A categorical approach to algebras and coalgebras

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Overview

- Category theory
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- Adjoint functors
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- Monads and comonads
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- Monads and comonads
- Adjoints and (co)monads
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- Adjoints and (co)monads
- Adjoint endofunctors
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- Adjoint endofunctors
- Distributive laws
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- Adjoint and (co)monads
- Adjoint endofunctors
- Distributive laws
- Hopf monads
Category theory

Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. AMS 1945
Category theory

Eilenberg - Mac Lane,
General Theory of natural equivalences
Trans. AMS 1945

Category $\mathcal{A}$: objects and morphisms $\text{Mor}_\mathcal{A}(A, A')$

- functors $F : \mathcal{A} \rightarrow \mathcal{B}$;
- $f : A \rightarrow A'$ sent to $F(f) : F(A) \rightarrow F(A')$ of $\mathcal{B}$,
Category theory

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Category $\mathbb{A}$: objects and morphisms $\text{Mor}_\mathbb{A}(A, A')$

- functors $F : \mathbb{A} \rightarrow \mathbb{B}$;
- $f : A \rightarrow A'$ sent to $F(f) : F(A) \rightarrow F(A')$ of $\mathbb{B}$,
- $\varphi_F : \text{Mor}_\mathbb{A}(A, A') \rightarrow \text{Mor}_\mathbb{B}(F(A), F(A'))$,
- $\varphi_F$ monomorph (epimorph) def. $F$ faithful (full)
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Category $\mathbb{A}$: objects and morphisms $\text{Mor}_\mathbb{A}(A, A')$

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Natural transformations $\psi : F \to G$, $F, G : \mathbb{A} \to \mathbb{B}$

\[
\begin{array}{ccc}
A & \xrightarrow{F(A)} & G(A) \\
\downarrow h & & \downarrow G(h) \\
A' & \xrightarrow{F(A')} & G(A')
\end{array}
\]

$\xrightarrow{\psi_A} \quad \xrightarrow{\psi_{A'}}$
Adjoint pair of functors $F : A \to B$, $G : B \to A$, bijection

$$\text{Mor}_B(F(A), B) \xrightarrow{\sim} \text{Mor}_A(A, G(B)),$$
| **Adjoint pair of functors** | **F : A → B, G : B → A, bijection** |
|-----------------------------|-------------------------------------|
| $\text{Mor}_B(F(A), B) \cong \text{Mor}_A(A, G(B))$, unit $\eta : 1 \to GF$ | counit $\varepsilon : FG \to 1$ |
Adjoint pair of functors $F : A \rightarrow B$, $G : B \rightarrow A$, bijection

$$\text{Mor}_B(F(A), B) \cong \text{Mor}_A(A, G(B)),$$

unit $\eta : 1 \rightarrow GF$

counit $\varepsilon : FG \rightarrow 1$

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = 1_F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = 1_G$$

$F$ preserves epimorphisms and coproducts

$G$ preserves monomorphisms and products

$\eta$ and $\varepsilon$ isomorphisms

$F$ is equivalence

$\varepsilon$ epi-(iso-)morphism

$G$ faithful (and full)

$\varepsilon$ split epimorphism

$G$ is separable

$\eta$ mono-(iso-)morphism

$F$ faithful (and full)

$\eta$ split monomorphism

$F$ is separable

$\eta$ extr. epi-, $\varepsilon$ monomorph

$(F, G)$ pair of $\ast$-functors
### Adjoint pair of functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, bijection

| bijection | $\text{Mor}_\mathcal{B}(F(A), B) \xrightarrow{\sim} \text{Mor}_\mathcal{A}(A, G(B))$ | unit $\eta : 1 \to GF$ | counit $\varepsilon : FG \to 1$ |
|-----------|---------------------------------|----------------.|-----------------|

| $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = 1_F$, | $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = 1_G$ |

- $F$ preserves epimorphisms and coproducts
Eilenberg-Moore: Adjoint functors and triples, 1965

Adjoint pair of functors $F : A \to B$, $G : B \to A$, bijection

$\text{Mor}_B(F(A), B) \cong \text{Mor}_A(A, G(B))$, unit $\eta : 1 \to GF$

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$$\text{Mor}_B(F(A), B) \xrightarrow{\cong} \text{Mor}_A(A, G(B)),$$

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\[
F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = 1_F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = 1_G
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$F$ preserves epimorphisms and coproducts

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$\eta$ and $\varepsilon$ isomorphisms $F$ is equivalence
Adjoint pair of functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, bijection

$$\text{Mor}_\mathcal{B}(F(A), B) \cong \text{Mor}_\mathcal{A}(A, G(B)),$$

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$G$ is separable
Adjoint pair of functors $F : A \to B$, $G : B \to A$, bijection

\[ \text{Mor}_B(F(A), B) \xrightarrow{\sim} \text{Mor}_A(A, G(B)), \quad \text{unit} \quad \eta : 1 \to GF, \]
\[ \text{counit} \quad \varepsilon : FG \to 1 \]

\[ F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = 1_F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = 1_G \]

- $F$ preserves epimorphisms and coproducts
- $G$ preserves monomorphisms and products

- $\eta$ and $\varepsilon$ isomorphisms
- $\varepsilon$ epi-(iso-)morphism
- $\varepsilon$ split epimorphism
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- $F$ is equivalence
- $G$ faithful (and full)
- $G$ is separable
- $F$ faithful (and full)
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Adjoint pair of functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, bijection

$$\text{Mor}_\mathcal{B}(F(A), B) \cong \text{Mor}_\mathcal{A}(A, G(B)), \quad \text{unit } \eta : 1 \to GF$$

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$$F \xrightarrow{F \eta} FGF \xrightarrow{\varepsilon F} F = 1_F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G \varepsilon} G = 1_G$$

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$F$ is equivalence

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$G$ is separable

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$F$ is separable

$(F, G)$ pair of $*$-functors
Adjoint functors

Module categories

\[ R M \otimes_S - : S M \to R M, \quad \text{Hom}_R(M,-) : R M \to S M \]
Adjoint functors

Module categories

\[ R \mathcal{M} \otimes_S - : \mathcal{S} \mathcal{M} \to R \mathcal{M}, \quad \text{Hom}_R(M, -) : R \mathcal{M} \to \mathcal{S} \mathcal{M} \]

- **counit** \( \varepsilon_X : M \otimes_S \text{Hom}_R(M, X) \to X \),
- **unit** \( \eta_Y : Y \to \text{Hom}_R(M, M \otimes_S Y) \).

\( U = \text{Hom}_R(M, \mathbb{Q}) \), \( \mathbb{Q} \) cogenerator in \( \text{Gen}(M) \)
## Adjoint functors

### Module categories

| Module category | Expression |
|-----------------|------------|
| \( R^M \otimes_S - : S^M \to R^M \), \( \text{Hom}_R(M, -) : R^M \to S^M \) |

- **Counit** \( \varepsilon_X : M \otimes_S \text{Hom}_R(M, X) \to X \),
- **Unit** \( \eta_Y : Y \to \text{Hom}_R(M, M \otimes_S Y) \).

### Isomorphisms

- \( \eta \) and \( \varepsilon \) isomorphisms: \( R^M \) progenerator
  \( R^M \cong S^M \), Morita equivalence
- \( \varepsilon \) isomorphism: \( R^M \) generator
Adjoint functors

**Module categories**

\[ R^M \otimes_S - : S^M \to R^M, \quad \text{Hom}_R(M, -) : R^M \to S^M \]

- **counit** \( \varepsilon_X : M \otimes_S \text{Hom}_R(M, X) \to X \),
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- \( \eta \) and \( \varepsilon \) isomorphisms: \( R^M \) progenerator
  \( R^M \simeq S^M \), Morita equivalence
- \( \varepsilon \) isomorphism: \( R^M \) generator
- \( \eta \) isomorphisms: \( M_S \) faithfully flat
  \( S^M \simeq \text{Pres}(R^M) \), Sato equivalence
Adjoint functors

Module categories

\[ R\mathcal{M} \otimes_S \mathcal{M} \rightarrow \mathcal{R}\mathcal{M}, \quad \text{Hom}_R(M, -) : \mathcal{R}\mathcal{M} \rightarrow \mathcal{S}\mathcal{M} \]

- **Counit** \( \varepsilon_X : M \otimes_S \text{Hom}_R(M, X) \rightarrow X \),
- **Unit** \( \eta_Y : Y \rightarrow \text{Hom}_R(M, M \otimes_S Y) \).

- \( \eta \) and \( \varepsilon \) isomorphisms: \( R\mathcal{M} \) progenerator
  \( R\mathcal{M} \cong S\mathcal{M} \), Morita equivalence

- \( \varepsilon \) isomorphism: \( R\mathcal{M} \) generator

- \( \eta \) isomorphisms: \( M_S \) faithfully flat
  \( S\mathcal{M} \cong \text{Pres}(R\mathcal{M}) \), Sato equivalence

- \( \varepsilon \) monomorph and \( \eta \) epimorph (tilting theory)
  \( \text{Gen}(M) \cong \text{Cog}(S\mathcal{U}) \), Brenner-Butler equivalence
  \( U = \text{Hom}(M, Q) \), \( Q \) cogenerator in \( \text{Gen}(M) \)
Monads and comonads

**Monads**: $T : A \to A$ endofunctor

natural transformations: $m : TT \to T$, $e : 1_A \to T$,
with associativity and unitality conditions
Monads and comonads

**Monads:** \( T : \mathcal{A} \rightarrow \mathcal{A} \) endofunctor

natural transformations: \( m : TT \rightarrow T, \ e : 1_\mathcal{A} \rightarrow T, \) with associativity and unitality conditions

**\( T \)-modules:** \( \varrho : T(A) \rightarrow A \)

with associativity and unitality conditions.

\( T \)-morphisms \( (A, \varrho) \xrightarrow{f} (A', \varrho'), T(A) \xrightarrow{T(f)} T(A') \)

\( (\text{Eilenberg-Moore}) \) category of \( T \)-modules \( \mathcal{M}_T \)
Monads and comonads

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**$T$-modules:** $\varrho : T(A) \to A$

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$T$-morphisms $(A, \varrho) \xrightarrow{f} (A', \varrho')$, $T(A) \xrightarrow{T(f)} T(A')$

\[\begin{array}{ccc}
T(A) & \xrightarrow{T(f)} & T(A') \\
\varrho & \downarrow & \varrho' \\
A & \xrightarrow{f} & A'
\end{array}\]

(Eilenberg-Moore) category of $T$-modules $\mathcal{M}_T$

**Free and forgetful functor - adjoint**

$\phi_T : \mathcal{A} \to \mathcal{A}_T$, $A \mapsto (T(A), m_A : TT(A) \to T(A))$,

$U_T : \mathcal{A}_T \to \mathcal{A}$, $(A, \varrho_A) \mapsto A$. 
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**Free and forgetful functor - adjoint**

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$\text{Mor}_{\mathcal{A}_T}(\phi_T(A), B) \xrightarrow{\sim} \text{Mor}_\mathcal{A}(A, U_T(B))$, $f \mapsto f \circ \eta_A$. 
Monads and comonads

**Comonads:** \( T : A \rightarrow A \) endofunctor

natural transformations: \( \delta : T \rightarrow TT, \quad \varepsilon : T \rightarrow 1_A, \)
with coassociativity and counitality conditions
Monads and comonads

**Comonads:** $T : A \rightarrow A$ endofunctor

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**$T$-comodules:** $\omega : A \rightarrow T(A)$

with coassociativity and counitality conditions.

$T$-morphisms $(A, \omega) \xrightarrow{g} (A', \omega')$,

\[
\begin{array}{ccc}
A & \xrightarrow{g} & A' \\
\downarrow{\omega} & & \downarrow{\omega'} \\
T(A) & \xrightarrow{T(g)} & T(A')
\end{array}
\]

(Eilenberg-Moore) category of $T$-comodules $\mathbb{M}^T$
Monads and comonads

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**\( T \)-comodules:** \( \omega : A \to T(A) \)
with coassociativity and counitality conditions.

