Hyperbolic manifolds with convex boundary

Jean-Marc Schlenker∗

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Abstract

Let $(M, \partial M)$ be a 3-manifold, which carries a hyperbolic metric with convex boundary. We consider the hyperbolic metrics on $M$ such that the boundary is smooth and strictly convex. We show that the induced metrics on the boundary are exactly the metrics with curvature $K > -1$, and that the third fundamental forms of $\partial M$ are exactly the metrics with curvature $K < 1$, for which the closed geodesics which are contractible in $M$ have length $L > 2\pi$. Each is obtained exactly once.

Other related results describe existence and uniqueness properties for other boundary conditions, when the metric which is achieved on $\partial M$ is a linear combination of the first, second and third fundamental forms.

Résumé

Soit $(M, \partial M)$ une variété de dimension 3, qui admet une métrique hyperbolique à bord convexe. On considère l’ensemble des métriques hyperboliques sur $M$ pour lesquelles le bord est lisse et strictement convexe. On montre que les métriques induites sur le bord sont exactement les métriques à courbure $K > -1$, et que les troisièmes formes fondamentales du bord sont exactement les métriques à courbure $K < 1$ dont les géodésiques fermées qui sont contractiles dans $M$ sont de longueur $L > 2\pi$. Chacune est obtenue pour une unique métrique hyperbolique sur $M$.

D’autres résultats, reliés à ceux-ci, décrivent des propriétés d’existence et d’unicité pour d’autres conditions au bord, lorsque la métrique qu’on prescrit sur $\partial M$ est une combinaison linéaire des première, seconde et troisième formes fondamentales.

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∗Laboratoire Emile Picard, UMR CNRS 5580, UFR MIG, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4, France. schlenker@picard.ups-tlse.fr; http://picard.ups-tlse.fr/~schlenker.
Convex surfaces in $H^3$ Consider a compact, convex subset $\Omega$ of $H^3$. Its boundary $\partial \Omega$ is a convex sphere. Suppose that $\partial \Omega$ is smooth and strictly convex (i.e. its principal curvatures never vanish); its induced metric, which is also called its "first fundamental form" and denoted by $I$, then has curvature $K > -1$.

A well-known theorem, due mainly to Aleksandrov [Ale58a] and Pogorelov [Pog73], states that, conversely, each smooth metric with curvature $K > -1$ on $S^2$ is induced on a unique smooth, strictly convex surface in $H^3$. Analogous results also hold in the Euclidean 3-space [Nie53] and in the 3-sphere.

We consider here this question, in the more general setting where the sphere (which can be considered as the boundary of a ball) is replaced by the boundary of a rather arbitrary 3-manifold $M$. That is, we will show that the hyperbolic metrics on $M$ with smooth, strictly convex boundary are uniquely determined by the induced metrics on the boundary, and that, under mild topological assumptions on $M$, the possible induced metrics on the boundary are exactly the metrics with curvature $K > -1$. This statement was conjectured by Thurston, and the existence part was already proved by Labourie [Lab92].

The third fundamental form There is another metric, the third fundamental form, which is defined on a smooth, strictly convex surface $S$ in $H^3$. To define it, let $N$ be a unit normal vector field to $S$, and let $\nabla$ be the Levi-Civita connection of $H^3$; the shape operator $B : TS \to TS$ of $S$ is defined by: $Bx := -\nabla_x N$. It satisfies the Gauss equation, $\det(B) = K + 1$, and the Codazzi equation, $d\nabla B = 0$, where $\nabla$ is the Levi-Civita connection of the induced metric $I$, $K$ is its curvature, and $d\nabla B(x, y) := (\nabla_x B)y - (\nabla_y B)x$.

The second fundamental form of $S$ is defined by:

$$\forall s \in S, \forall x, y \in T_s S, \quad II(x, y) = I(Bx, y) = I(x, By),$$

and its third fundamental form by:

$$\forall s \in S, \forall x, y \in T_s S, \quad III(x, y) = I(Bx, By).$$

The third fundamental form is “dual” to the induced metric on surfaces, in a precise way that is explained in section 1. In particular, some questions about the third fundamental form of surfaces in $H^3$ can be translated into questions about isometric embeddings in a dual space, the de Sitter space, whose definition is also given in section 1.

For smooth, strictly convex spheres in $H^3$, the possible third fundamental forms are the metrics with curvature $K < 1$ and closed geodesics of lengths $L > 2\pi$, and each is obtained in exactly one way (see Sch94, Sch96). We will consider here an extension of this result to the setting of hyperbolic manifolds with convex boundary, and obtain a description of the possible third fundamental forms of hyperbolic manifolds with smooth, strictly convex boundary.

Main results In all the paper, we consider a compact, connected 3-manifold $M$ with boundary. We will suppose that $M$ admits a complete, hyperbolic, convex co-compact metric. This is a topological assumption, and could easily be stated in purely topological terms (see e.g. Thun84). It clearly implies that $M$ also admits a hyperbolic metric for which the boundary is a smooth, strictly convex surface, this can be seen by smoothing the set of points at distance 1 from the convex core in a complete, convex co-compact hyperbolic metric. In addition, we will suppose in most of the paper — until the second part of section 9 — that $M$ is not a solid torus; this is a special case that is treated separately at the end of section 9. Thus each boundary component of $M$ is a compact surface of genus at least 2, except in the simpler case where $M$ is a ball. The two main results are then as follows.

**Theorem 0.1.** Let $g$ be a hyperbolic metric on $M$ such that $\partial M$ is smooth and strictly convex. Then the induced metric $I$ on $\partial M$ has curvature $K > -1$. Each smooth metric on $\partial M$ with $K > -1$ is induced on $\partial M$ for a unique choice of $g$.

**Theorem 0.2.** Let $g$ be a hyperbolic metric on $M$ such that $\partial M$ is smooth and strictly convex. Then the third fundamental form $III$ of $\partial M$ has curvature $K < 1$, and its closed geodesics which are contractible in $M$ have length $L > 2\pi$. Each such metric is obtained for a unique choice of $g$.

**Known cases** Theorems 0.1 and 0.2 provide a new light on some previous results. The existence part of Theorem 0.1 is a result of Labourie [Lab92]. When $M$ is a ball, Theorem 0.1 is a consequence of the work, already mentioned, of Aleksandrov and Pogorelov, while Theorem 0.2 already stands in Sch94. Moreover,
Theorem 0.2 was also known in the special case where $M$ is “fuchsian” (i.e. it has an isometric involution fixing a closed surface), see [LS00].

When one considers polyhedra instead of smooth surfaces, the analog of Theorem 0.1 when $M$ is a ball was obtained by Aleksandrov [Ale58], while the analog of Theorem 0.2 was obtained by Rivin and Hodgson [Riv96, RH93], following work of Andreev [And70]. Going to the category of ideal polyhedra, we find results of Andreev [And71] and Rivin [Riv93]. Moreover, in this setting, results on the fuchsian case of the analog of Theorem 0.2 are hidden in works on circle packings (see [Thurston, chapter 13 or [CDV91]) or on similarity structures (see [Rivin]). The analog of Theorem 0.2 when the topology of $M$ is “general”, but when $\partial M$ is locally like an ideal polyhedron, is in [Sch01c]. Finally analogs of Theorems 0.1 and 0.2 also hold for hyperideal polyhedra when $M$ is a ball, see [Sch03a] and the result of Bao and Bonahon [BB02], and when $M$ is “fuchsian”, a result of Rousset [Rou04]. Here again the topologically general case of the analog of Theorem 0.2 holds, see [Sch02a]. This is also related to results on complete smooth surfaces, see [Sch03b]. Thus several important results on compact or ideal polyhedra can be seen as polyhedral versions of Theorems 0.1 and 0.2 in the special case where $M$ has no topology.

Another setting where results similar to Theorems 0.1 and 0.2 should hold is for the convex cores of hyperbolic convex co-compact manifolds. Their induced metrics are hyperbolic, and the fact that each hyperbolic metric can be obtained is a consequence either of Labourie’s result [Lab92] or of the known (weak) form of a conjecture of Sullivan, see [EM86] (or [BC93] when the boundary of $M$ is compressible). The uniqueness, however, remains elusive. The questions concerning the third fundamental form can be stated in terms of bending lamination, and existence results here were obtained recently by Bonahon and Otal, and by LeCuire [BO01, Lect02]; the uniqueness is known only when the bending lamination is supported by closed curves, in this case it is proved in [BO01].

**Other conditions** As a consequence of Theorems 0.1 and 0.2 we can fairly easily find corollaries which are also generalizations. They describe other metrics, defined from the extrinsic invariants of the boundary, that can be obtained is a consequence either of Labourie’s result [Lab92] or of the known (weak) form of a conjecture of Sullivan, see [EM86] (or [BC93] when the boundary of $M$ is compressible). The uniqueness, however, remains elusive. The questions concerning the third fundamental form can be stated in terms of bending lamination, and existence results here were obtained recently by Bonahon and Otal, and by LeCuire [BO01, Lect02]; the uniqueness is known only when the bending lamination is supported by closed curves, in this case it is proved in [BO01].

**Theorem 0.3.** Let $k_0 \in (0, 1)$. Let $h$ be a Riemannian metric on $\partial M$. There exists a hyperbolic metric $g$ on $M$, such that the principal curvatures of the boundary are between $k_0$ and $1/k_0$, and for which:

$$I - 2k_0 II + k_0^2 III = h$$

if and only if $h$ has curvature $K \geq -1/(1 - k_0^2)$, $g$ is then unique.

**Theorem 0.4.** Let $k_0 \in (0, 1)$. Let $h$ be a Riemannian metric on $\partial M$. There exists a hyperbolic metric $g$ on $M$, such that the principal curvatures of the boundary are between $k_0$ and $1/k_0$, and for which:

$$k_0^2 I - 2k_0 II + III = h$$

if and only if $h$ has curvature $K \leq 1/(1 - k_0^2)$, and its closed geodesics which are contractible in $M$ have length $L > 2\pi/\sqrt{1 - k_0^2}$, $g$ is then unique.

When one takes the limit $k_0 \to 0$ in those statements, one finds Theorems 0.1 and 0.2. The proofs, which are done in section 9, are based on a simple trick using normal deformations of the boundary.

In addition, one can also prescribe another “metric” on the boundary, the “horospherical metric” $I + 2II + III$, see [Sch02a]. This is done by very different methods, and has a different flavor.

Putting all the results above together and adding an elementary scaling argument, we could state a general result describing existence and uniqueness properties for boundary conditions where one prescribes a metric on $\partial M$ of the form:

$$aI - 2\sqrt{ab}II + bIII, \text{ with } a, b \geq 0 \text{ and } (a, b) \neq (0, 0).$$

On the other hand, one should not be too optimistic about what sort of boundary conditions one can prescribe. For instance, just by considering the spheres of radius $r \in (0, \infty)$ one can show that many metrics made from linear combination of the extrinsic invariants of the boundary do not lead to any infinitesimal rigidity result.
Outline of the proofs  We now give a brief outline of the proofs of Theorems 1.1 and 1.2. Both proofs — and they are closely related — use a deformation argument, and show that a natural map between two spaces is a homeomorphism.

Definition 0.5. We call:

- \( \mathcal{G} \) the space of hyperbolic metrics on \( M \), with strictly convex, smooth boundary, considered up to isotopy.
- \( \mathcal{H} \) the space of \( C^\infty \) metrics on \( \partial M \) with curvature \( K > -1 \), up to isotopy.
- \( \mathcal{H}^* \) the space of \( C^\infty \) metrics on \( \partial M \) with curvature \( K < 1 \), with closed geodesics of length \( L > 2\pi \) when they are contractible in \( M \), up to isotopy.
- \( \mathcal{F} : \mathcal{G} \to \mathcal{H} \) the map sending a hyperbolic metric on \( M \) to the induced metric on the boundary.
- \( \mathcal{F}^* : \mathcal{G} \to \mathcal{H}^* \) the map sending a hyperbolic metric on \( M \) to the third fundamental form of the boundary.

The main point will be that \( \mathcal{F} \) and \( \mathcal{F}^* \) are homeomorphisms. This will follow from the following facts:

- they are both Fredholm operators of index 0;
- they are both locally injective, i.e. their differential is injective at each point of \( \mathcal{G} \);
- they are proper;
- \( \mathcal{G} \) is connected, while both \( \mathcal{H} \) and \( \mathcal{H}^* \) are simply connected.

The main technical point of the paper is the local injectivity statement, which can naturally be formulated as infinitesimal rigidity: given a hyperbolic metric \( g \) on \( M \) with smooth, strictly convex boundary, one can not deform it infinitesimally without changing the induced metric on the boundary (resp. the third fundamental form of the boundary). There are several possible proofs of this for convex surfaces in \( H^3 \), for instance based on an index argument [Lab89], and for convex polyhedra in \( H^3 \) it can be proved using an argument already used by Legendre [LegII] and Cauchy [Cau13] in \( \mathbb{R}^3 \), see [RH93]. In our context, however, things are different since the “relative position” of the boundary components has to be taken into account.

Lemma 6.1. Let \( g \in \mathcal{G} \). For any non-trivial first-order deformation \( \dot{g} \in T_g \mathcal{G} \), the induced first-order variation of the induced metric on \( \partial M \) is non-zero.

Lemma 6.10. Let \( g \in \mathcal{G} \). For any non-trivial first-order deformation \( \dot{g} \in T_g \mathcal{G} \), the induced first-order variation of the third fundamental form of \( \partial M \) is non-zero.

Those lemmas are the subject of sections 1 through 6. Section 1 starts with some basic well-known results on hyperbolic 3-space, and also contains some less standard constructions, like the hyperbolic-de Sitter duality and the Pogorelov Lemma on isometric deformations of surfaces. Section 2 gives details about the geometry of hyperbolic manifolds with convex boundary and their universal covers, and explains how the infinitesimal rigidity question can be translated as a Euclidean geometry statement. Section 3 is concerned with some aspects of the rigidity of smooth, strictly convex surfaces in \( \mathbb{R}^3 \) (or in \( H^3 \)), and section 4 contains some estimates on geometric objects associated to a first-order deformation of the metric. All those elements are then used in sections 5 and 6 to prove Lemmas 6.1 and 6.10.

We also have to prove that \( \mathcal{F} \) and \( \mathcal{F}^* \) are proper; this can be formulated as a compactness property, more precisely as the following lemmas.

Lemma 7.1. Let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of hyperbolic metrics on \( M \) with smooth, strictly convex boundary. Let \( (h_n)_{n \in \mathbb{N}} \) be the sequence of induced metrics on the boundary. Suppose that \( (h_n) \) converges to a smooth metric \( h \) on \( \partial M \), with curvature \( K > -1 \). Then there is a sub-sequence of \( (g_n) \) which converges to a hyperbolic metric \( g \) with smooth, convex boundary.

Lemma 7.2. Let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of hyperbolic metrics on \( M \) with smooth, strictly convex boundary. Let \( (h_n)_{n \in \mathbb{N}} \) be the sequence of third fundamental forms of the boundary. Suppose that \( (h_n) \) converges to a smooth metric \( h \) on \( \partial M \), with curvature \( K < 1 \) and closed geodesics of length \( L > 2\pi \) when they are contractible in \( M \). Then there is a sub-sequence of \( (g_n) \) which converges to a hyperbolic metric \( g \) with smooth, convex boundary.
They are proved in section 7, using an approach previously developed in [Lab89, Sch96] for isometric embeddings of convex surfaces in Riemannian (resp. Lorentzian) 3-manifolds. Section 8 follows with the proof that \( \mathcal{G} \) is connected, and that \( \mathcal{H} \) and \( \mathcal{H}^* \) are simply connected.

The third ingredient is the fact that \( \mathcal{F} \) and \( \mathcal{F}^* \) are Fredholm:

**Lemma 9.1** \( \mathcal{F} \) and \( \mathcal{F}^* \) are Fredholm maps of index 0.

This is related to the elliptic nature of the PDE problem underlying this situation. The proof is in section 9. The end of the proofs of Theorems 0.1 and 0.2 stand also in section 9, and section 10 contains the proofs of Theorems 0.3 and 0.4 as well as some remarks and open questions.

1 Background on hyperbolic geometry

This section contains various constructions and results on hyperbolic 3-space. Some are completely classical, and given here only for reference elsewhere in the text, while other are less well-known.

**Hyperbolic 3-space and the de Sitter space** Hyperbolic 3-space can be constructed as a quadric in the Minkowski 4-space \( \mathbb{R}^{4,1} \), with the induced metric:

\[
H^3 := \{ x \in \mathbb{R}^{4,1} \mid \langle x, x \rangle = -1 \wedge x_0 > 0 \}.
\]

But \( \mathbb{R}^{4,1} \) also contains another quadric, which is the called the de Sitter space of dimension 3:

\[
S_3^3 := \{ x \in \mathbb{R}^{4,1} \mid \langle x, x \rangle = 1 \}.
\]

By construction, it is Lorentzian and has an action of \( \text{SO}(3,1) \) which is transitive on orthonormal frames, so it has constant curvature; one can easily check that its curvature is 1. \( S_3^3 \) contains many space-like totally geodesic 2-planes, each isometric to \( S^2 \) with its canonical round metric. Each separates \( S_3^3 \) into two "hemispheres", each isometric to a "model" which we call \( S^3_{1,+} \), and which we will take to be, in the quadric above, the set of points \( x \in S_3^3 \) with positive first coordinate.

**Projective models** Using the definitions of \( H^3 \) and \( S_3^3 \) as quadrics, given above, we can define the projective models of \( H^3 \) and of the de Sitter hemisphere \( S_{1,+}^3 \). We will call \( D^3 \) the ball of radius 1 and center 0 in \( \mathbb{R}^3 \). We will be using the classical projective model of hyperbolic space, which is well known, and the almost as classical projective model of the de Sitter space.

**Definition 1.1.** Let:

\[
\begin{align*}
\phi_H : & \quad H^3 \to D^3 \\
& \quad x \leadsto x/x_0,
\end{align*}
\]

\[
\begin{align*}
\phi_S : & \quad S_{1,+}^3 \to \mathbb{R}^3 \setminus \overline{D^3} \\
& \quad x \leadsto x/x_0,
\end{align*}
\]

\[
\begin{align*}
\phi : & \quad H^3 \cup S_{1,+}^3 \to \mathbb{R}^3 \\
& \quad x \leadsto \begin{cases} 
\phi_H(x) & \text{if } x \in H^3, \\
\phi_S(x) & \text{if } x \in S_{1,+}^3.
\end{cases}
\end{align*}
\]

The main property of those projective models is that they send geodesic segments to segments of \( \mathbb{R}^3 \). \( \phi_H \) sends a point at distance \( \rho \) from \( x_0 \) in \( H^3 \) to a point at distance \( \tanh(\rho) \) from 0 in \( \mathbb{R}^3 \). We call **radial** the direction of the geodesic from \( x \) to \( x_0 \), and **lateral** the directions orthogonal to the radial direction. Then \( d_{x,\phi_H} \):

- sends the radial direction to the radial direction, and the lateral directions to the lateral directions in \( \mathbb{R}^3 \).
- "contracts" the lateral directions by a factor \( 1/\cosh(\rho) \), i.e. it multiplies the metric in those directions by \( 1/\cosh^2(\rho) \).
- "contracts" the radial direction by a factor \( 1/\cosh^4(\rho) \), i.e. it multiplies the metric in those directions by \( 1/\cosh^4(\rho) \).
The description of $\phi_S$ can be done along similar lines, with the distance to $x_0$ replaced by the distance to the totally geodesic plane dual to $x_0$ (see below for the definition of the duality) and the $\cosh$ essentially replaced by a $\sinh$.

We will also need the following rather explicit description of the properties of the geodesic spheres.

**Proposition 1.2.** For each $\rho \in \mathbb{R}_+$, call $I_\rho$ and $I_\rho$ the induced metric and second fundamental form of the sphere of radius $\rho$ in $H^3$; for each $t \in (0,1)$, call $I_t$ and $I_t$ the induced metric and second fundamental form of the sphere of radius $t$ in $\mathbb{R}^3$. Then, for $t = \tanh(\rho)$, we have:

$$I_\rho = \cosh^2(\rho) I_t, \quad I_\rho = \cosh^2(\rho) I_t.$$  

Proof. Let $\text{can}_{S^2}$ be the canonical metric on $S^2$, then $I_\rho = \sinh^2(\rho) \text{can}_{S^2}$, $I_\rho = \sinh(\rho) \cosh(\rho) \text{can}_{S^2}$, $I_t = t^2 \text{can}_{S^2}$, $I_t = t \text{can}_{S^2}$: the result follows. $\square$

The important point here is that the scaling of $I$ and $I$ are the same. The same phenomenon also happens for the de Sitter space.

**Proposition 1.3.** For each $t > 1$, let $S_t \subset \mathbb{R}^3$ be the sphere of radius $t$ and center $0$, let $\rho := \coth^{-1}(t)$, and let $S_\rho := \phi_S^{-1}(S_t)$. Let $I_\rho, I_\rho, I_t, I_t$ be the induced metrics and second fundamental forms on $S_\rho$ and $S_t$ respectively. Then:

$$I_\rho = \sinh^2(\rho) I_t, \quad I_\rho = \sinh^2(\rho) I_t.$$  

Proof. It is a simple matter to check that:

$$I_\rho = \cosh^2(\rho) \text{can}_{S^2}, \quad I_t = t^2 \text{can}_{S^2} = \cosh^2(\rho) \text{can}_{S^2},$$

$$I_\rho = \cosh(\rho) \sinh(\rho) \text{can}_{S^2}, \quad I_t = t \text{can}_{S^2} = \cosh(\rho) \text{can}_{S^2},$$

and the result follows. $\square$

### Principal curvatures and projective transformations

An important point later on is that given a compact hyperbolic manifold with smooth, strictly convex boundary, the image in the projective model of its universal cover is a convex set bounded by a surface which has its principal curvatures bounded between two positive constants. This will follow from the following Lemma.

**Lemma 1.4.** Let $\Omega \subset H^3$ be a complete, non-compact convex domain containing $x_0$. Suppose that the principal curvatures of $\partial \Omega$ are between two constants $c_m$ and $c_M$, $c_M > c_m > 0$, and that the norm of the gradient of the distance to $x_0$ (restricted to $\partial \Omega$) is at most $1 - \epsilon$ for some fixed $\epsilon > 0$. Let $R > 0$. Then $\phi(\Omega) \subset D^3$ is a convex domain and, outside the ball of radius $R$, the principal curvatures of the boundary are bounded between $c_m$ and another constant $c_M$ depending on $c_M$, $R$ and $\epsilon$.

The proof of this lemma is based on the fact that, under a projective map, the second fundamental form of surfaces changes conformally, as stated in the following proposition. It is of course also valid in higher dimension, and could be stated in higher codimension too.

**Proposition 1.5.** Let $p : M \to \overline{M}$ be a projective diffeomorphism between two $3$-dimensional space-forms. Let $S \subset M$ be a surface, and let $\overline{S} := p(S)$. Let $N, \overline{N}$ be the unit normal vectors of $S$ and $\overline{S}$, and let $\overline{I}$ and $\overline{I}$ be the second fundamental forms of $S$ and $\overline{S}$, respectively. Then:

$$\overline{I} = \frac{1}{\langle N, p^* N \rangle_M} p_* I.$$  

Proof. Let $x \in S$, and let $\overline{x} := p(x)$. Let $P$ and $\overline{P}$ be the totally geodesic planes tangent to $S$ at $x$ and to $\overline{S}$ at $\overline{x}$, respectively — such planes exist since $M$ and $\overline{M}$ are space-forms. By definition of a projective transformation, $\overline{P} = p(P)$.

For each $y \in P$ close to $x$, consider the geodesic $g_y$ orthogonal to $P$ starting from $y$, parametrized at speed $1$. Let $u(y)$ be such that $g_y(u(y)) \subset S$. Then $\overline{I}$ at $x$ can be defined as the Hessian at $x$ of $u$. Note that the direction of $g_y$ does not really matter, because $S$ is tangent to $P$ at $x$, and we are only interested in the second-order behavior of $u$ at $x$: changing the direction of $g_y$ changes only higher-order terms, if the orthogonal projection of $g_y'(0)$ on the direction orthogonal to $P$ has unit norm.
We can take this definition to \( S \) using \( p, \) we obtain that \( p, \mathcal{I} \) is the Hessian at \( \mathcal{I} \) of the function defined as the parameter, at the intersection with \( S, \) of the geodesic \( \overline{g} \) starting from \( y \in P, \) with \( \overline{g}(0) \) equal to the image by \( p, \) of the unit normal vector of \( P. \) (The function considered here is different from \( u \circ \mathcal{I}^{-1}, \) because the parametrization of the geodesics is different, but the Hessian at \( \mathcal{I} \) remains the same).

