The derived dimensions of \((m, n)\)-Igusa-Todorov algebras \(^*\)\(^\dagger\)

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Abstract

We give an upper bound for the dimension of the bounded derived categories of \((m, n)\)-Igusa-Todorov algebras which is a generalization of \(n\)-Igusa-Todorov algebras, where \(m, n\) are two nonnegative integers. As an applications, we get a new upper bound for the dimension of bounded derived categories in terms of the projective dimensions of certain of simple modules as well as radical layer length of artin algebra \(\Lambda\).

1 Introduction

Given a triangulated category \(\mathcal{T}\), Rouquier introduced in \cite{17} the dimension \(\text{tri.dim} \ \mathcal{T}\) of \(\mathcal{T}\) under the idea of Bondal and van den Bergh in \cite{6}. This dimension and the infimum of the Orlov spectrum of \(\mathcal{T}\) coincide, see \cite{15, 3}. Roughly speaking, it is an invariant that measures how quickly the category can be built from one object. Many authors have studied the upper bound of \(\text{tri.dim} \ \mathcal{T}\), see \cite{3, 5, 7, 9, 12, 14, 17, 18, 22, 25, 23} and so on. There are a lot of triangulated categories having infinite dimension, for instance, Oppermann and St’ov’ıček proved in \cite{14} that all proper thick subcategories of the bounded derived category of finitely generated modules over a Noetherian algebra containing perfect complexes have infinite dimension.

Let \(\Lambda\) be an artin algebra. Let \(\text{mod} \ \Lambda\) be the category of finitely generated right \(\Lambda\)-modules and let \(D^b(\text{mod} \ \Lambda)\) be the bounded derived category of \(\Lambda\). For convenience, \(\text{tri.dim} \ D^b(\text{mod} \ \Lambda)\) also is said to be the derived dimension of artin algebra \(\Lambda\)(for example, see \cite{7}).

The dimension of triangulated category plays an important role in representation theory(\cite{3, 5, 7, 9, 12, 14, 17, 18, 23}). For example, it can be used to study the representation dimension of artin algebras (\cite{17, 13}). Similar to the dimension of triangulated categories, the (extension) dimension of an abelian category was introduced by Beligiannis in \cite{4}, also see \cite{8}. The size of the extension dimension reflects how far an artin algebra is from a finite representation type, some relate result can be see \cite{25, 22, 21} and so on. We denote \(\text{ext.dim} \ \text{mod} \ \Lambda\) by the extension dimension of \(\text{mod} \ \Lambda\), and \(\text{LL}(\Lambda)\) by the Loewy length of \(\Lambda\), \(\text{gl.dim} \ \Lambda\) by the global dimension of \(\Lambda\), \(\text{rep.dim} \ \Lambda\) by the representation dimension of \(\Lambda\)(see \cite{2}). The upper bounds for the dimensions of the bounded derived category of \(\text{mod} \ \Lambda\) can be given in terms of \(\text{LL}(\Lambda)\), \(\text{gl.dim} \ \Lambda\), \(\text{rep.dim} \ \Lambda\), and

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ext.\,dim \,\text{mod} \,\Lambda. \text{ Given an non-semisimple artin algebra } \Lambda, \text{ these dimensions have the following relation}
\[
\text{ext.\,dim \,\text{mod} \,\Lambda} \leq \max\{\text{LL}(\Lambda) - 1, \text{gl.\,dim} \,\Lambda, \text{rep.\,dim} \,\Lambda - 2\}.
\]

In [19], Wei introduced the notion of \text{n-Igusa-Todorov algebra}. The relation between extension dimension and 0-Igusa-Todorov algebra is that artin algebra \(\Lambda\) is 0-Igusa-Todorov algebra if and only if \text{ext.\,dim \,\text{mod} \,\Lambda} \leq 1 (see [25, Proposition 3.15]). On the other hand, the authors in [22] give an upper bound for the derived dimension of \text{n-Igusa-Todorov}. For the sake of convenience, we will introduce the notion of \((m, n)\)-Igusa-Todorov algebras, where \(m, n\) are two nonnegative integers, which is a generalization of \text{n-Igusa-Todorov} algebra. Moreover, we also give an upper bound for the dimension of bounded derived categories of \((m, n)\)-Igusa-Todorov algebras in terms of \(m\) and \(n\).

For a length-category \(C\), generalizing the Loewy length, Huard, Lanzilotta and Hernández introduced in [10, 11] the (radical) layer length associated with a torsion pair, which is a new measure for objects of \(C\). Let \(\Lambda\) be an artin algebra and \(\mathcal{V}\) a set of some simple modules in \text{mod} \,\Lambda. Let \(t_{\mathcal{V}}\) be the torsion radical of a torsion pair associated with \(\mathcal{V}\) (see Section 4 for details). We use \(\ell_{\ell_{\mathcal{V}}}(\Lambda)\) to denote the \(t_{\mathcal{V}}\)-radical layer length of \(\Lambda\). For a module \(M\) in \text{mod} \,\Lambda, we use \(\text{pd} \,M\) to denote the projective dimensions of \(M\); in particular, set \(\text{pd} \,M = -1\) if \(M = 0\). For a subclass \(\mathcal{B}\) of \text{mod} \,\Lambda, the \textbf{projective dimension} \(\text{pd} \,\mathcal{B}\) of \(\mathcal{B}\) is defined as
\[
\text{pd} \,\mathcal{B} = \begin{cases} 
\sup\{\text{pd} \,M \mid M \in \mathcal{B}\}, & \text{if } \mathcal{B} \neq \emptyset; \\
-1, & \text{if } \mathcal{B} = \emptyset.
\end{cases}
\]

Note that \(\mathcal{V}\) is a finite set. So, if each simple module in \(\mathcal{V}\) has finite projective dimension, then \(\text{pd} \,\mathcal{V}\) attains its (finite) maximum.

Now, let us list some results about the upper bound of the dimension of bounded derived categories.

