Center-symmetric algebras and bialgebras: relevant properties and consequences

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Abstract. Lie admissible algebra structures, called center-symmetric algebras, are defined. Main properties and algebraic consequences are derived and discussed. Bimodules are given and used to build a center-symmetric algebra on the direct sum of underlying vector space and a finite dimensional vector space. Then, the matched pair of center-symmetric algebras is established and related to the matched pair of sub-adjacent Lie algebras. Besides, Manin triple of center-symmetric algebras is defined and linked with their associated matched pairs. Further, center-symmetric bialgebras of center-symmetric algebras are investigated and discussed. Finally, a theorem yielding the equivalence between Manin triple of center-symmetric algebras, matched pairs of Lie algebras and center-symmetric bialgebras is provided.

Keywords. Lie-admissible algebra; Lie algebra; center-symmetric algebra; matched pair; Manin triple; bialgebra; cocycle.

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1. Introduction

Consider the algebra $(\mathcal{A}, \mu)$, i.e., a $\mathbb{K}$ vector space $\mathcal{A}$ endowed with a binary operation or law (bilinear homomorphism) $\mu$ defined as:

$$\mu : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

$$(x, y) \longmapsto \mu(x, y).$$

Define the associator of the binary product by a trilinear homomorphism on $\mathcal{A}$ as follows [4]:

$$\text{ass}_\mu : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

$$(x, y, z) \longmapsto \mu(\mu(x, y), z) - \mu(x, \mu(y, z)).$$

Let $\sigma \in \Sigma_3$ (symmetry group of degree $n$ ($n \in \mathbb{N}$)), acting on the associator as:

$$\sigma(x_1, x_2, x_3) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}).$$

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Definition 1.1.

The algebra \( \mathcal{A} = (\mathcal{A}, \mu) \) is called Lie admissible if the commutator of \( \mu \), denoted by \([\cdot, \cdot]_\mu\), defines on \( \mathcal{A} \) a Lie algebra structure, i.e., \([x, y]_\mu = \mu(x, y) - \mu(y, x)\) (bilinear and skew-symmetric) and \([x, y]_\mu + [z, x]_\mu, y]_\mu + [[y, z]_\mu, x]_\mu = 0\) (Jacobi identity).

Definition 1.2.

The algebra \( (\mathcal{A}, \mu) \) is called Lie-admissible if and only if \( \mu \) satisfies:

\[
\sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} \text{ass}_\mu \circ \sigma = 0,
\]

where \( \varepsilon \) is the signature of \( \sigma \).

Definition 1.3.

Let \( G \) be a subgroup of \( \Sigma_3 \). We say that the algebra law is \( G \)-associative if

\[
\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \text{ass}_\mu \circ \sigma = 0.
\]

The subgroups of \( \Sigma_3 \) are well known. We have: \( G_1 = \{\text{id}\}, G_2 = \{\text{id}, \tau_{12}\}, G_3 = \{\text{id}, \tau_{23}\}, G_4 = \{\text{id}, \tau_{13}\}, G_5 = \{A_3\} \) (Alternating group) and \( G_6 = \Sigma_3 \). \( \tau_{ij} \) is the transposition between \( i \) and \( j \), i.e., explicitly:

\[
\tau_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \tau_{13} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \tau_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \text{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.
\]

We deduce the following types of Lie admissible algebras:

1. If \( \mu \) is \( G_1 \)-associative, then \( \mu \) is associative law.
2. If \( \mu \) is \( G_2 \)-associative, then \( \mu \) is a law of Vinberg algebra [12]. If \( \mathcal{A} \) is finite-dimensional, then the associated Lie admissible algebra is provided with an affine structure.
3. If \( \mu \) is \( G_3 \)-associative, then \( \mu \) is a law of pre-Lie algebra (also called left-symmetric algebra).
4. If \( \mu \) is \( G_4 \)-associative, then \( \mu \) satisfies

\[
(xy)z - x(yz) = (yz)x - z(yx), \quad \forall x, y, z \in \mathcal{A}.
\]

We called the corresponding algebra center-symmetric algebra.
5. If \( \mu \) is \( G_5 \) associative, then \( \mu \) satisfies the generalized Jacobi condition i.e.

\[
(xy)z + (yz)x + (zx)y = x(yz) + y(zx) + z(xy).
\]

Moreover, if the law is antisymmetric, then it is a law of Lie algebra.
6. If \( \mu \) is \( G_6 \)-associative, then \( \mu \) is a Lie admissible Law.

This work aims at studying \( G_4 \)-associative structures, called center-symmetric algebras. Their algebraic properties are investigated. Related bimodule and matched pairs are given. Associated Manin triples built look like the Manin triple of Lie algebras [2]. Besides, from symmetry role of matched pairs, equivalent relations are established in the framework of center-symmetric bialgebras making some bridges with the Lie-bialgebra construction by Drinfeld [3].

Throughout this work, we consider \( \mathcal{A} \), a finite dimensional vector space over the field \( \mathbb{K} \) of characteristic zero (0) together with a bilinear product \( \cdot \) defined as \( \cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \) such that

\[
( x, y ) \mapsto x \cdot y.
\]

2. Basic properties: main definitions and algebraic consequences

In this section, we give the definition of the center-symmetric algebra, provide their basic properties and deduce relevant algebraic consequences, similarly to known framework of left-symmetric algebras [1].

