Comparison of Speeds of Convergence in Riesz-Type Families of Summability Methods. II*

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Abstract. Certain summability methods for functions and sequences are compared by their speeds of convergence. The authors are extending their results published in paper [9] for Riesz-type families \( \{A_\alpha\} (\alpha > \alpha_0) \) of summability methods \( A_\alpha \). Note that a typical Riesz-type family is the family formed by Riesz methods \( A_\alpha = (R, \alpha), \alpha > 0 \). In [9] the comparative estimates for speeds of convergence for two methods \( A_\gamma \) and \( A_\beta \) in a Riesz-type family \( \{A_\alpha\} \) were proved on the base of an inclusion theorem. In the present paper these estimates are improved by comparing speeds of three methods \( A_\gamma \), \( A_\beta \) and \( A_\delta \) on the base of a Tauberian theorem. As a result, a Tauberian remainder theorem is proved. Numerical examples given in [9] are extended to the present paper as applications of the Tauberian remainder theorem proved here.

Keywords: speed of convergence, Tauberian remainder theorem, Riesz-type family of summability methods, Riesz methods, generalized integral Nörlund methods, Borel-type methods.

AMS Subject Classification: 40C10; 40E05; 40G05; 40G10.

1 Introduction and Basic Notions

We continue comparing speeds of convergence in Riesz-type families of summability methods started in paper [9]. In the mentioned paper any two methods in a Riesz-type family were compared by speed of convergence. In the present paper we improve our estimates comparing by speed of convergence any three methods in a Riesz-type family.

1.1. We begin our paper recalling the basic notions used in [9]. Let us consider functions \( x = x(u) \) defined for \( u \geq 0 \), bounded and Lebesgue-measurable on every finite interval \([0, u_0] \). Let us denote the set of all such functions by \( X \).
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Suppose that $A$ is a transformation of functions $x = x(u)$ (or, in particular, of sequences $x = (x_n)$) into functions $Ax = y = y(u) \in \mathbb{X}$. If the limit $\lim_{u \to \infty} y(u) = s$ exists then we say that $x = x(u)$ is convergent to $s$ with respect to the summability method $A$, and write $x(u) \to s(A)$. If $y = y(u)$ is bounded then we say that $x$ is bounded with respect to $A$, and write $x(u) = O(A)$. We denote by $\omega A$ the set of all these functions $x$, where the transformation $A$ is applied, and by $cA$ and $mA$ the set of all functions $x$ which are convergent and bounded with respect to the method $A$, respectively. The method $A$ is said to be regular if $\lim_{u \to \infty} x(u) = s$ implies $\lim_{u \to \infty} y(u) = s$ whenever $x \in \mathbb{X}$. Further we use the notation $c_0$ for the set of all functions $x \in \mathbb{X}$ having $\lim_{u \to \infty} x(u) = 0$.

One of the most common summability method for functions $x$ is an integral method $A$ is defined with the help of transformation

$$y(u) = \int_0^\infty a(u,v)x(v)\,dv,$$

where $a(u,v)$ is a certain function of two variables $u \geq 0$ and $v \geq 0$. We say also that the integral method $A$ is defined by the function $a(u,v)$. An example of an integral summability method is the generalized integral Nörlund method $(N, P(u), Q(u))$ defined with the help of transformation

$$y(u) = \frac{1}{R(u)} \int_0^u P(u-v)Q(v)x(v)\,dv \quad (u > 0),$$

where $P = P(u)$ and $Q = Q(u)$ are non-negative functions from $\mathbb{X}$ such that

$$R(u) = \int_0^u P(u-v)Q(v)\,dv \neq 0 \quad \text{for} \quad u > 0.$$

In particular, if $Q(u) = 1$ and $P(u) = u^{\alpha - 1}$ for $u > 0$ and $\alpha > 0$, we get the Riesz method $(R, \alpha)$.

