A LÖWNER VARIATIONAL METHOD IN THE THEORY OF SCHLICHT FUNCTIONS

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ABSTRACT. A Löwner variational method is developed that allows to calculate arbitrary continuous coefficient functionals of the second, third and fourth coefficients of schlicht functions. Based on this method an improved lower bound for the Milin-constant (0.034856..) is given, as well as an improved lower bound for the maximal modulus of the seventh coefficient of odd schlicht functions (1.006763..). The extremals for the generating function \( \kappa(t) \) of the Löwner differential equation are determined precisely.

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First set up some notation, denote the unit disk by \( \mathbb{D} \), the set of schlicht functions by \( S \) and for an arbitrary \( n \in \mathbb{N} \) let \( \mathbb{V}_n \) denote the \( n \)-th coefficient region of the schlicht functions. Consider a continuous coefficient functional \( \Phi: \mathbb{V}_n \to \mathbb{C} \). For every point \( a \in \mathbb{V}_n \) there exists a function \( f \in S \) so that \( a = (a_2(f), \ldots, a_n(f)) \) where \( a_k(f) \) \( k = 2, \ldots, n \) denotes the \( k \)-th coefficient of \( f \in S \). Then since \( S \) and \( \mathbb{V}_n \) are compact (see [1], chapter 11)

\[
\max_{f \in S} \Re(\Phi(a)) \mid a \in \mathbb{V}_n = \max_{f \in S} \Re(\Phi(f))
\]

exists. For the sake of simplicity the notations in (1) will be used synonymously. The following results about the Löwner differential equation can be found for instance in [1] or [2]. For an arbitrary schlicht function \( f \in S \) which is a single slit mapping let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). Then \( f \) can be embedded as initial point into a Löwner chain \( f(z,t) = e^{t}z + \sum_{n=2}^{\infty} a_n(t) z^n \), i.e. \( f(z,0) = f(z), \ z \in D \) which satisfies the differential equation

\[
\dot{f}(z,t) = \frac{1 + \kappa(t)z}{1 - \kappa(t)z} z f'(z,t)
\]
The Löwner differential equation (2) for the schlicht function $f : \mathbb{D} \rightarrow \mathbb{C}$ yields the differential equation

$$(3) \quad \dot{a}_n(t) = na_n(t) + \sum_{k=1}^{n-1} 2ka_k(t)\kappa(t)^{n-k}$$

or in integrated form

$$(4) \quad a_n(t) = -e^{nt} \int_{t}^{\infty} e^{-ns} \sum_{k=1}^{n-1} 2ka_k(s)\kappa(s)^{n-k} \, ds$$

for the coefficients $a_n(t)$, $a_n(0) = a_n$ if $n \geq 2$. For a single slit mapping $\kappa : [0, \infty) \rightarrow \mathbb{C}$ is a continuous function with $|\kappa(t)| = 1$. Substituting $x = e^{-t}$ and putting $a_n(x) = a_n(t)$ yields the equivalent differential equation

$$(5) \quad -x\kappa'(x) = n\kappa(x) + \sum_{k=1}^{n-1} 2k\kappa(x)\mu(x)^{n-k}$$

for $n \geq 2$ where $\mu : (0,1) \rightarrow \mathbb{C}$ is a continuous function that satisfies $\mu(x) = \kappa(t)$. Let $g_n(x) = x^n\kappa_n(x)$ and $g_1(x) \equiv 1$ then (5) takes the form

$$(6) \quad g_n'(x) = -\sum_{k=1}^{n-1} 2kx^{n-k-1}g_k(x)\mu(x)^{n-k}$$

or in integrated form

$$(7) \quad g_n(x) = -\int_{0}^{x} \sum_{k=1}^{n-1} 2kt^{n-k-1}g_k(t)\mu(t)^{n-k} \, dt$$

if $n \geq 2$. Notice that $g_n(1) = a_n$ and $g_n(0) = 0$ if $n \geq 2$. These representations hold in particular if $\mu$ is continuous or a step function with a finite number of steps. In the sequel the function $\mu : (0,1) \rightarrow \mathbb{C}$ (and $\kappa$ synonymously) will be called a generating function of the schlicht function $f : \mathbb{D} \rightarrow \mathbb{C}$, since because of (7) and the identity theorem it uniquely determines the coefficients of the schlicht function $f$. No assumption whether the function $f \in \mathcal{S}$ uniquely determines the generating function is needed in the sequel.

Some further preparations are necessary in order to formulate Theorem 1, in particular a special class of step functions needs to be introduced. For every $m \in \mathbb{N}$ consider the equidistant partition of the interval $[0,1]$ into $m$ subintervals $I_k$, $k = 1, \ldots, m$ and define $s : [0,1] \rightarrow \mathbb{C}$ by $s(x) = c_k$ if $x \in I_k$ where $c_k \in \mathbb{C}$ and $|c_k| = 1$ for $k = 1, \ldots, m$. Let $I_k$ be defined by $I_k = [(k-1)/m, k/m)$ for $k = 1, \ldots, m-1$ and $I_m = [(m-1)/m, 1]$. Denote the set of these step functions by $T_m([0,1])$. Every such step function $s \in T_m([0,1])$, $m \in \mathbb{N}$ generates a Löwner chain because the function $p(w,t) = (1 + s(x(t))w)/(1 - s(x(t))w)$ where $x(t) = e^{-t}$ has positive real part for every $t \in [0,\infty)$ (Theorem 3.4). As a consequence every $s \in T_m([0,1])$ is a generating function of a schlicht function $f_s$ which can be embedded as the initial point $f(z,0) = f_s(z)$ into a Löwner chain $f(,t)$ that satisfies the differential equation (2). $f_s$ in turn generates a point $(a_2(f_s), \ldots, a_n(f_s)) \in \mathbb{V}_n$ through formula (7). Let $\mathbb{V}_n^m$ denote the subset of $\mathbb{V}_n$ that is generated by
step functions $s \in T_m([0,1])$ in this way.

