α-BALL DIVERGENCE AND ITS APPLICATIONS TO CHANGE-POINT PROBLEMS FOR BANACH-VALUED SEQUENCES *

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In this paper, we extend a measure of divergence between two distributions: Ball divergence, to a new one: α-Ball divergence. With this new notion, we propose its sample statistic which can be used to test whether two weakly dependent sequences of Banach-valued random vectors have the same distribution. The properties of α-Ball divergence and its sample statistic, as Ball divergence has, are inspected and shown to hold for random sequences which are functionals of some absolutely regular sequences. We further apply the sample statistic to change-point problems for a sequence of weakly dependent Banach-valued observations with multiple possible change-points. Our procedure does not require any assumptions on special change-point type. It could detect the number of change-points as well as their locations. We also prove the consistency of the estimated change-point locations. Extensive simulation studies and analyses of two interesting real data sets about wind direction and bitcoin price illustrate that our procedure has considerable advantages over other existing competitors, especially when observations are non-Euclidean or there are distributional changes in the variance.

1. Introduction. Stationarity is crucial in analyzing random sequences. Without stationarity, it is hard to do statistical inference which usually requests probabilistic mechanism constant underlying a segment of observations. Therefore, it is important to check whether changes occur in a sequence of observations prior to statistical inference. Such change-point
problems arise in many fields of real world applications: abrupt events in video surveillance [33]; deterioration of product quality in quality control [26]; credit card fraud in finance [4]; alteration of genetic regions in cancer research [15] and so on.

To detect changes in a sequence of observations, change-point analysis, as a branch of statistics, gives a research framework and provides an effective tool. There is a large and rapidly growing literature on it [2, 34, 38, 16]. Many methods rely on the assumed parametric models to detect special change types such as location, scale or presumed distribution family. Page [35] introduced a method by examining the ratio of log-likelihood functions. Lavielle and Teyssiere [27] detected change-points by maximizing a log-likelihood function. Yau and Zhao [42] proposed a likelihood ratio scan method for piecewise stationary autoregressive time series. The Bayesian change-point detection methods assume that the observations are normally distributed, and calculate the probability of change-point at each point [3, 40, 31]. We just name but a few. Since parametric methods potentially suffer from model misspecification, most current methods are developed to detect general distributional changes with more relaxed assumptions. Kawahara and Sugiyama [24] provided an algorithm which relied heavily on estimating the ratio of probability densities. Lung-Yut-Fong, Lévy-Leduc and Cappé [30] identified change-points via the well-known Wilcoxon rank statistic. Matteson and James [32] proposed a new nonparametric method using the concept of energy distance for independent observations. There are also some binary segmentation methods which are based on CUSUM-type statistics [16, 10, 14]. Two advantages of the binary segmentation procedure are its simplicity and computational efficiency, but one drawback is the inability to control the overall significance level, in other words, binary segmentation is a ‘greedy’ procedure. Zou et al. [43] introduced a nonparametric empirical likelihood approach to detect multiple change-points in the independent sequences. The locations of the change-points are estimated via the dynamic programming algorithm and the use of the intrinsic order structure of the likelihood function. Automatically detecting the number of change-points is also an important issue of the change-point detection method. Some methods are under the assumption that only one change-point exists [37]. Some methods suppose that the number of change-points are known but locations of them are unknown, such as Hawkins [18], Lung-Yut-Fong, Lévy-Leduc and Cappé [30], which is inconvenient in real data analysis.

With the increasing richness of data types, modern statistical applications are often faced with non-Euclidean data, such as shape data, functional data, spatial data and so on. One of two applications in this paper is to study
changes in monsoon direction which is defined in a simple Riemannian manifold: Circle. As illustrated in the section of real data analysis, traditional statistical tools developed in Hilbert spaces failed in detecting the changes with the wind direction data of Yunnan-Guizhou Plateau (105°E, 27°N) collected from 2015/06/01 to 2015/10/30. To our best knowledge, few methods have been presented to detect change-points in a Banach-valued sequence. Chen et al. [8] and Chu and Chen [11] proposed a series of graph-based non-parametric approaches that could be applied to non-Euclidean data with arbitrary dimension. However, their proposed methods have several shortcomings: they work only for iid observations; they can only detect one or two change-points, instead of estimating the number of change-points automatically; and they have not been verified to detect change-points consistently in theory. Therefore, it is still a challenge to develop more suitable methods of detecting arbitrary distributional changes, change-point locations as well as the number of change-points for Banach-valued sequences.

Ball Divergence [36] is a recently developed measure of the divergence between two probabilities in separable Banach spaces. Ball Divergence of two probabilities is zero if and only if these two probability measures are identical even for non-Euclidean data without any moment assumptions. Since its sample statistic is constructed by metric ranks, the test procedure is robust to heavy-tailed data or outliers. Also, the test procedure has been proved to be consistent against alternative hypothesis and works for imbalanced data. In this paper, we expect to develop a procedure to detect change-points for Banach-valued sequences in the spirit of Ball Divergence. We extend this notion to $\alpha$-Ball divergence such that its sample statistic is more suitable for weakly dependent sequences, and then we inspect the properties of $\alpha$-Ball divergence and its sample statistic as Ball divergence has. We find that the properties of Ball divergence continue to hold for Banach-valued sequences which are functionals of some absolutely regular sequences. We further develop a hierarchical algorithm for multiple change-point problems with the statistic of $\alpha$-Ball divergence. Summarily, our method has the following advantages: First, it works for a sequence of observations which could be multivariate, weakly dependent, or Banach-valued. Second, it does not require any assumptions on the distributions of observations, or any special change types. Simulation studies and real data analyses show that it works well for scale changes. Third, it is robust to heavy-tailed data or outliers. Finally, our hierarchical algorithm can detect change-points consistently without the assumption that the number of change-points is known.

The rest of the article is organized as follows. In section 2, we review the notion of Ball divergence and its sample statistic, and generalize them to
weakly dependent Banach-valued sequences. Under the mild conditions we show that their properties can be inherited. We address change-point problems and the application of new notion in section 3. In section 4, we show the performance of our method with comparison to some existing methods in various simulation settings. In section 5, two real data analyses illustrate the versatility and practicability of our proposed method. We make some concluding remarks in Section 6. All technical details are deferred to Appendix.

2. Ball divergence for random sequences.

2.1. A Review of Ball Divergence. Ball divergence (BD) is a measure of the difference between two probabilities in a separable Banach space \((A, \|\cdot\|)\), with the norm \(\|\cdot\|\). \(\forall u,v \in A\), the distance between \(u\) and \(v\) deduced from the norm is \(\rho(u,v) = \|u - v\|\). Denote by \(\bar{B}(u,r) = \{x|\rho(x,u) \leq r\}\) a closed ball and \(\mathcal{B}\) the smallest \(\sigma\)-algebra in \(A\) that contains all closed (or open) subsets of \(A\). Let \(\mu\) and \(\nu\) be two probabilities on \(\mathcal{B}\). Ball divergence is defined as follows.

**DEFINITION 2.1.** The Ball divergence of two Borel probabilities \(\mu\) and \(\nu\) in a separable Banach space \(A\) is defined as an integral of the square of the measure difference between \(\mu\) and \(\nu\) over arbitrary closed balls as following,

\[
D(\mu, \nu) = \int \int_{A \times A} (\mu - \nu)^2(\bar{B}(u,\rho(u,v)))(\mu(du)\mu(dv) + \nu(du)\nu(dv)).
\]

This new notion has a remarkable property as stated following:

**THEOREM 2.1.** Given two Borel probabilities \(\mu\) and \(\nu\) in a separable Banach space \(A\), then \(D(\mu, \nu) \geq 0\) where the equality holds if and only if \(\mu = \nu\).

