MASS OF THE $\eta'$ MESON IN THE CHIRAL LIMIT IN THE ZERO MOMENTUM MODES ENHANCEMENT QUANTUM MODEL OF THE QCD NONPERTURBATIVE VACUUM

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Abstract

Using the trace anomaly and low energy relations, as well as the Witten-Veneziano formula for the mass of the $\eta'$ meson, we have developed a formalism which makes it possible to express the gluon condensate, the topological susceptibility and the mass of the $\eta'$ meson as functions of the truly nonperturbative vacuum energy density, which is one of the most important characteristics of the QCD true ground state. It was directly applied to the numerical evaluation of the chiral QCD vacuum topological structure within its quantum zero momentum modes enhancement model. A rather good agreement with the phenomenological and experimental values of the above-mentioned quantities has been achieved. With the help of the Witten-Veneziano formula, we derived an absolute lower bounds for the pion decay constant and the mass of the $\eta'$ meson in the chiral limit. By introducing the most general parametrization of the gluon condensate, we also proposed how the correct $N_f$ (number of flavors) dependence of its phenomenologically extracted value could be restored.

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I. INTRODUCTION

It is well known that one of the most important aspects of the famous $U(1)$ problem [1,2] is the large mass of the $\eta'$ meson. It does not vanish in the chiral limit, so the $\eta'$ meson is not the Nambu-Goldstone (NG) boson. In Ref. [3] (see also Ref. [4]) by using the large $N_c$ limit technique the expression for the mass of the $\eta'$ meson was derived, namely

$$m_{\eta'}^2 = \frac{2N_f}{F_\pi^2} \chi_t + \Delta,$$  \hspace{1cm} (1.1)

where $\Delta = 2m_K^2 - m_{\eta'}^2$, $N_f$ is the number of light quarks and $F_\pi$ is the pion decay constant. However, the important quantity which enters this formula is the topological density operator (topological susceptibility), $\chi_t$ (for definition see section 3). In the chiral limit it is screened that is why it is defined for Yang-Mills (YM) fields, i.e., for pure gluodynamics ($N_f = 0$). It is one of the main characteristics of the QCD nonperturbative vacuum where it measures the fluctuations of the topological charge.

The precise validity of the Witten-Veneziano (WV) formula (1.1) is, of course, not completely clear because of its origin. Nevertheless, let us regard it as exact for simplicity (in any case we have nothing better than Eq. (1.1)). However, there are phenomenological reasons [5,6] as well as some lattice indications [7] to believe that QCD is close to $SU(\infty)$. Using now experimental values of all physical quantities entering this formula, one obtains that the phenomenological ("experimental") value of the topological susceptibility is

$$\chi_t^{phen} = 0.001058 \text{ GeV}^4 = (180.36 \text{ MeV})^4 = 0.1377 \text{ GeV}^4/\text{fm}^4.$$  \hspace{1cm} (1.2)

In the chiral limit $\Delta = 0$ since $K^\pm$ and $\eta$ particles are NG excitations. It is worth noting further that neither the mass of the $\eta'$ meson nor the pion decay constant in the chiral limit cannot exceed their experimental values. So the WV formula (1.1) provides an absolute lower bounds for the pion decay constant and the mass of the $\eta'$ meson in the chiral limit, namely

$$854 \leq m_{\eta'}^0 \leq 957.77 \text{ (MeV)},$$  \hspace{1cm} (1.3)

and

$$83.2 \leq F_\pi^0 \leq 93.3 \text{ (MeV)},$$  \hspace{1cm} (1.4)

respectively. They should be compared with their experimental values (upper bounds in the previous expressions). Let us note that the chiral perturbation theory value of the pion decay constant in the chiral limit, $F_\pi^0 = (88.3 \pm 1.1) \text{ MeV}$ [8], obviously satisfies these bounds, Eq. (1.4). Recent lattice result [9] (see also brief review [7]) for the mass of the $\eta'$ meson in the continuum chiral limit is $m_{\eta'}^0 = 863(86) \text{ MeV}$, satisfying thus bounds (1.3).

