ON TOPOLOGICAL UPPER-BOUNDS ON THE NUMBER OF SMALL CUSPIDAL EIGENVALUES

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Abstract. Let $S$ be a noncompact, finite area hyperbolic surface of type $(g,n)$. Let $\Delta_S$ denote the Laplace operator on $S$. As $S$ varies over the moduli space $M_{g,n}$ of finite area hyperbolic surfaces of type $(g,n)$, we study, adapting methods of Lizhen Ji [Ji] and Scott Wolpert [Wo], the behavior of small cuspidal eigenpairs of $\Delta_S$. In Theorem 2 we describe limiting behavior of these eigenpairs on surfaces $S_m \in M_{g,n}$ when $(S_m)$ converges to a point in $M_{g,n}$. Then we consider the $i$-th cuspidal eigenvalue, $\lambda_i(S)$, of $S \in M_{g,n}$. Since non-cuspidal eigenfunctions (residual eigenfunctions or generalized eigenfunctions) may converge to cuspidal eigenfunctions, it is not known if $\lambda_i(S)$ is a continuous function. However, applying Theorem 2 we prove that, for all $k \geq 2g-2$, the sets

$$C_{g,n}^k = \{ S \in M_{g,n} : \lambda_k(S) > \frac{1}{4} \}$$

are open and contain a neighborhood of $\bigcup_{i=1}^{n_1} M_{g,3} \cup M_{g-1,2}$ in $M_{g,n}$. Moreover, using topological properties of nodal sets of small eigenfunctions from [O], we show that $C_{g,n}^{2g-1}$ contains a neighborhood of $M_{0,n+1} \cup M_{g,1}$ in $M_{g,n}$. These results provide evidence in support of a conjecture of Otal-Rosas [O-R].

1. Introduction

In this paper a hyperbolic surface is a two dimensional complete Riemannian manifold $S$ with sectional curvature equal to $-1$. Such a surface is isomorphic to the quotient $\mathbb{H}/\Gamma$, of the Poincaré upper halfplane $\mathbb{H}$ by a Fuchsian group $\Gamma$, i.e. a discrete torsion-free subgroup of $\text{PSL}(2,\mathbb{R})$. The Laplace operator on $\mathbb{H}$ is the differential operator which associates to a $C^2$-function $f$ the function

$$\Delta f(z) = y^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

Since the action of $\text{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ leaves $\Delta$ invariant, $\Delta$ induces a differential operator on $S = \mathbb{H}/\Gamma$ which extends to a self-adjoint operator $\Delta_S$ densely defined on $L^2(S)$. It is a general fact that the Laplace operator is a non-positive operator whose spectrum is contained in the smallest interval $(-\infty, -\lambda_0(S)] \subset \mathbb{R}^-$ with $\lambda_0(S) \geq 0$.

Definition 1.1. Let $\lambda > 0$ be a real number and $f \in L^2(S)$ be a nonzero function on $S$. The pair $(\lambda, f)$ is called an eigenpair of $S$ if $\Delta_S f + \lambda f \equiv 0$ on $S$ where $\lambda$ and $f$ are respectively called an eigenvalue and an eigenfunction.

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(sometimes a $\lambda$-eigenfunction). When $0 < \lambda \leq 1/4$, we add the adjective small i.e. $(\lambda, f)$, $\lambda$ and $f$ are respectively called a small eigenpair, a small eigenvalue and a small eigenfunction.

We begin with a noncompact, finite area hyperbolic surface $S$ of type $(g,n)$ i.e. $S \in \mathcal{M}_{g,n}$. The Laplace spectrum of such a surface is composed of two parts: the discrete part and the continuous part $\mathcal{L}$. The continuous part covers the interval $[\frac{1}{2}, \infty)$ and is spanned by Eisenstein series with multiplicity $n$. Eisenstein series are not eigenfunctions although they satisfy

$$\Delta E(.,s) + s(1-s)E(.,s) = 0,$$

because they are not in $L^2$. For this reason, they are called generalized eigenfunctions. The discrete spectrum consists of eigenvalues. They are distinguished into two parts: the residual spectrum and the cuspidal spectrum. An eigenpair $(\lambda, f)$ is called residual if $f$ is a linear combination of residues of meromorphic continuations of Eisenstein series. Such $\lambda$ and $f$ are respectively called a residual eigenvalue and a residual eigenfunction. The residual spectrum is a finite set contained in $[0, \frac{1}{2})$. On the other hand, an eigenpair $(\lambda, f)$ is called cuspidal if $f$ tends to zero at each cusp. In this case $\lambda$ and $f$ are respectively called a cuspidal eigenvalue and a cuspidal eigenfunction. These eigenvalues with multiplicity are arranged by increasing order and we are respectively called a small cuspidal eigenvalue and a small cuspidal eigenfunction.

In $[O-R]$, Jean-Pierre Otal and Eulalio Rosas proved that the total number of small eigenvalues of any hyperbolic surface of type $(g,n)$ is at most $2g - 3 + n$. In the same paper they formulate the following:

**Conjecture.** Let $S$ be a noncompact, finite area hyperbolic surface of type $(g,n)$. Then $\lambda^2_{2g-2}(S) > \frac{5}{4}$.

This conjecture is motivated by the following two results

**Proposition 1.2. (Huxley [Hu], Otal [O])** Let $S$ be a finite area hyperbolic surface of genus 0 or 1. Then $S$ does not carry any small cuspidal eigenpair.

**Proposition 1.3. (Otal [O])** Let $S$ be a finite area hyperbolic surface of type $(g,n)$. Then the multiplicity of a small cuspidal eigenvalue of $S$ is at most $2g - 3$.

The set $\mathcal{M}_{g,n}$ carries a topology for which two surfaces $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Gamma'$ are close when the groups $\Gamma$ and $\Gamma'$ can be conjugated inside $\text{PSL}(2,\mathbb{R})$ so that they have generators which are close. With this topology $\mathcal{M}_{g,n}$ is not compact. However it can be compactified by adjoining $\cup_i \mathcal{M}_{g_i,n_i}$'s for each $(g_1, n_1), \ldots, (g_k, n_k)$ with $2\sum_{i}^k (g_i - 2) + \sum_{i}^k n_i = 2g - 2 + n$. In this compactification a sequence $(S_m) \in \mathcal{M}_{g,n}$ converges to $S_\infty \in \overline{\mathcal{M}}_{g,n}$ if and only if for any given $\epsilon > 0$ the $\epsilon$-thick part $(S_m]^{[\epsilon, \infty)}$ converges to $S_\infty]^{[\epsilon, \infty)}$ in the Gromov-Hausdorff topology. Recall that the $\epsilon$-thick part of a surface $S$ is the subset of those points of $S$ where the injectivity radius is at least $\epsilon$. 


Recall also that the injectivity radius of a point $p \in S$ is the radius of the largest geodesic disc that can be embedded in $S$ with center $p$.

For any $N \in \mathbb{N}$ and $t \in \mathbb{R}_{>0}$ we define the sets

$$C_{g,n}^t(N) = \{ S \in \mathcal{M}_{g,n} : \lambda_N^t(S) > t \}.$$ 

It is clear that $C_{g,n}^1(k) \subset C_{g,n}^1(k + 1)$ for $k \geq 1$. With this notation the conjecture can be formulated by saying that

$$C_{g,n}^1(2g - 2) = \mathcal{M}_{g,n}.$$ 

In this paper, we study the sets $C_{g,n}^1(k)$. The methods developed here are not sufficient to prove the conjecture but we show that the sets $C_{g,n}^1(2g - 2)$ and $C_{g,n}^1(2g - 1)$ ($C_{g,n}^1(2g - 2) \subseteq C_{g,n}^1(2g - 1)$) contains neighborhoods of certain strata in the compactification of $\mathcal{M}_{g,n}$.

**Theorem 1.4.** (i) For any integer $k$, $C_{g,n}^1(k)$ is an open subset of $\mathcal{M}_{g,n}$.

(ii) $C_{g,n}^1(2g - 2)$ contains a neighborhood of $\bigcup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}$ in $\overline{\mathcal{M}_{g,n}}$.

(iii) $C_{g,n}^1(2g - 1)$ contains a neighborhood of $\mathcal{M}_{0,n+1} \cup \mathcal{M}_{g,1}$ in $\overline{\mathcal{M}_{g,n}}$.

Observe that it is theoretically possible for a residual eigenfunction to converge to a cuspidal eigenfunction. Therefore indicating that $\lambda_{2g-1}^c$ may not be continuous. Also, the result [P-S] suggest that $\lambda_{2g-1}^c$ may not be continuous at those $S \in \mathcal{M}_{g,n}$ where it takes value strictly more than $\frac{1}{4}$. Therefore, the first assertion is not completely trivial.

The paper is organized as follows. In §1 we recall some preliminaries for convergence of hyperbolic surfaces in $\overline{\mathcal{M}_{g,n}}$. In §2 and §3 we study convergence properties of eigenpairs on converging hyperbolic surfaces. Similar study has already been carried out by Scott Wolpert [Wo], Lizhen Ji [Ji] and Christopher Judge [J]. We shall first make precise the notions of convergence in $\overline{\mathcal{M}_{g,n}}$ and the notion of convergence of a sequence of functions on a converging sequence of surfaces.

1.1. **Convergence of functions.** Let $(S_m)$ be a sequence of surfaces in $\mathcal{M}_{g,n}$ converging to a surface $S_\infty$ in the compactification $\overline{\mathcal{M}_{g,n}}$. Another way of understanding this convergence is as follows:

Let $S_m = \mathbb{H}/\Gamma_m$ and let $0 < c_0 < \epsilon_0$ ($\epsilon_0$ is the Margulis constant; see thick/thin decomposition for details) be a fixed constant. Let $x_m \in S_m^{[c_0, \infty)}$. Up to a conjugation of $\Gamma_m$ in PSL$(2, \mathbb{R})$, one may assume that $i \in \mathbb{H}$ is mapped to $x_m$ under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$. Then up to extracting a subsequence we may suppose that $\Gamma_m$ converges to some Fuchsian group $\Gamma_\infty$. We say that the pair $(\mathbb{H}/\Gamma_m, x_m)$ converges to $(\mathbb{H}/\Gamma_\infty, x_\infty)$ where $x_\infty$ is the image of $i \in \mathbb{H}$ under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_\infty$. Let $S_\infty$ be the hyperbolic surface of finite area whose connected components are the $\mathbb{H}/\Gamma_\infty$’s for different choices of base point $x_m$ in different connected components of $S_m^{[c_0, \infty)}$. The surface $S_\infty$ does not depend, up to isometry, on the choice of the base point $x_m$ in a fixed connected component of $S_m^{[c_0, \infty)}$ (i.e. if $y_m$ be
a point in the same connected component of $S_m^{[c_0, \infty)}$ as $x_m$ then the corresponding limiting surfaces are isometric). One can check that $(S_m) \to S_\infty$ in $\overline{M}_{g,n}$.

**Convergence of functions**

Fix an $\epsilon > 0$ and choose a base point $x_m \in S_m^{[\epsilon, \infty)}$ for each $m$. Assume that the pair $(\mathbb{H} / \Gamma_m, x_m)$ converges to $(\mathbb{H} / \Gamma_\infty, x_\infty)$ where, for each $m \in \mathbb{N} \cup \{\infty\}$, the point $i \in \mathbb{H}$ maps to $x_m$ under the projection $\mathbb{H} \to \mathbb{H} / \Gamma_m$.

