On “two important theorems” in canonical duality theory

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Abstract

In this short note we show, providing counterexamples, that the “two important theorems” in the recent paper [Y, Yuan, Global optimization solutions to a class of non-convex quadratic minimization problems with quadratic constraints, in Canonical Duality Theory, D.Y. Gao et al. (eds), (AMMA, volume 37), Springer, 2017] are false.

1 Introduction

The aim of [3] (the same as [2]) is to study “non-convex quadratic minimization problems with quadratic constraints”. The method used in that study is “a very powerful method, proposed by David Gao, called canonical duality”. It is our aim to show that both theorems stated in this paper are false.

To ease the reading of our note we present the statements, as well as their ingredients, making so the note self-contained. Of course, we encourage the reader to look also at [2] and/or [3].

2 Framework and statements of [3]

Let us quote the corresponding text from [3], split by some remarks. So, the

“Non-convex quadratic minimization problems with quadratic constraints \((P_{qq})\) can be formulated as follows ((\(P_{qq}\) in short)

\[
(P_{qq}): \min \left\{ P(x) = \frac{1}{2} x^T A x - f^T x : x \in \mathcal{X}_A \right\}, \quad (1)
\]

where \(A = A^T \in \mathbb{R}^{n \times n}\) is an indefinite matrix, and the feasible space \(\mathcal{X}_A\) is defined by

\[
\mathcal{X}_A \triangleq \left\{ x \in \mathbb{R}^n \mid \frac{1}{2} x^T Q_i x + b_i^T x \leq c_i, \ i = 1, \ldots, m \right\}, \quad (2)
\]

in which \(Q_i = Q_i^T \in \mathbb{R}^{n \times n} (i = 1, \ldots, m)\) are given nonsingular matrices, \(b_i \in \mathbb{R}^n (i = 1, \ldots, m)\) are given vectors which control the geometric centers. \(c_i (i = 1, \ldots, m) \in \mathbb{R}\) are given input constants.

In order to make sure that the feasible space \(\mathcal{X}_A\) is nonempty, the quadratic constraints must satisfy the Slater regularity condition, i.e., there exists one point \(x_0\) such that \(\frac{1}{2} x_0^T Q_i x_0 + b_i^T x_0 \leq c_i, \ i = 1, \ldots, m.\)”

First observe that the Slater regularity condition (in fact Slater’s constraint qualification, see [1, p. 243]) for problem \((P_{qq})\) is: there exists \(x_0 \in \mathbb{R}^n\) such that \(\frac{1}{2} x_0^T Q_i x_0 + b_i^T x_0 < c_i\) for all \(i \in \{1, m\}\).

One continues with:

“In this work, one hard restriction is given that \(f \neq 0 \in \mathbb{R}^n\). The restriction is very important to guarantee the uniqueness of global optimal solution of \((P_{qq})\)”. 
In fact the condition \( f \in \mathbb{R}^n \setminus \{0\} \) does not guarantee the uniqueness of a global optimal solution of \( (P_{qq}) \), as Example 1 (below) shows.

The following notation is used in the statement of Theorem 2 of [3]:

\[ \mathcal{G}_+(A) \triangleq \{ B \in \mathbb{R}^{n \times n} \mid A + B \succ 0 \} \quad (5) \]

On page 344 of [3] one finds:

"The canonical dual function of \( P(x) \) is defined by the following equation (referred to [8]1

\[ P^d(\sigma) = Q^h(\sigma) - I^*(\sigma), \quad (10) \]

where

\[ Q^h(\sigma) = \text{sta} \{ \varepsilon^T \sigma + \frac{1}{2} x^T A x - f^T x \} = -\frac{1}{2} F(\sigma)^T G(\sigma)^{-1} F(\sigma) - c^T \sigma, \quad (11) \]

in which the notation \( \text{sta} \{x \in \mathbb{R}^n\} \) is the operator to find out the stationary point in the space \( \mathbb{R}^n \), \( G(\sigma) \), \( F(\sigma) \) and \( c \) are defined by

\[ G(\sigma) = (A + \sum_{i=1}^{m} Q_i \sigma_i), \quad F(\sigma) = (f - \sum_{i=1}^{m} b_i \sigma_i), \quad c = (c_1, c_2, \ldots, c_m)^T, \quad (12) \]

where \( \sigma_i \) is the \( i \)th element of \( \sigma \).