**Theorem 1.1.** Let \(\Lambda\) be an artin algebra and \(\mathcal{V}\) a set of some simple modules in \text{mod} \,\Lambda. We have

1. ([18, Proposition 7.37]) \(\text{tri.\,dim} \,D^b(\text{mod} \,\Lambda) \leq \text{LL}(\Lambda) - 1\);
2. ([18, Proposition 7.4] and [12, Proposition 2.6]) \(\text{tri.\,dim} \,D^b(\text{mod} \,\Lambda) \leq \text{gl.\,dim} \,\Lambda\);
3. ([23, Theorem 3.8]) \(\text{tri.\,dim} \,D^b(\text{mod} \,\Lambda) \leq (\text{pd} \,\mathcal{V} + 2)(\ell_{\ell_{\mathcal{V}}}(\Lambda) + 1) - 2\);
4. ([22]) \(\text{tri.\,dim} \,D^b(\text{mod} \,\Lambda) \leq 2(\text{pd} \,\mathcal{V} + \ell_{\ell_{\mathcal{V}}}(\Lambda)) + 1\);
5. ([20]) if \(\ell_{\ell_{\mathcal{V}}}(\Lambda) \leq 2\), then \(\text{tri.\,dim} \,D^b(\text{mod} \,\Lambda) \leq \text{pd} \,\mathcal{V} + 3\).

The aim of this paper is to prove the following

**Theorem 1.2.** Let \(\Lambda\) be an \((m, n)\)-Igusa-Todorov algebra. Then \(\text{tri.\,dim} \,D^b(\text{mod} \,\Lambda) \leq 2m + n + 1\).

**Theorem 1.3.** Let \(\Lambda\) be an artin algebra and \(\mathcal{V}\) a set of some simple modules in \text{mod} \,\Lambda. If \(\ell_{\ell_{\mathcal{V}}}(\Lambda) \geq 2\), then \(\text{tri.\,dim} \,D^b(\text{mod} \,\Lambda) \leq 2\ell_{\ell_{\mathcal{V}}}(\Lambda) + \text{pd} \,\mathcal{V} - 1\).

By Theorem [1.15] and Theorem [1.3] we have
Corollary 1.4. Let $\Lambda$ be an artin algebra and $V$ a set of some simple modules in $\text{mod} \, \Lambda$. Then $\text{tri.dim} \, D^b(\text{mod} \, \Lambda) \leq \max \{2\ell^V(\Lambda_A) + \text{pd} \, V - 1, \text{pd} \, V + 3\}$.

We also give examples to explain our results. Sometimes, we may be able to get a better upper bound for the dimension of the bounded derived category of $\Lambda$.

Corollary 1.5. If artin algebra $\Lambda$ is $n$-Igusa-Todorov algebra. Then $\text{tri.dim} \, D^b(\text{mod} \, \Lambda) \leq n + 3$.

2 Preliminaries

2.1 $(m, n)$-Igusa-Todorov algebras

In order to illustrate our main results, we introduce the following definition.

Definition 2.1. For two nonnegative integers $m, n$. The artin algebra $\Lambda$ is said to be $(m, n)$-Igusa-Todorov algebra if there is a module $V \in \text{mod} \, \Lambda$ such that for any module $M$ there exists an exact sequence

$$0 \to V_m \to V_{m-1} \to \cdots \to V_1 \to V_0 \to \Omega^n(M) \to 0$$

where $V_i \in \{V\}_1$ for each $0 \leq i \leq m$. Such a module $V$ is said to be $(m, n)$-Igusa-Todorov module.

Remark 2.2. (1) By Definition 2.1, we know that $(1, n)$-Igusa-Todorov algebras is the same as $n$-Igusa-Todorov algebras (see [19]).

(2) $(0, n)$-Igusa-Todorov algebras is the same as $n$-syzygy-finite algebras (see [19]).

(3) $(m, n)$-Igusa-Todorov algebras are $(m+i, n-i)$-Igusa-Todorov algebras, also are $(m, n+i)$-Igusa-Todorov algebras, where $(n - i)$ is non-negative.

(4) If $\text{gl.dim} \, \Lambda < \infty$, then $\Lambda$ is $(\text{gl.dim} \, \Lambda, 0)$-Igusa-Todorov algebra.

(5) Set $\text{LL}(\Lambda) = n \geq 2$. By [24, Lemma 3.1], we know that, for each $M \in \text{mod} \, \Lambda$, we have the following exact sequence

$$0 \to M_{n-1} \to M_{n-2} \to \cdots \to M_1 \to M_0 \to M \to 0$$

be an exact sequence in $\text{mod} \, \Lambda$, where $M_i \in \text{add}(\Lambda / \text{rad}^{n-i} \Lambda)$. Then $\Lambda$ is $(\text{LL}(\Lambda) - 1, 0)$-Igusa-Todorov algebra by Definition 2.1. Since $M_0$ is projective, we can get that $\Lambda$ is $(\text{LL}(\Lambda) - 2, 1)$-Igusa-Todorov algebra. Then $\Lambda$ is $(\text{LL}(\Lambda) - 2, 2)$-Igusa-Todorov algebra by Remark 2.2(3).

2.2 The dimension of triangulated category

We recall some notions from [17, 18, 13]. Let $\mathcal{T}$ be a triangulated category and $\mathcal{I} \subseteq \text{Ob} \, \mathcal{T}$. Let $\langle \mathcal{I} \rangle_1$ be the full subcategory consisting of $\mathcal{T}$ of all direct summands of finite direct sums of shifts of objects in $\mathcal{I}$. Given two subclasses $\mathcal{I}_1, \mathcal{I}_2 \subseteq \text{Ob} \, \mathcal{T}$, we denote $\mathcal{I}_1 \ast \mathcal{I}_2$ by the full subcategory of all extensions between them, that is,

$$\mathcal{I}_1 \ast \mathcal{I}_2 = \{ X \mid X_1 \to X \to X_2 \to X_1[1] \text{ with } X_1 \in \mathcal{I}_1 \text{ and } X_2 \in \mathcal{I}_2 \}.$$ 

