Induced Representations of Quantum Groups

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Abstract

In this paper we show how to construct explicitly induced representations for bicrossproduct Hopf algebras with abelian kernels starting from one-dimensional characters of the commutative sector. We introduce this technique by means of two concrete physical examples: two quantum deformations of the (1 + 1) Galilei algebra.

1 Introduction

From its beginning in 1986 quantum groups and quantum algebras have largely attracted the attention of mathematicians and physicists. The main reason for this fascination is the very rich mathematical structure carried by these objects, which allows to mimic systematically useful constructions developed in many other well known branches of Mathematics, in particular, in Lie group theory. In consequence, there are a huge variety of potential applications of quantum groups ranging from integrable systems or quantum mechanics to conformal field theory (see, for instance, Ref. [2] and [3]).

The contribution of this paper is located within the applications of these new mathematical entities to the description of deformed symmetries (or \(q\)-symmetries) of physical systems as well as of the space-time. Quantum kinematical algebras and groups can be used for the study of \(q\)-symmetries of the deformed space-time (\(q\)-space-time), since the \(q\)-space-time can be considered as a non-commutative homogeneous space of the

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quantum kinematical groups. Consequently, we are interested in quantum groups as the adequate tool to describe the short range of the space-time structure, which looks to be non-commutative.

Another approach to quantum groups is closely related with the deformation of the commutative algebra of functions. One of the most interesting examples of it is related with the problem of the quantization of physical systems and with the deformation of phase spaces [4] (see Ref. [5] for a review and references therein).

On the other hand, the study of the representations of the quantum kinematical groups is an interesting problem, that can be useful for determining the behaviour of physical systems endowed with deformed symmetries. Obviously, it looks natural to construct the representations of quantum groups in the framework of non-commutative homogeneous spaces, which are the natural arena for quantum groups. This procedure fits in Connes’ program of noncommutative geometry [6].

Moreover, it is expected a rich interplay between $q$–spaces and representations, in particular, in relation with the $q$-analogous of the harmonic analysis and $q$–special functions.

Physically, as it is well known, a projective unitary irreducible representation of a symmetry group of a given physical system leads to a definition of quantum elementary physical system [7], and also gives a prescription for computing expected values (the observables are assumed to form the symmetry algebra we start with).

Kinematical groups like Poincar´e and Galilei are semidirect product of the translation group and the homogeneous group of rotations and boosts (Lorentz or homogeneous Galilei, respectively). Therefore, the most appropriated method to construct their unitary representations is the Mackey method for induced representations of semidirect products [8].

In this paper we obtain the induced representations of two non-equivalent quantum deformations of the $(1 + 1)$ Galilei algebra by using a generalization of Mackey’s method. In both cases included here the quantum $(1+1)$ Galilei algebra has a structure of bicrossproduct, which is a generalization of the semidirect product of Lie groups (or algebras) to Hopf algebras [9]. That constitutes the first approximation in order to get a quantum analogue of Mackey’s theory. Some attempts have been made to extend this technique to the quantum case from the mathematical [10] as well as from the physical [11, 12, ?] point of view. However, in all these cases the approach has been mainly focused on corepresentations of quantum groups, in other words, in representations of the coalgebra part. However, this paper deals with the dual case, closer to the classical one, constructing representations in the algebra part.

The organization of the paper is as follows. In Section 2 we review the algebraic structure related with the topics of Hopf algebra, quantum algebra and quantum group. The bicrossproduct structure is also described here. In next Section we introduce the basic elements of the theory of induced representations of quantum groups, which is connected with module theory, and build up induced representations for two non-equivalent defor-
mations of the Galilei algebra. Some comments and remarks on the results obtained here together with a collection of open problems close the paper.

2 Quantum groups and quantum algebras

As it is well known quantum groups and quantum algebras are neither Lie groups nor Lie algebras, but the mathematical structure underlying both kind of objects is that of Hopf algebra.

2.1 Hopf algebras

A Hopf algebra restores, in some sense, the symmetry lost when a product law is added to a (complex) vector space $V$ in order to get an algebra. The Hopf algebraic setting allows not only for the possibility to compose but also to “decompose” elements in $V$. More explicitly, on the linear space $V$ we have two linear mappings $m : V \otimes V \to V$, $\Delta : V \to V \otimes V$, referred to as the product and the coproduct, respectively. Both mappings are compatible in the sense that

$$\Delta \circ m = (m \otimes m) \circ (\text{id} \otimes \tau \circ \text{id}) \circ (\Delta \otimes \Delta),$$

where $\tau(v \otimes v') = v' \otimes v$ is the “flip” operator on $V \otimes V$. This compatibility means indeed that $\Delta$ (or $m$) is a morphism of algebras (or coalgebras) when a suitable definition of algebra (or coalgebra) is introduced on $V \otimes V$.

