ON THE SOLUTIONS OF UNIVERSAL DIFFERENTIAL EQUATION
BY NONCOMMUTATIVE PICARD-VESSIOT THEORY

V.C. BUI, V. HOANG NGOC MINH, Q.H. NGÔ, AND V. NGUYEN DINH

ABSTRACT. Basing on Picard-Vessiot theory of noncommutative differential equations and algebraic combinatorics on noncommutative formal series with holomorphic coefficients, various recursive constructions of sequences of group-like series converging to solutions of universal differential equation are proposed. Basing on monoidal factorizations, these constructions intensively use diagonal series and various pairs of bases in duality, in concatenation-shuffle bialgebra and in a Loday’s generalized bialgebra. As applications, the unique solution, satisfying asymptotic conditions, of universal Knizhnik-Zamolodchikov equation is provided by dévissage.

CONTENTS

1. Introduction 2
2. Combinatorial frameworks 8
  2.1. Algebraic combinatorics on formal power series 8
  2.2. Diagonal series in concatenation-shuffle bialgebra 13
  2.3. More about diagonal series in concatenation-shuffle bialgebra and in a Loday’s generalized bialgebra 18
3. Solutions of universal differential equation 20
  3.1. Iterated integrals and Chen series 20
  3.2. Noncommutative differential equations 25
  3.3. Explicit solutions of noncommutative differential equations 28
4. Application to Knizhnik-Zamolodchikov equations 31
  4.1. Noncommutative generating series of polylogarithms 31
  4.2. Noncommutative generating series of hyperlogarithms 33
  4.3. Knizhnik-Zamolodchikov equations 34
5. Conclusion 38
6. Appendices 41
  6.1. KZ_4, the simplest non-trivial case 41
  6.2. KZ_4, other simplest non-trivial case 42
References 44

2020 Mathematics Subject Classification. Primary 54C40, 14E20; Secondary 46E25, 20C20.
1. Introduction

The objective of this work, by providing more explanations concerning the short text [3] and continuing the work of [26] [33], consists of expliciting solutions of universal differential equation (see [14] below, when the solutions exist) using in particular Volterra expansions for the Chen series. Ultimately, applied to the universal Knizhnik-Zamolodchikov (see [4] below, [3] [28] [55]), this provides by dévissage the unique group type solution satisfying asymptotic conditions, i.e. solutions of $KZ_n$ are obtained by use of solutions of $KZ_{n-1}$ and the noncommutative generating series of hyperlogarithms [14] [51].

These solutions use a Picard-Vessiot theory of noncommutative differential equations [26] and various factorizations of Chen series, for which, in Section 3 below, one can define the ring of holomorphic functions over $\mathbb{C}$, denoted by $(\mathcal{H}(\mathcal{V}), 1_{\mathcal{H}(\mathcal{V})})$, or in

- the ring of holomorphic functions over $\mathcal{V}$, denoted by $(\mathcal{H}(\mathcal{V}), 1_{\mathcal{H}(\mathcal{V})})$,
- the wedge algebra of holomorphic forms over $\mathcal{V}$, denoted by $\Omega(\mathcal{V})$.

In $\mathcal{H}(\mathcal{V})(\langle T_n \rangle)$, the coefficients $\{(S \mid w)\}_{w \in T_n^*}$ of $S$ are holomorphic and the partial differentiations $\partial_i (S \mid w)$ are well defined. So is the following differential

$$d(S \mid w) = \partial_1 (S \mid w) dz_1 + \cdots + \partial_n (S \mid w) dz_n.$$  

Hence, in Sections 3, 4 below, one can define $dS$ over $\mathcal{H}(\mathcal{V})(\langle T_n \rangle)$ as follows

$$S = \sum_{w \in T_n^*} (S \mid w) w, \quad dS = \sum_{w \in T_n^*} (d(S \mid w)) w,$$

and then study the following first order noncommutative differential equation [26], so-called universal differential equation, over $\mathcal{H}(\mathcal{V})(\langle T_n \rangle)$,

$$dS = M_n S, \quad where \quad M_n := \sum_{1 \leq i < j \leq n} \omega_{i,j} t_{i,j} \in \text{Lie}_\Omega(\mathcal{V})(\langle T_n \rangle).$$

Universality can be seen as, firstly, the differential forms $\{(\omega_{i,j})_{1 \leq i < j \leq n}\}$, in (1), are not determined yet and, secondly, replacing each letter $t_{i,j} \in T_n$ by a constant matrix $\mathcal{M}(t_{i,j})$ (resp. a holomorphic vector field $\mathcal{V}(t_{i,j})$) and obtaining then a linear (resp. nonlinear) differential equation [12] [27] [37] [51] (resp. [13] [25] [26] [43]). Note also that one can use the following alphabet in bijection with $T_n$

$$X = \{x_k\}_{1 \leq k \leq N}, \quad with \quad N = n(n - 1)/2.$$
In this case, one uses the set of differential forms \( \{ \omega_i \}_{1 \leq i \leq N} \) in bijection with \( X \) and then (4) becomes (see (9)–(10) below)

\[
(6) \quad dS = M_n S, \quad \text{where} \quad M_n := \sum_{i=1}^{N} \omega_i x_j \in \text{Lie}_{s(V)}(X).
\]

In particular, to the partition \( T_n \), onto \( T_{n-1} \) and \( T_n \), corresponds the split of the universal connection \( M_n \), onto \( M_{n-1} \in \text{Lie}_{s(V)}(T_{n-1}) \) and \( M_n \in \text{Lie}_{s(V)}(T_n) \)

\[
(7) \quad T_n = T_n \sqcup T_{n-1}, \quad \text{where} \quad T_n := \{ t_{k,n} \}_{1 \leq k \leq n-1},
\]

\[
(8) \quad M_n = M_{n} + M_{n-1}, \quad \text{where} \quad M_{n} := \sum_{k=1}^{n-1} \omega_{k,n} t_{k,n}.
\]

Then, using the intermediate alphabet in (4), it follows that (see also Remark 15 below)

\[
(9) \quad M_n = \sum_{1 \leq i < j \leq n} \omega_{i,j} t_{i,j} = \sum_{1 \leq k \leq N} F_k x_k = \sum_{1 \leq k \leq N} U_l d z_l,
\]

where

\[
(10) \quad F_k = \sum_{1 \leq l \leq n} f_{i,k} d z_j \quad \text{and then} \quad U_l = \sum_{1 \leq k \leq N} f_{i,k} x_k.
\]

For any \( S \neq 0 \) belonging to the integral ring \( \mathcal{H}(V) \langle T_n \rangle \), if \( S \) is solution of (4) then, by (2) and (9)–(10), one might have

\[
(11) \quad dS = M_n S = \sum_{1 \leq l \leq n} (\partial_l S) d z_l, \quad \text{with} \quad \partial_l S = U_l S.
\]

Hence, for any \( 1 \leq i, j \leq n, \partial_j \partial_i S = ((\partial_j U_i) + U_i U_j) S \), and since \( \partial_j \partial_i S = \partial_j \partial_i S \) then \( ((\partial_j U_i) - (\partial_i U_j) + [U_i, U_j]) S = 0 \). It follows that (see also Remark 15 below)

\[
(12) \quad \forall 1 \leq i, j \leq n, \partial_j U_i - \partial_i U_j = [U_i, U_j], \quad \text{or equivalently}, \quad dM_n = M_n \wedge M_n.
\]

This could induce a Lie ideal, \( J_n \), of relations among \( \{ t_{i,j} \}_{1 \leq i < j \leq n} \) and solutions of (4) could be algorithmically computed over \( \mathcal{H}(V) \langle T_n \rangle \) and then \( \mathcal{H}(V) \langle T_n \rangle / J_n \), as explained in Section 3.4 below.

According to (12), \( M_n \) is said to be flat and (4) is said to be completely integrable.

With the topology in (11), solution of (4), when exists, can be usually computed by the following convergent Picard’s iteration over the topological basis \( \{ w \} \in \pi \mathcal{T}_n \)

\[
(13) \quad F_0(\zeta, z) = 1_{\mathcal{H}(V)}, \quad F_i(\zeta, z) = F_{i-1}(\zeta, z) + \int_{\zeta} M_n(s) F_{i-1}(s), \quad i \geq 1,
\]

and the sequence \( \{ F_k \}_{k \geq 0} \) admits the limit, also called Chen series (see (6)–(10) and their bibliographic) of the holomorphic 1-forms \( \{ \omega_{i,j} \}_{1 \leq i < j \leq n} \) and along a path \( \zeta \rightarrow z \) over \( V \), modulo \( J_n \), is viewed as the fundamental solution of (4).

More generally, by a Ree’s theorem (see (6)–(10) and their bibliographies) Chen series is grouplike, belongs to \( e^{\text{Lie}_{s(V)}(T_n)} \), and can be put in the MRS\(^3\) factorization form (20)–(35)–(43) (see Proposition 4 and Corollary 2 below). Moreover, since the rank of the module of solutions of (4) is at most equals 1 then, under the action of the Hausdorff group, i.e. \( e^{\text{Lie}_{s(V)}(T_n)} \) playing the rôle of the differential Galois group of (4), any grouplike solution of (4) can be computed by multiplying

\(^3\text{MRS is an abbreviation of G. Mélançon, C. Reutenauer and M.P. Schützenberger.}\)
on the right of the previous Chen series, modulo $\mathcal{J}_n$, by an element of Haussdorf group which contains the monodromy group of $[14, 114, 26, 39, 40, 41]$.

From these, in practice, infinite solutions of $[14]$ can be computed using convergent iterations of pointwise convergence over $\mathcal{H}(\mathcal{V})\langle \mathcal{T}_n \rangle$ and then $\mathcal{H}(\mathcal{V})\langle \mathcal{T}_n \rangle/\mathcal{J}_n$. A challenge is to explicitly and exactly compute (and to study) these limits of convergent sequences of (not necessarily grouplike) series on the dual topological ring and over various corresponding dual topological bases. For that, on the one hand, thanks to the algebraic combinatorics on noncommutative series (recalled in Section 3.1 below) and, on the other hand, by means of a noncommutative symbolic calculus (introduced in Section 3.3 below) and a Picard-Vessiot theory of noncommutative differential equations (outlined in Section 3.2 below), solutions of $[14]$ are explicitly computed (as in Section 3.3 below).

As application of $[14]$–$[18]$ and $[12]$–$[13]$, in Section 4.3 below, substituting $t_{i,j}/2\pi$ and specializing $\omega_{i,j}(z)$ to $d\log(z_i - z_j)$ and then $\mathcal{V}$ to the universal covering, $\mathcal{C}_n$, of the configuration space of $n$ points on the plane $[5, 10, 47, 48]$, $\mathcal{C}_n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$, various expansions of Chen series, over $\mathcal{H}(\mathcal{C}_n)\langle \mathcal{T}_n \rangle$, will provide solutions of the following differential equation

\begin{equation}
(14) \quad d\Omega = \Omega_n F, \quad \text{where} \quad \Omega_n(z) := \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{2\pi} d\log(z_i - z_j),
\end{equation}

so-called $KZ_n$ equation and $\Omega_n$ is called universal KZ connection form and is splitting as follows (Proposition 7 below will examine the flatness $\Omega_n$ and integrability conditions of $[14]$, see also Lemma 2 below)

\begin{equation}
(15) \quad \Omega_n = \bar{\Omega}_n + \Omega_{n-1}, \quad \text{where} \quad \bar{\Omega}_n(z) := \sum_{k=1}^{n-1} \frac{t_{k,n}}{2\pi} d\log(z_k - z_n).
\end{equation}

In particular, let $\Sigma_{n-2} = \{z_1, \ldots, z_{n-2}\} \cup \{0\}$ (one puts $z_{n-1} = 0$) be the set of singularities and $s = z_n$. For $z_n \to z_{n-1}$, the connection $\bar{\Omega}_n$ behaves as $(2i\pi)^{-1}N_{n-1}$, where $N_{n-1}$ is nothing but the connection of the differential equation satisfied by the noncommutative generating series of hyperlogarithms (see $[131]$–$[182]$ below)

\begin{equation}
(16) \quad N_{n-1}(s) := t_{n-1,n} \frac{ds}{s} - \sum_{k=1}^{n-2} t_{k,n} \frac{ds}{z_k - s} \in \text{Lie}_e \mathcal{H}(\mathcal{C}_n)\langle \mathcal{T}_n \rangle/\mathcal{J}_n.
\end{equation}

**Example 1.**

- If $n = 2$ then $\mathcal{T}_2 = \{t_{1,2}\}$ and $\Omega_2(z) = (t_{1,2}/2i\pi)d\log(z_1 - z_2)$.

A solution of $d\Omega = \Omega_2 F$ is $F(z_1, z_2) = e^{(t_{1,2}/2i\pi)\log(z_1 - z_2)} = (z_1 - z_2)^{t_{1,2}/2i\pi}$ and it belongs to $\mathcal{H}(\mathcal{C}_2)\langle \mathcal{T}_2 \rangle$.

- For $n = 3$, $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $\Omega_3(z) = \bar{\Omega}_3 + \Omega_3(z)$, where $\bar{\Omega}_3 = (t_{1,3}d\log(z_1 - z_3) + t_{2,3}d\log(z_2 - z_3))/2i\pi \in \text{Lie}_e \mathcal{H}(\mathcal{C}_3)\langle \mathcal{T}_3 \rangle/\mathcal{J}_3$, which behaves as $N_2(s) = (t_{1,2}s^{-1}ds - t_{2,3}(z_1 - s)^{-1}ds)/2i\pi$, by putting $z_2 = 0$ and $z_1 = 1$, see also Appendix B.1.

---

4The holomorphic 1-form $\omega_{i,j}$ is, in this cases, of the form $\omega_{i,j}(z) = (dz_i - dz_j)/(z_i - z_j)$, for $1 \leq i < j \leq n$, and the Chen series belongs to $\mathcal{H}(\mathcal{C}_n)\langle \mathcal{T}_n \rangle$.

5To ease this application, in all the sequel, the alphabet $\mathcal{T}_n$ is preferred to $X$.

6 $z_n$ is variate moving towards $z_n-1$ and $z_k = a_k$ is fixed and then $d(z_n - z_k) = dz_n = ds$. 

Example 2.  

- Solution of \( dF = \Omega_3 F \) can be computed as limit of the sequence \( \{F_i\}_{i \geq 0}, \) in \( \mathcal{H}(\mathbb{C}_n^3)/\langle T_0 \rangle \), by convergent Picard’s iteration as in (13)

\[
F_0(z^0, z) = 1, \quad F_i(z^0, z) = F_{i-1}(z^0, z) + \int_{z^0}^{z} \Omega_3(s)F_{i-1}(s), \quad i \geq 1.
\]

- Let us compute, by another way, a solution of \( dF = \Omega_3 F \) thanks to the sequence \( \{V_i\}_{i \geq 0}, \) in \( \mathcal{H}(\mathbb{C}_n^3)/\langle T_3 \rangle, \) satisfying the following recursion\(^7\)

\[
V_0(z) = e^{(t_{1,2}/2\pi i)\log(z_1 - z_2)},
\]

\[
V_i(z) = V_0(z) \int_{z}^{z} \frac{1}{21i} d\log(z_1 - z_4) + \frac{t_{2,3}}{21i} d\log(z_2 - z_3) V_{i-1}(s) - (t_{1,2}/2\pi i)\log(z_1 - z_2) \int_{z}^{z} e^{-(t_{1,2}/2\pi i)\log(z_1 - z_2)} \Omega_3(s) V_{i-1}(s).
\]

The Chen series, of the holomorphic 1-forms \( \{d\log(z_i - z_j)\}_{1 \leq i < j \leq n} \) and along the path \( z^0 \rightsquigarrow z \) over universal covering \( \tilde{\mathbb{C}}_n^\ast \), can be used to determine solutions of (14) and depends on the differences \( \{z_i - z_j\}_{1 \leq i < j \leq n} \), as will be treated in Section 4 below to illustrate our purposes. Furthermore, the universal KZ connection form \( \Omega_n \) satisfies the following identity \( (17) \) (see also Proposition 7 below)

\[
d\Omega_n - \Omega_n \wedge \Omega_n = 0
\]

then \( \Omega_n \) is flat and \( (13) \) is completely integrable. It turns out that \( (17) \) induces the relations associated to following relations on \( \{t_{i,j}\}_{1 \leq i < j \leq n} \) \( (15) \) \( (16) \) \( (17) \).

\[
R_n = \begin{cases} 
[t_{i,k} + t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i, j, k, 1 \leq i < j < k \leq n, \\
[t_{i,j} + t_{i,k}, t_{i,j}] = 0 & \text{for distinct } i, j, k, 1 \leq i < j < k \leq n, \\
[t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i, j, k, 1 \leq i < j \leq n, 1 \leq k < l \leq n,
\end{cases}
\]

(18) \( R_n \) generating the Lie ideal \( \mathcal{J}_{R_n}\), of \( \text{Lie}_H(\mathcal{V})/\langle T_n \rangle \), seemingly different to the relations associated to the infinitesimal braid relations on \( \{t_{i,j}\}_{1 \leq i \leq j \leq n} \) \( (17) \):

\[
R_n = \begin{cases} 
t_{i,j} = 0 & \text{for } i = j, \\
t_{i,j} = t_{j,i} & \text{for distinct } i, j, \\
[t_{i,k} + t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i, j, k, \\
[t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i, j, k, l.
\end{cases}
\]

(19) \( \mathcal{R}_n \)

Solutions of (14) will be then expected belonging to \( \mathcal{H}(\mathbb{C}_n^3)/\langle T_n \rangle/\mathcal{J}_{R_n} \) and the logarithm of grouplike solutions will be expected in \( \text{Lie}_H(\mathcal{V})/\langle T_n \rangle/\mathcal{J}_{R_n}\). These expressions will be explicitly computed (see Section 4 below).

Now, let us explain our strategy for solving (14) throughout the universal KZ equation (14). This involves in high energy physics \( (65) \) and has applications on representation theory of affine Lie algebra and quantum groups, braid groups, topology of hyperplane complements, knot theory, \( \ldots 6 \) \( 7 \) \( 8 \) \( 17 \) \( 18 \) \( 28 \) \( 29 \) \( 30 \) \( 31 \) \( 45 \) \( 46 \) \( 54 \) \( 55 \):

- According to (11), the Chen series \( C_{z \rightsquigarrow z} \), of \( \{d\log(z_i - z_j)\}_{1 \leq i < j \leq n} \) and along the concatenation of the paths \( z \rightsquigarrow z^0 \) and \( z^0 \rightsquigarrow z \) over \( \mathcal{V} \) is followed

\[
C_{z \rightsquigarrow z} = C_{z^0 \rightsquigarrow z}C_{z \rightsquigarrow z^0}, \quad \text{or equivalently,}
\]

(20) \( \forall w \in T_n^* \quad \langle C_{z \rightsquigarrow z} | w \rangle = \sum_{u,v \in T_n^*} \langle C_{z^0 \rightsquigarrow z} | u \rangle \langle C_{z \rightsquigarrow z^0} | u \rangle \).
On the other side, the coefficients of the Chen series, along 0 \to z and of \
\{d\log(z_i - z_j)\}_{1 \leq i < j \leq n}, are not well defined. For example, for any 1 \leq i < 
j \leq n, the integral \[ \int_0^z d\log(z_i - z_j) \] is not defined. In general, strategies 
that are widely used in the literature are tangential base points\(^8\)\(^9\). 

Hence, in Section 4 below, as an extension of the treatment on polylogarithms in \([12]\) (resp. hyperlogarithms in \([130]\)) we will construct an other 
grouplike series for computing solution of \((14)\), denoted by \(F_{KZ_n}\), such that 
\begin{equation}
F_{KZ_n}(z) = C_{z \to -z} F_{KZ_n}(z^0).
\end{equation}

\(F_{KZ_n}(z)\) will normalize \(C_{0 \to z}\) (see Definitions \([4]\) and \([8]\) Corollaries \([4]\)\([5]\) below) and, as a counter term, \(F_{KZ_n}(z^0)\) belongs to \(\{e^{C} \}_{C \in \mathbb{C}}\). 

These will be obtained as image of diagonal series over \(T_n = T_n \sqcup T_{n-1}\) 
(see Lemma \([4]\) Propositions \([12]\)\([11]\) and Theorem \([4]\) below) over the shuffle bialgebra \((\mathbb{Q}(T)_n, \text{conc}, 1_{T_n}, \Delta_{\mathbb{M}})\) (resp. \((\mathbb{Q}(T)_{n-1}, \text{conc}_{\mathbb{M}}, 1_{T_{n-1}}, \Delta_{\mathbb{M}})\)) endowed the pair of dual bases, \(\{P_i\}_{i \in \mathbb{C}yyT_n}\) and \(\{S_l\}_{l \in \mathbb{C}ynT_n}\) (resp. \(\{P_i\}_{i \in \mathbb{C}ynT_{n-1}}\) and \(\{S_l\}_{l \in \mathbb{C}ynT_{n-1}}\)), indexed by Lyndon words over \(T_n\) (resp. \(T_{n-1}\))\(^{[11]}\).

\begin{equation}
\mathcal{D}_{T_n} = \bigcap_{i \in \mathbb{C}ynT_n} e_{i} \otimes_{P_i} \mathcal{D}_{T_n} \quad \text{(decreasing lexicographical ordered product)}
\end{equation}

where \(\omega\) is the half-shuffle product\(^{[49]}\) and, for any \(w = t_1 \ldots t_m \in T_n^+\), 
\[ a(w) = (-1)^w t_m \ldots t_1 \text{ and } r(w) = \text{ad}_{t_1} \circ \ldots \circ \text{ad}_{t_m} t_m. \]

Furthermore, considering \(T_n\), the sub Lie algebra of \(\text{Lie}_\mathbb{Q}(\langle T_n \rangle)\) generated by \(\{ad_{-T_n} t\}_{t \in T_{n-1}}\), the enveloping algebra \(U(\mathbb{C}yT_n)\) and its dual \(U(\mathbb{C}yT_n)\)\(^\vee\) are generated by the dual bases (see Section \([2.3]\) below)
\begin{equation}
\mathcal{B} = \{\text{ad}_{-T_n} t_1 \ldots \text{ad}_{-T_n} t_p \}_{t_1 \ldots t_p \in T_{n-1}},
\end{equation}
\begin{equation}
\mathcal{B}^\vee = \{a(T_n t_1) \omega(\ldots \omega a(T_n t_p \ldots)) \}_{t_1 \ldots t_p \in T_{n-1}}.
\end{equation}

• With the previous diagonal series \(\mathcal{D}_{T_n}\), for \(z_n \to z_{n-1}\), grouplike solution 
of \((14)\) is of the form \(h(z_n)H(z_1, \ldots, z_{n-1})\) (see Proposition \([4]\)\([6]\) Theorems \([2]\)\([3]\) Corollary \([4]\) below) such that 
\(- \ h \text{ is solution of } df = (2\pi)^{-1}N_{n-1}f, \text{ where } N_{n-1} \text{ is the connection determined in } \([16]\). \text{ Hence, } h(z_n) \sim z_n \to z_{n-1} \ (z_{n-1} - z_n)^{t_{n-1,n}/2\pi}.\)

\text{\(i.e.\) simply connected regions in the neighborhood of the divisor at infinity.}

\text{\(\text{\(9\)See Note \([4]\)\).}
The organization of this paper is as follows.

1. In Section 2, some algebraic combinatorics of the diagonal series, on the KZ functional expansion of $KZ_{n-1}$ considering two different cases of starting condition, $V_0$, for (24):

- as the grouplike series $(\alpha^z \otimes \text{Id})D_{\mathcal{T}_n}$. In this case, $\{V_k\}_{k \geq 0}$ converges for (1) to the unique solution of (14) satisfying asymptotic conditions achieving the dévissage (using the decreasing lexicographical order product "$\prod$"):

$$F_{KZ_n} = \prod_{l \in \mathcal{L}_ynT_n} e^{F_{S_l}P_l} \times \left(1_{T_n} + \sum_{v_1, \ldots, v_k \in \mathcal{T}_n^*; k \geq 1 \atop t_1, \ldots, t_k \in T_{n-1}} F_{\alpha(v_1t_1) \ldots \alpha(v_kt_k)} r(v_1t_1) \ldots r(v_kt_k) \right)$$

$$= \prod_{l \in \mathcal{L}_ynT_n} e^{F_{S_l}P_l} \left( \prod_{l_2 \in \mathcal{L}_ynT_{n-1}; l_1 \in \mathcal{L}_ynT_n} e^{F_{S_{l_1}S_{l_2}}P_{l_1}P_{l_2}} \right) \prod_{l \in \mathcal{L}_ynT_n} e^{F_{S_l}P_l},$$

- as $(\alpha^z \otimes \text{Id})D_{\mathcal{T}_n} \mod [\text{Lie}_H(Y) \langle T_n \rangle, \text{Lie}_H(Y) \langle T_n \rangle]$ (see also Remarks 11 and 15 below). In this case, extending the treatment in 17 below, one gets an approximation of (26):

$$F_{KZ_n} = e^{\sum_{t \in T_n} F_{1t} \left(1_{T_n} \right)} + \sum_{v_1, \ldots, v_k \in \mathcal{T}_n^*; k \geq 1 \atop \tilde{t}_1, \ldots, \tilde{t}_k \in \mathcal{T}_{n-1}} F_{\alpha(v_1\tilde{t}_1) \ldots \alpha(v_k\tilde{t}_k)} r(v_1\tilde{t}_1) \ldots r(v_k\tilde{t}_k).$$

where, for $w = t_1 \ldots t_m \in \mathcal{T}_n^*$, $\tilde{w} = t_1 \ldots \omega t_m$.

