Twisting finite-dimensional modules for the $q$-Onsager algebra $O_q$ via the Lusztig automorphism

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Abstract

The $q$-Onsager algebra $O_q$ is defined by two generators $A, A^*$ and two relations, called the $q$-Dolan/Grady relations. Recently P. Baseilhac and S. Kolb found an automorphism $L$ of $O_q$, that fixes $A$ and sends $A^*$ to a linear combination of $A^*, A^2A^*, AA^*A, A^*A^2$. Let $V$ denote an irreducible $O_q$-module of finite dimension at least two, on which each of $A, A^*$ is diagonalizable. It is known that $A, A^*$ act on $V$ as a tridiagonal pair of $q$-Racah type, giving access to four familiar elements $K, B, K^\downarrow, B^\downarrow$ in $\text{End}(V)$ that are used to compare the eigenspace decompositions for $A, A^*$ on $V$. We display an invertible $H \in \text{End}(V)$ such that $L(X) = H^{-1}XH$ on $V$ for all $X \in O_q$. We describe what happens when one of $K, B, K^\downarrow, B^\downarrow$ is conjugated by $H$. For example $H^{-1}KH = a^{-1}A - a^{-2}K^{-1}$ where $a$ is a certain scalar that is used to describe the eigenvalues of $A$ on $V$. We use the conjugation results to compare the eigenspace decompositions for $A, A^*$, $L^{\pm1}(A^*)$ on $V$. In this comparison we use the notion of an equitable triple; this is a 3-tuple of elements in $\text{End}(V)$ such that any two satisfy a $q$-Weyl relation. Our comparison involves eight equitable triples. One of them is $aA - a^2K, M^{-1}, K$ where $M = (aK - a^{-1}B)(a - a^{-1})^{-1}$. The map $M$ appears in earlier work of S. Bockting-Conrad concerning the double lowering operator $\psi$ of a tridiagonal pair.

Keywords. $q$-Onsager algebra, Lusztig automorphism, tridiagonal pair, equitable triple.

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1 Introduction

In this paper we consider a certain kind of module for the $q$-Onsager algebra $O_q$. Before getting into detail, we briefly set some notation. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$. Let $F$ denote a field. Every vector space discussed in this paper is over $F$. Every algebra discussed in this paper is associative, over $F$, and has a multiplicative identity. Fix a nonzero $q \in F$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}.$$
For elements $X, Y$ in any algebra, their commutator and $q$-commutator are given by

$$[X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1}YX.$$ 

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3.$$ 

**Definition 1.1.** (See [3] Section 2, [32] Definition 3.9.) The $q$-Onsager algebra $O_q$ is defined by generators $A, A^*$ and relations

$$[A, [A, A, A^*]_q]_{q^{-1}} = (q^2 - q^{-2})^2 [A, A], \quad (1)$$

$$[A^*, [A^*, A, A]_q]_{q^{-1}} = (q^2 - q^{-2})^2 [A, A^*]. \quad (2)$$

The relations (1), (2) are called the $q$-Dolan/Grady relations.

The algebra $O_q$ comes from algebraic graph theory, or more precisely, the theory of $Q$-polynomial distance-regular graphs [1], [17]. For such a graph, the adjacency matrix and any dual adjacency matrix satisfy two relations [19] Lemma 5.4 called the tridiagonal relations [32] Definition 3.9. If the graph has $q$-Racah type [23] Section 1] then these tridiagonal relations become the $q$-Dolan/Grady relations after an appropriate normalization. For an overview of this topic see [25].

The algebra $O_q$ comes up in the theory of tridiagonal pairs. Roughly speaking, a tridiagonal pair is a pair of diagonalizable linear maps on a nonzero finite-dimensional vector space, that each act in a block-tridiagonal fashion on the eigenspaces of the other one [21] Definition 1.1. A finite-dimensional irreducible $O_q$-module on which the generators are diagonalizable is essentially the same thing as a tridiagonal pair of $q$-Racah type [32] Theorem 3.10. For more information on $O_q$ and tridiagonal pairs, see [19], [20], [24], [32], [35–37], [39–41].

The algebra $O_q$ has applications to quantum integrable models [2–5], [7–9], reflection equation algebras [6], [11], and coideal subalgebras [12], [28], [29]. There is an algebra homomorphism from $O_q$ into the algebra \( \Box_q \) [37] Proposition 5.6], and the universal Askey-Wilson algebra [35], Sections 9,10].

We will be discussing automorphisms of $O_q$. By an automorphism of $O_q$ we mean an algebra isomorphism $O_q \rightarrow O_q$. In [10] P. Baseilhac and S. Kolb introduced the following automorphism of $O_q$.

**Lemma 1.2.** (See [10] Section 2.) There exists an automorphism $L$ of $O_q$ such that

$$L(A) = A, \quad L(A^*) = A^* + \frac{[A, [A, A^*]_q]}{(q - q^{-1})(q^2 - q^{-2})}. \quad (3)$$

The inverse automorphism $L^{-1}$ satisfies

$$L^{-1}(A) = A, \quad L^{-1}(A^*) = A^* + \frac{[A, [A, A^*]_{q^{-1}}]}{(q - q^{-1})(q^2 - q^{-2})}. \quad (4)$$
The automorphism $L$ is roughly analogous to the Lusztig automorphism of the quantum group $U_q(\hat{sl}_2)$. Motivated by this, we call $L$ the Lusztig automorphism of $\mathcal{O}_q$ [39].

The contents of the present paper are summarized as follows. Let $V$ denote a finite-dimensional irreducible $\mathcal{O}_q$-module on which each of $A$, $A^*$ is diagonalizable. To avoid trivialities, we assume that $V$ has dimension at least 2. We mentioned below Definition [11] that $A$, $A^*$ act on $V$ as a tridiagonal pair of $q$-Racah type. We give a detailed proof of this fact in order to set the stage. We then display an invertible $H \in \text{End}(V)$ such that $L(X) = H^{-1}XH$ on $V$ for all $X \in \mathcal{O}_q$. The existence of $H$ means that the $\mathcal{O}_q$-module $V$ is isomorphic to the $\mathcal{O}_q$-module $V$ twisted via $L^{\pm 1}$. From another point of view, it means that the following act on $V$ as isomorphic tridiagonal pairs: (i) $A$, $A^*$; (ii) $A$, $L(A^*)$; (iii) $A$, $L^{-1}(A^*)$. We display some identities that are satisfied by the eigenvalues of $H$. These identities are obtained using the Chu/Vandermonde summation formula for basic hypergeometric series. Using the identities we express $H^{\pm 1}$ as a polynomial in $A$. We then consider some elements $K$, $B$, $K^\dagger$, $B^\dagger$ in $\text{End}(V)$. These elements are familiar in the theory of tridiagonal pairs; they are used to compare the eigenspace decompositions for $A$, $A^*$ on $V$. The element $K$ was introduced in [25, Section 1.1]. The element $B$ is obtained from $K$ by inverting the ordering of the eigenspace decomposition for $A$ on $V$. The elements $K^\dagger$, $B^\dagger$ are obtained from $K$, $B$ by inverting the ordering of the eigenspace decomposition for $A^*$ on $V$. In [14,15] S. Bockting-Conrad investigates $K$, $B$ in detail, as part of her work on the double lowering operator $\psi$ [13]. She obtains some relations involving $K$, $B$, $A$ that she uses to turn $V$ into a module for the quantum group $U_q(\hat{sl}_2)$. We describe what happens when one of $K$, $B$, $K^\dagger$, $B^\dagger$ is conjugated by $H$. We show that

$$H^{-1}BH = aA - a^2B^{-1}, \quad H^{-1}KH = a^{-1}A - a^{-2}K^{-1}, \quad (5)$$

$$H^{-1}B^\dagger H = aA - a^2(B^\dagger)^{-1}, \quad H^{-1}K^\dagger H = a^{-1}A - a^{-2}(K^\dagger)^{-1}, \quad (6)$$

where $a$ is a certain scalar that is used to describe the eigenvalues for $A$ on $V$; see Lemma [2,4] below. Reformulating the equations (5), (6) we obtain

$$HB^{-1}H^{-1} = a^{-1}A - a^{-2}B, \quad HK^{-1}H^{-1} = aA - a^2K, \quad (7)$$

$$H(B^\dagger)^{-1}H^{-1} = a^{-1}A - a^{-2}B^\dagger, \quad H(K^\dagger)^{-1}H^{-1} = aA - a^2K^\dagger. \quad (8)$$

We use (5)-(8) to compare the eigenspace decompositions of $A$, $A^*$, $L^{\pm 1}(A^*)$ on $V$. In this comparison we use the notion of an equitable triple; this is a 3-tuple $X$, $Y$, $Z$ of invertible elements in $\text{End}(V)$ such that

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I, \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = I.$$

Our comparison involves eight equitable triples; see Proposition 7.4 below. One of the triples is $aA - a^2K$, $M^{-1}$, $K$ where

$$M = \frac{aK - a^{-1}B}{a - a^{-1}}.$$

The element $M$ appears in the work of Bockting-Conrad mentioned above; see [15, Section 6].

