Cospans spanning characterizations of violator and co-violator spaces

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Abstract

Given a finite set $E$ and an operator $\sigma : 2^E \rightarrow 2^E$, two subsets $X,Y \subseteq E$ are cospans spanning if $\sigma(X) = \sigma(Y)$ (Korte, Lovasz, Schrader; 1991). We investigate cospans spanning relations on violator spaces. A notion of a violator space was introduced in (Gärtner, Matoušek, Rüst, Škvorčová; 2008) as a combinatorial framework that encompasses linear programming and other geometric optimization problems.

Violator spaces are defined by violator operators. We introduce co-violator spaces based on contracting operators known also as choice functions. Let $\alpha, \beta : 2^E \rightarrow 2^E$ be a violator operator and a co-violator operator, respectively. Cospans spanning characterizations of violator spaces allow us to obtain some new properties of violator operators, co-violator operators, and their interconnections. In particular, we show that uniquely generated violator spaces enjoy so-called Krein-Milman properties, i.e., $\alpha(\beta(X)) = \alpha(X)$ and $\beta(\alpha(X)) = \beta(X)$ for every $X \subseteq E$.

Keywords: cospans spanning relation, uniquely generated violator space, co-violator space.

1 Introduction

Each set operator determines the partition of sets to equivalence classes with equal value of the operator. Let us have some set operator $\alpha$. Following [8]
we call two sets $X, Y$ cospanning if $\alpha(X) = \alpha(Y)$. Thus each set operator generates the cospanning equivalence relation on sets. Our goal is to investigate cospanning relations on violator spaces. These spaces were introduced in order to develop a combinatorial framework encompassing linear programming and other geometric optimization problems [5]. Violator spaces are defined by violator operators, which generalize closure operators [7]. We also pay special attention to violator spaces with unique bases. In Section 2, we introduce co-violator spaces based on contracting operators known also as choice functions. In Section 3, we characterize the cospanning relation with regards to violator spaces and describe the equivalence classes of the relation for violator and co-violator spaces. Cospanning characterizations allow us to obtain some new properties of violator operators, co-violator operators and their interconnections. In particular, we show that uniquely generated violator spaces enjoy so-called Krein-Milman properties.

1.1 Violator spaces

Violator spaces are arisen as a generalization of Linear Programming problems. LP-type problems have been introduced and analyzed by Matoušek, Sharir and Welzl [9, 11] as a combinatorial framework that encompasses linear programming and other geometric optimization problems. Further, Matoušek et al. [5] define a simpler framework: violator spaces, which constitute a proper generalization of LP-type problems. Originally, violator spaces were defined for a set of constraints $E$, where each subset of constraints $G \subseteq E$ was associated with $\nu(G)$ - the set of all constraints violating $G$.

The classic example of an LP-type problem is the problem of computing the smallest enclosing ball of a finite set of points in $\mathbb{R}^d$. Here $E$ is a set of points in $\mathbb{R}^d$, and the violated constraints of some subset of the points $G$ are exactly the points lying outside the smallest enclosing ball of $G$.

**Definition 1.1** [5] A violator space is a pair $(E, \nu)$, where $E$ is a finite set and $\nu$ is a mapping $2^E \to 2^E$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:

- $V11$: $X \cap \nu(X) = \emptyset$ (consistency),
- $V22$: $(X \subseteq Y$ and $Y \cap \nu(X) = \emptyset) \Rightarrow \nu(X) = \nu(Y)$ (locality).

Let $(E, \nu)$ be a violator space. Define $\varphi(X) = E - \nu(X)$. In what follows, if $(E, \nu)$ is a violator space and $\varphi(X) = E - \nu(X)$, then $(E, \varphi)$ will be called a violator space as well.
Definition 1.2 ([7]) A violator space is a pair \((E, \varphi)\), where \(E\) is a finite set and \(\varphi\) is an operator \(2^E \rightarrow 2^E\) such that for all subsets \(X, Y \subseteq E\) the following properties are satisfied:

1. **V1**: \(X \subseteq \varphi(X)\) (extensivity),
2. **V2**: \((X \subseteq Y \subseteq \varphi(X)) \Rightarrow \varphi(X) = \varphi(Y)\) (self-convexity).

Each violator operator \(\varphi\) is idempotent. Indeed, extensivity implies \(X \subseteq \varphi(X) \subseteq \varphi(\varphi(X))\). Then, by self-convexity, we conclude with \(\varphi(\varphi(X)) = \varphi(X)\).