For sequences $x = (x_n)$ we focus ourselves on certain semi-continuous summability methods $A$ defined by transformations

$$y(u) = \sum_{n=0}^{\infty} a_n(u)\,x_n \quad (u \geq 0),$$

where $a_n(u)$ ($n = 0, 1, \ldots$) are some functions from $\mathbb{X}$. An example of a semi-continuous method is the Borel method $B$ defined by the transformation

$$y(u) = \frac{1}{e^u} \sum_{n=0}^{\infty} \frac{u^n}{n!} x_n. \quad (1.1)$$

1.2. One of the basic notions in this paper is the "speed of convergence". We use here definitions based on the definitions for sequences (see [4] and [5]) and extended for functions in [8] and [12]. Let $\lambda = \lambda(u)$ be a positive function from $\mathbb{X}$ such that $\lambda(u) \to \infty$ as $u \to \infty$. It is said that a function $x = x(u)$ is convergent to $s$ with speed $\lambda$ (shortly: $\lambda$-convergent) if the finite
1.2. \( \alpha \) is a constant depending on \( \gamma \).

We use the notations \( c^{\lambda} \) and \( m^{\lambda} \) for the sets of all \( \lambda \)-convergent and \( \lambda \)-bounded functions \( x \), respectively. It is said that \( x \) is convergent or bounded with speed \( \lambda \) with respect to the summability method \( A \) if \( Ax \in c^{\lambda} \) or \( Ax \in m^{\lambda} \), respectively.

1.3. The main subject of the paper is a Riesz-type family of summability methods ([8, 13]). Let \( \{A_{\alpha}\} \) be a family of summability methods \( A_{\alpha} \) where \( \alpha > \alpha_{1} \) and which are defined by transformations of functions \( x = x(u) \in \omega A_{\alpha} \subset X \) into \( A_{\alpha}x = y_{\alpha} = y_{\alpha}(u) \in X \). Suppose that for any \( \beta > \gamma > \alpha_{1} \) we have

\[
\omega A_{\gamma} \subset \omega A_{\beta}. \tag{1.2}
\]

Definition 1. ([8], Definition 1; [13], Definition 2) A family \( \{A_{\alpha}\} \) \( (\alpha > \alpha_{1}) \) is said to be a Riesz-type family if for every \( \beta > \gamma > \alpha_{1} \) the relation (1.2) holds and the methods \( A_{\gamma} \) and \( A_{\beta} \) are connected through

\[
y_{\beta}(u) = \frac{M_{\gamma, \beta}}{r_{\beta}(u)} \int_{0}^{u} (u-v)^{\beta-\gamma-1} r_{\gamma}(v) y_{\gamma}(v) \, dv \quad (u > 0), \tag{1.3}
\]

\[
r_{\beta}(u) = M_{\gamma, \beta} \int_{0}^{u} (u-v)^{\beta-\gamma} y_{\gamma}(v) \, dv \quad (u > 0), \tag{1.4}
\]

where \( r_{\gamma} = r_{\gamma}(u) \) and \( r_{\beta} = r_{\beta}(u) \) are some positive functions from \( X \) and \( M_{\gamma, \beta} \) is a constant depending on \( \gamma \) and \( \beta \).

Example 1. Let \( \{A_{\alpha}\} \) be the family of generalized Nörlund methods \( A_{\alpha} = (N, p_{\alpha}(u), q(u)) \) \( (\alpha > \alpha_{0}) \) defined by positive functions \( p = p(u) \in X \) and \( q = q(u) \in X \) and a number \( \alpha_{0} \) such that

\[
r_{\alpha}(u) = \int_{0}^{u} p_{\alpha}(u-v) q(v) \, dv > 0 \quad (u > 0, \alpha > \alpha_{0}),
\]

where \( p_{\alpha}(u) = \int_{0}^{u} (u-v)^{\alpha-1} p(v) \, dv \). It is known that relations (1.3) together with (1.4) and (1.6) hold here for any \( \beta > \gamma > \alpha_{0} \) (see [14]), and thus this family is a Riesz-type family.

Example 2. Consider the Borel-type methods \( A_{\alpha} = (B, \alpha, q_{n}) \) (see [13]). Let \( (q_{n}) \) be a non-negative sequence such that the power series \( \sum q_{n} u^{n} \) has the radius of convergence \( R = \infty \) and \( q_{n} > 0 \) at least for one \( n \in \mathbb{N} \). Denote

\[
r_{\alpha}(u) = \sum_{n=1}^{\infty} \frac{n! q_{n} u^{n} + \alpha - 1}{\Gamma(n + \alpha)} \tag{1.5}
\]

and define the methods \( (B, \alpha, q_{n}) \) \( (\alpha > -1/2) \) for converging sequences \( x = (x_{n}) \) with the help of transformation

\[
y_{\alpha}(u) = \frac{1}{r_{\alpha}(u)} \sum_{n=1}^{\infty} \frac{n! q_{n} u^{n} + \alpha - 1}{\Gamma(n + \alpha)} x_{n} \quad (u > 0).
\]

The notation \( \alpha > \alpha_{1} \) means that we consider parameter values \( \alpha > \alpha_{1} \) or \( \alpha \geq \alpha_{1} \) with some fixed number \( \alpha_{1} \).

