QUASI-STATIONARY BINARY INSPIRAL:
PROJECT OVERVIEW

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Abstract. I describe the current status of a collaboration with J. D. Romano, R. H. Price, and W. Krivan to model the geometry of and gravitational radiation emitted by a binary system of compact objects in the regime where non-perturbative gravitational effects exist, but the rate of inspiral is still small relative to the orbital frequency. The method of looking for a stationary spacetime which approximates the evolving solution is initially being tested on a simpler model with an additional translational symmetry. This report consists of a general description of the method, followed by summaries of three techniques in varying stages of development: the simplification of the Einstein equations in the presence of two commuting Killing vectors which form a non-orthogonally-transitive symmetry group, the boundary conditions appropriate to the balance of ingoing and outgoing radiation needed to reconcile a stationary radiating solution with conservation of energy, and the treatment of gravitational waves far from the sources as linearized perturbations to the Levi-Civita spacetime. The poster presentation with which this paper is associated is available online at http://www- itp.unibe.ch/~whelan/poster.ps.gz and the current status of the project is described at http://www-itp.unibe.ch/~whelan/qsbi.html

1. Introduction

1.1. Inspiral of Compact Object Binaries

A pair of compact objects (black holes or neutron stars) in binary orbit about one another is stable in Newtonian gravity. In general relativity, however, the system will emit gravitational radiation, causing the bodies to spiral in towards one another. The gravitational radiation given off by this system is a prime candidate for detection by upcoming gravitational wave telescopes such as VIRGO and LIGO.

When the compact objects are far apart, all gravitational effects are weak and one can use the Post-Newtonian approximation, expanding in powers of $GM/Rc^2$. The final “plunge”, where the objects cease to orbit and collide rapidly, can be modelled by full general relativistic three-plus-one supercomputer simulations. An intermediate late inspiral phase, where strong gravitational effects are important, but the fraction of total energy lost to gravitational radiation each orbit is small, cannot be handled by supercomputer evolutions, which become unstable after several orbits. To handle strong gravity and slow inspiral, we need to use an approximation scheme. Modelling this intermediate phase will allow us to determine the early gravitational wave signal as well as provide initial data for the final plunge simulations.
1.2. Quasi-Stationary Approximation

The idea, initially proposed by Steven Detweiler [1], is that if the inspiral is slow, the system is nearly periodic: after one orbit, the objects have returned almost to their original locations, and radiation which has moved out has been replaced with new radiation of approximately the same shape. If the objects’ orbits are circular rather than elliptical, the spacetime is nearly stationary. If the approximate orbital frequency is $\Omega$, moving forward in time by $\delta t$ and rotating the resulting spatial slice by $-\Omega \delta t$ will not change the picture very much.

Our approach is to replace the true spacetime, which has this approximate continuous symmetry, with another solution to Einstein’s equations in which the symmetry is exact. This “stationary quasi-solution”, which must somehow replace the energy lost in radiation to prevent inspiral, should approximate the physical “quasi-stationary solution” over some time interval. This replaces a three-plus-one-dimensional numerical evolution with a three-dimensional instantaneous solution, reducing greatly the computing power required and hopefully escaping some of the numerical instabilities associated with evolution.

To further simplify the problem, we will initially look for a spacetime with an additional translational symmetry orthogonal to the orbital plane. This toy model of co-orbiting cosmic strings further reduces the numerical problem to two dimensions while hopefully retaining some of the qualitative features.

The rest of this paper consists of summaries of three techniques being developed for this project, but which may prove useful in other settings as well. Section 2 describes research done in collaboration with Joseph D. Romano [2] on the refinement of a formalism developed by Geroch [3] to simplify the Einstein equations when the spacetime admits two commuting Killing vectors, as is the case with the co-orbiting cosmic string model. Section 3 discusses the construction of a stationary radiating solution to a wave equation without the use of external forces by finding a preferred solution containing a balance of incoming and outgoing radiation, the subject of a collaboration with William Krivan and Richard H. Price [4]. Section 4 briefly touches on the concept of describing small-amplitude gravitational radiation as a perturbation not to Minkowski spacetime but to the Levi-Civita solution [5] for a single cosmic string.

