Crossed products by Hecke pairs I: 
*-algebraic theory

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Abstract

We develop a theory of crossed products by “actions” of Hecke pairs \((G, \Gamma)\), motivated by applications in non-abelian \(C^*\)-duality. Our approach gives back the usual crossed product construction whenever \(G/\Gamma\) is a group and retains many of the aspects of crossed products by groups. In this first of two articles we lay the \*-algebraic foundations of these crossed products by Hecke pairs and we explore their representation theory.

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Introduction

The present work is the first of two articles whose goal is the development of a theory of crossed products by Hecke pairs with a view towards application in non-abelian $C^*$-duality.

A Hecke pair $(G, \Gamma)$ consists of a group $G$ and a subgroup $\Gamma \subseteq G$ for which every double coset $\Gamma g \Gamma$ is the union of finitely many left cosets. In this case $\Gamma$ is also said to be a Hecke subgroup of $G$. Examples of Hecke subgroups include finite subgroups, finite-index subgroups and normal subgroups. It is in fact many times insightful to think of this definition as a generalization of the notion of normality of a subgroup.

Given a Hecke pair $(G, \Gamma)$ the Hecke algebra $\mathcal{H}(G, \Gamma)$ is a $*$-algebra of functions over the set of double cosets $\Gamma \backslash G / \Gamma$, with a suitable convolution product and involution. It generalizes the definition of the group algebra $C(G/\Gamma)$ of the quotient group when $\Gamma$ is a normal subgroup.

Heuristically, a crossed product of an algebra $A$ by a Hecke pair $(G, \Gamma)$ should be thought of as a crossed product (in the usual sense) of $A$ by an “action” of $G/\Gamma$. The quest for a sound definition of crossed products by Hecke pairs may seem hopelessly flawed since $G/\Gamma$ is not necessarily a group and thus it is unclear how it should “act” on $A$. It is the goal of this article and its coming sequel to show that in some circumstances such a definition can be given in a meaningful way, recovering the original one whenever $G/\Gamma$ is a group.

The term “crossed product by a Hecke pair” was first used by Tzanev [18] in order to give another perspective on the work of Connes and Marcolli [2]. This point of view was later formalized by Laca, Larsen and Neshveyev in [12], where they defined a $C^*$-algebra which can be interpreted as a reduced $C^*$-crossed product of a commutative $C^*$-algebra by a Hecke pair.

It seems to be a very difficult task to define crossed products of any given algebra $A$ by a Hecke pair, and for this reason we set as our goal to define a crossed product by a Hecke pair in a generality that will cover the following aspects:

- existence of a canonical spanning set of elements in the crossed product;
- possibility of defining covariant representations;
- the Hecke algebra must be a trivial example of a crossed product by a Hecke pair;
- the classical definition of a crossed product must be recovered whenever $G/\Gamma$ is a group;
- our construction should agree with that of Laca, Larsen and Neshveyev, whenever they are both definable;
- our definition should be suitable for applications in non-abelian $C^*$-duality.

In this first article on this subject we focus on defining such crossed products on a purely $*$-algebraic level and on developing their representation theory. The subjects of $C^*$-completions and relations with non-abelian $C^*$-duality will be further explored in the second article on the subject.

We develop a theory of crossed products of certain algebras $A$ by Hecke pairs which takes into account the above requirements. Our approach makes sense
when $A$ is a certain algebra of sections of a Fell bundle over a discrete groupoid. To summarize our setup: we start with a Hecke pair $(G, \Gamma)$, a Fell bundle $A$ over a discrete groupoid $X$ and an action $\alpha$ of $G$ on $A$ satisfying some "nice" properties. From this we naturally give the space $A/\Gamma$ of $\Gamma$-orbits of $A$ a Fell bundle structure over the orbit space $X/\Gamma$, which under our assumptions on the action $\alpha$ is in fact a groupoid. We can then define a $\ast$-algebra

$$C_c(A/\Gamma) \times^a_{\alpha} G/\Gamma,$$

which can be thought of as the crossed product of $C_c(A/\Gamma)$ by the Hecke pair $(G, \Gamma)$. We should point out that a crossed product for us is simply a $\ast$-algebra, which we can then complete with different $C^*$-norms or an $L^1$-norm. Hence, and so that no confusion arises, the symbol $\times^a_{\alpha}$ will always be used when talking about the (uncompleted) $\ast$-algebraic crossed product.

Our construction gives back the usual crossed product construction when $\Gamma$ is a normal subgroup of $G$. Moreover, given any action of the group $G/\Gamma$ on a Fell bundle $B$ over a groupoid $Y$, the usual crossed product $C_c(B) \times^a_{\alpha} G/\Gamma$ can be obtained via our setup as a crossed product by the Hecke pair $(G, \Gamma)$.

Many of the features present in crossed products by discrete groups carry over to our setting. For instance, the role of the group $G/\Gamma$ is played by the Hecke algebra $\mathcal{H}(G, \Gamma)$, which embeds in a natural way in the multiplier algebra of $C_c(A/\Gamma) \times^a_{\alpha} G/\Gamma$. Additionally, just like a crossed product $A \times G$ by a discrete group is spanned by elements of the form $a \ast g$, with $a \in A$ and $g \in G$, our crossed products by Hecke pairs also admit a canonical spanning set of elements.

The representation theory of crossed products by Hecke pairs also has many similarities with the group case, but some distinctive new features arise. For instance, as it is well-known in the group case, there is a bijective correspondence between nondegenerate representations of a crossed product $A \times G$ and the so-called covariant representations of $A$ and $G$, which are certain pairs of unitary representations of $G$ and representations of $A$. We will show that something completely analogous occurs for Hecke pairs, but in this case one is obliged to consider pre-representations of the Hecke algebra, i.e. representations of $\mathcal{H}(G, \Gamma)$ as (possibly) unbounded operators. This consideration was unnecessary in the group case because unitary operators are automatically bounded.

As stated before, this theory of crossed products by Hecke pairs is intended for applications in non-abelian $C^*$-duality theory. One of the main motivations is the establishment of a Stone-von Neumann theorem for Hecke pairs that encompasses the work of an Huef, Kaliszewski and Raeburn [7] and expresses their results in the language of crossed products. Additionally, we envisage for future work a form of Katayama duality with respect to Echterhoff-Quigg’s “crossed product” [4] (a terminology used in [7]). In a succinct and non-rigorous way this would mean that there is a canonical isomorphism of the type:

$$A \times_\delta G/\Gamma \times_\delta \omega G/\Gamma \cong A \otimes K(\ell^2(G/\Gamma)),$$

where $A \times_\delta G/\Gamma$ is a crossed product by a coaction of the homogeneous space $G/\Gamma$, while the second crossed product should be by the “dual action” of the Hecke pair $(G, \Gamma)$ in our sense. Such a result would bring insight into the emerging theory of crossed products by coactions of homogeneous spaces [3].
This article is organized as follows. In Section 1 we set up the conventions and preliminary results to be used in the rest of the article.

Section 2 is dedicated to the development of the required setup for defining crossed products by Hecke pairs. Here we explain what type of actions are involved, how to define the orbit space groupoids $X/H$ and the orbit bundles $A/H$ out of $A$, and how all the algebras $C_c(A/H)$ are related with each other for different subgroups $H \subseteq G$.

Lastly, in Section 3 we introduce the notion of a crossed product by a Hecke pair, explore some of its algebraic aspects and develop its representation theory. In the last part of this section we show how many of the formulas become much simpler in the case of free actions.

The present work is based on the author’s Ph.D. thesis [13] written at the University of Oslo. There are a few differences between the present work and [13], notably the greater generality of the types of actions involved. This improvement follows a suggestion of Dana Williams and John Quigg.

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1 Preliminaries

In this section we set up the conventions, notation, and background results which will be used throughout this work. We indicate the references where the reader can find more details, but we also provide proofs for those results which we could not find in the literature.

**Convention.** The following convention for displayed equations will be used throughout this work: if a displayed formula starts with the equality sign, it should be read as a continuation of the previously displayed formula.

A typical example takes the following form:

\[
\text{(expression 1)} = \text{(expression 2)} = \text{(expression 3)}.
\]

By Theorem A and Lemma B it then follows that

\[
= \text{(expression 4)} = \text{(expression 5)}.
\]

Under our convention starting with the equality sign in the second array of equations simply means that (expression 3) is equal to (expression 4).
1.1 *-Algebras and (pre-)*-representations

Let \( V \) be an inner product space over \( \mathbb{C} \). Recall that a function \( T : V \to V \) is said to be adjointable if there exists a function \( T^* : V \to V \) such that

\[
\langle T \xi , \eta \rangle = \langle \xi , T^* \eta \rangle,
\]

for all \( \xi, \eta \in V \). Recall also that every adjointable operator \( T \) is necessarily linear and that \( T^* \) is unique and adjointable with \( T^{**} = T \). We will use the following notation:

- \( L(V) \) denotes the *-algebra of all adjointable operators in \( V \)
- \( B(V) \) denotes the *-algebra of all bounded adjointable operators in \( V \).

Of course, we always have \( B(V) \subseteq L(V) \), with both *-algebras coinciding when \( V \) is a Hilbert space (see, for example, [16, Proposition 9.1.11]).

Following [16, Def. 9.2.1], we define a pre-*-representation of a *-algebra \( A \) on an inner product space \( V \) to be a *-homomorphism \( \pi : A \to L(V) \) and a *-representation of \( A \) on a Hilbert space \( \mathcal{H} \) to be a *-homomorphism \( \pi : A \to B(\mathcal{H}) \). As in [15] Def. 4.2.1, a pre-*-representation \( \pi : A \to L(V) \) is said to be normed if \( \pi(A) \subseteq B(V) \), i.e. if \( \pi(a) \) is a bounded operator for all \( a \in A \).

**Definition 1.1** ([16], Def. 10.1.17). A *-algebra \( A \) is called a BG*-algebra if all pre-*-representations of \( A \) are normed.

We now introduce our notion of an essential ideal. Our definition is not the usual one, but this choice of terminology will be justified in what follows.

**Definition 1.2.** Let \( A \) be a *-algebra. An ideal \( I \subseteq A \) is said to be essential if \( aI \neq \{0\} \) for all \( a \in A \setminus \{0\} \).

The usual definition of an essential ideal states that \( I \) is essential if it has nonzero intersection with every other nonzero ideal. Our definition is stronger, but coincides with the usual one for the general class of semiprime *-algebras. We recall from [15] Definition 4.4.1 that a *-algebra is said to be semiprime if \( aAa = \{0\} \) implies \( a = 0 \), where \( a \in A \). The class of semiprime *-algebras is quite large, containing all *-algebras that have a faithful *-representation on a Hilbert space (in particular, all C*-algebras) and many other classes of *-algebras (see [16] Theorem 9.7.21).

**Proposition 1.3.** Let \( A \) be an algebra and \( I \subseteq A \) a nonzero ideal. We have

i) If \( I \) is essential, then \( I \) has a nonzero intersection with every other nonzero ideal of \( A \).

ii) The converse of i) is true in case \( A \) is semiprime.
**Proof:** i) Let \( I \) be an essential ideal of \( A \). Let \( J \subseteq A \) be a nonzero ideal and \( a \in J \setminus \{0\} \). Since \( a \) is nonzero, then \( aI \neq \{0\} \). Hence, \( J \cdot I \neq \{0\} \), and since \( J \cdot I \subseteq J \cap I \), we have \( J \cap I \neq \{0\} \).

ii) Suppose \( A \) is semiprime. Suppose also that \( I \) is not essential. Thus, there is \( a \in A \setminus \{0\} \) such that \( aI = \{0\} \). Let \( J_a \subseteq A \) be the ideal generated by \( a \). We have \( J_a \cdot I = \{0\} \). Since \( (J_a \cap I)^2 \subseteq J_a \cdot I \) we have \( (J_a \cap I)^2 = \{0\} \). Since \( A \) is semiprime this implies that \( J_a \cap I = \{0\} \) (see [15, Theorem 4.4.3]). Hence, \( I \) has zero intersection with a nonzero ideal.

For \( C^\ast \)-algebras the focus is mostly on closed ideals. In this setting we still see that our definition is equivalent to the usual one ([17, Definition 2.35]):

**Proposition 1.4.** Let \( A \) be a \( C^\ast \)-algebra and \( I \subseteq A \) a closed ideal. The following are equivalent:

i) \( I \) is essential.

ii) \( I \) has nonzero intersection with every other nonzero ideal of \( A \).

iii) \( I \) has nonzero intersection with every other nonzero closed ideal of \( A \).

**Proof:** i) \( \iff \) ii) This was established in Proposition [13], since \( C^\ast \)-algebras are automatically semiprime.

ii) \( \implies \) iii) This is obvious.

ii) \( \iff \) iii) Let \( J \) be a nonzero ideal of \( A \) and \( \overline{J} \) its closure. From iii) we have \( I \cap \overline{J} \neq \{0\} \). Since \( I \) and \( \overline{J} \) are both closed, and \( A \) is a \( C^\ast \)-algebra, we have \( I \cdot \overline{J} = I \cap \overline{J} \). Now, it is clear that \( I \cdot J = \{0\} \) if and only if \( I \cdot \overline{J} = \{0\} \). Hence, we necessarily have \( I \cdot J \neq \{0\} \), which implies \( I \cap J \neq \{0\} \).

We now introduce the notion of an essential \( \ast \)-algebra. The class of essential \( \ast \)-algebras seems to be the appropriate class of \( \ast \)-algebras for which one can define a multiplier algebra (as we shall see in Section 1.2).

**Definition 1.5.** A \( \ast \)-algebra \( A \) is said to be essential if \( A \) is an essential ideal of itself, i.e. if \( aA \neq \{0\} \) for all \( a \in A \setminus \{0\} \).

Any unital \( \ast \)-algebra is obviously essential. Also, it is easy to see that a semiprime \( \ast \)-algebra is essential. The converse is false, so that essential \( \ast \)-algebras form a more general class than that of semiprime \( \ast \)-algebras:

**Example 1.6.** Let \( \mathbb{C}[X] \) be the polynomial algebra in one selfadjoint variable \( X \). For any \( n \geq 2 \) the algebra \( \mathbb{C}[X]/\langle X^n \rangle \) is essential, because it is unital, but it is not semiprime because \( [X^{n-1}](\mathbb{C}[X]/\langle X^n \rangle)[X^{n-1}] = \{0\} \).
1.2 *-Algebraic multiplier algebras

Every $C^*$-algebra can be embedded in a unital $C^*$-algebra in a “maximal” way. These maximal unitizations of $C^*$-algebras enjoy a number of useful properties and certain concrete realizations of these algebras are commonly referred to as multiplier algebras. The reader is referred to [17] for an account.

The definition of a multiplier algebra is quite standard in $C^*$-algebra theory, but this notion is in fact more general and applicable for more general types of rings and algebras. For example, in [1, Section 1.1] it is explained how multiplier algebras can be defined for semiprime algebras.

In this section we are going to generalize this notion to the context of essential $*$-algebras and derive their basic properties. We believe that essential $*$-algebras are the appropriate class of $*$-algebras for which one can define multiplier algebras, since the property $aA = \{0\} \Rightarrow a = 0$, which characterizes an essential $*$-algebra, is constantly used in proofs.

Multiplier algebras are many times defined via the so-called double centralizers (see for example [1]), but since we are only interested in algebras with an involution a slightly simpler and more convenient approach can be given, analogue to the Hilbert $C^*$-module approach to $C^*$-multiplier algebras (presented in [17, Section 2.3]). This is the approach we follow.

**Definition 1.7.** Let $\mathcal{C}$ be a subclass of $*$-algebras. A $*$-algebra $A \in \mathcal{C}$ is said to have a maximal unitization in $\mathcal{C}$ if there exists a unital $*$-algebra $B \in \mathcal{C}$ (called the maximal unitization of $A$) and a $*$-embedding $i : A \hookrightarrow B$ for which $i(A)$ is an essential ideal of $B$ and such that for every other $*$-embedding $j$ of $A$ as an essential ideal of a unital $*$-algebra $C \in \mathcal{C}$, there is a unique $*$-homomorphism $\phi : C \to B$ such that

\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{j} & & \downarrow{\phi} \\
C & &
\end{array}
\]

commutes.

**Lemma 1.8.** In the above diagram the $*$-homomorphism $\phi$ is always injective (even if $C$ was not unital).

**Proof:** We have that $j(A) \cap \ker \phi = \{0\}$, because if $j(a) \in j(A) \cap \ker \phi$, then $0 = \phi(j(a)) = i(a)$ and hence $a = 0$ and therefore $j(a) = 0$. Hence, since $j(A)$ is an essential ideal of $C$, it follows from Proposition [13(i)] that $\ker \phi = \{0\}$.

For $C^*$-algebras, one might expect to replace “ideal” by “closed ideal”, in Definition 1.7. This condition, however, follows automatically since $i(A)$ and $j(A)$ are automatically closed. Hence, this definition encompasses the usual definition of a maximal unitization for a $C^*$-algebra.

**Definition 1.9.** Let $A$ be a $*$-algebra. By a right $A$-module we mean a vector space $X$ together with a mapping $X \times A \to X$ satisfying the usual consistency
conditions. An \( A \)-linear mapping \( T : X \to Y \) between \( A \)-modules is a linear mapping between the underlying vector spaces such that \( T(xa) = T(x)a \), for all \( x \in X \) and \( a \in A \). We will often use the notation \( Tx \), instead of \( T(x) \).

Every \( * \)-algebra \( A \) is canonically a right \( A \)-module, with the action of right multiplication. This is the example we will use thoroughly in what follows.

Let \( \langle \cdot, \cdot \rangle_A : A \times A \to A \) be the function

\[
\langle a, b \rangle_A := a^*b.
\]

The function \( \langle \cdot, \cdot \rangle_A \) is an \( A \)-linear form, in the sense that the following properties are satisfied:

\[
\begin{align*}
\text{a)} & \quad \langle a, \lambda_1 b_1 + \lambda_2 b_2 \rangle_A = \lambda_1 \langle a, b_1 \rangle_A + \lambda_2 \langle a, b_2 \rangle_A, \\
\text{b)} & \quad \langle \lambda_1 a_1 + \lambda_2 a_2, b \rangle_A = \lambda_1 \langle a_1, b \rangle_A + \lambda_2 \langle a_2, b \rangle_A, \\
\text{c)} & \quad \langle ab, c \rangle_A = \langle a, bc \rangle_A, \\
\text{d)} & \quad \langle ac, b \rangle_A = c^* \langle a, b \rangle_A, \\
\text{e)} & \quad \langle a, b \rangle_A^* = \langle b, a \rangle_A,
\end{align*}
\]

for all \( a, a_1, a_2, b, b_1, b_2 \in A \) and \( \lambda_1, \lambda_2 \in \mathbb{C} \).

If the \( * \)-algebra \( A \) is essential we also have:

\[
\text{f)} \quad \text{If } \langle a, b \rangle_A = 0 \text{ for all } b \in A, \text{ then } a = 0.
\]

**Definition 1.10.** Let \( A \) be a \( * \)-algebra. A function \( T : A \to A \) is called adjointable if there is a function \( T^* : A \to A \) such that

\[
\langle T(a), b \rangle_A = \langle a, T^*(b) \rangle_A,
\]

for all \( a, b \in A \).

**Proposition 1.11.** If \( A \) is an essential \( * \)-algebra, then every adjointable map \( T : A \to A \) is \( A \)-linear. Moreover, the adjoint \( T^* \) is unique and adjointable with \( T^{**} = T \).

**Proof:** Let \( T \) be an adjointable map in \( A \) and \( x_1, x_2, y \in A \). We have

\[
\begin{align*}
\langle T(\lambda_1 x_1 + \lambda_2 x_2), y \rangle_A &= \langle \lambda_1 x_1 + \lambda_2 x_2, T^*(y) \rangle_A \\
&= \lambda_1 \langle x_1, T^*(y) \rangle_A + \lambda_2 \langle x_2, T^*(y) \rangle_A \\
&= \lambda_1 \langle T(x_1), y \rangle_A + \lambda_2 \langle T(x_2), y \rangle_A \\
&= \langle \lambda_1 T(x_1) + \lambda_2 T(x_2), y \rangle_A.
\end{align*}
\]

Hence, we have \( \langle T(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T(x_1) - \lambda_2 T(x_2), y \rangle_A = 0 \). We can then conclude from f) that

\[
T(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T(x_1) - \lambda_2 T(x_2) = 0,
\]

for all \( \lambda_1, \lambda_2 \in \mathbb{C} \).
i.e. $T$ is a linear map. 

Let us now check that $T$ is $A$-linear. For any $x, y, a \in A$ we have

$$\langle T(xa), y \rangle_A = \langle xa, T^*(y) \rangle_A = a^* \langle x, T^*(y) \rangle_A = a^* \langle T(x), y \rangle_A = \langle T(x)a, y \rangle_A.$$ 

Hence, we have $\langle T(xa) - T(x)a, y \rangle_A = 0$. We can then conclude from f) that $T(xa) - T(x)a = 0$, i.e. $T$ is $A$-linear.

Let us now prove the uniqueness of the adjoint $T^*$. Suppose there was a function $S : A \to A$ such that

$$\langle x, T^*(y) \rangle_A = \langle x, S(y) \rangle_A.$$ 

for all $x, y \in A$. Then, $(T^*(y) - S(y), x) = 0$. We can then conclude from f) that $T^*(y) - S(y) = 0$, i.e. $T^* = S$.

It remains to prove that $T^*$ is adjointable with $T^{**} = T$. This follows easily from the equality

$$\langle T^* x, y \rangle_A = \langle y, T^* x \rangle_A = \langle T y, x \rangle_A = \langle x, Ty \rangle_A.$$ 

\[\square\]

**Definition 1.12.** Let $A$ be an essential $*$-algebra. The set of all adjointable maps on $A$ is called the **multiplier algebra** of $A$ and is denoted by $M(A)$.

The multiplier algebra is in fact a $*$-algebra, and the proof of this fact is standard.

**Proposition 1.13.** Let $A$ be an essential $*$-algebra. The multiplier algebra of $A$ is a unital $*$-algebra with the sum and multiplication given by pointwise sum and composition (respectively), and the involution given by the adjoint.

**Proposition 1.14.** Let $A$ be an essential $*$-algebra. There is a $*$-embedding $L : A \to M(A)$ of $A$ as an essential ideal of $M(A)$, given by

$$a \mapsto L_a$$

where $L_a : A \to A$ is the left multiplication by $a$, i.e. $L_a(b) := ab$.

**Proof:** It is easy to see that, for every $a \in A$, $L_a$ is adjointable with adjoint $L_a^*$, thus the mapping $L$ is well-defined. Also clear is the fact that $L$ is a $*$-homomorphism. Let us prove that it is injective: suppose $L_a = 0$ for some $a \in A$. Then, for all $b \in A$ we have $ab = L_a b = 0$ and since $A$ is essential this implies $a = 0$. Thus, $L$ is injective.

It remains to prove that $L(A)$ is an essential ideal of $M(A)$. Let us begin by proving that it is an ideal. Let $T \in M(A)$. For every $a, b \in A$ we have

$$TL_a(b) = T(ab) = T(a)b = L_T a (b),$$
and also
\[ L_a T(b) = aT(b) = \langle a^*, T(b) \rangle \]
\[ = \langle T^*(a^*), b \rangle = (T^*(a^*))^* b \]
\[ = L((T^*)^*(b)) \].

Hence we have
\[ TL_a = L T_a \quad \text{and} \quad L_a T = L((T^*)^*)^* \], (1)
from which it follows easily that \( L(A) \) is an ideal of \( M(A) \).

Let us now prove that this ideal is essential. Let \( T \in M(A) \) be such that \( TL(A) = \{0\} \). Then, in particular, \( TL_a = 0 \) for all \( a \in A \), but as we have seen before \( TL_a = L T_a \), and since \( L \) is injective we must have \( Ta = 0 \) for all \( a \in A \), i.e. \( T = 0 \).

**Remark 1.15.** According to Proposition 1.14, an essential \( * \)-algebra \( A \) is canonically embedded in its multiplier algebra \( M(A) \). We will often make no distinction of notation between \( A \) and its embedded image in \( M(A) \), i.e. we will often just write \( a \) to denote an element of \( A \) and to denote the element \( L(a) \) of \( M(A) \). No confusion will arise from this because the left equality in (1) simply means, in this notation, that \( T \cdot a = T(a) \).

**Theorem 1.16.** Let \( A \) be an essential \( * \)-algebra and \( L : A \to M(A) \) the canonical \( * \)-embedding of \( A \) in \( M(A) \). If \( j : A \to C \) is a \( * \)-embedding of \( A \) as an ideal of a \( * \)-algebra \( C \), then there exists a unique \( * \)-homomorphism \( \phi : C \to M(A) \) such that the following diagram commutes

\[
\begin{array}{ccc}
M(A) & \xrightarrow{\phi} & C \\
\downarrow^T & & \downarrow^j \\
A & \xrightarrow{j} & C
\end{array}
\]

Moreover, if \( j(A) \) is essential then \( \phi \) is injective.

**Proof:** For simplicity of notation let us assume, without any loss of generality, that \( A \) itself is an ideal of a \( * \)-algebra \( C \), so that we avoid any reference to \( j \) (or its inverse). Let \( \phi : C \to M(A) \) be the function defined by
\[
\phi(c) : A \to A \\
\phi(c)a := ca,
\]
for every \( c \in C \). It is a straightforward computation to check that \( \phi(c) \in M(A) \) and that \( \phi \) itself is a \( * \)-homomorphism. It is also easy to see that \( \phi(a) = L_a \), for every \( a \in A \). Hence, \( \phi \circ j \equiv L \). Let us now prove the uniqueness of \( \phi \) relatively to this property. Suppose \( \bar{\phi} : C \to M(A) \) is another \( * \)-homomorphism such that
\[ \tilde{\phi} \circ j = L. \] Then, for all \( c \in C \) and \( a \in A \) we have

\[
(\tilde{\phi}(c) - \phi(c))L_a = \tilde{\phi}(c)L_a - \phi(c)L_a \\
= \tilde{\phi}(c)\tilde{\phi}(a) - \phi(c)\phi(a) \\
= \tilde{\phi}(ca) - \phi(ca) \\
= L_{ca} - L_{ca} \\
= 0.
\]

Since \( L(A) \) is an essential ideal of \( M(A) \) it follows that \( \tilde{\phi}(c) = \phi(c) \) for all \( c \in C \), i.e. \( \tilde{\phi} = \phi \).

The last claim of the theorem, concerning injectivity of \( \phi \), was proven in Lemma 1.8.

**Corollary 1.17.** The multiplier algebra \( M(A) \) is a maximal unitization of \( A \) in the class of: essential \( \ast \)-algebras, semiprime \( \ast \)-algebras and \( C^\ast \)-algebras.

**Proof:** By Theorem 1.16 we only need to check that if \( A \) is an essential \( \ast \)-algebra (respectively, semiprime \( \ast \)-algebra or \( C^\ast \)-algebra), then the multiplier algebra has the same property.

Suppose \( A \) is an essential \( \ast \)-algebra. Let \( T \in M(A) \) be such that \( TM(A) = \{0\} \). Then, by the embedding of \( A \) in \( M(A) \) we have \( Ta = 0 \) for all \( a \in A \), i.e. \( T = 0 \). Hence, \( M(A) \) is also an essential \( \ast \)-algebra.

Suppose \( A \) is a semiprime \( \ast \)-algebra. Let \( T \in M(A) \) be such that \( TM(A)T = \{0\} \). Then, we also have that \( TL_aM(A)TL_a = \{0\} \) for any \( a \in A \), and therefore \( L_{T(a)}M(A)L_{T(a)} = \{0\} \). Thus, in particular, \( L_{T(a)}L(A)L_{T(a)} = \{0\} \), and since \( L \) is injective this means that \( T(a)AT(a) = \{0\} \). Since \( A \) is semi-prime we conclude that \( T(a) = 0 \), and therefore \( T = 0 \). Hence, \( M(A) \) is semiprime. It is well-known that \( M(A) \) is a \( C^\ast \)-algebra when \( A \) is a \( C^\ast \)-algebra.

An important feature of \( C^\ast \)-multiplier algebras is that a nondegenerate \( \ast \)-representation of \( A \) extends uniquely to \( M(A) \). This result does not hold in general for essential \( \ast \)-algebras. Nevertheless we can still extend a nondegenerate \( \ast \)-representation of \( A \) to a unique pre-\( \ast \)-representation of \( M(A) \):

**Theorem 1.18.** Let \( A \) be an essential \( \ast \)-algebra, \( \pi : A \to B(H) \) a nondegenerate \( \ast \)-representation of \( A \) on a Hilbert space \( \mathcal{H} \) and \( \mathcal{V} \subseteq \mathcal{H} \) the dense subspace

\[ \mathcal{V} := \pi(A)\mathcal{H} = \text{span} \{ \pi(a)\xi : a \in A, \xi \in \mathcal{H} \}. \]

Then there is a unique pre-\( \ast \)-representation

\[ \tilde{\pi} : M(A) \to L(\mathcal{V}) \]

such that \( \tilde{\pi}(a) = \pi(a)\mathcal{V} \) for every \( a \in A \).
Proof: We define the pre-*-representation $\tilde{\pi} : M(A) \to L(\mathcal{Y})$ by

$$\tilde{\pi}(T) \left[ \sum_{i=1}^{n} \pi(a_i)\xi_i \right] := \sum_{i=1}^{n} \pi(Ta_i)\xi_i,$$

for $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$. Let us first check that $\tilde{\pi}$ is well-defined. Suppose $\sum_{i=1}^{n} \pi(a_i)\xi_i = \sum_{j=1}^{m} \pi(b_j)\eta_j$. Then, for every $z \in A$ we have

$$\pi(z) \left( \sum_{i=1}^{n} \pi(Ta_i)\xi_i - \sum_{j=1}^{m} \pi(Tb_j)\eta_j \right) = \sum_{i=1}^{n} \pi(zTa_i)\xi_i - \sum_{j=1}^{m} \pi(zTb_j)\eta_j = \pi(zT) \left( \sum_{i=1}^{n} \pi(a_i)\xi_i - \sum_{j=1}^{m} \pi(b_j)\eta_j \right) = 0.$$

Since the *-representation $\pi$ is nondegenerate we necessarily have

$$\sum_{i=1}^{n} \pi(Ta_i)\xi_i - \sum_{j=1}^{m} \pi(Tb_j)\eta_j = 0,$$

which means that $\tilde{\pi}(T)$ is well-defined.