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The methods \( A_\alpha = (B, \alpha, q_n) \) satisfy relations (1.3) and (1.4) with \( r_\alpha(u) \) defined by (1.5) and \( M_{\gamma, \beta} = 1/\Gamma(\beta - \gamma) \) (see [13]) and form therefore a Riesz-type family. In particular, if \( q_n = \frac{1}{n!} \) we get the Borel-type methods \( (B, \alpha) = (B, \alpha, 1/n!) \) (see [1, 2]). If, in addition, \( \alpha = 1 \), we have the Borel method \( B = (B, 1) \).

**Example 3.** Consider the family of generalized Nörlund methods \( A_\alpha = (N, u^{\alpha-1}, q(u)) \) where \( \alpha > 0 \) and \( q = q(u) \) is a positive function from \( X \). These methods are defined by transformation of \( x \) into \( A_\alpha x = y_\alpha(u) \) with

\[
y_\alpha(u) = \frac{1}{r_\alpha(u)} \int_0^u (u - v)^{\alpha-1} q(v)x(v) \, dv \quad (u > 0),
\]

where \( r_\alpha = r_\alpha(u) = \int_0^u (u - v)^{\alpha-1} q(v) \, dv \). This family satisfies relations (1.3) and (1.4) with

\[
M_{\gamma, \beta} = \frac{\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta - \gamma)}
\]

(see [9], Example 1) and therefore it is a Riesz-type family. In particular, if \( q(u) = 1 \) \( (u \geq 0) \) we have Riesz methods \( (N, u^{\alpha-1}, 1) = (R, \alpha) \).

## 2 Preliminary Results

We need some results proved in [9].

**2.1.** Speeds of convergence of any two methods in a Riesz-type family were compared in [9] on the base of an inclusion theorem which will be formulated as the following proposition.

**Proposition 1.** Let \( \{A_\alpha\} (\alpha \geq \alpha_1) \) be a Riesz-type family. Then we have for functions \( x = x(u) \) and numbers \( s \) and \( \beta > \gamma \geq \alpha_1 \) that

i) \( x(u) = O(A_\gamma) \implies x(u) = O(A_\beta) \),  
ii) \( x(u) \to s(A_\gamma) \implies x(u) \to s(A_\beta) \),

provided in case ii) that \( \lim_{u \to \infty} \int_0^u r_\alpha(v) \, dv = \infty \) is satisfied if \( \gamma = \alpha_1 \) is included.

The next theorem (see [9], Theorem 1) describes how the speed of convergence changes if we go from one summability method in the family to a stronger one.

**Theorem A.** Let \( \{A_\alpha\} (\alpha > \alpha_0) \) be a Riesz-type family. Let some positive function \( \lambda = \lambda(u) \to \infty \) (as \( u \to \infty \)) from \( X \) and some number \( \gamma > \alpha_0 \) such that \( \frac{r_\gamma(u)}{\lambda(u)} \in X \) be given.

i) Then we have for functions \( x = x(u) \) and numbers \( s \) and \( \beta > \gamma \) that

\[
\lambda(u) [y_\gamma(u) - s] = O(1) \implies \lambda_\beta(u) [y_\beta(u) - s] = O(1),
\]

where the speeds are related through the formulas

\[
\lambda_\beta(u) = \frac{r_\beta(u)}{b_\beta(u)}; \quad b_\beta(u) = M_{\gamma, \beta} \int_0^u (u - v)^{\beta - \gamma - 1} b_\gamma(v) \, dv; \quad b_\gamma(u) = \frac{r_\gamma(u)}{\lambda(u)}.
\]
ii) Moreover, we have that

\[ \lambda(u) [y_\gamma(u) - s] \to t \iff \lambda_\beta(u) [y_\beta(u) - s] \to t, \]  

