The effect of swirl on confined co-axial flow

G.P. Benham 1†, I.J. Hewitt 1, C.P. Please 1, and P.A.D. Bird 2

1Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG United Kingdom
2VerdErg Renewable Energy Limited, 6 Old London Rd, Kingston upon Thames KT2 6QF, United Kingdom

(Received xx; revised xx; accepted xx)

We study the problem of mixing between core and annular flow in a pipe, examining the effect of a swirling core flow. Such flows are important across a range of applications, including jet pumps, combustion chambers and aerospace engineering. Previous studies show that swirl can increase shear layer growth rates and, in the case of confining walls, reduce flow separation. However, the effect of swirl on pressure loss in a confined flow is uncertain. To address this, we develop a simplified model that approximates the axial flow profile as a linear shear layer separating uniform-velocity core and annular streams. The azimuthal flow profile is approximated as a solid body rotation within the core region, and a parabolic mixing profile within the shear layer. This model shows good agreement with computational turbulence modelling, whilst its simplicity and low computational cost make it ideal for benchmark predictions and design purposes. Using this model, we confirm that a swirling core is useful for increasing shear layer growth rates, but find that it is detrimental to pressure recovery. This has important implications for the design of diffusers that incorporate swirling flows. We use the model to describe the slow recirculation region that can form along the pipe axis for sufficiently large swirl, by approximating it as a stagnant zone with zero velocity. The criteria for the development of such a region are established in terms of the pipe expansion angle and inflow velocity profile.

1. Introduction

There are many industrial applications where swirl is added to jet flows and annular flows to improve performance. Most commonly, swirl is used in combustion chambers to increase fuel mixing rates and to stabilise the flame (Lilley 1977; Lee et al. 2005). There have also been studies which indicate that swirl can be used to improve plasma jet cutting performance (Gonzalez-Aguilar et al. 1999), to increase the efficiency of a jet pump (Guillaume & Judge 2004), and to reduce flow separation in diffusers (Fox & McDonald 1971). In addition, swirl is present in a variety of propulsion systems, such as in jet engines and turbomachinery, and plays an important role in the interaction between aircraft wakes (Bilanin et al. 1977). Our goal in this study is to examine the effect of swirl on mixing of confined co-axial flows.

Existing experimental studies of unconfined jets have shown that swirl (in the jet) can significantly increase entrainment rates and consequently the growth rate of the jet, both for compressible and incompressible fluids (Naughton et al. 1997; Gilchrist & Naughton 2005). Besides other experimental and numerical studies of unconfined jets (Leschziner & Rodi 1984; Gibson & Younis 1986; Elsner & Kurzak 1987; Farokhi et al. 1989; Liang

† Email address for correspondence: benham@maths.ox.ac.uk
there have also been some that consider the effect of swirl on annular flows and confined flows.

In the case of annular flows, Lee et al. (2005) studied the effect of swirl on the characteristics of an unconfined co-axial annular flow. In their experiments, both the core flow and the annular flow have swirl components, not necessarily in the same direction. They show that if the swirl is sufficiently strong, a stagnation point and recirculation region can develop inside the core flow. Other studies (Chigier & Chervinsky 1967; Escudier & Keller 1985; Champagne & Kromat 2000; Nayeri 2000) have discussed this recirculation region and relate it to the phenomenon of vortex breakdown. In the context of combustion chambers, the recirculation region is important because it has a stabilising effect on the flame. In this region, the flame speed and flow velocity are equalised due to the relatively low velocity. Both co-swirling and counter-swirling annular flows have been investigated experimentally (Ribeiro & Whitelaw 1980; Durbin & Ballal 1996; Gupta et al. 2001; Merkle et al. 2003) with the general conclusion that counter-swirl is more stabilising.

In the case of confined flows, there are a number of studies which consider the effect of swirl on the performance of a diffuser. The experiments of Fox & McDonald (1971) showed that by adding a solid body rotation to the inflow of a diffuser with flow separation, it is possible to reduce separation and increase pressure recovery significantly. By contrast, for unseparated flows, it was shown that swirl makes little difference to pressure recovery. The numerical simulations of Hah (1983) investigated the effects of a Rankine-type rotation and a solid-body rotation at the inlet of a diffuser, finding similar results to Fox & McDonald. Hah suggests that swirl reduces flow separation by means of the centrifugal force, which presses the boundary layer to the pipe wall. Both Fox & McDonald (1971) and Hah (1983) show that performance decreases for large swirl intensities, and attribute this behaviour to the formation of a recirculation region when the swirl number is large. So (1967) studied the recirculation region inside a conical diffuser in detail and presented a simple model based on integral equations of mass, axial momentum, angular momentum, and moment of axial momentum. However, there is a significant discrepancy between the model predictions and experimental results, which is attributed to the assumption of constant viscosity. The experiments of Nayeri (2000) study a shear layer between confined swirling co-axial flows. Both co-swirling and counter-swirling flows were investigated and it was found that, in both cases, the shear layer growth rates are larger than in the absence of swirl, though the counter-rotating case has the largest growth rates overall. The formation of a recirculation region is not studied. Nor is the effect of different pipe geometries.

In this study we present a simple model for confined co-axial flow with a swirling core, and we use the model to investigate the effect of swirl on the flow characteristics in a variety of pipe geometries. We study the effect of a swirling core on the pressure recovery in a diffuser, which is not addressed in the literature. In particular, we show that a swirling core is useful for producing a more uniform flow, whilst it is detrimental to pressure recovery. In accordance with previous experimental studies (Gilchrist & Naughton 2005), we find that the addition of swirl increases shear layer growth rates. We also confirm that under certain flow conditions, a recirculation region can form along the pipe axis, and we use our model to characterise the onset and size of this region in terms of the inflow conditions and the pipe expansion angle.

The model is based on a previous model for non-swirling shear layers (Benham et al. 2017a), and uses a similar approach to So (1967), where the governing equations are integral equations of mass, axial momentum and angular momentum.

The model is described in Section 2 and compared to computational turbulence
modelling in Section 3. In Section 4 we use the model to address the question of how swirl affects mixing and pressure loss in diffusers of different shapes, and we close with a summary in Section 5.

2. Mathematical model

In this section we describe the flow scenario we consider and present a simple model, which is an extension of the model presented by Benham et al. (2017a) for the case of no swirl. The model in that study considers an inflow composed of inner and outer streams of different speeds, with a velocity jump between them that evolves into a shear layer downstream. The model presented here addresses the same inflow profile for an axisymmetric pipe but with the inner (core) stream rotating. This is achieved by introducing an angular velocity component which is governed by equations derived from integrating conservation of angular momentum. Furthermore, we account for radial pressure gradients induced by the swirl as a consequence of the conservation of radial momentum. The shear layer growth equation is modified to account for the additional effect of the azimuthal shear stress induced by the swirl.

We also present an extended version of the model which addresses the development of a turbulent boundary layer near the pipe wall, and a symmetry boundary layer near the pipe axis. This extended model is discussed later in this section and in Appendix A.

The flow scenario is shown in Figure 1, in which we illustrate our coordinate system \((x, r)\). We consider axisymmetric flow in a long thin cylindrical pipe \(0 < r < h(x)\), where the rate of change of the pipe radius is small. The inflow is composed of a core swirling flow, with axial velocity \(U_2(0)\) and solid body rotation at angular velocity \(\Omega(0)\), and an annular non-swirling flow with axial velocity \(U_1(0)\). A shear layer forms between the flows and grows downstream, as illustrated by red dashed lines in Figure 1. Throughout this study, we restrict our attention to the case where the axial velocity of the core flow is smaller than that of the annular flow \(U_2 < U_1\). The opposite case is prone to asymmetric instabilities, such as the Coanda effect (Tritton 2012), which we do not attempt to model here. The flow scenario in this study is motivated by a hydropower application, in which the core swirling flow emerges from a turbine, and it is desired to mix this flow with the annular flow with minimal pressure loss.

We approximate the axial flow profile \(u_x\) by decomposing it into two uniform-velocity plug regions separated by a shear layer in which the velocity varies linearly between \(U_1(x)\) and \(U_2(x)\). Therefore, the axial velocity profile is approximated by

\[
  u_x(x, r) = \begin{cases} 
  U_2 & : 0 < r < h_2, \\
  U_2 + \varepsilon_r (r - h_2) & : h_2 < r < h - h_1, \\
  U_1 & : h - h_1 < r < h,
  \end{cases}
\]

where \(h_1(x)\) and \(h_2(x)\) are the widths of the plug regions, \(\delta = h - h_1 - h_2\) is the width of the shear layer, and \(\varepsilon_r = (U_1 - U_2)/\delta\) is the shear rate.

