ON A CONJECTURE OF DUNFIELD, FRIEDL AND JACKSON

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ABSTRACT. In this short Note, we show that the twisted Alexander polynomial associated to a parabolic $SL(2, \mathbb{C})$-representation detects genus and fibering of the twist knots. As a corollary, a conjecture of Dunfield, Friedl and Jackson is proved for the hyperbolic twist knots.

1. INTRODUCTION

Let $K$ be a knot in the 3-sphere $S^3$ and denote its knot group by $G(K)$. That is, $G(K) = \pi_1 E(K)$ where $E(K)$ is the knot exterior $S^3 \setminus \text{int}(N(K))$ which is a compact 3-manifold with torus boundary. For a nonabelian representation $\rho : G(K) \to SL(2, \mathbb{C})$, the twisted Alexander polynomial $\Delta_{K,\rho}(t) \in \mathbb{C}[t^{\pm 1}]$ is defined up to multiplication by some $t^i$, with $i \in \mathbb{Z}$, see [10], [16] and [9] for details.

If $K$ is a hyperbolic knot, namely the interior of $E(K)$ admits the complete hyperbolic metric with finite volume, there is a discrete faithful representation $\rho_0 : G(K) \to SL(2, \mathbb{C})$, which is called the holonomy representation, corresponding to the hyperbolic structure.

The hyperbolic torsion polynomial $T_K(t) \in \mathbb{C}[t^{\pm 1}]$ was defined in [3] for hyperbolic knots as a suitable normalization of $\Delta_{K,\rho_0}(t)$. It is a symmetric polynomial in the sense that $T_K(t^{-1}) = T_K(t)$, which seems to contain geometric information. In fact Dunfield, Friedl and Jackson conjectured in [3] that $T_K$ determines the genus $g(K)$ and moreover, the knot $K$ is fibered if and only if $T_K$ is monic.

They show in [3] that the conjecture holds for all hyperbolic knots with at most 15 crossings. Our main theorem in this note is the following.

Theorem 1.1. For all hyperbolic twist knots $K$ (see Figure 1) $T_K$ determines the genus $g(K)$ and moreover, the knot $K$ is fibered if and only if $T_K$ is monic.

As far as we know, this is the first infinite family of knots for which the conjecture is verified. Since twist knots are 2-bridge knots (in particular alternating knots), their genus and fibering can be detected by the Alexander polynomial (see [2], [12], [13], [14]). However there seems to be no a priori reason that the same must be true for $T_K$. See [3] Section 7], [8] for twisted Alexander polynomials and character varieties of knot groups.

Recall that an $SL(2, \mathbb{C})$-representation $\rho$ is called parabolic if the meridian of $G(K)$ is sent to a parabolic element of $SL(2, \mathbb{C})$ and $\rho(G(K))$ is nonabelian. Since the holonomy representation of hyperbolic knots is parabolic, the above theorem is an immediate consequence of the following:

Theorem 1.2. Let $K$ be a twist knot and $\rho : G(K) \to SL(2, \mathbb{C})$ a parabolic representation. Then $\Delta_{K,\rho}(t)$ determines $g(K)$. Moreover $K$ is fibered if and only if $\Delta_{K,\rho}(t)$ is monic.

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In the next section, we quickly review twisted Alexander polynomials of twist knots for parabolic $SL(2, \mathbb{C})$-representations (see [11], Sections 3, 4) for details. The proof of Theorem 1.2 will be given in Section 3.

2. Twisted Alexander polynomials of twist knots

Let $K = J(\pm 2, p)$ be the twist knot $(p \in \mathbb{Z})$. It is known that $J(\pm 2, 2q + 1)$ is equivalent to $J(\mp 2, 2q)$ and $J(\pm 2, p)$ is the mirror image of $J(\mp 2, -p)$. Hence we only consider the case where $K = J(2, 2q)$ for $q \in \mathbb{Z}$ (see Figure 1). The knot $J(2, 0)$ presents the trivial knot, so that we always assume $q \neq 0$. The typical examples are the trefoil knot $J(2, 2)$ and the figure eight knot $J(2, -2)$.