Specializing the convergent case to (21), it will illustrate, in Section 4 with the cases of $KZ_4$ and, in a similar way, $KZ_3$ (achieving Example 2).

The organization of this paper is as follows

- In Section 2 some algebraic combinatorics of the diagonal series, on the concatenation-shuffle bialgebra and on a Loday’s generalized bialgebra, will be recalled briefly by Theorem 4. In particular, we will insist on the monoidal factorizations (by Lazard and by Schützenberger) leading to various dual topological bases on which will base the computations of the next sections.

---

The sum $\sum_{l \geq 0} V_l$ is called Volterra expansion (like) of $dF = \Omega_n F$ (see (11) (22)).
In Section 3 various expansions of Chen series will be provided by Propositions 1–5, Theorem 2, and Corollary 3 to obtain grouplike solutions of (4) in the factorized forms, over $H(V)\langle T_n \rangle$ and then over $H(V)\langle T_n \rangle / \langle J_{R_n} \rangle$. In particular, by (7)–(8), finite factorization is similar to dévissage of $KZ_n$.

**Example 3.** Grouplike solution of $KZ_3$ admits polylogarithms as local coordinates and solutions of $KZ_2$ (admitting elementary transcendental functions $\{\log(z_i − z_j)\}_{1 ≤ i ≤ j ≤ n}$ as coordinates) as in Example 1.

**Remark 1.** Historically, noncommutative series were introduced by Fließ in control theory to study functional expansions (in particular, the Volterra’s expansion) of nonlinear dynamical systems via so-called Fließ’ generating series of dynamical systems [22, 23] which is in duality with Chen series [34], viewed as series in noncommutative indeterminates (see Definitions 2, 3, Lemma 3, Proposition 4 below).

After that, Sussmann [62] obtained an infinite product for Chen series using the Hall basis [64] and also a noncommutative differential equation, analogous to (4).

In this context, with the controls $\{u_k\}_{1 ≤ k ≤ N}$, the differential 1-forms are of the form $ω_k(z) = u_k(z)dz$, for $k = 1, \ldots, N$ (see also (6)–(10)). These controls are encoded by the alphabet $X = \{x_k\}_{1 ≤ k ≤ N}$ (see also (15)) and are Lebesgue integrable real-valued functions on the interval $[0, T]$, $T ∈ R_0$, is so-called the duration of the controls and then the Chen series of $\{ω_k\}_{1 ≤ k ≤ N}$ belongs to $L^∞([0, T], R)\langle X \rangle$ [34].

More systematically, other finite and infinite products (see Corollary 3 below) were also proposed to obtain functional expansions [34, 35, 36, 37, 32] basing on monoidal factorizations (by Lazard and by Schützenberger) which were intensively studied earlier in [1, 61] and are widely exploited in the present work using notations of [1, 61]. Furthermore, on the one hand, results in Section 3 below can be considered as a generalization of results for controlled dynamical systems in [34] and for special functions in [42]. On the other hand, Section 4 below is an application of these results, in continuation of [25] (see also Examples 1–3 of [25]).

2. Combinatorial frameworks

2.1. Algebraic combinatorics on formal power series. Now, for fixed $n$ and for any $2 ≤ k ≤ n$, in virtue of (8) let us consider

$$T_k := \{t_{i,j} \mid 1 ≤ i < j ≤ k\} = T_k \cup T_{k−1}, \text{ where } T_k := \{t_{j,k} \mid 1 ≤ j ≤ k−1\}.$$  (28)

In terms of cardinality, $|T_n| = n(n−1)/2$ and $|T_n| = n−1$. If $n ≥ 4$ then $|T_{n−1}| ≥ |T_n|$.

**Example 4.**

1. $T_5 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{1,5}, t_{2,3}, t_{2,4}, t_{2,5}, t_{3,4}, t_{3,5}, t_{4,4}\}$, one has $T_5 = \{t_{1,5}, t_{2,5}, t_{3,5}, t_{4,5}\}$ and $T_4$.

2. $T_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$, one has $T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$ and $T_3$.

3. $T_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$, one has $T_3 = \{t_{1,3}, t_{2,3}\}$ and $T_2 = \{t_{1,2}\}$.

With notations in (28), let us consider the following total order $T_n$

$$T_2 \succ \ldots \succ T_n \text{ and, for } 2 ≤ k ≤ n, \ t_{1,k} \succ \ldots \succ t_{k−1,k}$$  (29)

and then over the sets of Lyndon words [53, 61] $\text{LynT}$ and $\text{LynT}_n$ as follows

$$\text{LynT}_2 \succ \ldots \succ \text{LynT}_n.$$  (30)

\[\text{Note 26 below and the description in the beginning of Section 1.}\]
According to the Chen-Fox-Lyndon theorem \[53, 61, 64\], with the ordering in \(29\)\-(\(30\)), there is a unique way to get the standard factorization of \(l \in \mathcal{L}ynT_n\), i.e.\[12\] so that \(st(l) = (l_1, l_2)\), where \(l_2\) is the longest nontrivial proper right factor of \(l\) or equivalently its smallest such for the lexicographic ordering \[53\]. Then

\[
\mathcal{L}ynT_{n-1} \triangleright \mathcal{L}ynT_n, \mathcal{L}ynT_{n-1} \triangleright \mathcal{L}ynT_n.
\]

More generally, for any \((t_1, t_2) \in T_{k_1} \times T_{k_2}, 2 \leq k_1 < k_2 \leq n\), one also has

\[
t_2 t_1 \in \mathcal{L}ynT_{k_2} \subset \mathcal{L}ynT_n \quad \text{and} \quad t_2 \preceq t_2 t_1 \preceq t_1.
\]

Hence, as consequences of \(29\)\-(\(31\)), one obtains

- If \(l \in \mathcal{L}ynT_{k-1}\) and \(t \in T_{k}, 2 \leq k \leq n\) then \(t l \in \mathcal{L}ynT_n\) and \(t < tl < l\).
- If \(l_1 \in \mathcal{L}ynT_{k_1}\) and \(l_2 \in \mathcal{L}ynT_{k_2}\) (for \(2 \leq k_1 < k_2 \leq n\)) then \(l_2 l_1 \in \mathcal{L}ynT_{k_2} \subset \mathcal{L}ynT_n\) and \(l_2 \preceq l_2 l_1 \preceq l_1\).
- If \(l_1 \in \mathcal{L}ynT_k\) and \(l_2 \in \mathcal{L}ynT_{k-1}\) (for \(2 \leq k_1 < k_2 \leq n\)) then \(l_1 l_2 \in \mathcal{L}ynT_n\) and \(l_1 \preceq l_1 l_2 \preceq l_2\).

In this Section, \(A\) is a commutative integral ring containing \(\mathbb{Q}\) and, by notations in \[11, 53, 61\], \((T_n^*, 1_{T_n})\) is the free monoid generated by \(T_n\), for the concatenation denoted by \(\concat\) (and it will be omitted when there is non ambiguity). The set of noncommutative polynomials (resp. series) over \(T_n\) is denoted by \(\mathcal{A}(T_n)\) (resp. \(\mathcal{A}^\langle T_n^\rangle\) and \(\mathcal{A}^{\langle T_n \rangle} = \mathcal{A}(T_n)^+\) (i.e \(\mathcal{A}^{\langle T_n \rangle}\) is dual to \(\mathcal{A}(T_n)\)), via the following pairing

\[
\mathcal{A}^\langle T_n \rangle \otimes_{\mathcal{A}} \mathcal{A}(T_n) \longrightarrow \mathcal{A}, \quad T \otimes_{\mathcal{A}} P \longmapsto \langle T \mid P \rangle := \sum_{w \in T_n^\ast} \langle T \mid w \rangle \langle P \mid w \rangle.
\]

In the sequel, all algebras, linear maps and tensor signs that appear in the following are over \(\mathcal{A}\) unless specified otherwise.

The set of Lie polynomials (resp. Lie series), over \(T_n\) with coefficients in \(A\), is denoted by \(\mathcal{L}i e_{\mathcal{A}}(T_n)\) (resp. \(\mathcal{L}i e_{\mathcal{A}}^\langle T_n^\rangle\)). For convenience, the set of exponentials of Lie series will be denoted by \(e^{\mathcal{L}i e_{\mathcal{A}}^\langle T_n \rangle} = \{ e_C \}_{C \in \mathcal{L}i e_{\mathcal{A}}^\langle T_n \rangle}\).

The smallest algebra containing \(\mathcal{A}(T_n)\) and closed by rational operations (i.e. addition, concatenation, Kleene star) is denoted by \(\mathcal{A}^{\mathcal{R}at}(T_n)\). Any \(S \in \mathcal{A}^{\mathcal{R}at}(T_n)\) is said to be rational and, by a Schützenberger’s theorem \[1\], there is a linear representation \((\beta, \mu, \eta)\) of dimension \(k \geq 0\) such that (and conversely)

\[
S = \beta((\text{Id} \otimes \mu)D_{T_n})\eta = \sum_{w \in T_n^\ast} (\beta \mu(w)\eta)w,
\]

where \(\mu\) is the morphism of monoids from \(X^*\) to \(\mathcal{M}_{k,k}(A)\), mapping each letter to a \(k \times k\)-matrix, \(\beta\) is a column matrix in \(\mathcal{M}_{k,1}(A)\) and \(\eta\) is a row matrix in \(\mathcal{M}_{1,k}(A)\).

**Example 5 \[50\].** To simplify, let \(X\) be the alphabet \(\{x_0, x_1\}\). The rational series \((t^2x_0x_1)^*\) and \((-t^2x_0x_1)^*\) admit, respectively, \((\nu_1, \{\mu_1(x_0), \mu_1(x_1)\}, \eta_1)\) and \((\nu_2, \{\mu_2(x_0), \mu_2(x_1)\}, \eta_2)\) as the linear representations given by

\[
\nu_1 = (1 \ 0), \quad \mu_1(x_0) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \mu_1(x_1) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

\[
\nu_2 = (1 \ 0), \quad \mu_2(x_0) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \quad \mu_2(x_1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

\[\text{12}\text{It leads to Lie brackets constructing basis of free Lie algebra and enveloping algebra in \(70\).}\]
Recall that $\mathcal{A}^{rat}(\langle T_n \rangle)$ is also closed by shuffle which is denoted by $\shuffle$ and defined recursively, for any letters $x, y \in T_n$ and words $u, v \in T_n^*$, as follows [11]

\begin{equation}
0 \omega 1 \omega = 1 \omega \omega = u = u \quad \text{and} \quad (xu) \omega (yv) = x(\omega xu) + y(\omega xu).
\end{equation}

**Example 6** ([40]). With the notations in Example 3 one has (see [40])

\begin{equation}
(-t^2x_0x_1)^\omega = (-t^2x_0^2x_1^2)^\omega = (-4t^4x_0^2x_1^2)^\omega
\end{equation}

and $(-4t^4x_0^2x_1^2)^\omega$ admits $(\nu, \{\mu(x_0), \mu(x_1)\}, \eta)$ as the linear representations given by

\begin{equation}
\nu = (1 \quad 0 \quad 0 \quad 0), \mu(x_0) = \begin{pmatrix} 0 & it & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & it \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mu(x_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ it & 0 & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & t & it & 0 \end{pmatrix}, \eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\end{equation}

By a Radford’s theorem [59, 61], the shuffle algebra, over $T_n$ and with coefficients in $A$, admits $\mathcal{L} T_n^*$ as pure transcendence basis and then

\begin{equation}
\text{Sh}_A(\langle T_n \rangle) := (\mathcal{A}(\langle T_n \rangle), \omega) \simeq (\mathcal{A}[\{t\}_{t \in \mathcal{L} T_n^*}], \omega).
\end{equation}

Recall also that the following co-products (of $\text{conc}$ and $\omega$) are defined respectively, for any $u, v, w \in T_n^*$, as follows

\begin{equation}
(\Delta_\text{conc} w | u \otimes v) = \langle w | uv \rangle \quad \text{and} \quad (\Delta_\omega w | u \otimes v) = \langle w | u \omega v \rangle.
\end{equation}

**Example 7.** For any $t_1$ and $t_2 \in T_n$, one has

\begin{equation}
\Delta_\text{conc}(t_1t_2) = t_1t_2 \otimes 1_{T_n^*} + t_1 \otimes t_2 + t_1t_2 \otimes 1_{T_n^*},
\end{equation}

\begin{equation}
\Delta_\omega(t_1t_2) = t_1t_2 \otimes 1_{T_n^*} + t_1 \otimes t_2 + t_2 \otimes t_1 + 1_{T_n^*} \otimes t_1t_2.
\end{equation}

It follows, for any $w \in T_n^*$, that [13, 9]

\begin{equation}
\Delta_\text{conc} w = \sum_{u, v \in T_n^*, uv = w} u \otimes v \quad \text{and} \quad \Delta_\omega w = \sum_{u, v \in T_n^*} \langle w | u \omega v \rangle u \otimes v
\end{equation}

In particular,

\begin{equation}
\Delta_\text{conc} 1_{T_n^*} = 1_{T_n^*} \otimes 1_{T_n^*} \quad \text{and} \quad \Delta_\omega 1_{T_n^*} = 1_{T_n^*} \otimes 1_{T_n^*}
\end{equation}

and, for any $t \in T_n$,

\begin{equation}
\Delta_\text{conc} t = t \otimes 1_{T_n^*} + 1_{T_n^*} \otimes t \quad \text{and} \quad \Delta_\omega t = t \otimes 1_{T_n^*} + 1_{T_n^*} \otimes t.
\end{equation}

Both the products $\text{conc}$ and $\omega$ and the co-products $\Delta_\text{conc}$ and $\Delta_\omega$ are extended, for any noncommutative series $S, R \in \mathcal{A}(\langle T_n \rangle)$, by

\begin{equation}
SR = \sum_{w \in T_n^*, uv \in T_n^*, uv = w} \langle S | u \rangle \langle R | v \rangle w \in \mathcal{A}(\langle T_n \rangle),
\end{equation}

\begin{equation}
S \omega R = \sum_{u, v \in T_n^*} \langle S | u \rangle \langle R | v \rangle u \omega v \in \mathcal{A}(\langle T_n \rangle),
\end{equation}

\begin{equation}
\Delta_\text{conc} S = \sum_{w \in T_n^*} \langle S | w \rangle \Delta_\text{conc} w \in \mathcal{A}(\langle T_n^* \otimes T_n^* \rangle),
\end{equation}

\begin{equation}
\Delta_\omega S = \sum_{w \in T_n^*} \langle S | w \rangle \Delta_\omega w \in \mathcal{A}(\langle T_n^* \otimes T_n^* \rangle).
\end{equation}

\footnote{It follows that letters are primitive, for $\Delta_\text{conc}$ and $\Delta_\omega$.}
Remark 2 ([36] [37] [43]). Let \((\beta, \mu, \eta)\) be a linear representation of dimension \(k\) of \(S \in A^\text{rat} \langle \langle T_n \rangle \rangle\) (see [33]) which is also associated to the linear representations \((\beta, \mu, \epsilon_i)\) and \((\epsilon_i, \mu, \eta)\) of dimension \(k\) of the rational series \(\{L_i\}_{1 \leq i \leq k}\) and \(\{R_i\}_{1 \leq i \leq k}\), where

\[
e_i \in M_{1,k}(A) \quad \text{and} \quad \epsilon_i = (0 \ldots 0 \ 1 \ 0 \ldots 0).
\]

By [33], it follows that, for any \(x, y \in T_n\), one has

\[
\langle S | xy \rangle = \beta \mu(x)\mu(y)\eta = \sum_{i=1}^{k} (\beta \mu(x)\epsilon_i)\epsilon_i\mu(y)\eta = \sum_{i=1}^{k} \langle L_i | x \rangle \langle R_i | y \rangle,
\]

\[
\langle \Delta_{\text{conc}} S | x \otimes y \rangle = \langle S | xy \rangle = \sum_{i=1}^{k} \langle L_i | x \rangle \langle R_i | y \rangle = \sum_{i=1}^{k} \langle L_i \otimes R_i | x \otimes y \rangle.
\]

With these products and co-products, any series \(S\) in \(A\langle\langle T_n \rangle\rangle\) is said to be

- A character for \(\text{conc}\) (resp. \(\omega\)) if and only if, for \(u, v \in T_n\),

\[
\langle S | uv \rangle = \langle S | u \rangle \langle S | v \rangle \quad \text{resp.} \quad \langle S | u \omega v \rangle = \langle S | u \rangle \langle S | v \rangle.
\]

Or equivalently, it is group-like series for \(\Delta_{\text{conc}}\) (resp. \(\Delta_{\omega}\)) if and only if

\[
\langle S | 1_{T_n} \rangle = 1 \quad \text{and} \quad \Delta_{\text{conc}}(S) = \Phi(S \otimes S) \quad \text{(resp.} \quad \Delta_{\omega}(S) = \Phi(S \otimes S))
\]

where the map \(\Phi: A\langle\langle T_n \rangle\rangle^\vee \otimes A\langle\langle T_n \rangle\rangle^\vee \to (A\langle\langle T_n \rangle\rangle \otimes A\langle\langle T_n \rangle\rangle)^\vee\) is injective. Any group-like series, for \(\Delta_{\text{conc}}\), equals to \(P^\ast\), for \(P \in A.T_n\), and vice versa.

- A infinitesimal character, for \(\text{conc}\) (resp. \(\omega\)) if and only, for \(w, v \in T_n\),

\[
\langle S | wv \rangle = \langle S | w \rangle \langle v | 1_{T_n} \rangle + \langle w | 1_{T_n} \rangle \langle S | v \rangle,
\]

\[
\langle S | w \omega v \rangle = \langle S | w \rangle \langle v | 1_{T_n} \rangle + \langle w | 1_{T_n} \rangle \langle S | v \rangle.
\]

Or equivalently, \(S\) is a primitive series for \(\Delta_{\text{conc}}\) (resp. \(\Delta_{\omega}\)) if and only if

\[
\Delta_{\text{conc}} S = 1_{T_n} \otimes S + S \otimes 1_{T_n} \quad \text{resp.} \quad \Delta_{\omega} S = 1_{T_n} \otimes S + S \otimes 1_{T_n}.
\]

By a Ree’s theorem [60], any Lie series is primitive for \(\Delta_{\omega}\), and vice versa. For \(\Delta_{\omega}\), when \(\Phi\) is injective, if \(S\) is group-like then log \(S\) is primitive and, conversely, if \(S\) is primitive then \(e^S\) is group-like. In particular, the sets of primitive polynomials, for \(\Delta_{\omega}\) is

\[
\text{Prim}_{\omega}(T_n) = \text{Lie}_A\langle T_n \rangle \quad \text{and} \quad \text{Prim}_{\text{conc}}(T_n) = A.T_n.
\]

Finally, on the one hand, by [43] CQMM theorem, one has (see [61]) \(H_{\text{conc}}(T_n) := (A\langle\langle T_n \rangle\rangle, \text{conc}, 1_{T_n}, \Delta_{\text{conc}}) \simeq U(\text{Lie}_A\langle T_n \rangle)\),

\[
H_{\omega}(T_n) := (A\langle\langle T_n \rangle\rangle, \omega, 1_{T_n}, \Delta_{\omega}) \simeq U(\text{Lie}_A\langle T_n \rangle)^\vee,
\]

and, on the other hand, the Sweedler’s dual of \(H_{\omega}(T_n)\) is followed [61]

\[
H^\ast_{\omega}(T_n) = (A^\text{rat} \langle\langle T_n \rangle\rangle, \omega, 1_{T_n}, \Delta_{\omega}).
\]

The last dual is defined, for any \(S \in A\langle\langle T_n \rangle\rangle\), as follows [61]

\[
S \in H^\ast_{\omega}(T_n) \iff \Delta_{\text{conc}}(S) = \sum_{i \in I} L_i \otimes R_i,
\]

where \(I\) is finite and, by Remark [2] \(\{L_i, R_i\}_{i \in I}\) can be selected in \(A^\text{rat} \langle\langle T_n \rangle\rangle\).
Remark 3. With notation in Remark 2 one also has

\[ S \in A^{rat}(\langle T_n \rangle) \iff \Delta_{conc}(S) = \sum_{i \in I} L_i \otimes R_i. \]

In all the sequel, any word \( v = t_1 \ldots t_m \in T_n^* \) can be associated to the following polynomials in \( A(\langle T_n \rangle) \)

\[ \bar{v} = t_1 \omega \ldots \omega t_m = |v|! \omega_{t \in T_n} t^{\bar{v}}, \quad \hat{v} = \frac{\bar{v}}{|v|!} = \omega_{t \in T_n} t^{\hat{v}}, \]

where \( \bar{v} \) is the mirrour of \( v \) (i.e. \( \bar{v} = t_m \ldots t_1 \)), |v| is the length of \( v \) and \(|v|_i\) is the number of occurrences of \( t \) in \( v \).

Let \( \alpha \) be the injective linear endomorphism defined by

\[ \forall v \in T_n^*, \quad \alpha(v) = (-1)^{|v|} \bar{v}. \]

It is involutive \( (\alpha(1_{T_n^*}) = 1_{T_n^*}) \) and is extended, over \( A(\langle T_n \rangle) \), as follows

\[ \forall S \in A(\langle T_n \rangle), \quad \alpha(S) = \sum_{w \in T_n^*} (S | w)\alpha(w) = \sum_{w \in T_n^*} (-1)^{|w|}(S | w)\hat{w} \]

and then

\[ \forall S, R \in A(\langle T_n \rangle), \quad \alpha(SR) = \alpha(R)\alpha(S), \quad \alpha(S \omega R) = \alpha(S) \omega \alpha(R). \]

Moreover, if \( S \) is such that \( (S | 1_{T_n^*}) = 1 \) then \( \alpha(S) \) is its inverse, \( S^{-1} \), for \( conc \).

\[ \omega a(S) = a(S)\omega = 1_{T_n^*} \quad \text{and then} \quad \forall L \in Lie_A(\langle T_n \rangle), \quad \alpha(e^L) = e^{-L}. \]

Ending this section, let us also consider the following product\(^{15}\) \( \omega \), defined for any \( t \in T_n, R \in A(\langle T_n \rangle), H \in A(\langle T_n \rangle) \), by (see \[54, 40, 11, 43])

\[ 1_{T_n^*} (tH) = 0 \quad \text{and} \quad (tH) \omega R = \begin{cases} tH & \text{if} \quad R = 1_{T_n^*}, \\ t(H \omega R) & \text{if} \quad R \neq 1_{T_n^*}. \end{cases} \]

Example 8. For \( T_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\} \), using the second part of \[59] one has (with \( t = t_{1,3}, H = t_{1,2} \) and \( R = t_{2,3} \))

\[ (t_{1,3}t_{1,2}) \omega t_{2,3} = t_{1,3}(t_{1,2} \omega t_{2,3}) = t_{1,3}(t_{1,3}t_{2,3} + t_{2,3}t_{1,2}) = t_{1,3}t_{2,3} + t_{1,3}t_{2,3}t_{1,2} \]

and (with \( t = t_{1,3}, H = t_{1,2}^*, R = t_{2,3} \))

\[ (t_{1,3}t_{1,2}^*) \omega t_{2,3} = t_{1,3}(t_{1,2}^* \omega t_{2,3}) = t_{1,3}(t_{1,2}^*t_{2,3} + t_{1,2}^*t_{2,3}t_{1,2}) = t_{1,3}t_{2,3}t_{1,2} \]

This product \( \omega \) is not associative but satisfies the following identity

\[ \forall R, S, T \in A(\langle T_n \rangle), \quad (R \omega S) \omega T = R \omega (S \omega T) + R \omega (T \omega S). \]

\(^{15}\) It is more general than the one used in \[54, 40, 11, 43\] (denoted by \( \circ \), for iterated integrals associated to polynomials) and is called \( half-shuffle \), denoted by \( \prec \) in \[59\] and \( demi-shuffle \) in \[54\] (see Corollary 2 below in which involve iterated integrals associated to series).

This product has origine from the chronological product in quantum electrodynamical \[27\]

\[ (g \ast h)(t) = \int_0^t g(s)h'(s)ds, \]

i.e. the "half integration by part" while the shuffle product encodes the integration by part \[11\].