Definition 2.1. \((\mathcal{A}, \cdot)\), (or simply \( \mathcal{A} \)), is said to be a center-symmetric algebra if \( \forall x, y, z \in \mathcal{A} \), the associator of the bilinear product \( \cdot \), defined by \( ( x, y, z ) := ( x \cdot y ) \cdot z - x \cdot ( y \cdot z ) \), is symmetric in \( x \) and \( z \), i.e.,

\[
( x, y, z ) = ( z, y, x ).
\]

As matter of notation simplification, we will denote \( x \cdot y \) by \( xy \) if not any confusion.
Remark 2.2. Any associative algebra is a center-symmetric algebra.

Proposition 2.3. The bilinear product (commutator) \([\cdot, \cdot] : A \times A \to A\),
\((x, y) \mapsto [x, y] = x \cdot y - y \cdot x\) gives a Lie bracket structure on \(A\), known as the sub-adjacent Lie algebra \(G(A) := (A, [\cdot, \cdot])\) of \((A, \cdot)\).

Proof: By definition of the commutator, \([\cdot, \cdot]\) is bilinear and skew symmetric. The Jacobi identity comes from a straightforward computation. \(\square\)

Thus, as in the case of left-symmetric algebras, \((A, \cdot)\) can be called the compatible center-symmetric algebra structure of the Lie algebra \(G(A)\).

Considering the representations of the left \(L\) and right \(R\) multiplication operations:

\[
L : A \to \mathfrak{gl}(A),
\]

\[
x \mapsto L_x : A \to A, \quad y \mapsto x \cdot y,
\]

\[
R : A \to \mathfrak{gl}(A),
\]

\[
x \mapsto R_x : A \to A, \quad y \mapsto y \cdot x,
\]

we infer the adjoint representation \(\text{ad} := L - R\) of the sub-adjacent Lie algebra \(G(A)\) of a center-symmetric algebra \(A\) as follows:

\[
\text{ad} : A \to \mathfrak{gl}(A),
\]

\[
x \mapsto \text{ad}_x : A \to A, \quad y \mapsto [x, y],
\]

such that \(\forall x, y \in A, \text{ad}_x(y) := (L_x - R_x)(y)\).

Proposition 2.4. Let \((A, \cdot)\) be a center-symmetric algebra, and \(L, \text{ (resp. R), be the linear representation of the left, (resp. right), multiplication operator. Then,}

(1) For all \(x, y \in A\) we have: \([L_x, R_y] = [L_y, R_x]\) and \(L_x y - L_y x = R_x R_y - R_y x\).

(2) \(\text{ad} = L - R\) is a linear representation of the sub-adjacent Lie algebra \(G(A)\) of \((A, \cdot)\), i.e.,

\(\text{ad}_{[x, y]} = \{\text{ad}_x, \text{ad}_y\}, \forall x, y \in A\).

Proof: It is immediate from the definitions of considered operators. \(\square\)

3. Bimodules and matched pairs

Definition 3.1. Let \(A\) be a center-symmetric algebra, \(V\) be a vector space. Suppose

\(l, r : A \to \mathfrak{gl}(V)\) be two linear maps satisfying: For all \(x, y \in A\),

\[
[l_x, r_y] = [l_y, r_x]
\]

\[
l_{xy} - l_{yx} = r_x r_y - r_y x.
\]

Then, \((l, r, V)\) (or simply \((l, r)\)) is called bimodule of the center-symmetric algebra \(A\).

In this case, the following statement can be proved by a direct computation.

Proposition 3.2. Let \((A, \cdot)\) be a center-symmetric algebra and \(V\) be a vector space over \(\mathbb{K}\). Consider two linear maps, \(l, r : A \to \mathfrak{gl}(V)\). Then, \((l, r, V)\) is a bimodule of \(A\) if and only if, the semi-direct sum \(A \oplus V\) of vector spaces is turned into a center-symmetric algebra by defining the multiplication in \(A \oplus V\) by \((x_1 + v_1) \ast (x_2 + v_2) = x_1 \cdot x_2 + (l_x v_2 + r_x v_1), \forall x_1, x_2 \in A, v_1, v_2 \in V\). We denote it by \(A \ltimes_{l, r} V\) or simply \(A \ltimes V\).

Furthermore, we derive the next result.

Proposition 3.3. Let \(A\) be a center-symmetric algebra and \(V\) be a vector space over \(\mathbb{K}\). Consider two linear maps, \(l, r : A \to \mathfrak{gl}(V)\), such that \((l, r, V)\) is a bimodule of \(A\). Then, the map:

\[l - r : A \to \mathfrak{gl}(V) x \mapsto l_x - r_x,\]

is a linear representation of the sub-adjacent Lie algebra of \(A\).
Example 3.4. According to the Proposition 2.4, one can deduce that \((L, R, V)\) is a bimodule of the center-symmetric algebra \(A\), where \(L\) and \(R\) are the left and right multiplication operator representations, respectively.

