Operator Expansion
in the Derivative and Multiplication by $x$

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Abstract

We generalize to several variables Kurbanov and Maksimov’s result that all linear polynomial operators can be expressed as a formal sum $\sum_{k=0}^{\infty} a_k(X) D^k$ in terms of the derivative $D$ (or any degree reducing operator) and multiplication by $x$. In contrast, we characterize those linear operators that can be expressed as $\sum_{k=0}^{\infty} f_k(D) X^k$ and give several examples. Generalizations to several variables and arbitrary degree reducing operators are considered.

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1 Introduction

In this paper, we will study linear operators on polynomials and give certain summation formulas. All operators are assumed to be linear and their domain and range are assumed to be the ring of polynomials $K[x]$ where $K$ is some field of characteristic zero.

An operator is shift-invariant if $Q E^a = E^a Q$ for all $a \in K$, where $E^a$ is the shift operator $E^a p(x) = p(x + a)$. A fundamental result of umbral calculus (see [8, Theorem 2, p. 691]) is that $Q$ is shift-invariant if and only if it can be expressed (as an infinite series) in the derivative $D$. Its coefficients are given by the D-expansion formula

$$Q = \sum_{k=0}^{\infty} a_k D^k$$

where $a_k = [Q x^k / k!]_{x=0}$.

It has been asked [8, Problem 12, p. 752] what operators can be expressed (as infinite series) in $D$ and in $X$ where $X$ represents multiplication by $x$. Pincherle and Amaldi [8] proved that all linear operators

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can be so expressed. Kurbanov and Maksimov give an explicit construction of such an expansion in the Proceedings of the Uzbekistan Academy of Science [5] (even with the added restriction that the derivatives lie to the right of multiplication by $x$)

$$Q = \sum_{n=0}^{\infty} a_n(X) D^n$$

(2)

where the polynomials $a_n$ are defined by the following generating function

$$\frac{Q \exp(x t)}{\exp(x t)} = \sum_{n \geq 0} a_n(x) t^n.$$  

(3)

(By abuse of notation, $Q$ above acts on the coefficients of $\exp(x t) \in (K[x])[[t]]$.) Such an expansion will be called an XD-expansion since it is an expansion in $D$ and $X$ with $X$ lying to the left of $D$. Note that XD-expansions are useful in that the derivative is easy to calculate both numerically and symbolically. Moreover, XD-expansions allow us to manipulate arbitrary linear operators with similar ease. For example, the XD-expansion of the integration operator gives an asymptotic formula for integration (6) which can also be found in [1, Sections 3.5 and 3.6]. Note that infinite sums such as (2) are always well defined since when applied to any given polynomial $p(x)$ only a finite number of terms make a nonzero contribution.

Equation (2) can be viewed as the XD-generalization of equation (1). In fact, many of the results of umbral calculus are direct corollaries of (2). In section 2, a generalization of equation (2) will be proven. As do Kurbanov and Maksimov, the derivative will be replaced by a general degree reducing operator $B$. We will further generalize equation (2) and consider multivariate polynomial operators.

The duality between the operators $D$ and $X$ is a recurrent theme in the umbral calculus (see for example [9, p. 695] and [11]). Thus, it is natural to consider the dual form of (2), namely, expansions of the form

$$Q = \sum_{n=0}^{\infty} a_n(D) X^n$$

(4)

called DX-expansions. In [4, theorem 2.1], we find DX-expansions of certain umbral operators. Do all operators have a DX-expansion? Surprisingly, the answer is “no”, and those that do will be called DX-operators.

In section 2, we will show that the set of DX-operators form a subalgebra of the algebra of linear operators. We will characterize DX-operators in several ways.

There are several surprising differences between DX and XD-expansions that will be explored in this paper:

- Although all operators have XD-expansions, not all have DX-expansions.
- Although all XD-expansions converge when applied to polynomials, not all DX-expansions do so.
- Although Maksimov and Kurbanov’s formula (3) generalizes well to multivariate polynomial operators, and arbitrary degree reducing operators $B$, the corresponding results concerning DX-expansions do not hold in the more general setting.

DX and XD-expansions for many important operators are given in section 4. Examples include several new operator expansions as applications to the umbral calculus; in particular, we will show that all umbral operators, and umbral shifts are DX-operators.

Some open problems are suggested in section 5.
2 XD-Expansion

The objective of this section is to prove the XD-expansion formula \(2\). However, we will first restate the formula in greater generality. Since equation \(2\) is a generalization of the D-expansion formula \(1\), we will as an introduction indicate the corresponding generalization of equation \(1\).

