On the vanishing of some mock theta functions at odd roots of unity

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Abstract

We consider the problem of whether or not certain mock theta functions vanish at the roots of unity with an odd order. We prove for any such function $f(q)$ that there exists a constant $C > 0$ such that for any odd integer $n > C$ the function $f(q)$ does not vanish at the primitive $n$-th roots of unity. This leads us to conjecture that $f(q)$ does not vanish at the primitive $n$-th roots of unity for any odd positive integer $n$.

Keywords: Vanishing sums of roots of unity, Mock theta functions, Q-series

Mathematics Subject Classification: 33D15, 05A10

1 Introduction

Throughout $q$ is a complex number with $|q| < 1$, $n$ is a nonnegative integer, and $p$ is an odd prime number. A complex number $\alpha$ is an $n$-th root of unity if $\alpha^n = 1$ and it is called a primitive $n$-th root of unity if $n$ is the smallest nonnegative integer such that $\alpha^n = 1$. If the order of $\alpha$ is odd we will sometimes say that $\alpha$ is an odd root of unity. The asymptotic behaviour at the roots of unity is an important feature in the theory of mock theta functions. Indeed, for a function $f(q)$ defined by $q$-series to be a mock theta function one of Ramanujan’s conditions states:

\((\ast)\) for every root of unity $\zeta$, there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q) = O(1)$ as $q \to \zeta$ radially.

The first examples of mock theta functions were given by Ramanujan [13] and there is nowadays an intensive literature dealing with these functions, relations among them, identities relating them to other $q$-series, and their asymptotic behaviour at root of unity according to the condition $(\ast)$. For an account of the mock theta functions see for example [1,2,15,16]. Folsom–Ono–Rhoades [5] went further with the condition $(\ast)$ as they also gave explicit formulas for $O(1)$ as a linear combination of the root of unity and its powers.

For other references focusing more on the asymptotic behaviour of mock theta functions at the roots of unity including explicit formulas for $O(1)$ in terms of these roots, we refer to [3] and [6].

Our current motivation is of a quite different flavour as we will consider the problem of whether or not certain mock theta functions have zeros at the odd roots of unity. By way of example, let $\phi(q)$ and $\sigma(q)$ be the sixth order mock theta functions of Ramanujan [13]
given by
\[
\phi(q) = \sum_{k \geq 0} \frac{(-1)^k q^{k^2} (q; q^2)_k}{(-q; q^2)_{2k}} \quad \text{and} \quad \sigma(q) = \sum_{k \geq 0} \frac{q^{k^2+1} (-q; q^2)_k}{(q; q^2)_{k+1}}
\]
where we have used the standard notation from the theory of basic hypergeometric series [7]:
\[
(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).
\]

For convenience we let
\[
(a_1, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n \quad \text{and} \quad (a_1, \ldots, a_k; q)_\infty = \prod_{j=1}^{k} (a_j; q)_\infty.
\]

Rather than \(\sigma(q)\) we will consider \(\sigma(-q)\) since the latter function is defined and it terminates at the odd roots of unity. Ramanujan [13] stated the following formula which was first proved by Andrews–Hickerson [2, (0.19)]
\[
\phi(q^2) + 2\sigma(-q) = (q; q^2)_\infty^2 (q^3; q^6)_\infty (q^6; q^6)_\infty.
\]

Then it is clear from the right-hand side of this identity that the function \(\phi(q^2) + 2\sigma(-q)\) vanishes at the \(n\)-th roots of unity for any odd \(n\). So, it is natural to ask the question whether or not the individual terms \(\phi(q)\) and \(\sigma(-q)\) vanish at these \(n\)-th roots of unity. Besides, a very important remark is that our functions when evaluated at an odd primitive root of unity \(\zeta\) become finite sums of the form \(f(\zeta) = \sum c_i \zeta^i\) with rational coefficients \(c_i\). This is in connection with the theme of vanishing sums of roots of unity as presented in the paper by Conway–Jones [4].

Conway–Jones [4] considered certain equations involving trigonometric functions with rational arguments which they called trigonometric Diophantine equations. The authors reduced solving these equations to the problem of finding all linear combinations with rational coefficients among the roots of unity and they were able to classify all such linear combinations of less than ten terms. In particular, they considered linear relations of the form
\[
\sum_{i=0}^{k} a_i \zeta^{n_i} = 0 \quad \text{where} \quad a_i \text{ are rational numbers and} \quad \zeta \text{ is some root of unity of order} \quad Q \text{ and showed that for every} \quad C > 1 \text{ one has}
\]
\[
\log Q \leq C \sqrt{k \log k} + O(1).
\]

As was indicated in [4], vanishing sums of roots of unity have been also looked at earlier by Rédei [14]. Lenstra [10] studied the asymptotic behaviour of the coefficients occurring in linear combinations with integer coefficients of the roots of unity. For instance, the author proved that if \(\sum_{i=1}^{k} a_i \zeta_i = 0\) where \(\zeta_i\) are roots of unity and \(a_i\) are integers such that \(\gcd(a_1, \ldots, a_k) = 1\) and if no proper subset of \(\{\zeta_1, \ldots, \zeta_k\}\) is linearly dependent over \(\mathbb{Q}\), then
\[
|a_i| \leq 2^{1-k} k^{k/2}.
\]

Zannier [17] went further and allowed complex coefficients as he investigated linear relations \(\sum_{i=0}^{k} a_i \zeta^{n_i} = 0\) where \(a_i\) are complex numbers and \(\zeta\) is some root of unity of order \(Q\). The author as a main result obtained bounds for the order \(Q\). Lam-Leung [9]
for a given positive integer \( m \) classified the positive integers \( n \) for which there exist \( m \)-th roots of unity \( \zeta_1, \ldots, \zeta_n \) such that \( \sum_{i=1}^{n} \zeta_i = 0 \).

