Path-Following Methods for Generalized Nash Equilibrium Problems

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Abstract

Building upon the results in [HSK15], generalized Nash equilibrium problems are considered, in which the feasible set of each player is influenced by the decisions of their competitors. This is realized via the existence of one (or more) state constraint(s) establishing a link between the players. Special emphasis is put on the situation of a state encoded in a possibly non-linear operator equation. First order optimality conditions under a constraint qualification are derived. Aiming at a practically meaningful method, an approximation scheme using a penalization technique leading to a sequence of (Nash) equilibrium problems without dependence of the constraint set on the other players’ strategies is established. An associated path-following strategy related to a value function is then proposed. This happens at first on the most abstract level and is subsequently established to a narrower framework geared to the presence of partial differential equations in the constraint. Our findings are illustrated with examples having distributed and boundary controls—both involving semi-linear elliptic PDEs.

1 Introduction

In recent years a growing research effort was focused on the theoretical as well as numerical treatment of (generalized) Nash equilibrium problems (abbr.: (G)NEPs). For problems in finite dimensions a high level of sophistication has been reached there. We refer to the overview articles [PF05, FK07] as well as the contributions [vK09, FFP09, DvKF13] and the references given therein. For GNEPs within a function space setting the level of research is far less complete, but is nonetheless an active working field (cf. [BK13, HS13, HSK15, KKSW19]). In this article we study a generalization of the techniques developed and used in [HSK15] as well as [HS13]. The approach developed therein focussed on a class

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of Nash games with tracking functionals involving a partial differential equation (abbr.: PDE) constraint along with control as well as state constraints. The latter condition establishes an influence of the players on their feasible strategy choices of the players’ strategy sets. The techniques have been developed and successfully applied to a selection of linear PDEs (elliptic, parabolic and hyperbolic). So far, non-linear PDEs have not been considered in this regard. One of the main reasons is the difficulty to establish suitable conditions for the existence of Nash equilibria.

Within the scope of this work we extend the methods in [HSK15] to include a class of GNEPs with possibly non-linear operator equations. We distinguish two different levels of abstraction compared to the setting therein: In the first step we are guided by the typical form discussed in (abstract) optimization as well as in the literature on Nash games and derive a path-following scheme. The obtained results are then applied to a framework similar to the one in [HSK15], specifically tailored to the treatment of (non-)linear PDE constraints.

The remaining sections are organized as follows. In section 2 we introduce notations and important definitions which are vital for the understanding of the remainder of the work. Section 3 is devoted to the introduction of the concept of (generalized) Nash equilibrium problems and the derivation of first order conditions for Nash equilibria in form of Lagrange multiplier systems. In section 4 a penalization and path-following technique is used to derive asymptotic convergence results. Our findings therein are utilized in Section 5 to the specialized framework to facilitate the application to PDE related problems. In Section 6 the results are illustrated by a selection of problems involving semi-linear elliptic PDEs with distributed as well as boundary controls.

2 Notation and Preliminaries

In the following, for a given Banach space $X$ denote by $X^*$ its topological dual space and the associated dual pairing $(\cdot, \cdot)_{X^*, X} : X^* \times X \to \mathbb{R}$ by $(x^*, x)_{X^*, X} := x^*(x)$. Oftentimes we simply denote $(\cdot, \cdot)$, if the corresponding spaces are clear from the context. Two elements $x^* \in X^*$ and $x \in X$ are called orthogonal, if $(x^*, x) = 0$ and we write $x^* \perp x$ or $x \perp x^*$. We write $x_n \to x$ for the strong convergence and $x_n \rightharpoonup x$ to denote the weak convergence.

A subset $C \subseteq X$ is called convex, if for all $t \in (0, 1)$ and $x_0, x_1 \in C$ holds $tx_1 + (1 - t)x_0 \in C$. A set $K \subseteq X$ is called a cone, if for all $t \in \mathbb{R}$, $t \geq 0$ and $x \in K$ also $tx \in K$ holds. The normal cone of a nonempty, convex, closed set $C$ is defined as

$$N_C(x) := \{x^* \in X^* : (x^*, x' - x)_{X^*, X} \leq 0 \text{ for all } x' \in C\}.$$ 

Consider a functional $f : X \to \mathbb{R} \cup \{\pm \infty\}$. It is called lower semi-continuous, if for all sequences $x_n \to x$ in $X$ also

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

holds true and it is called proper, if $f(x) > -\infty$ for all $x \in X$ holds and there exists $x \in X$ with $f(x) < +\infty$. The epigraph of $f$ is defined as the set

$$\text{epi}(f) := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}.$$
The sublevel set of \( f \) with respect to a threshold \( \alpha \in \mathbb{R} \) is defined as
\[
\text{lev}_\alpha f := \{ x \in X : f(x) \leq \alpha \}.
\]
The functional \( f \) is called convex, if for all \( x_0, x_1 \in X \) and \( t \in (0, 1) \) holds
\[
f(tx_1 + (1 - t)x_0) \leq tf(x_1) + (1 - t)f(x_0).
\]
In fact, many of the above defined analytical properties can be characterized geometrically. As proven in [BP12, Proposition 2.5] or [BS00, Proposition 2.5], the lower semi-continuity is equivalent to the closedness of the epigraph as well as to the closedness of all sublevel sets. The convexity of \( f \) is equivalent with the convexity of epi \((f)\) and implies the convexity of \(\text{lev}_\alpha f\) for all \(\alpha \in \mathbb{R}\) (see [BP12, Proposition 2.3], [BV04, Section 3.4.2]).

The closed unit ball of \( X \) is denoted as
\[
\mathbb{B}_X := \{ x \in X : \|x\| \leq 1 \}.
\]
The interior of a set \( M \subseteq X \) is defined as
\[
\text{int}(M) := \{ x \in M : \text{ there exists } \varepsilon > 0 : x + \varepsilon \mathbb{B}_X \subseteq M \}.
\]
For two sets \( M_1, M_2 \subseteq X \) their Minkowski sum is defined by
\[
M_1 + M_2 = \{ x_1 + x_2 : x_i \in M_i \text{ for } i = 1, 2 \}.
\]
For another given Banach space \( Y \), a function \( F : X \to \mathcal{P}(Y) \) is called a set-valued operator or correspondence and is denoted by \( F : X \rightrightarrows Y \). Its graph is defined by
\[
\text{gph}(F) := \{ (x, y) \in X \times Y : y \in F(x) \}
\]
and its domain by
\[
\mathcal{D}(F) := \{ x \in X : F(x) \neq \emptyset \}.
\]
The graph is called closed, if for all sequences \( (x_n, y_n)_{n \in \mathbb{N}} \subseteq \text{gph}(F) \) with \( (x_n, y_n) \to (x, y) \) in \( X \times Y \) also \( (x, y) \in \text{gph}(F) \) follows. It is called weakly closed, if for all sequences \( (x_n, y_n)_{n \in \mathbb{N}} \subseteq \text{gph}(F) \) with \( (x_n, y_n) \rightharpoonup (x, y) \) in \( X \times Y \) also \( (x, y) \in \text{gph}(F) \) follows.

An operator \( A : X \to Y \) is called bounded, if bounded subsets of \( X \) are mapped to bounded subsets of \( Y \). It is called continuous, if for a sequence \( x_n \to x \) in \( X \) the convergence \( A(x_n) \to A(x) \) in \( Y \) holds. The operator is called weakly continuous, if for a sequence \( x_n \rightharpoonup x \) in \( X \) the convergence \( A(x_n) \rightharpoonup A(x) \) in \( Y \) holds. Moreover, the operator \( A \) is called Fréchet differentiable in \( x \in X \), if there exists an operator \( DA(x) \in \mathcal{L}(X, Y) \) such that
\[
\lim_{h \to 0} \frac{\|A(x + h) - A(x) - DA(x)h\|_Y}{\|h\|_X} = 0
\]
holds. If the space \( X \) admits a product structure \( X := \prod_{i=1}^n X_i \) for some \( n \in \mathbb{N} \) and a family of Banach spaces \( (X_i)_{i=1}^N \), then we denote with \( \partial_i A(x) \in \mathcal{L}(X_i, Y) \) the partial derivative of \( A \) in the \( i \)-th component defined by \( DA(x)h := \sum_{i=1}^N \partial_i A(x)h_i \) with \( h = (h_i)_{i=1}^N \in X \).
Let \( d \in \mathbb{N} \setminus \{0\} \) and let \( \Omega \subseteq \mathbb{R}^d \) be a bounded, open domain. For \( p \in [1, \infty) \) denote the Lebesgue space as

\[
L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable : } \int_\Omega |u|^p \, dx < +\infty \right\}
\]

with its elements only identified up to null sets, i.e. sets of Lebesgue measure zero. This space equipped with the norm \( \|u\|_{L^p(\Omega)} := \left( \int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} \) is a Banach space for all \( p \in (1, \infty) \) and a reflexive Banach space for \( p \in (1, \infty) \). The Sobolev spaces \( W^{1,p}(\Omega) \) are defined as

\[
W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \nabla u \in L^p(\Omega; \mathbb{R}^d) \right\},
\]

where \( \nabla u \) denotes the distributional derivative of \( u \). Equipped with the norm \( \|u\|_{W^{1,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^d \|\partial_i u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \), the space \( W^{1,p}(\Omega) \) is a Banach space, and for \( p \in (1, \infty) \) a reflexive Banach space. In case of \( p = 2 \) one also denotes \( H^1(\Omega) = W^{1,2}(\Omega) \).

3 Generalized Nash Equilibrium Problems

After the introduction of notions and concepts from functional and convex analysis we devote our attention to Nash games. In the scope of this section we introduce first the notion of (generalized) Nash equilibrium problems as well as of related equilibrium problems. After the discussion of essential properties and relations between them, we turn our attention to the derivation of necessary first order conditions in form of a Lagrange multiplier systems.

3.1 Definition of (Generalized) Nash Equilibrium Problems and Variational Equilibrium Problems

Let a family of Banach spaces \( U_i \) be given. Define the space \( U := U_1 \times \cdots \times U_N \) as well as the strategy sets \( U_{ad}^i \subseteq U_i \) and the joint strategy set \( U_{ad} := \prod_{i=1}^N U_{ad}^i \) together with a family of real-valued functionals \( J_i : U_{ad} \to \mathbb{R} \) for \( i = 1, \ldots, N \). With the index \( -i \) we denote strategies, where the \( i \)-th component has been omitted. A joint strategy \( (u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_N) \in U \) is written as \( (v_i, u_{-i}) \) — with no change of the ordering.

Definition 1 (cf. [Nas50]). A point \( u \in U_{ad} \) is called a Nash equilibrium, if for all \( i = 1, \ldots, N \) the inclusion

\[
u_i \in \arg\min \left\{ J_i(v_i, u_{-i}) \text{ subject to } v_i \in U_{ad}^i \right\}
\]

holds. The problem of finding such a point is called a Nash equilibrium problem (abbr.: NEP). If, moreover, the sets \( U_{ad}^i \) are convex and the objectives \( v_i \mapsto J_i(v_i, u_{-i}) \) are convex on \( U_{ad}^i \), the NEP is called convex.

Nash equilibrium problems are an important concept to model a competition between entities. Therefore, we also refer to NEPs as Nash games. A possible generalization of this concept with practical relevance is the case, where the strategies of the players influence the set of feasible choices for each of the
other players. This mechanism can be modeled by the introduction of a strategy mapping \( C_i : U_{ad} \rightarrow U_{ad} \) for each player. The combination of all of them \( C : U_{ad} \rightarrow U_{ad} \) with \( C(u) := C_1(u_{-1}) \times \cdots \times C_N(u_{-N}) \) is called the joint strategy mapping. Subsequently, it is possible to generalize the concept given in Definition 1.

**Definition 2.** A point \( u \in U_{ad} \) is called a (generalized) Nash equilibrium, if for all \( i = 1, \ldots, N \) holds

\[ u_i \in \arg\min \{ J_i(v_i, u_{-i}) \text{ subject to } v_i \in C_i(u_{-i}) \} \]

The problem of finding such a point is called a generalized Nash equilibrium problem (abbr.: GNEP). If, moreover, the sets \( C_i(u) \) are convex for all choices \( u \in \text{dom}(C) \) and the objectives \( v_i \mapsto J_i(v_i, u_{-i}) \) are convex on range \( (C_i) \), the GNEP is called convex.

For a better understanding of the problems under investigation, we propose the subsequent example:

**Example 3.** Consider the following generalized Nash equilibrium problem governed by a semi-linear elliptic PDE with distributed control:

\[
\begin{align*}
\text{minimize} & \quad J_i(u_i, u_{-i}) := \frac{1}{2} \int_{\omega_i} (y - y^i)^2 dx + \frac{\alpha}{2} \int_{\omega_i} u_i^2 dx \\
\text{subject to} & \quad a_i \leq u_i \leq b_i \text{ a.e. on } \omega_i, \quad \underline{\psi} \leq y \leq \overline{\psi} \text{ a.e. on } \Omega \\
& \quad -\Delta y + y^3 = \sum_{i=1}^4 B_i u_i \text{ in } \Omega, \quad y = 0 \text{ on } \partial \Omega.
\end{align*}
\]

Here, \( a_i, b_i \in L^2(\Omega) \) and \( \underline{\psi}, \overline{\psi} \in H^2(\Omega) \) with \( \text{tr}_{\partial \Omega}(\underline{\psi}) < 0 < \text{tr}_{\partial \Omega}(\overline{\psi}) \) a.e. on \( \partial \Omega \) are given parameters and \( B_i \) denotes the extension of the strategy \( u_i \) to the whole domain \( \Omega \) by the value zero. In this example, neither the functional, nor the feasible set contain an explicit dependence on the controls of the other players. Instead, the coupling takes place via the state as the solution of the PDE leading to a spatial coupling between the control regions \( \omega_i \).

Often, as demonstrated in Example 3, the players’ decisions are coupled via a single condition that needs to be fulfilled by all players simultaneously. In this case, the strategy mapping can be modeled via the feasible set of all players.

**Definition 4.** A GNEP is called jointly constrained or is said to have shared constraints, if there exists a set \( F \) such that Rosen’s Law (cf. [ACM11, p. 484])

\[ v_i \in C_i(u_{-i}) \iff (v_i, u_{-i}) \in F \]

holds. If, moreover, a GNEP is convex and jointly constrained and the set \( F \) is convex as well, then the GNEP is called jointly convex.

In the jointly constrained case it is worthwhile investigating a modified solution concept based on the previously introduced joint constraint set \( F \).

**Definition 5** (see also [Ros65]). A point \( u \in U_{ad} \) is called a variational equilibrium, if for all \( i = 1, \ldots, N \) it holds that

\[ u \in \arg\min \left\{ \sum_{i=1}^N J_i(v_i, u_{-i}) \text{ subject to } v \in F \right\} . \]
The problem of finding such a point is called a \textit{variational equilibrium problem} (abbr.: VEP).

