STABILITY OF MARTINGALE OPTIMAL TRANSPORT AND WEAK OPTIMAL TRANSPORT

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Abstract. Under mild regularity assumptions, the transport problem is stable in the following sense: if a sequence of optimal transport plans \( \pi_1, \pi_2, \ldots \) converges weakly to a transport plan \( \pi \), then \( \pi \) is also optimal (between its marginals).

Alfonsi, Corbetta and Jourdain \cite{3} asked whether the same property is true for the martingale transport problem. This question seems particularly pressing since martingale transport is motivated by robust finance where data is naturally noisy. On a technical level, stability in the martingale case appears more intricate than for classical transport since optimal transport plans \( \pi \) are not characterized by a ‘monotonicity’-property of \( \text{supp}\pi \).

In this paper we give a positive answer and establish stability of the martingale transport problem. As a particular case, this recovers the stability of the left curtain coupling established by Juillet \cite{32}. An important auxiliary tool is an unconventional topology which takes the temporal structure of martingales into account. Our techniques also apply to the weak transport problem introduced by Gozlan et al.

Keywords: stability, martingale transport, weak transport, causal transport, weak adapted topology, robust finance.

1. Introduction and main results

Let \( X \) and \( Y \) be Polish spaces and consider a continuous function \( c : X \times Y \rightarrow [0, \infty) \). Given probability measures \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \), the classical transport problem is

\[
\inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y)\pi(dx, dy), \tag{OT}
\]

where \( \Pi(\mu, \nu) \) denotes the set of couplings with \( X \)-marginal \( \mu \) and \( Y \)-marginal \( \nu \). A classical result in optimal transport asserts that \( \pi \in \Pi(\mu, \nu) \) is optimal for (OT) iff its support \( \text{supp}\pi \) is \( c \)-monotone \cite{40, 41}. One useful consequence of this characterization of optimality is the stability of (OT) with respect to the marginals \( \mu, \nu \) as well as the cost function \( c \). Indeed, the link between monotonicity and stability becomes apparent once one realizes that the notion of monotonicity is itself stable.

In this article we consider the martingale optimal transport problem from the point of view of monotonicity and stability. In fact, since this problem is an instance of a weak optimal transport problem, we will likewise study the latter class of problems from this viewpoint.

1.1. Stability of martingale optimal transport. The martingale optimal transport problem is a variant of (OT) stemming from robust mathematical finance (cf. \cite{29, 14, 38, 21, 20, 17, 12, 8, 31, 32, 35, 19, 30, 22, 27} among many others). In order to define this problem, we take \( X = Y = \mathbb{R} \), suppose that \( \mu, \nu \) have finite first moments, and introduce the set \( \Pi_M(\mu, \nu) \) of martingale couplings with marginals \( \mu, \nu \). To be precise, a transport plan \( \pi \) is a martingale coupling iff

\[
\int_{\mathbb{R}} y\pi_x(dy) = x \quad \mu\text{-a.s.}
\]
By a famous result of Strassen, the set $\Pi_M(\mu, \nu)$ is non-empty iff $\mu$ is smaller than $\nu$ in convex order. The martingale optimal transport problem is given by

$$\inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y)\pi(dx, dy).$$

(MOT)

The main result of the article is the stability of martingale transport. This gives a positive answer to the question posed by Alfonsi, Corbetta and Jourdain in [3, Section 5.3] in the case $d = 1$.

We denote by $\mathcal{P}_1(\mathbb{R})$ the set of probability measures with finite first moments and by $\mathcal{W}^1$ the topology of 1-Wasserstein convergence on $\mathcal{P}_1(\mathbb{R})$, cf. [40].

**Theorem 1.1 (MOT Stability).** Let $c, c_k : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, $k \in \mathbb{N}$, be continuous cost functions such that $c_k$ converges uniformly to $c$. Let $(\mu_k)_k, (\nu_k)_k \subseteq \mathcal{P}_1(\mathbb{R})$, where $\mu_k$ and $\nu_k$ respectively converge to $\mu$ and $\nu$ in $\mathcal{W}^1$. Let $\pi^k \in \Pi_M(\mu_k, \nu_k)$ be optimizers of

$$\inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y)\pi(dx, dy).$$

If $c(x, y) \leq a(x) + b(y)$ with $a \in L^1(\mu)$, $b \in L^1(\nu)$, and

$$\lim_{k \to \infty} \int_{\mathbb{R}} c_k(x, y) d\pi^k < \infty,$$

then any accumulation point of $[\pi^k]_k$ is an optimizer of the mot for the cost function $c$. In particular if the latter has a unique optimizer $\pi$, then $\pi^k \to \pi$ weakly.

**Corollary 1.2.** Let $c, c_k : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, $k \in \mathbb{N}$, be continuous cost functions such that $c_k$ converges uniformly to $c$. Let $(\mu_k)_k, (\nu_k)_k \subseteq \mathcal{P}_1(\mathbb{R})$, where $\mu_k$ and $\nu_k$ converge respectively to $\mu$ and $\nu$ in $\mathcal{W}^1$, and $\mu_k$ is smaller in convex order than $\nu_k$. Suppose that

$$c(x, y) \leq r[1 + |x| + |y|],$$

for some $r$.

Then we have

$$\lim_{k \to \infty} \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y)\pi(dx, dy) = \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y)\pi(dx, dy).$$

We remark that Juillet has obtained in [32] the stability of the left-curtain coupling and hence stability for martingale transport for specific costs. These results are recovered as particular cases of our main result.

Guo and Oblój in [25] introduce and study the convergence of a computational method for martingale transport where the marginals are discretely approximated and the martingale constraint is allowed to fail with a vanishing error.

### 1.2. Stability of optimal weak transport

Gozlan et al. [23] proposed the following non-linear generalization of (OT). Given a cost function $C : X \times \mathcal{P}(Y) \to \mathbb{R}$ the optimal weak transport problem is

$$\inf_{\pi \in \Pi_M(\mu, \nu)} \int_X C(x, \pi)_\mu(dx),$$

(OWT)

The multidimensional version of the martingale transport problem is defined analogously, although the mathematical finance application is less clear.

Note that the updated version [26] (listed on arxiv.org on April 8th 2019) shows in Proposition 4.7 that the optimal value of (MOT) is continuous wrt $(\mu, \nu) \in \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$ provided that $\mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$ is equipped with $\mathcal{W}_2$-convergence and $c$ is assumed to be Lipschitz continuous.

The authors of the present article decided to post this article on arxiv.org concurrently to emphasize the independence of our work. We also note that the main focus of [25] [26] lies on the numerics of martingale transport. In contrast to Theorem [1.1] the proof of [25 Proposition 4.7] is based on the dual problem rather than stability of the optimal couplings.
where \{\pi_x\}_x denotes a regular disintegration of the second coordinate given the first one. Observe that one may consider cost functions of the form

\[
C_M(x, p) := \begin{cases} 
\int_{\mathbb{R}} c(x, y)p(dy) & \text{if } y = x, \\
+\infty & \text{else},
\end{cases}
\]

and in this way (MOT) is a special case of (OWT).

While the original motivation for (OWT) mainly stems from applications to geometric inequalities (cf. Marton [34, 33] and Talagrand [36, 37]), weak transport problems appear also in a number of further topics, including martingale transport [2, 4, 16, 8, 9], the causal transport problem [5, 1], and stability in mathematical finance [6]. In fact, recently some works have considered non-linear martingale transport problems for cost functional as in transport problem [5, 1], and stability in mathematical finance [6]. In fact, recently some

Theorem 1.3 (OWT Stability). Let \(C, C_k : X \times \mathcal{P}_r(Y) \to [0, \infty), k \in \mathbb{N}\), be continuous cost functions such that

(a) \(C(x, \cdot) : x \in X\) is an equicontinuous family of convex functions,

(b) \(C_k\) converges uniformly to \(C\).

Let \(\mu_k \subseteq \mathcal{P}(X)\) and \((\nu_k)_k \subseteq \mathcal{P}_r(Y)\) which converge respectively weakly to \(\mu\) and in \(\mathcal{W}_r\) to \(\nu\). Let \(\pi^k \in \Pi(\mu_k, \nu_k)\) be for each \(k\) an optimizer of the weak optimal transport problem (OWT) with cost function \(C_k\). If

\[
\lim inf_k \int_X C_k(x, \pi^k_y)\mu_k(dx) < \infty,
\]

then any accumulation point of \{\pi^k\}_k is an optimizer of (OWT) for the cost function \(C\). In particular if the latter has a unique optimizer \(\pi\), then \(\pi^k \to \pi\) weakly.

We now describe the main idea used in the proofs of Theorems 1.1 and 1.3.

1.3. Monotonicity and the correct topology on the set of couplings. The article [9] investigates the optimal weak transport problem by essentially enlarging the original state space \(X \times Y\) to \(X \times \mathcal{P}(Y)\). We briefly review this idea since it points to the right notion of monotonicity which will be useful in proving the above stability results.

First we introduce the embedding map

\[
J : \mathcal{P}(X \times Y) \to \mathcal{P}(X \times \mathcal{P}(Y)),
\]

\[
\pi \mapsto \delta_x(dp)\text{proj}_Y(\pi)(dx).
\]

(1.1)

Despite the fact that this map is seldom continuous, it does enjoy a key property: it preserves the relative compactness of a set. As a consequence, one can easily obtain optimizers for the following suitable extension of (OWT) as soon as \(C\) is lower semicontinuous:

\[
\inf_{P \in \Lambda(\mu, \nu)} \int_{X \times \mathcal{P}(Y)} C(x, p)P(dx, dp),
\]

(OWT')

where \(\Lambda(\mu, \nu)\) is the set of couplings \(P \in \mathcal{P}(X \times \mathcal{P}(Y))\) with \(X\)-marginal \(\mu\) and with the property that the mean of the \(\mathcal{P}(Y)\)-marginal equals \(\nu\). When \(C(x, \cdot)\) is furthermore convex, the extended problem (OWT) is equivalent to the original one, and in addition, (OWT) can be shown to admit an optimizer by means of the natural projection operator from \(P(X \times \mathcal{P}(Y))\) onto \(\mathcal{P}(X \times Y)\).

This idea of using an embedding (which preserves relative compactness) into a larger space can be appreciated in the following terms: On the original space \(\mathcal{P}(X \times Y)\) we consider the initial topology of \(J\) when the target space is given the weak topology. This initial topology has been studied in [7, 6] and given the name adapted weak topology. One
Definition 1.4 (C-monotonicity). A coupling \( \pi \in \Pi(\mu, \nu) \) is C-monotone iff there exists a \( \mu \)-full set \( \Gamma \subseteq X \) such that for any finite number of points \( x_1, \ldots, x_N \in \Gamma \) and \( q_1, \ldots, q_N \in \mathcal{P}(Y) \) with \( \sum_{i=1}^{N} \pi(x_i) = \sum_{i=1}^{N} q_i \), we have

\[
\sum_{i=1}^{N} C(x_i, \pi(x_i)) \leq \sum_{i=1}^{N} C(x_i, q_i).
\]

It was shown under mild assumptions in [9] that optimality of \( \pi \) for \( \text{OWT} \) implies C-monotonicity in the sense of Definition 1.4 above. The reverse implication was shown to be true under the additional assumption that the cost function (sufficiency) is uniformly \( \mathcal{W}_1 \)-Lipschitz. In the present article we will generalize this result (and largely simplify the arguments) in Theorem 2.2. Once we are equipped with this necessary and sufficient criterion for optimality, the stability result Theorem 1.3 becomes a consequence of the fact that the notion of C-monotonicity is itself stable.

