Stable finite difference methods for Kirchhoff-Love plates on overlapping grids

Longfei Li\textsuperscript{a,1,*}, Hangjie Ji\textsuperscript{b}, Qi Tang\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA.
\textsuperscript{b}Department of Mathematics, University of California Los Angeles, Los Angeles, CA 90095, USA.
\textsuperscript{c}Los Alamos National Laboratory, Los Alamos, NM 87545, USA.

Abstract

In this work, we propose and develop efficient and accurate numerical methods for solving the Kirchhoff-Love plate model in domains with complex geometries. The algorithms proposed here employ curvilinear finite-difference methods for spatial discretization of the governing PDEs on general composite overlapping grids. The coupling of different components of the composite overlapping grid is through numerical interpolations. However, interpolations introduce perturbation to the finite-difference discretization, which causes numerical instability for time-stepping schemes used to advance the resulted semi-discrete system. To address the instability, motivated by an upwind scheme for solving the wave equation, we propose to add a fourth-order hyper-dissipation to the spatially discretized system to stabilize its time integration; this additional dissipation term captures the essential upwinding effect of the original upwind scheme. The investigation of strategies for incorporating the upwind dissipation term into several time-stepping schemes (both explicit and implicit) leads to the development of four novel algorithms. For each algorithm, formulas for determining a stable time step and a sufficient dissipation coefficient on curvilinear grids are derived by performing a local Fourier analysis. Quadratic eigenvalue problems for a simplified model plate in 1D domain are considered to reveal the weak instability due to the presence of interpolating equations in the spatial discretization. This model problem is further investigated for the stabilization effects of the proposed algorithms. Carefully designed numerical experiments are carried out to validate the accuracy and stability of the proposed algorithms, followed by two benchmark problems to demonstrate the capability and efficiency of our approach for solving realistic applications. Results that concern the performance of the proposed algorithms are also presented.

Keywords: Kirchhoff-Love plate, complex geometry, overlapping grids, curvilinear finite difference, Newmark-Beta scheme, artificial hyper-dissipation
# Contents

1 Introduction ............................................. 3

2 Governing Equations ...................................... 4

3 Numerical Methods ........................................ 5
   3.1 Composite overlapping grids .......................... 5
   3.2 Spatial discretization on curvilinear grid ........... 6
   3.3 Time-stepping schemes ............................... 7

4 Stability Analysis ......................................... 11
   4.1 Dissipation coefficient determination .................. 11
   4.2 Time step determination ............................. 12

5 Model Problem in 1D ....................................... 14
   5.1 Problem setup ....................................... 14
   5.2 Stability issues ..................................... 16

6 Numerical results .......................................... 19
   6.1 Convergence study .................................... 19
      6.1.1 Same order vs. same stencil .................... 19
      6.1.2 Manufactured solutions ......................... 21
      6.1.3 Analytical solutions for circular plates ........ 22
   6.2 Plate with numerous holes ............................ 26
   6.3 Traveling pulse on a guitar soundboard .............. 28

7 Conclusions ................................................ 31

8 Acknowledgment ........................................... 31

Appendix A .................................................. 32
   Appendix A.1 Transformed PDE on reference domain ...... 32
   Appendix A.2 Formula of the Discrete transformation .... 32
1. Introduction

Kirchhoff-Love theory [1] concerns the small deflection of thin plates subject to external loadings, and it is widely used in structural engineering for determining the stresses and deformations of thin-walled structures. The theory simplifies the solid mechanics by assuming that a 3D plate can be represented by its 2D mid-surface. The dimension reduction of Kirchhoff-Love theory offers great convenience for studying plates both analytically and numerically. This paper aims for the development of accurate and efficient finite-difference schemes for solving the Kirchhoff-Love plate model in domains with complex geometries.

The presence of the biharmonic operator in the governing PDE of the Kirchhoff-Love model poses a significant challenge to its numerical methods. To avoid the complication from the biharmonic operator, numerical methods have been developed based on various reformulations of the system [2–4] or mixed finite elements [5]; however, for some reformulations, it is challenging to support the general boundary conditions (such as supported or free boundary conditions) beyond the essential (i.e., clamped) boundary conditions [2, 3]. Solving the Kirchhoff-Love model directly using conforming finite element method (FEM) requires the use of finite elements of class $C^1$. However, $C^1$ elements are difficult to construct in multi-dimensions [5, 6]; hence they are rarely used in practice except for beams (1D plates). To circumvent this issue, one could use non-conforming elements such as Morley elements [7–11] or a discontinuous Galerkin method [12] where high-order continuities along the element edges are only enforced weakly.

An alternative approach to handle the biharmonic operator is through finite differences. Based on direct finite-difference approximations of the biharmonic operator and all the other lower-order derivatives, we have recently developed efficient and accurate finite-difference methods for the Kirchhoff-Love plate equation [13, 14]. These methods support all the boundary conditions (i.e., clamped, supported and free) and are straightforward to implement, but are limited to simple domains due to constraints with respect to meshing using a single structured grid.

Supporting a general boundary condition is a key step towards fluid-structure interaction (FSI) solvers involving plates. For instance, in our previous work of [15], we derived a generalized interface coupling condition for the beam/shell coupled with an incompressible flow, which can be viewed as a generalization of a non-standard boundary condition. Such an interface condition has been extended to other regimes such as the rigid body [16–18] or elastic solids [19, 20] coupled with an incompressible flow. In this work, as a first step towards an FSI solver involving plates in complex moving and deforming domain, we focus on the stable and robust schemes for solving the Kirchhoff-Love plate with complex geometry along with different boundary conditions.

Many numerical methods have been developed based on solving alternative models that are closely related to the Kirchhoff-Love plate. For example, the Kirchhoff-Love plate model can be regarded as the thin plate limit of the Reissner–Mindlin theory [1], which concerns the rotation in addition to the deflection of plates. The Reissner-Mindlin theory is more accurate for thicker plates and is easier to solve numerically because only $C^0$ elements are needed to approximate its unknowns (deflection and rotation). Therefore, good numerical methods developed for the Reissner-Mindlin plate can be exploited to solve the Kirchhoff-Love limit. For this type of methods, one needs to pay special attention to address the shear and membrane locking phenomena [21–23] that are caused by the inconsistency between the Kirchhoff-Love plate and Reissner-Mindlin plate at zero thickness. Continuum based (CB) element method is another widely used method in commercial software and research for solving plate/shell models [6, 24, 25]. The essence of the CB methodology is to derive a simplified model (referred to as CB shell elements) for the thin structure at the discrete level by imposing kinematic assumptions to the discretization of the entire 3D solid.

Isogeometric analysis (IGA) that uses NURBS basis functions for more precise geometric representations was proposed as a generalization of classical finite element analysis [26, 27]. Since its introduction, IGA has gained increasing attention in engineering and applied sciences communities for simulating challenging PDEs in domains with complex geometries. Recently, IGA and its variations (e.g., isogeometric collocation or Galerkin) have been well developed and widely used for problems with higher order derivatives such as various plate and shell models; see for example [23, 28–33] and the references therein.

