On the $L^p$-Theory of Vector-Valued Elliptic Operators

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Abstract
In this paper, we study vector-valued elliptic operators of the form $L f := \text{div}(Q \nabla f) - F \cdot \nabla f + \text{div}(C f) - V f$ acting on vector-valued functions $f : \mathbb{R}^d \to \mathbb{R}^m$ and involving coupling at zero and first order terms. We prove that $L$ admits realizations in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, for $1 < p < \infty$, that generate analytic strongly continuous semigroups provided that $V = (v_{ij})_{1 \leq i, j \leq m}$ is a matrix potential with locally integrable entries satisfying a sectoriality condition, the diffusion matrix $Q$ is symmetric and uniformly elliptic and the drift coefficients $F = (F_{ij})_{1 \leq i, j \leq m}$ and $C = (C_{ij})_{1 \leq i, j \leq m}$ are such that $F_{ij}, C_{ij} : \mathbb{R}^d \to \mathbb{R}^d$ are bounded. We also establish a result of local elliptic regularity for the operator $L$, we investigate on the $L^p$-maximal domain of $L$ and we characterize the positivity of the associated semigroup. Moreover, we prove $(L^p - L^q)$-estimates and Gaussian upper bounds for kernels associated to the operator $L$.

Keywords
Elliptic operator · Semigroup · Sesquiliner form · System of PDE · Local elliptic regularity · Maximal domain

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1 Introduction

The present paper deals with a class of vector-valued elliptic operators of the form

\[ \mathcal{L} f = \text{div}(Q \nabla f) - F \cdot \nabla f + \text{div}(C f) - V f \]  

(1.1)

acting on vector-valued functions \( f : \mathbb{R}^d \to \mathbb{R}^m \), for some integers \( d, m \geq 1 \), and involving coupling through the first and zero order terms. More precisely, for \( f = (f_1, \ldots, f_m) : \mathbb{R}^d \to \mathbb{R}^m \), one has

\[ (\mathcal{L} f)_i = \text{div}(Q \nabla f_i) - \sum_{j=1}^{m} F_{ij} \cdot \nabla f_j + \sum_{j=1}^{m} \text{div}(f_j C_{ij}) - \sum_{j=1}^{m} v_{ij} f_j \]

for each \( i \in \{1, \ldots, m\} \).

We point out that the operator \( \mathcal{L} \) appears in the study of Navier-Stokes equations. More precisely, in [33, 34], H. Triebel used a reduced form of Navier–Stokes type equations on \( \mathbb{R}^n \) (where \( d = m = n \) in such case) that matches vector-valued semilinear parabolic evolution equations via the Leray/Helmoltz projector, see [33, Chapter 6] for details. Moreover, a similar reduction method were applied in [18, 19] to convert Navier-Stokes equation to a semilinear parabolic system. The linear operator in [18, 19] is more appropriate to our situation. Besides, parabolic systems appear also in the study of Nash equilibrium for stochastic differential games, see [14, 15, 26] and [1, Sect. 6].

The theory of elliptic operators, in the scalar case, is by now well developed, see for instance [28] and [23] for bounded and unbounded coefficients respectively. However, the situation is quite different in the vector-valued case. Especially when dealing with coupling terms and unbounded coefficients. Indeed, the interest into operators as in (1.1) in the whole space with possibly unbounded coefficients has started in 2009 by Hieber et al. [17] with coupling through the lower order term of the elliptic operator and the motivation were the Navier-Stokes equation. Afterwards, few papers appeared, see [1, 3, 4, 13, 21, 22, 24, 25]. In [1, 3, 13] the authors studied the associated parabolic equation in \( C_b \)-spaces, assuming, among others, that the coefficients of the elliptic operator are Hölder continuous. In [13], solution to the parabolic system has been extrapolated to the \( L^p \)-scale provided the uniqueness.

In what concerns a Schrödinger type operator \( \mathcal{A} = \text{div}(Q \nabla \cdot) - V \), which corresponds to \( F = C = 0 \) in (1.1), a study in \( L^p \)-spaces is developed in [21, 22, 24, 25] following the semigroup approach, which leads to interesting results. The study of Schrödinger operators with matrix potential has been motivated by its application in mathematical physics. Indeed, the Schrödinger operator \( \mathcal{A} \) acts as a (minus) Hamiltonian of a non adiabatic system of interacting adsorbate and substrate atoms in the borderline of the Born–Oppenheimer approximation, see for instance [7, 36, 37].

For the literature review about the \( L^p \)-theory of the operator \( \mathcal{A} \), we summarize the known results as follows: In [24], it has been associated a sesquilinear form to \( \mathcal{A} \), for symmetric potential \( V \), and it has been established a consistent \( C_0 \)-semigroup in \( L^p(\mathbb{R}^d, \mathbb{R}^m) \), \( p \geq 1 \), which is also analytic. This is done by assuming that \( V \) is
pointwisely semi-definite positive with locally integrable entries and $Q$ is symmetric, bounded and satisfies the well-known ellipticity condition. Moreover, the author investigated on compactness and positivity of the semigroup. In [22], the authors associated a $C_0$-semigroup, in $L^p$-spaces, which is not necessarily analytic, to the Schrödinger operator with typically nonsymmetric potential, provided that the diffusion matrix $Q$ is, in addition to the ellipticity condition, differentiable, bounded together with its first derivatives, $V$ is semi–definite positive and its entries are locally bounded. Here, the authors followed the approach adopted by Kato in [20] for scalar Schrödinger operators with complex potential. The main tool has been local elliptic regularity and a Kato’s type inequality for vector-valued functions, i.e.,

$$\Delta_Q |f| \geq \frac{1}{|f|} \sum_{j=1}^m f_j \Delta_Q f_j \chi_{\{f \neq 0\}},$$  

for smooth functions $f : \mathbb{R}^d \to \mathbb{R}^m$, where $\Delta_Q := \text{div}(Q \nabla \cdot)$, see [22, Proposition 2.3]. Further properties such as maximal domain and others have been also investigated. The papers [21, 25] focused on the domain of the operator and further regularity properties. So that, under growth and smoothness assumptions on $V$, the authors proved that the domain of $\mathcal{A}$ is continuously embedded in $W^{2,p}(\mathbb{R}^d, \mathbb{R}^m)$, for $p \in (1, \infty)$. Furthermore, analysis of the semigroup kernel and, in the case of a symmetric potential, asymptotic behavior of the eigenvalues have been studied in [25].

In this article, using form methods and extrapolation techniques, we give a general results of existence of analytic strongly continuous semigroup $\{S_p(t)\}_{t \geq 0}$ associated to suitable realizations of $\mathcal{L}$ in $L^p$-spaces, for $1 < p < \infty$, under mild assumptions on the coefficients of $\mathcal{L}$. Namely, we assume that $Q$ is bounded and elliptic, $F$ and $C$ are bounded with a semi–boundedness condition on their divergences and $V$ has locally integrable entries and satisfies the following pointwise sectoriality condition

$$|\text{Im} \langle V(x)\xi, \xi \rangle| \leq M |\text{Re} \langle V(x)\xi, \xi \rangle|,$$  

(1.2)

for all $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{C}^m$. We point out that $V$ is not necessarily bounded, nor symmetric. For further regularity, we assume that the entries of $Q$ are in $C^1_b(\mathbb{R}^d)$ and $V$ is locally bounded. Note that, in [22, Proposition 5.4], see also [25, Proposition 4.5], the inequality (1.2) has been stated as a sufficient condition for the analyticity of the semigroup generated by realizations of $\mathcal{A}$ in $L^p(\mathbb{R}^d, \mathbb{R}^m)$, $p \in (1, \infty)$. Moreover, by [21, Example 4.3], one can see that without such a condition one may not have an analytic semigroup. Note also that, even in the scalar case, the existence of a semigroup in $L^p$-spaces associated to elliptic operators with unbounded drift and/or diffusion terms is not a general fact, see [31] and [27, Proposition 3.4 and Proposition 3.5]. Finally, we point out that our approach does not allow to consider coupling through the diffusion (second order) term, since the associated semigroups are not $L^p$-contractive, see for instance [9].

In addition, we establish a result of local elliptic regularity for solutions to elliptic systems, see Theorem 4.2. Namely, for given two vector-valued locally $p$–integrable
functions \( f, g \in L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m) \) satisfying \( \mathcal{L}f = g \) in a weak sense (distribution sense). Then \( f \) belongs to \( W^{2,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m) \), for \( p \in (1, \infty) \). This result generalizes [2, Theorem 7.1] to the vector-valued case. Thanks to this result we prove that the domain \( D(L_p) \) of \( L_p \), for \( p \in (1, \infty) \), coincides with the maximal domain:

\[
D_{p,\text{max}}(\mathcal{L}) := \{ f \in L^p(\mathbb{R}^d, \mathbb{R}^m) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m) : \mathcal{L}f \in L^p(\mathbb{R}^d; \mathbb{R}^m) \}.
\]

We also characterize the positivity of the semigroup \( \{ S_p(t) \}_{t \geq 0} \). We prove that \( \{ S_p(t) \}_{t \geq 0} \) is positive if, and only if, the operator \( \mathcal{L} \) is coupled only through the potential term and the coupling coefficients \( v_{ij}, i \neq j \), are negative or null.

To complete our study, we prove that the semigroup is ultracontractive and has a (matrix) kernel \( K(z, \cdot, \cdot) := (k_{ij}(z, \cdot, \cdot))_{1 \leq i, j \leq m} \) with entries satisfying Gaussian upper estimates (roughly) given by

\[
| k_{ij}(z, x, y) | \leq Me^{\alpha \Re z}(|\Re z|)^{-d/2} \exp \left( -N \Re \frac{|x - y|^2}{4z} \right),
\]

\( a.e. x, y \in \mathbb{R}^d, 1 \leq i, j \leq m, \)

for \( z \) belonging to a well-specified open sector in \( \mathbb{C} \). These estimates are, in fact, crucial for many applications such as the boundedness of the (sectorial) \( H_{\infty} \)-functional calculus for the operator \( \mathcal{L} \), see [6] for the scalar case, and also in the definition of the class of admissible potentials for Schrödinger operators, see [32]. This might be the subject of future works.

The organization of this paper is as follows: in Sect. 2, we associate a sesquilinear form to the operator \( \mathcal{L} \) in \( L^2(\mathbb{R}^d, \mathbb{C}^m) \) and we deduce the existence of an analytic \( C_0 \)-semigroup \( \{ S_2(t) \}_{t \geq 0} \) associated to \( \mathcal{L} \). In Sect. 3, we prove that \( \{ S_2(t) \}_{t \geq 0} \) is quasi \( L^\infty \)-contractive and we extend \( \{ S_2(t) \}_{t \geq 0} \) to an analytic \( C_0 \)-semigroup in \( L^p(\mathbb{R}^d, \mathbb{C}^m) \) by extrapolation techniques. In Sect. 4, we establish a local elliptic regularity result and we show that the domain of the generator of \( \{ S_2(t) \}_{t \geq 0} \) coincides with the maximal domain of \( \mathcal{L} \) in \( L^p(\mathbb{R}^d, \mathbb{R}^m) \), for \( p \in (1, \infty) \). In Sect. 5 we investigate on the positivity of \( \{ S_2(t) \}_{t \geq 0} \). The last Sect. 6 is devoted to the \( L^p - L^q \) estimates and Gaussian kernel estimates for real and complex times respectively.

**Notation**

Let \( \mathbb{K} \) denote the fields \( \mathbb{R} \) or \( \mathbb{C} \), \( d, m \geq 1 \) any integers, \( \langle \cdot, \cdot \rangle \) the inner-product of \( \mathbb{K}^N, N = d, m \). So that, for \( x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in \mathbb{C}^N, \langle x, y \rangle = \sum_{i=1}^N x_i \bar{y}_i \) and \( x \cdot y = \sum_{i=1}^N x_i y_i \).