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We describe how the eight equitable triples are related to the eigenspace decompositions for $A$, $A^*$, $L^{±1}(A^*)$ on $V$. We find it illuminating to make this description using diagrams; see Theorems 8.1–8.4 below. These theorems are the main results of the paper.

The paper is organized as follows. In Section 2 we prove that $A$, $A^*$ act on $V$ as a tridiagonal pair of $q$-Racah type. In Section 3 we describe the Lusztig automorphism $L$ of $\mathcal{O}_q$, and display an invertible $H \in \text{End}(V)$ such that $L(X) = H^{-1}XH$ on $V$ for all $X \in \mathcal{O}_q$. We also obtain some identities involving the eigenvalues of $H$. In Section 4 we introduce some diagrams that are used to compare the eigenspace decompositions for $A$, $A^*$, $L^{±1}(A^*)$ on $V$. In Section 5 we recall the elements $K$, $B$, $K^\perp$, $B^\perp$ and discuss their basic properties. In Section 6 we describe what happens when one of $K$, $B$, $K^\perp$, $B^\perp$ is conjugated by $H$. In Section 7 we discuss the notion of an equitable triple, and display eight equitable triples that involve $K$, $B$, $K^\perp$, $B^\perp$. In Section 8 we describe how the eight equitable triples are related to the eigenspace decompositions for $A$, $A^*$, $L^{±1}(A^*)$ on $V$. This description is made using diagrams.

## 2 The $q$-Dolan/Grady relations and tridiagonal pairs

We now begin our formal argument. For the rest of this paper, $V$ denotes an irreducible $\mathcal{O}_q$-module with finite dimension at least 2, on which each of $A$, $A^*$ is diagonalizable. In this section we show that $A$, $A^*$ act on $V$ as a tridiagonal pair of $q$-Racah type.

We consider how $A$ and $A^*$ act on each other’s eigenspaces. Let $\text{End}(V)$ denote the algebra consisting of the $\mathbb{F}$-linear maps from $V$ to $V$. Let $\mathcal{D}$ denote the set of eigenvalues for $A$ on $V$. By construction $\mathcal{D}$ is nonempty. We emphasize that the elements of $\mathcal{D}$ are mutually distinct. For $\lambda \in \mathcal{D}$ define $E_\lambda \in \text{End}(V)$ to be the projection onto the $\lambda$-eigenspace for $A$ on $V$. The following holds on $V$: (i) $AE_\lambda = \lambda E_\lambda = E_\lambda A$ ($\lambda \in \mathcal{D}$); (ii) $E_\lambda E_\mu = \delta_{\lambda,\mu}E_\lambda$ ($\lambda, \mu \in \mathcal{D}$); (iii) $I = \sum_{\lambda \in \mathcal{D}} E_\lambda$. For $\lambda, \mu \in \mathcal{D}$, multiply each side of (11) on the left by $E_\lambda$ and the right by $E_\mu$. After some routine simplification, we obtain

\[ E_\lambda A^*E_\mu(\lambda - \mu)(q\lambda - q^{-1}\mu)(q^{-1}\lambda - q\mu) = E_\lambda A^*E_\mu(\mu - \lambda)(q^2 - q^{-2})^2. \]

Therefore

\[ E_\lambda A^*E_\mu(\lambda - \mu)P(\lambda, \mu) = 0, \]

where

\[ P(\lambda, \mu) = \lambda^2 - (q^2 + q^{-2})\lambda \mu + \mu^2 + (q^2 - q^{-2})^2. \]  

(9)

Consequently

\[ E_\lambda A^*E_\mu = 0 \quad \text{or} \quad \lambda = \mu \quad \text{or} \quad P(\lambda, \mu) = 0. \]  

(10)

Note that $P(\lambda, \mu) = P(\mu, \lambda)$. Call $\lambda, \mu$ adjacent whenever $\lambda \neq \mu$ and $P(\lambda, \mu) = 0$. The adjacency relation is symmetric.

We just defined a symmetric binary relation on $\mathcal{D}$, called adjacency. This relation turns $\mathcal{D}$ into an undirected graph. For the graph $\mathcal{D}$, each vertex is adjacent to at most two other vertices, since the polynomial $P$ has degree two. The graph $\mathcal{D}$ is connected, because $V$ is irreducible as an $\mathcal{O}_q$-module. By these comments, the graph $\mathcal{D}$ is either a path or a cycle. Shortly we will show that a cycle cannot occur.
Lemma 2.1. Referring to the graph $\mathcal{D}$, let $\lambda$ denote a vertex that is adjacent to two distinct vertices $\mu, \nu$. Then

$$\mu - (q^2 + q^{-2})\lambda + \nu = 0.$$  

**Proof.** The vertices $\lambda, \mu$ are adjacent, so $P(\lambda, \mu) = 0$. The vertices $\lambda, \nu$ are adjacent, so $P(\lambda, \nu) = 0$. By these comments and (9),

$$0 = \frac{P(\lambda, \mu) - P(\lambda, \nu)}{\mu - \nu} = \mu - (q^2 + q^{-2})\lambda + \nu. \tag{□}$$

Lemma 2.2. We have $|\mathcal{D}| \geq 2$.

**Proof.** By construction $|\mathcal{D}| \geq 1$. We assume that $|\mathcal{D}| = 1$ and get a contradiction. The action of $A$ on $V$ is diagonalizable with a single eigenvalue. So $A$ acts on $V$ as a scalar multiple of the identity. The action of $A^*$ on $V$ is diagonalizable, so there exists a nonzero $v \in V$ that is an eigenvector for $A^*$. The subspace $W = Fv$ is invariant under $A$ and $A^*$. Therefore $W = V$ since $V$ is irreducible as an $O_q$-module. Consequently $V$ has dimension one, a contradiction. We have shown that $|\mathcal{D}| \geq 2$. \(\square\)

Definition 2.3. Define $d = |\mathcal{D}| - 1$. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of $\mathcal{D}$ such that $\theta_{i-1}, \theta_i$ are adjacent for $1 \leq i \leq d$.

