An extension of RPA preserving energy weighted sum rules.  
An application to a 3-level Lipkin model.  

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Abstract  

A limitation common to all extensions of RPA including only particle-hole configurations is that they violate to some extent the Energy Weighted Sum Rules. Considering one such extension, the improved RPA (IRPA), already used to study the electronic properties of metallic clusters, we show how it can be generalized in order to eliminate this drawback. This is achieved by enlarging the configuration space, including also elementary excitations corresponding to the annihilation of a particle (hole) and the creation of another particle (hole) on the correlated ground state. The approach is tested within a solvable 3-level model.  

I. INTRODUCTION  

Collective excitations are a common feature of a large variety of many body systems. Their properties are intimately related to the stucture of the ground state upon which they are built. The simplest theory of excited states of a quantum system where correlations are taken into account to some extent is the Random Phase Approximation (RPA). In this theory one introduces a set of operators $Q_{\nu}^\dagger$, whose action on the ground state $|\Psi_0\rangle$
creates the collective excitations, while the ground state itself is the vacuum for the $Q_{\nu}$ operators. The latters are defined as linear superpositions of particle-hole ($ph$) creation and annihilation operators, occupied ($h$) and unoccupied ($p$) single particle states being defined with respect to the Hartree-Fock ground state $|HF\rangle$. The $X$ and $Y$ coefficients of these linear forms are solutions of equations which can be derived by using the equations of motion method $[1,2]$. If the hamiltonian contains one- and two-body terms, the solution of these equations would imply the evaluation of one- and two-body density matrices. Standard $RPA$ is obtained by replacing them by those calculated in the uncorrelated ground state $|HF\rangle$. This approximation introduces a visible inconsistency since, on one hand, the definition of $|\Psi_0\rangle$ as the vacuum of the $Q_{\nu}$ operators is used to derive the formal equations determining the $X$ and $Y$ amplitudes; while, on the other hand, $|HF\rangle$ is used instead of $|\Psi_0\rangle$ in calculating the expectation values appearing in those equations.

Various attempts have been made to eliminate this inconsistency. We quote the pioneering works $[3,4]$ where the renormalized $RPA$ ($RRPA$) was introduced. The $RRPA$ was applied to study the low-lying spectrum and the transition densities of vibrational nuclei $[5]$ and the double beta decay $[6-8]$ more recently. A very important contribution to the solution of this problem has been given in $[9]$, where a general scheme, the self consistent $RPA$ ($SCRPA$), was developed (see also $[10]$ and references therein). In reference $[8]$ a fully renormalized $RPA$ (fully $RRPA$) has been proposed, which shares some similarities with the approach we are going to present in this paper. In $[11,12]$ it was shown that by using the number operator method $[13]$ it is possible to get for the $X$ and $Y$ coefficients a closed set of equations having the same form as in $RPA$, where the density matrices in the correlated ground state are expressed in terms of the $X$ and $Y$ coefficients themselves. Thus the equations to solve are non linear and their solution requires a big computational effort. In order to appreciate how much a better treatment of correlations modifies the $RPA$ results, in the same paper a simplified version of the approach was proposed, the $IRPA$, based on the linearization of the equations of motion. The simplification consists in contracting the two-body terms appearing in the commutator of the hamiltonian with a one
body \( ph \) operator with respect to the correlated ground state. In this way only one body density matrices have to be evaluated: the so obtained equations are still non linear, but they are much easier to be solved, since only the one-body density matrix appears. When the latter is calculated in \(|HF\rangle\) rather than in the correlated \(|\Psi_0\rangle\), \( RPA \) is again obtained \cite{14}. This approach was applied in \cite{11,12} to the study of the electronic properties of some simple metal clusters, obtaining a better description than \( RPA \). However, the formulation is quite general and its applicability is by no means limited to such systems.

A limitation common to all extensions of \( RPA \) including only \( ph \) configurations is that they violate to some extent the Energy Weighted Sum Rules (\( EWSR \)). In the present paper we will show that this drawback can be eliminated by enlarging the configuration space, including also those configurations corresponding to the annihilation of a particle (hole) and the creation of another particle (hole) on the ground state. This is in the same spirit of \cite{15,16}, where, for the first time, the particle-particle and hole-hole configurations were included within the \( SCRPA \) approximation.

