EXISTENCE AND MULTIPlicity OF SOLUTIONS FOR A QUASILINEAR ELLIPTIC SYSTEM ON UNBOUNDED DOMAINS INVOLVING NONLINEAR BOUNDARY CONDITIONS

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\textbf{Abstract} We prove two existence results for the nonlinear elliptic boundary value system involving \( p \)-Laplacian over an unbounded domain in \( \mathbb{R}^N \) with noncompact boundary. The proofs are based on variational methods applied to weighted spaces.

\textbf{Keywords} Quasilinear elliptic systems, nonlinear boundary conditions, variational methods, weighted function spaces.

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1. Introduction and main results

The objective of this paper is to study the nonlinear elliptic boundary value system

\begin{equation}
\begin{aligned}
&-\text{div}(a(x)|\nabla u_i|^{p-2}\nabla u_i) = \lambda f(x)u_i|u_i|^{p-2} + F_{u_i}(x, u_1, \cdots, u_n), \quad x \in \Omega, \\
&b(x)u_i|u_i|^{p-2} = h(x, u_i), \quad x \in \partial \Omega,
\end{aligned}
\end{equation}

where \( \Omega \subseteq \mathbb{R}^N \) is an unbounded domain with noncompact smooth boundary \( \partial \Omega \), the outward unit normal to which is denoted by \( n \) with \( p > 1 \) and \( i = 1, \cdots, n \).

The growing attention for the study of the \( p \)-Laplacian operator in the last few decades is motivated by the fact that it arises in various applications. The \( p \)-Laplacian operator in (1.1) is a special case of the divergence form operator \(-\text{div}(a(x, \nabla u))\), which appears in many nonlinear diffusion problems, in particular in the mathematical modeling of non-Newtonian fluids, for a discussion of some physical background, see [9]. We also refer to Aronsson-Janfalk [1] for the

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The concept of Hele-Shaw flow refers to the flow between two closely-spaced parallel plates, close in the sense that the gap between the plates is small compared to the dimension of the plates. Quasilinear problems with a variable coefficient also appear in the mathematical model of the torsional creep. This study is based on the observation that a prismatic material rod object to a torsional moment, at sufficiently high temperature and for an extended period of time, exhibits a permanent deformation, called creep. The corresponding equations are derived under the assumptions that the components of strain and stress are linked by a power law referred to as the creep-law [12, 15, 16].

The boundary condition of the system (1.1) describes a flux through the boundary which depends in a nonlinear manner on the solution itself, for some physical motivation of such boundary conditions, for example see [11, 19]. Some related the elliptic type equations and $p$-Laplacian equations results, we refer the reader to [2, 4–8, 10, 13, 14, 17, 22, 25–40] and the references therein.

Let $\Omega \subseteq \mathbb{R}^N$ be an unbounded domain with smooth boundary $\partial \Omega$. We assume throughout that $1 < p < N$, $a_0 < a \in L^\infty(\Omega)$, for some positive constant $a_0$ and $b : \partial \Omega \to R$ is continuous function satisfying

\[ (B_1) \quad \frac{a}{(1 + |x|)^N} \leq b(x) \leq \frac{C}{(1 + |x|)^N}, \]

for some constants $0 < c < C$.

Let $C^\infty_0(\Omega)$ be the space of $C^\infty_0(\mathbb{R}^N)$-functions restricted on $\Omega$. We define the weighted Sobolev space $E$ as the completion of $C^\infty_0(\Omega)$ in the norm

\[ \|u\|_E = \left( \int_\Omega (|\nabla u|^p + w(x)|u|^p)dx \right)^{1/p}, \]

where $w(x) = \frac{1}{(1 + |x|)^\alpha}$, and we denote $n$ times product of this space by $X = E^n$ with respect to the norm

\[ \|(u_1, \ldots, u_n)\|_X = \left( \sum_{i=1}^n \|u_i\|^p_E \right)^{1/p}. \]