Definition 3.5. [2] Let \(G\) and \(H\) be two Lie algebras and let \(\mu : H \to \mathfrak{gl}(G)\) and \(\rho : G \to \mathfrak{gl}(H)\) be two Lie algebra representations satisfying: For all \(x, y \in G, a, b \in H\),

\[
\rho(x) [a, b] - [\rho(x)a, b] - [a, \rho(x)b] + \rho(\mu(x)a)b - \rho(\mu(x)b)a = 0, \tag{3.3}
\]

\[
\mu(a) [x, y] - [\mu(a)x, y] - [x, \mu(a)y] + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0. \tag{3.4}
\]

Then, \((G, H, \mu, \rho)\) is called a matched pair of the Lie algebras \(G\) and \(H\), denoted by \(H \bowtie_H G\). In this case, \((G \bowtie H, \ast)\) defines a Lie algebra with respect to the product \(\ast\) satisfying:

\[
(x + a) \ast (y + b) = [x, y] + \mu(a)y - \mu(b)x + [a, b] + \rho(x)b - \rho(y)a.
\]

Theorem 3.6. Let \((A, \cdot)\) and \((B, \circ)\) be two center-symmetric algebras. Suppose that \((l_A, r_A, B)\) and \((l_B, r_B, A)\) are bimodules of \(A\) and \(B\), respectively, obeying the relations:

\[
-r_A(x)(a \circ b) + r_A(l_B(b)x)a + a \circ (r_A(x)b) + l_A(r_B(b)x)a + (l_A(x)b) \circ a - l_A(x)(b \circ a) = 0, \tag{3.5}
\]

\[
-r_B(a)(x \cdot y) + r_B(l_A(y)a)x + x \cdot (r_B(a)y) + l_B(r_A(y)a)x + (l_B(a)y) \cdot x - l_B(a)(y \cdot x) = 0, \tag{3.6}
\]

\[
a \circ (l_A(x)b) + (r_A(x)b) \circ a - (r_A(x)a) \circ b - l_A(l_B(b)x)b + r_A(r_B(b)x)a + l_A(l_B(b)x)a +
- b \circ (l_A(x)a) - r_A(r_B(x)a)b = 0, \tag{3.7}
\]

\[
x \cdot (l_B(a)y) + (r_B(a)y) \cdot x - (r_B(a)x) \cdot y - l_B(l_A(x)a)y + r_B(r_A(x)a)x + l_B(l_A(y)a)x +
- y \cdot (l_B(a)x) - r_B(r_A(x)a)y = 0, \tag{3.8}
\]

for any \(x, y \in A\) and \(a, b \in B\). Then, there is a center-symmetric algebra structure on \(A \bowtie B\) given by: \((x + a) \ast (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a)\). We denote this center-symmetric algebra by \(A \bowtie_{l, r} B\), or simply by \(A \bowtie B\). Then \((A, B, l_A, r_A, l_B, r_B)\) satisfying the above conditions is called matched pair of the center-symmetric algebras \(A\) and \(B\).

Proof: Consider \(x, y \in A\) and \(a, b \in B\).

We have \((x + a) \ast (y + b) = (xy + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a),\)

and the associator takes the form:

\[
(x + a, y + b, z + c) = (x, y, z) + (a, b, c) + (r_B(c)(x \cdot y) + l_A(x \cdot y)c - x \cdot (r_B(c)y) - l_A(l_B(y)c) - r_B(l_A(y)c)x + \{r_B(c)(r_B(b)x) + l_A(r_B(b)x)c + r_B(b \circ c)x + (l_A(x)b) \circ c - l_A(x)(b \circ c)\} + \{(r_B(b)x) \cdot z - l_B(R_A(x)b)z + r_A(z)(l_B(b)z) - r_B(R_A(z)b)x - l_A(R_A(z)b)) +
\]

\[
\{l_B(a)y \cdot z + l_B(r_A(y)a)z + r_A(z)(r_A(y)a) - l_B(a)(y \cdot z) - r_A(y \cdot z)a + (l_B(c)(l_B(a)y) + (r_A(c)y)a) \circ c + l_A(l_B(a)y)c + l_B(a)(r_B(c)y)\} - a \circ (l_A(y)c) - r_A(r_B(c)y)a + \{l_B(a \circ b)z + r_A(z)(a \circ b) -
\]

\[
+l_B(a)(l_B(b)z) - a \circ (r_A(z)b) - r_A(l_B(b)z)a,\]

which can also be re-expressed as:

\[
(x + a, y + b, z + c) = (x, y, z) + (a, y, c) + (a, b, z) + (x, b, c) +
\]

\[
(a, y, z) + (a, y, c) + (a, b, c). \tag{3.9}
\]