The space \( L^p(\mathbb{R}^d, \mathbb{K}^m), 1 < p < \infty \), is the vector-valued Lebesgue space endowed with the norm

\[
\| \cdot \|_p : f = (f_1, \ldots, f_m) \mapsto \| f \|_p := \left( \int_{\mathbb{R}^d} \left( \sum_{j=1}^m |f_j|^2 \right)^{p/2} \, dx \right)^{\frac{1}{p}}.
\]
We denote by $\langle \cdot, \cdot \rangle_{p,p'}$ the duality product between $L^p(\mathbb{R}^d, K^m)$ and $L^{p'}(\mathbb{R}^d, K^m)$ for $1 < p < \infty$ where $p' = \frac{p}{p-1}$. For $p = 2$, we denote it simply by $\langle \cdot, \cdot \rangle_2$.

We write $f \in L^p_{\text{loc}}(\mathbb{R}^d, K^m)$ if $\chi_B f$ belongs to $L^p(\mathbb{R}^d, K^m)$ for every bounded $B \subset \mathbb{R}^d$, with $\chi_B$ is the indicator function of $B$.

For $k \in \mathbb{N}$, $W^{k,p}(\mathbb{R}^d, K^m)$ denotes the vector-valued Sobolev space constituted of vector-valued functions $f = (f_1, \ldots, f_m)$ such that $f_j \in W^{k,p}(\mathbb{R}^d)$, for all $j \in \{1, \ldots, m\}$, where $W^{k,p}(\mathbb{R}^d)$ is the classical Sobolev space of order $k$ over $L^p(\mathbb{R}^d)$. Note that all the derivatives are considered in the distribution sense. $W^{k,p}_{\text{loc}}(\mathbb{R}^d, K^m)$ is the set of all measurable functions $f$ such that the distributional derivative $\partial^\alpha f$ belongs to $L^p_{\text{loc}}(\mathbb{R}^d, K^m)$, for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq k$. For $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we write $y \geq 0$ if $y_j \geq 0$ for all $j \in \{1, \ldots, m\}$.

2 The Sesquilinear Form and the Semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$

We consider the following differential expression

$$
\mathcal{L} f = \text{div}(Q \nabla f) - F \cdot \nabla f + \text{div}(C f) - V f,
$$

(2.1)

where $f : \mathbb{R}^d \to \mathbb{R}^m$ and the derivatives are considered in the sense of distributions. Here, $Q = (q_{ij})_{1 \leq i, j \leq d}$ and $V = (v_{ij})_{1 \leq i, j \leq m}$ are matrices where the entries are scalar functions: $v_{ij}, q_{ij} : \mathbb{R}^d \to \mathbb{R}$, and $F = (F_{ij})_{1 \leq i, j \leq m}$ and $C = (C_{ij})_{1 \leq i, j \leq m}$ are matrix functions with vector-valued entries: $F_{ij}, C_{ij} : \mathbb{R}^d \to \mathbb{R}^d$. So that

$$
(\text{div}(Q \nabla f))_i = \text{div}(Q \nabla f_i),
$$

$$
(F \cdot \nabla f)_i = \sum_{j=1}^m (F_{ij}, \nabla f_j)
$$

and

$$
(\text{div}(C f))_i = \sum_{j=1}^m \text{div}(f_j C_{ij})
$$

and

$$
(V f)_i = \sum_{j=1}^m v_{ij} f_j
$$

for each $i \in \{1, \ldots, m\}$.

Actually, for $f = (f_1, \ldots, f_m) \in W^{1,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m)$ for some $1 < p < \infty$, $\text{div}(Q \nabla f)$, $F \cdot \nabla f$ and $\text{div}(C f)$ are vector-valued distributions and are defined as follow

$$
(\text{div}(Q \nabla f), \phi) = - \int_{\mathbb{R}^d} \sum_{i=1}^m (Q \nabla f_i, \nabla \phi_i) \, dx,
$$
\[(F \cdot \nabla f, \phi) = \sum_{j=1}^{m} \int_{\mathbb{R}^d} (F_{ij} \cdot \nabla f_j) \tilde{\phi}_i \, dx,\]

and

\[(\text{div}(C f), \phi) = -\sum_{j=1}^{m} \int_{\mathbb{R}^d} f_j \langle C_{ij}, \nabla \phi_i \rangle \, dx\]

for every \(\phi = (\phi_1, \ldots, \phi_m) \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)\).

Throughout this paper we make the following assumptions

**Hypotheses (H1):**

- \(Q: \mathbb{R}^d \to \mathbb{R}^{d \times d}\) is measurable such that, for every \(x \in \mathbb{R}^d\), \(Q(x)\) is symmetric and there exist \(\eta_1, \eta_2 > 0\) such that
  \[\eta_1 |\xi|^2 \leq \langle Q(x) \xi, \xi \rangle \leq \eta_2 |\xi|^2.\]  \tag{2.2}

- For all \(x, \xi \in \mathbb{R}^d\).
- \(F_{ij}, C_{ij} \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)\), for all \(i, j \in \{1, \ldots, d\}\).
- \(v_{ij} \in L^1_{\text{loc}}(\mathbb{R}^d)\), for every \(i \in \{1, \ldots, m\}\) and there exists \(M > 0\) such that
  \[|\text{Im} \langle V(x) \xi, \xi \rangle| \leq M |\text{Re} \langle V(x) \xi, \xi \rangle|,\]  \tag{2.3}

- For all \(x \in \mathbb{R}^d\) and all \(\xi \in \mathbb{C}^m\).

Let us define, for every \(x \in \mathbb{R}^d\), \(V_s(x) := \frac{1}{2}(V(x) + V^*(x))\) to be the symmetric part of \(V(x)\), where \(V^*(x)\) is the conjugate matrix of \(V(x)\). \(V_{as}(x) := V(x) - V_s(x)\) denotes the antisymmetric part of \(V(x)\).

We start by a technical lemma

**Lemma 2.1** Let \(x \in \mathbb{R}^d\) and assume \(V\) satisfying (2.3). Then

\[|\langle V(x) \xi_1, \xi_2 \rangle| \leq (1 + M)|V_s(x) \xi_1, \xi_1|^{1/2}(V_s(x) \xi_2, \xi_2)^{1/2}\]  \tag{2.4}

for every \(\xi_1, \xi_2 \in \mathbb{C}^m\). Moreover, the inequality holds true also when substituting \(V\) by \(V_{as}\).

*In particular,*

\[\left| \int_{\mathbb{R}^d} \langle V_{as}(x) f(x), g(x) \rangle \, dx \right| \leq (1 + M)\|V_s^{1/2} f\|_{L^2(\mathbb{R}^d, \mathbb{C}^m)}\|V_s^{1/2} g\|_{L^2(\mathbb{R}^d, \mathbb{C}^m)}\]  \tag{2.5}

for every measurable \(f\) and \(g\) such that \(V_s^{1/2} f, V_s^{1/2} g \in L^2(\mathbb{R}^d, \mathbb{C}^m)\).
Proof} For $x \in \mathbb{R}^d$, $\langle V(x), \cdot \rangle$ is a sesquilinear form over $\mathbb{C}^m$. Taking into account that, for every $\xi \in \mathbb{C}^m$, $\text{Re} \langle V(x)\xi, \xi \rangle = \langle V_s(x)\xi, \xi \rangle$. Then, (2.4) follows by (2.3) and [28, Proposition 1.8]. Moreover, (2.4) holds true also when taking $V_{as}$ instead of $V$ in the left hand side of the inequality. Now, Cauchy Schwartz inequality yields (2.5). □

Let us now consider the sesquilinear form a given by

$$a(f, g) := \sum_{i=1}^{m} \int_{\mathbb{R}^d} \langle Q \nabla f_i, \nabla g_i \rangle \, dx + \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} (F_{ij} \cdot \nabla f_j) \bar{g}_i \, dx$$

$$+ \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} f_j \langle C_{ij}, \nabla g_i \rangle \, dx + \int_{\mathbb{R}^d} \langle Vf, g \rangle \, dx,$$

with domain

$$D(a) = \{ f \in H^1(\mathbb{R}^d, \mathbb{C}^m) : \int_{\mathbb{R}^d} \langle V_s f, f \rangle \, dx < \infty \} := D(a_0),$$

where

$$a_0(f, g) = \sum_{j=1}^{m} \langle Q \nabla f_j, \nabla g_j \rangle + \int_{\mathbb{R}^d} \langle V_s(x) f(x), g(x) \rangle \, dx.$$

The form $a$ satisfies the following properties

**Proposition 2.2** Assume Hypotheses (H1) are satisfied. Then,

- $a$ is densely defined;
- there exists $\omega > 0$ such that $a_\omega := a + \omega$ is accretive: $\text{Re} a(f) + \omega \| f \|_2^2 \geq 0$, for all $f \in D(a)$;
- $a$ is continuous;
- $a$ is closed on $D(a)$.

**Proof** Clearly, $C^\infty_c(\mathbb{R}^d, \mathbb{C}^m) \subseteq D(a)$ and thus, $a$ is densely defined. Moreover, by application of Young’s inequality, one obtains, for every $f \in D(a)$ and every $\varepsilon > 0$,

$$\text{Re} a(f) = \sum_{i=1}^{m} \int_{\mathbb{R}^d} |\nabla f_i|^2 \, dx + \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \text{Re} \left( (F_{ij} \cdot \nabla f_j) \bar{f_i} \right) \, dx$$

$$+ \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \text{Re} \left( f_j \langle C_{ij}, \nabla f_i \rangle \right) \, dx + \int_{\mathbb{R}^d} \text{Re} \langle Vf, f \rangle \, dx$$

$$\geq \eta_1 \sum_{i=1}^{m} \int_{\mathbb{R}^d} |\nabla f_i|^2 \, dx - \| F \|_{\infty} + \| C \|_{\infty} \int_{\mathbb{R}^d} \sum_{i=1}^{m} |f_i| \sum_{i=1}^{m} |\nabla f_i| \, dx$$

$$\geq (\eta_1 - \varepsilon) \int_{\mathbb{R}^d} \sum_{i=1}^{m} |\nabla f_i|^2 \, dx - c_\varepsilon \int_{\mathbb{R}^d} \sum_{i=1}^{m} |f_i|^2 \, dx.$$
So by choosing $\varepsilon = \eta_1/2$ and $\omega \geq c\eta_1/2$, one obtains $\Re a(f) + \omega \|f\|^2_2 \geq 0$, which shows that $a_\omega$ is accretive.

On the other hand, according to [24, Proposition 2.1], $(D(a), \| \cdot \|_{a_0})$ is a Banach space, where

$$\| \cdot \|_{a_0} := \sqrt{\| \cdot \|_2^2 + a_0(\cdot)}.$$ 

It is then enough to show that $\| \cdot \|_a$ is equivalent to $\| \cdot \|_{a_0}$ to conclude the closedness of $a$, where $\| \cdot \|_a$ is the graph norm associated to $a$ and it is given by

$$\| \cdot \|_a := \sqrt{(1 + \omega)\| \cdot \|_2^2 + \Re a(\cdot)}.$$ 

Here $\omega$ is such that $a_\omega$ is accretive. Let us first prove that $\| \cdot \|_a \lesssim \| \cdot \|_{a_0}$. Let $f \in D(a)$, one has $a(f) = a_0(f) + b(f)$, where

$$b(f) := \sum_{i,j=1}^m \int_{\mathbb{R}^d} (F_{ij} \cdot \nabla f_j) \bar{f}_i \, dx + \sum_{i,j=1}^m \int_{\mathbb{R}^d} f_j (C_{ij}, \nabla f_i) \, dx.$$ 

The claim then follows by application of Young’s inequality when estimating $b$ as in the align above. Conversely, since $a_0(f) = a(f) - b(f)$, in a similar way one deduces that $\| \cdot \|_{a_0} \lesssim \| \cdot \|_a$.