\( T \)-morphisms \( (A, \omega) \xrightarrow{g} (A', \omega') \),

\[
\begin{array}{ccc}
A & \xrightarrow{g} & A' \\
\downarrow \omega & & \downarrow \omega' \\
T(A) & \xrightarrow{T(g)} & T(A')
\end{array}
\]

(Eilenberg-Moore) category of \( T \)-comodules \( \mathbb{M}^T \)

**Forgetful and free functor - adjoint**

\( \phi^T : \mathbb{A} \to \mathbb{A}^T, \quad A \mapsto (T(A), \delta_A), \)

\( U^T : \mathbb{A}^T \to \mathbb{A}, \quad (A, \omega) \mapsto A. \)
Monads and comonads

**Comonads:** $T : A \to A$ endofunctor

natural transformations: $\delta : T \to TT$, $\varepsilon : T \to 1_A$,

with coassociativity and counitality conditions

**$T$-comodules:** $\omega : A \to T(A)$

with coassociativity and counitality conditions.

$T$-morphisms $(A, \omega) \xrightarrow{g} (A', \omega')$, $A \xrightarrow{g} A'$

$\omega \downarrow \downarrow \omega' \quad T(A) \xrightarrow{T(g)} T(A')$

(Eilenberg-Moore) category of $T$-comodules $\mathbb{M}^T$

**Forgetful and free functor - adjoint**

$\phi^T : \mathbb{A} \to \mathbb{A}^T$, $A \mapsto (T(A), \delta_A)$,

$U^T : \mathbb{A}^T \to \mathbb{A}$, $(A, \omega) \mapsto A$.

$\text{Mor}^T(A, T(X)) \cong \text{Mor}_A(U^T(A), X)$, $h \mapsto \varepsilon_X \circ h$
Adjoints and (co)monads

**Adjoint pair:** $F : A \to B$, $G : B \to A$

$$\eta : 1_A \to GF, \quad \varepsilon : FG \to 1_B$$

Related monad on $A$ $T = GF$:

$$A \to A$$

Product $m : GFGF \to GF$,

Unit $\eta : 1_A \to GF$.

$$(T, m : TT \to T, \eta : 1_A \to T)$$ is a monad on $A$.

Related comonad on $B$ $K = FG$:

$$B \to B$$

Coproduct $\delta : FG \to FGFG$,

Counit $\varepsilon : FG \to 1_B$.

$$(K, \delta : K \to KK, \varepsilon : K \to 1_B)$$ is a comonad on $B$. 
Adjoints and (co)monads

Adjoint pair: \( F : \mathcal{A} \to \mathcal{B}, \ G : \mathcal{B} \to \mathcal{A} \)

\[ \eta : 1_{\mathcal{A}} \to GF, \quad \varepsilon : FG \to 1_{\mathcal{B}} \]

Related monad on \( \mathcal{A} \)

\( T = GF : \mathcal{A} \to \mathcal{A}, \) product \( m : GFGF \xrightarrow{G\varepsilon F} GF, \)

unit \( \eta : 1_{\mathcal{A}} \to GF. \)
Adjoints and (co)monads

Adjoint pair: \( F : A \to B, \ G : B \to A \)

\[ \eta : 1_A \to GF, \quad \varepsilon : FG \to 1_B \]

Related monad on \( A \)

\[ T = GF : A \to A, \quad \text{product} \quad m : GFGF \xrightarrow{G\varepsilon F} GF, \]

\[ \text{unit} \quad \eta : 1_A \to GF. \]

\((T, m : TT \to T, \eta : 1_A \to T)\) is a \textbf{monad} on \( A \).
## Adjoint pair: \( F: A \rightarrow B, \ G: B \rightarrow A \)

\[
\begin{align*}
\eta & : 1_A \rightarrow GF, \\
\varepsilon & : FG \rightarrow 1_B 
\end{align*}
\]

**Related monad on \( A \)**

\[
T = GF : A \rightarrow A, \quad \text{product} \quad m : GFGF \xrightarrow{G\varepsilon F} GF, \\
\text{unit} \quad \eta : 1_A \rightarrow GF.
\]

\((T, m : TT \rightarrow T, \eta : 1_A \rightarrow T)\) is a **monad** on \( A \).

**Related comonad on \( B \)**

\[
K = FG : B \rightarrow B, \quad \text{coproduct} \quad \delta : FG \xrightarrow{F\eta G} FGFG, \\
\text{counit} \quad \varepsilon : FG \rightarrow 1_B.
\]
Adjoints and (co)monads

Adjoint pair: \( F : A \to B, \ G : B \to A \)

\[ \eta : 1_A \to GF, \ \varepsilon : FG \to 1_B \]

Related monad on \( A \)

\( T = GF : A \to A, \) product \( m : GFGF \xrightarrow{G\varepsilon F} GF, \) unit \( \eta : 1_A \to GF. \)

\((T, m : TT \to T, \eta : 1_A \to T)\) is a monad on \( A.\)

Related comonad on \( B \)

\( K = FG : B \to B, \) coproduct \( \delta : FG \xrightarrow{F\eta G} FGFG, \) counit \( \varepsilon : FG \to 1_B. \)

\((K, \delta : K \to KK, \varepsilon : K \to 1_B)\) is a comonad on \( B.\)
Adjoint pair: $F \dashv G : \mathcal{B} \to \mathcal{A}$, $\eta : 1_{\mathcal{A}} \to GF$, $\varepsilon : FG \to 1_{\mathcal{B}}$
Adjoint pair: $F \vdash G : \mathcal{B} \to \mathcal{A}$, $\eta : 1_{\mathcal{A}} \to GF$, $\varepsilon : FG \to 1_{\mathcal{B}}$

$\varepsilon$ split epi ($G$ separable): monad $T = GF : \mathcal{A} \to \mathcal{A}$

coproduct $\delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF$, $m \circ \delta' = 1_{GF}$
Adjoints and (co)monads

Adjoint pair: $F \dashv G : \mathcal{B} \to \mathcal{A}$, $\eta : 1_{\mathcal{A}} \to GF$, $\varepsilon : FG \to 1_{\mathcal{B}}$

$\varepsilon$ split epi ($G$ separable) : monad $T = GF : \mathcal{A} \to \mathcal{A}$

Coproduct $\delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF$, $m \circ \delta' = 1_{GF}$

$(T, m, \delta')$ with Frobenius condition (without counit)

\[
\begin{array}{ccc}
TT & \xrightarrow{\delta T} & TTT \\
m \downarrow & & \downarrow \text{Tm} \\
T & \xrightarrow{\delta} & TT,
\end{array}
\quad
\begin{array}{ccc}
TT & \xrightarrow{T \delta} & TTT \\
m \downarrow & & \downarrow \text{mT} \\
T & \xrightarrow{\delta} & TT,
\end{array}
\]

$T$ separable monad

$\eta$ split mono ($F$ separable) : comonad $K = FG : \mathcal{B} \to \mathcal{B}$

Product $m' := FGFGF$, $m' \circ \delta = 1_{FG}$

$(K, \delta, m')$ with Frobenius condition (without unit)

\[
\begin{array}{ccc}
TT & \xrightarrow{\delta T} & TTT \\
m \downarrow & & \downarrow \text{Tm} \\
T & \xrightarrow{\delta} & TT,
\end{array}
\quad
\begin{array}{ccc}
TT & \xrightarrow{T \delta} & TTT \\
m \downarrow & & \downarrow \text{mT} \\
T & \xrightarrow{\delta} & TT,
\end{array}
\]
Adjoints and (co)monads

Adjoint pair: \( F \dashv G : \mathbb{B} \to \mathbb{A}, \ \eta : 1_\mathbb{A} \to GF, \ \varepsilon : FG \to 1_\mathbb{B} \)

\( \varepsilon \) split epi (\( G \) separable): monad \( T = GF : \mathbb{A} \to \mathbb{A} \)

Coproduct \( \delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF, \ m \circ \delta' = 1_{GF} \)

\((T, m, \delta')\) with Frobenius condition (without counit)

\[
\begin{align*}
T T & \xrightarrow{\delta T} T T T \\
m & \downarrow \quad T m & m & \downarrow \quad m T \\
T & \xrightarrow{\delta} T T, & T & \xrightarrow{\delta} T T,
\end{align*}
\]
Adjoints and (co)monads

**Adjoint pair:** \( F \dashv G : \mathcal{B} \to \mathcal{A}, \; \eta : 1_\mathcal{A} \to GF, \; \varepsilon : FG \to 1_\mathcal{B} \)

**\( \varepsilon \) split epi (\( G \) separable):** monad \( T = GF : \mathcal{A} \to \mathcal{A} \)

- Coproduct \( \delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF, \quad m \circ \delta' = 1_{GF} \)
- \((T, m, \delta')\) with Frobenius condition (without counit)

\[
\begin{array}{cccc}
TT & \xrightarrow{\delta T} & TTT & \xrightarrow{T\delta} & TTT \\
\downarrow m & & \downarrow Tm & & \downarrow mT \\
T & \xrightarrow{\delta} & TT, & & T & \xrightarrow{\delta} & TT,
\end{array}
\]

**\( \eta \) split mono (\( F \) separable):** comonad \( K = FG : \mathcal{B} \to \mathcal{B} \)

- Product \( m' := FGFG \xrightarrow{F\eta^{-1}G} FG, \quad m' \circ \delta = 1_{FG} \)
Adjoint pair: \( F \dashv G : B \to A, \) \( \eta : 1_A \to GF, \) \( \varepsilon : FG \to 1_B \)

\( \varepsilon \) split epi (\( G \) separable): monad \( T = GF : A \to A \)

coproduct \( \delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF, \) \( m \circ \delta' = 1_{GF} \)

\((T, m, \delta')\) with Frobenius condition (without counit)

\[
\begin{array}{ccc}
TT & \xrightarrow{\delta T} & TTT \\
\downarrow m & & \downarrow Tm \\
T & \xrightarrow{\delta} & TT,
\end{array}
\]

\[
\begin{array}{ccc}
TT & \xrightarrow{T\delta} & TTT \\
\downarrow m \downarrow mT & & \downarrow mT \\
T & \xrightarrow{\delta} & TT,
\end{array}
\]

\( T \) separable monad

\( \eta \) split mono (\( F \) separable): comonad \( K = FG : B \to B \)

product \( m' := FGFG \xrightarrow{F\eta^{-1}G} FG, \) \( m' \circ \delta = 1_{FG} \)

\((K, \delta, m')\) with Frobenius condition (without unit)
Adjoint pair: $F \dashv G : \mathbb{B} \rightarrow \mathbb{A}$, $\eta : 1_\mathbb{A} \rightarrow GF$, $\varepsilon : FG \rightarrow 1_\mathbb{B}$

$\varepsilon$ split epi ($G$ separable): monad $T = GF : \mathbb{A} \rightarrow \mathbb{A}$

Coproduct $\delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF$, $m \circ \delta' = 1_{GF}$

$(T, m, \delta')$ with Frobenius condition (without counit)

$$
\begin{array}{ccc}
TT & \xrightarrow{\delta T} & TTT \\
m \downarrow & & \downarrow Tm \\
T & \xrightarrow{\delta} & TT,
\end{array}
\quad
\begin{array}{ccc}
TT & \xrightarrow{T\delta} & TTT \\
m \downarrow & & \downarrow mT \\
T & \xrightarrow{\delta} & TT,
\end{array}
$$

$T$ separable monad

$\eta$ split mono ($F$ separable): comonad $K = FG : \mathbb{B} \rightarrow \mathbb{B}$

Product $m' : =: FGFG \xrightarrow{F\eta^{-1}G} FG$, $m' \circ \delta = 1_{FG}$

$(K, \delta, m')$ with Frobenius condition (without unit)