Given two independent samples, \(X(M/0) = \{X_1, \ldots, X_M\}\) with the associated probability \(\mu\), and \(Y(N/0) = \{Y_1, \ldots, Y_N\}\) with \(\nu\) in a Banach space, Pan et al. [36] constructed a metric rank statistic for Ball divergence. Denote the indicator function by \(I(\cdot)\). Let \(c(x, y; z) = I(z \in \bar{B}(x, \rho(x,y)))\), which identifies whether the point \(z\) falls into the closed ball \(\bar{B}(x, \rho(x,y))\) with \(x\) as the center and \(\rho(x,y)\) as the radius, and \(e(x, y, z_1, z_2) = c(x, y; z_1)c(x, y; z_2)\), which determines whether two points \(z_1\) and \(z_2\) fall into the ball \(\bar{B}(x, \rho(x,y))\) together. Then let

\[
C_{ij}^{XX} = \frac{1}{M} \sum_{u=1}^{M} c(X_i, X_j; X_u), C_{ij}^{XY} = \frac{1}{N} \sum_{v=1}^{N} c(X_i, X_j; Y_v),
\]
\[ C_{kl}^{XX} = \frac{1}{M} \sum_{u=1}^{M} c(Y_k, Y_l; X_u), \quad C_{kl}^{YY} = \frac{1}{N} \sum_{v=1}^{N} c(Y_k, Y_l; Y_v), \]

where \(C_{ij}^{XX}\) represents the proportion of samples from the probability measure \(\mu\) located in the closed ball \(\bar{B}(X_i, \rho(X_i, X_j))\) and \(C_{ij}^{YY}\) represents the proportion of samples from the probability measure \(\nu\) located in the closed ball \(\bar{B}(X_i, \rho(X_i, X_j))\). Meanwhile, \(C_{kl}^{YX}\) and \(C_{kl}^{YY}\) represent the proportions of samples from the corresponding proportions located in the closed ball \(\bar{B}(Y_k, \rho(Y_k, Y_l))\). Let \(C_{M,N}^{1}\) and \(C_{M,N}^{2}\) be the averages of the square of the proportion difference from two samples over the balls made from two samples respectively:

\[ C_{M,N}^{1} = \frac{1}{M^2} \sum_{i,j=1}^{M} (C_{ij}^{XX} - C_{ij}^{XY})^2, \quad C_{M,N}^{2} = \frac{1}{N^2} \sum_{k,l=1}^{N} (C_{kl}^{YX} - C_{kl}^{YY})^2. \]

The sample statistic of Ball divergence is defined as

\[ D_{M,N} = C_{M,N}^{1} + C_{M,N}^{2}. \]

2.2. Notion Extension. To develop a novel procedure to detect change-points for Banach-valued sequences with the concept of Ball divergence, we first extend Ball divergence to \(\alpha\)-Ball divergence as follows.

**DEFINITION 2.2.** The \(\alpha\)-Ball divergence of two Borel measures \(\mu\) and \(\nu\) in \(A\) is defined as

\[ D_{\alpha}(\mu, \nu) = \int_{A \times A} (\mu - \nu)^2(\bar{B}(u, \rho(u, v)))\omega_{\alpha}(du)\omega_{\alpha}(dv), \]

where the mixture distribution measure \(\omega_{\alpha} = \alpha \mu + (1 - \alpha) \nu\) with the mixed parameter \(\alpha \in [0, 1]\).

Ball divergence \(D(\mu, \nu)\) can be expressed as the sum of two special \(D_{\alpha}(\mu, \nu)\)s with \(\alpha = 0\) and \(\alpha = 1\):

\[ D(\mu, \nu) = D_{0}(\mu, \nu) + D_{1}(\mu, \nu). \]

Both Ball divergence and \(\alpha\)-Ball divergence measure the difference between the two probability measures \(\mu\) and \(\nu\). As in Pan et al. [36], the \(\alpha\)-Ball divergence also has the property of equivalence, which is the key stone of our detection procedure.
THEOREM 2.2. Given two Borel probabilities $\mu, \nu$ in a separable Banach space $A$, then $D_\alpha(\mu, \nu) \geq 0$ where the equality holds if and only if $\mu = \nu$.

From Definition 2.1 and 2.2, the disparity between Ball divergence and $\alpha$-Ball divergence is how to set weights of balls. Suppose that $S_\mu$ and $S_\nu$ are the support of $\mu$ and $\nu$, if $u \in S_\mu$, $v \in S_\nu$, the weight of the ball $\bar{B}(u, \rho(u, v))$ is zero in Ball divergence but positive in $\alpha$-Ball divergence. Thus, $\alpha$-Ball divergence could catch more information on the difference between two probability measures which is more suitable to sequences. As we will show in the next section, $\alpha$ can be selected as the ratio of the two sample sizes, with which we have better simulation performance and proper theoretical results.

2.3. $\alpha$-Ball Divergence’s statistic. Suppose that a sequence of observations $\{Z_i\}_{1 \leq i \leq T}$ is comprised of two multivariate stationary sequences $\{Z_i\}_{1 \leq i \leq M}$ with the probability $P_1$ and $\{Z_i\}_{M+1 \leq i \leq T}$ with $P_2$, in which both of $P_1$ and $P_2$ are unknown. We propose an estimate of $D_\alpha(P_1, P_2)$ with $\alpha = M/T$ based on $\{Z_i\}_{1 \leq i \leq T}$. Let $N = T-M$, $C_{ij}^1 = \frac{1}{M} \sum_{u=1}^{M} c(Z_i, Z_j; Z_u)$, $C_{ij}^2 = \frac{1}{N} \sum_{v=M+1}^{T} c(Z_i, Z_j; Z_v)$, the empirical $\alpha$-Ball divergence of $P_1$ and $P_2$ is defined as

$$D_{M,N} = \frac{1}{T^2} \sum_{i,j=1}^{T} (C_{ij}^1 - C_{ij}^2)^2.$$ 

Next we will rewrite the empirical $\alpha$-Ball divergence as the sum of four $V$-statistics, and then provide the asymptotic properties of the empirical $\alpha$-Ball divergence by using the sample theory of $V(U)$-statistics for absolutely regular sequences. Denote $e_{12,3} = c(Z_1, Z_2, Z_3)$ and $e_{12,34} = c_{12,3}c_{12,4}$ with $c(\cdot)$ and $e(\cdot)$ defined in last subsection, and then denote

$$d_{12,3456} = e_{12,34} + e_{12,56} - e_{12,35} - e_{12,46}.$$ 

Let

$$D_{M,N} = \frac{1}{M^2 N^2} \sum_{i,j,u,u'} \sum_{v,v'} T d_{ij,uu'vv'},$$

$$D_{M,N}^2 = \frac{1}{M^2 N^4} \sum_{u,u'=1}^{M} \sum_{i,j,v,v'} T d_{ij,uu'vv'},$$

$$D_{M,N}^3 = \frac{1}{M^2 N^6} \sum_{i,u,u'=1}^{M} \sum_{j,v,v'} T d_{ij,uu'vv'},$$

$$D_{M,N}^4 = \frac{1}{M^2 N^8} \sum_{i,u,u'=1}^{M} \sum_{j,v,v'} T d_{ij,uu'vv'}. $$
\[ D_{M,N}^4 = \frac{1}{M^3 N^3} \sum_{j,a,u'=1}^M \sum_{i,v,v'=1}^T d_{ij,uu'vv'}. \]

