The Self-Dual String Soliton

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ABSTRACT

We obtain a BPS soliton of the M theory fivebrane’s equations of motion representing a supersymmetric self-dual string. The resulting solution is then dimensionally reduced and used to obtain 0-brane and \((p - 2)\)-brane solitons on D-\(p\)-branes.
1. Introduction

In recent years it has become clear that the still rather mysterious M theory governs many aspects of the lower dimensional string theories. What little is known of M theory is powerful enough to lead us to new phenomena in string theory and indeed new string theories. In particular the M theory fivebrane is strongly believed hold a new kind of self-dual string theory on its worldvolume [1,2]. This new and somewhat elusive theory has also appeared in other contexts such as the compactification of type IIB string theory on $K^3$ [3], M(atrix) theory on $T^5$ [4] and the S-duality of $N = 4$, $D = 4$ super-Yang-Mills [3]. Thus one may hope that a greater understanding of this self-dual string may lead directly to a greater understanding of duality, string theory and M theory.

In this paper we shall seek a supersymmetric string soliton solution to the M theory fivebrane’s equations of motion [5,6]. We will use the six-dimensional covariant field equations of motion derived in [6]. Alternative formulations of the fivebrane were given in [7]. In [8] a smooth string soliton solution of the field equations for an interacting second rank tensor gauge field was found. However, it is known [9] that the fivebrane equations of motion of [6] reduce to those of [8] when only the second rank tensor gauge field is active and as a result the solution of [8] can be lifted to be a solution of the full fivebrane equations of motion. However the solution so obtained takes all the scalar fields to be constant and it cannot preserve any supersymmetries since there is nothing to cancel off the tensor field force (this will also be apparent from the supersymmetry transformation given below). Instead it represents a non-singular field configuration around a string, analogous to the Born-Infeld expression for the electric field of a charged point particle [8]. In the next section we recall the fivebrane equations of motion and derive the supersymmetry transformation of the fermion field. From this result we derive the fivebrane analogue of the Bogomol’nyi conditions which preserve half the supersymmetry. In section three we use this Bogomol’nyi condition to solve the field equations and find the solution corresponding to the self-dual string. In
section four we dimensionally reduce the string soliton to obtain solitons of the type II D-p-brane worldvolume theories. In the final section we conclude with some comments on our solutions.

2. The Fivebrane and Its Worldvolume Supersymmetry

We use the fivebrane equations and conventions of [6]. In this paper the fivebrane is embedded in flat eleven-dimensional Minkowski superspace. We must distinguish between world and tangent indices, fermionic and bosonic indices and indices associated with the target space $\underline{M}$ and the fivebrane worldvolume $M$. On the fivebrane worldvolume the bosonic tangent space indices are denoted by $a, b, ... = 0, 1, 2, ..., 5$ and bosonic world indices by $m, n, ... = 0, 1, 2, ..., 5$. For example, the inverse vielbein of the bosonic sector of the fivebrane worldvolume is denoted by $E^m_a$. The bosonic indices of the tangent space of the target space $\underline{M}$ are denoted by the same symbols, but underlined, i.e. the inverse vielbein in the bosonic sector is given by $\underline{E}^{\underline{m}}_{\underline{a}}$. The fermionic indices follow the same pattern, those in the tangent space are denoted by $\alpha$ and $\underline{\alpha}$ for worldvolume $M$ and target space $\underline{M}$ respectively, while the world spinor indices are denoted by $\mu$ and $\underline{\mu}$.

The fivebrane sweeps out a superspace $M$ in the target superspace $\underline{M}$ which is specified in local coordinates $Z^M = (X^m, \Theta^\mu)$, $m = 0, 1, ..., 10, \mu = 1, ..., 32$. These coordinates are functions of the worldvolume superspace parameterised by $z^M = (x^m, \theta^\mu)$, $m = 0, 1, ..., 5; \mu = 1, ..., 16$. The $\theta^\mu$ expansion of the $z^M$ contains $x^m$ dependent fields of which the only independent ones are their $\theta^\mu = 0$ components, also denoted $X^m$ and $\Theta^\mu$, and a self-dual tensor $h_{abc}$ which occurs as at level $\theta^\mu$ in $\Theta^\mu$. Despite the redundancy of notation it will be clear from the context when we are discussing the component fields and the superfields.

The bosonic target space indices of $\underline{M}$ may be decomposed as those that lie in the fivebrane worldvolume and those that lie in the space transverse to the fivebrane; we denote these indices by $a$ and $a'$ respectively (i.e. $a = (a, a'), a = 0, 1, ..., 5; a' = 1', ..., 5'$) with a similar convention for world indices. The initially
thirty-two component spinor indices $\alpha$ are split into a pair of sixteen component spinor indices (i.e. $\alpha = (\alpha, \alpha'), \alpha = 1, \ldots, 16; \alpha' = 1', \ldots, 16'$) corresponding to the breaking of half of the supersymmetries by the fivebrane.