Write $\mathcal{I}_1 \circ \mathcal{I}_2 := (\mathcal{I}_1 \ast \mathcal{I}_2)_1$. Then $(\mathcal{I}_1 \circ \mathcal{I}_2) \circ \mathcal{I}_3 = \mathcal{I}_1 \circ (\mathcal{I}_2 \circ \mathcal{I}_3)$ for any subclasses $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ of $\mathcal{T}$ by the octahedral axiom. Write

$$\langle \mathcal{I} \rangle_0 := 0, \langle \mathcal{I} \rangle_{n+1} := \langle \mathcal{I} \rangle_n \circ \langle \mathcal{I} \rangle_1 \text{ for any } n \geq 1.$$
Definition 2.3. ([17 Definition 3.2]) The dimension \( \text{tri.dim} \mathcal{T} \) of a triangulated category \( \mathcal{T} \) is the minimal \( d \) such that there exists an object \( M \in \mathcal{T} \) with \( \mathcal{T} = \langle M \rangle_{d+1} \). If no such \( M \) exists for any \( d \), then we set \( \text{tri.dim} \mathcal{T} = \infty \).

Lemma 2.4. ([16 Lemma 7.3]) Let \( \mathcal{T} \) be a triangulated category and let \( X, Y \) be two objects of \( \mathcal{T} \). Then
\[
\langle X \rangle_m \cap \langle Y \rangle_n \subseteq \langle X \oplus Y \rangle_{m+n}
\]
for any \( m, n \geq 0 \).

3 The dimension of the bounded derived category of \((m, n)\)-Igusa-Todorov algebras

Let \( M \in \text{mod } \Lambda \) and an integer \( i \in \mathbb{Z} \). We let \( S^0_i(M) \) denote the stalk complex with \( M \) in the \( i \)th place and 0 in the other places. From the proof of [9, Theorem], we have the following lemma.

Lemma 3.1. ([22]) For any bounded complex \( X \) over \( \text{mod } \Lambda \), we have
\[
X \in \langle \oplus_{i \in \mathbb{Z}} S^0_i(X) \rangle_1 \cap \langle \oplus_{i \in \mathbb{Z}} S^0_i(B_i(X)) \rangle_1
\]
in \( D^b(\text{mod } \Lambda) \), and \( \oplus_{i \in \mathbb{Z}} S^0_i(X) \) and \( \oplus_{i \in \mathbb{Z}} S^0_i(B_i(X)) \) have only finitely many nonzero summands.

Lemma 3.2. If we have the following exact sequence
\[
0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to X_{-1} \to 0
\]
in \( \text{mod } \Lambda \), then \( \langle S^0(X_{-1}) \rangle_1 \subseteq \langle \oplus_{i=0}^n S^0(X_i) \rangle_{n+1} \).

Proof. Set \( K_i := \text{Ker}(X_{i-1} \to X_{i-2}) \) for each \( 1 \leq i \leq n \), and \( K_0 := X_{-1} \). Now, for each \( 0 \leq i \leq n \), we can get the following short exact sequences
\[
0 \to K_{i+1} \to X_i \to K_i \to 0;
\]
moreover, we have the following triangles in \( D^b(\text{mod } \Lambda) \)
\[
S^0(K_{i+1}) \to S^0(X_i) \to S^0(K_i) \to S^0(K_{i+1})[1];
\]
and then, we have
\[
\langle S^0(K_i) \rangle_1 \subseteq \langle S^0(X_i) \rangle_1 \cap \langle S^0(K_{i+1})[1] \rangle_1 = \langle S^0(X_i) \rangle_1 \cap \langle S^0(K_{i+1}) \rangle_1.
\]
And by Lemma 2.4, we get
\[
\langle S^0(X_{-1}) \rangle_1 \subseteq \langle \oplus_{i=0}^n S^0(X_i) \rangle_{n+1}.
\]

Lemma 3.3. ([1 Lemma 3.1]) Let \( X \) be a bound complex of objects of \( \text{mod } \Lambda \). Suppose that the homology \( H_i(X) \) is a projective module for every integer \( i \). Then \( X \in \langle S^0(\Lambda) \rangle_1 \).

\qed
Theorem 3.4. Let $\Lambda$ be an $(m, n)$-Igusa-Todorov algebra. Then $\text{tri.dim} \ D^b(\text{mod } \Lambda) \leq 2m+n+1$.

Proof. Take a bounded complex $X$ in $D^b(\text{mod } \Lambda)$. By the construction in the proof of [1, Proposition 3.2] and [12, Lemma 2.5], we have the following complex

$$
\Omega(X) := \cdots \to \Omega(X_{i+1}) \oplus R_{i+1} \to \Omega(X_i) \oplus R_i \to \Omega(X_{i-1}) \oplus R_{i-1} \to \cdots
$$

where all $P^{X_j}, R_j$ are projective. Now consider the above construction, where all $R_i$ are projective. Observe the following commutative diagrams, where all $Q_i$ are projective.

we can find that

$$\text{Im}(\Omega(X_{i+1}) \oplus R_{i+1} \to \Omega(X_i) \oplus R_i = \Omega(B_i(X)) \in \Omega(\text{mod } \Lambda),$$

and

$$\text{Ker}(\Omega(X_{i+1}) \oplus R_{i+1} \to \Omega(X_i) \oplus R_i = \Omega(Z_i(X)) \oplus Q_i+1 \in \Omega(\text{mod } \Lambda).$$

Inductively, we can get the following triangles in $D^b(\text{mod } \Lambda)$

$$\Omega(X) \to P^X \to X \to \Omega(X)[1]$$

$$\Omega^2(X) \to P^{\Omega(X)} \to \Omega(X) \to \Omega^2(X)[1]$$

$$\vdots$$

$$\Omega^n(X) \to P^{\Omega^{n-1}(X)} \to \Omega^{n-1}(X) \to \Omega^n(X)[1]$$
Since all \( H_i(P^i(X))(i \in \mathbb{Z}, 0 \leq j \leq n - 1) \) are projective, we know that \( P^i(X) \in \langle S^0(\Lambda) \rangle_1 \) by Lemma 3.3. Hence, we have

\[ \langle X \rangle_1 \subseteq \langle P^X \rangle_1 \circ \langle \Omega(X)[1] \rangle_1 \subseteq \langle S^0(\Lambda) \rangle_1 \circ \langle \Omega(X) \rangle_1. \]

