Critical Behavior of the Ferromagnetic Ising Model on a Sierpiński Carpet: Monte Carlo Renormalization Group Study

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Abstract

We perform a Monte Carlo Renormalization Group analysis of the critical behavior of the ferromagnetic Ising model on a Sierpiński fractal with Hausdorff dimension \(d_f \simeq 1.8928\). This method is shown to be relevant to the calculation of the critical temperature \(T_c\) and the magnetic eigen-exponent \(\gamma_h\) on such structures. On the other hand, scaling corrections hinder the calculation of the temperature eigen-exponent \(\gamma_t\). At last, the results are shown to be consistent with a finite size scaling analysis.

Keyword: MCRG, fractal, phase transition, critical exponents.

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1 Introduction

Since Mandelbrot attracted people’s attention to the fractals in the 60’%s and 70’%s [1], scientists began to take auto-similarity and scaling invariance as fundamental rules of nature. In the description of second order phase transitions, Widom’s homogeneity hypothesis [2] and Kadanoff’s real space renormalization group scheme [3] are based on the invariance of the physical behavior under any change of scale at the criticality where the correlation length is divergent. The validity of these hypothesis and scheme has been verified in different systems with an integer dimension which possess a translational symmetry [4]. As a matter of fact, a system with translational symmetry is auto-similar and can be considered as a particular case of fractal. Fractals are natural candidates to represent systems with non-integer dimensions. It turns out to be of fundamental relevance to know if the hypothesis and the scheme mentioned above still work on a general fractal system, where the translational symmetry is lost and replaced by scale invariance.

The critical behavior of the Ising model on fractal lattices has been firstly studied by Gefen and his co-workers in the 80’%s [5]. Their works were based on the Migdal-Kadanoff bond-moving renormalization method and showed that the topological features of the fractals plays an important role in the determination of the critical behavior. As a main result, they found that a phase transition at non-zero temperature can occur only if the fractal has an infinite ramification order. Later, Bonnier et al. used an alternative decimation method [6] and high-temperature expansions [7] to study the phase transition on various fractals, namely Sierpiński carpets. They also found that the critical exponents depend on the geometrical properties of the fractal. In the above studies, the applied methods are approximative and the results are not always consistent. Moreover, in these theoretical analyses, the spins were placed at the corners of the occupied squares of the Sierpiński carpets; in these cases, the number of spins doesn’t follow a power law.
of the lattice size and the Hausdorff dimension should not be expected to enter in the
description of the critical phenomena on such systems. Recently, due to the progress in
the simulation methods and the growth of the computer power, the critical behavior of
the Ising model on fractals of Hausdorff dimension \(d_f\) between 1 and 3 has been studied
numerically in a much more precise way [8, 9, 10, 11]. The results showed that the finite-
size scaling (FSS) analysis works in the case of fractals, although the convergence towards
the thermodynamical limit can be very slow when \(d_f < 2\), and that the hyperscaling law
\(d_f = 2\beta/\nu + \gamma/\nu\) is verified. Moreover, discrepancies with the predictions of the \(\epsilon\)-
expansions [13] were observed. The universality of phase transitions on fractals is said
to be weak [5, 14]. It is worth noting that the spins were placed at the center of the
occupied squares of the Sierpiński carpets in these studies; consequently, the number
of the spins increases as a power law of the lattice size with an exponent equal to the
Hausdorff dimension \(d_f\). In this case, \(d_f\) takes the place of the space dimension in Widom’s
homogeneity hypothesis because of the conservation of the total free energy under a
change of length scale; the generalization of this hypothesis in the case of fractals is quite
straightforward [12].