The application $m$ satisfies some properties that have natural analogues for $\Delta$, which are systematically prefixed with “co”. For example, the product $m$ is associative, i.e.,

$$m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m),$$

while the coproduct is said to be coassociative

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta.$$

The product is required to have a unit, and correspondingly the coproduct must have a counit. Algebraically this means that we have two linear mappings $\eta : \mathbb{C} \to V$, $\epsilon : V \to \mathbb{C}$, satisfying

$$m \circ (\eta \otimes \text{id}) = \text{id} = m \circ (\text{id} \otimes \eta), \quad (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta.$$
The algebraic structure we have described so far is known as a bialgebra, which can be seen as combination of two triplets \((V, m, \eta)\) and \((V, \Delta, \epsilon)\) called algebra and coalgebra, respectively.

Hopf algebras are bialgebras characterized besides by the existence of a linear antismorphism \(\gamma : V \rightarrow V\) verifying

$$m \circ (\gamma \otimes \text{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\text{id} \otimes \gamma) \circ \Delta.$$  

The mapping \(\gamma\) is called antipode, and it is easy to show that if it exists then is unique.

As examples of this kind of structure we can mention: (finite) group algebras, the algebra of functions on a (finite, Lie) group, and enveloping algebras of Lie algebras. All these examples have the property of being commutative or cocommutative, i.e.,

$$m \circ \tau = m, \quad \text{or} \quad \tau \circ \Delta = \Delta,$$

we can also say that these Hopf algebras are non-deformed or “classical”.

### 2.2 Quantum algebras and quantum groups

Quantum groups and quantum algebras are examples of Hopf algebras which are neither commutative nor cocommutative. There is a usual definition of quantum algebra in the sense of Drinfel’d \[1\] and Jimbo \[14\], whereas there are several approaches for quantum groups \[15, 16, 17\].

Let \(g\) be a Lie algebra and \(\mathcal{U}(g)\) its universal enveloping algebra, which is a “classical” Hopf algebra with coproduct, counit and antipode defined by

\[
\Delta(X) = 1 \otimes X + X \otimes 1, \quad \Delta(1) = 1 \otimes 1; \\
\epsilon(X) = 0, \quad \epsilon(1) = 1; \\
\gamma(X) = -X,
\]

where \(X \in g\).

A quantization or deformation of \(\mathcal{U}(g)\) is obtained by means of a deformed Hopf structure on \(\mathcal{U}_z(g) \equiv \mathcal{U}(g) \hat{\otimes} \mathbb{C}[[z]]\), which is the associative algebra of formal power series in \(z\) and coefficients in \(\mathcal{U}(g)\), such that

\[
\mathcal{U}_z(g) / z \mathcal{U}_z(g) \simeq \mathcal{U}(g)
\]

as Hopf algebras (in other words, \(\mathcal{U}_z(g) \rightarrow \mathcal{U}(g)\) when \(z \rightarrow 0\)).

On the other hand, let \(G\) be a finite dimensional Lie group and \(g\) its Lie algebra. Let us consider the commutative and associative algebra of smooth functions of \(G\) on \(\mathbb{C}, \text{Fun}(G)\), with the usual product of functions (i.e., \((fg)(x) = f(x)g(x), f, g \in \text{Fun}(G), x, y \in G\)). This algebra has a Hopf structure as follows

\[(\Delta(f))(x, y) = f(xy), \quad \epsilon(f) = f(e), \quad (\gamma(f))(x) = f(x^{-1}),\]

where \(f \in \text{Fun}(G)\).
where \( f \in \text{Fun}(G) \), \( x, y \in G \), and \( e \) is the unit element of \( G \). Note that in general \( \text{Fun}(G) \otimes \text{Fun}(G) \subseteq \text{Fun}(G \times G) \). When the group is finite the equality is strict, but if \( G \) is not a finite group \( \Delta(f) \) may not belong to \( \text{Fun}(G) \otimes \text{Fun}(G) \). This problem can be solved by an adequate restriction of the space \( \text{Fun}(G) \).

Incidentally, \( \text{Fun}(G) \) is the Hopf algebra dual of \( \mathcal{U}(\mathfrak{g}) \) by means of a suitable duality (for more details see, for instance, Ref. [18]).

After deformation the above commutative Hopf algebra becomes non-commutative. On the other hand, \( \text{Fun}(G) \) is cocommutative if and only if \( G \) is abelian.

Examples of quantum algebras and quantum groups appear in Section 3.

### 2.3 Bicrossproduct Hopf algebras

As we mentioned before, a bicrossproduct Hopf algebra can be seen as a generalization of the semidirect product of groups [9, 19]. In the following we present the essentials about this concept.