We will use the super-reparameterisations of the worldvolume to choose the so called static gauge. In this gauge we identify the bosonic coordinates in the worldvolume with the bosonic coordinates on the worldvolume (i.e. $X^a = x^n, n = 0, 1, \ldots, 5$) and set the fermionic fields $\Theta^\alpha = 0, \alpha = 1, \ldots, 16$. For a flat background $\Theta^\mu = \Theta^{\alpha} \delta^\mu_\alpha$. Thus the component field content of the fivebrane is $X^{a'} (a' = 1', \ldots, 5'), \Theta^{\alpha'} (\alpha' = 1', \ldots, 16')$ and the self-dual field strength $h_{abc}$. These fields contribute $5$, $8$ and $3$ on-shell degrees of freedom respectively and belong to the tensor multiplet of six-dimensional $(2, 0)$ supersymmetry. We will discuss below the decomposition of the eleven-dimensional Lorentz and spin groups with respect to the fivebrane and its consequences for the spinor index notation.

We are interested in solutions to the field equations in a flat background and so are required to solve the equations when $\Theta^{a'} = 0$. In this limit the bosonic equations of the fivebrane are given by [6]

$$G^{mn} \nabla_m X^{a'} = 0,$$
$$G^{mn} \nabla_m H_{npq} = 0.$$  \hfill (2.1)

The field tensor $H_{npq}$ is related to $h_{abc}$ by $H_{mnp} = E_m^a E_n^b E_p^c m^d_m m^e_n h_{abc}$ where $m^b_a \equiv \delta^b_a - 2h_{acd}h^{bcd}$. Although $H_{npq}$ is not self-dual, it does obey the usual Bianchi identity and, as a consequence, is the curl of the two form gauge field $B_{mn}$ (i.e. $H_{npq} = \partial_{[n} B_{pq]}$). This is in contrast to $h_{abc}$, which is self-dual but which is not in general the curl of a gauge field. As such, we must in addition to solving (2.1) ensure that $H_{mnp}$ is closed and that the corresponding $h_{abc}$ is self-dual. The worldvolume vielbein $E_m^a$ is specified by the relation $E_m^a = e^b_m (m^{-1})^a_b$, where the vielbein $e^a_m$ is that more usually associated with branes and is defined by

$$e^a_m \eta_{ab} e^b_n = g_{mn} = \partial_n Z^N \partial_m Z^M E^a_M E^b_N \eta_{ab}.$$  \hfill (2.2)

In equation (2.1) $G_{mn} = E_m^a E_n^b \eta_{ab}$ and the covariant derivative $\nabla_m$ is constructed
from the Levi-Civita connection corresponding to the metric $g_{mn}$.

In addition to solving the fivebrane field equations we wish to find a solution which preserves half the supersymmetry. As a result, we are required to find the supersymmetry transformation of $\Theta^{\alpha'}$ and then the fivebrane analogue of the Bogomol’nyi equation. To this end, we recall some of the salient points of the super-embedding formalism. The frame vector fields on the target manifold $M$ and the fivebrane worldvolume submanifold $M$ are given by $E_A = E_A^M \partial_M$ and $E_A = E_A^M \partial_M$ respectively. The coefficients $E_A^A$ encode the relationship between the vector fields $E_A$ and $E_A$, i.e. $E_A = E_A^A E_A$. Applying this relationship to the coordinate $z^M$ we find the equation

$$E_A^A = E_A^N \partial_N Z^M E_A^A.$$  

(2.3)

It can be shown that the vector fields on the fivebrane can be chosen such that

$$E_{\alpha'}^\beta = u_{\alpha'}^\beta + h_{\gamma'}^{\beta} u_{\gamma'}^\alpha, \quad E_{\alpha'}^\beta = u_{\alpha'}^\beta.$$  

(2.4)

and

$$E_a^a = m_a^b u_b^a, \quad E_a^a = u_a^a.$$  

(2.5)

In equation (2.4) the tensor $h_{\alpha'}^{\beta'}$ is related to the three-form $h_{abc}$ [6] by

$$h_{\alpha'}^{\beta'} \rightarrow h_{\alpha \beta}^j = \frac{1}{6} \delta^j_i (\gamma^{abc})_{\alpha \beta} h_{abc}.$$  

(2.6)

The matrix $u_a^b \equiv (u^b_\alpha, u^b_{\alpha'})$ is an element of $SO(1,10)$ and the matrix $u_{\alpha'}^\beta \equiv (u_{\alpha'}^\beta, u_{\alpha'}^{\beta'})$ forms an element of $Spin(1,10)$. As is clear from the notation, the indices with an overbar take the same range as those with an underline. We recall that the connection between the Lorentz and spin groups is given by

$$u_{\alpha'}^\gamma u_{\beta'}^\delta (\Gamma_a^g)_{\alpha \beta} = (\Gamma_{\alpha'}^{\delta} y_{\alpha'}^\gamma u_{\beta'}^\gamma.$$  

(2.7)

In the presence of a flat superspace target space the super-reparameterisation invariance reduces to translations and rigid supersymmetry transformations. The
latter take the form

\[ \delta x^\mu = \frac{i}{2} \Theta \Gamma^\mu \epsilon, \quad \delta \Theta^\mu = \epsilon^\mu. \] (2.8)