Repeating this process, we can get

\[ \langle \Omega^i(X) \rangle_1 \subseteq \langle P^X \rangle_1 \circ \langle \Omega^{i+1}(X)[1] \rangle_1 \subseteq \langle S^0(\Lambda) \rangle_1 \circ \langle \Omega^{i+1}(X) \rangle_1. \]

Moreover, and by Lemma 3.1, we have

\[ \langle X \rangle_1 \subseteq \langle S^0(\Lambda) \rangle_n \circ \langle \Omega^n(X) \rangle_1. \]

And by Lemma 3.2 we can get

\[ \langle X \rangle_1 \subseteq \langle S^0(\Lambda) \rangle_n \circ \langle \bigoplus_{i \in \mathbb{Z}} S^0(Z_i(\Omega^n(X))) \rangle_1 \circ \langle \bigoplus_{i \in \mathbb{Z}} S^0(B_i(\Omega^n(X))) \rangle_1. \]

Set \( V \) is a \((m, n)\)-Igusa-Todorov module. Note that \( Z_i(\Omega^n(X)) = \Omega^n(Z_i(X)) \oplus Q_i \in \Omega^n(\text{mod } \Lambda) \), by Definition 2.3, we can get the following exact sequence

\[ 0 \rightarrow V_m \rightarrow V_{m-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V_0 \rightarrow Z_i(\Omega^n(X)) \rightarrow 0 \]

where \( V_j \in [V \oplus \Lambda]_1 \) for each \( 0 \leq j \leq m \). By Lemma 3.1 we have

\[ \langle S^0(Z_i(\Omega^n(X))) \rangle_1 \subseteq \langle \bigoplus_{j=0}^m S^0(V_j) \rangle_{m+1} \subseteq \langle S^0(V) \rangle_{m+1}. \]

Similarly, we have

\[ \langle S^0(B_i(\Omega^n(X))) \rangle_1 \subseteq \langle S^0(V) \rangle_{m+1}. \]

Moreover, by Lemma 2.3 we have

\[ \langle X \rangle_1 \subseteq \langle S^0(\Lambda) \rangle_n \circ \langle \bigoplus_{i \in \mathbb{Z}} S^0(Z_i(\Omega^n(X))) \rangle_1 \circ \langle \bigoplus_{i \in \mathbb{Z}} S^0(B_i(\Omega^n(X))) \rangle_1 \]

\[ \subseteq \langle S^0(\Lambda) \rangle_n \circ \langle S^0(V) \rangle_{m+1} \circ \langle S^0(V) \rangle_{m+1} \]

\[ \subseteq \langle S^0(\Lambda) \oplus S^0(V) \rangle_{2m+n+2}. \]

And then we get \( D^b(\text{mod } \Lambda) = \langle S^0(\Lambda) \oplus S^0(V) \rangle_{2m+n+2} \). By Definition 2.3, we have tri.dim \( D^b(\text{mod } \Lambda) \leq 2m + n + 1. \)

**Corollary 3.5.** If artin algebra \( \Lambda \) is \( n \)-Igusa-Todorov algebra. Then tri.dim \( D^b(\text{mod } \Lambda) \leq n + 3. \)

**Proof.** By Remark 2.2(1) and Theorem 3.3. \( \square \)

**4.** \( (\ell \ell^t \nu(\Lambda) - 2, \text{pd } \nu + 2)\)-Igusa-Todorov algebra

For a module \( M \in \text{mod } \Lambda \), we use rad \( M \) and top \( M \) to denote the radical and top of \( M \) respectively. We use add \( M \) to denote the subcategory of \( \text{mod } \Lambda \) consisting of direct summands of finite direct sums of module \( M \). Let \( \mathcal{V} \) be a subset of all simple modules, and \( \mathcal{V}' \) the set of all the others simple modules in \( \text{mod } \Lambda \). We write \( \mathfrak{F}(\mathcal{V}) := \{ M \in \text{mod } \Lambda \mid \text{ there exists a chain } 0 \subseteq M_0 \subseteq \)
$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M$ of submodules of $M$ such that each quotients $M_i/M_{i-1} \in \mathcal{V}$. $\mathfrak{F}(\mathcal{V})$ is closed under extensions, submodules and quotients modules. Then we have a torsion part $(T, \mathfrak{F}(\mathcal{V}))$, and the corresponding torsion radical is denoted by $t_\mathcal{V}$, and we set $q_{t_\mathcal{V}}(M) = M/t_\mathcal{V}(M)$ for each $M \in \text{mod } \Lambda$. Note that, rad, top, $t_\mathcal{V}$ and $q_{t_\mathcal{V}}$ are covariant additive functors.

**Definition 4.1.** ([10]) The $t_\mathcal{V}$-radical layer length is a function $\ell(t_\mathcal{V}) : \text{mod } \Lambda \rightarrow \mathbb{N} \cup \{\infty\}$ via

$$\ell(t_\mathcal{V})(M) = \inf\{i \geq 0 \mid t_\mathcal{V} \circ F^i_{t_\mathcal{V}}(M) = 0, M \in \text{mod } \Lambda\}$$

where $F_{t_\mathcal{V}} = \text{rad} \circ t_\mathcal{V}$.

**Lemma 4.2.** ([22]) Let $\mathcal{V}$ be a subset of the set of all pairwise non-isomorphism simple $\Lambda$-modules. If $M \in \mathfrak{F}(\mathcal{V})$, then $\text{pd } M \leq \text{pd } \mathcal{V}$. In particular, we have $\text{pd } q_{t_\mathcal{V}}(M) \leq \text{pd } \mathcal{V}$.