It remains to show that $a$ is continuous in $(D(A), \| \cdot \|_{a_0})$, that is

$$|a(f, g)| \leq c \|f\|_{a_0} \|g\|_{a_0}, \quad \forall f, g \in D(a).$$

In view of (2.5), Cauchy-Schwartz inequality and the continuity of $a_0$, c.f. [24, Proposition 2.1 (iii)], one gets

$$|a(f, g)| \leq |a_0(f, g)| + |b(f, g)| + \left| \int_{\mathbb{R}^d} (Vas f, g) \, dx \right|$$

$$\leq c_1 \|f\|_{a_0} \|g\|_{a_0} + c_2 \|f\|_{H^1(\mathbb{R}^d, \mathbb{C}^m)} \|g\|_{H^1(\mathbb{R}^d, \mathbb{C}^m)}$$

$$+ c_3 \|V^{1/2} f\|_2 \|V^{1/2} g\|_2$$

$$\leq c \|f\|_{a_0} \|g\|_{a_0}.$$

\[\Box\]

We, finally, conclude the main theorem of this section as an immediate consequence of [28, Proposition 1.51 and Theorem 1.52] and Proposition 2.2.

**Theorem 2.3** Assume Hypotheses (H1) are satisfied. Then, $\mathcal{L}$ admits a realization $L = L_2$ in $L^2(\mathbb{R}^d, \mathbb{C}^m)$, that generates an analytic $C_0$-semigroup $\{S_2(t)\}_{t \geq 0}$. Moreover, there exists $\omega \geq 0$ such that

$$\|S_2(t)\|_2 \leq \exp(\omega t), \quad \text{for every } t \geq 0.$$
3 Extrapolation of the Semigroup to the $L^p$–Scale

In this section we extrapolate \( \{S_2(t)\}_{t \geq 0} \) to an analytic strongly continuous semigroup in \( L^p(\mathbb{R}^d, \mathbb{R}^m) \). For that purpose, it suffices to prove that there exists \( \tilde{\omega} \in \mathbb{R} \) such that \( \{S_{2}^{\tilde{\omega}}(t) := \exp(-\tilde{\omega}t)S_2(t)\}_{t \geq 0} \) satisfies the following \( L^\infty \)-contractivity property:

\[
\|S_{2}^{\tilde{\omega}}(t)f\|_\infty \leq \|f\|_\infty, \quad \forall f \in L^2(\mathbb{R}^d, C^m) \cap L^\infty(\mathbb{R}^d, C^m). \tag{3.1}
\]

From now on, we use the following notation:

\[
\langle y, z \rangle_Q(x) := \langle Q(x)y, z \rangle
\]

and

\[
|y|_Q(x) := \sqrt{\langle Q(x)y, y \rangle},
\]

for every \( x, y, z \in \mathbb{R}^d \). We also drop the \( x \) and denotes simply \( \langle \cdot, \cdot \rangle_Q \) and \( |\cdot|_Q \) for the ease of notation.

In this section we make use of the following hypotheses

**Hypotheses (H2):**

- \( F_{ij}, C_{ij} \in W^{1,\infty}_{loc}(\mathbb{R}^d, \mathbb{R}^d) \), for all \( i, j \in \{1, \ldots, m\} \), and there exists \( \gamma \in \mathbb{R} \) such that

\[
\langle \text{div}(F)(x)\xi, \xi \rangle := \sum_{i,j=1}^{m} \text{div}(F_{ij})(x)\xi_i\xi_j \leq \gamma|\xi|^2 \tag{3.2}
\]

and

\[
\langle \text{div}(C)(x)\xi, \xi \rangle := \sum_{i,j=1}^{m} \text{div}(C_{ij})(x)\xi_i\xi_j \leq \gamma|\xi|^2 \tag{3.3}
\]

for every \( \xi \in \mathbb{R}^m \) and \( x \in \mathbb{R}^d \).

We state, now, the first result of this section

**Proposition 3.1** Assume Hypotheses (H1) and (H2). Then there exists \( \tilde{\omega} \in \mathbb{R} \) such that \( \{S_{2}^{\tilde{\omega}}(t) := \exp(-\tilde{\omega}t)S_2(t)\}_{t \geq 0} \) is \( L^\infty \)-contractive.

**Proof** According to the characterization of \( L^\infty \)-contractivity property given by [30, Theorem 1], it suffices to prove that: for \( \tilde{\omega} \geq 0 \) such that \( a_{\tilde{\omega}} \) is accretive, the following statements hold:

1. \( f \in D(a) \) implies \( (1 \wedge |f|)\text{sign}(f) \in D(a) \),
2. \( \text{Re} \ a_{\tilde{\omega}}(f, f - (1 \wedge |f|)\text{sign}(f)) \geq 0, \quad \forall f \in D(a) \),
where \( \text{sign}(f) := \frac{f}{|f|} \chi_{\{f \neq 0\}} \). The first item follows by [24, Lemma 3.2]. Let us show (2). Set \( \mathcal{P}_f := (1 \wedge |f|) \text{sign}(f) \) and let \( \tilde{\omega} \) be large enough, so that \( a_{\tilde{\omega}} \) is accretive and \( \tilde{\omega} \geq \gamma \). According to [24, Lemma 3.2], we claim that

\[
\nabla (\mathcal{P}_f)_i = \frac{1 + \text{sign}(1 - |f|) f_i}{2} \frac{|f|}{|f|} \chi_{\{f \neq 0\}} \nabla |f| + \frac{1 \wedge |f|}{|f|} (\nabla f_i - \frac{f_i}{|f|} \nabla |f|) \chi_{\{f \neq 0\}} \nabla |f| \nabla f_i + \left( \frac{1 + \text{sign}(1 - |f|)}{2} - \frac{1 \wedge |f|}{|f|} \right) \frac{f_i}{|f|} \chi_{\{f \neq 0\}} \nabla |f| \chi_{\{f \neq 0\}} \nabla |f| \nabla f_i
\]

(3.4)

for every \( i \in \{1, \ldots, m\} \). Therefore,

\[
a_{\tilde{\omega}}(f, (f - \mathcal{P}_f)) := \sum_{i=1}^{m} \int_{\mathbb{R}^d} \langle Q \nabla f_i, \nabla (f - \mathcal{P}_f)_i \rangle \, dx + \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} (F_{ij} \cdot \nabla f_j) (f - \mathcal{P}_f)_i \, dx + \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} f_j \langle C_{ij}, \nabla (f - \mathcal{P}_f)_i \rangle \, dx
\]

\[
+ \int_{\mathbb{R}^d} (V f, (f - \mathcal{P}_f)) \, dx + \tilde{\omega} (f, (f - \mathcal{P}_f))_2
\]

\[= a_0(f, f - \mathcal{P}_f) + b(f, f - \mathcal{P}_f) + \int_{\mathbb{R}^d} (V f, (f - \mathcal{P}_f)) \, dx + \tilde{\omega} (f, (f - \mathcal{P}_f))_2,
\]

where

\[
a_0(f, f - \mathcal{P}_f) := \sum_{i=1}^{m} \int_{\mathbb{R}^d} \langle Q \nabla f_i, \nabla (f - \mathcal{P}_f)_i \rangle \, dx
\]

and

\[
b(f, f - \mathcal{P}_f) := \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} (F_{ij} \cdot \nabla f_j) (f - \mathcal{P}_f)_i \, dx + \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} f_j \langle C_{ij}, \nabla (f - \mathcal{P}_f)_i \rangle \, dx.
\]

Now, one has

\[
\int_{\mathbb{R}^d} (V f, (f - \mathcal{P}_f)) \, dx = \int_{\mathbb{R}^d} \left( 1 - \frac{1 \wedge |f|}{|f|} \chi_{\{f \neq 0\}} \right) \langle V f, f \rangle \, dx.
\]
Consequently, since by (2.3), Re \( \langle Vf, f \rangle \geq 0 \) a.e., it follows that

\[
E_1 := \text{Re} \int_{\mathbb{R}^d} \langle Vf, (f - \mathcal{P}f) \rangle \, dx \geq 0. \quad (3.5)
\]

On the other hand,

\[
\tilde{a}_0(f, f - \mathcal{P}f) = \tilde{a}_0(f, f) - \tilde{a}_0(f, \mathcal{P}f)
\]

\[
= \sum_{i=1}^{m} \int_{\mathbb{R}^d} \left( 1 - \frac{1 \wedge |f|}{|f|} \chi_{\{|f| \neq 0\}} \right) \langle Q \nabla f_i, \nabla f \rangle \, dx
\]

\[
+ \sum_{i=1}^{m} \int_{\mathbb{R}^d} \left[ \frac{1 \wedge |f|}{|f|} \chi_{\{|f| \neq 0\}} - \left( \frac{1 + \text{sign}(1 - |f|)}{2} \right) \chi_{\{|f| \neq 0\}} \right] \beta(|f|)
\]

\[
\times \langle Q \nabla f_i, \frac{f}{|f|} \nabla |f| \rangle \, dx. \quad (3.6)
\]

Applying an integration by part, one obtains

\[
b(f, f - \mathcal{P}f) = \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \left( 1 - \frac{1 \wedge |f|}{|f|} \chi_{\{|f| \neq 0\}} \right) (F_{ij} \cdot \nabla f_j) \, \bar{f}_i \, dx
\]

\[
+ \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} f_j \langle C_{ij}, \nabla (f - \mathcal{P}f)_i \rangle \, dx.
\]

\[
= \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \left( 1 - \frac{1 \wedge |f|}{|f|} \chi_{\{|f| \neq 0\}} \right) (F_{ij} \cdot \nabla f_j) \, \bar{f}_i \, dx
\]

\[
- \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \text{div}(f_j C_{ij}) \langle \bar{f}, f - \mathcal{P}f \rangle_i \, dx
\]

\[
= \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \left( 1 - \frac{1 \wedge |f|}{|f|} \chi_{\{|f| \neq 0\}} \right) (F_{ij} \cdot \nabla f_j) \, \bar{f}_i \, dx
\]

\[
- \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} (C_{ij} \cdot \nabla f_j) \langle \bar{f}, f - \mathcal{P}f \rangle_i \, dx
\]

\[
- \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \text{div}(C_{ij}) f_j \langle \bar{f}, f - \mathcal{P}f \rangle_i \, dx
\]

\[
= \sum_{i,j=1}^{m} \int_{\mathbb{R}^d} \left( 1 - \frac{1 \wedge |f|}{|f|} \chi_{\{|f| \neq 0\}} \right) ((F_{ij} - C_{ij}) \cdot \nabla f_j) \, \bar{f}_i \, dx
\]

\[
- \langle \text{div}(C)f, f - \mathcal{P}f \rangle.
\]
Summing up one obtains

\[
\text{Re } a_\tilde{\omega}(f, (f - P_f)) = \text{Re } a_0(f, (f - P_f)) + \text{Re } b(f, f - P_f)
\]

\[
+ \text{Re } \int_{\mathbb{R}^d} ((V + \tilde{\omega}I_m) f, f - P_f) dx
\]

\[
= \int_{\mathbb{R}^d} \alpha(|f|) \sum_{i=1}^m |\nabla f_i|^2_Q dx + \int_{\mathbb{R}^d} \beta(|f|) \sum_{i=1}^m \text{Re } (\tilde{f}_i \nabla f_i, \nabla |f|)_Q dx
\]

\[
+ \sum_{i,j=1}^m \int_{\mathbb{R}^d} \left(1 - \frac{1}{|f|} \chi_{|f|0} \right) \text{Re } ((F_{ij} - C_{ij}) \cdot \nabla f_j) \tilde{f}_i dx
\]

\[
+ \text{Re } \int_{\mathbb{R}^d} ((V - \text{div}(C) + \tilde{\omega}I_m) f, f - P_f) dx
\]

\[
= \int_{\mathbb{R}^d} \alpha(|f|) J_1(f) dx + \int_{\mathbb{R}^d} \beta(|f|) J_2(f) dx
\]

\[
+ \text{Re } \int_{\mathbb{R}^d} ((V - \text{div}(C) + \tilde{\omega}I_m) f, f - P_f) dx
\]

where

\[
J_1(f) := \sum_{i=1}^m \text{Re } (Q \nabla f_i, \nabla f_i) + \sum_{i,j=1}^m \text{Re } ((F_{ij} - C_{ij}) \cdot \nabla f_j) \tilde{f}_i
\]

and

\[
J_2(f) := \frac{1}{|f|} \sum_{i=1}^m \text{Re } (\tilde{f}_i \nabla f_i, \nabla |f|)_Q.
\]

