Ideals Modulo a Prime

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The main focus of this paper is on the problem of relating an ideal \( I \) in the polynomial ring \( \mathbb{Q}[x_1, \ldots, x_n] \) to a corresponding ideal in \( \mathbb{F}_p[x_1, \ldots, x_n] \) where \( p \) is a prime number; in other words, the reduction modulo \( p \) of \( I \). We first define a new notion of \( \sigma \)-good prime for \( I \) which does depend on the term ordering \( \sigma \), but not on the given generators of \( I \). We relate our notion of \( \sigma \)-good primes to some other similar notions already in the literature.

Then we introduce and describe a new invariant called the universal denominator which frees our definition of reduction modulo \( p \) from the term ordering, thus letting us show that all but finitely many primes are good for \( I \). One characteristic of our approach is that it enables us to easily detect some bad primes, a distinct advantage when using modular methods.

Keywords: Ideals; Modular; Groebner Bases; Term Orderings.

2010 Mathematics Subject Classification: 13P25, 13P10, 13-04, 14Q10, 68W30

1. Introduction and Notation

There is a long tradition of using modular techniques for speeding up computations which involve polynomials with rational coefficients. Consequently, it is practically impossible to quote all the papers related to this topic; a few of them are [7], [12], [17], [18], [19], [20], [21], [22], and [23]. Two main interrelated obstacles to the success of this kind of approach are the existence of bad, good and lucky primes and the difficulty of reconstructing the correct rational coefficients possibly in the presence of undetected bad primes. We refer to [1] for a discussion of the second problem and to [4] and [8] for some new results in this direction and applications to the problem of the implicitization of hypersurfaces and of the computation of minimal polynomials.
The main focus of this paper is on the problem of relating an ideal $I$ in the polynomial ring $P = \mathbb{Q}[x_1, \ldots, x_n]$ to a corresponding ideal in $\mathbb{F}_p[x_1, \ldots, x_n]$ where $p$ is a prime number. In other words, we face the problem of defining a reduction modulo $p$ of $I$.

To date there are two typical approaches to this problem. One takes an arbitrary set of generators $F \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ of the ideal $I$, then works with the ideal $\langle \text{prim}(F) \rangle$ generated by the primitive parts $\text{prim}(F) \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ where each polynomial is scaled by a suitable rational so that its coefficients become integers and with no common factor. The reduction mod $p$ of $I$ is then defined to be the ideal in $\mathbb{F}_p[x_1, \ldots, x_n]$ generated by the reductions mod $p$ of the elements in $\text{prim}(F)$. This approach has the merit of being easy to compute, but its main drawback is that it depends on the chosen generators of $I$.

The other approach works with the projection of $I \cap \mathbb{Z}[x_1, \ldots, x_n]$ into the quotient ring $\mathbb{F}_p[x_1, \ldots, x_n]$. This definition has the merit of being intrinsic to the ideal $I$, but has the drawback of being not so easy to compute. For a nice discussion about this topic, see for instance [21].

Our idea is different from both of these approaches. We fix a term ordering $\sigma$, and let $G_\sigma \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ be the reduced $\sigma$-Gröbner basis of $I$. Solely for those primes $p$ which do not divide the denominator of any coefficient in $G_\sigma$, we define the reduction mod $p$ of $I$ to be the ideal in $\mathbb{F}_p[x_1, \ldots, x_n]$ generated by the reductions mod $p$ of the elements in $\langle \text{prim}(G_\sigma) \rangle$.

Our definition uses the ideal $\langle \text{prim}(G_\sigma) \rangle \subseteq \mathbb{Z}[x_1, \ldots, x_n]$. It turns out that $\langle \text{prim}(F) \rangle \subseteq \langle \text{prim}(G_\sigma) \rangle \subseteq I \cap \mathbb{Z}[x_1, \ldots, x_n]$. It is interesting to observe that both inclusions can be strict. For instance, the first inclusion is strict if we have $F = \{2x + y, y\}$; then $\langle F \rangle = \langle 2x + y, y \rangle = \langle 2x, y \rangle$ while $\langle G_\sigma \rangle = \langle \text{prim}(G_\sigma) \rangle = \langle x, y \rangle$. The second inclusion is strict if $I = \langle x - \frac{1}{2}z, y - \frac{1}{2}z \rangle \subseteq \mathbb{Q}[x_1, \ldots, x_n]$, and $\sigma$ is any term ordering with $x >_\sigma y >_\sigma z$; for then $G_\sigma = \{x - \frac{1}{2}z, y - \frac{1}{2}z\}$, and consequently, $\langle \text{prim}(G_\sigma) \rangle = \langle 2x - z, 2y - z \rangle$ while $I \cap \mathbb{Z}[x_1, \ldots, x_n] = \langle x - y, 2y - z \rangle$.

Our definition has the merit of being independent of some arbitrary choice of system of generators of $I$ and easily computable. One possible objection is that it depends on the term ordering chosen. But there is a nice way out, which uses the notion of the Gröbner fan of $I$, and frees the definition of the reduction modulo $p$ from the choice of $\sigma$.

Here we give a more detailed description of the paper. In Section 2 we use results proved in [3], and introduce the notions of $\sigma$-good and $\sigma$-bad primes for $I$ with respect to a given term ordering $\sigma$, which exploit the uniqueness of the reduced $\sigma$-Gröbner basis. Notions of good and bad primes in modular computations are ubiquitous; see for instance [9] for a fine discussion. However, in our opinion there is still room for improving the knowledge of this topic. As a first result, we prove Theorem 2.3 which relates the behaviour of good primes with respect to two different term orderings.

From the theory of Gröbner Fans (see [17]) it follows that for any ideal $I$ in $P$ all
but finitely many primes are good for all term orderings (see Remark 2.3). In other words there is an integer $\Delta$, called the universal denominator (see Definition 2.3), such that for every prime $p$ which does not divide $\Delta$ we can define the reduction of $I$ to an ideal in $\mathbb{F}_p[x_1, \ldots, x_n]$ which is independent of any term ordering (see Definition 2.4), and hence it depends only on $I$.

In the context of polynomial ideals there are several notions of good and bad primes in the mathematical literature, and Section 3 is devoted to understanding how they are interrelated. We recall the notion of a minimal strong $\sigma$-Gröbner basis for ideals in $\mathbb{Z}[x_1, \ldots, x_n]$ and, in Theorems 3.1 and 3.2, we highlight the close relationship to the reduced $\sigma$-Gröbner basis. Following [22], we say that $p$ is Pauer-lucky for a set of polynomials $F \subseteq P$, if it does not divide the leading coefficients of any polynomial in a minimal strong $\sigma$-Gröbner basis of $\langle \text{prim}(F) \rangle$; see Definition 3.2. Then, given a term ordering $\sigma$, the ideal $I = \langle F \rangle$, and its reduced $\sigma$-Gröbner basis $G_\sigma$, we use the results contained in Theorems 3.1 and 3.2 to show that if $p$ is Pauer-lucky for $\text{prim}(F)$ then $p$ is $\sigma$-good for $I$ (see Proposition 3.1), and that $p$ is Pauer-lucky for $\text{prim}(G_\sigma)$ if and only if it is $\sigma$-good for $I$ (see Corollary 3.1).

In Section 4 we address the problem of detecting $\sigma$-bad primes when the reduced $\sigma$-Gröbner basis (in $\mathbb{Q}[x_1, \ldots, x_n]$) is not known. In [8] E.A. Arnold restricted her investigation to the case of homogeneous ideals, and used suitable Hilbert functions to detect some bad primes. We describe a similar but more general strategy. The main new idea is to use the term ordering $\sigma$ to order tuples of power products. In particular, we prove Proposition 4.1 and the key Lemma 4.2 which pave the way for the proof of the main Theorem 4.1 and its Corollary 4.2 which gives a nice criterion for detecting relatively bad primes. In essence, given two term orderings $\sigma$ and $\tau$, and two primes $p$ and $q$ which are both $\sigma$-good, but only one is $\tau$-good, then we can determine which is $\tau$-good just doing modular computations.

Apart from the theoretical advances already illustrated, are there practical applications of the theoretical results proved in this paper? First experiments show that a modular approach for the computation of some Gröbner bases can benefit from our results.

Most examples described in the paper were computed using the computer algebra system CoCoA (see [2] and [5]). The computations of minimal strong Gröbner bases were performed with SINGULAR (see [10]).

Note

For the basic notation and definitions about the theory of Gröbner bases see [12], [13], and [15]. The monoid of power-products in $n$ indeterminates is denoted by $\mathbb{T}^n$. We use the convention that $\text{LT}_\sigma((0)) = (0)$. In particular, if $t = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{T}^n$ is a power-product and $c$ is a coefficient, we say that $t$ is a term and $ct$ is a monomial. Throughout this article, when we use the notation $G = \{g_1, \ldots, g_r\}$, we actually mean that the $r$ elements in $G$ are numbered and distinct. We use the symbol $\mathbb{Z}_\delta$ to represent the localization of $\mathbb{Z}$ at the multiplicative system generated by the integer...
\[ \delta. \] Sometimes in the literature the symbol \( \mathbb{Z}[\frac{1}{\delta}] \) is used instead of \( \mathbb{Z}_\delta \). If \( p \) is a prime number, the symbol \( \mathbb{Z}_{(p)} \) denotes the localization of \( \mathbb{Z} \) at the maximal ideal \( (p) \).

There are several instances in the paper where we compare the minimal set of generators of two monomial ideals in different rings. Hence we introduce the following definition. Let \( K \) be a field, let \( P = K[x_1, \ldots, x_n] \) be a polynomial ring over \( K \), let \( \sigma \) be a term ordering on \( \mathbb{T}^n \), and let \( I \) be an ideal in \( P \). The unique minimal set of generators of \( \text{LT}_\sigma(I) \) is denoted by \( \text{MinLT}_\sigma(I) \subseteq \mathbb{T}^n \). We observe that while \( \text{LT}_\sigma(I) \) is a monomial ideal in \( P \), the set \( \text{MinLT}_\sigma(I) \) is a subset of \( \mathbb{T}^n \).

Later we introduce the tuple \( \text{OrdMinLT}_\sigma(I) \) which contains the same elements as \( \text{MinLT}_\sigma(I) \) placed in increasing \( \sigma \)-order (see Definition 4.4).

