The absolute continuity of the invariant measure of random iterated function systems with overlaps

by

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Abstract. We consider iterated function systems on the interval with random perturbation. Let $Y_\varepsilon$ be uniformly distributed in $[1-\varepsilon, 1+\varepsilon]$ and let $f_i \in C^{1+\alpha}$ be contractions with fixpoints $a_i$. We consider the iterated function system $\{Y_\varepsilon f_i + a_i(1-Y_\varepsilon)\}_{i=1}^n$, where each of the maps is chosen with probability $p_i$. It is shown that the invariant density is in $L^2$ and its $L^2$ norm does not grow faster than $1/\sqrt{\varepsilon}$ as $\varepsilon$ vanishes.

The proof relies on defining a piecewise hyperbolic dynamical system on the cube with an SRB-measure whose projection is the density of the iterated function system.

1. Introduction and statements of results. Let $\{f_1, \ldots, f_l\}$ be an iterated function system (IFS) on the real line, where the maps are applied according to the probabilities $(p_1, \ldots, p_l)$, with the choice of the map random and independent at each step. We assume that for each $i$, $f_i$ maps $[-1, 1)$ into itself so that the image is bounded away from $-1$ and $1$, and $f_i \in C^{1+\alpha}([-1, 1))$. Let $\nu$ be the invariant measure of our IFS, i.e.

$$\nu = \sum_{i=1}^l p_i \nu \circ f_i^{-1}. \tag{1.1}$$

Let $\mu=(p_1, \ldots, p_l)^\mathbb{N}$ be a Bernoulli measure on the space $\Sigma=\{1, \ldots, l\}^\mathbb{N}$. Let $h(p) = -\sum_{i=1}^l p_i \log p_i$ be the entropy of the left-shift operator with respect to the Bernoulli measure $\mu$. It was proved in [7], for non-linear, contracting on average, iterated function systems (and later extended in [3]) that

$$\dim_H(\nu) \leq h/|\chi|,$$

where $\dim_H(\nu)$ is the Hausdorff dimension of the measure $\nu$, and $\chi$ is the Lyapunov exponent of the IFS associated to the Bernoulli measure $\mu$.

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One can expect that, at least “typically”, the measure $\nu$ is absolutely continuous when $h/|\chi| > 1$. Essentially the only known approach to this is transversality. For example, for the linear case with uniform contraction ratios, see [8] and [10]. For the linear case and non-uniform contraction ratios, see [5] and [6]. For the non-linear case, see for example [14] and [1]. We note that there is another direction in the study of iterated function systems with overlaps, which is concerned with concrete, but non-typical systems, often of arithmetic nature, for which there is a dimension drop (see for example [4]).

Throughout this paper we are interested in studying absolute continuity with density in $L^2$. We will study a modification of the problem, namely we consider a random perturbation of the functions. The linear case was studied by Peres, Simon and Solomyak in [9]. They proved absolute continuity for random linear IFS, with non-uniform contraction ratios and also $L^2$ and continuous density in the uniform case. We extend this result by proving $L^2$ density with non-uniform contraction ratios and in the non-linear case.

We consider two cases. First let us suppose that for each $i \in \{1, \ldots, l\}$, $f_i$ maps $[-1, 1)$ into itself, $f_i([-1, 1))$ is bounded away from $-1$ and $1$, $f_i \in C^{1+\alpha}([-1, 1))$ and (1.2) $0 < \lambda_{i,\text{min}} \leq |f_i'(x)| \leq \lambda_{i,\text{max}} < 1$
for every $x \in [-1, 1)$. Moreover suppose that for every $i$ the fixed point of $f_i$ is $a_i \in (-1, 1)$, and (1.3) $i \neq j \Rightarrow a_i \neq a_j$.

Let $Y_\varepsilon$ be uniformly distributed on $[1 - \varepsilon, 1 + \varepsilon]$. Denote the probability measure of $Y_\varepsilon$ by $\eta_\varepsilon$. Let
(1.4) $f_i,Y_\varepsilon(x) = Y_\varepsilon f_i(x) + a_i (1 - Y_\varepsilon)$
for every $i \in \{1, \ldots, l\}$. Then $f_i,Y_\varepsilon(x)$ is in $[-1, 1)$ for all values of $x \in [-1, 1)$ and $Y_\varepsilon$, provided $\varepsilon$ is sufficiently small. The iterated maps are applied randomly according to the stationary measure $\mu$, with the sequence of independent and identically distributed errors $y_1, y_2, \ldots$ distributed as $Y_\varepsilon$, independent of the choice of the function. The Lyapunov exponent of the IFS is defined by

$$\chi(\mu, \eta_\varepsilon) = \mathbb{E}(\log(Y_\varepsilon f'))$$
and it is easy to see that

$$\chi(\mu, \eta_\varepsilon) < \sum_{i=1}^l p_i \log((1 + \varepsilon) \lambda_{i,\text{max}}) < 0$$

for sufficiently small $\varepsilon > 0$. Let $Z_\varepsilon$ be the random variable
(1.5) $Z_\varepsilon := \lim_{n \to \infty} f_{i_1,y_1,\varepsilon} \circ \cdots \circ f_{i_n,y_n,\varepsilon}(0)$,
where the numbers $i_k$ are i.i.d., with distribution $\mu$ on $\{1, \ldots, l\}$, and $y_{k,\varepsilon}$ are pairwise independent with the distribution of $Y_\varepsilon$ and also independent of the choice of $i_k$. Let $\nu_\varepsilon$ be the distribution of $Z_\varepsilon$.

One can easily prove the following theorem.

**Theorem 1.1.** The measure $\nu_\varepsilon$ converges weakly as $\varepsilon \to 0$ to the measure $\nu$ satisfying (1.1).