$K$ coseparable comonad
### Module categories - \( R M_S \)

\[ M \otimes_S - : SM \rightarrow RM, \quad \text{Hom}_R(M, -) : RM \rightarrow SM \]
# Adjoint pairs

### Module categories - $RM_S$

| Homomorphism                  | Description                                      |
|------------------------------|--------------------------------------------------|
| $M \otimes_S -$ : $SM \to RM$ | Homomorphism                                     |
| $\text{Hom}_R(M, -) : RM \to SM$ | Homomorphism                                     |

### Monad and Comonad

- **Monad**
  - $\text{Hom}_A(M, M \otimes_S -) : SM \to SM$,
  - $M \otimes_S \text{Hom}_R(M, -) : RM \to RM$

- **Comonad**
  - $\eta : (-) \to \text{Hom}_R(M, M \otimes_S -)$,
  - $\varepsilon : M \otimes_S \text{Hom}_R(M, -) \to (-)$,
Adjoint pairs and (co)monads

Module categories \(-RM_S\)

\[ M \otimes_S - : S \mathcal{M} \to \mathcal{R} \mathcal{M}, \quad \text{Hom}_R(M, -) : \mathcal{R} \mathcal{M} \to S \mathcal{M} \]

\[ \eta : (-) \to \text{Hom}_R(M, M \otimes_S -), \]
\[ \varepsilon : M \otimes_S \text{Hom}_R(M, -) \to (-), \]

monad \( \text{Hom}_A(M, M \otimes_S -) : S \mathcal{M} \to S \mathcal{M}, \)
comonad \( M \otimes_S \text{Hom}_R(M, -) : \mathcal{R} \mathcal{M} \to \mathcal{R} \mathcal{M} \)

\( \varepsilon \) isomorphisms: \( R \mathcal{M} \) generator

\( M_S \) fin. gen., projective, \( R \simeq \text{End}(M_S) \): comonad

\[ M \otimes_S \text{Hom}_R(M, -) \simeq M \otimes_S M^* \otimes_R (-) : \mathcal{R} \mathcal{M} \xrightarrow{\sim} \mathcal{R} \mathcal{M} \]
## Adjoint pairs and (co)monads

### Module categories \(- \mathcal{M}_S\)

| Monad | \(\text{Hom}_A(M, M \otimes_S -) : \mathcal{M}_S \to \mathcal{M}_S\), |
|---|---|
| Comonad | \(M \otimes_S \text{Hom}_R(M, -) : \mathcal{M}_R \to \mathcal{M}_R\), |

\[\eta : (-) \to \text{Hom}_R(M, M \otimes_S -),\]
\[\varepsilon : M \otimes_S \text{Hom}_R(M, -) \to (-),\]

\[\varepsilon\text{ isomorphisms: } \mathcal{M}_R \text{ generator}\]

\(M_S\) fin. gen., projective, \(R \simeq \text{End}(M_S)\): comonad

\[M \otimes_S \text{Hom}_R(M, -) \simeq M \otimes_S M^* \otimes_R - : \mathcal{M}_R \xrightarrow{\sim} \mathcal{M}_R\]

### Adjoint pair for \(\mathcal{M}^*_R\): comonad

\[M^* \otimes_R \text{Hom}_S(M^*, -) \simeq M^* \otimes_R M \otimes_S - : \mathcal{M}_S \to \mathcal{M}_S\]
Adjoints and (co)monads

Module categories - $R M_S$

$M \otimes_S - : sM \to R M$, \quad $\text{Hom}_R(M, -) : R M \to sM$

$\eta : (-) \to \text{Hom}_R(M, M \otimes_S -)$, \quad $\varepsilon : M \otimes_S \text{Hom}_R(M, -) \to (-)$,

monad $\text{Hom}_A(M, M \otimes_S -) : sM \to sM$, \quad comonad $M \otimes_S \text{Hom}_R(M, -) : R M \to R M$

$\varepsilon$ isomorphisms: $R M$ generator

$M_S$ fin. gen., projective, $R \simeq \text{End}(M_S)$: comonad

$M \otimes_S \text{Hom}_R(M, -) \simeq M \otimes_S M^* \otimes_R - : R M \xrightarrow{\simeq} R M$

Adjoint pair for $sM^*_R$: comonad

$M^* \otimes_R \text{Hom}_S(M^*, -) \simeq M^* \otimes_R M \otimes_S - : sM \to sM$

$M \otimes_S M^*$ $R$-coring, \quad $M^* \otimes_R M$ $S$-coring
### Module categories - $\_RM_S$

| $\_M \otimes_S - : \_S\_M \to \_R\_M$ | $\text{Hom}_R(M, -) : \_R\_M \to \_S\_M$ |
## Adjoint and (co)monads

| Module categories - $\mathcal{R} \mathcal{M}_S$ |
|-----------------------------------------------|
| $\mathcal{M} \otimes_S - : \mathcal{S}\mathcal{M} \rightarrow \mathcal{R}\mathcal{M}$, $\text{Hom}_\mathcal{R}(\mathcal{M}, -) : \mathcal{R}\mathcal{M} \rightarrow \mathcal{S}\mathcal{M}$ |

| $\eta : (-) \rightarrow \text{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{M} \otimes_S -)$ isomorphism |
|-----------------------------------------------|
| $\mathcal{M} \otimes_S \text{Hom}_\mathcal{R}(\mathcal{M}, -) : \mathcal{R}\mathcal{M} \rightarrow \mathcal{R}\mathcal{M}$, idempotent comonad, |
Adjoint functors and (co)monads

Module categories - $R M_S$

\[ M \otimes_S - : S M \to R M, \quad \text{Hom}_R(M, -) : R M \to S M \]

\[ \eta : (-) \to \text{Hom}_R(M, M \otimes_S -) \text{ isomorphism} \]

\[ M \otimes_S \text{Hom}_R(M, -) : R M \to R M, \text{ idempotent comonad,} \]
\[ \text{left exact, commutes with products, for } Q \in \text{Pres}(M) \text{ cogenerator} \]

\[ M \otimes_S \text{Hom}_R(M, -) \cong \text{Hom}_R(M \otimes_S \text{Hom}_R(M, Q), -) \]
Adjoints and (co)monads

Module categories - \( R \mathcal{M}_S \)

\[ M \otimes_S - : \mathcal{M}_S \to R\mathcal{M}, \quad \text{Hom}_R(M, -) : R\mathcal{M} \to \mathcal{M}_S \]

\( \eta : (-) \to \text{Hom}_R(M, M \otimes_S -) \) isomorphism

\[ M \otimes_S \text{Hom}_R(M, -) : R\mathcal{M} \to R\mathcal{M}, \text{ idempotent comonad, left exact, commutes with products, for } Q \in \text{Pres}(\mathcal{M}) \text{ cogenerator} \]

\[ M \otimes_S \text{Hom}_R(M, -) \cong \text{Hom}_R(M \otimes_S \text{Hom}_R(M, Q), -) \]

\( R \) cogenerator in \( R\mathcal{M} \)

\[ \text{Hom}_R(M \otimes_S M^*, -) \cong \text{Hom}_R(M \otimes_S M^*, \text{Hom}_R(M \otimes_S M^*, -)) \]

\[ \cong \text{Hom}_R(M \otimes_S M^* \otimes_R M \otimes_S M^*, -) \]

\( M \otimes_S M^* \) is \( R \)-ring
Adjoint monads and comonads

Adjoint endofunctors $F : \mathbb{A} \to \mathbb{A}$, $G : \mathbb{A} \to \mathbb{A}$

$\text{Mor}_\mathbb{A}(F(X), Y) \xrightarrow{\varphi} \text{Mor}_\mathbb{A}(X, G(Y))$, \hspace{1em} $\eta : 1_\mathbb{A} \to GF$, \hspace{1em} $\varepsilon : FG \to 1_\mathbb{A}$. 
Adjoint monads and comonads

Adjoint endofunctors \( F : \mathcal{A} \to \mathcal{A}, \ G : \mathcal{A} \to \mathcal{A} \)

\[
\text{Mor}_\mathcal{A}(F(X), Y) \xrightarrow{\varphi} \text{Mor}_\mathcal{A}(X, G(Y)), \quad \eta : 1_\mathcal{A} \to GF,
\]

\[
\varepsilon : FG \to 1_\mathcal{A}.
\]

\( F \) monad, \( m : FF \to F, \ e : I_\mathcal{A} \to F \)

\[
\text{Mor}_\mathcal{A}(F(X), Y) \xrightarrow{\varphi_{X,Y}} \text{Mor}_\mathcal{A}(X, G(Y))
\]

\[
\begin{array}{c}
\text{Mor}(m_X, Y) \\
\downarrow
\end{array}
\]

\[
\text{Mor}_\mathcal{A}(FF(X), Y) \xrightarrow{\simeq} \text{Mor}_\mathcal{A}(X, GG(Y))
\]

\[
\begin{array}{c}
\text{Mor}(X, ?) \\
\downarrow
\end{array}
\]

implies \( G \) comonad, \( \delta : G \to GG, \ \varepsilon : G \to I_\mathcal{A} \).
Adjoint monads and comonads

**Adjoint endofunctors** \( F : \mathbb{A} \to \mathbb{A}, \ G : \mathbb{A} \to \mathbb{A} \)

\[
\text{Mor}_\mathbb{A}(F(X), Y) \xrightarrow{\phi} \text{Mor}_\mathbb{A}(X, G(Y)), \quad \eta : 1_\mathbb{A} \to GF, \\
\varepsilon : FG \to 1_\mathbb{A}.
\]

**F monad,** \( m : FF \to F, \ e : I_\mathbb{A} \to F \)

\[
\text{Mor}_\mathbb{A}(F(X), Y) \xrightarrow{\phi_{X,Y}} \text{Mor}_\mathbb{A}(X, G(Y)) \\
\text{Mor}(m_X,Y) \downarrow \quad \quad \quad \quad \quad \downarrow \text{Mor}(X,?) \\
\text{Mor}_\mathbb{A}(FF(X), Y) \xrightarrow{\sim} \text{Mor}_\mathbb{A}(X, GG(Y))
\]

implies **G comonad,** \( \delta : G \to GG, \ \varepsilon : G \to I_\mathbb{A} \)

\[
\delta : G \xrightarrow{\eta^G} GFG \xrightarrow{G\eta^G} GGFFG \xrightarrow{GGm^G} GGFG \xrightarrow{GG\varepsilon} GG, \\
\varepsilon : G \xrightarrow{e^G} FG \xrightarrow{\varepsilon} 1_\mathbb{A}.
\]
Adjoint endofunctors

(F, m, e) monad, F ⊣ G

(G, δ, ε) is comonad, equivalence of categories

\[ \mathbb{A}_F \simeq \mathbb{A}_G \]

\[ F(A) \xrightarrow{h} A \quad \leftrightarrow \quad A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A) \]
Adjoint endofunctors

\[ (F, m, e) \text{ monad, } F \dashv G \]

