Tilting objects in the stable category of vector bundles on the weighted projective line of type 
\((2, 2, 2; \lambda)\)

Jianmin Chen, Yanan Lin, Shiquan Ruan

Abstract: We construct a tilting object for the stable category of vector bundles on a weighted projective line \(X\) of type \((2, 2, 2; \lambda)\), consisting of five rank two bundles and one rank three bundle, whose endomorphism algebra is a canonical algebra associated with \(X\) of type \((2, 2, 2, 2)\).

Keywords: tilting object; weighted projective line; vector bundle; stable category.

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1 Introduction

The notion of weighted projective lines was introduced by Geigle and Lenzing [1] to give a geometric treatment to canonical algebras which were studied by Ringel [9]. A weighted projective line can be interpreted as obtained from the usual projective line by inserting finitely many weights. In [1], Geigle and Lenzing proved that the category of coherent sheaves on

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†Corresponding author, E-mail: shiquanruan@stu.xmu.edu.cn
weighted projective lines is derived equivalent to the category of finite dimensional modules over some canonical algebra, and the category of vector bundles on a weighted projective line, as additive category, is equivalent to the category of graded Cohen-Macaulay modules over its corresponding graded ring. So the category of vector bundles on a weighted projective line carries two "natural" exact structures in Quillen’s sense [8]. In particular, under the exact structure which is called distinguished exact, induced from the category of graded Cohen-Macaulay modules over its corresponding graded ring, the category of vector bundles is Frobenius with the indecomposable projective-injective objects as all line bundles (i.e., rank 1 vector bundles)[4]. Therefore, the stable category of vector bundles with respect to the distinguished exact structure is a triangulated category.

Let $X$ be a weighted projective line over an algebraically closed field, $\text{vect} X$ the category of vector bundles on $X$, and $\text{vect} X$ its stable category obtained from the vector bundles by factoring out all the line bundles. Recently, Kussin, Lenzing and Meltzer showed a series of interesting results for stable vector bundle categories for weighted projective lines of triple weight type[4-6]. Among other things, they give a tilting object in $\text{vect} X$ of type $(p_1, p_2, p_3)$ with endomorphism ring $k\overrightarrow{A}_{p_1-1} \otimes_k k\overrightarrow{A}_{p_2-1} \otimes_k k\overrightarrow{A}_{p_3-1}$, where $p_1, p_2, p_3$ are integers greater than or equal to 2, and $k\overrightarrow{A}_n$ denotes the path algebra of oriented quiver of type $A_n$.

For the weighted projective line of type $(2, 2, 2, 2; \lambda)$, Kussin, Lenzing and Meltzer [4] proved, independently by K. Ueda [11], that the triangulated categories $\text{vect} X$ and $D^b(\text{coh} X)$ are equivalent. Moreover, $\text{vect} X$ has tilting objects. But it is still difficult to find a tilting object in $\text{vect} X$ since we can’t describe an explicit correspondence between objects in $\text{vect} X$ and $D^b(\text{coh} X)$. This paper constructs tilting objects in the stable category of vector bundles on the weighted projective line of type $(2, 2, 2; \lambda)$.

The paper is organized as follows:

In section 2, we collect basic definitions and properties. Section 3 discusses projective covers (res. injective hulls) of vector bundles. Theorem 3.3 gives an explicit description of projective covers (res. injective hulls) of Auslander
bundles and extension bundles. Proposition 3.4 shows the close relationships between the middle term of an Auslander-Reiten sequence in vect$\mathcal{X}$ and the other two terms. We give a formula to compute the slopes of vector bundles under shift in section 4 and describe all the exceptional objects in vect$\mathcal{X}$ in section 5. Section 6 contains the main result, Theorem 6.2. We construct a tilting object consisting of five bundles of rank two and one bundle of rank three, whose endomorphism algebra is a canonical algebra of type (2,2,2,2); furthermore, we show that there doesn’t exist any tilting object consisting only of bundles of rank two such that whose endomorphism algebra is a canonical algebra.

2 weighted projective line of type (2, 2, 2; $\lambda$)

Throughout $k$ is an algebraically closed field, $\lambda$ is a closed point of $\mathbb{P}_1(k)$ different from 0, 1, $\infty$, and identify equivalences with identifications.

In this section, we recall some basic definitions and properties about weighted projective line of type (2, 2, 2; $\lambda$).

Let $\mathbb{L}$ be the rank 1 abelian group on generators $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ with relations

$$2\vec{x}_1 = 2\vec{x}_2 = 2\vec{x}_3 = 2\vec{x}_4 =: \vec{c}.$$ 

Then $\mathbb{L}$ is an ordered group whose cone of positive elements is $\sum_{i=1}^{4} \mathbb{N}\vec{x}_i$, and each $\vec{x} \in \mathbb{L}$ can be uniquely written in normal form

$$\vec{x} = \sum_{i=1}^{4} l_i \vec{x}_i + l \vec{c},$$

where $0 \leq l_i \leq 1$ and $l \in \mathbb{Z}$.

In addition, if $\vec{x} = \sum_{i=1}^{4} l_i \vec{x}_i + l \vec{c}$ is in normal form, one can define

$$\vec{x} \geq 0 \text{ if and only if } l_i \geq 0 \text{ for } i = 1, 2, 3, 4 \text{ and } l \geq 0,$$

then each $\vec{x} \in \mathbb{L}$ satisfies exactly one of the two possibilities $\vec{x} \geq 0$ or $\vec{x} \leq \vec{w} + \vec{c}$, where $\vec{w} = 2\vec{c} - \sum_{i=1}^{4} \vec{x}_i$. The element $\vec{c}$ is called the canonical element and $\vec{w}$ is called the dualizing element of $\mathbb{L}$. 

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Denote by $S$ the commutative algebra
\[ S = k[X_1, X_2, X_3, X_4]/I = k[x_1, x_2, x_3, x_4], \]
where $I$ is the homogeneous ideal generated by $f_1 = X_3^2 - (X_2^2 + X_1^2)$ and $f_2 = X_4^2 - (X_2^2 + \lambda X_1^2)$. Then $S$ carries an $L$-grading by setting $\deg x_i = -\bar{x}_i (i = 1, 2, 3, 4)$, i.e., $S = \bigoplus_{\bar{x} \in L} S_{\bar{x}}$, where $S_{\bar{x}} S_{\bar{y}} \subseteq S_{\bar{x} + \bar{y}}$ and $S_0 = k$.

By an $L$-graded version of the Serre construction \[10\], the category $\text{coh} X$ of coherent sheaves on the weighted projective line $X$ of type $(2, 2, 2, 2; \lambda)$ is given by
\[ \text{coh} X = \text{mod}^L S/\text{mod}^L_0 S, \]
where $\text{mod}^L S$ is the category of finitely generated $L$-graded $S$ modules and $\text{mod}^L_0 S$ is the category of finite dimensional $L$-graded $S$ modules.

Use the notation $\widetilde{M}$ for $M \in \text{mod}^L S$ under the quotient functor, and call the process $q : \text{mod}^L S \to \text{coh} X, M \mapsto \widetilde{M}$ sheafification, then it is easy to see that $\widetilde{M(\bar{x})} = \widetilde{M(\bar{x})}$. Call $\mathcal{O} = S$ the structure sheaf of $X$, and $\mathcal{O}(\bar{x})$ a line bundle for any $\bar{x} \in L$.

**Proposition 2.1**\([1]\) The category $\text{coh} X$ is a hereditary, abelian, $k$-linear, Hom-finite, Noetherian category with Serre duality, i.e., $\text{DExt}(X, Y) = \text{Hom}(Y, \tau X)$, where the $k$-equivalence $\tau : \text{coh} X \to \text{coh} X$ is the shift $X \mapsto X(\bar{x})$. In addition, $\text{coh} X = \mathcal{H}_+ \cup \mathcal{H}_0$, where $\mathcal{H}_+$ denotes the full subcategory of $\text{coh} X$ consisting of all objects not having a simple subobject, $\mathcal{H}_0$ denotes the full subcategory of $\text{coh} X$ consisting of all objects of finite length, $\cup$ means that each indecomposable object of $\text{coh} X$ is either in $\mathcal{H}_+$ or in $\mathcal{H}_0$, and there are no non-zero morphisms from $\mathcal{H}_0$ to $\mathcal{H}_+$.

Objects in $\mathcal{H}_+$ are called vector bundles, and $\mathcal{H}_+$ is also denoted by $\text{vect} X$. In particular, all objects with the form $\mathcal{O}(\bar{x})$ for $\bar{x} \in L$ belong to $\text{vect} X$.

More details about $\mathcal{H}_0$ as follows.

**Proposition 2.2**\([1]\) (1) The simple objects of $\text{coh} X$ are parametrized by the projective line $\mathbb{P}_1(k)$, where to each point $a \neq \{\infty, 0, 1, \lambda\}$ there is associated a tube with a unique simple $S_a$, called ordinary simple, where to each $a \in \{\infty, 0, 1, \lambda\}$ there is associated a tube with two simple ob-
jects, called exceptional simples. Moreover, each ordinary simple $S$ satisfies $\text{Ext}^1(S, S) = k$, while each exceptional simple $S$ satisfies $\text{Ext}^1(S, S) = 0$.

(2) $\mathcal{H}_0$ is exact, abelian, uniserial category and decomposes into a coproduct $\coprod_{a \in F_1(k)} U_a$ of connected uniserial subcategories, whose associated quivers are tubes with mouth simple objects of coh$X$.

As for the structure of vect$X$, there have

Proposition 2.3 ([1]) (1) For each $\overrightarrow{x}, \overrightarrow{y} \in L$, there has

\[ \text{Hom}(O(\overrightarrow{x}), O(\overrightarrow{y})) = S_{\overrightarrow{y} \rightarrow \overrightarrow{x}}. \]

In particular, $\text{Hom}(O(\overrightarrow{x}), O(\overrightarrow{y})) \neq 0$ if and only if $\overrightarrow{x} \leq \overrightarrow{y}$.

(2) Each vector bundle $E$ has a filtration by line bundles

\[ 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_r = E, \]

where each factor $L_i = E_i/E_{i-1}$ is a line bundle.

Denoted by $K_0(\mathcal{X})$ the Grothendieck group of coh$X$, then the classes $[O(\overrightarrow{x})]$ for $0 \leq \overrightarrow{x} \leq \overrightarrow{e}$ form a $\mathbb{Z}$-basis of $K_0(\mathcal{X})$, and there is a $\mathbb{Z}$-bilinear form $\langle - , - \rangle : K_0(\mathcal{X}) \times K_0(\mathcal{X}) \rightarrow \mathbb{Z}$ on $K_0(\mathcal{X})$ induced by

\[ \langle [X], [Y] \rangle = \dim \text{Hom}(X, Y) - \dim \text{Ext}^1(X, Y) \]

for objects $X, Y \in \text{coh}X$, which is called Euler form.

The following are two additive functions on $K_0(\mathcal{X})$ called rank and degree. The rank $\text{rk} : K_0(\mathcal{X}) \rightarrow \mathbb{Z}$ is characterized by $\text{rk}(O(\overrightarrow{x})) = 1$ for $\overrightarrow{x} \in L$ and $\text{rk}(S) = 0$ for each simple object $S$.