One can conclude in that the mass of the $\eta'$ meson remains large even in the chiral limit, which is real problem indeed. Thus the large mass of the $\eta'$ meson in the chiral limit is due to the phenomenological value of the topological susceptibility. In other words, it is clear that through the topological susceptibility (i.e., via the WV formula (1.1)) the large mass of the $\eta'$ meson even in the chiral limit is determined by the topological properties of the QCD ground state, its nonperturbative vacuum. It has a very rich dynamical and topological structure.
It is a very complicated medium and its dynamical and topological complexity means that its structure can be organized at various levels (quantum, classical). It can contain many different components and ingredients which may contribute to the truly nonperturbative vacuum energy density (VED). It is well known that the VED in general is badly divergent [13], however the truly nonperturbative VED is finite, automatically negative and it has no imaginary part (stable vacuum). For gauge-invariant definition and concrete examples see recent papers [14,15] (for brief description see also sections 2 and 3). Precisely this quantity is one of the main characteristics of the QCD true ground state and precisely it is related to the nonperturbative gluon condensate via the trace anomaly relation [16] (section 3) as well as to the above-mentioned topological susceptibility via the low energy ”theorem” (relation) derived by Novikov, Schifman, Vanshtein and Zakharov (NSVZ) a long time ego [17] and rederived quite recently by Halperin and Zhitnitsky (HZ) [18] (section 4). Let us remind that the truly nonperturbative VED is nothing else but the bag constant apart from the sign, by definition [13,14,19]. It is much more general quantity than the string tension because it is relevant for light quarks as well.

II. ZMME QUANTUM MODEL

Many models of the QCD vacuum involve some extra classical color field configurations (such as randomly oriented domains of constant color magnetic fields, background gauge fields, averaged over spin and color, stochastic colored background fields, etc.) and ingredients such as color-magnetic and Abelian-projected monopoles (see Refs. [10,11,20] and references therein). The relevance of center vortices for QCD vacuum by both lattice [21] and analytical methods [22] was recently investigated as well. However, the most elaborated classical models are the random and interacting instanton liquid models (RILM and IILM) of the QCD vacuum. They are based on the existence of the topologically nontrivial, instanton-type fluctuations of gluon fields there, which are nonperturbative, weak coupling limit solutions to the classical equations of motion in Euclidean space [23] (and references therein). That is instantons may be qualitatively responsible for the $\eta'$ mass for the first time has been pointed out by ’t Hooft [24]. Quantitatively this problem due to instantons was investigated in our previous work [25].

The formalism developed there will be generalized here in order to investigate this problem (the large mass of the $\eta'$ meson even in the chiral limit) quantitatively in quantum field theory as well in particular QCD within the above-mentioned zero momentum modes enhancement (ZMME) quantum model of the QCD nonperturbative vacuum [26,27].

However, a few general remarks in advance are in order. The quantum part of the VED is, in general, determined by the effective potential approach for composite operators [28] (see also Ref. [29]). It allows us to investigate the nonperturbative QCD vacuum, since in the absence of external sources the effective potential is nothing else but the VED. It gives the VED in the form of the loop expansion where the number of the vacuum loops (consisting of the confining quarks with dynamically generated quark masses and nonperturbative gluons properly regularized with the help of ghosts (if any)) is equal to the power of the Plank constant, $h$. As was underlined above, this quantity in general is badly divergent at least as fours power of the ultraviolet cutoff (for detail discussion see Shifman’s contribution in Ref. [11] as well as our paper [30]). This reflects the fact (situation) that real QCD vacuum
being nonperturbative, nevertheless contains excitations and fluctuations of the gluon field configurations there which are of pure perturbative character and magnitude. In order to deal with its true nonperturbative structure, all kind of perturbative contributions should be subtracted from the VED. In this way one obtains the truly nonperturbative VED which is precisely the one of main characteristics of the QCD true ground state in continuum theory. This is absolutely similar to lattice approach where by using different "smoothing" techniques such as "cooling" [31], "cycling" [32], etc. it is possible to "wash out" all types of the perturbative fluctuations and excitations of the gluon field configurations from the QCD vacuum in order to deal only with its true nonperturbative structure, i.e., free from all kinds of perturbative "contamination" ("noise").

We have already formulated [14,15] a general method how to correctly (in a manifestly gauge-invariant way) calculate the truly nonperturbative VED in the QCD quantum models of its ground state using the above-mentioned effective potential approach for composite operators [28]. The truly nonperturbative VED was defined as integrated out of the truly nonperturbative part of the full gluon propagator over the deep infrared (IR) region (soft momentum region). The nontrivial minimization procedure which can be done only by the two different ways (leading however to the same numerical value (if any) of the truly nonperturbative VED) makes it possible to determine the soft cutoff in terms of the corresponding nonperturbative scale parameter which is inevitably present in any nonperturbative model for the full gluon propagator. The analysis of the truly nonperturbative Yang-Mills (YM) VED after the scale factorization provides an exact criterion for the separation of "stable vs. unstable" vacuum models. If the chosen Ansatz for the full gluon propagator is a realistic one, then our method uniquely determines the truly nonperturbative YM VED, which is always finite, automatically negative and it has no imaginary part (stable vacuum). In the same way (apart from some details concerning the chiral limit physics) can be determined the contribution to the truly nonperturbative VED from confining quark degrees of freedom [26,27].