Denote by $L^p(\Omega, w_1)$, $L^q(\Omega, w_2)$ and $L^m(\partial \Omega, w_3)$ the weighted Lebesgue spaces with weight functions $w_i(x) = (1 + |x|)^{\alpha_i}$ for $i = 1, 2, 3$ and the norms defined by

\[ \|u\|_{p, w_1}^p = \int_\Omega w_1(x)|u|^pdx, \quad \|u\|_{q, w_2}^q = \int_\Omega w_2(x)|u|^qdx \]

and

\[ \|u\|_{m, w_3}^m = \int_{\partial \Omega} w_3(x)|u|^md\sigma, \]

where

\[ -N < \alpha_1 \leq -p \quad \text{if} \quad p < N, (\alpha_1 < -p \quad \text{when} \quad p \geq N), \]

\[ (H_1) \quad -N < \alpha_2 \leq \frac{N-p}{p} - N \quad \text{if} \quad p < N, (-N < \alpha_2 < 0 \quad \text{when} \quad p \geq N), \]

\[ -N < \alpha_3 \leq \frac{N-p}{p} - N + 1 \quad \text{if} \quad p < N, (-N < \alpha_3 < 0 \quad \text{when} \quad p \geq N). \]

Then we have the following embedding and trace theorem.
Lemma 1.1 ([20]). If \( p \leq q \leq \frac{pN}{N-p} = p^* \) and \(-N < \alpha_2 \leq q \frac{N-p}{p} - N\), then the embedding operator \( E^n \hookrightarrow (L^q(\Omega, w_2))^n \) is continuous. If the upper bound for \( q \) be strict, then the embedding is compact.

If \( p \leq m \leq \frac{(N-1)}{N-p} \) and \(-N < \alpha_3 \leq m \frac{n-p}{p} - N + 1\), then the trace operator \( E^n \hookrightarrow (L^m(\partial\Omega, w_3))^n \) is continuous. If the upper bound for \( m \) be strict, then the trace operator is compact.

Furthermore, one can show

Lemma 1.2 ([21]). The quantity

\[
\|u\|_b = \left( \int_{\Omega} a(x)|\nabla u|^p dx + \int_{\partial\Omega} b(x)|u|^p d\sigma \right)^{1/p}
\]

defines an equivalent norm on \( E \). Moreover

\[
\|(u_1, \ldots, u_n)\|_B = \left( \sum_{i=1}^{n} \|u_i\|_{b_i}^p \right)^{1/p}
\]

defines an equivalent norm on \( X \).

Because the lack of separability for the functions \( F \) and \( h \), we need to restrict the problem (1.1) to the following assumptions on \( f, F \) and \( h \):

The function \( f \) is nontrivial measurable satisfying

\((f_1)\) \( 0 \leq f(x) \leq C(1 + |x|)^{\alpha_1} \) for a.e. \( x \in \Omega \).

The mapping \( h : \partial\Omega \to R \) is a Caratheodory function which fulfills the assumptions

\((f_2)\) \( |h(x, u)| \leq h_0(x) + h_1(x)|u|^{m-1} \), where \( h_i : \partial\Omega \to R \), \( (i = 0, 1) \) are measurable functions satisfying \( h_0 \in L^{\frac{m}{m-1}}(\partial\Omega, w_3^{1-m}) \), \( 0 \leq h_i \leq C h_3 \) \( (i = 0, 1) \).

We also assume

\((H_2)\) \( \lim_{s \to 0} \frac{h(x, s)}{h_0(x)|s|^{m-1}} = 0 \), uniformly in \( x \).

\((H_3)\) There exists \( \mu \in (p, p^*] \) s.t. \( \mu H(x,t) \leq th(x,t) \) a.e. \( x \in \Omega, \forall t \in R \), where

\[ H(x,t) = \int_{0}^{t} h(x,s) ds. \]

\((H_4)\) There is a nonempty open set \( O \subset \partial\Omega \) with \( H(x,t) > 0 \) for \( (x,t) \in O \times (0, \infty) \).

Also we need the following assumptions on \( F \):

\((F_1)\) \( F : \Omega \times (R^+)^n \to R^+ \) is a \( C^1\)-function such that \( F(x, tu_1, \ldots, tu_n) = t^p F(x, u_1, \ldots, u_n) \) \( (t > 0) \) holds for all \( (x, u_1, \ldots, u_n) \in \Omega \times (R^+)^n \).

\((F_2)\) \( F(x, u_1, \ldots, u_n) = 0 \) if \( u_j = 0 \) for some \( j = 1, \ldots, n \) and \( u_i \in R^+ \) for \( i = 1, \ldots, n, i \neq j \).

