We obtain for the attractive Dirac $\delta$-function potential in two-dimensional quantum mechanics a renormalized formulation that avoids reference to a cutoff and running coupling constant. Dimensional transmutation is carried out before attempting to solve the system, and leads to an interesting eigenvalue problem in $N-2$ degrees of freedom (in the center of momentum frame) when there are $N$ particles. The effective Hamiltonian for $N-2$ particles has a nonlocal attractive interaction, and the Schrodinger equation becomes an eigenvalue problem for the logarithm of this Hamiltonian. The 3-body case is examined in detail, and in this case a variational estimate of the ground-state energy is given.

I. Introduction

Scale invariance, and the ultraviolet divergences for which it is responsible, is an essential feature of the quantum field theories, QCD and electroweak, that comprise the standard model of elementary particle interactions. The divergences that are obtained in the process of quantizing the classical field theories on which the quantum dynamics are based can be removed via the mathematical procedure of renormalization. In condensed matter physics renormalization techniques are used to obtain mathematical models of physical phenomena (e.g. phase transitions) which, despite the presence of a characteristic scale, provided by a lattice spacing, are scale independent in nature. In contrast, renormalization in the context of elementary particle interactions is necessary to make the fundamental theory physically sensible.

That an awkward renormalization procedure is necessary to permit the quantum field theory paradigm to successfully describe elementary particle in-
interactions might be viewed as evidence that quantum field theory is not the proper framework for this problem. Some other exotic and finite theory might be more conceptually accurate and less mathematically cumbersome. On the other hand, one may take Wilson’s point of view, [1], on the role of renormalization in particle physics: a renormalizable quantum field theory may be viewed as an *effective model* which approximates for low enough energies (or long enough distances) a more fundamental and comprehensive theory. In addition, then, to seeking a more fundamental theory underlying renormalizable quantum field theories, we may aspire to a deeper understanding of the very successful effective theories we have.

Toward this end we take the point of view that renormalizable interactions might be given a *finite formulation*, [2], which avoids the need for renormalization altogether, without the necessity of discarding the framework of quantum field theory. The *renormalized interactions* that would be part of this formulation could not be scale-invariant; the scaling symmetry would be broken explicitly at the outset, and not through the renormalization procedure. Other properties of the erstwhile “fundamental” interactions might be modified as well.

This approach to finding a finite, effective theory of particle interactions would be tantamount to a reordering of the conventional analysis. It would require renormalizing the theory completely before embarking on efforts to solve it, rather than renormalizing in parallel with the finding of solutions. No nonphysical cutoffs or running parameters would appear in the formulation or solutions of the renormalized theory. Rather, such a theory could be formulated as a well-posed mathematical problem: a set of differential equations with
appropriate boundary conditions for example. If this point of view is the correct one, then the reason we cannot write down such a formulation of QCD or electroweak theory is not because they are only low-energy effective theories (though they may be) but rather because we have not yet achieved a deep enough understanding of these theories to write them down in the simplest way.

Although we believe finding a finite formulation of renormalizable interactions to be a worthwhile goal, attacking this problem directly, in say QCD, appears too formidable at this time. We have in the past, though, found examples of simpler, asymptotically free renormalizable theories that lend themselves to this approach. In [3] we renormalized the large-$N$ limit of the 1+1 dimensional non-Abelian Thirring (or Gross-Neveu) model before finding some exact solutions. The scale-breaking renormalized interaction manifested itself as a restriction on the domain of the Hamiltonian operator. A similar role for the Hamiltonian domain was found in [4] in the study of a quantum mechanical system of two particles in two dimensions attracted by a Dirac $\delta$-function potential. The same system was also examined in the path integral picture, where it was found that the renormalized interaction appeared as a subtle modification to the Wiener measure.

In this paper we continue with our investigation of quantum mechanical particles interacting through an attractive Dirac $\delta$-function potential in two dimensions. We first work out the case of three particles in detail, and eventually extend our ideas to the N-body case.
II. Review of the Two-Body Problem

Before embarking in the next section on the three-body problem, let us recall the renormalized formulation, in the Hamiltonian picture, of the two-body problem. In the two-body case, after separating out the center of mass coordinate, the original Schrodinger equation in configuration space is:

\[-2\Delta \Psi_\lambda(\bar{x}) - g\delta^2(\bar{x})\Psi_\lambda(\bar{x}) = \lambda \Psi_\lambda(\bar{x})\]  \hspace{1cm} (1)

where $\Delta$ is the two-dimensional Laplacian, and $g$ is a positive, dimensionless coupling constant. We have taken the masses of the two particles to be $m_1 = m_2 = 1/2$ and chosen units such that $\hbar = 1$. In momentum space the Schrodinger equation reads:

\[2p^2\Psi_\lambda(\bar{p}) - \frac{g}{(2\pi)^2} \int d^2p \Psi_\lambda(\bar{p}) = \lambda \Psi_\lambda(\bar{p})\]  \hspace{1cm} (2)

This eigenvalue problem, however, is nonphysical since, due to scale invariance, the presence of even a single negative energy solution implies a continuum of negative energy states extending down to minus infinity. On the other hand, if one attempts to restrict the domain of the Hamiltonian to the positive energy sector only, one finds that a complete set of eigenstates cannot be found: there will be no zero angular momentum eigenstate, \cite{2}. The Hamiltonian cannot be self-adjoint under such a choice of domain. (The situation is analogous to the presence of solutions to the Dirac equation of unbounded negative energy, and the impossibility of ignoring or "throwing away" these states).

This system’s illness can be cured via renormalization. First one regularizes the system by introducing a momentum cutoff (upper bound). Then the
coupling constant, $g$, is required to depend on the cutoff in such a way that
the ground-state energy of the regularized system remains finite as the cutoff
is removed, i.e. taken to infinity. This procedure removes $g$ from the problem,
replacing it with a parameter having dimensions of energy, a trade sometimes
called ‘dimensional transmutation’. This new parameter is arbitrary, character-
izes the strength of the interaction, and can be taken to be the ground-state
energy of the two-body system.

