X_2(u_o, z) \equiv X^{(1)}(u_o, z), \quad X_3(u_o, z) \equiv X^{(1+m)}(u_o, z).$$

Hence, we deduce

$$X_2(u_o, z) = X^{(0)}(u_o, z)K_0 = X^{(-1)}(u_o, z)K_{-1}K_0 = \cdots = X^{(1-m)}(u_o, z)K_{1-m} \cdots K_{-1}K_0.$$

From Equation (4.10), we deduce

$$S_1 = K_{1-m} \cdots K_{-1}K_0.$$

Analogously, we have

$$S_2 = K_1 \cdots K_m.$$

Up to now, we have considered a “static” picture, at the initial time $t = 0$ of the transformation $\beta_i$. By letting the time parameter $t$ varies, the Stokes rays and/or the ray $e^t$ rotate. In particular, immediately before the collision of Stokes and oriented rays, we have $R^{(h)}_{i,i+1} = K^{(1)}$ for a suitable $h \in \mathbb{Z}$. After the collision, we have $R^{(h)}_{i,i+1} \equiv R^{(0)}$. Hence, after the transformation $\beta_i$, we have the following transformation of Stokes matrices:

$$S_1 \mapsto S'_1 = K_{-m} \cdots K_{0}K_1 = K_{1-m}^{-1}S_1K_1,$$

$$S_2 \mapsto S'_2 = K_1 \cdots K_mK_{m+1} = K_1^{-1}S_2K_{1+m}.$$

Similarly, the central connection matrix transforms as follows:

$$C \mapsto C' = C K_1^{-1}$$

The only nonzero off-diagonal entries of $K_1, K_{1-m}, K_{1+m}$ are

$$(K_{1-m})_{i,i+1} = (S_1)_{i,i+1}, \quad (K_1)_{i+1,i} = (S_2)_{i+1,i},$$

$$(K_{1+m})_{i,i+1} = [e^{-2\pi i} \sqrt{-1} \Lambda S_1 e^{2\pi i} \sqrt{-1} \Lambda]_{i,i+1}.$$

The last identity follows from point (4) of Theorem 4.14. Finally, we also need to recover the lexicographical order, which is lost after the transformation $\beta_i$. By applying the permutation $i \leftrightarrow i + 1$, we complete the proof. 

Remark 5.5. Theorem 5.4 generalizes the braid group action in Dubrovin’s analytic theory of Frobenius manifolds, see [38, Th. 4.8]. The action of the braid group on $U_n \times L_n \times t$ is just the simplest case of a more general picture described in [15, 16]. The starting point is the observation that the “monodromy manifold”\(^1\) $U_n \times L_n \times t$ is isomorphic to the dual Poisson–Lie group.

\(^1\) In [15, 16], meromorphic connections on a closed disk $D \subseteq \mathbb{C}$, with an irregular singularity only, are studied. We have a connections on $\mathbb{C}^*$ with two singularities, and we need a further piece of information for a global description of the monodromy: the central connection matrix.
Action of the center $Z(B_n)$. Consider the shift of the admissible direction $\tau \mapsto \tau + 2\pi$. We have the following facts:

1. In the generic case (i.e., for canonical coordinates in general position), the number of Stokes rays in any sector of $\mathbb{C}^*$ of width $2\pi$ equals $n(n-1)$. An elementary braid acts whenever the line $\ell_\tau$ crosses a Stokes ray. So, in total, we expect that a complete rotation of $\ell_\tau$ corresponds to the product of $n(n-1)$ elementary braids.

2. The effect of the shift $\tau \mapsto \tau + 2\pi$ on the monodromy data can be identified with a transformation of different nature, namely, a different choice of normalization (2). This consists in a different choice of the branch of the logarithm. From this, it follows that the braid corresponding to $\tau \mapsto \tau + 2\pi$ must commute with any other braids.

From point (2), we deduce that the braid corresponding to $\tau \mapsto \tau + 2\pi$ is an element of the center $Z(B_n) \cong \mathbb{Z} \cong \text{Deck}(\hat{\mathbb{C}}^*)$.

The center $Z(B_n)$ is the cyclic group generated by the braid $\beta = (\beta_1 \ldots \beta_{n-1})^n$. From point (1), and the action of $\text{Deck}(\hat{\mathbb{C}}^*)$, we deduce the following result.

**Proposition 5.6.** The braid corresponding to the shift $\tau \mapsto \tau + 2\pi$ of the admissible direction is the generator $\beta$ of the center $Z(B_n)$. It acts as follows:

$$
(S_1, S_2, \Lambda) \beta = \left(e^{-2\pi \sqrt{-1} S_1} e^{2\pi \sqrt{-1} S_2}, e^{-2\pi \sqrt{-1} S_1} e^{2\pi \sqrt{-1} S_2}, \Lambda \right)
$$

$$
C \mapsto M_0^{-1} C e^{2\pi \sqrt{-1} \Lambda}.
$$

This proposition extends a computation (for $n = 3$) of the Example of Section 5.3.

**Corollary 5.7.** The generator $\beta = (\beta_1 \ldots \beta_{n-1})^n$ of the center $Z(B_n)$ acts on $U_n \times L_n \times \mathfrak{t}$ as follows:

$$
g^\beta = \left(e^{-2\pi \sqrt{-1} g_3} g_1 e^{2\pi \sqrt{-1} g_3}, e^{-2\pi \sqrt{-1} g_3} g_2 e^{2\pi \sqrt{-1} g_3}, g_3 \right).
$$

(5.4)

**Proof.** Let $g'$ be the r.h.s. of (5.4). Assume that there exists $\tilde{\beta} \in B_n$ such that $g^\beta = g'$. A simple computation shows that $B^{(j)}_i(g) e^{2\pi \sqrt{-1} g_3} = B^{(j)}_i(g')$ for $i = 1, \ldots, n-1$ and $j = 1, 2, 3$. Thus, we have $(g^\beta)^\beta_i = (g^{\tilde{\beta}})^\beta_i$ for any $i = 1, \ldots, n-1$ and any $g \in U_n \times L_n \times \mathfrak{t}$. Denote by $\mathcal{E}_{U_n \times L_n \times \mathfrak{t}}$ the group of bijections of $U_n \times L_n \times \mathfrak{t}$ and by $\iota : B_n \to \mathcal{E}_{U_n \times L_n \times \mathfrak{t}}$ the group morphism defining the
action. We have \( \iota(\hat{\beta}) \in \iota(Z(B_n)) \). Hence, there exists \( k \in \mathbb{Z} \) such that \( g^{\hat{\beta}} = g^{	heta k} \) for any \( g \in U_n \times L_n \times \mathfrak{t} \). We deduce \( k = 1 \), by Proposition 5.6.

\[ \square \]

5.5 Analytic continuation of the flat \( F \)-structure

There is a more global point of view from which one can reinterpret the results of the previous sections. It both makes transparent the appearance of a braid group action on the monodromy data, and clarifies the “duality” of settings (I) and (II) of the previous section. Moreover, it also describes the analytic continuation of the flat \( F \)-manifold structure.

Admissibility sets \( A_n, \mathcal{A}_n^O, \mathcal{A}_n^Z, \tilde{A}_n \). Consider the configuration space \( \text{Conf}_n(C) := (\mathbb{C}^n \setminus \Delta)/\mathfrak{S}_n \) of \( n \) points in the plane, together with the following covering space:

- the ordered \( \mathfrak{S}_n \)-covering \( \text{Conf}_n^O(C) := \mathbb{C}^n \setminus \Delta \),
- the covering \( \text{Conf}_n^Z(C) \) associated with the center \( Z(\pi_1(\text{Conf}_n(C))) \cong \mathbb{Z} \), and
- the universal cover \( \text{Conf}_n(C) \).

The fundamental group \( \pi_1(\text{Conf}_n^O(C)) \) is isomorphic to the group \( \mathfrak{P}_n \) of pure braids. The deck transformation group of the covering \( \text{Conf}_n^O(C) \to \text{Conf}_n(C) \) is isomorphic to \( \mathfrak{S}_n \cong \mathfrak{B}_n/\mathfrak{P}_n \).

The fundamental group of the space \( \text{Conf}_n^Z(C) \) is isomorphic to \( Z(\pi_1(\text{Conf}_n(C))) \cong \mathbb{Z} \). The deck transformation group of the covering \( \text{Conf}_n^Z(C) \to \text{Conf}_n(C) \) is isomorphic to \( \mathfrak{B}_n/\mathbb{Z}(\mathfrak{B}_n) \cong M_n(\mathbb{R}^2) \), the mapping class group of the \( n \)-punctured plane, see [14].

Notice that \( Z(B_n) = Z(P_n) \cong \mathbb{Z} \). The coverings spaces above fit into the chain

\[ \text{Conf}_n(C) \to \text{Conf}_n^Z(C) \to \text{Conf}_n^O(C) \to \text{Conf}_n(C). \]

Remark 5.8. The space \( \text{Conf}_n(C) \) has been described for the first time by S. Kaliman [50–52]; in loc. cit., it is proved that it is isomorphic to \( \mathbb{C}^2 \times \mathcal{T}(0, n+1) \), where \( \mathcal{T}(0, n+1) \) denotes the Teichmüller space of the Riemann sphere with \( n+1 \) punctures. The space \( \mathcal{T}(0, n+1) \) is homeomorphic to \( \mathbb{R}^{2n-4} \), and it is biholomorphic to a holomorphically convex Bergmann domain in \( \mathbb{C}^{n-2} \). For further details, see the interesting paper [60].

Given \( p = \{p_1, \ldots, p_n\} \in \text{Conf}_n(C) \), define the set \( \mathcal{S}(p) \subseteq \mathbb{R} \) by

\[ \mathcal{S}(p) := \{ \text{Arg}[\sqrt{-1}(p_i - p_j)] + 2k\pi \colon k \in \mathbb{Z}, \quad \text{Arg}(z) \in [-\pi, \pi] \}. \]

Any number \( \tau \in \mathbb{R} \setminus \mathcal{S}(p) \) is said to be an admissible direction at \( p \).

Introduce the smooth \((2n+1)\)-dimensional real manifolds

\[ \mathcal{X}_n := \text{Conf}_n(C) \times \mathbb{R}, \quad \mathcal{X}_n^O := \text{Conf}_n^O(C) \times \mathbb{R}, \quad \mathcal{X}_n^Z := \text{Conf}_n^Z(C) \times \mathbb{R}, \quad \mathcal{X}_n^\infty := \text{Conf}_n(C) \times \mathbb{R}, \]

Define the admissibility open subset \( A_n \subseteq \mathcal{X}_n \) by

\[ A_n := \{(p, \tau) \in \mathcal{X}_n : \tau \in \mathbb{R} \setminus \mathcal{S}(p) \}. \]
Analogously, define the open subsets \( A_n^O \subseteq X_n^O \), \( A_n^Z \subseteq X_n^Z \), and \( \widehat{A}_n \subseteq \widehat{X}_n \) as the preimages of \( A_n \) along the projections \( \widehat{X}_n \to X_n^Z \to X_n^O \to X_n \). We have the following commutative diagram:

\[
\begin{array}{ccccccc}
\widehat{A}_n & \to & A_n^Z & \to & A_n^O & \to & A_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widehat{X}_n & \to & X_n^Z & \to & X_n^O & \to & X_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Conf}_n(C) & \to & \text{Conf}_n^Z(C) & \to & \text{Conf}_n^O(C) & \to & \text{Conf}_n(C)
\end{array}
\]

**Homotopy groups of** \( A_n, A_n^O, A_n^Z, \widehat{A}_n \). Consider the subspace \( A'_n \) of admissibility set \( A_n \) defined by

\[
A'_n := \{(p,0) \in X_n : 0 \text{ is admissible at } p\}.
\]

**Lemma 5.9.** The subspace \( A'_n \) is a strong deformation retract of \( A_n \).

**Proof.** Let \( (p,\tau) \in A_n \). If \( p = \{p_1, \ldots, p_n\} \), denote by \( p^b := \frac{1}{n} \sum_{j=1}^n p_j \) the barycenter of the configuration. Let \( F : [0,1] \times A_n \to A_n \) be defined by a rotation w.r.t. the barycenter

\[
F(t, p, \tau) := \left( \left\{ e^{\sqrt{-1}\tau}(p_j - p^b) + p^b : j = 1, \ldots, n \right\}, \tau(1-t) \right).
\]

For all \( (p, \tau) \in A_n, (a,0) \in A'_n \) and \( t \in [0,1] \), we have \( F(0, p, \tau) = (p, \tau), F(1, p, \tau) \in A'_n \), and \( F(t, a, 0) = (a, 0) \). □

**Lemma 5.10.** The space \( A'_n \) is contractible.

**Proof.** We show that the point \( \{1, \ldots, n\}, 0 \) is a strong deformation retract of \( A'_n \). Given \( (p,0) \in A'_n \), with \( p = \{p_1, \ldots, p_n\} \), without loss of generality, we may assume that the \( p_j \)'s are labeled in 0-lexicographical order. Consider the continuous map \( F : [0,1] \times A'_n \to A'_n \) defined by

\[
F(t, p) = \begin{cases}
\left( \left\{ (1-2t)p_j + 2t \text{Re}(p_j) : j = 1, \ldots, n \right\}, 0 \right), & 0 \leq t \leq \frac{1}{2}, \\
\left( \left\{ 2t \text{Re}(p_j) + (2t-1)j : j = 1, \ldots, n \right\}, 0 \right), & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

The map \( F \) defines a strong deformation retraction of \( A'_n \) onto \( \{1, \ldots, n\}, 0 \). □

**Theorem 5.11.** We have

\[
\pi_i(A_n) = 0, \quad i = 0, 1, 2, \ldots,
\]

\[
\pi_0(A_n^O) = \mathbb{S}_n, \quad \pi_i(A_n^O) = 0, \quad i = 1, 2, 3, \ldots,
\]

\[
\pi_0(A_n^Z) = M_n(\mathbb{R}^2), \quad \pi_i(A_n^Z) = 0, \quad i = 1, 2, 3, \ldots,
\]

\[
\pi_0(\widehat{A}_n) = B_n, \quad \pi_i(\widehat{A}_n) = 0, \quad i = 1, 2, 3, \ldots,
\]

where the homotopy groups are based at an arbitrary point.
Proof. All homotopy groups of $\mathcal{A}_n$ vanish, because $\mathcal{A}_n$ is contractible by Lemmata 5.9 and 5.10. Since $\mathcal{A}_n$ is simply connected, we have the homeomorphisms $\mathcal{A}_n^O \cong \mathcal{A}_n \times \mathbb{G}_n, \mathcal{A}_n^Z \cong \mathcal{A}_n \times M_n(\mathbb{R}^2)$, and $\mathcal{A}_n \cong \mathcal{A}_n \times B_n$. Here $\mathbb{G}_n, \mathbb{Z},$ and $B_n$ are equipped with the discrete topology. The claim follows.

Relative homotopy groups. Given a triple $(A, B, c)$ of pointed topological spaces, with $c \in B \subseteq A$, the relative homotopy group $\pi_k(A, B, c)$, with $k \geq 1$, is the set of homotopy classes of continuous maps $f : (\mathbb{D}^k, \mathbb{S}^{k-1}, s_0) \to (A, B, c)$, with $s_0 \in \mathbb{S}^{k-1}$. In particular, the set $\pi_1(A, B, c)$ is the set of homotopy classes of paths $f : [0,1] \to A$ such that $f(0) \in B, f(1) = c$.

Fix a point $x \in \mathcal{A}_n$ and three points $x^0 \in \mathcal{A}_n^O, x^Z \in \mathcal{A}_n^Z, \hat{x} \in \hat{\mathcal{A}}_n$ over it.

Theorem 5.12. We have

\[
\begin{align*}
\pi_1(\mathcal{A}_n^O, \mathcal{A}_n^O, x) & \cong \pi_1(\mathcal{A}_n^O, \mathcal{A}_n^Z, x^Z) \cong \pi_1(\mathcal{A}_n^O, \mathcal{A}_n^Z, x^Z) \cong \pi_0(\mathcal{A}_n, \hat{x}) & \cong B_n, \\
\pi_k(\mathcal{A}_n^O, \mathcal{A}_n^O, x) & \cong \pi_k(\mathcal{A}_n^O, \mathcal{A}_n^Z, x^Z) \cong \pi_k(\mathcal{A}_n^O, \mathcal{A}_n^Z, x^Z) \cong \pi_k(\mathcal{A}_n, \hat{x}) & \cong 0, \quad k \geq 2.
\end{align*}
\]

Proof. With the morphisms of triples of topological spaces

\[
(\mathcal{X}_n, \mathcal{A}_n, \hat{x}) \to (\mathcal{X}_n^Z, \mathcal{A}_n^Z, \hat{x}) \to (\mathcal{X}_n^O, \mathcal{A}_n^O, x) \to (\mathcal{X}_n, \mathcal{A}_n, x),
\]

we can associate the following commutative diagram of relative homotopy groups:

\[
\begin{array}{cccccccc}
... & \rightarrow & \pi_1(\hat{\mathcal{A}}_n, \hat{x}) & \rightarrow & \pi_1(\mathcal{X}_n, \hat{x}) & \rightarrow & \pi_1(\hat{\mathcal{A}}_n, \hat{\mathcal{A}}_n, \hat{x}) & \rightarrow & \pi_0(\mathcal{A}_n, \hat{x}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
... & \rightarrow & \pi_1(\mathcal{A}_n^Z, x^Z) & \rightarrow & \pi_1(\mathcal{X}_n^Z, x^Z) & \rightarrow & \pi_1(\mathcal{X}_n^Z, \mathcal{A}_n^Z, x^Z) & \rightarrow & \pi_0(\mathcal{A}_n^Z, x^Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
... & \rightarrow & \pi_1(\mathcal{A}_n^O, x^O) & \rightarrow & \pi_1(\mathcal{X}_n^O, x^O) & \rightarrow & \pi_1(\mathcal{X}_n^O, \mathcal{A}_n^O, x^O) & \rightarrow & \pi_0(\mathcal{A}_n^O, x^O) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
... & \rightarrow & \pi_1(\mathcal{A}_n, x) & \rightarrow & \pi_1(\mathcal{X}_n, x) & \rightarrow & \pi_1(\mathcal{X}_n, \mathcal{A}_n, x) & \rightarrow & \pi_0(\mathcal{A}_n, x) \\
\end{array}
\]

The rows are the long exact relative homotopy sequences for each triples, and columns are the maps induced by (5.5). The maps $\alpha_1, \alpha_2, \alpha_3$ are bijections: this follows from the unique lifting property of paths for coverings. The claim then follows from Theorem 5.11.

