Natural Inflation From Fermion Loops

William H. Kinney* and K.T. Mahanthappa†

University of Colorado, Boulder

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Abstract

“Natural” inflationary theories are a class of models in which inflation is driven by a pseudo-Nambu-Goldstone boson. In this paper we consider two models, one old and one new, in which the potential for inflation is generated by loop effects from a fermion sector which explicitly breaks a global $U(1)$ symmetry. In both models, we retrieve the “standard” natural inflation potential, $V(\theta) = \Lambda^4 [1 + \cos (\theta/\mu)]$, as a limiting case of the exact one-loop potential, but we carry out a general analysis of the models including the limiting case. Constraints from the COBE DMR observation and from theoretical consistency are used to limit the parameters of the models, and successful inflation occurs without the necessity of fine-tuning the parameters.
I. INTRODUCTION

Inflationary cosmologies consider an epoch in the very early universe in which the energy density of the universe was dominated by vacuum energy, resulting in a period of exponential expansion of the universe \[ I \ 4 \]. This vacuum energy is typically generated in particle physics models by a scalar field potential associated with a broken symmetry. One attractive class of inflationary models is “Natural” inflation, which uses the vacuum energy associated with a pseudo-Nambu-Goldstone boson (PNGB) to drive inflation \[ I \ 5 \]. In this paper, we consider two models in which the PNGB is the result of a fermion sector which explicitly breaks a global \( U(1) \) symmetry: one earlier model involving chiral fermions \[ I \] and a new model involving non-chiral fermions. Section \[ II \] gives a review of “slow-roll” inflationary models with particular application to natural inflation. Section \[ III \] discusses the two models in detail, and Section \[ IV \] presents conclusions. We apply three conditions to the models discussed: (i) the model must satisfy standard observational constraints on inflationary theories, particularly those from the COBE DMR observation, (ii) the model must be weakly coupled, so that perturbation theory is valid, and (iii) the PNGB must acquire a nonzero mass at the minimum of its potential. These conditions allow us to place constraints on the parameters of the models, and we find that successful inflation occurs without the necessity of fine-tuning. In both models, we find that the inflationary potential used in Refs. \[ I \ 4 \] arises as a limiting case of the exact one-loop potential.

II. SLOW-ROLL AND NATURAL INFLATION

Inflationary cosmologies explain the observed flatness and homogeneity of the universe by postulating the existence of an epoch during which the energy density of the universe was dominated by vacuum energy, resulting in a period of exponential increase in the scale factor of the universe

\[
a(t) \propto e^{Ht}
\]  

(1)
The Hubble parameter $H$ is given by

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3m_{pl}^2} \rho_{vac} \simeq \text{const.} \quad (2)$$

where $m_{pl} \sim 10^{19}$GeV is the Planck mass. Nonzero vacuum energy is introduced into particle physics models by including a scalar field $\phi$, the *inflaton*, with a potential $V(\phi)$. During inflation, the inflaton is displaced from the minimum of its potential, resulting in a nonzero vacuum energy, and evolves to the minimum with equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \quad (3)$$

So-called “new” inflationary models consider potentials which contain at least one region flat enough that the evolution of the field is friction dominated

$$3H\dot{\phi} + V'(\phi) = 0 \quad (4)$$

This is known as the *slow-roll* approximation. This approximation can be shown to be valid if the *slow-roll parameters* $\epsilon$ and $|\eta|$ [6] are both much less than 1, where:

$$\epsilon(\phi) \equiv \frac{m_{pl}^2}{16\pi} \left[ \frac{V'(\phi)}{V(\phi)} \right]^2 \quad (5)$$

$$\eta(\phi) \equiv \frac{m_{pl}^2}{8\pi} \left[ \frac{V''(\phi)}{V(\phi)} - \frac{1}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \right] \quad (6)$$

For an inflationary theory to correspond to the observed universe, it must at minimum satisfy two conditions: (i) The scale factor of the universe must increase by a factor of at least $e^{60}$ during the inflationary epoch, in order to explain the observed thermal equilibrium of the cosmic microwave background radiation (CMB). (ii) Quantum fluctuations in the inflaton field give rise to primordial density fluctuations, with calculable amplitude $\delta$ and spectral index $n_s$ [7–10], which can be compared to the COBE DMR observation $\delta \sim 10^{-5}$, $n_s = 1.1 \pm 0.5$ [11,12]. These conditions can be used to place constraints on the form of the potential. Consider a scalar field initially at some value $\phi$. The field evolves according to equation (4) to the minimum of the potential, where it oscillates and decays into other
particles (*reheating*). Slow-roll ends and reheating commences at a field value $\phi_f$ where the slow-roll parameter $\epsilon(\phi_f)$ is unity [6]:

$$
\epsilon(\phi_f) \equiv \frac{m_{pl}^2}{16\pi} \left[ \frac{V'(\phi_f)}{V(\phi_f)} \right]^2 = 1
$$

