Gauge invariance of the Chern-Simons action in noncommutative geometry

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Abstract

In complete analogy with the classical case, we define the Chern-Simons action functional in noncommutative geometry and study its properties under gauge transformations. As usual, the latter are related to the connectedness of the group of gauge transformations. We establish this result by making use of the coupling between cyclic cohomology and K-theory and prove, using an index theorem, that this coupling is quantized in the case of the noncommutative torus.

1 Introduction

It is nowadays well admitted that the major difficulty encountered in developing a quantum theory of gravity lies in our current conception of geometry. Indeed, it is known that if we want to observe a particle localized in a very small region of space-time of size $\Delta x$, we have to use another particle with momentum $\Delta p$ such that $\Delta x \Delta p \geq \hbar$. Thus, the smaller the uncertainty on the position is, the larger the momentum $\Delta p$ is, so that compatibility with Einstein’s equations shows that a singularity appears in the limit $\Delta p \to 0$.

As a consequence, one has to give up the standard notion of localization in space-time itself and not only in phase space, as taught by quantum mechanics. To proceed, one may impose non trivial commutation relations between the space-time coordinates in order to obtain suitable uncertainty relations. The development of geometrical concepts within this context leads us directly to noncommutative geometry, which may be defined as the geometry of spaces whose coordinates fail to commute.

This new area of mathematics ranges from operators algebras to quantum groups, the latter appearing, for instance, when one tries to describe the symmetries of the quantum plane, whose coordinates satisfy $xy = qyx$ with $q \in \mathbb{C}^*$. Here, we will be mainly interested in the theory developed by A. Connes which relies on the use of operator algebraic concepts. Roughly speaking, it may be summed up by the two following steps. First, one tries to formulate a geometrical theory like measure theory or topology using a suitable subalgebra of the algebra of complex valued functions on the space $X$ under consideration. Then one extends the previous theory to more general algebras that are not necessarily commutative; these algebras are to be thought as algebras of coordinates on the ”quantum space” replacing $X$. These algebras $\mathcal{A}$ are always subalgebras of the algebras of bounded operators on a given Hilbert space $\mathcal{H}$; for instance Von Neumann algebras and $C^*$-algebras.

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are respectively relevant in the study of the noncommutative extension of measure theory and topology.

However, the notions relevant to physics are of differential nature, so that one must develop differential geometry in the noncommutative setting. Borrowing ideas from quantum mechanics, one replaces the derivative of a function by a commutator with a suitable operator $D$ acting on $\mathcal{H}$, so that one may define the differential of a "function" $a \in A$ by $da = [D, \pi(a)]$, $\pi$ being a representation of $A$ as operators on $\mathcal{H}$. Accordingly, it turns out that the relevant notion is that of a spectral triple $(A, \mathcal{H}, D)$ which is supposed to satisfy additional requirements given below. In the commutative case, if we assume that $\mathcal{M}$ is a compact manifold endowed with a spin structure, one can reconstruct all differential geometric notions from the spectral triple $(A, \mathcal{H}, D)$, where $A$ is the algebra of smooth function on $\mathcal{M}$ represented by multiplication on the Hilbert space of square integrable sections of the spinor bundle and $D$ is the standard Dirac operator.

In the general case, starting with such a triple one can reconstruct the analogue of gauge theory, even with nontrivial topological properties. Furthermore, one can built in all these cases a Yang-Mills action functional which exhibits all standard properties of a bona fide Yang-Mills action: positivity, gauge invariance, etc... Here, our main concern will be the construction of the Chern-Simons action, which has proved to be relevant, in the classical case, to many areas of mathematics and physics. Because of the nontrivial properties of this action under gauge transformation, we will have to use the machinery of noncommutative geometry including the coupling of cyclic cohomology to K-theory [3] and the index theorem [4].

## 2 Spectral triples and differential forms

To construct differential geometric objects in noncommutative geometry, the relevant notion is that of a spectral triple that we already have introduced above. Let us now precise its definition [2].

**Definition 2.1** A spectral triple $(A, \mathcal{H}, D)$ consists in an involutive algebra $A$ together with a faithful representation $\pi$ of $A$ by bounded operators on a separable Hilbert space $\mathcal{H}$. $D$ is an unbounded self-adjoint operator with compact resolvent and such that $[D, \pi(a)]$ is bounded for any $a \in A$.

Furthermore, one may assume that the triple $(A, \mathcal{H}, D)$ satisfies additional requirements stated in order to recover spin geometry from commutative spectral triples [3]. Amongst all these requirements commonly referred to as "axioms of noncommutative geometry", we will only make use of the following two.

**Axiom 1 (Dimension)** There is a positive integer $n$ such that the decreasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ of the eigenvalues of the compact operator $ds = |D|^{-1}$ satisfies

$$\lambda_k = O\left(\frac{1}{k^{1/n}}\right)$$

when $k \to \infty$.