In this paper, we propose to develop finite-difference based numerical methods for solving Kirchhoff-Love plates on composite overlapping grids. A composite overlapping grid refers to the collection of logically rectangular curvilinear component grids that cover the entire domain and overlap where they meet. Overlapping grids, also known as overset or Chimera grids, are often used for the efficient and accurate solution of PDEs.
The novel algorithms presented here are based on the common spatial discretization of the PDE on composite overlapping grid, which involves curvilinear finite-difference approximations for spatial derivatives on each component grid and interpolating formulas for coupling solutions on overlapping component grids. Four time-stepping methods, both explicit and implicit, are considered for the temporal integration of the spatially discrete system. One numerical challenge of our approach lies in the weak instability caused by the presence of interpolating equations in the discrete system, which breaks the nice symmetric property of the finite-difference discretization on structured component grids. Motivated by an upwind scheme that was developed for solving the wave equation on overlapping grids \[35, 36\], we propose to add a fourth-order hyper-dissipation to the spatially discretized system to stabilize its time integration; this additional dissipation term captures the essential upwinding effect of the original upwind scheme. Analysis and numerical tests are carefully carried out to validate the numerical properties of the methods. Finally, two benchmark problems involving plates with complex geometries are presented to illustrate that our finite-difference based approach is well-suited for solving plate models arising in realistic applications.

The remainder of the paper is organized as follows. In Section 2, we describe the governing equation and boundary conditions for the problem considered, followed by a detailed presentation of the proposed numerical algorithms in Section 3. In Section 4, we analyze the stability of our algorithms and derive formulas for determining stable time steps and sufficient dissipation that can be used in actual computations. In Section 5, a simplified 1D model problem is considered to illustrate the unstable modes caused by the interpolating equations, and to investigate the effects of our stabilization strategies. Numerical results for convergence studies and benchmark simulations are discussed in Section 6. Finally, some concluding remarks are given in Section 7.

2. Governing Equations

We consider the Kirchhoff-Love plate model for isotropic and homogeneous material that incorporates various physics including bending, tension and linear restoration (i.e., the Winkler foundation known in engineering community \[37, 38\]). Let \(w(x, y, t)\) with \((x, y) \in \Omega \subset \mathbb{R}^2\) denote the transverse displacement of the plate, then the Kirchhoff-Love model can be described by an initial-boundary value problem (IBVP). Specifically, the governing PDE for the displacement is given by

\[
\rho H \frac{\partial^2 w}{\partial t^2} = -Kw + T \nabla^2 w - D \nabla^4 w + f, \tag{1}
\]

where \(f = f(x, y, t)\) is a given external forcing, \(\rho\) is density, \(H\) is thickness, \(K\) the linear stiffness coefficient that acts as a linear restoring force, \(T\) is the tension coefficient, and \(D = EH^3/(12(1 - \nu^2))\) represents the flexural rigidity with \(\nu\) and \(E\) being the Poisson’s ratio and Young’s modulus, respectively.

On any boundary of the domain, \(\forall (x, y) \in \partial \Omega\), we may impose one of the following boundary conditions,

\[
\begin{align*}
\text{clamped:} \quad w &= 0, \quad \frac{\partial w}{\partial \mathbf{n}} = 0; \tag{2} \\
\text{supported:} \quad w &= 0, \quad \frac{\partial^2 w}{\partial \mathbf{n}^2} + \nu \frac{\partial^2 w}{\partial t^2} = 0; \tag{3} \\
\text{free:} \quad \frac{\partial^2 w}{\partial \mathbf{n}^2} + \nu \frac{\partial^2 w}{\partial t^2} = 0, \quad \frac{\partial}{\partial \mathbf{n}} \left[ \frac{\partial^2 w}{\partial \mathbf{n}^2} + (2 - \nu) \frac{\partial^2 w}{\partial t^2} \right] = 0, \tag{4}
\end{align*}
\]

where \(\partial/\partial \mathbf{n}\) and \(\partial/\partial t\) are the normal and tangential derivatives defined on the boundary of the domain. It is important to point out that, at corners between two free boundaries (4), a corner condition that imposes zero forcing, \(\partial^2 w/\partial x \partial y = 0\), must be included \[14, 39\].

The initial state of the plate is defined by

\[
w(x, y, 0) = w_0(x, y) \quad \text{and} \quad \frac{\partial w}{\partial t}(x, y, 0) = v_0(x, y), \tag{5}
\]

where \(w_0\) and \(v_0\) prescribe the initial displacement and velocity, respectively.
3. Numerical Methods

We aim to develop efficient and accurate numerical methods for solving Kirchhoff-Love plates with general geometries using composite overlapping grids. First, we discuss composite overlapping grids and the associated discretization approach, followed by the presentation of four time-stepping schemes for the stable integration of the spatially discretized system. We are also interested in the velocity $v(x, y, t) = \partial_t w(x, y, t)$ and acceleration $a(x, y, t) = \partial_t^2 w(x, y, t)$ of plates; therefore, all of our numerical methods are designed to solve for $v$ and $a$ to the same accuracy as the displacement solution $w$. Note that $v$ and $a$ are crucial information for multi-physics problems such as FSI applications; the accurate computation of these quantities is essential for any future development of FSI solvers involving Kirchhoff-Love plates.

3.1. Composite overlapping grids

Composite overlapping grids are efficient and powerful techniques that are often used for the solution of PDEs on domains with complicated shapes [34]. In Figure 1, we show an example of composite overlapping grid for a square plate with a circular hole cut at the center. In general, a composite overlapping grid, $\mathcal{G}$, consists of a set of structured component grids, $\{\mathcal{G}^{(n)}\}$, $g = 1, \ldots, N$, that cover the entire plate domain $\Omega$; the component grids overlap where they meet. Solutions on the different component grids in the overlapped region are coupled by interpolation. Typically, boundary-fitted curvilinear grids are used near the boundaries to resolve the shape of a plate, while one or more background Cartesian grids are used to handle the bulk of the domain for efficiency. Each component grid $\mathcal{G}^{(n)}$ is a logically rectangular, curvilinear grid in 2D, and is defined by a smooth mapping that maps the unit square into the subdomain covered by $\mathcal{G}^{(n)}$; i.e., $x = g(r)$ with $r = (r, s) \in [0, 1]^2$ and $x = (x, y) \in \mathbb{R}^2$. The original $(x, y)$-space is referred to as physical space, and the unit square $(r, s)$-space is called the reference space.

![Figure 1: Composite overlapping grid $\mathcal{G}_1$ for a square plate with a circular hole cut at the center. The side of the square is 4 and the radius of the circle is 1/2. Interpolation points on both component grids are highlighted.](image)

Grid points on a composite overlapping grid are classified as discretization points, interpolation points or unused points. On the discretization points of each component grid, curvilinear finite-difference methods are used to discretize the spatial derivatives of the PDE and boundary conditions; ghost points are used to aid the discretization of physical boundary conditions. Numerical solutions between different component grids are coupled together with interpolating equations. The interpolation points of the example composite grid are highlighted in Figure 1.