The dual feasible space is defined by

\[ S \triangleq \{ \sigma \in \mathbb{R}^m \mid \sigma \geq 0 \in \mathbb{R}^m, \quad \det(G(\sigma)) \neq 0 \} \quad (13)\].

Above, \( I^*(\sigma) \) is given by \( I^*(\sigma) = 0 \) if \( \sigma \geq 0 \), \( I^*(\sigma) = +\infty \) otherwise, and so

\[ P^d(\sigma) = -\frac{1}{2} F(\sigma)^T G(\sigma)^{-1} F(\sigma) - c^T \sigma \quad \text{if} \quad \sigma \geq 0, \quad P^d(\sigma) = -\infty \quad \text{otherwise}. \]

Notice that \( P^d(\sigma) \) is not defined if \( \sigma \geq 0 \) and \( \det(G(\sigma)) = 0! \) Also, it is quite strange that an “operator that is used to find the stationary point in the space \( \mathbb{R}^n \)” could be a real number.

One continues with:

"The canonical dual problem (\( P^d \) in short) associated with \( (P_{qq}) \) can be eventually formulated as follows

\[ (P^d) : \max_{\sigma \in S} \{ P^d(\sigma) \}. \quad (14) \]

2.3 Two important theorems

In order to show that there is no duality gap, the following theorem is presented.

**Theorem 1.** If \( A_i, Q_i, b_i, f_i, c_i, i = 1, 2, \ldots, m \), are given with definitions in \( (P_{qq}) \) such that

the dual feasible space

\[ \mathcal{Y} \triangleq \{ \sigma \in \mathcal{S} \mid G(\sigma)^{-1} F(\sigma) \in \mathcal{X} \} \quad (15) \]

is not empty, then the problem

\[ (P^d) : \max_{\sigma \in \mathcal{Y}} \{ P^d(\sigma) \}, \quad (16) \]

is canonically (perfectly) dual to \( (P_{qq}) \). In another words, if \( \overline{\sigma} \) is a solution of the dual problem

\[ \overline{\sigma} = G(\overline{\sigma})^{-1} F(\overline{\sigma}) \quad (17) \]

is a solution of \( (P_{qq}) \) and

\[ P(\overline{\sigma}) = P^d(\overline{\sigma}). \quad (18) \]

After the proof of this “important theorem”, one continues with:

"In order to get the optimization solution of \( (P_{qq}) \), we introduce the following subset

\[ \mathcal{S}_+= \{ \sigma \in \mathcal{S} \mid G(\sigma) \text{ is positive definite} \}. \quad (23) \]

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1The reference [8] is “Gao, DY. Canonical dual transformation method and generalized triality theory in nonsmooth global optimization. J Global Optim 2000; 17(1–4): 127–160”. 

2
In order to hold on the uniqueness of the optimal duality solution, the following existence theorem is presented.

**Theorem 2.** For any given symmetrical matrices \( A, Q_1, Q_2 \in \mathbb{R}^{n \times n} \), \( G_+(A) \) (defined by (5)) is the complementary positive definite matrix group of \( A \), if the following two conditions are satisfied

\[ C_1: \sum_{i=1}^{m} Q_i \in G_+(A); \]

\[ C_2: \text{there must exist one } k(1 \leq k \leq m) \text{ such that } Q_k \text{ is positive-definite and } Q_k \in G_+(A), \]

\[ \| D_k A^{-1} f \| > \| b_k^T D_k^{-1} \| + \sqrt{\| b_k^T D_k^{-1} \|^2 + 2|c_k|}, \] (24)

where \( Q_k = D_k^T D_k \) and \( \| \cdot \| \) is some vector norm.

Then, the canonical duality problem (16) has a unique nonzero solution \( \tilde{\sigma} \) in the space \( S_+ \).

First of all observe that there are two dual problems, (14) and (16). However, both theorems refer to problem (16). Moreover, probably the author intended to write \( X_i \) instead of \( X \), and \( f \) instead of \( f_i \) in the statement of Theorem 1. Because \( Q_k \) is positive-definite in Theorem 2, \( D_k := Q_k^{1/2} \) is positive-definite (hence symmetric) and \( D_k D_k = Q_k \); moreover, the norm has to be the Euclidean norm to have correct inequalities in the proof of Theorem 2. Furthermore, I suppose that the notion of solution of the problem (\( P_{qq} \)) is in the sense from [11, p. 2]; similarly for solution of problem (\( P^d \)).

