Radiative decays of quarkonium states, momentum operator expansion and nilpotent operators

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Abstract

We present the method of calculation of the radiative decays of composite quark–antiquark $Q\bar{Q}$ systems, with different $J^P_C$: $(Q\bar{Q})_{in} \to \gamma(Q\bar{Q})_{out}$. The method is relativistic invariant, it is based on the double dispersion relation integrals over the masses of the composite mesons, it can be used for the high spin particles and provides us with the gauge invariant transition amplitudes. We apply this method to the case when the photon is emitted by a constituent in the intermediate state (additive quark model). We perform the momentum operator expansion of the spin amplitudes for the decay processes. The problem of the nilpotent spin operators is discussed.

1 Introduction

In this paper, we provide a consistent presentation of the method of calculation of the radiative decays of the composite $Q\bar{Q}$ systems, with arbitrary $J^P_C$. We consider the radiative decays which are realised through the transition $(Q\bar{Q})_{in} \to \gamma(Q\bar{Q})_{out}$ shown in Fig. 1 (additive quark model approach). The method is based on the spectral integration over the masses of composite particles $(Q\bar{Q})_{in}$ and $(Q\bar{Q})_{out}$, it is relativistic and gauge invariant. The obtained amplitude is determined by the quark wave functions of the composite systems $(Q\bar{Q})_{in}$ and $(Q\bar{Q})_{out}$. The method, in its substantial part, uses the spin operator expansion technique developed previously in [1].

The consideration of triangle diagrams in terms of the spectral integral over the mass of a composite particle, or interacting system, has a long history. Triangle diagrams appear at the rescattering of the three-particle systems, and the energy dependences of corresponding amplitudes (on either total energy or one of pair energies) were studied rather long ago, though in nonrelativistic approximation, in the dispersion relation technique applied to the analysis of the threshold singularities (see [2] and references therein). The relativistic approximation was used for the extraction of logarithmic singularity of the triangle diagram, e.g.,
Figure 1: a) Triangle diagram for the radiative transition \((Q\bar{Q})_{in} \to \gamma(Q\bar{Q})_{out}\) with \(p^2 = M^2_{in}\), \(p'^2 = M^2_{out}\) and \((p-p')^2 = q^2\). b) Cuttings of the triangle diagram for the double discontinuity of the spectral integral with \(P^2 = s\), \(P'^2 = s'\) and \((P-P')^2 = q^2\).

see [3, 4] and references therein. Double dispersion relation representation of the triangle diagram without accounting for the spin structure was written in [5].

In the consideration of radiative decays of the spin particles, one of the most important point is a correct construction of the gauge invariant spin operators that allows us to perform both the expansion of the decay amplitude (written in terms of external variables) and write down the double discontinuity of the spectral integral (written in terms of the composite particle constituents). Such a procedure had been realised for the deuteron in [6, 7], correspondingly, for the elastic scattering and photodisintegration amplitude. A generalization of the method for the composite quark systems has been performed in [8, 9, 10].

There are two principal points which should be accounted for the processes shown in Fig. 1 considered in terms of the spectral integration technique:
(i) The amplitude of the process \((Q\bar{Q})_{in} \to \gamma(Q\bar{Q})_{out}\) should be expanded in a series in respect to a full set of spin operators, and this expansion should be done in a uniform way for both internal (quark) and external boson states. The spin operators should be orthogonal, and the spectral integrals are to be written for the amplitudes related to these orthogonal operators.
(ii) It should be taken into account that in the processes with the real photons (with photon four-momentum \(q^2 \to 0\)) the nilpotent spin operators appear, their norm being equal to zero [11]. Because of that, the representation of the amplitudes may be different, that does not affect the calculation result for partial widths.

The present paper was initiated by the study of the quarkonium systems in terms of
the spectral integral Bethe–Salpeter equation — the formulation of this equation for the quark–antiquark systems as well as the discussion of its properties may be found in [12]. Since the quark–antiquark interactions at moderately large and large distances cannot be considered as well-established, the treatment of quark–antiquark states should be based on not only the knowledge on their masses but also on the wave functions, and the source of this latter information is the radiative decay processes. Therefore, we present the formulae for the radiative transition amplitudes in the form convenient for their simultaneous analysis using the spectral integral Bethe–Salpeter equation [12].

The paper is organized as follows.

Section 2 is the introductory one. As an example, we use the transitions $Q\bar{Q}(J^{PC}=0^{-+}) \to \gamma + Q\bar{Q}(J^{PC}=1^{--})$ and $Q\bar{Q}(J^{PC}=0^{++}) \to \gamma + Q\bar{Q}(J^{PC}=1^{--})$ studied before in [8, 9, 10, 11] and recall the method of the amplitude representation in terms of the double spectral integral. We also formulate the problem of the momentum expansion of the spin amplitude in the case of nilpotent operators.

In Section 3, the method is applied to the transition $Q\bar{Q}(2^{++}) \to \gamma + Q\bar{Q}(1^{--})$ and in Section 4 it is applied to $Q\bar{Q}(1^{++}) \to \gamma + Q\bar{Q}(1^{--})$.

In Conclusion, we emphasise that the cases of $Q\bar{Q}(2^{++}) \to \gamma + Q\bar{Q}(1^{--})$ and $Q\bar{Q}(1^{++}) \to \gamma + Q\bar{Q}(1^{--})$ are rather general and can be used as a pattern for the consideration of the spectral integral representation of the amplitudes $(Q\bar{Q})_{in} \to \gamma + (Q\bar{Q})_{out}$ for the $Q\bar{Q}$ states with arbitrary spin.

2 Radiative transitions $P \to \gamma(q)V$ and $S \to \gamma(q)V$

This Section is the introductory one: we consider here the meson radiative transitions $Q\bar{Q}(0^{-+}) \to \gamma(q) + Q\bar{Q}(1^{--})$ and $Q\bar{Q}(0^{++}) \to \gamma(q) + Q\bar{Q}(1^{--})$ (below the massive mesons with $J^{PC}=0^{-+}, 1^{--}, 0^{++}$ are denoted as $P,V,S$, correspondingly). These mesons were investigated previously within the spectral integration technique [9, 10]. Using these processes as examples, we demonstrate the basic principles of the technique used in subsequent sections for the more complicated transitions. In parallel, we discuss the problem of the nilpotent operators that arise in the reactions with the photon emission.

2.1 Transition $P \to \gamma(q)V$

Here, we consider the transition $P \to \gamma(q)V$ for the virtual photon. We write down the spin operator for both initial mesons and quark intermediate states in the triangle diagram, with the cuttings shown in Fig. 1b. Then, we extract the invariant part of the amplitude (form factor) and present it for the emission of real photon expressed through the dispersion relation integral.
2.1.1 Polarization vectors of the massive vector particle $V$ and photon

The polarizations of the vector meson, $\epsilon^{(V)}_\beta$, and virtual photon, $\epsilon^{(\gamma(q))}_\alpha$, are the transverse vectors:

$$\epsilon^{(V)}_\beta p'_\beta = 0, \quad \epsilon^{(\gamma(q))}_\alpha q_\alpha = 0.$$  \hspace{0.5cm} (1)

Here, $q$ is the photon four-momentum and $p'$ is that of the vector meson. The pseudoscalar meson momentum is denoted as $p = q + p'$. The polarisation of the vector meson obeys the completeness condition as follows:

$$-\sum_{a=1,2,3} \epsilon^{(V)}_\alpha(a) \epsilon^{(V)_+}_\beta(a) = g^{\perp}_{\alpha\beta},$$

$$g^{\perp}_{\alpha\beta} = g_{\alpha\beta} - \frac{p'_\alpha p'_\beta}{p'^2}.$$  \hspace{0.5cm} (2)

Here, $g^{\perp}_{\alpha\beta}$ is the metric tensor operating in the space orthogonal to the vector-meson momentum $p'$.

The polarisation vector of the real photon ($q^2 = 0$) denoted as $\epsilon^{(\gamma)}_\alpha$ has two independent components only, they are orthogonal to the reaction plane:

$$\epsilon^{(\gamma)}_\alpha q_\alpha = 0, \quad \epsilon^{(\gamma)}_\alpha p_\alpha = 0.$$  \hspace{0.5cm} (3)

Correspondingly, the completeness condition for the real photon reads:

$$-\sum_{a=1,2} \epsilon^{(\gamma)}_\alpha(a) \epsilon^{(\gamma)_+}_\alpha(a) = g^{\perp\perp}_{\alpha\alpha'},$$

$$g^{\perp\perp}_{\alpha\alpha'} = g_{\alpha\alpha'} - \frac{p_\alpha p_{\alpha'}}{p^2} - \frac{q^\perp_\alpha q^\perp_{\alpha'}}{q^\perp^2}.\hspace{0.5cm} (4)$$

Here, $q^\perp$ is the orthogonal component of the photon momentum:

$$q^\perp_\alpha = g^{\perp}_{\alpha\alpha'} q_{\alpha'} = q_\alpha - \frac{(pq)}{p^2} p_\alpha,$$

$$g^{\perp}_{\alpha\alpha'} = g_{\alpha\alpha'} - \frac{p_\alpha p_{\alpha'}}{p^2}.$$  \hspace{0.5cm} (5)

For virtual photon, ($q^2 \neq 0$), the completeness condition for polarisation vectors is written in three-dimensional space:

$$-\sum_{a=1,2,3} \epsilon^{(\gamma^*)}_\alpha(a) \epsilon^{(\gamma^*)_+}_\alpha(a) = g^{\perp\gamma^*}_{\alpha\alpha'},$$

$$g^{\perp\gamma^*}_{\alpha\alpha'} = g_{\alpha\alpha'} - \frac{q_\alpha q_{\alpha'}}{q^2}.\hspace{0.5cm} (6)$$
2.1.2 Angular momenta of outgoing particles

The angular momentum of outgoing particles depends on the relative momenta of particles in their centre-of-mass system:

\[ g_{\alpha'\alpha} \cdot \frac{1}{2} (q - p')_{\alpha'} = g_{\alpha'\alpha} \left( q - \frac{1}{2} p \right)_{\alpha'} = q_{\alpha}^\perp, \]  

(7)

Following [1], we determine the angular momenta with the help of operators \( X_{\mu_1 \ldots \mu_L} (q^\perp) \).