Very recently a paper \cite{17} came to our knowledge, where the same problem is tackled and studied within a solvable 4-level model with a separable residual interaction. As we will show below, there are several differences with the present paper:

1) we explicitly show that the \( EWSR \) is exactly satisfied when the configuration space is enlarged;

2) by comparison with the exact solutions of the model we can judge about the quality of the results obtained in \( IRPA \) and its enlarged version, with respect to the \( RPA \) ones;

3) this comparison allows us to point out that, besides the merit of solving the \( EWSR \) problem, the approach has the shortcoming that spurious solutions appear. This problem is not discussed in \cite{17} where, indeed, probably because a separable residual interaction is used, only one collective state is found despite the fact that 3 elementary excitation modes are present in the model. In this context, it is worth mentioning that spurious solutions are also found in \cite{8}, where they are interpreted as ”new excitation modes”.

The paper is organized as follows. In Section II we shortly recall the main \( IRPA \) equa-
tions, pointing to the origin of the \textit{EWSR} violations. Then we show how this problem is solved when the enlarged space is considered. In Section III we illustrate the approach by applying it to a solvable 3-level model [19–21] and comparing the different approximations among themselves and with the exact results.

II. FORMULATION OF THE APPROACH

In this section we recall the main steps leading to the IRPA equations, presented in detail in [11,12], and illustrate why, being limited to \textit{ph} excitations, the IRPA approximation violates the \textit{EWSR} [2]. Then we show that, enlarging the space by including also \textit{pp} and \textit{hh} configurations, this difficulty is overcome. In this respect, our approach is similar to the fully RRP A [8].

A. IRPA and the EWSR problem

Let $|\Psi_0\rangle$ be the ground state of the system and $|\Psi_\nu\rangle$ its excited states. Assuming that the latters are linear combinations of \textit{ph} and \textit{hp} configurations built upon $|\Psi_0\rangle$ one writes:

$$
|\Psi_\nu\rangle \equiv Q_\nu^\dagger |\Psi_0\rangle \equiv \sum_{ph} [X_{ph}^\nu B_{ph}^\dagger - Y_{ph}^\nu B_{ph}] |\Psi_0\rangle ,
$$

where $p$ ($h$) denotes the quantum numbers of an unoccupied ($p$) and occupied ($h$) single particle state in the uncorrelated Hartree-Fock reference state $|HF\rangle$. In eq.(1) we have introduced renormalized \textit{ph} creation ($B_{ph}^\dagger$) and annihilation ($B$) operators. In [11,12] it is shown that in the basis diagonalizing the one-body density matrix they can be written as:

$$
B_{ph}^\dagger = D_{ph}^{-\nu} a_p^\dagger a_h ,
$$

with:

$$
D_{ph} \equiv n_h - n_p ,
$$

where $n_h$ and $n_p$ are respectively the hole and particle occupation numbers in the correlated ground state $|\Psi_0\rangle$. Assuming that $|\Psi_0\rangle$ is the vacuum of the $Q_\nu$ operators,
\[ Q_\nu |\Psi_0\rangle = 0 , \]  

(4)

the orthonormality conditions for the excited states leads to:

\[ \delta_{\nu\nu'} = \langle \Psi_\nu | \Psi_{\nu'} \rangle = \sum_{ph} [X_{ph}^{\nu*} Y_{ph}^{\nu'} - Y_{ph}^{\nu*} X_{ph}^{\nu'}] . \]  

(5)

The equations determining the \( X^{\nu} \) and \( Y^{\nu} \) amplitudes and the excitation energies \( E_\nu \) of the states \( |\Psi_\nu\rangle \) are obtained by using the equations of motion method \[1,2\]. They read:

\[
\begin{pmatrix}
A & B \\
B^* & A^*
\end{pmatrix}
\begin{pmatrix}
X^{\nu} \\
Y^{\nu}
\end{pmatrix} = E_\nu
\begin{pmatrix}
X^{\nu} \\
-Y^{\nu}
\end{pmatrix} ,
\]  

(6)

with the \( A \) and \( B \) matrices given by:

\[ A_{ph,p'h'} = \langle \Psi_0 | [B_{ph}, H, B_{p'h'}^\dagger] |\Psi_0\rangle \]  