Lemma 1.3 ([7]) Let \((E, \varphi)\) be a violator space. Then

\[
\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cup Y) = \varphi(X) = \varphi(Y)
\]

and

\[
(X \subseteq Y \subseteq Z) \land (\varphi(X) = \varphi(Z)) \Rightarrow \varphi(X) = \varphi(Y) = \varphi(Z)
\]

for every \(X, Y, Z \subseteq E\).

Since the second property deals with all sets lying between two given sets, following [10] we call the property convexity.

1.2 Uniquely generated violator spaces

Let \((E, \alpha)\) be an arbitrary space with the operator \(\alpha : 2^E \rightarrow 2^E\). \(B \subseteq E\) is a generator of \(X \subseteq E\) if \(\alpha(B) = \alpha(X)\). For \(X \subseteq E\), a basis (minimal generator) of \(X\) is a inclusion-minimal set \(B \subseteq E\) (not necessarily included in \(X\)) with \(\alpha(B) = \alpha(X)\). A space \((E, \alpha)\) is uniquely generated if every set \(X \subseteq E\) has a unique basis.

Proposition 1.4 [7] A violator space \((E, \varphi)\) is uniquely generated if and only if for every \(X, Y \subseteq E\)

\[
\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cap Y) = \varphi(X) = \varphi(Y)
\]

We can rewrite the property (3) as follows: for every set \(X \subseteq E\) of a uniquely generated violator space \((E, \varphi)\), the basis \(B\) of \(X\) is the intersection of all generators of \(X\):

\[
B = \bigcap\{Y \subseteq E : \varphi(Y) = \varphi(X)\}.
\]

One of the known examples of a not uniquely generated violator space is the violator space associated with the smallest enclosing ball problem. A basis of a set of points is a minimal subset with the same enclosing ball. In
particular, all points of the basis are located on the ball’s boundary. For \( \mathbb{R}^2 \) the set \( X \) of the four corners of a square has two bases: the two pairs of diagonally opposite points. Moreover, one of these pairs is a basis of the second pair. Thus the equality (4) does not hold.

For each arbitrary space \((E, \alpha)\) with the operator \(\alpha : 2^E \to 2^E\), an element \(x\) of a subset \(X \subseteq E\) is an extreme point of \(X\) if \(x \notin \alpha(X - x)\). The set of extreme points of \(X\) is denoted by \(\text{ex}(X)\).

**Proposition 1.5** [7] Let \((E, \varphi)\) be a violator space. Then
\[
\text{ex}(X) = \bigcap \{B \subseteq X : \varphi(B) = \varphi(X)\}.
\]

**Proposition 1.6** [7] Let \((E, \varphi)\) be a violator space. Then
\[
\text{ex}(\varphi(X)) \subseteq \text{ex}(X).
\]

**Theorem 1.7** [7] Let \((E, \varphi)\) be a violator space. Then \((E, \varphi)\) is uniquely generated if and only if for every set \(X \subseteq E\), \(\varphi(X) = \varphi(\text{ex}(X))\).

**Corollary 1.8** [7] Let \((E, \varphi)\) be a uniquely generated violator space. Then for every \(X \subseteq E\) the set \(\text{ex}(X)\) is the unique basis of \(X\).

2 Co-violator spaces

Theorem 1.7 and Proposition 1.6 show that there is some duality between extensive \((X \subseteq \varphi(X))\) and contracting \((\text{ex}(X) \subseteq X)\) operators. To study this connection we introduce a new type of spaces.

**Definition 2.1** A co-violator space is a pair \((E, c)\), where \(E\) is a finite set and \(c\) is an operator \(2^E \to 2^E\) such that for all subsets \(X, Y \subseteq E\) the following properties are satisfied:

- **CV1**: \(c(X) \subseteq X\),
- **CV2**: \((c(X) \subseteq Y \subseteq X) \Rightarrow c(X) = c(Y)\).

Operators satisfying the property **CV1** are called contracting operators.

In social sciences, contracting operators are called choice functions, usually adding a requirement that \(c(X) \neq \emptyset\) for every \(X \neq \emptyset\). The property **CV2** is called the *outcast property* or the *Aizerman property* [10].

The properties of co-violator spaces correspond to the corresponding ("mirrored") properties of violator spaces. For instance, every co-violator operator \(c\) is idempotent. Indeed, since \(c\) is contracting \((c(X) \subseteq c(X) \subseteq X)\). Then, **CV2** implies \(c(c(X)) = c(X)\).

Lemma 1.3 is converted to the following.
Lemma 2.2 Let \((E, c)\) be a co-violator space. Then

\[
c(X) = c(Y) \Rightarrow c(X \cap Y) = c(X) = c(Y)
\]

(5)

and

\[
(X \subseteq Y \subseteq Z) \land (c(X) = c(Z)) \Rightarrow c(X) = c(Y) = c(Z)
\]

(6)

for every \(X, Y, Z \subseteq E\).