Lemma 2.4. The graph $\mathcal{D}$ is a path. Moreover, there exists $0 \neq a \in F$ such that

$$\theta_i = a q^{d-2i} + a^{-1} q^{2i-d} \quad (0 \leq i \leq d).$$

**Proof.** For notational convenience define $n = |\mathcal{D}|$. For the graph $\mathcal{D}$ let $e$ denote the number of (undirected) edges. So $e = d$ if $\mathcal{D}$ is a path, and $e = n$ if $\mathcal{D}$ is a cycle. If $\mathcal{D}$ is a cycle then define $\theta_n = \theta_0$. For the graph $\mathcal{D}$ the vertices $\theta_{i-1}, \theta_i$ are adjacent for $1 \leq i \leq e$. By this and Lemma 2.1

$$\theta_{i-1} - (q^2 + q^{-2})\theta_i + \theta_{i+1} = 0 \quad (1 \leq i \leq e - 1).$$

For this recurrence the characteristic polynomial is

$$1 - (q^2 + q^{-2})x + x^2 = (x - q^2)(x - q^{-2}).$$

The roots $q^2, q^{-2}$ are distinct since $q$ is not a root of 1. By these comments the recurrence has general solution

$$\theta_i = a q^{d-2i} + \alpha q^{2i-d} \quad (0 \leq i \leq e), \tag{11}$$

where $a, \alpha \in F$. Using this general solution,

$$0 = P(\theta_0, \theta_1) = (q^2 - q^{-2})^2(1 - a\alpha).$$
The scalar $q^2 - q^{-2}$ is nonzero since $q$ is not a root of unity, so $a\alpha = 1$. Therefore $a \neq 0$ and $\alpha = a^{-1}$. It remains to show that the graph $\mathcal{D}$ is not a cycle. Assume that $\mathcal{D}$ is a cycle. By construction $\theta_n = \theta_0$. Also by construction $n \geq 3$, so $d = n - 1 \geq 2$. Using (11) with $\alpha = a^{-1}$,

$$0 = \frac{\theta_n - \theta_0}{\theta_d - \theta_1} = \frac{q^{2n} - 1}{q^{2d} - q^2}.$$ 

Therefore $q^{2n} = 1$, contradicting our assumption that $q$ is not a root of unity. Consequently $\mathcal{D}$ is not a cycle, and the result follows. 

For notational convenience, abbreviate $E_i = E_{\theta_i}$ for $0 \leq i \leq d$. By the construction and (10),

$$E_iA^*E_j = 0 \quad \text{if} \quad |i - j| > 1 \quad (0 \leq i, j \leq d). \quad (12)$$

For $0 \leq i \leq d$ let $V_i$ denote the $\theta_i$-eigenspace for $A$ on $V$. So $V_i = E_iV$.

**Lemma 2.5.** We have

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \quad (13)$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

**Proof.** We have

$$A^*V_i = IA^*E_iV = \sum_{\ell=0}^{d} E_\ell A^*E_iV.$$ 

For $0 \leq \ell \leq d$, $E_\ell A^*E_i = 0$ if $|\ell - i| > 1$. Also for $0 \leq \ell \leq d$, $E_\ell A^*E_iV \subseteq E_\ell V = V_\ell$. The result follows. 

Lemma 2.5 shows how $A^*$ acts on the eigenspaces for $A$ on $V$. Interchanging the roles of $A$, $A^*$ we see that $A$ acts on the eigenspaces for $A^*$ on $V$ in the following way.

**Lemma 2.6.** There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces for $A^*$ on $V$ such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta), \quad (14)$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

The concept of a tridiagonal pair was introduced in [21, Definition 1.1]. By Lemmas 2.5, 2.6 the elements $A, A^*$ act on the $\mathbb{O}_q$-module $V$ as a tridiagonal pair.

By [21] Lemma 4.5] the integers $d, \delta$ from Lemmas 2.5, 2.6 are equal; we call this common value the **diameter** of the $\mathbb{O}_q$-module $V$. For $0 \leq i \leq d$ let $\theta_i^*$ denote the eigenvalue associated with the eigenspace $V_i^*$ for $A^*$ on $V$.

**Lemma 2.7.** With the above notation, there exists $0 \neq b \in \mathbb{F}$ such that

$$\theta_i^* = bq^{d-2i} + b^{-1}q^{2i-d} \quad (0 \leq i \leq d).$$
Proof. Interchange the roles of $A$ and $A^*$ in Lemma 2.4.

We have a comment.

**Lemma 2.8.** Neither of $a^2, b^2$ is among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$.

Proof. Use Lemmas 2.4, 2.7 and the fact that the scalars \{\(\theta_i\)\}_{i=0}^d are mutually distinct and the scalars \{\(\theta_i^*\)\}_{i=0}^d are mutually distinct.

We mentioned earlier that the elements $A, A^*$ act on the $O_q$-module $V$ as a tridiagonal pair. By the form of the eigenvalues \{\(\theta_i\)\}_{i=0}^d (resp. \{\(\theta_i^*\)\}_{i=0}^d) in Lemma 2.4 (resp. Lemma 2.7), we see that this tridiagonal pair has $q$-Racah type in the sense of [38, Section 1].

3 The Lusztig automorphism of $O_q$

We continue to discuss the $O_q$-module $V$ from Section 2. Recall the Lusztig automorphism $L$ of $O_q$ from Lemma 1.2. In [39, Section 8] we briefly described the action of $L(A^*)$ on $V$. In the present section we describe this action in greater detail. Referring to the equation on the right in (3), for $0 \leq i, j \leq d$ we multiply each side on the left by $E_i$ and the right by $E_j$. This yields

$$E_iL(A^*)E_j = E_iA^*E_j t_{ij}, \quad (15)$$

where

$$t_{ij} = 1 + \frac{\theta_i - \theta_j}{q - q^{-1}} \frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}}. \quad (16)$$

If $|i - j| > 1$ then each side of (15) is equal to zero, in view of (12). We now consider (15) for $|i - j| \leq 1$. Note that

$$t_{ii} = 1 \quad (0 \leq i \leq d). \quad (17)$$

**Lemma 3.1.** For $0 \leq i, j \leq d$ such that $|i - j| = 1$,

$$\frac{q\theta_i - q^{-1}\theta_j}{q^2 - q^{-2}} \frac{q\theta_j - q^{-1}\theta_i}{q^2 - q^{-2}} = 1.$$

Proof. This is a reformulation of the equation $P(\theta_i, \theta_j) = 0$. \qed

**Lemma 3.2.** For $0 \leq i, j \leq d$ such that $|i - j| = 1$,

$$t_{ij}t_{ji} = 1. \quad (18)$$

Proof. To verify (18), evaluate the left-hand side using (16) and simplify the result using Lemma 3.1. \qed

**Definition 3.3.** Define $t_i = t_{01}t_{12} \cdots t_{i-1,i}$ for $0 \leq i \leq d$. We interpret $t_0 = 1$.

**Lemma 3.4.** We have $t_i \neq 0$ for $0 \leq i \leq d$.  


Proof. By Definition 3.3 and since each of \(t_{01}, t_{12}, \ldots, t_{i-i-1, i}\) is nonzero by Lemma 3.2.

Lemma 3.5. For \(0 \leq i, j \leq d\) such that \(|i - j| \leq 1\),

\[ t_{ij} = \frac{t_j}{t_i} \]

Proof. For \(i = j\) use (17). For \(|i - j| = 1\) use Lemma 3.2 and Definition 3.3.

Define \(H \in \text{End}(V)\) by

\[ H = \sum_{i=0}^{d} t_i E_i, \tag{19} \]

where \(\{t_i\}_{i=0}^{d}\) are from Definition 3.3. The map \(H\) is invertible and

\[ H^{-1} = \sum_{i=0}^{d} t_i^{-1} E_i. \tag{20} \]

Lemma 3.6. The following hold on \(V\):

(i) \(L(A) = H^{-1}AH\);

(ii) \(L(A^*) = H^{-1}A^*H\).

Proof. (i) We show that each side is equal to \(A\). By (3) we have \(L(A) = A\). For \(0 \leq i \leq d\) we have \(AE_i = \theta_i E_i = E_i A\), so \(E_i\) commutes with \(A\). By this and (19) we see that \(H\) commutes with \(A\). Therefore \(H^{-1}AH = A\).

(ii) Define \(\Delta = L(A^*) - H^{-1}A^*H\). We show that \(\Delta = 0\). Using \(I = E_0 + \cdots + E_d\) we obtain

\[ \Delta = I \Delta I = \sum_{i=0}^{d} \sum_{j=0}^{d} E_i \Delta E_j. \]

For \(0 \leq i, j \leq d\) we show that \(E_i \Delta E_j = 0\). Using (15) and (19), (20) we obtain

\[ E_i \Delta E_j = E_i A^* E_j (t_{ij} - t_i^{-1} t_j). \]

If \(|i - j| > 1\) then \(E_i A^* E_j = 0\) by (12). If \(|i - j| \leq 1\) then \(t_{ij} = t_i^{-1} t_j\) by Lemma 3.5. Therefore \(E_i \Delta E_j = 0\). We have shown that \(E_i \Delta E_j = 0\) for \(0 \leq i, j \leq d\), so \(\Delta = 0\). The result follows.

