Gauge-Invariant Renormalization Group at Finite Temperature

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I describe an application of Wilson Renormalization group to the real time formalism of finite temperature field theory. The approach has two nice features: 1) the RG flow equations describe non-perturbatively the effect of thermal fluctuations only, and, 2) the flow is gauge invariant. I then describe the application of the method to the study of the gluon self-energy in SU(N), and present results for the computation of the Debye and magnetic screening masses.

1 Thermal coarse graining

In this talk, I will describe an application of Wilson Renormalization Group (RG) to high temperature field theory. The main purpose is to construct a tool for the non-perturbative resummation of thermal fluctuations at larger and larger length scales applicable also to gauge theories [1, 2]. In the case of non-abelian gauge theories, the ultimate goal is to go below the scale at which the Hard Thermal Loop (HTL) effective field theory breaks down.

Our approach is the following. We assume the quantum field theory in the vacuum is known, that is, we have some approximation scheme (e.g. perturbation theory, lattice computations, ...) that we trust in order to do computations and relate the physical observables at $T = 0$ to the parameters of the theory. We are interested in the purely thermal effects arising when the theory is put in a thermal bath. Moreover, in many physical applications, we are mainly interested in long wavelength thermal fluctuations, since the short wavelength ones ($\lambda$ much smaller than $1/T$) can be treated perturbatively. Then, in the RG philosophy, we will integrate out all the ‘irrelevant’ degrees of freedom which, in this context, are given by all the quantum fluctuations and the thermal fluctuations of short wavelength. We will call this procedure ‘thermal coarse graining’.

The appropriate framework for this program is the real time formalism of high temperature field theory (see ref. [3] for a thorough discussion). Consider the free propagator of a scalar particle,

$$D^{11}(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi N(|k_0|)\delta(k^2 - m^2),$$

(1)
(where I have written down the 1-1 component only). It has two important features, which will play a crucial role in the following: 1) there is a clear separation between the ‘vacuum’ contribution, and the ‘thermal’ one, which contains the Bose-Einstein distribution function \( N(|k_0|) \), and, 2) the ‘thermal’ part involves on-shell particles only.

Proceeding in close analogy to what is done in the application of the RG to \( T = 0 \) quantum field theory,\(^4\) we modify the free propagator by introducing a cut-off function. The peculiarity of this approach is that the cut-off acts on the thermal part only of the propagator, i.e.

\[
D(k) = D^0(k) + D^T(k) \rightarrow D_\Lambda(k) = D^0(k) + \theta(|\vec{k}|, \Lambda)D^T(k),
\]

where \( \theta(|\vec{k}|, \Lambda) \) tends to 1 for \( \Lambda \ll |\vec{k}| \) and to 0 for \( \Lambda \gg |\vec{k}| \). Inserting the modified propagator in the path integral expression for the generating functional, we obtain a \( \Lambda \)-dependent generating functional and, by the usual manipulations, a \( \Lambda \)-dependent effective action. These objects reduce to their counterparts for the \( T = 0 \) renormalized quantum field theory in the \( \Lambda \rightarrow \infty \) limit, and to the ones of the quantum field theory in thermal equilibrium at the temperature \( T \) in the \( \Lambda \rightarrow 0 \) limit.

The interpolation between the two limits is described by RG equations which, in the case of one particle irreducible vertices can be obtained by the following simple recipe: 1) take the expression for the one loop correction to the desired vertex; 2) substitute the tree level propagators and vertices in it with the full, cut-off dependent ones; 3) take the derivative with respect to the explicit cut-off dependence (i.e. derive only the cut-off function in the propagators). To be more explicit, the RG equation for the tadpole in the scalar theory is given by

\[
\Lambda \frac{\partial}{\partial \Lambda} \Gamma^{(1)}_\Lambda = -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \delta(|\vec{k}| - \Lambda) N(|k_0|) \rho_\Lambda(k) \epsilon(k_0) \Gamma^{(3)}_\Lambda(k, -k, 0),
\]

where we recognise the Bose-Einstein distribution function, the full three-point function, and the spectral function \( \rho_\Lambda(k) \), defined as the discontinuity of the full propagator across the real axis.

The physical meaning of the flow equation (3) is evident: the new thermal modes coming into thermal equilibrium at the scale \( \Lambda \) are weighted by the full spectral function, induced by all the quantum and thermal modes already integrated out.

2 Gauge invariance

A strong limitation to the applicability of RG methods to gauge theories comes from the fact that the introduction of a momentum cut-off generally leads to
a breaking of BRS invariance. More precisely, new Λ-dependent contributions to the Slavnov-Taylor (ST) identities appear, which vanish only in the limit in which the cut-off is removed and the full theory is recovered. This is actually the situation in the applications of the RG both at $T = 0$ and at $T \neq 0$ in the imaginary time formalism [5]. In these cases, both on-shell and off-shell modes are cut-off and this unavoidably spoils BRS invariance.

On the other hand, in our formulation we are cutting thermal fluctuations only, which live in the on-shell sector of the theory, as we noticed after eq. (1). As a consequence, BRS invariance is preserved, and it can be shown explicitly that the extra contributions to the ST identities vanish identically [2].

From a computational point of view, this allows us to use BRS invariance as a powerful constraint in approximating the exact evolution equations. From a physical point of view, we have a tool to derive an effective, gauge invariant, field theory, even for a non-zero value of the cut-off.

