The Satisfactory Partition Problem

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\textbf{Abstract.} The Satisfactory Partition problem consists in deciding if the set of vertices of a given undirected graph can be partitioned into two nonempty parts such that each vertex has at least as many neighbours in its part as in the other part. This problem was introduced by Gerber and Kobler [European J. Oper. Res. 125 (2000) 283-291] and further studied by other authors, but its parameterized complexity remains open until now. It is known that the Satisfactory Partition problem, as well as a variant where the parts are required to be of the same cardinality, are NP-complete. We enhance our understanding of the problem from the viewpoint of parameterized complexity by showing that (1) the problem is FPT when parameterized by the neighbourhood diversity of the input graph, (2) it can be solved in $O(n^{c_\omega})$ where $c_\omega$ is the clique-width, (3) a generalized version of the problem is W[1]-hard when parameterized by the treewidth.

\textbf{Keywords:} Parameterized Complexity · FPT · W[1]-hard · treewidth · clique-width

\section{Introduction}

Gerber and Kobler \cite{Gerber2000} introduced the problem of deciding if a given graph has a vertex partition into two non-empty parts such that each vertex has at least as many neighbours in its part as in the other part. A graph satisfying this property is called partitionable. For example, complete graphs, star graphs, complete bipartite graphs with at least one part having odd size are not partitionable, whereas some graphs are easily partitionable: cycles of length at least 4, trees that are not star graphs \cite{Cappell2013}.

Given a graph $G = (V, E)$ and a subset $S \subseteq V(G)$, we denote by $d_S(v)$ the degree of a vertex $v \in V$ in $G[S]$, the subgraph of $G$ induced by $S$. For $S = V$, the subscript is omitted, hence $d(v)$ stands for the degree of $v$ in $G$. In this paper, we study the parameterized complexity of Satisfactory Partition and Balanced Satisfactory Partition problems. We define these problems as follows:
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**Satisfactory Partition**

**Input:** A graph $G = (V, E)$.

**Question:** Is there a nontrivial partition $(V_1, V_2)$ of $V$ such that for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq d_{V_{3-i}}(v)$?

A variant of this problem where the two parts have equal size is:

**Balanced Satisfactory Partition**

**Input:** A graph $G = (V, E)$ on an even number of vertices.

**Question:** Is there a nontrivial partition $(V_1, V_2)$ of $V$ such that $|V_1| = |V_2|$ and for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq d_{V_{3-i}}(v)$?

Given a partition $(V_1, V_2)$, we say that a vertex $v \in V_i$ is **satisfied** if $d_{V_i}(v) \geq d_{V_{3-i}}(v)$, or equivalently if $d_{V_i}(v) \geq \lceil \frac{d(v)}{2} \rceil$. A graph admitting a non-trivial partition where all vertices are satisfied is called **satisfactory partitionable**, and such a partition is called **satisfactory partition**.

A problem with input size $n$ and parameter $k$ is said to be ‘fixed-parameter tractable (FPT)’ if it has an algorithm that runs in time $O(f(k)n^c)$, where $f$ is some (usually computable) function, and $c$ is a constant that does not depend on $k$ or $n$. What makes the theory more interesting is a hierarchy of intractable parameterized problem classes above FPT which helps in distinguishing those problems that are not fixed parameter tractable. Closely related to fixed-parameter tractability is the notion of preprocessing. A reduction to a problem kernel, or equivalently, problem kernelization means to apply a data reduction process in polynomial time to an instance $(x, k)$ such that for the reduced instance $(x', k')$ it holds that $|x'| \leq g(k)$ and $k' \leq g(k)$ for some function $g$ only depending on $k$. Such a reduced instance is called a problem kernel. We refer to [8] for further details on parameterized complexity.

**Our results:** Our main results are the following:

- The Satisfactory Partition and Balanced Satisfactory Partition problems are fixed parameter tractable (FPT) when parameterized by neighbourhood diversity.
- The Satisfactory Partition and Balanced Satisfactory Partition problems can be solved in polynomial time for graphs of bounded clique-width.
- A generalized version of the Satisfactory Partition problem is W[1]-hard when parameterized by treewidth.

**Previous work:** In the first paper on this topic, Gerber and Kobler [12] considered a generalized version of this problem by introducing weights for the vertices and edges and showed that a general version of the problem is strongly NP-complete. For the unweighted version, they presented some sufficient conditions for the existence of a solution. This problem was further studied in [11, 3, 14]. The Satisfactory Partition problem is NP-complete and this implies that Balanced Satisfactory Partition problem is also NP-complete via a simple reduction in which we add new dummy vertices and dummy edges to the
Both problems are solvable in polynomial time for graphs with maximum degree at most $4$ [4]. They also studied generalizations and variants of this problem when a partition into $k \geq 3$ nonempty parts is required. Bazgan, Tuza, and Vanderpooten [13] studied an “unweighted” generalization of Satisfactory Partition, where each vertex $v$ is required to have at least $s(v)$ neighbours in its own part, for a given function $s$ representing the degree of satisfiability. Obviously, when $s = \lceil \frac{d}{2} \rceil$, where $d$ is the degree function, we obtain satisfactory partition. They gave a polynomial-time algorithm for graphs of bounded treewidth which decides if a graph admits a satisfactory partition, and gives such a partition if it exists.

