Control and stabilization of degenerate wave equations*

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Abstract

We study a wave equation in one space dimension with a general diffusion coefficient which degenerates on part of the boundary. Degeneracy is measured by a real parameter $\mu_a > 0$. We establish observability inequalities for weakly (when $\mu_a \in [0,1]$) as well as strongly (when $\mu_a \in [1,2]$) degenerate equations. We also prove a negative result when the diffusion coefficient degenerates too violently (i.e. when $\mu_a > 2$) and the blow-up of the observability time when $\mu_a$ converges to 2 from below. Thus, using the HUM method we deduce the exact controllability of the corresponding degenerate control problem when $\mu_a \in [0,2]$. We conclude the paper by studying the boundary stabilization of the degenerate linearly damped wave equation and show that a suitable boundary feedback stabilizes the system exponentially. We extend this stability analysis to the degenerate nonlinearly damped wave equation, for an arbitrarily growing nonlinear feedback close to the origin. This analysis proves that the degeneracy does not affect the optimal energy decay rates at large time. We apply the optimal-weight convexity method of [1, 2] together with the results of the previous section, to perform this stability analysis.

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Abbreviated title. Control of degenerate wave equations

1 Introduction

Control and inverse problems for degenerate PDE’s arise in many applications such as cloaking (building of devices that lead to invisibility properties from observation) [16], climatology [14], population genetics [6], and vision [13]. Such a variety of applications has given birth to challenging mathematical problems for degenerate PDE’s. A common feature of these problems is that they involve operators with variable diffusion coefficients that are not uniformly elliptic in the space domain, even though they are in general uniformly elliptic in compact subsets of the space domain, provided that these subsets are at a positive distance from the degeneracy. This degeneracy may occur either on a part of the boundary or on a sub-manifold of the space domain.

The loss of uniform ellipticity rises new questions related to the well-posedness of the evolution equations in suitable functional spaces as well as new estimates for the underlying elliptic equations. Similarly, in the degenerate case, new tools are necessary for the analysis of observability/nonobservability as well as stabilization.

Control issues for degenerate parabolic equations have received a lot of attention in the last ten years or so (see, for instance, [9, 10, 11, 12], [4], [19], and [8, 7]). New Carleman estimates with adapted weight functions, compared to the usual ones for nondegenerate parabolic equations, have been used to derive observability inequalities for the corresponding dual problems.

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Although degenerate wave equations have received less attention so far, we do believe that time has now come for a complete analysis and deeper understanding of these problems. Therefore, the purpose of this paper is to study controllability and observability issues for degenerate wave equations of the form

\[ u_{tt} - (a(x)u_x)_x = 0 \quad \text{in} \quad \Re > 0, \infty \times [0,1], \tag{1.1} \]

where \( a \) is positive on \([0,1]\) but vanishes at zero. Moreover, if stabilization is usually irrelevant in the parabolic case due to the intrinsic dissipation of diffusion models, it remains an important question for degenerate wave equations and will be addressed in this paper.

The degeneracy of (1.1) at \( x = 0 \) is measured by the parameter \( \mu_a \) defined by

\[ \mu_a := \sup_{0 < x \leq 1} \frac{x|a'(x)|}{a(x)}, \tag{1.2} \]

and one says that (1.1) degenerates weakly if \( \mu_a \in [0,1] \), strongly if \( \mu_a > 1 \). Here we assume \( \mu_a < 2 \) because, like in the parabolic case, observability no longer holds true if \( \mu_a \geq 2 \) as we show in Section 3.3 of this paper.

By determining suitable multipliers linked to the coefficient \( \mu_a \) of the degeneracy and proving refined trace theorems, we prove boundary observability inequalities for (1.1) in a sufficiently large time. This approach and tools are new in the context of degenerate wave equations, as far as we know. It is worth noting that, in problems involving cloaking which, obviously, is incompatible with observability, the degeneracy of the coefficients is quadratic (see [16]). So, our results are consistent with such a property. Moreover, we also study the behavior of the controllability (or observability) time as \( \mu_a \) converges to 2, appealing to Bessel functions to show that such a time blows up as \( \mu_a \) approaches to 2 from below.

For a certain class of weakly degenerate wave equations, an interesting result with \( x = 0 \) as observation region was obtained by Gueye [17] by using the explicit description of the spectrum of the corresponding elliptic operator to treat the related moment problem. As a consequence, an exact controllability result with Dirichlet boundary controls located at the degeneracy point was deduced for degenerate wave equations and then extended to degenerate parabolic equations, giving a first answer to a question that had been open for quite some time. The viewpoint of this paper differs from the one of [17]. Indeed, we allow coefficients to degenerate either weakly or strongly on the boundary and we obtain observability or controllability from the nondegenerate part of the boundary. Moreover, we employ direct techniques such as multipliers and sharp trace results.

Finally, we devote a substantial part of the paper to the study of boundary stabilization for (1.1) when \( \mu_a \in [0,1] \). We consider both the linear feedback

\[ u_t(t,1) + u_x(t,1) + \beta u(t,1) = 0, \tag{1.3} \]

and the nonlinear damping

\[ \rho(u_t(t,1)) + u_x(t,1) + \beta u(t,1) = 0, \tag{1.4} \]

where \( \beta > 0 \), and \( \rho \) is a nondecreasing function such that \( \rho(0) = 0 \). Thanks to the dominant energy approach together with suitable elliptic estimates, we prove that (1.3) stabilizes exponentially the corresponding solution of the degenerate wave equation. For the nonlinear feedback (1.4), we use the optimal-weight convexity method of [1, 2] to establish a quasi-optimal energy decay rate using the multipliers we have determined in the linear case. We also discuss several explicit examples of decay corresponding to different feedbacks. We recall that, for finite dimensional models, the optimality of the decay rates provided by the optimal-weight convexity method is proved in [2]. Moreover, our results show that, under the action of a nonlinear boundary damping, degenerate wave equations enjoy the same stability properties as the corresponding nondegenerate equations, in the sense that both models have the same decay rates of the energy.

We would like to point out that one can reformulate all the above results on nonlinear stabilization by replacing integral inequalities with a Lyapunov function technique. As we explain in Remark 5.8 below, such an operation is essentially of no use.

The paper is organized as follows. In section 2, we introduce our notations, define the degeneracy parameter \( \mu_a \), functional spaces and assumptions. We also prove Poincaré’s type inequalities and some key trace results for functions in weighted Sobolev spaces. In section 3, we consider the dual problem, prove well-posedness, and prove the direct inequality as well as the boundary observability property for \( \mu_a \in [0,2] \). We prove non-observability...
for \( \mu_a > 2 \) and the blow-up of the observability time when \( \mu_a \) converges to 2 from below. We conclude this section by proving exact boundary controllability for the controlled system when \( \mu_a \in [0, 2] \). We consider the boundary stabilization problem in section 4 and prove its well-posedness, together with its exponential stability. We extend this stability analysis to the nonlinear boundary stabilization problem in section 5.

## 2 Assumptions and preliminaries

### 2.1 Assumptions

Let \( a \in C([0,1]) \cap C^1([0,1]) \) be a function satisfying the following assumptions:

\[
\begin{cases}
(i) & a(x) > 0 \quad \forall x \in [0,1], \quad a(0) = 0, \\
(ii) & \mu_a := \sup_{0 < x \leq 1} \frac{x|a'(x)|}{a(x)} < 2, \quad \text{and} \\
(iii) & a \in C^{[\mu_a]}([0,1]),
\end{cases}
\]

where \([\cdot]\) stands for the integer part.

**Remark 2.1** Assumption (2.1) subsumes similar hypotheses that were formulated to treat degenerate parabolic equations (see, for instance, [4, 11, 19]). We list below some simple consequences of (2.1).

1. By integrating the inequality \( sa'(s) \leq \mu_a a(s) \quad \forall s \in [0,1] \)
   over \([x, 1]\) we obtain \( a(x) \geq a(1)x^\mu_a \quad \forall x \in [0,1] \).
   Consequently, \( 1/a \in L^1(0,1) \) when \( \mu_a \in [0, 1] \).

2. Observe that condition (2.1) (iii) is equivalent to require that \( a \in C^1([0,1]) \) when \( \mu_a \in [1, 2] \) (no extra assumption is imposed when \( \mu_a \in [0, 1] \)). In this case of strong degeneracy, we have that \( 1/a \notin L^1(0,1) \).
   Indeed, since \( a \in C^1([0,1]) \), we have that \( \frac{a(x)}{x} < 1 + |a'(0)| \)
   in some neighborhood of 0. So, \( 1/a \notin L^1(0,1) \).

### 2.2 Function spaces

We now introduce some weighted Sobolev spaces that are naturally associated with degenerate operators, see [13]. We denote by \( V_a^1(0,1) \) the space of all functions \( u \in L^2(0,1) \) such that

\[
\begin{cases}
(i) & u \text{ is locally absolutely continuous in } [0,1], \text{ and} \\
(iii) & \sqrt{a}u_x \in L^2(0,1).
\end{cases}
\]

It is easy to see that \( V_a^1(0,1) \) is an Hilbert space with the scalar product

\[
\langle u, v \rangle_{1,a} = \int_0^1 \left(a(x)u'(x)v'(x) + a(x)v(x)\right)dx, \quad \forall u, v \in V_a^1(0,1)
\]

and associated norm

\[
\|u\|_{1,a} = \left\{ \int_0^1 \left(a(x)|u'(x)|^2 + |u(x)|^2\right)dx \right\}^{\frac{1}{2}}, \quad \forall u \in V_a^1(0,1).
\]

Let us also set

\[
|u|_{1,a} = \left\{ \int_0^1 a(x)|u'(x)|^2dx \right\}^{\frac{1}{2}} \quad \forall u \in V_a^1(0,1).
\]
Actually, $| \cdot |_{1,\alpha}$ is an equivalent norm on the closed subspace of $V_{\alpha,0}^1(0,1)$ defined as

$$V_{\alpha,0}^1(0,1) = \{ u \in V_{\alpha}^1(0,1) : u(1) = 0 \}.$$

This fact is a simple consequence of the following version of Poincaré’s inequality.

**Proposition 2.2** Assume (2.1). Then

$$\|u\|_{L^2(0,1)}^2 \leq C_{\alpha} |u|_{1,\alpha}^2 \quad \forall u \in V_{\alpha,0}^1(0,1), \quad (2.4)$$

where

$$C_{\alpha} = \frac{1}{a(1)} \min \left\{ 4, \frac{1}{2 - \mu_\alpha} \right\}. \quad (2.5)$$

**Proof.** Let $u \in V_{\alpha,0}^1(0,1)$. We will prove two different bounds for $\|u\|_{L^2(0,1)}^2$ in terms of $|u|_{1,\alpha}^2$. The conclusion (2.4) will follow by taking the minimum of the two corresponding constants.

First, we use a direct argument. For any $x \in [0,1]$ we have that

$$|u(x)| = \left| \int_x^1 u'(s) ds \right| \leq |u|_{1,\alpha} \left\{ \int_x^1 \frac{ds}{a(s)} \right\}^\frac{1}{2}.$$

Therefore, by Fubini’s theorem,

$$\int_0^1 |u(x)|^2 dx \leq |u|_{1,\alpha}^2 \int_0^1 dx \int_x^1 \frac{ds}{a(s)} = |u|_{1,\alpha}^2 \int_0^1 \frac{ds}{a(s)} \int_0^x ds = |u|_{1,\alpha}^2 \int_0^1 \frac{s}{a(s)} ds. \quad (2.6)$$

By (2.2) we deduce that

$$\int_0^1 \frac{s}{a(s)} ds \leq \frac{1}{a(1)} \int_0^1 s^{1-\mu_\alpha} ds.$$

Together with (2.6), the above inequality yields the first bound we mentioned above, that is,

$$\|u\|_{L^2(0,1)}^2 \leq \frac{|u|_{1,\alpha}^2}{a(1)(2 - \mu_\alpha)} \quad \forall u \in V_{\alpha,0}^1(0,1). \quad (2.7)$$

Next, as an alternative proof, we adapt a reasoning that can be used to prove Hardy’s inequality. Observe that, for all $x \in [0,1]$,

$$0 \leq \int_x^1 \left( su'(s) + \frac{1}{2} u(s) \right)^2 ds = \int_x^1 \left( s^2 |u'(s)|^2 + \frac{1}{4} |u(s)|^2 + s u(s) u'(s) \right) ds = \int_x^1 \left( s^2 |u'(s)|^2 - \frac{1}{4} |u(s)|^2 \right) ds - \frac{1}{2} x |u(x)|^2.$$

Therefore, taking the limit as $x \downarrow 0$, by (2.2) we obtain the announced second bound:

$$\int_0^1 |u(s)|^2 ds \leq 4 \int_0^1 s^2 |u'(s)|^2 ds \leq 4 \int_0^1 s^{\mu_\alpha} |u'(s)|^2 ds \leq \frac{4}{a(1)} \int_0^1 a(s) |u'(s)|^2 ds \quad \forall u \in V_{\alpha,0}^1(0,1). \quad (2.8)$$

The conclusion follows from (2.7) and (2.8). \qed

**Example 2.3** The following are examples of functions $a$ satisfying assumption (2.1).

1. Let $\theta \in [0,2]$ be given. Define

$$a(x) = x^\theta \quad \forall x \in [0,1]. \quad (2.9)$$

In this case, we have

$$\|u\|_{L^2(0,1)}^2 \leq \min \left\{ 4, \frac{1}{2 - \theta} \right\} |u|_{1,\alpha}^2 \quad \forall u \in V_{\alpha,0}^1(0,1). \quad (2.10)$$
2. Let \( \theta \in [0, 2] \) be given and let \( \alpha \in [0, 1 - \theta/2] \). Then the function

\[
a(x) = \begin{cases} 
x^\theta (1 + \sin^2(\log x^\alpha)) & \forall x \in [0, 1] \\
0 & x = 0
\end{cases}
\]  

(2.11)

satisfies (2.1). Indeed,

\[
a'(x) = \theta x^{\theta - 1} (1 + \sin^2(\log x^\alpha)) + 2\alpha x^{\theta - 1} \sin(\log x^\alpha) \cos(\log x^\alpha) & \forall x \in [0, 1],
\]  

so that \( \mu_a \leq \theta + 2\alpha < 2 \). Notice that (2.10) is still valid for this weight function because

\[
a(x) \geq x^\theta & \forall x \in [0, 1],
\]

which is what is really needed for the proof of Proposition 2.2.