The degree $\text{deg} : K_0(\mathcal{X}) \rightarrow \mathbb{Z}$ is uniquely determined by the following properties:

1. $\text{deg}(O(\overrightarrow{x})) = \delta(\overrightarrow{x})$ for $\overrightarrow{x} \in L$, where $\delta : L \rightarrow \mathbb{Z}$ is the group homomorphism defined on generators by $\delta(\overrightarrow{x}_i) = 1$ ($i = 1, 2, 3, 4$);
2. $\text{deg}(O) = 0$;
3. $\text{deg}(S) = 2$ for each ordinary simple object, and $\text{deg}(S) = 1$ for each exceptional simple object.

For each $F \in \text{coh}X$, define the slope of $F$ as $\mu(F) = \text{deg}(F)/\text{rk}(F)$. It is an element in $\mathbb{Q} \cup \{\infty\}$. And according to [1], each object in vect$X$ has rank
> 0, then the slope belongs to \( \mathbb{Q} \); and each object in \( \mathcal{H}_0 \) has rank 0, then the slope \( \infty \).

In particular,

**Theorem 2.4 (Riemann-Roch formula)** ([7]) For each \( X, Y \in \text{coh} X \), there has

\[
\langle [X] \oplus [\tau X], [Y] \rangle = \text{rk}(X)\deg(Y) - \deg(X)\text{rk}(Y) = \text{rk}(X)\text{rk}(Y)(\mu(Y) - \mu(X)).
\]

Denote by \( \mathcal{H}^q = \text{add}(\text{indcoh}^q(X)) \), where \( \text{indcoh}^q(X) \) consisting of all indecomposable vector bundles of slope \( q \in \mathbb{Q} \cup \{ \infty \} \).

**Proposition 2.5** ([7])

1. \( \mathcal{H}^q \) is an exact Abelian subcategory of \( \text{coh} X \) which is closed under extension;

2. \( \text{coh} X = \bigvee_{q \in \mathbb{Q} \cup \{ \infty \}} \mathcal{H}^q \);

3. There are non-zero morphisms from \( \mathcal{H}^q \) to \( \mathcal{H}^r \) if and only if \( q \leq r \);

4. \( \mathcal{H}^\infty \) is just \( \mathcal{H}_0 \), and each \( \mathcal{H}^q \) is equivalent to \( \mathcal{H}_0 \).

We are interested in \( \text{vect} X \).

**Theorem 2.6** ([1]) Sheafification \( q: \text{mod}^L S \rightarrow \text{coh} X \) induces an equivalence

\[
\text{vect} X = \text{CM}^L S,
\]

where \( \text{CM}^L S \) consists of all \( M \in \text{mod}^L S \) satisfying \( \text{Hom}(E, M) = 0 = \text{Ext}^1(E, M) \), for each simple \( L \)-grade \( S \)-module \( E \).

**Remark:** The category \( \text{vect} X \) is fully embedded as an extension-closed subcategory into two different abelian categories

\[
\text{coh} X \hookrightarrow \text{vect} X = \text{CM}^L S \hookrightarrow \text{mod}^L S.
\]

So \( \text{vect} X \) carries two different "natural" exact structures.

Now we have the exact structure on \( \text{vect} X \) induced from \( \text{CM}^L S \). A sequence \( \eta: 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \) in \( \text{vect} X \) is called distinguished exact if and only if \( \text{Hom}(L, \eta) \) is exact for each line bundle \( L \).

By Serre duality, a sequence \( \eta: 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \) is distinguished exact if and only if \( \text{Hom}(\eta, L) \) is exact for each line bundle \( L \). Moreover, each distinguished exact sequence is exact in \( \text{coh} X \).
Proposition 2.7 ([6]) The category vect$X$ with the structure of distinguished exact is a Frobenius category, whose indecomposable projective-injective objects are just all the line bundles.

Due to Happel [3], the stable category of vect$X$ with respect to the distinguished exact structure is a triangulated category, denoted by vect$X$.

Proposition 2.8 ([6]) (1) Let $X$ be a vector bundle without direct summand which is a line bundle, $IX$ be the injective hull of $X$. There exists an exact sequence $0 \to X \to IX \to X'' \to 0$ in the Frobenius category vect$X$, then $X[1] = X''$ in vect$X$;

(2) The stable category vect$X$ is triangulated, Hom-finite, Krull-Schmidt $k$-category with Serre duality $\text{Hom}(X, Y[1]) = \text{DHom}(Y, \tau X)$, where $\tau$ is induced by $X \mapsto X(\overrightarrow{\omega})$. Moreover, vect$X$ is homologically finite, that is, for any $X, Y \in$ vect$X$, $\text{Hom}(X, Y[n]) = 0$, for $|n| \gg 0$;

(3) There is an action of the Picard group $\mathbb{L}$ on vect$X$ by the shift, i.e., any $\overrightarrow{x} \in \mathbb{L}$ sends $X \in$ vect$X$ to $X(\overrightarrow{x})$;

(4) The stable category vect$X$ has Auslander-Reiten sequences induced from the Auslander-Reiten sequences in vect$X$.

Furthermore,

Theorem 2.9 ([4]) The stable category vect$X$ and $D^b(\text{coh}X)$ are equivalent as triangulated categories.

3 Projective cover and injective hull

In order to describe tilting objects in vect$X$, we should firstly consider the projective covers and injective hulls of vector bundles in vect$X$. We consider the indecomposable bundles of rank two in vect$X$ since all line bundles are zero in vect$X$.

Proposition 3.1 For each indecomposable vector bundle $E$ of rank two, there exists a line bundle $L$ and a non-split exact sequence

$0 \to L(\overrightarrow{\omega}) \to E \to L(\overrightarrow{x}) \to 0$ with $0 \leq \overrightarrow{x} \leq \overrightarrow{\omega}$.

Proof: Recalling the first step in the proof of Theorem 4.8 in [6], and
noticing that $2\overrightarrow{x} = 0$ for the weighted projective line of $(2, 2, 2; \lambda)$, the result follows easily.□

Now we extend the notions of Auslander bundles and extension bundles in the weighted projective lines of triple weight type in [6] to the type $(2, 2, 2, 2; \lambda)$.

**Definition 3.2** Let $L$ be a line bundle on $X$. We call the middle term of the Auslander-Reiten sequence $0 \rightarrow L(\overrightarrow{x}) \rightarrow E \rightarrow L \rightarrow 0$ Auslander bundle associated with $L$, and denote it by $E = E_L$. For $0 \leq \overrightarrow{x} < \overrightarrow{c}$, let $\eta : 0 \rightarrow L(\overrightarrow{x}) \rightarrow E \rightarrow L(\overrightarrow{x}) \rightarrow 0$ be a non-split exact sequence. The middle term $E = E_L(\overrightarrow{x})$, which is uniquely defined up to isomorphism, is called the extension bundle with the data $(L, \overrightarrow{x})$.

**Remark** The definition of extension bundle is a little different from [6] since $\dim \text{Ext}^1(L(\overrightarrow{x}), L(\overrightarrow{c})) = 1$ for $0 \leq \overrightarrow{x} < \overrightarrow{c}$ and $\dim \text{Ext}^1(L(\overrightarrow{c}), L(\overrightarrow{x})) = 2$ in the weighted projective line of $(2, 2, 2, 2; \lambda)$.

Now we pay attention to the projective cover $PE$ of extension bundles $E$.

Noticing that $0 < \overrightarrow{x} < \overrightarrow{c}$ implies $\overrightarrow{x} = \overrightarrow{x}_i$ for some $i = 1, 2, 3, 4$, we only need to consider the cases of $E_L$ and $E_L(\overrightarrow{x}_i), 1 \leq i \leq 4$.

**Theorem 3.3** Let $L$ be a line bundle,

(1) if $E = E_L$, then $PE = L(\overrightarrow{x}) \bigoplus_{i=1}^4 L(-\overrightarrow{x}_i)$;

(2) if $E = E_L(\overrightarrow{x}_i)$, then $PE = L(\overrightarrow{x}) \bigoplus \bigoplus_{j=1, j \neq i}^4 L(\overrightarrow{x}_i - \overrightarrow{x}_j)$.

**Proof:** (1) Applying $\text{Hom}(L(-\overrightarrow{x}_i), -)$ to the exact sequence

$$0 \rightarrow L(\overrightarrow{x}) \xrightarrow{\alpha} E \xrightarrow{\beta} L \rightarrow 0,$$

we have $\dim \text{Hom}(L(-\overrightarrow{x}_i), E) = \dim \text{Hom}(L(-\overrightarrow{x}_i), L) = \dim S_{\overrightarrow{x}_i} = 1$. Assuming $\text{Hom}(L(-\overrightarrow{x}_i), E) = \langle \varphi_i \rangle$, then $\beta \varphi_i = x_i$. For each $\varphi \in \text{Hom}(L(\overrightarrow{x}), E)$, we claim that there exists a morphism $\theta \in \text{Hom}(L(\overrightarrow{x}), E)$, which factors

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through $\bigoplus_{i=1}^{4} L(-\mathfrak{x}_i)$, such that $\beta(\varphi - \theta) = 0$. In fact, if $\beta\psi = 0$, we can choose $\theta = 0$; if $\beta\psi \neq 0$, then $-\mathfrak{x} > 0$. We write $-\mathfrak{x}$ in normal form $\sum_{i=1}^{1} l_i \mathfrak{x}_i + l \mathfrak{x}$, and discuss it in two cases:

Case 1: $l_i \neq 0$ for some $i = 1, 2, 3, 4$. Then we have

$$\text{dim} \text{Hom}(L(-\mathfrak{x}), L(-\mathfrak{x}_i)) = \text{dim} \text{Hom}(L(\mathfrak{x}), L) = l + 1.$$ 

So $x_i : L(-\mathfrak{x}_i) \to L$ induces an isomorphism

$$\text{Hom}(L(-\mathfrak{x}), L(-\mathfrak{x}_i)) = \text{Hom}(L(\mathfrak{x}), L).$$

Hence there exists some $\theta_i \in \text{Hom}(L(-\mathfrak{x}), L(-\mathfrak{x}_i))$ such that

$$\beta\psi = x_i \theta_i = \beta\varphi \theta_i,$$

then we can choose $\theta = \varphi_i \theta_i$.

Case 2: $l_i = 0$ for each $i = 1, 2, 3, 4$, i.e., $-\mathfrak{x} = l \mathfrak{x}$. Then

$$\text{dim} \text{Hom}(L(-\mathfrak{x}_i), L(-\mathfrak{x})) = l.$$ 

Assume $\text{Hom}(L(\mathfrak{x}), L(-\mathfrak{x}_i)) = \{\theta_i | 1 \leq t \leq l\}$, then $\{x_i \theta_i | 1 \leq t \leq l\}$ are linearly independent in the space $\text{Hom}(L(\mathfrak{x}), L)$ since $x_i : L(-\mathfrak{x}_i) \to L$ is injective. Moreover, for each $j \neq i$, we know that $x_j^2 : L(\mathfrak{x}) \to L$ can not factor through $L(-\mathfrak{x}_i)$, that is, $x_j^2 \notin \langle x_i \theta_i | 1 \leq t \leq l \rangle$. So $\{x_i \theta_i | 1 \leq t \leq l\}$ forms a basis of $\text{Hom}(L(\mathfrak{x}), L)$. Hence, there exist $k_{t} \in k$ for $1 \leq t \leq l+1$, such that $\beta\psi = \sum_{t=1}^{l} k_t x_i \theta_i^t + k_{l+1} x_j^{2l-1} = \beta(\sum_{t=1}^{l} k_t \varphi_i \theta_i^t + k_{l+1} \varphi_j x_j^{2l-1})$. In this case, we can choose $\theta = \sum_{t=1}^{l} k_t \varphi_i \theta_i^t + k_{l+1} \varphi_j x_j^{2l-1}$. This finishes the proof of the claim.

