Early universe thermostatistics in curved momentum spaces

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The theories known as doubly special relativity are introduced in order to take into account an observer-independent length scale and the speed of light in the framework of special relativity. These theories can be generally formulated on the de Sitter and also recently proposed anti-de Sitter momentum spaces. In the context of these theories, we study the statistical mechanics and to do this, we consider the natural measure on the corresponding extended phase space. The invariant measure on the space of distinct microstates is obtained by restriction of the natural measure of the extended phase space to the physical phase space through the disintegration theorem. Having the invariant measure, one can study the statistical mechanics in an arbitrary ensemble for any doubly special relativity theory. We use the constructed setup to study the statistical properties of four doubly special relativity models. Applying the results to the case of early universe thermodynamics, we show that one of these models that is defined by the cosmological coordinatization of anti-de Sitter momentum space, implies a finite total number of microstates. Therefore, without attribution to any ensemble density and quite generally, we obtain entropy and internal energy bounds for the early radiation dominated universe. We find that while these results cannot be supported by the standard Friedmann equations, they indeed are in complete agreement with the nonsingular effective Friedmann equations that arise in the context of loop quantum cosmology.

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I. INTRODUCTION

Existence of a minimum measurable length scale, preferably of the order of the Planck length, is a common feature of the quantum gravity proposal which is suggested by quantum gravity candidates such as loop quantum gravity and string theory [1, 2]. Although a complete theory of quantum gravity is not yet made, it is widely believed that purely minimal length effects may be appreciable when gravity is negligible but the energy scale is very high [3]. In other words, as general relativity reduces to the special relativity at the weak gravity limit, it is natural to expect that a full theory of quantum gravity will be reduced to a deformed special relativity in which the issues of the existence of an invariant minimal length and the speed of light are supported [4]. Such a deformed special relativity will reduce to the standard special relativity at the low energy regime in light of the correspondence principle. In the absence of a full theory of quantum gravity, one may do this in reverse: starting from standard special relativity and deforming it in such a way that in addition to the speed of light the theory contains a minimal observer-independent length scale. This is the main idea of the doubly special relativity (DSR) which was proposed by Amelino-Camelia in Ref. [5]. Lorentz symmetry can be considered as an approximate symmetry that will be broken at the ultraviolet (UV) regime. However, it is also possible to construct a DSR theory that preserves Lorentz symmetry by nonlinear action of the Lorentz group on the momentum space [6]. Indeed, it was realized later that there are many DSR theories and all of them can be understood as different bases of the \( \kappa \)-Poincaré algebra on the non-commutative Minkowski spacetime [7]. Although there is not a unique DSR theory, there are some main features which are common between all DSR theories. For instance, the spacetime structure naturally turns out to be noncommutative [8] in agreement with the seminal work of Snyder on quantized Lorentz-invariant spacetime [9]. Interestingly, it is also shown that different DSR theories can be understood as different coordinate systems on de Sitter (dS) momentum space [10]. Recently, the anti-de Sitter (AdS) momentum space was also implemented in the context of the DSR theories as a complementary to the dS space [11, 12]. A maximal momentum or maximal energy corresponding to the universal observer-independent length scale then arises in these setups. Therefore, the expressions such as dispersion relations and invariant measures on the momentum space will be modified which in turn results modifications to the density of states [13]. The density of state determines the number of microstates, and therefore, the thermodynamical properties of the statistical systems will be significantly affected at the high temperature limit. The statistical mechanics in the DSR framework was first
studied in Ref. [16]. Thermodynamics of some statistical systems in the DSR framework are also studied in Ref. [17]. It is however natural to expect that some fundamental aspects of the early universe thermodynamics may be addressed in the DSR setup. In this paper, after a brief review on the role of curved dS and AdS momentum spaces in DSR theories, we introduce four different DSR models in section II. In section III, we obtain an invariant measure on the space of distinct microstates by means of which one can formulate the statistical mechanics for any DSR model in any ensemble. In section IV, we explore the cosmological applications of the setup when applied to the thermodynamics of the early radiation dominated universe. Section V is devoted to the summary and conclusions.