Similarly to the axial velocity, we approximate the azimuthal velocity profile by decomposing it into a core swirling region and an annular region with no swirl separated by an azimuthal shear layer in which the azimuthal velocity varies quadratically (motivated by CFD calculations which are discussed later). The approximate azimuthal velocity profile
Figure 1: Schematic diagram of our simple model for a swirling slower central flow mixing with a non-swirling faster outer flow, showing axial profiles of axial velocity $u_x$, azimuthal velocity $u_\theta$ and pressure $p$. We indicate the position of the shear layer with red dashed lines. The aspect ratios of the diagrams are exaggerated for illustration purposes.

is taken to be

$$u_\theta(x,r) = \begin{cases} \Omega r & : 0 < r < h_2, \\ (h - h_1 - r)(\frac{h_2 \Omega}{\delta} + \kappa(h_2 - r)) & : h_2 < r < h - h_1, \\ 0 & : h - h_1 < r < h, \end{cases} \quad (2.2)$$

where $\kappa(x)$ is the curvature of the velocity profile within the shear layer. From boundary layer theory, conservation of radial momentum indicates that radial pressure gradients must balance the centrifugal force from the swirling flow

$$\frac{\partial p}{\partial r} = \rho \frac{u_\theta^2}{r}, \quad (2.3)$$

where $\rho$ is the density, which is assumed to be constant. Therefore, from (2.2) and (2.3), the approximate pressure profile is

$$p(x,r) = \begin{cases} p_2 + P & : 0 < r < h_2, \\ p_1 + P & : h_2 < r < h - h_1, \\ P & : h - h_1 < r < h, \end{cases} \quad (2.4)$$

where $p_1(x,r)$ and $p_2(x,r)$ are lengthy functions (which are given in Appendix A) found by integrating (2.3) using the approximation (2.2), and $P(x)$ is the pressure in the non-swirling region of flow. Note that $p(x,r)$ is both continuous and once differentiable with respect to $r$.

In order to model the growth of the shear layer, we modify the model of Benham et al. (2017a) to account for the additional effect of azimuthal shear. Thus, we assume that the shear rate $\varepsilon_r$ decays according to

$$\frac{U_1 + U_2}{2} \frac{d\varepsilon_r}{dx} = -S_c \varepsilon_r^2 \frac{|u_1 - u_2|}{U_1 - U_2}, \quad (2.5)$$

where $u_1 = (U_1, 0)$ and $u_2 = (U_2, \Omega h_2)$ are the velocities at either side of the layer, and $S_c$ is an empirical spreading parameter which has been recorded to take values $S_c = 0.11 - 0.18$ (Benham et al. 2017b). Equation (2.5) can be derived from an entrainment argument using a two-dimensional analogy, which is detailed in Appendix A.

By invoking conservation of mass, axial momentum and angular momentum, we can derive equations describing how $U_1, U_2, h_1, h_2, \kappa$ and $P$ evolve. Integrated across the
pipe radius, conservation of mass and momentum equations are

\[ \frac{d}{dx} \int_0^h r u_x \, dr = 0, \quad (2.6) \]

\[ \rho \frac{d}{dx} \int_0^h r u_x \, dr + \int_0^h r \frac{\partial p}{\partial x} \, dr = h \tau_{wr}, \quad (2.7) \]

\[ \rho \frac{d}{dx} \int_0^h r^2 u_\theta u_x \, dr = 0, \quad (2.8) \]

where \( \tau_{wr} \) is the shear stress in the axial direction at the wall. We assume that the effect of shear stress in the azimuthal direction at the wall is small compared to that in the axial direction, so we do not include any stress terms on the right hand side of (2.8). Here, we parameterise the effect of the wall stress in the axial direction using an empirical friction factor \( f \), such that

\[ \tau_{wr} = -\frac{1}{8} f \rho U_1^2. \quad (2.9) \]

By choosing a parabolic profile for the azimuthal velocity (2.2) (instead of a linear profile), there is an additional degree of freedom \( \Omega(x) \) which is not determined by (2.8). This is accounted for by imposing conservation of angular momentum within the core region

\[ \rho \frac{d}{dx} \int_0^{h_2} r^2 u_\theta u_x \, dr = \rho \Omega h_2^2 \frac{d}{dx} \int_0^{h_2} r u_x \, dr. \quad (2.10) \]

We provide a derivation of (2.10), as well as (2.6) - (2.8), from the turbulent boundary layer equations in Appendix A.

Finally, we neglect energy dissipation within the plug regions, since it is small compared with that near the walls and in the shear layer. Therefore, we impose Bernoulli’s equation along streamlines in the annular plug region, and for the core region, since pressure varies radially, we only impose Bernoulli’s equation along the central non-swirling streamline, such that

\[ \frac{d}{dx} \left( P + \frac{1}{2} \rho U_1^2 \right) = 0, \quad \text{or} \quad h_1 = 0, \quad (2.11) \]

\[ \frac{d}{dx} \left( P + p_2(x,0) + \frac{1}{2} \rho U_2^2 \right) = 0, \quad \text{or} \quad h_2 = 0, \quad \text{or} \quad U_2 = 0, \quad (2.12) \]

where we ignore radial velocity components since they are small. Equations (2.11) and (2.12) are written in this form so that if either plug region is entrained by the shear layer \( (h_1 = 0 \text{ or } h_2 = 0) \), or if the core region slows to zero velocity \( (U_2 = 0) \), then the respective Bernoulli’s equation no longer holds downstream of that point. The latter modelling assumption is based on the observation that if the swirl intensity in the core region is sufficiently large, a slow-moving recirculation region can form along the pipe axis, as shown by Lee et al. (2005). A similar recirculation region has been observed for non-swirling flows if the flow is very non-uniform, or if the pipe expansion angle is too large (Benham et al., 2017b). In our model, we do not attempt to resolve the recirculation within such a region, but approximate it as a stagnant zone with zero axial velocity \( U_2 = 0 \) (see Figure 2). Later, in Section 3, we show that this approximation shows good agreement with CFD calculations.

To summarise, our model describes the development of the axial velocity \( u_x(x,r) \), given by (2.1), the azimuthal velocity \( u_\theta(x,r) \), given by (2.2), and the pressure \( p(x,r) \), given by (2.4), in a cylindrical pipe. Equations (2.5)-(2.12) govern the variables
Figure 2: Schematic diagram of a region of stagnation within a pipe. If the swirl in the core region is sufficiently strong, the flow is sufficiently non-uniform, or if the pipe expansion angle is sufficiently large, the slower core flow can reach zero axial velocity. The aspect ratio here is exaggerated for illustration purposes.

$U_1, U_2, \Omega, P, \delta, h_1, h_2, \varepsilon_r$ and $\kappa$, which are all functions of $x$. We can solve these equations given initial data at $x = 0$, numerical values for $f$ and $S_c$, and a channel shape $h(x)$. For all of the examples we consider here, the shear layer forms at $x = 0$, such that initial conditions for $\delta$ and $\varepsilon_r$ are taken as $\delta(0) = 0$ and $\varepsilon_r(0) = \infty$ (in practice, we solve for the reciprocal of the shear rate which satisfies $1/\varepsilon_r(0) = 0$). Furthermore, pressure is measured with respect to the reference pressure in the annular region at the origin $P(0) = 0$, without loss of generality. All other initial conditions form a set of parameters which are displayed in Table 1, where we write all dimensional parameters in terms of the velocity and length scalings $U_0 = U_1(0)$ and $h_0 = h(0)$. The non-dimensional swirl number (Naughton et al. 1997) of the core region, $S_w$, is defined as the ratio of angular momentum to axial momentum, measured at the inlet, and is given by

\[
S_w = \frac{\rho \int_0^{h_2} r^2 u_\theta u_x \, dr}{h_2 \int_0^{h_2} r \left( \rho u_x^2 + p \right) \, dr} \bigg|_{x=0} = \frac{2h_2(0)\Omega(0)U_2(0)}{h_2(0)^2\Omega(0)^2 - 4U_2(0)^2}. \tag{2.13}
\]

Our primary interest is understanding the effect of this swirl parameter on the flow development, which is discussed in Section 4.

As described earlier, we have also developed an extended version of this model, which accounts for the development of a turbulent boundary layer near the pipe wall and a symmetry boundary layer near the pipe axis (see Figure 3). Here we briefly summarise the extended model, whilst the complete details are given in Appendix A. For the turbulent boundary layer, following Schlichting et al. (1960), we introduce a $1/7$ power law for the axial velocity profile within a region of width $h_b$ near the pipe wall. This velocity profile satisfies the no-slip boundary condition at the wall and matches with the annular plug velocity, $U_1$, at the edge of the boundary layer. The width of the layer, $h_b(x)$, grows according to an equation that derives from integrating conservation of axial momentum over the boundary layer. The azimuthal velocity profile is given by a similar power law, satisfying the no-slip condition at the wall and matching with the azimuthal velocity at the edge of the boundary layer, which we denote $U_\theta(x)$. The velocity $U_\theta$ is assumed to be 0 until the shear layer reaches the edge of the boundary layer ($h_1 = 0$), at which point the swirling flow comes into contact with the boundary layer. In such situations, $U_\theta$ is determined by an equation which derives from conservation of angular momentum within the boundary layer.

