Kato’s inequalities for admissible functions to quasilinear elliptic operators $A$

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Abstract

Let $1 < p < \infty$ and let $\Omega$ be a bounded domain of $\mathbb{R}^N (N \geq 1)$. In this paper, we consider a class of second order quasilinear elliptic operators $A$ in $\Omega$ including the $p$-Laplace operator $\Delta_p$. First we establish various type of Kato’s inequalities for $A$ when $Au$ is a Radon measure. Then we prove the inverse maximum principle and describe the strong maximum principle. For this purpose it is crucial to introduce a notion of admissible class for the operator $A$ and use it effectively.  

1. Introduction

Let $1 < p < \infty$ and let $\Omega$ be a bounded domain of $\mathbb{R}^N (N \geq 1)$. In this paper we consider a class of second order operators $A$ in $\Omega$ with a growth order of degree $p - 1$, which include the so-called $p$-Laplace operator $\Delta_p$ defined by

$$\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u),$$

(1.1)

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N)$.

More precisely, we consider the following quasilinear degenerate elliptic operators $A$ of divergent form defined by

$$Au = \text{div} A(x, \nabla u),$$

(1.2)

where $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, is a mapping which satisfies the following assumptions for some positive numbers $c_1$ and $c_2$:

1. the function $x \mapsto A(x, \xi)$ is bounded measurable for all $\xi \in \mathbb{R}^N$, 

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2. the function $\xi \mapsto A(x, \xi)$ is continuous and satisfies $A(x, 0) = 0$ for a.e. $x \in \Omega$.

3. 
   $$|A(x, \xi) - A(x, \eta)| \leq c_1(|\xi| + |\eta|)^{p-2} |\xi - \eta|$$ 
   for all $\xi, \eta \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

4. 
   $$(A(x, \xi) - A(x, \eta)) \cdot (\xi - \eta) \geq c_2(|\xi| + |\eta|)^{p-2} |\xi - \eta|^2$$ 
   for all $\xi, \eta \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

5. 
   $$A(x, \lambda \xi) = \lambda |\lambda|^{p-2} A(x, \xi),$$ 
   for all $\lambda \in \mathbb{R} \setminus \{0\}$ and a.e. $x \in \Omega$.

Here we give simple remarks and examples concerning the nonlinear operator $A$.

**Remark 1.1.**

1. It follows from the assumptions 1, 2, 3 and 4 that we have
   $$|A(x, \xi)| \leq c_1 |\xi|^{p-1}$$ 
   for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, 

   $$(1.3)$$

   $$A(x, \xi) \cdot \xi \geq c_2 |\xi|^p$$ 
   for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$. 

   $$(1.4)$$

2. If the function $\xi \mapsto A(x, \xi)$ is continuously differentiable for a.e. $x \in \Omega$ and
   satisfies the estimates (1.5) and (1.6), then one can check that $A = A(x, \xi)$ satisfies the assumptions 3 and 4 (c.f. [8]).

   $$\sum_{j,k=1}^{N} \left| \frac{\partial A_j}{\partial \xi_k}(x, \xi) \right| \leq C|\xi|^{p-2}$$ 
   for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and a.e. $x \in \Omega$, 

   $$(1.5)$$

   $$\sum_{j,k=1}^{N} \frac{\partial A_j}{\partial \eta_k}(x, \eta) \xi_j \eta_k \geq C|\eta|^{p-2} |\xi|^2$$ 
   for all $\eta \in \mathbb{R}^N \setminus \{0\}, \eta \in \mathbb{R}^N$ and a.e. $x \in \Omega$, 

   $$(1.6)$$

   where $C$ is some positive number independent of each $a$ and $\xi$.

Before we go into the detail, we give some historical remarks. First let us recall some fundamental results on Kato’s inequality (see [14]). The classical version of Kato’s inequality for a Laplacian asserts that given any function $u \in \text{L}^1_{\text{loc}}(\Omega)$ such that

$$\Delta u \in \text{L}^1_{\text{loc}}(\Omega),$$

then $\Delta (u^+) \geq \chi_{\{u \geq 0\}} \Delta u$ in $\text{D}'(\Omega)$, 

$$(1.7)$$

where $u^+ = \max\{u, 0\}$. A similar inequality holds when $\Delta u$ is replaced by $\Delta_p u$ under additional assumptions on distributional derivatives of $u \in \text{L}^1_{\text{loc}}(\Omega)$ (see e.g. [11, 12]).

In [4, 5], H.Brezis and A.Ponce intensively studied Kato’s inequalities with $\Delta u$ being a Radon measure and established the strong maximum principle, the improved Kato’s inequalities and the inverse maximum principle. In [15, 16], we extended most of their results to the case where $\Delta_p u$ is a Radon measure. In these arguments it was crucial to introduce and make use of the notion of admissible class for the $p$-Laplace operator $\Delta_p$. 
In this paper we further extend these results to the operators \( A \) as a generalization of \( \Delta_p \) by introducing the proper admissible class in Definition 3.1. Then we will establish improved Kato’s inequalities for the operators \( A \), which leads us to various improved inequalities of Kato type as its corollaries. We also establish not only the strong maximum principle but also the inverse maximum principle which sharpens Kato’s inequalities for \( A \).

This article is organized in the following way. In Section 2 we collect definitions which are basic in this paper. In Section 3 we define the admissible class for our operators \( A \). In Section 4 we describe our main results (Theorem 1, Theorem 2 and Theorem 3) together with corollaries and remarks for the operators \( A \). In Section 5 we prepare some lemmas for the proofs of main results. Section 6 is devoted to establish Theorem 1 and Theorem 2 and Section 7 is devoted to establish Corollary 1 and Corollary 2. Finally in Appendix we present a result on the admissibility as Proposition 8.1, which proves a partial inverse of assertion 2 in Proposition 3.1.

2. Preliminaries

In this section we collect fundamental definitions in the present article.

Definition 2.1. (A p-capacity relative to \( \Omega \))
Let \( 1 < p < \infty \). For each compact set \( K \subset \Omega \) we define a \( p \)-capacity of \( K \) relative to \( \Omega \) by

\[
C_p(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p : \varphi \in C_c^\infty(\Omega), \varphi \geq 1 \text{ in some neighborhood of } K \right\}.
\]

Here we note that by the standard theory one can define the capacity \( C_p(A, \Omega) \) for any Borel set \( A \).

Definition 2.2.

1. By \( M(\Omega) \) we denote the space of all Radon measures on \( \Omega \).
2. By \( M_b(\Omega) \) we denote the space of all Radon measures \( \mu \in M(\Omega) \) having bounded variation on \( \Omega \).

