Front-door adjustment is a classic technique to estimate causal effects from a specified directed acyclic graph (DAG) and observed data. The advantage of this approach is that it uses observed mediators to identify causal effects, which is possible even in the presence of unobserved confounding. While the statistical properties of the front-door estimation are quite well understood, its algorithmic aspects remained unexplored for a long time. Recently, Jeong, Tian, and Barenboim [NeurIPS 2022] have presented the first polynomial-time algorithm for finding sets satisfying the front-door criterion in a given DAG, with an $O(n^3(n+m))$ run time, where $n$ denotes the number of variables and $m$ the number of edges of the graph. In our work, we give the first linear-time, i.e. $O(n+m)$, algorithm for this task, which thus reaches the asymptotically optimal time complexity, as the size of the input is $\Omega(n+m)$. We also provide an algorithm to enumerate all front-door adjustment sets in a given DAG with delay $O(n(n+m))$. These results improve the algorithms by Jeong et al. [2022] for the two tasks by a factor of $n^3$, respectively.

1 Introduction

One of the fundamental tasks in empirical science as well as in AI and ML is to establish cause-effect relationships. A possible approach to analyzing such relations is the so-called gold standard of experimentation – the randomized controlled trial. In practice, however, experimentation is not always possible due to costs, ethical constraints, or technical feasibility – e.g. an autonomous car should not need to crash to recognize that this has negative consequences. The goal of causal inference is to determine cause-effect relationships by combining observed and interventional data with existing knowledge. In this paper, we focus on the problem of deciding when causal effects can be identified from a graphical model and observed data and, if possible, how to estimate the strength of the effect.

The model is typically represented as a directed acyclic graph (DAG), whose edges encode direct causal influences between the random variables of interest. To analyze the causal effects in such models, the concept of the do-operator invented by Pearl [1995, 2009] is used, which performs a hypothetical intervention forcing exposure variables $X$ to take some values $x$. This allows to define the (total) causal effect of $X$ on outcome variables $Y$, denoted as $P(y \mid do(x))$, as the probability distribution of variables $Y$ after the intervention$^1$. The fundamental task in the causal analysis is to decide whether $P(y \mid do(x))$ can be expressed using only standard (i.e., do-operator free) probabilities involving observed variables. A variable is considered observed if it can be measured by the researcher.

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$^1$In this paper, by $P(x)$ we denote, for a random variable $X$, that $P(X = x)$. By bold capital letters $X, Y, \text{etc.}$, we denote sets/sequences of variables, and the corresponding sets/sequences of values are denoted by bold lowercase letters $x, y$. 
Figure 1: A DAG, with unobserved variables $V_1, V_2, V_3$, where the causal effect of $X = \{X_1, x_2, X_3\}$ on $Y = \{Y_1, Y_2, Y_3\}$ is identifiable by a simple formula, yet the IDC algorithm returns the complicated expression $P(y_1, y_2, y_3 \mid do(x_1, x_2, x_3)) = \sum_{v_0}[P(y_1|x_1) P(v_0|x_1) P(y_3|x_1, y_1, v_0, y_2) P(y_2|x_1, v_0)]$ [van der Zander et al., 2019].

It is well known that the prominent do-calculus [Pearl, 1995] (see also [Pearl, 2009, Chapter 3.4.2]) allows solving the identifiability problem in a sound and complete way [Huang and Valtorta, 2006, Shpitser and Pearl, 2006b] in the sense that, a causal effect of $X$ on $Y$ is identifiable if, and only if, there exists a sequence of applications of the do-calculus rules that transforms $P(y \mid do(x))$ into a formula that only includes observational quantities. Based on the do-calculus, Shpitser and Pearl [2006a] proposed the IDC algorithm to compute such a formula in polynomial time, or output that identification is impossible in case none exists. As such, researchers could potentially apply IDC to decide identifiability. However, two drawbacks affect the widespread use of the algorithm: Firstly it runs in polynomial time of high degree and, thus, it does not scale well. This precludes computations for graphs involving a reasonable amount of variables. Secondly, the IDC algorithm computes complex expressions, even in case of small DAGs for which a simple formula exists. For example, given the DAG in Fig. 1 with 10 variables, IDC provides $P(y_1, y_2, y_3 \mid do(x_1, x_2, x_3)) = \sum_{v_0}[P(y_1|x_1) P(v_0|x_1) P(y_3|x_1, y_1, v_0, y_2) P(y_2|x_1, v_0)]$ though for this instance a simple formula, namely, $P(y_1, y_2, y_3 \mid do(x_1, x_2, x_3)) = P(y_1, y_2, y_3 \mid x_1, x_2, x_3)$, is sufficient (for a more detailed discussion, see van der Zander et al., 2019).

