The existence of a path-factor without small odd paths

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Abstract
In this paper, we show that if a graph $G$ satisfies $c_1(G - X) + \frac{2}{3}c_3(G - X) \leq \frac{4}{3}|X| + \frac{1}{3}$ for all $X \subseteq V(G)$, then $G$ has a $\{P_2, P_5\}$-factor, where $c_i(G - X)$ is the number of components $C$ of $G - X$ with $|V(C)| = i$.

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1 Introduction
In this paper, all graphs are finite and simple. Let $G$ be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. For $u \in V(G)$, we let $N_G(u)$ and $d_G(u)$ denote the neighborhood and the degree of $u$, respectively. For $U \subseteq V(G)$, we let $N_G(U) = (\bigcup_{u \in U} N_G(u)) - U$. For disjoint sets $X, Y \subseteq V(G)$, we let $E_G(X, Y)$ denote the set of edges of $G$ joining a vertex in $X$ and a vertex in $Y$. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of $G$ induced by $X$. For two graphs $H_1$ and $H_2$, we let $H_1 + H_2$ denote the join of $H_1$ and $H_2$. Let $P_n$ denote the path of order $n$. For terms and symbols not defined here, we refer the reader to [2].

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For a set $\mathcal{H}$ of connected graphs, a spanning subgraph $F$ of a graph is called an $\mathcal{H}$-factor if each component of $F$ is isomorphic to a graph in $\mathcal{H}$. A path-factor of a graph is a spanning subgraph whose components are paths of order at least 2. Since every path of order at least 2 can be partitioned into paths of orders 2 and 3, a graph has a path-factor if and only if it has a $\{P_2, P_3\}$-factor. Akiyama, Avis and Era [1] gave a necessary and sufficient condition for the existence of a path-factor (here $i(G)$ denotes the number of isolated vertices of a graph $G$).

Theorem A (Akiyama, Avis and Era [1]) A graph $G$ has a $\{P_2, P_3\}$-factor if and only if $i(G - X) \leq 2|X|$ for all $X \subseteq V(G)$.

Now we consider a path-factor with additional conditions. For example, one may require a path-factor to consist of components of large order. Concerning such a problem, Kaneko [3] gave a necessary and sufficient condition for the existence of a path-factor whose components have order at least 3. On the other hand, for $k \geq 4$, it is not known that whether the existence problem of a path-factor whose components have order at least $k$ is polynomially solvable or not, though some results about such a factor have been obtained (see, for example, Kano, Lee and Suzuki [4] and Kawarabayashi, Matsuda, Oda and Ota [5]).

In this paper, we study a different type of path-factor problem. Specifically, we focus on the existence of a $\{P_2, P_{2k+1}\}$-factor ($k \geq 2$).

There are two motivations to study such factors. One of the motivations is related the notion of a hypomatchable graph. A graph $H$ is hypomatchable if $H - x$ has a perfect matching for every $x \in V(H)$. A graph is a propeller if it is obtained from a hypomatchable graph $H$ by adding new vertices $a, b$ together with edge $ab$, and joining $a$ to some vertices of $H$. Loebal and Poljak [6] proved the following theorem.

Theorem B (Loebal and Poljak [6]) Let $H$ be a connected graph. If either $H$ has a perfect matching, or $H$ is hypomatchable, or $H$ is a propeller, then the existence problem of a $\{P_2, H\}$-factor is polynomially solvable. The problem is NP-complete for all other graphs $H$.

In particular, for $k \geq 2$, the existence problem of a $\{P_2, P_{2k+1}\}$-factor is NP-complete. Because of this fact, existence problems concerning $\{P_2, P_{2k+1}\}$-factors seem to have unjustly been ignored. However, in general, the fact that a problem is NP-complete in terms of algorithm does not mean that one cannot obtain a theoretical result concerning the problem. From this viewpoint, in this paper, we
prove a theorem on the existence of a \( \{P_2, P_5\}\)-factor which, we hope, will serve as an initial attempt to develop the theory of \( \{P_2, P_{2k+1}\}\)-factors.

The other motivation is the fact that a \( \{P_2, P_{2k+1}\}\)-factor is a useful tool for finding large matchings. It is easy to see that if a graph \( G \) has a \( \{P_2, P_{2k+1}\}\)-factor, then \( G \) has a matching \( M \) with \( |M| \geq \frac{k}{2k+1}|V(G)| \). Thus the existence of a \( \{P_2, P_{2k+1}\}\)-factor helps to find large matchings.

In order to state our theorem, we need some more definitions. For a graph \( H \), we let \( \mathcal{C}(H) \) be the set of components of \( H \), and for \( i \geq 1 \), let \( \mathcal{C}_i(H) = \{ C \in \mathcal{C}(H) \mid |V(C)| = i \} \) and \( c_i(H) = |\mathcal{C}_i(H)| \). Note that \( c_1(H) \) is the number of isolated vertices of \( H \) (i.e., \( c_1(H) = i(H) \)). If a graph \( G \) has a \( \{P_2, P_5\}\)-factor, then \( c_1(G-X) + \frac{1}{2} c_3(G-X) \leq \frac{2}{3}|X| \) for all \( X \subseteq V(G) \) (see Section 2). Thus if a condition concerning \( c_1(G-X) \) and \( c_3(G-X) \) for \( X \subseteq V(G) \) assures us the existence of a \( \{P_2, P_3\}\)-factor, then it will make a useful sufficient condition.

The main purpose of this paper is to prove the following theorem.

**Theorem 1.1** Let \( G \) be a graph. If \( c_1(G-X) + \frac{2}{3} c_3(G-X) \leq \frac{4}{3}|X| + \frac{1}{3} \) for all \( X \subseteq V(G) \), then \( G \) has a \( \{P_2, P_5\}\)-factor.

We prove Theorem 1.1 in Sections 3 and 4. In Subsection 5.1 we show that the bound \( \frac{4}{3}|X| + \frac{1}{3} \) in Theorem 1.1 is best possible.

In our proof of Theorem 1.1 we make use of the following fact.

**Fact 1.1** Let \( G \) be a graph. Then \( G \) has a \( \{P_2, P_3\}\)-factor if and only if \( G \) has a path-factor \( F \) with \( \mathcal{C}_3(F) = \emptyset \).

We conclude this section with a conjecture concerning \( \{P_2, P_{2k+1}\}\)-factors with \( k \geq 3 \). By Theorems 1.1 and 1.3 for \( k \in \{1, 2\} \), there exists a constant \( a_k > 1 \) such that the condition \( \sum_{0 \leq i \leq k-1} c_{2i+1}(G-X) \leq a_k|X| \ (X \subseteq V(G)) \) assures us the existence of a \( \{P_2, P_{2k+1}\}\)-factor (one can take \( a_1 = 2 \) and \( a_2 = \frac{4}{3} \)). Thus one may expect that there exists a similar constant \( a_k > 1 \) for \( k \geq 3 \). However, when we consider the case where \( k \geq 3 \) with \( k \equiv 0 \) (mod 3), the situation changes drastically; that is, there exist infinitely many graphs \( G \) having no \( \{P_2, P_{2k+1}\}\)-factor such that
\[
\sum_{0 \leq i \leq k-1} c_{2i+1}(G-X) \leq \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3} \text{ for all } X \subseteq V(G) \text{ (see Subsection 5.2).}
\]
Thus we pose the following conjecture.

**Conjecture 1** Let \( k \geq 3 \), and let \( G \) be a graph. If \( \sum_{0 \leq i \leq k-1} c_{2i+1}(G-X) \leq \frac{4k+6}{8k+3}|X| \) for all \( X \subseteq V(G) \), then \( G \) has a \( \{P_2, P_{2k+1}\}\)-factor.
2 A necessary condition for a \( \{P_2, P_5\} \)-factor

In this section, we give a necessary condition for the existence of a \( \{P_2, P_5\} \)-factor in terms of invariants \( c_1 \) and \( c_3 \). We show the following proposition.

**Proposition 2.1** If a graph \( G \) has a \( \{P_2, P_5\} \)-factor, then \( c_1(G-X) + \frac{1}{2} c_3(G-X) \leq \frac{3}{2} |X| \) for all \( X \subseteq V(G) \).