Now the definition of \( \mathcal{I} \) described above applies also to \( \overline{I}, \) and it shows that \( \overline{I} \) differs from \( p, \mathcal{I} \) only by a factor equal to \( \langle p, N, \overline{N} \rangle_{\mathcal{I}}. \) The first equality follows. The second equality can be checked by a simple computation, or by applying the first formula to \( p^{-1}. \)

\[ \frac{1}{\cosh^2(\rho)} (\phi_H)_* I \leq \mathcal{T} \leq \frac{C_r}{\cosh^2(\rho)} (\phi_H)_* I. \]

Moreover, \( (\phi_H)_* N \) is easily estimated by the action of \( d\phi_H; \) its lateral component has length at most \( 1/\cosh(\rho), \) and the length of its radial component is between \( 1/(C_r \cosh^2(\rho)) \) and \( 1/\cosh^2(\rho). \) Those bounds on the radial and lateral components of \( (\phi_H)_* N, \) along with the bound on the lateral component of \( \overline{N}, \) show that:

\[ \frac{1}{C_r \cosh^2(\rho)} \leq \langle (\phi_H)_* N, \overline{N} \rangle \leq \frac{C_r}{\cosh^2(\rho)} \]

Proposition \ref{key_proposition} then shows that:

\[ \frac{1}{C_r \cosh^2(\rho)} (\phi_H)_* \mathcal{I} \leq \overline{I} \leq \frac{C_r}{\cosh^2(\rho)} (\phi_H)_* \mathcal{I}. \]

Thus \( I \) and \( \mathcal{I} \) scale by the same factor, up to a coefficient \( C_r^2, \) and so the principal curvatures of \( \partial \Omega \) and \( \phi_H(\partial \Omega) \) are the same, up to the same factor \( C_r^2. \)

\textbf{The hyperbolic-de Sitter duality}  

The study of the third fundamental form of surfaces in \( H^3 \) relies heavily on an important duality between \( H^3 \) and the de Sitter space \( S^3_1. \) Although this duality has been known for some time (see e.g. 
[1983], Chapter 2), it was first used in a deep and fundamentally new way by Rivin and Hodgson \[RH93\]; other approaches to this duality can be found elsewhere, e.g. in \[Sch98a. \] It associates to each point \( x \in H^3 \) a space-like, totally geodesic plane in \( S^3_1, \) and to each point \( y \in S^3_1 \) an oriented totally geodesic plane in \( H^3. \)

It can be defined using the quadric models of \( H^3 \) and \( S^3_1. \) Let \( x \in H^3; \) define \( d_x \) as the line in \( R^3 \) going through 0 and \( x, \) \( d^*_x \) is a time-like line; call \( d^*_x \) the orthogonal space in \( R^3, \) which is a space-like 3-plane. So \( d^*_x \) intersects \( S^3_1 \) in a space-like totally geodesic 2-plane, which we call \( x^*, \) and which is the dual of \( x. \) Conversely, given a space-like totally geodesic plane \( p \subset S^3_1, \) it is the intersection with \( S^3_1 \) of a space-like 3-plane \( P \ni 0. \) Let \( d \) be its orthogonal, which is a time-like line; the dual of \( p \) is the intersection \( d \cap H^3. \)

The same construction works in the opposite direction. Given a point \( y \in S^3_1, \) we call \( d_y \) the oriented line going through 0 and \( y, \) \( d^*_y \) its orthogonal, which is an oriented time-like 3-plane. Then \( y^* := d^*_y \cap H^3 \) is an oriented totally geodesic plane. Forgetting the orientation yields a duality between planes in \( H^3 \) and points in \( S^3_1^+. \)

We can then define the duality on surfaces. Given a smooth, oriented surface \( S \subset H^3, \) its dual \( S^* \) is the set of points in \( S^3_1 \) which are the duals of the oriented planes which are tangent to \( S. \) Conversely, given a smooth, space-like surface \( \Sigma \subset S^3_1, \) its dual is the set \( \Sigma^* \) of points in \( H^3 \) which are the duals of the planes tangent to \( \Sigma. \)

\textbf{Proposition 1.6.}  

\begin{itemize}
  \item If \( S \subset H^3 \) (resp. \( S^3_1 \)) is smooth and strictly convex (resp. smooth, space-like and strictly convex), its dual is smooth and strictly convex.
  \item The induced metric on \( S^* \) is the third fundamental form of \( S, \) and conversely.
  \item For any smooth, strictly convex surface \( S \subset H^3 \) (resp. space-like surface \( S \subset S^3_1 \)), \( (S^*)^* = S. \)
\end{itemize}
The proof of the various points, in the smooth or the polyhedral context, can be found e.g. in \cite{RH93, Sch96, LS00, Sch98a, Sch00}.

There is a purely projective definition of this duality, or more precisely of the closely related duality between \(H^3\) and the de Sitter hemisphere \(S^3_{1,+}\); the duality is now between points in \(S^3_{1,+}\) and non-oriented planes in \(H^3\).

To define it, let \(p\) be a totally geodesic plane in \(H^3\); its image \(\phi_H(p)\) is a disk \(D^3\) in \(\mathbb{R}^3\). The image by \(\phi_S\) of the dual of \(p\) is the unique point \(p^*\) in \(\mathbb{R}^3\) such that the lines going through \(p^*\) and \(\partial p\) are tangent to \(\partial D^3\).

Using it, we can — and we sometimes will — use the duality between surfaces in \(D^3\) and in \(\mathbb{R}^3 \setminus D^3\).

The duality between \(H^3\) and \(S^3_{1,+}\) is not primarily about surfaces, but about couples \((x, P)\), where \(P \subset T_x H^3\) is a plane containing \(x\). Given such a couple, we can define its dual as a couple \((x^*, P^*)\), where \(x^* \subset S^3_{1,+}\) is the dual of \(P\), and \(P^* \subset T_{x^*} S^3_{1,+}\) is the dual of \(x^*\).

Let \((x, P)\) be such a couple, and let \((x^*, P^*)\) be its dual. Given an element \(u \in \text{so}(3, 1)\), it defines a Killing vector field \(X_u\) on \(H^3\), and a Killing vector field \(X_u^*\) on \(S^3_{1,+}\). Moreover, the first-order deformation of \((x, P)\) under the Killing field \(X_u\) induces, through the duality, a first-order displacement \(Y \in T_{x^*} S^3_{1,+}\) of \(x^*\).

**Remark 1.7.** \(Y = X^*(x^*)\).

**Proof.** Consider \(x\), \(x^*\) and the normal vectors \(N\) and \(N^*\) to \(P\) and \(P^*\), respectively, as vectors in \(\mathbb{R}^4\); then \(N = x^*\), while \(x = N^*\). Moreover, if we consider \(u\) as a linear map on \(\mathbb{R}^4\), we see that \(X_u = u(x)\) as a vector in \(\mathbb{R}^4\). The first-order variation of \(x^* = N\) under \(X_u\) is just \(u(N) = u(x^*)\), so that it is equal to \(X^*(x^*)\). \(\square\)

**The Pogorelov map** There is an interesting extension of the projective model of \(H^3\) (and of \(S^3_{1,+}\)). It was first defined and used by Pogorelov \cite{Pog73}, at least for \(H^3\). Pogorelov actually defined a slightly more complicated map, between \(H^3 \times H^3\) and \(\mathbb{R}^3 \times \mathbb{R}^3\), but the version which we will use here is simply its linearization along the diagonal. The reader is referred to \cite{Sch98a} for another, more geometric approach of those questions. A slightly different but similar map is also used in \cite{LS00, Sch98a, Sch01a}.

**Definition 1.8.** We define \(\Phi_H : TH^3 \to TD^3\) as the map sending \((x, v) \in TH^3\) to \((\phi_H(x), w)\), where:

- the lateral component of \(w\) is the image by \(d\phi_H\) of the lateral component of \(v\).
- the radial component of \(w\) has the same direction and the same norm as the radial component of \(v\).

Here are the main properties of this map. The proof which we give here is elementary; it might appear simpler to what can be found e.g. in \cite{LS00, Sch98a}, where the arguments are of a more geometric nature.

**Lemma 1.9.** Let \(S\) be a smooth submanifold in \(H^3\), and let \(v\) be a vector field of \(H^3\) defined on \(S\); then \(v\) is an isometric deformation of \(S\) if and only if \(\Phi_H(v)\) is an isometric deformation of \(\phi_H(S)\). In particular, if \(v\) is a vector field on \(H^3\), then \(v\) is a Killing field if and only if \(\Phi_H(v)\) is a Killing field of \(D^3\).

**Proof.** The second part follows from the first by taking \(S = H^3\).

To prove the first part, we have to check that the Lie derivative of the induced metric on \(S\) under \(v\) vanishes if and only if the Lie derivative of the induced metric on \(\phi_H(S)\) under \(\Phi_H(v)\) vanishes. In other terms, if \(x, y\) are vector fields defined on \(S\), if we call \(g\) and \(\overline{g}\) the metrics on \(H^3\) and \(\mathbb{R}^3\) respectively, and \(\overline{x} = d\phi_H(x), \overline{y} = d\phi_H(y)\), then we have to prove that:

\[
(\mathcal{L}_v g)(x, y) = 0 \iff (\mathcal{L}_{\Phi_H(v)} \overline{g})(\overline{x}, \overline{y}) = 0 .
\]

We will actually prove that the two sides are proportional:

\[
(\mathcal{L}_v g)(x, y) = \cosh^2(\rho)(\mathcal{L}_{\Phi_H(v)} \overline{g})(\overline{x}, \overline{y}) .
\]

We decompose \(v\) into the radial component, \(fN\), where \(N\) is the unit radial vector, and the lateral component \(u\). Abusing notations a little, we also call \(f\) the function \(f \circ \phi_H^{-1}\), defined on the unit ball in \(\mathbb{R}^3\). Then \(\Phi_H(v) = f\overline{N} + \overline{u}\), where \(\overline{N}\) is the unit radial vector in \(\mathbb{R}^3\), and \(\overline{u} := d\phi_H(u)\).

By linearity, it is sufficient to prove equation (1) in the cases where \(x\) and \(y\) are non-zero, and each is either radial or lateral. We consider each case separately, and call \(\nabla\) and \(\overline{\nabla}\) the Levi-Civita connections of \(H^3\) and \(\mathbb{R}^3\) respectively.
1st case: $x$ and $y$ are both radial. Then:

$$(\mathcal{L}_v g)(x, y) = (\mathcal{L}_u g)(x, y) + (\mathcal{L}_{fN} g)(x, y) .$$

But:

$$(\mathcal{L}_u g)(x, y) = g(\nabla_x u, y) + g(x, \nabla_y u) = 0 ,$$

because both $\nabla_x u$ and $\nabla_y u$ are lateral. Moreover:

$$(\mathcal{L}_{fN} g)(x, y) = g(\nabla_x fN, y) + g(x, \nabla_y fN) = df(x)g(y, N) + df(y)g(x, N) .$$

The same computation applies in $\mathbb{R}^3$. In addition, $df(x) = df(\tau)$ by definition of $\tau$, so the scaling comes only from $g(x, N)$ versus $g(\tau, N)$. So equation 1 holds in this case.

2nd case: $x$ and $y$ are both lateral. Then, at a point at distance $\rho$ from $x_0$, with $t := \tanh(\rho)$, we have with Proposition 1.12 that:

$$(\mathcal{L}_u g)(x, y) = (\mathcal{L}_u I_\rho)(x, y) = \cosh^{2}(\rho)(\mathcal{L}_{\tau I_\rho})(\tau, \overline{\tau}) = \cosh^{2}(\rho)(\mathcal{L}_{\overline{\tau}})(\tau, \overline{\tau}) .$$

Moreover:

$$(\mathcal{L}_{fN} g)(x, y) = g(\nabla_x fN, y) + g(x, \nabla_y fN) = -2f N_p(x, y) ,$$

and since the same computation applies in $\mathbb{R}^3$, we see with Proposition 1.12 that:

$$(\mathcal{L}_{fN} g)(x, y) = \cosh^{2}(\rho)(\mathcal{L}_{fN})(\tau, \overline{\tau}) ,$$

and:

$$(\mathcal{L}_v g)(x, y) = \cosh^{2}(\rho)(\mathcal{L}_{\overline{\tau}})(\tau, \overline{\tau}) ,$$

so that equation 1 also holds in this case.

3rd case: $x$ is lateral, while $y$ is radial. We choose an arbitrary extension of $x$ and $y$ as vector fields which remain tangent, resp. orthogonal, to the spheres $\{ \rho = \text{const} \}$. Then:

$$(\mathcal{L}_u g)(x, y) = u.g(x, y) - g([u, x], y) - g(x, [u, y]) = -g(x, [u, y]) ,$$

because $[u, x]$ is tangent to the sphere of radius $\rho$. So, since the Lie bracket does not depend on the metric, we have with Proposition 1.12 that:

$$(\mathcal{L}_u g)(x, y) = -g(x, [u, y]) = -\cosh^{2}(\rho)(\mathcal{L}_{\overline{\tau}})(\tau, \overline{\tau}) = \cosh^{2}(\rho)(\mathcal{L}_{\overline{\tau}})(\tau, \overline{\tau}) .$$

In addition:

$$(\mathcal{L}_{fN} g)(x, y) = g(\nabla_x fN, y) + g(x, \nabla_y fN) = df(x)g(y, N) =$$

$$= df(x) \cosh^{2}(\rho)(\mathcal{L}_{\overline{\tau}})(N, \overline{\tau}) = \cosh^{2}(\rho)(\mathcal{L}_{\overline{\tau}})(\tau, \overline{\tau}) ,$$

again because the same computation applies in $\mathbb{R}^3$. As a consequence, we have again that:

$$(\mathcal{L}_v g)(x, y) = \cosh^{2}(\rho)(\mathcal{L}_{\overline{\tau}})(\tau, \overline{\tau}) ,$$

and 1 still holds.

Remark 1.10. • the proof only uses Proposition 1.12 more precisely the fact that I and II scale in the same way when one goes from $H^3$ to $\mathbb{R}^3$ by $\phi_H$.

• the same proof can be used in different dimensions, and also in spaces of other signatures.

In particular, we can then define in the same way a Pogorelov map for the de Sitter space, as done elsewhere, see e.g. [Sch90, Sch03]. Note that there is another related possible map for the de Sitter space, with values in the Minkowski space, see [Gajb90, LS00].

Definition 1.11. We define $\Phi_S : TS^3_{1, +} \to T(\mathbb{R}^3 \setminus \mathbb{D}^3)$ as the map sending $(x, v) \in TS^3_{1, +}$ to $(\phi_S(x), w)$, where:

• the lateral component of $w$ is the image by $d\phi_S$ of the lateral component of $v$. 

• the radial component of \( w \) has the same direction and the same norm as the radial component of \( v \).

**Lemma 1.12.** Let \( S \) be a smooth submanifold in \( S^3_{1,+} \), and let \( v \) be a vector field of \( S^3_{1,+} \) defined on \( S \); then \( v \) is an isometric deformation of \( S \) if and only if \( \Phi_S(v) \) is an isometric deformation of \( \phi_S(S) \). In particular, if \( v \) is a vector field on \( S^3_{1,+} \), then \( v \) is a Killing field if and only if \( \Phi_S(v) \) is a Killing field of \( \mathbb{R}^3 \).

The proof is the same as for Lemma 1.9 using Proposition 1.3 instead of Proposition 1.2.

Given an element \( u \in \text{so}(3,1) \), we have already seen that it defines a Killing vector field \( X_u \) on \( H^3 \) and a Killing field \( X_u^* \) on \( S^3_1 \). An important remark, which follows directly from the definition of the Pogorelov maps for \( H^3 \) and \( S^3_1 \), is that the images are “compatible”.

**Remark 1.13.** \( \Phi_H(X_u) \) and \( \Phi_S(X_u^*) \) are the restriction to \( D^3 \) and to \( \mathbb{R}^3 \setminus D^3 \), respectively, of the same Killing field of \( \mathbb{R}^3 \).

### Rigidity as a projective property

A direct but striking consequence of the previous paragraph is that infinitesimal rigidity is a purely projective property.

**Lemma 1.14.** Let \( P \) be a smooth surface (resp. a polyhedron) in \( H^3 \) (resp. \( S^3_{1,+} \)). \( P \) is infinitesimally rigid if and only if \( \phi(P) \subset \mathbb{R}^3 \) is infinitesimally rigid.

**Proof.** Suppose that \( P \subset H^3 \) is not infinitesimally rigid. Let \( v \) be a non-trivial infinitesimal isometric deformation of \( P \) in \( H^3 \). Then \( \Phi_H(v) \) is a non-trivial infinitesimal deformation of \( \phi_H(P) \) by Lemma 1.12. The same works in the other direction, and also in \( S^3_{1,+} \) by Lemma 1.12.

The proof also works for geometric objects that are more general than compact polyhedra in \( H^3 \) or space-like compact polyhedra in \( S^3_1 \), this is used for instance in \cite{Sch98a, Sch01a, Sch00}.

One can formulate results like Lemma 1.14 as follows. Consider a polyhedron \( P \in \mathbb{R}^3 \), and consider \( \mathbb{R}^3 \) as a projective space (one could as well replace \( \mathbb{R}^3 \) by \( \mathbb{R}P^3 \), but this would slightly cloud the issue). One can then put a constant curvature metric on \( \mathbb{R}^3 \), which is compatible with the projective structure: it could be a Euclidean metric, or a spherical metric (isometric to a hemisphere of \( S^3 \)), or the structure corresponding to hyperbolic \( 3 \)-space in a ball and \( S^3_{1,+} \) outside. Then the fact that \( P \) is, or is not, infinitesimally rigid, is independent of the constant curvature metric chosen. In particular, convex polyhedra — and this is a projective property — are always infinitesimally rigid, a result of Legendre \cite{LegII} and Cauchy \cite{Cau13}.

#### Necessary conditions on \( \mathbb{I} \)

One should remark that the length condition in Theorem 0.2 is necessary. This is a consequence of the next lemma.

**Lemma 1.15.** Let \( S \) be a strictly convex surface in \( H^3 \), and let \( S^* \) be the dual surface in \( S^3_1 \). The closed geodesics of \( S^* \) have length \( L > 2\pi \).

We refer the reader to \cite{CD95, RH93, Sch08a, Sch98a} for various proofs. We can also remark that there is a dual condition for surfaces in \( S^3_1 \) which are not space-like, see \cite{Sch01a}.

## 2 Hyperbolic manifolds with boundary

After defining, in the previous section, the basic tools that we will need, we will in this section study the hyperbolic and de Sitter surfaces that can be associated to a hyperbolic metric on \( M \).

### The extension of a manifold with convex boundary

First remark that, given a hyperbolic manifold with convex boundary, there is essentially a unique way to embed it isometrically into a complete hyperbolic manifold.

**Definition 2.1.** Let \( M \) be a compact hyperbolic manifold with convex boundary. There exists a unique complete, convex co-compact hyperbolic manifold \( E(M) \), which we call the **extension** of \( M \), in which \( M \) can be isometrically embedded so that the natural map \( \pi_1(M) \to \pi_1(E(M)) \) is an isomorphism.

As a consequence, \( \pi_1 M \) has a natural action on \( H^3 \).
Surfaces in the projective models  Let $g$ be a hyperbolic metric with smooth, strictly convex boundary on $M$. $g$ defines a metric, which we still call $g$, on the universal cover $\tilde{M}$ of $M$. Since $(\tilde{M}, g)$ is simply connected and has convex boundary, it admits a unique (up to global isometries of $H^3$) isometric embedding into $H^3$, and the image is a convex subset $\Omega$, with smooth, strictly convex boundary. Of course $\Omega$ is not compact; its boundary at infinity is the limit set $\Lambda$ of the action of $\pi_1 M$ on $H^3$ which is obtained by considering the extension of the action of $\pi_1 (M)$ on $\Omega$.

In the sequel, we will call $S := \partial \Omega$ the boundary of the image of $\tilde{M}$ in $H^3$. So $S$ is a complete, smooth, strictly convex surface in $H^3$, with $\partial_+ S = \Lambda$. We will also call $S^*$ the dual surface in the de Sitter space $S^3_1$; $S^*$ is a space-like, metrically complete, smooth, strictly convex surface in $S^3_{1,+}$, and its boundary at infinity is also $\Lambda$.

Boundary behavior  It is then necessary to understand more precisely the behavior of $S$ near the boundary, in particular in the projective model of $H^3$; we choose the center $x_0$ of the projective model as a point in $\Omega$. Recall that a geodesic ray starting from $x_0$ is a half-geodesic $\gamma : \mathbb{R}_+ \to H^3$ with $\gamma(0) = x_0$.

**Proposition 2.2.** There exists a constant $c_0 > 1$ such that:

- for any geodesic ray $\gamma$ starting from $x_0$ and ending on $\Lambda$, the distance from the points of $\gamma$ to $S$ varies between $1/c_0$ and $c_0$.
- for any point $x \in \partial \Omega$, there exists a geodesic ray $\gamma \subset \Omega$ starting from $x_0$ and ending on $\Lambda$ which is at distance at most $c_0$ from $x$.

**Proof.** Since $\partial M$ is compact and strictly convex, its principal curvatures are bounded from below by a strictly positive constant $\kappa$. An elementary geometric argument then shows that there exists a constant $\delta > 0$ such that any maximal geodesic segment in $M$ going through a point $x \in M$ at distance at most $\delta$ from $\partial M$ meets $\partial M$ at distance at most $1/\delta$ from $x$. This implies the existence of a lower bound on the distance from the geodesic rays to $\partial M$. The upper bound is obvious since $M$ is compact, so has a bounded diameter.

For the second point note that each point $x \in \partial M$ is at bounded distance from a point $x'_0$ in the compact core $C(M)$ of $M$; and each point $x'_0 \in \partial C(M)$ is either in a complete geodesic which is on $\partial C(M)$, or in an ideal triangle on $\partial C(M)$. In both cases $x'_0$ is at bounded distance from a geodesic $c$ which is on $\partial C(M)$, and which lifts to a complete geodesic in $\Omega$ with both endpoints on $\Lambda$. Going to the universal cover, we see that there exists a constant $C$ such that each point of $\Omega$ is at distance at most $C$ from a complete geodesic $c$ having both endpoints on $\Lambda$.

Let $x_1$ and $x_2$ be the endpoints of $c$ in $\Lambda$. There are geodesic rays $c_1$ and $c_2$ starting at $x_0$ and ending on $x_1$ and $x_2$, respectively. A basic property of hyperbolic triangles is that, for some universal constant $C'$, each point of $c$ is at distance at most $C'$ from either $c_1$ or $c_2$. Therefore, each point of $\Omega$ is at distance at most $C + C'$ from a ray starting at $x_0$ and ending on $\Lambda$.

The convexity of $\partial M$ then allows us to go from the previous uniform estimate to a $C^1$ estimate on the behavior of $S$.

**Corollary 2.3.** On $S$, the gradient of the distance $\rho$ to $x_0$ is bounded from above by a constant $\delta_0 < 1$.

**Proof.** Let $x \in \partial \Omega$. By Proposition 2.2, there exists a geodesic ray $\gamma_0$ starting from $x_0$, at distance at most $c_0$ from $x$. Let $P$ be the totally geodesic plane containing $\gamma_0$ and $x$. By construction, $P \cap \Omega$ is a convex, non-compact subset of $P \cong H^2$.

Let $\gamma$ be the geodesic in $P$ tangent to $\partial (P \cap \Omega)$ at $x$. By convexity, $\partial (P \cap \Omega)$ remains on the side of $\gamma$ containing $\gamma_0$. Let $\gamma_1$ be the geodesic ray starting at $x_0$ and going through $x$. An elementary fact of plane hyperbolic geometry is that there exists a constant $\delta_0(c_0)$ such that, if the angle between $\gamma_1$ and $\gamma$ at $x$ is smaller than $\delta_0(c_0)$, then some point of $\gamma$ is at distance smaller than $1/c_0$ from $\gamma_0$. So some point of $\partial \Omega$ would have to be at distance smaller than $1/c_0$ from $\gamma_0$, and this would contradict the first part of Proposition 2.2. Thus the angle between $\gamma_1$ and $\gamma$ is at least $\delta_0(c_0)$.

We will call $S$ the image by $\phi_H$ of $S$, and we will also call $\Lambda$ — with some abuse of notations — the image on the unit sphere $S^2 \subset \mathbb{R}^3$ of the limit set of $\pi_1 M$. By construction, $S \cup \Lambda = \partial \phi_H(\Omega)$ is a convex, closed surface in $\mathbb{R}^3$ — in particular it is topologically a sphere — and we will see below that it has some degree of smoothness even on $\Lambda$.

Using the projective model to go from $H^3$ to $D^3$, the corollary above translates into a simple estimate on the gradient of the distance $r$ to $0$ on $S$. We state it in the next lemma, along with some important estimates on the behavior on $S$ of the distance along $S$ to the limit set $\Lambda$, which we will call $\delta$ here and later on.
Lemma 2.4. There exists a constant $c_1 > 1$ such that, at all points of $\overline{S}$ where the distance $r$ to $0$ is close enough to 1, we have:

1. the angle $\theta$ between the normal to $\overline{S}$ and the radial direction is bounded from above by: $\theta \leq c_1 \sqrt{1 - r}$.
2. the distance $\delta$ on $\overline{S}$ to the limit set $\Lambda$ satisfies:

$$\frac{\sqrt{1 - r}}{c_1} \leq d_{\overline{S}}(x, \Lambda) \leq c_1 \sqrt{1 - r}.$$ 

Proof. Consider a point $x \in H^3$ far enough from $x_0$. Let $u \in T_x S$ be the unit vector on which $dp$ is maximal. Then $dp(u) \leq 1$. But $r = 1 - \tanh(\rho)$, so that $|d(u)| \leq \cosh^{-2}(\rho)$. Let $\overline{\phi} := \phi_H(x)$ and $\overline{\tau} := d\phi_H(u) \in T_{\overline{x}} \overline{S}$. Then, by Corollary 2.3 and the definition of $\phi_H$, there exists a constant $c > 0$ (depending on the constant $\delta_0$ of Corollary 2.3) such that $\|\overline{\tau}\| \geq c/\cosh(\rho)$. Moreover, it is clear that $\overline{\tau}$ is in the direction of greatest variation of $r$. Therefore, for another constant $c > 0$, $\|d\overline{\tau}\| \leq c/\cosh(\rho)$. Since $r = \tanh(\rho), 1/\cosh(\rho) = \sqrt{1 - r^2}$, and this proves the first point.

For the second point, let $\overline{x} \in \overline{S}$, and let $x := \phi_H^{-1}(\overline{x}) \in H^3$. Let $\gamma$ be a geodesic ray starting at $x_0$ and ending on $\Lambda$, such that the distance from $x$ to $\gamma$ is minimal. According to Proposition 2.2 this distance is between $1/c_0$ and $c_0$.