Let us now check that $\tilde{\pi}(T) \in L(\mathcal{Y})$, i.e. that $\tilde{\pi}(T)$ is indeed an adjointable operator in $\mathcal{Y}$. We will in fact prove that $\tilde{\pi}(T)^* = \tilde{\pi}(T^*)$, which follows from the following equality

$$\left\langle \tilde{\pi}(T) \sum_{i=1}^{n} \pi(a_i)\xi_i, \sum_{j=1}^{m} \pi(b_j)\eta_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \pi(Ta_i)\xi_i, \pi(b_j)\eta_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \xi_i, \pi(a_i^*T^*)\eta_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \xi_i, \pi(a_i^*T^*b_j)\eta_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \pi(a_i)\xi_i, \pi(T^*b_j)\eta_j \rangle = \left\langle \sum_{i=1}^{n} \pi(a_i)\xi_i, \tilde{\pi}(T^*) \sum_{j=1}^{m} \pi(b_j)\eta_j \right\rangle.$$

It is straightforward to see that $\tilde{\pi}$ is linear, multiplicative and, as we have seen, $\tilde{\pi}(T^*) = \tilde{\pi}(T)^*$, hence $\tilde{\pi}$ is a pre-*-representation of $M(A)$ on $\mathcal{Y}$.

It is also clear that, for any $a \in A$, $\tilde{\pi}(a)$ is just $\pi(a)$ restricted to $\mathcal{Y}$, because of the equality

$$\tilde{\pi}(a) \sum_{i=1}^{n} \pi(a_i)\xi_i = \sum_{i=1}^{n} \pi(aa_i)\xi_i = \pi(a) \sum_{i=1}^{n} \pi(a_i)\xi_i.$$

Let us now prove the uniqueness of $\tilde{\pi}$. Suppose $\phi : M(A) \to L(\mathcal{Y})$ is a pre-*-representation such that $\phi(a) = \pi(a)|_{\mathcal{Y}}$. Then, for every $z \in A$ and $v \in \mathcal{Y}$ we
have
\[ \pi(z)(\phi(T)v - \tilde{\pi}(T)v) = \pi(z)\phi(T)v - \pi(z)\tilde{\pi}(T)v = \phi(z)v - \tilde{\pi}(z)v = \pi(z)v - \pi(z)v = 0. \]

Since the \(^*\)-representation \(\pi\) is nondegenerate, we necessarily have
\[ \phi(T)v - \tilde{\pi}(T)v = 0, \]
which means that \(\phi(T) = \tilde{\pi}(T)\), i.e. \(\phi = \tilde{\pi}\).

**Remark 1.19.** Theorem \[1.18\] can be interpreted in the following way: every nondegenerate \(^*\)-representation \(\pi : A \to B(\mathcal{H})\) can be extended to \(M(A)\) by possibly unbounded operators, defined on the dense subspace \(\pi(A)\mathcal{H}\).

**Definition 1.20.** Let \(A\) be an essential \(^*\)-algebra. We will denote by \(M_B(A)\) the subset of \(M(A)\) consisting of all the elements \(T \in M(A)\) such that \(\tilde{\pi}(T) \in B(\mathcal{Y})\) for all nondegenerate \(^*\)-representations \(\pi : A \to B(\mathcal{H})\), where \(\mathcal{Y} := \pi(A)\mathcal{H}\) and \(\tilde{\pi}\) is the unique pre-\(^*\)-representation extending \(\pi\) as in Proposition \[1.18\].

As stated in the next result, \(M_B(A)\) is a \(^*\)-subalgebra of \(M(A)\). The advantage of working with \(M_B(A)\) over \(M(A)\) is that nondegenerate \(^*\)-representations of \(A\) always extend to \(^*\)-representations of \(M_B(A)\). Easy examples of elements of \(M_B(A)\) that might not belong to \(A\) are the projections and unitaries of \(M(A)\).

**Proposition 1.21.** Let \(A\) be an essential \(^*\)-algebra. The set \(M_B(A)\) is a \(^*\)-subalgebra of \(M(A)\) containing \(A\). Moreover, if \(\pi : A \to B(\mathcal{H})\) is a nondegenerate \(^*\)-representation of \(A\), then there exists a unique \(^*\)-representation of \(M_B(A)\) on \(\mathcal{H}\) that extends \(\pi\).

**Proof:** Let \(T, S \in M_B(A)\). Let \(\pi : A \to B(\mathcal{H})\) be any nondegenerate \(^*\)-representation of \(A\) and \(\tilde{\pi}\) its extension to \(L(\mathcal{Y})\), in the sense of Theorem \[1.18\]. By definition, \(\tilde{\pi}(T), \tilde{\pi}(S) \in B(\mathcal{Y})\), and therefore \(\tilde{\pi}(T + S), \tilde{\pi}(TS), \tilde{\pi}(T^*) \in B(\mathcal{Y})\), since \(B(\mathcal{Y})\) is a \(^*\)-algebra. Hence, \(M_B(A)\) is a \(^*\)-algebra of \(M(A)\). Moreover, \(A \subseteq M_B(A)\) since \(\tilde{\pi}(a) = \pi(a)|_{\mathcal{Y}} \in B(\mathcal{Y})\).

Let us now prove the last claim of this proposition. Let \(\pi : A \to B(\mathcal{H})\) be a nondegenerate \(^*\)-representation and \(\tilde{\pi} : M(A) \to L(\mathcal{Y})\) its extension as in Theorem \[1.18\]. Then we obtain by restriction a pre-\(^*\)-representation \(\tilde{\pi} : M_B(A) \to L(\mathcal{Y})\). By definition of \(M_B(A)\) we actually have \(\tilde{\pi}(M_B(A)) \subseteq B(\mathcal{Y})\). Hence \(\tilde{\pi}\) gives rise to a \(^*\)-representation \(\tilde{\pi} : M_B(A) \to B(\mathcal{H})\), since \(\mathcal{Y}\) is dense in \(\mathcal{H}\).

Let us now prove the uniqueness claim. Suppose \(\varphi\) is another representation of \(M_B(A)\) that extends \(\pi\). For \(T \in M_B(A), a \in A\) and \(\xi \in \mathcal{H}\) we have
\[ \varphi(T)\pi(a)\xi = \varphi(T)\varphi(a)\xi = \varphi(Ta)\xi = \pi(Ta)\xi = \tilde{\pi}(T)\pi(a)\xi. \]
By linearity and density it follows that \( \varphi(T) = \tilde{\pi}(T) \), i.e. \( \varphi = \tilde{\pi} \).

The above result is a generalization of the well-known result for \( C^* \)-algebras which states that any nondegenerate \( * \)-representation can be extended to the multiplier algebra (see for example [17, Corollary 2.51]), because \( M(A) = M_B(A) \) for any \( C^* \)-algebra \( A \).

### 1.3 Hecke algebras

We start by establishing some notation and conventions concerning left coset spaces and double coset spaces and we prove two results which will be useful later on.

Let \( G \) be a group, \( B, C \) subgroups of \( G \) and \( e \in G \) the identity element. The double coset space \( B \backslash G / C \) is the set

\[
B \backslash G / C := \{ BgC \subseteq G : g \in G \}.
\]

It is easy to see that the sets of the form \( BgC \) are either equal or disjoint, or in other words, we have an equivalence relation defined in \( G \) whose equivalence classes are precisely the sets \( BgC \).

The left coset space \( G / C \) is the set

\[
G / C := \{ e \} \backslash G / C = \{ gC \subseteq G : g \in G \}.
\]

Given an element \( g \in G \) and a double coset space \( B \backslash G / C \) (which can in particular be a left coset space by taking \( B = \{ e \} \)) we will denote by \( [g] \) the double coset \( BgC \). Thus, \( [g] \) denotes the whole equivalence class for which \( g \in G \) is a representative.

If \( A \) is a subset of \( G \) we define the double coset space \( B \backslash A / C \) as the set of double cosets in \( B \backslash G / C \) which have a representative in \( A \), i.e.

\[
B \backslash A / C := \{ BaC \subseteq G : a \in A \}.
\]

**Proposition 1.22.** Let \( A, B \) and \( C \) be subgroups of a group \( G \). If \( C \subseteq A \), then the following map is a bijective correspondence between the double coset spaces:

\[
B \backslash A / C \longrightarrow (B \cap A) \backslash A / C
\]

\[
[a] \mapsto [a].
\]

Similarly, if \( B \subseteq A \), then the following map is a bijective correspondence:

\[
B \backslash A / C \longrightarrow B \backslash A / (A \cap C)
\]

\[
[a] \mapsto [a].
\]
Proof: We first need to show that the map \((7)\) is well defined, i.e. if \(B a_1C = B a_2C\), for some \(a_1, a_2 \in A\), then \((B \cap A)a_1C = (B \cap A)a_2C\). If \(B a_1C = B a_2C\) then there exist \(b \in B\) and \(c \in C\) such that \(a_1 = ba_2c\), from which it follows that \(b = a_1c^{-1}a_2^{-1}\). Since \(A\) is a subgroup and \(C \subseteq A\), it follows readily that \(b \in B \cap A\), and therefore \(a_1 \in (B \cap A)a_2C\), i.e. \((B \cap A)a_1C = (B \cap A)a_2C\).

The map defined in \((5)\) is clearly surjective. It is also injective because if \((B \cap A)a_1C = (B \cap A)a_2C\), then clearly \(Ba_1C = Ba_2C\).

A completely analogous argument shows that map defined in \((6)\) is a bijection.

Suppose a group \(G\) acts (on the right) on a set \(X\) and let \(x \in X\). We will henceforward denote by \(S_x\) the stabilizer of the point \(x\), i.e.

\[ S_x := \{ g \in G : xg = x \}. \] (7)

Given a subset \(Z \subseteq X\) and a subgroup \(H \subseteq G\) we denote by \(Z/H\) the set of \(H\)-orbits which have representatives in \(Z\), i.e.

\[ Z/H := \{ zH : z \in Z \}. \]

Suppose now that \(H, K \subseteq G\) are subgroups and let \(x \in X\) be a point. The following result establishes a correspondence between the set of \(H\)-orbits \((xK)/H\) and the double coset space \(S_x \backslash K/H\):

**Proposition 1.23.** Let \(G\) be a group which acts (on the right) on a set \(X\). Let \(x \in X\) be a point and \(H, K \subseteq G\) be subgroups. We have a bijection

\[ (xK)/H \longrightarrow S_x \backslash K/H, \]

given by \(xgH \mapsto S_xgH\), where \(g \in K\).

Proof: Let us first prove that the map \(xgH \mapsto S_xgH\) is well defined, i.e. if \(xg_1H = xg_2H\), then \(S_xg_1H = S_xg_2H\). If \(xg_1H = xg_2H\), then there exists \(h \in H\) such that \(xg_1 = xg_2h\), which implies that \(x = xg_2h g_1^{-1}\), from which it follows that \(g_2h g_1^{-1} \in S_x\). Thus we see that

\[ S_xg_1H = S_xg_2h g_1^{-1}g_1H = S_xg_2H. \]

We conclude that the map is well-defined. The map is obviously surjective. It is also injective because if \(S_xg_1H = S_xg_2H\), then there exists \(r \in S_x\) and \(h \in H\) such that \(g_1 = rg_2h\), from which it follows that \(xg_1H = xrg_2hH = xg_2H\).

We will mostly follow [10] and [8] in what regards Hecke pairs and Hecke algebras and refer to these references for more details.

We start by establishing some notation which will be useful later on. Given a group \(G\), a subgroup \(\Gamma \subseteq G\) and \(g \in G\), we will denote by \(\Gamma^g\) the subgroup

\[ \Gamma^g := \Gamma \cap g \Gamma g^{-1}. \] (8)

We now recall the definition of a Hecke pair:
**Definition 1.24.** Let $G$ be a group and $\Gamma$ a subgroup. The pair $(G, \Gamma)$ is called a **Hecke pair** if every double coset $\Gamma g \Gamma$ is the union of finitely many right (and left) cosets. In this case, $\Gamma$ is also called a **Hecke subgroup** of $G$.

Given a Hecke pair $(G, \Gamma)$ we will denote by $L$ and $R$, respectively, the left and right coset counting functions, i.e.

$$L(g) := |\Gamma g \Gamma / \Gamma| = [\Gamma : \Gamma^g] < \infty$$

$$R(g) := |\Gamma \backslash \Gamma g \Gamma| = [\Gamma : \Gamma^{g^{-1}}] < \infty.$$  

We recall that $L$ and $R$ are $\Gamma$-biinvariant functions which satisfy $L(g) = R(g^{-1})$ for all $g \in G$. Moreover, the function $\Delta : G \to \mathbb{Q}^+$ given by

$$\Delta(g) := \frac{L(g)}{R(g)},$$

is a group homomorphism, usually called the **modular function** of $(G, \Gamma)$.

**Definition 1.25.** Given a Hecke pair $(G, \Gamma)$, the **Hecke algebra** $\mathcal{H}(G, \Gamma)$ is the $\ast$-algebra of finitely supported $\mathbb{C}$-valued functions on the double coset space $\Gamma \backslash G / \Gamma$ with the product and involution defined by

$$(f_1 \ast f_2)(\Gamma g \Gamma) := \sum_{h \in G / \Gamma} f_1(\Gamma h \Gamma) f_2(\Gamma h^{-1} g \Gamma),$$

$$f^\ast(\Gamma g \Gamma) := \Delta(g^{-1}) \overline{f(\Gamma g^{-1} \Gamma)}.$$  

Equivalently, we can define $\mathcal{H}(G, \Gamma)$ as the $\ast$-algebra of finitely supported $\Gamma$-left invariant functions $f : G / \Gamma \to \mathbb{C}$ with the product and involution operations given by

$$(f_1 \ast f_2)(g \Gamma) := \sum_{h \in G / \Gamma} f_1(h \Gamma) f_2(h^{-1} g \Gamma),$$

$$f^\ast(g \Gamma) := \Delta(g^{-1}) \overline{f(g^{-1} \Gamma)}.$$  

**Remark 1.26.** Some authors, including Krieg [10], do not include the factor $\Delta$ in the involution. Here we adopt the convention of Kaliszews, Landstad and Quigg [8] in doing so, as it gives rise to a more natural $L^1$-norm. We note, nevertheless, that there is no loss (or gain) in doing so, because these two different involutions give rise to $\ast$-isomorphic Hecke algebras.

The Hecke algebra has a natural basis, as a vector space, given by the characteristic functions of double cosets. We will henceforward identify a characteristic function of a double coset $1_{\Gamma g \Gamma}$ with the double coset $\Gamma g \Gamma$ itself.
1.4 Fell bundles over discrete groupoids

Let $X$ be a discrete groupoid. We will denote by $X^0$ the unit space of $X$ and by $s$ and $r$ the source and range functions $X \to X^0$, respectively.

We will essentially follow [11] when it comes to Fell bundles over groupoids. All the groupoids in this work are assumed to be discrete, so that the theory of Fell bundles admits a few simplifications. Basically a Fell bundle over a discrete groupoid $X$ consists of:

- a space $A$ together with a surjective map $p : A \to X$, such that each fiber $A_x := p^{-1}(x)$ is a Banach space, for every $x \in X$;
- a multiplication operation between fibers over composable elements of the groupoid, which we suggestively write as $A_x \cdot A_y \subseteq A_{xy}$;
- an involution $a \mapsto a^*$ which takes $A_x$ into $A_x^{-1}$.

These operations and norms satisfy some consistency properties which are described in [11, Section 2]. As it is well-known, each fiber over a unit element is naturally a $C^*$-algebra.

Standing Assumption 1.27. Given a Fell bundle $A$ over a discrete groupoid $X$ we will always assume that the fibers over units are non-trivial, i.e. $A_u \neq \{0\}$ for all $u \in X^0$.

Assumption 1.27 is not very restrictive. In fact, removing from the groupoid $X$ all the units $u \in X^0$ for which $A_u = \{0\}$ and also all the elements $x \in X$ such that $s(x)$ or $r(x)$ is $u$, we obtain a subgroupoid $Y$ for which the assumption holds (relatively to the restriction $A|_Y$ of $A$ to $Y$). Moreover, and this is the important fact, the algebras of finitely supported sections are canonically isomorphic, i.e. $C_c(A|_Y) \cong C_c(A)$.

The reason for us to follow Assumption 1.27 is because it will make our theory slightly simpler. Since we are interested mostly in algebras of sections, this assumption does not reduce the generality of the work in any way, as we observed in the previous paragraph.

Definition 1.28. Let $A$ be a Fell bundle over a discrete groupoid $X$. An automorphism of $A$ is a bijective map $\beta : A \to A$ which preserves the bundle structure, i.e. such that

i) $\beta$ takes any fiber onto another fiber;

ii) $\beta$ takes fibers over composable elements of $X$ to fibers over composable elements;

iii) As a map between (two) fibers, $\beta$ is a linear map;

iv) $\beta(a \cdot b) = \beta(a) \cdot \beta(b)$, whenever multiplication is defined;

v) $\beta(a^*) = \beta(a)^*$. 
The set of all automorphisms of $A$ forms a group under composition and will be denoted by $\text{Aut}(A)$.

It follows easily from $i)$ and $ii)$ above that every automorphism $\beta$ of $A$ entails a groupoid automorphism $\beta_0$ of $X$ such that $\beta_0(p(a)) = p(\beta(a))$. We also note that, by being a groupoid automorphism, $\beta_0$ takes units into units.

**Remark 1.29.** The restricted map $\beta : A_x \rightarrow A_{\beta_0(x)}$ is an isometric linear map. Linearity was required in condition $iii)$, but the fact that the map is an isometry follows from the other axioms. To see this we note that

$$\|\beta(a)\| = \|\beta(a^* \beta(a))\|^\frac{1}{2} = \|a^* a\|^\frac{1}{2}.$$  

Now $a^* a \in A_{s(x)}$ and $s(x) \in X^0$. Thus, we also have $\beta_0(s(x)) \in X^0$ and therefore both $A_{s(x)}$ and $A_{\beta_0(s(x))}$ are $C^*$-algebras. It follows from $iii)$, $iv)$ and $v)$ that the restricted map $\beta : A_{s(x)} \rightarrow A_{\beta_0(s(x))}$ is a $C^*$-isomorphism and is therefore isometric. Hence we have

$$\|\beta(a)\| = \|\beta(a^* a)\|^\frac{1}{2} = \|a^* a\|^\frac{1}{2} = \|a\|,$$

which shows that $\beta : A_x \rightarrow A_{\beta_0(x)}$ is an isometry.

Given a Fell bundle $A$ over a discrete groupoid $X$ we will denote by $C_c(A)$ its corresponding $*$-algebra of finitely supported sections. The following notation will be used throughout the rest of this work: for $x \in X$ and $a \in A_x$ the symbol $a_x$ will always denote the element of $C_c(A)$ such that

$$a_x(y) := \begin{cases} a, & \text{if } y = x \\ 0, & \text{otherwise}. \end{cases} \quad (16)$$

According to the notation above we can then write any $f \in C_c(A)$ uniquely as a sum of the form

$$f = \sum_{x \in X} (f(x))_x. \quad (17)$$

We recall that the operations of multiplication and involution in $C_c(A)$ are determined by

$$a_x \cdot b_y = \begin{cases} (ab)_{xy}, & \text{if } s(x) = r(y) \\ 0, & \text{otherwise}, \end{cases}$$

$$(a_x)^* = (a^*)_{x^{-1}},$$

where $x, y \in X$ and $a \in A_x$, $b \in A_y$. 

18
2 Orbit space groupoids and Fell bundles

In this section we present the basic set up which will enable us to define crossed products by Hecke pairs later in Section 3.

Our construction of a (\(*\)-algebraic) crossed product \(A \times^\text{alg} G/\Gamma\) of an algebra \(A\) by a Hecke pair \((G, \Gamma)\) will make sense when \(A\) is a certain algebra of sections of a Fell bundle over a discrete groupoid. In this section we show in detail what type of algebras \(A\) are involved in the crossed product and how they are obtained.

2.1 Group actions on Fell bundles

Throughout this section \(G\) will denote a discrete group. One of our ingredients for defining crossed products by Hecke pairs consists of a group action on a Fell bundle over a groupoid (a concept we borrow from [9, Section 6]). Such actions always carry an associated action on the corresponding groupoid (by groupoid automorphisms). Since we are primarily interested in right actions on groupoids, we start by recalling what they are:

**Definition 2.1.** Let \(X\) be a groupoid. A right action of \(G\) on \(X\) is a mapping
\[
X \times G \to X
\]
\[(x, g) \mapsto xg,
\]
which is a right action of \(G\) on the underlying set of \(X\), meaning that
1) \(xe = x\), for all \(x \in X\),
2) \(x(g_1g_2) = (xg_1)g_2\), for all \(x \in X\), \(g_1, g_2 \in G\),
which is compatible with the groupoid operations, meaning that
3) if \(x\) and \(y\) are composable in \(X\), then so are \(xg\) and \(yg\), for all \(g \in G\), and moreover
\[ (xy)(yg) = (xy)g, \]
4) \((xg)^{-1} = x^{-1}g\), for all \(x \in X\) and \(g \in G\).

In other words, a right action of \(G\) on \(X\) is a right action on the set \(X\) performed by groupoid automorphisms.

**Lemma 2.2.** Let \(X\) be a groupoid endowed with a right \(G\)-action. For every \(x \in X\) and \(g \in G\) we have
\[ s(xg) = s(x)g \quad \text{and} \quad r(xg) = r(x)g. \]
In particular, \(G\) restricts to an action on the unit space \(X^0\).
Proof: It follows easily from the definition of a right $G$-action that
\[ s(x)g = (x^{-1}x)g = (x^{-1}g)(xg) = (xg)^{-1}(xg) = s(xg), \]
and similarly for the range function.

Remark 2.3. Given elements $x, y$ in a groupoid $X$ endowed with a right $G$-action and given $g \in G$, we will often drop the brackets in expressions like $(xg)y$ and simply use the notation $xgy$. No confusion arises from this since $G$ is only assumed to act on the right. On the other hand, we will never write an expression like $xyg$ without brackets, since it can be confusing on whether it means $x(yg)$ or $(xy)g$.

Definition 2.4. [6, Section 6] Let $G$ be a group and $A$ a Fell bundle over a discrete groupoid $X$. An action of $G$ on $A$ consists of a homomorphism $\alpha: G \to \text{Aut}(A)$.

As observed in Section 1.4, each automorphism of $A$ carries with it an associated automorphism of the underlying groupoid $X$. Hence, an action of a group $G$ on $A$ entails an action of $G$ on $X$ by groupoid automorphisms. Since we are interested only in right actions on groupoids, we just ensure that these associated actions are on the right simply by taking inverses. Moreover, even though we will typically denote by $\alpha$ the action of $G$ on $A$, we will simply write $(x, g) \mapsto xg$ to denote its associated action on $X$ and it will be always assumed that this action comes from $\alpha$. To summarize what we have said so far: given an action $\alpha$ of $G$ on a Fell bundle $A$ over a groupoid $X$, there is an associated right $G$-action $(x, g) \mapsto xg$ on $X$ such that
\[ p(\alpha_g(a)) = p(a)g^{-1}. \] (18)

Remark 2.5. Typically one would require the mapping $(a, g) \mapsto \alpha_g(a)$ to be continuous, but this is not necessary here since both $G$ and $X$ are discrete.

Proposition 2.6. Let $\alpha$ be an action of a group $G$ on a Fell bundle $A$ over a groupoid $X$. We have an associated action $\overline{\alpha}: G \to \text{Aut}(C_c(A))$ of $G$ on $C_c(A)$ given by
\[ \overline{\alpha}_g(f)(x) := \alpha_g(f(xg)), \]
for $g \in G$, $f \in C_c(A)$ and $x \in X$.

Proof: Let us first prove that the action is well-defined, i.e. $\overline{\alpha}_g(f) \in C_c(A)$. The fact that $\overline{\alpha}_g(f)$ is finitely supported is obvious, so the only thing one needs to check is that $\overline{\alpha}_g(f)$ is indeed a section of the bundle, i.e. $\alpha_g(f(xg)) \in A_x$ for all $x \in X$, which is clear because $\alpha_g(A_y) = A_{yg}$. 

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Let us now check that \( \alpha_g \) is indeed a \( ^* \)-homomorphism for all \( g \in G \). Linearity of \( \alpha_g \) is obvious. Let \( f, f_1, f_2 \in \text{C}_c(A) \). We have

\[
\begin{align*}
\alpha_g(f_1 \cdot f_2)(x) &= \alpha_g((f_1 \cdot f_2)(xg)) \\
&= \sum_{y, z \in X \atop yz = x} \alpha_g(f_1(y))\alpha_g(f_2(z)) \\
&= \sum_{y, z \in X \atop \langle yg \rangle^{-1}(zg^{-1}) = x} \alpha_g(f_1(yg))\alpha_g(f_2(zg)) \\
&= \sum_{y, z \in X \atop \langle yg \rangle^{-1} = x} \alpha_g(f_1)(y)\alpha_g(f_2)(z) \\
&= (\alpha_g(f_1) \cdot \alpha_g(f_2))(x).
\end{align*}
\]

Hence, \( \alpha_g(f_1 \cdot f_2) = \alpha_g(f_1) \cdot \alpha_g(f_2) \). Also,

\[
\begin{align*}
\alpha_g(f^*)(x) &= \alpha_g(f^*(xg)) = \alpha_g(f(x^{-1}g))^* \\
&= (\alpha_g(f)(x^{-1}))^* = (\alpha_g(f))^*(x).
\end{align*}
\]

Hence, \( \alpha_g(f^*) = (\alpha_g(f))^* \). The fact that \( \alpha_{g_1g_2} = \alpha_{g_1} \circ \alpha_{g_2} \) for every \( g_1, g_2 \in G \) is also easily checked.

**Definition 2.7.** Let \( \alpha \) be a group action of \( G \) on a Fell bundle \( A \) over a groupoid \( X \) and let \( H \) be a subgroup of \( G \). We will say that the \( G \)-action is \( H \)-good if for any \( x \in X \) and \( h \in H \) we have

\[
s(x)h = s(x) \implies \alpha_{h^{-1}}(a) = a \quad \forall a \in A_x. \quad (19)
\]

Also, a right \( G \)-action on a groupoid \( X \) is said to be \( H \)-good if for any \( x \in X \) and \( h \in H \) we have

\[
s(x)h = s(x) \implies xh = x. \quad (20)
\]

It is clear from the definitions that if the action \( \alpha \) of \( G \) on \( A \) is \( H \)-good, then its associated right \( G \)-action on the underlying groupoid \( X \) is also \( H \)-good. We will mostly use actions on Fell bundles. However, some of our results (namely Proposition 2.10) are about groupoids only, and this is the reason for defining \( H \)-good actions for groupoids as well.

We now give equivalent definitions of a \( H \)-good action. For that we recall from [7] that given an action of \( G \) on a set \( X \) we denote by \( S_x \) the stabilizer of the point \( x \in X \). We will also denote by \( S(A_x) \) the set \( S(A_x) := \{ g \in G : \alpha_{g^{-1}}(a) = a, \forall a \in A_x \} \).

**Proposition 2.8.** Let \( \alpha \) be an action of \( G \) on a Fell bundle \( A \) over a groupoid \( X \). The following statements are equivalent:

\[\text{\ding{53}}\]
i) The action $\alpha$ is $H$-good.

ii) For every $x \in X$ we have that $S_{s(x)} \cap H = S(A_x) \cap H$.

iii) For any $x \in X$ we have

$$S_{s(x)} \cap H = S_x \cap H = S_{\tau(x)} \cap H = S(A_{s(x)}) \cap H = S(A_x) \cap H = S(A_{\tau(x)}) \cap H. \quad (21)$$

iv) The stabilizers of the $H$-actions on $X$ and on the fibers of $A$ are the same on composable pairs, i.e. if $x \in X$ and $y \in Y$ are composable, then

$$S_x \cap H = S_y \cap H = S(A_x) \cap H = S(A_y) \cap H.$$

Proof: i) $\implies$ ii) Since the action is $H$-good we have, by definition, that $S_{s(x)} \cap H \subseteq S(A_x) \cap H$. Also, if $h \in S(A_x) \cap H$, then we necessarily have $xh = x$, and therefore by Lemma 2.2 we get $s(x) = s(xh) = s(x)h$, from which we conclude that $h \in S_{s(x)} \cap H$. Hence we have $S_{s(x)} \cap H = S(A_x) \cap H$.

ii) $\implies$ iii) Repeating a little bit of what we did above: if $h \in S(A_x) \cap H$, then we necessarily have that $xh = h$, and therefore $h \in S_x \cap H$. Moreover, if $h \in S_x \cap H$, then it follows that by Lemma 2.2 that $h \in S_{s(x)} \cap H$. Thus, we have that

$$S(A_x) \cap H = S_x \cap H = S_{s(x)} \cap H.$$

Since $s(s(x)) = s(x)$, we also have, directly by our assumption of ii), that $S_{s(x)} \cap H = S(A_{s(x)}) \cap H$.