(7)

and

\[ B_{ph,p'h'} = -\langle \Psi_0 | [B_{p'h'}^\dagger, H, B_{ph}^\dagger] |\Psi_0\rangle . \]  

(8)

In (7) and (8) \( H \) is the Hamiltonian of the system and:

\[ [A, B, C] \equiv \frac{1}{2}([A, [B, C]] + [[A, B], C]) \]  

(9)

The standard RPA equations can be obtained by putting \( n_h=1, n_p=0 \) in the expressions for the operators \( B \) and \( B^\dagger \) (2) and by replacing the correlated ground state \( |\Psi_0\rangle \) appearing in (7) and (8) with the Hartree-Fock one \( |HF\rangle \). In [14] it is shown that the RPA equations can equivalently be obtained by linearizing the commutator \([H, B_{p'h'}^\dagger]\) in (7) and (8), i.e. by contracting it with respect to \( |HF\rangle \). A better approximation is done in IRPA, where the linearization is made by contraction in \( |\Psi_0\rangle \). In a loose notation, this means:

\[ [H, a_p^\dagger a_h] \rightarrow a_h^\dagger a + a^\dagger a_h a a \]

\[ \sim a_h^\dagger a + \langle \Psi_0 | a^\dagger a |\Psi_0\rangle a_h^\dagger a . \]  

(10)

Therefore, the occupation numbers in the correlated ground state appear in the IRPA expressions, while those in \( |HF\rangle \) (i.e. 0 or 1) appear in standard RPA. This procedure leads to:
\[ A_{ph,p'h'} = \frac{1}{2}(D^{1/2}_{ph} D^{-1/2}_{p'h'} + D^{1/2}_{p'h'} D^{-1/2}_{ph})(\epsilon_{p'p}\delta_{hh'} - \epsilon_{hh'}\delta_{pp'}) \]
\[ + D^{1/2}_{ph} D^{1/2}_{p'h'}(hp'|H_2|ph') , \]  
\[ (11) \]

where:
\[ \epsilon_{p'p} \equiv (p'|H_1|p) + \sum_{\alpha} n_{\alpha}(p'\alpha|H_2|p\alpha) \]  
\[ (12) \]
and
\[ \epsilon_{hh'} \equiv (h|H_1|h') + \sum_{\alpha} n_{\alpha}(\alpha h|H_2|\alpha h') . \]  
\[ (13) \]

For the matrix B one gets:
\[ B_{ph,p'h'} = D^{1/2}_{ph} D^{1/2}_{p'h'}(hh'|H_2|pp') . \]  
\[ (14) \]

In the above equations \( H_1 \) is the one-body term of the hamiltonian and \( H_2 \) its two-body part. We denote by \( \alpha \) a generic single particle state (occupied or unoccupied in \(|HF\rangle\)).

As shown in Appendix A of [11], using the number operator method [13] the occupation numbers appearing in the \( A \) and \( B \) matrices can be expressed in terms of the \( X \) and \( Y \) amplitudes as:
\[ n_p = \sum_{h\nu\nu'}(\delta_{\nu\nu'} - \frac{1}{2} \sum_{p_1h_1} D_{p_1h_1} X_{p_1h_1}^{\nu'} X_{p_1h_1}^{\nu'} D_{ph} Y_{ph}^{\nu'} Y_{ph}^{\nu'} , \]  
\[ (15) \]
\[ n_h = 1 - \sum_{p\nu\nu'}(\delta_{\nu\nu'} - \frac{1}{2} \sum_{p_1h_1} D_{p_1h_1} X_{p_1h_1}^{\nu'} X_{p_1h_1}^{\nu'} D_{ph} Y_{ph}^{\nu'} Y_{ph}^{\nu'} . \]  
\[ (16) \]

Therefore eq.s (8) are non linear. They have been solved iteratively in the case of metallic clusters [11,12]. It is, however, apparent that the approach is quite general and can be applied to any many body system. The matrices \( A \) and \( B \) in IRPA, eq.s(11) and (14), are different from those in standard RPA. On one side the Hartree-Fock single particle energies appearing in the \( A \) matrix of RPA are replaced by the quantities appearing in the first line of eq.(11). On the other side, the residual interaction in the expressions for \( A \) and \( B \) is now renormalized by the factors \( D^{1/2} \)'s. In RRPA only the latter modification is present. This latter modification is present also in RRPA.
A serious problem arises with respect to the EWSR. As it is well known, if $|\Psi_0\rangle$ and $|\Psi_\nu\rangle$ are a complete set of exact eigenstates of the hamiltonian, with eigenvalues $E_0$ and $E_\nu$, the following identity holds:

$$\sum_\nu (E_\nu - E_0) |\langle \Psi_\nu | F | \Psi_0 \rangle|^2 = \frac{1}{2} \langle \Psi_0 | [F, [H, F]] | \Psi_0 \rangle ,$$

