Immersion in $\mathbb{S}^n$ by complex spinors

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Abstract

Since the first work of Thomas Friedrich showing that isometric immersions of Riemann surfaces are related to spinors and the Dirac equation, various works appeared generalizing this approach to more general Spin-manifolds, in particular the case of submanifolds of Spin-manifolds of constant curvature. In the present work we investigate the case of submanifolds of Spin$^C$-manifolds of constant curvature.

Keywords: Immersion, Spinors, Clifford Algebras

1 Introduction

A century long classical problem on Differential Geometry is the study of isometric immersions of riemannian manifolds. Classically, this problem is studied using generalized forms of the Gauss-Codazzi equations. But in the special case of riemann surfaces there is the approach of the Weierstrass map using complex analysis. Recently, this problem gained a new impetus when Friedrich, [5], discovered that the Weierstrass map can be described using spinors.

Since than, numerous works appeared, [13, 8, 9, 10, 2, 3, 4], showing how Dirac equations, spinors, Gauss-Codazzi equations and isometric immersions are related. In particular, Bayard et al, [4], showed how to generalize de concept of the spinorial Weierstrass map to arbitrary dimensional spin manifolds.

On [12], we argued that in certain contexts, particularly for complex manifolds, the hypothesis of a Spin-structure is somewhat restrictive, being more natural to consider Spin$^C$-structures, and showed how the Weierstrass map constructed by Bayard can be adapted to this case.

On [4], spinor techniques are also used to investigate the more general problem of isometric immersions of manifolds on manifolds of constant curvature. As usual, to do follow the spinor approach we must assume that the manifolds involved carry a Spin-structure and, again, there are some cases, like complex manifolds, where this assumption is more restrictive than the assumption of a Spin$^C$-structure. In the present work we will consider Spinorial Representation of Submanifolds in $\mathbb{S}^n$. In particular we prove the following theorem:

Theorem 1. Let $M$ $p$-dimensional manifold, $E \to M$ a vector bundle of rank $q$, assume that $TM$ and $E$ are oriented and Spin$^C$. Suppose that $B : TM \times TM \to E$ is symmetric and bilinear. The following are equivalent:

1. There exist a section $\varphi \in \Gamma(S \sum^C)$ such that

$$\nabla^c_X \varphi = -\frac{1}{2} \sum_{i=1}^p e_i \cdot B(X, e_i) \cdot \varphi + \frac{1}{2} X \cdot \nu \cdot \varphi + \frac{1}{2} iA_l(X) \cdot \varphi, \forall X \in TM.$$

2. There exist an isometric immersion $F : M \to \mathbb{S}^n$ with normal bundle $E$ and second fundamental form $B$.

Furthermore, $F = \langle \nu \cdot \varphi, \varphi \rangle \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$.

2 Recalling some concepts

In the previous work, [12], we showed how to generalize de Weierstrass map obtained by Bayard et all to the case of Spin$^C$-manifolds, in particular every almost complex manifold. In this work we are interested in understanding if the Spin$^C$-hypothesis is also true the case of submanifolds of manifolds of constant curvature. Therefore, in this section we will recall some concepts already presented in [12].

2.1 Adapted structures

Let $H \to M$ be a hermitian vector bundle over $M$. A Spin$^C$-structure on $H$ is defined by the following double covering
where Spin$^C$ is the group defined by

$$\text{Spin}^C_n = \frac{\text{Spin}_n \times S^1}{\{(-1, -1)\}},$$

and $S^1 = U(1) \subset \mathbb{C}$ is understood as the unitary complex numbers. As usual, a Spin$^C$-structure can be viewed as a lift of the transition functions of $H$, $g_{ij}$, to the group Spin$^C$, $\tilde{g}_{ij}$, but now the transition functions are classes of pairs $\tilde{g}_{ij} = [(h_{ij}, z_{ij})]$, where $h_{ij} : U_i \cap U_j \rightarrow \text{Spin}_n$ and $z_{ij} : U_i \cap U_j \rightarrow S^1 = U(1)$.

The identity on Spin$^C$ is the class $\{(1, 1), (-1, -1)\}$. Because of this, neither $h_{ij}$ or $z_{ij}$ must satisfy the cocycle condition, only the class of the pair. But, $z_{ij}$ satisfies the cocycle condition and defines a complex line bundle $L$, associated with the $P_{S^1}$ principal bundle in the above diagram, called the determinant of the Spin$^C$-structure.

The description using transition functions is useful to make clear that Spin$^C$-structures are more general than Spin-structures. In fact, given a Spin-structure $P_{\text{Spin}}(H) \rightarrow P_{\text{SO}}(H)$ we immediately get a Spin$^C$-structure by considering $z_{ij} = 1$, in other words, by considering the trivial bundle as the determinant bundle of the structure. On the other hand [7], a Spin$^C$-structure produces a Spin-structure iff the determinant bundle has a square root, that is, the functions $z_{ij}$ satisfies the cocycle condition.

Another way where Spin$^C$-structures are natural is when we consider an almost complex manifold $(M, g, J)$. In this case the tangent bundle can be viewed as an $U(n)$ bundle, and the natural inclusion $U(n) \rightarrow SO(2n)$ produces a canonical Spin$^C$-structure on the tangent bundle [6] [14]. For this canonical structure the determinant bundle is identified with $\wedge^{0,n} M$ and the spinor bundle constructed using an irreducible complex representation of $\mathbb{C}l(n)$ is isomorphic with $\wedge^{0,k} M$. So, various structures on spinors can be described using know structures of $M$.