This axiom only gives us a lower bound on $n$, but together with the other axioms it defines the dimension of a spectral triple. Here, by dimension of a spectral triple we essentially mean an integer satisfying the previous axiom.
Since $\mathcal{D}$ has compact resolvent, its kernel is finite dimensional and $ds = |\mathcal{D}|^{-1}$, with $|\mathcal{D}| = \sqrt{\mathcal{D}^2}$ is well defined on the orthogonal complement of ker $\mathcal{D}$. This axiom implies that the sequence of eigenvalues of $ds^n = |\mathcal{D}|^{-n}$ is bounded by the sequence $C/k$ for $C > 0$ and $k$ large enough, so that the Dixmier trace $\text{Tr}_\omega(T|\mathcal{D}|^{-n})$ is well defined for any bounded operator $T$ [2]. In the commutative case one can recover the ordinary integral of a function from the Dixmier trace since we have
\[
\text{Tr}_\omega \left( \pi(f) |\mathcal{D}|^{-n} \right) = \frac{2^n - [n/2]}{\pi^{n/2} \Gamma(n/2 + 1)} \int_M d^n x \sqrt{g} f,
\] for any smooth function $f$ on a compact Riemannian manifold of dimension $n$. Accordingly, we define for any spectral triple of dimension $n$ and any bounded operator $T$ the analogue of the integral
\[
\int Tds^n = \frac{\pi^{n/2} \Gamma(n/2 + 1)}{2^{[n/2] - n}} \text{Tr}_\omega \left( T|\mathcal{D}|^{-n} \right),
\] bearing in mind that all operators in the algebra generated by $\pi(A)$ and $[\mathcal{D}, \pi(A)]$ are bounded operators.

The other axiom we will need is the regularity axiom.

**Axiom 3 (Regularity)** Any element $b$ of the algebra generated by $\pi(A)$ and $[\mathcal{D}, \pi(A)]$ lies in the domains of the powers of the derivation defined by $\delta(b) = [\mathcal{D}, b]$.

From this axiom, one deduces the following result [3].

**Proposition 2.1** Let $\mathcal{B}$ be the algebra generated by $\pi(A)$ and $[\mathcal{D}, \pi(A)]$. Then the map $b \mapsto \text{Tr}_\omega (b|\mathcal{D}|^{-n})$ is a trace on $\mathcal{B}$.

This trace property proves to be of primary importance when we study the behavior of the action functional under gauge invariance.

Let us now tackle the question of the construction of differential forms. We first introduce a formal construction called universal differential algebra.

**Definition 2.2** Let $\mathcal{A}$ be a unital algebra. The universal differential algebra over $\mathcal{A}$ is the graded algebra $\Omega(\mathcal{A}) = \bigoplus_{k \in \mathbb{N}} \Omega^k(\mathcal{A})$, where $\Omega^k(\mathcal{A})$ is the vector space generated by $a_0 \delta a_1 \ldots \delta a_k$ (4) for any $a_0, a_1, \ldots, a_k \in \mathcal{A}$. The product is obtained by simple juxtaposition together with the relations $\delta(ab) = \delta a b + a \delta b$ for any $a, b \in \mathcal{A}$ as well as $\delta(1) = 0$. The exterior derivative $d : \Omega^k(\mathcal{A}) \to \Omega^{k+1}(\mathcal{A})$ is the linear map defined by
\[
d(a_0 \delta a_1 \ldots \delta a_k) = \delta a_0 \delta a_1 \ldots \delta a_k
\] for all $a_0, a_1, \ldots, a_k \in \mathcal{A}$.

It fulfills the standard properties of a differential algebra.

**Proposition 2.2** The exterior derivative $d$ is nilpotent and fulfills the graded Leibniz rule $d(\omega \xi) = d\omega \xi + (-1)^p \omega d\xi$ for all $\omega \in \Omega^p(\mathcal{A})$ and $\xi \in \Omega^q(\mathcal{A})$.
However, this construction is a rather formal one and has to be represented at the level of the Hilbert space by replacing the derivative by a commutator. Accordingly, we define a representation of $\Omega(A)$ by

$$\pi(a_0)\delta a_1 \ldots \delta a_k = \pi(a_0)[D, \pi(a_1)] \ldots [D, \pi(a_k)].$$  

(6)

Although this map defines a representation of $\Omega(A)$ as an algebra, it fails to be a representation of the differential structure. Indeed, this requires that we define the differential of $\pi(\omega)$ as $d\pi(\omega) = \pi(d\omega)$ for any $\omega \in \Omega(A)$, which is not possible as soon as $\ker \pi \neq \{0\}$.

To proceed, let us define the ideal $J = \ker \pi + d(\ker \pi)$. The image of the quotient $\Omega(A)/J$ admits a well defined differential structure [2].

**Proposition 2.3** The graded algebra $\Omega_D(A)$ defined by

$$\Omega_D(A) = \pi(\Omega(A)/J) = \pi(\Omega(A))/\pi(d \ker \pi)$$

(7)

admits an exterior derivative $d$ such that $d\pi(\omega)$ is a representative of the class defined by $\pi(d\omega)$, for any $\omega \in \Omega(A)$. This exterior derivative is nilpotent and satisfies the graded Leibniz rule.