Although martingale optimal transport is a particular case of optimal weak transport, in this work we treat the two problems separately. The reason is twofold. On the one hand, for martingale optimal transport we will employ some arguments which at the moment only work in dimension one. On the other hand, we will also need to refine the notion of C-monotonicity when it comes to martingale couplings. We define

Definition 1.5 (Martingale C-monotonicity). A coupling \( \Pi_M(\mu, \nu) \) is martingale C-monotone iff there exists a \( \mu \)-full set \( \Gamma \subseteq \mathbb{R}^d \) such that for any finite number of points \( x_1, \ldots, x_N \in \Gamma \) and \( q_1, \ldots, q_N \in \mathcal{P}_1(\mathbb{R}^d) \) with \( \sum_{i=1}^{N} \pi(x_i) = \sum_{i=1}^{N} q_i \) and \( \int_{\mathbb{R}^d} yq_i(\text{d}y) = x_i \), we have

\[
\sum_{i=1}^{N} C(x_i, \pi(x_i)) \leq \sum_{i=1}^{N} C(x_i, q_i).
\]

The key to proving Theorem 1.1 boils down to two arguments: that martingale C-monotonicity is sufficient for optimality, and that this notion of monotonicity is itself stable.

2. On the weak transport problem

Let us complement Definition 1.4 with a more complete list of monotonicity properties:

Definition 2.1.

1. We call \( \Gamma \subseteq X \times \mathcal{P}(Y) \) C-monotone iff for any finite number of points

\[(x_1, p_1), \ldots, (x_N, p_N) \in \Gamma \text{ and } q_1, \ldots, q_N \in \mathcal{P}(Y) \text{ with } \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i,
\]

we have

\[
\sum_{i=1}^{N} C(x_i, p_i) \leq \sum_{i=1}^{N} C(x_i, q_i).
\]

2. A probability measure \( P \in \mathcal{P}(X \times \mathcal{P}(Y)) \), which is concentrated on a C-monotone set, is called C-monotone.

It was shown under mild assumptions in [9] that optimality of \( \pi \) for \( \text{OWT} \) implies C-monotonicity in the sense of Definition 1.4. The reverse implication is true under the additional assumption that the cost function is uniformly \( \mathcal{W}_1 \)-Lipschitz. The next theorem greatly extends this result. We recall that \( \mathcal{P}_r(Y) \) denotes the space of probability measures on \( Y \) with finite \( r \)-th moment, i.e., \( p \in \mathcal{P}_r(Y) \) iff \( p \in \mathcal{P}(Y) \) and

\[
\int_Y d(y, y_0)^r p(\text{d}y) < \infty
\]
Defining by standard separability arguments, we find then the law of large numbers implies almost surely

\[ W_r(p, q) = \inf_{\pi \in \Pi(p, q)} \int_{Y \times Y} d(y_1, y_2) \pi(dy_1, dy_2). \]

Recall the definition of \( \Lambda(\mu, \nu) \) given after (OWT).

**Theorem 2.2.** Let \( \mu \in \mathcal{P}(X), \nu \in \mathcal{P}_r(Y), C : X \times \mathcal{P}_r(Y) \to \mathbb{R} \) measurable. Assume either of the following conditions:

(a) \( C(x, \cdot) \) is \( W_r \)-uniformly continuous uniformly in \( x \in X \), i.e. there is a modulus of continuity \( \theta : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[ |C(x, p) - C(x, q)| \leq \theta(W_r(p, q)) \quad \forall x \in X. \]

(b) \( C \) is jointly continuous and there is \( K \in \mathbb{R} \) and \( (\chi_n)_{n \in \mathbb{N}} \) jointly continuous such that \( \Lambda(\mu, \nu) \) is optimal for (OWT). Similarly, if \( p \mapsto C(x, p) \) is convex, then \( \pi \in \Pi(\mu, \nu) \) is optimal for (OWT) if \( \pi \) is C-monotone.

**Proof.** Let \( P \) be C-monotone with C-monotone set \( \Gamma \). Fix \( P' \in \Lambda(\mu, \nu) \). We argue as in [1] for classical (linear) optimal transport. Take any iid sequences \((X_n)_{n \in \mathbb{N}}\) of \( X \)-valued random variables, and any iid sequences \((Y_n)_{n \in \mathbb{N}}\) of \( Y \)-valued random variables, on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with

\[ (X_n, Y_n) \sim P, \quad (X_n, Z_n) \sim P'. \]

In particular by the law of large numbers, we find \( \mathbb{P} \)-almost surely

\[ \int C(x, p)P(dx, dp) - \int C(x, p)P(dx, dp) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} C(X_n, Z_n) - C(X_n, Y_n). \]

Note that for any function \( g \in C(Y) \), which is majorized by \( y \mapsto d(y, y_0)^r \), we have

\[ \mathbb{E}[Y_n(g)] = v(g). \]

Then the law of large numbers implies almost surely

\[ \frac{1}{N} \sum_{n=1}^{N} Y_n(g) = v(g) = \frac{1}{N} \sum_{n=1}^{N} Z_n(g). \]

By standard separability arguments, we find \( \mathbb{P} \)-almost surely

\[ \frac{1}{N} \sum_{n=1}^{N} Y_n = v = \frac{1}{N} \sum_{n=1}^{N} Z_n \quad \mathbb{P} \text{-a.s.} \]

where convergence holds in \( W_r \). Let \( \omega \in \Omega \) be in a \( \mathbb{P} \)-full set s.t.

\[ \lim_{N \to \infty} \mathcal{W}_r \left( \frac{1}{N} \sum_{n=1}^{N} Y_n(\omega), \frac{1}{N} \sum_{n=1}^{N} Z_n(\omega) \right) = 0, \]

and \((X_n, Y_n(\omega))_{n \in \mathbb{N}} \) for all \( n \in \mathbb{N} \). From now on we omit the \( \omega \) argument. For each \( N \in \mathbb{N} \), we denote the \( W_r \)-optimal coupling in \( \Pi(\frac{1}{N} \sum_{n=1}^{N} Z_n, \frac{1}{N} \sum_{n=1}^{N} Y_n) \) by \( \chi^N \). We denote by \( \{\chi^N_z\}_{z \in \Gamma} \) a regular disintegration of \( \chi \) given its projection in the first coordinate (marginal). Defining \( \sum_{n=1}^{N} \chi^N Z_n(dy) := \int_{\mathcal{Z}} Z_n(dz)\chi^N_z(dy) \), we find

\[ \frac{1}{N} \sum_{n=1}^{N} \mathcal{W}_r(\chi^N Z_n, Z_n)^r \leq \frac{1}{N} \sum_{n=1}^{N} \int_{\mathcal{Z}} d(y, z)^r \chi^N_z(dy) Z_n(dz) = \mathcal{W}_r \left( \frac{1}{N} \sum_{n=1}^{N} Y_n, \frac{1}{N} \sum_{n=1}^{N} Z_n \right)^r. \]

Moreover, \( \sum_{n=1}^{N} \chi^N Z_n = \sum_{n=1}^{N} Y_n \), so by C-monotonicity

\[ \frac{1}{N} \sum_{n=1}^{N} C(Y_n, \chi^N Z_n) - C(X_n, Y_n) \geq 0. \]
Without loss of generality, we can assume that the modulus of continuity of $C$, here denoted by $\theta: \mathbb{R}^+ \to \mathbb{R}^+$, is concave and increasing. Hence, by Jensen’s inequality,

$$
\frac{1}{N} \sum_{i=1}^{N} \vartheta(W_i(\chi^n Z_n, Z_n)) \leq \vartheta \left( \frac{1}{N} \sum_{i=1}^{N} W_i(\chi^n Z_n, Z_n) \right)
$$

$$
\leq \vartheta \left( \int \frac{1}{N} \sum_{n=1}^{N} W_i(\chi^n Z_n, Z_n) \right)
$$

$$
\leq \vartheta \left( \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} Y_n, \frac{1}{N} \sum_{i=1}^{N} Z_n \right) \right).
$$

which vanishes as $N \to +\infty$. Using $C$-monotonicity of $P$ and uniform continuity, we obtain $\mathbb{P}$-almost surely

$$
\frac{1}{N} \sum_{n=1}^{N} C(X_n, Z_n) - C(X_n, Y_n) = \frac{1}{N} \sum_{n=1}^{N} C(X_n, Z_n) - C(X_n, \chi^n Z_n) + \frac{1}{N} \sum_{n=1}^{N} C(X_n, \chi^n Z_n) - C(X_n, Y_n)
$$

$$
\geq -\frac{1}{N} \sum_{n=1}^{N} \vartheta(W_i(Z_n, \chi^n Z_n))
$$

$$
\geq -\vartheta \left( \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} Y_n(\omega), \frac{1}{N} \sum_{i=1}^{N} Z_n(\omega) \right) \right) \to 0.
$$

Hence, $\mathbb{P}$-almost surely

$$
\int C(x, p)P'(dx, dp) - \int C(x, p)P(dx, dp) \geq \lim \inf_{N} \frac{1}{N} \sum_{n=1}^{N} C(X_n, Z_n) - C(X_n, Y_n) \geq 0.
$$

2.1. **Stability of C-Monotonicity.** Recall the embedding of (1.1)

$$
J: \mathcal{P}(X \times Y) \to \mathcal{P}(X \times \mathcal{P}(Y)),
$$

$$
\pi \mapsto \delta_{\pi_s}(dp) \text{proj}_X(\pi)(dx).
$$

The intensity $I(Q)$ of some measure $Q \in \mathcal{P}(\mathcal{P}(Y))$ is uniquely defined as the probability measure $I(Q) \in \mathcal{P}(Y)$ with

$$
I(Q)(f) = \int_{\mathcal{P}(Y)} \int_{Y} f(y)p(dy)Q(dp) \quad \forall f \in C_0(Y).
$$

**Remark 2.3.** In the light of this embedding it appears to be natural to consider $C$-monotonicity on the enhanced space $X \times \mathcal{P}(Y)$.

(a) If $\pi \in \Pi(\mu, \nu)$ is $C$-monotone then $J(\pi)$ is $C$-monotone: Indeed, due to $C$-monotonicity of $\pi$ it is possible to find a measurable set $\Gamma \subseteq X$ such that $\mu(\Gamma) = 1$ and define the $C$-monotone set

$$
\Gamma = \{(x, p) \in \hat{\Gamma} \times \mathcal{P}(Y) : p = \pi_x \}.
$$

Therefore, equivalently to Definition 2.1, we can demand that there exists a $C$-monotone set $\Gamma \subseteq X \times \mathcal{P}(Y)$ such that $(x, \pi_x) \in \Gamma$ for $\mu$-almost every $x \in X$.