Unlike other formulations, the super-embedding approach of [5,6] is invariant under super-reparameterisations of the worldvolume, that is, invariant under

\[ \delta z^M = -v^M, \] (2.9)

where \( v^M \) is a supervector field on the fivebrane worldvolume. The corresponding motion induced on the target space \( M \) is given by

\[ \delta Z^B = v^A E^B_A, \] (2.10)

where \( v^M = v^A E^M_A \) and rather than use the embedding coordinates \( Z^N \) we referred them to the background tangent space, i.e. \( Z^B \equiv Z^M E^B_M \). We are interested in supersymmetry transformations and so consider \( v^a = 0, \ v^\alpha \neq 0 \); with this choice and including the rigid supersymmetry transformation of the target space of equation (2.8) the transformation of \( \Theta^\alpha \) is given by [6]

\[ \delta \Theta^\alpha = v^\beta E^\alpha_\beta + \epsilon^\alpha. \] (2.11)

The local supersymmetry transformations \( v^\alpha \) are used to set \( \Theta^\alpha = 0 \) which is part of the static gauge choice. However, by combining these transformations with those of the rigid supersymmetry of the target space \( \epsilon^\alpha \) we find a residual rigid worldvolume supersymmetry which is determined by the requirement that the gauge choice \( \Theta^\alpha = 0 \) is preserved. Consequently, we require \( v^\beta = -\epsilon^\alpha (E^{-1})^\beta_\alpha \).

We then find the supersymmetry transformation for the fermions is given by

\[ \delta \Theta^\alpha' = \epsilon^\alpha (E^{-1})^\beta_\alpha E^\beta_\alpha', \] (2.12)

where we have chosen \( \epsilon^\alpha' = 0 \).
To evaluate this expression we are required to find $E_A$, or equivalently the $u$’s of $SO(1,10)$ and $Spin(1,10)$, in terms of the component fields in the limit $\Theta^\alpha = 0$. Using equation (2.3), the Lorentz condition $u_a^b \eta_{ab} u_d^b = \eta_{cd}$ and the static gauge choice $X^n = x^n$ we find that

$$ (u_a^b, u_a^{b'}) = (e^a_n \delta_n^b, e^a_n \partial_n x^{b'}) . \quad (2.13) $$

Using the remaining Lorentz conditions we find, up to a local $SO(5)$ rotation, that the full lorentz matrix $u_{\alpha}^{\beta}$ is given by

$$ u = \begin{pmatrix} e^{-1} & e^{-1} \partial X \\ -d^{-1}(\partial X)^T (\eta_1)^T & d^{-1} \end{pmatrix} , \quad (2.14) $$

where the matrix $d$ is defined by the condition $dd^T = I + (\partial X)^T \eta_1 (\partial X)$, $(\partial X)^T$ is the transpose of the matrix $(\partial_n X^{a'})$ and $\eta_1$ is the Minkowski metric on the fivebrane and is given by $\eta_1 = \text{diag}(-1,1,1,1,1,1)$.

The $u_{\alpha}^{\beta} \in Spin(1,10)$ corresponding to the above $u_a^b \in SO(1,10)$ are found using equation (2.7). It is instructive to carry out this calculation at the linearized level. To this order $u_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}$, $u_{\alpha'}^{\beta'} = \delta_{\alpha'}^{\beta'}$ while $u_{\alpha}^{\beta}$ and $u_{\alpha'}^{\beta'}$ are linear in the fields. Taking $(\alpha, \beta) = (\alpha, \beta)$ equation (2.7) becomes

$$ u_{\alpha}^{\gamma'} (\Gamma^{d'})_{\gamma' \beta} + (\alpha \leftrightarrow \beta) = (\Gamma^{a})_{\alpha \beta} \partial X^{d'} . \quad (2.15) $$

In order to analyse this equation we discuss the decomposition of the spinor indices in more detail. We recall that the bosonic indices of the fields on the fivebrane can be decomposed into longitudinal and transverse indices i.e. $\underline{a} = (a, a')$ according to the decomposition of the Lorentz group $SO(1,10)$ into $SO(1,5) \times SO(5)$. The corresponding decomposition of the spin groups is $Spin(1,10) \rightarrow Spin(1,5) \times USp(4)$. The spinor indices of the groups $Spin(1,5)$ and $USp(4)$ are denoted by $\alpha, \beta, ... = 1, ..., 4$ and $i, j, ... = 1, ..., 4$ respectively. Six-dimensional
Dirac spinor indices normally take eight values, however the spinor indices we use for Spin(1, 5) correspond to Weyl spinors. Although we began with spinor indices $\alpha$ that took thirty-two dimensional values and were broken into two pairs of indices each taking sixteen values $\alpha = (\alpha, \alpha')$, in the final six-dimensional expressions the spinor indices are further decomposed according to the above decomposition of the spin groups and we take $\alpha \rightarrow \alpha i$ and $\alpha' \rightarrow \alpha^i$ when appearing as superscripts and $\alpha \rightarrow \alpha i$ and $\alpha' \rightarrow \alpha^i$ when appear as subscripts [6]. It should be clear whether we mean $\alpha$ to be sixteen or four dimensional depending on the absence or presence of $i, j, \ldots$ indices respectively. For example, we will write $\Theta^\alpha \rightarrow \Theta^i_\alpha$.