(7)

where $\epsilon(\phi) < 1$ during the slow-roll period. The number of e-folds of inflation which occur when the field evolves from $\phi$ to $\phi_f$ is

$$
N(\phi) = \frac{8\pi}{m_{pl}^2} \int_{\phi}^{\phi_f} \frac{V(\phi')}{V'(\phi')} d\phi'
$$

(8)

We determine the upper limit on the initial field value $\phi \leq \phi_i$ for which sufficient inflation occurs by requiring $N(\phi_i) = 60$. Quantum fluctuations in the inflaton field when $\phi \sim \phi_i$ produce density fluctuations on scales of current astrophysical interest. We can constrain the potential by requiring that the magnitude of the density fluctuations $\delta(\phi_i)$ produced during the inflationary period match the observation by COBE [13]:

$$
\delta(\phi_i) = \left( \frac{2}{\pi} \right)^{1/2} \frac{[V(\phi_i)]^{3/2}}{m_{pl}^3 V'(\phi_i)} \{ 1 - \epsilon(\phi_i) + (2 - \ln 2 - \gamma) (2\epsilon(\phi_i) - \eta(\phi_i)) \} \sim 10^{-5}
$$

(9)

where $\gamma \simeq 0.577$ is Euler’s constant. In addition, it is possible to calculate the spectral index $n_s$ of the density fluctuations. The fluctuation power per logarithmic interval $P(k)$ is defined in terms of the density fluctuation amplitude $\delta_k$ on a scale $k$ as $P(k) \equiv |\delta_k|^2$. The spectral index $n_s$ is defined by assuming a simple power-law dependence of $P(k)$ on $k$,

$$
P(k) \propto k^{n_s}
$$

The spectral index $n_s$ of density fluctuations is given in terms of the slow-roll parameters $\epsilon$ and $\eta$ [13]:

$$
n_s \simeq 1 - 4\epsilon(\phi_i) + 2\eta(\phi_i)
$$

(10)

For $\epsilon, |\eta| \ll 1$ during inflation, inflationary theories predict a nearly scale-invariant power spectrum, $n_s \sim 1$. The value of the spectral index derived from the first year of COBE data is $n_s = 1.1 \pm 0.5$ [11,12]. The COBE two year results are also available [14]. However, different statistical methods used in analyzing the data lead to different bounds on the spectral index [13]. For the purposes of this paper, we will take $n_s \geq 0.6$. 

4
“Natural” inflationary theories use a pseudo-Nambu-Goldstone boson to drive the inflationary expansion. The basic scenario consists of the following: A spontaneous symmetry breaking phase transition occurs at a scale $\mu$, and temperature $T \sim \mu$. In this paper we will be considering symmetry breaking involving a single complex scalar field $\phi$, with a potential

$$V (\phi) = \lambda \left[ \phi^* \phi - \mu^2 \right]^2$$  \hspace{1cm} (11)

which is symmetric under a global $U(1)$ transformation $\phi \rightarrow e^{i\alpha} \phi$. At the minimum of the potential $V (\phi)$, we can parameterize the scalar field as $\phi = \sigma \exp [i\theta/\mu]$. The radial field $\sigma$ has a mass $M^2_\sigma \sim \lambda \mu^2$. The field $\theta$ is a Nambu-Goldstone boson, and is massless at tree level. If the $U(1)$ symmetry of the potential $V (\theta)$ is preserved by the rest of the Lagrangian, $\theta$ will remain massless with loop corrections. But if the $U(1)$ is broken by other terms in the Lagrangian, $\theta$ acquires a potential $V_1 (\theta)$ from loop corrections, leading to a nonzero mass, and is called a pseudo-Nambu-Goldstone boson (PNGB). Assuming the mass of $\theta$ is much less than that of the radial mode, $M^2_\theta \ll M^2_\sigma$, the field $\theta$ will be effectively massless near the original symmetry breaking scale $T \sim \mu$. As the temperature of the universe decreases, $T \ll M_\sigma$, excitations of the heavy $\sigma$ field will be damped, so that we can take $\sigma = \mu = \text{const}$. The only remaining degree of freedom will be $\theta$, and we can parameterize $\phi$ as:

$$\phi = \mu e^{i\theta/\mu}$$  \hspace{1cm} (12)

At temperatures $T \gg M_\theta$, the effective potential $V_1 (\theta)$ is negligible. When the universe cools to $T \sim M_\theta$, $V_1 (\theta)$ becomes important. The field $\theta$ will roll down the potential to its minimum, resulting in inflationary expansion during the period in which the energy density of the universe is dominated by vacuum energy. Natural inflationary models typically assume a potential for the PNGB of the form

$$V_1 (\theta) = \Lambda^4 \left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right]$$  \hspace{1cm} (13)

