Graded contractions of representations of Lie algebras

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Abstract. The concept of graded contractions for representations of Lie algebras will be presented. An explicit example of graded contractions of all irreducible finite-dimensional representations of \( \mathfrak{sl}(2, \mathbb{C}) \) to representations of Euclidean algebra \( \mathfrak{e}(2) \) will be described in detail.

1. Introduction
The classification of finite–dimensional representations of Lie algebras is still an open problem. Complete results are known only for semisimple Lie algebras. Solvable Lie algebras are not classified [1], much less their representations. The classification of representations of a given solvable Lie algebra presents an unsolved problem even in the case of Euclidean Lie algebras \( \mathfrak{e}(2) \) [2, 3].

The well known concept of Inönü–Wigner contractions [4] of Lie algebras allows one to construct solvable Lie algebras from simple ones as well as their representations. We describe an alternative way to these continuous contractions, so called graded contractions of representations [5], which allows us to obtain a class of representations of contracted solvable Lie algebras.

Irreducible representations play a key role in the representation theory of simple Lie algebras, since any representation of a simple Lie algebra is a direct sum of irreducible representations. However, the only irreducible representations of solvable Lie algebras are one–dimensional. Therefore, we substitute the irreducibility requirement by the indecomposability requirement. Thus, we will contract mainly irreducible representations of simple Lie algebras and in the search for representations of contracted Lie algebras we will demand indecomposability, faithfulness and nonequivalency of resulting contracted representations.
In following sections, we present a transparent systematical overview of definitions of graded contractions for Lie algebras [6] and their representations [5]. We consider only finite–dimensional representations of finite–dimensional complex Lie algebras.

Let us recall basic definitions concerning gradings of Lie algebras. A decomposition \( L = \bigoplus_{i \in I} L_i, \ L_i \neq 0 \), is called a grading of a Lie algebra \( \mathcal{L} = (L, [,.,]) \), if for any pair of indices \( i, j \in I \) there exists \( k \in I \) such that \([L_i, L_j] \subseteq L_k\). A pair of indices \( i, j \in I \) is called relevant if \([L_i, L_j] \neq 0\). For any relevant pair of indices \( i, j \in I \) we define \( i + j := k \in I \iff 0 \neq [L_i, L_j] \subseteq L_k\).

There are several types of gradings. A grading \( \Gamma \) is called finest grading, if \( \dim(L_i) = 1 \) for all \( i \in I \). If \( g \) is an automorphism of \( \mathcal{L} \) then gradings \( \Gamma \) and \( \Gamma^g : L = \bigoplus_{i \in I} g(L_i) \) are equivalent.

2. Graded contractions of Lie algebras

Definition 2.1. A Lie algebra \( \mathcal{L}^\varepsilon = (L, [,.,]_\varepsilon) \) is called a \( \Gamma \)-graded contraction of a Lie algebra \( \mathcal{L} = (L, [,.,]) \) if there exist a grading \( \Gamma : L = \bigoplus_{i \in I} L_i \) of \( \mathcal{L} \) and contraction parameters \( \varepsilon_{i,j} \in \mathbb{C} \) such that for all \( x \in L_i, y \in L_j, i, j \in I \) it holds \([x, y]_\varepsilon = \varepsilon_{i,j}[x, y]\).

Contraction parameters \( \varepsilon_{i,j} \) are called irrelevant if \([L_i, L_j] = 0 \) or relevant if \([L_i, L_j] \neq 0\). The irrelevant contraction parameters can be chosen arbitrarily, thus we define \( \varepsilon_{i,j} := 0 \). On the other hand, the relevant contraction parameters have to ensure that \([.,.,]_\varepsilon \) is a Lie bracket. Therefore, they must satisfy \( \varepsilon_{i,j} = \varepsilon_{j,i} \) and the system of contraction equations

\[
\varepsilon_{jk}\varepsilon_{i,j+k}[x, [y, z]] + \varepsilon_{ki}\varepsilon_{j,k+i}[y, [z, x]] + \varepsilon_{ij}\varepsilon_{k,i+j}[z, [x, y]] = 0 \tag{1}
\]