Monte Carlo renormalization group (MCRG) has been shown to be a powerful tool
in the study of critical phenomena [15, 16, 17, 18, 19]. It combines Monte Carlo (MC)
simulation techniques with the analysis of the real space renormalization group (RSRG).
It has been, so far, applied to systems with translational symmetry. A RSRG analysis in-
volves an infinite number of couplings when dealing with the Hamiltonian. Theoreticians
usually truncate the number of couplings to a finite one in order to make the calculations
tractable. It, therefore, makes RSRG an approximate method and the accuracy of the
results is linked to the number of the couplings considered. Even if only few couplings
are considered, the calculations associated with the process of decimation remain tedious.
With the aid of MC simulation, a renormalization-group calculation with a larger number
of couplings becomes possible and the results are expected to be more reliable. However,
some crucial problems are encountered in a MCRG study:

(1) The critical slowing down:

Hypothetically, the accuracy of the thermodynamical averages can be improved by
performing a larger number of MC steps to reduce the statistical errors. In fact, due
to the divergence of the correlation length at the criticality, local MC algorithms
(for e.g., the Metropolis one) suffer from critical slowing down, which hinders the
computation of accurate thermodynamical averages. Fortunately, the use of cluster
algorithms, for e.g., the Swendsen-Wang algorithm [20] and the Wolff algorithm
[21], enables to improve significantly the efficiency of the simulations.

(2) The finite size effects:

RSRG method involves an infinite size lattice, whereas computer simulations can
only be performed on finite size lattices. One should perform the simulation on
several large systems and then extrapolate the results to the infinite one.

The MCRG method, according to which one plays the game of changing the length scale,
should be a natural approach to explore the critical behavior of phase transitions on
hierarchical lattices. The purpose of this paper is to study the critical behavior of the
Ising model on a Sierpiński carpet using the MCRG method. As far as we know, it is
the first time that the MCRG method is used in the case of fractal lattices. The paper
is organized as follows: In Sec. II, we present the model and recall briefly the MCRG
method. We explain how the simulation is setup in Sec. III. Sec. IV is devoted to the
calculation of the critical temperature $T_c$, the temperature eigen-exponent $y_t$, and the
magnetic eigen-exponent $y_h$. A FSS analysis is set out in Sec. V for a consistency check.

2 Sierpiński carpet and MCRG method

A 3 by 3 square lattice with its central subsquare removed is chosen as “generating cell”.
The Sierpiński carpet we deal with is constructed iteratively from this generating cell:

1. We take the generating cell as the carpet at the first iteration step and denote it
   $SC(3, 1, 1)$.

2. The carpet at the $(k+1)$-th iteration step $SC(3, 1, k+1)$ is constructed by enlarging
   three times the size of the carpet at the $k$-th iteration step and replacing each
enlarged un-removed subsquare by the whole generating cell. This carpet becomes a “true” fractal in the mathematical sense when \( k \) tends to infinity.

The size of the lattice \( SC(3, 1, k) \) is equal to \( L_k = 3^k \). A spin is set at each center of an un-removed subsquare (so-called occupied square). The number of spins on \( SC(3, 1, k) \) is therefore equal to \( N_k = 8^k \). We notice that \( N_k \) is equal to \( L_k^{d_f} \) where \( d_f = \log 8 / \log 3 \simeq 1.892789 \) is the Hausdorff dimension. The critical behavior of the ferromagnetic Ising model on this Sierpiński carpet has been recently studied independently by P. Monceau et al. \[8, 11\] and J. Carmona et al. \[9\] by means of MC simulation. Thus, estimations of the critical temperature are available and the comparison between the two methods will be possible.

Let us briefly recall the MCRG method. The geometrical structure of a Sierpiński carpet remains invariant during a RSRG process involving a change in the length scale from 1 to \( b \), provided that \( b \) is equal to an integer power of the size of its generating cell. In our case, we set \( b \) equal to 3 and increase the length scale by this factor at each renormalization step. The majority rule is used to decimate the spin blocks: a new spin is assigned to a given block according to the sign of the summation of the spin states of the 8 occupied sites in this block; if the sum is zero, a spin state +1 or −1 is assigned to the block with the same probability. The reduced Hamiltonian after \( n \) renormalization steps (\( n \) is a positive integer) reads as

\[
\mathcal{H}^{(n)}(\{K^{(n)}_\alpha\}_{\alpha=1}^\infty) = \sum_{\alpha=1}^\infty K^{(n)}_\alpha S^{(n)}_\alpha,
\]

