Asymptotic integration of a certain second-order linear delay differential equation

Pavel Nesterov

Received: 4 September 2015 / Accepted: 3 October 2016 / Published online: 12 October 2016 © Springer-Verlag Wien 2016

Abstract We construct the asymptotic formulas for solutions of a certain linear second-order delay differential equation as independent variable tends to infinity. When the delay equals zero this equation turns into the so-called one-dimensional Schrödinger equation at energy zero with Wigner–von Neumann type potential. The question of interest is how the behaviour of solutions changes qualitatively and quantitatively when the delay is introduced in this dynamical model. We apply the method of asymptotic integration that is based on the ideas of the centre manifold theory in its presentation with respect to the systems of functional differential equations with oscillatory decreasing coefficients.

Keywords Asymptotics · Delay differential equation · Oscillating coefficients · Oscillation of solutions · Levinson’s theorem · Method of averaging · Schrödinger equation

Mathematics Subject Classification 34K06 · 34C29 · 34K25

1 Problem statement

The behaviour of solutions of the second-order linear differential equation

\[ \ddot{x} - q(t)x = 0 \]  

(1.1)
as $t \to \infty$ was studied in many papers. Significant number of them are devoted to the problem of oscillation or nonoscillation of solutions of Eq. (1.1) (see, e.g., [2,13,15,16]). Among the studies of Eq. (1.1) are also those that deal with the asymptotics for solutions of this equation (see for instance [6,7]). In [4,12,17], Eq. (1.1) is considered as one-dimensional Schrödinger equation at energy zero. The authors construct the asymptotics for solutions of this equation as $t \to \infty$ for some special potentials $q(t)$.

The covered cases include the so-called Wigner–von Neumann like potential:

$$q(t) = \frac{p(t)}{t^\rho}, \quad \rho > 0. \quad (1.2)$$

Here $p(t)$ is a real trigonometric polynomial, having zero mean value, i.e.,

$$p(t) = \sum_{j=-N}^{N} p_j e^{i\omega_j t}, \quad p_{-j} = \bar{p}_j, \quad \omega_{-j} = -\omega_j, \quad (1.3)$$

and

$$M[p(t)] = \lim_{T \to \infty} \frac{1}{T} \int_0^T p(t) dt = p_0 = 0. \quad (1.4)$$

Notation $\bar{a}$ stands for complex conjugate of $a$.

The oscillatory nature of the function $q(t)$ complicates the study of dynamics of Eq. (1.1) and, therefore, to construct the asymptotic formulas for solutions various methods are used. It appears that the dynamics of Eq. (1.1) in the case of the function $q(t)$, having the form (1.2), (1.3), is determined by the quantity

$$a = M \left[ \left( \int_0^t p(s) ds \right)^2 \right] - \left( M \left[ \int_0^t p(s) ds \right] \right)^2 = \sum_{j=-N}^{N} \frac{|p_j|^2}{\omega_j^2} \geq 0, \quad (1.5)$$

and also by the parameter $\rho$ that characterizes the decrease rate of the oscillation amplitude of the function $q(t)$ (see Table 1 in Sect. 4). In this paper we consider the delay differential equation

$$\ddot{x} - q(t)x(t-h) = 0, \quad h > 0, \quad t \geq t_0 > 0 \quad (1.6)$$

under conditions (1.2)–(1.4). Equations of the form (1.6) are usually considered in the context of the oscillation theory for functional differential equations. We just mention monographs [1,10,14], where the reader can find further references on the topic, and also papers [3,20,21], concerning the oscillation problem for second-order delay differential equations. We also note that the function $q(t)$ is assumed generally to be of constant sign for $t \geq t_0$. If function $q(t)$ oscillates, the known methods, as a rule, fail. We will construct the asymptotic formulas for solutions of Eq. (1.6) as $t \to \infty$.

The question of interest for us concerns the quantitative and qualitative differences in dynamics of Eqs. (1.1) and (1.6) with the same function $q(t)$. 

Springer
2 Description of the asymptotic integration method

We rewrite Eq. (1.6) as a system

\[
\dot{y} = B_0 y_t + G(t, y_t), \quad y(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \tag{2.1}
\]

where \(y_t(\theta) = y(t + \theta) (-h \leq \theta \leq 0)\) denotes the element of the space \(C_h = C([-h, 0], \mathbb{C}^2)\) consisting of all continuous functions defined on \([-h, 0]\) and acting to \(\mathbb{C}^2\). Further, \(B_0\) is a bounded linear functional acting from \(C_h \rightarrow \mathbb{C}^2\) that is defined by the formula

\[
B_0 \varphi(\theta) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \varphi(0), \quad \varphi(\theta) \in C_h. \tag{2.2}
\]

Functional \(G(t, \varphi(\theta))\), acting from \(C_h \rightarrow \mathbb{C}^2\), has the form

\[
G(t, \varphi(\theta)) = q(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \varphi(-h). \tag{2.3}
\]

The characteristic equation

\[
\det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda I - B_0(e^{\lambda \cdot I}) = \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix}, \tag{2.4}
\]

constructed for the linear autonomous system

\[
\dot{y} = B_0 y_t, \tag{2.5}
\]

has two coinciding roots \(\lambda_1 = \lambda_2 = 0\). Since the characteristic equation has a countable set of roots, we may assume formally that all other roots of Eq. (2.4) satisfy the inequality \(\text{Re } \lambda < -\beta\) for every real number \(\beta > 0\). This fact allows us to apply the method proposed in [19] to construct the asymptotics for solutions of Eq. (2.1). We will now describe the essence of this method. It is known (see, e.g., [11]) that linear autonomous Eq. (2.5) generates in \(C_h\) for \(t \geq 0\) a strongly continuous semigroup \(T(t): C_h \rightarrow C_h\). The solution operator \(T(t)\) of Eq. (2.5) is defined by \(T(t) \varphi = y^\varphi_t(\theta)\), where \(\varphi \in C_h\) and \(y^\varphi_t(\theta)\) is a unique solution of Eq. (2.5) with initial value \(y^\varphi_0(\theta) = \varphi\). The infinitesimal generator \(A\) of this semigroup is defined by \(A \varphi = \varphi'(\theta)\) for \(\varphi \in D(A)\). The domain of \(A\)

\[
D(A) = \{ \varphi \in C_h \mid \varphi'(\theta) \in C_h, \quad \varphi'(0) = B_0 \varphi \}
\]

is dense in \(C_h\). We introduce the Riesz representation of \(B_0\):

\[
B_0 \varphi = \int_{-h}^{0} d\eta(\theta) \varphi(\theta),
\]
where \( \eta(\theta) \) is \((2 \times 2)\)-matrix function of bounded variation on \([-h, 0]\). We can associate with (2.5) the transposed equation

\[
\dot{y}_* = -\int_{-h}^0 y_*(t - \theta) d\eta(\theta),
\]

(2.6)

where \( y_*(t) \) is a 2-dimensional complex row vector. The phase space for Eq. (2.6) is \( C'_h \equiv C([0, h], \mathbb{C}^{2*}) \), where \( \mathbb{C}^{2*} \) is the space of 2-dimensional row vectors. For \( \psi \in C'_h \) and \( \varphi \in C_h \) we define the bilinear form

\[
(\psi(\xi), \varphi(\theta)) = \psi(0)\varphi(0) - \int_{-h}^0 \int_0^\theta \psi(\xi - \theta) d\eta(\theta)\varphi(\xi) d\xi.
\]

(2.7)

If

\[
\Lambda = \{\lambda_1, \lambda_2\},
\]

then we can decompose \( C_h \) into a direct sum

\[
C_h = P_\Lambda \oplus Q_\Lambda.
\]

(2.8)