(17)

where $F$ is any hermitian single particle operator. The equality (17) is in general violated to some extent when $|\Psi_0\rangle$, $|\Psi_\nu\rangle$ and $E_\nu$ are calculated within some approximation. To which extent it is satisfied is a measure of the adequacy of the approximation. A very important feature of RPA is that eq.(17) is satisfied for any one-body operator if, in calculating its two sides, one considers $|HF\rangle$ instead of $|\Psi_0\rangle$ and the solutions of RPA for $|\Psi_\nu\rangle$ and $(E_\nu - E_0)$ [18]. This feature follows from the fact that, when $|HF\rangle$ is used in (17) instead of $|\Psi_0\rangle$ only particle-hole matrix elements remain in the r.h.s. It is easy to show [11] that, if the transition operator $F$ has only $p-h$ matrix elements, the two sides of eq.(17) are equal also within IRPA. However, this is not the case in general.

Let us consider separately the two sides of eq.(17) in IRPA, with a general one-body hermitian operator $F$:

$$F = \sum_{\alpha\beta} f_{\alpha\beta} a_\alpha^\dagger a_\beta .$$

(18)

The l.h.s. is easily calculated and gives:

$$\sum_\nu (E_\nu - E_0) |\langle \Psi_\nu | F | \Psi_0 \rangle|^2 = \sum_\nu (E_\nu - E_0) |\langle \Psi_0 | Q_\nu F | \Psi_0 \rangle|^2$$

$$= \sum_\nu (E_\nu - E_0) |\langle \Psi_0 | [Q_\nu, F] | \Psi_0 \rangle|^2$$

$$= \sum_\nu (E_\nu - E_0) \sum_{ph} f_{ph} D_{ph}^{1/2} (X_{ph}^\nu + Y_{ph}^\nu)^2 ,$$

(19)

which is formally equal to the RPA result, apart from the factor $D_{ph}^{1/2}$. Therefore, only the $ph$ components of $F$ enter. This is due to the fact that the excited states are described as superpositions of $ph$ configurations only. Starting from eq.s(3) and using the properties of the $X$ and $Y$ amplitudes, eq.(19) can be written as:
\[
\sum_{\nu} (E_{\nu} - E_0) \left| \langle \Psi_{\nu} | F | \Psi_0 \rangle \right|^2 = \sum_{ph} D_{ph}^{1/2} f_{ph} \sum_{p' h'} D_{p' h'}^{1/2} f_{p' h'} (A_{ph, p'h'} - B_{ph, p'h'}) .
\]

In order to evaluate the r.h.s. of eq. (17) one can use for the commutator \([H, F]\) the same linearization procedure already used in deriving eq.s (11) and (14). It is easy to realize that the result of such calculation cannot be equal to (20) since not only the \(ph\) matrix elements of the residual interaction will appear in it, but also other terms if they are present in the one body operator \(F\). This happens because the expectation value of the double commutator is taken in the correlated ground state \(|\Psi_0\rangle\). We will show this in the next subsection, where an enlarged configuration space, including also \(pp\) and \(hh\) components, will be used to express the excited states. Of course, if the correlations present in \(|\Psi_0\rangle\) are small and the occupation numbers do not differ too much from 0 and 1, the violations of the EWSR are small. But, in general, this is not the case. For example, for Na clusters, the discrepancy was found \([11, 12]\) to be about 25%.