Proof. Prove (5). Let \(c(X) = c(Y)\). \textbf{CV1} implies that \(c(X) \subseteq X\) and \(c(Y) = c(X) \subseteq Y\). Then \(c(X) \subseteq X \cap Y \subseteq X\), that gives (by \textbf{CV2}) \(c(X \cap Y) = c(X)\).

To prove (6) let \((X \subseteq Y \subseteq Z) \land (c(X) = c(Z))\). \textbf{CV1} yields \(c(Z) = c(X) \subseteq X \subseteq Y\). Then outcast property allows us to get \(c(Z) \subseteq Y \subseteq Z \Rightarrow c(Y) = c(X) = c(Z)\).

It is easy to see that all the properties of violator spaces hold in their dual interpretation for co-violator spaces. Since a co-violator operator is a choice function with outcast properties, the connection between these two types of spaces may result in better understanding of two theories and in new findings in each of them.

Connections between contracting and extensive operators were studied in many works, while most of them were dedicated to connections between choice functions and closure operators \([1, 3, 10]\). Naturally, extreme point operators were considered as choice functions. But, as we will see in Proposition 3.9, the extreme point operator of a violator space satisfies the outcast property, and so it forms a co-violator space, if and only if the violator space is uniquely generated. We also consider choice functions investigated in [4].

The interior operator (well-known in topology) is dual to a closure operator. Given an extensive operator \(\varphi : 2^E \rightarrow 2^E\), one can get a contracting operator \(c : c(X) = E - \varphi(E - X)\) or \(c(X) = \varphi(X)\).

Proposition 2.3 \((E, \varphi)\) is a violator space if and only if \((E, c)\) is a co-violator space, where \(c(X) = \varphi(X)\).

Proof. It is easy to see that \(\varphi\) is an extensive operator if and only if \(c\) is a contracting operator. To prove that \(c\) satisfies the outcast property if and only if \(\varphi\) is self-convex one has just to pay attention that:

\[
c(X) \subseteq Y \subseteq X \Leftrightarrow \overline{X} \subseteq \overline{Y} \subseteq c(\overline{X}) \Leftrightarrow \overline{X} \subseteq \overline{Y} \subseteq \varphi(X) \Rightarrow \varphi(\overline{X}) = \varphi(\overline{Y}) \Leftrightarrow c(X) = c(Y).
\]

The opposite direction is proved completely analogously.
3 Cospanning relations of violator and co-violator spaces

Let \( E = \{x_1, x_2, ..., x_d\} \). The graph \( H(E) \) is defined as follows. The vertices are the finite subsets of \( E \), two vertices \( A \) and \( B \) are adjacent if and only if they differ in exactly one element. Actually, \( H(E) \) is the hypercube on \( E \) of dimension \( d \), since the hypercube is known to be equivalently considered as the graph on the Boolean space \( \{0, 1\}^d \) in which two vertices form an edge if and only if they differ in exactly one position.

Let \( (E, \varphi) \) be a violator space. The two sets \( X \) and \( Y \) are equivalent (or cospanning) if \( \varphi(X) = \varphi(Y) \). In what follows, \( P \) denotes a partition of \( H(E) \) (or \( 2^E \)) into equivalence classes with regard to this relation, and \( [A]_\varphi := \{X \subseteq E : \varphi(X) = \varphi(A)\} \).

Remark 3.1 Note, that the cospanning relation associated with a violator operator \( \varphi \) coincides with the cospanning relation associated with an original violator mapping \( \nu \).

The following theorem characterizes cospanning relations in violator spaces.

**Theorem 3.2** Let \( E \) be a finite set and \( R \subseteq 2^E \times 2^E \) be an equivalence relation on \( 2^E \). Then \( R \) is the cospanning relation of a violator space if and only if the following properties hold for every \( X, Y, Z \subseteq E \):

- **R1**: if \( (X, Y) \in R \), then \( (X, X \cup Y) \in R \)
- **R2**: if \( X \subseteq Y \subseteq Z \) and \( (X, Z) \in R \), then \( (X, Y) \in R \).