Since we have $(xg)^{-1} = x^{-1}g$, it follows easily that $S_x = S_{x^{-1}}$. Similarly, since $\alpha_g(a^*) = \alpha_g(a^*)$, it follows easily that $S(A_x) = S(A_{x^{-1}})$. Observing that $s(x^{-1}) = \tau(x)$, equality (21) immediately yields that

$$S_x \cap H = S_y \cap H = S(A_x) \cap H = S(A_y) \cap H.$$

iii) $\implies$ iv) Suppose $x \in X$ and $y \in X$ are composable. Then, $s(x) = \tau(y)$ and equality (21) immediately yields that

$$S_x \cap H = S_y \cap H = S(A_x) \cap H = S(A_y) \cap H.$$

iv) $\implies$ i) Let $h \in H$ and $x \in X$ be such that $s(x)h = s(x)$. From iv) it follows that $h \in S(A_x) \cap H$. This means that the action is $H$-good.

It is easy to see that any $H$-good action is also $gHg^{-1}$-good for any conjugate $gHg^{-1}$, and also $K$-good for any subgroup $K \subseteq H$.

The following property will also be important for defining crossed products by Hecke pairs:

**Definition 2.9.** Let $X$ be a groupoid endowed with a right $G$-action and let $H$ be a subgroup of $G$. We will say that the action has the $H$-intersection property if

$$uH \cap ugHg^{-1} = uH^g, \quad (22)$$
for every unit $u \in X^0$ and $g \in G$.

An action of $G$ on a Fell bundle $\mathcal{A}$ is said to have the $H$-intersection property if its associated right $G$-action on the underlying groupoid has the $H$-intersection property.

We defer examples of $H$-good actions and actions with the $H$-intersection property for the next section. We now introduce one of the important ingredients for our definition of crossed products by Hecke pairs: the orbit space groupoid.

Let $G$ be a group, $H \subseteq G$ a subgroup and $X$ a groupoid endowed with a $H$-good right $G$-action. Then, the orbit space $X/H$ becomes a groupoid in a canonical way which we will now describe. For that, and throughout this text, we will use the following notation: given elements $x, y$ we define the set

$$H_{x,y} := \{ h \in H : s(x)h = r(y) \}. \quad (23)$$

The groupoid structure on $X/H$ is described as follows:

- A pair $(xH, yH) \in (X/H)^2$ is composable if and only if $H_{x,y} \neq \emptyset$, or equivalently, $r(y) \in s(x)H$. This property is easily seen not to depend on the choice of representatives $x, y$ from the orbits $xH, yH$ respectively.

- Given a composable pair $(xH, yH) \in (X/H)^2$, their product is

$$xH \cdot yH := x\tilde{h}yH, \quad (24)$$

where $\tilde{h}$ is any element of $H_{x,y}$. It will follow from the fact the action is $H$-good that $x\tilde{h}$ does not depend on the representative $\tilde{h}$ chosen from $H_{x,y}$. The result of the product $xH \cdot yH$ also does not depend on the choice of representatives $x, y$. We will prove this in the next result.

- The inverse of the element $xH$ is simply the element $x^{-1}H$. It is also easy to see that this does not depend on the choice of representative $x$.

**Proposition 2.10.** Let $G$ be a group, $H \subseteq G$ a subgroup and $X$ a groupoid endowed with a $H$-good right $G$-action. The operations above give the orbit space $X/H$ the structure of a groupoid. Moreover, the unit space $(X/H)^0$ of this groupoid is $X^0/H = \{ uH : u \in X^0 \}$, where $X^0$ is the unit space of $X$, and the range and source functions satisfy

$$s(xH) = s(x)H \quad \text{and} \quad r(xH) = r(x)H.$$

**Proof:** Let us first prove that the product is well-defined. Let $(xH, yH) \in (X/H)^2$ be a composable pair. The fact that $x\tilde{h}$ does not depend on the representative $\tilde{h}$ chosen from $H_{x,y}$ follows from the assumption that the action is $H$-good, since if $h_1, h_2 \in H_{x,y}$ then we have

$$s(x)h_1 = r(y) = s(x)h_2,$$

and therefore $s(x)h_1h_2^{-1} = s(x)$, and because the action is $H$-good $xh_1h_2^{-1} = x$, i.e. $xh_1 = xh_2$. 

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Let us now prove that \( X/H \) is a groupoid with the operations above. We check associativity first. Suppose \( xH, yH, zH \in X/H \) are such that \((xH, yH)\) is composable and \((yH, zH)\) is composable. We want to prove that \((xHyH, zH)\) and \((xH, yHzH)\) are also composable and moreover \((xHyH)zH = xHyHzH\). We have by definition that \( xHyH = xh_1yH \) and \( yHzH = yh_2zH \), where \( h_1 \) is any element of \( H_{x,y} \) and \( h_2 \) is any element of \( H_{y,z} \). We now notice that

\[
H_{xh_1y,z} = \{ h \in H : s(xh_1y) = r(z) \} = \{ h \in H : s(y)h = r(z) \} = H_{y,z}.
\]

Since \( H_{y,z} \neq \emptyset \) it follows that \( H_{xh_1y,z} \neq \emptyset \), and therefore \((xh_1yH, zH)\) is composable. Similarly,

\[
H_{x,yh_2z} = \{ h \in H : s(x)h = r(yh_2z) \} = \{ h \in H : s(x)h = r(y)h_2 \} = \{ h \in H : s(x)h = \tilde{r}(y) \} = H_{x,y} \tilde{h}_2.
\]

Hence, since \( H_{x,y} \neq \emptyset \) it follows that \( H_{x,yh_2z} \neq \emptyset \), and therefore \((xH, yh_2zH)\) is composable.

As we saw above \( H_{xh_1y,z} = H_{y,z} \), and since \( \tilde{h}_2 \in H_{y,z} \), we can write

\[
(xHyH)zH = xh_1yHzH = (xh_1y)h_2zH = xh_1h_2y\tilde{h}_2zH.
\]

Also seen above, we have that \( H_{x,y} \tilde{h}_2 = H_{x,y} \tilde{h}_2 \), so that \( \tilde{h}_1\tilde{h}_2 \in H_{x,y} \tilde{h}_2 \).

Hence, we conclude that

\[
(xHyH)zH = xH(yHzH).
\]

We now check that for any element \( xH \in X/H \) we have that \((xH, x^{-1}H)\) and \((x^{-1}H, xH)\) are composable pairs. We have that

\[
H_{x,x^{-1}} = \{ h \in H : s(x)h = r(x^{-1}) \} = \{ h \in H : s(x)h = s(x) \},
\]

and the identity element \( e \) obviously belongs to the latter set. Hence we conclude that \( H_{x,x^{-1}} \neq \emptyset \), and therefore \((xH, x^{-1}H)\) is composable. A similar observation shows that \((x^{-1}H, xH)\) is also composable.

To prove that \( X/H \) is a groupoid it now remains to prove the inverse identities \( xHyH^{-1}H = xH \) and \( y^{-1}HxH = xH \), in case \((xH, yH)\) is composable (for the first identity) and \((yH, xH)\) is composable (for the second identity). We first show that \( yH^{-1}yH^{-1} = yH^{-1}yH \). We have that \( yH^{-1}yH = yly^{-1}H \) for any element \( \tilde{h} \in H_{y,y^{-1}} \). Since, as we observed above, we always have \( e \in H_{y,y^{-1}} \), it follows that we can take \( \tilde{h} \) as \( e \). Thus, we get

\[
yH^{-1}yH = yy^{-1}H = r(y)H. \tag{25}
\]

From this it follows that

\[
xHyH^{-1}H = xHr(y)H = xh_1r(y)H,
\]

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where $\tilde{h}_1$ is any element of $H_{x'r(y)}$. By definition, $\tilde{h}_1$ is such that $r(y) = s(x)\tilde{h}_1 = s(x\tilde{h}_1)$. Hence we have that $x\tilde{h}_1 r(y) = x\tilde{h}_1$, and therefore

$$xHyH y^{-1}H = x\tilde{h}_1 H = xH.$$ 

The other identity $y^{-1}HyH xH = xH$ is proven in a similar fashion. Hence, we conclude that $X/H$ is a groupoid.

From equality (25) it follows easily that the units of $X/H$ are precisely the elements of the form $uH$ where $u \in X^0$, so that we can write $(X/H)^0 = X^0/H$. Also from (25) it follows that the range function in $X/H$ satisfies:

$$r(xH) = r(x)H.$$ 

The analogous result for the source function is proven in a similar fashion. Hence, we conclude that $X/H$ is a groupoid.

Let $\alpha$ be an action of $G$ on a Fell bundle $A$ over a groupoid $X$. Assume that the action is $H$-good, where $H$ is a subgroup of $G$. We will now define a new Fell bundle $A/H$ over the groupoid $X/H$. First we set some notation. The set of $H$-orbits of the action $\alpha$ on $A$ gives us a partition of $A$ into equivalence classes. We will denote by $[a]$ the equivalence class of the element $a \in A$, i.e.

$$[a] := \{\alpha_h(a)\}_{h \in H}.$$ 

We define $A/H$ as the set of all the $H$-orbits in $A$, i.e.

$$A/H := \{[a] : a \in A\},$$

which, as we will now see, is a Fell bundle over $X/H$ in a natural way.

**Proposition 2.11.** Let $\alpha$ be an action of a group $G$ on a Fell bundle $A$ over a groupoid $X$ and $H \subseteq G$ be a subgroup for which the $G$-action is $H$-good. The set of $H$-orbits $A/H$ forms a Fell bundle over the groupoid $X/H$ in the following way:

- The associated projection $p_H : A/H \to X/H$ is defined by $p_H([a]) := p(a)H$, where $p$ is the associated projection of the bundle $A$.
- The vector space structure on each fiber $(A/H)_{xH}$ is defined in the following way: if $a, b \in A_x$ then $[a] + [b] := [a + b]$, and if $\lambda \in \mathbb{C}$ then $\lambda [a] := [\lambda a]$.
- The norm on $A/H$ is defined by $\| [a] \| := \| a \|$.
- The multiplication maps $(A/H)_{xH} \times (A/H)_{yH} \to (A/H)_{xHyH}$, for a composable pair $(xH, yH)$, are defined in the following way: if $a \in A_x$ and $b \in A_y$, then

$$[a][b] = [\alpha_{\tilde{h}^{-1}}(a)b],$$

where $\tilde{h}$ is any element of $H_{x'y}$.
- The involution map is defined by $[a]^* := [a^*]$. 

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Lemma 2.12. Let $\alpha$ be an action of $G$ on a Fell bundle $A$ over a groupoid $X$ and $H \subseteq G$ be a subgroup for which the $G$-action is $H$-good. Let $x \in X$ and $a \in A_x$. Given any $y \in xH$ there exists a unique representative $b$ of $[a]$ such that $b \in A_y$.

Proof: Given an element $y \in xH$ we have that $y = xh$ for some $h \in H$. The element $\alpha_{h^{-1}}(a)$ is then a representative of $[a]$ such that $\alpha_{h^{-1}}(a) \in A_xh = A_y$, thus existence is established.

The uniqueness claim follows from the fact the action is $H$-good. Suppose we have two representatives $b$ and $c$ of $[a]$ such that both $b$ and $c$ belong to $A_y$. Being representatives of $[a]$ means that there are elements $h_1, h_2 \in H$ such that $b = \alpha_{h_1}(a)$ and $c = \alpha_{h_2}(a)$. Hence we have that

$$\alpha_{h_2h_1^{-1}}(b) = c,$$

and therefore $h_2h_1^{-1}$ takes $A_y$ into $A_y$. This means that $yh_1h_2^{-1} = y$ and therefore $s(y)h_1h_2^{-1} = s(y)$. Since the action is $H$-good it follows that $\alpha_{h_2h_1^{-1}}(b) = b$, and therefore $b = c$.

Proof of Proposition 2.11: First, it is clear that the vector space structure on each fiber $(A/H)_{xH}$ is well-defined. By this we mean two things: first, given two representatives $b$ and $c$ of $[a]$ such that $b, c \in A_y$, for a given representative $x$ of the orbit $xH$ (Lemma 2.12); second, the sum $[a + b]$ still lies in $(A/H)_{xH}$ and does not depend on the choice of representatives $a$ and $b$ (provided only that $a$ and $b$ are in the same fiber).

The norm on $A/H$ is also easily seen to be well-defined, i.e. independent of the choice of representative. This is true because any other representative of $[a]$ is of the form $\alpha_h(a)$ for some $h \in H$, and by Remark 1.29 we know that $\alpha_h$ gives an isometry between fibers. It is also clear that each fiber $(A/H)_{xH}$ is a Banach space under this norm.

The multiplication map is also easily seen to be well-defined: using the fact the $G$-action on $A$ is $H$-good we know that $\alpha_{h^{-1}}(a)b$ does not depend on the choice of element $h \in H_{x,y}$. Moreover, $\alpha_{h^{-1}}(a)b \in A_{xh}y$ and therefore $[\alpha_{h^{-1}}(a)b] \in A_{xh}yH$. The fact that the multiplication map does not depend on the chosen representatives of the orbits $[a]$ and $[b]$ is also easily checked.

It follows from a routine computation that map $(A/H)_{xH} \times (A/H)_{yH} \to (A/H)_{xH}yH$ is bilinear. Moreover, for $[a] \in (A/H)_{xH}$ and $[b] \in (A/H)_{yH}$, where we assume without loss of generality that $a \in A_x$ and $b \in A_y$, we have that

$$
\|[a][b]\| = \|\alpha_{h^{-1}}(a)b\| \leq \|\alpha_h^{-1}(a)||[b]| = \|[a]\|[b]|.
$$

We will now check associativity of the multiplication maps. Let $(xH, yH)$ and $(yH, zH)$ be two composable pairs in $X/H$, and let $[a] \in (A/H)_{xH}$, $[b] \in (A/H)_{yH}$ and $[c] \in (A/H)_{zH}$, where we assume without loss of generality that $a \in A_x$, $b \in A_y$ and $c \in A_z$. By definition, we have $[a][b] = [\alpha_{h_1}^{-1}(a)b]$, where
\( \tilde{h}_1 \) is any element of \( H_{x,y} \). Thus, we have
\[
\left( [a][b] \right) [c] = [\alpha_{h_1^{-1}}(a)b][c] = [\alpha_{h_2^{-1}}(\alpha_{h_1^{-1}}(a)b)c] = [\alpha_{h_2^{-1}h_1^{-1}}(a)\alpha_{h_2^{-1}}(b)c],
\]
where \( \tilde{h}_2 \) is any element of \( H_{x\tilde{h}_1y,z} \). One can easily check (or see the proof of Proposition 2.10 where this is done) that \( H_{x\tilde{h}_1y,z} = H_{y,z} \) and moreover that \( \tilde{h}_1\tilde{h}_2 \in H_{x,y\tilde{h}_2} \). From this observations it follows that
\[
\left( [a][b] \right) [c] = [\alpha_{h_2^{-1}h_1^{-1}}(a)\alpha_{h_2^{-1}}(b)c] = [a][\alpha_{h_2^{-1}}(b)c] = [a](b)[c].
\]
Hence, the multiplication maps are associative.

The involution on \( \mathcal{A}/H \) is also easily seen not to depend on choice of representative of the orbit, since the maps \( \alpha_h \) preserve the involution of \( \mathcal{A} \). Moreover, it is easily checked that: if \( [a] \in (\mathcal{A}/H)_x \), then \( [a]^* \in (\mathcal{A}/H)_{x^{-1}H} \) is conjugate linear, and \( [a]^{**} = [a] \). Let us now check that \( ([a][b])^* = [b]^*[a]^* \), whenever the multiplication is defined. Let us assume that \( a \in \mathcal{A}_x \) and \( b \in \mathcal{A}_y \) and that \( (xH,yH) \) is composable. We have that
\[
\left( [a][b] \right)^* = [\alpha_{h_1^{-1}}(a)b]^* = [b^*\alpha_{h_2^{-1}}(a^*)] = [\alpha_{h_2^{-1}}(b^*)a^*],
\]
where \( \tilde{h} \) is any element of \( H_{x,y} \). It is easily seen that \( \tilde{h}^{-1} \in H_{y^{-1},x^{-1}} \), so that
\[
\left( [a][b] \right)^* = [\alpha_{\tilde{h}^{-1}}(b^*)a^*] = [b^*][a]^* = [b]^*[a]^*.
\]
We also need to prove that \( \| [a]^*[a] \| = \| [a] \|^2 \). This is also easy because
\[
\| [a]^*[a] \| = \| [a^*][a] \| = \| [a^*a] \| = \| [a^*a] \| = \| a \|^2 = \| [a] \|^2.
\]
The last thing we need to check is that if \( [a] \in (\mathcal{A}/H)_x \), then \( [a]^*[a] \) is a positive element of \( (\mathcal{A}/H)_{yH} \) (seen as a \( C^* \)-algebra). We have that \( [a]^*[a] = [a]^*[a] \). We can assume without loss of generality that \( a \in \mathcal{A}_x \), so that \( a^*a \in \mathcal{A}_{x(x)} \). Since \( \mathcal{A} \) is a Fell bundle we have that \( a^*a \) is a positive element of \( \mathcal{A}_{x(x)} \) (seen as a \( C^* \)-algebra). Hence, there exists an element \( b \in \mathcal{A}_{x(x)} \) such that \( a^*a = b^*b \). Moreover, \( [b] \in (\mathcal{A}/H)_{s(x)} \) and it is now clear that
\[
[a]^*[a] = [a^*a] = [b^*b] = [b]^*[b],
\]
i.e. \( [a]^*[a] \) is a positive element of \( (\mathcal{A}/H)_{s(x)} \). This finishes our proof that \( \mathcal{A}/H \) is a Fell bundle.

**Convention.** For simplicity we will henceforward make the following convention. Given an orbit Fell bundle \( \mathcal{A}/H \) as described in Proposition 2.11, if we write that an element \( [a] \) belongs to some fiber \( (\mathcal{A}/H)_{xH} \), we will always assume that the representative \( a \) of \([a]\) belongs to fiber over the representative \( x \) of \( xH \). In other words, if we write that \( [a] \in (\mathcal{A}/H)_{xH} \), then we are implicitly assuming that \( a \in \mathcal{A}_x \). This is possible and unambiguous by Lemma 2.7.

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We apply this convention also for elements of $C_c(A/H)$, meaning that a canonical element $[a]_{xH} \in C_c(A/H)$ is always assumed to be written in a way that $a \in A_x$.

It is a straightforward fact that any function in $C_c(X/H)$ can also be seen as a complex-valued ($H$-invariant) function in $X$. This function in $X$ is in general no longer finitely supported, but it still makes sense as a function in $C(X)$, the vector space of all complex-valued functions in $X$. We will now see that something analogous can be said for the elements of $C_c(A/H)$.

Given an element $f \in C_c(A/H)$ we define a function $\iota(f) \in C(A)$, where $C(A)$ is the vector space of all sections of $A$, by the following rule:

$$\iota(f)(x) := R_x(f(xH)),$$

(28)

where $R_x(f(xH))$ is the unique representative of $f(xH)$ such that $R_x(f(xH)) \in A_x$, which is well-defined according to Lemma 2.12. It is then easy to see that the map $\iota$ is an injective linear map from $C_c(A/H)$ to $C(A)$.

For ease of reading we will henceforward drop the symbol $\iota$ and use the same notation both for elements of $C_c(A/H)$ and for their correspondents in $C(A)$.

Under this convention we can then write, for any $f \in C_c(A/H)$ and $x \in X$, that $[f(x)] = f(xH)$. Moreover, the decomposition (17) of $f \in C_c(A/H)$ as a sum of elements of the form $[a]_{xH}$ can now be written as:

$$f = \sum_{xH \in X/H} (f(xH))_{xH} = \sum_{xH \in X/H} [f(x)]_{xH}.$$

(29)

### 2.2 Examples

In this section we give some examples of $H$-good actions and actions satisfying the $H$-intersection property. For the rest of the section we assume that $A$ is a Fell bundle over a groupoid $X$ where a group $G$ acts and $H \subseteq G$ denotes a subgroup.

The first two examples (2.13 and 2.14) show that $H$-good actions that satisfy the $H$-intersection property are present in actions that have completely opposite behaviours, such as free actions and actions that fix every point.

**Example 2.13.** If the restricted action of $H$ on the unit space $X^0$ is free, then the action is $H$-good and satisfies the $H$-intersection property.

**Example 2.14.** If the restricted action of $H$ on $A$ fixes every point, then the action is $H$-good and satisfies the $H$-intersection property.

The following example is one of the examples that motivated the development of this theory of crossed products by Hecke pairs. This example, and the study of the crossed products associated to it, seems to be valuable for obtaining a form of Katayama duality with respect to crossed products by “coactions” of
Example 2.15. Suppose $X$ is the transformation groupoid $G \times G$. We recall that the multiplication and inversion operations on this groupoid are given by:

$$ (s, tr)(t, r) = (st, r) \quad \text{and} \quad (s, t)^{-1} = (s^{-1}, st). $$

Recall also that the source and range functions on $G \times G$ are defined by

$$ s(s, t) = (e, t) \quad \text{and} \quad r(s, t) = (e, st). $$

We observe that there is a natural right $G$-action on $G \times G$, given by

$$ (s, t)g := (s, tg). $$

Let $\delta$ be a coaction of $G$ on a $C^*$-algebra $B$ and $B$ the associated Fell bundle. Following [5, Section 3], we will denote by $A := B \times G$ the corresponding Fell bundle over the groupoid $G \times G$. Elements of $A$ have the form $(b_s, t)$, where $b_s \in B_s$ and $s, t \in G$. Any such element lies in the fiber $A_{(s, t)}$ over $(s, t)$.

It is easy to see that there is a canonical action $\alpha$ of $G$ on $A$, given by

$$ \alpha_g(b_s, t) := (b_s, tg^{-1}). $$

This action of $G$ on $A$ entails the natural right action of $G$ on $G \times G$, as described in (30). This $G$-action on $G \times G$ is free and therefore the action $\alpha$ is $H$-good and satisfies the $H$-intersection property with respect to any subgroup $H \subseteq G$.

The orbit space groupoid $(G \times G)/H$ can be canonically identified with the groupoid $G \times G/H$ of [4], whose operations are given by:

$$ (s, trH)(t, rH) = (st, rH) \quad \text{and} \quad (s, tH)^{-1} = (s^{-1}, stH). $$

Moreover, the orbit Fell bundle $A/H$ is canonically identified with the Fell bundle $B \times G/H$ over $G \times G/H$ defined in [4], and in this way $C_c(A/H)$ is canonically isomorphic with the Echterhoff-Quigg algebra $C_c(B \times G/H)$, also from [4].

2.3 The algebra $M(C_c(A))$

We will assume for the rest of this section that $G$ is a group, $H \subseteq G$ is a subgroup and $A$ is a Fell bundle over a groupoid $X$ endowed with a $G$-action $\alpha$. We also assume that the action $\alpha$ is $H$-good. We recall that $A/H$ stands for the orbit Fell bundle over the groupoid $X/H$, as defined in [29].

For the purpose of defining crossed products by Hecke pairs it is convenient to have a “large” algebra which contains the algebras $C_c(A/H)$ for different subgroups $H \subseteq G$. In this way we are allowed to multiply elements of $C_c(A/H)$ and $C_c(A/K)$, for different subgroups $H, K \subseteq G$, in a meaningful way. This large algebra will be the multiplier algebra $M(C_c(A))$. This section is thus devoted to show how algebras such as $C_c(A/H)$ and $C_c(X^0/H)$ embed in $M(C_c(A))$ in a canonical way.

Our first result shows that there is a natural inclusion $C_c(A/H) \subseteq M(C_c(A))$. 


Theorem 2.16. There is an embedding \( \iota \) of \( C_c(A/H) \) into \( M(C_c(A)) \) determined by the following rule: for any \( x, y \in X, a \in A_z \) and \( b \in A_y \) we have

\[
\iota([a]_{xH})b := \begin{cases} 
(a_{\tilde{h}-1}(a)b)_{x\tilde{h}y}, & \text{if } H_{x,y} \neq \emptyset \\
0, & \text{otherwise},
\end{cases}
\tag{31}
\]

where \( \tilde{h} \) is any element of \( H_{x,y} \).

Remark 2.17. The above result allows us to see \( C_c(A/H) \) as a \( * \)-subalgebra of \( M(C_c(A)) \). We shall henceforward drop the symbol \( \iota \) and make no distinction of notation between an element of \( C_c(A/H) \) and its correspondent multiplier in \( M(C_c(A)) \).

Proof of Theorem 2.16: Let us first show that expression (31) does indeed define an element of \( M(C_c(A)) \). For this it is enough to check that 

\[
\langle \iota([a]_{xH})b_y, c_z \rangle = \langle b_y, \iota([a]_{x^{-1}H})c_z \rangle ,
\]

for all \( b \in A_y \) and \( c \in A_z \), with \( y, z \in X \). For \( \iota([a]_{xH})b_y \) to be non-zero, we must necessarily have \( H_{x,y} \neq \emptyset \), and in this case \( \iota([a]_{xH})b_y = (\alpha_{\tilde{h}-1}(a)b)_{x\tilde{h}y} \), where \( \tilde{h} \in H_{x,y} \). Now,

\[
\langle \iota([a]_{xH})b_y, c_z \rangle = \langle (\alpha_{\tilde{h}-1}(a)b)_{x\tilde{h}y}, c_z \rangle = \langle b^* \alpha_{\tilde{h}-1}(a)^* \rangle_{y^{-1}(x^{-1})\tilde{h}} c_z
\]

For \( \alpha_{\tilde{h}-1}(a)^*c_z \) to be non-zero we must necessarily have \( r(z) = s(x^{-1})\tilde{h} \), i.e. \( \tilde{h} \in H_{x^{-1},z} \). So, to summarize, for \( \langle [a]_{xH}b_y, c_z \rangle \) to be non-zero we must have \( H_{x,y} \cap H_{x^{-1},z} \neq \emptyset \) and in this case we obtain

\[
\langle \iota([a]_{xH})b_y, c_z \rangle = b^*_{y^{-1}} \alpha_{\tilde{h}-1}(a)^* c_z ,
\]

where \( \tilde{h} \) is any element of \( H_{x,y} \cap H_{x^{-1},z} \). A similar computation for \( \langle b_y, \iota([a]_{x^{-1}H})c_z \rangle \) yields the exact same result.

Recall from (17) that any \( f \in C_c(A/H) \) can be written as

\[
f = \sum_{xH \in X/H} (f(xH))_{xH} .
\]

From this we are able to define a multiplier \( \iota(f) \in M(C_c(A)) \), simply by extending expression (31) by linearity.

We want to show that \( \iota \) is an injective \( * \)-homomorphism. First, we claim that given \( [a]_{xH}, [b]_{yH} \in C_c(A/H) \) we have

\[
\iota([a]_{xH})\iota([b]_{yH}) = \iota([a]_{xH}[b]_{yH}) .
\]

This amounts to proving that

\[
\iota([a]_{xH})\iota([b]_{yH}) = \begin{cases} 
\iota([\alpha_{\tilde{h}-1}(a)b]_{x\tilde{h}yH}), & \text{if } H_{x,y} \neq \emptyset \\
0, & \text{otherwise}
\end{cases}
\]

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with $\tilde{h}$ being any element of $H_{x,y}$. To see this, let $c_z \in \mathcal{A}_z$, with $z \in X$. We have

$$\iota([a]_{xH})\iota([b]_{yH})c_z = \begin{cases} \iota([a]_{xH})(\alpha_{h_0^{-1}}(b)c)_{yha_0z}, & \text{if } H_{y,z} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\alpha_{h_1^{-1}}(a)\alpha_{h_0^{-1}}(b)c)_{xh_1yha_0z}, & \text{if } H_{y,z} \neq \emptyset \text{ and } H_{x,yha_0z} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

with $h_0 \in H_{y,z}$ and $h_1 \in H_{x,yha_0z}$. But $H_{x,yha_0z} = H_{x,yha} = H_{x,y}h_0$, hence the above can be written as

$$= \begin{cases} (\alpha_{h_0^{-1}}(a)\alpha_{h_0^{-1}}(b)c)_{xh_0yha_0z}, & \text{if } H_{y,z} \neq \emptyset \text{ and } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\alpha_{h_0^{-1}}(a)\alpha_{h_0^{-1}}(b)c)_{xh_0yha_0z}, & \text{if } H_{y,z} \neq \emptyset \text{ and } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

where $\tilde{h} \in H_{x,y}$. Also, $H_{y,z} = H_{xh_0,y}$. Thus, we obtain

$$= \begin{cases} (\alpha_{h_0^{-1}}(a)\alpha_{h_0^{-1}}(b)c)_{xh_0yha_0z}, & \text{if } H_{xh_0,y} \neq \emptyset \text{ and } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \iota([a]_{xH})\iota([b]_{yH})c_z, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Since $\iota$ is linear and multiplicative on the elements of the form $[a]_{xH}$, it is necessarily an homomorphism. Now the fact that $\iota([a]_{xH})^* = \iota(([a]_{xH})^*) = \iota([a]_{x^{-1}H})^*$ follows directly from the computations in the beginning of this proof. Hence, $\iota$ is a $^*$-homomorphism.