Below, we give an explicit form of these operators for the lowest states considered here:

\[ L = 0 : \quad X^{(0)}_{\perp} = 1, \]  

(8)

\[ L = 1 : \quad X^{(1)}_{\mu} (q^\perp) = q^\perp_{\mu}, \]  

\[ L = 2 : \quad X^{(2)}_{\mu_1 \mu_2} (q^\perp) = \frac{3}{2} \left( q_{\mu_1}^\perp q_{\mu_2}^\perp - \frac{1}{3} g_{\mu_1 \mu_2} q_2^\perp \right). \]  

2.1.3 Amplitude for the decay \( P \to \gamma V \)

The decay amplitude \( P \to \gamma V \) (for the sake of definiteness, the real photon is considered) is written as a product of the spin structure and form factor:

\[ A_{P \to \gamma V} = \epsilon_\alpha^{(\gamma)} \epsilon_\beta^{(V)} A_{\alpha \beta}^{(P \to \gamma V)}, \]  

(9)

where

\[ A_{\alpha \beta}^{(P \to \gamma V)} = e \epsilon_{\alpha \beta \mu \nu} q_{\mu}^\perp p_\nu F_{P \to \gamma V}(0). \]  

(10)

In (10), the electron charge is singled out, and \( \epsilon_{\alpha \beta \mu \nu} \) is the wholly antisymmetric tensor. This expression works for the virtual photon transition \( (\gamma \to \gamma^*) \) with corresponding substitution: \( F_{P \to \gamma V}(0) \to F_{P \to \gamma V}(q^2) \).

In what follows the important role is played by the spin operator which enters (10): \( \epsilon_{\alpha \beta \mu \nu} q_{\mu}^\perp p_\nu \) (or, in an abridged form, \( \epsilon_{\alpha \beta q \mu} \)). Since \( \epsilon_{\alpha \beta pp} = 0 \), the spin operator can be represented as follows:

\[ S_{\alpha \beta}^{(P \to \gamma V)}(p, q) = \epsilon_{\alpha \beta qp}. \]  

(11)

Let us stress once more that this spin operator is valid for the reaction with both real and virtual photons.

2.1.4 Partial widths for \( P \to \gamma V \) and \( V \to \gamma P \)

The partial width for the decay \( P \to \gamma V \) is determined as follows:

\[ M_{F} \Gamma_{P \to \gamma V} = \int d\Phi_2(p; q, q') \left| \sum_{\alpha \beta} A_{\alpha \beta}^{(P \to \gamma V)} \right|^2 = \frac{\alpha}{8} \frac{M_{P}^2 - M_{V}^2}{M_{P}^2} \left| F_{P \to \gamma V}(0) \right|^2, \]  

\[ d\Phi_2(p; q, q') = \frac{1}{2} \frac{d^3 q}{(2\pi)^3} \frac{d^3 p'}{2 q_0 (2\pi)^3} \frac{d^3 p'}{2 p'_0} (2\pi)^4 \delta^{(4)}(p - q - q'), \]  

5
where \( M_V \) and \( M_P \) are the meson masses. The summation is carried out over the photon and vector meson polarisations; in the final expression \( \alpha = e^2/4\pi = 1/137 \).

The same form factor gives us the partial width for the decay \( V \to \gamma P \):

\[
M_V \Gamma_{V\to\gamma P} = \frac{1}{3} \int d\Phi_2(p; q, p') | \sum_{\alpha\beta} A_{\alpha\beta}^{(V\to\gamma P)}|^2 = \frac{\alpha}{24} \frac{M_P^2 - M_P^2}{M_V^2} |F_{P\to\gamma V}(0)|^2. \tag{13}
\]

### 2.1.5 Double spectral integral representation of the triangle diagram

To derive double spectral integral for the form factor \( F_{P\rightarrow\gamma V}(0) \), one needs to calculate the double discontinuity of the triangle diagram of Fig. 1b, where the cuttings are shown by dotted lines. In the dispersion representation, the invariant energy in the intermediate state differs from those of the initial and final states. Because of that, in the double discontinuity \( P \neq p \) and \( P' \neq p' \). The following requirements are imposed on the momenta in the diagram of Fig. 1b [6, 8]:

\[
(k_1 + k_2)^2 = P^2 > 4m^2, \quad (k'_1 + k_2)^2 = P'^2 > 4m^2 \tag{14}
\]

at fixed \( q^2 \):

\[
(P' - P)^2 = (k'_1 - k_1)^2 = q^2. \tag{15}
\]

Furthermore, in the spirit of the dispersion relation representation, we denote \( P^2 = s \), \( P'^2 = s' \).

When we begin with Feynman diagram, the propagators should be substituted by the residues in the poles, that is equivalent to the replacement as follows: \( (m^2 - k_1^2)^{-1} \to \delta(m^2 - k_2^2) \). Then, the double discontinuity of the amplitude \( A_{\alpha\beta}^{(P\to\gamma V)} \) becomes proportional to the three factors:

\[
disc_{\alpha}disc_{\beta}A_{\alpha\beta}^{(P\to\gamma V(L))} \sim Z_{P\to\gamma V}G_P(s)G_V(L)(s') \times \frac{\alpha}{24} \frac{M_P^2 - M_P^2}{M_V^2} |F_{P\to\gamma V}(0)|^2. \tag{16}
\]

The first factor in the right-hand side of (16) includes the vertices: the quark charge factor \( Z_{P\to\gamma V} \) (for the one-flavour states \( Z_{P\to\gamma V} = e_Q \)) as well as transition vertices \( P \to Q\bar{Q} \) and \( V \to Q\bar{Q} \) which are denoted as \( G_P(s) \) and \( G_V(L)(s') \) (transition \( V \to Q\bar{Q} \) is characterised by two angular momenta \( L = 0, 2 \)).

The second factor includes the space volumes of the two-particle states: \( d\Phi_2(P; k_1, k_2) \) and \( d\Phi_2(P'; k'_1, k'_2) \) that correspond to two cuts in the diagram of Fig. 1b (the space volume is determined in (12)). The factor \( (2\pi)^3 2k_20\delta^3(\vec{k}_2 - \vec{k}_2') \) takes into account the fact that one quark line is cut twice.
The third factor in (16) is the trace coming from the summation over the quark spin states. Since the spin factor in the transition $V \rightarrow Q\bar{Q}$ may be of two types (with dominant $S$- or dominant $D$-wave), we have the following variants for $\hat{G}^{(S,L,J)}_{\beta}(k')$:

\begin{align}
L = 0 : & \quad \hat{G}^{(1,0,1)}_{\beta}(k') = \gamma_1 V, \\
L = 2 : & \quad \hat{G}^{(1,2,1)}_{\beta}(k') = \sqrt{2}\gamma_\beta \lambda_{\beta}(k').
\end{align}

Here, $k' = (k'_1 - k'_2)/2$ is the momentum of outgoing quarks: $k' \perp P' = k'_1 + k'_2$.

The total vertex $\hat{G}^{V}_{\beta}(k')$ of the vector state is the sum of vertices for the components with $L = 0$ and $L = 2$:

$$
\hat{G}^{V}_{\beta}(k') = \hat{G}^{(1,0,1)}_{\beta}(k')G_{V(0)}(s') + \hat{G}^{(1,2,1)}_{\beta}(k')G_{V(2)}(s').
$$

Correspondingly, there are two traces for two different transitions: $P \rightarrow \gamma V(0)$ and $P \rightarrow \gamma V(2)$:

\begin{align}
S_{\alpha'\beta'}^{(P \rightarrow \gamma V(0))} &= -S_{\alpha\beta}^{(1,0,1)}(k')\gamma_\alpha^L(\hat{k}_1 + m)\gamma_\beta^L(-\hat{k}_2 + m) , \\
S_{\alpha'\beta'}^{(P \rightarrow \gamma V(2))} &= -S_{\alpha\beta}^{(1,2,1)}(k')\gamma_\alpha^L(\hat{k}_1 + m)\gamma_\beta^L(-\hat{k}_2 + m) .
\end{align}

Recall that $\gamma_\alpha^{L+} = \gamma_\alpha^L g_{\alpha'\alpha'}$, see Eq. (6).

To calculate the invariant form factor $F_{P \rightarrow \gamma V(L)}(q^2)$, we should extract from (19) the spin factor analogous to $S_{\alpha\beta}^{(P \rightarrow \gamma V)}(q, p)$ given by Eq. (11). The total form factor is the sum as follows: $F_{P \rightarrow \gamma V}(q^2) = F_{P \rightarrow \gamma V(0)}(q^2) + F_{P \rightarrow \gamma V(2)}(q^2)$.

For the $QQ$ quark states, this operator reads:

$$
S_{\alpha'\beta'}^{(0^- \rightarrow 1^- \rightarrow)}(\bar{q}, P') = \varepsilon_{\alpha'\beta'}q_P' ,
$$

where $\bar{q} = P' - P$, while $P' = k'_1 + k'_2$ and $P = k_1 + k_2$. Therefore, we have:

$$
S_{\alpha'\beta'}^{(P \rightarrow \gamma V(L))} = S_{\alpha'\beta'}^{(0^- \rightarrow 1^- \rightarrow)}(\bar{q}, P')S_{P \rightarrow \gamma V(L)}(s, s', q^2) ,
$$

where

$$
S_{P \rightarrow \gamma V(L)}(s, s', q^2) = \frac{\left(S_{\alpha'\beta'}^{(P \rightarrow \gamma V(L))}S_{\alpha\beta}^{(0^- \rightarrow 1^- \rightarrow)}(\bar{q}, P')\right)}{\lambda(s, s', q^2)} .
$$

As a result, we obtain:

$$
S_{P \rightarrow \gamma V(0)}(s, s', q^2) = 4m ,
$$

$$
S_{P \rightarrow \gamma V(2)}(s, s', q^2) = \frac{m}{\sqrt{2}} \left[2m^2 + s\right] - \frac{6ss'q^2}{\lambda(s, s', q^2)} ,
$$

7
with

\[ \lambda = (s - s')^2 - 2q^2(s + s') + q^4. \]  

(24)

The double discontinuity of the amplitude (16) is equal to

\[ \text{disc}_s \text{disc}_q A_{\alpha \beta}^{(P \to \gamma V(L))} = S_{\alpha \beta}^{(0+ \to \gamma^1-)}(q, P') \text{disc}_s \text{disc}_q F_{P \to \gamma V(L)}(s, s', q^2), \]

(25)

where

\[ \text{disc}_s \text{disc}_q F_{P \to \gamma V(L)} = Z_{P \to \gamma V} G_P(s) G_{V(L)}(s') d\Phi_2(P; k_1, k_2) d\Phi_2(P'; k'_1, k'_2) \times \]

\[ \times (2\pi)^3 2\kappa_2 \delta^3(\vec{k}'_2 - \vec{k}_2) S_{P \to \gamma V(L)}(s, s', q^2). \]

(26)

It defines the form factor through the dispersion integral as follows:

\[ F_{P \to \gamma V(L)}(q^2) = \int_4 \frac{ds}{m^2} \int_4 \frac{ds'}{m^2} \text{disc}_s \text{disc}_q F_{P \to \gamma V(L)}(s, s', q^2) \]

\[ \frac{(s - M_P^2)(s' - M_{V(L)}^2)}{(s - M_P^2)(s' - M_{V(L)}^2)}. \]

(27)

We have written the expression for \( F_{P \to \gamma V(L)}(q^2) \) without subtraction terms, assuming that the convergence of (27) is guaranteed by the vertices \( G_P(s) \) and \( G_{V(L)}(s') \), see Eq. (26). Furthermore, we define the wave functions of the \( Q \bar{Q} \) systems:

\[ \psi_P(s) = \frac{G_P(s)}{s - M_P^2}, \quad \psi_{V(L)}(s) = \frac{G_{V(L)}(s')}{s' - M_{V(L)}^2}. \]

(28)

After integrating over the momenta, in accordance with (26), one can represent (27) in the following form:

\[ F_{P \to \gamma V(L)}(q^2) = Z_{P \to \gamma V(L)} \int_4 \frac{dsds'}{16\pi^2} \psi_P(s) \psi_{V(L)}(s') \frac{\Theta(-ss'q^2 - m^2\lambda(s, s', q^2))}{\sqrt{\lambda(s, s', q^2)}} S_{P \to \gamma V(L)}(s, s', q^2), \]

(29)

where \( \Theta(X) \) is the step-function: \( \Theta(X) = 1 \) at \( X \geq 0 \) and \( \Theta(X) = 0 \) at \( X < 0 \).