Since by [25, Lemma 2.4], one has

\[
\nabla |f| = \frac{1}{|f|} \sum_{j=1}^m \text{Re } (\tilde{f}_j \nabla f_j)
\]

\[
\nabla |f| = \frac{1}{|f|} \text{Re } (\tilde{f}_i \nabla f_i, \nabla |f|)_Q \chi_{|f|0},
\]

then,

\[
J_2(f) = \frac{1}{|f|} \sum_{i=1}^m \text{Re } (\tilde{f}_i \nabla f_i, \nabla |f|)_Q
\]

\[
= (\nabla |f|, \nabla |f|)_Q \geq 0.
\]
Therefore,

\[
\int_{\mathbb{R}^d} \beta(|f|) J_2(f) \, dx \geq 0. \tag{3.7}
\]

Moreover one gets

\[
\text{Re} \int_{\mathbb{R}^d} \left( (-\text{div}(C) + \tilde{\omega} I_m) f, f - \mathcal{P}_f \right) \, dx \\
= \int_{\mathbb{R}^d} \left( 1 - \frac{1}{|f|} \chi_{\{f \neq 0\}} \right) \left( (-\text{div}(C) + \tilde{\omega}) f, f \right) \, dx \\
\geq (\tilde{\omega} - \gamma) \int_{\mathbb{R}^d} \left( 1 - \frac{1}{|f|} \chi_{\{f \neq 0\}} \right) |f|^2 \, dx \\
= (\tilde{\omega} - \gamma) \int_{\mathbb{R}^d} \alpha(|f|) |f|^2 \, dx. \tag{3.8}
\]

Now, taking in consideration (3.5), (3.7) and (3.8), one obtains

\[
\text{Re} a_{\tilde{\omega}}(f, (f - \mathcal{P}_f)) \geq \int_{\mathbb{R}^d} \alpha(|f|) J_1(f) \, dx + (\tilde{\omega} - \gamma) \int_{\mathbb{R}^d} \alpha(|f|) |f|^2 \, dx.
\]

Moreover, in view of Young’s inequality, for every \(\varepsilon > 0\) there exists \(c_\varepsilon > 0\) such that

\[
J_1(f) \geq \eta_1 \sum_{i=1}^m |\nabla f_i|^2 - \sum_{i,j=1}^m |(F_{ij} - C_{ij}), \nabla f_j||f_i| \\
\geq \eta_1 \sum_{i=1}^m |\nabla f_i|^2 - \sup_{i,j} \|F_{ij} - C_{ij}\|_{\infty} \sum_{i,j=1}^m |\nabla f_j||f_i| \\
\geq \eta_1 \sum_{i=1}^m |\nabla f_i|^2 - \varepsilon \sum_{i=1}^m |\nabla f_i|^2 - c_\varepsilon \sum_{i=1}^m |f_i|^2 \\
= (\eta_1 - \varepsilon) \sum_{i=1}^m |\nabla f_i|^2 - c_\varepsilon |f|^2.
\]

Consequently, for \(\varepsilon\) being such that \(\eta_1 > \varepsilon\), say \(\varepsilon = \eta_1/2\), and \(\tilde{\omega} > c_{\eta_1/2} + \gamma\), one gets

\[
\text{Re} a_{\tilde{\omega}}(f, (f - \mathcal{P}_f)) \geq \int_{\mathbb{R}^d} \alpha(|f|) \left[ (\eta_1 - \varepsilon) \sum_{i=1}^m |\nabla f_i|^2 + (\tilde{\omega} - \gamma - c_\varepsilon) |f|^2 \right] \, dx \\
\geq 0
\]

and this ends the proof. \(\Box\)

Hence, we have the following main result of this section.
Theorem 3.2 Let $1 < p < \infty$ and assume Hypotheses (H1) and (H2). Then, $\mathcal{L}$ has a realization $L_p$ in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ that generates an analytic $C_0$-semigroup $\{S_p(t)\}_{t \geq 0}$.

Proof Let $2 < p < \infty$. Instead of considering $\min(\omega, \tilde{\omega})$, we assume $\omega > \tilde{\omega}$.

In view of Theorem 2.3 and Proposition 3.1, the semigroup $\{S^0_p(t)\}_{t \geq 0}$ is analytic in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ and $L^\infty$-contractive. Therefore, using the Riesz-Thorin interpolation Theorem, $\{S^0_p(t)\}_{t \geq 0}$ has a unique analytic bounded extension $\{\tilde{S}^0_p(t)\}_{t \geq 0}$ to $L^p(\mathbb{R}^d, \mathbb{C}^m)$. Moreover, for every $f \in L^2(\mathbb{R}^d, \mathbb{C}^m) \cap L^\infty(\mathbb{R}^d, \mathbb{C}^m)$, one claims

$$
\|S^0_p(t)f - f\|_p \leq \|S^0_p(t)f - f\|_2^\theta \|S^0_p(t)f - f\|_2^{1-\theta} \leq 2^{1-\theta}\|f\|_\infty \|S^0_p(t)f - f\|_2^\theta,
$$

where $\theta = \frac{2}{p}$. Since by Theorem 2.3, the semigroup $\{S^0_p(t)\}_{t \geq 0}$ is strongly continuous in $L^2(\mathbb{R}^d, \mathbb{C}^m)$, it follows directly from (3.9) that $\{S^0_p(t)\}_{t \geq 0}$ is strongly continuous in $L^p(\mathbb{R}^d, \mathbb{C}^m)$.

For the case $1 < p < 2$, we argue by duality. Indeed, the adjoint semigroup $\{S^*(t)\}_{t \geq 0}$ is associated to $\mathcal{L}^*$, the formal adjoint of $\mathcal{L}$, where

$$
\mathcal{L}^*f := \text{div}(Q\nabla f) - C^* \cdot \nabla f + \text{div}(F^* f) - V^* f.
$$

Since the coefficients of $\mathcal{L}^*$ satisfy Hypotheses (H1) and (H2), similarly to $\mathcal{L}$, then $\{S^*(t)\}_{t \geq 0}$ is an analytic $C_0$-semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$ which is quasi $L^\infty$-contractive. Consequently, $\{S(t)\}_{t \geq 0}$ is quasi contractive in $L^1(\mathbb{R}^d, \mathbb{C}^m)$. So, the same interpolation arguments yield an extrapolation of $\{S(t)\}_{t \geq 0}$ to a holomorphic $C_0$-semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^m)$, for $1 < p < 2$. \hfill \Box

Remark 3.3 a) The semigroups $\{S_p(t)\}_{t \geq 0}$, $1 < p \leq 2$, can be extrapolated to a strongly continuous semigroup in $L^1(\mathbb{R}^d, \mathbb{C}^m)$. This follows, according to [35], as a consequence of the consistency and the quasi-contractivity of $\{S_p(t)\}_{t \geq 0}$, $1 < p \leq 2$.

b) If there exists a nonnegative locally bounded function $\mu : \mathbb{R}^d \to \mathbb{R}^+$ such that $\lim_{|x| \to \infty} \mu(x) = +\infty$ and

$$
\langle V_s(x)\xi, \xi \rangle \geq \mu(x)|\xi|^2, \quad \forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^m.
$$

Then, for every $1 < p < \infty$, $L_p$ has a compact resolvent and thus $\{S_p(t)\}_{t \geq 0}$ is compact. The proof of this claim is identical to [24, Proposition 4.3].

4 Local Elliptic Regularity and Maximal Domain of $L_p$

Since the coefficients of $\mathcal{L}$ are real, from now on, we consider vector-valued functions with real components. Thus, $L_p$ acts on $D(L_p) \subset L^p(\mathbb{R}^d, \mathbb{C}^m)$, for every $p \in (1, \infty)$.
and its associated semigroup \( \{S_p(t)\}_{t \geq 0} \) acts on \( L^p(\mathbb{R}^d, \mathbb{R}^m) \). Moreover, we assume that \( C \equiv 0 \) and thus
\[
\mathcal{L} f = \text{div}(Q \nabla f) - F \cdot \nabla f - V f. \tag{4.1}
\]

Throughout this section, we use the notation \( \Delta_Q := \text{div}(Q \nabla \cdot) \) and, in addition to Hypotheses (H1), we assume the following

**Hypotheses (H3):**

- \( q_{ij} \in C^1_b(\mathbb{R}^d) \), for all \( i, j \in \{1, \ldots, d\} \).
- \( v_{ij} \in L^\infty_{\text{loc}}(\mathbb{R}^d) \), for all \( i, j \in \{1, \ldots, m\} \).

**Remark 4.1** The assumption \( C \equiv 0 \) is actually without loss of generalities. Indeed, for every \( f \in C^\infty_c(\mathbb{R}^d, \mathbb{R}^m) \), one has
\[
\tilde{\mathcal{L}} f := \text{div}(Q \nabla f) - F \cdot \nabla f + \text{div}(C f) - V f.
\]

Hence, \( \tilde{\mathcal{L}} - \gamma \) has the same expression of (4.1) and the matrices \( Q \), \( \tilde{F} := F - C \) and \( \tilde{V} := V - \text{div}(C) + \gamma I_m \) satisfy Hypotheses (H1) and (H2) as well.

### 4.1 Local Elliptic Regularity

Here we give a regularity result for weak solutions to systems of elliptic equations. The following theorem generalizes [2, Theorem 7.1] to the vector valued case.

**Theorem 4.2** Let \( p \in (1, \infty) \) and assume Hypotheses (H1)–(H3). Let \( f \) and \( g \) belong to \( L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m) \) such that \( \mathcal{L} f = g \) in the distributional sense. Then, \( f \in W^{2;p}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m) \).