Proposition 3.7. For \(X \in \mathcal{O}_q\) the following hold on \(V\):

\[ L(X) = H^{-1}XH, \quad L^{-1}(X) = HXH^{-1}. \tag{21} \]

Proof. By Lemma 3.6 and since the algebra \(\mathcal{O}_q\) is generated by \(A, A^*\).

We will discuss the implications of Proposition 3.7 after reviewing a few concepts.
Definition 3.8. Let $\mathcal{A}$ denote an algebra and let $W, W'$ denote $\mathcal{A}$-modules. By an isomorphism of $\mathcal{A}$-modules from $W$ to $W'$ we mean an $\mathbb{F}$-linear bijection $\gamma : W \to W'$ such that $\xi \gamma = \gamma \xi$ on $W$ for all $\xi \in \mathcal{A}$. The $\mathcal{A}$-modules $W, W'$ are said to be isomorphic whenever there exists an isomorphism of $\mathcal{A}$-modules from $W$ to $W'$.

Definition 3.9. Let $\sigma$ denote an automorphism of an algebra $\mathcal{A}$. Let $W$ denote an $\mathcal{A}$-module. There exists an $\mathcal{A}$-module structure on $W$, called $W$ twisted via $\sigma$, that behaves as follows: for all $\xi \in \mathcal{A}$ and $w \in W$, the vector $\xi.w$ computed in $W$ twisted via $\sigma$ coincides with the vector $\sigma^{-1}(\xi).w$ computed in the original $\mathcal{A}$-module $W$. Sometimes we abbreviate $\sigma W$ for $W$ twisted via $\sigma$.

Proposition 3.10. The following $\mathcal{O}_q$-modules are isomorphic:

(i) the $\mathcal{O}_q$-module $V$;
(ii) the $\mathcal{O}_q$-module $V$ twisted via $L$;
(iii) the $\mathcal{O}_q$-module $V$ twisted via $L^{-1}$.

Moreover, the map $H$ from (19) is an isomorphism of $\mathcal{O}_q$-modules from (i) to (ii) and from (iii) to (i).

Proof. It suffices to prove the last assertion in the proposition statement. By the construction and Proposition 3.7 the map $H : V \to V$ is an $\mathbb{F}$-linear bijection such that $L^{-1}(X)H = HX$ and $HL(X) = XH$ on $V$ for all $X \in \mathcal{O}_q$. By this and Definitions 3.8, 3.9 we get the last assertion in the proposition statement.

We mentioned below Lemma 2.6 that $A, A^*$ act on the $\mathcal{O}_q$-module $V$ as a tridiagonal pair. Next we express Proposition 3.10 in terms of tridiagonal pairs. The notion of isomorphism for tridiagonal pairs is given in [30, Definition 3.1].

Proposition 3.11. The following are isomorphic tridiagonal pairs:

(i) the action of $A, A^*$ on $V$;
(ii) the action of $A, L(A^*)$ on $V$;
(iii) the action of $A, L^{-1}(A^*)$ on $V$.

Moreover, the map $H$ from (19) is an isomorphism of tridiagonal pairs from (ii) to (i) and from (i) to (iii).

We emphasize a few points about $L(A^*)$ and $L^{-1}(A^*)$.

Lemma 3.12. For $\varepsilon \in \{1, -1\}$ the element $L^\varepsilon(A^*)$ is diagonalizable on $V$, with eigenvalues $\{\theta_i^\varepsilon\}_{i=0}^d$ and $\theta_i^\varepsilon$-eigenspace $H^{-\varepsilon}V_i^*$ for $0 \leq i \leq d$.

Proof. By construction the element $A^*$ is diagonalizable on $V$, with eigenvalues $\{\theta_i^*\}_{i=0}^d$ and $\theta_i^*$-eigenspace $V_i^*$ for $0 \leq i \leq d$. The result follows from this and Proposition 3.7.
Definition 3.13. For notational convenience, we abbreviate $V_i^+ = H^{-1}V_i^*$ and $V_i^- = HV_i^*$ for $0 \leq i \leq d$.

Our next goal for this section is to develop some formulas concerning $H$ that will be used in later sections.

Lemma 3.14. We have
\[ t_{i-1,i} = a^2 q^{2(i-2i+1)} \quad (1 \leq i \leq d). \] (22)

Proof. Use Lemma 2.4 and (16).

Lemma 3.15. We have
\[ t_i = a^2 i q^{2i} (d-i) \quad (0 \leq i \leq d). \] (23)

Proof. Use Definition 3.3 and Lemma 3.14.

We recall some notation. For $z, t \in \mathbb{F}$,
\[ (z, t)_n = (1 - z)(1 - zt) \cdots (1 - zt^{n-1}) \quad (n \in \mathbb{N}). \]

We will be discussing basic hypergeometric series, using the notation of [18, 27].

Lemma 3.16. For $0 \leq r \leq s \leq d$,
\[ \frac{t_s}{t_r} = \sum_{i=0}^{s-r} a^i q^{i(d-2r)}(\theta_s - \theta_r)(\theta_s - \theta_{r+1}) \cdots (\theta_s - \theta_{r+i-1}) \frac{(q^2; q^2)_i}{(q^2; q^2)_i}, \] (24)
\[ \frac{t_r}{t_s} = \sum_{i=0}^{s-r} a^{-i} q^{i(2r-d)}(\theta_s - \theta_r)(\theta_s - \theta_{r+1}) \cdots (\theta_s - \theta_{r+i-1}) \frac{(q^{-2}; q^{-2})_i}{(q^{-2}; q^{-2})_i}. \] (25)

Proof. To verify (24), evaluate the left-hand side using Lemma 3.15 and the right-hand side using Lemma 2.4. The result becomes a special case of the basic Chu/Vandermonde summation formula [18, p. 354]:
\[ a^{2s-2r} q^{2(s-r)(d-r-s)} = 2\phi_1 \left( q^{2s-2r}, a^2 q^{2d-2r-2s} \middle| q^{-2}, q^{-2} \right). \]

We have verified (24). To obtain (25) from (24), replace $q \mapsto q^{-1}$ and $a \mapsto a^{-1}$.

Recall the eigenspaces $\{ V_i \}_{i=0}^d$ for $A$ on $V$.

Proposition 3.17. For $0 \leq r \leq d$ the following holds on $V_r + V_{r+1} + \cdots + V_d$:
\[ H = t_r \sum_{i=0}^{d-r} a^i q^{i(d-2r)}(A - \theta_r I)(A - \theta_{r+1} I) \cdots (A - \theta_{r+i-1} I) \frac{(q^2; q^2)_i}{(q^2; q^2)_i}, \] (26)
\[ H^{-1} = t_r^{-1} \sum_{i=0}^{d-r} a^{-i} q^{i(2r-d)}(A - \theta_r I)(A - \theta_{r+1} I) \cdots (A - \theta_{r+i-1} I) \frac{(q^{-2}; q^{-2})_i}{(q^{-2}; q^{-2})_i}. \] (27)
Proof. To verify (26), use (19), (24) to see that for \( r \leq s \leq d \) the \( V_s \)-eigenvalue for either side of (25) is equal to \( t_s \). We have verified (26). To verify (27), use (20), (25) to see that for \( r \leq s \leq d \) the \( V_s \)-eigenvalue for either side of (27) is equal to \( t_s^{-1} \). We have verified (27). \( \square 

Proposition 3.18. The following holds on \( V \):

\[
H = \sum_{i=0}^{d} a^i q^{id}(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I) \frac{1}{(q^2; q^2)_i},
\]

\[
H^{-1} = \sum_{i=0}^{d} a^{-i} q^{-id}(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I) \frac{1}{(q^{-2}; q^{-2})_i}.
\]

Proof. Set \( r = 0 \) in Proposition 3.17. \( \square 

We mention a variation on Lemma 3.16 and Propositions 3.17, 3.18.