Let \(x = (x_1, x_2, \ldots)\) be a finite or infinite set of variables. A monomial over \(x\) is a product \(x^n = x_1^{n_1}x_2^{n_2} \cdot \cdot \cdot\). Here and in the sequel \(n, m, j, \ldots\) denote sequences of nonnegative integers with finite support. A polynomial is a finite \(K\)-linear combination of monomials. We denote the ring of polynomials over \(x\) by \(K[x]\).

Let \(D_i\) be the derivative with respect to \(x_i\). Thus, \(D^k = D_1^{k_1}D_2^{k_2} \cdot \cdot \cdot\). Define \(n! = n_1!n_2! \cdot \cdot \cdot\), \((n)_k = n!/(n-k)!\), and \((n)_k^k = (n)_k / (n-k)!\) with the motivation that \(D^k x^n = (n)_k x^{n-k}\) and \((x + y)^n = \sum_k (\binom{n}{k}) x^k y^{n-k}\).

An operator is shift-invariant if \(QE^a = E^a Q\) where \(E^a = \exp(a \cdot D)\) is the shift operator \(E^a p(x) = p(x+a)\). We can now generalize the D-expansion formula \(1\).

Proposition 1 (D-Expansion Formula) A linear operator \(Q: K[x] \rightarrow K[x]\) is shift-invariant if and only if it can be expressed (as an infinite series) in the derivative \(D\)

\[ Q = \sum_k a_k D^k \]

where \(a_k = [Q x^k/k!]_{x=0}\).

Another generalization of the D-expansion formula \(1\) is to replace the derivative \(D\) with an arbitrary degree reducing operator \(B\). Suppose \(B: K[x] \rightarrow K[x]\) is such that for all nonconstant polynomials \(p, (\deg Bp) + 1 = \deg p\), and for all constants \(a, Ba = 0\). Then \(B\) is called degree reducing and there exists a unique sequence of polynomials \(b_n(x)\) (called the divided power sequence for \(B\)) such that \(Bb_n(x) = b_{n-1}(x)\) and \(b_0(0) = \delta_{00}\). For example, if \(B = D\), then \(b_n(x) = x^n/n!\). The sequence \(n!b_n(x)\) is called the basic family of \(B\) by Markowsky \(7\).

We then have the “B-expansion formula.”

Proposition 2 (B-Expansion Formula) A linear operator \(Q: K[x] \rightarrow K[x]\) commutes with \(B\) if and only if it can be expressed (as an infinite series) in \(B\).

\[ Q = \sum_{k=0}^{\infty} a_k B^k \]

where \(a_k = [Q b_k(x)]_{x=0}\).

It is of course possible to carry out both generalizations simultaneously. Let \(\{b_n(x): n\text{ finite support}\}\) be a basis of \(K[x]\) such that \(b_n(0) = \delta_{n0}\). Define the operator \(B_i\) by \(B_i b_n(x) = b_{n-e_i}(x)\) where \(e_i\) is the \(i\)th unit vector \((e_i)_j = \delta_{ij}\). Denote by \(B\) the sequence of operators \((B_1, B_2, \ldots)\). For example, if \(b_n(x) = x^n/n!\), then \(B = D\).

Note that the reader interested only in the univariate case may skip all the definitions involving bold-faced symbols with the assurance that in the univariate case they reduce to their univariate counterparts. In particular, \(e_i\) below should then be read as “1.”
Proposition 3 (B-Expansion Formula) A linear operator $Q: K[x] \rightarrow K[x]$ commutes with all $B_i$ if and only if it can be expressed (as a formal power series) in $B$.

$$Q = \sum_k a_k B_k$$

where $a_k = [Qb_k(x)]_{x=0}$.

A formal power series is an infinite linear combination of monomials. The ring of formal power series with coefficients in $K$ is denoted $K[[x]]$. The generating function $b(x, t)$ for the sequence of polynomials $b_n(x)$ is

$$b(x,t) = \sum_n b_n(x) t^n \in (K[x])[[t]].$$

For example, if $B = D$, then $b(x,t) = \exp(x \cdot t)$.

By abuse of notation, we allow operators on $K[x]$ to act on the coefficients of $b(x, t)$. For example,

$$B_i b(x, t) = \sum_n (B_i b_n(x)) t^n = \sum_n b_{n - e_i}(x) t^n = \sum_m b_m(x) t^{m + e_i} = t_i b(x, t).$$

Let $X_i$ be the operator of multiplication by $x_i$, and let $X = (X_1, X_2, \ldots)$. All linear operators can be expressed in terms of $X$ and $B$.