(b) If $\Gamma \subseteq X \times \mathcal{P}(Y)$ is $C$-monotone, and $C: X \times \mathcal{P}(Y) \to \mathbb{R}$ is convex in the second argument, i.e.,

$$
\forall x \in X : p \mapsto C(x, p) \text{ is convex},
$$

Then the enlarged set

$$
\hat{\Gamma} := \Gamma \cup \left\{ \left( x, \frac{1}{k} \sum_{i=1}^{k} p_i \right) : x \in X, (x, p_i) \in \Gamma, i = 1, \ldots, k \in \mathbb{N} \right\}
$$
is also $C$-monotone. If $C(x, \cdot)$ is continuous for all $x \in X$, then

$$\hat{\Gamma} := \left\{ (x, p) \in X \times \mathcal{P}(Y) : x \in X, p \in \mathcal{C}(\Gamma_x) \right\},$$

is also $C$-monotone.

(c) We observe that the set of probability measures $\Lambda(\mu, \nu)$ can be characterized by a family of continuous functions $\mathcal{F} \subseteq C(X \times \mathcal{P}_1(Y))$: $P \in \Lambda(\mu, \nu)$ if and only if

$$\int_{X \times \mathcal{P}(Y)} f(x)P(dx, dp) = \int_X f(x)\mu(dx), \quad \forall f \in C_b(X),$$

$$\int_{X \times \mathcal{P}(X)} \int_Y g(y)P(dy, dx) = \int_Y g(y)\nu(dy), \quad \forall g \in C_b(Y).$$

As a further observation we have the equivalence of $C$-monotonicity as in Definition 2.1 and $C$-finite optimality under the linear constraints $\mathcal{F}$

$$\mathcal{F} = \{ f \in C_b(X \times Y) : \exists g \in C_b(X), \ h \in C_b(Y) \ s.t. f(x, y) \equiv g(x) \ or \ f(x, y) = h(y) \},$$

which was introduced in [13] Definition 1.2.

Theorem 2.4. Let $C : X \times \mathcal{P}_1(Y) \to \mathbb{R}$ be measurable and $P^* \in \mathcal{P}_1(X \times \mathcal{P}_1(Y))$ optimal for $(\text{OWT})$ with finite value. Then $P^*$ is $C$-monotone. Especially, if $C$ is lower bounded and satisfies for all $x \in X$ and $Q \in \mathcal{P}(\mathcal{P}(Y))$

$$C(x, I(Q)) \leq \int_{\mathcal{P}(Y)} C(x, p)Q(dp),$$

then any optimizer $\pi^*$ of $(\text{OWT})$ with finite value is $C$-monotone.

Proof. The first assertion is a consequence of Remark 2.3(c) and [13] Theorem 1.4. To show the second assertion, let $P \in \Lambda(\mu, \nu)$. Then $I(P_2)\mu(dx) \in \Pi(\mu, \nu)$ and by (2.1)

$$\int_{X \times \mathcal{P}(Y)} C(x, p)P(dx, dp) \geq \int_{X \times \mathcal{P}(Y)} C(x, I(P_2))\mu(dx).$$

Hence, $J(\pi^*)$ is optimal for $(\text{OWT})$. By the previously shown we deduce $C$-monotonicity.

The assumption that $C$ is lower bounded is as a matter of fact not necessary to deduce $C$-monotonicity in the classical optimal weak transport setting, cf. [9] Theorem 5.2. Note that (2.1) holds when $C(x, \cdot)$ is lower semicontinuous and convex.

Lemma 2.5. Let $P_i, m_i \in \mathcal{P}_1(Y)$, $i = 1, \ldots, N$, with $\sum_{i=1}^N P_i = \sum_{i=1}^N m_i$, and $(p_i^k)_{k \in \mathbb{N}}$ be a sequence on $\mathcal{P}_1(Y)$ such that

$$p_i^k \to p_i \text{ in } \mathcal{W}_r.$$

Then there exist approximative sequences $(m_i^k)_{k}$ of competitors, i.e.,

$$\sum_{i=1}^k p_i^k = \sum_{i=1}^k m_i^k, \quad m_i^k \to m_i \text{ in } \mathcal{W}_r.$$

Proof. Since $\sum_{i=1}^N m_i = \sum_{i=1}^N P_i$, we find sub-probability measures $m_{i,j}$ with

$$m_j = \sum_{i=1}^N m_{i,j}, \quad p_i = \sum_{j=1}^N m_{i,j}, \quad \text{and } m_{i,j} \leq p_i \wedge m_j.$$

Denote by $(\chi_{i,j})_{i,j \in [1,N]}$ a regular disintegration of a $\mathcal{W}_r$-optimal transport plan $\chi_{i,j} \in \Pi(p_i, p_j)$ wrt. its first marginal $p_i$. Let $i, j \in \{1, \ldots, N\}$ and define

$$m_{i,j}^k(dy) := \int_Y \chi_{i,j}^k(dy)m_{i,j}(dz), \quad m_i^k := \sum_{j=1}^N m_{i,j}^k.$$
Since
\[ \sum_j W_j(m_{i,j},m_{i,j})' \leq \int \sum_j d(z,y)\chi_z^k(dy)m_{i,j}(dz) \leq \int d(z,y)\chi_z^k(dy)p_i(dz) = W_i(p_i',p_i'), \]
we deduce the convergence of \( m_{i,j}^k \) to \( m_{i,j} \), and in consequence, the convergence of \( m_i^k \) to \( m_i \). Finally observe that
\[ \sum_i m_i^k = \sum_i \sum_j m_{i,j}^k = \sum_i \int_Y \chi_z^k(dy)p_i(dz) = \sum_i p_i^k, \]
so indeed \( (m_i^k)_{i=1}^\infty \) are feasible competitors of \( (p_i^k)_{i=1}^\infty \) and converge to \( (m_i)_{i=1}^\infty \) in \( \mathcal{W}_r \). \( \Box \)

**Lemma 2.6.** Let \( C \in C(X \times \mathcal{P}_r(Y)) \), \( \varepsilon \geq 0 \), and \( N \in \mathbb{N} \). Then the set
\[ \Gamma_N^\varepsilon := \left\{ (x_i, p_i)_{i=1}^N \in (X \times \mathcal{P}_r(Y))^N \mid m_1, \ldots, m_N \in \mathcal{P}_r(Y) \text{ s.t. } \sum_{i=1}^N p_i = \sum_{i=1}^N m_i, \right. \]
\[ \left. \text{we have } \sum_{i=1}^N C(x_i, p_i) \leq \sum_{i=1}^N C(x_i, m_i) + \varepsilon \right\} \]
(2.2)
is a closed subset of \( (X \times \mathcal{P}_r(Y))^N \).

**Proof.** Take any convergent sequence \( (x_i^k, p_i^k)_{i=1}^N \in \Gamma_N^\varepsilon \), \( k \in \mathbb{N} \), such that
\[ x_i^k \to x_i \text{ in } X, \quad p_i^k \to p_i \text{ in } \mathcal{W}_r. \]
Assume that \( (x_i, m_i)_{i=1}^N \) is a competitor, i.e., \( \sum_{i=1}^N p_i = \sum_{i=1}^N m_i \). Lemma 2.5 provides an approximative sequence of competitors, and by continuity of \( C \) we conclude. \( \Box \)

The key ingredient towards stability of \( \text{(OWT)} \) is the following result concerning stability of the notion of \( C \)-monotonicity.

**Theorem 2.7.** Let \( C, C_k \in C(X \times \mathcal{P}_r(Y)) \), \( k \in \mathbb{N} \), and \( C_k \) converges uniformly to \( C \). If \( P, P^k \in \mathcal{P}_r(X \times \mathcal{P}_r(Y)) \), \( k \in \mathbb{N} \), such that
\( (a) \) for all \( k \in \mathbb{N} \) the measure \( P^k \) is \( C_k \)-monotone,
\( (b) \) the sequence \( (P^k)_{k \in \mathbb{N}} \) converges to \( P \),
then \( P \) is \( C \)-monotone. Moreover, if \( \pi, \pi^k \in \mathcal{P}_\mathcal{F}(X \times Y) \) and \( C^k \) is convex in the second argument, \( k \in \mathbb{N} \), such that
\( (a') \) for all \( k \in \mathbb{N} \) the measure \( \pi^k \) is \( C_k \)-monotone,
\( (b') \) the sequence \( (\pi^k)_{k \in \mathbb{N}} \) converges to \( \pi \),
then \( \pi \) is \( C \)-monotone.

**Proof.** The aim is to construct a \( C \)-monotone set \( \Gamma \) on which \( P \) is concentrated. So, we write \( P^{k,\otimes N} \) and \( p^{\otimes N} \) for the \( N \)-fold product measure of \( P^k \) and \( P \) where \( N \in \mathbb{N} \). By \( C_k \)-monotonicity and uniform convergence we find for any \( \varepsilon > 0 \) a natural number \( k_0 \) such that \( P^{k,\otimes N} \), \( k \geq k_0 \), is concentrated on \( \Gamma_N^\varepsilon \), see (2.2). Lemma 2.6 combined with the Portmanteau theorem yield that \( p^{\otimes N} \) is concentrated on \( \Gamma_N^\varepsilon \):
\[ 1 = \limsup_k P^{k,\otimes N}(\Gamma_N^\varepsilon) \leq P^{\otimes N}(\Gamma_N^\varepsilon) = 1. \]
As a consequence, we find that \( P^{\otimes N} \) gives full measure to the closed set \( \Gamma_N := \Gamma_N^0 \). Hence, we can cover the open set \( \Gamma_N \) by countably many sets of the form \( \bigotimes_{i=1}^N O_i \) where \( O_i \) is open in \( X \times \mathcal{P}_r(Y) \),
\[ \Gamma_N^\varepsilon = \bigcup_{k \in \mathbb{N}} \bigotimes_{i=1}^N O_{i,k}. \]
In particular, we deduce for any \( k \in \mathbb{N} \)
\[
0 = P^{\otimes N} \left( \bigotimes_{i=1}^{N} O_{i,k} \right) = \prod_{i=1}^{N} P(O_{i,k}).
\]
We find open sets \( A_N \) such that
\[
A_N := \bigcup_{k \in \mathbb{N}, c \in \{1, \ldots, N\}, P(O_{i,k}) = 0} O_{i,k}, \quad P(A_N) = 0,
\]
\[
\Gamma_N^c \subseteq \bigcup_{i=1}^{N} (X \times \mathcal{P}_Y(Y))^{i-1} \times A_N \times (X \times \mathcal{P}_Y(Y))^{N-i}.
\]
Since \( N \in \mathbb{N} \) was arbitrary we define the closed and \( C \)-monotone set
\[
\Gamma := \left( \bigcup_{N \in \mathbb{N}} A_N \right)^c, \quad P(\Gamma) = 1, \quad \Gamma^N \subseteq \Gamma_N.
\]
With the taken precautions it poses no challenge to verify that \( P \) is \( C \)-monotone on \( \Gamma \).
To show the second assertion, we embed \( \pi_k \in \mathcal{P}(X \times Y) \) into \( \mathcal{P}(X \times \mathcal{P}(Y)) \) owing to the map \( J \). Then, by compactness of \( \Lambda(\mu, \nu) \), we find an accumulation point \( P \in \mathcal{P}(X \times \mathcal{P}(Y)) \) of \( (J(\pi_k))_{k \in \mathbb{N}} \). Note that
\[
\mu(dx)I(P_x) =: \pi \in \Pi(\mu, \nu),
\]
determines a coupling, which is likewise an accumulation point of \( (\pi_k)_{k \in \mathbb{N}} \). Since \( P \) is concentrated on the \( C \)-monotone set \( \Gamma \), we find for any \( x \in X \) such that \( P_x(\Gamma_i) = 1 \) a sequence of measures \( p_i^x \in \Gamma_x \subseteq \mathcal{P}(Y) \), \( i \in \mathbb{N} \), with
\[
q_n^x := \frac{1}{n} \sum_{i=1}^{n} p_i^x \rightharpoonup I(P_x) = \pi_x, \quad n \to \infty, \quad \text{in } \mathcal{W}_r.
\]
By Remark 2.3(b) we know that \( (x, q_n^x) \) is contained in the \( C \)-monotone set \( \Gamma \). By closure of \( \Gamma \) we conclude \( (x, \pi_x) \in \Gamma \) for \( \mu \)-a.e. \( x \), and \( C \)-monotonicity of \( \pi \).