Then we have
\[
D_{M,N} = \frac{1}{T^2} \sum_{i,j=1}^T (C_{ij}^1 - C_{ij}^2)^2
\]
\[
= \frac{1}{T^2 M^2 N^2} \sum_{i,j=1}^T \sum_{u,u'=1}^M \sum_{v,v'=1}^T (c_{ij,u} c_{ij,u'} + c_{ij,v} c_{ij,v'} - c_{ij,u} c_{ij,v} - c_{ij,u'} c_{ij,v'})
\]
\[
= \frac{1}{T^2 M^2 N^2} \sum_{i,j=1}^T \sum_{u,u'=1}^M \sum_{v,v'=1}^T d_{ij,uu'vv'}
\]
\[
+ \frac{1}{M^2 T^2 N^2} \sum_{i,j=1}^T \sum_{u,u'=1}^M \sum_{v,v'=1}^T d_{ij,uu'vv'} + \frac{1}{M^2 T^2 N^2} \sum_{u,u'=1}^M \sum_{i,j=1}^T \sum_{v,v'=1}^T d_{ij,uu'vv'}
\]
\[
= \frac{M^2}{T^2} D_{M,N}^1 + \frac{N^2}{T^2} D_{M,N}^2 + \frac{M N}{T^2} D_{M,N}^3 + \frac{M N}{T^2} D_{M,N}^4.
\]

Notice that \( D_{M,N}^1, D_{M,N}^2, D_{M,N}^3, D_{M,N}^4 \) are two sample V-statistics of the order \((4,2), (2,4), (3,3)\) and \((3,3)\). Therefore, their asymptotic behaviors are conveniently obtained by the large sample theory of U-statistics for weakly dependent sequences under mild conditions in Aaronson et al. [1]. We introduce two concepts of the random sequence: absolutely regular and ergodic stationary sequence.

Given the probability space \((\Omega, \mathcal{F}, P)\) and two sub-\(\sigma\)-fields \(\mathcal{A}\) and \(\mathcal{B}\) of \(\mathcal{F}\), let
\[
\beta(\mathcal{A}, \mathcal{B}) = \sup \sum_{i=1}^m \sum_{j=1}^n |P(A_i \cap B_j) - P(A_i)P(B_j)|,
\]
where the supremum is taken over all partitions of \(\Omega\) into sets \(A_1, \ldots, A_m \in \mathcal{A}\), all partitions of \(\Omega\) into sets \(B_1, \ldots, B_m \in \mathcal{B}\) and all \(m, n \geq 1\). A stochastic sequence \(\{Z_i\}_{i \in \mathbb{Z}}\) is called absolutely regular ([13], also called weakly Bernoulli [1]), if
\[
\beta(l) = \sup \beta(\mathcal{F}_l^n, \mathcal{F}_{n+l}) \to 0,
\]
as \(l \to \infty\). Here the \(\mathcal{F}_i^n\) denotes the \(\sigma\)-field generated by the random variables \(Z_i, \ldots, Z_j\). In this paper, we suppose that \(\beta(l) = O(l^{-1-r})\) for any \(r > 0\). The
concept of absolutely regular sequence is wide enough to cover all relevant examples from statistics except for long memory sequences.

Recall that an ergodic, stationary sequence (ESS) \[1\] is a random sequence \(\{Z_i\}_{1 \leq i \leq T}\) of form \(Z_i = f \circ G^i\) where \(G^i\) is an ergodic, probability-preserving transformation in the probability space \((\Omega, \mathcal{F}, P)\), and \(f\) is a measurable function. In essence, an ESS implies that the random sequence will not change its statistical properties with time (stationarity) and that its statistical properties can be deduced from a single, sufficiently long sample of the sequence (ergodicity).

We have the following theorem for an absolutely regular sequence comprised of two ergodic stationary sequences:

**THEOREM 2.3.** Suppose that \(\{Z_i\}_{1 \leq i \leq T}\) is an absolutely regular sequence, \(\{Z_i\}_{1 \leq i \leq M}\) and \(\{Z_i\}_{M+1 \leq i \leq T}\) are both ergodic stationary sequence (ESS), and \(M/T \to \alpha_0\) for some \(\alpha_0 \in [0, 1]\), then

\[
D_{M,N} \xrightarrow{\text{a.s.}} D_{\alpha_0}(P_1, P_2).
\]

Theorem 2.3 shows that the empirical \(\alpha\)-Ball divergence converges to \(D_{\alpha_0}(P_1, P_2)\) almost surely. We further investigate the asymptotic distribution of \(D_{M,N}\). Under the null hypothesis, the empirical \(\alpha\)-Ball divergence is the sum of four degenerate V-statistics. As in \[36\], we denote \(Q(x, y; x', y')\) as the second component of the H-decomposition. Then we have the spectral decomposition:

\[
Q(x, y; x', y') = \sum_{k=1}^{\infty} \lambda_k f_k(x, y) f_k(x', y'),
\]

where \(\lambda_k\) and \(f_k\) are the eigenvalues and eigenfunctions of \(Q(x, y; x', y')\), \(Z_{1k}, Z_{2k}\) are \(iid\) \(N(0, 1)\), and

\[
d_k^2(\alpha_0) = (1 - \alpha_0) E_{P_1}[E_{P_2} f_k(Z_1, Z_{M+1})]^2
+ 2 \sum_{n=1}^{\infty} E_{P_1}[E_{P_2} f_k(Z_1, Z_{M+1}) f_k(Z_1, Z_{M+n+1})],
\]

\[
b_k^2(\alpha_0) = \alpha_0 E_{P_2}[E_{P_1} f_k(Z_1, Z_{M+1})]^2
+ 2 \sum_{n=1}^{\infty} E_{P_2}[E_{P_1} f_k(Z_1, Z_{M+1}) f_k(Z_{n+1}, Z_{M+1})],
\]

\[
\theta = 2E[E(\delta(Z_i, Z_j, Z_1)(1 - \delta(Z_i, Z_j, Z_{M+1}))|Z_i, Z_j)],
\]

for \(k = 1, 2, \ldots\).
THEOREM 2.4. Under null hypothesis $H_0 : P_1 = P_2$, $\{Z_i\}_{1 \leq i \leq T}$ is a stationary absolutely regular sequence with coefficients satisfying $\beta(l) = O(l^{-1-r})$ for $r > 0$, if $M, N \to \infty$, $M/T \to \alpha_0$ for some $\alpha_0 \in [0, 1]$, we have

$$\frac{MN}{T} D_{M,N} \xrightarrow{D} \sum_{k=1}^{\infty} 2 \lambda_k [(a_k(\alpha_0) Z_{1k} + b_k(\alpha_0) Z_{2k})^2 - (a_k^2(\alpha_0) + b_k^2(\alpha_0))] + \theta.$$

Under the alternative hypothesis, the empirical $\alpha$-Ball divergence is asymptotically normal because it is a sum of non-degenerate V-statistics. Let $g^{(1,0)}(Z_\mu)$ and $g^{(0,1)}(Z_\nu)$ be the first component of $H$-decomposition and $\delta_{1,0}^2 = \text{Var}(g^{(1,0)}(Z_\mu))$ and $\delta_{0,1}^2 = \text{Var}(g^{(0,1)}(Z_\nu))$. We can obtain the asymptotic distribution under the alternative hypothesis.