In addition to the boundary layer at the wall, we also introduce a symmetry boundary layer near the pipe axis. The symmetry boundary layer arises in situations where the
shear layer entrains the slower plug region, such that \( h_2 = 0 \), and the axial velocity profile (2.1) ceases to satisfy the symmetry condition \( \partial u_x / \partial r = 0 \) at \( r = 0 \). In this case, we introduce a symmetry boundary layer of width \( h_0 \) near the pipe axis, which has a quadratic axial velocity profile that satisfies the symmetry condition at \( r = 0 \), whilst matching with the velocity at the edge of the shear layer \( U_2 \). The width of this layer, \( h_0(x) \), grows according to an equation that derives from conservation of axial momentum.

In summary, the extended model introduces two boundary layers, three extra variables, \( h_b, U_\theta \) and \( h_0 \), and three equations which govern these new variables. In the next section, we compare results from both the simple model and the extended model to results from a \( k-\epsilon \) turbulence model.

### 3. Comparison with a \( k-\epsilon \) turbulence model

In this section we compare results from our simple model and our extended model with those from a steady \( k-\epsilon \) turbulence model [Lauder & Spalding, 1974], using the open-source software package OpenFoam [Weller et al., 1998]. As examples, we first present comparisons of velocity and pressure profiles for co-axial flow in a straight pipe at swirl number \( S_w = 0.67 \). Then we compare predictions of the recirculation region that forms along the axis of an expanding pipe at the same swirl number. Finally, we compare pressure recovery \( C_p \) and the outlet kinetic energy flux profile factor \( K(L) \) over a range of swirl numbers.
Figure 4: Comparison of our swirl model and our extended model (solid black line) with a \( k-\epsilon \) RANS model (blue circles) for a swirling slower central flow \( (S_w = 0.67, U_2(0)/U_0 = 0.5) \) mixing with a non-swirling faster outer flow in a straight channel. Axial profiles are shown at evenly spaced locations for the axial velocity \( u_x \), azimuthal velocity \( u_\theta \) and pressure \( p \). The position of the shear layer, calculated using the simple model, is overlaid in red dashed lines. In the case of the extended model, the turbulent boundary layer and the symmetry boundary layer are also overlaid in purple dotted and green dash-dotted lines, respectively.

As an example geometry for our first comparison, we choose a straight axisymmetric pipe with non-dimensional length \( L/h_0 = 20 \). The computations are performed using an axisymmetric geometry formed of 30000 cells, with 300 elements in the \( x \) direction and 100 elements in the \( r \) direction. A convergence check was performed and it was found that this mesh resolution was sufficient for capturing the details of the flow. The inflow velocity ratio is \( U_2(0)/U_0 = 0.5 \), the inflow core region radius is given by \( h_2(0) = h_1(0) = h_0/2 \), and the swirl number is \( S_w = 0.67 \). We use the free-stream boundary conditions for the turbulence variables at the inlet, which are \( k = I^2 \times 3/2 |u|^2 \) and \( \epsilon = 0.09k^{3/2} \ell \), with turbulence intensity \( I = 5\% \) and mixing length \( \ell = 0.1h_0 \). All of the turbulence parameters in the \( k-\epsilon \) model are taken as their standard values, which are given by Launder & Spalding [1974].

If we take typical length and velocity scales as \( h_0 \) and \( U_0 \), and we consider the viscosity of water \( \nu = 10^{-6} \text{ m}^2/\text{s} \), then the Reynolds number for the inlet flow is \( Re = 10^6 \). For comparison with our simple model and our extended model, we estimate the friction factor with the Blasius relationship [Blasius 1913; McKeon et al. 2005] for flow in smooth pipes \( f = 0.316Re^{-1/4} \), giving a value of \( f = 0.01 \). For the spreading parameter \( S_c \), we find that \( S_c = 0.11 \) gives the best agreement, which falls within the range of reported values [Benham et al. 2017b]. We have also compared our model to other geometries and at different Reynolds numbers, though we do not display the results here, and we find that \( S_c = 0.11 \) is consistently an appropriate value for a good fit.

In Figure 4 we compare profiles of axial velocity \( u_x \), azimuthal velocity \( u_\theta \) and pressure \( p \) at evenly spaced locations along the channel. The position of the shear layer is indicated by dashed red lines. The comparison shows that there is good agreement between the \( k-\epsilon \) model and both our simple model and our extended model, though the extended
model has better agreement overall. In fact, the mean relative error between the simple model and the $k$-$\epsilon$ model is 9.5%, compared to 6% for the extended model. We can see that for both the simple model and the extended model, the width of the shear layer, the maximum and minimum velocities and the parabolic swirl profile (and consequently the pressure) in the shear layer match closely to $k$-$\epsilon$. Since our simple model does not explicitly resolve the development of boundary layers, the comparison is less close near the pipe walls. However, pressure variations in the axial direction, which are controlled by $f$, agree well (see Figure 4), indicating that the effect of the boundary layers is captured by the simple model using the friction factor parameterisation. Furthermore, we can see that the comparison is also less close downstream, near $r = 0$, since the simple model does not satisfy the symmetry condition, $\partial u_x / \partial r = 0$ at $r = 0$, when the shear layer comes into contact with the pipe axis. In the case of the extended model, the development of the turbulent boundary layer, which is shown by a purple dotted line, matches closely with $k$-$\epsilon$, as does the development of the symmetry boundary layer, which is shown by a green dash-dotted line.

The extended model has roughly the same computational cost as the original swirl model, though it contains more variables and equations, and it is perhaps more difficult to implement. Therefore, the original model serves as a convenient tool for benchmark predictions, whilst the extended model is appropriate for more detailed and accurate calculations of flow development, which take longer to implement.

Next, we compare predictions of the stagnation region which can form along the pipe axis. In Figure 5 an example of a stagnation region is shown for an expanding pipe with expansion angle $1.4^\circ$ and non-dimensional length $L/h_0 = 20$. The swirl number is $S_w = 0.67$ and the inlet profile is given by $U_2(0)/U_0 = 0.5$, $h_1(0) = h_2(0) = h_0/2$. Velocity colour plots and streamlines are displayed for both the simple model and the $k$-$\epsilon$ model, where in the case of the streamlines, the radial velocity is calculated from the axial velocity,

$$u_r = -\frac{1}{r} \int_0^r r \frac{\partial u_x}{\partial x} dr.$$  \hspace{1cm} (3.1)

The size and position of the stagnation region show close agreement, indicating that the simple model has good predictive capabilities. Various studies (Chigier & Chervinsky 1967; Escudier & Keller 1985; Champagne & Kromat 2000; Nayeri 2000) relate this region to the phenomenon of vortex breakdown. In our model, it can be explained by Equations (2.3), (2.5) and (2.12). From (2.5) it is apparent that the presence of swirl induces a greater mixing rate between the flows. Consequently, due to (2.3), as the swirl profile spreads out, the pressure along the pipe axis $p_2(x,0)$ rises over a shorter distance. Hence, due to Bernoulli’s equation (2.12), the speed of the core region must drop over a shorter distance. If the effect is sufficiently pronounced then the core region will stagnate completely.