**Theorem 1.2.** Let $\nu_\varepsilon$ be the distribution of the limit (1.5). Assume that (1.2) and (1.3) hold, and

$$\sum_{i=1}^l p_i^2 \lambda_{i,\max} / \lambda_{i,\min}^2 < 1.$$  \hspace{1cm} (1.6)

Then for every sufficiently small $\varepsilon > 0$, $\nu_\varepsilon$ is absolutely continuous with respect to Lebesgue measure, with density in $L^2$, and there exists a constant $C$ such that the density of $\nu_\varepsilon$ satisfies

$$\|\nu_\varepsilon\|_2 \leq C/\sqrt{\varepsilon}.$$  \hspace{1cm} (1.7)

**Remark 1.** Let

$$C'_\varepsilon = \sqrt{32 \left(1 - \sum_{i=1}^l p_i^2 \lambda_{i,\max} / ((1+\varepsilon)\lambda_{i,\min})^2\right)} C''_\varepsilon,$$

$$C''_\varepsilon = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1 - \varepsilon^2} \right\}.$$  \hspace{1cm} (1.8)

The proof of Theorem 1.2 will show that $\|\nu_\varepsilon\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$. Hence we can choose any $C > \lim_{\varepsilon \to 0} C''_\varepsilon$.

**Remark 2.** Actually the proof of Theorem 1.2 shows that $Z_\varepsilon$ conditioned on the perturbations $y_{1,\varepsilon}, y_{2,\varepsilon}, \ldots$ has density in $L^2$ for almost all $y_{1,\varepsilon}, y_{2,\varepsilon}, \ldots$.

We can state an easy corollary of the theorem.

**Corollary 1.3.** Let $\{\lambda_i Y_\varepsilon x + a_i(1 - \lambda_i Y_\varepsilon)\}_{i=1}^l$ be a random iterated function system. Assume that (1.3) holds, and

$$\sum_{i=1}^l p_i^2 / \lambda_i < 1.$$  \hspace{1cm} (1.9)

Then for every sufficiently small $\varepsilon > 0$, $\nu_\varepsilon$ is absolutely continuous with respect to Lebesgue measure with density in $L^2$, and there exists a constant $C$ such that

$$\|\nu_\varepsilon\|_2 \leq C/\sqrt{\varepsilon}.$$  \hspace{1cm} (1.10)

We study another case of random perturbation, namely let $\tilde{\lambda}_{i,\varepsilon}$ be uniformly distributed on $[\lambda_i - \varepsilon, \lambda_i + \varepsilon]$. Let $\{\tilde{\lambda}_{i,\varepsilon} x + a_i(1 - \tilde{\lambda}_{i,\varepsilon})\}_{i=1}^l$ be our random
iterated function system, where \( a_i \neq a_j \) for every \( i \neq j \). Let \( \Lambda = (\lambda_1, \ldots, \lambda_l) \), and \( X_{\Lambda,\varepsilon} \) be the random variable

\[
X_{\Lambda,\varepsilon} = \sum_{k=1}^{\infty} \left( a_{i_k} (1 - \tilde{\lambda}_{i_k,\varepsilon}) \right) \prod_{j=1}^{k-1} \tilde{\lambda}_{i_j,\varepsilon}
\]

where the numbers \( i_k \) are i.i.d. with distribution \( \mu \) on \( \{1, \ldots, l\} \), and \( \tilde{\lambda}_{i_k,\varepsilon} \) are pairwise independent. Let \( \nu_{\Lambda,\varepsilon} \) denote the distribution of \( X_{\Lambda,\varepsilon} \). Moreover let \( \nu_{\Lambda} \) be the invariant measure of the iterated function system \( \{\lambda_i x + a_i (1 - \lambda_i)\}_{i=1}^{l} \) according to \( \mu \).

**Theorem 1.4.** The measure \( \nu_{\Lambda,\varepsilon} \) converges weakly to \( \nu_{\Lambda} \) as \( \varepsilon \to 0 \).

To have a statement similar to Theorem 1.2 we need a technical assumption

\[
\min_{i \neq j} \left| \frac{a_j \lambda_i - a_i \lambda_j}{\lambda_i - \lambda_j} \right| > 1.
\]

**Theorem 1.5.** Suppose that (1.9) and (1.3) hold, and moreover

\[
\sum_{i=1}^{l} \frac{p_i^2}{\lambda_i} < 1.
\]

Then for every sufficiently small \( \varepsilon > 0 \), the measure \( \nu_{\Lambda,\varepsilon} \) is absolutely continuous with respect to Lebesgue measure, with density in \( L^2 \), and there exists a constant \( C \) such that

\[
\| \nu_{\Lambda,\varepsilon} \|_2 \leq C/\sqrt{\varepsilon}.
\]

**Remark 3.** Let

\[
C'_\varepsilon = \sqrt{32 \frac{1}{(1 - \sum_{i=1}^{l} p_i^2/\lambda_i^2)} C''_\varepsilon},
\]

\[
C''_\varepsilon = \sigma \min_{i \neq j} \left| \frac{a_i \lambda_j - a_j \lambda_i - |\lambda_i - \lambda_j|}{\lambda_i \lambda_j} \right|
\]

where \( 0 < \sigma < 1 \). As in Remark 1, the proof of Theorem 1.5 will show that \( \| \nu_{\Lambda,\varepsilon} \|_2 \leq C'_\varepsilon/\sqrt{\varepsilon} \) for small \( \varepsilon \).

The main difference between Theorem 1.5 and Corollary 1.3 is the random perturbation. Namely, in Theorem 1.5 we choose the contraction ratio uniformly in the \( \varepsilon \)-neighborhood of \( \lambda_i \), while in Corollary 1.3 we choose the contraction ratio uniformly in the \( \lambda_i \varepsilon \)-neighborhood of \( \lambda_i \).

Throughout this paper we will use the method of 11.
2. Proof of Theorem 1.2. Let $Q = [-1, 1]^3$ and $m \in \mathbb{N}$. We partition the cube $Q$ into the rectangles \( \{Q_{1,k}, \ldots, Q_{l,k}\}_{k=0}^{2^m-1} \), where

\[
Q_{i,k} = \left\{ (x, y, z) \in Q : -1 + 2 \sum_{j=1}^{i-1} p_j \leq y < -1 + 2 \sum_{j=1}^{i} p_j, \right. \\
\left. -1 + k2^{-m+1} \leq z < -1 + (k+1)2^{-m+1} \right\},
\]

where we use the convention that an empty sum is 0. Hence we slice $Q$ into $2^m$ slices along the $z$-axis and $l$ slices along the $y$-axis. We thereby get $2^m l$ pieces which we call $Q_{i,k}$, according to the definition above.