(2.3)

provided that

\[ \lim_{u \to \infty} \int_0^u b_\gamma(v) \, dv = \infty. \]  

(2.4)

Under restriction (2.4) the condition \( \lambda(u) \to \infty \) implies \( \lambda_\beta(u) \to \infty \) in Theorem A (see [9], Remark 2). We note also that Theorem A can be considered as a generalization of case A) of Theorem 1 from [12], which was proved for matrix case. Certain evaluations for speed of convergence for Riesz and Nörlund matrix methods in Banach spaces were proved in recent papers [6] and [7].

2.2. The speeds \( \lambda = \lambda(u) \) and \( \lambda_\beta = \lambda_\beta(u) \) defined in Theorem A can be compared by the inequalities.

Let \( a = a(u) \) and \( b = b(u) \) be two positive functions from \( X \). If there exist positive numbers \( c_1, c_2 \) and \( u_0 \) such that the condition

\[ c_1 b(u) \leq a(u) \leq c_2 b(u) \]  

(2.5)

holds for every \( u > u_0 \), we write \( a(u) \approx b(u) \). If \( b = b(u) \) is nondecreasing and condition (2.5) is satisfied with some positive \( c_1 \) and \( c_2 \) for any \( u > 0 \), then we say that \( a = a(u) \) is almost nondecreasing.

The following proposition is proved in [9] (see [9], Propositions 2 and 3).

**Proposition 2.** Let a Riesz-type family \( \{ A_\alpha \} (\alpha > \alpha_0) \) and a positive function \( \lambda = \lambda(u) \in X \) be given. Fix some \( \gamma > \alpha_0 \) and suppose that \( \lambda_\beta = \lambda_\beta(u) \) \((\beta > \gamma > \alpha_0)\) is defined through (2.2). Then for \( \beta > \gamma > \alpha_0 \) we have:

i) \( \lambda_\beta(u) \leq L \lambda(u) \) \((u > 0)\) provided that \( \lambda = \lambda(u) \) is almost nondecreasing,

ii) \( \lambda_\beta(u) \geq \frac{K r_\beta(u)}{r_\gamma(u) u^{\beta - \gamma}} \lambda(u) \) \((u > 0)\) provided that \( b_\gamma(u) = r_\gamma(u) / \lambda(u) \) is almost nondecreasing, where \( L \) and \( K \) are some positive constants independent from \( u \).

Previous result state that switching to a stronger method, the speed of convergence can not be improved but also it cannot become too much worse. This is consistent with results known for matrix methods (see e.g. [4, 6, 12]).

3 Main Results. A Tauberian Remainder Theorem

First we prove a convexity theorem.

**Theorem 1.** Let \( \{ A_\alpha \} (\alpha \geq \alpha_1) \) be a Riesz-type family satisfying the condition

\[ r_\beta(u) / r_\alpha(u) \approx u^{\alpha - \alpha} \]  

(3.1)

for all \( \beta > \alpha > \alpha_1 \). Then we have for functions \( x = x(u) \) and numbers \( s \) and \( \beta > \delta \geq \gamma \geq \alpha_1 \) that

\[ x(u) = O(A_\gamma), \quad x(u) \to s(A_\beta) \iff x(u) \to s(A_\delta). \]  

(3.2)

