The improvement wave equations of relativistic and non-relativistic quantum mechanics

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Abstract

In this work, we follow the idea of the De Broglie’s matter waves and the analogous method that Schrödinger founded wave equation, but we apply the more essential Hamilton principle instead of the minimum action principle of Jacobi which was used in setting up Schrödinger wave equation. Thus, we obtain a novel non-relativistic wave equation which is different from the Schrödinger equation, and relativistic wave equation including free and non-free particle. In addition, we get the spin $\frac{1}{2}$ particle wave equation in potential field.

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1. Introduction

We know De Broglie suggested that not only does light have a dual nature but material particles also require a wave-particle description during 1922-23, and he further noticed correspondences between the classical theory of light and the classical theory of mechanics. He thought that we may obtain the wave equation of material particles by comparing the classical theory of light with the classical theory of material particle [1]. Schrödinger used De Broglie’s idea to obtain the wave equation of material particles, i.e., Schrödinger equation [2]. In the following, we also apply De Broglie’s suggestion to research the wave equation of material particles [3]. It’s well known that Schrödinger compared the Fermat principle with the minimum action principle of Jacobi, than, from the wave equation of electromagnetic wave obtained the wave equation of matter particles, i.e., Schrödinger equation. Obviously, the two principles have the similar mathematics form, but they are different in physics content. So, we begin with the more essential Hamilton principle, and we find that the principle can describe not only the geometrical optics but also the classical mechanics. Applying Hamilton principle instead of the minimum action principle of Jacobi, we can obtain new wave equations including time-independent and time-dependent, which are different from Schrödinger equation. With the covariant Hamilton principle, we can get the relativistic wave equation including free and external field particle, which extend the Klein-Gordon equation. Meanwhile, we obtain the relativistic wave equation of spin $\frac{1}{2}$ particle in potential field.

2. Non-relativistic wave equation

The time-independent wave equation of electromagnetic wave which frequency is $\nu$ is:

$$\nabla^2 \Psi(\vec{r}) + \frac{4\pi^2 n^2 \nu^2}{c^2} \Psi(\vec{r}) = 0,$$

(1)

where $n$ is refracting power, $c$ is light velocity and $\Psi(\vec{r})$ is a component of electromagnetic field $\vec{E}$ and $\vec{B}$. The Eq. (1) describes the wave nature of light such as interference and diffraction phenomenon of light. When light transmits at straight line it is described by the geometrical optics and the geometrical optics is a limiting case of wave theory of light. Fermat had reduced the laws of geometrical optics to the principles of ‘least-time’. That
is, a light ray follows the path requiring the least time. The Fermat principle is

$$\delta \int nds = 0,$$

(2)

For a classical material particle, when it moves in potential energy $V(r)$ it can be described by the Hamilton principle

$$\delta \int Ldt = \delta \int \frac{T-V(r)}{v}ds = 0,$$

(3)

where $L = T - V(r)$ is Lagrangian function, $T$, $V(r)$ and $v = \sqrt{\frac{2m(E-V(r))}{m}}$ are kinetic energy, potential energy and the velocity of material particle, $E$ is the total energy. From Hamilton principle, we can deduce Fermat principle, but we can not get Fermat principle from the minimum action principle of Jacobi, which is:

$$\delta \int \sqrt{2m(E-V(r))}ds = 0,$$

(4)

In order to obtain the wave equation of material particle we compare Eq. (2) with (3) not Eq. (2) with (4) which was applied by Schrödinger. Since the Eq. (2) and (3) is concordant we can think that material particle wave equation is similar to Eq. (1). In Eq. (1), when $n$ is replaced with $\frac{(T-V(r)m)}{\sqrt{2m(E-V(r))}}$ and the time-independent wave equation of material particle can be written as follows:

$$\nabla^2\Psi(\vec{r}) + A[\frac{(T-V(r)m)}{\sqrt{2m(E-V(r))}}]^2\Psi(\vec{r}) = 0,$$