The relationship between these concepts will be investigated next. For this sake, the concept of the \textit{Nikaido-Isoda functional} (cf. \cite{NI55}) is introduced.

\textbf{Definition 6.} Let a family of functionals $\mathcal{J}_i : U \to \mathbb{R}$ on a Banach space $U$ be given. The \textit{Nikaido–Isoda} functional $\Psi : U \times U \to \mathbb{R}$ is defined as

\[ \Psi(u, v) := \sum_{i=1}^{N} (\mathcal{J}_i(u_i, u_{-i}) - \mathcal{J}_i(v_i, u_{-i})). \]

\textbf{Theorem 7} (compare to \cite[Lemma 3.1]{NI55} and \cite[Theorems 2.1.2 and 2.2.3]{vH09}). Consider the following value functions associated with the Nikaido–Isoda functional:

\[ V(u) = \sup_{v \in C(u)} \Psi(u, v), \text{ and } \hat{V}(u) = \sup_{v \in \mathcal{F}} \Psi(u, v), \]

\textit{corresponding to the} GNEP \textit{respectively} VEP. \textit{Then} $V(u) = 0$ \textit{holds, if and only if} $u$ \textit{is a Nash equilibrium}, and $\hat{V}(u) = 0$, \textit{if and only if} $u$ \textit{is a variational equilibrium}.

\textit{Proof.} For the proof we refer to references given in Theorem 7 to \cite[Theorem 2]{HS21} in combination with the discussion given in Subsection 2.2 therein.

From the definition of the value functions $V$ and $\hat{V}$ one sees that the condition $v \in C(u)$ is substituted by the feasible set and hence is restricted to the fixed point set of $C$. The relationship between these two solution concepts is discussed in the following theorem.

\textbf{Theorem 8.} \textit{In a jointly constrained} GNEP \textit{every variational equilibrium is a} Nash equilibrium.

\textit{Proof.} Let $u \in \mathcal{F}$ be a variational equilibrium. Then $u \in C(u)$, and selecting an arbitrary $v_i \in C_i(u_{-i})$ and setting $\tilde{v} = (v_i, u_{-i}) \in \mathcal{F}$ yields

\[ \mathcal{J}_i(u_i, u_{-i}) - \mathcal{J}_i(v_i, u_{-i}) = \Psi(u, \tilde{v}) \leq \hat{V}(u) = 0 \]

and hence $\mathcal{J}_i(u_i, u_{-i}) \leq \mathcal{J}_i(v_i, u_{-i})$.

Within the scope of this work we do not address existence of equilibria explicitly. We just refer to \cite{Dut13} for the finite dimensional case and to \cite{HS21}, where this issue is discussed. It is just mentioned here, that the presence of (quasi-)convexity in the objective plays a crucial role for providing existence. In the following, we do not only want to discuss the Nash equilibrium problem, but also want to discuss the variational equilibrium problem in the presence of shared constraints. Hence, we discuss both of them in parallel in the scope of our investigation. As the proofs turn out to be very similar, we proceed most of the time with the statement formulated for both cases, but only proven for the Nash equilibrium problem.
3.2 First Order Conditions

After setting up the basic definitions we proceed with the characterization of solutions. For this sake the optimization based structure is used to derive first order optimality conditions in the form of a Lagrange-multiplier system for each player. The combined system serves as a first order necessary condition for a point to be a Nash equilibrium. Therefore we want to specify the framework used in the sections and propose the following set of assumptions.

Assumption 9.

(i) Let the strategy mapping \( C : U_{ad} \Rightarrow U_{ad} \) admit the form

\[
C_i(u_{-i}) = \{ u_i \in U_{ad}^i : g_i(u_i, u_{-i}) \in K_i \}
\]

involving a family of Fréchet differentiable mappings \( g_i : U \rightarrow X_i \) together with \( X_i \) a Banach space and \( K_i \subseteq X_i \) a non-empty, closed, convex cone.

(ii) Let in the above Assumption (i) the mappings \( g_i \) coincide with a Fréchet differentiable \( g : U \rightarrow X \) and let the constraint sets coincide with a \( K \subseteq X \). The joint constraint set of the resulting jointly constrained Nash game is denoted as

\[
F := \{ u \in U_{ad} : g(u) \in K \}.
\]

The part Assumption 9(i) corresponds solely to the case of a jointly constrained GNEP. In view of Nash games involving partial differential equations in the sense of Example 3, the operator \( g \) can be identified with a composition \( g = G \circ S \), where \( S \) denotes the solution mapping of an operator equation and \( G \) is another mapping establishing the state constraint in terms of a cone condition, see also Section 6. Having this consideration in mind, we will in the following refer to the conditions \( g_i(u) \in K_i \) respectively \( g(u) \in K \) as state constraint(s). Next, we derive a first order necessary condition for Nash equilibria.

Theorem 10. Consider a Nash equilibrium \( u \in U_{ad} \). Let Assumption 9(ii) hold and assume the objectives \( v_i \mapsto J_i(v_i, u_{-i}) \) to be Fréchet differentiable. Moreover, let the following (RZK)-condition

\[
\partial_i g_i(u) U_{ad}^i(u_i) - K_i(g_i(u)) = X_i \quad \text{for all } i = 1, \ldots, N \tag{RZK_Nash}
\]

hold. Then, there exist Lagrange multipliers \( \mu_i \in X_i^* \), such that the complementarity system

\[
\begin{align*}
0 = q_i - \partial_i g_i(u)^* \mu_i + \lambda_i & \quad \text{in } U_i^*, \quad \text{(1.1)} \\
q_i = \partial_i J_i(u) & \quad \text{in } U_i^*, \quad \text{(1.2)} \\
K_i^+ \ni \mu_i \perp g_i(u) & \subseteq K_i, \quad \text{(1.3)} \\
\lambda_i \in N_{U_{ad}^i}(u_i) & \quad \text{in } U_i^* \quad \text{(1.4)}
\end{align*}
\]

is fulfilled.
Proof. In essence we utilize [ZK79]. Since $u \in U_{ad}$ is a Nash equilibrium every player solves the minimization problem

$$\text{minimize } J_i(v_i, u_{-i}) \text{ over } v_i \in U_{ad}^i,$$

subject to \( g_i(v_i, u_{-i}) \in K_i \).

The strategy $u_i$ is the minimizer of this problem. The constraint qualification reads as $\partial_i g_i(u) U_{ad}^i(u_i) - K_i(g_i(u)) = X_i$ and leads by [ZK79, Theorem 4.1] to the existence of a Lagrange multiplier $\mu_i \in K_i^+$ such that the system

$$0 = \langle \mu_i, g_i(u) \rangle,$$
$$0 \in \partial_i J_i(u_i, u_{-i}) - \partial_i g_i(u)^* \mu_i + N_{U_{ad}^i}(u_i)$$

is fulfilled. Combining the collective first order systems of each player together with defining $q_i = \partial_i J_i(u)$ and setting $\lambda_i = \partial_i g_i(u)^* \mu_i - q_i \in N_{U_{ad}^i}(u_i)$ leads to (1).

The discussion of the first order system for variational equilibria follows the same pattern and is addressed in the following theorem.

**Theorem 11.** Consider a variational equilibrium $u \in U_{ad}$. Let Assumption 9(ii) hold and assume the objectives $v_i \mapsto J_i(v_i, u_{-i})$ to be Fréchet differentiable. Moreover, let the following (RZK)-condition

$$Dg(u) U_{ad}(u) - K(g(u)) = X \quad (\text{RZK}_{\text{Var}})$$

hold. Then, there exists a Lagrange multiplier $\mu \in X^*$, such that the complementarity system

$$0 = q_i - \partial_i g_i(u)^* \mu + \lambda_i \text{ in } U_i^*, \quad (3.1)$$
$$q_i = \partial_i J_i(u) \text{ in } U_i^*, \quad (3.2)$$
$$K^+ \ni \mu \perp g(u) \in K, \quad (3.3)$$
$$\lambda_i \in N_{U_{ad}^i}(u_i) \text{ in } U_i^* \quad (3.4)$$

is fulfilled.

Proof. The proof is essentially the same as the one of Theorem 10 and therefore omitted.

To deepen our understanding of the interconnection between Nash and variational equilibria, we compare the two systems derived in Theorem 10 and Theorem 11. Instead of one multiplier for each player in (1), we only have a single multiplier for all players arising form the joint state constraint.

**Remark 12.** For the jointly constrained case it holds: $(\text{RZK}_{\text{Nash}}) \Rightarrow (\text{RZK}_{\text{Var}})$. Let $u \in U_{ad}$ be a Nash equilibrium and assume $(\text{RZK}_{\text{Nash}})$ to be satisfied. Choose $x \in X$ arbitrarily. By assumption there exist $\alpha_i, \beta_i \geq 0$ together with $v_i \in U_{ad}^i$ and $k_i \in K$ such that

$$\alpha_i \partial_i g(u)(v_i - u_i) - (k_i - \beta_i g(u)) = x \text{ for all } i = 1, \ldots, N.$$
Without loss of generality we assume $\alpha_i > 0$. Otherwise we can as well write
\[ \partial_i g(u)(u_i - u_i) - (k_i - \beta_i g(u)) = x. \]
Thus the above equation is fulfilled with $v_i = u_i$ and $\alpha_i = 1$.

Set $\alpha := \left( \sum_{i=1}^N \frac{1}{\alpha_i} \right)^{-1}$ and multiply each equation with $\alpha_i^{-1}$. By subsequent addition we obtain
\[ \sum_{i=1}^N \partial_i g(u)\alpha(v_i - u_i) - \left( \sum_{i=1}^N k_i - \left( \sum_{i=1}^N \frac{\alpha_i \beta_i}{\alpha_i} \right) g(u) \right) = \left( \sum_{i=1}^N \frac{\alpha_i}{\alpha_i} \right) x = x. \]

Since $K$ is a convex cone, it holds $k := \sum_{i=1}^N k_i \in K$ and with $\beta := \sum_{i=1}^N \frac{\alpha_i \beta_i}{\alpha_i} > 0$ we get
\[ \alpha Dg(u)(v - u) - (k - \beta g(u)) = x. \]
Thus, (RZKVar) is fulfilled.

This observation tells us, that we have traded a stronger solution concept with a potentially weaker constraint qualification. A similar observation has been made in [HSK15] for the Slater condition proposed therein.

## 4 Penalization and Path-Following Technique

In the previous section we introduced and discussed the Nash equilibrium problem. One of the major analytic challenges is the presence of the other players decision in the constraint set. This has been partially addressed by the introduction of the variational equilibrium problem in the jointly constrained case. In view of numerical calculations however one may be interested in performing an approximation. For this sake, we relax the state constraint using a penalty functional and leave the control constraint realized by $U_{i}^{ad}$ unchanged. Again, we will derive first order necessary conditions. Subsequently, a convergence analysis not only regarding the solution of the penalized problems but also the multipliers occurring in the first order system is provided together with a practically applicable update strategy for the underlying penalty parameter.

### 4.1 Definition of Penalized Equilibrium Problems

Returning to the aforementioned dependence of the constraint set, the idea behind a penalty approach is the relaxation of the constraint and shifting the dependence into the objectives. For this sake, we use the approach derived in [HSK15] and mentioned in [HS21, Sections 3.1 and 3.2] and introduce a penalty functional. Therefore, we introduce the following two problems:

**Definition 13.**

(i) Let Assumption [HSK15] hold and let convex, lower semi-continuous penalty functionals $\beta_i : X_i \to [0, +\infty)$ with $\beta_i(x_i) = 0$, if and only if $x_i \in K_i$ be given together with a penalty parameter $\gamma > 0$. Define the penalized Nash equilibrium problem (GNEP$_\gamma$) as
\[ u_i \in \arg\min \{ J_i(v_i, u_{-i}) + \gamma \beta_i(g_i(v_i, u_{-i})) \} \] subject to $v_i \in U_{i}^{ad}$
for all $i = 1, \ldots, N$. 

(ii) Let Assumption[3][ii] hold and let a convex, lower semi-continuous penalty functional $\beta : X \to [0, +\infty)$ with $\beta(x) = 0$, if and only if $x \in K$ be given together with a penalty parameter $\gamma > 0$. Define the penalized variational equilibrium problem $\text{VEP}_\gamma$ as

$$u \in \arg\min \left\{ \sum_{i=1}^{N} J_i(v_i, u_{-i}) + \gamma \beta(g(v)) \right\} \text{ subject to } v \in U_{ad}.$$  

In the light of [HS21], we defined the penalty functionals $\pi_C(v, u) := \sum_{i=1}^{N} \beta_i(g_i(v, u_{-i}))$ and $\pi_F(v) := \beta(g(v))$ using the notation given therein. To meet the conditions for the $\Gamma$-convergence discussed therein, it is sufficient to assume the weak continuity of $v_i \mapsto g_i(v, u_{-i})$ and $g$, respectively and use the convexity of $\beta_i$, respectively $\beta$ to obtain the weak lower semi-continuity and subsequent of its composition.

### 4.2 First Order Conditions for the Penalized Equilibrium Problems

One way to achieve the convergence of a sequence of penalized equilibrium problems has been addressed in [HS21]. Therein, $\Gamma$-convergence has been employed to achieve this goal. Within the scope of this work however, we seek to obtain a more detailed view and establish our convergence theory via the first order system. Hence, of particular interest is the approximation of the multipliers in (1) (respectively (3)). We proceed by the derivation of the first order systems for the penalized problems.

**Theorem 14.** Let Assumption[3][ii] hold and let the objectives $U_{ad} \ni v_i \mapsto J_i(v_i, u_{-i}) \in \mathbb{R}$ as well as the penalty mappings $\beta_i : X_i \to [0, +\infty)$ and the operators $U_i \ni u_i \mapsto g_i(u_i, u_{-i}) \in X_i$ be Fréchet differentiable in their respective spaces.

Then, every Nash equilibrium $u^\gamma \in U_{ad}$ of $\text{GNEP}_\gamma$ fulfills the following first order system

$$0 = q^\gamma_i - \partial_i g_i(u^\gamma)^* \mu^\gamma_i + \lambda^\gamma_i \in U_i^*,$$

$$q^\gamma_i = \partial_i J_i(u^\gamma) \in U_i^*,$$

$$\mu^\gamma_i = -\gamma D \beta_i(g_i(u^\gamma)) \in X_i^*,$$

$$\lambda^\gamma_i \in N_{U_{ad}}(u^\gamma_i) \text{ in } U_i^* \quad (4.4)$$

for all $i = 1, \ldots, N$.