Similarly,

\[
(z + c, y + c, x + a) = (z, y, x) + (z, y, a) + (z, b, x) + (z, b, a) +
\]

\[
(c, y, x) + (c, b, a) + (c, y, a) + (c, b, x). \tag{3.10}
\]
Using the fact that \((l_A, r_A)\) is a bimodule of \(A\) and \((l_B, r_B)\) is a bimodule of \(B\), one arrives at the following result:

\[
(x + a, y + b, z + c) = (z + c, y + b, x + a) \iff \begin{cases}
(x, y, z) &= (z, y, x) \\
(x, y, c) &= (c, y, x) \\
(x, b, z) &= (z, b, x) \\
(x, b, c) &= (c, b, x) \\
(a, y, z) &= (z, y, a) \\
(a, y, c) &= (c, y, a) \\
(a, b, z) &= (z, b, a) \\
(a, b, c) &= (c, b, a)
\end{cases}
\]

\[
\begin{pmatrix}
l_A, r_A, B \\
l_B, r_B, A
\end{pmatrix}, \quad \begin{pmatrix}
(x, y) &= (c, y, x) \\
(x, b) &= (z, b, x) \\
(x, b) &= (c, b, x) \\
(a, y) &= (c, y, a)
\end{pmatrix}
\]

This last relation ends the proof. \(\square\)

Moreover, every center - symmetric algebra which is a direct sum of the underlying spaces of two center - symmetric sub - algebras can be obtained in the above way.

**Corollary 3.7.** Let \((A, B, l_A, r_A, l_B, r_B)\) be a matched pair of center - symmetric algebras. Then, \((\mathcal{G}(A), \mathcal{G}(B), l_A - r_A, l_B - r_B)\) is a matched pair of sub-adjacent Lie algebras \(\mathcal{G}(A)\) and \(\mathcal{G}(B)\).

**Proof:** By using the Proposition 3.3 and the bimodules \((l_A, r_A, B)\) and \((l_B, r_B, A)\), we have: \(\text{ad}_A := l_A - r_A\) and \(\text{ad}_B := l_B - r_B\) are the linear representations of the sub-adjacent Lie algebras \(\mathcal{G}(A)\) and \(\mathcal{G}(B)\) of the center-symmetric algebras \(A\) and \(B\), respectively. Then, the statement that \(\mathcal{G}(A) \cong \text{ad}_A \mathcal{G}(B)\) is a matched pair of the Lie algebras \(\mathcal{G}(A)\) and \(\mathcal{G}(B)\) follows from Theorem 3.6. Hence, \((\mathcal{G}(A), \mathcal{G}(B), \text{ad}_A, \text{ad}_B)\) is a matched pair of sub-adjacent Lie algebras \(\mathcal{G}(A)\) and \(\mathcal{G}(B)\). \(\square\)

**Definition 3.8.** Let \((l, r, V)\) be a bimodule of a center - symmetric algebra \(A\), where \(V\) is a finite dimensional vector space. The dual maps \(l^*, r^*\) of the linear maps \(l, r\) are defined, respectively, as: \(l^* : A \rightarrow \mathfrak{gl}(V^*)\) such that:

\[
l^* : A \rightarrow \mathfrak{gl}(V^*)
\]

\[
x \mapsto l^*_x : u^* \rightarrow l^*_x u^* : V \quad \mapsto \mathbb{K} \quad \langle l^*_x u^*, v \rangle := (u^*, l_x v),
\]

\[
r^* : A \rightarrow \mathfrak{gl}(V^*)
\]

\[
x \mapsto r^*_x : u^* \rightarrow r^*_x u^* : V \quad \mapsto \mathbb{K} \quad \langle r^*_x u^*, v \rangle := (u^*, r_x v),
\]

for all \(x \in A, u^* \in V^*, v \in V\).

**Proposition 3.9.** Let \(A\) be a center - symmetric algebra and \(l, r : A \rightarrow \mathfrak{gl}(V)\) be two linear maps, where \(V\) is a finite dimensional vector space. The following conditions are equivalent:

1. \((l, r, V)\) is a bimodule of \(A\).
2. \((r^*, l^*, V^*)\) is a bimodule of \(A\).

**Proof:** It stems from the Definition 3.8.

**Theorem 3.10.** Let \((\mathcal{A}, \cdot)\) be a center - symmetric algebra. Suppose that there exists a center - symmetric algebra structure \(" o \"\) on its dual space \(\mathcal{A}^*\). Then, \((\mathcal{A}, \mathcal{A}^*, R^*, L^*, R^*_0, L^*_0)\) is a matched pair of center - symmetric algebras \(A\) and \(A^*\) if and only if \((\mathcal{G}(A), \mathcal{G}(A^*), - \text{ad}_{l^*}, - \text{ad}_{l^*_0})\) is a matched pair of Lie algebras \(\mathcal{G}(A)\) and \(\mathcal{G}(A^*)\).
Proof: By considering the Theorem 3.10 setting \( t_A := R^*, r_A := L^*, t_B := R^*, r_B := L^* \), and exploiting the Definition 3.6 with \( G := G(A), H := G(A^*), \rho := R - L^*, \mu := R^* - L^* \), and the relations 3.11 and 3.12, we get the equivalences.

Proposition 3.11. Let \( G \) be a Lie algebra. Suppose \( \rho : G \to \mathfrak{gl}(V) \) and \( \mu : G \to \mathfrak{gl}(W) \) be two linear representations of \( G \), where \( V \) and \( W \) are two vector spaces. Then, the linear map \( \rho \otimes 1 + 1 \otimes \mu : G \to \mathfrak{gl}(V \otimes W) \) given by \((\rho \otimes 1 + 1 \otimes \mu)(v, w) := \rho(x)v \otimes w + v \otimes \mu(x)v \) is also a representation of \( G \).