Let us now prove injectivity of $\iota$. Suppose $f \in C_c(\mathcal{A}/H)$ is such that $\iota(f) = 0$. Decomposing $f$ as a sum of elements of the form $[a]_{xH}$, following (29), we get

$$0 = \iota(f) = \sum_{xH \in X/H} \iota([f(xH)]_{xH}) = \sum_{xH \in X/H} \iota([f(x)]_{xH}).$$

For any $y \in X$ we then have

$$0 = \sum_{xH \in X/H} \iota([f(x)]_{xH})(f(y)^*)_{y^{-1}}$$

$$= \sum_{xH \in X/H \atop *y \in \mathcal{A}(x)/H} \iota([f(x)]_{xH})(f(y)^*)_{y^{-1}}$$

$$= \sum_{xH \in X/H \atop *y \in \mathcal{A}(x)/H} (\alpha_{h_y^{-1}}(f(x))f(y)^*)_{xh_y^{-1}},$$

where $\tilde{h}_y$ is any element of $H_{x,y}$. Now the elements $x_1h_x^{-1}$ in the sum above are all different, because if we had $x_1h_x^{-1} = x_2h_x^{-1}$, then we would have
\[ x_1 \tilde{h}_{x_1} = x_2 \tilde{h}_{x_2} \] and therefore \( x_1 H = x_2 H \). Therefore each of the summands in the above sum is zero, and in particular we must have

\[ 0 = \langle \tilde{h}_{y^{-1}} f(y) f(y)^* \rangle \big|_{y \in \mathcal{A}} \]

and therefore \( f(y)f(y)^* = 0 \). Hence we get \( f(y) = 0 \), and since this is true for any \( y \in X \), we have \( f = 0 \), i.e. \( \iota \) is injective.

**Proposition 2.18.** There is an embedding \( \iota \) of \( C_b(X^0) \) into \( M(C_c(A)) \) defined by

\[ \iota(f) \cdot y := f(r(y))y \] for every \( f \in C_b(X^0) \), \( y \in X \) and \( b \in \mathcal{A}_y \).

**Remark 2.19.** The above result allows us to see \( C_b(X^0) \) as a \( * \)-subalgebra of \( M(C_c(A)) \). We shall henceforward drop the symbol \( \iota \) and make no distinction of notation between an element of \( C_b(X^0) \) and its correspondent multiplier in \( M(C_c(A)) \).

**Proof of Proposition 2.18:** It is easy to see that \( \langle \iota(f) \cdot y, c_z \rangle = \langle b_y, \iota(f)^*c_z \rangle \) for any \( y, z \in X \), \( b \in \mathcal{A}_y \) and \( c \in \mathcal{A}_z \), so that the expression (32) does define an element of \( M(C_c(A)) \).

Hence we get a linear map \( \iota : C_b(X^0) \rightarrow M(C_c(A)) \). Given two elements \( f_1, f_2 \in C_b(X^0) \), we have that

\[ \iota(f_1)\iota(f_2) \cdot y = f_1(r(y))f_2(r(y))y = \iota(f_1f_2) \cdot y \]

for any \( y \in X \) and \( b \in \mathcal{A}_y \), so that \( \iota \) is a \( * \)-homomorphism. Hence, we only need to prove that \( \iota \) is injective. This is not difficult to see: given \( f \in C_b(X^0) \) such that \( \iota(f) = 0 \) we have, for any unit \( u \in X^0 \) and \( b \in \mathcal{A}_u \), that

\[ 0 = \iota(f) \cdot u = f(u)b_u \]

Hence, \( f(u) = 0 \) because each fiber \( \mathcal{A}_u \) is non-zero by our assumption on Fell bundles (see Assumption 1.27). Since this is true for any \( u \in X^0 \) we get \( f = 0 \), i.e. \( \iota \) is injective.

Recall, from Lemma 2.2, that the action of \( G \) on \( X \) restricts to an action of \( G \) on the set \( X^0 \). Thus it makes sense to talk about the commutative \( * \)-algebra

\[ C_c(X^0/H) \subset C_b(X^0) \]

Since there is a canonical embedding, given by Proposition 2.18 of \( C_b(X^0) \) into \( M(C_c(A)) \), we have in particular an embedding of \( C_c(X^0/H) \) into \( M(C_c(A)) \) which identifies an element \( f \in C_c(X^0/H) \) with the multiplier \( f \in M(C_c(A)) \) given by:

\[ f \cdot y := f(r(y))H \cdot y \]
Moreover Proposition 2.18 applied to the groupoid \( X/H \) and the Fell bundle \( \mathcal{A}/H \) shows that there is a canonical embedding of \( C_\beta(X^0/H) \) into \( M(C_\beta(\mathcal{A}/H)) \), which identifies an element \( f \in C_\beta(X^0/H) \) with the multiplier \( f \in M(C_\beta(\mathcal{A}/H)) \) given by

\[
f[b]_{yH} := f(r(y)H)[b]_{yH}.
\] (33)

Since both \( C_\beta(X^0/H) \) and \( C_\beta(\mathcal{A}/H) \) are canonically embedded in \( M(C_\beta(\mathcal{A})) \), it is convenient to understand what happens (inside \( M(C_\beta(\mathcal{A})) \)) when one multiplies an element of \( C_\beta(X^0/H) \) by an element \( C_\beta(\mathcal{A}/H) \). Perhaps unsurprisingly, this product is given exactly by expression (33), which models the action of \( C_\beta(X^0/H) \) on \( C_\beta(\mathcal{A}/H) \) as multipliers of the latter algebra. In other words, it makes no difference to view \( C_\beta(X^0/H) \) inside \( M(C_\beta(\mathcal{A}/H)) \) or inside \( M(C_\beta(\mathcal{A})) \) when it comes to multiplication by elements of \( C_\beta(\mathcal{A}/H) \).

We will now show how the multiplication of elements of \( C_\beta(\mathcal{A}/H) \) by elements of \( C_\beta(X^0) \) is determined (inside \( M(C_\beta(\mathcal{A})) \)). Before we proceed we will first introduce some notation that will be used throughout this work: Given a set \( A \subset X^0 \) we will denote by \( 1_A \in C_\beta(X^0) \) the characteristic function of \( A \). In case \( A \) is a singleton \( \{a\} \) we will simply write \( 1_a \).

**Proposition 2.20.** Inside \( M(C_\beta(\mathcal{A})) \) we have that, for \( x \in X, \ a \in \mathcal{A}_x \) and \( u \in X^0 \),

\[
[a]_{xH}1_u = \begin{cases} \alpha_{\widetilde{h}^{-1}}(a)_{x\widetilde{h}}, & \text{if } H_{x,u} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}
\]

where \( \widetilde{h} \) is any element of \( H_{x,u} \).

**Proof:** Let \( y \in X \) and \( b \in \mathcal{A}_y \). For the product \( [a]_{xH}1_u b_y \) to be non-zero we must necessarily have \( u = r(y) \) (from (32)), and in this case we obtain

\[
[a]_{xH}1_u b_y = [a]_{xH}b_y = (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}} = \alpha_{\widetilde{h}^{-1}}(a)_{x\widetilde{h}}b_y,
\]

where \( \widetilde{h} \) is any element of \( H_{x,y} \). Since \( u = r(y) \), we have \( H_{x,y} = H_{x,u} \), and this concludes the proof.

It will be of particular importance to know how to multiply, inside \( M(C_\beta(\mathcal{A})) \), elements of \( C_\beta(\mathcal{A}/H) \) with elements of \( C_\beta(\mathcal{A}/K) \) when \( K \subseteq H \) is an arbitrary subgroup. It turns out that the algebra \( C_\beta(\mathcal{A}/K) \) is preserved by multiplication by elements of \( C_\beta(\mathcal{A}/H) \), as we show in the next result:

**Proposition 2.21.** Let \( K \subseteq H \) be any subgroup. We have that

\[
[a]_{xH}[b]_{yK} = \begin{cases} \alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}yK}, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}
\] (34)

where \( x, y \in X, \ a \in \mathcal{A}_x \) and \( b \in \mathcal{A}_y \). In particular \( C_\beta(\mathcal{A}/K) \) is invariant under multiplication by elements of \( C_\beta(\mathcal{A}/H) \).
Proof: First we observe that since the action is assumed to be $H$-good, it is automatically $K$-good, so that we can form the groupoid $X/K$ and the Fell bundle $A/K$.

Let $z \in X$ and $c \in A_z$. We have that

$$[a]_{xH} [b]_{hkK} c_z = \begin{cases} [a]_{xH} (\alpha_{k^{-1}} (b) c)_{ykz}, & \text{if } K_{y,z} \neq \emptyset; \\ 0, & \text{otherwise}, \end{cases}$$

where $k$ is any element of $K_{y,z}$ and $h$ is any element of $H_{x,ykz}$. Now, since $H_{x,ykz} = H_{x,y} = H_{x,yk}$, it follows that $hk^{-1} \in H_{x,y}$, and moreover since $K_{y,z} = K_{xhk^{-1}y,z}$, we conclude that

$$= \begin{cases} (\alpha_{k^{-1}} (\alpha_{k^{-1}} (a) b) c)_{xhk^{-1}y} z_k, & \text{if } H_{x,y} \neq \emptyset \text{ and } K_{xhk^{-1}y,z} \neq \emptyset; \\ 0, & \text{otherwise}, \end{cases}$$

Thus (34) follows immediately (the element $\tilde{h}$ in (34) is simply the element denoted by $hk^{-1}$ above).

In case the subgroup $K$ has finite index in $H$ we can strengthen Proposition 2.21 in the following way:

**Proposition 2.22.** Let $K \subseteq H$ be a subgroup such that $[H : K] < \infty$. Inside $M(C_c(A))$ we have that

$$[a]_{xH} = \sum_{[h] \in S_x \setminus H/K} [\alpha_{h^{-1}} (a)]_{xhkK}, \quad (35)$$

for any $x \in X$ and $a \in A_x$. In particular, inside $M(C_c(A))$ we have that $C_c(A/H)$ is a *-subalgebra of $C_c(A/K)$.

**Proof:** First we notice that since $[H : K] < \infty$ we have that the right hand side of (35) is a finite sum and therefore does indeed define an element of $C_c(A/K)$. To prove this result it suffices to show that

$$[a]_{xH} b_y = \sum_{[h] \in S_x \setminus H/K} [\alpha_{h^{-1}} (a)]_{xhkK} b_y, \quad (36)$$

for all $y \in X$ and $b \in A_y$. First we notice that both the right and left hand sides of (36) are zero unless $r(y) \in s(x)H$. In case $r(y) \in s(x)H$ we have

$$[a]_{xH} b_y = (\alpha_{k^{-1}} (a) b)_{xhk},$$

for all $y \in X$ and $b \in A_y$. First we notice that both the right and left hand sides of (36) are zero unless $r(y) \in s(x)H$. In case $r(y) \in s(x)H$ we have
where $\tilde{h}$ is any element of $H_x y$.

Recall from Proposition 1.23 that there is a bijective correspondence between the set of $K$-orbits $(x H)/K$ and the double coset space $S_x H/K$. It is clear that $a [a] x h K b y = (a \tilde{h}^{-1} (a) b) x \tilde{h} y$. Moreover, for all the classes $[h] \neq [\tilde{h}]$ in $S_x H/K$ we have $r(y) \notin s(x) h K$, because $r(y) \notin s(x) \tilde{h} K$. Hence, for all the classes $[h] \neq [\tilde{h}]$ in $S_x H/K$ we have $[a h^{-1} (a)] x h K b y = 0$. We conclude that

$$\sum_{[h] \in S_x H/K} [\alpha h^{-1} (a)] x h K b y = [\alpha \tilde{h}^{-1} (a)] x \tilde{h} K b y = (\alpha \tilde{h}^{-1} (a) b) x \tilde{h} y,$$

and equality (36) is proven.

**Remark 2.23.** In Proposition 2.22 the fact that $[H : K] < \infty$ was only used to ensure that the sum on the right hand side of (35) was finite. One could more generally just require that the sets $S_x H/K$ are finite for all $x \in X$, but this generality will not be used here.

As we saw in Proposition 2.6 we have an action $\overline{\alpha}$ of $G$ on $C_c(A)$. This action can be extended in a unique way to an action on $M(C_c(A))$, which we will still denote by $\overline{\alpha}$, by the following formula:

$$\overline{\alpha}_y (T) f := \overline{\alpha}_y (T \overline{\alpha}_y^{-1} (f)),$$

where $y \in G$, $T \in M(C_c(A))$ and $f \in C_c(A)$. We will now show what this action on $M(C_c(A))$ does to the algebras $C_0(X^0)$, $C_c(A/H)$ and $C_c(X^0/H)$.

**Proposition 2.24.** The extension of the action $\overline{\alpha}$ to $M(C_c(A))$, also denoted by $\overline{\alpha}$, satisfies the following properties:

(i) The restriction of $\overline{\alpha}$ to $C_0(X^0)$ is precisely the action that comes from the $G$-action on $X^0$.

(ii) For any $g \in G$ the automorphism $\overline{\alpha}_g$ takes $C_c(X^0/H)$ to $C_c(X^0/g H g^{-1})$, by

$$\overline{\alpha}_g (1_{x H}) = 1_{(x g^{-1}) (g H g^{-1})}.$$  

(iii) For any $g \in G$ the automorphism $\overline{\alpha}_g$ takes $C_c(A/H)$ to $C_c(A/g H g^{-1})$, by

$$\overline{\alpha}_g ([a] x H) = [\alpha (a)] (x g^{-1}) (g H g^{-1}).$$

(iv) Both $C_c(A/H)$ and $C_c(X^0/H)$ are contained in $M(C_c(A))^H$, the algebra of $H$-fixed points.

**Proof:** (i) Let $y \in X$, $b \in A_y$ and $f \in C_b(X^0)$. For any $g \in G$ let us denote by $f_g \in C_b(X^0)$ the function defined by $f_g (x) = f(x g)$. By definition of the extension of $\alpha$ to $M(C_c(A))$, we have

$$\overline{\alpha}_g (f) b_y = \overline{\alpha}_g (f \cdot \overline{\alpha}_g^{-1} (b)_y) = \overline{\alpha}_g (f \cdot \alpha g^{-1} (b) y y) = \overline{\alpha}_g (f (r(y) g) \alpha g^{-1} (b) y y) = \overline{\alpha}_g (f (r(y) g) b y) = f (r(y) g) b_y = f_g (r(y)) b_y = f_g b_y.$$

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Hence we conclude that $\alpha_g(f) = f_g$ and therefore the action $\alpha$ on $C_c(X^0)$ is just the action that comes from the $G$-action on $X^0$.

(ii) This follows directly from (i).

(iii) Let $y \in X$ and $b \in A_y$. By definition of the extension of $\pi$ to $M(C_c(A))$, we have

$$\pi_y([a]_{xH}) b_y = \pi_y([a]_{xH} \pi^{-1}_y(b_y)) = \pi_y([a]_{xH} \alpha^{-1}_y(b_y)).$$

Also, we can see that

$$\pi_y([a]_{xH} \alpha^{-1}_y(b)_{yg}) = \begin{cases} \pi_y((\alpha_{\tilde{h}^{-1}}(a) \alpha^{-1}_y(b))_{x\tilde{h}(yg)}) & \text{if } H_{x,yg} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{h} \in H_{x,yg}$. Now an easy computation shows that we have

$$H_{x,yg} = g^{-1}(gHg^{-1})_{xg^{-1},yg},$$

and thereby we obtain, for $t \in (gHg^{-1})_{xg^{-1},yg}$,

$$\pi_y([a]_{xH}) b_y = \begin{cases} (\alpha_{gg^{-1}t^{-1}g}(a)b)_{xg^{-1}yg} & \text{if } (gHg^{-1})_{xg^{-1},yg} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha_g(a) = (\alpha_{gg^{-1}t^{-1}g}(a)b)_{xg^{-1}yg}$. (iv) This follows directly from (ii) and (iii).

It is important to know how to multiply an element of $C_c(A/H)$ with an element of $C_c(X^0/gHg^{-1})$ inside $M(C_c(A))$. This is easy if we are under the assumption that $G$-action satisfies the $H$-intersection property. We recall from \(\S\) that $H^g$ stands for the subgroup $H \cap gHg^{-1}$.

**Proposition 2.25.** If the $G$-action moreover satisfies the $H$-intersection property, then for every $x \in X$ and $g \in G$ the following equality holds in $M(C_c(A))$:

$$[a]_{xH} 1_{s(x)gHg^{-1}} = [a]_{xH^s}.$$
Proof: For any $y \in X$ and $b \in A_y$ we have

$[a]xH 1_{s(x)gHg^{-1}} b_y =$

$\begin{cases} 
[a]xH b_y, & \text{if } r(y) \in s(x)gHg^{-1} \\
0, & \text{otherwise}
\end{cases}$

$= \begin{cases} 
(\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}, & \text{if } r(y) \in s(x)gHg^{-1} \text{ and } r(y) \in s(x)H \\
0, & \text{otherwise}
\end{cases}$

$= \begin{cases} 
(\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}, & \text{if } r(y) \in s(x)H \cap s(x)gHg^{-1} \\
0, & \text{otherwise}
\end{cases}$

where $\widetilde{h} \in H_{x,y}$. Now, by the $H$-intersection property, we obtain

$= \begin{cases} 
(\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}, & \text{if } r(y) \in s(x)H^g \\
0, & \text{otherwise}
\end{cases}$

Of course, we have $(H^g)_{x,y} \subseteq H_{x,y}$, and hence we can choose $\widetilde{h}$ as an element of $(H^g)_{x,y}$, thereby obtaining

$= [a]xH_y b_y,$

which finishes the proof. \(\square\)

3 **-Algebraic crossed product by a Hecke pair**

In this section we introduce our notion of a (**-algebraic) crossed product by a Hecke pair and we explore its basic properties and its representation theory. Throughout the rest of this work we impose the following standing assumption, based on the tools developed in Section 2.1.

Standing Assumption 3.1. We assume from now on that $(G, \Gamma)$ is a Hecke pair, $A$ is a Fell bundle over a groupoid $X$ endowed with a $\Gamma$-good right $G$-action $\alpha$ satisfying the $\Gamma$-intersection property.

3.1 Definition of the crossed product and basic properties

In this section we aim at defining the (**-algebraic) crossed product of $C_c(A/\Gamma)$ by the Hecke pair $(G, \Gamma)$. For that we are going to define some sort of a bundle over $G/\Gamma$, where the fiber over each $g\Gamma$ is precisely $C_c(A/g\Gamma)$. Recall that we denote by $\overline{\pi}$ the associated action of $G$ on $C_c(A)$ and also its extension to $M(C_c(A))$.

Definition 3.2. Let $B(A, G, \Gamma)$ be the vector space of finitely supported functions $f : G/\Gamma \to M(C_c(A))$ satisfying the following compatibility condition

$f(\gamma g\Gamma) = \overline{\pi}_{\gamma}(f(g\Gamma)), \quad (40)$

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for all $\gamma \in \Gamma$ and $g\Gamma \in G/\Gamma$.

**Lemma 3.3.** For every $f \in B(A, G, \Gamma)$ and $g \Gamma \in G/\Gamma$ we have

$$f(g\Gamma) \in M(C_c(A))^{\Gamma g}.$$ 

**Proof:** This follows directly from the compatibility condition (40), since for every $\gamma \in \Gamma$ we have $\alpha_{\gamma}(f(g\Gamma)) = f(\gamma g\Gamma)$.

**Definition 3.4.** The vector subspace of $B(A, G, \Gamma)$ consisting of the functions $f : G/\Gamma \to M(C_c(A))$ satisfying the compatibility condition (40) and the property

$$f(g\Gamma) \in C_c(A/\Gamma) \times_{\alpha} g\Gamma,$$ (41)

will be denoted by $C_c(A/\Gamma) \times_{\alpha} G/\Gamma$ and will be called the *-algebraic crossed product of $C_c(A/\Gamma)$ by the Hecke pair $(G, \Gamma)$.

It is relevant to point out that the definitions of the spaces $B(A, G, \Gamma)$ and $C_c(A/\Gamma) \times_{\alpha} G/\Gamma$ seem more suitable for Hecke pairs $(G, \Gamma)$, as in general a function in $B(A, G, \Gamma)$ could only have support on those elements $g\Gamma \in G/\Gamma$ such that $|\Gamma g\Gamma/\Gamma| < \infty$.

We now define a product and an involution in $B(A, G, \Gamma)$ by:

$$(f_1 * f_2)(g\Gamma) := \sum_{|h| \in G/\Gamma} f_1(h\Gamma) \pi_h(f_2(h^{-1}g\Gamma)),$$ (42)

$$(f^*)(g\Gamma) := \Delta(g^{-1})\pi_g(f(g^{-1}\Gamma))^*.$$ (43)

**Proposition 3.5.** $B(A, G, \Gamma)$ becomes a unital *-algebra under the product and involution defined above, whose identity element is the function $f$ such that $f(\Gamma) = 1$ and is zero in the remaining points of $G/\Gamma$.

**Proof:** First, we claim that the expression for the product defined above is well-defined in $B(A, G, \Gamma)$, i.e. for $f_1, f_2 \in B(A, G, \Gamma)$ the expression

$$(f_1 * f_2)(g\Gamma) := \sum_{|h| \in G/\Gamma} f_1(h\Gamma) \pi_h(f_2(h^{-1}g\Gamma))$$

is independent from the choice of the representatives $[h]$ and also that it has finitely many summands. Independence from the choice of the representatives $[h] \in G/\Gamma$ follows directly from the compatibility condition (40) and the fact that the sum is finite follows simply from the fact that $f_1$ has finite support.

Now we claim that $f_1 * f_2$ has also finite support, for $f_1, f_2 \in B(A, G, \Gamma)$. Let $S_1, S_2 \subseteq G/\Gamma$ be the supports of the functions $f_1$ and $f_2$ respectively. We
will regard $S_1$ and $S_2$ as subsets of $G$ (being finite unions of left cosets). It is easy to check that the function $G \times G \to M(C_c(A))$

$$(h, g) \mapsto f_1(h \Gamma) \overline{\mu}_h(f_2(h^{-1}g \Gamma))$$

has support contained in $S_1 \times (S_1 \cdot S_2)$. Since $(G, \Gamma)$ is a Hecke pair, the product $S_1 \cdot S_2$ is also a finite union of left cosets. Hence, $f_1 \ast f_2$ has finite support.

We also notice that $f_1 \ast f_2$ satisfies the compatibility condition \((\text{II})\), thus defining an element of $B(A, G, \Gamma)$, since for any $\gamma \in \Gamma$ we have

$$(f_1 \ast f_2)(\gamma \Gamma) = \sum_{[h] \in G/\Gamma} f_1(h \Gamma) \overline{\mu}_h(f_2(h^{-1} \gamma \Gamma)) = \sum_{[h] \in G/\Gamma} f_1(\gamma h \Gamma) \overline{\mu}_h(f_2(h^{-1} \Gamma)) = \sum_{[h] \in G/\Gamma} \overline{\alpha}_\gamma(f_1(h \Gamma)) \overline{\alpha}_\gamma \circ \overline{\mu}_h(f_2(h^{-1} \Gamma)) = \overline{\alpha}_\gamma((f_1 \ast f_2)(g \Gamma))).$$

In a similar way we can see that the expression that defines the involution is well-defined in $B(A, G, \Gamma)$. There are now a few things that need to be checked before we can say that $B(A, G, \Gamma)$ is an $^*$-algebra, namely that the product is associative and the involution is indeed an involution relatively to this product (the fact that the product is distributive and the properties concerning multiplication by scalars are obvious). The proofs of these facts are essentially just a mimic of the corresponding proofs for “classical” crossed products by groups. Thus, we can say that $B(A, G, \Gamma)$ is $^*$-algebra under this product and involution.

**Theorem 3.6.** $C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ is a $^*$-ideal of $B(A, G, \Gamma)$. In particular it is a $^*$-algebra for the above operations.

**Proof:** It is easy to see that the space $C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ is invariant for the involution, i.e.

$$f \in C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma \implies f^* \in C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma.$$

Thus, to prove that $C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ is a (two-sided) $^*$-ideal of $B(A, G, \Gamma)$ it is enough to prove that is a right ideal, i.e. if $f_1 \in B(A, G, \Gamma)$ and $f_2 \in C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ then $f_1 \ast f_2 \in C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$, because any right $^*$-ideal is automatically two-sided. Hence, all we need to prove is that $(f_1 \ast f_2)(g \Gamma) \in C_c(A/\Gamma^g)$, for every $f_1 \in B(A, G, \Gamma)$ and $f_2 \in C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$. The proof of this fact will follow the following steps:

1) Prove that: given a subgroup $H \subseteq G$, $f \in C_c(A/H)$ and a unit $u \in X^0$, we have $f \cdot 1_u \in C_c(A)$.

2) Let $T := (f_1 \ast f_2)(g \Gamma) = \sum_{[h] \in G/\Gamma} f_1(h \Gamma) \overline{\mu}_h(f_2(h^{-1}g \Gamma))$. Use 1) to show that $T \cdot 1_u \in C_c(A)$ for any unit $u \in X^0$. 

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3) Fix a unit \( u \in X^0 \). By 2) we have \( T1_u = \sum_i (a_i)_{x_i} \), where the elements \( x_i \in X \) are such that \( s(x_i) = u \). Show that \( T1_{u\Gamma^y} = \sum_i [a_i]_{x_i, \Gamma^y} \), and conclude that \( T1_{u\Gamma^y} \in C_c(A/\Gamma^y) \).

4) Prove that there exists a finite set of units \( \{ u_1, \ldots, u_n \} \subseteq X^0 \) such that \( T = \sum_{i=1}^n T1_{u_i\Gamma^y} \). Conclude that \( T \in C_c(A/\Gamma^y) \).

- **Proof of 1)**: This follows immediately from Proposition 2.20.

- **Proof of 2)**: We know that \( f_2(h^{-1}g\Gamma) \subseteq C_c(A/\Gamma^h \Gamma^g) \). Thus, from Proposition 2.23 we conclude that \( \overline{\mu}_h(f_2(h^{-1}g)) \subseteq C_c(A/\Gamma^h \Gamma^g) \Gamma^g \). Now, using 1), we see that \( \overline{\mu}_h(f_2(h^{-1}g))_1 \in C_c(A) \) and consequently \( f_1(h\Gamma)\overline{\mu}_h(f_2(h^{-1}g))_1 \in C_c(A) \). Hence, \( T1_u \in C_c(A) \).

- **Proof of 3)**: For any \( \gamma \in \Gamma^y \) we have, using Lemma 3.3

\[
T1_{u\gamma} = \overline{\mu}_{\gamma^{-1}}(T)1_{u\gamma} = \overline{\mu}_{\gamma^{-1}}(T\overline{\mu}_\gamma(1_{u\gamma})) = \overline{\mu}_{\gamma^{-1}}(T1_u) = \sum_i \alpha_{\gamma^{-1}}(a_i)_{x_i, \gamma}.
\]

Let \( y \in X \) and \( b \in A_y \). We have

\[
T1_{u\Gamma^y} b_y = \begin{cases} 
Tv_y, & \text{if } r(y) \in u\Gamma^y \\
0, & \text{otherwise}.
\end{cases}
\]

Assume now that \( r(y) \in u\Gamma^y \) and let \( \tilde{\gamma} \in \Gamma^y \) be such that \( r(y) = u\tilde{\gamma} \). We then have

\[
Tv_y = T1_{u\tilde{\gamma}} b_y = \sum_i \alpha_{\tilde{\gamma}^{-1}}(a_i)_{x_i, \tilde{\gamma}} b_y.
\]

Since \( s(x_i) = u \), we have \( s(x_i, \tilde{\gamma}) = u\tilde{\gamma} = r(y) \). Hence,

\[
Tv_y = \sum_i (\alpha_{\tilde{\gamma}^{-1}}(a_i)b)_{x_i, \tilde{\gamma}y}.
\]

We conclude that

\[
T1_{u\Gamma^y} b_y = \begin{cases} 
\sum_i (\alpha_{\tilde{\gamma}^{-1}}(a_i)b)_{x_i, \tilde{\gamma}y}, & \text{if } r(y) \in u\Gamma^y \\
0, & \text{otherwise}.
\end{cases}
\]

Thus, \( T1_{u\Gamma^y} \subseteq C_c(A/\Gamma^y) \).

- **Proof of 4)**: For easiness of reading of this last part of the proof we introduce the following definition: given \( F \in M(C_c(A)) \) we define the **support** of \( F \) to be the set \( \{ u \in X^0 : F1_u \neq 0 \} \). Notice in particular that the support of an element \([a]_{x'H} \), with \( a \neq 0 \), is the set \( s(x)H \).