Remark 2.4 We do not expect the constant \( C_a \) in (4.9) to be optimal. For instance, for the weight \( a \) in (2.9), we have that

\[
\min \left\{ 4, \frac{1}{2 - \theta} \right\} \to \frac{1}{2} \text{ as } \theta \downarrow 0,
\]

which is strictly greater than the minimal constant in the case \( \theta = 0 \), that is, \((2/\pi)^2\). On the other hand, (4.9) shows that \( C_a \) does not blow up as \( \mu_a \uparrow 2 \) because it is bounded above by 4.

Next, we define

\[
V^2_a(0,1) = \{ u \in V^1_a(0,1) : au' \in H^1(0,1) \},
\]

where \( H^1(0,1) \) denotes the classical Sobolev space of all functions \( u \in L^2(0,1) \) such that \( u' \in L^2(0,1) \). Notice that, if \( u \in V^2_a(0,1) \), then \( au' \) is continuous on \([0,1]\).

We collect below useful properties of the above functional spaces. Some of the following results are known, others are new. We prove all of them for completeness.

Proposition 2.5 Assume (2.1). Then the following properties hold true.

(I) For every \( u \in V^1_a(0,1) \)

\[
\lim_{x \downarrow 0} xu^2(x) = 0,
\]

(2.12)

\[
u^2(1) \leq \max \left\{ 2, \frac{1}{a(1)} \right\} \|u\|_{1,a}^2.
\]

(2.13)

Moreover, if \( \mu_a \in [0,1] \), then \( u \) is absolutely continuous in \([0,1]\).

(II) For every \( u \in V^2_a(0,1) \)

\[
\lim_{x \downarrow 0} x a(x) u'(x)^2 = 0.
\]

(2.14)

For all \( u \in V^2_a(0,1) \) and \( \phi \in V^1_a(0,1) \)

\[
\lim_{x \downarrow 0} a(x) \phi(x) u'(x) = 0,
\]

(2.15)

assuming, in addition, \( \phi(0) = 0 \) when \( \mu_a \in [0,1] \).

(III) If \( \mu_a \in [1,2] \), then for every \( u \in V^2_a(0,1) \)

\[
\lim_{x \downarrow 0} a(x) u'(x) = 0.
\]

(2.16)
Proof. (I) Let \( u \in V^1_a(0, 1) \). We will show that

\[
v(x) := \begin{cases} x u^2(x) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}
\]

is continuous on \([0, 1]\). Indeed, \( v \) is locally absolutely continuous in \([0, 1]\) and

\[
v'(x) = u^2(x) + 2 x u'(x) u(x) \quad \text{a.e. in } [0, 1].
\]

Now, the above right-hand side is in \( L^1(0, 1) \) because \( u \in L^2(0, 1) \) and, thanks to (2.2),

\[
\int_0^1 x^2 u'(x)^2 \, dx \leq \int_0^1 x^{\mu_a} |u'(x)|^2 \, dx \leq \frac{1}{a(1)} \int_0^1 a(x) |u'(x)|^2 \, dx.
\]

Then, the limit \( \lim_{x \downarrow 0} v(x) := L \) does exist and must vanish for otherwise \( u^2(x) \sim L/x \) (near zero) would not be summable. (2.12) is thus proved.

Next, we have that

\[
u^2(1) = v(1) = \int_0^1 (u^2(x) + 2 x u'(x) u(x)) \, dx \leq 2 \int_0^1 u^2(x) \, dx + \int_0^1 x^2 |u'(x)|^2 \, dx
\]

which, in turn, yields

\[
u^2(1) \leq 2 \int_0^1 u^2(x) \, dx + \frac{1}{a(1)} \int_0^1 a(x) |u'(x)|^2 \, dx
\]

in view of (2.17).

Now, suppose, in addition, that \( \mu_a \in [0, 1] \). Then

\[
u'(x) = \frac{1}{\sqrt{a(x)}} \sqrt{a(x)} u'(x) \quad \forall x \in [0, 1]
\]

is summable over \((0, 1)\) thanks to Remark 2.1-1 and (2.3)(iii). So, \( u \) is absolutely continuous in \([0, 1]\).

(II) Let \( u \in V^2_a(0, 1) \). We claim that

\[
v(x) := \begin{cases} x a(x) u'(x)^2 & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}
\]

is continuous on \([0, 1]\). Indeed, \( v \) is locally absolutely continuous in \([0, 1]\) and

\[
v'(x) = a(x) u'(x)^2 + x a'(x) u'(x)^2 + 2 x a(x) u'(x) u''(x)
\]

\[
= a(x) u'(x)^2 + 2 x u'(x) (a(x) u'(x) - x a'(x) u'(x)) \quad \text{a.e. in } [0, 1].
\]

Now, observe that the first term in the right-hand side above is summable over \([0, 1]\) in view of (2.3) (iii), and the same is true for second one because, by (2.2),

\[
x |u'(x)| \leq x^{\mu_a/2} |u'(x)| \leq \sqrt{\frac{a(x)}{a(1)}} |u'(x)| \quad \forall x \in [0, 1].
\]

As for the third term, owing to (2.1) (ii),

\[
x |a'(x)| u'(x)^2 \leq \mu_a a(x) u'(x)^2 \quad \forall x \in [0, 1]
\]

and the above right-hand side is summable in view of (2.2) (iii). Then, \( \lim_{x \downarrow 0} v(x) := L \) exists and must vanish, for otherwise \( a(x) u^2(x) \sim L/x \) (near zero) would not be summable. This concludes the proof of (2.14).

(III) Next, we proceed to prove (2.16) noting that \( \lim_{x \downarrow 0} a(x) u'(x) := L \) exists because \( u \in V^2_a(0, 1) \) and must vanish, for otherwise \( a(x) u'(x)^2 \sim L^2/a(x) \) (near zero) would not be summable in view of Remark 2.1-3.
Finally, in order to show (2.15), we begin by proving that the function

\[ w(x) := \begin{cases} 
  a(x) \phi(x) u'(x) & 0 < x \leq 1 \\
  0 & x = 0 
\end{cases} \]

is continuous on [0, 1]. This follows by the arguments as above, because

\[ w'(x) = a(x)\phi'(x)u'(x) + \phi(x)(a(x)u'(x))' \]

is summable over [0, 1]. Therefore, the limit \( \lim_{x \to 0} w(x) =: L \) exists and \( a(x)\phi(x)u'(x) \sim |L| \) near 0. We now have to distinguish two cases. If \( \mu_a \in [0,1] \) and \( \phi(0) = 0 \), then the conclusion is immediate. If, on the other hand, \( \mu_a \in [1,2] \), then, owing to (2.16),

\[ a(x)|u'(x)| = \left| \int_0^x (a(x)u'(x))' \, dx \right| \leq \sqrt{x} \| (a'u')' \|_{L^2(0,1)} \quad \forall x \in [0,1]. \]

Now, if \( L \neq 0 \), then, in a neighborhood of 0,

\[ \frac{|L|}{2} \leq a(x)\phi(x)u'(x) \leq \sqrt{x} \| (a'u')' \|_{L^2(0,1)}|\phi(x)|, \quad \forall x \in [0,1] \]

in contrast to the fact that \( \phi \in L^2(0,1) \).

\[ \square \]

3 Observability

Given a satisfying assumptions (2.1), let \( \mu_a \in [0,2] \) be the constant in assumption (ii). Consider the degenerate wave equation

\[ u_{tt} - (a(x)u_x)_x = 0 \quad \text{in } [0, \infty[, \times ]0,1[ \quad (3.1) \]

with

\[ \begin{cases} 
  \text{boundary conditions} & \, u(t,1) = 0 \quad \text{and} \quad u(t,0) = 0 \quad \text{if } \mu_a \in [0,1] \\
  \lim_{x \to 0} a(x) u_x(t, x) = 0 \quad \text{if } \mu_a \in [1,2] \\
  \text{initial conditions} & \, \begin{cases} 
  u(0,x) = u_0(x) \\
  u_t(0,x) = u_1(x) 
\end{cases} \quad x \in [0,1]. \quad (3.2) 
\end{cases} \]

We recall that, since equation (3.1) is degenerate, different boundary conditions have to be imposed at \( x = 0 \) depending on whether we are interested in:

- the \textit{weakly degenerate case} \( \mu_a \in [0,1] \), where, in view of Proposition 2.5-(I), we have that the Dirichlet boundary condition \( u(t,0) = 0 \) makes sense for any solution, and
- the \textit{strongly degenerate case} \( \mu_a \in [1,2] \), where, in view of Proposition 2.5-(II), we have that the Neumann boundary condition \( \lim_{x \to 0} a(x) u_x(t, x) = 0 \) is automatically satisfied by any classical solution.

In order to express the above boundary conditions in functional settings, we define \( H^2_{\alpha}(0,1) \) to be the closed subspace of \( V^2_{\alpha}(0,1) \) which consists of all \( u \in V^2_{\alpha}(0,1) \) satisfying \( u(0) = 0 \) when \( \mu_a \in [0,1] \). We also set

\[ H^2_{\alpha}(0,1) = V^2_{\alpha}(0,1) \cap H^1_{\alpha}(0,1). \]

Observe that all functions \( u \in H^2_{\alpha}(0,1) \) satisfy homogeneous boundary conditions at both \( x = 0 \) and \( x = 1 \). Such conditions are of Dirichlet type in the weakly degenerate case, whereas they are of Neumann/Dirichlet type at \( x = 0 \) and \( x = 1 \), respectively, when \( \mu_a \in [1,2] \).
3.1 Well-posedness

Let us recall the typical abstract set-up of semigroup theory which provides weak and classical notions of solutions for problem (3.1)-(3.2). Consider the Hilbert space \( H_0 = H^1(0, 1) \times L^2(0, 1) \) with the scalar product

\[
\langle (u, v), (\tilde{u}, \tilde{v}) \rangle = \int_0^1 (v(x)\tilde{v}(x) + a(x)u'(x)\tilde{u}'(x))\,dx \quad \forall (u, v), (\tilde{u}, \tilde{v}) \in H_0.
\]

Arguing as for the classical wave equation (see, for instance, [20]) one can show that the unbounded operator \( A : D(A) \subset H_0 \rightarrow H_0 \) defined by

\[
\begin{align*}
D(A) &= H^2_0(0, 1) \times H^1_0(0, 1) \\
A(u, v) &= (v, (au')') \quad \forall (u, v) \in D(A)
\end{align*}
\]

is maximal dissipative on \( H_0 \). Therefore, \( A \) is the generator of a contraction semigroup in \( H_0 \), denoted by \( e^{tA} \). For any \( U_0 := (u_0, v_0) \in H_0 \), \( U(t) := e^{tA}U_0 \) gives the so-called mild solution of the Cauchy problem

\[
\begin{align*}
\left\{ 
U'(t) &= AU(t) \quad (t \geq 0) \\
U(0) &= U_0.
\end{align*}
\]

When \( U_0 \in D(A) \), the above solution is classical in the sense that \( U \in C^1([0, \infty]; H_0) \cap \mathcal{C}([0, \infty]; D(A)) \) and the equation holds on \([0, \infty[\). In view of the above considerations, given \( (u_0, u_1) \in H^1_0(0, 1) \times L^2(0, 1) \), we say that the function

\[
u \in C^1([0, \infty[; L^2(0, 1)) \cap \mathcal{C}([0, \infty[; H^1_0(0, 1))
\]

is the mild (or weak) solution of problem (3.1)-(3.2) if \( (u(t), v(t)) = e^{tA}(u_0, v_0) \) for all \( t \geq 0 \). By the aforementioned regularity result for \( e^{tA} \), if \( (u_0, u_1) \in H^2_0(0, 1) \times H^1_0(0, 1) \), then \( u \) is the classical solution of (3.1)-(3.2) meaning that

\[
u \in C^2([0, \infty[; L^2(0, 1)) \cap \mathcal{C}^1([0, \infty[; H^1_0(0, 1)) \cap \mathcal{C}([0, \infty[; H^2_0(0, 1))
\]

and (3.1) is satisfied for all \( t \in [0, \infty[ \) and a.e. \( x \in [0, 1] \).