Our main objective in this work is to prove that for sufficiently large positive odd integer \( n \), the functions \( \phi(q) \) and \( \sigma(-q) \) along with a variety of other mock theta functions do not vanish at the \( n \)-th roots of unity. This leads us to conjecture that these functions do not vanish at the \( n \)-th roots of unity for any positive odd integer \( n \).

Furthermore, one can also consider for odd \( n \) sums

\[
\sum_{j} f(\alpha_j) : \alpha_j \text{ is an } n\text{-th root of unity}
\]

and ask the question for which odd positive integers \( n \) the sum (1) is equal to zero. It is of course tempting in the cases at hand to try to come up with closed formulas for \( \phi(\zeta) \) and \( \sigma(-\zeta) \) which yield answers to this question. However, letting \( \zeta \) be a \( p \)-th root of unity for the small prime values \( p = 3, 5, 7, 11 \), it is easy to verify that:

\[
\begin{align*}
\text{if } p = 3, & \quad \phi(\zeta) = -2\zeta \text{ and } \sigma(-\zeta) = \zeta^2, \\
\text{if } p = 5, & \quad \phi(\zeta) = \zeta^4 \text{ and } \sigma(-\zeta) = -\frac{\zeta^3}{2}, \\
\text{if } p = 7, & \quad \phi(\zeta) = 2 - \zeta \text{ and } \sigma(-\zeta) = -1 + \frac{\zeta^2}{2}, \\
\text{if } p = 11, & \quad \phi(\zeta) = \zeta - \zeta^2 - \zeta^4 + 3\zeta^8 \\
& \quad \text{and } \sigma(-\zeta) = \frac{-\zeta^2}{2} + \frac{\zeta^4}{2} - \frac{3\zeta^5}{2} + \frac{\zeta^8}{2},
\end{align*}
\]

which suggests that it would not be easy to find such closed formulas. In addition, observe from (2) that if \( p = 5 \) then the sum (1) vanishes for both \( f(q) = \phi(q) \) and \( f(q) = \sigma(-q) \) since

\[
\sum_{j} \phi(\alpha_j) : \alpha_j \text{ is a } 5\text{-th root of unity} = \phi(1) + \phi(\zeta) + \phi(\zeta^2) + \phi(\zeta^3) + \phi(\zeta^4) = 1 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta = 0
\]

and

\[
\sum_{j} \sigma(-\alpha_j) : \alpha_j \text{ is a } 5\text{-th root of unity} = \sigma(-1) + \sigma(-\zeta) + \sigma(-\zeta^2) + \sigma(-\zeta^3) + \sigma(-\zeta^4)
\]

\[
= \frac{1}{2} - \zeta^3 - \zeta - \frac{\zeta^4}{2} - \frac{\zeta^5}{2} = 0
\]

where \( \zeta \) is any primitive 5-th root of unity. However, for the cases \( p = 3, 7, 11 \) the sum (1) for these two functions does not vanish. We note that we do not know the answer to this question and it is not our purpose to discuss it in the present work.

In this paper we will pay attention to the vanishing problem at the odd roots of unity for the following sixth order mock theta functions of Ramanujan [13]

\[
\begin{align*}
\phi(q) &= \sum_{k \geq 0} \frac{(-1)^k q^{k^2}}{(q; q^2)_k}, \\
\psi(q) &= \sum_{k \geq 0} \frac{(-1)^k q^{(k+1)^2}}{(q; q^2)_{2k+1}}, \\
\lambda(q) &= \sum_{k \geq 0} \frac{(-1)^k q^{k^2}}{(q; q^2)_k}, \\
\mu(q) &= \sum_{k \geq 0} \frac{(-1)^k q^k}{(q; q)_k}, \\
\rho(q) &= \sum_{k \geq 0} \frac{q^{(k+1)^2}}{(q^2; q^2)_{k+1}}, \\
\sigma(q) &= \sum_{k \geq 0} \frac{q^{(k+2)^2}}{(q^2; q^2)_{k+1}},
\end{align*}
\]
the following eighth order mock theta functions of Gordon–McIntosh [8]

\[ S_0(q) = \sum_{k \geq 0} q^{k^2} (-q; q^2)_k, \quad S_1(q) = \sum_{k \geq 0} q^{(k+1)^2} (-q; q^2)_k, \]

\[ U_0(q) = \sum_{k \geq 0} q^{k^2} (q; q^2)_k, \quad U_1(q) = \sum_{k \geq 0} q^{(k+1)^2} (q; q^2)_k, \] (4)

the following fifth order mock theta functions of Ramanujan [12]

\[ \phi_0(q) = \sum_{k \geq 0} q^{k^2} (-q; q^2)_k \quad \text{and} \quad \phi_1(q) = \sum_{k \geq 0} q^{(k+1)^2} (-q; q^2)_k, \] (5)

and finally the following second-order mock theta function of Ramanujan [13]

\[ u(q) = \sum_{k \geq 0} \frac{(-1)^k q^{2k}}{(-q^2; q^2)_k}. \] (6)

We restricted ourselves to the mock theta functions in (3), (4), (5), and (6) because they satisfy the following properties required by our argument of proof. Firstly, either the series representing \( f(q) \) or \( f(-q) \) is both defined and terminating at the odd roots of unity. Observe that the terminating condition is guaranteed by the presence of \((q; q^2)_k\) and \((q; q)_k\) in the numerator of the \( k \)-th term of our series. So, for instance, our argument does not work for Ramanujan’s third order mock theta function

\[ f(q) = \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_{2n}} \]

as \( f(q) \) does not terminate at the odd roots of unity and \( f(-q) \) has singularities at these roots. A similar restriction applies for the fifth and the tenth order mock theta functions

\[ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q^2)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_n}. \]

Our third condition is that the function \( f(q) \) should be bounded at the odd roots of unity.