The customary way to obtain renormalized solutions to the problem, (see
[5], [6], [7], [8], [9]), then, is to solve the regularized system first, and then
take the limits of solutions (wavefunctions, scattering amplitudes, etc.) as the
cutoff is taken to infinity. In [4] we showed that an equivalent, but simpler,
formulation is given by the following two equations:

\[ 2p^2 \Psi_\lambda(\vec{p}) - \lim_{p \to \infty} 2p^2 \Psi_\lambda(\vec{p}) = \lambda \Psi_\lambda(\vec{p}) \]  

(3)

and

\[ \int d^2p (\Psi_\lambda(\vec{p}) - \frac{\eta_\lambda}{2p^2 + \mu^2}) = 0 \]  

(4)

where $\eta_\lambda \equiv \lim_{p \to \infty} 2p^2 \Psi_\lambda(\vec{p})$.

These two equations give an example of what we mean by a finite formulation of a renormalizable theory. The theory at this stage has been renormalized,
and can be treated as a well-posed mathematical problem. The first equation
is recognized as a renormalized version of the Schrödinger equation. The in-
teraction has become the term $-\eta_{\Psi_\lambda} = -\lim_{p \to \infty} 2p^2 \Psi(\vec{p})$ on the left hand
side. This term is the renormalized interaction. Wavefunctions for which $\eta_{\Psi_\lambda}$
is zero do not take part in the interaction. Interestingly, there is no adjustable parameter in the Hamiltonian operator appearing in this equation. In fact, the appropriate Hamiltonian in configuration space is the Laplacian with no interaction term at all: the deviation from the free theory is contained entirely in the specification of the Laplacian’s domain.

The second equation specifies the domain of the Hamiltonian. In configuration space it is a local condition that implies that wavefunctions with zero angular momentum diverge logarithmically at the origin; in configuration space, then, the interaction appears as this boundary condition on wavefunctions. The parameter $\mu^2$ has dimensions of energy, and can be picked arbitrarily. For any choice of $\mu^2 > 0$, the Hamiltonian will be self-adjoint (and in fact is a self-adjoint extension of the free Hamiltonian, [10]). Choosing $\mu^2$ corresponds to selecting the strength of the attractive interaction and it turns out that $-\mu^2$ is the ground-state energy, and that the free theory corresponds to letting $\mu^2 = 0$.

The above pair of equations can be solved completely, to yield all the energy eigenstates in the two-body case, [2] and [4]. Although we cannot analytically solve the N-body, or even the three-body, system, we can formulate the problem in a similar way, finding the analogs of equations (3) and (4), which can then be solved approximately.
III. Renormalizing the Three-Body Problem

The original three-body Schrödinger equation in configuration space is, when $m_1 = m_2 = m_3 = 1/2$ and $\hbar = 1$:

$$(-\Delta_1 - \Delta_2 - \Delta_3)\Psi_\lambda - g(\delta^2(\bar{x}_1 - \bar{x}_2) + \delta^2(\bar{x}_2 - \bar{x}_3) + \delta^2(\bar{x}_3 - \bar{x}_1))\Psi_\lambda = \lambda \Psi_\lambda \quad (5)$$

where $\Delta_i$ is the two-dimensional Laplacian in coordinate $\bar{x}_i$, $\Psi_\lambda = \Psi_\lambda(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, and $g$ is the positive, dimensionless coupling constant. In momentum space this becomes:

$$(k_1^2 + k_2^2 + k_3^2)\Psi_\lambda - \frac{g}{(2\pi)^2} \left( \int d^2k_{12} \Psi_\lambda + \int d^2k_{23} \Psi_\lambda + \int d^2k_{31} \Psi_\lambda \right) = \lambda \Psi_\lambda \quad (6)$$

where $\Psi_\lambda = \Psi_\lambda(\bar{k}_1, \bar{k}_2, \bar{k}_3)$ and $\bar{k}_{ij} = \frac{1}{2}(\bar{k}_i - \bar{k}_j)$.

As in the two-body problem, we work in momentum space, in the center of momentum frame where $\bar{K} \equiv \bar{k}_1 + \bar{k}_2 + \bar{k}_3 = 0$. Defining $\bar{p}_1 \equiv \frac{2}{3}(\frac{1}{2}(\bar{k}_2 + \bar{k}_3) - \bar{k}_1)$ (see [11] for a discussion of coordinate choices in the general three-body problem), the Schrödinger equation in this frame is:

$$(2k_{23}^2 + \frac{3}{2}p_1^2)\Psi_\lambda - \frac{g}{(2\pi)^2} \left( \int d^2k_{12} \Psi_\lambda + \int d^2k_{23} \Psi_\lambda + \int d^2k_{31} \Psi_\lambda \right) = \lambda \Psi_\lambda \quad (7)$$

where we choose the two independent coordinates to be $(\bar{k}_{23}, \bar{p}_1)$, such that $\Psi_\lambda = \Psi_\lambda(\bar{k}_{23}, \bar{p}_1)$, in which case $\bar{k}_{12}$ and $\bar{k}_{31}$ stand for $-\frac{1}{2}\bar{k}_{23} - \frac{2}{3}\bar{p}_1$ and $-\frac{1}{2}\bar{k}_{23} + \frac{3}{4}\bar{p}_1$ respectively. We could just as well have chosen $(\bar{k}_{31}, \bar{p}_2)$ or $(\bar{k}_{12}, \bar{p}_3)$ and

\[
(2k_{12}^2 + \frac{3}{2}p_2^2)\Psi_\lambda - \frac{g}{(2\pi)^2} \left( \int d^2k_{12} \Psi_\lambda + \int d^2k_{23} \Psi_\lambda + \int d^2k_{31} \Psi_\lambda \right) = \lambda \Psi_\lambda
\]