Unfortunately, we are now dealing with equivalence classes that usually admit more than one representative. In the simple case of a spectral triple obtained by tensoring the ordinary geometry of space-time by a matrix algebra, one can define a scalar product on $\pi(\Omega(A))$ by

$$\langle \pi(\omega), \pi(\eta) \rangle = \text{Tr}_\omega \left( \pi(\omega)^* \pi(\eta) |D|^{-n} \right)$$

(8)

for any $\omega, \eta \in \Omega^k(A)$, whereas forms of different degree are defined to be orthogonal [7]. Let us notice that $\pi(\omega)$ and $\pi(\eta)$ are bounded operators, so that the previous expression is well defined. However, it is not in general a scalar product because we cannot check that it is positive definite. Indeed, it could happen that the eigenvalues of $\pi(\omega)$ decrease sufficiently fastly so that the trace vanishes, even if $\pi(\omega) \neq 0$.

Since we are interested in defining the Chern-Simons action, we need a three dimensional spectral triple. Accordingly, we set $n = 3$ from now on but the discussion of what follows may generalized to other values of $n$.

The following condition defines the noncommutative analogue of a manifold without a boundary.

**Definition 2.3** A three dimensional spectral triple $(A, H, D)$ is said to satisfy the closedness condition if

$$\text{Tr}_\omega \left( [D, \pi(a_0)] \ldots [D, \pi(a_3)] |D|^{-3} \right) = 0$$

(9)

for any $a_0, \ldots, a_3 \in A$.

As a computational device, it allows us to use the rule of integration by parts,

$$\int (\pi(d\omega)\pi(\eta)) ds^3 = (-1)^{p+1} \int (\pi(\omega)\pi(d\eta)) ds^3$$

(10)

for any $\omega \in \Omega^p(A)$ and $\eta \in \Omega^q(A)$ with $p + q = 3$.

For later purposes, it is useful to relate this condition to cyclic and Hochschild cohomology. For completeness, we recall the following basic definitions [8].
Proposition 2.4 Let $\mathcal{A}$ an algebra and $\phi: \mathcal{A}^{n+1} \to \mathbb{C}$ a $(n+1)$-linear map. $\phi$ is said to be a Hochschild cocyle if it satisfies

$$
\sum_{i=0}^{n-1} (-1)^i \phi(a_0, \ldots, a_ia_{i+1}, \ldots, a_n) + (-1)^n \phi(a_n^2a_0, a_1, \ldots, a_{n-1}) = 0 \quad (11)
$$

for any $a_0, \ldots, a_n \in \mathcal{A}$. If in addition it fulfils

$$
\phi(a_0, a_1, \ldots, a_n) = (-1)^n \phi(a_1, \ldots, a_n, a_0), \quad (12)
$$

it is a cyclic cocycle.

There is an easy characterization of spectral triples fulfilling the closedness condition using cyclic cocycles.

Proposition 2.5 A spectral triple $$(\mathcal{A}, \mathcal{H}, D)$$ satisfies the closedness condition if and only if the map $\phi: \mathcal{A}^4 \to \mathbb{C}$ defined by

$$
\phi(a_0, a_1, a_2, a_3) = \text{Tr}_\omega \left( \pi(a_0) [D, \pi(a_1)] [D, \pi(a_2)] [D, \pi(a_3)] |D|^{-3} \right) \quad (13)
$$

is a cyclic cocycle.

Obviously, if $$(\mathcal{A}, \mathcal{H}, D)$$ is a spectral triple, then $$(M_N(\mathcal{A}), \mathcal{H} \otimes \mathbb{C}^N, D \otimes I_N)$$ is a spectral triple satisfying all requirements imposed to $$(\mathcal{A}, \mathcal{H}, D)$$. Moreover the former fulfils the closedness condition if only if the latter does. From now on, we shall work with the second one, whose differential forms are matrix valued forms.

Before applying this rather formal machinery to the construction of the Chern-Simons functional, let us adopt integral notations,

$$
\int \text{Tr} (\pi(\omega)) \, ds^n = \frac{\pi^{n/2} \Gamma(n/2 + 1)}{2^{n-[n/2]}} \text{Tr}_\omega (\pi(\omega) |D|^{-n}) \quad (14)
$$

for any matrix valued form $\omega \in M_N(\Omega(\mathcal{A}))$, as well as $da = [D, \pi(a)]$ for any $a \in M_N(\mathcal{A})$.

3 Gauge invariance of the Chern-Simons action

Before we come to grips with the noncommutative case, let us briefly recall some basic facts about Chern-Simons field theory [9]. If $\mathcal{M}$ is a compact and orientable three dimensional manifold and $G$ is a compact Lie group which may be chosen to be $SU(N)$ for concreteness, the Chern-Simons action is defined to be

$$
S_{CS}[A] = \frac{k}{4\pi} \int \text{Tr} \left( \text{AdA} + \frac{2}{3} A^3 \right), \quad (15)
$$

where $k \in \mathbb{R}$ is a coupling constant and $A$ is a 1-form with values in the Lie algebra of $G$. It is a remarkable fact that this action does not depend on a metric on $\mathcal{M}$ because we integrate a 3-form in dimension 3. This turns Chern-simons theory into a topological field theory [10], whose quantization yields non trivial topological invariants of the manifold $\mathcal{M}$ and allows us to recover the Jones polynomial of knot theory.
Under the gauge transformation given by the map $g$ from $\mathcal{M}$ into $G$, the gauge potential $A$ becomes $A^g = gAg^{-1} + gdg^{-1}$ and it is easy to show, using the standard properties of differential forms, that the Chern-Simons action is not gauge invariant,

$$S_{CS}[A^g] = S_{CS}[A] + \frac{k}{12\pi} \int \text{Tr} (gdg^{-1})^3. \quad (16)$$