Let $\overline{\gamma} := \phi_H(\gamma)$, then $\overline{\gamma}$ is a segment starting from 0 and ending on $\Lambda \subset S^2$, and among those segments it is at the smallest distance from $\overline{x}$. If the distance $\rho$ from $x_0$ to $x$ is large enough, the properties of the projective model show that:

$$\frac{1}{2c_0 \cosh(\rho)} \leq d(\overline{\gamma}, \overline{x}) \leq \frac{2c_0}{\cosh(\rho)}.$$ 

Since $1/\cosh(\rho) = \sqrt{1 - r^2} = \sqrt{(1 - r)(1 + r)}$, it follows that:

$$\frac{\sqrt{1 - r}}{2c_0} \leq d(\overline{\gamma}, \overline{x}) \leq 4c_0 \sqrt{1 - r},$$ 

and the estimate on $\delta$ follows. \qed

An important consequence of the first point is that, as $\overline{x} \to \overline{x}_0 \in \Lambda$ on $\overline{S}$, the unit normal vector to $\overline{S}$ at $\overline{x}$ converges to the unit normal vector to $S^2$ at $\overline{x}_0$. Thus $\overline{S}$ is "tangent" to $S^2$ along $\Lambda$.

Limit curvatures Using the previous estimates and the results of the previous section on the behavior of the projective model, we can find some higher-order estimates on $\overline{S}$, concerning its principal curvatures in the neighborhood of $\Lambda$.

Lemma 2.5. There exists a constant $c_2 > 1$ such that, at all points of $\overline{S}$ where the distance $r$ to $\partial D^3$ is small enough, the principal curvatures of $\overline{S}$ are bounded between $c_2$ and $1/c_2$.

Proof. Since $\partial M$ is compact and strictly convex, its principal curvatures are bounded between two constants $c_m$ and $c_M$, with $c_M \geq c_m > 0$. The results then follows from Lemma 1.4 and from Corollary 2.3. \qed

The dual surface The results obtained in this section for $\overline{S}$ are also valid for the dual surface $\overline{S}^*$. In particular, we have the following analogs of Lemma 2.4 and 2.5.

Lemma 2.6. There exists a constant $c_1^* > 1$ such that, at all points of $\overline{S}^*$ where the distance $r$ to 0 is close enough to 1, we have:

1. the angle $\theta$ between the normal to $\overline{S}^*$ and the radial direction is bounded from above: $\theta \leq c_1^* \sqrt{1 - r}$.
2. the distance $\delta$ on $\overline{S}^*$ to the limit set $\Lambda$ satisfies:

$$\frac{\sqrt{r - 1}}{c_1^*} \leq d_{\overline{S}^*}(x, \Lambda) \leq c_1^* \sqrt{r - 1}.$$ 

Sketch of the proof. Let $x \in \overline{S}$, and let $x^*$ be the dual point in $\overline{S}^*$, i.e. the point in $\mathbb{R}^3$ which is the dual of the plane tangent to $\overline{S}$ at $x$. By definition of the duality, the position of $x^*$ is determined by the position of the tangent plane to $\overline{S}$ at $x$, while the tangent plane to $\overline{S}^*$ at $x^*$ is determined by the position of $x$. The first point is thus a consequence of both points of Lemma 2.4. The second point is then a direct consequence of the first, as in the proof of Lemma 2.4. \qed
Lemma 2.7. There exists a constant $c_2^* > 1$ such that, at all points of $S^*$, where the distance $r$ to $\partial D^3$ is small enough, the principal curvatures of $S^*$ are bounded between $c_2^*$ and $1/c_2^*$.

Sketch of the proof. One possible way to prove this Proposition is through the estimates of Lemma 2.5 along with an explicit computation of the principal curvatures of the dual $\Sigma^*$ of a convex surface $\Sigma \subset D^3$.

Another way to prove it uses the same method as for Lemma 2.6 based on Proposition 1.5—in which the fact that the target manifold is Riemannian or Lorentzian plays no role—and on the de Sitter analog of Lemma 1.3. One only needs to control the angle between $S^*$ and the radial direction in $S^1_{1,+}$, which is precisely what Lemma 2.6 does.

3 Rigidity of smooth convex surfaces in $\mathbb{R}^3$

We now consider closely the problem of infinitesimal rigidity of a convex surface in the Euclidean (or the hyperbolic) 3-space. In the next sections, we will obtain some estimates on the deformations associated to a first-order deformation of a hyperbolic metric on $M$, until, in section 6, we use those estimates and the ideas developed here—which are mostly classical—to prove Lemma 6.1, dealing with the infinitesimal rigidity of hyperbolic manifolds with convex boundary.

It has been known for a long time—at least since the works of Cohn-Vossen [CV36] and Herglotz [Her79]—that the smooth, strictly convex closed surfaces in $\mathbb{R}^3$ are infinitesimally rigid, i.e., they admit no smooth infinitesimal deformation which does not change the metric at first order. The main rigidity result of this paper, however, is based on a small generalization of this statement, to surfaces which are not so smooth and for deformations which might also have singularities. It will therefore be necessary to have a good understanding of some of the ways in which the classical infinitesimal rigidity statement is proved. Note that, thanks to the statements of section 1, it does not make much difference to consider a compact (strictly convex) surface in $\mathbb{R}^3$, in $H^3$ or in $S^3_1$. We will however mostly remain in $\mathbb{R}^3$, although it is important to keep in mind that almost all applies in the same way in the hyperbolic 3-space.

We will describe in this section two formulations of the infinitesimal rigidity of smooth surfaces, in terms of some elliptic equations. The main point here is that the rigidity of a surface in $\mathbb{R}^3$ or $H^3$ goes hand in hand with another problem, which for convex surfaces in $H^3$ describes the infinitesimal deformations of the dual surface in $S^3_1$, and our main rigidity result (in section 6) will crucially use this fact. In this section, we will be less careful about the smoothness of the functions or vector fields which enter the picture, although the proofs used here are chosen so that they’re not too demanding in terms of regularity.

In all this section we will consider a smooth, strictly convex, closed surface $S \subset \mathbb{R}^3$ (or $S \subset H^3$). An infinitesimal deformation of $S$ is a vector field $u$ defined on $S$—formally, a section of the pull-back on $S$ by its immersion in $\mathbb{R}^3$ (or $H^3$) of the tangent bundle of $\mathbb{R}^3$ (or $H^3$). It is an infinitesimal isometry if the induced first-order variation of the induced metric of $S$ vanishes. $S$ is infinitesimally rigid if it has no non-trivial infinitesimal isometric deformation. There are three basic ways of stating this property:

1. by considering the component of $u$ which is normal to $S$; it is of the form $fN$, where $N$ is the unit normal vector field of $S$. If $u$ is an isometric deformation, $f$ is solution of a second-order elliptic PDE, whose principal symbol is given by the second fundamental form $\Pi$ of $S$.

2. in terms of the component $v$ of $u$ which is tangent to $S$; when $u$ is isometric, it is solution of a first-order elliptic PDE.

3. in terms of the first-order variation $\dot{B}$ of the shape operator $B$, which is also solution of a simple PDE which is elliptic and of first order.

We will consider the formulations (2) and (3); the main remark of this section is that those two formulations are ”adjoint” in a natural way. More precisely, each is ”adjoint” to a ”dual” of the other. This will be useful since, in the setting of the next section, the formulation (3) can not directly be used by lack of a priori smoothness of the deformations. The ”duality” which appears here will allow us to go from the formulation (2) to the formulation (3) and gain some smoothness in the process.

Extrinsic invariants of a convex surface The first point is to understand the first, second and third fundamental forms of $S$, as well as their Levi-Civita connections. The first proposition is rather classical, see e.g. [Sch90] [LS00].
Proposition 3.1. The Levi-Cività connection of $\mathcal{I}$ is:

$$\nabla_X Y = B^{-1}\nabla_X (BY) = \nabla_X Y + B^{-1}(\nabla_X B)Y.$$ 

Proof. By definition of the Levi-Cività connection, it is sufficient to prove that the connection $\tilde{\nabla}$ defined in the proposition is compatible with $\mathcal{I}$ and has no torsion. But, if $X, Y$ and $Z$ are vector fields on $S$:

$$X.\mathcal{I}(Y, Z) = X.\mathcal{I}(BY, BZ)$$

$$= \mathcal{I}(\nabla_X (BY), BZ) + \mathcal{I}(BY, \nabla_X (BZ))$$

$$= \mathcal{I}(B^{-1}\nabla_X (BY), Z) + \mathcal{I}(Y, B^{-1}\nabla_X (BZ))$$

$$= \mathcal{I}(\tilde{\nabla}_X Y, Z) + \mathcal{I}(Y, \tilde{\nabla}_X Z),$$

so that $\tilde{\nabla}$ is compatible with $\mathcal{I}$. Moreover:

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = B^{-1}\nabla_X (BY) - B^{-1}\nabla_X (BZ) - [X, Y]$$

$$= B^{-1}(\nabla_X (BY) - \nabla_Y (BX) - B[X, Y])$$

$$= B^{-1}(d\tilde{\nabla} B)(X, Y)$$

$$= 0,$$

and thus $\tilde{\nabla}$ has no torsion. □

Corollary 3.2. If $S$ is a surface in $\mathbb{R}^3$, the curvature of $\tilde{\nabla}$ is $\tilde{K} = 1$.

Proof. Let $(e_1, e_2)$ be an orthonormal moving frame for $\mathcal{I}$, and let $(\tilde{e}_1, \tilde{e}_2) := (B^{-1}e_1, B^{-1}e_2)$ be the corresponding orthonormal moving frame for $\mathcal{I}$. A direct computation using the previous proposition shows that the connection 1-forms of $(e_1, e_2)$ (for $\mathcal{I}$) and of $(\tilde{e}_1, \tilde{e}_2)$ (for $\mathcal{I}$) are the same. Therefore, their curvature 2-forms are also the same, so that:

$$K\text{da}_I = \tilde{K}\text{da}_\mathcal{I},$$

where $da_I$ and $da_\mathcal{I}$ are the area forms of $\mathcal{I}$ and $\mathcal{I}$, respectively. Thus the curvature $\tilde{K}$ is such that:

$$\tilde{K} = \frac{K}{\det(B)},$$

so that $\tilde{K} = 1$ by the Gauss formula. □

Another possible proof uses the fact that $\mathcal{I}$ is the pull-back by the Gauss map of the round metric on $S^2$.

Proposition 3.3. The Levi-Cività connection of $\mathcal{II}$ is:

$$\nabla_X Y = \frac{1}{2}(\nabla_X Y + \tilde{\nabla}_X Y) = \nabla_X Y + \frac{1}{2}B^{-1}(\nabla_X B)Y.$$ 

Proof. Again we have to prove that $\tilde{\nabla}$ is compatible with $\mathcal{II}$, and without torsion. Let $X, Y$ and $Z$ be 3 vector fields on $S$. Since $B$ is self-adjoint for $\mathcal{I}$, so is $\nabla_X B$, so that:

$$X.\mathcal{II}(Y, Z) = X.\mathcal{II}(BX, Y)$$

$$= \mathcal{II}(\nabla_X (BX), Z) + \mathcal{II}(BY, \nabla_X (Z))$$

$$= \mathcal{II}(\nabla_X (BX), Z) + \mathcal{II}(Y, \nabla_Z (BX))$$

$$= \mathcal{II}(\nabla_X Y, Z) + \mathcal{II}(Y, \nabla_Z X),$$

so that $\nabla$ is compatible with $\mathcal{II}$. The same computation as in the previous proposition shows that $\nabla$ has no torsion:

$$\nabla_X Y - \nabla_Y X - [X, Y] = \nabla_X Y - \nabla_Y X - \frac{1}{2}B^{-1}((\nabla_X B)Y - (\nabla_Y B)X) - [X, Y]$$

$$= \frac{1}{2}B^{-1}(d\nabla B)(X, Y) + (\nabla_X Y - \nabla_Y X - [X, Y])$$

$$= 0.$$

□
A consequence of those propositions is the following formula, which will be important in the sequel.

**Proposition 3.4.** Let \( X, Y \) and \( Z \) be 3 vector fields on \( S \). Then:

\[
X.\mathbb{I}(Y, Z) = \mathbb{I}(\nabla_X Y, Z) + \mathbb{I}(Y, \nabla_Y Z).
\]

**Proof.**

\[
\begin{align*}
X.\mathbb{I}(Y, Z) &= \mathbb{I}(\nabla_X Y, Z) + \mathbb{I}(Y, \nabla_X Z) \\
&= \mathbb{I}(\nabla_X Y, Z) - \frac{1}{2} \mathbb{I}(\nabla_X B Y, Z) + \frac{1}{2} \mathbb{I}(Y, \nabla_X B Z) + \mathbb{I}(Y, \nabla_X Z) \\
&= \mathbb{I}(\nabla_X Y, Z) - \frac{1}{2} \mathbb{I}(B^{-1}(\nabla_X B) Y, Z) + \frac{1}{2} \mathbb{I}(Y, B^{-1}(\nabla_X B) Z) + \mathbb{I}(Y, \nabla_X Z) \\
&= \mathbb{I}(\nabla_X Y, Z) + \mathbb{I}(Y, \nabla_X Z).
\end{align*}
\]

We will call \( \mathbb{J} \) the complex structure associated to \( \mathbb{I} \). There is a natural bundle of \((0, 1)\)-forms with values in the tangent bundle of \( S \).

**Definition 3.5.** We will call \( \Omega^{0,1}_\mathbb{I}(S) \) the sub-bundle of \( T^*S \times TS \) of 1-forms \( \omega \) with values in \( TS \) such that, for \( X \in TS \),

\[
\omega(JX) = -J\omega(X).
\]

The proof of the main rigidity result uses several differential operators defined using the connections \( \nabla \) and \( \tilde{\nabla} \) on \( S \), in spite of the fact that the metric which is mostly used is \( \mathbb{I} \). Recall that the usual \( \partial \) operator for \( \mathbb{I} \) is:

\[
(\partial_\mathbb{I} u)(X) := \nabla_X u + J\nabla_{\mathbb{J}X} u.
\]

Several elliptic operators, acting on vector fields on \( S \), have a principal symbol defined by \( \mathbb{I} \), in particular the following one, already considered by Labourie [Lab89]:

**Definition 3.6.** We will call:

\[
\begin{align*}
(\partial_J u)(X) &:= \nabla_X u + J\nabla_{\mathbb{J}X} u, \\
(\partial_{\mathbb{B}} u)(X) &:= \nabla_X u + \mathbb{J}\nabla_{\mathbb{J}X} u.
\end{align*}
\]

**Lemma 3.7.** \( \partial_J \) and \( \partial_{\mathbb{B}} \) are elliptic operators of index 6 acting from \( TS \) to \( \Omega^{0,1}_\mathbb{I}(S) \).

**Proof.** If \( u \) and \( X \) are vector fields on \( S \), we have:

\[
(\partial_J u)(\mathbb{J}X) = \nabla_{\mathbb{J}X} u + J\nabla_{\mathbb{J}X} u = -J\nabla_X u + \nabla_{\mathbb{J}X} u = -\nabla_X u + \mathbb{J}\nabla_{\mathbb{J}X} u,
\]

which shows that \( \partial_J \) has values in \( \Omega^{0,1}_\mathbb{I}(S) \). The same computation shows the same result for \( \partial_{\mathbb{B}} \).

The definition of \( \partial_J \) and \( \partial_{\mathbb{B}} \) shows that their principal symbol is the same as the principal symbol of the operator obtained by replacing \( \nabla \) (resp. \( \tilde{\nabla} \)) by \( \nabla \); this replacement leads to the usual \( \partial \) operator for \( \mathbb{I} \), which is well known to be elliptic of index 6. The lemma follows by the deformation invariance of the index.

**Tangent vector fields** The most natural way to consider the infinitesimal rigidity question for surfaces is in terms of the infinitesimal deformation vector fields. Given such a vector field, it is natural to consider its projection on the surface.

**Proposition 3.8.** Let \( u \) be an infinitesimal isometric deformation of \( S \), and let \( v \) be its component tangent to \( S \). Then the vector field \( w := B^{-1}v \) is solution of:

\[
\mathbb{B}w = 0.
\]
Proof. Let $X$ be a vector field on $S$.

\[
(\overline{\partial}_w X) = \nabla_X w + \overline{\nabla}_{JX} w = B^{-1}\nabla_X(Bw) + J\overline{\nabla}_{JX}(Bw) = B^{-1}\nabla_X v + \overline{J}B^{-1}\overline{\nabla}_{\overline{J}X} v.
\]

Since $u$ is isometric, there exists $\lambda \in \mathbb{R}$ such that, for all $x, y \in TS$, $I(\nabla_x v, y) + I(x, \nabla_y v) = 2\lambda I(x, y)$. Thus:

\[
I((\overline{\partial}_w X), X) = I(B^{-1}\nabla_X v + \overline{J}B^{-1}\overline{\nabla}_{JX} v, X) = I(\nabla_X v, X) - I(\overline{\nabla}_{JX} v, \overline{J}X) = \lambda I(X, X) - \lambda I(\overline{J}X, \overline{J}X) = 0,
\]

\[
I((\overline{\partial}_w X), \overline{J}X) = I(B^{-1}\nabla_X v + \overline{J}B^{-1}\overline{\nabla}_{JX} v, \overline{J}X) = I(\nabla_X v, \overline{J}X) + I(\overline{\nabla}_{JX} v, X) = 2\lambda I(X, \overline{J}X) = 0,
\]

and we obtain that $\overline{\partial}_w X = 0$. \qed

Remark 3.9. Let $w$ be a solution of equation (1) on $S$; there exists a unique infinitesimal first-order deformation of $S$ whose component tangent to $S$ is $Bw$.

Proof. Since $w$ is a solution of equation (1), a simple computation shows that the first-order deformation of the metric on $S$ induced by $Bw$ is proportional to $I$, i.e. it is of the form $\lambda I$, for some function $\lambda : S \to \mathbb{R}$. Thus, the vector field $Bw - \lambda N$ is an isometric first-order deformation of $S$. \qed

Of course, there are many solutions of equations (1); all "trivial" deformations of $S$ in $\mathbb{R}^3$ lead to one such solution, so that there is at least a 6-dimensional space of solutions. The point is that there is no other solution, a fact that we will prove by looking at a dual formulation of the problem.

Variations of the shape operator The second rather simple way of stating the infinitesimal rigidity of surfaces is in terms of first-order variations of the shape operator $B$. We will see that this approach is, in a precise sense, adjoint of the "tangent vector field" interpretation. We first recall the basics of the infinitesimal rigidity of convex surfaces in $\mathbb{R}^3$ (or $H^3$).

Proposition 3.10. Let $u$ be an isometric first-order deformation of $S$, and let $\dot{B}$ be the induced first-order variation of the shape operator $B$ of $S$. Then $\dot{B}$ is a solution of:

\[
\begin{cases}
\quad d\nabla \dot{B} = 0 \\
\quad \text{tr}(B^{-1}\dot{B}) = 0.
\end{cases}
\]

Proof. The Gauss and Codazzi equations on $B$ are:

\[
\begin{cases}
\quad d\nabla B = 0 \\
\quad \det(B) = K,
\end{cases}
\]

where $K$ is the curvature of the induced metric. Since the deformation $u$ is isometric, those equations remain true at first order, and the proposition follows by linearizing them. \qed

Proposition 3.11. Suppose that $S \subset \mathbb{R}^3$. Let $\dot{B}$ be a solution of (1). There exists an isometric first-order deformation of $S$, unique up to global infinitesimal isometries, whose induced deformation of $B$ is $\dot{B}$.

This is a direct consequence of the so called "fundamental theorem of the theory of surfaces", see e.g. [Spi75], and is also valid for surfaces in $H^3$. It is however interesting to check how the proof can be done directly, since we will need the same argument in a slightly more demanding context in the next sections; the proof given here is valid in $\mathbb{R}^3$.

We start from our surface $S$ and from an embedding $\phi : S \to \mathbb{R}^3$, and we call $N$ the unit normal vector to $\phi(S)$, which we consider as a $\mathbb{R}^3$-valued function on $S$. The shape operator $B$ of $S$ can be defined in this context by the equation:

\[
dN = -(d\phi) \circ B.
\]
We consider a first-order deformation $\dot{\phi}$ of $\phi$ which does not change the induced metric at first order, and call $\dot{N}$ the first-order deformation of $N$. Then the deformation $\dot{\phi}$ acts locally like a rigid motion, so we can define at each point $x \in S$ a vector $Y \in \mathbb{R}^3$, which is such that:

$$d\dot{\phi} = Y \wedge d\phi,$$

$$\dot{N} = Y \wedge N.$$  

Here, and everywhere in the paper, “$\wedge$” is the vector product in $\mathbb{R}^3$. Taking the first-order variation of $1$ shows that:

$$d\dot{N} = -(d\dot{\phi}) \circ B - (d\phi) \circ \dot{B} = -Y \wedge (d\phi \circ B) - d\phi \circ \dot{B},$$

while taking the differential of $1$ shows that:

$$d\dot{N} = dY \wedge N + Y \wedge dN = dY \wedge N - Y \wedge d\phi \circ B.$$  

Comparing those two equations, we have that:

$$dY \wedge N = -d\phi \circ \dot{B}. $$

Moreover, taking the differential of $1$ yields that, for any two vector fields $x$ and $y$ on $S$:

$$0 = d(d\dot{\phi})(x, y) = (dY \wedge d\phi)(x, y),$$

from which it follows that the image of $dY$ is always contained in the image of $d\phi$, and therefore orthogonal to $N$. Thus:

$$dY = -(dY \wedge N) \wedge N = (d\phi \circ \dot{B}) \wedge N.$$  

Taking the differential of this equations, we obtain that, for any two vector fields $x, y$ on $S$:

$$0 = dN(x) \wedge d\phi(\dot{B}y) - dN(y) \wedge d\phi(\dot{B}x) + N \wedge (x.\partial_\phi(\dot{B}y) - y.\partial_\phi(\dot{B}x)) - N \wedge d\phi(\dot{B}[x, y])$$

$$= -d\phi(\dot{B}x) \wedge d\phi(\dot{B}y) + d\phi(\dot{B}y) \wedge d\phi(\dot{B}x) +$$

$$+ N \wedge d\phi(\dot{B}y - \partial_y(\dot{B}x) - \dot{B}[x, y]) + N \wedge ((x.\partial_\phi(\dot{B}y) - (y.\partial_\phi(\dot{B}x)) + (y.\partial_\phi(\dot{B}y)) \wedge N = 0, $$

because $N \wedge N = 0$, so that:

$$-d\phi(\dot{B}x) \wedge d\phi(\dot{B}y) + d\phi(\dot{B}y) \wedge d\phi(\dot{B}x) + N \wedge d\phi(\dot{B}y)(x, y) = 0,$$

which can be written as, using the complex structure $J$ of $I$, as:

$$(-I(JBx, \dot{B}y) + I(JBy, \dot{B}x))N + N \wedge d\phi(d^N\dot{B})(x, y) = 0.$$  

Taking for $(x, y)$ an orthonormal basis of $TS$ shows that the component orthogonal to $d\phi(TS)$ vanishes if and only if $tr(B^{-1}\dot{B}) = 0$, while the component tangent to $TS$ is zero if and only if $d^N\dot{B} = 0$. So we recover equation $1$.

We can now prove Proposition $3.11$. Start from a solution $\dot{B}$ of $1$, and set:

$$\alpha := (d\phi \circ \dot{B}) \wedge N,$$

so that $\alpha$ is a $\mathbb{R}^3$-valued 1-form on $S$. Then the computation we’ve just done shows that $d\alpha = 0$, precisely because $\dot{B}$ is a solution of $1$. So we can integrate $\alpha$ over $S$, and find a function $Y : S \to \mathbb{R}^3$ such that $dY = \alpha$.

We can then define another $\mathbb{R}^3$-valued 1-form $\beta$ on $S$ by:

$$\beta := Y \wedge d\phi.$$  

Then:

$$d\beta = dY \wedge d\phi = ((d\phi \circ \dot{B}) \wedge N) \wedge d\phi.$$  

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Let \((x, y)\) be an orthonormal frame in which \(\hat{B}\) is diagonal, we see by computing \(d\beta(x, y)\) that \(d\beta = 0\), so that we can define a function \(\hat{\phi} : S \to \mathbb{R}^3\) such that \(d\hat{\phi} = \beta\). We then have, by definition of \(\beta\):

\[
d\hat{\phi} = Y \wedge d\phi ,
\]

so that \(\hat{\phi}\) is an isometric infinitesimal deformation of \(\phi(S)\). It also clearly follows from the computations made above that the first-order variation of \(B\) is indeed \(\hat{B}\), and this proves Proposition \ref{prop:infinitesimal}.

Now the key point, of course, is that there exists no such isometric infinitesimal deformation, i.e. the smooth, strictly convex surfaces are rigid. This result has been known for a long time, and the proof we will give is essentially well-known and classical, see e.g. \[Her79, LS00\]. There are several other classical proofs, see e.g. \[Yo62, Sp75\], but this one is well suited for our needs since it does not demand too much smoothness.

**Lemma 3.12.** Suppose that \(S \subset \mathbb{R}^3\). There is no non-trivial solution \(\hat{B}\) of \((\mathbb{I})\) on \(S\).

**Proof.** We consider the function \(f := r^2/2\) on \(\mathbb{R}^3\), where \(r\) is the distance to 0. Then:

\[
\text{Hess}(f) = g_0 ,
\]

where \(g_0\) is the canonical flat metric on \(\mathbb{R}^3\).

Let \(\text{Hess}(f)\) be the Hessian of the restriction of \(f\) to \(S\). Then:

\[
\text{Hess}(f) = I - df(N)\mathbb{I} ,
\]

where \(N\) is the unit exterior normal of \(S\). Thus, calling \(\hat{f}\) the first-order variation of the restriction of \(f\) to \(S\):

\[
\text{Hess}(\hat{f}) = -df(N)\mathbb{I} - df(N)\mathbb{I} .
\]

Now let \(\omega(X) := d\hat{f}(J\hat{B}X)\). Then:

\[
d\omega(X, Y) = X.d\hat{f}(J\hat{B}Y) - Y.d\hat{f}(J\hat{B}X) - d\hat{f}(J\hat{B}[X, Y]) = \text{Hess}(\hat{f})(X, J\hat{B}Y) - \text{Hess}(\hat{f})(Y, J\hat{B}X) + df(J\nabla_X(J\hat{B}Y) - J\nabla_Y(J\hat{B}X)) = \text{Hess}(\hat{f})(X, J\hat{B}Y) - \text{Hess}(\hat{f})(Y, J\hat{B}X) + df(J(dv^\nabla)B)(X, Y)) = -df(N)(\mathbb{I}(X, J\hat{B}Y) - \mathbb{I}(Y, J\hat{B}X)) - df(N)(\mathbb{I}(X, J\hat{B}Y) - \mathbb{I}(Y, J\hat{B}X)) .
\]

The second term vanishes because \((B^{-1}\hat{B}) = 0\), while it is simple to check — for instance by taking for \(X\) and \(Y\) eigenvectors of \(B\) — that:

\[
d\omega(X, Y) = 2df(N)\det(B)da(X, Y) .
\]

Since \((B^{-1}\hat{B}) = 0\), \(\det(B^{-1}\hat{B}) \leq 0\) and thus \(\det(B) \leq 0\) everywhere. Since the integral of \(d\omega\) over \(S\) is zero, \(\det(B) = 0\) everywhere, so that \(\hat{B} = 0\).