The energy of a mild solution \( u \) of (3.1) is the continuous function defined by

\[
E_u(t) = \frac{1}{2} \int_0^1 \left\{ u^2_t(t, x) + a(x)u^2_x(t, x) \right\} \,dx \quad \forall t \geq 0.
\] (3.3)

**Proposition 3.1** Assume (2.1) and let \( u \) be the mild solution of (3.1)-(3.2). Then

\[
E(t) = E(0) \quad \forall t \geq 0.
\] (3.4)

**Proof.** Suppose, first, that \( u \) is a classical solution of (3.1). Then, multiplying the equation by \( u_t \) and integrating by parts we obtain

\[
\begin{align*}
0 &= \int_0^1 u_t(t, x)\left\{ u_{tt}(t, x) - (a(x)u_x(t, x))_x \right\} \,dx \\
&= \int_0^1 \left\{ u_t(t, x)u_{tt}(t, x) + a(x)u_x(t, x)u_{tx}(t, x) \right\} \,dx - \left[ a(x)u_t(t, x)u_x(t, x) \right]_{x=0}^{x=1} \\
&= \frac{d}{dt}E_u(t)
\end{align*}
\]

By noting that the boundary terms vanish because of the boundary conditions in both the weakly and strongly degenerate cases, we conclude that the energy of \( u \) is constant. The same conclusion can be extended to any mild solution by an approximation argument. \( \square \)
3.2 Boundary observability

**Lemma 3.2** For any mild solution $u$ of (3.1) we have that $u_x(\cdot, 1) \in L^2(0, T)$ for every $T \geq 0$ and

$$a(1) \int_0^T u_x^2(t, 1) \, dt \leq \left( 6T + \frac{1}{\min\{1, a(1)\}} \right) E_u(0).$$

Moreover,

$$a(1) \int_0^T u_x^2(t, 1) \, dt = \int_0^T \int_0^1 \left\{ u_x^2(t, x) + (a(x) - xa'(x)) u_x^2(t, x) \right\} \, dx \, dt + 2 \left[ \int_0^1 xu_x(t, x) u_t(t, x) \, dx \right]_{t=0}^{t=T}.$$  (3.6)

**Proof.** Suppose first $(u_0, u_1) \in H_0^2(0, 1) \times H_0^1(0, 1)$ so that $u$ is a classical solution of (3.1). Then, by multiplying equation (3.1) by $x u_x$ and integrating over $]0, T[ \times [0, 1]$ we obtain

$$0 = \int_0^T \int_0^1 xu_x(t, x) \left( u_t(t, x) - (a(x)u_x(t, x))_x \right) \, dx \, dt$$

$$= \left[ \int_0^1 \int_0^1 xu_x(t, x) u_t(t, x) dx \right]_{t=0}^{t=T} - \int_0^T \int_0^1 xu_x(t, x) u_t(t, x) \, dx \, dt$$

$$- \int_0^T \int_0^1 \left( x a'(x) u_x^2(t, x) + a(x) u_x(t, x) u_{xx}(t, x) \right) \, dx \, dt$$

$$= \left[ \int_0^1 \int_0^1 xu_x(t, x) u_t(t, x) dx \right]_{t=0}^{t=T} - \int_0^T \int_0^1 x a'(x) u_x^2(t, x) \, dx \, dt$$

$$- \int_0^T \int_0^1 \left\{ x \left( \frac{u_x^2(t, x)}{2} \right)_x + a(x) \left( \frac{u_x^2(t, x)}{2} \right)_x \right\} \, dx \, dt.$$  (3.7)

We proceed to integrate by parts the last two terms above. We obtain

$$\int_0^T \int_0^1 x \left( \frac{u_x^2(t, x)}{2} \right)_x \, dx \, dt = -\frac{1}{2} \int_0^T \int_0^1 u_x^2(t, x) \, dx \, dt$$

(3.8)

because $x u_x^2(t, x)$ vanishes at $x = 1$ and, owing to (2.12), also at $x = 0$. Moreover, on account of (2.14) we have

$$\int_0^T \int_0^1 x a(x) \left( \frac{u_x^2(t, x)}{2} \right)_x \, dx \, dt = \frac{1}{2} \int_0^T a(1) u_x^2(t, 1) \, dt - \frac{1}{2} \int_0^T \int_0^1 (xa(x))' u_x^2(t, x) \, dx \, dt.$$  (3.9)

Then the identity (3.6) follows by inserting (3.8) and (3.9) into (3.7).

Next, recall (2.2) to obtain

$$\left| \int_0^1 xu_x(t, x) u_t(t, x) \, dx \right| \leq \frac{1}{2} \int_0^1 \left\{ u_x^2(t, x) + x^2 u_x^2(t, x) \right\} \, dx \leq \frac{E_u(0)}{\min\{1, a(1)\}} \quad \forall t \geq 0.$$  (3.10)

Now, we deduce (3.5) from (3.6), (3.10), the inequality $x a'(x) \leq 2a(x)$, and the constancy of the energy. The conclusion has thus been proved for classical solutions.

In order to extend (3.5) and (3.6) to the mild solution associated with the initial data $(u_0, u_1) \in H_0^1(0, 1) \times L^2(0, 1)$, it suffices to approximate such data by $(u_0^n, u_1^n) \in H_0^2(0, 1) \times H_0^1(0, 1)$ and use (3.5) to show that the normal derivatives of the corresponding classical solutions give a Cauchy sequence in $L^2(0, T)$.  \hfill \Box

**Lemma 3.3** For any mild solution $u$ of (3.1) we have that, for every $T \geq 0$,

$$\int_0^T \int_0^1 \left\{ a(x) u_x^2(t, x) - u_t^2(t, x) \right\} \, dt \, dx + \left[ \int_0^1 u(t, x) u_t(t, x) \, dx \right]_{t=0}^{t=T} = 0.$$  (3.11)
Proof. Once again we suppose \( u \) is a classical solution of (3.1). Multiplying equation (3.1) by \( u \) and integrating over \([0, T[\times]0, 1]\) we obtain

\[
0 = \int_0^T \int_0^1 u(t,x) \left( u_t(t,x) - (a(x)u_x(t,x))_x \right) dx dt
\]

\[
= \left[ \int_0^1 u(t,x)u_t(t,x)dx \right]_{t=0}^{t=T} - \int_0^T \int_0^1 u_x^2(t,x) dx dt
\]

\[
- \int_0^T \left[ a(x) u(t,x)u_x(t,x) \right]_{x=1}^{x=0} dx dt + \int_0^T \int_0^1 a(x) u_x^2(t,x) dx dt .
\]

The conclusion follows from the above identity because \( a(x) u(t,x)u_x(t,x) \) vanishes at \( x = 1 \) and, owing to (2.15), also at \( x = 0 \). An approximation argument allows to extend the conclusion to mild solutions. \( \square \)

**Theorem 3.4** Assume (2.1) and let \( u \) be the mild solution of (3.1)-(3.2). Then, for every \( T \geq 0 \),

\[
a(1) \int_0^T u_x^2(t,1) dt \geq \left\{ (2 - \mu_a) T - \frac{4}{\min\{1,a(1)\}} - 2 \mu_a \sqrt{C_a} \right\} E_u(0),
\]

where \( C_a \) is the constant in (4.9).

Proof. Suppose \( u \) is a classical solution of (3.1) (the general case can as usual be recovered by an approximation argument). By adding to the right-hand side of (3.6) the left side of (3.11) multiplied by \( \mu_a/2 \), we obtain

\[
a(1) \int_0^T u_x^2(t,1) dt = \int_0^T \int_0^1 \left\{ \left( 1 - \frac{\mu_a}{2} \right) u_x^2(t,x) + \left( 1 + \frac{\mu_a}{2} \right) a(x) - xa'(x) \right\} u_x^2(t,x) dt dx
\]

\[
+ 2 \left[ \int_0^1 x u_x(t,x)u_t(t,x)dx \right]_{t=0}^{t=T} + \frac{\mu_a}{2} \left[ \int_0^1 u(t,x)u_x(t,x)dx \right]_{t=0}^{t=T}
\]

\[
\geq (2 - \mu_a) T E_u(0) + 2 \left[ \int_0^1 x u_x(t,x)u_t(t,x)dx \right]_{t=0}^{t=T} + \frac{\mu_a}{2} \left[ \int_0^1 u(t,x)u_x(t,x)dx \right]_{t=0}^{t=T},
\]

where we have used the inequality \( xa'(x) \leq \mu_a a(x) \) and the constancy of the energy. The conclusion follows from the above inequality recalling (3.10) and observing that

\[
\frac{1}{2} \int_0^1 u(t,x)u_t(t,x)dx \leq \frac{1}{2} \int_0^1 \left( 1 \right) \frac{1}{\sqrt{C_a}} u_x^2(t,x) \right\} dx \leq \sqrt{C_a} E_u(0),
\]

where \( C_a \) is Poincaré’s constant in (4.9). \( \square \)

We recall that (3.1) is said to be observable (via the normal derivative at \( x = 1 \)) in time \( T > 0 \) if there exists a constant \( C > 0 \) such that for any \( (u_0, u_1) \in H_0^2(0,1) \times L^2(0,1) \) the mild solution of (3.1)-(3.2) satisfies

\[
\int_0^T u_x^2(t,1) dt \geq C E_u(0).
\]

Any constant satisfying (3.13) is called an observability constant for (3.1) in time \( T \). The supremum of all observability constants for (3.1) is denoted by \( C_T \). Equivalently, (3.1) is observable if

\[
C_T = \inf_{(u_0,u_1)\neq(0,0)} \frac{\int_0^T u_x^2(t,1) dt}{E_u(0)} > 0.
\]

The inverse \( c_T = 1/C_T \) is sometimes called the cost of observability (or the cost of control) in time \( T \).

**Corollary 3.5** Assume (2.1). Then (3.1) is observable in time \( T \) provided that

\[
T > T_a := \frac{4}{(2 - \mu_a) \min\{1,a(1)\}} + 2 \mu_a \sqrt{C_a},
\]

where \( C_a \) is defined in (4.9). In this case

\[
C_T \geq \frac{1}{a(1)} \left\{ (2 - \mu_a) T - \frac{4}{\min\{1,a(1)\}} - 2 \mu_a \sqrt{C_a} \right\}.
\]
Remark 3.6 Let \( a \) be any of the two functions in Example 2.3. Then we can apply the above to conclude that, defining
\[
T_\theta = \frac{1}{2 - \theta} \left( 4 + 2a \min \left\{ 2, \frac{1}{\sqrt{2 - \theta}} \right\} \right),
\]  
we have that
\[
C_T \geq (2 - \theta)(T - T_\theta) \quad \forall T \geq T_\theta.
\]  
Observe that \( T_\theta \to 2 \) as \( \theta \downarrow 0 \), which coincides with the classical observability time for the wave equation.

3.3 Failure of boundary observability

In this section, we shall see that boundary observability is no longer true when the constant \( \mu_a \) in (2.1) is greater than or equal to \( 2 \) and that, for \( \mu_a < 2 \), the controllability time blows up as \( \mu_a \uparrow 2 \). We discuss two examples with power-like coefficients.

Example 3.7 Given \( T > 0 \), consider the problem
\[
\begin{aligned}
&\begin{cases}
  u_{tt} - (x^2 u_x)_x = 0 \\
  \text{boundary conditions: } u(t, 1) = 0 \quad \text{and } \lim_{x \to 0} x^2 u_x(t, x) = 0 \\
  \text{initial conditions: } u(0, x) = u_0(x) \\
  u_t(0, x) = u_1(x)
\end{cases} & \text{in } [0, T] \times [0, 1[ \\
\end{aligned}
\]  
where \( u_0 \) and \( u_1 \) are smooth functions with compact support in \([0, 1[\), not identically zero. Observe that the so-called Liouville transform
\[
u(t, x) = \frac{1}{\sqrt{x}} u \left( t, \log \frac{1}{x} \right)\]
turns problem (3.16) into
\[
\begin{aligned}
&\begin{cases}
  v_{tt} - v_{yy} + \frac{1}{4} v = 0 \\
  v(t, 0) = 0 \\
  \text{initial conditions: } v(0, y) = e^{-y/2} u_0(e^{y}) := v_0(y) \\
  v_t(0, y) = e^{-y/2} u_1(e^{y}) := v_1(y)
\end{cases} & \text{in } [0, T] \times [0, \infty[ \\
\end{aligned}
\]  
Notice that \( v_0 \) and \( v_1 \) are, in turn, smooth functions with compact support in \([0, \infty[\) and the extreme point \( y = 0 \) of the \( y \)-domain corresponds to \( x = 1 \) in the \( x \)-domain. Since, for the wave equation (with a bounded potential) the support of the initial data propagates at finite speed (see, for instance, [15]), the normal derivative \( v_y(., 0) \) of the solution to (3.17) may well be identically zero on \([0, T]\) when the support of \( v_0 \) and \( v_1 \) is sufficiently far from \( y = 0 \). Consequently, problem (3.16) is not observable on \([0, T]\) via the normal derivative \( u_x(., 1) \).