2. QSBI I: Spacetimes with Two Killing Vectors
(Whelan and Romano [2])

2.1. Lack of a Block-Diagonal Coordinate Basis

The spacetime of two co-orbiting cosmic strings has two continuous symmetries, which are described by two Killing vector fields. One of these, which we call $K_0$, represents the combination of time translation and rotation about the orbital axis which leaves the spacetime unchanged. It can be thought of roughly in terms of traditional cylindrical coordinates as $\partial_t + \Omega \partial_\phi$. The other Killing vector simply corresponds to translation along the strings and can be thought of as $K_1 \sim \partial_z$.

In a numerical determination of the spacetime geometry, one seeks to fix the coordinate (i.e., gauge) information completely, and thus calculate the minimum number of quantities necessary to define the geometry. It is desirable, of course, to choose a gauge which takes advantage of the symmetries of the problem. Following from the example of static, axisymmetric spacetimes [6], we might wish to define two Killing co-
ordinates \( \{x^A\} \) \((x^0 \sim t, x^1 \sim z)\), supplemented by two other coordinates \( \{x^i\}\) \((x^2 \sim \rho, x^3 \sim \phi \sim \phi - \Omega t)\) on a subspace orthogonal to the two Killing vectors, and bring the line element into a block-diagonal form
\[
\lambda_{AB}(\{x^k\})dx^A dx^B + \gamma_{ij}(\{x^k\})dx^i dx^j. \tag{1}
\]

For our set of Killing symmetries, however, this fails because the symmetry group is not orthogonally transitive, which means that it is impossible to construct two-surfaces everywhere orthogonal to both Killing vector trajectories. The measure of this failure is a pair of scalar fields \(\{c_A\}\) given by
\[
c_A = * (K_0 \wedge K_1 \wedge dK_A). \tag{2}
\]

The co-rotational Killing vector \(K_0\) is not surface-forming \((K_0 \wedge dK_0 \neq 0)\), which leads to the non-vanishing of \(c_0\), indicating a lack of orthogonal transitivity.

### 2.2. Manifold of Killing Vector Orbits

Although the metric components in any coordinate basis will have non-vanishing “cross terms” \(\{g_{Ai}\}\) between the Killing and non-Killing directions, it still possible to describe the geometry in terms of two matrices \(\{\lambda_{AB}\}\) and \(\{\gamma_{ij}\}\) along with the two scalars \(c_0\) and \(c_1\). This done using a construction due to Geroch \[3\] which defines a two-manifold \(S\) of Killing vector orbits. The coordinates \(\{x^i| i = 2, 3\}\) are coordinates on this two-manifold, and any tensor on the original four-dimensional spacetime manifold \(M\) which has vanishing inner products with, and Lie derivatives along, the Killing vectors corresponds to a tensor on the two-manifold \(S\). In particular, the tensors and scalars on \(S\) which describe the spacetime geometry are

- The symmetric matrix of inner products \(\lambda_{AB} = K_A \cdot K_B\)
- The metric on \(S\), which has components \(\{\gamma_{ij}\}\) and corresponds to the projection tensor \(g_{\mu\nu} = \lambda^{AB} K_A\mu K_B\nu\) on \(M\) (where \(\lambda^{AB}\) is the inverse of \(\{\lambda_{AB}\}\))
- The two scalars \(c_A\) which define the lack of orthogonal transitivity.