**Proof.** Necessity follows immediately from Lemma 1.3.

Let us define an operator \( \varphi \) and prove that it satisfies extensivity and self-convexity. Since \( R \) is an equivalence relation, it defines a partition of \( 2^E \). Then, for each \( X \subseteq E \) there is only one class containing \( X \). Thus for every set \( X \), we define \( \varphi(X) \) as a maximal element in the class \( [X]_R \). Notice, that the property **R1** implies that each equivalence class has a unique maximal element, so the partition is well-defined. Hence, we obtain that \( X \subseteq \varphi(X) \) and \( \varphi(\varphi(X)) = \varphi(X) \). Then the self-convexity follows immediately from **R2**. It is easy to see that the cospanning relation w.r.t. \( \varphi \) coincides with \( R \).

In conclusion, each equivalence class of the cospanning relation of a violator space is closed under union (**R1**) and convex (**R2**).

The following theorem characterizes equivalence classes of co-violator spaces.
**Theorem 3.3** Let $E$ be a finite set and $R \subseteq 2^E \times 2^E$ be an equivalence relation on $2^E$. Then $R$ is the cospanning relation of a co-violator space if and only if the following properties hold for every $X, Y, Z \subseteq E$:

- **R3**: if $(X, Y) \in R$, then $(X, X \cap Y) \in R$
- **R2**: if $X \subseteq Y \subseteq Z$ and $(X, Z) \in R$, then $(X, Y) \in R$.

**Proof.** Necessity follows immediately from Lemma 2.2. By analogy with the proof of Theorem 3.2 we define $c(X)$ to be a minimal element in the class $[X]_R$. Since each class is closed under intersection (R3), the partition is well-defined. It is easy to see that operator $c$ is contracting, satisfies the outcast property, and its cospanning relation coincides with $R$. ■

Consider now both a violator operator $\varphi$ and a co-violator operator $c(X) = \overline{\varphi(X)}$.

**Proposition 3.4** There is a one-to-one correspondence between an equivalence class $[X]_\varphi$ of $X$ of the cospanning relation associated with a violator operator $\varphi$ and an equivalence class $[X]_c$ w.r.t. a co-violator operator $c$, i.e., $A \in [X]_\varphi$ if and only if $\overline{A} \in [X]_c$.

**Proof.** Indeed, $A \in [X]_\varphi \iff \varphi(X) = \varphi(A) \iff c(X) = c(A) \iff c(X) = c(A) \iff \overline{A} \in [X]_c$. ■

A uniquely generated violator space defines a cospanning relation with additional property R3 (see Proposition 1.4).

All in all, every uniquely generated violator space is a co-violator space as well. Each equivalence class of the cospanning relation of a uniquely generated violator space has an unique minimal element and an unique maximal element. More precisely, for the sets $A \subseteq B \subseteq E$, let us define the interval $[A, B]$ as $\{C \subseteq E : A \subseteq C \subseteq B\}$. Then each equivalence class of an uniquely generated violator space is an interval. We call a partition of $H(E)$ into disjoint intervals a **hypercube partition**. The following Theorem follows immediately from Theorem 3.2 and Proposition 1.4.

**Theorem 3.5** ([2]) (i) If $(E, \varphi)$ is a uniquely generated violator space, then $P$ is a hypercube partition of $H(E)$.

(ii) Every hypercube partition is the partition $P$ of $H(E)$ into equivalence classes of a uniquely generated violator space.

More specifically [7], $[A]_\varphi = [ex(A), \varphi(A)]$ for every set $A \subseteq E$.

Let us consider now a uniquely generated violator space $(E, \varphi)$ and the operator $ex$. Since each equivalence class $[A]_\varphi$ w.r.t. operator $\varphi$ is an interval
[ex(A), ϕ(A)]}, we can see that for each X ∈ [ex(A), ϕ(A)] not only ϕ(X) = ϕ(A), but ex(X) = ex(A) as well. Since P is a hypercube partition of \( H(E) \) we conclude with \([X]_ϕ = [X]_ex\). Thus the cospanning partition (quotient set) associated with an operator ϕ coincides with the cospanning partition associated with a contracting operator ex. Since \( ex(X) \) is a minimal element of \([X]\) we immediately obtain the following

**Proposition 3.6** If \((E, ϕ)\) is a uniquely generated violator space, then operator ex satisfies the following properties:

\[\begin{align*}
X1: & \quad ex(ex(X)) = ex(X) \\
X2: & \quad ex(X) = ex(Y) \Rightarrow ex(X \cup Y) = ex(X) = ex(Y) \\
X3: & \quad (X \subseteq Y \subseteq Z) \land (ex(X) = ex(Z)) \Rightarrow ex(X) = ex(Y) = ex(Z) \\
X4: & \quad ex(X) = ex(Y) \Rightarrow ex(X \cap Y) = ex(X) = ex(Y)
\end{align*}\]

If \((E, ϕ)\) is not a uniquely generated violator space, then the operator ex may or may not satisfy the properties X1-X4. Consider the two following examples.

