Chaos control applied to coherent states in transitional flows

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Abstract. Chaos control refers to a group of techniques by which an otherwise unstable dynamical state of a system can be maintained by small control forces. We here discuss their application to stabilizing the fixed points in a low dimensional model for shear flows. The simulations demonstrate a prototypical application of chaos control, show that control is almost always possible, and give insights into optimizing the control matrix from a design point of view.

1. Introduction

Chaotic dynamical systems come with a huge inventory of possible periodic and aperiodic motions. Typical trajectories within the attractor show a sensitive dependence on initial conditions, i.e. small deviations from the trajectories are amplified exponentially. The techniques developed within chaos control remove the amplification and allow to stabilize, in theory, any trajectory [1, 2]. Since the trajectories are part of the natural motions of the dynamical systems, the control forces vanish on the trajectory, so that control is achieved at minimal costs in the control forces.

Applications of these ideas to hydrodynamic flows include internal flows like Taylor-Couette [3] and wakes behind bluff bodies, where oscillations, rotations and other means are used to manipulate the shedding of vortices [4, 5, 6]. Similarly, the motion of particles around the cylinders can be influenced, thereby changing the trapping or release of particles in the wake [7, 8]. These ideas can extend the techniques available for controlling flows [9, 10, 11].

The application of ideas from chaos control which we consider here builds on the recent progress in transitional flows [12, 13, 14]. In pipe flow and plane Couette flow where turbulence appears without a linear instability of the laminar profile the transition is connected with the formation of a chaotic saddle in the state space of the system. The building blocks of the saddle are three-dimensional coherent structures that appear in Takens-Bogdanov bifurcations [15]. They typically are unstable, but appear transiently during the evolution of the flow [16, 17]. Applying the methods from chaos control should then make it possible to trap the flow on these states when the flow gets sufficiently close to them.

In section 2 we describe the basic ideas and formalism. In section 3 we explain the flow and the state that will be stabilized, together with the demonstration of chaos control. In section 4 we describe ideas that allow to reduce the dimension of the control matrix so that fewer modes need to be monitored and controlled. We conclude with a brief summary in section 5.
2. Basic elements of chaos control

Transitional flows are dominated by downstream vortices and streaks, which have been tracked in the form of travelling waves. In plane Couette flow and in low-dimensional models thereof, the states can be stationary because of the up-down symmetry [18, 19, 20, 21, 22].

Expanding the velocity field in terms of incompressible modes \( u_i(x, y, z) \) with amplitudes \( x_i(t) \), the Navier-Stokes equation then defines a dynamical system

\[
\dot{x} = f(x)
\]

(1)

where the functions \( f_i \) are nonlinear functions of the coefficients. We consider the situation where the fixed point \( x_T \) is known beforehand, say from explicit numerical solutions of the fixed point equation \( f(x_T) = 0 \). A linear control scheme [1, 2], appropriate when the trajectories are sufficiently close to the fixed point, then acts with a matrix \( A \) on the difference to the fixed point, so that the full system including the control becomes

\[
\dot{x} = f(x, t) + aA(x - x_T)
\]

(2)

In this representation an overall strength factor \( a \) has been pulled out of the matrix.

Linearizing about the fixed point we find, with \( Df \) the matrix of derivatives,

\[
\dot{x} = (Df(x_T) + aA)(x - x_T).
\]

(3)

The fixed point is stabilized if \( A \) and \( a \) are chosen such that the eigenvalues of \( (Df(x_T) + aA) \) are all stable. Introducing the set of eigenvalues \( \lambda_i \) and the associated left and right eigenvectors \( |r_i\rangle \) and \( \langle l_i| \), respectively, the matrix \( A \) can be constructed by projecting onto the unstable eigendirections only, i.e.

\[
A = \sum_{\lambda_i > 0} |r_i\rangle \langle l_i|
\]

(4)

and taking a parameter \( a \) such that

\[
a < -\max_{\lambda_i > 0} \lambda_i
\]

(5)

all eigenvalues are negative and the fixed point is stable. More negative \( a \) mean stronger damping and hence faster stabilization in the subspaces affected by \( A \). Since convergence to the fixed point is controlled by all eigenvalues, the initially unstable direction becomes less relevant and convergence is dominated by the largest of the remaining eigenvalues. With appropriate extensions of \( A \) also these eigenvalues can be influenced and the rate of convergence increased. However, as we will discuss in section 4, the number of entries in \( A \) is a stronger limiting factor than the eigenvalues.