**Proof.** Let $u^\gamma \in U_{ad}$ be a Nash equilibrium of $\text{GNEP}_\gamma$. Then, by Definition[3][ii] $u^\gamma_i$ is a solution of

$$u^\gamma_i \in \arg\min_{u_i \in U_i^*} \left\{ J_i(v_i, u_{-i}^\gamma) + \gamma \beta_i(g_i(v, u_{-i}^\gamma)) \right\}.$$  

Using the Fréchet differentiability this yields the following first order condition

$$0 \in \partial \cdot J_i(u^\gamma_i, u_{-i}^\gamma) + \gamma \partial_i g_i(u^\gamma)^* D \beta_i(g_i(u^\gamma)) + N_{U_{ad}}(u^\gamma_i) \text{ in } U_i^*.$$  

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Setting $q_i^γ := \partial_i J_i(u_i^γ, u_{-i}^γ)$ and $μ_i^γ = -γDβ_i(g_i(u^γ))$ there exists a $λ_i^γ ∈ N_{U_i^*} (u_i^γ)$ with

$$0 = q_i^γ - \partial_i g_i(u^γ)^∗μ_i^γ + λ_i^γ \text{ in } U_i^*$$

proving the assertion.

Comparing the system in Theorem 14 with the one given in Theorem 10, we observe that the role of the multiplier for the state constraint is (up to a sign) fulfilled by the derivative of the penalty function scaled by the penalty parameter. Next, we discuss the first order system for the penalized variational equilibrium problem.

**Theorem 15.** Let Assumption K(ii) hold and let the objectives $U_i^* ∈ v_i ↦ J_i(v_i, u_{-i}) ∈ ℝ$ as well as the penalty mapping $β : X → [0, +∞)$ and the operator $U ⊃ u ↦ g(u_i, u_{-i}) ∈ X$ be Fréchet differentiable.

Then, every equilibrium $u^γ ∈ U_∗$ of (VEPγ) fulfills the following first order system

$$0 = q_i^γ - \partial_i g_i(u^γ)^∗μ_i^γ + λ_i^γ \text{ in } U_i^*,$$  \hspace{1cm} (5.1)

$$q_i^γ = \partial_i J_i(u^γ) \text{ in } U_i^*,$$  \hspace{1cm} (5.2)

$$μ_i^γ = -γDβ_i(g(u^γ)) \text{ in } X^*,$$  \hspace{1cm} (5.3)

$$λ_i^γ ∈ N_{U_i^*} (u_i^γ) \text{ in } U_i^*,$$  \hspace{1cm} (5.4)

for all $i = 1, \ldots, N$.

**Proof.** Let $u^γ ∈ U_∗$ be an equilibrium of (VEPγ) (see Definition K(ii)), which means $u^γ$ fulfills

$$u^γ ∈ \arg\min_{u ∈ U_∗} \left\{ \sum_{i=1}^N J_i(v_i, u_{-i}) + γβ(g(v)) \right\}.$$

Using the Fréchet differentiability in combination with the product structure of the space $U$ and the set $U_∗$ yields the following first order condition

$$0 ∈ D \left( \sum_{i=1}^N J_i((•), u_{-i}) + γβ(•) \right) (u^γ) + N_{U_∗} (u^γ)$$

$$= (\partial_1 J_i(u^γ), \ldots, ∂N J_N(u^γ)) + γDg(u^γ)^∗Dβ(g(u^γ)) + N_{U_∗} (u^γ)$$

$$= \prod_{i=1}^N (\partial_i J_i(u^γ) + γ∂_i g(u^γ)^∗Dβ(g(u^γ)) + N_{U_i^*} (u_i^γ)) \text{ in } U_∗.$$  \hspace{1cm} (5.5)

Setting $q_i^γ := \partial_i J_i(u_i, u_{-i})$ with $μ_i^γ := -γDβ_i(g(u^γ))$ there exist $λ_i^γ ∈ N_{U_i^*} (u_i^γ)$ with

$$0 = q_i^γ - \partial_i g_i(u^γ)^∗μ_i^γ + λ_i^γ \text{ in } U_i^*$$

proving the assertion.

We observe that both—the penalized, jointly constrained Nash game and the penalized variational equilibrium problem—yield the same first order system. Hence, one might ask as well for the relation between the two concepts. To gain insight into this question we provide in analogy to Theorem 8 the following result.
Theorem 16. Let Assumption 9(ii) hold. Then every equilibrium of \((\text{VEP}_{\gamma})\) is an equilibrium of \((\text{GNEP}_{\gamma})\).

Proof. Let \(u^? \in U_{ad}\) be a variational equilibrium of \((\text{GNEP}_{\gamma})\). Take \(j \in \{1, \ldots, N\}\) and \(v_j \in U_{ad}^j\) arbitrarily and set \(\tilde{v} = (v_j, u_{-j}^\gamma)\). Then, we deduce

\[
\sum_{i=1}^{N} J_i(u_{i}^\gamma, u_{-i}^\gamma) + \gamma \beta(g(u^?)) \leq \sum_{i=1}^{N} J_i(v_i, u_{-i}^\gamma) + \gamma \beta(g(\tilde{v}))
\]

and hence

\[
J_j(u_j^\gamma, u_{-j}^\gamma) + \gamma \beta(g(u_j^\gamma, u_{-j}^\gamma)) \leq J_j(v_j, u_{-j}^\gamma) + \gamma \beta(g(v_j, u_{-j}^\gamma))
\]

proving \(u^?\) to be a Nash equilibrium for \((\text{GNEP}_{\gamma})\) since the choice of \(j\) was arbitrary. \(\square\)

In fact, in the case of a convex game with \(g\) being a concave vector-valued operator as in [HS20] and \(\beta\) being isotone, we obtain the convexity of the optimization problem and hence the first order system being a sufficient condition.

4.3 Asymptotic Behaviour of the Primal-Dual Path

After the derivation of the first order systems of the penalized equilibrium problems we investigate the behaviour of equilibria and multipliers as the penalty parameter \(\gamma\) goes to infinity. The naive expectation is the convergence of the multipliers in (4) and (5) towards the corresponding ones in (1) and (3) respectively. In fact, \(\Gamma\)-convergence of the equilibrium problems has been confirmed in [HS21] but this does not show the behavior of the multipliers yet. For the investigation of the latter we propose the following definition first.

Definition 17. Define the set-valued solution operator

\[
S : (0, \infty) \supseteq U \times U^* \times U^* \times \prod_{i=1}^{N} X_i, \gamma \mapsto (u^\gamma, q^\gamma, \lambda^\gamma, \mu^\gamma)
\]

by mapping \(\gamma\) to the set of solutions of (4). The set of primal-dual paths \(\mathcal{P}\) is defined as

\[
\mathcal{P} := \left\{((u^\gamma, q^\gamma, \lambda^\gamma, \mu^\gamma))_{\gamma>0} : (u^\gamma, q^\gamma, \lambda^\gamma, \mu^\gamma) \in S(\gamma)\right\}
\]

and every of its elements is called a (primal-dual) path for (4).

Define the set-valued solution operator

\[
\hat{S} : (0, \infty) \supseteq U \times U^* \times U^* \times X, \gamma \mapsto (u^\gamma, q^\gamma, \lambda^\gamma, \mu^\gamma)
\]

by mapping \(\gamma\) to the set of solutions of (5). Its set of primal-dual paths \(\hat{\mathcal{P}}\) is defined as

\[
\hat{\mathcal{P}} := \left\{((u^\gamma, q^\gamma, \lambda^\gamma, \mu^\gamma))_{\gamma>0} : (u^\gamma, q^\gamma, \lambda^\gamma, \mu^\gamma) \in \hat{S}(\gamma)\right\}.
\]

Next, we are investigating the asymptotic behaviour of the multipliers.
Lemma 18. Under the assumptions of Theorem 14 (respectively Theorem 15) assume additionally that for all \(i = 1, \ldots, N\) the mappings \(u \mapsto \partial_i \mathcal{J}(u_i, u_{-i})\) and \(u \mapsto \partial_i g_i(u)\) (respectively \(u \mapsto Dg(u)\)) are bounded (i.e.: images of bounded sets are bounded).

Moreover, let the following uniform Robinson-type condition hold:
There exists \(\varepsilon > 0\) such that for all \(u^\gamma\) in the path it holds that
\[
\varepsilon B_X \subseteq \partial_i g_i(u^\gamma)(U_{ad} - u^\gamma_i) - (K_i - g_i(u^\gamma)) \quad \text{for all } i = 1, \ldots, N \tag{6}
\]
(respectively \(\varepsilon B_X \subseteq Dg(u^\gamma)(U_{ad} - u^\gamma) - (K - g(u^\gamma))\)). \tag{7}

Then, the path \(\mathcal{P}\) (respectively \(\mathcal{P}'\)) is bounded.

Proof. (similar to [HSKL13], inspired by [HK06b])
The boundedness of the sets \(U_{ad}^\gamma\) for all \(i = 1, \ldots, N\) yields \(\|u^\gamma\|_U \leq C_U\) for some constant \(C_U > 0\). The assumed boundedness of the operators \(\partial_i \mathcal{J}_i\) yields the boundedness of \(q_i^\gamma\) for all \(i = 1, \ldots, N\). By the convexity of the penalty function \(\beta_i\) and the set \(K_i\) we obtain for all \(z_i \in K_i\), that
\[
0 = \beta_i(z_i) \geq \beta_i(g_i(u^\gamma)) + \langle D\beta_i(g_i(u^\gamma)), z_i - g_i(u^\gamma) \rangle \geq -\langle \mu^\gamma_i, z_i - g_i(u^\gamma) \rangle.
\]

To prove the boundedness of \(\mu^\gamma_i\) we utilize condition (6). Hence for all \(\gamma > 0\) and \(x_i \in X_i\) with \(\|x_i\|_{X_i} \leq \varepsilon\) there exist \(v_i^\gamma \in U_{ad}^\gamma\) as well as \(k_i^\gamma \in K_i\) fulfilling
\[
x_i = \partial_i g_i(u^\gamma)(v_i^\gamma - u_i^\gamma) - (k_i^\gamma - g_i(u^\gamma)).
\]

Applying \(\mu^\gamma_i\) yields
\[
\langle \mu^\gamma_i, x_i \rangle_{X_i^*, X_i} = \langle \mu^\gamma_i, \partial_i g_i(u^\gamma)(v_i^\gamma - u_i^\gamma) - (k_i^\gamma - g_i(u^\gamma)) \rangle_{X_i^*, X_i}
= \langle \partial_i g_i(u^\gamma)^* \mu^\gamma_i, v_i^\gamma - u_i^\gamma \rangle_{U^*_i, U_i} - \langle \mu^\gamma_i, k_i^\gamma - g_i(u^\gamma) \rangle_{X_i^*, X_i}
\leq \langle q_i^\gamma + \lambda_i^\gamma, v_i^\gamma - u_i^\gamma \rangle_{U^*_i, U_i} \leq \langle q_i^\gamma, v_i^\gamma - u_i^\gamma \rangle_{U^*_i, U_i}.
\]

The last expression is bounded by some constant \(C\) due to the boundedness of \(U_{ad}\) and the boundedness of \(q_i^\gamma\). Hence, we obtain
\[
\|\mu^\gamma_i\|_{X_i^*} \leq \frac{1}{\varepsilon} \sup_{\|x_i\|_{X_i} \leq \varepsilon} \langle \mu^\gamma_i, x_i \rangle_{X_i^*, X_i} \leq \frac{C}{\varepsilon}
\]
and conclusively also \(\lambda_i^\gamma = -\langle q_i^\gamma - \partial_i g_i(u^\gamma)^* \mu^\gamma_i \rangle\) is bounded. \(\square\)

Establishing a uniform Robinson condition implies the RZK condition for each \(\gamma\), but it is not clear, whether the opposite holds true, because [ZK79, Theorem 2.1] just guarantees the existence of a ball, but does not specify its size. Hence, the radius might vanish in the limit as the penalty parameter goes to infinity.

Remark 19. In analogy to Remark 12 we observe that also for the uniform Robinson-type conditions in Lemma 18 in the jointly constrained case it holds that (6) implies (7):
Since \(U_{ad}^\gamma\) is assumed to be convex, for all \(\gamma > 0\) and \(\lambda \geq 1\)
\[
\partial_i g_i(u^\gamma)(U_{ad} - u_i^\gamma) \subseteq \lambda \partial_i g_i(u^\gamma)(U_{ad}^\gamma - u_i^\gamma).
\]
Hence, we deduce with $\lambda = N$ the inclusion

$$\varepsilon B_X \subseteq \partial_i g(u^\gamma)(U^i_\text{ad} - u^i_\gamma) - (K - g(u^\gamma)) \subseteq N\partial_i g(u^\gamma)(U^i_\text{ad} - u^i_\gamma) - (K - g(u^\gamma))$$

for all $i = 1, \ldots, N$. Taking an arbitrary $x \in B_X$ yields hence the existence of $v^i_\gamma \in U^i_\text{ad}$ and $k^i_\gamma \in K$ such that

$$\varepsilon x = N\partial_i g(u^\gamma)(v^i_\gamma - u^i_\gamma) - (k^i_\gamma - g(u^\gamma))$$

for all $i = 1, \ldots, N$.

Summing over $i = 1, \ldots, N$ and dividing by $N$ yields

$$\varepsilon x = \frac{1}{N} \sum_{i=1}^N \partial_i g(u^\gamma)(v^i_\gamma - u^i_\gamma) - \left( \frac{1}{N} \sum_{i=1}^N k^i_\gamma - g(u^\gamma) \right)$$

with $k^\gamma := \frac{1}{N} \sum_{i=1}^N k^i_\gamma \in K$. Hence, we obtain the uniform Robinson-type condition.

As the non-linearity of the operator $g$ and its derivative play a role in (4) and (5) for the behaviour of the multipliers we need to discuss the influence of weak-to-weak as well as complete continuity on the first derivative. Therefore, we introduce the concept of uniform Fréchet differentiability.

**Definition 20.** (see [Lla86, Definition 5.1.2].) Let $X, Y$ be Banach spaces. A Fréchet differentiable operator $T : X \to Y$ is called uniformly Fréchet differentiable on a set $M \subseteq X$, if

$$\lim_{\|h\|_X \to 0, x \in M} \sup_{\|x\|_X} \frac{\|T(x + h) - T(x) - DT(x)h\|_Y}{\|h\|_X} = 0$$

holds.

As preparation for the upcoming convergence result in Theorem 26 we provide the following Lemmata containing properties of uniformly Fréchet differentiable. The proofs are given in the appendix.