From Theorems 2.2 and 2.7 we easily deduce the following corollary, which has Theorem 1.3 in the introduction as a particular case:

**Corollary 2.8.** Let \( C, C_k \in C(X \times \mathcal{P}_Y(Y)), k \in \mathbb{N} \), be non-negative cost functions such that
(a) \( C(x, \cdot) \) is equicontinuous uniformly in \( x \in X \),
(b) \( C_k \) converges uniformly to \( C \).
Given a sequence \((\mu_k)_k\) and \((\nu_k)_k\) of probability measures on \( \mathcal{P}(X) \) and \( \mathcal{P}_Y(Y) \), respectively, where \( \mu_k \) converges weakly to \( \mu \) and \( \nu_k \) converges in \( \mathcal{W}_r \) to \( \nu \). Let \( P^k \in \Lambda(\mu_k, \nu_k) \) be optimizers of the minimization problem \( (\text{OWT})_k \) with cost function \( C_k \). If
\[
\liminf_{k} P^k(\Gamma_k) < \infty,
\]
then any accumulation point of \( \{P^k\}_k \) is an optimizer of \( (\text{OWT}) \) for the cost \( C \).

If moreover \( C_k(x, \cdot) \) and \( C(x, \cdot) \) are convex, then an analogous statement holds in the case of \( (\text{OWT}) \).

3. Stability of martingale optimal transport

In this section we consider the martingale optimal transport problem \( (\text{MOT}) \), and \( X = Y = \mathbb{R}^d \). A generalization of \( c \)-cyclical monotonicity under additional linear constraints were suggested by [13] [42], which also encompass \( (\text{MOT}) \). Here, the set of linear constraints \( \mathcal{F}_M \subseteq C(\mathbb{R}^d \times \mathbb{R}^d) \) takes the shape
\[
\mathcal{F}_M := \left\{ f(x, y) \in C(\mathbb{R}^d \times \mathbb{R}^d) : f(x, y) = g(x)(y - x) \text{ and } g \in C_b(\mathbb{R}^d) \right\}.
\]

**Definition 3.1.**
(1) A measures \( \alpha' \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) is called a \( \mathcal{F}_M \)-competitor of \( \alpha \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) iff the marginals coincide and \( \alpha(f) = \alpha'(f) \) for all \( f \in \mathcal{F}_M \).

(2) We call \( \Gamma \subseteq \mathbb{R}^d \times \mathbb{R}^d \) \((c, \mathcal{F}_M)\)-monotone iff for any probability measure \( \alpha \), finitely supported on \( \Gamma \), and any competitor \( \alpha' \) we have \( \alpha(c) \leq \alpha'(c) \).

(3) A martingale coupling \( \pi \in \Pi_M \), which is supported on a \((c, \mathcal{F}_M)\)-monotone set, is then called \((c, \mathcal{F}_M)\)-monotone.

We will see in Lemma 3.6 that under given conditions \((c, \mathcal{F}_M)\)-monotonicity of a coupling is equivalent to martingale \( C \)-monotonicity (cf. Definition 3.2).

By [13], optimizers of \((\text{MOT})\) are concentrated on \((c, \mathcal{F}_M)\)-monotone sets. If \( c \) is continuous, then the reverse implication was shown in one dimension by Beiglböck and Juillet [15] and Griessler [24], but for arbitrary dimensions \( d \in \mathbb{N} \) it remains unanswered. Even more, the question of stability of \((\text{MOT})\) — which is completely understood for \((\text{OT})\) — is to the authors knowledge completely open even in one dimension. This open problem has been recently emphasized by Alfonsi, Corbetta and Jourdain in [3, Section 5.3].

Let us regard two natural generalizations of \((\text{MOT})\):

\[
\inf_{p \in \Lambda_M(\mu, \nu)} \int_{\mathbb{R}^d} C(x, \pi_x) p(dx), \quad \text{(MOWT)}
\]

\[
\inf_{\gamma \in \mathcal{P}(\mathbb{R})} \int_{\mathbb{R}^d \times \mathcal{P}(\mathbb{R})} C(x, p) p(dx, dp), \quad \text{(MOWT')}
\]

where \( \Lambda_M(\mu, \nu) \) is the set of all \( p \in \Lambda(\mu, \nu) \) giving full measure to \( \{(x, p) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}) : x = \int_{\mathbb{R}^d} y p(dy)\} \), i.e., \( p \in \Lambda_M(\mu, \nu) \) iff \( p \in \Lambda(\mu, \nu) \) and

\[
\int_{\mathbb{R}^d \times \mathcal{P}(\mathbb{R})} f(x, p) p(dx, dp) = 0 \quad \forall f \in \mathcal{F}_M,
\]

where the set of martingale constraints \( \tilde{\mathcal{F}}_M \) is given by

\[
\tilde{\mathcal{F}}_M := \left\{ f \in C_b(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)) : \exists g \in C_b(\mathbb{P}(\mathbb{R}^d)), \ h \in C_b(\mathcal{X}) \right\}.
\]

Definition 3.2.

(1) A measures \( \alpha' \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)) \) is called a \( \tilde{\mathcal{F}}_M \)-competitor of \( \alpha \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)) \) iff \( \alpha(f) = \alpha'(f) \) for all \( f \in \mathcal{F}_M \cup \tilde{\mathcal{F}} \).

(2) We call \( \Gamma \subseteq \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \) \((C, \tilde{\mathcal{F}}_M)\)-monotone iff for any probability measure \( \alpha \), finitely supported on \( \Gamma \), and any \( \tilde{\mathcal{F}}_M \)-competitor \( \alpha' \) we have \( \alpha(C) \leq \alpha'(C) \).

(3) A probability measure \( P \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)) \), which is supported on a \((C, \tilde{\mathcal{F}}_M)\)-monotone set, is then called \((C, \tilde{\mathcal{F}}_M)\)-monotone.

Again, by [13, Theorem 1.4] we find that \((C, \tilde{\mathcal{F}}_M)\)-monotonicity is a necessary optimality criterion.

Theorem 3.3. Let \( C : \mathcal{X} \times \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R} \) be measurable and \( P^* \in \Lambda_M(\mu, \nu) \) optimal for \((\text{MOWT'})\) with finite value, then \( P^* \) is \((C, \tilde{\mathcal{F}}_M)\)-monotone. Especially, if \( C \) is lower bounded and satisfies for all \( x \in \mathcal{X} \) and \( Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d)) \)

\[
C(x, H(Q)) \leq \int_{\mathcal{P}(\mathbb{R}^d)} C(x, p) Q(dp), \quad \text{(3.1)}
\]

then any optimizer \( \pi^* \) of \((\text{MOWT'})\) with finite value is \( C \)-monotone.

As before (3.1) holds when \( C(x, \cdot) \) is lower semicontinuous and convex, and in particular for \( C(x, p) = \int c(x, y)p(dy) \) if \( c \) is lower bounded and lower semicontinuous.

Proof. Since \((\text{MOWT'})\) is an optimal transport problem under additional linear constraint, the first statement is a consequence of [13, Theorem 1.4]. To show the second assertion, we note that any martingale coupling \( \pi \in \Pi(\mu, \nu) \) naturally induces an element in \( \Lambda_M(\mu, \nu) \).
Suppose that there is a sequence monotonicity.\(\square\)

Hence, \(J(\pi^*)\) is optimal for (MOWT'). By the previously shown we deduce \((\bar{F}_M, C)\)-monotonicity. \(\square\)

From here on we assume is this part that

\[ d = 1. \]

We think that our approach can also be adapted to cover higher dimensions.

**Lemma 3.4.** Let \(N \in \mathbb{N}\) and \(p_i \in \mathcal{P}_r(\mathbb{R})\) with \(\bar{F}_M\)-competitor \(q_i \in \mathcal{P}_r(\mathbb{R})\), \(i = 1, \ldots, N\), i.e.,

\[ \sum_{i=1}^N p_i = \sum_{i=1}^N q_i. \]

Suppose that there is a sequence \((p_{i,k})_{k \in \mathbb{N}}\) of measures in \(\mathcal{P}_r(\mathbb{R})\) such that \(\lim_{k} p_{i,k} = p_i\) for \(i = 1, \ldots, N\). Then for any \(k \in \mathbb{N}\) there exist \(\bar{F}_M\)-competitors \((q_{i,k})_{i=1}^N\) of \((p_{i,k})_{i=1}^N\) such that

\[ \lim_{k} q_{i,k} = q_i. \]

Since the proof of Lemma 3.4 is slightly demanding, we first give for convenience of the reader a more concrete version of the argument in the simpler setting of \(N = 2\):

**Proof of Lemma 3.4 for \(N = 2\).** Wlog \(q_1 \neq p_1\). Applying Lemma 2.5, we find a sequence \((q_{i,k})_k\) which converges to \(q_i\). We may further assume \(\int_\mathbb{R} yq_{1,k}(dy) < \int_\mathbb{R} yp_{1,k}(dy)\). We can decompose the measures \(q_1, q_2, p_1, p_2\) into sub-probability measures \(m_{i,j}, i, j \in \{1, 2\}\) such that

\[ p_i = m_{i,1} + m_{i,2}, \quad q_j = m_{1,j} + m_{2,j}. \]

By equality of the mean values of \(q_1\) and \(p_1\), we find that

\[ \int_\mathbb{R} ym_{1,2}(dy) = \int_\mathbb{R} ym_{2,1}(dy). \]

Thus, we find disjoint, open intervals \(I_1, I_2\) with \(\min(m_{1,2}(I_1), m_{2,1}(I_2)) > 0\) and \(\sup(I_2) < \inf(I_1)\). Similarly, we can decompose \(q_{1,k}, q_{2,k}, p_{1,k}, p_{2,k}\) in the same manner and obtain by the construction in Lemma 2.5 that \(m_{i,k}^{l}\) converges to \(m_{i,j}\) in \(\mathcal{W}_\epsilon\). Denote by \(\alpha_k > 0\) the constant such that

\[ \int_\mathbb{R} yq_k^l(dy) + \alpha_k \left( \frac{1}{m_{1,2}^l(I_1)} \int_{I_1} ym_k^l(dy) - \frac{1}{m_{2,1}^l(I_2)} \int_{I_2} ym_k^l(dy) \right) = \int_\mathbb{R} yp_k^l(dy). \]