We build a sequence of refined meshes denoted as $\{\mathcal{G}_n\}$ for each problem in practice, where $\mathcal{G}_1$ represents the base grid that has a target grid spacing $h = 1/10$. For curved grids (non-Cartesian), the grid spacings are not constant. So the target grid spacing is used as a guidance, and we try to maintain the cell volumes as uniform as possible. The grid shown in Figure 1 is the base grid $\mathcal{G}_1$ for this example grid. Finer grid $\mathcal{G}_n$ is obtained from refining the base grid by a factor of $n$; that is, the target grid spacing for $\mathcal{G}_n$ is $1/(10n)$. For accuracy studies, we typically perform convergence studies on the sequence of grids, $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_4, \mathcal{G}_8, \mathcal{G}_{16}\}$.
3.2. Spatial discretization on curvilinear grid

To discretize the governing equation (1) on a component grid, we first transform the problem in physical space to the reference space, and then approximate the partial derivatives in the \((r, s)\)-space using standard central difference formulas. The transformed derivatives are obtained using the derivatives of the mapping \(x = g(r)\) and chain rules.

Specifically, the Kirchhoff-Love plate equation (1) on the reference domain can be written as

\[
\rho H \frac{\partial^2 W}{\partial t^2} = \mathcal{L}W + F,
\]

where the plate operator \(\mathcal{L}\) is defined by

\[
\mathcal{L}W = -KW + a_1(r)W_r + a_2(r)W_s + b_{11}(r)W_{rr} + b_{12}(r)W_{rs} + b_{22}(r)W_{ss} \\
+ c_{11}(r)W_{r}^2 + c_{12}(r)W_{rs}^2 + c_{22}(r)W_{s}^2 + c_{222}(r)W_{sss},
\]

where \(W_r\) denotes the solution on grid \(r_{i,j}\), and \(h_r\) is the grid spacing in the \(r\)-direction of the unit square grid. Similar operators, \(D_{+s}\), \(D_{-s}\), and \(D_{0s}\), can be defined in the \(s\)-direction.

We approximate all the third- and fourth-order derivatives using 2nd-order accurate central finite difference formulas. Thus, the discrete operators for all the third-order derivatives are

\[
\begin{align*}
D_{rrr} &= D_{++}D_{--}D_{00}, & D_{rss} &= D_{+r}D_{--}D_{0s}, & D_{sss} &= D_{++}D_{-s}D_{0s}, \\
\end{align*}
\]

and those for all the fourth-order derivatives are

\[
\begin{align*}
D_{rrrr} &= (D_{++}D_{--})^2, & D_{rrss} &= D_{+r}D_{--}D_{00}D_{0s}, & D_{rrss} &= D_{++}D_{-s}D_{+s}D_{-s}, \\
D_{rsss} &= D_{0r}D_{0s}D_{+s}D_{-s}, & D_{ssss} &= (D_{++}D_{-s})^2.
\end{align*}
\]

Note that the difference operators defined in (7) and (8) utilize a stencil that takes 5 points in each direction of the \((r, s)\)-space as is shown in Figure 2.

![Figure 2: Stencil of the finite-difference operators.](image-url)
For the lower order (1st or 2nd) derivatives in (6), we have the freedom to approximate the derivatives either maintaining the same 2nd-order accuracy by taking a smaller stencil, or maintaining the same stencil size as all the other difference operators in (7) and (8) by implementing 4th-order accurate schemes. Specifically, the finite difference operators for the first and second order derivatives can be defined by

\[
D_r = D_{0r} - \delta \frac{h^2}{6} D_{0r} D_{+r} D_{-r}, \quad D_s = D_{0s} - \delta \frac{h^2}{6} D_{0s} D_{+s} D_{-s}, \quad (9)
\]

\[
D_{rr} = D_{+r} D_{-r} - \delta \frac{h^2}{12} (D_{+r} D_{-r})^2, \quad D_{ss} = D_{+s} D_{-s} - \delta \frac{h^2}{12} (D_{+s} D_{-s})^2, \quad D_{rs} = D_r D_s. \quad (10)
\]

These formulas are 2nd-order accurate if \( \delta = 0 \) (same order), and are 4nd-order accurate if \( \delta = 1 \) (same stencil); see Figure 2 for the stencils for \( \delta = 0 \) and \( \delta = 1 \).

Using the previously defined finite-difference operators, we are now ready to present the discrete equations for approximating (6) on a component grid, given by:

\[
\rho H \frac{d^2 W_{ij}}{dt^2} = \mathcal{L}_h W_{ij} + F_{ij}, \quad (11)
\]

where the discrete plate operator \( \mathcal{L}_h \) is given by

\[
\mathcal{L}_h W_{ij} = -Kw + a_1(r) D_r W_{ij} + a_2(r) D_s W_{ij} + b_{11}(r) D_{rr} W_{ij} + b_{12}(r) D_{rs} W_{ij} + b_{22}(r) D_{ss} W_{ij} + c_{111}(r) D_{rrr} W_{ij} + c_{112}(r) D_{rrs} W_{ij} + c_{122}(r) D_{rsv} W_{ij} + c_{222}(r) D_{sss} W_{ij} + d_{1111}(r) D_{rrrr} W_{ij} + d_{1112}(r) D_{rrrs} W_{ij} + d_{1212}(r) D_{rrss} W_{ij} + d_{1222}(r) D_{rass} W_{ij} + d_{2222}(r) D_{ssss} W_{ij}.
\]

Given the order of accuracy for all the difference operators defined in (7) – (10), we expect the truncation error of (11) to be 2nd-order.

If a physical boundary is present on this component grid, the discretized boundary condition can be readily derived by replacing the derivatives with the corresponding finite-difference approximations.

### 3.3. Time-stepping schemes

For numerical purposes, we propose to modify (11) by adding an artificial hyper-dissipation term. The modified plate equation is given by

\[
\frac{d^2 W_{ij}}{dt^2} = \frac{1}{\rho H} \left( \mathcal{L}_h W_{ij} + F_{ij} \right) + \nu_{ad} D_{ad} \frac{dW_{ij}}{dt}, \quad (12)
\]

where \( \nu_{ad} \) is the dissipation parameter and \( D_{ad} = -h^4 D_{rrrr} - h^4 D_{ssss} \) is the difference operator for the artificial hyper-dissipation. The dissipation term, inspired by the wave equation for overlapping grids [35, 36, 40], incorporates the essential upwinding effect into the problem and is indispensable for stabilizing the plate simulations on composite overlapping grids. It is important to point out that with \( \nu_{ad} = \mathcal{O}(1/h^2) \) the dissipation term is formally \( \mathcal{O}(h^2) \) so that its addition does not affect the overall accuracy of the spatial discretization; here \( h = \min(h_r, h_s) \).

In this paper, we propose the following time-stepping schemes to advance the modified discrete Kirchhoff-Love plate equations (12) in time. Let \( W^n_i, V^n_i, \) and \( A^n_i \) denote the numerical solutions of the plate displacement, velocity and acceleration at time \( t_n \), respectively. Here \( i = (i, j) \) is a multi-index for the grid point, and \( t_n = n\Delta t \) with a fixed time-step \( \Delta t \). The aim of the following time-stepping algorithms is to determine the numerical solutions at a new time given solutions at previous time levels. If boundary conditions for velocity or acceleration need to be applied by a particular scheme, they can be derived by taking appropriate time derivatives of the displacement boundary conditions given in (2) – (4).