**Proof** Let \( f = (f_1, \ldots, f_m) \) and \( g = (g_1, \ldots, g_m) \) belong to \( L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m) \) and assume that \( \mathcal{L} f = g \) in the sense of distributions. Hence,
\[
\Delta_Q f_i = g_i + \sum_{j=1}^m F_{ij} \cdot \nabla f_j + \sum_{j=1}^m v_{ij} f_j \tag{4.2}
\]

for each \( i \in \{1, \ldots, m\} \). Now, let \( \varphi \in C^2_c(\Omega) \) where \( \Omega \subset \mathbb{R}^d \) is any bounded set, and \( i \in \{1, \ldots, m\} \). A straightforward computation yields
\[
\Delta_Q (\varphi f_i) = \varphi \Delta_Q f_i + 2(Q \nabla \varphi) \cdot \nabla f_i + (\Delta_Q \varphi) f_i.
\]

Then, by (4.2) one gets
\[
\Delta_Q (\varphi f_i) = \varphi g_i + \sum_{j=1}^m \varphi F_{ij} \cdot \nabla f_j + 2(Q \nabla \varphi) \cdot \nabla f_i + \sum_{j=1}^m v_{ij} f_j \varphi + (\Delta_Q \varphi) f_i := \tilde{g}_i.
\]
Actually, \( \hat{g}_i \in W^{-1,p}(\mathbb{R}^d) := (W^{1,p'}(\mathbb{R}^d))' \). Indeed, since \( g_i \) and \( f_j \) belong to \( L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m) \), then \( \varphi g_i \), \( (\Delta \varphi f_i) \) and \( v_{ij} f_j \varphi \) lie in \( L^p(\mathbb{R}^d) \) and thus in \( W^{-1,p}(\mathbb{R}^d) \), for every \( j \in \{1, \ldots, m\} \). On the other hand, for every \( \psi \in C_c^\infty(\mathbb{R}^d) \), one has

\[
|(\varphi F_{ij} \cdot \nabla f_j, \psi)| = -\int_{\mathbb{R}^d} f_j \text{div}(\varphi \psi F_{ij}) \, dx \\
= \int_{\mathbb{R}^d} f_j \varphi \psi \text{div}(F_{ij}) \, dx + \int_{\mathbb{R}^d} f_j \psi \langle F_{ij}, \nabla \varphi \rangle \, dx + \int_{\mathbb{R}^d} f_j \varphi \langle F_{ij}, \nabla \psi \rangle \, dx \\
\leq (\|\text{div}(F_{ij})\varphi f_j\|_p + \|F_{ij}, \nabla \varphi\|_p) \|\psi\|_{p'} + \|F\|_\infty \|f_j \varphi\|_p \|\nabla \psi\|_{p'} \\
\leq (\|\text{div}(F_{ij})\varphi f_j\|_p + \|F_{ij}, \nabla \varphi\|_p \|f_j \varphi\|_p) \|\psi\|_{1,p'},
\]

which shows that \( \varphi F_{ij} \cdot \nabla f_j \in W^{-1,p}(\mathbb{R}^d) \), for every \( j \in \{1, \ldots, m\} \). Similarly, we get the claim for \( (Q \nabla \varphi) \cdot \nabla f_i \). Therefore, for all \( \lambda > 0 \),

\[
(\Delta \varphi - \lambda)(\varphi f_i) = \hat{g}_i - \lambda \varphi f_i \in W^{-1,p}(\mathbb{R}^d).
\]

Thus, according to [8, Proposition 2.2], \( \varphi f_i \in W^{1,p}(\mathbb{R}^d) \) and this is true for every \( \varphi \in C^2_c(\Omega) \), which implies that \( f_i \in W^{1,p}_{\text{loc}}(\mathbb{R}^d) \).

Now, going back to (4.2), one obtains \( \Delta \varphi f_i \in L^p_{\text{loc}}(\mathbb{R}^d) \). We then conclude by [2, Theorem 7.1] that \( f_i \) belongs to \( W^{2,p}_{\text{loc}}(\mathbb{R}^d) \). \( \square \)

### 4.2 \( L^p \)-Maximal Domain

The aim of this section is to coincide the domain \( D(L_p) \) of the generator of \( \{S_p(t)\}_{t \geq 0} \) with its maximal domain in \( L^p(\mathbb{R}^d, \mathbb{R}^m) \). We start by showing that \( C_c^\infty(\mathbb{R}^d; \mathbb{C}^m) \subset D(L_p) \).

**Lemma 4.3** Let \( p \geq 1 \) and assume Hypotheses (H1)–(H3). Then, \( C_c^\infty(\mathbb{R}^d, \mathbb{R}^m) \subset D(L_p) \) and \( L_p f = \mathcal{L} f \), for all \( f \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m) \).

**Proof** Let \( f \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m) \). One has \( \mathcal{L} f \in L^2(\mathbb{R}^d, \mathbb{R}^m) \) and integrating by parts, one claims \( \langle -\mathcal{L} f, g \rangle_2 = a(f, g) \), for all \( g \in D(a) \). Therefore, \( f \in D(L_2) \) and \( L_2 f = \mathcal{L} f \). Moreover, one has

\[
S_2(t) f - f = \int_0^t S_2(s) \mathcal{L} f \, ds, \quad \forall t > 0. \tag{4.3}
\]

Since \( \mathcal{L} f \in L^p(\mathbb{R}^d, \mathbb{R}^m) \), for all \( p \geq 1 \), and by consistency of the semigroups \( \{S_p(t)\}_{t \geq 0}, p \in [1, \infty) \), Equation (4.3) holds true in \( L^p(\mathbb{R}^d, \mathbb{R}^m) \), that is

\[
S_p(t) f - f = \int_0^t S_p(s) \mathcal{L} f \, ds, \quad \forall t > 0.
\]

By consequence, \( f \in D(L_p) \) and \( L_p f = \mathcal{L} f \) for all \( p \geq 1 \). \( \square \)
We next show that the space of test functions is a core for $L_p$, for $p \in (1, \infty)$. That is, $C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$ is dense in $D(L_p)$ by the graph norm.

**Proposition 4.4** Let $1 < p < \infty$ and assume Hypotheses (H1)–(H3). Then, the set of test functions $C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$ is a core for $L_p$.

**Proof** Fix $1 < p < \infty$ and let $\lambda > \gamma$ be large enough so that it belongs to $\rho(L_p)$. It suffices to prove that $(\lambda - L_p)C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ is dense in $L^{p'}(\mathbb{R}^d, \mathbb{R}^m)$. For this purpose, let $f \in L^{p'}(\mathbb{R}^d; \mathbb{R}^m)$ be such that $(\langle \lambda - \mathcal{L}, \varphi, f \rangle, p, p') = 0$, for all $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$. Then,

$$\lambda f - \Delta_Q f - F^* \cdot \nabla f + (V^* - \text{div}(F)) f = 0 \quad (4.4)$$

in the sense of distributions. By Theorem 4.2, one obtains $f_j \in W^{2,p'}_{\text{loc}}(\mathbb{R}^d)$ for all $j \in \{1, \ldots, m\}$. Then, (4.4) holds true almost everywhere on $\mathbb{R}^d$.

Now, consider $\xi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi_{B(1)} \leq \xi \leq \chi_{B(2)}$ and define $\xi_n(\cdot) = \xi(\cdot/n)$ for $n \in \mathbb{N}$. Assume $p' < 2$ and multiply (4.4) by $\xi_n(|f|^2 + \varepsilon^2)^{p'-2}f \in L^p(\mathbb{R}^d, \mathbb{R}^m)$ for $\varepsilon > 0$, $n \in \mathbb{N}$. Integrating by parts, one obtains

$$0 = \lambda \int_{\mathbb{R}^d} \xi_n(|f|^2 + \varepsilon^2)^{p'-2} |f|^2 dx - \sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla f_j, \nabla (\xi_n(|f|^2 + \varepsilon^2)^{p'-2} f_j) \rangle_Q dx$$

$$- \sum_{i,j=1}^m \int_{\mathbb{R}^d} \xi_n(|f|^2 + \varepsilon^2)^{p'-2} f_j (F_{ji}, \nabla f_j) dx$$

$$+ \int_{\mathbb{R}^d} \xi_n(|f|^2 + \varepsilon^2)^{p'-2} ((V^* - \text{div}(F^*)) f, f) dx$$

$$\geq (\lambda - \gamma) \int_{\mathbb{R}^d} \xi_n(|f|^2 + \varepsilon^2)^{p'-2} |f|^2 dx + \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla f_j| Q \xi_n(|f|^2 + \varepsilon^2)^{p'-2} f_j dx$$

$$+ \sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla f_j, \nabla \xi_n \rangle_Q (|f|^2 + \varepsilon^2)^{\frac{p'-2}{2}} f_j dx$$

$$- \|F\|_{\infty} \sum_{i,j=1}^m \int_{\mathbb{R}^d} \xi_n(|f|^2 + \varepsilon^2)^{p'-2} |f_j| |\nabla f_j| dx$$

$$+ (p' - 2) \sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla f_j, \nabla |f| \rangle_Q f_j |f| \xi_n(|f|^2 + \varepsilon^2)^{\frac{p'-4}{2}} dx$$

$$\geq (\lambda - \gamma) \int_{\mathbb{R}^d} \xi_n(|f|^2 + \varepsilon^2)^{p'-2} |f|^2 dx + \int_{\mathbb{R}^d} \sum_{j=1}^m |\nabla f_j|^2_Q \xi_n(|f|^2 + \varepsilon^2)^{p'-2} dx$$

$$+ \int_{\mathbb{R}^d} \langle Q \nabla |f|, \nabla \xi_n \rangle (|f|^2 + \varepsilon^2)^{p'-2} |f| dx$$
\[-\delta \sum_{j=1}^{m} \int_{\mathbb{R}^d} \xi_n (|f|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |\nabla f_j|_{Q}^2 \, dx - C_\delta \int_{\mathbb{R}^d} \xi_n (|f|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |f|^2 \, dx \]
\[+ (p' - 2) \int_{\mathbb{R}^d} |\nabla |f||_{Q}^2 \xi_n |f|^2 (|f|^2 + \varepsilon^2)^{\frac{p'-4}{2}} \, dx \]

for all \( \delta > 0 \) and some \( C_\delta > 0 \). Moreover, according to [25, Lemma 2.4], one has

\[|\nabla |f||_{Q}^2 \leq \sum_{j=1}^{m} |\nabla f_j|_{Q}^2.\]

So that, choosing \( \delta = \delta_p < p' - 1 \) and \( \lambda > \gamma + C_\delta_p \), one gets

\[0 \geq (\lambda - \gamma - C_\delta_p) \int_{\mathbb{R}^d} \xi_n (|f|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |f|^2 \, dx \]
\[+ \int_{\mathbb{R}^d} (Q \nabla |f|, \nabla \xi_n) (|f|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |f| \, dx \]
\[+ (p' - 1 - \delta) \int_{\mathbb{R}^d} |\nabla |f||_{Q}^2 \xi_n |f|^2 (|f|^2 + \varepsilon^2)^{\frac{p'-4}{2}} \, dx \]
\[\geq (\lambda - \gamma - C_\delta_p) \int_{\mathbb{R}^d} \xi_n (|f|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |f|^2 \, dx \]
\[+ \frac{1}{p'} \int_{\mathbb{R}^d} (Q \nabla (\xi_n (|f|^2 + \varepsilon^2)^{\frac{p'}{2}})), \nabla \xi_n) \, dx \]
\[= (\lambda - \gamma - C_\delta_p) \int_{\mathbb{R}^d} \xi_n (|f|^2 + \varepsilon^2)^{\frac{p'-2}{2}} |f|^2 \, dx \]
\[- \frac{1}{p'} \int_{\mathbb{R}^d} \Delta Q \xi_n (|f|^2 + \varepsilon^2)^{\frac{p'}{2}} \, dx.\]

Upon \( \varepsilon \to 0 \), one obtains

\[(\lambda - \gamma - C_\delta_p) \int_{\mathbb{R}^d} \xi_n |f|^{p'} \, dx - \frac{1}{p'} \int_{\mathbb{R}^d} \Delta Q \xi_n |f|^{p'} \, dx \leq 0.\]

A straightforward computation yields

\[\Delta \xi_n = \frac{1}{n} \sum_{i,j=1}^{m} \partial_i q_{ij} \partial_j \xi (\cdot / n) + \frac{1}{n^2} \sum_{i,j=1}^{m} q_{ij} \partial_i \partial_j \xi (\cdot / n).\]

So that \( \| \Delta Q \xi_n \|_{\infty} \) tends to 0 as \( n \to \infty \). Therefore, upon \( n \to \infty \), one claims

\[\int_{\mathbb{R}^d} |f|^{p'} \, dx \leq 0.\]

Hence, \( f = 0.\)
On the other hand, if $p' \geq 2$, multiplying (4.4) by $\xi_n \langle f \rangle^{p'-2} f$, performing the same computation, one gets $f = 0$ by letting $n$ tends to $\infty$. \hfill \Box

We show in the next that the domain $D(L_p)$ is equal to the $L^p$-maximal domain of $\mathcal{L}$.