\((F_3)\) \( F_{u_i}(x, u_1, \ldots, u_n) \) are strictly increasing functions about \( (u_1, \ldots, u_n) \) for all \( u_i > 0 \), \( i = 1, \ldots, n \).

Moreover, using Homogeneity property in \((F_1)\), we have the so-called Euler identity

\[
\left\{ \begin{array}{ll}
(u_1, \ldots, u_n) \cdot \nabla F(x, u_1, \ldots, u_n) = p^* F(x, u_1, \ldots, u_n), \\
F(x, u_1, \ldots, u_n) \leq K \left( \sum_{i=1}^{n} |u_i|^p \right)^{\frac{p}{p^*}} \text{ for some } K > 0.
\end{array} \right.
\] (1.2)
We say that $u = (u_1, \ldots, u_n)$ is a weak solution to the system (1.1) if $u = (u_1, \ldots, u_n) \in X$ and
\[
\sum_{i=1}^{n} \left\{ \int_{\Omega} a(x)|\nabla u_i|^{p-2}\nabla u_i \nabla v_i \, dx + \int_{\partial \Omega} b(x)|u_i|^{p-2}u_i v_i \, d\sigma - \lambda \int_{\Omega} f(x)|u_i|^{p-2}u_i v_i \, dx - \int_{\Omega} F_i(x, u_1, \ldots, u_n) v_i \, dx \right\} = 0,
\]
for any $(v_1, \ldots, v_n) \in X$.

The corresponding energy functional of the problem (1.1) is defined by
\[
J(\lambda, u_1, \ldots, u_n) = \frac{1}{p} \left[ \int_{\Omega} a(x) \sum_{i=1}^{n} |\nabla u_i|^p \, dx + \int_{\partial \Omega} b(x) \sum_{i=1}^{n} |u_i|^p \, d\sigma \right] - \lambda \int_{\Omega} f(x) \sum_{i=1}^{n} |u_i|^p \, dx - \int_{\partial \Omega} H(x, u_i) d\sigma - \int_{\Omega} F(x, u_1, \ldots, u_n) \, dx.
\]

Note that using Lemmas 1.1 and 1.2 we deduce that $J(\lambda)$ is well-defined on $X$.

Now we state our main results:

**Theorem 1.1.** Assume that the conditions $(f_1), (f_2), (H_1)-(H_4)$ and $(F_1)-(F_3)$ hold. Then the problem (1.1) has a nontrivial weak solution for every
\[
0 < \lambda < \Lambda = \inf_{(0, \ldots, 0) \neq (u_1, \ldots, u_n) \in X} \frac{\int_{\Omega} (a(x) \sum_{i=1}^{n} |\nabla u_i|^p) \, dx + \int_{\partial \Omega} (b(x) \sum_{i=1}^{n} |u_i|^p) \, d\sigma}{\int_{\Omega} (f(x) \sum_{i=1}^{n} |u_i|^p) \, dx}.
\]

**Theorem 1.2.** Assume that $h(x, s) \equiv 0$. Then the problem (1.1) has infinity many solutions for $0 < \lambda < \Lambda$.

## 2. Proof of Theorem 1.1

Let us consider $(H_9)$. We need the following proposition

**Proposition 2.1** ([20]). The corresponding Nemytskii operators
\[
N_h : L^m(\partial \Omega, w_3) \to L^{\frac{m}{m-r}}(\partial \Omega, w_3^{1-r}), \quad N_H : L^m(\partial \Omega, w_3) \to L^1(\partial \Omega)
\]
are bounded and continuous. Also if we set $\varphi(u) = f(x)|u|^{p-2}$, then the operators
\[
N_\varphi : L^p(\Omega, w_1) \to L^{\frac{p}{p-r}}(\Omega, w_1^{1-r}), \quad N_\phi : L^p(\partial \Omega, w_1) \to L^1(\Omega)
\]
are bounded and continuous, where $\phi$ denotes the primitive function of $\varphi$.

**Remark 2.1.** Note that $\lambda < \Lambda$ implies the existence of some $C_0 > 0$ such that
\[
\|u_1, \ldots, u_n\|_B^p - \lambda \int_{\Omega} (f(x) \sum_{i=1}^{n} |u_i|^p) \, dx \geq C_0 \|u_1, \ldots, u_n\|_B^p.
\]
Lemma 2.1. Under assumptions \((H_1) - (H_4)\) and \((F_1) - (F_3)\), \(J_\lambda\) is Fréchet differentiable on \(X\) and satisfies the Palais-Smale condition.