Using the corresponding decomposition of the spinor indices and the expressions

$$\left(\Gamma_a^{\alpha'}\right)_{\alpha'\beta} = -(\gamma_a^{\alpha'})^i_{ij} \delta^\alpha_{\beta i}, \quad \left(\Gamma^a\right)_{\alpha\beta} = -(\gamma^a)_{\alpha\beta} \eta_{ij},$$  \hspace{1cm} (2.16)

where $\eta_{ij}$ is the anti-symmetric invariant tensor of $USp(4)$ in equation (2.15) we find that

$$u^{\beta'}_{\alpha} \rightarrow u_{\alpha i\beta} = -\frac{1}{2}(\gamma^a)_{\alpha\beta}(\gamma^b_{\beta'})^i_j \partial_a X^b_{\alpha'}. \hspace{1cm} (2.17)$$

From equation (2.11) we find that the linearised supersymmetry transformation is given by

$$\delta_0 \Theta^{\beta}_{\beta} = \epsilon^{\alpha i} \left(\frac{1}{2}(\gamma^a)_{\alpha\beta}(\gamma^b_{\beta'})^i_j \partial_a X^b_{\alpha'} - \frac{1}{6}(\gamma^{abc})_{\alpha\beta} \delta^i_j h_{abc}\right). \hspace{1cm} (2.18)$$

We now wish to find the fivebrane analogue of the Bogomol’nyi condition. We will consider a static fivebrane whose world sheet lies in the $x^0, x^1$ directions and take all fields to be independent of $x^0$ and $x^1$. We take the transverse scalars $X^{a'}$, $a' = 1', 2', \ldots, 5'$ to lie only in the five direction. Denoting $X^{5'} = \phi$ and $h_{01a} = v_a$, $a = 2, \ldots, 5$ and taking $h_{0ab} = 0 = h_{1ab}$, $a, b = 2, \ldots, 5$ we find that
the preserved supersymmetries can be deduced from equation (2.18) to be

\[ \delta_0 \Theta^j_\alpha = 0 = \epsilon^{\alpha i} \left( \frac{1}{4} \delta^1_\alpha \gamma^k_i \partial^j_\alpha \phi \right) \]

(2.19)

Therefore if we take

\[ V_n \equiv H_{01n} = \pm \frac{1}{4} \partial_n \phi, \quad n = 2, \ldots, 5 \]

(2.20)

the solution will be invariant under linearised supersymmetries which satisfy

\[ \epsilon^\beta j_0 = \pm (\gamma^01)_\beta (\gamma_5)_j \epsilon^{\alpha i} \]

(2.21)

At the linearised level \( h_{abc} \) and \( H_{mnp} \delta^m_\alpha \delta^n_\beta \delta^p_\gamma \) are equal, however, we have taken the opportunity to write the Bogomol’nyi equation in such a way that it will be valid for the full supersymmetry transformations.

Finally, it remains to show that the full supersymmetry transformation (2.12) vanishes if (2.20) and (2.21) are satisfied. We note that the full supersymmetry variation of \( \Theta^\alpha' \) of equation (2.12) contains the matrix \((E^{-1})_\alpha \beta E_\beta \gamma'\). Supersymmetries are preserved when the determinant of this matrix vanishes. This will be the case if the determinant of the matrix \( E_\beta \gamma'(u^{-1})_\beta \gamma' \) vanishes and, since it is easier, we calculate this latter matrix. We find that

\[ E_\beta \gamma'(u^{-1})_\beta \gamma' = u_\alpha \beta(u^{-1})_\beta \gamma' + h_\alpha \delta \beta u_\delta'(u^{-1})_\beta \gamma' \]

(2.22)

To illustrate how to evaluate this expression we consider the first term which we may write as

\[ u_\alpha \beta(u^{-1})_\beta \gamma' = \frac{1}{2} u_\alpha \beta(1 - \Gamma_7 \frac{\delta}{\delta} (u^{-1})_\delta \gamma') = -\frac{1}{2} u_\alpha \beta(\Gamma_7 \frac{\delta}{\delta} (u^{-1})_\delta \gamma') \]

(2.23)

Writing \( \Gamma_7 = \frac{1}{60} \epsilon_{a_1a_2a_3a_4a_5a_6} \Gamma_{a_1a_2a_3a_4a_5a_6} \) and using equation (2.7) we find that the
first term becomes

\[-\frac{1}{2.6!} \epsilon_{a_1 a_2 a_3 a_4 a_5 a_6} b_1 b_2 b_3 b_4 b_5 b_6 \partial_{u a_1} \partial_{u a_2} \partial_{u a_3} \partial_{u a_4} \partial_{u a_5} \partial_{u a_6} (\Gamma_{b_1 b_2 b_3 b_4 b_5 b_6}) \gamma'. \]  

(2.24)

The final $\Gamma$-matrix in this last equation vanishes if the $b_i$ indices take values in the transverse direction an even number of times. However, for the field configuration of interest to us, it also vanishes if their are two or more of these indices in the transverse space. Implementing this restriction the first term can be written as

\[-\frac{1}{2} \det(e^{-1}) \partial_c X^{\gamma'} (\gamma^c)_{\alpha \beta} (\gamma^{\gamma'})_i^k. \]  

(2.25)

We have used the relation $(\Gamma_{b_1...b_5})^\gamma_\alpha = -(\gamma_{b_1...b_5})_{\alpha \gamma} (\gamma^{\gamma'})_i^k$ and the fact that $(\gamma^{abc})_{\alpha \beta}$ is anti-self-dual on its vector indices to carry out this last step.