Here \( P_\Lambda \) is the generalized eigenspace associated with \( \Lambda \) and \( Q_\Lambda \) is the complementary subspace of \( C_h \) such that \( T(t)Q_\Lambda \subseteq Q_\Lambda, t \geq 0 \). Let \( \Phi(\theta) \) be \((2 \times 2)\)-matrix whose columns are the generalized eigenfunctions \( \varphi_1(\theta), \varphi_2(\theta) \) of \( A \) corresponding to the eigenvalues from \( \Lambda \). Thus, the columns of \( \Phi(\theta) \) form the basis of \( P_\Lambda \). Moreover, let \( \Psi(\xi) \) be \((2 \times 2)\)-matrix whose rows \( \psi_1(\xi), \psi_2(\xi) \) form the basis of the generalized eigenspace \( P^T_\Lambda \) of the transposed Eq. (2.6), associated with \( \Lambda \). We can choose matrices \( \Phi(\theta) \) and \( \Psi(\xi) \) such that

\[
(\Psi(\xi), \Phi(\theta)) = \{(\psi_i(\xi), \varphi_j(\theta))\}_{1 \leq i, j \leq 2} = I.
\]

(2.9)

Since the columns of the matrix \( \Phi(\theta) \) form the basis of \( P_\Lambda \) and \( AP_\Lambda \subseteq P_\Lambda \), there exists \((2 \times 2)\)-matrix \( D \), whose spectrum is \( \Lambda \), such that \( A\Phi(\theta) = \Phi(\theta)D \). From the definition of \( A \) we deduce that

\[
\Phi(\theta) = \Phi(0)e^{D\theta}, \quad T(t)\Phi(\theta) = \Phi(\theta)e^{Dt} = \Phi(0)e^{D(t+\theta)},
\]

(2.10)

where \(-h \leq \theta \leq 0\) and \( t \geq 0 \). Analogously, for matrix \( \Psi(\xi) \) we have

\[
\Psi(\xi) = e^{-D\xi} \Psi(0),
\]

(2.11)

where \( 0 \leq \xi \leq h \). Matrices \( \Phi(0) \) and \( \Psi(0) \) are chosen in the following way. Since the columns of matrix \( \Phi(\theta) \) are the generalized eigenfunctions of \( A \), they should belong to \( D(A) \). This implies that
\[
\Phi'(0) = \Phi(0)D = B_0 \Phi = \int_{-h}^{0} d\eta(\theta) \Phi(0)e^{D\theta}.
\]

The same reasoning, using (2.6) and (2.11), yields that

\[
\Psi'(0) = -D\Psi(0) = -\int_{-h}^{0} e^{D\theta} \Psi(0)d\eta(\theta).
\]

Finally, the spaces \( P_\Lambda \) and \( Q_\Lambda \) from decomposition (2.8) of \( C_h \) may be defined as follows:

\[
P_\Lambda = \{ \varphi \in C_h \mid \varphi(\theta) = \Phi(\theta)a, \ a \in \mathbb{C}^2 \}, \]

\[
Q_\Lambda = \{ \varphi \in C_h \mid (\Psi, \varphi) = 0 \}.
\] (2.12)

Some easy calculations show that matrices \( \Phi(\theta) \), \( \Psi(\xi) \) and \( D \), when applied to Eq. (2.5) with operator (2.2), are defined by the following formulas:

\[
\Phi(\theta) = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \quad \Psi(\xi) = \begin{pmatrix} 1 & -\xi \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\] (2.13)

**Definition 2.1** Two-dimensional linear space \( \mathcal{W}(t) \subset C_h \) is said to be critical (or center-like) manifold of Eq. (2.1) for \( t \geq t_* \geq t_0 \) if the following conditions hold:

1. There exists continuous in \( t \geq t_* \) and \( \theta \in [-h, 0] \) (2 \times 2)-matrix \( H(t, \theta) \) whose columns belong to \( Q_\Lambda \) for \( t \geq t_* \) such that \( \|H(t, \cdot)\|_{C_h} \to 0 \) as \( t \to \infty \), where

\[
\|H(t, \cdot)\|_{C_h} = \sup_{-h \leq \theta \leq 0} |H(t, \theta)|
\]

and \( |\cdot| \) is some matrix norm;

2. The space \( \mathcal{W}(t) \) for \( t \geq t_* \) is defined by the formula

\[
\mathcal{W}(t) = \{ \varphi(\theta) \in C_h \mid \varphi(\theta) = \Phi(\theta)u + H(t, \theta)u, \ u \in \mathbb{C}^2 \}.
\] (2.14)

3. The space \( \mathcal{W}(t) \) is positively invariant for trajectories of Eq. (2.1) for \( t \geq t_* \), i.e., if \( x_T \in \mathcal{W}(T), \ T \geq t_* \), then \( x_t \in \mathcal{W}(t) \) for \( t \geq T \).

It follows from the results of [19] that for sufficiently large \( t \) there exists critical manifold of Eq. (2.1). Moreover, this manifold is attractive for all trajectories of Eq. (2.1) and the attraction rate is \( O(e^{-\beta t}) \), where \( \beta > 0 \) is an arbitrary real number. Trajectories lying for sufficiently large \( t \) in \( \mathcal{W}(t) \) are described by the formula

\[
y(t) = \Phi(\theta)u(t) + H(t, \theta)u(t), \ t \geq T, \ u(t) \in \mathbb{C}^2.
\] (2.15)
Here function $u(t)$ satisfies ordinary differential system

$$\dot{u} = \left[ D + \Psi(0)G(t, \Phi(\theta) + H(t, \theta)) \right] u, \quad t \geq T. \quad (2.16)$$

This system is referred to as projection of Eq. (2.1) on critical manifold $W(t)$ or, simply, as system on critical manifold. It can be shown (see [19]) that matrix $H(t, \theta)$ is a solution of the following partial differential problem:

$$\phi(\theta)\psi(0)G(t, \phi(\theta) + H(t, \theta)) + H(t, \theta) \left( D + \psi(0)G(t, \phi(\theta) + H(t, \theta)) \right) + \partial H = \begin{cases} \partial H, \quad -h \leq \theta < 0, \\ B_0 H + G(t, \phi(\theta) + H(t, \theta)), \quad \theta = 0. \end{cases} \quad (2.17)$$

Thus, if $u^{(1)}(t), u^{(2)}(t)$ are fundamental solutions of system on critical manifold (2.16) then all solutions of Eq. (2.1), due to (2.15), have the following asymptotic representation as $t \to \infty$:

$$y(t) = y_1(t) = \sum_{i=1}^{2} c_i \left( \phi(0) + H(t, 0) \right) u^{(i)}(t) + O(e^{-\beta t}), \quad t \to \infty, \quad (2.18)$$

where $c_1, c_2$ are arbitrary complex variables and $\beta > 0$ is an arbitrary real number. Further we will show that Eq. (2.16) in the considered case have the form

$$\dot{u} = \left[ D + A_1(t)t^{-\rho} + A_2(t)t^{-2\rho} + \cdots + A_k(t)t^{-k\rho} + R(t) \right] u, \quad u \in \mathbb{C}^2. \quad (2.19)$$

Here matrix $D$ is defined by (2.13), natural number $k$ is chosen in the way that $k\rho \leq 1 < (k+1)\rho$, and the entries of matrices $A_1(t), \ldots, A_k(t)$ are trigonometric polynomials, i.e.,

$$A_j(t) = \sum_{s=1}^{M} \beta_s^{(i)} e^{i\nu_s t}, \quad (2.20)$$

where $\beta_s^{(i)}$ are some constant complex matrices, and $\nu_s$ are real numbers. Finally, $R(t)$ is a certain absolutely integrable on $[t_\ast, \infty)$ matrix, i.e., matrix from the class $L_1[t_\ast, \infty)$. The coefficients of Eq. (2.19) have an oscillatory form and this fact complicates the process of asymptotics construction. Therefore, to construct the asymptotics for solutions of Eq. (2.19) first we introduce in this equation the so called averaging change of variable that makes it possible to exclude the oscillating coefficients from the main part of the system. The following theorem holds (see [18]).

**Theorem 2.1** For sufficiently large $t$, system (2.19) by the change of variable

$$u = \left[ I + Y_1(t)t^{-\rho} + Y_2(t)t^{-2\rho} + \cdots + Y_k(t)t^{-k\rho} \right] u_1 \quad (2.21)$$
Asymptotic integration of a certain second-order linear delay...

can be reduced to its averaged form

\[ \dot{u}_1 = \left[ D + A_1 t^{-\rho} + A_2 t^{-2\rho} + \cdots + A_k t^{-k\rho} + R_1(t) \right] u_1(t) \]  

(2.22)

with constant matrices \( A_1, \ldots, A_k \) and with matrix \( R_1(t) \) from \( L_1[\tau_*, \infty) \). In (2.21), \( I \) is the identity matrix and the entries of matrices \( Y_1(t), \ldots, Y_k(t) \) are trigonometric polynomials having zero mean value.