### 3 Examples

The first example shows that the condition \( f \neq 0 \in \mathbb{R}^n \) does not “guarantee the uniqueness of global optimal solution of (\( P_{qq} \)).

**Example 1** Take \( n = m = 1 \), \( P(x) := -x^2 + 2x \) and \( q(x) := -P(x) \) for \( x \in \mathbb{R} \). Clearly, the problem \( \min P(x) \text{ s.t. } q(x) \leq 0 \) has the solutions \( x_1 = 0 \) and \( x_2 = 2 \). In fact taking an arbitrary quadratic function \( P \) on \( \mathbb{R}^n \) and \( q := -P \), assuming that \( X_0 := \{ x \in \mathbb{R}^n \mid q(x) \leq 0 \} \neq \emptyset \), the sets of solutions of the problem \( \min P(x) \text{ s.t. } q(x) \leq 0 \) is \( \{ x \in \mathbb{R}^n \mid q(x) = 0 \} \).

The next example shows that \( \bar{x} \) provided in [3, Eq. (17)] is not, necessarily, a solution of the primal problem [3, Eq. (1)], contrary to what is stated in [3, Th. 1].

**Example 2** Let us take \( n = m = 1 \), \( P(x) := -x^2 - x \) and \( q(x) := \frac{1}{2}x^2 + x \) and \( c := 0 \); hence \( X_a = [-2, 0] \). With the notations in [3, Eq. (12)] we have that \( G(\sigma) = -2 + \sigma, F(\sigma) = 1 - \sigma \), and so \( P^d(\sigma) = -\frac{1}{2}(1-\sigma^2) \) (for \( \sigma \in S = [0, 2) \cup (2, \infty) \)). We have that \( Y = \{ \sigma \in S \mid q(\frac{\sigma - 2}{\sigma - 2}) \leq 0 \} = [0, 1] \cup [3, \infty) \). It is easy to verify that \( P^d(0) \geq P^d(\sigma) \) for all \( \sigma \in Y \). In fact \( \bar{\sigma} := 0 \) is the unique solution of problem (\( P^d \)). Indeed, we have that \( \bar{x} = F(\bar{\sigma})/G(\bar{\sigma}) = -\frac{1}{8}P(\bar{x}) > P(-2) = -2 \). Hence \( \bar{x} \) is not a solution of problem (\( P_{qq} \)); in fact \( P(\bar{x}) > P(x) \) for every \( x \in \mathbb{R} \setminus \{ x \} \).

The next example shows that under the hypothesis of [3, Th. 2], its conclusion that the dual problem [3, Eq. (16)] has a unique solution in \( S_+ \) can be false; in fact in this example the dual problem [3, Eq. (16)] has no solutions belonging to the set \( S_+ \).

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\(^2\)Observe that \( A \) is not assumed to be non-singular!
Example 3  Let $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Q := \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, $f := \begin{pmatrix} \sqrt{27} \\ 1 \end{pmatrix}$, $b := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $c = 52$. Clearly, $A + Q > 0$, and so condition $C_1$ holds. Moreover, $D^2 = Q$, where $D = \frac{1}{2}Q$, and $\|DA^{-1}f\| = \|(2\sqrt{27}, -2)\| = 4\sqrt{7} > 2\sqrt{26}$. Hence, [[3, Eq. (24)]] is verified, and so condition $C_2$ holds, too. The arguments for the claim that [[3, Eq. (16)]] has no solutions belonging to the set $S_+$ are provided below.