To calculate the integral at small \( q^2 \), we make the substitution:

\[ s = \Sigma + \frac{1}{2} zQ, \quad s' = \Sigma - \frac{1}{2} zQ, \quad q^2 = -Q^2, \]

(30)

thus representing the form factor as follows:

\[ F_{P \to \gamma V(L)}(-Q^2 \to 0) = Z_{P \to \gamma V(L)} \int_4 \frac{d\Sigma}{\pi} \psi_P(\Sigma) \psi_{V(L)}(\Sigma) \int_0^b \frac{dz}{\pi} \frac{S_{P \to \gamma V(L)}(\Sigma, z, -Q^2)}{16\sqrt{\Lambda(\Sigma, z, Q^2)}}, \]

\[ b = \sqrt{\frac{\Sigma}{m^2} - 4}, \quad \Lambda(\Sigma, z, Q^2) = (z^2 + 4\Sigma)Q^2. \]

(31)
After integrating over $z$ and substituting $\Sigma \to s$, the form factors for $L = 0, 2$ read:

$$
F_{P \to \gamma V(0)}(0) = Z_{P \to \gamma V(0)} \int_{4m^2}^{\infty} \frac{ds}{4\pi^2} \psi_P(s) \psi_V(s) \ln \frac{s + \sqrt{s(s - 4m^2)}}{s - \sqrt{s(s - 4m^2)}},
$$

$$
F_{P \to \gamma V(2)}(0) = Z_{P \to \gamma V(2)} \int_{4m^2}^{\infty} \frac{ds}{4\pi^2} \psi_P(s) \psi_V(s) \times
$$

$$
\times \left[ (2m^2 + s) \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - 3\sqrt{s(s - 4m^2)} \right].
$$

The form factors (32) are expressed through $\psi_P(s)$ and $\psi_V(s)$ by the integral over $s$: these expressions are sufficiently convenient for the combined fitting to the spectral integral Bethe–Salpeter equation [12] and radiative decay partial widths.

The total form factor is the sum as follows:

$$
F_{P \to \gamma V} = F_{P \to \gamma V(0)}(0) + F_{P \to \gamma V(2)}(0)
$$

The considered decay $P \to \gamma V$ offers rather simple case for the representation of the process of Fig. 1 in the form of the spectral integral, while the process $S \to \gamma V$ gives us a more complicated example — there we face the problem of the nilpotent spin operators.

### 2.2 Decay of the scalar meson $S \to \gamma V$

In the decay of the $0^{++}$ meson, the final particles can be in the $S$- and $D$-wave states and the corresponding spin factors read:

$$
S \text{ - wave : } (\epsilon^{(\gamma)} \epsilon^{(V)}),
$$

$$
D \text{ - wave : } X_{\alpha\beta}^{(2)}(q^\perp) \epsilon^{(\gamma)} \epsilon^{(V)}.
$$

For real photon $q^\perp \epsilon^{(\gamma)} = 0$, so, using $X_{\alpha\beta}^{(2)}(q^\perp)$ from Eq. (8), we see that only the $g^{\perp}_{\alpha\beta}$ term works in the $D$-wave. As a result, the $D$-wave operator,

$$
X_{\alpha\beta}^{(2)}(q^\perp) \epsilon^{(\gamma)} \epsilon^{(V)} = -\frac{1}{2} q^2 \epsilon^{(\gamma)} g_{\alpha\beta} \epsilon^{(V)} = -\frac{1}{2} q^2 (\epsilon^{(\gamma)} \epsilon^{(V)}),
$$

(35)
gives us the structure similar to that of the $S$-wave. Therefore, the amplitude of the real photon emission is determined by a single spin factor in the amplitude: $(\epsilon^{(\gamma)} \epsilon^{(V)})$. This factor may be represented as follows:

$$
\epsilon^{(\gamma)} g_{\alpha\beta} \epsilon^{(V)},
$$

(36)
where the metric tensor $g_{\alpha\beta}$ is given by (4). Thence, we have the following spin operator for the amplitude $S \to \gamma V$, when the real photon is emitted:

$$
S_{\alpha\beta}^{(S \to \gamma V)}(p, q) = g^{\perp}_{\alpha\beta}.
$$

(37)
This operator takes into consideration both $S$- and $D$-waves in the transition $S \rightarrow \gamma V$.

However, the representation of spin operator for the radiative transition $S \rightarrow \gamma V$ is not unique: along with (37), one may use a number of other forms.

### 2.2.1 Transition amplitude $S \rightarrow \gamma V$ and ambiguities in the representation of spin operator

Using the spin operator (37), we represent the amplitude $S \rightarrow \gamma V$ as follows:

$$A_{\alpha\beta}^{(S \rightarrow \gamma V)} = S_{\alpha\beta}^{(S \rightarrow \gamma V)}(p, q)F_{S \rightarrow \gamma V}(0), \tag{38}$$

where $F_{S \rightarrow \gamma V}(0)$ is the transition form factor. Below we demonstrate that the representation of the decay amplitude (38) is not unique.

At $q^2 = 0$, the spin operator (37) may be written as follows:

$$g^{1+}_{\alpha\beta}(0) = g_{\alpha\beta} + \frac{p'^2}{(p'q)^2} q_\alpha q_\beta - \frac{1}{(p'q)}(p'_\alpha q_\beta + q_\alpha p'_\beta). \tag{39}$$

Keeping in mind a consistent consideration of the case $q^2 \neq 0$, here we change the notation $g^{1+}_{\alpha\beta}(0) \rightarrow g^{1+}_{\alpha\beta}(0)$.

But it is also possible to apply another spin operator in (38), that is done rather often:

$$S_{\alpha\beta}^{(S \rightarrow \gamma V)}(p, q) \rightarrow \tilde{S}_{\alpha\beta} = g_{\alpha\beta} - \frac{p'_\alpha q_\beta}{(p'q)}, \tag{40}$$

Here, as above, the index $\alpha$ relates to the photon and $\beta$ to vector meson. At $q^2 = 0$, this operator obeys the requirement of transversality:

$$q_\alpha \tilde{S}_{\alpha\beta} = 0, \quad p'_\beta \tilde{S}_{\alpha\beta} = 0, \tag{41}$$

so it may be equally applied to the $S \rightarrow \gamma V$ amplitude.

The ambiguity in a representation of the amplitude is due to the existence of the nilpotent operator $L_{\alpha\beta}(0)$,

$$L_{\alpha\beta}(0)L_{\alpha\beta}(0) = 0, \tag{42}$$

which is orthogonal to $g^{1+}_{\alpha\beta}(0)$:

$$g^{1+}_{\alpha\beta}(0)L_{\alpha\beta}(0) = 0. \tag{43}$$

The operator $L_{\alpha\beta}(0)$ obeys the requirement of transversity,

$$q_\alpha L_{\alpha\beta}(0) = 0, \quad L_{\alpha\beta}(0)p'_\beta = 0 \tag{44}$$

This operator can be easily calculated using Eqs. (42) and (43) — it is equal to [11]

$$L_{\alpha\beta}(0) = \frac{p'^2}{(p'q)^2} q_\alpha q_\beta - \frac{1}{(p'q)} q_\alpha p'_\beta. \tag{45}$$
It is seen from (40) and (45) that
\[ g_{\alpha\beta}^{\perp\perp}(0) = \tilde{S}_{\alpha\beta} + L_{\alpha\beta}(0) . \]

Generally speaking, one can construct the spin operator using any linear combination of \( g_{\alpha\beta}^{\perp\perp}(0) \) and \( L_{\alpha\beta}(0) \):
\[ S^{(S\rightarrow\gamma V)}_{\alpha\beta} = g_{\alpha\beta}^{\perp\perp}(0) + C(p^2, p'^2) L_{\alpha\beta}(0) . \] (46)

Any of these operators may be equally applied to the construction of the transition amplitude \( S \rightarrow \gamma V \) for the emission of the real photon.

### 2.2.2 Transition \( S \rightarrow \gamma^* V \) in the case of virtual photon \( q^2 \neq 0 \)

For the virtual photon, \( q^2 \neq 0 \), the amplitude spin structure is described by the \( g_{\alpha\beta}^{\perp\perp} \) and \( L_{\alpha\beta} \) operators which should be generalised for this very case. In terms of \( p' \) and \( q \), these operators read:
\[ g_{\alpha\beta}^{\perp\perp}(p', q) = g_{\alpha\beta} + \frac{q^2}{(p'q)^2 - p'^2 q^2} p'_\alpha p'_\beta + \frac{p'^2}{(p'q)^2 - p'^2 q^2} q_{\alpha} q_{\beta} - \frac{(p'q)}{(p'q)^2 - p'^2 q^2} (q_{\alpha} p'_{\beta} + p'_{\alpha} q_{\beta}) , \] (47)

and
\[ L_{\alpha\beta}(p', q) = \frac{q^2}{(p'q)^2 - p'^2 q^2} p'_\alpha p'_\beta + \frac{p'^2}{(p'q)^2 - p'^2 q^2} q_{\alpha} q_{\beta} - \frac{(p'q)}{(p'q)^2 - p'^2 q^2} (q_{\alpha} p'_{\beta} + p'_{\alpha} q_{\beta}) . \] (48)

As is easy to see, these operators obey gauge invariance and are orthogonal to each other. At \( q^2 \rightarrow 0 \), the operators \( g_{\alpha\beta}^{\perp\perp}(p', q) \) and \( L_{\alpha\beta}(p', q) \) transform into the formulae (39) and (45), accordingly.