**A duality between the two formulations** An important element used in the next sections to prove a rigidity result is that the two formulations of the rigidity stated above — in terms of a vector field tangent to \(S\), or in terms of a bundle morphism of \(TS\) describing the first-order variation of \(B\) — are adjoint to each other. This follows from the next two propositions, which are valid in either \(\mathbb{R}^3\) or \(H^3\).

**Proposition 3.13.** The adjoint of \(\mathcal{J}^{-1}_S\) for \(\mathbb{I}\) is the operator defined by:

\[
(\mathcal{J}^{-1}_S)^* : \Omega^{0,1}_S \to TS , \quad h \mapsto -d^\nabla h(X, \mathcal{J}X) = -(\nabla_X h)(\mathcal{J}X) + (\nabla_{\mathcal{J}X} h)(X) ,
\]

where \(X\) is a unit vector field on \(S\). If \(\Omega\) is an open subset with smooth boundary of \(S\) and \(h\) is a smooth section of \(\Omega^{0,1}_S\), then:

\[
\int_\Omega (\mathcal{J}^{-1}_S v, h)_x da_x = \int_\Omega \mathbb{I}(v, -d^\nabla h) + \int_{\partial \Omega} \mathbb{I}(v, h(ds)) .
\]

Similarly, the adjoint of \(\mathcal{J}_T\) for \(\mathbb{I}\) is the operator \(-d^\nabla\) acting on sections of \(\Omega^{0,1}_S\), and:

\[
\int_\Omega (\mathcal{J}_T v, h)_x da_x = \int_\Omega \mathbb{I}(v, -d^\nabla h) + \int_{\partial \Omega} \mathbb{I}(v, h(ds)) .
\]
It is implicit in this proposition that the expression of the adjoint operators is actually independent of the choice of the unit vector $X$.

**Proof.** Let $h$ be a smooth section of $\Omega^0_1(S)$. Then, using Proposition 3.11

$$
\int_S \mathcal{B}(\mathcal{T}h, v, h) d\omega = \int_S \mathcal{B}(\mathcal{T} h, -\mathcal{T} h) d\omega
$$

$$
= \int_S \mathcal{B}(\mathcal{T} h)(X, h(\mathcal{T} X)) d\omega
$$

$$
= \int_S \mathcal{B}(\mathcal{T} h)v + \mathcal{T} h(\mathcal{T} X), h(\mathcal{T} X) d\omega
$$

$$
= \int_S \mathcal{B}(\mathcal{T} h)v, h(\mathcal{T} X)) d\omega
$$

$$
= \int_S \mathcal{B}(v, h(\mathcal{T} X)) - \mathcal{B}(v, h(X)) + \mathcal{B}(v, \mathcal{T} h(\mathcal{T} X)) d\omega
$$

$$
= \int_S \mathcal{B}(v, h(\mathcal{T} X)) - \mathcal{B}(v, h(X)) - \mathcal{B}(v, h(X)) - \mathcal{B}(v, \mathcal{T} h(\mathcal{T} X)) d\omega
$$

$$
= \int_S \omega(X, \mathcal{T} X) - \mathcal{B}(v, (d\mathcal{T} h)(X, \mathcal{T} X)) d\omega,
$$

with $\omega(X) := \mathcal{B}(v, h(X))$. The result follows for $\mathcal{T} h(\mathcal{T} X)^*$. The formula just obtained also shows the result concerning the integration over $\Omega$. A similar computation shows the corresponding results for $\mathcal{T} h(\mathcal{T} X)^*$. 

The point is that those adjoint operators are precisely those which describe the first-order deformations in terms of the variation of $\mathcal{B}$. The next two propositions are valid if $S \subset \mathbb{R}^3$ or $S \subset H^3$.

**Proposition 3.14.** Let $\hat{\mathcal{B}}$ be a bundle morphism from $TS$ to itself, which is self-adjoint for $I$. $\hat{\mathcal{B}}$ is a solution of equation (1) if and only if $h := B^{-1}\hat{\mathcal{B}}$ is a section of $\Omega^{0,1}_1(S)$ such that $d\mathcal{T} h = 0$.

**Proof.** Clearly, $h$ is self-adjoint for $\mathcal{B}$: if $X, Y$ are two vectors at a point $s \in S$, then

$$
\mathcal{B}(h(X), Y) = \mathcal{B}(B^{-1}\hat{\mathcal{B}}X, Y) = I(\hat{\mathcal{B}}X, Y) = I(X, B^{-1}\hat{\mathcal{B}}Y) = \mathcal{B}(X, B^{-1}\hat{\mathcal{B}}Y) = \mathcal{B}(X, h(Y)).
$$

Moreover $\hat{\mathcal{B}}$ satisfies the second equation of (1), i.e. $\text{tr}(B^{-1}\hat{\mathcal{B}}) = 0$, if and only if $\text{tr}(h) = 0$. But it is a simple matter of linear algebra that those two conditions are satisfied if and only if $h$ is a section of $\Omega^{0,1}_1(S)$, i.e. if and only if, for each $X \in TS$, $h(\mathcal{T} X) = B^{-1}h(X)$.

Similarly, the condition that $d\mathcal{T} h = 0$ can be written as:

$$
0 = (d\mathcal{T} h)(X, \mathcal{T} X)
$$

$$
= \mathcal{T} X(h(\mathcal{T} X)) - \mathcal{T} h(\mathcal{T} X) - h([X, \mathcal{T} X])
$$

$$
= B^{-1}(\mathcal{T} X(\hat{\mathcal{B}}X) - \mathcal{T} h(\mathcal{T} X) - \mathcal{T} h(\mathcal{T} X))
$$

so that it is equivalent to the first equation in (1).

The same proof can be used to show a "dual" statement:

**Proposition 3.15.** Let $\hat{\mathcal{A}}$ be a bundle morphism from $TS$ to itself. $\hat{\mathcal{A}}$ is a solution of:

$$
\begin{cases}
    d\mathcal{T} \hat{\mathcal{A}} = 0 \\
    \text{tr}(A^{-1}\hat{\mathcal{A}}) = 0
\end{cases}
$$

if and only if $h := A^{-1}\hat{\mathcal{A}}$ is a section of $\Omega^{0,1}_0(S)$ such that $d\mathcal{T} h = 0$. 

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Rigidity of a surface and its dual  Let’s sum up the remarks we have just made. Each infinitesimal isometric deformation of $S$ corresponds to a solution of $\overline{\partial}_g \varpi = 0$; the adjoint of this equation is $d^*\nu = 0$, and the solutions of this equation correspond to solutions $A$ of equations (1).

Suppose now that $S \subset H^3$, and let $S^* \subset S^1$ be the dual surface. Then the induced metric on $S^*$ is the third fundamental form of $S$ (and conversely) while the second fundamental forms of $S$ and $S^*$ are the same. Therefore, solutions of equation (1), and solutions of $d^*\nu = 0$ with $h$ a section of $\Omega^1_{\mathbf{g}}(S)$, describe isometric first-order deformations of $S^*$. Conversely, non-trivial infinitesimal isometric deformations of $S$ also correspond to non-zero solutions $B$ of (1), and thus to solutions $d^*\nu = 0$, and the adjoint equation is $\overline{\partial}_f \varpi = 0$, which also describes infinitesimal isometric deformations of $S^*$.

Duality and the Minkowski problem  The reader has probably by now realized that the duality between the induced metric and the third fundamental form plays an important role in the constructions made in this section — and it will also be central in the next sections. In the setting of the hyperbolic 3-space, the geometric meaning of the duality is clear: the third fundamental form of a surface $S \subset H^3$ is the induced metric on the dual surface in the de Sitter space. Then solutions of $\overline{\partial}_g \varpi = 0$, and of $d^*\nu = 0$ for $h \in \Omega^1_{\mathbf{g}}(S)$, correspond to infinitesimal isometric deformations of the surface $S$, while solutions of $\overline{\partial}_f \varpi = 0$ and of $d^*\nu = 0$ correspond to isometric deformations of the dual surface $S^*$.

Although it will not be used in this paper, it is interesting to point out that this duality retains a geometric meaning in $\mathbb{R}^3$, although the dual problem is not one of isometric deformations of surfaces. Indeed, solutions of $d^*\nu = 0$ correspond to variations $A$ of $A := B^{-1}$ which are solutions of (1), and we leave it to the interested reader to check that the non-existence of those in turns corresponds to the rigidity in the Minkowski problem, which deals with the existence of convex surfaces with a given push-forward, by the Gauss map, of its curvature measure.

4  First-order deformations

The developing map  We have seen in section 2 some details on the behavior of $\overline{\mathbf{S}}$ and $\overline{\mathbf{S}}^*$ near the limit set $\Lambda$. We will now study the boundary behavior of the vector fields on $\overline{\mathbf{S}}$ and $\overline{\mathbf{S}}^*$ corresponding to the first-order variations $\dot{g}$ of the hyperbolic metric $g$ on $M$. We will use this in the next two sections to prove the infinitesimal rigidity of hyperbolic manifolds with convex boundary.

The first point is to remark that one can associate to $\dot{g}$ a closed one-form $\omega$ on $M$ with value in $\mathfrak{sl}(2, \mathbb{C})$. This can be done along the ideas of Calabi [Ca61] and Weil [We60], but we will follow here an elementary approach. $\omega$ will not be canonically defined, but it will not be necessary.

To define $\omega$, we first recall the definition of the developing map of a hyperbolic metric on $M$.

**Definition 4.1.** Let $g$ be a hyperbolic metric with convex boundary on $M$. The universal cover $\hat{M}$ of $M$ is a simply connected hyperbolic manifold with convex boundary, which thus admits a unique isometric embedding into $H^3$. This defines a map $\text{dev}_g : \hat{M} \to H^3$, the **developing map** of $g$.

$\text{dev}_g$ is defined modulo the global isometries of $H^3$. Given a first-order deformation $\dot{g}$ of $g$, it determines a first-order deformation $\dot{\text{dev}}$ of $\text{dev}$. $\dot{\text{dev}}$ determines a vector field $u$ on $H^3$. $\dot{\text{dev}}$ is defined modulo the global Killing vector fields on $\Omega$; from here on we suppose that $\dot{\text{dev}}$ is normalized so that $u$ vanishes at $x_0$.

The natural $\mathfrak{sl}_2(\mathbb{C})$ bundle over $H^3$  Questions of infinitesimal deformations of hyperbolic 3-manifolds are classically associated to a natural $\mathfrak{sl}_2(\mathbb{C})$-bundle over $H^3$, and over hyperbolic manifolds. We recall some basic constructions concerning this aspect of the deformations. We consider here a convex subset $\Omega \subset H^3$. We will call $E$ the trivial $\mathfrak{sl}_2(\mathbb{C})$ bundle over $\Omega$. For each $x \in \Omega$ and each $\kappa \in E_x$, $\kappa$ is an element of $\mathfrak{sl}_2(\mathbb{C})$, in other terms a Killing vector field over $H^3$.

Given $x \in \Omega$ and $U \in T_x \Omega$, there is a unique Killing field $v$ over $\Omega$ which is a "pure translation" of vector $U$ at $x$; it is defined uniquely by the fact that $v(x) = U$ and that $\nabla v = 0$ at $x$. There is also a unique Killing field $v'$ which is a "pure rotation of vector $U"$ at $x$: it is uniquely defined by the fact that $v'(0) = 0$ and that $\nabla_w v' = U \wedge w$ for each $w \in T_x \Omega$. A standard fact is that $\mathfrak{sl}_2(\mathbb{C})$ is the direct sum of those two kinds of Killing fields, i.e. each Killing field on $\Omega$ has a unique decomposition as the sum of a pure translation at $x$ and a pure rotation at $x$. This yields a decomposition:

$$E_x = T_x \Omega \oplus T_x \Omega.$$
This decomposition depends on $x$, since a Killing field which is a pure translation (or rotation) at a point $x$ is not a pure translation (or rotation) at another point $y \neq x$.

So, given a point $x \in \Omega$, we have a natural identification of the vector space of Killing fields over $H^3$ with $T_x \Omega \oplus T_x \Omega$. We will often denote a Killing field $\kappa$ by a pair $(\tau_x, \sigma_x)$—depending on the choice of a point $x \in \Omega$. Then $\tau_x \in T_x \Omega$ will be the translation vector of the component of $\kappa$ which is a pure translation at $x$—so that $\tau_x = \kappa(x)$—while $\sigma_x \in T_x \Omega$ will be the rotation vector of the component of $\kappa$ which is a pure rotation at $x$.

The bundle $E$ comes equipped with a natural flat connection, which we will call $D$. Let $\kappa$ be a section of $E$, so that, for each $x \in \Omega$, $\kappa(x)$ is an element of $sl_2(\mathbb{C})$, i.e. a Killing field over $H^3$. Then, for each $x \in H^3$, $\kappa(x) = (\tau_x, \sigma_x) \in T_x \Omega \oplus T_x \Omega$. The connection $D$ is defined in those notations by:

$$D_w(\tau_x, \sigma_x) = (\nabla_w \tau_x + w \wedge \sigma_x, \nabla_w \sigma_x - w \wedge \tau_x).$$

The key point is the following Property of $D$.

**Property 4.2.** Let $\kappa$ be a section of $E$. $D\kappa = 0$ if and only if $\kappa$ corresponds, at each point $x \in H^3$, to the same Killing field over $H^3$.

From there it follows immediately that $D$ is a flat connection. However $D$ does not preserve the identification $E_x = T_x \Omega \oplus T_x \Omega$.

Now, given $dev'$, we wish to define a section $\kappa$ of $E$. We will do it in terms of the translation and rotation components of $\kappa$ at each point. For each $x \in \Omega$, we call $\tau_x$ the infinitesimal pure hyperbolic translation at $x$ of translation vector $u = dev'$. So $\tau_x$ is a Killing vector field on $H^3$, or, in other terms, an element of $sl_2(\mathbb{C})$. Therefore $\tau$ determines a section over $\Omega$ of the natural $sl_2(\mathbb{C})$ bundle.

Consider the vector field $w = u - \tau_x$ on $H^3$. By construction it vanishes at $x$, so we can consider its differential $dw : T_x \Omega \to T_x \Omega$. $dw$ decomposes uniquely as the sum $dw = a + s$, where $a, s : T_x \Omega \to T_x \Omega$ are respectively adjoint and skew-adjoint for the hyperbolic metric. There is a unique infinitesimal hyperbolic rotation of axis containing $x$ and differential $s$, we call it $\sigma_x$—and we also call $\sigma_x$ the rotation vector of that infinitesimal isometry. Again, $\sigma_x$ is a Killing field on $H^3$, or an element of $sl_2(\mathbb{C})$, and $\sigma$ defines another section over $\Omega$ of the natural bundle with fiber $sl_2(\mathbb{C})$. By choosing the right normalization for $dev'$ above, we decide that $\sigma_x = 0$.

In this way, we obtain from $dev'$ a section $\kappa$ of $E$, which we call $(\tau, \sigma)$ with the notations explained above. At each point $x \in \Omega$, $\tau_x, \sigma_x \in T_x \Omega$, and $\kappa_x$ is the Killing field over $H^3$ which is the sum of a pure translation of vector $\tau_x$ and a pure rotation of rotation vector $\sigma_x$ at $x$.

We can use $\kappa$ and $D$ to define a closed 1-form $\omega$ with values in the $sl_2(\mathbb{C})$-bundle over $\Omega$, as the exterior differential of $\kappa$:

$$\forall v \in T\Omega, \quad \omega(v) := (d^D\kappa)(v) = D_v \kappa.$$

$\kappa$ was defined geometrically, and its definition shows that, if one adds a Killing field $U$ to $dev'$, $\kappa$ is replaced by $\kappa + U$, and $\omega$ does not change. Thus the equivariance of $dev'$ implies that $\omega$ is invariant under the action of $\pi_1 M$, so it is defined over $M$.

Since $M$ is compact, $\omega$ is bounded: there exists a fixed constant $C_\omega$ such that, for any $x \in \Omega$ and any unit vector $v \in T_x \Omega$, both the translation and the rotation components, at $x$, of $\omega(v)$ are bounded by $C_\omega$. By “bounded” we mean here that the translation vector of $\tau_x$ at $x$, and the rotation speed of $\sigma_x$ at $x$, are bounded.

Using the identification of $E$ with $T\Omega \oplus T\Omega$, $\omega$ can be written as $\omega = (\omega_\tau, \omega_\sigma)$, where both $\omega_\tau$ and $\omega_\sigma$ are 1-forms with values in the tangent bundle $T\Omega$. Since $\omega$ is bounded, both $\omega_\tau$ and $\omega_\sigma$ are bounded. Moreover, it is a direct consequence of the definition of $\omega$ that, for each $w \in T\Omega$:

$$\nabla_w \tau = \omega_\tau(w) - w \wedge \sigma, \quad \nabla_w \sigma = \omega_\sigma(w) + w \wedge \tau.$$

We now sum up some relations between the deformation vector field $u$ and the section $\kappa$ which we have defined from $dev'$.

**Proposition 4.3.**

- Let $u$ be the vector field defined on $S$ by the deformation $\dot{g}$. Then $u(x) = \tau_x$. If $\dot{g}$ does not change the induced metric on $\partial M$, then $u$ is an infinitesimal isometric deformation of $S$.

- $\kappa$ defines a vector field $u^*$ on $S^*$. If $\dot{g}$ does not change the third fundamental form of $\partial M$, then $u^*$ is an infinitesimal isometric deformation of $S^*$.

**Proof.** $u$ is the vector field which determines the first-order deformation of $S$ under the deformation $\dot{g}$ of $g$, so it is clear that $u$ is an infinitesimal isometric deformation of $S$ if and only if the induced metric on $\partial M$ does not change. The definition of $u^*$ follows directly from the first-order deformation of $S^*$ through the duality with $S$. The same argument as for $u$ applies to the first-order variation of $\mathbf{III}$ under the deformation $u^*$, through the variation of the induced metric on $S^*$ and the Fokarev map for $S^3_{1,+}$. 

\[\square\]
Deformations of Euclidean metrics  We have seen above how the first-order deformations of a hyperbolic manifold can be described in terms of a 1-form with values in the canonical \( sl_2(\mathbb{C}) \)-bundle \( E \), and how \( E \) can be identified with \( T\Omega \oplus T\Omega \) at each point. We will now see how the same formalism applies — with minor differences — to deformations of a Euclidean manifold. The point will then be that the Pogorelov map allows one to move from the hyperbolic to the Euclidean formalism.

The space of Killing fields on \( \mathbb{R}^3 \) is a 6-dimensional vector space, which we will call \( E_0 \). We will consider here a domain \( \Omega \subset \mathbb{R}^3 \), which will later be the image in the projective model of \( \Omega \). We call \( E \) the trivial \( E_0 \)-bundle over \( \Omega \). Given \( \tau \in \Omega \), there is a canonical identification \( E_x = T_x\Omega \oplus T_x\Omega \), with the first factor corresponding to the pure translation component of the Killing field at \( \tau \), and the second factor corresponding to the rotation component at \( \tau \). In more explicit terms, to a Killing field \( \tau \), we associate \( (\tau, \sigma) \in T_{\tau}\Omega \oplus T_{\sigma}\Omega \), with:

\[
\tau := \tau(x) ,
\]
and, if \( \nabla \) is the flat connection on \( \Omega \), for each \( w \in T_{\sigma}\Omega \):

\[
\nabla_w \tau = \sigma \wedge w .
\]

As in the hyperbolic case, there is a flat canonical connection \( D \) on \( E \), defined as follows. Given a section of \( E \), defined by sections \( \tau \) and \( \sigma \) of \( T\Omega \) using the identification above, we have for all \( w \in T\Omega \):

\[
D_w(\tau, \sigma) = (\nabla_w \tau + w \wedge \sigma, \nabla_w \sigma) .
\]

It has the same basic property as in the hyperbolic setting.

**Property 4.4.** Let \( \kappa \) be a section of \( E \). \( D\kappa = 0 \) if and only if \( \kappa \) corresponds, at each point \( \tau \in \Omega \), to the same Killing field over \( \Omega \).

The Pogorelov map and infinitesimal isometries  An important point is that the Pogorelov map gives a way to translate things from the hyperbolic to the Euclidean formalism. We now consider the domain \( \Omega \subset H^3 \) corresponding to the universal cover of \( M \), and call \( \Omega \) its image in the projective model.

**Definition 4.5.** Let \( x \in \Omega \) and \( \kappa \in E_x \). \( \kappa \) is a Killing field on \( \Omega \), and the image of this vector field by the Pogorelov map is a Killing field on \( \Omega \), which we call \( \tau \). We call \( \Psi_H : E \to \overline{E} \) the bundle morphism sending \( \kappa \in E_x \) to \( \tau \), considered as an element of \( E_{\phi(x)} \).

Of course \( \Psi_H \) is just the Pogorelov map, we use a different notation from \( \Phi_H \) since it acts on sections of \( E \) instead of vector fields. The following statement is then a direct consequence of the properties of the Pogorelov map.

**Proposition 4.6.** Let \( \kappa \) be a section of \( E \). Then:

\[
\Psi_H(D\kappa) = D(\Psi_H(\kappa)) .
\]

**Proof.** The Pogorelov map \( \Phi_H \) sends hyperbolic Killing fields to Euclidean Killing fields. Therefore, \( \Psi_H \) sends flat sections of \( E \) for \( D \) to flat sections of \( \overline{E} \) for \( D \). It follows that it respects the connections \( D \) and \( \overline{D} \). \( \square \)

It is now time to express explicitly the action of \( \Psi_H \) on different components of a section of \( E \). From here on, we call \( R \) the unit radial vector on \( \Omega \) — in the direction opposite to \( x_0 \) — and \( \overline{R} \) the unit radial vector on \( \overline{\Omega} \) — also in the direction opposite to 0.

**Lemma 4.7.** Let \( x \in H^3 \setminus \{x_0\} \), and let \( v \in T_xH^3 \) be a unit lateral vector. Let \( \tau := \phi(x) \), with \( r := d(\tau, 0) \), and let \( \overline{\tau} \) be the unit lateral vector in the direction of \( \Phi_H(v) \). Then:

\[
\Psi_H(R, 0) = (\overline{R}, 0) , \quad \Psi_H(0, R) = (0, \overline{R}) ,
\]

\[
\Psi_H(v, 0) = \left( \left( \frac{1}{\sqrt{1-r^2}} \right) \overline{\tau} - \frac{r}{\sqrt{1-r^2}} \overline{R} \wedge \overline{\tau} \right) , \quad \Psi_H(0, v) = \left( 0, \frac{1}{\sqrt{1-r^2}} \overline{\tau} \right) .
\]
Proof. \((R, 0)\) corresponds to an infinitesimal translation at unit speed with axis the geodesic through \(x_0\) and \(x\). Therefore \(\Psi_{H}(R, 0)\) leaves globally invariant the segment through \(0\) and \(\overline{r}\). Checking its behavior at \(0\) shows that it is a translation at unit speed along this segment.

\((0, R)\) is an infinitesimal rotation at unit speed along the geodesic going through \(x_0\) and \(x\), so it leaves this geodesic pointwise invariant. So the same is true for \(\Psi_{H}(0, R)\) and the segment through \(0\) and \(\overline{r}\), and again considering the action at \(0\) shows that it is the infinitesimal rotation at unit speed and axis \((0, \overline{r})\), i.e. \((0, \overline{R})\).

\((v, 0)\) leaves globally invariant the totally geodesic plane containing the geodesic through \(x_0\) and \(x\), and tangent to \(v\). So \(\Psi_{H}(v, 0)\) also leaves invariant the plane containing \((0, \overline{r})\) and tangent to \(\overline{r}\). Moreover, \((v, 0)\) is a Killing field which is orthogonal to the geodesic through \(x_0\) and \(x\), so \(\Psi_{H}(v, 0)\) is orthogonal to \((0, \overline{r})\). Therefore \(\Psi_{H}(v, 0)\) is the sum of an infinitesimal translation of vector proportional to \(\overline{r}\) and of a rotation of axis parallel to \(R\) and \(\overline{r}\).

To identify the two coefficients, we compute its action at \(0\) and at \(\overline{r}\). At \(0\), it acts by a vector \(\cosh(\rho) \overline{r}\), where \(\rho = d(x_0, x)\). At \(\overline{r}\), the definition of the Pogorelov map shows that it acts by a vector \((1/\cosh(\rho)) \overline{r}\). So:

\[
\Psi_{H}(v, 0) = \begin{pmatrix}
\frac{1}{\cosh(\rho)} \overline{r} \\
\frac{1}{\cosh(\rho) r} \left( \frac{1}{\cosh(\rho)} - \cosh(\rho) \right) \overline{r} \wedge \overline{r}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\cosh(\rho)} \overline{r} \\
\frac{1}{\cosh(\rho) r} \left( -\frac{\sinh^2(\rho)}{\tanh(\rho) \cosh(\rho)} \right) \overline{r} \wedge \overline{r}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\cosh(\rho)} \overline{r} \\
\frac{1}{\cosh(\rho) r} \left( -\sinh(\rho) \overline{r} \wedge \overline{r} \right)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\sqrt{1 - r^2} \overline{r} \\
\sqrt{1 - r^2} \overline{r} \wedge \overline{r}
\end{pmatrix}.
\]

The argument is similar for \(\Psi_{H}(0, v)\); since \((0, v)\) is a rotation with axis the geodesic containing \(x\) and tangent to \(v\), \(\Psi_{H}(0, v)\) leaves globally invariant the segment starting from \(\overline{r}\) in the direction of \(\overline{r}\), so it is a rotation having this segment as its axis. To find the rotation angle, remark that the action of \(\Psi(0, v)\) at \(0\) is by a vector of norm \(\sinh(\rho)/r\), so that the rotation speed of \(\Psi(0, v)\) is \(\sinh(\rho)/r\), which is equal to \(1/\sqrt{1 - r^2}\). \(\square\)

**Upper bounds on \(\overline{\tau}\)** We will need later on to estimate the deformation of the Euclidean manifold \(\overline{\Omega}\). This deformation is described by the section \(\tau = \Psi_{H}(\kappa)\) of \(\overline{\Omega}\). By Proposition 4.13 \(\overline{\Omega} = \Psi_{H}(D\kappa)\), so that, if we set \(\overline{\tau} := d\overline{\Omega}\), we have: \(\overline{\tau} = \Psi_{H}(\omega)\).