As a final comparison between our simple model and CFD calculations, we make use of two commonly used measures of flow performance: the pressure recovery coefficient $C_p$ and the kinetic energy flux profile factor $K$ (Blevins 1984). The pressure recovery coefficient $C_p$ is a measure of the amount of kinetic energy in a flow which is converted into static pressure. There are various ways to define $C_p$, but we use the so-called mass-averaged pressure recovery (Filipenco et al. 1998), which is defined as

$$C_p = \frac{\int_0^h ru_x p dr|_{x=L} - \int_0^h ru_x p dr|_{x=0}}{\int_0^h \frac{1}{2} pru_x |u|^2 dr|_{x=0}},$$  \hspace{1cm} (3.2)

and takes values between $-\infty$ and 1, where $C_p = 1$ if all the inlet kinetic energy is
Figure 5: Comparison of the stagnation region in an expanding pipe with expansion angle 1.4°, displaying velocity colour plots and streamlines, calculated using our simple model and a k-\(\epsilon\) model. The swirl parameter is \(S_w = 0.67\) and the inlet profile is given by \(U_2(0)/U_0 = 0.5\), \(h_1(0) = h_2(0) = h_0/2\).

converted into static pressure. The kinetic energy flux profile factor \(K(x)\) is a measure of how non-uniform the axial velocity profile of a flow is, at a given position \(x\). For the purpose of this study, we consider \(K\) at the pipe outlet, which is given by

\[
K(L) = \frac{\frac{2}{\pi} \int_0^h r u_r^2 \, dr}{\left( \frac{2}{\pi} \int_0^h r u_x \, dr \right)^3} \bigg|_{x=L},
\]

and takes values between 1 and \(\infty\), where \(K(L) = 1\) corresponds to a uniform outflow. For the straight channel in Figure 4, we compare calculations of both \(C_p\) and \(K(L)\) using our simple model and the \(k-\epsilon\) model for values of the swirl parameter between \(S_w = 0\) and \(S_w = 1.71\). The results are plotted in Figure 6, where in the case of \(K(L)\), we also plot the inlet kinetic energy flux profile factor, \(K(0)\), for reference. Overall, there is good agreement, and both models show that increased swirl has the effect of reducing pressure recovery whilst making the flow slightly more uniform. The latter conclusion is consistent with other studies (Gilchrist & Naughton 2005), which show that swirl can increase shear layer growth rates. The former conclusion can be interpreted in the following way: Whilst swirl may mix the flows over a shorter length, it results in greater energy dissipation in doing so, which manifests in a loss of pressure.

Although we do not display the results here, we have also compared our simple model to other turbulence models, such as the \(k-\omega\) Shear Stress Transport (SST) model (Menter 1994), and in other pipe geometries, and we find similar close agreement. Furthermore, in Appendix B, we compare our model to some experimental data from other studies of an unconfined swirling jet. These comparisons give us confidence that our simple model has good predictive capabilities, whilst being computationally cheap. Therefore it is useful for examining the effect of swirl in a wide range of flow situations and exploring design spaces.

4. Model predictions

In this section we use our simple model to analyse the effect of swirl on different flow situations. First, we look at the effect on the behaviour of the shear layer, determining the distance downstream at which the shear layer has reached across the entire pipe
Figure 6: Comparison of the pressure recovery coefficient $C_p$ and the kinetic energy flux profile factor $K$ calculated using both our simple model and a $k$-$\epsilon$ model, where we vary the swirl number $S_w$. The kinetic energy flux profile factor is measured both at the outlet $K(L)$ and at the inlet $K(0)$, for reference. In these calculations, we consider the same flow geometry as in Figure 4. (this information is important for problems involving momentum transfer). Secondly, we establish which swirl numbers and pipe geometries result in a recirculation region forming along the pipe axis. Finally, we use our simple model to optimise the shape of a pipe for different swirl numbers, using both $C_p$ and $K(L)$ as design objectives.

4.1. Shear layer behaviour

One useful way to characterise the shear layer behaviour is by looking at the growth rate. However, since shear layer growth rates are not constant, this requires choosing a length scale over which one can take an average of the growth rate. The choice of this length scale is slightly arbitrary and will have a significant effect on growth rate measurements [Gilchrist & Naughton 2005]. In situations where the flow is confined, there is a more natural way to measure the shear layer growth rate, which is the distance it takes for the shear layer to reach across the pipe. Since the shear layer has an inner and an outer edge, which are not necessarily symmetric, it is useful to consider where each of these edges reaches their respective boundaries. The outer edge of the shear layer must reach the pipe wall $r = h$ at some distance $x = L_1$, whereas the inner edge of the shear layer reaches the pipe axis $r = 0$ at $x = L_2$ (see Figure 7(a)).

In Figure 7 we display colour plots of non-dimensional axial velocity $u_x/U_0$, and plots of the pressure recovery coefficient $C_p$, in both a straight channel and a widening channel (with expansion angle 1.4°), for different values of the swirl number. For this example we choose a channel with non-dimensional length $L/h_0 = 20$ and an inflow which is defined by $U_2(0)/U_0 = 0.5$ and $h_1(0) = h_2(0) = h_0/2$. By observation, it is apparent that the shear layer growth rate, which is non-linear, increases with swirl number. The corresponding values of $L_1$ and $L_2$ are plotted in Figure 8(a). We can clearly see that for both the straight channel and the widening channel, $L_1$ and $L_2$ decrease as the swirl number increases, confirming that shear layer growth rates are enhanced by swirl. We can also see that the values of $L_1$ and $L_2$ are smaller for the widening case, indicating that pipe expansion is another mechanism for enhanced shear layer growth rates.

4.2. Stagnation region

As is discussed by Lee et al. (2005), flows with large swirl numbers have the tendency to form a region of slowly moving recirculation. Here, we use our simple model to
characterise the onset and size of this region by approximating it as a stagnant zone with zero velocity, as described earlier.

In the examples in Figure 7 we can see a stagnation region forming for swirl number $S_w = 1.71$ in the straight pipe case, and for all displayed values of $S_w$ in the widening pipe case. In each case where the stagnation region forms, it then closes again some distance downstream. The results from our simple model indicate that the size of the stagnation region increases with $S_w$. Furthermore, it is evident that expanding pipes are more susceptible to stagnation than straight pipes. In the plots of pressure recovery $C_p$, given by (3.2), is plotted as a function of $x/h_0$ for each case. The distances, $L_1$ and $L_2$, at which the annular and core regions are entrained by the shear layer, respectively, are illustrated in (a).

Figure 7: The effect of increasing the swirl number $S_w$ between 0.27 and 1.71 on the axial velocity $u_x$ in the case of a straight channel (a, c, e, g) and a widening channel (b, d, f, h). The position of the shear layer is indicated in dashed black lines. Pressure recovery $C_p$, given by (3.2), is plotted as a function of $x/h_0$ for each case. The distances, $L_1$ and $L_2$, at which the annular and core regions are entrained by the shear layer, respectively, are illustrated in (a).
Figure 8: Effect of swirl on flow development properties. (a) $L_1/h_0$ and $L_2/h_0$ are the non-dimensional distances downstream at which the slower core and faster annular regions are completely entrained by the shear layer (see Figure 7 (a)). The inflow for this example is defined by $U_2(0)/U_0 = 0.5$ and $h_1(0) = h_2(0) = h_0/2$. (b) The minimum diffuser angle $\alpha_{\text{crit}}$ at which flow in a pipe with linearly expanding walls stagnates, for different values of the inlet velocity ratio $U_2(0)/U_0$ and swirl parameter $S_w$. The non-dimensional distance downstream at which stagnation first occurs $L_{\text{crit}}/h_0$ is indicated by the colour bar.

of the data points that demarcate the boundary between stagnation and no-stagnation.

We can see that the flow is less susceptible to stagnation for velocity ratios closer to 1, since the critical expansion angle $\alpha_{\text{crit}}$ (for which stagnation occurs) is larger, and the critical distance $L_{\text{crit}}/h_0$ is also larger.

The effect of increasing the swirl number is interesting and possibly counter-intuitive. For velocity ratios less than around $U_2(0)/U_0 \approx 0.7$, increasing $S_w$ causes a reduction in $\alpha_{\text{crit}}$, thereby increasing the range of parameter values for which stagnation occurs. This is expected and consistent with the findings of Lee et al. (2005). However, an unexpected result is that for velocity ratios close to 1, increasing $S_w$ reduces the range of parameter values for which stagnation occurs.

4.3. Diffuser shape optimisation

As a final investigation, we consider a shape optimisation problem in the presence of swirl. In order to illustrate the different effects of swirl, we consider optimising both $C_p$ and $K(L)$. Benham et al. (2017b) used a similar approach to maximise pressure recovery in a flow diffuser (in the absence of swirl) by manipulating the channel shape. In that study, a low-dimensional parameterisation of the shape is proposed, which consists of piecewise linear sections. Here, we make use of the same parameterisation, such that the pipe shape takes the form

$$h(x) = \begin{cases} h_0 : & 0 < x < x_1, \\ h_0 + \tan \alpha(x - x_1) : & x_1 < x < x_2, \\ h_L : & x_2 < x < L, \end{cases}$$

(4.1)

where $x_1$, $x_2$ are the shape division points, $L$ is the diffuser length, $h_0$, $h_L$ are the inlet and outlet radii, and $\alpha = \tan^{-1}(h_L - h_0)/(x_2 - x_1)$ is the expansion angle (see Figure 9 (a)). We consider $h_0$, $h_L$, and $L$ fixed, whilst we consider $x_1$ and $x_2$ as decision variables. For this class of shapes, we use our model to calculate both $C_p$ and $K(L)$ for all possible values of $x_1$ and $x_2$. The question of interest here is to what extent the
addition of swirl to the inlet flow is beneficial for achieving mixing and pressure recovery within such a diffuser. In each case, we choose a pipe which has non-dimensional length \( L/h_0 = 30 \) and expansion ratio \( h_L/h_0 = 1.5 \). The inflow is defined by \( U_2(0)/U_0 = 0.5 \) and \( h_1(0) = h_2(0) = h_0/2 \). We exclude values of \( x_1 \) and \( x_2 \) which produce a pipe shape with an expansion angle larger than 3.5° because it is expected that boundary layer separation can occur in such situations (Blevins 1984), which is not captured by our model.