Let us start recalling the definition of \( R \)–module. Let \( R \) be a unital ring, and \( \mathcal{X} \) a set equipped with an internal composition law denoted by +, and an external composition law, \( \triangleright \), with domain of operators in \( R \). We say that \((\mathcal{X}, +, \triangleright)\) is a left \( R \)–module if

i) \( (\mathcal{X}, +) \) is an abelian group,

ii) the external law \((R \times \mathcal{X} \to \mathcal{X})\) satisfies

\[
\alpha \triangleright (\beta \triangleright x) = (\alpha \beta) \triangleright x, \quad \forall \alpha, \beta \in R, \forall x \in \mathcal{X},
\]

\[
1 \triangleright x = x, \quad \forall x \in \mathcal{X},
\]

(2.1)

iii) the internal and the external law are compatible in the sense that

\[
\alpha \triangleright (x + y) = (\alpha \triangleright x) + (\alpha \triangleright y), \quad \forall \alpha \in R, \forall x, y \in \mathcal{X},
\]

\[
(\alpha + \beta) \triangleright x = (\alpha \triangleright x) + (\beta \triangleright x), \quad \forall \alpha, \beta \in R, \forall x \in \mathcal{X}.
\]

(2.2)

In the cases of interest for us we will consider the ring associated with the Hopf algebra \( H \), and the set \( \mathcal{X} \) is \textit{ab initio} a \( \mathbb{C} \)–vector space, denoted \( V \), hence an abelian group. So, we can rewrite the definition of module as follows.

The pair \((V, \rho)\), where \( \rho : H \otimes V \longrightarrow V \) is a linear map, is said to be an \( H \)–module if the external composition law (action) defined by

\[
h \triangleright v = \rho(h \otimes v)
\]

(2.3)

satisfies axioms (2.1). Note that the compatibility conditions (2.2) are now encoded in the linearity of \( \rho \). This mapping is also called a representation of \( H \) on \( V \) since it allows to represent the elements of \( H \) by endomorphisms of \( V \).
On the other hand, comodules are the dual objects to modules. The pair \((V, \beta)\), where \(\beta : V \rightarrow H \otimes V \ (\beta(v) = v^{(1)} \otimes v^{(2)})\) is a linear map, is said to be an \(H\)-comodule if the “external decomposition” law (coaction) defined by \(\beta\) satisfies

\[
(v^{(1)} \blacktriangleleft) \otimes v^{(2)} = \Delta(v^{(1)}) \otimes v^{(2)}, \quad \forall v \in V; \\
\epsilon(v^{(1)}) \otimes v^{(2)} = v, \quad \forall v \in V; 
\]

(2.4)

where we have written \(\beta(v) = v \blacktriangleleft\) to make the notation more symmetric between actions and coactions. Similarly to \(\rho\) the mapping \(\beta\) is called corepresentation.

Remark that for the last two definitions and do not take into account the whole Hopf algebra structure. Thus, only the algebra (coalgebra) sector is used for modules (comodules).

When a Hopf algebra \(H\) acts on an algebra \(A\) it is natural to demand some compatibility of the action with the algebraic structure. So, we say that \(A\) is a right \(H\)-module algebra if it is an \(H\)-module and the action satisfies

\[
(aa') \triangleright h = (a \triangleright h_{(1)})(a' \triangleright h_{(2)}), \\
1 \triangleright h = \epsilon(h),
\]

where \(\Delta(h) = \sum h_{(1)} \otimes h_{(2)}\).

A similar situation happens when \(H\) coacts on a coalgebra \(C\). Then, it is said that \(C\) is a left \(H\)-comodule coalgebra if it is an \(H\)-comodule and the coaction (\(\blacktriangleleft\)) satisfies

\[
c^{(1)} \triangleright c^{(2)} = 1_H \epsilon(c), \\
c^{(1)} \otimes c^{(2)}_{(1)} \otimes c^{(2)}_{(2)} = c^{(1)}_{(1)} c^{(1)}_{(2)} \otimes c^{(2)}_{(1)} \otimes c^{(2)}_{(2)},
\]

where \(c \triangleright = c^{(1)} \otimes c^{(2)}\).

It is also possible to define module coalgebras and comodule algebras (both with right and left-handed versions, of course) but we shall not need it.

The introduction of module algebras allows to go a step beyond the direct sum of algebras. If \(A\) is a right \(H\)-module algebra we can define an algebra structure on the tensor product \(H \otimes A\) by means of the composition law

\[
(h \otimes a)(h' \otimes a') = hh'_{(1)} \otimes (a \triangleright h'_{(2)})a'.
\]

It is immediate to check that the new algebra has \(1_H \otimes 1_A\) as unit element. This structure is called the semidirect product \(H \ltimes A\).