For a $C^\infty$ function $f$ on $S_m$ denote by $\tilde{f}$ the lift of $f$ under the projection $\mathbb{H} \to \mathbb{H} / \Gamma_m$. Let $(f_m)$ be a sequence of functions in $C^\infty(S_m) \cap L^2(S_m)$. One says that $(f_m)$ converges to a continuous function $f_\infty$ if $\tilde{f}_m$ converges, uniformly over compact subsets of $\mathbb{H}$, to $\tilde{f}_\infty$ for each choice of base points $x_m \in S_m^{[\epsilon, \infty)}$ and for each $\epsilon < c_0$.

With the above understanding of convergence of functions we shall prove the following theorem which has close resemblance with [Hi. Theorem 1.2] and [Wo] Theorem 4.2. However, our result does not follow from these. We would like to mention that a similar limiting theorem might not be true (see [Wo] p-71) if one considers $\lambda_m \geq \frac{1}{4}$ instead of $\lambda_m \leq \frac{1}{4}$ (see Theorem 2).

In the following, for a function $f \in L^2(S)$, we shall denote the $L^2$ norm of $f$ by $\|f\|$. Also, for $f \in L^2(V)$ and $U \subset V$ we denote the $L^2$-norm of the restriction of $f$ to $U$ by $\|f\|_U$. A function $f \in L^2(V)$ will be called normalized if $\|f\| = 1$. An eigenpair $(\lambda, \phi)$ will be called normalized if $\phi$ is normalized.

**Theorem 1.5.** Let $S_m \to S_\infty$ in $\overline{M}_{g,n}$. Let $(\lambda_m, \phi_m)$ be a normalized small cuspidal eigenpair of $S_m$. Assume that $\lambda_m$ converges to $\lambda_\infty$. Then one of the following holds:

1. There exist strictly positive constants $\epsilon, \delta$ such that $\lim \sup \|\phi_m\|_{S_m^{[\epsilon, \infty)}} \geq \delta$. Then, up to extracting a subsequence, $(\phi_m)$ converges to a $\lambda_\infty$-eigenfunction $\phi_\infty$ of $S_\infty$.

2. For each $\epsilon > 0$ the sequence $(\|\phi_m\|_{S_m^{[\epsilon, \infty)}})$ converges to 0. Then $S_\infty \in \partial M_{g,n}$ and $\lambda_\infty = \frac{1}{2}$. Moreover, there exist constants $K_m \to \infty$ such that, up to extracting a subsequence, $(K_m \phi_m)$ converges to a linear combination of Eisenstein series and (possibly) a cuspidal $\lambda_\infty$-eigenfunction of $S_\infty$.

**Remark 1.6.** For $s = \frac{1}{2}$, by Eisenstein series we understand a linear combination of the following two:

1. the classical (meromorphic continuation) Eisenstein series $E^i(., \frac{1}{2})$ corresponding to the cusps ($i$ is the index for cusps) on the surface,
2. the derivatives $\frac{d^k}{ds}E^i(., s)|_{s=\frac{1}{2}}$ of $E^i(., s)$ at $s = \frac{1}{2}$.

The first Fourier coefficient of such functions in any cusp have the form $\alpha y^{\frac{3}{2}} + \beta y^{\frac{2}{2}} \log y$. Each moderate growth $\frac{3}{2}$-eigenfunction is a linear combination of Eisenstein series, in the above sense, and (possibly) a cuspidal eigenfunction.

Theorem 2 will be applied to prove all three statements of Theorem 1. The first one is a direct application; in §4 we prove:

**Lemma 1** For any $k \geq 1$, $C_{g,n}(k)$ is an open subset of $M_{g,n}$.
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The second statement of Theorem 1 is also an easy application of Theorem 2 and the Buser construction [Bu]: we explain it now since the proof is short. We argue by contradiction and assume that there is a sequence \((S_m)\) in \(\mathcal{M}_{g,n}\) such that \(S_m\) converges to \(S_\infty \in \bigcup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}\) and \(\lambda_{2g-2}(S_m) \leq \frac{1}{4}\). Then \(S_\infty\) has exactly \(n + 1\) components of which exactly \(n\) are thrice punctured spheres. Observe that each component of \(S_\infty\) contains an old cusp i.e. cusps of \(S_\infty\) which are limits of cusps of \(S_m\) (see Proof of Theorem 2).

The construction used in the proof of [Bu] Theorem 8.1.3 implies that, for \(m\) large, \(S_m\) has at least \(n\) eigenvalues that converge to zero as \(m\) tends to infinity. Let us suppose by contradiction that one of the corresponding eigenfunctions \(\phi_m\) is cuspidal. Then by Theorem 2, \(\phi_m\) converges uniformly over compacta to a function \(\phi\) and \(\phi\) is an eigenfunction for the eigenvalue 0. So \(\phi\) is constant in each component of \(S_\infty\). On those components of \(S_\infty\), it contains an old cusp \(\phi\) is necessarily zero because \(\phi_m\) being cuspidal the average of \(\phi_m\) over any horocycle is zero. On the other component (the one that does not contain an old cusp) \(\phi\) is zero because the mean of \(\phi\) over \(S_\infty\) is equal to the mean of \(\phi_m\) over \(S_m\) which is zero (follows from Theorem 3.36). Therefore, \(\phi\) is the zero function which is a contradiction by Theorem 2. Hence, for large \(m\) each eigenfunction corresponding to any of the first \(n\) eigenvalues of \(S_m\) is necessarily residual. Now if \(\lambda_{2g-2}(S_m) \leq \frac{1}{4}\) then each \(S_m\) has at least \(2g-2+n\) small eigenvalues. This is a contradiction to [O-R] Theorem 2. Therefore we have proved that \(C^4_{g,n}(2g-2)\) contains a neighborhood of \(\bigcup_{i=1}^n \mathcal{M}_{0,3} \cup \mathcal{M}_{g-1,2}\) in \(\overline{\mathcal{M}_{g,n}}\).

In the last section we prove the last statement of Theorem 1. We consider \(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}\) as a subset of \(\partial \mathcal{M}_{g,n} = \overline{\mathcal{M}_{g,n}} \setminus \mathcal{M}_{g,n}\) and show the following

**Proposition 1.7.** There exists a neighborhood \(\mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1})\) of \(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}\) in \(\overline{\mathcal{M}_{g,n}}\) such that for each \(S \in \mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1})\): \(\lambda_{2g-1}(S) > \frac{1}{4}\)

i.e.,

\[
\mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}) \subset C^4_{g,n}(2g-1).
\]

Now we briefly sketch a proof of this proposition. We argue by contradiction and consider a sequence \((S_m)\) in \(\mathcal{M}_{g,n}\) that converges to \(S_\infty\) in \((\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}) \subset \partial \mathcal{M}_{g,n}\) such that \(\lambda^c_{2g-1}(S_m) \leq \frac{1}{4}\). Then, for \(1 \leq i \leq 2g-1\) and for each \(m\), we choose a small cuspidal eigenpair \((\lambda^c_{i,m}, \phi^c_{i,m})\) of \(S_m\) such that

(i) \(\{\phi^c_{i,m}\}_{i=1}^{2g-1}\) is an orthonormal family in \(L^2(S_m)\),

(ii) \(\lambda^c_{i,m}\) is the \(i\)-th eigenvalue of \(S_m\).

For \(1 \leq i \leq 2g-1\) let \(\lambda^c_{i,m}\) converges to \(\lambda^c_{i,\infty}\) as \(m \to \infty\). By Theorem 2 there are two possible types of behavior that the sequence \((\phi^c_{i,m})\) can exhibit. Either, for each \(1 \leq i \leq 2g-1\) the sequence \((\phi^c_{i,m})\) converges to a \(\lambda^c_{i,\infty}\)-eigenfunction \(\phi^c_{i,\infty}\) on \(S_\infty\), or for some \(i\) the sequence \((\lambda^c_{i,m}, \phi^c_{i,m})\) satisfies condition (2) in Theorem 2. However, in our case we have the following lemma:

**Lemma 2** For each \(i\), \(1 \leq i \leq 2g-1\), up to extracting a subsequence, the sequence \((\phi^c_{i,m})\) converges to a \(\lambda^c_{i,\infty}\)-eigenfunction \(\phi^c_{i,\infty}\) of \(S_\infty\). The limit functions \(\phi^c_{i,\infty}\) and \(\phi^c_{j,\infty}\) are orthogonal for \(i \neq j\) i.e. \(S_\infty\) has at least \(2g-1\) small eigenvalues. Moreover none of the \(\phi^c_{i,\infty}\) is residual.
Then we count the number of small eigenvalues of $S_\infty$ using [O-R] to conclude that at least one of $\phi_i^{\infty}$ is nonzero on the component of $S_\infty$ of type $(0,n+1)$. This leads to a contradiction by Huxley [Hu] or [O, Proposition 2].

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2. **Preliminaries**

In this section we shall recall some preliminary concepts that are important for our purpose. Metric convergence of a sequence $(S_m) \in \mathcal{M}_{g,n}$ to $S_\infty \in \mathcal{M}_{g,n}$ is one of the prime aspects of our study. We start by explaining the thick/thin decomposition of a hyperbolic surface which is convenient to understand the metric convergence.

2.1. **The thick / thin decomposition of a hyperbolic surface.** Let $S \in \mathcal{M}_{g,n}$. Recall that for any $\varepsilon > 0$, the $\varepsilon$-thin part of $S$, $S^{(0,\varepsilon)}$, is the set of points of $S$ with injectivity radius $< \varepsilon$. The complement of $S^{(0,\varepsilon)}$, the $\varepsilon$-thick part of $S$, denoted by $S^{[\varepsilon,\infty)}$, is the set of points where the injectivity radius of $S$ is $\geq \varepsilon$.

2.1.1. **Cylinders.** Let $\gamma$ be a simple closed geodesic on $S$. It can be viewed as the quotient of a geodesic in $\mathbb{H}$ by a hyperbolic isometry $\Upsilon$ fixing the geodesic. We may conjugate $\Upsilon$ such that the geodesic is the imaginary axis and the isometry is $\tau: z \rightarrow e^{2\pi l}z$, $2\pi l = l_\gamma$ being the length of the geodesic. We define the hyperbolic cylinder $C$ with core geodesic $\gamma$ as the quotient $\mathbb{H}/<\tau>$. Recall that the Fermi coordinates on $C$ assign to each point $p \in C$ the pair $(r,\theta) \in \mathbb{R} \times \{\gamma\}$ where $r$ is the signed distance of $p$ from $\gamma$ and $\theta$ is the projection of $p$ on $\gamma$ [Bu, p. 4]. These coordinates give a diffeomorphism of this hyperbolic cylinder to $\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}$. In terms of these coordinates the hyperbolic metric is given by:

$$ds^2 = dr^2 + l_\gamma^2 \cosh^2 r d\theta^2.$$ 

For $w \geq l$ we define the collar $C^w$ around $\gamma$ by

$$C^w = \{(r,\theta) \in C : l_\gamma \cosh r < w, 0 \leq \theta \leq 2\pi\}.$$ 

Then $C^w$ is diffeomorphic to an annulus whose each boundary component has length $w$. The Collar Theorem of Linda Keen [Ke] says that $C^1$ embeds in $S$ (more precisely, $C^{w(l_\gamma)}$ embeds in $S$ where $w(l_\gamma) = l_\gamma \cosh(\sinh^{-1}(\frac{1}{\sinh \frac{w}{2l_\gamma}})) > 1$ and $w(l_\gamma) \approx 2$).