Unlike the usual case for Spin-structures, a metric connection on $H$ is not enough to produce a connection on $P_{\text{Spin}^C}(H)$, for this, we also need a connection on the determinant bundle of the structure to get a connection on $P_{\text{SO}}(H) \times P_{S^1}(H)$ and be able to lift this connection to $P_{\text{Spin}^C}(H)$.

To understand the problem of immersions using the Dirac equation in the case of Spin$^C$-structures, and spinors associated to this structure, we need to understand adapted Spin$^C$-structures on submanifolds. The difference to the standard Spin case is that we need to keep track of the determinant bundle. Using the ideas of [11], we can describe the adapted structure.

Consider a Spin$^C$- $n$-dimensional manifold $Q$ and a isometrically immersed $p$-dimensional Spin$^C$ submanifold $M \rightarrow Q$. Let

$$\begin{align*}
P_{\text{Spin}^C_m}(Q) \xrightarrow{\Lambda^Q} P_{\text{SO}}_m(Q) \times P_{S^1}(Q) \\
P_{\text{Spin}^C_m}(Q) \big|_M \xrightarrow{\Lambda^Q} P_{\text{SO}}_m(Q) \big|_M \times P_{S^1}(Q) \\
P_{\text{Spin}^C_p}(M) \xrightarrow{\Lambda^M} P_{\text{SO}}_p(M) \times P_{S^1}(M)
\end{align*}$$

be the corresponding Spin$^C$-structures. And let the cocycles associated to this structures be, respectively, $\tilde{g}_{\alpha\beta}$, $\tilde{g}_{\alpha\beta}|_M$ and $\tilde{g}^1_{\alpha\beta}$, $\tilde{g}^2_{\alpha\beta}$. If we define the functions $\tilde{g}^2_{\alpha\beta}$ by

$$\tilde{g}^1_{\alpha\beta} \tilde{g}^2_{\alpha\beta} = \tilde{g}_{\alpha\beta}|_M$$

it is easy to see, using an adapted frame, that the two sets of functions $\tilde{g}^1_{\alpha\beta}$ and $\tilde{g}^2_{\alpha\beta}$ commutes. This implies that $\tilde{g}^2_{\alpha\beta}$ satisfies the cocycle condition, because both $\tilde{g}_{\alpha\beta}$ and $\tilde{g}^1_{\alpha\beta}$ satisfies. The cocycles $\tilde{g}^2_{\alpha\beta}$ are exactly the Spin$^C$-structure for the normal bundle $\nu(M)$. With this construction, if $L$, $L_1$ and $L_2$ denotes, respectively, the determinant bundle of the Spin$^C$-structure of $Q$, $M$ and $\nu(M)$ we have the relation

$$L = L_1 \otimes L_2$$

Knowing that $\nu(M)$ has a natural Spin$^C$-structure we can use the left regular representation of $\mathbb{C}l(n)$ on itself to construct
the following Spin$^C$-Clifford bundle (this bundles will act as spinor bundles)

\[
\Sigma^C Q := P_{Spin^C_n}(Q) \times_{\rho_n} Cl_n,
\]

\[
\Sigma^C Q|_M := P_{Spin^C_n}(Q)|_M \times_{\rho_n} Cl_n,
\]

\[
\Sigma^C M := P_{Spin^C_n}(M) \times_{\rho_p} Cl_p,
\]

\[
\Sigma^C \nu(M) := P_{Spin^C_n}(\nu(M)) \times_{\rho_q} Cl_q.
\]

Using the isomorphism $\mathbb{C}l_p \otimes \mathbb{C}l_q \simeq \mathbb{C}l_n$ and standard arguments, $[1]$, we get the relation

\[
\Sigma^C Q|_M \simeq \Sigma^C M \hat{\otimes} \Sigma^C \nu(M) := \Sigma^C.
\]

Let $\nabla^{\Sigma^C Q}$, $\nabla^{\Sigma^C M}$ and $\nabla^{\Sigma^C \nu}$ be the connection on $\Sigma^C Q$, $\Sigma^C M$ and $\Sigma^C \nu(M)$ respectively, induced by the Levi-Civita connections of $P_{SO_n}(Q)$, $P_{SO_p}(M)$, and $P_{SO_q}(\nu)$. We denote the connection on $\Sigma^C$ by

\[
\nabla^{\Sigma^C} = \nabla^{\Sigma^C M \otimes \Sigma^C \nu} := \nabla^{\Sigma^C M} \otimes Id + Id \otimes \nabla^{\Sigma^C \nu}.
\]

The connections on these bundle are linked by the following Gauss formula:

\[
\nabla^{\Sigma^C}_X \varphi = \nabla^X \varphi + \frac{1}{2} \sum_{i=1}^p e_i \cdot B(e_i, X) \cdot \varphi, \tag{1}
\]

where $B : TM \times TM \rightarrow \nu(M)$ is the second fundamental form and $\{e_1 \cdots e_p\}$ is a local orthonormal frame of $TM$. Here “$\hat{\otimes}$” is the Clifford multiplication on $\Sigma^C Q$.