THEOREM 2.5. $\{Z_i\}_{1 \leq i \leq T}$ is an absolutely regular sequence with coefficients satisfying $\beta(l) = O(l^{-1-r})$ for $r > 0$. Under $H_1 : P_1 \neq P_2$, if $M, N \to \infty$, and $M/T \to \alpha_0$ for some $\alpha_0 \in [0, 1]$, then we have

$$\sqrt{\frac{MN}{T}} (D_{M,N} - D_{\alpha_0}(P_1, P_2)) \xrightarrow{D} N(0, (1-\alpha_0)\delta_{1,0}^2 + \alpha_0 \delta_{0,1}^2).$$

Our new test can handle the problem of imbalanced sample sizes. We show that the test is consistent against general alternatives. As shown in the following theorem, the asymptotic power of the test does not go to zero no matter $\eta = \frac{M}{N}$ goes to 0 or $\infty$.

THEOREM 2.6. The test based on $D_{M,N}$ is consistent against any general alternative $H_1$. More specifically,

$$\lim_{(M,N) \to \infty} \text{Var}_{H_1}(D_{M,N}) = 0,$$

and

$$\Lambda := \liminf_{(M,N) \to \infty} (E_{H_1}D_{M,N} - E_{H_0}D_{M,N}) > 0.$$

3. Detection of change-points. In this section, using the notion $\alpha$-Ball divergence and its sample statistic, we propose a procedure to detect change-points in a sequence. Suppose there are $k$ hypothesized change-points located at $0 = T_0 < T_1 < \cdots < T_k < T_{k+1} = T$ in the sequence $\{Z_i\}_{1 \leq i \leq T}$. Let the $i$th cluster $Z(T_i/T_{i-1}) = \{Z_{T_{i-1}+1}, \ldots, Z_{T_i}\} \sim P_i$ where $P_i$ is the associated probability measure. We test

$$H_0 : P_i = P_{i+1} \text{ vs } H_A : P_i \neq P_{i+1}, \ \forall \ 1 \leq i \leq k.$$

If $H_0$ is rejected, we conclude that there is a change-point at $T_i$. Otherwise, there is no distributional change between $Z(T_i/T_{i-1})$ and $Z(T_{i+1}/T_i)$.
3.1. Hierarchical Algorithm. For simplicity, suppose that the sequence \( \{ Z_i \}_{1 \leq i \leq T} \) exists at most one change-point. Define the detection function for the sequence \( \{ Z_i \}_{1 \leq i \leq T} \) as
\[
V(Z(M/0), Z(T/M)) = \frac{MN}{T} D_{M,N}.
\]

The potential change-point location \( \hat{M}_T \) is then estimated by maximizing the detection function:
\[
\hat{M}_T = \arg \max_M V(Z(M/0), Z(T/M)).
\]

We use the bootstrap method to test whether \( V(Z(\hat{j}_i/\hat{T}_{i-1}), Z(\hat{l}_i/\hat{j}_i)) \) is significant or not. If it is significant, \( \hat{M}_T \) is the estimated change-point. Otherwise, there does not exist any change-point in the sequence.

Estimating multiple change-points is more complicated. Suppose that \( k-1 \) change-points have been estimated at locations \( 0 < \hat{T}_1 < \cdots < \hat{T}_{k-1} < T \), and \( \hat{T}_0 = 0, \hat{T}_k = T \). Those change-points partition the sequence into \( k \) clusters \( Z(\hat{T}_1/\hat{T}_0), \ldots, Z(\hat{T}_k/\hat{T}_{k-1}) \). Unlike in the one change-point case, for \( 1 \leq i \leq k-1 \), we introduce variable \( l_i \) in \( i \)-th cluster, and the detection function in the \( i \)-th cluster is defined as
\[
(\hat{j}_i, \hat{l}_i) = \arg \max_{\hat{T}_{i-1} < j_i < l_i \leq \hat{T}_i} V(Z(j_i/\hat{T}_{i-1}), Z(l_i/j_i)).
\]

Next we test whether \( V(Z(\hat{j}_i/\hat{T}_{i-1}), Z(\hat{l}_i/\hat{j}_i)) \) is significant or not for \( 1 \leq i \leq k-1 \). If it is significant, then \( \hat{j}_i \) is the \( k \)-th change-point. We repeat the above procedure until that the maximum value of the detection function in all clusters are not significant.

From (3.1), we can see that the introduction of \( l_i \) here is to alleviate a weakness of bisection algorithm [32]. Because in each cluster, there may exist multiple change-points. If we do not introduce \( l_i \), the value of \( V(Z(\hat{j}_i/\hat{T}_{i-1}), Z(\hat{l}_i/\hat{j}_i)) \) may be lower than \( V(Z(\hat{j}_i/\hat{T}_{i-1}), Z(\hat{l}_i/\hat{j}_i)) \) or even not significant.

The hierarchical algorithm which is used to estimate multiple change-points is illustrated as follows.

3.2. Hierarchical Significance Testing. In this section, we focus on the “significance test step” in Multiple change-points Algorithm.

Theorem 2.4 shows that the asymptotic null distribution of \( \frac{MN}{T} D_{M,N} \) is a mixture of \( \chi^2 \) distribution. In practice, it is difficult to directly take advantage of the asymptotic null distribution. Here we suggest using the moving block bootstrap [25] to obtain the p-values.
Algorithm 1 Multiple change-points Algorithm

Let the minimize sequence $\text{minimize} = m$, the change-points set $T = \{0, T\}$.

Suppose that $k - 1$ change-points have been estimated. This decomposes the observations into $k$ clusters. Set $\text{best} = -\infty$, $\text{location} = 0$, $\text{end} = 0$.

for each $i \in \{1, \ldots, k\}$ do

For the $i$th cluster $Z(T_i/T_{i-1})$: 

for $j_i = T_{i-1} + m, T_{i-1} + m + 1, \ldots, T_i - m$ do 

for $l_i = T_{i-1} + m, \ldots, T_i$ do 

Compute $V(Z(j_i/T_{i-1}), Z(l_i/j_i))$; 

if $V(Z(j_i/T_{i-1}), Z(l_i/j_i)) \geq \text{best}$ then 

$\text{best} = V(Z(j_i/T_{i-1}), Z(l_i/j_i)), \text{location} = j_i, \text{end} = l_i$; 

end if 

end for 

end for 

Significance testing for $Z_{\text{location}}$ within the cluster $Z(T_i/T_{i-1})$.

if significance then 

put location into $T$; 

else 

there does not exist new change-point in $Z(T_i/T_{i-1})$. 

end if 

end for

Given a set of observations \( \{Z_t\}_{1 \leq t \leq T} \) and the block size $b_T$, we draw a bootstrap resample \( \{Z^*_t\}_{1 \leq t \leq T} \) as follows: (i) define the $b_T$ dimensional vector $X_t = (Z_t, Z_{t-1}, \ldots, Z_{t-b_T+1})$; (ii) resample from block data \( \{X_t\}_{1 \leq t \leq T-b_T+1} \) with replacement to get pseudo data \( \{X^*_t\}_{1 \leq t \leq L} \) which satisfies $T = \lfloor L b_T \rfloor$, and denote the first $T$ elements of \( \{X^*_t\}_{1 \leq t \leq L} \) as the bootstrap resample \( \{Z^*_t\}_{1 \leq t \leq T} \); (iii) repeat steps (i) and (ii) $B$ times given each sample \( \{Z_t\}_{1 \leq t \leq T} \).