### B. The enlarged space

As shown in the previous subsection, the problem of violations of the EWSR arises because also in IRPA, as in RPA, the excited states are expressed as superpositions of \(ph\) configurations. Let us then consider the more general expansion:

\[
| \bar{\Psi}_{\nu} \rangle = \bar{Q}_{\nu}^\dagger | \bar{\Psi}_0 \rangle = \sum_{\alpha > \beta} (\bar{X}_{\alpha \beta}^\nu B_{\alpha \beta}^\dagger - \bar{Y}_{\alpha \beta}^\nu B_{\alpha \beta}) | \bar{\Psi}_0 \rangle ,
\]

where \(\alpha\) and \(\beta\) stand for any single particle state and \(\alpha > \beta\) means that we order these states according to decreasing occupation numbers, i.e. \(n_\alpha < n_\beta\). The operators \(B_{\alpha \beta}^\dagger\) and \(B_{\alpha \beta}\) are an obvious generalization of eq.s (2) and (3). As before we define \(| \bar{\Psi}_0 \rangle\) as the vacuum of the \(\bar{Q}_{\nu}\) operators:

\[
\bar{Q}_{\nu} | \bar{\Psi}_0 \rangle = 0 .
\]

In order to make simpler the notation, we will omit the bars in the collective operators and in the states, which, of course, are different from those considered in IRPA since now \(pp\) and

\[...\]
hh configurations are included, in addition to the ph ones. The derivation of the equations of motion can be done by following the same linearization procedure as before. They have the same form as in eq.(6), the matrices $A$ and $B$ being now:

$$A_{\alpha\beta,\gamma\delta} = \frac{1}{2}(D_{\alpha\beta}^{1/2}D_{\gamma\delta}^{-1/2} + D_{\gamma\delta}^{1/2}D_{\alpha\beta}^{-1/2})(\epsilon_{\alpha\gamma}\delta_{\beta\delta} - \epsilon_{\beta\delta}\delta_{\alpha\gamma})$$

$$+ D_{\alpha\beta}^{1/2}D_{\gamma\delta}^{1/2}(\beta\gamma|H_2|\alpha\delta)$$  \hspace{1cm} (23)

and

$$B_{\alpha\beta,\gamma\delta} = \frac{1}{2}(D_{\alpha\beta}^{1/2}D_{\gamma\delta}^{-1/2} - D_{\gamma\delta}^{1/2}D_{\alpha\beta}^{-1/2})(\epsilon_{\alpha\delta}\delta_{\beta\gamma} - \epsilon_{\beta\gamma}\delta_{\alpha\delta})$$

$$+ D_{\alpha\beta}^{1/2}D_{\gamma\delta}^{1/2}(\delta\beta|H_2|\gamma\alpha)$$ ,  \hspace{1cm} (24)

where:

$$\epsilon_{\alpha\beta} = (\alpha|H_1|\beta) + \sum_{\gamma} n_{\gamma}(\alpha\gamma|H_2|\beta\gamma) .$$  \hspace{1cm} (25)

Apart from the fact that in eq.s(23) and (24) the indices run over all single particle states, the main difference with eq.s(11) and (14) is the presence of the $\epsilon$ terms also in the $B$ matrix.

Coming back to the EWSR problem, eq.(20) is easily generalized to:

$$\sum_{\nu}(E_{\nu} - E_0)|\langle \Psi_{\nu}|F|\Psi_0 \rangle|^{2} = \sum_{\alpha>\beta} f_{\alpha\beta} D_{\alpha\beta}^{1/2} \sum_{\gamma>\delta} f_{\gamma\delta} D_{\gamma\delta}^{1/2}(A_{\alpha\beta,\gamma\delta} - B_{\alpha\beta,\gamma\delta}) ,$$

which, after some tedious manipulations, can be written as:

$$\sum_{\nu}(E_{\nu} - E_0)|\langle \Psi_{\nu}|F|\Psi_0 \rangle|^{2} = \frac{1}{2} \sum_{\alpha,\beta} f_{\alpha\beta} D_{\alpha\beta} \sum_{\gamma,\delta} f_{\gamma\delta}[\epsilon_{\alpha\gamma}\delta_{\beta\delta} - \epsilon_{\beta\delta}\delta_{\alpha\gamma} + (\beta\gamma|H_2|\alpha\delta)D_{\gamma\delta}] .$$  \hspace{1cm} (27)

The double commutator is easily calculated by using the same linearization procedure adopted to derive the equations of motion. Doing that one realizes that eq.(17) is indeed satisfied. Thus one obtains a kind of generalization of the Thouless theorem. Namely, eq.(17) is satisfied if one calculates its two sides by using the solutions of the equations of motion and by making the same approximations introduced in the derivation of the latters.
In principle the new approach does not appear to be more difficult than IRPA and its equations can be solved in realistic cases by the same iterative procedure used there. In practice, however, the computational effort is much heavier since the configuration space is much larger. For this reason we have decided to apply it to a solvable 3-level model \[19–21\].