Theorem 4 (XB-Expansion Formula) Let $B = (B_1, B_2, \ldots)$, and $b(x, t)$ be as above. Let $Q: K[x] \rightarrow K[x]$ be a linear operator. Then

$$Q = \sum_k a_k(X) B^k$$

where the polynomials $a_k$ are given by the generating function

$$\sum_k a_k(x) t^k = \frac{Qb(x, t)}{b(x, t)}.$$

Proof: Apply both sides of equation (5) to the basis $b_k(x)$, or equivalently to its generating function $b(x, t)$. The left-hand side gives $Bb(x, t)$ while the right-hand side gives

$$\sum_k a_k(X) B^k b(x \cdot t) = \sum_k a_k(x) t^k b(x \cdot t) = b(x \cdot t)^{-1} (Qb(x \cdot t)) b(x \cdot t) = Qb(x \cdot t).$$

Proposition 5 (XB-Uniqueness) The XB-expansion given in theorem 4 is unique.
Proof: It suffices to show that \( \sum_n a_n(X) B^n \) is the zero operator only if \( a_n(X) \) is zero for all \( n \). Suppose not, and let \( m \) be a minimal such \( n \). However, \( (\sum_n a_n(X) B^n) x^m = m! a_m(x) \neq 0 \). \( \square \)

We reserve most of our examples for section 3. However, we cannot resist giving a simple but important example here.

Let \( J \) denote the definite integral \( J p(x) = \int_0^1 p(u) du \). To express \( J \) in terms of \( D \) and \( X \), we apply \( J \) to \( \exp(x) \) which yields \( (\exp(x) - 1)/x \). We then divide by \( \exp(x) \) and replace \( x \) and \( t \) with \( X \) and \( D \) respectively, being sure to keep \( X \) on the left and \( D \) on the right since they do not commute. Thus,

\[
J = \sum_{n=0}^{\infty} (-1)^n X^{n+1} D^n/(n+1)!
\]

(6)

This single equation is essentially equivalent to the content of [1] sections 3.5 and 3.6.

However, note that the methods used here are purely formal. Thus, with equal ease, any linear operator in any number of dimensions can be expanded in terms of the derivative or any other “delta set” of degree reducing operators. For example, \( J \) can be expanded in terms of \( X \) and \( B = \Delta \) instead of \( X \) and \( D \) where \( \Delta \) is the forward difference operator \( \Delta p(x) = p(x+1) - p(x) \). The divided power sequence for \( B = \Delta \) is \( b_n(x) = \binom{x}{n} \) whose generating function is given by \( b(x,t) = (1+t)^x \). Thus, \( J = \sum_{n=0}^{\infty} a_n(X) B^n \) where \( a_n \) is given by the generating function \( \sum_{n=0}^{\infty} a_n(x)t^n = 1 - (1+t)^{-x} / \ln(1+t) \). In other words,

\[
J = X - X^2 \Delta / 2 + (X^2/4 + X^3/6) \Delta^2 - (X^2/6 + X^3/6 - X^4/24) \Delta^3 + \cdots.
\]

Recall that differentiation is simpler than integration, and that finite differences are even simpler still. Thus, formulas such as those above give simple algorithms by which one can approximate complicated linear operations. For more examples, see [3].

**Proposition 6 (XB-Convergence)** Any infinite sum of the form \( \sum_n a_n(X) B^n \) converges formally. In other words, only finitely many terms are nonzero when applied to any given polynomial.

**Proof:** Every polynomial can be expressed as a linear combination

\[
p(x) = \sum_{n \in S} c_n b_n(x)
\]

for some finite set \( S \). Let \( m_i = \max_{n \in S} (n_i) \). Then \( c_n(X) B^n p(x) \) is zero except when \( n \leq m \) componentwise, and there are only finitely many such \( n \). \( \square \)

Of course, the results above have only been proven for polynomials. One would hope to extend the expansion formulas by continuity to functions that are limits of polynomials in some sense. For example, \( J \cos x = \sin x \), and

\[
\sum_{n=0}^{\infty} \frac{(-1)^n X^{n+1} D^n}{n+1} \cos x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} (-1)^k \cos x + \sum_{k=0}^{\infty} \frac{x^{2k+2}}{(2k+2)!} (-1)^k \sin x
\]

\[= \sin x \cos x + (1 - \cos x) \sin x \]

\[= \sin x.
\]

This is an example of a direct verification. In general, one has to justify the interchange of a limit and a summation. The following proposition gives a sufficient condition to allow this interchange.