By duality, if we start with the left \(H\)-comodule coalgebra \(C\) we can construct a coalgebra structure on the tensor product \(C \otimes H\) defining the coproduct as

\[
\Delta(c \otimes h) = c_{(1)} \otimes c_{(2)}^{(1)} h_{(1)} \otimes c_{(2)}^{(2)} \otimes h_{(2)},
\]

and taking \(\epsilon_C \otimes \epsilon_H\) as antipode. The resulting coalgebra is the semidirect product \(C \rtimes H\).
Now, let us change the notation to consider simultaneously two Hopf algebras $K$ and $L$, such that $L$ is a right $K$–module algebra and $K$ is a left $L$–comodule coalgebra. According to the above semidirect product constructions $K \otimes L$ is equipped with a structure of algebra $(K \bowtie L)$ and other of coalgebra $(K \rhd L)$. The following five compatibility conditions \[\]

\begin{align*}
\epsilon(l \triangleleft k) &= \epsilon(l)\epsilon(k), \\
\Delta(l \triangleleft k) &= (l_{(1)} \triangleleft k_{(1)})k_{(2)}^{(1)} \otimes l_{(2)} \triangleleft k_{(2)}^{(2)}, \\
1 \triangleleft 1 &= 1 \otimes 1, \\
(kk') \triangleleft &= (k_{(1)} \triangleleft k'_{(1)})k'_{(2)}^{(1)} \otimes k_{(2)}^{(2)}, \\
k_{(1)}^{(1)}(l \triangleleft k_{(2)}) \otimes k_{(1)}^{(2)} &= (l \triangleleft k_{(1)})k_{(2)}^{(1)} \otimes k_{(2)}^{(2)},
\end{align*}

are sufficient conditions to guarantee that both structures fit adequately to form a bialgebra with antipode: the right-left bicrossproduct Hopf algebra $K \bowtie L$. The antipode is given by

\[\gamma(k \otimes l) = (1 \otimes \gamma(k^{(1)}l)) (\gamma(k^{(2)}) \otimes 1).\]

In analogy with the classical case we shall refer to $L$ as the kernel of the bicrossproduct. For our purposes we are interested in the case in which the kernel is commutative.

Since $K$ and $L$ generate $K \bowtie L$ we can construct a basis of $K \bowtie L$ using bases of $K$ and $L$.

When $K$ and $L$ are the universal enveloping algebras of Lie algebras $\mathfrak{f}$ and $\mathfrak{l}$, respectively, the right action of $K$ on $L$ is given by means of the Lie commutators, i.e., $l \triangleleft k = [l, k], \ l \in L, \ k \in \mathfrak{f}$.

The left-right version is constructed in a similar way. In this case one considers the right $K$–comodule coalgebra $\mathcal{L}$ and the left $\mathcal{L}$–module algebra $\mathcal{K}$. The new product and coproduct on $K \otimes \mathcal{L}$ are defined by

\begin{align*}
(k \otimes \lambda)(k' \otimes \lambda') &= \kappa(\lambda_{(1)} \triangleright k') \otimes \lambda_{(2)}\lambda', \\
\Delta(k \otimes \lambda) &= (\kappa_{(1)} \otimes \lambda_{(1)}^{(1)}) \otimes (\kappa_{(2)}\lambda_{(2)}^{(1)} \otimes \lambda_{(2)}).
\end{align*}

The unit and counit are as in the right-left case. The compatibility conditions read off as

\begin{align*}
\epsilon(\lambda \triangleright \kappa) &= \epsilon(\lambda)\epsilon(\kappa), \\
\Delta(\lambda \triangleright \kappa) \equiv (\lambda \triangleright \kappa)_{(1)} \otimes (\lambda \triangleright \kappa)_{(2)} &= (\lambda_{(1)}^{(1)} \triangleright \kappa_{(1)}) \otimes \lambda_{(2)}^{(1)}(\lambda_{(2)} \triangleright \kappa_{(2)}), \\
\triangleright (1) &\equiv 1^{(1)} \otimes 1^{(2)} = 1 \otimes 1, \\
\triangleright (kk') \equiv (kk')^{(1)} \otimes (kk')^{(2)} &= \kappa_{(1)}^{(1)}\kappa'}^{(1)} \otimes \kappa_{(2)}^{(1)}(\kappa_{(2)} \triangleright \kappa'_{(2)}), \\
\lambda_{(2)}^{(1)} \otimes (\lambda_{(1)} \triangleright \kappa)\lambda_{(2)}^{(2)} &= \lambda_{(1)}^{(1)} \otimes \lambda_{(1)}^{(2)}(\lambda_{(2)} \triangleright \kappa).
\end{align*}

The left-right bicrossproduct structure is denoted by $\mathcal{K} \rhd \mathcal{L}$.