In Figure 9 we display contour plots of pressure recovery \( C_p \) as a function of \( x_1 \) and \( x_2 \), for different values of the swirl parameter \( S_w = 0, 0.27, 0.67, 1.71 \). For each value of \( S_w \), we illustrate the optimum point with a black asterisk, and in Table 2 we display a table of the optimum points. As swirl increases, the optimum point changes slightly, with \( x_1 \) increasing and \( x_2 \) decreasing. This indicates that diffusers with swirling core flows should be designed with a longer initial straight section and a shorter expanding section (with a wider expansion angle) than for non-swirling flows. However, the value of \( C_p \) at the optimum point decreases with \( S_w \), suggesting that a swirling core is always detrimental to pressure recovery, which is consistent with the results from Figure 6 (a). We have also calculated \( C_p \) for other values of \( S_w \) and the maximum value indeed occurs at \( S_w = 0 \) (see Figure 9 (c)). Therefore, whilst swirl may improve pressure recovery when the flow is uniform, by reducing boundary layer separation (Fox & McDonald 1971; Hah 1983), it has the opposite effect on diffuser performance for non-uniform flow. It should be noted that our simple model does not account for boundary layer separation and, therefore, it cannot capture the positive effect of swirl on diffuser performance that is reported in
the literature. However, since we restrict our attention to diffusers with small expansion angles, $\alpha < 3.5^\circ$, there is little risk of boundary layer separation \(^{(\text{Blevins 1984})}\).

Similarly, we have calculated $K(L)$ as a function of $x_1$ and $x_2$ for different values of $S_w$. We do not show the corresponding contour plots here, but we display the minimum value of $K(L)$ for each value of $S_w$ in Figure 9 (f), and a table of the optimum points in Table 2. We find that the optimum shape is independent of swirl, having no initial straight section ($x_1 = 0$) and a very short widening section (with a large expansion angle). However, as $S_w$ increases the minimum value of $K(L)$ decreases, indicating that swirl enables the optimal shape to produce a more uniform outflow, which is consistent with the results from Figure 6 (b). We can conclude from these results that, whilst swirl makes the flow more uniform, it produces greater pressure losses in doing so.

## 5. Summary and concluding remarks

We have developed a simple model for confined co-axial flow with a core swirling region, which shows good agreement with CFD, whilst being computationally cheap. The model depends on two parameters, the friction factor $f$, which is well approximated with the Blasius relationship \(^{(\text{Blasius 1913, McKeon et al. 2005})}\), and the spreading parameter $S_c$, which is fitted to CFD calculations, giving a value $S_c = 0.11$ which is within the range of reported values \(^{(\text{Benham et al. 2017a})}\).

The model is useful for predicting essential aspects of the flow behaviour, such as the shear layer growth and pressure recovery. We use this model to characterise the criteria for which a recirculation region appears along the pipe axis, and at what distance downstream. We show that, whilst adding swirl to the core region is a good mechanism for increasing shear layer growth rates, it also increases pressure losses. Therefore, we conclude that a swirling core is advantageous for situations where mixing the flow over a short length scale is important, such as in combustion chambers, whereas it is disadvantageous for situations where pressure recovery is important, such as in flow diffusers.

We show that an extended version of the model, which includes a turbulent boundary layer at the pipe wall and a symmetry boundary layer at the pipe axis, shows even better agreement with CFD calculations than the simple model. Whilst unnecessary for the conclusions made using the simple model, the extended model provides more detailed and realistic flow predictions.

For future work, the effect of different swirl distributions for the core flow, such as a $q$ vortex \(^{(\text{Gilchrist & Naughton 2005})}\), could be investigated. Furthermore, this work

| $S_w$ | $x_1$  | $x_2$  | $C_p$  | $x_1$  | $x_2$  | $K(L)$  |
|-------|--------|--------|--------|--------|--------|--------|
| 0.27  | 3.4790 | 19.5918| 0.6695 | 0  | 4.2857 | 1.0284 |
| 0.67  | 4.8980 | 21.4286| 0.6511 | 0  | 4.2857 | 1.0274 |
| 1.71  | 4.8980 | 22.6531| 0.6196 | 0  | 4.2857 | 1.0264 |

Table 2: List of optimum shape configurations for different swirl numbers. The left hand table corresponds to the maximisation of the pressure recovery $C_p$, whereas the right hand table corresponds to the minimisation of the outlet kinetic energy flux profile factor $K(L)$. 

---

\(^{(\text{Blevins 1984})}\)
could be extended to situations in which the outer annular flow is also rotating, not necessarily in the same direction as the core flow.

This publication is based on work supported by the EPSRC Centre for Doctoral Training in Industrially Focused Mathematical Modelling (EP/L015803/1) in collaboration with VerdErg Renewable Energy Limited and inspired by their novel Venturi-Enhanced Turbine Technology for low-head hydropower.

Appendix A.

A.1. Shear layer growth

In this section we provide a derivation of Equation (2.5) based upon an entrainment assumption, using a two-dimensional analogy of the flow scenario. In the two-dimensional analogy, we replace the radial coordinate \( r \) with the two-dimensional coordinate \( y \), in each of the velocity profiles given by Equations (2.1) and (2.2). The azimuthal velocity \( u_\theta \) becomes the ‘transverse’ velocity \( v \) and the pressure gradients in the \( y \) direction vanish \( \partial p/\partial y = 0 \) (Schlichting et al. 1960).

We hypothesise that the shear layer entrains fluid from each of the plug regions. Similarly to a Morton-Taylor-Turner jet (Morton et al. 1956), we assume that the rate of entrainment is proportional to the magnitude of the velocity difference between the two plug regions \( E = 1/2S_c|u_1 - u_2| \), where \( S_c \) is an entrainment constant. Thus, the conservation of mass equations for each of the plug regions and the shear layer, are

\[
\frac{d}{dx} (U_1 h_1) = -\frac{1}{2} S_c |u_1 - u_2|, \quad (A 1)
\]

\[
\frac{d}{dx} (U_2 h_2) = -\frac{1}{2} S_c |u_1 - u_2|, \quad (A 2)
\]

\[
\frac{d}{dx} \left( \frac{1}{2} (U_1 + U_2) \delta \right) = S_c |u_1 - u_2|. \quad (A 3)
\]

In the case of a free shear layer with no transverse velocity component, such that \( v = 0 \) and \( U_1, U_2 \) are constant, Equation (A 3) is equivalent to the classic free shear layer equation

\[
\frac{d \delta}{dx} = 2S_c \frac{U_1 - U_2}{U_1 + U_2}. \quad (A 4)
\]

which is discussed by Schlichting et al. (1960). Using the definition of the shear rate within the shear layer, \( \varepsilon_y = (U_1 - U_2)/\delta \), and Equation (A 3), we find that

\[
\frac{U_1 + U_2}{2} \frac{d \varepsilon_y}{dx} = -S_c \varepsilon_y^2 |u_1 - u_2| + \frac{1}{\delta} \left( U_1 \frac{dU_1}{dx} - U_2 \frac{dU_2}{dx} \right). \quad (A 5)
\]

If we assume that Bernoulli’s equation (2.11) - (2.12) holds in each plug region (where \( p_2(x, 0) = 0 \) in the two-dimensional analogy) then we have

\[
\frac{d}{dx} \left( \frac{1}{2} U_1^2 - \frac{1}{2} U_2^2 \right) = 0. \quad (A 6)
\]

Therefore, combining (A 5) and (A 6), we arrive at the result

\[
\frac{U_1 + U_2}{2} \frac{d \varepsilon_y}{dx} = -S_c \varepsilon_y^2 \frac{|u_1 - u_2|}{U_1 - U_2}, \quad (A 7)
\]

which is equivalent to (2.5) if we replace \( \varepsilon_y \) with \( \varepsilon_r \), and \( v \) with \( u_\theta \).
A.2. Integrated mass and momentum equations