A property is said to hold almost everywhere(a.e.) on a set $X$ if it holds everywhere on $X$ except for a subset of measure zero. The gaussian function $[x]$, $x \in \mathbb{R}$ will also be needed, it denotes the largest integer $\leq x$.

**Theorem 1.** For $m, n \in \mathbb{N}$ let $\mathcal{V}_n$, $\mathcal{V}_n^m$ and $T_m((0,1])$ be defined as above and let $\Phi : \mathcal{V}_n \to \mathbb{C}$ denote a continuous functional. Suppose that for every $m \in \mathbb{N}$ there exists a function $f_m \in S$ such that

$$\sup \{|\Phi(a)| : a \in \mathcal{V}_n^m\} = \Phi(a_2(f_m), \ldots, a_n(f_m))$$ where $(a_2(f_m), \ldots, a_n(f_m)) \in \mathcal{V}_n^m$.

Then

$$\lim_{m \to \infty} \Phi(f_m) = \max_{f \in S} \Phi(f).$$

Furthermore if $s_m \in T_m((0,1])$ denotes a generating function of $f_m$, $m \in \mathbb{N}$ then there exists a subsequence $(s_m(k))_{k \in \mathbb{N}}$ of $(s_m)$ and a function $\tilde{\mu} \in L^\infty([0,1])$ so that $s_m(k) \to \tilde{\mu}$ uniformly a.e on $[0,1]$ for $k \to \infty$ and for almost every $x \in [0,1]$ the sequence $(f_m(k))_{k \in \mathbb{N}}$ converges locally uniformly in the unit disk to a function $\tilde{f} \in S$ that maximizes $\Phi$ and $\tilde{\mu}$ is a generating function of $\tilde{f}$.

**Proof.** Let $\epsilon > 0$ be arbitrary and let $\tilde{h} \in S$ denote a function that maximizes $\Phi$. Since the single-slit mappings lie dense in $S$ there exists a single slit mapping $h \in S$, $h(z) = z + \sum_{j=2}^{\infty} a_j z^j$ such that

$$|\Phi(a_2, \ldots, a_n) - \Phi(a_2(\tilde{h}), \ldots, a_n(\tilde{h}))| \leq |\Phi(a_2, \ldots, a_n) - \Phi(a_2(h), \ldots, a_n(h))| < \epsilon/2$$

Since $h$ is a single slit mapping there exists a continuous generating function $\mu : [0,1] \to \mathbb{C}$ of $h$. For $k \in \mathbb{N}$ and $x \in [0,1]$ define a sequence of step functions $t_k$ by $t_k(x) = \mu([kx] + 1)/k)$. If $x \in (0,1)$ is arbitrarily chosen, then since $|kx|/k \leq x < (|kx| + 1)/k)$ it follows that

$$0 < \frac{|kx| + 1}{k} - x = |x - \frac{|kx| + 1}{k}| \leq \frac{|kx| + 1}{k} - \frac{|kx|}{k} = \frac{1}{k}$$

Hence for every $\bar{\epsilon} > 0$ there exists a $N(x, \bar{\epsilon}) \in \mathbb{N}$ so that

$$|\mu(x) - t_k(x)| = |\mu(x) - \mu(\frac{|kx| + 1}{k})| < \bar{\epsilon}$$

if $k > N(x, \bar{\epsilon})$, i.e.

$$\lim_{k \to \infty} t_k(x) = \mu(x)$$

for $x \in (0,1)$. For every $k \in \mathbb{N}$ the step function $t_k$ generates a uniquely determined function $h_k \in S$, suppose that $h_k(z) = z + \sum_{j=2}^{\infty} a_j(z)^j$. By (7) there exist functions $g_{m(k)}(x)$ and $g_m(x)$ for $m \geq 2$ that have the representations

$$g_{m(k)}(x) = \int_0^x \sum_{j=1}^{m-1} 2 j s^{m-j-1} g_j(s) t_k(s)^{m-j} ds$$

and

$$g_m(x) = \int_0^x \sum_{j=1}^{m-1} 2 j s^{m-j-1} g_j(s) t(s)^{m-j} ds$$
and, since \( \mu \) is a generating function of \( h \)

\[
g_m(x) = -\sum_{j=1}^{m-1} 2j s^{m-j-1} g_j(s) \mu(s)^{m-j} ds.
\]

Then the relations \( g_m^{(k)}(1) = a_m^{(k)} = a_m(h_k) \) and \( g_m(1) = a_m = a_m(h) \) hold for \( m \in \mathbb{N} \). Applying the Lebesgue dominated convergence theorem, the product rule for limits and (11) to (12) by induction yields the relations

\[
l_{k} \to \infty b_{k}(x) = g_{m}(x)
\]

for every \( x \in (0, 1] \) and every \( m \geq 2 \). Since \( \Phi \) is continuous (14) with \( x = 1 \) implies that there exists a \( N(\varepsilon) \in \mathbb{N} \) so that for \( m > N(\varepsilon) \)

\[
|\text{Re}\Phi(a_{2}^{(m)}, \ldots, a_{n}^{(m)}) - \text{Re}\Phi(a_{2}, \ldots, a_{n})| \leq |\Phi(a_{2}^{(m)}, \ldots, a_{n}^{(m)}) - \Phi(a_{2}, \ldots, a_{n})| < \varepsilon/2.
\]