\( \forall x \in L_i, \forall y \in L_j, \forall z \in L_k, \forall i, j, k \in I \). The following reduced form of this system is often used in literature [6] if so called generic case, when all contraction parameters are relevant, occurs

\[
\varepsilon_{jk}\varepsilon_{i,j+k} = \varepsilon_{ki}\varepsilon_{j,k+i} = \varepsilon_{ij}\varepsilon_{k,i+j} \quad \forall i, j, k \in I.
\]

Solutions of the system of contraction equations are usually written in the form of a symmetric square matrix \( \varepsilon = (\varepsilon_{i,j}) \) called contraction matrix. A \( \Gamma \)-graded contraction \( \mathcal{L}^\varepsilon \) is called trivial if \( \mathcal{L}^\varepsilon \cong \mathcal{L} \) or \( \mathcal{L}^\varepsilon \) is abelian. An important example of a trivial contraction is given by a normalization matrix \( \alpha = (\alpha_{i,j}) \), where

\[
\alpha_{i,j} := \frac{a_i a_j}{a_{i+j}}, \quad a_k \in \mathbb{C} \setminus \{0\}, \forall k \in I.
\]

Normalization matrices allow us to define an equivalence on the set of all solutions of the system (1). Contraction matrices \( \varepsilon^1, \varepsilon^2 \) are called equivalent if there exists a normalization matrix \( \alpha = (\frac{a_i a_j}{a_{i+j}}) \) such that \( \varepsilon_{i,j}^1 = \frac{a_i a_j}{a_{i+j}} \varepsilon_{i,j}^2 \) for all \( i, j \in I \). Contracted Lie algebras given by equivalent contraction matrices are isomorphic.

We recall the definition of one–parametric continuous contractions and define continuous graded contractions which correspond to generalized Inönü–Wigner contractions.
Definition 2.2. Let \( \mathcal{L} = (L, [\cdot, \cdot]) \) be a Lie algebra and \( T : (0, 1] \rightarrow \text{GL}(L) : t \mapsto T(t) \) a continuous map. If there exists a limit

\[
[x, y]_T = \lim_{t \rightarrow 0} T(t)^{-1} [T(t)x, T(t)y] \quad \forall x, y \in L,
\]

then \( \mathcal{L}_T = (L, [\cdot, \cdot]_T) \) is called a continuous contraction of \( \mathcal{L} \). If there exists a basis in \( V \) such that \( T(t) = \text{diag}(t^{k_1}, t^{k_2}, \ldots, t^{k_n}), k_1, k_2, \ldots, k_n \in \mathbb{Z} \) then \( \mathcal{L}_T \) is called a generalized Inönu–Wigner contraction.

Definition 2.3. A \( \Gamma \)-graded contraction \( \mathcal{L}^\varepsilon \) is called continuous if there exist continuous functions \( a_i : (0, 1] \rightarrow \mathbb{C} \setminus \{0\}, i \in I \), such that for all relevant contraction parameters it holds

\[
\varepsilon_{i,j} = \lim_{t \rightarrow 0} \frac{a_i(t)a_j(t)}{a_{i+j}(t)}.
\]

3. Graded contractions of representations

Let us consider a \( \Gamma \)-graded contraction \( \mathcal{L}^\varepsilon \) of a Lie algebra \( \mathcal{L} \). In order to obtain representations of \( \mathcal{L}^\varepsilon \) we have to chose and contract some representation \( r \) of \( \mathcal{L} \). However, this representation \( r \) cannot be chosen arbitrarily, it has to fulfill the following compatibility condition.