Lemma 3.19. For \( 0 \leq r \leq s \leq d \),

\[
\frac{t_r}{t_s} = \sum_{i=0}^{s-r} \frac{a^i q^{i(2s-d)}(\theta_r - \theta_s)(\theta_r - \theta_{s-1}) \cdots (\theta_r - \theta_{s-i+1})}{(q^2; q^2)_i}, \quad (28)
\]

\[
\frac{t_s}{t_r} = \sum_{i=0}^{s-r} \frac{a^i q^{i(d-2s)}(\theta_r - \theta_s)(\theta_r - \theta_{s-1}) \cdots (\theta_r - \theta_{s-i+1})}{(q^{-2}; q^{-2})_i}. \quad (29)
\]

Proof. To verify (28), evaluate the left-hand side using Lemma 3.15 and the right-hand side using Lemma 2.4. The result becomes a special case of the basic Chu/Vandermonde summation formula [18, p. 354]:

\[
a^{2r-2s} q^{2(r-s)(d-r-s)} = 2\phi_1 \left(q^{2s-2r}, a^{-2} q^{2r+2s-2d}; q^2 \left| \frac{q^{-2}}{q^{-2}} \right. \right).
\]

We have verified (29). To obtain (29) from (28), replace \( q \mapsto q^{-1} \) and \( a \mapsto a^{-1} \). \( \square 

Proposition 3.20. For \( 0 \leq s \leq d \) the following holds on \( V_0 + V_1 + \cdots + V_s \):

\[
H = t_s \sum_{i=0}^{s} \frac{a^i q^{i(2s-d)}(A - \theta_s I)(A - \theta_{s-1} I) \cdots (A - \theta_{s-i+1} I)}{(q^2; q^2)_i},
\]

\[
H^{-1} = t_s^{-1} \sum_{i=0}^{s} \frac{a^{i} q^{i(d-2s)}(A - \theta_s I)(A - \theta_{s-1} I) \cdots (A - \theta_{s-i+1} I)}{(q^{-2}; q^{-2})_i}.
\]

Proof. Similar to the proof of Proposition 3.17. \( \square 

Proposition 3.21. The following holds on \( V \):

\[
H = t_d \sum_{i=0}^{d} \frac{a^{-i} q^{id}(A - \theta_d I)(A - \theta_{d-1} I) \cdots (A - \theta_{d-i+1} I)}{(q^2; q^2)_i},
\]

\[
H^{-1} = t_d^{-1} \sum_{i=0}^{d} \frac{a^{i} q^{-id}(A - \theta_d I)(A - \theta_{d-1} I) \cdots (A - \theta_{d-i+1} I)}{(q^{-2}; q^{-2})_i}.
\]

Proof. Set \( s = d \) in Proposition 3.20. \( \square 

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4 Some diagrams

We continue to discuss the $O_q$-module $V$ from Section 2. In this section we introduce some diagrams that will help us describe how the actions of $A$, $A^*$, $L^\pm 1(A^*)$ on $V$ are related.

Definition 4.1. By a \textit{decomposition of $V$} we mean a sequence $\{W_i\}_{i=0}^d$ of nonzero subspaces whose direct sum is $V$.

Example 4.2. The eigenspaces $\{V_i\}_{i=0}^d$ of $A$ (resp. $\{V_i^*\}_{i=0}^d$ of $A^*$) (resp. $\{V_i^+\}_{i=0}^d$ of $L(A^*)$) (resp. $\{V_i^-\}_{i=0}^d$ of $L^{-1}(A^*)$) form a decomposition of $V$.

Definition 4.3. Let $\{W_i\}_{i=0}^d$ denote a decomposition of $V$. Its \textit{inversion} is the decomposition $\{W_{d-i}\}_{i=0}^d$ of $V$.

Let $\{W_i\}_{i=0}^d$ denote a decomposition of $V$. We will describe this decomposition by the diagram

\[
• • • W_0 W_1 W_2 ⋯ W_{d-1} W_d
\]

The labels $W_i$ might be suppressed, if they are clear from the context.

Let $\{W_i\}_{i=0}^d$ and $\{W'_i\}_{i=0}^d$ denote decompositions of $V$. The condition

\[
W_0 + W_1 + ⋯ + W_i = W'_0 + W'_1 + ⋯ + W'_i \quad (0 \leq i \leq d)
\]

will be described by the diagram

\[
\begin{align*}
&W_0 & & W_1 & & W_2 & & ⋯ & & W_{d-1} & & W_d \\
&W'_0 & & W'_1 & & W'_2 & & ⋯ & & W'_{d-1} & & W'_d
\end{align*}
\]
Lemma 4.4. (See [21, Theorem 4.6].) There exist decompositions of $V$ that are related to \( \{V_i\}_{i=0}^d \) and \( \{V_i^*\}_{i=0}^d \) in the following way:

\[
\begin{array}{cccccccc}
& A^* & & & & & & \\
& V_0^* & V_1^* & V_2^* & \cdots & V_{d-2}^* & V_{d-1}^* & V_d^* \\
& & & & & & & \\
& V_0 & V_1 & V_2 & \cdots & V_{d-2} & V_{d-1} & V_d \\
\end{array}
\]

We are using the notation in Definition 3.13.

Lemma 4.5. There exist decompositions of $V$ that are related to \( \{V_i\}_{i=0}^d \) and \( \{V_i^+\}_{i=0}^d \) in the following way:

\[
\begin{array}{cccccccc}
& L(A^*) & & & & & & \\
& V_0^+ & V_1^+ & V_2^+ & \cdots & V_{d-2}^+ & V_{d-1}^+ & V_d^+ \\
& & & & & & & \\
& V_0 & V_1 & V_2 & \cdots & V_{d-2} & V_{d-1} & V_d \\
\end{array}
\]
Proof. By Proposition 3.11 and Lemma 4.4.

Lemma 4.6. There exist decompositions of $V$ that are related to $\{V_i\}_{i=0}^d$ and $\{V^-_i\}_{i=0}^d$ in the following way:

$$L^{-1}(A^*)$$

$$\begin{array}{cccccc}
V_0^- & V_1^- & V_2^- & \cdots & V_{d-2}^- & V_{d-1}^- & V_d^- \\
V_0 & V_1 & V_2 & \cdots & V_{d-2} & V_{d-1} & V_d \\
\end{array}$$

We are using the notation in Definition 3.13.

Proof. By Proposition 3.11 and Lemma 4.4.

5 The maps $K$, $B$, $K^\dagger$, $B^\dagger$

We continue to discuss the $\mathcal{O}_q$-module $V$ from Section 2. To aid in this discussion we bring in some maps $K$, $B$, $K^\dagger$, $B^\dagger$.

Definition 5.1. Let $\{W_i\}_{i=0}^d$ denote a decomposition of $V$. The corresponding map is the element $M \in \text{End}(V)$ such that $(M - q^{d-2i}I)W_i = 0$ for $0 \leq i \leq d$. In other words, $W_i$ is an eigenspace of $M$ with eigenvalue $q^{d-2i}$ for $0 \leq i \leq d$.

Example 5.2. Let $M$ denote a diagonalizable element in $\text{End}(V)$, with eigenvalues $\{q^{d-2i}\}_{i=0}^d$. For $0 \leq i \leq d$ let $W_i$ denote the eigenspace of $M$ for the eigenvalue $q^{d-2i}$. Then $\{W_i\}_{i=0}^d$ is a decomposition of $V$ and $M$ is the corresponding map.

Lemma 5.3. For a decomposition $\{W_i\}_{i=0}^d$ of $V$ the following maps are inverses:

(i) the map corresponding to $\{W_i\}_{i=0}^d$;

(ii) the map corresponding to $\{W_{d-i}\}_{i=0}^d$. 

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Let \( \{ W_i \}_{i=0}^d \) denote a decomposition of \( V \), with corresponding map \( M \). We will describe \( M \) using the diagram:

\[
M = W_0 \bullet W_1 \bullet W_2 \bullet \cdots \bullet W_{d-1} \bullet W_d
\]

We might suppress the labels \( W_i \) along with the \( \bullet \) notation, if they are clear from the context.