\((G, \delta, \varepsilon)\) is comonad, equivalence of categories

\[ \mathbb{A}_F \cong \mathbb{A}_G \]

\[ F(A) \xrightarrow{h} A \iff A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A) \]

\[ F \dashv F \text{ (Frobenius monad)} \Rightarrow \mathbb{A}_F \cong \mathbb{A}_F \]
Adjoint endofunctors

\[(F, m, e) \text{ monad, } F \dashv G\]

\[(G, \delta, \varepsilon) \text{ is comonad, equivalence of categories} \quad \mathbb{A}_F \simeq \mathbb{A}^G\]

\[F(A) \xrightarrow{h} A \iff A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)\]

\[F \dashv F \text{ (Frobenius monad)} \quad \Rightarrow \quad \mathbb{A}_F \simeq \mathbb{A}^F\]

\[(G, \delta, \varepsilon) \text{ comonad, } G \dashv F\]

\[(F, m, e) \text{ is monad, equival. of Kleisli categories} \quad \mathbb{A}^G \simeq \mathbb{A}_F\]
Adjoint endofunctors

\[(F, m, e) \text{ monad, } F \dashv G\]

\[(G, \delta, \varepsilon) \text{ is comonad, equivalence of categories } \mathcal{A}_F \simeq \mathcal{A}^G\]

\[F(A) \xrightarrow{h} A \iff A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)\]

\[F \dashv F \text{ (Frobenius monad) } \Rightarrow \mathcal{A}_F \simeq \mathcal{A}^F\]

\[(G, \delta, \varepsilon) \text{ comonad, } G \dashv F\]

\[(F, m, e) \text{ is monad, equival. of Kleisli categories } \tilde{\mathcal{A}}^G \simeq \tilde{\mathcal{A}}_F\]

\[\text{Mor}_{\tilde{\mathcal{A}}^G}(\phi^G(A), \phi^G(A')) \simeq \text{Mor}_A(G(A), A')\]
Adjoint endofunctors

\((F, m, e)\) monad, \(F \dashv G\)

\((G, \delta, \varepsilon)\) is comonad, equivalence of categories \(\mathbb{A}_F \simeq \mathbb{A}^G\)

\(F(A) \xrightarrow{h} A \iff A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)\)

\(F \dashv F\) (Frobenius monad) \(\Rightarrow \mathbb{A}_F \simeq \mathbb{A}^F\)

\((G, \delta, \varepsilon)\) comonad, \(G \dashv F\)

\((F, m, e)\) is monad, equival. of Kleisli categories \(\tilde{\mathbb{A}}^G \simeq \tilde{\mathbb{A}}_F\)

\(\text{Mor}_{\mathbb{A}^G}(\phi^G(A), \phi^G(A'))\) \(\simeq\) \(\text{Mor}_{\mathbb{A}}(G(A), A')\)

\(\simeq\) \(\text{Mor}_{\mathbb{A}}(A, F(A'))\)
Adjoint endofunctors

\((F, m, e)\) \text{ monad, } F \dashv G

\((G, \delta, \epsilon)\) is comonad, equivalence of categories \(\mathbb{A}_F \simeq \mathbb{A}^G\)

\[
F(A) \xrightarrow{h} A \quad \leftrightarrow \quad A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)
\]

\(F \dashv F\) (Frobenius monad) \(\Rightarrow\) \(\mathbb{A}_F \simeq \mathbb{A}^F\)

\((G, \delta, \epsilon)\) \text{ comonad, } G \dashv F

\((F, m, e)\) is monad, equival. of Kleisli categories \(\tilde{\mathbb{A}}^G \simeq \tilde{\mathbb{A}}_F\)

\[
\text{Mor}_{\mathbb{A}^G}(\phi^G(A), \phi^G(A')) \simeq \text{Mor}_{\mathbb{A}}(G(A), A') \\
\text{Mor}_{\mathbb{A}}(A, F(A')) \simeq \text{Mor}_{\mathbb{A}^F}(\phi_F(A), \phi_F(A'))
\]
## Module categories

### Adjoint endofunctors

\[ A \otimes_R (-), \text{Hom}_R(A, -) : \text{RMod} \to \text{RMod} \]

\[ \text{Hom}_R(A \otimes_R X, Y) \cong \text{Hom}_R(X, \text{Hom}_R(A, Y)) \]
Module categories

Adjoint endofunctors \( A \otimes_R - \), \( \text{Hom}_R(A, -) : R\text{M} \to R\text{M} \)

\[
\text{Hom}_R(A \otimes_R X, Y) \cong \text{Hom}_R(X, \text{Hom}_R(A, Y))
\]

\( A \otimes_R - \) monad \( \iff \) \( \text{Hom}_R(A, -) \) comonad, \( M_A \cong M^{\text{Hom}(A, -)} \)
## Module categories

**Adjoint endofunctors** \( A \otimes_R - \), \( \text{Hom}_R(A, -) : R\mathcal{M} \to R\mathcal{M} \)

\[
\text{Hom}_R(A \otimes_R X, Y) \xrightarrow{\sim} \text{Hom}_R(X, \text{Hom}_R(A, Y))
\]

\( A \otimes_R - \) monad \( \Leftrightarrow \) \( \text{Hom}_R(A, -) \) comonad, \( \mathcal{M}_A \simeq \mathcal{M}^{\text{Hom}(A,-)} \)

**Frobenius monad** \( A \otimes_R - \simeq \text{Hom}_R(A, -) \), \( \mathcal{M}_A \simeq \mathcal{M}^A \)
### Module categories

| Adjoint endofunctors | \( A \otimes_R - \), \( \text{Hom}_R(A, -) : R\text{M} \to R\text{M} \) |
|----------------------|-------------------------------------------------------------|
| \( \text{Hom}_R(A \otimes_R X, Y) \overset{\sim}{\to} \text{Hom}_R(X, \text{Hom}_R(A, Y)) \) |

| \( A \otimes_R - \) monad | \( \Leftrightarrow \) | \( \text{Hom}_R(A, -) \) comonad, \( \tilde{M}_A \simeq \tilde{M}^{\text{Hom}(A,-)} \) |

| Frobenius monad | \( A \otimes_R - \simeq \text{Hom}_R(A, -) \), \( \tilde{M}_A \simeq \tilde{M}^A \) |

| \( A \otimes_R - \) comonad (\( R \)-coring) | \( \Leftrightarrow \) | \( \text{Hom}_R(A, -) \) monad |
| \( \tilde{M}^A \simeq \tilde{M}_{\text{Hom}_R(A,-)} \), \( A \otimes_R X \leftrightarrow \text{Hom}_R(A, X) \) |
### Module categories

#### Adjoint endofunctors

\[
\text{Adjunct endofunctors } A \otimes_R -, \; \text{Hom}_R(A, -) : R^M \to R^M
\]

\[
\text{Hom}_R(A \otimes_R X, Y) \cong \text{Hom}_R(X, \text{Hom}_R(A, Y))
\]

#### Frobenius monad

\[
A \otimes_R - \text{ monad } \iff \text{Hom}_R(A, -) \text{ comonad}, \quad M_A \simeq M^{\text{Hom}(A, -)}
\]

#### Frobenius monad

\[
A \otimes_R - \simeq \text{Hom}_R(A, -), \quad M_A \simeq M^A
\]

#### Frobenius monad

\[
A \otimes_R - \text{ comonad (R-coring) } \iff \text{Hom}_R(A, -) \text{ monad}
\]

\[
\tilde{M}^A \simeq \tilde{M}^{\text{Hom}_R(A, -)}, \quad A \otimes_R X \mapsto \text{Hom}_R(A, X)
\]

#### RA fin. gen., projective

\[
\text{Hom}_R(A, -) \simeq A^* \otimes_R -
\]

\[
A \otimes_R - \text{ monad } \iff \quad A^* \otimes_R - \text{ comonad}
\]

\[
A \otimes_R - \text{ comonad } \iff \quad A^* \otimes_R - \text{ monad}
\]

Frobenius \quad A \simeq A^*, \quad M_A \simeq M^A
Hopf algebras

Heinz Hopf

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: $\mathbb{R}$-algebra ($A, m, \iota$), $\mathbb{R}$-coalgebra ($A, \delta, \varepsilon$)

$\delta: A \to A \otimes \mathbb{R}$,
$\varepsilon: A \to \mathbb{R}$

algebra morphisms

Hopf algebra: antipode

$S: A \to A \otimes A \delta \otimes A \xrightarrow{m^{-1}} A \otimes A = 1_{A \otimes A}$

equivalence

$A \otimes R \xrightarrow{M_R} A M_R$ (Hopf modules)
Hopf algebras

Heinz Hopf

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: \( R \)-algebra \((A, m, \iota)\), \( R \)-coalgebra \((A, \delta, \varepsilon)\)

\[ \delta : A \rightarrow A \otimes_R A, \quad \varepsilon : A \rightarrow R \quad \text{algebra morphisms} \]
Hopf algebras

Heinz Hopf

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: $R$-algebra $(A, m, ι)$, $R$-coalgebra $(A, δ, ε)$

$δ : A \to A \otimes_R A, \quad ε : A \to R$ algebra morphisms

Hopf algebra: antipode $S : A \to A$

$A \otimes A \xrightarrow{δ \otimes A} A \otimes A \otimes A \xrightarrow{A \otimes m} A \otimes A = 1_{A \otimes A}$
Heinz Hopf

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

Bialgebra: \( R \)-algebra \((A, m, \iota)\), \( R \)-coalgebra \((A, \delta, \varepsilon)\)

\[
\delta : A \rightarrow A \otimes_R A, \quad \varepsilon : A \rightarrow R \quad \text{algebra morphisms}
\]

Hopf algebra: antipode \( S : A \rightarrow A \)

\[
A \otimes A \xrightarrow{\delta \otimes A} A \otimes A \otimes A \xrightarrow{A \otimes m} A \otimes A = 1_{A \otimes A}
\]

equivalence \( A \otimes_R - : \text{\textbf{M}}_R \rightarrow \text{\textbf{A}}^\text{\textbf{M}} \) (Hopf modules)
Composition of monads and comonads

Tensor product of $R$-algebras $(A, m, e), (B, m', e')$

\[ A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \tau \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B \]

Distributive law: $\tau : B \otimes_R A \rightarrow A \otimes_R B$

\[ B \otimes B \otimes A \xrightarrow{m' \otimes A} B \otimes A \]

\[ B \otimes A \otimes B \xrightarrow{\tau \otimes B} A \otimes B \otimes B \xrightarrow{A \otimes m'} A \otimes B, \]

\[ A \xrightarrow{e' \otimes A} B \otimes A \]

\[ A \otimes e' \xrightarrow{A \otimes e'} A \otimes B. \]
Composition of monads and comonads

Tensorproduct of $R$-algebras $(A, m, e), (B, m', e')$

$$A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \tau \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B$$

Distributive law: $\tau : B \otimes_R A \rightarrow A \otimes_R B$

$$B \otimes B \otimes A \xrightarrow{m' \otimes A} B \otimes A \xrightarrow{\tau} B \otimes A \xrightarrow{e' \otimes A} A \otimes A \xrightarrow{\tau} A \otimes B.$$ $\tau = A \otimes B \otimes -$  

Lifting of endofunctors

$$B^M \xrightarrow{?} B^M \xrightarrow{U_B} M, A \otimes_R - \xrightarrow{A \otimes \tau} A \otimes B \otimes M \xrightarrow{U_B} M,$$
Composition of monads and comonads

**Liftings of endofunctors**

$F, G : A \to A$, $(F, m, e)$ monad, consider the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{GF} & A \\
\downarrow{U_F} & & \downarrow{U_F} \\
\text{A} & \xrightarrow{G} & \text{A}
\end{array}
$$

Questions:
- Does a lifting $G$ exist?
- $F$ and $G$ monads, when is $G$ a monad?
- $F$ monad, $G$ comonad, when is $G$ a comonad?

Beck, J., *Distributive laws*, 1969
Composition of monads and comonads

**Liftings of endofunctors**

\( F, G : \mathbb{A} \to \mathbb{A}, \ (F, m, e) \) monad, consider the diagram

\[
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\overline{G}} & \mathbb{A} \\
\downarrow U_F & & \downarrow U_F \\
\mathbb{A} & \xrightarrow{G} & \mathbb{A}
\end{array}
\]

**Questions**

- does a lifting \( \overline{G} \) exist?
Composition of monads and comonads

### Liftings of endofunctors

$F, G : A \rightarrow A$, $(F, m, e)$ monad, consider the diagram

\[
\begin{array}{ccc}
A_F & \xrightarrow{\overline{G}} & A_F \\
U_F \downarrow & & \downarrow U_F \\
A & \xrightarrow{G} & A,
\end{array}
\]

### Questions

- does a lifting $\overline{G}$ exist?
- $F$ and $G$ monads, when is $\overline{G}$ a monad?