The transition amplitude \( S \rightarrow \gamma^* V \) is determined by two form factors, namely,
\[ A^{(S\rightarrow\gamma^* V)}_{\alpha\beta} = g_{\alpha\beta}^{\perp\perp}(p', q) F_{transverse}(q^2) + L_{\alpha\beta}(p', q) F_{logitudinal}(q^2) . \] (49)

The first term in (49) corresponds to the process with the transverse polarised final state particles (\( \gamma^* \) and \( V \)), while the second one corresponds to the polarisations lying in the reaction plane.

The operators \( g_{\alpha\beta}^{\perp\perp}(p', q) \) and \( L_{\alpha\beta}(p', q) \) are singular. To avoid false kinematical singularities in the amplitude \( A^{(S\rightarrow\gamma^* V)}_{\alpha\beta} \), the poles in \( g_{\alpha\beta}^{\perp\perp}(p', q) \), \( L_{\alpha\beta}(p', q) \) should be cancelled by zeros of the amplitude.
Instead of Eq. (49), one can use another representation of the decay amplitude, for example, that is based on the terms given in (34). Then, we have

\[ S - \text{wave operator: } g_{\alpha \alpha'}^{\perp \perp} g_{\beta \beta'}^{V} ; \]
\[ D - \text{wave operator: } g_{\alpha \alpha'}^{\perp \gamma'} X_{\alpha \beta'}^{(2)}(q^{\perp}) g_{\beta \beta'}^{V} . \]  

(50)

Here, \( g_{\alpha \beta}^{V} \) is determined by Eq. (3) and \( g_{\alpha \alpha'}^{\perp \gamma'} \) by (6). Note that the convolution \( g_{\alpha \alpha'}^{\perp \gamma'} g_{\alpha \beta}^{V} \) does not coincide with \( g_{\alpha \beta}^{\perp \gamma}(p', q) \) given at \( q^{2} \neq 0 \) by Eq. (47). There is another, much more important difference between the operators given by Eq. (50) and those given by Eqs. (47) and (48): the operators (50) are not orthogonal to each other, in contrast to \( g_{\alpha \beta}^{\perp \gamma}(p', q) \) and \( L_{\alpha \beta}(p', q) \). Indeed, the convolution of operators (50) gives:

\[ g_{\alpha \alpha'}^{\perp \gamma'} g_{\alpha \beta}^{\perp \perp} g_{\alpha \alpha'}^{\perp \gamma'} X_{\alpha \beta'}^{(2)}(q^{\perp}) g_{\beta \beta'}^{V} = - \frac{q_{\perp}^{4}}{3q^{2}p'^{2}}(p^{2} + p'^{2} + q^{2}). \] 

(51)

Non-orthogonality of the operators (50) is due to the fact that the convolution has been performed with the help of the metric tensors \( g_{\alpha \alpha'}^{\perp \gamma'} \) and \( g_{\beta \beta'}^{V} \). Had we operated with the normal metric tensor, namely, had we substituted in (51) \( g_{\alpha \alpha'}^{\perp \gamma'} \rightarrow g_{\alpha \alpha'}^{\perp \perp} \) and \( g_{\beta \beta'}^{V} \rightarrow g_{\beta \beta'}^{\perp \perp} \), we would have the orthogonal \( S - \) and \( D - \)wave operators. The metric tensors \( g_{\alpha \alpha'}^{\perp \gamma'} \) and \( g_{\beta \beta'}^{V} \) in (50) allow us to fulfill the gauge invariance — in this way, just due to the gauge invariance, the orthogonality in the \( S - \) and \( D - \)wave operators (50) is broken. But, in the spectral representation of form factors of the composite systems, the orthogonal operators are needed to avoid the double counting. This is the reason why furthermore we deal with the orthogonal operators represented by formulae (47) and (48).

2.2.3 Analytical properties of the amplitude at \( q^{2} = 0 \)

Let us turn to the real photon and discuss analytical properties of the amplitudes, namely, the cancellation of kinematical singularities. The \( A_{\alpha \beta}^{(S \rightarrow \gamma V)} \) amplitude reads:

\[ A_{\alpha \beta}^{(S \rightarrow \gamma V)} = \left[ g_{\alpha \beta} + \frac{4m^{2}_{V}}{(m^{2}_{S} - m^{2}_{V})^{2}} q_{\alpha} q_{\beta} - \frac{2}{m^{2}_{S} - m^{2}_{V}} (p'_{\alpha} q_{\beta} + q_{\alpha} p'_{\beta}) \right] F_{\text{transverse}}(0) + \]
\[ \left[ \frac{4m^{2}_{V}}{(m^{2}_{S} - m^{2}_{V})^{2}} q_{\alpha} q_{\beta} - \frac{2}{m^{2}_{S} - m^{2}_{V}} q_{\alpha} p'_{\beta} \right] F_{\text{longitudinal}}(0). \]  

(52)

Here, we have used \( 2(p'q) = m^{2}_{S} - m^{2}_{V} \). To make nonsingular the term in front of \( q_{\alpha} q_{\beta} \) at \( m^{2}_{S} \rightarrow m^{2}_{V} \), it is necessary that

\[ [F_{\text{transverse}}(0) + F_{\text{longitudinal}}(0)]_{m^{2}_{S} \rightarrow m^{2}_{V}} \sim (m^{2}_{S} - m^{2}_{V})^{2}. \] 

(53)

This requirement is sufficient for the cancellation of kinematical singularity in front of \( q_{\alpha} p'_{\beta} \). However, to remove kinematical singularity in the term \( p'_{\alpha} q_{\beta} \), the following condition for \( F_{\text{transverse}}(0) \) should be fulfilled:

\[ F_{\text{transverse}}(0) \sim (m^{2}_{S} - m^{2}_{V}) \quad \text{at} \quad (m^{2}_{S} - m^{2}_{V}) \rightarrow 0. \] 

(54)
The constraint (53) is in fact the requirement imposed on \( F_{\text{longitudinal}}(0) \), but the \( F_{\text{longitudinal}}(0) \) itself, as was noted above, does not participate in the definition of the decay partial width of the process \( S \to \gamma V \).

The second constraint given by Eq. (54) for \( F_{\text{transverse}}(0) \) is the principal one for the decay physics — in quantum mechanics it is known as Siegert’s theorem [13], see also the discussion in [14, 15, 16].

### 2.2.4 Spin operator decomposition of the quark states in the triangle diagram

Now, let us consider the quark triangle diagram of Fig. 1. As was said above, in the \( Q\bar{Q} \) systems there are two possibilities to construct vector mesons with the angular momenta \( L = 0 \) and \( L = 2 \). For the transitions \( V \to Q\bar{Q}(L) \), we apply the vertices introduced in (17): \( \hat{G}^{(1,0,1)}_\beta \) and \( \hat{G}^{(1,2,1)}_\beta \). For the transition \( S \to Q\bar{Q}(L) \), we use, following [1], the spin operator \( mI \), where \( I \) is the unit matrix. The traces for two processes with the different vector-meson wave functions \( (L = 0,2) \) read:

\[
SP_{\alpha \beta}^{(S \to \gamma V(0))} = -Sp[\hat{G}^{(1,0,1)}_\beta (\hat{k}_1 + m)\gamma^{+\gamma^*}_\alpha (\hat{k}_1 + m) mI (\hat{k}_2 + m)] ,
\]

\[
SP_{\alpha \beta}^{(S \to \gamma V(2))} = -Sp[\hat{G}^{(1,2,1)}_\beta (\hat{k}_1 + m)\gamma^{+\gamma^*}_\alpha (\hat{k}_1 + m) mI (\hat{k}_2 + m)] .
\]

To calculate the invariant form factor for the transverse polarised final state particles (we denote it in an abridged form as \( F_{S \to \gamma V(L)}(q^2) \)), one should extract from (55) the corresponding spin factor. For the quark states, this operator reads:

\[
S^{(0^+ \to \gamma 1^-)}_{\alpha \beta}(\bar{q}, P') = g^{\perp \perp}_{\alpha \beta}(\bar{q}, P') .
\]

Recall that \( P = k_1' + k_2 \) and \( \bar{q} = P - P' = k_1 - k_1' \). We have:

\[
SP_{\alpha \beta}^{(S \to \gamma V(L))} = S^{(0^+ \to \gamma 1^-)}_{\alpha \beta}(\bar{q}, P') S_{S \to \gamma V(L)}(s, s', q^2) ,
\]

where

\[
S_{S \to \gamma V(L)}(s, s', q^2) = \frac{(SP_{\alpha \beta}^{(S \to \gamma V(L))} S^{(0^+ \to \gamma 1^-)}_{\alpha \beta}(\bar{q}, P'))}{(S^{(0^+ \to \gamma 1^-)}_{\alpha \beta}(\bar{q}, P') S^{(0^+ \to \gamma 1^-)}_{\alpha \beta}(\bar{q}, P'))} .
\]

The spin factors \( S_{S \to \gamma V(L)}(s, s', q^2) \) at \( L = 0, 2 \) are equal to

\[
S_{S \to \gamma V(0)}(s, s', q^2) = -2m[(s - s' + q^2 + 4m^2) - \frac{4s'q^4}{\lambda(s, s', q^2)}] ,
\]

\[
S_{S \to \gamma V(2)}(s, s', q^2) = -\frac{m}{2\sqrt{2}}[4m^4 - 2m^2(3s + s' - q^2) + s(s - s' + q^2) + \frac{2ss'q^2}{\lambda(s, s', q^2)}(16m^2 + 3q^2 - s - 3s')] ,
\]

13
with $\lambda(s, s', q^2)$ given by (24).