Proof. We use the notations

\[
I(u_1, \ldots, u_n) = \frac{1}{p} \|(u_1, \ldots, u_n)\|_B^p, \quad K_f(u_1, \ldots, u_n) = \frac{1}{p} \int_\Omega \left( f(x) \sum_{i=1}^n |u_i|^p \right) dx,
\]

\[
K_H(u_1, \ldots, u_n) = \int_{\partial \Omega} \sum_{i=1}^n H(x, u_i) d\sigma, \quad K_F(u_1, \ldots, u_n) = \int_\Omega F(x, u_1, \ldots, u_n) dx.
\]

Then the directional derivative of \(J_\lambda\) is

\[
\langle J'_\lambda(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = \langle I'(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle - \lambda \langle K'_f(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle - \langle K'_H(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle - \langle K'_F(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle,
\]

where

\[
\langle I'(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = \int_\Omega \left( a(x) \sum_{i=1}^n |\nabla u_i|^{p-2} \nabla u_i \nabla v_i \right) dx + \int_{\partial \Omega} \left( b(x) \sum_{i=1}^n |u_i|^{p-2} u_i v_i \right) d\sigma,
\]

\[
\langle K'_f(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = \int_\Omega \left( f(x) \sum_{i=1}^n |u_i|^{p-2} u_i v_i \right) dx,
\]

\[
\langle K'_H(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = \int_{\partial \Omega} \sum_{i=1}^n h(x, u_i) v_i d\sigma,
\]

\[
\langle K'_F(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = \int_\Omega \sum_{i=1}^n F_u(x, u_1, \ldots, u_n) v_i dx,
\]

for all \((v_1, \ldots, v_n) \in X\).

Clearly \(I'_\lambda : X \to X^*\) is continuous. The operator \(K'_H\) is a composition of the operators

\[
K'_H : X \to (L^m(\partial \Omega, w_3))^n \longrightarrow_{N_1 := (N_H \ldots N_H)} (L^{m-1}(\partial \Omega, w_3^{1/m}))^n \longrightarrow X^*
\]

where

\[
\langle l(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = \int_{\partial \Omega} \sum_{i=1}^n u_i v_i d\sigma.
\]

Since

\[
\sum_{i=1}^n \int_{\partial \Omega} |u_i v_i d\sigma \leq \sum_{i=1}^n \left( \int_{\partial \Omega} |u_i|^{m-1} w_3^{1/m} d\sigma \right)^{\frac{1}{m}} \left( \int_{\partial \Omega} |v_i|^m w_3 d\sigma \right)^{\frac{1}{m}},
\]

\(l\) is continuous by Lemma 1.1.

As a composition of continuous operators, \(K'_H\) is also continuous. Moreover using \((H_1)\), \(n\) product of trace operator \(X \to (L^m(\partial \Omega, w_3))^n\) is compact and \(K'_H\) is also compact.
In a similar way we obtain that the operator $K'_F$ is a composition of the operators

$$K'_F : X \to \left( L^p(\Omega, w_1) \right)^n \to \mathbb{N}(N,\ldots,N) \left( L^{\frac{1}{p}}(\Omega, w_1^{\frac{1}{p}}) \right)^n \to p^* X^*$$

where

$$\langle l'(u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = \int_{\Omega} \sum_{i=1}^{n} u_i v_i \, dx.$$  

Since

$$\sum_{i=1}^{n} \int_{\Omega} |u_i v_i| \, dx \leq \sum_{i=1}^{n} \left( \int_{\Omega} |u_i|^{p^*_p} w_1^{\frac{1}{p^*_p}} \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |v_i|^p w_1 \, dx \right)^{\frac{1}{p}},$$

$l'$ is continuous by Lemma 1.1. Again $K'_\phi$ is also continuous. In a similar way $K'_\phi$ is also compact.