2.1.2. **Cusps.** $S$ has $n$ ends called punctures. Cusps are particular neighborhood of the punctures. Denote by $i$ the parabolic isometry $i: z \rightarrow z + 2\pi$. For a choice of $t > 0$, a cusp $P^t$ is the half-infinite cylinder $\{z = x + iy : y > \frac{2t}{\pi}\} / <i>$. The boundary curve $\{y = \frac{2t}{\pi}\}$ is a horocycle of length $t$ that we identify with $\mathbb{R}/t\mathbb{Z}$. One can parametrize $P^t$ using the horocycle coordinates [Bu, p. 4] with respect to its boundary horocycle $\{y = \frac{2t}{\pi}\}$. The horocycle coordinates assigns to a point $p \in P^t$ the pair $(r,\theta) \in \mathbb{R}_{\geq 0} \times \{\mathbb{R}/t\mathbb{Z}\}$ where
$r$ is the distance from $p$ to the horocycle and $\theta$ the projection of $p$ on the horocycle. In terms of these coordinates the hyperbolic metric takes the form:

$$ds^2 = dr^2 + \left(\frac{t}{2\pi}\right)^2 e^{-2r}d\theta^2.$$ 

Recall that the cusp $\mathcal{P}^1$ (in fact $\mathcal{P}^2$) around each puncture embeds in $S$ and that those cusps corresponding to distinct punctures have disjoint interiors (ref. [Bu, Chapter 4]). We call them standard cusps. Observe that the area and boundary length of a standard cusp is equal to 1. For $t \leq 1$ denote the disjoint union $\bigcup_{c \in S} \mathcal{P}^t$ by $S^{c(0,1)}$ where $c$ ranges over distinct cusps in $S$.

2.1.3. The decomposition. By Margulis lemma there exists a constant $\epsilon_0 > 0$, the Margulis constant, such that for all $\epsilon \leq \epsilon_0$, $S^{(0,\epsilon)}$ is a disjoint union of embedded collars, one for each geodesic of length less than $2\epsilon$, and of embedded cusps, one for each puncture. The collar around a geodesic of length $\leq \epsilon$ is called a Margulis tube.

2.2. Metric degeneration of a collar to a pair of cusps. We describe how a collar around a geodesic of length $l_\gamma = 2\pi l$ converges as $l$ tends to zero to a pair of cusps. First shift the origin of the Fermi coordinates of $C_w(l_\gamma)$ to the right boundary of $C_w(l_\gamma)$ by making the change of variable $t = r - \sinh^{-1}\left(\frac{1}{\sinh l_\gamma^2}\right)$. In the shifted Fermi coordinates the metric on $C_w(l_\gamma)$ is equal to

$$ds^2 = dr^2 + l^2 \cosh^2(r - \sinh^{-1}\left(\frac{1}{\sinh l_\gamma^2}\right))d\theta^2.$$ 

For $r$ in a compact region we have the limiting

$$\lim_{l \to 0} l \cosh(r - \sinh^{-1}\left(\frac{1}{\sinh l_\gamma^2}\right)) = \frac{e^{-r}}{\pi}.$$ 

Now the hyperbolic metric on $\mathcal{P}^2$ is equal to

$$ds^2 = dr^2 + \frac{e^{-2r}}{\pi^2}d\theta^2$$ 

with respect to the boundary horocycle $\{y = \pi\}$ of $\mathcal{P}^2$.

Choose a base point $p_1$ on the right half of $C_w(l_\gamma)^{(e,\infty)}$. Then by above, as $l \to 0$, the pair $(C_w(l_\gamma), p_1)$ converges, up to extracting a subsequence, to $(\mathcal{P}^2, p)$ where $p \in \mathcal{P}^2^{(e,\infty)}$. Since one can choose the base point on the left half of $C_w(l_\gamma)$ also, the metric limit of $C_w(l_\gamma)$ is a pair of $\mathcal{P}^2$.

3. Mass distribution of small cuspidal functions over thin parts

Our goal is to study the behavior of sequences of small cuspidal eigenpairs $(\lambda_n, f_n)$ of $S_n \in M_{g,n}$ when $(S_n)$ converges to $S_\infty \in M_{g,n}$ and finally to prove Theorem 1. For this we need to understand how the mass ($L^2$ norm) of a small eigenfunction is distributed over the surface, and in particular how it is distributed with respect to the thin/thick decomposition. Let $S \in M_{g,n}$. Recall that for any $\epsilon \leq \epsilon_0$ the $\epsilon$-thin part, $S^{(0,\epsilon)}$, of $S$ consists of cusps.
and Margulis tubes. We separately study the mass distribution of a small cuspidal eigenfunction over these two different types of domains.

3.1. Mass distribution over cusps. For $2\pi \leq a < b$ consider the annulus $\mathcal{P}(a, b) = \{(x, y) \in \mathcal{P}^1 : a \leq y < b\}$ contained in a cusp $\mathcal{P}^1$ and bounded by two horocycles of length $\frac{2\pi}{a}$ and $\frac{2\pi}{b}$. We begin our study with the following lemma.

**Lemma 3.1.** For any $b > 2\pi$ there exists $K(b) < \infty$ such that for any small cuspidal eigenpair $(\lambda, f)$ of $\mathcal{P}^1$ one has

$$\|f\|_{\mathcal{P}(b, \infty)} < K(b) \|f\|_{\mathcal{P}(2\pi, b)}.$$  

If $\lambda < \frac{1}{2} - \eta$ for some $\eta > 0$ then there exists a constant $T(b, \eta) < \infty$ depending on $b$ and $\eta$ such that for any small eigenpair $(\lambda, f)$ one has

$$\|f\|_{\mathcal{P}(b, \infty)} < T(b, \eta) \|f\|_{\mathcal{P}(2\pi, b)}.$$  

Furthermore, $K(b), T(b, \eta) \to 0$ as $b \to \infty$.

**Proof.** We begin with the first part. Since $f$ is cuspidal inside $\mathcal{P}^1$ it can be expressed as

$$f(z) = \sum_{n \in \mathbb{Z}^*} f_n W_s(nz),$$  

where $s(1 - s) = \lambda$ and $W_s$ is the Whittaker function (see [I, Proposition 1.5]). The meaning of (3.4) is that the right hand series converges to $f$ in $L^2(\mathcal{P}^1)$ and that the convergence is uniform over compact subsets. Recall also that for $n \in \mathbb{Z}^*$ the Whittaker functions is defined by

$$W_s(nz) = 2(|n|y)^{\frac{1}{2}} K_{s-\frac{1}{2}}(|n|y)e^{inx}$$

where $K_\epsilon$ is the McDonald’s function and that for any $\epsilon$ (see [Le, p. 119])

$$K_\epsilon(y) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-y \cosh u - \epsilon u} du$$  

whenever the integral makes sense. From the expression it is clear that the functions $(W_s(n,))$ form an orthogonal family over $\mathcal{P}(a, b)$ (independent of the choices of $a$ and $b$). Hence (1) will follow from the following claim.

**Claim 3.6.** Let $s \in [\frac{1}{2}, 1]$. Then for any $b > 2\pi$ there exists $K(b) < \infty$ such that for all $n \in \mathbb{Z}^*$

$$\|W_s(nz)\|_{\mathcal{P}(b, \infty)} \leq K(b) \|W_s(nz)\|_{\mathcal{P}(2\pi, b)}.$$  

Furthermore, $K(b) \to 0$ as $b \to \infty$.

**Proof.** From the expression of $W_s$ we have

$$\|W_s(nz)\|_{\mathcal{P}(a, b)} = 2\pi \left( \int_a^b \left| n \right| y K_{s-\frac{1}{2}}(|n|y)^2 \frac{dy}{y^2} \right).$$

To prove the claim we may suppose that $n \geq 1$. Our next objective is to obtain bounds for the functions $K_{s-\frac{1}{2}}(y)$ for $s \in [\frac{1}{2}, 1]$. We start from the
above integral representation of $K_\epsilon(y)$. We write $K_\epsilon(y) = \frac{1}{2} \{ c(\epsilon, y) + d(\epsilon, y) \}$ where

$$c(\epsilon, y) = \int_{-1}^{1} e^{-y \cosh u - \epsilon u} du$$

and

$$d(\epsilon, y) = \int_{-\infty}^{-1} e^{-y \cosh u - \epsilon u} du + \int_{1}^{\infty} e^{-y \cosh u + \epsilon u} du.$$  \hspace{1cm} (3.7)

Now we treat $c(\epsilon, y)$ and $d(\epsilon, y)$ separately.

Bounding $c(\epsilon, y)$:
We have

$$c(\epsilon, y) = \int_{-1}^{1} e^{-y \cosh u - \epsilon u} du \leq e^\epsilon \int_{-1}^{1} e^{-y(1 + \frac{u^2}{2} + \frac{u^4}{4} + \ldots)} du$$

Since $\frac{uy}{2} > 1 + \frac{uy^2}{2}$ for $u > 0$, we have:

$$\int_{0}^{1} e^{-y \frac{u^2}{2}} du < \int_{0}^{1} \frac{du}{1 + \frac{uy^2}{2}} = \frac{2}{y} \tan^{-1} \left( \frac{y}{2} \right) \leq \frac{2}{y} \frac{\pi}{2}$$

Therefore

$$c(\epsilon, y) \leq 2\pi e^{2-\epsilon} \frac{e^{-y}}{y}.$$  \hspace{1cm} (3.8)

To obtain a lower bound, we write

$$\int_{-1}^{1} e^{-y \cosh u - \epsilon u} du \geq e^{-\epsilon} \int_{-1}^{1} e^{-y \cosh u} du = e^{-\epsilon} \int_{-1}^{1} e^{-y(1 + \frac{u^2}{2} + \frac{u^4}{4} + \ldots)} du$$

Since for all $u \in (0, 1]$ one has

$$\frac{u^2}{2!} + \frac{u^4}{4!} + \ldots < u \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \right) = u.$$  \hspace{1cm} (3.9)

Hence

$$c(\epsilon, y) \geq 2e^{-\epsilon} \cdot e^{-y} \int_{0}^{1} e^{-uy} du = 2e^{-\epsilon} \frac{e^{-y}}{y} (1 - e^{-y}).$$

Combining the above two inequalities

$$2e^{-\epsilon} \frac{e^{-y}}{y} (1 - e^{-y}) \leq c(\epsilon, y) \leq 2\pi e^{2-\epsilon} \frac{e^{-y}}{y}.$$  \hspace{1cm} (3.10)

Bounding $d(\epsilon, y)$:

$$d(\epsilon, y) = \int_{-\infty}^{-1} e^{-y \cosh u - \epsilon u} du + \int_{1}^{\infty} e^{-y \cosh u + \epsilon u} du$$

Now for any $u > 1$, \hspace{1cm} (3.11)

$$\frac{u^2}{2!} + \frac{u^4}{4!} + \ldots > \gamma_0 u^2 > \gamma_0 u$$
where \( \gamma_0 = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \).