In application, the choice of the block size $b_T$ involves a tradeoff. If block size becomes too small, the moving block bootstrap will destroy the time dependency of the data and the accuracy will decline. But if block size becomes too large, there will be few blocks to be used. In other word, increasing the block size reduces the bias and captures more persistent dependence, while decreasing the block size reduces the variance as more subsamples are available. Thus, some scholars consider the mean squared error as the optimal criterion to balance the bias and variance. For the linear time series, as proved in Carlstein [7], the value of the block size that minimizes MSE is 

$$b^*_T = \left( \frac{2|\rho|}{1 - \rho^2} \right)^{2/3} T^{1/3},$$

where $\rho$ is the first order autocorrelation. Because the construction of MSE depends on the knowledge of the underlying data generating sequence, no
optimal result is available for general sequence. In this paper, we follow Hong, Wang and Wang [22] and Xiao and Lima [41] to choose $b_T = \max\{p_T, \bar{p}_T\}$, where

$$p_T = \min \left\{ \left[ \left( \frac{3T}{2} \right)^{1/3} \left( \frac{2\hat{\rho}}{1 - \hat{\rho}^2} \right)^{2/3} \right], \left[ 8 \left( \frac{T}{100} \right)^{1/3} \right] \right\},$$

$[A]$ denotes the integer part of $A$ and $\hat{\rho}$ is the estimator of the first autocorrelation of $Z_t$, $\bar{p}_T$ is the same as (3.2) except replacing $\hat{\rho}$ by the estimated first order autocorrelation of $Z_t^2$. So the choice of $b_T$ considers the linear dependence and non-linear dependence.

3.3. Consistency. Next theorem shows the consistency of the estimated change-point locations under the following assumption.

ASSUMPTION 3.1. Suppose that we have a heterogeneous sequence of observations from two different strict stationary series. Let $\alpha_0 \in (0, 1)$ denotes the fraction of the observations belong to one of the observations, such that $Z(\lfloor \alpha_0 T \rfloor / 0) \sim P_1$ and $Z(T/\lfloor \alpha_0 T \rfloor) \sim P_2$. Finally, let $\delta_T$ be a sequence of positive number, such that $\delta_T \to 0$ and $T\delta_T \to \infty$ as $T \to \infty$.

THEOREM 3.1. Suppose Assumption 3.1 holds. Let

$$\hat{M}_T = \arg \max_M V(Z(M/0), Z(T/M))$$

be the estimated change-point location for a sample of size $T$. For all $\epsilon > 0$ and $T$ large enough such that $\alpha_0 \in [\delta_T, 1 - \delta_T]$, we have

$$P\left( \lim_{T \to \infty} \left| \frac{\hat{M}_T}{T} - \alpha_0 \right| < \epsilon \right) = 1.$$

This theorem shows that the consistency only requires the size of each cluster increases to $\infty$, but not necessarily at the same ratio. So as the Assumption 3.1, $\alpha_0$ can be close to 0 or 1 when $T \to \infty$, which is an imbalanced case.

4. Simulation. In this section, we present the numerical performance of the proposed method (BDCP) and compare it with several typical methods, including Bayesian method (BCP) [3], WBS method [16], the graph-based method-gSeg [8, 11] and energy distance based method (ECP) [32]. BCP, WBS, ECP and BDCP can estimate the number of change-points automatically while gSeg can detect only one change-point or an interval.
There are four commonly used criteria for the performance of those methods: the adjusted Rand index, the over segmentation error, the under segmentation error and the Hausdorff distance. Suppose that the true change-points set is \( T = \{0, T_1, \ldots, T_k, T\} \) and estimated change-points set is \( \hat{T} = \{0, \hat{T}_1, \ldots, \hat{T}_k, T\} \). Then denote the true clusters of series \( \{Z_t\}_{1 \leq t \leq T} \) by \( Z = \{Z(T_1/0), \ldots, Z(T/T_k)\} \) and the estimated clusters by \( \hat{Z} = \{Z(\hat{T}_1/0), \ldots, Z(T/\hat{T}_k)\} \). Consider the pairs of observation that fall into one of the following two sets: 

\( \{S_1\} = \{\text{pairs of observation in the same clusters under } Z \text{ and in same clusters under } \hat{Z}\} \);

\( \{S_2\} = \{\text{pairs of observation in different clusters under } Z \text{ and in different clusters under } \hat{Z}\} \). Denote \( \sharp S_1 \) and \( \sharp S_2 \) as the number of pairs of observation in each of these two sets. The Rand index \( RI \) is defined as

\[
RI = \frac{\sharp S_1 + \sharp S_2}{\binom{T}{2}}.
\]

One disadvantage of the Rand index is that it does not take on a constant value when comparing two random clustering [23]. Adjusted Rand index \( ARI \) is the corrected-for-chance version of the Rand index which is defined as

\[
ARI = \frac{RI - E(RI)}{1 - E(RI)},
\]

in which 1 corresponds to the maximum Rand index value.

On the other hand, we also calculate the distance between \( T \) and \( \hat{T} \) by

\[
\zeta(T | \hat{T}) = \sup_{b \in T} \inf_{a \in \hat{T}} |a - b| \quad \text{and} \quad \zeta(\hat{T} | T) = \sup_{b \in \hat{T}} \inf_{a \in T} |a - b|,
\]

which quantify the over-segmentation error and the under-segmentation error respectively [5, 43]. The Hausdorff distance [17] between \( T \) and \( \hat{T} \) is defined as

\[
\Delta(T, \hat{T}) = \sup \{\zeta(\hat{T} | T), \zeta(T | \hat{T})\}.
\]

To save space, we only report the results based on adjusted Rand index. The results under other criteria are deferred to the supplementary material.

Three scenarios are used for comparisons: univariate sequence, multivariate sequence and manifold sequence. In each scenario, we consider two types of examples, one without change-point, and one with two change-points as follow:

\[
\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m, X_{n+1}, X_{n+2}, \ldots, X_{2n}\}.
\]
The sample sizes are set to be \( n = 40, \ m = 40, 60, 80 \). We will repeat each model 400 times and the nominal significance level is at 0.05. To save space, some results of univariate sequences are available on the supplementary material.

4.1. *Multivariate sequence.* In this subsection, we consider the \( d = 3 \) dimensional sequences. Example 4.2.1-4.2.6 are the sequences with no change-point and Example 4.2.7-4.2.12 are the models with two change-points.

- **Example 4.1.1-4.1.3:**

\[
X_t = \epsilon_t,
\]

\( \epsilon_t \sim N(0, I_3) \) for Example 4.1.1, \( \epsilon_t \sim t_3(0, I_3) \) for Example 4.1.2 and \( \epsilon_t \sim \text{Cauchy}(0, I_3) \) for Example 4.1.3.

- **Example 4.1.4-4.1.5:**

\[
X_t = 0.5\epsilon_t + 0.5\epsilon_{t-1},
\]

\( \epsilon_t \sim N(0, I_3) \) for Example 4.1.4 and \( \epsilon_t \sim t_3(0, I_3) \) for Example 4.1.5.

- **Example 4.1.6-4.1.7:**

\[
X_t = \sigma_{X,t|t-1}\epsilon_t,
\]

\[
\sigma^2_{X,t|t-1} = 0.02 + 0.02\sigma^2_{X,t-1|t-2} + 0.05X_t^2,
\]

\( \epsilon_t \sim N(0, I_3) \) for Example 4.1.6 and \( \epsilon_t \sim t_3(0, I_3) \) for Example 4.1.7.