Let

\[
Q_i = \bigcup_{k=0}^{2^m-1} Q_{i,k}.
\]

We define a map $g_{\varepsilon,m} : Q \to Q$ so that each slice $Q_{i,k}$ is expanded as much as possible in the second and third coordinates. In the first coordinate it is mapped according to a perturbation of $f_i$, and hence contracted. Which perturbation is chosen depends on the third coordinate. There is a picture of this in Figure 1.

More precisely, we define $g_{\varepsilon,m} : Q \to Q$ by setting, for $(x, y, z) \in Q_{i,k}$,

\[
g_{\varepsilon,m} : (x, y, z) \mapsto \left( d(z)f_i(x) + a_i(1 - d(z)), \frac{1}{p_i}y + b(y), 2^m z + c(z) \right),
\]

where

\[
d(z) = 1 + 2^m \varepsilon (z - (-1 + (k + 1/2)2^{-m+1})) \quad \text{for} \ (x, y, z) \in Q_{i,k},
\]

\[
b(y) = 1 - \frac{1}{p_i} \left( -1 + 2 \sum_{j=1}^{i} p_j \right) \quad \text{for} \ (x, y, z) \in Q_{i,k},
\]

\[
c(z) = 2^m - 2k - 1 \quad \text{for} \ (x, y, z) \in Q_{i,k}.
\]

Hence $g_{\varepsilon,m}$ maps each of the pieces $Q_{i,j}$ so that it is contracted in the $x$-direction and fully expanded in the $y$- and $z$-directions.
Let $L_3$ be the normalised Lebesgue measure on $Q$. The measures
\[
\gamma_{\varepsilon,m,n} = \frac{1}{n} \sum_{k=0}^{n-1} L_3 \circ g_{\varepsilon,m}^{-k}
\]
converge weakly to an SRB-measure $\gamma_{\varepsilon,m}$ as $n \to \infty$ (see [12] and [13]). The measure $\gamma_{\varepsilon,m}$ is ergodic by the Hopf argument, since $g_{\varepsilon,m}$ is hyperbolic and the stable and unstable manifolds are parallel to the coordinate axes and have maximal extension in the box $Q$. Moreover, let $\nu_{\varepsilon,m}$ be the projection of $\gamma_{\varepsilon,m}$ onto the first coordinate. More precisely, if $E \subset [-1,1]$ is a measurable set, then we define $\nu_{\varepsilon,m}(E) = \gamma_{\varepsilon,m}(E \times [-1,1] \times [-1,1])$.

The measure $\nu_{\varepsilon,m}$ is the distribution of the limit
\[
\lim_{n \to \infty} f_{i_1,y_{1,\varepsilon}} \circ \cdots \circ f_{i_n,y_{n,\varepsilon}}(0),
\]
where $y_{i,\varepsilon}$ are uniformly distributed on $[1 - \varepsilon,1 + \varepsilon]$, but not independent. However, one can easily prove the following lemma.

**Lemma 2.1.** The measure $\nu_{\varepsilon,m}$ converges weakly to $\nu_{\varepsilon}$ as $m \to \infty$.

Let
\[
A_i = \{(i,0), (i,1), \ldots, (i, 2^m - 1)\} \quad \text{and} \quad A = \bigcup_{i=1}^l A_i.
\]
If $a = (i,k) \in A$ we will write $\hat{Q}_a$ for $Q_{i,k}$. With this notation we have
\[
Q = \bigcup_{a \in A} \hat{Q}_a \quad \text{and} \quad Q_i = \bigcup_{a \in A_i} \hat{Q}_a, \quad i = 0,1,\ldots,l.
\]

Let $\Theta_0 = A^{\mathbb{N} \cup \{0\}}$. If $p \in Q$ then there is a unique sequence $\rho_0(p) = \{\rho_0(p)_k\}_{k=0}^{\infty} \in \Theta_0$ such that
\[
g_{\varepsilon,m}^k(p) \in Q_{\rho_0(p)_k}, \quad k = 0,1,\ldots.
\]
The map $\rho_0: Q \to \Theta_0$ is not injective. We have $\rho_0(x,y,z) = \rho_0(x',y',z')$ if $y = y'$ and $z = z'$, but $\rho_0(x,y,z) \neq \rho_0(x',y',z')$ otherwise. Hence we can (and will) use the notation $\rho_0(y,z)$ instead of $\rho_0(x,y,z)$.

We will denote elements in $\Theta_0$ by $a$, $b$ and so on. We let $\sigma$ denote the left shift on $\Theta_0$, defined in the usual way.

We can transfer the measures $\gamma_{\varepsilon,m}$ to a measure $\gamma_{\Theta_0}$ by $\gamma_{\Theta_0} = \gamma_{\varepsilon,m} \circ \rho_0^{-1}$.

We let $\Theta$ denote the natural extension of $\Theta_0$. That is, $\Theta$ is the set of all two-sided infinite sequences such that any one-sided infinite subsequence of a sequence in $\Theta$ is a sequence in $\Theta_0$. The measure $\gamma_{\Theta_0}$ defines an ergodic measure $\gamma_{\Theta}$ on $\Theta$ in a natural way. If $\xi: \Theta \to \Theta_0$ is defined by $\xi\{i_k\}_{k \in \mathbb{Z}} = \{i_k\}_{k \in \mathbb{N} \cup \{0\}}$, then we define $\gamma_{\Theta}(\xi^{-1}E) = \gamma_{\Theta_0}(E)$. We can define a map $\rho^{-1}: \Theta \to Q$ such that $\rho^{-1}(\sigma(a)) = g_{\varepsilon,m}(\rho^{-1}(a))$ for any sequence $a \in \Theta$. 

