THE MERKURIEV-SUSLIN THEOREM FOR ANY SEMI-LOCAL RING

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Abstract. We introduce here a method which uses étale neighborhoods to extend results from smooth semi-local rings to arbitrary semi-local rings \( A \) by passing to the henselization of a smooth presentation of \( A \). The technique is used to show that étale cohomology of \( A \) agrees with Galois cohomology, the Merkuriev-Suslin theorem holds for \( A \), and to describe torsion in \( K_2(\mathbb{A}) \).

We introduce here a method which uses étale neighborhoods to extend results from smooth semi-local rings to arbitrary semi-local rings. Three applications are given. In the first and last, \( A \) is a connected, semi-local ring containing a field \( k \) while the second application holds for any connected, semi-local ring \( A \).

1) Let \( X = \text{Spec}(A) \), and let \( n \) an integer with \( (n, \text{char}(k)) = 1 \). If \( F \) is a finite, locally constant sheaf of \( \mathbb{Z}/n \)-modules for the étale site on \( X \), we show

\[
H^p(G(A_s/A), F(A_s)) \approx H^p(X, F)
\]

where \( A_s \) is the separable closure of \( A \), the left hand side is the Galois cohomology of \( A \) with coefficients in the \( G(A_s/A) \)-module \( F(A_s) \) and the right hand side is the étale cohomology group of the semi-local scheme \( X \) with coefficients in \( F \).

2) We extend the Merkuriev-Suslin theorem to a connected, semi-local ring \( A \); that is, for \( n \) relatively prime to the residue characteristics of \( A \), the Galois symbol map

\[
K_2(A)/n \approx H^2(A, \mathbb{Z}/n(2))
\]

is an isomorphism where, as usual, \( \mathbb{Z}/n(i) = \mu_n^\otimes i \). Since this implies the cup product map is surjective, we conclude that any Azumaya algebra of order \( n \) in \( Br(A) \) is similar to a tensor product of symbol algebras if \( A \) contains a primitive \( n^{th} \) root of unity.

3) We extend Suslin’s computation of the \( \ell \)-primary component of the torsion in \( K_2(k) \), \( k \) a field, to \( A \).

Fix notation as follows. \( H^p(X, F) \) (or, if \( X = \text{Spec}(A) \), \( H^p(A, F) \)) denotes the étale cohomology group of \( X \) with coefficients in the étale
sheaf $F$. For connected $A$, Galois cohomology will be indicated by $H^p(\pi_1(A, x), M)$ where $\pi_1(A, x)$ is the algebraic fundamental group of $A$ with base point $x : \text{Spec}(k_s) \to \text{Spec}(A)$ or, equivalently, by $H^p(G(A_s/A), M) (\pi_1(A, x) = G(A_s/A))$, the Galois group of the separable closure of $A$, where the base point is defined by embedding the residue field of $A$ into a separably closed field) when $M$ is an abelian group equipped with a continuous action. $X(d)$ stands for the set of generic points $x$ of irreducible components of $X$ of codimension $d$. If $M$ is a fixed abelian group, we let $M/n$ stand for $M/nM$, $M\{\ell\}$ stand for the $\ell$-primary component of $M$, and $\text{Div}(M)$ stand for the maximal divisible subgroup of $M$.

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1. Local functors

Let $(A, I)$ be a pair consisting of a commutative ring with 1 and an ideal $I \subset A$. An étale neighborhood of $(A, I)$ is a pair $(A', I')$ and an étale map $u : A \to A'$ such that $u(I)A' = I'$ and $u$ induces an isomorphism $\overline{u} : A/I \to A'/I'$. Geometrically étale neighborhoods of a closed set $W \subset X = \text{Spec}(A)$ look like a cartesian diagram where $\pi$ is étale:

$$
\begin{array}{ccc}
W' & \subset & X' \\
\downarrow & \cong & \downarrow \pi \\
W & \subset & X
\end{array}
$$

The set of all étale neighborhoods of $I$ in $A$ is a filtered category which we denote $\text{Et}(I)$. If we assume that $I \subseteq \text{rad}(A)$, then $\text{Et}(I)$ is used to define the henselization of the pair $(A, I)$ \cite{cite} by setting

$$
A_{/I}^h := \lim_{\rightarrow} A',
$$

Then the pair $(A_{/I}^h, I^h)$ is a hensel pair where $I^h = IA_{/I}^h$. If $I = m$ is maximal, we get the hensel local ring $A_m^h$.

Let $\mathcal{C}$ be a category containing $\text{Et}(I)$ as a full subcategory and $A_{/I}^h$. We introduce the definition of a local functor.

**Definition 1.** A covariant functor $F : \mathcal{C} \to \text{Ab}$ is said to be local (for the étale topology) with respect to a closed set $V(I)$ in $\text{Spec}(A)$ if
(1) \( I \) is a radical ideal in \( A \), \( \text{Et}(I) \) is a full subcategory of \( \mathcal{C} \), and \( A^h_{/I} \in \mathcal{C} \).

(2) the natural map

\[
\lim_{(A',I') \in \text{Et}(I)} F(A') \to F(A/I)
\]

is an isomorphism.

We say that \( F \) is local for the étale topology on \( X \) if, for all affine open subsets \( \text{Spec}(A) \subset X \) and all closed sets \( V(I) \subset \text{Spec}(A) \), \( F \) is local with respect to the closed set \( V(I) \subset \text{Spec}(A) \). In many cases of importance the limit condition above can be replaced with a condition involving the henselization of \( A \) along \( I \). Recall that a covariant functor \( F : \mathcal{C} \to \text{Ab} \) is said to be locally of finite presentation if for any filtered inductive limit \( A = \lim_{i \in I} A_i \), the natural map

\[
\lim_{i \in I} F(A_i) \to F(A)
\]

is an isomorphism.

For such functors we are only interested in a hensel pair condition.

**Definition 2.** A functor \( F : \mathcal{C} \to \text{Ab} \) is said to satisfy the hensel pair condition for \( I \), resp. epic hensel pair condition for \( I \), if \( F(A) \to F(A/I) \) is an isomorphism, resp. an epimorphism, for any hensel pair \((A, I)\). The functor satisfies the hensel pair condition, resp. epic hensel pair condition if it does so for any hensel pair \((A, I) \in \mathcal{C}\).

Then if \( F \) is locally of finite presentation and \( F \) satisfies the hensel pair condition, \( F \) is local for the closed set \( V(I) \). Thus in applying this definition we will first verify that \( F \) is locally of finite presentation and then that \( F(A^h) \to F(A^h/I^h) \) is an isomorphism when \( (A^h, I^h) \) is a hensel pair.

We are primarily interested in three examples of functors satisfying the hensel pair condition.

**Example 1.**

(1) \( H^i(\_ , F) \) where \( F \) is a locally constant sheaf of \( \mathbb{Z}/n \) modules on \( \text{Spec}(A)_{et} \) ([18] or [5])

(2) \( K_i(\_ ; \mathbb{Z}/n) \) where \( n \) is relatively prime to the residue characteristic of \( A \) ([1]) and

(3) \( K_2(\_)/n \) where \( n \) is relatively prime to the residue characteristic of \( A \).

The last example is easily seen to satisfy the hensel pair condition. Surjectivity of \( K_2(A)/n \to K_2(A/I)/n \) for a hensel pair \((A, I)\) follows.
immediately since $K_2$ is generated by symbols, and injectivity follows by a straightforward calculation done in [11, Appendix] or from Gabber's theorem [4]. Alternatively we could use the universal coefficient theorem, the second example, and note that the third term in the universal coefficient theorem is the $n$–torsion in $K_1$ which is the $n^{th}$ roots of unity. These examples will be discussed at greater length in the applications.