The twist knots are alternating knots and have genus one. The Alexander polynomial of $K = J(2, 2q)$ is given by $\Delta_K(t) = q - (2q - 1)t + qt^2$. Furthermore, it is known (see [14]) that $J(2, 2q)$ is fibered if and only if $|q| = 1$. It is also known that $J(2, 2q)$ is hyperbolic if $q \not\in \{0, 1\}$.

The knot group $G(J(2, 2q))$ has the presentation:

$$G(J(2, 2q)) = \langle x, y | w^q x = yw^q \rangle,$$

where $w = [y, x^{-1}]$. Suppose that $\rho : G(J(2, 2q)) \to SL(2, \mathbb{C})$ is a parabolic representation. After conjugating, if necessary, we may assume that

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}.$$

Let $\rho(w^q) = (a_{ij}(u))$ and write $\phi_q(u) = a_{11}(u)$. It is known that $\rho$ defines a group representation when $u$ satisfies $\phi_q(u) = 0$ (see [15], Theorem 2). We call $\phi_q(u)$ the Riley polynomial of the twist knot $J(2, 2q)$. By [11] Proposition 3.1, $\phi_q(u)$ has an explicit formula

$$\phi_q(u) = (1 - u) \frac{\lambda_+^q - \lambda_-^q}{\lambda_+ - \lambda_-} - \frac{\lambda_+^{q-1} - \lambda_-^{q-1}}{\lambda_+ - \lambda_-},$$

denote the eigenvalues of the matrix $\rho(w)$. Of course, the holonomy representation $\rho_0$ corresponds to one of the roots of $\phi_q(u) = 0$.

Lemma 2.1. The Riley polynomial $\phi_q(u)$ satisfies the following:
(1) The highest coefficient of \( \phi_q(u) \) is \( \pm 1 \).
(2) \( \phi_q(u) \in \mathbb{Z}[u] \) is irreducible.
(3) \( \deg \phi_q(u) = 2q - 1 \) \((q > 0)\) or \( 2|q| \) \((q < 0)\).

**Proof.** (1) See [15, Theorem 2]. (2), (3) See [7, Theorem 1]. \( \square \)

**Example 2.2.** We can easily check that \( \phi_1(u) = 1 - u, \phi_1(u) = 1 + u + u^2, \phi_2(u) = 1 - 2u + u^2 - u^3, \phi_2(u) = 1 + 2u + 3u^2 + u^3 + u^4 \) and \( \phi_3(u) = 1 - 3u + 3u^2 - 4u^3 + u^4 - u^5 \).

**Lemma 2.3.** For a parabolic representation \( \rho : G(K) \to SL(2, \mathbb{C}) \) of \( K = J(2,2q) \), the twisted Alexander polynomial \( \Delta_{K,\rho}(t) \) is given by

\[
\Delta_{K,\rho}(t) = \alpha \beta + \left\{ \alpha + \beta - 2\alpha \beta + \frac{\lambda_+ - \lambda_-}{2 + \lambda_+ + \lambda_-}(\alpha - \beta) \right\} t + \alpha \beta t^2,
\]

where \( \alpha = 1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{q-1} \) and \( \beta = 1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{q-1} \).

**Proof.** We only have to put \( s = 1 \) in the formula of [11, Theorem 4.1]. \( \square \)

**Example 2.4.** For \( K = J(2,2) \), there is just one parabolic representation up to conjugation and we have \( \Delta_{K,\rho}(t) = 1 + t^2 \). Similarly we obtain \( \Delta_{K,\rho}(t) = 1 - 4t + t^2 \) for any parabolic representation of \( K = J(2, -2) \).