Hence, in practice, total causal effects are estimated using other approaches. In particular, the vast majority is computed by covariate adjustment of the form $P(y \mid do(x)) = \sum_z P(y \mid x, z) P(z)$, which is valid if $Z$ satisfies the famous back-door criterion by Pearl. Shpitser et al. [2010] generalized the back-door criterion, providing the first sound and complete constructive back-door criterion, proposed by van der Zander et al. [2014], allows to find an adjustment set in linear-time $O(n + m)$, where $n$ denotes the number of variables and $m$ the number of edges of a given instance graph. Moreover, van der Zander et al. [2014] provided an algorithm to enumerate all adjustment sets with delay $O(n(n + m))$, which means that at most $O(n(n + m))$ time passes between two successive outputs.

However, while the constructive back-door criterion is guaranteed to find all instances in which a causal effect can be identified via covariate adjustment, it is well known that not all identifiable effects can be identified this way. For example, in a DAG: $U \triangleright X \leftleftharpoons M \triangleright Y$, with an unobserved variable $U$, the effect of $X$ on $Y$ is not identifiable via covariate adjustment, although $P(y \mid do(x))$ can be expressed as $\sum_m P(m \mid x) \sum_z P(y \mid x', m) P(x')$. This instance illustrates a use of another classic technique, known as front-door adjustment [Pearl, 1995], which is the main focus of this paper.

The advantage of this approach, as seen in the example, is that it leverages observed mediators to identify causal effects even in the presence of unobserved confounding. In the general case, if a set of variables $Z$ satisfies the front-door criterion\footnote{For a definition of front-door criterion, see Sec. 2} relative to $(X, Y)$ in a DAG $G$, the variables $Z$ are observed and $P(x, z) > 0$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y \mid do(x)) = \left\{ \begin{array}{ll} P(y) & \text{if } Z = \emptyset, \\ \sum_z P(z \mid x) \sum_{x'} P(y \mid x', z) P(x') & \text{otherwise.} \end{array} \right.$$ 

Front-door adjustment can be an effective alternative to standard covariate adjustments [Glynn and Kashin, 2018] and is met with increasing applications in real-world datasets [Bellemare et al., 2019, Gupta et al., 2021, Chino and Mayer, 2016, Cohen and Malloy, 2014]. The recent works [Kuroki, 2000, Glynn and Kashin, 2018, Gupta et al., 2021] have improved the understanding of the statistical properties of the front-door estimation and provided robust generalizations of this approach [Hünermund and Bareinboim, 2019, Fulcher et al., 2020].
The algorithmic aspects of front-door adjustment have been studied recently in [Jeong et al., 2022], which provided the first polynomial-time algorithm for finding a front-door adjustment set with an $O(n^3(n + m))$ run time. This amounts to $O(n^3)$ for dense graphs, which does not scale well even for a moderate number of variables. They also gave an algorithm for enumerating all front-door adjustment sets, which has delay $O(n^4(n + m))$.

In this work, we present the first linear-time, that is $O(n + m)$, algorithm, for finding a front-door adjustment set. This run time is asymptotically optimal, as the size of the input graph is $\Omega(n + m)$. For enumeration of front-door adjustment sets, we provide an $O(n(n + m))$-delay algorithm, thus matching the run times for finding and enumerating covariate adjustment sets.

2 Preliminaries

Basic graph definitions. A directed graph $G = (V, E)$ consists of a set of vertices (or variables) $V$ and a set of directed edges $E \subseteq V \times V$. In case of a directed edge $A \rightarrow B$, vertex $A$ is called a parent of $B$ and $B$ is a child of $A$. In case there is a causal path $A \rightarrow \cdots \rightarrow B$, then $A$ is called an ancestor of $B$ and $B$ is called a descendant of $A$. Vertices are descendants and ancestors of themselves, but not parents/children. The sets of parents, children, ancestors, and descendants of a vertex $V$ are denoted by $Pa(V), Ch(V), An(V),$ and $De(V)$, and they generalize to sets of vertices the natural way. We consider only acyclic graphs (DAGs), meaning if $B \in De(A)$, then there is no edge $B \rightarrow A$. The graph $G_{\emptyset}$ is the one, where the edges $\rightarrow S$ for every $S \in S$ are removed, and the graph $G_{S}$ is the one, where the edges $\leftrightarrow S$ for every $S \in S$ are removed.