**Proof.** Let \( F \) be a \( \{P_2, P_5\} \)-factor of \( G \), and let \( X \subseteq V(G) \). Then we can verify that

\[
c_1(P-X) + \frac{1}{2} c_3(P-X) \leq \frac{3}{2} |V(P) \cap X| \text{ for every } P \in \mathcal{C}(F). \tag{2.1}
\]

Since every component \( C \) of \( G-X \) with \( |V(C)| = 1 \) belongs to \( \bigcup_{P \in \mathcal{C}(F)} \mathcal{C}_1(P-X) \), we have

\[
|\mathcal{C}_1(G-X)| = \sum_{P \in \mathcal{C}(F)} |\mathcal{C}_1(P-X)| - \left| \left( \bigcup_{P \in \mathcal{C}(F)} \mathcal{C}_1(P-X) \right) - \mathcal{C}_1(G-X) \right|. \tag{2.2}
\]

Furthermore,

\[
|\mathcal{C}_3(G-X)| \leq \sum_{P \in \mathcal{C}(F)} |\mathcal{C}_3(P-X)| + \left| \mathcal{C}_3(G-X) - \left( \bigcup_{P \in \mathcal{C}(F)} \mathcal{C}_3(P-X) \right) \right|. \tag{2.3}
\]

Let \( C \) be a component of \( G-X \) with \( |V(C)| = 3 \) which does not belong to \( \bigcup_{P \in \mathcal{C}(F)} \mathcal{C}_3(P-X) \). Then \( C \) intersects with at least two components of \( F-X \). Since \( |V(C)| = 3 \), \( C \) contains a component of \( P-X \) of order 1 for some \( P \in \mathcal{C}(F) \). Since \( C \) is arbitrary, this implies that

\[
\left| \mathcal{C}_3(G-X) - \left( \bigcup_{P \in \mathcal{C}(F)} \mathcal{C}_3(P-X) \right) \right| \leq \left| \left( \bigcup_{P \in \mathcal{C}(F)} \mathcal{C}_1(P-X) \right) - \mathcal{C}_1(G-X) \right|. \tag{2.4}
\]
By (2.1)–(2.4),

\[
 c_1(G - X) + \frac{1}{2} c_3(G - X) \\
\leq \left( \sum_{P \in \mathcal{E}(F)} |\mathcal{E}_1(P - X)| - \left| \left( \bigcup_{P \in \mathcal{E}(F)} \mathcal{E}_1(P - X) \right) - \mathcal{E}_1(G - X) \right| \right) \\
+ \frac{1}{2} \left( \sum_{P \in \mathcal{E}(F)} |\mathcal{E}_3(P - X)| + |\mathcal{E}_3(G - X) - \left( \bigcup_{P \in \mathcal{E}(F)} \mathcal{E}_3(P - X) \right) | \right) \\
\leq \left( \sum_{P \in \mathcal{E}(F)} |\mathcal{E}_1(P - X)| - \left| \left( \bigcup_{P \in \mathcal{E}(F)} \mathcal{E}_1(P - X) \right) - \mathcal{E}_1(G - X) \right| \right) \\
+ \frac{1}{2} \left( \sum_{P \in \mathcal{E}(F)} |\mathcal{E}_3(P - X)| + \left| \left( \bigcup_{P \in \mathcal{E}(F)} \mathcal{E}_1(P - X) \right) - \mathcal{E}_1(G - X) \right| \right) \\
\leq \sum_{P \in \mathcal{E}(F)} |\mathcal{E}_1(P - X)| + \frac{1}{2} \sum_{P \in \mathcal{E}(F)} |\mathcal{E}_3(P - X)| \\
= \sum_{P \in \mathcal{E}(F)} \left( c_1(P - X) + \frac{1}{2} c_3(P - X) \right) \\
\leq \frac{3}{2} \sum_{P \in \mathcal{E}(F)} |\mathcal{V}(P) \cap X| \\
= \frac{3}{2} |X|.
\]

Thus we get the desired conclusion. \qed

3 \hspace{1em} A path-factor in bipartite graph

Let \( G \) be a bipartite graph with bipartition \((S, T)\). A subgraph \( F \) of \( G \) is \textit{S-central} if \( S \subseteq V(F) \) and \(|V(A) \cap T| \geq |V(A) \cap S|\) for every \( A \in \mathcal{C}(F)\).

In this section, we focus on the existence of a special path-factor in bipartite graphs, and show the following theorem, which will be used in our proof of Theorem 1.1.

**Theorem 3.1** Let \( S, T_1 \) and \( T_2 \) be disjoint sets with \( 1 \leq |S| \leq |T_1| + |T_2| \) and \(|T_1| + \frac{2}{3}|T_2| \leq \frac{4}{3}|S| + \frac{1}{3} \), and set \( T = T_1 \cup T_2 \). Let \( G \) be a bipartite graph with bipartition \((S, T)\) satisfying the property that for every \( X \subseteq V(G) \), we have either \(|N_G(X) \cap T_1| + \frac{2}{3}|N_G(X) \cap T_2| \geq \frac{4}{3}|X| \) or \( N_G(X) = T \). Then \( G \) has an S-central path-factor \( F \) such that \( V(A) \cap T_2 \neq \emptyset \) for every \( A \in \mathcal{C}_3(F) \).
Lemma 3.2 Let \( S, T_1, T_2, T \) and \( G \) be as in Theorem 3.1. Then \( G \) has an \( S \)-central path-factor.

Proof. Let \( X \subseteq S \). If \(|N_G(X) \cap T_1| + \frac{2}{3}|N_G(X) \cap T_2| \geq \frac{2}{3}|X|\), then \(|N_G(X)| \geq |N_G(X) \cap T_1| + \frac{2}{3}|N_G(X) \cap T_2| \geq \frac{2}{3}|X| \geq |X|\); if \( N_G(X) = T_1 \cup T_2 \), then \(|N_G(X)| = |T_1| + |T_2| \geq |S| \geq |X|\). In either case, we have \(|N_G(X)| \geq |X|\). Since \( X \) is arbitrary, \( G \) has a matching covering \( S \) by Hall’s marriage theorem. In particular, \( G \) has an \( S \)-central subgraph \( F \) such that every component of \( F \) is a path of order at least 2.

Choose \( F \) so that \(|V(F)|\) is as large as possible.

Suppose that \(|V(G) - V(F)| \neq \emptyset\). Note that \(|V(G) - V(F)| \subseteq T\). Now we define the set \( A \) of components of \( F \) as follows: Let \( A_1 \) be the set of components \( A \) of \( F \) with \( E_G(V(A) \cap S, V(G) - V(F)) \neq \emptyset \). For each \( i \geq 2 \), let \( A_i \) be the set of components \( A \) of \( F \) with \( A \notin \bigcup_{1 \leq j \leq i-1} A_j \) and \( E_G(V(A) \cap S, \bigcup_{A' \in A_{i-1}} (V(A') \cap T)) \neq \emptyset \). Let \( A = \bigcup_{i \geq 1} A_i \).

Claim 3.1 Every path belonging to \( A \) is isomorphic to \( P_3 \).

Proof. Suppose that \( A \) contains a path which is not isomorphic to \( P_3 \). Let \( i \) be the minimum integer such that \( A_i \) contains a path \( A_i = v_1^{(i)} \cdots v_l^{(i)} \) with \( A_i \neq P_3 \). By the minimality of \( i \), every path belonging to \( \bigcup_{1 \leq j \leq i-1} A_j \) is isomorphic to \( P_3 \). Hence by the definition of \( A_j \), there exists a vertex \( v_1^{(0)} \in V(G) - V(F) \) and there exist paths \( A_j = v_1^{(j)} v_2^{(j)} v_3^{(j)} \in A_j \) \((1 \leq j \leq i-1)\) such that \( E_G(V(A_1) \cap S, \{v_1^{(0)}\}) \neq \emptyset \) and \( E_G(V(A_{j+1}) \cap S, V(A_j) \cap T) \neq \emptyset \) for every \( j \) \((1 \leq j \leq i-1)\). For each \( j \) \((1 \leq j \leq i-1)\), by renumbering the vertices \( v_1^{(j)}, v_2^{(j)}, v_3^{(j)} \) of \( A_j \) backward (i.e., by tracing the path \( v_3^{(j)} v_2^{(j)} v_1^{(j)} \) backward and numbering the vertices accordingly) if necessary, we may assume that \( E_G(V(A_{j+1}) \cap S, \{v_1^{(j)}\}) \neq \emptyset \). Let \( m \) be an index such that \( v_1^{(m)} v_1^{(i-m)} \in E(G) \). Note that \( l \geq 2 \) and \( l \neq 3 \). Thus by renumbering the vertices \( v_1^{(i)}, \ldots, v_l^{(i)} \) of \( A_i \) backward if necessary, we may assume that \( m \neq 2 \) if \( l \) is odd, and \( m \) is odd if \( l \) is even. Let \( B_j = v_1^{(j-1)} v_2^{(j)} v_3^{(j)} \) \((1 \leq j \leq i-1)\), \( B_i = v_1^{(i-1)} v_2^{(i)} v_3^{(i)} v_{m+1}^{(i)} \cdots v_l^{(i)} \) and \( B_{i+1} = v_1^{(i)} \cdots v_{m-1}^{(i)} \) (note that \( B_{i+1} = \emptyset \) if and only if \( l \) is even and \( m = 1 \)). Then \(|V(B_j) \cap T| \geq |V(B_j) \cap S| \) for every \( j \) \((1 \leq j \leq i+1)\). Therefore \( F' = (F - (\bigcup_{1 \leq j \leq i} V(A_j))) \cup (\bigcup_{1 \leq j \leq i+1} B_j) \) is an \( S \)-central subgraph of \( G \) such that \( V(F') = V(F) \cup \{v_1^{(0)}\} \) and every component of \( F' \) is a path of order at least 2, which contradicts the maximality of \( F \).  