Therefore, there exists $\psi \in \text{Hom}(L(\mathfrak{x}), L(\mathfrak{x}))$ such that $\varphi - \theta = \alpha\psi$. So $\varphi = \theta + \alpha\psi$, i.e., $\varphi$ factors through $L(\mathfrak{x}) \bigoplus_{i=1}^{4} L(-\mathfrak{x}_i)$. It’s easy to see that there are no non-zero morphisms between two different direct summands of $L(\mathfrak{x}) \bigoplus_{i=1}^{4} L(-\mathfrak{x}_i)$. So $PE = L(\mathfrak{x}) \bigoplus_{i=1}^{4} L(-\mathfrak{x}_i)$. 

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(2) Applying $\text{Hom}(L(x_i - x_j), -)$ to $0 \to L(x_i) \to E \to L(x_i) \to 0$, we have $\text{dim} \text{Hom}(L(x_i - x_j), E) = \text{dim} \text{Hom}(L(x_i - x_j), L(x_i)) = 1$. Assuming $\text{Hom}(L(x_i - x_j), E) = \langle \phi_j \rangle$, then $\beta \phi_j = x_j$. Using similar arguments as in (1), we can prove that for each $\phi \in \text{Hom}(L(x_i), E)$, there exists a morphism $\theta \in \text{Hom}(L(x_i), E)$, which factors through $\bigoplus_{j \neq i} L(x_i - x_j)$, such that $\beta(\phi - \theta) = 0$. Therefore, there exists some morphism $\psi \in \text{Hom}(L(x_i), L(x_i))$ such that $\phi - \theta = \alpha \psi$, which means $\phi = \theta + \alpha \psi$ factors through $L(x_i) \bigoplus \bigoplus_{j \neq i} L(x_i - x_j)$. It’s easy to see that there are no non-zero morphisms between two different direct summands of $L(x_i) \bigoplus \bigoplus_{j \neq i} L(x_i - x_j)$.

Note: If $E$ is an indecomposable vector bundle of rank two but not an extension bundle, then $E$ fits into a non-split exact sequence $0 \to L(\omega) \to E \to L(c) \to 0$ for some line bundle $L$ and satisfies $E = E(\omega)$. Hence from [7] we see that $E$ is a quasi-simple object in some homogeneous tube with slope integer. We will give the projective cover $PE$ of $E$ in section 4.

Now we will show the relationships between projective covers of the vector bundle in the middle term of an Auslander-Reiten sequence and the other two terms.

**Proposition 3.4** Let $0 \to E(\omega) \to F \to E \to 0$ be an Auslander-Reiten sequence in $\text{vect}X$ with $E$ indecomposable of rank greater than or equal to two.

1. If $E[-1]$ is an Auslander bundle, i.e., $E[-1] = EL$ for some line bundle $L$, then $PE \bigoplus PE(\omega) = PF \bigoplus L(\omega)$.
2. If else, $PE \bigoplus PE(\omega) = PF$.

**Proof:** Since $E(\omega)[-1] = E[-1](\omega)$, there exists an Auslander-Reiten sequence in $\text{coh}(X)$

$$0 \to E(\omega)[-1] \to F' \to E[-1] \to 0.$$ 

Then we obtain a commutative diagram with distinguished exact sequences as follows:
We know that the natural projective morphism $\pi : PF \to F$ induces a morphism $\delta : PE(\omega) \oplus PE \to PF$. So we get a commutative diagram:

$$0 \to F' \xrightarrow{i} PE(\omega) \oplus PE \xrightarrow{\pi} F \xrightarrow{id} 0.$$

Then from the snake lemma in $\text{coh}(X)$, we know that $\delta'$ is surjective and there is an isomorphism $\rho : \text{Ker}(\delta') \to \text{Ker}(\delta)$.

Now we have the following exact commutative diagram:

$$0 \to \text{Ker}(\delta') \xrightarrow{\rho} \text{Ker}(\delta) \xrightarrow{\theta} 0 \to 0,$$

$$0 \to F' \xrightarrow{i} PE(\omega) \oplus PE \xrightarrow{\pi} F \xrightarrow{id} 0,$$

and $\delta$ is split surjective implies $\theta$ is split injective. Hence, there exists a morphism $\sigma : PE(\omega) \oplus PE \to \text{Ker}(\delta)$, such that $\sigma \theta = \text{id}_{\text{Ker}(\delta)}$. Let $\sigma' = \rho^{-1} \sigma i : F' \to \text{Ker}(\delta')$, then $\sigma' \theta' = \rho^{-1} \sigma i \theta' = \rho^{-1} \sigma \theta \rho = \text{id}_{\text{Ker}(\delta')}$, which implies $\theta'$ is split injective. So we obtain a split exact sequence:

$$0 \to \text{Ker}(\delta') \xrightarrow{\theta'} F' \xrightarrow{i} \text{F}[1] 0.$$
If $E[-1]$ is an Auslander bundle, say $E[-1] = E_L$, then there exists an Auslander-Reiten sequence in $\text{vect} \mathcal{X}$ as follows:

$$0 \to E[-1](\mathcal{O}) \to F'' \bigoplus L(\mathcal{O}) \to E[-1] \to 0,$$

where $F''$ is the unique indecomposable object with rank three and socle $L$. Hence $F' = F'' \bigoplus L(\mathcal{O})$ and $\text{Ker}(\delta') = L(\mathcal{O})$. Then it follows that $PE \bigoplus PE(\mathcal{O}) = PF \bigoplus L(\mathcal{O})$.

If else, $F'$ has no direct summands from line bundles. But $\text{Ker}(\delta') = \text{Ker}(\delta)$, which is a direct sum of line bundles. It follows that $\text{Ker}(\delta') = 0$.

Hence, $PE \bigoplus PE(\mathcal{O}) = PF$. □

By duality, we have analogous results of injective hull $IE$ for $E$ as follows:

**Theorem 3.5** For each line bundle $L$,

1. If $E = E_L$, then $IE = L \bigoplus (\bigoplus_{i=1}^{4} L(\mathcal{O} + \mathcal{O}_i))$;

2. If $E = E_L(\mathcal{O}_i)$, then $IE = L(\mathcal{O}_i) \bigoplus (\bigoplus_{j=1, j \neq i}^{4} L(\mathcal{O} + \mathcal{O}_j))$.

**Proposition 3.6** Let $0 \to E(\mathcal{O}) \to F \to E \to 0$ be an Auslander-Reiten sequence in $\text{vect} \mathcal{X}$ with $E$ indecomposable of rank greater than or equal to two.

1. If $E[-1]$ is an Auslander bundle, i.e., $E[-1] = E_L$ for some line bundle $L$, then $IE \bigoplus IE(\mathcal{O}) = IF \bigoplus L$.

2. If else, $IE \bigoplus IE(\mathcal{O}) = IF$.

**4 Slopes under shift**

To show an object is a tilting object, it is necessary to discuss the change of slope of a vector bundle under shift. In this section, we will give a formula to compute the slope of a vector bundle under shift.

**Lemma 4.1** Let $E$ be an indecomposable vector bundle with $\text{rk}(E) \geq 2$,

$$0 \to E(\mathcal{O}) \to F_1 \bigoplus F_2 \to E \to 0$$

be an Auslander-Reiten sequence with $F_1$ indecomposable and $\text{rk}(F_1) \geq 2$, then $\mu(E[-1]) = \mu(F_1[-1])$. 

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Proof: Since $\text{Hom}(E(-\omega), F_1) \neq 0$, and since the shift functor is an equivalence in $\text{vect} X$, we obtain that $\text{Hom}(E(-\omega)[-1], F_1[-1]) \neq 0$. Hence, $\mu(E(-\omega)[-1]) \leq \mu(F_1[-1])$. Similarly, we can obtain $\mu(F_1[-1]) \leq \mu(E[-1])$. Moreover, $E(-\omega)[-1] = E[-1](\omega)$ induces $\mu(E[-1]) = \mu(F_1[-1])$. So $\mu(E[-1]) = \mu(F_1[-1])$. \(\square\)

Remark: By Lemma 4.1, we know that two indecomposable objects $E, F \in \text{vect} X$ with rank $\geq 2$ are in the same tube in the Auslander-Reiten quiver of $\text{vect} X$ if and only if $E[-1], F[-1]$ are so.

Lemma 4.2 For each vector bundle $F$ and $x \in L$ with $\delta(x) = 0$,
(1) if $\mu(F) > 0$, then $\dim \text{Hom}(E_\omega(F), F) = \deg(F)$;
(2) if $\mu(F) < 0$, then $\dim \text{Hom}(F, E_\omega(F)) = -\deg(F)$.

Proof: (1) If $\mu(F) > 0$, applying $\text{Hom}(-, F)$ to $\eta: 0 \to O(\omega + x) \to E_\omega(F) \to O(x) \to 0$
and by the Riemann-Roch formula, we obtain
$$\dim \text{Hom}(E_\omega(F), F) = \dim \text{Hom}(O(\omega + x) \bigoplus O(x), F)$$
$$= \langle [O(\omega + x)] \bigoplus [O(x)], [F] \rangle$$
$$= \text{rk}(O(x))\deg(F) - \deg(O(x))\text{rk}(F)$$
$$= \deg(F).$$

(2) If $\mu(F) < 0$, applying $\text{Hom}(F, -)$ to $\eta$ we obtain
$$\dim \text{Hom}(F, E_\omega(F)) = -\deg(F).$$
\(\square\)

Now we can compute the slope of $E[-1]$ where $E$ is a vector bundle of rank $\geq 2$ and slope 0 or $\frac{1}{2}$.

Lemma 4.3 For each indecomposable object $E \in \text{vect} X$ with $\text{rk}(E) \geq 2$,
(1) if $\mu(E) = 0$, then $\mu(E[-1]) = -\frac{4}{3}$;
(2) if $\mu(E) = \frac{1}{2}$, then $\mu(E[-1]) = -\frac{1}{2}$.

Proof: (1) By Lemma 4.1, we can reduce to the case $\text{rk}(E) = 2$.
If $E(\omega) \neq E$, then according to the structure of the Auslander-Reiten quiver of $\text{vect} X$, $E$ is an Auslander bundle, i.e., there exists a line bundle
$L$ with $\mu(L) = 0$ such that $E = E_L$. Then $PE = L(\overrightarrow{\mathcal{J}}) \bigoplus (\bigoplus_{i=1}^{4} L(-\overrightarrow{x_i}))$, which implies $\deg(PE) = -4$ and $\text{rk}(PE) = 5$. From the distinguished exact sequence

$$0 \rightarrow E[-1] \rightarrow PE \rightarrow E \rightarrow 0,$$

we obtain $\deg(E[-1]) = -4$ and $\text{rk}(E[-1]) = 3$. Hence $\mu(E[-1]) = -\frac{3}{2}$.