Related to this, we recall that Andrews–Hickerson in [2] [Theorem 5.0] to confirm the asymptotic condition (*), needed to establish the boundedness of the two functions \( \phi(q) \) and \( \psi(q) \) at the odd roots of unity and Gordon-McIntosh [8, p. 332] needed to prove the boundedness of the functions \( S_0(q) \) and \( S_1(q) \) at such roots. It turns out that \( U_0(q) \) is also bounded at the odd roots of unity by virtue of the following formula of Gordon–McIntosh [8]:

\[ U_0(q) = S_0(q^2) + qS_1(q^2). \] (7)

On the other hand, as we are not aware whether it is well-known that the functions \( \phi_0(-q), \phi_1(-q), \) and \( u(q) \) are bounded at the odd roots of unity, we record this as a lemma which we will prove later in Sect. 6.

**Lemma 1** The functions \( \phi_0(-q), \phi_1(-q), \) and \( u(q) \) are bounded at the primitive odd roots of unity.
The following notation is justified.

**Definition 1**  For any odd primitive root $\zeta$ let

$$
M_{1,\zeta} = \sup\{\phi(\zeta)\}, \quad M_{2,\zeta} = \sup\{\psi(\zeta)\},
$$

$$
M_{3,\zeta} = \sup\{S_0(\zeta)\}, \quad M_{4,\zeta} = \sup\{S_1(\zeta)\}, \quad M_{5,\zeta} = \sup\{U_0(\zeta)\},
$$

$$
M_{6,\zeta} = \sup\{\phi_0(\zeta)\}, \quad M_{7,\zeta} = \sup\{\phi_1(\zeta)\}, \quad M_{8,\zeta} = \sup\{u(\zeta)\}.
$$

Furthermore, let for $j = 1, \ldots, 8$

$$
M_j = \inf\{M_j, \zeta: \zeta \text{ is a primitive root of unity of odd order}\}.
$$

The rest of the paper is organized as follows. In Sect. 2 we collect our main theorems and their corollaries along with proofs for these corollaries. Sects. 3–8 are devoted to proofs of the main theorems and in Sect. 6 we give the proof of Lemma 1. Finally, in Sect. 9 we shall give some comments and state our conjectures and open problems.

### 2 Main results

In this section we collect our main theorems along with their corollaries.

**Theorem 1**  If $n$ is an odd positive integer such that $n > M_1^2$, then the functions $\phi(q)$, $\mu(q)$, and $\sigma(-q)$ do not vanish at the primitive $n$-th roots of unity.

**Corollary 1**  If $n$ is an odd positive integer such that $n > M_1^2$, then the functions $\phi'(q)$, $\mu'(q)$, and $\sigma'(-q)$ defined by

$$
\phi'(q) = \sum_{k \geq 0} q^k(q;q^2)_k
$$

$$
\mu'(q) = 1 + \sum_{k \geq 1} \frac{q^{(k^2-1)/2}(q;q^2)_k}{(-q;q)_k}
$$

$$
\sigma'(-q) = \sum_{k \geq 0} \frac{(-1)^k (q;q)_k}{(-q;q^2)_{k+1}}
$$

do not vanish at the primitive $n$-th roots of unity.

**Proof**  As

$$
(q^{-1};q^{-2})_k = (1 - \frac{1}{q})(1 - \frac{1}{q^2}) \cdots (1 - \frac{1}{q^{2k-1}})
$$

$$
= (-1)^k q^{-k^2} (q;q^2)_k
$$

and

$$
(-q^{-1};q^{-1})_{2k} = (1 + \frac{1}{q})(1 + \frac{1}{q^2}) \cdots (1 + \frac{1}{q^{2k}})
$$

$$
= q^{-k(2k+1)} (-q;q)_{2k}
$$
we get
\[
\frac{(-1)^k q^{-k^2} (q^{-1}; q^{-2})_k}{(-q^{-1}; q^{-1})_{2k}} = \frac{q^k (q; q^2)_k}{(-q; q)_{2k}}
\]
and therefore we deduce that \(\phi'(q) = \phi(q^{-1})\). So, assuming that \(\phi' (\zeta) = 0\) for a primitive \(n\)-th root of unity means that \(\phi (\zeta^{-1}) = 0\) which contradicts Theorem 1 since \(\zeta^{-1}\) is also a primitive \(n\)-th root of unity. The same argument applies to \(\mu'(q) = \mu(q^{-1})\) and \(\sigma'(-q) = \sigma(q^{-1})\).

**Theorem 2** If \(n\) is an odd positive integer such that \(n > 4M^2_2\), then the functions \(\psi(q), \lambda(q), \) and \(\rho(-q)\) do not vanish at the primitive \(n\)-th roots of unity.

**Corollary 2** If \(n\) is an odd positive integer such that \(n > 4M^2_2\), then the functions \(\psi'(q), \lambda'(q), \) and \(\rho'(-q)\) defined by
\[
\psi'(q) = \sum_{k \geq 0} \frac{q^k (q; q^2)_k}{(-q; q)_{2k+1}}
\]
\[
\lambda'(q) = \sum_{k \geq 0} \frac{(-1)^k q^{k^2} (q; q^2)_k}{(-q; q)_k}
\]
\[
\rho'(-q) = \sum_{k \geq 0} \frac{(-1)^k q^{k+1} (q; q)_k}{(-q; q^2)_{k+1}}
\]
do not vanish at the primitive \(n\)-th roots of unity.