If we normalize the generators $T^a$ of the Lie algebra of $G$ such that $\text{Tr}(T^aT^b) = -\frac{1}{2} \delta^{ab}$, one has

$$\frac{1}{24\pi^2} \int \text{Tr} (gdg^{-1})^3 = n \quad (17)$$

where $n$ is an integer called ”winding number” of the map $g$ from $\mathcal{M}$ into $G$.

Accordingly, if $k$ is an integer, $e^{ikS_{CS}[A]}$ is gauge invariant and the partition function

$$Z[\mathcal{M}] = \int [\mathcal{D}A] e^{ikS_{CS}[A]} \quad (18)$$

is well defined as a gauge theory. It is worthwhile to notice that the quantization of the coupling constant is preserved in the one-loop analysis because it is just shifted by another integer \[\mathbb{Z}\].

Furthermore, this integer has a deep topological significance because it is a measure of the defect of connectedness of the group of gauge transformations. Indeed, if $g_0$ and $g_1$ are two gauge transformations connected by a path $t \in [0, 1] \mapsto g_t$, one can show that

$$\frac{d}{dt} \int \text{Tr} (g_tdg_t^{-1})^3 = 0. \quad (19)$$

Accordingly, the winding number is constant on each connected component of the group of gauge transformations.

Let us now come to the noncommutative case.

**Definition 3.1** Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple of dimension $3$ satisfying the closedness condition and let $A \in M_N(\Omega^1_\mathcal{D}(\mathcal{A}))$ a hermitian matrix of 1-forms. We define the Chern-Simons action as

$$S_{CS}[A] = \int \text{Tr} (K_1) ds^3, \quad (20)$$

where $K_1 \in M_N(\pi(\Omega^3(\mathcal{A})))$ is any representative of the class of the Chern-Simons form $K = \text{Ad}A + \frac{2}{3}A^3 \in M_N(\Omega^3_\mathcal{D}(\mathcal{A}))$.

To ensure self-consistency of this definition, we have to show that it only depends on the equivalence class of the Chern-Simons. If $K_2$ denotes an other representative of $K$, then, using the ideal properties of $J$, one has $\text{Tr}(K_1) - \text{Tr}(K_2) \in \pi(J) \cap \pi(\Omega^3(\mathcal{A}))$ so that

$$\text{Tr} (K_1) - \text{Tr} (K_2) = \sum_i da^i_1 da^i_2 da^i_3, \quad (21)$$

where $a^i_1$, $a^i_2$ and $a^i_3$ are elements of $\mathcal{A}$. From the closedness condition, we deduce that

$$\int (da^i_1 da^i_2 da^i_3) ds^3 = 0, \quad (22)$$
so that
\[ \int Tr \left( K_1 \right) ds^3 = \int Tr \left( K_2 \right) ds^3. \] (23)

At first sight, it is not clear whether this action is of topological nature or not. Indeed, if it was of topological nature, it should only depend on the choice of the Dirac operator in a weak form, because the latter also contains information pertaining to the metric structure. Anyway, it is easy to see that in the commutative case one recovers the standard Chern-Simons action, which is doubtless of topological nature.

Let us now the study the gauge invariance of this action.

**Theorem 3.1** Let \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) be a spectral triple of dimension 3 satisfying the closedness condition. Then, under the gauge transformation determined by a unitary element \(u\) of \(M_N(\mathcal{A})\), the Chern-Simons action becomes
\[ S_{\text{CS}}[uAu + udu^{-1}] = S_{\text{CS}}[A] + \Gamma[u], \] (24)
with
\[ \Gamma[u] = -\frac{1}{3} \int Tr \left( udu^{-1}udu^{-1}u \right) ds^3. \] (25)

**Proof:**

To proceed, let us introduce the curvature \(F = dA + A^2\) of \(A\). We have
\[ AdA + \frac{2}{3} A^3 = AF - \frac{1}{3} A^3, \] (26)
so that, if \(F_1\) is a representative of \(F\), the Chern-Simons action reads
\[ S_{\text{CS}}[A] = \int Tr \left( AF_1 - \frac{1}{3} A^3 \right) ds^3. \] (27)

Under a gauge transformation, \(A\) becomes \(uAu^{-1} + udu^{-1}\) and \(F\) transforms into \(uFu^{-1}\). Using the ideal structure of \(J\), it is clear that \(uF_1u^{-1}\) is a representative of \(uFu^{-1}\) and we have
\[ S_{\text{CS}}[uAu^{-1} + udu^{-1}] = \int Tr \left( (uAu^{-1} + udu^{-1}) uF_1u^{-1} - 1/3 (uAu^{-1} + udu^{-1}) \right) ds^3. \] (28)