3. Application to a 9-d model for shear flows

Following the ideas outlined in Waleffe [23] and the insights obtained from a Fourier mode model [24], a 9-dimensional model for shear flow turbulence was proposed and studied in [25, 26]. The geometry considered is a parallel shear flow, bounded by two free-slip walls and driven by a sinusoidal velocity profile. Fourier modes provide an efficient and useful representation of the flow. The 9 modes contained in the model are: the basic profile (mode 1) and its deformation (mode 9), the streaks (mode 2) and the vortices (mode 3), two transverse modes (mode 4 and 5), two normal vortices (mode 6 and 7) and a three-dimensional mode that couples to all modes (mode 8). The amplitudes \( x_i \) of these nine velocity fields then satisfy equations of the form

\[
\frac{dx_i}{dt} = \frac{\beta^2}{Re} \delta_{i1} - \frac{d_i}{Re} x_i + \sum_{j,k} N_{i,jk} x_j x_k
\]

(6)
Figure 1. Parallel shear flow with free-slip bounding surfaces (left) and bifurcation diagram (right) for the stationary points.

where the forcing $\delta_{i,1}$ drives the laminar shear profile, the terms $-d_{i}\frac{d}{Re}$ are the viscous damping rates and the $N$'s are the nonlinear couplings which are bilinear in the coefficients.

As regards the spatial degrees of freedom, we fix the height at $d = 2$ and consider the Nagata-Busse-Clever domain of length $L_x = 2\pi d$ and width $L_z = \pi d$. The system can be shown to have a saddle-node bifurcation at Reynolds number $Re = 308.16$, at which point two sets of fixed points are created each of which consists of four symmetry related fixed points. They are the only ones up to Reynolds number of about 1000 [27]. For the calculations shown here $Re = 400$.

Figure 2. The velocity field of the fixed point. Left: velocity field averaged in the streamwise direction in order to highlight the vortices. Right: velocity field in the midplane, i.e. for $y = 0$. In both cases in-plane velocity components are indicated by arrows, out of plane components in color.

The fixed points are associated to streamwise vortices and wavy streaks in the flow, as shown in Fig. 2. The upper branch solution has three unstable modes (one real with eigenvalue 0.021 and two complex conjugate ones with eigenvalues $0.006 \pm 0.021$), the lower branch solution has an additional real positive eigenvalues. Accordingly, the control matrix for the upper branch solution is built of the three left and right eigenvectors corresponding to the unstable directions. The complex eigendirections for the complex eigenvalues add up to real entries (alternatively, the complex entries can be avoided by working with 2-d real normal forms). For $Re = 400$ and
the upper branch solution, this yields the control matrix

\[
A = \begin{pmatrix}
0.3869 & 0.1340 & 0.6908 & -0.1383 & 0.2857 & -0.0967 & -0.5586 & -1.1141 & 0.1316 \\
-0.1268 & 0.0200 & -0.1007 & 0.3119 & -0.2385 & 0.0933 & -0.1224 & 1.1886 & 0.0027 \\
0.0010 & -0.0143 & 0.1859 & -0.0889 & 0.3308 & 0.0849 & 0.1481 & 0.4145 & 0.0142 \\
0.0544 & 0.0087 & 0.1709 & -0.0740 & 0.1940 & 0.0203 & 0.0052 & 0.0210 & 0.0223 \\
0.1914 & 0.0296 & 0.5961 & -0.2638 & 0.6801 & 0.0689 & 0.0213 & 0.0950 & 0.0774 \\
-0.0908 & -0.0150 & -0.0894 & 0.0959 & -0.0482 & 0.0573 & 0.0674 & 0.5880 & 0.0161 \\
-0.2148 & -0.1758 & -0.3383 & -0.3780 & 0.4126 & 0.0699 & 0.8865 & 0.0075 & -0.1106 \\
-0.0537 & 0.0163 & 0.0538 & 0.1544 & -0.0058 & 0.0849 & -0.0590 & 0.8354 & 0.0126 \\
0.1348 & 0.0045 & 0.1840 & -0.2274 & 0.2317 & -0.0621 & 0.0168 & -0.8568 & 0.0220
\end{pmatrix}
\]

The control has been tested with many initial conditions \( x(t = 0) = x_F \pm \epsilon y \), where \( y \) is a vector of random numbers between zero and one, and for different parameters \( a \) and \( \epsilon \). Table 1 shows the percentage of random initial conditions which are at the fixed point, i.e. \( x(t) - x_F < 10^{-8} \), after \( t \) time steps for \( a = -0.1 \) (which will move the most unstable eigenvalue from +0.021 to −0.079 and thus well below the other eigenvalues) and \( Re = 400 \).

The time series of the first two components of \( x(t) \) with and without control are shown in figure 3 and 4 for \( \epsilon = 0.1 \) and \( \epsilon = 1 \), respectively. The time evolution of the control force is shown in figure 5 as a semi-log plot for both examples. As expected it tends to zero when the trajectory approaches the fixed point.