2 FPT algorithm parameterized by neighbourhood diversity

In this section, we present an FPT algorithm for the Satisfactory Partition and Balanced Satisfactory Partition problems parameterized by neighbourhood diversity. For a vertex $v \in V(G)$, we use $N_G(v) = \{u : (u, v) \in E(G)\}$ to denote the (open) neighbourhood of vertex $v$ in $G$, and $N_G[v] = N_G(v) \cup \{v\}$ to denote the closed neighbourhood of $v$. The degree $d_G(v)$ of a vertex $v \in V(G)$ is $|N_G(v)|$. For a non-empty subset $S \subseteq V(G)$, we define its closed neighbourhood as $N_G[S] = \bigcup_{v \in S} N_G[v]$ and its open neighbourhood as $N_G(S) = N_G[S] \setminus S$. For a non-empty subset $A \subseteq V$ and a vertex $v \in V(G)$, $N_A(v)$ denotes the set of neighbours of $v$ in $A$, that is, $N_A(v) = \{u \in A : (u, v) \in E(G)\}$. We use $d_A(v) = |N_A(v)|$ to denote the degree of vertex $v$ in $G[A]$, the subgraph of $G$ induced by $A$. We say two vertices $u$ and $v$ in $G$ have the same type if and only if $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$. The relation of having the same type is an equivalence relation. The idea of neighbourhood diversity is based on this type structure.

**Definition 1.** [16] The neighbourhood diversity of a graph $G = (V, E)$, denoted by $\text{nd}(G)$, is the least integer $k$ for which we can partition the set $V$ of vertices into $k$ classes, such that all vertices in each class have the same type.

If neighbourhood diversity of a graph is bounded by an integer $k$, then there exists a partition $\{C_1, C_2, \ldots, C_k\}$ of $V(G)$ into $k$ type classes. It is known that such a minimum partition can be found in linear time using fast modular decomposition algorithms [20]. Notice that each type class could either be a clique or an independent set by definition. For algorithmic purpose it is often useful to consider a type graph $H$ of graph $G$, where each vertex of $H$ is a type class in $G$, and two vertices $C_i$ and $C_j$ are adjacent if there is complete bipartite clique between these type classes in $G$. It is not difficult to see that there will be either a complete bipartite clique or no edges between any two type classes. The key property of graphs of bounded neighbourhood diversity is that their type graphs have bounded size. In this section, we prove the following theorem:
Theorem 1. The Satisfactory Partition problem is fixed-parameter tractable when parameterized by the neighbourhood diversity.

Let $G$ be a connected graph such that $\text{nd}(G) = k$. Let $C_1, \ldots, C_k$ be the partition of $V(G)$ into sets of type classes. We assume $k \geq 2$ since otherwise the problem becomes trivial. We define $I_1 = \{C_i \mid C_i \subseteq V_1\}$, $I_2 = \{C_i \mid C_i \subseteq V_2\}$ and $I_3 = \{C_i \mid C_i \cap V_1 \neq \emptyset, C_i \cap V_2 \neq \emptyset\}$ where $(V_1, V_2)$ is a satisfactory partition. We next guess if $C_i$ belongs to $I_1$, $I_2$, or $I_3$. There are at most $3^k$ possibilities as each $C_i$ has three options: either in $I_1$, $I_2$, or $I_3$. We reduce the problem of finding a satisfactory partition to an integer linear programming optimization with $k$ variables. Since integer linear programming is fixed parameter tractable when parameterized by the number of variables [17], we conclude that our problem is FPT when parameterized by the neighbourhood diversity.

ILP Formulation: Given $I_1$, $I_2$ and $I_3$, our goal here is to answer if there exists a satisfactory partition $(V_1, V_2)$ of $G$ with all vertices of $C_i$ are in $V_1$ if $C_i \in I_1$, all vertices of $C_i$ are in $V_2$ if $C_i \in I_2$, and vertices of $C_i$ are distributed amongst $V_1$ and $V_2$ if $C_i \in I_3$. For each $C_i$, we associate a variable: $x_i$ that indicates $|V_1 \cap C_i| = x_i$. Because the vertices in $C_i$ have the same neighbourhood, the variables $x_i$ determine $(V_1, V_2)$ uniquely, up to isomorphism. We now characterize a satisfactory partition in terms of $x_i$. Note that $x_i = n_i = |C_i|$ if $C_i \in I_1$; $x_i = 0$ if $C_i \in I_2$.

Lemma 1. Let $C$ be a clique type class. Then $C$ is either in $I_1$ or $I_2$.