**Lemma 21.** Let $X, Y$ be Banach spaces and let furthermore $X$ be reflexive.

(i) Let $T : X \to Y$ be completely continuous and uniformly Fréchet differentiable on every bounded set. Then, for weakly convergent sequences $x_n \to x, h_n \to h$ in $X$ and weakly* convergent sequence $y_n \to^* y^*$ in $Y^*$ it holds that

$$DT(x_n)h_n \to DT(x)h \text{ in } Y \text{ and } DT(x_n)^*y^*_n \to DT(x)^*y^* \text{ in } X^*.$$ 

(ii) Let $T : X \to Y$ be weakly continuous and uniformly Fréchet differentiable on every bounded set. Then, for weakly convergent sequences $x_n \to x, h_n \to h$ in $X$ and weakly* convergent sequence $y_n \to^* y^*$ in $Y^*$ it holds that

$$DT(x_n)h_n \to DT(x)h \text{ in } Y \text{ and } DT(x_n)^*y^*_n \to^* DT(x)^*y^* \text{ in } X^*.$$
In view of the previous result let us point out that in the case of uniform Fréchet differentiability over every bounded set, continuity properties of the operator itself are inherited by its derivative as well as the pointwise dual derivative.

**Lemma 22.** Let $X,Y$ be Banach spaces with $X$ reflexive. Consider a weakly continuous and Fréchet differentiable operator $T : X \to Y$. Then, the first derivative $DT : X \to \mathcal{L}(X,Y)$ is a bounded operator.

Next, further properties of uniform Fréchet differentiable operators are discussed. Of interest is the discussion of properties of inverse operators as well as compositions.

**Lemma 23.** Let $T : X \to Y$ be uniformly Fréchet differentiable on every bounded subset as well as bijective, and let its inverse $S : Y \to X$ be continuously Fréchet differentiable. Moreover assume, that $S$ and the mapping $x \mapsto DT(x)^{-1}$ are bounded operators. Then, $S$ is uniformly Fréchet differentiable on every bounded subset of $Y$.

The corresponding result for compositions is addressed in the following lemma.

**Lemma 24.** Let $T_1 : X \to Y$ and $T_2 : Y \to Z$ be uniformly Fréchet differentiable on every bounded subset and let $T_1$ and $DT_1 : X \to \mathcal{L}(X,Y)$ as well as $DT_2 : Y \to \mathcal{L}(Y,Z)$ be bounded operators as well. Then, the composition $T_2 \circ T_1$ is uniformly Fréchet differentiable on every bounded subset of $Y$.

As an application of uniform Fréchet differentiability we discuss the relationship between the uniform RZK-condition \( \square \) and \( \text{(RZK}_{\text{Nash}}) \).

**Proposition 25.** Let a weakly convergent sequence $u^\gamma \to u$ in $U$ for $\gamma \to +\infty$ with $u^\gamma \in U_{\text{ad}}$ be given and assume $g_i : U_{\text{ad}} \to X_i$ to be weakly continuous and uniformly Fréchet differentiable on $U_{\text{ad}}$. Moreover assume, that the uniform Robinson-type condition \( \square \) is fulfilled. If $g_i(u) \in K_i$, then \( \text{(RZK}_{\text{Nash}}) \) is fulfilled.

**Proof.** Since $U_{\text{ad}}$ is a closed, convex subset it holds $u \in U_{\text{ad}}$. Taking an arbitrary $x_i \in X_i$ with $\|x_i\|_{X_i} \leq \varepsilon$. By \( \square \) there exist for all $\gamma > 0$ elements $v_i^\gamma \in U_{\text{ad}}^i$ and $k_i^\gamma \in K_i$ with

\[
x_i = \partial_i g_i(u^\gamma)(v_i^\gamma - u_i^\gamma) - (k_i^\gamma - g_i(u^\gamma)).
\]

Since $(v_i^\gamma)_{\gamma > 0} \subseteq U_{\text{ad}}^i$ is bounded and $U$ is reflexive, there exists a subsequence (not relabelled) with $v_i^{\gamma_k} \to v_i$ in $U_i$ and $v \in U_{\text{ad}}^i$. Then, we obtain

\[
k_i^\gamma = g_i(u^\gamma) - x_i - \partial_i g_i(u^\gamma)(v_i^\gamma - u_i^\gamma) \to g_i(u) - x_i - \partial_i g_i(u)(v_i - u_i) =: k_i \in K_i
\]

by the use of Lemma 22 ii as well as the convexity and closedness of $K_i$. Hence, we have found $v \in U_{\text{ad}}^i$ and $k_i \in K_i$ with

\[
x_i = \partial_i g_i(u)(v_i - u_i) - (k_i - g_i(u)),
\]

which implies $0 \in \text{int}(\partial_i g_i(u)(v_i - u_i) - (K_i - g_i(u)))$. Since by assumption $g_i(u) \in K_i$ holds true we conclude by \( \text{[ZK79]} \) see Equations (3.2)–(3.4) the relation \( \text{(RZK}_{\text{Nash}}) \).
We are now ready to return to Nash games and prove a convergence theorem for the first order condition.

**Theorem 26.** Let the conditions of Lemma 18 be fulfilled. Moreover, assume:

(i) For $i = 1, \ldots, N$, the operator $u \mapsto \partial_i J_i(u)$ has a weakly-weak* closed graph, (i.e. for all sequences $u^n \rightrightarrows u$ and $q_i^n := \partial_i J_i(u^n) \rightharpoonup^* q_i$ it holds that $q_i = \partial_i J_i(u)$). Let furthermore

$$
\langle \partial_i J_i(u), u_i \rangle_{U_i^*} \leq \limsup_{n \to \infty} \langle \partial_i J_i(u^n), u_i^n \rangle_{U_i^*}
$$

be fulfilled.

(ii) The mappings $g_i : U \to X_i$ are weakly continuous for all $i = 1, \ldots, N$, and for all sequences $u^n \rightrightarrows u$ in $U$ and $\mu_i^n \rightharpoonup^* \mu_i$ in $X_i^*$ it holds that

$$
\partial_i g_i(u^n)^* \mu_i^n \rightharpoonup \partial_i g_i(u)^* \mu_i \text{ in } U_i^*
$$

(respectively $g : U \to X$ is weakly continuous and for all sequences $u^n \rightrightarrows u$ in $U$ and $\mu^n \rightharpoonup^* \mu$ in $X^*$ it holds that $Dg(u^n)^* \mu^n \rightharpoonup Dg(u)^* \mu$ in $X^*$).

Then, every path has a limit point $(u, q, \lambda, \mu)$ and every such limiting point fulfils the first order system (14) in Theorem 14 for a GNEP (resp. the first order system (3) in Theorem 14 for a VEP).

**Proof.** Since the path induced by $\gamma_n \nearrow +\infty$ is bounded, there exist weakly convergent subsequences with

$$
u_i^{\gamma_n} \rightharpoonup u_i \text{ in } U_i, \quad q_i^{\gamma_n} \rightharpoonup q_i \text{ in } U_i^*, \quad \mu_i^{\gamma_n} \rightharpoonup^* \mu_i \text{ in } X_i^*, \quad \lambda_i^{\gamma_n} \rightharpoonup \lambda_i \text{ in } U_i^*
$$

due to the reflexivity of $U_i$ and the Banach-Alaoglu theorem for $X_i^*$. Due to the weak-weak* closedness of the graph of $\partial_i J_i$, we conclude by $(u^{\gamma_n}, q^{\gamma_n}) \in \text{gph}(\partial_i J_i)$ that $q_i \in \partial_i J_i(u)$. Applying the assumption of this theorem we deduce

$$
\partial_i g_i(u^{\gamma_n})^* \mu_i^{\gamma_n} \rightharpoonup \partial_i g_i(u)^* \mu_i \text{ in } U_i^*.
$$

For arbitrary $v_i \in U_i^{ad}$ we obtain

$$
\langle -\lambda_i, u_i \rangle_{U_i^*} u_i = \langle q_i - \partial_i g_i(u)^* \mu_i, u_i \rangle_{U_i^*} \leq \limsup_{n \to \infty} \langle q_i^{\gamma_n} - \partial_i g_i(u^{\gamma_n})^* \mu_i^{\gamma_n}, u_i^{\gamma_n} \rangle_{U_i^*} \leq \limsup_{n \to \infty} \langle -\lambda_i^{\gamma_n}, v_i \rangle_{U_i^*} = \langle q_i - \partial_i g_i(u)^* \mu_i, v_i \rangle_{U_i^*} \leq \langle -\lambda_i, v_i \rangle_{U_i^*} v_i,
$$

and hence $\lambda_i \in N_{U_i^{ad}}(u_i)$. Finally, we show $\mu_i \in N_{K_i}(g_i(u))$. For this purpose take an arbitrary $k_i \in K_i$. By the convexity of $\beta_i$ it holds that

$$
0 \leq \gamma_i \beta_i(g_i(u^{\gamma_n})) \leq \gamma_i \beta_i(k_i) - \gamma_i(D\beta_i(g_i(u^{\gamma_n})), k_i - g_i(u^{\gamma_n}))_{X_i^*} X_i = -\langle \mu_i^{\gamma_n}, k_i - g_i(u^{\gamma_n}) \rangle_{X_i^*} X_i.
$$

Due to the boundedness of $\mu_i^{\gamma_n}$ and $g_i(u^{\gamma_n})$ also $\gamma_i \beta_i(g_i(u^{\gamma_n}))$ is bounded, and therefore $\beta_i(g_i(u^{\gamma_n})) \to 0$. Since $g_i$ is weakly continuous, it holds that $0 \leq \beta_i(g_i(u)) \leq \liminf_{n \to \infty} \beta_i(g_i(u^n)) = 0$ and therefore $g_i(u) \in K_i$. Since $k_i \in K_i$
is a minimizer of $\beta_i$, one has $D\beta_i(k_i) = 0$. Using the monotonicity of $D\beta_i$ we observe that
\[
\langle \mu_i, k_i - g_i(u) \rangle_{X^*_i, X_i} = \lim_{n \to \infty} \langle \mu^*_i, k_i - g_i(u^*_n) \rangle_{X^*_i, X_i} \geq \lim_{n \to \infty} \langle -\gamma_n D\beta_i(k_i), k_i - g_i(u^*_n) \rangle_{X^*_i, X_i} = 0.
\]
This implies eventually $\mu_i \in K^+_i$ and $\langle \mu_i, g_i(u) \rangle_{X^*_i, X_i} = 0$. □

With these results at hand, we laid the theoretical foundation for the path-following strategy. The next step towards an applicable numerical method is the careful choice of an update strategy for the penalty parameter.

4.4 Path-Following Strategy

So far, we approximated GNEPs respectively VEPs by a sequence of equilibrium problems without moving sets. In practice however, the careful selection of the penalty parameter $\gamma$ is of crucial importance. This has several reasons: On the one hand, all results are inherently asymptotic and hence one is interested in a rapidly increasing sequence of penalty parameters. On the other hand, every selection of $\gamma$ leads to the solution of an equilibrium problem which—depending on the particular form of the problem—might be a challenging task in its own right and is addressed by iterative methods. The latter decisively depend on a careful selection of starting points. Often, a good choice is given by the solution of the previous iterate associated with its respective $\gamma$-value. Naturally, one is then interested that the new solution is not too far away from the previous one in order to enable fast convergence. The main aim of our path-following strategy is to balance these often conflicting interests by using a value function as an indication of a rapid change in the solution and hence a more moderate update of $\gamma$.

First, we prove the following rather general lemma on the differentiability of a certain type of merit function. For more on this compare to [BS00, Section 4.3].

**Lemma 27.** Let $U$ be a reflexive Banach space and $f : U \to \mathbb{R}$ be a proper functional. Consider a penalty functional $\pi : U \to [0, +\infty)$ and introduce the sequence of merit functionals
\[
f_{\gamma}(u) := f(u) + \gamma \pi(u),
\]
together with the (optimal) value function
\[
W : (0, +\infty) \to \mathbb{R}, \quad W(\gamma) := \inf_{u \in U} f_{\gamma}(u).
\]
Then, the following assertions hold true:

(i) The functional $W : (0, +\infty) \to \mathbb{R}$ is non-decreasing and concave.

(ii) Let $f, \pi$ be lower semi-continuous and the set of minimizers $\arg\min f_{\gamma}$ be non-empty for all $\gamma > 0$. Additionally define for $\gamma > 0$ the set
\[
R(\gamma) := \{u_\gamma \in \arg\min f_{\gamma} : \forall \eta \to 0 \exists u_{\gamma + \eta} \in \arg\min f_{\gamma + \eta} with u_{\gamma + \eta} \to u_\gamma\}.
\]
Then
\[
\sup_{u, \gamma} \pi(u, \gamma) \leq \liminf_{\eta \searrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta} \leq \limsup_{\eta \searrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta} \leq \inf_{u, \gamma} \pi(u, \gamma)
\]
holds. If \(\pi\) is continuous, then also
\[
\sup_{u, \gamma} \pi(u, \gamma) \leq \liminf_{\eta \nearrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta} \leq \limsup_{\eta \nearrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta} \leq \inf_{u, \gamma} \pi(u, \gamma)
\]
holds.

(iii) If \(f, \pi\) are lower semi-continuous and the solution mapping \(\gamma \mapsto \argmin f_{\gamma}\) is singleton and continuous, then \(W\) is differentiable with
\[
W'(\gamma) = \pi(u, \gamma).
\]

(iv) If \(\argmin f_{\gamma}\) is non-empty for all \(\gamma > 0\) and the functional \(W\) is differentiable in \(\bar{\gamma}\), then \(W'(\bar{\gamma}) = \pi(x, \bar{\gamma})\) holds for any \(x, \bar{\gamma} \in \argmin f_{\gamma}\).