3 Application: thermal masses in SU(N)

In this section I will illustrate some preliminary results on the gluon self-energy in SU(N) [6]. In particular I will concentrate on the longitudinal (or Debye) and transverse (or magnetic) masses, defined as

$$m^2_L = \Pi_L(q_0 = 0, |\vec{q}|^2 = -m^2_L),$$

$$m^2_T = \Pi_T(q_0 = 0, |\vec{q}|^2 = -m^2_T),$$

where $\Pi_{L,T}$ are obtained from the self-energy $\Pi^{\mu\nu}$ as $\Pi_L = \Pi^{00}$ and $\Pi_T = -1/2 \Pi^{ii}$. The definition [6] is gauge-independent, as shown in ref. [7]. Since the cut-off does not break BRS invariance, the same is true even for $\Lambda \neq 0$. In order to preserve this property, approximations to the full propagator and vertices appearing in the exact RG equations must respect ST identities to the required accuracy. In the spirit of using the RG to construct an effective theory valid at scales larger than $1/gT$, we will employ a ‘minimal approximation scheme’ in which: i) we use ‘HTL inspired’ propagator and vertices, and ii) we rotate to imaginary time and modify the zero mode only, using tree-level quantities for the other ones. In practice, we approximate the flow equation, which has the structure

$$\Lambda \frac{\partial}{\partial \Lambda} \Pi^{\mu\nu} = -i \int \frac{dk_0}{2\pi} N(k_0) \mathrm{Disc} F^{\mu\nu}(k_0),$$

with

$$T \sum_n F_0^{\mu\nu}(z = 2i\pi nT) - \int \frac{dk_0}{2\pi} F_0^{\mu\nu}(k_0) + T [F^{\mu\nu}_{HTL}(z = 0) - F_0^{\mu\nu}(z = 0)],$$

3
where in \( F_{HTL}^{\mu\nu} \) HTL-inspired propagator and vertices have been used, whereas in \( F_{0}^{\mu\nu} \) they appear at the tree level.

The ‘HTL-inspired’ propagator we need is

\[
\Delta_{\mu\nu}\big|_{k_0=0} = \frac{1}{|k|^2 + m_{L,A}^2} g_{\mu0}g_{\nu0} + \frac{1}{|k|^2 + m_{T,A}^2} \left( g_{\mu\nu} - g_{\mu0}g_{\nu0} + \frac{k_\mu k_\nu}{|k|^2} \right) - \alpha \frac{k_\mu k_\nu}{|k|^2} \bigg|_{k_0=0},
\]

where, compared to the true HTL propagator we have \( \Lambda \)-dependent \( m_L \) and \( m_T \) (from now on we will omit the \( \Lambda \)-dependence of the various variables).

The tree-level trilinear vertex does not satisfy the ST identity if \( m_T \neq 0 \). We correct it according to eq. (12) of ref. Moreover, it is easy to realize that, due to the breaking of Lorentz invariance in the thermal bath, there is a different running for the vertex with all the three space-like indices and those with at least one time-like index. Accordingly, we introduce two different couplings, \( g_L \) and \( g_T \).

Now we have a system of differential equations for \( m_L, m_T, g_L, g_T \), and the wave function renormalizations. In order to make contact with the results of perturbation theory and HTL, we first consider the running of the masses only.

In the picture on the left we plot \( m_L/T \) vs. \( \Lambda/T \). The coupling constants have been kept fixed to \( g = 0.8 \). For \( \Lambda/T \gg 1 \) there is no renormalization, since the thermal modes coming in equilibrium are Boltzmann suppressed. As \( \Lambda/T \sim 1 \) the HTL contribution is quickly built up (at leading order, \( m_{HTL}^{T}/T = g \) in SU(3)), and if we stop the running at \( \Lambda/T = O(g) \) we recover the result of the HTL effective field theory. Below this region the crucial question is the value of \( m_T \). In perturbation theory, a \( m_T = O(g^2T) \) is usually invoked as a infrared regulator. It corresponds to the lowest line, which exhibits decoupling for \( \Lambda < m_T \) and leads to a finite result in good agreement with the HTL one. On the other hand, if \( m_T = 0 \), \( m_L \) increases logarithmically in the infrared and the HTL result becomes completely unreliable at lower scales. The dashed line corresponds to the result obtained by coupling the RG equation for \( m_T \) to that for \( m_L \). Assuming \( m_T(T = 0) = O(\Lambda_{QCD}) \) a \( m_T \ll g^2T \) is generally found at finite temperature (for the parameters used in the plot we get \( m_T/g^2T = 5.3 \cdot 10^{-2} \)). As a consequence, the decoupling takes place ‘later’ and the finite result for \( m_L \) is sizably higher than the HTL one.

The running of the couplings is shown in the picture on the right. Notice the impressive difference in the thermal effects on \( g_L \) and \( g_T \). This is due to the fact that graphs with only transverse gluons in the loop contribute to the running of \( g_T \) but not to that of \( g_L \). The non-vanishing \( m_T \) leads to
decoupling in the infrared, but it takes place when the couplings have already reached highly non-perturbative values.

The failure of HTL perturbation theory in describing the long-distance physics in a non-abelian plasma is explicit in Fig. 1. The RG in the real time formalism provides a suitable framework to derive the effective theory for the long wavelength, non-perturbative modes.

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