II. CURVED MOMENTUM SPACES IN DSR THEORIES

As we have mentioned above, the DSR theories can be understood as different coordinate systems on the dS or AdS momentum spaces [10,12]. However, it should be noted that while the dS geometry of momentum space can be inspired by the standard structure of the \( \kappa \)-Poincaré Hopf algebra [2,8], such a quantum algebraic structure is not investigated for the case of the AdS momentum space (see however Ref. [14]). In fact, taking a minimal observer-independent length scale into account naturally leads us to deformed Lorentz transformations and curved momentum spaces [3]. The relevance of dS and AdS momentum spaces with the DSR theories may be easily realized when one notes that the Minkowski momentum space in the standard special relativity admits ten isometries and dS and AdS spaces are the only spaces (with Lorentzian signature in four dimension) that have the same number of isometries. Furthermore, the constant curvature of these spaces is consistent with the conjecture of observer independence of the quantum gravity scale. Thus, implementing these spaces naturally provides deformed Lorentz transformations that include an observer-independent quantum gravity scale. Also, since these spaces are asymptotically equivalent to the Minkowski spacetime (correspondence principle), the deformed Lorentz transformations reduce to their standard form in the flat limit (corresponding to the low energy regime). On the other hand, while the energy and momentum of a relativistic particle are defined in the usual way in the standard special relativity, we have freedom to define them in DSR theories on the curved momentum spaces such that there is no a clear reason to prefer one basis to another [7,18]. Apart from this feature which shows the importance of the local properties of the curved momentum spaces in DSR theories, it is also important to note the global topology of these spaces. The topologies of dS and AdS are \( \mathbb{R} \times S^3 \) and \( \mathbb{S}^1 \times \mathbb{R}^3 \) respectively. While a maximal momentum arises by a reasonable identification of \( \mathbb{R} \) with the space of energy and \( S^3 \) with the space of momenta in dS momentum space, a maximal energy arises when one identifies \( \mathbb{S}^1 \) with the space of energy and \( \mathbb{R}^3 \) with the space of momenta in AdS momentum space. In this respect, it seems that dS and AdS momentum spaces will be dual to each other. However, some interesting features arise in AdS momentum space which are not predicted by this expected duality, and therefore, these two spaces are qualitatively different (see Ref. [13]). These standard identifications lead to DSR theories with isotropic varying speed of light \( c = dE/dp \) while other identifications lead to the nonisotropic varying speed of light (see Ref. [13]). Indeed, the group of symmetries of dS space is \( SO(4,1) \) and the Lorentz symmetry can be preserved by identifying the Lorentz transformations with the six elements of the subgroup \( SO(3,1) \) and the four remaining generators with the positions [10]. In the same manner, the symmetry group of AdS space is \( SO(3,2) \), and the Lorentz invariance can be preserved through the identification of the Lorentz transformations with the subgroup \( SO(3,1) \) of \( SO(3,2) \). Interestingly, the commutation relations between the positions belonging to the two quotients of two algebras \( so(4,1)/so(3,1) \) in dS and \( so(3,2)/so(3,1) \) in AdS turn out to be noncommutative [10,14]. This feature is general for any DSR theory on different coordinate systems on dS or AdS momentum spaces. The space of four-momenta then will be the quotient spaces \( SO(4,1)/SO(3,1) \) and \( SO(3,2)/SO(3,1) \) in the case of dS and AdS momentum spaces respectively. Clearly, the corresponding extended eight-dimensional phase space is noncommutative with topology \( \mathbb{R}^4 \times \mathbb{dS} \) [11] and \( \mathbb{R}^4 \times \mathbb{AdS} \). At the flat low energy limit, the minimal observer-independent effects become negligible and both dS and AdS reduce to the Minkowski space with the standard phase space with \( \mathbb{R}^4 \times \mathbb{R}^4 \) topology. It is also interesting to note that while gravity can be understood as the curvature of the spacetime sector in a general theory of relativity, a minimal observer-independent length scale, as a universal UV cutoff, can be understood as a constant curvature of the momentum sector of the extended phase space in DSR theories, motivated by the effects of invariant quantum gravity scale on general relativity, known as gravity’s rainbow, also can be considered [19].

The curved four-momentum spaces in DSR theories then can be realized from the four-dimensional hypersurfaces [13]

\[-P_0^2 + P_1^2 + P_2^2 + P_3^2 = \pm l^{-2}, \tag{1}\]

which are embedded in five-dimensional flat spaces with signatures \((-\mathbb{+}+,\mathbb{+},\mathbb{+},\mathbb{+},\mathbb{+})\) and \((-\mathbb{-},\mathbb{+},\mathbb{+},\mathbb{+},\mathbb{+})\) for dS and AdS cases, respectively [20]. In relation (1), \( P_\Lambda \) terms with \( A = 0, \ldots, 4 \) are embedding coordinates and \( l \), with dimension of length, is the radius which signals an observer-independent length scale. In order to ensure that the quantum gravity effects become important just at the very high energy regime, the invariant length scale \( l \) is usually assumed to be of the order of the Planck length \( l = \beta_0 \ell_\text{Pl} \), where \( \beta_0 = \mathcal{O}(1) \) should be fixed by the
experiments \[21\]. The corresponding line elements then will be

\[ds^2 = -dP_0^2 + dP_1^2 + dP_2^2 + dP_3^2 \pm dP_4^2.\]  

(2)

In relations (1) and (2), the (+) and (−) signs denote the dS and AdS spaces respectively. One then can consider particular coordinate system on both dS and AdS momentum spaces by fixing the embedding coordinates \(P_A\) in terms of physical energy and momenta. As a common way one may solve the constraint relation (1) for \(P_4\) and then substitute the result into relation (2) which gives the metric of the four-dimensional curved momentum space. In this way, all the well-known DSR theories can be realized by a suitable fixing of the embedding coordinates \(P_A\). Interestingly, the Snyder algebra \[8\] can be derived from this setup \[13\], and therefore, it is nothing but a particular DSR theory. It is also possible to consider a deformed relativistic algebra in which the embedding coordinate \(P_4\) has not been removed and is present in the resultant associated four-dimensional algebra. Such an algebra, for instance, is investigated in Ref. \[22\] in the context of the stability theory of the Lie algebras in which \(P_4\) plays the role of nontrivial center of the resultant deformed algebra. In comparison with the Snyder and other DSR algebras such as bi-cross product algebra, this stable algebra cannot be considered as a closed algebra in four dimensions \[18\].