\(^{16}\) It uses also the classic identity \( a \omega b^* = b^*ab \), for \( a \) and \( b \in T_n \).
\((A\langle\mathcal{T}_n\rangle, \omega)\) is a Zinbiel algebra \([29]\) and \(\omega\) is a symmetrised product of \(\omega\), i.e. for any \(x, y \in \mathcal{T}_n, u, v \in \mathcal{T}_n^*\) and \(R,S,T \in A\langle\mathcal{T}_n\rangle\),
\begin{align*}
(xu) \omega (yv) = (xu) \omega (yv) + (yv) \omega (xu) \quad \text{and} \quad R \omega S = R \omega S + S \omega R.
\end{align*}

**Example 9.** For any \(t_1, t_2 \in \mathcal{T}_n, w_1, w_2 \in \mathcal{T}_n^+\), by the recursion \([35]\) one has
\begin{align*}
(t_1 w_1) \omega (t_2 w_2) &= t_1 (w_1 \omega (t_2 w_2)) + t_2 (w_2 \omega (t_1 w_1)) \\
&= (t_1 w_1) \omega (t_2 w_2) + (t_2 w_2) \omega (t_1 w_1),
\end{align*}
\begin{align*}
(t_1 w_1^*) \omega (t_2 w_2^*) &= t_1 (w_1^* \omega (t_2 w_2^*)) + t_2 (w_2^* \omega (t_1 w_1^*)) \\
&= (t_1 w_1^*) \omega (t_2 w_2^*) + (t_2 w_2^*) \omega (t_1 w_1^*).
\end{align*}

The Zinbiel bialgebra and its dual are Loday’s generalized bialgebras \([49]\), i.e.
\begin{align*}
Z_{\omega}(\mathcal{T}_n) &= (A\langle\mathcal{T}_n\rangle, \omega, 1_{\mathcal{T}_n}, \Delta_{\text{conc}}), \\
Z_{\text{conc}}(\mathcal{T}_n) &= (A\langle\mathcal{T}_n\rangle, \text{conc}, 1_{\mathcal{T}_n^*}, \Delta_{\omega}),
\end{align*}
where \(\Delta_{\omega} : A\langle\mathcal{T}_n\rangle \to A\langle\mathcal{T}_n\rangle \otimes A\langle\mathcal{T}_n\rangle\) is defined by \(\Delta_{\omega} 1_{\mathcal{T}_n^*} = 1_{\mathcal{T}_n} \otimes 1_{\mathcal{T}_n}^*\) and
\begin{itemize}
\item for any \(t \in \mathcal{T}_n, w \in \mathcal{T}_n^*, \Delta_{\omega} t = t \otimes 1_{\mathcal{T}_n^*}\) and \(\Delta_{\omega} tw = (\Delta_{\omega} t)(\Delta_{\omega} w)\),
\item for any \(P \in A\langle\mathcal{T}_n\rangle, \Delta_{\omega} P = \langle P | 1_{\mathcal{T}_n^*} \rangle 1_{\mathcal{T}_n^*} \otimes 1_{\mathcal{T}_n^*} + \sum_{w \in \mathcal{T}_n^+} \langle P | v \rangle \Delta_{\omega} v\).
\end{itemize}
for any \(P,Q \in A\langle\mathcal{T}_n\rangle\), one can check easily that
\begin{align*}
\Delta_{\text{conc}}(P \omega Q) = \Delta_{\text{conc}}(P) \omega \Delta_{\text{conc}}(Q) \quad \text{and} \quad \Delta_{\omega}(PQ) = \Delta_{\omega}(P)\Delta_{\omega}(Q).
\end{align*}

The co-product \(\Delta_{\omega}\) is also extended, for any \(S \in A\langle\mathcal{T}_n\rangle\), as follows
\begin{align*}
\Delta_{\omega} S = \sum_{w \in \mathcal{T}_n^*} \langle S | w \rangle \Delta_{\omega} w \in A\langle \mathcal{T}_n^* \otimes \mathcal{T}_n^* \rangle.
\end{align*}

### 2.2. Diagonal series in concatenation-shuffle bialgebra.
In all the sequel, by \([28]\), the characteristic series (see \([1]\)) of \(T_k\) and \(T_k^*\) (resp. \(T_n^*\) and \(T_n^*\)) are Lie polynomials, still denoted by \(T_k\) and \(T_k^*\) (resp. rational series \(T_k^*\) and \(T_k^*\)), for \(2 \leq k \leq n\).

Let \(\nabla S\) denote \(S - 1_{\mathcal{T}_k^*}\) (resp. \(S - 1_{\mathcal{T}_k} \otimes 1_{\mathcal{T}_n^*}\)), for \(S \in A(\mathcal{T}_k)\) (resp. \(A(\mathcal{T}_k) \otimes A(\mathcal{T}_k)\)).

If \(\langle S | 1_{\mathcal{T}_k^*} \rangle = 0\) (resp. \(\langle S | 1_{\mathcal{T}_k} \otimes 1_{\mathcal{T}_n^*} \rangle = 0\) then the Kleene star of \(S\) is defined by
\begin{align*}
S^* := 1 + S + S^2 + \cdots \quad \text{and} \quad S^+ := S^* S = SS^*.
\end{align*}
In the same way, for any \(2 \leq k \leq n\), the diagonal series is defined as follows
\begin{align*}
\mathcal{D}_{\mathcal{T}_k} = \mathcal{M}_{\mathcal{T}_k}^* \quad \text{and} \quad \mathcal{D}_{\mathcal{T}_k} = \mathcal{M}_{\mathcal{T}_k}^*, \quad \text{where} \quad \mathcal{M}_{\mathcal{T}_k} = \sum_{t \in \mathcal{T}_k} t \otimes t \quad \text{and} \quad \mathcal{M}_{\mathcal{T}_k} = \sum_{t \in \mathcal{T}_k} t \otimes t.
\end{align*}

One also defines
\begin{align*}
\mathcal{M}_{\mathcal{T}_k}^+ = \mathcal{D}_{\mathcal{T}_k} \mathcal{M}_{\mathcal{T}_k} = \mathcal{M}_{\mathcal{T}_k} \mathcal{D}_{\mathcal{T}_k} \quad \text{and} \quad \mathcal{M}_{\mathcal{T}_k}^+ = \mathcal{D}_{\mathcal{T}_k} \mathcal{M}_{\mathcal{T}_k} = \mathcal{M}_{\mathcal{T}_k} \mathcal{D}_{\mathcal{T}_k}
\end{align*}
and, expanding \([60]\), one also has
\begin{align*}
\mathcal{D}_{\mathcal{T}_k} = \sum_{w \in \mathcal{T}_k^*} w \otimes w = \sum_{w \in \mathcal{T}_k^*} w \otimes w, \quad \mathcal{D}_{\mathcal{T}_k} = \sum_{w \in \mathcal{T}_k^*} w \otimes w = \sum_{w \in \mathcal{T}_k^*} w \otimes w.
\end{align*}
If \( S \in \overline{A(T_k)} \) such that \( \langle S \mid 1_{T_k} \rangle = 0 \) then \( S^* \) is the unique solution of
\[
\nabla S = T_k S \quad \text{and} \quad \nabla S = ST_k.
\]
In the same way, for any \( 2 \leq k \leq n \), \( D_{T_k} \) (resp. \( D_{T_k} \)) is the unique solution of
\[
\nabla S = M_{T_k} S \quad \text{and} \quad \nabla S = SM_{T_k} \quad \text{for} \quad \nabla S = M_{T_k} S \quad \text{and} \quad \nabla S = SM_{T_k}.
\]
Let us recall, by (28) and in particular \( T_n = \bigcup T_{n-1} \) (for simplification), that
\[\text{For any } a_1, \ldots, a_{n-1} \in A, \text{ one has} \]
\[
( \sum_{i=1}^{n-1} a_i t_{i,n} )^* = \sum_{c_1, \ldots, c_{n-1} \geq 0} \left( \sum_{i=1}^{n-1} c_i t_{i,n} \right)^*.
\]
Thus, as \( A \)-modules, \( T_{n-1} \oplus T_n^* \) and \( T_n \oplus T_{n-1} \) are generated by the series of the following form (\( t_{i,j,1}, \ldots, t_{i,j,m} \) are the letters in \( T_{n-1} \))
\[
\left( \sum_{c_1, \ldots, c_{n-1} \geq 0} \left( \sum_{i=1}^{n-1} c_i t_{i,n} \right)^* \right) t_{i,j,1} \left( \sum_{c_1, \ldots, c_{n-1} \geq 0} \left( \sum_{i=1}^{n-1} c_i t_{i,n} \right)^* \right)
\]
and similarly for \( T_{n-1} \oplus T_m \) and \( T_n \oplus T_{n-1} \).

By Lazard factorization, i.e. \( T_n^* = T_n^*(T_{n-1}^*)^* = (T_{n-1}^*)^*T_n^* \), or equivalently, \( T_n^* = T_n^*(T_{n-1}^*)^* = (T_{n-1}^*)^*T_n^* \) [53] [64], one has
\[
T_n^* = \sum_{m \geq 0} T_{n-1} \oplus T_n^* = \sum_{m \geq 0} T_n^* \oplus T_{n-1}^*,
\]
and then, by (65), it follows that
\[
D_{T_n} = \sum_{m \geq 0} w \otimes w.
\]
Let the free Lie algebra \( Lie_A(T_n) \) be endowed the basis \( \{P_l\}_{l \in Lyn(T_n)} \) over which are constructed, for the enveloping algebra \( U(Lie_A(T_n)) \), the PBW basis \( \{P_w\}_{w \in T_n^*} \) and its dual, \( \{S_w\}_{w \in T_n^*} \) containing \( \{S_l\}_{l \in Lyn(T_n)} \) which is a pure transcendence basis of the shuffle algebra \( Sh_A(T_n) \) [61]:
\[
Lie_A(T_n) = \text{span}_A \{P_l\}_{l \in Lyn(T_n)}, \quad Sh_A(T_n) = A[\{S_l\}_{l \in Lyn(T_n)},
\]
\[\forall l, \lambda \in Lyn(T_n), \langle P_l \mid S_\lambda \rangle = \delta_{l,\lambda}, \quad \forall u, v \in T_n, \langle P_u \mid S_v \rangle = \delta_{u,v}.
\]
Homogenous in weight polynomial, \( P_w \) [53] \( S_w \) \( T_n \) \( T_n^* \) are constructed algorithmically and recursively (\( P_{1_{T_n}} = 1_{T_n^*} = S_{1_{T_n}} \)) as follows [53]
\[
\begin{cases}
P_t = t, & \text{for } t \in T_n, \\
P_l = [P_{l_1}, P_{l_2}], & \text{for } l \in Lyn(T_n \setminus T_n), \text{ st(l) } = (l_1, l_2), \\
P_w = P_{l_1} \cdots P_{l_k}, & \text{for } w = l_1 \cdots l_k, \text{ with } l_1, \ldots, l_k \in Lyn(T_n), l_1 \succ \ldots \succ l_k.
\end{cases}
\]

\[\text{For any } w \in T_n^*, \text{ the weight of } P_w \text{ and } S_w \text{ are equal to the length of } w, \text{ i.e. } |w|.
\]
Remark 5.  

\[ \begin{align*}
S_t &= t, \\
S_l &= tS_l', \quad \text{for } t \in T_n, \\
S_w &= \frac{S_{i_1}^{l_1} \ldots S_{i_k}^{l_k}}{i_1! \ldots i_k!}, \quad \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, \text{ with } l_1, \ldots, l_k \in LyN_T, l_1 \geq \ldots \geq l_k.
\end{align*} \]

(77)

Remark 4. Or equivalently, \( P_w = P_{l_1} \ldots P_{l_k} \) and \( S_w = S_{i_1} \ldots S_{i_k} \), for \( w = l_1 \ldots l_k \) with \( l_1 \geq \ldots \geq l_k \) and \( l_1, \ldots, l_k \in LyN_T \).

For any \( 2 \leq k \leq n \), by \( \ref{eq:80} \), one gets in the bialgebra \( H_{\pm}(T_k) \) \( \ref{eq:61} \) (and similarly in \( H_{\pm}(T_k) \))

\[ \begin{align*}
D_{T_k} &= \sum_{w \in T_k^*} S_v \otimes P_v = \sum_{l_1, \ldots, l_k \in LyN_T, l_1 \geq \ldots \geq l_k, n \geq 0} \frac{S_{i_1}^{l_1} \ldots S_{i_k}^{l_k}}{i_1! \ldots i_k!} \otimes P_{i_1}^{l_1} \ldots P_{i_k}^{l_k}, \\
\text{log} \ D_{T_k} &= \sum_{w \in T_k^*} w \otimes \pi_1(w),
\end{align*} \]

where \( \pi_1(w) \) is the projection on the set of primitive elements:

\[ \begin{align*}
\pi_1(w) &= \sum_{m \geq 1} (-1)^{m-1} \sum_{u_1, \ldots, u_m \in T_k^* \setminus \{1_{T_k^*}\}} \langle w \mid u_1 \ldots \ldots u_m \rangle u_1 \ldots u_m.
\end{align*} \]

(80)

2.3. More about diagonal series in concatenation-shuffle bialgebra and in a Loday’s generalized bialgebra. One defines the adjoint endomorphism, as being a derivation of \( \mathcal{L}ie_A\langle\langle T_n\rangle\rangle \), for any \( S \in \mathcal{L}ie_A\langle\langle T_n\rangle\rangle \), as follows

\[ \begin{align*}
ad_S : \mathcal{L}ie_A\langle\langle T_n\rangle\rangle &\rightarrow \mathcal{L}ie_A\langle\langle T_n\rangle\rangle, \\
R &\mapsto \ad_S R = [S, R]
\end{align*} \]

determining the so-called adjoint representation of Lie algebra \( \ref{eq:41} \) \( \ref{eq:51} \): \( \ad : \mathcal{L}ie_A\langle\langle T_n\rangle\rangle \rightarrow \text{End}(\mathcal{L}ie_A\langle\langle T_n\rangle\rangle) \), \( S \mapsto \ad_S \).

To ad corresponds to the right normed bracketing (bracketing from right to left) which is the injective linear endomorphism of \( A\langle\langle T_n\rangle\rangle \) defined by \( \ref{eq:63} \) \( r(1_{T_n}) = 0 \) and, for any \( t_1, \ldots, t_{m-1}, t_m \in T_n \), by \( \ref{eq:41} \) \( \ref{eq:61} \)

\[ \begin{align*}
r(t_1 \ldots t_{m-1} t_m) &= [t_1, [\ldots, [t_{m-1}, t_m] \ldots]] = \ad_{t_1} \circ \ldots \circ \ad_{t_{m-1}} t_m.
\end{align*} \]

(83)

Remark 5. (1) The coadjoint endomorphism is defined as follows

\[ \forall S \in \mathcal{L}ie_A\langle\langle T_n\rangle\rangle, \quad \text{coad}_S : \mathcal{L}ie_A\langle\langle T_n\rangle\rangle \rightarrow \mathcal{L}ie_A\langle\langle T_n\rangle\rangle, \quad R \mapsto \text{coad}_S R = [R, S]. \]

(2) The adjoint endomorphism of \( r \), denoted by \( \check{r} \), is defined by \( \ref{eq:41} \)

\[ \begin{align*}
\sum_{w \in T_n^+} w \otimes r(w) &= \sum_{w \in T_n^+} \check{r}(w) \otimes w,
\end{align*} \]

or equivalently, \( \langle r(v) \mid w \rangle = \langle v \mid \check{r}(w) \rangle \) \( (v, w \in T_n^+) \) satisfying

\[ \forall w \in T_n^+, \quad \langle w \mid w \rangle = \sum_{u,v \in T_n^+, uv=w} \check{r}(w) \otimes w. \]

It can be also defined recursively by \( \check{r}(1_{T_n}) = 0 \) and

\[ \forall t_1, t_2 \in T_n, w \in T_n^+, \quad \check{r}(t_1) = t_1, \quad \check{r}(t_1 w t_2) = t_1 \check{r}(w t_2) - t_2 \check{r}(t_1 w). \]

\[ \text{In } \ref{eq:41}, \text{r is denoted by } \varphi \text{ and is proved to be an isomorphism of Lie sub algebras.} \]
With Notations in \([72]\), let \( g \) be the endomorphism of \((\mathcal{A}(\mathcal{T}_n), \text{conc})\) defined by \( g(1_{\mathcal{T}_n}) = 1_{\mathcal{T}_n} \) and, for any \( w \in \mathcal{T}_n^+ \), by \( g(w) = a(w) \) such that
\[
(84) \quad \forall t \in \mathcal{T}_n, \quad g(w)(t) = -ta(w) = a(wt).
\]

Similarly, let us also associate \( r \) to \( f : (\mathcal{A}(\mathcal{T}_n), \text{conc}) \rightarrow (\text{End}(\text{Lie}_A(\langle \mathcal{T}_n \rangle)), \circ) \) defined by \( f(1_{\mathcal{T}_n}) = 1_{\text{End}(\text{Lie}_A(\langle \mathcal{T}_n \rangle))} \) and, for any \( t_1, \ldots, t_{m-1} \in \mathcal{T}_n \), as follows
\[
(85) \quad f(t_1 \ldots t_{m-1}) = \text{ad}_{t_1} \circ \ldots \circ \text{ad}_{t_{m-1}}.
\]

**Example 10.** Denoting, for any \( a, b \in \text{Lie}_A(\langle \mathcal{T}_n \rangle) \) and \( j > 0 \), \( \text{ad}_a^j b = b \) and \([4,53]\)
\[
\text{ad}_a^j b = [a, \text{ad}_a^{j-1} b] = \sum_{i=0}^j (-1)^i \binom{j}{i} a^i b a^{j-i} = r(a^j b) = f(a^j)(b),
\]

(1) one has, by the ordering \([29]-30\) and the dual bases in \([76]-[77]\), for any \( t \in \mathcal{T}_n \) and \( x \in \mathcal{T}_{n-1} \) and \( j \geq 0 \) (see \([28]\)), \( t < t \) and \( t^j x \in \mathcal{Lyn}_n \) and then, by induction, \( P_{t^j x} = \text{ad}_t^j x = f(t^j)(x) \) and \( S_{t^j x} = f(t^j) \).

(2) for \( T_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\} \), if \( t_{1,2} \prec t_{1,3} \prec t_{2,3} \) then \( t_{1,2}^j t_{1,3} \in \mathcal{Lyn}_3 \) and then \( P_{t_{1,2}^j t_{1,3}} = \text{ad}_{t_{1,2}}^j t_{1,3} = f(t_{1,2}^j)(t_{1,3}) \) and \( S_{t_{1,2}^j t_{1,3}} = t_{1,2}^j t_{1,3}, k \geq 0, i = 1 \) or \( 2 \).

Now, by the partitions in \([28]\), let \( \mathcal{I}_n \) be the sub Lie algebra of \( \text{Lie}_A(\langle \mathcal{T}_n \rangle) \) generated by \( \{\text{ad}_a^j \}_{j \geq 0} \). By the Lazard’s elimination \([4, 24, 50, 61]\), one has

- as Lie algebras and then by duality,
\[
(86) \quad \text{Lie}_A(\langle \mathcal{T}_n \rangle) = \text{Lie}_A(\langle \mathcal{T}_n \rangle) \times \mathcal{I}_n, \quad \text{Lie}_A(\langle \mathcal{T}_n \rangle)^\vee = \text{Lie}_A(\langle \mathcal{T}_n \rangle)^\vee \times \mathcal{I}_n^\vee,
\]

- as being modules and then by duality,
\[
(87) \quad \text{Lie}_A(\langle \mathcal{T}_n \rangle) = \text{Lie}_A(\langle \mathcal{T}_n \rangle) \oplus \mathcal{I}_n, \quad \text{Lie}_A(\langle \mathcal{T}_n \rangle)^\vee = \text{Lie}_A(\langle \mathcal{T}_n \rangle)^\vee \oplus \mathcal{I}_n^\vee,
\]

- and, by taking the enveloping algebras \([44]\) and then by duality,
\[
(88) \quad U(\text{Lie}_A(\langle \mathcal{T}_n \rangle)) = U(\text{Lie}_A(\langle \mathcal{T}_n \rangle)) U(\mathcal{I}_n), \quad U(\text{Lie}_A(\langle \mathcal{T}_n \rangle))^\vee = U(\text{Lie}_A(\langle \mathcal{T}_n \rangle))^\vee \ast U(\mathcal{I}_n)^\vee.
\]

\( \mathcal{I}_n \) can be also obtained as image by \( r \) of the free Lie algebra generated by \((-\mathcal{T}_n)^{\ast} \mathcal{T}_{n-1} \), on which the restriction of \( r \) is an isomorphism of free Lie algebras.

In other terms, let \( Y_{\mathcal{T}_n} := \{y_w\}_{w \in \mathcal{T}_n^{\ast} \mathcal{T}_{n-1}} \) be the new alphabet in which letters \( y_w \) are encoded by words \( w \) in \( \mathcal{T}_n^{\ast} \mathcal{T}_{n-1} \). Then, with this alphabet and the recursive constructions given in \([70]-[77]\), the families \( \{P_w\}_{w \in Y_{\mathcal{T}_n}^{\ast} \mathcal{T}_{n-1}} \) and \( \{S_w\}_{w \in Y_{\mathcal{T}_n}^{\ast} \mathcal{T}_{n-1}} \) form linear bases of \( U(\text{Lie}_A(Y_{\mathcal{T}_n}^{\ast} \mathcal{T}_{n-1})) \) and \( U(\text{Lie}_A(Y_{\mathcal{T}_n}^{\ast} \mathcal{T}_{n-1}))^\vee \), respectively, and their images form linear bases of \( U(\mathcal{I}_n) \) and \( U(\mathcal{I}_n)^\vee \).

**Example 11.** Let us illustrate this construction, as classically done in \([53]\), using the simple alphabet \( X = \{x_0, x_1\} = \{x_0\} \sqcup \{x_1\} \). Let \( Y_{x_0, x_1} \) be the new alphabet \( \{y_w\}_{w \in x_0^{\ast} x_1} \). After that, the linear bases \( \{P_w\}_{w \in Y^{\ast}} \) and \( \{S_w\}_{w \in Y^{\ast}} \) (or \( \{P_w\}_{w \in Y^{\ast}} \)) are constructed according to \([70]-[77]\). In particular, for \( s_1 > \cdots > s_r \), one has
\[
P_{x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1} = (\text{ad}_{x_0}^{s_1-1} x_1) \cdots (\text{ad}_{x_0}^{s_r-1} x_1) = r(x_0^{s_1-1} x_1) \cdots r(x_0^{s_r-1} x_1).
\]

Note also that each letter \( y_{x_0^{s_1-1} x_1} \) of \( Y_{x_0^{\ast} x_1} \) can be also encoded by the letter \( y_s \) of the alphabet \( Y = \{y_s\}_{s \geq 1} \) and then each word \( x_0^{s_1-1} x_1 \cdots x_0^{s_r-1} x_1 \) in \( X^{\ast} \) corresponds to the word \( y_{s_1} \cdots y_{s_r} \) in \( Y^{\ast} \) (see \([43]\)).
Example 12. For $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\} = T_3 \cup \mathcal{T}_2$, where $T_3 = \{t_{1,3}, t_{2,3}\}$ and $\mathcal{T}_2 = \{t_{1,2}\}$, let $T_3$ (resp. $\mathcal{T}_2$) play the role of $\{x_0\}$ (resp. $\{x_1\}$) of Example 11. In this case, the free monoid $\{t_{1,3}, t_{2,3}\}^*$ (equipping the set of Lyndon words $\operatorname{Lyn}(\{t_{1,3}, t_{2,3}\})$) plays the role of $x_0^*$. More generally, for the partition in (11) of the alphabet $T_n$, $T_n$ (resp. $T_n-1$) plays the role of $\{x_0\}$ (resp. $\{x_1\}$) of Example 11. In this case, the free monoid $T_n^*$ (equipping the set of Lyndon words $\operatorname{Lyn}(\mathcal{T}_n)$) plays the role of $x_0^*$.

Lemma 1. Let $\{b_i\}_{i \geq 0}$ and $\{\bar{b}_i\}_{i \geq 0}$ (resp. $\{c_i\}_{i \geq 0}$ and $\{\bar{c}_i\}_{i \geq 0}$) be a pair of (non necessary ordered) dual linear bases of $\mathcal{U}(\mathcal{I}_n)$ and $\mathcal{U}(\mathcal{I}_n)^\vee$ (resp. $\mathcal{U}(\operatorname{Lie}_A(\mathcal{T}_n))$ and $\mathcal{U}(\operatorname{Lie}_A(\mathcal{T}_n))^\vee$). Then the diagonal series is factorized as follows

$$D_{\mathcal{I}_n} = \left( \sum_{i \geq 0} \bar{c}_i \otimes c_i \right) \left( \sum_{i \geq 0} b_i \otimes \bar{b}_i \right).$$

Proof. The Lazard’s elimination described in (86)–(88), and $\{r(P_u)\}_{w \in \mathcal{Y}^*_{T_n^*}}$ and $\{r(S_v)\}_{w \in \mathcal{Y}^*_{T_{n-1}^*}}$ (resp. $\{P_w\}_{w \in T_n}$ and $\{S_w\}_{w \in T_{n-1}}$), generating freely $\mathcal{U}(\mathcal{I}_n)$ and $\mathcal{U}(\mathcal{I}_n)^\vee$ (resp. $\mathcal{U}(\operatorname{Lie}_A(\mathcal{T}_n))$ and $\mathcal{U}(\operatorname{Lie}_A(\mathcal{T}_n))^\vee$), yield the expected result.

