STABILITY AND COMPACTNESS
FOR COMPLETE $f$-MINIMAL SURFACES

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Abstract. Let $(M, g, e^{-f}d\mu)$ be a complete metric measure space with Bakry-Émery Ricci curvature bounded below by a positive constant. We prove that in $M$ there is no complete two-sided $L_f$-stable immersed $f$-minimal hypersurface with finite weighted volume. Further, if $M$ is a 3-manifold, we prove a smooth compactness theorem for the space of complete embedded $f$-minimal surfaces in $M$ with the uniform upper bounds of genus and weighted volume, which generalizes the compactness theorem for complete self-shrinkers in $\mathbb{R}^3$ by Colding-Minicozzi.

1. Introduction

Recall that a self-shrinker (for mean curvature flow in $\mathbb{R}^{n+1}$) is a hypersurface $\Sigma$ immersed in the Euclidean space $(\mathbb{R}^{n+1}, g_{can})$ satisfying that

$$H = \frac{1}{2} \langle x, \nu \rangle,$$

where $x$ is the position vector in $\mathbb{R}^{n+1}$, $\nu$ is the unit normal at $x$, and $H$ is the mean curvature of $\Sigma$ at $x$. Self-shrinkers play an important role in the study of singularity of mean curvature flow and have been studied by many people in recent years. We refer to [4], [5] and the references therein. In particular, Colding-Minicozzi [4] proved the following compactness theorem for self-shrinkers in $\mathbb{R}^3$.

**Theorem 1** ([4]). Given an integer $g \geq 0$ and a constant $V > 0$, the space $S(g, V)$ of smooth complete embedded self-shrinkers $\Sigma \subset \mathbb{R}^3$ with

- genus at most $g$,
- $\partial \Sigma = \emptyset$,
- $\text{Area}(B_R(x_0) \cap \Sigma) \leq VR^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$

is compact. Namely, any sequence of these has a subsequence that converges in the topology of $C^m$ convergence on compact subsets for any $m \geq 2$.

In this paper, we extend Theorem 1 to the space of complete embedded $f$-minimal surfaces in a 3-manifold. A hypersurface $\Sigma$ immersed in a Riemannian manifold $(M, g)$ is called an $f$-minimal hypersurface if its mean curvature $H$ satisfies that, for any $p \in \Sigma$,

$$H = \langle \nabla f, \nu \rangle,$$

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where $f$ is a smooth function defined on $M$ and $\nabla f$ denotes the gradient of $f$ on $M$. Here are some examples of $f$-minimal hypersurfaces:

- $f \equiv C$, an $f$-minimal hypersurface is just a minimal hypersurface.
- Self-shrinker $\Sigma$ in $\mathbb{R}^{n+1}$. $f = \frac{|x|^2}{4}$.
- Let $(M, g, f)$ be a shrinking gradient Ricci solitons; i.e. after a normalization, $(M, g, f)$ satisfies the equation $\text{Ric} + \nabla^2 f = \frac{1}{2} g$ or equivalently the Bakry-Émery Ricci curvature $\text{Ric}_f := \text{Ric} + \nabla^2 f = \frac{1}{2}$. We may consider $f$-minimal hypersurfaces in $(M, g, f)$. In particular, the previous example: a self-shrinker $\Sigma$ in $\mathbb{R}^{n+1}$ is $f$-minimal in Gauss shrinking soliton $(\mathbb{R}^{n+1}, g, \frac{|x|^2}{4})$.
- $M = \mathbb{H}^{n+1}(-1)$, the hyperbolic space. Let $r$ denote the distance function from a fixed point $p \in M$ and $f(x) = nar^2(x)$, where $a > 0$ is a constant. Now $\text{Ric}_f \geq n(2a - 1)$. The geodesic sphere of radius $r$ centered at $p$ is an $f$-minimal hypersurface if the radius $r$ satisfies $2ar = \coth r$.

An $f$-minimal hypersurface $\Sigma$ can be viewed in two ways. One is that $\Sigma$ is $f$-minimal if and only if $\Sigma$ is a critical point of the weighted volume functional $e^{-f} d\sigma$, where $d\sigma$ is the volume element of $\Sigma$. The other one is that $\Sigma$ is $f$-minimal if and only if $\Sigma$ is minimal in the new conformal metric $\tilde{g} = e^{-2f} g$ (see Section 2 and Appendix). $f$-minimal hypersurfaces have been studied before as even more general stationary hypersurfaces for parametric elliptic functionals; see for instance the work of White [14] and Colding-Minicozzi [7].

We prove the following compactness result:

**Theorem 2.** Let $(M^3, g, e^{-f} d\mu)$ be a complete smooth metric measure space and $\text{Ric}_f \geq k$, where $k$ is a positive constant. Given an integer $g \geq 0$ and a constant $V > 0$, the space $S_{g,V}$ of smooth complete embedded $f$-minimal surfaces $\Sigma \subset M$ with

- genus at most $g$,
- $\partial \Sigma = \emptyset$,
- $\int_{\Sigma} e^{-f} d\sigma \leq V$

is compact in the $C^m$ topology, for any $m \geq 2$. Namely, any sequence of $S_{g,V}$ has a subsequence that converges in the $C^m$ topology on compact subsets to a surface in $S_{g,V}$, for any $m \geq 2$.

Since the existence of the uniform scale-invariant area bound is equivalent to the existence of the uniform bound of the weighted area for self-shrinkers (see Remark 1 in Section 5), Theorem 2 implies Theorem 1. Also, in [2], we will apply Theorem 2 to obtain a compactness theorem for the space of closed embedded $f$-minimal surfaces with the upper bounds of genus and diameter.

To prove Theorem 2 we need to prove a nonexistence result on $L_f$-stable $f$-minimal hypersurfaces, which is of independent interest.

**Theorem 3.** Let $(M^{n+1}, g, e^{-f} d\mu)$ be a complete smooth metric measure space with $\text{Ric}_f \geq k$, where $k$ is a positive constant. Then there is no complete two-sided $L_f$-stable $f$-minimal hypersurface $\Sigma$ immersed in $(M, g)$ without boundary and with finite weighted volume (i.e. $\int_{\Sigma} e^{-f} d\sigma < \infty$), where $d\sigma$ denotes the volume element on $\Sigma$ determined by the induced metric from $(M, g)$.
Here we explain briefly the meaning of $L_f$ stability. For an $f$-minimal hypersurface $\Sigma$, the $L_f$ operator is

$$L_f = \Delta_f + |A|^2 + \text{Ric}_f(\nu, \nu),$$

where $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is the weighted Laplacian on $\Sigma$. In particular, for self-shrinkers, it is the so-called $L$ operator:

$$L = \Delta + |A|^2 - \frac{1}{2} \langle x, \nabla \cdot \rangle + \frac{1}{2}.$$

$L_f$-stability of $\Sigma$ means that its weighted volume $\int_{\Sigma} e^{-f} d\sigma$ is locally minimal; that is, the second variation of its weighted volume is nonnegative for any compactly supported normal variation. We leave more details about the definition of $L_f$-stability and some of its properties to Section 2 and the Appendix.

For self-shrinkers in $\mathbb{R}^{n+1}$, Colding-Minicozzi [6] proved that

**Theorem 4 ([6]).** There are no $L$-stable smooth complete self-shrinkers without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$.

Since the first and third authors [3] of the present paper proved that for self-shrinkers, properness, the polynomial volume growth, and finite weighted volume are equivalent, Theorem 3 implies Theorem 4.

In this paper, we also discuss the relationship among the properness, polynomial volume growth and finite weighted volume of $f$-minimal submanifolds (Propositions 3, 4 and 5). We obtain their equivalence when the ambient space $(M, \bar{g}, f)$ is a shrinking gradient Ricci solitons, i.e. $\bar{\text{Ric}} + \nabla^2 f = \frac{1}{2} \bar{g}$, with the condition that $f$ is a convex function with $|\nabla f|^2 \leq f$ (Corollary 1).