Definition 3.1. A representation \( r : \mathcal{L} \rightarrow \text{gl}(V) \) is called compatible with the grading \( \Gamma : L = \bigoplus_{i \in I} L_i \) if there exists a decomposition \( V = \bigoplus_{m \in M} V_m \) such that for any pair of indices \( i \in I, m \in M \) there exists \( n \in M \) such that

\[
r(L_i)V_m := \{ r(x)v \mid x \in L_i, v \in V_m \} \subseteq V_n.
\]

For indices \( i \in I, m \in M \) we define \( i \triangleright m := n \in M \Leftrightarrow 0 \neq r(L_i)V_m \subseteq V_n \).

Let us note that any finite dimensional representation of a simple Lie algebra \( \mathcal{L} \) is compatible with the Cartan grading of \( \mathcal{L} \) (root space decomposition) and the corresponding decomposition of the representation space \( V \) is its weight space decomposition.

Definition 3.2. Let \( \mathcal{L}^\varepsilon \) be a \( \Gamma \)-graded contraction of \( \mathcal{L} \), \( r : \mathcal{L} \rightarrow \text{gl}(V) \) be a representation of \( \mathcal{L} \) compatible with the grading \( \Gamma \), then any representation \( r^\psi : \mathcal{L}^\varepsilon \rightarrow \text{gl}(V) \) is called a graded contraction of \( r \) if there exist \( \psi_{i,m} \in \mathbb{C} \) such that for all \( x \in L_i, v \in V_m, i \in I, m \in M \) it holds

\[
r^\psi(x)v = \psi_{i,m} r(x)v.
\]

Contraction parameters \( \psi_{i,m} \) are called irrelevant if \( r(L_i)V_m = 0 \) or relevant if \( r(L_i)V_m \neq 0 \). The irrelevant contraction parameters can be chosen arbitrarily, thus we define \( \psi_{i,m} := 0 \). On the other hand, the relevant contraction parameters have to satisfy the following system of contraction equations

\[
\varepsilon_{i,j} \psi_{i+j,m} r([x, y]) v = \psi_{i,j \triangleright m} \psi_{j,m} r(x)r(y)v - \psi_{j,i \triangleright m} \psi_{i,m} r(y)r(x)v
\]

(2)
\( \forall x \in L_i, \forall y \in L_j, \forall v \in V_m, \forall i, j \in I, \forall m \in M. \) The following stronger condition, depending only on properties of the grading, is usually required in literature [5]

\[
\varepsilon_{i,j} \psi_{i+j,m} = \psi_{i,j\triangleright m} \psi_{j,m} = \psi_{j,i\triangleright m} \psi_{i,m}, \quad \forall i, j \in I, \forall m \in M. \quad (3)
\]

It allows to handle all Lie algebras and their representations with the same grading properties simultaneously. However, it usually reduces the number of possible solutions for a given representation of a given Lie algebra. The equivalence of the systems (2) and (3) is still an open question. Let us note that in the case of \( \mathbb{Z}_3 \)-graded \( \text{sl}(2, \mathbb{C}) \) considered further, these systems are not equivalent.

The solutions of (2) are written in the form of generally rectangular contraction matrices \( \psi = (\psi_{i,m}) \). The introduction of a normalization matrix \( \beta = (\beta_{i,m}) \), where \( \beta_{i,m} := \frac{b_m}{b_{i\triangleright m}}, b_m \in \mathbb{C} \setminus \{0\}, m \in M \), allows us to define an equivalence on the set of all solutions of system (2). Contraction matrices \( \psi^1, \psi^2 \) are called equivalent if there exists a normalization matrix \( \beta = (\beta_{i,m}) \) such that \( \psi^1_{i,m} = \frac{b_m}{b_{i\triangleright m}} \psi^2_{i,m} \) for all \( i \in I, m \in M \). Contracted representations corresponding to equivalent contraction matrices are equivalent.

We reformulate the definition of continuous contractions of representations [4] and define continuous graded contractions of representations accordingly.