Among all the possible coordinate systems on dS and AdS momentum spaces, the natural coordinate system on dS momentum space is inspired by the bi-cross product basis of \(\kappa\)-Poincaré algebra which is known as the cosmological coordinates since it corresponds to the cosmological rendition of dS space in position space. This DSR theory defined the deformed Lorentz transformation such that the Lorentz symmetry is preserved. On the other hand, its counterpart on AdS momentum space, i.e., the DSR theory defined by cosmological coordinates on AdS momentum space, breaks the Lorentz symmetry. The Lorentz invariant DSR theory is then found in static coordinatization of AdS momentum space \[13\]. Also, the static coordinatization of dS momentum space is investigated, which breaks the Lorentz invariance. Although the DSR theories are different from each other, as a candidate for the flat limit of ultimate quantum gravity theory, all of them are possible, and there is not a clear physical reason to prefer one over the other. In the next subsections, we therefore review the results of Ref. \[13\] for dS and AdS momentum spaces in both of the cosmological and static coordinates. Our task is to generally formulate the statistical mechanics in DSR theories defined on curved momentum spaces and then compare the different DSR theories (that arise from different coordinatization on momentum spaces) from the thermostatistical point of view.

### A. de Sitter momentum space

#### 1. Cosmological coordinate (dS-Cosm model)

The relations between the induced physical energy and momenta \((E, p_i)\) and the embedding coordinates in the cosmological coordinate system are defined as \[13\]

\[
P_0(E, \vec{p}) = \frac{1}{l} \sinh(lE) + \frac{l^2 p^2}{2} \exp(lE),
\]

\[
P_1(E, \vec{p}) = -p_i \exp(lE),
\]

\[
P_4(E, \vec{p}) = \frac{1}{l} \cosh(lE) + \frac{l^2 p^2}{2} \exp(lE),
\]

where \(p = |\vec{p}| = \sqrt{\delta^{ij} p_i p_j}\) with \(i, j = 1, 2, 3\). Rewriting line element (2) with a (+) sign (corresponds to ds space) in terms of the physical energy and momenta \((E, p_i)\) defined by the above relations and then removing \(P_4\) by means of constraint (1) (again with a (+) sign), the line element of dS momentum space works out to be

\[ds^2 = -dE^2 + \exp(2lE) \sum_{i=1}^{3} dp_i^2.\]  

(4)

The above line element gives the invariant integration measure \[22\]

\[
\frac{d\mu(E, \vec{p})}{4\pi} = \exp(3lE)dE p^2 dp,
\]

(5)

on the momentum space. The corresponding deformed mass-shell condition is determined by demanding \(P_4\) to be constant in (1) as \(-P_0^2 + P^2 = l^2 - P_4^2 = m^2\) with \(P_0 > 0\) and \(P_4 < 0\). For the massless case \(m = 0\), with which we are interested in this paper, it gives

\[C \left(1 + \frac{l^2 C}{4}\right) = 0,\]

(6)

where

\[C = -\frac{4}{l^2} \sinh^2(lE/2) + p^2 \exp(lE).\]

(7)

Solving the above constraint gives the modified dispersion relation \(E = -l^{-1} \ln(1 - lp)\). This dispersion relation also shows that there is a maximal momentum as \(p \leq 1/l\) which is the consequence of compact \(S^3\) topology of the space of momenta \[22\]. The energy \(E\), however, can take any positive value as \(E \in [0, \infty)\). In the flat low energy limit \(lE \ll E/p_i\), the line element reduces to the flat case \(ds^2 \approx -dE^2 + \sum_{i=1}^{3} dp_i^2\), the invariant measure \[6\] reduces to the standard well-known measure \(d\mu(E, \vec{p}) = 4\pi dE p^2 dp\), and the deformed mass-shell condition \[7\] also leads to the standard Einsteinian dispersion relation \(C = -E^2 + p^2\) for the massless particles with \(E, p \in [0, \infty)\). The modified dispersion relation \[7\] immediately leads to the varying speed of light \(c = \frac{dE}{dp} = \exp(-lE) = (1 - lp)^{-1}\) (see also Ref. \[13\]).
2. Static coordinate (dS-Stat model)

We do not repeat all the calculations for the DSR theory defined by the static coordinatization of dS momentum space and only review the main results we deal with throughout this paper (see Ref. [13] for more details).

The line element associated with the static coordinate system defined on dS momentum space is given by

$$ds^2 = -(1-l^2p^2)dE^2 + \frac{dp^2}{1-l^2p^2} + p^2dQ^2,$$  

(8)

where clearly $dQ^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the metric of the two-sphere with unit radius. The corresponding invariant integration measure is

$$\frac{d\mu(E, p)}{4\pi} = dEp^2 dp,$$  

(9)

which remains unchanged. For the massless particles, the Casimir is given by

$$C = -\frac{1}{l^2} \sinh^2(lE)(1-l^2p^2) + p^2 = 0,$$  

(10)

which gives the modified dispersion relation $E = l^{-1}\tan^{-1}(lp)$ with $p \leq 1/l$ and $E \in [0, \infty)$. In the low energy limit $lE \ll l/E_{\text{Pl}} \ll 1$, the line element correctly reduces to the flat Minkowski one and the deformed mass-shell condition (10) leads to the usual dispersion relation $C = -E^2 + p^2$ for the massless particles with $E, p \in [0, \infty)$. The modified dispersion relation (10) also leads to the varying speed of light as $c = \frac{dE}{dp} = (1-l^2p^2)^{-1}$ in this chart.