If $E(\overrightarrow{\mathcal{J}}) = E$, then $E$ is a quasi-simple object in some homogeneous tube with slope zero. For the line bundle $L = \mathcal{O}(-\overrightarrow{x_1})$, each direct summand of $IE_L(\overrightarrow{x_i})$ is with slope zero. Hence for $1 \leq i \leq 4$, we have $\text{Hom}(IE_L(\overrightarrow{x_i}), E) = 0$, because there are no non-zero morphisms between different tubes with the same slope. By the Riemann-Roch formula, we obtain that

$$\dim \text{Hom}(E_L(\overrightarrow{x_i}) \bigoplus E_L(\overrightarrow{x_i})(\overrightarrow{\mathcal{J}}), E)$$

$$= \langle [E_L(\overrightarrow{x_i})] \bigoplus [E_L(\overrightarrow{x_i})(\overrightarrow{\mathcal{J}})], [E] \rangle$$

$$= \text{rk}(E_L(\overrightarrow{x_i})) \deg(E) - \deg(E_L(\overrightarrow{x_i})) \text{rk}(E)$$

$$= 2.$$ 

Then it follows that

$$\dim \text{Hom}(E_L(\overrightarrow{x_i}), E) = \dim \text{Hom}(E_L(\overrightarrow{x_i})(\overrightarrow{\mathcal{J}}), E) = 1.$$ 

Hence, we obtain that $\dim \overline{\text{Hom}}(E_L(\overrightarrow{x_i}), E) = 1$, which implies

$$\mu(E[-1]) \geq \mu(E_L(\overrightarrow{x_i})[-1]) = -\frac{3}{2}.$$ 

So each direct summand of $PE$ is with slope $-1$, i.e., $PE \in \mathcal{H}^{-1}$. Moreover, for each $\overrightarrow{x} \in L$ with $\delta(\overrightarrow{x}) = -1$, we have

$$\dim \text{Hom}(\mathcal{O}(\overrightarrow{x}) \bigoplus \mathcal{O}(\overrightarrow{x} + \overrightarrow{\mathcal{J}}), E) = \deg E(-\overrightarrow{x}) = 2.$$ 

It follows that

$$\dim \text{Hom}(\mathcal{O}(\overrightarrow{x}), E) = \dim \text{Hom}(\mathcal{O}(\overrightarrow{x} + \overrightarrow{\mathcal{J}}), E) = 1.$$
Hence, \( PE = \bigoplus_{\delta(\vec{x}) = -1} \mathcal{O}(\vec{x}) \). So \( \deg(PE) = -8 \), and \( \text{rk}(PE) = 8 \). From the distinguished exact sequence

\[
0 \to E[-1] \to PE \to E \to 0,
\]

we obtain \( \deg(E[-1]) = -8 \) and \( \text{rk}(E[-1]) = 6 \). Therefore, \( \mu(E[-1]) = -\frac{4}{3} \).

(2) According to the Lemma 4.1, we can reduce to the case that \( E \) is a quasi-simple object in \( \text{vect} X \).

Indeed, if \( E(\vec{\omega}) \neq E \) then \( \text{rk}(E) = 2 \), and \( E = E_{\mathbb{L}}(\vec{\omega}) \) for some line bundle \( \mathbb{L} \) with \( \mu(\mathbb{L}) = 0 \) and \( 1 \leq i \leq 4 \). So \( PE = L(\vec{\omega}) \bigoplus \bigoplus_{j=1,j \neq i} L(\vec{x}_j) \).

From the distinguished exact sequence

\[
0 \to E[-1] \to PE \to E \to 0,
\]

we obtain \( \deg(E[-1]) = -1 \) and \( \text{rk}(E[-1]) = 2 \), hence \( \mu(E[-1]) = -\frac{1}{2} \).

If \( E(\vec{\omega}) = E \), then \( \deg(E) = 2 \) and \( \text{rk}(E) = 4 \). By the Riemann-Roch formula and noticing that \( \mu(E_{\mathcal{O}(-\vec{x}_1)}[1]) = \frac{1}{3} \), we obtain

\[
\dim \text{Hom}(E_{\mathcal{O}(-\vec{x}_1)}[1], E) = \dim \text{Hom}(E_{\mathcal{O}(-\vec{x}_1)}[1], E) = 1,
\]

which implies that

\[
\mu(E[-1]) \geq \mu(E_{\mathcal{O}(-\vec{x}_1)}) = -1.
\]

Moreover, \( E(\vec{\omega})[-1] = E[-1] \) implies \( \mu(E[-1]) \neq -1 \). Hence \( PE \in \mathcal{H}^0 \).

Furthermore, for each \( \vec{x} \in \mathbb{L} \) with \( \delta(\vec{x}) = 0 \), by arguments as above, we obtain \( \dim \text{Hom}(\mathcal{O}(\vec{x}), E) = 1 \). So we get \( PE = \bigoplus_{\delta(\vec{x}) = 0} \mathcal{O}(\vec{x}) \). Therefore, \( \deg(E[-1]) = -2 \) and \( \text{rk}(E[-1]) = 4 \). It follows that \( \mu(E[-1]) = -\frac{1}{2} \). \( \square \)

**Theorem 4.4** Let \( E, F \) be two non-isomorphism indecomposable objects in \( \text{vect} X \) with rank \( \geq 2 \), then \( \mu(E) = \mu(F) \) if and only if \( \mu(E[-1]) = \mu(F[-1]) \).

**Proof** : If \( \mu(E) = \mu(F) = n \), then \( \mu(E[-1]) = \mu(E(-n \vec{x}_1)[1]) + \delta(n \vec{x}_1) = n + \mu(E(-n \vec{x}_1)[-1]) = n - \frac{4}{3} = \mu(F[-1]). \)
If $\mu(E) = \mu(F) = n + \frac{q}{p}$ with $p > q$, according to Lemma 4.1, we can reduce to the following two cases:

Case 1, $\text{rk}(E) = \text{rk}(F) = p$. Notice that $E, E(\mathcal{V})$, and $E(\mathcal{V}_i - \mathcal{V}_j)(1 \leq i < j \leq 4)$ are all non-isomorphism vector bundles with rank $p$ and slope $n + \frac{q}{p}$, so there exists an element $\mathcal{V} \in \mathbb{L}$ satisfying $\delta(\mathcal{V}) = 0$, such that $F = E(\mathcal{V})$. Hence $\mu(F[-1]) = \mu(E(\mathcal{V})[-1]) = \mu(E[-1]) + \delta(\mathcal{V}) = \mu(E[-1])$.

Moreover, since $\mu$ is quasi-simple in some homogeneous tube, that is, deg $(E) = 2q$, $\text{rk } (E) = 2p$, and $E = E(\mathcal{V})$.

Proposition 4.5 For each indecomposable vector bundle $E$ with $\text{rk}(E) \geq 2$, assume $\mu(E) = n + \frac{q}{p}$, where $p, q, n \in \mathbb{Z}$, $0 \leq \frac{q}{p} < 1$ and $(p, q) = 1$.

(1) If $0 \leq \frac{q}{p} \leq \frac{1}{3}$, then $\mu(E[-1]) = n - \frac{4p - 11q}{3p - 8q}$.

(2) If $\frac{1}{3} < \frac{q}{p} < 1$, then $\mu(E[-1]) = n + \frac{q}{p - 4q}$.

Proof: By assumption, we have $\mu(E(-n \mathcal{V}_1)[-1]) = \mu(E[-1](-n \mathcal{V}_1)) = \mu(E[-1]) - n\delta(\mathcal{V}_1) = \mu(E[-1]) - n$, so $\mu(E[-1]) = n + \mu(E(-n \mathcal{V}_1)[-1])$. Moreover, $\mu(E(-n \mathcal{V}_1)) = \mu(E) - n\delta(\mathcal{V}_1) = \mu(E) - n = \frac{q}{p}$. Hence, we only need to show the result for $n = 0$. Moreover, according to Theorem 4.4, we can assume that $E$ is a quasi-simple object in some homogeneous tube, that is, deg $(E) = 2q$, $\text{rk } (E) = 2p$, and $E = E(\mathcal{V})$.

(1) Since $-\frac{4}{3} \leq \mu(E[-1]) \leq -1$, we have $PE = I(E[-1]) \in \mathcal{H}^0 \vee \mathcal{H}^{-1}$.

Suppose $\text{deg}(E[-1]) = -2d$ and $\text{rk}(E[-1]) = 2r$. For each $\mathcal{V} \in \mathbb{L}$ satisfying
\[\delta(\vec{x}) = 0,\] we have
\[
\dim \text{Hom}(O(\vec{x}) \bigoplus O(\vec{x} + \vec{w}), E) = \dim \text{Hom}(E_{O(\vec{x})}, E) = \deg(E) = 2q.
\]
It follows that
\[
\dim \text{Hom}(O(\vec{x}), E) = \dim \text{Hom}(O(\vec{x} + \vec{w}), E) = q.
\]
And for each \(\vec{y} \in L\) satisfying \(\delta(\vec{y}) = -1\), we have
\[
\dim \text{Hom}(E[-1], O(\vec{y}) \bigoplus O(\vec{y} + \vec{w})) = -\deg(E[-1]) - \text{rk}(E[-1]) = 2d - 2r,
\]
which implies
\[
\dim \text{Hom}(E[-1], O(\vec{y})) = \dim \text{Hom}(E[-1], O(\vec{y} + \vec{w})) = d - r.
\]
Hence,
\[
PE = I(E[-1]) = (\bigoplus_{\delta(\vec{x})=0} O(\vec{x})^q) \bigoplus (\bigoplus_{\delta(\vec{y})=-1} O(\vec{y})^{d-r}).
\]
It follows that \(\deg(PE) = -8(d - r)\) and \(\text{rk}(PE) = 8(d - r) + 8q\). On the other hand, from the exact sequence
\[
0 \longrightarrow E[-1] \longrightarrow PE \longrightarrow E \longrightarrow 0,
\]
we have \(\deg(PE) = \deg(E[-1]) + \deg(E) = 2q - 2d\) and \(\text{rk}(PE) = \text{rk}(E[-1]) + \text{rk}(E) = 2p + 2r\). Thus we obtain \(d = 4p - 11q, r = 3p - 8q\). Hence, \(\mu(E[-1]) = -\frac{4p-11q}{3p-8q}\).

(2) Since \(-1 < \mu(E[-1]) < -\frac{1}{5}\), we have \(PE \in \mathcal{H}^0\). For each \(\vec{x} \in L\) satisfying \(\delta(\vec{x}) = 0\), we have
\[
\dim \text{Hom}(O(\vec{x}) \bigoplus O(\vec{x} + \vec{w}), E) = \deg(E) = 2q.
\]
It follows that
\[
\dim \text{Hom}(O(\vec{x}), E) = \dim \text{Hom}(O(\vec{x} + \vec{w}), E) = q.
\]
Hence,
\[ PE = \bigoplus_{\delta(\overline{x})=0} \mathcal{O}(\overline{x})^q. \]
So \( \deg(PE) = 0 \) and \( \text{rk}(PE) = 8q \). Then from the exact sequence
\[ 0 \rightarrow E[-1] \rightarrow PE \rightarrow E \rightarrow 0, \]
we get \( \deg(E[-1]) = -2q \) and \( \text{rk}(E[-1]) = 8q - 2p \), which implies that \( \mu(E[-1]) = \frac{q}{p-4q} \).