Definition 2.3. (Quasi-everywhere and quasicontinuity)

1. If a property holds everywhere except possibly on a set of \( p \)-capacity zero, then it is said to hold quasi-everywhere.
2. We say that a function \( u : \Omega \to \mathbb{R} \) is quasicontinuous if there exists a sequence of open subsets \( \omega_n \) of \( \Omega \) such that \( \overline{\omega_n} \subset \Omega \), \( u|_{\Omega \setminus \omega_n} \) is continuous for \( n \geq 1 \) and \( C_p(\overline{\omega_n}, \Omega) \to 0 \) as \( n \to \infty \).

Definition 2.4. (Decomposition of Radon measure)
We recall that any Radon measure \( \mu \) can be uniquely decomposed as a sum of two Radon measures on \( \Omega \) (see e.g. [5, 10]) : \( \mu = \mu_a + \mu_s \) (\( \mu_a \) is the absolutely continuous part and \( \mu_s \) is the singular part of \( \mu \)), where

\[
\begin{cases}
\mu_a(A) = 0 & \text{for any Borel measurable set } A \subset \Omega \text{ such that } C_p(A, \Omega) = 0, \\
|\mu_s|(\Omega \setminus F) = 0 & \text{for some Borel measurable set } F \subset \Omega \text{ such that } C_p(F, \Omega) = 0. 
\end{cases}
\]
Here by \( C_p(A, \Omega) \) we denote a \( p \)-capacity of a Borel set \( A \) relative to \( \Omega \).

We note that \( (\mu_\alpha)^\pm = (\mu_\alpha^\pm)_\alpha \), \((\mu_\alpha)^\pm = (\mu_\alpha^\pm)_\alpha\), \( |\mu_\alpha| = |\mu_\alpha| \) and \( |\mu_\alpha| = |\mu_\alpha| \) hold by the definition, where \( \nu^+ = \max[\nu, 0] \), \( \nu^- = \max[-\nu, 0] \) and \( |\nu| = \nu^+ + \nu^- \) for a Radon measure \( \nu \) on \( \Omega \).

3. Admissibility

In this section, we introduce the admissible class for \( \mathcal{A} \). This notion is initially introduced in [16] when \( \mathcal{A} = \Delta_p \).

**Definition 3.1.** (Admissible class for the operator \( \mathcal{A} \) in \( W^{1,p^*}(\Omega) \)) Let \( 1 < p < \infty \) and \( p^* = \max[1, p - 1] \). A function \( u \) is said to be admissible for the operator \( \mathcal{A} \) in \( W^{1,p^*}(\Omega) \) if \( u \in W^{1,p^*}_0(\Omega) \), \( Au \in M(\Omega) \) and there exists a sequence \( \{u_n\}_{n=1}^\infty \subset W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) such that:

1. \( u_n \to u \) a.e. in \( \Omega \) and \( u_n \to u \) in \( W^{1,p^*}_0(\Omega) \) as \( n \to \infty \).
2. \( Au_n \in L^{1,p}_0(\Omega) \) \( (n = 1, 2, \ldots) \) and

\[
\sup_n |Au_n| < \infty \quad \text{for every } \omega \subset \subset \Omega. \tag{3.1}
\]

It follows directly from Definition 3.1 that if \( u \) is admissible for the operator \( \mathcal{A} \), then \( u \) can be approximated by a sequence of good functions not only in the sense of the distribution but also in the sense of measure. Moreover it follows from Lemma 3 in Section 5 that if \( u \) is admissible, then for every \( k > 0 \), \( T_k u \) (the truncation of \( u \) defined by (5.1)) should belong to \( W^{1,p^*}_{\text{loc}}(\Omega) \). From this fact it is possible even in the case of \( p = 2 \) to give a particular pair of operators \( \mathcal{A} \) and non-admissible functions for \( \mathcal{A} \) in \( W^{1,1}(\Omega) \) (c.f. [13, 21]). We also state some basic properties concerning the admissibility here, which are known when \( \mathcal{A} = \Delta_p \) (c.f. [15] and [16]). In Appendix we will establish the opposite result of the statement 2 as Theorem 8.1.

**Proposition 3.1.** Let \( N \geq 1 \), \( 1 < p < \infty \), \( p^* = \max[1, p - 1] \) and \( \Omega \) be a bounded domain of \( \mathbb{R}^N \).

1. Assume that a function \( u \) is admissible in \( W^{1,p^*}_0(\Omega) \). Then \( u^+ = \max[u, 0] \) and \( u^- = \max[-u, 0] \) are also admissible.
2. In Definition 3.1, the sequence \( \{u_n\} \) can be taken in \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

**Proof of 1:** Since \( u \in W^{1,p^*}_0(\Omega) \), \( Au, A(u^+) \) and \( A(u^-) \) are well-defined in \( D'(\Omega) \). Let \( \{u_n\} \) be an approximate sequence of \( u \) in the sense of Definition 3.1. It follows from the condition 1 that \( Au_n = A(u^+_n) - A(u^-_n) \) and \( Au_n \to Au \) (i.e. \( A(u^+_n) \to A(u^+) \) in \( D'(\Omega) \) as \( n \to \infty \). Moreover, it follows from the condition 2 and the weak compactness of measures that we have \( Au_n \to Au \) (i.e. \( A(u^+_n) \to A(u^+) \) ) up to subsequences in the sense of measures as \( n \to \infty \). Therefore if \( u \) is admissible, then \( u^+ \) and \( u^- \) are admissible as well.

**Proof of 2:** We assume that \( u \) is admissible for \( \mathcal{A} \) in \( W^{1,p^*}_0(\Omega) \). Then we have a sequence of functions \( \{u_n\} \subset W^{1,p^*}_0(\Omega) \cap L^\infty(\Omega) \) \( (n = 1, 2, \ldots) \) satisfying the properties 1 and 2 in Definition 3.1. By the previous step, we see that each \( u_n \) is also admissible for \( \mathcal{A} \) in \( W^{1,p^*}_0(\Omega) \) and approximated by a sequence of functions \( \{\xi_k\} \subset W^{1,p}_0(\Omega) \). Hence the assertion is now clear. \( \square \)
Example 1. (Admissible weak solutions) Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ containing the origin.

Let $u = |x|^\alpha$ for $\alpha = (p - N)/(p - 1)$. Then $u$ satisfies

$$\Delta_p u = \alpha |\alpha|^{p-2} c_N \delta \quad \text{in } D'(\Omega),$$

where $\delta$ denotes a Dirac mass and $c_N$ denotes the surface area of the $N$-dimensional unit ball $B_1$. It is easy to see that $u \in W^{1,p'}(\Omega)$ if and only if $p > 2 - 1/N$. When $2 - \frac{1}{N} < p < N$ with $N \geq 2$, $u$ is admissible in $W^{1,p'}_{\text{loc}}(\Omega)$. When $p \geq N$ with $N \geq 2$ or $p > 1$ with $N = 1$, we clearly have $u \in W^{1,p}(\Omega)$. Hence it follows from Proposition 3.1 (3) that $u$ is admissible in $W^{1,p'}_{\text{loc}}(\Omega)$.