Proof: It comes from a straightforward computation.

Theorem 3.12. Let \( A \) be a center - symmetric algebra with the product given by the linear map \( \beta^* : A \otimes A \to A \). Suppose there is a center - symmetric algebra structure \( " \circ ^* " \) on the dual space \( A^* \) provided by a linear map \( \alpha^* : A^* \otimes A^* \to A^* \). Then, \((G(A), G(A^*), -\operatorname{ad}^*, -\operatorname{ad}^*)\) is a matched pair of Lie algebras \( G(A) \) and \( G(A^*) \) if and only if \( \alpha : A \to A \otimes A \) is a 1 cocycle of \( G(A) \) associated to \((-\operatorname{ad}) \otimes 1 + 1 \otimes (-\operatorname{ad}) \) and \( \beta : A^* \to A^* \otimes A^* \) is a 1-cocycle of \( G(A^*) \) associated to \((-\operatorname{ad}) \otimes 1 + 1 \otimes (-\operatorname{ad}) \).

Proof: See Appendix.

4. Manin triple and center-symmetric bialgebras

In this section, similarly to the notion of Manin triple of Lie algebras introduced in [2], we first give the definition of Manin triple of a center - symmetric algebra and investigate its associated bialgebra structure. Then, we provide the basic definition and properties of center - symmetric bialgebras.

Definition 4.1. A Manin triple of center - symmetric algebras is a triple \((A, A^+, A^-)\) together with a nondegenerate symmetric bilinear form \( \mathcal{B}(\cdot, \cdot) \) on \( A \) which is invariant, i.e., \( \forall x, y, z \in A, \mathcal{B}(x \ast y, z) = \mathcal{B}(x, y \ast z) \), satisfying:

(1) \( A = A^+ \oplus A^- \) as \( \mathbb{K} \) - vector space;

(2) \( A^+ \) and \( A^- \) are center - symmetric subalgebras of \( A \);

(3) \( A^+ \) and \( A^- \) are isotropic with respect to \( \mathcal{B} \), i.e., \( \mathcal{B}(A^+, A^+) = 0 = \mathcal{B}(A^-, A^-) \).

Two Manin triples \((A_1, A^+_1, A^-_1, \mathcal{B}_1)\) and \((A_2, A^+_2, A^-_2, \mathcal{B}_2)\) of center - symmetric algebras \( A_1 \) and \( A_2 \) are homomorphically (isomorphically) if there is a homomorphism (isomorphism) \( \varphi : A_1 \to A_2 \) such that \( \varphi(A^+_1) \subset A^+_2 \), \( \varphi(A^-_1) \subset A^-_2 \), \( \mathcal{B}_1(x, y) = \varphi^* \mathcal{B}_2(\varphi(x), \varphi(y)) \). In particular, if \((A, \cdot)\) is a center - symmetric algebra, and if there exists a center - symmetric algebra structure on its dual space \( A^* \) denoted \((A^*, \cdot)\), then there is a center - symmetric algebra structure on the direct sum of the underlying vector space of \( A \) and \( A^* \) (see Theorem 3.6) such that \((A \oplus A^*, A, A^*)\) is the associated Manin triple with the invariant bilinear symmetric form given by \( \mathcal{B}_A(x \ast a^*, y \ast b^*) = \langle x, b^* \rangle \ast \langle y, a^* \rangle, \forall x, y \in A; a^*, b^* \in A^* \), called the standard Manin triple of the center - symmetric algebra \( A \).

Theorem 4.2. Let \((A, \cdot)\) and \((A^*, \cdot)\) be two center - symmetric algebras. Then, the sixtuple \((A, A^*, R, L, R^*, L^*)\) is a matched pair of center - symmetric algebras \( A \) and \( A^* \) if and only if \((A \oplus A^*, A, A^*)\) is their standard Manin triple.

Proof: By considering that \((A, A^*, R, L, R^*, L^*)\) is a matched pair of center - symmetric algebras, it follows that the bilinear product \( \ast \) defined in the Theorem 3.6 is center - symmetric on the direct sum of underlying vectors spaces, \( A \oplus A^* \). Computing and comparing the relations, we get: \( \mathcal{B}_A((x + a) \ast (y + b), (z + c)) = \mathcal{B}_A((x + a), (y + b) \ast (z + c)) \), \forall x, y, z \in A; a, b, c \in A^*, \) which expresses the invariance of the standard bilinear form on \( A \oplus A^* \). Therefore, \((A \oplus A^*, A, A^*)\) is the standard Manin triple of the center - symmetric algebras \( A \) and \( A^* \).

Definition 4.3. Let \( A \) be a vector space. A center - symmetric bialgebra structure on \( A \) is a pair of linear maps \((\alpha, \beta)\) such that \( \alpha : A \to A \otimes A, \beta : A^* \to A^* \otimes A^* \) satisfying:

(1) \( \alpha^* : A^* \otimes A^* \to A^* \) is a center - symmetric algebra structure on \( A^* \),
We also denote this center - symmetric bialgebra by \((A, A^*, \alpha, \beta)\) or simply \((A, A^*)\).