We show this application in the next section, where we compare the results of IRPA and of its enlarged version with the exact solutions of the model.

### III. THE MODEL AND THE RESULTS

Let us first of all illustrate the solvable model to which we applied the enlarged version of IRPA.

It consists of three levels, 0, 1 and 2, with energies $\epsilon_0$, $\epsilon_1$ and $\epsilon_2$, respectively. Let $2\Omega$ be the degeneracy of each level and $N = 2\Omega$ the total number of fermions in the system.

We define the operators:

$$ K_{ij} \equiv \sum_m a_{im}^\dagger a_{jm} , $$

where the indices $i$ and $j$ denote one of the three levels and the index $m$ runs over the $2\Omega$ substates of each of them. The operators $K$ satisfy the following commutation relations:

$$ [K_{ij}, K_{kl}] = \delta_{jk} K_{il} - \delta_{il} K_{kj} . $$

They are therefore the generators of the $U(3)$ algebra. The algebra becomes $SU(3)$ if we consider the additional relation,

$$ N = \sum_i K_{ii} , $$

that fixes the total number of particles.

We introduce the hamiltonian for our system as follows:

$$ H = \sum_{i \neq 0} \epsilon_i K_{ii} + V_0 \sum_{i,j \neq 0} K_{i0}K_{0j} + V_1 \sum_{i,j \neq 0} (K_{i0}K_{j0} + K_{0j}K_{0i}) $$

$$ + V_2 \sum_{i,j,k \neq 0} (K_{i0}K_{jk} + K_{kj}K_{0i}) + V_3 \sum_{i,j,k,l \neq 0} K_{ij}K_{kl} . $$

(31)
The terms with the \( V_0 \) and \( V_1 \) strengths describe, respectively, the \( phph \) and \( pphh \) parts of the interaction. The term with the \( V_2 \) strength is related to the \( ppph \) part, while the last term represents the \( pppp \) part. In standard \( RPA \) the only two-body terms of \( H \) that enter in the expressions for the matrices \( A \) and \( B \) of the equations of motion (3) are those with the strengths \( V_0 \) and \( V_1 \), i.e. the \( ph \) two-body terms. In the \( IRPA \) approach \cite{irpa1,irpa2} the ground state that is actually used in the calculations is correlated; so the single particle occupation numbers aren’t strictly 1 for hole states and 0 for particle states, as in \(|HF⟩\). In this case also the \( pppp \) term enters in the expressions for the matrices (actually only in the matrix \( A \)). In the \( IRPA \) approach with the enlarged configuration space all the terms contribute.

The exact results for the system can be obtained either by using the \( SU(3) \) symmetry of the model or by diagonalizing the hamiltonian (31) in the complete set of states:

\[
|n_1n_2⟩ ≡ C(K_{10})^{n_1}(K_{20})^{n_2}|0⟩ ,
\]

where \(|0⟩\) denotes the state in which all the particles are in the level 0, \( n_1 \) and \( n_2 \) are the numbers of particles in the levels 1 and 2, respectively, and \( C \) represents a normalization factor.

With the same set of parameters chosen for the exact calculation, after having performed a standard \( RPA \) calculation, we solved the equations of motion both in the \( IRPA \) approach of \cite{irpa1,irpa2} and in the new approach, with the enlarged configuration space. In the \( IRPA \) case the operators \( Q^\dagger_\nu \) are defined as linear combinations of \( ph \) (\( i0 \), with: \( i \neq 0 \)) and \( hp \) (\( 0i \), with \( i \neq 0 \)) configurations, as in eq.(1):

\[
Q^\dagger_\nu ≡ \sum_i (X^\nu_i K_{i0} - Y^\nu_i K_{0i}) ,
\]

while in the enlarged calculation they are defined as in eq.(21):

\[
Q^\dagger_\nu ≡ \sum_{i>j} (X^\nu_{ij} K_{ij} - Y^\nu_{ij} K_{ji}) ,
\]

where:

\[
\tilde{K}_{ij} ≡ \frac{1}{(2\Omega)^{1/2}} D_{ij}^{1/2} K_{ij} .
\]
With the definition (35) we get, for the excited states of the system, the same orthonormality conditions as given in eq.(3).