Let \( T = \{ t_1, t_2, \ldots, t_r \} \) be a set of power-products. We define the interreduction of \( T \) to be the unique maximal subset \( T' \) of \( T \) with the property that there is no pair \( (t_i, t_j) \) of distinct elements in \( T' \) such that \( t_i \mid t_j \). We say that \( T \) is interreduced if it is equal to its own interreduction.

The radical of a positive integer \( N \), \( \text{rad}(N) \), is the product of all primes dividing \( N \). Obviously from the definition we have \( p \mid N \iff p \mid \text{rad}(N) \) for any prime \( p \). For example, \( \text{rad}(240) = 30 \). Note that, for any positive integer \( N \), we have \( \mathbb{Z}_N = \mathbb{Z}_\delta \) where \( \delta = \text{rad}(N) \).

Let \( \delta \) be a positive integer, and \( p \) a prime number not dividing \( \delta \). We write \( \pi_p \) to denote the canonical homomorphism \( \mathbb{Z}_\delta \to \mathbb{F}_p \) and all its natural “coefficient-wise” extensions to \( \mathbb{Z}_\delta[x_1, \ldots, x_n] \to \mathbb{F}_p[x_1, \ldots, x_n] \); we call them all reduction homomorphisms modulo \( p \).

2. Reductions modulo \( p \)

In this section we analyse the concept of reduction modulo a prime \( p \). In particular, we give a definition for the reduction mod \( p \) of an ideal which is independent of the particular generators we have.

**Definition 2.1.** Let \( P = \mathbb{Q}[x_1, \ldots, x_n] \).

(a) Given a polynomial \( f \in P \), we define the denominator of \( f \), denoted by \( \text{den}(f) \), to be the positive least common multiple of the denominators of the coefficients of \( f \). In particular, we define \( \text{den}(0) = 1 \).

(b) Given a set of polynomials \( F \) in \( P \), we define the denominator of \( F \), denoted by \( \text{den}(F) \), to be the least common multiple of \( \{ \text{den}(f) \mid f \in F \} \). For completeness we define \( \text{den}(\emptyset) = 1 \) where \( \emptyset \) denotes the empty set.

(c) Given a term ordering \( \sigma \) and an ideal \( I \) in \( P \) with reduced \( \sigma \)-Gröbner basis \( G_\sigma \), we define the \( \sigma \)-denominator of \( I \) to be \( \text{den}_\sigma(I) = \text{den}(G_\sigma) \).

The following easy example shows that \( \text{den}_\sigma(I) \) generally depends on \( \sigma \).

**Example 2.1.** Let \( P = \mathbb{Q}[x, y] \) and let \( g = x + 2y \in P \), and let \( I = (g) \). Clearly \( \{ g \} \) is the reduced \( \sigma \)-Gröbner basis of \( I \) with respect to any term ordering \( \sigma \) with \( x >_\sigma y \).
Instead, the reduced $\tau$-Gröbner basis of $I$ with respect to any term ordering $\tau$ with $y >_\tau x$ is $\{y + \frac{1}{2}x\}$. Therefore we have $\text{den}_\sigma(I) = 1$ while $\text{den}_\tau(I) = 2$.

The following lemma collects some important results taken from [3] (see also [20]).

Lemma 2.1. Let $P = \mathbb{Q}[x_1, \ldots, x_n]$, and let $\sigma$ be a term ordering on $\mathbb{T}^n$. Let $f \in P \setminus \{0\}$, and $I$ be an ideal in $P$ with reduced $\sigma$-Gröbner basis $G_\sigma$. Furthermore, let $\delta \in \mathbb{N}^+$ be such that all coefficients of $f$ and $G_{\sigma, f}$ are in $\mathbb{Z}_\delta$, where $G_{\sigma, f}$ is the subset $\{g \in G_\sigma | \text{LT}_\sigma(g) \leq_{\sigma} \text{LT}_\sigma(f)\}$.

(a) Every intermediate step of rewriting $f$ via $G_\sigma$ has all coefficients in $\mathbb{Z}_\delta$.

(b) The polynomial $\text{NF}_{\sigma, I}(f)$ has all coefficients in $\mathbb{Z}_\delta$.

Proof. Follows easily from [3, Lemma 3.2] restricted to $G_{\sigma, f}$.

The following theorem is the foundation stone of our investigation. In particular, it sets the right context in which the reduction mod $p$ of a Gröbner basis is the Gröbner basis of the ideal it generates (claim 4).

Theorem 2.1. (Reduction modulo $p$ of Gröbner Bases)

Let $P = \mathbb{Q}[x_1, \ldots, x_n]$, let $\sigma$ be a term ordering on $\mathbb{T}^n$. Let $I$ be an ideal in $P$ with reduced $\sigma$-Gröbner basis $G_\sigma$. Let $p$ be a prime number which does not divide $\text{den}(I)$.

(a) The set $\pi_p(G_\sigma)$ is the reduced $\sigma$-Gröbner basis of the ideal $\langle \pi_p(G_\sigma) \rangle$.

(b) The set of the residue classes of the elements in $\mathbb{T}^n \setminus \text{LT}_\sigma(I)$ is an $\mathbb{F}_p$-basis of the quotient ring $\mathbb{F}_p[x_1, \ldots, x_n]/\langle \pi_p(G_\sigma) \rangle$.

(c) For every polynomial $f \in P$ such that $p \nmid \text{den}(f)$ we have the equality $\pi_p(\text{NF}_{\sigma, I}(f)) = \text{NF}_{\sigma, \langle \pi_p(G_\sigma) \rangle}(\pi_p(f))$.

Proof. See [3, Theorem 3.7].

2.1. Good Primes

Along the lines in [3], Theorem 2.1 motivates the following definitions.

Definition 2.2. Let $P = \mathbb{Q}[x_1, \ldots, x_n]$.

(a) Let $F$ be a finite set of polynomials in $P$. We say that a prime $p$ is bad for $F$ if $p \mid \text{den}(F)$, i.e. $p$ divides the denominator of at least one coefficient of at least one polynomial in $F$.

(b) Let $\sigma$ be a term ordering on $\mathbb{T}^n$, let $I$ be an ideal in $P$, and let $G_\sigma$ be the reduced $\sigma$-Gröbner basis of $I$. If $p$ is bad for $G_\sigma$ we say that $p$ is $\sigma$-bad for $I$. Otherwise we say that $p$ is $\sigma$-good for $I$.
(c) If \( p \) is a \( \sigma \)-good prime for \( I \) we define the \((p, \sigma)\)-reduction of \( I \) to be the ideal \( \pi_p(G_\sigma) = \langle \pi_p(G_\sigma) \rangle \subseteq \mathbb{F}_p[x_1, \ldots, x_n] \) generated by the reductions modulo \( p \) of the polynomials in \( G_\sigma \).

Now we can reinterpret Theorem 2.1(a) as follows.

**Remark 2.1.** Let \( P = \mathbb{Q}[x_1, \ldots, x_n] \), and \( \sigma \) a term ordering on \( \mathbb{T}^n \). Let \( I \) be an ideal in \( P \), and \( G_\sigma \) its reduced \( \sigma \)-Gröbner basis. For every \( \sigma \)-good prime \( p \) for \( I \) we have

(a) the set \( \pi_p(G_\sigma) \) is the reduced \( \sigma \)-Gröbner basis of \( I_{(p, \sigma)} \), i.e. the ideal it generates.

(b) \( \text{MinLT}_\sigma(I) = \text{MinLT}_\sigma(I_{(p, \sigma)}) \).

**Remark 2.2.** We observe that the apparently simplistic definition, stating that \( p \) is \( \sigma \)-good for \( I \) if and only if \( p \) does not divide \( \text{den}(G_\sigma) \), acquires a much deeper meaning after the above remark, and provides further support for the notation \( I_{(p, \sigma)} \) since the reduced \( \sigma \)-Gröbner basis of any ideal is unique.

Theorem 2.1 turns out to be the essential tool for proving the following result, which tells us that, for all but finitely many primes, we may take simply \( \langle \pi_p(F) \rangle \) as the reduction modulo \( p \) of the ideal \( \langle F \rangle \). Naturally the set of suitable primes depends on \( F \), the given system of generators. This dependence prompts us to prefer using reduced Gröbner bases as generating sets.

**Theorem 2.2.** Let \( \sigma \) be a term ordering on \( \mathbb{T}^n \), let \( P = \mathbb{Q}[x_1, \ldots, x_n] \), let \( I \) be an ideal in \( P \), and let \( G_\sigma \) be its reduced \( \sigma \)-Gröbner basis. Then let \( F \) be any finite set of polynomials in the ideal \( I \), and let \( \delta \) be a positive integer such that both \( G_\sigma \) and \( F \) are contained in \( \mathbb{Z}_\delta[x_1, \ldots, x_n] \). Let \( p \) be a prime number such that \( p \mid \delta \).

(a) We have \( \text{rad}(\text{den}_\sigma(I)) \mid \delta \).

(b) We have \( \langle \pi_p(F) \rangle \subseteq I_{(p, \sigma)} \subseteq \mathbb{F}_p[x_1, \ldots, x_n] \).

(c) If there exists a matrix \( M \) with entries in \( \mathbb{Z}_\delta[x_1, \ldots, x_n] \) such that \( G_\sigma = F \cdot M \), then we have \( \langle \pi_p(F) \rangle = I_{(p, \sigma)} \).

**Proof.** To prove claim (a) we observe that the minimal localization of \( \mathbb{Z} \) where \( G_\sigma \) is contained is \( \mathbb{Z}_{\text{den}_\sigma(I)}[x_1, \ldots, x_n] \), and the conclusion follows.

To prove claim (b) we observe that Theorem 2.1(a) implies that every element of \( F \) can be written as a linear combination of elements of \( G_\sigma \) where the “coefficients” are polynomials in \( \mathbb{Z}_\delta[x_1, \ldots, x_n] \). In general, there will be several ways to reduce each element of \( F \) by the basis \( G_\sigma \), we may pick any one, and use the corresponding linear combination. We can view \( F \) and \( G_\sigma \) as row-matrices by ordering their elements in some way. Then writing the linear combinations as columns, we obtain a matrix \( A \) over \( \mathbb{Z}_\delta[x_1, \ldots, x_n] \) (see Lemma 2.1(a)) satisfying \( F = G_\sigma \cdot A \). This implies that \( \pi_p(F) = \pi_p(G_\sigma) \cdot \pi_p(A) \), concluding the proof since \( \pi_p(G_\sigma) \) generates \( I_{(p, \sigma)} \).
Finally, we prove (c). By claim (a) the prime \( p \) is \( \sigma \)-good for \( I \), hence we have the equality \( I_{(p,\sigma)} = \langle \pi_p(G_\sigma) \rangle \). Moreover, we have \( \pi_p(G_\sigma) = \pi_p(F) \cdot \pi_p(M) \) hence the implication \( I_{(p,\sigma)} \subseteq \langle \pi_p(F) \rangle \) follows. The reverse inclusion follows from (b), and the proof is complete.