Monodromy data as functions on $\mathcal{A}_n^Z$. Let $M$ be a flat $F$-manifold with Euler field $E$, and denote by $M'$ the set of points $p \in M$ at which the spectrum $\text{spec}(E \circ p)$ is simple. We have the local biholomorphism

\[
u : M' \rightarrow \text{Conf}_n(\mathbb{C}), \quad p \mapsto \text{spec}(E \circ p).
\]

Consider the pulled-back fiber bundles on $M'$

\[
\mathcal{X}_M^Z := \nu^* \mathcal{X}_n^Z, \quad \mathcal{X}_M^O := \nu^* \mathcal{X}_n^O, \quad \mathcal{X}_M := \nu^* \mathcal{A}_n^O, \quad \hat{\mathcal{X}}_M := \nu^* \hat{\mathcal{A}}_n.
\]

\footnote{The group structure is well defined only for $k \geq 2$.}
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together with their open subsets

\[ A_M := (v^*)^{-1} A_n, \quad A^O_M := (v^*)^{-1} A^O_n, \quad A^Z_M := (v^*)^{-1} A^Z_n, \quad \hat{A}_M := (v^*)^{-1} \hat{A}_n. \]

Given \( p_0 \in M' \), the monodromy data of \((M', p_0)\) are well defined after fixing the choice of normalizations (1)-(6) of Section 5.1.

The choice of (5) only, that is, an ordering of canonical coordinates at \( p_0 \), is equivalent to the choice of a point of \( \text{Conf}^O_n(\mathbb{C}) \) over \( v(p_0) = \text{spec}(E\sigma p_0) \).

The choice of (6) only, that is, an admissible direction at \( p_0 \), is equivalent to the choice of a point of \( \mathbb{O}_M \) over \( p_0 \).

The choice of both (5) and (6) is equivalent to the choice of a point of \( \mathbb{Z}_M \) over \( p_0 \).

If choices of (1)–(4) are fixed; however, the 4-tuple \((S_1, S_2, \Lambda, C)\) is not well defined as a single-valued function on \( \mathbb{O}_M \). Indeed, if \((p_0, u_0, \tau)\) is a fixed point of \( \mathbb{O}_M \), for any \( k \in \mathbb{Z} \), there exist paths \( \gamma_k : [0, 1] \to \mathbb{A}^O_M \) such that \( \gamma(0) = (p_0, u_0, \tau) \) and \( \gamma_k(1) = (p_0, u_0, \tau + 2\pi k) \). Namely, \( \gamma_k \) are lifts of loops in the center \( Z(\pi_1(\text{Conf}^O_n(\mathbb{C}))) \).

Thus, the joint choice of (2), (5), and (6) is equivalent to the choice of a point of \( \mathbb{Z}_M \) over \( p_0 \).

Theorems 4.20 and 4.21 can be reformulated as follows.

**Theorem 5.13.** Fix a choice of normalizations (1), (3), (4), and a point in \( \mathbb{Z}_M \). The monodromy data \((S_1, S_2, \Lambda, C)\) are locally constant functions on \( \mathbb{Z}_M \).

In total, we have \( \text{card}(A^Z_M) \) possible values of the monodromy data at \( z = \infty \). Different values at different connected components of \( A^Z_M \) are labeled by paths in \( \pi_1(\mathbb{A}^Z_M, \hat{A}_M) \). The map (5.6) induces a morphism in homotopy

\[ v_* : \pi_1(\mathbb{A}^Z_M, \hat{A}_M) \to \pi_1(\mathbb{A}^Z_n, \hat{A}_n) \cong B_n. \]

The paths of settings (I) and (II) of Section 5.4 are representatives of homotopy classes in \( \pi_1(\mathbb{A}^O_M, \mathbb{A}^O_M) \cong \pi_1(\mathbb{A}^Z_M, \mathbb{A}^Z_M) \). More precisely, consider the double fibrations

Paths of setting (I) represent classes in \( \pi_1(\rho_2^{-1}(\tau), \rho_2^{-1}(\tau)) \) for fixed \( \tau \in \mathbb{R} \).

Paths of setting (II) represent classes in \( \pi_1(\rho_1^{-1}(p), \rho_1^{-1}(p)) \) for fixed \( p \in M' \).

Thus the “duality” mentioned in Section 5.4 reflects the underlying double fibrations above. In both cases (I) and (II), we have induced paths in \( \pi_1(\mathbb{A}^O_M, \mathbb{A}^O_M) \).

Following the terminology of [27], given \( \tau \in \mathbb{R} \), we call \( \tau\)-chamber of \( M \) any connected component of the open set \( p_1(\rho_2^{-1}(\tau)) \).

**Analytic continuation.** Let \( (M, p_0) \) be the germ of a semisimple analytic flat \( F \)-manifold with Euler vector field. Assume that \( p_0 \) is tame semisimple. The whole flat \( F \)-structure can be analytically continued. The picture described in this section gives an insight on this continuation procedure.
By shrinking $M$, we can assume that the germ is defined on a simply connected set, sufficiently small so that (5.6) is an embedding. We can thus identify $M'$ with $\nu(M') \subseteq \text{Conf}_n(\mathbb{C})$.

Fix a point $\hat{u}_0 \in \text{Conf}_n(\mathbb{C})$ over $\nu(p_0)$, we have an open embedding of $(M', p_0) \cong (\nu(M'), \nu(p_0)) \subseteq (\text{Conf}_n(\mathbb{C}), \hat{u}_0)$. In this way, one finds a maximal tame analytic continuation of the initial germ. Notice that the coefficients of the joint system of differential equations (3.23) continue to meromorphic functions of $u \in \text{Conf}_n(\mathbb{C})$: this is the Painlevé property of the solution $V_{\lambda}(u)$ of the isomonodromic differential equations (3.26). By fixing choices of normalizations (1), (3), and (4), the monodromy data $(S_1, S_2, \Lambda, C)$ of the system (3.23) can be seen as locally constant functions on the space $\tilde{\mathcal{A}}_n$. This space has countably many connected components in bijection with the braid group $B_n$. All possible values of $(S_1, S_2, \Lambda, C)$ are given by the action of the braid group $B_n$ of Theorem 5.4.

### 6 Riemann–Hilbert–Birkhoff Inverse Problem for Semisimple Flat $F$-Manifolds

#### 6.1 RHB problem $\mathcal{P}[\mathbf{u}, \tau, \mathfrak{M}]$ and the Malgrange–Sabbah theorem

**Admissible data.** Denote by $\text{Arg}(z) \in [-\pi, \pi]$ the principal branch of the argument of the complex number $z$. Let $u \in \mathbb{C}^n$, and set

$$
\delta(u) := \left\{ \text{Arg} \left(-\sqrt{-1}(u^i - u^j) + 2\pi k \right) : k \in \mathbb{Z}, i, j \text{ are s.t. } u^i \neq u^j \right\}.
$$

Any element $\tau \in \mathbb{R} \setminus \delta(u)$ will be said to be admissible at $u$.

**Definition 6.1.** Let $u \in \mathbb{C}^n$ and $\tau$ admissible at $u$. A $(u, \tau)$-admissible datum is a 6-tuple $\mathfrak{M} := (B, D, L, S_1, S_2, C)$ of matrices in $M_n(\mathbb{C})$ such that:

1. the matrix $B$ is diagonal, that is, $B = B'$,
2. $D = \text{diag}(D_1, \ldots, D_n)$ is a diagonal matrix of integers,
3. we have $L_{ij} = 0$ if $D_i - D_j \in \mathbb{Z}_{<0}$,
4. we have $\text{tr} B = \text{tr} D + \text{tr} L$. (6.1)
5. the matrices $S_1, S_2, C$ are invertible, with $\det S_1 = \det S_2 = 1$,
6. $(S_1)_{ii} = (S_2)_{ii} = 1$,
7. if $i \neq j$, then $(S_1^{-1})_{ij} = 0$ if $\text{Re}(e^{\sqrt{-1}(\tau - \pi)(u^i - u^j)}) > 0$,
8. if $i \neq j$, then $(S_2)_{ij} = 0$ if $\text{Re}(e^{\sqrt{-1}\tau(u^i - u^j)}) > 0$,
9. we have $S_1^{-1} e^{2\pi \sqrt{-1} B} S_2^{-1} = C^{-1} e^{2\pi \sqrt{-1} L} C$. (6.2)

If $u \in \Delta$, define the partition $\{1, \ldots, n\} = \bigsqcup_{r \in J} I_r$ such that for any $r \in J$, we have $\{i, j\} \subseteq I_r$ if and only if $u^i = u^j$. We then require the further vanishing condition

10. $(S_1^{-1})_{ij} = (S_2)_{ij} = 0$ if $i, j \in I_r$ for some $r \in J$. 

If $u \in \Delta$, define the partition $\{1, \ldots, n\} = \bigsqcup_{r \in J} I_r$ such that for any $r \in J$, we have $\{i, j\} \subseteq I_r$ if and only if $u^i = u^j$. We then require the further vanishing condition

10. $(S_1^{-1})_{ij} = (S_2)_{ij} = 0$ if $i, j \in I_r$ for some $r \in J$. 

\[\text{tr} B = \text{tr} D + \text{tr} L.\]
Lemma 6.2. Let $u_0 \in \mathbb{C}^n$ and $\tau$ admissible at $u_0$. If $\mathcal{M}$ is $(u_0, \tau)$-admissible, then there exists a sufficiently small neighborhood $\mathcal{V}$ of $u_0$ such that

1. $\tau$ is admissible at $u$, for all $u \in \mathcal{V}$,
2. $\mathcal{M}$ is $(u, \tau)$-admissible for all $u \in \mathcal{V}$.

Let $u \in \mathbb{C}^n$ and $\tau$ admissible at $u$. Consider the complex $z$-plane with a branch cut from 0 to $\infty$:

$$\tau - \pi < \arg z < \tau + \pi.$$ 

Let $r > 0$ and denote by $\Gamma = \Gamma(\tau, r)$ the union of the following oriented paths, see Figure 1:

1. the half-line $\Gamma_{-\infty}$ defined by $\arg z = \tau \pm \pi$, $|z| > r$, originating from $\infty$;
2. the half-line $\Gamma_{+\infty}$ defined by $\arg z = \tau$, $|z| > r$, ending to $\infty$;
3. the half-circle $\Gamma_1$ defined by $\tau - \pi < \arg z < \tau$, $|z| = r$, counterclockwise oriented;
4. the half-circle $\Gamma_2$ defined by $\tau < \arg z < \tau + \pi$, $|z| = r$, counterclockwise oriented.

The orientations uniquely define the + and − sides for each path $\Gamma_{\pm\infty}, \Gamma_1, \Gamma_2$. For $z \in \Gamma_{-\infty}$, we use the symbol $z_{\pm}$ if $\arg z = \tau \pm \pi$. Set $\Pi_0, \Pi_L, \Pi_R$ to be the components of complement $\mathbb{C} \setminus \Gamma$, and $T_1, T_2$ to be the two nodes of $\Gamma$, as in Figure 1. Let $\mathcal{M} := (B, D, L, S_1, S_2, C)$ be a $(u, \tau)$-admissible datum. Define two functions

$$Q(\cdot; u), H(\cdot; u) : \Gamma \to GL(n, \mathbb{C}),$$

by

$$Q(z; u) := U(u)z + B \log z, \quad U(u) := \text{diag}(u^1, \ldots, u^n),$$

$$H(z; u) :=
\begin{cases}
    e^{Q(z; u)}S_1^{-1}e^{-Q(z; u)}, & \text{along } \Gamma_{-\infty}, \\
    e^{Q(z; u)}S_2^{-1}e^{-Q(z; u)}, & \text{along } \Gamma_{+\infty}, \\
    e^{Q(z; u)}C^{-1}z^{-L}z^{-D}, & \text{along } \Gamma_1, \\
    e^{Q(z; u)}S_2^{-1}C^{-1}z^{-L}z^{-D}, & \text{along } \Gamma_2.
\end{cases}$$

Problem 6.3 (Problem $\mathcal{P}[u, \tau, \mathcal{M}]$). Find an analytic function $G : \mathbb{C} \setminus \Gamma \to M_n(\mathbb{C})$ such that

1. $G|_{\Pi_{\nu}}$ extends continuously to $\overline{\Pi_{\nu}}$ for $\nu = 0, L, R$;
(2) the nontangential limits $G_\pm : \Gamma \to M_n(\mathbb{C})$ of $G$ from the — and $+$ sides of $\Gamma$ exist, and are continuous;

(3) they are related by

$$G_+(z) = G_-(z)H(z; u);$$

(4) $G(z)$ tends to the identity matrix $I$ as $z \to \infty$.

**Theorem 6.4** [24, Section 3]. Let $u_0 \in \mathbb{C}^n$. Assume that the pair $(\tau, \mathcal{M})$ is admissible at each point of a sufficiently small open neighborhood $\mathcal{V}$ of $u_0$. If $\mathcal{P}[u_0, \tau, \mathcal{M}]$ is solvable, there exists an analytic set $\Theta \subseteq \mathcal{V} \setminus \{u_0\}$ such that $\mathcal{P}[u, \tau, \mathcal{M}]$ is solvable for all $u \in \mathcal{V} \setminus \Theta$. Moreover, the solution $G(z; u)$ is unique and holomorphic w.r.t. $u \in \mathcal{V} \setminus \Theta$.

**Remark 6.5.** In [24], we showed that Theorem 6.4 is essentially equivalent to a (weaker) extension, due to C. Sabbah [80, Th. 4.9], of a previous result of B. Malgrange [70]. For this reason, we refer to Theorem 6.4 as Malgrange–Sabbah theorem. The original result of Malgrange concerns the case $u_0 \in \mathbb{C}^n \setminus \Delta$. The result of Sabbah concerns the case $u_0 \in \Delta$.

### 6.2 Construction of semisimple flat $F$-manifolds via an RHB inverse problem

Let $u_0 \in \mathbb{C}^n, \tau$ be an admissible direction at $u_0$, and $\mathcal{M} = (B, D, L, S_1, S_2, C)$ be a $(u_0, \tau)$-admissible datum. Assume that the RHB boundary value problem $\mathcal{P}[u_0, \tau, \mathcal{M}]$ is solvable. Let $\mathcal{V}$ and $\Theta$ as in Malgrange–Sabbah Theorem 6.4: the problem $\mathcal{P}[u, \tau, \mathcal{M}]$ is well defined, solvable and with unique solution $G(z, u)$, holomorphic w.r.t. $u \in \mathcal{V} \setminus \Theta$. Consider the asymptotic expansions of $G(z, u)$ for $z \to 0$ and $z \to \infty$:

$$G(z, u) = 1 + z^{-1}F_1(u) + O(z^{-2}), \quad z \to \infty, \quad z \in \Pi_{L/R},$$

$$G(z, u) = G_0(u) + zG_1(u) + z^2G_2(u) + O(z^3), \quad z \to 0,$$

with coefficients $F_1, G_i$’s holomorphic w.r.t. $u$. Define the functions

$$X_{L/R}(z, u) := G(z, u)z^Bz^U, \quad z \in \Pi_{L/R},$$

$$X_0(z, u) := G(z, u)z^Dz^L, \quad z \in \Pi_0.$$

**Lemma 6.6.** The functions $X_0(z, u), X_{L/R}(z, u)$ are solutions of the joint system of differential equations

$$\frac{\partial}{\partial u^i}X = (zE_i - V_i(u)^T)X, \quad V_i(u) := [F_1(u)^T, E_i] \equiv -\left(\frac{\partial G_0}{\partial u^i} \cdot G_0^{-1}\right)^T, \quad (6.3)$$

$$\frac{\partial}{\partial z}X = \left(U - \frac{1}{z}V(u)^T\right)X, \quad V(u) := [F_1(u)^T, U] - B \equiv -(G_0^T)^{-1}(D + L')G_0^T. \quad (6.4)$$
Proof. We have \( X_L(z, u) = X_R(z, u)S_2 \) and \( X_R(z, u) = X_0(z, u)C \). It follows that \( \partial_z X_0 \cdot X_0^{-1} = \partial_z X_L \cdot X_L^{-1} = \partial_z X_R \cdot X_R^{-1} \), and the resulting function \( f(z, u) := \partial_z X_i(z, u)X_i(z, u)^{-1} \), with \( i = 0, R, L \), is analytic with respect to \( z \in \mathbb{C}^* \). We have

\[
\frac{\partial X_{L/R}}{\partial z} \cdot X_{L/R}^{-1} = \partial_z G \cdot G^{-1} + \frac{1}{z} GBG^{-1} + GUG^{-1}
\]

\[
= U + \frac{1}{z} ([F_1(u), A(u)] + B) + O\left(\frac{1}{z^2}\right), \quad z \to \infty,
\]

\[
\frac{\partial X_0}{\partial z} \cdot X_0^{-1} = \partial_z G \cdot G^{-1} + \frac{1}{z} (GDG^{-1} + Gz^D Lz^{-D} G^{-1})
\]

\[
= \frac{1}{z} G_0(u)(D + L')G_0(u)^{-1} + O(1), \quad z \to 0.
\]

The last equality follows from the identity \( z^D Lz^{-D} = L' + O(z) \), which is deduced from the admissibility condition (3) on the matrix \( L \). The matrices \( S_1, S_2, C \) are constant with respect to both \( u \) and \( z \); we deduce that the r.h.s. of the two equalities above are equal. This implies that \( X_{L/R} \) and \( X_0 \) are solutions of the differential equation (6.4). Similarly, we have

\[
\frac{\partial X_{L/R}}{\partial u_i} \cdot X_{L/R}^{-1} = \frac{\partial G}{\partial u_i} \cdot G^{-1} + zGE_iG^{-1} = zE_i + [F_1, E_i] + O\left(\frac{1}{z}\right),
\]

\[
\frac{\partial X_0}{\partial u_i} \cdot X_0^{-1} = \frac{\partial G}{\partial u_i} \cdot G^{-1} = \frac{\partial G_0}{\partial u_i} \cdot G_0^{-1} + O(z),
\]

where \((E_i)_{ab} = \delta_{ai}\delta_{bi}\). The matrices \( S_1, S_2, C \) being constant, we deduce that the r.h.s. of the two equalities above are equal. Hence, \( X_{L/R} \) and \( X_0 \) are solutions of the differential systems (6.3). \( \square \)

Remark 6.7. Notice that \( V(u) \) is diagonalizable with diagonal Jordan form \(-D - L'\). In particular, the eigenvalues do not depend on \( u \).