From here on, given a point \(x \in \Omega, x \neq x_0\), and a vector \(u \in T_x\Omega\), we call \(u^r\) and \(u^\perp\), respectively, the radial and the lateral components of \(u\). We also use the same notation in \(\overline{\Omega}\). Given a vector \(w \in T\overline{\Omega}\), we will also call \(\omega^r(w)\) and \(\omega^\perp(w)\), respectively, the components of \(\omega(w)\) in the first and second factors of \(\overline{\Omega} = T\overline{\Omega} \oplus T\overline{\Omega}\). Thus, for instance, \(\omega^r(w)\) is the radial component of the rotation vector of \(\omega(w)\), etc.

**Lemma 4.8.** Let \(\overline{\tau} \in \overline{\Omega} \setminus \{0\}\) with \(r = d(0, \overline{r}) > 0\), and let \(\tau \in T_x\overline{\Omega}\) be a lateral unit vector. Then:

\[
\begin{align*}
||\omega^r(\overline{R})|| & \leq C_{\omega} r, \\
||\omega^\perp(\overline{R})|| & \leq C_{\omega} r, \\
||\omega^{-r}(\overline{R})|| & \leq C_{\omega} r, \\
||\omega^{-\perp}(\overline{R})|| & \leq C_{\omega} r.
\end{align*}
\]

**Proof.** We already know that \(\omega^r\) and \(\omega^\perp\) are both bounded by \(C_{\omega}\). The upper bounds are consequences of two things:

- the behavior of the projective map \(\phi\), i.e. the fact that \(\overline{R} = \cosh^2(\rho) d\phi(R)\) and \(\overline{\tau} = \cosh(\rho) d\phi(v)\), where \(v\) is a unit vector.

- the properties of \(\Psi_{H}\), as described in Lemma 4.7.

In the first and second line, we consider the radial components, for which there is no scaling by Lemma 4.7, so that the only scaling which comes in is the one coming from \(\phi\), which adds a factor \(\cosh^2(\rho) = 1/(1 - r^2)\) for the radial vector \(\overline{R}\) and a factor \(\cosh(\rho) = 1/\sqrt{1 - r^2}\) for the lateral vector \(\overline{\tau}\). The estimates follow using the fact that \(1 - r^2 = (1 - r)(1 + r) \geq (1 - r)\).

On the third line, for \(\omega^{-r}(\overline{R})\), there is a factor \(\cosh^2(\rho)\) corresponding to the scaling of \(\overline{R}\), and a factor \(\sqrt{1 - r^2}\) coming from Lemma 4.7. For \(\omega^{-\perp}(\overline{R})\), there is a factor \(\cosh(\rho)\) corresponding to the scaling of \(\overline{\tau}\), which compensates a factor \(\sqrt{1 - r^2}\) coming from Lemma 4.7.
Finally, on the fourth line, one has to be a little more careful, because Lemma 4.7 shows that the lateral rotation component of $\omega$ comes from both the lateral rotation and the lateral translation component of $\omega$. The term $\omega^\perp (R)$ comes from:

- the term $\omega^\perp (R)$, with a coefficient $\cosh^2(\rho)$ coming from the scaling of $R$, and a factor $1/\sqrt{1-r^2}$ from Lemma 4.7.

Similarly, the term $\omega^\perp (v)$ comes from:

- the term $\omega^\perp (v)$, with a coefficient $\cosh(\rho)$ coming from the scaling of $R$, and a factor $1/\sqrt{1-r^2}$ from Lemma 4.7.

In the next corollary, and in the rest of the paper, we call $C$ a ”generic” constant, whose precise value can change from line to line. In each line, the value of $C$ might depend on $M$ and on $g$, but not on more local data.

**Corollary 4.9.** Let $\vec{x} \in \overline{\Omega} \setminus \{0\}$ with $r = d(0, \vec{x}) > 0$. Then, at $\vec{x}$:

\[ |\vec{\tau}(x)| \leq C|\log(1-r)|, \quad |\vec{\tau}^\perp(x)| \leq C\sqrt{1-r}, \]

\[ |\vec{\tau}(x)| \leq C|\log(1-r)|, \quad |\vec{\tau}^\perp(x)| \leq \frac{C}{\sqrt{1-r}}. \]

**Proof.** This is a direct consequence of the previous Lemma, obtained by integrating the inequalities of the first column on the segment $[0, \vec{x}]$. \qed

It is interesting to note that the estimate on $\vec{\tau}^\perp$ is better than what would have been obtained by a straightforward estimate in $H^3$. This can be explained using a more subtle estimate in $H^3$, very similar to the one used in the proof of Lemma 4.8 below.

## 5 Hölder bounds

Before giving, in the next section, the key infinitesimal rigidity Lemma, we need some precise estimates on the behavior, in the Euclidean ball, of the deformation vector fields on $\overline{\mathcal{S}}$ and $\overline{\mathcal{S}}^\ast$. They will follow from a careful analysis of the Killing vector fields induced by the first-order deformation $\dot{g}$.

Recall that $u$ is a vector field induced on $\mathcal{S}$ by the first-order deformation $\dot{g}$ of $g$. We call $\overline{\pi}$ its image by the Pogorelov transformation $\Phi_H$, so that $\overline{\pi}$ is an isometric first-order deformation of the surface $\overline{\mathcal{S}}$ which is the image of $\mathcal{S}$ in the projective model. We also call $\overline{\pi}$ the orthogonal projection of $\overline{\pi}$ on $\overline{\mathcal{S}}$.

**Bounds on $\nabla \overline{\pi}$.** Recall that we call $\delta$ the Euclidean distance, along $\overline{\mathcal{S}}$, to the limit set $\Lambda$. By Lemma 2.2,

$\sqrt{1-r}/c_1 \leq \delta \leq c_1\sqrt{1-r}$ near $\Lambda$, where $c_1 > 0$ is some constant. The first lemma we need is an upper bound, near the limit set $\Lambda$, on the norm of $\nabla \overline{\pi}$.

**Lemma 5.1.** There exists a constant $C > 0$ such that, on $\overline{\mathcal{S}}$:

\[ \|\nabla \overline{\pi}\| \leq C|\log(\delta)|. \]

**Proof.** We denote by $\Pi_{\pi}$ the orthogonal projection on $\overline{\mathcal{S}}$, seen as a morphism from $T_{\pi}\mathbb{R}^3$ to $T_{\pi}\overline{\mathcal{S}}$, for $\pi \in \overline{\mathcal{S}}$. Let $\overline{\mathcal{S}}$, and let $\overline{\pi} \in T_{\pi}\overline{\mathcal{S}}$ be a vector of norm at most 1. Then $\overline{\pi} = \overline{\pi}^\perp + \lambda R$, where $\overline{\pi}^\perp$ is the lateral component of $\overline{\pi}$; $\|\overline{\pi}^\perp\| \leq 1$ and $|\lambda| \leq C\delta$ by Lemma 2.2.

Now $\overline{\pi} = \Pi_{\pi}(\overline{\pi})$ by definition, so that:

\[ \nabla_{\nabla \overline{\pi}} = (\nabla_{\nabla \Pi})\overline{\pi} + \Pi(\nabla_{\nabla \overline{\pi}}), \]

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and we have by (**)  
\[ \nabla w^v = (\nabla w^v) - \Pi(w^\perp) + \Pi(w^\perp) \, (\nabla w^v). \]

By Corollary 4.9 we have:  
\[ \|\nabla w^v\| \leq C|\log(1 - r)|. \]

In addition, \( \|\nabla w^v\| \leq C \) by Lemma 2.5 (recall that, in this proof, \( C \) is a generic constant whose value can change from line to line). Putting both together and using that \( \delta \) behaves as \( \sqrt{1 - r} \), we find that:  
\[ \|\nabla w^v\| \leq C|\log(\delta)|. \]

For the second term in (1), note that, at \( \pi = \pi^\perp + \pi^\perp \), where, again by Corollary 4.9  
\[ \|\pi^\perp\| \leq C|\log(1 - r)| \leq C|\log(\delta)|, \|\pi^\perp\| \leq \frac{C}{\sqrt{1 - r}} \leq \frac{C}{\delta}. \]

Using the bounds \( \|\pi^\perp\| \leq 1 \) and \( |\lambda| \leq C\delta \), we find that:  
\[ \|\Pi(w^\perp \pi)\| \leq \|\Pi((w^\perp + \lambda R) \pi^\perp + \pi^\perp)\| \leq \|\Pi(w^\perp \pi^\perp)\| + \|\Pi(R \pi^\perp)\| + \|\Pi(w^\perp \pi^\perp)\| \leq C + C|\log(\delta)|, \]

where the upper bound on the first term uses that \( w^\perp \pi^\perp \) is radial (so that \( \Pi \) adds a factor \( \delta \)) while the upper bound on the second term uses that \( |\lambda| \leq C\delta \). From this, we see that:  
\[ \|\Pi(w^\perp \pi)\| \leq C|\log(\delta)|. \]

For the third term in equation (1), note that:  
\[ \Pi(w^\perp \pi) = \Pi(w^\perp (w^\perp)) + \lambda \Pi(w^\perp (w^\perp)) + \Pi(w^\perp (w^\perp)) + \lambda \Pi(w^\perp (w^\perp)) . \]

Lemma 4.8 shows that the first term is bounded by \( C \), the second by \( \lambda C/\delta \), and thus by \( C \), the third by \( \delta C/\delta = C \) because the projection induces a factor \( \delta \), and the last by \( \lambda \delta C/\delta^2 \) (again \( \Pi \) adds a factor \( \delta \)), and thus by \( C \). Thus all 4 terms are bounded by \( C \).

Using this along with (1) and (1) in equation (1) proves the Lemma.  

Hölder bounds on \( \pi \)  We now state the key technical Lemma, about the Hölder bounds on \( \pi \) near \( \Lambda \).

Lemma 5.2. For all \( \epsilon \in (0, 1) \), there exists a constant \( C_\epsilon > 0 \) such that, if \( \pi, \pi \in S \) are such that \( d(\pi, \pi) \leq 1/2 \), then \( \|\pi(\pi) - \pi(\pi)\| \leq C_\epsilon d(\pi, \pi)^\epsilon \).  

We will use another point, called \( \pi \), which we define as the point of \( \pi(\pi) \) which is closest to 0. We call \( r_0 := r(\pi) \), and let \( \delta_0 := d(\pi, \pi) \). The key properties of \( \pi \) are the following.

Proposition 5.3.  
1. if \( \pi \neq \pi \), the angle at \( \pi \) of the triangle \( (0, \pi, \pi) \) is between \( \pi/2 \) and \( \pi/2 + C \sqrt{1 - r_0} \). 
2. \( \delta_0 \leq C / \sqrt{1 - r_0} \).  

Proof. For the first point, note that, if \( \pi \neq \pi \), the angle at \( \pi \) of the triangle \( (0, \pi, \pi) \) — call it \( \alpha \) — is equal to \( \pi/2 \) since \( \pi \) is at minimal distance from 0. Otherwise, \( \pi = \pi \) is on \( \partial \Omega \), and, by convexity of \( \Omega \), the segment \( [\pi, \pi] \) is towards the interior of \( \Omega \). Therefore Lemma 2.3 shows that \( \alpha \) is at most \( \pi/2 + C \sqrt{1 - r_0} \).

For the second point, note that:  
\[ r(\pi)^2 = r(\pi)^2 + 2 \cos(\alpha) d(\pi, \pi) + d(\pi, \pi) \geq r(\pi)^2 + d(\pi, \pi)^2. \]

Since \( r(\pi) < 1 \), so that:  
\[ \sqrt{1 - r(\pi)^2} \geq d(\pi, \pi). \]

But \( 1 - r(\pi)^2 \leq 2(1 - r(\pi)) \), it follows that \( d(\pi, \pi) \leq C \sqrt{1 - r(\pi)} \). Since the same argument can be used with \( \pi \) replaced by \( \pi \), we obtain the result.  

We now call \( (\rho, \rho) \) the (unique) flat section of \( \rho \) which is such that \( \rho(\rho) = \rho(\rho) \) and that \( \rho(\rho) = \rho(\rho) \); then we call \( \delta \rho := \rho - \rho, \delta \rho := \rho - \rho \). Lemma 5.2 will follow from the following estimates on \( \delta \rho \) and on \( \delta \rho \).
Proposition 5.4. We have the following estimates:

\[
\|\delta\sigma(\tau)\| \leq \frac{C\delta_0}{1-r_0} + C \left( \frac{1}{\sqrt{1-r(\tau)}} - \frac{1}{\sqrt{1-r_0}} \right),
\]

\[
\|\delta\sigma'(\tau)\| \leq C \left| \log \left( \frac{1-r(\tau)}{1-r_0} \right) \right|,
\]

\[
\|\delta\sigma(\tau)\| \leq C \left| \log \left( \frac{1-r(\tau)}{1-r_0} \right) \right|,
\]

\[
\|\delta\sigma(\tau)\| \leq C \delta_0 + C \left( \sqrt{1-r_0} - \sqrt{1-r(\tau)} \right).
\]

Proof. We consider two paths, such that the union is a path going from \( \tau \) to \( \tau' \). The first, \( \gamma_\perp : [0, l] \to D^3 \), is a geodesic segment on the sphere at distance \( r_0 \) from 0, and goes from \( \tau \) to the point of \( [0, \tau'] \) at distance \( r_0 \) from 0. The second, \( \gamma_r : [0, r-r_0] \to D^3 \), goes from \( \gamma_\perp(l) \) to \( \tau' \) along the segment joining them.

We consider first the upper bound on \( \delta\sigma \). For all \( s \in [0, l] \), we have:

\[
\|\nabla\gamma_\perp(s)\delta\sigma\| = \|\sigma'(\gamma_\perp(s))\| \leq \frac{C}{1-r_0}
\]

by Lemma [Lemma 5.4] so that:

\[
\|\delta\sigma(\gamma_\perp(l))\| \leq \frac{C\delta_0}{1-r_0}.
\]

For each \( t \in [0, r-r_0] \), Lemma [Lemma 4.8] shows that:

\[
\|\nabla\gamma_r(t)\delta\sigma\| = \|\sigma'(\gamma_r(t))\| \leq \frac{C}{\sqrt{1-r(\gamma_r(t))}}
\]

so that, by integration:

\[
\|\delta\sigma(\tau)\| \leq \frac{C\delta_0}{1-r_0} + C \left( \frac{1}{\sqrt{1-r(\tau)}} - \frac{1}{\sqrt{1-r_0}} \right).
\]

To obtain the bound on \( \delta\sigma' \), note that, for all \( s \in [0, l] \):

\[
\|\nabla\gamma_\perp(s)\delta\sigma\| \leq \frac{1}{r_0} \|\delta\sigma(\gamma_\perp(s))\| + \|\sigma'(\gamma_\perp(s))\| \leq \frac{C}{\sqrt{1-r_0}}.
\]

Integrating this yields that:

\[
\|\delta\sigma'(\gamma_\perp(l))\| \leq C \frac{\delta_0}{\sqrt{1-r_0}} \leq C.
\]

Moreover, for all \( t \in [0, r-r_0] \):

\[
\|\nabla\gamma_r(t)\delta\sigma\| = \|\sigma'(\gamma_r(t))\| \leq \frac{C}{1-r(\gamma_r(t))},
\]

and, by integration:

\[
\|\delta\sigma'(\gamma_\perp(l))\| \leq C + C \left| \log \left( \frac{1-r(\gamma_r(t))}{1-r_0} \right) \right|,
\]

and the second point follows.

For the third point, we use that, for all \( s \in [0, l] \):

\[
\|\nabla\gamma_\perp(s)\delta\sigma\| \leq \|\sigma'(\gamma_\perp(s))\| + \|\delta\sigma\| \leq \frac{C}{\sqrt{1-r_0}} + \frac{C}{\sqrt{1-r_0}};
\]

using the estimates obtained in the proof of point (1). So, by integration:

\[
\|\delta\sigma(\gamma_\perp(l))\| \leq C \frac{\delta_0}{\sqrt{1-r_0}} \leq C.
\]
For all $t \in [0, r - r_0]$:

$$\|\nabla_{\gamma'(t)} \delta\| = \|\tau_{\gamma'}(\gamma'(t))\| + \|\delta\| \leq \frac{C}{1 - r(\gamma(t))} + \frac{C}{\sqrt{1 - r(\gamma(t))}} ,$$

again by the estimate on $\delta\tau$ obtained above, so that, by integration:

$$\|\delta\tau\| \leq C \left| \log \left( \frac{1 - r(x)}{1 - r_0} \right) \right| .$$

For the last point, we use that, for all $s \in [0, l]$:

$$\|\nabla_{\gamma'(s)} \delta\| = \|\tau_{\gamma'}(\gamma'(s))\| + \|\delta\| \leq C + C + C ,$$

where we have used also the upper bound in the proof of point (2) above. It follows by integration that:

$$\|\delta\tau_{\perp}(\gamma'(s))\| \leq C\delta_0 .$$

Moreover, for all $t \in [0, r - r_0]$:

$$\|\nabla_{\gamma'(t)} \delta\| = \|\tau_{\gamma'}(\gamma'(t))\| + \|\delta\| \leq \frac{C}{1 - r(\gamma(t))} + \frac{C}{\sqrt{1 - r(\gamma(t))}} ,$$

so that, by integration:

$$\|\delta\tau_{\perp}(\tau)\| \leq C\delta_0 + C \left( \sqrt{1 - r_0} - \sqrt{1 - r(x)} \right) .$$

Those bounds are basically what we need to prove Lemma 5.2 because of the next proposition.

**Proposition 5.5.** If $\delta_0$ is small enough, then:

$$\sqrt{1 - r_0} - \sqrt{1 - r(x)} \leq C\delta_0 ,$$

$$\sqrt{1 - r(x)} \left| \log \left( \frac{1 - r(x)}{1 - r_0} \right) \right| \leq \delta_0 |\log(\delta_0)| .$$

**Proof.** By the upper bound on the angle at $\tau$, we have:

$$r(x) \leq r_0 + C\delta_0 \sqrt{1 - r_0} + \delta_0^2 ,$$

so that:

$$1 - r(x) \geq (1 - r_0) - C\delta_0 \sqrt{1 - r_0} - \delta_0^2 .$$

If $\sqrt{1 - r_0} \leq C\delta_0$, then:

$$\sqrt{1 - r_0} - \sqrt{1 - r(x)} \leq \sqrt{1 - r_0} \leq C\delta_0 ,$$

and:

$$\sqrt{1 - r(x)} \left| \log \left( \frac{1 - r(x)}{1 - r_0} \right) \right| \leq \sqrt{1 - r_0} |\log(1 - r_0)| \leq C\delta_0 |\log(\delta_0)| ,$$

so we only have to consider the case where $\delta_0/\sqrt{1 - r_0}$ is small enough to ensure — through (1) — that $1 - r(x) \geq (1 - r_0)/2$.

Then:

$$\sqrt{1 - r_0} - \sqrt{1 - r(x)} = 2 \frac{1}{2} \int_{r_0}^{r(x)} ds \sqrt{1 - s} \leq \frac{r(x) - r_0}{2\sqrt{1 - r(x)}} \leq r(x) - r_0 ,$$

But the angle at $\tau$ of the triangle $[0, \tau, x]$ is at most $C\delta_0$, and, since $\delta_0 \leq \sqrt{1 - r_0}/C$, the angle between $[\tau, x]$ and the lateral directions is everywhere at most $C\sqrt{1 - r_0}$. Thus:

$$r(x) - r_0 \leq C\delta_0 \sqrt{1 - r_0} ,$$
and it follows that:
\[ \sqrt{1 - r_0} - \sqrt{1 - r(x)} \leq \frac{r(x) - r_0}{\sqrt{1 - r_0}} \leq C\delta_0. \]

Similarly:
\[ \left| \log \left( \frac{1 - r(x)}{1 - r_0} \right) \right| = 2 \left| \frac{\sqrt{1 - r(x)} - \sqrt{1 - r_0}}{\sqrt{1 - r_0}} \right| \leq 2 \int_{\sqrt{1 - r_0}}^{\sqrt{1 - r(x)}} \frac{ds}{s} \leq 2 \frac{\sqrt{1 - r_0} - \sqrt{1 - r(x)}}{\sqrt{1 - r_0}} \leq \frac{C\delta_0}{\sqrt{1 - r(x)}} , \]

which proves the second equation.

**Proof of Lemma 5.7.** Recall that $\overline{\tau}$ is the orthogonal projection on $\overline{S}$ of the vector field $\overline{\tau}$. Clearly the orthogonal projection of $\tau_0$ is Hölder, because $\tau_0$ is a Killing field and $\overline{S}$ is smooth enough by the results of section 2. So the result will follow from the fact that the orthogonal projection on $T_{\overline{\tau}}\overline{S}$ of $\delta\overline{\tau}(\overline{\tau})$ is bounded by $C\delta_0|\log(\delta_0)|$, with the same bound applying with $\overline{\tau}$ replaced by $\overline{y}$.

By definition, $\delta\overline{\tau}(\overline{\tau}) = \delta\overline{\tau}^\perp(\overline{\tau}) + \delta\overline{\tau}^\parallel(\overline{\tau})$. The orthogonal projection on $T_{\overline{\tau}}\overline{S}$ of $\delta\overline{\tau}^\perp(\overline{\tau})$ is bounded by $|\delta\overline{\tau}^\parallel(\overline{\tau})|$, and so by:
\[ C\delta_0 + C \left( \sqrt{1 - r_0} - \sqrt{1 - r(\overline{\tau})} \right) \]

by Proposition 5.3 and thus by $C\delta_0$ by Proposition 5.5. The orthogonal projection on $T_{\overline{\tau}}\overline{S}$ of $\delta\overline{\tau}^\parallel(\overline{\tau})$ is bounded by $C\sqrt{1 - r(\overline{\tau})}|\delta\overline{\tau}^\parallel(\overline{\tau})|$ by Lemma 2.4, so by:
\[ C\sqrt{1 - r(\overline{\tau})} \left| \log \left( \frac{1 - r(\overline{\tau})}{1 - r_0} \right) \right| \]

by Proposition 5.4 and therefore by $C\delta_0|\log(\delta_0)|$ by Proposition 5.5. This proves the Lemma.

**Hölder estimates on the dual surface** The argument given here extends with minor modifications to yield the corresponding estimates on the deformation vector $\overline{\tau}^*$ on the dual surface $\overline{S}^*$. We will outline the proof here, with emphasis on the small differences with the estimates on $\overline{S}$.

**Proposition 5.6.** Let $\overline{\tau}$, and let $\overline{\tau}^*$ be the dual point on $\overline{S}^*$. Let $\theta(\overline{\tau}^*)$ be the angle between $T_{\overline{\tau}}\overline{S}^*$ and the lateral directions, and let $(\overline{\tau} - \overline{\tau})^\perp$ and $(\overline{\tau} - \overline{\tau})^\parallel$ be the decomposition in lateral and radial components, at $\overline{\tau}$, of the vector $\overline{\tau}^* - \overline{\tau}$. Then, for some constant $C > 0$:

1. $|\overline{\tau}^* - \overline{\tau}| \leq C(1 - r(\overline{\tau}))$.
2. $|\overline{\tau} - \overline{\tau})^\perp| \leq C\sqrt{1 - r(\overline{\tau})}$.
3. $|\theta(\overline{\tau}^*)| \leq C\sqrt{1 - r(\overline{\tau})}$.
4. The map sending a point of $\overline{S}$ to its dual in $\overline{S}^*$ is Lipschitz.

**Proof.** The projective definition of the duality shows that $\overline{\tau}^*$ is in the plane dual to $\overline{\tau}$. This plane is exactly the set of points $m \in \mathbb{R}^3$ such that the orthogonal projection of $m$ on the line $(0, \overline{\tau})$ is at distance $1/(1 - r(\overline{\tau}))$ from 0, this proves the first point.

For the second point, let $\theta(\overline{\tau})$ be the angle between $T_{\overline{\tau}}\overline{S}$ and the lateral directions, then $\theta(\overline{\tau})$ is also — by the projective definition of the duality — the angle at 0 between the direction of $\overline{\tau}$ and of $\overline{\tau}^*$. This, along with the proof of the first point, shows the second point.

Finally, points (3) is a consequence of Lemma 2.4 using the definition of the duality, and point (4) follows from Lemma 2.4 and from Lemma 2.5.

There is also an analog of Lemma 5.4.

**Lemma 5.7.** There exists a constant $C > 0$ such that, on $\overline{S}^*$:
\[ |\nabla \overline{\tau}^*| \leq C|\log(\delta)| . \]

**Proof.** It is a consequence of Remark 1.7 that $\overline{\tau}^*$ is the orthogonal projection on $T_{\overline{\tau}^*}$ of $\overline{\tau}(\overline{\tau}) + \overline{\tau}(\overline{\tau}) \wedge (\overline{\tau}^* - \overline{\tau})$. The Lemma then follows from Lemma 4.8 along with point (4) of the previous proposition.

We can now state the dual of Lemma 5.2.