**Proof** Proofs follow easily from Theorem 2 and the relations
\[
\psi'(q) = \psi(q^{-1}), \lambda'(q) = \lambda(q^{-1}), \text{ and } \rho'(-q) = \rho(q^{-1}).
\]

**Theorem 3** Let \(n\) be an odd positive integer.

(a) If \(n > M^2_2\), then the function \(S_0(-q)\) does not vanish at the primitive \(n\)-th roots of unity.

(b) If \(n > M^2_2\), then the function \(S_1(-q)\) does not vanish at the primitive \(n\)-th roots of unity.

(c) If \(n > M^2_2\), then the functions \(U_0(-q)\) and \(U_1(-q)\) do not vanish at the primitive \(n\)-th roots of unity.

**Corollary 3** Let \(n\) be an odd positive integer and let
\[
S_0(q) = \sum_{k \geq 0} \frac{(-1)^k q^{k(k+1)} (-q; q^2)_k}{(-q^2; q^2)_k}
\]
\[
S_1(q) = \sum_{k \geq 0} \frac{(-1)^k q^{k(k-1)} (-q; q^2)_k}{(-q^2; q^2)_k}
\]
\[
U_0(q) = \sum_{k \geq 0} \frac{(-1)^k q^{2k(k+1)} (-q; q^2)_k}{(-q^4; q^4)_k}
\]
\[
U_1(q) = \sum_{k \geq 0} \frac{(-1)^{k+1} q^{2k(k+1)+1} (-q; q^2)_k}{(-q^2; q^4)_k}
\].
(a) If \( n > M^2_5 \), then the function \( S'_0(-q) \) does not vanish at the primitive \( n \)-th roots of unity.

(b) If \( n > M^2_5 \), then the function \( S'_1(-q) \) does not vanish at the primitive \( n \)-th roots of unity.

(c) If \( n > M^2_5 \), then the functions \( U'_0(-q) \) and \( U'_1(-q) \) do not vanish at the primitive \( n \)-th roots of unity.

Proof Part (a) follows by Theorem 3(a) combined with the identity \( S'_0(q) = S_0(q^{-1}) \), part (b) by Theorem 3(b) combined with the identity \( S'_1(q) = S_1(q^{-1}) \), and part (c) is a consequence of Theorem 3(c) and the relations \( U'_0(q) = U_0(q^{-1}) \) and \( U'_1(q) = U_1(q^{-1}) \).

\[ \square \]

**Theorem 4** Let \( n \) be an odd positive integer.

(a) If \( n > M^2_5 \), then the function \( \phi_0(-q) \) does not vanish at the primitive \( n \)-th roots of unity.

(b) If \( n > M^2_5 \), then the function \( \phi_1(-q) \) does not vanish at the primitive \( n \)-th roots of unity.

**Theorem 5** Let \( n \) be an odd positive integer. If \( n > M^2_5 \), then the function \( u(q) \) does not vanish at the primitive \( n \)-th roots of unity.

3 Proof of Theorem 1

Let \( n \) be an odd positive integer. It is easily verified that \( \phi(q) \) does not vanish at the \( n \)-th roots of unity for \( n = 1, 3, 5 \). Now suppose that \( n > \sup(5, M^2_5) \) and let \( \zeta \) be a primitive \( n \)-th root of unity. Then

\[
\phi(\zeta) = \sum_{k=0}^{n-1} (-1)^k \frac{\zeta k^2(\zeta; \zeta^2)^k}{(-\zeta; \zeta^2)_{2k}} = 1 - \frac{\zeta(1-\zeta)}{(1+\zeta)(1+\zeta^3)} + \frac{\zeta^4(1-\zeta)(1-\zeta^3)}{(1+\zeta)(1+\zeta^3)(1+\zeta^3)(1+\zeta^4)} + \ldots + (-1)^{\frac{n-1}{2}} \frac{\zeta(\frac{n-1}{2})^2(\zeta; \zeta^2)^{\frac{n-1}{2}}}{(-\zeta; \zeta^2)_{n-1}}.
\]

Since for any primitive \( n \)-th root of unity \( \alpha \) we have the following basic relation

\[
(-\alpha; \alpha)_{n-1} = (1+\alpha)(1+\alpha^2)\ldots(1+\alpha^{n-1}) = 1,
\]

the identity (8) becomes

\[
\phi(\zeta) = 1 - \zeta(1-\zeta)(1+\zeta^3)\ldots(1+\zeta^{n-1}) + \zeta^4(1-\zeta)(1-\zeta^3)(1+\zeta^5)\ldots(1+\zeta^{n-1}) + \ldots + (-1)^{\frac{n-1}{2}} \zeta(\frac{n-1}{2})^2(\zeta; \zeta^2)^{\frac{n-1}{2}}.
\]

From

\[
\zeta(1-\zeta^{n-1}) = \zeta - 1, \ \zeta^3(1-\zeta^{n-3}) = \zeta^3 - 1, \ldots, \zeta^{n-2}(1-\zeta^2) = \zeta^{n-2} - 1
\]

we get

\[
(1-\zeta^2)(1-\zeta^4)\ldots(1-\zeta^{n-1}) = (-1)^{\frac{n-1}{2}} \zeta^{-\frac{n-1}{2}} - \zeta^{-\frac{n+1}{2}}(1-\zeta)(1-\zeta^3)\ldots(1-\zeta^{n-2}).
\]
This combined with the elementary fact
\((\zeta; \zeta)_{n-1} = (1 - \zeta)(1 - \zeta^2)\cdots(1 - \zeta^{n-1}) = n\)
yields
\[
(1 - \zeta)(1 - \zeta^2)\cdots(1 - \zeta^{n-2})^2 (-1)^{\frac{n-1}{2}} \zeta^{-\left(\frac{n-1}{2}\right)^2} = n.
\]