By \(\mathcal{W}_1\)-convergence, we have on the one hand \(\lim_{k} m_{1,2}^l(I_1) > 0\) and \(\lim_{k} m_{2,1}^l(I_2) > 0\), and on the other,

\[ \lim_{k} \int_\mathbb{R} yq_k^l(dy) - \int_\mathbb{R} yp_k^l(dy) = 0, \]

implying that \(\alpha_k \to 0\). Therefore, there is an index \(k_0 \in \mathbb{N}\) such that

\[ q_k^1 = q_k^1 + \alpha_k \left( \frac{m_{1,2}^l(I_1)}{m_{1,2}^l(I_1) - m_{2,1}^l(I_2)} \right), \quad q_k^2 = q_k^2 - \alpha_k \left( \frac{m_{1,2}^l(I_1)}{m_{1,2}^l(I_1) - m_{2,1}^l(I_2)} \right), \]

are both probability measures for \(k \geq k_0\). Then \((q_k^1)_{k \geq k_0}, (q_k^2)_{k \geq k_0}\) are the desired sequences. \(\square\)
Proof of Lemma 3.3. Let \( d \in \mathbb{R} \) and define
\[
I_d^1 := \{ i \in \mathbb{N} : F_{p_i}(d) = 1 \}, \quad I_d^2 := \{ i \in \mathbb{N} : F_{q_i}(d) = 1 \}.
\]
Clearly, if there exists \( d \in \mathbb{R} \cup \{+\infty\} \) such that
\[
i \in (I_d^1)^c \implies p_i((-\infty, d)) = 0, \quad i \in (I_d^2)^c \implies q_i((-\infty, d)) = 0,
\]
\[
\sum_{i \in I_d^1} p_i = \sum_{i \in I_d^2} q_i,
\]
then \( I_d^1 = I_d^2 \). In this case we can split the problem into two parts: Finding sequences of competitors for the index sets \( I_d^1 \) and \( \{1, \ldots, N\} \setminus I_d^1 \). It is sufficient to show the existence of such a sequence for the sub problem \( I_d^1 \), where we also assume that \( d \) is minimal.

Thus, assume without loss of generality that \( d \) is minimal with \( d = +\infty \), and \( I_d^1 = I_d^2 = \{1, \ldots, N\}, \ N > 1 \). Applying Lemma 2.5 we find a sequence \( (q_i, k)_k \) which converges to \( q_i \) in \( W_r \), in particular,
\[
\lim_{k} \left| \int_{\mathbb{R}} y q_k^i(dy) - \int_{\mathbb{R}} y q_i(dy) \right| = 0.
\]
The convex hull of the support of \( q_i \) is denoted by
\[
S_i := \text{co}(\text{supp } q_i), \quad i = 1, \ldots, N.
\]
If \( \lambda(S_j \cap S_i) > 0 \) (\( \lambda \) denotes the Lebesgue measure) and \( S_j \subseteq \text{int}(S_i) \) then there are open intervals \( O_{i,t}^+, O_{i,t}^-, O_{j,t}^+, O_{j,t}^- \) such that
\[
q_i(O_{i,t}^+) > 0, \quad q_j(O_{j,t}^+) > 0, \quad t \in \{-, +\},
\]
\[
sup(O_{i,t}^-) < \inf(O_{j,t}^-), \quad sup(O_{j,t}^+) < \inf(O_{i,t}^+).
\]
By weak convergence of \( q_k^i \) to \( q_i \), the Portmanteau theorem implies \( \lim_{k} q_k^i(O_{i,t}^+) > 0 \) and \( \lim_{k} q_k^j(O_{j,t}^+) > 0 \). In particular, when \( k \) is sufficiently large it is possible to move mass between \( q_k^i \) and \( q_k^j \) and therefore, either adjust the barycenter of \( q_k^i \) or the one of \( q_k^j \) slightly. Our procedure starts with the smallest \( S_i \) wrt the following order:
\[
A, B \subseteq \mathbb{R}, \quad A \leq B \iff (\inf A, \sup A) \subseteq \text{int}(B, \sup B).
\]
By minimality of \( d \) we have \( \lambda(S_j) > 0 \).

Assume \( i \in \{1, \ldots, N\} \) is given, such that for all \( j \in \{1, \ldots, N\} \) with \( S_j \subseteq S_i \) the barycenters of \( q_k^i \) are already correct (when \( k \) is sufficiently large). For all \( j \neq i \) such that \( S_j \subseteq S_i \) and
\[
S_j \subseteq S_i : \lambda(S_j \cap S_j) > 0 \lor S_j \subseteq \text{int}(S_i),
\]
and \( k \) sufficiently large, we adequately adjust the barycenter of \( q_k^i \) possibly at the expense of the barycenter of \( q_k^j \). Now there are either two cases:

1. (3.3) held true for all \( j \neq i \) with \( S_j \subseteq S_i \), which would imply that (given \( k \) is sufficiently large), we have not only corrected the barycenter of \( q_k^i \) but also the one of \( q_k^j \), since
\[
\int_{\mathbb{R}} y q_k^i(dy) = \sum_{j=1}^{N} \int_{\mathbb{R}} y p_k^j(dy) - \sum_{j=1}^{N} \int_{\mathbb{R}} y q_k^j(dy) = \int_{\mathbb{R}} y q_k^i(dy).
\]
Hence, we have found the desired sequences.

2. Let \( S_i \) be the smallest interval wrt the order introduced in (3.2) such that \( S_j \geq S_i \) and (3.3) fails. By minimality of \( d \) we deduce \( \lambda(S_j \cap S_i) > 0 \), which allows now to adjust the barycenter of \( q_k^i \) at the expense of \( q_k^j \) (if \( k \) is sufficiently large).

We can repeat the reasoning with the thus found index \( l \). Since there are only finitely many elements, the procedure terminates.

The key ingredient of this part is the following stability result concerning the notion of \((C, F_M)\)-monotonicity:
Theorem 3.5. Let \( C, C_k \in C(\mathbb{R} \times \mathcal{P}_r(\mathbb{R})), k \in \mathbb{N} \), and \( C_k \) converges uniformly to \( C \). If \( P, P^k \in \mathcal{P}_r(\mathbb{R} \times \mathcal{P}_r(\mathbb{R})), k \in \mathbb{N} \), such that

(a) for all \( k \in \mathbb{N} \) the measure \( P^k \) is \((C_k, \tilde{F}_M)\)-monotone,
(b) the sequence \((P^k)_{k \in \mathbb{N}}\) converges to \( P \),

then \( P \) is \((C, \tilde{F}_M)\)-monotone. Moreover, if \( \pi, \pi^k \in \mathcal{P}_r(\mathbb{R} \times \mathbb{R}) \) and \( C \) is convex in the second argument, \( k \in \mathbb{N} \), such that

(a') for all \( k \in \mathbb{N} \) the measure \( \pi^k \) is \((C_k, \tilde{F}_M)\)-monotone,
(b') the sequence \((\pi^k)_{k \in \mathbb{N}}\) converges to \( \pi \),

then \( \pi \) is \((C, \tilde{F}_M)\)-monotone.

Proof. The proof runs parallel to the one of Theorem 2.7. Using Lemma 3.4 we can alter Lemma 2.6 such that for all \( \varepsilon \geq 0 \) and \( N \in \mathbb{N} \) the set

\[
\tilde{\Gamma}_N^\varepsilon := \left\{ (x_i, p_i)_{i=1}^N \in (\mathbb{R} \times \mathcal{P}_r(\mathbb{R}))^N \mid \forall m_1, \ldots, m_N \in \mathcal{P}_r(\mathbb{R}) \text{ s.t. } \sum_{i=1}^N p_i = \sum_{i=1}^N m_i \text{ and } \int_\mathbb{R} y p_i(dy) = \int_\mathbb{R} y p_i(dy) \right\}
\]

is closed. The aim is to construct a \((C, \tilde{F}_M)\)-monotone set \( \tilde{\Gamma} \) on which \( P \) is concentrated. So, we write \( P^k, \tilde{P} \) and \( P^N, \tilde{P}^N \) for the \( N \)-fold product measure of \( P^k \) and \( P \) where \( N \in \mathbb{N} \). By \((C_k, \tilde{F}_M)\)-monotonicity and uniform convergence we find for any \( \varepsilon > 0 \) a natural number \( k_0 \) such that \( P^k, k = k_0 \), is concentrated on \( \tilde{\Gamma}_N^\varepsilon \). By closure of \( \tilde{\Gamma}_N^\varepsilon \), the Portmanteau theorem yield that \( \tilde{P}^N \) is concentrated on \( \tilde{\Gamma}_N^\varepsilon \):

\[
1 = \limsup_k P^{k, \otimes \varepsilon}_N(\tilde{\Gamma}_N^\varepsilon) \leq P^N(\tilde{\Gamma}_N^\varepsilon) = 1.
\]

As a consequence, we find that \( P^N \) gives full measure to the closed set \( \tilde{\Gamma}_N := \tilde{\Gamma}_N^0 \). Hence, we can cover the open set \( \tilde{\Gamma}_N^\varepsilon \) by countably many sets of the form \( \bigotimes_{i=1}^N O_i \), where \( O_i \) is open in \( \mathbb{R} \times \mathcal{P}_r(\mathbb{R}) \).

\[
\tilde{\Gamma}_N^\varepsilon = \bigcup_{k \in \mathbb{N}} \bigotimes_{i=1}^N O_{i,k}.
\]

In particular, we deduce for any \( k \in \mathbb{N} \)

\[
0 = P^N\left( \bigotimes_{i=1}^N O_{i,k} \right) = \prod_{i=1}^N P(O_{i,k}).
\]

We find open sets \( A_N \) such that

\[
A_N := \bigcup_{k \in \mathbb{N}, i \in [1, \ldots, N]} O_{i,k}, \quad P(A_N) = 0,
\]

\[
\tilde{\Gamma}_N^\varepsilon \subseteq \bigcup_{i=1}^N (X \times \mathcal{P}_r(Y))^{i-1} \times A_N \times (X \times \mathcal{P}_r(Y))^{N-i}.
\]

Since \( N \in \mathbb{N} \) was arbitrary we define the closed and \((C, \tilde{F}_M)\)-monotone set

\[
\tilde{\Gamma} = \left( \bigcup_{N \in \mathbb{N}} A_N \right)^\varepsilon, \quad P(\tilde{\Gamma}) = 1, \quad \tilde{\Gamma}^\varepsilon \subseteq \tilde{\Gamma}.
\]

With the taken precautions it poses no challenge to verify that \( P \) is \((C, \tilde{F}_M)\)-monotone on \( \tilde{\Gamma} \).