B. Anti-de Sitter momentum space

1. Cosmological coordinate (AdS-Cosm model)

For the case of the cosmological coordinate system on AdS momentum space, the relations between induced physical energy and momenta $(E, p)$ and the embedding coordinate are defined as follows [13]:

$$P_0(E, p) = \frac{1}{l} \sin(lE),$$  

$$P_i(E, p) = p_i \cos(lE),$$  

$$P_4(E, p) = \frac{1}{l} \cos(lE)\sqrt{1+l^2p^2}. $$  

(11)

Similar to the case of dS momentum space, substituting from the above relations into the relation (2) and then removing $P_0$ from the (−) sign of constraint (11), the line element of AdS momentum space in terms of physical energy and momenta $(E, p)$ turns out to be

$$ds^2 = -dE^2 + \cos^2(lE) \left( \frac{dp^2}{1+l^2p^2} + p^2dQ^2 \right). $$  

(12)

The invariant measure on the momentum space then will be

$$\frac{d\mu(E, p)}{4\pi} = \cos^3(lE) dE \frac{p^2 dp}{\sqrt{1+l^2p^2}}. $$  

(13)

The associated mass-shell condition $-P_0^2 + P_i^2 = l^{-2} - P_4^2 = m^2$ then leads to

$$C = -\frac{1}{l^2} \sin^2(lE) + p^2 \cos^2(lE) = 0, $$  

(14)

for the massless case $m = 0$. By solving the above relation one can easily find the modified dispersion relation $E = l^{-1}\tan^{-1}(lp)$ which clearly implies a maximal energy $E \leq +\pi/2l$, and thus in relation (11) we have $0 \leq P_0 \leq +1/l$. Relations (12), (13), and (14) reduce to their standard counterparts in the flat low energy limit $lE \propto E/E_{\text{Pl}} \ll 1$. The modified dispersion relation (14) also implies the variation of the speed of light at high energy regime as $c = \frac{dE}{dp} = \cos^2(lE) = (1+l^2p^2)^{-1}$ [13].

2. Static coordinate (AdS-Stat model)

In static coordinatization of AdS momentum space, the line element takes the following form

$$ds^2 = -(1+l^2p^2)dE^2 + \frac{dp^2}{1+l^2p^2} + p^2dQ^2, $$  

(15)

and therefore the integration measure remains unchanged as

$$\frac{d\mu(E, p)}{4\pi} = dEp^2 dp. $$  

(16)

The Casimir invariant is given by

$$C = -\frac{1}{l^2} \sin^2(lE)(1+l^2p^2) + p^2 = 0,$$  

(17)

for the massless case which implies the modified dispersion relation $E = l^{-1}\tan^{-1}(lp)$ with $p \in [0, \infty)$ and $E \leq \pi/2l$. This modified dispersion relation implies the varying speed of light as $c = \frac{dE}{dp} = (1+l^2p^2)^{-1}$.

III. STATISTICAL MECHANICS: INVARIANT MEASURE

In this section, we generally formulate the statistical mechanics for DSR theories. The standard statistical mechanics is based on the invariant measure (density of states) of the physical phase space of a nonrelativistic system which is a six-dimensional symplectic manifold (for a system consisting of one particle) and determines the number of microstates at the semiclassical regime. In the case of relativistically formulated theories like DSR theories, we deal with an invariant measure on an extended
eight-dimensional phase space. In order to formulate the statistical mechanics for such systems, we should be able to count the number of distinct accessible microstates of the system by finding the corresponding appropriate measure on its physical phase space \[\Gamma_X\] \([23]\). Some attempts have been made in this direction to study the different statistical systems in the context of DSR theories (see Refs. \[16, 17\]). But, here, we would like to generally formulate the statistical mechanics of DSR theories in a more systematic way. The approach we introduce allows one to study the thermodynamical properties of statistical systems in any ensemble for any DSR theory.

To find the invariant measure on the six-dimensional physical phase space, we start with a natural measure on the eight-dimensional extended phase space \(\Gamma_X \equiv (t, \vec{x}; E, \vec{p})\) that is given by \[\mu_X = \int d\mu_X = \int d\mu(t, \vec{x})d\mu(E, \vec{p}), \tag{18}\]