The above-briefly-described general method can serve as a test of QCD vacuum different not only quantum, lattice [14,33] but classical models [15,34] as well. By applying our method to the classical theory, we thereby investigating the stability vs. instability of the vacuum of the classical models against quantum corrections. However, it is worth emphasizing that the only way to calculate the truly nonperturbative VED in quantum field theory in particular QCD from first principles [14,15] is the effective potential approach for composite operators [28] by substituting there a well-justified ansatz for the full gluon propagator since an exact solution(s) to its Schwinger-Dyson (SD) equation is not yet known. Moreover, it seems to us that it cannot be found in principle because of too complicated mathematical structure of the corresponding SD equation [2]. Fortunately, however, in order to calculate the truly nonperturbative VED it suffices to know its deep IR asymptotics which for realistic models usually coincides with its truly nonperturbative part [14].

In our previous works [14,26,27] we have formulated a new, quantum model of the QCD ground state (its nonperturbative vacuum): the ZMME model or simply ZME since we al-

\[ ^1 \text{Any deviation of the full gluon propagator from the free perturbative one automatically assumes its dependence on a characteristic mass scale parameter [30].} \]
ways work in the momentum space. It is based on the existence and importance of such kind of the nonperturbative, topologically nontrivial quantum excitations of the gluon field configurations (due to self-interaction of massless gluons only, i.e., without involving any extra degrees of freedom) which can be effectively, correctly described by the $q^{-4}$-type behaviour of the full gluon propagator in the deep IR domain. Such strong IR singular behavior of the full gluon propagator can be referred to as the strong coupling regime, the so-called ”infrared slavery” [2]. In realistic models for the full gluon propagator its truly nonperturbative part usually coincides with its deep IR asymptotics, emphasizing thus the strong intrinsic influence of the IR properties of the theory on its nonperturbative dynamics. By applying the above-described general method to this model, the nonperturbative chiral QCD vacuum was found stable, i.e., having a ”stationary” state, which can be a manifestation of a possible existence of a ground state in this model. Consequently, the corresponding truly nonperturbative YM VED is finite and negative.

Within the ZME quantum model of the QCD ground state, the truly nonperturbative VED depends on a scale at which the nonperturbative effects become important. If QCD itself is confining theory, such a characteristic scale should certainly exist. The confining quark part of the VED depends in addition on the constant of integration of the corresponding quark SD equation in the chiral limit. The numerical value of the nonperturbative scale as well as the above-mentioned constant of integration is obtained from the bounds (1.4) for the pion decay constant in the chiral limit by implementing a physically well-motivated scale-setting scheme [26,27]. We have obtained the following numerical results for the truly nonperturbative VED due to ZME quantum model:

\[
\begin{align*}
\epsilon_{ZME} &= -(0.01413 - N_f 0.00196) \, GeV^4, \\
\epsilon_{ZME} &= -(0.009 - N_f 0.00124) \, GeV^4,
\end{align*}
\]

(2.1)

where, obviously, the first and second values are due to upper and lower bounds in Eq. (1.4), respectively. Here and further on below $N_f$ is the number of light flavors. We see that the confining quark part is approximately one order of magnitude less than the YM part and it is of opposite sign.

It is instructive to compare our values (2.1) with those which are due to instantons in the chiral limit [25]. Within the above-mentioned RILM, for dilute ensemble, the truly nonperturbative VED is $\epsilon_I = -(b/4) \times n_0$. Here $b = 11 - (2/3)N_f$ is the first coefficient of the $\beta$ function (see below) and $n_0$ is the instanton number density in the chiral limit [23,25]. Due to all reasonable estimates of this quantity (which follows from phenomenology or lattice approach) its numerical value cannot exceed its phenomenological value, i.e., $n_0 \leq n = 1.0 \, fm^{-4}$. Thus, at maximum the truly nonperturbative VED due to RILM is

\[
\epsilon_I = -(0.004179 - N_f 0.00025) \, GeV^4.
\]

(2.2)

\[2\text{In nonperturbative QCD all numerical results depend on the numerical value of the nonperturbative (characteristic) scale and not on $\alpha_s$ as in phenomenological and perturbative QCD. This is due to phenomenon of ”dimensional transmutation” in field theories with spontaneous symmetry breaking [35,36] (see also discussion in Ref. [30]).}
It is clear that our values (2.1) approximately one order of magnitude bigger than those of
instantons can provide at all (2.2).