Since the assumptions $(F_1)$ and $(F_3)$ hold, we get $F_{u_i} \in C(\overline{\Omega} \times (R^+)^n, R^+)$ are positively homogeneous of degree $p^* - 1$. Moreover using the above fact, we get the existence of a positive constant $M$ such that

$$F_{u_i}(x, u_1, \ldots, u_n) \leq M \sum_{i=1}^{n} |u_i|^{p^*_p - 1}, \quad \forall x \in \overline{\Omega}, \forall (u_1, \ldots, u_n) \in (R^+)^n. \quad (2.1)$$

By the Sobolev embedding theorem, we derive that $K'_F$ is continuous and compact and the continuous differentiability of $J_\lambda$ follows.

Now let $U_m = (u_{1m}, \ldots, u_{nm}) \in X$ be a Palais-Smale sequence for the functional $J_\lambda$, i.e.,

$$|J'_\lambda(U_m)| \leq C, \quad \text{for all } m \quad (2.2)$$

and

$$||J'_\lambda(U_m)||_{X^*} \to 0 \quad \text{as } m \to \infty. \quad (2.3)$$

For $m$ large enough we have

$$|\langle J'_\lambda(U_m), U_m \rangle| \leq \mu ||U_m||_B.$$

This implies

$$C + ||U_m||_B \geq J_\lambda(U_m) - \frac{1}{\mu} \langle J'_\lambda(U_m), U_m \rangle. \quad (2.4)$$

Using a direct calculation we have

$$J_\lambda(U_m) - \frac{1}{\mu} \langle J'_\lambda(U_m), U_m \rangle = \left( \frac{1}{p} - \frac{1}{\mu} \right) \left( ||U_m||^p_B - \lambda \int_{\Omega} f(x) \left( \sum_{i=1}^{n} |u_{im}|^p \right) \, dx \right)$$

$$- \int_{\partial \Omega} \sum_{i=1}^{n} \left( H(x, u_{im}) - \frac{1}{\mu} h(x, u_{im}) u_{im} \right) \, d\sigma$$

$$- \int_{\Omega} (F(x, u_{1m}, \ldots, u_{nm})$$

$$- \frac{1}{\mu} \sum_{i=1}^{n} F_{u_i}(x, u_{1m}, \ldots, u_{nm}) u_{im} \, dx.$$
By \((H_4)\) we deduce that

\[
\sum_{i=1}^{n} \int_{\partial \Omega} H(x, u_{i_m}) d\sigma \leq \frac{1}{\mu} \sum_{i=1}^{n} \int_{\partial \Omega} h(x, u_{i_m}) u_{i_m} d\sigma.
\]

Also using the property \((F_4)\), we have

\[
\int_{\Omega} \left[ F(x, u_{1_m}, \ldots, u_{n_m}) - \frac{1}{\mu} \sum_{i=1}^{n} (u_{1_m}, \ldots, u_{n_m}) \cdot \nabla F(x, u_{1_m}, \ldots, u_{n_m}) dx \right] \\
= \int_{\Omega} \left[ (1 - \frac{p^*}{\mu}) F(x, u_{1_m}, \ldots, u_{n_m}) dx \right] < 0,
\]

since \(\mu \in (p, p^*)\). So we deduce that

\[
J_\lambda(U_m) - \frac{1}{\mu} \langle J_\lambda'(U_m), U_m \rangle \geq \left( \frac{1}{p} - \frac{1}{\mu} \right) C_0 ||U_m||_B^p.
\] (2.5)

Relations (2.4) and (2.5) yield \(C + ||U_m||_B \geq \left( \frac{1}{p} - \frac{1}{\mu} \right) C_0 ||U_m||_B^p\), and hence \(U_m\) is bounded.

To show that \(U_m\) contains a Cauchy sequence we use the following inequalities for \(\xi \in R^N\) (see Diaz [9, Lemma 4.10]):

\[
|\varepsilon - \xi|^p \leq C(|\varepsilon|^{p-2}|\varepsilon - \xi|)(\varepsilon - \xi), \text{ for } p \geq 2,
\] (2.6)

\[
|\varepsilon - \xi|^2(|\varepsilon| + |\xi|)^{2-p} \leq C(|\varepsilon|^{p-2}|\varepsilon - \xi|)(\varepsilon - \xi), \text{ for } 1 < p < 2.
\] (2.7)