The form factor of the considered process takes the form:

$$F_{S \rightarrow \gamma V(L)}(q^2) = Z_{S \rightarrow \gamma V} \frac{\int_{4m^2}^{\infty} ds' d\psi_S(s') \psi_{V(L)}(s') \theta(-ss'q^2 - m^2\lambda(s, s', q^2)) S_{S \rightarrow \gamma V(L)}(s, s', q^2)}{\sqrt{\lambda(s, s', q^2)}}.$$  \hspace{1cm} (60)

To calculate the integral at $q^2 \rightarrow 0$, we make, in a complete similarity with the calculations done in Eqs. (30) and (31), the following substitution: $q^2 = -Q^2$, $s = \Sigma + zQ/2$, $s' = \Sigma - zQ/2$ and represent the form factor at small $Q$ as follows:

$$F_{S \rightarrow \gamma V(L)}(-Q^2 \rightarrow 0) = Z_{S \rightarrow \gamma V} \int_{4m^2}^{\infty} d\Sigma \frac{\psi_S(\Sigma) \psi_{V(L)}(\Sigma)}{\pi} + b \int_{-b}^{b} dz \frac{S_{S \rightarrow \gamma V(L)}(\Sigma, z, -Q^2)}{16\sqrt{\lambda(\Sigma, z, Q^2)}},$$  \hspace{1cm} (61)

where $b = \sqrt{(\Sigma/m^2 - 4)}$ and $\lambda(\Sigma, z, Q^2) = (z^2 + 4\Sigma)Q^2$. After the integration over $z$ and substitution $\Sigma \rightarrow s$, we have:

$$F_{S \rightarrow \gamma V(0)}(0) = Z_{S \rightarrow \gamma V} \frac{m}{4\pi} \int_{4m^2}^{\infty} ds \frac{\psi_S(s) \psi_{V(0)}(s)}{\pi} I_{S \rightarrow \gamma V}(s),$$  \hspace{1cm} (62)

$$F_{S \rightarrow \gamma V(2)}(0) = Z_{S \rightarrow \gamma V} \frac{m}{2\pi} \int_{4m^2}^{\infty} ds \frac{\psi_S(s) \psi_{V(2)}(s)}{\pi} (s - 4m^2) I_{S \rightarrow \gamma V}(s),$$

$$I_{S \rightarrow \gamma V}(s) = \sqrt{s(s - 4m^2)} - 2m^2 \ln \frac{\sqrt{s + \sqrt{s - 4m^2}}}{\sqrt{s - \sqrt{s - 4m^2}}}. $$

The total form factor is equal to

$$F_{S \rightarrow \gamma V}(0) = F_{S \rightarrow \gamma V(0)}(0) + F_{S \rightarrow \gamma V(2)}(0).$$  \hspace{1cm} (63)

### 2.2.5 Partial widths for the decay processes with the emission of real photon

Similarly to the form factor calculations performed above, the partial width of the scalar meson decay $S \rightarrow \gamma V$ reads:

$$M_S \Gamma_{S \rightarrow \gamma V} = \int d\Phi_2(p, q, p') |\sum_{\alpha, \beta} A_{\alpha\beta}^{(S \rightarrow \gamma V)}|^2 = \frac{\alpha}{2} \frac{M_S^2 - M_V^2}{M_S^2} |F_{S \rightarrow \gamma V}(0)|^2. $$  \hspace{1cm} (64)

Recall that in the final expression $\alpha = e^2/4\pi = 1/137$. Likewise, partial width of the vector meson decay $V \rightarrow \gamma S$ is equal to:

$$M_V \Gamma_{V \rightarrow \gamma S} = \frac{\alpha}{6} \frac{M_S^2 - M_V^2}{M_V^2} |F_{S \rightarrow \gamma V}(0)|^2. $$  \hspace{1cm} (65)

Here, we do not specify the quark structure of the vector meson omitting the index $L$.  


2.2.6 Normalization conditions for wave functions of \(Q\bar{Q}\) states

It is convenient to write the normalization conditions for \(P\), \(S\) and \(V\) meson wave functions using the charge form factor of a meson:

\[
F_{\text{charge}}(0) = 1.
\]  

(66)

The amplitude of the charge factor is defined by the diagram of Fig. 1a, with \((Q\bar{Q})_\text{in} = (Q\bar{Q})_\text{out}\). For \(P\) and \(S\) mesons, the amplitude takes the form:

\[
A_\alpha(q) = e(p + p')_\alpha F_{\text{charge}}(q^2),
\]

(67)

while \(F_{\text{charge}}(q^2)\) can be calculated in the same way as the transition form factors considered above. For the meson \(V\), we consider the amplitude averaged over the spins of the massive vector particle. At \(q^2 = 0\), it takes a form:

\[
A^{(V)}(q \to 0) = 3e(p + p')_\alpha F^{(V)}_{\text{charge}}(0).
\]

(68)

The normalisation conditions based on the formula (67) for \(P\) and \(S\) mesons read:

\[
1 = \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_P^2(s) 2s \sqrt{s - 4m^2},
\]

(69)

\[
1 = \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_S^2(s) 2m^2 (s - 4m^2) \sqrt{s - 4m^2}.
\]

For the vector mesons \(V\), the normalisation condition reads as follows:

\[
1 = W_{00}^{[V]} + W_{02}^{[V]} + W_{22}^{[V]},
\]

(70)

\[
W_{00}^{[V]} = \frac{1}{3} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{V(0)}^2(s) 4(s + 2m^2) \sqrt{s - 4m^2}.
\]

(71)

\[
W_{02}^{[V]} = \frac{\sqrt{2}}{3} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{V(0)}(s) \psi_{V(2)}(s) (s - 4m^2)^2 \sqrt{s - 4m^2}.
\]

\[
W_{22}^{[V]} = \frac{2}{3} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{V(2)}^2(s) \frac{(8m^2 + s)(s - 4m^2)^2}{16} \sqrt{s - 4m^2}.
\]

In [9, 10], the calculations of charge form factors are explained in a more detail.

3 Transitions \(2^{++}-\text{meson} \rightarrow \gamma V\) and \(1^{++}-\text{meson} \rightarrow \gamma V\)

In this Section, using the decays of the \(2^{++}\)-meson (denoted as \(T\)) and \(1^{++}\)-meson (denoted as \(A\)), we perform the treatment which can be easily generalised for any spin particle.
3.1 Transition $T \to \gamma V$

To operate with the $2^+\text{-meson}$, we use the polarisation tensor $\epsilon_{\mu\nu}(a)$ with five components $a = 1, \ldots, 5$. This polarisation tensor, being symmetrical and traceless, obeys the completeness condition:

\[
\sum_{a=1,\ldots,5} \epsilon_{\mu\nu}(a)\epsilon_{\mu'\nu'}^+(a) = \frac{1}{2} \left( g_{\mu\nu}' g_{\mu'\nu'} + g_{\mu\nu} g_{\mu'\nu'}' - \frac{2}{3} g_{\mu\nu} g_{\mu'\nu'}' \right) = O_{\mu\nu}',
\]

\[
\sum_{a=1,\ldots,5} \epsilon_{\mu\nu}(a)\epsilon_{\mu'\nu'}^+(a) = 5.
\] (72)

Here, $O_{\mu\nu}'$ is a standard projection operator $O_{\mu'\nu'}^\mu$ for the angular momentum $L = 2$ which obeys the requirements: $O_{\mu'\nu'}^\mu O_{\mu'\nu'}^{\mu''} = O_{\mu'\nu'}^{\mu'}$ and $O_{\mu'\nu'}^\mu = 0$, see [1] for more detail.

In terms of the polarisation tensor $\epsilon_{\mu\nu}$ and vectors $\epsilon_{\alpha}^{(\gamma)}, \epsilon_{\beta}^{(V)}$, the independent spin structures for the transitions with virtual photon ($q^2 \neq 0$) read:

1. $S$-wave: $\epsilon_{\mu\nu} \epsilon_{\mu'}^{(\gamma)} \epsilon_{\nu'}^{(V)}$,
2. $D$-wave: $\epsilon_{\mu\nu} X_{\mu\nu}^{(2)} (q^\perp) (\epsilon^{(\gamma)} \epsilon^{(V)})$,
3. $D$-wave: $\epsilon_{\mu\nu} X_{\nu\beta}^{(2)} (q^\perp) \epsilon_{\mu}^{(\gamma)} \epsilon_{\beta}^{(V)}$,
4. $D$-wave: $\epsilon_{\mu\nu} X_{\nu\alpha}^{(2)} (q^\perp) \epsilon_{\mu}^{(\gamma)} \epsilon_{\alpha}^{(V)}$,
5. $G$-wave: $\epsilon_{\mu\nu} X_{\mu\nu\alpha\beta}^{(4)} (q^\perp) \epsilon_{\alpha}^{(\gamma)} \epsilon_{\beta}^{(V)}$.

(73)

Correspondingly, we have five independent form factors which describe the transition $2^{++}-\text{meson} \to \gamma^* V$. But, for the real photon ($q^2 = 0$), the number of independent form factors is reduced to three and, keeping this fact in mind, we consider below the production of the transverse polarised photon.

3.1.1 Spin operators for transverse polarised photon, $2^+-\text{meson} \to \gamma^* V$

Here, we consider the transverse polarised photon, though with $q^2 \neq 0$. We introduce the following spin operators, which correspond to the spin structures (73):

\[
S_{\mu,\nu,\alpha,\beta}^{(1)} = \frac{1}{q^2} O_{\mu,\nu}^{\gamma} g_{\mu,\alpha}^{\perp} g_{\nu,\beta}^{\perp} V,
\]

\[
S_{\mu,\nu,\alpha,\beta}^{(2)} = \frac{1}{q^2} O_{\mu,\nu}^{\gamma} X_{\nu,\alpha}^{(2)} (q^\perp) g_{\alpha,\beta}^{\perp} V = -\frac{1}{q^2} X_{\mu,\nu}^{(2)} (q^\perp) g_{\alpha,\beta}^{\perp},
\]

\[
S_{\mu,\nu,\alpha,\beta}^{(3)} = -\frac{1}{q^2} O_{\mu,\nu}^{\gamma} X_{\nu,\beta}^{(2)} (q^\perp) g_{\mu,\alpha}^{\perp} g_{\beta,\gamma}^{\perp},
\]

\[
S_{\mu,\nu,\alpha,\beta}^{(4)} = -\frac{1}{q^2} O_{\mu,\nu}^{\gamma} X_{\nu,\beta}^{(2)} (q^\perp) g_{\mu,\alpha}^{\perp} g_{\beta,\gamma}^{\perp},
\]

\[
S_{\mu,\nu,\alpha,\beta}^{(5)} = \frac{1}{q^2} O_{\mu,\nu}^{\gamma} X_{\nu,\alpha}^{(4)} (q^\perp) g_{\alpha,\beta}^{\perp} g_{\gamma,\delta}^{\perp} V.
\] (74)
Recall that here $q^\perp_\alpha = g^\perp_{\alpha\alpha'} q_{\alpha'} = q_\alpha - p_\alpha (pq) / p^2$ and $g^\perp_{\alpha\alpha'} = g_{\alpha\alpha'} - p_\alpha p_{\alpha'} / p^2$.

Let us demonstrate the method of construction of these operators by considering the $G$-wave spin structure from (73), $\epsilon^{\mu \nu}_i X^{(4)}_{\mu' \nu' \alpha' \beta'} (q^\perp) \epsilon^{(V)}_{\alpha'} \epsilon^{(V)}_{\beta'}$. It should be multiplied by the polarisations $\epsilon^{+}_\mu (a)$, $\epsilon^{(\gamma^+)}_{\alpha'_i} (b)$, $\epsilon^{(V)}_{\beta'} (c)$, with the summation over $a, b, c$:

$$
\sum_{a, b, c} \epsilon^{+}_\mu (a) \epsilon^{(\gamma^+)}_{\alpha'_i} (b) X^{(4)}_{\mu' \nu' \alpha' \beta'} (q^\perp) \epsilon^{(V)}_{\alpha'_i} \epsilon^{(V)}_{\beta'} (c) \epsilon^{(V)}_{\beta'} (c),
$$

that gives us $S^{(5)}_{\mu \nu, \alpha \beta}$, because the convolution $\epsilon^{(\gamma^+)}_{\alpha'_i} (b) \epsilon^{(\gamma^+)}_{\alpha'_i} (b)$ results in $g^\perp_{\alpha'_i \alpha'_i}$.