This is a limiting case of the exact one-loop potential of the models considered in this paper. $\mu$ is the original symmetry breaking scale, and $\Lambda$ is an independent energy scale characterizing the temperature at which the potential $V_1 (\theta)$ becomes significant.
We can use (8) and (9) to constrain the parameters $\mu$ and $\Lambda$ as follows [4]. Inflation ends at a field value $\theta_f$ where the slow-roll parameter $\epsilon(\theta_f) = 1$:

$$
\left(\frac{\theta_f}{\mu}\right) = \cos^{-1}\left[1 - 16\pi \left(\frac{\mu}{m_{pl}}\right)^2 \frac{1}{1 + 16\pi \left(\frac{\mu}{m_{pl}}\right)^2}\right]
$$

The number of e-folds for a given initial field value $\theta_i$ (8) is independent of the scale $\Lambda$, but depends on the scale $\mu$. Requiring $N(\theta_i) \geq 60$ results in an upper bound $\theta \leq \theta_i$ in terms of $\mu$:

$$
\left(\frac{\theta_i}{\mu}\right) = \cos^{-1}\left[1 - 2 \frac{16\pi \left(\frac{\mu}{m_{pl}}\right)^2}{1 + 16\pi \left(\frac{\mu}{m_{pl}}\right)^2} \exp\left[-\frac{60}{8\pi} \left(\frac{m_{pl}}{\mu}\right)^2\right]\right]
$$

so that $\theta_i$ decreases rapidly with decreasing scale $\mu$.

We can constrain the scale $\Lambda$ as a function of $\mu$ by using the COBE limit on the magnitude of density fluctuations (9):

$$
\delta(\theta_i) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{\mu \Lambda^2}{m_{pl}^3}\right) \left[1 + \cos\left(\frac{\theta_i}{\mu}\right)\right]^{\frac{3}{2}} \left[1 - \epsilon(\theta_i) + (2 - \ln 2 - \gamma) (2 \epsilon(\theta_i) - \eta(\theta_i))\right] \sim 10^{-5}
$$

where $\epsilon(\theta_i)$ and $\eta(\theta_i)$ are independent of the scale $\Lambda$. As shown in Fig. 1, $\Lambda$ is highly sensitive to the original symmetry breaking scale: for $\mu \sim m_{pl}$ we require $\Lambda \sim 10^{-3} m_{pl}$, and this drops rapidly as $\mu$ decreases. The mass of the PNGB at the minimum of the potential $(\theta/\mu) = \pi$ also decreases with $\mu$:

$$
M_{\theta}^2 \equiv V''((\theta/\mu) = \pi) = \frac{\Lambda^4}{\mu^2}
$$

We can constrain the value of the symmetry breaking scale $\mu$ using the spectral index of density fluctuations [10]. Fig. 2 shows the spectral index $n_s$ as a function of mass scale $\mu$. The COBE result $n_s \geq 0.6$ then constrains the symmetry breaking scale to $\mu \gtrsim 0.3 m_{pl}$. We will take the upper limit on the mass scale to be the Planck mass, $\mu = m_{pl}$, where the fluctuation power spectrum is nearly scale-invariant, with $n_s = 0.95$. A fluctuation spectrum consistent with cold dark matter (CDM) models requires $n_s \sim 0.7$, and the corresponding
symmetry breaking scale is \( \mu = 0.36 \, m_{pl} \). Natural inflation with \( \mu \leq m_{pl} \) is not consistent with a spectral index \( n_s > 1 \).

There is also the question of whether sufficient inflation will occur at all with significant probability, especially for mass scales \( \mu < m_{pl} \) where the initial field values for which sufficient inflation occurs, \( \theta \leq \theta_i \), represent a very small portion of the available space of initial conditions. As the universe cools to \( T \sim \Lambda \), we expect the field \( \theta \) and its derivative \( \dot{\theta} \) to take on different values in different regions of the universe; here we will assume that the field is to a good approximation uniform within any pre-inflation horizon volume. The universe just prior to inflation then consists of a large number of causally disconnected regions, each with independent initial conditions for \( \theta \) and \( \dot{\theta} \). Each independent region will inflate a different amount, perhaps not at all, depending on the conditions within that region. A successful model for inflation is a model in which the post-inflation universe is strongly dominated by regions in which \( N (\theta) \geq 60 \). It can be shown that the initial value of \( \dot{\theta} \) does not significantly affect the number of e-folds of inflation [10], and hence we consider here only the upper limit \( \theta \leq \theta_i \) in (13). Consider a pre-inflation horizon volume \( V_0 \) and initial field value \( \theta_i \): during inflation, this region will expand to a volume \( V = V_0 \exp [3N (\theta)] \). The fraction of the volume of the post-inflation universe for which \( N (\theta) \geq 60 \) is then [5]

\[
F (N \geq 60) = 1 - \left[ \frac{\int_{\theta_i}^{\pi \mu} \exp [3N (\theta)] \, d\theta}{\int_{H/2\pi}^{\pi \mu} \exp [3N (\theta)] \, d\theta} \right]
\]

(18)