where \(d\mu(t, \vec{x})\) and \(d\mu(E, \vec{p})\) are the standard invariant volume elements on the spacetime and momentum sector of \(\Gamma_X\) which are defined by the metrics on these spaces. The metric on the spacetime sector is flat since DSR theories are the flat limit of the ultimate quantum gravity theories (see Ref. \[3\], and therefore, the curvature of the spacetime sector will be zero \[23\]. It is, however, important to note that while the metric is flat, it is not defined on the standard (commutative) Minkowski spacetime. But, it is indeed defined on a noncommutative \(\kappa\)-Minkowski spacetime (\(\kappa \approx l^{-1}\) in our notation) dual to the corresponding curved momentum space \[8\]. For many-particle systems that one usually considers in field theory and statistical mechanics, there is not an appropriate (well-defined) measure on noncommutative spacetime which respects all the desired symmetries. More precisely, the standard Lebesgue measure \(d\mu(t, \vec{x}) = dt^4x = dt^4x\) respects the \(\kappa\)-Poincaré symmetries while it evidently cannot support the cyclicity of the action functional (see Ref. \[23\] for more details). Trying to recover the cyclicity of the action functional, one, however, should renounce the \(\kappa\)-Poincaré invariance of the theory and also the correspondence principle such that the standard commutative Minkowski spacetime would not be obtained from the corresponding \(\kappa\)-Minkowski spacetime in the low energy limit \(\kappa \rightarrow \infty\) (or equivalently \(l \rightarrow 0\)). The problem is not yet definitively answered. In this respect, we consider the standard Lebesgue measure \(d\mu(t, \vec{x}) = dt^4x = dt^4x\) for the configuration space of \(\Gamma_X\) which respects both the \(\kappa\)-Minkowski spacetime structure and correspondence principle. For the momentum sector, the measure is completely defined by the metric on curved dS or AdS momentum spaces as \(d\mu(E, \vec{p}) = 4\pi \sqrt{-g} dt^4p\) where \(g\) is the determinant of the metric of the associated curved momentum space.

Measure \(\mu_X\) has all the properties of a measure. The points on the constraint, however, are not totally distinct microstates. Indeed, the constraint is subdivided into the equivalent classes of microstates which are linked by the time evolution that is generated by the constraint itself (orbits). They are physically equivalent since the time evolution induced by the constraint is nothing but a gauge transformation in the relativistically formulated theories such as DSR theories. Taking the coordinate \(t\) to be time, we can parametrize microstates in each set by \(t\). Then, to obtain the space of the distinct microstates, we must consider only one microstate of each equivalent class (gauge fixing). This can be done by choosing the slice (that is a six-dimensional manifold) of \(t = t_0\) on the constraint that is appropriately intersecting with orbits. This space provides the physical phase space of the system and one can find a measure on it by \(\mu_C\) and the constraint \(\delta(t - t_0)\) through the disintegration theorem. Using again the disintegration theorem to fix the gauge, the appropriate measure on the physical phase space then turns out to be

\[
\mu_p = \int d\mu_p = \int \delta(C)\delta(t - t_0) d\mu_X \tag{20}
\]

\[
= \int \delta(C)\delta(t - t_0) d\mu(t, \vec{x})d\mu(E, \vec{p})\ .
\]

This is a natural measure on the space of distinct and physically relevant microstates for a statistical system. Measure \(\mu_p\) is indeed nothing but the density of states when one selects particular ensemble density such as Dirac delta function, Boltzmann factor, Bose-Einstein, or Fermi-Dirac ensemble densities for microcanonical, canonical, Bose-Einstein, and Fermi-Dirac statistics, respectively. Having measure \(\mu_p\) at hand, we are then adequately equipped to generally formulate the statistical mechanics for DSR theories in any ensemble.

\[\text{IV. EARLY UNIVERSE THERMODYNAMICS}\]

One of the most interesting features of the presented setup is its application to the thermodynamics of the early radiation dominated universe. One then should study the statistical mechanics of the effectively massless particles (bosons and fermions) which contribute to
the energy content of the radiation dominated universe. Measure (20), however, does not an analytical solution for the Bose-Einstein and Fermi-Dirac ensemble densities for four models that are introduced in this paper. Nevertheless, we will introduce an approach with which one can realize general features of any DSR theory, which is important for early universe thermodynamics, without attribution to any ensemble density.

A. Total number of microstates

To do so, we consider the total number of microstates for a particle in a DSR framework. In statistical mechanics, the density of states determines the number of accessible microstates for the system under consideration. The density of states is determined by measure (20) when one fixes a particular ensemble density. For example, one should consider the well-known Bose-Einstein and Fermi-Dirac ensemble densities to study the statistical mechanics of the early radiation dominated universe. But, what do all the ensemble densities do in statistical mechanics formalism? They indeed define the probability distribution over the set of all microstates by restricting the system to the subset of accessible microstates from the infinite set of all microstates that the system can potentially access. The number of total microstates is determined by measure (20) without attribution to any ensemble density. To be more precise, the particles in the early Universe (such as photons and electrons) are nonlocalized, and therefore, the spacetime part of the measure (20) simply reduces to the physical volume V (in which the particles are confined) as

$$\Omega = \frac{V}{\hbar^3} \int \delta(C) d\mu(E, \vec{p}). \quad (21)$$

in which, to take into account the Heisenberg uncertainty principle in semiclassical statistical mechanics, we have divided the measure (20) by $\hbar^3$, with $\hbar$ being the Planck constant. Note that although $h = 2\pi$ in our unites since $h = 1$, we explicitly work with $h$ rather than $2\pi$ to show its significance in the determination of the number of microstates. For the standard early universe thermodynamics, the bosons and fermions obey the usual Einsteinian dispersion relation $E = p$ with nondeformed measure, and the total number of microstates (21) is diverging as

$$\Omega(V) = \frac{4\pi V}{\hbar^3} \int_0^\infty E^2 dE \to \infty. \quad (22)$$