Consider incompressible flow in a cylindrical channel defined by $0 \leq x \leq L$, $0 \leq r \leq h(x)$, where $h \ll L$. Because the domain is long and thin, the boundary layer approximation to the Navier-Stokes equations is appropriate for modelling the flow in the whole domain (Schlichting et al. 1960). The axisymmetric turbulent boundary layer equations for the time-averaged velocities $(u_x, u_\theta, u_r)$ and pressure $p$, in cylindrical polar coordinates, are

\begin{align}
\frac{\partial u_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (ru_r) &= 0, \quad (A \, 8) \\
\frac{u_x}{r} \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left((\nu + \nu_t) r \frac{\partial u_x}{\partial r}\right), \quad (A \, 9) \\
\frac{u^2_\theta}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r}, \quad (A \, 10) \\
\frac{u_x}{r} \frac{\partial u_\theta}{\partial x} + u_r \frac{\partial}{\partial r} (ru_\theta) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left((\nu + \nu_t) r^2 \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right), \quad (A \, 11)
\end{align}

where $\rho$ is the density, $\nu$ is the viscosity and $\nu_t$ is the eddy viscosity. We impose the no-slip condition on the pipe wall $r = h(x)$, and the symmetry condition on the pipe axis $r = 0$, such that

\begin{align}
u_x = u_r = u_\theta &= 0, \quad \text{on} \quad r = h(x), \quad (A \, 12) \\
\frac{\partial u_x}{\partial r} = 0, \quad \text{and} \quad u_r, u_\theta &= \text{finite}, \quad \text{on} \quad r = 0. \quad (A \, 13)
\end{align}

Now we integrate Equations (A 8), (A 9) and (A 11) across the channel in the $r$ direction, using boundary conditions (A 12) - (A 13), to give

\begin{align}
d \frac{d}{dx} \int_0^h r u_x \, dr &= 0, \quad (A \, 14) \\
\rho d \frac{d}{dx} \int_0^h r u_x^2 \, dr &= -\int_0^h r \frac{\partial p}{\partial x} \, dr + h \tau_{w_r}, \quad (A \, 15) \\
\rho d \frac{d}{dx} \int_0^h r^2 u_\theta u_x \, dr &= h^2 \tau_{w_\theta}, \quad (A \, 16)
\end{align}

where the wall stress terms are

\begin{align}
\tau_{w_r} &= \left[\rho (\nu + \nu_t) \frac{\partial u_x}{\partial r}\right]_{y=h}, \quad (A \, 17) \\
\tau_{w_\theta} &= \left[\rho (\nu + \nu_t) \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right]_{y=h}. \quad (A \, 18)
\end{align}

We now approximate the solutions to these equations by decomposing the flow into an ‘outer flow’, comprising an annular non-swirling flow and a swirling core flow separated by a shear layer, as described in the main text, and narrow boundary layers adjacent to the walls that allow the no-slip conditions to be satisfied. With the exception of the extended model, which is discussed in Appendix A.4, the boundary layers are not resolved explicitly but their effect is parameterised by writing the wall stress terms (A 17) - (A 18) in terms of the outer velocity near the wall and a friction factor. Thus, the bulk of the flow profile is taken to be that given by (2.1), (2.2) and (2.4). Furthermore, since there are both axial and azimuthal components to the ‘outer’ velocity $u_{outer}$, the stress vector...
\textbf{Bourget \& Marichal 1990} takes the form

$$\tau_w = (\tau_{w_1}, \tau_{w_2}) = -\frac{1}{8} \rho |u_{outer}| u_{outer}.$$  \hfill (A 19)

However, from (2.1), the azimuthal component of the outer velocity is 0. Therefore, the wall stress terms on the right hand side of (A 15) and (A 16) are replaced with

$$h\tau_{w_1} = -\frac{1}{8} \rho U_1^2 h,$$  \hfill (A 20)

$$h^2 \tau_{w_2} = 0,$$  \hfill (A 21)

where $f$ is an empirical friction factor. In Appendix A.4 we revise some of these assumptions by including a boundary layer at the pipe wall, and including a generalised wall stress vector which contains both axial and azimuthal components.

Finally, Equation (2.10) can be derived by integrating (A 8) and (A 11) from $r = 0$ to $r = h_2$, using the symmetry boundary condition (A 13), and noting that $\partial u_\theta / \partial r - u_\theta / r = 0$ at $r = h_2$. Therefore, we find

$$\frac{d}{dx} \int_0^{h_2} ru_x dr - h_2 U_2 \frac{dh_2}{dx} + h_2 u_r |_{r=h_2} = 0,$$  \hfill (A 22)

$$\frac{d}{dx} \int_0^{h_2} r^2 u_\theta u_x dr - h_2^2 U_2 \frac{dh_2}{dx} + h_2^2 \Omega u_r |_{r=h_2} = 0.$$  \hfill (A 23)

By combining (A 22) and (A 23), we eliminate $u_r |_{r=h_2}$ to give the result

$$\frac{d}{dx} \int_0^{h_2} r^2 u_\theta u_x dr = h_2^2 \Omega \frac{d}{dx} \int_0^{h_2} ru_x dr.$$  \hfill (A 24)

\section*{A.3. Pressure terms}

The pressure terms $p_1$ and $p_2$ in (2.4) are found by integrating (2.3), using the azimuthal velocity profile (2.2). The result is

$$p_1 = P + \frac{1}{12 \delta^2} \left( \delta^2 \kappa^2 (h_2 - r + \delta)(25h_2^3 + h_2(13r - 5\delta)(r - \delta) \right.$$

$$- (r - \delta)^2(3r + \delta) + h_2^2(-23r + 31\delta)) + 4\kappa h_2 \delta \Omega (h_2 - r + \delta)(11h_2^2$$

$$+ 2(r - \delta)^2 + h_2(13\delta - 7r)) + 6h_2^2 \Omega^2 (h_2 - r + \delta)(3h_2 - r + 3\delta)$$

$$+ 12h_2^2(h_2 + \delta)^2(\kappa \delta + \Omega)^2 \log \frac{h_2}{h_2 + \delta} \right),$$  \hfill (A 25)

$$p_2 = P + \frac{1}{12 \delta^2} \left( \kappa^2 \delta^2 (2h_2 + \delta)(6h_2^2 + 6h_2 \delta - \delta^2) \right.$$

$$+ 4\kappa h_2 \delta \Omega (6h_2^2 + 9h_2 \delta + 2\delta^2) + 6\Omega^2 \delta (2h_2^3 + 2h_2^2 \delta + r^2 \delta)$$

$$+ 12h_2^2(h_2 + \delta)^2(\kappa \delta + \Omega)^2 \log \frac{h_2}{h_2 - \delta} \right).$$  \hfill (A 26)

\section*{A.4. Extended model}

We have already compared our simple model to a $k-\epsilon$ turbulence model in Section 3 and found good agreement. However, the agreement was less good near the pipe walls (see Figure 4), due to the fact that our model does not account for boundary layer development. Furthermore, when the core region is entrained by the shear layer (i.e. $h_2 = 0$), we also see less good agreement due to the fact that the velocity profile, given
by (2.1), does not satisfy the symmetry condition $\partial u_x/\partial r = 0$ at $r = 0$. In this section we propose two extensions to our model which address these discrepancies. These extensions, although they are not necessary to draw the conclusions we have made previously, are especially useful for more detailed flow predictions. By including these extensions we introduce several new variables, which we model using equations which derive from integrating the turbulent boundary layer equations (A8) - (A11) over the width of the boundary layer.

Firstly we incorporate into our model a turbulent boundary layer near the pipe wall, by making use of the $1/7$ power law, as is discussed by Schlichting et al. (1960). It was shown experimentally that an axial velocity profile within the turbulent boundary layer, of width $h_b$, which takes the form

$$u_x = U_1 \left(\frac{h - r}{h_b}\right)^{1/7},$$

(A 27)

exhibits good comparison with experimental data for a range of Reynolds numbers ($4 \times 10^3 < Re < 3.2 \times 10^6$ Schlichting et al. 1960). Therefore, we introduce an additional layer to our axial velocity profile (2.1) and use (A 27) for $h - h_b < r < h$ (see Figure 3). Note that, in reality, there is a viscous sub-layer and a log-law layer within the turbulent boundary layer but, for the purposes of this study, we neglect these details and assume that (A 27) applies all the way to the pipe wall. The boundary layer thickness $h_b$ is determined by an equation which derives from integrating conservation of axial momentum across the boundary layer, as described below.