On the other hand since \( (a_{2}^{(m)}, \ldots, a_{n}^{(m)}) \in \mathbb{V}_{m}^{n} \) the supposition (8) implies that for every \( m \in \mathbb{N} \) there exists a function \( f_{m} \in S \) such that

\[
\text{Re}\Phi(a_{2}^{(m)}, \ldots, a_{n}^{(m)}) \leq \text{Re}\Phi(a_{2} f_{m}, \ldots, a_{n} f_{m})
\]

and \( (a_{2}(f_{m}), \ldots, a_{n}(f_{m})) \in \mathbb{V}_{m}^{n} \). But \( h_{S} \subseteq S \) was chosen as an extremal function that maximizes \( \text{Re}\Phi \) and hence (10),(15) and (16) imply that

\[
0 \leq \text{Re}\Phi(h_{S}) - \text{Re}\Phi(a_{2}(f_{m}), \ldots, a_{n}(f_{m})) \leq \text{Re}\Phi(h_{S}) - \text{Re}\Phi(a_{2}^{(m)}, \ldots, a_{n}^{(m)}) < \varepsilon
\]

for \( m > N(\varepsilon) \). This proves (9). For an arbitrary \( m \in \mathbb{N} \) since \( (a_{2}(f_{m}), \ldots, a_{n}(f_{m})) \in \mathbb{V}_{m}^{n} \) the schlicht function \( f_{m} \) possesses a generating step function \( s_{m} \in T_{m}([0, 1]) \). The sequence \( (s_{m}) \) is uniformly bounded, \( s_{m} \in L^{\infty}([0, 1]) \) and since \( (L^1([0, 1]))' = L^{\infty}([0, 1]) \) the sequential Banach-Alaoglu Theorem asserts that the sequence \( (s_{m}) \) possesses a weak-*-convergent subsequence \( (s_{m(k)}) \). A continuous linear functional \( \Psi : L^{1}([0, 1]) \to \mathbb{C} \) can be represented in the form \( \Psi(f) = \int_{0}^{1} f(t) \mu(t) dt \) where \( \mu \in L^{\infty}([0, 1]) \) (Theorem 6.16). The norm of \( \Psi \) is defined by \( ||\Psi|| = \sup{|\Psi(f)|/||f||} \in L^{1}([0, 1]), ||f|| \leq 1 \). Weak-*- convergence in this context means that

\[
\lim_{k \to \infty} \Psi_{m(k)}(f) = \Psi(f)
\]

for every \( f \in L^{1}([0, 1]) \) where \( \Psi_{m(k)}(f) = \int_{0}^{1} f(t) s_{m(k)}(t) dt \). Given an arbitrary \( \varepsilon > 0 \) and an arbitrary \( f \in L^{1}([0, 1]) \) because of (18) there exists a \( N(f, \varepsilon) \in \mathbb{N} \) such that \( ||\Psi_{m(k)}(f) - \Psi(f)|| < \varepsilon \) if \( k > N(f, \varepsilon) \) and therefore \( ||\Psi_{m(k)} - \Psi|| \to 0 \) for \( k \to \infty \). Since the mapping \( \Psi \mapsto \mu \) is an isometric isomorphism and since the union of countable sets of measure zero is again a set of measure zero the subsequence \( (s_{m(k)}) \) of \( (s_{m}) \) converges uniformly a.e. on \( [0, 1] \). Moreover the subsequences \( (s_{m(k)}) \) and \( (f_{m(k)}) \) may be chosen in such a way that for the same index set \( \{m(k)\} \in \mathbb{N} \) both properties \( (f_{m(k)}) \) converges locally uniformly on \( D \) and \( (s_{m(k)}) \) converges uniformly a.e. on \( [0, 1] \) hold simultaneously. This proves the theorem.  \( \Box \)
For Theorem 1 to be useful explicit formulae of the coefficients of schlicht functions that are generated by step functions are needed. Lemma 1 provides these representations for the second, third and fourth coefficients.

**Lemma 1.** Let \( n \in \mathbb{N} \) be arbitrary, with the notation of \( T_n([0,1]) \) as above let \( s \in T_n([0,1]) \), \( s : [0,1] \to \mathbb{C} \) be defined by \( s(x) = c_k \) if \( x \in I_k \) where \( c_k \in \mathbb{C} \), for \( k = 1, \ldots, n \) and let \( c_0 \equiv 0 \). Suppose that \( s \) is a generating function of the schlicht function \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \). Then

\[
\begin{align*}
    a_2 &= -\frac{2}{n} \sum_{k=1}^{n} c_k \\
    a_3 &= a_2^2 - \sum_{k=1}^{n} \frac{2k-1}{n^2} c_k^2 \\
    a_4 &= 3a_2a_3 - 2a_2^3 - \sum_{k=1}^{n} c_k^2 \{ k^2 c_k + (2k-1) \sum_{j=0}^{k-1} c_j \}.
\end{align*}
\]

**Proof.** For \( x \in (0,1) \) the closed form representation \( s(x) = c_{[nx]+1} \) holds and \( s(1) = c_n \). Then (7) yields

\[
    g_2(x) = -\int_{0}^{x} 2s(t)dt = -\frac{2}{n} \sum_{k=0}^{[nx]} c_k - 2(x - \frac{[nx]}{n}) c_{[nx]+1}
\]

Taking the limit \( \lim_{x \to 1^-} g_2(x) \) in (22) proves (19). To prove (20) substitute \( g_2'(t) = -2s(t) \) (equation (6) with \( n = 2 \)) in (7). This yields

\[
g_3(1) = \int_{0}^{1} 2g_2(t)g_2'(t) \, dt = 2g_2(1)^2 - \sum_{k=1}^{n} \frac{k}{(k-1)/n} \int_{0}^{k/n} 2t c_k^2 \, dt = 2g_2(1)^2 - \sum_{k=1}^{n} c_k^2 \frac{2k-1}{n^2}.\]