Dually we obtain the following result which is similar to the previous one:

**Proposition 4.6** For each indecomposable vector bundle \( E \) with \( \text{rk}(E) \geq 2 \), assume \( \mu(E) = n + \frac{q}{p} \), where \( p, q, n \in \mathbb{Z} \), \( 0 \leq \frac{q}{p} < 1 \) and \( (p, q) = 1 \).

1. If \( 0 \leq \frac{q}{p} \leq \frac{2}{3} \), then \( \mu(E[1]) = n + \frac{4p-5q}{3p-4q} \).
2. If \( \frac{2}{3} < \frac{q}{p} < 1 \), then \( \mu(E[1]) = n + \frac{12p-19q}{5p-8q} \).

**Remark:** If \( E \) is an indecomposable vector bundle of rank two and \( E = E(\overline{x}) \), then there exists a line bundle \( L \) and a non-split exact sequence
\[ 0 \rightarrow L(\overline{x}) \rightarrow E \rightarrow L(\overline{c}) \rightarrow 0. \]
By the proof of Proposition 4.5 and 4.6, we can show that \( PE = \bigoplus_{\delta(\overline{x})=0} L(\overline{x}) \) and \( IE = \bigoplus_{\delta(\overline{x})=2} L(\overline{x}). \)

**Corollary 4.7** For each indecomposable object \( E \in \text{vect} \mathcal{X} \) with \( \text{rk}(E) \geq 2 \), if \( \mu(E) = n \in \mathbb{Z} \), then for each \( m \in \mathbb{N} \),
1. \( \mu(E[m]) = n + m + \frac{m}{2m+1} \);
2. \( \mu(E[-m]) = 2n - \mu(E[m]) = n - m - \frac{m}{2m+1} \).

**Proof:** (1)We prove the result by induction. For \( m = 1 \), \( \mu(E[1]) = n + \frac{4}{3} = n + 1 + \frac{1}{3} \). Suppose for \( m = k \), \( \mu(E[k]) = n + k + \frac{k}{2k+1} \). Then for \( m = k + 1 \), notice \( 0 \leq \frac{k}{2k+1} \leq \frac{2}{3} \), so \( \mu(E[k + 1]) = \mu(E[k][1]) = n + k + \frac{4(2k+1) - 5k}{3(2k+1) - 4k} = n + k + 1 + \frac{k+1}{2(k+1)+1} \). Therefore, for each \( m \in \mathbb{N} \), we have \( \mu(E[m]) = n + m + \frac{m}{2m+1} \).
(2) Analogously. \( \square \)

## 5 Exceptional objects

Direct summands of tilting objects are exceptional objects. This section is due to discuss exceptional objects in \( \text{vect} \mathcal{X} \). Theorem 5.5 shows all
exceptional objects in vectX.

**Lemma 5.1** ([6], Proposition 2.7) If X, Y are both exceptional in cohX, and [X] = [Y] in $K_0(X)$, then $X = Y$.

**Proposition 5.2** For each line bundle $L$, the vector bundle $E_L(\overrightarrow{x_i})$ is exceptional in cohX, and $E_L(\overrightarrow{x_i})[1] = E_L(\overrightarrow{x_i})(\overrightarrow{x_j})$, for each $j \neq i$.

**Proof:** For simplification, we write $E_L(\overrightarrow{x_i})$ by $E$ throughout the proof.

Applying $\text{Hom}(L(\overrightarrow{\omega}), -)$ to the exact sequence $\eta_i : 0 \rightarrow L(\overrightarrow{\omega}) \rightarrow E \rightarrow L(\overrightarrow{x_i}) \rightarrow 0$, we obtain

$$\text{Hom}(L(\overrightarrow{\omega}), E) = k$$

and

$$\text{Ext}^1(L(\overrightarrow{\omega}), E) = 0.$$

Similarly, applying $\text{Hom}(L(\overrightarrow{x_i}), -)$ to $\eta_i$ we obtain

$$\text{Hom}(L(\overrightarrow{x_i}), E) = \text{Ext}^1(L(\overrightarrow{x_i}), E) = 0.$$

Then applying $\text{Hom}(-, E)$ to $\eta_i$ we obtain a long exact sequence:

$$0 \rightarrow \text{Hom}(L(\overrightarrow{x_i}), E) \rightarrow \text{Hom}(E, E) \rightarrow \text{Hom}(L(\overrightarrow{\omega}), E)$$

$$\rightarrow \text{Ext}^1(L(\overrightarrow{x_i}), E) \rightarrow \text{Ext}^1(E, E) \rightarrow \text{Ext}^1(L(\overrightarrow{\omega}), E) \rightarrow 0.$$

It follows that

$$\text{Hom}(E, E) = \text{Hom}(L(\overrightarrow{\omega}), E) = k$$

and $\text{Ext}^1(E, E) = 0$.

Hence, $E$ is exceptional in cohX.

As a result, we only need to prove $E[1]$ and $E(\overrightarrow{x_j}) (j \neq i)$ have the same class in $K_0(X)$ since they are both exceptional in cohX.

For any $j \notin I \subseteq \{1, 2, 3, 4\}$, we have the following two exact sequences

$$0 \rightarrow L(\overrightarrow{\omega}) \rightarrow L(\overrightarrow{\omega} + \overrightarrow{x_j}) \rightarrow S_j \rightarrow 0$$

and

$$0 \rightarrow L(\overrightarrow{\omega} + \sum_{i \in I} \overrightarrow{x_i}) \rightarrow L(\overrightarrow{\omega} + \overrightarrow{x_j} + \sum_{i \in I} \overrightarrow{x_i}) \rightarrow S_j(\sum_{i \in I} \overrightarrow{x_i}) \rightarrow 0,$$
where $S_j$ denotes the unique simple sheaf concentrated in the point $x_j$ with $\text{Hom}(L, S_j) = k$. By noticing that $S_j(\sum_{i \in I} \mathcal{F}_i) = S_j$ since $j \notin I$, we obtain

$$\left[ L(\mathcal{O} + \sum_{i \in I} \mathcal{F}_i) \right] + \left[ L(\mathcal{O} + \mathcal{F}_j) \right] = \left[ L(\mathcal{O} + \mathcal{F}_j + \sum_{i \in I} \mathcal{F}_i) \right] + \left[ L(\mathcal{O}) \right].$$

Then from the exact sequence

$$0 \longrightarrow E \overset{\alpha}{\longrightarrow} IE \overset{\beta}{\longrightarrow} E[1] \longrightarrow 0,$$

where $IE = L(\mathcal{F}_i) \bigoplus (\bigoplus_{j \neq i} L(\mathcal{O} + \mathcal{F}_j))$, we obtain that $[E[1]] = [IE] - [E] = \sum_{j \neq i} [L(\mathcal{O} + \mathcal{F}_j)] - [L(\mathcal{O})] = [L(\mathcal{O} + \sum_{j \neq i} \mathcal{F}_j)] + [L(\mathcal{O})] = [L(\mathcal{O} + \mathcal{F}_j)] + [L(\mathcal{O})]$, and $[E(\mathcal{F}_j)] = [L(\mathcal{O} + \mathcal{F}_j)] + [L(\mathcal{O} + \mathcal{F}_j)]$. Now considering the following two exact sequences in $\text{coh}\mathcal{X}$:

$$0 \longrightarrow L(\mathcal{O}) \longrightarrow L(\mathcal{O} + \mathcal{F}_j) \longrightarrow S_j \longrightarrow 0$$

and

$$0 \longrightarrow L(\mathcal{F}_i + \mathcal{F}_j) \longrightarrow L(\mathcal{F}_i + \mathcal{F}_j) \longrightarrow S' \longrightarrow 0,$$

we obtain $S' = S_j(\mathcal{F}_i + \mathcal{F}_j - \mathcal{O}) = S_j$ for each $j \neq i$, which implies $[E[1]] = [E(\mathcal{F}_j)]$. □

**Remark** The proof of the first statement in Proposition 5.2 is an instance of mutations for an exceptional pair, compare [2].

**Corollary 5.3** For each line bundle $L$, we have

$$E_L(\mathcal{F}_i)[n] = E_L(\mathcal{F}_i)(\sum_{j=1, j \neq i}^{4} k_j \mathcal{F}_j),$$

where $k_j \in \mathbb{Z}$ satisfying $\sum_{j=1, j \neq i}^{4} k_j = n$. Then $\mu(E_L(\mathcal{F}_i)[n]) = n + \mu(E_L(\mathcal{F}_i))$.

**Proof:** We prove the result by induction.

For $n = 1$, there are two possibilities for $\sum_{j=1, j \neq i}^{4} k_j = 1$. In the first case, there exists $j \in \{1, 2, 3, 4\} \setminus \{i\}$ such that $\sum_{j=1, j \neq i}^{4} k_j \mathcal{F}_j = \mathcal{F}_j$, we have
\[ [E_L(\langle x_i \rangle)_1] = [L(\langle x_i + c \rangle)] + [L(\langle x \rangle)] = [E_L(\langle x_i \rangle)(\langle x_j \rangle)] \] from Proposition 5.2.

In the other case, \( \sum_{j=1, j \neq i}^4 k_j \langle x_j \rangle = \sum_{j \neq i} \langle x_j - c \rangle \), we have \( [E_L(\langle x_i \rangle)(\sum_{j \neq i} \langle x_j - c \rangle)] = [L(\langle x + \sum_{j \neq i} \langle x_j - c \rangle \rangle)] + [L(\langle x_i \rangle + \sum_{j \neq i} \langle x_j - c \rangle \rangle)] = [L(\langle x_i \rangle)] + [L(\langle x + c \rangle)] \).

Now considering the following two exact sequences in coh\( \mathcal{X} \):

\[
0 \rightarrow L(\langle x \rangle) \rightarrow L(\langle x + c \rangle) \rightarrow S(\langle x \rangle) \rightarrow 0
\]

and

\[
0 \rightarrow L(\langle x_i \rangle) \rightarrow L(\langle x_i + c \rangle) \rightarrow S(\langle x_i \rangle) \rightarrow 0,
\]

we obtain \( [S(\langle x \rangle)] = [S(\langle x_i \rangle)] = [S] \), which implies \( [E_L(\langle x_i \rangle)(\sum_{j \neq i} \langle x_j - c \rangle)] = [E_L(\langle x_i \rangle)[1]] \). Hence, \( E_L(\langle x_i \rangle)[1] = E_L(\langle x_i \rangle)(\sum_{j=1, j \neq i}^4 k_j \langle x_j \rangle) \).

Suppose for \( n = k \), the result holds. For \( n = k + 1 \), noticing that \( E_L(\langle x_i \rangle)(\langle x \rangle) = E_L(\langle x \rangle)(\langle x_i \rangle) \), we get

\[
E_L(\langle x_i \rangle)[k + 1] = E_L(\langle x_i \rangle)[k][1] = E_L(\langle x_i \rangle)(\sum_{j=1, j \neq i}^4 k_j \langle x_j \rangle),
\]

where \( k_j \in \mathbb{Z} \) satisfying \( \sum_{j=1, j \neq i}^4 k_j = k + 1 \). This finishes the proof. \( \square \)

**Lemma 5.4** For any indecomposable objects \( X, Y \in \text{vect}\mathcal{X} \), we have two exact sequences:

1. \( 0 \rightarrow \text{Hom}(X, Y[-1]) \rightarrow \text{Hom}(X, PY) \rightarrow \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Y) \rightarrow 0 \);
2. \( 0 \rightarrow \text{Hom}(X[1], Y) \rightarrow \text{Hom}(IX, Y) \rightarrow \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Y) \rightarrow 0 \).