where \( \{K^{(n)}_\alpha\}_{\alpha=1}^\infty \) is the coupling set after \( n \) renormalization steps and \( S^{(n)}_\alpha \)'s are the conjugate lattice sums of the spin products on the reduced network. The matrix \( \mathcal{T} \), which describes the flow of the couplings from the \( n \)-th renormalization step to the \( (n + 1) \)-th one, is defined by

\[
T^{(n+1,n)}_{\alpha\beta} = \frac{\partial K^{(n+1)}_\alpha}{\partial K^{(n)}_\beta}
\]

and satisfies the following relation:

\[
[\partial S]^{(n+1,n)}_{\gamma\beta} = \sum_{\alpha=1}^\infty [\partial S]^{(n+1,n+1)}_{\gamma\alpha} T^{(n+1,n)}_{\alpha\beta},
\]
where \( \partial S^{(n,m)}_{\alpha\beta} \equiv \partial (S^{(n)}_\alpha / \partial K^{(m)}_\beta) \) can be calculated by \( \langle S^{(n)}_\alpha S^{(m)}_\beta \rangle - \langle S^{(n)}_\alpha \rangle \langle S^{(m)}_\beta \rangle \). One can hence calculate the matrix \( \mathbf{T} \) by inverting the above relation. Two critical exponents enable to describe the static scaling properties of the system: the temperature eigen-exponent \( y_t \) and the magnetic one \( y_h \). They describe respectively the scaling behavior of the reduced temperature \( t = T/T_c - 1 \) (where \( T_c \) is the critical temperature) and the external magnetic field \( h \) near the criticality under a change of the length unit from 1 to \( b \):

\[
\begin{align*}
t & \rightarrow \ t' = b^{y_t} t , \\
h & \rightarrow \ h' = b^{y_h} h .
\end{align*}
\]

\( b^{y_t} \) and \( b^{y_h} \) are associated with the two relevant directions of the renormalization flows and can be obtained by finding the largest eigenvalues of the \( \mathbf{T} \) matrix in the even- and odd-coupling subspaces, respectively.

### 3 Simulation setup

Up to 4-spin couplings have been considered in our study and the interaction range has been restricted within a \( 3 \times 3 \) square. There are 25 even couplings and 11 odd couplings as shown in Fig.1(a) and Fig.1(b), respectively. The symmetry number obtained by rotating and reflecting a given coupling is indicated in the third column of the figure. These couplings are listed in decreasing order according to the importance factor proposed by Blöte et al. \[17\]:

\[
F = (2^{s/2} \bar{r})^{-1},
\]

where \( \bar{r} \) is the average distance between the \( s \) spins.

The value of \( F \) is given in the right column of the figure. The simulations have been carried out at two different temperatures, \( T = 1.4813 \) obtained by Carmona et al. \[9\] and \( T = 1.4795 \) obtained by Monceau et al. \[11\]. The size of the starting Sierpiński carpet varies between \( 3^4 = 81 \) and \( 3^8 = 6561 \). The simulation procedure is organized as follows:

1. An equilibrium spin configuration on the starting Sierpiński carpet \( SC(3,1,k) \) is generated by the Wolff algorithm \[21\] at the simulation temperature.

2. The lattice sums conjugated to the 25 even couplings and to the 11 odd ones are calculated.
3. A decimation is done, according to the majority rule, by dividing the size of the Sierpiński carpet by 3 and the lattice sums associated with the renormalized carpet are calculated.

4. The step 3 is repeated until the size of the renormalized Sierpiński carpet is smaller than 9.

A data sample of the lattice sums is built up from $10^5$ steps of the above simulation, with periodic boundary conditions. At each simulation temperature, for the five different sizes of the starting Sierpiński carpet, 10 independent data samples are collected. The errors bars are estimated from a standard statistical analysis of these data.