Example 3.8 Given \( T > 0 \) and \( \theta > 2 \), consider the problem
\[
\begin{aligned}
&\begin{cases}
  u_{tt} - (x^\theta u_x)_x = 0 \\
  \text{boundary conditions: } u(t, 1) = 0 \quad \text{and } \lim_{x \to 0} x^\theta u_x(t, x) = 0 \\
  \text{initial conditions: } u(0, x) = u_0(x) \\
  u_t(0, x) = u_1(x)
\end{cases} & \text{in } [0, T] \times [0, 1[ \\
\end{aligned}
\]  
where \( u_0 \) and \( u_1 \) are smooth functions with compact support in \([0, 1[\]. Define \( \varphi : [0, 1[ \to [0, \infty[ \) by
\[
\varphi(x) = \int_x^1 \frac{ds}{s^{\theta/2}} = \frac{2(x^{1-\theta/2} - 1)}{\theta - 2},
\]
and denote by \( \psi \) the inverse of \( \varphi \), that is,
\[
\psi(y) = \left( \frac{2}{2 + (\theta - 2)y} \right)^{2/\theta^2} \quad \forall y \in [0, \infty[.
\]
As in Example 3.7, the change of variable
\[ u(t, x) = \frac{1}{x^{\theta/4}} v(t, \varphi(x)) \]
transforms problem (3.18) into
\[
\begin{cases}
  v_{tt} - v_{yy} + \frac{c(\theta)}{[2 + (\theta - 2)y]^2} v = 0 & \text{in } [0, T] \times [0, \infty[
  \\
  v(t, 0) = 0 & 0 < t < T
  \\
  \text{initial conditions: } & \begin{cases}
    v(0, y) = v_0(y) & y \in ]0, \infty[, \\
    v_1(0, y) = v_1(y) & y \in ]0, \infty[, 
  \end{cases}
\end{cases}
\tag{3.19}
\]
where \( c(\theta) = \theta(3\theta - 4)/4 \).

Notice that, as before, \( v_0 \) and \( v_1 \) are smooth functions with compact support in \([0, \infty[\) and the extreme point \( y = 0 \) of the \( y \)-domain corresponds to \( x = 1 \) in the \( x \)-domain. Therefore, the finite speed of propagation of the support for the wave equation (with a bounded potential) implies that the normal derivative \( v_y(\cdot; 0) \) is identically zero on \([0, T] \) when the support of \( v_0 \) and \( v_1 \) is sufficiently far from \( y = 0 \). Consequently, problem (3.16) is not observable on \([0, T] \).

### 3.4 Blow-up of observability time

In this section, we will show that, for any fixed \( T > 0 \) the observability constant \( C_T(\theta) \) of (3.18), with \( 0 \leq \theta < 2 \), goes to zero as \( \theta \uparrow 2 \). We begin by recalling spectral results for the family of Sturm-Liouville eigenvalue problems

\[
\begin{cases}
  -(x^\theta y'(x))' = \lambda y(x) & x \in ]0, 1[ \\
  \lim_{x \downarrow 0} x^\theta y'(x) = 0 \text{ and } y(1) = 0 .
\end{cases}
\tag{3.20}
\]

For any \( \nu \geq 0 \), denote by \( J_\nu \) the Bessel function of the first kind of order \( \nu \), that is,
\[
J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left( \frac{x}{2} \right)^{2m+\nu},
\]
where \( \Gamma \) is Euler’s Gamma function. Let \( j_\nu \) be the first positive zero of \( J_\nu \).

**Proposition 3.9** Given \( \theta \in [1, 2] \) define
\[
\nu_0 = \frac{\theta - 1}{2 - \theta} \quad \text{and} \quad \kappa_0 = \frac{2 - \theta}{2}.
\tag{3.21}
\]

Then the first eigenvalue of (3.20) is given by \( \lambda_0 = \kappa_0^2 J_{\nu_0}^2 \) and the corresponding normalized eigenfunction is
\[
y_0(x) = \sqrt{2\kappa_0} \frac{J_{\nu_0}(j_{\nu_0} x^{\nu_0})}{|J_{\nu_0}(j_{\nu_0})|} \quad (0 < x < 1).
\]

See [21] for the proof.

**Theorem 3.10** For any fixed \( T > 0 \) the observability constant \( C_T(\theta) \) of (3.18), with \( 1 \leq \theta < 2 \), satisfies
\[
C_T(\theta) \leq (2 - \theta)T.
\tag{3.22}
\]

**Proof.** Define
\[
u_0(t, x) = \sin \left( \sqrt{\lambda_0} t \right) y_0(x) \quad (t, x) \in ]0, T[ \times ]0, 1[.
\]
Then \( u_\theta \) satisfies (3.18) with \( u_0 \equiv 0 \) and \( u_1(x) = \sqrt{\lambda_0} y_0(x) \). Now, straightforward computations lead to
\[
\frac{\int_0^T |\partial_x u_\theta|^2(t, 1) \, dt}{E_{u_\theta}(0)} = 2T \kappa_0 \left( 1 - \frac{\sin(2\sqrt{\lambda_0} T)}{2\sqrt{\lambda_0} T} \right) < (2 - \theta)T
\]
taking into account the definition of \( \kappa_0 \) in (3.21). The conclusion follows recalling the definition of \( C_T \). \( \square \)
Remark 3.11 Given $C > 0$, let $T^*_\theta(C)$ denote the infimum of all times $T > 0$ such that $C$ is an observability constant for (3.18) in time $T$. Then (3.22) yields

$$T^*_\theta(C) \geq \frac{C}{2 - \theta},$$

which means that the observability time $T^*_\theta(C)$ blows up, as $\theta \uparrow 2$, at essentially the same speed as $T_\theta$ in (3.14).

### 3.5 Controllability

We consider the following controlled degenerate system

$$y_{tt} - (a(x)y_x)_x = 0 \quad \text{in } [0, \infty] \times [0, 1]$$

(3.23)

with

- boundary conditions $y(t, 1) = f$ and $y(t, 0) = 0$ if $\mu_a \in [0, 1]$,
- initial conditions
  $$\begin{align*}
  y(0, x) &= y_0(x) \\
  y_t(0, x) &= y_1(x)
  \end{align*}$$

where $f \in L^2(0, T)$ is the control. The solution of this controlled system is defined by transposition. At this stage, we have to introduce some notation. Let us define the operator $A_0 : D(A_0) \subset H \mapsto H$ where $D(A_0) = H^1_a(0, 1)$ and $A_0u := -(au')'$ for $u \in D(A_0)$. We define $H^{-1}_a(0, 1)$ as the dual space of $H^1_a(0, 1)$ with respect to the pivot space $L^2(0, 1)$. Then, thanks to Proposition 2.2, one can prove that $A_0$ is an isomorphism from $H^1_a(0, 1)$ onto $H^{-1}_a(0, 1)$. In particular, we have $H^{-1}_a(0, 1) = A_0H^1_a(0, 1)$.

**Definition 3.12** Let $f \in L^2_{\text{loc}}(0, \infty)$ and let $(y_0, y_1) \in L^2(0, 1) \times H^{-1}_a(0, 1)$ be fixed arbitrarily. We say that $y$ is a solution by transposition of (3.23)-(3.24) if

$$y \in C^1([0, \infty]; H^{-1}_a(0, 1)) \cap C([0, \infty]; L^2(0, 1))$$

satisfies for all $T > 0$

$$\begin{align*}
(y'\! (T), w^0_T)_{H^{-1}_a(0, 1), H^1_a(0, 1)} - \int_0^1 y(T)w^1_T dx &= (y_1, w(0))_{H^{-1}_a(0, 1), H^1_a(0, 1)} - \int_0^1 y_0w'(0) dx \\
&+ \int_0^T f(t)w_x(t, 1) dt \quad \forall (w^0_T, w^1_T) \in H^1_a(0, 1) \times L^2(0, 1),
\end{align*}$$

(3.25)

where $w$ is the solution of the backward equation

$$w_{tt} - (a(x)w_x)_x = 0 \quad \text{in } [0, \infty] \times [0, 1]$$

(3.26)

with

- boundary conditions $w(t, 1) = 0$ and
- initial conditions
  $$\begin{align*}
  w(T, x) &= w^0_T(x) \\
  w_T(T, x) &= w^1_T(x)
  \end{align*}$$

(3.27)

Note that thanks to the change of variable $w(t, x) = w(T - t, x)$ and to our previous results, the backward problem (3.26) admits a unique solution $w \in C^1([0, \infty]; L^2(0, 1)) \cap C([0, \infty]; H^1_a(0, 1))$. Moreover, this solution depends continuously on $W^T := (w^0_T, w^1_T) \in H^1_a(0, 1) \times L^2(0, 1)$ and the energy $E_w$ of $w$ is conserved through time. Now thanks to the direct inequality (3.5), we have

$$\int_0^T w_T^2(t, 1) dt \leq D_T E_w(0) = D_T E_w(T).$$
Thus, the right hand side of (3.25) defines a continuous linear form with respect to \((w_T^0, w_T^1) \in H_a^1(0,1) \times L^2(0, 1)\). Moreover, this linear form depends continuously on \(T > 0\), for all \(T > 0\). Therefore, there is a unique solution by transposition \(w \in C^1([0, \infty; H_a^{-1}(0,1) \cap C([0, \infty; L^2(0,1)))\ of\ (3.25)\).

Let \((y_0, y_1) \in L^2(0, 1) \times H_a^{-1}(0,1)\), \((y_0^T, y_1^T) \in L^2(0, 1) \times H_a^{-1}(0,1)\) be given: then one wants to determine if there exists a control \(\beta \in \Lambda\), such that the solution of \((3.23)\) satisfies \((y_0, y_1)(T, \cdot) \equiv (y_0^T, y_1^T)(\cdot)\). If this is possible for every \((y_0, y_1) \in L^2(0, 1) \times H_a^{-1}(0,1)\) and \((y_0^T, y_1^T) \in L^2(0, 1) \times H_a^{-1}(0,1)\), one says that \((3.23)\) is exactly controllable in \(L^2(0, 1) \times H_a^{-1}(0,1)\).

By linearity and reversibility, it is easy to check that this property will hold as soon as it holds for arbitrary initial data \((y_0, y_1)\) and for a zero final state, that is for \((y_0^T, y_1^T)(\cdot) = (0, 0)\).

Let us consider the bilinear form \(\Lambda\) defined on \(H_a^1(0,1) \times L^2(0, 1)\) by

\[
\Lambda(W^T, \tilde{W}^T) = \int_0^T w_x(t,1)\tilde{w}_x(t,1)dt \quad \forall \ W^T, \tilde{W}^T \in H_a^1(0,1) \times L^2(0,1).
\]

Thanks to the direct inequality \(\Lambda\) is continuous on \(H_a^1(0,1) \times L^2(0,1)\). Moreover thanks to the observability inequality (3.12) \(\Lambda\) is coercive on \(H_a^1(0,1) \times L^2(0,1)\) for \(T > T_a\). We also define the continuous linear map

\[
\mathcal{L}(W^T) := \langle y_1, w(0)\rangle_{H_a^{-1}(0,1), H_a^1(0,1)} - \int_0^1 y_0 w'(0)dx \quad \forall \ W^T \in H_a^1(0,1) \times L^2(0,1).
\]

Since \(\Lambda\) is continuous and coercive on \(H_a^1(0,1) \times L^2(0,1)\), and \(\mathcal{L}\) is continuous on the Hilbert space \(H_a^1(0,1) \times L^2(0,1)\), we can apply the Lax-Milgram Lemma. This implies that there exists a unique \(W^T \in H_a^1(0,1) \times L^2(0,1)\) such that

\[
\Lambda(W^T, \tilde{W}^T) = -\mathcal{L}(\tilde{W}^T).
\]

We set \(f = w_x(t,1)\) and denote by \(y\) the solution by transposition of (3.23). Then we have

\[
\int_0^T f(t)\tilde{w}_x(t,1)dt = \int_0^T w_x(t,1)\tilde{w}_x(t,1)dt = \Lambda(W^T, \tilde{W}^T) = -\langle y_1, \tilde{w}(0)\rangle_{H_a^{-1}(0,1), H_a^1(0,1)} + \int_0^1 y_0 \tilde{w}'(0)dx \quad \forall \ (\tilde{w}_0, \tilde{w}_1) \in H_a^1(0,1) \times L^2(0,1).
\]

On the other hand, by definition of the transposition solutions, we have

\[
\int_0^T f(t)\tilde{w}_x(t,1)dt = \langle y'(T), \tilde{w}_0 \rangle_{H_a^{-1}(0,1), H_a^1(0,1)} - \int_0^1 y(T)\tilde{w}_1^Tdx = \langle y_1, \tilde{w}(0)\rangle_{H_a^{-1}(0,1), H_a^1(0,1)} + \int_0^1 y_0 \tilde{w}'(0)dx + \quad \forall \ (\tilde{w}_0, \tilde{w}_1) \in H_a^1(0,1) \times L^2(0,1), \quad (3.28)
\]

Hence, comparing these two last relations, we deduce that

\[
\langle y'(T), \tilde{w}_0 \rangle_{H_a^{-1}(0,1), H_a^1(0,1)} - \int_0^1 y(T)\tilde{w}_1^Tdx = 0 \quad \forall \ (\tilde{w}_0, \tilde{w}_1) \in H_a^1(0,1) \times L^2(0,1).
\]

Thus, we have

\[
\langle y, y' \rangle(T, \cdot) \equiv (0, 0) \quad \text{on} \quad (0, 1).
\]

4 Stabilization

4.1 Linear stabilization

Given \(a\) satisfying assumptions (2.1), let \(\mu_a \in [0, 2]\) be the constant in assumption (ii). Consider the degenerate wave equation with boundary damping

\[
u_{tt} - \left(\alpha(x)u_x\right)_x = 0 \quad \text{in} \quad [0, T[ \times [0,1]
\]

with

\[
u(t, 1) + u_x(t, 1) + \beta u(t, 1) = 0, \quad \left\{
\begin{array}{ll}
u(t, 0) = 0 & \text{if} \ \mu_a \in [0,1], \\
\lim_{x \to 0} \alpha(x) u_x(t, x) = 0 & \text{if} \ \mu_a \in [1,2],
\end{array}
\right.
\]

\[
u(0, x) = u_0(x), \quad u_1(0, x) = u_1(x)
\]

where \(\beta > 0\) is given.
4.2 Well-posedness

Let us denote by $W^1_0(0,1)$ the space $V^1_0(0,1)$ itself, if $\mu_a \in [1,2[$, and the closed subspace of $V^1_0(0,1)$ consisting of all the functions $u \in V^1_0(0,1)$ such that $u(0) = 0$, if $\mu_a \in [0,1[$. Moreover, we set $W^2_0(0,1) = V^2_0(0,1) \cap W^1_0(0,1)$.