**Proposition 4.5** Let $1 < p < \infty$ and assume Hypotheses (H1)–(H3). Then

$$D(L_p) = \{ f \in L^p(\mathbb{R}^d, \mathbb{R}^m) : \mathcal{L} f \in L^p(\mathbb{R}^d; \mathbb{R}^m) \} := D_{p,\text{max}}(\mathcal{L}).$$

**Proof** We first show that $D(L_p) \subseteq D_{p,\text{max}}(\mathcal{L})$. Let $f \in D(L_p)$, it suffices to show that the distribution $\mathcal{L} f$ coincides with $L_p f$, in order to conclude, thanks to Theorem 4.2, that $f$ lies in $D_{p,\text{max}}(\mathcal{L})$. Actually, according to Proposition 4.4, there exits a sequence $(f_n)_n$ in $C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ such that $f_n \to f$ and $\mathcal{L} f_n \to L_p f$ in $L^p(\mathbb{R}^d, \mathbb{R}^m)$. Then, a straightforward computation yields

$$\langle L f, g \rangle_{p, p'} = \lim_{n} \langle \mathcal{L} f_n, g \rangle_{p, p'} = \lim_{n} \langle f_n, \mathcal{L}^* g \rangle_{p, p'} = \langle f, \mathcal{L}^* g \rangle_{p, p'} = \langle \mathcal{L} f, g \rangle$$

for every $g \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$. Hence, $\mathcal{L} f = L_p f \in L^p(\mathbb{R}^d, \mathbb{R}^m)$ implies $f \in D_{p,\text{max}}(L_p)$ by Theorem 4.2. For the other inclusion, it is enough to prove that $\lambda - \mathcal{L}$ is one to one on $D_{p,\text{max}}(\mathcal{L})$, for some $\lambda > 0$. Indeed, this implies that $\lambda \in \rho(\mathcal{L}_{p,\text{max}}) \cap \rho(L_p)$, where $\mathcal{L}_{p,\text{max}}$ is the realization of $\mathcal{L}$ on $D_{p,\text{max}}(\mathcal{L})$. Since $D_{p,\text{max}}(\mathcal{L}) \subseteq D(L_p)$, thus $L_p = \mathcal{L}_{p,\text{max}}$. Now, let $f \in D_{p,\text{max}}(\mathcal{L})$ be such that $(\lambda - \mathcal{L}) f = 0$. Arguing similarly as in the proof of Proposition 4.4, one obtains $f = 0$ and this ends the proof. \hfill \Box

**Remark 4.6** It is relevant to have $D(L_p) \subseteq W^{2,p}(\mathbb{R}^d, \mathbb{R}^m)$, for $1 < p < \infty$, which is equivalent to the coincidence of domains $D(L_p) = W^{2,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p)$, where $D(V_p)$ refers to the maximal domain of multiplication by $V$ in $L^p(\mathbb{R}^d, \mathbb{R}^m)$. Actually, in [25, Sect. 3], it has been shown the following

$$\| f \|_{2,p} + \| V f \|_p \leq C(\| \Delta Q f - V f \|_p + \| f \|_p)$$

for all $f \in W^{2,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p)$, provided that $V = \hat{V} + v I_m$, with $0 \leq v \in W^{1,p}_{\text{loc}}(\mathbb{R}^d)$ such that $|\nabla v| \leq C v$ and $\hat{V}$ satisfies

$$\sup_{1 \leq j \leq m} \| (\partial_j \hat{V}) \hat{V}^{-\gamma} \|_\infty < \infty$$

for some $\gamma \in [0, 1/2)$. Now, taking into the account, the Landau’s inequality

$$\| \nabla f \|_p \leq \varepsilon \| \Delta Q f \|_p + M \| f \|_p,$$
for every \( \varepsilon > 0 \), one claims
\[
\|f\|_{2,p} + \|Vf\|_p \leq C' \left( \|L_pf\|_p + \|f\|_p \right).
\]
Therefore, \( D(L_p) = W^{2,p}(\mathbb{R}^d, \mathbb{R}^m) \cap D(V_p) \).

5 Positivity

In this section we characterize the positivity of the semigroup \( \{S_p(t)\}_{t \geq 0} \) for \( 1 < p < \infty \). Since the family of semigroups \( \{S_p(t)\}_{t \geq 0}, \ p \in [1, \infty) \), is consistent, i. e., \( S_p(t)f = S_q(t)f \), for every \( t \geq 0, \ 1 \leq p, q < \infty \) and all \( f \in L^p(\mathbb{R}^d, \mathbb{R}^m) \cap L^q(\mathbb{R}^d, \mathbb{R}^m) \), it suffices to characterize the positivity of \( \{S_2(t)\}_{t \geq 0} \). For this purpose, we endow \( \mathbb{R}^m \) with the usual partial order: \( x \geq y \) if and only if, \( x_i \geq y_i \), for all \( i \in \{1, \ldots, m\} \). As in Sect. 4, we assume that \( C \equiv 0 \). By positivity of \( \{S_2(t)\}_{t \geq 0} \) we mean \( S_2(t)f \geq 0 \) a.e., for every \( t \geq 0 \) and all \( f \in L^2(\mathbb{R}^d, \mathbb{R}^m) \) such that \( f \geq 0 \) a.e.

We apply the Ouhabaz’ criterion for invariance of closed convex subsets by semigroups, c.f. [30, Theorem 3] and [29]. We then get the following result

**Theorem 5.1** Assume Hypotheses (H1). Then, the semigroup \( \{S_2(t)\}_{t \geq 0} \) is positive, if and only if, \( F_{ij} = 0 \) and \( v_{ij} \leq 0 \) almost everywhere and for every \( i \neq j \in \{1, \ldots, m\} \).

**Proof** Let \( C = \{f \in L^2(\mathbb{R}^d, \mathbb{R}^m) : f \geq 0 \ \text{a.e.}\} \) and \( P_p f = f^+ = (f_i^+)_{1 \leq i \leq m} \), where \( f_i^+ = \max(0, f_i) \). Then, \( C \) is a closed convex subset of \( L^2(\mathbb{R}^d, \mathbb{R}^m) \) and \( P_p \) is the corresponding projection. Now, let \( \omega \geq 0 \) such that \( a_\omega \) is accretive. According to [30, Theorem 3 (iii)], \( \{S_2(t)\}_{t \geq 0} \) is positive if, and only if, the form a satisfies the following

- \( f \in D(a) \) implies \( f^+ \in D(a) \),
- \( a_\omega(f^+, f^-) \leq 0 \), for all \( f \in D(a) \), where \( f^- = f^+ - f \).

Now, assume that \( \{S_2(t)\}_{t \geq 0} \) is positive. Let \( i \neq j \in \{1, \ldots, m\}, n \in \mathbb{N} \) and \( 0 \leq \varphi \in C_c^\infty(\mathbb{R}^d) \). Set \( f = \xi_n e_j - \varphi e_i \). One has
\[
0 \geq a_\omega(f^+, f^-) = \frac{1}{n} \int_{\mathbb{R}^d} \langle F_{ij}, \nabla \xi_n(\cdot/n) \rangle \varphi \, dx + \int_{\mathbb{R}^d} v_{ij} \xi_n \varphi \, dx.
\]
Letting \( n \to \infty \), by dominated convergence theorem, one gets \( \int_{\mathbb{R}^d} v_{ij} \varphi \, dx \leq 0 \) for every \( 0 \leq \varphi \in C_c^\infty(\mathbb{R}^d) \), which implies that \( v_{ij} \leq 0 \) almost everywhere. On the other hand, considering, for every \( n \in \mathbb{N} \),
\[
g(x) = g^{(k,n)}(x) := \exp(nx_k)\varphi(x)e_j - \exp(-nx_k)\varphi(x)e_i,
\]
where \( x_k \) is the k-th component of \( x \in \mathbb{R}^d \), for every \( k \in \{1, \cdots, d\} \), then,
\[
\nabla g_j^+ = n \exp(nx_k)\varphi e_k + \exp(nx_k)\nabla \varphi.
\]
Therefore,
\[ 0 \geq \frac{1}{n} a_\omega (g^+, g^-) = \int_{\mathbb{R}^d} F_{ij}^{(k)} \varphi^2 \, dx + \frac{1}{n} \int_{\mathbb{R}^d} \langle F_{ij}, \nabla \varphi \rangle \varphi \, dx \]
\[ + \frac{1}{n} \int_{\mathbb{R}^d} v_{ij} \varphi^2 \, dx, \]
where \( F_{ij}^{(k)} \) indicates the \( k \)-th component of \( F_{ij} \). So, by letting \( n \to \infty \), one deduces that \( F_{ij}^{(k)} \leq 0 \) almost everywhere and for each \( k \in \{1, \cdots , d\} \). In a similar way, one gets \( F_{ij}^{(k)} \geq 0 \) a.e. by considering \( \tilde{g} \) instead of \( g \), where
\[ \tilde{g}(x) = \tilde{g}^{(k,n)}(x) := \exp(-nx_k)\varphi(x)e_j - \exp(nx_k)\varphi(x)e_i. \]
So that \( F_{ij} = 0 \) almost everywhere.

Conversely, assume \( F_{ij} = 0 \) and \( v_{ij} \leq 0 \) for all \( i \neq j \in \{1, \ldots , m\} \). Let \( f \in D(a) \), then, by [24, Theorem 4.2], one gets \( f^+ \in D(a) \). Furthermore, it follows, by [16, Theorem 7.9], that \( \nabla f_i^+ = \chi_{\{f_i>0\}} \nabla f_i \) and \( \nabla f_i^- = \chi_{\{f_i<0\}} \nabla f_i \). Let us now prove that \( a_\omega (f^+, f^-) \leq 0 \). One has
\[ a_\omega (f^+, f^-) = \sum_{i=1}^m \int_{\mathbb{R}^d} \langle \nabla f_i^+, \nabla f_i^- \rangle \, dx + \sum_{i=1}^m \int_{\mathbb{R}^d} \langle F_{ii}, \nabla f_i^+ \rangle f_i^- \, dx \]
\[ + \sum_{i,j=1}^m \int_{\mathbb{R}^d} v_{ij} f_j^+ f_i^- \, dx + \omega(f^+, f^-)_2 \]
\[ = \sum_{i \neq j}^m \int_{\mathbb{R}^d} v_{ij} f_j^+ f_i^- \, dx \]
\[ \leq 0. \]

This ends the proof. \( \square \)

6 Kernel Estimates

In this section we prove \( L^p-L^q \)-estimates and Gaussian kernel estimates for our consistent semigroup \( \{S(t)\}_{t \geq 0} \) in \( L^p(\mathbb{R}^d, \mathbb{C}^m) \) spaces for \( 1 \leq p < \infty \). The results of this section and some proofs are inspired by those in [11, Sect. 3.2] for the scalar case. Some proofs are presented here with details just for the sake of completeness of our work. However, we guarantee, to our knowledge, that the results established here are treated for the first time for vector-valued elliptic operators involving coupling first order terms. Furthermore, our assumptions in \( Q \) and \( V \) are weaker than those of [25] and [17]. First, we prove the following ultracontractivity result for \( \{S(t)\}_{t \geq 0} \).
Theorem 6.1 Assume that Hypotheses (H1)-(H2) are satisfied. Then, there exist \( M > 0 \) and \( \omega \in \mathbb{R} \) such that

\[
\|S(t)f\|_{\infty} \leq Me^{\omega t} t^{-\frac{d}{2}} \|f\|_{1}, \quad t > 0, \quad f \in L^{1}(\mathbb{R}^{d}, \mathbb{C}^{m}).
\] (6.1)