We note that the $L^2$ norm of the density $\nu_{\varepsilon,m}$ is not larger than twice that of the density of $\gamma_{\varepsilon,m}$. If $h_{\nu_{\varepsilon,m}}(x)$ and $h_{\gamma_{\varepsilon,m}}(x,y,z)$ denote the densities of $\nu_{\varepsilon,m}$ and $\gamma_{\varepsilon,m}$ respectively, then by Cauchy–Schwarz’s inequality

$$\|\nu_{\varepsilon,m}\|^2_2 \leq \int_{-1}^{1} h_{\nu_{\varepsilon,m}}(x)^2 \, dx = 32 \int_{-1}^{1} \left( \int_{-1}^{1} h_{\gamma_{\varepsilon,m}}(x,y,z) \frac{dy}{2} \frac{dz}{2} \right)^2 \, dx = 4 \|\gamma_{\varepsilon,m}\|^2_2.$$  

This proves that if $\gamma_{\varepsilon,m}$ has $L^2$ density, then so has $\nu_{\varepsilon,m}$, and

$$\|\nu_{\varepsilon,m}\|^2_2 \leq 2 \|\gamma_{\varepsilon,m}\|^2_2. \tag{2.1}$$

If $p$ is a point in $Q$, then we let $T_pQ$ denote the tangent space at $p$. For each $p$ in $Q$ we define the following cone in the tangent space $T_pQ$:

$$C_p = \left\{ (u,v,w) \in T_pQ : \frac{|u|}{w}, \frac{|v|}{w} < \frac{2^{m+1} \varepsilon}{2m - \lambda_{\max} \lambda_{\max}(1 + \varepsilon)} \right\},$$

where $\lambda_{\max} = \max_i \lambda_i$, $\lambda_{\max} = \max_i \sup_{x \in [-1,1]} |f_i'(x)|$. The following lemma states that the set of cones $C_p$ defines a family of unstable cones, and that images of certain curves intersect transversally. There is an illustration of the transversality in Figure 2.

**Lemma 2.2.** The cones $C_p$ make up a family of unstable cones, that is, $d_p g_{\varepsilon,m}(C_p) \subset C_{\varepsilon,m}(p)$.

Moreover, for sufficiently large $m$ and every $0 < \varepsilon < \min_{i \neq j} \frac{|a_i - a_j|}{2 + |a_i + a_j|}$, if $\xi_1 \subset Q_{\xi_1}$ and $\xi_2 \subset Q_{\xi_2}$ are two curve segments with tangents in $C_p$ such that $\xi_1 \subset A_i$ and $\xi_2 \subset A_j$, $i \neq j$, then if $g_{\varepsilon,m}(\xi_1)$ and $g_{\varepsilon,m}(\xi_2)$ intersect, and if $(u_1,v_1,1)$ and $(u_2,v_2,1)$ are tangents to $g_{\varepsilon,m}(\xi_1)$ and $g_{\varepsilon,m}(\xi_2)$ respectively, it follows that $|u_1 - u_2| > C_{\varepsilon,m}$, where

$$C_{\varepsilon,m} = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1 - \varepsilon^2} - \frac{4(1 + \varepsilon) \lambda_{\max} \lambda_{\max}}{2m - \lambda_{\max} \lambda_{\max}(1 + \varepsilon)} \right\}.$$
Proof. The Jacobian of \( g_{\varepsilon,m} \) is

\[
d_{p}g_{\varepsilon,m} = \begin{pmatrix}
    d(z)f'_{i}(x) & 0 & 2^{m}\varepsilon(f_{i}(x) - a_{i}) \\
    0 & 1/p_{i} & 0 \\
    0 & 0 & 2^{m}
\end{pmatrix}
\]

where \( p = (x, y, z) \in Q_{i,k} \). If \((u, v, w) \in C_{p} \), then

\[
d_{p}g_{\varepsilon,m}(u, v, w) = \begin{pmatrix}
    d(z)f'_{i}(x)u + 2^{m}\varepsilon(f_{i}(x) - a_{i})w \\
    (1/p_{i})v \\
    2^{m}w
\end{pmatrix}
\]

We just need to check that this vector is in \( C_{p} \), provided that \( m \) is large. This is easily checked, using that \(|d(z)| \leq 1 + \varepsilon\), \(|f'_{i}(x)| \leq \lambda_{i,\text{max}}\) and \(|f_{i}(x) - a_{i}| \leq 2\). Indeed,

\[
\frac{|d(z)f'_{i}(x)u + 2^{m}\varepsilon(f_{i}(x) - a_{i})w|}{2^{m}|w|} \leq \frac{(1 + \varepsilon)\lambda_{i,\text{max}}}{2^{m}} |u| + 2\varepsilon
\]

\[
\leq \frac{(1 + \varepsilon)\lambda_{i,\text{max}}}{2^{m}} \frac{2^{m+1}\varepsilon}{2^{m} - (1 + \varepsilon)\lambda_{\text{max,\text{max}}}} + 2\varepsilon \leq \frac{2^{m+1}\varepsilon}{2^{m} - (1 + \varepsilon)\lambda_{\text{max,\text{max}}}}
\]

and

\[
\frac{|(1/p_{i})v|}{2^{m}|w|} \leq \frac{1}{p_{i}2^{m}} \frac{2^{m+1}\varepsilon}{2^{m} - (1 + \varepsilon)\lambda_{\text{max,\text{max}}}} \leq \frac{2^{m+1}\varepsilon}{2^{m} - (1 + \varepsilon)\lambda_{\text{max,\text{max}}}}
\]

proves that \( d_{p}g_{\varepsilon,m}(C_{p}) \subset C_{g_{\varepsilon,m}(p)} \) if \( m \) is sufficiently large, so that \( 2^{m} - (1 + \varepsilon)\lambda_{\text{max,\text{max}}} > 0 \) and \( p_{i}2^{m} > 1 \).