Note that if we have a parallel spinor $\varphi$ in $\Sigma^C Q$, for example if $Q = \mathbb{R}^n$, then Eq. (1) implies the following generalized Killing equation

\[
\nabla^{\Sigma^C}_X \varphi = -\frac{1}{2} \sum_{i=1}^p e_i \cdot B(e_i, X) \cdot \varphi. \tag{2}
\]

### 2.2 A $\mathbb{C}l_n$-valued inner product

To obtaining an immersion using spinors that satisfies certain equations, we need the following $\mathbb{C}l_n$-valued inner product

\[
\tau : \mathbb{C}l_n \rightarrow \mathbb{C}l_n
\]

\[
\tau(a e_{i_1} e_{i_2} \cdots e_{i_k}) := (-1)^k \overline{\alpha} e_{i_k} \cdots e_{i_2} e_{i_1},
\]

\[
\langle \langle \cdot, \cdot \rangle \rangle : \mathbb{C}l_n \times \mathbb{C}l_n \rightarrow \mathbb{C}l_n
\]

\[
\langle \langle \xi_1, \xi_2 \rangle \rangle \mapsto \langle \langle \xi_1, \xi_2 \rangle \rangle = \tau(\xi_2)\xi_1.
\]

\[
\langle \langle (g \otimes s)\xi_1, (g \otimes s)\xi_2 \rangle \rangle = s \tau(\xi_2)\tau(g)\xi_1 = \tau(\xi_2)\xi_1 = \langle \langle \xi_1, \xi_2 \rangle \rangle,
\]

\[
ge \otimes s \in Spin^C_n \subset \mathbb{C}l_n,
\]

so the product is well defined on the Spin$^C$-Clifford bundles, i.e., Eq. (2.2) induces a $\mathbb{C}l_n$-valued map:

\[
\sum^{\Sigma^C} Q \times \sum^{\Sigma^C} Q \rightarrow \mathbb{C}l_n
\]

\[
(\varphi_1, \varphi_2) = ([p, [\varphi_1]], [p, [\varphi_2]]) \mapsto \langle \langle [\varphi_1], [\varphi_2] \rangle \rangle = \tau([\varphi_2])[\varphi_1],
\]

where $[\varphi_1], [\varphi_2]$ are the representative of $\varphi_1, \varphi_2$ in the $Spin^C_n$ frame $p \in P_{Spin^C_n}$.

**Lemma 2.** The connection $\nabla^{\Sigma^C} Q$ is compatible with the product $\langle \langle \cdot, \cdot \rangle \rangle$.

**Proof.** Fix $s = (e_1, \ldots, e_n) : U \subset M \subset Q \rightarrow P_{SO_n}$ a local section of the frame bundle, $l : U \subset M \subset Q \rightarrow P_{S^1}$ a local section of the associated $S^1$-principal bundle, $w^Q : T(P_{SO_n}) \rightarrow so(n)$ is the Levi-Civita connection of $P_{SO(n)}$ and $iA : TP_{S^1} \rightarrow i\mathbb{R}$ is an arbitrary connection on $P_{S^1}$, denote by $w^Q(ds(X)) = (w_{ij}(X)) \in so(n)$, $iA(dl(X)) = iA(X)$. 
If $\psi = [p, [\psi]]$ and $\psi' = [p, [\psi']]$ are sections of $\sum^C Q$ we have:

$$
\nabla_X^{\sum^C Q} \psi = \left[ p, X([\psi]) + \frac{1}{2} \sum_{i<j} w_{ij}(X)e_i e_j \cdot [\psi] + \frac{1}{2} iA^l(X)[\psi] \right],
$$

$$
\left\langle \left\langle \nabla_X^{\sum^C Q} \psi, \psi' \right\rangle \right\rangle = \bar{\psi'} \left( X([\psi]) + \frac{1}{2} \sum_{i<j} w_{ij}(X)e_i e_j \cdot [\psi] + \frac{1}{2} iA^l(X)[\psi] \right),
$$

$$
\left\langle \left\langle \psi, \nabla_X^{\sum^C Q} \psi' \right\rangle \right\rangle = \left( X([\psi']) + \frac{1}{2} \sum_{i<j} w_{ij}(X)e_i e_j \cdot [\psi'] + \frac{1}{2} iA^l(X)[\psi'] \right)[\psi] = \left( X([\psi']) + \frac{1}{2} \sum_{i<j} w_{ij}(X)e_i e_j - \frac{1}{2} iA^l(X)[\psi'] \right)[\psi],
$$

then

$$
\left\langle \left\langle \nabla_X^{\sum^C Q} \psi, \psi' \right\rangle \right\rangle + \left\langle \left\langle \psi, \nabla_X^{\sum^C Q} \psi' \right\rangle \right\rangle = \left[ \bar{\psi'} [X(\xi)] + X([\bar{\psi'}]) \right][\psi],
$$

$$
X(\langle \psi, \psi' \rangle) = X(\bar{\xi} \xi) = X(\bar{\xi}) \xi + \bar{\xi} X(\xi).
$$

\[\square\]