The following easy example shows that the inclusion in claim (b) can be strict even when \( F \) is a generating set.

**Example 2.2.** We follow the notation in the proof above.

```plaintext
/* */ use P ::= QQ[x,y,z], DegRevLex;
/* */ F := [x +2*z, x +2*y]; I := ideal(F);
/* */ G := ReducedGBasis(I); G;
[x +2*z, y -z]
/* */ [GenRepr(g, I) | g in G];
[[1, 0], [-1/2, 1/2]]
```

The “new prime” \( 2 \) shows up in the denominators of the coefficients representing the reduced \( \sigma \)-Gröbner basis elements as linear combinations of the original generators. Now we look at what happens modulo \( 2 \) when we create an ideal from the original generators, and when we create an ideal from the reduced Gröbner basis.

```plaintext
/* */ use P2 ::= ZZ/(2)[x,y,z], DegRevLex;
/* */ pi2 := PolyRingHom(P, P2, CanonicalHom(QQ, P2), indets(P2));
/* */ J2 := ideal.apply(pi2, F));
/* */ ReducedGBasis(J2);
[x]
/* */ I2 := ideal.apply(pi2, G));
/* */ ReducedGBasis(I2);
[y +z, x]
```

Here we see that the inclusion in Theorem 2.2.b can be strict even though the prime \( p = 2 \) is \( \text{DegRevLex} \)-good for \( I \). In the next section we shall see that \( 2 \) is not a “lucky prime” for \( F \).

Next we present the main result of this subsection. It examines the situation when a prime is good with respect to two different term orderings.

**Theorem 2.3.** Let \( \sigma \) and \( \tau \) be two term orderings on \( \mathbb{T}^n \), let \( P = \mathbb{Q}[x_1, \ldots, x_n] \), and let \( I \) be an ideal in \( P \). Then let \( G_\sigma \) and \( G_\tau \) be the reduced Gröbner bases of \( I \) with respect to \( \sigma \) and \( \tau \), and let \( p \) be a prime which is both \( \sigma \)-good and \( \tau \)-good for \( I \).

1. We have the equality \( I_{(p,\sigma)} = I_{(p,\tau)} \).
2. The reduced \( \tau \)-Gröbner basis of \( I_{(p,\sigma)} \) is \( \pi_p(G_\tau) \).

**Proof.** Since claim (b) follows immediately from (a) and Remark 2.1 it is sufficient to prove claim (a). Let \( \delta = \text{lcm}(\text{den}_\sigma(I), \text{den}_\tau(I)) \), so both \( G_\sigma \) and \( G_\tau \) are contained in the ring \( \mathbb{Z}_\delta[x_1, \ldots, x_n] \). From the assumption about \( p \) we may apply Theorem 2.2 with \( F = G_\tau \) to deduce that \( I_{(p,\tau)} = \langle \pi_p(G_\tau) \rangle \subseteq I_{(p,\sigma)} \). Applying Theorem 2.2...
again, after exchanging the roles of $\sigma$ and $\tau$, shows that $I(p,\sigma) = \langle \pi_p(G_\sigma) \rangle \subseteq I(p,\tau)$. This proves the claim. 

2.2. Universal Denominator

In this subsection we recall some facts from Gröbner Fan Theory (see [17]) and use them to define the universal denominator of an ideal.

Remark 2.3. It is well-known that the Gröbner fan of an ideal is finite (e.g. see [17]), hence for every ideal in $\mathbb{Q}[x_1,\ldots,x_n]$ there are only finitely many distinct reduced Gröbner bases. Each of these bases has its own corresponding denominator; thus any prime which does not divide any of these denominators is good for all term orderings.

This remark motivates the following definition.

Definition 2.3. Let $I$ be an ideal in $\mathbb{Q}[x_1,\ldots,x_n]$. Then the least common multiple of all $\text{den}_\sigma(I)$, as we vary $\sigma$, is called the universal denominator of $I$, and is denoted by $\Delta(I)$.

Remark 2.4. We now see a big advantage of our choice of using reduced Gröbner bases: the finiteness of the “fan” of the reduced Gröbner bases enables us to make this definition. If we allow more general generating sets then there is no finite “universal denominator”. For example, the ideal $\langle x, y \rangle$ admits generating sets such as $\{x + \frac{1}{p}y, y\}$ for any prime $p$.

Next we show the existence of a well-behaved notion of reduction of $I$ modulo $p$ which is independent of the term orderings.

Proposition 2.1. Let $I$ be an ideal in $P$, let $\Delta(I)$ be its universal denominator, and let $p$ be a prime not dividing $\Delta(I)$. Then $I(p,\sigma)$ does not depend on $\sigma$.

Proof. For any term orderings $\sigma$ and $\tau$, the prime $p$ is both $\sigma$-good and $\tau$-good. So, by Theorem 2.3 we have $I(p,\sigma) = I(p,\tau)$. 

This proposition motivates the following definition.

Definition 2.4. Let $I$ be a non-zero ideal in $P$, let $\Delta(I)$ be its universal denominator, and let $p$ be a prime not dividing $\Delta(I)$. Then the reduction of $I$ modulo $p$, denoted $I_p$, is the ideal $I(p,\sigma)$, for any choice of $\sigma$.

The main practical problem related to this definition is the computation of the universal denominator of $I$ which is, in general, not an easy task. Let us see some examples.

Example 2.3. Let $P = \mathbb{Q}[x,y,z]$ and let $I = \langle x^2 - y, xy + z + 1, z^2 + x \rangle$. It is a zero-dimensional ideal and its Gröbner fan consists of twelve cones.
So we have $\Delta(I) = 2^2 \cdot 7$. Consequently the reduction $I_p$ is defined for every prime $p$ other than 2 and 7, and is generated by the reduction modulo $p$ of any of these Gröbner bases.

Example 2.4. While many ideals do have relatively small universal denominators, a few seemingly simple ideals can have surprisingly large ones. This usually arises when the Gröbner fan comprises many cones, which can happen easily when there are many indeterminates. We exhibit two examples with few indeterminates which nevertheless have impressive denominators.

The first example in $\mathbb{Q}[x, y, z]$ is the ideal $\langle x^2 y + xy^2 + 1, y^3 + x^2 z, z^3 + x^2 \rangle$ whose universal denominator is larger than $2 \times 10^{404}$ and has 105 distinct prime factors (including all primes up to 100 except 79 and 89). The Gröbner fan of this ideal comprises 392 cones.

The second example is in the ring $\mathbb{Q}[x, y, z, w]$: it is the apparently innocuous ideal $\langle xyz + yzw + y, z^3 + x^2, y^2 z + w^3, x^3 + y^3 \rangle$. The Gröbner fan of this ideal comprises almost 37000 cones, and its universal denominator is larger than $2 \times 10^{379530}$. This number has at least 24539 distinct prime factors including more than $2^{15}$ of all primes less than $2^{15}$; in fact, the smallest prime not dividing the universal denominator is 4463. We checked the primeness of factors larger than 2 using the function `mpz_probab_prime_p` from the GMP library, specifying 25 iterations of the Miller–Rabin test (see [11]).

3. Good primes vs lucky primes

In this section we recall some notions of lucky primes which have a long history, and compare them with our notion of good primes. We restrict our attention to the case where the ring of coefficients is $\mathbb{Z}$, although the theory is more general (see for instance [6] and [22]). Several results described in this subsection are known, however we adapt them to our notation, and for some of them we provide new proofs.

In this section we fix a term ordering $\sigma$ on the monoid $T^n$ of the power-products in $n$ indeterminates, consequently we sometimes omit the symbol $\sigma$. Computations of minimal, strong Gröbner bases were performed by SINGULAR (see [10]).
The first important tool is the following definition (see [6], Definition 4.5.6).

**Definition 3.1.** Let \( g_1, \ldots, g_s \) be non-zero polynomials in \( \mathbb{Z}[x_1, \ldots, x_n] \). We say that \( G_{\sigma, \mathbb{Z}} = \{ g_1, \ldots, g_s \} \) is a strong \( \sigma \)-Gröbner basis for the ideal \( J = \langle G_{\sigma, \mathbb{Z}} \rangle \), if for each \( f \in J \) there exists some \( i \in \{1, \ldots, s\} \) such that \( \text{LM}_\sigma(g_i) \) divides \( \text{LM}_\sigma(f) \).

We say that \( G_{\sigma, \mathbb{Z}} \) is a minimal strong \( \sigma \)-Gröbner basis if it is a strong \( \sigma \)-Gröbner basis and \( \text{LM}_\sigma(g_i) \) does not divide \( \text{LM}_\sigma(g_j) \) whenever \( i \neq j \).

**Remark 3.1.** In [6] the theory of minimal strong Gröbner bases is fully developed, in particular it is stated that every non-zero ideal in \( \mathbb{Z}[x_1, \ldots, x_n] \) has a minimal strong Gröbner basis (see [6], Exercise 4.5.9).

It is well known that reduced Gröbner bases have the property that the leading terms of their elements are pairwise distinct. This also holds for minimal strong Gröbner bases in \( \mathbb{Z}[x_1, \ldots, x_n] \) because the coefficient ring \( \mathbb{Z} \) is a principal ideal domain.

The following easy examples show the difference between a minimal strong Gröbner basis of an ideal \( J \subseteq \mathbb{Z}[x_1, \ldots, x_n] \) and the reduced Gröbner basis of the extended ideal \( J \mathbb{Q}[x_1, \ldots, x_n] \). Note that, whereas the elements of the reduced Gröbner basis are monic, in a strong Gröbner basis the coefficients of the leading monomial play an essential role in divisibility checking.