Proof. (compare also to [BS00, Proposition 4.12])

ad (i) The functionals \(\gamma \mapsto f(u) + \gamma \pi(u)\) are non-decreasing and concave (affine) for every fixed \(u \in U\). Hence the functional \(\gamma \mapsto W(\gamma)\) is non-decreasing and concave as it is the pointwise infimum.

ad (ii) Choose \(\gamma, \eta > 0\) as well as \(u, \gamma \in \argmin f_{\gamma}\) arbitrarily. By definition holds
\[
W(\gamma + \eta) - W(\gamma) \leq f(\gamma + \eta)(u, \gamma) - f(\gamma)(u, \gamma) = \eta \pi(u, \gamma)
\]
and hence
\[
\limsup_{\eta \searrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta} \leq \inf_{u, \gamma} \pi(u, \gamma).
\]
In the same fashion we get for \(\eta < 0\), that
\[
\liminf_{\eta \nearrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta} \geq \sup_{u, \gamma} \pi(u, \gamma).
\]
Let now \(u, \gamma \in R(\gamma)\) and \(\eta_k \searrow 0\) be arbitrary. There exists a sequence \((u_k)_{k \in \mathbb{N}}\) with \(u_k \in \argmin f_{\gamma + \eta_k}\) and \(u_k \to u, \gamma\). Then, we get
\[
W(\gamma + \eta_k) - W(\gamma) \geq f(\gamma + \eta_k)(u_k) - f(\gamma)(u_k) = \eta_k \pi(u_k).
\]
By the lower semi-continuity we obtain
\[
\pi(u, \gamma) \leq \liminf_{k \to \infty} \pi(u_k) \leq \liminf_{k \to \infty} \frac{W(\gamma + \eta_k) - W(\gamma)}{\eta_k}.
\]
Hence, we see
\[ \sup_{u, \gamma \in R(\gamma)} \pi(u, \gamma) \leq \lim \inf_{\eta \searrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta}. \]

Using the continuity of \( \pi \) one obtains analogously
\[ \lim \sup_{\eta \nearrow 0} \frac{W(\gamma + \eta) - W(\gamma)}{\eta} \leq \inf_{u, \gamma \in R(\gamma)} \pi(u, \gamma). \]

ad (iii): If now the mapping \( \gamma \mapsto \text{argmin} f_\gamma \) is singleton and continuous, we obtain \( R(\gamma) = \text{argmin} f_\gamma \) and the assertion follows by the use of Statement (ii).

ad (iv): Assuming the differentiability of \( W \) in \( \bar{\gamma} \) choose an arbitrary \( u, \bar{\gamma} \in \text{argmin} f_\gamma \) and define the functional
\[ V : (0, +\infty) \to \mathbb{R}, \quad V(\gamma) := W(\gamma) - W(\gamma) + (\gamma - \bar{\gamma})\pi(u, \gamma). \]

Then \( V \) is differentiable in \( \bar{\gamma} \) and attains a global minimum there:
\[ V(\gamma) = f_\gamma(u, \gamma) - W(\gamma) + (\gamma - \bar{\gamma})\pi(u, \gamma) \geq f(u, \gamma) + \bar{\gamma}\pi(u, \gamma) - f(u, \gamma) - \gamma\pi(u, \gamma) + (\gamma - \bar{\gamma})\pi(u, \gamma) = 0 = V(\bar{\gamma}). \]

Thus \( 0 = V'(\bar{\gamma}) = \pi(u, \gamma) - W'(\bar{\gamma}) \) holds true, yielding the assertion.

Remark 28. The above theorem can directly be applied to cover the corresponding results in [HR15, Theorems 4.3 – 4.5], [AHS18, Theorem 2.1] and partly in [HK06b, Proposition 4.2] for the case of \( \lambda = 0 \) therein.

The update principle of \( \gamma \) is based on the expected influence after a change of this parameter. This is achieved via its directional derivative. If the increase is large, the \( \gamma \) update is chosen more conservatively, whereas with a small derivative a more aggressive increase is admitted. In contrast to the corresponding techniques for optimization problems (see [HIK02] and the references in Remark 28) the value functions do not only depend explicitly on the parameter \( \gamma \) itself but also on the calculated equilibrium. Therefore a sequence of value functions is considered — one for each \( \gamma \)-update.

Definition 29.

(i) Consider \( \text{(GNEP}_\gamma \text{)} \). Let for \( \bar{\gamma} > 0 \) a strategy \( u, \bar{\gamma} \in U_{\text{ad}} \) be given. Define for \( \text{(GNEP}_\gamma \text{)} \) its value function \( W(\gamma, u, \bar{\gamma}) : (0, +\infty) \to \mathbb{R} \) by
\[ W(\gamma, u, \bar{\gamma}) := \inf_{v \in U_{\text{ad}}} \left( \sum_{i=1}^{N} \left( J_i(v, u_{\bar{i}, \gamma}) + \gamma\beta(g(v)) \right) \right). \]

(ii) Consider \( \text{(VEP}_\gamma \text{)} \). Let for \( \bar{\gamma} > 0 \) an equilibrium \( u, \bar{\gamma} \in U_{\text{ad}} \) be given. Define for \( \text{(VEP}_\gamma \text{)} \) its value function \( \tilde{W}(\gamma, u, \bar{\gamma}) : (0, +\infty) \to \mathbb{R} \) by
\[ \tilde{W}(\gamma, u, \bar{\gamma}) := \inf_{v \in U_{\text{ad}}} \left( \sum_{i=1}^{N} J_i(v, u_{\bar{i}, \gamma}) + \gamma\beta(g(v)) \right). \]
Here, the value functions $W$ and $\hat{W}$ depend on the penalty parameter and the associated approximation of the equilibrium. In fact, a rapid change in these value functionals indicates in first place a change in the respective solution of the encoded minimization problem. The latter is also called the best response, see e.g. [HS21] and the references therein. This however can as well be interpreted as an indication of a change for the current iterate $u^\gamma$, as the condition to be an equilibrium translates into the iterate and the best-response need to coincide. Utilizing the abstract differentiability results in Lemma 27, we obtain for the setup introduced in Definition 29 the following corollary.

Corollary 30. Let $\gamma > 0$ be fixed and $u^\gamma \in U_{ad}$ be chosen. Let the functionals $v_i \mapsto J_i(v_i, u_{-i})$ be weakly lower semi-continuous. Moreover, assume the operators $g_i : U \to X_i$ (respectively the operator $g : U \to X$) to be weakly continuous. For the value function in Definition 29(ii) there holds

$$\limsup_{\eta \searrow 0} \frac{W(\bar{\gamma} + \eta, u^\gamma) - W(\bar{\gamma}, u^\gamma)}{\eta} \leq \sum_{i=1}^N \beta_i(g_i(u^\gamma)).$$

If for every fixed $u \in U_{ad}$ the best response mapping $B^\gamma$ defined by

$$B^\gamma_i(u_{-i}) := \arg\min_{v_i \in U_{ad}} (J_i(v_i, u_{-i}) + \gamma \beta_i(g_i(v_i)))$$

and $B^\gamma(u) := \prod_{i=1}^N B^\gamma_i(u_{-i})$ is a singleton and continuous in $\gamma$, then $W$ is directionally differentiable with $W'(\bar{\gamma}; 1) = \sum_{i=1}^N \beta_i(g_i(u^\gamma))$.

For the value function in Definition 29(iii) there holds

$$\limsup_{\eta \searrow 0} \frac{\tilde{W}(\bar{\gamma} + \eta, u^\gamma) - \tilde{W}(\bar{\gamma}, u^\gamma)}{\eta} \leq \beta(g(u^\gamma)).$$

If for every fixed $u \in U_{ad}$ the response mapping $\tilde{B}^\gamma$ defined by

$$\tilde{B}^\gamma(u) := \arg\min_{v \in U_{ad}} \left( \sum_{i=1}^N J_i(v_i, u_{-i}) + \gamma \beta(g(v)) \right)$$

is a singleton and continuous in $\gamma$, then $\tilde{W}$ is differentiable with $\tilde{W}'(\bar{\gamma}; 1) = \beta(g(u^\gamma))$.

Proof. We check the conditions of Lemma 27. For this purpose, define for chosen $u^\gamma \in U_{ad}$ the functionals $f(u) := \sum_{i=1}^N J_i(u_i, u_{-i}^\gamma) + I_{U_{ad}}(u)$ and $\pi(u) = \sum_{i=1}^N \beta_i(g_i(u, u_{-i}^\gamma))$. As $J_i(\cdot, u_{-i}^\gamma)$ is weakly lsc. for all $i = 1, \ldots, N$ and $U_{ad}$ is closed, convex and hence weakly closed, $f$ is weakly lsc. Due to the assumed weak continuity of $g_i$ and the weak lower semi-continuity of $\beta_i$ (see Definition 13), also the functional $\pi$ is weakly lower semi-continuous. By Lemma 27(iii) the functional $W(\cdot, u^\gamma)$ is non-decreasing and concave. Using the inequality for the right-sided difference quotient in Lemma 27(ii), proves the asserted inequality. For the response mapping holds $B^\gamma(u^\gamma) = \arg\min f^\gamma$. Hence, we can apply Lemma 27(iii) and obtain the directional differentiability in positive direction. □
As update strategy one approach is to establish a model function aiming at re-
sembling the behavior of the value function as successfully performed in [HK06b],
[HK06a], [HR15] and [AHS18]). Due to the presence of the computed equilib-
rium we use the simplified strategy in [HSK15]. The intention is to perform a
more conservative $\gamma$-update for larger changes in the solution and more aggres-
sive ones for small changes in the solution. As a way to indicate these changes
the value functions $W, \hat{W}$ respectively the estimates of their directional deriva-
tives are used. Is the estimate large, a smaller $\gamma$-update is performed. Is the
estimate small, a larger update is added. From this consideration we utilize the
technique in [HSK15] and recover the subsequent algorithms:

### Algorithm 1: Nikaido–Isoda-based path-following—GNEPs

**Data:** Choose $\gamma_0 > 0, C_{\text{Path}} > 0, \varepsilon > 0$.

**Result:** Approximation of a Nash Equilibrium.

1. for $k = 1, 2, \ldots$ do
2. \hspace{1em} Solve $[\text{GNEP}_\gamma]$ for $\gamma = \gamma_k$ with result $u^{\gamma_k}$.
3. \hspace{1em} if $\beta_i(g_i(u^{\gamma_k})) = 0$ for all $i = 1, \ldots, N$ then
4. \hspace{2em} return $u^{\gamma_k}$.
5. \hspace{1em} else
6. \hspace{2em} Set $\gamma_{k+1} := \gamma_k + \max \left( \frac{C_{\text{Path}}}{\sum_{i=1}^N \beta_i(g_i(u^{\gamma_k}))}, \varepsilon \right)$.
7. end
8. end

Here, the speed of the method is controlled by the two parameters $\varepsilon$ as a lower
bound imposed on the $\gamma$-update and the parameter $C_{\text{Path}} > 0$ controlling the
growth in the first order approximation

$$W(\gamma + \eta, u^\gamma) \approx W(\gamma, u^\gamma) + \eta \sum_{i=1}^N \beta_i(g_i(u^\gamma)).$$

With the condition $W(\gamma + \eta, u^\gamma) - W(\gamma, u^\gamma) \approx C_{\text{Path}}$ this leads to the estimate

$$\eta \approx \sum_{i=1}^N \beta_i(g_i(u^\gamma)).$$

In the same fashion variational equilibria can be treated as well.

### Algorithm 2: Nikaido–Isoda-based path-following—VEPs

**Data:** Choose $\gamma_0 > 0, C_{\text{Path}} > 0, \varepsilon > 0$.

**Result:** Approximation of a Variational Equilibrium.

1. for $k = 1, 2, \ldots$ do
2. \hspace{1em} Solve $[\text{VEP}_\gamma]$ for $\gamma = \gamma_k$ with result $u^{\gamma_k}$.
3. \hspace{1em} if $\beta(g(u^{\gamma_k})) = 0$ then
4. \hspace{2em} return $u^k$.
5. \hspace{1em} else
6. \hspace{2em} Set $\gamma_{k+1} := \gamma_k + \max \left( \frac{C_{\text{Path}}}{\beta(g(u^{\gamma_k}))}, \varepsilon \right)$.
7. end
8. end

With these results at hand we established a penalization technique suitable for
the treatment of GNEPs and VEPs, investigated the interconnections between
these two concepts, developed the convergence analysis as well as a practically
usable update strategy for the penalty parameter. Returning to the treatment
of equilibrium problems involving PDE constraints, we transfer our results to a specialized framework which is tailored to that particular setting.

5 Specialized Framework

In proximity to tracking-type objectives in optimal control we focus our attention on the following class of Nash equilibrium problems, similarly to the one discussed in [HSK15]:

\[
\begin{align*}
\text{minimize} & \quad J_1^i(y) + J_2^i(u_i) \quad \text{over} \quad (u_i, y) \in U_i \times Y \\
\text{subject to} & \quad u_i \in U^i_{\text{ad}}, \quad G(y) \in K, \\
& \quad A(y) = f + B(u_i, u_{-i}) \quad \text{in} \quad W.
\end{align*}
\]

The role of the operator \(g\) is then fulfilled by the composition of the solution operator and the operator \(G\) realizing, together with the set \(K\), the state constraint. Rewriting (8) using a reduced formulation yields

\[
\begin{align*}
\text{minimize} & \quad J_i(u_i, u_{-i}) := J_1^i(u_i) + J_2^i(S(u_i, u_{-i})) \quad \text{over} \quad u_i \in U_i \\
\text{subject to} & \quad u_i \in U^i_{\text{ad}} \quad \text{and} \quad g(u) \in K,
\end{align*}
\]

where \(S\) corresponds to the solution map of the operator equation in the constraints of (8). The reduced form fits directly into the framework of jointly constraint GNEPs (see Definition 4) as proposed in Assumption 9(ii). In this setting the operator \(g = G \circ S\), which is responsible for the state constraint, is effectively determined by the solution operator of the underlying PDE and hence only implicitly defined. To guarantee the properties required by the results in the previous section we propose the following group of standing assumptions, which are in proximity of the ones given in [HSK15, Assumption 2.1]:

Assumption 31.

(i) The spaces \(U_i\) are reflexive, separable Banach spaces, \(Y, W\) are reflexive Banach spaces, \(X\) is a Banach space.

(ii) The embedding \(Y \hookrightarrow X\) is continuous.

(iii) The space \(U := \prod_{i=1}^N U_i\) is equipped with the product topology.

(iv) The operator \(A : Y \to W\) is bijective and continuously Fréchet differentiable.

(v) The mapping \(B \in \mathcal{L}(U, W)\) is a bounded, linear operator and \(B_i \in \mathcal{L}(U_i, W)\) is defined by \(B_i u_i := B(u_i, 0_{-i})\).

(vi) The constraint set \(K \subseteq X\) is a non-empty, convex, closed cone.

(vii) The strategy \(U^i_{\text{ad}} \subset U_i\) is non-empty, bounded, closed, convex.

(viii) The solution operator \(S := A^{-1}(f + B \cdot) : U \to Y\) of the equation

\[ A(y) = f + Bu \]

is assumed to be completely continuous.

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(ix) The mapping $G : Y \to X$ is continuously Fréchet differentiable and we define $g : U \to X$ by $g = G \circ S$.

(x) The first derivative $DA(y) \in \mathcal{L}(Y, W)$ is invertible for all $y \in Y$ and the map $y \mapsto DA(y)^{-1}$ is bounded.

(xi) The set $F := \{u \in U_{ad} : g(u) \in K\}$ is non-empty.

(xii) The functionals $F_i : Y \to \mathbb{R}$ are completely continuous and $J_i^2 : U_i \to \mathbb{R}$ are continuous and weakly lower semi-continuous. Moreover, all of them are assumed to be bounded from below and continuously Fréchet differentiable.