Using the trace properties of the map \(\pi(\omega) \mapsto \int Tr(\pi(\omega))ds^3\) and the relation \(du^{-1}u + u^{-1}du = 0\), we get
\[ \int Tr \left( (uAu^{-1} + udu^{-1}) uF_1u^{-1} \right) ds^3 = \int Tr (AF) ds^3 - \int Tr (duFu^{-1}) ds^3, \] (29)
as well as
\[ -1/3 \int Tr \left( uAu^{-1} + udu^{-1} \right)^3 ds^3 = -\frac{1}{3} \int Tr (A)^3 ds^3 \]
\[ -\frac{1}{3} \int Tr (udu^{-1})^3 ds^3 - \int Tr (udu^{-1}uA^2u^{-1}) ds^3 + \int Tr (duAdu^{-1}) ds^3. \] (30)
Gathering all terms, we obtain

\[
S_{CS}[Au + udu^{-1}] = S_{CS}[A] + \Gamma[u] \\
+ \int \text{Tr} \left( udu^{-1}uF_{1}u^{-1} - udu^{-1}uA^{2}u^{-1} + duAdu^{-1} \right) \, ds^{3}.
\]  

(31)

The operator appearing on the left hand side is a representative of the 3-form

\[
udu^{-1}uF_{1}u^{-1} - udu^{-1}uA^{2}u^{-1} + duAdu^{-1} = udu^{-1}udAu^{-1} + duAdu^{-1} \\
= -dud(Au^{-1}) \\
= -d\left( ud(Au^{-1}) \right).
\]  

(32)

Because this form is exact, the integral of all its representatives vanishes by the closedness condition,

\[
\int \text{Tr} \left( udu^{-1}uF_{1}u^{-1} - udu^{-1}uA^{2}u^{-1} + duAdu^{-1} \right) \, ds^{3} = 0,
\]  

(33)

which proves that

\[
S_{CS}[Au + udu^{-1}] = S_{CS}[A] + \Gamma[u].
\]  

(34)

\[
\square
\]

Let us now try to understand the topological significance of \( \Gamma[u] \). We first have to recall the definition of the group \( K_{1}(\mathcal{A}) \).

**Definition 3.2** Let \( \mathcal{A} \) be a \( C^{*} \)-algebra (i.e. it is an involutive Banach algebra whose norm satisfies \( ||aa^{*}|| = ||a||^{2} \) for any \( a \in \mathcal{A} \)) and let us denote by \( U_{N}(\mathcal{A}) \) the group of unitary elements of \( M_{N}(\mathcal{A}) \). Then, using the embedding

\[
u \in U_{N}(\mathcal{A}) \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U_{N+1}(\mathcal{A})
\]  

(35)

we define

\[
U_{\infty}(\mathcal{A}) = \bigcup_{N=1}^{\infty} U_{N}(\mathcal{A}).
\]  

(36)

By definition, \( K_{1}(\mathcal{A}) \) is the group \( \pi_{0}(U_{\infty}(\mathcal{A})) \) of connected components of \( U_{\infty}(\mathcal{A}) \).

We refer to [11] for a general introduction to K-theory. It is important to point out that this definition works at the level of \( C^{*} \)-algebras, which corresponds to continuous functions, where the algebra \( \mathcal{A} \) appearing in the spectral triple \( (\mathcal{A}, \mathcal{H}, \mathcal{D}) \) forms the analogue of smooth functions. However, one can show, using holomorphic functional calculus, that this does not really matter [8]. Thus, we work with \( \mathcal{A} \) as if it was a \( C^{*} \)-algebra.

**Proposition 3.1** With the assumptions of the previous theorem, \( \Gamma[u] \) only depends on the class of \( u \) in the group \( K_{1}(\mathcal{A}) = \pi_{0}(U_{\infty}(\mathcal{A})) \).

\[
\square
\]
Proof:

This result relies on the coupling of cyclic cohomology to $K^1(\mathcal{A})$ [3]: if $\phi_{2m+1}$ is an odd dimensional cyclic cocycle and $u$ unitary, then $\phi_{2m+1}(u - 1, u^{-1} - 1, \ldots, u - 1, u^{-1} - 1)$ only depends on the class of $u$ in $K_1(\mathcal{A})$.

Thanks to the closedness condition, $\Phi$ defined by

$$\Phi(a_0, a_1, a_2, a_3) = \int (a_0 da_1 da_2 da_3) ds^3,$$  \hspace{1cm} (37)

is a cyclic cocycle on $\mathcal{A}$ (cf proposition 2.5) that we extend to a cyclic cocycle $\tilde{\Phi}$ on $M_N(\mathcal{A})$ using the trace by

$$\tilde{\Phi}(a_0, a_1, a_2, a_3) = \int \text{Tr} (a_0 da_1 da_2 da_3) ds^3.$$  \hspace{1cm} (38)

Consequently, $\tilde{\Phi}(u - 1, u^{-1} - 1, u, u^{-1} - 1)$ only depends on the class of $u$ in $K_0(\mathcal{A})$.