Lemma 6.8. The off-diagonal entries \( (F''_1)^T \) satisfy the Darboux–Egoroff equations (3.27) and (3.28), and the homogeneity conditions

\[
\sum_{k=1}^{n} u^k \delta_k F_1(u)^j = (b_i - b_j - 1)F_1(u)^j, \quad B = \text{diag}(b_1, \ldots, b_n).
\]

(6.5)

Proof. The compatibility condition \( \partial_i \partial_j = \partial_j \partial_i \) for the joint system (6.3),(6.4) reads

\[
[E_j, \partial_i F_1^T] - [E_i, \partial_j F_1^T] + [[E_i, F_1^T], [E_j, F_1^T]] = 0.
\]

This coincides with Equations (3.27) and (3.28). Let \( \kappa \in \mathbb{C}^* \), and set

\[
h(z) := z^D z^L C^{-1} \kappa^{-L} z^{-L} \kappa^{-D} z^{-D}.
\]

The piecewise analytic function \( \tilde{G} : (\Pi_0 \cup \Pi_L \cup \Pi_R) \times (\kappa V \setminus \kappa \Theta) \to \mathbb{C} \) defined by

\[
\tilde{G}(z; u) := \kappa^{-B} G(\kappa z; \kappa^{-1} u) h(z)^{-1}, \quad z \in \Pi_0,
\]

\[
\tilde{G}(z; u) := \kappa^{-B} G(\kappa z; \kappa^{-1} u) \kappa^B, \quad z \in \Pi_{L/R},
\]
solves the same RHB problem $\mathcal{P}[\mathbf{u}, \tau, \mathfrak{M}]$ as $G$. By uniqueness of solution, we have $\tilde{G} = G$. This implies that $F_1(\kappa^{-1}\mathbf{u}) = \kappa \cdot \kappa^{-B} F_1(\mathbf{u}) \kappa^B$, and (6.5) follows.

Define the off-diagonal matrix $\Gamma(\mathbf{u})$ by $\Gamma(\mathbf{u})_{ij} := F_1(\mathbf{u})_{ji}$.

**Corollary 6.9.** For any fixed $\mathbf{H}_o \in (\mathbb{C}^*)^n$, there exists a unique $\mathbf{H}(\mathbf{u}) = (H_1(\mathbf{u}), \ldots, H_n(\mathbf{u}))^T$, analytic in $\mathbb{V} \setminus \Theta$, satisfying

$$\partial_j H_i = \Gamma^j_i H_j, \quad i \neq j, \quad \partial_i H_i = - \sum_{k \neq i} \Gamma^j_k H_k,$$

and such that $\mathbf{H}(\mathbf{u}_o) = \mathbf{H}_o$. Moreover, the functions $H_i$ are never vanishing.

**Proof.** The linear Pfaffian system (6.6) is completely integrable, by Lemma 6.8. This ensures uniqueness and existence of solutions $H_i$. The nonvanishing of solutions is a standard result, see, for example, [44, Ch. 11].

If we compare Equations (6.6) with the equations of Proposition 3.18, we notice that an homogeneity condition of the functions $H_i(\mathbf{u})$ is missing. In general, for arbitrary choices of the initial datum $\mathbf{H}_o$, there are no constants $\delta_1, \ldots, \delta_n$ such that

$$\sum_{j=1}^n u^j \partial_j H_i = - \delta_i H_i, \quad i = 1, \ldots, n. \quad (6.7)$$

The following lemma will clarify how to recover (6.7).

**Lemma 6.10.** Let $\mathbb{V}$ be the linear space of (column) vector solutions $\mathbf{H}(\mathbf{u}) = (H_1(\mathbf{u}), \ldots, H_n(\mathbf{u}))^T$, analytic in $\mathbf{u} \in \mathbb{V} \setminus \Theta$, of the Pfaffian system (6.6). The matrix $V(\mathbf{u})$ acts on $\mathbb{V}$. Moreover, if $V(\mathbf{u})H(\mathbf{u}) = \ell H(\mathbf{u})$, with $\ell \in \mathbb{C}$, we have

$$\sum_{j=1}^n u^j \partial_j H_i = (\ell + b_i) H_i, \quad i = 1, \ldots, n.$$  

**Proof.** The matrices $V_i(\mathbf{u})$ defined in (6.3) have entries $(V_i)^j_h = \Gamma^j_i \delta_{ih} - \delta_{jh} \Gamma^j_i$. Thus, the joint system (6.6) can be written in matrix form $\partial_k \mathbf{H} = V_k \mathbf{H}$, for any $k = 1, \ldots, n$. We have

$$\partial_k(V\mathbf{H}) = (\partial_k V)\mathbf{H} + V \partial_k \mathbf{H} = [V_k, V] \mathbf{H} + VV_k \mathbf{H} = V_k VH,$$

where we have used the equation $\partial_k V = [V_k, V]$ (compatibility condition of the joint system (6.3),(6.4)). If $VH = \ell \mathbf{H}$, we deduce

$$\sum_{j=1}^n u^j \partial_j \mathbf{H} = \sum_{j=1}^n u^j V_j \mathbf{H} = \sum_{j=1}^n u^j [\Gamma, E_j] \mathbf{H} = [\Gamma, U] \mathbf{H} = (V + B)\mathbf{H} = (\ell \cdot 1 + B)\mathbf{H}.$$  

**Remark 6.11.** The incompatibility of the joint system of Equations (6.6) and (6.7), for arbitrary choices of $\mathbf{H}_o$ and $\delta_1, \ldots, \delta_n$, was already noticed in [62, Section 2], in the construction procedure.
of bi-flat $F$-manifold starting from a solution of the Darboux–Egoroff system (3.27), (3.28), and (3.29). Lemma 6.10 is essentially formulated in [62, Remark 2.3].

**Corollary 6.12.** Assume that

$$\prod_{j=1}^{n} [G_0(u_o)^{-1}]_j^1 \neq 0. \quad (6.8)$$

Then, the functions $H_i(u) := [G_0(u)^{-1}]_i^1$, with $i = 1, \ldots, n$, are nonvanishing in a sufficiently small neighborhood of $u_o$, and satisfy the joint system (6.6) and (6.7) with $\delta_i = D_1 + L_{11} - b$, for $i = 1, \ldots, n$.

**Proof.** Let $H(u)$ be the first column of $(G_0(u)^{-1})^T$. From (6.3), we deduce $\partial_k H = V_k H$ for any $k = 1, \ldots, n$. This is equivalent to (6.6). Moreover, from (6.4), we deduce $V(u)(G_0(u)^{-1})^T = (G_0(u)^{-1})^T(-D - L')$. By taking the first column of both sides, we obtain $VH = \ell H$ with $\ell = -D_1 - L_{11}$. We conclude by Lemma 6.10. □

Thus, if the nonvanishing assumption (6.8) holds, we have an automatic choice for a solution $H(u)$ of the joint system of Equations (6.6) and (6.7), which is entry-by-entry nonvanishing in a sufficiently small neighborhood of $u_o$. Such a choice is given by the first column of $[G_0(u)^{-1}]^T$. In what follows, we will assume (6.8) and we will fix such a choice for $H$.

**Lemma 6.13.** Let $G_0(u), H_i(u)$ as above. For any $\alpha = 1, \ldots, n$, the one form

$$\varpi_\alpha(u) := \sum_{k=1}^{n} G_0(u)_\alpha^k H_k(u) du^k$$

is closed.

**Proof.** Set $H := \text{diag}(H_1, \ldots, H_n)$. We have $\partial_j [G_0^T H]^\alpha_i = (G_0^T)^\alpha_i H_i^j + (G_0^T)^\alpha_i H_j^i = \delta_i [G_0^T H]^\alpha_j$. This can be easily seen by invoking Equations (6.3) and (6.6). □

**Lemma 6.14.** Let $G_1(u), H_i(u)$ as above. For any $\alpha = 1, \ldots, n$, the one form

$$\varphi_\alpha(u) := \sum_{k=1}^{n} G_1(u)_\alpha^k H_k(u) du^k$$

is closed.

**Proof.** First, notice that we have $\partial_i G_1 = E_i G_0 + \partial_i G_0 \cdot G_0^{-1} \cdot G_1$. This follows from the fact that $X_0$ is a solution of the joint system (6.3) and (6.4). Furthermore, we have the identity $\varphi_\nu(u) := \sum_\alpha [G_0(u)^{-1} G_1(u)]^\alpha_\nu \cdot \varpi_\alpha(u)$. Hence, we deduce

$$d \varphi_\nu = \sum_{\alpha,i,j} \delta_i [G_0(u)^{-1} G_1(u)]^\alpha_\nu du^i \wedge \varpi_\alpha = \sum_{\alpha,i,k} (G_0^{-1})^\alpha_i (G_0)^\nu_\alpha (G_0)_\alpha^k H_k du^i \wedge du^k = 0.$$ □
Consider a polydisc $D(u_0) \subseteq \mathcal{V} \setminus \Theta$ centered in $u_0$, and sufficiently small so that $H_i(u) \neq 0$ for $u \in D(u_0)$, and $i = 1, \ldots, n$. Define the functions
\begin{align*}
t^\alpha(u) &:= \int_{u_0}^u \omega^\alpha, \quad F^\alpha(u) := \int_{u_0}^u \varphi_\nu, \quad \alpha, \nu = 1, \ldots, n, \tag{6.9}
\end{align*}
where $u \in D(u_0)$. By definition, the functions $t = (t^\alpha)_{\alpha}$ define a system of coordinates on $D(u_0)$, with Jacobian matrix
\begin{align*}
\left( \frac{\partial t^\alpha}{\partial u^i} \right)_{\alpha, i} = G_0(u)^T H(u). \tag{6.10}
\end{align*}

Introduce the connection $\nabla$ on the holomorphic tangent bundle of $D(u_0)$ by
\begin{align*}
\nabla_\beta \frac{\partial}{\partial u^j} = \sum_{h=1}^n K^h_{ij} \frac{\partial}{\partial u^h}, \quad i, j = 1, \ldots, n, \tag{6.11}
\end{align*}
where the Christoffel symbols $K^h_{ij}$, with $i, j, h = 1, \ldots, n$, are defined by
\begin{align*}
K^h_{ij} &= 0, \quad i, j, k \text{ distinct}, \tag{6.12}
K^1_{ij} &= K^i_{ji} = -K^1_{jj} = \frac{H_j}{H_i} \Gamma^i_j, \quad i \neq j, \tag{6.13}
K^1_{ii} &= -\sum_{h \neq i} K^1_{ih}. \tag{6.14}
\end{align*}

Moreover, define the commutative and associative product $\circ$ of holomorphic vector fields on $D(u_0)$ by
\begin{align*}
\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \sum_{h=1}^n c^h_{ij} \frac{\partial}{\partial u^h}, \quad c^h_{ij} = \delta^h_i \delta^h_j, \quad i, j, h = 1, \ldots, n, \tag{6.15}
\end{align*}
with unit $e = \sum_{j=1}^n \frac{\partial}{\partial u^j}$.

**Theorem 6.15.** The datum $(D(u_0), \nabla, c, e)$ defines an analytic semisimple homogeneous flat $F$-manifold. More precisely:

1. The coordinates $t$ are flat coordinates, and the coordinates $u$ are canonical coordinates.
2. The coordinate $t^1$ is such that $\frac{\partial}{\partial t^1} = e$.
3. The functions $F^\alpha(u(t))$ are solutions of the oriented associativity Equations (2.2), and Equations (2.1) hold for $A^1 = 1, A^\mu = 0$ for $\mu \neq 1$.
4. The $\bar{\Psi}$-matrix is $\bar{\Psi}(u) = (G_0(u)^T H(u))^{-1}$.
5. The vector field $E = \sum_i u^i \frac{\partial}{\partial u^i}$ defines an Euler vector field.
6. The tensor $\mu^k$, with $\lambda = D_1 + L_{11}$, is given by $\mu^k = -D - L'$.
7. The homogeneous degrees $q_\alpha$ are given by $q_\alpha = -D_\alpha - L_{\alpha\alpha} + D_1 + L_{11}$ for $\alpha = 1, \ldots, n$.
8. The conformal dimensions are $\delta_j = D_1 + L_{11} - b_j$, for $j = 1, \ldots, n$. 
The monodromy data \((\mu^\lambda, [R], \hat{S}_1, \hat{S}_2, \Lambda, \hat{C})\) at \(u_0\), computed with respect to \(\lambda = D_1 + L_{1_1}, \tau\), the Lamé coefficients \((H^i)_{i=1}^n\), can be reconstructed from the admissible datum \(\mathcal{M}\):

\[
\mu^\lambda = -D - L', \quad [R] = [L''], \quad \hat{S}_1 = S_1, \quad \hat{S}_2 = S_2, \quad \Lambda = B, \quad \hat{C} = C.
\]

**Proof.** The proof that \((D(u_0), V, c, e)\) is a flat \(F\)-manifold is the same as in [2, Th. 2.2, Lem. 2.3]. More precisely, the flatness of \(V\) is checked via explicit computations of the Riemann curvature tensor. The identity \(\nabla e = 0\) is equivalent to the identities

\[
\Gamma^k_{ij} = -\Gamma^j_{ik}, \quad \text{for } i \neq j,
\]

and

\[
\Gamma^i_{ii} = -\sum_{h \neq i} \Gamma^i_{ih}
\]

for the Christoffel symbols. The symmetry of \(\nabla c\) in its covariant (i.e., lower) indices is equivalent to the identities

\[
\Gamma^h_{ij} = -\Gamma^j_{ih}, \quad \text{for } i \neq j,
\]

and

\[
\Gamma^h_{ij} = 0
\]

for \(i, j, k\) distinct. Furthermore, the flat \(F\)-structure is semisimple, and \(u\) are canonical coordinates, as (6.15) manifestly shows.

The matrix \(M(u) := G^T_0(u)H(u)\) satisfies

\[
\partial_i M = \partial_i G^T_0 \cdot H + G^T_0 \partial_i H
\]

by Equation (6.3)

\[
= -G^T_0 V_i H + G^T_0 \partial_i H
\]

\[
= M(-H^{-1}V_i H + H^{-1} \partial_i H).
\]

Arrange the Christoffel symbols \(K^h_{ij}\) into \(n\) matrices \(K_1, \ldots, K_n\) by setting \((K_i)^h = K^h_{ij}\), for \(i, j, h = 1, \ldots, n\). We claim that

\[
-H^{-1}V_i H + H^{-1} \partial_i H = K_i,
\]

for any \(i = 1, \ldots, n\). (6.16)

Indeed, the entry \((h, j)\) of the l.h.s. of (6.16) equals

\[
\frac{H_j}{H_h} \Gamma^h_{ij} + \frac{1}{H_h} \partial_i H_j \delta^h_j = \frac{H_j}{H_h} (\Gamma^h_i \delta_{ij} - \Gamma^h_j \delta_{ih}) + \frac{1}{H_h} \partial_i H_j \delta^h_j = \begin{cases} 0, & i, j, h \text{ distinct}, \\ \frac{H_j}{H_i} \Gamma^i_j, & h = i \neq j, \\ -\frac{H_i}{H_h} \Gamma^h_i, & h \neq i = j, \\ -\sum_{\ell \neq h} \frac{H_{\ell}}{H_i} \Gamma^h_{i\ell}, & i = j = h. \end{cases}
\]

This proves (6.16). Since \(M^\alpha_j = \frac{\partial u^\alpha}{\partial u^j}\), for \(\alpha, j = 1, \ldots, n\), the identity \(M^{-1} \partial_t M = K_i\) reads

\[
K^h_{ij} = \sum_{\alpha} \frac{\partial u^h}{\partial t^\alpha} \frac{\partial}{\partial u^i} \left( \frac{\partial t^\alpha}{\partial u^j} \right).
\]

By recalling the transformation rule for Christoffel symbols, this proves that in the coordinates \(t\), all the Christoffel symbols of \(V\) vanish. So, \(t\) are \(V\)-flat coordinates. Since the coefficients \(H_i\) are chosen as in Corollary 6.12, that is, \(H_i(\mathbf{u}) = [G_0(\mathbf{u})]^{-1}_i\), for \(i = 1, \ldots, n\), we have

\[
\frac{\partial u^i}{\partial t^1} = [H^{-1}(G_0^{-1})^T]_1^i = \frac{1}{H_i} (G_0^{-1})_i^1 = 1, \quad i = 1, \ldots, n,
\]

so that

\[
\frac{\partial}{\partial t^1} = \sum_{j=1}^n \frac{\partial}{\partial u^j} = e.
\]
The functions $F^\alpha(u(t))$ are the potentials, by Corollary 3.2. The validity of Equations (2.1), with $A^1 = 1$ and $A^\mu \neq 0$ for $\mu \neq 1$, is clear. Point (4) is clear from (6.10). All the remaining claims directly follow by inspection of the joint system (6.3) and (6.4) written in the coordinate frame $\left(\frac{\partial}{\partial t}\right)_{\alpha=1}^n$, Remarks 3.5 and 3.19, and by the definition of the monodromy data and of the RHB problem $P[u_o, \tau, \mathcal{M}]$. □

**Definition 6.16.** We denote by $F[u_o, \tau, \mathcal{M}]$ the germ (pointed at $u_o$) of the analytic flat $\mathcal{F}$-manifold $(D(u_o), \nabla, c, e)$ described in Theorem 6.15.