Our applications are a result of the following theorem.

**Theorem 1.** Let $A$ be a semi-local ring essentially of finite type over a base ring $k$. Suppose $A \cong B/J$ where $B$ is a smooth semi-local ring essentially of finite type over $k$. Let $C$ be a category of semi-local ring extensions of $B$ containing $A$, $B/h$, and the full subcategory $Et(J)$. Suppose $F_1, F_2 : C \to Ab$ are two covariant functors which are either both local for $V(J)$ or $F_2$ is local for $V(J)$ and $F_1$ is locally of finite presentation and satisfies the epic hensel pair condition. Let $\phi : F_1 \to F_2$ be a natural transformation such that $\phi(B')$ is an isomorphism if $B'$ is smooth and essentially of finite type over $k$. Then $\phi(A)$ is an isomorphism.

**Proof.** If $A$ is essentially of finite type over $k$, then $A$ has a presentation as $B/J$ where $B$ is essentially of finite type and smooth over $k$. Thus $\phi(B)$ is an isomorphism as is $\phi(B')$ for any $B' \in Et(J)$ and so, by (1), $\phi(A)$ must be an isomorphism in the first case. In the second case, $F_1(B/h) \to F_1(A)$ must be an isomorphism since $F_2$ is local for $V(J)$ and $\phi(B/h)$ is an isomorphism. Hensel pair. □

**Corollary 1.** In the above situation $F_1$ is local for $V(J)$ if it is locally of finite presentation and satisfies the epic hensel pair condition.

2. Applications

We have three applications of this perspective. They all rely on first establishing an isomorphism for semi-local rings smooth over a base scheme and then using the examples above to extend the result to arbitrary semi-local schemes over the base.

2.1. **Galois cohomology = Étale cohomology.** As a first application we consider the relationship between étale and Galois cohomology of semi-local rings. While it is a generally accepted fact for smooth local rings over a field, the details do not, as far as I know, appear in the literature although there is an argument, due to Bruno Kahn, when the generalized Kato conjecture holds [11]. We begin by reviewing and
reinterpreting Galois cohomology and then proving the isomorphism in this case.

Let \( S \) be an arbitrary connected scheme. Define a site \( S_{\text{ret}} \) by considering the category of schemes \( T \to S \) which are finite and étale (= revêtement étale) over \( S \). A covering morphism is a surjection of schemes over \( S \), \( T_2 \to T_1 \), which will necessarily be finite and étale. This is the same as considering the class \( E = (\text{ret}) \) of all finite, étale morphisms \([12\text{ Chapter II, Section 1}].\) \( S_{\text{ret}} \) is then the small site \((E/S)_{\text{Et}}\) where \( E \) consists of ”revêtements étales” as above. This site is discussed in \([12\text{ Chapter 1}]\) where it is called \( FEt \) and used to describe the fundamental group of \( S \). Cohomology in \( S_{\text{ret}} \) can be identified with Galois cohomology according to the following recipe.

Suppose \( F \) is a sheaf on \( S_{\text{ret}} \). Then \( F \) takes finite, disjoint unions of schemes to direct products. Thus the discussion in \([12\text{ Chapter III, Example 2.6}]\) applies to the covering \( T \to S \) in \( S_{\text{ret}} \) where \( T \) is Galois over \( S \) with group \( G \). Hence

\[
\tilde{H}^p(T/S, F) \cong H^p(G, F(T))
\]

(2)

where \( F(T) \) is a left \( G \)-module via the action of \( G \) on \( T \).

Let \( \iota : S(S_{\text{ret}}) \to \mathcal{P}(S_{\text{ret}}) \) be the forgetful functor that regards a sheaf as a contravariant functor defined on \( S_{\text{ret}} \). When we pass to the limit over all coverings \( T/S \) in \( S_{\text{ret}} \), the left hand side of (2) becomes \( \hat{H}^p(S_{\text{ret}}, F) \). If we fix a geometric point \( x : \text{Spec}(k_s) \to S \) where \( k_s \) is a separably closed field, then the theory of the fundamental group expresses any finite, étale covering \( T \to S \) as a quotient of a Galois covering. Consequently the right hand side of (2) is \( \varinjlim H^p(G_i, F(T_i)) \) where \( G_i \) ranges through the finite quotients of \( \pi_1(S, x) \) and \( T_i \to S \) is the corresponding étale covering with group \( G_i \). Let \( F(S_s) \) denote the abelian group \( \varinjlim F(T_i) \). If \( F \) is a locally constant sheaf on \( S \), then the subgroup of \( \pi_1(S, x) \) defining the covering \( T \to S \) such that \( F \mid_T \) is constant is of finite index in \( \pi_1(S, x) \) and acts trivially on \( F(S_s) \). Hence \( F(S_s) \) has a continuous \( \pi_1(S, x) \)-action. In general this need not be the case, but we have enough to conclude that for \( F \) a locally constant sheaf on \( S_{\text{ret}} \),

\[
H^p(S_{\text{ret}}, F) \cong H^p(\pi_1(S, x), F(S_s))
\]

(3)

(Note that a finite, locally constant sheaf for the étale topology is the same as a finite, locally constant sheaf on \( S_{\text{ret}} \).) In view of the isomorphism (2) on \( S_{\text{ret}} \), this is essentially the statement that sheaf cohomology coincides with Čech cohomology. Corollary 2.5 of Chapter III \([12]\) gives sufficient conditions for this; namely, for every surjection...
$F \to F''$ of sheaves, the map
\[
\lim_{\longrightarrow} \prod F(U_{i_0...i_p}) \to \lim_{\longrightarrow} \prod F''(U_{i_0...i_p})
\]
is surjective where the limit is over all coverings $\{\coprod U_i \to S\}$ of $S$ and $U_{i_0...i_p} = U_{i_0} \times_S \cdots \times_S U_{i_p}$. But if $T \to S$ is a Galois covering of $S$ with group $G$, then $F(T_{i_0...i_p}) = F(\coprod_{G^{\times p}} T) = \prod_{G^{\times p}} F(T)$. Thus taking limits of $F(T_{i_0...i_p})$ over all coverings means taking limits over all coverings of products of copies of $F(T)$ and so surjectivity is immediate.

This discussion reduces our first application to the following theorem about the change of sites morphism $\tau : S_{et} \to S_{ret}$.

**Theorem 2.** Let $A$ be a connected semi-local ring containing a field $k$. If $F$ is a finite, locally constant sheaf of $\mathbb{Z}/n$-modules for the étale site on $A$ where $(n, \text{char}(k)) = 1$, then
\[
H^p(A_{ret}, \tau^* F) \to H^p(A_{et}, F)
\]
is an isomorphism for all $p$.