Given these objects, it is possible to define a non-coordinate basis on the four-dimensional spacetime by \(e_A = K_A\) and \(e_i = \gamma_{ij} dx^j\). In this basis the metric has components
\[
g_{AB} = \lambda_{AB}, \quad g_{Ai} = 0, \quad g_{ij} = \gamma_{ij}; \tag{3}
\]
the \(\{c_A\}\) give the commutation coefficients of the non-commuting basis vectors:
\[
[e_2, e_3] \propto \lambda^{AB} c_A K_B. \tag{4}
\]

### 2.3. Einstein Equations

Geroch’s original paper contained four partial differential equations involving \(\{\gamma_{ij}\}, \lambda_{00}, \lambda_{01}, \lambda_{11}, c_0,\) and \(c_1\) which were equivalent to the vacuum Einstein equations \(G_{\mu\nu} = 0\). We have found a streamlined derivation which treats the indexed quantities \(\lambda_{AB}\) and \(c_A\) as single entities, rather than dealing with each component individually. We have also derived explicit expressions (given in \[2\]) for the components \(G_{AB}, G_{Ai},\) and \(G_{ij}\) which can be used even when the stress-energy tensor is non-vanishing.

In the case where the off-block-diagonal components \(\{T_{Ai}\}\) of the stress-energy tensor vanish, the Einstein equations \(G_{Ai} = 0\) simply say that the scalars \(c_A\) are
For the remaining six Einstein equations, a convenient choice of gauge is given by
\begin{align}
    c_0 &\equiv 2\Omega, \quad c_1 \equiv 0, \\
    \lambda_{00} &\equiv (\lambda + X(\rho, \varphi)^2)Z(\rho, \varphi)^{-1}, \quad \lambda_{01} = X(\rho, \varphi), \quad \lambda_{11} = Z(\rho, \varphi), \\
    \gamma_{22} &\equiv 1, \quad \gamma_{23} = 0, \quad \gamma_{33} = -\lambda(\rho, \varphi)^{-1}\rho^2 F(\rho, \varphi). \quad (5, 6, 7)
\end{align}

These four functions $X, Z, \lambda,$ and $F$ of the two coordinates $x^2 = \rho$ and $x^3 = \varphi$ provide a fixed gauge in which the problem can be solved numerically. There are six block-diagonal Einstein equations (for $\{G_{\alpha\beta}\}$ and $\{G_{ij}\}$), but only four of them are independent because of the contracted Bianchi identities $(dx^\mu)^{\mu} = 0$. These equations will be combined with a treatment of the cosmic string sources and the boundary conditions at infinity and the origin to produce a numerically determined spacetime.

### 3. QSBI II: Radiation-Balanced Boundary Conditions
(Whelan, Krivan and Price [4])

#### 3.1. Radiative Boundary Conditions

The true physical spacetime which we are ultimately modelling contains gravitational radiation at infinity which is outgoing. This loss of energy leads to decay of the orbits and inspiral of the compact objects in a non-stationary solution. In many radiative problems, a solution with outgoing radiation can be constrained to be stationary by an external force whose agent does not couple to the radiation. However, this is at odds with the idea that in General Relativity all matter gravitates, so we should look instead for a solution where there is no net energy loss to infinity due to the radiation. A naïve replacement for the outgoing radiation boundary conditions would be a standing wave condition. However, that turns out to be inappropriate, as standing waves require a node (Dirichlet) or extremum (Neumann) at a particular location, and a standing wave condition thus fails to converge to a well-defined limit as that location is moved out to infinity.

We have been investigating the question of how to implement a sensible boundary condition leading to a balance of radiation in the context of a simple theory: a nonlinear scalar field $\psi(t, \rho, \phi)$ in two-plus-one dimensions, where the source $\sigma$ and field $\psi$ are required to be co-rotating (i.e., depend only on $\rho$ and $\varphi = \phi - \Omega t$). This causes the field equation to take the form
\begin{equation}
    \Box^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \rho^{-1} \frac{\partial \psi}{\partial \rho} + \left(\rho^{-2} - \Omega^2 e^{-2}\right) \frac{\partial^2 \psi}{\partial \varphi^2} = \sigma + \lambda \psi^3. \quad (8)
\end{equation}