(5)

where $A$ is a constant and it can be obtained in the following. For a free material particle, its potential energy $V(r) = 0$ and total energy $\varepsilon = \frac{p^2}{2m}$, and it is associated with a plane wave

$$\Psi(\vec{r}, t) = \Psi(\vec{r})f(t) = e^{i\vec{p}\cdot\vec{r}-\varepsilon t},$$

(6)

Substitution of Eq. (6) into Eq. (5) gives

$$\left(\frac{i}{\hbar}\vec{p}\right)^2\Psi(\vec{r}) + A\frac{\varepsilon^2m^2}{2m\varepsilon}\Psi(\vec{r}) = 0,$$

(7)

The constant $A$ is

$$A = 4/\hbar^2,$$

(8)

From Eq. (5) and Eq. (8)

$$\nabla^2\Psi(\vec{r}) + \frac{4}{\hbar^2}[\frac{(T-V(r)m)}{\sqrt{2m(E-V(r))}}]^2\Psi(\vec{r}) = 0.$$
and
\[-\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}) = \frac{(E - 2V(r))^2}{E - V(r)} \Psi(\vec{r}).\]  
(9)

as
\[\frac{(E - 2V(r))^2}{E - V(r)} = \frac{E^2 - 4EV(r) - 4V^2(r)}{E - V(r)} = E - V(r) - 2V(r) + \frac{V^2(r)}{E - V(r)}\]  
(10)

Combining these two equations
\[\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}) = (E - V(r) - 2V(r) + \frac{V^2(r)}{E - V(r)}) \Psi(\vec{r})\]  
(11)

The Eq. (11) is the time-independent wave equation of material particle, and it is different from Schrödinger equation
\[-\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}) = (E - V(r)) \Psi(\vec{r})\]  
(12)

Comparing Eq. (11) with (12), we can find the Eq. (11) has an additional term \((-2V(r) + \frac{V^2(r)}{E - V(r)})\), and this term can be taken as perturbation in Coulomb field, but it is not sure in the other potential field. Meanwhile, from Eq. (11) we can find that the momenta \(p\) which in the external field can’t be made into operator. But for a free particle \(V(r) = 0\), the Eq. (11) can be written as
\[\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}) = E \Psi(\vec{r})\]  
(13)

Obviously, for a free particle, its momenta \(p\) can be made into operator \(-i\hbar \nabla\).

The time-dependent wave equation of material particle can be obtained by time-dependent wave equation of electromagnetic wave. The time-dependent wave equation of electromagnetic wave is
\[\nabla^2 \Psi(\vec{r}, t) - \frac{n^2}{c^2} \frac{\partial^2 \Psi(\vec{r}, t)}{\partial t^2} = 0,\]  
(14)

According to the same method at the above, we replace \(n\) with \(\frac{(T - V(r))m}{\sqrt{2m(E - V(r))}}\), then the time-dependent wave equation of material particle can be written as
\[\nabla^2 \Psi(\vec{r}, t) + A' \frac{(T - V(r))^2 m^2}{2m(E - V(r))} \frac{\partial^2 \Psi(\vec{r}, t)}{\partial t^2} = 0,\]  
(15)

For a free material particle, its potential energy \(V(r) = 0\) and total energy \(\varepsilon = \frac{p^2}{2m}\). Substituting the plane wave Eq. (6) into Eq. (15), we have
\[\left(\frac{i}{\hbar} \vec{p}\right)^2 \Psi(\vec{r}, t) + \frac{\varepsilon^2 m^2}{2m\varepsilon} (-\frac{i}{\hbar} \varepsilon)^2 \Psi(\vec{r}, t) = 0,\]  
(16)
The constant $A'$ is 

$$A' = -4/\varepsilon^2,$$  \hspace{1cm} (17)

From Eq. (15) and Eq. (17)

$$\nabla^2 \Psi(\vec{r}, t) - \frac{4}{\varepsilon^2} \cdot \frac{(T - V(r))^2 m^2}{2m(E - V(r))} \frac{\partial^2 \Psi(\vec{r}, t)}{\partial t^2} = 0,$$  \hspace{1cm} (18)

i.e.

$$\frac{\varepsilon^2}{2m} \nabla^2 \Psi(\vec{r}, t) = \frac{(E - 2V(r))^2}{E - V(r)} \frac{\partial^2 \Psi(\vec{r}, t)}{\partial t^2},$$  \hspace{1cm} (19)

The Eq. (19) is time-dependent wave equation of material particle in external field $V(r)$. Obviously, here the total energy $E$ can’t be made into operator.