Evaluating the second term of equation in a similar manner we find that the matrix of equation (2.22) becomes

\[E^{\beta'}_\alpha (u^{-1})^{\gamma'}_\beta = -\frac{1}{2} \det(e^{-1}) \partial_n \partial_c X^{\gamma'} (\gamma^c)_{\alpha \beta} (\gamma^{\gamma'})_i^k + \frac{1}{12} (1+\det(e^{-1})) h_{abc} (\gamma^{abc})_{\alpha \beta} \delta_i^k. \]  

(2.26)

Substituting in the field configuration discussed above we find that the Bogomol'nyi condition for the full non-linear supersymmetry is given by

\[v_c = \frac{1}{2} \frac{1}{1+\det(e)} \partial_n \phi. \]  

(2.27)

The preserved supersymmetries have a parameter $\epsilon^\alpha$ which satisfies equation (2.21) provided we replace $\epsilon^\alpha_0$ in that equation by $\epsilon^\alpha E^{\beta'}_\alpha$. We will shortly show that this condition is the same as that of equation (2.19).
3. The Self-Dual String as a Soliton

Let us look for a string soliton whose worldsheet lies in the $x^0, x^1$ plane. We take all fields to be independent of $x^0$ and $x^1$. In this section, we denote the six-dimensional worldvolume indices which take the full range with a hat, i.e. $\hat{a}, \hat{b}, ..., \hat{m}, \hat{n}, ... = 0, 1, ..., 5$, and denote the four coordinates transverse to $x^0, x^1$ as $a, b, ..., m, n, ... = 2, 3, 4, 5$. We take only one of the scalar fields $X^a'$ to be active thus breaking the $SO(5)$ symmetry to $SO(4)$. We choose this scalar field to be $X^5'$, the other scalar fields being constants. Consider the ansatz

$$X^5' = \phi,$$
$$h_{01a} = v_a,$$
$$h_{abc} = \epsilon_{abcd} v^d,$$  \hspace{1cm} (3.1)

with the other components of $h_{abc}$ vanishing. The reader may verify that the ansatz respects the self-duality of $h_{abc}$. The first step in solving the field equations is to calculate the geometry of the fivebrane for these field configurations. The matrix $m$ introduced previously takes the form

$$m_{\hat{a}\hat{b}} \equiv \delta_{\hat{a}\hat{b}} - 2h_{\hat{a}\hat{c}\hat{d}}h^{\hat{b}\hat{c}\hat{d}} = \begin{pmatrix} 1 + 4v^2 & 0 \\ 0 & 1 + 4v^2 \end{pmatrix} (1 - 4v^2)\delta_{\hat{a}\hat{b}} + 8v_a v_b.$$  \hspace{1cm} (3.2)

where $v^2 = v^a v_a$. We can also compute the two metrics $g_{\hat{m}\hat{n}}$ and $G_{\hat{m}\hat{n}}$ which occur in the fivebrane equations of motion (2.1). The former is the more usual brane metric and is given by

$$g_{\hat{m}\hat{n}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ \delta_{mn} + \partial_m \phi \partial_n \phi \end{pmatrix},$$  \hspace{1cm} (3.3)
while the latter metric on the fivebrane becomes

\[ G_{\hat{m}\hat{n}} = \eta_{\hat{a}\hat{b}} E_{\hat{a}}^{\hat{m}} E_{\hat{b}}^{\hat{n}} = \begin{pmatrix} -(1 + 4v^2)^2 & 0 \\ 0 & (1 + 4v^2)^2 \end{pmatrix} (1 - 4v^2)^2 g^{m'n} + 16v^a v^b e_a^m e_b^n \] (3.4)

The vielbein \( e_{\hat{m}}^{\hat{a}} \) associated with \( g_{\hat{m}\hat{n}} \) can be found to have the form

\[ e_{\hat{m}}^{\hat{a}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \delta^a_m + c\phi_m \phi^a \end{pmatrix}, \] (3.5)

where \( \phi_n \equiv \partial_n \phi \), \( c \equiv (|\phi_n|^2)^{-1}(-1 \pm \sqrt{1 + |\phi_n|^2}) \), \( |\phi_n|^2 \equiv \phi_n \phi_m \delta^m_n \) and we adopt the convention that the derivative of \( \phi \) only ever carries a lower world index (i.e. \( \phi_a = \delta^n_a \phi_n \)). We also find from equation (3.3) the following determinants

\[ \det g_{mn} = 1 + |\phi_n|^2, \quad \det e_m^a = \sqrt{1 + |\phi_n|^2}. \] (3.6)