Here a cutoff \( H/2\pi \), the magnitude of quantum fluctuations on the scale of a horizon size, has been introduced as the lower limit for the field value. A successful inflationary theory then has the characteristic that

\[
\Pi (\theta_i) \equiv \frac{\int_{\theta_i}^{\pi \mu} \exp [3N (\theta)] \, d\theta}{\int_{H/2\pi}^{\pi \mu} \exp [3N (\theta)] \, d\theta} \ll 1
\]

(19)

This condition is satisfied to a high degree for the range of symmetry breaking scales allowed by observational constraints, with \( \Pi (\theta_i) < 10^{-62} \) for \( \mu > 0.3 \, m_{pl} \).
We now consider two specific models of natural inflation, both of which generate the inflationary potential for the PNGB from fermion loop corrections. Both models have three parameters: the mass scale $\mu$, the bare fermion mass $m_0$ and a dimensionless Yukawa coupling $g$. As above, we can constrain the scale $\mu$ in terms of the spectral index of the density fluctuations created during inflation. For the purposes of this paper, we will be taking $m_0 < \mu$. But now the COBE constraint on the amplitude of density fluctuations results in a relation between the Yukawa coupling and the fermion mass parameter $g = g(m_0)$. In the cases considered here, the coupling $g$ increases with decreasing $m_0$, so no fine-tuning of couplings is necessary. In addition to the cosmological limits described above, we can also constrain the models by requiring: (i) that the coupling constant remain perturbative, $g < 1$, and (ii) that the PNGB have a nonzero mass at the minimum of the potential, to avoid the presence of undesirable long-range forces in the theory, and to make reheating possible. These constraints allow us to place lower limits on $m_0$ in both cases.

Chiral Fermions: Take a Lagrangian with a complex scalar field $\phi$ and fermions $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma^5)\psi$, of the form:

$$
L = L_{\text{kin}} - \bar{\psi}_L m_0 \psi_R - \bar{\psi}_R m_0 \psi_L - g \bar{\psi}_L \phi \psi_R - g \bar{\psi}_R \phi^* \psi_L - \lambda \left[ \phi^* \phi - \mu^2 \right]^2
$$

$$
L_{\text{kin}} = \partial_\mu \phi^* \partial^\mu \phi + i \bar{\psi} \gamma^\mu \partial_\mu \psi
$$

(20)

The fields transform under a global $U(1)$ symmetry as

$$
\phi \rightarrow e^{i\alpha} \phi
$$

$$
\psi_L \rightarrow e^{i\frac{\alpha}{2}} \psi_L
$$

$$
\psi_R \rightarrow e^{-i\frac{\alpha}{2}} \psi_R
$$

(21)

so that the global $U(1)$ is exact in the limit $m_0 \rightarrow 0$. The fermion mass term $-\bar{\psi}_L m_0 \psi_R - \bar{\psi}_R m_0 \psi_L$ explicitly breaks the $U(1)$ symmetry and results in the presence of a PNGB, which
will be the inflaton. Neglecting the one-loop shift in the vacuum expectation value of the \( \phi \) field, \( < \phi > = \mu + \text{(one-loop corrections)} \), we parameterize \( \phi \) as in (12): \( \phi = \mu \exp \left[ i \theta / \mu \right] \).

The tree level fermion mass is

\[
m^2(\theta) \equiv m^1(\theta) m(\theta) = m^2_0 + g^2 \mu^2 + 2g\mu m_0 \cos \left( \frac{\theta}{\mu} \right) \tag{22}
\]

Here we see that \( m_0 \) is the mass of the fermion at temperatures above the scale of spontaneous symmetry breaking, \( T > \mu \). In the spontaneously broken phase, the physical fermion mass is given by (22), and \( m_0 \) is treated as a parameter. The explicitly broken \( U(1) \) symmetry is reflected in the dependence of the fermion mass on the value of the PNGB \( \theta \). Quantum effects will generate a potential for \( \theta \), which we will use as an inflationary potential. The one-loop effective potential for \( \theta \) is [17]

\[
V_1(\theta) = -\frac{1}{64\pi^2} \left[ m^2(\theta) \right]^2 \ln \left[ \frac{m^2(\theta)}{\mu^2} \right] = -\frac{1}{16\pi^2} g^2 \mu^2 m_0^2 \left[ \frac{1}{2} \left( \frac{m_0}{g\mu} + \frac{g\mu}{m_0} \right) + \cos \left( \frac{\theta}{\mu} \right) \right]^2 \times \ln \left\{ \frac{2 g m_0}{\mu} \left[ \frac{1}{2} \left( \frac{m_0}{g\mu} + \frac{g\mu}{m_0} \right) + \cos \left( \frac{\theta}{\mu} \right) \right] \right\} \tag{23}
\]

The minimum of \( V_1(\theta) \) is at \( (\theta/\mu) = \pi \), and the physical fermion mass at the minimum of the potential is

\[
m[(\theta/\mu) = \pi] = \left\{ 2g\mu m_0 \left[ \frac{1}{2} \left( \frac{m_0}{g\mu} + \frac{g\mu}{m_0} \right) - 1 \right] \right\}^{\frac{1}{2}} \tag{24}
\]