Should the shear layer meet the edge of the boundary layer (i.e. $h_1 = 0$) then it will come into contact with the swirling flow. Therefore, in such situations we propose that there will be an additional swirl boundary layer of a similar form to (A 27). We wish to keep the model simple, so in order to calculate an analytical expression for the pressure in the boundary layer from Equation (2.3), we choose a $1/2$ exponent, instead of a $1/7$ exponent, for the swirl boundary layer profile. Therefore, the azimuthal velocity within the boundary layer is taken to have the form

$$u_\theta = U_\theta \left(\frac{h - r}{h_b}\right)^{1/2},$$

(A 28)

where $U_\theta(x)$ is the magnitude of the azimuthal velocity at the edge of the boundary layer. Note that $U_\theta$ is zero until the boundary layer comes into contact with the swirling shear layer (i.e. $h_1 = 0$), after which it is determined by an equation which derives from integrating conservation of angular momentum across the boundary layer.

By integrating the turbulent boundary layer equations (A 9) and (A 11), from $r = h - h_b$ to $r = h$, using the no-slip boundary condition (A 12), we get

$$\rho \frac{d}{dx} \int_{h - h_b}^{h} ru_x^2 \, dr + \rho (h - h_b) U_1^2 \frac{d}{dx} (h - h_b) - \rho (h - h_b) U_1 u_r \bigg|_{r = h - h_b} = - \int_{h - h_b}^{h} \frac{\partial p}{\partial x} \, dr + h \tau_{w_r} - (h - h_b) \tau_{b_r},$$

(A 29)

$$\rho \frac{d}{dx} \int_{h - h_b}^{h} r^2 u_\theta u_x \, dr + \rho (h - h_b)^2 U_1 U_\theta \frac{d}{dx} (h - h_b) - \rho (h - h_b)^2 U_\theta u_r \bigg|_{r = h - h_b} = h^2 \tau_{w_\theta} - (h - h_b)^2 \tau_{b_\theta},$$

(A 30)
where the shear stress terms are given by

\[ \tau_{br} = \left[ \rho (\nu + \nu_t) \frac{\partial u_x}{\partial r} \right]_{y=h-h_b}, \tag{A 31} \]

\[ \tau_{wr} = \left[ \rho (\nu + \nu_t) \frac{\partial u_x}{\partial r} \right]_{y=h}, \tag{A 32} \]

\[ \tau_{ba} = \left[ \rho (\nu + \nu_t) \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \right]_{y=h-h_b}, \tag{A 33} \]

\[ \tau_{wa} = \left[ \rho (\nu + \nu_t) \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \right]_{y=h}, \tag{A 34} \]

and the radial velocity at the edge of the boundary layer, \( u_r |_{r=h-h_b} \), is found by integrating the conservation of mass equation (A 8) from \( r = h - h_b \) to \( r = h \), giving

\[ (h - h_b) u_r |_{r=h-h_b} = \frac{d}{dx} \int_{h-h_b}^{h} ru_x \, dr + (h - h_b) U_1 \frac{d}{dx} (h - h_b). \tag{A 35} \]

Within the boundary layer, the axial and azimuthal velocities, (A 27) and (A 28), satisfy the no-slip condition at \( r = h \) and ensure the ‘outer’ velocities \( U_1 \) and \( U_\theta \) are continuous at the edge of the boundary layer \( r = h - h_b \). As before, we parameterise the wall stress terms (A 32) and (A 34) in terms of the velocity at the edge of the boundary layer, \( \mathbf{u}_{\text{outer}} = (U_1, U_\theta) \), and a friction factor (see Equation (A 19)). Note that, unlike the simple model, where \( U_\theta = 0 \) and therefore \( \tau_{wa} = 0 \), here both \( U_\theta \) and \( \tau_{wa} \) are not necessarily zero. Therefore, in addition, we use (A 19) to account for the stress term in (A 16) in terms of \( U_1 \) and \( U_\theta \). We ignore the axial shear stress between the edge of the boundary layer and the plug region (A 31), such that \( \tau_{br} = 0 \), since we expect it to be small. On the other hand, we do not ignore the azimuthal shear stress between the edge of the boundary layer and the plug region (A 33), since this is essentially the driving mechanism for the increase of \( U_\theta \) when the shear layer comes into contact with the boundary layer. Instead, we approximate (A 33) using a Prandtl mixing length (Schlichting et al. 1960) model

\[ \nu_t = \ell^2 \frac{\partial u_x}{\partial r}. \tag{A 36} \]

We take the mixing length as \( \ell = \beta \delta \) and the velocity gradient as \( \partial u_x / \partial r = (U_1 - U_2) / \delta \), in order to be consistent with the governing equation (2.5), which is based on the same Prandtl mixing length model (Schlichting et al. 1960, Jiménez 2004). Finally, since \( \nu \ll \nu_t \) away from the wall, (A 33) is approximated as

\[ \tau_{ba} = \rho \beta^2 (U_1 - U_2) (\kappa \delta^2 - h_2 \Omega). \tag{A 37} \]

The constant of proportionality \( \beta \) has been measured experimentally (Jiménez 2004, Pope 2000), finding \( \beta = 0.04 \) (\( \ell \) is given in terms of the momentum thickness \( \delta \), such that \( \ell = 0.26 \delta \approx 0.04 \delta \)).

Finally, by combining (A 29), (A 30) and (A 35), we eliminate \( u_r |_{r=h-h_b} \) to obtain the governing equations for \( h_b \) and \( U_\theta \), which are

\[ \rho \frac{d}{dx} \int_{h-h_b}^{h} ru_x^2 \, dr + \int_{h-h_b}^{h} r \frac{\partial p}{\partial x} \, dr = \rho U_1 \frac{d}{dx} \int_{h-h_b}^{h} ru_x \, dr + h \tau_{wr}, \tag{A 38} \]

\[ \rho \frac{d}{dx} \int_{h-h_b}^{h} r^2 u_\theta u_x \, dr = \rho U_\theta (h - h_b) \frac{d}{dx} \int_{h-h_b}^{h} ru_x \, dr + h^2 \tau_{wa} - (h - h_b)^2 \tau_{ba}. \tag{A 39} \]

Next we incorporate a ‘symmetry boundary layer’ into the model. If the core region
is entrained by the shear layer (i.e. $h_2 = 0$), then the axial velocity profile \( u_x \) does not have zero gradient at the pipe axis, $r = 0$, which shows disagreement with CFD calculations (see Figure 4). Therefore, we posit that there is a boundary layer along the pipe axis of width $h_0$, which satisfies the symmetry condition, $\partial u_x / \partial r = 0$ at $r = 0$. Motivated by comparison with CFD calculations, we propose a parabolic axial velocity profile within the symmetry boundary layer. The parabola function

\[
\ u_x = U_2 + \left( \frac{U_1 - U_2}{2h_0^2} \right) \left( r^2 - h_0^2 \right),
\]

(A 40)

is suitable, since it is both continuous and has continuous derivative ($\partial u_x / \partial r$) at $r = h_0$. Therefore, we introduce the symmetry boundary layer to our axial velocity profile (2.1) and use (A 40) for $0 < r < h_0$ (see Figure 3). The width of the symmetry boundary layer $h_0$ is determined by integrating the turbulent boundary layer equations (A 8) and (A 9) from $r = 0$ to $r = h_0$, using the symmetry boundary condition (A 13), giving

\[
\rho \frac{d}{dx} \int_0^{h_0} ru_x^2 \, dr - \frac{\rho h_0}{2} \left( U_1 - U_2 \right)^2 + \rho h_0 u_r |_{r=h_0} = - \int_0^{h_0} r \frac{\partial p}{\partial x} \, dr + h \tau_{0r},
\]

(A 42)

where the shear stress term is

\[
\tau_{0r} = \left[ \rho (\nu + \nu_t) \frac{\partial u_x}{\partial r} \right]_{y=h_0}.
\]

(A 43)

Similarly to the turbulent boundary layer, we approximate the stress term (A 43) using a Prandtl mixing length model, using the same values for $\ell$ and $\partial u_x / \partial r$, such that

\[
\tau_{0r} = \rho \beta^2 (U_1 - U_2)^2.
\]

(A 44)

Thus, by eliminating $u_r |_{r=h_0}$ from (A 41) and (A 42), we obtain the governing equation for $h_0$, the width of the symmetry layer, which is

\[
\rho \frac{d}{dx} \int_0^{h_0} ru_x^2 \, dr + \int_0^{h_0} r \frac{\partial p}{\partial x} \, dr = \rho U_2 \frac{d}{dx} \int_0^{h_0} ru_x \, dr + h_0 \tau_{0r}.
\]

(A 45)

To summarise, we have incorporated two additional boundary layers into our simple model: a turbulent boundary layer near the pipe wall satisfying the no-slip condition, and a symmetry boundary layer near the pipe axis satisfying the symmetry condition. We have introduced three new variables $h_b$, $h_0$, and $U_\theta$, which are all functions of $x$. These variables are governed by Equations (A 38), (A 39) and (A 45), in which we have used Prandtl mixing length arguments to model internal shear stress terms, and a friction factor parameterisation to model the wall stress terms.