**Example 3.1.** Let \( J = \langle x^2, 2x \rangle \) be an ideal in \( \mathbb{Z}[x] \). Then \( G_{\sigma, \mathbb{Z}} = \{ x^2, 2x \} \) is a minimal strong \( \sigma \)-Gröbner basis of \( J \), while \( \{ x \} \) is the reduced \( \sigma \)-Gröbner basis of the extended ideal \( J \mathbb{Q}[x] \).

Let \( F_\mathbb{Z} = \{ 2x, 3y \} \subseteq \mathbb{Z}[x, y] \). Then \( \{ 2x, 3y, xy \} \) is a minimal strong \( \sigma \)-Gröbner basis of the ideal \( \langle F_\mathbb{Z} \rangle \) for any term ordering \( \sigma \), while \( \{ x, y \} \) is the reduced \( \sigma \)-Gröbner basis of the extended ideal \( \langle F_\mathbb{Z} \rangle \mathbb{Q}[x, y] \).

The reduced \( \sigma \)-Gröbner basis is a unique, canonical choice amongst all \( \sigma \)-Gröbner bases; in contrast, a minimal strong \( \sigma \)-Gröbner basis is not unique.

**Example 3.2.** Let \( \sigma \) be the \texttt{DegRevLex} term ordering on \( \mathbb{T}^2 \). In the ring \( \mathbb{Z}[x, y] \) let \( G_{\sigma, \mathbb{Z}} = \{ y^2 - x, 2x \} \) and \( G'_{\sigma, \mathbb{Z}} = \{ y^2 + x, 2x \} \). Then clearly \( \langle G_{\sigma, \mathbb{Z}} \rangle = \langle G'_{\sigma, \mathbb{Z}} \rangle \) and both \( G_{\sigma, \mathbb{Z}} \) and \( G'_{\sigma, \mathbb{Z}} \) are minimal strong \( \sigma \)-Gröbner bases of this ideal. The unique reduced \( \sigma \)-Gröbner basis of the extended ideal is \( G = \{ x, y^2 \} \).

Although not unique, we shall now see that two minimal strong \( \sigma \)-Gröbner bases of an ideal \( J \) in \( \mathbb{Z}[x_1, \ldots, x_n] \) share the same leading monomials.

**Lemma 3.1.** Let \( J \) be an ideal in \( \mathbb{Z}[x_1, \ldots, x_n] \), and \( \sigma \) be a term-ordering on \( \mathbb{T}^n \). Let \( G_{\sigma, \mathbb{Z}} \) and \( G'_{\sigma, \mathbb{Z}} \) be two minimal strong \( \sigma \)-Gröbner bases of \( J \). Then \( \{ \text{LM}_\sigma(g) \mid g \in G_{\sigma, \mathbb{Z}} \} = \{ \text{LM}_\sigma(g') \mid g' \in G'_{\sigma, \mathbb{Z}} \} \). Consequently we have \( \# G_{\sigma, \mathbb{Z}} = \# G'_{\sigma, \mathbb{Z}} \) and \( \{ \text{LC}_\sigma(g) \mid g \in G_{\sigma, \mathbb{Z}} \} = \{ \text{LC}_\sigma(g') \mid g' \in G'_{\sigma, \mathbb{Z}} \} \).

**Proof.** This equality can be proved along the same lines as the proof of the uniqueness of the minimal generating set of a monomial ideal in \( \mathbb{K}[x_1, \ldots, x_n] \) where \( \mathbb{K} \) is a field – see for instance [13] Proposition 1.3.11.b].
Given a polynomial in $\mathbb{Q}[x_1,\ldots,x_n]$ we define its primitive integral part; it has integer coefficients with no common factor, so its modular reduction is non-zero for any prime $p$.

**Definition 3.2.** Let $f$ be a non-zero polynomial in $\mathbb{Q}[x_1,\ldots,x_n]$, and let $c$ be the integer content of $f \cdot \text{den}(f) \in \mathbb{Z}[x_1,\ldots,x_n]$. Then the **primitive integral part of $f$, denoted $\text{prim}(f)$, is the primitive polynomial $c^{-1}f \cdot \text{den}(f) \in \mathbb{Z}[x_1,\ldots,x_n]$. If $F$ is a set of non-zero polynomials in $\mathbb{Q}[x_1,\ldots,x_n]$ then $\text{prim}(F) = \{\text{prim}(f) \mid f \in F\}$.

For example, if $f = 2x + \frac{1}{3}$ then $\text{prim}(f) = 3x + 2$.

Let $G_{\sigma}$ be the reduced $\sigma$-Gröbner basis of an ideal in $\mathbb{Q}[x_1,\ldots,x_n]$. The following theorem shows some important properties of all minimal strong $\sigma$-Gröbner bases of the ideal generated by $\text{prim}(G_{\sigma})$ in $\mathbb{Z}[x_1,\ldots,x_n]$.

**Theorem 3.1.** Let $P = \mathbb{Q}[x_1,\ldots,x_n]$, and $\sigma$ be a term-ordering on $\mathbb{T}^n$. Let $I$ be a non-zero ideal in $P$, and let $G_{\sigma} = \{g_1,\ldots,g_r\}$ be its reduced $\sigma$-Gröbner basis, whose elements are indexed so that $\text{LT}_{\sigma}(g_1) <_{\sigma} \cdots <_{\sigma} \text{LT}_{\sigma}(g_r)$. Then let $G_{\sigma,\mathbb{Z}} = \{\hat{g}_1,\ldots,\hat{g}_s\}$ be a minimal strong $\sigma$-Gröbner basis of the ideal $J = \langle \text{prim}(G_{\sigma}) \rangle \subseteq \mathbb{Z}[x_1,\ldots,x_n]$.

(a) The elements in $G_{\sigma,\mathbb{Z}}$ can be indexed so that $\text{LT}_{\sigma}(\hat{g}_i) = \text{LT}_{\sigma}(g_i)$ for $i = 1,\ldots,r$ while for $i = r+1,\ldots,s$ each $\text{LT}_{\sigma}(\hat{g}_i)$ is a proper multiple of $\text{LT}_{\sigma}(g_k)$ for some $k \leq r$.

(b) The subset $\{\hat{g}_1,\ldots,\hat{g}_s\}$ is a minimal $\sigma$-Gröbner basis of $I$ in $P$.

(c) We have $\text{LC}_{\sigma}(\hat{g}_i) \mid \text{LC}_{\sigma}(\text{prim}(g_i))$ for $i = 1,\ldots,r$.

(d) If there exists a prime $p$ such that $p \mid \text{den}(g_i)$ but $p \nmid \text{den}(g_j)$ for every $j = 1,\ldots,i-1$ then $p \mid \text{LC}_{\sigma}(\hat{g}_i)$.

**Proof.** We start by proving claims (a) and (b). For each $i = 1,\ldots,r$ we have $\text{prim}(g_i) \in J$, hence there is at least one polynomial $\hat{g}_j \in G_{\sigma,\mathbb{Z}}$ such that $\text{LM}_{\sigma}(\hat{g}_j) \mid \text{LM}_{\sigma}(\text{prim}(g_i))$. Now $\hat{g}_j \in I$, hence there is at least one polynomial $g_k \in G_{\sigma}$ such that $\text{LT}_{\sigma}(g_k) \mid \text{LT}_{\sigma}(\hat{g}_j)$. Since $G_{\sigma}$ is a reduced Gröbner basis it follows that $k = i$, and then also $\text{LT}_{\sigma}(\hat{g}_j) = \text{LT}_{\sigma}(g_i)$. So by suitably renumbering we may assume $j = i$.

Now we consider $i > r$. Again we observe $\hat{g}_i \in I$, hence there is at least one polynomial $g_k \in G_{\sigma}$ such that $\text{LT}_{\sigma}(g_k) \mid \text{LT}_{\sigma}(\hat{g}_i)$. Since $G_{\sigma,\mathbb{Z}}$ is minimal and $\text{LT}_{\sigma}(\hat{g}_k) = \text{LT}_{\sigma}(g_k)$ we deduce from Remark [1] that $\text{LT}_{\sigma}(\hat{g}_i)$ must be a proper multiple of $\text{LT}_{\sigma}(g_k)$. We have now proved claims (a) and (b).

Next we prove claim (c). From claim (a) it follows that the two polynomials $\hat{g}_i$ and $\text{prim}(g_i)$ have the same leading term. Since $\text{prim}(g_i) \in J$ there is at least one polynomial $\hat{g}_j \in G_{\sigma,\mathbb{Z}}$ such that $\text{LM}_{\sigma}(\hat{g}_j) \mid \text{LM}_{\sigma}(\text{prim}(g_i))$. This implies that $\text{LT}_{\sigma}(\hat{g}_j) \mid \text{LT}_{\sigma}(\hat{g}_i)$, which in turn implies that $j = i$. Hence $\text{LC}_{\sigma}(\hat{g}_i) \mid \text{LC}_{\sigma}(\text{prim}(g_i))$.

Finally, we prove claim (d). Let $h = \hat{g}_i - \text{LC}_{\sigma}(\hat{g}_i) \cdot g_i$, and observe that $h \in I$. Using the fact that $\text{LM}_{\sigma}(\hat{g}_i) = \text{LC}_{\sigma}(\hat{g}_i) \cdot \text{LT}_{\sigma}(g_i)$ we can write
\[
h = (\hat{g}_i - \text{LM}_{\sigma}(\hat{g}_i)) - \text{LC}_{\sigma}(\hat{g}_i) \cdot (g_i - \text{LT}_{\sigma}(g_i)) = \hat{h}_i - \text{LC}_{\sigma}(\hat{g}_i) \cdot h_i
\]
where \( \hat{h}_i = \hat{g}_i - \text{LM}_\sigma(\hat{g}_i) \) and \( h_i = g_i - \text{LT}_\sigma(g_i) \). Now, since \( h \in I \), we have

\[
0 = \text{NF}_{\sigma,I}(h) = \text{NF}_{\sigma,I}(\hat{h}_i) - \text{LC}_\sigma(\hat{g}_i) \cdot \text{NF}_{\sigma,I}(h_i)
\]

Hence we have the equality \( \text{NF}_{\sigma,I}(\hat{h}_i) = \text{LC}_\sigma(\hat{g}_i) \cdot \text{NF}_{\sigma,I}(h_i) \). Given that \( g_i \in G_\sigma \), the reduced \( \sigma \)-Gröbner basis of \( I \), we have that \( \text{NF}_{\sigma,I}(h_i) = h_i \), which implies that \( \text{NF}_{\sigma,I}(\hat{h}_i) = \text{LC}_\sigma(\hat{g}_i) \cdot h_i \).