Notice that $W^2_0(0,1) = V^2_0(0,1)$ when $\mu_a \in [1,2[$.

Now, consider the Hilbert space $H_\beta = W^1_0(0,1) \times L^2(0,1)$ with the scalar product

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle = \int_0^1 \left( v(x)\tilde{v}(x) + a(x)u(x)\tilde{u}(x) \right) dx + a(1)\beta u(1)\tilde{u}(1) \quad \forall (u, v), (\tilde{u}, \tilde{v}) \in H_\beta$$

and the unbounded operator $A_\beta : D(A_\beta) \subset H_\beta \to H_\beta$ defined by

$$D(A_\beta) = \{(u, v) \in W^2_0(0,1) \times W^1_0(0,1) : u'(1) + v(1) + \beta u(1) = 0\}$$

$$A_\beta(u, v) = (v, (au')') \quad \forall (u, v) \in D(A_\beta).$$

Observe that $u'(1), v(1)$, and $\beta u(1)$ are well defined for all $(u, v) \in W^2_0(0,1) \times W^1_0(0,1)$ because of the classical Sobolev embedding theorem.

**Proposition 4.1** Assume (2.1). Then $A_\beta$ is a maximal dissipative operator on $H_\beta$.

**Proof.** Let $(u, v) \in D(A_\beta)$. Then

$$\langle A_\beta(u, v), (u, v) \rangle = \int_0^1 \left( (au')'v + av'v' \right) dx + a(1)\beta u(1)v(1)$$

$$= a(1)v(1)u'(1) + \beta u(1)) = -a(1)v^2(1) \leq 0.$$

Therefore, $A_\beta$ is dissipative.

In order to show that $A_\beta$ is maximal dissipative, it remains to check that $I - A_\beta$ is onto. Equivalently, given any $(f, g) \in H_\beta$, we have to solve the problem

$$\begin{cases}
(u, v) \in D(A_\beta) \\
v = u - f \\
u - (au')' = f + g.
\end{cases}$$

Consider the bilinear form $b : W^1_0(0,1) \times W^1_0(0,1) \to \mathbb{R}$ given by

$$b(u, \phi) = \int_0^1 \left( u\phi + au'\phi' \right) dx + (\beta + 1)a(1)u(1)\phi(1),$$

and the linear form $L : W^1_0(0,1) \to \mathbb{R}$ given by

$$L\phi = \int_0^1 (f + g)\phi dx + a(1)\phi(1)f(1).$$

In view of Proposition 2.5, $b$ is a continuous bilinear form on $W^1_0(0,1) \times W^1_0(0,1)$ and $L$ is a continuous linear functional on $W^1_0(0,1)$. Moreover since $\beta \geq 0$, $b$ is also coercive on $W^1_0(0,1) \times W^1_0(0,1)$. So, by the Lax-Milgram Theorem there exists a unique solution $u \in W^1_0(0,1)$ of the variational problem

$$b(u, \phi) = L\phi \quad \forall \phi \in W^1_0(0,1).$$

We prove that $(u, v) \in D(A_\beta)$ and solves (4.3) as follows. We denote by $C^\infty_c(0,1)$ the space of functions which are in $C^\infty(0,1)$ with compact support in $(0,1)$. Since $C^\infty_c(0,1) \subset W^1_0(0,1)$, we have

$$\int_0^1 \left( u\phi + au'\phi' \right) dx = \int_0^1 (f + g)\phi dx \quad \forall \phi \in C^\infty_c(0,1).$$
Hence by duality, we have \( u - (au')' = f + g \) in the sense of distributions. Thus \( u \in W^2_a(0, 1) \) and
\[
u - (au')' = f + g \quad \text{a.e in } (0, 1).
\]

Thus, we deduce after an integration by parts together with (2.15) that
\[
\int_0^1 u \phi' \, dx + \int_0^1 a u' \phi' \, dx - a(1) u'(1) \phi(1) = \int_0^1 (f + g) \phi \, dx \quad \forall \phi \in W^1_a(0, 1).
\]

This combined with (4.4) yields
\[
a(1) \phi(1) \left( u'(1) + (\beta + 1) u(1) - f(1) \right) = 0 \quad \forall \phi \in W^1_a(0, 1).
\]

Since \( a(1) > 0 \) and the function \( \phi \) defined by \( \phi(x) = x \) for all \( x \in (0, 1) \) is in \( W^1_a(0, 1) \), we deduce that
\[
u'(1) + (\beta + 1) u(1) - f(1) = 0.
\]

Setting \( v = u - f \), we check that \((u, v) \in D(A_\beta)\) and solves (4.3).

Therefore, \( A_\beta \) is the generator of a contraction semigroup in \( \mathcal{H}_\beta \), denoted by \( e^{tA_\beta} \). For any \( U_0 := (u_0, u_1) \in \mathcal{H}_\beta \), \( U(t) := e^{tA_\beta} U_0 \) can be viewed as the weak solution of the Cauchy problem
\[
\begin{aligned}
&U'(t) = A_\beta U(t) \quad t > 0 \\
&U(0) = U_0.
\end{aligned}
\]

Moreover, the above solution is classical when \( U_0 \in D(A_\beta) \). We thus have the following result.

**Corollary 4.2** Assume (2.1). Then, for any \( U_0 = (u_0, u_1) \in D(A_\beta) \), problem (4.5) has a unique solution
\[
U \in C^1([0, \infty); \mathcal{H}_\beta) \cap C([0, \infty); D(A_\beta))
\]
given by \( U(t) = e^{tA_\beta} U_0 \). Moreover, setting \( U(t) = (u(t), v(t)) \), we have that

\begin{itemize}
  \item \( u \) is the unique solution of problem (4.1)-(4.2) such that
    \[
u \in C^2([0, \infty); L^2(0, 1)) \cap C^1([0, \infty); W^1_a(0, 1)) \cap C([0, \infty); W^2_a(0, 1)),
    \]
  \item the energy of \( u \) defined by
    \[
    E_u(t) := \frac{1}{2} \int_0^1 \left( u_t^2 + au_x^2 \right) \, dx + \beta a(1) u_x^2(t, 1)
    \]
    satisfies
    \[
    \frac{dE_u(t)}{dt} = -a(1) u_t^2(t, 1) \leq 0 \quad \forall t \geq 0.
    \]
\end{itemize}

We shall need the following results in the sequel.

**Proposition 4.3** Assume (2.1). Then
\[
\| u \|_{2, a}^2(0, 1) \leq 2 |u(1)|^2 + C'_a \| u \|_{1, a}^2 \quad \forall u \in W^1_a(0, 1),
\]
where
\[
C'_a = \frac{1}{a(1)} \min \left\{ 4, \frac{2}{2 - \mu_a} \right\}.
\]

Moreover assume that \( \beta > 0 \). Then, denoting by \( \| \cdot \|_{1, a} \) the norm defined by
\[
\| u \|_{1, a} = \left( |u|_{1, a}^2 + \beta a(1) u^2(1) \right)^{1/2} \quad u \in W^1_a(0, 1),
\]

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we have
\[ ||u||_{1,a}^2 \geq \alpha_a ||u||_{L^2(0,1)}^2 \quad \forall u \in W^1_a(0,1). \] (4.10)

where
\[ \alpha_a = \min \left( \frac{1}{C_a}, \frac{\beta a(1)}{2} \right) > 0. \]

Moreover we also have
\[ \frac{\alpha_a}{\alpha_a + 1} \left( ||u||_{1,a}^2 + \beta a(1)u^2(1) \right) \leq ||u||_{1,a}^2 \leq \gamma_a ||u||_{1,a}^2 \quad \forall u \in W^1_a(0,1), \] (4.11)

where
\[ \gamma_a = \max \left( 2\beta a(1), 1 + \frac{2\beta}{2 - \mu_a} \right). \]

**Proof.** Let \( u \in W^1_a(0,1) \). We follow the proof of Proposition 2.2. We give two different bounds for \( ||u||_{L^2(0,1)}^2 \) in terms of \( ||u||_{1,a}^2 \) and \( u^2(1) \). The conclusion (4.8) will follow by taking the minimum of the two corresponding constants.

First, for any \( x \in [0,1] \) we have that
\[ |u(x) - u(1)| = \left| \int_x^1 u'(s)ds \right| \leq ||u||_{1,a} \left\{ \int_x^1 \frac{ds}{a(s)} \right\}^{\frac{1}{2}}. \]

Therefore, proceeding as in the proof of (2.6), we have
\[ \int_0^1 |u(x) - u(1)|^2dx \leq ||u||_{1,a}^2 \int_0^1 dx \int_x^1 \frac{ds}{a(s)} = \frac{1}{2} \int_0^1 s \frac{ds}{a(s)} \leq \frac{1}{a(1)(2 - \mu_a)}. \] (4.12)

Since
\[ \int_0^1 |u(x)|^2dx \leq 2|u(1)|^2 + 2 \int_0^1 |u(x) - u(1)|^2dx, \]

we deduce by (4.12) the first bound we mentioned above, that is,
\[ ||u||_{L^2(0,1)}^2 \leq 2|u(1)|^2 + \frac{2||u||_{1,a}^2}{a(1)(2 - \mu_a)} \quad \forall u \in W^1_a(0,1). \] (4.13)

Next, observe that, for all \( x \in [0,1], \)
\[ 0 \leq \int_x^1 \left( su'(s) + \frac{1}{2}u(s) \right)^2ds \]
\[ = \int_x^1 \left( s^2|u'(s)|^2 + \frac{1}{4}|u(s)|^2 + su(s)u'(s) \right)ds \]
\[ = \int_x^1 \left( s^2|u'(s)|^2 - \frac{1}{4}|u(s)|^2 \right)ds + \frac{1}{2}|u|^2(1) - \frac{1}{2}x|u(x)|^2. \]

Therefore, taking the limit as \( x \downarrow 0 \), by (2.2) and (2.12) we obtain the announced second bound:
\[ \int_0^1 |u(s)|^2ds \leq 2u^2(1) + 4 \int_0^1 s^2|u'(s)|^2ds \leq 2u^2(1) + \frac{4}{a(1)} \int_0^1 a(s)|u'(s)|^2ds \quad \forall u \in W^1_a(0,1). \] (4.14)

The inequality (4.8) follows from (4.13) and (4.14).

We have
\[ ||u||_{1,a}^2 \geq \alpha_a \left( \frac{1}{C_a}, \frac{\beta a(1)}{2} \right) (2u^2(1) + C_a' ||u||_{1,a}^2) \geq \alpha_a ||u||_{L^2(0,1)}^2 \quad \forall u \in W^1_a(0,1). \]

Writing \( 1 = \frac{\alpha_a}{\alpha_a + 1} + \frac{1}{\alpha_a + 1} \) and using the above inequality, we obtain
\[ ||u||_{1,a}^2 \geq \frac{\alpha_a}{\alpha_a + 1} ||u||_{1,a}^2 + \frac{\alpha_a}{\alpha_a + 1} ||u||_{L^2(0,1)}^2 \quad \forall u \in W^1_a(0,1). \]
This gives the left hand side of (4.11). On the other hand, since
\[ |u(1)|^2 \leq 2 \int_0^1 |u(x)|^2 \, dx + 2 \int_0^1 |u(x) - u(1)|^2 \, dx \leq 2\|u\|_{L^2(0,1)}^2 + \frac{2\|u\|_{L^1(2, \mu_+)}^2}{a(1)(2 - \mu_+)} \quad \forall u \in W^1_0(0,1). \]
This inequality yields
\[ \|\|u\|_{1,a}^2 \leq 2\beta a(1)\|u\|_{L^2(0,1)}^2 + (1 + \frac{2\beta}{2 - \mu_+})\|u\|_{1,a}^2 \leq \max \left(2\beta a(1), 1 + \frac{2\beta}{2 - \mu_+}\right)\|u\|_{1,a}^2 \quad \forall u \in W^1_0(0,1). \]
This gives the right inequality in (4.11).

**Proposition 4.4** Assume (2.1) and that \( \beta > 0 \) is given. Then the variational problem
\[ \int_0^1 az'\phi' \, dx + \beta a(1)z(1)\phi(1) = \lambda a(1)\phi(1) \quad \forall \phi \in W^1_0(0,1). \] 

admits a unique solution \( z \in W^1_0(0,1) \) which satisfies the elliptic estimates
\[ \|\|z\|_{1,a}^2 \leq \frac{a(1)}{\beta} \lambda^2, \quad \|z\|_{L^2(0,1)}^2 \leq \frac{a(1)}{\beta a_0} \lambda^2. \] 

Moreover \( z \in W^2_0(0,1) \) and solves
\[ \begin{cases} -(az')' = 0, \\ z'(1) + \beta z(1) = \lambda. \end{cases} \] 

**Proof.** We denote by \( \tilde{b} \) the bilinear form on \( W^1_0(0,1) \) defined by
\[ \tilde{b}(z, \phi) = \int_0^1 az'\phi' \, dx + \beta a(1)z(1)\phi(1) \quad z, \phi \in W^1_0(0,1). \]

Thanks to Proposition 4.3 (see (4.11)), \( \tilde{b} \) is a symmetric continuous and coercive bilinear form on \( W^1_0(0,1) \) and the linear form \( L \) defined by \( L\phi =: \lambda a(1)\phi(1) \) for \( \phi \in W^1_0(0,1) \) is continuous. Hence thanks to the Lax-Milgram’s Theorem, the above variational problem admits a unique solution \( z \in W^1_0(0,1) \). Hence we have
\[ \|\|z\|_{1,a}^2 = \tilde{b}(z, z) = \lambda a(1)z(1) \leq \sqrt{\frac{a(1)}{\beta}} |\lambda| \|z\|_{1,a} \cdot \]

Hence we have
\[ \|\|z\|_{1,a}^2 \leq \frac{a(1)}{\beta} \lambda^2. \]

This, together with (4.10) yields
\[ \|z\|_{L^2(0,1)}^2 \leq \frac{a(1)}{\beta a_0} \lambda^2. \]

Proceeding as in the proof of Proposition 5.3, we show that \( z \in W^2_0(0,1) \) and solves (4.17).