4. Reduced control matrices

Since the eigenvectors are not aligned with the basis vectors, the control matrices \( A \) typically are fully occupied, even if only a single unstable direction has to be controlled. The consequences for practical applications are that all modes have to be monitored and all modes have to be controlled. Therefore, it is of interest to find control matrices with a smaller number of non-zero elements. As an example, found by trial and error, stabilization has been achieved with the matrix

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.1859 & 0 & 0.3308 & 0 & 0.1481 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5961 & 0 & 0.6801 & 0 & 0.0213 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.3383 & 0 & 0.4126 & 0 & 0.8865 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and \( a = -0.1 \). With this reduced matrix it is only necessary to measure and to influence three components: the vortices, one transverse flow and one normal vortex mode. The control still
Figure 3. Time series of the first two components of $x(t)$ for $Re = 400$ and $\epsilon = 10^{-1}$ (blue lines). The red lines indicate the values of the fixed point. The time series without control ($a = 0$) and with control ($a = -0.1$) are shown in the left and right columns, respectively.

| $\epsilon$ | 1800 | 1900 | 2000 | 2100 | 2200 | 2300 | 2400 | 2500 |
|------------|------|------|------|------|------|------|------|------|
| 0.1        | 4.1  | 14.4 | 16.3 | 50.1 | 92.8 | 100.0|      |      |
| 1.0        | 0.3  | 1.3  | 5.4  | 21.4 | 54.2 | 84.2 | 90.6 | 95.6 |

Table 2. Percentage of initial conditions which arrive at the fixed point after $t$ time steps for control with the reduced matrix.

works, as documented in Table 2.

The more general question of how small a matrix suffices to control the system does not allow for an immediate and complete solution. Within linear perturbation theory one can calculate for each eigenvalue the rate at which it changes, i.e.

$$\lambda_i(a) \approx \lambda_i(a = 0) + a \langle i | A | i \rangle.$$  \hfill (9)

Fixing a subset of matrix elements, the aim is to find entries for the matrix so that the eigenvalue with the largest real is negative and as strongly damped as possible. For a given set of coefficients, this can be determined by linear programming (e.g. the simplex method) [28].
Figure 4. Similar to Fig. 3 but for larger initial deviations from the fixed point, $\epsilon = 10^0$.

Figure 5. Time series of the Euclidean norm of the control force for the small perturbation from Fig. 3 (left frame) and the larger one from Fig. 4 (right frame).
Figure 6. Variation of the real parts of the eigenvalues over a larger range in $a$ (left) and near $a = 1$ (right) for the control with three non-zero matrix elements. The parabolic segments arise near a coalescence of two real eigenvalues and the formation of a pair of complex conjugate ones.

In order to take into account that the perturbations may be so large that linear perturbation theory for the eigenvalues does not apply any more, we turn to a stochastic optimization procedure: It consist in repeatedly taking a matrix element in $A$, calculating the eigenvalues for about 1000 values in the interval $[-1.3, 1.3]$ and then selecting the value for which the maximal eigenvalue is smallest. The selection of the next matrix element is not restricted to the matrix elements that have not be optimized before: we therefore have the opportunity to optimize some matrix elements several times. The total number repeats of this loop defines the maximal number of matrix elements considered. Moreover, we excluded mode number 8 because of its complex spatial structure and mode number 9, the deformation of the basis profile, because it is a consequence of the turbulent modes.

This procedure gives matrices with very few non-zero matrix elements that allow to stabilize the system sufficiently close to the fixed point. Unexpectedly, we also find that for larger perturbations the system has other fixed points, different from the original one. For them the control, of course, does not vanish. In order to avoid these spurious solutions one therefore has to limit the control to trajectories that get sufficiently close.

The smallest matrix found has only two entries, $A(7,3) = -0.427$ and $A(4,7) = 0.014$. However, the neighborhood that is stabilized has a radius of $10^{-3}$ only. As a function of the parameter $a$, the eigenvalues change as shown in Figure 6. Since the matrix is obtained by sequentially optimizing two matrix elements, it is not optimal in a global sense and can in principle be improved further with a combined optimization in both matrix elements.

In another example, stabilization was found for a matrix with only three entries: $A(2,2) = -0.3$, $A(3,3) = -0.48$ and $A(6,1) = 0.65$. It requires to measure and to manipulate the vortex and the streak, and to change a normal vortex in response to the downstream velocity. Scaling the matrix with the parameter $a$ changes the eigenvalues in the manner shown in Figure 6. One notices that at the optimal point two sets of eigenvalues cross: this was observed in other cases as well and seems to be a generic feature.

5. Conclusions
The study of chaos control in this low-dimensional setting has revealed a few promising features. First, it is interesting to see that even though the control is based on a study of the linearized equations, it extends to a large part of the state space, and essentially all initial conditions are attracted towards this fixed point if the full matrix $A$ is used. Secondly, it is possible to achieve
\[
\begin{array}{lcccccc}
\varepsilon/\gamma & 1500 & 1600 & 1700 & 1800 & 1900 \\
0.1 & 20.3 & 67.6 & 88.0 & 99.8 & 100 \\
\end{array}
\]

**Table 3.** Percentage of initial conditions which arrive at the fixed point after \( t \) time steps for control with the reduced matrix with only three elements. For larger initial amplitudes spurious fixed points appear and the control has to be restricted to the immediate neighborhood of the fixed point.

stabilization with a reduced number of entries in the matrix \( A \). However, very often the system develops spurious fixed points and the control has to be limited to the neighborhood of the fixed point. Thirdly, a stochastic search for matrices with a smaller number of non-zero entries is feasible and gives control matrices with as few as two entries. However, in some cases the fraction of state space in which control is achieved is rather small, and additional spurious fixed points of the full control system can appear.

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