One of the main purposes in this paper is to generalize a formalism, developed earlier in
our paper [25], in order to directly calculate the gluon condensate as a function of the truly
nonperturbative VED, the topological susceptibility and the mass of the η′ meson in the
chiral limit as a functions of the truly nonperturbative YM VED. It is based on using the
trace anomaly (section 3) and low energy (section 4) relations, as well as the WV formula
for the mass of the η′ meson in the chiral limit (section 5). It was directly applied to the
ZMME quantum model of the QCD nonpertubative vacuum. In section 6 we present our
discussion and conclusions. Our numerical results are shown in Tables I-III.

III. THE TRACE ANOMALY RELATION

The truly nonperturbative VED is important in its own right as one of the main char-
acteristics of the QCD nonperturbative vacuum. Furthermore it assists in estimating such
an important phenomenological parameter as the gluon condensate, introduced in the QCD
sum rules approach to resonance physics [37]. The famous trace anomaly relation [16] in the
general case (nonzero current quark masses $m_f^0$) is

$$\Theta_{\mu\mu} = \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^a_{\mu\nu} + \sum_f m_f^0 \bar{q}_f q_f.$$  (3.1)

where $\Theta_{\mu\mu}$ is the trace of the energy-momentum tensor and $G^a_{\mu\nu}$ being the gluon field strength
tensor while $\alpha_s = g^2/4\pi$. Sandwiching Eq. (3.1) between vacuum states and on account of
the obvious relation $\langle 0 | \Theta_{\mu\mu} | 0 \rangle = 4\epsilon_t$, one obtains

$$4\epsilon_t = \langle 0 | \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^a_{\mu\nu} | 0 \rangle + \sum_f m_f^0 \langle 0 | \bar{q}_f q_f | 0 \rangle,$$  (3.2)

where $\langle 0 | \bar{q}_f q_f | 0 \rangle$ is the quark condensate and $\epsilon_t$ is the sum of all possible independent, truly
nonperturbative contributions to the VED (the total VED). In quantum theory it consists of
two parts: the truly nonperturbative gluon part properly regularized with the help of ghosts
(if any), $\epsilon_g$, the so-called YM part at $N_f = 0$, and confining quark part with dynamically
generated quark masses, $\epsilon_q$, i.e.,

$$\epsilon_t = \epsilon_g + N_f \epsilon_q.$$  (3.3)

In general case it is impossible to use the weak coupling limit solution to the $\beta$ function,
so it is convenient to introduce the gluon condensate as follows:

$$\bar{G}_2 \equiv \langle \bar{G}^2 \rangle \equiv -\langle \frac{\beta(\alpha_s)}{4\alpha_s} G^2 \rangle \equiv -\langle 0 | \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^a_{\mu\nu} | 0 \rangle.$$  (3.4)

This is nothing else but the most general parametrization of the gluon correlation function,$\langle 0 | G^a_{\mu\nu} G^a_{\mu\nu} | 0 \rangle$ (for details, see sect. VI below) which comes from the trace anomaly ralation
(3.2) itself. If confinement happens then the $\beta$ function is always in the domain of attraction
(i.e., always negative) without IR stable fixed point [2]. Thus the gluon condensate,
parametrized as in Eq. (3.4), is always positive as it should be. The trace anomaly relation (3.2) then can be written as follows:

\[ 4 \epsilon_t = -\tilde{G}_2 + \sum_f m_f^0 \langle 0 | \bar{q}_f q_f | 0 \rangle. \]  

(3.5)

Since the gluon condensate \( \tilde{G}_2 \) is always positive and finite while quark condensate is always negative and also finite, then the truly nonperturbative total VED \( \epsilon_t \) is always negative and finite without having imaginary part since \( \tilde{G}_2 \) and quark condensate are real finite numbers as it has been emphasized above. As was underlined in the preceding section, the confining quark part of the total vacuum energy density \( \epsilon_q \) (which is always positive) as usual an order of magnitude less than the corresponding YM part \( \epsilon_g \) (which is always negative), so the total truly nonperturbative VED \( \epsilon_t \) is always negative in complete agreement with Eq. (3.5). On account of Eq. (3.3), it gives the gluon condensate (3.4) as a function of \( N_f \) if somebody knows, of course, how to calculate \( \epsilon_t \) and quark condensate from first principles (see below and our papers [14,26,27]).