- **Example 4.1.8:**

\[
X_t = 0.5\epsilon_t + 0.5\epsilon_{t-1},
\]

\[
Y_t = \mu + 0.5\epsilon_t + 0.5\epsilon_{t-1},
\]

\( \mu = (4, 4, 4), (6, 6, 6), (8, 8, 8) \), \( \epsilon_t \sim N(0, I_3) \).

- **Example 4.1.9:**

\[
X_t = 0.5\epsilon_t + 0.5\epsilon_{t-1},
\]

\[
Y_t = \mu + 0.5\epsilon_t + 0.5\epsilon_{t-1},
\]

\( \mu = (4, 4, 4), (6, 6, 6), (8, 8, 8) \), \( \epsilon_t \sim t_3(0, I_3) \).
• Example 4.1.10:

\[ X_t = \epsilon_t, \]
\[ Y_t = \mu + \epsilon_t, \]
\[ \mu = (4, 4, 4), (6, 6, 6), (8, 8, 8), \epsilon_t \sim Cauchy(0, I_3). \]

• Example 4.1.11:

\[ X_t = 0.5\epsilon_t + 0.5\epsilon_{t-1}, \]
\[ Y_t = 0.5\epsilon_t\sigma + 0.5\epsilon_{t-1}\sigma, \]
\[ \sigma = (3, 3, 3), (5, 5, 5), (7, 7, 7), \epsilon_t \sim N(0, I_3). \]

• Example 4.1.12:

\[ X_t = 0.5\epsilon_t + 0.5\epsilon_{t-1}, \]
\[ Y_t = 0.5\epsilon_t\sigma + 0.5\epsilon_{t-1}\sigma, \]
\[ \sigma = (3, 3, 3), (5, 5, 5), (7, 7, 7), \epsilon_t \sim t_3(0, I_3). \]

• Example 4.1.13:

\[ X_t = \epsilon_t, \]
\[ Y_t = \epsilon_t\sigma, \]
\[ \sigma = (9, 9, 9), (16, 16, 16), (25, 25, 25), \epsilon_t \sim Cauchy(0, I_3). \]

• Example 4.1.14-4.1.15:

\( X, Y \sim CCC - GARCH(1, 1) \), let

Case 1:
\[ \omega_Y = 2\omega_X, A_Y = 4A_X, B_Y = 5B_X, \]

Case 2:
\[ \omega_Y = 3\omega_X, A_Y = 5A_X, B_Y = 5B_X. \]

Case 3:
\[ \omega_Y = 4\omega_X, A_Y = 6A_X, B_Y = 5B_X. \]

\( \epsilon_t \sim N(0, 1) \) for Example 4.1.14 and \( \epsilon_t \sim t(df = 3) \) for Example 4.1.15.
Table 1 shows that ECP and BDCP can handle the multivariate stationary series. BCP works well when the distribution are normal but has lower adjusted Rand index in t distribution and Cauchy distribution. We do not consider the WBS method which can not handle the multivariate cases. Table 2-4 show the results of those examples with two change-points. All the four methods have excellent performance in multivariate normal distribution and multivariate t distribution with location shift while ECP and gSeg are better. Example 4.1.11-4.1.13 consider the scale shift case and Example 4.1.14-4.1.15 are the popular GARCH models which are also the scale shift case. We can see from Table 3-4 that BDCP has the best performance in almost all the scale shift cases.

4.2. Manifold-valued sequence. In this subsection, we show some manifold-valued examples where ECP can not detect the change-points but BDCP works well. Consider the distribution in an unit circle and let

\begin{align*}
P_1 &= \text{Unif}([-\pi/6, \pi/6] \cup [11\pi/6, 13\pi/6]), \\
P_2 &= \text{Unif}([\pi/3, 2\pi/3] \cup [7\pi/3, 8\pi/3]), \\
P_3 &= \text{Unif}([5\pi/6, 7\pi/6] \cup [17\pi/6, 19\pi/6]), \\
P_4 &= \text{Unif}([4\pi/3, 5\pi/3] \cup [10\pi/3, 11\pi/3]), \\
P_5 &= \text{Unif}([0, 4\pi]).
\end{align*}

We calculate the circular distance which is defined by

\begin{equation}
d(X_i, X_j) = \min(|X_i - X_j|, 2\pi - |X_i - X_j|).
\end{equation}

We simulate four examples with 0, 1, 2, 3 change-points respectively. Let \( n = 40, m = 40, 60, 80 \).

- **Example 4.2.1:**
  \[\{X_1, \ldots, X_{3n}\} \sim P_5.\]

- **Example 4.2.2:**
  \[\{X_1, \ldots, X_n\} \sim P_1, \{Y_1, \ldots, Y_m\} \sim P_3.\]

- **Example 4.2.3:**
  \[\{X_1, \ldots, X_n\} \sim P_1, \{Y_1, \ldots, Y_m\} \sim P_3, \{X'_1, \ldots, X'_n\} \sim P_2.\]
Example 4.2.4:

\[
\{X_1, \ldots, X_n\} \sim P_1, \{Y_1, \ldots, Y_m\} \sim P_3,
\]

\[
\{X'_1, \ldots, X'_n\} \sim P_2, \{Y'_1, \ldots, Y'_m\} \sim P_4.
\]

Table 5 shows that BCP, WBS, ECP all perform well when there is no change-point. However, they do not work when the sequences have change-points (Table 6). That is because BCP is based on normal distribution, and WBS is a CUSUM statistics which does not work in a circular distribution. For ECP, that is because the circular distance is not of strong negative type (Theorem 9.1 in Hjorth et al. [21]). Method gSeg can detect change-points when the number of change-points is one or two but not perform well in Example 4.2.4. BDCP has remarkable performance in those examples.

5. Real data analysis.

5.1. Wind direction of Yunnan-Guizhou Plateau. In subsection 4.2, BDCP shows the remarkable performance in the manifold-valued sequences. Actually, lots of real data such as the monsoon direction data are distributed in a manifold rather than in the Euclidean space. Monsoon is used to describe seasonal changes in atmospheric circulation and precipitation associated with the asymmetric heating of the land and sea. The major monsoon systems in the world consist of West African Monsoon (WAM), Indian summer monsoon (ISM), East Asian Monsoon (EAM) and so on. In this subsection, we analyze the wind direction data of Yunnan-Guizhou Plateau (105°E, 27°N) from 2015/06/01 to 2015/10/30. The data is available in R package rWind. Yunnan-Guizhou Plateau is located in southwest China, with local climate influenced by both ISM and EAM [39][29](Fig. 1A and S1). Strict spatial boundaries between the ISM and the ASM are difficult to define [9] though previous researchers have suggested 103°E as the dividing line on the basis of summer prevailing winds.

Daily wind directions are shown in the top-left of Figure 1. Note that degree 0 represents due North, π/2 represents due East, π represents due South and 3π/2 represents due West. We can see that the wind directions are distributed in almost all directions. In the beginning, the most widely distributed direction is the southwest wind from India Ocean, and then turns smoothly to southeast, which is from Pacific Ocean. In particular, Yunnan-Guizhou Plateau was mostly influenced by ISM on June and July. After July, the influence of EAM gradually increased [28].

To detect the change-point in the wind direction series, in this part we calculate the circular distance between the daily direction which is defined
The performance of the five methods are shown in Figure 1. BCP, WBS and ECP can not detect any change-point, which is the same as the simulation study in subsection 4.2. gSeg detects an interval between “2015/07/17” and “2016/07/28”. BDCP estimates four change-points located at “2015/07/01”, “2015/07/29”, “2015/09/02” and “2015/09/26”.