To prove the other statement of the lemma, assume that \( p = (x_{p}, y_{p}, z_{p}) \in Q_{i} \) and \( q = (x_{q}, y_{q}, z_{q}) \in Q_{j}, i \neq j \), are such that \( g_{\varepsilon,m}(p) = g_{\varepsilon,m}(q) = (x, y, z) \). Then, if \( p \in Q_{i} \),

\[
d_{p}g_{\varepsilon,m} : (u, v, 1) \mapsto 2^{m}\left(\frac{d(z_{p})f'_{i}(x_{p})}{2^{m}}u + (f_{i}(x_{p}) - a_{i})\varepsilon, \frac{v}{p_{i}}, 1\right).
\]

Then

\[
f_{i}(x_{p}) = \frac{x - a_{i}(1 - d(z_{p}))}{d(z_{p})} \quad \text{and} \quad f_{j}(x_{q}) = \frac{x - a_{j}(1 - d(z_{q}))}{d(z_{q})}.
\]

Without loss of generality, assume that \( a_{i} > a_{j} \). For simplicity we study the case \( x \geq a_{i} > a_{j} \). The proofs for \( a_{i} \geq x \geq a_{j} \) and \( a_{i} > a_{j} \geq x \) are similar. Then

\[
d_{p}g_{\varepsilon,m}(C_{p}) \subset \left\{ w(u, v, 1) : \frac{x - a_{i}}{1 + \varepsilon} \varepsilon - \Delta_{i}\varepsilon \leq u \leq \frac{x - a_{i}}{1 - \varepsilon} \varepsilon + \Delta_{i}\varepsilon \right\},
\]
where $\Delta_i = \frac{2(1+\varepsilon)\lambda_{i,\max}}{2m-\lambda_{\max,\max}(1+\varepsilon)}$. Therefore

$$|u_2 - u_1| \geq \frac{x - a_j \varepsilon - x - a_i \varepsilon - (\Delta_i + \Delta_j)\varepsilon}{1 + \varepsilon} - \frac{\varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} - 2\max_i \Delta_i \varepsilon$$

for every $x \geq a_i > a_j$. Let $\Delta_{\max} = \max_i \Delta_i$. Since $0 < \varepsilon < \min_{i \neq j} \frac{|a_i - a_j|}{2 + |a_i + a_j|}$, we have

$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} > 0.$$ 

Therefore

$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} - 2\Delta_{\max} > 0,$$

for sufficiently large $m$. By similar methods we have for $a_i \geq x \geq a_j$,

$$|u_2 - u_1| \geq \left(\frac{a_i - a_j}{1 + \varepsilon} - 2\Delta_{\max}\right)\varepsilon,$$

and for $a_i > a_j \geq x$,

$$|u_2 - u_1| \geq \left(\frac{a_i - a_j - \varepsilon(a_i + a_j + 2)}{1 - \varepsilon^2} - 2\Delta_{\max}\right)\varepsilon.$$

Therefore we can choose

$$C_{\varepsilon,m} = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1 - \varepsilon^2} - 2\Delta_{\max} \right\}. \quad \blacksquare$$

The rest of the section follows Tsujii’s article [15].

**Proof of Theorem 1.2.** For any $r > 0$ we define the bilinear form $(\cdot, \cdot)_r$ of signed measures on $\mathbb{R}$ by

$$(\rho_1, \rho_2)_r = \int_{\mathbb{R}} \rho_1(B_r(x))\rho_2(B_r(x)) \, dx$$

where $B_r(x) = [x - r, x + r]$. It is easy to see that if

$$\liminf_{r \to 0} \frac{1}{r^2}(\rho, \rho)_r < \infty$$

then the measure $\rho$ has density in $L^2$ (see [15]). Moreover

$$\|\rho\|_2^2 \leq \liminf_{r \to 0} \frac{1}{r^2}(\rho, \rho)_r.$$

Let $\gamma_z$ denote the conditional measure of $\gamma_{\varepsilon,m}$ on the set $R_z = \{(u, v, w) \in Q : v = y, w = z\}$. Since the one-dimensional Lebesgue measure is invariant under the action of $g_{\varepsilon,m}$ projected to the second coordinate, we conclude
that \( \gamma_z \) is independent of \( y \) almost everywhere. It follows that

\[
(2.2) \quad \| \gamma_{\varepsilon,m} \|_2^2 = \int_{-1}^1 \| \gamma_z \|_2^2 \, dz.
\]

Let

\[
J(r) := \frac{1}{r^2} \int_{-1}^1 (\gamma_z, \gamma_z)_r \, dz.
\]

By the invariance of \( \gamma_{\varepsilon,m} \) it follows that

\[
(2.3) \quad \gamma_z = 2^{-m} \sum_{i=1}^l p_i \sum_{a \in A_i} \gamma_{g_{\varepsilon,m}^{-a}(z)} \circ g_{\varepsilon,m}^{-a},
\]

where \( g_{\varepsilon,m}^{-a} \) denotes the inverse branch of \( g_{\varepsilon,m} \) with image in \( \hat{Q}_a \). Recall that

\( a \in A_i \) means that \( a = (i, k) \) for some \( k \), so that \( \hat{Q}_a = Q_{i,k} \) for some \( k \). We denote the measure \( \gamma_{g_{\varepsilon,m}^{-a}(z)} \circ g_{\varepsilon,m}^{-a} \) by \( \sigma_{a,z} \). Then by (2.3) and the definition of \( J(r) \),

\[
(2.4) \quad J(r) = \frac{1}{2^{2m} r^2} \sum_{i=1}^l \sum_{j=1}^l p_i p_j \sum_{a \in A_i} \sum_{b \in A_j} \int_{-1}^1 (\sigma_{a,z}, \sigma_{b,z})_r \, dz.
\]

For fixed \( a, b \in A_i \),

\[
(2.5) \quad (\sigma_{a,z}, \sigma_{b,z})_r \leq (\sigma_{a,z}, \sigma_{a,z})_r^{1/2}(\sigma_{b,z}, \sigma_{b,z})_r^{1/2}
\]

\[
\leq (1 + \varepsilon) \lambda_{\max}(\gamma_{g_{\varepsilon,m}^{-a}(z)}, \gamma_{g_{\varepsilon,m}^{-b}(z)})^{1/2}(1 - \varepsilon)^{r\lambda_{\min}} \times (\gamma_{g_{\varepsilon,m}^{-b}(z)}, \gamma_{g_{\varepsilon,m}^{-a}(z)})^{1/2}(1 - \varepsilon)^{r\lambda_{\min}}
\]

\[
\leq (1 + \varepsilon) \lambda_{\max}(\gamma_{g_{\varepsilon,m}^{-a}(z)}, \gamma_{g_{\varepsilon,m}^{-b}(z)})^{1/2}(1 - \varepsilon)^{r\lambda_{\min}} + (\gamma_{g_{\varepsilon,m}^{-a}(z)}, \gamma_{g_{\varepsilon,m}^{-b}(z)})^{1/2}(1 - \varepsilon)^{r\lambda_{\min}}.
\]