The statement $(A \perp B \mid C)_G$ in DAG $G$ holds for disjoint sets of vertices $A, B, C \subseteq V$ if $A$ and $B$ are d-separated in $G$ given $C$ — that is, if there is no open path from any vertex $A \in A$ to a vertex $B \in B$ given $C$. A path is open if, for any collider $Y$ (that is $X \rightarrow Y \leftarrow Z$), we have $De(Y) \cap C \neq \emptyset$, and, for any non-collider $Y$, we have $Y \notin C$.

In this work, we sometimes consider ways instead of paths. A way may contain a vertex up to two times, and is open given $C$ if, for any collider $Y$, we have $Y \in C$ and, for any non-collider $Y$, we have $Y \notin C$. In case there is an open way between two sets of vertices, there is also an open path and vice versa. Hence, ways can be used to determine d-separation and they make up the traversal sequence of the Bayes-Ball algorithm [Shachter, 1998] for testing d-separation in linear time.

A path (or way) from $A$ to $B$ is called a back-door path (or way) if it starts with the edge $A \leftarrow$. For a set of vertices $A$, we often speak of proper back-door paths (or ways), which are such that the path $A \leftarrow \ldots B$ for $A \in A$ and $B \in B$ does not contain any other vertex in $A$.

Front-door adjustment sets. Let $I \subseteq R, X$ and $Y$ be sets of vertices $(X, Y, R)$ and $Y$ are pairwise disjoint). A set of variables $Z$ with $I \subseteq Z \subseteq R$ satisfies the front-door criterion relative to $(X, Y)$ in a DAG $G$ if [Pearl, 1995]:

1. $Z$ intercepts all directed paths from $X$ to $Y$.
2. There is no unblocked proper back-door path from $X$ to $Z$, i.e., $(Z \perp \perp X)_{G_{\emptyset}}$.
3. All proper back-door paths from $Z$ to $Y$ are blocked by $X$, i.e., $(Z \perp \perp Y \mid X)_{G_{\emptyset}}$.

As we state this criterion for sets of vertices $X$ and $Y$, it is necessary to speak of proper back-door paths. The set $I$ consists of variables that must be included in the front-door adjustment set, the set of variables $R$ consists of the ones that can be used. We consider acyclic graphs which consist only of directed edges (DAGs). Bidirected edges $A \leftrightarrow B$ are sometimes used to represent unobserved confounding and could be replaced by $A \leftarrow U \rightarrow B$, where $U$ is a new variable, which is not in $R$, in order to make use of the algorithms presented here.

The following algorithmic idea for finding a set $Z$ satisfying the front-door criterion (if such a set exists) was recently given by [Jeong et al., 2022]:

(i) Let $Z_{(i)} \subseteq R$ be the set of all variables $Z \subseteq R$, which satisfy $(Z \perp \perp X)_{G_{\emptyset}}$.
(ii) Let $Z_{(ii)} \subseteq Z_{(i)}$ be the set of variables $Z \subseteq Z_{(i)}$, for which $\exists S \subseteq Z_{(i)}$ such that $(\{Z\} \cup S \perp \perp Y | X)_{G_{\emptyset}}$.
(iii) If $I \subseteq Z_{(ii)}$ and $Z_{(iii)}$ intercepts all causal paths from $X$ to $Y$, then output $Z_{(ii)}$, else output $\perp$.

This algorithm is correct because all vertices not in $Z_{(ii)}$ cannot be in any set satisfying the front-door criterion (as they are not in $R$ or would violate item 2. and/or item 3. of the front-door criterion). It follows from this maximality

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Footnotes:

1Paths can only contain each vertex a single time.
2Essentially, by traversing the causal path from a collider to the descendant in $C$ and back, one can construct an open way for an open path.
of \( Z_{(ii)} \) that if \( I \) is not a subset of \( Z_{(ii)} \) or if \( Z_{(ii)} \) does not satisfy item 1. of the front-door criterion (which is tested in step (iii)), then no set does (if some set \( Z \) satisfies item 1., then any superset does as well). [Jeong et al. 2022] showed that step (i) can be performed in time \( O(n(n+m)) \), step (ii) in time \( O(n^3(n+m)) \) and step (iii) in time \( O(n+m) \).