4 MCRG Results

4.1 Critical temperature $T_c$

The size of the lattice after $n + p$ renormalization steps from the starting Sierpiński carpet of size $L_{k+p}$ is equal to the size of the lattice after $n$ renormalization steps from the starting Sierpiński carpet of size $L_k$. At the fixed point $K^*$, the thermodynamical averages of the lattice sums calculated from renormalized carpets with the same size should be independent of the size of the starting Sierpiński carpet. Let us assume that the initial coupling vector $K^{(0)}$ is close to $K^*$. The difference of the lattice sums on the reduced lattices of equal size, derived from the small displacement $\delta K = K^* - K^{(0)}$, should satisfy the following relation:

$$\left(\langle S^{(n+p)}_\alpha \rangle_{L_{k+p}} - \langle S^{(n)}_\alpha \rangle_{L_k}\right)_{K^{(0)}} \simeq - \sum_{\beta} \left( [\partial S_{L_{k+p}}^{(n+p,0)}]_{\alpha\beta} - [\partial S_{L_k}^{(n,0)}]_{\alpha\beta} \right)_{K^{(0)}} \delta K_{\beta},$$

where the suffix $L_{k+p}$ or $L_k$ indicates that the physical quantity is obtained from the starting Sierpiński carpet of size $L_{k+p}$ or $L_k$. In the even coupling subspace, the nearest neighborhood coupling $K_1$ represents the inverse of the temperature. If we neglect the contribution from the directions other than the nearest neighborhood coupling and set $\alpha = 1$ in Eq. (4), the critical temperature can be calculated by

$$T_c^{[k+p,k]}(n) = \left( K_1^* \right)^{-1} = \left[ K_1^{(0)} + \delta K_1 \right]^{-1}.$$
\[
K_1^{(0)} - \frac{\left( (S^{(n+p)})_{L_k} - (S^{(n)})_{L_k} \right)_{K_1^{(0)}}}{\left( [\partial S_{L_k+p}]_{1}^{(n+p,0)} - [\partial S_{L_k}]_{1}^{(n,0)} \right)_{K_1^{(0)}}}^{-1}
\]
where the symbol \( T_{c}^{[k+p,k]}(n) \) indicates that the temperature is obtained from two starting carpets of size \( 3^{k+p} \) and \( 3^k \) after \( n + p \) and \( n \) renormalization steps respectively. Fig. 2 shows the evolution of \( T_{c}^{[k+p,k]}(n) \) calculated from pairs of the starting Sierpiński carpets at the simulation temperatures \( T_{sim} = \left[ K_1^{(0)} \right]^{-1} = 1.4813 \) and \( T_{sim} = \left[ K_1^{(0)} \right]^{-1} = 1.4795 \). These results deserve the following comments:

i) At the two simulation temperatures, \( T_{c}^{[k+p,k]}(n) \) decreases as \( n \) increases and tends to converge to some value.

ii) If we fix \( p = 1 \), the value of \( T_{c}^{[k+p,k]}(n) \) decreases as \( k \) increases. We can see that \( T_{c}^{[5,4]}(n) > T_{c}^{[6,5]}(n) > T_{c}^{[7,6]}(n) > T_{c}^{[8,7]}(n) \) and the curves \( T_{c}^{[7,6]}(n) \) and \( T_{c}^{[8,7]}(n) \) are very close to each other.

iii) \( T_{c}^{[k+p,k]}(n) \) decreases as \( k \) decreases if the size of the largest starting Sierpiński carpet \( 3^{k+p} \) is fixed. One can observe that \( T_{c}^{[8,7]}(n) > T_{c}^{[8,6]}(n) > T_{c}^{[8,5]}(n) > T_{c}^{[8,4]}(n) \) where the size of the largest starting carpet is fixed at 6561. These curves tend to converge to some temperature while \( n \) increases.

The best estimation for the critical temperature can be obtained from the 6-data points \( T_{c}^{[8,7]}(n) \) for \( n = 0, 1, \cdots, 5 \), by performing the following three-parameter \((T_c, A_0, n_0)\) fit:

\[
T_{c}^{[k+p,k]}(n) = T_c + A_0 3^{-n/n_0}.
\]  

The results of the fits at the two simulation temperatures lead to:

at \( T_{sim} = 1.4813 \), \((T_c, A_0, n_0) = (1.47959(15), 0.02718(28), 0.839(25))\) with the reliability \( R^2 = 0.99968 \) and

at \( T_{sim} = 1.4795 \), \((T_c, A_0, n_0) = (1.47933(27), 0.02827(49), 0.868(42))\) with the reliability \( R^2 = 0.99911 \).