**References**

Beck, J., *Distributive laws*, 1969
Composition of monads and comonads

**Liftings of endofunctors**

$F, G : \mathbb{A} \rightarrow \mathbb{A}$, $(F, m, e)$ monad, consider the diagram

\[
\begin{array}{ccc}
\mathbb{A}F & \xrightarrow{\overline{G}} & \mathbb{A}F \\
U_F \downarrow & & \downarrow U_F \\
\mathbb{A} & \xrightarrow{G} & \mathbb{A},
\end{array}
\]

**Questions**

- does a lifting $\overline{G}$ exist?
- $F$ and $G$ monads, when is $\overline{G}$ a monad?
- $F$ monad, $G$ comonad, when is $\overline{G}$ a comonad?
Composition of monads and comonads

**Liftings of endofunctors**

$F, G : \mathbb{A} \to \mathbb{A}$, $(F, m, e)$ monad, consider the diagram

$$
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\bar{G}} & \mathbb{A} \\
\downarrow{U_F} & & \downarrow{U_F} \\
\mathbb{A} & \xleftarrow{G} & \mathbb{A}
\end{array}
$$

**Questions**

- does a lifting $\bar{G}$ exist ?
- $F$ and $G$ monads, when is $\bar{G}$ a monad ?
- $F$ monad, $G$ comonad, when is $\bar{G}$ a comonad ?

**Beck, J., *Distributive laws*, 1969**

$F, G : \mathbb{A} \to \mathbb{A}$ natural transformations $\lambda : FG \to GF$
Mixed distributive law (entwining): \((F, m, e), (G, \delta, \varepsilon)\)

lifting \(\overline{G}\) comonad \(\iff\ \lambda : FG \to GF\) with comm. diagrams
Mixed distributive law (entwining): \((F, m, e), (G, \delta, \varepsilon)\)

lifting \(\overline{G}\) comonad \(\iff\) \(\lambda : FG \to GF\) with comm. diagrams

\[
\begin{align*}
FFG & \xrightarrow{m_G} FG & FG & \xrightarrow{F\delta} FGG & \xrightarrow{\lambda_G} GFG \\
F\lambda & \downarrow & \lambda & \downarrow & G\lambda \\
FGF & \xrightarrow{\lambda_F} GFF & \xrightarrow{GM} GF, & GF & \xrightarrow{\delta_F} GGF, \\
\end{align*}
\]
Mixed distributive law (entwining): \((F, m, e), (G, \delta, \varepsilon)\)

lifting \(\overline{G}\) comonad \(\iff\) \(\lambda : FG \to GF\) with comm. diagrams

\[
\begin{align*}
FFG \xrightarrow{m_G} FG & \quad FG \xrightarrow{F\delta} FGG \xrightarrow{\lambda_G} GFG \\
F\lambda \downarrow & \quad \lambda \downarrow & \quad \lambda \downarrow \\
FGF \xrightarrow{\lambda_F} GFF \xrightarrow{Gm} GF, & \quad GFG \xrightarrow{G\lambda} GGF, \\
G \xrightarrow{e_G} FG & \quad FG \xrightarrow{F\varepsilon} F \\
Ge \downarrow & \quad \varepsilon_F \downarrow \\
GF, & \quad GF.
\end{align*}
\]
Mixed distributive law (entwining): \((F, m, e), (G, \delta, \varepsilon)\)

lifting \(\overline{G}\) comonad \(\iff\ \lambda : FG \to GF\) with comm. diagrams

**Diagram:**

\[
\begin{align*}
FFG & \xrightarrow{m_G} FG & FG & \xrightarrow{F\delta} FGG & \xrightarrow{\lambda_G} GFG \\
F\lambda & \downarrow & \lambda & \downarrow & G\lambda \\
FGF & \xrightarrow{\lambda_F} GFF & G & \xrightarrow{G\varepsilon} GGF, \\
F\varepsilon & \downarrow & \varepsilon_F & \downarrow & \\
G & \xrightarrow{e_G} FG & G & \xrightarrow{\delta_F} GGF, \\
Ge & \downarrow & \lambda & \downarrow & \\
& GF, & & GF. \\
\end{align*}
\]

Mixed modules: \(\varrho_A : F(A) \to A, \ \varrho : A \to G(A)\), category \(\Delta^G_F\)

\[
\begin{align*}
F(A) & \xrightarrow{\varrho_A} A & \xrightarrow{\varrho^A} G(A) \\
F(\varrho^A) & \downarrow & \lambda_A & \downarrow \varrho(\varrho_A) \\
FG(A) & \xrightarrow{\lambda_A} GF(A). \\
\end{align*}
\]
Bimonad $B : \mathbb{A} \to \mathbb{A}$, $(B, m, e), (B, \delta, \varepsilon)$

**mixed distributive law** $\lambda : BB \to BB$, compatibility

$$
\begin{array}{ccc}
BB & \xrightarrow{m} & B \\
\downarrow{B\delta} & & \downarrow{\delta} \\
BBB & \xrightarrow{\lambda_B} & BBB.
\end{array}
$$

$B(A) \in \mathbb{A}^B_B$

$\eta : 1_{\mathbb{A}} \to B$ monad morphism, $\varepsilon : B \to 1_{\mathbb{A}}$ comonad morphism
Bimonad $B : \mathbb{A} \to \mathbb{A}, (B, m, e), (B, \delta, \varepsilon)$

**mixed distributive law** $\lambda : BB \to BB$, compatibility

\[ \begin{align*}
BB & \xrightarrow{m} B \xrightarrow{\delta} BB \\
\downarrow B\delta & \quad & \quad \downarrow Bm \\
BBB & \xrightarrow{\lambda_B} BBB.
\end{align*} \]

$B(\mathfrak{A}) \in \mathbb{A} B^B$

$\eta : 1_\mathbb{A} \to B$ monad morphism, $\varepsilon : B \to 1_\mathbb{A}$ comonad morphism

**Category of (mixed) $B$-bimodules $\mathbb{A}^B_B$ - free functor**

$\phi^B_B : \mathbb{A} \to \mathbb{A}^B_B$, $A \leftrightarrow BB(\mathfrak{A}) \xrightarrow{m_A} B(\mathfrak{A}) \xrightarrow{\delta_A} BB(\mathfrak{A})$. 
Bimonad $B : \mathbb{A} \to \mathbb{A}$, $(B, m, e), (B, \delta, \varepsilon)$

**mixed distributive law** $\lambda : BB \to BB$, compatibility

\[
\begin{align*}
BB & \xrightarrow{m} B \xrightarrow{\delta} BB \\
B\delta & \downarrow \quad Bm \quad \lambda_B \\
BBB & \xrightarrow{\lambda_B} BBB.
\end{align*}
\]

$\eta : 1_{\mathbb{A}} \to B$ monad morphism, $\varepsilon : B \to 1_{\mathbb{A}}$ comonad morphism

**Category of (mixed) $B$-bimodules** $\mathbb{A}_B^B$ - free functor

$\phi_B^B : \mathbb{A} \to \mathbb{A}_B^B$, $A \mapsto BB(A) \xrightarrow{m_A} B(A) \xrightarrow{\delta_A} BB(A)$.

full and faithful by

$\text{Mor}_B^B(B(A), B(A')) \simeq \text{Mor}_B(B(A), A') \simeq \text{Mor}_A(A, A')$
Bimonad $B : \mathbb{A} \rightarrow \mathbb{A}$, $(B, m, e), (B, \delta, \varepsilon)$

**mixed distributive law** $\lambda : BB \rightarrow BB$, compatibility

$$
\begin{align*}
BB & \xrightarrow{m} B \xrightarrow{\delta} BB \\
B\delta & \downarrow \\
BBB & \xrightarrow{\lambda_B} BBB.
\end{align*}
$$

$\eta : 1_\mathbb{A} \rightarrow B$ monad morphism, $\varepsilon : B \rightarrow 1_\mathbb{A}$ comonad morphism

**Category of (mixed) $B$-bimodules $\mathbb{A}^B_B$ - free functor**

$$
\phi_B^B : \mathbb{A} \rightarrow \mathbb{A}^B_B, \quad A \mapsto BB(A) \xrightarrow{m_A} B(A) \xrightarrow{\delta_A} BB(A).
$$

full and faithful by

$$
\text{Mor}_B^B(B(A), B(A')) \simeq \text{Mor}_B(B(A), A') \simeq \text{Mor}_A(A, A')
$$

**Hopf monads (antipode $S : B \rightarrow B$)**

$$
\phi_B^B \text{ equivalence} \iff \quad BB \xrightarrow{B\delta} BBB \xrightarrow{mB} BB
$$
Bimonads

**Bimonad on Set**

- $G \times - : \text{Set} \to \text{Set}, A \mapsto G \times A$

Mixed $G \times -$-modules $\text{Set}$

$G \times -$ is a monad, $G \times -$ is a comonad,

$\delta : G \to G \times G, g \mapsto (g, g)$

entwining $\psi : G \times G \to G \times G, (g, h) \mapsto (gh, g)$

Hopf monads on $\text{Set}$

For bimonad $G \times -$ there are equivalent:

$\varphi_G : \text{Set} \to \text{Set}$ is an equivalence;

$G \times -$ has an antipode;

$G$ is a group.
Bimonads

Bimonad on Set

- $G \times - : \text{Set} \to \text{Set}, A \mapsto G \times A$
- $G \times -$ monad, $G$ is monoid
Bimonads

**Bimonad on Set**

- $G \times - : \mathbf{Set} \to \mathbf{Set}$, $A \mapsto G \times A$
- $G \times -$ is monad, $G$ is monoid
- $G \times -$ is comonad, $\delta : G \to G \times G$, $g \mapsto (g, g)$
Bimonads

| Bimonad on Set |
|----------------|
| $G \times - : \textbf{Set} \to \textbf{Set}$, $A \mapsto G \times A$ |
| $G \times -$ monad, $G$ is monoid |
| $G \times -$ comonad, $\delta : G \to G \times G$, $g \mapsto (g, g)$ |
| entwining $\psi : G \times G \to G \times G$, $(g, h) \mapsto (gh, g)$ |
# Bimonads

## Bimonad on Set

- \( G \times - : Set \to Set, A \mapsto G \times A \)
- \( G \times - \) monad, \( G \) is monoid
- \( G \times - \) comonad, \( \delta : G \to G \times G, g \mapsto (g, g) \)
- entwining \( \psi : G \times G \to G \times G, (g, h) \mapsto (gh, g) \)

## Mixed \( G \times - \)-modules \( Set_G^G \)

\[ A \in Set, \quad G \times A \to A \to G \times A \]
Bimonads

Bimonad on Set

- $G \times - : \mathbf{Set} \to \mathbf{Set}$, $A \mapsto G \times A$
- $G \times -$ monad, $G$ is monoid
- $G \times -$ comonad, $\delta : G \to G \times G$, $g \mapsto (g, g)$
- Entwining $\psi : G \times G \to G \times G$, $(g, h) \mapsto (gh, g)$

Mixed $G \times -$-modules $\mathbf{Set}_G$

$A \in \mathbf{Set}$, $G \times A \to A \to G \times A$

Hopf monads on Set

For bimonad $G \times -$ there are equivalent:

- $\phi^G : \mathbf{Set} \to \mathbf{Set}_G^G$ is an equivalence;
### Bimonads

- **Bimonad on Set**
  - $G \times - : \text{Set} \rightarrow \text{Set}, \ A \mapsto G \times A$
  - $G \times -$ monad, $G$ is monoid
  - $G \times -$ comonad, $\delta : G \rightarrow G \times G, \ g \mapsto (g, g)$
  - Entwining $\psi : G \times G \rightarrow G \times G, \ (g, h) \mapsto (gh, g)$