**Proof.** We wish to apply our extension theorem. We begin by treating the case of a smooth, semi-local ring $R$ essentially of finite type over $k$, an algebraically closed field first. We will show that $R^q \tau_* F = 0$ for $q > 0$ and any finite, locally constant sheaf $F$ of $\mathbb{Z}/n$-modules. Suppose $x \in H^q(R, F)$ with $q > 0$. We may assume $F = C_X$ is a constant sheaf of $\mathbb{Z}/n$-modules with value $C$ since $F$ becomes constant after a finite, etale extension of $R$. By assumption there is a smooth, connected variety $X$ over $k$, closed points $t_i \in X, 1 \leq i \leq m$, such that $X \cong O_{X,t_1...t_m}$ and an element $x' \in H^q(X, C_X)$ such that $x' |_{\text{Spec}(O_{X,t_1...t_m})} = x$. We construct a finite, étale covering $\rho : X' \to X$ such that $\rho^*(x') = 0$ using induction on $\dim(X)$ and the existence of Artin neighborhoods. If $\dim(X) = 0$, the assertion is obvious. If $\dim(X) = d$, the existence of an Artin neighborhood relative to $k$ means there is a diagram ([12, p. 117] or [16])

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \overline{X} \\
\downarrow f & & \downarrow \overline{f} \\
Y & \leftarrow & X_0
\end{array}
\]

in which

1. $j$ is an open immersion, dense in each fibre and $Y = \overline{X} - X$;
2. $\overline{f}$ is smooth and projective with geometrically irreducible fibres of dimension one;
3. $g$ is finite and étale and each fibre of $g$ is non-empty.

(Although the statement of the existence of Artin neighborhoods given in [12] refers only to a local ring, the proof as given in [16] Expose
XI, Section 3] clearly extends to the semi-local case. In fact, the proof of Proposition 3.3, the key statement, is footnoted to that effect. Since base extension of a local ring from an arbitrary field \( k \) to its separable closure \( k_s \) may produce a semi-local ring we need this more general result.

Now \((Y, X)\) is a smooth \( X_0\)-pair. The Gysin sequence \[12\] Chapter VI, Corollary 5.3 and Remark 5.4 (a) becomes
\[
0 \to R^1 f_* C_X \to R^1 f_* C_X \to g_*(C_Y \otimes T_{Y/X}) \to R^2 f_* C_X \to R^2 f_* C_X \to 0,
\]
and \( R^p f_* C_X \cong R^p f_* C_X \) for \( p > 2 \). The base change maps for the proper morphisms \( f \) and \( g \) are isomorphisms. Consequently the base change map for \( f \) is also an isomorphism. Since the fibres of \( f \) are non-complete curves, we find \( R^p f_* C_X = 0 \) for \( p > 1 \). Moreover \( f \) is smooth and so \( R^1 f_* C_X \) is finite, locally constant on \( X \) and \( g_* (C_Y \otimes T_{Y/X}) \) is locally constant on \( Y \).

Therefore we conclude that \( R^p f_* C_X \) is a finite, locally constant sheaf of \( \mathbb{Z}/n \)-modules for \( p = 0 \) or \( 1 \).

This information shows that the Leray spectral sequence for \( f \) degenerates into the long exact sequence
\[
\ldots \to H^q(X_0, f_* C_X) \to H^q(X, C_X) \to H^{q-1}(X_0, R^1 f_* C_X) \to H^{q+1}(X_0, f_* C_X) \to \ldots
\]
and we may apply the induction hypothesis. Then for any
\[
y \in H^{q-1}(X_0, R^1 f_* C_X),
\]
there is, after shrinking \( X_0 \) if necessary, a finite, étale covering space \( \rho' : X'_0 \to X_0 \) such that \( \rho'^*(y) = 0 \). Now by using the pullbacks of appropriate coverings of \( X_0 \) to \( X \) and a diagram chase, it is a straightforward matter to produce a finite, étale covering \( \rho : X' \to X \) such that \( \rho^*(x') = 0 \).

This argument is, of course, the essence of Remark 3.16 \[12\] Chapter III and shows that \( R^q \tau_* F = 0 \) for \( q > 0 \). Hence the Leray spectral sequence for the change of site morphism \( \tau : X_{et} \to X_{ret} \) collapses, and we conclude
\[
H^p(R_{ret}, \tau_* F) \cong H^p(R_{\acute{e}t}, F)
\]

Now suppose that \( R \) is a semi-local ring which is smooth and essentially of finite type over \( k \), a not necessarily algebraically closed field. Then for any scheme \( X \) there is a Hochschild-Serre spectral sequence \[12\] Chapter III, Remark 2.21(b)]
\[
H^p(G(k_s/k), H^q(X_s, F)) \Rightarrow H^p(X, F)
\]
where $X_s = X \times_k \text{Spec}(k_s)$. Similarly the theory of algebraic fundamental groups constructs a profinite group extension
\[
1 \to G((R \otimes_k k_s)/R) \to G(k_s/k) \to 1
\]
and so there is a corresponding Hochschild-Serre spectral sequence
\[
H^p(G(k_s/k), H^q(G((R \otimes_k k_s)/R \otimes_k k_s), F((R \otimes_k k_s)_s))) \Rightarrow H^n(G(Rs/R), F(R_s))
\]
(7)
The change of site morphism defines a homomorphism $\Psi$ from (6) to (7).

Since $F$ is a sheaf of $\mathbb{Z}/n$-modules where $n$ is relatively prime to $\text{char}(k)$, we conclude, as usual, that $H^p(X \times_k \text{Spec}(k_s), F) \cong H^p(X \times_k \text{Spec}(\overline{k}), F)$ where $\overline{k}$ is the algebraic closure of $k$. Consequently, by the first case, $\Psi$ is an isomorphism on the $E_2^{p,q}$ terms. This shows that (11) is an isomorphism when $A$ is essentially smooth over a not necessarily algebraically closed field $k$.

Some preparation is required in order to apply Theorem 1 to an arbitrary semi-local ring $A$ whether we work with the small or large étale site. Our hypothesis on $F$ shows that $F$ is representable in either the small or large site by a semi-local scheme $\mathbb{F}$ which is finite and étale over $A$. Now $A$ is a filtered limit of connected semi-local rings $A_i$ essentially of finite type over $k$. Consequently there is a finite, étale covering space $\mathbb{F}_{i_0}$ over $A_{i_0}$ for a sufficiently large $A_{i_0}$ such that $\mathbb{F}_{i_0} \times_{A_{i_0}} A \cong \mathbb{F}$. We use $\mathbb{F}_{i_0}$ to define a locally constant sheaf $F_{i_0}$ over $A_{i_0}$ whose restriction to $A$ is $F$. Next find a presentation of $A_{i_0}$ by a smooth connected semi-local ring $B_{i_0}$ essentially of finite type over $k$ so that $A_{i_0} \cong B_{i_0}/I$. Since any étale covering space of $A_{i_0}$ may be lifted to an étale covering space of $B_{i_0}$ (by the theorem of the primitive element all one has to do is lift a separable polynomial), we may assume there is a finite, locally constant sheaf $G$ on $\text{Spec}(B_{i_0})$ whose restriction to $\text{Spec}(A)$ is $F$.

Let $C$ be the category of connected semi-local rings over $B$, $F_1 = H^p(-_{\text{ret}}, \tau_*G)$, and $F_2 = H^p(-, G)$ where now we consider $G$ on the large site over $\text{Spec}(B)$. Note that $G$ is represented by a finite étale covering space $B'$ of $B$ in $B_{\text{et}}$ and so $G$ is also represented by $B'$ on the big étale site over $B$. Since $B'$ also defines a covering in $B_{\text{ret}}$ the same observation applies to the sheaf $\tau_*G$ in the big site $B_{\text{ret}}$. After showing that the hypotheses of our extension theorem are met we can conclude that (14) is an isomorphism for any $B$-algebra $A$ with the property that $G|_A = F$ and $\tau_*G|_A = \tau_*F$.