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Lemma 5.8. For all $\epsilon \in (0, 1)$, there exists a constant $C_\epsilon > 0$ such that, if $\tau^*, \gamma^* \in \mathcal{S}$ are such that $d(\tau^*, \gamma^*) \leq 1/2$, then $\|\tau^*(\tau^*) - \gamma^*(\gamma^*)\| \leq C_\epsilon d(\tau^*, \gamma^*)$. 

Proof of Lemma 5.8. We use the same setting as in the proof of Lemma 5.2, i.e., $\tau$ is the point of $[\tau, \gamma]$ which is closest to 0, and we call $(\tau_0, \sigma_0)$ the flat section of $E$ such that $\tau_0(\tau) = \tau(\tau)$ and that $\sigma_0(\tau) = \sigma(\tau)$, and call $\delta \tau = \tau - \tau_0, \delta \sigma = \sigma - \sigma_0$. Clearly the orthogonal projection on $\mathcal{T} \mathcal{S}$ of the Killing field corresponding to $(\tau_0, \sigma_0)$ is Hölder, so the Lemma will follow as in the proof of Lemma 5.2 — from the fact that the orthogonal projection on $\mathcal{T}_\tau \mathcal{S}$ of $\delta \tau(\tau) + \delta \sigma(\tau) \wedge (\tau^* - \tau)$ is bounded by $C_{\delta_0} \| \log(\delta_0) \|$.

The bound on the orthogonal projection of $\delta \tau(\tau)$ is obtained exactly like in the proof of Lemma 5.2 using the angle bound in Proposition 5.6. To bound the other term, use Proposition 5.4 to obtain that:

$$|| (\delta \tau(\tau) \wedge (\tau^* - \tau)) \| = || \delta \tau(\tau) \wedge (\tau^* - \tau) + \delta \sigma(\tau) \wedge (\tau^* - \tau) || \leq \left( \frac{C_{\delta_0}}{1 - r_0} + C \left( \frac{1}{\sqrt{1 - r(\tau)}} - \frac{1}{\sqrt{1 - r_0}} \right) \right) (1 - r(\tau)) + C \sqrt{1 - r(\tau)} \left| \log \left( \frac{1 - r(\tau)}{1 - r_0} \right) \right| \right).$$

Using Proposition 5.5 we find that:

$$|| (\delta \tau(\tau) \wedge (\tau^* - \tau)) \| \leq C_{\delta_0} \| \log(\delta_0) \|.$$ 

In a similar way:

$$|| (\delta \sigma(\tau) \wedge (\tau^* - \tau)) \| = || \delta \sigma(\tau) \wedge (\tau^* - \tau) \| \leq \left( \frac{C_{\delta_0}}{1 - r_0} + C \left( \frac{1}{\sqrt{1 - r(\tau)}} - \frac{1}{\sqrt{1 - r_0}} \right) \right) \sqrt{1 - r(\tau)},$$

so the orthogonal projection of this term on $\mathcal{T}_\tau \mathcal{S}$ is bounded by:

$$C_{\delta_0} + C \left( \frac{1 - r_0}{\sqrt{1 - r(\tau)}} - \frac{1}{\sqrt{1 - r_0}} \right),$$

which is bounded by $C_{\delta_0} \| \log(\delta_0) \|$ by Proposition 5.5. \hfill $\square$

6 Infinitesimal rigidity

We now have all the tools necessary to prove the infinitesimal rigidity Lemma for hyperbolic manifolds with convex boundary. We first consider the rigidity with respect to the induced metric, then with respect to the third fundamental form.

Lemma 6.1. Let $g \in \mathcal{G}$. For any non-trivial first-order deformation $\hat{g} \in T_g \mathcal{G}$, the induced first-order variation of the induced metric on $\partial M$ is non-zero.

We give first a broad outline of the proof, then some additional estimates, and finally the proof.

Outline of the proof. The idea underlying the rigidity lemmas is that, once the problem has been translated into a rigidity statement in $\mathbb{R}^3$, the known results concerning the rigidity of convex surfaces in Euclidean 3-space should hold, or at least their proofs should apply. In practice, things are not quite as simple, because the vector fields $\tau$ and $\tau^*$ obtained in $\mathbb{R}^3$ are indeed infinitesimal deformations of $\mathcal{S}$ and $\mathcal{S}^*$, but they might diverge on $\Lambda$; the results of [Log73] can therefore not be applied directly.

Recall that $\tau$ is the orthogonal projection of $\tau$ on $\mathcal{S}$, and that $\tau := B^{-1} \tau$. We have seen that $\tau$ is solution of $\mathcal{G} \tau = 0$, and that it satisfies a Hölder estimate on $\mathcal{S}$. This estimate will allow us to prove that it is actually in the Sobolev space $H^1$ on the whole sphere (including the limit set). To prove that this equation has no solution in $H^1$ other than those coming from the global infinitesimal isometries, we wish to consider the adjoint equation, $d^* \hat{h} = 0$.

To have a good interpretation of this adjoint equation, however, it helps to consider the equation $\mathcal{G} \tau = 0$ on a hyperbolic, rather than a Euclidean, surface, since then the adjoint equation describes the infinitesimal
rigidity of the dual surface in the de Sitter space. Therefore we will use another projective model of \( H^3 \), different from the one used to define \( \mathcal{S} \) from \( S \), but with the same “center”; we decide that in this model the image of \( H^3 \) is a ball of radius 2 in \( \mathbb{R}^3 \).

We call \( \mathcal{S}_h \) the inverse image of \( \mathcal{S} \) by this model, and \( \Lambda_h \) the set corresponding to the limit set of \( M \). Thus \( \mathcal{S}_h \cup \Lambda_h \) is a closed, convex hyperbolic surface — it does not get close to the boundary at infinity of \( H^3 \).

The estimates in section 2 indicate that \( \mathcal{S} \) has principal curvatures bounded between two positive constants. Since \( \mathcal{S}_h \) is away from the boundary of \( H^3 \), the projective model we use does not distort the principal curvatures much, so that \( \mathcal{S}_h \) also has principal curvatures bounded between two positive constants. This means that its second fundamental form is quasi-Lipschitz to its induced metrics, and thus to the canonical metric on \( S^2 \).

Let \( \overline{\nu}_h \) be the image of \( \overline{\nu} \) on \( \mathcal{S}_h \) by the Pogorelov map, and let \( \nu_h \) be the orthogonal projection of \( \overline{\nu}_h \) on \( \mathcal{S}_h \). By the basic property of the Pogorelov map, \( \overline{\nu}_h \) is an isometric infinitesimal deformation of \( \mathcal{S}_h \); therefore, calling \( B_h \) the shape operator of \( \mathcal{S}_h \), \( \nu_h := B_h^{-1} \overline{\nu}_h \) is a solution of \( \mathcal{S}_h \mathcal{G}_h \nu_h = 0 \) on \( \mathcal{S}_h \). The results of section 5 will be used to show that \( \nu_h \) behaves rather well on \( \Lambda_h \), where it satisfies a Hölder estimate.

The key of the argument will be that this fact implies that \( \overline{\nu}_h \) is in the Sobolev space \( H^1 \) (for the metric \( \mathcal{G} \)), so that it is actually not only a pointwise solution of \( \mathcal{H}_h \mathcal{G}_h \nu_h = 0 \) on \( \mathcal{S}_h \), but also a global solution on \( \mathcal{S}_h \cup \Lambda_h \), which is topologically a sphere. This point uses the fact that the Hausdorff dimension of \( \Lambda \) is strictly less than 2 (a result of Sullivan [Sul79]).

Since \( \mathcal{G}_h \) is elliptic, it acts continuously from \( H^1 \) to \( L^2 \). We then move from \( \mathcal{G}_h \) to its adjoint \( q^\mathcal{G} \); considering the dual surface (in the de Sitter space) we show that it has no solution. This implies that the co-kernel of \( \mathcal{G}_h \) restricted to \( H^1 \) is empty, and therefore that the dimension of its kernel is equal to its index, namely 6. Since there is a 6-dimensional space of \( H^1 \) solutions which come from trivial isometric deformations, all solutions are of this kind, and the conclusion follows.

### Analytical properties of \( \nu_h \)
We now need some precise informations on the analytical properties of \( \nu_h \), or more precisely on the vector field \( \nabla \nu_h := B_h^{-1} \nabla \overline{\nu}_h \). Note that the smoothness properties of \( \nabla \nu \) will be with respect to the metric \( \mathcal{G} \), and to the connection \( \nabla \).

Note first that \( \nu_h \) satisfies the same Hölder estimate, near \( \Lambda_h \), as the estimate on \( \nu \) given by Lemma 5.2. This follows from the estimates of section 5 on the radial and the lateral components of \( \overline{\nu} \); since we have kept the same “center” in the projective model, and since \( \mathcal{S}_h \) is compact, the radial and lateral components of \( \overline{\nu}_h \) satisfies the same estimates (up to constants) as the corresponding components of \( \overline{\nu} \), and the Hölder estimate on \( \nu_h \) follows.

#### Definition 6.2.
We call \( H^1 \mathcal{G}(T \mathcal{S}_h) \) the space of sections of \( T \mathcal{S}_h \) which are in the Sobolev space \( H^1 \) for the third fundamental form \( \mathcal{G} \) of \( \mathcal{S}_h \).

Here \( H^1 \) is the space of functions \( u \) such that the integral of the norm of \( du \) converges. We first state an elementary technical proposition useful later on.

#### Proposition 6.3.
Let \( \delta \mathcal{G} \) be the distance to \( \Lambda_h \), for \( \mathcal{G} \), on \( \mathcal{S}_h \). Then \( \log(\delta \mathcal{G}) \) is in \( L^2 \) for the area form of the third fundamental form of \( \mathcal{S}_h \).

**Proof.** By a result of Bishop [Bis96] that the Minkowski dimension of \( \Lambda \) (as a subset of \( S^2 \)) is strictly smaller than 2 (it was earlier proved by Sullivan for the Hausdorff dimension). It follows that the Minkowski dimension of \( \Lambda \) in \( \mathcal{S} \) is also strictly smaller than 2. Since the projective map we use does not distort the distance much on \( \mathcal{S} \), it follows that the Minkowski dimension of \( \Lambda_h \) in \( \mathcal{S}_h \cup \Lambda_h \) is also strictly smaller than 2, and, since the principal curvature are bounded between two positive constants, the same holds for the third fundamental form. The result then follows by an elementary integration argument.

Note that the proof could also be done using the fact that the Hausdorff dimension of \( \Lambda \) is strictly smaller than 2, but it is then technically more tricky.

#### Corollary 6.4.
The integral of the square of the norm of \( \nabla \nu_h \) over \( \mathcal{S}_h \) converges, and, for all \( \alpha \in (0, 1) \), \( \nabla \nu \) is \( C^\alpha \) Hölder on \( \mathcal{S}_h \cup \Lambda_h \).

**Proof.** We have seen that \( \nu_h \) satisfies the same Hölder estimate as \( \nu \) in Lemma 5.2, but \( \nabla \nu_h := B_h^{-1} \nabla \overline{\nu}_h \), so that, for any \( X \in T \mathcal{S}_h \):

\[
\nabla_X \nabla \nu_h = B_h^{-1} \nabla_X (B_h \nu_h) = B_h^{-1} \nabla_X \overline{\nu}_h,
\]

so (since the eigenvalues of \( B_h \) are bounded and \( \mathcal{G} \) is quasi-Lipschitz to \( \mathcal{I} \)) \( \nabla \nu_h \) is also \( C^\alpha \)-Hölder, but for \( \mathcal{G} \).
Moreover, \( \|\nabla \pi\| \) is bounded by \( C \log(\delta) \) by Lemma \ref{lem:log-estimate}, so, since the radial and lateral components of \( \pi_h \) behave as the corresponding components of \( \pi \), the same estimate applies to \( \pi_h \): it follows that \( \|\nabla \pi_h\| \mathbf{\mathbf{\leq}} C \log(\delta) \). Lemma \ref{lem:log-estimate} thus shows that the integral of the square of the norm of \( \nabla \pi_h \) over \( \Sigma_h \) converges. □

This corollary will be helpful thanks to the following simple lemma. It is probably well-known to the specialists — and can be extended in various directions — but we have included a proof for completeness.

**Lemma 6.5.** Let \( \Lambda \subset \Sigma \) be a subset of Hausdorff dimension \( \delta > 1 \) in a closed surface. Let \( u : \Sigma \to \mathbb{R} \) be a \( L^2 \) function such that:

- \( u \) is everywhere \( C^\alpha \), for some \( \alpha \in (\delta - 1, 1) \).
- \( u \) is smooth outside \( \Lambda \).
- we have:
  \[
  \int_{\Omega \setminus \Lambda} \| du \|^2 \, da < \infty .
  \]

Then \( u \) is in the Sobolev space \( H^1(\Sigma) \).

**Proof.** The statement is essentially local, so it is sufficient to prove the same result in \( \Omega := [0, 1] \times [0, 1] \), with the support of \( u \) in \((0,1) \times (0,1)\). Let \( v \) be a smooth vector field on \( \Omega \).

Choose \( \epsilon > 0 \). By definition of the Hausdorff dimension of \( \Lambda \), there exists a covering of \( \Lambda \) by geodesic balls \( B_1, \cdots, B_N \), with \( B_i \) of center \( x_i \) and radius \( r_i \leq \epsilon \), such that \( \sum_i r_i^{1+\alpha} \leq \epsilon \). Let \( B := \Omega \cap (\cup_i B_i) \). We replace some of the balls \( B_i \) by sub-domains, and find a covering of \( B \) by a finite number of convex domains \( D_i \), with disjoint interiors, so that \( D_i \subset B_i \) for each \( i \). Then:

\[ \forall i \in \{1, \cdots, N\}, \quad L(\partial D_i) \leq L(\partial B_i) \leq 2\pi r_i . \]

Let \( v \) be a smooth vector field on \( S^2 \). Then:

\[
\left| \int_{\Omega \setminus B} u \text{div}(v) + du(v) \, da \right| = \left| \int_{\Omega \setminus B} \text{div}(uv) \, da \right|
= \left| \int_{\partial B} u \langle v, n \rangle \, dx \right|
= \left| \sum_i \int_{\partial D_i} u \langle v, n \rangle \, dx \right|
\leq \sum_i \left| \int_{\partial D_i} (u(x_i) + (u(x) - u(x_i))) \langle v(x_i) + (v(x) - v(x_i)), n \rangle \, dx \right| .
\]

Note that:

\[ \int_{\partial D_i} \langle v(x_i), n \rangle = \int_{D_i} \nabla \langle v(x_i) \rangle \, dx = 0 , \]

so that:

\[
\left| \int_{\Omega \setminus B} u \text{div}(v) + du(v) \, da \right| \leq \sum_i \left| \int_{\partial D_i} u(x_i) \langle v(x_i), n \rangle \right|
+ \int_{\partial D_i} |u(x)(v(x) - v(x_i), n) + (u(x) - u(x_i)) \langle v(x_i), n \rangle | \, dx \leq 2\pi \sum_i C_1 \max |u| r_i^2 + C_2 \max \| v \| r_i^{1+\alpha} ,
\]

where \( C_1 \) is the maximum of \( |D_i| \) and \( C_2 \) is the Hölder modulus of \( u \).

This means that, if \( \epsilon \) is small enough, we have for some \( C_3 > 0 \):

\[
\left| \int_{\Omega \setminus B} u \text{div}(v) + du(v) \, da \right| \leq C_3 \sum_i r_i^{1+\alpha} \leq C_3 \epsilon .
\]

Taking the limit as \( \epsilon \to 0 \), we see that:

\[
\int_{\Omega \setminus \Lambda} u \text{div}(v) \, da = - \int_{\Omega \setminus \Lambda} du(v) \, da .
\]
Since the $H^1$ norm of $u$ over $\Omega \setminus \Lambda$ is finite, we see that:

$$\left| \int_{\Omega} u \text{div}(v) \, da \right| = \left| \int_{\Omega \setminus \Lambda} du(v) \, da \right| \leq \|u\|_{H^1(\Omega \setminus \Lambda)} \|v\|_{L^2}.$$ 

This inequality, which is true for all smooth vector field $v$, shows that $u$ is in $H^1(\Omega)$, as needed. 

As a direct consequence of Corollary 6.4 and of Lemma 6.3 we have the following important statement.

**Lemma 6.6.** $\omega_h$ is in $H^1_0(T(\mathbb{S}_h \cup \Lambda_h))$, and it is a solution of $\mathcal{D}_h \omega_h = 0$ on $\mathbb{S}_h \cup \Lambda_h$.

This is important because $H^1_0(T(\mathbb{S}_h \cup \Lambda_h))$ is a space on which $\mathcal{D}_h$ acts well.

**Lemma 6.7.** $\mathcal{D}_h$ acts continuously from $H^1_0$ to $L^2(\Omega^{0,1}_h(\mathbb{S}_h \cup \Lambda_h))$.

**Proof.** Let $\tau \in H^1_0$. Let $U \subset \mathbb{S}_h \cup \Lambda_h$ be an open subset, and let $X$ be a smooth unit vector field on $U$. Then, by definition of $H^1_0$, both $\nabla_X \tau$ and $\nabla_T X \tau$ are $L^2$ sections of $TU$. So $\mathcal{D}_h \tau$ is a $L^2$ section of $TU$. Since this is true for any simply connected open subset $U \subset \mathbb{S}_h \cup \Lambda_h$, $\mathcal{D}_h \tau$ is a $L^2$ section of $\Omega^{0,1}_h(\mathbb{S}_h \cup \Lambda_h)$.

$L^2$ solutions of $d\nabla$ The other important point of the proof is that there is no $L^2$ solution of $d\nabla h = 0$ in $\Omega^{0,1}_h(\mathbb{S}_h \cup \Lambda_h)$. We first prove the corresponding result for $d\nabla h = 0$ on Euclidean surfaces, following closely the ideas used in section 3; we only have to keep closely track of the smoothness issue.

**Lemma 6.8.** Let $S$ be a closed, strictly convex surface in $\mathbb{R}^3$, with principal curvatures bounded between two strictly positive constants. There is no non-trivial solution of $d\nabla h = 0$ with $h \in L^2(\Omega^{0,1}_h(S))$.

**Proof.** We will follow the proof of Proposition 3.11 and Lemma 3.12 but we will have to be more careful because we have only a limited degree of smoothness — actually only the hypothesis that $h$ is $L^2$.

First note that, according to Proposition 6.14, it is equivalent to show that there is no non-trivial $L^2$ solution of equation (11), because, since $B$ has its eigenvalues bounded between two positive constants, $\tilde{B}$ is in $L^2$ if and only if $B^{-1} \tilde{B}$ is. So we consider a $L^2$ solution $\tilde{B}$ of (11), and will prove that it is 0.

We now follow the proof of Proposition 3.11. We consider $S$ as the image of a map $\phi : S \to \mathbb{R}^3$, and its unit normal vector field as a map $N : S \to \mathbb{R}^3$. We define a $\mathbb{R}^3$-valued 1-form $\alpha$ by $\alpha := (d\phi \circ \tilde{B}) \wedge N$. Then, as in the proof of Proposition 3.11, $d\alpha = 0$ because $\tilde{B}$ is a solution of (11), and we can integrate it to obtain a $H^1$ function $Y : S \to \mathbb{R}^3$. We can then define another $\mathbb{R}^3$-valued 1-form $\beta$ on $S$ by: $\beta := Y \wedge d\phi$, check that it is closed, so that it can be integrated as a function $\phi : S \to \mathbb{R}^3$, which defines an isometric first-order deformation of $\phi$ inducing the first-order deformation $\tilde{B}$ of $B$.

We now consider the function $f := r^2/2$ on $\phi(S)$, and its pull-back by $\phi$ on $S$ (which we still call $f$). The infinitesimal deformation $\phi$ induces a first-order variation $\tilde{f}$ of $f$, which is in $H^1$ because $Y$ is in $H^1$. It is then possible to follow through the proof of Proposition 3.11: the key point is that, since $\tilde{B}$ is $L^2$, its determinant is $L^1$, so that $d\omega$ is also $L^1$ and the integration argument works.

We now use this result to prove that there is no non-trivial $L^2$ solution of $d\nabla h = 0$ on $\mathbb{S}_h \cup \Lambda_h$. The proof uses the duality with the de Sitter space, and the Pogorelov map.

**Corollary 6.9.** There is no $L^2$ solution of $d\nabla h = 0$ with $h \in \Omega^{0,1}_h(\mathbb{S}_h \cup \Lambda_h)$.

**Proof.** Consider the surface $S^*$ in the de Sitter space which is dual to $\mathbb{S}_h \cup \Lambda_h$. Since $\mathbb{S}_h \cup \Lambda_h$ is closed, strictly convex, and has its second fundamental form bounded between two strictly positive constants, $S^*$ is a closed, space-like, strictly convex surface with its second fundamental form bounded between two positive constant. Since $\mathbb{S}_h \cup \Lambda_h$ is a closed surface in $H^3$, we can choose the projective model in such a way that the image of $S^*$ in this model remains in a compact subset of $\mathbb{R}^3$, and away from the unit ball containing the projective model of $H^3$.

As pointed above, a solution of $d\nabla h = 0$ on $\mathbb{S}_h \cup \Lambda_h$ with $h \in L^2(\Omega^{0,1}_h(\mathbb{S}_h \cup \Lambda_h))$ translates as a solution of $d\nabla h = 0$ on $S^*$, with $h \in L^2(\Omega^{0,1}_h(S^*))$. If $B^*$ is the shape operator of $S^*$, then $\tilde{B}^* := B^* h$ is a solution of equation (11) on $S^*$, which describes a first-order infinitesimal deformation of $S^*$.

Consider the image $S^*_e$ of $S^*$ in the projective model of $S^*_e$: as mentioned above, we choose the projective model so that $S^*_e$ is a closed surface in $\mathbb{R}^3$. $S^*_e$ remains outside the unit sphere, so the projective model does not change the principal curvatures much, and therefore the second fundamental form of $S^*_e$ is bounded between two positive constants. Applying the Pogorelov map $\Phi_S$ (see Definition 11.11) to the first-order deformation of $S^*$ obtained above, we get a first-order isometric deformation of $S^*_e$. Since $S^*_e$ is in a compact set of $\mathbb{R}^3$ away from the unit one ball, this first-order deformation again corresponds to a solution of equation (11) on $S^*_e$ which is in $L^2$. But there is no non-trivial such solution by the previous lemma.
Proof of the Rigidity Lemma for 1 Consider an infinitesimal deformation \( \dot{g} \) of the hyperbolic metric on \( M \), which does not change the induced metric on \( \partial M \). The construction of section 6.1 leads to a vector field \( u \) on the boundary \( S \) of the universal cover of \( M \), seen as a surface in \( H^3 \). Using the Pogorelov map, it translates as an infinitesimal deformation \( \overline{\tau} \) of the surface \( S \) in \( \mathbb{R}^3 \). Recall that \( S \) is a convex surface with second fundamental form bounded between two positive constants, which is tangent to the unit sphere along the image of the limit set of \( M \).

We now take the orthogonal projection \( \overline{\tau} \) of \( \tau \) on \( \overline{S} \), and set \( \overline{\omega} := B^{-1} \overline{\tau} \). Then \( \overline{\omega} \) is a solution of \( \overline{\partial} \overline{\omega} = 0 \) on \( \overline{S} \). We then call \( \overline{\omega}_h \) be the image of \( \overline{\omega} \) on \( \overline{S}_h \) by the Pogorelov transformation, so \( \overline{\omega}_h \) is a first-order isometric deformation of \( \overline{S}_h \). Consider the orthogonal projection \( \overline{\tau}_h \) of \( \overline{\tau}_h \) on \( \overline{S}_h \), and let \( \overline{\omega}_h := B_h^{-1} \overline{\tau}_h \), where \( B_h \) is the shape operator of \( \overline{S}_h \). Then, from Lemma 6.13 \( \overline{\omega}_h \) is in \( H^1 \) for \( \partial \overline{S} \) and is a solution of \( \partial_{h} \overline{\omega}_h = 0 \) on \( \overline{S}_h \cup \Lambda_h \).

By Lemma 6.14 \( \overline{\partial} \overline{\omega} \) is an elliptic operator of index 6 on \( \overline{S}_h \cup \Lambda_h \); it acts (continuously by Lemma 6.17) from the space of \( H^2 \) sections of \( T(\overline{S}_h \cup \Lambda_h) \) (for \( \partial \overline{S} \)) to the space of \( L^2 \) sections of \( \Omega^0_{\overline{S}}(\overline{S}_h \cup \Lambda_h) \). By Proposition 8.13 its adjoint (or rather the adjoint of \( \overline{\partial} \overline{\omega} \)) is \(-d^\nabla\). But we know (by Corollary 5.9) that the kernel of \(-d^\nabla\) is trivial. In particular, \( \overline{\omega}_h \) is a trivial deformation, so that \( \overline{\omega} \) is trivial, and thus \( \dot{g} \) is trivial. This proves Lemma 6.1.

The dual rigidity The proof of the infinitesimal rigidity with respect to the induced metric also applies, with minor modifications, for the third fundamental form. The most important difference is that one has to use the hyperbolic-de Sitter duality, and the problem is moved to \( S^3 \), its adjoint (or rather the adjoint of \( \overline{\partial} \)). Consider an infinitesimal deformation \( \dot{g} \in T_g \mathcal{G} \), the induced first-order variation of the third fundamental form of \( \partial M \) is non-zero.

The proof follows the same steps as the proof of Lemma 6.11. The key point is of course Lemma 8.9. The proof of Lemma 6.11 with minor changes, also shows the same result for the dual surface. Of course, we call \( H^1_{\overline{S}}(T \overline{S}) \) the space of sections of \( T \overline{S} \) which are in the Sobolev space \( H^1 \) for the third fundamental form of \( \overline{S} \). We let \( \overline{\omega} := (B^*)^{-1} \overline{\tau} \), and call \( \overline{\partial} \overline{\omega} \) the analog of \( \overline{\partial} \overline{\omega} \) on the dual surface \( \overline{S}^* \cup \Lambda \).

Lemma 6.11. \( \overline{\omega} \) is in \( H^1_{\overline{S}}(T \overline{S}) \), and it is a solution of \( \overline{\partial} \overline{\omega} = 0 \) on \( \overline{S} \).