Solving (12) in the second-order form directly leads to the algorithms referred to by us as the C2 and UPC2 schemes, which are listed in Algorithms 1 and 2, respectively. Both schemes advance the system in time by approximating the second-order time derivative with the second-order centered finite-difference formula (C2). The difference between C2 and UPC2 lies in the treatment of \( \partial_t W_i \) in the artificial dissipation term. In particular, we approximate \( \partial_t W_i \) with a backward time difference for the C2 scheme, while we include the upwind dissipation term using a predictor-corrector scheme for the UPC2 scheme, so that both
Algorithm 1: C2 time-stepping scheme

**Input:** displacement at two previous time levels; i.e., $W^n_i$ and $W^{n-1}_i$

**Output:** new displacement $W^{n+1}_i$, and current velocity and acceleration $V^n_i, A^n_i$

**Procedures:**

Advance the equation using centered time difference scheme

$$A^n_i = \frac{1}{\rho H} \left( L_h W^n_i + F^n_i \right)$$

$$W^{n+1}_i = 2W^n_i - W^{n-1}_i + \Delta t^2 A^n_i + \Delta t \nu_{ad} \left( D_{ad} W^n_i - D_{ad} W^{n-1}_i \right)$$  \hspace{1cm} (13)

$$V^n_i = \frac{W^{n+1}_i - W^{n-1}_i}{2\Delta t}$$

**Remark:** $V^n_i$ and $A^n_i$ are obtained from post-processing the displacement solutions, which are not involved in the main updating formula (13); therefore, boundary conditions are enforced for $W^{n+1}_i$ only to fill in the values at ghost and/or boundary grid points. Interpolation routines are called at the end of each step to fill in solutions for the interpolation points of composite overlapping grids.

Algorithm 2: UPC2 time-stepping scheme

**Input:** displacement at two previous time levels; i.e., $W^n_i$ and $W^{n-1}_i$

**Output:** new displacement $W^{n+1}_i$, and current velocity and acceleration $V^n_i, A^n_i$

**Procedures:**

**Stage I:** predict the solution using C2 scheme without artificial dissipation

$$A^n_i = \frac{1}{\rho H} \left( L_h W^n_i + F^n_i \right)$$

$$W^p_i = 2W^n_i - W^{n-1}_i + \Delta t^2 A^n_i$$  \hspace{1cm} (14)

**Stage II:** correct the solution with the artificial dissipation

$$W^{n+1}_i = W^p_i + \frac{1}{2} \Delta t \nu_{ad} \left( D_{ad} W^p_i - D_{ad} W^{n-1}_i \right)$$  \hspace{1cm} (15)

$$V^n_i = \frac{W^{n+1}_i - W^{n-1}_i}{2\Delta t}$$

**Remark:** $V^n_i$ and $A^n_i$ are obtained from post-processing the displacement solutions, which are not involved in the main updating formulas (14) and (15); therefore, boundary conditions are enforced for $W^{n+1}_i$ only to fill in the values at ghost and/or boundary grid points. Interpolation routines are called at the end of both stages to fill in solutions for the interpolation points of composite overlapping grids.

Schemes remain explicit with or without the artificial dissipation. But how we deal with the dissipation leads to different time step restrictions that is to be discussed in Section 4.2.

We have previously developed two numerical methods referred to as the PC22 and NB2 schemes to solve a generalized Kirchhoff-Love model written in first order form [13]. However, both schemes become unstable if applied directly for solving the plate equation on composite overlapping grids due to the presence of interpolating equations in the spatial discretization. In this paper, we propose to stabilize the schemes by
solving the first order form of the modified plate equation (12),

\[
\begin{align*}
\frac{dW_{ij}}{dt} &= V_{ij} \\
\frac{dV_{ij}}{dt} &= A_{ij} + \nu_{ad} \mathcal{D}_{ad} V_{ij} \\
\end{align*}
\]

with \( A_{ij} = \frac{1}{\rho H} (\mathcal{L}_h W_{ij} + F_{ij}) \).

(16)

For each step of the PC22 time-stepping scheme, the algorithm advances (16) by taking a second-order accurate Adams-Bashforth predictor, followed by a second-order Adams-Moulton corrector. Specifically, the scheme is summarized in Algorithm 3.

**Algorithm 3: PC22 time-stepping scheme**

**Input:** solutions at two previous time levels; i.e., \((W^n_i, V^n_i, A^n_i)\) and \((W^{n-1}_i, V^{n-1}_i, A^{n-1}_i)\)

**Output:** solutions at the new time level; i.e., \((W^{n+1}_i, V^{n+1}_i, A^{n+1}_i)\)

**Procedures:**

*Stage I: predict solutions using a second-order Adams-Bashforth (AB2) predictor*

\[
W^p_i = W^n_i + \Delta t \left( \frac{3}{2} V^n_i - \frac{1}{2} V^{n-1}_i \right) \\
V^p_i = V^n_i + \Delta t \left[ \frac{3}{2} (A^n_i + \nu_{ad} \mathcal{D}_{ad} V^n_i) - \frac{1}{2} (A^{n-1}_i + \nu_{ad} \mathcal{D}_{ad} V^{n-1}_i) \right] \\
A^p_i = \frac{1}{\rho H} \left( \mathcal{L}_h W^p_{ij} + F^{n+1}_{ij} \right).
\]

*Stage II: correct solutions using a second-order Adams-Moulton (AM2) corrector*

\[
W^{n+1}_i = W^n_i + \Delta t \left( \frac{1}{2} V^n_i + \frac{1}{2} V^p_i \right) \\
V^{n+1}_i = V^n_i + \Delta t \left[ \frac{1}{2} (A^n_i + \nu_{ad} \mathcal{D}_{ad} V^n_i) + \frac{1}{2} (A^p_i + \nu_{ad} \mathcal{D}_{ad} V^p_i) \right] \\
A^{n+1}_i = \frac{1}{\rho H} \left( \mathcal{L}_h W^{n+1}_{ij} + F^{n+1}_{ij} \right).
\]

Remark: Boundary conditions are applied after both the predictor and corrector stages to fill in the solutions of \(W\) and \(V\) at ghost and/or boundary grid points. We do not apply boundary conditions for \(A\) since it is derived from post-processing \(W\) solutions and is not part of the main updating formulas. Interpolation routines are called at the end of both stages to fill in solutions for the interpolation points of composite overlapping grids.

The NB2 scheme takes advantage of a second-order accurate version of the well-known Newmark-Beta scheme [41] and implements it in a predictor-corrector fashion. The complete algorithm for the NB2 scheme is given in Algorithm 4.
Algorithm 4: NB2 time-stepping scheme

**Input:** solutions at the previous time level; i.e., \((W_n^i, V_n^i, A_n^i)\)

**Output:** solutions at the new time level; i.e., \((W_n+1^i, V_n+1^i, A_n+1^i)\)

**Procedures:**

Stage I. compute a first-order prediction for displacement and velocity

\[
W_p^i = W_n^i + \Delta t V_n^i + \frac{\Delta t^2}{2} (1 - 2\beta) A_n^i \\
V_p^i = V_n^i + \Delta t (1 - \gamma) A_n^i
\]

Stage II. solve a system of equations for acceleration at \(t_{n+1}\)

\[
(\rho H - \beta \Delta t^2 \mathcal{L}_h - \gamma \Delta t \nu_{ad} \mathcal{D}_{ad}) A_{n+1}^i = \mathcal{L}_h W_p^i + F_{n+1}^i + \nu_{ad} \mathcal{D}_{ad} V_p^i
\]

Stage III. correct the displacement and velocity using the new acceleration solution

\[
W_{n+1}^i = W_n^i + \Delta t V_n^i + \frac{\Delta t^2}{2} [(1 - 2\beta) A_n^i + 2\beta A_{n+1}^i] \\
V_{n+1}^i = W_n^i + \Delta t [(1 - \gamma) a_n^i + \gamma A_{n+1}^i]
\]

**Remark:** A second-order accurate Newmark-Beta scheme with \(\beta = 1/4\) and \(\gamma = 1/2\) is used here. Boundary conditions are applied after stages I and III to fill in the solutions of \(W\) and \(V\) at ghost and/or boundary grid points; for composite overlapping grids, interpolation routines are called as well. For stage II, equations for acceleration at ghost and boundary nodes are given by boundary conditions, while equations at the interpolation points are given by a 4th order polynomial interpolation formula.
4. Stability Analysis

Strategies for determining the dissipation coefficient $\nu_{\text{ad}}$ in (12) and a stable time step $\Delta t$ for each of the algorithms proposed in Section 3.3 are discussed here. Formulas for curvilinear grids are derived in the usual way by freezing coefficients and using a local Fourier analysis.