**Proposition 7** Let \( Q \) be a shift-operator with representation \( Q = \sum_n a_n(X) D^n \) and let \( f \) be the pointwise limit of a sequence of polynomials \( (p_k)_{k \geq 0} \) for all \( x \) in some set \( V \). If there exists \( \varphi_n(x) \) such that \( |a_n(x) D^n p_k(x)| \leq \varphi_n(x) \) and \( \sum_{n=0}^{\infty} \varphi_n(x) < \infty \) for all \( n, k \in \mathbb{N} \) and \( x \in V \), then

\[
\lim_{k \to \infty} \sum_{n=0}^{\infty} a_n(X) D^n p_k(x) = \sum_{n=0}^{\infty} a_n(X) D^n f(x).
\]

**Proof:** This follows from Lebesgue’s dominated convergence theorem, since series are integrals with respect to the measure \( \sum_n \delta_n \), where \( \delta_n \) is the Dirac measure at \( n \). \( \square \)
3 DX-Expansions

3.1 Commutation Rules

When an operator has a DX-expansion, the coefficient of $D^k X^n$ in (2) is not necessarily the same as that of $X^m D^k$ in (4) since $D$ and $X$ do not commute (cf. proposition 15). In fact, their commutator $DX - XD$ is the identity. Applying theorem 5, we obtain

$$D^j X^i = \sum_{k=0}^{j} (i_k) (j) k D^{i-k} X^{j-k} / k! \tag{7}$$

where the lower factorial $(x)_n = x(x-1)(x-2) \ldots (x-n+1)$. For another proof, see [3, Section 2.2].

Let $a_{nm} = D^n X^m / n! m!$ and $b_{nm} = X^m D^n / n! m!$. Then we have $a_{nm} = \sum_{t} b_{n-t,m-t} / t!$, or fixing the difference between $n$ and $m$ and indexing only by $n$, we have $a_n = \sum_{t} b_{n-t} / t!$. This sum is easily inverted, for the vector $a = M b$ where $M$ is the upper triangular infinite Toeplitz matrix with $\frac{i}{i!}$ on the $i$th diagonal. However, the inverse of this matrix has diagonals $(-1)^i / i!$ as can be seen from the identity $\sum_{i+j=k} (-1)^i / i! j! = (1 - 1)^k / k! = \delta_{k0}$. Thus, we obtain

$$X^i D^j = \sum_{k=0}^{j} (-1)^k (i_k) (j) k D^{i-k} X^{j-k} / k! \tag{8}$$

More generally, we have the following result.

**Proposition 8 ([D,X]-Commutation)**

$$f(D)p(X) = \sum_{k=0}^{\infty} p^{(k)}(X) f^{(k)}(D) / k! \tag{9}$$

$$p(X)f(D) = \sum_{k=0}^{\infty} (-1)^k f^{(k)}(D) p^{(k)}(X) / k! \tag{10}$$

**Proof:** Consider linear combinations of (7) and (8). □

In so far as possible, we will derive a “B” analog and a multivariate analog of each result. These may be omitted by the reader essentially interested in the DX-expansions.

In this case, an arbitrary operator $B$ may have virtually any commutator $BX - XB$. Thus, no corresponding “[B,X]-commutation” result can be given. However, in the multivariate case, we obtain the following result.

**Proposition 9 ([D,X]-Commutation)**

$$f(D)p(X) = \sum_{k} p^{(k)}(X) f^{(k)}(D) / k!$$

$$p(X)f(D) = \sum_{k} (-1)^{|k|} f^{(k)}(D) p^{(k)}(X) / k!$$

where $|k| = k_1 + k_2 + \cdots$ and $f^{(k)}(D)$ is given by the multivariate Pincherle derivative $f^{(k)}(x) = D^k f(x)$.

**Proof:** Apply $[D,X]$-commutation to each variable. Recall that $D_i$ and $x_j$ commute for $i \neq j$. □
3.2 Convergence of $DX$-Expansions

In section 2, we have seen that all operators have XD-expansions, and even XB, XD, BD-expansions. One might suspect that the commutation rules of the previous subsection would lead to corresponding DX (et al)-expansions. Nevertheless, we will see below that certain operators (for example $J$) lack a corresponding DX-expansion. (See remark after theorem 13.) The finite commutation rules of the previous subsection do not generalize to infinite commutation rules, for if we substitute (8) into (3), the resulting sum is not necessarily well defined. For example, each coefficient of $X^nD^n$ in an XD-expansion makes its own contribution to the coefficient of $X^0D^0$ in the corresponding DX-expansion. Since there are no conditions on the coefficients of an XD-expansion, the sum obtained by direct substitution need not converge. That is to say, when an operator is applied to a polynomial of degree $n$, the sum in (3) is well defined since only the first $n + 1$ terms make nonzero contributions. On the other hand, all terms of (4) make potential contributions. The obvious corresponding result is false in the multivariate case. Consider for example the sum

\[ \sum_{k=0}^{\infty} f_k(D) X^k \]

The sum converges formally if and only if when applied to any polynomial there are only finitely many contributions. These two constraints correspond exactly to the condition above.