3 A Linear-Time Algorithm for Finding Front-Door Adjustment Sets

We show how step (i) and (ii) can be performed in linear-time \( O(n+m) \), which leads to the first linear-time algorithm for finding front-door adjustment sets. We begin by demonstrating that, using the Bayes-Ball algorithm, step (i) can be executed in time \( O(n+m) \).

**Lemma 1.** It is possible to find \( Z_{(i)} \subseteq \mathbb{R} \), i.e., all vertices in \( \mathbb{R} \) satisfying \( Z \perp \perp X \) in time \( O(n+m) \).

**Proof.** Start the Bayes-Ball [Shachter, 1998] algorithm at \( X \) in the DAG \( G_X \). Precisely the vertices \( U \) not reached by the algorithm satisfy \( Z \perp \perp X \) in \( G_X \). Hence, \( Z_{(i)} = U \cap \mathbb{R} \).

It remains to show how we can execute step (ii), i.e., to compute \( Z_{(ii)} \) in time \( O(n+m) \). Thus, it is our task, given a set \( Z_{(ii)} \subseteq \mathbb{R} \) disjoint with \( X \) and \( Y \), which contains all vertices satisfying (i), to decide for all \( Z \in Z_{(ii)} \) if there exists a set \( S \subseteq Z_{(ii)} \) with \( (\{Z\} \cup S \perp \perp Y \mid X)_{G_{(Z)} \cup S} \).

First, let us define the notion of a forbidden vertex \( v \):

**Definition 1.** A vertex \( V \) is forbidden if it is not in \( Z_{(ii)} \). Hence, by definition this is the case if (a) \( V \notin Z_{(ii)} \) or (b) there exists no \( S \subseteq Z_{(ii)} \) for \( V \) such that \( (\{V\} \cup S \perp \perp Y \mid X)_{G_{(V)} \cup S} \).

Our goal will be to find all forbidden vertices. The remaining vertices then make up the sought after set \( Z_{(ii)} \). We utilize the following lemma.

**Lemma 2.** Vertex \( V \) is forbidden if, and only if,

(A) \( V \notin Z_{(ii)} \),

(B) \( V \leftarrow Y \), or

(C) there exists an open back-door way \( \pi \) (consisting of at least three variables) from \( V \) to \( Y \) given \( X \), and all its non-terminal vertices are forbidden.

**Proof.** We show two directions. Firstly, if \( V \in Z_{(ii)} \), there is no edge \( V \leftarrow Y \) and there exists no open back-door way with only forbidden vertices from \( V \) to \( Y \) given \( X \), then there exists a set \( S \subseteq Z_{(ii)} \), for which \( (\{V\} \cup S \perp \perp Y \mid X)_{G_{(V)} \cup S} \) holds. It can be constructed by choosing, for every open way, a non-forbidden vertex \( U \) and its set \( S_U \) (which fulfills \( (\{U\} \cup S_U \perp \perp Y \mid X)_{G_{(U)} \cup S_U} \) and taking the union \( \bigcup_U (\{U\} \cup S_U) = S \). In particular, by taking \( U \) into \( S \), we close the open way it is on as \( U \) is a non-collider (it is not in \( X \) by definition) and hence the way is cut, due to the removal of outgoing edges from \( U \). By adding all vertices in \( S_U \), \( U \) and the vertices in \( S_U \) have no open back-door way to \( Y \) given \( X \). For this, note that if \( (\{U\} \cup S_U \perp \perp Y \mid X)_{G_{(U)} \cup S_U} \) holds, then we also have for every set \( S' \supseteq \{U\} \cup S_U \) that \( (\{U\} \cup S' \perp \perp Y \mid X)_{G_{(U)} \cup S_U} \). Hence, taking the union \( \bigcup_U (\{U\} \cup S_U) \) will not open any previously closed ways.

Secondly, if \( V \) is not in \( Z_{(ii)} \), then \( V \) is forbidden by definition (part (a) of the definition) and if there exists an open back-door way \( \pi \) with only forbidden vertices, a set \( S \) satisfying (b) can never be found (the way could only be closed by adding one of its vertices to \( S \), but all of them are forbidden).