To graphically support the result of BDCP, Figure 2 shows the wind rose plot for the five periods detected by BDCP. We can see the significant changes of the direction distribution especially between 2015/07/29 - 2015/09/01 and 2015/09/01 - 2015/09/25. The wind directions are almost southwest or west in June and July, then turn to southeast in September. As mentioned in Hillman et al. [20], 75% of the average annual precipitation falls in the months of June-September associated with the ISM, and the ISM gets weaker during June and July because insolation decreases by 2-3%. BDCP can perfectly detect the change of influence between ISM and
5.2. **Bitcoin price.** Bitcoin is the most popular form of cryptocurrency in recent years. According to research of Cambridge University in 2017 [19], there are 2.9 to 5.8 million unique cryptocurrency wallet users, most of whom use bitcoin. One of the known features of bitcoin is its high volatility. Bitcoin is not a denominated flat currency and there is no central bank overseeing the issuing of bitcoin, its price is thus driven solely by the investors. Using the weekly data over 2010-2013 period, Brière, Oosterlinck and Szafarz [6] showed that bitcoin investment had some high distinctive features, including exceptionally high average return and volatility. Hence, accurately fitting its variation is important [12].

Bitcoin can be exchanged for other currencies, products, and services in legal or black markets. Chu, Nadarajah and Chan [12] measured the volatility of bitcoin exchange rate against six major currencies. They found that the behavior of bitcoin was sharply different from those currencies; its interquartile range was much wider, its skewness was much more negative, its kurtosis
was much more peaked and its variance was much larger. Bitcoin showed the highest annualized volatility of percentage change in daily exchange rates. In this subsection, we detect the change-points of daily log-return of bitcoin using methods BCP, WBS, gSeg, ECP and BDCP. The datasets are available on http://api.bitcoincharts.com/v1/csv/bitstampUSD.csv.gz. Figure 3 shows the exchange rate of bitcoin and daily log-returns during 2011/09/13 - 2012/12/31.

Figure 4 shows the performance of the five methods. BCP and WBS can not handle the severe volatility at the beginning of the sequence. gSeg detects one change-point at “2012-02-23”, and ECP estimates a change-point at “2012-02-09”. BDCP detects four change-points at “2012/02/11”, “2012/04/16”, “2012/05/19”, “2012/08/21”.

On February 11, 2012, Paxum, an online payment service and popular means for exchanging bitcoin announced it would cease all dealings related to the currency due to the concerns of its legality. Two days later, regulatory issues surrounding money transmission compelled the popular bitcoin
exchange and service firm TradeHill to terminate its business and immediately began selling its bitcoin assets to refund its customers and creditors. Bitcoin trading started to cool down during that period.

After May 19, the price of bitcoin had increased from $5.07 to the maximum $14.14 on August 17 and kept at that level after that. The reasons for the rise were plenty. Lots of online articles on this subject expressed the same message: Bitcoin was now going mainstream. WordPress, ranked by Alexa as the 21st most popular site in the world, started to accept Bitcoin for payment in November, 2012.

The variances of these five stages detected by BDCP are: 0.1054, 0.0182, 0.0059, 0.0458, 0.0177. The daily log-return sequence was very flat and the price almost did not change during period 2012/04/16 - 2012/05/18. But it was violent during other periods from Figure 4.

6. Conclusion. We developed a change-point detection procedure for weakly dependent Banach-valued sequences. The core of our proposed method is a novel measure of divergence: α-Ball divergence. α-Ball divergence is zero if and only if two weakly dependent sequences have the same distributions. We also verified the asymptotic properties of its sample statistic for absolutely regular sequences. Simulation studies illustrated that our method had remarkable performance with comparison to other existing methods in various settings. Two real data analyses indicated that our method could deal with non-Euclidean sequences or scale changes very well. Besides, our method is robust since our test statistic is rank-based.

We will further investigate α-Ball divergence and its relative techniques either in theory or in application. For example, the current computational complexity of our proposed algorithm is \(O(kT^2 \log T)\), where \(k\) is the number of change-points, and \(T\) is the length of the sequence. It is quite interesting and useful to find an algorithm with less computational complexity.

SUPPLEMENTARY MATERIAL

SUPPLEMENTARY MATERIAL FOR: α-Ball divergence and its applications to change-point problems for Banach-valued sequences
(http://www.e-publications.org/ims/support/download/imsart-ims.zip). Owing to the space constraints, we have moved many of the technical proofs and details to the Appendix, which is contained in the supplementary document References.
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The performance of adjusted Rand index for multivariate series with no change-point. The highest average adjusted Rand index is highlighted in bold. The last four columns refer to the adjusted Rand index ratio between the four methods to BDCP.

| Example | BDCP | BCP/BDCP | gSeg/BDCP | ECP/BDCP |
|---------|------|----------|-----------|----------|
| 4.1.1   | 0.955| 1.042    | 0.000     | 0.963    |
| 4.1.2   | 0.945| 0.048    | 0.000     | 0.995    |
| 4.1.3   | 0.930| 0.005    | 0.000     | 0.984    |
| 4.1.4   | 0.955| 0.984    | 0.000     | 1.047    |
| 4.1.5   | 0.825| 0.248    | 0.000     | 1.206    |
| 4.1.6   | 0.965| 0.969    | 0.000     | 0.979    |
| 4.1.7   | 0.920| 0.087    | 0.000     | 0.995    |
Table 2
The performance of adjusted Rand index for Example 4.1.8 - 4.1.10 with change in mean. The highest average adjusted Rand index is highlighted in bold. The last four columns refer to the adjusted Rand index ratio between the four methods to BDCP.

| Example | m  | µ  | BDCP | BCP/BDCP | gSeg/BDCP | ECP/BDCP | ECP/BDCP |
|---------|----|----|------|----------|-----------|----------|----------|
| 4.1.8   |    |    |      |          |           |          |          |
| 40      | 4  | 0.962 | 0.947 | 0.964 | 1.018 |
|         | 6  | 0.996 | 0.999 | 0.995 | 1.004 |
|         | 8  | 0.995 | 1.005 | 1.005 | 1.005 |
| 60      | 4  | 0.970 | 0.941 | 0.969 | 1.008 |
|         | 6  | 0.994 | 0.999 | 1.000 | 1.006 |
|         | 8  | 0.987 | 1.013 | 1.013 | 1.013 |
| 80      | 4  | 0.969 | 0.981 | 0.979 | 1.009 |
|         | 6  | 0.991 | 1.005 | 1.005 | 1.005 |
|         | 8  | 0.987 | 1.013 | 1.013 | 1.012 |
| 4.1.9   |    |    |      |          |           |          |          |
| 40      | 4  | 0.979 | 0.853 | 0.996 | 1.015 |
|         | 6  | 0.987 | 0.902 | 1.004 | 1.012 |
|         | 8  | 0.990 | 0.938 | 1.006 | 1.010 |
| 60      | 4  | 0.972 | 0.823 | 1.001 | 1.020 |
|         | 6  | 0.973 | 0.898 | 1.016 | 1.026 |
|         | 8  | 0.988 | 0.914 | 1.009 | 1.009 |
| 80      | 4  | 0.961 | 0.838 | 1.023 | 1.027 |
|         | 6  | 0.971 | 0.884 | 1.026 | 1.025 |
|         | 8  | 0.966 | 0.900 | 1.033 | 1.032 |
| 4.1.10  |    |    |      |          |           |          |          |
| 40      | 4  | 0.979 | 0.444 | 0.993 | 0.836 |
|         | 6  | 0.987 | 0.496 | 0.999 | 0.949 |
|         | 8  | 0.988 | 0.514 | 0.999 | 0.975 |
| 60      | 4  | 0.981 | 0.437 | 1.000 | 0.848 |
|         | 6  | 0.985 | 0.469 | 1.002 | 0.937 |
|         | 8  | 0.986 | 0.506 | 1.004 | 0.989 |
| 80      | 4  | 0.975 | 0.389 | 1.012 | 0.842 |
|         | 6  | 0.980 | 0.420 | 1.012 | 0.950 |
|         | 8  | 0.983 | 0.476 | 1.010 | 0.983 |
Table 3
The performance of adjusted Rand index for Example 4.1.11 - 4.1.13 with change in scale. The highest average adjusted Rand index is highlighted in bold. The last four columns refer to the adjusted Rand index ratio between the four methods to BDCP.