The mass of the PNGB at the minimum of the potential is

\[
M_\theta^2 \equiv V_1''[(\theta/\mu) = \pi] = -\frac{1}{16\pi^2} g^2 \mu^2 m_0^2 \left[ \frac{1}{2} \left( \frac{m_0}{g\mu} + \frac{g\mu}{m_0} \right) - 1 \right] \left\{ 1 + 2 \ln \left[ 2 \left( \frac{g m_0}{\mu} \right) \left[ \frac{1}{2} \left( \frac{m_0}{g\mu} + \frac{g\mu}{m_0} \right) - 1 \right] \right] \right\} \tag{25}
\]

Note that both the fermion mass [24] and the PNGB mass [25] vanish when \( (m_0/g\mu) = 1 \), which we exclude because a massless \( \theta \) field will result in the presence of undesirable long-range forces in the theory. In the limit of weak coupling, \( (m_0/g\mu) \gg 1 \), the effective potential becomes
\[ V_1 (\theta) \simeq -\frac{1}{32\pi^2} g \mu m_0^3 \left\{ 1 + 2 \ln \left( \left( \frac{m_0}{\mu} \right)^2 \right) \right\} \left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right] \]  

(26)

This is of the form (13), with

\[ \Lambda^4 = -\frac{1}{32\pi^2} g \mu m_0^3 \left\{ 1 + 2 \ln \left( \left( \frac{m_0}{\mu} \right)^2 \right) \right\} \]  

(27)

Note that this is a positive quantity for \((m_0/\mu) < e^{-(1/4)}\). Remembering that \(\Lambda\) is constrained in terms of the mass scale \(\mu\) by (16), this results in a relationship between the Yukawa coupling \(g\) and the fermion mass parameter \(m_0\):

\[ g = -32\pi^2 \left( \frac{\Lambda}{\mu} \right)^4 \left( \frac{\mu}{m_0} \right)^3 \left\{ 1 + 2 \ln \left( \left( \frac{m_0}{\mu} \right)^2 \right) \right\}^{-1} \]  

(28)

The important feature to note is that the coupling constant \(g\) increases with decreasing \(m_0\). In this limit, the physical fermion mass (24) is dominated by the parameter \(m_0\):

\[ m \left[ \left( \theta/\mu \right) = \pi \right] \simeq m_0 \]  

(29)

In the limit of weak explicit symmetry breaking \((m_0/g\mu) \ll 1\), the effective potential (23) becomes

\[ V_1 (\theta) \simeq -\frac{1}{32\pi^2} g^3 \mu^3 m_0 \left[ 1 + 2 \ln \left( g^2 \right) \right] \left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right] \]  

(30)

This is also of the form (13), with

\[ \Lambda^4 = -\frac{1}{32\pi^2} g^3 \mu^3 m_0 \left[ 1 + 2 \ln \left( g^2 \right) \right] \]  

(31)

The physical fermion mass (24) is dominated by the symmetry breaking scale \(\mu\):

\[ m \left[ \left( \theta/\mu \right) = \pi \right] \simeq g \mu \]  

(32)

In this case the coupling constant \(g\) also increases with decreasing \(m_0\), so the necessity to fine-tune the coupling present in most inflationary theories is entirely avoided. In fact, we have the opposite problem: for very small \(m_0\), the theory becomes strongly coupled and the perturbative analysis above is inconsistent. Fig. 3 shows the coupling constant as a function
of $m_0$ for the case $\mu = m_{pl}$. We can place a constraint on $m_0$ by requiring that the Yukawa coupling be perturbative, $g < 1$. For $g > e^{-1/4}$, the right-hand side of (31) becomes negative, which corresponds to exchanging the location of the maxima and minima of the potential (23). Thus, for $g < 1$, we have

$$\Lambda^4 = \left| \frac{1}{32\pi^2} g^3 \mu^3 m_0 \left[ 1 + 2 \ln \left( g^2 \right) \right] \right|$$  \hspace{1cm} (33)

Setting $g = 1$ in (33), the lower limit on $m_0$ is then

$$m_0 \geq 32\pi^2 \mu \left( \frac{\Lambda}{\mu} \right)^4$$  \hspace{1cm} (34)

This is a limit based on theoretical consistency and not on phenomenology. However, for $m_0$ below this limit, the entire procedure of calculating the inflationary potential (23) in perturbation theory is seen to be invalid. For a scale-invariant power spectrum, $n_s \sim 1$, we require a symmetry breaking scale near the Planck mass, $\mu \sim m_{pl}$. In this case, the lower limit on $m_0$ is $m_0 \gtrsim 10^{10}$GeV. For a CDM power spectrum, $n_s = 0.7$, the lower limit drops to $m_0 \gtrsim 10^5$GeV. For the lowest value of $n_s = 0.6$ allowed by COBE, $\mu = 0.32 m_{pl}$, the lower limit is $m_0 \gtrsim 10^5$GeV. However, for very small values of $m_0$, the hierarchy of scales $m_0 \ll \mu$ will not in general be maintained when the parameters are renormalized, and one must invoke an additional symmetry of the Lagrangian to preserve the difference in scales. Ref. [5] contains a description of one possible mechanism. Fig. 4 shows the lower limit on $m_0$ as a function of the spectral index. We can also calculate the fermion mass $m$ (24) and the PNGB mass (25) as a function of $m_0$ (Fig. 5). Typical fermion masses are in the range $10^{16}$-$10^{19}$GeV, with PNGB masses in the range $10^{13}$-$10^{14}$GeV. For all values of the parameters, the fermion mass is greater than the mass of the PNGB, so that the decay $\theta \to \bar{\psi} \psi$ is forbidden, and reheating must take place in some other, unspecified, sector of the theory.