Both $F_1$ and $F_2$ are locally of finite presentation. For $F_2$ this is [12, Chapter III, Lemma 1.16] suitably interpreted. For $F_1$ we need
a different argument. Suppose \( R = \lim_{\rightarrow} R_i \) with \( R_i \) a connected semi-local ring over \( B \) where \( i \in I \), a filtered set. Any étale covering space of \( \lim R_i \) comes from an étale covering space defined over one of the \( R_i \) and homomorphisms between any two étale covering spaces of \( \lim R_i \) descend to a homomorphism between covering spaces over one of the \( R_i \). Thus we have an isomorphism

\[
\lim_{R_i} \lim_{T_i/R_i} \check{H}^p(T_i/R_i, G) \to \lim_{T/R} \check{H}^p(T/R, G)
\]

since the Čech cohomology groups depend only on the Galois group of the covering by (2) and \( G \) is locally constant. Thus, using the Galois cohomology interpretation of \( F_1 \), we also have

\[
H^p(R_{\text{ret}}, \tau_* G) \cong \lim_{R_i \to R} H^p(R_{\text{ret}}/R_i, \tau_* G)
\]

and so \( F_1 \) is locally of finite presentation.

Now we turn to the hensel pair condition. Suppose \((R, J)\) is a hensel couple and \( G \) is a locally constant sheaf of \( \mathbb{Z}/n - \)modules on \( \text{Spec}(R) \). Then \( G \) is represented by a finite, étale group scheme over \( R \), and by Example 1 we have an isomorphism for a hensel couple \((R, J)\) and such a \( G \),

\[
H^p(R_{\text{ét}}, G) \cong H^p(R/J_{\text{ét}}, G|_{R/J}).
\]

Thus \( F_2 \) is a local functor for the étale topology satisfying the hensel pair condition. As for \( F_1 \), there is a one-to-one correspondence between étale covering spaces of \( R/J \) and étale covering spaces of \( R \), and so we have an isomorphism

\[
H^p(R_{\text{ret}}, \tau_* G) \cong H^p(R/J_{\text{ret}}, \tau_*(G|_{R/J})).
\]

Hence \( F_1 \) is a local functor for the étale topology and also satisfies the hensel pair condition. We can now apply Theorem 1 to conclude the proof. \( \square \)

**Corollary 2.** Let \( A \) be a semi-local ring containing a field \( k \). Suppose \( F \) is a finite, locally constant sheaf of \( \mathbb{Z}/n - \)modules where \( (n, \text{char}(k)) = 1 \). Then for any \( x \in H^p(A, F) \), \( p > 0 \), there is a Galois extension \( A'/A \) such that \( x |_{A'} = 0 \in H^p(A', F) \).

2.2. **Merkurjev-Suslin theorem.** Our second application is an extension of the Mercuriyev-Suslin theorem to an arbitrary semi-local ring.

For a semi-local ring \( A \) and any \( n \) which is relatively prime to the residue characteristics of \( A \), Kummer theory provides a natural isomorphism \( K_1(A)/n \to H^1(A, \mathbb{Z}/n(1)) \). The Galois symbol map \( [20] \) is obtained by extending this map multiplicatively to \( K_1(A)/n \to H^1(A, \mathbb{Z}/n(i)) \).
Theorem 3. Let $A$ be a semi-local ring. Then if $n$ is relatively prime to the residue characteristics of $A$, the Galois symbol map
\begin{equation}
K_2(A)/n \to H^2(A, \mathbb{Z}/n(2))
\end{equation}
is an isomorphism.

Corollary 3. Let $A$ be a semi-local ring, and suppose $n$ is an integer which is relatively prime to the residue characteristics of $A$. Then the cup product map
\[ H^1(A, \mathbb{Z}/n(1)) \times H^1(A, \mathbb{Z}/n(1)) \to H^2(A, \mathbb{Z}/n(2)) \]
is onto. In particular, if $A$ contains a primitive $n^{th}$ root of unity, the $n$-torsion in the Brauer group of $A$ is generated by symbol algebras $(a, b)_n$.

Corollary 4. Let $X$ be a scheme, and suppose $n$ is an integer which is relatively prime to the residue characteristics of $X$. The symbol map defines an isomorphism of Zariski sheaves
\[ K_2/n \to H^2(\mathbb{Z}/n(2)). \]

The proof requires some preliminary material on K-theory with coefficients, Chern classes, and a discussion of results of Gillet which will be used to prove the theorem for semi-local rings smooth and essentially of finite type over a semi-local Dedekind ring.

Fix a ring $A$, a prime $\ell$ that is a unit in $A$, and an integer $n$ which is at least 2 if $\ell = 2$ (See [21, Proposition 2.4] for a detailed discussion of the case $\ell = 2$.) Algebraic K-theory with coefficients, $K_r(A; \mathbb{Z}/\ell^n)$, is a bifunctor in $A$ and the cyclic group $\mathbb{Z}/\ell^n$, and, for a pair of integers $1 \leq i, 0 \leq j \leq 2i$, there is a theory of Chern classes given by natural transformations
\[ c^{\ell^n}_{i,j} : K_{2i-j}(A; \mathbb{Z}/\ell^n) \to H^j(A, \mathbb{Z}/\ell^n(i)) \]
satisfying certain properties. Among the properties these objects satisfy are:

1. (universal coefficient theorem) There is a natural short exact sequence
\begin{equation}
0 \to K_r(A)/\ell^nK_r(A) \to K_r(A; \mathbb{Z}/\ell^n) \to \ell^nK_{r-1}(A) \to 0
\end{equation}

2. (functoriality) For any $n > m$, the diagram
\[ \begin{array}{ccc}
K_{2i-j}(A; \mathbb{Z}/\ell^m) & \to & K_{2i-j}(A; \mathbb{Z}/\ell^n) \\
\downarrow c^{\ell^m}_{ij} & & \downarrow c^{\ell^n}_{ij} \\
H^j(A, \mathbb{Z}/\ell^m(i)) & \to & H^j(A, \mathbb{Z}/\ell^n(i))
\end{array} \]
defined from the inclusion $\mathbb{Z}/\ell^m \hookrightarrow \mathbb{Z}/\ell^n$ commutes.
(3) (naturality) For any valuation ring $O_v$ and prime $\ell$ distinct from the residue characteristic of $O_v$ and any $n$, we have a commutative diagram

$$
\begin{array}{ccc}
K_{2i-j}(F; \mathbb{Z}/\ell^m) & \xrightarrow{\partial} & K_{2i-j-1}(\kappa(v); \mathbb{Z}/\ell^n) \\
\downarrow c_{ij}^{\ell^n} & & \downarrow (1-i)c_{i-1,j-1}^{\ell^n} \\
H^j(F; \mathbb{Z}/\ell^n(i)) & \xrightarrow{\partial} & H^{j-1}(\kappa(v), \mathbb{Z}/\ell^n(i-1))
\end{array}
$$

where $\kappa(v)$ is the residue field of $O_v$ and $F$ is its field of fractions. In particular the Chern classes fit together to define a map from the Quillen-Gersten complex for $K_i(\mathbb{Z}/\ell^m)$ to the Bloch-Ogus complex for $H^i(\mathbb{Z}/\ell^n(\cdot))$. (Quillen’s argument in [14] applies unchanged to $K_i(\mathbb{Z}/\ell^n)$.)

Details of the above properties can be found in [17].