It is important to note that in the standard statistical mechanics of the early Universe, the Bose-Einstein and Fermi-Dirac ensemble densities select a finite subset of microstates from the infinite total number of microstates (22) as the accessible microstates for the system. But, however, the system can access more and more microstates by increasing the energy. The reason for which we have considered the total number of microstates will become clear when one is interested in DSR theories which predict an upper bound for the total energy of the system. Let us calculate (21) for the four different DSR models that are introduced in this paper. Using the results of section II in relation (21), leads to the following results

$$\Omega(V,l) = \left\{ \begin{array}{ll}
\frac{16\pi V}{\hbar^3} \int_0^\infty e^{2lE} \sin^2(lE/2) dE \to \infty, & \text{dS-Cosm} \\
\frac{4\pi V}{\hbar^3} \int_0^\infty \tanh^2(lE) dE \to \infty, & \text{dS-Stat} \\
\frac{\pi V}{\hbar l} \int_0^\pi \sin^2(2lE) dE = \frac{\pi^3 V}{l^3}, & \text{AdS-Cosm} \\
\frac{4\pi V}{\hbar^3} \int_0^\pi \tanh^2(2lE) dE \to \infty, & \text{AdS-Stat} 
\end{array} \right. \quad (23)$$

The above results show that the number of total microstates for a particle is infinite for dS-Cosm, dS-Stat, and AdS-Stat models while it is finite for the case of the AdS-Cosm model. Let us elaborate more on the results (23). By increasing the kinematical energy $E$, a statistical system can access more and more microstates. For DSR theories with maximal energy such as AdS-Cosm and AdS-Stat, this process cannot infinitely continue since there is an upper bound $E \in [0, \pi/2l]$, while in standard special relativity we have $(E \in [0, \infty])$. The compact $S^1$ topology for the DSR theories that are defined on AdS momentum space, makes the total volume of the energy space always finite. Measure (21) or (20) is however defined on the whole of the momentum space including the space of momenta $\vec{p}$ that is not compact for the AdS case. Therefore, having just compact energy space cannot make the total number of microstates (21) finite. The space of momenta $\vec{p}$ affects the total number of microstates (21) from constraint $C$. In this respect, apart from the compact topology of the energy space which is a necessary condition to have a finite number of microstates, we should also explore another enough condition which would explain how the number of total microstates is finite in the AdS-Cosm model while it is infinite for the AdS-Stat case. The key is indeed the dimensional reduction at the UV regime which leads to the reduction of the number of microstates in this regime. This is the common feature of almost all quantum gravity candidates, and the DSR models also predict dimensional reduction at the UV regime. More precisely, by parametrizing the constraint by integer $\gamma$ as $C(1 + l^2 \gamma^2)$, one can realize dynamical dimensional reduction for the Hausdorff dimension of the momentum space. Following Ref. 31, one also identifies the Hausdorff dimension of the momentum space with the spectral dimension of the spacetime sector and then interprets the Hausdorff dimensional reduction of the momentum space as the spectral dimension reduction in the spacetime sector. For the four models which we have considered in this paper with the standard Hausdorff dimension of momentum space equal to 4 in the IR regime, the Hausdorff
dimension at the UV regime runs as

\[ d_H(4, \gamma) = \begin{cases} 
\frac{6}{1+\gamma}, & \text{dS-Cosm} \\
\frac{3}{1+\gamma}, & \text{dS-Stat} \\
\frac{3}{1+\gamma}, & \text{AdS-Cosm} \\
\frac{4}{1+\gamma}, & \text{AdS-Stat} 
\end{cases} \] (24)

From the results of section II, it is clear that \( \gamma = 1 \) for the dS-Cosm model and \( \gamma = 0 \) for the three other models. According to (24), the dS-Cosm, dS-Stat and AdS-Cosm models imply the dynamical dimensional reduction by 1 at the UV regime while the AdS-Stat model does not. Note that the AdS-Cosm model is the only model that has both the necessary and enough conditions: compact energy space and also dimensional reduction in the UV regime. We conjecture that these conditions are sufficient to have a finite total number of microstates for the statistical systems. The results (23) and (24) show that the existence of a finite total number of microstates immediately leads to an entropy bound for the system under consideration which is also a common feature of quantum gravitational systems such as black holes [34].

**B. Entropy and energy density bounds**

To obtain the entropy bound, we note that in the AdS-Cosm model with a finite total number of microstates [24] for one particle, the total number of microstates for the system consisting of \( N \) such particles will be \( \Omega_N = \Omega^N / N! \) where the Gibbs factor is also considered since the particles are indistinguishable. The associated maximum entropy \( S_{\text{max}} = \ln \Omega_N \) then takes the following form

\[ \frac{S_{\text{max}}}{N} = \ln \left( \frac{V}{h^3 l^3} \right) + \ln \left( \frac{\pi^2}{4N} \right) + 1, \] (25)

in which we have used (23) and also the Stirlings approximation \( \ln N! = N \ln N - N \). Taking the fact that \( hl \sim l_{\text{Pl}} \) into account in relations (23) and (24), one can see that the total number of microstates for the universe is precisely determined by the factor \( V/l_{\text{Pl}}^3 \). This result shows that the fundamental volume of microstates for the quantum gravitational statistical system will be proportional to \( l_{\text{Pl}}^3 \) with which the physical volume \( V \) is quantized. We note however that the fundamental volume of microstates in the standard statistical system (for a particle) is \( h^3 \) with which the phase space volume is quantized. In some senses this feature is similar to the case of Bekenstein-Hawking entropy of black holes where the number of microstates is determined by the factor \( A/l_{\text{Pl}}^2 \) with \( A \) being the horizon area of the black hole [31, 32].