What DX-expansions converge?

To answer this question, we define the order $\text{ord}(f)$ of a nonzero formal power series $f(t) = \sum_{k=0}^{\infty} c_k t^k$ to be the smallest $j$ such that $c_j \neq 0$. If $\text{deg}(p) = n$ and $\text{ord}(f) = k$, then $f(D)p(x)$ is of degree $n - k$ if $n \geq k$, and $f(D)p(x) = 0$ otherwise. The order of the zero series is taken to be $+\infty$.

**Proposition 10 (DX-Convergence)** (1) The formal sum of operators $\sum_{k=0}^{\infty} f_k(D) X^k$ converges in the discrete topology if and only if $\lim_{k \to \infty} [\text{ord}(f_k) - k] = +\infty$.

(2) The formal sum of operators $\sum_{k=0}^{\infty} D^k a_k(X)$ converges in the discrete topology if and only if $\lim_{k \to \infty} [k - \text{deg}(a_k)] = +\infty$.

**Proof:** We will prove only (1) since (2) follows by similar reasoning.

(Only if) By hypothesis, the polynomial sequence $\left( \sum_{k=0}^{K} f_k(D) X^k p(x) \right)_{K \geq 0}$ is eventually constant for all polynomials $p(x) \in K[x]$. Thus, $\sum_{k=0}^{\infty} f_k(D) x^k p(x)$ is eventually zero. Hence, $\text{ord}(f_k) - k$ is eventually greater than $\text{deg}(p)$. Since $p(x)$ may have any degree, $\lim_{k \to \infty} \text{ord}(f_k) - k = +\infty$.

(If) Let $p(x)$ be a polynomial of degree $n$. Since $\lim_{k \to \infty} \text{ord}(f_k) - k = +\infty$, there exists a $k_0$ beyond which $\text{ord}(f_k) - k > n$. Thus, the sequence $\left( \sum_{k=0}^{K} f_k(D) X^k \right)_{K \geq 0}$ is constant for $K > k_0$.

The “$B$-analog” of proposition 10 is very easy to state. In fact, its proof identical to that of proposition mutatis mutandis.

**Proposition 11 (BX-Convergence)** (1) The formal sum of operators $\sum_{k=0}^{\infty} f_k(B) X^k$ converges in the discrete topology if and only if $\lim_{k \to \infty} [\text{ord}(f_k) - k] = +\infty$.

(2) The formal sum of operators $\sum_{k=0}^{\infty} B^k a_k(X)$ converges in the discrete topology if and only if $\lim_{k \to \infty} [k - \text{deg}(a_k)] = +\infty$.

The obvious corresponding result is false in the multivariate case. Consider for example the sum $\sum_{k=0}^{\infty} D_1^k X_2^k$ which does converge formally. The correct necessary and sufficient conditions are more complex than in the univariate case.

**Proposition 12 (DX-Convergence)** The operator sum $Q = \sum_{k,n} c_{k,n} D^k X^n$ converges formally if and only if for all $j$, there exist only finitely many triples $m, k, n$ such that $m + n - k = j$ and $c_{k,n} \neq 0$.

**Proof:** The sum converges formally if and only if when applied to any polynomial there are only finitely many contributions. In other words, $Q x^m$ has only contributions to finitely many terms $x^j$, and only finitely many contributions to each such term. These two constraints correspond exactly to the condition above.

We will not state the obvious BX-analog of proposition 12.
3.3 Characterization of DX-Operators

What operators can be represented by DX-expansions?

A simple criterion is given in theorem 13 based on the matricial representation of an operator. Any operator can be represented as an infinite matrix \( C = (c_{nk})_{n,k} \geq 0 \) with respect to, for example, the basis \( \{x^n : n \geq 0 \} \),

\[
Qx^n = \sum_{k=0}^{\infty} c_{nk}x^k.
\]

Let \( q_t \) be the \( t \)-th diagonal of the matrix \( C \): \( q_t(n) = c_{n,n+t} \) where \( n \) is a nonnegative integer, and \( t \) is an arbitrary integer under the convention that \( c_{nk} = 0 \) for \( k < 0 \).

**Theorem 13 (DX-Characterization)** Using the above notation, \( Q \) has a DX-expansion if and only if \( q_t \in K[n] \) for all integers \( t \).