Remark 3.1. When $1 < p \leq 2 - 1/N$ holds, we can consider $u$ as a renormalized solution. We remark that if $p \leq 2 - 1/N$, then we cannot expect the solution of an equation of the form $\Delta_p u = f \in M(\Omega)$ to be in $W^{1,1}_{\text{loc}}(\Omega)$. For the detail, see e.g. [1, 2, 7, 18, 19].

4. Main results

In this section, we describe our main theorems concerning improved Kato’s inequalities, the inverse maximum principle and the strong maximum principle. In Theorem 1 and Corollary 1, we extend Kato’s inequality involving $\Delta_p$ to the case where $\mathcal{A}u \in M(\Omega)$, where $M(\Omega)$ denotes the space of Radon measures on $\Omega$.

Theorem 1. (Improved Kato’s inequalities) Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p - 1]$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. Let $\Phi$ be a $C^1$ convex function in $\mathbb{R}$ such that $\Phi' \geq 0$ in $\mathbb{R}$ and $\Phi' \in L^\infty(\mathbb{R})$.

Assume that $u$ is admissible for the operator $\mathcal{A}$ in $W^{1,p'}_{\text{loc}}(\Omega)$ in the sense of Definition 3.1.

Then we have

$$\mathcal{A}\Phi(u) \geq \Phi'(u)\alpha^{-1}(\mathcal{A}u)_a - ||\Phi'||_{L^\infty(\mathbb{R})}^{-1}(\mathcal{A}u)_a^{-1} \quad \text{in } D'(\Omega). \quad (4.1)$$

From this theorem it follows that we have

Corollary 1. Assume the same assumptions in Theorem 1. Then it holds that

$$\mathcal{A}(u^+) \geq \chi_{[u \geq 0]}(\mathcal{A}u)_a - (\mathcal{A}u)_a^{-1} \quad \text{in } D'(\Omega), \quad (4.2)$$

$$\mathcal{A}|u| \geq \text{sgn}(u)(\mathcal{A}u)_a - |\mathcal{A}u|_a^{-1} \quad \text{in } D'(\Omega), \quad (4.3)$$

where $\text{sgn}(t) = 1$ for $t > 0$, $\text{sgn}(t) = -1$ for $t < 0$, and $\text{sgn}(0) = 0$.

Remark 4.1.

1. Let $u$ be the fundamental solution to $\Delta_p$ in Example 1 for $p > 2 - 1/N$. Then the equality holds in both inequalities (4.2) and (4.3).

2. Note that the right-hand sides of (4.2) and (4.3) are well-defined because $u$ is quasicontinuous (see Definition 2.3). More precisely it follows from Lemma 1 that there exists $u : \Omega \rightarrow \mathbb{R}$ quasicontinuous and $u = \tilde{u}$ a.e. in $\Omega$. In (4.2) and (4.3), we identify $u$ with its quasicontinuous representative. Then we see that $\chi_{[u \geq 0]}$ and $\text{sgn}(u)$ are locally integrable in $\Omega$ with respect to $|(\mathcal{A}u)_a|$ (see also [5, 16]).
Secondly, we establish the inverse maximum principle for the nonlinear operator $A$ including a corollary. We remark that the inverse maximum principle for $\Delta$ was initially proved in [5, 6], and then it was extended in [15] for the $p$-Laplace operator $\Delta_p$. Though the idea of the proof is similar to the case of $\Delta_p$, we will give a proof for the sake of self-containedness.

**Theorem 2. (Inverse maximum principle)** Let $N \geq 1$, $1 < p < \infty$, $p^* = \max\{1, p-1\}$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. Assume that $u$ is admissible for the operator $A$ in $W^{1,p}_\text{loc}(\Omega)$ in the sense of Definition 3.1 and $u \geq 0$ a.e. in $\Omega$. Then we have

$$(-Au)_+ \geq 0 \quad \text{in } \Omega. \quad (4.4)$$

From this theorem together with the previous one we have:

**Corollary 2.** Let $N \geq 1$, $1 < p < \infty$, $p^* = \max\{1, p-1\}$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. Assume that $u$ is admissible in the sense of Definition 3.1. Then we have

$$(-A(u^+))_+ = (-Au)_+ \quad \text{in } \Omega, \quad (4.5)$$

$$(-A(u^-))_+ = (-Au)_-, \quad (-A|u|)_+ = |Au|_+ \quad \text{in } \Omega. \quad (4.6)$$

Lastly we describe the result on the strong maximum principle for our operator $A$.

**Definition 4.1.** Let $1 < p < \infty$ and let $Q(\cdot)$ be a nonlinear term satisfying the following properties $[Q_0]$ and $[Q_1]$.

$[Q_0]$: $Q(t)$ is a continuous increasing function on $[0, \infty)$ with $Q(0) = 0$.

$[Q_1]$: \[ \limsup_{t \to +0} \frac{Q(t)}{t^{p-1}} < \infty. \quad (4.7) \]

In [16] we have established the strong maximum principle for $-\Delta_p u + aQ(u)$ where $0 \leq a \in L^1_{\text{loc}}(\Omega)$ and $Q$ satisfies the conditions $[Q_0]$ and $[Q_1]$. According to the argument in [16] using Definition 3.1 and Proposition 3.1 instead, we can extend the strong maximum principle to allow operators $A + aQ(\cdot)$ and admissible functions $u \in W^{1,p}_{\text{loc}}(\Omega)$ in the same strategy. Hence we describe the corresponding result without a proof. For the detailed argument see [16].

**Theorem 3. (Strong Maximum Principle)** Let $N \geq 1$, $1 < p < \infty$, $p^* = \max\{1, p-1\}$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. Assume that $Q$ satisfies the conditions $[Q_0]$ and $[Q_1]$. Assume that $u \in W^{1,p}_{\text{loc}}(\Omega)$, $u \geq 0$ a.e. in $\Omega$, $Q(u) \in L^1_{\text{loc}}(\Omega)$ and $u$ is admissible in the sense of Definition 3.1.

Then we have the following:

1. There exists a quasicontinuous function $\tilde{u} : \Omega \mapsto \mathbb{R}$ such that $u = \tilde{u}$ a.e. in $\Omega$.

2. Let $a \in L^1_{\text{loc}}(\Omega), a \geq 0$ a.e. in $\Omega$. Assume that

$$-Au + a(x)Q(u) \geq 0 \quad \text{in the sense of measures}, \quad (4.8)$$

i.e.,

$$\int_E Au \leq \int_E aQ(u) \quad \text{for every Borel set } E \subseteq \Omega. \quad (4.9)$$

If $\tilde{u} = 0$ on a set of positive $p$-capacity in $\Omega$, then $u = 0$ a.e. in $\Omega$. 

Remark 4.2.  
1. The definition p-capacity denoted by $C_p(E, \Omega)$ is given in Definition 2.3 in connection with quasicontinuity of functions.
2. It follows from (4.9) that the positive part $(Au)^+$ should be absolutely continuous with respect to the Lebesgue measure.