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Proof. Suppose first that $\gamma > \alpha_1$. Without a loss of generality we may take $\beta = \gamma + 1$ and $s = 0$. Suppose that
\[
y_{\gamma+1}(u) \to 0 \text{ as } u \to \infty, \quad y_{\gamma}(u) = O(1) \tag{3.3}
\]
for a function $x = x(u)$ and some value $\gamma$ of the parameter, and show that
\[
y_{\delta}(u) \to 0 \text{ as } u \to \infty \tag{3.4}
\]
for any $\delta$ such that $\gamma < \delta < \gamma + 1$. By relation (1.3) we have that
\[
y_{\delta}(u) = \frac{M_{\gamma, \delta}}{r_{\delta}(u)} \int_{0}^{u} (u - v)^{\delta-\gamma-1} r_{\gamma}(v)y_{\gamma}(v) \, dv \quad (u > 0).
\]
Choose some $\theta \in (1/2, 1)$ and divide $y_{\delta}(u)$ into two parts:
\[
y_{\delta}(u) = \frac{M_{\gamma, \delta}}{r_{\delta}(u)} \int_{0}^{\theta u} (u - v)^{\delta-\gamma-1} r_{\gamma}(v)y_{\gamma}(v) \, dv
\]
\[
+ \frac{M_{\gamma, \delta}}{r_{\delta}(u)} \int_{0}^{\theta u} [(\delta - \gamma - 1)(u - v)^{\delta-\gamma-2} \int_{0}^{v} r_{\gamma}(t)y_{\gamma}(t) \, dt] \, dv = I_{1}(u, \theta) + I_{2}(u, \theta). \tag{3.5}
\]
Thus we have the equality $y_{\delta}(u) = I_{1}(u, \theta) + I_{2}(u, \theta)$. Note that $I_{1}(u, \theta)$ and $I_{2}(u, \theta)$ depend also on $\gamma, \delta$. Integrating by parts, we get for $I_{1}(u, \theta)$ the following form:
\[
I_{1}(u, \theta) = \frac{M_{\gamma, \delta}}{r_{\delta}(u)} \left( (u - v)^{\delta-\gamma-1} \int_{0}^{\theta u} r_{\gamma}(v)y_{\gamma}(v) \, dv \right)_{0}^{\theta u}
\]
\[
+ \frac{M_{\gamma, \delta}}{r_{\delta}(u)} \int_{0}^{\theta u} [(\delta - \gamma - 1)(u - v)^{\delta-\gamma-2} \int_{0}^{v} r_{\gamma}(t)y_{\gamma}(t) \, dt] \, dv = I_{1}^{1}(u, \theta) + I_{1}^{\prime}(u, \theta),
\]
where
\[
I_{1}^{1}(u, \theta) = \frac{M_{\gamma, \delta}}{r_{\delta}(u)} \left( (u - v)^{\delta-\gamma-1} \int_{0}^{\theta u} r_{\gamma}(v)y_{\gamma}(v) \, dv \right)_{0}^{\theta u} = \frac{M_{\gamma, \delta}}{r_{\delta}(u)} (u - \theta u)^{\delta-\gamma-1}
\]
\[
\times \int_{0}^{\theta u} r_{\gamma}(t)y_{\gamma}(t) \, dt = \frac{M_{\gamma, \delta}}{M_{\gamma, \gamma+1}} \frac{(u - \theta u)^{\delta-\gamma-1}}{r_{\delta}(u)} r_{\gamma+1}(\theta u)y_{\gamma+1}(\theta u)
\]
and
\[
I_{1}^{\prime}(u, \theta) = \frac{M_{\gamma, \delta}}{M_{\gamma, \gamma+1}} \frac{1}{r_{\delta}(u)} \int_{0}^{\theta u} [(\delta - \gamma - 1)(u - v)^{\delta-\gamma-2} \int_{0}^{v} r_{\gamma}(t)y_{\gamma}(t) \, dt] \, dv
\]
\[
= \frac{M_{\gamma, \delta}}{M_{\gamma, \gamma+1}} \frac{1}{r_{\delta}(u)} \int_{0}^{\theta u} (\delta - \gamma - 1)(u - v)^{\delta-\gamma-2} r_{\gamma+1}(v)y_{\gamma+1}(v) \, dv.
\]
Using conditions (3.1) and (3.3) we get
\[
I_{1}^{1}(u, \theta) = O(1) \frac{(u - \theta u)^{\delta-\gamma-1}}{r_{\delta}(u)} r_{\gamma+1}(u)y_{\gamma+1}(u) = O(1) u^{\gamma+1-\delta} u^{\delta-\gamma-1}
\]
\[
\times (1 - \theta)^{\delta-\gamma-1} y_{\gamma+1}(u) = o(1)(1 - \theta)^{\delta-\gamma-1} = o_{\theta}(1) \text{ as } u \to \infty.
\]
Thus we have $I'_1(u, \theta) = o_\theta(1)$ as $u \to \infty$. Let us show that also $I''_1(u, \theta) = o_\theta(1)$ as $u \to \infty$. Denoting