We continue with the proof of the lemma. Let \( X_0 = (\bigcup_{A \in A} V(A)) \cap S \) and
\(Y_0 = ((\bigcup_{A \in \mathcal{A}} V(A)) \cap T) \cup (V(G) - V(F))\). Since \(V(G) - V(F) \neq \emptyset\) and \(\mathcal{A} \subseteq \mathcal{C}_3(F)\) by Claim 3.1, we have
\[
|Y_0 \cap T_1| + \frac{2}{3}|Y_0 \cap T_2| \geq \frac{2}{3}|Y_0| \geq \frac{2}{3}(2|X_0| + 1). \tag{3.1}
\]
By the definition of \(\mathcal{A}\), \(N_G(S - X_0) \cap Y_0 = \emptyset\). In particular, \(N_G(S - X_0) \neq T\), and hence \(|N_G(S - X_0) \cap T_1| + \frac{2}{3}|N_G(S - X_0) \cap T_2| \geq \frac{4}{3}|S - X_0|\). This together with (3.1) implies that
\[
|T_1| + \frac{2}{3}|T_2| \geq (|Y_0 \cap T_1| + |N_G(S - X_0) \cap T_1|) + \frac{2}{3}(|Y_0 \cap T_2| + |N_G(S - X_0) \cap T_2|) \\
\geq \frac{2}{3}(2|X_0| + 1) + \frac{4}{3}|S - X_0| \\
= \frac{4}{3}|S| + \frac{2}{3},
\]
which contradicts the assumption that \(|T_1| + \frac{2}{3}|T_2| \leq \frac{4}{3}|S| + \frac{1}{3}\), completing the proof of the lemma. \(\square\)

We here outline the proof of Theorem 3.1. We choose an \(S\)-central path-factor \(F_0\) so that \(F_0\) will satisfy certain minimality conditions (see the paragraph following the proof of Claim 3.3). We then introduce operations which turn \(F_0\) into a new path-factor (see the paragraphs following Claim 3.5 and Claim 3.6), and show that the new path-factor contradicts our choice of \(F_0\).

**Proof of Theorem 3.1.** We start with some definitions. Let \(F\) be an \(S\)-central path-factor of \(G\). For each integer \(i \geq 2\), let \(\mathcal{C}_i^{(1)}(F) = \{A \in \mathcal{C}(F) \mid V(A) \cap T = \emptyset\}\) and \(\mathcal{C}_i^{(2)}(F) = \mathcal{C}_i(F) - \mathcal{C}_i^{(1)}(F)\). If there is no fear of confusion, we simply write \(\mathcal{C}_i\) and \(\mathcal{C}_i^{(h)}\) (\(h \in \{1, 2\}\)) instead of \(\mathcal{C}_i(F)\) and \(\mathcal{C}_i^{(h)}(F)\), respectively.

Let \(D_F\) be the digraph defined by \(V(D_F) = \mathcal{C}(F)\) and \(E(D) = \{AB \mid E_G(V(A) \cap S, V(B) \cap T) \neq \emptyset\}\). For each edge \(AB \in E(D_F)\), we fix an edge \(\varphi_F(AB)\) in \(E_G(V(A) \cap S, V(B) \cap T)\), and let \(\sigma_F(AB) \in V(G)\) be the vertex of \(A\) incident with \(\varphi_F(AB)\) and \(\tau_F(AB) \in V(G)\) be the vertex of \(B\) incident with \(\varphi_F(AB)\) (see Figure 1).

For a path \(A = x_1x_2 \cdots x_7 \in \mathcal{C}_7\), the vertex \(x_4\) is called the *center* of \(A\). A directed path \(P = A_1A_2 \cdots A_l (l \geq 2)\) of \(D_F\) is *admissible* if \(A_i \in \mathcal{C}(F) - (\mathcal{C}_3 \cup \mathcal{C}_5^{(1)})\) and \(A_i \in \mathcal{C}_3^{(2)} \cup \mathcal{C}_5^{(1)}\) for every \(i\) (\(2 \leq i \leq l - 1\)). An admissible path \(P = A_1A_2 \cdots A_l\) of \(D_F\) is weakly admissible if either

**(W1)** \(A_1 \in \mathcal{C}_5^{(2)}\) and \(|V(A_1) \cap T_2| = 1\), or

**(W2)** \(A_1 \in \mathcal{C}_7\) and \(\sigma_F(A_1A_2)\) is the center of \(A_1\).
An admissible path $P$ of $D_F$ is strongly admissible if $P$ is not weakly admissible.

A path system with respect to $F$ is a sequence $(P_1, \ldots, P_m)$ ($m \geq 0$) of admissible paths such that

(P1) for each $i$ ($1 \leq i \leq m$), when we write $P_i = A_1A_2\cdots A_l$, $\{A_j \mid 1 \leq j \leq l - 1\} \cap (\bigcup_{1 \leq j \leq i - 1} V(P_j)) = \emptyset$ and $A_l \in C^{(1)}_3 \cup (\bigcup_{1 \leq j \leq i - 1} V(P_j))$, and

(P2) for each $i$ ($1 \leq i \leq m - 1$), $P_i$ is weakly admissible.

A path system $(P_1, \ldots, P_m)$ with respect to $F$ is complete if $m \geq 1$ and $P_m$ is strongly admissible.

By straightforward calculations, we get the following claim (and we omit its proof).

Claim 3.2 Let $F$ be an $S$-central path-factor of $G$. Then the following hold.

(i) For $A \in C^{(1)}_3(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| = 2 = \frac{2}{3}|V(A) \cap S| + \frac{2}{3}$.

(ii) For $A \in C^{(2)}_3(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| \geq \frac{4}{3}|V(A) \cap S|$.

(iii) For $A \in C^{(1)}_5(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| > \frac{4}{3}|V(A) \cap S|$.

(iv) For $A \in C^{(2)}_5(F)$ with $|V(A) \cap T_2| = 1$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| = \frac{4}{3}|V(A) \cap S|$.

(v) For $A \in C^{(1)}_7(F)$, $|V(A) \cap T_1| + \frac{2}{3}|V(A) \cap T_2| = \frac{4}{3}|V(A) \cap S|$.

The following claim plays a key role in the proof of the theorem.

Claim 3.3 Let $F$ be an $S$-central path-factor of $G$ with $C^{(1)}_3(F) \neq \emptyset$, and let $(P_1, \ldots, P_m)$ be a path system with respect to $F$ ($m \geq 0$). Then the system can be extend to a complete path system $(P_1, \ldots, P_m, P_{m+1}, \ldots, P_{m'})$ with respect to $F$. 

Figure 1: Edge $\varphi_F(AB)$ and vertices $\sigma_F(AB)$ and $\tau_F(AB)$
Proof. We take a maximal path system \((\mathcal{P}_1, \ldots, \mathcal{P}_m, \mathcal{P}_{m+1}, \ldots, \mathcal{P}_{m'})\) with respect to \(F\). We show that \((\mathcal{P}_1, \ldots, \mathcal{P}_{m'})\) is a complete path system. Suppose that \((\mathcal{P}_1, \ldots, \mathcal{P}_{m'})\) is not a complete path system. Then \(\mathcal{P}_i\) is weakly admissible for each \(i\) with \(1 \leq i \leq m'\) (this includes the case where \(m' = 0\)).

Set \(\mathcal{A}_1 = \bigcup_{1 \leq i \leq m'} V(\mathcal{P}_i)\) (note that \(\mathcal{A}_1 = \emptyset\) if and only if \(m' = 0\)). Let \(X = (\bigcup_{A \in \mathcal{A}_1} V(A)) \cap S\) and \(Y_h = (\bigcup_{A \in \mathcal{A}_1} V(A)) \cap T_h\) \((h \in \{1, 2\})\). Then by the definition of a weakly admissible path (and the definition of a path system), \(\mathcal{A}_1 \subseteq \mathcal{C}_3 \cup \mathcal{C}_5 \cup \mathcal{C}_7^{(1)}\), and if \(A \in \mathcal{A}_1 \cap \mathcal{C}_5^{(2)}\), then \(|V(A) \cap T_1| = 1\). Furthermore, by condition (P1) in the definition of a path system, \(\mathcal{A}_1 \neq \emptyset\) if and only if \(\mathcal{A}_1 \cap \mathcal{C}_3^{(1)} \neq \emptyset\). Hence by Claim 3.2

\[
|Y_1| + \frac{2}{3}|Y_2| \geq \frac{4}{3}|X| \tag{3.2}
\]

and

\[
|Y_1| + \frac{2}{3}|Y_2| \geq \frac{4}{3}|X| + \frac{2}{3} \text{ if } \mathcal{A}_1 \neq \emptyset \tag{3.3}
\]

Let \(\mathcal{A}_2 = \mathcal{C}_3^{(1)} - \mathcal{A}_1\), \(X^* = (\bigcup_{A \in \mathcal{A}_2} V(A)) \cap S\) and \(Y_h^* = (\bigcup_{A \in \mathcal{A}_2} V(A)) \cap T_h\) \((h \in \{1, 2\})\). By Claim 3.2(i),

\[
|Y_1^*| + \frac{2}{3}|Y_2^*| \geq \frac{4}{3}|X^*| \tag{3.4}
\]

and

\[
|Y_1^*| + \frac{2}{3}|Y_2^*| = \frac{4}{3}|X^*| + \frac{2}{3} \text{ if } \mathcal{A}_2 \neq \emptyset \tag{3.5}
\]