As a rule, to construct the asymptotics for solutions of Eq. (2.22) we need to compute only a few constant matrices. Hence, we give the explicit formulas only for the following matrices:

\[ A_1 = M[A_1(t)], \]  

(2.23)

\[ A_2 = M[A_2(t) + A_1(t)Y_1(t)], \]  

(2.24)

\[ A_3 = M[A_3(t) + A_2(t)Y_1(t) + A_1(t)Y_2(t)]. \]  

(2.25)

Matrices \( Y_1(t), Y_2(t) \) with zero mean value are the solutions of the following matrix differential equations:

\[ \dot{Y}_1 - DY_1 + Y_1 D = A_1(t) - A_1, \]  

(2.26)

\[ \dot{Y}_2 - DY_2 + Y_2 D = A_2(t) + A_1(t)Y_1(t) - Y_1(t)A_1 - A_2. \]  

(2.27)

The subsequent transformation of the averaged system (2.22) is intended to reduce it to the form

\[ \dot{u}_2 = \left[ A_0 + V(t) \right] t^{-\alpha} u_2 + R_2(t) u_2, \quad \alpha > 0, \]  

(2.28)

where \( A_0 \) is a constant matrix with distinct eigenvalues, \( V(t) \to 0 \) as \( t \to \infty \) and \( \dot{V}(t), R_2(t) \in L_1[\tau_*, \infty) \). The following lemma holds (see, e.g., [2,8,9]).

**Lemma 2.1** (Diagonalization of variable matrices) *Suppose that all eigenvalues of the matrix \( A_0 \) are distinct. Moreover, suppose that the matrix \( V(t) \to 0 \) as \( t \to \infty \) and \( \dot{V}(t) \in L_1[\tau_*, \infty) \). Then for sufficiently large \( t \) there exists a nonsingular matrix \( C(t) \) such that*

\( i) \) the columns of this matrix are the eigenvectors of the matrix \( A_0 + V(t) \) and \( C(t) \to C_0 \) as \( t \to \infty \). The columns of the constant matrix \( C_0 \) are the eigenvectors of the matrix \( A_0 \);

\( ii) \) the derivative \( \dot{C}(t) \in L_1[\tau_*, \infty) \);

\( iii) \) it brings the matrix \( A_0 + V(t) \) to diagonal form, i.e.,

\[ C^{-1}(t) \left[ A_0 + V(t) \right] C(t) = \hat{\Lambda}(t), \]

where \( \hat{\Lambda}(t) = \text{diag} \left( \hat{\lambda}_1(t), \hat{\lambda}_2(t) \right) \) and \( \hat{\lambda}_1(t), \hat{\lambda}_2(t) \) are the eigenvalues of the matrix \( A_0 + V(t) \).
In (2.28), we make the change of variable
\[ u_2(t) = C(t)u_3(t), \]  
(2.29)
where \( C(t) \) is the matrix from Lemma 2.1. This change of variable brings system (2.28) to what is called \( L \)-diagonal form:
\[ \dot{u}_3 = \left[ \Lambda(t) + R_3(t) \right] u_3, \]  
(2.30)
where \( \Lambda(t) = \text{diag}(\lambda_1(t), \lambda_2(t)), \lambda_j(t) = \hat{\lambda}_j(t)t^{-\alpha} \) \((j = 1, 2)\) and
\[ R_3(t) = -C^{-1}(t)\dot{C}(t) + C^{-1}(t)R_2(t)C(t). \]

The properties (i) and (ii) of the matrix \( C(t) \) imply that matrix \( R_3(t) \) belongs to \( L_1[t_*, \infty) \). To construct the asymptotics for solutions of \( L \)-diagonal system (2.30) as \( t \to \infty \) the well-known Levinson’s theorem can be used. Suppose that the following dichotomy condition holds for the entries of the matrix \( \Lambda(t) \): either the inequality
\[ \int_{t_1}^{t_2} \text{Re}(\lambda_i(s) - \lambda_j(s))ds \leq K_1, \quad t_2 \geq t_1 \geq t_*, \]  
(2.31)
or the inequality
\[ \int_{t_1}^{t_2} \text{Re}(\lambda_i(s) - \lambda_j(s))ds \geq K_2, \quad t_2 \geq t_1 \geq t_*, \]  
(2.32)
is valid for each pair of indices \((i, j)\), where \( K_1, K_2 \) are some constants. What follows is Levinson’s fundamental theorem (see, for instance, [8,9]).

**Theorem 2.2** (Levinson) Let the dichotomy condition (2.31), (2.32) be satisfied. Then the fundamental matrix of Eq. (2.30) has the following asymptotics as \( t \to \infty \):
\[ U(t) = (I + o(1)) \exp\left\{ \int_{t_1}^{t} \Lambda(s)ds \right\}. \]  
(2.33)

**3 Construction of the asymptotic formulas**

With account of (2.3) and (2.13), equation (2.17) for matrix \( H(t, \theta) \), describing the critical manifold \( \mathcal{W}(t) \), takes the following form:
Asymptotic integration of a certain second-order linear delay…

\[
q(t) \begin{pmatrix} \theta - \theta h \\ 1 \\ -h \end{pmatrix} + q(t) \begin{pmatrix} \theta \\ 0 \\ 1 \\ 0 \end{pmatrix} H(t, -h) + H(t, \theta) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + q(t)H(t, \theta) \begin{pmatrix} 0 \\ 0 \\ 1 \\ -h \end{pmatrix}
\]

\[
+ q(t)H(t, \theta) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} H(t, -h) + \frac{\partial H}{\partial t}
\]

\[
= \begin{cases} 
\frac{\partial H}{\partial \theta}, & -h \leq \theta < 0, \\
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} H(t, 0) + q(t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ -h \end{pmatrix} + q(t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} H(t, -h), & \theta = 0.
\end{cases}
\]

(3.1)

System on critical manifold (2.16) has the form

\[
\dot{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + q(t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ -h \end{pmatrix} + q(t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} H(t, -h) \]

(3.2)

Since

\[
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} H(t, -h) = \begin{pmatrix} 0 \\ h_{11}(t, -h) h_{12}(t, -h) \end{pmatrix}, \quad H(t, \theta) = \begin{pmatrix} h_{11}(t, \theta) h_{12}(t, \theta) \\ h_{21}(t, \theta) h_{22}(t, \theta) \end{pmatrix},
\]

we need to determine from Eq. (3.1) only the entries $h_{11}(t, \theta)$ and $h_{12}(t, \theta)$ of the matrix $H(t, \theta)$. Due to [19], solution of Eq. (3.1) may be written in the form

\[
H(t, \theta) = H_1(t, \theta)t^{-\rho} + H_2(t, \theta)t^{-2\rho} + \cdots + H_k(t, \theta)t^{-k\rho} + Z(t, \theta),
\]

(3.3)

where the natural number $k$ is chosen in the way that $(k + 1)\rho > 1$, $\|Z(t, -\cdot)\|_{C_b} \in L_1[t_*, \infty)$ and the entries of matrices $H_i(t, \theta)$ are trigonometric polynomials in variable $t$, whose coefficients depend sufficiently smoothly on $\theta \in [-h, 0]$. We substitute (3.3) into Eq. (3.1) and recall (1.2). Collecting terms with factor $t^{-\rho}$ and omitting the terms from $L_1[t_*, \infty)$, we get the following equation for matrix $H_1(t, \theta):

\[
p(t) \begin{pmatrix} \theta - \theta h \\ 1 \\ -h \end{pmatrix} + H_1(t, \theta) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\partial H_1}{\partial t}
\]

\[
= \begin{cases} \frac{\partial H_1}{\partial \theta}, & -h \leq \theta < 0, \\
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} H_1(t, 0) + p(t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ -h \end{pmatrix}, & \theta = 0.
\end{cases}
\]

(3.4)

We seek the solution of this equation in the form

\[
H_1(t, \theta) = \sum_{j=-N}^{N} \beta_j(\theta)e^{i\omega_j t}.
\]