The operators (74) should be orthogonalised as follows:

$$
S^{(1)}_{\mu \nu, \alpha \beta} (p', q) = S^{(1)}_{\mu \nu, \alpha \beta},
$$

$$
S^{(2)}_{\mu \nu, \alpha \beta} (p', q) = S^{(2)}_{\mu \nu, \alpha \beta} - S^{(1)}_{\mu \nu, \alpha \beta} (p', q) \frac{S^{(2)}_{\mu \nu, \alpha' \beta'} (p', q) S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q)}{S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q) S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q)},
$$

$$
S^{(3)}_{\mu \nu, \alpha \beta} (p', q) = S^{(3)}_{\mu \nu, \alpha \beta} - S^{(1)}_{\mu \nu, \alpha \beta} (p', q) \frac{S^{(3)}_{\mu \nu, \alpha' \beta'} (p', q) S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q)}{S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q) S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q)},
$$

$$
- S^{(2)}_{\mu \nu, \alpha \beta} (p', q) \frac{S^{(2)}_{\mu \nu, \alpha' \beta'} (p', q) S^{(3)}_{\mu \nu, \alpha' \beta'} (p', q)}{S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q) S^{(1)}_{\mu \nu, \alpha' \beta'} (p', q)}.
$$

In this way, we construct three operators, $i = 1, 2, 3$. The operators $S^{(4)}_{\mu \nu, \alpha \beta}$ and $S^{(5)}_{\mu \nu, \alpha \beta}$ are nilpotent at $q^2 = 0$, so we do not present explicit expressions for them here but concentrate our attention on the calculation of the amplification for the emission of the real photon.

The results of calculation of the spin operator convolutions are presented in Appendix 2. Here, we give the orthogonalised operator norms, which determine the decay partial width:

$$
S^{(1)}_{\mu \nu, \alpha \beta} (p', q) S^{(1)}_{\mu \nu, \alpha \beta} (p', q) = z_{11} (M_T^2, M_V^2, q^2),
$$

$$
z_{11} (M_T^2, M_V^2, 0) = \frac{3M_T^4 + 3M_T^2 M_V^2 + 3M_V^4}{12 M_T^2 M_V^2},
$$

$$
z_{22} (M_T^2, M_V^2, 0) = 9 \frac{M_T^4 + 10 M_T^2 M_V^2 + M_V^4}{3 M_T^2 + 34 M_T^2 M_V^2 + 3 M_V^2},
$$

$$
z_{33} (M_T^2, M_V^2, 0) = 9 \frac{(M_T^2 + M_V^2)^2}{2 M_T^2 + 10 M_T^2 M_V^2 + M_V^4}.
$$

### 3.1.2 Calculation of the transition amplitude $T(L) \rightarrow \gamma V(L')$ for the emission of the real photon

The transition amplitude of the $T \rightarrow \gamma V$ decay can be written using the operators (75) as follows:

$$
A^{(T(L) \rightarrow \gamma V(L'))}_{\mu \nu, \alpha \beta} = \sum_{i=1,2,3} S^{(1)}_{\mu \nu, \alpha \beta} (p', q) F^{(i)}_{T(L) \rightarrow \gamma V(L')} (0),
$$

17
where \( F_{T(L)\rightarrow\gamma V(L')}^{i}(0) \) are the form factors at \( q^2 = 0 \).

Considering the double discontinuity related to Fig. 1b, we should expand the following traces in a series over the spin operators:

\[
S_{\mu\nu,\alpha\beta}^{(T(1)\rightarrow\gamma V(0))} = -Sp \left[ \tilde{G}_{\beta}^{(1,0,1)}(\hat{k}_{1} + m)\gamma^{\dagger}\gamma(\hat{k}_{1} + m)T_{\mu\nu}^{(1)}(k)(-\hat{k}_{2} + m) \right], \tag{78}
\]

\[
S_{\mu\nu,\alpha\beta}^{(T(1)\rightarrow\gamma V(2))} = -Sp \left[ \tilde{G}_{\beta}^{(1,2,1)}(\hat{k}_{1} + m)\gamma^{\dagger}\gamma(\hat{k}_{1} + m)T_{\mu\nu}^{(1)}(k)(-\hat{k}_{2} + m) \right],
\]

\[
S_{\mu\nu,\alpha\beta}^{(T(3)\rightarrow\gamma V(0))} = -Sp \left[ \tilde{G}_{\beta}^{(1,0,1)}(\hat{k}_{1} + m)\gamma^{\dagger}\gamma(\hat{k}_{1} + m)T_{\mu\nu}^{(3)}(k)(-\hat{k}_{2} + m) \right],
\]

\[
S_{\mu\nu,\alpha\beta}^{(T(3)\rightarrow\gamma V(2))} = -Sp \left[ \tilde{G}_{\beta}^{(1,2,1)}(\hat{k}_{1} + m)\gamma^{\dagger}\gamma(\hat{k}_{1} + m)T_{\mu\nu}^{(3)}(k)(-\hat{k}_{2} + m) \right].
\]

Recall that here we have used the notations \( \gamma^{\dagger}\gamma = g_{\alpha\alpha'}\gamma^{\dagger}\gamma', k = (k_{1} - k_{2})/2, k' = (k'_{1} - k_{2})/2 \) and vertices \( \tilde{G}_{\beta}^{(1,0,1)}, \tilde{G}_{\beta}^{(1,2,1)} \) given in Eq. (17).

The operators for the transitions \( 2^{++}\text{-meson} \rightarrow Q\bar{Q}(L) \) for \( L = 1, 3 \) read:

\[
T_{\mu\nu}^{(1)}(k) = \frac{3}{\sqrt{2}} \left[ k_{\mu}\gamma_{\nu} + k_{\nu}\gamma_{\mu} - \frac{2}{3}g_{\mu\nu}^{\dagger}\hat{k} \right], \tag{79}
\]

\[
T_{\mu\nu}^{(3)}(k) = \frac{5}{\sqrt{2}} \left[ k_{\mu}k_{\nu}\hat{k} - \frac{1}{5}k^{2}(g_{\mu\nu}^{\dagger}\hat{k} + \gamma_{\mu}\gamma_{\nu} + k_{\mu}k_{\nu}) \right].
\]

To calculate invariant form factors \( F_{T\rightarrow\gamma V(L)}^{(i)}(q^{2}) \), we should expand the traces of (78) in a series over the spin operators for the quark states \( S_{\mu\nu,\alpha\beta}^{(i)}(P', \bar{q}) \) and extract the invariant factors:

\[
S_{\mu\nu,\alpha\beta}^{(T(L)\rightarrow\gamma V(L'))} = \sum_{i=1,2,3} S_{\mu\nu,\alpha\beta}^{(i)}(P', \bar{q})S_{T(L)\rightarrow\gamma V(L')}(s, s', q^{2}), \tag{80}
\]

Let us emphasise that in (80) the spin operators depend on the intermediate-state quark variables, \( P' \) and \( \bar{q} \).

The invariant spin factors read:

\[
S_{T(L)\rightarrow\gamma V(L')}(s, s', q^{2}) = \frac{\left( S_{\mu\nu,\alpha\beta}^{(T(L)\rightarrow\gamma V(L'))} S_{\mu\nu,\alpha\beta}^{(i)}(P', \bar{q}) \right)}{\left( S_{\mu\nu,\alpha\beta}^{(i)}(P', \bar{q}) S_{\mu\nu,\alpha\beta}^{(i)}(P', \bar{q}) \right)}, \tag{81}
\]

where \( i = 1, 2, 3 \). Invariant spin factors determine the form factors in a standard way:

\[
F_{T(L)\rightarrow\gamma V(L')}(q^{2}) = Z_{T\rightarrow\gamma V} \int_{4m^{2}}^{\infty} \frac{dsds'}{16\pi^{2}} \psi_{T(L)}(s)\psi_{V(L')}(s') \times \frac{\theta(-ss'q^{2} - m^{2}\lambda(s, s', q^{2}))}{\sqrt{\lambda(s, s', q^{2})}} S_{T(L)\rightarrow\gamma V(L')}(s, s', q^{2}). \tag{82}
\]

To calculate the integral at \( q^{2} \rightarrow 0 \), we make, as before (see Eqs. (30) and (31)), the following substitution: \( q^{2} = -Q^{2}, s = \Sigma + zQ/2, s' = \Sigma - zQ/2 \) and perform the integration over \( z \).
We have:

$$F_{T(L)\to \gamma V(L')}^{(i)} = Z_{T(L)\to \gamma V(L')} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{T(L)}(s) \psi_{V(L')}(s) J_{T(L)\to \gamma V(L')}^{(i)}(s).$$

(83)

Here,

$$S_{T(1)\to \gamma V(0)}^{(1)}(s) = -\frac{\sqrt{3}}{5} (8m^2 + 3s) I_{T\to \gamma V}^{(1)}(s),$$

$$S_{T(1)\to \gamma V(0)}^{(2)}(s) = \frac{2}{3} S_{T(1)\to \gamma V(0)}^{(3)}(s) = -\frac{2}{3\sqrt{3}} I_{T\to \gamma V}^{(2)}(s),$$

$$S_{T(1)\to \gamma V(2)}^{(1)}(s) = -\frac{\sqrt{6}}{40} (16m^2 - 3s)(4m^2 - s) I_{T\to \gamma V}^{(1)}(s),$$

$$S_{T(1)\to \gamma V(2)}^{(2)}(s) = \frac{2}{3} S_{T(1)\to \gamma V(2)}^{(3)}(s) = -\frac{2\sqrt{2}}{12\sqrt{3}} (8m^2 + s) I_{T\to \gamma V}^{(2)}(s),$$

$$S_{T(3)\to \gamma V(0)}^{(1)}(s) = -\frac{3\sqrt{2}}{20} (4m^2 - s)^2 I_{T\to \gamma V}^{(1)}(s),$$

$$S_{T(3)\to \gamma V(0)}^{(2)}(s) = \frac{2}{3} S_{T(3)\to \gamma V(0)}^{(3)}(s) = -\frac{\sqrt{2}}{18} (6m^2 + s) I_{T\to \gamma V}^{(2)}(s),$$

$$S_{T(3)\to \gamma V(2)}^{(1)}(s) = -\frac{3}{80} (4m^2 - s)^2 (8m^2 + s) I_{T\to \gamma V}^{(1)}(s),$$

$$S_{T(3)\to \gamma V(2)}^{(2)}(s) = \frac{2}{3} S_{T(3)\to \gamma V(2)}^{(3)}(s) = -\frac{1}{72} (16m^2 - 3s)(4m^2 - s) I_{T\to \gamma V}^{(2)}(s),$$