The rest of this paper is organized as follows: In Section 2 some definitions, notation and facts are given as preliminaries. In Section 3 we prove Propositions 3, 4 and 5. In Section 4 we prove Theorem 3. In Section 5 we prove Theorem 2. In the Appendix we calculate the second variation of the volume functional of $f$-minimal submanifolds and discuss some properties of $L_f$-stability for $f$-minimal submanifolds.

2. Preliminaries

In general, a smooth metric measure space, denoted by $(M^m, \bar{g}, e^{-f} d\mu)$, is an $m$-dimensional Riemannian manifold $(M^m, \bar{g})$ together with a weighted volume form $e^{-f} d\mu$ on $M$, where $f$ is a smooth function on $M$ and $d\mu$ is the volume element induced by the metric $\bar{g}$. In this paper, unless otherwise specified, we denote by a bar all quantities on $(M, \bar{g})$, for instance by $\nabla$ and $\bar{\text{Ric}}$, the Levi-Civita connection and the Ricci curvature tensor of $(M, \bar{g})$ respectively. For $(M, \bar{g}, e^{-f} d\mu)$, an important and natural tensor is the $\infty$-Bakry-Émery Ricci curvature tensor $\bar{\text{Ric}}_f$ (for simplicity, Bakry-Émery Ricci curvature), which is defined by

$$\bar{\text{Ric}}_f := \bar{\text{Ric}} + \nabla^2 f,$$

where $\nabla^2 f$ is the Hessian of $f$ on $M$. If $f$ is constant, $\bar{\text{Ric}}_f$ is the Ricci curvature $\bar{\text{Ric}}$ on $M$ respectively.

A Riemannian manifold with Bakry-Émery Ricci curvature bounded below by a positive constant has some properties similar to a Riemannian manifold with Ricci curvature bounded below by a positive constant. For instance, see the work of...
Wei-Wylie [13], Munteanu-Wang [11,12] and the references therein. In this paper, we will use the following proposition by Morgan [10] (see also its proof in [13]).

**Proposition 1.** If a complete smooth metric measure space \((M, g, e^{-f} du)\) has \(Ric_f \geq k\), where \(k\) is a positive constant, then \(M\) has finite weighted volume (i.e. \(\int_M e^{-f} d\mu < \infty\)) and finite fundamental group.

Now, let \(i : \Sigma^n \to M^m, n < m\), be an \(n\)-dimensional smooth immersion. Then \(i : (\Sigma^n; i^*\bar{g}) \to (M^m, \bar{g})\) is an isometric immersion with the induced metric \(i^*\bar{g}\). For simplicity, we still denote \(i^*\bar{g}\) by \(\bar{g}\) whenever there is no confusion. We will denote for instance by \(\nabla, \text{Ric}, \Delta\) and \(d\sigma\), the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the volume element of \((\Sigma, \bar{g})\) respectively.

The function \(f\) induces a weighted measure \(e^{-f} d\sigma\) on \(\Sigma\). Thus we have an induced smooth metric measure space \((\Sigma^n, \bar{g}, e^{-f} d\sigma)\).

The associated weighted Laplacian \(\Delta_f\) on \((\Sigma, \bar{g})\) is defined by

\[
\Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle.
\]

The second order operator \(\Delta_f\) is a self-adjoint operator on the space of square integrable functions on \(\Sigma\) with respect to the measure \(e^{-f} d\sigma\) (however the Laplacian operator in general does not have this property).

The second fundamental form \(A\) of \((\Sigma, \bar{g})\) is defined by

\[
A(X, Y) = (\nabla_X Y)^\perp, \quad X, Y \in T_p \Sigma, p \in \Sigma,
\]

where \(\perp\) denotes the projection to the normal bundle of \(\Sigma\). The mean curvature vector \(H\) of \(\Sigma\) is defined by \(H = \text{tr} A = \sum_{i=1}^n \langle \nabla e_i, e_i \rangle^\perp\).

**Definition 1.** The weighted mean curvature vector of \(\Sigma\) with respect to the metric \(\bar{g}\) is defined by

\[
H_f = H + (\nabla f)^\perp.
\]

The immersed submanifold \((\Sigma, \bar{g})\) is called \(f\)-minimal if its weighted mean curvature vector \(H_f\) vanishes identically, or equivalently if its mean curvature vector satisfies

\[
H = -(\nabla f)^\perp.
\]

**Definition 2.** The weighted volume of \((\Sigma, \bar{g})\) is defined by

\[
V_f(\Sigma) := \int_{\Sigma} e^{-f} d\sigma.
\]

It is well known that \(\Sigma\) is \(f\)-minimal if and only if \(\Sigma\) is a critical point of the weighted volume functional. Namely, it holds that

**Proposition 2.** If \(T\) is a compactly supported variational vector field on \(\Sigma\), then the first variation formula of the weighted volume of \((\Sigma, \bar{g})\) is given by

\[
\frac{d}{dt} V_f(\Sigma_t) \bigg|_{t=0} = -\int_\Sigma \langle T^\perp, H_f \rangle e^{-f} d\sigma.
\]

On the other hand, an \(f\)-minimal submanifold can be viewed as a minimal submanifold under a conformal metric. Precisely, define the new metric \(\tilde{g} = e^{-\frac{f}{2}} \bar{g}\) on \(M\), which is conformal to \(\bar{g}\). Then the immersion \(i : \Sigma \to M\) induces a metric \(i^*\tilde{g}\)
on Σ from \((M, \tilde{g})\). In the following, \(i^* \tilde{g}\) is still denoted by \(\tilde{g}\) for simplicity. The volume of \((\Sigma, \tilde{g})\) is

\[ V(\Sigma) := \int_{\Sigma} d\tilde{\sigma} = \int_{\Sigma} e^{-f} d\sigma = V_f(\Sigma). \]

Hence Proposition 2 and (5) imply that

\[ \int_{\Sigma} \left\langle T^\perp, \tilde{H} \right\rangle d\tilde{\sigma} = \int_{\Sigma} \left\langle T^\perp, H_f \right\rangle e^{-f} d\sigma, \]

where \(d\tilde{\sigma} = e^{-f} d\sigma\) and \(\tilde{H}\) denote the volume element and the mean curvature vector of \(\Sigma\) with respect to the conformal metric \(\tilde{g}\) respectively.

Identity (6) implies that \(\tilde{H} = e^{2f} H_f\). Therefore \((\Sigma, \tilde{g})\) is \(f\)-minimal in \((M, \bar{g})\) if and only if \((\Sigma, \bar{g})\) is minimal in \((M, \bar{g})\).

Now suppose that \(\Sigma^n\) is a hypersurface immersed in \(M^{n+1}\). Let \(p \in \Sigma\) and \(\nu\) be a unit normal at \(p\). The second fundamental form \(A\) and the mean curvature \(H\) of \((\Sigma, g)\) are as follows:

\[ A: T_p \Sigma \to T_p \Sigma, A(X) = \nabla_X \nu, X \in T_p \Sigma, \]

\[ H = \text{tr} A = -\sum_{i=1}^n (\nabla e_i, e_i, \nu). \]

Hence the mean curvature vector \(H\) of \((\Sigma, g)\) satisfies \(H = -H \nu\). Define the weighted mean curvature \(H_f\) of \((\Sigma, \tilde{g})\) by \(H_f := -H_f \nu\). Then

\[ H_f = H - \left\langle \nabla f, \nu \right\rangle. \]

**Definition 3.** A hypersurface \(\Sigma\) immersed in \((M^{n+1}, \bar{g}, e^{-f} d\mu)\) with the induced metric \(\bar{g}\) is called an \(f\)-minimal hypersurface if it satisfies

\[ H = \left\langle \nabla f, \nu \right\rangle. \]

For a hypersurface \((\Sigma, \bar{g})\), the \(L_f\) operator is defined by

\[ L_f := \Delta_f + |A|^2 + \text{Ric}_f(\nu, \nu), \]

where \(|A|^2\) denotes the square of the norm of the second fundamental form \(A\) of \(\Sigma\).