In the finite dimensional case it is easy to show that $(K \bowtie L)^* = K^* \rhd L^*$. For the cases we are interested in, although they are infinite dimensional, a similar result holds, provided that “duality” is changed for “dually paired algebras” (see Ref. \[\] or \[\]).
On the other hand, one can prove that given a bicrossproduct Hopf algebra, \( H = K \rhd L, \) with dual \( H^* = K\triangleright L^* \), a nondegenerate dual pairing between \( H \) and \( H^* \) can be defined in terms of nondegenerate pairings \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) for the pairs \( (K, K^*) \) and \( (L, L^*) \), respectively, by
\[
\langle kl, \kappa \lambda \rangle = \langle k, \kappa \rangle_1 \langle l, \lambda \rangle_2.
\]
An immediate consequence of this statement, that we shall use later, is that with the pairings defined above, if \( \{k_m\} \) and \( \{\kappa^m\} \) are dual basis for \( K \) and \( K^* \), and \( \{l_n\} \) and \( \{\lambda^n\} \) are dual basis for \( L \) and \( L^* \), respectively, then \( \{k_m l_n\} \) and \( \{\kappa^m \lambda^n\} \) are dual basis for \( H \) and \( H^* \).

### 3 Induced representations

In the theory of representations of Hopf algebras the symmetry played by the algebra and coalgebra structures is broken. On the one hand, the algebra structure of a Hopf algebra, \( H \), leads to a ring structure on \( H \) and, hence, to (in general non-commutative) module theory, but on the other hand, the coalgebra structure allows a tensor product of \( H \)-modules turning this category into a monoidal one.

Induced representations are precisely extensions of scalars from the point of view of module theory. Effectively, let us consider the unital associative algebra \( A \) as a ring with unit, let \( B \) be a subalgebra of \( A \) containing the unit and \( V \) a right \( B \)-module. The algebra \( A \) can be considered as a left \( B \)-module (by means of left regular translations), and therefore the tensor product (on \( B \)) \( V \otimes_B A \) makes sense. In this last \( B \)-module, \( V \otimes_B A \), we can extend the scalars to \( A \), and then we say that \( V^\uparrow = V \otimes_B A \) is the \( A \)-module induced from the \( B \)-module \( V \).

A similar construction can be carried out by replacing \( A \) by its linear dual \( A^* \), and looking at it as the right \( B \)-module associated with the left regular action on \( A \). In the literature this construction is referred to as coinduced representations \([20]\) or produced representations \([21]\).

It is worthy to note that the terminology about representations in quantum group literature is a bit confusing. So, terms like “representation” and “induced representation” have been also used to denote “corepresentation” and “induced corepresentation” \([11] \). Hoping not introduce more confusion we will speak of induced representations in the sense of the preceding paragraph.

The ideas introduced above allow us to develop the construction of the induced representations for two interesting physical examples, corresponding to two non-equivalent deformations of the one-dimensional Galilei algebra.
3.1 Standard quantum (1 + 1) Galilei algebra

The standard quantum (1 + 1) Galilei algebra for which we calculate the induced representations is a contraction of the $\kappa$-Poincaré in (1 + 1) dimensions \[22\].

3.1.1 Algebraic structure

The Hopf algebra structure of the standard quantum (1 + 1) Galilei algebra, $U_\omega[\mathfrak{g}(1,1)]$, is defined by

\[
\begin{align*}
[H, K] &= -P, \quad [P, K] = \omega P^2, \quad [H, P] = 0; \\
\Delta H &= H \otimes 1 + 1 \otimes H, \quad \Delta X = X \otimes 1 + \exp(-2\omega H) \otimes X, \quad X \in \{P, K\}; \\
\epsilon(X) &= 0, \quad X \in \{H, P, K\}; \\
\gamma(H) &= -H, \quad \gamma(X) = -e^{2\omega H} X, \quad X \in \{P, K\}.
\end{align*}
\]

The Hopf algebra $U_\omega[\mathfrak{g}(1,1)]$ has a bicrossproduct structure \[22\] given by $U_\omega[\mathfrak{g}(1,1)] = U[\mathbb{R}] \bowtie U_\omega[t_2]$, where $U[\mathbb{R}] = \langle K \rangle$ (“boost sector”) and $U_\omega[t_2]$ is the deformed subalgebra generated by $P$ and $H$ (“translation sector”). The right action of $U[\mathbb{R}]$ on $U_\omega[t_2]$ is given by

\[ P \triangleright K = \omega P^2, \quad H \triangleright K = -P, \]

and the left coaction of $U_\omega[t_2]$ on $U[\mathbb{R}]$ is

\[
K \triangleright = e^{-2\omega H} \otimes K.
\]