Then
\[
(-1)^{\frac{n-1}{2}} \zeta^{\left(\frac{n-1}{2}\right)^2} (\zeta; \zeta^2)_{\frac{n}{2}} = (-1)^{\frac{n-1}{2}} \zeta^{\left(\frac{n-1}{2}\right)^2} \sqrt{n}
\]
and thus
\[
|(-1)^{\frac{n-1}{2}} \zeta^{\left(\frac{n-1}{2}\right)^2} (\zeta; \zeta^2)_{\frac{n}{2}}| = \sqrt{n}.
\]
Using (10) and (12) we achieve
\[
|\phi(\zeta)| = \left| \sum_{k=0}^{\frac{n-3}{2}} (-1)^k \xi_k^2 (\zeta; \zeta^2)_k + (-1)^{\frac{n-1}{2}} \xi^{\left(\frac{n-1}{2}\right)^2} (\zeta; \zeta^2)_{\frac{n}{2}} \right|
\]
\[
\geq \left| \sum_{k=0}^{\frac{n-3}{2}} (-1)^k \xi_k^2 (\zeta; \zeta^2)_k \right| - |(-1)^{\frac{n-1}{2}} \xi^{\left(\frac{n-1}{2}\right)^2} (\zeta; \zeta^2)_{\frac{n}{2}}|
\]
\[
= \sqrt{n} \left[ \sum_{k=0}^{\frac{n-3}{2}} (-1)^k \xi_k^2 (\zeta; \zeta^2)_k \right]
\]
\[
\geq \sqrt{n} - M_1
\]
\[
> 0.
\]
Consequently, \(\phi(q)\) does not vanish at the primitive \(n\)-th roots of unity for any odd \(n > M_1^2\). As for \(\mu(q)\), we need the following identity of Ramanujan [13] which was first confirmed by Andrews-Hickerson [2, (0.20)g]
\[
2\phi(q^2) - 2\mu(q) = (-q; q^2)_\infty^2(-q^3; q^6)_\infty^2(q^6; q^6)_\infty.
\]
Replacing \(q\) by \(-q\) in the previous relation we get
\[
2\phi(q^2) - 2\mu(q) = (q; q^2)_\infty^2(q^3; q^6)_\infty^2(q^6; q^6)_\infty.
\]
Observe that the difference \(\phi(q^2) - \mu(q)\) vanishes at the \(n\)-th roots of unity since the right hand-side of (13) clearly does. Now suppose for a contradiction that \(\mu(\zeta) = 0\) for some primitive \(n\)-th roots of unity. Then so does \(\phi(\zeta^2)\) by the previous observation. But this contradicts the first statement of the theorem since \(\zeta^2\) is also a primitive \(n\)-th root of unity. Similarly, for the statement regarding \(\sigma(q)\) we appeal the following identity of Ramanujan [13] which was first proved by Andrews-Hickerson [2, (0.19)g]
\[
\phi(q^2) + 2\sigma(q) = (-q; q^2)_\infty^2(-q^3; q^6)_\infty^2(q^6; q^6)_\infty.
\]
Upon replacing \(q\) by \(-q\) we find
\[
\phi(q^2) + 2\sigma(-q) = (q; q^2)_\infty^2(q^3; q^6)_\infty^2(q^6; q^6)_\infty
\]
from which it follows that \( \phi(q^2) + 2\sigma(-q) \) vanishes at the \( n \)-th roots of unity. Thus assuming that \( \sigma(-\zeta) = 0 \) at some primitive \( n \)-th root of unity implies that \( \phi(\zeta^2) = 0 \) which is impossible by the first statement of the theorem.

### 4 Proof of Theorem 2

We proceed in the same way as in the proof of Theorem 1. Let \( n \) be an odd positive integer. It is easy to check that \( \psi(q) \) does not vanish at the \( n \)-th roots of unity for \( n = 1, 3, 5 \). Let \( n > \text{sup}(5, 4M_2) \) and let \( \zeta \) be a primitive \( n \)-th root of unity. We have with the help of (9) and (11)

\[
\psi(\zeta) = \sum_{k=0}^{n-1} (-1)^k \zeta^{(k+1)^2/2} (\zeta; \zeta^2)_k \frac{(-\zeta; \zeta)_{2k+1}}{(-\zeta; \zeta)_{2k+1}}
\]

\[
= \frac{1}{1+\zeta} - \frac{\zeta^4(1 - \zeta)}{(1 + \zeta)(1 + \zeta^2)(1 + \zeta^3)}
\]

\[
+ \frac{\zeta^9(1 - \zeta)(1 - \zeta^5)}{(1 + \zeta)(1 + \zeta^2)(1 + \zeta^3)(1 + \zeta^4)(1 + \zeta^5)}
\]

\[
+ \cdots + (-1)^{n-1} \frac{\zeta^{(n-1)^2/2} (\zeta; \zeta^2)_{n-1}}{(-\zeta; \zeta)_{n-1}(1 + 1)}
\]

\[
= (1 + \zeta^2) \cdots (1 + \zeta^{n-1}) - \zeta^4(1 - \zeta)(1 + \zeta^3)(1 + \zeta^5) \cdots (1 + \zeta^{n-1})
\]

\[
+ \cdots + (-1)^{n-1} \frac{\zeta^{(n-1)^2/2} (\zeta; \zeta^2)_{n-1}}{(-\zeta; \zeta)_{n-1}(1 + 1)}
\]

\[
= (1 + \zeta^2) \cdots (1 + \zeta^{n-1}) - \zeta^4(1 - \zeta)(1 + \zeta^4)(1 + \zeta^5) \cdots (1 + \zeta^{n-1})
\]

\[
+ \cdots + \sqrt{n} \frac{1}{2} (-1)^{n-1} \frac{\zeta^{(n+1)^2/2}}{n^2 - \frac{n}{2}}.
\]