The inequality constraint is $q(x_1, x_2) = \frac{1}{2}(4x_1^2 + 4x_2^2) - 52 \leq 0$

With the notations in [[3, Eq. (12)]] we have that $G(\sigma) = \text{diag}(4\sigma + 1, 4\sigma - 1)$, $F(\sigma) = f$, $c = 52$, and so $P^d(\sigma) = -\frac{1}{2}\left(\frac{27}{4\sigma + 1} + \frac{1}{4\sigma - 1}\right) - 52\sigma$ (for $\sigma \in S = [0, \frac{1}{4}) \cup (\frac{1}{4}, \infty)$). Moreover, $S_+ = (\frac{1}{4}, \infty)$. Observe that

$$\psi(\sigma) := q(G(\sigma)^{-1}F(\sigma)) = q\left(\frac{\sqrt{27}}{4\sigma + 1}, \frac{1}{4\sigma - 1}\right) = 2\left[\frac{27}{(4\sigma + 1)^2} + \frac{1}{(4\sigma - 1)^2} - 26\right]$$

for $\sigma \in \mathbb{R} \setminus \{-\frac{1}{4}, \frac{1}{4}\}$ . Let us study the functions $\psi$ and $\varphi$ defined by

$$\varphi : \mathbb{R} \setminus \{-\frac{1}{4}, \frac{1}{4}\} \to \mathbb{R}, \quad \varphi(\sigma) := -\frac{1}{2}\left(\frac{27}{4\sigma + 1} + \frac{1}{4\sigma - 1}\right) - 52\sigma.$$

Clearly

$$\varphi'(\sigma) = 2\left[\frac{27}{(4\sigma + 1)^2} + \frac{1}{(4\sigma - 1)^2} - 26\right] = \psi(\sigma), \quad \varphi''(\sigma) = -16\left[\frac{27}{(4\sigma + 1)^3} + \frac{1}{(4\sigma - 1)^3}\right]$$

for $\sigma \in \mathbb{R} \setminus \{-\frac{1}{4}, \frac{1}{4}\}$. First, we have the following table of variation for $\psi = \varphi'$:

| $\sigma$ | $-\infty$ | $-\frac{1}{4}$ | $0$ | $\frac{1}{4}$ | $\infty$ |
|----------|------------|---------------|----|--------------|----------|
| $\psi'$  | 0          | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $-\infty$|
| $\psi''$ | +          | +             | +  | +            | +        |

where we have taken into consideration that $\psi'(1/8) = 0$, $\psi(1/8) = -20 < 0$, and

$$\lim_{\sigma \to \pm \frac{1}{4}} |\psi'(\sigma)| = \lim_{\sigma \to \pm \frac{1}{4}} \psi(\sigma) = \infty, \quad \lim_{\sigma \to \pm \infty} \psi'(\sigma) = 0, \quad \lim_{\sigma \to \pm \infty} \psi(\sigma) = -52.$$

Taking into account the variation of $\varphi' = \psi$, we get the following table for the variation of $\varphi$ on the interval $[0, \infty)$:

| $\sigma$ | 0    | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | $\infty$ |
|----------|------|---------------|---------------|---------------|----------|
| $\varphi'$ | 4    | 0             | $-\infty$     | $+\infty$     | $+\infty$ |
| $\varphi''$ | -13  | $-\infty$     | $+\infty$     | $-\infty$     | $-\infty$ |

We observe that $\chi_{\sigma} = \{x \in \mathbb{R}^2 \mid \|x\| \leq \sqrt{26}\}$, $S = [0, \frac{1}{4}) \cup (\frac{1}{4}, \infty)$, $Y = [\sigma_2, \sigma_3] \cup [\sigma_4, \infty)$. Since $\varphi$ is decreasing on each one of the intervals $[\sigma_2, \sigma_3]$ and $[\sigma_4, \infty)$, $\max_{\sigma \in Y} P^d(\sigma) = \max\{\varphi(\sigma_2), \varphi(\sigma_4)\}$. Clearly, $\varphi(\sigma_2) \geq \varphi(\frac{1}{2}) = -29/2$. So, it is sufficient to prove that $\varphi(\sigma) \leq -29/2$ on $(\frac{1}{4}, \infty)$. Replacing $4\sigma$ by $t$, the preceding inequality is equivalent to each one of the following inequalities $\frac{27}{t+1} + \frac{1}{t-1} + 26t \geq 29$, $\chi(t) := 26t^3 - 29t^2 + 2t + 3 \geq 0$ for $t > 1$. But $\chi'(t) = 78t^2 - 58t + 2 = 27(t(9t - 29) + 2 > 0$ for $t \geq 1$, whence $\chi(t) \geq \chi(1) = 2$ for $t \geq 1$. Hence $\sigma_2 \in Y \setminus S_+$ is the only solution of problem (16). This shows that the conclusion of Theorem 2 in [[3]] is false.
References

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