Thus
\[
d(\epsilon, y) = e^{-y} \int_1^\infty \left\{ e^{-y\left(\frac{u^2}{\pi} + \frac{u^4}{4\pi^2} + \ldots \right)} - e^{-y\left(\frac{u^2}{\pi} + \frac{u^4}{4\pi^2} + \ldots \right) + \epsilon u} \right\} du
\]
\[
\leq e^{-y} \int_1^\infty \left\{ e^{-y\gamma_0 u - \epsilon u} + e^{-y\gamma_0 u + \epsilon u} \right\} du
\]
\[
= \frac{e^{-y(\epsilon y + \gamma_0)}}{\gamma_0 + \frac{\epsilon}{y}} + \frac{e^{-y(\epsilon y - \gamma_0)}}{\gamma_0 - \frac{\epsilon}{y}}.
\]

Thus combining the estimates for \( c(\epsilon, y) \) and \( d(\epsilon, y) \) we obtain
\[
2e^{-\epsilon} \frac{e^{-y}}{y} (1 - e^{-y}) < K_\epsilon(y) < 2\pi e^{-\epsilon} \frac{e^{-y}}{y} + e^{-y} \frac{e^{-y(\gamma_0 + \epsilon)}}{\gamma_0 + \frac{\epsilon}{y}} + e^{-y(\gamma_0 - \epsilon)} \frac{\epsilon}{\gamma_0 - \frac{\epsilon}{y}}.
\]

Let
\[
\delta(\epsilon, y) = \frac{e^{-y(\gamma_0 + \epsilon)}}{\gamma_0 + \frac{\epsilon}{y}} + \frac{e^{-y(\gamma_0 - \epsilon)}}{\gamma_0 - \frac{\epsilon}{y}}.
\]

Observe that for \( \epsilon < 1 \) and \( y \geq \frac{2}{\gamma_0} \)
\[
\delta(\epsilon, y) < \frac{4 \cosh 1}{\gamma_0} e^{-\gamma_0 y} = \delta_0(y).
\]

So, for \( y \geq \frac{2}{\gamma_0} \) large enough
\[
2e^{-\epsilon} \frac{e^{-y}}{y} < K_\epsilon(y) < \frac{e^{-y}}{y} \left( 2\pi e^\epsilon + \delta_0(y) \right). \tag{3.9}
\]

Going back to the expression of \( W_s \), for \( s \in \left[ \frac{1}{2}, 1 \right] \), we find:
\[
\frac{1}{2\pi} \left\| W_s(nz) \right\|^2_{L^2(2\pi, b)} = \int_{2\pi} b 4\pi K_{s-\frac{1}{2}}(ny)^2 dy = \int_{2\pi} b 4n K_{s-\frac{1}{2}}(ny)^2 dy
\]
\[
\geq \int_{2\pi} b 4n K_{s-\frac{1}{2}}(ny)^2 dy \geq \frac{16ne^{1-2s}}{b} \int_{2\pi} b e^{-2ny} (ny)^2 dy = \frac{16ne^{1-2s}}{nb^2} \int_{2\pi} b e^{-2ny} (ny)^2 dy
\]
\[
= \frac{16ne^{1-2s}}{nb^2} \left( \int_{2\pi} \frac{b}{y^2} e^{-2ny} dy + \int_{\frac{b}{2}}^b \frac{b}{y^2} e^{-2ny} dy \right) > \frac{16ne^{1-2s}}{nb^2} \frac{b}{y^2} e^{-2ny} dy
\]
\[
= \frac{16e^{1-2s}}{nb} e^{-nb} \frac{1 + O(e^{-nb} + \frac{1}{b})}{y^2} \tag{3.10}
\]

i.e.
\[
\left\| W_s(nz) \right\|^2_{L^2(2\pi, b)} > 2\pi \frac{16e^{1-2s}}{nb} e^{-nb} \left\{ 1 + O(e^{-nb} + \frac{1}{b}) \right\}
\]

Also,
\[
\frac{1}{2\pi} \left\| W_s(nz) \right\|^2_{L^2(b, \infty)} = \int_{1}^{\infty} 4ny K_{s-\frac{1}{2}}(ny)^2 dy = \int_{1}^{\infty} 4n K_{s-\frac{1}{2}}(ny)^2 dy
\]
\[
\leq \int_{b}^{\infty} 4n \frac{b}{y} K_{s-\frac{1}{2}}(ny)^2 dy \leq \frac{4n(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2}{b} \int_{1}^{\infty} e^{-2ny} (ny)^2 dy
\]
\[
= \frac{4(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2}{nb} e^{-2nb} \left\{ 1 + O(\frac{1}{b}) \right\}
\]
i.e.
\[ \|W_s(nz)\|_{P(b,\infty)}^2 \leq 2\pi \frac{2(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2 e^{-2nb}}{nb^2} \{1 + O\left(\frac{1}{b}\right)\} \]  
(3.11)

In the last inequality, we used the following estimate from [Le, Section 3.2]:
\[ \int_{t_1}^{t_2} \frac{e^{-2\alpha y}}{y^2} dy = \frac{e^{-2\alpha t_1}}{2\alpha t_1} \{1 + O(e^{2(t_1-t_2)} + t_1^{-1})\} \]
with an absolute constant for the \(O\)-term for \(\alpha > 1\).

Comparing (3.10) and (3.11) we get, for any \(n \in \mathbb{Z}^*\)
\[ \|W_s(nz)\|_{P(b,\infty)} \leq K(b)\|W_s(nz)\|_{P(2\pi, b)} \]  
(3.12)

where
\[ K^2(b) = \frac{e^{2s-1}}{8} \frac{1}{(2\pi e^{s-\frac{1}{2}} + \delta_0(b))^2 e^{-|n|b} \{1 + O(\frac{1}{b})\}} \]

From the expression it is clear that \(K\) is bounded independent of \(n, b\) (once \(b\) is large enough) and \(s \in \left[\frac{1}{2}, 1\right]\). So we obtain the claim by choosing some \(b > \frac{2}{\delta_0}\) sufficiently large (once and for all) such that the \(O\)-terms in the expression of \(T\) are small enough. It is also clear from the expression that when \(b \to \infty\), \(K(b) \to 0\). This proves the Claim 3.6 and hence the first part of Lemma 3.1.

Now we prove the second part. Let \(\lambda < \frac{1}{2} - \eta\) for some \(\eta > 0\) and let \((\lambda, f)\) be a residual eigenpair. The Fourier expansion of \(f\) inside \(P^1\) has the form
\[ f(z) = f_0 y^s + \sum_{n \in \mathbb{Z}^*} f_n W_s(nz) = f_0 y^s + g(z) \]  
(3.13)

where \(s(1 - s) = \lambda\), \(s \in (0, \frac{1}{2})\) (see [I]) and \(g(z) = \sum_{n \in \mathbb{Z}^*} f_n W_s(nz)\). Since \(f_0 y^s\) and \(g\) are orthogonal and since the first part can be applied to \(g\), one needs only to prove the lemma for the term \(f_0 y^s\). So we calculate:
\[ \int_{-c}^{c} \frac{y^{2s}}{y^2} \frac{dy}{y^2} = \frac{1}{1-2s} \left( \frac{1}{a^{1-2s}} - \frac{1}{c^{1-2s}} \right). \]

Therefore, for \(b > 2\pi\),
\[ \|f_0 y^s\|_{P(b,\infty)}^2 = \frac{1}{(\frac{b}{2\pi})^{1-2s} - 1} \|f_0 y^s\|_{P(2\pi, b)}^2. \]  
(3.14)

The lemma is satisfied by \(T_2(b, \eta)\) such that
\[ T_2^2(b, \eta) = \max \left( K^2(b), \frac{1}{(\frac{b}{2\pi})^{1-2s} - 1} \right). \]

From the expression it is clear that \(T_2(b, \eta)\) depends only on two quantities: \(b\) and \(\frac{1}{2} - s\). Since \(\frac{1}{2} - s > \sqrt{\eta} > 0\), \(\frac{1}{(\frac{b}{2\pi})^{1-2s} - 1} \to 0\) when \(b \to \infty\). This proves the second part.
3.2. Mass distribution over Margulis tubes. Now we study the distribution of the mass of a small eigenfunction over Margulis tubes. Let $\gamma$ be a simple closed geodesic of length $l_\gamma = 2\pi l$. Recall that $C^0$ denotes the collar around $\gamma$ bounded by two equidistant curves of length $a$. Any $f \in L^2(C^1)$ can be written as a Fourier series in the $\theta$-coordinate:

$$f(r, \theta) = a_0(r) + \sum_{j=1}^{\infty} \left( a_j(r) \cos j\theta + b_j(r) \sin j\theta \right). \quad (3.15)$$

The functions $a_j = a_j(r)$ and $b_j = b_j(r)$ are defined on $[-\cosh^{-1}(\frac{1}{2}l_\gamma), \cosh^{-1}(\frac{1}{2}l_\gamma)]$ and are called the $j$-th Fourier coefficients of $f$ (in $C^1$). When $f$ is a $\lambda$-eigenfunction, $a_j$ and $b_j$ are solutions of the differential equation

$$\frac{d^2 \phi}{dr^2} + \tanh r \frac{d \phi}{dr} + \left( \lambda - \frac{j^2}{l_\gamma^2 \cosh^2 r} \right) \phi = 0. \quad (3.16)$$

We set $[f]_0 = a_0(r)$ and $[f]_1 = f - [f]_0$. The following lemma concerns the distribution of masses of $[f]_0$ and $[f]_1$ inside $C^1$.

**Lemma 3.17.** For any $l_\gamma < \epsilon \leq \epsilon_0$ there exist constants $T_1(\epsilon), T_2(\epsilon) < \infty$, depending only on $\epsilon$, such that for any small eigenpair $(\lambda, f)$ of $C^1$ the following inequalities hold:

$$\| [f]_1 \|_{C^*} < T_1(\epsilon) \| [f]_1 \|_{C^1 \setminus C^*} \quad (3.18)$$

and

$$\| [f]_0 \|_{C^0 \setminus C^*} < T_2(\epsilon) \| [f]_0 \|_{C^1 \setminus C^0}. \quad (3.19)$$

Therefore, for any $l_\gamma < \epsilon \leq \epsilon_0$ and any small eigenpair $(\lambda, f)$ of $C^1$ one has

$$\| f \|_{C^0 \setminus C^*} < \max \{ T_1(\epsilon_0), T_2(\epsilon) \} \| f \|_{C^1 \setminus C^0}. \quad (3.20)$$

If $\lambda < \frac{1}{2} - \eta$ for some $\eta > 0$ then there exists a constant $T_0(\epsilon, \eta) < \infty$, depending only on $\eta$ and $\epsilon$, such that

$$\| [f]_0 \|_{C^*} < T_0(\epsilon, \eta) \| [f]_0 \|_{C^1 \setminus C^*}. \quad (3.21)$$

Furthermore, $T_1(\epsilon), T_0(\epsilon, \eta) \to 0$ as $\epsilon \to 0$.

Before starting the proof of the above lemma we make a few observations about the solutions of (3.16). The change of variable $u(r) = \cosh^{\frac{1}{2}}(r) \phi(r)$ transforms (3.16) into

$$\frac{d^2 u}{dr^2} = \left( \frac{1}{4} - \lambda \right) u + \frac{1}{4 \cosh^2 r} u + \frac{j^2}{l_\gamma^2 \cosh^2 r} u. \quad (3.22)$$

Let $s_j$ (resp. $c_j$) be the solution of (3.22) satisfying the conditions: $s_j(0) = 0$ and $s'_j(0) = 1$ (resp. $c_j(0) = 1$ and $c'_j(0) = 0$). Since (3.22) is invariant under $r \to -r$ one has: $s_j(-r) = -s_j(r)$ and $c_j(-r) = c_j(r)$ for all $j \geq 0$. Therefore there exists $t > 0$ such that $s_j > 0$ and $c'_j > 0$ on $(0, t]$. Now we prove the following claim.