Now combine (15) with (12) to obtain

\[
|\psi(\zeta)| = \sum_{k=0}^{n-1} (-1)^k \zeta^{(k+1)^2/2} (\zeta; \zeta^2)_k \frac{(-\zeta; \zeta)_{2k+1}}{(-\zeta; \zeta)_{2k+1}}
\]

\[
+ (-1)^{n-1} \frac{\zeta^{(n+1)^2/2} (\zeta; \zeta^2)_{n-1}}{(-\zeta; \zeta)_{n-1}(1 + 1)}
\]

\[
\geq \frac{\sqrt{n}}{2} - \sum_{k=0}^{n-1} (-1)^k \zeta^{(k+1)^2/2} (\zeta; \zeta^2)_k \frac{(-\zeta; \zeta)_{2k+1}}{(-\zeta; \zeta)_{2k+1}}
\]

\[
\geq \frac{\sqrt{n}}{2} - M_2
\]

\[
> 0.
\]

This shows that \( \psi(q) \) does not vanish at the primitive \( n \)-th roots of unity. Regarding the function \( \lambda(q) \) we first make an appeal to the following formula which was stated by Ramanujan [13] and proved by Andrews–Hickerson [2, (0.21)]

\[
2q^{-1}\psi(q^2) + \lambda(-q) = (-q; q^2)_\infty^2 (-q, -q^2, q^5; q^6)_\infty.
\]

Replacing \( q \) by \( -q \) we get

\[
-2q^{-1}\psi(q^2) + \lambda(q) = (q; q^2)_\infty^2 (q, q^5, q^6; q^6)_\infty
\]

(16)
from which we deduce that \(-2q^{-1}\psi(q^2) + \lambda(q)\) vanishes at the primitive roots of unity of odd order. So, assuming that \(\lambda(\xi) = 0\) for some primitive \(n\)-th root of unity \(\xi\) implies that \(\xi^{-1}\psi(\xi^2) = 0\). A contradiction since \(\xi^2\) is also a primitive \(n\)-th root of unity.

As for the function \(\rho(q)\), we combine similar ideas with the following identity of Ramanujan [13] which was confirmed in [2, (0.18)]

\[
q^{-1}\psi(q^2) + \rho(q) = (-q; q^2)_\infty^2(-q, -q^5; q^6)_\infty.
\] (17)

5 Proof of Theorem 3

Let \(n\) be an odd positive integer. As the result is evident for \(n = 1, 3, 5\) we assume that \(n \geq \sup\{5, M^2\}\). Let \(\xi\) be a primitive \(n\)-th root of unity. From

\[
S_0(-q) = \sum_{k \geq 0} \frac{(-1)^k q^{k^2}}{(-q^2; q^2)_k},
\]

we clearly see that the sum on the right hand-side terminates at \(\xi\). Since \(\xi^2\) is also a primitive \(n\)-th root of unity, we obtain with the help of (9)

\[
S_0(-\zeta) = \sum_{k = 0}^{n^2-1} \frac{(-1)^k \zeta^{k^2} (\zeta; \zeta^2)_k}{(-\zeta^2, \zeta^2)_k} + \frac{(-1)^{n-1} \zeta^{(n-1)^2} (\zeta; \zeta^2)_{n-1}}{(-\zeta^2, \zeta^2)_{n-1}}
\]

\[
= 1 - \zeta(1 - \zeta)(1 + \zeta^4)(1 + \zeta^6) \cdots (1 + \zeta^{2n-2})
\]

\[
+ \zeta(1 - \zeta)^2(1 + \zeta^3)(1 + \zeta^6) \cdots (1 + \zeta^{2n-2})
\]

\[
+ \cdots + (-1)^{n-1} \zeta^{(n-1)^2} (\zeta; \zeta^2)_{n-1} (1 + \zeta^{n-1})(1 + \zeta^{n+1}) \cdots (1 + \zeta^{2n-2})
\]

\[
+ \frac{(-1)^{n-1} \zeta^{(n-1)^2} (\zeta; \zeta^2)_{n-1}}{(-\zeta^2, \zeta^2)_{n-1}}.
\]

We now claim that the norm of the \((n - 1)/2\)-term of \(S_0(-\zeta)\) is

\[
\left| \frac{(-1)^{n^2-1} \zeta^{(n^2-1)^2} (\zeta; \zeta^2)_{n^2-1}}{(-\zeta^2, \zeta^2)_{n^2-1}} \right| = \sqrt{n}.
\]