**Definition 3.3.** Let \( \mathcal{L}_T \) be a continuous contraction of \( \mathcal{L} \), \( r : \mathcal{L} \to \text{gl}(V) \) a representation of \( \mathcal{L} \) and \( U : (0, 1] \to \text{GL}(V) : t \mapsto U(t) \) a continuous map. If for all \( x \in \mathcal{L} \) there exists a limit

\[
r_0(x) = \lim_{t \to 0} U(t)^{-1} r(T(t)x)U(t)
\]

then \( r_0 \) is called a **continuous contraction of representation** \( r \).

**Definition 3.4.** Let \( \mathcal{L}^\varepsilon \) be a continuous graded contraction of \( \mathcal{L} \) such that \( \varepsilon_{i,j} = \lim_{t \to 0} \frac{a_i(t)a_j(t)}{a_{i+j}(t)}, r^\psi : \mathcal{L}^\varepsilon \to \text{gl}(V) \) a graded contraction of the representation \( r : \mathcal{L} \to \text{gl}(V) \) then \( r^\psi \) is called continuous if there exist continuous functions \( b_m : (0, 1] \to \mathbb{C} \setminus \{0\}, m \in M \) such that for all relevant contraction parameters it holds

\[
\psi_{i,m} = \lim_{t \to 0} \frac{a_i(t)b_m(t)}{b(t)_{i\triangleright m}}.
\]

4. **Examples**

In the following examples we will contract all irreducible representations (with dimension \( n \geq 3 \)) of the simple Lie algebra \( \text{sl}(2, \mathbb{C}) \) to representations of the Euclidean algebra \( e(2) \). We consider all possible gradings of \( \text{sl}(2, \mathbb{C}) \) and compare the outcomes.

The Lie algebra \( \text{sl}(2, \mathbb{C}) \) is given as the set of all \( 2 \times 2 \) square traceless complex matrices. Choosing a basis of \( \text{sl}(2, \mathbb{C}) \)

\[
X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
we can describe all its nonequivalent gradings as follows:

- $\mathbb{Z}_2$-grading $sl(2, \mathbb{C}) = L_0 \oplus L_1$, $L_0 = \mathbb{C}X_0, L_1 = \text{span}_\mathbb{C}\{X_1, X^{-1}\}$
- $\mathbb{Z}_3$-grading $sl(2, \mathbb{C}) = L_0 \oplus L_1 \oplus L_2$, $L_0 = \mathbb{C}X_0, L_1 = \mathbb{C}X_1, L_2 = \mathbb{C}X^{-1}$
- $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading $sl(2, \mathbb{C}) = L_{11} \oplus L_{01} \oplus L_{10}$, $L_{11} = \mathbb{C}X_0, L_{01} = \mathbb{C}(X_1 + X^{-1}), L_{10} = \mathbb{C}(X_1 - X^{-1})$

Any $n$–dimensional irreducible representation $r$ of $sl(2, \mathbb{C})$ on the vector space $V = \text{span}_\mathbb{C}(v_1, v_2, \ldots, v_n)$ is given by

\[ r(X_0)v_j = (n + 1 - 2j)v_j, \quad r(X_{-1})v_j = j(n - j)v_{j-1}, \quad r(X_1)v_j = v_{j+1}. \]

4.1. $\mathbb{Z}_2$–graded $sl(2, \mathbb{C})$

There exists only one contraction of $\mathbb{Z}_2$–graded $sl(2, \mathbb{C})$ to algebra $e(2)$.

\[
\begin{align*}
&\text{sl}(2, \mathbb{C}) &\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\text{e}(2) \\
&[X_0, X_{-1}] = 2X_{-1} & [X_0, X_{-1}]_e = 2X_{-1} \\
&[X_0, X_1] = -2X_1 & [X_0, X_1]_e = -2X_1 \\
&[X_{-1}, X_1] = X_0 & [X_{-1}, X_1]_e = 0
\end{align*}
\]