The dual algebra $F_\omega[G(1,1)]$ is generated by the local coordinates $v, x, t$. The commutators, coproduct, counit and antipode are given by \[22\]

\[
\begin{align*}
[t, x] &= -2\omega x, \quad [x, v] = \omega v^2, \quad [t, v] = -2\omega v; \\
\Delta t &= t \otimes 1 + 1 \otimes t, \quad \Delta x = x \otimes 1 + 1 \otimes x - t \otimes v, \quad \Delta v = v \otimes 1 + 1 \otimes v; \\
\epsilon(f) &= 0, \quad f \in \{v, t, x\}; \\
\gamma(v) &= -v, \quad \gamma(x) = -x - tv, \quad \gamma(t) = -t.
\end{align*}
\]

The bicrossproduct structure $F_\omega[G(1,1)] = \langle v \rangle \bowtie \langle x, t \rangle$ is encoded in the left action

\[ x \triangleright v = \omega v^2, \quad t \triangleright v = -2\omega v, \]

and right coaction

\[ \triangleright x = x \otimes 1 - t \otimes v, \quad \triangleright t = t \otimes 1. \]

The duality pairing between the Hopf algebras $U_\omega[\mathfrak{g}(1,1)]$ and $F_\omega[G(1,1)]$ is given explicitly by

\[
\langle K^m P^n H^p, v^q x^r t^s \rangle = m! n! p! \delta_q^m \delta_r^n \delta_s^p.
\]
3.1.2 Induced representations

The representations of $U^{[g(1,1)]}$ are induced using (left–)characters of the translation sector

$$1 \rightarrow P^n H^p = (ia)^n(ib)^p, \quad n, p \in \mathbb{N}, \quad (3.2)$$

where $\rightarrow$ stands for the action of $U^{[g]}$ on $C$. These induced representations have as carrier space $C^{\uparrow}$ the space $\text{Hom}_{U^{[g]}}(U^{[g(1,1)]}, C)$, which is contained in the space $\text{Hom}_C(U^{[g(1,1)]}, C) = U^{[g(1,1)]*}$. This later can be identified with $F^{[G(1,1)]}$. Since the elements of $C^{\uparrow}$ are $U^{[g]}$–morphisms they are characterized by the “equivariance condition”

$$f(XP^n H^p) = f(X) \rightarrow P^n H^p. \quad (3.3)$$

A generic element of $F^{[G(1,1)]}$ will be

$$f = f_{q,r,s}v^q x^r t^s, \quad (3.4)$$

then, using the pairing $(3.1)$ and imposing the equivariance condition $(3.3)$, in order to have $f$ contained in $C^{\uparrow}$, we get

$$q!r!s!f_{q,r,s} = \langle f, K^q P^r H^s \rangle = \langle f, K^q \rangle \rightarrow P^r H^s = q!f_{q,0,0}(ia)^r(ib)^s. \quad (3.5)$$

Introducing this last relation $(3.5)$ in expression $(3.4)$ we obtain that $C^{\uparrow}$ is the subspace of $F^{[G(1,1)]}$ whose elements are of the form $\phi(v)e^{iax}e^{ibt}$.

Let us consider the basis $v^m e^{iax}e^{ibt}$ of $C^{\uparrow}$. The action of the elements $X$ of $U^{[g(1,1)]}$ on it will be given in terms of the $C$–numbers $[X]^m_{q,r,s}$ by means of the expression

$$v^m e^{iax}e^{ibt} \rightarrow X = [X]^m_{q,r,s}v^q x^r t^s, \quad (3.6)$$

where $\rightarrow$ also denotes the action of $U^{[g(1,1)]}$ on $C^{\uparrow}$. The evaluation of $[X]^m_{q,r,s}$ is made using the pairing $(3.1)$

$$q!r!s![X]^m_{q,r,s} = \langle v^m e^{iax}e^{ibt} \rightarrow X, K^q P^r H^s \rangle = \langle v^m e^{iax}e^{ibt}, XK^q P^r H^s \rangle. \quad (3.7)$$