**Proposition 4.4.** Let \((A, \cdot)\) be a center - symmetric algebra and \((A^*, \circ)\) be a center - symmetric algebra structure on its dual space \(A^*\). Then the following conditions are equivalent:

1. \((A \otimes A^*, A, A^*)\) is the standard Manin triple of considered center - symmetric algebras;
2. \((G(A), G(A^*), -\text{ad}^*, -\text{ad}^*_{\circ})\) is a matched pair of sub-adjacent Lie algebras;
3. \((A^*, R, L^*, R^*_{\circ}, L^*_{\circ})\) is a matched pair of center - symmetric algebras;
4. \((A, A^*)\) is a center - symmetric bialgebra.

**Proof:** From Theorem 3.10 (2) \(\iff\) (3), while from Theorem 3.12 (2) \(\iff\) (4). Theorem 4.2 shows that (1) \(\iff\) (3).

## 5. Concluding remarks

In this work, we have defined Lie admissible algebra structures, called center - symmetric algebras for which main properties and algebraic consequences have been derived and discussed. Bimodules have been given and used to build a center - symmetric algebra on the direct sum of a center - symmetric algebra and a vector space. Then, we have established the matched pair of center - symmetric algebras, which has been related to the matched pair of sub-adjacent Lie algebras. Besides, we have defined the Manin triple of center - symmetric algebras and linked it with their associated matched pairs. Further, we have investigated and discussed center - symmetric bialgebras of center - symmetric algebras. Finally, we have provided a theorem yielding the equivalence between Manin triple of center - symmetric algebras, matched pairs of Lie algebras and center - symmetric algebras, and center - symmetric bialgebras.

**Appendix**

**Proof of the Theorem 3.12**

Let \(\{e_1, e_2, \cdots, e_n\}\) be a basis of \(A\) and \(\{e_1^*, e_2^*, \cdots, e_n^*\}\) its dual basis. Consider \(e_i \cdot e_j = \sum_{k=1}^{n} c_{ij}^k e_k\) and \(e_i^* \circ e_j^* = \sum_{k=1}^{n} f_{ij}^k e_k^*\), where \(c_{ij}^k, f_{ij}^k \in K\) are structure constants associated to \(\cdot\) and \(\circ\), respectively.

Then, it follows that:

\[
\alpha(e_k) = \sum_{i,j=1}^{n} f_{ij}^k e_i \otimes e_j, \quad \beta(e_k^*) = \sum_{i,j=1}^{n} c_{ij}^k e_i^* \otimes e_j^*,
\]

and

\[
\alpha([e_i, e_j]) = \sum_{m,l=1}^{n} \{ (c_{ij}^k - c_{ji}^k) f_{ml}^k \} e_m \otimes e_l,
\]

and

\[
\beta([e_i^*, e_j^*]) = \sum_{m,l=1}^{n} \{ (f_{ij}^k - f_{ji}^k) c_{ml}^k \} e_m^* \otimes e_l^*,
\]

and we get:

\[
\{(\text{ad} \cdot)(e_i) \otimes 1 + 1 \otimes (\text{ad} \circ)(e_i)\} \cdot \alpha(e_j) = \{(\text{ad} \cdot)(e_j) \otimes 1 + 1 \otimes (\text{ad} \circ)(e_j)\} \cdot \alpha(e_i) = \sum_{m,l=1}^{n} \sum_{k=1}^{n} \left\{ -f_{kl}(c_{ik}^m - c_{ki}^m) + f_{kl}(c_{jk}^m - c_{kj}^m) - f_{ml}^k (c_{ik}^j - c_{ki}^j) + f_{ml}^j (c_{jk}^i - c_{kj}^i) \right\} e_m \otimes e_l
\]

Taking into account the fact that \(\alpha\) is a 1-cocycle of \(G(A)\) associated to \((-\text{ad}) \otimes 1 + 1 \otimes (-\text{ad})\), and using the relations 5.1 and 5.2 yield:

\[
\sum_{k=1}^{n} (c_{ij}^k - c_{ji}^k) f_{ml}^k = \sum_{k=1}^{n} \left\{ f_{kl}(c_{ik}^m - c_{ki}^m) - f_{ml}^k (c_{ik}^j - c_{ki}^j) + f_{ml}^j (c_{jk}^i - c_{kj}^i) \right\},
\]

and

\[
\sum_{k=1}^{n} (f_{ij}^k - f_{ji}^k) c_{ml}^k = \sum_{k=1}^{n} \left\{ -f_{kl}(c_{ik}^m - c_{ki}^m) + f_{kl}(c_{jk}^m - c_{kj}^m) - f_{ml}^k (c_{ik}^j - c_{ki}^j) + f_{ml}^j (c_{jk}^i - c_{kj}^i) \right\}.
\]
Besides, we obtain:

\[
\{(-\text{ad}_{\alpha})(e^*_i) \otimes 1 + 1 \otimes (-\text{ad}_{\alpha})(e^*_j)\} \beta(e^*_k) = \{(-\text{ad}_{\alpha})(e^*_j) \otimes 1 + 1 \otimes (-\text{ad}_{\alpha})(e^*_i)\} \beta(e^*_k) = \\
\sum_{m, l = 1}^{n} \sum_{k = 1}^{n} \left\{ -c^l_{kl}(f^m_{ik} - f^m_{ik}) + c^l_{kl}(f^m_{jk} - f^m_{jk}) - c^l_{mk}(f^l_{ik} - f^l_{ik}) + c^l_{mk}(f^l_{jk} - f^l_{jk}) \right\} (e^*_m \otimes e^*_l). \quad (5.5)
\]