To show the second assertion, we embed \( \pi_k \in \mathcal{P}(\mathbb{R} \times \mathbb{R}) \) into \( \mathcal{P}(\mathbb{R} \times \mathcal{P}(\mathbb{R})) \) owing to the map \( J \). Then, by compactness of \( \Lambda_M(\mu, \nu) \), we find an accumulation point \( P \in \mathcal{P}(\mathbb{R} \times \mathcal{P}(\mathbb{R})) \) of \((J(\pi_k))_{k \in \mathbb{N}}\). Note that \( P \) gives full measure to \( \{(x, p) \in \mathbb{R} \times \mathcal{P}_r(\mathbb{R}) : x = \int_{\mathbb{R}} y p(dy)\} \), and

\[
\mu(dx) I(P_x) =: \pi \in \Pi_M(\mu, \nu)
\]
determines a martingale coupling, which is likewise an accumulation point of \((\pi^k)_{k \in \mathbb{N}}\).

Since \(P\) is concentrated on the \((C, \tilde{F}_M)\)-monotone set \(\Gamma\), we find for any \(x \in X\) such that 
\[P_\lambda(\Gamma_x) = 1\] 
a sequence of measures \(p^i_n \in \Gamma_x \subseteq \mathcal{P}_1(\mathbb{R})\), \(i \in \mathbb{N}\), with
\[q^i_n := \frac{1}{n} \sum_{i=1}^{n} p^i_n \rightarrow I(p) = \pi, \quad n \rightarrow \infty, \text{ in } \mathcal{W}_r.\]

By convexity of \(C\), we find that \((x, q^i_n)\) is contained in the \((C, \tilde{F}_M)\)-monotone set \(\hat{\Gamma}\). By closure of \(\hat{\Gamma}\) we conclude \((x, \pi) \in \hat{\Gamma}\) for \(\mu\)-a.e. \(x\), and \((C, \tilde{F}_M)\)-monotonicity of \(\pi\).

\(\square\)

**Lemma 3.6.** Let \(c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\), \(c(x, \cdot)\) be upper semicontinuous for all \(x \in \mathbb{R}\), and \(\pi \in \Pi_M(\mu, \nu)\). Then \(\pi\) is \((c, \tilde{F}_M)\)-monotone if and only if \(\pi\) is \((C, \tilde{F}_M)\)-monotone (with \(C(x, p) := \int_{\mathbb{R}} c(x, y)p(dy)\)).

**Proof.** Let \(\hat{\Gamma} \subseteq \mathbb{R} \times \mathbb{R}\) be \((c, \tilde{F}_M)\)-monotone. Consider the set
\[\Gamma = \{(x, p) \in X \times \mathcal{P}(Y) : x \in \text{proj}_1(\hat{\Gamma}), p(\hat{\Gamma}_x) = 1, p \in \mathcal{P}_1(\mathbb{R}), c(x, \cdot) \in L^1(p)\}.

Take any sequence \((x_1, p_1), \ldots, (x_N, p_N) \in \Gamma\) with competitors \(q_1, \ldots, q_N \in \mathcal{P}_1(\mathbb{R})\), i.e.,
\[\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i, \quad \int_{\mathbb{R}} y p_i(dy) = \int_{\mathbb{R}} y q_i(dy).

We find for any \((x_i, p_i) \in \Gamma\), a sequence of finitely supported measures \((p^i_k)_{k \in \mathbb{N}}\) where the support of \(p^i_k\) is contained in \(\hat{\Gamma}_x\) for all \(k \in \mathbb{N}\) and
\[\lim_k \int c(x_i, y)p^i_k(dy) = \int c(x_i, y)p_i(dy).

Thus, Lemma 3.4 provides sequences of feasible and finitely supported competitors \(q^i_1, \ldots, q^i_N\) with corresponding limit points \(q_1, \ldots, q_N\). Then \((c, \tilde{F}_M)\)-monotonicity yields
\[\sum_{i=1}^{N} \int c(x_i, y)q^i_k(dy) = \lim_k \sum_{i=1}^{N} c(x_i, y)p^i_k(dy) \leq \liminf_k \sum_{i=1}^{N} \int c(x_i, y)q_i(dy) \leq \sum_{i=1}^{N} \int c(x_i, y)q_i(dy).

Now, let \(\pi\) be \((C, \tilde{F}_M)\)-monotone on an analytically measurable set \(\Gamma \subseteq \mathbb{R} \times \mathcal{P}(\mathbb{R})\). Denote by \(\hat{\Gamma} \subseteq X \times Y\) the set introduced in Lemma 4.2. Then we find that \(\pi\) is concentrated on \(\hat{\Gamma}\). To see that \(\hat{\Gamma}\) is \((c, \tilde{F}_M)\)-monotone, take a finite subset of \(\hat{\Gamma}\), i.e., \(G := \{(x_1, y_1), \ldots, (x_N, y_N)\} \subseteq \hat{\Gamma}\).

Let \(\alpha\) be supported on \(G\) and \(\beta\) be a competitor, i.e.,
\[\alpha \circ \text{proj}_1 = \beta \circ \text{proj}_1, \quad \alpha \circ \text{proj}_2 = \beta \circ \text{proj}_2,\]
\[\int_{\mathbb{R}} c x_i(dy) = \int_{\mathbb{R}} c y_i(dy), \quad i = 1, \ldots, N.

As in the proof of Proposition 4.4 for each \((x_i, y_i)\) denote by \(K_i\) the set obtained by Lemma 4.2
\[a^k(dx, dy) := \sum_{i=1}^{N} \delta_{x_i}(dx) \frac{\alpha^k(x_i, y_i)}{p_i(K_i \cap B_{\bar{t}_i}(y_i))}(p_i)_{|K_i \cap B_{\bar{t}_i}(y_i)}(dy), \quad k \in \mathbb{N}\]

Then \(a^k\) converges to \(\alpha\), and by Lemma 3.4 we find a sequence \(\beta^k\), which converges to \(\beta\).

Thus, we have
\[\alpha(c) = \lim_k \alpha^k(c) \leq \liminf_k \beta^k(c) \leq \beta(c),\]
where we use \((C, \tilde{F}_M)\)-monotonicity and upper semicontinuity of \(c\).

\(\square\)

**Proof of Theorem 3.5.** By Lemma 3.4 we find that \(\pi_k\) is \((C_k, \tilde{F}_M)\)-monotone, which is again preserved under this limit by Theorem 3.5. Then, by [24, Theorem 1.3] we have optimality of \(\pi\) for \([\text{MOT}]\) with cost \(c\).

\(\square\)
Proof of Corollary 4.2. Let \( \pi_k \) optimal for (MOT) for the cost function \( c_k \) and marginal measures \( \mu_k, \nu_k \). We may apply Theorem 1.1 showing that every accumulation point (with respect to weak convergence) of \( \{ \pi_k \} \) is an optimizer for (MOT) for the cost function \( c \) and marginal measures \( \mu, \nu \). On \( \mathbb{R}^2 \) we may choose the \( \ell^1 \)-metric \( D(x, y) = |x| + |y| \) in order to define the \( 1 \)-Wasserstein metric on \( \mathcal{P}(\mathbb{R}) \). Then
\[
\int Dd\pi_k = \int |x|d\mu_k + \int |y|d\nu_k \to \int |x|d\mu + \int |y|d\nu = \int Dd\pi,
\]
for any coupling \( \pi \) with marginals \( \mu, \nu \). It follows that the accumulation points of \( \{ \pi_k \} \) under the weak topology or under the \( \mathcal{W}_1 \) topology coincide. The following inequality is immediate
\[
\liminf_{k \to \infty} \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y)\pi_k(dx, dy) \geq \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y)d\pi(dx, dy).
\]
In order to prove
\[
\limsup_{k \to \infty} \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y)\pi_k(dx, dy) \leq \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y)d\pi(dx, dy),
\]
it suffices to observe that if (for some subsequence which we do not track) \( \pi_k \to \pi \) in \( \mathcal{W}_1 \), then \( \int c_d\pi_k \to \int c_d\pi \), since \( \pi \) must be optimal for the r.h.s. But this is clear since \( c_k \to c \) uniformly and \( c \) has growth dominated by \( D \).

\[
\square
\]

4. The relation of OT and OWT

Theorem 4.1. Let \( \pi \in \mathcal{P}(X \times Y) \) and \( c \in L^1(\pi) \). Then \( c \)-cyclically monotonicity of \( \pi \) implies \( C \)-monotonicity (with \( C(x, p) := \int X c(x, y)p(dy) \)). If \( c(x, \cdot) \) is upper semicontinuous for all \( x \in X \), then \( C \)-monotonicity of \( \pi \) implies \( c \)-cyclic monotonicity.

Lemma 4.2. Let \( \Gamma \subseteq X \times \mathcal{P}(Y) \) be analytically measurable, and \( c : X \times Y \to \mathbb{R} \) be Borel measurable, then there exists an analytically measurable set \( \tilde{\Gamma} \subseteq X \times Y \) with the following property: For any \( (x, y) \in \tilde{\Gamma} \) we find \( (x, p) \in \Gamma \) such that for any \( \varepsilon > 0 \) there is a Borel measurable set \( K \subseteq Y \) and

\[
\begin{align*}
(1) & \quad p(K) \geq 1 - \varepsilon, \\
(2) & \quad c \text{ restricted to the fibre } \{ x \} \times K \text{ is continuous}, \\
(3) & \quad \text{for all } y \in \text{supp}(p|_K) \text{ we have } \\
& \quad \int_B(c(x, z)p(B \cap y \cap \tilde{K}) \to c(x, y) \text{ for } \delta \searrow 0.
\end{align*}
\]

Proof. Without loss of generality, we can assume that \( c \) is bounded. Denote by \( (U_n)_{n \in \mathbb{N}} \) a countable basis of the topology on \( \mathbb{R} \). Let \( \alpha \in \mathbb{R}_+^N \) and define the \( \mathcal{B}(X \times Y) \)-measurable set
\[
A(\alpha) := \bigcup_{n \in \mathbb{N}} \{ (x, y) \in X \times Y : \exists (x, z) \in c^{-1}(U_n), dy(y, z) < \alpha_n \} \setminus c^{-1}(U_n).
\]
First of all, we see that \( c \) restricted to \( A(\alpha) \) is continuous on the fibre \( \{ x \} \times Y \):
\[
c^{-1}(U_n) \setminus A(\alpha) \subseteq \{ (x, y) \in X \times Y : \exists (x, z) \in c^{-1}(U_n), dy(y, z) < \alpha_n \} \setminus c^{-1}(U_n) \subseteq c^{-1}(U_n).
\]
Intersecting all sets with \( A(\alpha)^c \cap \{ x \} \times Y \) we obtain equality in the above chain of inclusions. On top of this, the sets are open restricted to \( \{ x \} \times Y \). Since \( (U_n)_{n \in \mathbb{N}} \) forms a basis of the topology on \( \mathbb{R} \) we find that \( c|_{A(\alpha)^c} \) is continuous on the fibre \( \{ x \} \times Y \) for all \( x \in X \).