**Example 3.7** Let \( E = \{1, 2, 3\} \). Define \( ϕ(X) = X \) for each \( X \subseteq E \) except \( ϕ(\{2\}) = \{2, 3\} \) and \( ϕ(\{1, 2\}) = \{1, 3\} = \{1, 2, 3\} \). It is easy to check that \((E, ϕ)\) is a violator space and the operator ex satisfies X1, X2, and X4, but while \( ex(\{1\}) = ex(\{1, 2, 3\}) = \{1\} \), \( ex(\{1\}) \neq ex(\{1, 2\}) \), i.e., the operator ex is not convex.

**Example 3.8** Let \( E = \{1, 2, 3, 4, 5, 6\} \). Define \( ϕ(X) = X \) for each \( X \subseteq E \) except \( ϕ(\{1\}) = \{1, 2\} \), \( ϕ(\{1, 2\}) = ϕ(\{1, 2, 3\}) = \{1, 2, 4\} = \{1, 2, 3, 4\} \) and \( ϕ(\{1, 2, 5\}) = ϕ(\{1, 2, 6\}) = \{1, 2, 5, 6\} \). It is easy to check that \((E, ϕ)\) is a violator space. In addition, \( ex(\{1, 2, 3, 4\}) = ex(\{1, 2, 5, 6\}) = \{1, 2\} \), while \( ex(\{1, 2\}) = \{1\} \). Hence, ex is not idempotent (X1) and does not satisfy X4. Since \( ex(\{1, 2, 3, 4, 5, 6\}) = \{1, 2, 3, 4, 5, 6\} \) the operator ex does not satisfy X2 as well, but, compared to the previous example, ex is convex.

**Proposition 3.9** Let \((E, ϕ)\) be a violator space. The following assertions are equivalent:

(i) \((E, ϕ)\) is uniquely generated
(ii) \( X5: \quad ex(X) \subseteq Y \subseteq X \Rightarrow ex(X) = ex(Y) \) (the outcast property)
(iii) \( X6: \quad ϕ(ex(X)) = ϕ(X) \)
(iv) \( X7: \quad ex(ϕ(X)) = ex(X) \)

**Proof.** If \((E, ϕ)\) is a uniquely generated violator space, then operator ex satisfies X5, X6 and X7, since \([X]_ϕ = [X]_ex = [ex(X), ϕ(X)]\).
Before we continue with the proof, it is important to mention that from the definition of the operator $ex$ it follows that $ex(B) = B$ for each basis $B$.

Further we prove that if a violator space $(E, \varphi)$ satisfies the property $X5$, then it is uniquely generated. Suppose that there is a set $X \subseteq E$ with two bases $B_1$ and $B_2$. Then $\varphi(X) = \varphi(B_1) = \varphi(B_2) = \varphi(B_1 \cup B_2)$. Thus Proposition 1.5 implies $ex(B_1 \cup B_2) \subseteq B_1 \cap B_2$. Then we have $ex(B_1 \cup B_2) \subseteq B_1 \subseteq B_1 \cup B_2$ and $ex(B_1 \cup B_2) \subseteq B_2 \subseteq B_1 \cup B_2$, but $ex(B_1) = B_1 \neq ex(B_2) = B_2$. In other words, we see that $ex$ does not satisfy the outcast property.

$(iii) \Rightarrow (i)$ follows from Theorem 1.7.

Now, it is only left to prove that if a violator space $(E, \varphi)$ satisfies the property $X7$, then it is uniquely generated. Suppose there is a set $X \subseteq E$ with two bases $B_1 \neq B_2$. Then $\varphi(X) = \varphi(B_1) = \varphi(B_2)$, and so $ex(\varphi(B_1)) = ex(\varphi(B_2))$. Since $ex(B_1) = B_1 \neq ex(B_2) = B_2$, we conclude that the property $X7$ does not hold.

It is worth reminding that $X6$ and $X7$ are called Krein-Milman properties. In other words, every uniquely generated violator space is a Krein-Milman space [6].

4 Conclusion

Many combinatorial structures are described using operators defined on their ground sets. For instance, closure spaces are defined by closure operators, and violator spaces are described by violator operators. In this paper, we introduced co-violator spaces based on contracting operators known also as choice functions. Cospanning characterizations of violator spaces allowed us to obtain some new properties of violator operators, co-violator operators and their interconnections. In further research, our intent is to extend this "cospanning" approach to a wider spectrum of combinatorial structures closure spaces, convex geometries, antimatroids, etc.

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