$$c'_{\gamma, \delta}(u, v) = \begin{cases} \frac{1}{r_\delta(u)}(u - v)^{\delta-\gamma-2}r_{\gamma+1}(v), & \text{if } 0 \leq v \leq \theta u, \\ 0, & \text{if } v > \theta u, \end{cases}$$

we will show that the integral transformation defined by $c'_{\gamma, \delta}(u, v)$ is a $c_0 \to c_0$ type transformation. We use Theorem 6 from [3] which gives the sufficient conditions for the regularity of integral methods. Let us prove first that

$$\int_0^{v_0} c'_{\gamma, \delta}(u, v) \, dv = o_\theta(1) \quad \text{as } u \to \infty,$$

assuming that $v_0$ is a fixed positive number and $v < v_0 < \theta u$. We get:

$$\int_0^{v_0} c'_{\gamma, \delta}(u, v) \, dv = \frac{1}{r_\delta(u)} \int_0^{v_0} (u - v)^{\delta-\gamma-2}r_{\gamma+1}(v) \, dv \leq \frac{r_{\gamma+1}(u)}{r_\delta(u)} \int_0^{v_0} (u - v)^{\delta-\gamma-2} \, dv = O(1)u^{\gamma+1-\delta}(u - v)^{\delta-\gamma-1} \bigg|_0^{v_0} = O(1)\left[u^{\gamma+1-\delta}(u - v_0)^{\delta-\gamma-1} - 1\right] = O(1)\left[(1 - \frac{v_0}{u})^{\delta-\gamma-1} - 1\right] = o_\theta(1) \quad \text{as } u \to \infty.$$

Following Theorem 6 from [3] it remains to show that the condition

$$\int_0^{\theta u} c'_{\gamma, \delta}(u, v) \, dv = O_\theta(1) \quad (u > 0)$$

is also fulfilled. With the help of (3.1) we get:

$$\int_0^{\theta u} \frac{r_{\gamma+1}(v)}{r_\delta(u)}(u - v)^{\delta-\gamma-2} \, dv \leq \frac{r_{\gamma+1}(u)}{r_\delta(u)} \int_0^{\theta u} (u - v)^{\delta-\gamma-2} \, dv = O(1)u^{\gamma+1-\delta}(u - \theta u)^{\delta-\gamma-1} = O_\theta(1).$$

Thus we have shown that the integral transformation defined by $c'_{\gamma, \delta}(u, v)$ is of type $c_0 \to c_0$ for every $\theta \in (1/2; 1)$, and therefore condition $I''_1(u, \theta) = o_\theta(1)$ is satisfied. By the obtained relations we have that

$$I_1(u, \theta) = I'_1(u, \theta) + I''_1(u, \theta) = o_\theta(1) \quad \text{as } u \to \infty. \quad (3.6)$$

Next we evaluate the quantity $I_2(u, \theta)$ using relations (3.1) and (3.3):

$$I_2(u, \theta) = O(1) \int_{\theta u}^u (u - v)^{\delta-\gamma-1}r_{\gamma+1}(v) \, dv \leq O(1)\frac{r_{\gamma+1}(u)}{r_\delta(u)\theta u} \int_{\theta u}^u (u - v)^{\delta-\gamma-1} \, dv = O(1)\left[u^{\gamma-\delta}(u - v)^{\delta-\gamma}\right]_u^{\theta u} = O(1)(1 - \theta)^{\delta-\gamma}. \quad (3.7)$$

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Now we are able to complete our proof showing that (3.4) is true for every $\gamma < \delta < \gamma + 1$. We choose $\varepsilon > 0$ and afterwards $\theta_\varepsilon \in (1/2, 1)$ so, that

$$I_2(u, \theta_\varepsilon) = O(1)(1 - \theta_\varepsilon)^{\delta - \gamma} < \frac{\varepsilon}{2} \text{ for any } u > 0$$

(see (3.7)). Next we choose $U = U_{\theta_\varepsilon}$ so, that $|I_1(u, \theta_\varepsilon)| < \varepsilon/2$ for all $u > U$ (see (3.6)). It follows from (3.5) that $|y_\delta(u)| < \varepsilon$ when $u > U$, i.e., (3.4) holds. Thus we have shown that implication (3.2) is true for all $\beta > \delta > \gamma > \alpha_1$.