**Theorem 4.5** Assume (2.1) and that \( \beta > 0 \) is given. Then for any \((u_0, u_1) \in H_\beta\), the solution of (4.1)-(4.2) satisfies the uniform exponential decay
\[ E_u(t) \leq E_u(0)e^{1/t^{M_{\beta}}}, \quad \forall t \in [M_{\beta} + \infty). \] 

where \( M_{\beta} > 0 \) is given in (4.32) and is independent of \((u_0, u_1)\).
Proof. Let \( U_0 = (u_0, u_1) \in D(A_2) \) be given, and \( U \) be the corresponding solution of problem (4.5). Then we recall that setting as above \( U(t) = (u(t), v(t)) \), we have that \( u \) is the solution of problem (4.1)-(4.2). We multiply (4.1) by \( xu_x \) and integrate the resulting equation over \((S, T) \times (0, 1)\). This gives after suitable integrations by parts

\[
\int_S^T \int_0^1 \left( - x \left( \frac{u_0^2}{2} \right)_x + a(x)u_2^2 + ax(x) \left( \frac{u_0^2}{2} \right)_x \right) dxdt + \left[ \int_0^1 xu_x u_1 dx \right]_S^T - \int_S^T [xu_2^1]_0^1 dt = 0 \quad \forall \ 0 \leq S \leq T.
\]

We integrate by parts twice again. This gives

\[
\int_S^T \int_0^1 \left( \frac{u_0^2}{2} + (a - xa') \frac{u_2^2}{2} \right) dxdt + \left[ \int_0^1 xu_x u_1 dx \right]_S^T - \frac{1}{2} \int_S^T \left( a(1)u_2^2(t, 1) + u_2^2(t, 1) \right) dt = 0 \quad \forall \ 0 \leq S \leq T. (4.19)
\]

We multiply (4.1) by \( u \) and integrate the resulting equation over \((S, T) \times (0, 1)\). This gives after a suitable integration by parts.

\[
\int_S^T \int_0^1 \left( - u_1^2 + au_2^2 \right) dxdt + \left[ \int_0^1 u_1 udx \right]_S^T - \int_S^T [au_x u_1^1]_0^1 dt = 0 \quad \forall \ 0 \leq S \leq T.
\]

Using now Proposition 2.5 (see (III)), we have

\[
(a(x)u(t, x)u_x(t, x))_{|x=0} = 0,
\]

so that

\[
\int_S^T \int_0^1 \left( - u_1^2 + au_2^2 \right) dxdt + \left[ \int_0^1 u_1 udx \right]_S^T - \int_S^T a(1)u_1(t, 1)u(t, 1)dt = 0 \quad \forall \ 0 \leq S \leq T. (4.20)
\]

We now combine (4.19) multiplied by 2 with (4.20) multiplied \( \frac{\mu_a}{2} \). This gives

\[
\int_S^T \int_0^1 \left[ (2 - \mu_a) \frac{u_0^2}{2} + [2(a - xa') + a\mu_a] \frac{u_2^2}{2} \right] dxdt + \frac{2 - \mu_a}{2} \beta a(1) \int_S^T u^2(t, 1)dt =
\]

\[
- 2 \left( \int_0^1 xu_x u_1 dx \right)_S^T - \frac{\mu_a}{2} \left[ \int_0^1 u_1 udx \right]_S^T + \int_S^T h(t)dt \quad \forall \ 0 \leq S \leq T, (4.21)
\]

where the function \( h \) is given by

\[
h(t) = (1 + a(1))u_0^2(t, 1) + a(1)(1 + \beta - \mu_a)u^2(t, 1) + (2\beta - \frac{\mu_a}{2})a(1)u_1(t, 1)u(t, 1) \quad t \in (S, T).
\]

By definition of \( \mu_a \), we have

\[
(2 - \mu_a)a \leq 2(a - xa') + a\mu_a.
\]

This, together with (4.21), gives

\[
(2 - \mu_a) \int_S^T E_u(t)dt \leq - \left[ \int_0^1 2xu_x u_1 + \frac{\mu_a}{2} u_1 udx \right]_S^T + \int_S^T h(t)dt \quad \forall \ 0 \leq S \leq T. (4.23)
\]
On the other hand, we have
\[
h(t) \leq \eta_1 u_t^2(t, 1) + \eta_2 a(1) u^2(t, 1) \quad \forall \ t \in (S, T),
\] (4.24)
where
\[
\eta_1 = (1 + \frac{3}{2} a(1)), \ \eta_2 = \left[ \beta(1 + \beta - \mu_a) + \frac{1}{2} (2\beta - \frac{\mu_a}{2})^2 \right].
\]
We also have
\[
\int_0^1 \left| 2xu_x u_t + \frac{\mu_a}{2} u_i u_i dx \right| \leq \int_0^1 \left[ x^2 u_x^2 + (1 + \frac{\mu_a}{4}) u_t^2 + \frac{\mu_a}{4} u_i^2 \right] dx.
\]
Using (2.17) together with (4.8), we deduce that
\[
\int_0^1 \left| 2xu_x u_t + \frac{\mu_a}{2} u_i u_i dx \right| \leq \int_0^1 \left[ (1 + \frac{\mu_a}{4}) u_t^2 + (\frac{1}{a(1)} + \frac{\mu_a}{4} C'\alpha) u_x^2 + \frac{\mu_a}{2} u^2(1) \right] dx \leq C''_a E_a(t) \quad \forall \ t \in [S, T],
\]
where
\[
C''_a = 2 \max \left( 1 + \frac{\mu_a}{4}, \frac{1}{a(1)} + \frac{\mu_a}{4} C'\alpha, \frac{\mu_a}{2\beta a(1)} \right).
\]
Using this inequality together with (4.24) in (4.23), we obtain
\[
(2 - \mu_a) \int_S^T E_u(t) dt \leq C''_a \left( E_u(S) + E_u(T) \right) + \eta_1 \int_S^T u_t^2(t, 1) dt + \eta_2 \int_S^T a(1) u^2(t, 1) dt.
\] (4.25)
Using the dissipation relation (4.7), we deduce that
\[
(2 - \mu_a) \int_S^T E_u(t) dt \leq C''_a \left( E_u(S) + E_u(T) \right) + \frac{\eta_1}{a(1)} \left( E_u(S) - E_u(T) \right) + \eta_2 \int_S^T a(1) u^2(t, 1) dt \leq \left( 2C''_a + \frac{\eta_1}{a(1)} \right) E_u(S) + \eta_2 \int_S^T a(1) u^2(t, 1) dt.
\] (4.26)
We now estimate the last term of this inequality as follows. Set \( \lambda = u(t, 1) \) and denote by \( z \) the solution of the degenerate elliptic problem (4.17). We multiply (4.1) by \( z \) and integrate the resulting equation over \( (S, T) \times (0, 1) \). This gives after suitable integrations by parts,
\[
\int_0^T a(1) u^2(t, 1) dt = \int_0^T \int_0^1 u_i z dx \ d t - a(1) \int_0^T u_t(t, 1) z(t, 1) dt - \left[ u_i(t, 1) z dx \right]_0^T.
\] (4.27)
We now estimate the terms of the right hand side in this inequality, as follows. First, thanks to the second inequality in (4.16), we have
\[
||z||_{L^2(0, 1)}^2 \leq \frac{a(1)}{\beta \alpha} u_i(t, 1)^2.
\] (4.28)
Moreover, thanks to the first inequality in (4.16) and to the definition of \( \| \cdot \|_{1, a} \), we have
\[
\beta a(1) z^2(t, 1) \leq \| z \|_{L^2(0, 1)}^2 \leq \frac{a(1)}{\beta} u^2(t, 1),
\]
so that
\[
z^2(t, 1) \leq \frac{2}{\beta \alpha} u^2(t, 1) \leq \frac{2}{\beta \alpha} E_u(t).
\] (4.29)
On the other hand, we have, thanks to the second inequality in (4.16)
\[
\left| \int_0^1 u_i(t, x) z(t, x) dx \right| \leq \frac{1}{\beta \alpha} \left( \int_0^1 u_i^2 dx + \frac{\beta a(1)}{2} u^2(t, 1) \right) \leq \frac{1}{\beta \alpha} E_u(t) \quad \forall \ t \in [S, T].
\] (4.30)
We now use (4.28)-(4.30) in (4.27). This gives

\[
\int_S T a(1)u^2(t,1)dt \leq \delta \left(1 + \frac{1}{\beta^3}\right) \int_S T E_u(t)dt + \frac{1}{2\delta} \left(1 + \frac{1}{\beta\alpha_a}\right) \int_S T a(1)u^2(t,1)dt + \frac{1}{\beta\sqrt{\alpha_a}}(E_u(S) + E_u(T)).
\]

Using now (4.7) in this estimate, we obtain

\[
\int_S T a(1)u^2(t,1)dt \leq \delta \left(1 + \frac{1}{\beta^3}\right) \int_S T E_u(t)dt + \frac{1}{2\delta} \left(1 + \frac{1}{\beta\alpha_a}\right)(E_u(S) - E_u(T)) + \frac{1}{\beta\sqrt{\alpha_a}}(E_u(S) + E_u(T)).
\]

We now choose \( \delta = \frac{2 - \mu_a}{2\eta_2(1 + \frac{1}{\beta^3})} \) in the above inequality and combine the resulting inequality in (4.26) to obtain

\[
\int_S T E_u(t)dt \leq M_{a,\beta}E_u(S),
\]

where

\[
M_{a,\beta} = \frac{2}{(2 - \mu_a)} \left[2C_a + \frac{\eta_1}{a(1)} + \frac{\eta_2^2}{2 - \mu_a} \left(1 + \frac{1}{\beta\alpha_a}\right) + \frac{2\eta_3}{\beta\sqrt{\alpha_a}}\right].
\]

Now we use the following well-known result (see [18, Theorem 8.1]).

**Lemma 4.6** Assume that \( E : [0, +\infty) \rightarrow [0, +\infty) \) is a non-increasing function and that there is a constant \( M > 0 \) such that

\[
\int_{t}^{\infty} E(s)ds \leq ME(t), \quad \forall t \in [0, +\infty).
\]

Then we have

\[
E(t) \leq E(0)e^{1-t/M}, \quad \forall t \in [M, +\infty).
\]

Applying this result on \( E = E_u \) which is nonnegative, nonincreasing on \([0, \infty)\) and satisfies (4.31), we have

\[
E_u(t) \leq E_u(0)e^{1-t/M_{a,\beta}}, \quad \forall t \in [M_{a,\beta}, +\infty).
\]

\[ \square \]

## 5 Nonlinear stabilization

In the previous section we considered the case of a linear boundary feedback. Here we extend our stability analysis to one-dimensional degenerate wave equations damped by a nonlinear boundary feedback with arbitrary growth. For this, we combine our results for the linear case with the optimal-weight convexity method of [1, 2].

Let \( \rho : \mathbb{R} \rightarrow \mathbb{R} \) be a nondecreasing continuous function such that \( \rho(0) = 0 \) and assume there exist constants \( c_1 > 0, c_2 > 0 \) and an odd, continuously differentiable, strictly increasing function \( g \) on \([-1, 1]\) such that

\[
c_1g(|s|) \leq |\rho(s)| \leq c_2g^{-1}(|s|) \quad \forall |s| \leq 1,
\]

\[
c_1|s| \leq |\rho(s)| \leq c_2|s| \quad \forall |s| \geq 1.
\]

As before, let \( a \) be given such that assumptions (2.1) hold, and let \( \mu_a \in [0, 2] \) be the constant in assumption (ii). Consider the degenerate wave equation

\[
u_{tt} - (a(x)u_x)_x = 0 \quad \text{in } [0, T[ \times ]0, 1[\]
\]

with the nonlinear boundary damping

\[
\begin{cases}
\rho(u_t(t, 1)) + u_x(t, 1) + \beta u(t, 1) = 0, \\
u(t, 0) = 0 \quad \text{if } \mu_a \in [0, 1], \\
\lim_{x \uparrow 0} a(x)u_x(t, x) = 0 \quad \text{if } \mu_a \in [1, 2], \\
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)
\end{cases}
\]

where \( \beta \geq 0 \) is given.
Remark 5.1 Typical examples for \( g \) are:

- the linear case \( g(x) = cx \) on \( \mathbb{R} \),
- the polynomial case \( g(x) = |x|^{p-1}x \) with \( p > 1 \) in a neighborhood of \( x = 0 \),
- \( g(x) = |x|^{p-1}x \ln^q(\frac{1}{|x|}) \) with \( p > 1, q > 0 \) in a neighborhood of \( x = 0 \),
- \( g(x) = \text{sign}(x)e^{-1/x^2} \) in a neighborhood of \( x = 0 \),
- \( g(x) = \text{sign}(x)e^{-\ln^p(\frac{1}{|x|})} \) with \( 1 < p < 2 \) in a neighborhood of \( x = 0 \).