**Lemma 3.** The map $\langle \cdot, \cdot \rangle : \sum^C Q \times \sum^C Q \to \mathbb{C}l_n$ satisfies:

1. $\langle X \cdot \psi, \varphi \rangle = -\langle \langle \psi, X \cdot \varphi \rangle, \psi, \varphi \in \sum^C Q, \ X \in TQ.$
2. $\tau \langle \langle \psi, \varphi \rangle \rangle = \langle \langle \varphi, \psi \rangle \rangle$, $\psi, \varphi \in \sum^C Q$

**Proof.** This is an easy calculation:

1. $\langle X \cdot \psi, \varphi \rangle = \tau[\varphi][X \cdot \psi] = \tau[\varphi][X][\psi] = -\tau[\varphi][X][\psi] = \langle \langle \psi, X \cdot \varphi \rangle \rangle$
2. $\tau \langle \langle \psi, \varphi \rangle \rangle = \tau(\tau[\varphi][\psi]) = \tau[\varphi][\psi] = \langle \langle \varphi, \psi \rangle \rangle$.

\[\square\]

Note the same idea, product and properties are valid for the bundles $\sum^C Q, \sum^C M, \sum^C \nu(M), \sum^C M \hat{\otimes} \sum^C \nu(M)$.

### 3 Spinorial Representation of Submanifolds in $\mathbb{S}^n$

#### 3.1 Adapted $Spin^C$ groups

Fix $n = p + q$ and consider the decomposition

$$
\mathbb{R}^p \oplus \mathbb{R}^q = \mathbb{R}^n \hookrightarrow \mathbb{R}^n \oplus \mathbb{R} = \mathbb{R}^{n+1}.
$$

We have natural inclusions

$$
SO(p) \times SO(q) \rightarrow SO(n) \rightarrow SO(n+1),
$$

$$
Spin_p^C \times Spin_p^C \overset{i_1}{\rightarrow} Spin_n^C \overset{i_2}{\rightarrow} Spin_{(n+1)}^C \subset Cl_{(n+1)}.
$$

Since $Spin_{(n+1)}^C$ acts naturally on $Cl_{(n+1)}$ by left multiplication and by adjoint representation

$$
\begin{align*}
I : Spin_{(n+1)}^C & \rightarrow End_C Cl_{(n+1)} \\
Ad_{(n+1)} : Spin_{(n+1)}^C & \rightarrow End_C Cl_{(n+1)}
\end{align*}
$$

we can define the following representations

$$
\begin{align*}
\rho_1 := I \circ i_2 : Spin_n^C & \rightarrow End_C Cl_{(n+1)} \\
\rho := I \circ i_2 \circ i_1 : Spin_p^C \times Spin_p^C & \rightarrow End_C Cl_{(n+1)}, \\
Ad := Ad_{(n+1)} \circ i_2 \circ i_1 : Spin_p^C \times Spin_p^C & \rightarrow End_C Cl_{(n+1)}.
\end{align*}
$$
3.2 Adapted Spinor Bundles

Since \( \mathbb{R}^{n+1} \) is oriented and \( \text{Spin}^C \) it induces a canonical \( \text{Spin}^C \) structure on \( S^n \). Denote by \( P_{SO(n+1)}(\mathbb{R}^{n+1}) \) the orthonormal frame bundle of \( \mathbb{T} \mathbb{R}^{n+1} \) and by \( (P_{SO(n+1)}(\mathbb{R}^{n+1}))_{|S^n} \) the adapted orthonormal frame bundle of the isometric immersion \( S^n \to \mathbb{R}^{n+1} \). The respective \( \text{Spin}^C \) structures are expressed by:

\[
\Lambda^C : P_{\text{Spin}^C_{n+1}}(\mathbb{R}^{n+1}) \to P_{SO(n+1)}(\mathbb{R}^{n+1}) \times P_{S^1}(\mathbb{R}^{n+1}),
\]

\[
\Lambda^C_{|S^n} : \left( P_{\text{Spin}^C_{n+1}}(\mathbb{R}^{n+1}) \right)_{|S^n} \to \left( P_{SO(n+1)}(\mathbb{R}^{n+1}) \right)_{|S^n} \times \left( P_{S^1}(\mathbb{R}^{n+1}) \right)_{|S^n}.
\]

Let \( M \) a \( p \)-dimensional manifold, \( E \to M \) a real vector bundle of rank \( q \), assume that \( TM \) and \( E \) are oriented and \( \text{Spin}^C \). Denote by \( P_{SO(p)}(M) \) the frame bundle of \( TM \) and by \( P_{SO(q)}(E) \) the frame bundle of \( E \). The respective \( \text{Spin}^C \) structures are defined as

\[
\Lambda^{1C} : P_{\text{Spin}^C_p}(M) \to P_{SO(p)}(M) \times P_{S^1}(M),
\]

\[
\Lambda^{2C} : P_{\text{Spin}^C_q}(E) \to P_{SO(q)}(E) \times P_{S^1}(E).
\]

Finally we are able to define the following spinor bundles:

\[
\sum^C \mathbb{R}^{n+1} := \left( P_{\text{Spin}^C_{n+1}}(\mathbb{R}^{n+1}) \right) \times \mathbb{Cl}(n+1),
\]

\[
\sum^C S^n := \left( P_{\text{Spin}^C_{n+1}}(\mathbb{R}^{n+1}) \right)_{|S^n} \times \rho \mathbb{Cl}(n+1),
\]

\[
\sum^C := \left( P_{\text{Spin}^C_p} \times_M P_{\text{Spin}^C_q} \right) \times \rho \mathbb{Cl}(n+1),
\]

\[
S \sum^C := \left( P_{\text{Spin}^C_p} \times_M P_{\text{Spin}^C_q} \right) \times \rho \text{Spin}^C_{n+1}.
\]