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Since $\overline{\alpha}(f_2(h^{-1}g\Gamma)) \in C_c(A/h\Gamma h^{-1} \cap g\Gamma g^{-1})$, there exists a finite number of units $v_1, \ldots, v_k \in X^0$ such that $\overline{\alpha}(f_2(h^{-1}g\Gamma))$ has support in

$$\bigcup_{i=1}^k v_i(h\Gamma h^{-1} \cap g\Gamma g^{-1}) \subseteq \bigcup_{i=1}^k v_i g\Gamma g^{-1}.$$

Hence, there is a finite number of units $w_1, \ldots, w_l \in X^0$ such that $T$ has support contained in

$$\bigcup_{i=1}^l w_i g\Gamma g^{-1}.$$

Therefore, $T$ has support contained in

$$\bigcup_{i=1}^l \bigcup_{j=1}^m \theta_i \Gamma^g,$$

where $\theta_1, \ldots, \theta_m$ are representatives of the classes of $g\Gamma g^{-1}/\Gamma^g$ (being a finite number because $(G, \Gamma)$ is a Hecke pair). Thus, we have proven that there is a finite number of units $u_1, \ldots, u_n \in X^0$ such that $T$ has support inside $\bigcup_{i=1}^n u_i \Gamma^g$. Moreover, we can suppose we have chosen the units $u_1, \ldots, u_n$ such that the corresponding orbits $u_i \Gamma^g$ are mutually disjoint. It is now easy to see that we have $T = \sum_{i=1}^n T 1_{u_i \Gamma^g}$. Indeed, given $y \in X$ and $b \in A_y$, if $r(y) \notin \bigcup_{i=1}^n u_i \Gamma^g$, then

$$T b_y = T 1_{r(y)} b_y = 0 = \sum_{i=1}^n T 1_{u_i \Gamma^g} b_y,$$

and if $r(y) \in \bigcup_{i=1}^n u_i \Gamma^g$, then $r(y)$ belongs to precisely one of the orbits, say $u_\alpha \Gamma^g$, and we have

$$\sum_{i=1}^n T 1_{u_i \Gamma^g} b_y = T 1_{u_\alpha \Gamma^g} b_y = T b_y.$$

Hence, we must have $T = \sum_{i=1}^n T 1_{u_i \Gamma^g}$, and by 3) we conclude that $T \in C_c(A/\Gamma^g)$. \hfill \Box

As it is well-known, when working with crossed products $A \times G$ by discrete groups, one always has an embedded copy of $A$ inside the crossed product. Something analogous happens in the case of crossed products by Hecke pairs, where $C_c(A/\Gamma)$ is canonically embedded in $C_c(A/\Gamma) \times^\alpha_{\mu} G/\Gamma$, as is stated in the next result (whose proof amounts to routine verification).

**Proposition 3.7.** There is a natural embedding of the $^*$-algebra $C_c(A/\Gamma)$ in $C_c(A/\Gamma) \times^\alpha_{\mu} G/\Gamma$, which identifies an element $f \in C_c(A/\Gamma)$ with the function $\iota(f) \in C_c(A/\Gamma) \times^\alpha_{\mu} G/\Gamma$ such that

$$\iota(f)(\Gamma) = f \quad \text{and} \quad \iota(f) \text{ is zero elsewhere}.$$
Remark 3.8. The above result says that we can identify $C_c(A/\Gamma)$ with the functions of $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ with support in $\Gamma$. We shall, henceforward, make no distinctions in notation between an element of $C_c(A/\Gamma)$ and its correspondent in $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$.

Theorem 3.9. $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is an essential *-ideal of $B(A,G,\Gamma)$. In particular, $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is an essential *-algebra. Moreover, there are natural embeddings

$$C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma \hookrightarrow B(A,G,\Gamma) \hookrightarrow M(C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma) ,$$

that make the following diagram commute

\[ \begin{array}{ccc}
C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma & \longrightarrow & M(C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma) \\
\downarrow & & \\
B(A,G,\Gamma)
\end{array} \]

Proof: We have already proven that $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is a *-ideal of $B(A,G,\Gamma)$, thus we only need to check that this ideal is in fact essential. Suppose $f \in B(A,G,\Gamma)$ is such that $f \ast (C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma) = \{0\}$. Then, in particular, using Proposition \ref{prop:essential-ideal}, we must have $f \ast (C_c(A/\Gamma)) = \{0\}$. Let $g \in G$ and take $[a]_{x\Gamma} \in C_c(A/\Gamma)$, we then have

$$0 = (f \ast [a]_{x\Gamma})(g\Gamma) = f(g\Gamma)\pi_g([a]_{x\Gamma}) = f(g\Gamma)[\alpha_g(a)]_{xg^{-1}g^{-1}} .$$

Thus, multiplying by $1_{\pi(x)g^{-1}} \in M(C_c(A))$ we get

$$0 = f(g\Gamma)[\alpha_g(a)]_{xg^{-1}g^{-1}} 1_{\pi(x)g^{-1}} = f(g\Gamma)\alpha_g(a)_{xg^{-1}} = f(g\Gamma)\pi_g(a_x) .$$

Since this true for all $a \in A_x$ and $x \in X$ and given that $\alpha$ takes fibers of $A$ bijectively into fibers of $A$, we must have $f(g\Gamma)b_y = 0$ for all $b \in A_y$ and $y \in X$. Hence, we must have $f(g\Gamma) = 0$. Thus, $f = 0$ and we conclude that $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is an essential *-ideal of $B(A,G,\Gamma)$.

Since $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is a *-subalgebra of $B(A,G,\Gamma)$, we immediately conclude that $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is an essential *-algebra.

The embedding of $B(A,G,\Gamma)$ in $M(C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma)$ then follows from the universal property of multiplier algebras, Theorem \ref{thm:universal-property-multiplier-algebra}.

In the theory of crossed products $A \times G$ by groups, one always has an embedded copy of the group algebra $C(G)$ inside the multiplier algebra $M(A \times G)$. Something analogous happens in the case of crossed products by Hecke pairs, where the Hecke algebra $H(G,\Gamma)$ is canonically embedded in the multiplier algebra $M(C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma)$, as is stated in the next result (whose proof amounts to routine verification).
Proposition 3.10. The Hecke $\ast$-algebra $\mathcal{H}(G, \Gamma)$ embeds in $B(A, G, \Gamma)$ in the following way: an element $f \in \mathcal{H}(G, \Gamma)$ is identified with the element $\tilde{f} \in B(A, G, \Gamma)$ given by
\[
\tilde{f}(g\Gamma) := f(g\Gamma)1,
\]
where $1$ is the unit of $M(C_c(A))$.

The next result does not typically play an essential role in the case of crossed products by groups, but will be extremely important for us in case of crossed products by Hecke pairs. The proof is also just routine verification.

Proposition 3.11. The algebra $C_c(X_0^0/\Gamma)$ embeds in $B(A, G, \Gamma)$ in the following way: an element $f \in C_c(X_0^0/\Gamma)$ is identified with the function $\iota(f) \in B(A, G, \Gamma)$ given by
\[
\iota(f)(\Gamma) = f \quad \text{and} \quad \iota(f) \text{ is zero elsewhere}.
\]

Remark 3.12. Propositions 3.10 and 3.11 allow us to view both the Hecke $\ast$-algebra $\mathcal{H}(G, \Gamma)$ and $C_c(X_0^0/\Gamma)$ as $\ast$-subalgebras of $B(A, G, \Gamma)$. We shall henceforward make no distinctions in notation between an element of $\mathcal{H}(G, \Gamma)$ or $C_c(X_0^0/\Gamma)$ and its correspondent in $B(A, G, \Gamma)$.

The purpose of the following diagram is to illustrate, in a more condensed form, all the canonical embeddings we have been considering so far:

\[
\begin{array}{c}
\xymatrix{ C_c(A/\Gamma) & C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma \ar[l] \ar[r] & \mathcal{H}(G, \Gamma) \\
& B(A, G, \Gamma) \ar[l] & M(C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma)) \ar[l] \\
& C_c(X_0^0/\Gamma) \ar[l] }
\end{array}
\]

Remark 3.13. The reason for considering the algebra $B(A, G, \Gamma)$ is two-fold. On one side $B(A, G, \Gamma)$ made it easier to make sure the convolution product (42) was well-defined in $C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma$. On the other (perhaps more important) side, the fact that both $\mathcal{H}(G, \Gamma)$ and $C_c(X_0^0/\Gamma)$ are canonically embedded in $B(A, G, \Gamma)$ allows us to treat the elements of $\mathcal{H}(G, \Gamma)$ and $C_c(X_0^0/\Gamma)$ both as multipliers in $M(C_c(A/\Gamma) \times_{\alpha}^{alg} G/\Gamma))$, but also allows us to operate these elements with the convolution product and involution expressions (42) and (43), as these are defined in $B(A, G, \Gamma)$.
As it is well-known in the theory of crossed products by discrete groups, a \((\ast\)-algebraic) crossed product \(A \times G\) is spanned by elements of the form \(a \ast g\), where \(a \in A\) and \(g \in G\) (here \(g\) is seen as an element of the group algebra \(C(G) \subseteq M(A \times G)\)). We will now explore something analogous in the case of crossed products by Hecke pairs. It turns out that \(C_c(A/\Gamma)^{\ast\text{alg}} G/\Gamma\) is spanned by elements of the form \([a]_{x\Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma}\), where \(x \in X\), \(a \in A_s\) and \(g \Gamma \in G/\Gamma\), as we show in the next result.

**Theorem 3.14.** For any \(f \in C_c(A/\Gamma)^{\ast\text{alg}} G/\Gamma\) we have

\[
f = \sum_{[g] \in G/\Gamma} \sum_{x \Gamma^\gamma \in X/\Gamma^\gamma} \left[ f(g \Gamma(x)) \right]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma}.
\]

(44)

In particular, \(C_c(A/\Gamma)^{\ast\text{alg}} G/\Gamma\) is spanned by elements of the form

\([a]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma}\),

with \(x \in X\), \(a \in A_s\) and \(g \Gamma \in G/\Gamma\).

The following lemma is needed in order to prove the above result:

**Lemma 3.15.** Let \(x \in X\), \(a \in A_s\) and \(g \Gamma \in G/\Gamma\). We have

\([a]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma} (h \Gamma) = \begin{cases} [\alpha, \gamma(a)]_{x \gamma^{-1} \Gamma^\gamma}, & \text{if } h \Gamma = \gamma g \Gamma, \text{ with } \gamma \in \Gamma \\ 0, & \text{otherwise} \end{cases}
\]

In particular,

\([a]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma} (g \Gamma) = [a]_{x \Gamma^\gamma}.
\]

**Proof:** An easy computation yields

\([a]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma} (h \Gamma) = [a]_{x \Gamma} \ast g \Gamma(h \Gamma) \ast \bar{\pi}_h(1_{s(x)g \Gamma}),
\]

from which we conclude that \([a]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma}\) is supported in the double coset \(\Gamma g \Gamma\). Now, evaluating at the point \(g \Gamma \in G/\Gamma\) we get

\([a]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma} (g \Gamma) = \begin{cases} [a]_{x \Gamma} \ast g \Gamma(g \Gamma) \ast \bar{\pi}_g(1_{s(x)g \Gamma}) \\ [a]_{x \Gamma} \ast \bar{\pi}_g(1_{s(x)g \Gamma}) \\ [a]_{x \Gamma} \ast 1_{s(x)g \Gamma}^{-1} \\ [a]_{x \Gamma^\gamma} \end{cases}
\]

where the last equality comes from Proposition 2.25. From the compatibility condition \((40)\) and Proposition 2.25 it then follows that, for \(\gamma \in \Gamma\),

\([a]_{x \Gamma} \ast g \Gamma \ast 1_{s(x)g \Gamma} (\gamma g \Gamma) = \bar{\pi}_\gamma([a]_{x \Gamma^\gamma}) = [\alpha, \gamma(a)]_{x \gamma^{-1} \Gamma^\gamma}.
\]

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Proof of Theorem 3.14: Let us first prove that the expression on the right hand side of (44) is well-defined. It is easy to see that for every \( g \in G \), the expression

\[
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ f(g \Gamma)(x) \right]_{x \Gamma^g} \ast \Gamma g \Gamma \ast 1_{s(x)g \Gamma} \]

does not depend on the choice of the representative \( x \) of \( x \Gamma^g \). Now, let us see that it also does not depend on the choice of the representative \( g \) in \( \Gamma g \Gamma \). Let \( \gamma g \theta \), with \( \gamma, \theta \in \Gamma \), be any other representative. We have

\[
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ f(\gamma g \theta \Gamma)(x) \right]_{x \Gamma^g} \ast \Gamma \gamma g \theta \Gamma \ast 1_{s(x)\gamma g \theta \Gamma} =
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ f(\gamma g \Gamma)(x) \right]_{x \Gamma^g} \ast \Gamma g \Gamma \ast 1_{s(x)\gamma g \Gamma} =
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ \alpha_{\gamma}(f(g \Gamma))(x) \right]_{x \Gamma^g} \ast \Gamma g \Gamma \ast 1_{s(x)\gamma g \Gamma} =
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ \alpha_{\gamma}(f(g \Gamma)(x \gamma)) \right]_{x \Gamma^g} \ast \Gamma g \Gamma \ast 1_{s(x)\gamma g \Gamma}
\]

We notice that there is a well-defined bijective correspondence \( X/\Gamma^g \rightarrow X/\Gamma^{\gamma g} \) given by \( x \Gamma^g \mapsto x \gamma^{-1} \Gamma^{\gamma g} \). Thus, we get

\[
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ \alpha_{\gamma}(f(g \Gamma)(x)) \right]_{x \gamma^{-1} \Gamma^{\gamma g}} \ast \Gamma g \Gamma \ast 1_{s(x)\gamma^{-1} \Gamma^{\gamma g}} =
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ f(g \Gamma)(x) \right]_{x \Gamma^g} \ast \Gamma g \Gamma \ast 1_{s(x)g \Gamma}.
\]

Hence, the expression in (44) is well-defined. Let us now prove the decomposition in question. For any \( t \Gamma \in G/\Gamma \) we have

\[
\sum_{[g] \in G/\Gamma} \sum_{x \Gamma^g \in X/\Gamma^g} \left[ f(g \Gamma)(x) \right]_{x \Gamma^g} \ast \Gamma g \Gamma \ast 1_{s(x)g \Gamma} \ast (t \Gamma) =
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ f(t \Gamma)(x) \right]_{x \Gamma^g} \ast \Gamma t \Gamma \ast 1_{s(x)\Gamma t \Gamma} \ast (t \Gamma).
\]

By Lemma 3.15 it follows that

\[
\sum_{x \Gamma^g \in X/\Gamma^g} \left[ f(t \Gamma)(x) \right]_{x \Gamma^g} =
\sum_{x \Gamma^t \in X/\Gamma^t} \left[ f(t \Gamma)(x) \right]_{x \Gamma^t}
\]

and this finishes the proof. \( \square \)

In the following result we collect some useful equalities concerning products in \( C_\infty(A/\Gamma) \times \alpha^{\Gamma \Gamma} G/\Gamma \), which will be useful later on. One should observe the similarities between the equalities (47) and (48) and the equalities obtained by an
In particular, from (46) we see that elements of the form $C^1_{\Gamma}$, if in their setting one was allowed to somehow “drop” the representations. The similarity is more than a coincidence and will be addressed in the sequel of this article.

Let us now prove equality (47). An easy computation yields

$$[a]_{x\Gamma} \ast \Gamma g \Gamma = \sum_{[\gamma] \in S \setminus \Gamma \Gamma^{-1}} [a]_{x\Gamma} \ast \Gamma g \Gamma \ast [a]_{x\gamma g\Gamma}.$$  

In particular, from (48) we see that $C_c(A/\Gamma) \times \alpha^G G/\Gamma$ is also spanned by all elements of the form $1_{r(x)\Gamma} \ast \Gamma g \Gamma \ast [a]_{x\Gamma}$, with $g \in G$, $x \in X$ and $a \in A_x$.

**Proof:** Let us first prove equality (45). First we notice that

$$([a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\Gamma})^s = \Delta(g) \ast 1_{s(x)\Gamma} \ast \Gamma g^{-1} \ast 1_{s(x^{-1})\Gamma},$$

which means that $([a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\Gamma})^s$ has support in the double coset $\Gamma g^{-1} \Gamma$. Now evaluating this element on $g^{-1} \Gamma$ we get,

$$([a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\Gamma})^s \ast (g^{-1} \Gamma) =$$

$$= \Delta(g) \ast 1_{s(x)\Gamma} \ast \Gamma g^{-1} \ast 1_{s(x^{-1})\Gamma}.$$

Let us now prove equality (46). We have

$$1_{r(x)\Gamma} \ast \Gamma g \Gamma \ast [a g^{-1}(a)]_{xg\Gamma} = \Delta(g) \ast 1_{s(x)\Gamma} \ast \Gamma g^{-1} \ast 1_{r(x)\Gamma}$$

which together with (46) yields

$$\Delta(g) \ast [a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\Gamma} =$$

$$= [a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\Gamma}.$$
from which we conclude that \([a]_{x\Gamma} \ast \Gamma g \Gamma \) has support in \(\Gamma g \Gamma\). Evaluating this element on the point \(g \Gamma\) we get

\[
[a]_{x\Gamma} \ast \Gamma g \Gamma (g \Gamma) = [a]_{x\Gamma} \cdot \Gamma g \Gamma (g \Gamma) = [a]_{x\Gamma}.
\]

From Proposition 2.22 one always has the following decomposition

\[
[a]_{x\Gamma} = \sum_{[\gamma] \in S_x \setminus \Gamma \Gamma^g} [\alpha_{\gamma^{-1}}(a)]_{x\gamma \Gamma^g}.
\]

Together with Lemma 3.15 we get

\[
[a]_{x\Gamma} \ast \Gamma g \Gamma (g \Gamma) = [a]_{x\Gamma} = \sum_{[\gamma] \in S_x \setminus \Gamma \Gamma^g} [\alpha_{\gamma^{-1}}(a)]_{x\gamma \Gamma^g} = \sum_{[\gamma] \in S_x \setminus \Gamma \Gamma^g} [a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\gamma \Gamma^g} (g \Gamma),
\]

and equality (47) is proven.

Equality (48) follows easily from (47) by taking the involution and using the fact that \(S_x = S_{x^{-1}}\).

The last claim of this proposition follows simply from (46) and Proposition 3.14.

In the theory of crossed products \(A \times G\) by discrete groups one has a “covariance relation” of the form \(g \ast a \ast g^{-1} = \alpha_g(a)\). This relation is essential in the passage from covariant representations of the system \((A, G, \alpha)\) to representations of the crossed product. More generally, the following relation holds in \(A \times G\):

\[
g \ast a \ast h = \alpha_g(a) \ast gh.
\]

We will now explore how this generalizes to the setting of crossed products by Hecke pairs. What we are aiming for is a description of how products of the form \(\Gamma g \Gamma \ast [a]_{x\Gamma} \ast \Gamma s \Gamma\) can be expressed by the canonical spanning set of elements of the form \([b]_{y\Gamma} \ast \Gamma \Lambda \Gamma \ast 1_{s(x)h\Gamma}\) (according to Theorem 3.14). This will be achieved in Corollary 3.19 below and will play an important role in the representation theory of crossed products by Hecke pairs, particularly in the definition of covariant representations. One should observe the similarities between the expressions we obtain both in Theorem 3.17 and Corollary 3.19 and the expression provided by an Huef, Kaliszewski and Raeburn in [7, Definition 1.1] (if one “forgets” the representations in their setting). Once again, this is more than a coincidence as we will see in in the sequel to this article. In fact, an Huef, Kaliszewski and Raeburn’s definition served as a guiding line for our results below and for the definition of a covariant representation (Definition 3.24) which we shall present in the next section.

Before we establish the results we are aiming for we need to establish some notation, which will be used throughout this work. For \(w, v \in G\) and a unit
More generally, if \( \tilde{y} \) and the numbers \( y \in X^0 \) we define the sets

\[
\begin{align*}
n_{u,v}^y &:= \{ [r] \in \Gamma u \Gamma / \Gamma : r^{-1} w v \Gamma \subseteq \Gamma r \Gamma \text{ and } y w^{-1} \in y r \Gamma^{-1} \}, \\
\delta_{u,v}^y &:= \{ [r] \in \Gamma u \Gamma / \Gamma : r^{-1} w v \Gamma \subseteq \Gamma r \Gamma \text{ and } y w^{-1} \in y r \Gamma^{-1} \Gamma w v \}.
\end{align*}
\]

and the numbers

\[
\begin{align*}
n_{u,v}^y &:= \# n_{u,v}^y, \\
d_{u,v}^y &:= \# \delta_{u,v}^y, \\
N_{u,v}^y &:= \frac{n_{u,v}^y}{d_{u,v}^y}.
\end{align*}
\]

We will also denote by \( E_{u,v}^y \) the double coset space

\[
E_{u,v}^y := S_y \backslash \Gamma / (u \Gamma u^{-1} \cap v \Gamma v^{-1}).
\]

**Theorem 3.17.** Let \( g, s \in G \) and \( y \in X^0 \). We have that

\[
\Gamma g \Gamma * 1_{y \Gamma} * \Gamma s \Gamma = \sum_{[u] \in \Gamma u \Gamma / \Gamma} \sum_{[v] \in \Gamma v \Gamma / \Gamma} \frac{N_{u,v}^y}{L(uv)} (1_{y \gamma u^{-1} \Gamma} * \Gamma v u \Gamma * 1_{y \gamma v \Gamma})
\]

\[
= \sum_{[v] \in \Gamma v \Gamma / \Gamma} \sum_{[\gamma] \in E_{v}^y} \frac{L(g) N_{u,v}^y}{L(gv)} (1_{y \gamma g^{-1} \Gamma} * \Gamma g v \Gamma * 1_{y \gamma v \Gamma})
\]

\[
= \sum_{[u] \in \Gamma u \Gamma / \Gamma} \sum_{[v] \in \Gamma v \Gamma / \Gamma} \frac{\Delta(g) N_{u,v}^y}{L(u^{-1} v)} (1_{y \gamma u \Gamma} * \Gamma u^{-1} v \Gamma * 1_{y \gamma v \Gamma}).
\]

In order to prove the above result we will need the following lemma, which gives some properties of the numbers \( n_{u,v}^y \) and \( d_{u,v}^y \).

**Lemma 3.18.** Let \( w, v, \in G, \theta \in \Gamma \) and \( y \in X^0 \). The numbers \( n_{w,v}^y \) and \( d_{w,v}^y \) satisfy the following properties:

1. \( n_{w,v}^y \theta = n_{w,v}^y \)
2. \( n_{w,v}^y = n_{v,w}^y \)
3. \( n_{w,v}^y \theta^{-1} v = n_{w \theta^{-1}, v}^y \)
4. \( d_{w,v}^y \theta \sigma = d_{w,v}^y \)
5. \( d_{w,v}^y \cdot \gamma = d_{w^\gamma v}^y \)

More generally, if \( \tilde{w}, \tilde{v} \in G \) and \( \tilde{y} \in X^0 \) are such that \( \Gamma \tilde{w} \Gamma = \Gamma w \Gamma, \Gamma \tilde{v} \Gamma = \Gamma v \Gamma, \tilde{y} \Gamma = y \Gamma, \tilde{w} \Gamma = w \Gamma \) and \( \tilde{y} \Gamma w^{-1} \Gamma w v = y w^{-1} \Gamma w v \), then

1. \( n_{w,v}^y \theta = n_{w^\theta v}^y \)
2. \( d_{w,v}^y \theta = d_{w^\theta v}^y \)
Proof: Assertions i) and i') are obvious.

Assertion ii) follows from the observation that \([r] \mapsto [\theta^{-1} r]\) establishes a bijection between the sets \(n_{w,v}^\theta\) and \(n_{\theta w,v}^\theta\).

Assertion ii') is proven in a similar fashion as assertion ii).

To prove assertion iv), let \(\theta \in \Gamma^{uw}\) be such that \(\tilde{y} \theta^{-1} = y \theta^{-1}\). We have

\[
\begin{align*}
n_{\tilde{w},\tilde{v}}^\theta &= \{ [r] \in \Gamma \tilde{w} \Gamma / \Gamma : r^{-1} \tilde{w} \tilde{v} \Gamma \subseteq \Gamma \tilde{v} \Gamma \text{ and } \tilde{y} \theta^{-1} \in \tilde{y} \Gamma r^{-1} \} \\
&= \{ [r] \in \Gamma w \Gamma / \Gamma : r^{-1} w v \Gamma \subseteq \Gamma v \Gamma \text{ and } y \theta^{-1} \in y \Gamma r^{-1} \}.
\end{align*}
\]

Since \(\theta \in \Gamma^{uw}\) we have \(\theta w v \Gamma = w v \Gamma\), so that

\[
\begin{align*}
\{ [r] \in \Gamma \theta^{-1} w \Gamma / \Gamma : r^{-1} \theta^{-1} w v \Gamma \subseteq \Gamma v \Gamma \text{ and } y \theta^{-1} \in y \Gamma r^{-1} \} \\
= n_{\theta^{-1} w,v}^\theta.
\end{align*}
\]

Now, from assertion ii), it follows that \(n_{\tilde{w},\tilde{v}}^\theta = n_{\theta^{-1} w,v}^\theta = n_{w,v}^\theta\).

As for assertion iv'), taking \(\theta \in \Gamma^{uw}\) again as such that \(\tilde{y} \theta^{-1} = y \theta^{-1}\), we notice that

\[
\begin{align*}
n_{\tilde{w},\tilde{v}}^\theta &= \{ [r] \in \Gamma \tilde{w} \Gamma / \Gamma : r^{-1} \tilde{w} \tilde{v} \Gamma \subseteq \Gamma \tilde{v} \Gamma \text{ and } \tilde{y} \theta^{-1} \in \tilde{y} \Gamma r^{-1} \Gamma \tilde{w} \Gamma \} \\
&= \{ [r] \in \Gamma w \Gamma / \Gamma : r^{-1} w v \Gamma \subseteq \Gamma v \Gamma \text{ and } y \theta^{-1} \in y \Gamma r^{-1} \Gamma w v \Gamma \} \\
&= \{ [r] \in \Gamma w \Gamma / \Gamma : r^{-1} w v \Gamma \subseteq \Gamma v \Gamma \text{ and } y \theta^{-1} \in y \Gamma r^{-1} \Gamma w v \Gamma \} \\
&= n_{w,v}^\theta.
\end{align*}
\]

Assertions iii) and iii') are a direct consequence of iv) and iv').

Proof of Theorem 3.17: We have

\[
\begin{align*}
\Gamma g \Gamma \ast 1_{y \Gamma} \ast \Gamma s \Gamma \ast (t \Gamma) &= \sum_{[w] \in \Gamma^{G / \Gamma}} \Gamma g \Gamma ([w] \Gamma) \overline{x}_{w}((1_{y \Gamma} \ast \Gamma s \Gamma) (w^{-1} t \Gamma)) \\
&= \sum_{[w] \in \Gamma^{G / \Gamma}} \overline{x}_{w}((1_{y \Gamma} \ast \Gamma s \Gamma) (w^{-1} t \Gamma)) \\
&= \sum_{[w] \in \Gamma^{G / \Gamma}} \overline{x}_{w}(1_{y \Gamma} \cdot \Gamma s \Gamma (w^{-1} t \Gamma)) \\
&= \sum_{[w] \in \Gamma^{G / \Gamma}} \overline{x}_{w}(1_{y \Gamma}) \\
&= \sum_{[w] \in \Gamma^{G / \Gamma}} 1_{y \Gamma}^{-1}.
\end{align*}
\]

We now claim that

\[
\sum_{[w] \in \Gamma^{G / \Gamma}} 1_{y \Gamma}^{-1} = \sum_{[w] \in \Gamma^{G / \Gamma}} \sum_{[\gamma] \in E_{w^{-1} \Gamma}^{\gamma}} N_{w,v}^\gamma \cdot 1_{y \Gamma}^{-1} \cdot 1_{y \Gamma} \ast (t \Gamma) \cdot (1_{y \Gamma} \ast \Gamma s \Gamma) (w^{-1} t \Gamma) \cdot (1_{y \Gamma} \ast \Gamma s \Gamma) (w^{-1} t \Gamma)
\]

To see this, we will evaluate both the right and left expressions above on all points \(x \in X^y\) and see that we obtain the same value. First, we note that
if \( x \in X^0 \) is not of the form \( y \theta \tilde{w}^{-1} \), for some \( \theta \in \Gamma \) and \( \tilde{w} \in \Gamma g \Gamma \) such that \( \tilde{w}^{-1} \Gamma \subseteq \Gamma s \Gamma \), then both expressions are zero. Suppose now that \( x = y \theta \tilde{w}^{-1} \) for some \( \tilde{w} \in \Gamma g \Gamma \) such that \( \tilde{w}^{-1} \Gamma \subseteq \Gamma s \Gamma \). Evaluating the left expression we get

\[
\sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1}} (y \theta \tilde{w}^{-1}) = \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1}} (y \theta \tilde{w}^{-1}) = n_{\tilde{w}, \tilde{w}^{-1} t}^{y \theta}.
\]

As for the right expression, first we observe that if \( y \theta \tilde{w}^{-1} \in y \gamma w^{-1} \Gamma' \), then by Lemma 3.18 (iv) and (iv') we have \( N_{\tilde{w}, \tilde{w}^{-1} t}^{y \theta} = N_{w, w^{-1} t}^{y \gamma} \). Thus, evaluating the right expression we get

\[
\sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1}} \gamma w^{-1} \Gamma' = \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1}} (y \theta \tilde{w}^{-1}) = 1_{y \Gamma w^{-1}} \Gamma',
\]

from which we obtain that,

\[
N_{\tilde{w}, \tilde{w}^{-1} t}^{y \theta} \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1} \gamma w^{-1} \Gamma' \Gamma w^{-1} \Gamma} = N_{\tilde{w}, \tilde{w}^{-1} t}^{y \theta} \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1} \gamma w^{-1} \Gamma' \Gamma w^{-1} \Gamma} = N_{\tilde{w}, \tilde{w}^{-1} t}^{y \theta} \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1} \gamma w^{-1} \Gamma' \Gamma w^{-1} \Gamma} = N_{\tilde{w}, \tilde{w}^{-1} t}^{y \theta} \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1} \gamma w^{-1} \Gamma' \Gamma w^{-1} \Gamma}.
\]

So, equality 3.18 is established.