**Proposition 6.17.** Let $h = \text{diag}(h_1, ..., h_n) \in GL(n, \mathbb{C})$. If $\mathcal{M} = (B, D, L, S_1, S_2, C)$ is $(u_o, \tau)$-admissible, then also the datum

$$h \mathcal{M} := (B, \quad D, \quad L, \quad h^{-1}S_1h, \quad h^{-1}S_2h, \quad Ch)$$

is $(u_o, \tau)$-admissible. If $P[u_o, \tau, \mathcal{M}]$ is solvable, then so is $P[u_o, \tau, h \mathcal{M}]$. Moreover, if the solution of $P[u_o, \tau, \mathcal{M}]$ satisfies the assumption (6.8), then also the solution of $P[u_o, \tau, h \mathcal{M}]$ does. The resulting pointed flat $\mathcal{F}$-manifolds $F[u_o, \tau, \mathcal{M}]$ and $F[u_o, \tau, h \mathcal{M}]$ coincide, being defined by the same potentials $(F^1(t), ..., F^n(t))$, up to linear terms.

**Proof.** The admissibility of $h \mathcal{M}$ is easily checked. Denote by $H(z, u)$ and $H'(z, u)$ the coefficients of the problems $P[u, \tau, \mathcal{M}]$ and $P[u, \tau, h \mathcal{M}]$, respectively. We have

$$H' := \begin{cases} 
  h^{-1}H, & \text{along } \Gamma_{-\infty}, \\
  h^{-1}H, & \text{along } \Gamma_{+\infty}, \\
  h^{-1}H, & \text{along } \Gamma_1, \\
  h^{-1}H, & \text{along } \Gamma_2.
\end{cases}$$

If $G(z, u)$ denotes the solution of the problem of $P[u, \tau, \mathcal{M}]$, with $u$ varying in a sufficiently small neighborhood of $u_o$, then the function

$$G'(z, u) := \begin{cases} 
  h^{-1}G(z, u)h, & \text{on } \Pi_{L/R}, \\
  h^{-1}G(z, u), & \text{on } \Pi_0
\end{cases}$$

is the solution of $P[u, \tau, h \mathcal{M}]$. We then have the following rescaling:

$$G_0(u) \mapsto G'_0(u) = h^{-1}G_0(u),$$

$$G_1(u) \mapsto G'_1(u) = h^{-1}G_1(u),$$

$$H_j(u) \mapsto H'_j(u) = h_jH_j(u), \quad j = 1, ..., n.$$

The forms $\phi_\alpha(u)$ and $\phi'_\alpha(u)$ are invariant. □
**Proposition 6.18.** Let $G = \text{diag}(g_1, \ldots, g_n) \in GL(n, \mathbb{C})$. If $\mathcal{M} = (B, D, L, S_1, S_2, C)$ is $(u_0, \tau)$-admissible, then also the datum

$$G \mathcal{M} := (B, D, G^{-1}LG, S_1, S_2, G^{-1}C)$$

is $(u_0, \tau)$-admissible. If $P[u_0, \tau, \mathcal{M}]$ is solvable, then so is $P[u_0, \tau, G \mathcal{M}]$. Moreover, if the solution of $P[u_0, \tau, \mathcal{M}]$ satisfies the assumption (6.8), then also the solution of $P[u_0, \tau, G \mathcal{M}]$ does. The resulting pointed flat $F$-manifolds $\mathcal{F}[u_0, \tau, \mathcal{M}]$ and $\mathcal{F}[u_0, \tau, G \mathcal{M}]$ are isomorphic: if $(\tilde{F}^\alpha(t))^{\alpha=1}_{\alpha=1}$ and $(\tilde{F}^\alpha(t))^{\alpha=1}_{\alpha=1}$ are the potentials defining the structure $\mathcal{F}[u_0, \tau, \mathcal{M}]$ and $\mathcal{F}[u_0, \tau, G \mathcal{M}]$, respectively, we have

$$\tilde{F}^\alpha = \frac{g_\alpha}{g_1} t^\alpha + c^\alpha, \quad \tilde{F}^\alpha(t) = \frac{g_\alpha}{g_1} F^\alpha(t) + \text{linear terms in } t, \quad \alpha = 1, \ldots, n.$$  

In particular, $\frac{\partial}{\partial t^1} = \frac{\partial}{\partial \tilde{t}^1} = e$.

**Proof.** The admissibility of $G \mathcal{M}$ is easily checked. Denote by $H(z, u)$ and $H'(z, u)$ the coefficients of the problems $P[u, \tau, \mathcal{M}]$ and $P[u, \tau, G \mathcal{M}]$, respectively. We have

$$H' = H \quad \text{along } \Gamma_{\pm \infty}, \quad H' = HG \quad \text{along } \Gamma_1, \Gamma_2.$$  

If $G(z, u)$ denotes the solution of the problem of $P[u, \tau, \mathcal{M}]$, with $u$ varying in a sufficiently small neighborhood of $u_0$, then the function

$$G'(z, u) := G(z, u) \quad \text{on } \Pi_{L/R}, \quad G'(z, u) = G(z, u)G \quad \text{on } \Pi_0,$$

is the solution of $P[u_0, \tau, G \mathcal{M}]$. We then have the following rescaling:

$$G_0(u) \mapsto G'_0(u) = G_0(u)G,$$

$$G_1(u) \mapsto G'_1(u) = G_1(u)G,$$

$$H_j(u) \mapsto H'_j(u) = g_1^{-1} H_j(u), \quad j = 1, \ldots, n.$$  

The forms $\varpi_\alpha(u)$ and $\varphi_\alpha(u)$ transform as follows: if $G = \text{diag}(g_1, \ldots, g_n)$, then

$$\varpi_\alpha(u) \mapsto \varpi'_\alpha(u) = \frac{g_\alpha}{g_1} \varpi_\alpha(u), \quad \varphi_\alpha(u) \mapsto \varphi'_\alpha(u) = \frac{g_\alpha}{g_1} \varphi_\alpha(u), \quad \alpha = 1, \ldots, n.$$  

The claim follows. \hfill \Box

### 6.3 Reconstruction of admissible germs of semisimple flat $F$-manifold

In this section, we prove that all admissible germs of analytic flat $F$-manifolds are of the form $\mathcal{F}[u_0, \tau, \mathcal{M}]$. Let $(M, p)$ be an admissible germ of an analytic flat $F$-manifold. Without loss of generality, we fix flat local coordinates $t$ so that $\frac{\partial}{\partial t^1} = e$. Fix choices of normalizations (1)–(6) of Section 5.1. We then have a well-defined system $(\lambda, \mu^1, R, S_1, S_2, \Lambda, C)$ of monodromy data at $p$, computed w.r.t. the chosen ordering of $u_0 = u(p)$, an admissible direction $\tau$, and the normalization $(H_{\alpha,1}, \ldots, H_{\alpha,n}) \in (\mathbb{C}^*)^n$ of Lamé coefficients at $p$. 
Lemma 6.19. The matrix $\mu^\lambda = \text{diag}(q_1, \ldots, q_n) - \lambda \cdot \mathbf{1}$ has a unique decomposition $\mu^\lambda = D^\lambda + S^\lambda$ with

$$D^\lambda = \text{diag}(d_1, \ldots, d_n), \quad d_i \in \mathbb{Z}, \quad i = 1, \ldots, n$$

$$S^\lambda = \text{diag}(\rho_1, \ldots, \rho_n), \quad \text{Re}(\rho_i) \in [0, 1[, \quad i = 1, \ldots, n.$$

We have

$$[D^\lambda, S^\lambda] = 0, \quad [R, S^\lambda] = 0. \quad (6.17)$$

Proof. The uniqueness of the decomposition and the first commutation relation of (6.17) are clear. For any pair $(i, j)$, we have $[S^\lambda, R]_{ij} = (\rho_i - \rho_j)R_{ij} = 0$. Indeed, for $i \neq j$, we have two possibilities: if $q_i - q_j \not\in \mathbb{Z}_{<0}$, then $R_{ij} = 0$; if $q_i - q_j \in \mathbb{Z}_{<0}$, then $\rho_i - \rho_j = 0$. □

Remark 6.20. It follows that $z^{-\mu^\lambda} z^R = z^{-D^\lambda} z^R - S^\lambda$.

Proposition 6.21. The 6-tuple $\mathcal{M} = (\Lambda, -D^\lambda, R - S^\lambda, S_1, S_2, C)$ is $(\mathbf{u}_o, \tau)$-admissible. The problem $\mathcal{P}[\mathbf{u}_o, \tau, \mathcal{M}]$ is solvable. Moreover, the solution $G(z, \mathbf{u}_o)$ satisfies the nonvanishing assumption (6.8).

Proof. Points (1)–(4) of Definition 6.1 directly follow from the definitions and properties of $R, \Lambda, D^\lambda, S^\lambda$. Points (5)–(10) of Definition 6.1 follow from Propositions 4.16 and 4.17. Let

- $\Xi_0(t, z)$ be the fixed solution of the joint system (3.2) in Levelt normal form,
- $X_i(\mathbf{u}, z)$, with $i = 1, 2, 3$, be the solutions of the joint system (3.23) uniquely defined by the asymptotics (4.6).

The (unique) solution of the problem $\mathcal{P}[\mathbf{u}_o, \tau, \mathcal{M}]$ is

$$G(z, \mathbf{u}_o) = \begin{cases} (\Xi(t_o)H_o^{-1})^T \Xi_0(t_o, z)z^{-R}z^{\mu^\lambda}, & z \in \Pi_0, \\ X_2(\mathbf{u}_o, z)e^{-zU_o}z^{-\Lambda}, & z \in \Pi_R, \\ X_3(\mathbf{u}_o, z)e^{-zU_o}z^{-\Lambda}, & z \in \Pi_L. \end{cases}$$

Here, we set $t_o := t(p)$, $U_o := \text{diag}(u_1^o, \ldots, u_n^o)$, and $H_o := \text{diag}(H_{o,1}, \ldots, H_{o,n})$. Notice that assumption (6.8) holds, since $\Xi(t_o)_1 = 1$ for any $i = 1, \ldots, n$, see Remark 3.6. □

Remark 6.22. In [24], it is proved that solutions of $\mathcal{P}[\mathbf{u}, \tau, \mathcal{M}]$ can be factorized via two auxiliary RHB problems $\mathcal{P}_1[\mathbf{u}, \tau, \mathcal{M}]$ and $\mathcal{P}_2[\mathbf{u}, \tau, \mathcal{M}]$. The problem $\mathcal{P}_1[\mathbf{u}, \tau, \mathcal{M}]$ is shown to admit unique solution $\Psi(z, \mathbf{u})$ holomorphically depending on $\mathbf{u}$ varying in a neighborhood of $\mathbf{u}_o$, see [24, Th. 3.7]. Given $\Psi(z, \mathbf{u})$, the problem $\mathcal{P}_2[\mathbf{u}, \tau, \mathcal{M}]$ is formulated, and it is shown to be locally uniquely solvable, see [24, proof of Th. 3.13].

If $\mathbf{u}_o \in \Delta$, assumption (10) of Definition 6.1 is crucial for the proof of the unique solvability of $\mathcal{P}_1[\mathbf{u}, \tau, \mathcal{M}]$, see [24, proof of Lem. 3.6]. If $(M, p)$ is not an admissible germ, then the monodromy data are not well defined. Indeed, Theorem 4.14 does not hold, solutions $X_i(\mathbf{u}_o, z)$, with $i = 1, 2, 3$, are not unique. With each such a triple of solutions, we can associated a pair $(S_1, S_2)$ of Stokes matrices. In general these, Stokes matrices do not satisfy Proposition 4.17.
Theorem 6.23. The analytic germ $F[u, \tau, \mathfrak{M}]$ of flat $F$-manifold is isomorphic to the original admissible germ $(M, p)$. They are defined by the same oriented associativity potentials (modulo linear terms).

In the light of the construction of Section 6.2, Theorem 6.23 follows from the following crucial result.

Lemma 6.24. Let one of the following assumptions hold:

1. $u_o \in \mathbb{C}^n \setminus \Delta$,
2. $u_o \in \Delta$, and if $u'_o = u'_o$ with $i, j \in \{1, \ldots, n\}, i \neq j$, then $\delta_i - \delta_j \notin \mathbb{Z} \setminus \{0\}$.

Let

$$F(u) = F_0 + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} F^{(\ell)} \prod_{j=1}^{k} \tilde{u}_{\ell_j}, \quad \tilde{u}_i := u_i - u_{0,i},$$

be a matrix-valued formal power series whose off-diagonal entries $F^i_j$ are formal solutions of the Darboux–Egoroff system (3.27)–(3.31). The off-diagonal entries of the coefficients $F^{(\ell)}$ can be uniquely reconstructed from the off-diagonal entries of $F_0$.

Proof. We have to show that the derivatives $\partial_{i_1} \ldots \partial_{i_N} F^i_j(u_o)$ can be computed from the only knowledge of the numbers $F^i_j(u_o)$. We proceed by induction on $N$. Let us start with the case $N = 1$.

**Step 1.** For $i, j, k$ distinct, by expanding both sides of $\partial_k F^i_j = F^i_k F^k_j$ in power series, and equating the coefficients, one reconstructs the coefficients of $\partial_k F^i_j(u_o)$.

**Step 2.** From the identities (3.30) and (3.31) for $F^i_j$, one can compute $\partial_i F^i_j(u_o)$ and $\partial_j F^i_j(u_o)$ provided that $u_{0,i} \neq u_{0,j}$.

**Step 3.** Assume that $u_{0,i} = u_{0,j}$. By taking the $\partial_i$-derivative of both sides of (3.30), we obtain

$$(\delta_j - \delta_i - 2) \partial_i F^i_j + (u^j - u^i) \partial^2_i F^i_j = \sum_{k \neq i, j} (u^k - u^j) [\partial_i F^i_k F^k_j + F^i_k \partial_i F^k_j ].$$

By evaluating (6.18) at $u = u_o$, we can compute all the numbers $\partial_i F^i_j(u_o)$, namely,

$$\partial_i F^i_j(u_o) = \frac{1}{\delta_j - \delta_i - 2} \sum_{k \neq i, j} (u^k_o - u^j_o) \left[ \partial_i F^i_k(u_o) F^k_j(u_o) + F^i_k(u_o) \partial_i F^k_j(u_o) \right].$$

Notice that the only terms $\partial_i F^i_k(u_o)$ appearing in this sum are those computed in Step 2.

**Step 4.** If $u_{0,j} = u_{0,j}$, the numbers $\partial_j F^i_j(u_o)$ can be computed similarly as in Step 3, by invoking Equation (3.31):

$$\partial_j F^i_j(u_o) = \frac{1}{\delta_j - \delta_i - 2} \sum_{k \neq i, j} (u^k_o - u^j_o) \left[ F^i_j(u_o) F^i_k(u_o) F^k_j(u_o) + F^i_k(u_o) \partial_j F^k_j(u_o) \right].$$

This proves that all the first derivatives $\partial_i F^i_j(u_o)$ can be computed.

**Inductive step.** Assume to know all the $N$th derivatives $\partial_{i_1} \ldots \partial_{i_N} F^i_j(u_o)$. We show how to compute the number $\partial_{h_1} \ldots \partial_{h_{N+1}} F^i_j(u_o)$ for any $(N + 1)$-tuple $(h_1, \ldots, h_{N+1})$. 


Step 1. Assume that there exists \( \ell' \in \{1, \ldots, N+1\} \) such that \( h_{\ell'} \neq i, j \). We have

\[
\partial_{h_1} \ldots \partial_{h_{N+1}} F_{i \ j} = \partial_{h_1} \ldots \partial_{h_{\ell'-1}} \partial_{h_{\ell'+1}} \ldots \partial_{h_{N+1}} [\partial_{h_{\ell'}} F_{i \ j}] = \partial_{h_1} \ldots \partial_{h_{\ell'-1}} \partial_{h_{\ell'+1}} \ldots \partial_{h_{N+1}} [F_{i \ j}^{h_{\ell'}}].
\]

By evaluation at \( u = u_0 \), we can compute all the numbers \( \partial_{h_1} \ldots \partial_{h_{N+1}} F_{i \ j}(u_0) \).

Now we need to compute the mixed derivatives \( \partial_i \partial_j^{N+1-p} F_{i \ j}(u_0) \), with \( 0 \leq p \leq N + 1 \).

Step 2. Assume \( p > 0 \) and \( u_{o,i} \neq u_{o,j} \). Take the \( \partial_i^{p-1} \partial_j^{N+1-p} \)-derivative of both sides of (3.30): by evaluation at \( u = u_0 \), we can compute the numbers \( \partial_i^{p} \partial_j^{N+1-p} F_{i \ j}(u_0) \).