Proof. Let $C$ be a clique type class. Let $u, v \in C$ and $N(u) \setminus \{v\} = N(v) \setminus \{u\} = \{w_1, \ldots, w_m\}$. For the sake of contradiction, suppose $C$ is in $I_3$, that is, there exists a satisfactory partition $(V_1, V_2)$ such that $u \in V_1$ and $v \in V_2$. Assume that $m$ is an even integer, that is, $m = 2\ell$ for some integer $\ell$. As $u \in V_1$, at least $\ell + 1$ vertices from the set $\{w_1, \ldots, w_{2\ell}\}$ must lie in $V_1$ in order to satisfy $u$. Similarly, as $v \in V_2$, at least $\ell + 1$ vertices from the set $\{w_1, \ldots, w_{2\ell}\}$ must lie in $V_2$ in order to satisfy $v$. This is a contradiction, as there are only $2\ell$ vertices in the set. A similar argument holds when $m$ is odd. This proves the lemma.

Now we consider the following four cases:

Case 1: Suppose $v$ belongs to a clique type class $C_j$ in $I_1$. Then the number of neighbours of $v$ in $V_1$, that is,

$$\sum_{i:C_i \in N_H(C_j) \cap I_1} n_i + \left( \sum_{i:C_i \in N_H(C_j) \cap I_3} x_i \right) - 1.$$

The number of neighbours of $v$ in $V_2$, that is,

$$\sum_{i:C_i \in N_H(C_j) \cap I_2} n_i + \sum_{i:C_i \in N_H(C_j) \cap I_3} (n_i - x_i).$$
Therefore, vertex $v$ is satisfied if and only if

$$
\sum_{i: C_i \in N_H(C_j) \cap I_1} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_2} 2x_i \geq 1 + \sum_{i: C_i \in N_H(C_j) \cap I_3} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_1} n_i
$$

(1)

**Case 2:** Suppose $v$ belongs to a clique type class $C_j$ in $I_2$. Then similarly, $v$ is satisfied if and only if

$$
\sum_{i: C_i \in N_H(C_j) \cap I_2} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_3} n_i \geq 1 + \sum_{i: C_i \in N_H(C_j) \cap I_1} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_3} 2x_i
$$

(2)

**Case 3:** Suppose $v$ belongs to an independent type class $C_j$ in $V_1$, that is, $C_j \in I_1 \cup I_3$. Then the number of neighbours of $v$ in $V_1$, that is,

$$
d_{V_1}(v) = \sum_{i: C_i \in N_H(C_j) \cap I_1} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_3} x_i.
$$

Note that if $C_j \in I_3$, then only $x_j$ vertices of $C_j$ are in $V_1$ and the remaining $y_j$ vertices of $C_j$ are in $V_2$. The number of neighbours of $v$ in $V_2$, that is,

$$
d_{V_2}(v) = \sum_{i: C_i \in N_H(C_j) \cap I_2} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_3} (n_i - x_i).
$$

Therefore, $v$ is satisfied if and only if

$$
\sum_{i: C_i \in N_H(C_j) \cap I_1} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_3} 2x_i \geq 1 + \sum_{i: C_i \in N_H(C_j) \cap I_1} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_2} n_i
$$

(3)

**Case 4:** Suppose $v$ belongs to an independent type class $C_j$ in $V_2$, that is, $C_j \in I_2 \cup I_3$. Similarly, vertex $v$ is satisfied if and only if

$$
\sum_{i: C_i \in N_H(C_j) \cap I_2} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_3} n_i \geq 1 + \sum_{i: C_i \in N_H(C_j) \cap I_1} n_i + \sum_{i: C_i \in N_H(C_j) \cap I_3} 2x_i
$$

(4)

We now formulate ILP formulation of satisfactory partition, for given $I_1$, $I_2$ and $I_3$. The question is whether there exist $x_j$ under the conditions $x_j = n_j$ if $C_j \in I_1$, $x_j = 0$ if $C_j \in I_2$, $x_j \in \{1, 2, \ldots, n_j - 1\}$ if $C_j \in I_3$ and the additional conditions described below:

- Inequality 1 for all clique type classes $C_j \in I_1$
- Inequality 2 for all clique type classes $C_j \in I_2$
- Inequality 3 for all independent type classes $C_j \in I_1$
- Inequality 4 for all independent type classes $C_j \in I_2$
\[ \sum_{C_i \in N_H(C_j) \cap I_2} n_i + \sum_{C_i \in N_H(C_j) \cap I_3} n_i = \sum_{C_i \in N_H(C_j) \cap I_1} n_i + \sum_{C_i \in N_H(C_j) \cap I_3} 2x_i \]

for all independent type classes \( C_j \in I_3 \).

For Balanced Satisfactory Partition problem, we additionally ask that

\[ \sum_{i: C_i \in I_1} n_i + \sum_{i: C_i \in I_3} x_i = \sum_{i: C_i \in I_1} (n_i - x_i) + \sum_{i: C_i \in I_2} n_i. \]

**Solving the ILP:** Lenstra [17] showed that the feasibility version of \( p \)-ILP is FPT with running time doubly exponential in \( p \), where \( p \) is the number of variables. Later, Kannan [14] designed an algorithm for \( p \)-ILP running in time \( p^{O(p)} \).