This proves (20). To prove (21) substitute \( g_3'(t) = -2s(t) \) in (7) for \( n = 4 \). Then

\[
g_4(1) = \int_{0}^{1} \frac{1}{4} t^2 g_2^3(t) - t g_2(t) g_2'(t)^2 + 3g_3(t)g_2'(t) \, dt
\]

Apply integration by parts to (23) and substitute \( g_3'(t) = 2g_2(t)g_2'(t) - \frac{1}{2} t g_2'(t)^2 \) to obtain

\[
g_4(1) = 3g_3(1)g_2(1) - 2g_2(1)^3 + \int_{0}^{1} \frac{1}{4} t^2 g_2^3(t) + \frac{1}{2} t g_2(t) g_2'(t)^2 \, dt
\]

Now use (22) to evaluate the integral expression in (24) on the subinterval \( (k-1)/n, k/n \) where \( 1 \leq k \leq n \)

\[
\begin{align*}
    \int_{(k-1)/n}^{k/n} \frac{1}{4} t^2 g_2^3(t) + \frac{1}{2} t g_2(t) g_2'(t)^2 \, dt &= -\int_{(k-1)/n}^{k/n} 2t^2 c_k^3 + 2t^2 c_k^2 \left( \sum_{j=0}^{k-1} c_j + 2(t - \frac{k-1}{n}) c_k \right) \, dt
\end{align*}
\]
Summing these expressions and substituting into (24) yields (21).  

Lemma 1 allows to express the real and imaginary parts of the second, third and fourth coefficients in (19)-(21) as trigonometric polynomials, since the \(c_k\) have representations \(c_k = \cos \varphi_k + i \sin \varphi_k\) for \(k = 1, .., m\). As a consequence the real part \(Re \Phi : \mathbb{V}^m \to \mathbb{R}\) of the continuous functional \(\Phi\) restricted to \(\mathbb{V}^m\) can be expressed as a continuous function \(\Theta_m : [0, 2\pi]^m \to \mathbb{R}\) \(\Theta_m(\varphi_1, .., \varphi_m) = Re \Phi(a_2, .., a_n)\). Therefore the Heine-Borel Theorem ensures that \(\Theta_m\) attains its maximum on \([0, 2\pi]^m\). It follows that condition (8) in Theorem 1 holds for every continuous coefficient functional of the second, third and fourth coefficients.

Maximizing sequences were calculated, the partition of the interval \([0, 2\pi]\) was always chosen as an equipartition with a varying number of subintervals \(m\). A method of successive refinements was used and the computer program used for the calculations is available from the author.

Comparison with well-known values of functionals (Fekete-Szegö theorem, third column, Table 1) indicates that by the method used here the functionals can be calculated with a precision of approximately 0.000005. Table 1 shows the results for some functionals.

**Table 1: Values of some functionals depending on the number \(m\) of subintervals**

| \(|\gamma_1|^2 + 2|\gamma_2|^2 - \frac{5}{2}\)| | \(|\gamma_1|^2 + 2|\gamma_2|^2 + 3|\gamma_3|^2 - \frac{11}{2}\)| | \(\frac{1}{2} a_3 - \frac{1}{3} a_2^2\)| | \(\frac{1}{2} a_4 - \frac{1}{2} a_3 a_2 + \frac{1}{10} a_2^3\)| |
|---|---|---|---|
| \(m = 50\) | 0.034815 | 0.029473 | 1.013393 | 1.006727 |
| \(m = 100\) | 0.034845 | 0.029566 | 1.013412 | 1.006755 |
| \(m = 200\) | 0.034853 | 0.029591 | 1.013414 | 1.006762 |
| \(m = 400\) | 0.034854 | 0.029596 | 1.013415 | 1.006763 |

Here the \(\gamma_k, k = 1, 2, 3\) denote the logarithmic coefficients of schlicht functions (see [I], chapter 5 for a definition of the logarithmic coefficients). The first two functionals of Table 1 are associated with the Milin-constant (see [I], chapter 5 for a definition of these functionals and their relation to the Milin-constant), the third functional of Table 1 is the modulus of the fifth
coefficient of odd schlicht functions and the fourth functional is the modulus of the seventh coefficient of odd schlicht functions. In particular the calculations reveal that Leemans result(5) that 1090/1083 ≈ 1.006463 is the maximum of the modulus of the seventh coefficient of odd schlicht functions which are typically real does not extend to the whole class of schlicht functions.

Numerical evidence indicates that the maximum of the functional in the first column might be attained by a typically real function. The next Lemma supports this assumption.

**Lemma 2.** Let T denote the class of typically real functions and S the class of schlicht functions. Then

\[
\max_{f \in S \cap T} |\gamma_1|^2 - 1 + 2|\gamma_2|^2 - \frac{1}{2} = 2\lambda_0^4 e^{-4\lambda_0} + (3\lambda_0^2 + 2\lambda_0 + 1)e^{-2\lambda_0} - 1 \approx 0.03485611
\]

where \(\lambda_0\) is the zero of the equation \(0 = 4e^{-2\lambda}(\lambda^2 - \lambda^3) - 3\lambda + 1\) in \([0, \infty)\), i.e. \(\lambda_0 = 0.39004568\)....