**Claim 3.23.** Let $L > 0$. Let $g : [0, L] \to \mathbb{R}$ be a $C^2$-function which satisfies the inequality:

$$\frac{d^2 g}{dr^2} > \delta^2 g$$

where $\delta > 0$ is a small constant. Then $\gamma$ is simple closed geodesic of length $l_\gamma$. Recall that $C^0$ denotes the collar around $\gamma$ bounded by two equidistant curves of length $a$. Any $f \in L^2(C^1)$ can be written as a Fourier series in the $\theta$-coordinate:
for some $\delta > 0$. If $g^{'}(0) \geq 0$ then $\frac{g(r)}{\cosh \delta r}$ is a monotone increasing function of $r$ in $(0, L]$.

Proof. Observe that

$$\left( \frac{g(r)}{\cosh \delta r} \right)^{'} = \frac{g^{'}(r) \cosh \delta r - \delta g(r) \sinh \delta r}{\cosh^{2}(\delta r)}.$$ 

Consider the function $H$ defined on $[0, L]$ by

$$H(r) = g^{'}(r) \cosh \delta r - \delta g(r) \sinh \delta r.$$ 

Since $g$ is a $C^2$ function $H$ is continuous on $[0, L]$. Observe that the claim follows if $H(r) > 0$ in $(0, L]$. Now for any $r \in (0, L]$

$$H^{'}(r) = g^{''}(r) \cosh \delta r - \delta^2 g(r) \cosh \delta r = (g^{''}(r) - \delta^2 g(r)) \cosh \delta r > 0.$$ 

Therefore for $r > 0$, $H(r) > H(0) = g^{'}(0) \geq 0$. Hence the claim.$\square$

Proof of Lemma 3.17 We need to estimate, for $l_{\gamma} \leq t < w \leq 1$, the quantities:

$$\|f\|_{1, \mathcal{C}' \setminus \mathcal{C}^t} = l_{\gamma} \int_{-Lw}^{-Lt} \left( \sum_{j=1}^{\infty} \alpha_j^2 + \beta_j^2 \right) dr + l_{\gamma} \int_{Lt}^{Lw} \left( \sum_{j=1}^{\infty} \alpha_j^2 + \beta_j^2 \right) dr$$

and

$$\|f\|_{0, \mathcal{C}' \setminus \mathcal{C}^t} = l_{\gamma} \int_{-Lw}^{-Lt} \alpha_0^2 dr + l_{\gamma} \int_{Lt}^{Lw} \alpha_0^2 dr$$

where $\alpha_0(r) = \cosh^{\frac{1}{2}}(r) a_0(r)$, $\alpha_j(r) = a_j(r) \cosh^{\frac{1}{2}}(r)$, $\beta_j(r) = b_j(r) \cosh^{\frac{1}{2}}(r)$ and $L_u = \cosh^{-1}(\frac{u}{L}).$ Since $s_j$ is odd and $c_j$ is even, for any symmetric subset $U \subset [-L_1, L_1]$, $s_j$ and $c_j$ are orthogonal in $L^2(U)$. Now $\alpha_j$ and $\beta_j$ are linear combinations of $s_j$ and $c_j$ for $j \geq 1$ and $\alpha_0$ is a linear combination of $s_0$ and $c_0$. Therefore, since $s_j$ and $c_j$ are orthogonal, it is enough to prove the lemma with $s_j$ and $c_j$ instead of $[f]_1$ and with $s_0$ and $c_0$ instead of $[f]_0$. We detail the computations for $s_j$. The computations for $c_j$ are similar. Let us choose $\epsilon$ such that $l_{\gamma} < \epsilon < \epsilon_0$. The lemma reduces to find $K_1(\epsilon), K_2(\epsilon) < \infty$, depending on $\epsilon$, and $K_0(\epsilon, \eta) < \infty$, depending on $\epsilon, \eta \geq 0$, such that

$$\|s_j\|_{\mathcal{C}' \setminus \mathcal{C}^t} < K_1(\epsilon)\|s_j\|_{\mathcal{C}^t \setminus \mathcal{C}'}, \|s_0\|_{\mathcal{C}^t \setminus \mathcal{C}^t} < K_2(\epsilon)\|s_0\|_{\mathcal{C}^t \setminus \mathcal{C}^0}$$

and

$$\|s_0\|_{\mathcal{C}^t} < K_0(\epsilon, \eta)\|s_0\|_{\mathcal{C}^t \setminus \mathcal{C}^t}.$$ 

Let $\eta < \epsilon < \lambda$ and set $\delta_0 = \sqrt{\eta}$ and set for $j \geq 1$, $\delta_j = 1$. Notice that $l \cosh r < 1$ on $[0, L_1)$. Hence by (3.22) $s_j : [0, L_1) \to \mathbb{R}$ satisfies the inequality:

$$\frac{d^2 s_j}{dr^2} > \delta_j^2 s_j.$$ 

Hence by Claim 3.23, $h_j(r) = \frac{s_j(r)}{\cosh r}$, for $j \geq 1$, is strictly increasing on $(0, L_1)$. The same is true for $h_0 = \frac{s_0(r)}{\cosh r}$ (even when $\delta_0 = 0$).
We begin with the proof of the second part of the Lemma. So we assume $\eta > 0$. For $0 \leq a < b$ consider the integral:
\[
\int_a^b h_0^2(r)dr = \int_a^b h_0^2(r)\cosh^2(\delta_0 r)dr.
\]
Since $h_0$ is strictly increasing we have
\[
h_0^2(a)\int_a^b \cosh^2(\delta_0 r)dr < \int_a^b s_0^2(r)dr < h_0^2(b)\int_a^b \cosh^2(\delta_0 r)dr. \tag{3.24}
\]
Now choosing $a = 0$ and $b = L_\epsilon$ the last inequality in (3.24) gives
\[
\|s_0\|_{C^2 \setminus C_\epsilon}^2 < 2L_\epsilon h_0^2(L_\epsilon)\int_0^{L_\epsilon} \cosh^2(\delta_0 r)dr. \tag{3.25}
\]
Next choosing $a = L_\epsilon$ and $b = L_{\gamma}$ the first inequality in (3.24) gives
\[
\|s_0\|_{C^4 \setminus C_\epsilon}^2 > 2L_\gamma h_0^2(L_{\gamma})\int_{L_\epsilon}^{L_{\gamma}} \cosh^2(\delta_0 r)dr. \tag{3.26}
\]
Therefore
\[
\|s_0\|_{C^4} < T_0\|s_0\|_{C^4 \setminus C_\epsilon}, \tag{3.27}
\]
where
\[
T_0^2 = \frac{\sinh 2\delta_0 L_\epsilon + 2\delta_0 L_\epsilon}{\sinh 2\delta_0 L_\epsilon - \sinh 2\delta_0 L_\epsilon + 2\delta_0 (L_1 - L_\epsilon)}. \tag{3.28}
\]
We see that $T_0$ depends only on $\epsilon, \delta_0$ and $L_{\gamma}$. Now $L_\epsilon = \cosh^{-1}(\frac{\epsilon}{L_{\gamma}}) = \log(\frac{\epsilon}{L_{\gamma}} + \sqrt{(\frac{\epsilon}{L_{\gamma}})^2 - 1})$. Therefore, for $\epsilon$ and $\delta_0 = \eta > 0$ fixed, and $L_{\gamma}$ small
\[
T_0^2 < K_0 \frac{1}{\epsilon - 2\delta_0 - 1},
\]
and the constant $K_0$ is independent of $L_{\gamma}$ as soon as $L_{\gamma}$ is small compared to $\epsilon$. Thus we can choose $T_0(\epsilon, \eta)$ independent of $L_{\gamma}$ satisfying (3.27). This proves (3.21).

For $s_j$, $j \geq 1$, exactly the same computations for $s_0$ work with $\delta_0$ replaced by $\delta_j = 1$. Hence in this case our constant,
\[
T_1^2(\epsilon) < K_1 \frac{1}{\epsilon^2 - 1},
\]
depends only on $\epsilon$. This proves (3.18).

Now we prove (3.20). Since $s_0 : [0, L_1] \to \mathbb{R}^+$ is strictly increasing we have:
\[
\int_{L_\epsilon}^{L_{\epsilon_0}} s_0^2(r)dr < s_0^2(L_{\epsilon_0})(L_{\epsilon_0} - L_\epsilon) \text{ and } \int_{L_{\epsilon_0}}^{L_1} s_0^2(r)dr > s_0^2(L_{\epsilon_0})(L_1 - L_{\epsilon_0}).
\]
Combining the two inequalities we obtain
\[
\|s_0\|_{C^4 \setminus C_0} < T_2(\epsilon)\|s_0\|_{C^4 \setminus C_0} \tag{3.29}
\]
where
\[
T_2^2(\epsilon) = \frac{L_{\epsilon_0} - L_\epsilon}{L_1 - L_{\epsilon_0}} < K_2 \left(\frac{\log \frac{1}{\epsilon}}{\log \frac{1}{\epsilon_0} - 1}\right). \tag{3.30}
\]
The constant $K_2$ is independent of $L_{\gamma}$ as soon as $L_{\gamma}$ is small compared to $\epsilon$. Thus we can choose $T_2(\epsilon)$ independent of $L_{\gamma}$ satisfying (3.29). This proves (3.20).
3.3. Applications. Let $S$ be a finite area hyperbolic surface with $n$ punctures. Denote by $P_i$ the standard cusp around the $i$-th puncture. Recall that $P_i$’s have disjoint interiors and that each of them is isometric to the half-infinite annulus $P_i$ (see 2.1.2). Applying Lemma 3.1 in each $P_i$ separately we obtain the following corollary which will be useful in our analysis.

**Corollary 3.31.** For any $0 < \epsilon < \epsilon_0$ there exists $T(\epsilon) < \infty$, depending only on $\epsilon$, such that for any small cuspidal eigenpair $(\lambda, f)$ of $S$ one has

$$\|f\|_{S^{[0,\epsilon]}} < T(\epsilon)\|f\|_{S^{[0,1]} \setminus S^{[0,\epsilon]}}. \quad (3.32)$$

If $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ then for any $0 < \epsilon < \epsilon_0$ there exists $T_1(\epsilon, \eta) < \infty$, depending only on $\epsilon$ and $\eta$, such that for any $\lambda$-eigenfunction $f$ of $S$ one has

$$\|f\|_{S^{[0,\epsilon]}} < T_1(\epsilon, \eta)\|f\|_{S^{[0,1]} \setminus S^{[0,\epsilon]}}. \quad (3.33)$$

Furthermore, $T(\epsilon)$ and $T_1(\epsilon, \eta)$ tends to zero as $\epsilon \to 0$.

Using this corollary and (3.20) we deduce the following

**Corollary 3.34.** For any $0 < \epsilon < \epsilon_0$ there exists a constant $L(\epsilon) < \infty$, depending only on $\epsilon$, such that for any small cuspidal eigenfunction $f$ of $S$ one has

$$\|f\|_{S^{[\epsilon,\infty]}} < L(\epsilon)\|f\|_{S^{[0,\epsilon]}}. \quad (3.35)$$

Now we give a new proof of the following theorem of D. Hejhal [H].