| Example | m | \(\sigma\) | BDCP | BCP/BDCP | gSeg/BDCP | ECP/BDCP | ECP/BDCP |
|---------|---|----------|------|---------|----------|---------|---------|
| 4.1.11  | 40| 3        | 0.786 | 0.384   | 0.888    | 0.052   |
|         |   | 5        | 0.931 | 0.622   | 0.911    | 0.632   |
|         |   | 7        | 0.956 | 0.663   | 0.949    | 0.941   |
|         | 60| 3        | 0.834 | 0.206   | 0.787    | 0.036   |
| 4.1.12  | 60| 5        | 0.942 | 0.408   | 0.908    | 0.646   |
|         |   | 7        | 0.954 | 0.471   | 0.985    | 0.971   |
|         | 80| 3        | 0.822 | 0.155   | 0.787    | 0.052   |
|         |   | 5        | 0.941 | 0.248   | 0.919    | 0.624   |
|         |   | 7        | 0.955 | 0.272   | 0.978    | 0.981   |
|         | 40| 3        | 0.521 | 0.810   | 0.964    | 0.123   |
|         |   | 5        | 0.838 | 0.621   | 0.885    | 0.443   |
|         |   | 7        | 0.906 | 0.603   | 0.905    | 0.715   |
|         | 60| 3        | 0.597 | 0.616   | 0.915    | 0.101   |
|         |   | 5        | 0.833 | 0.475   | 0.870    | 0.459   |
|         |   | 7        | 0.893 | 0.477   | 0.920    | 0.776   |
|         | 80| 3        | 0.640 | 0.480   | 0.780    | 0.086   |
|         |   | 5        | 0.842 | 0.359   | 0.809    | 0.469   |
|         |   | 7        | 0.898 | 0.356   | 0.893    | 0.751   |
|         | 40| 9        | 0.686 | 0.618   | 0.914    | 0.058   |
|         |   | 16       | 0.854 | 0.488   | 0.874    | 0.109   |
|         |   | 25       | 0.907 | 0.492   | 0.883    | 0.141   |
|         | 60| 9        | 0.707 | 0.497   | 0.932    | 0.054   |
|         |   | 16       | 0.885 | 0.424   | 0.884    | 0.089   |
|         |   | 25       | 0.918 | 0.406   | 0.908    | 0.120   |
|         | 80| 9        | 0.736 | 0.443   | 0.841    | 0.067   |
|         |   | 16       | 0.883 | 0.356   | 0.900    | 0.053   |
|         |   | 25       | 0.927 | 0.350   | 0.924    | 0.061   |
Table 4
The performance of adjusted Rand index for Example 4.1.14 and 4.1.15 with change in parameters of GARCH model. The highest average adjusted Rand index is highlighted in bold. The last four columns refer to the adjusted Rand index ratio between the four methods to BDCP.

| Example | m | case | BDCP  | BCP/BDCP | gSeg/BDCP | ECP/BDCP |
|---------|---|------|-------|----------|-----------|----------|
| 4.1.14  | 40| 1    | 0.786 | 0.622    | 0.771     | 0.052    |
|         |   | 2    | 0.931 | 0.663    | 0.925     | 0.941    |
|         |   | 3    | 0.956 |          |           |          |
| 60      | 1 |      | 0.834 | 0.206    | 0.689     | 0.036    |
|         |   | 2    | 0.942 | 0.408    | 0.827     | 0.646    |
|         |   | 3    | 0.954 | 0.471    | 0.948     | 0.971    |
| 80      | 1 |      | 0.822 | 0.155    | 0.619     | 0.052    |
|         |   | 2    | 0.941 | 0.248    | 0.811     | 0.624    |
|         |   | 3    | 0.955 | 0.272    | 0.938     | 0.981    |
| 40      | 1 |      | 0.521 | 0.810    | 1.086     | 0.123    |
|         |   | 2    | 0.838 | 0.621    | 0.834     | 0.443    |
|         |   | 3    | 0.906 | 0.603    | 0.877     | 0.715    |
| 60      | 1 |      | 0.597 | 0.616    | 0.859     | 0.101    |
|         |   | 2    | 0.833 | 0.475    | 0.801     | 0.459    |
|         |   | 3    | 0.893 | 0.477    | 0.897     | 0.776    |
| 80      | 1 |      | 0.640 | 0.480    | 0.705     | 0.086    |
|         |   | 2    | 0.842 | 0.359    | 0.787     | 0.469    |
|         |   | 3    | 0.898 | 0.356    | 0.880     | 0.751    |

Table 5
The performance of adjusted Rand index for manifold series with no change-point. The highest average adjusted Rand index is highlighted in bold. The last four columns refer to the adjusted Rand index ratio between the four methods to BDCP.

| Example | T  | BDCP  | BCP/BDCP | WBS/BDCP | gSeg/BDCP | ECP/BDCP |
|---------|----|-------|----------|----------|-----------|----------|
| 4.2.1   | 120| 0.915 | 1.087    | 1.011    | 0.000     | 1.027    |
| 4.2.1   | 140| 0.945 | 1.048    | 1.005    | 0.000     | 1.011    |
| 4.2.1   | 160| 0.940 | 1.064    | 1.016    | 0.000     | 1.005    |
Table 6

The performance of adjusted Rand index for manifold series with 1, 2, 3 change-points. The highest average adjusted Rand index is highlighted in bold. The last four columns refer to the adjusted Rand index ratio between the four methods to BDCP.

| Example | m  | BDCP | BCP/BDCP | WBS/BDCP | gSeg/BDCP | ECP/BDCP |
|---------|----|------|----------|----------|-----------|----------|
| 4.2.2   | 40 | 0.989| 0.007    | 0.029    | 0.961     | 0.123    |
|         | 60 | 0.986| 0.012    | 0.016    | 0.972     | 0.074    |
|         | 80 | 0.985| 0.003    | 0.019    | 0.980     | 0.037    |
| 4.2.3   | 40 | 0.994| 0.004    | 0.054    | 0.901     | 0.053    |
|         | 60 | 0.993| 0.007    | 0.088    | 1.004     | 0.082    |
|         | 80 | 0.983| 0.008    | 0.123    | 1.017     | 0.071    |
| 4.2.4   | 40 | 0.995| 0.003    | 0.317    | 0.714     | 0.096    |
|         | 60 | 0.994| 0.005    | 0.414    | 0.730     | 0.114    |
|         | 80 | 0.994| 0.009    | 0.482    | 0.756     | 0.110    |
Fig 4. The results of BCP, WBS, ECP, BDCP on the daily log-returns of bitcoin series from 2011/09/13-2012/12/31.