In the chiral limit \( (m_f^0 = 0) \) things drastically simplify. From Eq. (3.5) one obtains, \( \tilde{G}_2^0 = -4 \epsilon_t^0 \), indeed, where superscript "0" means the chiral limit. Thus in this limit the truly nonperturbative total VED is nothing else but the gluon condensate apart from the overall numerical factor. In what follows we will saturate the total VED in this equation by our values (2.1), i.e., to put \( \epsilon_t^0 = \epsilon_{ZME} + \ldots \). The numerical results are shown in Table I.

### IV. THE TOPOLOGICAL SUSCEPTIBILITY

One of the main characteristics of the QCD nonperturbative vacuum is the topological density operator (topological susceptibility) in gluodynamics \( (N_f = 0) \) [3]

\[ \chi_t = \lim_{q \to 0} i \int d^4x e^{iqx} \frac{1}{N_c} \langle 0 | T \{ q(x) q(0) \} | 0 \rangle, \]  

(4.1)

where \( q(x) \) is the topological charge density, defined as \( q(x) = (\alpha_s/4\pi) F(x) \tilde{F}(x) = (\alpha_s/4\pi) F_{\mu\nu}^a(x) \tilde{F}_{\mu\nu}^a(x) \) and \( F_{\mu\nu}^a(x) = (1/2) \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a (x) \) is the dual gluon field strength tensor while \( N_c \) is the number of different colors. In the definition of the topological susceptibility (4.1) it is assumed that the corresponding regularization and subtraction of all types of the perturbative contributions have been already done in order Eq. (4.1) to stand for the renormalized, finite and the truly nonperturbative topological susceptibility (see Refs. [3,17,18,38]). The anomaly equation in the WV notations is

\[ \partial_\mu J_5^\mu = N_f (2/N_c) (\alpha_s/4\pi) F \tilde{F}. \]  

(4.2)

The topological susceptibility can be related to the nonperturbative gluon condensate via the low energy ”theorem” in gluodynamics proposed by NSVZ [17] (by using the dominance of self-dual fields hypothesis in the YM vacuum) as follows:

\[ \lim_{q \to 0} i \int d^4x e^{iqx} \langle 0 | T \left\{ \frac{\alpha_s}{8\pi} G \tilde{G}(x) \frac{\alpha_s}{8\pi} G \tilde{G}(0) \right\} | 0 \rangle = -\xi^2 \frac{\beta(\alpha_s)}{4\alpha_s} G^2. \]  

(4.3)
Quite recently it was discussed by HZ in Ref. [18] (see also references therein) who noticed that it is not precisely a Ward identity, but rather is a relation between the corresponding correlation functions, indeed. That is why in what follows we call Eq. (4.3) as low energy relation or NSVZ-HZ relation. Thus there exist two proposals how to fix the numerical value of the coefficient $\xi$. The value $\xi = 2/b$, where here and everywhere below (apart from section 6) $b = 11$, was suggested by NSVZ [17] who used the above-mentioned dominance of self-dual fields hypothesis in the YM vacuum. A second one, $\xi = 4/3b$, was advocated very recently by HZ using a one-loop connection between the conformal and axial anomalies in the theory with auxiliary heavy fermions [18] (and references therein). For completeness we will use both values for the $\xi$ parameter.

Let us note only that there exists an obvious relation between these two values, namely $\xi_{HZ} = (2/3)\xi_{NSVZ}$.

The anomaly equation in the NSVZ-HZ notations is

$$\partial_\mu J_5^\mu = N_f (\alpha_s/4\pi) G \tilde{G},$$

with $N_f = 3$. Thus in order to get the topological susceptibility in the WV form from the relation (4.3), it is necessary to make a replacement in its left hand side as follows: $G \tilde{G} \rightarrow (2/N_c)F \tilde{F}$ in accordance with the anomaly equations (4.2) and (4.4). Then the WV topological susceptibility (4.1) finally becomes

$$\chi_t = -\xi^2 \frac{\beta(\alpha_s)}{4\alpha_s} G^2 = -(2\xi)^2 \epsilon_{YM},$$

where the second equality comes from Eqs. (3.4) and (3.6) by denoting the truly nonperturbative VED at $N_f = 0$ as $\epsilon_{YM}$. The significance of this formula is that it gives the topological susceptibility as a function of the truly nonperturbative VED for pure gluodynamics, $\epsilon_{YM}$. For numerical results in the NSVZ mode see Table II. In order to obtain numerical results for the topological susceptibility in the HZ mode, it suffices in this Table to multiply numbers in $GeV$ units by factor $(2/3)^2$ and numbers in $MeV$ units by factor $\sqrt{2/3}$.