The \(L_f\)-stability of \(\Sigma\) is defined as follows:

**Definition 4.** A two-sided \(f\)-minimal hypersurface \(\Sigma\) is said to be \(L_f\)-stable if for any compactly supported smooth function \(\varphi \in C^\infty(\Sigma)\), it holds that

\[ -\int_{\Sigma} \varphi L_f \varphi e^{-f} d\sigma = \int_{\Sigma} |\nabla \varphi|^2 - (|A|^2 + \text{Ric}_f(\nu, \nu)) \varphi^2 | e^{-f} d\sigma \geq 0. \]

It is known that an \(f\)-minimal hypersurface \((\Sigma, \bar{g})\) is \(L_f\)-stable if and only if \((\Sigma, \tilde{g})\) is stable as a minimal surface with respect to the conformal metric \(\tilde{g} = e^{-f} \bar{g}\). See more details in the Appendix of this paper.

In this paper, for closed hypersurfaces, we choose \(\nu\) to be the outer unit normal.
3. Properness, polynomial volume growth 
and finite weighted volume of \(f\)-minimal hypersurfaces

In [3], the first and third authors of the present paper proved that the finite weighted volume of a self-shrinker \(\Sigma^n\) immersed in \(\mathbb{R}^m\) implies it is properly immersed. In [9], Ding-Xin proved that a properly immersed self-shrinker must have the Euclidean volume growth. Combining these two results, it was proved in [3] that for immersed self-shrinkers, properness, polynomial volume growth and finite weighted volume are equivalent.

In this section we study the relationship among the properness, polynomial volume growth and finite weighted volume of \(f\)-minimal submanifolds, some of which will be used later in this paper.

Let \(\Sigma\) be an \(n\)-dimensional submanifold in a complete manifold \(M^m, n < m\). \(\Sigma\) is said to have polynomial volume growth if, for a \(p \in M\) fixed, there exist constants \(C\) and \(d\) so that for all \(r \geq 1\),

\[
\text{Vol}(B^M_r(p) \cap \Sigma) \leq Cr^d,
\]

where \(B^M_r(p)\) is the extrinsic ball of radius \(r\) centered at \(p\) and \(\text{Vol}(B^M_r(p))\) denotes the volume of \(B^M_r(p) \cap \Sigma\). When \(d = n\) in (9), \(\Sigma\) is said to be of Euclidean volume growth.

Before proving the following Proposition 3, we recall an estimate implied by the Hessian comparison theorem (cf., for instance, [6], Lemma 7.1).

**Lemma 1.** Let \((M, g)\) be a complete Riemannian manifold with bounded geometry, that is, \(M\) has sectional curvature bounded by \(k\) (\(|K_M| \leq k\)), and injectivity radius bounded below by \(i_0 > 0\). Then the distance function \(r(x)\) satisfies

\[
|\nabla^2 r(V, V) - \frac{1}{r} |V - (V, \nabla r) \nabla r|^2 | \leq \sqrt{k},
\]

for \(r < \min\{i_0, \frac{1}{\sqrt{k}}\}\) and any unit vector \(V \in T_xM\).

Using this estimate we will prove

**Proposition 3.** Let \(\Sigma^n\) be a complete noncompact \(f\)-minimal submanifold immersed in a complete Riemannian manifold \(M^m\). If \(\Sigma\) has finite weighted volume, then \(\Sigma\) is properly immersed.

**Proof.** We argue by contradiction. Since the argument is local, we may assume that \((M, g)\) has bounded geometry. Suppose that \(\Sigma\) is not properly immersed. Then there exist a number \(2R < \min\{i_0, \frac{1}{\sqrt{k}}\}\) and \(o \in M\) so that \(\overline{B}^M_R(o) \cap \Sigma\) is not compact in \(\Sigma\), where \(\overline{B}^M_R(o)\) denotes the closure of the (open) ball \(B^M_R(o)\) in \(M\) of radius \(R\) centered at \(o\). Then for any \(a > 0\), there is a sequence \(\{p_k\}\) of points in \(B^M_R(o) \cap \Sigma\) with dist\(\Sigma(p_k, p_j) \geq a > 0\) for any \(k \neq j\). So \(B^\Sigma_{\frac{a}{2}}(p_k) \cap B^\Sigma_{\frac{a}{2}}(p_j) = \emptyset\) for any \(k \neq j\), where \(B^\Sigma_{\frac{a}{2}}(p_k)\) and \(B^\Sigma_{\frac{a}{2}}(p_j)\) denote the intrinsic balls in \(\Sigma\) of the radius \(\frac{a}{2}\) centered at \(p_k\) and \(p_j\) respectively. Choose \(a < 2R\). Then \(B^\Sigma_{\frac{a}{2}}(p_j) \subset B^M_{2R}(o)\). If
p ∈ B_{\frac{a}{2}}(p_j), the extrinsic distance function \( r_j(p) = \text{dist}_M(p, p_j) \) from \( p_j \) satisfies
\[
\Delta r_j = \sum_{i=1}^{n} \nabla^2 r_j(e_i, e_i) + \langle H, \nabla r_j \rangle \\
\geq \frac{n}{r_j} |\nabla r_j|^2 - n\sqrt{k} - \langle \nabla f^+, \nabla r_j \rangle \\
\geq \frac{n}{r_j} |\nabla r_j|^2 - c,
\]
where \( c = n\sqrt{k} + \sup_{B_{2R}(0)} |\nabla f| \). Lemma I is used above. Hence
\[
\Delta r_j^2 \geq 2n - 2cr_j.
\]
Choosing \( a \leq \min\{\frac{n}{2c}, 2R\} \), we have for \( 0 < \mu \leq \frac{a}{2} \),
\[
\int_{B_{\mu}^\Sigma(p_j)} (2n - 2cr_j) d\sigma \leq \int_{B_{\mu}^\Sigma(p_j)} \Delta r_j^2 d\sigma \\
= \int_{\partial B_{\mu}^\Sigma(p_j)} \langle \nabla r_j^2, \nu \rangle d\sigma \\
\leq 2\mu A(\mu),
\]
where \( \nu \) denotes the outward unit normal vector of \( \partial B_{\mu}^\Sigma(p_j) \) and \( A(\mu) \) denotes the area of \( \partial B_{\mu}^\Sigma(p_j) \). Using the co-area formula in (10), we have
\[
\int_0^\mu (n - cs) A(s) ds \leq \int_0^\mu \int_{d_\Sigma(p,p_j)=s} (n - cr_j) d\sigma \leq \mu A(\mu).
\]
This implies
\[
(n - c\mu)V(\mu) \leq V'(\mu),
\]
where \( V(\mu) \) denotes the volume of \( B_{\mu}^\Sigma(p_j) \). So
\[
\frac{V'(\mu)}{V(\mu)} \geq \frac{n}{\mu} - c.
\]
Integrating (12) from \( \varepsilon > 0 \) to \( \mu \), we have
\[
\frac{V(\mu)}{V(\varepsilon)} \geq (\frac{\mu}{\varepsilon})^n e^{-c(\mu - \varepsilon)}.
\]
Since \( \lim_{s \to 0^+} \frac{V(s)}{s^n} = \omega_n \),
\[
\frac{V(\mu)}{V(\varepsilon)} \geq \omega_n \mu^n e^{-c\mu}.
\]
Thus we conclude
\[
\int_\Sigma e^{-f} d\sigma \geq \sum_{j=1}^{\infty} \int_{B_{\frac{a}{2}}^\Sigma(p_j)} e^{-f} d\sigma \geq \inf_{B_{2R}(0)} (e^{-f}) \sum_{j=1}^{\infty} \int_{B_{\frac{a}{2}}^\Sigma(p_j)} d\sigma = \infty.
\]
This contradicts the assumption of the finite weighted volume of \( \Sigma \). \( \square \)