**Proof:** Applying \( \text{Hom}(X, -) \) to the distinguished exact sequence

\[
0 \longrightarrow Y[-1] \overset{\alpha}{\longrightarrow} PY \overset{\beta}{\longrightarrow} Y \longrightarrow 0,
\]

we obtain an exact sequence:

\[
0 \longrightarrow \text{Hom}(X, Y[-1]) \overset{\alpha^*}{\longrightarrow} \text{Hom}(X, PY) \overset{\beta^*}{\longrightarrow} \text{Hom}(X, Y) \).
\]

For each \( \varphi \in \text{Hom}(X, Y) \), we have \( \overline{\varphi} = 0 \in \text{Hom}(X, Y) \) if and only if \( \varphi \) factors through a direct sum of line bundles and this is equivalent to
the fact that \( \varphi \) factors through \( PY \) since \( \beta : PY \twoheadrightarrow Y \) is distinguished surjective. So \( \text{Cok}(\beta) = \text{Hom}(X,Y)/\text{Im}\beta = \text{Hom}(X,Y) \). We obtain the exact sequence (1).

Similarly, applying \( \text{Hom}(-,Y) \) to the distinguished exact sequence

\[
0 \longrightarrow X \xrightarrow{\alpha} IX \xrightarrow{\beta} X[1] \longrightarrow 0,
\]

we obtain the exact sequence (2).

Now we describe all the exceptional objects in \( \text{vect} X \).

**Theorem 5.5** A vector bundle \( E \) is exceptional in \( \text{vect} X \) if and only if \( E \) is an Auslander bundles or a vector bundle with \( \text{rk}(E) = p \) and \( \mu(E) = \frac{q}{p} \notin \mathbb{Z}, (p,q) = 1 \).

**Proof:** Let \( E \) be an indecomposable object in \( \text{vect} X \). Then for any \( n \geq 2 \) or \( n \leq -1 \), we have \( \text{Hom}(E, E[n]) = D\text{Hom}(E[n-1], E(\overline{w})) = 0 \). So \( E \) is exceptional in \( \text{vect} X \) if and only if \( \text{Hom}(E, E(\overline{w})) = 0 \) and \( \text{Hom}(E, E) = k \).

Notice that if \( E(\overline{w}) = E \), then \( \text{Hom}(E, E(\overline{w})) \neq 0 \), and then \( E \) can not be exceptional. We only need to consider the following two cases:

Case 1: \( \mu(E) = \frac{q}{p} \notin \mathbb{Z}, (p,q) = 1 \). Suppose \( \text{deg}(E) = qr \) and \( \text{rk}(E) = pr \), then \( \text{Hom}(IE, E(\overline{w})) = 0 \). According to the knowledge of the hammocks, we have

\[ \dim \text{Hom}(E, E(\overline{w})) = \dim \text{Hom}(E, E(\overline{w})) = \left\lfloor \frac{r}{2} \right\rfloor, \]

here, \( \left\lfloor \frac{r}{2} \right\rfloor \) means the integral part of \( \frac{r}{2} \). So \( \dim \text{Hom}(E, E(\overline{w})) = 0 \) if and only if \( r = 1 \). In this case, \( \dim \text{Hom}(E, E) = \dim \text{Hom}(E, E) = \left\lfloor \frac{r+1}{2} \right\rfloor = 1 \). Hence \( E \) is exceptional. So if \( \mu(E) = \frac{q}{p} \notin \mathbb{Z}, (p,q) = 1 \), then \( E \) is exceptional if and only if \( \text{rk}E = p \).

Case 2: \( \mu(E) = n \). Suppose \( \text{rk}(E) = r \), we have \( \dim \text{Hom}(E, E) = \left\lfloor \frac{r+1}{2} \right\rfloor \) and \( \dim \text{Hom}(IE, E) = \frac{1+(-1)^{r+1}}{2} \), so \( \dim \text{Hom}(E, E) = \dim \text{Hom}(E, E) - \dim \text{Hom}(IE, E) = \left\lfloor \frac{r+1}{2} \right\rfloor - \frac{1+(-1)^{r+1}}{2} \). Then \( \dim \text{Hom}(E, E) = 1 \) if and only if \( r = 2 \). Hence \( \dim \text{Hom}(E, E(\overline{w})) = \left\lfloor \frac{r}{2} \right\rfloor = 1 \) and \( \dim \text{Hom}(IE, E(\overline{w})) = \frac{1+(-1)^{r}}{2} = 1 \). Then we get \( \dim \text{Hom}(E, E(\overline{w})) = \dim \text{Hom}(E, E(\overline{w})) - \dim \text{Hom}(IE, E(\overline{w})) = 0 \). Hence \( E \) is exceptional. Therefore, if \( \mu(E) = n \),
then $E$ is exceptional if and only if $E(\mathcal{W}) \neq E$ and $r = 2$, that is, $E$ is an Auslander bundle. \hfill \Box

6 Tilting objects

Recall that an object $T$ in a triangulated category $\mathcal{D}$ is called a tilting object if \(\text{Hom}_{\mathcal{D}}(T, T[n]) = 0\) for $n \neq 0$, and $T$ generates $\mathcal{D}$ as a triangulated category, that is, the smallest subcategory $\langle T \rangle$ of $\mathcal{D}$, closed under shift $[1]$ and $[-1]$, direct sums and direct summands, third terms of triangles, equal to $\mathcal{D}$.

As shown in [6], $T = \bigoplus_{0 \leq x \leq c} \mathcal{O}(\mathcal{W} + \mathcal{W}_1)$ is a tilting object in $D^b(\text{coh} X)$. Then under the auto-equivalence $\rho$ of $D^b(\text{coh} X)$ which acting on slopes $q$ by $q \mapsto \frac{q}{1+q}$, the image $\rho T$ is a tilting object in $D^b(\text{coh} X)$. Then $\rho T$ is also a tilting object in $\text{vect} X$ since all of the indecomposable direct summands of $\rho T$ are not line bundles.

Our motivation is to look for some tilting objects directly in $\text{vect} X$.

From now on, we are going to construct a tilting object in $\text{vect} X$, and always use the notation $E = E_\mathcal{O}, E_i = E_\mathcal{O}(\mathcal{W}_i)$. 

**Lemma 6.1** For any $n \in \mathbb{Z}$, we have

1. $\text{Hom}(E_i, E[n]) = 0$;
2. $\text{Hom}(E, E_i[n]) = \delta_{n,0} k$.

**Proof:**

(1) If $n \leq 0$, then $\mu(E[n]) \leq \mu(E) = 0 < \frac{1}{2} = \mu(E_i)$ implies $\text{Hom}(E_i, E[n]) = 0$, so $\text{Hom}(E_i, E[n]) = 0$. If $n \geq 2$, then $\mu(E[n-1]) \geq \mu(E[1]) = \frac{2}{3} > \frac{1}{2} = \mu(E_i(\mathcal{W}))$, so $\text{Hom}(E[n-1], E_i(\mathcal{W})) = 0$. By Serre duality, $\text{Hom}(E_i, E[n]) = D\text{Hom}(E[n-1], E_i(\mathcal{W})) = 0$. If $n = 1$, then $\text{dim Hom}(E, E_i(\mathcal{W})) = \text{deg}(E_i(\mathcal{W})) = 1$, and $\text{dim Hom}(IE, E_i(\mathcal{W})) = \text{dim Hom}(\mathcal{O}, E_i(\mathcal{W})) = 1$, which implies $\text{Hom}(E, E_i(\mathcal{W})) = 0$. By Serre duality, we have $\text{Hom}(E_i, E[1]) = D\text{Hom}(E, E_i(\mathcal{W})) = 0$.

(2) For any $n \neq 0$, we obtain $\text{Hom}(E, E_i[n]) = 0$ as shown in (1). For $n = 0$, we have $\text{dim Hom}(E, E_i) = \text{deg}(E_i) = 1$ and $\text{dim Hom}(IE, E_i) = \text{dim Hom}(\mathcal{O}, E_i) = 0$, so $\text{Hom}(E, E_i) = 1$. \hfill \Box
Applying the functor \( \text{Hom}(-, \mathcal{O}(\mathcal{J})) \) to the Auslander-Reiten sequence

\[
\xi : 0 \rightarrow \mathcal{O}(\mathcal{J}) \rightarrow E(\mathcal{J} + \mathcal{I}) \rightarrow \mathcal{O}(\mathcal{J} + \mathcal{I}) \rightarrow 0,
\]

we obtain an exact sequence

\[
0 \rightarrow \text{Ext}^1(\mathcal{O}(\mathcal{J} + \mathcal{I}), \mathcal{O}(\mathcal{J})) \rightarrow \text{Ext}^1(E(\mathcal{J} + \mathcal{I}), \mathcal{O}(\mathcal{J})) \rightarrow \text{Ext}^1(\mathcal{O}(\mathcal{J}), \mathcal{O}(\mathcal{J})) \rightarrow 0.
\]

Then \( \text{Ext}^1(\mathcal{O}(\mathcal{J} + \mathcal{I}), \mathcal{O}(\mathcal{J})) = S_{\mathcal{J} + \mathcal{I}} = 0 \) and \( \text{Ext}^1(\mathcal{O}(\mathcal{J}), \mathcal{O}(\mathcal{J})) = S_{\mathcal{J}} = k \), which imply \( \text{Ext}^1(E(\mathcal{J} + \mathcal{I}), \mathcal{O}(\mathcal{J})) = k \). Hence, there is a vector bundle \( F \) fitting into the following exact sequence

\[
\zeta : 0 \rightarrow \mathcal{O} \rightarrow F \rightarrow E(\mathcal{J} + \mathcal{I}) \rightarrow 0.
\]

It is easy to see that \( \text{deg}(F) = 2 \), and \( \text{rk}(F) = 3 \). Moreover, \( F \) is indecomposable since there is no line bundle \( L \) satisfying \( \text{Hom}(\mathcal{O}(\mathcal{J}), L) \neq 0 \) and \( \text{Hom}(L, E(\mathcal{J} + \mathcal{I})) \neq 0 \).

**Theorem 6.2** Let \( T = E \bigoplus \bigoplus_{i=1}^{4} E_i \bigoplus F \), then

1. \( T \) is a tilting object in vect\( \mathcal{X} \);
2. \( \text{End}(T) \) is a canonical algebra of type (2,2,2).

**Proof:** (1) First we show that \( \text{Hom}(T, T[n]) = 0 \) for any \( n \neq 0 \) and that \( E, E_1, E_2, E_3, E_4, F \) forms an exceptional sequence.

By comparing slopes and using the Serre duality, we obtain:

(i) For \( n \neq 0 \), \( \text{Hom}(E_i, F[n]) = D\text{Hom}(F[n - 1], E_i(\mathcal{J})) = 0 \) and \( \text{Hom}(E, F[n]) = D\text{Hom}(F[n - 1], E(\mathcal{J})) = 0 \).