**Definition 5.4.** Referring to the diagram in Lemma 4.4, let \( K, B, K^\dagger, B^\dagger \) denote the maps that correspond to the non-horizontal decompositions as shown below:

Below we cite some results about \( K \) and \( B \); similar results hold for \( K^\dagger \) and \( B^\dagger \). By [25, Section 1.1],

\[
\frac{qKA - q^{-1}AK}{q - q^{-1}} = aK^2 + a^{-1}I, \quad \frac{qBA - q^{-1}AB}{q - q^{-1}} = a^{-1}B^2 + aI. \quad (30)
\]

By [14, Theorem 9.9],

\[
aK^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}}KB - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}BK + a^{-1}B^2 = 0. \quad (31)
\]

The equations (30), (31) can be reformulated as follows. By [16, Lemma 12.12],

\[
\frac{qAK^{-1} - q^{-1}K^{-1}A}{q - q^{-1}} = a^{-1}K^{-2} + aI, \quad \frac{qAB^{-1} - q^{-1}B^{-1}A}{q - q^{-1}} = aB^{-2} + a^{-1}I. \quad (32)
\]
By [14, Theorem 9.10],

\[
a^{-1}K^{-2} - \frac{a^{-1}q -aq^{-1}}{q - q^{-1}}K^{-1}B^{-1} - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}B^{-1}K^{-1} + aB^{-2} = 0. \tag{33}
\]

We clarify why (31), (33) are equivalent. They each assert that the maps

\[
\frac{a^{-1}(q - q^{-1})}{a^{-1} - a}K^{-1}B - \frac{a^{-1}q -aq^{-1}}{a^{-1} - a}I, \quad \frac{a^{-1}(q - q^{-1})}{a^{-1} - a}BK^{-1} - \frac{aq - a^{-1}q^{-1}}{a^{-1} - a}I
\]

are inverses, and also that the maps

\[
\frac{a(q - q^{-1})}{a - a^{-1}}B^{-1}K - \frac{aq - a^{-1}q^{-1}}{a - a^{-1}}I, \quad \frac{a(q - q^{-1})}{a - a^{-1}}KB^{-1} - \frac{aq - a^{-1}q^{-1}}{a - a^{-1}}I
\]

are inverses.

\section{How conjugation by $H$ affects $K$, $B$, $K^\dagger$, $B^\dagger$}

We continue to discuss the $\mathcal{O}_q$-module $V$ from Section 2. In this section we describe what happens when one of $K$, $B$, $K^\dagger$, $B^\dagger$ is conjugated by $H$.

\begin{proposition}
The maps $K$, $B$, $K^\dagger$, $B^\dagger$ from Definition 5.4 satisfy

\[
H^{-1}BH = aA - a^2B^{-1}, \quad H^{-1}KH = a^{-1}A - a^{-2}K^{-1}, \tag{34}
\]

\[
H^{-1}B^\dagger H = aA - a^2(B^\dagger)^{-1}, \quad H^{-1}K^\dagger H = a^{-1}A - a^{-2}(K^\dagger)^{-1}. \tag{35}
\]
\end{proposition}

\begin{proof}
We first obtain the equation on the right in (34). Since $H$ commutes with $A$, it suffices to show that

\[
HK^{-1} - K^{-1}H = a(A - aK - a^{-1}K^{-1})H. \tag{36}
\]

For $0 \leq i \leq d$ let $U_i$ denote the eigenspace of $K$ for the eigenvalue $q^{d-2i}$. Thus $\{U_i\}_{i=0}^d$ is a decomposition of $V$, and $K$ is the corresponding map. For $0 \leq i \leq d$ the following holds on $U_i$:

\[
aK + a^{-1}K^{-1} = \theta_i I. \tag{37}
\]

By [21, Theorem 4.6],

\[
(A - \theta_i I)U_i \subseteq U_{i+1} \quad (0 \leq i \leq d - 1), \quad (A - \theta_d I)U_d = 0. \tag{38}
\]

Define

\[
R = A - aK - a^{-1}K^{-1}.
\]

By (37), for $0 \leq i \leq d$ the following holds on $U_i$:

\[
R = A - \theta_i I. \tag{39}
\]
By (38) and (39),
\[ RU_i \subseteq U_{i+1} \quad (0 \leq i \leq d - 1), \quad RU_d = 0. \tag{40} \]

By (40) and the construction,
\[ RK = q^2 KR. \]

For \(0 \leq r \leq d\) we show that (36) holds on \(U_r\). Using (39), (40) we find that for \(0 \leq i \leq d - r\) the following holds on \(U_r\):
\[ R^i = (A - \theta_r I)(A - \theta_{r+1} I) \cdots (A - \theta_{r+i-1} I). \tag{41} \]

Also by (40) we have \(R^{d-r+1} = 0\) on \(U_r\). By Definition 5.4 and the discussion above Lemma 4.4,
\[ U_r + U_{r+1} + \cdots + U_d = V_r + V_{r+1} + \cdots + V_d. \]

The above subspace contains \(U_r\), so by Proposition 3.17 and (41) the following holds on \(U_r\):
\[ H = t_r \sum_{i=0}^{d-r} a^i R^i K^i \left( \frac{q^2}{q^2} \right)_i. \]

We may now argue that on \(U_r\),
\[ HK^{-1} - K^{-1}H = t_r \sum_{i=0}^{d-r} a^i R^i K^i \left( \frac{q^2}{q^2} \right)_i (1 - q^2 i) \]
\[ = t_r \sum_{i=1}^{d-r} a^i R^i K^{i-1} \left( \frac{q^2}{q^2} \right)_{i-1} \]
\[ = t_r \sum_{i=0}^{d-r-1} a^{i+1} R^{i+1} K^i \left( \frac{q^2}{q^2} \right)_i \]
\[ = t_r \sum_{i=0}^{d-r} a^{i+1} R^{i+1} K^i \left( \frac{q^2}{q^2} \right)_i \]
\[ = aRH \]
\[ = a(A - aK - a^{-1}K^{-1})H. \]

We have obtained (36), which implies the equation on the right in (34). The remaining equations in (34), (35) are obtained by the following observations. In (34), the equation on the left is obtained from the equation on the right by replacing the decomposition \(\{V_i\}_{i=0}^d\) by its inversion \(\{V_{d-i}\}_{i=0}^d\). Moreover (35) is obtained from (34) by replacing the decomposition \(\{V_i^*\}_{i=0}^d\) by its inversion \(\{V_{d-i}^*\}_{i=0}^d\). 

\[ \square \]
Corollary 6.2. Referring to the diagram in Lemma 4.5, the maps that correspond to the non-horizontal decompositions are shown below:

\[ L(A^*) \]

\[ aA - a^2 B^{-1} \]
\[ a^{-1} A - a^{-2} K^{-1} \]
\[ aA - a^2 (B^\dagger)^{-1} \]
\[ a^{-1} A - a^{-2} (K^\dagger)^{-1} \]

\[ A \]

Proof. By Propositions 3.7, 6.1.

The following result is a reformulation of Proposition 6.1.

Proposition 6.3. The maps \( K, B, K^\dagger, B^\dagger \) from Definition 5.4 satisfy

\[ HB^{-1}H^{-1} = a^{-1} A - a^{-2} B, \quad HK^{-1}H^{-1} = aA - a^2 K, \] (42)
\[ H(B^\dagger)^{-1}H^{-1} = a^{-1} A - a^{-2} B^\dagger, \quad H(K^\dagger)^{-1}H^{-1} = aA - a^2 K^\dagger. \] (43)

Proof. These equations are reformulations of the equations in (34), (35).
**Corollary 6.4.** Referring to the diagram in Lemma 4.6, the maps that correspond to the non-horizontal decompositions are shown below:

\[ L^{-1}(A^*) \]

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
a^{-1}A - a^{-2}B
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
a - a^{-1}K - a^{-2}B^\dagger
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
aA - a^2K^\dagger
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
a^{-1}A - a^{-2}B^\dagger
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
a^{-1}A - a^{-2}B
\end{array}
\end{array} \]

\[ A \]

**Proof.** By Propositions 3.7, 6.3.

Shortly, we will make use of the following four maps:

\[ \frac{aK - a^{-1}B}{a - a^{-1}}, \quad \frac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}. \]  