There is only one possible decomposition of the representation space which ensures the compatibility of any $n$–dimensional irreducible representation with $\mathbb{Z}_2$ grading

\[ V = V_0 \oplus V_1, \quad V_0 = \text{span}_\mathbb{C}\{v_j \mid j \text{ even}\}, \quad V_1 = \text{span}_\mathbb{C}\{v_j \mid j \text{ odd}\}. \]  

All contraction parameters are relevant and contraction matrix $\psi$ and normalization matrix $\beta$ have a form

\[ \psi = \begin{pmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{pmatrix}, \quad \beta = \begin{pmatrix} \frac{a_0}{b_0} & \frac{a_1}{b_1} \\ \frac{b_0}{a_0} & \frac{b_1}{a_1} \end{pmatrix}. \]

The system of contraction equations

\[ \psi_{11}\psi_{10} = 0, \quad \psi_{10} = \psi_{10}\psi_{01} = \psi_{10}\psi_{00}, \quad \psi_{11} = \psi_{11}\psi_{01} = \psi_{11}\psi_{00}, \]

has the following solutions:

\[ \psi^1 = \begin{pmatrix} \psi_{00} & \psi_{01} \\ 0 & 0 \end{pmatrix}, \quad \psi^2 = \begin{pmatrix} 1 & 1 \\ \psi_{10} & 0 \end{pmatrix}, \quad \psi^3 = \begin{pmatrix} 1 & 1 \\ 0 & \psi_{11} \end{pmatrix}, \]

where $\psi_{00}$ and $\psi_{01}$ are arbitrary complex numbers while $\psi_{10}$ and $\psi_{11}$ are nonzero complex numbers. Since $k$–th row in the matrix $\psi$ determines the action of $k$–th grading subspace, we see that the solution $\psi^1$ corresponds to an unfaithful representation. Using the normalization matrix $\beta$ we normalize the remaining solutions to the form

\[ \psi^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

These solutions correspond to nonequivalent faithful indecomposable (as we will see further) representations of $e(2)$. Using Kronecker delta symbol, these representations are given as follows

\[ r^{\psi^2}(X_0)v_j = (n+1-2j)v_j, \quad r^{\psi^2}(X_{-1})v_j = \delta_{0,i}j(n-j)v_{j-1}, \quad r^{\psi^2}(X_1)v_j = \delta_{0,i}v_{j+1} \]

\[ r^{\psi^3}(X_0)v_j = (n+1-2j)v_j, \quad r^{\psi^3}(X_{-1})v_j = \delta_{1,i}j(n-j)v_{j-1}, \quad r^{\psi^3}(X_1)v_j = \delta_{1,i}v_{j+1} \]

where $i = j \mod 2$. 

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4.2. $\mathbb{Z}_2 \times \mathbb{Z}_2$–graded $\mathfrak{sl}(2, \mathbb{C})$

There are three possible contractions of $\mathbb{Z}_2 \times \mathbb{Z}_2$–graded $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{e}(2)$. Except the 3–dimensional case we were able to find only one decomposition (4) of the representation space $V$ which provides the compatibility of representations of $\mathfrak{sl}(2, \mathbb{C})$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading. All resulting representations of $\mathfrak{e}(2)$ for all three possible contractions are, up to equivalency, the same as in the case of $\mathbb{Z}_2$ grading.

4.3. $\mathbb{Z}_3$–graded $\mathfrak{sl}(2, \mathbb{C})$

There exists only one contraction of $\mathbb{Z}_3$–graded $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{e}(2)$.

$$\mathfrak{sl}(2, \mathbb{C}) \xrightarrow{\varepsilon = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \mathfrak{e}(2)$$

contraction

$$\begin{aligned}
[X_0, X_{-1}] &= 2X_{-1} \\
[X_0, X_1] &= -2X_1 \\
[X_{-1}, X_1] &= X_0
\end{aligned}$$

$\mathbb{Z}_3$–grading corresponds to the root space decomposition of $\mathfrak{sl}(2, \mathbb{C})$, thus we choose the weight space decomposition of the representation space $V$. In fact, there are many possible decompositions of $V$ which provides the compatibility of representation, however all of them can be obtained from this one by merging the weight spaces.