5. Lemmas

Let $N \geq 1$, $1 < p < \infty$, $p^* = \max[1, p - 1]$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. We prepare a fundamental lemma (for the proof see e.g. [20], Chapter 2).

Lemma 1.  
1. If $u \in W^{1,p}_0(\Omega)$, then $u$ can be redefined almost everywhere in $\Omega$ so as to be quasicontinuous.
2. If $u \in W^{1,p}_{loc}(\Omega)$, then there exists a sequence of smooth functions $\{\varphi_n\}$ such that $\varphi_n \to u$ pointwise quasi-everywhere in $\Omega$ as $n \to \infty$.

Let us prepare more lemmas. Given $k > 0$, we denote a truncation function $T_k: \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) = \max[-k, \min[k, s]]. \quad (5.1)$$

Since $T_k|_{R_+}$ is concave, we have the following lemma in the spirit of the standard $L^1$-version of Kato’s inequality (see [14]).

Lemma 2. Assume that $v \in W^{1,p}_{loc}(\Omega)$, $Av \in L^1_{loc}(\Omega)$ and $v \geq 0$ a.e. in $\Omega$. Then, for any $k > 0$ we have

$$\mathcal{A}T_k(v) \leq t_k(v)Av \quad \text{in } D'(\Omega), \quad (5.2)$$

where the function $t_k: \mathbb{R}_+ \to \{0, 1\}$ is given by

$$t_k(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq k, \\ 0 & \text{if } s > k. \end{cases}$$

Proof of Lemma 2: Let $\Phi$ be a $C^2$ concave function in $\mathbb{R}$ such that $\Phi(t) = t$ if $t \leq 0$, $|\Phi(t)| < 1$ if $t > 0$, $0 \leq \Phi' \leq 1$ in $\mathbb{R}$ and $\lim_{t \to \infty} \Phi'(t) = 0$. Set $\Phi_n(t) = k + \Phi(n(t - k))/n$ for $t \in \mathbb{R}$ and $n = 1, 2, \ldots$. Then we see that $\{\Phi_n\}$ is a sequence of $C^2$ concave functions in $\mathbb{R}$ such that $\Phi_n(t) = t$ if $t \leq k$, $|\Phi_n(t) - k| < \frac{1}{n}$ if $t > k$, $0 \leq \Phi'_n \leq 1$ in $\mathbb{R}$ and $\lim_{n \to \infty} \Phi_n'(t) = 0$ if $t > k$. Then we define

$$\Phi_{n, \eta}(t) = \Phi_n(t) + \eta t \quad \text{for } \eta > 0 \text{ and } t \in \mathbb{R}. \quad (5.3)$$

We may assume that $v$ is smooth by the approximation argument. By a direct calculation

$$\mathcal{A}\Phi_{n, \eta}(v) = \Phi'_{n, \eta}(v)^{p-1}Av + (p - 1)\Phi''_{n, \eta}(v)^{p-2}\Phi''_{n, \eta}(v)A(x, \nabla v) \cdot \nabla v$$

$$\leq \Phi'_{n, \eta}(v)^{p-1}Av \quad (\Phi''_{n, \eta} = \Phi'' \leq 0 \text{ by concavity of } \Phi_n)$$

Letting $\eta \to 0$, we clearly have

$$\mathcal{A}\Phi_n(v) \leq \Phi'_n(v)^{p-1}Av \quad \text{in } \Omega. \quad (5.4)$$

As $n \to \infty$, we finally get the inequality (5.2). \qed

Recall the following standard estimate for a truncation of admissible function $u$ (see, e.g., Lemma 1 in [4], Lemma 2 and Lemma 3 in [16]):
Lemma 3. Let \( \Omega \subset \mathbb{R}^N \) be an open set. Assume that \( u \) is admissible for the operator \( A \) in \( W^{1,p}_\text{loc}(\Omega) \). Then

\[
T_k(u) \in W^{1,p}_\text{loc}(\Omega), \quad \text{for every } k > 0.
\] (5.5)

Moreover, given \( \omega' \subset \omega \subset \Omega \), there exists positive constant \( C \) such that

\[
\int_\omega |\nabla T_k(u)|^p \leq C k \left( \int_{\omega'} |Au| + \int_{\omega'} |\nabla u|^{p-1} \right),
\] (5.6)

where positive constant \( C \) are independent on each \( u \). Moreover, there exists a quasi-continuous function \( \tilde{u} : \Omega \to \mathbb{R} \) such that \( u = \tilde{u} \) a.e. in \( \Omega \). We recall that: A function \( u : \Omega \to \mathbb{R} \) is said to be quasicontinuous if there exists a sequence of open subsets \( \omega_n \) of \( \Omega \) such that \( \overline{\omega_n} \subset \Omega, u|_{\omega \setminus \omega_n} \) is continuous for \( n \geq 1 \) and \( C_p(\overline{\omega_n}, \Omega) \to 0 \) as \( n \to \infty \).

The next lemma is seen in [3]; Theorem 2.1.

Lemma 4. Let \( \nu \in M(\Omega) \). Then \( \nu \in L^1(\Omega) + W^{-1,p'}(\Omega) \) if and only if \( \nu \) is an absolutely continuous measure with respect to \( p \)-capacity (i.e. \( \nu_s = 0 \)). Here \( p' = p/(p-1) \) and \( W^{-1,p'}(\Omega) \) is the dual space of \( W^{1,p}_0(\Omega) \).

When \( p = 2 \), the next lemma is seen in [5]; Lemma 2.1.

Lemma 5. Assume that \( \nu \in M(\Omega) \) is an absolutely continuous measure with respect to \( p \)-capacity (i.e. \( \nu_s = 0 \)). Let \( \{v_n\} \) be a sequence in \( L^\infty(\Omega) \cap W^{1,p}_0(\Omega) \) such that \( \sup_n ||v_n||_{L^\infty(\Omega)} < \infty \) and \( v_n \rightharpoonup v \) weakly in \( W^{1,p}_0(\Omega) \) as \( n \to \infty \). Then

\[
v_n \rightharpoonup v \quad \text{in} \quad L^1_{\text{loc}}(\Omega; d\nu) \quad \text{as} \quad n \to \infty.
\]

Equivalently, there exists a subsequence \( \{v_{n_k}\} \) converging to \( v \) \( |\nu| \)-a.e. in \( \Omega \). Here \( L^1_{\text{loc}}(\Omega; d\nu) = \{f : \int_\omega |f| d\nu < \infty \} \) for any \( \omega \subset \subset \Omega \).