V. THE $U(1)$ PROBLEM

The topological susceptibility (4.1) assists in the resolution of the above-mentioned $U(1)$ problem [1,2] via the WV formula for the mass of the $\eta'$ meson (1.1). Within our notations it is expressed (in the chiral limit) as follows: $f_{\eta'}^2 m_{\eta'}^2 = 4N_f \chi_t$, where $f_{\eta'}$ is the $\eta'$ residue defined in general as $\langle 0 | \sum_{q=u,d,s} \bar{q} \gamma_\mu \gamma_5 q | \eta' \rangle = i \sqrt{N_f} f_{\eta'} p_\mu$ and $\langle 0 | N_f \frac{\alpha_s}{4\pi} F \tilde{F} | \eta' \rangle = (N_c \sqrt{N_f}/2) f_{\eta'} m_{\eta'}^2$ [3]. Using the normalization relation $f_{\eta'} = \sqrt{2F_0^{\eta'}}$, one finally obtains

$$f_{\eta'} = \sqrt{2F_0^{\eta'}},$$

In the weak coupling limit solution to the $\beta$ function (see, expression (6.2) below), the NSVZ-HZ low energy relation (4.3) for the NSVZ value of the $\xi$ parameter is reduced (as it should be) to that which was used in our previous paper [25] where the instanton vacuum in the chiral limit in various modifications was investigated. At the same time, the functional dependence of the topological susceptibility on the vacuum energy density is not, of course, changed (compare expression (3.5) of Ref. [25] with Eq. (4.5) below, on account of the corresponding values of the $\xi$ parameter).
\[ F_\pi^2 m_{\eta'}^2 = 2N_f \chi_t. \]  

(5.1)

Eq. (4.5) then implies

\[ m_{\eta'}^2 = -2N_f \left( \frac{2\xi}{F_\pi} \right)^2 \epsilon_{YM}, \]  

(5.2)

which expresses the mass of the \( \eta' \) meson as a function of the truly nonperturbative YM VED. In previous expressions we omit for simplicity the superscript "0" in the pion decay constant as well as in \( m_{\eta'}^2 \). Our numerical results for the mass of the \( \eta' \) meson in the chiral limit (5.2) are shown in Table III.

VI. DISCUSSION AND CONCLUSIONS

A. \( N_f \) dependence of the gluon condensate

Let us emphasize, that the general parametrization of the gluon condensate, introduced in Eq. (3.4), can be formally applied to any \( \beta \) function: to its weak coupling limit solution which we certainly know (see below) or to its strong coupling limit one which we certainly do not know yet. Also it is remains relevant whether one considers the chiral limit case or not. In phenomenology, however, the oftenly used parametrization of the gluon condensate is \[ G_2 \equiv \langle G^2 \rangle \equiv \langle \frac{\alpha_s}{\pi} G^2 \rangle \equiv \langle 0 | \frac{\alpha_s}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a | 0 \rangle, \]  

(6.1)

which in what follows we will call standard (or phenomenological) parametrization of the gluon condensate. Of course, the gluon condensate in principle can be parametrized by different ways. Here it makes sense to remind that by parametrization it is understood which numerical factor (\( \alpha_s/\pi \), \( \alpha_s \), etc.) is chosen to be associated with the gluon correlation function \( \langle 0 | G_{\mu\nu}^a G_{\mu\nu}^a | 0 \rangle \). By comparing these two parametrizations of the same gluon correlation function, it becomes clear that the general parametrization \( \bar{G}_2 \), given in Eq. (3.4), is useful in the nonperturbative calculations from first principles as it has been explained above. At the same time, it is useless in phenomenology since we do not know the \( \beta \) function there. However, in phenomenology in many important cases it is legitimated to use the weak coupling limit solution to the \( \beta \) function,

\[ \beta(\alpha_s) = -b \frac{\alpha_s^2}{2\pi} + 0(\alpha_s^3), \quad b = 11 - \frac{2}{3}N_f, \]  

(6.2)

for example in instanton calculus [23,25]. Then the general parametrization (3.4) is reduced to the phenomenological one, Eq. (6.1), as follows:

\[ \bar{G}_2 = \frac{b}{8} \times G_2 = (1.375 - 0.083N_f) \times G_2. \]  

(6.3)

This relation makes it possible to resolve the old-standing problem in QCD phenomenology. It is well known that the phenomenologically extracted value of the gluon condensate does
not explicitly depend on \( N_f \). It is usually nonexplicitly assumed that it is relevant for all \( N_f \) by making from heavy to light quarks [37]. Though in general it should be \( N_f \) explicitly dependent quantity as it follows from the trace anomaly relation (3.5) on account of Eq. (3.3). This is true in the weak coupling regime and in the chiral limit as well.