Let \( (x, p, \alpha) \in X \times \mathcal{P}(Y) \times \mathbb{R}_+^N \) and set \( K := \text{proj}_2(A(\alpha)^c \cap \{ x \} \times Y) \). Then we find for any \( y \in \text{supp}(p|_K) \) that if \( \delta \searrow 0 \)
\[
\frac{1}{p|_{K \cap B(y)}(K \cap B(y))} p|_{K \cap B(y)} \to \delta,
\]
and, since $c$ is bounded,

$$\int_{K \cap B(\varepsilon)} \frac{c(x, z)}{K \cap B(y)} p(dz) \to c(x, y).$$

For any $B \in \mathcal{B}(Y)$ the map $(x, p, \alpha) \mapsto \delta_\alpha \otimes p(B \setminus A(\alpha))$ is Borel measurable, from which we derive measurability of the map $(x, p, \alpha) \mapsto p_{A(\alpha)}$. Then the set

$$M^\varepsilon := \{(x, p, \alpha) \in X \times \mathcal{P}(Y) \times \mathbb{R}_+^N : \delta \otimes p(A(\alpha)) < \varepsilon\}$$

is again Borel measurable. In particular, for any $(x, p) \in X \times \mathcal{P}(Y)$, we find there is $\alpha \in \mathbb{R}_+^N$ such that $(x, p, \alpha) \in M^\varepsilon$. The Jankov-von Neumann uniformization theorem provides an analytically measurable selection of $M^\varepsilon$

$$f^\varepsilon : X \times \mathcal{P}(Y) \to \mathbb{R}_+^N.$$

Hence, $(x, p) \mapsto p_{A(f^\varepsilon(x, p))}$ yields an analytically measurable function.

Since the topology on $Y$ is Polish, there is a countable basis $(O_k)_{k \in \mathbb{N}}$ of its topology. Note that $y \in \text{supp}(p)$ is equivalent to: $\forall k \in \mathbb{N}$, either $U_k$ has positive probability under $p$ or $y$ is not element of $U_k$. Therefore,

$$\hat{f}^\varepsilon := \text{proj}_{13} \left( \Gamma \times Y \cap \bigcap_{k} \{ (x, p, y) \in X \times \mathcal{P}(Y) \times Y : p_{A(f^\varepsilon(x, p))}(O_k) > 0 \text{ or } y \in U_k \} \right)$$

is analytically measurable, and $\hat{f} := \bigcup_{j \in \mathbb{N}} \hat{f}^\varepsilon_j$, where $e_j$ is some sequence on $\mathbb{R}_+$ with $\lim_j e_j = 0$, satisfies the required properties.

\begin{lemma}
Let $c : X \times Y \to \mathbb{R}$, $C(x, p) := \int_y c(x, y)p(dy)$, $\pi \in \Pi(\mu, \nu)$, and $c \in L^1(\pi)$. If $A \subseteq X \times Y$ is measurable, and $\pi$ is $C$-monotone, then $\pi|_A$ is also $C$-monotone.
\end{lemma}

\begin{proof}
The fibre of $x \in X$ on $A$ is measurable and denoted by $A_x := \{y \in Y : (x, y) \in A\}$. Let $x_1, \ldots, x_n \in \Gamma \cap \{x \in X : \pi(A_x) > 0\}$, where $\Gamma$ is a $C$-monotone subset of $X \times \mathcal{P}(Y)$ with $(x, \pi_x) \in \Gamma$ for $\mu$-a.e. $x$, $N \in \mathbb{N}$, and define

$$p_i := \pi_x|_{A_x}, \quad \tilde{p}_i := \frac{1}{p_i(A_x)} p_i, \quad i = 1, \ldots, N.$$

Without loss of generality we can assume $\tilde{p}_i = (\pi|_{A_x})_i$, for all $i = 1, \ldots, N$. There exists a natural number $n > 1$, such that its reciprocal value is smaller than $\min_j p_j(A_x)$. Then,

$$r_i = \frac{n\pi_x - \tilde{p}_i}{n - 1} \in \mathcal{P}(Y),$$

and, by linearity of $C$ in the second argument,

$$n \sum_{i=1}^N C(x_i, \tilde{p}_i) = \sum_{i=1}^N C(x_i, \tilde{p}_i) + (n - 1)C(x_i, r_i).$$

To show $C$-monotonicity, let $\tilde{q}_1, \ldots, \tilde{q}_N \in \mathcal{P}(Y)$ with $\sum_{i=1}^N \tilde{q}_i = \sum_{i=1}^N \tilde{p}_i$,

$$q_{i, j} := \begin{cases} \tilde{q}_i & j = 1, \\ r_j & 2 \leq j \leq n, \end{cases} \quad \sum_{i=1}^N \sum_{j=1}^n q_{i, j} = n \sum_{i=1}^N \pi_x$$

By $C$-monotonicity of $\pi$ we have

$$n \sum_{i=1}^N C(x_i, \tilde{p}_i) \leq \sum_{i=1}^N \sum_{j=1}^n C(x_i, q_{i, j}),$$

which is (again by linearity of $C(x, \cdot)$) equivalent to

$$\sum_{i=1}^N C(x_i, \tilde{p}_i) \leq \sum_{i=1}^N C(x_i, \tilde{q}_i).$$

\end{proof}
Proposition 4.4. Let \( c : X \times Y \to \mathbb{R} \), \( c(x, \cdot) \) be upper semicontinuous. If \( \Gamma \subseteq X \times \mathcal{P}(Y) \) a \( C \)-monotone (where \( C(x, p) := \int Y c(x, y)p(dy) \)), analytically measurable set, and for all \((x, p) \in \Gamma \) we have \( c(x, \cdot) \in L^1(p) \), then \( \Gamma \subseteq X \times Y \), see Lemma 4.2, is \( c \)-cyclically monotone. If a \( c \)-cyclically monotone set \( \Gamma \subseteq X \times Y \) is given, then
\[
\Gamma := \left\{ (x, p) \in X \times \mathcal{P}(Y) : p(\Gamma_x) = 1, c(x, \cdot) \in L^1(p) \right\}
\]
is \( C \)-monotone.

Proof. Suppose that \( \Gamma \) is \( C \)-monotone. Given any finite number of points \((x_1, y_1), \ldots, (x_N, y_N)\) in \( \Gamma \) we find \((x_i, p_i) \in \Gamma \) and \( p_i(\Gamma_x) = 1 \). For each \( y_i \) there is a (Borel measurable) set \( K_i \subseteq Y \) with \( y_i \in \supp(p_i|_{K_i}) \) and \( c|_{(x, K_i)} \) is continuous. Hence,
\[
p_i^c := \frac{1}{p_i(B_\varepsilon(y_i) \cap K_i)} p_i|_{B_\varepsilon(y_i) \cap K_i}, \quad \lim_{\varepsilon \to 0} p_i^c = \delta_{y_i}, \quad \int_{K_i \setminus B_\varepsilon(y_i)} p_i(K_i \cap B_\varepsilon(y_i)) \, dz \to c(x_i, y_i).
\]
By \( C \)-monotonicity, the restriction property (Lemma 4.3), and upper continuity of \( c \), we conclude (with the convention that \( N + 1 = 1 \)):
\[
\sum_{i=1}^N c(x_i, y_i) = \lim_{\varepsilon \to 0} \sum_{i=1}^N c(x_i, p_i^c) \leq \lim_{\varepsilon \to 0} \sum_{i=1}^N c(x_i, p_i(\Gamma_x)) \leq \sum_{i=1}^N c(x_i, y_{i+1}).
\]
Now, let \( \tilde{\Gamma} \) be a \( c \)-cyclically monotone set. Due to \( c(x, \cdot) \) being in \( L^1(p) \) for all \((x, p) \in \Gamma \), by the law of large numbers, we have
\[
\frac{1}{n} \sum_{i=1}^n \delta_{y_i} \to p, \quad \frac{1}{n} \sum_{i=1}^n c(x, Y_i) \to \int_Y c(x, y)p(dy).
\]
Thus, we can approximate \( p \) by discrete measures concentrated on \( \tilde{\Gamma} \) and obtain \( C \)-monotonicity of \( \Gamma \). Let \((x_1, p_1), \ldots, (x_N, p_N) \in \Gamma \) and \( q_1, \ldots, q_N \in \mathcal{P}(Y) \) with \( \sum_{i=1}^n p_i = \sum_{i=1}^N q_i \), using \( c \)-cyclic monotonicity and the sequence of competitors constructed in Lemma 2.5, we find
\[
\sum_{i=1}^N C(x_i, p_i) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n c(x_i, y_j) \leq \sum_{i=1}^N \int_Y c(x_i, y)q_i(dy) \leq \sum_{i=1}^N C(x_i, q_i).
\]

5. The barycentric OWT problem

Theorem 5.1. Let \( \theta : \mathbb{R}^d \to \mathbb{R} \) be convex and have a maximal growth of order \( r \geq 1 \), i.e.,
\[
\exists c_0 > 0 : \forall x \in X \quad |\theta(x)| \leq c_0(1 + |x|^r), \quad (5.1)
\]
Let \( \mu \in \mathcal{P}_r(\mathbb{R}^d) \) and \( \nu \in \mathcal{P}_r(\mathbb{R}^d) \). A coupling \( \pi \in \Pi(\mu, \nu) \) is optimal for
\[
V_\theta(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} C_\theta(x, \pi_x) \mu(dx), \quad C_\theta(x, p) := \theta \left( \int_{\mathbb{R}^d} \gamma p(dy) \right),
\]
if and only if \( \pi \) is \( C_\theta \)-monotone.

Proof. By Theorem 2.4 optimality already implies \( C_\theta \)-monotonicity. Let \( \pi \) be \( C_\theta \)-monotone and \((X_n, n \geq 2)\) be an iid sequence distributed according to \( \mu \). Then by the law of large numbers we find a sequence of points \((x_n, n \geq 2)\) in \( \text{proj}_Y(\Gamma \cap \{(x, \pi_x) : x \in X\}) \), where \( \Gamma \) is a \( C \)-monotone set, such that (again by standard separability arguments)
\[
\mu_N := \frac{1}{N} \sum_{n=1}^N \delta_{x_n} \to \mu \quad \text{in } \mathcal{W}_r, \quad \nu_N := \frac{1}{N} \sum_{n=1}^N \pi_{x_n} \to \nu \quad \text{in } \mathcal{W}_1,
\]
and
\[
\frac{1}{N} \sum_{n=1}^N C_\theta(x_n, \pi_{x_n}) \to \int_X C_\theta(x, \pi_x) \mu(dx).
\]
C-monotonicity entails that
\[ V_0(\mu_N, \nu_N) = \frac{1}{N} \sum_{i=1}^{N} C(x_i, \pi_{x_i}). \]
Then the stability result [10] Theorem 1.5] shows convergence of the optimal values, i.e.,
\[ \frac{1}{N} \sum_{i=1}^{N} C(x_i, \pi_{x_i}) = V_0(\mu, \nu) \rightarrow V_0(\mu, \nu) \quad \text{as } N \rightarrow +\infty, \]
which completes the proof. \(\square\)