Furthermore, according to [49], as Lie algebra, $\mathcal{I}_n$ is obviously a Leibniz algebra generated by $\{\text{ad}_{-\mathcal{T}_n} t_v\}_{v \in T_n}$ and $\mathcal{I}_n^\vee$ is the Zinbiel subalgebra of $(\mathcal{A}(\mathcal{T}_n), \omega)$ generated by $\{-t_{T_n}^{k_0} t_{\mathcal{T}_n-1}\}_{k_0 \geq 0}$. These constitute the Zinbiel bialgebra $Z_{\omega}(\mathcal{T}_n)$. Indeed,

For any $k \geq 1$, let $T_n^k$ denote the characteristic series of $\hat{\mathcal{I}} \in T_n^*$, $|v| = k$, i.e. $\hat{\mathcal{I}}$

$$T_n^k = \sum_{w \in \{v \in T_n^*, |v| = k\}} \hat{w}.$$

(89)

Definition 1. One define

$$\mathcal{B} := \{\text{ad}_{-\mathcal{T}_n} t_1 \ldots \text{ad}_{-\mathcal{T}_n} t_p \} \}_{t_1, \ldots, t_p \in T_{n-1}}.$$

$$\mathcal{B}^\vee := \left\{ (\ldots -t_{-p} T_{k_p} - \ldots t_{-1} T_{k_1}) \right\}_{k_1, \ldots, k_p \geq 0, p \geq 1}.$$

Remark 6. For any $k \geq 0$, expanding $T_n^k$ and $\hat{T}_n^k$, it is immediate that

$$\mathcal{B} = \left\{ (\ldots -v_{-p} t_{-p} \ldots) \right\}_{v_{-1}, \ldots, v_{-p} \in T_{n-1}}.$$

$$\mathcal{B}^\vee = \left\{ (\ldots -t_{-p} u_{-p} \ldots) \right\}_{u_{-1}, \ldots, u_{-p} \in T_{n-1}}.$$

$$\hat{\mathcal{B}} = \left\{ (\ldots -\hat{v}_{-p} \ldots) \right\}_{\hat{v}_{-0}, \ldots, \hat{v}_{-p} \in T_{n-1}}.$$

In the sequel, computations do appear $\text{ad}_{-\mathcal{T}_n}$ (for antipodes by $a$ in [53]–[55]), instead of $\text{ad}_{\mathcal{T}_n}$, and the descriptions of $\mathcal{B}^\vee$ and of $\mathcal{U}(\mathcal{I}_n)^\vee$ will be simpler as in, respectively, Remark 6 and Proposition 3 below (see also [53]).
Definition 2. (dual bases).

(1) \(\langle v_1 t_1 | r(v_2 t_2) \rangle = \delta_{v_1,v_2} \delta_{t_1,t_2}\), for \(v_1, v_2 \in T_n^*\) and \(t_1, t_2 \in T_{n-1}\). Hence, as modules, \(\mathcal{I}_n \simeq \langle \text{span}_A \{r(v) \} \rangle_{v \in T_{n-1}} \langle I \rangle\) and, by duality, \(\mathcal{I}_n^\vee \simeq \langle \text{span}_A \{-t u \} \rangle_{u \in T_{n-1}} \langle I \rangle\) and, by duality, \(\mathcal{I}_n^\vee \simeq \langle \text{span}_A \{a(v) \} \rangle_{v \in T_{n-1}} \langle I \rangle\).

(2) \(\langle v_1 t_1 | \cdots | a(v_p t_p) \rangle = 1\), for \(v_1, \ldots, v_p \in T_n^*\) and \(t_1, \ldots, t_p \in T_{n-1}\). Hence,

\[
\mathcal{U}(\mathcal{I}_n) \simeq \langle \text{span}_A \{(-1)^{k_1, \cdots, k_p} r(v_1 t_1) \cdots r(v_p t_p) \}^{p \geq 1} \rangle_{t_1, \ldots, t_p \in T_{n-1}}.
\]

\[
\mathcal{U}(\mathcal{I}_n)^\vee \simeq \langle \text{span}_A \{a(v_1) \cdots a(v_p) \}^{p \geq 1} \rangle_{t_1, \ldots, t_p \in T_{n-1}}.
\]

Proposition 1 (dual bases).

Proof. (1) By [31], for any \(u = \tilde{v} \in T_n^*\), one has \(-tv = (-1)^{|u|} a(u t)\) and then \(\{\text{ad}_{T_n^*}^{T_n^*} t \}_{t \in T_{n-1}} = \{(-1)^{|u|} r(v) \}_{v \in T_{n-1}}\) and \(\{(-1)^{|u|} r(v) \}_{v \in T_{n-1}} = \{-T_n^* \}_{t \in T_{n-1}} = \langle a(u t) \rangle_{u \in T_{n-1}} \langle I \rangle\).

(2) Since \(\{(-1)^{|u|} r(v) \}_{v \in T_{n-1}}\) is \(A\)-linearly free and any \(r(v)\) is primitive for \(\Delta_{T_n^*}\) (by definition) then, basing on previous item and using PBW and CQMM theorems, \(\mathcal{B}\) and \(\mathcal{B}^\vee\) generate freely \(\mathcal{U}(\mathcal{I}_n)\) and \(\mathcal{U}(\mathcal{I}_n)^\vee\). It follows then the expected results (see also Remark 6).

(3) It is a consequence of the Lazard’s elimination described in [86–88].

Definition 2. (dual bases).

(1)\(\lambda_r : (A(\langle T_{n-1} \rangle, \text{conc}) \rightarrow (A(\mathbb{T}_n), \text{conc})\) be the conc-morphism of algebras defined over letters by

\[
\lambda_r(t) = r((-T_n^*)^* t) = \sum_{v \in T_n^*} (-1)^{|v|} r(v t).
\]

(2) Let \(\lambda_t\) be the isomorphism, from the Cauchy algebra \((A(\langle T_{n-1} \rangle, \text{conc})\) to the Zinbiel algebra \((A(\mathbb{T}_n), \text{conc})\) defined over letters by

\[
\lambda_t(t) = a((-T_n^*)^* t) = \sum_{v \in T_n^*} (-1)^{|v|} a(v t), \quad \hat{\lambda}_t(t) = \sum_{v \in T_n^*} (-1)^{|v|} a(v \hat{t}).
\]

(3)\(\hat{\lambda} : (A(\langle T_{n-1} \rangle, \text{conc}) \rightarrow (A(\mathbb{T}_n), \text{conc})\) be the isomorphism of algebras defined over letters by

\[
\lambda(t \otimes t) = \text{diag}(\lambda_t \otimes \lambda_r)(t \otimes t) = \sum_{v \in T_n^*} a(v t \otimes \text{conc} r(v t),
\]

\[
\hat{\lambda}(t \otimes t) = \text{diag}(\hat{\lambda}_t \otimes \lambda_r)(t \otimes t) = \sum_{v \in T_n^*} a(v \hat{t} \otimes \text{conc} r(v t).
\]

\[\text{Using } \text{conc} \text{ (resp. } \text{conc} \text{ on the left and } \text{conc} \text{ on the right of } \otimes.
\]

For convenience, they are also denoted by \(\otimes\).
Proposition 2. (1) With the notations in (66–71) and (81)–(85), one has
\[ \lambda = (g \otimes f)D_{T_n} = \sum_{w \in \mathcal{T}_n^*} g(w) \otimes f(w) = \prod_{l \in \mathcal{L}_{yn} T_n} e^{g(S_l) \otimes f(P_l)} = \prod_{l \in \mathcal{L}_{yn} T_n} e^{a(S_l) \otimes a(P_l)}. \]

(2) With the notations in Proposition 1, one also has
\[ \lambda(M^+_T_{T_n-1}) = (\lambda(M_{T_n-1})^+) \quad \text{and} \quad \hat{\lambda}(M^+_T_{T_n-1}) = (\hat{\lambda}(M_{T_n-1})^+). \]

Proof. (1) By (80) and (81), the restrictions of \( g \) and \( f \) on, respectively, \( Sh_A(T_n) \) and \( \mathcal{L}ie_A(T_n) \) are morphisms of algebras. Then \( \lambda(t \otimes t) = ((g \otimes f)D_{T_n})(t \otimes t) \), for \( t \in T_{T_n-1} \).

(2) By the previous item, one deduces the expected expressions for \( \lambda(M_{T_n-1}) \) and \( \lambda(M^+_T_{T_n-1}) \) (and similarly for \( \hat{\lambda}(M_{T_n-1}) \) and \( \hat{\lambda}(M^+_T_{T_n-1}) \)):
\[ \lambda(M_{T_n-1}) = \lambda \left( \sum_{t \in T_{T_n-1}} t \otimes t \right) = \sum_{t \in T_{T_n-1}} \lambda(t \otimes t), \]
\[ \lambda(M^+_T_{T_n-1}) = (\lambda(M_{T_n-1}))^+ = \left( \sum_{v \in T_{T_n-1}^*, t \in T_{T_n-1}} a(vt)_{\otimes} \otimes_{\mathcal{L}ie} r(vt) \right)^+. \]

□

Theorem 1 (factorized diagonal series). With the bases in (66–71), Definitions 1 and Propositions 2, the diagonal series \( D_{T_n} \) is factorized as follows
\[ D_{T_n} = \prod_{l \in \mathcal{L}_{yn} T_n} e^{S_l \otimes P_l} = D_{T_{T_n-1}} \left( \prod_{l_1 < l_2} e^{S_{l_1} \otimes P_{l_2}} \right) D_{T_n}, \]
\[ D_{T_n} = D_{T_n} \left( 1_{T_n^*} \otimes 1_{T_n^*} + \sum_{k \geq 1} \sum_{\substack{v_1, \ldots, v_k \in T_n^* \to T_n, \ t_1, \ldots, t_k \in T_{T_n-1} \to T_n \to \mathcal{L}_{yn} T_n}} a(v_1 t_1) \otimes (\ldots \otimes a(v_k t_k) \ldots) \otimes r(v_1 t_1) \ldots r(v_k t_k) \right). \]

From now on any \( S \in A(\mathcal{T}_k), 2 \leq k \leq n, \) can be expressed as image by \( S \otimes \text{Id} \) of \( D_{T_n} \) (resp. \( \log D_{T_k} \)) by (and similarly in \( A(\mathcal{T}_k) \))
\[ S = \sum_{w \in \mathcal{T}_k^*} (S | S_w) P_w \]
\[ = \left( \sum_{w \in \mathcal{T}_k^*} (S | w) w \right). \]
3.1. Iterated integrals and Chen series. In all the sequel, \( \mathcal{V} \) is the simply connected manifold on \( \mathbb{C}^n \). The pushforward (resp. pullback) of any diffeomorphism \( g \) on \( \mathcal{V} \) is denoted by \( g_* \) (resp. \( g^* \)). The ring of holomorphic functions over \( \mathcal{V} \) is denoted by \( (\mathcal{H}(\mathcal{V}), *, 1_{\mathcal{H}(\mathcal{V})}) \) and the differential ring \((\mathcal{H}(\mathcal{V}), \partial_1, \ldots, \partial_n)\) by \( \mathcal{A} \).

- \( \mathcal{C} \) denotes the sub differential ring of \( \mathcal{A} \) (i.e. \( \partial_i \mathcal{C} \subset \mathcal{C} \), for \( 1 \leq i \leq n \)).
- \( d \) denotes the total differential defined by

\[
\forall f \in \mathcal{H}(\mathcal{V}), \quad df = (\partial_1 f)dz_1 + \ldots + (\partial_n f)dz_n,
\]
where \( \partial_i \), for \( i = 1, \ldots, n \), denotes the partial derivative operator \( \partial / \partial z_i \) defined, for any \( a = (a_1, \ldots, a_n) \in \mathcal{H}(\mathcal{V}) \), as follows

\[
(\partial_i f)(a) = \frac{\partial f(a)}{\partial z_i} = \lim_{z \to a} \frac{f(z_1, \ldots, z_i, \ldots, z_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{z_i - a_i}.
\]

**Example 13.** For any \( u \in \mathcal{H}(\mathcal{V}) \), if \( f \) satisfies the differential equation \( \partial_i f = uf \) then \( f = Ce^{\log u} \in \mathcal{H}(\mathcal{V}) \), where \( C \) is a constant.

- \( \Omega(\mathcal{V}) \) denotes the space of holomorphic forms over \( \mathcal{V} \) being graded as follows

\[
\Omega(\mathcal{V}) = \bigoplus_{p \geq 0} \Omega^p(\mathcal{V}),
\]

where \( \Omega^p(\mathcal{V}) \) (specially, \( \Omega^0(\mathcal{V}) = \mathcal{H}(\mathcal{V}) \)) is the space of holomorphic \( p \)-forms over \( \mathcal{V} \). Equipped the wedge product, \( \wedge \), \( \Omega \) is a graded algebra such that

\[
\forall \omega_1 \in \Omega^p_1, \omega_2 \in \Omega^p_2, \quad \omega_1 \wedge \omega_2 = (-1)^{p_1p_2} \omega_2 \wedge \omega_1.
\]

- Over \( A(\langle \mathcal{T}_n \rangle) \) (resp. \( \Omega^p(\mathcal{V})|\langle \mathcal{T}_n \rangle \)), \( p \geq 0 \), the derivative operators \( d, \partial_1, \ldots, \partial_n \) are extended as follows (see also (99)).

\[
\forall S = \sum_{w \in \mathcal{T}_n^*} (S \mid w) w, \quad dS = \sum_{w \in \mathcal{T}_n^*} (d(S \mid w)) w = \sum_{i=1}^n (\partial_i S) dz_i.
\]

**Example 14.** Let \( t_{i,j} \in \mathcal{T}_n \) and \( U_{i,j}(z) = t_{i,j}(z_i - z_j)^{-1} \), for \( 0 \leq i < j \leq n \). Any solution of \( \partial F = U_{i,j} F \) is of the form \( F(z) = e^{t_{i,j}(\log(z_i - z_j))^{-1}} C = (z_i - z_j)^{-t_{i,j}} C \), where \( C \in \mathbb{C}[\langle \mathcal{T}_n \rangle] \) (see also Example 13).

- \( \varsigma \rightsquigarrow z \) denotes a path (with fixed endpoints, \( (\varsigma, z) \)) over \( \mathcal{V} \), i.e. the parametrized curve \( \gamma : [0, 1] \to \mathcal{V} \) such that \( \gamma(0) = \varsigma = (\varsigma_1, \ldots, \varsigma_n) \) and \( \gamma(1) = z = (z_1, \ldots, z_n) \).

For any \( i, j \in \mathbb{N}, 1 \leq i < j \leq n \), let \( \xi_{i,j} \in \mathcal{C} \) and let \( \omega_{i,j} := d\xi_{i,j} \) be holomorphic 1-form belonging to \( \Omega^1(\mathcal{V}) \). By (99), one also has

\[
d\xi_{i,j} = \sum_{k=1}^n (\partial_k \xi_{i,j}) dz_k.
\]

**Example 15.** For \( \xi_{i,j} = \log(z_i - z_j) \), for \( 1 \leq i < j \leq n \), let us denote the sub differential ring, of \( \mathbb{C}(z) \), \( \mathcal{C}[\{ (\partial_i \xi_{i,j})^{\pm 1}, \ldots, (\partial_n \xi_{i,j})^{\pm 1} \}_{1 \leq i < j \leq n}] \) by \( \mathcal{C}_0 \).

Over \( \mathcal{V} \), the holomorphic function \( \xi_{i,j} \in \mathcal{H}(\mathcal{V}) \) is called a primitive for \( \omega_{i,j} \) which is said to be an exact form and then is a closed form (i.e. \( d\omega_{i,j} = 0 \)). It follows then the iterated integral and the Chen series, of \( \{ \omega_{i,j} \}_{1 \leq i < j \leq n} \) and along \( \varsigma \rightsquigarrow z \), in Definition 4 below are a homotopy invariant.

**Definition 3.**

1. Let \( a \in \mathbb{Q} \) and \( \chi_a \) be a real morphism \( \mathcal{T}_n^* \to \mathbb{R}_{\geq 0} \). The series \( S \in A(\langle \mathcal{T}_n \rangle) \) is said satisfy the \( \chi_a \)-growth condition if and only if, choosing a compact \( K \) on \( A \),

\[
\exists c \in \mathbb{R}_{\geq 0}, k \in \mathbb{N}, \quad \forall w \in \mathcal{T}_n^{\geq k}, \quad \| (S \mid w) \|_K \leq c \chi(w) \| w \|^{-a}.
\]

\(^{21}\) If \( f \in \mathcal{H}(\mathcal{V}) \equiv \Omega^0(\mathcal{V}) \) and \( \omega \in \Omega^1(\mathcal{V}) \) then \( \omega \wedge f \in \Omega^1(\mathcal{V}) \) and \( d(\omega \wedge f) = (d\omega) \wedge f + \omega \wedge (df) \).
Proposition 3. \(1\) For \(i = 1\) or \(2\), let \(S_i \in A\langle T_n \rangle\) and \(K_i\) be a compact on \(A\) such that

\[
\sum_{w \in T_n^*} \| \langle S_1 \mid w \rangle \|_{K_1} \| \langle S_2 \mid w \rangle \|_{K_2} < +\infty.
\]

Then one defines

\[
\langle S_1 \mid S_2 \rangle := \sum_{w \in T_n^*} \langle S_1 \mid w \rangle \langle S_2 \mid w \rangle.
\]

Lemma 2. Let \(a_1, a_2 \in \mathbb{Q}\) such that \(a_1 + a_2 < 1\). Let \(\chi_{a_1}, \chi_{a_2}\) be morphisms of monoids \(T_n^* \rightarrow \mathbb{R}_{\geq 0}\). For any \(i = 1, 2\), let \(S_i \in A\langle T_n \rangle\) satisfying the \(\chi_{a_i}\)-growth condition. If \(\sum_{t \in T_n} \chi_{a_1}(t) \chi_{a_2}(t) < 1\) then \(\langle S_1 \mid S_2 \rangle\) is well defined.

Proof. It is due to the fact that

\[
\| \sum_{w \in T_n^*} \langle S_1 \mid w \rangle \langle S_2 \mid w \rangle \| \leq \sum_{w \in T_n^*} \| \langle S_1 \mid w \rangle \|_{K_1} \| \langle S_2 \mid w \rangle \|_{K_2} \leq c_1 c_2 \sum_{w \in T_n^*} \frac{1}{|w|^{a_1 + a_2}} \chi_{a_1}(w) \chi_{a_2}(w) \leq c_1 c_2 \sum_{w \in T_n^*} \chi_{a_1}(w) \chi_{a_2}(w) = c_1 c_2 \left( \sum_{t \in T_n} \chi_{a_1}(t) \chi_{a_2}(t) \right)^*.
\]

By assumption, it follows then the expected result. \(\Box\)

Remark 7. With Notations in Lemma 2, one has

\[
\left( \sum_{t \in T_n} \chi_{a_1}(t) \chi_{a_2}(t) \right)^* = \sum_{w \in T_n^*} \chi_{a_1}(w) \chi_{a_2}(w) = \sum_{k \geq 0} \sum_{w \in T_n^*} \chi_{a_1}(w) \chi_{a_2}(w) < +\infty,
\]

meaning also that \(S_1 \in \text{Dom}(S_2)\) and \(S_2 \in \text{Dom}(S_1)\), where \(\text{Dom}(S_i) := \{ R \in A\langle T_n \rangle \mid \sum_{k \geq 0} \langle S_i \mid [R]_k \rangle \text{ converges in } K_i \}, \ [R]_k = \sum_{w \in T_n^*} \langle R \mid w \rangle w\).

Note also that \(\text{Dom}(S_i)\) can be void.

Definition 4. \(1\) The iterated integral, along the path \(\varsigma \rightarrow z\) over \(\mathcal{V}\) and of the holomorphic 1-forms \(\{ \omega_{i,j} \}_{1 \leq i < j \leq n}\), is given by \(\alpha^{\varsigma}_1(1_{T_n^*}) = 1_{\mathcal{H}(\mathcal{V})}\) and, for any \(w = t_{i_1, j_1}, t_{i_2, j_2}, \ldots, t_{i_k, j_k} \in T_n^*\), by

\[
\alpha^{\varsigma}_1(w) = \int_{\varsigma}^{z} \omega_{i_1, j_1}(s_1) \int_{\varsigma}^{s_1} \omega_{i_2, j_2}(s_2) \ldots \int_{\varsigma}^{s_{k-1}} \omega_{i_k, j_k}(s_k) \in \mathcal{H}(\mathcal{V}),
\]

where \((\varsigma, s_1, \ldots, s_k, z)\) is a subdivision of the path \(\varsigma \rightarrow z\) over \(\mathcal{V}\).

\(2\) The Chen series, along the path \(\varsigma \rightsquigarrow z\) over \(\mathcal{V}\) and of the holomorphic 1-forms \(\{ \omega_{i,j} \}_{1 \leq i < j \leq n}\), is the following noncommutative generating series

\[
C_{\varsigma \rightsquigarrow z} := \sum_{w \in T_n^*} \alpha^{\varsigma}_1(w) w \in A\langle T_n \rangle.
\]

Proposition 3. \(1\) The Chen series, along the path \(\varsigma \rightsquigarrow z\) over \(\mathcal{V}\) and of the holomorphic 1-forms \(\{ \omega_{i,j} \}_{1 \leq i < j \leq n}\), satisfies the \(\chi_{a}\)-growth condition.
(2) Let \( (\beta, \mu, \eta) \) be linear representation of \( S \in A^\text{rat} \langle \langle T_n \rangle \rangle \). Then \( (C_{\zeta \to z} \mid S) \) is well defined and then
\[
(C_{\zeta \to z} \mid S) = \alpha^z(S) = \sum_{w \in T_n^*} (\beta \mu(w) \eta) \alpha^z(w).
\]

(3) Let \( S_i \in A^\text{rat} \langle \langle T_n \rangle \rangle \), for \( i = 1, 2 \). Then \( \alpha^z(S_1 \cup S_2) = \alpha^z(S_1) \alpha^z(S_2) \).

**Proof.**
1) By induction on the length of \( w \in T_n^* \) and by use the length of the path \( \zeta \to z \), denoted by \( \ell \). one proves that \( C_{\zeta \to z} \) satisfies the \( \chi_1 \)-growth condition, with \( \chi_1(y) = \ell \), for \( t \in T_n \).
2) Since \( (S \mid w) = \beta \mu(w) \eta \), for \( w \in T_n^* \), then \( S \) satisfies the \( \chi_2 \)-growth condition, with \( \chi_2(t) = ||\mu(t)|| \), for \( t \in T_n \) (using of norm on matrices with coefficients in \( A \)). By Lemma 2, it follows then the expected result.
3) The recursion (35) yields \( \alpha^z(u \cup v) = \alpha^z(u) \alpha^z(v) \), for \( u, v \in T_n^* \) (a Chen’s lemma, (11)) and then the expected result, by extending to \( A^\text{rat} \langle \langle T_n \rangle \rangle \).

**Definition 5.** Let \( K \) denote the algebra generated by \( \{\alpha^z(R)\}_{R \in C^\text{rat} \langle \langle T_n \rangle \rangle} \) and then \( C \subset A \subset K \).