Step 3. Assume \( p > 0 \) and \( u_{o,i} = u_{o,j} \). Take the \( \partial_i^{p-1} \partial_j^{N+1-p} \)-derivative of both sides of (6.18), to obtain

\[
(\delta_j - \delta_i - p - 1) \partial_i^{p} \partial_j^{N+1-p} F_{i \ j} + (N + 1 - p) \partial_i^{p+1} \partial_j^{N-p} F_{i \ j} + (u_i - u_j) \partial_i^{p+1} \partial_j^{N+1-p} F_{i \ j}
\]

\[
= \partial_i^{p-1} \partial_j^{N+1-p} \sum_{k \neq i, j} (u_k - u_j) [\partial_i F_{i \ k} F_{k \ j} + F_{i \ k} \partial_i F_{k \ j}].
\]

(6.19)

Specialize (6.19) for \( p = N + 1 \): by evaluation at \( u = u_0 \) of both sides, we can compute the derivative \( \partial_i^{N+1} F_{i \ j}(u_0) \).

Specialize (6.19) for \( p = N \): by evaluation at \( u = u_0 \) of both sides, we can compute the derivative \( \partial_i^{N} \partial_j F_{i \ j}(u_0) \).

Repeating this procedure, by decreasing \( p \mapsto p - 1 \) at each step, we can compute all the mixed derivatives \( \partial_i^{p} \partial_j^{N+1-p} F_{i \ j}(u_0) \).

Step 4. Assume \( p = 0 \). The derivative \( \partial_j^{N+1} F_{i \ j}(u_0) \) can be computed, as in Steps 2 and 3, by invoking Equation (3.31).

This proves that all the \( (N+1) \)th derivatives \( \partial_{h_1} \ldots \partial_{h_{N+1}} F_{i \ j}(u_0) \) can be computed. \( \square \)

6.4 Convergence of semisimple admissible formal flat \( F \)-manifolds

We are now ready to prove the following result.

Theorem 6.25. Let \((H, \Phi)\) be an admissible formal semisimple flat \( F \)-manifold over \( \mathbb{C} \), with Euler field \( E \). The oriented associativity potentials \( \Phi = (\Phi_1, \ldots, \Phi^n) \) have a nonempty common domain of convergence.

Proof. Fix one ordering \( u_i \in \mathbb{C}^n \) of the eigenvalues of the operator \( U(t) \) at \( t = 0 \). We have \( n \times n \) matrix-valued (a priori) formal power series in \( u \)

\[
V(u) = v_0 + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} V^{(\ell)} \prod_{j=1}^{k} u_{\ell_j}, \quad V_i(u) = v_{i,0} + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} v_i^{(\ell)} \prod_{j=1}^{k} u_{\ell_j},
\]

\[
\Psi(u) = \psi_0 + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} \psi^{(\ell)} \prod_{j=1}^{k} u_{\ell_j}, \quad \Gamma(u) = \gamma_0 + \sum_{k=1}^{\infty} \sum_{\ell_1, \ldots, \ell_k=1}^{n} \frac{1}{k!} \gamma^{(\ell)} \prod_{j=1}^{k} u_{\ell_j},
\]

where \( \overline{u}_i := u_i - u_{o,i} \) for \( i = 1, \ldots, n \). These power series are well defined by the semisimplicity assumption, and they satisfy properties described in Propositions 3.20 and 3.21.
Set $H_o := \Psi_o \bar{\Psi}_o^{-1}$, where $\bar{\Psi}_o := (\frac{\partial u^i}{\partial t^\alpha}|_{t=0})_{i,\alpha=1}^n$.

After fixing choices of normalizations of Section 5.1, we can introduce a system of monodromy data $(\lambda, \mu^\lambda, R, S_1, S_2, \Lambda, C)$ for the formal flat $F$-structure, computed w.r.t. an admissible direction $\tau$ at $u_o$, and the normalization $H_o$ of Lamé coefficients at the origin. Proposition 6.21 holds true, with the same proof. We can set the RHB problem $\mathcal{F}[u_o, \tau, \mathcal{M}]$. This problem is solvable w.r.t. $u$ on an open neighborhood $V \setminus \Theta$ of $u_o$, by Theorem 6.4. The unique solution $G(z; u)$ is holomorphic in $u \in V \setminus \Theta$, and with expansion

$$G(z; u) = I + \frac{1}{z} F_{an}^1(u) + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad z \in \Pi_{L/R},$$

$$G(z; u) = G_0(u) + G_1(u)z + G_2(u)z^2 + G_3(u)z^3 + O(z^4), \quad z \to 0.$$ 

Here, the superscript “an” stands for analytic. As output of Section 6.2, we also obtain a compatible joint system of differential equations (with analytic coefficients in $u$, not just formal) of the form

$$\frac{\partial}{\partial u^i} X = (zE_i - V_i(u)^T) X, \quad \frac{\partial}{\partial z} X = \left(U - \frac{1}{z} V(u)^T\right) X,$$

where

$$V_{an}(u) := [F_{an}^1(u)^T, U] - \Lambda,$$

$$V_{an}^i(u) := [F_{an}^1(u)^T, E_i] \equiv - \left(\frac{\partial G_0}{\partial u^i} \cdot G_0^{-1}\right)^T = \left(\frac{\partial}{\partial u^i} (G_0^T)^{-1}\right) \cdot G_0^T.$$ 

We also have

$$V_{an}(u_o) = V_o, \quad G_0(u_o) = (\Psi_o^{-1})^T, \quad \partial_i G_0 = V_{an}^i G_0, \quad i = 1, \ldots, n.$$ 

From the equality $[F_{an}^1(u_o)^T, E_i] = V_o = [\Gamma_o, E_i]$, we deduce that $[F_{an}^1(u_o)^T]' = \Gamma_o$. Moreover, by Lemma 6.8, the off-diagonal matrix $[F_{an}^1(u)^T]'$ solves Equations (3.27)–(3.31). By Lemma 6.24, we obtain $\Gamma(u) = [F_{an}^1(u)^T]'$. In particular, $\Gamma(u)$ is convergent. It follows that $\Psi(u) = (G(u)^{-1})^T$, $V_i(u) = V_{an}^i(u)$, and $V(u) = V_{an}(u)$ are convergent.

The oriented associativity potentials $\Phi^1, \ldots, \Phi^n$ can be reconstructed via formulas (6.9). The original formal structure $(H, \Phi)$ turns out to be equivalent to the analytic flat $F$-manifold $\mathcal{F}[u_o, \tau, \mathcal{M}]$.

**Open question**: Does it exist a semisimple and strictly doubly resonant germ of flat $F$-structure that is purely formal?

A positive answer would imply the optimality of Theorem 6.25. The study of the strictly doubly resonant germs goes beyond the general theory developed in [26].

**Remark 6.26**: Consider a trivial vector bundle $E$ on $\mathbb{P}^1$, equipped with a meromorphic connection $\nabla^0$ with connection matrix $\Omega$ given by

$$\Omega = -\left[U_o + \frac{1}{z} (\Lambda + [(F''_o)^T, U_o])\right] dz, \quad U_o = \text{diag}(u^1_o, \ldots, u^n_o), \quad \Lambda = \text{diag}(\lambda - \delta_1, \ldots, \lambda - \delta_n).$$
Malgrange’s theorem [69, 71] asserts that if \( u_0 \in \mathbb{C}^n \setminus \Delta \), the connection \( \nabla^o \) has a germ of universal deformation. Its connection matrix is

\[
-d(zU) - ([F''(u)^T, U] + \Lambda) \frac{dz}{z} - [F''(u)^T, dU],
\]

where \( F''(u) \) is the unique off-diagonal solution of the Darboux–Egoroff equations of Lemma 6.24.

If \( u_0 \in \Delta \) and \( \delta_i - \delta_j \in \mathbb{Z} \setminus \{0\} \), C. Sabbah proved that \( \nabla^o \) admits an integrable deformation of the form (6.21), see [80]. In this case, the deformation is not universal. Nevertheless, one can prove that there exists a class \( \mathfrak{F} \) of integrable deformations of \( \nabla^o \) that are induced by pull-back (via a unique map) by the integrable deformation constructed by Sabbah. We refer the interested reader to [25], where the class \( \mathfrak{F} \) and its generic elements are described. In both cases (\( u_0 \notin \Delta \), or \( u_0 \in \Delta \) with \( \delta_i - \delta_j \notin \mathbb{Z} \setminus \{0\} \)), the function \( F''(u) \) is analytic in a neighborhood of \( u_0 \).

6.5  On the number of monodromy local moduli

Consider all \( n \)-dimensional germs of homogenous semisimple flat \( F \)-manifolds, modulo local isomorphisms.

**Theorem 6.27.** The local isomorphism classes of \( n \)-dimensional germs of homogeneous semisimple flat \( F \)-manifolds generically depend on \( n^2 - n \) parameters. The local isomorphism classes of \( n \)-dimensional germs of homogeneous semisimple Frobenius manifolds generically depend on \( \frac{1}{2}(n^2 - n) \) parameters.

**Proof.** We show that germs of flat \( F \)-manifolds are identifiable with points of a “stratified space” \( X \), whose generic dimension is \( n^2 - n \).

Let \((M, p_0)\) be a pointed semisimple homogeneous flat \( F \)-manifold. For generic \( p_0 \), the germ \((M, p_0)\) is admissible. After fixing

1. a system of local flat coordinates \( t \) (such that \( \frac{\delta}{\delta t} = e \)),
2. \( \lambda_0 \in \mathbb{C} \),
3. an admissible direction \( \tau \) at \( p_0 \),
4. the initial value \( H_0 \) of the Lamé coefficients,

we can introduce a tuple of monodromy data \( \mathcal{M} = (\mu^{\lambda_0}, R, S_1, S_2, \Lambda_0, C) \). Conversely, from the knowledge of \((u(p_0), \tau, \mathcal{M})\), we can reconstruct the germ of the structure together with the above choices (1), (2), and (4): the original germ is indeed isomorphic to \( F[u(p_0), \tau, \mathcal{M}] \). So, we can use the data \( \mathcal{M} \) to parametrize the germs of flat \( F \)-structures on \((\mathbb{C}^n, u(p_0))\).

For generic germs, the matrix \( \mu^{\lambda_0} \) has simple spectrum, and with eigenvalues not differing by integers, so that \( R = 0 \). So, generically, we have a tuple of matrices \((\mu^{\lambda_0}, \Lambda_0, S_1, S_2, C)\) for a total of \( 2n^2 + n \) entries.

We need to impose several constraints, and to cut out redundancies in the counting. The matrices \((\mu^{\lambda_0}, \Lambda_0, S_1, S_2, C)\) must satisfy Equation (4.12), for a total of \( n^2 \) constraints. Moreover, given \( h_1, h_2 \in GL(n, \mathbb{C}) \) diagonal, the flat \( F \)-manifolds
\[ \mathcal{F}[\mathbf{u}(p_0), \tau, \mathcal{M}], \mathcal{F}[\mathbf{u}(p_0), \tau, \mathcal{M}'], \mathcal{F}[\mathbf{u}(p_0), \tau, \mathcal{M}''] \] with

\[ \mathcal{M}' = (\mu^{k_0}, \Lambda_0, h_1^{-1}S_1h_1, h_1^{-1}S_2h_1, Ch_1), \quad \mathcal{M}'' = (\mu^{k_0}, \Lambda_0, S_1, S_2, h_2^{-1}C), \]

coincide up to isomorphisms, by Propositions 6.17 and 6.18. In total, the number of free parameters equals \( 2n^2 + n - n^2 - 2n = n^2 - n \).

How to choose a set of \( n^2 - n \) independent parameters out of \( (\mu^{k_0}, \Lambda_0, S_1, S_2, C) \)? By Equation (4.12), we have

\[ S_1^{-1}e^{2\pi \sqrt{-1}\Lambda_0}S_2^{-1} \in \mathcal{O}(e^{2\pi \sqrt{-1}\mu^{k_0}}), \]

where \( \mathcal{O}(e^{2\pi \sqrt{-1}\mu^{k_0}}) \) denotes the similarity orbit of \( e^{2\pi \sqrt{-1}\mu^{k_0}} \). The codimension of the orbit \( \mathcal{O}(e^{2\pi \sqrt{-1}\mu^{k_0}}) \) in \( M(n, \mathbb{C}) \) equals the dimension of the centralizer

\[ \dim \{ A \in M_n(\mathbb{C}) : [A, e^{2\pi \sqrt{-1}\mu^{k_0}}] = 0 \}, \]

see [1, §2.4]. Hence, for generic \( \mu^{k_0} \), we have

\[ \dim \mathcal{O}(e^{2\pi \sqrt{-1}\mu^{k_0}}) = n^2 - n. \]

Moreover, it is easy to see that if a matrix \( A \) admits an lower-diagonal-upper (LDU)-decomposition\(^1\)

\[ A = G_1G_2G_3, \quad G_1 \in L_n, \quad G_3 \in U_n, \quad G_2 = \text{diag}(g_1, \ldots, g_n), \]

then such a decomposition is unique, see, for example, [49]. From this, it follows that \( (S_1, S_2) \) can be used as coordinates on \( \mathcal{O}(e^{2\pi \sqrt{-1}\mu^{k_0}}) \). The total number of parameters \( (\mu^{k_0}, S_1, S_2) \) equals \( n^2 \).

Alternatively, one can choose \( (\mu^{k_0}, C) \) as coordinates on \( \mathcal{O}(e^{2\pi \sqrt{-1}\mu^{k_0}}) \), provided that \( C \) is defined up to left multiplication by diagonal invertible matrices. The point corresponding to \( (\mu^{k_0}, C) \) is

\[ C^{-1}e^{2\pi \sqrt{-1}\mu^{k_0}}C \in \mathcal{O}(e^{2\pi \sqrt{-1}\mu^{k_0}}). \]

In total, we get \( n + (n^2 - n) = n^2 \) parameters. For parametrizing germs of flat \( F \)-manifolds:

- in the tuple \( (\mu^{k_0}, S_1, S_2) \), the matrices \( (S_1, S_2) \) should be taken up to conjugation \( (S_1, S_2) \mapsto (h^{-1}S_1h, h^{-1}S_2h) \) with \( h \) diagonal;
- in the tuple \( (\mu^{k_0}, C) \) should be taken up to right multiplication \( C \mapsto Ch \) with \( h \) diagonal and invertible.

In total, we have \( n^2 - n \) free parameters. At nongeneric points, the space \( X \) can get additional strata.

Let us also consider the subspace \( X_{\text{Frob}} \subseteq X \) of all \( n \)-dimensional pointed germs of homogeneous semisimple Frobenius manifolds.

For Frobenius manifolds, we have standard choices \( \lambda_0 = \frac{d}{2} \) (with \( d \) the charge of the manifold) and \( H_{i,j} = \eta(\partial_i, \partial_j)^{\frac{1}{2}} \) for \( i = 1, \ldots, n \). The choice \( \lambda_0 = \frac{d}{2} \) further reduces the set of monodromy

\(^1\) Here, as in Section 5.3, \( L_n \) (resp. \( U_n \)) denotes the group of lower (resp. upper) triangular unipotent \( n \times n \)-matrices.
data, because we automatically have $\Lambda_0 = 0$. Moreover, the pair $(S_1, S_2)$ must satisfy the relation $S_1^{-1} = S_2^T$, see Remark 4.19. Hence, due to the constraint (4.12), the matrices $\mu^{k_0}$ and $C$ can be reconstructed from the knowledge of $S_1$ only.

The condition $S_1^{-1} = S_2^T$ imposes $\frac{1}{2}(n^2 - n)$ constraints. Notice that we do not need to quotient by the action $(S_1, h) \mapsto h^{-1} S_1 h$, since the normalization of the Lamé coefficients is fixed. We have a total amount of free $\frac{1}{2}(n^2 - n)$ parameters out of $(\mu^{k_0}, S_1, S_2, C)$. This proves that generic stratum of $X_{\text{Frob}}$ has dimension $\frac{1}{2}(n^2 - n)$. □

Remark 6.28. We underline that the tuple $(\mu^{k_0}, R, S_1, S_2, \Lambda, C)$ of monodromy data actually provide two equivalent systems of “essential parameters” classifying germs. For generic germs, one system is $(\mu^{k_0}, [S_1], [S_2])$, the other is $(\mu^{k_0}, [C])$. Here, $[S_i]$ with $i = 1, 2$, denote the conjugacy classes $h^{-1} S_i h$ ($h \in GL(n, \mathbb{C})$ diagonal), and $[C]$ denotes the equivalence class $h_1 C h_2$ ($h_1, h_2 \in GL(n, \mathbb{C})$ diagonal). In both, cases have a total of $n^2 - n$ essential parameters.

Remark 6.29. There is another possible way to deduce that the local isomorphism class of a semisimple homogeneous flat $\mathbb{F}$-manifold structure depends on $n^2 - n$ moduli, namely, by parametrizing the structure with the initial datum $\bar{\Gamma}_o$ of the Darboux–Tsarev equations (3.14), (3.15), and (3.19). This joint system of differential equations, indeed, is complete in the sense of Darboux [31, Livre III, Ch. I, pag. 332], which is of the form

$$\frac{\partial \bar{\Gamma}_i}{\partial u^h} = f_{ijh}(u, \bar{\Gamma}), \quad \bar{\Gamma} = (\bar{\Gamma}_{ij})_{i,j=1}^n, \quad i, j, h = 1, \ldots, n, \quad i \neq j,$$

whose coefficients $f_{ijh}(u, \bar{\Gamma})$ satisfy the compatibility conditions

$$\frac{\partial f_{ijh}}{\partial u^k} + \sum_{\ell, m} \frac{\partial f_{ijh}}{\partial \bar{\Gamma}_{\ell m}} f_{\ell m k} = \frac{\partial f_{ijk}}{\partial u^h} + \sum_{\ell, m} \frac{\partial f_{ijk}}{\partial \bar{\Gamma}_{\ell m}} f_{\ell m h}.$$

A solution $\bar{\Gamma} = (\bar{\Gamma}_{ij})_{i,j=1}^n$ is uniquely determined by its initial datum at an arbitrary point $u_o$, see [31, Livre III, Ch. I, Th. II]. For the proof of the compatibility conditions above see [3, App. 1, arXiv version]. I thank P. Lorenzon for many useful discussions on this point.