While one can see that an operator has a DX-expansion simply by exhibiting such an expansion, the only convenient way to see that an operator does not have such an expansion is to apply the criterion of theorem 13. For example, consider the definite integration operator \( J \) defined above. For \( Q = J \), \( Qx^n = x^{n+1}/(n+1) \). Thus, \( q_t(n) = 1/(n+1) \notin K[n] \), so \( J \) does not have a DX-expansion. (See propositions 24 and 23 for other examples.)

Theorem 13 is all the more surprising since it has no obvious “BX-analogue.” For example, let \( Bx^n = \sin(n)x^{n-1} \). Now, \( Q = B \) obviously has a BX-expansion, namely \( B \) itself, yet \( q_{-1}(n) = \sin(n) \notin K[n] \).

On the other hand, theorem 13 has the multivariate analog below which we will prove in the place of theorem 13.

**Theorem 14 (DX-Characterization)** Let \( Q : K[x] \to K[x] \) be a linear operator. Let \( q_t(n) = c_{n,n+t} \) where \( Qx^n = \sum_k c_{nk}x^k \). Then \( Q \) has a DX-expansion \( Q = \sum_{n} q_n(D)\mathbf{x}^n \) if and only if \( q_t(n) \in K[n] \) for all integer vectors \( t \) (with finite support).

**Proof:** (Only if) The only terms which contribute to \( q_t \) are \( D^k\mathbf{x}^{t+k} \). By (1) of proposition 12, there are finitely many such terms. Each makes a contribution of \( (n+t+k) \in K[n] \). Thus, \( q_t(n) \in K[n] \).

(If) Define the operator \( Q_t \) by the relation \( Q_t\mathbf{x}^n = q_t(n)\mathbf{x}^{n+t} \). Then \( Q = \sum_t Q_t \) is a convergent expansion of \( Q \) since \( q_t \) has finite support, so it will suffice to show that each \( Q_t \) is a DX-operator.

The polynomial \( p_k(n) = (n+t+k) \) has leading term \( n^k \). Thus, \( p_k \) is a basis for \( K[n] \), and \( q_t = \sum a_k p_k \). Thus, \( Q_t = \sum a_k D^k\mathbf{x}^{t+k} \). (Note that if \( t_i < 0 \), then \( a_k \) is necessarily zero for \( k_i < -t_i \).)

Note that since \( p_k \) is a basis, the choice of \( a_k \) is unique. Thus, we have the following result.

**Proposition 15 (XD-Uniqueness)** XD-expansions and XD-expansions are unique.
3.4 Closure of $\mathcal{D}X$-Operators

The main result of this section is the fact that the set $\mathcal{D}X$ of operators with $\mathcal{D}X$-expansions is closed under composition. The following corollary of proposition 10 is crucial to the proof of theorem 17.

Lemma 16 Using the above notation, if $Q \in \mathcal{D}X$, then $q_t$ is identically zero for $t$ large.

Proof: Since $\lim_{k \to \infty} \operatorname{ord}(f_k) - k = +\infty$, it follows that $k - \operatorname{ord}(f_k)$ takes a maximum value $T$. Hence, $q_t$ is identically zero for $t > T$. $\square$

It can similarly be shown that if $Q$ has a $\mathcal{B}X$-expansion, then $q_t$ is identically zero for $t$ large.

Theorem 17 The set $\mathcal{D}X$ of $\mathcal{D}X$-operators forms a (noncommutative, unitary) $K$-subalgebra of the algebra $L(K[x], K[x])$ of all linear operators equipped with the standard operations

\[
(R \circ P)p = R(Pp) \\
(cQ)p = c(Qp) \\
(R + P)p = Rp + Pp.
\]

Proof: $\mathcal{D}X$ is trivially closed under multiplication by constants, and under addition. We will now show that $\mathcal{D}X$ is closed under composition.

Let $Q$ and $R$ be $\mathcal{D}X$-operators. Thus, as defined above $q_t$ and $r_s$ are polynomials by theorem 13. Moreover, by lemma 16, $q_t$ is identically zero for $t > T$, and $r_s$ is identically zero for $s > S$. Define $Q = R \circ P$.

It suffices to show that $q_u$ is a polynomial.

\[
Qx^n = \sum_{t \in \mathbb{Z}} p_t(n) Rx^{n+t} \\
= \sum_{s,t \in \mathbb{Z}} p_t(n) r_s(n + t) x^{n+t+s} \\
q_u(n) = \sum_{s+t=u}^T p_t(n) r_s(n + t) \\
= \sum_{t=u-S}^T p_t(n) r_{u-t}(n + t).
\]

Thus, $q_u(n) \in K[n]$ since it is a finite sum of polynomials. $\square$

Note that $q_u$ is identically zero for $u > S + T$.