**Proof.** The first and second logarithmic coefficients \(\gamma_1\) and \(\gamma_2\) by (3) satisfy the equations

\[
\dot{\gamma}_1(t) = \gamma_1(t) + \kappa(t) \quad \text{and} \quad \dot{\gamma}_2(t) = 2\gamma_2(t) + 2\gamma_1(t)\kappa(t) + \kappa(t)^2.
\]

Then

\[
\gamma_1(t)e^{-t} = -\int_t^\infty \kappa(s)e^{-s} ds
\]

and the second equation is equivalent to the equation

\[
\frac{d}{dt}(\gamma_2(t)e^{-2t}) = -2(\gamma_1(t)e^{-t})\frac{d}{dt}(\gamma_1(t)e^{-t}) + e^{-2t}\kappa(t)^2.
\]

Let \(\gamma_1(0) = \gamma_1\) and \(\gamma_2(0) = \gamma_2\) then integrating the last equation yields

\[
\gamma_2 = \gamma_1^2 - \int_0^\infty e^{-2t}\kappa(t)^2 dt
\]

Suppose that \(Im\gamma_2 = 0\) and \(Im\gamma_3 = 0\), then also \(Im\gamma_2 = 0\) and \(Im\gamma_1 = 0\) and

\[
2|\gamma_2|^2 + |\gamma_1|^2 = 2|\gamma_1|^2 - Re\int_0^\infty e^{-2t}\kappa(t)^2 dt + |\gamma_1|^2.
\]

Let \(R\kappa(t) = \cos\theta(t)\) then

\[
2|\gamma_2|^2 + |\gamma_1|^2 = 2|\gamma_1|^2 - \int_0^\infty e^{-2t}\cos2\theta(t) dt + |\gamma_1|^2
\]

\[
= 2|\gamma_1|^2 + \frac{1}{2} - 2\int_0^\infty (e^{-t}\cos\theta(t))^2 dt + \gamma_1^2.
\]

If \(\int_0^\infty (e^{-t}\cos\theta(t))^2 dt = (\lambda + \frac{1}{2})e^{-2\lambda}\) then the Valiron-Landau Lemma(11, chapter 3) yields \(\gamma_1^2 \leq (\lambda + 1)^2 e^{-2\lambda}\). Hence

\[
2|\gamma_2|^2 + |\gamma_1|^2 \leq 2(\lambda^2 e^{-2\lambda} + \frac{1}{2})^2 + (\lambda + 1)^2 e^{-2\lambda} = F(\lambda).
\]

A necessary condition that \(F(\lambda)\) attains its maximum is \(0 = 4e^{-2\lambda}(\lambda^2 - \lambda^3) - 3\lambda + 1\). The solution of the equation is \(\lambda_0 \approx 0.39004568\). The existence of a schlicht function for which equality holds
in (25) is proved almost identical as in the proof of Theorem 3.22 in [1]. Since the second and
the third coefficients were chosen to be real Goodman’s Theorem[5] ensures the existence of
a typically real function for which equality holds in (25).

Furthermore, since numerical calculation of the Milin-Functional \( \delta_3 := |\gamma_1|^2 - 1 + 2|\gamma_2|^2 - \frac{1}{2} + 3|\gamma_3|^3 - \frac{1}{3} \) for \( n = 3 \) shows that \( \delta_3 < 0.03 \) the value given in the Lemma might well be the Milin-
constant.

Theorems similar to Theorem 1 were given by Tammi([6], chapter 3.4). The proof here is some-
what different and Tammi only considered the second and the third coefficients of schlicht
functions(see [6], chapter 3.3). In particular he did not use his results to systematically derive
Euler-Lagrange-type equations for the generating functions. It is however this kind of application
of Theorem 1 and Lemma 1 given in the next two theorems that makes the method a true
variational method.

**Theorem 2.** Let \( k_0(z) = z/(1 - e^{it}z)^2, \) \( z \in \mathbb{D}, \) \( \theta \in \mathbb{R} \) denote the rotations of the Koebe function
and suppose \( \Phi : \Delta_\epsilon \rightarrow \mathbb{R} \) is continuously differentiable in \( \Delta_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4 + \epsilon \} \) for
some \( \epsilon > 0. \) Let \( D_1 \Phi \) and \( D_2 \Phi \) denote the partial derivatives of \( \Phi(x, y) \) with respect to \( x \) and \( y \)
resp. Then there exists a schlicht function \( f(z) = z + \sum_{k=2}^{\infty} \alpha_k z^k \) that maximizes the functional
\( Re(a_3) + \Phi(Rea_2, Ima_2) \) and a generating function \( \kappa : [0, \infty) \rightarrow \mathbb{C} \) of \( f \) that satisfies the equation
(26)
\[
0 = 2e^{-t}Re(t)Im\kappa(t) + c_1 Im\kappa(t) + c_2 Re\kappa(t)
\]
a.e. on \([0, \infty)\) where \( c_1 = Rea_2 + \frac{1}{2} D_1 \Phi(Rea_2, Ima_2) \) and \( c_2 = Ima_2 - \frac{1}{2} D_2 \Phi(Rea_2, Ima_2) \). Furthermore if \( c_1c_2 \neq 0 \) then \( Re(t)Im\kappa(t) \neq 0 \) for \( t \in [0, \infty) \) and
(27)
\[
\lim_{t \to \infty} e^{-t} \kappa(t) = e^{i\varphi} \quad (0, 2\pi)
\]
where \( tan\varphi = -c_2/c_1. \) If the extremal function \( f \in S \) is embedded into a Löwner chain \( f(., t) \) as
the initial point, i.e. \( f(z, 0) = f(z) \) for \( z \in \mathbb{D} \) then the image domains \( f(\mathbb{D}, t) \) satisfy
(28)
\[
e^{-t} f(\mathbb{D}, t) \rightarrow k_{\varphi + \pi}(\mathbb{D}) \quad t \to \infty \quad \text{in the sense of the kernel if} \quad c_1c_2 \neq 0
\]
If \( c_1 \neq 0 \) or \( c_2 \neq 0 \) and \( c_1c_2 = 0 \) then there exists a \( t_0 \in [0, \infty) \) such that
(29)
\[
e^{-t} f(\mathbb{D}, t) = k_0(\mathbb{D}) \quad \text{or} \quad e^{-t} f(\mathbb{D}, t) = k_\pi(\mathbb{D}) \quad \text{if} \quad c_1 \neq 0 \quad \text{and} \quad c_2 = 0
\]
(30)
\[
e^{-t} f(\mathbb{D}, t) = k_{\pi/2}(\mathbb{D}) \quad \text{or} \quad e^{-t} f(\mathbb{D}, t) = k_{3\pi/2}(\mathbb{D}) \quad \text{if} \quad c_1 = 0 \quad \text{and} \quad c_2 \neq 0
\]
holds for \( t \geq t_0. \)