- **Mixed $G \times -$-modules $\text{Set}_G^G$**
  - $A \in \text{Set}, \ G \times A \rightarrow A \rightarrow G \times A$

- **Hopf monads on Set**
  - For bimonad $G \times -$ there are equivalent:
    - $\phi_G^G : \text{Set} \rightarrow \text{Set}_G^G$ is an equivalence;
    - $G \times -$ has an antipode;
### Bimonads

**Bimonad on Set**
- \( G \times - : \text{Set} \to \text{Set}, A \mapsto G \times A \)
- \( G \times - \) monad, \( G \) is monoid
- \( G \times - \) comonad, \( \delta : G \to G \times G, g \mapsto (g, g) \)
- entwining \( \psi : G \times G \to G \times G, (g, h) \mapsto (gh, g) \)

**Mixed \( G \times - \)-modules \( \text{Set}_G \)**
- \( A \in \text{Set}, \quad G \times A \to A \to G \times A \)

**Hopf monads on Set**
For bimonad \( G \times - \) there are equivalent:
- \( \phi^G_G : \text{Set} \to \text{Set}_G^G \) is an equivalence;
- \( G \times - \) has an antipode;
- \( G \) is a group.
Finite dimensional algebras: $\text{Hom}_K(A, -) \simeq A^* \otimes_K -$

- **Adjoint pair**: $(A \otimes_K -, A^* \otimes_K -)$,
- **Counit**: $\varepsilon: A \otimes_K A^* \to K$, $a \otimes f \mapsto f(a)$,
- **Unit**: $\eta: K \to A^* \otimes_K A$, $1 \mapsto \sum a_i^* \otimes a_i$ (dual basis).
Finite dimensional algebras: $\text{Hom}_K(A, -) \simeq A^* \otimes_K -$

- **Adjoint pair**: $(A \otimes_K -, A^* \otimes_K -)$,
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- **Unit**: $\eta : K \to A^* \otimes_K A$, $1 \mapsto \sum a_i^* \otimes a_i$ (dual basis).

- $A \otimes_K -$ monad $\iff A^* \otimes_R -$ comonad, $A M \simeq A^* M$
Finite dimensional algebras: $\text{Hom}_K(A, -) \simeq A^* \otimes_K -$ 

adjoint pair $(A \otimes_K -, A^* \otimes_K -)$,
counit $\varepsilon : A \otimes_K A^* \rightarrow K$, $a \otimes f \mapsto f(a)$,
unit $\eta : K \rightarrow A^* \otimes_K A$, $1 \mapsto \sum a_i^* \otimes a_i$ (dual basis).

$A \otimes_K -$ monad $\iff A^* \otimes_R -$ comonad, $A^!M \simeq A^*M$

entwining $\tau : A \otimes_K A^* \rightarrow A^* \otimes_K A$, $A^!M \simeq A^* \otimes A^!M$,
$A$-coring $A^* \otimes_K A \otimes_A -$: $A^!M \rightarrow A^!M$,
$A^*$-ring $A \otimes_K A^* \otimes A^*$ $-$: $A^!M \rightarrow A^!M$
Finite dimensional algebras: $\text{Hom}_K(A, -) \simeq A^* \otimes_K -$ 

adjoint pair $(A \otimes_K -, A^* \otimes_K -)$,
counit $\varepsilon : A \otimes_K A^* \to K$, $a \otimes f \mapsto f(a)$,
unit $\eta : K \to A^* \otimes_K A$, $1 \mapsto \sum a_i^* \otimes a_i$ (dual basis).

$A \otimes_K -$ monad $\iff A^* \otimes_R -$ comonad, $A M \simeq A^* M$

entwining $\tau : A \otimes_K A^* \to A^* \otimes_K A$, $A^* M \simeq A^* \otimes A M$,
$A$-coring $A^* \otimes_K A \otimes A -$: $A M \to A M$,$A^*$-ring $A \otimes_K A^* \otimes A^* -$: $A^* M \to A^* M$

$A \otimes_K -$ comonad $\iff A^* \otimes_K -$ monad, $A \tilde{M} \simeq A^* \tilde{M}$
Finite dimensional algebras: $\text{Hom}_K(A, -) \simeq A^* \otimes_K -$ 

adjoint pair $(A \otimes_K -, A^* \otimes_K -),$ 
counit $\varepsilon: A \otimes_K A^* \to K, a \otimes f \mapsto f(a),$ 
unit $\eta: K \to A^* \otimes_K A, 1 \mapsto \sum a_i^* \otimes a_i$ (dual basis).

$A \otimes_K -$ monad $\iff A^* \otimes_R -$ comonad, $A \tilde{M} \simeq A^* M$

entwining $\tau: A \otimes_K A^* \to A^* \otimes_K A,$ $A \tilde{M} \simeq A^* \otimes A \tilde{M},$
A-coring $A^* \otimes_K A \otimes_A -: A \tilde{M} \to A \tilde{M},$
A*-ring $A \otimes_K A^* \otimes A^* -: A^* \tilde{M} \to A^* \tilde{M}$

$A \otimes_K -$ comonad $\iff A^* \otimes_K -$ monad, $A \tilde{\tilde{M}} \simeq A^* \tilde{\tilde{M}}$

entwining $\tau: A \otimes_K A^* \to A^* \otimes_K A,$ $A \tilde{\tilde{M}} \simeq A^* \tilde{\tilde{M}},$
monad $A \tilde{\tilde{M}} \to A \tilde{\tilde{M}},$ $A \otimes X \mapsto A \otimes A^* \otimes X$
Finite dimensional algebras: \( \text{Hom}_K(A, -) \cong A^* \otimes_K - \)

| Adjoint pair | \((A \otimes_K -, A^* \otimes_K -)\) |
|--------------|-----------------------------------|
| Counit \( \varepsilon \) | \( A \otimes_K A^* \to K, a \otimes f \mapsto f(a) \) |
| Unit \( \eta \) | \( K \to A^* \otimes_K A, 1 \mapsto \sum a_i^* \otimes a_i \) (dual basis) |

\( A \otimes_K - \) monad \( \iff \) \( A^* \otimes_R - \) comonad, \( \tilde{A}M \cong A^* M \)

| Entwining \( \tau \) | \( A \otimes_K A^* \to A^* \otimes_K A, \quad \tilde{A}M \cong A^* \tilde{M} \) |
|----------------|----------------------------------|
| A-coring       | \( A^* \otimes_K A \otimes A \dashv: \quad \tilde{A}M \to A\tilde{M} \) |
| A*-ring        | \( A \otimes_K A^* \otimes A^* \dashv: \quad \tilde{A}M \to A^* \tilde{M} \) |

\( A \otimes_K - \) comonad \( \iff \) \( A^* \otimes_K - \) monad, \( \tilde{A}\tilde{M} \cong A^* \tilde{M} \)

| Entwining \( \tau \) | \( A \otimes_K A^* \to A^* \otimes_K A, \quad \tilde{A}\tilde{M} \cong A^* \tilde{M} \) |
|----------------|----------------------------------|
| Monad          | \( \tilde{A}\tilde{M} \to \tilde{A}\tilde{M}, \quad A \otimes X \mapsto A \otimes A^* \otimes X \) |
| Comonad        | \( A^* \tilde{M} \to A^* \tilde{M}, \quad A^* \otimes Y \mapsto A^* \otimes A \otimes Y \) |
Finite dimensional algebras: $\text{Hom}_K(A, -) \cong A^* \otimes_K -$ 

Nakayama functors, $D(-) := (-)^*, \ D(A) = A^*$ 

\[
\begin{align*}
\nu(-) & \coloneqq D \text{Hom}_A(-, AA) : \ A - \text{mod} \to A - \text{mod} \\
\nu^-( -) & \coloneqq \text{Hom}_A(D(-), A_A) : \ A - \text{mod} \to A - \text{mod} \\
\nu(X) & \coloneqq D \text{Hom}_A(X, AA) \cong D(A) \otimes_A X, \\
\nu^-(X) & \coloneqq \text{Hom}_A(D(X), A_A) \cong \text{Hom}_A(D(A), X),
\end{align*}
\]
Finite dimensional algebras: $\text{Hom}_K(A, -) \simeq A^* \otimes_K -$

**Nakayama functors,** $D(-) := (-)^*, D(A) = A^*$

| Function | Description | Domain | Codomain |
|----------|-------------|--------|----------|
| $\nu(-)$ | $D \text{Hom}_A(-, AA)$ | $A - \text{mod}$ | $A - \text{mod}$ |
| $\nu^\perp(-)$ | $\text{Hom}_A(D(-), AA)$ | $A - \text{mod}$ | $A - \text{mod}$ |
| $\nu(X)$ | $D \text{Hom}_A(X, AA)$ | $\simeq$ | $D(A) \otimes_A X,$ |
| $\nu^\perp(X)$ | $\text{Hom}_A(D(X), AA)$ | $\simeq$ | $\text{Hom}_A(D(A), X),$ |

**Adjoint pair**

$\text{Hom}_A(D(A), -)$: $A - \text{mod} \to A - \text{mod},$

$D(A) \otimes_A -$ : $A - \text{mod} \to A - \text{mod}$
Thank you!
Bimonads

**Antipode - natural transformation** $S : B \to B$

- $B \xrightarrow{\varepsilon} I \xrightarrow{\eta} B$
- $BB \xrightarrow{S_B} BB$
- $\delta : B \to I$
- $\mu : BB \to B$

Equivalent:
- (a) $B$ has an antipode;
- (b) $\gamma$ is an isomorphism.
Bimonads

**Antipode - natural transformation** $S : B \rightarrow B$

$$
\begin{array}{ccc}
B & \xrightarrow{\varepsilon} & I \\
\downarrow{\delta} & & \uparrow{\mu} \\
BB & \xrightarrow{S_B} & BB
\end{array}
$$

**Natural map**

$$
\gamma : BB \xrightarrow{\delta_B} BBB \xrightarrow{B_m} BB
$$

Equivalent:

(a) $B$ has an antipode;

(b) $\gamma$ is an isomorphism.
Bimonads

**Antipode - natural transformation** $S : B \to B$

$$
\begin{array}{ccc}
B & \xrightarrow{\varepsilon} & I \\
\downarrow{\delta} & & \downarrow{\eta} \\
BB & \xrightarrow{S_B} & BB \\
\end{array}
$$

**Natural map**

$$\gamma : BB \xrightarrow{\delta_B} BBB \xrightarrow{Bm} BB$$

**Equivalent**

(a) $B$ has an antipode;
(b) $\gamma$ is an isomorphism.
Braided bimonads

Consider $\mathcal{B} = (B, m, e, \delta, \varepsilon)$, where $B : A \to A$ is such that $\underline{B} = (B, m, e)$ is a monad and $\overline{B} = (B, \delta, \varepsilon)$ is a comonad.

**Double entwinings**

natural transformation $\tau : BB \to BB$ such that

1. $\tau$ is a mixed distributive law from the monad $\underline{B}$ to the comonad $\overline{B}$;
2. $\tau$ is a mixed distributive law from the comonad $\overline{B}$ to the monad $\underline{B}$. 

These conditions are obviously equivalent to

3. $\tau$ is a monad distributive law for the monad $\underline{B}$;
4. $\tau$ is a comonad distributive law for the comonad $\overline{B}$.
Braided bimonads

Consider $\mathcal{B} = (B, m, e, \delta, \varepsilon)$, where $B : \mathbb{A} \to \mathbb{A}$ is such that $\underline{B} = (B, m, e)$ is a monad and $\overline{B} = (B, \delta, \varepsilon)$ is a comonad.

**Double entwinings**

natural transformation $\tau : BB \to BB$ such that

(i) $\tau$ is a mixed distributive law from the monad $\underline{B}$ to the comonad $\overline{B}$;

(ii) $\tau$ is a mixed distributive law from the comonad $\overline{B}$ to the monad $\underline{B}$.