In the calculations that follow it is important to distinguish between the two possible tangent space bases associated with the vielbeins \( e_{\hat{a}}^{\hat{m}} \) and \( E_{\hat{a}}^{\hat{m}} \) which are in turn defined in terms of the two metrics \( g_{\hat{m}\hat{n}} \) and \( G_{\hat{m}\hat{n}} \) respectively. We recall that they are related by \( e_{\hat{a}}^{\hat{m}} = (m^{-1})_{\hat{a}}^{\hat{b}} E_{\hat{b}}^{\hat{m}} \). For example \( h_{\hat{a}\hat{d}\hat{e}} \) has its indices refered to the \( E_{\hat{a}}^{\hat{m}} \) frame vector fields and is related to the curl of the gauge field by

\[ H_{\hat{m}\hat{n}\hat{p}} = E_{\hat{m}}^{\hat{a}} E_{\hat{n}}^{\hat{b}} E_{\hat{p}}^{\hat{c}} m_{\hat{b}}^{\hat{d}} m_{\hat{c}}^{\hat{e}} h_{\hat{a}\hat{d}\hat{e}} = e_{\hat{m}}^{\hat{a}} e_{\hat{n}}^{\hat{b}} e_{\hat{p}}^{\hat{c}} (m^{-1})_{\hat{c}}^{\hat{e}} h_{\hat{a}\hat{d}\hat{e}}. \] We now calculate the tensor \( H_{\hat{m}\hat{n}\hat{p}} \). The only non-vanishing components are

\[ H_{01m} \equiv V_m = \frac{1}{(1 + 4v^2)^2} e_m^a v_a, \] (3.7)

\[ H_{mnq} = \frac{\det e}{(1 - 4v^2)^2} \epsilon_{mnq} e^a q v_a. \]

Using the equations (3.7) and (3.5) the Bogomol'nyi condition of equation
(2.19) \((V_n = \phi_n)\) can be written as

\[
v_a = \frac{1}{2} \frac{\phi_a}{1 + \sqrt{1 + |\phi_n|^2}}.
\]

(3.8)

Sustituting the expressions for the determinants of equation (3.6) we find that the Bogomol’nyi condition of equation (2.19) does indeed coincide with that of equation (2.27). A short calculation using the result of equation (3.8) shows that

\[
H_{mnp} = \pm \frac{1}{4} \epsilon_{mnpq} \phi_r \delta^{rq},
\]

(3.9)

and that

\[
G^{mn} = \delta^{mn} \frac{4}{(|\phi_n|^2)^2} \left(1 \pm \sqrt{1 + |\phi_n|^2}\right)^2 \propto \delta^{mn}.
\]

(3.10)

Let us now solve the equations of motion (2.1). For the ansatz of equation (3.1) the scalar equation becomes

\[
G^{mn} \nabla_m \phi_n = 0,
\]

(3.11)

while the equation for the gauge field leads to the two equations

\[
G^{mn} \nabla_m V_n = 0,
\]

(3.12)

and

\[
G^{mn} \nabla_m H_{mpq} = 0.
\]

(3.13)

Clearly, the equation of motion for \(H_{01m} \equiv V_n\) now reduces to that of \(\phi\) if we use the Bogomol’nyi condition. Using equation (3.10) we find that the scalar equation (3.11) becomes

\[
\delta^{mn} \partial_m \partial_n \phi = 0.
\]

(3.14)

The equation for \(H_{mnp}\) becomes an identically satisfied.
We must also ensure that $H_{mnp}$ is closed. We find that the equation

$$\epsilon^{mnpq} \partial_m H_{npq} = 0 ,$$  \hspace{1cm} (3.15)

reduces to the scalar equation (3.14).

Therefore we find the solutions are given by harmonic functions on the flat transverse space $R^4$. For $N$ strings located at $y^m_I$, ($I = 0, 1, \ldots, N - 1$) the solution (3.7) reduces to

$$H_{01m} = \pm \frac{1}{4} \partial_m \phi ,$$

$$H_{mnp} = \pm \frac{1}{4} \epsilon_{mnpq} \delta^{qr} \partial_r \phi ,$$ \hspace{1cm} (3.16)

$$\phi = \phi_0 + \sum_{I=0}^{N-1} \frac{2Q_I}{|x - y^I|^2} ,$$

where $\phi_0$ and $Q_I$ are constants. Note that the solution is smooth everywhere except at the centres of the strings. Furthermore the presence of the conformal factor in (3.10) implies that the equations of motion are satisfied even at the poles of $\phi$, so that no source terms are required and the solution is truly solitonic. Clearly the string soliton (3.16) is self-dual even though in general the tensor $H_{mnp}$ need not be. If we consider a single string then we find it has the same electric and magnetic charges

$$Q_E = \frac{1}{Vol(S^3_1)} \int_{S^3_\infty}^{} *H = \mp Q_0 ,$$

$$Q_M = \frac{1}{Vol(S^3_1)} \int_{S^3_\infty}^{} H = \mp Q_0 ,$$ \hspace{1cm} (3.17)

where $S^3_\infty$ is the transverse sphere at infinity, $S^3_1$ the unit sphere and $*$ is the flat six-dimensional Hodge star. To calculate the mass per unit length of this string one could wrap it around a circle of unit circumference and reinterpret the string as a 0-brane in five dimensions. The mass per unit length is then identified with
the mass of the 0-brane. As we will see below this definition leads to an infinite
tension.