Proof of Lemma 5: We may assume that \( |\nu|(\Omega) < \infty \). By Lemma 4, we can assume that \( \nu = f - \text{div} G \) in \( D'(\Omega) \) for some \( f \in L^1(\Omega) \) and \( G = (g_1, g_2, \ldots , g_N) \in (L^{p'}(\Omega))^N \). By a density argument, we have

\[
\int_\Omega w \varphi \, d\nu = \int_\Omega w \varphi f + \int_\Omega G \cdot \nabla \varphi \quad \text{for any} \quad \varphi \in C_0^\infty(\Omega), w \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega).
\] (5.7)

Since the sequence \( \{|v_n - v|\} \) is bounded in \( W^{1,p}_0(\Omega) \), by Rellich’s theorem, we have \( |v_n - v| \to 0 \) in \( L^p(\Omega) \) as \( n \to \infty \). Therefore \( |v_n - v| \to 0 \) weakly in \( W^{1,p}_0(\Omega) \) as \( n \to \infty \). For a given \( \varepsilon > 0 \), we choose \( \omega \subset \subset \Omega \) such that \( |\nu|(\Omega \setminus \omega) < \varepsilon \). Then we fix \( \varphi_0 \in C_0^\infty(\Omega) \) such that \( 0 \leq \varphi_0 \leq 1 \) in \( \Omega \) and \( \varphi_0 = 1 \) in \( \omega \). By using (5.7) with \( w = |v_n - v| \) and \( \varphi = \varphi_0 \), we have

\[
\int_\Omega |v_n - v| \, d\nu = \int_\omega |v_n - v| \, d\nu + \int_{\Omega \setminus \omega} |v_n - v| \, d\nu
\]

\[
\leq \int_\omega |v_n - v| \, d\nu + 2C |\nu|(\Omega \setminus \omega) \leq \int_\Omega |v_n - v| \varphi_0 \, d\nu + 2C \varepsilon
\]

\[
= \int_\Omega |v_n - v| \varphi_0 f + \int_\Omega \nabla (|v_n - v| \varphi_0) \cdot G + 2C \varepsilon.
\]

By weak convergence of \( |v_n - v| \varphi_0 \) in \( W^{1,p}_0(\Omega) \), we see immediately that \( \int_\Omega \nabla (|v_n - v| \varphi_0) \cdot G \to 0 \) as \( n \to \infty \). Since \( f \in L^1(\Omega) \) and \( |v_n - v| \to 0 \) in measure as \( n \to \infty \), it
6. Proofs of Theorem 1 and Theorem 2

Let us recall the assumption in Theorem 1 that \( \Phi \) is a \( C^1 \) convex function in \( R \), \( \Phi' \geq 0 \) in \( R \) and \( \Phi' \in L^\infty(R) \). Without loss of generality, we assume that \( \Phi \in C^2(R) \), \( 0 \leq \Phi' \leq 1 \) and \( \Phi'' \) has compact support in \( R \) because the general case can be deduced by approximation. First we prepare the next.

**Lemma 6.** Assume that \( u \) is admissible for \( A \) in \( W^{1,p}_0(\Omega) \). Assume
\[
\sup_{t \in R} (\Phi'(t))^{p-2} \Phi''(t) < \infty \quad \text{if } 1 < p < 2.
\]

Let \( \{u_n\} \) be an approximate sequence of \( u \) in the sense of Definition 1.1.

Then \( \Phi'(u_n)^{p-1} \) is bounded in \( W^{1,p}_0(\Omega) \) and \( \Phi'(u_n)^{p-1} \varphi \to \Phi'(u)^{p-1} \varphi \) weakly in \( W^{1,p}_0(\Omega) \) as \( n \to \infty \).

**Proof of Lemma 6:** Since \( \Phi'(u_n)^{p-1} \varphi \to \Phi'(u)^{p-1} \varphi \) in \( L^1(\Omega) \) as \( n \to \infty \) and \( \varphi \) has compact support in \( \Omega \), it suffices to show that \( \{\Phi'(u_n)^{p-1}\} \) is bounded in \( W^{1,p}_0(\Omega) \). Let \( M > 0 \) be such that \( \text{supp} \Phi'' \subset [-M,M] \). Noting that \( \nabla u_n = \nabla T_M(u_n) \) for \( u_n \in \text{supp} \Phi'' \), we have
\[
\nabla (\Phi'(u_n)^{p-1}) = (p-1) \Phi'(u_n)^{p-2} \Phi''(u_n) \nabla T_M(u_n) \quad \text{in } \Omega.
\]

Let \( \omega \subset \subset \omega' \subset \subset \Omega \). From (5.6) it follows that for some constant \( C > 0 \) independent of \( n \) and \( M \)
\[
\int_\omega |\nabla T_M(u_n)|^p \leq CM \int_\omega (|Au_n| + |\nabla u_n|^{p-1}),
\]
which implies
\[
\sup_n \int_\omega |\nabla T_M(u_n)|^p < \infty,
\]
because \( \{u_n\}, \{|\nabla u_n|^{p-1}\} \) and \( \{|Au_n|\} \) are bounded in \( L^1(\omega') \) by the assumption on \( u_n \). Hence by (6.2), (6.4) and (6.1) we obtain
\[
\sup_n \int_\omega |\nabla (\Phi'(u_n)^{p-1})|^p \leq (p-1)^p \left( \sup_{t \in R} (\Phi'(t))^{p-2} \Phi''(t) \right) \int_\omega |\nabla T_M(u_n)|^p < \infty.
\]

This concludes the proof. \( \square \)

**Proof of Theorem 1:**

Let \( \{u_n\} \) be an approximate sequence of \( u \) in the sense of Definition 3.1. By direct calculation we have
\[
\mathcal{A} \Phi(u_n) = \Phi'(u_n)^{p-1} Au_n + (p-1) \Phi'(u_n)^{p-2} \Phi''(u_n) |\nabla u_n|^p \quad \text{in } \Omega.
\]

First we assume that \( p \geq 2 \). Since \( \Phi' > 0, \Phi'' \geq 0 \) and \( \Phi' \in L^\infty(R) \), it follows from (6.5) that
\[
\mathcal{A} \Phi(u_n) \geq \Phi'(u_n)^{p-1} Au_n = \Phi'(u_n)^{p-1} Au + \Phi'(u_n)^{p-1} (Au_n - Au) \geq \Phi'(u_n)^{p-1} (Au) + \Phi'(u_n)^{p-1} (Au_n - Au)
\]
\[
= I_n - ||\Phi'||_{L^\infty(R)} (Au)_a^+ + J_n \quad \text{in } D'(\Omega),
\]
where
\[
I_n = \int_D \Phi'(u_n)^{p-1} Au_n \quad \text{and} \quad J_n = \int_D \Phi'(u_n)^{p-1} (Au_n - Au) \quad \text{in } D'(\Omega).
\]
where \( I_n = \Phi'(u_n)^{-1}(Au)_a \) and \( J_n = \Phi'(u_n)^{-1}(Au_n - Au) \). Let \( \varphi \in C^\infty_0(\Omega) \) with \( \varphi \geq 0 \) in \( \Omega \). Noting that \( \varphi' \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), \( \varphi'(u_n) \to \varphi'(u) \) a.e. in \( \Omega \) and that \( A(x, \nabla u_n) \to A(x, \nabla u) \) in \( L^1_{\text{loc}}(\Omega) \) as \( n \to \infty \) from the assumption on \( u_n \), we have