\[
(2k_{31}^2 + \frac{3}{2}p_3^2)\Psi_\lambda - \frac{g}{(2\pi)^2} \left( \int d^2k_{12} \Psi_\lambda + \int d^2k_{23} \Psi_\lambda + \int d^2k_{31} \Psi_\lambda \right) = \lambda \Psi_\lambda
\]
would have found the same Schrödinger equation, as $2k_{23}^2 + \frac{3}{2}p_1^2 = 2k_{31}^2 + \frac{3}{2}p_2^2 = 2k_{12}^2 + \frac{3}{2}p_3^2$ since the three particles are identical. From now on we will use this symmetry to write this combination simply as $2k^2 + \frac{3}{2}p^2$. The particle singled out by the choice of coordinates (e.g. particle “1” with the choice $(\vec{k}_{23}, \vec{p}_1)$) is sometimes called the “spectator” particle. Then the other two may be referred to as the “interacting pair”, even though in reality all three particles interact with each other.

For the same reasons that (2) was found to be nonphysical, (6) also represents an ill-posed physical problem. In this case if one tries to restrict the Hamiltonian domain to positive energy states, one will find that states with zero angular momentum in the $\vec{k}_{23}$ variable are missing, [2]. A renormalization of the system is again necessary. By carrying out the renormalization before, and independent of, solving the system, we can cast the problem in a form analogous to (3) and (4) above.

The first step in our renormalization program is to regularize the system by reconsidering the problem in an artificial momentum space wherein the momenta are bounded above by a cutoff. From Wilson’s point of view, this cutoff represents some scale beyond which we are not entitled to apply our low-energy effective theory. Here it is convenient to impose this condition by requiring $2k^2 + \frac{3}{2}p^2 < \Lambda^2$. We must also allow the coupling constant, $g$, to depend on $\Lambda$. By doing this we will be able to specify that $g$ change with $\Lambda$ in such a way that the physics predicted by the effective theory we derive is independent of the cutoff. The resulting Schrödinger equation is:

$$
(2k^2 + \frac{3}{2}p^2 - \lambda)\Psi_{\lambda} = \frac{g(\Lambda/\nu)}{(2\pi)^2} \sum_i \int_0^\Lambda d^2k_i \Psi_{\lambda}^i
$$

(8)
where \( \nu \) is an arbitrary parameter with dimensions of momentum, \( \int^\Lambda \) indicates the the restriction \( 2k^2 + \frac{3}{2}p^2 < \Lambda^2 \), the sum has three terms, and from now on \( k_1, k_2 \) and \( k_3 \) will correspond to, and be used interchangeably with, \( k_{23}, k_{31} \) and \( k_{12} \) respectively.

The presence of \( \Lambda \) breaks scale invariance, such that there is no longer the instability problem preventing us from allowing negative energy states. Also, if we define \( f^\Lambda_{\lambda,i}(\bar{p}_i) \equiv \frac{g(\Lambda/\nu)}{(2\pi)^{2}} \int^\Lambda d^2k_i \Psi^\Lambda_{\lambda,i} \), then (8) can be written:

\[
(2k^2 + \frac{3}{2}p^2 - \lambda)\Psi^\Lambda_{\lambda} = \sum_i f^\Lambda_{\lambda,i}(\bar{p}_i) \tag{9}
\]

If we decide to consider \( \Psi^\Lambda_{\lambda} \) to be a function of \( \bar{k}_{23} \) and \( \bar{p}_1 \), then it is also useful to note:

\[
\bar{p}_2 \equiv \frac{2}{3}(\frac{1}{2}(\bar{k}_3 + \bar{k}_1) - \bar{k}_2) = -\bar{k}_{23} - \frac{1}{2}\bar{p}_1 \tag{10}
\]

and

\[
\bar{p}_3 \equiv \frac{2}{3}(\frac{1}{2}(\bar{k}_1 + \bar{k}_2) - \bar{k}_3) = \bar{k}_{23} - \frac{1}{2}\bar{p}_1 \tag{11}
\]

It is left to determine what \( g(\Lambda/\nu) \) must be in order that this regularized system will have a well-defined limit as we let \( \Lambda \to \infty \). By integrating (8) over any of the \( k_i \) it is not difficult to show, [2], that \( g(\Lambda/\nu) \) must satisfy, for large enough \( \Lambda/\nu \):
\[ \frac{(2\pi)^2}{g(\Lambda/\nu)} = \int_{k<\Lambda} \frac{d^2k}{2k^2 + \nu^2} \]  

(12)

where we keep in mind that \( \nu^2 \) is arbitrary. (12) implies that \( g(\Lambda/\nu) \to 0 \) as we let \( \Lambda \to \infty \), indicating that the interaction we consider is *asymptotically free*. This expression for \( g \) is, with \( \nu^2 \) replaced by \( \mu^2 \), precisely what was found during the renormalization of the two-body system, [4]. This is a result we should expect. In fact it would be reasonable to fix \( g(\Lambda/\nu) \) using the two-body result rather than to rederive it in the three-body case. This correspondence allows us to identify \( -\nu^2 = -\mu^2 \) as the binding energy of the two particle bound state, i.e. the ground-state energy of the two-body system. From now on we will use \( -\mu^2 \) to denote this quantity.