**Lemma 4.3.** ([22]) Let $\mathcal{V}$ be a subset of the set of all pairwise non-isomorphism simple $\Lambda$-modules. For $M \in \text{mod } \Lambda$ and positive integer $n$. We have the following exact sequences

$$0 \rightarrow t_\mathcal{V}F^i_{t_\mathcal{V}}(M) \rightarrow F^i_{t_\mathcal{V}}(M) \rightarrow q_{t_\mathcal{V}}F^i_{t_\mathcal{V}}(M) \rightarrow 0$$

$$0 \rightarrow F^{i+1}_{t_\mathcal{V}}(M) \rightarrow t_\mathcal{V}F^i_{t_\mathcal{V}}(M) \rightarrow \text{top } t_\mathcal{V}F^i_{t_\mathcal{V}}(M) \rightarrow 0,$$

for $0 \leq i \leq n - 1$.

**Lemma 4.4.** ([10]) Let $\mathcal{V} \subseteq \{\text{simple right } \Lambda\text{-modules}\}$ and $M \in \text{mod } \Lambda$. If $t_\mathcal{V}(M) \neq 0$ then

$$\ell(t_\mathcal{V})(\Omega t_\mathcal{V}(M)) \leq \ell(t_\mathcal{V})(\Omega t_\mathcal{V}(\Lambda)) - 1.$$

By [23, Lemma 2.6], we have

**Lemma 4.5.** Let $\mathcal{V} \subseteq \{\text{simple right } \Lambda\text{-modules}\}$ and $M \in \text{mod } \Lambda$. If $\ell(t_\mathcal{V})(M) \leq k$, then $\ell(t_\mathcal{V})(F^k_{t_\mathcal{V}}(M)) = 0$.

The following result should be well known, for convenience, we give a proof.

**Lemma 4.6.** Let $\Lambda$ be an artin algebra. Given the exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in mod $\Lambda$, we can get the following exact sequence

$$0 \rightarrow \Omega^{i+1}(C) \rightarrow \Omega^i(A) \oplus Q_i \rightarrow \Omega^i(B) \rightarrow 0$$

for some projective module $Q_i$ in mod $\Lambda$, where $i \geq 0$.

**Proof.** Consider the following pullback

0 \rightarrow 0 \rightarrow \Omega(C) \rightarrow \Omega(C) \rightarrow 0

0 \rightarrow A \rightarrow A \oplus P \rightarrow P \rightarrow 0

0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,

(4.1)
where $P$ is the projective cover of $C$. Applying horseshoe lemma to the middle column exact sequence of the above diagram, we can get the following exact sequences

$$0 \to \Omega^{i+1}(C) \to \Omega^i(A) \oplus Q_i \to \Omega^i(B) \to 0$$

for some projective module $Q_i$ in $\text{mod} \Lambda$, where $i \geq 0$.

Now, we can get the following main results.

**Theorem 4.7.** Let $\Lambda$ be an artin algebra. $\mathcal{V}$ is the set of some simple modules with finite projective dimension. If $\ell^\mathcal{V}(\Lambda) \geq 2$, then $\Lambda$ is a $(\ell^\mathcal{V}(\Lambda) - 2, \text{pd} \mathcal{V} + 2)$-Igusa-Todorov algebra.

**Proof.** Let $\ell^\mathcal{V}(\Lambda) = m$ and $\text{pd} \mathcal{V} = n$. Let $M \in \text{mod} \Lambda$ and $N = \Omega t_\mathcal{V}(M)$. From the following exact sequence

$$0 \to t_\mathcal{V}(M) \to M \to q_\mathcal{V}(M) \to 0,$$

and by horseshoe lemma, we can get

$$\Omega^{n+2}(M) \cong \Omega^{n+2}(t_\mathcal{V}(M)) = \Omega^{n+1}(N). \quad (4.2)$$

By Lemma 4.3 we have the following exact sequences

$$0 \to t_\mathcal{V}F_{t_\mathcal{V}}^i(N) \to F_{t_\mathcal{V}}^i(N) \to q_{t_\mathcal{V}}F_{t_\mathcal{V}}^i(N) \to 0 \quad (4.3)$$

$$0 \to F_{t_\mathcal{V}}^{i+1}(N) \to t_\mathcal{V}F_{t_\mathcal{V}}^i(N) \to \text{top} t_\mathcal{V}F_{t_\mathcal{V}}^i(N) \to 0, \quad (4.4)$$

By horseshoe lemma, and short exact sequences (4.3) and (4.4), we can obtain the following exact sequences

$$\Omega^{n+1}(t_\mathcal{V}F_{t_\mathcal{V}}^i(N)) \cong \Omega^{n+1}(F_{t_\mathcal{V}}^i(N)) \quad (4.5)$$

$$0 \to \Omega^{n+1}(F_{t_\mathcal{V}}^{i+1}(N)) \to \Omega^{n+1}(t_\mathcal{V}F_{t_\mathcal{V}}^i(N)) \oplus P_i \to \Omega^{n+1}(\text{top} t_\mathcal{V}F_{t_\mathcal{V}}^i(N)) \to 0, \quad (4.6)$$

where all $P_i$ are projective. Moreover, by (4.5), (4.6), for each $i \geq 0$, we get the following exact sequences

$$0 \to \Omega^{n+1}(F_{t_\mathcal{V}}^{i+1}(N)) \to \Omega^{n+1}(F_{t_\mathcal{V}}^i(N)) \oplus P_i \to \Omega^{n+1}(\text{top} t_\mathcal{V}F_{t_\mathcal{V}}^i(N)) \to 0. \quad (4.7)$$