\[
\langle A\Phi(u_n), \varphi \rangle = - \int_\Omega \Phi'(u_n)^{-1} A(x, \nabla u_n) \cdot \nabla \varphi \tag{6.7}
\]

by the dominated convergence theorem. Noting that \( \varphi' \) is bounded in \( W^{1,p}_0(\Omega) \) as \( n \to \infty \) by Lemma 6, we can apply Lemma 5 with \( v_n = \Phi'(u_n)^{-1} \varphi, v = \Phi'(u)^{-1} \varphi \) and \( \nu = (Au)_a \), so that we get

\[
\langle J_n, \varphi \rangle = \langle \Phi'(u_n)^{-1}(Au)_a, \varphi \rangle \to \langle \Phi'(u)^{-1}(Au)_a, \varphi \rangle \quad \text{as} \quad n \to \infty \tag{6.8}
\]

by Lemma 4 and the dominated convergence theorem. As for \( J_n \) we have

\[
\langle J_n, \varphi \rangle = \langle \Phi'(u_n)^{-1}(Au_n - Au), \varphi \rangle
\]

\[
= - \int_\Omega \Phi'(u_n)^{-1} (A(x, \nabla u_n) - A(x, \nabla u)) \cdot \nabla \varphi
\]

\[
- \int_\Omega \nabla (\Phi'(u_n)^{-1}) \cdot (A(x, \nabla u_n) - A(x, \nabla u)) \varphi. \tag{6.9}
\]

From \( \Phi' \in L^\infty(\mathbb{R}) \) and \( |A(x, \nabla u_n) - A(x, \nabla u)| \to 0 \) in \( L^1_{\text{loc}}(\Omega) \) as \( n \to \infty \) it follows that

\[
\int_\Omega \Phi'(u_n)^{-1} (A(x, \nabla u_n) - A(x, \nabla u)) \cdot \nabla \varphi \to 0 \quad \text{as} \quad n \to \infty. \tag{6.10}
\]

We show

\[
\int_\Omega \nabla (\Phi'(u_n)^{-1}) \cdot (A(x, \nabla u_n) - A(x, \nabla u)) \cdot \varphi \to 0 \quad \text{as} \quad n \to \infty. \tag{6.11}
\]

Noting that \( \Phi'' \) has compact support, we see that \( \{|\nabla u_n|\} \) is bounded in \( L^p(\omega) \) with \( \text{supp} \varphi \subset \subset \omega \) by virtue of Lemma 3. In fact, since \( \nabla u_n = \nabla_T M(u_n) \) for \( u_n \in \text{supp} \Phi'' \), with a constant \( M > 0 \) satisfying \( \text{supp} \Phi'' \subset [-M, M] \), it follows from (6.4) that \( \sup_n \int_\omega |\nabla u_n|^p \leq \infty \). Hence we may suppose \( |\nabla u| \in L^p(\omega) \) and \( |\nabla u_n - \nabla u| \to 0 \) in \( L^p(\omega) \) as \( n \to \infty \) by Fatou’s lemma. Then we also see that \( \{|\nabla u_n| + |\nabla u|^2 - |\nabla u_n - \nabla u| \to 0 \) in \( L^p(\omega) \) as \( n \to \infty \). Therefore (6.11) follows, since \( \{|\nabla (\Phi'(u_n)^{p-1})|\} \) is bounded in \( L^p(\omega) \) by Lemma 6. From (6.7), (6.8) and (6.11) it follows that

\[
\langle J_n, \varphi \rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{6.12}
\]

Consequently, using (6.7), (6.8) and (6.12), from (6.6) we obtain

\[
\langle A\Phi(u), \varphi \rangle \geq \langle \Phi'(u)^{-1}(Au)_a, \varphi \rangle - \|\Phi''\|_{L^\infty(\mathbb{R})} \langle (Au)_a, \varphi \rangle, \tag{6.13}
\]

which implies (4.1).

We proceed to the case where \( 1 < p < 2 \). We set \( \Phi^\eta(t) := \Phi(t) + \eta t \) for \( t \in \mathbb{R} \) with \( \eta > 0 \). Then we see that for each \( \eta > 0 \)

\[
\sup_{t \in \mathbb{R}} (\Phi^\eta)'(t)^{p-2}(\Phi^\eta)'(t) = \sup_{t \in \mathbb{R}} (\Phi'(t) + \eta)^{p-2} \Phi'(t) \leq \eta^{p-2} \sup_{t \in \mathbb{R}} \Phi'(t) < \infty. \tag{6.14}
\]
Hence we can apply Lemma 6 with \( \Phi^\prime \) instead of \( \Phi \), so that in a quite similar way we reach to the inequality (6.13) replaced \( \Phi \) by \( \Phi^\prime \), that is, we have

\[
\langle A\Phi^\prime(u), \varphi \rangle \geq \langle (\Phi^\prime)'(u)^{p-1}(Au)_a, \varphi \rangle - \|\,(\Phi^\prime)'\,\|^{p-1}_{L^\infty(R)}(\langle Au_\alpha^-\rangle, \varphi) \tag{6.15}
\]

\[
= \langle (\Phi(u) + \eta)^{p-1}(Au)_a, \varphi \rangle - \|\,(\Phi^\prime)\,\|^{p-1}_{L^\infty(R)}(\langle Au_\alpha^-\rangle, \varphi)
\]

for any \( \varphi \in C_0^{\infty}(\Omega) \) with \( \varphi \geq 0 \) in \( \Omega \). Letting \( \eta \to 0 \), we have

\[
\langle A\Phi^\prime(u), \varphi \rangle = -\int_\Omega (\Phi^\prime)'(u)^{p-1}A(x, \nabla u) \cdot \nabla \varphi
\]

\[
\to -\int (\Phi^\prime(u)^{p-1}A(x, \nabla u) \cdot \nabla \varphi = \langle A\Phi(u), \varphi \rangle. \tag{6.16}
\]

Since \( A(x, \nabla u) \in L^1_{\text{loc}}(\Omega) \), and also we have

\[
\langle (\Phi^\prime(u) + \eta)^{p-1}(Au)_a, \varphi \rangle \to \langle \Phi^\prime(u)^{p-1}(Au)_a, \varphi \rangle \tag{6.17}
\]

because \( (\Phi^\prime(u) + \eta)^{p-1} \) is locally integrable in \( \Omega \) with respect to \( |(Au)_a| \) uniformly for \( \eta > 0 \). Hence (6.13) follows from (6.15), (6.16) and (6.17). This completes the proof.

\[\square\]