Using the lemma above, we can find all forbidden vertices in linear time. We mark all vertices \( V \notin Z_{(ii)} \) as forbidden. We then start a graph search (similar to Bayes-Ball) at \( Y \) visiting only forbidden vertices. For this, when handling a vertex \( V \), we iterate over its neighbors. If a neighbor (defined according to Bayes-Ball rules; that is depending on the edge types which lead to the vertex and its membership in the conditioning set \( X \)) is also forbidden, we visit it. If the neighbor is not yet forbidden, but a child \( U \) of \( V \), we mark it as forbidden and also visit it (we have (B) that \( U \to Y \) or (C) an open back-door way from \( U \) to \( Y \) via forbidden vertices and hence it is forbidden). If it is a parent \( P \) and not yet forbidden, we store for the vertex that it is reachable via a non-back-door way over forbidden vertices (but we do not mark it forbidden). If it later becomes forbidden, we continue the Bayes-Ball search also including the incoming direction \( P \to \) (that is, we follow edges \( 
\rightarrow P \) as well).
All vertices visited during the algorithm in addition to the ones that are not in \( Z_{(i)} \) (that is the ones satisfying part (a) of the definition), are exactly the forbidden vertices \( F \). Hence \( V \setminus F \) gives us \( Z_{(ii)} \).

```
input : A DAG \( G = (V, E) \) and sets \( X, Y, Z_{(ii)} \subseteq V \).
output: Set \( Z_{(ii)} \).

1 Initialize visited[\( V, \text{inc} \)] , visited[\( V, \text{out} \)] and continuelylater[\( V \)] with \text{false} for all \( V \in V \).
2 Set forbidden[\( V \)] = \text{true} if \( V \not\in Z_{(i)} \), else to \text{false}.

3 function visit(\( G, V, \text{edgetype} \))
   4 visited[\( V, \text{edgetype} \)] = \text{true}
   5 forbidden[\( V \)] = \text{true}
   6 if \( V \not\in X \) then
      7   foreach \( U \in \text{Ch}(V) \) do
         8      if !visited[\( U, \text{inc} \)] then
         9         visit(\( G, U, \text{inc} \))
        10   end
        11 end
       12 if (edgetype == \text{inc} \land V \in X) \lor (edgetype == \text{out} \land V \not\in X) then
          13     foreach \( U \in \text{Pa}(V) \) do
          14        if forbidden[\( U \)] \land !visited[\( U, \text{out} \)] then
          15          visit(\( G, U, \text{out} \))
          16        else if !visited[\( U, \text{out} \)] then
          17          continuelylater[\( U \)] = \text{true}
          18        end
          19 end
          20 if continuelylater[\( V \)] \land !visited[\( V, \text{out} \)] then
          21     visit(\( G, V, \text{out} \))
          22 end
       23 end
       24 foreach \( Y \in Y \) do
          25        if !visited[\( Y, \text{out} \)] then
          26          visit(\( G, Y, \text{out} \))
          27        end
          28 if !visited[\( Y, \text{inc} \)] then
          29        visit(\( G, Y, \text{inc} \))
          30 end
       31 end
       32 return \( V \setminus \{ V \mid \text{forbidden}[V] = \text{true} \} \)

Algorithm 1: Finding the set \( Z_{(ii)} \) when given \( Z_{(i)} \) in time \( O(n + m) \).

**Theorem 1.** Given a DAG and sets \( X, Y, Z_{(i)} \), Algorithm 7 computes the set \( Z_{(ii)} \) in time \( O(n + m) \).

**Proof.** The correctness follows from Lemma 2 and the fact that a vertex \( V \) is marked forbidden, if, and only if, (A) it is not in \( Z_{(i)} \) or (B) it is a child of \( Y \) or (C) there is an open back-door way from \( V \) to \( Y \) given \( X \) over forbidden vertices. For this, we first observe that the algorithm is sound, i.e., a vertex marked forbidden is actually forbidden. Moreover, the completeness with regard to (A) and (B) is clear, hence, it remains to show it with regard to (C).

Consider the situation when the algorithm has terminated and assume there are forbidden vertices, which have not been visited by the algorithm and are not marked forbidden (let these be \( U = \{ U_1, \ldots, U_k \} \) and note that these are forbidden due to (C), else they would have been marked). We show that there exists a open back-door way \( \pi \) over forbidden vertices from some \( U_i \) to \( Y \) which does not contain any other \( U_j \). If this were not the case consider the last \( U_j \) on an open back-door way from \( U_i \) to \( Y \) (such a way has to exist as \( U_i \) is forbidden, it is only not marked forbidden). If it occurs as \( U_j \not\rightarrow \) the existence of \( \pi \) would follow. Hence, it has to occur as \( U_j \rightarrow \). Then it would hold that \((U \not\perp \perp Y \mid X \land U_j)\) and the vertices in \( U \) would not be forbidden. Thus, there is such a \( \pi \) and it starts with the edge \( U_i \rightarrow P \) with \( P \) being visited by the algorithm (on \( \pi \), every vertex is marked forbidden and it is open, hence, the
graph search must have reached $P$ by correctness of Bayes-Ball). It follows that $U_i$ as a child of $P$ is also visited and marked forbidden, which leads to a contradiction.

The run-time bound can be derived from the fact that for each vertex we call visit at most two times (for edgetype $==$ inc and edgetype $==$ out) and as every such call has cost $O(|Ne(v)|)$, every edge is visited only a constant amount of times. Hence, we obtain a run time of $O(n + m)$. \qed

The full algorithm for obtaining a front-door adjustment set $Z$ is given in Algorithm 2.