For a free particle, its potential energy $V(r) = 0$ and total energy $\varepsilon = E = \frac{p^2}{2m}$

$$\frac{\varepsilon}{2m} \nabla^2 \Psi(\vec{r}, t) = \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t),$$  \hspace{1cm} (20)

i.e.

$$\varepsilon \left(-\frac{\hbar^2 \nabla^2}{2m}\right) \Psi(\vec{r}, t) = \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t),$$  \hspace{1cm} (21)

According to our conclusion in the front, we know that the momenta $p$ of a free particle can be made into operator $-i\hbar \nabla$, so

$$-\frac{\hbar^2 \nabla^2}{2m} = \frac{p^2}{2m} = \varepsilon$$  \hspace{1cm} (22)

and so

$$\varepsilon^2 \Phi(\vec{r}, t) = -\hbar^2 \frac{\partial^2}{\partial t^2} \Phi(\vec{r}, t),$$  \hspace{1cm} (23)

so that

$$\varepsilon \Phi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \Phi(\vec{r}, t),$$  \hspace{1cm} (24)

Obviously, Eq. (24) is the time-dependent wave equation of free particle. It is accordant with the result of Schrödinger and the energy $\varepsilon$ of free particle can be made into operator

$$\varepsilon \rightarrow i\hbar \frac{\partial}{\partial t}$$  \hspace{1cm} (25)

From Eq. (19), we can find that the total energy $E$ of the non-free particle can’t be made into operator. If the potential energy does not explicitly depend on the time, the Eq. (19) may be separated by writing

$$\Psi(\vec{r}, t) = \Psi(\vec{r}) f(t)$$  \hspace{1cm} (26)
The separated equations are
\[
\frac{\varepsilon^2}{2m} \frac{E - V(r)}{(E - 2V(r))^2} \frac{1}{\Psi(r)} \nabla^2 \Psi(r) = C, \tag{27}
\]
\[
\frac{d^2 f(t)}{dt^2} = Cf(t), \tag{28}
\]
\(C\) is a separation constant independent of \(\vec{r}\) and \(t\). From Eq. (27), we can get
\[
\nabla^2 \Psi(r) = C \frac{2m (E - 2V(r))^2}{\varepsilon^2 (E - V(r))} \Psi(r) \tag{29}
\]
and
\[
-\frac{\hbar^2}{2m} \nabla^2 \Psi(r) = C \frac{(-\hbar^2) (E - 2V(r))^2}{\varepsilon^2 (E - V(r))} \Psi(r), \tag{30}
\]
When the coefficient \(C\) is
\[
C = -\frac{\varepsilon^2}{\hbar^2}, \tag{31}
\]
then the Eq. (30) become into the Eq. (9). Substitution of \(C\) into Eq. (28) gives
\[
\frac{d^2 f(t)}{dt^2} = -\frac{\varepsilon^2}{\hbar^2} f(t), \tag{32}
\]
The solution of Eq. (32) is
\[
f(t) = B_1 e^{i\frac{\varepsilon}{\hbar} t} + B_2 e^{-i\frac{\varepsilon}{\hbar} t}, \tag{33}
\]
The complete solution of Eq. (19) is
\[
\Psi(\vec{r}, t) = \sum_n \Psi_n(\vec{r}) (B_1 e^{i\frac{\varepsilon}{\hbar} t} + B_2 e^{-i\frac{\varepsilon}{\hbar} t}), \tag{34}
\]
Where \(\Psi_n(\vec{r})\) is the eigenfunction of the Eq. (19), \(\varepsilon = \frac{p^2}{2m}\) is the total energy when the potential energy \(V(r)\) is vanished. Now, we can obtain the conclusions in the following:

(1) The new wave equations of material particle are different from the Schrödinger wave equations.