So, the computation of $[X]^m_{q,r,s}$ has been reduced to the problem of writing the monomial $XK^q P^r H^s$ in the “normal ordering” defined by the above basis of $C^{\uparrow}$. When $X = K$ the task is trivial, for the other two generators, $P$ and $H$, we use the following results whose proof is made by induction

$$P \triangleleft K^k = k!\omega^k P^{k+1}, \quad PK^q = \sum_{k=0}^{q} \frac{q!}{(q-k)!} \omega^k K^{q-k} P^{k+1},$$

$$H \triangleleft K^{k+1} = -k!\omega^k P^{k+1}, \quad HK^q = K^q H - \sum_{k=0}^{q-1} \frac{q!}{(k+1)(q-k-1)!} \omega^k K^{q-k-1} P^{k+1}.$$
Thus, we get that

\[ q!r!s!q^{m}_{q,r,s} = m!\delta_{q+1}^{m}(ia)^{r}(ib)^{s}, \]

\[ q!r!s!P^{m}_{q,r,s} = m!\sum_{k=0}^{q} q! \omega^{k} \delta_{q-k}^{m}(ia)^{k+1+r}(ib)^{s}, \]

\[ q!r!s!H^{m}_{q,r,s} = m!\delta_{q}^{m}(ia)^{r}(ib)^{s+1} \]

\[ -m!\sum_{k=0}^{q-1} q! \omega^{k} \delta_{q-k-1}^{m}(ia)^{k+1+r}(ib)^{s}. \]

Now substituting expressions (3.7) in (3.6) we obtain the desired action of the generators on the basis of \( \mathbb{C}^{\dagger} \). Finally, in order to have meaningful expressions for the actions of the generators of \( U_\omega[g(1,1)] \) it is necessary “to complete” the space \( \mathbb{C}^{\dagger} \). In consequence, we shall work with the space of formal series in \( v, \mathbb{C}[[v]] \).

Summarizing, the induced representations of \( U_\omega[g(1,1)] \) determined by the (left–)characters of the translation sector (3.2) have as support space the space \( \mathbb{C}[[v]] \). The explicit form of these representations is

\[ \phi(v) \vdash K = \phi'(v), \]

\[ \phi(v) \vdash P = \phi(v) \frac{ia}{1-\omega av}, \]

\[ \phi(v) \vdash H = \phi(v) [ib + \frac{1}{\omega} \ln(1 - i\omega av)]. \]

The representation is labeled by two real parameters \( a \) and \( b \), however, by the transformation \( H \rightarrow H - ib \) the coefficient \( b \) vanishes. Therefore, the representations labeled by \((a, b)\) are pseudoequivalent (i.e., equivalent up to a phase) to those with \((a, 0)\).

It is worthy to note that in the limit \( \omega \rightarrow 0 \) we recover a unitary irreducible representation of the nondeformed Galilei group provided that \( a \) is a real parameter. Effectively, taking the limit \( \omega \rightarrow 0 \) in (3.8) we get the infinitesimal action of the infinitesimal generators of the group

\[ \phi(v) \vdash K = \phi'(v), \]

\[ \phi(v) \vdash P = ia\phi(v), \]

\[ \phi(v) \vdash H = -iav\phi(v), \]

\[ K = d/dv, \]

\[ P = ia, \]

\[ H = -iav. \]

On the other hand, the unitary irreducible representation up to a phase obtained by Mackey’s method that correspond to the above one \((m = 0, C \neq 0)\) is

\[ U_{\rho}(t, x, v)\psi)(\xi) = e^{i(px-\xi t)}\psi(\xi + vp), \]

\[ p \in \mathbb{R}^{*}. \]

If one computes the infinitesimal action associated with the above representation (3.10) it coincides with (3.8) when \( p = a \) and after the variable change \( v \leftrightarrow \xi = va \). Obviously, the functions to be consider in the limit will be of integrable square.
So, the space of formal series in $v$ is too large for the study of the unitarity as we have just mentioned. For that, we can reduce the support space by considering the space of polynomials in $v$. Let us define

$$v_k = (e^{iax})^{-k} v (e^{iax})^k = \frac{v}{1 - k i \omega v}, \quad k \in \mathbb{Z}.$$  

The action (3.8) in the space of the polynomials in $v_k$ is as follows

$$(v_k)^n \triangleleft K = n (v_k)^{n-1} (1 + k i \omega v_k)^2;$$

$$(v_k)^n \triangleleft P = i a (v_k)^n (1 + k i \omega v_1),$$

$$ (v_k)^n \triangleleft e^{\omega H} = i a (v_k)^n (1 - k i \omega v_0), \quad (3.11)$$

where $v_0 = v$. We see that the representation (3.11) is reducible but not completely reducible, and $\mathbb{C} \oplus \mathbb{P}[v_0] \oplus \mathbb{P}[v_1]$ determines an irreducible subspace for this representation, where $\mathbb{P}[v_k] = v_k \mathbb{C}[v_k]$ and $\mathbb{C}[v_k]$ is the space of polynomials in $v_k$. An open problem is to construct a Haar measure in such a way that this representation becomes unitary, and in the limit we can recover the space of integrable square functions. A solution of this problem for the quantum $(1+1)$ extended Galilei group is given in Ref. [?].