**Proof of Theorem 2:**

When \( A = \Delta \), the proof of Theorem is seen in [10]; Theorem 3. We remark that the proof in the general case is accomplished in the same line under the assumption of admissibility. It follows from Lemma 3 that \( T_k(u) \in W^{1,p}_{\text{loc}}(\Omega) \) and \( AT_k(u) \in M(\Omega) \) for any \( k > 0 \). Moreover \( AT_k(u) \leq (Au)^{+} \) in \( D'(\Omega) \). Let us simply denote \( Au \) by \( \mu \in M(\Omega) \). Let \( E \) be a Borel measurable subset of \( \Omega \) such that \( C_\mu(E, \Omega) = 0 \) and \( |\mu_a|(\Omega \setminus E) = 0 \), and let \( K \) be a compact subset of \( E \). Then we immediately have

\[
AT_k(u) \leq \mu^{+} \quad \text{in } D'(\Omega \setminus K). \tag{6.18}
\]

Take and fix an arbitrary \( \varphi \in C_0^{\infty}(\Omega) \) with \( \varphi \geq 0 \) in \( \Omega \). Let \( \{\varphi_n\} \subset C_0^{\infty}(\Omega \setminus K) \) be such that \( 0 \leq \varphi_n \leq \varphi \) in \( \Omega \) and \( \varphi_n \to \varphi \) in \( W^{1,p}_{0}(\Omega) \) as \( n \to \infty \). Then we see as \( n \to \infty \) that

\[
\langle AT_k(u), \varphi_n \rangle = -\int A(x, \nabla T_k(u)) \cdot \nabla \varphi_n
\]

\[
\to -\int A(x, \nabla T_k(u)) \cdot \nabla \varphi = \langle A(T_k(u)), \varphi \rangle. \tag{6.19}
\]

Since it follows from (6.18) that for any \( n \geq 1 \)

\[
\langle AT_k(u), \varphi_n \rangle \leq \int_{\Omega \setminus K} \varphi_n \,d\mu^{+} \leq \int_{\Omega \setminus K} \varphi \,d\mu^{+}, \tag{6.20}
\]

by (6.19) and (6.20) we get

\[
\langle AT_k(u), \varphi \rangle \leq \int_{\Omega \setminus K} \varphi \,d\mu^{+},
\]

which implies

\[
AT_k(u) \leq \chi_{\Omega \setminus K}\mu^{+} \quad \text{in } D'(\Omega).
\]

As \( k \to \infty \) we get

\[
\mu = Au \leq \chi_{\Omega \setminus K}\mu^{+} \quad \text{in } D'(\Omega).
\]
Therefore,
\[ \mu_s|_K = \mu|_K \leq 0 \quad \text{in } D'(\Omega). \]
Since \( K \) is an arbitrary compact subset of \( E \) and \( \mu \) is (Borel) regular as a Radon measure, we conclude
\[ \mu_s \leq 0 \quad \text{in } D'(\Omega). \]

7. Proofs of Corollary 1 and Corollary 2

**Proof of Corollary 1:** Let \( \Phi \) be a \( C^1 \) convex function in \( R \) such that \( \Phi(t) = t \) if \( t \geq 0, |\Phi(t)| < 1 \) if \( t < 0, 0 < \Phi'/1 \) in \( R \) and \( \lim_{t \to -\infty} \Phi'(t) = 0 \). Set \( \Phi_n(t) = \Phi(nt)/n \) for \( t \in R \) and \( n = 1, 2, \ldots \). Then we see that \( \{ \Phi_n \} \) is a sequence of \( C^1 \) convex functions in \( R \) such that
\[ \Phi_n(t) = t \text{ if } t \geq 0, |\Phi_n(t)| < \frac{1}{n} \text{ if } t < 0, 0 \leq \Phi'_n \leq 1 \text{ in } R. \]
It follows from Theorem 1 that
\[ A\Phi_n(u) \geq \Phi'_n(u)^{p-1}(Au)_a - (Au)_a^- \quad \text{in } D'(\Omega). \tag{7.1} \]
First we see that \( A\Phi_n(u) \to A(u) \) in \( D'(\Omega) \) as \( n \to \infty \). It follows from the remark just after Corollary 1 that \( \chi_{[u \geq 0]} \) is locally integrable in \( \Omega \) with respect to \( |(Au)_a| \).
Therefore, as \( n \to \infty \) in (7.1) we have
\[ A(u) \geq \chi_{[u \geq 0]}(Au)_a - (Au)_a^- \quad \text{in } D'(\Omega), \tag{7.2} \]
which is (4.2). Since for any \( \varphi \in C_0^\infty(\Omega) \)
\[ \langle A|u|, \varphi \rangle = \langle \text{div}(A(x, \nabla|u|)), \varphi \rangle = - \int_\Omega A(x, \nabla(u^+ + u^-)) \cdot \nabla \varphi \]
\[ = - \int_\Omega A(x, \nabla u^+ \cdot \nabla \varphi - \int_\Omega A(x, \nabla u^-) \cdot \nabla \varphi \]
\[ = \langle A(u^+) + A(u^-), \varphi \rangle, \]
we have
\[ A|u| = A(u^+) + A(u^-) \quad \text{in } D'(\Omega). \tag{7.3} \]
Noting \( u^- = (-u)^+ \) and applying (7.2) to \(-u\) instead of \( u \), we have
\[ A(u^-) = A((-u)^+) \geq \chi_{[-u \geq 0]}(A(-u))_a - (A(-u))_a^- \tag{7.4} \]
\[ = - \chi_{[u \leq 0]}(Au)_a - (Au)_a^+ \quad \text{in } D'(\Omega). \]
From (7.2), (7.3) and (7.4) it follows that
\[ A|u| \geq \text{sgn}(u)(Au)_a - |Au|_a \quad \text{in } D'(\Omega), \]
which shows (4.3). \qed

**Proof of Corollary 2:** Since \( u^+ \) and \( u^- \) are admissible (see [15] Proposition1.1), we can apply Theorem 2 to \( u^+ \) and \( u^- \) respectively, so that we have
\[ (-A(u^+))_a \geq 0, \quad (-A(u^-))_a \geq 0 \quad \text{in } D'(\Omega). \tag{7.5} \]
In a similar way as (7.3), we see
\[ Au = A(u^+) - A(u^-) \quad \text{in } D'(\Omega), \] (7.6)
which implies
\[ (-Au)_s = (-A(u^+))_s - (-A(u^-))_s \quad \text{in } D'(\Omega). \] (7.7)

By (7.5) and (7.7) we have \((-Au)_s \leq (-A(u^+))_s\) and \((-Au^+)_s \geq 0\) in \(D'(\Omega)\), therefore we obtain
\[ (-Au^+)_s \leq (-A(u^+))_s \quad \text{in } D'(\Omega). \] (7.8)
On the other hand, from (4.2) it follows that
\[ (Au)_s = (A(u^+))_s \quad \text{in } D'(\Omega), \] (7.9)

From (7.8) and (7.9) we see (4.5).