In the case \(p \geq 2\):

\[
||U_m - U_k||_B^p \\
= ||(u_{1_m} - u_{1_k}, \ldots, u_{n_m} - u_{n_k})||_B^p \\
= \sum_{i=1}^{n} ||u_{i_m} - u_{i_k}||_B^p \sum_{i=1}^{n} \left[ \int_{\Omega} a(x)||\nabla u_{i_m} - \nabla u_{i_k}||^p dx + \int_{\partial \Omega} b(x)|u_{i_m} - u_{i_k}||^p d\sigma \right] \\
\leq C \sum_{i=1}^{n} \left[ \int_{\Omega} a(x)||\nabla u_{i_m}||^{p-2}\nabla u_{i_m} \cdot (u_{i_m} - u_{i_k}) - ||\nabla u_{i_k}||^{p-2}\nabla u_{i_k} \cdot (u_{i_m} - u_{i_k}) dx \\
+ \int_{\partial \Omega} b(x)||u_{i_m}||^{p-2}u_{i_m} (u_{i_m} - u_{i_k}) - ||u_{i_k}||^{p-2}u_{i_k} (u_{i_m} - u_{i_k}) d\sigma \right] \\
= C(\langle J_\lambda'(U_m), (U_m - U_k) \rangle - \langle J_\lambda'(U_k), (U_m - U_k) \rangle) \\
= C(\langle J_\lambda'(U_m), (U_m - U_k) \rangle - \langle J_\lambda'(U_k), (U_m - U_k) \rangle) \\
+ \lambda \langle K_f'(U_m), (U_m - U_k) \rangle - \lambda \langle K_f'(U_k), (U_m - U_k) \rangle \\
+ \langle K_H'(U_m), (U_m - U_k) \rangle - \langle K_H'(U_k), (U_m - U_k) \rangle \\
+ \langle K_F'(U_m), (U_m - U_k) \rangle - \langle K_F'(U_k), (U_m - U_k) \rangle) \\
\leq C(||J_\lambda'(U_m) - J_\lambda'(U_k)||_{X^*} + ||K_f'(U_m) - K_f'(U_k)||_{X^*} + ||K_H'(U_m) - K_H'(U_k)||_{X^*} + ||K_F'(U_m) - K_F'(U_k)||_{X^*}) ||U_m - U_k||_B
\]
\[ \leq C(||J'_\lambda(U_m)||_{X^*} + ||J'_\lambda(U_k)||_{X^*} + \lambda(||K'_\lambda(U_m) - K'_\lambda(U_k)||_{X^*}) + ||K'_{B'}(U_m) - K'_{B'}(U_k)||_{X^*} + ||K'_{B'}(U_m) - K'_{B'}(U_k)||_{X^*})||U_m - U_k||_B. \]

This concludes that there exists a subsequence of \(U_m\) which converges in \(X\) because of \(J'_\lambda(U_m) \to 0\) and \(K'_\lambda\) is compact for \(\gamma \in \{f, H, F\}\).

If \(1 < p < 2\), modifying the proof of [18, Lemma 3], we can easily deduce that
\[ ||U_m - U_k||_B^2 \leq C(||J'(U_m), (U_m - U_k)|| - ||J'(U_k), (U_m - U_k)||)||U_m||_B^{2-p} + ||U_k||_B^{2-p}. \]

Since \(||U_m||_B\) is bounded, the same arguments as the case \(p \geq 2\), lead to a convergent subsequence.

\[ \square \]

**Proof of Theorem 1.1.** We shall use the mountain pass lemma to obtain a solution. In what follows, we notice two points to verify the geometric assumptions of the mountain pass theorem. From assumptions \((f_2)\) and \((H_2)\), for every \(\epsilon_i > 0\) there is a \(C_{\epsilon_i} > 0\) such that
\[ |H(x, u_i)| \leq \epsilon_i b(x)|u_i|^p + C_{\epsilon_i} w_3(x)|u_i|^m. \]

Thus using \((B_1)\) and Lemma 1.1, we have
\[ \sum_{i=1}^n \int_{\partial \Omega} H(x, u_i) d\sigma \leq \sum_{i=1}^n \epsilon_i \int_{\partial \Omega} b(x)|u_i|^p d\sigma + \sum_{i=1}^n C_{\epsilon_i} \int_{\partial \Omega} w_3(x)|u_i|^m d\sigma \leq \epsilon C_1 \sum_{i=1}^n ||u_1, \ldots, u_n||_B^p + C_2 ||u_1, \ldots, u_n||_B^2, \]
where \(\epsilon = \max \{\epsilon_i; i = 1, \ldots, n\}\) and \(C_\epsilon = \max \{C_{\epsilon_i}; i = 1, \ldots, n\}\).