**Proposition 4.** Let \( (M^n, \overline{g}, e^{-f} d\mu) \) be a complete smooth metric measure space with \( \overline{Ric}_f = k \), where \( k \) is a positive constant. Assume that \( f \) is a convex function. If \( \Sigma^n \) is a complete noncompact properly immersed \( f \)-minimal submanifold in \( M \), then \( \Sigma \) has finite weighted volume and Euclidean (hence polynomial) volume growth.
Proof. Since \((M, g, f)\) is a gradient shrinking Ricci soliton, it is well known that, by a scaling of the metric \(g\) and a translating of \(f\), still denoted by \(g\) and \(f\) respectively, we may normalize the metric so that \(k = \frac{1}{2}\) and the following identities hold:

\[
\bar{R} + |\nabla f|^2 - f = 0, \\
\bar{R} + \bar{\Delta} f = \frac{m}{2}, \\
\bar{R} \geq 0.
\]

From these equations, we have that

\[
\bar{\Delta} f - |\nabla f|^2 + f = \frac{m}{2} \quad \text{and} \quad |\nabla f|^2 \leq f.
\]

It was proved by Cao and the third author [1] that there is a positive constant \(c\) so that

\[
\frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2
\]

for any \(x \in M\) with \(r(x) = \text{dist}_M(p, x) \geq r_0\), where \(p\) is a fixed point in \(M\) and \(c, r_0\) are positive constants that depend only on \(m\) and \(f(p)\).

By [14], we know that \(f\) is a proper function on \(M\). Since \(\Sigma\) is properly immersed in \(M\) and \(f\) is proper in \(M\), \(f|\Sigma\) is also a proper smooth function on \(\Sigma\). Note that with the scaling metric and translating \(f\), \(\Sigma\) is still \(f\)-minimal. Hence

\[
\Delta f - |\nabla f|^2 + f = (\bar{\Delta} f -\sum_{\alpha=n+1}^m f_{\alpha\alpha} - |\nabla f^\perp|^2) - |\nabla f^\top|^2 + f
\]

\[
= \bar{\Delta} f - |\nabla f|^2 + f -\sum_{\alpha=n+1}^m f_{\alpha\alpha}
\]

\[
\leq \frac{m}{2}.
\]

Also we have

\[
|\nabla f|^2 = |\nabla f^\top|^2 \leq |\nabla f|^2 \leq f.
\]

By Theorem 1.1 of [3], \(\Sigma\) has finite weighted volume and the Euclidean volume growth of the sub-level set of \(f\) with respect to the scaling metric and the translating \(f\), and hence with respect to the original metric and \(f\). Moreover, by the estimate [14], we have that \(\Sigma\) has the Euclidean volume growth.

Next we prove the following:

**Proposition 5.** Let \((M^m, \bar{g}, e^{-f}d\mu)\) be a complete smooth metric measure space with \(\text{Ric}_f \geq k\), where \(k\) is a positive constant. Assume that \(|\nabla f|^2 \leq 2kf\). If \(\Sigma^n\) is a complete submanifold (not necessarily \(f\)-minimal) with polynomial volume growth, then \(\Sigma\) has finite weighted volume.

**Proof.** By a scaling of the metric, we may assume that \(k = \frac{1}{2}\). The proof follows from an estimate of \(f\). Munteanu-Wang [11] extended the estimate [14] to \((M^m, \bar{g}, e^{-f}d\mu)\) with \(\text{Ric}_f \geq \frac{1}{2}\) and \(|\nabla f|^2 \leq f\). Combining the assumption that \(\Sigma\)
has polynomial volume growth with the estimate \((13)\), we have
\[
\int_{\Sigma} e^{-f} d\sigma = \int_{\Sigma \cap B_{r_0}(p)} e^{-f} d\sigma + \sum_{i=0}^{\infty} \int_{\Sigma \cap (B_{r_0+i+1}(p) \setminus B_{r_0+i}(p))} e^{-f} d\sigma
\]
\[
\leq C_1 \text{Vol}(\Sigma \cap B_{r_0}(p)) + C \sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2} \text{Vol}(\Sigma \cap B_{r_0+i+1}(p))
\]
\[
\leq C \left[ r_0^d + \sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2} (r_0 + i + 1)^d \right]
\]
\[
< \infty.
\]

By Propositions \(3\), \(4\) and \(5\) we have the following.

**Corollary 1.** Let \((M^m, \bar{g}, f)\) be a complete shrinking gradient Ricci soliton with \(\bar{Ric}_f = \frac{1}{2}\). Assume that \(f\) is a convex function. If \(\Sigma\) is a complete \(f\)-minimal submanifold immersed in \(M\), then for \(\Sigma\) the properness, polynomial volume growth, and finite weighted volume are equivalent.

### 4. Nonexistence of \(L_f\) stable \(f\)-minimal hypersurfaces

In this section, we prove Theorem \(3\) which is a key to proving the compactness theorem in Section \(5\).

**Theorem 5 (Theorem \(3\)).** Let \((M, \bar{g}, e^{-f} d\mu)\) be a complete smooth metric measure space with \(\bar{Ric}_f \geq k\), where \(k\) is a positive constant. Then there is no two-sided \(L_f\)-stable complete \(f\)-minimal hypersurface \(\Sigma\) immersed in \((M, g)\) without boundary and with finite weighted volume (i.e. \(\int_{\Sigma} e^{-f} d\sigma < \infty\)).

**Proof.** We argue by contradiction. Suppose that \(\Sigma\) is an \(L_f\)-stable complete \(f\)-minimal hypersurface immersed in \((M, g)\) without boundary and with finite weighted volume. Recall that a two-sided hypersurface \(\Sigma\) is \(L_f\)-stable if the following inequality holds, that is, for any compactly supported smooth function \(\varphi \in C^\infty_0(\Sigma),\)
\[
\int_{\Sigma} \left[ | \nabla \varphi |^2 - (|A|^2 + \bar{Ric}_f(\nu, \nu)) \varphi^2 \right] e^{-f} d\sigma \geq 0.
\]

Observe that any closed hypersurface cannot be \(L_f\)-stable. This is because the assumption \(\bar{Ric}_f \geq k > 0\) implies that \((15)\) cannot hold for \(\varphi \equiv c\) on \(\Sigma\). Hence, \(\Sigma\) must be noncompact.

Let \(\eta\) be a nonnegative smooth function on \([0, \infty)\) satisfying
\[
\eta(s) = \begin{cases} 
1 & \text{if } s \in [0, 1) \\
0 & \text{if } s \in [2, \infty) 
\end{cases}
\]
and \(|\eta'| \leq 2\).

Fix a point \(p \in \Sigma\) and let \(r(x) = \text{dist}_{\Sigma}(p, x)\) denote the (intrinsic) distance function on \(\Sigma\). Define a sequence of functions \(\varphi_j(x) = \eta(\frac{r(x)}{j}), j \geq 1\). Then
\[|\nabla \varphi_j|^2 \leq 1 \text{ for } j \geq 2. \] Substituting \(\varphi_j, j \geq 2\) for \(\varphi\) in (15):
\[
\int_{\Sigma} \left[ |\nabla \varphi_j|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu))\varphi_j^2 \right] e^{-f} d\sigma \\
\leq \int_{\Sigma} \left( |\nabla \varphi_j|^2 - k\varphi_j^2 \right) e^{-f} d\sigma \\
= \int_{B_j^\Sigma(p) \setminus B_{j,1}^\Sigma(p)} |\nabla \varphi_j|^2 e^{-f} d\sigma - \int_{B_j^\Sigma(p)} k\varphi_j^2 e^{-f} d\sigma \\
\leq \int_{B_j^\Sigma(p) \setminus B_{j,1}^\Sigma(p)} e^{-f} d\sigma - k \int_{B_j^\Sigma(p)} \varphi_j^2 e^{-f} d\sigma \\
\leq \int_{B_j^\Sigma(p) \setminus B_{j,1}^\Sigma(p)} e^{-f} d\sigma - k \int_{B_j^\Sigma(p)} e^{-f} d\sigma,
\]
where \(B_j^\Sigma(p)\) is the intrinsic geodesic ball in \(M\) of radius \(j\) centered at \(p\). Since \(\Sigma\) has finite weighted volume, we have, when \(j \to \infty\),
\[
\int_{B_{j,1}^\Sigma(p) \setminus B_j^\Sigma(p)} e^{-f} d\sigma \to 0.
\]
Choosing \(j\) large enough, we have that \(\varphi_j\) satisfies
\[
\int_{\Sigma} \left( |\nabla \varphi_j|^2 - (|A|^2 + \overline{\text{Ric}}_f(\nu, \nu))\varphi_j^2 \right) e^{-f} d\sigma < -\frac{k}{2} \int_{B_j^\Sigma(p)} e^{-f} d\sigma < 0.
\]
This contradicts the fact that \(\Sigma\) is \(L_f\)-stable. \(\square\)

5. Compactness of \(f\)-minimal Surfaces

Before proving Theorem \(\ref{thm:compactness}\), we give some facts.