From the formulas

\[
\zeta^2(1 + \zeta^{n-2}) = 1 + \zeta^2, \quad \zeta^4(1 + \zeta^{n-4}) = 1 + \zeta^4, \ldots, \zeta^{n-1}(1 + \zeta^2) = 1 + \zeta^{n-1}
\]

we deduce that

\[
(1 + \zeta)(1 + \zeta^3) \cdots (1 + \zeta^{n-2}) = \zeta^{-\frac{\zeta^{n-1}}{2}}(1 + \zeta^2)(1 + \zeta^4) \cdots (1 + \zeta^{n-1})
\]

which combined with (9) gives

\[
1 = \left((1 + \zeta^2)(1 + \zeta^4) \cdots (1 + \zeta^{n-1})\right)^2 \zeta^{-\frac{\zeta^{n-1}}{2}}
\]

or equivalently,

\[
(-\zeta^2, \zeta^2)_{n-1} = \zeta^{\frac{\zeta^{n-1}}{2}}.
\] (18)
Now use the foregoing formula and (12) to derive the claim. Then

\[
|S_0(-\xi)| = \frac{\sum_{k=0}^{n/2} (-1)^k \xi^k (\xi; \xi^2)_k}{(\xi^2; \xi^2)_k} + \frac{(-1)^{n-1} \xi^{(n+1)/2} (\xi; \xi^2)_{n-1}}{(-\xi^2; \xi^2)_{n-1}}
\]

\[
\geq \sqrt{n} - \sum_{k=0}^{n/2} (-1)^k \xi^k (\xi^2; \xi^2)_k
\]

\[
\geq \sqrt{n} - M_3
\]

\[
> 0
\]

which means that \(S_0(-\xi) \neq 0\) for any primitive \(n\)-th root of unity \(\xi\). This proves part (a). Proofs for parts (b) and (c) regarding the functions \(S_1(-q)\) and \(U_1(-q)\) are omitted as they follow similarly. As for \(U_1(-q)\), we need the following formula of Gordon–McIntosh [8, (1.9)] (with \(q\) replaced by \(-q\))

\[
U_0(-q) + 2U_1(-q) = (q; q^2)_\infty (q^2; q^4)_\infty (q^4; q^4)_\infty.
\]  (19)

Clearly the sum \(U_0(-q) + 2U_1(-q)\) vanishes at the \(n\)-th roots of unity. Assume for a contradiction that \(U_1(-\xi) = 0\) for some primitive \(n\)-th root of unity. Then obviously \(2U_1(-\xi) = 0\). Now combining with (19) gives \(U_0(-\xi) = 0\), which is absurd.

6 Proof of Lemma 1

The idea of proof for all the three functions is essentially the same as the proof of [2, Theorem 5.0] but we establish only the statement on the function \(u(q)\) as it requires more effort. Consider the following eighth order mock theta functions from [8]

\[
T_0(q) = \sum_{k \geq 0} q^{(k+1)(k+2)} (-q^2; q^2)_k / (-q; q^2)_{k+1}
\]

\[
T_1(q) = \sum_{k \geq 0} q^{k(k+1)} (-q^2; q^2)_k / (-q; q^2)_{k+1}
\]

We note that in [8, p. 332] it has been shown that \(T_0(q)\) and \(T_1(q)\) are both bounded at the even roots of unity. We claim that \(T_1(q)\) is bounded at the primitive odd roots of unity. Let \(\xi\) be a primitive \(N\)-th root of unity for an odd positive integer \(N\) and write \(q = r\xi\) for \(0 \leq r \leq 1\). As the claim is clear for \(r = 0\), we assume that \(0 < r \leq 1\). Let

\[
u_n(r) = \frac{q^{n(n+1)} (-q^2; q^2)_n}{(-q; q^2)_{n+1}}.
\]

Then it is easy to see that

\[
|\nu_n(r)| = q^{2Nn+N(N+1)} \left| (-q^{2n+2}; q^2)_N \right| |u_n(r)|.
\]  (20)
By an application of [2, Lemma 5.2] to the foregoing identity with \( R = r^{2n+2k} \) and \( R' = r^{2n+2N} \), we get

\[
|(-q^{2n+2}; q^2)_N| = \prod_{k=1}^{N} |1 + r^{2n+2k} \zeta^{2n+2k}|
\]

\[
\leq \prod_{k=1}^{N} r^{k-N} |1 + r^{2n+2N} \zeta^{2n+2k}|
\]

\[
= r^{N(1-N)/2} |1 + r^{2n+2N} \zeta^{2n+2k}|.
\]

Since \( \zeta \) is a primitive \( N \)-th root of unity and \( k \) runs from 1 to \( N \), we have that \( \zeta^{2n+2k} \) runs through the \( N \)-th roots of unity and therefore \( 1 + r^{2n+2N} \zeta^{2n+2k} \) runs through the solutions of the equation

\[
(x - 1)^N = r^{N(2n+2N)}.
\]

Then by Vieta’s formula, the product of these solutions is \((-1)^N (1 - r^{N(2n+2N)})\) and thus we obtain

\[
|(-q^{2n+2}; q^2)_N| \leq r^{N(1-N)/2} (1 - r^{N(2n+2N)}).
\]

In addition, it has been shown in [2, p. 95] that

\[
|(q^{2n+3}; q^2)_N| \geq r^{N(N-1)/2}.
\]

Then a combination of the triangular inequality and the previous inequality yields

\[
|(-q^{2n+3}; q^2)_N| \geq |(q^{2n+3}; q^2)_N| \geq r^{N(N-1)/2}.
\]