**Theorem 3.36.** Consider a sequence $(S_m) \in M_{g,n}$ converging to $S_{\infty} \in \mathcal{M}_{g,n}$. Let $(\lambda_m, \phi_m)$ be a normalized small eigenpair of $S_m$ such that $\lambda_m \to \lambda_{\infty}$. If $\lambda_{\infty} < \frac{1}{4}$ then, up to extracting a subsequence, $\phi_m$ converges to a normalized $\lambda_{\infty}$-eigenfunction $\phi_{\infty}$ of $S_{\infty}$.

D. Hejhal’s proof uses convergence of Green’s functions of $S_m$ to that of $S_{\infty}$. Our approach is more elementary and uses the above estimates on the mass distribution of eigenfunctions over thin part of surfaces.

**Proof of Theorem 3.36.** First we prove that, up to extracting a subsequence, $\phi_m$ converges to a $\lambda_{\infty}$-eigenfunction $\phi_{\infty}$ of $S_{\infty}$. By Theorem 2 (which will be proven in §3) it is enough to prove that there exist $\epsilon, \delta > 0$ such that $\|\phi_m\|_{S^{[\epsilon,\infty]}} \geq \delta$ up to extracting a subsequence. We argue by contradiction. Suppose that for any $\epsilon > 0$ the sequence $\|\phi_m\|_{S^{[\epsilon,\infty]}} \to 0$ as $m \to \infty$. Let $\eta > 0$, such that $\lambda_m < \frac{1}{4} - \eta$ for all $m \geq 1$. By Lemma 3.17 we have

$$\|\phi_m\|_{C^\epsilon} < \max\{T_0(\epsilon, \eta), T_1(\epsilon)\}\|\phi_m\|_{C^1 \setminus C_\epsilon}. \quad (3.37)$$

Therefore from (3.33) and (3.37) we have

$$\|\phi_m\|_{S^{[0,\epsilon]}} < \max\{T_0(\epsilon, \eta), T_1(\epsilon)\}\|\phi_m\|_{S^{[0,\epsilon]} \setminus S^{[0,\infty]}}. \quad (3.38)$$

Hence if $\|\phi_m\|_{S^{[\epsilon,\infty]}} \to 0$ as $m \to \infty$ then $\|\phi_m\| \to 0$ as $m \to \infty$. This is a contradiction to the fact that each $\phi_m$ is normalized i.e. $\|\phi_m\| = 1$.

Next we prove that $\|\phi_{\infty}\| = 1$. By uniform convergence over compacta, in each cusp and in each pinching collar, the Fourier coefficients of $\phi_m$ will converge to the corresponding Fourier coefficients of $\phi_{\infty}$. Therefore, by
\[ (3.18), (3.21) \text{ and } (3.33), \phi_m' \text{s are uniformly integrable: for any } \delta > 0 \text{ there exist } \epsilon > 0 \text{ such that for all large values of } m \]

\[ \|\phi_m\|_{S_{m,\epsilon}} > 1 - \delta. \quad (3.39) \]

Hence \( \|\phi_{\infty}\| = 1 \). This finishes the proof. \( \square \)

4. PROOF OF THEOREM 2

Let \( (S_m) \) be a sequence in \( \mathcal{M}_{g,n} \) which converges in \( \overline{\mathcal{M}_{g,n}} \) to \( S_{\infty} \). Let \( \Gamma_m, \Gamma_{\infty} \) be such that \( S_m = \mathbb{H}/\Gamma_m \text{ and } S_{\infty} = \mathbb{H}/\Gamma_{\infty} \). Recall that the convergence \( S_m \to S_{\infty} \) means that for any fixed positive constant \( \epsilon_1 \leq \epsilon_0 \) (\( \epsilon_0 \) is the Margulis constant) and a choice of base point \( p_m \in S_{m,\epsilon}^{(1,\infty)} \), after conjugating \( \Gamma_m \) so that the projection \( \mathbb{H} \to \mathbb{H}/\Gamma_m \) maps \( i \) to \( p_m \), \( (\mathbb{H}/\Gamma_m, p_m) \) converges to a component \( (\mathbb{H}/\Gamma_{\infty}, p_{\infty}) \) of \( S_{\infty} \). We begin by fixing some \( \epsilon < \epsilon_0 \) and \( p_m \in S_{m,\epsilon}^{(\epsilon,\infty)} \). In the following we assume that \( \epsilon_1, p_m, \Gamma_m, p_{\infty}, \) and \( \Gamma_{\infty} \) satisfy the previous statement.

To simplify notations we shall assume that only one closed geodesic \( \gamma_m \) gets pinched as \( S_m \to S_{\infty} \in \partial \mathcal{M}_{g,n} \). In particular the limit surface \( S_{\infty} \) (which may be disconnected) has two new cusps. Denote the standard cusps of \( S_m \) by \( \mathcal{P}_1(m), \mathcal{P}_2(m), \ldots, \mathcal{P}_n(m) \) and the limits of these in \( S_{\infty} \in \partial \mathcal{M}_{g,n} \) by \( \mathcal{P}_1^{(\infty)}, \ldots, \mathcal{P}_n^{(\infty)} \) and denote by \( \mathcal{P}_{n+1}(\infty), \mathcal{P}_{n+2}(\infty) \) the new cusps which arise due to the pinching of \( \gamma \). The cusps \( \mathcal{P}_i(\infty) \) for \( 1 \leq i \leq n \) will be called old cusps.

Recall that we have a sequence of small cuspidal eigenpairs \( (\lambda_m, \phi_m) \) of \( S_m = \mathbb{H}/\Gamma_m \) such that the \( L^2 \)-norm of \( \phi_m \) is 1 and \( \lambda_m \to \lambda_{\infty} \leq \frac{1}{4} \).

**Notation 4.1.** In what follows \( d\mu_m \) will denote the area measure on \( S_m \) for \( m \in \mathbb{N} \cup \{\infty\} \) and \( d\mu_{\mathbb{H}} \) will denote the area measure on \( \mathbb{H} \). The lift of \( f \in L^2(S_m) \) to \( \mathbb{H} \) under the projection \( \mathbb{H} \to \mathbb{H}/\Gamma_m \), defined as above, will be denoted by \( \tilde{f} \).

By Green’s formula one has:

\[ \int_{S_m} |\nabla \phi_m|^2 d\mu_m = \lambda_m \int_{S_m} |\phi_m|^2 d\mu_m = \lambda_m. \]

Let \( K \subset \mathbb{H} \) be compact. One can cover \( K \) by finitely many geodesics balls of radius \( \rho \). If \( \rho \) is sufficiently small then each of these balls maps injectively to \( S_m \) since \( \Gamma_m \to \Gamma_{\infty} \). Therefore, since \( \|\phi_m\| = 1 \) \( \|\tilde{\phi}_m|_{K}\| \) is bounded depending only on \( K \). From the mean value formula [F] Corollary 1.3 there exists a constant \( \Lambda(\lambda_{\infty}, \rho) \) such that for \( \lambda_m \) close to \( \lambda_{\infty} \),

\[ |\tilde{\phi}_m(q)| \leq \Lambda(\lambda_{\infty}, \rho) \int_{N(K, \frac{\rho}{2})} |\tilde{\phi}_m|d\mu_{\mathbb{H}} \]

for each \( q \in K \) where \( N(K, r) \) denotes the closed neighborhood of radius \( r \) of \( K \) in \( \mathbb{H} \). Next we use the \( L^p \)-Schauder estimates [B-J-S, Theorem 4, Sect. II.5.5] to obtain a uniform bound for \( \nabla \tilde{\phi}_m \) on \( N(K, \frac{\rho}{2}) \). This makes \( (\tilde{\phi}_m|_{K}) \) an equicontinuous family. So, by Arzela-Ascoli theorem, up to extracting a subsequence, \( (\phi_m) \) converges to a continuous function \( \phi_{\infty} \) on \( K \). By a diagonalization argument one may suppose that the sequence works for all compact subsets of \( \mathbb{H} \). Therefore, up to extracting a subsequence, \( \phi_m \to \phi_{\infty} \).
uniformly over compacta. By this uniform convergence it is clear that \( \tilde{\phi}_\infty \) is a weak solution of the Laplace equation: \( \Delta u + \lambda_\infty u = 0 \). Therefore, by elliptic regularity, \( \tilde{\phi}_\infty \) indeed a smooth and satisfies
\[
\Delta \tilde{\phi}_\infty + \lambda_\infty \tilde{\phi}_\infty = 0.
\]
Also by the convergence \( \tilde{\phi}_\infty \) induces a function \( \phi_\infty \) on \( S_\infty \) that satisfies
\[
\Delta \phi_\infty + \lambda_\infty \phi_\infty = 0.
\]
However, \( \phi_\infty \) may not be an eigenfunction since it could be the zero function. In order to discuss this point, we shall consider two cases according to whether the \( L^2 \)-norm \( \| \phi_m \|_{S_m^{(\epsilon, \infty)}} \) of the restriction of \( \phi_m \) to \( S_m^{(\epsilon, \infty)} \) is bounded below by a positive constant or not.

**Case 1:** \( \exists \epsilon, \delta > 0 \) such that \( \lim \sup \| \phi_m \|_{S_m^{(\epsilon, \infty)}} \geq \delta \). We may assume that \( \lim \| \phi_m \|_{S_m^{(\epsilon, \infty)}} \geq \delta \). Then by the uniform convergence of \( \tilde{\phi}_m \to \tilde{\phi}_\infty \) over compacta,
\[
\int_{S_m^{(\epsilon, \infty)}} \phi_m^2 d\mu_m = \lim_{m \to \infty} \int_{S_m^{(\epsilon, \infty)}} \phi_m^2 d\mu_m \geq \delta > 0.
\]
Therefore \( \phi_\infty \) is not the zero function and its \( L^2 \) norm is less than 1. Therefore it is a \( \lambda_\infty \)-eigenfunction.

**Case 2:** For any \( \epsilon > 0 \) the sequence \( \| \phi_m \|_{S_m^{(\epsilon, \infty)}} \to 0 \). Then we will prove the following statements:
(i) \( S_\infty \in \partial \mathcal{M}_{g,n} \),
(ii) \( \lambda_\infty = \frac{1}{4} \) and
(iii) \( \exists \) constants \( K_m \) such that, up to extracting a subsequence, \( (K_m \tilde{\phi}_m) \) converges uniformly to a function which is a linear combination of Eisenstein series and (possibly) a \( \frac{1}{4} \)-cuspidal eigenfunction.

(i) Suppose by contradiction that \( S_\infty \in \mathcal{M}_{g,n} \). Then all the cusps of \( S_\infty \) are old cusps. Let \( s(S_\infty) \) denote the systole of \( S_\infty \). Then, for \( 0 < \epsilon < \frac{s(S_\infty)}{2} \) and for \( m \) large enough, we have \( S_m^{(0, \epsilon)} \subset \bigcup_{i=1}^n \mathcal{P}_i(m) \). Therefore, applying Corollary 3.31, the assumption \( \| \phi_m \|_{S_m^{(\epsilon, \infty)}} \to 0 \) implies that \( \| \phi_m \| \to 0 \). This is a contradiction since each \( \phi_m \) is normalized. Thus \( S_\infty \in \partial \mathcal{M}_{g,n} \).

(ii) follows from Theorem 3.36.