On the other hand, the internal energy \( U \) is the average of the kinematical energy \( E \), and the total potentially accessible internal energy for the statistical system consisting of \( N \) particles in this setup then will be

\[ U_{\text{tot}} = N \times \left( \frac{\int E \, d\mu_p}{\int d\mu_p} \right). \] (26)

Substituting the Einsteinian dispersion relation into the above relation, one realizes that this relation is diverging in the framework of standard special relativity. From the statistical point of view, this is because of the fact that the system can access more and more microstates by increasing the kinematical energy \( E \). This possibility in standard special relativity leads to the well-known cosmological feature that there is not an upper bound for the energy density of the early radiation dominated universe (the standard Stefan-Boltzmann law). Calculating relation (26) for the four presented DSR models, we deduce that it is diverging for all the models except the AdS-Cosm model. In this case, relation (26) converges to

\[ \frac{U_{\text{max}}}{N} = \frac{4l}{\pi} \int_0^{\pi} \sin^2(2lE)E dE = \frac{\pi}{4l}. \] (27)

The above result shows that the existence of the upper bound \( E \leq \pi/2l \) together with the dimensional reduction at the UV regime (which drastically reduces the number of microstates) leads to the nontrivial upper bound (27) for the internal energy as \( U \leq U_{\text{max}} \sim E_{\text{Pl}} \). This bound together with the entropy bound (25) leads to an upper bound for the energy density of the early radiation dominated universe as \( \rho \leq \rho_{\text{max}} \) with \( \rho_{\text{max}} = U_{\text{max}} / V = (U_{\text{max}} / N) / (V / N) \) which after substituting \( (V / N) \) from (24) and some manipulations is given by

\[ \rho_{\text{max}} = \frac{e \gamma^3}{16h^3 l^4} \exp \left( - \frac{S_{\text{max}}}{N} \right). \] (28)

The above relation shows that \( \rho \leq \rho_{\text{max}} \sim T_{\text{Pl}}^4 \).

**V. COSMOLOGY: DSR VERSUS LQC**

What is the cosmological implication of bounds (25) and (28)? Consider the standard Friedmann equation

\[ H^2 = \frac{8\pi G}{3} \rho, \] (29)

where \( H = \dot{a} / a \) is the Hubble parameter with \( a \) being the scale factor. At first glance, one can deduce that the existence of the upper bound (28) for the energy density implies an upper bound for the Hubble parameter through the Friedmann equation (29). To be more precise, one should note that the geometric part of the Friedmann
equation is completely determined by the classical Einstein’s equations which do not predict any upper bound for the Hubble parameter. Indeed, it is the standard statistical mechanics that is consistent with the standard classical Einstein’s equations such that both the Hubble parameter and energy density of the radiation dominated universe diverge at a big bang leading to the so-called big bang singularity problem. We however notice that the upper bound that arises for the energy density of the radiation dominated universe in our setup is due to the quantum gravitational (minimal length) effects. Thus, the inconsistency between the right- and left-hand sides of the Friedmann equation arises when one applies the quantum gravitational effects for the matter content while considering the geometric part to be purely classical. We should therefore explore quantum gravitational effects for the geometric part which support the energy density bound \(28\) and also entropy bound \(25\) that we have obtained in the DSR framework. Very interestingly, the modified Friedmann equations that are suggested by loop quantum cosmology (LQC) \([36]\) predict an upper bound for the energy density. The modified Friedmann equation for the flat early radiation dominated universe is given by \([37]\)

\[
H^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_c}\right),
\]

where

\[
\rho_c = \frac{3}{(8\pi G a_0 \gamma^2 l_{\text{Pl}}^2)},
\]

where \(\gamma\) is the Barbero-Immirzi parameter which should be fixed by black hole entropy calculations \([38]\) and \(a_0 = 4\sqrt{3}\pi\gamma\) is a numerical parameter. In relation \(30\), \(\rho \leq \rho_c\) and therefore \(H \leq \left(\frac{2\pi G}{3\rho_c}\right)^{1/2}\). Now, with looking at the result \(27\) we may modify the effective Friedmann equation \(30\) by following the identification of the energy density bounds,

\[
\rho \leq \rho_{\text{max}} = \rho_c.
\]

This identification shows the relevant correspondence between DSR theory, on one side and LQC on the other side. Relation \(32\) also determines the numerical value of the entropy bound \(25\). The natural identification \(32\) leads to the upper bound for the Hubble parameter,

\[
H \leq H_{\text{max}} = \left(\frac{2\pi G}{3\rho_{\text{max}}}\right)^{1/2},
\]

through the effective Friedmann equation \(30\). Note that the bound for the energy density (or Hubble parameter) in LQC is obtained from the holonomy-flux algebra through the quantization of the flat FRW geometry by the method of loop quantum gravity while bound \(25\) is obtained in a very different manner \textit{i.e.} by statistical considerations of the particles in the early Universe in the DSR framework. However, these two different pictures match each other in a fascinating manner. This result also confirms that DSR can prepare a suitable framework for the (semiclassical) flat limit of quantum gravity.