(ii) For \( n \neq 1 \), \( \text{Hom}(F, E_i[n]) = D\text{Hom}(E_i[n - 1], F(\mathcal{J})) = 0 \) and \( \text{Hom}(F, E[n]) = D\text{Hom}(E[n - 1], F(\mathcal{J})) = 0 \). According to Lemma 6.1, in order to prove \( T \) is a tilting object in vect\( \mathcal{X} \), we only need to show that \( \text{Hom}(F, E_i[1]) = 0 \) for each \( i = 1, 2, 3, 4 \) and \( \text{Hom}(F, E[1]) = 0 \).

Applying \( \text{Hom}(E_i, -) \) to the exact sequence

\[
\zeta(\mathcal{J}) : 0 \rightarrow \mathcal{O} \rightarrow F(\mathcal{J}) \rightarrow E(\mathcal{J}) \rightarrow 0,
\]
we obtain an exact sequence:

\[ 0 \longrightarrow \text{Hom}(E_i, F(\mathcal{W})) \longrightarrow \text{Hom}(E_i, E(\mathcal{X}_j)) \longrightarrow \text{Ext}^1(E_i, \mathcal{O}) \longrightarrow 0. \]

Then \( \text{Hom}(E_i, F(\mathcal{W})) = 0 \) provided by

\[ \dim \text{Ext}^1(E_i, \mathcal{O}) = \dim \text{DHom}(\mathcal{O}(\mathcal{W}), E_i) = 1 \]

and

\[ \dim \text{Hom}(E_i, E(\mathcal{X}_j)) = \dim \text{Hom}(E_i(-\mathcal{X}_j), E) = -\deg(E_i(-\mathcal{X}_j)) = 1. \]

Therefore, \( \text{Hom}(F, E_i[1]) = \text{DHom}(E_i, F(\mathcal{W})) = 0 \) for each \( i = 1, 2, 3, 4. \)

Applying \( \text{Hom}(\mathcal{O}, -) \) to the exact sequence \( \zeta \) and \( \xi \), we obtain two exact sequences:

\[ 0 \longrightarrow \text{Hom}(\mathcal{O}, F) \longrightarrow \text{Hom}(\mathcal{O}, E(\mathcal{W} + \mathcal{X}_j)) \longrightarrow \text{Ext}^1(\mathcal{O}, \mathcal{O}(\mathcal{W})) \longrightarrow 0, \]

and

\[ 0 \rightarrow \text{Hom}(\mathcal{O}(\mathcal{W}), F) \rightarrow \text{Hom}(\mathcal{O}(\mathcal{W} + \mathcal{X}_j)) \rightarrow \text{Hom}(\mathcal{O}(\mathcal{W} + \mathcal{X}_j)) \rightarrow 0. \]

So it is easy to show that \( \dim \text{Hom}(\mathcal{O}, F) = 0 \). Then it follows that \( \dim \text{Hom}(\mathcal{O}(\mathcal{W}), F) = 2 \) since \( \dim \text{Hom}(\mathcal{O}(\mathcal{W}) \bigoplus \mathcal{O}(\mathcal{W}), F) = \deg(F) = 2. \)

Therefore,

\[ \dim \text{Hom}(IE, F(\mathcal{W})) = \dim \text{Hom}(\mathcal{O}(\mathcal{W}), F(\mathcal{W})) = \dim \text{Hom}(\mathcal{O}(\mathcal{W}), F) = 2. \]

By the fact that \( \dim \text{Hom}(E, F(\mathcal{W})) = \deg(F(\mathcal{W})) = 2 \), we have

\[ \text{Hom}(F, E[1]) = \text{DHom}(E, F(\mathcal{W})) = 0. \]

Next, we need to show \( T \) generates \( \text{vect} \mathcal{X} \). Since \( E, E_1, E_2, E_3, E_4, F \) is an exceptional sequence, it suffices to prove that for each indecomposable vector bundle \( X \) with \( \text{rk}(X) \geq 2 \), there exists some \( n \in \mathbb{Z} \), such that \( \text{Hom}(T, X[n]) \neq 0. \)

Indeed, if we fix an indecomposable vector bundle \( X \) with \( \text{rk}(X) \geq 2 \), then there exists some \( m \in \mathbb{Z} \) such that \( m \leq \mu(X) < m + 1 \). By Corollary 4.7, we
We have $-\frac{m}{2m+1} = \mu(E_{\mathcal{O}(m\mathcal{F})}) - m - \frac{m}{2m+1} = \mu(E_{\mathcal{O}(m\mathcal{F})}[m]) \leq \mu(X[-m]) < \\
\mu(E_{\mathcal{O}((m+1)\mathcal{F})}[m]) = \mu(E_{\mathcal{O}((m+1)\mathcal{F})}) - m - \frac{m}{2m+1} = \frac{m+1}{2m+1}$. Moreover, if $-\frac{m}{2m+1} \leq \mu(X[-m]) < 0$, then by Proposition 4.6, we have $\frac{m+1}{2m+1} \leq \mu(X[-m][1]) < \frac{4}{3}$. Hence, there exists a suitable integer $n_1 \in \mathbb{Z}$ such that $0 \leq \mu(X[n_1]) < \frac{4}{3}$. So we only need to show that for each indecomposable vector bundle $X$ with $\text{rk}(X) \geq 2$ and $0 \leq \mu(X) < \frac{4}{3}$, there exists some $n \in \mathbb{Z}$, such that $\text{Hom}(T,X[n]) \neq 0$.

(i) If $0 < \mu(X) < 1$, then

$$\text{Hom}(E,X) = \text{Hom}(\mathcal{O}(\mathcal{G})) \bigoplus \mathcal{O},X) - \text{Hom}(IE,X) = \text{Hom}(\mathcal{O}(\mathcal{G}),X).$$

If $\text{Hom}(\mathcal{O}(\mathcal{G}),X) \neq 0$, then $\text{Hom}(E,X) \neq 0$. If $\text{Hom}(\mathcal{O}(\mathcal{G}),X) = 0$, then $\text{Ext}^1(X,\mathcal{O}) = 0$. Applying $\text{Hom}(X,-)$ to the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow F(\mathcal{G}) \longrightarrow E(\mathcal{G}_j) \longrightarrow 0,$$

we have

$$0 \longrightarrow \text{Hom}(X,F(\mathcal{G})) \longrightarrow \text{Hom}(X,E(\mathcal{G}_j)) \longrightarrow \text{Ext}^1(X,\mathcal{O}) = 0.$$

So $\text{dim} \text{Hom}(X,F(\mathcal{G})) = \text{dim} \text{Hom}(X,E(\mathcal{G}_j)) = -\text{deg}(X(\mathcal{G}_j)) = \text{rk}(X) - \text{deg}(X) = \text{rk}(X)(1-\mu(X)) > 0$. Hence, $\text{Hom}(F,X[1]) = \text{DHom}(X,F(\mathcal{G})) = \text{DHom}(X,F(\mathcal{G})) \neq 0$.

(ii) If $\mu(X) = 1$, applying $\text{Hom}(-,X)$ to the exact sequence

$$0 \longrightarrow \mathcal{O}(\mathcal{G}) \longrightarrow F \longrightarrow E(\mathcal{G} + \mathcal{G}_j) \longrightarrow 0,$$

we have

$$0 \longrightarrow \text{Hom}(E(\mathcal{G} + \mathcal{G}_j),X) \longrightarrow \text{Hom}(F,X) \longrightarrow \text{Hom}(\mathcal{O}(\mathcal{G}),X) \longrightarrow \text{Ext}^1(E(\mathcal{G} + \mathcal{G}_j),X) \longrightarrow 0.$$

Notice that

$$\text{dim} \text{Hom}(E(\mathcal{G} + \mathcal{G}_j),X) - \text{dim} \text{Ext}^1(E(\mathcal{G} + \mathcal{G}_j),X)$$

$$= \langle [E(\mathcal{G} + \mathcal{G}_j)], [X] \rangle$$

$$= \langle [\mathcal{O}(\mathcal{G} + \mathcal{G}_j)] + [\mathcal{O}(\mathcal{G}_j)], [X] \rangle$$

$$= \text{rk}(\mathcal{O}(\mathcal{G}_j)) \text{deg}(X) - \text{deg}(\mathcal{O}(\mathcal{G}_j)) \text{rk}(X)$$

$$= \text{deg}(X) - \text{rk}(X) = 0.$$
We get \( \dim \text{Hom}(F, X) = \dim \text{Hom}(\mathcal{O}(\mathfrak{f}), X) \geq 1 \). Moreover, since

\[
IF = \mathcal{O}(\mathfrak{f}) \bigoplus \bigoplus_{i=1}^{4} \mathcal{O}(\mathfrak{f} + \mathfrak{x}_i),
\]

we get

\[
\dim \text{Hom}(IF, X) = \dim \text{Hom}(\bigoplus_{i=1}^{4} \mathcal{O}(\mathfrak{f} + \mathfrak{x}_i), X) \leq 1.
\]

Hence, \( \text{Hom}(F, X) = 0 \) if and only if \( X = E\mathcal{O}(\mathfrak{x}_i) \). In this case,

\[
\dim \text{Hom}(E_i, E\mathcal{O}(\mathfrak{x}_i)) = \dim \text{Hom}(E_i(-\mathfrak{x}_i), E) = -\deg E_i(-\mathfrak{x}_i) = 1
\]

and

\[
\text{Hom}(IE_i, E\mathcal{O}(\mathfrak{x}_i)) = 0
\]

imply that \( \text{Hom}(E_i, E\mathcal{O}(\mathfrak{x}_i)) \neq 0 \).

(iii) If \( \mu(X) = 0 \), then by Serre duality,

\[
\dim \text{Hom}(E_i, X[1]) = \dim D\text{Hom}(X, E_i(\mathfrak{f}))
\]

\[
= \dim D\text{Hom}(X, E_i(\mathfrak{f})) - \dim D\text{Hom}(X, PE_i(\mathfrak{f})).
\]

Applying \( \text{Hom}(X, -) \) to the exact sequence

\[
0 \rightarrow \mathcal{O} \rightarrow E_i(\mathfrak{f}) \rightarrow \mathcal{O}(\mathfrak{f} + \mathfrak{x}_i) \rightarrow 0,
\]

we obtain

\[
0 \rightarrow \text{Hom}(X, \mathcal{O}) \rightarrow \text{Hom}(X, E_i(\mathfrak{f})) \rightarrow \text{Hom}(X, \mathcal{O}(\mathfrak{f} + \mathfrak{x}_i))
\]

\[
\rightarrow \text{Ext}^1(X, \mathcal{O}) \rightarrow 0.
\]

Notice that \( P(E_i(\mathfrak{f})) = \mathcal{O} \bigoplus (\bigoplus_{j \neq i} \mathcal{O}(\mathfrak{f} + \mathfrak{x}_i - \mathfrak{x}_j)) \). We obtain that

\[
\dim \text{Hom}(X, E_i(\mathfrak{f}))
\]

\[
= \dim \text{Hom}(X, E_i(\mathfrak{f})) - \dim \text{Hom}(X, P(E_i(\mathfrak{f})))
\]

\[
= \dim \text{Hom}(X, \mathcal{O}(\mathfrak{f} + \mathfrak{x}_i)) - \sum_{j \neq i} \dim \text{Hom}(X, \mathcal{O}(\mathfrak{f} + \mathfrak{x}_i - \mathfrak{x}_j))
\]

\[
- \dim \text{Ext}^1(X, \mathcal{O}) \geq 0.
\]
Moreover, \( \text{Hom}(X,E_i(\overline{\omega})) = 0 \) if and only if \( X = E \) or \( X = E_{\mathcal{O}(\overline{\omega} + \overline{x}_i - \overline{x}_j)} \) for some \( j \neq i \). So we only need to consider the case \( X = E_{\mathcal{O}(\overline{\omega} + \overline{x}_i - \overline{x}_j)} \).