Now the other tool we need is a reformulation and extension to a semi-local ring of a result of Gillet-Levine [7] and Gillet [6].

**Theorem 4** (Gillet-Levine, Gillet). Let $B$ be a connected, semi-local ring with quotient field $K$ which is smooth and essentially of finite type over a semi-local Dedekind ring $D$. Then, for any integer $n$ relatively prime to the residue characteristics of $B$ and any $q \geq 0$, we have a Gersten-Quillen resolution of $K_q(B; \mathbb{Z}/n)$:

$$
0 \to K_q(B; \mathbb{Z}/n) \to K_q(K; \mathbb{Z}/n) \to \bigoplus_{x \in B^{(1)}} K_{q-1}(\kappa(x); \mathbb{Z}/n) \to \cdots
$$

**Proof.** The argument of Gillet and Levine and Gillet immediately extends to semi-local rings as is clear from going through their arguments. \qed

We also need a weak version of this result for etale cohomology. This has been demonstrated by Gillet but remains unpublished [8]. A proof for $n$ an odd integer relatively prime to the residue characteristics of $B$ is given in an appendix at the end of this paper.

**Lemma 1.** Let $B$ be a connected, semi-local ring which is smooth and essentially of finite type over a semi-local Dedekind ring, $D$. Let $K$ be the quotient field of $B$. Then, for any integer $n$ which is relatively prime to the residue characteristic of $D$,

$$
H^2(B, \mathbb{Z}/n(2)) \to H^2(K, \mathbb{Z}/n(2))
$$

is a monomorphism.

We will need the universal coefficient theorem to produce a Gersten-Quillen sequence for $K_2(-)/n$ using work of Gillet and Gillet-Levine.
and then the compatibility of the Chern class map with the Quillen-Gersten complex and the Bloch-Ogus complex will do the rest in the smooth case.

Proof. We deal with a semi-local algebra $A$ over a mixed characteristic Dedekind ring $\mathcal{D}$ first. We may assume $A$ is connected. Both $K_2(\mathbb{Z}/n)$ and $H^2(\mathbb{Z}/n(2))$ are locally of finite presentation over $\text{Spec}(\mathcal{D})$, and so we may assume $A$ is essentially of finite type over $\mathcal{O}_\mathcal{D}$. Let $C$ be the category of connected semi-local rings essentially of finite presentation over $\mathcal{D}$. We wish to apply Theorem 1 to this situation with $k = \mathcal{D}$, $F_1 = K_2(\mathbb{Z}/n)$, $F_2 = H^2(\mathbb{Z}/n(2))$, and $\phi : F_1 \rightarrow F_2$ being the Galois symbol map.

First observe that $K_2(\mathbb{Z}/n)$ satisfies the epic hensel pair condition since it is generated by symbols and $H^2(\mathbb{Z}/n(2))$ satisfies the hensel pair condition by Gabber or Strano’s result ([5] or [18]). It remains to show that $\phi(B)$ is an isomorphism if $B$ is smooth and essentially of finite type over $\mathcal{D}$. The exactness of

$$K_2(B;\mathbb{Z}/n) \hookrightarrow K_2(K;\mathbb{Z}/n) \rightarrow \bigsqcup_{x \in (\text{Spec}B)^{(1)}} K_1(\kappa(x);\mathbb{Z}/n)$$

where $K$ is the quotient field of $B$ was shown by Gillet ([6] using work of Gillet-Levine [7] (see Theorem 2.6). We see that $K_1(\kappa(x);\mathbb{Z}/n) \cong K_1(\kappa(x))/n$, $K_2(B;\mathbb{Z}/n)$ is an extension of $\mu_n(B)(\cong K_1(\mathbb{Z}/n(2)))$ by $K_2(B)/n$, and similarly for $K_2(K;\mathbb{Z}/n)$. We can now reinterpret (10) as the exact sequence of the first line below

$$K_2(B)/n \hookrightarrow K_2(K)/n \rightarrow \bigsqcup_{x \in (\text{Spec}B)^{(1)}} K_1(\kappa(x))/n \begin{array}{c}\downarrow \approx \\ \downarrow \approx \end{array} \begin{array}{c}H^2(\mathbb{Z}/n(2)) \hookrightarrow H^2(K,\mathbb{Z}/n(2)) \rightarrow \bigsqcup_{x \in (\text{Spec}B)^{(1)}} H^1(\kappa(x),\mathbb{Z}/n(1)) \end{array}$$

Here the bottom row is not necessarily exact but is a complex, and the first map is a monomorphism by Lemma 1, the first vertical isomorphism is the Merkuriev-Suslin theorem for fields, and the second vertical isomorphism is the observation that both groups are isomorphic to $\bigsqcup_{x \in (\text{Spec}B)^{(1)}} \kappa(x)^*/\kappa(x)^{*n}$. The diagram commutes by the naturality condition above. Since the bottom row is a complex we conclude that the first vertical map is an isomorphism as desired. Theorem 1.1 now finishes this case.

The case of a semi-local ring containing a field is similar but simpler since we can use Grayson’s version of the Gersten-Quillen sequence ([9] and the Bloch-Ogus sequence ([3] in place of the argument involving algebraic K-theory with coefficients and Lemma 1). □
Suppose the Bloch-Kato conjecture holds for fields; that is, the Galois symbol map
\[(11) \quad K^M_i(K)/n \to H^i(K, \mathbb{Z}/n)i)\]
is an isomorphism where \(K^M_i(K)\) is the \(i\)th Milnor K-group of the field \(K\) and \(n\) is relatively prime to \(\text{char}(K)\). Then the same argument applies since the Gersten-Quillen and Bloch-Ogus sequences hold for \(i\) as well as 2, at least up to \((i-1)!\) torsion and Gillet’s unpublished result. Thus the Bloch-Kato conjecture for fields in degree \(i\) and \(i-1\) would show that \((11)\) is an isomorphism for smooth local rings essentially of finite type over \(k\). The rest of the argument is then identical, and so we would conclude that \(K^M_i(A)/n \to H^i(A, \mathbb{Z}/n)i)\) is an isomorphism if \(A\) is a semi-local ring containing \(k\) and \((n, (i-1)!\text{char}(k)) = 1\).

2.3. Torsion in \(K_2\). The final application was suggested by C. Weibel. Recall that Suslin in [7] used Chern classes to construct an isomorphism
\[K_2(F)\{\ell\} \to H^1(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/\text{Div}(H^1(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)))\]
where \(F\) is either a field of positive characteristic and \(\ell\) is a prime distinct from \(\text{char}(F)\), or \(F\) is a field of finite type over \(\mathbb{Q}\). Note that Suslin is using continuous Galois cohomology since he uses the identification
\[H^1(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/\text{Div}(H^1(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) = H^2_{\text{cont}}(F, \mathbb{Z}_\ell(2))\{\ell\}\]
where continuous cochains must be used to correctly identify \(H^2_{\text{cont}}(F, \mathbb{Z}_\ell(2))\{\ell\}\) as \(H^1(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))\) modulo its maximal divisible subgroup.

**Theorem 5.** Let \(A\) be a semi-local ring containing a field \(k\) and suppose that \(A\) is either essentially of finite type over \(\mathbb{Q}\) or \(\text{char}(k) > 0\). Let \(\ell\) be a prime distinct from \(\text{char}(k)\). Then the Chern class \(c_{2,1}\) induces an isomorphism
\[K_2(A)\{\ell\} \to H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/\text{Div}(H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \to H^2_{\text{cont}}(A, \mathbb{Z}_\ell(2))\{\ell\}\]
where \(H^2_{\text{cont}}(A, \mathbb{Z}_\ell(2))\) is continuous étale cohomology [10].