Note that in (34) the indices $i$ and $j$ run over all the 3 single particle levels of the model. In both cases (33) and (34) we have solved the non linear problem of eq.s(6) by means of an iterative procedure. We fixed the number of particles $N$ equal to 10. In this case the number of exact eigenstates of the hamiltonian is 66. The $RPA$ and $IRPA$ calculations will give two excited states, since their configuration space is composed only by the two configurations $(1,0)$ and $(2,0)$. The enlarged $IRPA$ will give three states, since its configuration space is made by the three configurations $(1,0)$, $(2,1)$ and $(2,0)$.

We tested various values for the four parameters $V_0$, $V_1$, $V_2$ and $V_3$ and for the energies of the levels, the results being qualitatively the same. In Fig. 1 we show one case, where:

$$\epsilon_0 = 0, \quad \epsilon_1 = \epsilon, \quad \epsilon_2 = 2.5\epsilon, \quad (36)$$

$$V_0 = -\chi, \quad V_1 = \chi, \quad V_2 = \frac{-\chi}{2}, \quad V_3 = \frac{\chi}{10}. \quad (37)$$

Both the $\epsilon$ and $\chi$ parameters have the units of an energy. In the figure the excitation energies are represented versus the increasing values of the strength $\chi$. Dashed lines refer to exact values of energies. Among all the 66 exact eigenvalues the two represented ones are those with energies equal to 1 and 2.5 at $\chi = 0$; i.e. those which correspond to the two $RPA$ and $IRPA$ excited states. Dotted lines refer to $RPA$ and dot-dashed lines to $IRPA$ values. The three values corresponding to the enlarged $IRPA$ approach are represented by full lines.

The collapse point of $RPA$, where its first excitation energy becomes imaginary, appears at $\chi = 0.024$. We observe that both the $IRPA$ and the enlarged $IRPA$ calculations push the collapse point towards greater values of the strength parameter $\chi$; so, in this regard, both methods improve the $RPA$ results. Moreover it can be seen that the two exact values are better approximated by the first and the third states obtained in the enlarged $IRPA$ approach, than by the two $IRPA$ states. This is especially evident for the higher state.

It is interesting to focus the attention on the presence of the additional state that the
enlarged IRPA gives, with respect to RPA and IRPA. Actually this state doesn’t correspond to any of the found exact states. This fact seems to indicate that it is a spurious state. On the other hand its energy is not zero or very small, as it happens normally for spurious states. Its energy always starts, when \( \chi \) starts from zero, from the energy difference between the levels 1 and 2, and so it depends on how we fix the values \( \epsilon_1 \) and \( \epsilon_2 \). This would mean that, if we applied our approach in a realistic calculation, the spurious states that would appear wouldn’t be easily recognized and eliminated, not having in principle zero or very small energies. This could cause problems in the interpretation of the calculated spectrum of excitations. A similar situation is encountered in RPA at finite temperature and was also found in [8]. Let us look at the transition probabilities related to this state. Figure 2 shows the transition probabilities related to the obtained states, for RPA and EIRPA calculations, for different \( \chi \) strengths. Let’s observe in the figure the transition probability related to the spurious state,

\[
P_{sp} = |\langle \Psi_{\nu sp} | F | \Psi_0 \rangle|^2 ,
\]

where |\( \Psi_0 \rangle\) is the ground state, |\( \Psi_{\nu sp} \rangle\) is the spurious state and \( F \) the one body operator [18] with all the \( f_{\alpha\beta} \)'s equal to one. We can see that \( P_{sp} \) is very small, with respect to the other two transition probabilities, only when \( \chi \) is far from the collapse point; when \( \chi \) approaches the collapse point the transition probability (38) becomes appreciable (see the case \( \chi = 0.04 \) in the figure). The same trend is found for other sets of parameters.

This means that in the evaluation of any physical quantity, in a realistic calculation, the existence of spurious states would have some influence and it would be important to recognize and eliminate them from the calculation. The problem of how to recognize them is still open, as they don’t have in general small energies and/or small transition probabilities.