7 | APPLICATIONS TO LM-CohFTs, F-CohFTs, AND OPEN WDVV EQUATIONS

7.1 | Losev–Manin moduli spaces and LM-CohFTs

A $n$-pointed chain of projective lines $(C; s_0, s_\infty; s_1, \ldots, s_n)$ consists of the following data:

1. a nodal curve $C = C_1 \cup \cdots \cup C_m$ (over $\mathbb{C}$) whose irreducible components $C_j$ are projective lines;
2. each component $C_j$ is equipped with two marked points $p_j^\pm$, called poles;
3. $C_i$ and $C_j$ intersect only if $|i - j| = 1$;
4. $C_i$ and $C_{i+1}$ intersect transversally in $p_i^+ = p_{i+1}^-$;
5. $s_0 = p_1^- \in C_1$ and $s_\infty = p_m^+ \in C_m$ are called white points;
6. $s_1, \ldots, s_n \in C \setminus \{p_1^\pm, \ldots, p_m^\pm\}$ are called black points.
A $n$-pointed chain of projective lines is stable if there is at least one black point on each irreducible components. Notice that black points are allowed to coincide. Two $n$-pointed chains of projective lines $(C; s_0, s_\infty; s_1, ..., s_n)$ and $(C'; s'_0, s'_\infty; s'_1, ..., s'_n)$ are isomorphic if there exists an isomorphism $\varphi : C \to C'$ such that $\varphi(s_j) = s'_j$ for $j = 0, 1, ..., n, \infty$ (Figure 2).

**The spaces $L_n$.** The Losev–Manin moduli space $\overline{L}_n$, with $n \geq 1$, is defined as the fine-moduli space of stable $n$-pointed chains of projective lines [66].

The space $\overline{L}_n$ is a $(n-1)$-dimensional smooth toric variety (over $\mathbb{C}$): it contains the open dense torus

$$L_n = \{(\mathbb{P}^1; 0, \infty; s_1, ..., s_n)\}/\text{iso} \cong (\mathbb{C}^*)^n/\mathbb{C}^* \cong (\mathbb{C}^*)^{n-1}.$$ 

The space $\overline{L}_n$ is the toric variety associated with the convex polytope in $\mathbb{C}^n$ called permutohedron, defined as the convex hull of the $\mathfrak{S}_n$-orbit of the point $(1, 2, ..., n) \in \mathbb{C}^n$, see [66]. Such a toric variety can be constructed via iteration of blow-ups of $\mathbb{P}^{n-1}$. As a first step, blow-up $n$ points $p_1, ..., p_n$ in general position in $\mathbb{P}^{n-1}$. Subsequently, blow-up the strict transforms of the $\frac{1}{2}n(n-1)$ lines passing through the pairs $(p_i, p_j)$ for all $i, j = 1, ..., n$. Continue this blowing-up procedure up to $(n-3)$-dimensional hyperplanes, see [54, §4.3].

Alternatively, $\overline{L}_n$ is the toric variety defined by the fan formed by the Weyl chambers of the roots system of type $A_{n-1}$, with $n \geq 2$, [11].

The group $\mathfrak{S}_2 \times \mathfrak{S}_n$ naturally acts on $\overline{L}_n$ by permuting white and black points, respectively.

The cohomology ring $H^*(\overline{L}_n, \mathbb{Q})$ was studied in [66, 73]. It is algebraic: all odd cohomology groups vanish, and $H^*(\overline{L}_n, \mathbb{Q})$ is isomorphic to the Chow ring $A^*(\overline{L}_n, \mathbb{Q})$, [66, Th. 2.7.1]. See also [13] where the groups $H^*(\overline{L}_n, \mathbb{Q})$ are determined as representation of $\mathfrak{S}_2 \times \mathfrak{S}_n$.

Given $n_1, n_2 \geq 1$, we have a natural morphism $\overline{L}_{n_1} \times \overline{L}_{n_2} \to \overline{L}_{n_1+n_2}$, defined by concatenation of white points. Furthermore, each boundary divisor of $\overline{L}_n$ is isomorphic to $\overline{L}_{n_1} \times \overline{L}_{n_2}$ with $n_1 + n_2 = n$.

Let $\overline{M}_{0,n+2}$ be the moduli space of stable $(n + 2)$-pointed trees of projective lines. We have a surjective birational morphism $p_n : \overline{M}_{0,n+2} \to \overline{L}_n$ for any choice of two different labels $i, j$ in $(1, ..., n + 2)$ (the chosen white points).

**Losev–Manin cohomological field theories.** Let $V_1, V_2$ be two complex vector spaces of finite dimensions. An LM-cohomological field theory (for short, LM-CohFT), on the pair $(V_1, V_2)$, is the datum of polylinear maps

$$\alpha_n : V_1^* \otimes V_1 \otimes V_2^\otimes n \to H^*(\overline{L}_n, \mathbb{C}), \quad \text{with } n \geq 1,$$
such that, for any chosen bases† $(v_1, \ldots, v_{N_1})$ of $V_1$ and $(w_1, \ldots, w_{N_2})$ of $V_2$, the following properties are satisfied:

1. $\alpha_n$ is $\mathfrak{S}_n$-covariant w.r.t. the natural actions of $\mathfrak{S}_n$ on both $V_2^\otimes n$ and $H^*(\overline{L}_n, \mathbb{C})$,
2. for any partition $I \bigsqcup J = \{1, \ldots, n\}$ with $|I| = n_1$ and $|J| = n_2$, we have†

$$\text{gl}^* \alpha_n (v_i^\vee \otimes v_h^\otimes \bigotimes_{i \in I} w_{\rho_i}) = \alpha_{n_1} (v_i^\vee \otimes v_{\mu}^\otimes \bigotimes_{i \in I} w_{\rho_i}) \otimes \alpha_{n_2} (v_{\mu}^\vee \otimes v_h^\otimes \bigotimes_{i \in J} w_{\rho_i}),$$

where $1 \leq i, h \leq N_1$ and $1 \leq \rho_1, \ldots, \rho_n \leq N_2$, and $\text{gl} : \overline{L}_{n_1} \times \overline{L}_{n_2} \to \overline{L}_{n_1+n_2}$ is the gluing map.

Remark 7.1. The spaces $\overline{L}_n$ and $\overline{M}_{0,n}$, and their higher genus analogs, are two examples of moduli spaces of weighted stable pointed curves constructed in [45], corresponding to two different choices of weights. Losev–Manin CohFT’s fit in a more general construction developed in [12], in the setting of moduli spaces of curves and maps with weighted stability conditions. We borrow the terminology “Losev–Manin CohFT” from [82].

**Commutativity equations.** Consider two complex vector spaces $V_1, V_2$ of dimension $N_1, N_2$, respectively. Fix a basis $(w_1, \ldots, w_{N_2})$ of $V_2$ and let $t := (t^1, \ldots, t^{N_2})$ be the dual coordinates.

The Losev–Manin commutativity equation for $B \in \mathbb{C}[t] \otimes \text{End}(V_1)$ is given by

$$dB \wedge dB = 0. \quad (7.1)$$

In coordinates $t$, Equation (7.1) is equivalent to the commutation relations

$$\left[ \frac{\partial B}{\partial t^i}, \frac{\partial B}{\partial t^j} \right] = 0, \quad i, j = 1, \ldots, N_2.$$

Fix a basis $(v_1, \ldots, v_{N_1})$ of $V_1$, and let $(v_1^\vee, \ldots, v_{N_1}^\vee)$ be the dual basis of $V_1^\vee$.

Given an LM-CohFT $(\alpha_n)_{n \geq 1}$, define the formal power series $B_j^j \in \mathbb{C}[t]$, with $i, j = 1, \ldots, N_1$, by

$$B_j^j (t) := \sum_{m=1}^{\infty} \sum_{\rho_1, \ldots, \rho_m=1}^{N_2} \frac{t^{\rho_1} \ldots t^{\rho_m}}{m!} \int_{\overline{L}_m} \alpha_m \left( v_i^\vee \otimes v_j^\vee \bigotimes_{\epsilon=1}^m w_{\rho_{\epsilon}} \right). \quad (7.2)$$

The matrix $B := (B_j^j)_{1 \leq j \leq N_1}$ represents an element of $\mathbb{C}[t] \otimes \text{End}(V_1)$ in the basis $v_1, \ldots, v_{N_1}$ of $V_1$.

**Theorem 7.2.** The matrix $B$ is a solution of the commutativity equation (7.1). Vice versa, any solution $B$ of (7.1), such that $B(0) = 0$, has the form (7.2) for a unique LM-CohFT $(\alpha_n)_{n \geq 1}$.

**Proof.** This is an equivalent reformulation of [66, Th. 3.3.1, Prop. 3.6.1] and [67, Th. 5.1.1]. □

---

† In the following paragraphs, if $(e_1, \ldots, e_N)$ is a basis of a vector space $V$, then $(e_1^\vee, \ldots, e_N^\vee)$ denotes the dual basis of $V^\vee$.

† Einstein’s summation rule over repeated Greek indices is used.
7.2 \textit{F}-cohomological field theories

Let $V$ be a complex vector space of finite dimension $N$. Denote by $\overline{M}_{g,n}$ the Deligne–Mumford moduli space of genus $g$ stable curves with $n$ marked points, defined for $g, n \geq 0$ in the stable regime $2g - 2 + n > 0$.

As in [20], an \textit{F}-cohomological field theory (for short $F$-CohFT) is the datum of

- polynomial maps $c_{g,n+1}: V^* \otimes V \otimes \otimes_{i=1}^{n} e_{\rho_i} \to H^{ev}(\overline{M}_{g,n+1}, \mathbb{C})$, for $2g - 1 + n > 0$, and
- a distinguished vector $e_1 \in V$,

such that, for any chosen basis $(e_1, ..., e_N)$ of $V$, the following properties are satisfied:

1. $c_{g,n+1}$ is $\mathfrak{S}_n$-covariant w.r.t. the natural actions of $\mathfrak{S}_n$ on both $V^* \otimes V \otimes \otimes_{i=1}^{n} e_{\rho_i}$ (permutation of the $n$ copies of $V$) and on $H^{ev}(\overline{M}_{g,n+1}, \mathbb{C})$ (permutation of the last $n$ marked points);
2. $\pi^* c_{g,n+1}(e_{\rho_0} \otimes \otimes_{i=1}^{n} e_{\rho_i}) = c_{g,n+2}(e_{\rho_0} \otimes \otimes_{i=1}^{n} e_{\rho_i} \otimes e_1)$, for $1 \leq \rho_1, ..., \rho_n \leq N$, where $\pi: \overline{M}_{g,n+2} \to \overline{M}_{g,n+1}$ is the map forgetting the last marked point;
3. $c_{0,3}(e_{\alpha} \otimes e_{\beta} \otimes e_1) = \delta_{\alpha \beta}$, for $1 \leq \alpha, \beta \leq N$;
4. for any partition $I \coprod J = \{1, ..., n\}$ with $|I| = n_1$ and $|J| = n_2$, we have\(^\dagger\)

$$
\text{gl} c_{g_1+g_2,n_1+n_2+1}(e_{\rho_0} \otimes \otimes_{i=1}^{n} e_{\rho_i}) = c_{g_1,n_1+2}(e_{\rho_0} \otimes \otimes_{i\in I} e_{\rho_i} \otimes e_{\mu}) \otimes c_{g_2,n_2+1}(e_{\mu} \otimes \otimes_{j\in J} e_{\rho_j}),
$$

for $1 \leq \rho_1, ..., \rho_n \leq N$, and $\text{gl}: \overline{M}_{g_1,n_1+2} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2+1}$ is the corresponding gluing map.

The \textit{genus 0 sector} (or tree-level) of a given $F$-CohFT is the datum of the maps $(c_{0,n})_{n \geq 2}$ and the distinguished vector $e_1 \in V$ only.

Given a tree-level $F$-CohFT, fix a basis $(e_1, ..., e_N)$ of $V$, and denote by $t := (t^1, ..., t^N)$ the dual coordinates. Define the formal power series $F^\alpha \in \mathbb{C}[\llbracket t \rrbracket]$, for $\alpha = 1, ..., N$, by

$$
F^\alpha(t) := \sum_{n=2}^{\infty} \sum_{\rho_1, ..., \rho_n=1}^{N} \frac{t^{\rho_1} ... t^{\rho_n}}{n!} \int_{M_{0,n+1}} c_{0,n+1}(e_{\alpha} \otimes \otimes_{i=1}^{n} e_{\rho_i}).
$$

(7.3)

**Theorem 7.3.** The functions $F^\alpha(t)$ are solution of the oriented associativity equations

$$
\frac{\partial^2 F^\alpha}{\partial t^\alpha \partial t^\beta} = \delta^\alpha_{\beta}, \quad \alpha, \beta = 1, ..., N,
$$

(7.4)

$$
\frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\beta} \frac{\partial^2 F^\mu}{\partial t^\gamma \partial t^\delta} = \frac{\partial^2 F^\alpha}{\partial t^\mu \partial t^\gamma} \frac{\partial^2 F^\mu}{\partial t^\beta \partial t^\delta}, \quad \alpha, \beta, \gamma, \delta = 1, ..., N,
$$

(7.5)

and thus, define a formal flat $F$-manifold structure on $V$ with unit $e_1$.

Vice versa, any solution $(F^1, ..., F^N)$ of (7.4)–(7.5), with $F^\alpha(0) = 0$ and $\partial^\alpha_{\beta} F^\alpha(0) = 0$ for all $\alpha, \beta = 1, ..., N$, is of the form (7.3) for a unique tree-level $F$-CohFT $(c_{0,n})_{n \geq 2}$.

\(^\dagger\) Einstein’s summation rule over repeated Greek indices is used.
Proof. The first part of the statement follows from a simple computation, invoking properties (1)–(4) above. For a proof of the second part of the statement, see Appendix B.

Remark 7.4. The first part of Theorem 7.3 was already formulated in [20, Sec. 2.1]. The second part of the statement, however, was not addressed in loc. cit.

7.3 From tree-level $\mathcal{F}$-CohFT to LM-CohFT, and vice versa

Given a tree-level $\mathcal{F}$-CohFT on $V$, an LM-CohFT is naturally defined on the pair $(V_1, V_2) = (V, V)$. For any $n \geq 1$, define

$$\alpha_n := (p_n)_* \circ c_{0,n+2} : V^* \otimes V^\otimes(n+1) \to H^*(\overline{L}_n, \mathbb{C}),$$

where $p_n : \overline{M}_{0,n+2} \to \overline{L}_n$ is the surjective birational morphism defined by a choice of two white points.

Proposition 7.5. The polylinearn maps $(\alpha_n)_{n \geq 1}$ define an LM-CohFT on $(V, V)$.

Proof. The $\mathfrak{S}_n$-covariance of $\alpha_n$ follows from the $\mathfrak{S}_{n+1}$-covariance of $c_{0,n+2}$. For $n_1 + n_2 = n$, we have the following commutative diagram:

$$
\begin{array}{ccc}
\overline{M}_{0,n_1+2} \times \overline{M}_{0,n_2+2} & \xrightarrow{\text{gl}} & \overline{M}_{0,n+2} \\
p_{n_1} \times p_{n_2} \downarrow & & \downarrow p_n \\
\overline{L}_{n_1} \times \overline{L}_{n_2} & \xrightarrow{\text{gl}} & \overline{L}_n
\end{array}
$$

with proper vertical arrows and local complete intersections as horizontal arrows. The gluing property of $\alpha_n$ then follows from the gluing property of $c_{0,n+2}$ and the excess intersection formula [39, Prop. 6.6 and Prop. 17.4.1]. Notice that the excess bundle $\mathbb{E}$ has rank 0 (both gl and $\tilde{\text{gl}}$ have codimension 1), hence

$$\tilde{\text{gl}}^* (p_n)_* x = (p_{n_1} \times p_{n_2})_* \text{gl}^* x,$$

for all $x \in H^c(\overline{M}_{0,n+2}, \mathbb{C}) \cong A^*(\overline{M}_{0,n+2})_\mathbb{C}$. □

Vice versa, given an LM-CohFT $(\alpha_n)_{n \geq 1}$ on $(V_1, V_2)$, we can reconstruct a formal flat $\mathcal{F}$-manifold, provided that

- $\dim V_1 = \dim V_2 = N$,
- we are given an extra amount of data, namely, a primitive vector.

Definition 7.6. Let $B \in \mathbb{C}[\overline{t}] \otimes \text{End}(V_1)$ be a solution of commutativity equations (7.1). A vector $h \in V_1$ is primitive for $B$ if the vectors

$$\frac{\partial B}{\partial \overline{t}_1} \bigg|_{t=0} \cdot h, \quad ..., \quad \frac{\partial B}{\partial \overline{t}_N} \bigg|_{t=0} \cdot h$$
define a basis of $V_1$. Equivalently, $h$ is primitive if, for any chosen basis $(v_1, \ldots, v_N)$ of $V_1$, we have

$$
\det \left( \frac{\partial B^i_\mu}{\partial t^k} \bigg|_{0} \right)_{i,k=1}^N \neq 0, \quad \text{where } h = h^\mu v_\mu.
$$

If $B$ admits a primitive vector $h$, we can identify $V_1$ and $V_2$ via the isomorphism $w_k \mapsto \frac{\partial B}{\partial t^k} |_0 h$, for $k = 1, \ldots, N$. Under such an identification, $t$ can be thought as coordinates on $V_1 \cong V_2$.