By (4.7), let $i = 0, 1$, and consider the following push out,

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{n+1}(F_{t_\mathcal{V}}^2(N)) & \longrightarrow & \Omega^{n+1}(F_{t_\mathcal{V}}^1(N)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{n+1}(F_{t_\mathcal{V}}(N)) \oplus P_1 & \longrightarrow & \Omega^{n+1}(N) \oplus P_0 \oplus P_1 & \longrightarrow & \Omega^{n+1}(\text{top} t_\mathcal{V}(N)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{n+1}(\text{top} t_\mathcal{V}F_{t_\mathcal{V}}(N)) & \longrightarrow & D_0 & \longrightarrow & \Omega^{n+1}(\text{top} t_\mathcal{V}(N)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & \\
\end{array}
\]
by the diagram (4.8), we can get the following two exact sequences

\[
0 \rightarrow \Omega^{n+1}(F^2_{tv}(N)) \rightarrow \Omega^{n+1}(N) \oplus P_0 \oplus P_1 \rightarrow D_0 \rightarrow 0, \tag{4.9}
\]

\[
0 \rightarrow \Omega^{n+1}(\text{top } t_VF_{tv}(N)) \rightarrow D_0 \rightarrow \Omega^{n+1}(\text{top } t_V(N)) \rightarrow 0. \tag{4.10}
\]

By (4.9) and (4.7) (for the case \(i = 2\)), and consider the following push out,

\[
\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
\Omega^{n+1}(F^3_{tv}(N)) & \rightarrow & \Omega^{n+1}(F^3_{tv}(N)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Omega^{n+1}(F^2_{tv}(N)) \oplus P_2 \\
& & \downarrow \\
& & \Omega^{n+1}(N) \oplus P_0 \oplus P_1 \oplus P_2 \\
& & \downarrow \\
& & D_0 \\
\end{array}
\tag{4.11}
\]

by the diagram (4.11), we can get the following two exact sequences

\[
0 \rightarrow \Omega^{n+1}(F^3_{tv}(N)) \rightarrow \Omega^{n+1}(N) \oplus P_0 \oplus P_1 \oplus P_2 \rightarrow D_1 \rightarrow 0, \tag{4.12}
\]

\[
0 \rightarrow \Omega^{n+1}(\text{top } t_VF^2_{tv}(N)) \rightarrow D_1 \rightarrow D_0 \rightarrow 0. \tag{4.13}
\]

Continue this process, we can get the following exact sequences

\[
0 \rightarrow \Omega^{n+1}(F^j_{tv}(N)) \rightarrow \Omega^{n+1}(N) \oplus (\oplus_{i=0}^{j-1} P_i) \rightarrow D_{j-2} \rightarrow 0, \tag{4.14}
\]

\[
0 \rightarrow \Omega^{n+1}(\text{top } t_VF^{j-1}_{tv}(N)) \rightarrow D_{j-2} \rightarrow D_{j-3} \rightarrow 0, \tag{4.15}
\]

where \(j \geq 2\) and \(D_{-1} := \Omega^{n+1}(\text{top } t_V(N))\).

From (4.14) (4.14), we list the following exact sequences

\[
\begin{align*}
0 & \rightarrow \Omega^{n+1}(\text{top } t_VF^1_{tv}(N)) \rightarrow D_0 \rightarrow D_{-1} \rightarrow 0 \\
0 & \rightarrow \Omega^{n+1}(\text{top } t_VF^2_{tv}(N)) \rightarrow D_1 \rightarrow D_0 \rightarrow 0 \\
0 & \rightarrow \Omega^{n+1}(\text{top } t_VF^3_{tv}(N)) \rightarrow D_2 \rightarrow D_1 \rightarrow 0 \\
\vdots \\
0 & \rightarrow \Omega^{n+1}(\text{top } t_VF^{m-2}_{tv}(N)) \rightarrow D_{m-3} \rightarrow D_{m-4} \rightarrow 0 \\
0 & \rightarrow \Omega^{n+1}(F^{m-1}_{tv}(N)) \rightarrow \Omega^{n+1}(N) \oplus (\oplus_{i=0}^{m-2} P_i) \rightarrow D_{m-3} \rightarrow 0.
\end{align*}
\]

On the other hand, by Lemma 4.4, we have \(\ell\ell t_V(N) = \ell\ell t_V(\Omega t_V(M)) \leq \ell\ell t_V(\Lambda) - 1 = m - 1\).

By Lemma 4.5, we have \(\ell\ell t_V(F^{m-1}_{tv}(N)) = 0\), that is, \(F^{m-1}_{tv}(N) \in \mathfrak{F}(V)\). And by Lemma 4.2
\[ \text{pd} F_{t_v}^{m-1}(N) \leq \text{pd} \mathcal{V} = n. \] Now, by Lemma 4.6 and the above exact sequences, we can get the following

\[ \begin{array}{c}
0 \rightarrow \Omega^{m-2}(D_{-1}) \rightarrow \Omega^{n+m-2}(\text{top} t_v F_{t_v}(N)) \oplus Q_{m-2} \rightarrow \Omega^{m-3}(D_0) \rightarrow 0 \\
0 \rightarrow \Omega^{m-3}(D_0) \rightarrow \Omega^{n+m-3}(\text{top} t_v F_{t_v}(N)) \oplus Q_{m-3} \rightarrow \Omega^{m-4}(D_1) \rightarrow 0 \\
0 \rightarrow \Omega^{m-4}(D_1) \rightarrow \Omega^{n+m-4}(\text{top} t_v F_{t_v}(N)) \oplus Q_{m-4} \rightarrow \Omega^{m-5}(D_2) \rightarrow 0 \\
\vdots \\
0 \rightarrow \Omega(D_{m-4}) \rightarrow \Omega^{n+1}(\text{top} t_v F_{t_v}^{m-2}(N)) \oplus Q_1 \rightarrow D_{m-3} \rightarrow 0 \\
\Omega^{n+1}(N) \oplus (\oplus_{i=0}^{m-2} P_i) \cong D_{m-3},
\end{array} \]

where \( Q_i \) is projective for each \( 1 \leq i \leq m - 2 \). Consider the following pullback diagram:

\[ \begin{array}{c}
0 \\
0 \rightarrow \Omega(D_{m-4}) \rightarrow W \rightarrow \Omega^{n+1}(N) \rightarrow 0 \\
\downarrow \\
0 \rightarrow \Omega(D_{m-4}) \rightarrow \Omega^{n+1}(\text{top} t_v F_{t_v}^{m-2}(N)) \oplus Q_1 \rightarrow \Omega^{n+1}(N) \oplus (\oplus_{i=0}^{m-2} P_i) \rightarrow 0 \\
\downarrow \\
\oplus_{i=0}^{m-2} P_i \\
0 \rightarrow 0
\end{array} \]

The short exact sequence of the middle column in the above diagram is split since \( \oplus_{i=0}^{m-2} P_i \) is projective. That is, \( W \in \text{add}(\Omega^{n+1}(\text{top} t_v F_{t_v}^{m-2}(N)) \oplus \Lambda) \).