To be specific, the discrete Fourier transformation of the homogeneous version of (12) for a given wave number pair $\omega = (\omega_r, \omega_s)$ is found to be

$$
\frac{d^2\hat{W}_\omega}{dt^2} = \frac{1}{\rho H} \hat{Q}(\xi_r, \xi_s; \mathbf{r})\hat{W}_\omega - \nu_{\text{ad}} \left(16 \sin^4(\xi_r/2) + 16 \sin^4(\xi_s/2)\right) \frac{d\hat{W}_\omega}{dt},
$$

where $\hat{Q}(\xi_r, \xi_s; \mathbf{r})$ denotes the Fourier transform (symbol) of the discrete operator $\mathcal{L}_h$, and $\hat{W}_\omega$ is the Fourier coefficient of $W_{jk}$ that is defined by $W_{jk} = \hat{W}_\omega e^{i\xi_r h_r} e^{i\xi_s h_s}$ with $\xi_r = 2\pi \omega_r h_r$ and $\xi_s = 2\pi \omega_s h_s$. The specific expression of $\hat{Q}(\xi_r, \xi_s; \mathbf{r})$ is given in Appendix A.2 to save space. A particular time-stepping scheme is stable for solving (12) provided it is stable for (17) $\forall (\xi_r, \xi_s) \in [-\pi, \pi]^2$ and $\forall \mathbf{r} \in [0, 1]^2$. A sufficient condition for the stability of the scheme can be derived by considering the worst case scenario of (17).

With a slight abuse of notations, we consider the following test problem to study the stability issues for solving the plate problem,

$$
\frac{d^2w}{dt^2} = Qw - \mu \frac{\partial w}{\partial t} \quad \text{with} \quad \mu \geq 0,
$$

where $Q = \hat{Q}_M/(\rho H)$ such that $|\hat{Q}_M| = \max(|\hat{Q}(\xi_r, \xi_s; \mathbf{r})|)$ and $\mu = 32 \nu_{\text{ad}}$. Note that $\hat{Q}_M$ and $\mu$ are attained at $\xi_r = \pm \pi$ and $\xi_s = \pm \pi$ which correspond to the plus-minus modes in each direction of the $(r, s)$-space.

4.1. Dissipation coefficient determination

First, we look at the test problem (18) analytically, whose general solution is given by

$$
w(t) = c_1 e^{\xi_r t} + c_2 e^{\xi_s t} \quad \text{with} \quad \xi_s = \frac{-\mu \pm \sqrt{\mu^2 + 4Q}}{2}.
$$

Here $c_1$ and $c_2$ are constants to be determined by the initial conditions. Since the plate operator $\mathcal{L}$ in (6) is self-adjoint, the Fourier transform (or symbol) of the corresponding discrete operator $\mathcal{L}_h$ on a single grid should be a real negative number (c.f., Appendix A.2). In this case, the test problem is well-posed in the sense that the solution (19) is bounded over time.

However, on composite overlapping grids, the symmetry of the differentiation matrix associated with $\mathcal{L}_h$ is spoiled by the interpolating equations. As a result, for some modes, the symbol of the differential matrix, is perturbed off the real axis that can cause the solution (19) to grow exponentially if there is no sufficient dissipation. Thus, we investigate imposing an artificial hyper-dissipation to stabilize the perturbation induced by interpolation. For this purpose, we consider the test problem (18) with perturbed $Q$. In particular, we assume $Q = m + in$, where $m = Q_M < 0$ and $n \neq 0$ is induced by perturbation of interpolation. In this case, $\xi_s$ in the general solution (19) can be written as

$$
\xi_s = -\mu \pm \sqrt{r e^{i\theta}/2} = -\mu \pm \frac{\sqrt{r \cos(\theta/2) + i\sqrt{r} \sin(\theta/2)}}{2},
$$

where

$$
r = \sqrt{(\mu^2 + 4m)^2 + 16n^2}, \quad \theta = \arctan 2 \left(4n, \mu^2 + 4m\right).
$$

An artificial dissipation is sufficient to stabilize the problem if $\Re(\xi_s) \leq 0$; therefore, a lower bound for $\mu$ can be derived from this restriction that is given by

$$
\mu \geq \sqrt{r} |\cos(\theta/2)|.
$$

It is not straightforward to use (21) to determine the strength of dissipation to be added in the algorithms since it is hard to get an estimate for $n$ and $\theta$ without numerically computing the eigenvalues, which can be very expensive computationally.
On the other hand, we do not want the dissipation to be too strong to damp out the oscillatory nature of the solution (19), so we restrict \( \mu^2 + 4m < 0 \). Given that \( m = Q_M < 0 \) and \( \mu = 32\nu_{ad} \), we may specify the dissipation coefficient as

\[
\nu_{ad} = C_{df} \sqrt{-\frac{Q_M}{16}},
\]

where \( C_{df} \in [0, 1] \) is a dissipation factor that can be used to control the strength of the artificial dissipation in practice.

4.2. Time step determination

We now analyze the stability for the various time-stepping methods using the test problem (18) with perturbation \( (Q = m + in) \) and dissipation taken into consideration. An estimate of a stable time step \( \Delta t \) is obtained for each time-stepping scheme following the stability analysis.

C2 scheme.

The test problem if advanced with the C2 scheme leads to the following difference equation,

\[
\frac{w^{n+1} - 2w^n + w^{n-1}}{\Delta t^2} = Qw^n - \mu \frac{w^n - w^{n-1}}{\Delta t},
\]

whose characteristic equation is given by

\[
\zeta^2 - (2 + \Delta t^2Q - \mu\Delta t)\zeta + (1 - \mu\Delta t) = 0.
\]

The stability of the scheme depends on the root condition \( |\zeta_\pm| \leq 1 \).

Without dissipation, we find that the scheme is nondissipative \( (|\xi_\pm| = 1) \) and works only for the unperturbed case \( (Q = Q_M) \) subject to a time step restriction given by

\[
\Delta t \leq \frac{2}{\sqrt{-Q_M}}.
\]

With dissipation, the time step constraint derived from solving the root condition \( |\zeta_\pm| \leq 1 \) is

\[
\Delta t \leq \min \left\{ \frac{4\sqrt{2}}{\mu + \sqrt{\mu^2 - 8\sqrt{2}Q_M}}, \frac{2}{\mu} \right\},
\]

where the assumption that the perturbation induced imaginary part is much smaller in magnitude (i.e., \( |n| \ll |m| \)) has been used.