The best value of the critical temperature can be estimated by taking the average of the results of the two fits; we hence have \( T_c = 1.47946(16) \).
In the following subsections, the eigen-exponents will be calculated from simulations carried out at $T_{\text{sim}} = 1.4795$, since it lies closer to the best estimate of the critical temperature $T_c = 1.47946(16)$. The simulations carried out at $T_{\text{sim}} = 1.4813$ will provide a consistency check. It is well known that the exponents provided by the MCRG method are not very sensitive to small deviations from the critical temperature.

### 4.2 Temperature eigen-exponent $y_t$

We work in the even-coupling subspace and consider in turn the first 1 to 25 couplings shown in Fig. 1(a). We calculate the $T$ matrix in the subspace at each renormalization level by inverting Eq. (3) with the help of the Gauss-Jordan elimination method (§2.1 in Ref. [22]). The dimension of $T$ is $N_{\text{coupling}} \times N_{\text{coupling}}$ where $N_{\text{coupling}}$ varies between 1 and 25. The largest eigenvalue $\lambda_t$ of $T$ is obtained by applying the eigenvalue searching method for a real non-symmetric square matrix given in §11.5 and §11.6 of Ref. [22]. The temperature eigen-exponent is then calculated by $y_t = \log_3 \lambda_t$. Fig. 3 shows the evolution of the eigen-exponent $y_t$ as a function of the number of even couplings $N_{\text{coupling}}$, obtained from starting Sierpiński carpets of different sizes and at different renormalization levels. One can see that $y_t$ tends to converge towards stable values when $N_{\text{coupling}}$ increases, within the statistical errors, excepted at the highest renormalization levels where the sizes of the reduced lattices are $3^3$ and $3^2$. We estimate $y_t$ in the infinite couplings limit by taking the average of the values of $y_t$ obtained from 16 to 25 even couplings. The average value at each renormalization level is reported in Table 1 where we have disregarded the unreliable results at the highest renormalization levels. We find that the finite-size effect is not significant because at the same renormalization level, the values of $y_t$ obtained from the starting Sierpiński carpets of different sizes are about the same. A similar situation has been observed in the study of the 3D Ising model on regular lattices [17, 18]. In order to extrapolate the value of $y_t$ on the starting Sierpiński carpet of infinite size, we plot in the Fig. 4 the value of $\lambda_t$ obtained at level $n - (n + 1)$ from the starting carpet $SC(3, 1, n + 4)$ with respect to $n$. We find that, up to the renormalization levels which were performed, $\lambda_t$ seems not to be convergent. This behavior is quite different from the one observed in the case of the 3D Ising model on regular lattices [17, 18] where
\( \lambda_t \) tends to converge to some value while \( n \) increases. We, hence, cannot estimate the convergence by performing the three-parameter fit \((\lambda_t^*, a_t, \omega_t)\) proposed in Ref. 18:

\[
\lambda_t = \lambda_t^* + a_t 3^{-\omega t n}.
\]  

(7)

The lack of convergence in the case of the Sierpiński carpet may be due to a small value of the scaling correction exponent \( \omega_t \). At the present stage, we are only able to provide an upper bound for the Sierpiński carpet of infinite size: \( \lambda_t < 1.781(10) \) or \( y_t < 0.5254(51) \). The same situation is also observed at \( T_{\text{sim}} = 1.4813 \). The difficulty in estimating the value of \( y_t \) has already been observed in the study based on FSS analysis [11]. According to Widom’s homogeneity hypothesis, \( y_t \) is equal to the inverse of the correlation-length exponent \( \nu \). Monceau et al. [11] have shown that the maxima values of the logarithmic derivatives \( \Phi_i \) (for \( i = 1, 2 \)), defined by

\[
\Phi_i = \frac{\langle E \rangle - \langle E|M|_i \rangle}{\langle |M|^i \rangle}
\]

(8)

where \( E \) is the total energy and \( M \) the total magnetic moment, are affected by scaling corrections. Instead of scaling properly as a power law, the maxima values of \( \Phi_i \) (for \( i = 1, 2 \)) appear to exhibit a slight concavity in a log-log plot with respect to the lattice size. Monceau et al. could only provide an upper bound for \( y_t \) from their MC study: \( 1/\nu < 1/1.565 \approx 0.639 \). Our estimation is consistent with their result.