(3.5)

We now substitute (3.5) into (3.4) and collect terms with like exponentials $e^{i\omega_j t}$. Taking into account formula (1.3), we obtain the following problems for matrices $\beta_j(\theta), j = -N, \ldots, N$:
\[ p_j \begin{pmatrix} \theta & -\theta h \\ 1 & -h \end{pmatrix} + \beta_j(\theta) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + i \omega_j \beta_j(\theta) \]

\[ = \begin{cases} \frac{d\beta_j}{d\theta}, & -h \leq \theta < 0, \\ \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \beta_j(0) + p_j \left( \begin{array}{cc} 0 & 0 \\ 1 & -h \end{array} \right), & \theta = 0. \end{cases} \] (3.6)

Let

\[ \beta_j(\theta) = \begin{pmatrix} \beta_{11}(\theta) & \beta_{12}(\theta) \\ \beta_{21}(\theta) & \beta_{22}(\theta) \end{pmatrix}, \] (3.7)

where we omit temporarily the dependence of the entries of the matrix \( \beta_j(\theta) \) on index \( j \). We, therefore, may rewrite problem (3.6) as a system of the first-order ordinary differential equations

\[ \begin{cases} \beta_{11}'(\theta) = p_j \theta + i \omega_j \beta_{11}(\theta), \\ \beta_{12}'(\theta) = -p_j \theta h + \beta_{11}(\theta) + i \omega_j \beta_{12}(\theta), \\ \beta_{21}'(\theta) = p_j + i \omega_j \beta_{21}(\theta), \\ \beta_{22}'(\theta) = -p_j h + \beta_{21}(\theta) + i \omega_j \beta_{22}(\theta) \end{cases} \] (3.8)

with initial conditions that are defined from the algebraic system

\[ \begin{cases} i \omega_j \beta_{11}(\theta) = \beta_{21}(0), \\ \beta_{11}(0) + i \omega_j \beta_{12}(0) = \beta_{22}(0), \\ i \omega_j \beta_{21}(0) = 0, \\ \beta_{21}(0) + i \omega_j \beta_{22}(0) = 0. \end{cases} \] (3.9)

From (3.9) we deduce that \( \beta_{ij}(0) = 0 \) for all \( i, j = 1, 2 \). Solving (3.8) with zero initial conditions at the point \( \theta = 0 \), we get

\[ \beta_{11}^{(j)}(\theta) := \beta_{11}(\theta) = -\frac{p_j}{\omega_j^2} e^{i \omega_j \theta} - \frac{p_j}{\omega_j} \theta + \frac{p_j}{\omega_j^2}, \] (3.10)

\[ \beta_{12}^{(j)}(\theta) := \beta_{12}(\theta) = \left( \frac{p_j h}{\omega_j^2} + \frac{2 p_j}{i \omega_j^3} \right) e^{i \omega_j \theta} - \frac{p_j}{\omega_j^2} \theta e^{i \omega_j \theta} + \left( \frac{p_j h}{i \omega_j} - \frac{p_j}{\omega_j^2} \right) \theta - \left( \frac{p_j h}{\omega_j^2} + \frac{2 p_j}{i \omega_j^3} \right). \] (3.11)

Finally, as it follows from [19], we have the following estimation for the matrix \( Z(t, \theta) \) in (3.3) as \( t \to \infty \):

\[ \| Z(t, \cdot) \| H = O\left( \frac{d}{dt} (t^{-\rho}) \right) + O(t^{-(k+1)\rho}) = O(t^{-(\rho+1)}) + O(t^{-(k+1)\rho}). \] (3.12)

Thus, using (1.2), (3.3), (3.5), (3.7), (3.10), (3.11), (3.12), we obtain the following representation for system on critical manifold (3.2):

\[ \ddot{u} = \left[ D + A_1(t) t^{-\rho} + A_2(t) t^{-2\rho} + \cdots + A_{k+1}(t) t^{-(k+1)\rho} + R(t) \right] u. \] (3.13)
Asymptotic integration of a certain second-order linear delay...

Here matrix \( D \) is defined in \( (2.13) \) and matrices \( A_1(t), A_2(t) \) have the following expressions:

\[
A_1(t) = p(t) \begin{pmatrix} 0 & 0 \\ 1 & -h \end{pmatrix}, \quad A_2(t) = p(t) \sum_{j=-N}^{N} \begin{pmatrix} 0 & 0 \\ \beta_{11}(j)(-h) & 0 \end{pmatrix} e^{k_{12}j}. \quad (3.14)
\]

We do not need the explicit formulas for matrices \( A_i(t), i = 3, \ldots, k+1 \), for the sequel. We only note that the entries of these matrices are real trigonometric polynomials

\[
A_i(t) = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix},
\]

located in positions, marked by the symbol "\(*\).\) Matrix \( R(t) \) is a certain matrix from \( L_1[t_*, \infty) \) that admits the following asymptotic estimation as \( t \to \infty \):

\[
R(t) = \begin{pmatrix} 0 & 0 \\ (t^{-(k+2)\rho}) + O(t^{-(2\rho+1)}) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (t^{-(k+2)\rho}) + O(t^{-(2\rho+1)}) \end{pmatrix}. \quad (3.15)
\]

In Eq. \( (3.13) \) we make the averaging change of variable

\[
u = \left[ I + Y_1(t) t^{-\rho} + \cdots + Y_{k+1}(t) t^{-(k+1)\rho} \right] u_1.
\]

Substituting the above expression in Eq. \( (3.13) \) and keeping the term of the order \( O(t^{-(\rho+1)}) \) (despite the fact that this term belongs to \( L_1[t_*, \infty) \)), by Theorem 2.1, we obtain the averaged system

\[
\dot{u}_1 = \left[ D + A_1 t^{-\rho} + A_2 t^{-2\rho} + A_3 t^{-3\rho} + \cdots + A_{k+1} t^{-(k+1)\rho} + \rho Y_1(t) t^{-(\rho+1)} + R_1(t) \right] u_1. \quad (3.16)
\]

Here \( A_i, i = 1, \ldots, k+1, \) are certain real constant matrices. Matrices \( A_1, A_2 \) and \( A_3 \) are calculated according to formulas \( (2.23)-(2.25) \). Matrix \( Y_1(t) \), whose entries are trigonometric polynomials with zero mean value, is defined from Eq. \( (2.26) \) and has the following form:

\[
Y_1(t) = \left( \int \int \int p(t) \, dt \right)^2 - 2 \int \int \int p(t) \, dt \, \int \int p(t) \, dt \, dt - h \int \int p(t) \, dt \, dt - h \int \int p(t) \, dt \, dt
\]

Symbol \( \int \) denotes the antiderivative having zero mean value. Matrix \( R_1(t) \in L_1[t_*, \infty) \) and has the asymptotic order \( O(t^{-(k+2)\rho}) + O(t^{-(2\rho+1)}) \) as \( t \to \infty. \)

Due to \( (1.4), (2.23) \) and \( (3.14) \), matrix \( A_1 \) is a zero matrix. When calculating matrix \( A_2 \) according to formula \( (2.24) \), we note that

\[
M[A_1(t) Y_1(t)] = \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix},
\]

\( \odot \) Springer
where \( a = M \left[ \left( \int p(t)dt \right)^2 \right] \) is exactly the quantity, defined in (1.5). Moreover, to get the above expression we also used the following relations that can be proved simply by integration by parts:

\[
M \left[ p(t) \int p(t)dt \right]^2 = - \left[ \left( \int p(t)dt \right)^2 \right], \quad M \left[ p(t) \int p(t)dt \right] = 0,
\]

\[
M \left[ p(t) \int p(t)dt \int p(t)(dt)^2 \right] = - M \left[ \int p(t)dt \int \int p(t)(dt)^2 \right] = 0.
\]