Moreover, if \( a \in A_i \) and \( b \in A_j \), \( i \neq j \), then

\[
(\sigma_{a,z}, \sigma_{b,z})_r = \int \sigma_{a,z}(B_r(x)) \sigma_{b,z}(B_r(x)) \, dx
\]

\[
= \int \int \int \mathbb{1}_{\{s: |s-x| < r\}}(s) \mathbb{1}_{\{t: |t-x| < r\}}(t) \, d\sigma_{a,z}(s) \, d\sigma_{b,z}(t) \, dx
\]

\[
\leq \int 2r \mathbb{1}_{\{s,t: |s-t| < 2r\}}(s,t) \, d\sigma_{a,z}(s) \, d\sigma_{b,z}(t)
\]

\[
= \int \mathbb{1}_{\{(c,d): \rho^{-1}(\cdots c_{-2} c_{-1} a \rho_0(z)) - \rho^{-1}(\cdots d_{-2} d_{-1} b \rho_0(z)) < 2r\}}(c,d) \, d\gamma_{\Theta}(c) \, d\gamma_{\Theta}(d).
\]

Let us comment on the notation \( \rho_0(z) \). Actually \( \rho_0(z) \) is not defined, but rather \( \rho_0(x, y, z) \). Recall that \( \rho_0(x, y, z) \) is independent of \( x \) and that we therefore have introduced the notation \( \rho_0(y, z) \). Moreover, as noticed above, the measures \( \gamma_z \), and therefore also \( \sigma_{a,z} \), are independent of \( y \). Hence we can choose arbitrary \( x, y \) and let \( \rho_0(z) \) denote \( \rho_0(x, y, z) = \rho_0(y, z) \). Since all
the estimates below will be independent of this choice of $y$, we will use the notation $\rho_0(z)$ instead of $\rho_0(x, y, z)$.

By a change of order of integration we get

$$\int_{-1}^{1} (\sigma_{a, z}, \sigma_{b, z})_r dz \leq 2r \int \mathcal{L}_1(\{ z : |\rho^{-1}(\cdots c_{-2} c_{-1} a \rho_0(z)) - \rho^{-1}(\cdots d_{-2} d_{-1} b \rho_0(z))| < 2r \}) d\gamma_\Theta(c) d\gamma_\Theta(d).$$

We will now apply Lemma 2.2 to (2.6). Note that}

$$z \mapsto \rho^{-1}(\cdots c_{-2} c_{-1} a \rho_0(z)),$$

and

$$z \mapsto \rho^{-1}(\cdots d_{-2} d_{-1} b \rho_0(z))$$

are two curves with tangents in the cones $C_p$. Lemma 2.2 states that these curves have a transversal intersection, if they intersect, so that

$$\mathcal{L}_1(\{ z : |\rho^{-1}(\cdots c_{-2} c_{-1} a \rho_0(z)) - \rho^{-1}(\cdots d_{-2} d_{-1} b \rho_0(z))| < 2r \}) \leq 4r/C_{\varepsilon, m}.$$

Hence

$$1 - \frac{1}{2} \int_{-1}^{1} (\sigma_{a, z}, \sigma_{b, z})_r dz \leq \frac{8r^2}{C_{\varepsilon, m}}.$$

By using (2.4) we have

$$J(r) = \frac{1}{2^{2m} r^2} \sum_{i=1}^{l} p_i^2 \sum_{a, b \in A_i} \int_{-1}^{1} (\sigma_{a, z}, \sigma_{b, z})_r dz$$

$$+ \frac{1}{2^{2m} r^2} \sum_{i \neq j} p_i p_j \sum_{a \in A_i} \sum_{b \in A_j} \int_{-1}^{1} (\sigma_{a, z}, \sigma_{b, z})_r dz.$$

We first give an upper bound for the first summand in (2.8), using (2.5) and an integral transformation. By (2.5) we have

$$\sum_{a, b \in A_i} \int_{-1}^{1} (\sigma_{a, z}, \sigma_{b, z})_r dz$$

$$\leq (1 + \varepsilon) \lambda_{i, \max} 2^m \sum_{a \in A_i} \int_{-1}^{1} (\gamma_{g_{\varepsilon, m}^{-a}(z)})^{\frac{r}{(1-\varepsilon) \lambda_{i, \min}}} dz$$

$$= (1 + \varepsilon) \lambda_{i, \max} 2^m \sum_{k=0}^{2^m-1} \int_{-1+k2^{m-1}}^{1+(k+1)2^{m-1}} (\gamma_z, \gamma_z)^{\frac{r}{(1-\varepsilon) \lambda_{i, \min}}} dz.$$

Hence

$$\frac{1}{2^{2m} r^2} \sum_{i=1}^{l} p_i^2 \sum_{a, b \in A_i} \int_{-1}^{1} (\sigma_{a, z}, \sigma_{b, z})_r dz$$

$$\leq \frac{1}{2^{2m} r^2} \sum_{i=1}^{l} p_i^2 (1 + \varepsilon) \lambda_{i, \max} 2^m \sum_{k=0}^{2^m-1} \int_{-1+k2^{m-1}}^{1+(k+1)2^{m-1}} (\gamma_z, \gamma_z)^{\frac{r}{(1-\varepsilon) \lambda_{i, \min}}} dz.$$
\[
\begin{align*}
&\leq \sum_{i=1}^{l} p_i^2 \frac{1}{((1-\varepsilon)\lambda_{i,\text{min}})^2} \left( \frac{1}{(1-\varepsilon)\lambda_{i,\text{min}}} \right)^2 \int_{-1}^{1} (\gamma_z, \gamma_z) \frac{r}{(1-\varepsilon)\lambda_{i,\text{min}}} \, dz \\
&\leq \max_i J \left( \frac{r}{\lambda_{i,\text{min}}(1-\varepsilon)} \right) \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\text{max}}}{((1-\varepsilon)\lambda_{i,\text{min}})^2}.
\end{align*}
\]