Finally, let us notice that $\Gamma[u] = 1/3 \tilde{\Phi}(u - 1, u^{-1} - 1, u - 1, u^{-1} - 1)$ because of

$$(udu^{-1})^2 = -udu^{-1} duu^{-1} = -ud(u^{-1} - 1) du - (u^{-1} - 1).$$  \hspace{1cm} (39)

Using the closedness condition, we get

$$\tilde{\Phi}(1, u^{-1}, u, u^{-1}) = \int \text{Tr} \left( 1, u^{-1}, u - 1, u^{-1} - 1 \right) ds^3 = 0,$$  \hspace{1cm} (40)

which proves that

$$\Gamma[u] = \frac{1}{3} \tilde{\Phi}(u - 1, u^{-1} - 1, u - 1, u^{-1} - 1).$$  \hspace{1cm} (41)

Accordingly, $\Gamma[u]$ only depends on the class of $u$. \hfill $\Box$

Therefore, it is clear that $\Gamma[u]$ is constant on the connected components of $U_\infty(\mathcal{A})$ and the non invariance of the Chern-Simons functional is due to the defect of connectedness of the group $U_N(\mathcal{A})$ of gauge transformations, as in the classical case.

Although we prove that $\Gamma[u]$ only depends on the connected component in which $u$ lies, we have not shown that it is an integer up to a multiplicative constant. To proceed further, we have to relate it to the index of a given Fredholm operator.

4 An application of the index theorem

In the general case, it is always possible to associate to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a class of Fredholm operators (i.e. it is a bounded operator with finite dimensional kernel and cokernel) whose index can be computed using a local formula involving cyclic cocycles [4].

To state this result, we need the following definition.

Definition 4.1 Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a spectral triple of dimension $n$ and let us denote by $\delta$ the derivation $\delta(b) = [\mathcal{D}, b]$ for any $b$ in the algebra generated by $\pi(\mathcal{A})$ and $[\mathcal{D}, \pi(\mathcal{A})]$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is said to have discrete dimension spectrum if there is a discrete subset $\Sigma \in \mathbb{C}$ such that the functions

$$\zeta_{\delta}(z) = \text{Tr}(b|\mathcal{D}|^z),$$  \hspace{1cm} (42)

holomorphic for $\Re(z)$ large enough, extend holomorphically to $\mathbb{C} - \Sigma$ for any $b$ belonging to the algebra generated by the elements of $\mathcal{B}$ and their images through $\delta^k$. 


When \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) has discrete dimension spectrum, we define on the algebra generated by \(\mathcal{B}\) and \(|\mathcal{D}|^z, z \in \mathbb{C}\), a sequence \((\tau_k)_{k \in \mathbb{N}}\) of linear functional by
\[
\tau_k(b) = \text{res}_{z=0} z^k \text{Tr} \left( b |\mathcal{D}|^{-2z} \right).
\] (43)

In general, these functionals fail to be traces. However, when all the poles are simple, only \(\tau_0\) is nontrivial and one can show that it is a trace which generalizes to the noncommutative case the Wodzicki residue.

Let us now consider a spectral triple of dimension 3 and let us define on \(\mathcal{A}\) two cochains by
\[
\phi_3(a_0, a_1, a_2, a_3) = \frac{1}{12} \tau_0 \left( a_0 da_1 da_2 da_3 |\mathcal{D}|^{-3} \right) - \frac{1}{6} \tau_1 \left( a_0 da_1 da_2 da_3 |\mathcal{D}|^{-3} \right)
\] (44)
and
\[
\phi_1(a_1, a_2) = \tau_0 (a_0 da_1) \|dd|^{-1}) - \frac{1}{4} \tau_0 \left( a_1 \nabla(da_1) |\mathcal{D}|^{-3} \right)
- \frac{1}{2} \tau_1 \left( a_1 \nabla(da_1) |\mathcal{D}|^{-3} \right) + \frac{1}{8} \tau_0 \left( a_1 \nabla^2(da_1) |\mathcal{D}|^{-5} \right)
+ \frac{1}{3} \tau_1 \left( a_1 \nabla^2(da_1) |\mathcal{D}|^{-5} \right) + \frac{1}{12} \tau_2 \left( a_1 \nabla^2(da_1) |\mathcal{D}|^{-5} \right).
\]

where we have used \(\nabla(b) = [\mathcal{D}^2, b]\) for any \(b \in \mathcal{B}\) and we write \(da = [\mathcal{D}, \pi(a)]\) and \(a\) instead of \(\pi(a)\) for any \(a \in \mathcal{A}\).

Let us also introduce the unitary operator \(F\) defined by \(\mathcal{D} = |\mathcal{D}|F\) on the orthogonal complement of the finite dimensional kernel of \(\mathcal{D}\), as well as its positive part \(P = \frac{1+\mathcal{F}}{2}\). Of course, we extend this construction to the spectral triple \((\mathcal{M}_N(\mathcal{A}), \mathcal{H} \otimes \mathbb{C}^N, \mathcal{D} \otimes I_N)\) so that we can deal with matrices over \(\mathcal{A}\).

In dimension 3, we can formulate the index theorem as follows [4].