By taking the \( k_{23} \to \infty \) limit of (8) we can make the identification:

\[
\lim_{k_{23} \to \infty} 2k^2_{23} \Psi^\Lambda_{\lambda} = \frac{g(\Lambda/\mu)}{(2\pi)^2} \int k_{23}^2 \Psi^\Lambda_{\lambda} = f^\Lambda_{\lambda,1}(\bar{p}) \]  

(13)

This equation has counterparts with \( \bar{k}_{23} \) replaced by \( \bar{k}_{31} \) or \( \bar{k}_{12} \), such that (8) could also be written:

\[
(2k^2 + \frac{3}{2}p^2 - \lambda)\Psi^\Lambda_{\lambda} = \sum \lim_{k_{i} \to \infty} 2k^2_{i} \Psi^\Lambda_{\lambda} \]  

(14)

This equation does not contain \( g \), and will remain valid as we let \( \Lambda \to \infty \). In this limit, (14) becomes the three-body renormalized Schrodinger equation, comparable to (3) in the two-body case. Again, no coupling parameter appears in the renormalized Hamiltonian. The parameter characterizing the interaction
strength, which we continue to take to be the two-body binding energy, will appear in the domain (boundary condition) equation.

We can find the domain equation by combining (12) and (13). Together, as $\Lambda \to \infty$, these equations impose the following conditions on wavefunctions:

$$\int d^2k_i(\Psi_{\Lambda}(\vec{k}_i, \vec{p}_i) - \frac{f_{\Lambda,i}(\vec{p}_i)}{2k_i^2 + \mu^2}) = 0 \quad (15)$$

where $f_{\Lambda}(\vec{p}_i) = \lim_{k_i \to \infty} 2k_i^2 \Psi_{\Lambda}$. One way of understanding the choice of $g(\Lambda/\nu) = g(\Lambda/\mu)$ in (12) is that $g$ must have the form which ensures that (13) will continue to hold true as we let $\Lambda \to \infty$.

(15) is in fact three equations, one each for $i = 1, 2, 3$. They are integrals over momentum space, but in configuration space they are local conditions dictating that the wavefunctions either go to zero or diverge logarithmically, at a rate determined by $\mu$, whenever the coordinates of two particles are made to coincide. That the wavefunction can blow up under these circumstances is unusual, but not proscribed since the singularities are square-integrable.

We thus find that the three-body analogs of the two-body equations (4) and (3) are (15) and the three-body renormalized Schrodinger equation:

$$(2k^2 + \frac{3}{2}p^2 - \lambda)\Psi_{\Lambda} = \sum_i f_{\Lambda,i}(\vec{p}_i) \quad (16)$$

The equations (15) and (16) comprise the renormalized, finite formulation of the three-body problem we seek. The form of these equations indicates that the unknown parts of the wavefunctions are essentially their limits as the relative momenta of particles are taken to infinity, i.e. the functions $f_{\Lambda,i}$,
\( i = 1, 2, 3. \) We will require the particles to satisfy Bose statistics, in which case \( \Psi_\lambda \) must be symmetric under permutations of particle indices. Then the \( f_{\lambda,i} \) must all be the same function, which we will call \( f_\lambda \). \( f_\lambda \) is a function of one momentum variable rather than two, and so the eigenvalue problem has already been simplified. The analogous quantity, \( \eta_{\Psi_\lambda} \equiv \lim_{p \to \infty} 2p^2 \Psi_\lambda(\bar{p}) \), in the two-body case was simply a constant which could be fixed by normalization, so that the eigenvalue problem could be trivially solved. In the N-body case we will find an eigenvalue problem in \( N - 2 \) degrees of freedom in the center of momentum frame.

Although the crux of our problem in solving the system is finding the functions \( f_\lambda \) for all eigenvalues \( \lambda \), (15) and (16) contain not only \( f_\lambda \) but the wavefunction \( \Psi_\lambda \) as well. It would be convenient to eliminate \( \Psi_\lambda \) from the problem altogether, and obtain the equation satisfied by \( f_\lambda \) alone. This we can do by employing the Lippmann-Schwinger formulation, [12], of the Schrödinger equation. The Lippmann-Schwinger approach is conventionally reserved for scattering states, but here we will find it a convenient starting point in both the scattering and bound state sectors of the theory.

The Lippmann-Schwinger equation for our three-body system can be written down by inspection using (16). It is:

\[
\Psi_\lambda = g(\bar{p}_1, \bar{k}_{23}) \delta(2k^2 + \frac{3}{2}p^2 - \lambda) + \frac{\sum_i f_\lambda^\pm(\bar{p}_i)}{2k^2 + \frac{3}{2}p^2 - \lambda \mp i\epsilon} \tag{17}
\]

We choose for concreteness to write (17) in the basis \((\bar{k}_{23}, \bar{p}_1)\). \( g(\bar{p}_1, \bar{k}_{23}) \) is arbitrary, excepting the fact that it must be symmetric under permutations of particle indices.
The energy \( \epsilon > 0 \) in (17) is infinitesimally small. It must appear when \( \lambda > 0 \) to make the division by the singular operator \( 2k^2 + \frac{3}{2}p^2 - \lambda \) well-defined. The choice of sign, \( \mp \), in the denominator corresponds to a choice of boundary conditions: the upper sign implying outgoing scattered waves (which is ordinarily the physical case), and the lower sign meaning converging scattered waves. \( \epsilon \) can eventually be taken to zero, but will serve to regulate otherwise divergent integrals in intermediate steps. We add the label \( \pm \) to \( f_{\lambda}^\pm \) to signify this choice of boundary conditions.

The first term in (17) represents the kernel of the operator \( 2k^2 + \frac{3}{2}p^2 - \lambda \), which appears on the left hand side of (16). It is a solution to the free Schrodinger equation. In the case \( \lambda > 0 \) it represents the unscattered portion of the wave, and in configuration space gives the wavefunction its asymptotic behavior as particles become infinitely separated. When \( \lambda < 0 \) this term disappears. Negative energy states are comprised of only the second term, and their form is specified by (17) up to the unknown function \( f_{\lambda} \). All information contained in (16) is also in (17).