Let us now consider the zero modes of a single string soliton, i.e. $N = 1$. Given
that we preserve the asymptotic form of the solution, there are four bosonic zero
modes $y_0^m$ coming from the location of the centre of the string. It can be seen
that there are no other bosonic zero modes since these would come from non-zero
expressions for $h_{0mn}$ and $h_{1mn}$ and would ruin the block diagonal form of $m_{\hat{a}}^\hat{b}$
which was crucial for solving the field equations. This can also be seen from the
fact that the solution preserves eight supercharges and so there is no room for any
additional bosonic zero modes in the two dimensional worldsheet supermultiplet.

The fermionic zero modes are generated by the broken supersymmetries $\epsilon^j_\alpha$
which satisfy

$$\epsilon^\beta j = \mp (\gamma^{01})_\alpha^\beta (\gamma_{5'})^j_\alpha \epsilon^{\alpha i}. \quad (3.18)$$

If we call spinors with $\gamma^{01}\epsilon = \epsilon (\gamma^{01}\epsilon = -\epsilon)$ left (right) moving then we clearly
have four left and four right moving fermion zero modes on the string world sheet,
correlated with the eigenvalue of $\gamma_{5'}$.

At first sight the above counting appears to miss four bosonic zero modes
coming from the scalars $X_0^1, X_0^2, X_0^3, X_0^4$. However these scalars do not lead to
additional zero modes since they are fixed by their asymptotic values. This is
analogous to the BPS monopole solutions in $N = 4, D = 4$ super-Yang-Mills
where there are six scalars with an $SO(6)$ symmetry relating them. A given BPS
monopole will choose a particular scalar and break this symmetry down to $SO(5)$.
The monopoles obtained using different scalars are to be viewed as distinct, forming
an $SO(6)$ multiplet of monopoles. Similarly, here we obtain a $SO(5)$ multiplet of
self-dual strings.

Thus the string has a $(4,4)$ supermultiplet of zero modes in agreement with
that found in [2]. However, contrary to our solution, the strings described there
do not carry any charge with respect to the $H$ field. Therefore the string soliton
found here might represent a D-string in the six-dimensional self-dual string theory. From the M theory point of view the strings obtained here should be interpreted as the ends of infinitely extended membranes. The scalar \( X^5' = \phi \) then corresponds to the direction of the membrane transverse to the fivebrane. Clearly the \( SO(5) \) symmetry rotates the choice of this direction. The infinite mass per unit length can be seen as arising from the infinite length of this membrane.

4. Solitons on D-branes

It was shown in [6] that M theory fivebrane’s equations of motion can be double dimensionally reduced to the Dirac-Born-Infeld equations of the type IIA D-fourbrane. It follows then that the self-dual string soliton constructed here can also be reduced to a 0-brane or 1-brane solution on the D-fourbrane worldvolume. By T-duality it follows that all of the type II D-brane worldvolume theories admit 0-brane and \((p - 3)\)-brane solitons preserving half of the supersymmetry, which can be obtained from the D-fourbrane solutions. Here we shall content ourselves with the \( p > 3 \) branes since the lower dimensional branes do not admit asymptotically free solutions. In what follows below the background type II supergravity fields are those of flat Minkowski space and we set all but one of the scalar fields on the D-\( p \)-brane to be constant. As with the previous section we use hatted indices to denote all of the \( p + 1 \) worldvolume coordinates and unhatted indices for the transverse space, which is \( p \) dimensional for 0-branes or three dimensional for \((p - 3)\)-branes.

First let us wrap the string around the compact dimension \( x^1 \) to produce a 0-brane BPS soliton on the D-fourbrane worldvolume with the two-form field-strength \( F_{\bar{m}\bar{n}} = H_{\bar{m}\bar{n}1} \) [6]. Note that because of the self-duality condition the other components \( H_{\bar{m}\bar{n}\bar{p}} \) do not appear in the D-fourbrane’s effective action as an independent field [6]. In this case we obtain

\[
F_{0m} = \mp \frac{1}{4} \partial_m \phi , \\
\phi = \phi_0 + \sum_{I=0}^{N-1} \frac{2Q_I}{|x - y_I|^2} .
\]  

(4.1)
We could also dimensionally reduce the self-dual string soliton by compactifying along $x^5$. In this case we obtain a string soliton in the D-fourbrane worldvolume. Defining $F_{\hat{m}\hat{n}} = H_{\hat{m}\hat{n}5}$ we obtain

$$F_{mn} = \pm \frac{1}{4} \epsilon_{mnp} \delta^{pq} \partial_q \phi,$$

$$\phi = \phi_0 + \sum_{I=0}^{N-1} \frac{4Q_I}{|x - y_I|}.$$  \hspace{1cm} (4.2)

The generalisation to $p > 4$ branes is now straightforward. One has 0-branes with

$$F_0m = \pm \frac{1}{4} \partial_m \phi,$$

$$\phi = \phi_0 + \frac{4}{p-2} \sum_{I=0}^{N-1} \frac{Q_I}{|x - y_I|^{p-2}},$$  \hspace{1cm} (4.3)

and $(p-3)$ branes with

$$F_{mn} = \pm \frac{1}{4} \epsilon_{mnp} \delta^{pq} \partial_q \phi,$$

$$\phi = \phi_0 + \sum_{I=0}^{N-1} \frac{4Q_I}{|x - y_I|}.$$ \hspace{1cm} (4.4)