**Theorem 4.1** If \(u \in \mathcal{M}_N(\mathcal{A})\) is unitary, then \(PuP\) is a Fredholm operator on \(PH\) whose index is given by
\[
\text{Ind}(PuP) = \phi_1(u, u^{-1}) - \phi_3(u, u^{-1}, u, u^{-1}).
\] (45)

Since \(\text{Ind}(PuP) = \dim \ker PuP - \dim \ker Pu^*P\) is an integer, we always have
\[
\phi_1(u, u^{-1}) - \phi_3(u, u^{-1}, u, u^{-1}) \in \mathbb{Z}.
\] (46)

Unfortunately, even in the case of simple spectrum, only \(\phi_3\) can be related to \(\Gamma[u]\) and thus the index theorem does not prove the integrality of \(\Gamma[u]\) in the general case.

However, besides the commutative case we can construct a simple example in which this integrality result holds. Let us define the three dimensional noncommutative torus [12] as the algebra \(\mathcal{A}_\theta\) of power series of the form
\[
\sum_{p_1, p_2, p_3 \in \mathbb{Z}} a_{p_1, p_2, p_3} U_1^{p_1} U_2^{p_2} U_3^{p_3},
\] (47)
where \(U_1, U_2\) and \(U_3\) are unitary elements fulfilling the relations
\[
U_i U_j = e^{2i\pi \theta_{ij}} U_j U_i,
\] (48)
with \( \theta \in M_3(\mathbb{R}) \) an antisymmetric matrix. Moreover, we always assume that the sequence \((a_{p_1,p_2,p_3})_{(p_1,p_2,p_3)\in \mathbb{Z}^3}\) decreases faster than any polynomial, which characterizes the analogue of "smooth functions" on the three dimensional noncommutative torus.

On \( A_\theta \) we define a trace by

\[
\int \left( \sum_{p_1,p_2,p_3 \in \mathbb{Z}} a_{p_1,p_2,p_3} U_1^{p_1} U_2^{p_2} U_3^{p_3} \right) = a_{0,0,0},
\]

which is completely similar to the usual integral since it singles out the constant mode in the Fourier expansion.

Moreover, we define three derivations \( \partial_1, \partial_2 \) and \( \partial_3 \) by their actions on the generators

\[
\partial_i U_j = 2i\pi \delta_{ij} U_j,
\]

where \( \delta_{ij} \) equals 1 if \( i = j \) and 0 otherwise. These derivations are analogous to the derivations with respect to the standard coordinates on the usual torus.

This construction yields a 3-dimensional spectral triple \((A, \mathcal{H}, D)\), where \( A \) is the algebra \( A_\theta \) acting by multiplication on the Hilbert space \( \overline{\mathcal{A}}_\theta \otimes \mathbb{C}^2 \), where \( \overline{\mathcal{A}}_\theta \) is the completion of \( A_\theta \) for the scalar product defined by the trace. The Dirac operator is \( D = i\sigma_\mu \partial_\mu \), where \( \sigma_\mu, \mu = 1, 2, 3 \) denote the Pauli matrices and, as usual, summation over repeated greek indices \( \lambda, \mu, \nu, \ldots \) ranging from 1 to 3 is self-understood. The choice of the Pauli matrices means that we take the analogue of the euclidean metric on the noncommutative torus but other constant metrics \( g^{\mu\nu} \) may be taken. In this case, one shows that the Chern-Simons action is independent of \( g^{\mu\nu} \).

For the sake of brevity we do not give here any detailed calculation and refer to [13] for a more thorough account. Let us simply state that this spectral triple fulfils the closedness condition and that the Chern-Simons action is

\[
S_{CS}[A_\mu] = \frac{k}{4\pi} \int \epsilon_{\lambda\mu\nu} \text{Tr} \left( A_\lambda \partial_\mu A_\nu + \frac{2}{3} A_\lambda A_\mu A_\nu \right),
\]

where \( A_\mu \) is a hermitian matrix with entries in \( A_\theta \), \( \epsilon_{\lambda\mu\nu} \) is the completely antisymmetric tensor with \( \epsilon_{123} = 1 \) and \( k \in \mathbb{R} \) is a coupling constant.

Under the gauge transformation determined by the unitary \( u \), we have \( A_\mu \rightarrow uA_\mu u^{-1} + u\partial_\mu u^{-1} \), and the Chern-Simons action becomes

\[
S_{CS}[uA_\mu u^{-1} + u\partial_\mu u^{-1}] = S_{CS}[A_\mu] + \Gamma[u],
\]

with

\[
\Gamma[u] = \frac{k}{12\pi} \int \epsilon_{\lambda\mu\nu} \text{Tr} \left( \partial_\lambda u \partial_\mu u^{-1} \partial_\nu u \right).
\]