Noting that \( m = Q_M \), we propose the following formula for determining \( \Delta t \) in Algorithm 1,

\[
\Delta t = \begin{cases} 
C_{sf} \frac{2}{\sqrt{-Q_M}}, & \text{if } \mu = 0, \\
C_{sf} \min \left\{ \frac{4\sqrt{2}}{\mu + \sqrt{\mu^2 - 8\sqrt{2}Q_M}}, \frac{2}{\mu} \right\}, & \text{if } \mu \neq 0.
\end{cases}
\]

Here \( C_{sf} \in (0, 1] \) is a stability factor (sf) that multiplies an estimate of the largest stable time step based on the above analysis, and \( \mu = 32\nu_{ad} \) with the dissipation coefficient defined in (22).

UPC2 scheme.

The UPC2 scheme applied to the test problem leads to

\[
\frac{w^{p} - 2w^n + w^{n-1}}{\Delta t^2} = Qw^n, \\
\frac{w^{n+1} - w^p}{\Delta t^2} = -\mu \frac{w^p - w^{n-1}}{2\Delta t}.
\]
Combining the predictor and corrector steps by eliminating \( w^p \), the UPC2 scheme is essentially a 3-step scheme given by
\[
\begin{align*}
w^{n+1} - (1 - \frac{1}{2} \mu \Delta t)(2 + \Delta t^2 Q) w^n + (1 - \mu \Delta t) w^{n-1} &= 0.
\end{align*}
\]
Similarly, the root condition \( |\zeta| \leq 1 \) for the characteristic equation implies the following time step restriction
\[
\Delta t \leq \min \left\{ \frac{2}{\sqrt{-m}}, \frac{2}{\mu} \right\}.
\]
Therefore, we use the following time step for Algorithm 2,
\[
\Delta t = C_{sf} \min \left\{ \frac{2}{\sqrt{-Q_M}}, \frac{2}{\mu} \right\},
\]
where \( C_{sf} \in (0, 1] \) and \( \mu = 32 \nu_{ad} \) with \( \nu_{ad} \) given in (22).

**PC22 scheme.**

The PC22 scheme solves the test problem in the first-order form,
\[
\frac{d}{dt} u = A u \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ Q & -\mu \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} w \\ v \end{bmatrix}.
\]
The time-stepping scheme is stable for the first order system provided the scheme is stable for all the eigenvalues of \( A \). The 3-step updating formula for the PC22 scheme is found to be
\[
u^{n+1} = (1 + z + \frac{3}{4} z^2) \nu^n + \frac{1}{4} z^2 \nu^{n-1} = 0,
\]
where \( z = \Delta t \lambda_A \) and \( \lambda_A = \left( -\mu \pm \sqrt{\mu^2 + 4Q} \right) / 2 \) are the eigenvalues of \( A \).

Following [13], by requiring \( z \) to be inside a super-ellipse that approximates the stability region of the scheme, a stable time step for Algorithm 3 can be derived; that is,
\[
\Delta t = C_{sf} \left( \left| \mu \right| \frac{4}{2^\frac{1}{2} r_a} + \left| \frac{\sqrt{-\mu^2 - 4Q_M}}{2 r_b} \right| \right)^{\frac{1}{2}} \quad \text{with} \quad r_a = 1.75, \ r_b = 1.2.
\]
Similarly, we may take \( C_{sf} \in (0, 1] \) and \( \mu = 32 \nu_{ad} \) with \( \nu_{ad} \) given in (22).

**NB2 scheme.**

The NB2 scheme for the test problem (18) can be written as
\[
\begin{bmatrix} w^{n+1} \\ v^{n+1} \end{bmatrix} = A^{-1} B \begin{bmatrix} w^n \\ v^n \end{bmatrix}
\]
where
\[
A = \begin{bmatrix} 1 - Q \beta \Delta t^2 & \beta \Delta t^2 \mu \\ -Q \Delta t \gamma & \Delta t \gamma + 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 - Q \Delta t^2 (2 \beta - 1) & \frac{\mu (2 \beta - 1) \Delta t^2}{4} + \Delta t \\ -Q \Delta t (\gamma - 1) & \Delta t \mu \gamma (\gamma - 1) + 1 \end{bmatrix}.
\]
The scheme is stable if \( |\text{eig}(A^{-1} B)| \leq 1 \). Noting that \( \beta = 1/4 \) and \( \gamma = 1/2 \) for the NB2 scheme, the eigenvalues of \( A^{-1} B \) if \( \mu = 0 \) are found to be
\[
\lambda_1 = \frac{-\frac{3Q \Delta t^2}{2} - 2 \Delta t \sqrt{\frac{9Q^2 \Delta t^2}{16} + 4Q + 4} + 4Q + 4}{4 \left( \frac{Q \Delta t^2}{4} - 1 \right)} \quad \text{and} \quad \lambda_2 = \frac{-2 \Delta t \sqrt{\frac{9Q^2 \Delta t^2}{16} + 4Q + 3Q \Delta t^2 + 4}}{4 \left( \frac{Q \Delta t^2}{4} - 1 \right)}.
\]
We can see that the NB2 scheme is nondissipative since $|\lambda_1| = |\lambda_2| = 1$ if there is no perturbation and no artificial dissipation added, i.e., $Q = Q_M < 0$ and $\mu = 0$. As a result, any perturbation caused by interpolation introduces weak instability to the scheme (see the next section for more details on weak instability). Given sufficient artificial dissipation to suppress the weak instability, the NB2 scheme is implicit in time and stable for any time step. For accuracy reasons, we choose its time step based on the condition for the explicit PC22 scheme (25), but with a much larger stability factor. Typically, we choose $C_{sf} \in [1, 50]$ in applications.

For convenience, we summarize the time step and dissipation formulas in Table 1. Except for the implicit NB2 scheme, all the other schemes are explicit. We normally set $C_{sf} = 0$ for the explicit schemes. From our experience, it is generally sufficient to set $C_{df} = 0.1$ to damp out the weak instability caused by the inclusion of interpolating equations in the discretization on composite overlapping grids.

| Name  | Formula | Comments |
|-------|---------|----------|
| C2    | $\Delta t = \begin{cases} 2 \sqrt{-Q_M} & \text{if } \mu = 0 \\ C_{sf} \min \left\{ \frac{4\sqrt{2}}{\mu + \sqrt{\mu^2 - 8\sqrt{2}Q_M}}, \frac{2}{\mu} \right\} & \text{if } \mu \neq 0 \end{cases}$ | Without dissipation, the scheme is nondissipative and works only for the single grid. With dissipation, the scheme has a reduced time-step. We choose $C_{sf} \in (0, 1]$. |
| UPC2  | $\Delta t = C_{sf} \min \left\{ \frac{2}{\sqrt{-Q_M}}, \frac{2}{\mu} \right\}$ | The scheme has no reduction in time-step with artificial dissipation. We choose $C_{sf} \in (0, 1]$. |
| PC22  | $\Delta t = C_{sf} \left( \frac{\mu^{\frac{3}{2}}}{2r_a} + \sqrt{-\mu^2 - 4Q_M} \right) \left( \frac{2}{2r_b} \right)^{\frac{3}{2}}$ | The radii of the super ellipse are $r_a = 1.75, r_b = 1.2$. We choose $C_{sf} \in (0, 1]$. |
| NB2   | $\Delta t = C_{sf} \left( \frac{\mu^{\frac{3}{2}}}{2r_a} + \sqrt{-\mu^2 - 4Q_M} \right) \left( \frac{2}{2r_b} \right)^{\frac{3}{2}}$ | The time step is chosen to be the same as the PC22 scheme but with a larger $C_{sf}$; typically, we choose $C_{sf} \in [1, 50]$ for accuracy reasons. |
| Dissipation | $\mu = 32\nu_{ad}$ with $\nu_{ad} = C_{df} \sqrt{-Q_M} / 16$ | $C_{df} \in [0, 1]$ is the dissipation factor used to control the strength of the artificial dissipation. |

5. Model Problem in 1D

A simplified model plate that consists of bending only in 1D is considered here to demonstrate the numerical properties of the proposed schemes. On a simple composite overlapping grid, we illustrate the weak instability caused by the presence of interpolating equations, and investigate the stabilizing effects of the various strategies of incorporating artificial dissipation to the system.