At this point we must part ways with the usual Lippmann-Schwinger analysis. No potential appears in (17), and we cannot use the self-consistency of this equation alone to solve for the wavefunction. In the place of usual term involving the potential stands the unknown function \( f_{\lambda}^\pm \). The only further information we can glean about \( f_{\lambda}^\pm \) from (17) comes from taking the limit of this equation as \( k_{23} \to \infty \). Since \( f_{\lambda}^\pm (\vec{p}_1) = \lim_{k_{23} \to \infty} 2k_{23}^2 \Psi_{\lambda} \), taking this limit and using (10) and (11) gives:

\[
\lim_{k_{23} \to \infty} (f_{\lambda}(\vec{k}_{23}) + f_{\lambda}(-\vec{k}_{23})) = 0
\] (18)
However, the bosonic symmetry requires that $f_\lambda(-\vec{k}_{23}) = f_\lambda(\vec{k}_{23})$, so that (18) becomes the simple asymptotic condition:

$$\lim_{k_{23} \to \infty} f_\lambda(\vec{k}_{23}) = 0 \quad (19)$$

To gain more information about $f_\lambda$, we need to use (15). This once more highlights the fact that in this system, the interaction between particles is encoded in the boundary conditions on wavefunctions rather than in a conventional potential energy term. Inserting (17) into (15) and performing integrations where possible yields the following implicit equation for $f_\lambda$:

$$\left( \ln \left( \frac{1}{\mu^2} |\lambda - \frac{3}{2} p^2| \right) \mp i\pi \Theta(\lambda - \frac{3}{2} p^2) \right) f_\lambda^\pm(\vec{p}) - \frac{2}{\pi} \int d^2 k \frac{f_\lambda^\pm(\vec{k})}{(k^2 + p^2 + \vec{p} \cdot \vec{k} - \frac{\lambda}{2} \mp \frac{it}{2})} = \frac{4}{\pi^2} \int d^2 k \ g(\vec{p}, \vec{k}) \delta(2k^2 - \frac{3}{2} p^2 - \lambda) \quad (20)$$

This ungainly looking equation contains all the information about wavefunctions and energy eigenvalues we need. Imposing the requirement that $f_\lambda^\pm$ has asymptotic behavior such that the integral in (20) converges will ensure (18) holds true. When $\lambda > 0$, this equation is an inhomogeneous linear equation in $f_\lambda^\pm$. Operating on $f_\lambda^\pm$ on the left hand side of (20) is a symmetric integral operator, which must be inverted to get $f_\lambda^\pm$ for any given $g$. When $\lambda < 0$, the inhomogeneous term proportional to $g$ vanishes identically, leaving an eigenvalue problem for $f_\lambda^\pm$ and $\lambda$. In the following section we partially solve the problem of finding $f_\lambda^\pm$ for positive and negative energies, and show that
the negative energy sector yields an explicit form for a renormalized, nonlocal effective Hamiltonian.
IV. Toward Solving the Three-Body Problem

All solutions to the free Schrödinger equation have positive energy and can be categorized by the following choice of basis:

\[ \Psi^\text{Free}_\lambda = g(\vec{p}_1, \vec{k}_{23}) \delta(2k^2 + \frac{3}{2}p^2 - \lambda) \] (21)

where

\[ g(\vec{p}_1, \vec{k}_{23}) = \sum_{\Pi} \delta^2(\vec{k}_{23} - \frac{1}{2}(\vec{\kappa}_{\Pi(2)} - \vec{\kappa}_{\Pi(3)}))\delta^2(\vec{p}_1 - (\vec{\kappa}_{\Pi(2)} + \vec{\kappa}_{\Pi(3)})) \] (22)

The constant vectors \( \vec{\kappa}_1, \vec{\kappa}_2 \) and \( \vec{\kappa}_3 \) are the momenta of the three particles and label the states. There are no restrictions on them other than they represent the particles in the center of momentum frame wherein \( \vec{\kappa}_1 + \vec{\kappa}_2 + \vec{\kappa}_3 = 0 \). The energy is \( \lambda = \kappa^2_1 + \kappa^2_2 + \kappa^2_3 \). The sum in (22) is over the six permutations of \((1, 2, 3)\), and gives \( g \) the bosonic symmetry we require. For our purposes it is convenient to absorb the \( k_{23} \) dependence into \( p_1 \) using the delta function in (21) and to diagonalize the angular momentum in the angle, \( \theta_{23} \), of \( \vec{k}_{23} \). In this case \( g \) becomes:

\[ g_n(\vec{p}_1, \theta_{23}) = \sum_{\Pi} e^{i n \theta_{23}} e^{-i n \theta_{\Pi(2)}, \Pi(3)} \delta(k^2_{23} - \frac{1}{4}(\vec{\kappa}_{\Pi(2)} - \vec{\kappa}_{\Pi(3)})^2)\delta^2(\vec{p}_1 - (\vec{\kappa}_{\Pi(2)} + \vec{\kappa}_{\Pi(3)})) \] (23)
where $n$ is an integer, and $\theta_{i,j}$ denotes the angle of $\frac{1}{2}(\vec{\kappa}_i - \vec{\kappa}_j)$.

Recall that in the two-body problem only those states carrying zero angular momentum participated in the interaction. The wavefunctions with nonzero angular momentum were just the free ones. We find a similar situation in the three-body case. The free wavefunctions

$$\Psi_{n,\lambda} = g_n(\bar{p}_1, \theta_{23}) \delta(2k^2 + \frac{3}{2}p^2 - \lambda)$$

(24)

for $n \neq 0$ are the positive energy solutions we seek in the sector of the Hilbert space where the angular momenta of any pair of particles is nonzero. Clearly (24) solves (16) with $f_\lambda = 0$, for any $n$. For $n = 0$, due to the integration over $\theta_{23}$, these wavefunctions also satisfy (15) for $f_\lambda = 0$. Note that the right hand side of (20) becomes zero in this case. This is consistent with the linear operator on the left hand being nonsingular.