Wei-Wylie (\cite{13}, Theorem 7.3) used the mean curvature comparison theorem to give a distance estimate for two compact hypersurfaces \(\Sigma_1\) and \(\Sigma_2\) in a smooth metric measure space \((M, g, e^{-f} d\mu)\) with \(\overline{\text{Ric}}_f \geq k\), where \(k\) is a positive constant. Observe that for two complete properly immersed hypersurfaces \(\Sigma_1\) and \(\Sigma_2\), if at least one of them is compact, there is a minimizing geodesic segment joining \(\Sigma_1\) and \(\Sigma_2\) and realizing their distance. Hence the proof of Theorem 7.3 \(\cite{13}\) can be applied to obtain the following.

**Proposition 6.** Let \((M, g, e^{-f} d\mu)\) be an \((n+1)\)-dimensional complete smooth metric measure space with \(\overline{\text{Ric}}_f \geq k\), where \(k\) is a positive constant. If \(\Sigma_1\) and \(\Sigma_2\) are two complete properly immersed hypersurfaces, at least one of which is compact, then the distance \(d(\Sigma_1, \Sigma_2)\) satisfies

\[
d(\Sigma_1, \Sigma_2) \leq \frac{1}{k} \left( \max_{x \in \Sigma_1} |H_f^{\Sigma_1}(x)| + \max_{x \in \Sigma_2} |H_f^{\Sigma_2}(x)| \right),
\]
where \(H_f^{\Sigma_i}, i = 1, 2\), denotes the weighted mean curvatures of \(\Sigma_i\) respectively.

**Corollary 2.** Let \((M, g, e^{-f} d\mu)\) be as in Proposition \(\ref{prop:compactness}\). Then there is a closed ball \(\overline{B}^M\) of \(M\) satisfying that any complete properly immersed \(f\)-minimal hypersurface \(\Sigma\) must intersect it.

**Proof.** Fix \(p \in M\) and a geodesic sphere \(S^M_r(p)\) of \(M\). By Proposition \(\ref{prop:compactness}\)
\[
d(S^M_r(p), \Sigma) \leq \frac{1}{k} \max_x |H_f^{S^M_r}(x)| = C,
\]
where $C$ is independent of $\Sigma$. Therefore there is a closed ball $\overline{B}^M$ of $M$ with radius big enough so that any $\Sigma$ must intersect it.

We need the following fact:

**Proposition 7.** Let $M$ be a simply connected Riemannian manifold. If a hypersurface $\Sigma$ is complete, not necessarily connected, properly embedded, and has no boundary, then every component of $\Sigma$ separates $M$ into two components and thus is two-sided. Therefore $\Sigma$ has a globally defined unit normal.

*Proof.* Suppose $\Sigma_j$ is a component of $\Sigma$. By contrast, $M \setminus \Sigma_j$ has one component. Since $\Sigma$ is a properly embedded $f$-minimal hypersurface, for any $p \in \Sigma_j$ there is a neighborhood $W$ of $p$ in $M$ so that $W \cap \Sigma_j = W \cap \Sigma$ only has one piece (i.e. it is a graph above a connected domain in the tangent plane of $p$). Thus we have a simply closed curve $\gamma$ passing $p$, transversal to $\Sigma_j$ at $p$, and $\Sigma_j \cap \gamma = p$. Since $M$ is simply connected, we have a disk $D$ with the boundary $\gamma$. Again since $\Sigma$ is proper, the intersection of $\Sigma_j$ with $\partial D = \gamma$ cannot be one point, which is a contradiction. \qed

Combining Proposition 3 in Section 3 with Proposition 7, we obtain

**Proposition 8.** Let $(M, \overline{g}, e^{-f}d\mu)$ be a simply connected complete smooth metric measure space. If a complete $f$-minimal hypersurface has finite weighted volume, then every component of $\Sigma$ separates $M$ into two components and thus is two-sided. Therefore $\Sigma$ has a globally defined unit normal.

We will take the same approach as in Colding-Minicozzi’s paper [4] to prove Theorem 2, a smooth compactness theorem for complete $f$-minimal surfaces. First we recall a well known local singular compactness theorem for embedded minimal surfaces in a Riemannian 3-manifold.

**Proposition 9** (cf. [4], Proposition 2.1). Given a point $p$ in a Riemannian 3-manifold $M$, there exists an $R > 0$ such that the following holds: Let $\Sigma_j$ be embedded minimal surfaces in $B_{2R}(p) \subset M$ with $\partial \Sigma_j \subset \partial B_{2R}(p)$. If each $\Sigma_j$ has area at most $V$ and genus at most $g$ for some fixed $V, g$, then there exist a finite collection of points $x_k$, a smooth embedded minimal surface $\Sigma \subset B_R(p)$ with $\partial \Sigma \subset \partial B_R(p)$ and a subsequence of $\{\Sigma_j\}$ that converges in $B_R(p)$ (with finite multiplicity) to $\Sigma$ away from the set $\{x_k\}$.

Here and in the following, we denote by $B_R$ the ball $B_R^M$ in $M$ for simplicity.

It is known that $\Sigma$ is $f$-minimal with respect to metric $\overline{g}$ if and only if $\Sigma$ is minimal with the conformal metric $\tilde{g} = e^{-f}g$ (see Appendix). Using this fact and applying Proposition 9, we may prove a global singular compactness theorem for $f$-minimal surfaces.

**Proposition 10.** Let $M$ be a complete 3-manifold and $(M, \overline{g}, e^{-f}d\mu)$ a smooth metric measure space. Suppose that $\Sigma_i \subset M$ is a sequence of smooth complete embedded $f$-minimal surfaces with genus at most $g$, without boundary, and with weighted area at most $V$, i.e.

\[
\int_{\Sigma_i} e^{-f}d\sigma \leq V < \infty.
\]

Then there are a subsequence, still denoted by $\Sigma_i$, a smooth embedded complete non-trivial $f$-minimal surface $\Sigma \subset M$ without boundary, and a locally finite collection
of points $S \subset \Sigma$ so that $\Sigma_i$ converges smoothly (possibly with multiplicity) to $\Sigma$ off of $S$. Moreover, $\Sigma$ satisfies $\int_\Sigma e^{-f}d\sigma \leq V$ and is properly embedded.

Here a set $S \subset M$ is said to be locally finite if $B_R(p) \cap S$ is finite for every $p \in M$ and for all $R > 0$.

**Proof.** Consider the conformal metric $\tilde{g} = e^{-f} g$ on $M$. For a point $p \in M$, let $\tilde{B}_R(p) \subset M$ denote the ball in $(M, \tilde{g})$ of radius $2R$ centered at $p$. Then the area of $\tilde{B}_R(p) \cap \Sigma_j$ satisfies

$$\tilde{\text{Area}}(\tilde{B}_R(p) \cap \Sigma_j) \leq \int_{\Sigma_j} d\tilde{\sigma} = \int_{\Sigma_j} e^{-f}d\sigma \leq V. \quad (18)$$

Also, it is clear that the genus of $\tilde{B}_R(p) \cap \Sigma_j$ remains at most $g$. Then by Proposition[9] there exist an $R > 0$ and a finite collection of points $x_k$, a smooth embedded minimal surface $\Sigma \subset \tilde{B}_R(p)$, with $\partial \Sigma \subset \partial \tilde{B}_R$ and a subsequence of $\{\Sigma_j\}$ that converges in $\tilde{B}_R(p)$ (with finite multiplicity) to $\Sigma$ away from the set $\{x_k\}$.