We note that, although we obtain a potential of the form (13) in both limits $(m_0/g\mu) \gg 1$ and $(m_0/g\mu) \ll 1$, the values obtained for the parameter $\Lambda$

$$\Lambda^4 \sim g\mu m_0^3 \text{ for } (m_0/g\mu) \gg 1$$
\[ \Lambda^4 \sim g^3 \mu^3 m_0 \text{ for } (m_0/g\mu) \ll 1 \]  

(35)

differ from the result of \( \Lambda^4 \sim (g\mu m_0)^2 \) in Ref. \([5]\).

Non-Chiral Fermions: We now construct a model in which the global \( U(1) \) is preserved by the fermion mass term, but broken by the Yukawa coupling:

\[
L = L_{\text{kin}} - \bar{\psi} m_0 \psi - g \bar{\psi} \phi \psi - g \bar{\psi} \phi^* \psi - \lambda \left( \phi^* \phi - \mu^2 \right)^2
\]

(36)

The fields transform under a global \( U(1) \) symmetry as \( \phi \to e^{i\alpha} \phi, \psi \to e^{i\alpha} \psi \), so that the \( U(1) \) symmetry is exact when \( g \to 0 \). The Yukawa couplings explicitly break the \( U(1) \) symmetry and result in the presence of a PNGB, which will be the inflaton. Parameterizing \( \phi \) as in \([12]\), \( \phi = \mu \exp[i\theta/\mu] \), the tree level fermion mass is

\[
m^2(\theta) = \left[ m_0 + 2g\mu \cos\left(\frac{\theta}{\mu}\right) \right]^2
\]

(37)

The one-loop effective potential for the PNGB is

\[
V_1(\theta) = -\frac{1}{64\pi^2} \left[ m^2(\theta) \right]^2 \ln \left[ \frac{m^2(\theta)}{\mu^2} \right]
\]

\[
= -\frac{1}{4\pi^2} g^4 \mu^4 \left[ \left( \frac{m_0}{2g\mu} \right) + \cos\left( \frac{\theta}{\mu} \right) \right]^4 \ln \left\{ 4g^2 \left[ \left( \frac{m_0}{2g\mu} \right) + \cos\left( \frac{\theta}{\mu} \right) \right]^2 \right\}
\]

(38)

When \( (m_0/2g\mu) \leq 1 \), the effective potential has two equivalent minima

\[
\left( \frac{\theta_{\text{min}}}{\mu} \right) = \cos^{-1} \left[ -\left( \frac{m_0}{2g\mu} \right) \right]
\]

(39)

However, the PNGB is massless in the true vacuum, as \( V''(\theta_{\text{min}}) = 0 \). We must therefore exclude this parameter range, since there will be long-range forces associated with the field \( \theta \). In the parameter range \( (m_0/2g\mu) > 1 \), the potential \( V_1(\theta) \) has a single minimum at \( \theta/\mu = \pi \). The mass of \( \theta \) at the minimum of the potential is:

\[
M^2_\theta \equiv V''(\theta = \pi\mu) = -\frac{1}{2\pi^2} g^4 \mu^2 \left[ \left( \frac{m_0}{2g\mu} \right) - 1 \right]^3 \left\{ 1 + 2 \ln \left( 4g^2 \left[ \left( \frac{m_0}{2g\mu} \right) - 1 \right]^2 \right) \right\}
\]

(40)

The physical fermion mass is

\[
m \left[ (\theta/\mu) = \pi \right] = 2g\mu \left[ \left( \frac{m_0}{2g\mu} \right) - 1 \right]
\]

(41)
We can simplify the form of the potential in the limit where $M_\theta$ is large:

\[
\left( \frac{m_0}{2g\mu} \right) \gg 1
\]  

(42)

Introducing a cosmological constant so that $V_1 (\theta = \pi\mu) = 0$, the potential becomes

\[
V_1 (\theta) = \frac{1}{16\pi^2}g\mu m_0^3 \left[ 4 \ln \left( \frac{\mu}{m_0} \right) - 1 \right] \left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right]
\]  

(43)

This is of the form (13), with

\[
\Lambda^4 = \frac{1}{16\pi^2}g\mu m_0^3 \left[ 4 \ln \left( \frac{\mu}{m_0} \right) - 1 \right]
\]  