Let \((B_1, \ldots, B_l)\) \((l \geq 0)\) be a sequence such that for each \(i\) \((1 \leq i \leq l)\), \(B_i \in (\mathcal{C}_3^{(2)} \cup \mathcal{C}_5^{(1)}) - (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{B_j \mid 1 \leq j \leq i - 1\})\) and there exists an edge of \(D_F\) from \(B_i\) to an element in \(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \{B_j \mid 1 \leq j \leq i - 1\}\). We choose \((B_1, \ldots, B_l)\) so that \(l\) is as large as possible. Let \(\mathcal{A}_3 = \{B_i \mid 1 \leq i \leq l\}\), \(X^{**} = (\bigcup_{A \in \mathcal{A}_3} V(A)) \cap S\) and \(Y_h^{**} = (\bigcup_{A \in \mathcal{A}_3} V(A)) \cap T_h\) \((h \in \{1, 2\})\). By Claim 3.2(ii)(iii),

\[
|Y_1^{**}| + \frac{2}{3}|Y_2^{**}| \geq \frac{4}{3}|X^{**}| \tag{3.6}
\]

Let \(X^0 = X \cup X^* \cup X^{**}\) and \(Y^0_h = Y_h \cup Y_h^* \cup Y_h^{**}\) \((h \in \{1, 2\})\). If \(m' \geq 1\), then \(\mathcal{A}_1 \neq \emptyset\); if \(m' = 0\) (i.e., \(\mathcal{A}_1 = \emptyset\)), then \(\mathcal{A}_2 \neq \emptyset\) because \(\mathcal{C}_3^{(1)} \neq \emptyset\). Thus by (3.3) and (3.5), either \(|Y_1| + \frac{2}{3}|Y_2| \geq \frac{4}{3}|X| + \frac{2}{3}\) or \(|Y_1^*| + \frac{2}{3}|Y_2^*| = \frac{4}{3}|X^*| + \frac{2}{3}\). This together with (3.2), (3.4) and (3.6) leads to

\[
|Y_1^0| + \frac{2}{3}|Y_2^0| \geq \frac{4}{3}|X^0| + \frac{2}{3} \tag{3.7}
\]
Since $|T_1| + \frac{2}{3}|T_2| \leq \frac{4}{3}|S| + \frac{1}{3}$, this implies $X^0 \neq S$ and hence $C(F) - (A_1 \cup A_2 \cup A_3) \neq \emptyset$.

Let $\tilde{A} = C(F) - (A_1 \cup A_2 \cup A_3)$, $\tilde{X} = (\bigcup_{A \in \tilde{A}} V(A)) \cap S$ and $\tilde{Y}_h = (\bigcup_{A \in \tilde{A}} V(A)) \cap T_h$ ($h \in \{1, 2\}$). Note that $S$ is the disjoint union of $X^0$ and $\tilde{X}$ and, for $h \in \{1, 2\}$, $T_h$ is the disjoint union of $Y_h$ and $\tilde{Y}_h$. If $|\tilde{Y}_1| + \frac{2}{3}|\tilde{Y}_2| \geq \frac{4}{3}|\tilde{X}|$, then by (3.7), $|\tilde{Y}_1| + \frac{2}{3}|\tilde{Y}_2| = (|Y_1| + |\tilde{Y}_1|) + \frac{2}{3}(|Y_2| + |\tilde{Y}_2|) \geq \frac{4}{3}|X^0| + \frac{4}{3} + \frac{4}{3}|\tilde{X}| = \frac{4}{3}|S| + \frac{4}{3}$, which is a contradiction. Thus $|\tilde{Y}_1| + \frac{2}{3}|\tilde{Y}_2| < \frac{4}{3}|\tilde{X}|$. On the other hand, since $A_1 \cup A_2 \neq \emptyset$, we have $Y_1 \cup Y_2 \neq \emptyset$, and hence $\tilde{Y}_1 \cup \tilde{Y}_2 \neq T$. Consequently $N_G(\tilde{X}) \subseteq \tilde{Y}_1 \cup \tilde{Y}_2$ by the assumption of the theorem, which implies that there exists a vertex $x \in \tilde{X}$ with $N_G(x) \cap (Y_1 \cup Y_2) \neq \emptyset$.

Let $\tilde{A} \in \tilde{A}$ be the path containing $x$. By the definition of $A_2$ and $\tilde{A}$, $\tilde{A} \not\in C_3^{(1)}$. By the maximality of $(B_1, \ldots, B_t)$, $\tilde{A} \not\in C_3^{(2)} \cup C_3^{(1)}$. Thus $\tilde{A} \in C(F) - (C_3 \cup C_5^{(1)})$. By the definition of $(B_1, \ldots, B_t)$ and $x$, there exists a directed path $P' = \tilde{A}_1 \cdots \tilde{A}_p$ of $D_F$ such that $\tilde{A}_1 = \tilde{A}$, $\tilde{A}_i \in A_3$ ($2 \leq i \leq p - 1$) and $\tilde{A}_p \in A_1 \cup A_2$. Then $P'$ is an admissible path of $D_F$. Now the sequence $(P_1, \ldots, P_m, P')$ is a path system with respect to $F$, which contradicts the maximality of $(P_1, \ldots, P_m)$. This contradiction completes the proof of the claim. \[\square\]

We turn to the proof of Theorem 3.1. By way of contradiction, suppose that $C_3^{(1)}(F) \neq \emptyset$ for every $S$-central path-factor $F$ of $G$. By Lemma 3.2, $G$ has an $S$-central path-factor $F_0$. Note that an empty sequence is a path system with respect to $F_0$. Hence by Claim 3.3 there exists a complete path system $(P_1, \ldots, P_m)$ with respect to $F_0$. Choose $F_0$ and $(P_1, \ldots, P_m)$ so that

(F1) $|C_3^{(1)}(F_0)|$ is as small as possible, and

(F2) subject to (F1), $(|V(P_1)|, \ldots, |V(P_m)|)$ is lexicographically as small as possible.

For each $i$ ($1 \leq i \leq m$), write $P_i = A_1^{(i)} \cdots A_t^{(i)}$. Then $\bigcup_{1 \leq i \leq m} P_i$ contains a directed path $B_1B_2 \cdots B_p$ of $D_{F_0}$ with $B_1 = A_1^{(m)}$ and $B_p \in C_3^{(1)}(F_0)$. For each $i$ ($1 \leq i \leq p$), write $B_i = v_{i,1}v_{i,2} \cdots v_{i,q_i}$. For $i$ ($1 \leq i \leq p - 1$), let $s_i$ be the integer with $v_{i,s_i} = \sigma_F(B_iB_{i+1})$, and for $i$ ($2 \leq i \leq p$), let $t_i$ be the integer with $v_{i,t_i} = \tau_{F_0}(B_{i-1}B_i)$. As in the proof of Claim 3.1 by renumbering the vertices of some of the $B_i$ backward if necessary, we may assume that

(B1) $s_1 \geq \frac{q_1+1}{2}$ if $q_1$ is odd,

(B2) $\{v_{1,1}, v_{1,3}\} \cap T_2 \neq \emptyset$ if $B_1 \in C_7^{(2)}(F_0)$ and $s_1 = 4,$

(B3) $s_1$ is odd if $q_1$ is even,
(B4) $t_i < s_i$ for each $i$ $(2 \leq i \leq p - 1)$, and

(B5) $t_p = q_p$ $(= 3)$.

Note that (B3) means that when $q_1$ is even, the vertices of $B_1$ are numbered so that $v_{1,q_1} \in T$. Thus $v_{i,q_i} \in T$ for each $i$ $(1 \leq i \leq p)$. We can divide the type of $B_1$ into three possibilities as follows:

Claim 3.4 One of the following holds:

(1) $|V(B_1)|$ is even and $s_1$ is odd;

(2) $B_1 \in \mathcal{C}_5^{(2)}(F_0) \cup \mathcal{C}_7^{(2)}(F_0)$, $s_1 = 4$ and $\{v_{1,1}, v_{1,3}\} \cap T_2 \neq \emptyset$; or

(3) $|V(B_1)| \geq 7$ and $s_1 \geq 6$.

Proof. If $|V(B_1)|$ is even, then (1) holds by (B3). Thus we may assume $|V(B_1)|$ is odd. Then by the definition of a strongly admissible path, $B_1 \in \mathcal{C}_5^{(2)}(F_0)$ and $|V(B_1) \cap T_2| \geq 2$, or $B_1 \in \mathcal{C}_7^{(1)}(F_0)$ and $s_1 \neq 4$, or $B_1 \in \mathcal{C}_7^{(2)}(F_0)$, or $|V(B_1)| \geq 9$. If $B_1 \in \mathcal{C}_5^{(2)}(F_0)$ and $|V(B_1) \cap T_2| \geq 2$, then (2) holds by (B1). If $B_1 \in \mathcal{C}_7^{(1)}(F_0) \cup \mathcal{C}_7^{(2)}(F_0)$ and $s_1 \neq 4$, then (3) holds by (B1). If $B_1 \in \mathcal{C}_7^{(2)}(F_0)$ and $s_1 = 4$, then (2) holds by (B2). If $|V(B_1)| \geq 9$, then (3) holds by (B1). □

As for $B_i$ with $2 \leq i \leq p - 1$, the following claim follows immediately from the definition of a weakly admissible path.