See e.g. [18] for the linear and polynomial cases and [2] for the other cases and the references therein, and [5] for the last example when \( p > 2 \).

5.1 Well-posedness

We keep the functional spaces introduced in the previous section (for linear stabilization). However, we now need to deal with the nonlinear unbounded operator. We define the functional

\[
J = \int_{\mathbb{R}} \left[ a(1)v(1) + \rho(v(1)) + \beta u(1) \right] \, dx
\]

for the linear and polynomial cases and

\[
\int_{\mathbb{R}} \left[ a(1)v(1) + \rho(v(1)) - \rho(z(1)) \right] \, dx
\]

for the other cases and the references therein, and \( J \) is onto. Equivalently, given any \( (f, g) \in \mathcal{H}_\beta \), we have to solve the problem

\[
\begin{cases}
(u, v) \in D(A_{\beta}^{nl}) \\
v = u - f \\
u - (au')' = f + g .
\end{cases}
\]

Remark 5.2 Note that the set \( D = \{(u, v) \in W_a^2(0, 1) \times H_0^1(0, 1) : u'(1) + \beta u(1) = 0\} \) is a subset of \( D(A_{\beta}^{nl}) \) and is dense in \( W_a^1(0, 1) \times L^2(0, 1) \). Therefore \( D(A_{\beta}^{nl}) \) is dense in \( W_a^1(0, 1) \times L^2(0, 1) \).

Proposition 5.3 Assume (2.1) and the above assumptions on \( \rho \). Then \( A_{\beta}^{nl} \) is a maximal dissipative operator on \( \mathcal{H}_\beta \).

Proof. Consider \( (u, v), (w, z) \in D(A_{\beta}^{nl}) \). Then

\[
\langle A_{\beta}^{nl}(u,v), (w,z) \rangle = \int_0^1 \left( (a(u-w)')'(v-z) + a(u-w)'(v-z)' \right) dx + a(1)\beta(u-w)(v-z)(1) =
\]

\[
a(1)(v-z)(1) \left[ (u-w)'(1) + \beta(u-w)(1) \right] = -a(1)(v-z)(1) \left[ \rho(v(1)) - \rho(z(1)) \right] \leq 0 .
\]

Therefore, \( A_{\beta}^{nl} \) is dissipative. Let us now prove that \( I - A_{\beta}^{nl} \) is onto. Equivalently, given any \( (f, g) \in \mathcal{H}_\beta \), we have to solve the problem

\[
\begin{cases}
(u, v) \in D(A_{\beta}^{nl}) \\
v = u - f \\
u - (au')' = f + g .
\end{cases}
\]

(5.4)

Let us define

\[
R(s) = \int_0^s \rho(\tau)d\tau \quad \forall \, s \in \mathbb{R} .
\]

(5.5)

We define the functional \( J_{\beta} : W_a^1(0, 1) \to \mathbb{R} \) by

\[
J(u) = \frac{1}{2} \left[ \int_0^1 \left( u^2(x) + a(x)u'^2(x) \right) \, dx + \beta a(1)u^2(1) + a(1)R(u(1) - f(1)) - \int_0^1 (f + g)(x)u(x) \, dx \right] .
\]

(5.6)

Then one can check that \( J \) is continuously differentiable on \( W_a^1(0, 1) \) and its differential is given by

\[
J'(u)\phi = \int_0^1 (u\phi + au'\phi') \, dx + \beta a(1)u(1)\phi(1) + a(1)\rho(u(1) - f(1))\phi(1) - \int_0^1 (f + g)\phi \, dx \quad \forall \, u, \phi \in W_a^1(0, 1) .
\]

(5.7)
Moreover, since \( \rho \) is nondecreasing on \( \mathbb{R} \), we deduce that \( J \) is a strictly convex function and

\[
J(u) = \frac{1}{2} \| u \|^2_{1,a} - \| f + g \|_{L^2(0,1)} \| u \|_{L^2(0,1)} \geq \| u \|_{1,a} \left( \frac{1}{2} \| u \|_{1,a} - \frac{1}{\alpha_a} \| f + g \|_{L^2(0,1)} \right) \quad \forall u \in W^1_a(0,1).
\]

Hence since the norm \( \| \cdot \|_{1,a} \) is equivalent to the norm \( \| \cdot \|_{1} \) on \( W^1_a(0,1) \), \( J(u) \to +\infty \) as \( \| u \|_{1,a} \to +\infty \). Hence \( J \) is coercive and strictly convex on \( W^1_a(0,1) \) and thus \( J \) attains a minimum at some unique point \( u \in W^1_a(0,1) \), which satisfies the Euler equation

\[
J'(u) = 0.
\]

Thus \( u \in W^1_a(0,1) \) is the unique solution of

\[
\int_0^1 (u\phi + au'\phi')dx + \beta a(1)u(1)\phi(1) + a(1)\rho(u(1) - f(1))\phi(1) - \int_0^1 (f + g)\phi dx = 0 \quad \forall \phi \in W^1_a(0,1). \tag{5.8}
\]

In particular for all \( \phi \in C_0^\infty(0,1) \), we have

\[
\int_0^1 (u\phi + au'\phi')dx = \int_0^1 (f + g)\phi dx \quad \forall \phi \in C_0^\infty(0,1).
\]

Hence by duality, we have \( u - (au')' = f + g \) in the sense of distributions. Thus \( u \in W^2_a(0,1) \) and

\[
u - (au')' = f + g \quad \text{a.e in } (0,1).
\]

This yields

\[
a(1)\phi(1) [u'(1) + \beta u(1) + \rho(u(1) - f(1))] = 0 \quad \forall \phi \in W^1_a(0,1).
\]

Since \( a(1) > 0 \) and the function \( \phi \) defined by \( \phi(x) = x \) for all \( x \in (0,1) \) is in \( W^1_a(0,1) \) we deduce that

\[
u'(1) + \beta u(1) + \rho(u(1) - f(1)) = 0.
\]

Setting \( v = u - f \), we check that \((u,v) \in D(A_\beta^n)\) and solves (5.4). \( \square \)

Hence thanks to classical results on nonlinear maximal monotone operators (see e.g. \([?, ?]\)), we have

**Corollary 5.4** Assume \((2.1)\) and that \( \rho \) satisfies the above assumptions. Then, for any \( U_0 = (u_0, u_1) \in D(A_\beta^n) \),

problem \((5.2)-(5.3)\) has a unique solution \( u \) such that

\[
u \in W^{2,\infty}([0,\infty); L^2(0,1)) \cap W^{1,\infty}([0,\infty); W^1_a(0,1)) \cap L^{\infty}([0,\infty); W^2_a(0,1)).
\]

Moreover the energy of \( u \) defined by \((4.6)\) satisfies the dissipation relation

\[
\frac{dE_u}{dt}(t) = -a(1)u_t(t,1)\rho(u_t(t,1)) \leq 0 \quad \forall t \geq 0. \tag{5.9}
\]

**5.2 Nonlinear stability analysis**

We now follow the optimal-weight convexity method introduced in \([1]\) and simplified in \([2]\) (see also \([3]\)). For this, we need to introduce several functions. We first define a function \( H : [0,r_0^2] \to [0,\infty) \) by

\[
H(x) = \sqrt{x} g(\sqrt{x}) \quad x \in [0,r_0^2], \tag{5.10}
\]

where \( r_0 \leq 1 \) is assumed to be sufficiently small. We assume that \( H \) is strictly convex on \([0,r_0^2]\). We extend \( H \) to a function \( \tilde{H} \) on \([0,\infty)\) by setting \( \tilde{H}(x) = +\infty \) when \( x \notin [0,r_0^2] \). We then define a function \( L \) on \([0,\infty)\) by

\[
L(y) = \begin{cases} \tilde{H}^*(y) \quad & \text{if } y > 0, \\ 0 \quad & \text{if } y = 0, \end{cases} \tag{5.11}
\]

where \( \tilde{H}^* \) is the dual function of \( \tilde{H} \).
where $\hat{H}^*$ stands for the convex conjugate of $\hat{H}$ defined by $\hat{H}^*(y) = \sup_{x \in \mathbb{R}} \{xy - \hat{H}(x)\}$. One can show that $L$ is a continuous increasing, one-to-one and onto function from $[0, \infty)$ on $[0, r_0^2)$. Moreover $L$ is continuously differentiable on $(0, \infty)$ and

$$0 < L(H'(r_0^2)) < r_0^2,$$

holds (see [1, 2] for more details). Finally we define a function $\Lambda_H$ on $[0, r_0^2]$

$$\Lambda_H(x) = \frac{H(x)}{xH'(x)}$$

(5.13)

Note that $\Lambda_H([0, r_0^2]) \subset [0, 1]$ thanks to our convexity assumptions.

**Theorem 5.5** We assume the above hypotheses on $a$ and on $\rho$, $g$ and $H$, and that $\beta > 0$ is given. Let $(u_0, u_1) \in \mathcal{H}_\beta$ be given such that $E_u(0) > 0$, and $u$ be the corresponding solution of (5.2)-(5.3). Let $\gamma > \max(\frac{E_u(0)}{2L(H'(r_0^2))}, C_6)$ (where $C_6$ is an explicit constant appearing in (5.29)) then the energy $E_u$ of $u$ satisfies the following estimate:

$$E_u(t) \leq 2\gamma L\left(\frac{1}{\psi_0(x)}\right), \quad \forall \ t \geq \frac{M}{H'(r_0^2)}.$$  

(5.14)

where

$$\psi_0(x) = \frac{1}{H'(r_0^2)} + \int_{1/x}^{H'(r_0^2)} \frac{1}{y^2(1 - \Lambda_H((H')^{-1}(\theta)))} \, dy.$$  

(5.15)

Furthermore, if $\limsup_{x \to 0^+} \Lambda_H(x) < 1$, then $E$ satisfies the following simplified decay rate

$$E_u(t) \leq 2\gamma \left(\frac{M}{t}\right)^{-\frac{\kappa}{l}}.$$  

(5.16)

for $t$ sufficiently large, and where $\kappa > 0$ is a constant independent of $E(0)$.

**Remark 5.6** The above theorem shows that the solutions of the boundary degenerate nonlinearly damped wave equation above have the same stability properties as the corresponding nondegenerate nonlinearly damped wave equation, that is both have the same decay rates of their energies. In particular,

- For the polynomial case for which $g(x) = |x|^{p-1}x$ in a neighborhood of $x = 0$ with $p > 1$,

$$E_u(t) \leq C_{E_u(0)} \gamma t^{-\frac{2}{p^2}}$$

for sufficiently large $t$.

- For $g(x) = |x|^{p-1}x \ln^q(\frac{1}{|x|})$ in a neighborhood of $x = 0$ with $p > 1, q > 0$,

$$E_u(t) \leq C_{E_u(0)} \gamma t^{-\frac{2}{p^2}} \ln(t)^{-2q/(p-1)}$$

for sufficiently large $t$.

- For $g(x) = \text{sign}(x)e^{-1/x^2}$ in a neighborhood of $x = 0$,

$$E_u(t) \leq C_{E_u(0)} \gamma \ln^{-1}(t)$$

for sufficiently large $t$.

- For $g(x) = \text{sign}(x)e^{-\ln^p(\frac{1}{|x|})}$ in a neighborhood of $x = 0$ with $p > 2$,

$$E_u(t) \leq C_{E_u(0)} \gamma e^{-2(\ln(t))^{1/p}}$$

for sufficiently large $t$.