The end of the proof also follows closely the argument given above, with \( S \) replaced by \( \overline{S} \); in the final step, one should associate to \( \overline{S} \) a closed hyperbolic surface \( S_h \), as \( S_h \) for \( S \). We leave the details to the reader.

7 Compactness

We move here to the proof of the properness of \( \mathcal{F} \) and \( \mathcal{F}^* \). As mentioned in the introduction, those properties are equivalent to the following lemmas, which we will prove at the end of the section.

Lemma 7.1. Let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of hyperbolic metrics on \( M \) with smooth, convex boundary. Let \( (h_n)_{n \in \mathbb{N}} \) be the sequence of induced metrics on the boundary. Suppose that \( (h_n) \) converges to a smooth metric \( h \) on \( \partial M \), with curvature \( K > -1 \). Then there is a sub-sequence of \( (g_n) \) which converges to a hyperbolic metric \( g \) with smooth, convex boundary.

Lemma 7.2. Let \( (g_n)_{n \in \mathbb{N}} \) be a sequence of hyperbolic metrics on \( M \) with smooth, convex boundary. Let \( (h_n)_{n \in \mathbb{N}} \) be the sequence of third fundamental forms of the boundary. Suppose that \( (h_n) \) converges to a smooth metric \( h \) on \( \partial M \), with curvature \( K < 1 \) and closed geodesics which are of length \( L > 2\pi \) when they are contractible in \( M \). Then there is a sub-sequence of \( (g_n) \) which converges to a hyperbolic metric \( g \) with smooth, convex boundary.

Convex surfaces and pseudo-holomorphic curves We will state below some compactness results for isometric embeddings of convex surfaces in \( H^3 \) or in \( S^3 \). Although we will not prove those lemmas here, it should be pointed out that the proofs rest on the use of a compactness result for pseudo-holomorphic curves. Namely, given a surface \( \Sigma \) with a Riemannian metric of curvature \( K > K_0 \) (resp. \( K < K_0 \)) and a Riemannian (resp. Lorentzian) 3-manifold \( M \) of sectional curvature \( K < K_0 \) (resp. \( K > K_0 \)), the isometric embeddings of \( \Sigma \) in \( M \) lift to pseudo-holomorphic curves in a natural space of 1-jets of infinitesimal isometric embeddings of \( \Sigma \) in \( M \), which comes equipped with an almost-complex structure defined on a distribution of 4-planes. This idea was outlined by Gromov in [Gro85], and developed in [Lab88] [Lab97] [Sch90] [Sch94].
Compactness of surfaces in $H^3$. The compactness lemma for $I$ is based on the following statement.

**Lemma 7.3** ([Lab97]). Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of metrics on $D^2$ with curvature $K > -1$, converging to a smooth metric $h$ with curvature $K > -1$. Suppose that $d_h(0, \partial D^2) \geq 1$. Let $x_0 \in H^3$, and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of isometric embeddings, $f_n : (D^2, h_n) \to H^3$, with $f_n(0) = x_0$. Then:

- either there is a subsequence of $(f_n)$ which converges $C^\infty$ to an isometric embedding of $(D^2, h)$ in $H^3$.
- or there exists a geodesic segment $\gamma \ni 0$ on which a subsequence of $(f_n)$ converges to an isometry to a geodesic segment of $H^3$; and, in addition, the integral of the mean curvature $H_n$ on each segment transverse to $\gamma$ diverges as $n \to \infty$.

We now come back to the setting of Lemma 7.1 and consider a sequence $(g_n)_{n \in \mathbb{N}}$ of hyperbolic metrics on $M$. We suppose that the sequence of induced metrics $(h_n)$ on $\partial M$ converges to a limit $h$ which is smooth, with curvature $K > -1$. Note that we can apply a sequence of global isometries of $H^3$ to fix one point of $\partial M$.

Using the next proposition, this means that, after taking a subsequence, we can suppose that each connected component of $\partial M$ has a lift containing a point which converges in $H^3$.

**Proposition 7.4.** Under the hypothesis of Lemma 7.1, the diameter of $(M, g_n)$ remains bounded.

**Proof.** Since the induced metrics on $\partial M$ converge, the diameter of each boundary component, for the induced metrics, is bounded. But the closest-point projection from the boundary components of $M$ to the corresponding components of the boundary $\partial C(M)$ of the convex core of $M$ is a contraction, so that the induced metrics on $\partial C(M)$ have bounded diameter. Since those metrics are hyperbolic, this means that they remain in a compact subset of Teichmüller space.

Now the map sending the conformal structure at infinity on $\partial_\infty E(M)$ to the conformal structure of the induced metric on $\partial C(M)$ is proper (see [BC03], and [EM86] when $M$ is incompressible boundary). So the conformal structures on $\partial_\infty E(M)$ also remain bounded, and, by the Ahlfors-Bers theorem, the sequence of metrics on $E(M)$ remains in a compact set. In particular, the diameter of the convex cores of $(M, g_n)$ remain bounded.

Now an elementary argument shows that, if the maximal distance to $C(M)$ went to infinity on a connected component of $\partial M$, then the diameter of this connected component would go to infinity. Therefore, the distance of the points of $\partial M$ to $C(M)$ remains bounded, and the diameter of $(M, g_n)$ is also bounded.

We can now use Lemma 7.3 to state a compactness result for the lifts in $H^3$ of the boundary components of $M$.

**Corollary 7.5.** For each connected component $S_i$ of the boundary of the universal cover $\tilde{M}$ of $M$, there exists a subsequence of $(g_n)$ and a sequence $(\rho_n)$ of isometries such that the image by $\rho_n$ of $S_i$ converges $C^\infty$ on compact subsets to an equivariant surface in $H^3$.

**Proof.** Choose a point $x_0 \in S_i$, and apply Lemma 7.3 to a disk $D_0$ of radius 1 for $h$, centered at $x_0$. Suppose that the second alternative of the lemma applies, and consider a small neighborhood of $\gamma$. The integral of $H_n$ on segments transverse to $\gamma$ would diverge, and this would contradict the fact that $\partial M$ — and thus $S_i$ — is locally convex and embedded. So the first case of Lemma 7.3 applies.

Now choose a point $x_1$ on the boundary of $D_0$, and apply again Lemma 7.3 to a disk of radius 1 centered at $x_1$. The same argument as above shows that, after taking a subsequence, $S_i$ converges $C^\infty$ on $D_0 \cup D_1$.

Applying recursively the same argument leads to an increasing sequence of compact subsets of $S_i$ on which the sequence converges $C^\infty$.

The proof of the main compactness lemma for the induced metric follows.

**Proof of Lemma 7.4.** Corollary 7.3 shows that, for each end of $E(M)$, the corresponding boundary component of $\tilde{M}$ converges $C^\infty$, so that its representation converges. This shows that each end of $E(M)$ converges, so that the hyperbolic metric on $E(M)$ converges. Since each boundary component of $M$ converges in $E(M)$, the hyperbolic metrics on $M$ clearly converge. □
Compactness in $S^3_1$. We outline here the proof of the compactness lemma with respect to the third fundamental form; it follows the proof of Lemma 7.1 with some natural modifications. It is based on a compactness result for sequences of isometric embeddings of surfaces in Lorentzian 3-manifolds.

Lemma 8.1. Both $\partial M$ and $\partial M$ are connected and simply connected.

Proof. We first prove that $\partial M$ is connected. Let $h_0, h_1 \in \mathcal{H}$. Since the space of metrics on $\partial M$ is connected, there is a one-parameter family $(h_t)_{t \in [0, 1]}$ of metrics on $\partial M$ connecting $h_0$ to $h_1$. But then there exists a constant $C > 0$ such that, for all $t \in [0, 1]$, $h_t$ has curvature $K > -C^2/2$. In addition, there is another constant $\epsilon > 0$ such that, for all $t \in [0, \epsilon] \cup [1 - \epsilon, 1]$, $h_t$ has curvature $K > -1$. We define a new family $(\overline{h}_t)_{t \in [0, 1]}$ as follows:

- for $t \in [0, \epsilon]$, $\overline{h}_t = h_t(1 + (C - 1)t/\epsilon)^2$.
- for $t \in [\epsilon, 1 - \epsilon]$, $\overline{h}_t = C^2 h_t$.
- for $t \in [1 - \epsilon, 1]$, $\overline{h}_t = h_t(1 + (C - 1)(1 - t)/\epsilon)^2$.

It is then clear that, for each $t \in [0, 1]$, $\overline{h}_t$ has curvature $K > -1$, and this proves that $\mathcal{H}$ is connected.

We proceed similarly to prove that $\mathcal{H}^*$ is simply connected. Let $h : S^1 \to \mathcal{H}$ be a closed path in $\mathcal{H}$. Since the space of metrics (defined up to isotopy) on $\partial M$ is contractible, there is a map $f$ from $D^2$ to the space of smooth metrics on $\partial M$ such that, for each point $x \in S^1$, $f(x) = h(x)$. Then:

- there exists a constant $C > 0$ such that, for each $y \in D^2$, $f(y)$ has curvature $K > -C^2/2$.}

8 Spaces of metrics

The last step of the proof is to set up deformation arguments for $\mathcal{F}$ and for $\mathcal{F}^*$. The main point of this section is that the space of metrics which enter the picture are topologically simple.

Lemma 8.1. Both $\mathcal{H}$ and $\mathcal{H}^*$ are connected and simply connected.

Proof. We first prove that $\mathcal{H}$ is connected. Let $h_0, h_1 \in \mathcal{H}$. Since the space of metrics on $\partial M$ is connected, there is a one-parameter family $(h_t)_{t \in [0, 1]}$ of metrics on $\partial M$ connecting $h_0$ to $h_1$. But then there exists a constant $C > 0$ such that, for all $t \in [0, 1]$, $h_t$ has curvature $K > -C^2/2$. In addition, there is another constant $\epsilon > 0$ such that, for all $t \in [0, \epsilon] \cup [1 - \epsilon, 1]$, $h_t$ has curvature $K > -1$. We define a new family $(\overline{h}_t)_{t \in [0, 1]}$ as follows:

- for $t \in [0, \epsilon]$, $\overline{h}_t = h_t(1 + (C - 1)t/\epsilon)^2$.
- for $t \in [\epsilon, 1 - \epsilon]$, $\overline{h}_t = C^2 h_t$.
- for $t \in [1 - \epsilon, 1]$, $\overline{h}_t = h_t(1 + (C - 1)(1 - t)/\epsilon)^2$.

It is then clear that, for each $t \in [0, 1]$, $\overline{h}_t$ has curvature $K > -1$, and this proves that $\mathcal{H}$ is connected.

We proceed similarly to prove that $\mathcal{H}^*$ is simply connected. Let $h : S^1 \to \mathcal{H}$ be a closed path in $\mathcal{H}$. Since the space of metrics (defined up to isotopy) on $\partial M$ is contractible, there is a map $f$ from $D^2$ to the space of smooth metrics on $\partial M$ such that, for each point $x \in S^1$, $f(x) = h(x)$. Then:

- there exists a constant $C > 0$ such that, for each $y \in D^2$, $f(y)$ has curvature $K > -C^2/2$.
there is another constant \( \epsilon > 0 \) such that, for each \( y \in D^2 \) at distance at most \( \epsilon \) from \( S^1 = \partial D^2 \), \( f(y) \) has curvature \( K > -1 \).

We define another map \( \tilde{f} \) from \( D^2 \) to the space of smooth metrics on \( \partial M \) as follows:

- for \( y \in D^2 \) at distance at most \( \epsilon \) from \( \partial D^2 \), \( \tilde{f}(y) = f(y)/(1 + (C - 1)d(y, \partial D^2)/\epsilon)^2 \).
- for \( y \in D^2 \) at distance at least \( \epsilon \) from \( \partial D^2 \), \( \tilde{f}(y) = C^2 f(y) \).

It is then clear that \( \tilde{f} \) has values in \( \mathcal{H} \), so that \( \mathcal{H} \) is simply connected.

To proof that \( \mathcal{H}^* \) is connected, we proceed as in the proof that \( \mathcal{H} \) is connected, with a minor difference; given \( h_0, h_1 \in \mathcal{H}^* \), we choose a path \( (h_t)_{t \in [0,1]} \) connecting them, and we pick \( C > 0 \) such that, for each \( t \in [0,1] \), \( h_t \) has curvature \( K < C^2/2 \) and closed geodesics of length \( L > 4\pi/C \). We also choose \( \epsilon > 0 \) such that, for \( t \in [0,\epsilon] \cup [1 - \epsilon,1] \), \( h_t \in \mathcal{H}^* \). We then define \( (\tilde{h}_t)_{t \in [0,1]} \) by scaling exactly as above, and check that, for all \( t \in [0,1] \), \( \tilde{h}_t \in \mathcal{H}^* \). So \( \mathcal{H}^* \) is connected.

Finally, proving that \( \mathcal{H}^* \) is simply connected is similar to proving that \( \mathcal{H} \) is simply connected. Given a closed path \( h : S^1 \to \mathcal{H}^* \), we choose a map \( f \) from \( D^2 \) to the space of smooth metrics on \( \partial M \) which coincides with \( f \) on \( \partial D^2 \). We then pick a constant \( C > 0 \) such that, for all \( y \in D^2 \), \( f(y) \) has curvature \( K < C^2/2 \), and has closed geodesics of length \( L > 4\pi/C \). The scaling argument is then the same as for \( \mathcal{H} \) above.

We can also remark that \( \mathcal{G} \) is connected; this is a direct consequence of the Ahlfors-Bers theorem, which shows that the space of complete, convex co-compact metrics on \( E(M) \) is connected.

### 9 Proof of the main theorems

We now have most of the tools necessary to prove that \( \mathcal{F} \) and \( \mathcal{F}^* \) are homeomorphisms. Since we already know that their differentials are injective, this will basically follow from the fact that both \( \mathcal{F} \) and \( \mathcal{F}^* \) are Fredholm of index 0. This fact is essentially a consequence of the ellipticity of the underlying PDE problems. Actually there are two ways to consider our geometric problem in an elliptic setting.

The most natural way is to describe Theorems 0.1 and 0.2 as statements about the boundary value problem of finding a hyperbolic metric on \( M \) with some conditions on \( \partial M \). The linearization is then an elliptic boundary value problem on \( (M, \partial M) \), and it should be possible to prove that its index is 0 by deforming it to a more tractable elliptic problem. This is done, for the statement concerning the induced metric, in [Sch01b] (in higher dimensions, in the setting of Einstein metrics of negative curvature).

We will rather use another description, which seems somehow simpler to expound, and also better suited to the study of the third fundamental form. The point will be to remark that, given a hyperbolic manifold with convex boundary, each boundary component has an equivariant embedding in \( H^3 \). Such equivariant embeddings for the various boundary components usually do not "fit" so as to be obtained from a hyperbolic metric on \( M \), but, since the space of complete hyperbolic metrics on \( E(M) \) is finite dimensional, the obstruction space is also finite dimensional. Thus the index of the problem concerning hyperbolic metrics on \( M \) can be computed rather simply from the index of the problems concerning equivariant embeddings of surfaces in \( H^3 \), which are indicated by the Riemann-Roch theorem.

The Nash-Moser theorem There is a technical detail of some importance that should be pointed out; namely, the basic analytical tool that we will use here is the Nash-Moser inverse function theorem. One could think that a simpler approach using Banach spaces should be sufficient, and should lead to results in the \( C^r \) or \( H^s \) categories, rather than only in the \( C^\infty \) category which appears in the statements of Theorems 0.1 and 0.2.

The reason the \( C^\infty \) category, and the Nash-Moser theorem, are necessary, is that there is a "loss of derivatives" phenomenon at work here, which is typical of some isometric embedding questions, and which also appears in questions on Einstein manifolds with boundary [Sch01b]. Namely, when one deforms a surface \( \Sigma \) in \( H^3 \) through a vector field \( v \), the induced variation of the induced metric has two components:

- the normal component of \( v \) induces a deformation proportional to the second fundamental form of \( S \), which is of order 0 with respect to \( v \);
- the tangent component of \( v \) induces a deformation which is of order 1 with respect to \( v \).

Since those two kinds of deformations have different orders in \( v \), a Banach-space approach runs into difficulties; the \( C^\infty \) category is required in our proof, as well as the Nash-Moser Theorem. It remains however likely that an analogous statement is true in the \( C^r \) or the \( H^s \) categories; proving it would demand additional efforts.
**Main lemma**  The key point is the following statement.

**Lemma 9.1.** \( \mathcal{F} \) and \( \mathcal{F}^* \) are local homeomorphisms.

We first consider the map \( \mathcal{F} \). We will decompose it as follows:

\[
\mathcal{F} : \mathcal{G} \xrightarrow{\mathcal{F}_1} \mathcal{E} \xrightarrow{\mathcal{F}_2} \mathcal{H} \times \mathcal{R} \xrightarrow{\Pi_1} \mathcal{H},
\]

where:

- \( \partial_1 M, \partial_2 M, \ldots, \partial_N M \) are the connected components of \( \partial M \).
- For each \( i \in \{1, \ldots, N\} \), \( \mathcal{E}_i \) is the space of smooth, strictly convex equivariant embeddings (see the definition below) of \( \partial_i M \) in \( \mathbb{H}^3 \), considered modulo right composition by an isotopy of \( \partial_i M \) and modulo the global isometries of \( \mathbb{H}^3 \), and \( \mathcal{E} := \prod_{i=1}^N \mathcal{E}_i \).
- \( \mathcal{F}_1 \) is the map sending a hyperbolic metric \( g \in \mathcal{G} \) to the equivariant embeddings of the \( \partial_i M \) corresponding to the boundary components of \( \tilde{M} \), seen as a convex subset of \( \mathbb{H}^3 \).
- For each \( i \in \{1, \ldots, N\} \), \( \mathcal{H}_i \) is the space of smooth metrics on \( \partial_i M \) with curvature \( K > -1 \), considered up to isotopy, and \( \mathcal{H} := \prod_{i=1}^N \mathcal{H}_i \).
- For each \( i \in \{1, \ldots, N\} \), \( \mathcal{R}_i \) is the space of irreducible representations of \( \pi_1(\partial_i M) \) in \( \text{PSL}(2, \mathbb{C}) \), considered up to global isometry (see below) and \( \mathcal{R} := \prod_{i=1}^N \mathcal{R}_i \).
- \( \mathcal{F}_2 \) is the natural map sending an \( N \)-uple of equivariant embeddings of the \( \partial_i M \) to the \( N \)-uples of their induced metrics and of their representations.
- \( \Pi_1 : \mathcal{H} \times \mathcal{R} \to \mathcal{H} \) and \( \Pi_2 : \mathcal{H} \times \mathcal{R} \to \mathcal{R} \) are the projections on the first and on the second factor, respectively.

We will give some details on various elements of this decomposition before proving Lemma 9.1.

**Representations**  For each \( i \in \{1, \ldots, N\} \), denote by \( g_i \) the genus of \( \partial_i M \).

**Definition 9.2.** For each \( i \in \{1, \ldots, N\} \), we call \( \mathcal{R}_i \) the space of irreducible representations of \( \pi_1(\partial_i M) \) in \( \text{Isom}(\mathbb{H}^3) \), modulo the global isometries, i.e.:

\[
\mathcal{R}_i := \frac{\text{Hom}_{\text{irr}}(\pi_1(\partial_i M), \text{Isom}(\mathbb{H}^3))}{\text{Isom}(\mathbb{H}^3)}.
\]

We also define \( \mathcal{R} \) as their product, \( \mathcal{R} := \prod_{i=1}^N \mathcal{R}_i \). We call \( \mathcal{R}_M \) the subset of \( \mathcal{R} \) of elements which come from an element \( g \in \mathcal{G} \), i.e. a hyperbolic metric with smooth, strictly convex boundary on \( M \).

The dimension of those representation spaces can be computed simply since the \( \pi_1(\partial_i M) \) are finitely generated groups. We only outline the proof here, and refer the reader to e.g. [Kap01] for details.

**Proposition 9.3.**  For each \( i \in \{1, \ldots, N\} \), \( \dim \mathcal{R}_i = 12g_i - 12 \). So \( \dim \mathcal{R} = \sum_{i=1}^N (12g_i - 12) \). \( \mathcal{R}_M \) is a submanifold of \( \mathcal{R} \) of dimension \( \sum_{i=1}^N (6g_i - 6) \).

**Heuristics of the proof.** For each \( i \in \{1, \ldots, N\} \), \( \pi_1(\partial_i M) \) has a presentation with \( 2g_i \) generators and one relation. Since \( \dim \text{Isom}(\mathbb{H}^3) = 6 \), the dimension of \( \text{Hom}(\pi_1(\partial_i M), \text{Isom}(\mathbb{H}^3)) \) is \( 12g_i - 6 \). Quotienting by the global isometries removes 6 dimensions.

The elements \( \rho \in \mathcal{R}_M \) are in one-to-one correspondence with the complete, convex co-compact hyperbolic metrics on \( M \). By the Ahlfors-Bers theorem (see e.g. [Ah66]) those metrics are in one-to-one correspondence with the conformal metrics on \( \partial M \). But this space is the product of the Teichmüller spaces of the \( \partial_i M \), so it has dimension \( \sum_{i=1}^N (6g_i - 6) \). \( \square \)
**Equivariant embeddings**  We will also need the notion of equivariant embedding of a surface in $H^3$. In particular, given a hyperbolic metric on $M$, each connected component of $\partial M$ inherits a natural equivariant embedding in $H^3$.

**Definition 9.4.** Let $\Sigma$ be a compact surface. An equivariant embedding of $\Sigma$ in $H^3$ is a couple $(\phi, \rho)$, where:

- $\phi: \tilde{\Sigma} \to H^3$ is an embedding.
- $\rho: \pi_1 \Sigma \to \text{Isom}(H^3)$ is a morphism.
- For each $x \in \tilde{\Sigma}$ and each $\gamma \in \pi_1 \Sigma$:
  \[
  \phi(\gamma x) = \rho(\gamma) \phi(x) .
  \]

$\rho$ is the representation of the equivariant embedding $(\phi, \rho)$.

As already mentioned, given a hyperbolic metric $g \in G$ on $M$, its developing map defines a family of equivariant embeddings of the boundary components of $M$ into $H^3$.

**Definition 9.5.** For each $i \in \{1, \cdots, N\}$, we call $E_i$ the space of smooth, strictly convex equivariant embeddings of $\partial_i M$ in $H^3$, considered modulo right composition with an isotopy, and modulo the global isometries. $E := \prod_{i=1}^N E_i$.

As already mentioned, given a hyperbolic metric $g \in G$ on $M$, its developing map defines a family of equivariant embeddings of the boundary components of $M$ into $H^3$.

**Definition 9.6.** We call:

\[
F_1 : G \to E
\]

the map sending a hyperbolic metric on $M$ with smooth, strictly convex boundary, to the $N$-uple of convex equivariant embeddings of the boundary components of $M$.

Given an equivariant embedding of a surface $\Sigma$ in $H^3$, it induces a metric and a third fundamental form on the universal cover of $\Sigma$, but also, by the equivariance which is in the definition, on $\Sigma$ itself. For each $i \in \{1, \cdots, N\}$, we call $H_i$ the space of smooth metrics on $\partial_i M$ with curvature $K > -1$, and $H_i^*$ the space of smooth metrics on $\partial_i M$ with curvature $K < 1$. Then $\Pi_{i=1}^N H_i = H$, while $\Pi_{i=1}^N H_i^* \subset H^*$.

**Definition 9.7.** For each $i \in \{1, \cdots, N\}$, we call $F_{2,i}$ the map from $E_i$ to $H_i \times R_i$ sending an equivariant embedding of $\partial_i M$ to its induced metric and its representation. Then we call:

\[
F_2 : E \to H \times R \quad \text{with} \quad (\phi_i)_{1 \leq i \leq N} \mapsto (F_{2,i}(\phi_i))_{1 \leq i \leq N}.
\]

$F_1$ is a map between infinite dimensional spaces, but it is Fredholm and has a finite index. The index computation basically uses the fact that $R$ is finite dimensional, and the codimension of $R_M$ in $R$. More precisely, the proof will use the following facts.

**Proposition 9.8.** 1. $\Pi_2 \circ F_2 : E \to R$ is a submersion.

2. Therefore, $E_M := (\Pi_2 \circ F_2)^{-1}(R_M)$ has codimension $\sum_i (6g_i - 6)$ in $E$.

3. $F_1$ is a local homeomorphism between $G$ and $E_M$.

**Proof.** Point (1) is elementary, since each first-order deformation of the representation of an equivariant embedding of a compact surface can be obtained by a first-order deformation of the equivariant embedding. Point (2) is a consequence since $R_M$ has codimension $\sum_i (6g_i - 6)$ in $R$.

Point (3) is almost as elementary. Clearly, only elements of $E_M$ are in the image of $F_1$, this is by definition of $E_M$. Conversely, given an $N$-uple of equivariant convex embeddings of the $\partial_i M$ such that their representations come from a complete, convex co-compact hyperbolic metric on $M$ (the definition of $R_M$), it is clear that the equivariant embeddings come from a hyperbolic metric $g \in G$ on $M$. The fact that the relation is one-to-one is a consequence of the fact that the boundary surfaces — seen as equivariant surfaces in $H^3$ — uniquely determine $g$.

The next step is that $dF_2$ is Fredholm, and its index can be computed easily using the Riemann-Roch Theorem.
Proposition 9.9. For each \( i \in \{1, \cdots, N\} \), \( dF_{2,i} \) is a Fredholm map of index \(-6g_i -6\). Therefore, \( dF_2 \) is a Fredholm map of index \(-\sum_{i=1}^{N} (6g_i + 6)\).

Proof. The second part clearly follows from the first. To prove the first part, we choose \( i \in \{1, \cdots, N\} \).