Additionally, we recall the following result:
For all \(s \in (0, \infty)\) there is a constant \(C_s > 0\) such that
\[ (x + y)^s \leq C_s (x^s + y^s) \quad \text{for all} \quad x, y \in (0, \infty). \]

Now using the estimate (1.2) and Lemma 1.1 we get
\[ \int_{\Omega} F(x, u_1, \ldots, u_n) dx \leq K \int_{\Omega} \left( \sum_{i=1}^n |u_i|^p \right)^{\frac{p}{r^*}} dx \]
\[ = K \int_{\Omega} \left( |u_1|^p + \ldots + |u_n|^p \right)^{\frac{p}{r^*}} dx \leq K C_p \int_{\Omega} \left( |u_1|^p (p^*/p) + \ldots + |u_n|^p (p^*/p) \right) dx \]
\[ \leq K C_p C_3 ||u_1, \ldots, u_n||_B^{p^*}. \]

Consequently this two facts and Remark 2.1 imply that
\[ J_\lambda(u_1, \ldots, u_n) = \frac{1}{p} ||(u_1, \ldots, u_n)||_B^p - \frac{\lambda}{p} \sum_{i=1}^n \int_{\Omega} f(x)|u_i|^p dx \]
\[ - \sum_{i=1}^n \int_{\partial \Omega} H(x, u_i) d\sigma - \int_{\Omega} F(x, u_1, \ldots, u_n) dx \geq \frac{1}{p} C_0 ||(u_1, \ldots, u_n)||_B^p - \lambda \epsilon C_1 ||(u_1, \ldots, u_n)||_B^p \]
\[ - C_2 ||(u_1, \ldots, u_n)||_B^p - K C_p C_3 ||(u_1, \ldots, u_n)||_B^{p^*}. \]
For $\epsilon > 0$ and $R > 0$ small enough, we deduce that for every $(u_1, \ldots, u_n) \in X$ with $\| (u_1, \ldots, u_n) \|_B = R$, the righthand side is strictly greater than $0$.

It remains to show that there exists $V = \{ v_1, \ldots, v_n \} \in X$ with $\| (v_1, \ldots, v_n) \|_B > R$ such that $J_\lambda(\nu_1, \ldots, \nu_n) \leq 0$. Choose $\psi \in C_\infty^\infty(\Omega), \psi \geq 0$ such that $\text{Supp} \psi \cap \partial \Omega \subset O$. From $(H_3)$ we see that $H(x, t) \geq C_4 t^\mu - C_5$ on $O \times (0, \infty)$. Then using $(F_2)$, for $t > 0$, we have

$$J_\lambda(t \psi, 0, \ldots, 0) = \frac{t^p}{p} \| (t \psi, 0, \ldots, 0) \|_B^p - \lambda \int_\Omega f(x) \psi^p \, dx$$

$$- \int_{\partial \Omega} H(x, t \psi) \, d\sigma - \int_\Omega F(x, t \psi, 0, \ldots, 0) \, dx$$

$$\geq \frac{t^p}{p} \| (\psi, 0, \ldots, 0) \|_B^p - C_4 t^\mu \int_0^\infty \psi^\mu \, d\sigma + C_5 |O|.$$

Since $\mu > p$ the righthand side tends to $-\infty$ as $t \to \infty$ and for sufficiently large $t_0$, $V = (t\psi, 0, \ldots, 0)$ has the desired property.

Since $J_\lambda$ satisfies the Palais-Smale condition and $J_\lambda(0, \ldots, 0) = 0$, the mountain pass lemma shows that there is a nontrivial critical point of $J_\lambda$ in $X$ with critical value

$$c = \inf_{g \in G} \max_{t \in [0, 1]} J_\lambda(g(t)) > 0$$

where $G = \{ g \in C([0, 1], X); g(0) = (0, \ldots, 0), g(1) = V \}$. \hfill $\square$

### 3. Proof of Theorem 1.2

We recall here a version of the Ljusternik-Schnirelman principle in Banach spaces which was discussed by Browder [3], Zeidler [41], Rabinowitz [23] and Szulkin [24]. We then shall apply the principle to establish the existence of a sequence of solutions for the problem (1.1).