**Proposition 7.7** [67, Prop. 5.3.3]. If $B$ admits a primitive vector $h$, then there exists a formal flat $F$-manifold structure on $V_1 \cong V_2$ with flat identity $h$. The oriented associativity potentials $\mathbf{F} = (F_1, \ldots, F^N)$ satisfy

$$
B_\alpha^\beta = \frac{\partial F_\alpha}{\partial t^\beta}.
$$

**Proof.** Let $(v_1, \ldots, v_N)$ be a basis of $V_1$ with $v_1 = h$. By assumption, we have $\det \left( \frac{\partial B^i}{\partial t^k} |_{0} \right) \neq 0$. Hence, up to change of basis $(w_1, \ldots, w_N)$ of $V_2$, we can assume that in the coordinates $t$, we have

$$
\frac{\partial B^i}{\partial t^k}(t) = \delta^i_k \Rightarrow B^i_1(t) = t^i + c.
$$

Consider the $\mathbb{C}[\llbracket t \rrbracket] \otimes V_1$-valued differential form $dB \wedge d(Bh)$. In the bases $(w_i)_{i=1}^N$ and $(v_i)_{i=1}^N$ chosen as above, it has components

$$
[dB \wedge d(Bh)]^i = \left( \frac{\partial B^i_\lambda}{\partial t^\nu} dt^\nu \right) \wedge \left( \delta^\lambda_\mu dt^\mu \right) = \frac{\partial B^i_\mu}{\partial t^\nu} dt^\nu \wedge dt^\mu.
$$

On the other hand, $dB \wedge d(Bh) = (dB \wedge dB)h = 0$, since $h$ is $t$-independent and $B$ a solution of (7.1). Hence,

$$
\frac{\partial B^i_\mu}{\partial t^\nu} - \frac{\partial B^i_\nu}{\partial t^\mu} = 0, \quad \mu, \nu = 1, \ldots, N.
$$

This implies the existence of $F^i_j \in \mathbb{C}[\llbracket t \rrbracket]$ such that $B^i_j = \partial_j F^i$.

**Theorem 7.8.** The following notions are equivalent:

1. formal flat $F$-manifold,
2. tree-level $F$-cohomological field theory,
3. LM-cohomological field theory with primitive element.

**Proof.** It follows from Theorems 7.2 and 7.3 and Propositions 7.5 and 7.7.

**Remark 7.9.** Black marked points of stable $n$-pointed chains of projective lines are allowed to coincide. It would be tempting to compare these coincidences of black points with the coalescence phenomenon at irregular singularities of ordinary differential equations studied in [26, 30]. Any contingent relation deserves further investigations. I thank Y.I. Manin for pointing out such an analogy in a private communication.
### 7.4 Homogeneous $F$-CohFTs

An $F$-CohFT $(c_{g,n+1})_{g,n}$ is said to be **homogeneous** if

1. The vector spaces $V$ and $V^*$ are graded, with homogeneous bases $(e_1, ..., e_N)$ and $(e^\vee_1, ..., e^\vee_N)$, such that
   \[ \deg e_\alpha = -\deg e^\vee_\alpha = q_\alpha, \quad \alpha = 1, ..., N \]
   \[ \deg e = 0; \]

2. There exist $r_1, ..., r_N, \gamma \in \mathbb{C}$ such that
   \[ \deg c_{g,n+1}(e^\vee_\alpha \otimes \bigotimes_{i=1}^n e_{\alpha_i}) + \pi_* c_{g,n+2}(e^\vee_{\alpha_0} \otimes \bigotimes_{i=1}^n e_{\alpha_i} \otimes r^\lambda e^\lambda) \]
   \[ = \left( \sum_{i=1}^n q_{\alpha_i} - q_{\alpha_0} + \gamma g \right) c_{g,n+1}(e^\vee_{\alpha_0} \otimes \bigotimes_{i=1}^n e_{\alpha_i}), \quad (7.6) \]

where $\deg : H^*(\overline{M}_{g,n}) \to H^*(\overline{M}_{g,n})$ rescales a $k$th degree class by a factor $\frac{k^2}{2}$, and $\pi : \overline{M}_{g,n+2} \to \overline{M}_{g,n+1}$ is the morphism forgetting the last marked point.

**Proposition 7.10.** If a tree-level $F$-CohFT is homogeneous, then the associated formal flat $F$-manifold with potentials (7.3) is homogeneous, with the Euler vector field

\[ E = \sum_{\alpha=1}^N \left( (1 - q_\alpha)t^\alpha + r^\alpha \right) \frac{\partial}{\partial t^\alpha}. \]

*Proof.* A simple computation shows that Equations (7.6), specialized at $g = 0$, imply Equations (2.4) for the potentials (7.3).

---

**Theorem 7.11.** Let $(H, \Phi)$ be a formal semisimple flat $F$-manifold over $\mathbb{C}$, with $\dim_\mathbb{C} H = N$. Let $(c_{0,n+1})_{n \geq 2}$ and $(\alpha_n)_{n \geq 1}$ be the underlying tree-level $F$-CohFT and $LM$-CohFT, respectively. If $(H, \Phi)$ is admissible, then there exist real positive constants $m, k_1, ..., k_N \in \mathbb{R}_+$ such that

\[ \left| \int_{\overline{M}_{0,|n|+1}} c_{0,|n|+1} \left( \bigotimes_{j=1}^N \Delta_\beta^\vee \otimes \bigotimes_{j=1}^N \Delta_j^\otimes n_j \right) \right| \leq m n! \prod_{j=1}^N k_j^{n_j}, \quad n \in \mathbb{N}^N, \quad \beta = 1, ..., N, \]

\[ \left| \int_{\overline{M}_{|n|}} \alpha_{|n|} \left( \bigotimes_{j=1}^N \Delta_\beta^\vee \otimes \Delta_\gamma \otimes \bigotimes_{j=1}^N \Delta_j^\otimes n_j \right) \right| \leq m n! \prod_{j=1}^N k_j^{n_j}, \quad n \in \mathbb{N}^N, \quad \beta, \gamma = 1, ..., N, \]

where we set $n! := \prod_{j=1}^N n_j$, and $|n| := \sum_{j=1}^N n_j$. 

---
7.5 Open WDVV equations

Let $k$ be a commutative $\mathbb{Q}$-algebra. Consider a formal Frobenius manifold over $k$, with Euler vector field $E$, defined by the solution $F \in k[[t^1, \ldots, t^n]]$ of the WDVV equations:

$$
\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^\mu_{\alpha\beta} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\mu \partial t^\beta} \eta^\mu_{\alpha\beta},
$$

(7.7)

$$
\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta} = \text{const.}, \quad \eta = (\eta_{\alpha\beta})_{\alpha,\beta}, \quad \eta^{-1} = (\eta^{\alpha\beta})_{\alpha,\beta},
$$

(7.8)

$$
E^\nu \frac{\partial F}{\partial t^\nu} = (3 - d)F + Q(t), \quad E^\nu = (1 - q^\nu t^\nu + r^\nu,
$$

(7.9)

where $\alpha, \beta, \gamma \in \{1, \ldots, n\}, d \in k$ is the conformal dimension (or charge) of the Frobenius manifold, $q^\nu, r^\nu \in k$, and $Q(t) \in k[t]$ is a quadratic polynomial in $t$.

The open WDVV equations (OWDVV) are the following overdetermined system of PDEs for $F^o \in k[[t^1, \ldots, t^n, s]]$:

$$
\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^\mu_{\alpha\beta} \frac{\partial^2 F^o}{\partial t^\nu \partial t^\gamma} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta} \frac{\partial^2 F^o}{\partial s \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\gamma \partial t^\beta \partial t^\mu} \eta^\mu_{\alpha\beta} \frac{\partial^2 F^o}{\partial t^\nu \partial t^\alpha} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta} \frac{\partial^2 F^o}{\partial s \partial t^\gamma},
$$

(7.10)

$$
\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^\mu_{\alpha\beta} \frac{\partial^2 F^o}{\partial t^\nu \partial s} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta} \frac{\partial^2 F^o}{\partial s^2} = \frac{\partial^2 F^o}{\partial s \partial t^\beta} \frac{\partial^2 F^o}{\partial s \partial t^\gamma},
$$

(7.11)

$$
\frac{\partial^2 F^o}{\partial t^1 \partial t^\alpha} = 0, \quad \frac{\partial^2 F^o}{\partial t^1 \partial s} = 1,
$$

(7.12)

$$
E^\nu \frac{\partial F^o}{\partial t^\nu} + \left(1 - \frac{d}{2}s + r^{n+1}\right) \frac{\partial F^o}{\partial s} = \frac{3 - d}{2} F^o + L(t),
$$

(7.13)

where $\alpha, \beta, \gamma \in \{1, \ldots, n\}, r^{n+1} \in k$, and $L(t) \in k[t]$ is a linear polynomial in $t$.

The OWDVV equations first appeared in [48, Th. 2.7], in the context of open Gromov–Witten theory. These equations subsequently appeared in [18, 19, 78]: although not explicitly mentioned in loc. cit., the OWDVV equations follow from the open topological recursion relations equations, see [78, Th. 1.5], [18, Th. 4.1], [19, Lem. 3.6], [17, Sec. 4], and [10, Sec. 1]. The OWDVV equations play a central role in the general theory of relative quantum cohomology developed in [85].

**Proposition 7.12** (P. Rossi [10]). The following conditions are equivalent:

1. $(F, F^o)$ is a solution of WDVV and OWDVV equations (7.7)–(7.13),
2. $(\frac{\partial F}{\partial t^\nu} \eta^\nu_1, \ldots, \frac{\partial F}{\partial t^\nu} \eta^\nu_n, F^o)$ is a solution of the oriented associativity equations (2.1)–(2.2) in the coordinates $(t, s)$, and the corresponding formal flat $F$-manifold is homogenous.

**Proof.** The claim follows by a direct check. □
We will refer to the formal flat $F$-structure of point (2) of Proposition 7.12 as the formal flat $F$-manifold underlying the pair $(F, F^o)$. As a corollary of Theorem 6.25, we deduce the following result.

**Theorem 7.13.** Let $F \in \mathbb{C}[[t]]$, $F^o \in \mathbb{C}[[t, s]]$ be solutions of the WDVV and OWDVV equations. If the underlying formal flat $F$-manifold is semisimple, and it is not strictly doubly resonant, then both $F$ and $F^o$ are convergent.

**Remark 7.14.** According to a conjecture of B. Dubrovin, the monodromy data of the quantum cohomology of a smooth projective variety encode information about the derived category $D^b(X)$, see [22, 23, 28, 37]. It would be interesting to look for analog relations starting from the monodromy data, as defined here, of flat $F$-structures given by relative quantum cohomologies of [85]. This will be addressed in a future project of the author.

**APPENDIX A: PROOF OF THEOREM 2.20**

Let $M$ be an analytic homogeneous semisimple flat $F$-manifold, and let $E_1, E_2 \in \Gamma(TM)$ be two Euler vector fields.

**Lemma A.1.** We have $[E_1, E_2] = E_1 - E_2$.

**Proof.** By Proposition 2.14, we can choose canonical coordinates so that $E_1 = \sum_j u^j \partial_j$ and $E_2 = \sum_j (u^j + c^j) \partial_j$. We have

$$[E_1, E_2] = \sum_j (E^h_1 \partial_h E^i_2 - E^h_2 \partial_h E^i_1) \partial_j = -\sum_j c^j \partial_j = E_1 - E_2.$$

\[\square\]

**Lemma A.2.** We have $\nabla E_1 = \nabla E_2$.

**Proof.** Since $\nabla$ is torsionless, we have $\nabla E_1 E_2 = \nabla E_2 E_1 + [E_1, E_2]$. For arbitrary vector field $X$, we have

$$\nabla_X \nabla E_1 E_2 = \nabla_X \nabla E_2 E_1 + \nabla_X [E_1, E_2].$$

\[\square\]

**Proof of Theorem 2.20.** Introduce the operators $U_1(X) := E_1 \circ X$ and $U_2(X) := E_2 \circ X$, and

$$\mu(X) := X - \nabla_X E_1 = X - \nabla_X E_2, \quad X \in \Gamma(TM).$$

Choose canonical coordinates so that $E_1 = \sum_j u^j \partial_j$ and $E_2 = \sum_j (u^j + c^j) \partial_j$. Set $\Psi^i_\alpha = \frac{\partial u^i}{\partial t^\alpha}$. The matrix $\Psi$ diagonalizes both $U_1$ and $U_2$:

$$U_1 = \Psi U_1 \Psi^{-1} = \text{diag}(u^1, \ldots, u^n),$$
\( \mathbf{U}_2 = \tilde{\Psi} \mathbf{U}_2 \tilde{\Psi}^{-1} = \text{diag}(u^1 + c^1, \ldots, u^n + c^n) \).

Set \( \tilde{\mathbf{V}} := \tilde{\Psi} \mathbf{U}_2 \tilde{\Psi}^{-1} \), and also introduce an off-diagonal matrix \( \tilde{\Gamma} = (\tilde{\Gamma}^j_i) \) by

\[
\tilde{\Gamma}^i_j := - (\tilde{\mathbf{V}}_i)_j, \quad i \neq j.
\]

The matrix \( \tilde{\Gamma} \) is a solution of the Darboux–Tsarev equations. Moreover, we have

\[
\tilde{\mathbf{V}} = \tilde{\mathbf{V}}' + [\tilde{\Gamma}, \mathbf{U}_1], \quad \text{and also} \quad \tilde{\mathbf{V}} = \tilde{\mathbf{V}}' + [\tilde{\Gamma}, \mathbf{U}_2],
\]

(A.1)

where \( \mathbf{V}' \) denotes the diagonal part of \( \mathbf{V} \). This follows from Propositions 3.7 and 3.9.

Let \( j \neq k \), and take the \( (j, k) \) entry of both Equations (A.1). We have

\[
\tilde{\mathbf{V}}^i_k(u) = \tilde{\Gamma}^i_k(u)(u^k - u^j) = \tilde{\Gamma}^i_k(u)(u^k - u^j) + \tilde{\Gamma}^i_k(u)(c^k - c^j).
\]

Hence, we have

\[
\tilde{\Gamma}^i_k(u)(c^k - c^j) = 0, \quad \text{for any } j \neq k.
\]

This means that

\[
c^j \neq c^k \implies \tilde{\Gamma}^i_k = \tilde{\Gamma}^i_k = 0.
\]

Introduce the partition \( \bigsqcup_{r=1}^N I_r = \{1, \ldots, n\} \) s.t. \( c^i = c^j \) only if \( i, j \) are in a same block, that is, \( i, j \in I_r \) for some \( r \).

It follows that \( \tilde{\Gamma}^i_j = 0 \) unless \( i, j \in I_r \). Take \( i, j \in I_r \) and \( k \notin I_r \). We have

\[
\partial_k \tilde{\Gamma}^i_j = - \tilde{\Gamma}^i_k \tilde{\Gamma}^k_j + \tilde{\Gamma}^i_k \tilde{\Gamma}^j_k = 0.
\]

We have proved that

\begin{itemize}
  \item the function \( \tilde{\Gamma}^i_j \) is not identically zero only if the indices \( i, j \) are in the same block \( I_r \);
  \item the function \( \tilde{\Gamma}^i_j \) only depends on coordinates \( u^k \) with \( i, j, k \) in the same block \( I_r \).
\end{itemize}

It follows that all the matrices \( \tilde{\Gamma}, \tilde{\mathbf{V}}, \tilde{\mathbf{V}}_i, \tilde{\Psi} \) admit a direct sum decomposition

\[
\tilde{\Gamma} = \bigoplus_{r=1}^N \tilde{\Gamma}^{(r)}, \quad \tilde{\mathbf{V}} = \bigoplus_{r=1}^N \tilde{\mathbf{V}}^{(r)}, \quad \tilde{\mathbf{V}}_i = \bigoplus_{r=1}^N \tilde{\mathbf{V}}_i^{(r)}, \quad \tilde{\Psi} = \bigoplus_{r=1}^N \tilde{\Psi}^{(r)},
\]

and each summand \( \tilde{\Gamma}^{(r)}, \tilde{\mathbf{V}}^{(r)}, \tilde{\mathbf{V}}_i^{(r)}, \tilde{\Psi}^{(r)} \) only depends on canonical coordinates \( u^k \) with \( k \in I_r \). The original flat \( F \)-manifold \( M \) locally decomposes into \( N \) corresponding pieces:

\[
M \equiv \bigoplus_{j=1}^N M^{(j)}.
\]

The flat \( F \)-manifold \( M \) is irreducible if and only if \( N = 1 \). This completes the proof. \( \square \)
In order to complete the proof of Theorem 7.3, we need to recall some preliminary known results on the (co)homology groups $H_\ast(M_{0,n}, \mathbb{C})$ and $H^\ast(M_{0,n}, \mathbb{C})$.

**Graphs.** In what follows, a *graph* $\tau$ is an ordered family $(V_\tau, H_\tau, \partial_\tau, j_\tau)$ where

- $V_\tau$ is a finite set of *vertices*,
- $H_\tau$ is a finite set of *half-edges*, equipped with a vertex assignment function $\partial_\tau : H_\tau \to V_\tau$, and an involution $j_\tau : H_\tau \to H_\tau$.

The set $E_\tau$ of 2-cycles of $j_\tau$ is the set of *edges* of $\tau$. The set $S_\tau$ of fixed points of $j_\tau$ is the set of *tails* of $\tau$. The datum of $j_\tau$ is thus equivalent to the datum of $E_\tau$ and $S_\tau$. For each vertex $v \in V_\tau$, define the set $H_\tau(v)$ of half-edges attached at $v$ by

$$H_\tau(v) := \partial_\tau^{-1}(v),$$

the set $E_\tau(v)$ of edges attached to $v$ by

$$E_\tau(v) := \{ \{f_1, f_2\} \in E_\tau : \partial_\tau(f_1) = v \text{ or } \partial_\tau(f_2) = v \},$$

and the set $S_\tau(v)$ of tails attached to $v$ by

$$S_\tau(v) := \{ f \in S_\tau : \partial_\tau f = v \}.$$

We clearly have a partition $H_\tau(v) \cong E_\tau(v) \coprod S_\tau(v)$.