Let \( (\phi_i, \rho_i) \in E_i \), so that \( \phi_i \) is a convex equivariant embedding of \( \partial_i M \) into \( H^3 \), and that \( \rho_i \) is its representation. Let \( E_i(\rho_i) \) be the subspace of \( E_i \) of convex equivariant embeddings of \( \partial_i M \) with the same representation \( \rho_i \). The tangent space \( T_{(\phi_i, \rho_i)}E_i \) is naturally associated to a space of vector fields along \( \phi_i(\partial_i M) \) with some equivariance property under the action of \( \rho_i \); the tangent space of \( E_i(\rho_i) \), however, is simply associated to the space of vector fields along \( \phi_i(\partial_i M) \) which are invariant under the action of \( \rho_i \), i.e. with the space of smooth sections of the pull-back by \( \phi_i \) on \( \partial_i M \) of \( TH^3 \).

Let \( u \) be a smooth section of this bundle, it has a natural decomposition as \( u = \lambda N + v \), where \( N \) corresponds (under the pull-back map) to the unit normal vector of \( \phi_i(\partial_i M) \), while \( v \) is tangent to \( \partial_i M \). The first-order variation of the induced metric on \( \partial_i M \) induced by \( \lambda N \) is simply \( \lambda I \), while the component orthogonal to \( I \) of the first-order variation of the induced metric associated to \( v \) is given by \( \overline{\partial}_I v \), where the operator \( \overline{\partial}_I \), defined in section \( \mathcal{E} \), has the same principal symbol as the \( \overline{\partial} \) operator of the second fundamental form of \( \partial_i M \).

By the Riemann-Roch theorem (or the index theorem), the \( \overline{\partial} \) operator acting on vector fields on \( \partial_i M \) is Fredholm and has index \(-6g_i -6\). So \( \overline{\partial}_I \) is Fredholm with the same index. But, from the way \( u \) acts on the induced metric, this means that the map sending a section of \( E_i(TH^3) \) to the induced variation of the induced metric is also Fredholm with index \(-6g_i -6\). This map is the differential at \( \phi_i \) of the restriction of \( F_{2,i} \), to \( E_i(\rho_i) \), seen as a map from \( E_i(\rho_i) \) to \( H_i \times \{ \rho_i \} \). But the codimension of \( E_i(\rho_i) \) in \( E_i \) is equal to the codimension of \( H_i \times \{ \rho_i \} \) in \( H_i \times \mathcal{R}_i \). So the index of \( F_{2,i} \) is still \(-6g_i -6\). \( \square \)

We can now move to the key point.

Proposition 9.10. Let \( g \in \mathcal{G} \). In the neighborhood of \( F_1(g) \in E_M \), the restriction of \( F_2 \) to \( E_M \), composed with the projection \( \Pi_1 \), is a local homeomorphism onto \( \mathcal{H} \).

Proof. The proof uses the Nash-Moser theorem, and we will use results from \( \text{[Ham82]} \). To keep as close as possible from this text, we first define versions of \( \mathcal{E} \) and \( \mathcal{H} \) without taking a quotient by the group of isotopies of \( \partial_i M \). So, for each \( i \in \{1, \cdots, N\} \), we call \( \mathcal{E}_i \) the space of smooth, strictly convex equivariant embeddings of \( \partial_i M \) into \( H^3 \), up to global isometries, and let \( \mathcal{E} := \sum_{i=1}^{N} \mathcal{E}_i \). We also call \( \mathcal{E}_M \) the space of elements of \( \mathcal{E} \) which project to \( E_M \). It follows from \( \text{[Ham82]} \) (see Corollary II.2.3.2, p. 146) that \( \mathcal{E} \) is a smooth, tame Fréchet manifold.

In the same way, for each \( i \in \{1, \cdots, N\} \), we call \( \mathcal{H}_i \) the space of smooth metrics on \( \partial_i M \) with \( K > -1 \), and let \( \mathcal{H} := \sum_{i=1}^{N} \mathcal{H}_i \). \( \mathcal{H} \) is an open subset in a smooth, tame Fréchet space, by Theorem II. 2.3.1 of \( \text{[Ham82]} \), p. 146.

There is a natural map \( F_2 : \mathcal{E} \to \mathcal{H} \times \mathcal{R} \). We will change it a little. We now restrict our attention to a neighborhood of some fixed \( g \in \mathcal{G} \), and of its images in \( \mathcal{E} \) and in \( \mathcal{H} \times \mathcal{R} \). Since \( \mathcal{R}_M \) is a submanifold of codimension \( \sum_i (6g_i + 6) \) of \( \mathcal{R} \), we can find a smooth submersion \( \Sigma : \mathcal{R} \to N \), where \( N \) is a vector space of dimension \( \sum_i (6g_i + 6) \), such that \( \mathcal{R}_M = \Sigma^{-1}(0) \). Let \( F_2 : \mathcal{E} \to \mathcal{H} \times \mathcal{R} \) be the map obtained from \( F_2 \) by composing it with \( \Sigma \) on the \( \mathcal{R} \) factor. As a consequence of Proposition \( 9.9 \), we see that, at each point, \( dF_2 \) is Fredholm with index 0. Moreover, it is injective by Lemma \( 6.1 \).

Let \( i \in \{1, \cdots, N\} \). Given an element of \( (h,n) \in \mathcal{T}H \times \mathcal{N} \), with \( h = (h_1, \cdots, h_N) \), one can find its inverse images by \( dF_2 \), which are equivariant vector fields along the image of the \( \partial_i M \), in a 2-steps process, repeated for all \( i \in \{1, \cdots, N\} \):

- the components of the vector fields normal to the surfaces are given by the components of the \( h_i \) along the second fundamental forms of the \( \partial_i M \).
- their components tangent to the surfaces are solutions of \( \overline{\partial}_I v = h'_i \), where the \( h'_i \) are given by the component of the \( h_i \) orthogonal to the second fundamental forms of the \( \partial_i M \).

The second step involves the resolution of an elliptic PDE.

We can thus apply Theorem II.3.3.3 of \( \text{[Ham82]} \), p. 158; it shows that, in the neighborhood of \( F_1(g) \), the differentials of \( F_2 \) are invertible, and that the inverses define a smooth, tame family of linear maps. So we can apply the Nash-Moser inverse function theorem (\( \text{[Ham82]} \), Theorem III.1.1.1, p. 171) and see that \( F_2 \) is a local homeomorphism. Restricting our attention to \( \mathcal{E}_M \), it follows that the restriction of \( F_2 \) to \( \mathcal{E}_M \) is a local homeomorphism onto \( \mathcal{H} \).

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We can now take the quotient on both sides by the isotopy group of $\partial M$, and obtain that the restriction of $F_2$ to $E_M$ is a local homeomorphism onto $H$.

**Proof that $F$ is a local homeomorphism**  It is obviously a consequence of the third point of Proposition 9.8 and of Proposition 9.10.

**The third fundamental form**  The proof given above extends with minor differences to the third fundamental form. We need a slightly different definition to replace $F_2$.

**Definition 9.11.** For each $i \in \{1, \ldots, N\}$, we call $F_{*i}^2$ the map from $E_i$ to $H_i^* \times R_i$ sending a convex equivariant embedding of $\partial_i M$ to its third fundamental form and its representation. Then:

$$F_{*i}^2 : \mathcal{E} \to \mathcal{H}^* \times R \quad (\phi_i)_{1 \leq i \leq N} \mapsto (F_{*i}^2(\phi_i))_{1 \leq i \leq N}.$$

The following analog of Proposition 9.9 then holds, and its proof is the same as the proof of Proposition 9.9, except that one has to use the hyperbolic-de Sitter duality and to operate in $S_1^3$.

**Proposition 9.12.** For each $i \in \{1, \ldots, N\}$, $F_{*i}^2$ is a Fredholm map of index $-(6g_i - 6)$. Therefore, $F_{*i}^2$ is a Fredholm map of index $-\sum_{i=1}^N (6g_i - 6)$.

Now, as for $F$, $F^*$ is a proper map from $G \to H^*$, and the same argument as above shows that $F^*$ is a local homeomorphism between $G$ and $H^*$.

**Main proofs**  We now have all the tools to prove Theorems 0.1 and 0.2 when $M$ is not a solid torus.

**Proof of Theorem 0.1.** According to Lemma 9.1, $F$ is a local homeomorphism between $G$ and $H$. Since it is also proper by Lemma 7.1, it is a covering. But since $G$ is connected and $H$ is connected and simply connected (by Lemma 8.1), $F$ is a global homeomorphism, and Theorem 0.1 follows.

**Proof of Theorem 0.2.** The proof is exactly the same as that of Theorem 0.1, using Lemmas 6.10 and 7.2 instead of 6.1 and 7.1.

**The solid torus**  We now suppose that $M$ is a solid torus. The proofs of Theorems 0.1 and 0.2 are then mostly simpler than in the general case, but a little different. One reason is that the boundary is a torus — while in the other cases the boundary components are always surfaces of higher genus. Another is that the limit set has now only two points, so that the proof of the infinitesimal rigidity is a little different (but rather simpler).

The proof is based on the same series of lemmas as when $M$ is not the solid torus; we repeat them here, with sketches of proofs only when they differ from the general case detailed above.

**Lemma 9.13.** The map $F : G \to H$ (resp. $F^* : G \to H^*$) sending a hyperbolic metric on $M$ to the induced metric (resp. third fundamental form) of the boundary is Fredholm.

This is proved as in Lemma 9.1. There are differences, however, in the computation of the index (although it is still 0). The key points are as follows.

**Lemma 9.14.**

1. The map sending an equivariant smooth, strictly convex embedding of $T^2$ into $H^3$ to its induced metric is Fredholm, with index 2.

2. The space of representations of $Z^2$ in $SO(3,1)$ has dimension 4.

3. The space of representations of $Z$ in $SO(3,1)$ has dimension 2.

**Proof.** The first point is a direct consequence of the Riemann-Roch theorem, as in the proof of Proposition 9.9. The second is a simple computation using the relationship between the images of the two generators (they commute) while the third uses the fact that a hyperbolic isometry without fixed point is uniquely determined, up to conjugation, by its translation length and rotation angle.

**Corollary 9.15.** $F$ and $F^*$ are Fredholm operators of index 0.
Proof. This follows from the previous lemma, because \(2 = 4 - 2\).

**Lemma 9.16.** \(F\) and \(F^*\) have injective differentials at each point of \(G\).

**Proof.** It follows the proof of Lemmas 9.10 and 6.10. The main difference is that the limit set \(\Lambda\) now has only two points, so the analytic arguments are made rather easier.

**Lemma 9.17.** \(F\) and \(F^*\) are proper.

The proof uses the same tools as for Lemmas 7.1 and 7.2.

**Lemma 9.18.** \(G\) is connected, and \(H\) and \(H^*\) are simply connected.

**Proof.** The connectedness of \(G\) can be obtained in an elementary way, by using the connectedness of the space of actions without fixed point of \(\mathbb{Z}\) on \(H^3\). To show that \(H\) and \(H^*\) are simply connected, one can use the same trick as in the proof of Lemma 8.1.

The proofs of Theorems 9.1 and 9.2 follow from those lemmas, when \(M\) is a solid torus, just as in the other cases (when \(M\) is not a solid torus) above, because \(F\) and \(F^*\) are local covering, which then have to be homeomorphisms.

## 10 Other conditions on \(\partial M\)

We will prove in this section theorems 10.1 and 10.2. They are simple consequences of Theorems 9.1 and 9.2 along with some well-known properties of equidistant surfaces in hyperbolic 3-space, and in particular of the following statement.

**Proposition 10.1.** Let \(S\) be a (complete) convex surface in \(H^3\). For \(t > 0\), let \(S_t\) be the equidistant surface at distance \(t\) from \(S\). If \(I_t\) and \(\mathcal{I}_t\) are the induced metric and third fundamental form, respectively, of \(S_t\), then:

\[
I_t = \cosh^2(t)I_0 + 2\cosh(t)\sinh(t)\mathcal{I}_0 + \sinh^2(t)\mathcal{I}_0,
\]

\[
\mathcal{I}_t = \sinh^2(t)I_0 + 2\cosh(t)\sinh(t)\mathcal{I}_0 + \cosh^2(t)\mathcal{I}_0.
\]

Moreover, the principal curvatures of \(S_t\) are bounded between \(\tanh(t)\) and \(1/\tanh(t)\).

The proof, which is a simple computation, is left to the reader. As a consequence, we have the following analogous statement, which we prove as well.

**Proposition 10.2.** Let \(k_0 \in (0, 1)\). Let \(S\) be a (complete) convex surface in \(H^3\), with principal curvatures in \((k_0, 1/k_0)\). For each \(t > 0\) with \(\tanh(t) \leq k_0\), the set \(S_{-t}\) of points at distance \(t\) from \(S\), on the normal to \(S\), in the direction of the convex side of \(S\), is a smooth, convex surface. Its induced metric and third fundamental form, respectively, are:

\[
I_{-t} = \cosh^2(t)I_0 - 2\cosh(t)\sinh(t)\mathcal{I}_0 + \sinh^2(t)\mathcal{I}_0,
\]

\[
\mathcal{I}_{-t} = \sinh^2(t)I_0 - 2\cosh(t)\sinh(t)\mathcal{I}_0 + \cosh^2(t)\mathcal{I}_0.
\]

We can now prove Theorem 10.3, which we recall for the reader’s convenience.

**Theorem 10.3.** Let \(k_0 \in (0, 1)\). Let \(h\) be a Riemannian metric on \(\partial M\). There exists a hyperbolic metric \(g\) on \(M\), such that the principal curvatures of the boundary are between \(k_0\) and \(1/k_0\), and for which:

\[
I - 2k_0I + k_0^2\mathcal{I} = h
\]

if and only if \(h\) has curvature \(K > -1/(1 - k_0^2)\). \(g\) is then unique.

**Proof.** Let \(h\) be a smooth metric on \(\partial M\) with curvature \(K > -1/(1 - k_0^2)\). Let \(t_0 := \tanh^{-1}(k_0)\), and set \(\overline{h} := \cosh^2(t_0)h\). Since \(\cosh^2(t_0) = 1/(1 - k_0^2)\), \(\overline{h}\) has curvature \(K > -1\), so by Theorem 9.1, there exists a unique hyperbolic metric \(\overline{g}\) with smooth convex boundary such that the induced metric on \(\partial M\) is \(\overline{h}\).

Now consider the extension \(E(M)\) of \(M\), and let \(g\) be the hyperbolic metric on \(M\) obtained from \(\overline{g}\) by replacing each connected component of the boundary of \(M\) by the equidistant surface at distance \(t_0\) in the corresponding end of \(E(M)\). Proposition 10.1 shows that, for this metric, the principal curvatures of the
boundary are bounded between $k_0$ and $1/k_0$, and the extrinsic invariants $I$, $II$ and $III$ of the boundary are such that:

$$\overline{h} = \cosh^2(t_0)I - 2 \cosh(t_0) \sinh(t_0) II + \sinh^2(t_0) III .$$

By definition of the scaling used to define $\overline{h}$ from $h$, we see that:

$$h = I - 2 \tanh(t_0) II + \tanh^2(t_0) III = I - 2k_0 II + k_0^2 III ,$$

as needed.

Conversely, let $g'$ be a hyperbolic metric with convex boundary such that the principal curvatures of the boundary are bounded between $k_0$ and $1/k_0$, and such that, on the boundary, the extrinsic invariants $I'$, $II'$ and $III'$ are such that $I' - 2k_0 II' + k_0^2 III' = h$. Apply proposition 10.2 with $t_0 := \tanh^{-1}(k_0)$. We obtain a hyperbolic metric $\overline{g}'$ with smooth, locally convex boundary, such that the induced metric on the boundary is:

$$I = \cosh^2(t_0)I' - 2 \cosh(t_0) \sinh(t_0) II' + \sinh^2(t_0) III'$$
$$= \cosh^2(t_0) h .$$

So $\overline{g}'$ corresponds to the metric $\overline{g}$ constructed above, and therefore $g' = g$, and this proves the uniqueness part of the statement.

The proof of Theorem 0.4 is analogous, using Theorem 0.2 instead of Theorem 0.1.

**Theorem 0.4** Let $k_0 \in (0,1)$. Let $h$ be a Riemannian metric on $\partial M$. There exists a hyperbolic metric $g$ on $M$, such that the principal curvatures of the boundary are between $k_0$ and $1/k_0$, and for which:

$$k_0^2 I - 2k_0 II + III = h$$

if and only if $h$ has curvature $K < 1/(1-k_0^2)$, and its closed, contractible geodesics have length $L > 2\pi \sqrt{1-k_0^2}$, $g$ is then unique.

**Proof.** Let $h$ be a smooth metric on $\partial M$, with curvature $K < 1/(1-k_0^2)$ and with closed geodesics of length $L > 2\pi \sqrt{1-k_0^2}$. Let $t_0 := \tanh^{-1}(k_0)$, and set $\overline{h} := \cosh^2(t_0) h$. Since $\cosh^2(t_0) = 1/(1-k_0^2)$, $\overline{h}$ has curvature $K < 1$, and its closed geodesics have length $L > 2\pi$. So by Theorem 0.2 there exists a unique hyperbolic metric $\overline{g}$ with smooth convex boundary such that the third fundamental of $\partial M$ is $\overline{h}$.

Now consider the extension $E(M)$ of $M$, and let $g$ be the hyperbolic metric on $M$ obtained from $\overline{g}$ by replacing each connected component of the boundary of $M$ by the equidistant surface at distance $t_0$ in the corresponding end of $E(M)$. Proposition 10.1 shows that, for this metric, the principal curvatures of the boundary are bounded between $k_0$ and $1/k_0$, and the third fundamental for $\overline{g}$ of the boundary is:

$$\overline{h} = \sinh^2(t_0)I - 2 \cosh(t_0) \sinh(t_0) II + \cosh^2(t_0) III .$$

Using the scaling used to define $\overline{h}$ from $h$, we see that:

$$h = \tanh^2(t_0)I - 2 \tanh(t_0) II + III = k_0^2 I - 2k_0 II + III ,$$

as needed.

Conversely, let $g'$ be a hyperbolic metric with convex boundary such that the principal curvatures of the boundary are bounded between $k_0$ and $1/k_0$, and such that, on the boundary, the extrinsic invariants $I'$, $II'$ and $III'$ are such that $k_0^2 I' - 2k_0 II' + III' = h$. Apply proposition 10.2 with $t_0 := \tanh^{-1}(k_0)$. We obtain a hyperbolic metric $\overline{g}'$ with smooth, locally convex boundary, such that the third fundamental for the boundary is:

$$I = \sinh^2(t_0)I' - 2 \cosh(t_0) \sinh(t_0) II' + \cosh^2(t_0) III'$$
$$= \cosh^2(t_0) h .$$

So $\overline{g}'$ corresponds to the metric $\overline{g}$ constructed above, and therefore $g' = g$, and this proves the uniqueness part of the statement.
Some open questions  The smoothness assumptions in Theorems 0.1 and 0.2 are not quite satisfactory. It should be possible to state results also in the $C^k$ category. It is not done here because of a well-known ”loss of derivatives” phenomenon, already apparent for isometric embeddings of surfaces.

As already mentioned in the introduction, the phenomena behind Theorems 0.1 and 0.2 should not be limited to manifolds with smooth boundaries, but should rather apply to general convex boundaries with no smoothness assumption, for instance to polyhedral boundaries. This is indicated by the case when $M$ is simply a ball, where general results without smoothness were obtained by Pogorelov [Pog73] for the induced metric, and by G. Moussong [Mou02] for the third fundamental form. Moreover, in the case of the third fundamental form, results do hold when the boundary looks locally like an ideal [Sch01c] or hyperideal [Sch02a] polyhedron.

It would in particular be interesting to know whether analogs of Theorems 0.1 and 0.2 hold for the convex cores of hyperbolic manifolds. The induced metric on the boundary is then hyperbolic, while the third fundamental form is replaced by the bending lamination. It is still not known whether an infinitesimal rigidity result holds for either the induced metric or the bending lamination, although it is known that those two questions are equivalent [Bon96]. Moreover, it is known that any hyperbolic metric can be achieved as the induced metric on the boundary of the convex core of a hyperbolic 3-manifold [Lab92], and a characterization of the possible bending laminations (without uniqueness) was also obtained recently [BO01, Lec02].

Some questions also remain concerning smooth surfaces, in particular those which are not compact but have “ends”. This is formally analogous to the hyperideal polyhedra, where one has to use the condition that, near each end, all faces are orthogonal to a hyperbolic plane. In the smooth case, however, it looks like the right condition might be that the boundary at infinity of each end of infinite area is a circle (in the Möbius geometry of the boundary at infinity). When one considers constant curvature metrics on the boundary, analogs of Theorems 0.1 and 0.2 hold in the topologically simplest case (see [Sch98a]) and, in a wider setting, for existence results concerning manifolds with a given induced metric on the boundary [Sch03a].

It is probably not completely necessary to restrict one’s attention to convex co-compact manifolds in Theorems 0.1 and 0.2. I would guess that similar results also hold for geometrically finite manifolds. An interesting and simple case would be a manifold with one cusp, and one boundary component which is topologically a torus.

Higher dimensions  In my opinion, it would be quite interesting to understand whether the results described here for hyperbolic manifolds with convex boundary have some sort of extension in the setting of Einstein manifolds with boundary, maybe in dimension 4. A first step is taken in [Sch01b], but it is far from satisfactory. The main problem is the lack of a strong enough infinitesimal rigidity statements for Einstein manifolds with boundary (note that one such statement, similar to the one in [Sch01b], is obtained in [RS99] by very different methods).

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References

[Ahl66] L. V. Ahlfors. Lectures on quasiconformal mappings. D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10.

[Ale58a] A. D. Aleksandrov. Vestnik Leningrad Univ., 13(1), 1958.

[Ale58b] A. D. Alexandrow. Konvexe polyeder. Akademie-Verlag, Berlin, 1958.
[LS00] F. Labourie and Jean-Marc Schlenker. Surfaces convexes fuchsiennes dans les espaces lorentziens à courbure constante. *Math. Annalen*, 316:465–483, 2000.

[Mou02] Gabor Moussong. Personal communication. July 2002.

[Nir53] L. Nirenberg. The Weyl and Minkowski problem in differential geometry in the large. *Comm. Pure Appl. Math*, 6:337–394, 1953.

[Pog73] A. V. Pogorelov. *Extrinsic Geometry of Convex Surfaces*. American Mathematical Society, 1973. Translations of Mathematical Monographs. Vol. 35.

[RH93] I. Rivin and C. D. Hodgson. A characterization of compact convex polyhedra in hyperbolic 3-space. *Invent. Math.*, 111:77–111, 1993.

[Riv86] I. Rivin. *Thesis*. PhD thesis, Princeton University, 1986.

[Riv93] I. Rivin. On the geometry of ideal polyhedra in hyperbolic 3-space. *Topology*, 32:87–92, 1993.

[Riv94] I. Rivin. Euclidean structures on simplicial surfaces and hyperbolic volume. *Annals of Math.*, 139:553–580, 1994.

[Riv96] I. Rivin. A characterization of ideal polyhedra in hyperbolic 3-space. *Annals of Math.*, 143:51–70, 1996.

[Rou04] Mathias Rousset. Sur la rigidité de polyèdres hyperboliques en dimension 3 : cas de volume fini, cas hyperidéal, cas fuchsien. *Bull. Soc. Math. France*, 132:233–261, 2004.

[RS99] Igor Rivin and Jean-Marc Schlenker. The Schlafli formula in Einstein manifolds with boundary. *Electronic Research Announcements of the A.M.S.*, 5:18–23, 1999.

[Sch94] Jean-Marc Schlenker. Surfaces elliptiques dans des espaces lorentziens à courbure constante. *Compte Rendus de l'Académie des Sciences, Série A*, 319:609–614, 1994.

[Sch96] Jean-Marc Schlenker. Surfaces convexes dans des espaces lorentziens à courbure constante. *Commun. Anal. and Geom.*, 4:285–331, 1996.

[Sch98a] Jean-Marc Schlenker. Métries sur les polyèdres hyperboliques convexes. *J. Differential Geom.*, 48(2):323–405, 1998.

[Sch98b] Jean-Marc Schlenker. Représentations de surfaces hyperboliques complètes dans $H^3$. *Annales de l'Institut Fourier*, 48(3):837–860, 1998.

[Sch00] Jean-Marc Schlenker. Dihedral angles of convex polyhedra. *Discrete Comput. Geom.*, 23(3):409–417, 2000.

[Sch01a] Jean-Marc Schlenker. Convex polyhedra in Lorentzian space-forms. *Asian J. of Math.*, 5:327–364, 2001.

[Sch01b] Jean-Marc Schlenker. Einstein manifolds with convex boundaries. *Commentarii Math. Helvetici*, 76(1):1–28, 2001.

[Sch01c] Jean-Marc Schlenker. Hyperbolic manifolds with polyhedral boundary. [math.GT/0111136](http://arxiv.org/abs/math.GT/0111136) available at [http://picard.ups-tlse.fr/~schlenker](http://picard.ups-tlse.fr/~schlenker), 2001.

[Sch02a] Jean-Marc Schlenker. Hyperideal polyhedra in hyperbolic manifolds. Preprint [math.GT/0212355](http://arxiv.org/abs/math.GT/0212355), 2002.

[Sch02b] Jean-Marc Schlenker. Hypersurfaces in $H^n$ and the space of its horospheres. *Geom. Funct. Anal.*, 12:395–435, 2002.

[Sch03a] Jean-Marc Schlenker. Hyperbolic manifolds with constant curvature boundaries. In preparation, 2003.

[Sch03b] Jean-Marc Schlenker. A rigidity criterion for non-convex polyhedra. [math.DG/0301333](http://arxiv.org/abs/math.DG/0301333) to appear in *Discrete Comput. Geom.*, 2003.
[Spi75] M. Spivak. *A comprehensive introduction to geometry, Vol.I-V.* Publish or perish, 1970-1975.

[Sul79] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.

[Thu80] William P. Thurston. Three-dimensional geometry and topology. Recent version available on [http://www.msri.org/publications/books/gt3m/](http://www.msri.org/publications/books/gt3m/) 1980.

[Vek62] I. N. Vekua. *Generalized analytic functions.* Pergamon Press, London, 1962.

[Wei60] A. Weil. On discrete subgroups of Lie groups. *Annals of Math.*, 72(1):369–384, 1960.