Free wavefunctions with quantum number $n$ not equal to zero do not scatter. Nontrivial, i.e. interacting, wavefunctions can therefore be taken to have the form (17), with $g(\bar{p}_1, \bar{k}_{23}) \rightarrow g_0(\bar{p}_1)$. $g_0$ is the symmetrized ($g_0(\bar{p}_1) = g_0(\bar{p}_2) = g_0(\bar{p}_3)$) function, for $n = 0$, given in (23). It is independent of $\theta_{23}$, so the integral on the right hand side of (20) becomes trivial, and this equation becomes:

$$\left( \ln \left( \frac{1}{\mu^2} |\lambda - \frac{3}{2}p^2| \right) \mp i\pi \Theta(\lambda - \frac{3}{2}p^2) \right) f^\pm(\bar{p}) - \frac{2}{\pi} \int d^2k \frac{f^\pm(k)}{(k^2 + p^2 + \bar{p} \cdot \bar{k} - \frac{\lambda}{2} \mp \frac{i\epsilon}{2})}$$

$$= \frac{2}{\pi} \Theta(\lambda - \frac{3}{2}p^2) g_0(\bar{p})$$

(25)

Positive energy scattering solutions, $f^\pm_\lambda$ exist for all $\lambda > 0$ and all $g_0$. Note
that the operator acting on $f^\pm_\lambda$ in (25) is rotationally invariant. One can let $f^\pm_\lambda(\bar{p}) \sim e^{i l\theta}$, where the integer $l$ is the total angular momentum of the three particle system, and solve each angular momentum sector independently. It is also worth noting that a rotation in the angle $\theta_{23}$ is not a symmetry of the interacting Hamiltonian; although scattered wavefunctions have a free part independent of this angle, the scattered components of these states will in general depend on $\theta_{23}$. Equation (25) is as far as we will take our analysis in the positive energy sector.

For the consideration of negative energy states, let us take $\lambda = -\eta^2$. In this case, (17) gives the form of the wavefunctions as:

$$\Psi_\lambda = \frac{\sum_i f_\lambda(\bar{p}_i)}{2k^2 + \frac{3}{2}p^2 + \eta^2}$$  \hspace{1cm} (26)

The infinitesimal parameter $\epsilon$ can be set to zero in this case. When $\lambda = -\eta^2$ (25) becomes an eigenvalue problem for $f_\lambda$ and $\lambda$:

$$\ln\left(\frac{1}{\mu^2(\eta^2 + \frac{3}{2}p^2)}\right)f_\lambda(\bar{p}) - \frac{2}{\pi} \int d^2k f_\lambda(\bar{k}) \frac{f_\lambda(\bar{k})}{(k^2 + p^2 + \bar{p} \cdot \bar{k} + \frac{\eta^2}{2})} = 0$$  \hspace{1cm} (27)

For a separable attractive potential (i.e. one having the form $<\bar{k}|V|\bar{k}'>$ = $-v(k)v(k')$) the Faddeev integral equation for the T-matrix, \[13\], written as an equation for the wavefunction, can be reduced to an equation in just one variable. Bruch and Tjon, \[14\], have shown that, in the limit in which such a potential is made to have zero range in real space, this equation is precisely the eigenvalue equation we have found in (27).
Our eigenvalue problem can be brought to a more conventional form by writing it in terms of dimensionless variables. We let \( \bar{p} = \eta \bar{x} \), \( \bar{k} = \eta \bar{y} \), and \( f_\lambda(\bar{p}) = u_\lambda(\bar{x}) \). Then (27) becomes

\[
\ln\left(1 + \frac{3}{2} \bar{x}^2\right) u_\lambda(\bar{x}) - \frac{2}{\pi} \int d^2y \frac{u_\lambda(\bar{y})}{(\bar{y}^2 + \bar{x}^2 + \bar{x} \cdot \bar{y} + \frac{1}{2})} = \ln\left(\frac{\mu^2}{\eta^2}\right) u_\lambda(\bar{x}) \tag{28}
\]

Denoting the operator on the left hand side of (28) by \( W \), we can rewrite that equation as:

\[
W u_\lambda = \ln\left(\frac{\mu^2}{\eta^2}\right) u_\lambda \tag{29}
\]

We find, therefore, a linear eigenvalue problem for the eigenfunction \( u_\lambda \) and the eigenvalue \( \ln\left(\frac{\mu^2}{\eta^2}\right) \). Note that \( W \) is a symmetric, nonlocal operator. Its eigenfunctions, \( u_\lambda \), give us the undetermined part (see (26)) of the wavefunction, \( \Psi_\lambda \).

Since the energy is \( -\eta^2 \), can identify the operator

\[
H \equiv -\mu^2 e^{-W} \tag{30}
\]

as the Hamiltonian, with one degree of freedom (in addition to the center of mass) effectively integrated out, of the renormalized system in the negative energy sector. We thus find that the logarithm of the effective Hamiltonian of the spectator particle, in momentum space, is an integral operator. The form of the integral operator \( W \) reveals that the renormalized, effective "spectator Hamiltonian" includes an attractive potential which is nonlocal. It is important
to realize, however, that the nonlocality of this Hamiltonian arises because we have effectively integrated out one degree of freedom, a simplification we could make because the actual interaction is of zero range. A positive attribute of the formulation in terms of this one-particle Hamiltonian is that the boundary conditions, and therefore the interactions, are built in, and do not come in through some supplementary condition on wavefunctions.

The operator $W$ consists of two terms: an unbounded positive kinetic-energy-like operator (a multiplication by $\ln(\frac{3}{2}x^2 + 1)$ in momentum space) and an interaction term (the integral operator in (28)). In light of numerical and variational evidence we conjecture, but have not proved, that the interaction part of $W$ is bounded below. Certainly the Hamiltonian including both the “kinetic” and “potential” energy terms is bounded below. For a proof of this see [14].