**Appendix B.**

In this appendix we compare our simple model to the experimental results of Gilchrist & Naughton (2005), Chigier & Chervinsky (1967) and Elsner & Kurzak (1987) for an incompressible swirling jet. Unlike the examples of confined flow we consider in this study, all of these experimental studies are for unconfined flow. Furthermore, they consider a swirling jet entering an ambient fluid, unlike the co-axial annular flow we consider. Therefore, in order to compare with our model, we set $U_1 = 0$ and take the limit as $h(x)$, $h_1(x) \to \infty$. Furthermore, the pressure term $P(x)$ in Equation (2.4) becomes the constant hydrostatic pressure, which we take as $P = 0$, without loss of generality.
Figure 10: Normalised jet growth rates $\frac{w'}{w'_0}$ for a swirling jet in an ambient fluid, where we vary the inlet swirl number $S_w$. Simple swirl model compared against experimental results from the literature.

As a metric for our comparison, we use the growth rate of the ‘jet’ width. In terms of our formulation, the width of the jet corresponds to the combined width of the core region and the shear layer

$$w = h_2 + \delta. \quad (B\ 1)$$

The jet growth rate is, of course, non-linear, so we must take an average over a length scale $\Delta$. Therefore, the average jet width growth rate is defined as

$$\overline{w'} = \frac{1}{\Delta} \left( h_2(\Delta) + \delta(\Delta) - h_2(0) - \delta(0) \right). \quad (B\ 2)$$

In Figure 10 we display a comparison of $\overline{w'}$, normalised with respect to the jet growth rate for non-swirling flow $w'_0$, calculated using our simple model and the experimental results in the literature (with $\Delta = 5h_2(0)$). The experimental data is quite scattered and has large error bars, so it is difficult to conclude how good the comparison is. Furthermore, the results of Gilchrist & Naughton (2005) involve two different types of swirl distribution, a solid body rotation and a $q$ vortex, whereas we only consider solid body rotation. However, our model seems to agree well with the all the experimental data for small $S_w$ and with the results of Elsner & Kurzak (1987), which are for a slightly heated jet, for large $S_w$.

REFERENCES

Benham, GP, Castrejon-Pita, AA, Hewitt, IJ, Please, CP, Style, RW & Bird, P 2017a Turbulent shear layers in confining channels. arXiv preprint arXiv:1705.01046.

Benham, GP, Hewitt, IJ, Please, CP & Bird, P 2017b Optimal control of diffuser shapes for confined turbulent shear flows. arXiv preprint arXiv:1710.07950.

Bilanin, AJ, Teske, ME & Williamson, G 1977 Vortex interactions and decay in aircraft wakes. AIAA Journal 15 (2), 250–260.

Bilancius, H 1913 Das ähnlichkeitsgesetz bei reibungsvorgängen in flüssigkeiten. In Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, pp. 1–41. Springer.

Blevins, RD 1984 Applied fluid dynamics handbook. New York, Van Nostrand Reinhold Co., 1984, 568 p. 1.

Bourget, P-L & Marichal, D 1990 Remarks about variations in the drag coefficient of circular cylinders moving through water. Ocean Engineering 17 (6), 569–585.
Champagne, FH & Kromat, S 2000 Experiments on the formation of a recirculation zone in swirling coaxial jets. *Experiments in Fluids* **29** (5), 494–504.

Chigier, NA & Chervinsky, A 1967 Experimental investigation of swirling vortex motion in jets. *Journal of Applied Mechanics* **34** (2), 443–451.

Durbin, MD & Ballal, DR 1996 Studies of lean blowout in a step swirl combustor. *Journal of Engineering for Gas Turbines and Power* **118** (1), 72–77.

Elsner, JW & Kurzak, L 1987 Characteristics of turbulent flow in slightly heated free swirling jets. *J Fluid Mech* **180**, 147–169.

Escudier, MP & Keller, J 1985 Recirculation in swirling flow—a manifestation of vortex breakdown. *AIAA Journal* **23** (1), 111–116.

Farokhi, S, Taghavi, R & Rice, EJ 1989 Effect of initial swirl distribution on the evolution of a turbulent jet. *AIAA Journal* **27** (6), 700–706.

Filipenco, VG, Deniz, S, Johnston, JM, Greitzer, EM & Cumpsty, NA 1998 Effects of inlet flow field conditions on the performance of centrifugal compressor diffusers: Part 1 discrete passage diffuser. *Journal of Turbomachinery* **122**, 1–10.

Fox, RW & McDonald, AT 1971 Effects of swirling inlet flow on pressure recovery in conical diffusers. *AIAA Journal* **9** (10), 2014–2018.

Gibson, MM & Younis, BA 1986 Calculation of swirling jets with a Reynolds stress closure. *Physics of Fluids* **29** (1), 38–48.

Gilchrist, RT & Naughton, JW 2005 Experimental study of incompressible jets with different initial swirl distributions: mean results. *AIAA journal* **45** (4), 741–751.

Gonzalez-Aguilar, J, Sanjurjo, CP, Rodriguez-Yunta, A & Calderón, MAG 1999 A theoretical study of a cutting air plasma torch. *IEEE Transactions on Plasma Science* **27** (1), 264–271.

Guillaume, DW & Judge, TA 2004 Improving the efficiency of a jet pump using a swirling primary jet. *Review of Scientific Instruments* **75** (2), 553–555.

Gupta, AK, Lewis, MJ & Daurer, M 2001 Swirl effects on combustion characteristics of premixed flames. *Journal of Engineering for Gas Turbines and Power* **123** (3), 619–626.

Hah, C 1983 Calculation of various diffuser flows with inlet swirl and inlet distortion effects. *AIAA Journal* **21** (8), 1127–1133.

Jiménez, J 2004 Turbulence and vortex dynamics. *Notes for the Polytechnique course on turbulence, École Polytechnique, Palaiseau, France*.

Launder, BE & Spalding, DB 1974 The numerical computation of turbulent flows. *Computer Methods in Applied Mechanics and Engineering* **3** (2), 269–289.

Lee, KHI, Setoguchi, T, Matsuo, S & Kim, HD 2005 An experimental study of the under-expanded swirling jet with secondary coaxial stream. *Shock Waves* **14** (1-2), 83–92.

Leschziner, MA & Rodi, W 1984 Computation of strongly swirling axisymmetric free jets. *AIAA Journal* **22** (12), 1742–1747.

Liang, H & Maxworthy, T 2005 An experimental investigation of swirling jets. *J Fluid Mech* **525**, 115–159.

Lilley, DG 1977 Swirl flows in combustion: a review. *AIAA Journal* **15** (8), 1063–1078.

McKeon, BJ, Zagarola, MV & Smits, AJ 2005 A new friction factor relationship for fully developed pipe flow. *J Fluid Mech* **538**, 429.

Menter, FR 1994 Two-equation eddy-viscosity turbulence models for engineering applications. *AIAA Journal* **32** (8), 1598–1605.

Merkle, K, Haessler, H, Büchner, H & Zarzalis, N 2003 Effect of co-and counter-swirl on the isothermal flow-and mixture-field of an airlblast atomizer nozzle. *International Journal of Heat and Fluid Flow* **24** (4), 529–537.

Morton, BR, Taylor, GI & Turner, JS 1956 Turbulent gravitational convection from maintained and instantaneous sources. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 234, pp. 1–23.

Naughton, JW, Cattafesta III, LN & Settles, GS 1997 An experimental study of compressible turbulent mixing enhancement in swirling jets. *J Fluid Mech* **330**, 271–305.

Nayeri, C 2000 Investigation of the three-dimensional shear layer between confined coaxial jets with swirl. PhD thesis, Technischen Universität Berlin.

Oberleithner, K, Paschereit, CO & Wygnanski, I 2014 On the impact of swirl on the growth of coherent structures. *J Fluid Mech* **741**, 156–199.
Pope, S B 2000 Turbulent flows. Cambridge University Press.
Ribeiro, MM & Whitelaw, JH 1980 Coaxial jets with and without swirl. *J Fluid Mech* **96** (4), 769–795.
Schlichting, H, Gersten, K, Krause, E, Oertel, H & Mayes, K 1960 Boundary-layer theory. Springer.
So, KL 1967 Vortex phenomena in a conical diffuser. *AIAA Journal* **5** (6), 1072–1078.
Tritton, DJ 2012 Physical fluid dynamics. Springer Science & Business Media.
Weller, HG, Tabor, G, Jasak, H & Fureby, C 1998 A tensorial approach to computational continuum mechanics using object-oriented techniques. *Computers in Physics* **12** (6), 620–631.