Here $\gamma$ is as in Theorem 5.5 (see e.g., [18] for the linear and polynomial cases and [2] for the other cases and the references therein, and also [5] for the last example in the case $p > 2$).
Proof. Thanks to the density of $D(A^w_β)$ in $H_β$, and since $A^w_β$ is a maximal dissipative operator, it is sufficient to consider smooth initial data $(u_0, u_1)$. Hence, let $U_0 = (u_0, u_1) ∈ D(A^w_β)$ be given, and $u$ be the corresponding solution of problem (5.2)-(5.3). Let $γ > \frac{E_u(0)}{2L(U_0)}$ which will be precise later on in the proof and define the optimal-weight function as

$$w(s) = L^{-1} \left( \frac{E_u(s)}{2γ} \right) \; \forall \; s ≥ 0.$$  

(5.17)

We multiply (5.2) by $w(E_u(t)) xu_x$ and integrate the resulting equation over $(S, T) × (0, 1)$. After suitable integrations by parts as in the previous section, this gives for all $0 ≤ S ≤ T$

$$\int_S^T w(E_u(t)) \int_0^1 \left( -x \left( \frac{u^2}{2} \right)_x + a(x) u_x^2 + x a(x) \left( \frac{u^2}{2} \right) \right) dx dt + \left[ w(E_u(t)) \int_0^1 u_x u_t dx \right]_S^T - \int_S^T w(E_u(t)) \left[ u_x u^2 \right]_0^1 dt - \int_S^T w'(E_u(t)) E_u'(t) \int_0^1 u_x u_t dx dt = 0.$$  

(5.18)

We multiply (5.2) by $w(E_u(t)) u$ and integrate the resulting equation over $(S, T) × (0, 1)$. This gives after a suitable integration by parts and thanks to our trace results,

$$\int_S^T w(E_u(t)) \int_0^1 \left( -u^2 + au_x^2 \right) dx dt + \left[ w(E_u(t)) \int_0^1 u_t u dx \right]_S^T - \int_S^T w(E_u(t)) a(1) u_x(t, 1) u(t, 1) dt - \int_S^T w'(E_u(t)) E_u'(t) \int_0^1 u_t dx dt = 0 \; \forall \; 0 ≤ S ≤ T.$$  

(5.19)

We now combine (5.18) multiplied by 2 with (5.19) multiplied $\frac{μ_a}{2}$. This gives

$$\int_S^T \left[ w(E_u(t)) \int_0^1 \left( 2 - μ_a \right) u_x^2 + \left[ 2(a-xa') + aμ_a \right] \frac{u_x^2}{2} \right] dx dt + \frac{2 - μ_a}{2} β a(1) \int_S^T w(E_u(t)) u^2(t, 1) dt =$$

$$-2 \left[ w(E_u(t)) \int_0^1 u_x u_t dx \right]_S^T - \frac{μ_a}{2} \left[ w(E_u(t)) \int_0^1 u_t u dx \right]_S^T + \int_S^T w'(E_u(t)) E_u'(t) \int_0^1 \left( 2u_x + \frac{μ_a}{2} u \right) u_t dx dt + \int_S^T w(E_u(t)) \tilde{h}(t) dt \; \forall \; 0 ≤ S ≤ T,$$  

(5.20)

where the function $\tilde{h}$ is given by

$$\tilde{h}(t) = u_x^2(t, 1) + a(1) γ (u(t, 1))^2 + a(1) β (1 + β - μ_a) u^2(t, 1) + \left( 2β - \frac{μ_a}{2} \right) a(1) γ (u(t, 1)) u(t, 1) \; t ∈ (S, T).$$  

(5.21)

By definition of $μ_a$, we have

$$(2 - μ_a) a ≤ 2(a - xa') + aμ_a.$$  

This, together with (5.20), gives

$$(2 - μ_a) \int_S^T E_u(t) dt ≤ - \left[ w(E_u(t)) \int_0^1 2u_x u_t + \frac{μ_a}{2} u_t u dx \right]_S^T + \int_S^T w'(E_u(t)) E_u'(t) \int_0^1 \left( 2u_x + \frac{μ_a}{2} u \right) u_t dx dt +$$

$$\int_S^T \tilde{h}(t) dt \; \forall \; 0 ≤ S ≤ T.$$  

(5.22)
On the other hand, we have
\[ \tilde{h}(t) \leq \eta_3 u_t^2(t, 1) + \eta_4 \rho(u_t(t, 1))^2 + \eta_5 a(1) u^2(t, 1) \quad \forall t \in (S, T), \]  
(5.23)
where \( \eta_i \) for \( i = 3, 4, 5 \) are positive constants which do not depend on the weight function \( w \) nor on \( E(t) \). The two first terms in (5.22) are estimates as in the linear stabilization case. This, together with the properties that \( w \) is nondecreasing whereas \( E \) is non increasing yield
\[
(2 - \mu_a) \int_S^T E_u(t) dt \leq K_a w(E_u(S)) E_u(S) + \eta_3 \int_S^T w(E_u(t)) u_t^2(t, 1) dt + \eta_4 \int_S^T w(E_u(t)) \rho(u_t(t, 1))^2 dt + \eta_5 \int_S^T w(E_u(t)) a(1) u^2(t, 1) dt. \quad (5.24)
\]
where \( K_a \) is a positive constant which do not depend on the weight function \( w \) nor on \( E(t) \). We now estimate the last term of this inequality as in the linear stabilization case, using once again in addition our optimal weight function. Set \( \lambda = u(t, 1) \) and denote by \( z \) the solution of the degenerate elliptic problem (4.17). We multiply (5.2) by \( w(E_u(t))z \) and integrate the resulting equation over \( (S, T) \times (0, 1) \). This gives after suitable integrations by parts.
\[
\int_S^T a(1) w(E_u(t)) u^2(t, 1) dt = \int_S^T w(E_u(t)) \int_0^1 u_t z dx dt + \int_S^T w(E_u(t)) E_u(t) \int_0^1 u_t z dx dt - \int_S^T w(E_u(t)) \rho(u_t(t, 1)) z(t, 1) dt - \left[ w(E_u(t)) \int_0^1 u_t z dx \right]_S^T. \quad (5.25)
\]
We now estimate the terms of the right hand side in this inequality, as follows. Using (4.28)-(4.30) in (5.25), we obtain for all \( \delta > 0 \)
\[
\int_S^T a(1) w(E_u(t)) u^2(t, 1) dt \leq \delta \int_S^T w(E_u(t)) E_u(t) dt + C_1 w(E_u(S)) E_u(S) + C_2 \left( 1 + \frac{1}{\delta} \right) \int_S^T w(E_u(t)) \left( \rho^2(u_t(t, 1)) + u_t^2(t, 1) \right) dt,
\]
where \( C_1, C_2 \) are positive constants which do not depend on the weight function \( w \) nor on \( E(t) \). Choosing \( \delta = \frac{2 - \mu_a}{2\eta_5} \) in the above inequality and combining the resulting inequality in (5.24) yield
\[
\int_S^T w(E_u(t)) E_u(t) dt \leq C_3 w(E_u(S)) E_u(S) + C_4 \int_S^T w(E_u(t)) \left( \rho^2(u_t(t, 1)) + u_t^2(t, 1) \right) dt, \quad (5.26)
\]
where \( C_3, C_4 \) are positive constants which do not depend on the weight function \( w \) nor on \( E(t) \). It remains to estimate the last term on the right hand side of the above inequality. We further proceed as in [1, 2]. That is we fix \( t \geq 0 \). Assume first that \( |u_t(t, 1)| \leq \varepsilon_0 \) where \( \varepsilon_0 = \min(1, g(r_0)) \). Hence, thanks to our assumption on \( \rho \), we have
\[
\left| \frac{\rho(u_t(t, 1))}{c_2} \right|^2 \leq \left| g^{-1}(u_t(t, 1)) \right|^2 \leq \left| g^{-1}(\varepsilon_0) \right|^2 \leq r_0^2.
\]
On the other hand, we have
\[
H \left( \left| \frac{\rho(u_t(t, 1))}{c_2} \right|^2 \right) = \frac{|\rho(u_t(t, 1))|}{c_2} g \left( \left| \frac{\rho(u_t(t, 1))}{c_2} \right| \right) \leq \frac{1}{c_2} u_t(t, 1) \rho(u_t(t, 1)).
\]
Hence, since \( H \) is nondecreasing, we have whenever \( t \) is such that \( |u_t(t, 1)| \leq \varepsilon_0 \)
\[
w(E_u(t)) |\rho(u_t(t, 1))|^2 \leq c_2^2 w(E_u(t)) H^{-1} \left( \frac{1}{c_2} u_t(t, 1) \rho(u_t(t, 1)) \right) \leq c_2^2 H^*(w(E_u(t))) + c_2 u_t(t, 1) \rho(u_t(t, 1)). \quad (5.27)
\]
We now assume that $t$ is such that $|u_t(t,1)| \geq \varepsilon_0$, then up to a change in the constants $c_1$ and $c_2$ in (5.1), we can assume
\[ |\rho(u_t(t,1))| \leq c_2 |u_t(t,1)|, \]
so that
\[ \int_{t \in [S,T],|u_t(t,1)| \geq \varepsilon_0} w(E_u(t))|\rho(u_t(t,1))|^2 \leq \frac{c_2}{a(1)} w(E_u(S))E_u(S). \]
Combining this last estimate together with (5.27), we obtain
\[ \int_S^T w(E_u(t))|\rho(u_t(t,1))|^2 dt \leq c_2^2 \int_S^T \hat{H}^*(w(E_u(t))) + \frac{c_2}{a(1)} E_u(S) (1 + w(E_u(S))) . \tag{5.28} \]
We similarly estimate the term $\int_S^T w(E_u(t))u_t^2(t,1)dt$ proceeding as in [1, 2]. That is, we fix $t \geq 0$. We consider first the case for which $|u_t(t,1)| \leq \varepsilon_1$ where $\varepsilon_1 = \min\{r_0, g(r_1)\}$ where $r_1$ is defined by
\[ r_1^2 = H^{-1}\left(\frac{c_2}{c_2} H(r_0^2)\right). \]
Thanks to our assumptions on $\rho$, we have
\[ H\left(|u_t(t,1)|^2\right) \leq \frac{1}{c_1} u_t(t,1) \rho(u_t(t,1). \]
Hence, we have
\[ w(E_u(t))|u_t(t,1)|^2 \leq w(E_u(t))H^{-1}\left(\frac{1}{c_1} u_t(t,1) \rho(u_t(t,1))\right) \leq \hat{H}^*(w(E_u(t))) + \frac{1}{c_1} u_t(t,1) \rho(u_t(t,1). \tag{5.29} \]
Assume now that $t$ is such that $|u_t(t,1)| \geq \varepsilon_1$, then up to a change in the constants $c_1$ and $c_2$ in (5.1), we can assume
\[ |\rho(u_t(t,1))| \geq c_1 |u_t(t,1)|, \]
so that
\[ \int_{t \in [S,T],|u_t(t,1)| \geq \varepsilon_1} w(E_u(t))|u_t(t,1)|^2 \leq \frac{1}{c_1 a(1)} w(E_u(S))E_u(S). \]
Combining this last estimate together with (5.27), we obtain
\[ \int_S^T w(E_u(t))|u_t(t,1)|^2 dt \leq \int_S^T \hat{H}^*(w(E_u(t))) dt + \frac{1}{c_1 a(1)} E_u(S) (1 + w(E_u(S))) . \tag{5.30} \]
On the other hand, we recall that $\gamma$ satisfies (5.12), thus we have
\[ w(E_u(S)) \leq L^{-1}\left(\frac{E_u(0)}{2\gamma}\right) < H(r_0^2) \quad \forall \ S \geq 0. \]
Inserting the estimates (5.28) and (5.30) in (5.26), and using the above estimate, we obtain
\[ \int_S^T w(E_u(t))E_u(t) dt \leq C_5 E_u(S) + C_6 \int_S^T \hat{H}^*(w(E_u(t))) dt, \tag{5.31} \]
where $C_5, C_6$ are positive constants which do not depend on the weight function $w$ nor on $E(t)$. Thanks to our choice of weight function $w$
\[ L(w(E_u(t))) = \frac{E_u(t)}{2\gamma} \quad \forall \ t \geq 0, \]
so that we have
\[ \int_S^T w(E_u(t))E_u(t) dt \leq C_5 E_u(S) + \frac{C_6}{2\gamma} \int_S^T w(E_u(t))E_u(t) dt, \]
Choosing $\gamma \geq C_6$ in addition to (5.12), we obtain that
\[ \int_S^T w(E_u(t))E_u(t) dt \leq M E_u(S) \quad \forall \ 0 \leq S \leq T., \tag{5.32} \]
where $M = 2C_5$. Then proof can be completed applying the following result (see [2, Theorem 2.3]). □
Theorem 5.7 Let $H$ be a strictly convex function on $[0, r_0^2]$ such that $H(0) = H'(0) = 0$ and define $L$ and $\Lambda_H$ as above. Let $E$ be a given nonincreasing, absolutely continuous function from $[0, +\infty)$ on $[0, +\infty)$ with $E(0) > 0$ satisfying the following weighted nonlinear inequality

$$
\int_S^T L^{-1}(E(t)) \frac{E(t)}{2\gamma} dt \leq ME(S), \quad \forall 0 \leq S \leq T.
$$

(5.33)

where $M > 0$ and where $\gamma > \frac{E(0)}{2L(H'(r_0^2))}$. Then $E$ satisfies the following estimate:

$$
E(t) \leq 2\gamma L\left(\frac{1}{\psi_0^{-1}(\frac{1}{H})}\right), \quad \forall t \geq \frac{M}{H'(r_0^2)}.
$$

(5.34)

where $\psi_0$ is defined in (5.15). Furthermore, if $\lim \sup_{x \to 0^+} \Lambda_H(x) < 1$, then $E$ satisfies the following simplified decay rate

$$
E(t) \leq 2\gamma \left(\frac{\kappa M}{t}\right),
$$

(5.35)

for $t$ sufficiently large, and where $\kappa > 0$ is a constant independent of $E(0)$.

Remark 5.8 It should be noted that one can also reformulate, with no mathematical originality and no gain with respect to applications and research, all our results on the nonlinear stabilization of degenerate equations of this section by means of a “Lyapunov” presentation. In this case, it is sufficient to track all the steps of our proof, remove all the integrations with respect to time (from $S$ to $T$) and multiply afterwards the resulting inequality by a weight function, which can be a weaker (and less good) weight function than in the original method introduced for the first time in [1] (see also [2]). This weaker weight function can easily be deduced by dropping in the original computations of [1], the negative part in the convex conjugate of the strictly convex function $H^*$ defined in (5.10). Namely, this consists in replacing $H^*(y) = y(H^')^{-1}(y) - H((H^')^{-1}(y))$ for $y \in [0, c]$ (for a suitable $c > 0$) in the original paper by the function $H^*_\kappa(y) = y(H^')^{-1}(y)$ for $y \in [0, c]$. The results would also be weaker and destroy some nice and further properties proved later on in [2] which lead to simplified and optimal energy decay rates.

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