If the effective potential energy term alone is in fact bounded below it indicates that there are a finite number of normalizable bound states, as well as a continuum of negative energy scattering states wherein a bound state of two particles scatters from the third particle, and the kinetic energy of scattering is less in magnitude than the two-particle binding energy, $-\mu^2$. These states will have counterparts in the positive energy sector for which the kinetic energy dominates. Also in the positive energy sector will be scattering states of three unbound fundamental particles. In fact Bruch and Tjon, [14], have carried out a numerical study of (28) and concluded that there are just two three-particle bound states, a result consistent with the above picture.

The expression for $H$ in (30), in combination with the explicit form of $W$, is one of our key results. It gives us a rare glimpse at an explicit renormalized
Hamiltonian, that in most theories can at best be asymptotically approached order by order in perturbation theory. In this system the entire interaction appears in the domain of the renormalized Hamiltonian. The fact that the interaction is of zero range allows us to obtain a compact form for the eigenvalue problem by effectively integrating out one degree of freedom. That the simplest way to write the resulting Hamiltonian is in terms of its logarithm is interesting, but should not surprise us in light of the prevalence of logarithmic dependences in asymptotically free theories. We will find in the N-body case as well that the Hamiltonian is best written in terms of its logarithm. Before doing this, however, we will analyze the three-body case a little further, finding in the next section an approximation to the ground state and ground state energy.
V. Approximating the Three-Body Ground State

Our system of three particles is invariant under rotations, so that we may use this symmetry to simultaneously diagonalize the Hamiltonian and the total angular momentum. Letting the integer \( l \) be the angular momentum quantum number, the form of negative energy wavefunctions becomes:

\[
\Psi_{\lambda,l} = \frac{\sum_i e^{i l \theta_i} f_{\lambda,l}(\bar{p}_i)}{2k^2 + \frac{3}{4}p^2 + \eta^2}
\]  

(31)

where \( \theta_i \) is the angle of \( \bar{p}_i \).

Letting \( u_{\lambda,l}(x) = f_{\lambda,l}(p) \), the eigenvalue equation, (28), can be decomposed into separate angular momentum sectors, giving us:

\[
\ln \left( \frac{3}{2}x + 1 \right) u_{\lambda,l}(x) - 2 \int_0^\infty dy \frac{(xy)^{l/2}}{(xy)^{l/2} - x - (x + y + \frac{1}{2})^2 - xy} u_{\lambda,l}(y) = \ln \left( \frac{\mu^2}{\eta^2} \right) u_{\lambda,l}(x)
\]

(32)

The ground state will correspond to the (or one of the) \( l = 0 \) negative energy state(s). Letting \(-\gamma^2\) be the ground state energy, the form of this state, which we will call \( \Psi_0 \), is:

\[
\Psi_0 = \frac{\sum_i f_0(p_i)}{2k^2 + \frac{3}{4}p^2 + \gamma^2}
\]

(33)

with \( f_0(p) = u_0(x) \) being the eigenstate corresponding to the lowest eigenvalue \( \ln \left( \frac{\mu^2}{\eta^2} \right) \) solving the equation:
\begin{align*}
\ln\left(\frac{3}{2}x + 1\right)u(x) - 2\int_0^\infty dy \frac{u(y)}{\sqrt{(x + y + \frac{1}{2})^2 - xy}} &= \ln\left(\frac{\mu^2}{\eta^2}\right)u(x) \quad (34)
\end{align*}

By definition \( u_0 \) solves this equation for the value \( \eta^2 = \gamma^2 \). Equivalently we can regard \( u_0 \) as the function which minimizes the quadratic functional:

\begin{align*}
Q[u] &\equiv \int_0^\infty \ln\left(\frac{3}{2}x + 1\right)u^2(x) - 2\int_0^\infty dx \int_0^\infty dy \frac{u(x)u(y)}{\sqrt{(x + y + \frac{1}{2})^2 - xy}} \quad (35)
\end{align*}

subject to the constraint \( \int_0^\infty dxu^2(x) = 1 \).

One way of obtaining an approximation to the ground state wavefunction is to numerically solve a discretized version of (34). That is, we can diagonalize the matrix version of the operator \( W \), \( W_{ij} \), obtained by replacing the continuous variables \( 0 < x, y < \infty \) by discrete ones \( 0 < i\Delta, j\Delta < N\Delta \) for integers \( i, j \) and \( N \) and positive \( \Delta \). The task then is to diagonalize \( W_{ij} \). Doing this amounts to making a discrete approximation to the regularized problem within which there is a high momentum cutoff. For large \( N\Delta \) the unique positive eigenvector should give a useful picture of the ground state of the renormalized problem.

Using this method to obtain a numerical estimate of the ground state wavefunction shows that \( u_0 \) has a long, power law type tail. In fact \( u_0(x) \) is quite well approximated by a function of the form \( \sqrt{b/(b + x)} \). This rational function is normalized such that the integral of its square on the positive real line is unity, and therefore can be taken as a variational ansatz for \( u_0 \). Inserting
this form in the quadratic functional $Q[u]$ and minimizing with respect to the parameter $b$ yields the upper bound on the ground state energy:

$$\lambda_0 = -\gamma^2 < -\mu^2 e^{2.6} \approx -13.5\mu^2$$

Recalling that $-\mu^2$ has the meaning of the two-body ground state energy, we see that the three-body energy is significantly less, which is physically reasonable. Also, this bound is consistent with the estimate $\lambda_0 \approx -16.1\mu^2$ found by Bruch and Tjon using a numerical diagonalization of the Hamiltonian. In addition they find one other bound state, with spectator particle angular momentum $l = 1$, having energy $\approx -1.25\mu^2$. 
VI. The N-Body Problem