Surprisingly, theorem 17 does not even generalize to two variables. Consider the $\mathcal{D}X$-operators $R = \sum_{k=0}^\infty D_1^k X_2^k$, and $P = \sum_{k=0}^\infty D_2^k X_1^k$. Now, let $Q = R \circ P$. Recall that $q_{00}(i, j)$ is the coefficient of $x_1^i x_2^j$ in $Qx_1^ix_2^j$. Thus, $q_{00}(n, n)$ is given by the finite hypergeometric function $\binom{3}{0, 1, n - 1} = \sum_{k=0}^n (n)_k (n+k)_k$. All of the terms of the sum are nonnegative, and the term corresponding to $k = 0$ gives a lower bound of $(n!)^2$. This guarantees that $q_{00}$ is not polynomial. Thus, $Q$ is not a $\mathcal{D}X$-operator by theorem 14.
3.5 Coefficients of $DX$-Expansions

Note that (3) gives an efficient method of calculating the XD-expansion of an operator $Q$. Given the action of $Q$ on a few polynomials $Qx^0, Qx^1, \ldots, Qx^n$, the first few terms of its XD-expansion can be automatically calculated [2]. On the other hand, the coefficients $f_k(D)$ of a DX-expansion (4) depend on the action of $Q$ on all the powers of $x$. No efficient means of calculating DX-expansions is known in general. (Various special techniques are used for each case treated in section 4.) Is there an analog of (3) for DX-expansions? The only results in this direction found so far are the following propositions.

**Proposition 18** Let $Q \in DX$. If

$$Q = \sum_{n=0}^{\infty} c_n(D) X^n,$$

then

$$\frac{Q \exp(xt)}{\exp(xt)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} c_n^{(k)}(t) x^{n-k}.$$

**Proof:** By (3),

$$Q \exp(xt) = \sum_{n=0}^{\infty} c_n(D) X^n \exp(xt) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} X^{n-k} c_n^{(k)}(D) \exp(xt) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} c_n^{(k)}(t) \exp(xt).$$

$$\frac{Q \exp(xt)}{\exp(xt)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} c_n^{(k)}(t). \quad \square$$

By similar reasoning, we obtain the following proposition.

**Proposition 19** Let $Q \in DX$.

$$Q = \sum_{n=0}^{\infty} D^n a_n(X),$$

then

$$\frac{Q \exp(xt)}{\exp(xt)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a_n^{(k)}(x) t^{n-k}. \quad (11)$$

4 Examples

In this section we present example of DX- and XD-expansions.
4.1 Umbral Operators

A delta operator $P$ is a shift-invariant degree reducing operator. Note that by the D-expansion formula $P = f(D)$ where ord$(f) = 1$. Let $p_n(x)$ be the divided power sequence of $P$. The operator $U_P: p_n(x) \to x^n$ is the umbral operator associated to $P$, and the operator $\sigma_P: p_n(x) \to (n+1)p_{n+1}(x)$ is the umbral shift associated to $P$. For example, $U_D = I$ and $\sigma_D = X$.

**Proposition 20** The umbral operator $U_P$ is a $DX$-operator if and only if $P x = 1$.

**Proof:** (If) Let $P = p(D)$, and let $r(t) = t - p(t)$. By [4, theorem 2.1],

$$U_P = \sum_{k=0}^{\infty} p'(D) r(D)^k X^k / k!.$$ 

By proposition [10], the sum above converges since ord$(p'^k r^k) = 2k$ and $2k - k \to +\infty$.

(Only If) Let $a = P x \in K$ and $Q = U_P$. The leading coefficient of $p_n(x)$ is $1/a^n$. The result follows from theorem [3] since $q_0(n) = a^n$ is not a polynomial with respect to $n$ unless $a = 1$. 

On the other hand, all umbral operators have the following XD-expansion.

**Proposition 21** Let $P$ be a delta operator. Then its umbral operator $U_P$ can be expressed as

$$U_P = \sum_{k=0}^{\infty} X^k (P - D)^k / k!.$$ 

**Proof:** Let $\overline{p}_k(x)$ be the conjugate sequence of polynomials for $P = p(D)$ defined by the generating function

$$\sum_{k=0}^{\infty} \overline{p}_k(x) t^k / k! = \exp(xp(t)).$$ 

Note by [3, theorem 7, p. 708], $U_P x^k = \overline{p}_k(x)$. We now calculate the coefficients $c_n(x)$ of the XD-expansion by applying (3).

$$\sum_{n=0}^{\infty} c_n(x) t^n = \frac{U_P \exp(xt)}{\exp(xt)} = \exp(-xt) \sum_{k=0}^{\infty} (U_P x^k) t^k / k! = \exp(-xt) \sum_{k=0}^{\infty} \overline{p}_k(x) t^k / k! = \exp(x(p(t) - t)).$$

Hence, $U_P = \sum_{k=0}^{\infty} X^k (P - D)^k / k!$. 