Now, from (4.2) and the above exact sequences, we can get the following long exact

\[ 0 \rightarrow \Omega^{m-2}(D_{-1}) \rightarrow \Omega^{n+m-2}(\text{top} t_v F_{t_v}(N)) \oplus Q_{m-2} \rightarrow \Omega^{n+m-3}(\text{top} t_v F_{t_v}^2(N)) \oplus Q_{m-3} \rightarrow \cdots \rightarrow W \rightarrow \Omega^{n+2}(M) \rightarrow 0, \]

where

\[ \Omega^{m-2}(D_{-1}), \Omega^{n+m-2}(\text{top} t_v F_{t_v}(N)) \oplus Q_{m-2}, \cdots, W \in \text{add}(\Lambda \oplus (\oplus_{i=m+1}^{n+1} Q^i(\Lambda/\text{rad} \Lambda))). \]

By Definition 2.1 we know that \( \Lambda \) is \((m-2, n+2)\)-Igusa-Todorov algebra, that is, \((\ell \ell t_v(\Lambda) - 2, \text{pd} \mathcal{V} + 2)\)-Igusa-Todorov algebra. \( \square \)

**Theorem 4.8.** Let \( \Lambda \) be an artin algebra. \( \mathcal{V} \) is the set of some simple modules with finite projective dimension. If \( \ell \ell t_v(\Lambda) \geq 2 \). Then \( \text{tri.dim} D^b(\mod \Lambda) \leq 2\ell \ell t_v(\Lambda) + \text{pd} \mathcal{V} - 1 \).

**Proof.** By Theorem 3.4 and Theorem 4.7 we have

\[ \text{tri.dim} D^b(\mod \Lambda) \leq 2(\ell \ell t_v(\Lambda) - 2) + (\text{pd} \mathcal{V} + 2) + 1 = 2\ell \ell t_v(\Lambda) + \text{pd} \mathcal{V} - 1. \]

\( \square \)
5 Examples

Example 5.1. ([23]) Consider the bound quiver algebra \( \Lambda = kQ/I \), where \( k \) is an algebraically closed field and \( Q \) is given by

\[
\begin{array}{ccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & \cdots & m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
m + 1 & \rightarrow & m + 2 & \rightarrow & m + 3 & \rightarrow & m + 4 & \rightarrow & \cdots & m + (m + 1)
\end{array}
\]

and \( I \) is generated by \( \{ \alpha_1^2, \alpha_1 \alpha_{m+1}, \alpha_1 \alpha_{m+2}, \alpha_1 \alpha_2, \alpha_2 \alpha_3, \ldots, \alpha_m \} \) with \( m \geq 10 \). Then the indecomposable projective \( \Lambda \)-modules are

\[
P(1) = 3 \
P(2) = 4 \
P(3) = 4 \
P(m + 1) = m + 1, \ P(m + 2) = m + 2
\]

and \( P(i + 1) = \text{rad} P(i) \) for any \( 2 \leq i \leq m - 1 \).

We have

\[
\text{pd} S(i) = \begin{cases} 
\infty, & \text{if } i = 1; \\
1, & \text{if } 2 \leq i \leq m - 1; \\
0, & \text{if } m \leq i \leq m + 2.
\end{cases}
\]

So \( S^\infty = \{ S(1) \} \) and \( S^{<\infty} = \{ S(i) \mid 2 \leq i \leq m + 2 \} \).

Let \( \mathcal{V} := \{ S(i) \mid 3 \leq i \leq m - 1 \} \subseteq S^{<\infty} \). Then \( \text{pd} \mathcal{V} = 1 \) and \( \ell \mathcal{V}(\Lambda) = 2 \) (see [23 Example 4.1])

(1) By Theorem LI.1, we have \( \text{tridi}m D^b(\text{mod} \Lambda) \leq \text{LL}(\Lambda) - 1 = m - 2 \).

(2) By Theorem LI.3, we have \( \text{tridi}m D^b(\text{mod} \Lambda) \leq (\text{pd} \mathcal{V} + 2)(\ell \mathcal{V}(\Lambda) + 1) - 2 = 7 \).

(3) By Theorem LI.4, we have \( \text{tridi}m D^b(\text{mod} \Lambda) \leq 2(\text{pd} \mathcal{V} + \ell \mathcal{V}(\Lambda)) + 1 = 7 \).

(4) By Theorem LI.5, \( \text{tridi}m D^b(\text{mod} \Lambda) \leq \text{pd} \mathcal{V} + 3 = 4 \).

(5) By Theorem LI.8 \( \dim D^b(\text{mod} \Lambda) \leq 2 \ell \mathcal{V}(\Lambda) + \text{pd} \mathcal{V} - 1 = 4 \). In fact, if \( \ell \mathcal{V}(\Lambda) = 2 \), then the upper bound \( 2 \ell \mathcal{V}(\Lambda) + \text{pd} \mathcal{V} - 1 \) is equal to \( \text{pd} \mathcal{V} + 3 \).