(2) Comparing Eq. (11) with (12), we can find the time-independent wave equation (Eq. (11)) has an additional term \((-2V(r) + \frac{V^2(r)}{E - V(r)})\), and this term can be taken as perturbation in Coulom field, but it is not sure in the other potential field.

(3) When we compare the time-dependent wave equation (Eq. (19)) with the Schrödinger’s time-dependent wave equation, we can find the relation between state and time is \((C_1 e^{\frac{\varepsilon}{\hbar} t} + C_2 e^{-\frac{\varepsilon}{\hbar} t})\) in the Schrödinger wave equation, but the relationship is \((B_1 e^{\frac{\varepsilon}{\hbar} t} + B_2 e^{-\frac{\varepsilon}{\hbar} t})\) in
our time-dependent wave equation, i.e., different state according to different evaluation factor of time in the Schrödinger wave equation, different state according to the same evaluation factor of time in our wave equation.

(4) We find that the energy $\varepsilon$ and moment $p$ of free particle can be made into operator $i\hbar \frac{\partial}{\partial t}$ and $-i\hbar \nabla$. However, the total energy $E$ and momenta $P$ of the particle in external field cannot be made into operator $i\hbar \frac{\partial}{\partial t}$ and $-i\hbar \nabla$.

3. Relativistic wave equation

In the following, we will give the relativistic wave equations of material particle. Firstly, we should extend the Hamilton principle to covariant form. The Hamilton principle is

$$\delta \int L dt = 0,$$

(35)

where $L$ is Lagrange function. The Eq. (35) can be written by $dt = \gamma d\tau$ and $ds = cd\tau$ as

$$\delta \int \frac{\gamma}{c} L ds = 0,$$

(36)

where $\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$, $u$ is particle velocity, $c$ is light velocity, $d\tau$ is static time and $ds$ is four-dimension differential interval. Obviously, the Eq. (36) is covariant when $\gamma L$ is Lorentz-invariant.

For a free particle [5]

$$\gamma L = -m_0 c^2$$

(37)

and for photon, we suppose

$$\gamma L = -E_s = constant$$

(38)

where $E_s$ is the static energy of a photon. From Eq. (36) and Eq. (38)

$$\delta \int ds = 0,$$

(39)

The Eq. (39) is covariant form of the Fermat principle ($n = 1$), and none but $n = 1$, the time-dependent wave equation of electromagnetic wave is covariant.

From the Eq. (1), when $n(n = 1)$ is replaced with $-m_0 c^2$, we can obtain relativistic wave equation of free particle

$$\nabla^2 \Psi(\vec{r}) + B(-m_0 c^2)^2 \Psi(\vec{r}) = 0,$$

(40)
For a free particle
\[ \Psi(\vec{r}, t) = \Psi(\vec{r}) f(t) = e^{i(\vec{p} \cdot \vec{r} - Et)}, \tag{41} \]
where \( E \) is the energy of free particle. From Eq. (40) and Eq. (41)
\[ (\frac{i}{\hbar} \vec{p})^2 + Bm_0^2c^4 = 0, \tag{42} \]
The constant \( B \) is
\[ B = \frac{p^2}{\hbar^2 m_0^2 c^4} = \frac{E^2 - E_0^2}{E_0^2 \hbar^2 c^2} \tag{43} \]
Substitution of Eq. (43) into Eq. (40) gives
\[ \nabla^2 \Psi(\vec{r}) + \frac{E^2 - E_0^2}{E_0^2 \hbar^2 c^2} E_0^2 \Psi(\vec{r}) = 0, \tag{44} \]
i.e.
\[ (E_0^2 - \hbar^2 c^2 \nabla^2) \Psi(\vec{r}) = E^2 \Psi(\vec{r}), \tag{45} \]
Comparing with
\[ (E_0^2 + c^2 p^2) \Psi(\vec{r}) = E^2 \Psi(\vec{r}), \tag{46} \]
Making momentum \( p \) into operator \(-i\hbar \nabla\), we can obtain the Eq. (45) by Eq. (46).