(xiii) The operator $D J_i^2 : U_i \to U_i^*$ has a weak-weak* closed graph.

Following closely the steps of the previous section, we discuss the path-following scheme in the specialized framework. Hence, we proceed with the derivation of the first order system.

**Corollary 32.** Let $u \in U_{ad}$ be a Nash equilibrium together with $y = S(u)$ and let the following (RZK) constraint qualification

$$
(DG(y)DA(y)^{-1}B_i)U_{ad}(u_i) - K(G(y)) = X
$$

hold. Then there exist $p_i \in W^*$, $\mu_i \in X^*$ and $\lambda_i \in U_i^*$ for $i = 1, \ldots, N$ fulfilling the following necessary first order system

\begin{align}
0 &= DJ_i^2(u_i) + B_i^* p_i + \lambda_i \quad \text{in } U_i^*, \\
A(y) &= f + Bu \quad \text{in } W, \\
DA(y)^* p_i &= DJ_i^1(y) - DG(y)^* \mu_i \quad \text{in } Y^*, \\
\lambda_i &\in N_{U_{ad}}(u_i) \quad \text{in } U_i^*, \\
K^+ &\ni \mu_i \perp G(y) \in K.
\end{align}

Let $u \in U_{ad}$ be a variational equilibrium together with $y = S(u)$ and let the following (RZK) constraint qualification

$$
(DG(y)DA(y)^{-1}B)U_{ad}(u) - K(G(y)) = X
$$

hold. Then there exist $p_i \in W^*$, $\lambda_i \in U_i^*$ for $i = 1, \ldots, N$ and $\mu \in X^*$ fulfilling the following necessary first order system

\begin{align}
0 &= DJ_i^2(u_i) + B_i^* p_i + \lambda_i \quad \text{in } U_i^*, \\
A(y) &= f + Bu \quad \text{in } W, \\
DA(y)^* p_i &= DJ_i^1(y) - DG(y)^* \mu \quad \text{in } Y^*, \\
\lambda_i &\in N_{U_{ad}}(u_i) \quad \text{in } U_i^*, \\
K^+ &\ni \mu \perp G(y) \in K.
\end{align}

**Proof.** Let $u \in U_{ad}$ be a Nash equilibrium. Using Assumption 31 one calculates the following derivatives

\begin{align}
\partial_i S(u) &= \partial_i \left( A^{-1}(f + B \cdot) \right)(u) = (DA^{-1}(f + Bu))B_i = DA(y)^{-1}B_i, \\
\partial_i g(u) &= \partial_i (G \circ S)(u) = DG(y)\partial_i S(u) = DG(y)DA(y)^{-1}B_i, \\
\partial_i J_i(u) &= \partial_i \left( J_i^1(S(\cdot, u_{-i})) + J_i^2(\cdot) \right)(u_i) = \partial_i S(u)^* DJ_i^1(y) + DJ_i^2(u_i) \\
&= B_i^* DA(y)^{-*} DJ_i^1(y) + DJ_i^2(u_i).
\end{align}
Then we obtain for \( RZK_{\text{Nash}} \) that

\[
X = \partial_i g(u) U_{ad}^i(u_i) - K(g(u)) = (D \mathcal{G}(y) D A(y)^{-1} B_i) U_{ad}^i(u_i) - K(\mathcal{G}(y))
\]

and for \( RZK_{\text{Val}} \) that

\[
X = D g(u) U_{ad}(u) - K(g(u)) = (D \mathcal{G}(y) D A(y)^{-1} B) U_{ad}(u) - K(\mathcal{G}(y)).
\]

Applying Theorem 10 and Theorem 11 yields the existence of \( \mu_i \in K^+ \) and \( \lambda_i \in N_{U_{ad}}^i(u_i) \) with

\[
0 = \langle \mu_i, g(u) \rangle_X, \quad X = \langle \mu_i, \mathcal{G}(y) \rangle_X
\]

and

\[
0 = \partial_i J_i(u) - \partial_i g(u)^* \mu_i + \lambda_i
\]

\[
= B_i^* DA(y)^{-*} D J_i^1(y) + DJ_i^2(u_i) - B_i^* DA(y)^{-*} \mathcal{G}(y)^* \mu_i
\]

\[
= D J_i^2(u_i) + B_i^* DA(y)^{-*} (D J_i^1(y) - \mathcal{G}(y)^* \mu_i) .
\]

Setting \( p_i = DA(y)^{-*} (D J_i^1(y) - \mathcal{G}(y)^* \mu_i) \) and using \( y = S(u) \) yields the system

\[
0 = D J_i^2(u_i) + B_i^* p_i + \lambda_i,
\]

\[
A(y) = f + Bu.
\]

\[
DA(y)^* p_i = D J_i^1(y) - \mathcal{G}(y)^* \mu_i
\]

and thus the assertion. The proof for the variational equilibrium reads essentially the same and is hence omitted.

Next, the penalty technique is applied. This leads to the following penalized NEP

\[
\text{minimize } \quad J_i^1(y_i) + J_i^2(v_i) + \gamma \beta(\mathcal{G}(y)) \text{ over } v_i \in U_i,
\]

\[
\text{subject to } \quad v_i \in U_{ad}^i, \quad A(y_i) = f + B(v_i, u_{-i}) \text{ in } W.
\]

(13)

Analogously one obtains for \( VEP_{\gamma} \), the problem

\[
\text{minimize } \quad \sum_{i=1}^N (J_i^1(y_i') + J_i^2(v_i)) + \gamma \beta(\mathcal{G}(y_i')) \text{ over } v \in U,
\]

\[
\text{subject to } \quad v \in U_{ad}, \quad A(y_i') = f + B(v_i, u_{-i}) \text{ in } W \text{ for } i = 1, \ldots, N,
\]

\[
\text{and } A(y_i') = f + Bu_i' \text{ in } W.
\]

(14)

It is worth noting that the latter equilibrium problem requires another solution of the PDE involving feasible test strategies \( v_i \) only.

The first order system for both problems is formulated in the following corollary.

**Corollary 33.** Let \( u^\gamma \in U_{ad} \) be a solution of \( (13) \) or \( (14) \) together with states \( y^\gamma = S(u^\gamma) \). Then there exist \( p_i^\gamma \in W^* \), \( \lambda_i^\gamma \in U_i^* \) for \( i = 1, \ldots, N \) and \( \mu^\gamma \in X^* \) fulfilling the following necessary first order system

\[
0 = D J_i^2(u_i^\gamma) + B_i^* p_i^\gamma + \lambda_i^\gamma \text{ in } U_i^* ,
\]

\[
A(y^\gamma) = f + Bu^\gamma \text{ in } W,
\]

\[
DA(y^\gamma)^* p_i^\gamma = D J_i^1(y^\gamma) - \mathcal{G}(y^\gamma)^* \mu^\gamma \text{ in } Y^* ,
\]

\[
\lambda_i^\gamma \in N_{U_{ad}}^i(u_i^\gamma) \text{ in } U_i^* ,
\]

\[
\mu = -\gamma D \beta(\mathcal{G}(y^\gamma)) \text{ in } X^* .
\]

(15.1) (15.2) (15.3) (15.4) (15.5)
Proof. Using the calculations in the proofs of Corollary 32, Theorem 14 and Theorem 15 one obtains (15).

6 Examples

In the remainder of this work we apply the developed tools and methods to a selection of examples involving semi-linear elliptic PDEs. In this context, we establish two cases of Nash games with players having tracking-type objectives. In the first example we will see an instance of distributed control and in the second one an example of boundary control. In addition to the illustration of the numerical methods developed within this work, this is taken as an opportunity to study the influence of the non-cooperative aspect of Nash games in comparison to an associated cooperative optimization problem.

As a common setup we take as domain the unit square $\Omega = (0,1)^2 \subseteq \mathbb{R}^2$ and establish a partition into four subdomains as depicted in Figure 1. Each of these domains serves as area of interest for exactly one player. The tracking type functional as well as each player’s control depend on this player’s domain only. However, their strategies have an effect on a state being the solution of semi-linear elliptic PDE. By this, a spatial coupling is established leading to an interaction between these players. Additionally, a box constraint for the state is established, which can be rewritten as a cone constraint in a product space. In the cooperative case however, the sum over all objectives is taken and minimized with respect to the combined strategies of all players simultaneously. As desired state we take a piecewise constant function with values $y_i = y_i^d$ a.e. on $\omega_i$ with constants $y_1^d = 0.1$, $y_2^d = 0.2$, $y_3^d = 0$ and $y_4^d = 0.3$. The desired state is plotted in Figure 1 as well. As constraints for the controls as well as the state we use box constraints.

![Figure 1: Left: Decomposition of $\Omega = (0,1)^2$ into subdomains, Right: Plot of the desired state.](image)

For the solution of the corresponding first order systems we use in both cases a semi-smooth Newton method (cf. [CNQ00, HIK02]). Therefore, the semi-smoothness of the operator $\max(\cdot, 0) : L^{r_1}(\Omega) \to L^{r_2}(\Omega)$ for $r_1 > r_2$ is utilized. Since Newton-type methods only guarantee convergence, when the starting point is in a sufficiently small neighbourhood from a solution. This issue is
addressed in two ways: On the one hand by the selection of parameters for the update strategy to avoid big changes in the solutions between the iterates. Then the solution with respect to the previous parameter can be used as a starting value.

On the other hand, one way to extend the basin of attraction is the use of a very simple damping technique. In our case, this is established by trying at first the iteration without damping and 25 as the maximal number of iterations. Whenever the method fails to achieve both—the relative tolerance of $10^{-10}$ as well as the absolute tolerance of $10^{-10}$, the damping factor is halved, the maximal iteration number is doubled and the Newton method is attempted with the same starting value again. In case of success, the new settings are kept for all subsequent $\gamma$-iterates. Clearly, this procedure is heuristic only, but it performed well in our experiments. For more sophisticated damping strategies we just refer to [Deu05, Chapter 3] and for other techniques to [DF95].

As discretization method we use in all examples the finite element method (FEM) and for its implementation the package FEniCS (cf. [ABH+15]). We discretize the domain using a crossed mesh with $2^7$ segments along every side of the unit square. For the discretization of the state as well as each adjoint state we take the finite element spaces

$$S^1(T) := \{ z \in H^1(\Omega) : v|_T \in P_1(T) \text{ for all } T \in \mathcal{T} \}$$

and

$$S^1_0(T) := S^1(T) \cap H^1_0(\Omega).$$

We are making use of the variational discretization technique as presented in [HPUU08, Section 3.2.5], see also the references therein. By that, the control remains undiscretized and the state equation is discretized by a Galerkin approach with the above introduced spaces. The derivation of the first order system leads to the corresponding discretization of the adjoint equation. The elimination of the control via the adjoint state leads to a non-linear coupled system on the discrete level and is hence numerically accessible.

Example 34. First, we return to the introductory Example 3, which is restated for convenience, and consider the following generalized Nash equilibrium problem governed by a semi-linear elliptic PDE with distributed control:

$$\text{minimize } J_i(u_i, u_{-i}) := \frac{1}{2} \int_{\omega_i} (y - y_{i}^d)^2 dx + \frac{\alpha}{2} \int_{\omega_i} u_i^2 dx \text{ over } u_i \in L^2(\omega_i)$$

subject to $a_i \leq u_i \leq b_i$ a.e. on $\omega_i$, $\underline{\psi} \leq y \leq \overline{\psi}$ a.e. on $\Omega$ and

$$- \Delta y + y^3 = \sum_{i=1}^{4} B_i u_i \text{ in } \Omega, \ y = 0 \text{ on } \partial\Omega.$$  

Let $a_i = -32$ and $b_i = 32$ and choose the regularization parameter $\alpha = 10^{-5}$. We use the subdomain partition depicted in Figure 1. As upper and lower obstacle for the state we take $\underline{\psi} = 0$ together with $\overline{\psi} = 0.3$ being the upper and lower pointwise bounds of $y_{i,d}$ on all of $\Omega$. To use the results of this work we formulate this example in a way suitable to the framework established in Assumption 31. The operators $B_i \in \mathcal{L}(L^2(\omega_i), H^{-1}(\Omega))$ are just the extension of a function on $\omega_i$ by zero on the whole domain combined with the embedding.
of $L^2(\Omega)$ in $H^{-1}(\Omega)$. The operator $A$ will be defined via the relation
\[
\langle A(y), z \rangle_{H^0_0(\Omega), H^{-1}(\Omega)} = (\nabla y, \nabla z)_{L^2(\Omega, \mathbb{R}^d)} + \langle y, z \rangle_{L^2(\Omega)}
\]
for sufficiently regular arguments and test functions. It is straightforward to show existence and uniqueness of a solution in $H^1_0(\Omega)$ with test functions in $H^{-1}(\Omega)$. However, to discuss the choice of suitable function spaces, we use regularity theory for PDEs. Therefore, set $Y := W^{1,r}(\Omega)$ and $X := C(\bar{\Omega})$. Then, we obtain the continuous embedding $X \hookrightarrow Y$. To establish the complete continuity of the solution operator $S : U \rightarrow W^{1,r}_0(\Omega)$ we observe by Theorem 3.2.1.2] that there exists a constant $C > 0$ with
\[
\|z\|_{H^2(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)} \right)
\]
where $f \in L^2(\Omega)$ and $z \in H^2(\Omega) \cap H^1_0(\Omega)$ is the solution of the Poisson problem
\[
-\Delta z = f \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \partial \Omega.
\]
In the following we will adapt the notation $a \lesssim b$ for $a, b > 0$ if there exists a constant $c > 0$ such that $a \leq c \cdot b$ holds.