For the second summand in (2.8), we use (2.7) to prove that it is bounded by

\[
(2.10) \quad \frac{1}{2m^2} m \sum_{i \neq j} p_i p_j \sum_{a \in A_i} \sum_{b \in A_j} \int_{-1}^{1} (\sigma_{a,z}, \sigma_{b,z}) \, r \, dz \\
\leq \frac{1}{2m} \sum_{i \neq j} p_i p_j \sum_{a \in A_i} \sum_{b \in A_j} \frac{8r^2}{C_{\varepsilon,m}} \leq \frac{8}{C_{\varepsilon,m}}.
\]

By combining (2.9) and (2.10) we have

\[
(2.11) \quad J(r) \leq \frac{8}{C_{\varepsilon,m}} + \beta \max_i J \left( \frac{r}{\lambda_{i,\text{min}}(1-\varepsilon)} \right)
\]

where \( \beta = \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\text{max}}}{((1-\varepsilon)\lambda_{i,\text{min}})^2} \) is less than 1 by (1.6).

We define recursively a strictly decreasing sequence \( r_k \). Let \( r_0 < 1/2 \) be fixed. Assume that \( r_{k-1} \) has been defined. Then we define \( r_k = (1-\varepsilon)\lambda_{i_k,\text{min}} r_{k-1} \), where \( i_k \) is chosen such that

\[
\max_i J \left( \frac{r_k}{(1-\varepsilon)\lambda_{i,\text{min}}} \right) = J \left( \frac{r_k}{(1-\varepsilon)\lambda_{i_k,\text{min}}} \right) = J(r_{k-1}).
\]

Hence \( r_k = r_0 (1-\varepsilon)^k \prod_{n=1}^{k} (\lambda_{i_n,\text{min}}) \).

We note that \( r_k \) is a well defined sequence. By induction and (2.11), we have

\[
(2.12) \quad J(r_k) \leq \frac{8}{C_{\varepsilon,m}} + \beta^k J(r_0)
\]

for every \( k \geq 1 \). Hence by (2.1), (2.2) and (2.12) we get

\[
(2.13) \quad \|\nu_{\varepsilon,m}\|_2^2 \leq 4 \lim_{r \to 0} \inf J(r) \leq 4 \lim_{k \to \infty} \inf J(r_k) \leq \frac{32}{C_{\varepsilon,m}} \frac{1}{1 - \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\text{max}}}{((1-\varepsilon)\lambda_{i,\text{min}})^2}}.
\]

We now use the fact that a closed ball in the Hilbert space \( L^2 \) is compact in the weak topology. (See for instance Theorem V.2.1 in Yosida’s book [16].) Hence, if \( h_{\nu_{\varepsilon,m}} \) is the density of \( \nu_{\varepsilon,m} \), then \( h_{\nu_{\varepsilon,m}} \) is in \( L^2 \), and from the above we know that there is a constant \( C'_\varepsilon \) such that \( \|h_{\nu_{\varepsilon,m}}\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon} \).
By the compactness statement above, there is an $h$ with $\|h\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$ such that some subsequence of $h_{\nu_{\varepsilon,m}}$ converges weakly to $h$. Moreover $h$ defines a probability measure since $1 = \int 1 \cdot h_{\nu_{\varepsilon,m}} d\mathcal{L}_3 \to \int 1 \cdot h d\mathcal{L}_3$.

Since $\nu_{\varepsilon,m}$ converges weakly to $\nu_\varepsilon$ it follows that $\nu_\varepsilon$ has density in $L^2$ and

$$\|\nu_\varepsilon\|_2 \leq \frac{1}{\sqrt{\varepsilon}} C'_\varepsilon,$$

where

$$C'_\varepsilon = \sqrt{\frac{32}{(1 - \sum_{i=1}^l p_i^2 (1+\varepsilon)\lambda_{i,\max}})} C''_\varepsilon,$$

$$C''_\varepsilon = \lim_{m \to \infty} C_{\varepsilon,m} = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon (-|a_i + a_j| - 2)}{1 - \varepsilon^2} \right\}.$$

3. Proof of Theorem 1.5. We do not give the whole proof of Theorem 1.5 because it is similar to the proof of Theorem 1.2. We only prove a modification of Lemma 2.2, which is important as it proves transversality.

First we define a new dynamical system $\tilde{g}_{\varepsilon,m} : Q \to Q$, similar to the dynamical system $g_{\varepsilon,m} : Q \to Q$. Let $Q_{i,k}$ and $A_{i,k}$ be as in Section 2. Let $\tilde{g}_{\varepsilon,m} : Q \to Q$ be defined by

$$\tilde{g}_{\varepsilon,m} : (x, y, z) \mapsto \left( \tilde{d}(z)x + a_i(1 - \tilde{d}(z)), \frac{1}{p_i}y + b(y), 2^mz + c(z) \right)$$

for $(x, y, z) \in Q_i$, where

$$\tilde{d}(z) = \lambda_i + 2^m\varepsilon(z - (-1 + (k + 1/2)2^{-m+1})) \quad \text{for } (x, y, z) \in Q_{i,k},$$

$$b(y) = 1 - \frac{1}{p_i} \left( -1 + 2 \sum_{j=1}^i p_j \right) \quad \text{for } (x, y, z) \in Q_{i,k},$$

$$c(z) = 2^m - 2k - 1 \quad \text{for } (x, y, z) \in Q_{i,k}.$$

Hence the only difference between $\tilde{g}_{\varepsilon,m}$ and $g_{\varepsilon,m}$ is in the first coordinate, where the perturbation of $f_i$ is made. Figure 1 also serves to visualise the action of $\tilde{g}_{\varepsilon,m}$.

We define the cones

$$C_p = \left\{ (u, v, w) \in T_p Q : \left| \frac{u}{w} \right|, \left| \frac{v}{w} \right| < \frac{2^m+1\varepsilon}{2^m - \lambda_{\max} - \varepsilon} \right\},$$

where $p \in Q$ and $\lambda_{\max} = \max_i \lambda_i$. Similar to Lemma 2.2, we show that these cones define a family of unstable cones, and that a certain transversality property holds.