Let $\{\tilde{B}_R(p_i)\}$ be a countable cover of $(M, \tilde{g})$ of small balls such that $\{\tilde{B}_R(p_i)\}$ is still a cover of $(M, \tilde{g})$. On each $\tilde{B}_R(p_i)$, applying the previous local convergence and then passing to a diagonal subsequence, we obtain that there are a subsequence of $\Sigma_i$, still denoted by $\Sigma_i$, a smooth embedded minimal surface $\Sigma$ (with respect to the metric $\tilde{g}$) without boundary, and a locally finite collection of points $S \subset \Sigma$ so that $\Sigma_i$ converges smoothly (possibly with multiplicity) to $\Sigma$ off of $S$. Since $\Sigma$ has no boundary, it is complete in the original metric $g$. Thus we obtain the smooth convergence of the subsequence to the smooth embedded complete $f$-minimal surface $\Sigma$ off of $S$.

By Corollary[2] $\Sigma$ is nontrivial. The convergence of $\Sigma_i$ to $\Sigma$ and (17) imply $\int_\Sigma e^{-f}d\sigma \leq V$. By Proposition[8] $\Sigma$ is properly embedded. □

We need to show that the convergence is smooth across the points in $S$. To prove this, we need the following.

**Proposition 11.** Assume that the ambient manifold $M$ in Proposition[10] is simply connected. If the convergence of the sequence $\{\Sigma_i\}$ has multiplicity greater than one, then $\Sigma$ is $L_f$-stable.

**Proof.** By Proposition[8] we know that $\Sigma_i$ and $\Sigma$ are orientable. We may have two ways to prove the proposition. The first is to use the known fact on minimal surfaces. It is known that (cf. [6], Appendix A) if the multiplicity of the convergence of a sequence of embedded orientable minimal surfaces in a simply connected 3-manifold is not one, then the limit minimal surface is stable. Under the conformal metric $\tilde{g}$, a sequence $\{\Sigma_i\}$ of minimal surfaces converges to a smooth embedded orientable minimal surface $\Sigma$ and thus $\Sigma$ is stable. Also, the conclusion that $\Sigma$ is stable with respect to the conformal metric $\tilde{g}$ is equivalent to saying that $\Sigma$ is $L_f$-stable under the original metric $g$ (see Appendix).

The second way is to prove it directly. We may prove that $L_f$ is the linearization of the $f$-minimal equation by a proof similar to the one in [4], Appendix A. By arguing as in Proposition 3.2 in [4], we can find a smooth positive function $u$ on $\Sigma$ satisfying

$$L_fu = 0. \quad (19)$$

This implies that $\Sigma$ is $L_f$-stable. □
Proof of Theorem 2. By the assumption on $\text{Ric}_f$ and Proposition 1, $M$ has finite fundamental group. After passing to the universal covering, we may assume that $M$ is simply connected. Given a sequence of smooth complete embedded $f$-minimal surfaces $\{\Sigma_i\}$ with genus $g$, $\partial \Sigma_i = \emptyset$, and the weighted area at most $V$, by Proposition 10 there is a subsequence, still denoted by $\{\Sigma_i\}$, that converges in the topology of smooth convergence on compact subsets to a smooth embedded complete $f$-minimal surface $\Sigma$ away from a locally finite set $\mathcal{S} \subset \Sigma$ (possibly with multiplicity). Moreover, the limit surface $\Sigma \subset M$ is complete, properly embedded, $\int_\Sigma e^{-f}d\sigma \leq V$, has no boundary and has a well-defined unit normal $\nu$. We also have the equivalent convergence under the conformal metric $\bar{g}$.

If $\mathcal{S}$ is not empty, Allard’s regularity theorem implies that the convergence has multiplicity greater than one. Then by Proposition 11 we conclude that $\Sigma$ is $L_f$-stable. But Proposition 5 says that there is no such $\Sigma$. This contradiction implies that $\mathcal{S}$ must be empty. We complete the proof of the theorem. \[\square\]

Remark 1. For self-shrinkers, the condition that the scale-invariant uniform area bound exists (i.e. there is a uniform bound $V_1$: Area$(B_R(x_0) \cap \Sigma) \leq V_1 R^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$) implies that the uniform bound $V$ of weighted area (i.e. $\int_\Sigma e^{-f}d\sigma < V$) exists (cf. the proof of Proposition 5). The converse is also true by the conclusion that the entropy of a self-shrinker can be achieved by $F_{0,1}$ for self-shrinkers with polynomial volume growth (see Section 7 of [5]). Therefore Theorem 2 generalizes the result of Colding-Minicozzi (Theorem 1) for self-shrinkers.

Remark 2. Combining Theorem 2 with the upper bound estimate of weighted area for closed embedded $f$-minimal surfaces of fixed genus in a complete 3-manifold with $\text{Ric}_f \geq k > 0$, we may obtain the smooth compactness theorem for the space of closed embedded $f$-minimal surfaces of fixed topological type and with diameter bound. We discuss it in [2].

APPENDIX

In this appendix, we discuss the $L_f$-stability properties of $f$-submanifolds. With the same notation as in Section 2 let $(M^m, \bar{g})$ be an $m$-dimensional Riemannian manifold and $i: \Sigma^n \to M^m$, $n < m$, be an immersion. Let $\bar{g} = e^{-\frac{f}{2}}g\bar{g}$ denote the new conformal metric on $M$. Therefore $i$ may induce two isometric immersions of $\Sigma$: $(\Sigma, \bar{g}) \to (M, \bar{g})$ and $(\Sigma, \bar{g}) \to (M, \bar{g})$ respectively.

When $(\Sigma, \bar{g})$ is minimal, it is well known that the second variation of the volume of $(\Sigma, \bar{g})$ is given by

**Proposition 12** (cf. [6]). Let $(\Sigma, \bar{g})$ be a minimal submanifold in $(M, \bar{g})$. If $T$ is a normal compactly supported variational vector field on $\Sigma$ (that is, $T = T^\perp$), then the second variational formula of the volume $\bar{V}$ of $(\Sigma, \bar{g})$ is given by

$$
\left. \frac{d^2}{dt^2} \bar{V}(\Sigma_t) \right|_{t=0} = -\int_{\Sigma} \langle T, JT \rangle_{\bar{g}} d\bar{\sigma},
$$

where the stability operator (or Jacobi operator) $J$ is defined on a normal vector field $T$ to $\Sigma$ by

$$
JT = \Delta_{(\Sigma, \bar{g})} T + tr_{(\Sigma, \bar{g})} [\bar{Rm}(\cdot, T)\cdot]_{\perp} + \bar{B}(T).
$$
Here $\Delta^\perp_{(\Sigma, g)} T = \sum_{i=1}^{n}(\nabla^\perp_{e_i} \nabla^\perp_{e_i} T - \nabla^\perp_{\nabla_{e_i} e_i} T)$ is the Laplacian determined by the normal connection $\nabla^\perp$ of $(\Sigma, g)$, $\tilde{Rm}$ is the curvature tensor on $(M, \tilde{g})$, $\text{tr}(\Sigma, g)[\tilde{Rm}(\cdot, T)]^\perp = \sum_{i=1}^{n} [\tilde{Rm}(\tilde{e}_i, T)\tilde{e}_i]^\perp$, $\tilde{A}$ denotes the second fundamental form of $(\Sigma, \tilde{g})$, $\tilde{B}(T) = \sum_{i,j=1}^{n} \langle \tilde{A}(\tilde{e}_i, \tilde{e}_j), T \rangle \tilde{A}(\tilde{e}_i, \tilde{e}_j)$, and $\{\tilde{e}_i\}$, $i = 1, \ldots, n$, is a local orthonormal base of $(\Sigma, \tilde{g})$.