In what follows we define as \( \nu \) the unit vector field into adapted tangent Bundle

\[
T^C := TM \oplus E \oplus \nu := \left( P_{\text{Spin}^C_p} \times_M P_{\text{Spin}^C_q} \right) \times \text{Ad} \mathbb{R}^{n+1}
\]
as the one that in any spinor frame \( p \in \left( P_{\text{Spin}^C_p} \times_M P_{\text{Spin}^C_q} \right) \) is written as

\[
\nu = [p, f_{n+1}],
\]

where \( f_{n+1} \) is the constant unit vector of a basis \( \{f_1, \cdots f_n\} \cup \{f_{n+1}\} \) of decomposition mentioned before \( \mathbb{R}^n \oplus \mathbb{R} \).

The next map will be important to us latter

\[
\xi : TM \oplus E \oplus \nu \to \mathbb{Cl}_{n+1}
\]

\[
\xi(X) := \langle \langle X \cdot \varphi, \varphi \rangle \rangle.
\]

3.2.1 Connection on \( P_{S^1} \) Bundle

We can define the bundle \( P_{S^1} \) as the one with transition functions defined by product of transition functions of \( P_{S^1}(M) \) and \( P_{S^1}(E) \). It is not difficult to see that there is a canonical bundle morphism:

\[
\Phi : P_{S^1}(M) \times_M P_{S^1}(E) \to P_{S^1}
\]
such that, in any local trivialization, the following diagram commute:

\[
\begin{tikzcd}
P_{S^1}(M) \times_M P_{S^1}(E) \arrow[r, \Phi] \arrow[d]\ & P_{S^1} \\
U_\alpha \times S^1 \times S^1 \arrow[r, \phi_\alpha] \ & U_\alpha \times S^1
\end{tikzcd}
\]

where \( \phi_\alpha(x, r, s) = (x, rs), x \in U_\alpha, r, s \in S^1 \).

Denote arbitrary connections on \( P_{S^1}(M) \) and \( P_{S^1}(E) \) by

\[
iA^1 : TP_{S^1}(M) \to \mathbb{i}\mathbb{R}, \ iA^2 : TP_{S^1}(E) \to \mathbb{i}\mathbb{R}.
\]
Express local sections by

\[ s = (e_1, \cdots, e_p) : U \to P_{SO_p}(M), \]
\[ l_1 : U \to P_{S^1}(M), \quad l_2 : U \to P_{S^1}(E), \quad l = \Phi(l_1, l_2) : U \to P_{S^1}. \]

Now \( iA : TP_{S^1} \to i\mathbb{R} \) is the connection defined by

\[ iA(d\Phi(l_1, l_2)) = iA_1(dl_1) + iA_2(dl_2), \]
\[ iA(dl(X)) := iA^l(X), \quad X \in TM. \]

## 3.3 Main Theorem

Established the notation we have the following:

**Theorem 4.** Let \( M \) be a \( p \)-dimensional manifold, \( E \to M \) a vector bundle of rank \( q \), assume that \( TM \) and \( E \) are oriented and \( Spin^C \). Suppose that \( B : TM \times TM \to E \) is symmetric and bilinear. The following are equivalent:

1. There exist a section \( \varphi \in \Gamma(S \sum^C) \) such that

\[ \nabla_X^C\varphi = -\frac{1}{2} \sum_{i=1}^p e_i \cdot B(X, e_i) \cdot \varphi + \frac{1}{2} X \cdot \nu \cdot \varphi + \frac{1}{2} iA^l(X) \cdot \varphi, \quad \forall X \in TM. \]

2. There exist an isometric immersion \( F : M \to \mathbb{S}^n \) with normal bundle \( E \) and second fundamental form \( B \).

Furthermore, \( F = (\langle \nu \cdot \varphi, \varphi \rangle) \in \mathbb{S}^n \subset \mathbb{R}^{n+1} \).

**Proof.** 2) \( \Rightarrow \) 1) Since \( \mathbb{R}^{n+1} \) is contractible there exists a global section

\[ s : \mathbb{R}^{n+1} \to P_{Spin^C_{n+1}}(\mathbb{R}^{n+1}), \]

with corresponding parallel orthonormal basis

\[ h = (E_1, \cdots, E_{n+1}) : \mathbb{R}^{n+1} \to P_{SO_{n+1}}(\mathbb{R}^{n+1}), \]
\[ l : \mathbb{R}^{n+1} \to P_{S^1}(\mathbb{R}^{n+1}), \quad \Lambda^{n+1}_{n+1}(s) = (h, l). \]

Fix the constant \( 1 = [\varphi] \in Spin^C_{n+1} \subset Cl_{n+1} \) and define the spinor field

\[ \varphi = [s, [\varphi]] \in \sum^C \mathbb{R}^{n+1} := \left( P_{Spin^C_{n+1}}(\mathbb{R}^{n+1}) \right) \times Cl_{n+1}. \]