**\( p \)-Variable Integer Linear Programming Feasibility (\( p \)-ILP):** Let matrices \( A \in \mathbb{Z}^{m \times p} \) and \( b \in \mathbb{Z}^{p \times 1} \) be given. The question is whether there exists a vector \( x \in \mathbb{Z}^{p \times 1} \) satisfying the \( m \) inequalities, that is, \( A \cdot x \leq b \). We use the following result:

**Lemma 2.** [17,14,10] \( p \)-ILP can be solved using \( O(p^{2.5p+o(p)} \cdot L) \) arithmetic operations and space polynomial in \( L \). Here \( L \) is the number of bits in the input.

In the formulation for Satisfactory Partition problem, we have at most \( k \) variables. The value of any variable in the integer linear programming is bounded by \( n \), the number of vertices in the input graph. The constraints can be represented using \( O(k^2 \log n) \) bits. Lemma 2 implies that we can solve the problem with the given guess \( I_1, I_2 \) and \( I_3 \) in FPT time. There are at most \( 3^k \) choices for \( (I_1, I_2, I_3) \), and the ILP formula for a guess can be solved in FPT time. Thus Theorem 1 holds.

### 3 Graphs of bounded clique-width

This section presents a polynomial time algorithm for the Satisfactory Partition and Balanced Satisfactory Partition problems for graphs of bounded clique-width. The clique-width of a graph \( G \) is a parameter that describes the structural complexity of the graph; it is closely related to treewidth, but unlike treewidth it can be bounded even for dense graphs. In a vertex-labeled graph, an \( i \)-vertex is a vertex of label \( i \).

A \( c \)-expression is a rooted binary tree \( T \) such that

- each leaf has label \( o_i \) for some \( i \in \{1, \ldots, c\} \),
- each non-leaf node with two children has label \( \cup \), and
- each non-leaf node with only one child has label \( \rho_{i,j} \) or \( \eta_{i,j} \) (\( i, j \in \{1, \ldots, c\}, i \neq j \)).

Each node in a \( c \)-expression represents a vertex-labeled graph as follows:
a $\rho_i$ node represents a graph with one $i$-vertex;

– a $\cup$-node represents the disjoint union of the labeled graphs represented by its children;

– a $\eta_{ij}$-node represents the labeled graph obtained from the one represented by its child by replacing the labels of the $i$-vertices with $j$;

– a $\eta_{ij}$-node represents the labeled graph obtained from the one represented by its child by adding all possible edges between $i$-vertices and $j$-vertices.

A $c$-expression represents the graph represented by its root. A $c$-expression of a $n$-vertex graph $G$ has $O(n)$ vertices. The clique-width of a graph $G$, denoted by $\text{cw}(G)$, is the minimum $c$ for which there exists a $c$-expression $T$ representing a graph isomorphic to $G$.

A $c$-expression of a graph is irredundant if for each edge $\{u, v\}$, there is exactly one node $\eta_{ij}$ that adds the edge between $u$ and $v$. It is known that a $c$-expression of a graph can be transformed into an irredundant one with $O(n)$ nodes in linear time [7]. Here we use irredundant $c$-expression only.

Computing the clique-width and a corresponding $c$-expression of a graph is NP-hard [9]. For $c \leq 3$, we can compute a $c$-expression of a graph of clique-width at most $c$ in $O(n^2m)$ time [10], where $n$ and $m$ are the number of vertices and edges, respectively. For fixed $c \geq 4$, it is not known whether one can compute the clique-width and a corresponding $c$-expression of a graph in polynomial time. On the other hand, it is known that for any fixed $c$, one can compute a $(2^{c+1} - 1)$-expression of a graph of clique-width $c$ in $O(n^3)$ time [13]. For more details see [15]. Now we prove the following theorem.

**Theorem 2.** Given an $n$-vertex graph $G$ and an irredundant $c$-expression $T$ of $G$, the satisfactory partition and balanced satisfactory partition problems are solvable in $O(n^{3c})$ time.

For each node $t$ in a $c$-expression $T$, let $G_t$ be the vertex-labeled graph represented by $t$. We denote by $V_t$ the vertex set of $G_t$. For each $i$, we denote the set of $i$-vertices in $G_t$ by $V_t^i$. For each node $t$ in $T$, we construct a table $dp_t(x, \mathbf{r}, \mathbf{s}, \mathbf{s}) \in \{\text{true, false}\}$ with indices $x : \{1, \ldots, c\} \to \{0, \ldots, n\}$, $\mathbf{r} : \{1, \ldots, c\} \to \{0, \ldots, n\}$, $\mathbf{s} : \{1, \ldots, c\} \to \{-n + 1, \ldots, n - 1\} \cup \{\infty\}$, and $\mathbf{\bar{s}} : \{1, \ldots, c\} \to \{-n + 1, \ldots, n - 1\} \cup \{\infty\}$ as follows. We set $dp_t(x, \mathbf{r}, \mathbf{s}, \mathbf{s}) = \text{true}$ if and only if there exists a partition $(S, \bar{S})$ of $V_t$ such that for all $i \in \{1, 2, \ldots, c\}$

- $\mathbf{r}(i) = |S \cap V_t^i|$;
- $\mathbf{r}(i) = |\bar{S} \cap V_t^i|$;
- if $S \cap V_t^i \neq \emptyset$, then $\mathbf{s}(i) = \min_{v \in S \cap V_t^i} \left\{|N_{G_t}(v) \cap S| - |N_{G_t}(v) \setminus S|\right\}$, otherwise $\mathbf{s}(i) = \infty$;
- if $\bar{S} \cap V_t^i \neq \emptyset$, then $\mathbf{\bar{s}}(i) = \min_{v \in \bar{S} \cap V_t^i} \left\{|N_{G_t}(v) \cap \bar{S}| - |N_{G_t}(v) \setminus \bar{S}|\right\}$, otherwise $\mathbf{\bar{s}}(i) = \infty$.