As \(\beta\) is the 1-cocycle issued from \((\text{ad}_{\alpha}) \otimes 1 + 1 \otimes (-\text{ad}_{\alpha})\) and using the relations \((5.2)\) and \((5.5)\), we obtain:

\[
\sum_{k = 1}^{n} (f^k_{ij} - f^k_{ji}) e^*_m = \sum_{k = 1}^{n} \left\{ c^l_{kl}(f^m_{ik} - f^m_{ik}) - c^l_{kl}(f^m_{jk} - f^m_{jk}) + c^l_{mk}(f^l_{ik} - f^l_{ik}) - c^l_{mk}(f^l_{jk} - f^l_{jk}) \right\}. \quad (5.6)
\]

Now, let us find the relations associated to the equations \((3.3)\) - \((3.4)\) of the matched pair of Lie algebras \(\mathcal{G}(A)\) and \(\mathcal{G}(A^*)\). We have \(\forall i, j, k:\)

\[
\langle (-\text{ad}^*)(e_i) e^*_j, e_k \rangle = -\left\langle \sum_{k = 1}^{n} (c^l_{ik} - c^l_{kj}) e^*_k, e_k \right\rangle,
\]

providing

\[
(-\text{ad}^*)(e_i) e^*_j = - \sum_{k = 1}^{n} (c^l_{ik} - c^l_{kj}) e^*_k. \quad (5.7)
\]

Similarly,

\[
(-\text{ad}^*)(e^*_j) e_j = - \sum_{k = 1}^{n} (f^l_{ik} - f^l_{ki}) e^*_i,
\]

\[
(-\text{ad}^*)(e^*_m)[e_i, e_j] = \sum_{k = 1}^{n} (c^k_{ij} - c^k_{ji}) (-\text{ad}^*)(e^*_m) e_k = - \sum_{l = 1}^{n} \sum_{k = 1}^{n} (c^l_{ik} - c^l_{kj}) (f^l_{m} - f^l_{jm}) e_i.
\]

Then,

\[
(-\text{ad}^*)(e^*_m)[e_i, e_j] = - \sum_{l = 1}^{n} \sum_{k = 1}^{n} (c^k_{ij} - c^k_{ji}) (f^k_{m} - f^k_{jm}) e_i, \quad (5.9)
\]

\[
- \text{ad}^*_m(\text{ad}^*(e_i) e^*_m) e_j - [e_i, \text{ad}^*_m(\text{ad}^*(e_j) e^*_m)] - [\text{ad}^*_m(\text{ad}^*(e_j) e^*_m), e_i, e_j]
\]

\[
= \sum_{k = 1}^{n} \sum_{l = 1}^{n} \left\{ -(c^m_{ik} - c^m_{ki})(f^l_{mk} - f^l_{km}) - (f^m_{ik} - f^m_{ki})(c^l_{jk} - c^l_{kj}) \right\}.
\]

Using the fact that \((\mathcal{G}(A), \mathcal{G}(A^*), \text{ad}^*, \text{ad}^*_m)\) is a bimodule of Lie algebras, we have

\[
\sum_{k = 1}^{n} (c^k_{ij} - c^k_{ji}) (f^k_{m} - f^k_{jm}) = \sum_{k = 1}^{n} (c^m_{ik} - c^m_{ki})(f^l_{mk} - f^l_{km}) + (f^m_{ik} - f^m_{ki})(c^l_{jk} - c^l_{kj}) + (c^m_{jk} - c^m_{kj})(f^l_{mk} - f^l_{km}) + (f^m_{jk} - f^m_{kj})(c^l_{ij} - c^l_{ji}), \quad (5.10)
\]

that is,

\[
\sum_{k = 1}^{n} (c^k_{ij} - c^k_{ji}) f^k_{m} + \sum_{k = 1}^{n} (c^m_{ik} - c^m_{ki}) f^j_{ik} + (c^l_{jk} - c^l_{kj}) f^m_{jk} = \sum_{k = 1}^{n} (c^m_{jk} - c^m_{kj}) f^l_{ik} + (c^l_{ij} - c^l_{ji}) f^m_{jk} + (c^m_{jk} - c^m_{kj}) f^i_{mk} + (c^m_{jk} - c^m_{kj}) f^k_{jm}.
\]