Now, by Proposition 3.18 we see that

\[
\sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1}} \gamma w^{-1} \Gamma' = \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1}} \gamma w^{-1} \Gamma' \Gamma w^{-1} \Gamma = \sum_{[w] \in \Gamma g \Gamma / \Gamma} 1_{y \Gamma w^{-1}} \gamma w^{-1} \Gamma' \Gamma w^{-1} \Gamma (t \Gamma)
\]

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Now, using the fact that condition $w^{-1}t\Gamma \subseteq Gs\Gamma$ means that there exists a (necessarily unique) element $[v] \in Gs\Gamma/G$ such that $w^{-1}t\Gamma = v\Gamma$, or equivalently, $t\Gamma = wv\Gamma$, we obtain

$$\sum_{[w]\in Gs\Gamma/G} \sum_{[v]\in Gs\Gamma/G} N^y_{w,v} (1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma)$$

$$= \sum_{[w]\in Gs\Gamma/G} \sum_{[v]\in Gs\Gamma/G} N^y_{w,v} (1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma)$$

We now claim that

$$\sum_{[w]\in Gs\Gamma/G} \sum_{[v]\in Gs\Gamma/G} N^y_{w,v} (1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma) = \sum_{[w]\in Gs\Gamma/G} \sum_{[v]\in Gs\Gamma/G} N_{w,v} (1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma)$$

To prove this we note that, given any $[w] \in Gs\Gamma/G$ and $[v] \in Gs\Gamma/G$, the element $(1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma)$ is nonzero if and only if $t\Gamma = \Gamma w \ast 1_{y\gamma w^{-1}r}$, so that we can write

$$\sum_{[v]\in Gs\Gamma/G} \sum_{[v]\in Gs\Gamma/G} N^y_{w,v} (1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma) = \sum_{[v]\in Gs\Gamma/G} \sum_{[v]\in Gs\Gamma/G} N^y_{w,v} (1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma)$$

$$= \sum_{[v]\in Gs\Gamma/G} \sum_{[v]\in Gs\Gamma/G} N^y_{w,v} (1_{y\gamma w^{-1}r} \ast \Gamma w \ast 1_{y\gamma w^{-1}r})(t\Gamma)$$

By Lemma 3.18 (ii) and (ii'), we know that $N^y_{w,v} = N^y_{w,v}$, hence
Moreover, we also have
\[
\frac{N_{w,v}^{y}}{L(wv)} (1_{y_{\gamma_w-1}} \ast \Gamma_{wv} \ast 1_{y_{\gamma_v}})(t{\Gamma}).
\]

Hence, we have proven that
\[
\Gamma_{g\Gamma} \ast 1_{y_{\Gamma}} \ast \Gamma_{s\Gamma} = \sum_{[\theta] \in \Gamma_{g\Gamma}/\Gamma} \sum_{[v] \in \Gamma_{s\Gamma}/\Gamma} \frac{N_{w,v}^{y}}{L(wv)} (1_{y_{\gamma_w-1}} \ast \Gamma_{wv} \ast 1_{y_{\gamma_v}}).
\]

Also,
\[
\Gamma_{g\Gamma} \ast 1_{y_{\Gamma}} \ast \Gamma_{s\Gamma} = \sum_{[w] \in \Gamma_{g\Gamma}/\Gamma} \sum_{[\gamma] \in \Gamma_{s\Gamma}/\Gamma} \frac{N_{w,v}^{y}}{L(wv)} (1_{y_{\gamma_w-1}} \ast \Gamma_{wv} \ast 1_{y_{\gamma_v}}).
\]

Moreover, we also have
\[
\sum_{[\gamma] \in \Gamma_{s\Gamma}/\Gamma} \sum_{[v] \in \Gamma_{s\Gamma}/\Gamma} \frac{L(g)N_{g,v}^{y}}{L(gv)} (1_{y_{\gamma_w-1}} \ast \Gamma_{g\Gamma} \ast 1_{y_{\gamma_v}}).
\]

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but since there is a well-defined bijection $E^y_{g^{-1},v} \to E^y_{g^{-1},v}$ given by $[\gamma] \mapsto [\gamma \theta]$, we obtain

\[
\sum_{[\theta] \in \Gamma / \Gamma^{s-1}} \sum_{[v] \in \Gamma \setminus \Gamma} \frac{\Delta(g) N^{g \theta}_{,g \theta^{-1},v}}{L(g^2)} \left( 1_{y \gamma g^{-1},v} \ast \Gamma g^2 \ast 1_{y \gamma g^{-1},v} \right)
\]

and from Lemma 3.18 we get $N^{y \gamma g}_{,g \theta^{-1},v} = N^{y \gamma g}_{,g \theta^{-1},v}$, thus

\[
\sum_{[\theta] \in \Gamma / \Gamma^{s-1}} \sum_{[v] \in \Gamma \setminus \Gamma} \frac{\Delta(g) N^{g \theta}_{,g \theta^{-1},v}}{L(g^2)} \left( 1_{y \gamma g^{-1},v} \ast \Gamma g^2 \ast 1_{y \gamma g^{-1},v} \right)
\]

\[
\sum_{[u] \in \Gamma / \Gamma^{s-1}} \sum_{[v] \in \Gamma \setminus \Gamma} \frac{\Delta(g) N^{y \gamma g}_{,u^{-1},v}}{L(u^2)} \left( 1_{y \gamma u^{-1},v} \ast \Gamma u^{-1} \ast 1_{y \gamma u^{-1},v} \right).
\]

\[\square\]

**Corollary 3.19.** Similarly, for $a \in A_x$ with $x \in X$, we have

\[
\Gamma g \ast [a] \ast \Gamma s \ast = \sum_{[u] \in \Gamma / \Gamma^{s-1}} \sum_{[v] \in \Gamma \setminus \Gamma} \frac{N^{s(x) \gamma}_{u^{-1},v}}{L(u v)} \left( 1_{x \gamma u^{-1},v} \ast \Gamma v \ast 1_{x \gamma u^{-1},v} \right)
\]

\[
\sum_{[u] \in \Gamma / \Gamma^{s-1}} \sum_{[v] \in \Gamma \setminus \Gamma} \frac{L(g) N^{s(x) \gamma}_{u^{-1},v}}{L(g v)} \left( 1_{x \gamma u^{-1},v} \ast \Gamma v \ast 1_{x \gamma u^{-1},v} \right)
\]

\[
\sum_{[u] \in \Gamma / \Gamma^{s-1}} \sum_{[v] \in \Gamma \setminus \Gamma} \frac{\Delta(g) N^{y \gamma g}_{,x^{-1},v}}{L(x^2)} \left( 1_{y \gamma x^{-1},v} \ast \Gamma x^{-1} \ast 1_{y \gamma x^{-1},v} \right).
\]

**Proof:** According to equality (18) in Proposition 3.16 we have

\[
\Gamma g \ast [a] \ast \Gamma s \ast = \sum_{[\theta] \in \Gamma / \Gamma^{s-1}} 1_{\tau x \theta g^{-1},v} \ast \Gamma g \ast [a] \ast \Gamma s \ast \Gamma
\]

\[
\sum_{[\theta] \in \Gamma / \Gamma^{s-1}} 1_{\tau x \theta g^{-1},v} \ast \Gamma g \ast [a] \ast \Gamma s \ast \Gamma
\]

and by (16) in the same proposition we get

\[
\sum_{[\theta] \in \Gamma / \Gamma^{s-1}} [\alpha_{g^{-1}}(a)]_{x \gamma g^{-1},v} \ast \Gamma g \ast 1_{s(x)} \ast \Gamma s \ast \Gamma.
\]

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and by Theorem 3.17, we obtain

\[
\sum_{[\theta] \in \mathcal{S}_c \setminus \Gamma^{\mathcal{S}_c^{-1}}} \frac{N_{w,v}^{[\theta]}}{L(wv)} \left[ a_{\theta^{-1}(a)} \right]_{x \theta y^{-1} \Gamma} \ast 1_{\mathfrak{s}(x) \gamma w^{-1} \Gamma} \ast \Gamma w v \Gamma \ast 1_{\mathfrak{s}(x) \gamma y \Gamma}.
\]

For each fixed \( w, v \) and \( \gamma \) all the summands in the expression

\[
\sum_{[\theta] \in \mathcal{S}_c \setminus \Gamma^{\mathcal{S}_c^{-1}}} \frac{N_{w,v}^{[\theta]}}{L(wv)} \left[ a_{\theta^{-1}(a)} \right]_{x \theta y^{-1} \Gamma} \ast 1_{\mathfrak{s}(x) \gamma w^{-1} \Gamma} \ast \Gamma w v \Gamma \ast 1_{\mathfrak{s}(x) \gamma y \Gamma},
\]

are zero except precisely for one summand and we have

\[
\sum_{[\theta] \in \mathcal{S}_c \setminus \Gamma^{\mathcal{S}_c^{-1}}} \frac{N_{w,v}^{[\theta]}}{L(wv)} \left[ a_{\theta^{-1}(a)} \right]_{x \theta y^{-1} \Gamma} \ast 1_{\mathfrak{s}(x) \gamma w^{-1} \Gamma} \ast \Gamma w v \Gamma \ast 1_{\mathfrak{s}(x) \gamma y \Gamma} =
\]

\[
\frac{N_{w,v}^{[\theta]}}{L(wv)} \left[ a_{w^{-1} \gamma^{-1}(a)} \right]_{x \gamma w^{-1} \Gamma} \ast \Gamma w v \Gamma \ast 1_{\mathfrak{s}(x) \gamma y \Gamma}.
\]

Hence we obtain

\[
\Gamma g \Gamma \ast a_{x \Gamma} \ast \Gamma s \Gamma =
\]

\[
= \sum_{[\theta] \in \mathcal{S}_c \setminus \Gamma^{\mathcal{S}_c^{-1}}} \frac{N_{w,v}^{[\theta]}}{L(wv)} \left[ a_{w^{-1} \gamma^{-1}(a)} \right]_{x \gamma w^{-1} \Gamma} \ast \Gamma w v \Gamma \ast 1_{\mathfrak{s}(x) \gamma y \Gamma}.
\]

The remaining equalities in the statement of this corollary are proven in a similar fashion.

\[\square\]

3.2 Basic Examples

Example 3.20. We will now show that when \( \Gamma \) is a normal subgroup of \( G \) our notion of a crossed product by the Hecke pair \((G, \Gamma)\) is precisely the usual crossed product by the group \( G/\Gamma \). Normality of the subgroup \( \Gamma \) implies that the \( G \)-action \( \overline{\sigma} \) on \( M(C_c(A)) \) gives rise to an action of \( G/\Gamma \) on \( C_c(A/\Gamma) \). Moreover, we have \( \Gamma^g = \Gamma \) for all \( g \in G \), and it follows easily from the definitions that \( C_c(A/\Gamma) \times_{\text{alg}} G/\Gamma \) is nothing but the usual crossed product by the action of the group \( G/\Gamma \).

It is also interesting to observe that any usual crossed product \( C_c(B) \times_{\text{alg}} G/\Gamma \) coming from an action of the group \( G/\Gamma \) on a Fell bundle \( B \) over a groupoid \( Y \) is actually a crossed by the Hecke pair \((G, \Gamma)\) in our sense. To see this we note that the action of \( G/\Gamma \) on \( B \) lifts to an action of \( G \) on \( B \). In this lifted action the subgroup \( \Gamma \) acts trivially, so that the action is \( \Gamma \)-good. Moreover, since \( \Gamma \) is normal in \( G \), the \( \Gamma \)-intersection property is also trivially satisfied. It is clear that \( Y/\Gamma \) is just \( Y \) and \( B/\Gamma \) coincides with \( B \). Thus, forming the crossed product by the Hecke pair \((G, \Gamma)\) will give nothing but the usual crossed product by \( G/\Gamma \), i.e. \( C_c(B/\Gamma) \times_{\text{alg}} G/\Gamma \equiv C_c(B) \times_{\text{alg}} G/\Gamma \).
Example 3.21. We will now explain how the Hecke algebra $\mathcal{H}(G, \Gamma)$ is an example of a crossed product by a Hecke pair, namely $\mathcal{H}(G, \Gamma) \cong \mathbb{C} \times_{\text{alg}}^G G/\Gamma$, just like group algebras are examples of crossed products by groups.

We start with a groupoid $X$ consisting of only one element, i.e., $X = \{\ast\}$, and we take $\mathbb{C}$ as the Fell $\mathcal{A}$ bundle over $X$, i.e., $\mathcal{A}_\ast = \mathbb{C}$. We take also the trivial $G$-action on $\mathcal{A}$. Since the $G$-action fixes every element of $\mathcal{A}$, it is indeed $\Gamma$-good and in this case we have $X/\Gamma = X = \{\ast\}$. For the orbit bundle we have that $\mathcal{A}/\Gamma = \mathcal{A}$, and moreover

$$C_c(\mathcal{A}/\Gamma) \cong C_c(X/\Gamma) \cong C_c(X) \cong \mathbb{C}.$$ 

Hence, we are in the conditions of the Standing Assumption 3.11 and we can form the crossed product $C_c(\mathcal{A}/\Gamma) \times_{\text{alg}}^G G/\Gamma$, which we will simply write as $\mathbb{C} \times_{\text{alg}}^G G/\Gamma$.

Since $\mathbb{C}$ is unital the definitions of $B(\mathcal{A}, G, \Gamma)$ and $\mathbb{C} \times_{\text{alg}}^G G/\Gamma$ coincide in this case. Moreover Definition 3.4 reads that $\mathbb{C} \times_{\text{alg}}^G G/\Gamma$ is the set of functions $f : G/\Gamma \to \mathbb{C}$ satisfying the compatibility condition (40). Since the action $\overline{\sigma}$ is trivial, the compatibility condition simply says that $\mathbb{C} \times_{\text{alg}}^G G/\Gamma$ consists of all the functions $f : G/\Gamma \to \mathbb{C}$ which are left $\Gamma$-invariant. Moreover, the product and involution expressions become respectively

$$(f_1 \ast f_2)(g\Gamma) := \sum_{[h] \in G/\Gamma} f_1(h\Gamma) f_2(h^{-1}g\Gamma),$$

$$(f^\ast)(g\Gamma) := \Delta(g^{-1}) f(g^{-1}\Gamma).$$

Hence, it is clear that $\mathbb{C} \times_{\text{alg}}^G G/\Gamma$ is nothing but the Hecke algebra $\mathcal{H}(G, \Gamma)$.

It follows from this that the product $\Gamma g\Gamma \ast 1_{\ast \Gamma} \ast \Gamma s\Gamma$ is just the product of the double cosets $\Gamma g\Gamma$ and $\Gamma s\Gamma$ inside the Hecke algebra, since $1_{\ast \Gamma}$ is the identity element. It is interesting to note in this regard that the expression for this product described in Theorem 3.17 is a familiar expression for the product $\Gamma g\Gamma \ast \Gamma s\Gamma$ in $\mathcal{H}(G, \Gamma)$. To see this, we note that the stabilizer $S_e$ of $\ast$ is the whole group $G$, and therefore $E_{u,v}$ consists only of the class $[e]$. Moreover, the numbers $n_{u-1,v}$ and $d_{u-1,v}'$ defined in (51) and (52), are equal, so that $N_{u-1,v} = 1$. Thus, the expression described in Theorem 3.17 is just the usual expression

$$\Gamma g\Gamma \ast \Gamma s\Gamma = \sum_{[u] \in \Gamma g^{-1}\Gamma/\Gamma} \frac{\Delta(g)}{L(u^{-1}v)} \Gamma u^{-1}v\Gamma.$$ 

Example 3.22. As a generalization of Example 3.21 we will now show that if the $G$-action fixes every element of the bundle $\mathcal{A}$, then $C_c(\mathcal{A}/\Gamma) \times_{\text{alg}}^G G/\Gamma$ is isomorphic to the $^\ast$-algebraic tensor product of $C_c(\mathcal{A}/\Gamma)$ and $\mathcal{H}(G, \Gamma)$. This result also has a known analogue in the theory of crossed products by groups.

Proposition 3.23. If the $G$-action fixes every element of $\mathcal{A}$, then we have

$$C_c(\mathcal{A}/\Gamma) \times_{\text{alg}}^G G/\Gamma \cong C_c(\mathcal{A}/\Gamma) \odot \mathcal{H}(G, \Gamma),$$

where $\odot$ is the symbol that denotes the $^\ast$-algebraic tensor product.
Proof: Given that we have canonical embeddings of $C_c(A/\Gamma)$ and $\mathcal{H}(G, \Gamma)$ into $M(C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma)$ we have a canonical linear map from $C_c(A/\Gamma) \otimes \mathcal{H}(G, \Gamma)$ to $M(C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma)$ determined by

$$f_1 \otimes f_2 \mapsto f_1 * f_2,$$

(56)

where $f_1 \in C_c(A/\Gamma)$ and $f_2 \in \mathcal{H}(G, \Gamma)$. Standard arguments can be used to show that this mapping is injective (since the mappings from both $C_c(A/\Gamma)$ and $\mathcal{H}(G, \Gamma)$ into the multiplier algebra of the crossed product are injections). It is also clear that the image of the map determined by (56) is contained in $C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$. Let us now check that this mapping is surjective. First we will show that the elements of $C_c(A/\Gamma)$ commute with elements of $\mathcal{H}(G, \Gamma)$ inside $M(C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma)$. It follows from expressions (17) and (18) that

$$[a]_{x\Gamma} * \Gamma g \Gamma = \sum_{[\gamma] \in S_x \setminus \Gamma / \Gamma^g} [a]_{x\Gamma} * \Gamma g \Gamma * 1_{s(x)g\Gamma}$$

$$= \sum_{[\gamma] \in S_x \setminus \Gamma / \Gamma^g} 1_{r(x)\Gamma} * \Gamma g \Gamma * [\alpha_{g^{-1} \gamma^{-1}(a)}]_{x\gamma g\Gamma}.$$

Since every point of $X$ is fixed by the associated $G$-action on $X$, we have that $S_x = G$, and therefore $S_x \setminus \Gamma / \Gamma^g$ consists only of the class $[e]$, so that we can write

$$= 1_{r(x)\Gamma} * \Gamma g \Gamma * [\alpha_{g^{-1}(a)}]_{x\gamma g\Gamma}.$$

Moreover, since the $G$-actions on $A$ and $X$ are trivial we can furthermore write

$$= 1_{r(x)g^{-1} \Gamma} * \Gamma g \Gamma * [a]_{x\Gamma}.$$

Now, by the same reasoning as above and using expression (18) we have

$$= \sum_{[\gamma] \in S_x \setminus \Gamma / \Gamma^g} 1_{r(x)\Gamma g^{-1} \Gamma} * \Gamma g \Gamma * [a]_{x\Gamma}$$

$$= \Gamma g \Gamma * [a]_{x\Gamma}.$$

Thus we conclude that $[a]_{x\Gamma} * \Gamma g \Gamma = \Gamma g \Gamma * [a]_{x\Gamma}$. By Theorem 3.14 we know that elements of the form $[a]_{x\Gamma} * \Gamma g \Gamma * 1_{s(x)g\Gamma}$ span $C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$, and from the commutation relation we just proved it follows that

$$[a]_{x\Gamma} * \Gamma g \Gamma * 1_{s(x)g\Gamma} = \Gamma g \Gamma * [a]_{x\Gamma} * 1_{s(x)g\Gamma}$$

$$= \Gamma g \Gamma * [a]_{x\Gamma} * 1_{s(x)\Gamma}$$

$$= \Gamma g \Gamma * [a]_{x\Gamma}$$

$$= [a]_{x\Gamma} * \Gamma g \Gamma,$$

so that $C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ is spanned by elements of the form $a_{x\Gamma} * \Gamma g \Gamma$. We now conclude that the image of the map (56) is the whole $C_c(A/\Gamma) \times^{alg}_{\alpha} G/\Gamma$.

The fact that this map is a $^*$-homomorphism also follows directly from the commutation relation proved above. \qed
3.3 Representation theory

In this section we develop the representation theory of crossed products by Hecke pairs. We will introduce the notion of a covariant pre-representation and show that there is a bijective correspondence between covariant pre-representations and representations of the crossed product, in a similar fashion to the theory of crossed products by groups.

Recall from Proposition 1.21 that every nondegenerate *-representation \( \pi : C_c(A/\Gamma) \to B(\mathcal{H}) \) extends uniquely to a *-representation

\[
\tilde{\pi} : M_B(C_c(A/\Gamma)) \to B(\mathcal{H}).
\]

We will use the notation \( \tilde{\pi} \) to denote this extension throughout this section, many times without any reference. Since \( C_c(X^0/\Gamma) \) is a \( BG^* \)-algebra (it is spanned by projections) we naturally have \( C_c(X^0/\Gamma) \subseteq M_B(C_c(A/\Gamma)) \).

Definition 3.24. Let \( \pi \) be a nondegenerate *-representation of \( C_c(A/\Gamma) \) on a Hilbert space \( \mathcal{H} \) and \( \tilde{\pi} \) its unique extension to a *-representation of \( M_B(C_c(A/\Gamma)) \). Let \( \mu \) be a unital pre-*-representation of \( H(G, \Gamma) \) on the inner product space \( \mathcal{W} := \pi(C_c(A/\Gamma)) \mathcal{H} \). We say that \((\pi, \mu)\) is a covariant pre-*-representation if the following equality

\[
\mu(\Gamma g)\pi([a]_{s\Gamma})\mu(\Gamma s\Gamma) = \sum_{[u] \in \Gamma g^{-1} \Gamma/\Gamma} \sum_{[a] \in \mathcal{E}_c(\gamma)} \frac{\Delta(g) N^{\gamma}(x, y)}{L(u^{-1} v)} \tilde{\pi}([\alpha_{u^{-1} \gamma^{-1}(a)]x\gamma u\Gamma}) \mu(\Gamma u^{-1} v) \tilde{\pi}(1_{s(x)} \gamma v \Gamma),
\]  

holds on \( L(\mathcal{W}) \), for all \( g, s \in G \) and \( x \in X \).

Condition (57) simply says that the pair \((\pi, \mu)\) must preserve the structure of products of the form \( \Gamma g \Gamma \star [a]_{s\Gamma} \star \Gamma s\Gamma \), when expressed in terms of the canonical spanning set of elements of the form \([b]_{s\Gamma} \star \Gamma d\Gamma \star 1_{s(y)}d\Gamma\), as explicitly described in Corollary 3.19.

The reader should note the similarity between our definition of a covariant pre-*-representation and the covariant pairs of an Hufn, Kaszieskiw and Raceburn in [7] Definition 1.1). Their notion of covariant pairs served as a motivation for us and is actually a particular case of our Definition 3.24 as we shall see in the sequel to this article.

The operators \( \tilde{\pi}([a_{u^{-1} \gamma^{-1}(a)]x\gamma u\Gamma}) \mu(\Gamma u^{-1} v) \tilde{\pi}(1_{s(x)} \gamma v \Gamma) \) in expression (57) are all bounded, as we will now show, and are therefore defined in the whole Hilbert space \( \mathcal{H} \).

Theorem 3.25. Let \( \pi : C_c(A/\Gamma) \to B(\mathcal{H}) \) be a nondegenerate *-representation and \( \mu : H(G, \Gamma) \to L(\mathcal{W}) \) a pre-*-representation on the inner product space \( \mathcal{W} := \pi(C_c(A/\Gamma)) \). Every element of the form

\[
\pi([a]_{s\Gamma})\mu(\Gamma g\Gamma)\tilde{\pi}(1_{s(x)} \gamma g \Gamma),
\]

is a bounded operator on \( \mathcal{W} \) and therefore extends uniquely to the whole Hilbert space \( \mathcal{H} \).

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We will need some preliminary facts and lemmas in order to prove Theorem 3.25. These auxiliary results will also be useful later in this section.

Let \( \pi : C_c(\mathcal{A}/\Gamma) \to B(\mathcal{H}) \) be a nondegenerate \( \ast \)-representation and \( \tilde{\pi} \) its extension to \( M_B(C_c(\mathcal{A}/\Gamma)) \). For any unit \( u \in X^0 \) the operator \( \tilde{\pi}(1_{u\Gamma}) \in B(\mathcal{H}) \) is a projection, and therefore \( \tilde{\pi}(1_{u\Gamma}) \mathcal{H} \) is a Hilbert subspace. The fiber \( (\mathcal{A}/\Gamma)_{u\Gamma} \) is a \( C^\ast \)-algebra which we can naturally identify with the \( \ast \)-subalgebra

\[
\{ [a]_{u\Gamma} \in C_c(\mathcal{A}/\Gamma) : [a] \in (\mathcal{A}/\Gamma)_{u\Gamma} \} \subseteq C_c(\mathcal{A}/\Gamma),
\]

under the identification given by

\[
(\mathcal{A}/\Gamma)_{u\Gamma} \ni [a] \mapsto [a]_{u\Gamma} \in C_c(\mathcal{A}/\Gamma).
\]

The \( \ast \)-representation \( \tilde{\pi} \) when restricted to \( (\mathcal{A}/\Gamma)_{u\Gamma} \), under the above identification, leaves the subspace \( \tilde{\pi}(1_{u\Gamma}) \mathcal{H} \) invariant, because

\[
\tilde{\pi}([a]_{u\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi = \tilde{\pi}([a]_{u\Gamma}) \xi = \tilde{\pi}(1_{u\Gamma}) \tilde{\pi}([a]_{u\Gamma}) \xi.
\]

The following lemma assures that this restriction is nondegenerate.

**Lemma 3.26.** Let \( \pi : C_c(\mathcal{A}/\Gamma) \to B(\mathcal{H}) \) be a nondegenerate \( \ast \)-representation and \( \tilde{\pi} \) its unique extension to \( M_B(C_c(\mathcal{A}/\Gamma)) \). The \( \ast \)-representation of \( (\mathcal{A}/\Gamma)_{u\Gamma} \) on the Hilbert space \( \tilde{\pi}(1_{u\Gamma}) \mathcal{H} \), as above, is nondegenerate.

**Proof:** Let \( \tilde{\pi}(1_{u\Gamma}) \xi \) be an element of \( \tilde{\pi}(1_{u\Gamma}) \mathcal{H} \) such that

\[
\tilde{\pi}([a]_{u\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi = 0,
\]

for all \([a] \in (\mathcal{A}/\Gamma)_{u\Gamma}\). We want to prove that \( \tilde{\pi}(1_{u\Gamma}) \xi = 0 \). To see this, let \( x \in X \) and \([b] \in (\mathcal{A}/\Gamma)_{x\Gamma}\). We have two alternatives: either \( s(x) \Gamma \neq u \Gamma \) or \( s(x) \Gamma = u \Gamma \). In the first case we see that

\[
\tilde{\pi}([b]_{x\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi = \tilde{\pi}([b]_{x\Gamma} \cdot 1_{u\Gamma}) \xi = 0,
\]

whereas for the second we see that

\[
\| \tilde{\pi}([b]_{x\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi \|^2 = \langle \tilde{\pi}([b]_{x\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi, \tilde{\pi}([b]_{x\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi \rangle = \langle \tilde{\pi}([b^* b]_{s(x)\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi, \tilde{\pi}(1_{u\Gamma}) \xi \rangle = \langle \tilde{\pi}([b^* b]_{u\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi, \tilde{\pi}(1_{u\Gamma}) \xi \rangle = 0,
\]

by assumption. Thus, in any case we have \( \tilde{\pi}([b]_{x\Gamma}) \tilde{\pi}(1_{u\Gamma}) \xi = 0 \) for all \( x \in X \) and \([b] \in (\mathcal{A}/\Gamma)_{x\Gamma}\). By nondegeneracy of \( \pi \), this implies that \( \tilde{\pi}(1_{u\Gamma}) \xi = 0 \), as we wanted to prove. \( \square \)

**Lemma 3.27.** Let \( \pi \) be a nondegenerate \( \ast \)-representation of \( C_c(\mathcal{A}/\Gamma) \) on a Hilbert space \( \mathcal{H} \). We have that \( \pi(C_c(\mathcal{A}/\Gamma)) \mathcal{H} = \tilde{\pi}(C_c(X^0/\Gamma)) \mathcal{H} \).
Proof: It is clear that $\pi(C_c(A/\Gamma))\mathcal{H} \subseteq \tilde{\pi}(C_c(X^0/\Gamma))\mathcal{H}$ since for any element of the form $[a]_{x\Gamma}$ in $C_c(A/\Gamma)$ and $\xi \in \mathcal{H}$ we have $\pi([a]_{x\Gamma})\xi = \pi(1_{\Gamma x})[a]_{x\Gamma}\xi = \tilde{\pi}(1_{\Gamma x})\xi$.

Let us now prove that $\tilde{\pi}(C_c(X^0/\Gamma))\mathcal{H} \subseteq \pi(C_c(A/\Gamma))\mathcal{H}$. Let $u\Gamma \in X^0/\Gamma$ and $\xi \in \mathcal{H}$. We know, by Lemma 3.26, that $\pi$ gives a nondegenerate $\ast$-representation of $(A/\Gamma)_{u\Gamma}$ on $\tilde{\pi}(1_{u\Gamma})\mathcal{H}$. Since $(A/\Gamma)_{u\Gamma}$ is a $C^\ast$-algebra we have, by the general version of Cohen’s factorization theorem ([15, Theorem 5.2.2]), that there exists $[c] \in (A/\Gamma)_{u\Gamma}$ and $\eta \in \tilde{\pi}(1_{u\Gamma})\mathcal{H}$ such that

$$\tilde{\pi}(1_{u\Gamma})\xi = \pi([c]_{u\Gamma})\eta,$$

which means that $\tilde{\pi}(1_{u\Gamma})\xi \in \pi(C_c(A/\Gamma))\mathcal{H}$. This finishes the proof.