Since, due to (1.3) and (3.10),

\[
M \left[ p(t) \sum_{j=-N}^N \beta_{11}^{(j)}(-h)e^{i\omega_j t} \right] = a - \sum_{j=-N}^N \frac{|p_j|^2}{\omega_j^2} e^{i\omega_j h},
\]

we finally obtain the following formula for matrix \( A_2 \):

\[
A_2 = \begin{pmatrix} 0 & 0 \\ -a(h) & \nu(h) \end{pmatrix}, \quad (3.17)
\]

where

\[
a(h) = \sum_{j=-N}^N \frac{|p_j|^2}{\omega_j^2} e^{i\omega_j h} = 2 \sum_{j=1}^N \frac{|p_j|^2}{\omega_j^2} \cos(\omega_j h), \quad a(0) = a, \quad (3.18)
\]

and, according to (1.3), (3.11),

\[
\nu(h) = M \left[ p(t) \sum_{j=-N}^N \beta_{12}^{(j)}(-h)e^{i\omega_j t} \right] = 2 \sum_{j=-N}^N \frac{|p_j|^2}{\omega_j^2} \left( \frac{h}{\omega_j^2} - \frac{1}{i\omega_j^3} \right) e^{i\omega_j h}
\]

\[
= \sum_{j=1}^N \frac{4|p_j|^2}{\omega_j^2} \left( h \cos(\omega_j h) - \frac{1}{\omega_j} \sin(\omega_j h) \right) = 2\nu(h) - 4 \sum_{j=1}^N \frac{|p_j|^2}{\omega_j^3} \sin(\omega_j h).
\]

(3.19)

By (2.25), we get the following expression for matrix \( A_3 \):

\[
A_3 = \begin{pmatrix} 0 & 0 \\ \phi(h) & \psi(h) \end{pmatrix}.
\]

Here \( \varphi(h) \) and \( \psi(h) \) are some real quantities that do not influence the main part of the asymptotics. To improve the estimation of the absolutely integrable term we introduce in Eq. (3.16) the averaging change of variable

\[
u_1 = \left[ I + V(t)T^{-(\rho+1)} \right] u_2.
\]
Matrix $V(t)$, whose entries are trigonometric polynomials with zero mean value, is the solution of the matrix differential equation

$$\dot{V} - DV + V \dot{D} = \rho Y_1(t).$$

Since $M[Y_1(t)] = 0$, we get the averaged system

$$\dot{u}_2 = \left[ D + A_2 t^{-2\rho} + A_3 t^{-3\rho} + \cdots + A_{k+1} t^{-(k+1)\rho} + R_2(t) \right] u_2,$$

where $R_2(t) = O\left(t^{-(k+2)\rho}\right) + O\left(t^{-(2\rho+1)}\right) + O(t^{-\rho+2})$ as $t \to \infty$. We continue to simplify system on critical manifold by introducing in Eq. (3.20) the so-called shearing transformation

$$u_2 = \begin{pmatrix} t^{\frac{\rho}{2}} & 0 & 0 \\ 0 & t^{-\frac{\rho}{2}} & 0 \end{pmatrix} u_3. \tag{3.21}$$

We get the system

$$\dot{u}_3 = \left[ B_0 t^{-1} + B_1 t^{-\rho} + B_2 t^{-2\rho} + \cdots + B_k t^{-k\rho} + R_3(t) \right] u_3, \tag{3.22}$$

where

$$B_0 = \frac{\rho}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ -a(h) & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ \varphi(h) & \nu(h) \end{pmatrix}, \tag{3.23}$$

and $B_i (i = 3, \ldots, k)$ are some real matrices. Matrix $R_3(t)$ from $L_1[t_*, \infty)$ has the asymptotic order $O(t^{-(k+1)\rho}) + O(t^{-\rho+1}) + O(t^{-2})$ as $t \to \infty$. We can rewrite system (3.22) in the form (2.28). Hence, to construct the asymptotics for its fundamental solutions as $t \to \infty$ we can use Lemma 2.1 together with Levinson’s theorem. Several variation intervals of the parameter $\rho$ should be considered further. First, we assume that

$$\rho > 1. \tag{3.24}$$

This is the case when all functions $t^{-j\rho}$, $j = 1, \ldots, k$, belong to $L_1[t_*, \infty)$. Since $B_0$ is the diagonal matrix, Eq. (3.22) has $L$-diagonal form (2.30). It follows from Theorem 2.2 that the fundamental matrix of Eq. (3.22) has the following asymptotics as $t \to \infty$:

$$U_3(t) = \left[ I + o(1) \right] \begin{pmatrix} t^{-\frac{\rho}{2}} & 0 \\ 0 & t^{\frac{\rho}{2}} \end{pmatrix}.$$ 

With account of (3.21) we get the following asymptotic representation for the fundamental matrix of Eq. (3.20):

$$U_2(t) = \begin{pmatrix} 1 + o(1) & o(t^{\rho}) \\ o(1) & 1 + o(1) \end{pmatrix}, \quad t \to \infty. \tag{3.25}$$
Unfortunately, this formula does not give the leading term for the first component of the second fundamental solution of Eq. (3.20). This can be done in the following way. Let

\[ u_2(t) = \begin{pmatrix} u_{12}(t) \\ u_{22}(t) \end{pmatrix} = \begin{pmatrix} u_{12}(t) \\ 1 + o(1) \end{pmatrix}. \]

Then, taking into account (3.24), we deduce from (3.20) that

\[
\dot{u}_{12} = u_{22}(t) + O(t^{-(\rho+2)})u_{12}(t) + O(t^{-(\rho+2)})u_{22}(t)
\]

\[
= O(t^{-(\rho+2)})u_{12}(t) + 1 + o(1).
\]

Since \( O(t^{-(\rho+2)}) \in L_1[t_*, \infty) \), integrating this equation yields

\[
u_{12}(t) = \exp \left\{ \int_{t_0}^t O(s^{-(\rho+2)}) \, ds \right\} u_{12}(t_*) + \int_{t_0}^t \exp \left\{ \int_s^t O(\tau^{-(\rho+2)}) \, d\tau \right\} \times (1 + o(1)) \, ds = (1 + o(1))t, \quad t \to \infty.
\]

Hence, the fundamental solutions of Eq. (3.20) (and, consequently, the fundamental solutions of Eq. (3.2) as well) have the following asymptotics as \( t \to \infty \):

\[
u^{(1)}(t) = \begin{pmatrix} 1 + o(1) \\ o(1) \end{pmatrix}, \quad \nu^{(2)}(t) = \begin{pmatrix} (1 + o(1))t \\ 1 + o(1) \end{pmatrix}.
\]

It follows from (2.13), (3.3) and (3.12) that

\[
\Phi(0) + H(t, 0) = I + o(1), \quad t \to \infty. \tag{3.26}
\]

Therefore, recalling (2.1), we deduce from (2.18), that all solutions of Eq. (1.6) have the following asymptotics as \( t \to \infty \), provided inequality (3.24) holds:

\[
x(t) = c_1(1 + o(1))t + c_2(1 + o(1)) + O(e^{-\beta t}).
\]

Here \( c_1, c_2 \) are arbitrary real constants and \( \beta > 0 \) is an arbitrary real number. Suppose now that \( \rho = 1 \).

We can rewrite Eq. (3.22) in the following form:

\[
\dot{u}_3 = \left[ A_0 t^{-1} + R_4(t) \right] u_3, \tag{3.27}
\]

where

\[
A_0 = B_0 + B_1 = \begin{pmatrix} -\frac{1}{2} & 1 \\ -a(h) & \frac{1}{2} \end{pmatrix},
\]

\( \mathcal{S} \) Springer
and \( R_4(t) = B_2 t^{-2\rho} + \cdots + B_k t^{-k\rho} + R_3(t) = O(t^{-2}) \in L_1[t_*, \infty). \) The eigenvalues of the matrix \( A_0, \) and, consequently, the asymptotics for solutions of Eq. (3.27) will differ depending on the sign of the quantity \( a(h) - \frac{1}{4}. \)

- \( a(h) > \frac{1}{4}. \) The eigenvalues of the matrix \( A_0 \) are

\[
\lambda_{1,2} = \pm i \sqrt{a(h) - \frac{1}{4}}.
\]

Since the eigenvalues of the matrix \( A_0 \) are distinct, the change of variable \( u_3 = Cu_4, \) where matrix \( C \) brings matrix \( A_0 \) to diagonal form, reduces Eq. (3.27) to \( L- \) diagonal form (2.30) with matrix \( \Lambda(t) = \text{diag}(\lambda_1, \lambda_2)^t \). By applying Levinson’s theorem, we get the following asymptotics for the fundamental matrix of Eq. (3.22) as \( t \to \infty: \)

\[
U_3(t) = \left[ \begin{array}{cc} \exp[\lambda_1 \ln t] & \lambda_1 \exp[\lambda_2 \ln t] \\ 0 & \exp[\lambda_2 \ln t] \end{array} \right] \left( \frac{1}{2} + i \sqrt{a(h) - \frac{1}{4}} \right) + o(1)
\]

If we now return to the initial Eq. (1.6) and recall (2.1), (2.18) and (3.26), we get the following asymptotics for its solutions as \( t \to \infty: \)

\[
x(t) = c_1 (1 + o(1)) t^{\frac{1}{2}} \exp \left\{ i \sqrt{a(h) - \frac{1}{4}} \ln t \right\} + c_2 (1 + o(1)) t^{\frac{1}{2}} \exp \left\{ -i \sqrt{a(h) - \frac{1}{4}} \ln t \right\} + O(e^{-\beta t}).
\]

Here \( c_1, c_2 \) are arbitrary complex constants and \( \beta > 0 \) is an arbitrary real number.