5.1. Problem setup

Specifically, the model equation reads

$$\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^4 w}{\partial x^4}, \quad x \in [0, 1].$$

(26)
and the corresponding boundary conditions at the end points \((x = 0, x = 1)\) are given by

\[
\text{clamped : } \quad w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \tag{27}
\]
\[
\text{supported : } \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \tag{28}
\]
\[
\text{free : } \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial x^3} = 0. \quad \tag{29}
\]

Note that in 1D domain, the Kirchhoff-Love plate is the same as the well-known Euler-Bernoulli beam model.

To demonstrate the weak instability due to the presence of interpolating equations in the discretized system, we consider the model equation (26) subject to the supported boundary condition (28) on both ends of the domain as an example, and discretize the IBVP using both a single grid \((G_s)\) and a composite overlapping grid \((G_c = G^{(1)} \cup G^{(2)})\) for comparison; a diagram for the mesh setup is shown in Figure 3. For the composite grid, interpolation points of each component grid are identified using solid-square markers, and are connected with their donor points from the other component grid. Note that, for consistency with the finite-difference discretization, a five-point stencil is used for interpolation (i.e., 4th-order polynomial interpolation). We also investigate the stabilizing effects in all the algorithms introduced in Section 3 using this model problem. Thus, the artificial dissipation term is incorporated into the simplified model problem (26), and are spatially discretized on the grids \(G_s\) and \(G_c\) as well.

Let \(W_i(t)\) be the approximation of \(w(x_i, t)\), then the discrete system on \(G_s\) is given by

\[
\frac{d^2 W_i}{dt^2} = -(D_{+x} D_{-x})^2 W_i - \nu_{ad} h_1^4 (D_{+x} D_{-x})^2 \frac{dW_i}{dt}, \quad i = 1, 2, \ldots, N - 1; \quad j = 1, 2,
\]
\[
W_{i_b} = D_{+x} D_{-x} W_{i_b} = 0, \quad i_b = 0, N,
\]
\[
D_{+x}^5 W_{i_g} = 0, \quad i_g = -2, N + 2.
\]

The last equation corresponds to the fifth-order extrapolation for the second ghost line, where the extrapolation operator is either \(D_{+x}^5\) or \(D_{-x}^5\) so that it extrapolates into the interior of the grid.

On the composite overlapping grid \(G_c\) and denoting \(W_i^{(j)} \approx w \left( x_i^{(j)}, t \right)\), the model problem is discretized.
\[
\frac{d^2 W^{(j)}_i}{dt^2} = -(D_{xx} D_{xx})^2 W^{(j)}_i - \nu_{ad}h_x^4 (D_{xx} D_{xx})^2 \frac{dW^{(j)}_i}{dt}, \quad i = 1, 2, \ldots, N_j - 1; j = 1, 2,
\]
\[
W^{(j)}_{ib} = D_{xx} D_{xx} W^{(j)}_{ib} = 0, \quad i_b = 0 \text{ if } j = 1; \quad i_b = N_2 \text{ if } j = 2,
\]
\[
D_{xx} W^{(j)}_{ig} = 0, \quad i_g = -2 \text{ if } j = 1; \quad i_g = N_2 + 2 \text{ if } j = 2,
\]
\[
W^{(1)}_{ip} - \sum_{l=1}^{5} c_{2,l}(i_p) W^{(2)}_{d_{2,l}(i_p)} = 0, \quad i_p = N_1 + 1, N_1 + 2,
\]
\[
W^{(2)}_{ip} - \sum_{l=1}^{5} c_{1,l}(i_p) W^{(1)}_{d_{1,l}(i_p)} = 0, \quad i_p = -1, -2.
\]

Here the last two conditions are the interpolating equations that couple the numerical solutions on the two component grids together. In the interpolating equations, \(d_{j,l}(i_p)\) for \(l = 1, \ldots, 5\) represent the indices of the donor points on \(G^{(j)}\) for interpolating the grid function at \(x^{(k)}_{ip}\) on the other component grid \(G^{(k)}\); the corresponding interpolating coefficients are denoted by \(c_{j,l}(i_p)\). The relationship between the interpolation points and their donor points are schematically depicted in Figure 3.

If we denote \(W\) as the solution vector, where \(W = [W_{-2}, \ldots, W_{N+2}]^T\) for the single grid case and \(W = [W^{(1)}_{-2}, \ldots, W^{(1)}_{N_1+2}, W^{(2)}_{-2}, \ldots, W^{(2)}_{N_2+2}]^T\) for the composite grid case, then the above spatially discretized systems, including the boundary and interpolation conditions, can be concisely written into the following matrix form,

\[
M_t \frac{d^2 W}{dt^2} = M_L W + \nu_{ad} M_{ad} \frac{dW}{dt}.
\]  

(30)

Here \(M_t\) is identity matrix except for the rows corresponding to the ghost, boundary and interpolation points, which are set to be zero to accommodate the boundary and interpolation conditions. Following (22), the dissipation coefficient \(\nu_{ad}\) for this 1D model problem is specifically given by

\[
\nu_{ad} = C_{ad} \frac{1}{2h_x^2}.
\]  

(31)

5.2. Stability issues

For the purpose of demonstrating the stability issues of the various time-stepping methods introduced in Section 3.3, we consider the spatially discretized system (30) obtained from the one-dimensional mesh setup shown in Figure 3 with \(N_1 = 15\), \(N_2 = 9\) and \(N = 20\).

C2 and UPC2 schemes.

The matrix equation (30), if solved with the C2 or UPC2 scheme, leads to the following general updating formula,

\[
Q_2 W^{n+1} + Q_1 W^n + Q_0 W^{n-1} = 0,
\]  

(32)

where \(W^n\) is the numerical solution at time level \(t_n\). The coefficient matrices in the updating formula (32) for the C2 scheme are given by

\[
Q_2 = M_t, \quad Q_1 = -2M_t - \Delta t^2 M_L - \Delta t \nu_{ad} M_{ad}, \quad Q_0 = M_t + \Delta t \nu_{ad} M_{ad},
\]

while those for the UPC2 scheme are given by

\[
Q_2 = M_t, \quad Q_1 = - \left( M_t + \frac{\Delta t}{2} \nu_{ad} M_{ad} \right) \left( 2M_t + \Delta t^2 M_L \right), \quad Q_0 = M_t + \Delta t \nu_{ad} M_{ad}.
\]

Note that C2 and UPC2 schemes are the same scheme if no artificial dissipation is added (i.e., \(\nu_{ad} = 0\)).