Proof Let \( f \in C_{c}^{\infty}(\mathbb{R}^{d}, \mathbb{C}^{m}) \) and set \( \alpha(t) = \|e^{-\omega t} S(t)f\|_{2}^{-\frac{1}{2}} \) for every \( t > 0 \). Now, going back to the proof of Proposition 2.2, one can claim that there exists \( \omega \in \mathbb{R} \) such that

\[
-\langle (L - \omega) g, g \rangle \geq \frac{\eta_1}{2} \int_{\mathbb{R}^{d}} \sum_{i=1}^{m} |\nabla g_{i}|^{2} \, dx := \frac{\eta_1}{2} \|\nabla g\|_{2}^{2} \quad \text{for all } g \in D(a). \quad (6.2)
\]

Moreover, since the semigroup \( \{S(t)\}_{t \geq 0} \) is analytic, then \( S(t)f \in D(L) \subset D(a) \), for all \( t > 0 \). So that the above applied to \( g = e^{-\omega t} S(t)f, t > 0 \), yields

\[
\alpha'(t) = \frac{d}{dt} \left( \langle e^{-\omega t} S(t)f, e^{-\omega t} S(t)f \rangle \right)^{-\frac{1}{2}}
= -\frac{4}{d} \|e^{-\omega t} S(t)f\|_{2}^{-\frac{1}{2} - 1} \langle (L - \omega)e^{-\omega t} S(t)f, e^{-\omega t} S(t)f \rangle
\geq \frac{2\eta_1}{d} \|e^{-\omega t} S(t)f\|_{2}^{-\frac{1}{2} - 1} \|\nabla(e^{-\omega t} S(t)f)\|_{2}^{2}.
\]

Furthermore, using Nash’s inequality [11, Theorem 2.4.6-(ii)] and the fact that \( \|e^{-\omega t} S(t)f\|_{1} \leq \|f\|_{1} \) one shows that

\[
\alpha'(t) \geq \frac{2\eta_1}{d\bar{N}} \|e^{-\omega t} S(t)f\|_{1}^{-\frac{4}{d}} \geq \frac{2\eta_1}{d\bar{N}} \|f\|_{1}^{-\frac{4}{d}},
\]

for some \( \bar{N} > 0 \). Therefore,

\[
\alpha(t) \geq \int_{0}^{t} \frac{d}{ds} \alpha(s) \, ds \geq \frac{2\eta_1}{d\bar{N}} t \|f\|_{1}^{-\frac{4}{d}}.
\]

Then,

\[
\|S(t)f\|_{2} \leq \left( \frac{2\eta_1}{d\bar{N}} \right)^{-\frac{d}{4}} e^{\omega t} t^{-\frac{d}{4}} \|f\|_{1}, \quad t > 0. \quad (6.3)
\]

This proves the \( L^{1} - L^{2} \) estimates for the semigroup \( \{S(t)\}_{t \geq 0} \). To conclude, it suffices to show the \( L^{2} - L^{\infty} \) estimates for \( \{S(t)\}_{t \geq 0} \). To do so, let us consider the adjoint semigroup \( \{S^{*}(t)\}_{t \geq 0} \), associated to \( L^{*} \), the adjoint operator of \( L \), which is given by

\[
L^{*}f := \text{div}(Q\nabla f) - C^{*} \cdot \nabla f + \text{div}(F^{*} f) - V^{*} f.
\]
Note that the operator $L^*$ has the same properties as $L$, since the coefficients of $L^*$ satisfy Hypotheses (H1) and (H2), similarly to $L$. Then, using the same arguments as above, one obtains

$$
\|S^*(t)f\|_2 \leq \left(\frac{2\eta_1}{d\tilde{N}}\right)^{-\frac{d}{4}} e^{\omega t - \frac{d}{4}} \|f\|_1, \quad t > 0.
$$

On the other hand, let $g \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$. Then

$$
\int_{\mathbb{R}^d} |\langle S(t)f(x), g(x) \rangle| \, dx = |\langle S(t)f, g \rangle_2| \\
= |\langle f, S^*(t)g \rangle_2| \\
\leq \|S^*(t)g\|_2 \|f\|_2 \\
\leq \left(\frac{2\eta_1}{d\tilde{N}}\right)^{-\frac{d}{4}} e^{\omega t - \frac{d}{4}} \|f\|_2 \|g\|_1, \quad t > 0.
$$

This implies that $S(t)f \in L^\infty(\mathbb{R}^d, \mathbb{C}^m)$. Furthermore

$$
\|S(t)f\|_\infty \leq \left(\frac{2\eta_1}{d\tilde{N}}\right)^{-\frac{d}{4}} e^{\omega t - \frac{d}{4}} \|f\|_2, \quad t > 0. \quad (6.4)
$$

Consequently $S(t)f \in L^\infty(\mathbb{R}^d, \mathbb{C}^m)$, for every $f \in L^1(\mathbb{R}^d, \mathbb{C}^m)$, in view of (6.3) and (6.4). Moreover, one has

$$
\|S(t)f\|_\infty = \|S(t/2)S(t/2)f\|_\infty \leq \left(\frac{2\eta_1}{d\tilde{N}}\right)^{-\frac{d}{4}} e^{\omega t - \frac{d}{2}} \|S(t/2)f\|_2 \\
\leq \left(\frac{d\tilde{N}}{\eta_1}\right)^{\frac{d}{2}} e^{\omega t - \frac{d}{4}} \|f\|_1.
$$

This proves the result by taking $M = \left(\frac{d\tilde{N}}{\eta_1}\right)^{\frac{d}{2}}$. \hfill \Box

As a consequence of Theorem 6.1, we obtain the following $L^p$–$L^q$-estimates.

**Corollary 6.2** Let $1 \leq p < q \leq \infty$ and assume that Hypotheses (H1)-(H2) hold. Then, there exist $\tilde{M} > 0$ and $\omega \in \mathbb{R}$ such that

$$
\|S(t)f\|_q \leq \tilde{M} e^{\omega t - \frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_p, \quad t > 0, \quad f \in L^p(\mathbb{R}^d, \mathbb{R}^m). \quad (6.5)
$$

**Proof** Let $1 \leq q \leq \infty$. Then, using (6.1), the $L^1$-estimate

$$
\|e^{-\omega t} S(t)f\|_1 \leq \|f\|_1, \quad t > 0, \quad f \in L^1(\mathbb{R}^d, \mathbb{R}^m)
$$

...
and, in view of Riesz-Thorin interpolation, Theorem [11, Sect. 1.1.5] (i.e., for $(1, 1) \hookrightarrow (1, q) \hookrightarrow (1, \infty)$) it follows that
\[
\| S(t)f \|_q \leq \tilde{M} e^{\omega t} t^{-\frac{d}{2}(1-\frac{1}{q})} \| f \|_1, \quad t > 0, \ f \in L^1(\mathbb{R}^d, \mathbb{R}^m). \tag{6.6}
\]
Moreover, since for all $1 \leq q \leq \infty$, one has
\[
\| e^{-\omega t} S(t)f \|_q \leq \| f \|_q, \quad t > 0, \ f \in L^q(\mathbb{R}^d, \mathbb{R}^m). \tag{6.7}
\]
Then, using the same argument and by interpolation between (6.6) and (6.7), i.e., for $(1, q) \hookrightarrow (p, q) \hookrightarrow (q, q)$, one obtains
\[
\| S(t)f \|_q \leq \tilde{M} e^{\omega t} t^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| f \|_p, \quad t > 0, \ f \in L^p(\mathbb{R}^d, \mathbb{R}^m). \tag{6.8}
\]

\[\Box\]

**Proposition 6.3** Let $1 \leq p < \infty$ and assume (H). Then, for each $t > 0$ there exists a matrix kernel $K(t, \cdot, \cdot) := (k_{i,j}(t, \cdot, \cdot))_{1 \leq i,j \leq m}$ is provided by Dunford–Pettis Theorem [5, Theorem 1.3], see also [25, Sect. 5] for its adaptation to vector-valued functions. For the estimation (6.9), it is a direct consequence of Theorem 3.1.

\[\Box\]

**Theorem 6.4** (Gaussian estimates) Assume that Hypotheses (H1)-(H2) hold. Then, there exist $N_1, N_2 > 0$ such that
\[
|k_{i,j}(t, x, y)| \leq N_1 e^{\alpha^* t} t^{-\frac{d}{2}} \exp \left( -N_2 \frac{|x-y|^2}{4t} \right), \quad t > 0, \ a.e. x, y \in \mathbb{R}^d, 1 \leq i,j \leq m \tag{6.9}
\]
for some $\alpha^* \in \mathbb{R}$.

**Proof** For each $t > 0$, the fact that the semigroup $\{S(t)\}_{t \geq 0}$ is defined by the integral operator in $L^p(\mathbb{R}^d, \mathbb{C}^m)$ which is governed by $K(t, x, y) := (k_{i,j}(t, x, y))_{1 \leq i,j \leq m}$, is provided by Dunford–Pettis Theorem [5, Theorem 1.3], see also [25, Sect. 5] for its adaptation to vector-valued functions. For the estimation (6.9), it is a direct consequence of Theorem 3.1.

\[\Box\]

**Theorem 6.4** (Gaussian estimates) Assume that Hypotheses (H1)-(H2) hold. Then, there exist $N_1, N_2 > 0$ such that
\[
|k_{i,j}(t, x, y)| \leq N_1 e^{\alpha^* t} t^{-\frac{d}{2}} \exp \left( -N_2 \frac{|x-y|^2}{4t} \right), \quad t > 0, \ a.e. x, y \in \mathbb{R}^d, 1 \leq i,j \leq m \tag{6.10}
\]
for some $\alpha^* \in \mathbb{R}$ and all $t > 0$, a.e. $x, y \in \mathbb{R}^d, 1 \leq i,j \leq m$.