- \( a(h) < \frac{1}{4}. \) The eigenvalues of the matrix \( A_0 \) are defined by the formula

\[
\lambda_{1,2} = \pm \sqrt{\frac{1}{4} - a(h)}.
\]

Acting in the same manner as in the previous case we obtain the following asymptotics for the fundamental matrix of Eq. (3.22) as \( t \to \infty: \)

\[
U_3(t) = \left[ \begin{array}{cc} \frac{1}{2} + \sqrt{\frac{1}{4} - a(h)} & \frac{1}{2} - \sqrt{\frac{1}{4} - a(h)} \\ 0 & \frac{1}{2} + \sqrt{\frac{1}{4} - a(h)} \end{array} \right] + o(1) \left( \begin{array}{cc} t^{\lambda_1} & 0 \\ 0 & t^{\lambda_2} \end{array} \right)
\]

The behaviour of solutions of Eq. (1.6) as \( t \to \infty \) is described by the asymptotic formula

\[
x(t) = c_1 (1 + o(1)) t^{1/2+\sqrt{1/4-a(h)}} + c_2 (1 + o(1)) t^{1/2-\sqrt{1/4-a(h)}} + O(e^{-\beta t}),
\]
where $c_1, c_2$ are arbitrary real constants and $\beta > 0$ is an arbitrary real number. Consider now the case
\[
\frac{1}{2} < \rho < 1.
\] (3.28)

We rewrite Eq. (3.22) in the form
\[
\dot{u}_3 = \left[ B_1 + V(t) \right] t^{-\rho} u_3 + R_4(t) u_3,
\] (3.29)
where $V(t) = B_0 t^{\rho - 1}$ and the matrix $R_4(t) = B_2 t^{-2\rho} + \cdots + B_k t^{-k\rho} + R_3(t) = O(t^{-2\rho})$ belongs to $L_1[t_*, \infty)$. Note, that Eq. (3.29) has the form (2.28), and, hence, asymptotics for its solutions can be constructed by the use of Lemma 2.1 and Theorem 2.2. The asymptotics will differ depending on the sign of the quantity $a(h)$.

- $a(h) > 0$. The eigenvalues of the matrix $\left[ B_1 + V(t) \right] t^{-\rho}$ have the following expression:
\[
\lambda_{1,2}(t) = \pm i \sqrt{a(h)} t^{-\rho} + O(t^{\rho - 2}), \quad t \to \infty.
\]

Since the term $O(t^{\rho - 2})$, due to (3.28), belongs to $L_1[t_*, \infty)$, the fundamental matrix of Eq. (3.29) has the following asymptotic representation as $t \to \infty$:
\[
U_3(t) = \begin{bmatrix}
1 & 1 \\
i \sqrt{a(h)} & -i \sqrt{a(h)}
\end{bmatrix} + o(1) \times \begin{bmatrix}
\exp\left\{i \sqrt{a(h)} (1 - \rho)^{-1} t^{1-\rho}\right\} & 0 \\
0 & \exp\left\{-i \sqrt{a(h)} (1 - \rho)^{-1} t^{1-\rho}\right\}
\end{bmatrix}.
\]

Analogous to the cases considered above, we construct the asymptotics for solutions of the initial Eq. (1.6) as $t \to \infty$. We obtain the following asymptotic formula:
\[
x(t) = c_1 (1 + o(1)) t^{\frac{\rho}{2}} \exp\left\{\frac{i \sqrt{a(h)} (1 - \rho)^{-\frac{3}{2}}}{1 - \rho} t^{1-\rho}\right\}
+ c_2 (1 + o(1)) t^{\frac{\rho}{2}} \exp\left\{-\frac{i \sqrt{a(h)} (1 - \rho)^{-\frac{3}{2}}}{1 - \rho} t^{1-\rho}\right\} + O(e^{-\beta t}),
\]
where $c_1, c_2$ are arbitrary complex constants and $\beta > 0$ is an arbitrary real number.

- $a(h) < 0$. The eigenvalues of the matrix $\left[ B_1 + V(t) \right] t^{-\rho}$ are
\[
\lambda_{1,2}(t) = \pm \sqrt{-a(h)} t^{-\rho} + O(t^{\rho - 2}), \quad t \to \infty.
\]
We, thus, have the following asymptotic representation for the fundamental matrix of Eq. (3.29) as $t \to \infty$:

$$U_3(t) = \left[ \begin{pmatrix} 1 \sqrt{-a(h)} & 1 \sqrt{-a(h)} \\ -1,1 \end{pmatrix} + o(1) \right] + \frac{1}{t^\rho} \left[ \begin{pmatrix} \exp\{\sqrt{-a(h)}(1 - \rho) t^1\rho\} & 0 \\ 0 & \exp\{-\sqrt{-a(h)}(1 - \rho) t^1\rho\} \end{pmatrix} \right].$$

Solutions of Eq. (1.6) have the following asymptotics as $t \to \infty$:

$$x(t) = c_1 \left( 1 + o(1) \right) t^{\frac{\nu(h)}{2}} \exp\left\{ \frac{\sqrt{-a(h)}}{1 - \rho} t^1\rho \right\} + c_2 \left( 1 + o(1) \right) t^{\frac{\nu(h)}{2}} \exp\left\{ -\frac{\sqrt{-a(h)}}{1 - \rho} t^1\rho \right\} + O(e^{-\beta t}),$$

where $c_1, c_2$ are arbitrary real constants and $\beta > 0$ is an arbitrary real number.

Finally, suppose that $\rho \leq \frac{1}{2}$.

System (3.22) may be written in the form

$$\dot{u}_3 = \left[ B_1 + V(t) \right] t^{-\rho} u_3 + R_3(t) u_3, \quad (3.31)$$

where $V(t) = B_2 t^{-\rho} + \cdots + B_k t^{-(k-1)\rho} + B_0 t^{\rho}$. As before, asymptotics will differ depending on the sign of the quantity $a(h)$.