Representing the connection forms by

\[ w^{\mathbb{R}^{n+1}}(dh(X)) = (w^{h}_{ij}(X)) \in so(n + 1), \quad iA(dl(X)) = iA^l(X) \in i\mathbb{R}, \]

we have

\[ \nabla_X^{C \mathbb{R}^{n+1}} \varphi = \left[ s, X([\varphi]) + \left\{ \frac{1}{2} \sum_{i<j} w^{h}_{ij}(X) E_i E_j + \frac{1}{2} iA^l(X) \right\} \cdot [\varphi] \right] = \left[ s, \frac{1}{2} iA^l(X) \cdot [\varphi] \right], \]
\[ = \frac{1}{2} iA^l(X) \cdot \varphi. \]

If \( \nu \) is the normal vector field of the immersion \( \mathbb{S}^n \subset \mathbb{R}^{n+1} \), consider a local adapted orthonormal frame

\[ \{ f_1, \cdots, f_n, \nu \} : U \to P_{SO(n+1)}(\mathbb{R}^{n+1}) \bigg|_{\mathbb{S}^n}. \]

Denote by \( B^{\mathbb{S}^n} : T\mathbb{S}^n \times \mathbb{S}^n \to \nu(\mathbb{S}^n) \) the second fundamental form of the immersion \( \mathbb{S}^n \subset \mathbb{R}^{n+1} \).
Restricting \( \varphi \) in Eq.(3) to \( \sum^C S^n \) and applying the gauss formula Eq.(1) we obtain

\[
\nabla_X^{\sum^C S^n} \varphi - \nabla_X^{S^n} \varphi = \frac{1}{2} \sum_{i=1}^{n} f_i \cdot B^{s^n}(X, f_i) \cdot \varphi
\]

\[
\frac{1}{2} i A^f(X) \cdot \varphi - \nabla_X^{S^n} \varphi = -\frac{1}{2} X \cdot \nu \cdot \varphi
\]

\[
\nabla_X^{S^n} \varphi = \frac{1}{2} X \cdot \nu \cdot \varphi + \frac{1}{2} i A^f(X) \cdot \varphi.
\]

Furthermore, now we can restrict \( \varphi \) in Eq.(3) to \( S \sum^C \) and apply again the gauss formula Eq.(1):

\[
\nabla_X^{\sum^C S^n} \varphi - \nabla_X^{S^n} \varphi = \frac{1}{2} \sum_{i=1}^{p} e_i \cdot B(X, e_i) \cdot \varphi
\]

\[
\frac{1}{2} X \cdot \nu \cdot \varphi + \frac{1}{2} i A^f(X) \cdot \varphi - \nabla_X^{S^n} \varphi = \frac{1}{2} \sum_{i=1}^{p} e_i \cdot B(X, e_i) \cdot \varphi
\]

Then we prove the first part of the theorem

\[
\nabla_X^{S^n} \varphi = -\frac{1}{2} \sum_{i=1}^{p} e_i \cdot B(X, e_i) \cdot \varphi + \frac{1}{2} X \cdot \nu \cdot \varphi + \frac{1}{2} i A^f(X) \cdot \varphi.
\]

1) \( \Rightarrow \) 2) The idea here is to prove that \( F = \langle (\nu \cdot \varphi, \varphi) \rangle \in \mathbb{S}^n \subset \mathbb{R}^n \) gives us an immersion preserving the metric, the second fundamental form and the normal connection. For this purpose, we will present the following lemmas:

**Lemma 5.** Let \( \varphi = [p, \varphi] \in \Gamma(S \sum^C) \) a section satisfying Eq.(1). Then:

1. \( F : M \to \mathbb{S}^n \subset \mathbb{R}^{n+1} \).

2. \( dF(X) = \xi(X) = \langle (X \cdot \varphi, \varphi) \rangle \), \( \forall X \in TM \).

**Proof.** 1. This follow from

\[
F = \langle (\nu \cdot \varphi, \varphi) \rangle = \tau[\varphi] f_{n+1}[\varphi] = Ad(\tau[\varphi])(f_{n+1}) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}.
\]

2. First note that, since

\[
\nu = [p, f_{n+1}] \in \Gamma \left( \left(P^{\text{Spin}_p} \times_M P^{\text{Spin}_n} \right) \times_{Ad} \mathbb{R}^{n+1} \right)
\]

we have

\[
\nabla_X^{T\mathbb{S}^n} \nu = [p, X(f_{n+1}) + Ad_*(\omega^C(dp(X)))(f_{n+1})] = 0,
\]

since \( f_{n+1} \) is constant and

\[
Ad(p)(f_{n+1}) = f_{n+1}, \quad \forall p \in P^{\text{Spin}_p} \times_M P^{\text{Spin}_n}.
\]

\[
\omega^C : T \left(P^{\text{Spin}_n} \times_M P^{\text{Spin}_n} \right) \to \text{so}(n) \oplus i\mathbb{R} \subset \mathcal{Cl}_n.
\]