In general, the degree of the twisted Alexander polynomial gives a lower bound for the knot genus \( g(K) \). In fact, for every nonabelian representation \( \rho : G(K) \to SL(2, \mathbb{C}) \), the following inequality holds (see [4]):

\[(2.1) \quad 4g(K) - 2 \geq \deg \Delta_{K,\rho}(t).\]

When the equality holds in \((2.1)\), we say \( \Delta_{K,\rho}(t) \) determines the knot genus. For a fibered knot \( K \), it is known that \( \Delta_{K,\rho}(t) \) determines \( g(K) \) and is a monic polynomial (see [11, 4, 5, 6, 9]).

3. **Proof of Theorem [1.2]**

First we denote the highest coefficient of \( \Delta_{K,\rho}(t) \) in Lemma 2.3 by \( \gamma_q(u) \), namely \( \gamma_q(u) = \alpha \beta \). Moreover we put \( \tau_q(u) = tr \rho(w^q) = \lambda_+^q + \lambda_-^q \). By [11, Corollary 4.1], \( \tau_q(u) = \tau_{-q}(u) \) is a monic polynomial in \( \mathbb{Z}[u] \) and \( \deg \tau_q(u) = 2|q| \).

**Example 3.1.** Since \( \tau_1(u) = u^2 + 2 \), we obtain \( \tau_{1\pm 2}(u) = \tau_1^2 - 2 = u^4 + 4u^2 + 2 \) and \( \tau_{1\pm 3}(u) = \tau_1^3 - 3\tau_1 = u^6 + 6u^4 + 9u^2 + 2 \).

Now an easy calculation shows that

\[
\gamma_q(u) = (1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{q-1})(1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{q-1})
\]

\[
= \tau_{q-1}(u) + (\text{some polynomial in } \tau_1, \ldots, \tau_{q-2}).
\]

Thus we have \( \deg \gamma_q(u) = 2|q| - 2 \). By Lemma [2.1](1), (2), if \( \gamma_q(u) = 0 \) for a complex number \( u \) satisfying \( \phi_q(u) = 0 \), then the Riley polynomial \( \phi_q(u) \) divides \( \gamma_q(u) \). But this contradicts the fact that

\[
\deg \phi_q(u) = 2|q| - \max\{\text{sign}(q), 0\} > 2|q| - 2 = \deg \gamma_q(u).
\]

Hence \( \gamma_q(u) \) never vanishes for the parabolic representations. Thus \( \deg \Delta_{K,\rho}(t) = 2 \) and hence it determines the genus.
A similar argument applied to $\gamma_q(u) - 1$ shows that $\Delta_{K,\rho}(t)$ is not a monic polynomial for the nonfibered twist knot $K = J(2, 2q)$ with $|q| > 1$. This completes the proof of Theorem 1.2.

**Remark 3.2.** For the 3830 nonfibered 2-bridge knots $K(a, b)$ with $b < a \leq 287$, Dunfield, Friedl and Jackson numerically compute the twisted Alexander polynomials for the parabolic representations. In fact, it is shown in [3] Section 7.6 that $\Delta_{K,\rho}(t)$ is nonmonic and determines the knot genus in every case.

**Remark 3.3.** Let $\delta_q(u)$ be the second coefficient of $\Delta_{K,\rho}(t)$ in Lemma 2.3. As we saw in Example 2.4, $\delta_{\pm 1}(u)$ are integers for the fibered twist knots $J(2, \pm 2)$. It is not so hard to show that $\delta_q(u) \in \mathbb{Z}[u]$ and $\deg \delta_q(u) = 2q - 4$ for $q > 1$. Therefore we can conclude that $\delta_q(u)$ with $q > 2$ is not a rational number for the parabolic representations. On the other hand, we have $\Delta_{K,\rho}(t) = (u^2 + 4) - 4t + (u^2 + 4)t^2$ for the hyperbolic twist knot $K = J(2, 4)$. In particular, we see that the second coefficient $\delta_2(u)$ is an integer for the holonomy representation, although $J(2, 4)$ is nonfibered (see [3] Section 6.5).

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