Claim 3.5 Let $2 \leq i \leq p - 1$. Then one of the following holds:

(1) $B_i \in \mathcal{C}_3^{(2)}(F_0)$ and $s_i = 2$;

(2) $B_i \in \mathcal{C}_5(F_0)$ and $s_i = 2$ or $4$; or

(3) $B_i \in \mathcal{C}_7^{(1)}(F_0)$ and $s_i = 4$. □

Let $i_0$ be the minimum integer $i$ $(\geq 2)$ satisfying one of the following two conditions:

(I1) $i = p$; or

(I2) $2 \leq i \leq p - 1$ and $t_i = 1$.  

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Set $B'_i = v_{1,1}v_{1,2} \cdots v_{1,s_i-1}$ and, for each $i$ ($2 \leq i \leq i_0$), set

$$B'_i = v_{i-1,q_i-1}v_{i-1,q_i-2} \cdots v_{i-1,s_i-1}v_{i,t_i}v_{i,t_i-1} \cdots v_{i,1}$$

(see Figure 2). Let $2 \leq i \leq i_0 - 1$. By the definition of $i_0$, $t_i \geq 3$. On the other hand, $s_i \leq 4$ by Claim 3.5. Hence $t_i = s_i - 1$. Since $i$ ($2 \leq i \leq i_0 - 1$) is arbitrary, it follows that

$$B'_1, \ldots, B'_{i_0}$$

are vertex-disjoint paths of $G$. Let $2 \leq i \leq i_0 - 1$. By the definition of $i_0$, $t_i \geq 3$. On the other hand, $s_i \leq 4$ by Claim 3.5. Hence $t_i = s_i - 1$. Since $i$ ($2 \leq i \leq i_0 - 1$) is arbitrary, it follows that

$$B'_1, \ldots, B'_{i_0}$$

are vertex-disjoint paths of $G$ (3.8)

and

$$\bigcup_{1 \leq i \leq i_0} V(B'_i) = \bigcup_{1 \leq i \leq i_0} V(B_i) - \{v_{i_0,j} | t_{i_0} + 1 \leq j \leq q_{i_0}\}. (3.9)$$

Furthermore,

$$|V(B'_i) \cap T| \geq |V(B'_i) \cap S|$$

for each $i$ ($2 \leq i \leq i_0$) (3.10) because $v_{i-1,q_i-1} \in T$. If $B'_1 \neq \emptyset$, then $v_{1,s_1-1} \in T$, and hence

$$|V(B'_1) \cap T| \geq |V(B'_1) \cap S|$$

(3.11) (if $B'_1 = \emptyset$, then (3.11) trivially holds). Also

$$|V(B'_i) \cap V(B_{i-1})|$$

is even and $|V(B'_i) \cap V(B_{i-1})| \geq 2$ for each $i$ ($2 \leq i \leq i_0$) (3.12)

because $v_{i-1,s_i-1} \in S$ and $v_{i-1,q_i-1} \in T$. It follows from (3.12) that

$$|V(B'_i)| \geq 5$$

for each $i$ ($2 \leq i \leq i_0 - 1$) (3.13) because $|V(B'_i) \cap V(B_i)| = t_i \geq 3$. Since $|V(B'_1)| = s_1 - 1$, we see from Claim 3.4 that

$$|V(B'_i)|$$

is even or $|V(B'_1)| \geq 3$, (3.14)

and

$$V(B'_i) \cap T_2 \neq \emptyset$$

if $|V(B'_i)| = 3$. (3.15)

Combining (3.10) through (3.15), we get the following claim.

Claim 3.6 (i) For each $i$ with $1 \leq i \leq i_0$, we have $|V(B'_i) \cap T| \geq |V(B'_i) \cap S|$. 12
(ii) For each \( i \) with \( 1 \leq i \leq i_0 - 1 \),

(a) \(|V(B'_i)|\) is even or \(|V(B'_i)| \geq 3\), and

(b) \( V(B'_i) \cap T_2 \neq \emptyset \) if \( B'_i \simeq P_3 \). \( \square \)

Suppose that \( i_0 = p \). Then

\[
|V(B'_p)| \geq 5 \tag{3.16}
\]

by (3.12) and (B5). Let \( F_1 = (F_0 - (\bigcup_{1 \leq i \leq p} V(B_i))) \cup (\bigcup_{1 \leq i \leq p} B'_i) \). Then by Claim 3.6 (3.16), (3.8), (3.9) and (B5), \( F_1 \) is an \( S \)-central path-factor of \( G \), and \( B'_i \notin C_3^{(1)}(F_1) \) for each \( i \) \((1 \leq i \leq p)\). Since \( B_1 \in C_3^{(1)}(F_0) \), we have \(|C_3^{(1)}(F_1)| < |C_3^{(1)}(F_0)|\), which contradicts the minimality of \(|C_3^{(1)}(F)|\). Thus \( 2 \leq i_0 \leq p - 1 \). Then by the definition of \( i_0 \), \( t_{i_0} = 1 \). Hence \( B''_{i_0} = B_{i_0} \cup B'_0 \) is a path of \( G \) with \(|V(B''_{i_0}) \cap T| \geq |V(B''_i) \cap S|\) (see Figure 3). Set \( F_2 = (F_0 - (\bigcup_{1 \leq i \leq i_0} V(B_i))) \cup (\bigcup_{1 \leq i \leq i_0 - 1} B'_i) \cup B''_{i_0} \). Then by Claim 3.6 (3.8) and (3.9), \( F_2 \) is an \( S \)-central path-factor of \( G \), and \( B'_i \notin C_3^{(1)}(F_1) \) for each \( i \) \((1 \leq i \leq i_0 - 1)\). Furthermore,

\[
|V(B''_{i_0})| = |V(B_{i_0})| + |V(B'_{i_0}) \cap V(B_{i_0 - 1})|. \tag{3.17}
\]

Since \(|V(B'_{i_0}) \cap V(B_{i_0 - 1})| \geq 2 \) by (3.12), this implies \(|V(B''_{i_0})| \geq 5\), and hence we also have \( B''_{i_0} \notin C_3^{(1)}(F_1) \). Thus \(|C_3^{(1)}(F_2)| = |C_3^{(1)}(F_0)|\).

Set \( k_0 = \min\{k \mid B_{i_0} \in V(P_k)\} \), and write \( B_{i_0} = A^{(k_0)}_{j_0} \). If \( B_p \in V(P_{k_0}) \), then the fact that \( B_{i_0} \neq B_p \) implies that \( j_0 \leq l_{k_0} - 1 \); if \( B_p \notin V(P_{k_0}) \), then the minimality of \( k_0 \) implies that \( j_0 \leq l_{k_0} - 1 \). In either case, we have \( j_0 \leq l_{k_0} - 1 \).
Case 1: \( j_0 = 1 \).

Since \( B_{i_0} = A_{i_0}^{(k_0)} \) and \( i_0 \geq 2 \), \( B_1 \in \bigcup_{k_0+1 \leq i \leq m} V(P_i) \). In particular, \( k_0 \leq m-1 \) and \( P_{k_0} \) is weakly admissible. Hence \( B_{i_0} \in \mathcal{C}_3^{(2)}(F_0) \cup \mathcal{C}_7^{(1)}(F_0) \). This together with (3.17) and (3.12) implies that \( B''_{i_0} \in \mathcal{C}_3^{(2)}(F_2) \) or \( |V(B''_{i_0})| \geq 7 \). Thus the directed path \( \mathcal{P}''_{k_0} = B''_{i_0} A_{i_0}^{(k_0)} \cdots A_{i_0}^{(k_0)} \) of \( \mathcal{D}_F \) is strongly admissible. Consequently \( (\mathcal{P}_1, \ldots, \mathcal{P}_{k_0-1}, \mathcal{P}''_{k_0}) \) is a complete path system with respect to \( F_2 \). Since \( k_0 \leq m-1 \) and \( |V(P_{k_0})| = |V(P_{k_0})| \), we see that \( (|V(P_1)|, \ldots, |V(P_{k_0-1})|, |V(P_{k_0})|) \) is lexicographically less than \( (|V(P_1)|, \ldots, |V(P_{k_0-1})|, |V(P_{k_0})|, \ldots, |V(P_m)|) \), which contradicts the minimality of \( (|V(P_1)|, \ldots, |V(P_m)|) \).

Case 2: \( 2 \leq j_0 \leq l_{k_0} - 1 \).