**Corollary 5.** Suppose \(A\) is a unibranch, e.g. normal, local ring containing a field \(k\) with \(\text{char}(k) = p > 0\), and let \(\ell\) be as above. Then
\[K_2(A)\{\ell\} \approx H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))\]

The conclusion of Theorem 5 is independent of the characteristic of \(k\), but the argument required in the two cases is different. We begin by recalling the structure of Suslin’s argument in [19] and the role of
continuous cohomology. He first observes (in the proof of Proposition 3.8) that \( c_{2,1} \) appears in a commutative, exact diagram

\[
\begin{array}{ccc}
K_3(F) & \rightarrow & K_3(F; \mathbb{Z}/\ell^n) \xrightarrow{c_{2,1}} H^1(F, \mathbb{Z}/\ell^n(2)) \\
\downarrow & & \downarrow \\
H^1_{cont}(F, \mathbb{Z}_\ell(2)) & \rightarrow & H^1_{cont}(F, \mathbb{Q}_\ell(2)) \rightarrow H^1_{cont}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \rightarrow H^2_{cont}(F, \mathbb{Z}_\ell(2))\{\ell\} \rightarrow 0
\end{array}
\]

where the right hand vertical arrow comes from the coefficient sequence defined by multiplication by \( \ell^n \) on \( \mathbb{Q}_\ell/\mathbb{Z}_\ell(2) \). Continuous cohomology is used here to make the bottom sequence exact (the first arrow is defined on \( \ell \)-adic sheaves by sending \( x \mapsto \ell^{-n}x \)) and to identify the image of \( H^1_{cont}(F, \mathbb{Q}_\ell(2)) \) with the maximal divisible subgroup of \( H^1_{cont}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) and the image of \( H^1_{cont}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) with the \( \ell \)-primary torsion in \( H^2_{cont}(F, \mathbb{Z}_\ell(2)) \). Now by exactness and the universal coefficient theorem \( c_{2,1} \) factors through \( K_2(F)\{\ell\} \) as

\[
\tau_{2,1} : K_2(F)\{\ell\} \rightarrow H^1_{cont}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/\text{Image}(H^1_{cont}(F, \mathbb{Q}_\ell(2))) \\
\cong H^2_{cont}(F, \mathbb{Z}_\ell(2))\{\ell\}.
\]

Suslin has already shown that \( H^1_{cont}(F, \mathbb{Z}_\ell(2)) \) is finite when \( \text{char}(F) > 0 \) \[19\] Corollary 2.8, and so \( H^1_{cont}(F, \mathbb{Q}_\ell(2)) = 0 \) and

\[
H^1_{cont}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \cong H^2_{cont}(F, \mathbb{Z}_\ell(2))\{\ell\}
\]

in this case. He then shows that \( \tau_{2,1} : K_2(F)\{\ell\} \rightarrow H^1_{cont}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \) is an isomorphism \[19\] Theorem 3.9] by an induction argument. In characteristic zero \( H^1_{cont}(F, \mathbb{Q}_\ell(2)) \) is only zero if \( F_0 \), the algebraic closure of \( \mathbb{Q} \) in \( F \), has only real embeddings in \( \mathbb{C} \), but he can reduce the theorem to \( F_0 \) \[19\] Proposition 3.3] and here the desired result was proven earlier by Tate. Thus extending this result to a semi-local ring containing a field which is essentially of finite type over \( \mathbb{Q} \) in characteristic zero requires first defining the factorization of \( c_{2,1} \) through \( \tau_{2,1} \) (which will require continuous étale cohomology \[10\]) and then showing that \( \tau_{2,1} \) is an isomorphism.

\( c_{2,1} \) is defined for arbitrary semi-local rings \( A \) (at least if \( n > 2 \) when \( \ell = 2 \)) and Suslin’s factorization argument applies which allows us to define

\[
\tau_{2,1}(A) : K_2(A)\{\ell\} \rightarrow H^1_{cont}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/Div^1(A)
\]

where, for simplicity, \( Div^1(A) \) denotes the maximal \( \ell \)-divisible subgroup of \( H^1_{cont}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \). The exact sequence needed for this factorization requires using continuous étale cohomology which Jannsen
The Merkuriev-Suslin theorem for any semi-local ring

\[ \text{Theorem 3, while different in characteristic 0 case is essentially the same but is complicated etc. for typographical reasons. If } \text{char}(A) \text{ is positive, then } H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \text{ is finite and so } H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \text{ which maps onto the maximal divisible subgroup of } H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \text{ vanishes. Thus the bottom sequence in (14) is just the Bloch-Ogus sequence and so } \tau_{2,1}(A) \text{ is an isomorphism. If } \text{char}(A) = 0 \text{ and } A \text{ is essentially of finite type over } \mathbb{Q}, \text{ then } H^0(\kappa(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \text{ has no divisible part for any } x \in \text{Spec}(A)^{(1)}. \text{ Thus } \text{Div}^1(A) = \text{Div}^1(F), \text{ and so the bottom sequence is an exact quotient of the Bloch-Ogus sequence. Thus in either case } \tau_{2,1}(A) \text{ is an isomorphism.}

Now suppose } A \text{ is an arbitrary semi-local ring of positive characteristic. Since both } K_2(A) \text{ and } H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \text{ commute with direct limits of rings we may assume that } A \text{ is a semi-local ring essentially of finite type over a field. We can then use Theorem 1 since } H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \text{ is local for the étale topology with respect to closed sets by 1) in Example (11) and } K_2(A)\{\ell\} \text{ is locally of finite presentation and satisfies the epic hensel pair condition. This latter condition follows since } K_3(A; \mathbb{Z}/\ell^n) \text{ is local for closed sets by 2) in Example (11) and so } \nu_0 K_2(A), \text{ being a quotient, satisfies the epic hensel pair condition. Note that the Corollary to Theorem 1 shows that } K_2(A)\{\ell\} \text{ is then local for the étale topology with respect to closed sets.}

The characteristic 0 case is essentially the same but is complicated by the fact that } H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/\text{Div}^1(A) \text{ does not behave nicely with...
respect to limits of rings. However Jannsen has shown that
\[ H^1(A, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2))/Div^1(A) \cong H^2_{cont}(A, \mathbb{Z}_\ell(2)) \{ \ell \}. \]
In addition he shows that the sequence,
\[ 0 \rightarrow \lim_{\leftarrow} H^2(B, \mathbb{Z}/\ell^n(2)) \rightarrow H^2_{cont}(B, \mathbb{Z}_\ell(2)) \rightarrow \lim_{\leftarrow} H^2(B, \mathbb{Z}/\ell^n(2)) \rightarrow 0, \]
is exact where \( \lim_{\leftarrow} \) refers to the first left derived functor of the system \( \{ H^2(B, \mathbb{Z}/\ell^n(2)) \} \). Consequently
\[ H^2_{cont}(B^h, \mathbb{Z}_\ell(2)) \cong H^2_{cont}(A, \mathbb{Z}_\ell(2)) \]
if \( B^h \) is the henselization with respect to \( I \) of a presentation \( A \cong B/I \) as a quotient of a semi-local ring \( B \) smooth over \( k \). This and the corresponding isomorphism
\[ H^1(B^h, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2)) \cong H^1(A, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2)) \]
then allow us to conclude that
\[ H^2_{cont}(B^h, \mathbb{Z}_\ell(2)) \{ \ell \} \cong H^2_{cont}(A, \mathbb{Z}_\ell(2)) \{ \ell \}. \]
Now we may apply Theorem to show \( \overline{\sigma}_{2,1} : K_2(A) \{ \ell \} \rightarrow H^2_{cont}(A, \mathbb{Z}_\ell(2)) \{ \ell \} \) is an isomorphism.