Now we look at the denominators of \( \text{NF}_{\sigma,I}(\hat{h}_i) \) and \( \text{LC}_\sigma(\hat{g}_i) \cdot h_i \). Notice that \( \hat{h}_i \) has integer coefficients; then using the fact that \( \text{LT}_\sigma(g_k) >_\sigma \text{LT}_\sigma(h_i) \) for all \( k \geq i \) we can apply Lemma 2.1 to conclude that \( \text{NF}_{\sigma,I}(\hat{h}_i) \in \mathbb{Z}[x_1, \ldots, x_n] \) where \( \delta' = \text{lcm}(\text{den}(g_1), \ldots, \text{den}(g_{i-1})) \). By hypothesis we know that \( p \nmid \delta' \), thus \( p \nmid \text{den}(\text{NF}_{\sigma,I}(\hat{h}_i)) \). Again, by hypothesis \( p \mid \text{den}(\hat{g}_i) \). Also, since \( g_i \) is an element of a reduced Gröbner basis, it is monic; and the existence of \( p \) implies it is not a monomial. Thus we have \( p \mid \text{den}(g_i) \). By the equality of the two normal forms we know that \( p \mid \text{den}(\text{LC}_\sigma(\hat{g}_i) \cdot h_i) \), hence we necessarily have \( p \mid \text{LC}_\sigma(\hat{g}_i) \).

The following example illustrates claim (d).

**Example 3.3.** Let \( \sigma = \text{DegRevLex} \) on \( \mathbb{T}^2 \) and let \( g_1 = y - \frac{1}{3}, g_2 = x - \frac{1}{6} \in \mathbb{Q}[x, y] \). Then \( G_\sigma = \{ g_1, g_2 \} \) is the reduced \( \sigma \)-Gröbner basis of \( I = \langle G_\sigma \rangle \), and \( G_{\sigma, \mathbb{Z}} = \{ 3y - 1, 2x - y, xy + y^2 - x \} \) is a minimal strong \( \sigma \)-Gröbner basis of \( J = \langle \text{prim}(G_\sigma) \rangle \subseteq \mathbb{Z}[x, y] \) indexed according to claim (a). As stated in claim (d):

- since \( 3 \mid \text{den}(g_1) \), we therefore have \( 3 \mid \text{LC}_\sigma(\hat{g}_1) \); indeed \( \text{LC}_\sigma(\hat{g}_1) = 3 \).
- since \( 2 \mid \text{den}(g_2) \) and \( 2 \nmid \text{den}(g_1) \), we therefore have \( 2 \mid \text{LC}_\sigma(\hat{g}_2) \); indeed \( \text{LC}_\sigma(\hat{g}_2) = 2 \).

The following example illustrates the fact that simply sorting the elements of a minimal strong Gröbner basis by increasing \( \text{LT}_\sigma \) may not satisfy claim (a).

**Example 3.4.** Let \( P = \mathbb{Q}[x, y, z] \) with term ordering \( \sigma = \text{DegRevLex} \) on \( \mathbb{T}^3 \). Let \( g_1 = y - \frac{1}{3}, g_2 = x - \frac{1}{2} \) and \( g_3 = z^3 \). Then \( G_\sigma = \{ g_1, g_2, g_3 \} \) is the reduced Gröbner basis of the ideal \( I = \langle G \rangle \), and we have \( \text{LT}_\sigma(g_1) <_\sigma \text{LT}_\sigma(g_2) <_\sigma \text{LT}_\sigma(g_3) \). A minimal strong Gröbner basis of the ideal \( J = \langle \text{prim}(G_\sigma) \rangle \subseteq \mathbb{Z}[x, y, z] \) with elements indexed according to claim (a) is \( G_{\sigma, \mathbb{Z}} = \{ 3y - 1, 2x - 1, z^3, xy - x + y \} \), but clearly we have \( \text{LT}_\sigma(\hat{g}_3) >_\sigma \text{LT}_\sigma(\hat{g}_4) \).

Since the set of leading coefficients is independent of the specific choice of minimal strong Gröbner basis of \( J \), we make the following definition.

**Definition 3.3.** Given a finite set \( F_\mathbb{Z} \) of non-zero polynomials in \( \mathbb{Z}[x_1, \ldots, x_n] \), we define \( \text{lcm}_\sigma(F_\mathbb{Z}) = \text{lcm}\{ \text{LC}_\sigma(f) \mid f \in F_\mathbb{Z} \} \in \mathbb{Z} \), the least common multiple of all the leading coefficients in \( F_\mathbb{Z} \). Given an ideal \( J \in \mathbb{Z}[x_1, \ldots, x_n] \) we define \( \text{lcm}_\sigma(J) = \text{lcm}_\sigma(G_{\sigma, \mathbb{Z}}) \), where \( G_{\sigma, \mathbb{Z}} \) is one of its minimal strong \( \sigma \)-Gröbner bases.

Now we apply Theorem 3.1 to show that the primes appearing in \( \text{den}_\sigma(I) \) are the same as those appearing in the leading coefficients of any minimal strong \( \sigma \)-
Gröbner basis of the ideal generated by the primitive integral parts of the reduced \(\sigma\)-Gröbner basis of \(I\).

**Theorem 3.2.** Let \(\sigma\) be a term ordering on \(\mathbb{T}^n\), let \(I\) be a non-zero ideal in \(\mathbb{Q}[x_1, \ldots, x_n]\), and let \(G_\sigma\) be its reduced \(\sigma\)-Gröbner basis. Then \(\text{rad}(\text{den}(G_\sigma)) = \text{rad}(\text{lcm}(J))\) where \(J = \langle \text{prim}(G_\sigma) \rangle\).

**Proof.** Let \(G_{\sigma, \mathbb{Z}}\) be a minimal strong Gröbner basis of the ideal \(J \subseteq \mathbb{Z}[x_1, \ldots, x_n]\). The conclusion follows from the following two claims.

Claim (1): We have \(\text{lcm}(\langle J \rangle) \mid \text{den}(\langle J \rangle)\) and hence \(\text{rad}(\text{lcm}(\langle J \rangle)) \mid \text{rad}(\text{den}(\langle J \rangle))\).

Claim (2): We have \(\text{rad}(\text{den}(\langle J \rangle)) \mid \text{rad}(\text{lcm}(\langle J \rangle))\).

Let \(G_\sigma = \{g_1, \ldots, g_r\}\) be any prime. The set \(\{g_1, \ldots, g_r\}\) is indexed according to Theorem 3.1.a.

Let us prove claim (1). From Theorem 3.1.c we get \(\text{LC}_{\sigma}(\langle \tilde{g}_i \rangle) \subseteq \text{LC}_{\sigma}(\text{prim}(g_i))\) for every \(i = 1, \ldots, r\). Moreover, it is clear that \(\text{LC}_{\sigma}(\text{prim}(g_1)) = \text{den}(g_1)\) and \(\text{LC}_{\sigma}(\text{prim}(g_i)) = \text{den}(g_i)\) for every \(i = 1, \ldots, r\). Consequently, to finish the proof of claim (1) we show that \(\text{lcm}(J) = \text{lcm}(\langle \tilde{g}_1, \ldots, \tilde{g}_r \rangle)\).

Claim (2) follows easily from Theorem 3.1.d.

The following example shows that in Theorem 3.2 it is not sufficient that \(G_\sigma\) is just a minimal \(\sigma\)-Gröbner basis of \(I\).

**Example 3.5.** Let \(P = \mathbb{Q}[x, y, z]\), and term ordering \(\sigma = \text{DegRevLex}\) on \(\mathbb{T}^3\). Let \(I = \langle yz - z^2, xy - z^2 \rangle\) be an ideal in \(P\), then \(G_\sigma = \{yz - z^2, xy - z^2, xz^2 - z^3\}\) is its reduced \(\sigma\)-Gröbner basis.

Let \(p\) be any prime. The set \(G_{\text{min}} = \{yz - z^2, xy - z^2, xz^2 - z^3 + \frac{1}{p}(yz - z^2)\}\) is a minimal, but not reduced, \(\sigma\)-Gröbner basis of \(I\). Clearly \(\text{den}(G_{\text{min}}) = p\). On the other hand, a minimal strong \(\sigma\)-Gröbner basis of the ideal \(\langle \text{prim}(G_{\text{min}}) \rangle\) is \(G_{\sigma, \mathbb{Z}} = \{yz - z^2, xy - z^2, xz^2 - z^3\}\), hence \(\text{lcm}(\langle \text{prim}(G_{\text{min}}) \rangle) = 1\).

The following example shows that under the assumptions of Theorem 3.2 we do not necessarily have the equality \(\text{den}(G_\sigma) = \text{lcm}(\langle \text{prim}(G_\sigma) \rangle)\).

**Example 3.6.** Let \(\sigma = \text{DegRevLex}\), let \(I = \langle 2x - y, 2y - z \rangle \subseteq \mathbb{Q}[x, y, z]\). Its reduced \(\sigma\)-Gröbner basis is \(G_\sigma = \{y - \frac{1}{2}z, x - \frac{1}{2}z\}\), hence \(\text{den}(G_\sigma) = 4\). A minimal strong Gröbner basis of the ideal \(\langle \text{prim}(G_\sigma) \rangle\) is \(G_{\sigma, \mathbb{Z}} = \{2y - z, 2x - y, xz - y^2\}\), hence \(\text{lcm}(G_{\sigma, \mathbb{Z}}) = 2\).

**Remark 3.2.** We note that we can make claim 3.1.d stronger: if \(p\) is a prime satisfying the conditions in 3.1.d then the greatest power of \(p\) dividing \(\text{den}(g_i)\) is the same as the greatest power dividing \(\text{LC}_{\sigma}(\tilde{g}_i)\). Observe that Example 3.6 does not contradict this stronger claim.
Next we recall, using our setting and language, the definition of lucky primes according to [22]. Franz Pauer described lucky ideals (in \( R \)) when the coefficient ring \( R \) of the polynomial ring is very general. Then he considered the case where \( R \) is a principal ideal domain. We rephrase his definition for the case \( R = \mathbb{Z} \).