$$V = \bigoplus_{k \in M} V_k, \quad V_k = \mathbb{C}v_k, \quad k \in M = \{1, 2, \ldots, n\}.$$

In the case of an even dimension $n$ there are only two irrelevant contraction parameters $\psi_{-1,1} = \psi_{1,n} = 0$ while in the case of an odd dimension $n$ there are three irrelevant parameters $\psi_{-1,1} = \psi_{1,n} = \psi_{0,n(n-1)/2} = 0$. The contraction matrix and the normalization matrix have the following form

$$\psi = \begin{pmatrix} \psi_{01} & \psi_{02} & \psi_{03} & \ldots & \psi_{0n} \\ 0 & \psi_{-12} & \psi_{-13} & \ldots & \psi_{-1n} \\ \psi_{11} & \psi_{12} & \psi_{13} & \ldots & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & b_2 & b_3 & \ldots & 0 \\ b_1 & b_2 & b_3 & \ldots & b_{n-1} \end{pmatrix}.$$

The contraction system consists of the following three types of equations

$$\psi_{1,k} \psi_{-1,k+1} = 0, \quad \psi_{-1,k+1}(2 - (n + 1 - 2k)\psi_{0,k} + (n - 1 - 2k)\psi_{0,k+1}) = 0, \quad k = 1, 2, \ldots n - 1,$$

$$\psi_{-1,k+1}(2 - (n + 1 - 2k)\psi_{0,k} + (n - 1 - 2k)\psi_{0,k+1}) = 0.$$
of contracted representation, we add the following necessary condition for indecomposability $\psi_{1,k} + \psi_{-1,k+1} = 1$ for all $k = 1, \ldots, n - 1$. This demand enables us to rewrite the rest of the contraction system in the form

$$2 - (n + 1 - 2k)\psi_{0,k} + (n - 1 - 2k)\psi_{0,k+1} = 0, \quad k = 1, 2, \ldots n - 1.$$  

Solutions of this recurrent equation depend on the dimension of representation. For $n$ odd we get

$$\psi_{0,k} = 1, \quad \forall k \neq \frac{n + 1}{2},$$

while for $n$ even

$$\psi_{0,k} = \frac{(n - 1)\psi_{0,1} - 2(k - 1)}{(n - 1) - 2(k - 1)}, \quad \forall k = 2, \ldots, n.$$  

Overall, we get $2^{(n - 1)}$ indecomposable nonequivalent representations

$$r^{\psi}(X_{-1})v_k = \psi_{-1,k}k(n - k)v_{k-1}, \quad r^{\psi}(X_1)v_k = \psi_{1,k}v_{k+1},$$

given by the possible choices of $\psi_{1,k}$, $\psi_{-1,k}$ in $\{0, 1\}$. Two of these choices lead to the unfaithful representations while the rest leads to the faithful representations. In the case of an odd dimension $n$, the weights are preserved

$$r^{\psi}(X_0)v_k = (n + 1 - 2k)v_k.$$  

Let us note that all these representations are continuous contractions of irreducible representations of $\text{sl}(2, \mathbb{C})$. In the case of an even dimension $n$, each choice of $\psi_{1,k}$, $\psi_{-1,k}$ corresponds to one–parametric set of nonequivalent representations depending on the complex parameter $\psi_{0,1}$

$$r^{\psi}(X_0)v_k = ((n - 1)\psi_{0,1} - 2(k - 1))v_k.$$  

The representations given by $\psi_{0,1} = 1$ are continuous contractions of irreducible representations of $\text{sl}(2, \mathbb{C})$.

The results obtained for $\mathbb{Z}_2$–grading are given by the choices $\psi_{0,1} = 1$ and $\psi_{-1,k} = \psi_{1,k} = \delta_{0,k \text{mod} 2}$, for all $k = 1, 2, \ldots, n$ or $\psi_{-1,k} = \psi_{1,k} = \delta_{1,k \text{mod} 2}$, for all $k = 1, 2, \ldots, n$.