Example 5.2. ([22]) Let \( k \) be an algebraically closed field and \( \Lambda = kQ/I \), where \( Q \) the quiver

\[
\begin{array}{ccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mu_1 & \rightarrow & \mu_2 & \rightarrow & \mu_3 & \rightarrow & \mu_4
\end{array}
\]

and \( I \) is generated by \( \{ \alpha m, \alpha \beta, \gamma_1 \delta - \gamma_2 \delta, \rho_1 \mu_1 \alpha, \rho_2 \mu_2 \alpha, \mu_1 \beta - \mu_2 \beta \} \), \( n > 2m + 1, m \geq 5 \).
Then the indecomposable projective \( \Lambda \)-modules are

![Diagram](image)

and \( P(i + 1) = \text{rad} P(i) \) for each \( 3 \leq i \leq n + 2 \).

It is straightforward to verify that

\[
pd S(i) = \begin{cases} 
\infty, & \text{if } i = 1, n + 3; \\
2, & \text{if } i = 2, n + 4; \\
1, & \text{if } 3 \leq i \leq n + 2.
\end{cases}
\]

Let \( \mathcal{V} := \{S(i) \mid 3 \leq i \leq n + 2 \text{ or } i = n + 2\} \). Then \( \text{pd} \mathcal{V} = 2 \). Let \( \mathcal{V}' \) be all the others simple modules in \( \text{mod} \Lambda \), that is, \( \mathcal{V}' = \{S(1), S(2), S(n + 3), S(n + 4)\} \).

Because \( \Lambda = \oplus_{i=1}^{n+4} P(i) \), we have

\[\ell_{\mathcal{V}}(\Lambda) = \max\{\ell_{\mathcal{V}}(P(i)) \mid 1 \leq i \leq n + 4\}\]

by \([10, \text{Lemma 3.4(a)}]\).

In order to compute \( \ell_{\mathcal{V}}(P(1)) \), we need to find the least non-negative integer \( i \) such that \( t_{\mathcal{V}} F_{t_{\mathcal{V}}}(P(1)) = 0 \). Since \( \text{top} P(1) = S(1) \in \text{add} \mathcal{V}' \), we have \( t_{\mathcal{V}}(P(1)) = P(1) \) by \([10, \text{Proposition 5.9(a)}]\). Thus

\[F_{t_{\mathcal{V}}}(P(1)) = \text{rad} t_{\mathcal{V}}(P(1)) = \text{rad}(P(1)) = T_{m-1} \oplus P(2)\]

where \( T_{m-1} = \begin{array}{c} 1 \\
\vdots \\
1 \end{array} \) (the number of 1 is \( m - 1 \)).
Since top $T_{m-1} = S(1) \in \mathcal{V'}$, we have $t_V(T_{m-1}) = T_{m-1}$ by \cite[Proposition 5.9(a)]{10}. Similarly, $t_V(P(2)) = P(2)$. We have
\[ t_V F_{t_V}(P(1)) = t_V(T_{m-1} \oplus P(2)) = t_V(T_{m-1}) \oplus t_V(P(2)) = T_{m-1} \oplus P(2) \]
and
\[ F_{t_V}^2(P(1)) = \text{rad} t_V F_{t_V}(P(1)) = \text{rad}(T_{m-1} \oplus P(2)) = \text{rad}(T_{m-1}) \oplus \text{rad}(P(2)) = T_{m-2} \oplus M. \]

where $M = \begin{pmatrix} 3 \\ 4 \\ 5 \\ \vdots \\ n+2 \\ n+3 \\ n+4 \\ 1 \\ 1 \end{pmatrix}$

And
\[ t_V F_{t_V}^2(P(1)) = t_V(T_{m-2} \oplus M) = t_V(T_{m-2}) \oplus t_V(M) = T_{m-2} \oplus P(n+3). \]

Repeat the process, we have can get that $S(1)$ is a direct summand of $t_V F_{t_V}^{m-1}(P(1))$, that is $t_V F_{t_V}^{m-1}(P(1)) \neq 0$; and $t_V F_{t_V}^m(P(1)) = 0$. Moreover, $\ell \ell^{t_V}(P(1)) = m$. Similarly, we have
\[ \ell \ell^{t_V}(P(i)) = \begin{cases} 4, & \text{if } 2 \leq i \leq n; \\ 3, & \text{if } 3 \leq i \leq n+3; \\ m+1, & \text{if } i = n+4. \end{cases} \]

Consequently, we conclude that $\ell \ell^{t_V}(\Lambda_\Lambda) = \max\{\ell \ell^{t_V}(P(i)) \mid 1 \leq i \leq n+4\} = m + 1$. We have $\ell \ell^{t_V}(\Lambda_\Lambda) = m$.

Note that LL($\Lambda$) = $n + 5$ and gl.dim $\Lambda = \infty$.

(1) By Theorem \ref{1.1}(1), we have
\[ \text{tri.dim } D^b(\text{mod } \Lambda) \leq \text{LL}(\Lambda) - 1 = n + 4. \]

(2) By Theorem \ref{1.1}(3), we have
\[ \text{tri.dim } D^b(\text{mod } \Lambda) \leq (\text{pd } \mathcal{V} + 2)(\ell \ell^{t_V}(\Lambda) + 1) - 2 = (1 + 2)(m + 1 + 1) - 2 = 3m + 4. \]

(3) By Theorem \ref{1.1}(4), we have
\[ \text{tri.dim } D^b(\text{mod } \Lambda) \leq 2(\text{pd } \mathcal{V} + \ell \ell^{t_V}(\Lambda)) + 1 = 2 \times (2 + m) + 1 = 2m + 5. \]
(4) By Theorem 4.8, $\text{tri.dim } D^b(\text{mod } \Lambda) \leq 2\ell^V(\Lambda) + \text{pd } V - 1 = 2m + 1 - 1 = 2m$.

Since $n > 2m + 1$ and $m \geq 5$, we get that

$$2m < 2m + 5 = \inf\{2(\text{pd } V + \ell^V(\Lambda)) + 1, \text{gl.dim } \Lambda, LL(\Lambda) - 1, (\text{pd } V + 2)(\ell^V(\Lambda) + 1) - 2\}.$$ 

That is, our new upper bound for the dimension of the dimension of $D^b(\text{mod } \Lambda)$ can get a better upper bound of bounded derived categories sometimes.

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