**Remark 8.**
1) Definition 3, Lemma 2 are extensions of the ones in [19] [34].
2) Using (31), for any \( S \in \text{Lie}_K \langle \langle T_n \rangle \rangle \), let \( \varphi_S = e^{\text{ad}_S} \). One has
\[
\forall R \in \text{Lie}_A \langle \langle T_{n-1} \rangle \rangle, \quad \varphi_S(R) = e^{\text{ad}_S} R = \sum_{k \geq 0} \frac{1}{k!} \text{ad}_S^k R \in \text{Lie}_K \langle \langle T_n \rangle \rangle.
\]
In particular, for \( S \in \text{Lie}_K \langle T_n \rangle, R \in \text{Lie}_K \langle T_{n-1} \rangle \) and then \( S \in T_n, R \in T_{n-1} \). Using (70), if \( \varphi_{P \ell} = e^{\text{ad}_{P \ell}} \) with \( \ell \in \text{Lyn} T_n \) then, for \( q = P \ell \) with \( \ell \in T_{n-1} \), and using (29) (32), one obtains \( \ell \in \text{Lyn} T_n \) and then (see (76))
\[
\varphi_{P \ell}(P_k) = e^{\text{ad}_{P \ell}} P_k = \sum_{k \geq 0} \frac{1}{k!} \text{ad}_{P \ell}^k P_k = \sum_{k \geq 0} \frac{1}{k!} P_k \ell^k.
\]
In particular, if \( P_1 = P = \ell \in T_{n-1} \) then (see (76)–(77))
\[
\varphi_{P \ell}(\ell) = e^{\text{ad}_{P \ell}} \ell = \sum_{k \geq 0} \frac{1}{k!} \text{ad}_{P \ell}^k \ell = \sum_{k \geq 0} \frac{r(k \ell)}{k!} \ell \quad \text{and by duality } \quad \varphi_{P \ell}(\ell) = \sum_{k \geq 0} \frac{1}{k!} \ell^k = e^{\ell}.
\]

**Corollary 1.** With the morphism \( \alpha^z : (C^\text{rat} \langle \langle T_n \rangle \rangle, \omega, 1_{T_n^*}) \longrightarrow (K, \times, 1_K) \), one has
1) For any \( t_{i,j} \in T_n \) and \( k \geq 1 \),
\[
\alpha^z(t_{i,j}^k) = (\alpha^z(t_{i,j}))^k / k!, \quad \text{and then } \quad \alpha^z(t_{i,j}) = \exp(\alpha^z(t_{i,j})).
\]
2) For any \( t_{i,j} \in T_{n-1} \) and \( R \in C^\text{rat} \langle \langle T_n \rangle \rangle \) and \( H \in C^\text{rat} \langle \langle T_n \rangle \rangle \),
\[
\alpha^z(H \omega_{i,j} R) = \left\{ \begin{array}{ll}
\alpha^z(t_{i,j} H) & \text{if } R = 1_{T_n^*}, \\
\int_{S} \omega_{i,j}(s) \alpha^z(H) \alpha^z(R) & \text{if } R \neq 1_{T_n^*}.
\end{array} \right.
\]
3) For any \( x_1, \ldots, x_k \in T_n \) and \( a_1, \ldots, a_k \in C \) (\( k \geq 1 \)),
\[
\alpha^z\left( \left( \prod_{p=1}^{k} (x_{p}^{i_p} t_{i,j} w) \right) \right) = \prod_{p=1}^{k} \sum_{i_p, j_p, \ell_p} \alpha^z(x_{p}^{i_p}) \alpha^z(t_{i,j} x_{p}^{i_p} \omega_{i,j} R) \prod_{p=1}^{k} \alpha^z(x_{p}^{i_p} t_{i,j} w).
\]

\[
\alpha^*_\zeta \left( \left( \sum_{p=1}^{k} a_p x_p \right)^* t_{i,j} w \right) = \prod_{p=1}^{k} \alpha^*_\zeta((a_p x_p)^*) \alpha^*_\zeta((-a_p x_p)^*) \frac{w}{2}.
\]

**Proof.** By Proposition 3 and

- Since \( t_{i,j}^k = t_{i,j}^{-k} / k! \) then it follows the expected results.
- By (3), it follows the expected result.
- By the following facts (proved by induction), it follows the expected results

\[
\alpha^*_\zeta \left( \left( \sum_{p=1}^{k} a_p x_p \right)^* t_{i,j} w \right) = \int \omega_{i,j}(s) \left( \prod_{p=1}^{k} \frac{[\alpha^*_\zeta(x_p) - \alpha^*_\zeta(x_p)]^p}{l_p!} \right) \alpha^*_\zeta(w)
\]

\[
= \prod_{p=1}^{k} \omega_{i,j}(s) \sum_{p=1}^{l_p} \alpha^*_\zeta(x_p^p) \int \omega_{i,j}(s) \alpha^*_\zeta(-x_p^p) \alpha^*_\zeta(w),
\]

\[
= \int \omega_{i,j}(s) \left( \prod_{p=1}^{k} e^{e_p(x_p) - \alpha^*_\zeta(x_p)} \right) \alpha^*_\zeta(w)
\]

\[
= \prod_{p=1}^{k} e^{e_p(x_p) - \alpha^*_\zeta(x_p)} \alpha^*_\zeta(w).
\]

\[
\square
\]

**Remark 9** ([22] [40]). Developing the idea of universality, for simplification, let \( C_{\zeta \rightarrow z} \) be the Chen series, along \( \zeta \rightarrow z \) and of \( \omega_0(z) = dz/\zeta \) and \( \omega_1(z) = dz/(1 - z) \).

Let \( a, b, c \) be real parameters and let \( S \in \mathbb{C}^{\text{rat} \| x_0, x_1 \|} \) be the rational series admitting the triplet \( (\beta, \mu, \eta) \) as parametrized linear representation [25]:

\[
\beta = \begin{pmatrix} \eta \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu(x_0) = - \begin{pmatrix} 0 \\ ab \end{pmatrix}, \quad \mu(x_1) = - \begin{pmatrix} 0 \\ c \end{pmatrix}, \quad \mu(x_2) = - \begin{pmatrix} 1 \\ c - a - b \end{pmatrix}.
\]

One can consider the following hypergeometric equation

\[
z(1 - z) \dot{y}(z) + [c - (a + b + 1)] z \ddot{y}(z) - ab \dot{y}(z) = 0,
\]

in which putting \( q_1(z) = -y(z) \) and \( q_2(z) = (1 - z) \ddot{y}(z) \), the state vector \( q \) satisfies the following linear differential equation associated to \( (\beta, \mu, \eta) \) [22] [23]

\[
\dot{q}(z) = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \mu(x_0) \\ \mu(x_1) \end{pmatrix} \left( \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right), \quad \left( \begin{pmatrix} q_0 \\ q_0 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Or equivalently, considering two following parametrized linear vector fields [22] [23]

\[
A_0 = -(abq_1 + cq_2) \frac{\partial}{\partial q_2} \quad \text{and} \quad A_1 = -q_2 \frac{\partial}{\partial q_1} - (c - a - b)q_2 \frac{\partial}{\partial q_2},
\]

\( q \) satisfies then the following differential equation [22] [23]

\[
\dot{q}(z) = \frac{1}{z} A_0(q) + \frac{1}{1 - z} A_1(q) \quad \text{and} \quad y(z) = -q_1(z).
\]

By Proposition 3, one has \( \langle C_{0 \rightarrow z} \| S \rangle = \alpha^*_0(S) = q_1(z) = -y(z) \).
3.2. Noncommutative differential equations. Getting back to (14), let us consider the Chen series $C_{g, z}$, of the holomorphic 1-forms $\{\omega_{ij}\}_{1 \leq i, j \leq n}$ and along the path $z \mapsto z$ over the simply connected manifold $V$. Let $g$ be a diffeomorphism on $V$ and $C_{g, z}$ be the Chen series, of $\{g^*\omega_{ij}\}_{1 \leq i, j \leq n}$ and along $z \mapsto z$, or equivalently, of $\{\omega_{ij}\}_{1 \leq i, j \leq n}$ and along $g, z \mapsto z$ [11]:

$$C_{g, z} = \sum_{m \geq 0} \sum_{t_{i_1, j_1}, \ldots, t_{i_m, j_m} \in T_n} \int_{\xi} g^*\omega_{i_1, j_1}(s_1) \cdots \int_{\xi} g^*\omega_{i_m, j_m}(s_m)$$

(105)

$$= \sum_{w \in T_n} \alpha_{g(z)}(w).$$

$C_{g, z}$ is obtained by the Picard’s iteration, as in [13], and convergent for (1)

$$F^*_0(\varsigma, z) = 0,$$

(106) $F^*_1(\varsigma, z) = F^*_0(\varsigma, z) + \int_{\xi} M^*_n(s)F^*_1(s), i \geq 1,$

where

$$M^*_n := g^*M_n \quad \text{associated to} \quad dS = M^*_n S.$$

Definition 6. To the partitions in [28] and with Definition 2, one defines

$$G := \{\text{C-linearly free}\} \in \mathbf{Lie}(\mathfrak{T}_n).$$

For any $\phi \in G$, let $\hat{\phi}$ be its adjoint to $\phi$ and let us consider the Picard’s iterations with initial condition $F^*_0$, according to following recursion similar to [13] (for $i \geq 1$):

$$F^*_i(\varsigma, z) = F^*_0(\varsigma, z) + \int_{\xi} M^*_n(s)F^*_i(s), i \geq 1,$$

(108) $M^*_n := \phi(M_n-1) \quad \text{associated to} \quad dF^* = M^*_n-1 F.$$

Remark 10. In (108), $\{F^*_i\}_{i \geq 1}$ is image by $\phi$ of $\{F_i\}_{i \geq 0}$ (given in [13]), being viewed as a generalization on noncommutative variable of the Fredholm like transformation, so-called functional rotation of sequence (of orthogonal functions) with the kernel of rotation $K(s, t)$ [13], and $M^*_n$ is also a generalization of such kernel:

$$\phi(s) = f(s) + \int_a^b K(s, t)f(t)dt.$$

Proposition 4. Let $S \in \mathcal{A}(\mathfrak{T}_n)$ be a grouplike solution of (1). Then

(1) If $H \in \mathcal{A}(\mathfrak{T}_n)$ is another grouplike solution for (1) then there exists $C \in \mathcal{L}ie(\mathfrak{T}_n)$ such that $S = He^C$ (and conversely).

(2) The following assertions are equivalent

(a) The family $\{\langle S | w \rangle \}_{w \in T_n}$ is $C$-linearly free.

(b) The family $\{\langle S | t \rangle \}_{t \in \mathfrak{T}_n}$ is $C$-algebraically free.

(c) The family $\{\langle S | t \rangle \}_{t \in \mathfrak{T}_n}$ is $C$-algebraically free.

(d) The family $\{\langle S | t \rangle \}_{t \in \mathfrak{T}_n \cup \{1\}}$ is $C$-linearly free.

(e) The family $\{\omega_{ij}\}_{1 \leq i, j \leq n}$ is such that, for any $(c_{i,j})_{1 \leq i, j \leq n} \in \mathbb{C}(\mathfrak{T}_n)$ and $f \in \text{Frac}(\mathbb{C})$, one has

$$\sum_{1 \leq i, j \leq n} c_{i,j} \omega_{i,j} = df \implies (\forall 1 \leq i < j \leq n) (c_{i,j} = 0).$$
(f) \{\omega_{i,j}\}_{1 \leq i < j \leq n} is C-free and \text{dFrac}(C) \cap \text{span}_C \{\omega_{i,j}\}_{1 \leq i < j \leq n} = \{0\}.

Sketch. (1) The proof is similarly treated in [40]: since

\[ d(\text{SS}^{-1}) = d(\text{Id}) = 0 \]  

then, applying the Liebniz rule, \( (\text{dS})S^{-1} + S(\text{dS}^{-1}) = 0 \) and then (see also (58) and (92)) \( dS^{-1} = -S^{-1}(\text{dS})S^{-1} = -S^{-1}(M_nS)^{-1} = -S^{-1}M_n(\text{SS}^{-1}) = -S^{-1}M_n \). One also has

\[ d(S^{-1}H) = S^{-1}(\text{dH}) + (\text{dS}^{-1})H = S^{-1}(M_nH) - (S^{-1}M_n)H = 0. \]

Thus, \( S^{-1}H \) is a constant series. Since the inverse and the product of grouplike elements are grouplike then it follows the expected result.

(2) This is a group-like version of the abstract form of Theorem 1 of [14]. It goes as follows

• due to the fact that \( A \) is without zero divisors, using the fields of fractions of \( C \) and \( A \), we have the embeddings \( C \subset \text{Frac}(C) \subset \text{Frac}(A) \). Frac\( (A) \) is a differential field, and its differential operator can still be denoted by \( d \) as it induces the previous one on \( A \). The same holds for \( A\langle \langle T_\alpha \rangle \rangle \subset \text{Frac}(A)\langle \langle T_\alpha \rangle \rangle \) and \( d \). Hence, equation (14) can be transported in \( \text{Frac}(A)\langle \langle T_\alpha \rangle \rangle \) and \( M_n \) satisfies the same condition as previously.

• Equivalence between 2a 2d comes from the fact that \( C \) is without zero divisors and then, by denominator chasing, linear independences with respect to \( C \) and \( \text{Frac}(C) \) are equivalent. In particular, supposing condition 2a the family \( \{(S \mid x)\}_{x \in T_\alpha \cup \{T_\alpha^2\}} \) (basic triangle) is \( \text{Frac}(C) \)-linearly independent which imply, by Theorem 1 of [14], condition 2a.

• Still by Theorem 1 of [14], 2e 2f are equivalent and then \( \{(S \mid w)\}_{w \in T_\alpha^2} \) is \( \text{Frac}(C) \)-linearly free which induces \( C \)-linear dependence (i.e. 2a).

In the sequel, with the notations in Definition 5 let

• \( F(S) := \text{span}_C \{(S \mid w)\}_{w \in T_\alpha^2}, \) for \( S \in A\langle \langle T_\alpha \rangle \rangle, \)

• \( g \) be the diffeomorphism on \( \mathcal{V} \) acting by pullback on \( \{\omega_{i,j}\}_{1 \leq i < j \leq n} \) as follows

\[ g^* \omega_{i,j} = \sum_{1 \leq k < l \leq n} \omega_{k,l} h_{i,j}^{k,l}, \text{ for } h_{i,j}^{k,l} \in K, \]

• \( \psi \) be the morphism of algebras \( (C\langle \langle T_\alpha \rangle \rangle, \text{conc}) \rightarrow (C^\text{rat}\langle \langle T_\alpha \rangle \rangle, \text{rat}) \) defined, for any \( t_{i,j} \in T_\alpha, \) as follows \( 22 \) (see also [50] for the half-shuffle)

\[ \psi(t_{i,j}) = \sum_{1 \leq k < l \leq n} t_{k,l} H_{i,j}^{k,l}, \text{ for } H_{i,j}^{k,l} \in C^\text{rat}\langle \langle T_\alpha \rangle \rangle. \]

Example 16. With \( T_\alpha = \{t_{1,2}, t_{1,3}, t_{2,3}\}, \) let \( \omega_{1,2}(z) = -d \log(z_1 - z_2) \) and \( \omega_{1,3}(z) = -d \log(z_1 - z_3) \) and \( \omega_{2,3}(z) = -d \log(z_2 - z_3). \) Then let

(1) \( g \) be the diffeomorphism on \( \mathcal{C}^*_3 \) as follows

\[ g^* \begin{pmatrix} \omega_{1,2} \\ \omega_{1,3} \\ \omega_{2,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (z_1 - z_2)^{-1} \log((z_2 - z_3)^{-1}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_{1,2} \\ \omega_{1,3} \\ \omega_{2,3} \end{pmatrix}, \]

\[ 22 \psi(t_{1,2}) \ldots t_{v,r} \psi(t_{1,3}) \psi(t_{2,3}) = \psi(t_{1,2}) \omega(\psi(t_{2,3})) \frac{1}{2} (\omega(\psi(t_{v,r}))). \]
(2) \( \psi : (C(T_\omega), \text{conc}) \to (C^{\text{rat}}(\langle T_\omega \rangle), \omega) \) be the morphism of algebras defined by
\[
\psi(t_{1,2}) = t_{1,2}t_{1,2}^* \text{ and } \psi(t_{1,3}) = t_{1,3}t_{1,2}^* \text{ and } \psi(t_{2,3}) = t_{2,3}t_{2,3}^*.
\]

(3) With the data in previous items, by Example 6 and Proposition 1 one has
\[
\alpha_z^+ (\psi(t_{1,3}) \omega t_{2,3}) = \alpha_z^+ (\psi(t_{1,3}t_{1,2}^* \omega t_{2,3})) = \int_{z_1} z^* \omega_{1,3} = \alpha_{g(z)}^+(t_{1,3}).
\]

Proposition 2 holds, in particular, for \( C_{z \leadsto z} \). Hence, one deduces that

Corollary 2. (1) The following assertions are equivalent:23
\begin{enumerate}[(a)]
    
    \item The restricted \( \omega \)-morphism \( \alpha_z^+ \), on \( C(T_n) \), is injective.
    
    \item The family \( \{\alpha_z^+(w)\}_{w \in T_n} \) is \( C \)-linearly free.
    
    \item The family \( \{\alpha_z^+(l)\}_{l \in \mathcal{E} \cap T_n} \) is \( C \)-algebraically free.
    
    \item The family \( \{\alpha_z^+(t)\}_{t \in T_n} \) is \( C \)-algebraically free.
    
    \item The family \( \{\alpha_z^+(l)\}_{l \in \mathcal{E} \cap T_n} \) is \( C \)-linearly free.
    
    \item \( \forall E \in e^{\text{Cerf}}(\langle T_n \rangle), \exists \phi \in \text{Aut}(F(C_{z \leadsto z})), \phi(C_{z \leadsto z}) = C_{z \leadsto z} E. \)
\end{enumerate}

(2) The following assertions are equivalent (see Notations in 105, 110–111)
\begin{enumerate}[(a)]
    
    \item For any \( 1 \leq i < j \leq n \) and \( 1 \leq k < l \leq n \), one has \( h_{i,j}^k \omega_{k,l} = \alpha_z^+(H_{i,j}^k) \).
    
    \item The restricted \( \omega \)-morphism \( \alpha_z^+ \), on \( C(T_n) \), is injective.
    
    \item The Chen series, of \( \{\omega_{i,j}\}_{1 \leq i < j \leq n} \) and along \( g_z \leadsto z \), satisfies
\[
C_{g_z \leadsto z} = \sum_{w \in T_n^*} \alpha_z^+(\psi(w))w = C_{z \leadsto z} E, \text{ where } E \in e^{\text{Cerf}}(\langle T_n \rangle).
\]
\end{enumerate}

(3) For any \( \phi \in G \), there exists a diffeomorphism \( g \) on \( \mathcal{V} \) such that the Chen series, of \( \{\omega_{i,j}\}_{1 \leq i < j \leq n} \) and along \( g \leadsto z \), can be expressed as follows
\[
C_{g \leadsto z} := \sum_{w \in T_{n-1}^*} \alpha_{g(z)}^+(w) = \sum_{w \in T_{n-1}^*} \alpha_{g(z)}^+(w) \phi_{(\omega_{i,j})}(w).
\]

Proof. The first item is a consequence of Proposition 4. Applying Propositions 3 and Corollary 1 one gets the second item. By duality, one gets
\[
\sum_{w \in T_{n-1}^*} \alpha_z^+(w) \phi_{(\omega_{i,j})}(w) = \sum_{w \in T_{n-1}^*} \alpha_z^+(w) \phi_{(\omega_{i,j})}(w).
\]

Applying the second item with \( \psi = \tilde{\phi} \), it follows the last item. \( \square \)

In Proposition 4, the Hausdorff group of \( H_{\omega}(T_n) \) plays the rôle of the differential Galois group of \( C_{z \leadsto z} \) + grouplike solutions, i.e. \( \text{Gal}(M_n) = e^{\text{Cerf}}(\langle T_n \rangle) \), mapping grouplike solution to another grouplike solution and then leading to the definitions, on the one hand, of the system fundamental of \( C_{z \leadsto z} \) as \( \{C_{z \leadsto z}\} \) and, on the other hand, of the PV extension related to \( C_{z \leadsto z} \) as \( C \overset{\mathcal{T}_n}{\to} \{C_{z \leadsto z}\} \). [23]

23In particular, \( C = C_0 \) (see Example 12) yielding \( F_{KZ_n} \) in Definition 8 Corollaries 4 below.
3.3. Explicit solutions of noncommutative differential equations. In the sequel, \(\{V_k\}_{k \geq 0}\) and \(\{\tilde{V}_k\}_{k \geq 0}\) denote the sequences of series in \(\mathcal{A}(\langle T_n\rangle)\), satisfying the recursion in (25) with the following starting conditions being grouplike series:

\[
V_0(\varsigma, z) := (\alpha_\varsigma \otimes \text{Id})D_{T_n} = \prod_{l \in \mathbb{Z}^{\mathbb{N}} T_n} e^{\alpha_l(S_l)P_l} \quad \text{(decreasing lexicographical ordered product)}.
\]

\[
\tilde{V}_0(\varsigma, z) := \sum_{t \in \mathcal{E}_n} \alpha_t(\varsigma, t) = V_0(\varsigma, z) \mod [\mathcal{L}_{\mathcal{A}}(\langle T_n\rangle), \mathcal{L}_{\mathcal{A}}(\langle T_n\rangle)].
\]

**Remark 11.**

- \(V_0\) is the Chen series, of \(\{\omega_{k,n}\}_{1 \leq k \leq n-1}\) and along \(\varsigma \rightsquigarrow z\), and satisfies the \(\chi_{\alpha}\)-growth condition (see by Proposition (3)). It can be obtained by using the following Picard’s iteration, analogous to (14), which is convergent for the discrete topology in (1) but does not mean that \(V_0\) satisfies \(dS = M_n S\) (see Remark 12 below)

\[
F_0(\varsigma, z) = 1_{\mathcal{U}(V)}, \quad F_i(\varsigma, z) = F_{i-1}(\varsigma, z) + \int_\varsigma^z \hat{M}_n(s)F_{i-1}(s), \quad i \geq 1.
\]

- With data in (136) below, \(V_0\) will behave, for \(z_n \to z_{n-1}\), as the noncommutative generating series of hyperlogarithms (see (131)–(132) below) and, of course, as the noncommutative generating series of polylogarithms for \(n = 3\) (see (125) below).

- \(\tilde{V}_0\) satisfies the partial differential equation \(\partial_nf = \hat{M}_n f\) and (113) is equivalent to a nilpotent structural approximation of order 1 of \(V_0\) [35], i.e. \(\log \tilde{V}_0 = \log V_0 \mod [\mathcal{L}_{\mathcal{A}}(\langle T_n\rangle), \mathcal{L}_{\mathcal{A}}(\langle T_n\rangle)]\) (see also Remark 17 below).

**Definition 7.**

1. Let \(\varphi_{T_n} \in G\) be the conc-morphisms of \(\mathcal{A}(\langle T_n\rangle)\), depending \(\varsigma \rightsquigarrow z\) subdived by \((\varsigma, s_1, \ldots, s_k, z)\), such that, over \(T_{n-1}\),

\[
\varphi_{T_n} \equiv \varphi_n \equiv \text{Id}
\]

and, over \(T_{n-1}\), by \(24\)

\[
\varphi_{T_n}^{(\varsigma,z)} = \prod_{l \in \mathbb{Z}^{\mathbb{N}} T_n} e^{\text{ad}_{-\alpha_\varsigma(S_l)P_l}} \quad \text{and} \quad \varphi_{T_n}^{(\varsigma)} = e^{\sum_{t \in \mathcal{E}_n} \text{ad}_{-\alpha_\varsigma(S_t)P_t}}
\]

and they are chronologically defined as follows (for \(t_{i_1, j_1} \ldots t_{i_k, j_k} \in T_{n-1}\))

\[
\varphi_{T_n}^{(\varsigma)}(t_{i_1, j_1} \ldots t_{i_k, j_k}) = \varphi_{T_n}^{(\varsigma, s_1)}(t_{i_1, j_1}) \ldots \varphi_{T_n}^{(\varsigma, s_k)}(t_{i_k, j_k}),
\]

\[
\varphi_{T_n}^{(\varsigma,z)}(t_{i_1, j_1} \ldots t_{i_k, j_k}) = \varphi_{T_n}^{(\varsigma, s_1)}(t_{i_1, j_1}) \ldots \varphi_{T_n}^{(\varsigma, s_k)}(t_{i_k, j_k}).
\]

2. Let \(\varphi_n\) and \(\hat{\varphi}_n\) be the morphisms of \(\mathcal{A}(\langle T_n\rangle)\) defined, for any \(t \in T_n\), by

\[
\varphi_n(t) = \varphi_{T_n}(t) \mod \mathcal{H}_n \quad \text{and} \quad \hat{\varphi}(t) = \hat{\varphi}_{T_n}(t) \mod \mathcal{H}_n,
\]

where \(\mathcal{H}_n\) is the ideal of relators on \([t_{i,j}]_{1 \leq i < j \leq n}\) induced by (12).

**Proposition 5.** With the notations in Definitions (3) and (112)–(113), one has

\[
\varphi_{T_n}^{(\varsigma,z)}(t_{i_k, j_k}) = e^{\text{ad}_{-\varphi_0(\varsigma, s_k)}t_{i_k, j_k}} \quad \text{and} \quad \hat{\varphi}_{T_n}^{(\varsigma,z)}(t_{i_k, j_k}) = e^{\text{ad}_{-\varphi_0(\varsigma, s_k)}t_{i_k, j_k}}.
\]

24See Note 3
25For any \(a, b \in \mathcal{L}_{\mathcal{A}}(\langle T_n\rangle)\), one has \(e^{-a}be^{a} = e^{\text{ad}_{-a}b}\).
Hence, there exists \( H \) and \( \dot{H} \in \mathcal{A}(\mathcal{T}_n) \) satisfying the differential equation in (109) such that

\[
\sum_{k \geq 0} V_k = V_0H \quad \text{and} \quad \sum_{k \geq 0} \dot{V}_k = \dot{V}_0\dot{H}
\]

and \( \kappa_w = V_0\varphi_{T_n}(w) \) and \( \dot{\kappa}_w = \dot{V}_0\dot{\varphi}_{T_n}(w) \), for \( w \in \mathcal{T}_{n-1}^* \), such that

\[
V_k(\varsigma, z) = \sum_{w = t_{i_1,j_1} \cdots t_{i_k,j_k} \in \mathcal{T}_{n-1}^*} \int_{\varsigma}^z \omega_{i_1,j_1}(s_1) \cdots \int_{\varsigma}^{s_{k-1}} \omega_{i_k,j_k}(s_k) \kappa_w(z, s),
\]

\[
\dot{V}_k(\varsigma, z) = \sum_{w = t_{i_1,j_1} \cdots t_{i_k,j_k} \in \mathcal{T}_{n-1}^*} \int_{\varsigma}^z \omega_{i_1,j_1}(s_1) \cdots \int_{\varsigma}^{s_{k-1}} \omega_{i_k,j_k}(s_k) \dot{\kappa}_w(z, s).
\]

Reducing by \( \mathcal{J}_n \), one gets analogous results using respectively \( \varphi_n \) and \( \dot{\varphi}_n \) (and then, in this case, one has \( \kappa_w = V_0\varphi_n(w) \) and \( \dot{\kappa}_w = \dot{V}_0\dot{\varphi}_n(w) \), for \( w \in \mathcal{T}_{n-1}^* \)).