A graph $\tau$ is conveniently identified with its associated topological space $||\tau||$. Vertices of $\tau$ are identified with $|V_\tau|$ distinct points $\{p(v)\}_{v \in V_\tau}$ on the curve $C := \{(t, t^2, t^3) : t \in \mathbb{R}\} \subseteq \mathbb{R}^3$, an edge $\{f_1, f_2\} \in E_\tau$ is identified with the segment joining the points $p(\partial_\tau(f_1))$ and $p(\partial_\tau(f_2))$, tails at $v$ are identified with a star of $|S_\tau(v)|$ small segments originating from $p(v)$, intersecting neither edges nor other tails at other vertices. The space $||\tau||$ is the union of all these vertices and segments, equipped with the topology induced from $\mathbb{R}^3$. The graph $\tau$ is a *tree* if $||\tau||$ is connected and $H_1(||\tau||, \mathbb{Z}) = 0$. A tree with $n$ tails will be called an *$n$-tree*.

**Dual stable graphs.** To each point $[C, (x)] \in M_{0,n}$, we can attach a dual stable graph $\tau$ as follows:

1. the vertices of $\tau$ are in 1-1 correspondence with the irreducible components of $C$,
2. each node of $C$ is replaced by an edge connecting the vertices corresponding to the two sides of the node,
3. for each $i = 1, \ldots, n$, attach an *tail* with label $i$ to the vertex corresponding to the irreducible component containing $x_i$.

The resulting graph $\tau$ is always an $n$-tree satisfying the following stability condition: each vertex has valence $|H_\tau(v)|$ at least 3. We say that $[C, (x)]$ has combinatorial type $\tau$.

Vice versa, for any stable $n$-tree $\tau$, there exists a locally closed irreducible subscheme $D(\tau) \subseteq \overline{M}_{0,n}$ parametrizing curves of combinatorial type $\tau$. The stratum $D(\tau)$ uniquely identifies the isomorphism class of $\tau$, and its codimension equals the number of edges $|E_\tau|$.

---

1. By Vandermonde determinant, any four points $p(v_1), \ldots, p(v_4)$ are not coplanar, and they are vertices of a tetrahedron. This argument shows that segments representing edges intersect only at the appropriate vertices.
Isomorphism classes of one edges $n$-trees are parametrized by stable unordered 2-partitions $\sigma = \{S_1, S_2\}$ of $\{1, \ldots, n\}$.

For example, the $n$-tree with one vertex corresponds to the open stratum $\mathcal{M}_{0,n}$. The strata of codimension one are labeled by isomorphism classes of one-edge stable $n$-trees $\sigma$. Each such class can be identified with a stable unordered 2-partition of the set $\{1, \ldots, n\}$. This consists of a set $\sigma = \{S_1, S_2\}$ such that $\{1, \ldots, n\} = S_1 \coprod S_2$, and $|S_i| \geq 2$ for $i = 1, 2$. See Figure B.1.

Given $(i, j, k, l) \in \{1, \ldots, n\}^4$ and a stable unordered 2-partition $\sigma$, we write $i\sigma jkl$ if $i, j$ and $k, l$ belong to the two different elements of $\sigma$.

**Keel’s theorem.** Introduce commuting indeterminates $D_\sigma$, indexed by stable unordered 2-partitions $\sigma$ of $\{1, \ldots, n\}$. Consider the ideal $I_n \subseteq \mathbb{C}[(D_\sigma)_\sigma]$ generated as follows:

1. For each $(i, j, k, l) \in \{1, \ldots, n\}^4$, set
   \[ R_{ijkl} := \sum_{i\sigma jkl} D_\sigma - \sum_{k jri l} D_\tau \in I_n; \]

2. If $\sigma$ and $\tau$ are such that $i\sigma jkl$ and $ij\tau kl$ for some $(i, j, k, l) \in \{1, \ldots, n\}^4$, then set
   \[ D_\sigma D_\tau \in I_n. \]

**Theorem B.1.** We have an isomorphism of rings
\[
\mathbb{C}[(D_\sigma)_\sigma]/I_n \cong H^*(\mathcal{M}_{0,n}, \mathbb{C}) \cong A^*(\overline{\mathcal{M}_{0,n}}; \mathbb{C})
\]
defined by
\[
D_\sigma \mapsto \text{dual of the closed cycle } \overline{D(\sigma)}.
\]

In particular, all odd cohomology groups vanish.

**Good monomials.** Consider a stable $n$-tree $\tau$. Any edge $e \in E_\tau$ defines a stable unordered 2-partition $\sigma(e)$ of $\{1, \ldots, n\}$: by cutting $e$, we obtain two trees, whose tails (halves of $e$ excluded) form the two parts of $\sigma(e)$.

For each stable $n$-tree $\tau$, define the monomial $m(\tau) := \prod_{e \in E_\tau} D_{\sigma(e)}$. This is a monomial in $\mathbb{C}[(D_\sigma)_\sigma]$ of degree $|E_\tau|$. Monomials of this form are called good monomials.

Under Keel isomorphism (B.3), $m(\tau)$ is the dual of the class $[\overline{D(\tau)}] \in H_*(\overline{\mathcal{M}_{0,n}}; \mathbb{C})$. This follows from the fact that boundary components intersect transversally.

**Theorem B.2** [72, Ch.III, §3.6]. Good monomials modulo $I_n$ span $\mathbb{C}[(D_\sigma)_\sigma]/I_n$. Equivalently, the classes $[\overline{D(\tau)}]$ span $H_*(\overline{\mathcal{M}_{0,n}}; \mathbb{C})$. 
**Manin’s relations in higher codimensions.** We need information about linear relations among all good monomials of fixed degree, generalizing Keel’s relations (B.1).

Let \( n \geq 4 \), and \( \tau \) to be an \( n \)-tree. Given

1. \( \nu \in V_\tau \) with valence \( |H_\tau(\nu)| \geq 4 \),
2. \( (i, j, k, l) \in H_\tau(\nu)^4 \) pairwise distinct half-edges,

set \( T := H_\tau(\nu) \setminus \{i, j, k, l\} \). For any ordered 2-partition \( \alpha = (T_1, T_2) \) of \( T \) (the case \( T_i = \emptyset \) is allowed), we can define two trees \( \tau'(\alpha) \) and \( \tau''(\alpha) \), with \(|E_\tau| + 1 \) edges each.

The tree \( \tau'(\alpha) \) is defined by replacing the vertex \( \nu \) with a new edge \( e \), at whose vertices we have half-edges \( \{i, j\} \cup T_1 \) and \( \{k, l\} \cup T_2 \), respectively.

The tree \( \tau''(\alpha) \) is defined by replacing the vertex \( \nu \) with a new edge \( e \), at whose vertices we have half-edges \( \{k, j\} \cup T_1 \) and \( \{i, l\} \cup T_2 \), respectively. See Figure B.2.

For each system \( (\tau, \nu, i, j, k, l) \) as above, define the polynomial

\[
R(\tau, \nu, i, j, k, l) := \sum_\alpha m(\tau'(\alpha)) - m(\tau''(\alpha)) \in \mathbb{C}[\{D_\sigma\}].
\]

**Theorem B.3.** We have \( R(\tau, \nu, i, j, k, l) \in I_n \). Moreover, all linear relations modulo \( I_n \) between good polynomials of degree \( r + 1 \) are spanned by all the relations \( R(\tau, \nu, i, j, k, l) \) with \(|E_\tau| = r\).

For a proof, see [72, Ch. III, Prop. 4.7.1, Th. 4.8].

**Proof of Theorem 7.3.** We are now able to complete the proof. Given potentials

\[
F^\alpha(t) = \sum_{n \geq 2} \sum_{\rho_1, \ldots, \rho_n=1}^N \frac{t^{\rho_1} \cdots t^{\rho_n}}{n!} c_\rho^\alpha \prod_{\rho=1}^{\rho_n} c_\rho^\alpha \in \mathbb{C}, \quad \alpha = 1, \ldots, N, \tag{B.4}
\]

Equations (7.4) and (7.5) are

\[
c_1^\alpha = \delta_1^\alpha, \quad c_1^\beta_1 \cdots \rho_n = 0 \quad \text{for } n > 0, \tag{B.5}
\]

\[
c_\mu^\alpha_\rho_1 \cdots \rho_n \gamma_{\rho_1 \cdots \rho_n} = c_\mu^\alpha_\rho_1 \cdots \rho_n \beta_{\rho_1 \cdots \rho_n} \gamma_{\rho_1 \cdots \rho_n}, \tag{B.6}
\]

Notice that the lower indices of the coefficients \( c \)'s can be arbitrarily permuted, that is, \( c_\rho^\alpha \) is uniquely identified by \( \alpha \) and the set \( \{\rho_1, \ldots, \rho_n\} \). We need to prove that the potentials \( F^\alpha_\rho \) are of the form (7.3) for a unique existing tree-level \( F \)-CohFT \( \{c_{0,n+1}\}_{n \geq 2} \). We first prove the uniqueness, and hence the existence of such an \( F \)-CohFT.
Uniqueness. Assume that there exists a tree-level $F$-CohFT $(c_{0,n+1})_{n \geq 2}$ such that
\[
\int_{\mathcal{M}_{0,n+1}} c_{0,n+1} \left( e^\vee_{\rho_1} \bigotimes_{i=2}^{n+1} e_{\rho_i} \right) = c_{\rho_2 \cdots \rho_n}, \tag{B.7}
\]
for any $1 \leq \rho_1, \ldots, \rho_{n+1} \leq N$ and any $n > 1$. We claim that it is then possible to compute the numbers
\[
\int_{D(\tau)} c_{0,n+1} \left( e^\vee_{\rho_1} \bigotimes_{i=2}^{n+1} e_{\rho_i} \right), \tag{B.8}
\]
for all stable $(n+1)$-trees $\tau$. The homology classes $[D(\tau)]$ span $H_*(\mathcal{M}_{0,n+1}, \mathbb{C})$, by Theorem B.2. Hence, the datum of all possible numbers (B.8), for fixed indices $\rho_1, \ldots, \rho_{n+1} \in \{1, \ldots, N\}$, uniquely defines the cohomology class $c_{0,n+1}(e^\vee \otimes \bigotimes_{i=2}^{n+1} e_{\rho_i})$ as linear functional on $H_*(\mathcal{M}_{0,n+1}, \mathbb{C})$. In other words, if we are able to compute all possible numbers (B.8), then any $F$-CohFT $(c_{0,n+1})_{n \geq 2}$ satisfying (B.7) is unique.

Given $\tau$, the number (B.8) can be computed as follows, by iteration of the gluing property (4) of $F$-CohFT's. Denote by $v_0 \in V_\tau$ the vertex of $\tau$ such that $1 \in S_\tau(v_0)$, that is, at which the tail 1 is attached. Orient the edges $e \in E_\tau$ in such a way that $v_0$ becomes an “attractor.” In this way, at each vertex $v \in V_\tau \setminus \{v_0\}$, there is a single edge with outward orientation, all other edges being with inward orientation. At $v_0$, all edges are inward. Denote by $E^\text{in}_\tau(v)$ the set of inward edges at $v$, and by $E^\text{out}_\tau(v)$ the set of outward edges at $v$.

Consider now the following monomials attached to $\tau$. Each such monomial is product of coefficients $c$'s in (B.4). In total, we have $|V_\tau|$ factors $c$'s, one for each vertex $v \in V_\tau$. An index that is repeated inside the monomial (once upper and once lower) is said to be saturated.

The factor corresponding to $v \in V_\tau$ will have a total number of indices (upper and lower) equal to $|H_v|$, that is, in bijection with half-edges. We will have

1. a total number of $|E^\text{out}_\tau(v)| \in \{0, 1\}$ upper saturated indices,
2. a total number of $|E^\text{in}_\tau(v)|$ lower saturated indices, and
3. a total number of $|S_\tau(v)|$ lower indices selected from $(\rho_1, \ldots, \rho_{n+1})$.

The saturation of indices is dictated by edges: indices labeled by two halves of the same edge are saturated (one is up, the other is down). Nonsaturated indices are dictated by the sets $S_\tau(v)$: the factor corresponding to $v \in V_\tau$ will have lower indices $\rho_i$ with $i \in S_\tau(v)$. The vertex $v_0$ is the only vertex whose corresponding factor $c$ has upper index $\rho_1$.

The number (B.8) equals the sum of all such monomials, over all possible values (ranging in $\{1, \ldots, N\}$) of all saturated indices, according to Einstein's summation rule. For example, if $n = 7$, and $\tau_1, \tau_2$ are the graphs of Figure B.3, then
\[
\int_{D(\tau_1)} c_{0,8} \left( e^\vee_{\rho_1} \bigotimes_{j=2}^{8} e_{\rho_j} \right) = c_{\rho_2 \rho_4 \rho_5} c_{\rho_3 \rho_8} c_{\rho_2 \rho_7},
\]
\[
\int_{D(\tau_2)} c_{0,8} \left( e^\vee_{\rho_1} \bigotimes_{j=2}^{8} e_{\rho_j} \right) = c_{\rho_2 \rho_4 \rho_5} c_{\rho_3 \rho_8} c_{\rho_2 \rho_5} c_{\rho_3 \rho_8}.
\]

\[\text{For later notational convenience, we slightly changed the labelings: } \alpha \mapsto \rho_1 \text{ and } \rho_i \mapsto \rho_{i+1}, \text{ for } i = 1, \ldots, n.\]
This is just an iteration of gluing rule (4) of an $F$-CohFT, which can be seen as a special instance of computation of (B.8) for a one-edge $(n+1)$-tree. This proves uniqueness.

**Existence.** In the previous part of the proof, we described an algorithm. For any fixed $\rho = (\rho_1, ..., \rho_{n+1})$, the algorithm associates with any stable $(n+1)$-tree $\tau$, a complex number $Y_\rho(\tau) \in \mathbb{C}$, a polynomial expression in the coefficients $c'$s in (B.4). If we show that all linear relations between the homology classes $[D(\tau)]$ are preserved by the map $\tau \mapsto Y_\rho(\tau)$, then we would have a well-defined linear functional

$$\tilde{Y}_\rho : H_*(\overline{M}_{0,n+1}, \mathbb{C}) \to \mathbb{C}, \quad D(\tau) \mapsto Y_\rho(\tau),$$

that is, a cohomology class. This would lead to a candidate as $F$-CohFT,

$$c_{0,n+1} : V^* \otimes V^{\otimes n} \to H_*(\overline{M}_{0,n+1}, \mathbb{C}), \quad \left( e_\rho^v \otimes \bigotimes_{i=2}^{n+1} e_{\rho_i} \right) \mapsto \tilde{Y}_\rho.$$ 

Indeed, the properties (1)–(3) of $F$-CohFT for $c_{0,n+1}$ would follow from the symmetry of $e_{\rho_2 \cdots \rho_{n+1}}$ in the lower indices, and Equations (B.5) and (B.6). Also, the gluing property (4) would follow from the definition of the numbers $Y_\rho(\tau)$. This would complete the proof.

By Theorem B.3, it is then sufficient to prove that, for any fixed system $(\tau, v, i, j, k, l)$, all the relations $R(\tau, v, i, j, k, l)$ are preserved by $Y_\rho$, that is,

$$\sum_\alpha Y_\rho(\tau'(\alpha)) = \sum_\beta Y_\rho(\tau''(\beta)). \quad (B.9)$$

The trees $\tau', \tau''$ are obtained from $\tau$ by replacing the vertex $v \in E_\tau$ with an edge. There are many ways to do this, labeled by 2-partitions of $H_*(v)$. They induced a 2-partition of $i, j, k, l$. We put on the l.h.s. of (B.9) those which split $\{i, j, k, l\}$ in two pieces $\{i, j\} \coprod \{k, l\}$, and we put on the r.h.s. of (B.9) those which split $\{i, j, k, l\}$ in two pieces $\{k, j\} \coprod \{i, l\}$. The remaining partitions do not contribute.

We have in total $2^5$ possible cases to consider, according whether $v$ coincides with the marked vertex $v_0$ or not, and whether each of $i, j, k, l$ is an edge or a tail.

Consider, for example, the case in which $v = v_0$ and each of $i, j, k, l$ is a tail. In the l.h.s. of (B.9), we have the contributions coming from two possible graphs, according to the resulting position of the distinguished tail labeled by 1. Analogously, in the r.h.s., we have the contributions coming from two graphs. See Figure B.4.
Equations (B.9) thus reduce to an identity of the form

\[
\sum_{\alpha} \left( c_{\rho_1, \rho_j, \ldots, \rho_k, \rho_l, \ldots}^\alpha + c_{\rho_j, \rho_l, \ldots, \rho_k, \rho_1, \ldots}^\alpha \right) = \sum_{\beta} \left( c_{\rho_1, \rho_k, \ldots, \rho_j, \rho_l, \ldots}^\beta + c_{\rho_j, \rho_l, \ldots, \rho_k, \rho_1, \ldots}^\beta \right),
\]

where dots stand for all possible partitions of indices, induced by \(\alpha\) and \(\beta\). Both red terms and black terms in this equation cancel due to Equations (B.6).

The reader can check that all other 31 possible cases can be handled similarly. One can always recognize in Equation (B.9) a linear combination of identities (B.6), whose left and right sides correspond to the inserted edge in \(\tau'\) and \(\tau''\), respectively.

This completes the proof.

ACKNOWLEDGEMENTS
The author is thankful to D. Guzzetti, C. Hertling, A.R. Its, P. Lorenzoni, Yu.I. Manin, D. Masero, A.T. Ricolfi, P. Rossi, V. Roubtsov, A. Varchenko, C. Sabbah, M. Smirnov, and D. Yang for several valuable discussions. The author is thankful to the Hausdorff Research Institute for Mathematics (HIM) in Bonn, Germany, where this project was started, for providing excellent working conditions during the JTP “New Trends in Representation Theory.” This research was supported by HIM (Bonn, Germany), and by the FCT Projects UIDB/00208/2020, PTDC/MAT-PUR/30234/2017, 2021.01521.CEECIND, and 2022.03702.PTDC (GENIDE). The author is a member of the COST Action CA21109 CaLISTA.

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