**Definition 3.4.** Let \( \sigma \) be a term ordering on \( \mathbb{T}^n \), and let \( F_\mathbb{Z} \subseteq \mathbb{Z}[x_1, \ldots, x_n] \) be a set of non-zero polynomials. Let \( G_{\sigma, \mathbb{Z}} \) be a minimal strong \( \sigma \)-Gröbner basis of the ideal \( \langle F_\mathbb{Z} \rangle \subseteq \mathbb{Z}[x_1, \ldots, x_n] \). A prime \( p \) is called \( \sigma \)-Pauer-lucky for \( F_\mathbb{Z} \) (or simply Pauer-lucky for \( F_\mathbb{Z} \) if \( \sigma \) is clear from the context) if \( p \) does not divide the leading coefficient of any polynomial in \( G_{\sigma, \mathbb{Z}} \).

In [22] Proposition 6.1] Pauer proved the following relation between Pauer-lucky and good primes.

**Proposition 3.1.** Let \( \sigma \) be a term ordering on \( \mathbb{T}^n \), let \( F \subseteq \mathbb{Q}[x_1, \ldots, x_n] \) be a set of non-zero polynomials, and let \( p \) be a prime number. If \( p \) is Pauer-lucky for \( \text{prim}(F) \) then \( p \) is \( \sigma \)-good for \( \langle F \rangle \).

The inclusion stated in this proposition can be strict, as the following examples show.

**Example 3.7.** Recalling Example 2.2 the prime 2 is good for the ideal \( \langle F \rangle \). However, the minimal strong Gröbner basis of the ideal \( \langle \text{prim}(F) \rangle \) is \( \{2y - 2z, x + 2z\} \), hence 2 is not Pauer-lucky for \( \text{prim}(F) \).

**Example 3.8.** Let \( \sigma = \text{DegRevLex} \), and let \( F = \{x^2y - \frac{7}{2}y, xy^2 - \frac{3}{2}x\} \subseteq \mathbb{Q}[x, y] \). The reduced \( \sigma \)-Gröbner basis of the ideal \( \langle F \rangle \) is \( G_\sigma = \{xy^2 - \frac{3}{2}x, x^2y - \frac{35}{6}y^2, y^3 - \frac{3}{2}y\} \). Now we consider the two ideals \( \langle \text{prim}(F) \rangle \) and \( \langle \text{prim}(G_\sigma) \rangle \) in \( \mathbb{Z}[x, y] \). A minimal strong \( \sigma \)-Gröbner basis of \( \langle \text{prim}(G_\sigma) \rangle \) is

\[
\{6x^2 - 35y^2, 5y^3 - 3y, 5xy^2 - 3x, x^2y^2 - 3x^2 + 14y^2\}
\]

A minimal strong \( \sigma \)-Gröbner basis of \( \langle \text{prim}(F) \rangle \) is

\[
\{6x^2 - 35y^2, 35y^3 - 21y, 5xy^2 - 3x, 2x^2y - 7y, x^2y^2 - 3x^2 + 14y^2\}
\]

Hence \( \text{den}_\sigma(\langle F \rangle) = \text{den}(G_\sigma) = \text{lcm}_\sigma(\langle \text{prim}(G_\sigma) \rangle) = 2 \cdot 3 \cdot 5 \cdot 7 \), in accordance with Theorem 3.2 while \( \text{lcm}_\sigma(\langle \text{prim}(F) \rangle) = 2 \cdot 3 \cdot 5 \cdot 7 \). Consequently the prime 7 is not Pauer-lucky for \( \text{prim}(F) \), while it is a good prime for the ideal \( \langle F \rangle \).

In view of the notion of Pauer-luckyness we can rephrase Theorem 3.2 as follows, which generalizes the implication in Proposition 3.1 (originally [22] Proposition 6.1) into an equivalence when \( F \) is a reduced \( \sigma \)-Gröbner basis.

**Corollary 3.1.** Let \( \sigma \) be a term ordering on \( \mathbb{T}^n \), let \( F \subseteq \mathbb{Q}[x_1, \ldots, x_n] \) be a set of non-zero polynomials, let \( G_\sigma \) be the reduced \( \sigma \)-Gröbner basis of the ideal \( \langle F \rangle \). Then a prime number \( p \) is \( \sigma \)-Pauer-lucky for \( \text{prim}(G_\sigma) \) if and only if it is \( \sigma \)-good for the ideal \( \langle F \rangle \).
We conclude the section by mentioning another important paper which deals with a notion of lucky primes.

**Remark 3.3.** In the paper [8], Elisabeth Arnold considered the case where the polynomials in $F$ are homogeneous with respect to the standard grading, and proves that, if $G_\sigma$ is the reduced $\sigma$-Gröbner basis of $\langle F \rangle$, a prime $p$ is Pauer-lucky for $\text{prim}(F)$ if and only if the reduced Gröbner basis of $\langle \pi_p(\text{prim}(F)) \rangle$ is $\pi_p(G_\sigma)$. Moreover, this is also equivalent to $p$ being Hilbert-lucky and good for $\langle F \rangle$.

For a reformulation of this result and a nice example see [9], Theorem 5 and Example 6.

4. Detecting Bad Primes

With the fundamental help of Theorem 2.3 we have seen the nice relation between ideals generated by the reductions modulo $p$ of two reduced Gröbner bases of $I$ when $p$ is good for both term orderings. But what happens when $p$ is good for one and bad for the other? We point out that the situation of knowing whether a prime is good or bad for some particular term ordering does arise in some useful circumstances: for instance, in implicitization where the generators of the eliminating ideal (see [3]) naturally form a reduced Gröbner basis with respect to an elimination term ordering for the dependent variables (i.e. the coordinate indeterminates to be used for expressing the implicit form).

In the following we shall find it convenient to order finite sets of distinct power-products. For this reason we introduce the following definition.

**Definition 4.1.** Let $\sigma$ be a term ordering on $T^n$ and let $P = K[x_1, \ldots, x_n]$.

(a) A tuple $(t_1, t_2, \ldots, t_r)$ of distinct power-products in $T^n$ is called $\sigma$-ordered if we have $t_1 <_\sigma t_2 <_\sigma \cdots <_\sigma t_r$. The empty tuple is $\sigma$-ordered.

(b) Let $F$ be a set or tuple of non-zero polynomials in $P$. The $\sigma$-ordered tuple of the interreduction of $\text{LT}_\sigma(F)$ is denoted by $\text{OrdMinLT}_\sigma(F)$.

(c) Let $I$ be an ideal in $P$. Then the $\sigma$-ordered tuple of the leading terms of any minimal $\sigma$-Gröbner basis of $I$ is denoted by $\text{OrdMinLT}_\sigma(I)$. In particular, if $I$ is the zero ideal then $\text{OrdMinLT}_\sigma(I)$ is the empty tuple.

**Example 4.1.** Let $P = \mathbb{Q}[x, y]$ and let $\sigma = \text{DegRevLex}$. We consider the set of polynomials $F = \{x+y+1, x^2+2x+y+1, y^3\}$. Observe that $\text{LT}_\sigma(F) = \{x, x^2, y^3\}$ is not interreduced; interreduction produces $\text{OrdMinLT}_\sigma(F) = (x, y^3)$. In contrast, working with the ideal $I = \langle F \rangle$ gives $\text{OrdMinLT}_\sigma(I) = (y, x)$ since the reduced Gröbner basis is $\{x+1, y\}$.

We define a total ordering on the $\sigma$-ordered tuples of distinct power-products.

**Definition 4.2.** Let $\sigma$ be a term ordering on the monoid $T^n$, and let $T = (t_1, \ldots, t_r)$ and $T' = (t'_1, \ldots, t'_r)$ be $\sigma$-ordered tuples of distinct power-products.
in \( \mathbb{T}^n \). We say that \( T' \) \( \sigma \)-precedes \( T \) and write \( T' \prec_{\sigma} T \) if either \( T \) is a proper prefix of \( T' \), i.e. \( r < r' \) and \( t_i = t'_i \) for all \( i = 1, \ldots, r \), or there exists an index \( k \in \{1, \ldots, \min(r, r')\} \) such that \( t_i = t'_i \) for every \( i = 1, \ldots, k-1 \) and \( t'_k <_{\sigma} t_k \).

We write \( T' \preceq_{\sigma} T \) to mean either \( T' \prec_{\sigma} T \) or \( T' = T \).

**Remark 4.1.** We observe that “\( \sigma \)-precedes” is just the “\( \sigma \)-lexicographical” ordering on the \( \sigma \)-ordered tuples \( (T, x^\infty) \) where \( T \) is a \( \sigma \)-ordered tuple of distinct powers, and \( x^\infty \) is \( \sigma \)-greater than any power-product. For instance, we now easily see that every non-empty tuple \( \sigma \)-precedes the empty tuple.

**Example 4.2.** Let \( \sigma = \text{Lex} \) on \( \mathbb{T}^3 \) with \( x >_{\sigma} y >_{\sigma} z \). We compare these tuples:

- \((z, y, x) \prec_{\sigma} (z, y)\) — since \( x <_{\sigma} x^\infty \), equivalently, \((z, y)\) is a proper prefix
- \((z, y) \prec_{\sigma} (z, y^2, x)\) — since \( y <_{\sigma} y^2 \)

**Proposition 4.1.** Let \( P = K[x_1, \ldots, x_n] \), and \( \sigma \) be a term ordering on \( \mathbb{T}^n \). Let \( J \) be an ideal in \( P \), and let \( F \) be a set of non-zero polynomials in \( J \).

(a) \( \text{OrdMinLT}_{\sigma}(J) = \text{OrdMinLT}_{\sigma}(F) \) if and only if \( F \) is a \( \sigma \)-Gröbner basis of \( J \).

(b) \( \text{OrdMinLT}_{\sigma}(J) \prec_{\sigma} \text{OrdMinLT}_{\sigma}(F) \) if \( F \) is not a \( \sigma \)-Gröbner basis of \( J \).

**Proof.** By definition, \( F \subseteq J \) is a \( \sigma \)-Gröbner basis of \( J \) if and only if \( \text{LT}_{\sigma}(F) \) generates \( \text{LT}_{\sigma}(J) \). Hence claim (a) follows. Now we prove claim (b).