(84)

where

$$I_{T\to \gamma V}^{(1)}(s) = 2m^2 \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - \sqrt{s(s - 4m^2)},$$

$$I_{T\to \gamma V}^{(2)}(s) = m^2(m^2 + s) \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - \frac{1}{12} \sqrt{s(s - 4m^2)(s + 26m^2)}.$$

Total form factor is a sum over four terms:

$$F_{T\to \gamma V}^{(i)} = \sum_{L,L'} F_{T(L)\to \gamma V(L')}^{(i)}.$$ 

(86)

### 3.1.3 Normalisation conditions and partial widths

The normalisation condition for tensor mesons reads:

$$1 = W_{11}[T] + W_{13}[T] + W_{33}[T],$$

$$W_{11}[T] = \frac{1}{5} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_{T(1)}^2(s) \frac{1}{2} (8m^2 + 3s)(s - 4m^2) \sqrt{1 - \frac{4m^2}{s}},$$

(87)
\[ W_{13}[T] = \frac{1}{5} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_T(1)(s) \psi_T(3)(s) \frac{\sqrt{3}}{2\sqrt{2}} (s - 4m^2)^3 \sqrt{1 - \frac{4m^2}{s}}, \]
\[ W_{33}[T] = \frac{1}{5} \int_{4m^2}^{\infty} \frac{ds}{16\pi^2} \psi_T(3)(s) \frac{1}{16} (6m^2 + s) (s - 4m^2)^3 \sqrt{1 - \frac{4m^2}{s}}. \]

Partial width of the decay \( T \to \gamma V \) is equal to:
\[
m_T \Gamma_{T \to \gamma V} = e^2 \int d\Phi_2(p_i,q,p_f) \frac{1}{5} \sum_{\mu,\alpha,\beta} |A_{\mu,\alpha,\beta}|^2 = \frac{\alpha}{20} \frac{m_T^2 - m_V^2}{m_T^2} \left[ z_{11}^4 (M_T^2, M_V^2, 0) (F_T^{(1)}_{\gamma V})^2 + z_{22}^4 (M_T^2, M_V^2, 0) (F_T^{(2)}_{\gamma V})^2 + z_{33}^4 (M_T^2, M_V^2, 0) (F_T^{(3)}_{\gamma V})^2 \right].
\]

The same block of form factors determines the partial width for \( V \to \gamma T \):
\[
m_V \Gamma_{V \to \gamma T} = e^2 \int d\Phi_2(p_i,q,p_f) \frac{1}{5} \sum_{\mu,\alpha,\beta} |A_{\mu,\alpha,\beta}|^2 = \frac{\alpha}{12} \frac{m_V^2 - m_T^2}{m_V^2} \left[ z_{11}^4 (M_T^2, M_V^2, 0) (F_T^{(1)}_{\gamma V})^2 + z_{22}^4 (M_T^2, M_V^2, 0) (F_T^{(2)}_{\gamma V})^2 + z_{33}^4 (M_T^2, M_V^2, 0) (F_T^{(3)}_{\gamma V})^2 \right].
\]

Let us emphasize that the factors \( z_{aa}^4 (M_T^2, M_V^2, 0) \) are symmetrical with respect to \( T \leftrightarrow V \) permutation: \( z_{aa}^4 (M_T^2, M_V^2, 0) = z_{aa}^4 (M_V^2, M_T^2, 0) \).

### 3.2 Transition \( A \to \gamma V \)

For the reaction \( 1^{++} \text{-meson} \to \gamma 1^{--} \text{-meson} \) (or \( A \to \gamma V \)), one can write three partial states: the \( S \)-wave state with the total spin \( S = s_\gamma + s_V = 1 \) and two \( D \)-wave states with \( S = s_\gamma + s_V = 1 \). Generally, we have three spin structures, but only two of them survive in the case of transverse polarised photon (below, as before, \( p \) is the momentum of the decaying particle and \( q \) is that of the outgoing photon):
\[
S^{(1)}_{\mu,\alpha,\beta}(p,q) = g^{\perp\perp}_{\alpha\beta} g^{\perp\perp}_{\gamma} \varepsilon_{\mu\alpha_q^p} q^\perp \varepsilon_{\beta q^p} q^\perp \varepsilon_{\gamma q^p} q^\perp \varepsilon_{\delta q^p},
\]
\[
S^{(2)}_{\mu,\alpha,\beta}(p,q) = \frac{1}{q^\perp} q^\perp q^{\perp\perp}_{\alpha \mu} g^{\perp\perp}_{\alpha \gamma} g^{\perp\perp}_{\beta \delta} \varepsilon_{\mu\alpha_q^p} q^\perp \varepsilon_{\beta q^p} q^\perp \varepsilon_{\gamma q^p} q^\perp \varepsilon_{\delta q^p} = \frac{1}{q^\perp} q^\perp q^{\perp\perp}_{\alpha \mu} g^{\perp\perp}_{\alpha \gamma} g^{\perp\perp}_{\beta \delta} \varepsilon_{\mu\alpha_q^p} q^\perp \varepsilon_{\beta q^p} q^\perp \varepsilon_{\gamma q^p} q^\perp \varepsilon_{\delta q^p} = 0.
\]

Here, as previously, we use the abridged form \( \varepsilon_{\mu\alpha,\beta}q_\xi \equiv \varepsilon_{\mu\alpha q^p} \). The vanishing of \( S^{(3)}_{\mu,\alpha,\beta} \) is due to the equality \( q_\xi^\perp g^{\perp\perp}_{\alpha \xi} = 0 \).
3.2.1 Spin operators and decay amplitude

The operators $S_{\mu,\alpha\beta}^{(a)}(p, q)$ should be orthogonised:

$$S_{\mu,\alpha\beta}^{(\perp 1)}(p, q) = S_{\mu,\alpha\beta}^{(1)}(p, q),$$

$$S_{\mu,\alpha\beta}^{(\perp 2)}(p, q) = S_{\mu,\alpha\beta}^{(2)}(p, q) - S_{\mu,\alpha\beta}^{(\perp 1)}(p, q) \frac{S_{\mu',\alpha'\beta'}^{(1)}(p, q) S_{\mu',\alpha'\beta'}^{(2)}(p, q)}{S_{\mu,\alpha\beta}^{(\perp 1)}(p, q) S_{\mu',\alpha'\beta'}^{(2)}(p, q)}.$$  (92)

We determine the convolutions

$$S_{\mu,\alpha\beta}^{(\perp a)}(p, q) S_{\mu,\alpha\beta}^{(\perp b)}(p, q) \equiv z_{ab}^\perp(M_A^2, M_V^2, q^2).$$  (93)

At $q^2 = 0$ (see Appendix 3 for the details), they are equal to:

$$z_{11}^\perp(M_A^2, M_V^2, 0) = -\frac{(M_A^4 + 6M_A^2M_V^2 + M_V^4)}{2M_V^2},$$

$$z_{22}^\perp(M_A^2, M_V^2, 0) = -\frac{2M_A^4(M_A^4 + M_V^2)^2}{(M_A^4 + 6M_A^2M_V^2 + M_V^4)}.$$  (94)

The transition amplitude $A \rightarrow \gamma V$ reads:

$$A_{\mu,\alpha\beta}^{(A \rightarrow \gamma V(L))} = \sum_{i=1,2} S_{\mu,\alpha\beta}^{(\perp i)}(p, q) F_{A \rightarrow \gamma V(L)}^{(i)}(0),$$  (95)

being determined by two form factors $F_{A \rightarrow \gamma V}^{(i)}(0) (i = 1, 2)$.

3.2.2 Calculation of the quark triangle diagram of Fig. 1 for the emission of the real photon

The diagram of Fig. 1b for the processes $A \rightarrow \gamma V(L) (L = 0, 2)$ is determined by the following traces:

$$S_{\mu,\alpha\beta}^{(A \rightarrow \gamma V(0))} = -Sp \left[ \hat{G}_{\beta}^{(1,0,1)}(\hat{k}_1 + m)\gamma^{\perp \gamma}(\hat{k}_1 + m)T_{\mu}(k)(-\hat{k}_2 + m) \right],$$

$$S_{\mu,\alpha\beta}^{(A \rightarrow \gamma V(2))} = -Sp \left[ \hat{G}_{\beta}^{(1,2,1)}(\hat{k}_1 + m)\gamma^{\perp \gamma}(\hat{k}_1 + m)T_{\mu}(k)(-\hat{k}_2 + m) \right],$$  (96)

where the transition $A \rightarrow Q\bar{Q}$ is equal to:

$$T_{\mu}(k) = \sqrt{\frac{2}{3s}} \epsilon_{\mu k\gamma P}.$$  (97)

Recall that $k = (k_1 - k_2)/2$ and $P = k_1 + k_2$.

To calculate the invariant form factor $F_{A \rightarrow \gamma V(L)}(q^2)$, we should expand (96) in a series with respect to the spin operators $S_{\mu,\alpha\beta}^{(\perp i)}(P, \tilde{q})$ where $\tilde{q} = P - P'$:

$$S_{\mu,\alpha\beta}^{(A \rightarrow \gamma V(L))} = \sum_{i=1,2} S_{\mu,\alpha\beta}^{(\perp i)}(P, \tilde{q}) S_{A \rightarrow \gamma V(L)}^{(i)}(s, s', q^2),$$  (98)
that gives us

$$S_{A \rightarrow \gamma V(L)}^{(i)}(s, s', q^2) = \frac{(S_p^{(A \rightarrow \gamma V(L))} S_{\mu, \alpha}^{(i)}(P, \bar{q}))}{(S_{\mu, \alpha}(P, \bar{q}) S_p^{(i)}(P, \bar{q}))},$$  \hspace{1cm} (99)$$

where $i = 1$ or $2$.