Finally:

\[
dF(X) = X(F) = X \left( \langle \nu \cdot \varphi, \varphi \rangle \right)
\]

\[
= \left\langle \langle \nu \cdot \nabla_X^{S^n} \varphi, \varphi \rangle \right\rangle + \left\langle \langle \nu \cdot \varphi, \nabla_X^{S^n} \varphi \rangle \right\rangle
\]

\[
= (Id - \tau) \langle \nu \cdot \nabla_X \varphi, \varphi \rangle
\]

\[
= -\frac{1}{2} (Id - \tau) \sum_{i=1}^{p} \langle \nu \cdot e_i \cdot B(X, e_i) \cdot \varphi, \varphi \rangle
\]

\[
+ \frac{1}{2} (Id - \tau) \langle \nu \cdot X \cdot \nu \cdot \varphi, \varphi \rangle
\]

\[
+ \frac{1}{2} (Id - \tau) \langle i A^f(X) \nu \cdot \varphi, \varphi \rangle.
\]
Using that $\nu, e_i, B(X, e_i)$ are mutually orthogonal we get

$$\langle\langle \nu \cdot e_i \cdot B(X, e_i) \cdot \varphi, \varphi \rangle \rangle = \tau \left( \langle\langle \nu \cdot e_i \cdot B(X, e_i) \cdot \varphi, \varphi \rangle \rangle \right)$$

$$\langle\langle X \cdot \varphi, \varphi \rangle \rangle = -\tau \left( \langle\langle X \cdot \varphi, \varphi \rangle \rangle \right)$$

$$\langle\langle iA'(X)\nu \cdot \varphi, \varphi \rangle \rangle = \tau \left( \langle\langle iA'(X)\nu \cdot \varphi, \varphi \rangle \rangle \right)$$

Then:

$$dF(X) = \langle\langle X \cdot \varphi, \varphi \rangle \rangle, \ \forall X \in TM.$$

\[\square\]

**Lemma 6.** With notations above the following statements are valid

1. The map $F : M \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$, is an isometry.

2. The map

$$\Phi_E : E \rightarrow F(M) \times \mathbb{R}^{n+1}$$

$$X \in E_m \mapsto (F(m), \xi(X))$$

is an isometry between $E$ and the normal bundle of $F(M)$ into $\mathbb{S}^n$, preserving connections and second fundamental forms.

**Proof.**

1. Let $X, Y \in \Gamma(TM \oplus E \oplus \nu \mathbb{R})$, consequently

$$\langle\langle \xi(X), \xi(Y) \rangle \rangle = -\frac{1}{2} \langle\langle \xi(X)\xi(Y) + \xi(Y)\xi(X) \rangle \rangle$$

$$= -\frac{1}{2} \left( \tau[\varphi][X][\varphi][\varphi][Y][\varphi] + \tau[\varphi][Y][\varphi][\varphi][X][\varphi] \right)$$

$$= -\frac{1}{2} \tau[\varphi] ([X][Y] + [Y][X])[\varphi] = \tau[\varphi] ((X,Y))[\varphi]$$

$$= \langle\langle X,Y \rangle \rangle \tau[\varphi][\varphi] = \langle\langle X,Y \rangle \rangle.$$

This implies that $F$ is an isometry, and that $\Phi_E$ is a bundle map between $E$ and the normal bundle of $F(M)$ into $\mathbb{S}^n$ which preserves the metrics of the fibers. Note that $(F(m), \xi(\nu))$ is orthogonal to $\mathbb{S}^n$.

2. Denote by $B_F$ and $\nabla_{iF}$ the second fundamental form and the normal connection of the immersion $F$. We want to show that:

$$i) \xi(B(X,Y)) = B_F(\xi(X), \xi(Y)),$$

$$ii) \xi(\nabla_X \eta) = (\nabla_{iF}^\xi(\xi)\xi)(\eta),$$

for all $X, Y \in \Gamma(TM)$ and $\eta \in \Gamma(E)$.

$i)$ First note that:

$$B^F(X,Y) := \{\nabla^F_{\xi(X)}(\xi(Y))\} = \{X(\xi(Y))\} \perp,$$

where the superscript $\perp$ means that we consider the component of the vector which is normal to the immersion and tangent to $\mathbb{S}^n$.

Suppose that in $x_0 \in M$, $\nabla^M X = \nabla^M Y = 0$, to simplify write $\nabla^E_X \varphi = \nabla_X \varphi$ and $\nabla^M X = \nabla X$,

$$X(\xi(Y)) = \langle\langle Y \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle\langle Y \cdot \varphi, \nabla_X \varphi \rangle \rangle$$

$$= (id - \tau) \langle\langle Y \cdot \varphi, \nabla_X \varphi \rangle \rangle$$