That is, $\mathbf{r}(i)$ denotes the number of the $i$-vertices in $S$; $\mathbf{\bar{s}}(i)$ denotes the number of the $i$-vertices in $\bar{S}$; $\mathbf{s}(i)$ is the “surplus” at the weakest $i$-vertex in $S$ and $\mathbf{\bar{s}}(i)$ is the “surplus” at the weakest $i$-vertex in $\bar{S}$.
Let $\tau$ be the root of the $c$-expression $T$ of $G$. Then $G$ has a satisfactory partition if there exist $r, \tilde{r}, s, \tilde{s}$ satisfying

1. $dp_r(r, \tilde{r}, s, \tilde{s}) = true$;
2. $\min\{|s(i), \tilde{s}(i)| \geq 0$.

For the Balanced Satisfactory Partition problem, we additionally ask that $\sum_{i=1}^{c} r(i) = \sum_{i=1}^{c} \tilde{r}(i)$. If all entries $dp_r(r, \tilde{r}, s, \tilde{s})$ are computed in advance, then we can verify above conditions by spending $O(1)$ time for each tuple $(r, \tilde{r}, s, \tilde{s})$.

In the following, we compute all entries $dp_t(r, \tilde{r}, s, \tilde{s})$ in a bottom-up manner. There are $(n+1)^c \cdot (n+1)^c \cdot (2n)^c \cdot (2n)^c = O(n^{4c})$ possible tuples $(r, \tilde{r}, s, \tilde{s})$. Thus, to prove Theorem 2, it is enough to prove that each entry $dp_t(r, \tilde{r}, s, \tilde{s})$ can be computed in time $O(n^{4c})$ assuming that the entries for the children of $t$ are already computed.

**Lemma 3.** For a leaf node $t$ with label $o_i$, $dp_t(r, \tilde{r}, s, \tilde{s})$ can be computed in $O(1)$ time.

**Proof.** Observe that $dp_t(r, \tilde{r}, s, \tilde{s}) = true$ if and only if $r(j) = 0$, $\tilde{r}(j) = 0$, $s(j) = 0$, and $\tilde{s}(j) = 0$ for all $j \neq i$ and either

- $r(i) = 0$, $\tilde{r}(i) = 1$, $s(i) = \infty$, $\tilde{s}(i) = 0$, or
- $r(i) = 1$, $\tilde{r}(i) = 0$, $s(i) = 0$, $\tilde{s}(i) = \infty$.

The first case corresponds to $S = \emptyset$, $\tilde{S} = V_t^i$, and the second case corresponds to $S = V_t^i$, $\tilde{S} = \emptyset$. These conditions can be checked in $O(1)$ time.

**Lemma 4.** For a $\cup$-node $t$, $dp_t(r, \tilde{r}, s, \tilde{s})$ can be computed in $O(n^{4c})$ time.

**Proof.** Let $t_1$ and $t_2$ be the children of $t$ in $T$. Then $dp_t(r, \tilde{r}, s, \tilde{s}) = true$ if and only if there exist $r_1, \tilde{r}_1, s_1, \tilde{s}_1$ and $r_2, \tilde{r}_2, s_2, \tilde{s}_2$ such that $dp_t(r_1, \tilde{r}_1, s_1, \tilde{s}_1) = true$, $dp_t(r_2, \tilde{r}_2, s_2, \tilde{s}_2) = true$, $r(i) = r_1(i) + r_2(i)$, $\tilde{r}(i) = \tilde{r}_1(i) + \tilde{r}_2(i)$, $s(i) = \min\{s_1(i), s_2(i)\}$ and $\tilde{s}(i) = \min\{\tilde{s}_1(i), \tilde{s}_2(i)\}$ for all $i$. The number of possible pairs for $(r_1, r_2)$ is at most $(n+1)^c$ as $r_2$ is uniquely determined by $r_1$; the number of possible pairs for $(\tilde{r}_1, \tilde{r}_2)$ is at most $(n+1)^c$ as $\tilde{r}_2$ is uniquely determined by $\tilde{r}_1$. There are at most $2^c(2n)^c$ possible pairs for $(s_1, s_2)$ and for $(\tilde{s}_1, \tilde{s}_2)$ each. In total, there are $O(n^{4c})$ candidates. Each candidate can be checked in $O(1)$ time, thus the lemma holds.