Since \( B_{i_0} = A_{i_0}^{(k_0)} \), \( B_1 \in \mathcal{C}_3^{(2)}(F_0) \cup \mathcal{C}_7^{(1)}(F_0) \). This together with (3.17) and (3.12) implies that \( B''_{i_0} \in \mathcal{C}_3^{(2)}(F_2) \) or \( |V(B''_{i_0})| \geq 7 \). Thus the directed path \( \mathcal{P}''_{k_0} = B''_{i_0} A_{i_0}^{(k_0)} A_{i_0+1}^{(k_0)} \cdots A_{i_0+l_{k_0}}^{(k_0)} \) of \( \mathcal{D}_F \) is admissible. Consequently \( (\mathcal{P}_1, \ldots, \mathcal{P}_{k_0-1}, \mathcal{P}''_{k_0}) \) is a path system with respect to \( F_2 \). By Claim 3.3, the system can be extend to a complete path system \( (\mathcal{P}_1, \ldots, \mathcal{P}_{k_0-1}, \mathcal{P}''_{k_0}, Q_1, \ldots, Q_\alpha) \) with respect to \( F_2 \) (it is possible that \( \alpha = 0 \)). Since \( j_0 \geq 2 \), \( |V(P_{k_0})| = l_{k_0} - j_0 + 1 < l_{k_0} = |V(P_{k_0})| \), and hence \( (|V(P_1)|, \ldots, |V(P_{k_0-1})|, |V(P_{k_0})|, |V(P_{k_0})|, \ldots, |V(Q_\alpha)|) \) is lexicographically less than \( (|V(P_1)|, \ldots, |V(P_{k_0-1})|, |V(P_{k_0})|, \ldots, |V(P_m)|) \), which contradicts the minimality of \( (|V(P_1)|, \ldots, |V(P_m)|) \).

This completes the proof of Theorem 3.1. \( \square \)
4 Proof of Theorem 1.1

Let $G$ be as in Theorem 1.1. By assumption, we have $c_1(G) + \frac{2}{3}c_3(G) \leq \frac{4}{3}|\emptyset| + \frac{1}{3} = \frac{1}{3}$. Hence $c_1(G) = c_3(G) = 0$.

We now proceed by induction on $|V(G)| + |E(G)|$. We may assume $V(G) \neq \emptyset$. Note that if $E(G) = \emptyset$, then $c_1(G) = |V(G)| \geq 1$, which is a contradiction. This means that the theorem holds for graphs $G$ with $E(G) = \emptyset$ in the sense that the assumption is not satisfied. We henceforth assume that $E(G) \neq \emptyset$ and the theorem holds for graphs $G'$ with $|V(G')| + |E(G')| < |V(G)| + |E(G)|$.

Let $S = \{X \subseteq V(G) \mid c_1(G - X) + c_3(G - X) \geq 1\}$. Since $c_1(G - N_G(x)) \geq 1$ for $x \in V(G)$, $S \neq \emptyset$. Set

$$\beta = \min_{X \in S} \left\{ \frac{4}{3}|X| + \frac{1}{3} - c_1(G - X) - \frac{2}{3}c_3(G - X) \right\}.$$ 

Claim 4.1 If $\beta \geq 2$, then $G$ has a $\{P_2, P_3\}$-factor.

Proof. Let $e \in E(G)$, and suppose that $C_1(G - e) \cup C_3(G - e) \neq \emptyset$. Take $C \in C_1(G - e) \cup C_3(G - e)$. Since $c_1(G) = c_3(G) = 0$, $e$ joins a vertex in $V(C)$ and a vertex $y$ in $V(G) - V(C)$. This implies $C \in C_1(G - y) \cup C_3(G - y)$, and hence $\frac{4}{3}|\{y\}| + \frac{1}{3} - (c_1(G - y) + \frac{2}{3}c_3(G - y)) \leq \frac{4}{3} + \frac{1}{3} - \frac{2}{3} = \frac{1}{3}$, which contradicts the assumption that $\beta \geq 2$. Thus $c_1(G - e) = c_3(G - e) = 0$ for all $e \in E(G)$. From the fact that $c_1(G - e) = \emptyset$ for all $e \in E(G)$, it follows that $d_G(x) \geq 2$ for all $x \in V(G)$. Assume for the moment that $d_G(x) = 2$ for all $x \in V(G)$. Then each component of $G$ is a cycle. Since $c_3(G) = 0$, this implies that $G$ has a path-factor $F$ with $C_3(F) = \emptyset$. Hence by Fact 1.1, $G$ has a $\{P_2, P_3\}$-factor. Thus we may assume that there exists $x_0 \in V(G)$ such that $d_G(x_0) \geq 3$.

Fix an edge $e^* = x_0y_0 \in E(G)$ incident with $x_0$, and let $G' = G - e^*$. By an assertion in the first paragraph of the proof the claim, $c_1(G') = c_3(G') = 0$. Let $X \subseteq V(G')$. We show that $\frac{4}{3}|X| + \frac{1}{3} - c_1(G' - X) - \frac{2}{3}c_3(G' - X) \geq 0$. We have $\frac{4}{3}|\emptyset| + \frac{1}{3} - c_1(G') - c_3(G') = \frac{1}{3} > 0$. Thus we may assume $X \neq \emptyset$. Note that $|\{C_1(G' - X) - C_1(G - X)| + |\{C_3(G' - X) - C_3(G - X)| \leq 2$, and hence

$$c_1(G' - X) + \frac{2}{3}c_3(G' - X) \leq c_1(G - X) + \frac{2}{3}c_3(G - X) + 2. \quad (4.1)$$

Furthermore, if equality holds in (4.1), then $x_0, y_0 \notin X$ and $\{x_0\}, \{y_0\} \in C_1(G' - X)$. If $c_1(G - X) + c_3(G - X) \geq 1$, then by the definition of $\beta$, $\frac{4}{3}|X| + \frac{1}{3} - c_1(G - X) - \frac{2}{3}c_3(G - X) \geq \beta \geq 2$ which, together with (4.1), leads to $\frac{4}{3}|X| + \frac{1}{3} - (c_1(G' - X) + \frac{2}{3}c_3(G' - X)) \geq \frac{4}{3}|X| + \frac{1}{3} - (c_1(G - X) + \frac{2}{3}c_3(G - X) + 2) \geq \beta - 2 \geq 0$. Thus
we may assume that $c_1(G - X) + c_3(G - X) = 0$. By \(\mathbb{I.I}\), $c_1(G' - X) + c_3(G' - X) \leq c_1(G - X) + c_3(G - X) + 2 = 2$. By way of contradiction, suppose that $\frac{4}{3}|X| + \frac{1}{3} - (c_1(G' - X) + \frac{2}{3}c_3(G' - X)) < 0$. Then $\frac{4}{3}|X| + \frac{1}{3} - 2 < 0$. Since $X \neq \emptyset$, this forces $|X| = 1$ and $c_1(G' - X) + \frac{2}{3}c_3(G' - X) = 2$. Hence equality in \(\mathbb{I.I}\), which implies $\{x_0\} \in \mathbb{C}(G' - X)$. Consequently $d_G(x_0) \leq |X \cup \{y_0\}| = 2$, which contradicts the fact that $d_G(x_0) \geq 3$. Thus we have $\frac{4}{3}|X| + \frac{1}{3} - c_1(G' - X) - \frac{2}{3}c_3(G' - X) \geq 0$ for all $X \subseteq V(G')$. By the induction assumption, $G'$ has a $\{P_2, P_3\}$-factor. Therefore $G$ also has a $\{P_2, P_3\}$-factor. \(\square\)

By Claim \(\mathbb{I.I}\) we may assume that $\beta \leq \frac{5}{3}$.

Let $S \in \mathbb{S}$ be a maximum set with $\frac{4}{3}|S| - c_1(G - S) - \frac{2}{3}c_3(G - S) + \frac{1}{3} = \beta$.

**Claim 4.2** Let $C$ be a component of $G - S$.

(i) If $|V(C)| \notin \{1, 3\}$, then $C$ has a $\{P_2, P_3\}$-factor.

(ii) If $|V(C)| = 3$, then $C$ is complete.

**Proof.**

(i) Suppose that $C$ has no $\{P_2, P_3\}$-factor. Then by the induction assumption, there exists a set $S' \subseteq V(C)$ with $\frac{4}{3}|S'| + \frac{1}{3} - c_1(C - S') - \frac{2}{3}c_3(C - S') < 0$.

Set $S_0 = S \cup S'$. Since $\mathbb{C}_1(G - S_0) = \mathbb{C}_1(G - S) \cup \mathbb{C}_1(C - S')$, $\mathbb{C}_3(G - S_0) = \mathbb{C}_3(G - S) \cup \mathbb{C}_3(C - S')$ and $\mathbb{C}_1(C - S') \cup \mathbb{C}_3(C - S') \neq \emptyset$, we have $S_0 \in \mathbb{S}$. We also get $\frac{4}{3}|S_0| + \frac{1}{3} - c_1(G - S_0) - \frac{2}{3}c_3(G - S_0) = (\frac{4}{3}|S| + \frac{1}{3} - c_1(G - S) - \frac{2}{3}c_3(G - S)) + (\frac{4}{3}|S'| - c_1(C - S') - \frac{2}{3}c_3(C - S')) < \beta$. This contradicts the definition of $\beta$.