Replacing \(l\) (resp. \(m\)) by \(m\) (resp. \(l\)) in the right-hand side of the equality leads to:

\[
\sum_{k = 1}^{n} (c^k_{ij} - c^k_{ji}) f^k_{m} = \sum_{k = 1}^{n} \left\{ -(c^m_{ik} - c^m_{ki}) f^l_{mk} + (c^m_{jk} - c^m_{kj}) f^i_{mk} + (c^m_{jk} - c^m_{kj}) f^k_{jm} + (c^m_{jk} - c^m_{kj}) f^l_{jm} \right\}. \quad (5.11)
\]
which is identical to the equation (5.3). Besides,

\[
(-\text{ad}^*(e_m))[e^*_i, e^*_j] = -\sum_{l=1}^{n} \sum_{k=1}^{n} \{(f^k_{ij} - f^k_{ji})(e^k_{ml} - c^k_{lm})\} e^*_l
\]  \hspace{1cm} (5.12)

\[\begin{align*}
&- \text{ad}^*(\text{ad}^*(e_m))e^*_m - [e^*_i, \text{ad}^*(e_m)e^*_j] + \text{ad}^*(\text{ad}^*(e_j))e^*_m - [\text{ad}^*(e_m)e^*_i, e^*_j] = \\
&\sum_{l=1}^{n} \sum_{k=1}^{n} \{(f^k_{ik} - f^k_{ki})(c^k_{kl} - c^k_{lk}) - (c^k_{mk} - c^k_{km})(f^l_{ik} - f^l_{ki}) + (f^m_{jk} - f^m_{kj})(c^k_{lk} - c^k_{lk}) - \\
&(c^k_{mk} - c^k_{km})(f^l_{jk} - f^l_{kj})\} e^*_l.
\end{align*}\]

Then, with \(G(A) \ni \text{ad}^* \rightarrow \text{ad}^* G(A)\) and the relation (5.12), we obtain

\[\sum_{k=1}^{n} (f^k_{ij} - f^k_{ji})(c^k_{ml} - c^k_{lm}) = \sum_{k=1}^{n} (f^m_{ik} - f^m_{ki})(c^k_{kl} - c^k_{lk}) - (c^k_{mk} - c^k_{km})(f^l_{ik} - f^l_{ki}) + \]

\[+ (f^m_{jk} - f^m_{kj})(c^k_{lk} - c^k_{lk}) + (c^k_{mk} - c^k_{km})(f^l_{jk} - f^l_{kj}),\]

i.e.,

\[\begin{align*}
&\sum_{k=1}^{n} (f^k_{ij} - f^k_{ji})c^k_{ml} + \sum_{k=1}^{n} c^k_{kl}(f^m_{ik} - f^m_{ki}) + c^k_{kl}(f^m_{ik} - f^m_{ki}) - c^k_{mk}(f^l_{ik} - f^l_{ki}) - c^k_{mk}(f^l_{ik} - f^l_{ki}) = \\
&\sum_{k=1}^{n} (f^k_{ij} - f^k_{ji})c^k_{lm} + \sum_{k=1}^{n} c^k_{lk}(f^m_{ik} - f^m_{ki}) + c^k_{lk}(f^m_{ik} - f^m_{ki}) - c^k_{km}(f^l_{ik} - f^l_{ki}) - c^k_{km}(f^l_{ik} - f^l_{ki}),
\end{align*}\]

Replacing \(l\), (resp. \(m\)), by \(m\), (resp. \(l\)), in the right-hand side of the equality leads to

\[\sum_{k=1}^{n} (f^k_{ij} - f^k_{ji})c^k_{ml} = \sum_{k=1}^{n} (f^k_{ij} - f^k_{ji})c^k_{ml} - c^k_{kl}(f^m_{ik} - f^m_{ki}) + c^k_{kl}(f^m_{ik} - f^m_{ki}) - c^k_{mk}(f^l_{ik} - f^l_{ki}) + c^k_{mk}(f^l_{ik} - f^l_{ki}),(5.13)\]

which is identical to the equation (5.6). \(\square\)

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References

[1] C. Bai, Left-symmetric bialgebras and an analogue of the classical Yang-Baxter equation, Com. in Contemp. Math. 10, No. 2 (2008) pp. 221 - 260.
[2] V. Chari and A. Pressley, A guide to Quantum Groups (Cambridge University Press), Cambridge (1995).
[3] V. G. Drinfeld, Quantum group, Proceedings of the international congress for the mathematicians, Berkeley, California USA, (1986) pp. 798-820.
[4] M. Goze and E. Remm, Lie-admissible algebras and operads, Journal of Algebra 273 (2004) pp. 129 - 152.
[5] S. Majid, Foundation of Quantum Group Theory, Cambridge University Press, Cambridge (1995).
[6] S. Majid, Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations, Pacific J. Math. 141 No.2 (1990) pp. 311-322.
[7] X. Ni and C. Bai, Poisson bialgebras, J. Math. Phys. 54, 023515 (2013); 10.1063/1.4792668.
[8] E. Remm, Opérades Lie-admissible, C. R. Math. Acad. Sci. Paris 334 (12) (2002) pp. 1047 - 1050.
[9] R. M. Santilli, Supplemento al Nuovo cimento, Series I, vol. 6. (1968) pp. 1225-1249.
[10] R. D. Schafer, An introduction to nonassociative algebras, Stillwater, Oklahoma, (1961).
[11] Y. Kosmann - Schwarzbach, Lie bialgebras, Poisson Lie groups and dressing transformations, Integrability of Nonlinear Systems, Second edition, Lecture Notes in Physics 638, Springer-Verlag, (2004), pp. 107-173.
[12] E. B. Vinberg, Conve homogeneous cones, Tranl. of Moscow Math. Soc. 12 (1963) pp. 340 403.
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