**Lemma 5.** For a $\eta_{ij}$-node $t$, $dp_t(r, \tilde{r}, s, \tilde{s})$ can be computed in $O(1)$ time.

**Proof.** Let $t'$ be the child of $t$ in $T$. Then, $dp_t(r, \tilde{r}, s, \tilde{s}) = true$ if and only if $dp_t(r, \tilde{r}, s', \tilde{s}') = true$ for some $s', \tilde{s}'$ with the following conditions:

- $s(h) = s'(h)$ and $\tilde{s}(h) = \tilde{s}'(h)$ hold for all $h \notin \{i, j\}$;
- $s(i) = s'(i) + 2r(j) - |V_t^i|$ and $s(j) = s'(j) + 2r(i) - |V_t^j|$;
- $\tilde{s}(i) = \tilde{s}'(i) + 2\tilde{r}(j) - |V_t^i|$ and $\tilde{s}(j) = \tilde{s}'(j) + 2\tilde{r}(i) - |V_t^j|$.
We now explain the condition for \( s(i) \). Recall that \( T \) is irredundant. That is, the graph \( G_T \) does not have any edge between the \( i \)-vertices and the \( j \)-vertices. In \( G_T \), an \( i \)-vertex has exactly \( r(j) \) more neighbours in \( S \) and exactly \(|V'_T| - r(j)\) more neighbours in \( S \). Thus we have \( s(i) = s'(i) + 2r(j) - |V'_T| \). The lemma holds as there is only one candidate for each \( s'(i), s''(j), s'(i), s''(j) \).

**Lemma 6.** For a \( \rho_{ij} \)-node \( t \), \( dp_t(r, \bar{s}, s, \bar{s}) \) can be computed in \( O(n^4) \) time.

**Proof.** Let \( t' \) be the child of \( t \) in \( T \). Then, \( dp_t(r, \bar{s}, s, \bar{s}) = \text{true} \) if and only if there exist \( r', \bar{s}', s', \bar{s}' \) such that \( dp_{t'}(r', \bar{s}', s', \bar{s}') = \text{true} \), where:

\[
\begin{align*}
- & r(i) = 0, r(j) = r'(i) + r'(j), \text{ and } r(h) = r'(h) \text{ if } h \notin \{i, j\}; \\
- & s(i) = \infty, s(j) = \min\{s'(i), s'(j)\}, \text{ and } s(h) = s'(h) \text{ if } h \notin \{i, j\}; \\
- & \bar{s}(i) = \infty, \bar{s}(j) = \min\{\bar{s}'(i), \bar{s}'(j)\}, \text{ and } \bar{s}(h) = \bar{s}'(h) \text{ if } h \notin \{i, j\}.
\end{align*}
\]

The number of possible pairs for \((r'(i), r'(j))\) is \( O(n) \) as \( r'(j) \) is uniquely determined by \( r'(i) \); similarly the number of possible pairs for \((\bar{s}'(i), \bar{s}'(j))\) is \( O(n) \) as \( \bar{s}'(j) \) is uniquely determined by \( \bar{s}'(i) \). There are at most \( O(n) \) possible pairs for \((s'(i), s'(j))\) and for \((s'(i), s'(j))\). In total, there are \( O(n^4) \) candidates. Each candidate can be checked in \( O(1) \) time, thus the lemma holds.

### 4 W[1]-hardness parameterized by treewidth

In this section we show that a generalization of Satisfactory Partition is W[1]-hard when parameterized by treewidth. We consider the following generalization of Satisfactory Partition, where some vertices are forced to be in the first part \( V_1 \) and some other vertices are forced to be in the second part \( V_2 \). 

| Satisfactory Partition<sup>FS</sup> |
|---|
| **Input:** A graph \( G = (V, E) \), a set \( V_\Delta \subseteq V(G) \), and a set \( V_\Box \subseteq V(G) \). |
| **Question:** Is there a satisfactory partition \((V_1, V_2)\) of \( V \) such that (i) \( V_\Delta \subseteq V_1 \) (ii) \( V_\Box \subseteq V_2 \). |

In this section, we prove the following theorem:

**Theorem 3.** The Satisfactory Partition<sup>FS</sup> is W[1]-hard when parameterized by the treewidth of the graph.

We prove W[1]-hardness by using techniques from a hardness result for the problem of finding minimum defensive alliance in a graph [3]. Let \( G = (V, E) \) be an undirected and edge weighted graph, where \( V, E \), and \( w \) denote the set of nodes, the set of edges and a positive integral weight function \( w : E \rightarrow \mathbb{Z}^+ \), respectively. An orientation \( \Lambda \) of \( G \) is an assignment of a direction to each edge \( \{u, v\} \in E(G) \), that is, either \( (u, v) \) or \( (v, u) \) is contained in \( \Lambda \). The weighted outdegree of \( u \) on \( \Lambda \) is \( w^\text{out}_u = \sum_{\{u, v\} \in \Lambda} w(\{u, v\}) \). We define Minimum Maximum Outdegree problem as follows:
Lemma 7. The Satisfactory Partition\textsuperscript{FSC} is $W[1]$-hard when parameterized by the treewidth of the primal graph.