**Proof.** Let \( x = e^{-t} \) and let \( (f_n) \) be a maximizing sequence that satisfies (8), which exists by
Theorems 1, 2. For \( n \in \mathbb{N} \) with the notations as above suppose that \( s_n \in T_n([0,1]) \) is a generating
step function of \( f_n \) defined by
(31)
\[
s_n(x) = c_k^{(n)}, \quad x \in I_k, \quad |c_k^{(n)}| = 1
\]
for $k = 1, \ldots, n$. If a function $h \in S$ satisfies (8) and has representations (19), (20) for $a_2$ and $a_3$ where $c_j^{(n)} = \exp(i\varphi_j^{(n)})$, $j = 1, \ldots, n$ then the differentiability-criterion for stationary points from multivariable differential calculus can be applied and yields the equations

$$0 = \frac{\partial}{\partial \varphi_j}( (\text{Re}a_2)^2 - (\text{Im}a_2)^2 - \sum_{j=1}^{n} \frac{2j-1}{n^2} \cos(2\varphi_j^{(n)}) + \Phi(\text{Re}a_2, \text{Im}a_2))$$

$$= \frac{\partial \text{Re}a_2}{\partial \varphi_j}(2\text{Re}a_2 + D_1\Phi(\text{Re}a_2, \text{Im}a_2)) - \frac{\partial \text{Im}a_2}{\partial \varphi_j}(2\text{Im}a_2 - D_2\Phi(\text{Re}a_2, \text{Im}a_2)) +$$

$$\frac{2j-1}{n^2} \sin\varphi_j^{(n)} \cos\varphi_j^{(n)}$$

for $1 \leq j \leq n$. Use (19) to obtain the equivalent system

$$0 = (\text{Re}a_2 + \frac{1}{2}D_1\Phi(\text{Re}a_2, \text{Im}a_2)) \sin\varphi_j^{(n)} + (\text{Im}a_2 - \frac{1}{2}D_2\Phi(\text{Re}a_2, \text{Im}a_2)) \cos\varphi_j^{(n)} +$$

$$\frac{2j-1}{n} \sin\varphi_j^{(n)} \cos\varphi_j^{(n)}$$

for $1 \leq j \leq n$. By Theorem 1 there exists a subsequence $(s_{n(k)})$ of $(s_n)$ which converges uniformly a.e. on $[0, 1]$ to a function $\mu \in L^\infty([0, 1])$ and additionally the subsequence $(f_{n(k)})$ of $(f_n)$ also converges locally uniformly in $D$ to an extremal function $f$. Choose an arbitrary but fixed $x \in (0, 1]$, $x \notin \mathbb{Q}$ such that $\lim_{k \to \infty} s_{n(k)}(x)$ exists. Then for every $n \in \mathbb{N}$ there exists a number $m(n) \in \mathbb{N}$, $1 \leq m(n) \leq n$ such that $(m(n) - 1)/n < x < m(n)/n$. Hence for $j = m(n)$

$$0 = (\text{Re}a_2 + \frac{1}{2}D_1\Phi(\text{Re}a_2, \text{Im}a_2)) \sin\varphi_j^{(n)} + (\text{Im}a_2 - \frac{1}{2}D_2\Phi(\text{Re}a_2, \text{Im}a_2)) \cos\varphi_j^{(n)} +$$

$$\frac{2m(n) - 1}{n} \sin\varphi_{m(n)}^{(n)} \cos\varphi_{m(n)}^{(n)}.$$  

By the Cantor intersection theorem

$$x = \bigcap_{n \in \mathbb{N}} \left[ \left( (m(n) - 1)/n, m(n)/n \right) \right]$$

and hence $\lim_{n \to \infty} (m(n)/n) = x$. Since $\lim_{k \to \infty} s_{n(k)}(x) = \mu(x)$, since $D_1\Phi$ and $D_2\Phi$ are continuous in $\Delta_\varepsilon$ and since $\lim_{k \to \infty} a_2(f_{n(k)}) = a_2$ taking the limit $k \to \infty$ for the subsequence $(n(k))$ in the last equation yields

$$0 = (\text{Re}a_2 + \frac{1}{2}D_1\Phi(\text{Re}a_2, \text{Im}a_2)) \text{Im}\mu(x) +$$

$$(\text{Im}a_2 - \frac{1}{2}D_2\Phi(\text{Re}a_2, \text{Im}a_2)) \text{Re}\mu(x) +$$