Let a sequence $(w^n)_{n \in \mathbb{N}} \subseteq U = \prod_{i=1}^n L^2(\omega_i)$ with $u^n \rightharpoonup u$ be given and let $(y^n)_{n \in \mathbb{N}}$ denote the corresponding sequence of solutions of the state equations in the sense of $H^1_0(\Omega)$. Then we obtain the following estimate
\[
\|y^n\|_{H^2(\Omega)} \lesssim \left\| \sum_{i=1}^4 B_i u^n_i - (y^n)^\ast \right\|_{L^2(\Omega)} \leq \sum_{i=1}^4 \|u_i^n\|_{L^2(\omega_i)} + \|y^n\|_{H^2(\Omega)}^3 \sim \sum_{i=1}^4 \|u_i^n\|_{L^2(\omega_i)} \left( \sum_{i=1}^4 \|u_i^n\|_{L^2(\omega_i)} \right)^3,
\]
where we used the continuous embedding $H^2(\Omega) \hookrightarrow L^6(\Omega)$. Therefore the boundedness of $(y^n)_{n \in \mathbb{N}}$ in $H^2(\Omega)$ holds and for every subsequence we can extract another subsequence with weak limit point $y^* \in H^2(\Omega)$. Since $H^2(\Omega) \hookrightarrow H^1_0(\Omega)$ we obtain the strong convergence in $H^1_0(\Omega)$ and by the continuity of $A$ that $A(y_n) \rightharpoonup A(y^*)$ in $H^{-1}(\Omega)$. By $A(y^n) = \sum_{i=1}^4 B_i u_i^n \rightharpoonup \sum_{i=1}^4 B_i u_i^*$ in $H^{-1}(\Omega)$ one observes $A(y^*) = \sum_{i=1}^4 B_i u_i^*$. Consequently, since the choice of the first subsequence was arbitrary, we obtain the weak convergence of the whole sequence in $H^2(\Omega)$. Since the embedding in $H^2(\Omega) \hookrightarrow W^{1,1}(\Omega)$ is compact, we obtain eventually the complete continuity of the solution operator $S : \prod_{i=1}^n L^2(\omega_i) \rightarrow W^{1,r}_0(\Omega)$. As space $W$ we use $W = W^{-1,r}(\Omega)$ being the dual of $W^{1,\alpha}(\Omega)$ with $\frac{1}{r} + \frac{1}{\alpha} = 1$. We further define the mapping $G : W^{1,\alpha}_0(\Omega) \rightarrow W^{1,r}(\Omega; \mathbb{R}^2)$ by $G(y) := (y - \psi - \bar{\psi} - y)$ with the cone
\[
K := \{(z_1, z_2) \in W^{1,r}(\Omega; \mathbb{R}^2) : z_1, z_2 \geq 0 \text{ a.e. on } \Omega\},
\]
and take $J_1^1(y) := \frac{\alpha}{2} \|y - y_d\|_{L^2(\omega)}^2$ and $J_2^1(u_i) := \frac{\alpha}{2} \|u_i\|_{L^2(\omega)}^2$. Then, $DJ_2^1(u_i) = 0u_i$ is even a weakly continuous mapping.

The fulfillment of the constraint qualification is a delicate question and is in the following assumed within the scope of this work. In the light of the embedding $W^{1,r}(\Omega) \hookrightarrow C(\Omega)$ for $r > d$ a formulation in the space of continuous functions
might be of advantage.
Hence, we establish the penalization technique discussed in Section 4 and Section 5. For this sake we introduce as penalty function the Moreau–Yosida regularization with respect to the $L^2$-norm of the set \( \{ z \in L^2(\Omega) : \psi \leq z \leq \overline{\psi} \} \) corresponding to the state constraint. This leads to
\[
\beta(y) := \frac{1}{2} \int_\Omega (y - \overline{\psi})^2 + \frac{1}{2} \int_\Omega (\overline{\psi} - y)^2 \, dx.
\]
As first order system, we derive by the use of Theorem 26 the following system:
\[
u_i = \text{Proj}_{U_{ad}} \left( -\frac{1}{\alpha} 1_{\omega_i} p_i \right) \quad \text{ in } \Omega, \quad (16.1)
\]
\[-\Delta y + y^3 = \sum_{i=1}^4 B_i u_i \quad \text{ in } \Omega, \quad (16.2)
\]
\[y = 0 \quad \text{ on } \partial \Omega, \quad (16.3)
\]
\[-\Delta p_i + 3y^2 p_i = 1_{\omega_i} (y - y_d) + \gamma (y - \overline{\psi})^+ - \gamma (\overline{\psi} - y)^+ \quad \text{ in } \Omega, \quad (16.4)
\]
\[p_i = 0 \quad \text{ on } \partial \Omega. \quad (16.5)
\]
In this setup the cooperative comparison problem reads as
\[
\text{minimize } \sum_{i=1}^4 \left( \tilde{J}_i^1(y) + \tilde{J}_i^2(u_i) \right) = \frac{1}{2} \| y - y_d \|^2_{L^2(\Omega)} + \frac{1}{2} \sum_{i=1}^4 \alpha_i \| u_i \|^2_{L^2(\omega_i)}
\]
subject to \( u_i \in L^2(\omega_i), a_i \leq u_i \leq b_i, \psi \leq y \leq \overline{\psi} \) a.e. on \( \Omega \) and
\[-\Delta y + y^3 = \sum_{i=1}^4 B_i u_i \text{ in } \Omega, \quad y = 0 \text{ on } \partial \Omega. \quad (17)
\]
The first order system can be derived by standard techniques in optimization or by the results in this paper for the case \( N = 1 \) with exactly one player. This leads to
\[
u_i = \text{Proj}_{U_{ad}} \left( -\frac{1}{\alpha} 1_{\omega_i} p_i \right) \quad \text{ in } \Omega, \quad (18.1)
\]
\[-\Delta y + y^3 = \sum_{i=1}^4 B_i u_i \quad \text{ in } \Omega, \quad (18.2)
\]
\[y = 0 \quad \text{ on } \partial \Omega, \quad (18.3)
\]
\[-\Delta p + 3y^2 p = (y - y_d) + \gamma (y - \overline{\psi})^+ - \gamma (\overline{\psi} - y)^+ \quad \text{ in } \Omega, \quad (18.4)
\]
\[p = 0 \quad \text{ on } \partial \Omega. \quad (18.5)
\]
Comparing (18) and (16) it is worth noting that \( (y - y_d) = \sum_{i=1}^4 (y - y_d^i) 1_{\omega_i} \) by our partition of the domain \( \Omega \). In the following numerical experiments, we choose as parameters \( C_{\text{Path}} = 10^{-5} \) and \( \varepsilon = 10 \) for the path-following strategy for both — the cooperative as well as the non-cooperative case.
The plots of the results are depicted below. Since the controls are basically truncations of the adjoint states (see (18)) we restrict ourselves to the depiction of the controls (in Figure 2) and states (in Figure 3).
To illustrate the controls we depicted the combined control $u = \sum_{i=1}^{4} \mathbb{1}_{\omega_i} u_i$. In both cases the different control regions $\omega_i$ can be clearly recognized. The cooperative case admits a solution that is in $H^1_0(\Omega)$, whereas in the non-cooperative case only piecewise $H^1$-regularity for the combined controls is guaranteed. In the latter case the boundaries between the control regions are clearly visible due to activity of the box constraint.

In Figure 4 the update history for both cases is depicted. Both iterations are terminated as soon as the penalty function drops below a threshold of $10^{-15}$ or the penalty parameter $\gamma$ exceeds the value $10^8$. The direct comparison of the two graphs indicate that the update mechanism is less aggressive in the non-cooperative case. On the one hand, this might be related to the explicit presence of the computed equilibrium in the value function (see also Definition 29). On the other hand, the game leads the player to compete in their decisions near boundary regions with another player. Hence, the benefit from the competition rewarded via the tracking term might be more valuable than the punishment implied by the penalty functional. This phenomenon also translates to the states: As it can be seen from Figure 5 the interfaces between the different regions are more distinct in the non-cooperative case. However, in the regions $\omega_1$ and $\omega_2$ the state falls below respectively exceeds the desired values near the regions $\omega_4$ and $\omega_3$. 
The behaviour of the sum of objective values (without the penalty function) is depicted in Figure 5. They both follow the very same pattern that is well
known from optimization problems. A frequently used concept in game theory related to discrete mathematics, e.g., in the context of selfish routing, is called the \textit{price of anarchy} (see e.g. [NRTV07, Definition 20.4]) defined by

$$\text{PoA} = \frac{\sup_{u \in \text{Equil}} \sum_{i=1}^{N} J_i(u)}{\min_{u \in F} \sum_{i=1}^{N} J_i(u)}.$$

Here, $F$ denotes again the joint constraint set for the problem, and ‘Equil’ denotes the set of all Nash equilibria. A calculation with respect to the last iterate yields $\text{PoA} \geq 1.1641$ as a lower approximation. For the Newton iterates we note, that the application of the damping strategy has only been applied in the cooperative case and only for the last two $\gamma$-iterations. The reason for this behaviour lies in the more conservative update behaviour of the penalty parameter for the Nash game.

\textbf{Example 35.} Consider the following generalized Nash equilibrium problem governed by a semi-linear elliptic PDE with boundary control

\begin{align*}
\text{minimize} & \quad J_i(u_i, u_{-i}) := \frac{1}{2} \int_{\omega_i} (y - y_i^d)^2 dx + \frac{\alpha}{2} \int_{\Gamma_i} u_i^2 dS \text{ over } u_i \in L^2(\Gamma_i) \\
\text{subject to} & \quad a_i \leq u_i \leq b_i \text{ a.e. on } \Gamma_i, \psi \leq y \leq \bar{\psi} \text{ a.e. on } \Omega \text{ and} \\
& \quad -\Delta y + y = 0, \quad \text{in } \Omega, \quad \frac{\partial y}{\partial \nu} + y^3 = \sum_{i=1}^{N} B_i u_i \text{ on } \partial \Omega.
\end{align*}

We choose again $a_i = -32$ and $b_i = 32$ and as regularization parameter $\alpha = 10^{-5}$. Together with the subdomain partition depicted in Figure 1, we define as control region for player $i$ the set $\Gamma_i := \partial \Omega \cap \partial \omega_i$. As upper and lower obstacle for the state again $\bar{\psi} = 0$ and $\psi = 0.3$ are taken.

To utilize our framework we check the conditions in Assumption 31. For this sake, we define again $X = C(\bar{\Omega})$ as well as $Y := W^{1,r}(\Omega)$ with $r \in (2, 3)$.

Using the regularity result [Sav98, Theorem 4] one can deduce from its proof for $s \in (-\frac{1}{2}, \frac{1}{2})$ that for $f \in L^2(\Omega)$ and $g \in H^{-\frac{s}{2}+}(\Omega)$ the solution of the PDE

$$-\Delta z + z = f \text{ in } \Omega, \quad \frac{\partial z}{\partial \nu} = g \text{ in } \partial \Omega,$$

satisfies the estimate

$$\|z\|_{H^{\frac{s}{2}+}(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|g\|_{H^{-\frac{s}{2}+}(\partial \Omega)},$$

with a constant depending on $s$ as well as the domain $\Omega$.

Taking a weakly convergent sequence $(u^n)_{n \in \mathbb{N}} \subseteq \prod_{i=1}^{4} L^2(\Gamma_i)$ with limit $u^* \in U$ leads to a sequence of states $(y^n)_{n \in \mathbb{N}} \subseteq H^1(\Omega)$ with $\bar{y}^n = S(u^n)$. By using the embedding $H^{\frac{1}{2}}(\partial \Omega) \hookrightarrow L^p(\partial \Omega)$ for all $p \in (1, +\infty)$ we infer that $(\text{tr}_{\partial \Omega} y)^3 \in$
\[ L^2(\partial \Omega) \hookrightarrow H^{-\frac{1}{2}}(\partial \Omega). \] With \( s = \frac{1}{3} \) we obtain
\[
\| y^n \|_{H^\frac{2}{3}(\Omega)} \lesssim \left\| \sum_{i=1}^{4} B_i u^n_i - (\text{tr}_{\partial \Omega} y^n)^3 \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} \lesssim \left\{ \sum_{i=1}^{4} \| u^n_i \|_{L^2(\Gamma_i)} + \| \text{tr}_{\partial \Omega} y^n \|_{L^6(\partial \Omega)} \right\} \lesssim \left\{ \sum_{i=1}^{4} \| u^n_i \|_{L^2(\Gamma_i)} + \| y^n \|_{H^\frac{1}{2}(\Omega)} \right\}^3.
\]

From the boundedness of the sequence \((u^n)_{n \in \mathbb{N}}\) we also deduce the boundedness of \((y^n)_{n \in \mathbb{N}}\) in \( H^\frac{2}{3}(\Omega) \). By reflexivity we further infer along every subsequence the existence of weakly convergent subsequence in \( H^\frac{2}{3}(\Omega) \). Using the embedding \( H^\frac{2}{3}(\Omega) \hookrightarrow L'^r(\Omega) \) for all \( r \in (2, 3) \). Hence we obtain \( H^\frac{2}{3}(\Omega) \hookrightarrow W^1^r(\Omega) \) as well and deduce the strong convergence in the latter space as well as in \( H^\frac{1}{2}(\Omega) \). By the continuity of the operator \( A : H^1(\Omega) \rightarrow H^{-1}(\Omega) \) we infer \( A(y^n) \rightarrow A(y) \) in \( H^{-1}(\Omega) \) and by the weak convergence of \( u^n \) as well \( \left\{ \sum_{i=1}^{4} u^n_i, \text{tr}_{\partial \Omega}(\cdot) \right\}_{H^{-\frac{1}{2}}(\Omega), H^\frac{1}{2}(\Omega)} \rightarrow \left\{ \sum_{i=1}^{4} u_i, \text{tr}_{\partial \Omega}(\cdot) \right\}_{H^{-\frac{1}{2}}(\Omega), H^\frac{1}{2}(\Omega)} \) in \( H^{-1}(\Omega) \). Hence, \( y \) is the unique solution of the state equation with respect to \( u \). Thus, the whole sequence \((y^n)_{n \in \mathbb{N}}\) converges weakly in \( H^\frac{2}{3}(\Omega) \) and strongly in \( W^1^r(\Omega) \) for \( r \in (2, 3) \). Eventually, we deduce the complete continuity of the solution operator.