(44)

\[ \frac{aK^\dagger - a^{-1}B^\dagger}{a - a^{-1}}, \quad \frac{a^{-1}(K^\dagger)^{-1} - a(B^\dagger)^{-1}}{a^{-1} - a}. \]  

(45)

The map on the left in (44) first appeared in [15, Definition 6.1]. Both maps in (44) appear in [16, Section 12]. By (34),

\[ H^{-1} \frac{aK - a^{-1}B}{a - a^{-1}} H = \frac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}. \]  

(46)

By (35),

\[ H^{-1} \frac{aK^\dagger - a^{-1}B^\dagger}{a - a^{-1}} H = \frac{a^{-1}(K^\dagger)^{-1} - a(B^\dagger)^{-1}}{a^{-1} - a}. \]  

(47)

**Lemma 6.5.** Each map in (44), (45) is diagonalizable, with eigenvalues \( \{q^{d-2i}\}^{d}_{i=0} \).

**Proof.** By [15, Lemma 8.1], the result holds for the map on the left in (44). By this and (46), the result holds for the map on the right in (44). Replacing the decomposition \( \{V_i^*\}^{d}_{i=0} \) by its inversion \( \{V_{d-i}^*\}^{d}_{i=0} \), we obtain the result for the maps (45). □
7 Equitable triples

We continue to discuss the $\mathcal{O}_q$-module $V$ from Section 2. As we compare the eigenspace decompositions for $A, A^*, L^\pm(A^*)$ on $V$, we will use the concept of an equitable triple. We will define this concept after some preliminary remarks.

Let $X, Y$ denote elements in $\text{End}(V)$ such that

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I.$$ \hspace{1cm} (48)

We call (48) the $q$-Weyl relation.

Lemma 7.1. For the above maps $X, Y$ the following (i), (ii) hold:

(i) Let $u \in V$ denote an eigenvector for $X$ with nonzero eigenvalue $\lambda$. Then the vector $(Y - \lambda^{-1}I)u$ is either zero, or an eigenvector for $X$ with eigenvalue $q^{-2}\lambda$.

(ii) Let $v \in V$ denote an eigenvector for $Y$ with nonzero eigenvalue $\mu$. Then the vector $(X - \mu^{-1}I)v$ is either zero, or an eigenvector for $Y$ with eigenvalue $q^2\mu$.

Proof. (i) One checks that

$$(X - q^{-2}\lambda I)(Y - \lambda^{-1}I)u = 0.$$

(ii) Similar to the proof of (i) above.

Corollary 7.2. (See [22, Lemma 11.4].) For the above maps $X, Y$ the following are equivalent:

(i) $X$ is diagonalizable with eigenvalues $\{q^{d-2i}\}_{i=0}^d$;

(ii) $Y$ is diagonalizable with eigenvalues $\{q^{d-2i}\}_{i=0}^d$.

Assume that (i), (ii) hold. Then the eigenspace decompositions of $X, Y$ are described by the following diagram:

\[ \begin{array}{c}
  Y \\
  \downarrow \\
  X 
\end{array} \]

Proof. This is a routine consequence of Lemma 7.1.

Definition 7.3. An equitable triple on $V$ is a 3-tuple $X, Y, Z$ of invertible elements in $\text{End}(V)$ such that

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I, \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = I.$$
The references [26], [36] describe how equitable triples are related to the quantum group $U_q(\mathfrak{sl}_2)$.

**Proposition 7.4.** For each row in the table below, the given 3-tuple $X, Y, Z$ is an equitable triple on $V$.

| example | $X$        | $Y$        | $Z$        |
|---------|------------|------------|------------|
| 1       | $aA - a^2 K$ | $M^{-1}$   | $K$        |
| 2       | $a^{-1}A - a^{-2}B$ | $M^{-1}$   | $B$        |
| 3       | $aA - a^2 K^\dagger$ | $(M^\dagger)^{-1}$ | $K^\dagger$ |
| 4       | $a^{-1}A - a^{-2}B^\dagger$ | $(M^\dagger)^{-1}$ | $B^\dagger$ |
| 5       | $K^{-1}$   | $N^{-1}$   | $a^{-1}A - a^{-2}K^{-1}$ |
| 6       | $B^{-1}$   | $N^{-1}$   | $aA - a^2 B^{-1}$ |
| 7       | $(K^\dagger)^{-1}$ | $(N^\dagger)^{-1}$ | $a^{-1}A - a^{-2}(K^\dagger)^{-1}$ |
| 8       | $(B^\dagger)^{-1}$ | $(N^\dagger)^{-1}$ | $aA - a^2 (B^\dagger)^{-1}$ |

In the above table we abbreviate

$$M = \frac{aK - a^{-1}B}{a - a^{-1}}, \quad N = \frac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}, \quad (49)$$

$$M^\dagger = \frac{aK^\dagger - a^{-1}B^\dagger}{a - a^{-1}}, \quad N^\dagger = \frac{a^{-1}(K^\dagger)^{-1} - a(B^\dagger)^{-1}}{a^{-1} - a}. \quad (50)$$

**Proof.** For examples 1, 2 the $q$-Weyl relations from Definition 7.3 are a routine consequence of (30), (31). They also follow from [16, Lemmas 11.5, 12.8, Theorem 12.5]. Examples 3, 4 are obtained from examples 1, 2 by replacing the decomposition $\{V^*_i\}_{i=0}^d$ by its inversion $\{V^*_i\}_{i=0}^d$. For examples 5, 6 the $q$-Weyl relations from Definition 7.3 are a routine consequence of (32), (33). They also follow from [16, Lemmas 11.5, 12.16, Theorem 12.14]. Examples 7, 8 are obtained from examples 5, 6 by replacing the decomposition $\{V^*_i\}_{i=0}^d$ by its inversion $\{V^*_i\}_{i=0}^d$.

8 The main results

We continue to discuss the $O_q$-module $V$ from Section 2. In this section we display some diagrams that illustrate how the eight equitable triples from Proposition 7.4 are related to the eigenspace decompositions for $A, A^*, L^{\pm 1}(A^*)$ on $V$.

In the next result, we compare the diagrams in Definition 5.4 and Corollary 6.2. In order to make the comparison, we reflect the diagram in Corollary 6.2 about the horizontal line segment labelled $A$.
Theorem 8.1. We have

\[ A^* \]

\[ L(A^*) \]

Proof. Recall the abbreviations \( N, N^\downarrow \) from (49), (50). The diagram in the theorem statement is making some assertions about \( N \) and \( N^\downarrow \). We now verify these assertions, starting with \( N \). By Lemma 6.5, \( N \) is diagonalizable with eigenvalues \( \{q^d_{-2i}\}_{i=0}^d \). By Proposition 7.4 one finds that the q-Weyl relation is satisfied by the pair \( K^{-1}, N^{-1} \) and also the pair \( N^{-1}, a^{-1}A - a^{-2}K^{-1} \). In the diagram of the theorem statement, the vertical line segment on the left represents the \( N \)-eigenspace decomposition of \( V \). This line segment is properly attached in the diagram, due to Corollary 7.2 and our above findings. We have verified the assertions about \( N \). The assertions about \( N^\downarrow \) are verified in a similar manner. \( \square \)
The following theorem contains a diagram. This diagram is obtained from the diagram in Theorem 8.1 by inverting the orientation for some of the edges.

**Theorem 8.2.** For each oriented 3-cycle in the diagram below, the corresponding maps form an equitable triple.

\[ A^* \]

\[ A \]

\[ L(A^*) \]

*Proof.* The above diagram contains four oriented 3-cycles. For each one, the corresponding maps form an equitable triple by examples 5–8 in the table of Proposition 7.4. \(\square\)
In the next result, we compare the diagrams in Definition 5.4 and Corollary 6.4. In order to make the comparison, we reflect the diagram in Corollary 6.4 about the horizontal line segment labelled $A$.