$$2x \text{Re}\mu(x) \text{Im}\mu(x) = 0.$$  

This proves (26). If $c_1 \neq 0$ and $c_2 = 0$ then by (26) $\text{Im}\mu(t) \equiv 0$ or $\text{Re}\mu(t) = -(1/2)c_1 e^t$. Therefore there exists a $t_0 \in [0, \infty)$ such that $\text{Im}\mu(t) \equiv 0$ and $\text{Re}\mu(t) = \pm 1$ if $t \geq t_0$. To prove (29) observe
that for \( t \in [t_0, \infty) \) by (4) the second coefficient \( a_2(f(., t)) \) of the function \( f(., t) \) is given by
\[
a_2(f(., t)) = -e^{2t} \int_1^\infty e^{-2s} e^{k(s)} ds = \pm e^{2t} \int_1^\infty 2e^{-s} ds = \pm 2e^t
\]
and hence the second coefficient of the normalized function \( e^{-t} f(., t) \in S \) is \( \pm 2 \). By Bieberbachs theorem (29) follows. (30) is proved similarly. Finally if \( c_1 \neq 0 \) and \( c_2 \neq 0 \) then \( \text{Re}u(x) \neq 0 \) and \( \text{Im}u(x) \neq 0 \) for \( x \in (0, 1] \). Suppose on the contrary this were not true, and suppose that \( \text{Re}u(x) = 0 \). Then since \( (\text{Re}u(x))^2 + (\text{Im}u(x))^2 = 1 \) (26) implies that \( c_1 = 0 \), a contradiction. Similarly suppose that \( \text{Im}u(x) = 0 \) then (26) implies that \( c_2 = 0 \). Therefore
\[
0 \neq -c_2 = \lim_{x \to 0} \frac{\text{Im}u(x)}{\text{Re}u(x)} = \tan \theta
\]
where \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and
\[
(33) \quad \lim_{x \to 0^+} \mu(x) = e^{i\varphi}
\]
where \( \varphi = \theta \) or \( \varphi = \theta + \pi \). Let \( \varepsilon > 0 \) be arbitrarily chosen. Then by (5)-(7) \( x^2a_2(x) = g_2(x) = -\int_0^x 2\mu(t)dt \) and hence by (33)
\[
(34) \quad \left| \frac{e^{-2t}a_2(f(., t))}{e^{-t}} - (-2e^{i\varphi}) \right| = \left| \frac{-\int_0^x 2\mu(t) - 2e^{i\varphi} dt}{x} \right| \leq 2 \sup_{t \in [0, x]} |\mu(t) - e^{i\varphi}| < 2\varepsilon.
\]
if \( 0 < x < \delta \). Therefore (34) implies that every accumulation point of the normal family \( \{e^{-t}f(., t)|t \in [0, \infty]\} \) of the form
\[
h(z) = \lim_{t_n \to \infty} e^{-t_n} f(z, t_n) \quad z \in \mathbb{D}
\]
has second coefficient \( a_2(h) = -2e^{i\varphi} \) where \( (t_n) \) is a sequence such that \( \lim_{n \to \infty} t_n = \infty \). Hence by Bieberbachs theorem and by the Carathéodory kernel theorem \( h(z) = k_{\varphi+\pi}(z) \).

Theorem 2 together with elementary results from the theory of schlicht functions allows for a geometric characterization of the image domains of extremal functions. Since the complement in the extended plane \( S^2 - f(\mathbb{D}) \) of an extremal function \( f \in S \) is compact and connected (3, Theorem 13.11) and since \( f(\mathbb{D}) \) lies dense in \( \mathbb{C} \) (1, Theorem 9.4) it follows that \( S^2 - f(\mathbb{D}) \) consists of a collection of Jordan arcs such that \( S^2 - f(\mathbb{D}) \) is connected. Since in the cases (28)-(30) the normalized family \( e^{-t} f(., t) \) converges in the sense of the kernel to a rotation of the Koebe-function \( k_\varphi, \varphi \in [0, 2\pi) \) there exists an asymptotic direction at infinity defined by \( t(-e^{-i\varphi}) = t\kappa(t), t \in [0, \infty) \). The generating function of \( k_\varphi \) is given by \( \kappa(t) \equiv -e^{i\varphi} \) and therefore the asymptotic direction at infinity is defined either by \( -t(c_1 + ic_2), t \in [0, \infty) \) or by \( t(c_1 + ic_2), t \in [0, \infty) \).

The proof given shows that the method even allows for geometrical characterizations of extremal functions, information which is usually only accessible by quadratic differential methods. Whereas the results given in Theorem 2 were already well-known(by other methods) for
some time, the results given in the next theorem are new and since $\max_{f \in S} \text{Re}\{a_4 - \frac{1}{2}a_2a_3 + \frac{i}{2}a_2^2\} = \max_{f \in \mathbb{S}} |a_4 - \frac{1}{2}a_2a_3 + \frac{i}{2}a_2^2|$ there exists an extremal that maximizes the modulus of the seventh coefficient of odd schlicht functions and satisfies the Euler-Lagrange-type equation given in this theorem.

**Theorem 3.** With the notations as above let the functional $\Phi : \mathbb{V}_4 \to \mathbb{C}$ be defined by $\Phi(a_2, a_3, a_4) = a_4 + a_2a_3 + \beta a_2^3$ for some $a, \beta \in \mathbb{R}$. Then there exists an extremal function $f \in S$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that maximizes $\text{Re} \Phi$ and a Löwner chain $f(z, t) = e^{t}z + \sum_{n=2}^{\infty} a_n(t) z^n$ with $f(z, 0) = f(z)$, $z \in \mathbb{D}$ such that the generating function $\mu(x) = \kappa(t)$ associated with the Löwner chain $f(z, t)$ satisfies the equation

$$0 = c_2 \text{Re} \mu(x) + c_1 \text{Im} \mu(x) +$$

$$\text{Im} \{2x(\alpha + 3)a_2 \mu(x)^2 + 3x^2 \mu(x)^3 + 4x \mu(x)^2 \} \int_0^x \mu(t) dt - \mu(x) \left\{\int_0^x 2t \mu(t)^2 dt\right\}$$

a.e. on $(0, 1]$ where $c_1 = \text{Re}\{(1 + 2\alpha + 3\beta)a_2^2 + (\alpha + 2)a_3\}$ and $c_2 = \text{Im}\{(1 + 2\alpha + 3\beta)a_2^2 + (\alpha + 2)a_3\}$. If moreover $\lim_{x \to 0^+} \mu(x) = e^{i\varphi}$, $\varphi \in [0, 2\pi)$ exists and $c_1 \neq 0$ or $c_2 \neq 0$ then $\tan \varphi = -(c_2/c_1)$ and the image domains $e^{-t}f(\mathbb{D}, t)$ converge in the sense of the kernel to $k_{\varphi + n}(\mathbb{D})$ as $t \to \infty$.