(44)

In this limit, the physical fermion mass is dominated by the parameter $m_0$:

\[
m \left[ (\theta/\mu) = \pi \right] \simeq m_0
\]  

(45)

We again have a relationship between the Yukawa coupling $g$ and $m_0$:

\[
g = 16\pi^2 \left( \frac{\Lambda}{\mu} \right)^4 \left( \frac{\mu}{m_0} \right)^3 \left[ 4 \ln \left( \frac{\mu}{m_0} \right) - 1 \right]^{-1}
\]  

(46)

In this limit, the mass of $\theta$ is, as usual, $M_\theta^2 = (\Lambda^4/\mu^2)$. Again, successful inflation requires the coupling to become stronger with decreasing $m_0$. Note, however, that the approximation (12) breaks down for $g \sim 1, m_0 < \mu$, and we cannot use the requirement of small coupling to limit $m_0$. In addition, successful inflation occurs in a significantly large parameter space which does not correspond to the limiting case (13). We must consider the case $(m_0/2g\mu) \sim O(1)$ separately, since it requires solution of the inflationary constraints for the exact potential (13). Numerical solution of the inflationary constraints indicates that the theory never becomes strongly coupled, and in fact $g$ “turns over” and begins to decrease for small $m_0$ (Fig. 3). However, the mass of $\theta$ drops from its value in the limit (12), $M_\theta^2 = (\Lambda^4/\mu^2)$, to zero as $(m_0/2g\mu)$ approaches 1. The requirement that $M_\theta$ be greater than zero allows us to place a lower limit on $m_0$. Taking $(m_0/2g\mu) = 1$, the potential is

\[
V_1 (\theta) = -\frac{1}{4\pi^2}g^4\mu^4 \left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right]^4 \ln \left\{ 4g^2 \left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right]^2 \right\}
\]  

(47)
The field value at the end of inflation $\theta_f$ is defined in terms of the first-order slow-roll parameter $\epsilon(\theta)$:

$$\epsilon(\theta_f) \equiv \frac{m_{pl}^2}{16\pi} \left[ \frac{V_1' (\theta_f)}{V_1 (\theta_f)} \right]^2 = 1$$

(48)

where

$$\frac{m_{pl}V_1' (\theta)}{V_1 (\theta)} = -2 \left( \frac{m_{pl}}{\mu} \right) \left[ \frac{\sin \left( \frac{\theta}{\mu} \right)}{1 + \cos \left( \frac{\theta}{\mu} \right)} \right] \left[ 1 + 2 \ln \left\{ 4g^2 \left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right]^2 \right\} \right]$$

(49)

For a small coupling $g$, $\epsilon(\theta)$ is approximately

$$\epsilon(\theta) \approx \frac{1}{\pi} \left( \frac{m_{pl}}{\mu} \right)^2 \frac{\sin \left( \frac{\theta}{\mu} \right)}{\left[ 1 + \cos \left( \frac{\theta}{\mu} \right) \right]^2}$$

(50)

Then for $\epsilon(\theta_f) = 1$

$$\cos \left( \frac{\theta_f}{\mu} \right) = \frac{1 - \pi (\mu/m_{pl})^2}{1 + \pi (\mu/m_{pl})^2}$$

(51)

The upper limit on the initial field value $\theta \leq \theta_i$ is defined such that the number of e-folds of inflation $N(\theta_i) = 60$, where

$$N(\theta) \equiv \frac{8\pi}{m_{pl}^2} \int_{\theta_f}^{\theta_i} \frac{V_1 (\theta)}{V_1' (\theta)} d\theta = 2\pi \left( \frac{\mu}{m_{pl}} \right)^2 \ln \left[ \frac{1 - \cos \left( \frac{\theta_f}{\mu} \right)}{1 - \cos \left( \frac{\theta_i}{\mu} \right)} \right]$$

(52)

Taking $\theta_i$ such that $N(\theta_i) = 60$,

$$\cos \left( \frac{\theta_i}{\mu} \right) = 1 - \left[ \frac{2\pi (\mu/m_{pl})^2}{1 + \pi (\mu/m_{pl})^2} \right] \exp \left[ -\frac{30}{\pi} \left( \frac{m_{pl}}{\mu} \right)^2 \right]$$

(53)

The coupling constant $g = (m_0/2\mu)$ can then be determined as a function of the symmetry breaking scale $\mu$ by numerical solution of the COBE constraint (1). For $\mu \sim m_{pl}$, the spectral index is $n_s \simeq 0.85$ and the lower limit on $m_0$ is $m_0 \gtrsim 10^{16}\text{GeV}$. We get $m_0 \gtrsim 10^{15}\text{GeV}$ for the CDM power spectrum, $n_s = 0.7$. For the lowest value of $n_s = 0.6$ allowed by COBE, $m_0 \gtrsim 10^{14}\text{GeV}$. Fig. [1] shows the lower limit on $m_0$ as a function of the spectral index $n_s$.