The original Schrodinger equation of the N-body problem in momentum space is:

$$\sum_{i=1}^{N} k_i^2 \Psi_{\lambda} - \frac{g}{(2\pi)^2} \sum_{i<j} \int d^2 k_{ij} \Psi_{\lambda} = \lambda \Psi_{\lambda}$$  \hspace{1cm} (37)

where we again take $\hbar = 1$ and the particle masses to be $m_i = 1/2$. $\bar{k}_{ij} = \frac{1}{2}(\bar{k}_i - \bar{k}_j)$ are, as in the three-body case, the relative momenta of pairs of particles. As before the system is ill-defined due to scale invariance. We regularize the problem with a momentum cutoff $\Lambda$ such that $\sum_i k_i^2 < \Lambda^2$, and define for each pair of particles the functions:

$$f_{\lambda,ij} = g(\Lambda/\mu) (2\pi)^2 \int_{\Lambda} d^2 k_{ij} \Psi_{\lambda}$$  \hspace{1cm} (38)

$g(\Lambda/\mu)$ is as given in (12), and the superscript $\Lambda$ on the integral indicates the restriction $\sum_i k_i^2 < \Lambda^2$. Recall that $-\mu^2$ is the two-body ground state energy.

As $\Lambda$ is taken to $\infty$ in (38), $g(\Lambda/\mu)$ is driven to zero, and picks out only the logarithmically divergent part of the integral over the relative momenta of particles. The renormalized Schrodinger equation then becomes:

$$\sum_{i=1}^{N} k_i^2 \Psi_{\lambda} - \lambda \Psi_{\lambda} = \sum_{i<j} f_{\lambda,ij}$$  \hspace{1cm} (39)

The functions $f_{\lambda,ij}$ are the limits as $\Lambda \to \infty$ of the $f_{\lambda,ij}^\Lambda$, and in this limit (38) and (12) ensure that $f_{\lambda,ij} = \lim_{\bar{k}_{ij} \to \infty} 2k_{ij}^2 \Psi_{\lambda}$. $f_{\lambda,ij}$ is independent, then, of $\bar{k}_{ij}$, and if we continue to consider the particles to be bosons, then the
necessary permutation symmetry will imply that only one of these functions need be specified. The others will follow by permutation of particle indices.

In a similar manner as in the two- and three-body problems, we can show that the renormalized Schrödinger equation, (39), must be supplemented by the domain equations:

\[ \int d^2 k_{ij} (\Psi_\lambda - f_{\lambda,ij} \frac{f_{\lambda,ij}}{2k_{ij}^2 + \mu^2}) = 0 \]  

(40)

There is no difficulty in using the Lippmann-Schwinger form of (39) in combination with (40) to derive an equation like (20) for the N-body problem. The full equation is not very illuminating, however, so we do not reproduce it here. Rather we confine ourselves to the negative energy sector and again let \( \lambda = -\eta^2 \). The wavefunctions then have the form:

\[ \Psi_\lambda = \frac{\sum_{i<j} f_{\lambda,ij}}{\sum_{i=1}^N k_i^2 + \eta^2} \]  

(41)

Inserting this form into (40), taking, for example, the integration to be over the relative momentum \( \vec{k}_{12} \), and transforming to dimensionless variables \( \bar{x}_i = \frac{\vec{k}_i}{\eta} \), yields the eigenvalue equation:

\[ \ln\left(1 + \frac{1}{2}(\bar{x}_1 + \bar{x}_2)^2 + \sum_{i=3}^N x_i^2 \right) u_{\lambda,ij} - \frac{2}{\pi} \int d^2 x_{12} \frac{\sum_{i<j} u_{\lambda,ij}}{\sum_{i=1}^N x_i^2 + 1} = \ln\left(\frac{\mu^2}{\eta^2}\right) u_{\lambda,ij} \]  

(42)

where \( u_{\lambda,ij}(\bar{x}_1, ..., \bar{x}_N) = f_{\lambda,ij}(\vec{k}_1, ..., \vec{k}_N) \). One must keep in mind that \( u_{\lambda,ij} \) is independent of \( \frac{1}{2}(\vec{x}_i - \vec{x}_j) \), so that in the center of momentum frame this
function depends on $N-2$ two-dimensional vectors. The linear operator on the left hand side of (42) is the N-body version of the operator $W$. We may, as in the three-body system, identify the renormalized spectator Hamiltonian in the negative energy sector as the exponential of this operator, i.e. $H = -\mu^2 e^{-W}$.

We conclude that the N-body problem can be given a renormalized formulation that is in principle no more complicated than for the three-body problem. Of course it is much more difficult to solve, or even to estimate its ground state and ground state energy. Our most intriguing result, however, that the simplest renormalized formulation is in terms of the logarithm of the Hamiltonian which is an integral operator in momentum space, continues to hold in the negative energy sector of the N-body problem for arbitrary $N$.

We have found, therefore, for all $N$, a renormalized, i.e. finite, formulation of the system of $N$ particles interacting through an attractive Dirac $\delta$-function potential.

An undesirable feature of our formulation is the schism between the treatment of positive and negative energy states. States for all positive values of energy exist, and are found by inverting a linear integral operator in momentum space. Negative energy states, in contrast, come as solutions to an eigenvalue problem. This is the price we pay for combining equations (16) and (15) into a formulation of the problem in which the boundary condition on wavefunctions is built in, and not a supplementary condition.

In a sense, however, all the essential features of the system are included in the negative energy sector. In this sector are contained normalizable bound states, scattering amongst these composite particles, as well as scattering of the composite particles with the elementary excitations of the theory. It is
possible, in fact, that as we allow $N \to \infty$, keeping the ground state energy finite, that it is only this sector which evolves into the resulting field theory. The field theory found this way should admit a finite formulation along the lines presented here, and is a natural direction for future research.
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