In the numerical solution for the non-linear equation (8), one uses a finite co-ordinate grid and specifies a set of boundary conditions at the large finite radius $\rho = R$ which marks the end of the grid. Table 3.1 shows several possible boundary conditions. We call, e.g., the solution resulting from the application of the outgoing boundary condition at a particular $R$, $\psi_{\text{out}}^R$. The local conditions defining outgoing or ingoing radiation each produce well-defined limits (which we call simply $\psi_{\text{out}}$ and $\psi_{\text{in}}$) as $R$ is taken to infinity, but the standing wave solutions do not converge to any limit.
Table 1. Some large-distance boundary conditions. Purely ingoing or outgoing radiation has a well-defined limit as the radius $R$ at which it is applied is taken to infinity, but standing wave conditions (e.g., Neumann or Dirichlet) do not.

\[
\begin{array}{|c|c|c|}
\hline
\text{Outgoing} & \text{BCs at } \rho = R & R \to \infty \\
\hline
\text{ingoing} & (\partial_{\rho} + c^{-1} \partial_t) \psi_R^\text{in} = 0 & \psi^\text{in} \sim e^{im(\Omega c^{-1} \rho - \varphi)} \sim e^{im\Omega(c^{-1} \rho + t)} \\
\text{Neumann SW} & \partial_\rho \psi^N_R = 0 & \text{N/A} \\
\text{Dirichlet SW} & \psi^D_R = 0 & \text{N/A} \\
\hline
\end{array}
\]

Taking the average of the $R \to \infty$ limits $\psi^\text{out}$ and $\psi^\text{in}$ defines a function

\[
\psi^\text{avg} = \frac{1}{2}(\psi^\text{in} + \psi^\text{out}) 
\]

which has a balance of ingoing and outgoing radiation without reference to any particular radius $R$. In the linear ($\lambda = 0$) theory, where the principle of superposition holds, it is a solution to the field equations, but in the non-linear theory it is not. It is also not defined as the limit of any local boundary conditions (in particular, the is average of out- and ingoing solutions is not produced by an average of out- and ingoing boundary conditions).

3.2. Green’s Function Solution

The tasks of defining a solution analogous to $\psi^\text{avg}$ and numerically determining that solution in the absence of a local boundary condition can both be accomplished by a Green’s function method.

If we define a Green’s function $G(\rho, \varphi|\rho_0, \varphi_0)$ such that

\[
\square^2 G = (\rho_0)^{-1} \delta(\rho - \rho_0) \delta(\varphi - \varphi_0),
\]

the differential equation (10) can be converted into an integral equation

\[
\psi(\rho, \varphi) = \int\rho_0 \, d\rho_0 \, d\varphi_0 \, G(\rho, \varphi|\rho_0, \varphi_0) [\sigma(\rho_0, \varphi_0) + \lambda \psi^3(\rho, \varphi)].
\]

Solving to the Green’s function equation (11) requires the application of boundary conditions at infinity, so there are actually a family of Green’s functions with boundary conditions “built into” them. For example, using the retarded Green’s function $G^\text{out}$ in (11) gives the outgoing solution $\psi^\text{out}$, using the advanced Green’s function $G^\text{in}$ gives the ingoing solution $\psi^\text{in}$, and similarly for any of the boundary conditions defined at finite radii in Table 1.

Since the Green’s function equation (11) is linear, even when the wave equation (8) is not, we can always average the retarded and advanced Green’s functions to give a time-symmetric Green’s function

\[
G^\text{sym} = \frac{1}{2}(G^\text{in} + G^\text{out})
\]

which defines a radiation-balanced solution $\psi^\text{sym}$.