(ii) Suppose that $|V(C)| = 3$ and $C$ is not complete (i.e., $C$ is a path of order three). Let $x \in C$ be the vertex with $d_C(x) = 2$. Then $c_1(C - x) = 2$ and $c_3(C - x) = 0$. Set $S_1 = S \cup \{x\}$. Since $\mathbb{C}_1(G - S_1) = \mathbb{C}_1(G - S) \cup \mathbb{C}_1(C - x)$, $\mathbb{C}_3(G - S_1) = \mathbb{C}_3(G - S) - \{C\}$ and $\mathbb{C}_1(C - x) \neq \emptyset$, we have $S_1 \in \mathbb{S}$. We also get $\frac{4}{3}|S_1| + \frac{1}{3} - c_1(G - S_1) - \frac{2}{3}c_3(G - S_1) = (\frac{4}{3}|S| + \frac{1}{3}) + \frac{1}{3} - (c_1(G - S) + 2) - \frac{2}{3}c_3(G - S) - 1 = \beta$. This contradicts the maximality of $S$. \(\square\)

Set $T_1 = \mathbb{C}_1(G - S)$, $T_2 = \mathbb{C}_3(G - S)$ and $T = T_1 \cup T_2$. Now we construct a bipartite graph $H$ with bipartition $(S, T)$ by letting $uC \in E(H)$ ($u \in S, C \in T$) if and only if $N_G(u) \cap V(C) \neq \emptyset$.  

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Claim 4.3 The following hold.

(i) $|T_1| + \frac{2}{3}|T_2| \leq \frac{4}{3}|S| + \frac{1}{3}$.

(ii) $1 \leq |S| \leq |T_1| + |T_2|$.

(iii) For every $X \subseteq S_0$, either $|N_H(X) \cap T_1| + \frac{2}{3}|N_H(X) \cap T_2| \geq \frac{4}{3}|X|$ or $N_H(X) = T$.

Proof.

(i) By the assumption of the theorem, $|T_1| + \frac{2}{3}|T_2| = c_1(G - S) + \frac{2}{3}c_3(G - S) \leq \frac{4}{3}|S| + \frac{1}{3}$.

(ii) Since $c_1(G) + c_3(G) = 0$ and $c_1(G - S) + c_3(G - S) \geq 1$, $S \neq \emptyset$ (i.e., $|S| \geq 1$).

Since $\frac{4}{3}|S| + \frac{1}{3} - |T_1| - \frac{2}{3}|T_2| = \frac{4}{3}|S| + \frac{1}{3} - c_1(G - S) - \frac{2}{3}c_3(G - S) = \beta \leq \frac{1}{3}$, we get $|S| \leq \frac{4}{3}|T_1| + \frac{2}{3}|T_2| + 1 \leq \frac{4}{3}|T| + 1 < |T| + 1$, and hence $|S| \leq |T| = |T_1| + |T_2|$.

(iii) Suppose that there exists a set $X \subseteq S$ such that $|N_H(X) \cap T_1| + \frac{2}{3}|N_H(X) \cap T_2| < \frac{4}{3}|X|$ and $N_H(X) \neq T$. Since $T - N_H(X) \subseteq \mathcal{C}_1(G - (S - X)) \cup \mathcal{C}_3(G - (S - X))$ by the definition of $H$, we have $S - X \notin S$. We also get $c_1(G - (S - X)) + \frac{2}{3}c_3(G - (S - X)) \geq (|T_1| - |N_H(X) \cap T_1|) + \frac{2}{3}(|T_2| - |N_H(X) \cap T_2|) = (c_1(G - S) + \frac{2}{3}c_3(G - S)) - (|N_H(X) \cap T_1| + \frac{2}{3}|N_H(X) \cap T_2|).$ Consequently $\frac{4}{3}|S - X| + \frac{1}{3} - c_1(G - (S - X)) - \frac{2}{3}c_3(G - (S - X)) \leq (\frac{4}{3}|S| + \frac{1}{3} - c_1(G - S) - \frac{2}{3}c_3(G - S)) - (\frac{4}{3}|X| - |N_H(X) \cap T_1| - \frac{2}{3}|N_H(X) \cap T_2|) < \beta$, which contradicts the definition of $\beta$. \qed

By Claim 4.3 and Theorem 3.1, $H$ has an $S$-central path-factor $F$ such that $V(A) \cap T_2 \neq \emptyset$ for every $A \in \mathcal{C}_3(F)$. For $A \in \mathcal{C}(F)$, let $U_A = V(A) \cap S$, $L_{A,h} = V(A) \cap T_h$ ($h \in \{1, 2\}$), and $L_A = L_{A,1} \cup L_{A,2}$. Let $G_A$ be the graph obtained from $G[U_A \cup (\bigcup_{C \in L_A} V(C))]$ by deleting all edges of $G[U_A]$.

Claim 4.4 For each $A \in \mathcal{C}(F)$, $G_A$ has a $\{P_2, P_3\}$-factor.

Proof. Since $A$ is a path of $H$, there exists a path $Q_A$ of $G_A$ such that $U_A \subseteq V(Q_A)$ and $V(Q_A) \cap V(C) \neq \emptyset$ for every $C \in L_A$. Choose $Q_A$ so that $|V(Q_A)|$ is as large as possible. Then for each $C \in L_{A,2}$ (i.e., $C \in L_A$ with $|V(C)| = 3$), since $C$ is complete by Claim 4.2(ii), it follows that either $V(C) \subseteq V(Q_A)$ or $|V(C) \cap V(Q_A)| = 1$.
and

if $C \in \mathcal{L}_A$ is an endvertex of the path $A$ of $H$, then $V(C) \subseteq V(Q_A)$.

Recall that $\mathcal{L}_{A,2} \neq \emptyset$ if $|V(A)| = 3$. Consequently $|V(Q_A)| \geq |V(A)|$ and, in the case where $|V(A)| \geq 3$, we have $|V(Q_A)| = 5$ or 7. Since $|V(A)| \geq 2$, this means that $|V(Q_A)| \geq 2$ and $|V(Q_A)| \neq 3$. Furthermore, for each $C \in \mathcal{L}_A$, $C - V(Q_A)$ is either empty or a path of order two. Therefore if we set $F_A = Q_A \cup (\bigcup_{C \in \mathcal{L}_A} (C - V(Q_A)))$, then $F_A$ is a path-factor of $G_A$ with $\mathcal{C}_3(F_A) = \emptyset$. By Fact 1.1, $G_A$ has a $\{P_2, P_5\}$-factor. □

By Claims 4.2(i) and 4.4, $G$ has a $\{P_2, P_5\}$-factor.

This completes the proof of Theorem 1.1.

5 Examples

In this section, we construct graphs having no $\{P_2, P_{2k+1}\}$-factor.

5.1 Graphs without $\{P_2, P_5\}$-factor

Let $n \geq 1$ be an integer. Let $Q_0$ be a path of order 3, and let $a$ be an endvertex of $Q_0$. Let $Q_1, \ldots, Q_n$ be disjoint paths of order 7, and for each $i$ ($1 \leq i \leq n$), let $b_i$ be the center of $Q_i$. Let $H_n$ denote the graph obtained from $\bigcup_{0 \leq i \leq n} Q_i$ by joining $a$ to $b_i$ for every $i$ ($1 \leq i \leq n$) (see Figure 4).

Suppose that $H_n$ has a $\{P_2, P_5\}$-factor $F$. Since $Q_0$ does not have a $\{P_2, P_5\}$-factor, $F$ contains $ab_i$ for some $i$ ($1 \leq i \leq n$). Since $d_F(b_i) \leq 2$, this requires that at least one of the components of $Q_i - b_i$ should have a $\{P_2, P_5\}$-factor, which is
impossible because each component of \(Q_i - b_i\) is a path of order 3. Thus \(H_n\) has no \(\{P_2, P_3\}\)-factor.

**Lemma 5.1** For all \(X \subseteq V(H_n)\), \(c_1(H_n - X) + \frac{2}{3} c_3(H_n - X) \leq \frac{4}{3} |X| + \frac{2}{3}\).