These conditions are obviously equivalent to

(iii) $\tau$ is a monad distributive law for the monad $\underline{B}$;

(iv) $\tau$ is a comonad distributive law for the comonad $\overline{B}$. 
Braided bimonads

Consider $\mathcal{B} = (B, m, e, \delta, \varepsilon)$, where $B : A \to A$ is such that $\underline{B} = (B, m, e)$ is a monad and $\overline{B} = (B, \delta, \varepsilon)$ is a comonad.

**Double entwinings**

natural transformation $\tau : BB \to BB$ such that

(i) $\tau$ is a mixed distributive law from the monad $\underline{B}$ to the comonad $\overline{B}$;

(ii) $\tau$ is a mixed distributive law from the comonad $\overline{B}$ to the monad $\underline{B}$. 
Braided bimonads

Consider $\mathcal{B} = (B, m, e, \delta, \varepsilon)$, where $B : \mathbb{A} \to \mathbb{A}$ is such that $\underline{B} = (B, m, e)$ is a monad and $\overline{B} = (B, \delta, \varepsilon)$ is a comonad.

Double entwinings

natural transformation $\tau : BB \to BB$ such that

(i) $\tau$ is a mixed distributive law from the monad $\underline{B}$ to the comonad $\overline{B}$;

(ii) $\tau$ is a mixed distributive law from the comonad $\overline{B}$ to the monad $\underline{B}$.

These conditions are obviously equivalent to

(iii) $\tau$ is a monad distributive law for the monad $\underline{B}$;

(iv) $\tau$ is a comonad distributive law for the comonad $\overline{B}$.
Braided bimonads

Let $\tau : BB \to BB$ be a double entwining - commutative

$$BB \xrightarrow{m} B \xrightarrow{\delta} BB$$

$$\delta \delta$$

$$BBBB \xrightarrow{B\tau B} BBBB,$$

Then $\overline{\tau} : BB \xrightarrow{\delta B} BBB \xrightarrow{B\tau} BBB \xrightarrow{mB} BB$ is a **mixed distributive law** from the monad $B$ to the comonad $\overline{B}$. 

$$BB \xrightarrow{B\varepsilon} B$$

$$m \downarrow \quad \varepsilon \downarrow \quad e \downarrow \quad \delta$$

$$B \xrightarrow{\varepsilon} 1,$$

$$1\xleftarrow{e} B \xrightarrow{\varepsilon} 1,$$

$$1 \xrightarrow{e} B \xrightarrow{\varepsilon} 1.$$
Braided bimonads

\( \tau^2 = 1 \) and \( \tau \) satisfies the Yang-Baxter equation

\[
\begin{array}{c}
\text{BBB} \xrightarrow{\tau B} \text{BBB} \xrightarrow{B\tau} \text{BBB} \\
\downarrow B\tau \quad \quad \quad \quad \downarrow \tau B
\end{array}
\]

BBB then

\[
\begin{array}{c}
\text{BBB} \xrightarrow{\tau B} \text{BBB} \xrightarrow{B\tau} \text{BBB} \\
\downarrow B\tau \quad \quad \quad \quad \downarrow \tau B
\end{array}
\]

BBB
Braided bimonads

\(\tau^2 = 1\) and \(\tau\) satisfies the Yang-Baxter equation

\[
\begin{align*}
BBB & \xrightarrow{\tau B} BBB \xrightarrow{B\tau} BBB \\
& \xrightarrow{B\tau} BBB \xrightarrow{\tau B} BBB
\end{align*}
\]

then

\[
\begin{align*}
BBB & \xrightarrow{\tau B} BBB \xrightarrow{B\tau} BBB \\
& \xrightarrow{B\tau} BBB \xrightarrow{\tau B} BBB
\end{align*}
\]

\(BB\) is a bimonad

with multiplication, comultiplication and entwining structure

\[
\begin{align*}
BBBB & \xrightarrow{B\tau B} BBBBB \xrightarrow{mm} BB \\
BB & \xrightarrow{\delta\delta} BBBBB \xrightarrow{B\tau B} BBBBB, \\
BBBB & \xrightarrow{B\tau B} BBBBB \xrightarrow{\tau\tau} BBBBB \xrightarrow{B\tau B} BBBBB
\end{align*}
\]
Given $\tau$ as above, an opposite bimonad $B^{\text{op}}$ can be defined for $B$ with multiplication

$$m \cdot \tau : BB \xrightarrow{\tau} BB \xrightarrow{m} B$$

and comultiplication

$$\tau \cdot \delta : B \xrightarrow{\delta} BB \xrightarrow{\tau} BB.$$  

If $B$ has an antipode $S$, then $S : B^{\text{op}} \to B$ is a bimonad morphism provided that

$$\tau \cdot BS = SB \text{ and } \tau \cdot BS = SB.$$
Module categories

**Coalgebras - comultiplication and counit,** $C \otimes_R \cdot : R M \to R M$

$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$

with coassociativity and counitality conditions.
Module categories

**Coalgebras - comultiplication and counit,** $C \otimes_R - : R\text{M} \rightarrow R\text{M}$

$\Delta : C \rightarrow C \otimes_R C, \quad \varepsilon : C \rightarrow R,$

with coassociativity and counitality conditions.

**C-comodules - category $^C\text{M}$**

$q^M : M \rightarrow C \otimes_R M,$

with compatibility conditions.
Module categories

Coalgebras - comultiplication and counit, $C \otimes_R - : R^M \rightarrow R^M$

$\Delta : C \rightarrow C \otimes_R C$, $\varepsilon : C \rightarrow R$, with coassociativity and counitality conditions.

C-comodules - category $^C_M$

$q^M : M \rightarrow C \otimes_R M$, with compatibility conditions.

Adjoint functors - $M \in R^M$, $X \in ^C_M$

$U^C : ^C_M \rightarrow R^M$, $C \otimes_R - : R^M \rightarrow ^C_M$, $Hom_C(M, C \otimes_R X) \rightarrow Hom_R(M, X)$, $f \mapsto \varepsilon_X \circ f$. 
Module categories

Coalgebras - comultiplication and counit, $C \otimes_R - : \mathcal{R}M \to \mathcal{R}M$

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

with coassociativity and counitality conditions.

C-comodules - category $\mathcal{C}M$

$$\rho^M : M \to C \otimes_R M,$$

with compatibility conditions.

Adjoint functors - $M \in \mathcal{R}M, X \in \mathcal{C}M$

$$U^C : \mathcal{C}M \to \mathcal{R}M, \quad C \otimes_R - : \mathcal{R}M \to \mathcal{C}M,$$

$$\text{Hom}^C(M, C \otimes_R X) \to \text{Hom}_R(M, X), \quad f \mapsto \varepsilon_X \circ f.$$
Modules and comodules on $R^M$

Equivalent for $C \in R^M$:

(a) $C \otimes_R - : R^M \to R^M$ is a comonad ($C$ is an $R$-coalgebra);
(b) $\text{Hom}_R(C, -) : R^M \to R^M$ is a monad.
## Modules and comodules on $\mathbb{R}^M$

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**The category $\mathbb{C}^M$**

1. $\mathbb{C}^M$ has colimits, coproducts and cokernels;
Modules and comodules on $R^M$

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The category $C^M$

(1) $C^M$ has colimits, coproducts and cokernels;
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Modules and comodules on $R^M$

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(1) $C^M$ has colimits, coproducts and cokernels;
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(3) monomorphisms need not be injective maps.
Modules and comodules on $\mathbb{R}^{\mathbb{M}}$

**Equivalent for $C \in \mathbb{R}^{\mathbb{M}}$:**

(a) $C \otimes_{\mathbb{R}} - : \mathbb{R}^{\mathbb{M}} \to \mathbb{R}^{\mathbb{M}}$ is a comonad ($C$ is an $\mathbb{R}$-coalgebra);

(b) $\text{Hom}_{\mathbb{R}}(C, -) : \mathbb{R}^{\mathbb{M}} \to \mathbb{R}^{\mathbb{M}}$ is a monad.

**The category $\mathbb{C}^{\mathbb{M}}$:**

(1) $\mathbb{C}^{\mathbb{M}}$ has colimits, coproducts and cokernels;

(2) $\mathbb{C}^{\mathbb{M}}$ is abelian provided $C_{\mathbb{R}}$ is flat;

(3) monomorphisms need not be injective maps.

**The category $\mathbb{M}[C, -]$:**

(1) $\mathbb{M}[C, -]$ has limits, products and kernels;
Modules and comodules on $R^M$

Equivalent for $C \in R^M$:

(a) $C \otimes_R - : R^M \to R^M$ is a comonad ($C$ is an $R$-coalgebra);
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The category $^C M$

(1) $^C M$ has colimits, coproducts and cokernels;
(2) $^C M$ is abelian provided $C_R$ is flat;
(3) monomorphisms need not be injective maps.

The category $M[C, -]$

(1) $M[C, -]$ has limits, products and kernels;
(2) $M[C, -]$ is abelian provided $C_R$ is projective;
## Modules and comodules on $R^M$

### Equivalent for $C \in R^M$:

(a) $C \otimes_R - : R^M \to R^M$ is a comonad ($C$ is an $R$-coalgebra);
(b) $\text{Hom}_R(C, -) : R^M \to R^M$ is a monad.

### The category $C^M$

1. $C^M$ has colimits, coproducts and cokernels;
2. $C^M$ is abelian provided $C_R$ is flat;
3. monomorphisms need not be injective maps.

### The category $M[C,-]$

1. $M[C,-]$ has limits, products and kernels;
2. $M[C,-]$ is abelian provided $C_R$ is projective;
3. epimorphisms need not be surjective maps.
Monads and Comonads in $R^M$

**Correspondence of categories**

$$\text{Hom}^C(C, -) : {^CM} \rightarrow {^M[C, -]}, \quad M \mapsto \text{Hom}^C(C, M).$$

has left adjoint (contratensor product).
Monads and Comonads in $\mathcal{R}^\mathcal{M}$

**Correspondence of categories**

$\text{Hom}^C(C, -) : \mathcal{C}^\mathcal{M} \to \mathcal{M}_{[C, -]}$, $M \mapsto \text{Hom}^C(C, M)$.

has left adjoint (contratensor product).