On the other hand, it is easily seen that the \((A, \mathcal{H}, D)\) is a three dimensional spectral triple with simple dimension spectrum, so that one can apply the index theorem. For \( \Re(z) > 3/2 \), the trace in the full Hilbert space \( \mathcal{H} \otimes \mathbb{C}^N \) given by

\[
\text{Tr} \left( u\partial_\mu u^{-1} |D|^{2z} \right) = i\text{Tr} \left( u\sigma_\mu \partial_\mu u^{-1} |D|^{2z} \right)
\]

vanishes identically because it involves a trace on a single Pauli matrix. Accordingly, its residue is 0 and we have

\[
\tau_0 \left( u\partial_\mu u^{-1} |D|^{-1} \right) = 0.
\]
The same result holds for \( \tau_0(u \nabla (du^{-1})|D|^{-1}) \) and for \( \tau_0(u \nabla^2 (du^{-1})|D|^{-1}) \), so that we have
\[
\phi_1(u, u^{-1}) = 0. \tag{56}
\]

Finally, let us compute
\[
\phi_3(u, u^{-1}, u, u^{-1}) = \frac{1}{12} \text{res}_{z=0} \text{Tr} \left( u[D, u^{-1}][D, u][D, u^{-1}]|D|^{-3-2z} \right). \tag{57}
\]

Because of the relation \( \sigma_\lambda \sigma_\mu \sigma_\nu = i \epsilon_{\lambda \mu \nu} \), the trace over Pauli matrices simply yields \( 2i \epsilon_{\lambda \mu \nu} \) and we get, bearing in mind that the scalar product on \( \mathcal{H} \) is given by
\[
\int_{\mathcal{H}} \text{Tr} \left( u\partial_\lambda u^{-1} \partial_\mu u \partial_\nu u^{-1} \right) \text{Tr} \Delta^{-3/2-z}, \tag{58}
\]
where \( \Delta \) denotes the standard 3-dimensional Laplacian on the commutative torus. It is important to notice that the LHS of the previous equation involves two different traces: the first trace denotes a trace on \( \mathcal{M}(\mathcal{A}) \) whereas the second one is to be taken over all non-zero modes of the Laplacian \( \Delta \).

Using the relation
\[
\sum_{k \in \mathbb{Z}} e^{-tk^2} \sim \sqrt{\pi/t}, \tag{59}
\]
we obtain (see [14] for a detailed account)
\[
\text{res}_{z=0} \text{Tr} \left( \Delta^{-3/2-z} \right) = \text{res}_{z=3/2} \text{Tr} \left( \Delta^{-z} \right) = \frac{1}{4\pi^2}. \tag{60}
\]

Gathering everything together, we get
\[
\phi_3(u, u^{-1}, u, u^{-1}) = \frac{1}{24\pi^2} \int \epsilon_{\lambda \mu \nu} \text{Tr} \left( u\partial_\lambda u^{-1} \partial_\mu u \partial_\nu u^{-1} \right). \tag{61}
\]

Because \( \phi_1 \) vanishes identically, the index theorem shows that
\[
\frac{1}{24\pi^2} \int \epsilon_{\lambda \mu \nu} \text{Tr} \left( u\partial_\lambda u^{-1} \partial_\mu u \partial_\nu u^{-1} \right) \in \mathbb{Z}, \tag{62}
\]
which is completely analogous to the classical case, even for such a highly noncommutative "manifold". Accordingly, \( \Gamma[u] \) belongs to \( 2i\pi \mathbb{Z} \) as soon as \( k \in \mathbb{Z} \). This quantization of the coupling constant makes the partition function
\[
Z(\mathcal{A}_\theta) = \int [DA_\mu] e^{\frac{ik}{12\pi} S_{CS}[A_\mu]} \tag{63}
\]
well defined after gauge fixing, where the measure \( [DA_\mu] \) has to be understood as a product of all the one-dimensional measures pertaining to the Fourier modes.

To show that this index is actually non-trivial, let us construct a simple example using the Power-Rieffel [14]. Let \( \theta \) be the deformation matrix given by
\[
\theta = \begin{pmatrix}
0 & \alpha & 0 \\
-\alpha & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \alpha \in [0, 1]. \tag{64}
\]
In the algebra $\mathcal{A}_\theta$, we can construct a hermitian projection $e$ by

$$e = (U_1 f(U_2))^* + g(U^2) + U_1 f(U_2), \quad (65)$$

where $f_1$ and $f_2$ are suitable smooth functions on $S^1$, that we may always choose such that $\int f e = \theta$ and

$$\frac{1}{2i\pi} \int e(\partial_1 e \partial_2 e - \partial_2 e \partial_1 e) = 1. \quad (66)$$

From this projection, let us construct

$$U = \frac{U_3 + U_3^*}{2} + (2e - 1) \frac{U_3 - U_3^*}{2}. \quad (67)$$

Because $e$ is a hermitian projection, $U$ is unitary and we have

$$\begin{align*}
\partial_1 U &= 2\partial_1 e \frac{U_3 - U_3^*}{2} \\
\partial_2 U &= 2\partial_2 e \frac{U_3 - U_3^*}{2} \\
U\partial_3 U^{-1} &= 2i\pi(2e - 1). \quad (68)
\end{align*}$$

This yields, after a lengthy but straightforward computation,

$$\int \epsilon_{\lambda\mu\nu} \text{Tr} \left( u \partial_\lambda u^{-1} \partial_\mu u \partial_\nu u^{-1} \right) = 24\pi^2. \quad (69)$$

Furthermore, if we replace $U$ by $U^n$, we obtain an index equal to $n$.

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