The transition form factor is determined by the standard formula:

$$F_{A \rightarrow \gamma V(L)}^{(i)}(q^2) = Z_{A \rightarrow \gamma V} \int_0^\infty \frac{ds ds'}{4m^2} \psi_A(s) \psi_{V(L)}(s') \times$$

$$\times \frac{\theta(-ss'q^2 - m^2\lambda(s, s', q^2))}{\sqrt{\lambda(s, s', q^2)}} S_{A \rightarrow \gamma V(L)}^{(i)}(s, s', q^2),$$

In the limit $q^2 \rightarrow 0$, performing calculations as in the cases considered above, we obtain:

$$F_{A \rightarrow \gamma V(L)}^{(i)} = Z_{A \rightarrow \gamma V} \int_0^\infty \frac{ds}{4m^2} \psi_A(s) \psi_{V(L)}(s) J_{A \rightarrow \gamma V(L)}^{(i)}(s),$$ \hspace{1cm} (101)$$

$$J_{A \rightarrow \gamma V(0)}^{(1)}(s) = -\sqrt{3} I_{A \rightarrow \gamma V}(s),$$

$$J_{A \rightarrow \gamma V(2)}^{(2)}(s) = \frac{\sqrt{3}}{8} (4m^2 - s) I_{A \rightarrow \gamma V}(s),$$

where

$$I_{A \rightarrow \gamma V}(s) = \sqrt{s} \left(2m^2 \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{\sqrt{s} - \sqrt{s - 4m^2}} - \sqrt{s(s - 4m^2)} \right).$$ \hspace{1cm} (102)$$

The total form factor is equal to:

$$F_{A \rightarrow \gamma V}(0) = F_{A \rightarrow \gamma V(0)}(0) + F_{A \rightarrow \gamma V(2)}(0)$$ \hspace{1cm} (103)$$

### 3.2.3 Normalisation condition and partial widths

The normalisation condition for the $1^{++}$ meson wave function reads:

$$1 = \frac{1}{2} \int_0^\infty \frac{ds}{4m^2} \psi_A^2(s) s(s - 4m^2) \sqrt{1 - \frac{4m^2}{s}}.$$ \hspace{1cm} (104)$$

The partial width of the decay $A \rightarrow \gamma V$ is equal to

$$m_A \Gamma_{A \rightarrow \gamma V} = e^2 \int d\Phi_2(p, q, p') \frac{1}{3} \sum_{\mu, \alpha} |A_{\mu, \alpha}|^2$$

$$= \frac{\alpha}{12} \frac{m_A^2 - m_V^2}{m_A^2} \left[ z_{11}^2(M_A^2, M_V^2, 0) \left(F^{(1)}(0)\right)^2 + z_{12}^2(M_A^2, M_V^2, 0) \left(F^{(2)}(0)\right)^2 \right].$$
For the partial width of the decay $V \to \gamma A$, one has:

$$m_V \Gamma_{V \to \gamma A} = \frac{\alpha m_V^2 - m_A^2}{12 m_V^2} \left[ z_{11}^\perp (M_V^2, M_A^2, 0) \left( F^{(1)}(0) \right)^2 + \right. \left. z_{22}^\perp (M_V^2, M_A^2, 0) \left( F^{(2)}(0) \right)^2 \right].$$ \hspace{1cm} (106)

Let us emphasise that $z_{aa}^\perp (M_V^2, M_A^2, 0) \neq z_{aa}^\perp (M_A^2, M_V^2, 0)$.

### 4 Conclusion

The considerations of the tensor meson decays (Section 3) and pseudovector meson decay (Section 4), being in fact general cases, can be easily expanded for any mesons.

Actually, the tensor meson decay is a pattern for an amplitude, where the parity of initial meson coincides with the parity of final state. For this case, we construct the spin scalars from the polarisation and angular momentum functions $X_{\mu_1 \cdots \mu_L}^{(L)} (k^\perp)$, see Eq. (73) for tensor meson. With a completeness condition for the vector and tensor polarisations, we construct gauge invariant spin operators ((74) for the tensor mesons. The orthogonalisation of these operators for the case of the real photon emission allows us to single out the operators with nonzero norm, of the type of Eq. (75)), and nilpotent operators. The operators of the first kind are used in the expansion of the amplitude in a series in respect to external operators (Eq. (77)), as well as for the quark triangle diagram (Eqs. (80) and (81)). The spectral integrals are written for the invariant form factors, which are the coefficients in front of the orthogonalised operators.

The decay of axial meson provides us with an example, where the parity changes sign from initial to final state. In this case, there is only one difference: in the first step, we construct the pseudoscalars from the particle polarisations and angular momentum function $X_{\mu_1 \cdots \mu_L}^{(L)} (k^\perp)$. Further consideration is carried out following the scheme common to that of tensor mesons.

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Appendix 1: Convolutions of spin operators in the decay $T \rightarrow \gamma V$

In the calculation of the spin factors we need the convolutions

$$S^{(a)}_{\mu\nu,\alpha\beta}(P', \tilde{q})S^{(b)}_{\mu\nu,\alpha\beta}(P', \tilde{q}) \equiv z_{ab}(s, s', q^2) .$$  \hfill (107)

They are as follows:

\[
\begin{align*}
    z_{11}(s, s', q^2) &= \frac{3s^2 - 6s q^2 + 34 s' + 3q^4 - 6q^2 s' + 3s'^2}{12ss'}, \\
    z_{22}(s, s', q^2) &= 3, \\
    z_{33}(s, s', q^2) &= \frac{3s^2 - 6sq^2 + 13ss' + 3q^4 - 6q^2 s' + 3s'^2}{12ss'}, \\
    z_{44}(s, s', q^2) &= \frac{3s^2 - 6sq^2 + 34ss' + 3q^4 - 6q^2 s' + 3s'^2}{48ss'}, \\
    z_{12}(s, s', q^2) &= 1, \\
    z_{13}(s, s', q^2) &= \frac{-3s^2 - 6s q^2 - 8ss' + 3q^4 - 6q^2 s' + 3s'^2}{12ss'}, \\
    z_{14}(s, s', q^2) &= \frac{3s^2 - 6sq^2 + 34ss' + 3q^4 - 6q^2 s' + 3s'^2}{24ss'}, \\
    z_{23}(s, s', q^2) &= \frac{1}{2}, \\
    z_{24}(s, s', q^2) &= \frac{1}{2}, \\
    z_{34}(s, s', q^2) &= \frac{-3s^2 + 6sq^2 + 8ss' - 3q^4 + 6q^2 s' - 3s'^2}{24ss'}, \\
    z_{55}(s, s', q^2) &= \frac{2s^2 - 4sq^2 + 11ss' + 2q^4 - 4q^2 s' + 2s'^2}{8ss'}, \\
    z_{15}(s, s', q^2) &= \frac{-s^2 + 2sq^2 + 2ss' - q^4 + 2q^2 s' - s'^2}{4ss'}, \\
    z_{25}(s, s', q^2) &= \frac{3}{2}, \\
    z_{35}(s, s', q^2) &= \frac{s^2 - 2sq^2 + 4ss' + q^4 - 2q^2 s' + s'^2}{4ss'}, \\
    z_{45}(s, s', q^2) &= \frac{-s^2 - 2sq^2 - 2ss' + q^4 - 2q^2 s' + s'^2}{8ss'}. \\
\end{align*}
\hfill (109)

For the orthogonal operators, we have:

$$S^{(\perp a)}_{\mu\nu,\alpha\beta}(P', \tilde{q})S^{(\perp b)}_{\mu\nu,\alpha\beta}(P', \tilde{q}) \equiv z_{ab}^\perp(s, s', q^2)$$  \hfill (110)

with

$$z_{11}^\perp(s, s', q^2) = \frac{3s^2 - 6s q^2 + 34 s' + 3q^4 - 6q^2 s' + 3s'^2}{12ss'} .$$  \hfill (111)
Spin factors given by $\Sigma, z$ variables in the limit $Q \to 0$ read:

\[
S_{T(1)}^{(3)}(\Sigma, z, Q \to 0) = -\frac{2}{3} \frac{2}{S_{T(L)}^{(2)}(\Sigma, z, Q \to 0)} = \frac{2}{S_{T(L)}^{(2)}(\Sigma, z, Q \to 0)}.
\]

**Appendix 2: Convolutions of spin operators in the decay $A \to \gamma V$**

We denote the convolutions of the spin operators at $q^2 \neq 0$ as follows:

\[
S_{\mu\nu\alpha\beta}^{(a)}(P', \bar{q})S_{\mu\nu\alpha\beta}^{(b)}(P', \bar{q}) \equiv z_{ab}(s, s', q^2),
\]

\[
(113)
\]

where

\[
\begin{align*}
z_{11}(s, s', q^2) &= \frac{-(s^2 + 6ss' + s'^2) + q^2(2s + 2s' - q^2)}{2s'}, \\
z_{22}(s, s', q^2) &= \frac{(s + s' - q^2)^2}{2s'}, \\
z_{12}(s, s', q^2) &= \frac{(s + s' - q^2)^2}{2s'}.
\end{align*}
\]

(114)
For the orthogonalised operators,
\[ S_{\mu\alpha\delta}^{(\perp a)}(P', \tilde{q}) S_{\mu\alpha\delta}^{(\perp b)}(P', \tilde{q}) \equiv z_{\alpha\beta}^{(\perp)}(s, s', q^2), \] (115)

one obtains:

\[ z_{11}^{(\perp)}(s, s', q^2) = \frac{-(s^2 + 6ss' + s'^2) + q^2(2s + 2s' - q^2)}{2s'}, \]
\[ z_{22}^{(\perp)}(s, s', q^2) = -\frac{2s(s + s' - q^2)^2}{(s^2 + 6ss' + s'^2) - q^2(2s + 2s' - q^2)}. \] (116)

For the calculation of the spin factors in the double dispersion integral, we use variables \( s = \Sigma + \frac{1}{2}zQ, \ s' = \Sigma - \frac{1}{2}zQ, \) and \( q^2 = -Q^2. \) Using these variables, in the limit \( Q \to 0, \) we have for spin factors:

\[ S_{A \to \gamma V(0)}^{(1)}(\Sigma, z, Q \to 0) = \frac{5\sqrt{2}}{\sqrt{3}} \frac{m^2z^2 + 4m^2\Sigma - \Sigma^2}{\sqrt{\Sigma}(z^2 + 4\Sigma)}, \] (117)
\[ S_{A \to \gamma V(0)}^{(2)}(\Sigma, z, Q \to 0) \to Qz \left[ -\frac{5}{\sqrt{6}} \frac{(m^2z^2 + 4m^2\Sigma - 3\Sigma^2)}{\sqrt{\Sigma}(z^2 + 4\Sigma)} \right] \to 0, \]
\[ S_{A \to \gamma V(2)}^{(1)}(\Sigma, z, Q \to 0) = \frac{\sqrt{2}}{\sqrt{3}} \frac{4m^2\Sigma + m^2z^2 - \Sigma^2(\Sigma - 4m^2)}{\sqrt{\Sigma}(z^2 + 4\Sigma)}, \]
\[ S_{A \to \gamma V(2)}^{(2)}(\Sigma, z, Q \to 0) \to Qz \left[ -\frac{1}{2\sqrt{6}} \frac{(32m^4\Sigma + 8m^4z^2 + 4m^2z^2 - 7m^2\Sigma z^2 - 3\Sigma^3)}{\sqrt{\Sigma}(z^2 + 4\Sigma)} \right] \to 0. \]

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