### 4.3 Magnetic eigen-exponent \( y_h \)

We now focus on the odd-coupling subspace. We consider in turn the first \( N_{\text{coupling}} \) odd couplings shown in Fig.1(b) where \( N_{\text{coupling}} = 1, 2, \ldots, 11 \). The \( T \) matrix in the odd-coupling subspace is obtained from Eq. (3) by considering only the lattice sums conjugated to the odd couplings. We denote \( \lambda_h \) the largest eigenvalue of this matrix, and the magnetic eigen-exponent is calculated by \( y_h = \log_3 \lambda_h \). Fig. 5 shows the evolution of \( y_h \) as a function of the number of odd couplings at each renormalization level obtained from the starting Sierpiński carpet \( SC(3, 1, k) \) with \( k = 4, 5, \ldots, 8 \). We can see that \( y_h \) tends to be stable as \( N_{\text{coupling}} \) increases. Larger fluctuations are observed at the highest renormalization level where \( y_h \) is extracted from the reduced carpets of sizes \( 3^3 \) and \( 3^2 \); we will disregard
these values. We estimate $y_h$ in the infinite couplings limit by taking the average value of them when 7 to 11 odd couplings are considered. The results are reported in Table 2. The size effect of the starting Sierpiński carpet on the evolution of $y_h$ with respect to the renormalization level is more significant than it is on the evolution of $y_t$. However, the value of $y_h$ at level $n - (n + 1)$ from the starting carpet $SC(3,1,n+4)$ tends to converge to some value as $n$ increases (see Fig. 6). We hence perform the three-parameter fit suggested by Baillie et al. \cite{18}:

$$\lambda_h = \lambda_h^* + a_h 3^{-\omega_h n}.$$ (9)

The fit yields $(\lambda_h^*, a_h, \omega_h) = (7.38298(66), -0.0737(12), 1.86(12))$ with a reliability equal to $R^2 = 0.99949$. According to these results, the magnetic eigen-exponent $y_h$ takes the value $1.81973(9)$ and the associated scaling correction exponent $\omega_h$ the value $1.86(12)$. $y_h$ is slightly larger than the one obtained by Monceau et al. \cite{11}. They found that $\gamma/\nu = 1.732(4)$ and $\beta/\nu = 0.075(10)$. According to the relations: $\gamma/\nu = 2y_h - d_f$ and $\beta/\nu = d_f - y_h$ \cite{12}, their results correspond to $y_h = 1.812(2)$ and $y_h = 1.818(10)$, respectively. No significant discrepancy can be brought out from these results, since the relative difference between the exponent provided respectively by the MCRG method and MC simulations remains smaller than 0.5 percent.

5 Results from FSS analysis

The lattice sums associated with the first odd and the first even couplings represent respectively the total magnetic moment $M$ and minus the total energy $E$. We can, hence, perform a FSS analysis for a consistency check. According to this analysis, the thermodynamical average of the total magnetic moment at $T_c$ should scale as $\langle |M| \rangle \sim L^{d_f - \beta/\nu} = L^{y_h}$ and the derivative of the Binder’s cumulant $U = 1 - \langle M^4 \rangle / (3 \langle M^2 \rangle^2)$ at the critical temperature $T_c$ scales as

$$\frac{dU}{d\beta_B} = -(1 - U)(\Phi_4 - 2\Phi_2) \sim L^{1/\nu} = L^{y_n},$$ (10)