Corollary 4 is obtained by noting that Suslin [19, Corollary 2.8] also proved that \( H^1(F, \mathbb{Z}_\ell(2)) \) is finite if \( F \) is a field of positive characteristic. But if \( A \) is a unibranch local ring, e.g. a normal local ring, then \( H^1(A, \mathbb{Z}_\ell(2)) \hookrightarrow H^1(F, \mathbb{Z}_\ell(2)) \) is also finite. Consequently \( H^1(A, \mathbb{Q}_\ell(2)) = 0 \), and so we have an isomorphism \( H^1(A, \mathbb{Q}_\ell / \mathbb{Z}_\ell(2)) \cong H^2(A, \mathbb{Z}_\ell(2)) \{ \ell \}. \)

**Appendix A.**

In this appendix \( \zeta_n \) will always stand for a primitive \( n^{th} \) root of unity. We wish to prove the following theorem. (Note that this is part of [11, Theorem 1] and is proved without assuming the generalized Kato conjecture.)

**Theorem 6.** Let \( A \) be a connected, regular semi-local ring with quotient field \( K \), and \( n \) an odd positive integer relatively prime to the residue characteristics of \( A \). Then
\[ H^2(A, \mathbb{Z}/n(2)) \overset{j}{\rightarrow} H^2(K, \mathbb{Z}/n(2)) \]
is injective.

We begin with a series of lemmas to calculate the cohomology of the extension obtained by adjoining an \( n^{th} \) root of 1. Fix the following situation in order to describe \( \mathbb{Z}/n(k) \) and the action of the cyclotomic character on this module. Fix an integer \( n \) such that \( n = dm \) and
\( m = de \) for some positive integers \( d, e \), and let \( A \) be a connected ring with \( \zeta_m \in A \) and \( d \) a unit in \( A \). Assume \( A[\zeta_n] \) is connected. Then

\[
X^d - \zeta_m = \prod_{i=1}^{d} \left( X - \zeta_n (\zeta_m^i) \right)
\]

in \( A[\zeta_n] \), and \( A[\zeta_n] \) is a Galois extension of \( A \) with cyclic Galois group \( G_d := \mathbb{Z}/d = \langle \sigma \rangle \subset (\mathbb{Z}/n)^{\times} \).

Construct a dictionary for the action of \( G_d \) on \( \mathbb{Z}/n(\zeta_n) \) by \( \sigma \) sends \( \zeta_n \mapsto \zeta_n (\zeta_m^e) \).

Since \( n \mid m^2 \), we have

\[
G_d \text{ on } \mathbb{Z}/n\left(\zeta_n, \zeta_m = \zeta_d, \zeta_m^e \right) \in \mathbb{Z}/n(1)^{\times 3} \quad (1, d, m = de) \in \mathbb{Z}/n^{\times 3}
\]

\[
\sigma(\zeta_n) = \zeta_n (\zeta_m^e) = \zeta_n^{1+m} \in \mathbb{Z}/n(1) \quad \sigma(1) = 1 + m \in \mathbb{Z}/n
\]

\[
\sigma^k(\zeta_n) = (\zeta_n^{1+m})^k = \zeta_n^{1+km} \in \mathbb{Z}/n(\zeta_n) \quad \sigma^k(1) = (1 + m)^k = 1 + km \in \mathbb{Z}/n
\]

This dictionary makes the computation of the cohomology of \( G_d \) on \( \mathbb{Z}/n(\zeta_n) \) straightforward.

**Lemma 2.** \( H^0(G_d, \mathbb{Z}/n(\zeta_n)) = \mathbb{Z}/(m(d, k)) \), and

\[
H^{2r}(G_d, \mathbb{Z}/n(\zeta_n)) = \begin{cases} 
\mathbb{Z}/(d, k) & \text{if } d \text{ is odd or } k \text{ is even or } 4 \nmid m \\
\mathbb{Z}/2(d, k) & \text{if } d \text{ is even and } k \text{ is odd and } 4 \nmid m 
\end{cases}
\]

and

\[
H^{2r+1}(G_d, \mathbb{Z}/n(\zeta_n)) = \begin{cases} 
\mathbb{Z}/(d, k) & \text{if } d \text{ is odd or } k \text{ is even or } 4 \nmid m \\
\mathbb{Z}/2(d, k) & \text{if } d \text{ is even and } k \text{ is odd and } 4 \nmid m 
\end{cases}
\]

**Proof.** The standard resolution used to calculate the cohomology of a cyclic group \( G_d \) on \( \mathbb{Z}/n(\zeta_n) \) is used to carry out this calculation. Our dictionary allows us to calculate \( N_k = \sum_{i=0}^{d-1} \sigma^ik : \mathbb{Z}/n \to \mathbb{Z}/n \) and \( T_k = 1 - \sigma^k : \mathbb{Z}/n \to \mathbb{Z}/n \) as

\[
T_k(j) = j - j(1 + km) = -kmj
\]

\[
N_k(j) = j \left( \sum_{i=0}^{d-1} (1 + ikm) \right)
\]

\[
= j \left( d + km \frac{d(d-1)}{2} \right)
\]

\[
= \begin{cases} 
d \quad & \text{if } d \text{ is odd or } k \text{ is even} \\
d(k - \frac{d}{2}) \quad & \text{if } d \text{ is even and } k \text{ is odd}
\end{cases}
\]

In the case when \( d \) is even and \( k \) is odd, we note that

\[
(m, 1 - \frac{m}{2}) = \begin{cases} 
2 & \text{if } 4 \nmid m \\
1 & \text{otherwise}
\end{cases}
\]
Then the sequence that calculates the cohomology groups is
\[
\cdots \mathbb{Z}/n \xrightarrow{N_k} \mathbb{Z}/n \xrightarrow{T_k} \mathbb{Z}/n \xrightarrow{N_k} \mathbb{Z}/n \xrightarrow{T_k} \mathbb{Z}/n \cdots ,
\]
and we find
\[
\text{Ker } T_k = \frac{d}{(d, k)} \mathbb{Z}/n \cong \mathbb{Z}/(m(d, k))
\]
\[
\text{Im } T_k = km\mathbb{Z}/n = (d, k) \cdot m\mathbb{Z}/n
\]
\[
\text{Ker } N_k = \begin{cases} m\mathbb{Z}/n & \text{if } d \text{ is odd or } k \text{ is even or } 4 \mid m \\ \frac{m}{2}\mathbb{Z}/n & \text{if } d \text{ is even, } k \text{ is odd, and } 4 \nmid m \end{cases}
\]
\[
\text{Im } N_k = \begin{cases} d\mathbb{Z}/n & \text{if } d \text{ is odd or } k \text{ is even or } 4 \mid m \\ 2d\mathbb{Z}/n & \text{if } d \text{ is even, } k \text{ is odd, and } 4 \nmid m \end{cases}
\]
The rest is straightforward. □

**Corollary 6.** Let \( A \) be a regular, connected semi-local domain, and let \( n \) be an odd positive integer relatively prime to the residue characteristics of \( A \). Suppose that \( m \) is the largest divisor of \( n \) such that \( \zeta_m \in A \). Let \( A_n = A[\zeta_n] \), and let \( G = \text{Gal}(A_n/A) \). Then \( H^p(G, \mathbb{Z}/n(2)) = \begin{cases} \mathbb{Z}/m(2) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases} \)

**Proof.** If \( m = n \), we are done. Suppose that \( n = \ell n' \) where \( \ell \) is the smallest prime divisor of \( \frac{n}{m} \). Then there is a cyclic subgroup \( H < G \) such that \( A_{n'} := A^H = A[\zeta_{n'}] \), and \( A_{n'} \) is a Galois extension of \( A \) with abelian Galois group \( G/H \).