We present now, in table 1, the results obtained for the EWSR, in IRPA and in enlarged IRPA cases. These results refer again to the choice (36) and (37) for the parameters. The violation of the EWSR, in the IRPA case, increases with the increasing values of the \( \chi \) parameter and is about of 30% for \( \chi = 0.03 \). In the enlarged IRPA case the EWSR is
always exactly satisfied, as expected.

This is an important achievement of the present method, because, as stressed in Section 1, all the methods that have been proposed so far in order to go beyond the $RPA$, by avoiding the inconsistency of the Quasi Boson Approximation, always violate the $EWSR$ identity.

IV. CONCLUSIONS

In conclusion, we have presented an extension of $RPA$ which avoids the use of the Quasi Boson Approximation and, at variance with many other attempts made in the same direction, preserves exactly the $EWSR$. This is obtained as a generalization of a previously studied approach by enlarging the configuration space with respect to that commonly used, which contains only particle-hole elementary excitations. The approach has been tested on a 3-level solvable model.

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REFERENCES

[1] D.J. Rowe, *Nuclear Collective Motion*, (Methuen, London, 1970).

[2] P. Ring and P. Schuck, *The Nuclear Many-Body Problem*, (Springer, New York, 1980).

[3] K. Hara, Progr. Theor. Phys. **32** (1964) 88.

[4] D.J. Rowe, Revs. Mod. Phys. **40** (1968) 153.

[5] D. Karadjov, V.V. Voronov and F. Catara, Phys. Lett. **B306** (1993) 197; J. Phys. **G20** (1994) 1431.

[6] J. Toivanen and J. Suhonen, Phys. Rev. Lett. **75** (1995) 410.

[7] J. Suhonen et al., Z. Physik **A358** (1997) 358.

[8] A.A. Raduta et al., Nucl. Phys. **A634** (1998) 497.

[9] P. Schuck and S. Ethofer, Nucl. Phys. **A212** (1973) 269.

[10] P. Schuck, J. Dukelsky and G. Röpke, *Many Body Theory of Correlated Fermion Systems*, ed. J.M. Arias, M.I. Gallardo and M. Lozano (World Scientific, Singapore, 1998), p. 223.

[11] F. Catara, G. Piccitto, M. Sambataro and N. Van Giai, Phys. Rev. **B54** (1996-II) 17536.

[12] F. Catara, M. Grasso, G. Piccitto and M. Sambataro, Phys. Rev. **B58** (1998-II) 16070.

[13] D.J. Rowe, Phys. Rev. **175** (1968) 1283.

[14] G.E. Brown, *Unified Theory of Nuclear Models*, (North Holland, Amsterdam, 1964).

[15] J. Dukelski, and P. Schuck, Phys. Lett. **B387** (1996), 233.

[16] S. Schafer, and P. Schuck Phys. Rev. **B59** (1999-I), 1712.

[17] N.D. Dang and A. Arima, RIKEN preprint (March 2000) and to appear in Phys. Rev. C.
[18] D. J. Thouless, Nucl. Phys. 21 (1960) 225.

[19] S.Y. Li, A. Klein, and R.M. Dreizler, J.Math. Phys. 11, 975 (1970).

[20] G. Holzwarth, and T. Yukawa, Nucl. Phys. A219, 125 (1974).

[21] M. Sambataro, Phys. Rev. C60 (1999), 1.
FIGURES

FIG. 1. Excitation energies versus $\chi$, with the $(36)-(37)$ parameters. The energies in $Y$ axis are expressed in units of $\epsilon$.

FIG. 2. Transition probabilities for four values of $\chi$, with the $(36)-(37)$ parameters. The energies in $X$ axis are expressed in units of $\epsilon$. 
TABLES

| $\chi/\epsilon$ | l.h.s. IRPA | l.h.s. Enl. IRPA | r.h.s.     |
|----------------|-------------|------------------|-----------|
| 0.012          | 1.84957     | 2.1877893606     | 2.1877893605 |
| 0.03           | 0.89411     | 1.3607871867     | 1.3607871868 |
| 0.04           | 0.67858     | 1.5861229231     | 1.5861229230 |

TABLE I. The l.h.s of $EWSR$ in the $IRPA$ and in the enlarged $IRPA$ cases, together with the r.h.s., for different values of $\chi/\epsilon$ and with the parameters (33)-(34).