$$= (id - \tau) \left( \varphi, \frac{1}{2} \sum_{j=1}^p Y \cdot e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} Y \cdot X \cdot \nu \cdot \varphi - \frac{1}{2} iA'(X)Y \cdot \varphi \right)$$

```
\[ (id - \tau) \left\langle \left[ \sum_{j=1}^{p} \sum_{k=1}^{p} y^j e_k \cdot e_j \cdot B(X, e_j) - \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} y^j x^j e_k \cdot e_j \cdot \nu \cdot \varphi - iA'(X)Y \right], \varphi \right\rangle \]

\[ = (id - \tau) \left\langle \left[ - \sum_{j=1}^{p} y^j \cdot B(X, e_j) + \sum_{j=1}^{p} y^j x^j \nu + \sum_{j=1}^{p} \sum_{k=1, k \neq j}^{p} y^j e_k \cdot e_j \cdot (B(X, e_j) - x^j \nu) - iA'(X)Y \right], \varphi \right\rangle \]

\[ = (id - \tau) \left\langle \left[ \frac{1}{2} (-B(X, Y) + D) \cdot \varphi \right] \right\rangle + \langle Y, X \rangle \langle \langle \varphi, \nu \cdot \varphi \rangle \rangle, \]

where

\[ D = \sum_{j=1}^{p} \sum_{k=1, k \neq j}^{p} y^j e_k \cdot e_j \cdot (B(X, e_j) - x^j \nu) - iA'(X)Y \]

\[ \tau[D] = [D]. \]

Consequently

\[ X(\xi(Y)) = \frac{1}{2} (id - \tau) \left\langle \left[ \sum_{j=1}^{p} \sum_{k=1}^{p} y^j e_k \cdot e_j \cdot B(X, e_j) - \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} y^j x^j e_k \cdot e_j \cdot \nu \cdot \varphi - iA'(X)Y \right], \varphi \right\rangle \]

\[ = -\tau[\varphi]\tau[B(X, Y)][\varphi] + \langle Y, X \rangle \langle \langle \varphi, \nu \cdot \varphi \rangle \rangle \]

\[ = \langle \langle \varphi, B(X, Y) \cdot \varphi \rangle \rangle + \langle Y, X \rangle \langle \langle \varphi, \nu \cdot \varphi \rangle \rangle \]

\[ = \xi(B(X, Y)) + \langle Y, X \rangle \xi(\nu). \]

Therefore we conclude

\[ B^F(\xi(X), \xi(Y)) := \{ \nabla^F_{\xi(X)} \xi(Y) \}^\perp = \{ X(\xi(Y)) \}^\perp \]

\[ = \{ \xi(B(X, Y)) + \langle Y, X \rangle \xi(\nu) \}^\perp. \]

here we used the fact that \( F \) is an isometry

\[ B(X, Y) \in E, \xi(B(X, Y)) \in TF(M)^\perp, \xi(\nu) \in \{ TF_S^n \}^\perp. \]

Then i) follows.

ii) First note that

\[ \nabla^F_{\xi(X)} \xi(\eta) = \{ X(\xi(\eta)) \}^\perp = \{ X \langle \langle \eta \cdot \varphi, \varphi \rangle \rangle \}^\perp \]

\[ = \langle \langle \nabla_X \eta \cdot \varphi, \varphi \rangle \rangle^\perp + \langle \langle \eta \cdot \nabla_X \varphi, \varphi \rangle \rangle^\perp. \]

I will show that:

\[ \langle \langle \eta \cdot \nabla_X \varphi, \varphi \rangle \rangle^\perp + \langle \langle \eta \cdot \varphi, \nabla_X \varphi \rangle \rangle^\perp = 0. \]

In fact

\[ \langle \langle \eta \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle \eta \cdot \varphi, \nabla_X \varphi \rangle \rangle = (id - \tau) \langle \langle \eta \cdot \nabla_X \varphi, \varphi \rangle \rangle \]

\[ = (-id + \tau) \left\langle \left[ \frac{1}{2} \sum_{j=1}^{p} \eta \cdot e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} \sum_{j=1}^{p} \eta \cdot X \cdot \nu \cdot \varphi - \frac{1}{2} \sum_{j=1}^{p} \eta \cdot \xi A'(X) \eta \cdot \varphi \right], \varphi \right\rangle \]

\[ = (-id + \tau) \left\langle \left[ \frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{q} a^s \eta \cdot \xi e_j \cdot f_s \cdot f_k - \frac{1}{2} \eta \cdot X \cdot \nu - \frac{1}{2} \sum_{j=1}^{p} \eta \cdot \xi A'(X) \eta \right], \varphi \right\rangle \]

\[ = (-id + \tau) \left\langle \left[ \frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{q} a^s \eta \cdot \xi e_j \cdot f_s \cdot f_k - \frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{q} a^s \eta \cdot \xi e_j \cdot f_s \cdot f_k - \frac{1}{2} \sum_{j=1}^{p} \eta \cdot X \cdot \nu - \frac{1}{2} \sum_{j=1}^{p} \eta \cdot \xi A'(X) \eta \right], \varphi \right\rangle , \]

from what

\[ \langle \langle \eta \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle \eta \cdot \varphi, \nabla_X \varphi \rangle \rangle \]

\[ = 2\tau[\varphi] \left[ \sum_{j=1}^{p} \sum_{s=1}^{q} a^s \xi e_j \cdot f_s \cdot f_k \right] \varphi \]

\[ = \tau[\varphi] \xi(\cdot)[\varphi] = \xi(\cdot) \in TF(M) \]

\[ \Rightarrow \langle \langle \eta \cdot \nabla_X \varphi, \varphi \rangle \rangle^\perp + \langle \langle \eta \cdot \varphi, \nabla_X \varphi \rangle \rangle^\perp = 0. \]
In conclusion
\[ \nabla_{\xi(X)} \xi(\eta) = \langle (\nabla_X \eta \cdot \varphi, \varphi) \rangle = (\xi(\nabla_X \eta)) = \xi(\nabla_X \eta). \]

At the end ii) follows.

With these Lemmas the theorem is proved.

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