Now combining (20), (21), and (22) we find

\[
|u_{n+N}(r)| \leq q^{2Nn+2N} (1 - r^{N(2n+2N)}) |u_n(r)|,
\]

which by virtue of [2, Lemma 5.3] yields

\[
|u_{n+N}(r)| \leq \frac{N(2n + 2N)}{2n^2 + 2N + N(2n + 2N)} |u_n(r)|
\]

\[
\leq \frac{n + N}{2n} |u_n(r)|
\]

\[
\leq \frac{8}{11} |u_n(r)|,
\]

provided \( n \geq N \). Then by [2, Lemma 5.1], \( \sum_{n \geq 0} |u_n(r)| \) is bounded and thus \( \sum_{n \geq 0} u_n(r) \) is bounded too. This confirms the claim. Similarly one can prove that \( T_0(q) \) is bounded too at the odd primitive roots of unity. Moreover, as by Gordon–McIntosh [8]

\[
U_1(q) = T_0(q^2) + qT_1(q^2),
\]

we see that \( U_1(q) \) is also bounded at the odd roots of unity. Recalling that \( U_0(q) \) is bounded at these roots of unity and combining with the following relation of McIntosh [11]

\[
u(q) = U_0(q) - 2U_1(q),
\]

we deduce that \( u(q) \) is also bounded at the odd roots of unity. This completes the proof.
7 Proof of Theorem 4

We use the same approach as in Section 3. From the identity

\[
\phi_0(-\zeta) = \sum_{k=0}^{n-1} (-1)^k \zeta^k (\zeta; \zeta^2)_k
\]

\[
= 1 - \zeta (1 - \zeta) + \zeta^4 (1 - \zeta)(1 - \zeta^3) + \cdots + (-1)^{n-1} \zeta^{(n-1)/2} (\zeta; \zeta^2)^{n-1}/2
\]

and (12) we get

\[
|\phi_0(-\zeta)| = \left| \sum_{k=0}^{n-1} (-1)^k \zeta^k (\zeta; \zeta^2)_k + (-1)^{n-1} \zeta^{(n-1)/2} (\zeta; \zeta^2)^{n-1}/2 \right|
\]

\[
\geq \left| \sum_{k=0}^{n-3} (-1)^k \zeta^k (\zeta; \zeta^2)_k \right| - \left| (-1)^{n-1} \zeta^{(n-1)/2} (\zeta; \zeta^2)^{n-1}/2 \right|
\]

\[
= \sqrt{n} - \sum_{k=0}^{n-3} (-1)^k \zeta^k (\zeta; \zeta^2)_k
\]

\[
\geq \sqrt{n} - M_6
\]

> 0.

This shows that \(\phi_0(-q)\) does not vanish at the primitive \(n\)-th roots of unity for any odd \(n > M_6^2\). The proof for the statement on \(\phi_1(-q)\) is omitted as it follows similarly.

8 Proof of Theorem 5

Noticing the resemblance between \(u(q)\) and \(S_0(-q)\) and using the same steps as in Section 5, we get

\[
u_0(\zeta) = \sum_{k=0}^{n-3} \frac{(-1)^k \zeta^k (\zeta; \zeta^2)_k}{(-\zeta^2; \zeta^2)^k} + \frac{(-1)^{n-1} \zeta^{(n-1)/2} (\zeta; \zeta^2)^{n-1}/2}{{(-\zeta^2; \zeta^2)^{n-1}/2}}
\]

\[
= 1 - \zeta (1 - \zeta)\left(1 + \zeta^4(1 + \zeta^6)\cdots(1 + \zeta^{2n-2})\right)^2
\]

\[
+ \zeta^4 (1 - \zeta)(1 - \zeta^3)\left(1 + \zeta^6(1 + \zeta^8)\cdots(1 + \zeta^{2n-2})\right)^2
\]

\[
+ \cdots + (-1)^{n-1} \zeta^{(n-1)/2} (\zeta; \zeta^2)^{n-1}/2 \left(1 + \zeta^{n-1}(1 + \zeta^{n+1})\cdots(1 + \zeta^{2n-2})\right)^2
\]

\[
+ \frac{(-1)^{n-1} \zeta^{(n-1)/2} (\zeta; \zeta^2)^{n-1}/2}{{(-\zeta^2; \zeta^2)^{n-1}/2}}.
\]

Appealing to (12) and (18) we see that the norm of the \((n-1)/2\)-term of \(u(\zeta)\) is

\[
\left| \frac{(-1)^{n-1} \zeta^{(n-1)/2} (\zeta; \zeta^2)^{n-1}/2}{{(-\zeta^2; \zeta^2)^{n-1}/2}} \right| = \sqrt{n}.
\]
This yields

\[ |u(\zeta)| = \left| \sum_{k=0}^{n-3} (-1)^k \xi^{k^2} (\zeta; \xi^2)_k \right| + \left| \sum_{k=0}^{n-1} (-1)^{k+1} \xi^{k^2} (\zeta; \xi^2)_k \right| \]

\[ \geq \sqrt{n} - \sum_{k=0}^{n-3} (-1)^k \xi^{k^2} (\zeta; \xi^2)_k \]

\[ \geq \sqrt{n} - M_8 > 0. \]

This completes the proof.

9 Concluding remarks

In this section we let \( f(q) \) denote any one of the functions in the statements of Theorems 1–5, along with the statements of the listed corollaries. Based on our theorems and their corollaries stating that for sufficiently large odd \( n \), the function \( f(q) \) does not vanish at the \( n \)-th roots of unity, we are led to the following conjecture.

**Conjecture 1** For any odd positive integer and any primitive \( n \)-th root of unity \( \zeta \) we have \( f(\zeta) \neq 0 \).

As we observed earlier in the comments just after (2), the sum (1) for \( f(q) = \phi(q) \) and \( f(q) = \sigma(-q) \) vanishes at \( p = 3 \) but not at \( p = 3, 7, 11 \). We have the following open problems.

**Open 1** Find prime numbers for which the sum (1) is zero.

**Open 2** Is the number of primes for which the sum (1) vanishes finite or infinite?

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