In the linear theory, this solution $\psi^\text{sym}$ will be equal to the average $\psi^\text{avg}$ of the ingoing and outgoing solutions. In the non-linear theory, we can calculate the two numerically (by iterating (11)) and compare them. We are ultimately trying to use
Figure 1. Comparison between $\psi_{\text{sym}}$ and $\psi_{\text{avg}}$ in the wave zone for a source of two particles of opposite charge at $\rho/2 = .25$, $\varphi = (\pi \pm \pi)/2$, with a rotational frequency of $\Omega = 1/2$ and an orbital velocity of one-fourth the speed of light. The fields are plotted versus $\rho/2$ for the range of angles $\varphi \in [1, \pi/2]$. The amplitude of $\psi_{\text{avg}}$ is around 20% smaller and the phase is shifted 40 degrees relative to $\psi_{\text{sym}}$. This agreement is only middling because the sources are still somewhat relativistic.

the outgoing piece of $\psi_{\text{sym}}$ as an approximation for $\psi_{\text{out}}$ (and thus as a further approximation for the actual inspiralling solution), and we expect that approximation to be good when $\psi_{\text{sym}} \approx \psi_{\text{avg}}$.

3.3. Numerical Results

Numerical calculations [4] for a strongly non-linear ($\lambda = 20$) theory have shown that the approximation becomes good as the orbital velocity of the sources becomes non-relativistic. Figures 1 and 2 show the comparison between $\psi_{\text{sym}}$ and $\psi_{\text{avg}}$ for orbital velocities of one-fourth and one-eighth the speed of light, respectively.

This agreement occurs even when the theory is highly nonlinear because, even though $\psi_{\text{in}}$ and $\psi_{\text{out}}$ differ in the wave zone, they are approximately the same in the inner, strong-field region, as illustrated in Figure 3.

The final step in the scalar field theory analysis will be to numerically extract the “outgoing part” from the time-symmetric solution $\psi_{\text{sym}}$ and compare it directly to $\psi_{\text{out}}$; we expect that the two will agree when $\psi_{\text{sym}}$ is approximately equal to $\psi_{\text{avg}}$.

4. QSBI III: Gravitational Waves on a Cosmic String Background

4.1. Schwarzschild vs. Levi-Civita Backgrounds

In the three-plus-one binary inspiral problem, with non-extended sources, gravitational waves at infinity can be treated as perturbations to Minkowski spacetime. This is because at large distances, where higher multipoles of the non-radiative field can be ignored, the non-radiative part of the geometry is associated with the Schwarzschild
Figure 2. Comparison between $\psi^{\text{sym}}$ and $\psi^{\text{avg}}$ in the wave zone for a source of two particles of opposite charge at $\rho/2 = 0.125$, $\phi = (\pi \pm \pi)/2$, with a rotational frequency of $\Omega = 1/2$ and an orbital velocity of one-eighth the speed of light. The fields are plotted versus $\rho/2$ for the range of angles $\phi \in [1, \pi/2]$. With these less relativistic sources, the agreement is better than in Figure 1, with approximately a 9% smaller amplitude for $\psi^{\text{avg}}$ and 7 degree phase shift.

The metric

$$-(1 - r_s/r) c^2 dt^2 + (1 - r_s/r)^{-1} dr^2 + r^2 d\Omega^2,$$

which contains a length scale $r_s = 2GM/c^2$; taking $r/r_s \to \infty$ reduces the Schwarzschild line element to the Minkowski one, which is just another way of saying that the Schwarzschild solution is asymptotically flat.

With infinitely extended sources, this is not possible. Far from the strings, where the internal structure is not felt, the non-radiative effects are approximated by the Levi-Civita solution \[1\] of a single cosmic string at the origin with mass-per-unit length $\Lambda$, whose line element is given for small $\Lambda$ by

$$-(\rho^2 c^2 dt^2 + \rho^{-2c} dz^2 + d\rho^2 + \rho^2 d\phi^2).$$

The mass parameter $C = 2G\Lambda/c^2$ is now dimensionless; this lack of a length scale means that Levi-Civita spacetime is not asymptotically flat and we must use Levi-Civita rather than Minkowski as our background metric.