Let $Y$ be a real reflexive Banach space and $\Sigma$ the collection of all symmetric subsets of $Y - \{ 0 \}$ which are closed in $X$ ($A$ is symmetric if $A = -A$). A nonempty set $A \in \Sigma$ is said to be of genus $k$ (denoted by $\gamma(A) = k$) if $k$ is the smallest integer with the property that there exists an odd continuous mapping from $A$ to $R^k - \{ 0 \}$. If there is no such $k$, $\gamma(A) = \infty$, and if $A = \emptyset$, $\gamma(A) = 0$.

In order to continue the proof we shall need the following proposition.

**Proposition 3.1** ([23, Corollary 4.1]). Suppose that $M$ is a closed symmetric $C^1$-submanifold of a real Banach space $Y$ and $0 \notin M$. Suppose also that $J \in C^1(M, \mathbb{R})$ is even and bounded below. Define

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} J(x),$$

where $\Gamma_j = \{ A \subset M : A \in \Sigma, \gamma(A) \geq j \text{ and } A \text{ is symmetric} \}$. If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if $J$ satisfies $(PS)_c$ for all $c = c_j, j = 1, \ldots, k$, then $J$ has at least $k$ distinct pairs of critical points.

Define on $X$ the even functional

$$J_\lambda(u_1, \ldots, u_n) = \frac{1}{p} \| (u_1, \ldots, u_n) \|_B^p - \lambda \sum_{i=1}^n \int_\Omega f(x) |u_i|^p \, dx,$$
on the closed symmetric $C^1$-manifold

$$S_F = \{(u_1, \ldots, u_n) \in X; K_F(u_1, \ldots, u_n) = 1\}.$$  

By our hypotheses of $f$, $F$ and $h$, Lemma 2.2 and Proposition 3.1, we claim that $\tilde{J}_\lambda|_{S_F}$ possesses at least $\gamma(S_F)$ pairs of distinct critical points. Since $F : \Omega \times (R^+)^n \to R^+$ is a $C^1$-function, there exists a nonempty open set $\tilde{O} \subset \Omega$ such that $F(x, t_1, \ldots, t_n) > 0$ for all $(x, t_1, \ldots, t_n) \in \tilde{O} \times (R^+)^n$. Using the properties of the genus it follows that $\gamma(\tilde{O}) \geq \gamma(B_\delta)$, where $B_\delta$ is the unit ball of $W_0^{1,p}(\tilde{O}) \subset X$. On the other hand it is well known that the genus of the unit ball of an infinite dimensional Banach space is infinity, so $\gamma(S_F) = \infty$. Therefore we conclude that there exists a sequence $\{(u_{1_m}, \ldots, u_{n_m})\} \subset X$ such that any $(u_{1_m}, \ldots, u_{n_m})$ is a constrained critical point of $\tilde{J}_\lambda$ on $S_F$.

By the Lagrange multipliers rule, there exists a sequence $\{\lambda_m\} \subset R$ such that

$$||(u_{1_m}, \ldots, u_{n_m})||_B^p - \lambda \sum_{i=1}^n \int_{\Omega} f(x)|u_{i_m}|^p dx = \lambda_m K_F(u_{1_m}, \ldots, u_{n_m}). \tag{3.1}$$

Since $(u_{1_m}, \ldots, u_{n_m}) \in S_F$ and $0 < \lambda < \Lambda$, so the right hand side of (3.1) is positive and so $\lambda_m > 0$. Setting

$$v_{i_m} = \lambda_m^{-\frac{1}{p-\sigma}} u_{i_m},$$

we have the following equation

$$\lambda_m^{-\frac{p}{p-\sigma}} ||(v_{1_m}, \ldots, v_{n_m})||_B^p - \lambda \lambda_m^{-\frac{p}{p-\sigma}} \sum_{i=1}^n \int_{\Omega} f(x)|v_{i_m}|^p dx = \lambda_m \lambda_m^{-\frac{p}{p-\sigma}} K_F(v_{1_m}, \ldots, v_{n_m}).$$

Since $\lambda_m \neq 0$, we derive

$$||(v_{1_m}, \ldots, v_{n_m})||_B^p - \lambda \sum_{i=1}^n \int_{\Omega} f(x)|v_{i_m}|^p dx = K_F(v_{1_m}, \ldots, v_{n_m}).$$

This proves the theorem. \qed

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