**Corollary 5.2.** Let \(\theta, \theta_k : \mathbb{R}^d \rightarrow \mathbb{R}, k \in \mathbb{N},\) be convex functions such that
(a) \(\theta\) has maximal growth of order \(r,\) see [5.1],
(b) \(C_0\) converges uniformly to \(C_\theta.\)
Given a sequence \((\mu_k)_k\) and \((\nu_k)_k\) of probability measures on \(\mathcal{P}_r(\mathbb{R}^d)\) and \(\mathcal{P}_1(\mathbb{R}^d),\) respectively, where \(\mu_k\) converges to \(\mu\) in \(\mathcal{W}_r,\) and \(\nu_k\) converges to \(\nu\) in \(\mathcal{W}_1.\) Let \(\pi^k \in \Pi(\mu_k, \nu_k)\) be optimizers of \(V_{\theta_k}.\) If
\[ \lim \inf_{k} V_{\theta_k}(\mu_k, \nu_k) < \infty \]
then there exists \(\pi \in \Pi(\mu, \nu)\) such that \(\pi^k\) converges weakly (up to extraction of a subsequence) to \(\pi,\) and \(\pi\) is (finite) optimizer of \(V_\theta(\mu, \nu).\)

6. Epilogue

Convexity is a natural assumption in the setting of weak transport. It is known that convexity of \(C(x, \cdot)\) is necessary to obtain general existence of minimizers in the space of couplings, see [4] Example 3.2. Similarly, convexity is required for C-monotonicity to be a necessary optimality criterion:

**Example 6.1.** Let \(X = \{0\}, Y = \{0, 1\}, \mu = \delta_0, \nu = \frac{1}{2}(\delta_0 + \delta_1),\) and \(C(x, p) = \min(p(0)), p(1))).\) Then \(C\) is continuous and concave on \(X \times \mathcal{P}(Y),\) but the only (and therefore optimal) coupling \(\mu \otimes \nu \in \Pi(\mu, \nu)\) is not C-monotone. Indeed,
\[ 2C(0, \nu) = 1 > 0 = C(0, \delta_0) + C(0, \delta_1). \]

In the classical optimal transport setting c-cyclical monotonicity implies optimality already when the cost function \(c\) is bounded from below and real valued, see [11]. A similar conclusion cannot be drawn in optimal weak transport as C-monotonicity does not even imply optimality when \(C\) is lower continuous:

**Example 6.2.** Let \(X = [0, 1], Y = [0, 1],\) and \(C(x, p) = p_{x, \lambda}([0, 1]) = p([0, 1 \setminus \{x\}),\) which is a measurable cost function (c.f. Proposition 6.4) and lower semicontinuous for fixed \(x \in [0, 1],\) Given a weakly convergent sequence \(p_k \rightharpoonup p \in \mathcal{P}(Y),\) the Portmanteau theorem yields
\[ \lim \inf_{k} C(x, p_k) = \lim \inf_{k} p([0, 1 \setminus \{x\}) \geq p([0, 1 \setminus \{x\}) = C(x, p). \]
The product coupling \(\pi := \lambda \otimes \lambda \in \Pi(\lambda, \lambda)\) where \(\lambda\) denotes the uniform distribution on \([0, 1)\) is in fact C-monotone: Since \(\pi_x = \lambda (\lambda\text{-almost surely}),\) we have for any \(x_1, \ldots, x_N \in [0, 1]\) and \(q_1, \ldots, q_N \in \mathcal{P}(Y)\) with
\[ \sum_{i=1}^{N} \pi_{x_i} = N \lambda = \sum_{i=1}^{N} q_i \]
that \(q_i\) is absolutely continuous to \(\lambda,\) hence,
\[ \sum_{i=1}^{N} C(x_i, \pi_{x_i}) = N = \sum_{i=1}^{N} C(x_i, q_i). \]
But the unique optimizer is \(\pi^*(dx, dy) := \lambda(dx)\delta_\lambda(dy)\) and
\[ \int_{[0,1]} C(x, \pi^*_x) \lambda(dx) = \int_{[0,1]} C(x, \pi_x) \lambda(dx). \]
Even when $C$ is given as the integral with respect to some $c: X \times Y \to \mathbb{R}$, we cannot hope that $C$-monotonicity implies optimality and/or $c$-cyclical monotonicity, as the next example shows.

**Example 6.3.** Let $X = [0, 1]$, $Y = [0, 1]$ and $C(x, p) = \int_{[0,1]} c(x, y)p(dy)$ with $c(x, y) := \mathbb{1}_{[0,1]}(y)$. As in the previous example, the product coupling $\pi = A \otimes \lambda$ is $C$-monotone, but not optimal, whereas $\pi'(dx, dy) = \lambda(dx)\delta_t(dy)$ is optimal and in particular $c$-cyclical monotone.

The failure of $C$-monotonicity to provide optimality in these simple settings (the cost function $C$ is even bounded and lower semicontinuous) is caused by the manner it varies over $X \times Y$: The variation over $X$ is pointwise (similar to $c$-cyclical monotonicity), whereas over $Y$ variations are taken in a weak sense, i.e., we require that the $Y$-intensities of the two competing sequences agree. Here, $C$-monotonicity is unable to detect the jump from 1 to 0 at $x$, and we could argue that $C$-monotonicity yields optimality for all couplings $\pi \in \Pi(\lambda, \lambda)$ such that $\pi_x \ll \lambda$ for $\lambda$-almost all $x \in [0, 1]$. To be able to compare with all competing couplings, more regularity of $C$ in ‘$Y$-direction’ is necessary, e.g., upper semicontinuity as in Theorem 4.1 or uniform equicontinuity as in Theorem 2.2 but the question remains open precisely how much regularity is required.

Notably, Example 6.2, when taking $C := -C$ as cost, provides an upper semicontinuous cost function which is convex (in fact linear) in the following sense: Let $Q \in \mathcal{P}(\mathcal{P}([0, 1]))$, then

$$\int_{\mathcal{P}([0, 1])} \tilde{C}(x, p)Q(dp) = \tilde{C}\left( x, \int_{\mathcal{P}([0, 1])} pQ(dp) \right).$$

Therefore, lower semicontinuity together with convexity is a strictly stronger assumption than the convexity property stated above together with measurability, as demanded in Theorem 2.4.

We conclude this section by showing the following measurable variant of the Lebesgue decomposition theorem.

**Proposition 6.4.** Let $\mathcal{M}_a(X)$ the space of all finite measures on $X$ be equipped with the topology of weak convergence of measures. Then the map

$$T: \mathcal{M}_a(X) \times \mathcal{M}_a(X) \to \mathcal{M}_a(X) \times \mathcal{M}_a(X),$$

$$(p, q) \mapsto (q_{ac,p}, q_{s,p}),$$

where $T(p, q)$ is the unique Lebesgue decomposition of $q$ wrt $p$, i.e.,

$$q_{ac,p} + q_{s,p} = q, \quad q_{ac,p} \ll p, \quad q_{s,p} \perp p,$$

is measurable.

**Proof.** This proof is an adaptation of the one presented in [39] to obtain additional measurability of the decomposition wrt the considered measures. The idea of this proof is to define for any $\delta > 0$ and $A \in \mathcal{B}(X)$ a measurable function

$$F_{\delta, A}: \mathcal{M}_a(X) \times \mathcal{M}_a(X) \times C_b(X) \to \mathbb{R},$$

$$(p, q, g) \mapsto \begin{cases} g(\mathbb{1}_A) & p(A) < \delta, \\ q(A) & \text{else}, \end{cases}$$

(6.1)

which in turn allows us to define $T(p, q)(A)$ as a countable infimum of measurable function. Since $X$ is Polish there exists a countable family $\mathcal{O}$ of open sets on $X$ such that $\sigma(\mathcal{O}) = \mathcal{B}(X)$. Denote by $\mathcal{R}(\mathcal{O})$ the (set-theoretic) ring generated by $\mathcal{O}$, which is again countable. Then, for any $\varepsilon > 0, A \in \mathcal{B}(X)$ and $p \in \mathcal{P}(X)$ there is a set $A_\varepsilon \in \mathcal{R}(\mathcal{O})$ such that

$$p(A \setminus A_\varepsilon \cup A_\varepsilon \setminus A) < \varepsilon.$$

By a simple approximation argument, we find

$$\inf_{(B \in \mathcal{B}(X): \ p(B) = 0)} q(A \setminus B) = \inf_{\delta \searrow 0} \inf_{(B \in \mathcal{R}(\mathcal{O}): \ p(B) < \varepsilon)} q(A \setminus B).$$

(6.2)
In the consecutive step we demonstrate that it is possible to replace the sets $\mathcal{R}(O)$ by using a countable family of continuous functions: For any $O \in \mathcal{O}$ we find a sequence of continuous functions $0 \leq f_n^O \leq 1$, $n \in \mathbb{N}$ on $X$ such that $f_n \nearrow \mathbb{1}_O$. Denote by $\mathcal{F}$ the countable family of continuous functions $\mathcal{F} := \left\{ 1 \lor (0 \land \sum_{k=1}^{n} (-1)^k f_k^O) \mid k \in \mathbb{N}, \forall i: j_i \in \{-1, 1\}, O_i \in \mathcal{O}, k_i \in \mathbb{N} \right\}.$ Then

$$\inf_{\{f \in \mathcal{F} : p(f) < \delta\}} q(A \setminus B) \geq \inf_{\{f \in \mathcal{F} : p(f) < \delta\}} q(\mathbb{1}_A - f)^2. \quad (6.3)$$

At the same time, if $f \in \mathcal{F}$ and $p(f) < \delta$ then we find a measurable set $C \in \mathcal{B}(X)$ such that $p(C) = p(f)$. $q(\mathbb{1}_{A \cap C}) \geq q(\mathbb{1}_A f), q(\mathbb{1}_{A \setminus C}) \geq q(\mathbb{1}_{A \setminus f}).$

Thus we find

$$\inf_{\{f \in \mathcal{F} : p(f) < \delta\}} q(\mathbb{1}_A - f)^2 \geq \inf_{\{f \in \mathcal{F} : p(f) < \delta\}} q(A \setminus B). \quad (6.4)$$

By putting $(6.2)$, $(6.3)$ and $(6.4)$ together, we conclude

$$\inf_{\{f \in \mathcal{F} : p(f) < \delta\}} q(A \setminus B) = \inf_{\delta > 0} \inf_{\{f \in \mathcal{F} : p(f) < \delta\}} q(\mathbb{1}_A - f)^2. \quad (6.5)$$

Thus $F_A : \mathcal{M}_1(X) \times \mathcal{M}_1(X) \to \mathbb{R}$ defined by

$$F_A(p, q) = \inf_{\delta > 0} \inf_{\{f \in \mathcal{F} : p(f) < \delta\}} q(\mathbb{1}_A - f)^2 = \inf_{k \in \mathbb{N}, \{f \in \mathcal{F}\}} F_A(p, q, f) \quad (6.6)$$

is measurable. As in [9] the set function $A \mapsto q_{a,p}(A) := F_A(p, q)$ defines a finite measure which is absolutely continuous with respect to $p$ and singular to $p$. Further, $q_{a,p}$ and $q_{a,-p} := q - q_{a,p}$ are the unique Lebesgue decomposition of $q$ with respect to $p$. By the measurability of $F_A$, we find that $(p, q) \mapsto (q_{a,p}, q_{a,-p})$ is measurable. \hfill \Box

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