Proof of Theorem 3.25: The operator $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\tilde{\pi}(1_{s(x)g\Gamma})$ is clearly defined on the inner product space $\pi(C_c(A/\Gamma))\mathcal{H}$. By Lemma 3.27 this operator is then defined on the space $\tilde{\pi}(C_c(X^0/\Gamma))\mathcal{H}$. Since

$$\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\tilde{\pi}(1_{s(x)g\Gamma}) = \pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\tilde{\pi}(1_{s(x)g\Gamma}) \tilde{\pi}(1_{s(x)g\Gamma}),$$

it follows that the operator $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\tilde{\pi}(1_{s(x)g\Gamma})$ is actually defined in the whole Hilbert space $\mathcal{H}$ (or in other words, it extends canonically to $\mathcal{H}$).

A similar argument shows that $\tilde{\pi}(1_{s(x)g\Gamma})\mu((\Gamma g\Gamma)^\ast)x_{x^{-1}\Gamma}$ is also defined in the whole Hilbert space $\mathcal{H}$ and it is easy to see that $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\tilde{\pi}(1_{s(x)g\Gamma})$ is an adjointable operator on $\mathcal{H}$, whose adjoint is $\tilde{\pi}(1_{s(x)g\Gamma})\mu((\Gamma g\Gamma)^\ast)x_{x^{-1}\Gamma}$. Since adjointable operators on a Hilbert space are necessarily bounded (see [16, Proposition 9.1.11]), it follows that $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\tilde{\pi}(1_{s(x)g\Gamma})$ is a bounded operator.

The striking feature that we actually have to consider pre-representations of $\mathcal{H}(G, \Gamma)$, and not just representations, was not present in the theory of crossed products by groups because a group algebra $C(G)$ of a discrete group is always a $BG^\ast$-algebra and therefore all of its pre-representations come from true representations (see further Remark 3.31).

It will be useful to distinguish between covariant pre-$\ast$-representations and covariant $\ast$-representations, so we will treat them in separate definitions. As will be discussed below we will see covariant $\ast$-representations as a particular type of covariant pre-$\ast$-representations.

Definition 3.28. Let $\pi$ be a nondegenerate $\ast$-representation of $C_c(A/\Gamma)$ on a Hilbert space $\mathcal{H}$ and $\mu$ a unital $\ast$-representation of $\mathcal{H}(G, \Gamma)$ on $\mathcal{H}$. We say that $(\pi, \mu)$ is a covariant $\ast$-representation if equality (57) holds in $B(\mathcal{H})$ for all $g, s \in G$ and $x \in X$.

Lemma 3.29. Let $(\pi, \mu)$ be a covariant $\ast$-representation on a Hilbert space $\mathcal{H}$. Then $\mu$ leaves the subspace $\mathcal{W} := \pi(C_c(A/\Gamma))\mathcal{H}$ invariant.

Proof: Consider elements of the form $\pi([a]_{x\Gamma})\xi$, whose span gives $\mathcal{W}$. Using
the fact that $\mu$ is unital and the covariance relation (57) we see that
\[
\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\xi = \\
= \mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma)\xi \\
= \sum_{[a]_{x\Gamma}} \sum_{\gamma \in \mathbb{E}_{\alpha,\gamma}(x)} \frac{\Delta(g)\Lambda_{\alpha_1\gamma_1}(x)}{L(u)} \tilde{\pi}([a_{\mu^{-1}\gamma_1}(a)]_{x\gamma x\Gamma}) \mu(\Gamma u^{-1}\Gamma) \tilde{\pi}(1_{s(x)\gamma\Gamma})\xi.
\]
Hence, $\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\xi \in \mathbb{W}$, and consequently $\mu(\Gamma g\Gamma)$ leaves $\mathbb{W}$ invariant. This finishes the proof. \qed

From a covariant $^*$-representation $(\pi, \mu)$ one can obtain canonically a covariant pre-$^*$-representation $(\pi, \mu)$, just by restricting $\mu$ to the dense subspace $\mathbb{W} := \pi(C_c(\mathcal{A}/G))\mathbb{H}$ (which is an invariant subspace by Lemma 3.29). So we can regard covariant $^*$-representations as a special kind of covariant pre-$^*$-representations: they are exactly those for which $\mu$ is normed. As we shall see later in Example 3.41 there are covariant pre-$^*$-representations which are not covariant $^*$-representations, thus in general the latter form a proper subclass of the former. We shall also see examples where they actually coincide.

Remark 3.30. Equivalently, one could define covariant (pre-)$^*$-representation using any other of the equalities in Corollary 3.19 and substituting with the appropriate (pre-)$^*$-representations. It is easy to see, using completely analogous arguments as in the proof of Corollary 3.19 or Theorem 3.17 that all three expressions yield the same result.

Remark 3.31. Even though it might not be entirely clear from the start, when $\Gamma$ is a normal subgroup of $G$ the definition of a covariant pre-representation is nothing but the usual definition of covariant representation of the system $(C_c(\mathcal{A}/G), G/\Gamma)$. We recall that a covariant representation of $(C_c(\mathcal{A}/G), G/\Gamma)$ is a pair $(\pi, U)$ consisting of a nondegenerate $^*$-representation $\pi$ of $C_c(\mathcal{A}/\Gamma)$ and a unitary representation $U$ of $G/\Gamma$ satisfying the relation
\[
\pi(\mathcal{A}/\Gamma)(f) = U_\Gamma\pi(f)U^{-1}_\Gamma,
\]
for all $f \in C_c(\mathcal{A}/G)$ and $g\Gamma \in G/\Gamma$. Now, as it is well known, every unitary representation $U$ of $G/\Gamma$ is associated in a canonical way to a unital $^*$-representation $\mu$ of the group algebra $\mathbb{C}(G/\Gamma)$, so that we can write the covariance condition as $\pi(\mathcal{A}/\Gamma)(f) = \mu(g\Gamma)\pi(f)\mu(g^{-1}\Gamma)$. As a consequence we have that for any $g\Gamma, s\Gamma \in G/\Gamma$, $x \in X$ and $a \in \mathcal{A}_x$:
\[
\mu(g\Gamma)\pi([a]_{x\Gamma})\mu(s\Gamma) = \pi([a_{gs^{-1}}(a)]_{xg^{-1}s\Gamma})\mu(g^{-1}s\Gamma).
\]
We want to check that covariant representations of the system $(C_c(\mathcal{A}/G), G/\Gamma)$ are the same as covariant pre-$^*$-representations as in Definition 3.24.

Given a covariant pre-$^*$-representation $(\pi, \mu)$ on some Hilbert space $\mathbb{H}$ in the sense of Definition 3.24 we have that $\mu$ is a pre-$^*$-representation of $\mathbb{C}(G/\Gamma)$, which is normed since any group algebra of a discrete group is a $BG^*$-algebra,
and thus we can see $\mu$ as a true $^\ast$-representation on $A$. We then have that

$$
\mu(g\Gamma)\pi([a]_{x\Gamma})\mu(g^{-1}\Gamma) = \\
\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma g^{-1}\Gamma) = \\
\sum_{[u] \in \Gamma^{-1}g\Gamma} \sum_{[\gamma] \in E_{g^{-1}}(x)} \frac{\Delta(g)}{L(u^{-1}v)} \tilde{\pi}([a_{u^{-1}g\gamma-1}], \mu(\Gamma u^{-1}e\Gamma) \tilde{\pi}(1_{s(x)g\gamma\Gamma})
$$

$$
= \sum_{[\gamma] \in E_{g^{-1}}(x)} N_{g^{-1}}(s(\gamma)) \tilde{\pi}([\alpha_g(a)]_{xg^{-1}\Gamma}) \mu(\Gamma(gg^{-1}\Gamma) \tilde{\pi}(1_{s(x)g^{-1}\Gamma})
$$

$$
= \sum_{[\gamma] \in E_{g^{-1}}(x)} N_{g^{-1}}(s(\gamma)) \tilde{\pi}([\alpha_g(a)]_{xg^{-1}\Gamma}) \pi(1_{s(x)g^{-1}\Gamma})
$$

It is clear from the normality of $\Gamma$ that $E_{g^{-1}}(x)_{g^{-1}}$ consists only of the class $[c]$ and moreover $N_{g^{-1}}(s(x))_{g^{-1}} = 1$, so that

$$
\mu(g\Gamma)\pi([a]_{x\Gamma})\mu(g^{-1}\Gamma) = \pi([\alpha_g(a)]_{xg^{-1}\Gamma}).
$$

By linearity it follows that $\mu(g\Gamma)\pi(f)\mu(g^{-1}\Gamma) = \pi(M_{g\Gamma}(f))$ for any $f \in C_c(A, G/\Gamma)$. Thus, with $U$ being the unitary representation of $G/\Gamma$ associated to $\pi$, we see that $(\pi, U)$ is covariant representation of the system $(C_c(A, G/\Gamma), G/\Gamma)$.

For the other direction, let $(\pi, U)$ be a covariant representation of the system $(C_c(A, G/\Gamma), G/\Gamma)$ and let $\mu$ be the $^\ast$-representation of $C_c(A, G/\Gamma)$ associated to $U$, which we restrict to the inner product space $\pi(C_c(A, G/\Gamma))$. We want to prove that $(\pi, \mu)$ is a covariant pre-$^\ast$-representation in the sense of Definition 3.24. We have

$$
\mu(g\Gamma)\pi([a]_{x\Gamma})\mu(g^{-1}\Gamma) = \\
\pi([\alpha_g(a)]_{xg^{-1}\Gamma}) \mu(gsG)
$$

$$
= \sum_{[u] \in \Gamma^{-1}g\Gamma} \sum_{[\gamma] \in E_{g^{-1}}(x)} \frac{\Delta(g)}{L(u^{-1}v)} \tilde{\pi}([a_{u^{-1}g\gamma-1}], \mu(\Gamma u^{-1}e\Gamma) \tilde{\pi}(1_{s(x)g\gamma\Gamma}),
$$

where the last equality is obtained following analogous computations as those above. Thus, $(\pi, \mu)$ is a covariant pre-$^\ast$-representation in the sense of Definition 3.24.

The following result makes it clear that some of the relations we have inside the crossed product (see Proposition 3.10) are preserved upon taking covariant pre-$^\ast$-representations. This is expected since, as we stated before, we will prove that covariant pre-representations give rise to representations of the crossed product, and this result is the first step in that direction:
Proposition 3.32. Let \((\pi, \mu)\) be a covariant pre-\(^{-1}\)-representation. The following two equalities hold:

\[
\pi(1_{\tau(x)\Gamma})\mu(\Gamma g\Gamma)\pi([a_{\gamma^{-1}}(a)]_{xg\Gamma}) = \pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\pi(1_{\kappa(x)\Gamma}). \tag{58}
\]

\[
\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma}) = \sum_{\gamma \in E_{g^{-1}, e}^{(x)}} \pi(1_{\tau(x)\gamma_{g^{-1}}\Gamma})\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma}). \tag{59}
\]

**Proof:** Since \((\pi, \mu)\) is a covariant pre-\(^{-1}\)-representation we have

\[
\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma}) = \mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma) = \sum_{\gamma \in E_{g^{-1}, e}^{(x)}} N_{g, e}^{(x)} \pi([a_{\gamma^{-1}}(a)]_{x\gamma_{g^{-1}}\Gamma})\mu(\Gamma g\Gamma)\pi(1_{\kappa(x)\Gamma})
\]

where the last equality comes from the fact that \(r_{g, e}^{(x)} \gamma = 1 = d_{g, e}^{(x)} \gamma\), and thus \(N_{g, e}^{(x)} \gamma = 1\). From this it follows that

\[
\pi(1_{\tau(x)g^{-1}\Gamma})\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma}) = \\
= \sum_{\gamma \in E_{g^{-1}, e}^{(x)}} \pi(1_{\tau(x)g^{-1}\Gamma})\pi([a_{\gamma^{-1}}(a)]_{x\gamma_{g^{-1}}\Gamma})\mu(\Gamma g\Gamma)\pi(1_{\kappa(x)\Gamma})
\]

Now the product \(1_{\tau(x)g^{-1}\Gamma} \cdot [a_{\gamma^{-1}}(a)]_{x\gamma_{g^{-1}}\Gamma}\) is nonzero only when \(r(x)g^{-1}\Gamma = r(x)\gamma_{g^{-1}}\Gamma\), from which one readily concludes that \(r(x)\gamma \in r(x)g^{-1}\Gamma\). Since one trivially has \(r(x)\gamma \in r(x)\Gamma\) we conclude that

\[
r(x)\gamma \in r(x)\Gamma \cap r(x)g^{-1}\Gamma,
\]

and by the \(\Gamma\)-intersection property we have \(r(x)\gamma \in r(x)\Gamma^{g^{-1}}\). From Proposition 1.23 this means that \([\gamma] = [e]\) in \(E_{g^{-1}, e}^{(x)}\). We recall that \(E_{g^{-1}, e}^{(x)} = S_{r(x)} \setminus \Gamma^{g^{-1}}\), and since \(\Gamma^{g^{-1}} \subseteq \Gamma\) we have by Proposition 1.22 that \([\gamma] \rightarrow [\gamma]\) defines a canonical bijection between \(E_{g^{-1}, e}^{(x)}\) and \((S_{r(x)} \cap \Gamma) \setminus \Gamma^{g^{-1}}\). Since the \(G\)-action is \(\Gamma\)-good we necessarily have \(S_{r(x)} \cap \Gamma = S_e \cap \Gamma = S_{r(x)} \cap \Gamma\), and therefore using Proposition 1.22 one more time we can say that \(E_{g^{-1}, e}^{(x)} = E_{g^{-1}, e}^{(x)}\). Hence, we can say that \([\gamma] = [e]\) in \(E_{g^{-1}, e}^{(x)}\). We conclude that

\[
\pi(1_{\tau(x)g^{-1}\Gamma})\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma}) = \pi(1_{\tau(x)g^{-1}\Gamma} \cdot [a_{\gamma}(a)]_{xg^{-1}\Gamma})\mu(\Gamma g\Gamma)\pi(1_{\kappa(x)\Gamma})
\]

\[
= \pi([a_{\gamma}(a)]_{xg^{-1}\Gamma})\mu(\Gamma g\Gamma)\pi(1_{\kappa(x)\Gamma}).
\]
Since the last expression is valid for any $x \in X$ and $[a] \in (\mathcal{A}/\Gamma)_{\Gamma \Gamma}$, if we take $x$ to be $xg$ and $[a]$ to be $[a_{g^{-1}}(a)]$ we obtain the desired equality \[58\]:
\[
\check{\pi}(1_{r(x)\Gamma})\mu(\Gamma g \Gamma)\check{\pi}([a_{g^{-1}}(a)]xg\Gamma) = \check{\pi}([a]_{x\Gamma})\mu(\Gamma g \Gamma)\check{\pi}(1_{s(x)g\Gamma}).
\]

Let us now prove equality \[59\]. Using the equality in beginning of this proof and equality \[58\] which we have just proven, we get precisely
\[
\mu(\Gamma g \Gamma)\check{\pi}([a]_{x\Gamma}) = \sum_{|\gamma| \in E_{\pi(\pi)}^{(x)}} \check{\pi}([a_{g\gamma^{-1}}(a)]x\gamma g\Gamma - 1 \Gamma) \mu(\Gamma g \Gamma)\check{\pi}(1_{s(x)\Gamma}g \Gamma)
\]
\[
= \sum_{|\gamma| \in E_{\pi(\pi)}^{(x)}} \check{\pi}(1_{r(x)\Gamma}g \Gamma - 1 \Gamma) \mu(\Gamma g \Gamma)\check{\pi}([a_{g^{-1}}(a)]x\Gamma)
\]
\[
= \sum_{|\gamma| \in E_{\pi(\pi)}^{(x)}} \check{\pi}(1_{r(x)\Gamma}g \Gamma - 1 \Gamma) \mu(\Gamma g \Gamma)\check{\pi}(a_{x})
\]

This finishes the proof.

The passage from a covariant pre-representation $(\pi, \mu)$ to a representation of $C_{c}(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is done via the so-called integrated form $\pi \times \mu$, which we now describe:

**Definition 3.33.** Let $(\pi, \mu)$ be a covariant pre-* -representation on a Hilbert space $\mathcal{H}$. We define the integrated form of $(\pi, \mu)$ as the function $\pi \times \mu : C_{c}(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma \rightarrow B(\mathcal{H})$ defined by
\[
[\pi \times \mu](f) := \sum_{[g] \in \mathcal{G} / G / \Gamma} \sum_{x \Gamma \gamma \in X / \Gamma} \check{\pi}\left([f(g \Gamma)(x)]_{x \Gamma}\right) \mu(\Gamma g \Gamma)\check{\pi}(1_{s(x)g \Gamma}).
\]

**Remark 3.34.** For $f$ of the form $f = a_{x \Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)g \Gamma}$ we have
\[
[\pi \times \mu](f) = \check{\pi}([a]_{x \Gamma}) \mu(\Gamma g \Gamma)\check{\pi}(1_{s(x)g \Gamma}).
\]
Moreover, from equality \[58\], for $f'$ of the form $f' = 1_{r(x)\Gamma} \ast \Gamma g \Gamma \ast [a_{g^{-1}}(a)]xg\Gamma$ we have
\[
[\pi \times \mu](f') = \check{\pi}(1_{r(x)\Gamma}) \mu(\Gamma g \Gamma)\check{\pi}([a_{g^{-1}}(a)]xg\Gamma).
\]

**Proposition 3.35.** The integrated form $\pi \times \mu$ of a covariant pre-* -representation $(\pi, \mu)$ is a well-defined nondegenerate * -representation.

**Proof:** First we need to check that the expression that defines $[\pi \times \mu](f)$ for a given $f \in C_{c}(\mathcal{A}/\Gamma) \times_{\alpha}^{alg} G/\Gamma$ is well-defined. This is proven in an entirely
analogous way as in the proof that the expression \( f_{(44)} \) in Proposition 3.14 is well-defined. Secondly, we need to show that \([\pi \times \mu](f)\) makes sense as an element of \( B(\mathcal{H})\). From Theorem 3.25 we have that

\[
\tilde{\pi}\left([f(g\Gamma)(x)]_{x\Gamma}\right)\mu(\Gamma g \Gamma)\tilde{\pi}(1_{s(x)\Gamma}) \in B(\mathcal{H}),
\]

thus, it follows that \([\pi \times \mu](f) \in B(\mathcal{H})\), and therefore \([\pi \times \mu](f)\) admits a unique extension to \( B(\mathcal{H})\).

Now, it is obvious that \( \pi \times \mu\) is a linear transformation. Let us check that it preserves the involution. It is then enough to check it for elements of the form \( f = [a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\Gamma} \). Since \((\pi, \mu)\) is a covariant pre-*-representation we have, by Propositions 3.32 and 3.10

\[
(\pi \times \mu)(f) = \Delta(g) \tilde{\pi}(1_{s(x)\Gamma}) \mu(\Gamma g^{-1} \Gamma) \tilde{\pi}([a^*]_{x^{-1}\Gamma})
\]

First, we notice that for the product \( f_1 \ast f_2 \) we will start by proving that

\[
[\pi \times \mu](f_1 \ast f_2) = [\pi \times \mu](f_1) \times [\pi \times \mu](f_2),
\]

for \( f_1 := [a]_{x\Gamma} \ast \Gamma g \Gamma \ast 1_{s(x)\Gamma} \) and \( f_2 := [b]_{y\Gamma} \ast \Gamma s \Gamma \ast 1_{s(y)\Gamma} \). Let us compute the expression on the left side of (60). First, we notice that for the product \( f_1 \ast f_2 \) to be non-zero one must have \( \text{r}(y) \in s(x)g\Gamma \), and in this case we obtain

\[
f_1 \ast f_2 = [a]_{x\Gamma} \ast \Gamma g \Gamma \ast [b]_{y\Gamma} \ast \Gamma s \Gamma \ast 1_{s(y)\Gamma},
\]

which by Corollary 3.19 gives

\[
= \sum_{[u] \in \Gamma^{-1} \Gamma / \Gamma} \Delta(g) N_{u^{-1}u}^{s(y)\gamma} L(u^{-1}v) [a]_{x\Gamma} \ast [\alpha_{u^{-1},\gamma,1}(b)]_{u^{-1}u\Gamma} \ast \Gamma u^{-1}v \Gamma \ast 1_{s(y)\gamma u\Gamma} \ast 1_{s(y)\Gamma}
\]

\[
= \sum_{[v] \in \Gamma^{-1} \Gamma / \Gamma} \Delta(g) N_{u^{-1}u}^{s(y)\gamma} L(u^{-1}v) [a]_{x\Gamma} \ast [\alpha_{u^{-1},\gamma,1}(b)]_{u^{-1}u\Gamma} \ast \Gamma u^{-1}v \Gamma \ast 1_{s(y)\gamma u\Gamma}
\]

\[
= \sum_{[v] \in \Gamma^{-1} \Gamma / \Gamma} \Delta(g) N_{u^{-1}u}^{s(y)\gamma} L(u^{-1}v) [a]_{x\Gamma} \ast [\alpha_{u^{-1},\gamma,1}(b)]_{u^{-1}u\Gamma} \ast \Gamma u^{-1}v \Gamma \ast 1_{s(y)\gamma u\Gamma}.
\]

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The product $[a]_x\Gamma * [\alpha_{u-1\gamma^{-1}1}(b)]_{y\gamma u\Gamma}$ is always either zero or of the form $[c]_{(x\theta)(y\gamma u)}\Gamma$, for some $\theta \in \Gamma$ and $c \in A_{(x\theta)(y\gamma u)}$. The point is that $s((x\theta)(y\gamma u)) = s(y\gamma u)$, so that each non-zero summand in the last sum above is actually of the form $[c]_{z\Gamma} \ast \Gamma d\Gamma \ast 1_{s(x)\delta\Gamma}$, for appropriate $[c] \in (A/\Gamma)_{z\Gamma}$, $z \in X$ and $d \in G$. Thus, by linearity of $\pi \times \mu$ and Remark 3.34 we obtain

$$[\pi \times \mu](f_1 * f_2) =$$

$$= \sum_{[u] \in \Gamma y^{-1}\Gamma / \Gamma} \Delta(g) N_{u^{-1}v}^{s(y)\gamma} \frac{\bar{\pi}([u]_{x\Gamma} \cdot [\alpha_{u^{-1}\gamma^{-1}1}(b)]_{y\gamma u\Gamma}) \mu(\Gamma u^{-1}v\Gamma) \bar{\pi}(1_{s(y)\gamma v\Gamma})}{L(u^{-1}v)}.$$

Let us now compute the expression on the right side of (60). We have

$$[\pi \times \mu](f_1) [\pi \times \mu](f_2) = \bar{\pi}([a]_{x\Gamma} \Gamma \Gamma) \bar{\pi}(1_{s(x)\delta\Gamma}) \bar{\pi}([b]_{y\gamma u\Gamma} \Gamma \Gamma) \bar{\pi}(1_{s(y)\gamma v\Gamma}).$$

For $1_{s(x)\delta\Gamma} \Gamma \Gamma \cdot [b]_{y\gamma u\Gamma}$ to be non-zero we must have $r(y) \in s(x)_{y\Gamma}$, and in this case we obtain, using the definition of a covariant pre-$\ast$-representation,

$$[\pi \times \mu](f_1) [\pi \times \mu](f_2) =$$

$$= \sum_{[u] \in \Gamma y^{-1}\Gamma / \Gamma} \Delta(g) N_{u^{-1}v}^{s(y)\gamma} \frac{\bar{\pi}([u]_{x\Gamma} \cdot [\alpha_{u^{-1}\gamma^{-1}1}(b)]_{y\gamma u\Gamma}) \mu(\Gamma u^{-1}v\Gamma) \bar{\pi}(1_{s(y)\gamma v\Gamma})}{L(u^{-1}v)}.$$

Hence, we have proven equality (60) for the special case of $f_1$ and $f_2$ being $f_1 := [a]_{x\Gamma} \Gamma \Gamma \cdot 1_{s(x)\delta\Gamma}$ and $f_2 := [b]_{y\gamma u\Gamma} \Gamma \Gamma \cdot 1_{s(y)\gamma v\Gamma}$. Using this we will now show that equality (60) holds for any $f_1, f_2 \in C_c(A/\Gamma) \times \alpha_{g\Gamma} G/\Gamma$. In fact, by Proposition 3.13 $f_1$ and $f_2$ can be written as sums

$$f_1 = \sum_i v_i, \quad f_2 = \sum_j w_j,$$

where each $v_i$ and $w_j$ is of the form $[a]_{x\Gamma} \Gamma \Gamma \cdot 1_{s(x)\delta\Gamma}$, for some $g\Gamma \in G/\Gamma$, $x \in X$ and $a \in A_x$. Since $\pi \times \mu$ is a linear mapping we have

$$[\pi \times \mu](f_1 * f_2) = [\pi \times \mu]\left(\left(\sum_i v_i\right) \ast \left(\sum_j w_j\right)\right)$$

$$= [\pi \times \mu]\left(\sum_{i,j} v_i \ast w_j\right)$$

$$= \sum_{i,j} [\pi \times \mu](v_i \ast w_j).$$
and by the special case of equality \([f_1 \ast f_2] = 0\) we have just proven we get
\[
[f_1 \ast f_2] = \sum_{i,j} [\pi \times \mu](v_i)[\pi \times \mu](w_j)
\]
\[
= \left( \sum_i [\pi \times \mu](v_i) \right) \left( \sum_j [\pi \times \mu](w_j) \right)
\]
\[
= [\pi \times \mu]\left( \sum_i v_i \right)[\pi \times \mu]\left( \sum_j w_j \right)
\]
\[
= [\pi \times \mu](f_1)[\pi \times \mu](f_2).
\]
Hence, \(\pi \times \mu\) is a *-representation. To finish the proof we now only need to show that \(\pi \times \mu\) is nondegenerate. The restriction of \(\pi \times \mu\) to the *-subalgebra \(C_c(A/\Gamma)\) is precisely the representation \(\pi\). Since \(\pi\) is assumed to be nondegenerate it follows that \(\pi \times \mu\) must be nondegenerate as well.

The next result shows how from a representation of the crossed product one can naturally form a covariant pre-representation.

**Proposition 3.36.** Let \(\Phi : C_c(A/\Gamma) \times_{\alpha}^\text{alg} G/\Gamma \to B(\mathcal{H})\) be a nondegenerate *-representation. Consider the pair \((\Phi|, \omega_\Phi)\) defined by

- \(\Phi| = \text{the restriction of } \Phi \text{ to } C_c(A/\Gamma)\).
- \(\Phi|\) be the extension of \(\Phi\) to a pre-*-representation (via Proposition 1.18) of \(M(C_c(A/\Gamma) \times_{\alpha}^\text{alg} G/\Gamma)\) on the inner product space \(\Phi(C_c(A/\Gamma) \times_{\alpha}^\text{alg} G/\Gamma) \mathcal{H}\). We define \(\omega_\Phi\) to be the restriction of \(\tilde{\Phi}\) to \(\mathcal{H}(G, \Gamma)\).

The pair \((\Phi|, \omega_\Phi)\) is a covariant pre-*-representation.

We will need some preliminary lemmas in order to prove Proposition 3.36.

**Lemma 3.37.** If \(\Phi : C_c(A/\Gamma) \times_{\alpha}^\text{alg} G/\Gamma \to B(\mathcal{H})\) is a nondegenerate *-representation, then its restriction to \(C_c(A/\Gamma)\) is also nondegenerate.

**Proof:** Let \(\xi \in \mathcal{H}\) be such that \(\Phi(C_c(A/\Gamma))\xi = \{0\}\). We want to show that \(\xi = 0\). Since \(\Phi\) is nondegenerate, it is then enough to prove that \(\Phi(C_c(A/\Gamma) \times_{\alpha}^\text{alg} G/\Gamma)\xi = \{0\}\). Thus, by virtue of Proposition 3.16 it suffices to prove that
\[
\Phi(1_{r(x)\Gamma} \ast \Gamma \alpha_{-1}(a)_{xg\Gamma})\xi = 0 \text{ for all } g \in G, x \in X, a \in A_x.
\]
We have
\[
\|\Phi(1_{r(x)\Gamma} \ast \Gamma \alpha_{-1}(a)_{xg\Gamma})\xi\|^2 = \Delta(g)\langle\Phi(1_{r(x)\Gamma} \ast \Gamma \alpha_{-1}(a)_{xg\Gamma})\xi, \xi\rangle = \Delta(g)\langle\Phi(1_{r(x)\Gamma} \ast \Gamma \alpha_{-1}(a)_{xg\Gamma})\xi, \xi\rangle = 0.
\]
Hence \(\xi = 0\) and therefore \(\Phi\) restricted to \(C_c(A/\Gamma)\) is nondegenerate.
Lemma 3.38. Let $\Phi : C_c(A/G) \times_{\alpha^g} G/G \to B(\mathcal{H})$ be a nondegenerate $^*$-representation and $\tilde{\Phi}$ its unique extension to $M_B(C_c(A/G) \times_{\alpha^g} G/G)$ (via Proposition [1.21]). Let $\Phi|$ be the restriction of $\Phi$ to $C_c(A/G)$ and $\tilde{\Phi}|$ its unique extension to $M_B(C_c(A/G))$. We have that

$$\tilde{\Phi}(f) = \tilde{\Phi}|(f),$$

for all $f \in C_c(X^0/G)$. In other words, the two $^*$-representations $\tilde{\Phi}$ and $\tilde{\Phi}|$ are the same in $C_c(X^0/G)$.