**Proof.** The first result is a consequence of (108) and (112)–(114). According to (25), iterative computations by (108) yield the expected expressions with

\[
H(\varsigma, z) = 1_{\mathcal{T}_n} + \sum_{k \geq 1} t_{i_1,j_1} \cdots t_{i_k,j_k} \in \mathcal{T}_{n-1}^* \int_{\varsigma}^z \omega_{i_1,j_1}(s_1) \varphi_{T_n}(s_1, t_{i_1,j_1}) \cdots \int_{\varsigma}^{s_{k-1}} \omega_{i_k,j_k}(s_k) \varphi_{T_n}(s_k, t_{i_k,j_k})
\]

\[
\dot{H}(\varsigma, z) = 1_{\mathcal{T}_n} + \sum_{k \geq 1} t_{i_1,j_1} \cdots t_{i_k,j_k} \in \mathcal{T}_{n-1}^* \int_{\varsigma}^z \omega_{i_1,j_1}(s_1) \dot{\varphi}_{T_n}(s_1, t_{i_1,j_1}) \cdots \int_{\varsigma}^{s_{k-1}} \omega_{i_k,j_k}(s_k) \dot{\varphi}_{T_n}(s_k, t_{i_k,j_k})
\]

Theorem 2 (Volterra expansion like for Chen series). With the notations in Definitions 2 and Theorems 1 and Propositions 2–3, one has

\[
H(\varsigma, z) = (\alpha_i^z \otimes \text{Id})\lambda(\mathcal{M}_{T_{n-1}}) = (\alpha_i^z \otimes \text{Id})\text{diag}((\lambda_t \otimes \lambda_r)(\mathcal{M}_{T_{n-1}})),
\]

\[
\dot{H}(\varsigma, z) = (\alpha_i^z \otimes \text{Id})\dot{\lambda}(\mathcal{M}_{T_{n-1}}) = (\alpha_i^z \otimes \text{Id})\text{diag}((\dot{\lambda}_t \otimes \lambda_r)(\mathcal{M}_{T_{n-1}})),
\]

and \( C_{\varsigma \to z} = V_0(\varsigma, z)H(\varsigma, z) \).

Reducing by \( \mathcal{J}_n \), one gets analogous results using respectively \( \varphi_n \) and \( \dot{\varphi}_n \).

**Proof.** By Proposition 2, the images by \( \alpha_i^z \otimes \text{Id} \) of \( \lambda(t \otimes t) \) and \( \dot{\lambda}(t \otimes t) \), for \( t \in \mathcal{T}_{n-1} \), are respectively followed (see also Notations in (54), (57) and (83))

\[
\int_{\varsigma}^z \omega_{i,j}(s)\varphi_{T_n}(s, t) = (\alpha_i^z \otimes \text{Id})\lambda(t \otimes t) = \sum_{v \in \mathcal{T}_n} \alpha_v(z)(a(t))r(vt),
\]
\[
\int_{\varsigma} \omega_{i,j}(\varsigma; z) \mathcal{P}_{T_n}(t) = (\alpha_{\varsigma} \otimes \text{Id}) \hat{\lambda}(t \otimes t) = \sum_{v \in T_n} \alpha_{\varsigma}^v(a(\hat{v}t)) r(\hat{v}t).
\]

Hence, for any \( t_{i_1,j_1} \cdots t_{i_k,j_k} \in T_{n-1} \), one iteratively obtains
\[
\int_{\varsigma} \omega_{i_1,j_1}(\varsigma; z) \cdots \int_{\varsigma} \omega_{i_k,j_k}(\varsigma; z) \mathcal{P}_{T_n}(t_{i_1,j_1} \cdots t_{i_k,j_k}) = \sum_{v_1, \ldots, v_k \in T_n} \alpha_{\varsigma}^v(a(v_1t_1) \cdots a(v_kt_k)) r(v_1t_1) \cdots r(v_kt_k),
\]
\[
\int_{\varsigma} \omega_{i_1,j_1}(\varsigma; z) \cdots \int_{\varsigma} \omega_{i_k,j_k}(\varsigma; z) \mathcal{P}_{T_n}(t_{i_1,j_1} \cdots t_{i_k,j_k}) = \sum_{v_1, \ldots, v_k \in T_n} \alpha_{\varsigma}^v(a(\hat{v}t_1) \cdots a(\hat{v}t_k)) r(v_1t_1) \cdots r(v_kt_k).
\]

By Propositions 5 and 5 summing for \( k \) on \( \mathbb{N} \), it follows the expected expressions:
\[
H(\varsigma, z) = 1_{T_n} + \sum_{k \geq 1} \sum_{v_1, \ldots, v_k \in T_n} \alpha_{\varsigma}^v(a(v_1t_1) \cdots a(v_kt_k)) r(v_1t_1) \cdots r(v_kt_k),
\]
\[
\hat{H}(\varsigma, z) = 1_{T_n} + \sum_{k \geq 1} \sum_{v_1, \ldots, v_k \in T_n} \alpha_{\varsigma}^v(a(\hat{v}t_1) \cdots a(\hat{v}t_k)) r(v_1t_1) \cdots r(v_kt_k).
\]

\[\square\]

**Corollary 3.** With the notations in Definition 4 and in Theorem 3, one has the following

1. **Infinite factorization of Chen series:**
\[
C_{\varsigma \rightarrow z} = \prod_{t \in \text{Dyn} T_n} e^{\alpha_{\varsigma}^t(S_t) P_t} \in e^{\text{Lie}_{\text{A}}(T_n)} \quad \text{(decreasing lexicographical ordered product)}.
\]

2. **Finite factorization of Chen series** (see also [12] and Remark 11),
\[
C_{\varsigma \rightarrow z} = V_0(\varsigma, z) H(\varsigma, z)
\]
and then \( H(\varsigma, z) \in e^{\text{Lie}_{\text{A}}(T_n)} \), being \( V_0^{-1}(\varsigma, z) C_{\varsigma \rightarrow z} \) and satisfying (109).

**Proof.** These are classic for Chen series (see [34] for example), using
1. Proposition 13 the Friedrichs criterion [61] and [62],
2. Theorem 2 and then (112).

\[\square\]

**Remark 12.** Replacing letters, in [11]–[18], by vector fields or matrices (see also Remark 2), the following sum is **Volterra expansion of solution of** [11]–[18] [34] [42]
\[
\sum_{m \geq 0} V_m = V_0 H, \quad \text{with the Volterra kernels} \quad \left\{ \sum_{w \in T_n} \kappa_w \right\}_{m \geq 0}.
\]

In particular, the sequence \( \{ F_i \}_{i \geq 0} \) with matrices in [13] yields the so-called Dyson series associated to [11] [6] [27]. It was applied in the disturbance (or noise) rejection (important problem in control theory. Corollary 2 corresponds to a change of controls for rejection of disturbances (or noises) by Lazard elimination [34]) and

---

26This means also *dévissage.*
4. Application to Knizhnik-Zamolodchikov equations

4.1. Noncommutative generating series of polylogarithms. For $KZ_3$ (see Examples [12]), essentially interested in solutions of (115) over $|0,1|$ and via the involution $s \mapsto 1-s$, Drinfeld proposed the following solution in $\mathcal{H}(\mathbb{C}^2_i)\langle \langle T_3 \rangle \rangle$ [17]:

$$F(z) = (z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\pi i}G((z_3 - z_2)/(z_1 - z_2)),$$

where $G$, belonging to $\mathcal{H}(\mathbb{C} \setminus \{0,1\})\langle \langle t_{1,2}, t_{2,3} \rangle \rangle$, satisfies the noncommutative differential using the connection $N_2$ determined in Example [11]

$$dG(s) = N_2(s)G(s).$$

Without explaining any method to obtain, he stated that (115) admits a unique solution, $G_0$ (resp. $G_1$), satisfying the following asymptotic condition [17]

$$G_0(s) \sim e^{-x_0 \log(s)} = s^{x_0} \quad \text{and} \quad G_1(s) \sim 1 \quad (1-s)^{-x_1},$$

and there is unique grouplike series $\Phi_{KZ} \in \mathbb{R}\langle \langle X \rangle \rangle$ such that $G_0 = G_1\Phi_{KZ}$. This series satisfies a system of algebraic relations (duality, hexagonal and pentagonal) [8] [17], so-called Drinfeld’s series [32] or Drinfeld’s associator [35].

In [17], the coefficients $\{c_{k,l}\}_{k,l \geq 0}$ of $\log \Phi_{KZ}$ are identified as follows

- Setting $A := t_{1,2}$, $B := t_{2,3}$ and supposing that $[A, B] = 0$, Drinfeld proposed $z^A/2\pi i (1-z)^B/2\pi i$ as solution of (115), over $|0,1|$, satisfying standard asymptotic conditions [16]. Such approximation solution of $KZ_3$ (a group series on $\mathcal{H}(\mathbb{C}^2_i)\langle \langle T_3 \rangle \rangle$) for which the logarithm belongs then to the following partial abelianization (see also Remark [15] below) and will be examined, as application of (25) and (113), in Section 4.3.

$$\text{Lie}_{\mathcal{H}(\mathbb{C}^2_i)}\langle \langle t_{1,2}, t_{1,3}, t_{2,3} \rangle \rangle/\text{Lie}_{\mathcal{H}(\mathbb{C}^2_i)}\langle \langle t_{1,2}, t_{2,3} \rangle \rangle, \text{Lie}_{\mathcal{H}(\mathbb{C}^2_i)}\langle \langle t_{1,2}, t_{2,3} \rangle \rangle].$$

- Then setting $\tilde{A} = A/2\pi i$ and $\tilde{B} = B/2\pi i$, he also proposed, over $|0,1|$, the standard solutions $G_0 = z^\tilde{A}(1-z)^\tilde{B}V_0(z)$ and $G_1 = z^\tilde{A}(1-z)^\tilde{B}V_1(z)$, where $V_0, V_1$ have continuous extensions to $|0,1|$ and is group-like solution of the following noncommutative differential equation, in the topological free Lie algebra, $p := \text{span}\{dA \text{ad}_A^t \text{ad}_B[A, B]\}_{k,l \geq 0}$, with $V_0(0) = 1, V_1(1) = 1$.

$$dS(z) = Q(z)S(z) \quad \text{where} \quad Q(z) := e^{d_{-\log(1-z)}^t}e^{d_{-\log(z)}^t}B \quad \frac{B}{z - 1} \in p.$$
Since $G_0 = G_1 \Phi_{KZ}$ then $\Phi_{KZ} = V(0)V(1)^{-1}$, where $V$ is a solution of (118) and then, by identification in the abelianization $\mathfrak{p}/\{p, p\}$, as follows

$$
\log \Phi_{KZ} = \sum_{k,l \geq 0} c_{k,l} B^{k+l+1} A^{l+1} = \int_0^1 Q(z) dz \pmod{[p, p]}
$$

$$
= \int_0^1 e^{ad - \log(1-z)B} e^{ad - \log(x)A} \tilde{B} dx \pmod{[p, p]}
$$

(119)

and by serial expansions of exponentials, one deduces that

$$
\log \Phi_{KZ} = \sum_{k,l \geq 0} \frac{1}{l!k!} \int_0^1 \log \left( \frac{1}{1-z} \right) dz \left( B^k A^l \tilde{B} \right) dx \pmod{[p, p]}
$$

(120)

The following divergent (iterated) integral is regularized\(^{30}\) by

$$
(121) c_{k,l} = \frac{1}{(2\pi)^{k+l+2(k+1)!}} \int_0^1 \log \left( \frac{1}{1-z} \right) dz \left( B^k A^l \tilde{B} \right) (2\pi)^{k+l+1}
$$

and, by a Legendre’s formula\(^3\). Drinfeld stated that previous process is equivalent to the following identification\(^2\)\[17\]:

$$
1 + \sum_{k,l \geq 0} c_{k,l} B^{k+l+1} A^{l+1} = \exp \sum_{n \geq 2} (\zeta(n) (B^n + A^n - (B + A)^n)).
$$

(122)

$\Phi_{KZ}$ is completely studied in \[33\] thanks to the polylogarithms defined by

$$
L_{1,x} = \log(\mathcal{H}(\mathbb{C} \setminus \{0, 1\})), \quad L_{x,0} = \log(s), \quad L_{x,1} = \log(1-s),
$$

(123)

$$
L_{x,w} = \int_0^s \omega_i(s) L_i(w), \quad \text{where} \quad \begin{cases} x_i w \in \mathcal{L}_{\mathbb{C} \setminus \{0, 1\}}, \\ \omega_0(s) = s^{-1}, \\ \omega_1(s) = (1-s)^{-1}, \\ \{x_i w \in \mathcal{L}_{\mathbb{C} \setminus \{0, 1\}} \}
$$

(124)

$$
\text{where} \quad (X^*, X^+) \text{ is the monoid generated by} \ X = \{x_0, x_1\} \text{ (ordered by} \ x_0 < x_1). \text{ In particular,} \ \{L_i\}_{\mathcal{L}_{\mathbb{C} \setminus \{0, 1\}}} \text{ (resp.} \ \{L_{x,w}\}_{w \in X^*} \text{) is algebraically (resp. linearly) free, over} \ \mathbb{C}, \text{ and the noncommutative series of} \ \{L_{x,w}\}_{w \in X^+} \text{ is grouplike (see Proposition} \[3\], as being the actual solution over} \ \mathcal{H}(\mathbb{C} \setminus \{0, 1\}) \langle X \rangle \text{ of} \ (115) \text{ satisfying the asymptotic conditions} \ (116), \ [39], \ [43]:
$$

(125)

$$
\Phi_{KZ} := \prod_{t \in \mathcal{L}_{\mathbb{C} \setminus \{0, 1\}}} e^{\xi_i t \cdot \Phi_{KZ}}, \quad \text{with} \quad \begin{cases} x_0 = \frac{t_{1,3}}{2i\pi}, \\ x_1 = -\frac{t_{2,3}}{2i\pi}, \\ \{P_i\}_{\mathcal{L}_{\mathbb{C} \setminus \{0, 1\}}} \text{ (resp.} \ \{S_i\}_{\mathcal{L}_{\mathbb{C} \setminus \{0, 1\}}} \text{) is linear basis of} \ \mathcal{L}_{\mathbb{C} \setminus \{0, 1\}} \text{ (resp.} \ Sh_{\mathbb{C} \setminus \{0, 1\}} \text{) and}
$$

(126)

$\Phi_{KZ}$ is given by the above product.

30 The readers are invited to consult \[13\] for a comparison of these regularized values yielding expressions of $\Phi_{KZ}$ and $\log \Phi_{KZ}$, in which involve polyzetas.

31 i.e. the Taylor expansion of $\log(1 - z)$ involving only the real numbers $\{\zeta(k)\}_{k \geq 2}$ and $\gamma$ (as regularized value of the harmonic series $1 + 2^{-1} + 3^{-1} + \ldots$).

32 Note that the summation on right side starts with $n = 2$ and then $\gamma$ could not be appeared in the regularization proposed in \[17\].
admitting \{\text{Li}_i(1)\}_{i \in \mathcal{Lyn} X \setminus X} as convergent coordinates and the coordinates \{\langle \Phi_{KZ} | w \rangle \}_{w \in X^*} as the finite part of the singular expansions at \(z = 1\) of \{\text{Li}_w \}_{w \in X^*}. In the comparison scale \((1 - z)^{-a} \log^b(1 - z))_{a, b \in \mathbb{N}}\) (see \((125)\)). Moreover, in virtue of \((126)\), \(L((z_3 - z_2)/(z_1 - z_3))\) is grouplike solution of \(KZ_3\). So does \((114)\), for which any other grouplike solution of \(KZ_3\) can be deduced by right multiplication by constant grouplike series as treated in Appendix 6.1 below.

4.2. Noncommutative generating series of hyperlogarithms. Recall also that, after \(KZ_4\), Drinfeld proposed asymptotic solutions, for \(KZ_4\), on different zones in the region \(\{z \in \mathbb{R}^4| z_1 < z_2 < z_3 < z_4\}\) and exact solutions, as in \((114)\), are not provided yet. It was a break with respect to the strategy in previous cases. Several works tried to advance on the resolution of \(KZ_m\) (for \(n \geq 4\)). Indeed, it was studied the Dirichlet functions \{\text{Di}_w(F; s)\}_{w \in X} (and their parametrization) indexed by words in \(X = \{x_i\}_{0 \leq i \leq N}\) (totally ordered by \(x_0 < \ldots < x_N\), i.e. iterated integrals of the following holomorphic 1-forms \(20, 36, 37\).

\[(127)\] \(\omega_i(s) = \frac{ds}{s}, \quad \omega_i(s) = F_i(s)ds, \quad \text{where } F_i(s) = \sum_{k \geq 1} f_{i,k} z^k, 0 \leq i \leq N.\)

In particular, for singularities in \(\Sigma_N = \{0, a_1, \ldots, a_N\}\) (in bijection with \(X\)) and

\[(128)\] \(F_i(s) = (s - a_i)^{-1}, \quad 0 \leq i \leq N,\)

these correspond to Lappo-Danilevsky’s hyperlogarithms \(\text{Li}_{x_i}(s) = \log(s - a_i), \quad 1 \leq i \leq N,\)

and, for any Lyndon word \(x_i w \in \mathcal{Lyn} X \setminus X\), by

\[(130)\] \(\text{Li}_{x_i w}(s) = \int_0^s \omega_i(\sigma) \text{Li}_w(\sigma), \quad \text{where } \omega_i(s) = \frac{ds}{s - a_i}.\)

These hyperlogarithms \(\{\text{Li}_i\}_{i \in \mathcal{Lyn} X}\) (resp. \(\{\text{Li}_w\}_{w \in X^*}\)) are algebraically (resp. linearly) free over \(\mathbb{C}\) \(14\), i.e. the character \(\text{Li}_w\) of \((\mathbb{C}(X), \omega, 1_{X^*})\) (see \((130)\)) is injective and its graph, viewed as noncommutative generating series, is grouplike and can be put in the MRS form as follows \(\text{Li}_w\) (see also Proposition 4 below)

\[(131)\] \(L := \sum_{w \in X^*} \text{Li}_w w = \prod_{i \in \mathcal{Lyn} X} e^{\text{Li}_{w_i} P_i}.\)

This series belongs to \(\mathcal{H}(\mathbb{C} \setminus \Sigma_N)/\mathcal{O}(X)\) (while, as already said, solutions of \((14)\) belong to \(\mathcal{H}(\mathcal{C}_x)/\mathcal{O}(T_n)\)) and, by \((127) - (128)\), satisfies the following differential equation

\[(132)\] \(dL(s) = (x_0 \omega_0(s) + x_1 \omega_1(s) + \ldots x_N \omega_N(s))L(s),\)

and quite involves in the resolution of \((14)\) according to \((15) - (16)\).

Indeed, taking \(N = n - 2\), \(a_k = z_k\), for \(1 \leq k \leq n - 2\), and substituting

\[(133)\] \(x_0 = t_{n-1,n}/2\pi\) and \(\forall k = 1, \ldots, n - 2, x_k = -t_{k,n}/2\pi,\)

\(33\)For this point, Lyndon words are more efficient for checking the convergence of \(\{\text{Li}_w(1)\}_{w \in X^*}\) (see \((13)\)) using a Radford’s theorem \(59, 61\).

\(35\)These coefficients are convergent and regularized divergent polyzetas \(35, 32\).

\(36\)put one can also obtain directly as shows in appendices in Section 6 below.

\(36\)and, of course, colored polylogarithms for the case of roots of unity, i.e \(a_i = e^{2\pi i/N}\) \(41\).
\(\tilde{M}_n\) (given in (139)) induces the following simpler expression for \(N_{n-1}\) (given in (16)) as the connection of \(\tilde{\omega} \) satisfied by \(L\) (given in (130)–(131)):

\[
N_{n-1}(z) = x_0 \frac{ds}{s} + \sum_{k=1}^{n-2} x_k \frac{ds}{\alpha_k - s} \quad \text{and then} \quad dL(z) = N_{n-1}(s)L(s). 
\]

This showed, in fact, the grouplike series \(\tilde{\omega}\) in (131) (resp. (126)) is not but normalizes the Chen series, of \(\{\omega_i\}_{0 \leq i \leq N}\) in (128) (resp. \(\{\omega_i\}_{0 \leq i \leq 1}\) in (124) and along \(0 \to z\), in which the integral \(\int \tilde{\omega}(s)\), for example, is not defined.}

**4.3. Knizhnik-Zamolodchikov equations.** Ending this note, let \(p\) be the projection \(\mathbb{C}_n^\times \to \mathbb{C}_n^\times\) and let us consider the following affine plans

\[
(P_{i,j}) : z_i - z_j = 1, \quad \text{for} \quad 1 \leq i < j \leq n.
\]

Let us consider

\[
\begin{align*}
\omega_{i,j}(z) &= (z_i - z_j)^{-1}, \quad \text{for} \quad 1 \leq i, j \leq n, \\
\omega_{i,j}(z) &= \omega_{i,j}(d(z_i - z_j)), \quad \text{for} \quad 1 \leq i < j \leq n,
\end{align*}
\]

and then the Chen series \(C_{\omega_{i,j}}\), of the holomorphic 1-forms \(\{d\log(z_i - z_j)\}_{1 \leq i < j \leq n}\) and along the path \(z^0 \to z\) over \(V := \mathbb{C}_n^\times\). As in Section 1 let \(A := \mathcal{H}(V)\).

**Remark 13.** Let \(k \geq 1, t_{i,j} \in T_n, z^0 \in P_{i,j}\). Then \(\alpha_x^k(t_{i,j}) = \log^k(z_i - z_j)/k!\).

**Definition 8** (normalized Chen series). Let \(F^\bullet : (\mathbb{C}(T_n), \omega, T_n^\times) \to (A, \ast, 1_A)\) is the character defined by

\[
F^T = 1_A, \quad F^T_{t_{i,j}}(z) = \log(z_i - z_j), \quad \text{for} \quad t_{i,j} \in T_n,
\]

and, for any \(t_{i,j}w \in \mathcal{L}nT_n \setminus T_n\) and \(z^0\) moving towards 0, by

\[
F_{t_{i,j}}(w)(z) = \int_{z^0}^{z} \omega_{t_{i,j}}(s)F_w(s).
\]

Let \(F_{KZ_n}\) be the graph of \(F^\bullet\) (i.e. the noncommutative generating series of \(\{F_w\}_{w \in T_n^\times}\)).

**Remark 14.**

1. If \(F \in A\) and \(F\) is expanded as follows\(\text{[38]}\)

\[
F(z) = \sum_{1 \leq i < j \leq n}^n f(n_{i,j}; 1 \leq i < j \leq n) \prod_{1 \leq i < j \leq n} (z_i - z_j)^{n_{i,j}}
\]

then, for any \((i_0, j_0)\) such that \(1 \leq i_0 < j_0 \leq n\), one has, for any \(k \geq 0\),

\[
\lim_{z_{j_0} \to z_{i_0}} (z_{i_0} - z_{j_0})^k F(z) = 0.
\]

2. By a Radford’s theorem\(\text{[59] [61]}\), \(F_w, w \in T_n^\times\), is polynomial on \(\{F_i\}_{i \in \mathcal{L}nT_n}\) and depends on the differences \(\{z_i - z_j\}_{1 \leq i < j \leq n}\). In particular, for \(w \in T_n^\times\), by induction on \(|w|\), \(F_w\) can be expanded by (see the previous item)

\[
F_w(z) = \sum_{1 \leq i < j \leq n}^n f_w(n_{i,j}; 1 \leq i < j \leq n) \prod_{1 \leq i < j \leq n} (z_i - z_j)^{n_{i,j}}
\]

and \(F^k_{t_{i,j}}(z) = \alpha_{x}^k(t_{i,j})\), for \(z^0 \in P_{i,j}, t_{i,j} \in T_n, k \geq 1\) (see also Remark\(\text{[39]}\)).

---

\(\text{[37]}\) Coefficients are quite defined over the algebraic basis indexed \(\mathcal{L}nX\) as in (124) (resp. (150)).

\(\text{[38]}\) \(\log(z_i - z_j) = \sum_{k \geq 1} (-1)^{k-1}((z_i - z_j) - 1)^k/k\), for \(|z_i - z_j| < 1\).

\(\text{[39]}\) The coefficients \(f(n_{i,j}; 1 \leq i < j \leq n)\)'s are indexed by integers \(n_{i,j} > 0\), for \(1 \leq i < j \leq n\).
(3) By (10) and Proposition 4 multiplying on the right of the Chen series, of \( \{d \log(z_i - z)\}_{1 \leq i \leq L} \) and along \( z^0 \mapsto z \) over \( \mathbb{C}_v^2 \), by \( F_{\mathbb{K}Z_n}(z^0) \in \{e^C\}_{C \in \text{Lie}(\langle T_0 \rangle)} \), \( F_{\mathbb{K}Z_n}(z) \) normalizes \( C_{z^0 \rightarrow z} \) and satisfies (14).