In the following, we give the time-dependent relativistic wave equation of free particle by replacing \( n(n = 1) \) with \(-m_0 c^2\) in Eq. (13), then
\[ \nabla^2 \Psi(\vec{r}, t) + B'(-m_0 c^2)^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) = 0, \tag{47} \]
Similarly, substitution of the Eq. (40) into Eq. (47) gives
\[ B' = -\frac{p^2}{E_0^2 E_0^2} = -\frac{p^2}{E_0^2 (E_0^2 + c^2 p^2)} \tag{48} \]
from Eq. (48) and Eq. (47)
\[ \nabla^2 \Psi(\vec{r}, t) - \frac{p^2}{(E_0^2 + c^2 p^2)} \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) = 0, \tag{49} \]
Making \( p \) into operator \(-i\hbar \nabla\), we can obtain the quantum wave equation as following:
\[ -\hbar^2 \nabla^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) = (-c^2 \hbar^2 \nabla^2 + m_0^2 c^4) \nabla^2 \Psi(\vec{r}, t), \tag{50} \]
define the wave function \( \Phi(\vec{r}, t) \)
\[ \Phi(\vec{r}, t) = \nabla^2 \Psi(\vec{r}, t), \tag{51} \]
as $\nabla^2$ and $\frac{\partial^2}{\partial t^2}$ are compatible and so

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \Phi(\vec{r}, t) = (-c^2 \hbar^2 \nabla^2 + m_0^2 c^4) \Phi(\vec{r}, t), \quad (52)$$

Obviously, the Eq. (52) which we obtain is the same as Klein-Gordon equation.

In the following, we consider a particle in the external potential field $V(x_\mu)$. In order to get time-independent relativistic wave equation, $\gamma L$ should be Lorentz-invariant, and it can be constructed

$$\gamma L = -m_0 c^2 - V(x_\mu), \quad (53)$$

where $V(x_\mu)$ is the Lorentz-invariant potential. Replacing $\gamma L$ with $\gamma L$ in Eq. (1), we can get

$$\nabla^2 \Psi(\vec{r}) + D(-m_0 c^2 - V(x_\mu))^2 \Psi(\vec{r}) = 0, \quad (54)$$

For a free particle

$$V(x_\mu) = 0,$$

$$\Psi(\vec{r}, t) = \Psi(\vec{r}) f(t) = e^{i(\vec{p} \cdot \vec{r} - \epsilon t)}, \quad (55)$$

Substitution of Eq. (55) into Eq. (54) gives

$$\left(\frac{\bar{\hbar}}{\hbar^2 m_0^2 c^4}\right)^2 = 0, \quad (56)$$

then

$$D = \frac{\epsilon^2 - E_0^2}{\hbar^2 c^2 E_0^2}, \quad (57)$$

where $\epsilon = E - V(x_\mu) = \sqrt{c^2 p^2 + E_0^2}$, and $E$ is the total energy of particle in the potential field.

From Eq. (54) and Eq. (57)

$$\nabla^2 \Psi(\vec{r}) + \frac{(\epsilon^2 - E_0^2)}{\hbar^2 c^2 E_0^2}(E_0 + V(x_\mu))^2 \Psi(\vec{r}) = 0, \quad (58)$$

and

$$-c^2 \hbar^2 \nabla^2 \psi(\vec{r}) = \left((E - V(x_\mu))^2 - E_0^2\right)[1 + \frac{V(x_\mu)}{E_0}] \Psi(\vec{r}), \quad (59)$$

the Eq. (59) is the time-independent wave equation of the material particle in the external potential field.
In the following, we can get time-dependent relativistic wave equation of the material particle in the external potential field by replacing \( n(n = 1) \) with \( \gamma L \) in Eq. (13).

\[
\nabla^2 \Psi(\vec{r}, t) + D'(-m_0c^2 - V(x_\mu))^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) = 0,
\]

(60)

from Eq. (55) and Eq. (60)

\[
\left(\frac{i}{\hbar} \vec{p}\right)^2 + D' m_0^2 c^4 \left(-\frac{i}{\hbar} \varepsilon\right)^2 = 0,
\]