Proof: By Lemma [3.37] the subspace $\Phi(C_c(A/G))\mathcal{H}$ is dense in $\mathcal{H}$, so that it is enough to check that $\tilde{\Phi}(f)\tilde{\Phi}(f_2)\xi = \tilde{\Phi}|(f)\tilde{\Phi}|(f_2)\xi$, for all $f_2 \in C_c(A/G)$ and $\xi \in \mathcal{H}$. By definition of the extension $\tilde{\Phi}$ (see Proposition [1.21]) we have

$$\tilde{\Phi}(f)\tilde{\Phi}(f_2)\xi = \tilde{\Phi}|(f)\tilde{\Phi}|(f_2)\xi,$$

where $f \ast f_2$ is the product of $f$ and $f_2$, which lies inside $C_c(A/G) \times_{\alpha^g} G/G$. Since both $f$ and $f_2$ are elements of $B(A,G,G)$ we see the product $f \ast f_2$ as taking place in $B(A,G,G)$. By definition of the embeddings of $C_c(X^0/G)$ and $C_c(A/G)$ in $B(A,G,G)$ we have that $f \ast f_2$ is nothing but the element $f \cdot f_2$, where the product is just the product of $f$ and $f_2$ inside $M(C_c(A))$. As we observed in Section [2.3] this product is exactly same as the product of $f$ and $f_2$ in $M(C_c(A/G))$. Thus, the following computation makes sense:

$$\tilde{\Phi}(f)\tilde{\Phi}(f_2)\xi = \tilde{\Phi}|(f)\tilde{\Phi}|(f_2)\xi = \Phi(f \cdot f_2)\xi.$$

This finishes the proof. ⊓⊔

Lemma 3.39. Let $\Phi : C_c(A/G) \times_{\alpha^g} G/G \to B(\mathcal{H})$ be a nondegenerate $^*$-representation. We have that

$$\Phi(C_c(A/G))\mathcal{H} = \Phi(C_c(A/G) \times_{\alpha^g} G/G)\mathcal{H}.$$

Proof: The inclusion $\Phi(C_c(A/G))\mathcal{H} \subseteq \Phi(C_c(A/G) \times_{\alpha^g} G/G)\mathcal{H}$ is obvious. To check the converse inclusion it is enough to prove that

$$\Phi([a]_{A^G} \ast \Gamma g \Gamma * 1_{s(\alpha)G})\xi \in \Phi(C_c(A/G))\mathcal{H},$$

for all $x \in X$, $a \in A_x$, $g \in G$ and $\xi \in \mathcal{H}$. Let $\tilde{\Phi} : M_B(C_c(A/G) \times_{\alpha^g} G/G) \to B(\mathcal{H})$ be the unique extension of $\Phi$ to a $^*$-representation of $M_B(C_c(A/G) \times_{\alpha^g} G/G)$, as in Proposition [1.21]. We then get

$$\Phi([a]_{A^G} \ast \Gamma g \Gamma * 1_{s(\alpha)G})\xi = \Phi(1_{r(\alpha)G} * [a]_{A^G} \ast \Gamma g \Gamma * 1_{s(\alpha)G})\xi = \tilde{\Phi}(1_{r(\alpha)G})\Phi([a]_{A^G} \ast \Gamma g \Gamma * 1_{s(\alpha)G})\xi.$$

Denoting by $\Phi|$ the restriction of $\Phi$ to $C_c(A/G)$ we have, by Lemma 3.38 that

$$= \tilde{\Phi}|(1_{r(\alpha)G})\Phi([a]_{A^G} \ast \Gamma g \Gamma * 1_{s(\alpha)G})\xi,$$
Thus, it remains to be checked that
\[ \Phi([a]_{x\Gamma} \ast \Gamma \Gamma \ast 1_{x(x)_{2\Gamma}}) \xi \in \Phi((C_c(X/\Gamma)).H). \] By Lemma 3.27 it then follows that
\[ \Phi([a]_{x\Gamma} \ast \Gamma \Gamma \ast 1_{x(x)_{2\Gamma}}) \xi \in \Phi((C_c(A/\Gamma)).H). \]

**Proof of Proposition 3.36**: First of all, by Lemma 3.37 \( \Phi \) is indeed a nondegenerate *-representation of \( C_c(A/\Gamma) \). Secondly, from Lemma 3.39 we have
\[ \Phi(C_c(A/\Gamma)).H = \Phi(C_c(A/\Gamma) \times^a G/\Gamma).H. \]
Thus, \( \omega_\Phi \) is a pre-*-representation of \( H(G, \Gamma) \) on \( \mathcal{H} := \Phi(C_c(A/\Gamma)).H \). We now only need to check covariance. We have
\[
\begin{align*}
\omega_\Phi(\Gamma \Gamma \Phi)(([a]_{x\Gamma}) \omega_\Phi(\Gamma \Gamma) = \\
= \Phi(\Gamma \Gamma) \Phi([a]_{x\Gamma}) \Phi(\Gamma \Gamma) = \\
= \Phi(\Gamma \Gamma \ast [a]_{x\Gamma} \ast \Gamma \Gamma) = \\
= \tilde{\Phi} \left( \sum_{[u] \in \Gamma^{-1} \Gamma / \Gamma} \sum_{[v] \in \Gamma^{-1} \Gamma / \Gamma} \frac{\Delta(u) \Lambda(u \gamma v, v)}{L(u^{-1} v)} \left[ [a_{u^{-1} \gamma v, v}]_{x \Gamma} \ast \Gamma \Gamma \ast 1_{x(x)_{2\Gamma}} \right] \right),
\end{align*}
\]
Denoting by \( \tilde{\Phi} \) the unique extension of \( \Phi \) to \( M_B(C_c(A/\Gamma)) \) we have, by Lemma 3.38 that
\[
\begin{align*}
= \sum_{[u] \in \Gamma^{-1} \Gamma / \Gamma} \sum_{[v] \in \Gamma^{-1} \Gamma / \Gamma} \frac{\Delta(u) \Lambda(u \gamma v, v)}{L(u^{-1} v)} \tilde{\Phi}([a_{u^{-1} \gamma v, v}]_{x \Gamma} \ast \Gamma \Gamma \ast \Gamma \Gamma \ast 1_{x(x)_{2\Gamma}}) \omega_\Phi(\Gamma \Gamma \ast \Gamma \Gamma \ast 1_{x(x)_{2\Gamma}}). \end{align*}
\]
This finishes the proof.

**Theorem 3.40.** There is a bijective correspondence between nondegenerate *-representations of \( C_c(A/\Gamma) \times^a G/\Gamma \) and covariant pre-*-representations. This bijection is given by \( (\pi, \omega) \mapsto \pi \times \mu \), with inverse given by \( \Phi \mapsto (\Phi, \omega_\Phi) \).

**Proof:** We have to prove that the composition of these maps, in both orders, is the identity.

Let \( (\pi, \mu) \) be a covariant pre-*-representation and \( \pi \times \mu \) its integrated form. We want to show that
\[ (\pi \times \mu)|_\mathcal{H} = (\pi, \mu). \]
By definition of the integrated form we readily have \( (\pi \times \mu)|_\mathcal{H} = \pi \). This also implies, via Lemma 3.37 that the inner product spaces on which \( \mu \) and \( \omega_{\pi \times \mu} \) are defined are actually the same. Thus, it remains to be checked that \( \omega_{\pi \times \mu} = \mu \). Let \( \pi([a]_{x\Gamma}) \xi \) be one of the generators of \( \pi(C_c(A/\Gamma)).H \). We have
\[
\begin{align*}
\omega_{\pi \times \mu}(\Gamma \Gamma \pi)([a]_{x\Gamma}) \xi = \\
= \pi \times \mu(\Gamma \Gamma \pi)([a]_{x\Gamma}) \xi = \\
= \pi \times \mu(\Gamma \Gamma \pi \ast [a]_{x\Gamma}) \xi,
\end{align*}
\]
and using Proposition 3.16 Remark 3.34 and Proposition 3.32 we obtain
\[
\begin{align*}
&= [\pi \times \mu] \left( \sum_{[\gamma] \in E^{(\alpha)}_{g-1, a}} 1_{r(x)\gamma \Gamma} \star \Gamma g \Gamma \star [a]_{\gamma \Gamma} \right) \xi \\
&= \sum_{[\gamma] \in E^{(\alpha)}_{g-1, a}} \bar{\pi}(1_{r(x)\gamma \Gamma}) \mu(\Gamma g \Gamma) \bar{\pi}([a]_{\gamma \Gamma}) \xi \\
&= \mu(\Gamma g \Gamma) \pi([a]_{\gamma \Gamma}) \xi 
\end{align*}
\]
Hence, we conclude that \( \omega_{\pi \times \mu} = \mu \).

Now let \( \Phi \) be a \( ^* \)-representation of \( C_c(A/\Gamma) \times_{\alpha}^G G/\Gamma \) and \( (\Phi(1, \psi)) \) its corresponding covariant \( ^* \)-representation. We want to prove that
\[
\Phi(1, \psi) \times \omega_{\Phi} = \Phi.
\]
Let \( 1_{r(x)\Gamma} \star \Gamma g \Gamma \star [\alpha g^{-1}(a)]_{\gamma \Gamma} \) be one of the spanning elements of \( C_c(A/\Gamma) \times_{\alpha}^G G/\Gamma \) and \( \xi \in \mathcal{H} \). We have
\[
[\Phi(1, \psi) \times \omega_{\Phi}] (1_{r(x)\Gamma} \star \Gamma g \Gamma \star [\alpha g^{-1}(a)]_{\gamma \Gamma}) \xi = \bar{\Phi}(1_{r(x)\Gamma}) \omega_{\Phi}(\Gamma g \Gamma) \bar{\Phi}([\alpha g^{-1}(a)]_{\gamma \Gamma}) \xi,
\]
which by Lemma 3.33 gives that
\[
\begin{align*}
&= \bar{\Phi}(1_{r(x)\Gamma}) \bar{\Phi}(\Gamma g \Gamma) \bar{\Phi}([\alpha g^{-1}(a)]_{\gamma \Gamma}) \xi \\
&= \Phi(1_{r(x)\Gamma} \star \Gamma g \Gamma \star [\alpha g^{-1}(a)]_{\gamma \Gamma}) \xi.
\end{align*}
\]
Thus, \( \Phi(1, \psi) \times \omega_{\Phi} = \Phi. \) \( \square \)

### 3.4 More on covariant pre-\( ^* \)-representations

In the previous section we introduced the notion of covariant pre-\( ^* \)-representations of \( C_c(A/\Gamma) \times_{\alpha}^G G/\Gamma \) (Definition 3.24) and a particular instance of these which we called covariant \( ^* \)-representations (Definition 3.28).

In this section we will see that the class of covariant pre-\( ^* \)-representations is in general strictly larger than the class of covariant \( ^* \)-representations. It is thus unavoidable, in general, to consider pre-representations of the Hecke algebra in the representation theory of crossed products by Hecke pairs. We shall also see, nevertheless, that in many interesting situations every covariant pre-\( ^* \)-representation is actually a covariant \( ^* \)-representation.

**Example 3.41.** Let \((G, \Gamma)\) be a Hecke pair such that its corresponding Hecke algebra \( \mathcal{H}(G, \Gamma) \) does not have an enveloping \( C^* \)-algebra (it is well known that such pairs exist, as for example \((G, \Gamma) = (SL_2(Q), SL_2(Z_n))\) as discussed in [6]). The fact that the Hecke algebra does not have an enveloping \( C^* \)-algebra implies that there is a sequence of \( ^* \)-representations \( \{\mu_n\}_{n \in \mathbb{N}} \) of \( \mathcal{H}(G, \Gamma) \) on Hilbert spaces \( \mathcal{H}_n \) and an element \( f \in \mathcal{H}(G, \Gamma) \) such that \( \|\mu_n(f)\| \to \infty \).

Let \( \mathcal{Y} \) be the inner product space \( \mathcal{Y} := \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n \) and \( \mu : \mathcal{H}(G, \Gamma) \to L(\mathcal{Y}) \) the diagonal pre-\( ^* \)-representation
\[
\mu := \bigoplus_{n \in \mathbb{N}} \mu_n,
\]
which of course is not normed. Let \( X = \{x_1, x_2, \ldots \} \) be an infinite countable set, with the trivial groupoid structure, i.e. \( X \) is just a set. We consider the Fell bundle \( \mathcal{A} \) over \( X \) whose fibers are the complex numbers, i.e. \( \mathcal{A}_x = \mathbb{C} \) for every \( x \in X \), and we consider the trivial action of \( G \) on \( \mathcal{A} \), i.e. the action that fixes every element of \( \mathcal{A} \). Thus, the action is \( \Gamma \)-good and has the \( \Gamma \)-intersection property. We also have that

\[
C_c(\mathcal{A}/\Gamma) = C_c(X) = C_c(X^0/\Gamma).
\]

Let \( \pi : C_c(X) \to B(\mathcal{F}) \) be the \(*\)-representation on the Hilbert space completion \( \mathcal{F} \) of \( \mathcal{V} \) such that \( \pi(1_{x_n}) \) is the projection onto the subspace \( \mathcal{H}_{x_n} \).

We claim that \((\pi, \mu)\) is a covariant \(*\)-representation of \( C_c(X) \times^{alg}_G \mathcal{G}/\Gamma \). To see this, first we notice that \( \pi \) is obviously nondegenerate and moreover \( \pi(C_c(X))\mathcal{F} = \mathcal{V} \), which is the inner product space where \( \mu \) is defined. Next we notice that for every \( x_n \in X \) and \( g \in G \), the operators \( \pi(1_{x_n}) \) and \( \mu(\Gamma g \Gamma) \) commute. Moreover, we have

\[
\pi(1_{x_n})\mu(\Gamma g \Gamma)\pi(1_{x_n}) = \mu_n(\Gamma g \Gamma),
\]

on the subspace \( \mathcal{H}_{x_n} \). Also we have

\[
\begin{align*}
\mu(\Gamma g \Gamma)\pi(1_{x_n})\mu(\Gamma s \Gamma) & = \\
\mu(\Gamma g \Gamma)\mu(\Gamma s \Gamma)\pi(1_{x_n}) & = \\
\sum_{[u] \in \Gamma g^{-1} \Gamma/\Gamma} \frac{\Delta(g)}{L(u^{-1} v)} \mu(\Gamma u^{-1} v \Gamma) \pi(1_{x_n}) & = \\
\sum_{[u] \in \Gamma g^{-1} \Gamma/\Gamma} \frac{\Delta(g)}{L(u^{-1} v)} \pi(1_{x_n}) \mu(\Gamma u^{-1} v \Gamma) \pi(1_{x_n}) & = \\
\sum_{[u] \in \Gamma g^{-1} \Gamma/\Gamma} \sum_{[v] \in \Gamma x_n \Gamma} \frac{\Delta(g)}{L(u^{-1} v)} \frac{\Delta^2(v)}{L(u^{-1} v)} \pi(1_{x_n, \gamma u}) \mu(\Gamma u^{-1} v \Gamma) \pi(1_{x_n, \gamma v}),
\end{align*}
\]

where the last equality comes from the fact that since \( S_{x_n} = \mathcal{G} \) we must have that \( E_{u,v} \) consists only of the class \([e]\), \( N_{u,v}^{x_n} = 1 \) and also that \( x_n, \gamma u = x_n = 1_{x_n, \gamma v} \).

So we have established that \((\pi, \mu)\) is indeed a covariant \(*\)-representation. Nevertheless, \( \mu \) is not normed, so that \((\pi, \mu)\) is not a covariant \(*\)-representation.

It is worth noting that here we are in the conditions of Example 3.22 so that

\[
C_c(X) \times^{alg}_G \mathcal{G}/\Gamma \cong C_c(X) \otimes \mathcal{H}(G, \Gamma).
\]

Example 3.41 shows that there can be more covariant \(*\)-representations than covariant \(*\)-representations. Nevertheless, the two classes actually coincide in many cases. One such case is when \( C_c(\mathcal{A}/\Gamma) \) has an identity element:

**Proposition 3.42.** If the crossed product \( C_c(\mathcal{A}/\Gamma) \times^{alg}_G \mathcal{G}/\Gamma \) has an identity element (equivalently, if \( C_c(\mathcal{A}/\Gamma) \) has an identity element), then every covariant \(*\)-representation is a covariant \(*\)-representation.
Proof: Let us assume that $C_c(A/\Gamma) \times_{\alpha}^a G/\Gamma$ has an identity element (equivalently, $C_c(A/\Gamma)$ has an identity element).

Let $(\pi, \mu)$ be a covariant pre-$\ast$-representation. As it was shown in Theorem 3.40, the integrated form $\pi \times \mu$ is a $\ast$-representation of $C_c(A/\Gamma) \times_{\alpha}^a G/\Gamma$ such that $\mu = \omega_{\pi \times \mu}$, where $\omega_{\pi \times \mu}$ is the pre-$\ast$-representation which is obtained by extending $\pi \times \mu$ to the multiplier algebra $M(C_c(A/\Gamma) \times_{\alpha}^a G/\Gamma)$ and then restricting it to $H(G, \Gamma)$. Since the crossed product $C_c(A/\Gamma) \times_{\alpha}^a G/\Gamma$ has an identity element, we have

$$M(C_c(A/\Gamma) \times_{\alpha}^a G/\Gamma) = C_c(A/\Gamma) \times_{\alpha}^a G/\Gamma,$$

and therefore $\omega_{\pi \times \mu}$ is just the restriction of $\pi \times \mu$ to the Hecke algebra $H(G, \Gamma)$. Hence, $\mu = \omega_{\pi \times \mu}$ is a true $\ast$-representation.

Another interesting situation where covariant pre-$\ast$-representations coincide with covariant $\ast$-representations is when $H(G, \Gamma)$ is a $BG^\ast$-algebra. This is known to be the case for many classes of Hecke pairs $(G, \Gamma)$ as we proved in [14]. Actually, most of the classes of Hecke pairs for which a full Hecke $C^\ast$-algebra is known to exist are such that $H(G, \Gamma)$ is $BG^\ast$-algebra.

Proposition 3.43. If $H(G, \Gamma)$ is a $BG^\ast$-algebra, then every covariant pre-$\ast$-representation is a covariant $\ast$-representation.

Proof: If $H(G, \Gamma)$ is a $BG^\ast$-algebra, then every pre-$\ast$-representation of $H(G, \Gamma)$ is automatically normed and hence arises from a true $\ast$-representation.

3.5 Crossed product in the case of free actions

In this section we will see that when the associated $G$-action on $X$ is free the expressions for the products of the form $\Gamma g \Gamma \ast [a]_{x \Gamma} \ast \Gamma s \Gamma$, described in Corollary 3.19, as well as the definition of a covariant pre-$\ast$-representation become much simpler and even more similar to the notion of covariant pairs of [7].

Theorem 3.44. If the action of $G$ on $X$ is free, then

$$\Gamma g \Gamma \ast 1_{y \Gamma} \ast \Gamma s \Gamma = \sum_{[u] \in \Gamma y^{-1} \Gamma / \Gamma} 1_{yu^{-1} e \Gamma} \ast 1_{yu \Gamma} \tag{61}$$

and similarly,

$$\Gamma g \Gamma \ast [a]_{x \Gamma} \ast \Gamma s \Gamma = \sum_{[u] \in \Gamma y^{-1} \Gamma / \Gamma} [\alpha_{u^{-1}(a)}]_{xu \Gamma} \ast \Gamma u^{-1} e \Gamma \ast 1_{s(x) e \Gamma} \tag{62}.$$
Lemma 3.45. If the action of $G$ on $X$ is free, then

$$n^y_{w,v} = 1 \quad \text{and} \quad d^y_{w,v} = [\Gamma^{uw} : \Gamma^{uw} \cap w\Gamma w^{-1}] .$$

**Proof:** We have

$$n^y_{w,v} = \{ [r] \in \Gamma w\Gamma / \Gamma : r^{-1}wv \Gamma \subseteq \Gamma v \Gamma \text{ and } yw^{-1} \in y\Gamma r^{-1} \}$$

$$= \{ [r] \in \Gamma w\Gamma / \Gamma : r^{-1}wv \Gamma \subseteq \Gamma v \Gamma \text{ and } w^{-1} \in \Gamma r^{-1} \}$$

$$= \{ [r] \in \Gamma w\Gamma / \Gamma : r^{-1}wv \Gamma \subseteq \Gamma v \Gamma \text{ and } r\Gamma = w\Gamma \}$$

$$= \{ w\Gamma \} .$$

Thus, $n^y_{w,v} = 1$. Also,

$$\vartheta^y_{w,v} = \{ [r] \in \Gamma w\Gamma / \Gamma : r^{-1}wv \Gamma \subseteq \Gamma v \Gamma \text{ and } yw^{-1} \in y\Gamma r^{-1} \Gamma^{uw} \}$$

$$= \{ [r] \in \Gamma w\Gamma / \Gamma : w^{-1} \in \Gamma r^{-1} \Gamma^{uw} \} .$$

Now we notice that in the above set the condition $r^{-1}wv \Gamma \subseteq \Gamma v \Gamma$ is automatically satisfied from the second condition $w^{-1} \in \Gamma r^{-1} \Gamma^{uw}$, because the latter means that $r^{-1} = \theta_1 w^{-1} \theta_2$ for some $\theta_1 \in \Gamma$ and $\theta_2 \in \Gamma^{uw}$. Thus, we get

$$\vartheta^y_{w,v} = \{ [r] \in \Gamma w\Gamma / \Gamma : w^{-1} \in \Gamma r^{-1} \Gamma^{uw} \}$$

$$= \{ [r] \in \Gamma w\Gamma / \Gamma : r \in \Gamma^{uw} w\Gamma \}$$

$$= \Gamma^{uw} w\Gamma / \Gamma .$$

Thus, we obtain $d^y_{w,v} = [\Gamma^{uw} w\Gamma / \Gamma] = [\Gamma^{uw} : \Gamma^{uw} \cap w\Gamma w^{-1}] . \quad \square$

**Proof of Theorem 3.44:** We have seen in Theorem 3.17 that

$$\Gamma g\Gamma \ast 1_y \ast \Gamma s\Gamma = \sum_{[u] \in \Gamma_2 / \Gamma} \sum_{[\gamma] \in E_{u,v}} \Delta(g) N^y_{u^{-1},v} (1_y \gamma u \Gamma \ast \Gamma u^{-1} v \Gamma \ast 1_y v \Gamma)$$

It follows from Lemma 3.45 that

$$N^y_{u^{-1},v} = \frac{1}{[\Gamma^{u^{-1}v} : \Gamma^{u^{-1}v} \cap u^{-1} \Gamma u]} .$$

Moreover, freeness of the action also implies that

$$E^y_{u,v} = S_y \backslash (v\Gamma v^{-1} \cap u\Gamma u^{-1})$$

$$= \Gamma / (v\Gamma v^{-1} \cap u\Gamma u^{-1}) .$$

Now, we have the following well-defined bijective correspondence

$$\Gamma / (\Gamma^u \cap \Gamma^v) \longrightarrow \Gamma / (u\Gamma v^{-1} \cap u\Gamma u^{-1})$$

$$[\theta] \mapsto [\theta],$$

given by Proposition 1.22. Note that $\Gamma^u \cap \Gamma^v$ is simply the subgroup $u\Gamma u^{-1} \cap v\Gamma v^{-1} \cap \Gamma$, but in the following we will take preference on the notation $\Gamma^u \cap \Gamma^v$ for being shorter.
Consider now the action of $\Gamma$ on $G/\Gamma \times G/\Gamma$ by left multiplication and denote by $O_{h_1, h_2}$ the orbit of the element $(h_1 \Gamma, h_2 \Gamma) \in G/\Gamma \times G/\Gamma$. It is easy to see that the map

$$\Gamma/(\Gamma^{h_1} \cap \Gamma^{h_2}) \rightarrow O_{h_1, h_2}$$

$$[\theta] \mapsto (\theta h_1 \Gamma, \theta h_2 \Gamma)$$

is also well-defined and is a bijection. We will denote by $C$ the set of all orbits contained in $\Gamma g^{-1} G/\Gamma \times G \Gamma/\Gamma$ (note that this set is $\Gamma$-invariant, so that it is a union of orbits). We then have

$$\Gamma g \Gamma * 1\Gamma * \Gamma s \Gamma =$$

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y \gamma u \Gamma} * \Gamma u^{-1} v \Gamma * 1_{y \gamma v \Gamma})$$

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y \gamma u \Gamma} * \Gamma u^{-1} v \Gamma * 1_{y \gamma v \Gamma})$$

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y \gamma u \Gamma} * \Gamma u^{-1} \gamma u \Gamma * 1_{y \gamma v \Gamma})$$

where the last equality comes from the fact that $N_{u-1, v}^{y \Gamma} = N_{u-1, \gamma u \Gamma}^{y \Gamma}$, which is a consequence of Lemma \[3.13\] \[iii\], or simply by Lemma \[3.15\]. Using now the bijection between $\Gamma/(\Gamma^{u} \cap \Gamma^{v})$ and the orbit space $O_{u, v}$ as described above, we obtain

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y \gamma u \Gamma} * \Gamma r^{-1} \gamma u \Gamma * 1_{y \gamma v \Gamma})$$

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y \gamma u \Gamma} * \Gamma r^{-1} \gamma u \Gamma * 1_{y \gamma v \Gamma})$$

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y \gamma u \Gamma} * \Gamma r^{-1} \gamma u \Gamma * 1_{y \gamma v \Gamma})$$

where $\#O$ denotes the total number of elements of the given orbit $O$. Changing the names of the variables ($r$ to $u$ and $t$ to $v$) we get

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\#O \Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y u \Gamma} * \Gamma u^{-1} v \Gamma * 1_{y v \Gamma})$$

$$= \sum_{\{u\} \in \Gamma y^{-1} G/\Gamma} \sum_{\{v\} \in \Gamma y \Gamma} \frac{\#O \Delta(g)N_{u-1, v}^{y \Gamma}}{L(u^{-1} v)} (1_{y u \Gamma} * \Gamma u^{-1} v \Gamma * 1_{y v \Gamma})$$

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We are now going to prove that the coefficients satisfy

\[ \frac{\#O_{u,v} \Delta(g) N_{u^{-1},v}^y}{L(u^{-1}v)} = 1. \]

This follows from the following computation:

\[
\frac{\#O_{u,v} N_{u^{-1},v}^y \Delta(g)}{L(u^{-1}v)} = \frac{[\Gamma : \Gamma^u \cap \Gamma^v]}{[\Gamma : \Gamma^u]} \cdot \frac{1}{[\Gamma : \Gamma^u \cap \Gamma^v \cap u^{-1}1\Gamma u]} \cdot \frac{[\Gamma : \Gamma^u^{-1}]}{[\Gamma : \Gamma^u]} \cdot \frac{\Gamma^u : \Gamma^u \cap \Gamma^v}{\Gamma} \cdot \frac{\Gamma^u^{-1} : \Gamma^u \cap \Gamma^v}{\Gamma^u}
\]

\[
\frac{[u\Gamma^{-1} : \Gamma^u \cap \Gamma^v]}{[\Gamma : \Gamma^u \cap \Gamma^v]} = \frac{[u\Gamma^{-1} : \Gamma^u \cap \Gamma^v]}{[\Gamma : \Gamma^u \cap \Gamma^v]} = 1.
\]

This finishes the first claim of the theorem. The second claim, concerning the product \( \Gamma g \Gamma * [a]_{x\Gamma} * \Gamma s \Gamma \), is proven in a completely similar fashion. 

**Proposition 3.46.** Let \( \pi : C_c(A/\Gamma) \to B(\mathcal{H}) \) be a nondegenerate \(^*\)-representation, \( \mu : \mathcal{H}(G, \Gamma) \to L(\pi(C_c(A/\Gamma), \mathcal{H})) \) a unital pre-\(^*\)-representation, and let us assume that the associated G-action on \( X \) is free. The pair \( (\pi, \mu) \) is a covariant pre-\(^*\)-representation if and only if the following equality

\[
\mu(\Gamma g \Gamma) \pi([a]_{x\Gamma}) \mu(\Gamma s \Gamma) = \sum_{[u] \in \Gamma^{-1} \Gamma \Gamma} \pi([\alpha_{u^{-1}}(a)]_{xu\Gamma}) \mu(\Gamma u^{-1} v \Gamma) \overline{\pi}(1_{s(x)v\Gamma}).
\]

(63)

holds for all \( g, s \in G \), \( x \in X \) and \( a \in A_x \).

**Proof:** \((\Rightarrow)\) Assume that \((\pi, \mu)\) is a covariant pre-\(^*\)-representation. Then we have

\[
\mu(\Gamma g \Gamma) \pi([a]_{x\Gamma}) \mu(\Gamma s \Gamma) = [\pi \times \mu]((\Gamma g \Gamma) * [a]_{x\Gamma} * \Gamma s \Gamma)
\]

\[
= [\pi \times \mu] \left( \sum_{[u] \in \Gamma^{-1} \Gamma \Gamma} [\alpha_{u^{-1}}(a)]_{xu\Gamma} * \Gamma u^{-1} v \Gamma * 1_{s(x)v\Gamma} \right)
\]

\[
= \sum_{[u] \in \Gamma^{-1} \Gamma \Gamma} \pi([\alpha_{u^{-1}}(a)]_{xu\Gamma}) \mu(\Gamma u^{-1} v \Gamma) \overline{\pi}(1_{s(x)v\Gamma}).
\]

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In order to prove equality (57) one just needs to show that
\[
\sum_{[u] \in \Gamma^{-1} \Gamma / \Gamma} \sum_{[\gamma] \in \mathcal{E}(x)} \frac{\Delta(g) N_{u^{-1}v}^{\Gamma} \gamma}{L(u^{-1}v)^{x}} \tilde{\pi}([\alpha_{u^{-1}v}]_{x} \mu(\Gamma u^{-1}v) \tilde{\pi}(1_{s(x)} v) \Gamma)
\]
and this is proven in a completely analogous way as in the proof of Theorem 3.44.
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