To obtain a spectral index $n_s > 0.85$, we must take $(m_0/2g\mu) > 1$. The nearly scale-invariant limit, $n_s = 0.95$, occurs for $m_0 \gtrsim 10^{17}\text{GeV}$.

Fig. [8] shows the fermion mass $m$ (11) and the PNGB mass $M_\theta$ (10) as functions of the parameter $m_0$. Here, as in the chiral model, the PNGB is lighter than the fermions for all values of the parameters.
IV. CONCLUSIONS

We have considered two models for inflation in which a PNGB acquires a potential as a result of fermion loop corrections in a Lagrangian with an explicitly broken global $U(1)$ symmetry. Although the $U(1)$ symmetry considered in these models is very simple, the mechanisms of the explicit symmetry breaking are quite general. In both models, successful inflation occurs without fine-tuning of parameters, and the potential (13) arises as a limiting case of the exact one-loop effective potential for the PNGB. In the case of the chiral model, our result $\Lambda^4 \sim g\mu m_0^3$ for $(m_0/g\mu) \gg 1$ and $\Lambda^4 \sim g^2\mu^2m_0$ for $(m_0/g\mu) \ll 1$ differs from the result of $\Lambda^4 \sim (g\mu m_0)^2$ in Ref. [5]. In the case of the non-chiral model, successful inflation occurs in parameter regimes for which the limiting case (13) is not applicable.

It is also natural to ask similar questions about models in which the explicit symmetry breaking takes place in the gauge sector rather than the fermion sector. This is the subject of continuing work.

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FIGURES

FIG. 1. $\Lambda$ vs. symmetry breaking scale $\mu$ in natural inflation.

FIG. 2. Spectral index $n_s$ vs. symmetry breaking scale $\mu$ in natural inflation.

FIG. 3. Coupling constant $g$ vs. $m_0$ with $\mu = m_{pl}$ (chiral model). The solid lines are for the approximations $(m_0/g\mu) \gg 1$ and $(m_0/g\mu) \ll 1$. The dashed line is a numerical solution for the exact potential (23) in the parameter regime $(m_0/g\mu) \sim O(1)$.

FIG. 4. Lower limit on $m_0$ vs. spectral index $n_s$ (chiral model).

FIG. 5. Fermion mass $m$ and PNGB mass $M_\theta$ vs. $m_0$ (chiral model), with $\mu = 0.6m_{pl}, 0.8m_{pl}$ and $m_{pl}$, where the higher particle masses correspond to larger $\mu$. The solid lines are for the limits $(m_0/g\mu) \gg 1$ and $(m_0/g\mu) \ll 1$, and the dashed lines are for $(m_0/g\mu) \sim O(1)$.

FIG. 6. Coupling constant $g$ vs. $m_0$, with $\mu = m_{pl}$ (non-chiral model). The solid line is the result for the exact potential (38), and the dashed line is the result for the limit $(m_0/2g\mu) \gg 1$.

FIG. 7. Lower limit on $m_0$ vs. spectral index $n_s$ with $(m_0/2g\mu) = 1$ (non-chiral model).

FIG. 8. Fermion mass $m$ and PNGB mass $M_\theta$ vs. $m_0$, for the case $\mu = m_{pl}$ (non-chiral model).
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FIG. 1: $\Lambda$ vs. symmetry breaking scale $\mu$ in natural inflation.

FIG. 2: Spectral index $n_s$ vs. symmetry breaking scale $\mu$ in natural inflation.
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This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9503331v3
FIG. 3: Coupling constant $g$ vs. $m_0$ with $\mu = m_{pl}$ (chiral model). The solid lines are for the approximations $(m_0/g\mu) \approx 1$ and $(m_0/g\mu) \ll 1$. The dashed line is a numerical solution for the exact potential (24) in the parameter regime $(m_0/g\mu) = O(1)$.

FIG. 4: Lower limit on $m_0$ vs. spectral index $n_s$ (chiral model).
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This figure "fig1-5.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9503331v3
FIG. 5: Fermion mass $m$ and PNGB mass $M_\theta$ vs. $m_0$ (chiral model), with $\mu = 0.6 \, m_{pl}$, $0.8 \, m_{pl}$ and $m_{pl}$ where the higher particle masses correspond to larger $\mu$. The solid lines are for the limits $(m_0/g\mu) \gg 1$ and $(m_0/g\mu) \ll 1$, and the dashed lines are for $(m_0/g\mu) \approx O(1)$.

FIG. 6: Coupling constant $g$ vs. $m_0$, with $\mu = m_{pl}$ (non-chiral model). The solid line is the result for the exact potential (39), and the dashed line is the result for the limit $(m_0/2g\mu) \gg 1$. 
This figure "fig1-6.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9503331v3
FIG. 7: Lower limit on $m_0$ vs. spectral index $n_s$ with $(m_0/2g\mu) = 1$ (non-chiral model).

FIG. 8: Fermion mass $m$ and PNGB mass $M_\theta$ vs. $m_0$, for the case $\mu = m_{pl}$ (non-chiral model).