- $a(h) > 0$. It follows from formulas (3.23) for matrices $B_1$ and $B_2$ that the eigenvalues of the matrix $[B_1 + V(t)] t^{-\rho}$ are complex conjugate for sufficiently large $t$ and have the form

$$\lambda_{1,2}(t) = \left( \frac{\nu(h)}{2} + O(t^{-\rho}) \right) t^{-2\rho} \pm i\sqrt{-a(h)} t^{-\rho} \left( 1 + O(t^{-\rho}) \right), \quad t \to \infty. \quad (3.32)$$

Here quantity $\nu(h)$ is defined by formula (3.19). We remark that the term $O(t^{-3\rho})$ in (3.32) belongs to $L_1[t^\ast, \infty)$ if $\rho > \frac{1}{3}$. By using Lemma 2.1 together with Levinson’s theorem, we obtain the following asymptotic formula for the fundamental matrix of Eq. (3.31) as $t \to \infty$:

$$U_3(t) = \left[ \begin{pmatrix} \frac{1}{i\sqrt{a(h)}} & \frac{1}{i\sqrt{-a(h)}} \\ 1,1 \end{pmatrix} + o(1) \right] \begin{pmatrix} \exp\{g_c(t)\} & 0 \\ 0 & \exp\{-g_c(t)\} \end{pmatrix},$$

where

$$g_c(t) = f(t) + i\sqrt{a(h)}(1 - \rho)^{-1} t^{1-\rho} \left( 1 + o(1) \right) \quad (3.33)$$
and

\[
  f(t) = \begin{cases}
    \frac{\nu(h)}{2} \ln t, & \rho = \frac{1}{2}, \\
    \frac{\nu(h)}{2(1 - 2\rho)} t^{1-2\rho}, & \frac{1}{3} < \rho < \frac{1}{2}, \\
    \left(\frac{\nu(h)}{2} + o(1)\right) t^{1-2\rho} (1 - 2\rho), & \rho \leq \frac{1}{3}.
  \end{cases}
\]  

(3.34)

We get the following asymptotics for solutions of Eq. (1.6) as \( t \to \infty \):

\[
  x(t) = c_1 (1 + o(1)) t^{\frac{\rho}{2}} \exp \left( f(t) \pm \frac{i\sqrt{\nu(h)}}{1 - \rho} t^{1-\rho} (1 + o(1)) \right)
  + c_2 (1 + o(1)) t^{\frac{\rho}{2}} \exp \left( f(t) - \frac{i\sqrt{\nu(h)}}{1 - \rho} t^{1-\rho} (1 + o(1)) \right) + O(e^{-\beta t}),
\]

(3.35)

where \( c_1, c_2 \) are arbitrary complex constants and \( \beta > 0 \) is an arbitrary real number.

In particular, if \( \rho = \frac{1}{2} \), according to (3.34) the asymptotic representation (3.35) gets the following form:

\[
  x(t) = c_1 (1 + o(1)) t^{1/4 + \nu(h)/2} \exp \left( 2i\sqrt{\nu(h)} t^{1/2} (1 + o(1)) \right)
  + c_2 (1 + o(1)) t^{1/4 + \nu(h)/2} \exp \left( -2i\sqrt{\nu(h)} t^{1/2} (1 + o(1)) \right) + O(e^{-\beta t}).
\]

\( a(h) < 0 \). The eigenvalues of the matrix \( B_1 + V(t) \) in this case are real for sufficiently large \( t \):

\[
  \lambda_{1,2}(t) = \pm \sqrt{-a(h)} t^{-\rho} (1 + O(t^{-\rho})), \quad t \to \infty.
\]

Fundamental matrix of Eq. (3.31) has the following asymptotics as \( t \to \infty \):

\[
  U_3(t) = \left( \begin{array}{cc}
  \frac{1}{\sqrt{-a(h)}} & \frac{1}{\sqrt{-a(h)}} + o(1) \\
  \exp\{g_r(t)\} & 0 \\
  \exp\{-g_r(t)\} & \end{array} \right),
\]

where

\[
  g_r(t) = \sqrt{-a(h)} (1 - \rho)^{-1} t^{1-\rho} (1 + o(1)).
\]  

(3.36)

Hence, we have the following asymptotic representation for solutions of Eq. (1.6) as \( t \to \infty \):

\[
  x(t) = c_1 (1 + o(1)) t^{\frac{\rho}{2}} \exp \left( \frac{-a(h)}{1 - \rho} t^{1-\rho} (1 + o(1)) \right)
  + c_2 (1 + o(1)) t^{\frac{\rho}{2}} \exp \left( -\frac{-a(h)}{1 - \rho} t^{1-\rho} (1 + o(1)) \right) + O(e^{-\beta t}),
\]

where \( c_1, c_2 \) are arbitrary real constants and \( \beta > 0 \) is an arbitrary real number.
4 Comparison of the asymptotics for solution of Eqs. (1.1) and (1.6)

In this section we compare the behaviour of solutions of Eq. (1.1) and Eq. (1.6) with function $q(t)$ having the form (1.2) as $t \to +\infty$. Asymptotic formulas for fundamental solutions of Eq. (1.1) for different values of parameter $\rho$ are given in Table 1.

In Table 2 the asymptotic formulas for solutions of Eq. (1.6), obtained in the previous section, are collected. Therein $c_1$, $c_2$ are arbitrary real (or complex) constants, $\beta > 0$ is an arbitrary real number, quantity $a(h)$ is defined by formula (3.18) and functions $g_c(t)$, $g_r(t)$ are described by formulas (3.33), (3.36).

### Table 1  Asymptotic formulas for fundamental solutions of Eqs. (1.1), (1.2) as $t \to +\infty$ ($a \neq 0$)

| $\rho$ | $x_1(t)$ | $x_2(t)$ |
|------|----------|----------|
| $\rho > 1$ | $x_1(t) = t(1 + o(1))$, $x_2(t) = 1 + o(1)$ |
| $\rho = 1$ | $a > \frac{1}{4}$, $x_1, x_2(t) = \frac{1}{4} \exp\{\pm i\sqrt{a} - 1/4 \ln t\} (1 + o(1))$ |
| $\rho = 1$ | $a < \frac{1}{4}$, $x_1, x_2(t) = \frac{1}{4} \pm i\sqrt{4a-1} (1 + o(1))$ |
| $\frac{1}{2} < \rho < 1$ | $x_1, x_2(t) = \frac{1}{2} \exp\{\pm \sqrt{a}(1-\rho)^{-1} t^{1-\rho}\} (1 + o(1))$ |
| $\rho \leq \frac{1}{2}$ | $x_1, x_2(t) = \frac{1}{2} \exp\{\pm \sqrt{a}(1-\rho)^{-1} t^{1-\rho} (1 + o(1))\} (1 + o(1))$ |

### Table 2  Asymptotic formulas for solutions of Eq. (1.6) as $t \to +\infty$

| $\rho$ | $x(t)$ |
|------|----------|
| $\rho > 1$ | $x(t) = c_1 (1 + o(1)) t + c_2 (1 + o(1)) + O(e^{-\beta t})$ |
| $\rho = 1$ | $a(h) > \frac{1}{4}$, $x(t) = c_1 (1 + o(1)) t^{1/2} \exp\{\pm \sqrt{a(h)} - 1/4 \ln t\} + c_2 (1 + o(1)) t^{1/2} \exp\{-i\sqrt{a(h)} - 1/4 \ln t\} + O(e^{-\beta t})$ |
| $\rho = 1$ | $a(h) < \frac{1}{4}$, $x(t) = c_1 (1 + o(1)) t^{1/2} + \sqrt{4a(h)} + c_2 (1 + o(1)) t^{1/2} - \sqrt{4a(h)} + O(e^{-\beta t})$ |
| $\frac{1}{2} < \rho < 1$ | $a(h) > 0$, $x(t) = c_1 (1 + o(1)) t^{\rho} \exp\{\sqrt{a(h)} (1-\rho)^{-1} t^{1-\rho}\} + c_2 (1 + o(1)) t^{\rho} \exp\{-\sqrt{a(h)} (1-\rho)^{-1} t^{1-\rho}\} + O(e^{-\beta t})$ |
| $\frac{1}{2} < \rho < 1$ | $a(h) < 0$, $x(t) = c_1 (1 + o(1)) t^{\rho} \exp\{\frac{\sqrt{a(h)}}{1-\rho} (1-\rho)^{-1} t^{1-\rho}\} + c_2 (1 + o(1)) t^{\rho} \exp\{\frac{-\sqrt{a(h)}}{1-\rho} (1-\rho)^{-1} t^{1-\rho}\} + O(e^{-\beta t})$ |
| $\rho \leq \frac{1}{2}$ | $a(h) > 0$, $x(t) = c_1 (1 + o(1)) t^{\rho} \exp\{g_c(t)\} + c_2 (1 + o(1)) t^{\rho} \exp\{g_c(t)\} + O(e^{-\beta t})$ |
| $\rho \leq \frac{1}{2}$ | $a(h) < 0$, $x(t) = c_1 (1 + o(1)) t^{\rho} \exp\{g_r(t)\} + c_2 (1 + o(1)) t^{\rho} \exp\{-g_r(t)\} + O(e^{-\beta t})$ |
We now compare the behaviour of solutions of Eq. (1.1) and Eq. (1.6) as $t \to \infty$ by analyzing the asymptotic formulas in Tables 1 and 2.