From (4.5) and (7.7) we have
\[ (Au)_s = (A(u^+))_s = (Au^+)_s = (Au^-)_s \quad \text{in } D'(\Omega), \] (7.10)
and by (4.5), (7.3) and (7.10) we have
\[ (-Au)_s = (-A(u^+))_s + (-A(u^-))_s = (Au^+)_s + (Au^-)_s = (A\rho)_s \quad \text{in } D'(\Omega), \] (7.11)
which shows (4.6). □

8. Appendix

In this section we prove

**Proposition 8.1.** Let \( N \geq 1, 1 < p < \infty, p^* = \max\{1, p - 1\} \) and \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). Then a function \( u \in W^{1,p}(\Omega) \) is admissible in \( W^{1,p^*}_{\text{loc}}(\Omega) \), if \( Au \in M(\Omega) \).

**Proof of Proposition 8.1:** To use a diagonal argument, we choose and fix a family of open set \( \{\omega_n\} \) such that
\[ \omega_1 \subset \omega_2 \subset \cdots \subset \omega_n \subset \cdots \subset \omega_{n+1} \subset \cdots \subset \Omega \quad \text{and } \Omega = \bigcup_{n=0}^{\infty} \omega_n. \] (8.1)

Let \( \{u_n\} \subset W^{1,p}_{\text{loc}}(\Omega) \) be the sequence in the Definition 3.1. Let \( \rho \in C^\infty_0(B_1) \) be a radial, nonnegative and decreasing mollifier. By extending \( v \in L^1(\Omega) \) to the whole space \( \mathbb{R}^N \) so that \( v \equiv 0 \) outside \( \Omega \), we define a mollification of \( v \) with \( \varepsilon > 0 \) by
\[ v^\varepsilon(x) := \rho_\varepsilon * v(x) = \int_\Omega \rho_\varepsilon(x-y)v(y)dy \quad \text{for } x \in \Omega \quad \text{and } n = 1, 2, \ldots, \] (8.2)

First we prove that \( u \in W^{1,p}_{0}(\Omega) \) is admissible in \( W^{1,p^*}_{\text{loc}}(\Omega) \), if \( Au \) is a Radon measure on \( \Omega \). Again by extending \( u \in W^{1,p}_{0}(\Omega) \) to the whole space \( \mathbb{R}^N \) so that \( u \equiv 0 \) outside \( \Omega \), Similarly we extend \( A(x, \nabla u) \) to the whole space \( \mathbb{R}^N \) so that \( A(x, \nabla u) \equiv 0 \) outside \( \Omega \). Let \( w_n \in W^{1,p}_{0}(\Omega) \) be the unique weak solution of the following boundary value
problem for the monotone operator $A$ (see e.g. [17]): For $n = 1, 2, \cdots$ and $\varepsilon_1 > \varepsilon_2 > \cdots \varepsilon_n > \cdots \to 0$, we set

$$
\begin{cases}
A w_n = \text{div } A^\varepsilon_n (x, \nabla u) & \text{in } \Omega, \\
w_n = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(8.3)

where $A(x, \nabla u) \in (L^{p'}(R^N))^N$ with $p' = p/(p-1)$ and $A^\varepsilon_n (x, \nabla u) \in (C^\infty(R^N))^N$ is a mollification of $A(x, \nabla u)$ defined by (8.2). It follows from the standard regularity theorem that $w_n \in L^\infty(\Omega)$ for $n = 1, 2, \cdots$ (see e.g. [9]). Let us set $A u = \text{div } A(x, \nabla u) = \mu$. We note that $|\mu(\omega)| < \infty$ for any $\omega \subset \subset \Omega$. Then we have

$$
\text{div } A^\varepsilon_n (x, \nabla u) = (\text{div } A(x, \nabla u))^\varepsilon_n = (\mu)^\varepsilon_n = \mu^\varepsilon_n \text{ in } \omega \text{ provided that } \varepsilon_n \text{ is sufficiently small. Hence we clearly have}
$$

$$
|A w_n(\omega)| = |\text{div } A^\varepsilon_n (x, \nabla u)(\omega)| = |\mu^\varepsilon_n(\omega)| \to |\mu(\omega)| \text{ as } n \to \infty.
$$

This proves the condition 2. Next we show

$$
w_n \to u \text{ in } W_0^{1, p}(\Omega) \text{ as } n \to \infty.
$$

(8.4)

Then the condition 1 is clearly satisfied, since one may choose a subsequence. By using $w_n - u \in W_0^{1, p}(\Omega)$ as a test function, we have

$$
- (A w_n - A u, w_n - u) = \int_\Omega (A(x, w_n) - A(x, \nabla u)) \cdot \nabla (w_n - u)
\geq c_2 \int_\Omega |\nabla (w_n - u)|^p \quad \text{(by (4))},
$$

(8.5)

In the left-hand side, using Young’s inequality for $\delta > 0$ we have

$$
- (A w_n - A u, w_n - u) \leq \int_\Omega (A^\varepsilon_n (x, \nabla u_n) - A(x, \nabla u)) \cdot \nabla (w_n - u)
\leq C(\delta) \int_\Omega |A^\varepsilon_n (x, \nabla u) - A(x, \nabla u)|^{p'} + \delta \int_\Omega |\nabla (w_n - u)|^p,
$$

(8.6)

where $C(\delta) > 0$ is a constant depending only on $\delta$.

By the assumption 3 we have $||A^\varepsilon_n (x, \nabla u) - A(x, \nabla u)||_{L^{p'}(\Omega)} \to 0$ as $n \to \infty$. It follows from (8.5) and (8.6) that $\nabla w_n \to \nabla u$ in $(L^p(\Omega))^N$ as $n \to \infty$, which implies (8.4). Then, taking a subsequence if necessary, we may assume that $w_n \to u$ a.e. in $\Omega$ as $n \to \infty$.

Lastly we treat the case where $u \in W_0^{1, p}(\Omega)$. For each $n$ we choose $\eta_n \in C^\infty_c (\omega_{n+1})$ such that $0 \leq \eta_n \leq 1$ and $\eta_n = 1$ in some neighborhood of $\partial \omega_n$. Let us set $v_n = \eta_n u$ ($n = 1, 2, 3, \cdots$). Then we see that $v_n \in W_0^{1, p}(\omega_{n+1})$, $v_n \to u$ in $W_0^{1, p}(\Omega)$ as $n \to \infty$ and $A v_n \in W^{-1, p'}(\Omega) \cap L^1(\omega_{n+1})$. Moreover we have $|A v_k(\omega_n)| = |A u(\omega_n)|$ for any $k \geq n$. By the previous step with obvious modification, one can approximate each $v_k$ inductively by $\xi_k \in W_0^{1, p}(\Omega) \cap L^\infty(\Omega)$ such that $\xi_k \to u$ in $W_0^{1, p'(\Omega)}$ as $k \to \infty$ and $||A \xi_k(\omega_n) - |A u(\omega_n)| < \frac{1}{k}$ for $k \geq n$. Then the assertion is now clear. □

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