If $\gamma = \alpha_1$, then we choose some $\gamma < \gamma_1 < \delta$ and get that $x(u) = O(A_\gamma)$ implies $x(u) = O(A_{\gamma_1})$. To finish the proof, it remains to apply implication (3.2), already proved, with $\gamma_1$ instead of $\gamma$. □

Note that Theorem 1 was formulated (but not proved) in [13] as Proposition 4 with a hint on analogy with matrix case (see [10, 11]). The following Tauberian remainder theorem extends Theorem A.

**Theorem 2.** Let $\{A_\alpha\} (\alpha > \alpha_0)$ be a Riesz-type family. Let some positive function $\lambda = \lambda(u) \to \infty$ (as $u \to \infty$) from $X$ and some number $\gamma > \alpha_0$ such that $r_\gamma(u)/\lambda(u) \in X$ be given. Suppose that $b_\beta(u)$ and $\lambda_\beta(u)$ are defined through (2.2). Suppose also that the following condition

$$b_\beta(u)/b_\alpha(u) \approx u^{\beta - \alpha} \quad (u > 0) \quad (3.8)$$

is satisfied for any $\beta > \alpha > \gamma$. Then we have for functions $x = x(u)$ and numbers $s$ and $\beta > \delta > \gamma$ that

$$\lambda(u)[y_\gamma(u) - s] = O(1), \lambda_\beta(u)[y_\beta(u) - s] \to t \implies \lambda_\delta(u)[y_\delta(u) - s] \to t. \quad (3.9)$$

*Proof.* We set $\alpha_1 = \gamma$ and construct another family $\{B_\alpha\} (\alpha \geq \gamma)$ on the base of relations (2.2). Namely, we define the methods $B_\alpha$ by the transformations of a function $y = y(u) \in X$ into $\eta_\alpha = \eta_\alpha(u)$ with

$$\eta_\alpha(u) = \frac{M_{\gamma, \alpha}}{b_\alpha(u)} \int_0^u (u - v)^{\beta - \gamma - 1}b_\gamma(v)y(v) \, dv \quad (\alpha > \gamma)$$

and $\eta_\gamma(u) = y(u)$, i.e., $B_\gamma = I$. The family $\{B_\alpha\} (\alpha \geq \gamma)$ is a Riesz-type family (see Example 3) satisfying the presumptions of Theorem 1. Let us apply methods $B_\alpha$ to $y = \lambda(u)[y_\gamma(u) - s]$ and realize that $B_\alpha y = \eta_\alpha(u) = \lambda_\alpha(u)[y_\alpha(u) - s]$ for any $\alpha > \gamma$. Thus, implication (3.9) holds by Theorem 1 for any $\beta > \delta > \gamma$ as (3.2) in the form

$$y(u) = O(B_\gamma), \quad y(u) \to t(B_\beta) \implies y(u) \to t(B_\delta).$$

□

An analogous Tauberian remainder theorem for "matrix case" was proved in [12] as Theorem 2. Some Tauberian remainder theorems for Nörlund and Riesz matrix methods in Banach spaces were proved recently in [6] and [7]. Some estimates for speeds in a Riesz-type family (weaker than here) can be found also in [8].
4 Examples on Comparison of Speeds of Convergence

Here we give some numerical examples on application of Theorem 2 for comparison of speeds of convergence in special Riesz-type families. More precisely, we extend Examples 5, 7 and 9 from [9], where Theorem A was applied. In mentioned examples comparative evaluations (2.1) and (2.3) for speeds of any two methods $A_\gamma$ and $A_\beta$ in Riesz-type families $\{A_n\}$ are presented. Here we improve these results, comparing any three methods $A_\gamma$, $A_\beta$ and $A_\delta$ with the help of implication (3.9).