The scalar field only depends on the transverse coordinates and $\epsilon$ is the flat volume form on the three dimensional transverse space to the $(p-3)$-brane. These solutions preserve half of the worldvolume supersymmetry and in the case of a single soliton carry the electric charge $\pm Q_0$ (for (4.3)) or magnetic charge $\pm Q_0$ (for (4.4)) with respect to $F_{\hat{m}\hat{n}}$. These solutions represent an infinitely long string or $(p-2)$-brane respectively, stretching out from the D-$p$-brane along the direction specified by the scalar $\phi$. As with the self-dual string this transverse direction is acted on by the $SO(9-p)$ symmetry which rotates the scalars. These solutions are infinitely massive, similar to a point-like charge in Maxwell’s theory. This is because the Dirac-Born-Infeld expression for the energy does not regulate the divergences of the fields at the soliton centres. However this is to be expected given the interpretation as the end of an infinitely long string or $(p-2)$-brane.
One can also explicitly verify that these solitons preserve half the worldvolume supersymmetry at the linearised order by compactifying $D = 10$ super-Maxwell theory to $p+1$ dimensions, viewed as the lowest order approximation to the D-brane effective action. However our construction from the self-dual string solution shows that (4.3) and (4.4) are solutions to the non-linear Dirac-Born-Infeld equations of motion preserving half of the supersymmetry to all orders.

The D-threebrane is a special case and so we shall treat it separately here. By wrapping the self-dual string on a torus to four dimensions we obtain dyonic 0-brane solitons on the D-threebrane worldvolume. Now $x^1$ and $x^5$ are compact so that $\phi$ is depends only on $x^m = x^2, x^3, x^4$. Let us consider new coordinates $(u, v)$ related to $(x^1, x^5)$ by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^1 \\ x^5 \end{pmatrix},$$

(4.5)

with $ad - bc = 1$. If we define the two-form field strength $F_{\hat{m}\hat{n}} = H_{\hat{m}\hat{n}u}$ then we find the dyonic solutions

$$F = aF^E + cF^M,
\phi = \phi_0 + \sum_{I=0}^{N-1} \frac{4Q_I}{|x - y_I|},$$

(4.6)

where $F^E$ and $F^M$ are the purely electric and purely magnetic field strengths with the components

$$F^E_{0m} = \pm \frac{1}{4} \partial_m \phi,
F^M_{mn} = \pm \frac{1}{4} \varepsilon_{mnp} \delta^{pq} \partial_q \phi.$$

(4.7)

The solution corresponding to a single dyon then has the electric and magnetic charges

$$(Q_E, Q_M) = \pm Q_0(a, c).$$

(4.8)
As is well known the $SL(2,\mathbb{Z})$ S-duality symmetry of $N = 4$, $D = 4$ theory acts as
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} A, \tag{4.9}
\]
where $A \in SL(2,\mathbb{Z})$ and is simply the modular group acting on the torus [10,11].

5. Discussion

In this paper we have obtained a self-dual string soliton as a BPS solution of
the M theory fivebrane’s equation of motion. Furthermore we showed that the
full non-linear equations of motion and Bogomol’nyi equation were satisfied to all
orders. We also used the self-dual string soliton to obtain soliton states on type II
D-$p$-branes for $p > 2$. Finally we would like to close with some comments.

It is interesting to note that at the linearised level, where we the two metrics
$G^{mn}$ and $g^{mn}$ are flat and the tensor $m^a_b$ is the identity, the field equations (2.1)
become non-interacting. Thus even the theory of a free tensor multiplet contains
a non-trivial BPS soliton in its spectrum. In fact this statement deserves some
qualification since the theory isn’t completely free at the linearised level because
the fermions still interact with the bosons (although the fermions have been set to
zero in the solution). However, this does help to clarify why the analysis of the M
theory fivebrane’s BPS states carried out in [2] was so successful, even though it
assumed that one could treat the self-dual string theory as non-interacting.

This unusual situation is what one might expect given the interpretation of
the self-dual string as limit in which a type IIB D-threebrane wrapped around a
2-cycle of $K3$ becomes light and decouples from gravity [3]. If one considers the
supergravity field equations for a generic $p$-brane there are equations for the field
strength and for a scalar field, similar to (2.1) (with $G^{mn} = g^{mn}$ now identified
with the spacetime metric).* However, in supergravity what makes these equations

* The type IIB threebrane (and hence the related six-dimensional self-dual string also) is
actually unique in this respect because the dilaton is constant [12].
interacting and non-linear is the fact that all these fields couple to the spacetime metric, even though they don’t couple directly to each other (at least to lowest order in the supergravity effective action). Thus in a limit where gravity is decoupling one may expect the lowest order bosonic field equations to become those of a free theory.

While this paper was in the final stages of completion we received a copy of [13] and learnt of [14] which overlap with the contents of section three.

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