Our proposal is to consider (to interpret) \( \bar{G}_2 \) as a real phenomenological condensate which on account of any numerical value of the standard gluon condensate \( G_2 \), through the relation (6.3), becomes explicitly dependent on \( N_f \), indeed. In other words, our proposal allows one to restore the explicit \( N_f \) dependence of the phenomenologically determined gluon condensate. The relation (6.3) in fact prescribes which numerical factor, explicitly depending on \( N_f \), should be multiplied on \( G_2 \) in order to fix the correct \( N_f \) dependence of the gluon condensate in phenomenology. This interpretation is in agreement with the trace anomaly relation, of course.

Another problem is the numerical value of the standard gluon condensate \( G_2 \) itself. It can be taken either from phenomenology or lattice simulations (see below). The phenomenological analysis of QCD sum rules for the standard gluon condensate implies, \( G_2 \simeq 0.012 \, \text{GeV}^4 \), which can be changed within a factor of two [37]. However, it has been already pointed out [39] that QCD sum rules substantially underestimate the value of the standard gluon condensate. The most recent phenomenological calculation of the standard gluon condensate is given by Narison in Ref. [40], where a brief review of many previous calculations is also presented. His analysis leads to the update average value as \( G_2 = (0.0226 \pm 0.0029) \, \text{GeV}^4 \). In Ref. [41] from the families of \( J/\Psi \) and \( \Upsilon \) mesons substantially larger values were recently derived, namely \( 0.04 \leq G_2 \leq 0.105 \, (\text{GeV}^4) \). Comparable with these bounds, the values for the standard gluon condensate were reported in recent lattice simulations [42].

Obviously that the standard gluon condensate in the chiral limit cannot be equal to its any phenomenological (empirical) value. Apparently it should be less, i.e., \( G^0_2 \simeq \nu \times G_2 \), where \( \nu < 1 \) is some real number. Then the relation (6.3) in the chiral limit becomes \( \bar{G}^0_2 = (b/8) \times G^0_2 \simeq (\nu b/8) \times G_2 = (1.375 - 0.083 N_f) \times \nu \times G_2 \). In Ref. [17] it has been argued indeed that the gluon condensate in the chiral limit is approximately two times less than its any phenomenological (empirical) value, \( G^0_2 \simeq 0.5 G_2 \), i.e., \( \nu \simeq 0.5 \). Then the previous relation becomes \( \bar{G}^0_2 \simeq (b/16) \times G_2 = (0.6875 - 0.0415 N_f) \times G_2 \), whatever numerical value of \( G_2 \) is.

Our values for the gluon condensate (3.4) in the chiral limit (Table I) substantially larger than it is possible to estimate from phenomenology at all with the help of the above-derived relations. This difference is not only due to different physical observables as was noticed in Ref. [23]. It seems to us that it reflects rather different underlying physics as well. Our gluon condensate (which was calculated from first principles) is the strong coupling regime result and reflects the nontrivial topology of the true QCD vacuum where quantum excitations of the gluon fields play an important role. Precisely these types of gluon field configurations are mainly responsible for quark confinement and dynamical breakdown of chiral symmetry [14,26,27]. At the same time, phenomenologically extracted gluon condensate being the weak coupling limit result can be associated with classical instanton-type fluctuations in the QCD vacuum which by themselves do not confine quarks [32,43,44].

Concluding this subsection, let us note that the confining quark condensate contribution to the trace anomaly relation (3.5) vanishes in the chiral limit. However, due to all reasonable estimates of light quark masses, numerically its contribution is at 20% and thus comparable...
with the systematic error in any phenomenological determination of the standard gluon condensate itself [37,45].

B. Some general remarks on the trace anomaly relation

It is widely believed that the truly nonperturbative VED is determined by the trace anomaly relation (3.2) or equivalently (3.5). This may be so in phenomenology but not, of course, in nonperturbative QCD, where the situation is completely opposite (see discussion below). If by some phenomenological analysis it is possible to extract numerical values of the gluon and quark condensates then substituting them into the right hand side of the trace anomaly relation (3.5), one is able to estimate the total VED, indeed. However, this number such obtained tells a little (or even nothing) about detail structure of the QCD vacuum. It is not surprising since the gluon and quark condensates being its average (global) characteristics cannot account for its microscopic, detail structure.