(iii) Fix some \( \epsilon, 0 < \epsilon < \epsilon_0 \). Choose constants \( K_m \geq 1 \) such that
\[
\int_{S_m^{(\epsilon, \infty)}} |K_m \phi_m|^2 d\mu_m = 1.
\]
Therefore the sequence \( (K_m) \) must diverge to \( \infty \). Using mean value formula [F, Corollary 1.3], \( L^p \)-Schauder estimates [B-J-S] and elliptic regularity, as earlier, and Corollary 3.34 we obtain that, up to extracting a subsequence, \( (K_m \phi_m) \) converges, uniformly over compacta, to a \( C^\infty \) function \( \phi_\infty \) that satisfies
\[
\Delta \phi_\infty + \frac{1}{4} \phi_\infty = 0.
\]
Moreover, \( \tilde{\phi}_\infty \) induces a function \( \phi_\infty \) on \( S_\infty \) that satisfies
\[
\Delta \phi_\infty + \frac{1}{4} \phi_\infty = 0.
\]
Using the uniform convergence over compacta we have
\[
\int_{S_{\infty}} |\phi_\infty|^2 d\mu_\infty = \lim_{m \to \infty} \int_{S_m} K_m \phi_m^2 d\mu_m = 1.
\]
Therefore \( \phi_\infty \) is not the zero function. From Lemma 3.1 and Lemma 3.17 (3.18) we deduce that \( \phi_\infty \) satisfies moderate growth condition \([Wo]\) p. 80 in each cusp. It is known that for any \( \lambda \geq \frac{1}{2} \) the space of moderate growth \( \lambda \)-eigenfunctions of \( S_\infty \) is spanned by Eisenstein series and (possibly) \( \lambda \)-cuspidal eigenfunctions (see §3 in \([Wo]\)). In particular, \( \phi_\infty \) is a linear combination of Eisenstein series and (possibly) a cuspidal eigenfunction. This finishes the proof of \((iii)\). □

5. Proof of Theorem 1

We begin by proving Lemma 1 which says that \( C_{g,n}(k) \) is open in \( M_{g,n} \).

5.1. Proof of Lemma 1. Empty set is open by convention. Therefore, we argue by contradiction and assume that there exists a \( S \in C_{g,n}(k) \) such that every neighborhood of \( S \) contains points from \( M_{g,n} \setminus C_{g,n}(k) \). In other words, there exists a sequence \( (S_m) \subseteq M_{g,n} \) that converges to \( S \) and, for all \( m \), \( \lambda_k(S_m) \leq \frac{1}{4} \). For \( 1 \leq i \leq k \), let us denote by \( \phi^i_m \) a normalized \( \lambda^i(S_m) \)-cuspidal eigenfunction such that \( \{\phi^i_m\}^k_{i=1} \) is an orthonormal family in \( L^2(S_m) \). Since we are considering small eigenvalues, up to extracting a subsequence, the sequence \( \{\lambda_k(S_m)\} \) converges. For simplicity we assume that, for \( 1 \leq i \leq k \), the sequence \( \{\lambda^i(S_m)\} \) converges and denote by \( \lambda^i_\infty \) its limit. Observe that, for \( 1 \leq i \leq k \), \( \lambda^i_\infty \leq \frac{1}{4} \). Now, since \( S \in M_{g,n} \) by Theorem 2, up to extracting a subsequence, \( \{\phi^i_m\} \) converge to \( \lambda^i_\infty \)-eigenfunction \( \phi^i_\infty \) of \( S \). Moreover, by the result about uniform integrability inside cusps in Corollary 3.31 \( \|\phi^i_\infty\| = 1 \). Hence \( \{\phi^i_\infty\}^k_{i=1} \) is an orthonormal family in \( L^2(S) \) so that the \( k \)-th cuspidal eigenvalue \( \lambda^i_\infty(S) \) of \( S \) is below \( \frac{1}{4} \). This is a contradiction because by our assumption \( \lambda^i_\infty(S) \geq \frac{1}{4} \) as \( S \in C_{g,n}(k) \). □

Now we give a proof of Proposition 1.7 which says

**Proposition 1.7** There exists a neighborhood \( N(M_{g,1} \cup M_{0,n+1}) \) of \( M_{g,1} \cup M_{0,n+1} \) in \( M_{g,n} \) such that for each \( S \in N(M_{g,1} \cup M_{0,n+1}) \): \( \lambda^i_{2g-1}(S) > \frac{1}{4} \)
i.e.
\[
N(M_{g,1} \cup M_{0,n+1}) \subseteq C_{g,n}(2g - 1).
\]

5.2. Proof of Proposition. We argue by contradiction and assume that there is a sequence \( S_m \in M_{g,n} \) converging to \( S_\infty \) in \( M_{g,1} \cup M_{0,n+1} \subset \partial M_{g,n} \) such that \( \lambda^i_{2g-1}(S_m) \leq \frac{1}{4} \). For \( 1 \leq i \leq 2g - 1 \) and for each \( m \) we choose small cuspidal eigenpairs \( (\lambda^i_m, \phi^i_m) \) of \( S_m \) such that

(i) \( \{\phi^i_m\}^{2g-1}_{i=1} \) is an orthonormal family in \( L^2(S_m) \),

(ii) \( \lambda^i_m \) is the \( i \)-th eigenvalue of \( S_m \).
Theorem 2 provides two possible behaviors of the sequence $(\phi_m^i)$. However in our case we have Lemma 2:

**Lemma 2** For each $i$, $1 \leq i \leq 2g - 1$, up to extracting a subsequence, the sequence $(\phi_m^i)$ converges to a $\lambda_m^\infty$-eigenfunction $\phi_m^\infty$ of $S^\infty$. The limit functions $\phi_m^\infty$ and $\phi_m^\infty$ are orthogonal for $i \neq j$ i.e. $S^\infty$ has at least $2g - 1$ small eigenvalues. Moreover none of the $\phi_m^\infty$ is residual.

5.2.1. Proof of Lemma 2. By uniform convergence of $\phi_m^i$ to $\phi_m^\infty$, we have $\|\phi_m^\infty\| \leq 1$. To prove the first two statements of the lemma it is enough to prove that, for $1 \leq i \leq 2g - 1$, $\|\phi_m^i\| = 1$ because this will imply that $\phi_m^i$ is not the zero function and that $(\phi_m^i)$ is uniformly integrable over the thick parts: for any $t > 0$ there exists $\epsilon$ such that for all $m$ one has,

$$\|\phi_m^\infty\|_{S_m^{(t,\infty)}} > 1 - t.$$  

To prove that, for each $1 \leq i \leq 2g - 1$, $\|\phi_m^i\| = 1$ we argue by contradiction and assume that for some $1 \leq i \leq 2g - 1$, $\|\phi_m^i\| = 1 - \delta$. To simplify the notation, denote the sequence $(\lambda_m, \phi_m^i)$ by $(\lambda_m, \phi_m)$ and the limit $(\lambda_m^\infty, \phi_m^\infty)$ by $(\lambda_m^\infty, \phi_m^\infty)$. By Corollary 3.31 the functions $\phi_m$ are uniformly integrable over the union of cusps of $S_m$: for any $t > 0$ there exists $\epsilon > 0$ such that for all $m$ one has:

$$\|\phi_m\|_{S_m^{(t,\infty)}} < t.$$  

Since $S_m \in \mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}$ there is only one closed geodesic, $\gamma_m \subset S_m$, whose length $l_{\gamma_m}$ tends to zero. For any $l \leq 1$ and for $m$ large enough such that $l_{\gamma_m} < l$ denote by $\mathcal{C}_m \subset S_m$ the collar around $\gamma_m$ bounded by two equidistant curves of length $l$. In view of the uniform integrability inside cusps (5.1), there exists $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ there exists $m(\epsilon)$ such that for $m \geq m(\epsilon)$ we have:

$$\|\phi_m\|_{\mathcal{C}_m} > \frac{\delta}{2}.$$  

(5.2)

Now we distinguish again two cases depending on whether $\lambda_m^\infty < \frac{1}{4}$ or $\lambda_m^\infty = \frac{1}{4}$. If $\lambda_m^\infty < \frac{1}{4}$ then we have a contradiction since $\|\phi_m^\infty\| = 1$ by Theorem 3.36. Hence we may suppose that $\lambda_m^\infty = \frac{1}{4}$. So, by Theorem 2 either $\phi_m^\infty$ is the zero function or, for instance by Theorem 3.2], $\phi_m^\infty$ is cuspidal. Now recall that by lemma 3.17 we have uniform integrability of $|\phi_m^i|$: for any $t$ there exists $\epsilon$ such that for all $m$:

$$\|\phi_m^i\|_{\mathcal{C}_m} < t.$$  

Hence by (5.2), there exists $\epsilon_1$ such that for any $\epsilon \leq \epsilon_1$ the exists $m_1(\epsilon)$ such that for $m \geq m_1(\epsilon)$ one has:

$$\|\phi_m^i\|_{\mathcal{C}_m} > \frac{\delta}{4}.$$  

(5.3)

In particular, if $c(\epsilon, m) = \sup_{x \in \mathcal{C}_m^\epsilon} |\phi_m^i|_0$ then, since area of $\mathcal{C}_m^\epsilon$ is less than 1, we have for any $\epsilon \leq \epsilon_1$ and $m \geq m_1(\epsilon)$:

$$c(\epsilon, m) > \frac{\delta}{4}.$$  

(5.4)

Now we prove that $|\phi_m^i|$ is uniformly small inside $\mathcal{C}_m^\epsilon$. More precisely,
Lemma 5.5. Let \( \epsilon > 0 \) be such that \( 0 < \epsilon < 1 \). There exists a constant \( K < \infty \), independent of \( \epsilon \), and \( m_2(\epsilon) \in \mathbb{N} \) such that for \( m \geq m_2(\epsilon) \) and \( z \in \mathcal{C}_m^1 \):

\[
||\phi_m||_1(z) < K \frac{\epsilon^2}{1-\epsilon}.
\]

**Proof.** Consider the expansion of \( \phi_m \) inside \( \mathcal{C}_m^1 \) with respect to the Fermi coordinates (see 2.1.1):

\[
\phi_m(r, \theta) = a_0^m(r) + \sum_{j=1}^{\infty} \left( a_j^m(r) \cos j\theta + b_j^m(r) \sin j\theta \right). \tag{5.6}
\]

Here, for each \( j \geq 0 \), \( (a_j^m, b_j^m) \) are the \( j \)-th Fourier coefficients of \( \phi_m \) inside \( \mathcal{C}_m^1 \) and are defined for all \( |r| \leq L_{1,m} \). Recall that, for any \( \epsilon \in [l_{\gamma_m}, 1] \) we denote by \( L_{\epsilon,m} \) the number \( \cosh^{-1}(\frac{r}{\epsilon}) \). Recall also that since \( \phi_m \) is a \( \lambda_m \)-eigenfunction, \( a_j^m \) and \( b_j^m \) satisfy (3.16) with \( 2\pi l = l_{\gamma_m} \) and \( \lambda = \lambda_m \).

Therefore, for \( j \geq 1 \), one can express:

\[
(1) \quad a_j^m(r) = a_{m,j}s_{m,j}(r) + b_{m,j}c_{m,j}(r)
\]

\[
(2) \quad b_j^m(r) = a_{m,j}'s_{m,j}(r) + b_{m,j}'c_{m,j}(r) \tag{5.7}
\]

where \( s_{m,j}(r) \) and \( c_{m,j}(r) \) are the two linearly independent solutions of (3.16) with \( l = l(\gamma_m) \) and \( \lambda = \lambda_m \).