3.2 Non-standard quantum $(1+1)$ Galilei algebra

This non-standard quantum $(1+1)$ Galilei algebra is a contraction [22] of the non-standard Poincaré algebra [24, 25].

3.2.1 Algebraic structure

The structure of $U_\rho[\mathfrak{g}(1,1)]$ is given by

$$[H, K] = -\frac{1}{4\rho} (1 - \exp(-4\rho P)), \quad [P, K] = 0, \quad [H, P] = 0;$$

$$\Delta P = P \otimes 1 + 1 \otimes P, \quad \Delta X = X \otimes 1 + \exp(-2\rho P) \otimes X, \quad X \in \{H, K\};$$

$$\epsilon(X) = 0, \quad X \in \{H, P, K\};$$

$$\gamma(P) = -P, \quad \gamma(X) = -e^{2\rho P} X, \quad X \in \{H, K\}.$$  

The quantum group $F_\rho[G(1,1)]$ is determined by

$$[t, v] = 0, \quad [x, v] = -2\rho v, \quad [t, x] = 2\rho t;$$

$$\Delta t = t \otimes 1 + 1 \otimes t, \quad \Delta x = x \otimes 1 + 1 \otimes x - t \otimes v, \quad \Delta v = v \otimes 1 + 1 \otimes v;$$

$$\epsilon(f) = 0, \quad f \in \{t, x, v\};$$

$$\gamma(v) = -v, \quad \gamma(x) = -x - tv, \quad \gamma(t) = -t.$$
We can make similar considerations to those made for the standard case about the bicrossproduct structure, except that now the duality between both algebras is given by
\[ \langle K^m H^n P^p, e^i H^r x^p \rangle = m! n! p! \delta_m^r \delta^n_0 \delta^p_0. \] (3.12)
Note also that now the order of $H, P$ and $t, x$ has been changed with respect to the order taken in (3.1).

### 3.2.2 Induced representations

Let us consider the following representation of the translation sector $L$ on $\mathbb{C}^1$:
\[ 1 \mapsto H^n P^p = (ib)^n (ia)^p, \quad n, p \in \mathbb{N}. \] (3.13)
The support space of the induced representation, denoted by $\mathbb{C}^\uparrow$, is the subspace of $F_\rho[G(1,1)]$ whose elements are like $\phi(v)e^{ibx}e^{iax}$. This subspace is isomorphic to $\mathbb{C}[[v]]$. The explicit action of the generators of $U_\rho[\mathfrak{g}(1,1)]$ over the elements of $\mathbb{C}[[v]]$ is
\begin{align*}
\phi(v) \mapsto K &= \phi'(v), \\
\phi(v) \mapsto P &= \phi(v) ia, \\
\phi(v) \mapsto H &= \phi(v) [ib + \frac{1}{4\rho}(1 - e^{-4ia})v].
\end{align*}
(3.14)
The computation of this representation is based on the following result
\begin{align*}
H \triangleleft K^k &= H \delta^k_0 - \frac{1}{4\rho}(1 - e^{-4a})\delta^k_1, \\
HK^q &= K^q H - \frac{1}{4\rho} q K^{q-1}(1 - e^{-4a})P.
\end{align*}

Similarly to the above case the representation labeled by $(a, b)$ is equivalent to that with $(a, 0)$. The irreducibility of the representation follows from the fact that $K$ and $H$ can be interpreted as ladder operators acting on the space of polynomials in $v$. The above result is not all surprising if one takes into account that the algebra (only the algebra, not the whole Hopf structure) $U_\rho[\mathfrak{g}(1,1)]$ contains the oscillator algebra.

### 4 Conclusions

We have constructed induced representations of two quantum groups, and seen that only the algebra structure has been relevant in our procedure. The coalgebra structure helps in the computation of some expressions but is not essential. However, the coalgebra structure is crucial to allow the tensor product of representations of the Hopf algebra.
Both examples presented here have a bicrossproduct structure, which provides technical facilities, for example, in evaluating pairings, nevertheless it has not been essential in the induction process.

The mechanism of induced representations looks to be a systematic way of construction of representations, while some times in the literature the construction of representations or corepresentations has been made by ad hoc procedures.

There are some open problems to establish a complete theory of induced representations of quantum groups. We can mentioned, for instance, the definition of equivalence criteria of representations; the irreducibility of the representations, that is, to know the conditions to construct irreducible representations; the unitarity of the induced representations, and if this procedure allows to obtain all the irreducible representations. A solution for the unitarity problem is connected with the construction of a quantum analogue of the Haar measure. Work on these questions is in progress, and the results will be published elsewhere.

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