If \( \zeta_\ell \notin A_{n'} \), then \( A_n \) and \( A_{n'} \) satisfy the hypothesis of the lemma.

Hence
\[
H^p(H, \mathbb{Z}/n(2)) = \begin{cases} \mathbb{Z}/m(2) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}
\]

If \( \zeta_\ell \notin A_{n'} \), then \([A_n : A_{n'}] = \ell - 1 \) is relatively prime to \( n' \). Hence
\[
H^p(H, \mathbb{Z}/n(2)) = \begin{cases} \mathbb{Z}/n'(2) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}
\]

Now by using the Hochschild-Serre spectral sequence,
\[
E_2^{p,q} = H^p(G/H, H^q(H, \mathbb{Z}/n(2))) \Rightarrow H^{p+q}(G, \mathbb{Z}/n(2)),
\]
and the above calculations, we conclude that \( H^p(G/H, H^0(H, \mathbb{Z}/n(2))) \cong H^p(G, \mathbb{Z}/n(2)) \). Since \( H^0(H, \mathbb{Z}/n(2)) = \mathbb{Z}/n'(2) \), we may apply an induction hypothesis and thus prove the corollary. □
Lemma 3. Let $A$ be a regular, connected, semi-local domain with quotient field $K$, $n$ a positive integer relatively prime to the residue characteristics of $A$. Let $B$ be a Galois extension of $A$ with quotient field $L$, and let $G = \text{Gal}(B/A)$. Suppose $\zeta_n \in B$. Then

$$H^1(G, H^1(B, \mathbb{Z}/n(2))) \to H^1(G, H^1(L, \mathbb{Z}/n(2)))$$

is injective.

Proof. Since $A$ is a UFD, we have an exact sequence

$$0 \to A^* \to K^* \prod_{x \in A^{(1)}} \mathbb{Z}/\text{ord}_x \to 0$$

where $A^{(1)} = \{ x \in \text{Spec}(A) / \text{ht}(x) = 1 \}$ and $\text{ord}_x : K^* \to \mathbb{Z}/\mathbb{Z}$ calculates the order of $f \in K^*$ at the discrete valuation associated to $x$, and $i_x : \text{Spec}(k(x)) \to \text{Spec}(A)$ is the inclusion of the residue field of $A_x$. If we map this sequence to itself by multiplication by $n$, the cokernels form the short exact sequence

$$0 \to A^*/A^{(1)} \to K^*/K^{(1)} \prod_{x \in A^{(1)}} \mathbb{Z}/n \to 0.$$

There is a sequence of $G$-modules obtained by twisting the sequence for $B$ with the cyclotomic character

$$0 \to B^*/B^{(1)} \to L^*/L^{(1)} \prod_{y \in B^{(1)}} \mathbb{Z}/n \to 0,$$

relating $B^*/B^{(1)}(1) = H^1(B, \mathbb{Z}/n(2))$ to $L^*/L^{(1)}(1) = H^1(L, \mathbb{Z}/n(2))$ since $\zeta_n \in L^*$. Thus we need only show that

$$H^0(G, L^*/L^{(1)}(1)) \to H^0(G, \prod_{y \in B^{(1)}} i_y \mathbb{Z}/n(1))$$

is onto to complete the proof of the lemma. But

$$H^0(G, \prod_{y \in B^{(1)}} i_y \mathbb{Z}/n(1)) \cong \prod_{x \in A^{(1)}} H^0(D_{y|x}, i_y \mathbb{Z}/n(1)) \cong \prod_{x \in A^{(1)}} i_x \mathbb{Z}/m_x(1)$$

where $D_{y|x}$ is the decomposition group of a choice of $y \in B^{(1)}$ lying over $x \in A^{(1)}$ and $\mathbb{Z}/m_x(1)$ is the group of roots of unity of order $n$ in $k(y)$ fixed by $D_{y|x}$ (expressed in the above notation).

Any element $\alpha \in \prod_{x \in A^{(1)}} i_x \mathbb{Z}/m_x(1)$ can be written as

$$\alpha = \sum_{x \in A^{(1)}} \delta_x \cdot \zeta_n^r$$

where the sum is finite and $\delta_x = 0$ for $x' \in A^{(1)}$ if $x' \neq x$ and $\delta_x = 1 \in i_x \mathbb{Z}/n$ at $x \in A^{(1)}$. Then there is $f_x \in K^*/K^{(1)}$ with $\text{ord}_x(f_x) = \delta_x$.  


and \( \text{ord}_{x'}(f_x) = 0 \) if \( x' \neq x \). Hence
\[
\sum_{x \in A^{(1)}} \left( \sum_{\sigma \in G/D_{y|x}} f_x \sigma(\zeta_n x) \right)
\]
has image \( \alpha \).

\[ \square \]

**Remark 1.** The proof amounts to first reducing to the case of a dvr \( A \) by using the UFD sequence and then solving the problem at the completion where \( D_{y|x} \) becomes the Galois group and finally spreading that solution around using the transitivity of the action of \( G \) on \( \{ y \in B^{(1)}/y \text{ lies over a fixed } x \in A^{(1)} \} \).

**Proof.** If \( \zeta_n \in A \), the theorem reduces to the well known result \( Br(A) \subset Br(K) \) since \( H^2(A, \mathbb{Z}/n(2)) \cong H^2(K, \mathbb{Z}/n(1)) \cong nBr(A) \) and similarly for \( K \). We reduce the theorem to this case by analyzing the cohomology of a cyclic Galois covering \( A[\zeta_n]/A \) with Galois group \( G \). There is a spectral sequence for this covering
\[
E_{p,q}^{2} = H^p(G, H^q(A[\zeta_n], \mathbb{Z}/n(2))) \Rightarrow H^n(A, \mathbb{Z}/n(2)).
\]

Suppose \( x \in \text{Ker} \left[ H^2(A, \mathbb{Z}/n(2)) \to H^2(K, \mathbb{Z}/n(2)) \right] \). Then \( x = 0 \) in \( H^2(A[\zeta_n], \mathbb{Z}/n(2))^G \) by the above remarks. But Lemma 3 shows that \( x = 0 \) also in \( E_{\infty}^{1,1} \subset E_{\infty}^{1,1} \subset H^1(G, H^1(A[\zeta_n], \mathbb{Z}/n(2))) \). Finally Corollary 5 shows that \( x \in E_{\infty}^{2,0} \subset H^2(G, \mathbb{Z}/m(2)) \) where \( \zeta_m \in A \). But \( H^2(G, \mathbb{Z}/m(2)) \) is the same for the Galois covering \( A[\zeta_n]/A \) and \( K[\zeta_n]/K \). Hence \( x = 0 \) and the theorem follows. \( \square \)

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