According to (20)–(21) and Theorem 1 the image of \( D_{T_n} \) by \( F_\ast \otimes \text{Id} \) yields

**Proposition 6** (factorizations of normalized Chen series).

1. (One has

\[
F_{\mathbb{K}Z_n} = \bigg( \prod_{l \in \text{Lyn} T_{n-1}} e^{F_{S_l} P_l} \bigg) \bigg( \prod_{l = 1}^{n-1} e^{F_{S_l} P_l} \bigg) \bigg( \prod_{l \in \text{Lyn} T_n} e^{F_{S_l} P_l} \bigg) = \sum_{l \in \text{Lyn} T_n} e^{F_{S_l} P_l} \times \left( 1 + \sum_{v_1, \ldots, v_k \in T_{n-1}} F_a(v_1 t_1, t_1 \omega_1 + \ldots + \omega_m a(v_i t_i)) r(v_1 t_1) \ldots r(v_k t_k) \right),
\]

and, as image by \( F_\ast \otimes \text{Id} \) of \( D_{T_n} \) in (14), \( \log F_{\mathbb{K}Z_n} \) is primitive, for \( \Delta_{\omega} \).

2. Modulo \( \{\text{Lie}_{1,\mathbb{A}}(\langle T_n \rangle), \text{Lie}_{1,\mathbb{A}}(\langle T_n \rangle)\} \), one also has

\[
F_{\mathbb{K}Z_n} = e^{\sum_{l \in \text{Lyn} T_n} F_{S_l} P_l} \left( 1 + \sum_{k \geq 1} \sum_{v_1, \ldots, v_k \in T_{n-1}} F_{a(v_1 t_1, \ldots, v_k t_k)} r(v_1 t_1) \ldots r(v_k t_k) \right).
\]

**Corollary 4.** With Notation in Example 17 one has

1. The morphism \( F_\ast : (C_0(T_n), \omega) \rightarrow (\text{span}_{w \in T_n} \{F_w\}) \) is injective.

2. Let \( K \subset K_{n-1} \) be the algebras generated, respectively, by \( \{F_l\}_{l \in \text{Lyn} T_n} \) and \( \{F_l\}_{l \in \text{Lyn} T_{n-1}} \). Then \( K_{T_n} \) and \( K_{T_{n-1}} \) are \( C_0 \)-algebraically disjoint.

3. There exists \( E \in \text{Lie}_{\mathbb{K}Z_n}(\langle T_{n-1} \rangle) \) such that, for \( z^0 \rightarrow 0 \),

\[
F_{\mathbb{K}Z_{n-1}}(z^0 E) = 1 + \sum_{k \geq 1} \sum_{v_1, \ldots, v_k \in T_{n-1}} \int_{z^0}^{z} \omega_{i_1, j_1} (s_1) \ldots \int_{z^0}^{s_{k-1}} \omega_{i_k, j_k} (s_k) e^{\frac{\Delta_{\omega} z^0}{2}} (v_1 t_1 \ldots v_k t_k).
\]

\[
F_{\mathbb{K}Z_n} = \bigg( \prod_{l \in \text{Lyn} T_{n-1}} e^{F_{S_l} P_l} \bigg) F_{\mathbb{K}Z_{n-1}} E.
\]

4. \( \{\text{ad}_{-T_n}^{t_1} t_1 \ldots \text{ad}_{-T_{n-1}}^{t_p} t_p \}_{i_1, \ldots, k \geq 0, p \geq 1} \) of \( \mathcal{U}(\mathbb{I}_N) / \{\text{Lie}_{1,\mathbb{A}}(\langle T_n \rangle), \text{Lie}_{1,\mathbb{A}}(\langle T_n \rangle)\} \) is dual to \( \{(-t_1 \hat{T}_n)^{i_1} \ldots (-t_k \hat{T}_n)^{i_k} \}_{i_1, \ldots, k \geq 0, p \geq 1} \) of \( \mathcal{U}(\mathbb{I}_N) \).

**Proof.** These are consequences of Propositions 4, 5, Corollary 2, and Theorem 2.

In order to examine grouplike solutions of \( K \subset \mathfrak{z} \) with asymptotic conditions by \( \text{dévissage} \), let us consider again the alphabet \( T_n = \{t_{i,j}\}_{1 \leq i, j \leq n} \) satisfying (19) and (20).

\[
U_i := \sum_{j=1, j \neq i}^{n} t_{i,j} u_{i,j}, \quad 1 \leq i \leq n.
\]
With the split \( (8) \), i.e. \( M_n = \bar{M}_n + M_{n-1} \), and the data in \( (136) \), one has
\[
\bar{M}_n = \sum_{k=1}^{n-1} t_{k,n} \frac{d(z_k - z_n)}{z_k - z_n}, \quad M_n = \sum_{1 \leq i < j \leq n} t_{i,j} \frac{d(z_j - z_i)}{z_j - z_i} = \sum_{i=1}^{n} U_i(z) \, dz_i.
\]
Moreover, as in \( (15) - (16) \), \( \bar{M}_n \) behaves, for \( n \to n-1 \), as the following connection
\[
N_{n-1}(s) = t_{n-1,n} \frac{ds}{s} - \sum_{k=1}^{n-2} t_{k,n} \frac{ds}{a_k - s}, \quad \text{with} \quad \left\{ \begin{array}{l}
\ s = z_n, \\
\ a_k = z_k.
\end{array} \right.
\]

**Proposition 7.**

1. The family \( \{U_i\}_{1 \leq i \leq n} \) satisfies
\[
\sum_{i=1}^{n} U_i = 0, \quad \sum_{i=1}^{n} z_i U_i(z) = \sum_{1 \leq i < j \leq n} t_{i,j}, \quad \partial_i U_j - \partial_j U_i = [U_i, U_j] = 0.
\]
2. If \( G \) is solution of \( (14) \) then it satisfies the following identities
\[
\sum_{i=1}^{n} \partial_i G(z) = 0 \quad \text{and} \quad \sum_{i=1}^{n} z_i \partial_i G(z) = \sum_{1 \leq i < j \leq n} t_{i,j} G(z)
\]
and the partial differential equations \( \partial_i G = U_i G \), for \( i = 1, \ldots, n \).
3. One has \( M_n \wedge M_n = 0 \) and \( dM_n = 0 \) and then \( dM_n = 0 \).
4. One has \( d\Omega_n - \Omega_n \wedge \Omega_n = 0 \) (see \( (17) \)) and \( d\Omega_n = 0 \).

**Proof.**

1. Since \( u_{i,j} = -u_{j,i} \) then
\[
\sum_{i=1}^{n} U_i = \sum_{i=1}^{n} \sum_{1 \leq j \leq n} (t_{i,j} - t_{j,i}) u_{i,j}.
\]

By the infinitesimal braid relations given in \( (19) \), we get the first identity.

For the second identity, using a change of indices as follows
\[
\sum_{i=1}^{n} z_i U_i(z) = \sum_{i=1}^{n} t_{i,j} \left( \sum_{1 \leq j < i \leq n} \frac{z_i}{z_i - z_j} - \sum_{1 \leq i < j \leq n} \frac{z_i}{z_j - z_i} \right)
= \sum_{i=1}^{n} t_{i,j} \left( \sum_{1 \leq i < j \leq n} \frac{z_i}{z_i - z_j} - \frac{z_j}{z_i - z_j} \right) = \sum_{1 \leq i < j \leq n} t_{i,j}.
\]

The third identity is obtained by direct calculations:
\[
\partial_i U_j - \partial_j U_i = \sum_{1 \leq j \leq n} t_{j,1} (\partial_i u_{j,1}) - \sum_{1 \leq k \leq n} t_{i,k} (\partial_j u_{i,k})
= -t_{j,1} (z_j - z_i)^{-2} + t_{i,j} (z_i - z_j)^{-2}
\]
\[
[U_i, U_j] = \sum_{1 \leq k \leq n} t_{i,k} t_{j,l} u_{i,k} u_{j,l} + \sum_{1 \leq k \leq n} t_{i,k} t_{j,l} u_{i,k} u_{j,l}
+ \sum_{1 \leq k \leq n} t_{i,k} [t_{j,l}] u_{i,k} u_{j,l} + \sum_{1 \leq k \leq n} t_{i,k} [t_{j,l}] u_{i,k} u_{j,l}
= \sum_{1 \leq k \leq n} t_{i,k} [t_{j,l}] u_{i,k} u_{j,l} + \sum_{1 \leq k \leq n} (z_i t_{j,k} + t_{j,k}) u_{i,k} u_{j,l}
+ \sum_{1 \leq j \leq n} (z_{j,i} t_{k,i} + t_{k,i}) u_{i,k} u_{j,l}.
\]

\( \text{Note 6.} \)
By infinitesimal braid relations in (19), one gets \( \partial_t U_j - \partial_j U_t = [U_i, U_j] = 0 \).

(2) The first identities are consequences of the item 1. By \( t_{ij} = \Omega_{ij} \) and the item 1 of Proposition 7, one obtains

\[
dG(z) = \left( \sum_{i=1}^{n} U_i(z) \, dz_i \right) G(z) = \sum_{i=1}^{n} (U_i(z)G(z)) \, dz_i = \sum_{i=1}^{n} (\partial_t G(z)) \, dz_i
\]

and by (103), one obtains the last result.

(3) By (138) and the item 1 of Proposition 7, one obtains

\[
M_n(z) \land M_n(z) = \sum_{i,j=1}^{n} U_i(z)U_j(z) \, dz_i \land dz_j
\]

\[
d \land M_n(z) = \sum_{i,j=1}^{n} \partial_j U_j(z) - \partial_j U_i(z) \, dz_i \land dz_j = 0.
\]

and, on the other hand, \( d \land M_n = d(M_n - M_{n-1}) = dM_n - dM_{n-1} = 0 \).

(4) Substituting \( t_{ij} \) by \( t_{i,j}/2\pi \) on \( M_n \) and \( \bar{M}_n \), one gets the expected results. In all the sequel, as for (17), the letters in \( T_n \) satisfy now (18). \( \square \)

Remark 15. With data in (139) and by Proposition 7, one has \( \Omega_n \) is flat and \( dS = \Omega_n S \) is completely integrable (see also (17)) and, on the other side, \( \Omega_n \) is not flat and \( dS \) is not completely integrable. Indeed, one has \( d \land \bar{M}_n = 0 \) and

\[
M_n' \land M_n' = \sum_{1 \leq i, j, k \leq n-1} t_{i,n} t_{j,n} \, d\log(z_i - z_n) \land d\log(z_j - z_n).
\]

Getting flatness of \( M_n \), one could further assume that \( \{t_{i,n}, t_{i,n}\} = 0 \), as in the definition of \( V_0 \) in (113) and then in Definition 7 thanks to \( \varphi \) and \( \psi \) which are used in Propositions 5–6 and Theorem 2 (see also (114)).

Now, we are in situation back to (14) and its solutions with asymptotic conditions, by Definitions 7,8 and Propositions 9,10 to achieve our application.

Theorem 3 (dévissage). With Definition 7 and data in (139), grouplike solution of (14) can be put in the form \( h(z_n)H(z_1, \ldots, z_{n-1}) \) such that, for \( z_n \to z_{n-1} \),

(1) \( h \) is solution of \( df = N_{n-1} f \), where \( N_{n-1} \) is the connection determined in (139). Hence, \( h(z_n) \sim z_n \to z_{n-1} (z_{n-1} - z_n)^{t_{n-1,n}} \).

(2) \( H(z_1, \ldots, z_{n-1}) \) satisfies \( dS = M_{n-1}^\varphi S \), i.e. (104) with \( \phi = \varphi_n \) and

\[
M_{n-1}^\varphi(z) = \sum_{1 \leq i < j \leq n-1} d\log(z_i - z_j) \, \varphi_n(z)^{(i,j)}(t_{i,j}),
\]

\[
\varphi_n^{(i,j)}(t_{i,j}) \sim z_n \to z_{n-1} e^{ad_{-\log(z_{n-1} - z_n)}} t_{n-1,n} t_{i,j} \mod J_{n-1}.
\]

Moreover, \( M_{n-1}^\varphi \) exactly coincides with \( M_{n-1} \) in \( \bigcap_{1 \leq k < n-1} (P_k, n-1) \).

---

42 Observed by B. Enriquez, using the \( \mathbb{C} \)-linear independence of \( \{\log(z_i - z_n)\}_{1 \leq i \leq n-1} \).

43 For \( 1 \leq i < j \leq n \), changing \( t_{ij} \) by \( t_{ij}/2\pi \) (thus \( M_n \) and \( M_{n-1} \) become \( \Omega_n \) and \( \Omega_{n-1} \), respectively), one deduces results for (13).

44 See Note 54 and Remark 14.
Conversely, for \( z_n \to z_{n-1} \), if \( h \) satisfies \( df = N_{n-1} f \) and \( H(z_1, \ldots, z_{n-1}) \) satisfies (109) then \( h(z_n)H(z_1, \ldots, z_{n-1}) \) is solution of (1).

**Proof.** For \( z_n \to z_{n-1} \), on the one hand, \( h \equiv V_0 \) and it behaves as generating series of hyperlogarithms (i.e. iterated integrals of holomorphic forms \( \{ds/(s_s)\}_{1 \leq k < n} \), with the singularities \( s_k = z_n - z_k \), see Remarks 9 and 12). It follows then the first assertion. On the other hand, with \( \varphi_n = \varphi_{T_n} \mod J_{R_n} \) as in Definition 7 the Picard’s iteration (108) converges, for the discrete topology, to a solution of (109) having the expected connection:

\[
H(z_1, \ldots, z_{n-1}) = \sum_{m \geq 0} \sum_{t_{i_1,j_1} \cdots t_{i_m,j_m} \in T_{n-1}} \int_{z^0}^z d\log(s_{i_1} - s_{j_1}) \varphi_n(z_{i_1,j_1}) \cdots \int_{z^0}^z d\log(s_{i_m} - s_{j_m}) \varphi_n(z_{i_m,j_m}),
\]

\[
\varphi_n(z_{i,j}) = \prod_{i < j} e^{ad_{F_{S_{T_n}(z)}}} \mod J_{R_n}
\]

Conversely, let \( C \in \mathbb{C} \langle 1, \tau_{n-1} \rangle / J_{R_{n-1}} \) such that \( \langle C \mid 1 \tau_{n-1} \rangle = 1_A \). If \( HC \) satisfies (109) then, by Propositions 4, \( V_0 HC \) satisfies (1).

Theorem 3 is established for \( z_n \to z_{n-1} \) and, for dévissage, can be performed recursively. Up to a permutation of \( \mathfrak{S}_n \), it can be adapted for other cases. Hence

**Corollary 5** (solution of \( KZ_n \) satisfying asymptotic condition). \( F_{KZ_n} \) is unique group-like solution of (1) satisfying

\[
F_{KZ_n}(z) \sim \sum_{1 \leq i \leq n} (z_{i-1} - z_i)^{t_{i-1,i}} G_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)
\]

in \( A \langle 1, \tau \rangle / J_{R_n} \) and \( G_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \) satisfies (109).

Moreover, for \( y_1 = z_1, \ldots, y_{i-1} = z_{i-1}, y_i = z_{i+1}, \ldots, y_n = z_n \), the connection \( M_{n-1}^{z_{i-1}} \) is expressed as follows

\[
M_{n-1}^{z_{i-1}}(y) = \sum_{1 \leq i < j \leq n} d\log(y_i - y_j) e^{ad_{-\log(y_i - y_j)}} t_{i,j} \mod J_{R_n}
\]

and exactly coincides with \( M_{n-1} \) in \( \bigcap_{1 \leq k < n-1} (P_{k,n-1}) \).

5. Conclusion

Basing on the Lazard and Schützenberger factorizations over the monoid generated by the alphabet \( T_n = \langle t_{i,j} \rangle_{1 \leq i < j \leq n} = T_{n-1} \cup T_n \) \( (T_n = \langle t_{k,n} \rangle_{1 \leq k \leq n-1}) \) and, on the other side, the noncommutative symbolic calculus on \( \mathcal{H}(\mathcal{V}) \langle 1, \tau_n \rangle \) (i.e. the ring of noncommutative series over \( T_n \), with holomorphic coefficients in \( \mathcal{H}(\mathcal{V}) \) [12], various combinatorics on Chen series, of the holomorphic 1-forms \( \{\omega_{i,j}\}_{1 \leq i < j \leq n} \) and along a path \( \zeta \rightsquigarrow z \) over the simply connected manifold \( \mathcal{V} \),

\[
C_{\zeta \to z} = \sum_{w \in T_n} a_\zeta^w w
\]

were obtained, by extending [13], over \( \mathcal{H}(\mathcal{V}) \langle 1, \tau_n \rangle \) and then over \( \mathcal{H}(\mathcal{V}) \langle 1, \tau_n \rangle / J_n \), where \( J_n \) is the ideal of relators on \( \langle t_{i,j} \rangle_{1 \leq i < j \leq n} \) induced by (12). These are
used in order to compute, by iterations over $\mathcal{H}(\mathcal{V})\langle\mathcal{T}_n\rangle$, solution of the universal differential equation $dS = M_n S$ and its Galois differential group, where

$$M_n = M_n + M_{n-1}, \quad M_n := \sum_{1 \leq i < j \leq n} \omega_{i,j} t_{i,j}, \quad M_n := \sum_{k=1}^{n-1} \omega_{k,n} t_{k,n}.$$  

More precisely, it was focus on the sequences of series in $\mathcal{H}(\mathcal{V})\langle\mathcal{T}_n\rangle$, $\{V_k\}_{k \geq 0}$ and $\{\hat{V}_k\}_{k \geq 0}$ satisfying the recursion

$$S_k(\zeta, z) = S_0(\zeta, z) \sum_{t_{i,j} \in \mathcal{T}_{n-1}} \int_{\zeta}^{z} \omega_{i,j}(s) S_0^{-1}(\zeta, s) t_{i,j} S_{k-1}(\zeta, s),$$  

with the starting conditions being grouplike series, for $\Delta_{\mathcal{H}}$,

- $V_0(\zeta, z) = \prod_{i \in \mathcal{L} \, \mathcal{Y}_n \mathcal{T}_n} e^{\alpha_i(S_i \, P_i)}$,
- $\hat{V}_0 = V_0 \mod [\mathcal{L} \mathcal{I} \mathcal{E}(\mathcal{V})\langle\mathcal{T}_n\rangle, \mathcal{L} \mathcal{I} \mathcal{E}(\mathcal{V})\langle\mathcal{T}_n\rangle]$.

Technically and intensively, in Section 2 using the pairs of dual bases (introduced in (76)–(77) and in Definition 1) and then applying Lemma 1 Propositions 1–2 and Theorem 1 various expansions of diagonal series (given in (66)) were provided, in the concatenation-shuffle bialgebra and in a Loday’s generalized bialgebra:

$$D_{\mathcal{T}_n} = D_{\mathcal{T}_{n-1}} \left( \prod_{i_2 \in \mathcal{L} \, \mathcal{Y}_n \mathcal{T}_n} e^{S_i \, P_i} \right) D_{\mathcal{T}_n}$$

$$= D_{\mathcal{T}_n} \left( 1_{\mathcal{T}_n^*} \otimes 1_{\mathcal{T}_n^*} + \sum_{k \geq 1} v_1, \ldots, v_k \in \mathcal{T}_n^* \right.$$  

$$a(v_1 t_1) \omega(\cdots \omega a(v_k t_k) \ldots) \otimes r(v_1 t_1) \ldots r(v_k t_k) \bigg).$$

After that, in Sections 3–4 basing on Chen series (see Definition 3 and their properties (established in Propositions 3–4 and Corollary 1 for our needs) and then applying Propositions 5–6, Theorems 2–3 and Corollaries 4–5, it was proved that

1. $\sum_{k \geq 0} V_k$ converges, for the discrete topology, to $C_{\zeta \rightarrow z}$, i.e. the limit of the following iteration

$$F_0(\zeta, z) = 1_{\mathcal{H}(\mathcal{V})}, \quad F_i(\zeta, z) = F_{i-1}(\zeta, z) + \int_{\zeta}^{z} M_n(s) F_{i-1}(s), i \geq 1.$$  

2. Specializing $\alpha_i j = d \log(z_i - z_j)$ and then $\mathcal{V} = \mathcal{C}_n^0$ and reducing by $\mathcal{J}_\mathcal{R}_n$, for $z_n \rightarrow z_{n-1}$, grouplike solution of (4) is of the form $h(z_n) H(z_1, \ldots, z_{n-1})$ such that

(a) $h$ is solution of $dh = N_{n-1} f$, where $N_{n-1}$ is the connection determined in (139). Hence, $h(z_n) \sim z_n \rightarrow z_{n-1} (z_{n-1} - z_n)^{f_{n-1}.n}$.  

(b) $H(z_1, \ldots, z_{n-1})$ satisfies $dS = M_{n-1} S$, where

$$\varphi_{n}^{(0, z)}(z) = \sum_{1 \leq i < j \leq n-1} d \log(z_i - z_j) \varphi_{n}^{(0, z)}(t_{i,j}),$$

$$\varphi_{n}^{(0, z)}(t_{i,j}) \sim z_n \rightarrow z_{n-1} e^{ad_{-d \log(z_{n-1} - z_n)} t_{n-1, n}} t_{i,j} \mod \mathcal{J}_\mathcal{R}_n.$$  

\[45\text{See Note} 4\]
(3) The normalized Chen series (see Definition 8) provides by dévissage, over 
\( H(\tilde{C}_n)\langle \langle T_n \rangle \rangle \) and then over \( H(\tilde{C}_n)\langle \langle T_n \rangle \rangle / J_{R_n} \), the unique solution of (14) satisfying asymptotic conditions, obtained as image of \( D T_n \),

\[
F_{KZ_n} = \prod_{\ell \in \text{Lyn}T_n} e^{F_{S_\ell} P_\ell} \times \left( 1_{T_n} + \sum_{v_1, \ldots, v_k \in T_n^*, k \geq 1 \atop t_1, \ldots, t_k \in T_{n-1}} F_{a(v_1 t_1) \ldots a(v_k t_k)} r(v_1 t_1) \ldots r(v_k t_k) \right)
\]

functional expansion of \( KZ_{n-1} \)

\[
= \prod_{\ell \in \text{Lyn}T_n} e^{F_{S_\ell} P_\ell} \left( 1_{T_n} + \sum_{v_1, \ldots, v_k \in T_n^*, k \geq 1 \atop t_1, \ldots, t_k \in T_{n-1}} F_{a(v_1 t_1) \ldots a(v_k t_k)} r(v_1 t_1) \ldots r(v_k t_k) \right).
\]

(4) On the other hand, since \( \hat{V}_0 \) is a nilpotent approximation of order 1 of \( V_0 \) (see Remark 11) then, by the families of polynomials, in Definition 1, the series on \( \{\hat{V}_k\}_{k \geq 0} \) approximates \( C_\varsigma \rightsquigarrow z \) yielding then an approximation solution of \( KZ_n \), as extension of a treatment in [17] or in (117):

\[
F_{KZ_n} \equiv e^{\sum_{t \in T_n} F_t t} \left( 1_{T_n} + \sum_{v_1, \ldots, v_k \in T_n^*, k \geq 1 \atop t_1, \ldots, t_k \in T_{n-1}} F_{a(\hat{v}_1 t_1) \ldots a(\hat{v}_k t_k)} r(v_1 t_1) \ldots r(v_k t_k) \right).
\]

In the forthcoming works, these results will be completed by following directions

(1) To compare the solution of (14) satisfying asymptotic conditions, obtained by dévissage in \( H(\tilde{C}_n)\langle \langle T_n \rangle \rangle / J_{R_n} \) (see Theorem 3), with the one, obtained by Kohno-Drinfel’d decomposition.

(2) To study a sub algebra of noncommutative formal power series, over \( T_n \), containing \( S \) such that the following sum converges in (semi)-norm (see also Definition 3, Remarks 9 and 12)

\[
\langle F_{KZ_n}, S \rangle := \sum_{w \in T_n^*} \langle F_{KZ_n} | w \rangle \langle S | w \rangle
\]

and to provide a representation for such series \( S \) (see also Lemma 2).

(3) Other integrability criteria for (14), than (17), are also outlined in [54] and will examined by Proposition 6, Theorem 5, Corollaries 4–5.

(4) As already said in Section 1

(a) on the one hand, results in Section 3 is a generalization of results for controlled dynamical systems in [34] and for special functions in [42] then it could be great to get back to applications in controlled systems,

(b) on the other hand, Section 4 is an application of these results, in continuation of [25] (see Examples 1–3 in [25]) then it could be also great to get more applications in Physics.