Proof. Let $G = (V, E, w)$ and a positive integer $r$ be an instance of Minimum Maximum Outdegree. We construct an instance of Satisfactory Partition\textsuperscript{FSC} as follows. An example is given in Figure 1. For each vertex $v \in V(G)$, we introduce a set of new vertices $H_v = \{h_{v1}^\Delta, \ldots, h_{v2}^\Delta\}$. For each edge $(u, v) \in E(G)$, we introduce the set of new vertices $V_{uv} = \{u_1^v, \ldots, u_{w(u,v)}^v\}$, $V'_{uv} = \{v_1^v, \ldots, v_{w(u,v)}^v\}$, $V''_{uv} = \{w_1^{u,v}, \ldots, w_{w(u,v)}^{u,v}\}$. We now define the graph $G'$ with

$$V(G') = \bigcup_{v \in V(G)} H_v \bigcup_{(u, v) \in E(G)} (V_{uv} \cup V''_{uv} \cup V'_v \cup V''_v).$$
and

\[ E(G') = \{(v, h) | v \in V(G), h \in H_v \} \]

\[ \bigcup \left\{ (u, x) | (u, v) \in E(G), x \in V_{uv} \cup V_{\square} \right\} \]

\[ \bigcup \left\{ (x, v) | (u, v) \in E(G), x \in V_{uv} \cup V_{\square} \right\} \]

\[ \bigcup \left\{ (u_i^v, u_{i+1}^v), (v_i^v, v_{i+1}^v), (v_i^u, v_{i+1}^u) | (u, v) \in E(G), 1 \leq i \leq w(u, v) \right\}. \]

We define the complementary vertex pairs

\[ C = \left\{ (u_i^v, u_{i+1}^v), (u_i^v, v_{i+1}^v), (v_i^u, v_{i+1}^u), (u_i^v, v_{i+1}^u) | (u, v) \in E(G), 1 \leq i \leq w(u, v) \right\}. \]

Complementary vertex pairs are shown in dashed lines in Figure 1. Finally we define \( V_\square = V(G) \bigcup \cup_{v \in V(G)} H_v \) and \( V_{\square} = \bigcup_{(u, v) \in E(G)} (V_{\square} \cup V_{\square} \cup V_{\square} \cup V_{\square}). \)

We use \( I \) to denote \((G', V_\square, V_{\square}, C)\) which is an instance of SATISFACTORY PARTITION FSC.

Clearly, it takes polynomial time to compute \( I \). We now prove that the treewidth of the primal graph \( G' \) of \( I \) is bounded by a function of the treewidth of \( G \). We do so by modifying an optimal tree decomposition \( \tau \) of \( G \) as follows:

- For each \((u, v) \in E(G)\), we take an arbitrary node whose bag \( B \) contains both \( u \) and \( v \) and add to it a chain of nodes \( 1, 2, \ldots, w(u, v) - 1 \) such that the bag of node \( i \) is \( B \cup \{u_i^v, u_{i+1}^v, v_i^u, v_{i+1}^u, u_i^v, u_{i+1}^v, v_i^u, v_{i+1}^u\} \).
- For each \((u, v) \in E(G)\), we take an arbitrary node whose bag \( B \) contains \( u \) and add to it a chain of nodes \( 1, 2, \ldots, w(u, v) \) such that the bag of node \( i \) is \( B \cup \{u_i^v, u_{i+1}^v\} \).
- For each \((u, v) \in E(G)\), we take an arbitrary node whose bag \( B \) contains \( v \) and add to it a chain of nodes \( 1, 2, \ldots, w(u, v) \) such that the bag of node \( i \) is \( B \cup \{v_i^u, v_{i+1}^u\} \).
- For each \( v \in V(G) \), we take an arbitrary node whose bag \( B \) contains \( v \) and add to it a chain of nodes \( 1, 2, \ldots, 2r \) such that the bag of node \( i \) is \( B \cup \{v_i^u, v_{i+1}^u\} \).

Clearly, the modified tree decomposition is a valid tree decomposition of the primal graph of \( I \) and its width is at most the treewidth of \( G \) plus eight.

Let \( D \) be the directed graph obtained by an orientation of the edges of \( G \) such that for each vertex the sum of the weights of outgoing edges is at most \( r \).

Consider the partition

\[ V_1 = V_{\square} \bigcup \bigcup_{(u, v) \in E(D)} (V_{uv} \cup V_{vu}^\square) \bigcup H_v \bigcup \bigcup_{(u, v) \in E(D)} (V_{uv} \cup V_{vu}^\square) \]