Since \( F \) is not a \( \sigma \)-Gröbner basis of \( J \), we have \( \text{OrdMinLT}_{\sigma}(F) \neq \text{OrdMinLT}_{\sigma}(J) \). If it happens that \( \text{OrdMinLT}_{\sigma}(F) \) is a proper prefix of \( \text{OrdMinLT}_{\sigma}(J) \), the conclusion follows immediately. So we assume that \( \text{OrdMinLT}_{\sigma}(F) \) is not a proper prefix. Note that \( \text{OrdMinLT}_{\sigma}(J) \) cannot be a proper prefix of \( \text{OrdMinLT}_{\sigma}(F) \) as otherwise this would imply that there is \( f \in F \subseteq J \) with \( \text{LT}_{\sigma}(f) \notin \text{LT}_{\sigma}(J) \).

Let \( \text{OrdMinLT}_{\sigma}(F) = (t_1, t_2, \ldots) \), let \( \text{OrdMinLT}_{\sigma}(J) = (t'_1, t'_2, \ldots) \), and let \( k \) be the first index such that \( t_k \neq t'_k \). Since \( F \subseteq J \) we know that \( t_k \in \text{LT}_{\sigma}(J) \), and hence \( t_k \) is a multiple of some element of \( \text{OrdMinLT}_{\sigma}(J) \). Since \( \text{OrdMinLT}_{\sigma}(F) \) is interreduced, \( t_k \) is not a multiple of any of the other \( t_j \), and thus specifically not a multiple of any of \( t'_1, \ldots, t'_{k-1} \). Hence \( t_k \) can only be a non-trivial multiple of \( t'_k \) or a multiple of \( t'_j \) for some index \( j > k \). Either way \( t_k >_{\sigma} t'_k \), and so \( \text{OrdMinLT}_{\sigma}(J) \prec_{\sigma} \text{OrdMinLT}_{\sigma}(F) \) as claimed.

The next example illustrates the importance of \( \text{OrdMinLT}_{\sigma}(F) \) being interreduced.

**Example 4.3.** Let \( P = K[x, y] \) and let \( \sigma = \text{degRevLex} \). Let \( J = \langle x, y^3 \rangle \) and consider the \( \sigma \)-ordered tuple \( T = (x, x^2, y^3) \); the elements of \( T \) are clearly non-zero polynomials in \( J \). We observe that \( \text{OrdMinLT}_{\sigma}(J) = \text{OrdMinLT}_{\sigma}(T) = \langle x, y^3 \rangle \). However, the tuple \( T \) is not interreduced, and we have \( T \prec_{\sigma} \text{OrdMinLT}_{\sigma}(J) \).
We recall here a standard result from the theory of Gröbner bases; for the sake of completeness we include the proof.

**Lemma 4.1.** Let \( P = K[x_1, \ldots, x_n] \), let \( \sigma \) be a term ordering on \( T^n \), and let \( I, J \) be ideals in \( P \). If \( I \subseteq J \) then \( \text{LT}_\sigma(I) \subseteq \text{LT}_\sigma(J) \).

**Proof.** Since \( I \subseteq J \) we clearly have \( \text{LT}_\sigma(I) \subseteq \text{LT}_\sigma(J) \). Let \( f \in J \setminus I \) with minimal \( \sigma \)-leading term, thus \( \text{LT}_\sigma(f) \in \text{LT}_\sigma(J) \). However, by the minimality of \( \text{LT}_\sigma(f) \) we see that \( f \) cannot be head-reduced by any element of a \( \sigma \)-Gröbner basis of \( I \). Hence we conclude that \( \text{LT}_\sigma(f) \notin \text{LT}_\sigma(I) \). \( \square \)

We are ready to prove the following interesting result.

**Corollary 4.1.** Let \( P = K[x_1, \ldots, x_n] \), let \( \sigma \) be a term ordering on \( T^n \), and let \( I, J \) be ideals in \( P \). If \( I \subseteq J \) then \( \text{OrdMinLT}_\sigma(J) \prec_\sigma \text{OrdMinLT}_\sigma(I) \).

**Proof.** Let \( G_\sigma \) be a \( \sigma \)-Gröbner basis of \( I \). Thus \( G_\sigma \) is a set of non-zero polynomials in \( J \). By Proposition 4.1 we have \( \text{OrdMinLT}_\sigma(J) \preceq_\sigma \text{OrdMinLT}_\sigma(G_\sigma) = \text{OrdMinLT}_\sigma(I) \). From Lemma 4.1 and the assumption that \( I \subseteq J \) the conclusion follows. \( \square \)

Next we prove another useful result.

**Lemma 4.2.** Let \( \sigma \) be a term ordering on \( T^n \). Let \( T = (t_1, t_2, \ldots, t_r) \) be an interreduced \( \sigma \)-ordered tuple of elements in \( T^n \), and let \( T' \) be another set of elements in \( T^n \). Assume that there exist \( t' \in T' \) and an index \( k \) such that:

1. \( t_1, \ldots, t_{k-1} \in T' \)
2. \( t_k \succ_\sigma t' \)
3. \( t' \) is not divisible by any \( t_i \in T \)

Then \( \text{OrdMinLT}_\sigma(T') \prec_\sigma T \), and \( T \) is not a proper prefix of \( \text{OrdMinLT}_\sigma(T') \).

**Proof.** Let \( t'_\min = \min_\sigma \{ \bar{t} \in T' \mid \bar{t} \not\in I \} \), and let \( j \) be the smallest index such that \( t_j \succ_\sigma t'_\min \). These definitions imply that \( t'_\min \succ_\sigma t_{j-1} \), and also \( j \leq k \), so we know that \( t_1, \ldots, t_{j-1} \in T' \).

Now we define the tuple \( \overline{T'} = (t_1, \ldots, t_{j-1}, t'_\min) \), which is clearly \( \sigma \)-ordered. Furthermore, we have that \( \overline{T'} \prec_\sigma T \) because \( t_j \succ_\sigma t'_\min \) by construction.

Next we prove that \( \overline{T'} \) is a prefix of \( \text{OrdMinLT}_\sigma(T') \), and therefore satisfies \( \text{OrdMinLT}_\sigma(T') \preceq_\sigma \overline{T'} \preceq_\sigma T \).

The set of power-products in \( \overline{T'} \) is interreduced: we already know that \( \{t_1, \ldots, t_{j-1}\} \) is interreduced, and \( t'_\min \) is not divisible by any of them; on the other hand, we see that \( t'_\min \) cannot divide any of them because it is the \( \sigma \)-greatest element.

Now, it suffices to show that each element of \( T' \) (or, equivalently, of \( T' \setminus \overline{T'} \)) is either \( \succ_\sigma t'_\min \) or a multiple of an element of \( \overline{T'} \). Let \( s' \in T' \); we shall argue depending on whether \( s' \) is divisible by some element of the tuple \( T \). First we consider the case where \( s' \) is not divisible by any \( t_i \in T \). By definition of \( t'_\min \)
we see that $s' \geq_{\sigma} t'_{\min}$; if $s' = t'_{\min}$ it is trivially a multiple of an element of $T'$, otherwise $s' >_{\sigma} t'_{\min}$ as claimed. We address now the case where $s'$ is a multiple of some $t_i \in T$. If $i < j$ then $s'$ is clearly a multiple of an element of $T'$; otherwise $i \geq j$, so $s' \geq_{\sigma} t_i \geq_{\sigma} t_j >_{\sigma} t'_{\min}$.

In conclusion, $\text{OrdMinLT}_{\sigma}(T') \prec_{\sigma} T$, and $T$, not containing $t'_{\min}$, is not a proper prefix of $\text{OrdMinLT}_{\sigma}(T')$.

The following example illustrates the steps in this proof.

**Example 4.4.** Let $\sigma = \text{DegRevLex}$. Consider the interreduced $\sigma$-ordered tuple $T$ and the set $T'$:

$$T = (xyz, x^3, x^2y^2, xz^4, y^6, z^7)$$

$$T' = \{xyz, x^3, x^2z^2, xy^2, y^7, x^2y^8\}.$$  

We take $k = 3$, so $t_k = x^2y^2$, and $t' = xy^2$, which is not a multiple of any power-product in $T$: these choices satisfy the hypotheses of the lemma. Following through the proof we have $t'_{\min} = t'$, $j = 2$ and $T' = (xyz, xy^2)$, and we see clearly that $T' \prec_{\sigma} T$. We compute the tuple $\text{OrdMinLT}_{\sigma}(T') = (xyz, xy^2, x^3, x^2z^2, y^7)$, and observe that $T'$ appears as a (proper) prefix. Consequently $\text{OrdMinLT}_{\sigma}(T') \prec_{\sigma} T'$.

The following easy example shows the importance of the non-divisibility assumption in the lemma.

**Example 4.5.** Let $\sigma = \text{DegRevLex}$ and let $T = (x, y^3)$, an interreduced $\sigma$-ordered tuple. Now let $T' = \{x, x^2, y^4, z^4\}$. For $k = 1$ there is no $t' \in T'$ with $t' \prec_{\sigma} t_k$; and for $k = 2$ the only elements of $T'$ which are $\sigma$-less-than $t_k$ are $x$ and $x^2$, but both are divisible by $t_1$. So we cannot apply the lemma. Indeed, $\text{OrdMinLT}_{\sigma}(T') = (x, y^4, z^4)$, and we have $T \prec_{\sigma} \text{OrdMinLT}_{\sigma}(T')$.

Now we are ready to prove the main theorem of this section.

**Theorem 4.1.** Let $P = \mathbb{Q}[x_1, \ldots, x_n]$, let $\sigma, \tau$ be two term orderings on $\mathbb{N}^n$, let $I$ be a non-zero ideal in $P$, and let $p$ be a prime which is $\sigma$-good for $I$.

(a) If $p$ is $\tau$-good for $I$, we have $\text{OrdMinLT}_{\sigma}(I(p, \sigma)) = \text{OrdMinLT}_{\tau}(I)$.

(b) If $p$ is $\tau$-bad for $I$, we have $\text{OrdMinLT}_{\tau}(I(p, \sigma)) \prec_{\sigma} \text{OrdMinLT}_{\tau}(I)$, and also that $\text{OrdMinLT}_{\tau}(I)$ is not a proper prefix of $\text{OrdMinLT}_{\tau}(I(p, \sigma))$.

**Proof.** Let $G_{\tau}$ be the reduced $\tau$-Gröbner basis of $I$, and $G_{\sigma}$ be the reduced $\sigma$-Gröbner basis for $I$.