```
input : A DAG $G = (V, E)$ and sets $X$, $Y$, $I \subseteq R \subseteq V$.
output : Set $Z$ with $I \subseteq Z \subseteq R$ or $\bot$.
1 Start the Bayes-Ball [Shachter, 1998] algorithm at $X$ in $G_X$. Let $U$ be the set of nodes not visited.
2 Let $Z_{(i)} = U \cap R$.
3 Let $Z_{(ii)}$ given by Algorithm 1.
4 Start a breadth-first search at $X$ following only directed edges until nodes in $Z_{(ii)} \cup Y$ are visited. Let $W$ be the set of visited nodes.
5 if $I \subseteq Z_{(ii)}$ and $W \cap Y = \emptyset$ then
6     return $Z_{(ii)}$
7 else
8     return $\bot$
9 end

Algorithm 2: Finding a front-door adjustment set $Z$ relative to $(X, Y)$ in time $O(n + m)$.
```

**Theorem 2.** Given a DAG and sets $X$ and $Y$, Algorithm 2 finds a front-door adjustment set $Z$ relative to $(X, Y)$ with $I \subseteq Z \subseteq R$, or decides that such a set does not exist, in time $O(n + m)$.

**Proof.** According to Lemma 1 and Theorem 1, the algorithm finds $Z_{(i)}$ and $Z_{(ii)}$. By following directed edges, the algorithm verifies the conditions that $I \subseteq Z_{(ii)}$ and that $Z_{(ii)}$ intercepts all causal paths from $X$ to $Y$. As Jeong et al. [2022] have shown, this makes $Z_{(ii)}$ a front-door adjustment set. \qed

**Testing.** Given a DAG and sets $X$, $Y$, and $Z$, one can verify whether the set $Z$ is a front-door adjustment set $Z$ relative to $(X, Y)$ by setting $I = R = Z$ and calling the finding algorithm. The algorithm finds $Z$ if and only if $Z$ is a front-door adjustment set.

**Enumeration.** With standard techniques developed by van der Zander et al. [2014] and applied by Jeong et al. [2022] to the front-door criterion, it is possible to enumerate all front-door sets with delay $O(n \cdot \text{find}(n, m))$, where $\text{find}(n, m)$ is the time it takes to find a front-door set in a graph with $n$ vertices and $m$ edges. Hence, with the linear-time algorithm for finding front-door adjustment sets presented in this work, an $O(n(n + m))$ delay enumeration algorithm follows directly. This improves the previous best run-time of $O(n^4(n + m))$ by Jeong et al. [2022] by a factor of $n^3$.

**Corollary 1.** There exists an algorithm for enumerating all front-door adjustment sets in a DAG $G = (V, E)$ with delay $O(n(n + m))$.

A linear-time delay algorithm appears to be out-of-reach because even for the simpler tasks of enumerating d-separators and back-door adjustment sets, the best known delay is again $O(n(n + m))$ [van der Zander et al., 2014].

**4 Conclusion**

We have shown that front-door adjustment sets can be found in linear time $O(n + m)$, which is asymptotically optimal. Moreover, we provided an $O(n(n + m))$ delay algorithm for enumerating all front-door adjustment sets. Thus, for the tasks of finding and enumerating, we match the time complexity for the back-door adjustment setting.

We also provide implementations of our algorithms, for finding and enumerating front-door adjustment sets, in the programming language Julia [Bezanson et al., 2017] at https://github.com/mwien/frontdoor-adjustment.
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