Recall that the weighted volume of $(\Sigma, g)$ is defined by
\begin{equation}
V_f(\Sigma) = \int_{\Sigma} e^{-f} d\sigma.
\end{equation}

By a direct computation similar to that of (20), we may prove the second variation formula of the weighted volume of $f$-minimal submanifold $(\Sigma, g)$.

**Definition 5.** For any normal vector field $T$ on $(\Sigma, g)$, the second order operator $\Delta^\perp_f$ is defined by
\begin{equation}
\Delta^\perp_f T := \Delta^\perp T - tr[\nabla f \otimes \nabla^\perp T(\cdot, \cdot)]
= \sum_{i=1}^{n} (\nabla^\perp_{e_i} \nabla^\perp_{e_i} T - \nabla^\perp_{\nabla_{e_i} e_i} T) - \sum_{i=1}^{n} (e_i f)(\nabla^\perp_{e_i} T).
\end{equation}

The operator $L_f$ on $(\Sigma, g)$ is defined by
\begin{equation}
L_f T = \Delta^\perp_f T + R(T) + B(T) + F(T).
\end{equation}

In the above, $\nabla^\perp$ denotes the normal connection of $(\Sigma, g)$; $\{e_i\}$, $i = 1, \ldots, n$, is a local orthonormal base of $(\Sigma, g)$; $B(T) = \sum_{i,j=1}^{n} \langle A(e_i, e_j), T \rangle A(e_i, e_j)$, where $A$ denotes the second fundamental form of $(\Sigma, g)$; $R(T) = tr(\Sigma, g)[\tilde{Rm}(\cdot, T)]^\perp = \sum_{i=1}^{n} [\tilde{Rm}(e_i, T)e_i]^\perp$, where $\tilde{Rm}$ denotes the Riemannian curvature tensor of $(M, \tilde{g})$; and $F(T) = [\nabla^2 f(T)]^\perp = \sum_{\alpha=n+1}^{m} \nabla^2 f(T, e_\alpha)e_\alpha$, where $\{e_\alpha\}$, $\alpha = n+1, \ldots, m$, is a local orthonormal normal vector field on $(\Sigma, \tilde{g})$.

**Proposition 13.** Let $(\Sigma, g)$ be an $f$-minimal submanifold in $(M, \bar{g})$. If $T$ is a normal compactly supported variational vector field on $\Sigma$ (that is, $T = T^\perp$), then the second variation of the weighted volume of $(\Sigma, \bar{g})$ is given by
\begin{equation}
\frac{d^2}{dt^2} V_f(\Sigma_t) \bigg|_{t=0} = - \int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma.
\end{equation}

**Proof.** Let $\psi(\cdot, t) \in (-\varepsilon, \varepsilon)$ be a compactly supported variation of $\Sigma$ so that $T = d\psi(\frac{\partial}{\partial x_i})$ is the variational vector field, $\Sigma_t = \psi(\Sigma, t)$, $\Sigma_0 = \Sigma$. Choose a normal coordinate system $\{x_1, \ldots, x_n\}$ at a point $p \in \Sigma$. We can consider $\{x_1, \ldots, x_n, t\}$ to be a coordinate system of $\Sigma \times (-\varepsilon, \varepsilon)$ near the point $(p, 0)$. Denote $e_i = d\psi(\frac{\partial}{\partial x_i})$ for $i = 1, \ldots, n$. The induced metric on $\Sigma_t$ from $(M, \bar{g})$ is given for $g_{ij} = \langle e_i, e_j \rangle$. 


Hence \( g_{ij}(p, 0) = \delta_{ij} \) and \( \nabla e_i e_j(p, 0) = 0 \). Denote by \( d\sigma_t \) the volume element of \( \Sigma_t \). Then \( d\sigma_t = J(x, t)d\sigma_0 \), where \( d\sigma_0 = d\sigma \) and the function \( J(x, t) \) is given by

\[
J(x, t) = \frac{\sqrt{G(x, t)}}{\sqrt{G(x, 0)}},
\]

with \( G(x, t) = \det(g_{ij}(x, t)) \). Denote by \( d(\sigma_f)_t \) the weighted volume element of \( \Sigma_t \). Then \( d(\sigma_f)_t = J_f(x, t)d\sigma_0 \), where \( J_f(x, t) = J(x, t)e^{-f(x, t)} \), \( f(x, t) = f(\psi(x, t)) \).

Since \( \frac{\partial J}{\partial t} = \sum_{i, j=1}^n g^{ij} \langle \nabla e_i, e_j \rangle J, \frac{\partial J_f}{\partial t} = (\sum_{i, j=1}^n g^{ij} \langle \nabla e_i, e_j \rangle - \langle \nabla f, T \rangle)J_f \). Note that \( T \) is a normal vector field. A direct computation gives, at \((p, 0)\),

\[
\left. \frac{\partial^2 J_f}{\partial t^2} \right|_{t=0} = \left[ -2 \sum_{i, j=1}^n \langle A_{ij}, T \rangle^2 + \langle R(e_i, T)e_i \rangle \right. \\
+ \sum_{i=1}^n \langle \nabla e_i, \nabla_T T, e_i \rangle + \sum_{i=1}^n \langle \nabla e_i, \nabla_{e_i} T \rangle \\
- \langle \nabla^2 f(T, T) - \langle \nabla f, \nabla_T T \rangle \\
\left. \left. + \sum_{i=1}^n \langle \nabla e_i, e_i \rangle - \langle \nabla f, T \rangle \right\rangle \left( \sum_{j=1}^n \langle \nabla e_j, e_j \rangle - \langle \nabla f, T \rangle \right) \right] J_f.
\]

By

\[
\sum_{i=1}^n \langle \nabla e_i, \nabla_T T, e_i \rangle = \sum_{i, j=1}^n \langle \nabla e_i, e_j \rangle^2 + \sum_{i=1}^n \sum_{\alpha=n+1}^m \langle \nabla e_i, e_\alpha \rangle^2 \\
= \sum_{i, j=1}^n \langle A_{ij}, T \rangle^2 + \sum_{i=1}^n \langle \nabla_{e_i} T, \nabla_{e_i} T \rangle \\
= |\langle A(\cdot, \cdot), T \rangle|^2 + |\nabla_T T|^2
\]

and \( \sum_{i=1}^n \langle \nabla e_i, \nabla_T T, e_i \rangle = \text{div}(\nabla_T T)^\top - \langle (\nabla_T T)^\top, H \rangle \) we have that, at \( p \),

\[
\left. \frac{\partial^2 J_f}{\partial t^2} \right|_{t=0} = \left[ -|\langle A(\cdot, \cdot), T \rangle|^2 - \sum_{i=1}^n \langle R(e_i, T)e_i, T \rangle + |\nabla_T T|^2 + \text{div}(\nabla_T T)^\top \\
- \langle (\nabla_T T)^\top, H \rangle - \langle \nabla^2 f(T, T) - \langle \nabla f, \nabla_T T \rangle + \langle T, H_f \rangle^2 \rangle e^{-f}. \right.
\]