As in the previous Example 34 we take \( \mathcal{G} : W^{1,r}_0(\Omega) \rightarrow W^{1,r}(\Omega; \mathbb{R}^2) \) defined by \( \mathcal{G}(y) := (y - \psi, \overline{\nu} - y) \) along with
\[
K := \{ (z_1, z_2) \in W^{1,r}(\Omega; \mathbb{R}^2) : z_1, z_2 \geq 0 \text{ a.e. on } \Omega \}.
\]

And again, the constraint qualification is assumed to hold true. For the penalization of the state constraint we introduce
\[
\beta(y) := \frac{1}{2} \int_{\Omega} (y - \psi)^{2+} \, dx + \frac{1}{2} \int_{\Omega} (\psi - y)^{2+} \, dx
\]
as well. The first order system of the penalized problem can be derived with the help of Corollary 33 as in the previous example:
\[
u_i = \text{Proj}_{U_{ad}} \left( -\frac{1}{\alpha} \mathbb{I}_{\Gamma_i} \text{tr}_{\partial \Omega} p_i \right) \quad \text{in } \Omega, \quad (19.1)
\]
\[
-\Delta y + y = 0 \quad \text{in } \Omega, \quad (19.2)
\]
\[
\frac{\partial y}{\partial \nu} + y^3 = \sum_{i=1}^{N} B_i u_i \quad \text{on } \partial \Omega, \quad (19.3)
\]
\[
-\Delta p_i + p_i = \mathbb{I}_{\omega_i}(y - y_d) + \gamma (y - \psi)^+ - \gamma (\psi - y)^+ \quad \text{in } \Omega, \quad (19.4)
\]
\[
\frac{\partial p_i}{\partial \nu} + 3y^2 p_i = 0 \quad \text{on } \partial \Omega. \quad (19.5)
\]
Hence, the cooperative comparison problem reads as

\[
\begin{align*}
\text{minimize} \quad & \sum_{i=1}^{4} (J_1^i(y) + J_2^i(u_i)) = \frac{1}{2} \| y - y_d \|^2_{L^2(\Omega)} + \sum_{i=1}^{4} \frac{\alpha}{2} \| u_i \|^2_{L^2(\Gamma_i)} \\
\text{subject to} \quad & u_i \in L^2(\Gamma_i), a_i \leq u_i \leq b_i \text{ a.e. on } \Gamma_i, \psi \leq y \leq \overline{\psi} \text{ a.e. on } \Omega \text{ and} \\
& -\Delta y + y = 0 \text{ in } \Omega, \quad \frac{\partial y}{\partial \nu} + y^3 = \sum_{i=1}^{N} B_i u_i \text{ on } \partial \Omega.
\end{align*}
\]

The first order system can as in the previous example. This leads to

\[
\begin{align*}
\frac{\partial y}{\partial \nu} + y^3 = \sum_{i=1}^{4} B_i u_i & \quad \text{in } \Omega, \quad (21.3) \\
-\Delta y + y = 0 & \quad \text{in } \Omega, \quad (21.2) \\
\frac{\partial p}{\partial \nu} + 3y^2p = 0 & \quad \text{on } \partial \Omega. \quad (21.5)
\end{align*}
\]

Below, the results of the experiments are depicted. For better orientation in Figure 7 the distinction of the domain \(\Omega\) has been added in the colouring of the desired state of Figure 1. Similarly to the previous example the competition between the players induces in activity of the control constraints near the boundary points between the control boundaries \(\Gamma_i\). In contrast, the cooperative case does not reach the bounds. (Note the different scaling.) This drastic difference in behaviour affects the states as well. As one can see in Figure 8 the desired state is not even nearly met on \(\omega_1\) and \(\omega_2\) in the non-cooperative case, but it is observably better met in the cooperative one.

Figure 7: Plot of the controls. Left: Results for Example 35 with values between \(-32\) and \(32\). Right: Results for the cooperative version in (20) with values between \(-6.5\) and \(5.5\).
The influence of this competition can also be seen in the update behaviour. As before, the non-cooperative case has less aggressive updates, but when compared to distributed control the difference is even more profound. The behaviour of the summed objectives is in the light of the previous example counterintuitive. However, in the optimization case it is clear that due to imposing the state constraint the objective values are a non-decreasing sequence. For Nash equilibria such a result is not available and is in fact not a mandatory behaviour as it can be seen in this example. This observation can therefore be interpreted as an instance of a Braess paradoxon (cf. [Bra68]). Using the final iterates of the algorithm we obtain for the price of anarchy \( \text{PoA} \geq 1.29890 \) as lower estimate. So in comparison to Example 34 the price of anarchy is considerably bigger, which indicates a more competitive environment and might hence be an explanation for the behaviour observed in Figure 10. For the Newton iterations the established damping rule has only been used towards the end for large \( \gamma \)-updates.

Figure 8: Plot of the states. Left: Results for Example 35. Right: Results for the cooperative version (20).

Figure 9: Plot of the \( \gamma \)-updates. Left: Results for Example 35. Right: Results for the cooperative version (20).
7 Conclusion

In the present work a framework for the analytical and numerical treatment of generalized Nash equilibrium problems has been developed for different levels of abstraction. Special emphasis has been put to the application of our results to games derived from optimal control problems involving non-linear partial differential equations. As approximation technique a penalization concept for equilibria has been discussed along with the treatment of the multipliers in the first order systems. The results have been applied to two GNEPs with optimal control problems with distributed respectively boundary controls. The comparison of these games with their cooperative counterparts has been used to inspect and visualize the influence of competition as well as the enforcement of joint constraints to the choice of strategies of the players.
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A Appendix

Proof of Lemma 21. Consider for $t \in \mathbb{R}\setminus\{0\}$ the sequence of mappings $T_t : X \times X \rightarrow Y$ by

$$T_t(x, h) := \frac{T(x + th) - T(x)}{t}.$$ 

Using the uniform Fréchet differentiability one obtains that for all bounded subsets $M \subseteq X$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{\|T(x' + h') - T(x') - DT(x')h'\|_Y}{\|h'\|_X} \leq \varepsilon$$

for all $x' \in M$ and $h' \in X$ with $\|h'\|_X \leq \delta$. Hence, taking $R > 0$ and $t > 0$ sufficiently small yields

$$\sup_{x' \in M, h \in X, \|h\|_X \leq R} \|T_t(x, h) - DT(x)h\|_Y = \sup_{x' \in M, h \in X, \|h\|_X \leq R} \frac{\|T(x + th) - T(x) - tDT(x)h\|_Y}{|t|} \leq \varepsilon R,$$

which implies the uniform convergence of $(x, h) \mapsto T_t(x, h)$ towards $(x, h) \mapsto DT(x)h$ on every bounded subset of $X \times X$ for $t \rightarrow 0$ (cf. also [Ehr62, Chapter 10, footnote 5]).

Let first $T$ be weakly continuous. Take sequences $x_n \rightarrow x$ and $h_n \rightarrow h$ in $X$ as well as an arbitrary $y^* \in Y^*$. By the boundedness of the sequences $(x_n)_{n \in \mathbb{N}}, (h_n)_{n \in \mathbb{N}}$ and the uniform Fréchet differentiability we obtain

$$|\langle y^*, DT(x_n)h_n - DT(x)h \rangle_{Y^*, Y}| \leq |\langle y^*, DT(x_n)h_n - T_t(x_n, h_n) \rangle_{Y^*, Y}| + |\langle y^*, T_t(x_n, h_n) - T_t(x, h) \rangle_{Y^*, Y}| + |\langle y^*, DT(x, h) - T_t(x, h) \rangle_{Y^*, Y}| \leq 2\varepsilon + |\langle y^*, T_t(x_n, h_n) - T_t(x, h) \rangle_{Y^*, Y}|.$$

Using the weak continuity of $T$ one obtains for fixed $t \in \mathbb{R}\setminus\{0\}$ also the weak continuity of $T_t$ by its definition. This yields

$$0 \leq \limsup_{n \rightarrow \infty} |\langle y^*, DT(x_n)h_n - DT(x)h \rangle_{Y^*, Y}| \leq 2\varepsilon$$

for all $\varepsilon > 0$ and, thus, the weak convergence of $DT(x_n)h_n$.

Taking arbitrary sequences $x_n \rightarrow x$ in $X$ and $y^*_n \rightarrow y^*$ in $Y^*$ we see for all $h \in X$ that

$$(DT(x_n)y^*_n, h)_{X^*, X} = \langle y^*_n, DT(x_n)h \rangle_{Y^*, Y} \rightarrow \langle y^*, DT(x)h \rangle_{Y^*, Y}$$

using the previously proven weak convergence $DT(x_n)h \rightarrow DT(x)h$ in $Y$.

In the case of $T$ being completely continuous take again sequences $x_n \rightarrow x$ and
$h_n \to h$ in $X$ as well as an arbitrary $y^* \in Y^*$. Using the uniform Fréchet differentiability we see
\[
\|DT(x_n)h_n - DT(x)h\|_Y \leq \|DT(x_n)h_n - T_t(x_n, h_n)\|_Y
\]
\[
+ \|T_t(x_n, h_n) - T_t(x, h)\|_Y + \|DT(x)h - T_t(x, h)\|_Y
\]
\[
\leq 2\varepsilon + \|T_t(x_n, h_n) - T_t(x, h)\|_Y.
\]
The complete continuity of $T$ yields as well for fixed $t \in \mathbb{R}\setminus\{0\}$ the complete continuity of $T_t$ and hence we obtain
\[
0 \leq \limsup_{n \to \infty} \|DT(x_n)h_n - DT(x)h\|_Y \leq 2\varepsilon
\]
for $\varepsilon > 0$, thus yielding the strong convergence.
For the remaining assertion we use an indirect proof. Assume, there exist $x_n \to x$ and $y_n^* \to^* y^*$ and $\varepsilon > 0$ such that $\|DT(x_n)^*y_n^* - DT(x)^*y^*\|_{X^*} \geq 2\varepsilon$ holds for all $n \in \mathbb{N}$. Then, for every $n$ there exists a $v_n \in X$ with $\|v_n\|_X \leq 1$ such that
\[
\langle DT(x_n)^*y_n^* - DT(x)^*y^*, v_n \rangle_{X^*, X} \geq \frac{1}{2}\|DT(x_n)^*y_n^* - DT(x)^*y^*\|_{X^*} \geq \varepsilon.
\]
By the Banach-Alaoglu theorem we extract a weakly convergent subsequence from $(v_n)_{n \in \mathbb{N}}$ with limit $v$ (not relabeled). Using the previously shown complete continuity of $(x, h) \mapsto DT(x)h$ yields
\[
\varepsilon \leq \langle DT(x_n)^*y_n^* - DT(x)^*y^*, v \rangle_{X^*, X} = \langle y_n^*, DT(x_n)v_n \rangle_{Y^*, Y} - \langle y^*, DT(x)v \rangle_{Y^*, Y} - \langle y^*, DT(x)v \rangle_{Y^*, Y} = 0
\]
and thus the requested contradiction.

\textbf{Proof of Lemma 23} Suppose the contrary. Then, there exists a positive bound $R > 0$ together with a sequence $(x_n)_{n \in \mathbb{N}}$, $\|x_n\|_X \leq R$ for all $n \in \mathbb{N}$ and $\|DT(x_n)\|_{\mathcal{L}(X, Y)} \geq 2n$. Hence, there exists $(h_n)_{n \in \mathbb{N}}$ with $\|h_n\|_X \leq 1$ and $\|DT(x_n)h_n\|_Y \geq n$. By the reflexivity of $X$ one extracts weakly convergent subsequences $x_n \to x$ and $h_n \to h$ in $X$. Lemma 22 yields the weak convergence of $DT(x_n)h_n \to DT(x)h$ in $Y$ and thus the boundedness of $\|DT(x_n)h_n\|_Y$—a contradiction.

\textbf{Proof of Lemma 23} Let a bounded subset $L \subset Y$ be given. Since $S$ is assumed to be a bounded operator, the set $M := S(L + B_Y) \subset X$ is bounded as well. Using the assumption on the inverses of the first derivatives, let
\[
0 < B := \sup_{x \in M} \|DT(x)^{-1}\|_{\mathcal{L}(Y, X)} < +\infty.
\]
By the uniform Fréchet differentiability there exists for all $\varepsilon > 0$ a positive number $\delta \in (0, 1)$ such that for all $x \in M$ and $h \in X$ with $\|h\|_X \leq \delta$ it holds that
\[
\frac{1}{\|h\|_X} \|T(x + h) - T(x) - DT(x)h\|_Y \leq \frac{\varepsilon}{B + 1}.
\]
Let now $y \in L$ and $d \in Y$ with $\|d\|_Y \leq \frac{\delta}{B + 1}$ be chosen arbitrarily. Set $x := S(y) \in M$ and $h := S(y + d) - S(y)$. Then, using the relation $DS(y') = $
Hence, we deduce the following estimate:

\[
\|h\|_X = \|S(y + d) - S(y)\|_X = \left\| \int_0^1 DS(y + td)dtdt \right\|_X \\
\leq \int_0^1 \|DT(S(y + td))^{-1}\|_{\mathcal{L}(Y, X)}\|d||Ydt \leq B\|d||Y \leq \frac{B}{B + 1}\delta < \delta.
\]

This implies the uniform Fréchet differentiability of the map \(S\) on every bounded subset. \(\square\)

**Proof of Lemma 24** Let a bounded subset \(M \subseteq X\) be given and take \(\varepsilon > 0\) arbitrarily. By the boundedness of \(T_1\) the set \(L := \text{conv}(T_1(M + B_X))\) is bounded. Using the boundedness of \(DT_1\) respectively \(DT_2\) define

\[
B_1 := \sup_{x \in M + B_X} \|DT_1(x)\|_{\mathcal{L}(X, Y)} < +\infty,
\]

and

\[
B_2 := \sup_{y \in L + B_Y} \|DT_2(y)\|_{\mathcal{L}(Y, Z)} < +\infty.
\]

By the uniform Fréchet differentiability of \(T_1\) there exist \(\varepsilon > 0\) and \(\delta \in (0, 1)\) with

\[
\frac{1}{\|h\|_X} \|T_1(x + h) - T_1(x) - DT_1(x)h\|_Y \leq \frac{\varepsilon}{B_1 + B_2 + 1}
\]

for all \(h \in X\) with \(\|h\|_X \leq \frac{\delta}{B_1 + B_2 + 1}\) and all \(x \in M\) and

\[
\frac{1}{\|k\|_X} \|T_2(y + k) - T_2(y) - DT_2(y)k\|_Y \leq \frac{\varepsilon}{B_1 + B_2 + 1}
\]

for all \(k \in Y\) with \(\|k\|_Y \leq \delta\) and all \(y \in L\). Then, we obtain for all \(y', y \in L + B_Y\)

\[
\|T_2(y') - T_2(y)\|_Z = \left\| \int_0^1 DT_2(y + t(y' - y))(y' - y)dtdt \right\|_Z \\
\leq \int_0^1 \|DT_2(y + t(y' - y))\|_{\mathcal{L}(Y, Z)}\|y' - y\|_Y dt \leq B_2\|y' - y\|_Y.
\]

(22)
For $x \in M$ and $h \in X$ with $\|h\|_X \leq \frac{x}{B_1+1}$ we get $\|DT_1(x)h\|_Y \leq \delta$ and hence using the uniform Fréchet differentiability the estimate \(22\) yields

$$
\|T_2(T_1(x+h)) - T_2(T_1(x)) - DT_2(T_1(x))DT_1(x)h\|_Z
\leq \|T_2(T_1(x+h)) - T_2(T_1(x) + DT_1(x)h)\|_Z
+ \|T_2(T_1(x) + DT_1(x)h) - T_2(T_1(x)) - DT_2(T_1(x))DT_1(x)h\|_Z
\leq B_2 \|T_1(x+h) - T_1(x) - DT_1(x)h\|_Y + \varepsilon \|DT_1(x)h\|_Y
\leq \frac{\varepsilon}{B_1 + B_2 + 1} B_2 \|h\|_X + \frac{\varepsilon}{B_1 + B_2 + 1} B_1 \|h\|_X
\leq \varepsilon \|h\|_X,
$$

which ends the proof.

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