(61)

then

\[
D' = -\frac{p^2}{E_0^2 \varepsilon^2} = -\frac{p^2}{E_0^2 (E_0^2 + c^2 p^2)},
\]

(62)

Substitution of Eq. (62) into Eq. (60) gives

\[
\nabla^2 \Psi(\vec{r}, t) - \frac{(E_0 + V(x_\mu))^2}{E_0^2 (E_0^2 + c^2 p^2)} h^2 \nabla^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) = 0,
\]

(63)

Making momentum \( p \) into operator \(-i\hbar \nabla\), we can obtain

\[
(E_0^2 - c^2 h^2 \nabla^2) \nabla^2 \Psi(\vec{r}, t) + \frac{(E_0 + V(x_\mu))^2}{E_0^2} h^2 \nabla^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) = 0,
\]

(64)

define the wave function \( \Phi(\vec{r}, t) \)

\[
\Phi(\vec{r}, t) = \nabla^2 \Psi(\vec{r}, t),
\]

(65)

and so

\[
(E_0^2 - c^2 h^2 \nabla^2) \Phi(\vec{r}, t) = -(1 + \frac{V(x_\mu)}{E_0})^2 h^2 \frac{\partial^2}{\partial t^2} \Phi(\vec{r}, t),
\]

(66)

The Eq. (66) is time-dependent relativistic wave equation of the material particle in the external potential field. Obviously, when \( V(x_\mu) = 0 \) the equation becomes the Klein-Gordon equation.

Now, we give an example for a charged particle in electromagnetic field. Firstly, we give its time-independent relativistic wave equation and its \( \gamma L \) is [5]

\[
\gamma L = -m_0c^2 + eA_\mu U_\mu,
\]

(67)

with

\[
A_\mu = (\vec{A}, \frac{i}{c} \varphi), U_\mu = \gamma(\vec{v}, ic),
\]
where \( A_\mu \) is electromagnetic four-vector, \( U_\mu \) is four-velocity.

and so

\[
V(x_\mu) = -eA_\mu U_\mu = -e\gamma(\vec{A} \cdot \vec{v} - \varphi),
\]

(68)

also

\[
\vec{p} = \gamma m_0 \vec{v}, \quad \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}},
\]

(69)

then

\[
\vec{v} = \frac{e\vec{p}}{\sqrt{p^2 + m_0^2 c^2}} = \frac{e^2 \vec{p}}{\epsilon},
\]

(70)

and so

\[
V(x_\mu) = -\frac{e}{E_0}(e^2 \vec{A} \cdot \vec{p} - \epsilon \varphi),
\]

(71)

Making \( \vec{A} \cdot \vec{p} \) into operator

\[
\vec{A} \cdot \vec{p} \rightarrow \vec{A} \cdot \hat{\vec{p}} + \hat{\vec{p}} \cdot \vec{A}
\]

(72)

and

\[
\hat{\vec{p}} \cdot \vec{A} - \vec{A} \cdot \hat{\vec{p}} = -i\hbar \nabla \cdot \vec{A},
\]

(73)

so

\[
\hat{\vec{p}} \cdot \vec{A} = -i\hbar \nabla \cdot \vec{A} + \vec{A} \cdot \hat{\vec{p}},
\]

(74)

combining the Eq. (72) with Eq. (74) gives

\[
\vec{A} \cdot \hat{\vec{p}} \rightarrow \vec{A} \cdot \hat{\vec{p}} - \frac{i\hbar}{2} \nabla \cdot \vec{A} = -i\hbar(\vec{A} \cdot \nabla + \frac{1}{2} \nabla \cdot \vec{A}),
\]

(75)

from the Eq. (71) and Eq. (75)

\[
V(x_\mu) = -\frac{e}{E_0}[-i\hbar c^2(\vec{A} \cdot \nabla + \frac{1}{2} \nabla \cdot \vec{A}) - \epsilon \varphi],
\]