Example 4. We consider the Riesz methods $A_\alpha = (R, \alpha)$ ($\alpha > 0$). Choose the speed of convergence $\lambda(u) = (u + 1)^\rho$ ($\rho > 0$) and fix some number $\gamma > 0$. Suppose that $x = x(u)$ is a function having a given speed of convergence $\lambda(u)$ with respect to the method $A_\gamma = (R, \gamma)$ and define with the help of formulas (2.2) the function $b_\beta(u)$ and afterwards the speed of convergence $\lambda_\beta(u)$ of $x = x(u)$ with respect to the methods $A_\beta = (R, \beta)$ for $\beta > \gamma$. In Example 5 in [9] the following estimates for $b_\beta(u)$ and $\lambda_\beta(u)$ were proved for any $\beta > \gamma$ if $u \to \infty$:

\[
\begin{align*}
  b_\beta(u) &\approx M_{\gamma, \beta} B(\beta - \gamma, \gamma - \rho + 1) u^{\beta - \rho}/\gamma, & \text{if } \rho < \gamma + 1, \\
  b_\beta(u) &\approx \left\{ \begin{array}{ll} 
    u^{\beta - \gamma - 1} \log u, & \text{if } \rho = \gamma + 1, \\
    u^{\beta - \gamma - 1}, & \text{if } \rho > \gamma + 1,
  \end{array} \right. \\
  \lambda_\beta(u) &\approx \frac{\Gamma(\gamma + 1)\Gamma(\beta - \rho + 1)}{\Gamma(\beta + 1)\Gamma(\gamma - \rho + 1)} u^{\rho} \sim \frac{\Gamma(\gamma + 1)\Gamma(\beta - \rho + 1)}{\Gamma(\beta + 1)\Gamma(\gamma - \rho + 1)} \lambda(u), & \text{if } \rho < \gamma + 1,
\end{align*}
\]

Estimates (4.1)–(4.2) show that condition (3.8) is satisfied for all $\beta > \alpha > \gamma$. Thus Theorem 2 applies, and implication (3.9) is true for any $\beta > \delta > \gamma$ where speeds $\lambda_\beta$ and $\lambda_\delta$ obey evaluates (4.3) and (4.4).

Example 5. Let us consider the Borel-type methods $A_\alpha = (B, \alpha, 1/n!) = (B, \alpha)$ ($\alpha > -1/2$). Suppose that $\lambda(u) = (u + 1)^\rho e^u$, fix some $\gamma > -1/2$ and find $\lambda_\beta(u)$ for $\beta > \gamma$ through (2.2) again. In Example 7 in [9] for $\beta > \gamma$ the following estimates were proved:

\[
\begin{align*}
  b_\beta(u) &\approx \left\{ \begin{array}{ll} 
    u^{\beta - \gamma - 1}, & \text{if } \rho > 1, \\
    u^{\beta - \gamma - 1} \log u, & \text{if } \rho = 1, \\
    u^{\beta - \gamma - \rho}, & \text{if } \rho < 1,
  \end{array} \right. \\
  \lambda_\beta(u) &\approx \left\{ \begin{array}{ll} 
    e^u / u^{\beta - \gamma - 1} \sim \frac{\lambda(u)}{u^{\beta - \gamma + \rho - 1}}, & \text{if } \rho > 1, \\
    e^u / u^{\beta - \gamma - 1} \log u \sim \frac{\lambda(u)}{u^{\beta - \gamma} \log u}, & \text{if } \rho = 1, \\
    e^u / u^{\beta - \gamma - \rho} \sim \frac{\lambda(u)}{u^{\beta - \gamma}}, & \text{if } \rho < 1.
  \end{array} \right.
\end{align*}
\]

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Condition (3.8) is satisfied for all $\beta > \alpha > \gamma$ by relations (4.5). Therefore, Theorem 2 applies again, and implication (3.9) is true for any $\beta > \delta > \gamma$ where speeds $\lambda_{\beta}$ and $\lambda_{\delta}$ obey evaluates (4.6).

Example 6. Suppose that $A_{\alpha} = (N, u^{\alpha-1}, e^{u^\varphi})$ ($\alpha > 0$) where $0 < \varphi < 1$ is some fixed number. Suppose that $\lambda(u) = e^{u^\varphi}$. It was shown in Example 9 in [9] that $b_{\beta}(u) \approx u^{\beta-\gamma+(1-\varphi)\gamma}$ and $\lambda_{\beta}(u) \approx e^{u^\varphi(u^{\varphi(\gamma-\beta)})} = u^{\varphi(\gamma-\beta)}\lambda(u)$ for $\beta > \gamma$. We see that (3.8) is satisfied for any $\beta > \alpha > \gamma$. Therefore, implication (3.9) is true for any $\beta > \delta > \gamma$ by Theorem 2.

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