**Theorem 8.3.** We have

$$A^*$$

$$L^{-1}(A^*)$$

$${aK-a^{-1}B \over a-a^{-1}}$$

$${aK^\perp-a^{-1}B^\perp \over a-a^{-1}}$$

$${a^{-1}A-a^{-2}B \over a^{-2}}$$

$$aA-a^2K$$

$$a^{-1}A-a^{-2}B^\perp$$

$${aA-a^2K^\perp \over a^{-1}}$$

Proof. Recall the abbreviations $M, M^\perp$ from (19), (50). The diagram in the theorem statement is making some assertions about $M$ and $M^\perp$. We now verify these assertions, starting with $M$. By Lemma 6.5, $M$ is diagonalizable with eigenvalues $\{q^{d-2i}\}_{i=0}d$. By Proposition 7.4 one finds that the $q$-Weyl relation is satisfied by the pair $aA-a^2K, M^{-1}$ and also the pair $M^{-1}, K$. In the diagram of the theorem statement, the vertical line segment on the left represents the $M$-eigenspace decomposition of $V$. This line segment is properly attached in the diagram, due to Corollary 7.2 and our above findings. We have verified the assertions about $M$. The assertions about $M^\perp$ are verified in a similar manner. $\Box$
The following theorem contains a diagram. This diagram is obtained from the diagram in Theorem 8.3 by inverting the orientation for some of the edges.

**Theorem 8.4.** For each oriented 3-cycle in the diagram below, the corresponding maps form an equitable triple.

![Diagram](attachment:image.png)

**Proof.** The above diagram contains four oriented 3-cycles. For each one, the corresponding maps form an equitable triple by examples 1–4 in the table of Proposition 7.4.

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References

[1] E. Bannai and T. Ito. *Algebraic combinatorics I. Association schemes*. The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984.

[2] P. Baseilhac. An integrable structure related with tridiagonal algebras. *Nuclear Phys. B* 705 (2005) 605–619.

[3] P. Baseilhac. Deformed Dolan-Grady relations in quantum integrable models. *Nuclear Phys. B* 709 (2005) 491–521.

[4] P. Baseilhac and S. Belliard. Generalized $q$-Onsager algebras and boundary affine Toda field theories. *Lett. Math. Phys.* 93 (2010) 213–228.

[5] P. Baseilhac and S. Belliard. The half-infinite XXZ chain in Onsager's approach. *Nuclear Phys. B* 873 (2013) 550–584.

[6] P. Baseilhac and S. Belliard. An attractive basis for the $q$-Onsager algebra; arXiv:1704.02950.

[7] P. Baseilhac and K. Koizumi. A new (in)finite dimensional algebra for quantum integrable models. *Nuclear Phys. B* 720 (2005) 325–347.

[8] P. Baseilhac and K. Koizumi. A deformed analogue of Onsager’s symmetry in the XXZ open spin chain. *J. Stat. Mech. Theory Exp.* 2005, no. 10, P10005, 15 pp. (electronic).

[9] P. Baseilhac and K. Koizumi. Exact spectrum of the XXZ open spin chain from the $q$-Onsager algebra representation theory. *J. Stat. Mech. Theory Exp.* 2007, no. 9, P09006, 27 pp. (electronic).

[10] P. Baseilhac and S. Kolb. Braid group action and root vectors for the $q$-Onsager algebra. *Transformation Groups* (2020) https://doi.org/10.1007/s00031-020-09555-7.

[11] P. Baseilhac and K. Shigechi. A new current algebra and the reflection equation. *Lett. Math. Phys.* 92 (2010) 47–65.

[12] S. Belliard and N. Crampe. Coideal algebras from twisted Manin triples. *J. Geom. Phys.* 62 (2012) 2009–2023.

[13] S. Bockting-Conrad. Two commuting operators associated with a tridiagonal pair. *Linear Algebra Appl.* 437 (2012) 242–270.

[14] S. Bockting-Conrad. Tridiagonal pairs of $q$-Racah type, the double lowering operator $\psi$, and the quantum algebra $U_q(\mathfrak{sl}_2)$. *Linear Algebra Appl.* 445 (2014) 256–279.

[15] S. Bockting-Conrad. Some $q$-exponential formulas involving the double lowering operator $\psi$ for a tridiagonal pair; arXiv:1907.01157v2.

[16] S. Bockting-Conrad and P. Terwilliger. The algebra $U_q(\mathfrak{sl}_2)$ in disguise. *Linear Algebra Appl.* 459 (2014) 548–585.
[17] A. E. Brouwer, A. Cohen, A. Neumaier. *Distance-regular graphs*. Springer-Verlag, Berlin 1989.

[18] G. Gasper and M. Rahman. *Basic hypergeometric series. With a foreword by Richard Askey. Second edition*. Encyclopedia of Mathematics and its Applications, 96, Cambridge University Press, Cambridge, 2004.

[19] T. Ito. TD-pairs and the $q$-Onsager algebra. *Sugaku Expositions*. 32 (2019) 205–232.

[20] T. Ito, K. Nomura, P. Terwilliger. A classification of sharp tridiagonal pairs. *Linear Algebra Appl.* 435 (2011) 1857–1884.

[21] T. Ito, K. Tanabe, P. Terwilliger. Some algebra related to $P$- and $Q$-polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192.

[22] T. Ito and P. Terwilliger. The $q$-tetrahedron algebra and its finite-dimensional irreducible modules. *Comm. Algebra* 35 (2007) 3415–3439.

[23] T. Ito and P. Terwilliger. Distance-regular graphs of $q$-Racah type and the $q$-tetrahedron algebra. *Michigan Math. J.* 58 (2009) 241–254.

[24] T. Ito and P. Terwilliger. Tridiagonal pairs of $q$-Racah type. *J. Algebra* 322 (2009) 68–93.

[25] T. Ito and P. Terwilliger. The augmented tridiagonal algebra. *Kyushu J. Math.* 64 (2010) 81–144.

[26] T. Ito, P. Terwilliger, C.H. Weng. The quantum algebra $U_q(sl_2)$ and its equitable presentation. *J. Algebra* 298 (2006) 284–301.

[27] R. Koekoek, P. A Lesky, R. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. With a foreword by Tom H. Koornwinder. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[28] S. Kolb. Quantum symmetric Kac-Moody pairs. *Adv. Math.* 267 (2014) 395–469.

[29] S. Kolb and J. Pellegrini. Braid group actions on coideal subalgebras of quantized enveloping algebras. *J. Algebra* 336 (2011), 385–416.

[30] K. Nomura and P. Terwilliger. Sharp tridiagonal pairs. *Linear Algebra Appl.* 429 (2008) 79–99.

[31] P. Terwilliger. The subconstituent algebra of an association scheme. III. *J. Algebraic Combin.* 2 (1993) 177–210.

[32] P. Terwilliger. Two relations that generalize the $q$-Serre relations and the Dolan-Grady relations. In *Physics and Combinatorics 1999 (Nagoya)*, 377–398, World Scientific Publishing, River Edge, NJ, 2001.
[33] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. Linear Algebra Appl. 330 (2001) 149–203.

[34] P. Terwilliger. An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006.

[35] P. Terwilliger. The universal Askey-Wilson algebra. SIGMA Symmetry Integrability Geom. Methods Appl. 7 (2011) Paper 069, 24 pp.

[36] P. Terwilliger. Finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules from the equitable point of view. Linear Algebra Appl. 439 (2013) 358–400.

[37] P. Terwilliger. The $q$-Onsager algebra and the positive part of $U_q(\mathfrak{sl}_2)$. Linear Algebra Appl. 521 (2017) 19–56.

[38] P. Terwilliger. Tridiagonal pairs of $q$-Racah type, the Bockting operator $\psi$, and $L$-operators for $U_q(L(\mathfrak{sl}_2))$. Ars Math. Contemp. 14 (2018) 55–65.

[39] P. Terwilliger. The Lusztig automorphism of the $q$-Onsager algebra. J. Algebra 506 (2018) 56–75.

[40] P. Terwilliger. The $q$-Onsager algebra and the universal Askey-Wilson algebra. SIGMA Symmetry Integrability Geom. Methods Appl. 14 (2018) Paper No. 044, 18 pp.

[41] P. Terwilliger and R. Vidunas. Leonard pairs and the Askey-Wilson relations. J. Algebra Appl. 3 (2004) 411–426.

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