**Proof** Consider the following set
\[
\Sigma := \{ \phi \in C_b^\infty(\mathbb{R}^d) : \| \nabla \phi \|_\infty \leq 1 \}. 
\]
Let \( \lambda \in \mathbb{R} \) and \( \phi \in \Sigma \). Consider the semigroup \( \{ S_{\phi, \lambda}(t) \}_{t \geq 0} \) in \( L^p(\mathbb{R}^d, C^m) \) (for \( 1 \leq p < \infty \)) given by

\[
S_{\phi, \lambda}(t)f := e^{-\lambda \phi} S(t) e^{\lambda \phi} f
\]

Then, \( \{ S_{\phi, \lambda}(t) \}_{t \geq 0} \) defines a \( C_0 \)-semigroup \( L^p(\mathbb{R}^d, C^m) \) for every \( 1 \leq p < \infty \). Let us denote by \( L_{\phi, \lambda} \) its generator. Let \( f \in C^\infty_c(\mathbb{R}^d, C^m) \), then

\[
L_{\phi, \lambda} f := \text{div}(Q \nabla f) - \tilde{F} \cdot \nabla f + \text{div}(Cf) - \tilde{V} f
\]

where

\[
\tilde{F} = F - 2\lambda Q \nabla \phi
\]

and

\[
\tilde{V} = - \left[ \lambda \text{div}(Q \nabla \phi) + \lambda \text{div}(C \phi) + \lambda F \cdot \nabla \phi + \lambda^2 \left| \nabla \phi \right|^2_Q - V \right]
\]

Hence, one can observe that the generator \( L_{\phi, \lambda} \) has the same properties as \( L \). Now, by integrating by parts and using Young’s inequality it holds that for all \( \varepsilon > 0 \) there exists \( c_\varepsilon > 0 \) such that

\[
\langle -L_{\phi, \lambda} f, f \rangle
\]

\[
= \sum_{i=1}^m \int_{\mathbb{R}^d} \langle Q \nabla f_i, \nabla f_i \rangle \, dx + \sum_{i,j=1}^m \int_{\mathbb{R}^d} \text{Re} \left( \langle \tilde{F}_{ij} \cdot \nabla f_j \rangle \tilde{f}_i \right) \, dx
\]

\[
+ \sum_{i,j=1}^m \int_{\mathbb{R}^d} \text{Re} \left( f_j \langle C_{ij}, \nabla f_i \rangle \right) \, dx + \int_{\mathbb{R}^d} \text{Re} \left( \tilde{V} f, f \right) \, dx
\]

\[
= \sum_{i=1}^m \int_{\mathbb{R}^d} |\nabla f_i|^2_Q \, dx + \sum_{i,j=1}^m \int_{\mathbb{R}^d} \text{Re} \left( \langle \tilde{F}_{ij} \cdot \nabla f_j \rangle \tilde{f}_i \right) \, dx
\]

\[
- 2\lambda \sum_{i=1}^m \int_{\mathbb{R}^d} \text{Re} \left( \langle (Q \nabla \phi) \cdot \nabla f_i \rangle f_i \right) \, dx + \sum_{i,j=1}^m \int_{\mathbb{R}^d} \text{Re} \left( f_j \langle C_{ij}, \nabla f_i \rangle \right) \, dx
\]

\[
- \int_{\mathbb{R}^d} \left[ \lambda \text{div}(Q \nabla \phi) + \lambda \text{div}(C \phi) + \lambda F \cdot \nabla \phi + \lambda^2 \left| \nabla \phi \right|^2_Q \right] |f|^2 \, dx
\]

\[
+ \int_{\mathbb{R}^d} \text{Re} \left( \tilde{V} f, f \right) \, dx
\]

\[
= \sum_{i=1}^m \int_{\mathbb{R}^d} |\nabla f_i|^2_Q \, dx + \sum_{i,j=1}^m \int_{\mathbb{R}^d} \text{Re} \left( \langle \tilde{F}_{ij} \cdot \nabla f_j \rangle \tilde{f}_i \right) \, dx
\]

\[
+ \sum_{i,j=1}^m \int_{\mathbb{R}^d} \text{Re} \left( f_j \langle C_{ij}, \nabla f_i \rangle \right) \, dx + 2\lambda \sum_{i,j=1}^m \int_{\mathbb{R}^d} \text{Re} \left( \langle (C \phi) \cdot \nabla f_j \rangle f_i \right) \, dx
\]
\[
2\lambda \int_{\mathbb{R}^d} \sum_{i,j=1}^{m} f_j ((F\phi), \nabla f_i) \, dx + \lambda^2 \int_{\mathbb{R}^d} |\nabla \phi|_Q^2 \, |f|^2 \, dx + \int_{\mathbb{R}^d} \text{Re} \langle Vf, f \rangle \, dx \\
\quad \geq \eta_1 \sum_{i=1}^{m} \int_{\mathbb{R}^d} |\nabla f_i|^2 \, dx - (1 + 2|\lambda|)(\|F\|_\infty + \|C\|_\infty) \sum_{i=1}^{m} \sum_{i,j=1}^{m} |\nabla f_i| \, dx \\
- \lambda^2 \int_{\mathbb{R}^d} |\nabla \phi|_Q^2 \, |f|^2 \, dx \\
\quad \geq (\eta_1 - \epsilon) \sum_{i=1}^{m} \int_{\mathbb{R}^d} |\nabla f_i|^2 \, dx - (c_\epsilon + \eta_2 \lambda^2) \sum_{i=1}^{m} |f_i|^2 \, dx.
\]

Now, if we set \( \epsilon = \frac{\eta_1}{2} \), then,

\[
\langle -L_{\phi,\lambda} f, f \rangle \geq \frac{\eta_1}{2} \sum_{i=1}^{m} |\nabla f_i|^2 \, dx - (c_\epsilon + \eta_2 \lambda^2) \sum_{i=1}^{m} |f_i|^2 \, dx.
\]

By taking \( \omega = \eta_2 \lambda^2 \) and \( \alpha^* = c_{\eta_1/2} \) one obtains

\[
-\langle (L_{\phi,\lambda} - \omega - \alpha^*) f, f \rangle \geq \frac{\eta_1}{2} \|\nabla f\|_2^2.
\]

Finally, using the same argument as in the proof of Theorem 6.1, we obtain the following \( L_1-L_\infty \)-estimate for \( \{S_{\phi,\lambda}(t)\}_{t \geq 0} \):

\[
\|S_{\phi,\lambda}(t)f\|_\infty \leq M e^{(\alpha^* + \omega)t} t^{-\frac{d}{4}} \|f\|_1, \quad t > 0, \ f \in L^1(\mathbb{R}^d, \mathbb{R}^m). \tag{6.11}
\]

Therefore,

\[
|e^{\lambda(\phi(x) - \phi(y))} k_{i,j}(t, x, y)| \leq M e^{(\alpha^* + \omega)t} t^{-\frac{d}{4}},
\]

that is,

\[
|k_{i,j}(t, x, y)| \leq M e^{\alpha^* t} t^{-\frac{d}{4}} \exp \left( \eta_2 \lambda^2 t + \lambda (\phi(x) - \phi(y)) \right). \tag{6.12}
\]

for every \( t > 0 \), almost every \( x, y \in \mathbb{R}^d \), and each \( 1 \leq i, j \leq m \). Optimizing in \( \lambda \), one claims

\[
|k_{i,j}(t, x, y)| \leq M e^{\alpha^* t} t^{-\frac{d}{4}} \exp \left( -\frac{|\phi(x) - \phi(y)|^2}{4\eta_2 t} \right), \quad t > 0, \ a.e. x, y \in \mathbb{R}^d, \ 1 \leq i, j \leq m.
\]

Furthermore, by the well–known result of the equivalence of the geodesic distance to the euclidean one, one claims the existence of two constants \( c_1, c_2 > 0 \) such that
\[ c_1|x - y| \leq \sup_{\phi \in \Sigma} \{\phi(x) - \phi(y)\} \leq c_2|x - y|, \ x, y \in \mathbb{R}^d. \]

Thus, one can find \( M > 0 \) such that
\[ |k_{i,j}(t, x, y)| \leq M e^{\alpha^* t} t^{-\frac{d}{2}} \exp \left( -N \frac{|x - y|^2}{4t} \right). \]

This proves the Gaussian upper estimates as wanted. \( \square \)

Finally, we extend our Gaussian kernel estimates (6.10) for complex times, that is, for the complexes \( z \) in the sector \( S_\theta \) of analyticity of the semigroup \( \{S(t)\}_{t \geq 0} \) in \( L^2(\mathbb{R}^d, \mathbb{C}^m) \). Note that \( S_\theta := \{z \in \mathbb{C} : |\arg(z)| < \theta, \text{ and } z \neq 0\} \) for some \( \theta \in (0, \pi/2) \). Now, we state the following preliminary results.

**Proposition 6.5** Let (H1)-(H2) be satisfied. Then, there exists \( \hat{M} > 0 \) such that
\[ \left\| \tilde{S}(z) f \right\|_\infty \leq \hat{M} \exp(\omega \text{Re}(z))(\text{Re}(z))^{-\frac{d}{2}} \| f \|_1, \text{ for all } f \in L^1(\mathbb{R}^d, \mathbb{C}^m), \] (6.13)
for every \( z \in S_\theta \). Moreover,
\[ |k_{i,j}(z, x, y)| \leq \hat{M} \exp(\omega \text{Re}(z))(\text{Re}(z))^{-\frac{d}{2}}, \ z \in S_\theta, \text{ a.e.} x, y \in \mathbb{R}^d, \ 1 \leq i, j \leq m. \] (6.14)

**Proof** Let \( z \in S_\theta \). Write \( z = \varepsilon \text{Re}(z) + (1 - \varepsilon) \text{Re}(z) + i \text{Im}(z) := \varepsilon \text{Re}(z) + \tilde{z}, \) with \( \varepsilon > 0 \) small enough so that \( \tilde{z} \in S_\theta \). Now, writing \( S(z) = S(\frac{\varepsilon z}{2} \text{Re}(z)) S(z) S(\frac{\varepsilon}{2} \text{Re}(z)) \) one claims that
\[ L^1(\mathbb{R}^d, \mathbb{R}^m) \overset{S(\frac{\varepsilon}{2} \text{Re}(z))}{\hookrightarrow} L^2(\mathbb{R}^d, \mathbb{R}^m) \overset{S(z)}{\hookrightarrow} L^2(\mathbb{R}^d, \mathbb{R}^m) \overset{S(\frac{\varepsilon}{2} \text{Re}(z))}{\hookrightarrow} L^\infty(\mathbb{R}^d, \mathbb{R}^m). \]

Thus,
\[ \left\| S(z) \right\|_{1, \infty} \leq \left\| S(\frac{\varepsilon}{2} \text{Re}(z)) \right\|_{1, 2} \left\| S(z) \right\|_{2, 2} \left\| S(\frac{\varepsilon}{2} \text{Re}(z)) \right\|_{2, \infty} \leq \hat{M} \exp(\omega \text{Re}(z))(\text{Re}(z))^{-\frac{d}{2}}. \]

According to the Dunford–Pettis Theorem [5, Theorem 1.3], for each \( z \in S_\theta \), there exists a matrix kernel \( K(z, \cdot, \cdot) := (k_{i,j}(z, \cdot, \cdot))_{1 \leq i, j \leq m} \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^{m \times m}) \) such that
\[ S(z) f(x) = \int_{\mathbb{R}^d} K(z, x, y) f(y) dy, \ f \in L^p(\mathbb{R}^d, \mathbb{C}^m). \]

Therefore each component \( k_{i,j}(z, \cdot, \cdot) \) satisfies
\[ |k_{i,j}(z, x, y)| \leq \hat{M} \exp(\omega \text{Re}(z))(\text{Re}(z))^{-\frac{d}{2}}, \ z \in S_\theta, \text{ a.e.} x, y \in \mathbb{R}^d, \ 1 \leq i, j \leq m. \] \( \square \)
Next, we prove the Gaussian kernel estimates for complex times using Phragmén-Lindelöf type theorems for sectors, introduced in [10, Sect. 2].

**Theorem 6.6** Let (H1)-(H2) be satisfied. Then, there exists $M, N > 0$ such that

$$\left|k_{i,j}(z, x, y)\right| \leq M e^{a^* \text{Re} z (\text{Re} (z)) - \frac{d}{2} \exp \left(-N \text{Re} \frac{|x - y|^2}{4z}\right)}$$

for every $z \in S_\theta$.

**Proof** Without loss of generalities, we assume $\theta = \pi/2$, then $S_\theta = \mathbb{C}^+$ (think to a unitary analytic transformation from $S_\theta$ to $\mathbb{C}^+$). Now, since $z \mapsto K(z, x, y)$ is analytic on $S_\theta$ as a kernel of a holomorphic semigroup, see for instance [12, Section 2]. Then, from the kernel estimates (6.10) and (6.14), one claims the Gaussian bound for complex time kernel (6.6) by a Phragmén-Lindelöf type interpolation theorem, see [10, Proposition 2.2].

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