Recall that \( s_{m,j}(r)\cosh^{\frac{1}{2}}(r) \) and \( c_{m,j}(r)\cosh^{\frac{1}{2}}(r) \) satisfy:

\[
\frac{d^2u}{dr^2} = \left( \frac{1}{4\cosh^2 r} + \frac{j^2}{\cosh^2 r} \right) u.
\]

Since, for \( r \leq L_{\epsilon,m}, l^2\cosh^2 r \leq 1 \) by Claim 3.23, for each \( j \geq 1 \), there exists strictly increasing functions \( h_{m,j} : [0, L_{1,m}] \to \mathbb{R}_{>0} \) and \( k_{m,j} : [0, L_{1,m}] \to \mathbb{R}_{>0} \) such that

\[
(i) \quad s_{m,j}(r)\sqrt{\cosh(r)} = h_{m,j}(r) \cosh jr
\]

\[
(ii) \quad c_{m,j}(r)\sqrt{\cosh(r)} = k_{m,j}(r) \cosh jr. \tag{5.8}
\]

We denote by \( \mathcal{P}_{n+1}(\infty) \) and \( \mathcal{P}_{n+2}(\infty) \) the two new cusps of \( S_\infty \) that appear as the limit of \( \mathcal{C}_m^1 \) as \( m \to \infty \). Now, let us assume:

\[
\sup_{2 \in \partial \mathcal{P}_{n+1}(\infty) \cup \partial \mathcal{P}_{n+2}(\infty)} |\phi_\infty|(z) < \frac{1}{4}.
\]

Then, by the uniform convergence of \( \phi_m \) to \( \phi_\infty \) over compacta, we have a \( N \in \mathbb{N} \) such that for \( m \geq N \) and \( z \in \partial \mathcal{C}_m^1 \):

\[
|\phi_m|(z) < \frac{1}{4}.
\]

By (5.6) for any \( j \geq 1 \):

\[
|a_j^m|(|\pm L_{1,m}|) = \frac{1}{\pi} \int_0^{2\pi} \phi_m(\pm L_{1,m}, \theta) \cos j\theta d\theta \leq \frac{1}{2}. \tag{5.9}
\]

Similar calculations for \( b_j^m \) provide: \( |b_j^m|(|\pm L_{1,m}|) \leq \frac{t}{2} \). Recall that \( s_{m,j} \) is odd and \( c_{m,j} \) is even. So by (5.2.1) and (5.2.1):

\[
(i) \quad a_j^m(L_{1,m}) + a_j^m(-L_{1,m}) = 2b_{m,j}k_j(L_{1,m}) \cosh jL_{1,m} \frac{\sqrt{\cosh L_{1,m}}}{\cosh L_{1,m}}.
\]
\[(ii) \ a_j^m(L_{1,m}) - a_j^m(-L_{1,m}) = 2a_{m,j}h_j(L_{1,m}) \frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}}. \quad (5.10)\]

Therefore, by (5.9) and (5.2.1):
\[(i) \ |b_{m,j}|k_j(L_{1,m}) \frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}} < \frac{t}{2} \]
\[(ii) \ |a_{m,j}|h_j(L_{1,m}) \frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}} < \frac{t}{2}. \quad (5.11)\]

Therefore, for any \(r \leq L_{1,m}:\)
\[|a_j^m(r)| = |a_{m,j}s_{m,j}(r) + b_{m,j}c_{m,j}(r)| < |a_{m,j}|s_{m,j}(r) + |b_{m,j}|c_{m,j}(r).\]
The last term of the inequality is
\[|a_{m,j}|h_j(L_{1,m}) \frac{\cosh jr}{\sqrt{\cosh r}} + |b_{m,j}|k_j(L_{1,m}) \frac{\cosh jr}{\sqrt{\cosh r}} < t \frac{\cosh jL_{1,m}}{\sqrt{\cosh r} \ \cosh jL_{1,m}} \]

since \(h_{m,j}\) and \(k_{m,j}\) are strictly increasing functions (by (5.2.1)). Similarly,
\[|b_j^m(r)| < t \frac{\cosh jr}{\sqrt{\cosh r} \ \cosh jL_{1,m}}.\]

Hence
\[||\phi_m||_1(r, \theta) < 2t \sum_{j=1}^{\infty} \frac{\cosh jr}{\sqrt{\cosh r} \ \cosh jL_{1,m}}. \quad (5.12)\]

Since, for \(j \geq 1\), the function \(\frac{\cosh jr}{\sqrt{\cosh r}}\) is strictly increasing, for any \(r \leq L_{\epsilon,m}:\)
\[\sum_{j=1}^{\infty} \frac{\cosh jr}{\sqrt{\cosh r} \ \cosh jL_{1,m}} < \sum_{j=1}^{\infty} \frac{\cosh jL_{\epsilon,m}}{\sqrt{\cosh L_{\epsilon,m} \ \cosh jL_{1,m}}} \quad (5.13)\]

Now fix an \(\epsilon\) such that \(0 < \epsilon < 1\). Observe that \(L_{\epsilon,m} = \log(\frac{\epsilon}{\epsilon_m} + \sqrt{(\frac{\epsilon}{\epsilon_m})^2 - 1}).\) So, for \(m\) large such that \(l_{\gamma_m}\) is small compared to \(\epsilon:\)
\[\sum_{j=1}^{\infty} \frac{\cosh jL_{\epsilon,m}}{\sqrt{\cosh L_{\epsilon,m} \ \cosh jL_{1,m}}} < K' \sum_{j=1}^{\infty} e^{j-\frac{1}{2}} = K' \frac{e^\frac{1}{2}}{1 - \epsilon} \quad (5.14)\]

where the constant \(K'\) can be chosen independently of \(\epsilon\) as soon as \(m\) is larger than some number \(m_2(\epsilon) \in \mathbb{N}\. \) Therefore, by (5.12) and (5.14), for \(m \geq m_2(\epsilon)\) and \((r, \theta) \in C_m^\epsilon\)
\[||\phi_m||_1(r, \theta) < 2t K' \frac{e^\frac{1}{2}}{1 - \epsilon}. \quad (5.15)\]

This proves the lemma. □

Now fix \(\epsilon < \epsilon_1\) (see (5.3)) such that \(K' \frac{e^\frac{1}{2}}{1 - \epsilon'} < \frac{\delta}{4}\) and choose \(m \geq \max\{m_1(\epsilon), m_2(\epsilon)\}\). Then by Lemma 5.5 and (5.4): for each \(z \in C_m^\epsilon\)
\[c(\epsilon, m) > ||\phi_m||_1|(z). \quad (5.16)\]

So the parallel curve \(\alpha_m\) with distance \(r_0 \leq L_{\epsilon,m}\) from \(\gamma_m\) such that \(c = ||\phi_m||_0(r_0)\) has the property that \(\phi_m\) has constant sign on it. In other words, the nodal set \(Z(\phi_m)\) does not intersect this curve. This is a contradiction to the next lemma.
Lemma 5.17. Let $S$ be a noncompact, finite area hyperbolic surface of type $(g, n)$. Let $\gamma$ be a simple closed geodesic that separates $S$ into two connected components $T_1$ and $T_2$ such that $T_1$ is topologically a sphere with $n + 1$ punctures and $T_2$ is topologically a genus $g$ surface with one puncture. Let $f$ be a small cuspidal eigenfunction of $S$. Then the zero set $Z(f)$ of $f$ intersects every curve homotopic to $\gamma$.

**Proof.** Recall that $Z(f)$ is a locally finite graph \[\text{[Cl]}\]. Let us assume that $Z(f)$ does not intersect some curve $\tau$ homotopic to $\gamma$. We have $S \setminus \tau = T_1 \cup T_2$ and all the punctures of $S$ are contained in $T_1$. Consider the components of $T_1 \setminus Z(f)$. Recall that since $f$ is cuspidal $Z(f)$ contains all the punctures of $S$ and therefore these components give rise to a cell decomposition of a once punctured sphere. The Euler characteristic of the component $F$ containing $\tau$ as a puncture is either negative or zero (since $\gamma$ and each component of $Z(f)$ are essential; see \[\text{[O]}\]). Each component of $T_1 \setminus Z(f)$ other than $F$ (at least one such exists since $g$ changes sign in $T_1$) is a nodal domain of $f$ and hence has negative Euler characteristic \[\text{[O]}\]. Also $Z(f)$ being a graph has non-positive Euler characteristic. Let $C^+$ (resp. $C^-$) be the union of the nodal domains contained in $T_1$ which are different from $F$ and where $f$ is positive (resp. negative). Denote by $\chi(X)$ the Euler characteristic of the topological space $X$. Since the Euler characteristic of a once punctured sphere is 1, by the Euler-Poincaré formula one has:

$$1 = \chi(F) + \chi(C^+) + \chi(C^-) + \chi(Z(f)).$$

This is a contradiction because the right hand side of the equality is strictly negative. \ \[\square\]

Now we prove that $\phi_\infty$ is not a residual eigenfunction. It is clear from the uniform convergence that $\phi_\infty$ is cuspidal at the old cusps. If $\phi_\infty$ is a residual eigenfunction then the only possibility is that $\phi_\infty$ is not cuspidal at one of the two new cusps. Let us assume that $\phi_\infty$ is residual in $P_{n+1}$. Then, for sufficiently large $t$, $\phi_\infty$ has constant sign in $P_{n+1}^t$. Therefore, by the uniform convergence $\phi_m|_{S_{m,\infty}} \to \phi_\infty|_{S_{\infty,\infty}}$ it follows that, for all $m$ large, $\phi_m$ has...
constant sign on a component of \( \partial C^1 \). Since this component is homotopic to \( \gamma_m \) this leads to a contradiction to Lemma 5.17 as well. This finishes the proof of Lemma 2.

\[ \square \]

\[ 5.2.2. \textit{Continuation of Proof of Proposition.} \]

Let us denote the two components of \( S_\infty \) by \( N_1 \) and \( N_2 \) such that \( N_1 \in \mathcal{M}_{g,1} \) and \( N_2 \in \mathcal{M}_{0,n+1} \). Lemma 2 says that \( S_\infty \) must have at least \( 2g-1 \) many small cuspidal eigenvalues. By [O-R Théorème 0.2] the number of non-zero small eigenvalues of \( N_1 \) is at most \( 2g-2 \). In particular, the number of small cuspidal eigenvalues of \( N_1 \) is at most \( 2g-2 \). Thus for some \( i, 1 \leq i \leq 2g-1 \), \( \phi^i_\infty \) is not the zero function when restricted to \( N_2 \) i.e. \( \phi^i_\infty \) is a cuspidal eigenfunction of \( N_2 \). This is a contradiction because \( N_2 \) does not have any small cuspidal eigenfunction by \[ H \] or \[ O \].

\[ \square \]

\[ \textbf{Remark 5.18.} \] The arguments in the proof of Proposition are applicable to more general settings. In particular, let \((S_m)\) be a sequence in \( \mathcal{M}_{g,n} \) that converges to \( S_\infty \in \partial \mathcal{M}_{g,n} \). Let \((\lambda_m, \phi_m)\) be a normalized small eigenpair of \( S_m \). Let \( \lambda_m \to \lambda_\infty \) as \( m \) tends to infinity. The arguments show the following: If \( \lim \inf_{m \to \infty} \| \phi_m \| < 1 \) then there exists a curve \( \alpha_m \), homotopic to a geodesic of length tending to zero, on which, up to extracting a subsequence, \( \phi_m \) has constant sign.

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