Using \( \text{div}(e^{-f}(\nabla_T T)^\top) = e^{-f}\text{div}(\nabla_T T)^\top - e^{-f}\langle (\nabla_T T)^\top, \nabla f \rangle \), we have at \( p \):

\[
(25) \quad \left. \frac{\partial^2 J_f}{\partial t^2} \right|_{t=0} = \left[ |\nabla_T T|^2 - |\langle A(\cdot, \cdot), T \rangle|^2 - \sum_{i=1}^n \langle R(e_i, T)e_i, T \rangle - \langle \nabla^2 f(T, T) \\
- \parallel (\nabla_T T)^\top, H_f \rangle + \langle T, H_f \rangle^2 \parallel e^{-f} + \text{div}(e^{-f}(\nabla_T T)^\top) \right].
\]
Observe that the right-hand side of (25) is independent of the choice of coordinates. Hence (25) holds on $\Sigma$. By integrating (25) and using the fact that $\Sigma$ is $f$-minimal (i.e. $H_f = 0$), we obtain

$$
\left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = \int_{\Sigma} \left( |\nabla^\perp T|^2 - |\langle \cdot \rangle, T \rangle|^2 - \langle R(T), T \rangle - \nabla^2 f(T, T) \right) e^{-f} d\sigma
$$

$$
= -\int_{\Sigma} \langle T, \nabla_f^\perp T + A(T) + R(T) + F(T) \rangle e^{-f} d\sigma
$$

$$
= -\int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma.
$$

Substituting $e^{-f}T$ for $T$ in the identity $\int_{\Sigma} |\nabla^\perp T|^2 d\sigma = -\int_{\Sigma} \langle T, \Delta_f^\perp T \rangle d\sigma$, we have

$$
\int_{\Sigma} |\nabla^\perp T|^2 e^{-f} d\sigma = -\int_{\Sigma} \langle T, \Delta_f^\perp T \rangle e^{-f} d\sigma.
$$

Thus we have the second variation formula of the weighted volume of $\Sigma$:

$$
\left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = -\int_{\Sigma} \langle T, \Delta_f^\perp T + A(T) + R(T) + F(T) \rangle e^{-f} d\sigma
$$

$$
= -\int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma.
$$

□

**Definition 6.** An $f$-minimal submanifold $(\Sigma, g)$ is called $L_f$-stable if the second variation of the weighted volume of $\Sigma$ given by (24) is nonnegative for any normal compactly supported variational vector field $T$ on $\Sigma$.

Observe that for an $f$-minimal submanifold $\Sigma$ and its normal compactly supported variation, it holds that $V_f(\Sigma_t) = \tilde{V}(\Sigma_t)$. Then

$$
\left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = \left. \frac{d^2}{dt^2} \tilde{V}(\Sigma_t) \right|_{t=0}.
$$

By (20), (24), and (26), we have

$$
\int_{\Sigma} \langle T, JT \rangle d\sigma = \int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma.
$$

This implies that

$$
\int_{\Sigma} e^{-\frac{2}{\lambda_f}} \langle T, JT \rangle e^{-f} d\sigma = \int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma.
$$

By (28), the following equality holds.

**Corollary 3.** For any normal vector field $T$ on $\Sigma$,

$$
JT = e^{\frac{2}{\lambda_f}} L_f T.
$$

The operator $L_f$ corresponds to a symmetric bilinear form $B_f(T, T)$ for the space of normal compactly supported vector fields on $\Sigma$:

$$
B_f(T, T) := -\int_{\Sigma} \langle T, L_f T \rangle e^{-f} d\sigma.
$$
We define the $L_f$-index, denoted by $L_f$-ind, of $(\Sigma, g)$ by the maximum of the dimensions of negative definite subspaces of $B_f$. Hence $(\Sigma, g)$ is $L_f$-stable if and only if its $L_f$-ind = 0.

On the other hand, for minimal $(\Sigma, \tilde{g})$, it is well known that the stability operator $J$ also defines a symmetric bilinear form $\tilde{B}(T,T)$,

\[(30) \quad \tilde{B}(T,T) := -\int_\Sigma \langle T, JT \rangle \tilde{g}d\tilde{\sigma}.\]

There are also the concepts of index and stability of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \tilde{g})$ is stable if and only if the index $\text{ind}(\Sigma, \tilde{g}) = 0$. Since $B_f(T,T) = \tilde{B}(T,T)$, it holds that

**Proposition 14.** $L_f$-ind of $(\Sigma, g)$ is equal to the index of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, g)$ is $L_f$-stable if and only if $(\Sigma, \tilde{g})$ is stable in $(M, \tilde{g})$.

Now if $\Sigma$ is a two-sided hypersurface, that is, if there is a globally-defined unit normal $\nu$ on $(\Sigma, g)$, take $T = \varphi \nu$. Then the second variation (24) implies that

**Proposition 15.** Let $\Sigma$ be a two-sided $f$-minimal hypersurface in $(M^{n+1}, \bar{g})$. If $\varphi$ is a compactly supported smooth function on $\Sigma$, then the second variation of the weighted volume of $(\Sigma, g)$ is given by

\[(31) \quad \left. \frac{d^2}{dt^2} V_f(\Sigma_t) \right|_{t=0} = -\int_\Sigma \varphi L_f(\varphi)e^{-f}d\sigma,\]

where $\nu$ denotes the unit normal of $(\Sigma, \bar{g})$ and the operator $L_f$ is defined by $L_f = \Delta_f + |A|_{\bar{g}}^2 + \bar{\text{Ric}}_f(\nu, \nu)$.

**Definition 7.** The operator $L_f = \Delta_f + |A|_{\bar{g}}^2 + \bar{\text{Ric}}_f(\nu, \nu)$ is called the $L_f$-stability operator of hypersurface $(\Sigma, \bar{g})$.

A bilinear form on space $C_{c}^\infty(\Sigma)$ of compactly supported smooth functions on $\Sigma$ is defined by

\[(32) \quad B_f(\varphi, \varphi) := -\int_\Sigma \varphi L_f \varphi e^{-f}d\sigma = \int_\Sigma [||\nabla \varphi||^2 - (|A|_{\bar{g}}^2 + \bar{\text{Ric}}_f(\nu, \nu))\varphi^2]e^{-f}d\sigma.\]

The $L_f$-index, denoted by $L_f$-ind, of $(\Sigma, \bar{g})$ is defined to be the maximum of the dimensions of negative definite subspaces of $B_f$. Hence $(\Sigma, \bar{g})$ is $L_f$-stable if and only if $L_f$-ind = 0. Clearly the definition of $L_f$-index is equivalent to the corresponding definition using the variational vector field $T$ as before.

Also, for minimal hypersurface $i : (\Sigma, \bar{g}) \rightarrow (M^{n+1}, \tilde{g})$, it is well known that if $\psi$ is a compactly supported smooth function on $\Sigma$, then the second variation of the volume $\bar{V}$ of $(\Sigma, i^*\tilde{g})$ is given by

\[(33) \quad \left. \frac{d^2}{dt^2} \bar{V}(\Sigma_t) \right|_{t=0} = -\int_\Sigma \psi J(\psi)d\tilde{\sigma},\]
where $\tilde{\mathbb{A}}$ denotes the second fundamental form of $(\Sigma, \tilde{g})$, $\tilde{\nu}$ denotes the unit normal of $(\Sigma, \tilde{g})$, and $J = \triangle_{\tilde{g}} + |\tilde{\mathbb{A}}|_{\tilde{g}}^2 + \text{Ric}(\tilde{\nu}, \tilde{\nu})$ is the stability operator (or the Jacobi operator) of $(\Sigma, \tilde{g})$.

The following holds, from (28).

Proposition 16. Let $(\Sigma^n, g)$ be an $f$-minimal hypersurface immersed in $(M, g)$. Then for all $\varphi \in C_c^\infty(\Sigma)$,

\[
\int_{\Sigma} (e^{-\frac{f^n}{2}} \varphi) J(e^{-\frac{f^n}{2}} \varphi) e^{-f} d\sigma = \int_{\Sigma} \varphi L_f(\varphi) e^{-f} d\sigma.
\]

Corollary 4. For $\varphi \in C^\infty(\Sigma)$, $J(e^{-\frac{f^n}{2}} \varphi) = e^{\frac{f^n}{2}} L_f(\varphi)$.

Corollary 5. $L_f$-ind of $(\Sigma, g)$ is equal to the index of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \tilde{g})$ is $L_f$-stable if and only if $(\Sigma, \tilde{g})$ is stable in $(M, \tilde{g})$.

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