1. $\rho > 1$.
   The main parts of the asymptotic formulas in this case coincide with each other. Thus, the delay does not significantly influence the dynamics of solutions.

2. $\rho = 1$.
   Unlike Eq. (1.1), the behaviour of solutions of Eq. (1.6) is defined by quantity $a(h)$. The variation of delay results in quantitative and qualitative changes in dynamics of solutions. Let us remark the following feature. It follows from (3.18) that $|a(h)| \leq a$ for all $h \geq 0$. Consequently, if solutions of Eq. (1.1) are oscillating ($a > \frac{1}{4}$), then the solutions of Eq. (1.6) may become nonoscillating for those $h$ that satisfy the inequality $a(h) < \frac{1}{4}$. If solutions of Eq. (1.1) are nonoscillating ($a < \frac{1}{4}$), then the solutions of Eq. (1.6) are also nonoscillating, since in this case $a(h) < \frac{1}{4}$ for all $h \geq 0$. Finally, we note that both Eqs. (1.1) and (1.6) have unbounded solutions in this case.

3. $\frac{1}{2} < \rho < 1$.
   The main difference in dynamics of solutions of Eq. (1.6) and solutions of Eq. (1.1) is the following. Quantity $a(h)$, that defines the dynamics of solutions of Eq. (1.6), unlike quantity $a$, that defines the dynamics of solutions Eq. (1.1), may take negative values. In particular, solutions of Eq. (1.1) are oscillating if $\rho \in \left(\frac{1}{2}, 1\right)$, but solutions of Eq. (1.6) may become nonoscillating for those $h$ that satisfy inequality $a(h) < 0$. We also note that both Eqs. (1.1) and (1.6) have unbounded solutions provided that $\rho \in \left(\frac{1}{2}, 1\right)$.

4. $\rho \leq \frac{1}{2}$.
   First we note that as in the previous case solutions of Eq. (1.6), unlike solutions of Eq. (1.1), may be nonoscillating for the values of $h$ such that $a(h) < 0$. The significant difference in dynamics of solutions of Eq. (1.1) and solutions of Eq. (1.6) may be also observed for the values of $h$ that satisfy inequality $a(h) > 0$. We recall now formulas (3.33), (3.34) and also formula (3.19), describing quantity $v(h)$. We conclude that the behaviour of solutions of Eq. (1.6) as $t \to \infty$ is defined by the sign of the quantity $\frac{1}{4} + \frac{v(h)}{2}$, if $\rho = \frac{1}{2}$, or by the sign of quantity $v(h)$, if $\rho < \frac{1}{2}$. So, if $\rho < \frac{1}{2}$, all nonzero solutions of Eq. (1.1) are unbounded, but all solutions of Eq. (1.6) tend to zero as $t \to \infty$ for those values of parameter $h$ that satisfy inequalities $a(h) > 0$ and $v(h) < 0$. We note that the quantity $v(h)$ is negative for all sufficiently small $h > 0$, since

$$v(h) = -\frac{4h^3}{3} \left(1 + O(h^2)\right) \sum_{j=1}^{N} |p_j|^2, \quad h \to 0.$$  

It appears that differences in dynamics of solutions of Eqs. (1.1) and (1.6), stated above, have much in common with differences in dynamics of Eq. (1.1) and its difference analog.
Asymptotic integration of a certain second-order linear delay...

\[ x(n + 2) - 2x(n + 1) + \left(1 - \frac{p(n)}{n^\rho}\right)x(n) = 0, \quad n \geq n_0, \]

studied in \([5]\).

In conclusion we remark that absolutely analogous to how it was done in the present paper we may construct the asymptotics for solutions of equation

\[ \ddot{x} - \xi(t)p(t)x(t - h) = 0, \quad h > 0, \quad t \geq t_0. \]

Here function \(p(t)\) is defined by formulas \((1.3), (1.4)\), the real function \(\xi(t)\) is of constant sign for sufficiently large and \(\xi(t) \to 0\) as \(t \to \infty\). Moreover, \(\xi'(t) \in L_1[t_0, \infty)\) and there exists nonnegative integer \(K\) such that \(\xi^K(t) \notin L_1[t_0, \infty)\) but \(\xi^{K+1}(t) \in L_1[t_0, \infty)\).

Acknowledgements
This research was supported by the Grant of the President of the Russian Federation No. MK-4625.2016.1.

References

1. Agarwal, R.P., Bohner, M., Li, W.-T.: Nonoscillation and Oscillation: Theory for Functional Differential Equations. Dekker, New York (2004)
2. Bellman, R.: Stability Theory of Differential Equations. McGraw-Hill, New York (1953)
3. Berezansky, L., Braverman, E.: Some oscillation problems for a second order linear delay differential equation. J. Math. Anal. Appl. 220(2), 719–740 (1998)
4. Bodine, S., Lutz, D.A.: Asymptotic analysis of solutions of a radial Schrödinger equation with oscillating potential. Math. Nachr. 279(15), 1641–1663 (2006)
5. Burd, V., Nesterov, P.: Asymptotic behaviour of solutions of the difference Schrödinger equation. J. Differ. Equ. Appl. 17(11), 1555–1579 (2011)
6. Cassell, J.S.: The asymptotic behaviour of a class of linear oscillators. Quart. J. Math. 32(3), 287–302 (1981)
7. Cassell, J.S.: The asymptotic integration of some oscillatory differential equations. Quart. J. Math. 33(2), 281–296 (1982)
8. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955)
9. Eastham, M.S.P.: The Asymptotic Solution of Linear Differential Systems. Clarendon Press, Oxford (1989)
10. Erbe, L.H., Kong, Q., Zhang, B.G.: Oscillation Theory for Functional Differential Equations. Dekker, New York (1995)
11. Hale, J., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations, vol. 99. Springer, New York (1993). (Appl. Math. Sci.)
12. Its, A.R.: The asymptotic behavior of solutions to the radial Schrödinger equation with oscillating potential at energy zero. Selecta Math. Soviet. 3, 291–300 (1984)
13. Kondrat’ev, V.A.: Elementary derivation of a necessary and sufficient condition for non-oscillation of the solutions of a linear differential equation of second order [Elementarnyj vyvod neobhodimogo i dostatochnogo usloviya nekoleblemnosti reshenij linejnogo differencial’nogo uravnenija vtorogo porjadka]. Uspekhi Mat. Nauk 12(3), 159–160 (1957). [in Russian]
14. Ladde, G.S., Lakshmikantham, V., Zhang, B.G.: Oscillation Theory of Differential Equations with Deviating Arguments. Dekker, New York (1987)
15. Levin, A.Y.: Integral criteria for the equation \(\ddot{x} + q(t)x = 0\) to be nonoscillatory [Integral’nyj kriterij neoscialljacionnosti dlja uravnenija \(\ddot{x} + q(t)x = 0\)]. Uspekhi Mat. Nauk 20(2), 244–246 (1965) [in Russian]
16. Levin, A.J.: Behavior of the solutions of the equation \(\ddot{x} + p(t)\ddot{x} + q(t)x = 0\) in the nonoscillatory case. Math. USSR Sbornik 4(1), 33–55 (1968)
17. Nesterov, P.N.: Construction of the asymptotics of the solutions of the one-dimensional Schrödinger equation with rapidly oscillating potential. Math. Notes. 80(2), 233–243 (2006)
18. Nesterov, P.N.: Averaging method in the asymptotic integration problem for systems with oscillatory-decreasing coefficients. Differ. Equ. 43, 745–756 (2007)
19. Nesterov, P.: Asymptotic integration of functional differential systems with oscillatory decreasing coefficients: a center manifold approach. Electron. J. Qual. Theory Differ. Equ. 33, 1–43 (2016)
20. Opluštil, Z., Šremr, J.: Some oscillation criteria for the second-order linear delay differential equation. Math. Bohem. 136(2), 195–204 (2011)
21. Opluštil, Z., Šremr, J.: Myshkis type oscillation criteria for second-order linear delay differential equations. Monatsh. Math. 178(1), 143–161 (2015)