The indecomposability of the obtained representations follows from the fact that $r^{\psi}(X_0)$ is diagonalizable and all its eigenvalues have multiplicity one. According to [7] representation $r$ of $\mathcal{L}$ is indecomposable if and only if the only nontrivial idempotent in the set $\{A \in gl(V) \mid Ar(X) = r(X)A, \forall X \in \mathcal{L}\}$ is identity mapping. The only operator which commutes with $r^{\psi}(X_0)$ is given by $A_0 = A_kv_k, \quad A_k \in \mathbb{C}, \quad k = 1, 2, \ldots, n$. In order to commute with $r^{\psi}(X_{-1})$ and $r^{\psi}(X_1)$, $A$ must satisfy $\psi_{-1,k}A_k = A_{k-1}\psi_{-1,k}$, for all $k = 2, \ldots, n$ and $\psi_{1,k}A_k = A_{k+1}\psi_{1,k}$, for all $k = 1, \ldots, n - 1$, i.e.

$$\psi_{-1,k+1}(A_{k+1} - A_k) = 0 = \psi_{1,k}(A_{k+1} - A_k), \quad \forall k = 1, 2, \ldots, n - 1.$$  


Considering the necessary indecomposability condition $\psi_{1,k} + \psi_{-1,k} = 1$, we get $A_k = A_{k+1}$ for all $k = 1, 2, \ldots, n - 1$. Thus, the only operator which commutes with the whole $\mathfrak{r}^\psi(\mathfrak{e}(2))$ is a complex multiple of identity. Therefore, the representation $\mathfrak{r}^\psi$ is indecomposable.

Two representations $r_1, r_2$ of a Lie algebra $\mathcal{L}$ on the vector space $V$ are equivalent if there exists an automorphism $A : V \to V$ such that $r_1(X)A = Ar_2(X)$ for all $X \in \mathcal{L}$. Any two representations given by different values of the parameter $\psi_{0,1}$ are clearly nonequivalent since they differ in the trace of the operator $\mathfrak{r}^\psi(X_0)$. Let us consider whether representations $\mathfrak{r}^{\psi^1}, \mathfrak{r}^{\psi^2}$ with $\psi_{0,1}^1 = \psi_{0,1}^2$ given by the different choice of $\psi_{-1,k}$ and $\psi_{1,k}$ are equivalent. Since $\mathfrak{r}^{\psi^1}(X_0) = \mathfrak{r}^{\psi^2}(X_0)$, the automorphism $A$, which provides the equivalency, has the form $Av_k = A_kv_k, A_k \in \mathbb{C}, k = 1, 2, \ldots, n$. There exists $j$ such that, for example, $1 = \psi_{1,j}^1 \neq \psi_{1,j}^2 = 0$. Thus, we have $A_j = \psi_{1,j}^1A_j = A_{j+1}\psi_{1,j}^1 = 0$ which is in contradiction with the fact that $A$ is an automorphism.

5. Conclusion

We have described the concept of graded contraction of representations of Lie algebras. We have shown the example, where all irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ are contracted to representations of Euclidean algebra $\mathfrak{e}(2)$. In the case of $\mathbb{Z}_3$–graded $\mathfrak{sl}(2, \mathbb{C})$, each $n$–dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ is contracted to $2^{(n-1)}-2$ representations of $\mathfrak{e}(2)$, if $n$ is odd, and $2^{(n-1)}-2$ one–parametric families of representations of $\mathfrak{e}(2)$, if $n$ is even. All these representations are indecomposable, faithful and mutually nonequivalent. All obtained representations are so called string representations [2] of $\mathfrak{e}(2)$, i.e. representations with all weight multiplicities (eigenvalues of operator $\mathfrak{r}(X_0)$) equal to 1. In fact, in the case of even dimension $n$ we have obtained all indecomposable string representations, up to equivalence.

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