Contrast with the triple sum expansion given in [12].
4.2 Umbral Shifts

Proposition 22 All umbral shifts $\sigma_P$ are DX-operators.

Proof: Given $c \in K^*$, the umbral shift $\sigma_P$ is identical to $c\sigma_{cP}$. Thus, without loss of generality, we may suppose that $Px = 1$. Moreover, $\sigma_P = (U_P)^{-1}UXP$. Note that $(U_P)^{-1} = U_R$ where $R = r(D), P = p(D), r(p(t)) = p(r(t)) = t$ (cf. [3] theorem 7, p. 708). By the Lagrange inversion formula, $Rx = 1$. Thus, $\sigma_P$ is the composition of three DX-operators: $U_R, X, U_P$. Hence, by theorem 17, $\sigma_P$ is a DX-operator. $\blacksquare$

Alternate Proof: The following XD-expansion of $\sigma_P$ is a restatement of Rodrigues’ formula [3, theorem 4, p. 695]:
\[
\sigma_P = X \frac{1}{P'}
\]
where $P' = PX - XP$ is the shift-invariant operator called the Pincherle derivative of $P$ [3, Section 4, p. 694]. We then deduce
\[
\sigma_P = \frac{1}{P'} X - \left( \frac{1}{P'} \right)' = \frac{1}{P'} X + \frac{P''}{(P')^2}
\]
as an explicit DX-expansion of $\sigma_P$. $\blacksquare$

Note that $X$ appears with exponent at most one in expansions (12) and (13).

4.3 Endomorphisms

Proposition 23 The only endomorphisms of $K[x]$ with DX-expansions are the translation operators.

Proof: Let $Q$ be an endomorphism of $K[x]$, $Qp(x) = p(q(x))$. If $\text{deg}(q) > 1$, then $q_t$ is not identically zero for $t$ large, and by lemma [4], $Q$ can not be a DX-operator.

If $\text{deg}(q) = 0$, then without loss of generality (compose $Q$ with a translation if necessary), $Q$ is evaluation at zero. Thus, $q_0(n) = 1$ if $n = 0$, and 0 otherwise. This function is not a polynomial. Hence, by theorem [3], $Q$ is not a DX-operator.

If $\text{deg}(q) = 1$, then $q(x) = ax + b$. Without loss of generality, $q(x) = ax$. Thus, $q_0(n) = a^n$ which is not polynomial unless $a = 1$. $\blacksquare$

On the other hand, by [3], all endomorphisms $Q$ of $K[x]$ have the following XD-expansion
\[
Q = \sum_{k=0}^{\infty} (q(X) - X)^k D^k / k!.
\]
where $Q(p(x)) = p(q(x))$. Notice the similarity to Taylor’s formula.

In the multivariate case, let $q_i(x) = Qx_i$. Then
\[
Q = \sum_k (q(X) - X)^k D^k / k!.
\]
5 Open Problems

a) By theorem 17, the product of two DX operators is again a DX operator. Given the DX-expansion of $Q$ and $R$, is there an explicit formula for the DX-expansion of their product? Similarly, is there an explicit formula for the XD-expansion of any two operators given their XD-expansions.

b) If $K$ is a topological field, then we have a weaker notion of convergence than the discrete topology, and thus more DX-operators. How can they be classified?

c) Is the product of two BX-operators also a BX-operator? This problem is especially difficult since we have no criteria analogous to theorem 13 to tell whether a product is a BX-operator. In fact, it remains to be seen whether BX-expansions are unique (when they exist).

d) Find a formula more explicit than propositions 18 and 19 by which to calculate DX-expansions. Is there a BX-analog of propositions 18 and 19?

e) Let $B$ be a degree lowering operator as above, and let $Y$ be a degree raising operator. For what $B$ and $Y$ can all linear operators be expressed by a (unique) $YB$-expansion: $\sum_{k=0}^{\infty} B^k a_k(Y)$.

f) Characterize operators $Q$ with identical XD and DX-expansions. That is,

$$Q = \sum_{i,j} c_{i,j} X^i D^j = \sum_{i,j} c_{i,j} D^j X^i.$$ 

Conjecture: $Q$ must be of the form $p(X) + f(D)$ for some polynomial $p$ and formal power series $f$.

g) Extensions of the Umbral Calculus from polynomials to inverse Laurent series (negative powers of $x$) and Artinian series (fractional powers of $x$) are known [10, 6]. Can these be used to derive XD expansions of operators that act not on polynomials, but on Laurent or Artinian series?

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