Since \( PF = \mathcal{O}(\overline{\omega})^2 \bigoplus_{i \neq j} \mathcal{O}(\overline{x}_i - \overline{x}_j) \) and \( \text{deg}(F(\overline{\omega})) = 2 \), we have

\[
\dim \text{Hom}(F,E_{\mathcal{O}(\overline{\omega} + \overline{x}_i - \overline{x}_j)}[1]) = \dim \text{DHom}(E_{\mathcal{O}(\overline{\omega} + \overline{x}_i - \overline{x}_j)}, F(\overline{\omega})) = 1.
\]

(iv) If \( 1 < \mu(X) < \frac{4}{3} \), then \( \text{Hom}(X, \mathcal{O}(\overline{\omega})) \neq 0 \). Notice that

\[
\text{Hom}(F,X) = \text{DHom}(X(\overline{\omega}), F[1]) = \text{DHom}(X(\overline{\omega}), E_{\mathcal{O}(\overline{\omega} + \overline{\omega})}).
\]

It follows that

\[
\dim \text{Hom}(F,X) = \dim \text{Hom}(X(\overline{\omega}), E_{\mathcal{O}(\overline{\omega} + \overline{\omega})}) - \dim \text{Hom}(X(\overline{\omega}), PE_{\mathcal{O}(\overline{\omega} + \overline{\omega})})
\]

\[
= \dim \text{Hom}(X(\overline{\omega}), \mathcal{O}(\overline{\omega} + \overline{\omega}) \bigoplus \mathcal{O}(\overline{\omega})) - \dim \text{Hom}(X(\overline{\omega}), \mathcal{O}(\overline{\omega}))
\]

\[
= \dim \text{Hom}(X(\overline{\omega}), \mathcal{O}(\overline{\omega} + \overline{\omega})) - \dim \text{Hom}(X, \mathcal{O}(\overline{\omega})) \neq 0.
\]

(2) Since \( \dim \text{Hom}(E,F) = \text{deg}(F) = 2 \) and

\[
\dim \text{Hom}(IE,F) = \dim \text{Hom}(\mathcal{O},F) = 0,
\]

we have \( \dim \text{Hom}(E,F) = 2 \). Furthermore, for each \( i = 1,2,3,4 \), applying \( \text{Hom}(\cdot,F) \) to the exact sequence

\[
\eta_i : 0 \rightarrow \mathcal{O}(\overline{\omega}) \rightarrow E_i \rightarrow \mathcal{O}(\overline{x}_i) \rightarrow 0,
\]

we obtain an exact sequence

\[
0 \rightarrow \text{Hom}(E_i,F) \rightarrow \text{Hom}(\mathcal{O}(\overline{\omega}), F) \rightarrow \text{Ext}^1(\mathcal{O}(\overline{x}_i), F) \rightarrow 0;
\]

and applying \( \text{Hom}(\mathcal{O}(\overline{x}_i),\cdot) \) to the exact sequence \( \zeta \), we obtain an exact sequence

\[
0 \rightarrow \text{Hom}(\mathcal{O}(\overline{x}_i), E(\overline{\omega} + \overline{x}_j)) \rightarrow \text{Ext}^1(\mathcal{O}(\overline{x}_i), \mathcal{O}(\overline{\omega}))
\]

\[
\rightarrow \text{Ext}^1(\mathcal{O}(\overline{x}_i), F) \rightarrow \text{Ext}^1(\mathcal{O}(\overline{x}_i), E(\overline{x}_j + \overline{\omega})) \rightarrow 0.
\]

Using that \( \text{Ext}^1(\mathcal{O}(\overline{x}_i), E(\overline{\omega} + \overline{x}_j)) = \text{DHom}(E(\overline{x}_j), \mathcal{O}(\overline{x}_i)) \), we can easily get that

\[
\dim \text{Hom}(\mathcal{O}(\overline{x}_i), E(\overline{\omega} + \overline{x}_j)) = \dim \text{Ext}^1(\mathcal{O}(\overline{x}_i), E(\overline{\omega} + \overline{x}_j)) = \delta_{i,j},
\]

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which implies that
\[ \dim \operatorname{Ext}^1(\mathcal{O}(h_i), F) = \dim \operatorname{Ext}^1(\mathcal{O}(\overline{x}_i), \mathcal{O}(\overline{\omega})) = \dim \mathcal{S}_{\overline{x}_i} = 1. \]

Hence, we can obtain \( \dim \operatorname{Hom}(E_i, F) = 1 \) since \( \dim \operatorname{Hom}(\mathcal{O}(\overline{\omega}), F) = 2 \).

Moreover, \( \dim \operatorname{Hom}(IE_i, F) = 0 \) implies \( \dim \operatorname{Hom}(E_i, F) = 1 \).

Now we describe generators and the relations. Assume \( 1 \leq i \leq 4 \) throughout the rest of the proof.

Applying \( \operatorname{Hom}(-, \mathcal{O}(\overline{\tau})) \) to the exact sequence
\[
0 \rightarrow \mathcal{O}(\overline{x}_i) \overset{\eta_i}{\rightarrow} E(\overline{\omega} + \overline{x}_i) \rightarrow \mathcal{O}(\overline{\omega} + \overline{x}_i) \rightarrow 0,
\]
then \( \eta_i \) induces an isomorphism \( \operatorname{Hom}(E(\overline{\omega} + \overline{x}_i), \mathcal{O}(\overline{\tau})) \cong \operatorname{Hom}(\mathcal{O}(\overline{x}_i), \mathcal{O}(\overline{\tau})) \) sending the generator \( \theta_i \) to \( x_i \), that is, the following diagram (I) commutes:

\[
\begin{array}{ccc}
\mathcal{O}(\overline{x}_i) & \overset{\eta_i}{\rightarrow} & E(\overline{\omega} + \overline{x}_i) \\
\downarrow{\scriptstyle x_i} & & \downarrow{\scriptstyle \theta_i} \\
\mathcal{O}(\overline{\tau}) & & \mathcal{O}(\overline{\tau})
\end{array}
\quad
\begin{array}{ccc}
F & \overset{\pi_i}{\rightarrow} & E(\overline{\omega} + \overline{x}_i) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \theta_i} \\
\mathcal{O}(\overline{\tau}) & & \mathcal{O}(\overline{\tau})
\end{array}
\]

(I) (II)

Analogously, applying \( \operatorname{Hom}(-, \mathcal{O}(\overline{\tau})) \) to the exact sequence
\[
0 \rightarrow \mathcal{O}(\overline{\omega}) \rightarrow F \overset{\pi_i}{\rightarrow} E(\overline{\omega} + \overline{x}_i) \rightarrow 0,
\]
then \( \pi_i \) induces an isomorphism \( \operatorname{Hom}(E(\overline{\omega} + \overline{x}_i), \mathcal{O}(\overline{\tau})) \cong \operatorname{Hom}(F, \mathcal{O}(\overline{\tau})) \) sending the generator \( \theta_i \) to \( \gamma \), that is, the above diagram (II) commutes.

Now from the following commutative diagram,
where $f_i$ and $g_i$ are obtained from pullbacks, we have

$$
\gamma(g_i f_i) = \theta_i \pi_i g_i f_i = (\theta_i \eta_i) x_i \pi = x_i^2 \pi.
$$

Hence, the fact that $\{x_i^2 \mid 1 \leq i \leq 4\}$ are pairwise linearly independent implies that $\{g_i f_i \mid 1 \leq i \leq 4\}$ are pairwise linearly independent since $\pi$ is surjective.

Next, we claim that $\theta_i : E(\overline{\omega} + \overline{x}_i) \to O(\overline{\tau})$ is surjective. Otherwise, $\operatorname{Im}(\theta_i) \subseteq O(\overline{\tau})$ is a line bundle satisfying $\operatorname{Hom}(E(\overline{\omega} + \overline{x}_i), \operatorname{Im}(\theta_i)) \neq 0$, which implies $\operatorname{Im}(\theta_i) = O(\overline{\omega} + \overline{x}_i)$. And it follows that $\operatorname{Ker}(\theta_i) = O(\overline{x}_i)$, which is a contradiction.

Moreover, assuming the generators of $\operatorname{Hom}(O(\overline{\omega}), F)$ are $h_1, h_2$, one checks easily that $\operatorname{Ker}(\gamma) = O(\overline{\omega})^2$, and there has the following exact sequence:

$$
0 \longrightarrow O(\overline{\omega})^2(\overline{h_1,h_2}) \longrightarrow F \xrightarrow{\gamma} O(\overline{\tau}) \longrightarrow 0.
$$

Then by applying $\operatorname{Hom}(E,-)$ to this exact sequence one obtains that $\gamma$ induces an monomorphism

$$
\gamma : \operatorname{Hom}(E, F) \longrightarrow \operatorname{Hom}(E, O(\overline{\tau})); \quad g_i f_i \longmapsto \gamma(g_i f_i).
$$

Hence the relations that

$$
x_3^2 = x_2^2 + x_1^2 \text{ and } x_4^2 = x_2^2 + \lambda x_1^2
$$

imply that

$$
g_3 f_3 = g_2 f_2 + g_1 f_1 \text{ and } g_4 f_4 = g_2 f_2 + \lambda g_1 f_1.
$$
Therefore, End\((T)\) is the canonical algebra of type \(2,2,2,2\) on generators \(\{g_if_i|1 \leq i \leq 4\}\) subject to the relations
\[g_3f_3 = g_2f_2 + g_1f_1 \text{ and } g_4f_4 = g_2f_2 + \lambda g_1f_1.\]
\[\square\]

Similarly, we can obtain that:

**Theorem 6.3** Let \(T' = F[-1]\langle \overrightarrow{J} \rangle \bigoplus E \bigoplus \bigoplus_{i=1}^{4} E_i\), then

1. \(T'\) is a tilting object in \(\text{vect} \overrightarrow{X}\);
2. \(\text{End}(T')\) is an algebra given by the following quiver with relations:

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2
\end{array}
\xrightarrow{\beta_i} \begin{array}{c}
\beta_i \\
\beta_3 \\
\beta_4
\end{array}
\]
\[\beta_i \alpha_1 = \beta_i \alpha_2, \quad \beta_i \alpha_1 = \beta_i \alpha_2, \quad i \in \{1, 2, 3, 4\};\]

**Remark:** For any two distinct indecomposable vector bundles \(E_1, E_2\) of rank two, we have \(\mu E_1, \mu E_2 \in \mathbb{Z} \bigcup \left(\frac{1}{2} \mathbb{Z} \right)\) and \(\dim \text{Hom}(E_1, E_2) \neq 0\) implies \(\mu E_1 \leq \mu E_2 \leq \mu(E_1[1])\). So one can easily check that \(\dim \text{Hom}(E_1, E_2) \leq 1\) case by case. Hence there doesn’t exist any tilting object only consisting of rank two bundles whose endomorphism algebra is a canonical algebra. In other words, our result Theorem 6.2 can not be much more simple.

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