(76)

substitution of the Eq. (76) into Eq. (59) gives

\[
-c^2 \hbar^2 \nabla^2 \Psi(\vec{r}) = \{[E - \frac{e}{E_0}(i\hbar c^2(\vec{A} \cdot \nabla + \frac{1}{2} \nabla \cdot \vec{A}) + \epsilon \varphi)]^2 - E_0^2 \} \{1 + \frac{e^2}{E_0^2}[i\hbar c^2(\vec{A} \cdot \nabla + \frac{1}{2} \nabla \cdot \vec{A}) + \epsilon \varphi]^2 \}^2 \Psi(\vec{r}),
\]

(77)

The Eq. (77) is time-independent relativistic wave equation of a charged particle in electromagnetic field.
In order to get the time-dependent relativistic wave equation of the material particle in the external potential field, we substitution of the Eq. (76) into Eq. (66)

\[ (E_0^2 - c^2\hbar^2\nabla^2)\Psi(\vec{r}, t) = -\{1 + \frac{e}{E_0} [i\hbar c^2 (\vec{A} \cdot \nabla + \frac{1}{2}\nabla \cdot \vec{A}) + \varepsilon \varphi]\}^2\hbar^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t), \]  

(78)

The Eq. (78) is the time-dependent relativistic wave equation of a charged particle in electromagnetic field.

4. the relativistic wave equation of spin \(\frac{1}{2}\) particle

In the following, we give the relativistic wave equation of spin \(\frac{1}{2}\) particle in potential field. We know Dirac resolved the Klein-Gordon wave equation which is second order in space-time

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) - \nabla^2 \Psi(\vec{r}, t) + \frac{m_0^2c^2}{\hbar^2} \Psi(\vec{r}, t) = 0, \]  

(79)

into first order in space-time

\[ \frac{1}{c} \frac{\partial}{\partial t} \Psi + \vec{\alpha} \cdot \frac{\partial}{\partial \vec{x}} \Psi + \frac{im_0c}{\hbar} \beta \Psi = 0, \]  

(80)

with

\[ \Psi = \begin{pmatrix} \Psi_1(r, t) \\ \Psi_2(r, t) \\ \Psi_3(r, t) \\ \Psi_4(r, t) \end{pmatrix} \]  

(81)

The Eq. (80) is the famous Dirac equation. As the same method, we resolve the time-dependent relativistic wave equation of the material particle in the external potential field

\[ (E_0^2 - c^2\hbar^2\nabla^2)\Psi(\vec{r}, t) = -(1 + \frac{V(x_\mu)}{E_0})^2\hbar^2 \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t), \]

i.e.

\[ (1 + \frac{V(x_\mu)}{E_0})\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) - \nabla^2 \Psi(\vec{r}, t) + \frac{m_0^2c^2}{\hbar^2} \Psi(\vec{r}, t) = 0, \]  

(82)

into

\[ (1 + \frac{V(x_\mu)}{E_0}) \frac{1}{c} \frac{\partial}{\partial t} \Psi + \vec{\alpha} \cdot \frac{\partial}{\partial \vec{x}} \Psi + \frac{im_0c}{\hbar} \beta \Psi = 0, \]  

(83)

Premultiply this equation by

\[ (1 + \frac{V(x_\mu)}{E_0}) \frac{1}{c} \frac{\partial}{\partial t} \Psi - (\vec{\alpha} \cdot \frac{\partial}{\partial \vec{x}} \Psi + \frac{im_0c}{\hbar} \beta) \Psi = 0, \]  

(84)
Here, the momentum’s operator \( \frac{\partial}{\partial x_i} \) can’t act on the potential function \( V(x(\mu)) \), and so

\[
(1 + \frac{V(x_{\mu})}{E_0})^2 \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Psi - \sum_{ik} \alpha_i \alpha_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \Psi + \frac{m_0^2 c^2}{\hbar^2} \beta^2 \Psi - \frac{imc}{\hbar} \sum_i (\alpha_i \beta + \beta \alpha_i) \frac{\partial}{\partial x_i} \Psi = 0, \tag{85}
\]