where $L$ is the lattice size and $\beta_B = (k_B T)^{-1}$ is the inverse of temperature. In Fig. 7, we plot the values of $\langle |M| \rangle$ and $dU/d\beta_B$ at $T = 1.4795$ for the 5 different lattice sizes from
3^4 to 3^8 in logarithmic coordinates. We find that the values of \langle |M| \rangle line up along straight lines, with the lattice size covering several order of magnitude. The slope provided by a least-square fit from the 5 above points, is 1.8198(11), with a reliability \( R^2 = 0.99994 \). It turns out that this value is consistent with the value of \( y_h \) obtained from MCRG method in the subsection 4.3. The behavior of \( dU/d\beta_B \) is not affected by scaling corrections, (see Fig.7), although the \( \Phi_i \) (for \( i = 1, 2 \)) exhibit a slight concavity as a function of the lattice size in a log-log plot. The reason why \( dU/d\beta_B \) is not affected by scaling corrections may be a kind of “magic” cancellation of these scaling correction effects in the difference between \( \Phi_4 \) and \( 2\Phi_2 \). The slope measured from \( dU/d\beta_B \) is 0.449(6) with the fitting reliability \( R^2 = 0.99948 \). This value is consistent with the result obtained in the subsection 4.2 where \( y_t < 0.5254(51) \).

6 Conclusion

The MCRG method has been shown to provide reliable values of the critical temperature \( T_c \) and the magnetic eigen-exponent \( y_h \), for the Ising model in a case where the underlying network has a fractal structure. The scaling correction exponent \( \omega_h \) associated with \( y_h \) can also be calculated. Moreover, difficulties are encountered in the estimation of the temperature eigen-exponent \( y_t \) where, unlike regular lattices, the convergence of the renormalization flow in this direction seems to be very slow; nevertheless, an upper bound for \( y_t \) can be provided. A similar slowness in the convergence of the logarithmic derivatives of the magnetization \( \Phi_i \) to the thermodynamical limit has already been brought out from MC simulations in the case of the same Sierpiński fractal [9] [11]. In reference 11, scaling corrections have been shown to affect strongly the behavior of the \( \Phi_i \)’s and only an upper bound for \( y_t \) has been calculated from their finite size behavior. The difficulties in calculating \( y_t \) for fractal structures arise in the two methods, but emerge differently. At last, the results provided by the MCRG method have been shown to be consistent with the one obtained by Monceau et al. [11] and with an additional FSS analysis of the MCRG results.
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**Figure captions**

Fig.1 (a) 25 even couplings and (b) 11 odd couplings, considered in our MCRG study. They are ordered according to the Blöte’s importance factor $F$ whose value is given in the right column. The symmetry number is given in the next to last right column.

Fig.2 Evolution of $T_{[k+p,k]}(n)$ with respect to the different pairs of the starting Sierpiński carpets $SC(3,1,k+p)$ and $SC(3,1,k)$ (denoted by $[k+p,k]$) obtained at (a) $T_{sim} = 1.4813$, (b) $T_{sim} = 1.4795$.

Fig.3 Variation of $y_t$ with respect to the number of even couplings $N_{coupling}$ at different renormalization levels from the starting Sierpiński carpet $SC(3,1,k)$ with $k = 4, 5, \cdots, 8$.

Fig.4 $\lambda_t$ obtained at level $n - (n + 1)$ from the starting carpet $SC(3,1,n + 4)$.

Fig.5 Variation of $y_h$ with respect to the number of odd couplings $N_{coupling}$ at different renormalization levels from the starting Sierpiński carpet $SC(3,1,k)$ with $k = 4, 5, \cdots, 8$.

Fig.6 $\lambda_h$ obtained at level $n - (n + 1)$ from the starting carpet $SC(3,1,n + 4)$.