\(^{46}\)Up to our knowledge, such solution is not provided yet.
6.1. **KZ\textsubscript{3}, the simplest non-trivial case.** With the notations given in Example 2, a solution of KZ\textsubscript{3} is explicit as $F = V_0 G$, where $V_0(z) = (z_1 - z_2)^{t_1/2\pi i}$ and, similarly as in Proposition 5, $G$ is expanded via Corollary 1 as follows

$$
G(z) = \sum_{m \geq 0} t_{i_1,j_1} \ldots t_{i_m,j_m} \in \{t_{1,3}, t_{2,3}\} \omega_{i_1,j_1}(s_1) \varphi^{s_1}(t_{i_1,j_1}) \ldots \int_0^z \omega_{i_m,j_m}(s_m) \varphi^{s_m}(t_{i_m,j_m}),
$$

where $\omega_{1,3}(z) = d \log(z_1 - z_3)$ and $\omega_{2,3}(z) = d \log(z_2 - z_3)$ and

$$
\varphi^z = e^{ad_{(t_{1,2}/2\pi i)\log(z_1 - z_2)}} = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2\pi i)^k k!} ad_{t_{1,2}}^k.
$$

One also has $\varphi^{(s_1)}(t_{i_1,j_1}) \ldots \varphi^{(s_m)}(t_{i_m,j_m}) = V_0(z)^{-1} \Delta(t_{i_1,j_1} \ldots t_{i_m,j_m}, s_1, \ldots, s_m)$. Moreover, Example 10 (equipping the ordering $t_{1,2} < t_{1,3} < t_{2,3}$), one has

$$
\varphi^z(t_{1,3}) = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2\pi i)^k k!} P_{t_{1,3}}^k, \quad \hat{\varphi}^z(t_{1,3}) = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2\pi i)^k k!} S_{t_{1,3}}^k,
$$

where $\hat{\varphi}$ is the adjoint to $\varphi$ and is defined by

$$
\hat{\varphi}^z = \sum_{k \geq 0} \frac{\log^k(z_1 - z_2)}{(-2\pi i)^k k!} t_{1,2} = e^{-(t_{1,2}/2\pi i) \log(z_1 - z_2)}.
$$

Hence, belonging to $\mathcal{H}(\tilde{\mathbb{C}}_+^3)\|\mathcal{T}_3\|$, $G$ satisfies $dG(z) = \tilde{\Omega}_2(z) G(z)$, where

$$
\tilde{\Omega}_2(z) = (\varphi^z(t_{1,3}) d \log(z_1 - z_3) + \varphi^z(t_{2,3}) d \log(z_2 - z_3))/2\pi i.
$$

In the affine plane $(P_{1,2}) : z_1 - z_2 = 1$, one has $d \log(z_1 - z_2) = 0$ and then $\varphi \equiv \text{Id}$. Changing $x_0 = t_{1,3}/2\pi i, x_1 = -t_{2,3}/2\pi i$ and setting $z_1 = 1, z_2 = 0, z_3 = s$, $dG(z) = \tilde{\Omega}_2(z) G(z)$ is similar to (115), i.e.

$$
\tilde{\Omega}_2(z) = \frac{1}{2\pi i} \left( t_{1,3} \frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3} \frac{d(z_2 - z_3)}{z_2 - z_3} \right) = x_1 \omega_1(s) + x_0 \omega_0(s),
$$

and admits the noncommutative generating series of polylogs as the actual solution satisfying the asymptotic conditions in (116). Thus, by L given in (125), and the homographic substitution $g : z_3 \mapsto (z_3 - z_2)/(z_1 - z_2)$, mapping $\{z_2, z_1\}$ to $\{0, 1\}$ (see Examples 12), a particular solution of KZ\textsubscript{3}, in $(P_{1,2})$, is $L_{\frac{z_3 - z_2}{z_1 - z_2}}^{(z_1 - z_2)}(t_{1,3} + t_{2,3} + t_{3,3})/2\pi i$.

To end with KZ\textsubscript{3}, by quadratic relations relations given in (18), one has $[t_{1,2} + t_{2,3} + t_{3,3}, t] = 0$, for $t \in \mathcal{T}_3$, meaning that $t$ commutes with $(z_1 - z_2)(t_{1,2} + t_{2,3} + t_{3,3})/2\pi i$ and then $(z_1 - z_2) \mathcal{A}(\mathcal{T}_3)$ commutes with $\mathcal{A}(\mathcal{T}_3)$. Thus, KZ\textsubscript{3} also admits

$$
(z_1 - z_2)^{(t_{1,2} + t_{2,3} + t_{3,3})/2\pi i} L_{\frac{z_3 - z_2}{z_1 - z_2}}^{(z_3 - z_2)}(z_3 - z_2) \mathcal{A}(\mathcal{T}_3)\|\mathcal{T}_3\| = e^{(t_{1,2} + t_{2,3} + t_{3,3})/2\pi i} log(z_1 - z_2),
$$

which is grouplike and independent on the variable $z_3 = s$, and then belongs to the differential Galois group of KZ\textsubscript{3}.

---

47 Generally, $s \mapsto (s - a)(c - b)(s - b)^{-1}(c - a)^{-1}$ maps the singularities $\{a, b, c\} \in \{0, +\infty, 1\}$. 48 Note also that these solutions could not be obtained by Picard’s iteration in Example 2, $(z_1 - z_2)^{(t_{1,2} + t_{2,3} + t_{3,3})/2\pi i}$, which is grouplike and independent on the variable $z_3 = s$, and then belongs to the differential Galois group of KZ\textsubscript{3}. 

---

**UNIVERSAL DIFFERENTIAL EQUATION BY NONCOMMUTATIVE P-V THEORY 41**
6.2. $KZ_4$, other simplest non-trivial case. For $n = 4$, one has $T_1 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$ and then $T_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$. Then, by Proposition 48

$$\varphi_{T_4}^{(c, z)} = e^{ad - \sum_{t \in T_4} \omega(t) \frac{\alpha_t}{\alpha_t}} \quad \text{and} \quad \forall t_{i,j} \in T_4, \varphi_{T_4}^{(c, z)}(t_{i,j}) = \varphi_{T_4}^{(c, z)}(t_{i,j}).$$

If $z_4 \to z_3$ then

$$F(z) = V_0(z)G(z_1, z_2, z_3), \quad \text{where} \quad V_0(z) = e^{\sum_{1 \leq i \leq 4} t_{i,4} \log(z_i - z_4)}$$

and $G(z_1, z_2, z_3)$ satisfies $dS = M_3^{s,4}S$ with

$$M_3^{s,4}(z) = \varphi_{t_{1,2}}^{(z, z)}(t_{1,2})d\log(z_1 - z_2) + \varphi_{t_{1,3}}^{(z, z)}(t_{1,3})d\log(z_1 - z_3)$$

$$+ \varphi_{t_{2,3}}^{(z, z)}(t_{2,3})d\log(z_2 - z_3).$$

In the intersection $(P_{1,3}) \cap (P_{2,3})$, one has $\log(z_1 - z_3) = \log(z_2 - z_3) = 0$ and $\varphi_{t_{1,2}} \equiv \text{Id}$ and then $M_3^{s,4}$ exactly coincides with $M_3$.

$$F = V_0G$$

is solution with $V_0(z) = (z_3 - z_4)^{t_{3,4}/2\pi^2}$ and similarly to Proposition 48

$$G(z) = \sum_{m \geq 0} \sum_{t_{i_1,j_1} \cdots t_{i_m,j_m} \in \{t_{1,2}, t_{1,3}, t_{2,3}, t_{3,4}, t_{2,4}\}^*} \int_0^z \left[ \omega_{i_1,j_1}(s_1)\varphi^{t_{i_1,j_1}}(t_{i_1,j_1}) \cdots \int_0^{s_{m-1}} \omega_{i_m,j_m}(s_m)\varphi^{t_{i_m,j_m}}(t_{i_m,j_m}) \right].$$

where $\omega_{i,j}(z) = d\log(z_i - z_j), 1 \leq i < j \leq 4$ and

$$\varphi^z = e^{ad - (t_{3,4}/2\pi^2)\log(z_3 - z_4)} = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} ad_{t_{3,4}}^k.$$

One also has $\varphi^{(c,s_1)}(t_{i_1,j_1}) \cdots \varphi^{(c,s_m)}(t_{i_m,j_m}) = V_0(z)^{-1}h_{t_{i_1,j_1} \cdots t_{i_m,j_m}}(z, s_1, \cdots, s_m).$

Moreover, by equipping the ordering $t_{1,2} \succ t_{1,3} \succ t_{2,3} \succ t_{1,4} \succ t_{2,4} \succ t_{3,4}$ in $[29]$ and $[50]$, one has

$$\varphi^z(t_{1,2}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} P_{t_{3,4}, t_{1,2}}^k, \quad \hat{\varphi}(t_{1,2}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} S_{t_{3,4}, t_{1,2}}^k,$$

$$\varphi^z(t_{1,3}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} P_{t_{3,4}, t_{1,3}}^k, \quad \hat{\varphi}(t_{1,3}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} S_{t_{3,4}, t_{1,3}}^k,$$

$$\varphi^z(t_{2,3}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} P_{t_{3,4}, t_{2,3}}^k, \quad \hat{\varphi}(t_{2,3}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} S_{t_{3,4}, t_{2,3}}^k,$$

$$\varphi^z(t_{1,4}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} P_{t_{3,4}, t_{1,4}}^k, \quad \hat{\varphi}(t_{1,4}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} S_{t_{3,4}, t_{1,4}}^k,$$

$$\varphi^z(t_{2,4}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} P_{t_{3,4}, t_{2,4}}^k, \quad \hat{\varphi}(t_{2,4}) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} S_{t_{3,4}, t_{2,4}}^k,$$

where $\hat{\varphi}$ is the adjoint to $\varphi$ and is defined by

$$\hat{\varphi}(c, z) = \sum_{k \geq 0} \frac{\log^k(z_3 - z_4)}{(-2i\pi)^k k!} t_{3,4}^k = e^{-(t_{3,4}/2\pi^2)\log(z_3 - z_4)}.$$
Hence, belonging to $\mathcal{H}(\tilde{C}_4^4)\langle\langle T_4\rangle\rangle$, $G$ satisfies $dG(z) = \Omega_4(z)G(z)$, where
\[
\Omega_4(z) = \frac{1}{2\pi i} \left( \phi^{(z)}(t_{1,2}) d\log(z_1 - z_2) + \phi^{\ast}(t_{1,3}) d\log(z_1 - z_3) \right. \\
+ \left. \phi^{(z)}(t_{2,3}) d\log(z_2 - z_3) + \phi^{(z)}(t_{1,4}) d\log(z_1 - z_4) \right.
\]
\[
\left. + \phi^{(z)}(t_{2,4}) d\log(z_2 - z_4). \right)
\]
In the affine plane $(P_{3,4}) : z_3 - z_4 = 1$, one has $\log(z_3 - z_4) = 0$ and then $\phi \equiv 1d$. By the cubic coordinate system on the moduli space $\mathcal{M}_{0,5}$ [6, 29] we can put $z_1 = xy, z_2 = y, z_3 = 1, z_4 = 0$, one has
\[
\Omega_3(xy, y, 1, 0) = \frac{1}{2\pi i} (t_{12} d\log(y(1 - x)) + t_{13} d\log(1 - xy) \\
+ \sum_{i<j} t_{14} d\log(x) + (t_{12} + t_{14} + t_{24}) d\log y).
\]
The differential equation
\[
dG(x, y) = \Omega_3(xy, y, 1, 0) G(x, y)
\]
admits the unique solution $G(x, y)$ [15] satisfying the asymptotic condition
\[
G(x, y) \sim (0, 0) x^{(2\pi i)^{-1} t_{1,4} y^{(2\pi i)^{-1} (t_{1,4} + t_{24})}}.
\]
Thus, by the homographic substitution
\[
g : \left\{ \begin{array}{c} z_1 \mapsto (z_1 - z_4)/(z_2 - z_4) \\
         z_2 \mapsto (z_2 - z_4)/(z_3 - z_4) \end{array} \right.,
\]
mapping $\{z_3, z_4\}$ to $\{1, 0\}$, a particular solution of $KZ_4$ is $G\left( \begin{array}{c} z_1 - z_4 \\
         z_2 - z_4 \\
         z_3 - z_4 \end{array} \right) (z_3 - z_4)^{(2\pi i)^{-1} \sum_{i<j} t_{i,j}}$. In $(P_{3,4})$. Now does $G\left( \begin{array}{c} z_1 - z_4 \\
         z_2 - z_4 \\
         z_3 - z_4 \end{array} \right)$ commute with $(z_3 - z_4)^{(2\pi i)^{-1} \sum_{i<j} t_{i,j}}$ and then $(z_3 - z_4)^{(2\pi i)^{-1} \sum_{i<j} t_{i,j}}$ commutes with $A(\langle\langle T_4\rangle\rangle)$. Thus, $KZ_4$ also admits $(z_3 - z_4)^{(2\pi i)^{-1} \sum_{i<j} t_{i,j}} G\left( \begin{array}{c} z_1 - z_4 \\
         z_2 - z_4 \\
         z_3 - z_4 \end{array} \right)$ as a particular solution in $(P_{3,4})$.

Acknowledgements. The authors would like to thank D. Barsky, G.H.E. Duchamp and B. Enriquez for fruitful interactions and improving suggestions and also J.Y. Enjalbert, G. Koshevoy, L. Pourrin and C. Tollu for discussions.

We also thank the anonymous reviewers for their constructive criticism and generous suggestions.

\footnote{\[ (z_3 - z_4)^{(2\pi i)^{-1} \sum_{i<j} t_{i,j}} = e^{(2\pi i)^{-1} \log(z_3 - z_4) \sum_{i<j} t_{i,j}}, \] which is grouplike and independent on the variables $z_1 = xy, z_2 = y$, and then belongs to the differential Galois group of $KZ_4$.}
References

1. J. Berstel & C. Reutenauer.– Rational series and their languages, Springer-Verlag, 1988.
2. V.C. Bui, G.H. E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, C. Tollu.– (Pure) transcendence bases in $\varphi$-deformed shuffle bialgebras, Journal Electronique du Séminaire Lotharingien de Combinatoire, 74 (2018).
3. V.C. Bui, V. Hoang Ngoc Minh, Q.H. Ngo and V. Nguyen Dinh.– On the solutions of universal differential equations, Journal of Physics: Conference Series 2667 (2023).
4. N. Bourbaki.– Lie Groups and Lie Algebras, Hermann Paris, France 1975.
5. F. Brown.– Périodes des espaces de modules $\mathbb{M}_{0,n}$ et valeurs zêtas multiples, thèse (2006).
6. P. Cartier.– Jacobienne généralisées, monodromie unipotente et intégrales itérées, Séminaire Bourbaki, 687 (1987), 31–52.
7. P. Cartier.– Développements récents sur les groupes de tresses. Applications à la topologie et à l’algèbre, Séminaire Bourbaki, 716 (1990), 17–57.
8. P. Cartier.– Fonctions polylogarithmes, nombres polyzetas et groupes pro-unipotents.– Séminaire BOURBAKI, 53ème, n°885, 2000-2001.
9. P. Cartier.– A Primer of Hopf Algebras, Frontiers in Number Theory, Physics, and Geometry II (2007).
10. S. Carr.– Multizeta values: Lie algebras and periods on $\mathbb{M}_{0,n}$, thèse (2008).
11. K.-T. Chen.– Iterated integrals and exponential homomorphisms, Proc. Lond. Mathem. Soc. (3) 4 (1954) 502–512.
12. P. Deligne.– Equations Différentielles à Points Singuliers Réguliers, Lecture Notes in Math, 163, Springer-Verlag (1970).
13. J. Delsarte.– Les rotations fonctionnelles, Annales de la Faculté des sciences de Toulouse : Mathématiques, Série 3, Tome 20 (1928), pp. 47-127.
14. M. Deneufchâtel, G.H.E. Duchamp, Hoang Ngoc Minh, A.I. Solomon.– Independence of hyperlogarithms over function fields via algebraic combinatorics, dans Lec. N. in Comp. Sc. (2011), Volume 6742/2011, 127-139.
15. J. Dixmier– Algèbres enveloppantes, Paris, Gauthier-Villars 1974.
16. V. Drinfel’d– Quantum groups, Proc. Int. Cong. Math., Berkeley, 1986.
17. V. Drinfel’d– Quasi-Hopf Algebras, Len. Math. J., 1, 1419-1457, 1990.
18. V. Drinfel’d– On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J., 4, 829-860, 1991.
19. G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, K. Penson, P. Simonnet.– Mathematical renormalization in quantum electrodynamics via noncommutative generating series, in Applications of Computer Algebra, Springer Proceedings in Mathematics and Statistics, pp. 59-100 (2017).
20. J.Y. Enjalbert, V. Hoang Ngoc Minh.– Analytic and combinatoric aspects of Hurwitz polyzetas, Journal de Théorie des Nombres de Bordeaux, Tome 19 (2007) no. 3, pp. 595-640.
21. P.I. Etingof, I. Frenkel, A.A. Kirillov.– Lectures on representation theory and Knizhnik-Zamolodchikov, American Mathematical Society (1998).
22. M. Fliess.– Fonctionnelles causales non linéaires et indéterminées non commutatives, Bull. Soc. Math. France, N°109, 1981, pp. 3-40.
23. M. Fliess.– Réalisation des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices, Inv. Math., 71, 1983, pp. 521-537.
24. G. Duchamp, D. Krob.– The free partially commutative Lie algebra: Bases and ranks, Advances in Mathematics, Volume 95, Issue 1, pp. 92-126 (1992).
25. G. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo, K. Penson, P. Simonnet.– Mathematical renormalization in quantum electrodynamics via noncommutative generating series, in Applications of Computer Algebra, Springer Proceedings in Mathematics and Statistics, pp. 59-100 (2017).
26. G. Duchamp, V. Hoang Ngoc Minh, V. Nguyen Dinh.– Towards a noncommutative Picard-Vessiot theory, In preparation, [arXiv:2008.10872]
27. Dyson.– F.J., The radiation theories of Tomonaga, Schwinger and Feynman, Physical Rev, vol 75, (1949), pp. 486-502.
28. Furusho, H.– Pentagon and hexagon equations, Ann. of Math., Vol. 171 (2010), No. 1, 545-556.
29. Furusho, H.– Double shuffle relation for associators, Ann. of Math., Vol. 174 (2011), No. 1, 341-360.
30. Jimbo, M.– Quantum R-matrix for the generalized Todda system, Comm. Math. Phys. 102 (1986) 537-547.
31. Jimbo, M.– Quantum R-matrix related to the generalized Todda lattice, Lect. Notes Phys., Springer Verlag 246 (1986).
32. J. Gonzalez-Lorca.– Série de Drinfel’d, monodromie et algèbres de Hecke, Ph. D., Ecole Normale Supérieure, Paris, 1998.
33. R. Hain.– Iterated integrals and mixed Hodge structures on homotopy groups, Lecture Notes in Math., 1246, Springer, Berlin, 1987, 75â€“83.
34. Hoang Ngoc Minh.– Contribution au développement d'outils informatiques pour résoudre des problèmes d'automatique non linéaire, Thèse, Lille, 1990.
35. V. Hoang Ngoc Minh, G. Jacob, N. Oussous.– Input/Output Behaviour of Nonlinear Control Systems : Rational Approximations, Nilpotent structural Approximations, in Analysis of controlled Dynamical Systems, Progress in Systems and Control Theory, Birkhäuser, 1991, pp. 253-262.
36. Hoang Ngoc Minh.– Fonctions de Dirichlet d'ordre n et de paramètre t, Discrete Math., 180, 1998, pp. 221-241.
37. V. Hoang Ngoc Minh, G. Jacob.– Symbolic Integration of meromorphic differential equation via Dirichlet functions, Disc. Math. 210, pp. 87-116, 2000.
38. V. Hoang Ngoc Minh & M. Petitot.– Lyndon words, polylogarithmic functions and the Riemann ζ function, Discrete Math., 217, 2000, pp. 273-292.
39. V. Hoang Ngoc Minh, M. Petitot and J. Van der Hoeven.– Polylogarithms and Shuffle Algebra, Proceedings of FPSAC’98, 1998.
40. V. Hoang Ngoc Minh.– Differential Galois groups and noncommutative generating series of polylogarithms, Automata, Combinatorics & Geometry, World Multi-conf. on Systemics, Cybernetics & Informatics, Florida, 2003.
41. V. Hoang Ngoc Minh.– Shuffle algebra and differential Galois group of colored polylogarithms, Automata, Combinatorics & Geometry, Nuclear Physics B - Proceedings Supplements, vol. 135, 2004.
42. V. Hoang Ngoc Minh.– Calcul symbolique non commutatif, Presse Académique Francophone, Saarbrücken 2014.
43. V. Hoang Ngoc Minh, On the solutions of universal differential equation with three singularities, in Confluentes Mathematici, Tome 11 (2019) no. 2, p. 25-64.
44. E. Jurisich & R. Wilson.– A Generalization of Lazard’s Elimination Theorem, Communications in Algebra , pp. 4037-4041 (2004).
45. Kolno T.– Monodromy representations of braid groups and Yang-Baxter equations, Annales de l’Institut Fourier, Tome 37 (1987) no. 4, p. 139-160
46. Kolno T.– Linear representations of braid groups and classical Yang-Baxter equations, in Joan S. Birman, Anatoly Libgober (eds.) Braids Cont. Math. 78 (1988), 329-363.
47. Kolno T.– Série de Poincaré-Koszul associée aux groupes de tresses pure Invent. math. 82, 57-75 (1985).
48. Kolno T.– Loop spaces of configuration spaces and finite type invariants in Invariants of knots and 3-manifolds, Geometry & Topology Monographs, Volume 4, (2002), pp. 143–160
49. Loday, J-L.– Cup-product for Leibniz cohomology and dual Leibniz algebras, Math. Scand., 77 (1995), no. 2, pp. 189-196.
50. M. Lazard.– Groupes, anneaux de Lie et problème de Burnside, In: Zappa, G. (eds) Gruppi, anelli di Lie e teoria della coomologia. C.I.M.E. Summer Schools, vol 20, Springer, Berlin, Heidelberg.
51. J.A. Lappo-Danilevsky.– Théorie des systèmes des équations différentielles linéaires, Chelsea, New York, 1953.
52. T.Q.T. Lê & J. Murakami.– Kontsevich’s integral for Kauffman polynomial, Nagoya Math., pp 39-65, 1996.
53. M. Lothaire.– Combinatorics on Words, Encyclopedia of Mathematics and its Applications, Addison-Wesley, 1983.
54. O. Mathieu.– Equations de Knizhnik-Zamolodchikov et théorie des représentations.– Séminaire BOURBAKI, 77ème, n°227, 1995, pp. 47-67.
55. H. Nakamura.– Demi-shuffle duals of Magnus polynomials in free associative algebra arXiv:2109.14070
56. S. Oi and K. Ueno.– *KZ equation on the moduli space* $\mathcal{M}_{0,5}$ *and harmonic product of multiple polylogarithms*, Proceedings of the London Math. Soc., Vol. 105 (2012), Issue 5, 983-1020.
57. S. Oi and K. Ueno.– *Connection problem of Knizhnik-Zamolodchikov equation on moduli space* $\mathcal{M}_{0,5}$, preprint (2011), arXiv:math.QA/1109.0715.
58. G. Racinet.– *Séries génératrices non-commutatives de polyzetas et associateurs de Drinfel’d*, thèse (2000).
59. D.E. Radford.– *A natural ring basis for the shuffle algebra and an application to group schemes*, Journal of Algebra, 58:432–454, 1979.
60. R. Ree.– *Lie elements and an algebra associated with shuffles*, Ann. Math 68 210–220, 1958.
61. Reutenauer C.– *Free Lie Algebras*, London Math. Soc. Monographs (1993).
62. H. Sussmann.– *A product expansion of the Chen series*, in Theory and Applications of Nonlinear Control Systems, C.I. Byrnes and A. Lindquist (editors), 1986.
63. T. Terasoma.– *Selberg integral and Multiple Zeta Values*, Compositio Mathematica 133 (2022), 1–24.
64. G. Viennot.– *Algèbres de Lie libres et monoïdes libres*, Lecture Notes in Mathematics, Springer-Verlag, 691, 1978.
65. S. Weinzierl.– *Feynman Integrals: A Comprehensive Treatment for Students and Researchers*, UNITEXT for Physics, Springer Nature (2022)

University of Sciences, Hue University, 77 - Nguyen Hue street - Hue city, Vietnam.

*Current address*: University of Sciences, Hue University, 77 - Nguyen Hue street - Hue city, Vietnam.

*Email address*: bvchien@hueuni.edu.vn

Université de Lille, 1 Place Déliot, 59024 Lille, France.

*Current address*: Université de Lille, 1 Place Déliot, 59024 Lille, France.

*Email address*: vincent.hoang-ngoc-minh@univ-lille.fr

Hanoi University of Science and Technology, 1 Dai Co Viet, Hai Ba Trung, Ha Noi, Viet Nam.

*Current address*: Hanoi University of Science and Technology, 1 Dai Co Viet, Hai Ba Trung, Ha Noi, Viet Nam.

*Email address*: hoan.ngoquoc@hust.edu.vn

University of Science and Technology of Hanoi, 18 Hoang Quoc Viet, Cau Giay, Ha Noi, Viet Nam.

*Current address*: University of Science and Technology of Hanoi, 18 Hoang Quoc Viet, Cau Giay, Ha Noi, Viet Nam.

*Email address*: nguyen-dinh.vu@usth.edu.vn