As the same method by which Dirac obtained his equation, we can get the same matrix

\[
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{86}
\]

from the Eq. (83), we can obtain

\[
i\hbar \frac{\partial}{\partial t} \Psi = \frac{E_0}{E_0 + V(x_{\mu})} (-i \hbar c \vec{\alpha} \cdot \frac{\partial}{\partial \vec{x}} + m_0 c^2 \beta) \Psi = \hat{H} \Psi, \tag{87}
\]

with

\[
\hat{H} = \frac{E_0}{E_0 + V(x_{\mu})} (-i \hbar c \vec{\alpha} \cdot \frac{\partial}{\partial \vec{x}} + m_0 c^2 \beta) \tag{88}
\]

The operator \( \hat{H} \) is Hamilton operator of spin \( \frac{1}{2} \) particle in external field.

For a stationary state

\[
\Psi = \Psi(r) e^{-iEt/\hbar} \tag{89}
\]

The Eq. (87) becomes

\[
\frac{E_0}{E_0 + V(x_{\mu})} (-i \hbar c \vec{\alpha} \cdot \frac{\partial}{\partial \vec{x}} + m_0 c^2 \beta) \Psi = E \Psi \tag{90}
\]

The Eq. (90) is the eigenvalue equation of energy \( E \) of spin \( \frac{1}{2} \) particle in external field.

In the following, we will give the relativistic wave equation of spin \( \frac{1}{2} \) and \( m_0 = 0 \) particle in external potential field. From the Eq. (82), we can get the time-dependent relativistic wave equation of the material particle in the external potential field for \( m_0 = 0 \)

\[
(1 + \frac{V(x_{\mu})}{E_0})^2 \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi - \nabla^2 \Phi = 0, \tag{91}
\]

The relativistic wave equation of spin \( \frac{1}{2} \) and \( m_0 = 0 \) particle is

\[
(1 + \frac{V(x_{\mu})}{E_0}) \frac{1}{c} \frac{\partial}{\partial t} \Phi + \sum_{i=1}^{3} \sigma_i \frac{\partial}{\partial x_i} \Phi = 0, \tag{92}
\]

Premultiply this equation by

\[
-(1 + \frac{V(x_{\mu})}{E_0}) \frac{1}{c} \frac{\partial}{\partial t} \Phi + \sum_{k=1}^{3} \sigma_k \frac{\partial}{\partial x_k} \Phi = 0, \tag{93}
\]
and so
\[
\{-\left(1 + \frac{V(x_\mu)}{E_0}\right)^2 \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \sum_{i,k} \left(\sigma_k \sigma_i \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i}\right)\} \Phi = 0,
\] (94)
Symmetrizing the Eq. (94)
\[
\{-\left(1 + \frac{V(x_\mu)}{E_0}\right)^2 \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{1}{2} \sum_{i,k} \left(\sigma_k \sigma_i + \sigma_k \sigma_i\right) \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i}\} \Phi = 0,
\] (95)
Comparing the Eq. (95) with Eq. (91), we can get
\[
\frac{1}{2} (\sigma_k \sigma_i + \sigma_k \sigma_i) = \delta_{ik}, (i, k = x, y, z)
\] (96)
so
\[
\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1,
\]
\[
\sigma_x \sigma_y = -\sigma_y \sigma_x,
\]
where \(\sigma\) is the Pauli matrices. Then the Eq. (93) can be written as
\[
(1 + \frac{V(x_\mu)}{E_0}) \frac{1}{c} \frac{\partial}{\partial t} \Phi = -\vec{\sigma} \cdot \vec{\nabla} \Phi,
\] (97)
also
\[
\frac{i\hbar}{\partial t} \Phi = \frac{-i\hbar c \vec{\sigma} \cdot \vec{\nabla}}{(1 + \frac{V(x_\mu)}{E_0})} \Phi
\] (98)
In our work, we give non-relativistic wave equations which are different from the Schrödinger wave equation, and extend the Klein-Gordon equation, which includes the external field. Meanwhile, we obtain the relativistic wave equation of spin \(\frac{1}{2}\) particle in external field. We think our theory maybe have some effects to quantum field theory.

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