Fig.7 $\langle |M| \rangle$ and $dU/d\beta_B$ at $T = 1.4795$ on SC(3,1,k) with $k = 4, 5, \cdots, 8$ in logarithmic coordinates.
Table 1: Average value of $y_t$ calculated from the starting Sierpiński carpets at different renormalization levels. It is obtained by considering 16 to 25 even couplings at the simulation temperature $T_{sim} = 1.4795$. At the highest renormalization level, these average value are disregarded (denoted by the symbol “—”) because the eigen-exponent is not stable with respect to the number of the even couplings.

| level | $SC(3, 1, 8)$ | $SC(3, 1, 7)$ | $SC(3, 1, 6)$ | $SC(3, 1, 5)$ | $SC(3, 1, 4)$ |
|-------|---------------|---------------|---------------|---------------|---------------|
| 0 – 1 | 0.7497(17)    | 0.7437(20)    | 0.7474(14)    | 0.7506(26)    | 0.7483(26)    |
| 1 – 2 | 0.7186(27)    | 0.7235(14)    | 0.7225(19)    | 0.7221(17)    | —             |
| 2 – 3 | 0.6465(26)    | 0.6459(32)    | 0.6466(25)    | —             | —             |
| 3 – 4 | 0.5807(19)    | 0.5883(23)    | —             | —             | —             |
| 4 – 5 | 0.5254(51)    | —             | —             | —             | —             |
| 5 – 6 | —             | —             | —             | —             | —             |

Table 2: Average value of $y_h$ calculated from the starting Sierpiński carpets at different renormalization levels. It is obtained by considering 7 to 11 odd couplings at the simulation temperature $T_{sim} = 1.4795$. At the highest renormalization level, these average values are disregarded (denoted by the symbol “—”) because the eigen-exponent is not stable with respect to the number of the odd couplings.

| level | $SC(3, 1, 8)$ | $SC(3, 1, 7)$ | $SC(3, 1, 6)$ | $SC(3, 1, 5)$ | $SC(3, 1, 4)$ |
|-------|---------------|---------------|---------------|---------------|---------------|
| 0 – 1 | 1.80753(8)    | 1.80762(3)    | 1.80799(4)    | 1.80861(5)    | 1.81060(8)    |
| 1 – 2 | 1.81602(2)    | 1.81621(3)    | 1.81673(3)    | 1.81857(4)    | —             |
| 2 – 3 | 1.81713(1)    | 1.81759(3)    | 1.81948(4)    | —             | —             |
| 3 – 4 | 1.81807(1)    | 1.81985(2)    | —             | —             | —             |
| 4 – 5 | 1.81967(3)    | —             | —             | —             | —             |
| 5 – 6 | —             | —             | —             | —             | —             |
(1.)
(2.)
(3.)
(4.)
(5.)
(6.)
(7.)
(8.)
(9.)
(10.)
(11.)
(12.)
(13.)
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(15.)
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(17.)
(18.)
(19.)
(20.)
(21.)
(22.)
(23.)
(24.)
(25.)

2 0.5000
2 0.3536
2 0.2500
4 0.2236
1 0.2197
4 0.1916
4 0.1860
2 0.1768
8 0.1734
8 0.1655
4 0.1613
1 0.1553
8 0.1516
8 0.1456
4 0.1456
2 0.1432
4 0.1432
4 0.1400
4 0.1400
8 0.1356
4 0.1355
8 0.1349
4 0.1327
8 0.1219
1 0.1098
|   |   |   |   |
|---|---|---|---|
| 1 | inf | 2 | 0.1875 |
| 2 | 4 | 0.3107 | 4 | 0.1802 |
| 3 | 2 | 0.2652 | 8 | 0.1749 |
| 4 | 8 | 0.2281 | 4 | 0.1639 |
| 5 | 4 | 0.2197 | 4 | 0.1553 |
| 6 | 8 | 0.2026 |   |   |
$T_{\text{sim}} = 1.4795$

- $\langle |M| \rangle$
- $\Delta \frac{dU}{d\beta_B}$
$T_{\text{sim}} = 1.4795$
$T_{sim} = 1.4795$
$T_{\text{sim}} = 1.4795$
\[ T_{\text{sim}} = 1.4795 \]
$T_{sim} = 1.4795$

$T_{c^{[k+p,k]}(n)}$ vs $n$
$T_{c [k+p,k]}(n)$

$T_{sim} = 1.4813$

- [5,4]
- [6,5]
- [7,6]
- [8,7]
- [8,6]
- [8,5]
- [8,4]