It should also be noted that the scale factor takes a minimum nonzero value in the context of LQC. This feature of LQC can also be realized from our setup through the well-known adiabatic condition for the universe (see also Ref. \([34]\)),

\[
S a^3 = \text{cons.},
\]

where \(S\) is the entropy of the radiation dominated universe. The existence of the entropy bound \(25\) implies that there is a nonzero minimum value for the scale factor as

\[
a \geq a_{\text{min}} = \left(\frac{\text{cons.}}{S_{\text{max}}}\right)^{1/3}.
\]

Using relation \(25\) in the above relation and also applying the fact that \(N/V\) is constant for \(N, V \to \infty\), as one usually assumed in standard statistical mechanics, one can show that \(a_{\text{min}} \sim l_{\text{Pl}}\). Therefore, the consistency between the statistical mechanics in the AdS-Cosm DSR model on the one hand and the results of loop quantum cosmology on the other hand is completed: The energy density and Hubble parameter approach the maximum values \(22\) and \(33\) when the scale factor approaches the minimum nonzero value \(25\). Thus, the singularity resolution in radiation dominated universe is completely understood from both the geometrical (LQC) and thermodynamical (statistical mechanics in DSR) sides.

In summary, the standard thermodynamical results of a radiation dominated universe match the classical (usual) Friedmann equations and cannot support the modified Friedmann equation \(30\). On the other hand, the statistical mechanics based on the AdS-Cosm DSR model matches the effective Friedmann equation \(30\) and cannot match the standard Friedmann equation \(29\). However, it is important to note that although the LQC geometry and the DSR statistical mechanics are qualitatively consistent, the geometry and matter parts were not obtained from a unique setup. One then attempts to explore a bridge between these two setups which seem to be mathematically and conceptually very different. Nevertheless, any DSR theory with modified dispersion relation leads to the modification of geometry such that the spacetime metric becomes energy dependent \([19]\). In this respect, it seems possible to reobtain the LQC geometry from a DSR theory (or maybe a class of them) in the context of gravity’s rainbow. We are going to study such a setup for the early radiation dominated universe in the next research program.

While there is not a clear physical reason to prefer one DSR theory over the other, thermostatistical consideration suggests that the theories with a finite total number of microstates such as AdS-Cosm are more admissible. It should be noted that the AdS-Cosm model breaks the Lorentz invariance while dS-Cosm and AdS-Stat do not.
Although it is not clear that the Lorentz symmetry will be broken or deformed at the UV regime, it is interesting to study a DSR theory which preserves the Lorentz invariance and also supports the existence of finite total number of microstates. Such a DSR theory is relevant from both the kinematical and thermodynamical point of views.

VI. SUMMARY AND CONCLUSIONS

Existence of a minimum length scale, below which no other length scales can be probed, is the main feature of quantum gravity candidates such as string theory and loop quantum gravity. Although a complete theory of quantum gravity is not yet formulated, it is natural to expect that a nongravitational theory which supports the existence of a minimal length scale arises at the flat limit (weak gravity limit but high energy regime) of the ultimate quantum theory of gravity. The DSR theories are then investigated in order to take into account a minimal observer-independent length scale in special relativity. These theories are formulated on the dS and AdS momentum spaces. There are various kinds of DSR theories which can be realized from the different coordinatization of these curved momentum spaces. Since the topology of the dS and AdS spaces are $\mathbb{R} \times S^3$ and $S^1 \times \mathbb{R}^3$ respectively, a maximal momentum and maximal energy naturally arise in dS and AdS momentum spaces, respectively, by demanding an isotropic (varying) speed of light.

In order to study the associated statistical mechanics, we first introduced a natural measure on the extended phase space. In light of the disintegration theorem, we can easily study the thermostatistics of any DSR theory in any ensemble. Without attribution to any ensemble density, and quite generally, we have studied the general statistical properties of four DSR models: dS-Cosm, dS-Stat, AdS-Cosm, and AdS-Stat. Applying the setup to the statistical mechanics of the early radiation dominated universe we have shown that the total number of microstates for the AdS-Cosm model is finite. We conjecture that this result emerges for two reasons: having compact energy space and dimensional reduction at the UV regime. The AdS-Cosm model is the only model that has both of these properties. We then calculated the corresponding entropy and internal energy bounds in this model, and we have explored the cosmological implications of these results. We found that the geometry of the standard Friedmann equations is no longer applicable to respect these results since they cannot support the existence of an upper bound for the energy density of the radiation dominated universe. We have shown that the AdS-Cosm DSR model respects the geometry of effective Friedmann equations that arise from the context of loop quantum cosmology. The existence of a minimum nonzero scale factor that arises in loop quantum cosmology can also be understood by means of the resultant entropy bound in the DSR setup through the adiabatic condition for the universe. Finally, it seems that the DSR theories which predict a finite total number of microstates, such as the AdS-Cosm model, are more relevant from the thermostatistical point of view, and they can be considered as good candidates for the flat limit of the quantum gravity proposal.

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In the context of triply special relativity, the measure of position spacetime may be changed even if one neglects gravity. The curvature of spacetime in such non-gravitational theories will be due to an invariant IR scale and the cosmological constant may be interpreted as a universal IR scale without purely gravitational origin.

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