The quantity which is responsible for its detail structure is precisely the total truly nonperturbative VED which, by definition, is the sum of all possible, independent contributions. In this way it reflects the complexity and variety of the quantum, dynamical degrees of freedom (quarks and gluons) in the QCD vacuum. It may contain contributions even from classical field configurations (instantons, for example). Precisely the detail structure of the QCD vacuum determines its global characteristics, and not vice versa. We have already explained in section 2 how the truly nonperturbative VED should be calculated from first principles in nonperturbative QCD. It is well known how to calculate from first principles the quark condensate as integrated out of the trace of the truly nonperturbative quark propagator over the deep IR region. Then the trace anomaly relation (3.5) becomes simply an algebraic equation for calculating the gluon condensate and in the chiral limit it is nothing else but the truly nonperturbative VED itself.

The contribution to the right hand side of the trace anomaly relation (3.5) from the confining quark condensate vanishes in the chiral limit. However, it does not mean that the contribution from confining quarks with dynamically generated quark masses also vanishes in the left hand side of the trace anomaly relation. Even in the chiral limit the vacuum quark loops provide nonzero contribution into the total VED. Thus this quantity contains much more information about QCD vacuum than its global characteristics can provide at all. Reflecting its detail structure the total VED is the main characteristics of the QCD nonperturbative vacuum and thereby is responsible for its global characteristics as well.

C. Summary

Using the trace anomaly relation (3.2), The NSVZ-HZ low energy relation (4.3) and WV formula for the mass of the $\eta'$ meson (5.2) in the chiral limit, we have developed a well-justified formalism in the most general form in order to express the topological susceptibility, the gluon condensate and the mass of the $\eta'$ meson in the chiral limit as a functions of the truly nonperturbative VED. The crucial role in this belongs to the NSVZ-HZ low-energy relation (4.3). It allows one to relate the important topological quantities such as the gluon...
condensate, the truly nonperturbative VED, the topological susceptibility, etc. to each other in a self-consistent way by providing well-justified coefficients between them.

This formalism has been immediately applied to the investigation of the chiral QCD vacuum structure within its quantum ZMME model. Our numbers are collected in Tables I-III. Our values for the topological susceptibility in the NSVZ mode (Table II) slightly overestimate its phenomenological value while in the HZ mode underestimate it since recalling $\xi_{HZ} = (2/3)\xi_{NSVZ}$ (see also text at the very end of section 4). The same situation takes place with the mass of the $\eta'$ meson in the chiral limit (Table III). However, it is worth emphasizing that the precise validity neither of the WV formula (1.1) nor the NSVZ-HZ low energy relation (4.3) is not known. That is why we cannot address the question of systematic error bars one might assign to the final numerical values summarized in Tables II and III. The topological susceptibility to leading order in the large $N_c$ limit is of order $N_c^0$. The next-to-leading correction of order $N_c^{-1}$, however might be not very small at $N_c = 3$ which was used in our calculations.

Any way, the ZMME values of the truly nonperturbative VED (2.1) are of the necessary order of magnitude to account for the phenomenological value of the topological susceptibility and therefore to saturate the large mass of the $\eta'$ meson. In other words, the IR singularities likely to be presented in the QCD nonperturbative vacuum and summarized by the corresponding behavior of the full gluon propagator in the deep IR domain are mainly responsible for the mass of the $\eta'$ meson and consequently for the phenomenological value of the topological susceptibility. Precisely this type of gluon field configurations is closely related to quark confinement as well [14,26,27]. Also our proposal how to restore the correct $N_f$ dependence of the phenomenologically extracted value of the standard gluon condensate seems to be important in general and for instanton calculus [23] in particular. It is well known that the gluon condensate due to instantons does not explicitly depend on $N_f$ [23,25,37].

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### TABLE I. ZMME model values for the gluon condensate.

| $F_\pi^0$ (MeV) | $\bar{G}_2^0$ (GeV$^4$) | $N_f = 0$ | $N_f = 1$ | $N_f = 2$ | $N_f = 3$ |
|----------------|-------------------------|-----------|-----------|-----------|-----------|
| 93.3           | 0.05652                 | 0.04936   | 0.04084   | 0.0330    |
| 83.2           | 0.0360                 | 0.03104   | 0.02608   | 0.02112   |

### TABLE II. ZMME model values for the topological susceptibility.

| $F_\pi^0$ (MeV) | $\chi_t$ (GeV$^4$) | $\chi_t^{1/4}$ (MeV) |
|----------------|---------------------|----------------------|
| 93.3           | 0.00187             | 207.9                |
| 83.2           | 0.00119             | 185.7                |

### TABLE III. ZMME model values for the mass of the $\eta'$ meson in the chiral limit (MeV units).

| $F_\pi^0$ | $m_{\eta'}$ (NSVZ) | $m_{\eta'}$ (HZ) |
|-----------|---------------------|------------------|
| 93.3      | 1135                | 756.86           |
| 83.2      | 1015                | 677              |