**Proof.** Let \(X \subseteq V(H_n)\). Then we can verify that

\[
c_1(Q_0 - X) + \frac{2}{3} c_3(Q_0 - X) \leq \frac{4}{3} |V(Q_0) \cap X| + \frac{2}{3}\]

and

\[
c_1(Q_i - X) + \frac{2}{3} c_3(Q_i - X) \leq \frac{4}{3} |V(Q_i) \cap X| \text{ for every } i \ (1 \leq i \leq n)
\]

Since every component \(C\) of \(H_n - X\) with \(|V(C)| = 1\) belongs to \(\bigcup_{0 \leq i \leq n} C_1(Q_i - X)\), we have

\[
|C_1(H_n - X)| = \sum_{0 \leq i \leq n} |C_1(Q_i - X)| - \left| \left( \bigcup_{0 \leq i \leq n} C_1(Q_i - X) \right) - C_1(H_n - X) \right|. \tag{5.3}
\]

Furthermore,

\[
|C_3(H_n - X)| \leq \sum_{0 \leq i \leq n} |C_3(Q_i - X)| + \left| C_3(H_n - X) - \left( \bigcup_{0 \leq i \leq n} C_3(Q_i - X) \right) \right|. \tag{5.4}
\]

Let \(C\) be a component of \(H_n - X\) with \(|V(C)| = 3\) which does not belong to \(\bigcup_{0 \leq i \leq n} C_3(Q_i - X)\). Then \(C\) intersects with at least two of the \(Q_i\) (\(0 \leq i \leq n\)). Since \(|V(C)| = 3\), \(C\) contains a component of \(Q_i - X\) of order 1 for some \(i \ (0 \leq i \leq n)\). Since \(C\) is arbitrary, this implies that

\[
\left| C_3(H_n - X) - \left( \bigcup_{0 \leq i \leq n} C_3(Q_i - X) \right) \right| \leq \left| \left( \bigcup_{0 \leq i \leq n} C_1(Q_i - X) \right) - C_1(H_n - X) \right|. \tag{5.5}
\]
By (5.1)–(5.5),
\[
    c_1(H_n - X) + \frac{2}{3} c_3(H_n - X) \\
    \leq \left( \sum_{0 \leq i \leq n} |\mathcal{E}_1(Q_i - X)| - \left( \bigcup_{0 \leq i \leq n} \mathcal{E}_1(Q_i - X) \right) - \mathcal{E}_1(H_n - X) \right) \\
    + \frac{2}{3} \left( \sum_{0 \leq i \leq n} |\mathcal{E}_3(Q_i - X)| + |\mathcal{E}_3(H_n - X) - \left( \bigcup_{0 \leq i \leq n} \mathcal{E}_3(Q_i - X) \right) | \right) \\
    \leq \left( \sum_{0 \leq i \leq n} |\mathcal{E}_1(Q_i - X)| - \left( \bigcup_{0 \leq i \leq n} \mathcal{E}_1(Q_i - X) \right) - \mathcal{E}_1(H_n - X) \right) \\
    + \frac{2}{3} \left( \sum_{0 \leq i \leq n} |\mathcal{E}_3(Q_i - X)| + \left( \bigcup_{0 \leq i \leq n} \mathcal{E}_1(Q_i - X) \right) - \mathcal{E}_1(H_n - X) \right) \\
    \leq \sum_{0 \leq i \leq n} |\mathcal{E}_1(Q_i - X)| + \frac{2}{3} \sum_{0 \leq i \leq n} |\mathcal{E}_3(Q_i - X)| \\
    = \sum_{0 \leq i \leq n} \left( c_1(Q_i - X) + \frac{2}{3} c_3(Q_i - X) \right) \\
    \leq \frac{4}{3} \sum_{0 \leq i \leq n} |V(Q_i) \cap X| + \frac{2}{3} \\
    = \frac{4}{3} |X| + \frac{2}{3}.
\]
Thus we get the desired conclusion. □

From Lemma 5.1 we get the following proposition, which implies that Theorem 1.1 is best possible.

**Proposition 5.2** There exist infinitely many graphs \(G\) having no \(\{P_2, P_5\}\)-factor such that \(c_1(G - X) + \frac{2}{3} c_3(G - X) \leq \frac{4}{3} |X| + \frac{2}{3}\) for all \(X \subseteq V(G)\).

### 5.2 Graphs without \(\{P_2, P_{2k+1}\}\)-factor for \(k \geq 3\)

Let \(k \geq 3\) be an integer with \(k \equiv 0 \pmod{3}\), and write \(k = 3m\). Let \(n \geq 1\) be an integer. Let \(R_0\) be a complete graph of order \(n\). For each \(i\) \((1 \leq i \leq 2n + 1)\), let \(K_i\) be a complete graph of order \(2m - 1\), and let \(R_i\) denote the graph obtained from \(K_i\) by joining each vertex of the union of \(2m + 1\) disjoint paths of order 2 to all vertices of \(K_i\). Let \(H'_n = R_0 + \left( \bigcup_{1 \leq i \leq 2n + 1} R_i \right)\) (see Figure 5).

Since \(|V(R_i)| = 2k + 1\) and \(R_i\) does not contain a path of order \(2k + 1\), \(R_i\) has no \(\{P_2, P_{2k+1}\}\)-factor. Suppose that \(H'_n\) has a \(\{P_2, P_{2k+1}\}\)-factor \(F\). Then for
each \( i \) \( (1 \leq i \leq 2n + 1) \), \( F \) contains an edge joining \( V(R_i) \) and \( V(R_0) \). Since \( 2n + 1 > 2|V(R_0)| \), this implies that there exists \( x \in V(R_0) \) such that \( d_F(x) \geq 3 \), which is a contradiction. Thus \( H_n' \) has no \( \{P_2, P_{2k+1}\} \)-factor.

**Lemma 5.3** For all \( X \subseteq V(H_n') \), \( \sum_{0 \leq j \leq k-1} c_{2j+1}(H_n' - X) \leq \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3} \).

**Proof.** Let \( X \subseteq V(H_n') \).

**Claim 5.1** For each \( i \) \( (1 \leq i \leq 2n + 1) \), \( \sum_{0 \leq j \leq k-1} c_{2j+1}(R_i - X) \leq \frac{4k+6}{8k+3}|V(R_i) \cap X| + \frac{2k+3}{8k+3} \).

**Proof.** We first assume that \( V(K_i) \not\subseteq X \). Then \( R_i - X \) is connected. Clearly we may assume that \( \sum_{0 \leq j \leq k-1} c_{2j+1}(R_i - X) = 1 \). Then \( |V(R_i) \cap X| \geq 2 \) because \( |V(R_i)| = 2k + 1 \). Hence \( \sum_{0 \leq j \leq k-1} c_{2j+1}(R_i - X) = 1 < \frac{4k+6}{8k+3} \cdot 2 < \frac{4k+6}{8k+3}|V(R_i) \cap X| + \frac{2k+3}{8k+3} \). Thus we may assume that \( V(K_i) \subseteq X \).

Let \( \alpha \) be the number of components of \( R_i - V(K_i) \) intersecting with \( X \). Since \( \alpha \leq 2m + 1 \), we have \( (8m + 1)\alpha \leq (4m + 2)(2m - 1 + \alpha) + 2m + 1 \), and hence

\[
\alpha \leq \frac{4m+2}{8m+1}(2m-1+\alpha) + \frac{2m+1}{8m+1} = \frac{4k+6}{8k+3}(2m-1+\alpha) + \frac{2k+3}{8k+3}.
\]

Furthermore, \( \sum_{0 \leq j \leq k-1} c_{2j+1}(R_i - X) = c_1(R_i - X) \leq \alpha \) and \( |V(R_i) \cap X| = |V(K_i)| + |(V(R_i) - V(K_i)) \cap X| \geq 2m-1+\alpha \). Consequently we get \( \sum_{0 \leq j \leq k-1} c_{2j+1}(R_i - X) \leq \frac{4k+6}{8k+3}|V(R_i) \cap X| + \frac{2k+3}{8k+3} \). \( \square \)

Assume for the moment that \( V(R_0) \not\subseteq X \). Then \( H_n' - X \) is connected. Clearly we may assume that \( \sum_{0 \leq j \leq k-1} c_{2j+1}(H_n' - X) = 1 \). Then \( |X| \geq 2 \) because \( |V(H_n')| \geq 2k + 1 \). Hence \( \sum_{0 \leq j \leq k-1} c_{2j+1}(H_n' - X) = 1 < \frac{4k+6}{8k+3} \cdot 2 < \frac{4k+6}{8k+3}|X| + \frac{2k+3}{8k+3} \). Thus we may assume that \( V(R_0) \subseteq X \). Then clearly

\[
|c_{2j+1}(H_n' - X)| = \sum_{1 \leq i \leq 2n+1} |c_{2j+1}(R_i - X)|. \quad (5.6)
\]
By Claim 5.1 and (5.6),

\[
\sum_{0 \leq j \leq k-1} c_{2j+1}(H'_n - X) = \sum_{0 \leq j \leq k-1} \left( \sum_{1 \leq i \leq 2n+1} c_{2j+1}(R_i - X) \right) \\
\leq \sum_{1 \leq i \leq 2n+1} \left( \frac{4k+6}{8k+3} |V(R_i) \cap X| + \frac{2k+3}{8k+3} \right) \\
= \frac{4k+6}{8k+3} (|X| - |V(R_0)|) + \frac{2k+3}{8k+3} (2n+1) \\
= \frac{4k+6}{8k+3} |X| + \frac{2k+3}{8k+3}.
\]

Thus we get the desired conclusion. □

From Lemma 5.3, we get the following proposition, which implies that if Conjecture 1 is true, then the coefficient of $|X|$ in the conjecture is best possible.

**Proposition 5.4** For an integer $k \geq 3$ with $k \equiv 0 \pmod{3}$, there exist infinitely many graphs $G$ having no $\{P_2, P_{2k+1}\}$-factor such that

\[
\sum_{0 \leq j \leq k-1} c_{2j+1}(G - X) \leq \frac{4k+6}{8k+3} |X| + \frac{2k+3}{8k+3} 
\]

for all $X \subseteq V(G)$.

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