**Proof.** Let $x = e^{-t}$ and let $(f_n)$ be a maximizing sequence that satisfies (8), which exists by Theorem 1 and Lemma 1. For $n \in \mathbb{N}$ with the notations above suppose that $s_n \in T_n([0, 1])$ is a generating step function of $f_n$ defined by

$$s_n(x) = c_k^{(n)}, \quad x \in I_k, \quad |c_k^{(n)}| = 1$$

for $k = 1, \ldots, n$. Then the functions $f_n \in S$ satisfy (8) and have representations (19), (20) and (21) for $a_2(f_n)$, $a_3(f_n)$ and $a_4(f_n)$ with $c_j^{(n)} = \text{exp}(i \varphi_j^{(n)})$, $j = 1, \ldots, n$. Hence, since $\Phi$ is continuously differentiable in a neighbourhood of $\mathbb{V}_4$ it follows that

$$0 = \frac{\partial}{\partial \varphi_j^{(n)}} \text{Re}\{a_4(f_n) + a_2(f_n)a_3(f_n) + \beta a_2(f_n)^3\}$$

$$= \text{Re}\frac{\partial}{\partial \varphi_j^{(n)}} [(\alpha + 3)a_2(f_n)a_3(f_n) + (\beta - 2)a_2(f_n)^3$$

$$- \frac{2}{n^2} \sum_{k=1}^{n} \left[c_k^{(n)}]^2[k^2 c_k^{(n)}] + (2k - 1) \sum_{m=0}^{k-1} c_m^{(n)}\right]\}$$

$$= \text{Re}\{-\alpha + 3)[a_2(f_n)(2a_2(f_n)^2 - 2c_j^{(n)} + \frac{2j-1}{n}2(c_j^{(n)})^3 + a_3(f_n) - \frac{2}{n}c_j^{(n)}]\}$$

$$+ (\beta - 2)a_2(f_n)^2 i \frac{2}{n}c_j^{(n)}$$

$$- \frac{2}{n^2} \left[3i j^2(c_j^{(n)})^3 + 2i(2j-1)(c_j^{(n)})^2 \sum_{m=0}^{j-1} c_m^{(n)} + ic_j^{(n)} \sum_{m=j+1}^{n} (2m-1)(c_m^{(n)})^2\right]$$
for $1 \leq j \leq n$. By Theorem 1 there exists a subsequence $(s_{n(k)})$ of $(s_n)$ which converges uniformly a.e. on $[0, 1]$ to a function $\mu \in L^\infty([0, 1])$ and additionally the subsequence $(f_{n(k)})$ of $(f_n)$ also converges locally uniformly in $\mathbb{D}$ to an extremal function $f$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Suppose that $x \in [0, 1], x \notin \mathbb{Q}$ and that $\lim_{k \to \infty} s_{n(k)}(x)$ exists. Then for every $n \in \mathbb{N}$ there exists a number $m(n) \in \mathbb{N}, 1 \leq m(n) \leq n$ such that $(m(n) - 1)/n < x < m(n)/n$ and for $j = m(n)$ the last equation is equivalent to the equation

$$0 = \text{Im}(2\alpha + 3\beta)c_{m(n)}^{(n)}a_2(f_n)^2 + (\alpha + 3)(a_3(f_n)c_{m(n)}^{(n)} + \frac{2m(n) - 1}{n}(c_{m(n)}^{(n)})^2 a_2(f_n))$$

$$+ \frac{1}{n^2}[3m(n)^2(c_{m(n)}^{(n)})^3 + 2(2m(n) - 1)(c_{m(n)}^{(n)})^2 \sum_{l=0}^{m(n)-1} c_{l}^{(n)} + c_{m(n)}^{(n)} \sum_{l=m(n)+1}^{n} (2l - 1)(c_{l}^{(n)})^2].$$

By the Cantor intersection theorem

$$(36) \quad x = \bigcap_{n \in \mathbb{N}} \left( [(m(n) - 1)/n, m(n)/n]\right)$$

and hence $\lim_{n \to \infty}(m(n)/n) = x$. Since $\lim_{k \to \infty} s_{n(k)} = \mu(x)$ and $\lim_{k \to \infty} a_j(f_{n(k)}) = a_j(f) = a_j$ for $j \in \mathbb{N}$ and since the sums in the last equation can be represented as integrals of step functions, the Lebesgue dominated convergence theorem can be applied and letting $k \to \infty$ for the subsequence $(n(k))$ in the last equation yields

$$0 = \text{Im}(2\alpha + 3\beta)\mu(x)a_2^2 + (\alpha + 3)(\mu(x)a_3 + 2x\alpha_2\mu(x)^2)$$

$$+ 3x^2\mu(x)^3 + 4x\mu(x)^2\int_0^x \mu(t)dt + \mu(x)\int_x^1 2t\mu(t)^2 dt.$$

Substituting $g_2'(x) = -2\mu(x)$ into (7) for $n = 3$ yields $g_3(x) = g_2(x)^2 - \int_0^x 2t\mu(t)^2 dt$ and hence

$$0 = \text{Im}(2\alpha + 3\beta)\mu(x)a_2^2 + (\alpha + 3)(\mu(x)a_3 + 2x\alpha_2\mu(x)^2)$$

$$+ 3x^2\mu(x)^3 + 4x\mu(x)^2\int_0^x \mu(t)dt - \mu(x)\int_0^x 2t\mu(t)^2 dt - \mu(x)(a_3 - a_2^3).$$

This proves the equation for the extremals. The second part is proved almost identical as in Theorem 2( arguments following (33)).

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