**Equivalence of Kleisli categories**

For any $X \in \mathcal{R}^\mathcal{M}$,

$C \otimes_R X \leftrightarrow \text{Hom}^C(C, C \otimes_R X) \simeq \text{Hom}_R(C, X)$,

$\text{Hom}_R(C, X) \leftrightarrow C \otimes_{[C, -]} \text{Hom}_R(C, X) \simeq C \otimes_R X$. 
The monads $\mathcal{R}C^* \otimes_R -$ and $[C, -]$

$\beta_M : C^* \otimes_R M \to \text{Hom}_R(C, M), \quad f \otimes m \mapsto [c \mapsto f(c)m].$
The monads $\mathcal{R}(C^* \otimes_R -)$ and $[C, -]$

$\beta_M : C^* \otimes_R M \to \text{Hom}_R(C, M), \quad f \otimes m \mapsto [c \mapsto f(c)m].$

This yields a functor

$$F : \mathcal{M}[C, -] \to C^* \mathcal{M}$$

$$\text{Hom}_R(C, M) \to M \ni [c \mapsto f(c)m] \mapsto C^* \otimes_R M \xrightarrow{\beta_M} \text{Hom}_R(C, M) \to M.$$
Modules and comodules in $\mathcal{M}$

**The monads $R C^* \otimes_R -$ and $[C, -]$**

\[ \beta_M : C^* \otimes_R M \to \text{Hom}_R(C, M), \quad f \otimes m \mapsto [c \mapsto f(c)m]. \]

This yields a functor

\[ F : \mathcal{M}[C, -] \longrightarrow C^* \mathcal{M} \]

\[ \text{Hom}_R(C, M) \to M \quad \mapsto \quad C^* \otimes_R M \xrightarrow{\beta_M} \text{Hom}_R(C, M) \to M. \]

**The comonads $C \otimes_R -$ and $\text{Hom}_R(C^*, -)$**

\[ \alpha_M : C \otimes_R M \to \text{Hom}_R(C^*, M), \quad c \otimes m \mapsto [f \mapsto f(c)m]. \]
Modules and comodules in $\mathcal{M}$

**The monads $\mathcal{R}C^* \otimes \mathcal{R} -$ and $[C, -]$**

$\beta_M : C^* \otimes \mathcal{R} M \to \text{Hom}_\mathcal{R}(C, M), \quad f \otimes m \mapsto [c \mapsto f(c)m].$

This yields a functor

$F : \mathcal{M}_{[C,-]} \longrightarrow C^*\mathcal{M}$

$\text{Hom}_\mathcal{R}(C, M) \to M \quad \mapsto \quad C^* \otimes \mathcal{R} M \xrightarrow{\beta_M} \text{Hom}_\mathcal{R}(C, M) \to M.$

**The comonads $C \otimes \mathcal{R} -$ and $\text{Hom}_\mathcal{R}(C^*, -)$**

$\alpha_M : C \otimes \mathcal{R} M \to \text{Hom}_\mathcal{R}(C^*, M), \quad c \otimes m \mapsto [f \mapsto f(c)m].$

This yields a functor

$G : \mathcal{M}^C \longrightarrow \mathcal{M}^{(C^*, -)} \cong C^*\mathcal{M}$

$M \to C \otimes \mathcal{R} M \quad \mapsto \quad M \to C \otimes \mathcal{R} M \xrightarrow{\alpha_M} \text{Hom}_\mathcal{R}(C^*, M).$
Module categories

Adjoint functors between module categories \( R^M, S^M \) by \( R^P_S \)

\[
P \otimes_S - : S^M \to R^M, \quad \text{Hom}_R(P, -) : R^M \to S^M.
\]
Module categories

Adjoint functors between module categories $\mathcal{R}\mathcal{M}$, $\mathcal{S}\mathcal{M}$ by $\mathcal{R}\mathcal{P}\mathcal{S}$

$P \otimes_S - : \mathcal{S}\mathcal{M} \to \mathcal{R}\mathcal{M}$, $\operatorname{Hom}_R(P, -) : \mathcal{R}\mathcal{M} \to \mathcal{S}\mathcal{M}$.

Adjunction - $N \in \mathcal{R}\mathcal{M}$, $X \in \mathcal{S}\mathcal{M}$

$\operatorname{Hom}_R(P \otimes_S X, N) \to \operatorname{Hom}_S(X, \operatorname{Hom}_R(P, N))$, $f \mapsto [x \mapsto f(- \otimes x)]$. 
Module categories

**Adjoint functors between module categories** $R^M$, $S^M$ by $RP_S$

\[ P \otimes_S - : S^M \to R^M, \quad \text{Hom}_R(P, -) : R^M \to S^M. \]

**Adjunction** - $N \in R^M$, $X \in S^M$

\[ \text{Hom}_R(P \otimes_S X, N) \to \text{Hom}_S(X, \text{Hom}_R(P, N)), \]
\[ f \mapsto [x \mapsto f(- \otimes x)]. \]

**Count unit and unit**

\[ \varepsilon_M : P \otimes_S \text{Hom}_R(P, M) \to M, \quad p \otimes f \mapsto f(p), \]
\[ \eta_X : X \to \text{Hom}_R(P, P \otimes_S X), \quad x \mapsto [p \mapsto p \otimes x]. \]
Module categories

equivalent

(a) $\varepsilon_M$ is an epi(iso)morphism for all $M \in R^M$;
(b) $\text{Hom}_R(P, -) : R^M \to S^M$ is faithful;
(c) $P$ is a generator in $R^M$. 
## Module categories

**equivalent**

(a) $\varepsilon_M$ is an epi(iso)morphism for all $M \in R\mathbb{M}$;
(b) $\text{Hom}_R(P, -) : R\mathbb{M} \to S\mathbb{M}$ is faithful;
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**equivalent**

(a) $\eta_X$ is an isomorphism for all $X \in S\mathbb{M}$;
(b) $P \otimes_S -$ is full and faithful (faithfully flat).
### Module categories

**equivalent**

- (a) $\varepsilon_M$ is an epi(iso)morphism for all $M \in R\mathcal{M}$;
- (b) $\text{Hom}_R(P, -) : R\mathcal{M} \to S\mathcal{M}$ is faithful;
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**equivalent**

- (a) $\eta_X$ is an isomorphism for all $X \in S\mathcal{M}$;
- (b) $P \otimes_S -$ is full and faithful (faithfully flat).

**equivalent**

- (a) $\eta$ and $\varepsilon$ are isomorphisms;
- (b) $P \otimes_S -$ is an equivalence (with inverse $\text{Hom}_R(P, -)$);
- (c) $R\!P$ is a finitely generated, projective generator and $S = \text{End}_R(P)$. 
equivalent if $S = \text{End}_R(P)$

(a) $\varepsilon_M$ is a monomorphism for all $M \in R \mathcal{M}$,  
$\eta_X$ is an epimorphism for all $X \in S \mathcal{M}$;

(b) $(P \otimes_S -, \text{Hom}_R(P, -))$ induces an equivalence

\[ P \otimes_S - : S \mathcal{M} \text{Hom}_R(P, P \otimes_S -) \longrightarrow R \mathcal{M} P \otimes_S \text{Hom}_R(P, -). \]
Module categories

**equivalent if** $S = \text{End}_R(P)$

(a) $\varepsilon_M$ is a monomorphism for all $M \in \mathcal{R}M$, $\eta_X$ is an epimorphism for all $X \in \mathcal{S}M$;

(b) $(P \otimes_S -, \text{Hom}_R(P, -))$ induces an equivalence

$$P \otimes_S - : \mathcal{S}M\text{Hom}_R(P, P \otimes_S-) \rightarrow \mathcal{R}M \text{Hom}_R(P, -).$$

In this case $P$ is a $\ast$-module (in the sense of Menini-Orsatti).
Module categories

Equivalent if $S = \text{End}_R(P)$

(a) $\varepsilon_M$ is a monomorphism for all $M \in \mathcal{R}\mathcal{M}$,
$\eta_X$ is an epimorphism for all $X \in \mathcal{S}\mathcal{M}$;

(b) $(P \otimes_S - , \text{Hom}_R(P, -))$ induces an equivalence

\[ P \otimes_S - : \mathcal{S}\mathcal{M}\text{Hom}_R(P, P \otimes_S -) \longrightarrow \mathcal{R}\mathcal{M}P \otimes_S \text{Hom}_R(P, -). \]

In this case $P$ is a $\ast$-module (in the sense of Menini-Orsatti).

Tilting modules $P$

$P$ $\ast$-module and for any $M \in \mathcal{R}\mathcal{M}$, there is monomorphism
$M \rightarrow P \otimes_S X$, for some $X \in \mathcal{S}\mathcal{M}$.

(implies Brenner-Butler equivalence)
Module categories

| $\mathbb{R}P$ finitely generated and projective | $\text{Hom}_\mathbb{R}(P, -) \simeq P^* \otimes \mathbb{R} -$ |
|---|---|
| adjoint pair | $(P \otimes \mathbb{R} - , P^* \otimes \mathbb{R} - )$ |
| counit | $\varepsilon : P \otimes \mathbb{R} P^* \rightarrow \mathbb{R}, p \otimes f \mapsto f(p)$ |
| unit | $\eta : \mathbb{R} \rightarrow P^* \otimes \mathbb{R} P, 1 \mapsto \sum p_i^* \otimes p_i$ (dual basis). |
**Module categories**

\[ \mathcal{R}P \text{ finitely generated and projective} \]

\[ \text{Hom}_\mathcal{R}(P, -) \simeq P^* \otimes_{\mathcal{R}} - \]

- **Adjoint pair** \((P \otimes_{\mathcal{R}} -, P^* \otimes_{\mathcal{R}} -)\),
  - **Counit** \(\varepsilon: P \otimes_{\mathcal{R}} P^* \to \mathcal{R}, \ p \otimes f \mapsto f(p),\)
  - **Unit** \(\eta: \mathcal{R} \to P^* \otimes_{\mathcal{R}} P, \ 1 \mapsto \sum p_i^* \otimes p_i \) (dual basis).

\[ P^* \otimes_{\mathcal{R}} P \otimes_S - \text{ is a monad on } \text{Sm} (S\text{-ring}) \]

\[ P^* \otimes_{\mathcal{R}} P \overset{\sim}{\longrightarrow} \text{End}_\mathcal{R}(P), \ f \otimes p \mapsto [x \mapsto f(x)p]. \]
$R_P$ finitely generated and projective\hspace{1cm}Hom$_R(P, -) \simeq P^* \otimes_R -$

adjoint pair\hspace{1cm}$(P \otimes_R -, P^* \otimes_R -)$,

\begin{itemize}
  \item counit $\varepsilon : P \otimes_R P^* \to R, \ p \otimes f \mapsto f(p)$,
  \item unit $\eta : R \to P^* \otimes_R P, \ 1 \mapsto \sum p^*_i \otimes p_i$ (dual basis).
\end{itemize}

$P^* \otimes_R P \otimes_S -$ is a monad on $S^\mathbb{M}$ (S-ring)

$P^* \otimes_R P \xrightarrow{\sim} \text{End}_R(P), \ f \otimes p \mapsto [x \mapsto f(x)p].$

$P \otimes_S P^* \otimes_R -$ is a comonad on $R^\mathbb{M}$ (R-coring) - coproduct

$P \otimes_S P^* \to P \otimes_S P^* \otimes_R P \otimes_S P^*, \ p \otimes f \mapsto \sum p \otimes p^*_i \otimes p_i \otimes f.$
Module categories

| $P = R^n$ | $\text{End}_R(R^n) \simeq M_n(R)$, $R$ commutative |
|-----------|---------------------------------------------------|
| **adjoint pair** | $(R^n \otimes_R -, (R^n)^t \otimes_R -)$, |
| **counit** | $\varepsilon : R^n \otimes_R (R^n)^t \rightarrow R$, evaluation, |
| **unit** | $\eta : R \rightarrow (R^n)^t \otimes_R R^n$, $1 \mapsto \sum e_i^* \otimes e_i$. |
### Module categories

| $P = R^n$ | $\text{End}_R(R^n) \cong M_n(R)$, $R$ commutative |
|-----------|--------------------------------------------------|
| **adjoint pair** | $(R^n \otimes_R - , (R^n)^t \otimes_R -)$, |
| **counit** | $\varepsilon : R^n \otimes_R (R^n)^t \to R$, evaluation, |
| **unit** | $\eta : R \to (R^n)^t \otimes_R R^n$, $1 \mapsto \sum e_i^* \otimes e_i$. |

$M_n(R)$ is $R$-algebra and $R$-coalgebra.
Module categories

\[ P = R^n \quad \text{End}_R(R^n) \cong M_n(R), \ R \text{ commutative} \]

adjoint pair \[ (R^n \otimes_R -, (R^n)^t \otimes_R -), \]

counit \[ \varepsilon : R^n \otimes_R (R^n)^t \to R, \ \text{evaluation}, \]

unit \[ \eta : R \to (R^n)^t \otimes_R R^n, \ 1 \mapsto \sum e_i^* \otimes e_i. \]

\[ M_n(R) \text{ is } R\text{-algebra and } R\text{-coalgebra}. \]

Sweedler coring: \[ h : R \to A, \ P = AA_R, 1975 \]

\[ \delta : A \otimes_R A \to A \otimes_R A \otimes_A A \otimes_R A, \]

\[ a \otimes b \mapsto a \otimes 1 \otimes_A 1 \otimes b; \]

\[ \varepsilon = m : A \otimes_R A \to A, \]

\[ a \otimes b \mapsto ab. \]
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