**Lemma 3.1.** Suppose that (1.9) holds. The cones $C_p$ form a family of unstable cones, that is, $d_p \tilde{g}_{\varepsilon,m}(C_p) \subset C'_{\tilde{g}_{\varepsilon,m}(p)}$. 
Moreover, for sufficiently large \( m \) and every sufficiently small \( \epsilon > 0 \), if \( \zeta_1 \subset Q_{\xi_1} \) and \( \zeta_2 \subset Q_{\xi_2} \) are two line segments with tangents in \( C_p \) such that \( \xi_1 \in A_i \) and \( \xi_2 \in A_j \), \( i \neq j \), then if \( \tilde{g}_{\epsilon,m}(\zeta_1) \) and \( \tilde{g}_{\epsilon,m}(\zeta_2) \) intersect, and if \((u_1,v_1,1)\) and \((u_2,v_2,1)\) are tangents to \( \tilde{g}_{\epsilon,m}(\zeta_1) \) and \( \tilde{g}_{\epsilon,m}(\zeta_2) \) respectively, there exists a constant \( C_{\epsilon,m} \), depending on \( \epsilon \) and \( m \), but bounded away from 0 and infinity, such that \( |u_1 - u_2| > C_{\epsilon,m}\epsilon \).

**Proof.** The Jacobian of \( \tilde{g}_{\epsilon,m} \) is

\[
d_p\tilde{g}_{\epsilon,m} = \begin{pmatrix}
\tilde{d}(z) & 0 & 2^m\epsilon(x - a_i) \\
0 & 1/p_i & 0 \\
0 & 0 & 2^m
\end{pmatrix},
\]

where \( p = (x, y, z) \in Q_{i,k} \). If \( (u, v, w) \in C_p \), then

\[
d_p\tilde{g}_{\epsilon,m}(u, v, w) = \begin{pmatrix}
\tilde{d}(z)u + 2^m\epsilon(x - a_i)w \\
(1/p_i)v \\
2^m w
\end{pmatrix}.
\]

The estimate

\[
\frac{|\tilde{d}(z)u + 2^m\epsilon(x - a_i)w|}{|2^m w|} \leq \frac{\tilde{d}(z)|u|}{2^m|w|} + 2\epsilon \leq \frac{\lambda_i + \epsilon}{2^m} \cdot \frac{2^{m+1}\epsilon}{2^m - \lambda_{\max} - \epsilon} + 2\epsilon \leq \frac{2^{m+1}\epsilon}{2^m - \lambda_{\max} - \epsilon}
\]

shows that \( d_p\tilde{g}_{\epsilon,m}(C_p) \subset C_{\tilde{g}_{\epsilon,m}(p)} \). Now we prove the other statement of the lemma. Assume that \( p = (x_p, y_p, z_p) \in Q_i \) and \( q = (x_q, y_q, z_q) \in Q_j \), \( i \neq j \), are such that \( \tilde{g}_{\epsilon,m}(p) = \tilde{g}_{\epsilon,m}(q) = (x, y, z) \). Then

\[
p \in Q_i \Rightarrow d_p\tilde{g}_{\epsilon,m} : (u, v, 1) \mapsto 2^m\left(\frac{\tilde{d}(z_p)}{2^m}u + (x_p - a_i)\epsilon, \frac{v}{p_i}, 1\right),
\]

and

\[
x_p = \frac{x - a_i(1 - \tilde{d}(z_p))}{\tilde{d}(z_p)}, \quad x_q = \frac{x - a_j(1 - \tilde{d}(z_q))}{\tilde{d}(z_q)}.
\]

Let \( \tilde{\Delta}_i = \frac{2(\lambda_i + \epsilon)}{2^m - \lambda_{\max} - \epsilon} \). Then

\[
d_p\tilde{g}_{\epsilon,m}(C_p) \subset \left\{ w(u, v, 1) : \frac{x - a_i}{d(z_p)}\epsilon - \tilde{\Delta}_i\epsilon \leq u \leq \frac{x - a_i}{d(z_p)}\epsilon + \tilde{\Delta}_i\epsilon \right\}.
\]

Therefore

\[
|u_2 - u_1| \geq \left( \left| \frac{x - a_i}{d(z_p)} - \frac{x - a_j}{d(z_q)} \right| - (\tilde{\Delta}_i + \tilde{\Delta}_j) \right) \epsilon.
\]
The term
\[ \left| \frac{x - a_i}{\tilde{d}(z_p)} - \frac{x - a_j}{\tilde{d}(z_q)} \right| \]
can be estimated by
\[ \left| \frac{x - a_i}{\tilde{d}(z_p)} - \frac{x - a_j}{\tilde{d}(z_q)} \right| \geq \left| \frac{\tilde{d}(z_p) - \tilde{d}(z_q)}{\tilde{d}(z_p)\tilde{d}(z_q)} \right| \left| \frac{\tilde{d}(z_p) - \tilde{d}(z_q)}{\tilde{d}(z_p)\tilde{d}(z_q)} \right| \left| \frac{\tilde{d}(z_p) - \tilde{d}(z_q)}{\tilde{d}(z_p)\tilde{d}(z_q)} \right| \]

Hence, this term is positive provided that
\[ |a_j\tilde{d}(z_p) - a_i\tilde{d}(z_q)| > |\tilde{d}(z_p) - \tilde{d}(z_q)|. \]

Since \( \lambda_i - \varepsilon \leq \tilde{d}(z_p) \leq \lambda_i + \varepsilon \) and \( \lambda_j - \varepsilon \leq \tilde{d}(z_q) \leq \lambda_j + \varepsilon \), this is implied by (1.9) if \( \varepsilon \) is sufficiently small.

If we let
\[ C_{\varepsilon,m} = \frac{1}{2} \min_{i \neq j} \frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i\lambda_j}, \]
then
\[ |u_2 - u_1| \geq C_{\varepsilon,m} \varepsilon, \]
provided that \( \varepsilon \) is small and \( m \) large.

In fact we can let
\[ C_{\varepsilon,m} = \sigma \min_{i \neq j} \frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i\lambda_j} \]
for some \( 0 < \sigma < 1. \]

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