and

\[ V_2 = \bigcup_{(u, v) \in E(D)} (V_{uv} \cup V_{vu}^\square \cup V_{uv} \cup V_{vu}^\square) \bigcup \bigcup_{(u, v) \in E(D)} (V_{uv} \cup V_{vu}^\square). \]
To prove that \((V_1, V_2)\) is a satisfactory partition, first we prove that \(d_{V_1}(x) \geq d_{V_2}(x)\) for all \(x \in V_1\). If \(x\) is a vertex in \(H_v\) or \(V_{uv} \cup V'_{uv}\), then clearly all neighbours of \(x\) are in \(V_1\), hence \(x\) is satisfied. Suppose \(x \in V(G)\). Let \(w_{out}^x\) and \(w_{in}^x\) denote the sum of the weights of outgoing and incoming edges of vertex \(x\), respectively. Hence \(d_{V_1}(x) = 2r + w_{in}^x\) and \(d_{V_2}(x) = 2w_{out}^x + w_{in}^x\) in \(G'\). This shows that \(x\) is satisfied as \(w_{out}^x \leq r\). Now we prove that \(d_{V_2}(x) \geq d_{V_1}(x)\) for all \(x \in V_2\). If \(x\) is a vertex in \(V_{uu} \cup V_{uv} \cup V'_{uv}\) then \(x\) has one neighbour in \(V_1\) and one neighbour in \(V_2\). If \(x \in V'_{uu} \cup V_{uv} \cup V'_{uv}\) then \(x\) has one neighbour in \(V_2\) and no neighbours in \(V_1\). Thus the vertices in \(V_2\) are satisfied.

Conversely, suppose \((V_1, V_2)\) is a satisfactory partition of \(I\). For every \((u, v) \in E(G)\), either \(V_{uv} \cup V'_{uv} \in V_1\) or \(V_{vu} \cup V'_{vu} \in V_1\) due to the complementary vertex pairs. We define a directed graph \(D\) by \(V(D) = V(G)\) and

\[
E(D) = \left\{ (u, v) \mid V_{vu} \cup V'_{vu} \in V_1 \right\} \cup \left\{ (v, u) \mid V_{uv} \cup V'_{uv} \in V_1 \right\}.
\]
Suppose there is a vertex $x$ in $D$ for which $w^x_{\text{out}} > r$. Clearly $x \in V_1$. We know $d_{V_1}(x) = 2r + w^x_{\text{in}}$ and $d_{V_2}(x) = 2w^x_{\text{out}} + w^x_{\text{in}}$. Then $d_{V_2}(x) > d_{V_1}(x)$, as by assumption $w^x_{\text{out}} > r$, a contradiction to the fact that $(V_1, V_2)$ is a satisfactory partition of $G'$. Hence $w^x_{\text{out}} \leq r$ for all $x \in V(D)$.

Next we prove the following result which eliminates complementary pairs.

**Lemma 8.** Satisfactory Partition$^{\text{FS}}$, parameterized by the treewidth of the graph, is W[1]-hard.

**Proof.** Let $I = (G, V\Box, V\triangle, C)$ be an instance of Satisfactory Partition$^{\text{FSC}}$. Consider the primal graph of $I$, that is the graph $G^p$ where $V(G^p) = V(G)$ and $E(G^p) = E(G) \cup C$. From this we construct an instance $I' = (G', V'_\Box, V'_\triangle)$ of Satisfactory Partition$^{\text{FS}}$ problem. For each $(a, b) \in C$ in the primal graph $G^p$, we introduce two new vertices $\triangle_{ab}$ and $\Box_{ab}$ and four new edges in $G'$. We now define the $G'$ with

$$V(G') = V(G) \cup \bigcup_{(a, b) \in C} \{\triangle_{ab}, \Box_{ab}\}$$

and

$$E(G') = E(G) \cup \bigcup_{(a, b) \in C} \{(a, \triangle_{ab}), (a, \Box_{ab}), (b, \triangle_{ab}), (b, \Box_{ab})\}.$$  

Finally, we define the sets $V'_\triangle = V\triangle \cup (a, b) \in C \{\triangle_{ab}\}$ and $V'_\Box = V\Box \cup (a, b) \in C \{\Box_{ab}\}$.

We illustrate our construction in Figure 2. It is easy to see that we can compute $I'$ in polynomial time and its treewidth is linear in the treewidth of $I$.

![Fig. 2. Gadget for a pair of complementary vertices $(a, b)$ in the reduction from Satisfactory Partition$^{\text{FSC}}$ to Satisfactory Partition$^{\text{FS}}$.](image)

The following holds for every solution $(V'_1, V'_2)$ of $I'$: $V'_1$ contains $\triangle_{ab}$ for every $(a, b) \in C$, so it must also contain $a$ or $b$. It cannot contain both $a$ and $b$ for any $(a, b) \in C$, because $\Box_{ab} \in V'_2$. Restricting $(V'_1, V'_2)$ to the original vertices thus is a solution to $I$. Conversely, for every solution $(V_1, V_2)$ of $I$, the partition $(V'_1, V'_2)$ where $V'_1 = V_1 \cup (a, b) \in C \{\triangle_{ab}\}$ and $V'_2 = V_2 \cup (a, b) \in C \{\Box_{ab}\}$, is a solution of $I'$.

This proves Theorem 3.
5 Conclusion

In this work we proved that the Satisfactory Partition and Balanced Satisfactory Partition problems are FPT when parameterized by neighbourhood diversity; the problems are polynomial time solvable for graphs of bounded clique width, a generalized version of the Satisfactory Partition problem is \( W[1] \)-hard when parameterized by treewidth. The parameterized complexity of the Satisfactory Partition problem remains unsettled when parameterized by other important structural graph parameters like clique-width, modular width and treedepth.

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