4.2. Perturbations to Levi-Civita

Suppose the metric is given by the Levi-Civita metric plus a small perturbation $h_{\mu\nu}(t, \rho, \phi)$. (In our problem the $t$ and $\phi$ dependence will further be restricted to a function of $\phi - \Omega t$, but we are thinking here of more general considerations.) We can divide the components \{\$h_{\mu\nu}\$\} into those which are odd under inversion of the $z$ coordinate ($h_{zt}$, $h_{zp}$, and $h_{z\phi}$) and those which are even ($h_{zz}$, $h_{tt}$, $h_{t\rho}$, $h_{t\phi}$, $h_{\rho\rho}$, $h_{\rho\phi}$, and $h_{\phi\phi}$). In the linearized theory, the odd and even sectors are uncoupled.
Figure 3. Illustration of the agreement between $\psi^\text{out}$ and $\psi^\text{in}$ in the strong-field region. The parameters are the same as in Figure 2, i.e., oppositely-charged source particles at $\rho/2 = 0.125$, $\varphi = (\pi \pm \pi)/2$, a rotational frequency of $\Omega = 1/2$ and an orbital velocity of one-eighth the speed of light. The fields are plotted against $\rho/2$ for $\varphi = 3\pi/8$. Note that while the two solutions differ in the outer, wave zone, they are nearly equal in the inner regions which are the source of the most of the nonlinear effects, leading to the approximate agreement of $\psi^\text{sym}$ and $\psi^\text{avg}$ (which are identical in the linear theory) illustrated in Figure 2.

To count the physical degrees of freedom, we consider the effects of a gauge transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - 2\xi_{(\mu;\nu)}$$

by a gauge parameter $\xi_{(t,\rho,\phi)}$, as well as the Hamiltonian constraint $H$ and the three components $\{H_z, H_\rho, H_\phi\}$ of the momentum constraint.

The three odd components of the metric perturbation consist of one gauge degree of freedom $\xi_z$, one constraint $H_z$, and one physical degree of freedom, which can be thought of as the “cross polarization” (since it involves $h_{z\phi}$).

The seven even components include three gauge degrees of freedom $\xi_t$, $\xi_\rho$, and $\xi_\phi$, three constraints $H$, $H_\rho$, and $H_\phi$, and one physical “plus polarization” involving $h_{zz}$ and $h_{\phi\phi}$.

The next step in a treatment akin to that done by Moncrief [7] for waves on a Schwarzschild background will be to derive explicit wave equations for these two gauge-invariant physical degrees of freedom. For the purposes of the quasi-stationary binary inspiral program, it may suffice to work with linearized versions of the gauge-fixed equations derived in [2], but the picture of general perturbations to Levi-Civita will at least be useful in interpreting the quantities involved.
5. Future Outlook

The work described in Section 2 has produced non-linear differential equations which can be numerically implemented given a suitable treatment of the sources and the boundary conditions on the radiation at the outer boundary. The study of radiation balance in the context of non-linear scalar field theory in Section 3 will be completed by the extraction of the outgoing part of the time-symmetric Green’s function solution and the comparison between that and the actual outgoing solution (which we expect to be unavailable in the gravitational case). Application of these methods to General Relativity will be complicated by the fact that it is probably not possible to formulate a (linear) Green’s Function solution to the non-linear wave equation in that case. The analysis of perturbations to Levi-Civita spacetime described in Section 4 is still in the early stages, and its full development may or may not be necessary to the understanding of our problem, which possesses the additional co-rotational Killing vector.

Finally, should the technique of finding a stationary solution with a balance of ingoing and outgoing radiation prove successful in the essentially two-dimensional problem of orbiting cosmic strings, it can be applied to the computationally more intensive three-dimensional problem.

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