We start by proving claim (a). By hypothesis, $p$ is both $\sigma$-good and $\tau$-good for $I$, hence Theorem 2.3 implies that the reduced $\tau$-Gröbner basis of $I(p, \sigma)$ is $\pi_p(G_{\tau})$ which has the same leading terms as $G_{\tau}$, and the conclusion follows immediately from Remark 2.1b.
Now we prove claim (b). Let \( G_\tau = \{ g_1, \ldots, g_r \} \) where the elements are indexed so that \( \text{LT}_\tau(g_i) <_\tau \text{LT}_\tau(g_{i+1}) \) for \( i = 1, \ldots, r-1 \). For \( i = 1, \ldots, r \), let \( \tilde{g}_i = \text{prim}(g_i) \) so in particular \( \pi_p(\tilde{g}_i) \neq 0 \). Define the following \( \tau \)-ordered tuple \( T \) and set \( T' \)

\[
T = (\text{LT}_\tau(\tilde{g}_1), \ldots, \text{LT}_\tau(\tilde{g}_r))
\]

\[
T' = \{ \text{LT}_\tau(\pi_p(\tilde{g}_1)), \ldots, \text{LT}_\tau(\pi_p(\tilde{g}_r)) \}
\]

By definition of \( \pi_p \) we have \( \text{LT}_\tau(\pi_p(\tilde{g}_i)) \leq_\tau \text{LT}_\tau(\tilde{g}_i) \) for all \( i \). Since \( p \) is \( \tau \)-bad, there is at least one index \( j \) such that \( p \) divides the denominator of \( g_j \in G_\tau \), hence it divides also the leading coefficient of \( \tilde{g}_j \). Therefore \( \text{LT}_\tau(\pi_p(\tilde{g}_j)) <_\tau \text{LT}_\tau(\tilde{g}_j) \); let \( k \) be the smallest such index. Moreover, since \( G_\tau \) is a reduced Gröbner basis, \( \text{LT}_\tau(\pi_p(\tilde{g}_k)) \) is not a multiple of any element of \( \text{LT}_\tau(G_\tau) \). Therefore we can apply Lemma 4.2 to \( T \) and \( T' \) with the above value of \( k \) and deduce that \( \text{OrdMinLT}_\tau(T') \prec_\tau T = \text{OrdMinLT}_\tau(I) \), and that \( \text{OrdMinLT}_\tau(I) \) is not a proper prefix of \( \text{OrdMinLT}_\tau(T') \).

Let \( F = \{ \pi_p(\tilde{g}_1), \ldots, \pi_p(\tilde{g}_r) \} \). By Lemma 2.11b we deduce that \( F \subseteq I_{(p, \sigma)} \).

Hence Proposition 4.1 implies that \( \text{OrdMinLT}_\tau(I_{(p, \sigma)}) \leq_\tau \text{OrdMinLT}_\tau(F) = \text{OrdMinLT}_\tau(T') \).

Combining the two inequalities, the conclusion follows.

This theorem enables us to detect some bad primes without having to compute the reduced \( \tau \)-Gröbner basis of \( I \) over the rationals.

**Corollary 4.2.** Let \( P = \mathbb{Q}[x_1, \ldots, x_n] \), let \( \sigma \) and \( \tau \) be two term orderings on \( \mathbb{T}^n \), and let \( I \) be a non-zero ideal in \( P \). Let \( p \) and \( q \) be \( \sigma \)-good primes for \( I \).

If \( \text{OrdMinLT}_\tau(I_{(q, \sigma)}) \prec_\tau \text{OrdMinLT}_\tau(I_{(p, \sigma)}) \) then \( q \) is \( \tau \)-bad for \( I \).

**Proof.** By Theorem 4.1 we know that \( \text{OrdMinLT}_\tau(I_{(p, \sigma)}) \leq_\tau \text{OrdMinLT}_\tau(I) \).

Hence \( \text{OrdMinLT}_\tau(I_{(q, \sigma)}) \prec_\tau \text{OrdMinLT}_\tau(I) \), so Theorem 4.1(a) implies that the prime \( q \) is \( \tau \)-bad for \( I \).

**Example 4.6.** In the polynomial ring \( \mathbb{Q}[x, y, z] \) with term ordering \( \sigma = \text{DegRevLex} \), let \( F = \{ x^2y + 7xy^2 - 2, y^3 + x^2z, z^3 + x^2 - y \} \) and let \( I = \langle F \rangle \). It turns out that all primes are \( \sigma \)-good for \( I \), and we have

\[
\text{OrdMinLT}_\sigma(I) = \text{OrdMinLT}_\sigma(I_{(p, \sigma)}) = (z^3, y^3, x^2y, x^4z, x^6) \quad \text{for all primes } p.
\]

Now we consider the term ordering \( \tau = \text{Lex} \). It turns out that the set of \( \tau \)-bad primes for \( I \) is \( S = \{ 2, 7, 11, 55817, p \} \) where \( p \approx 1.8 \times 10^{65} \). We have

\[
\text{OrdMinLT}_\tau(I) = \text{OrdMinLT}_\tau(I_{(p, \tau)}) = (z^{26}, y, x) \quad \text{for all primes } p \notin S.
\]

For the bad primes we obtain:

- \( \text{OrdMinLT}_\tau(I_{(2, \tau)}) = (z^{17}, yz, y^3, xz^6, xy^2, x^2) \)
- \( \text{OrdMinLT}_\tau(I_{(7, \tau)}) = (z^{13}, y, x^2) \)
- \( \text{OrdMinLT}_\tau(I_{(11, \sigma)}) = \text{OrdMinLT}_\tau(I_{(55817, \sigma)}) = \text{OrdMinLT}_\tau(I_{(p, \sigma)}) = (z^{25}, yz, y^2, x) \)
To conclude the paper we show an example illustrating the merits of Corollary 4.2

**Example 4.7.** Let $P = \mathbb{Q}[x, y, z, w, s, t]$, let $\sigma$ be a term ordering where each of $x, y, z, w$ is $\sigma$-greater than every power product in $s$ and $t$, and let $\tau$ be any elimination ordering for $[s, t]$ which restricts to $\text{DegRevLex}$ on $T(x, y, z, w)$. Let $f_1 = t^3, f_2 = st^2 - 2s^2, f_3 = s^2t - 5, f_4 = s^3 - 7t$, and let $J = \langle x - f_1, y - f_2, z - f_3, w - f_4 \rangle$.

The given generators of $J$ form a reduced $\sigma$-Gröbner basis; so clearly every prime is $\sigma$-good for $J$. However, we do not know which ones are $\tau$-good.

Let us now look at $\text{OrdMinLT}_\tau(I_{(p, \sigma)})$ for the primes 2, 3, 5 and 7.

For $p = 2$ we compute $\text{OrdMinLT}_\tau(I_{(p, \sigma)})$ obtaining the following tuple

$$[y^2, z^3, y^4, ts, tx, tsz, sz, t^2, st, sz, s^2t, s^3]$$

The best (and only!) tuple we have seen so far is $\text{OrdMinLT}_\tau(I_{(2, \sigma)})$.

For $p = 3$ we compute $\text{OrdMinLT}_\tau(I_{(p, \sigma)})$ to be the following

$$[z^3, yz^3, y^2z^3, x^3y^3, y^4, y^4t, z^3y^3w, xz^3w^2, x^2y^2w, x^2y^3w, x^3z^2w, sz, sy, sx, tz^2, ty^2, t^2, t^3, sw^2, tz^2, tyw^2, t^2, st, s^3]$$

If we compare $\text{OrdMinLT}_\tau(I_{(3, \sigma)})$ with the best tuple, namely $\text{OrdMinLT}_\tau(I_{(2, \sigma)})$, we see from the very first elements of the tuples that $\text{OrdMinLT}_\tau(I_{(2, \sigma)}) \prec \tau \text{OrdMinLT}_\tau(I_{(3, \sigma)})$. So the prime 2 is surely $\tau$-bad. And the best tuple we have seen so far is now $\text{OrdMinLT}_\tau(I_{(3, \sigma)})$.

Next we choose $p = 5$. For $\text{OrdMinLT}_\tau(I_{(p, \sigma)})$ we get the following tuple

$$[z^3, yz^3, y^2z^3, x^3y^3, y^4, y^4t, z^3y^3w, xz^3w^2, x^2y^2w, x^2y^3w, x^3z^2w, sz, sy, sx, tz^2, ty^2, t^2, t^3, sw^2, tz^2, tyw^2, t^2, st, s^3]$$

If we compare $\text{OrdMinLT}_\tau(I_{(5, \sigma)})$ with the best tuple seen so far we see that the tuples agree up to the boxed elements, but then we find that $\text{OrdMinLT}_\tau(I_{(3, \sigma)}) \prec \tau \text{OrdMinLT}_\tau(I_{(5, \sigma)})$. So the prime 3 is surely $\tau$-bad, and we have a new best tuple, namely $\text{OrdMinLT}_\tau(I_{(5, \sigma)})$.

For the prime $p = 7$ we compute the tuple $\text{OrdMinLT}_\tau(I_{(p, \sigma)})$ to be the following

$$[z^3, yz^3, y^2z^3, y^4, sz, sy, sx, tz^2, sw^2, tz, ty^2, t^2, t^3, sw, s^2t, s^3]$$

If we compare $\text{OrdMinLT}_\tau(I_{(7, \sigma)})$ with the best tuple seen so far, just from comparing the very first elements, we find that $\text{OrdMinLT}_\tau(I_{(7, \sigma)}) \prec \tau \text{OrdMinLT}_\tau(I_{(5, \sigma)})$. So the prime 7 is surely $\tau$-bad, and $\text{OrdMinLT}_\tau(I_{(5, \sigma)})$ remains the best tuple.

If we try further primes, we find that they produce tuples equal to $\text{OrdMinLT}_\tau(I_{(5, \sigma)})$. At this point we are inclined to believe that 5 is a good prime, but have no actual proof of this.

For this very small example, we can just compute directly the reduced $\tau$-Gröbner basis of $J$; this will then confirm that 5 is indeed good. The main point is that once we have seen one good prime, the test from Corollary 4.2 gives us a sure way of distinguishing good primes from bad ones. However, without some “outside information” we cannot know whether the best prime seen so far is actually good; it may be just “less bad” than other primes tried.
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