The solution of the difference equation (32) is related to the quadratic eigenvalue problem (also known as the characteristic equation),

\[
Q_2 \lambda^2 + Q_1 \lambda + Q_0 = 0,
\]  

(33)
which is derived from seeking separable solutions of the form $W^n = \lambda^n W_0$. The stability of the time-stepping scheme requires that $|\lambda| \leq 1$ for all the eigenvalues of (33). Therefore, in Figure 4, we plot the numerically computed eigenvalues for the schemes with and without artificial dissipation on a complex plane together with a unit circle $|z| = 1$ as reference; a scheme would be unstable if eigenvalues are observed outside of the unit circle.

![Figure 4: Eigenvalues of C2 scheme with no artificial dissipation (left), C2 scheme with artificial dissipation (middle), UPC2 scheme (right).](image)

If no artificial dissipation is included in the scheme, the C2 scheme is non-dissipative and stable on the single grid $G_s$; this is seen in the left image of Figure 4 that all the eigenvalues for the single grid case remain on the unit circle. Due to the perturbation caused by the interpolating equations, we can see in the same image that, for the composite grid case, there are complex conjugate pairs of eigenvalues with one pair lying inside the unit circle and the other outside that causes weak instability for the scheme.

The weak instability can be stabilized by including artificial dissipation in an explicit manner as the C2 scheme described in Algorithm 1 or in an upwind predictor-corrector approach as the UPC2 scheme described in Algorithm 2. For both schemes, setting the dissipation $\nu_{ad}$ in (31) with $C_{df} = 0.1$ is sufficient for stabilization. The eigenvalues for the C2 scheme with dissipation and the UPC2 scheme are shown in the middle and right images of Figure 4, respectively. It is clear that all the eigenvalues, especially the high frequency modes, are damped so that no more unstable modes are observed outside the unit circle.

### PC22 scheme.

To solve (30) with the PC22 time-stepping scheme, we need to convert the equation into the first-order form,

$$\mathcal{I} \frac{dU}{dt} = \mathcal{M}U,$$

(34)

where

$$U = \begin{bmatrix} W \\ V \end{bmatrix}, \quad \mathcal{I} = \begin{bmatrix} M_I & 0 \\ 0 & M_I \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} 0 & M_v \\ M_L & \nu_{ad} M_{ad} \end{bmatrix}.$$

Here $M_I$ equals the identity matrix except for the rows corresponding to the ghost, boundary and interpolation points, which are given by the boundary conditions of the velocity component. Note that the boundary conditions for velocity are derived from the time derivative of the displacement ones.

Integrating (34) using the PC22 time-stepping scheme leads to a three step updating formula,

$$Q_2 U^{n+1} + Q_1 U^n + Q_0 U^{n-1} = 0,$$

where

$$Q_2 = \mathcal{I}, \quad Q_1 = -\mathcal{I} - \Delta t \mathcal{M} - \frac{3}{4} \Delta t^2 \mathcal{M}^2, \quad Q_0 = \frac{1}{4} \Delta t^2 \mathcal{M}^2.$$

The stability issues and the stabilizing effects of this scheme are then investigated by solving the corresponding eigenvalue problem with and without artificial dissipation; the results are shown in Figure 5.
From the results for $\nu_{ad} = 0$ that are shown in the left image of Figure 5, we can see that PC22 is a dissipative scheme, and it is stable for solving the model problem on the single grid. However, the dissipation of the scheme itself is not enough to stabilize the algorithm applied on the composite overlapping grid $G_c$; extra artificial dissipation are essential for stability. With $C_{df} = 0.1$, it is then observed in the right image of Figure 5 that the unstable modes are further damped by the artificial dissipation such that the corresponding eigenvalues fall inside of the unit circle.

**Figure 5: Eigenvalues of PC22 scheme.**

**NB2 scheme.**

The NB2 scheme also solves the model problem in the first-order form (34). It is a two-step scheme and has the following updating formula,

$$Q_2 U^{n+1} + Q_1 U^n = 0,$$

where

$$Q_2 = \begin{bmatrix} M_I - \Delta t^2 \beta M_L & -\Delta t^2 \beta \nu_{ad} M_{ad} \\ -\Delta t \gamma M_L & M_I - \Delta t \gamma \nu_{ad} M_{ad} \end{bmatrix} \quad \text{and} \quad Q_1 = -\begin{bmatrix} M_I + \frac{\Delta t^2}{\Delta t (1 - \gamma)} M_L & \Delta t M_I + \frac{\Delta t^2}{\Delta t (1 - \gamma)} \nu_{ad} M_{ad} \\ M_I + \Delta t (1 - \gamma) \nu_{ad} M_{ad} & M_I + \Delta t (1 - \gamma) \nu_{ad} M_{ad} \end{bmatrix}.$$

**Figure 6: Eigenvalues of NB2 scheme.**

Computed eigenvalues for the NB2 scheme are plotted in Figure 6. We can see from the plot for $\nu_{ad} = 0$ (left image) that NB2 scheme is nondissipative and is stable for the single grid $G_s$. The presence of interpolation in the discretization formulas on the composite grid $G_c$ perturbs the eigenvalues so that some modes become unstable. In the right image of Figure 6, we demonstrate that incorporating the artificial dissipation with $C_{df} = 0.1$ is sufficient to stabilize these unstable modes.
6. Numerical results

Numerical results for a series of test problems designed to validate the properties and performances of the proposed algorithms are now presented. Mesh refinement studies using methods of manufactured solutions and problems with known analytical solutions are first conducted to verify the accuracy and stability of all the algorithms. Using a test case of a plate with multiple holes, we then demonstrate the efficiency and capability of our approach for solving problems with complicated geometrical configurations. Finally, to illustrate that the proposed algorithms can be applied in realistic applications, we explore the responding vibrations of a guitar soundboard subject to an initial pulse. Results for comparing the performance of the proposed algorithms for solving the guitar problem are also presented.

6.1. Convergence study

In this section, we conduct convergence study for a square plate using the method of manufactured solutions (MMS), a technique often used to construct exact solutions for numerical validations [42]. In particular, the forcing term in (1) is specified so that a chosen (manufactured) function becomes an exact solution to the forced equations. The accuracy of our proposed schemes are also carefully verified using a simple circular plate model, whose analytical solutions are available for certain boundary conditions.

For all the test problems with known exact solutions, errors of the numerical solutions are measured in the maximum norm. Mathematically, we define the error of the displacement and its maximum norm at time level \( t_n \) as

\[
E(w) = w_e(x_i, t_n) - W^n_i \quad \text{and} \quad ||E(w)|| = \max_{x_i \in \Omega} |E(w)|,
\]

where \( w_e \) is the exact displacement. Similar definitions for the velocity and the acceleration solutions are given accordingly.

6.1.1. Same order vs. same stencil

In the first test, we aim to settle the \( \delta \) value to be used in formulas (9) and (10) for approximating the 1st- and 2nd-order derivatives involved in the IBVP of the Kirchhoff-Love model. For this purpose, we compare the accuracy of the numerical results obtained using the same order (\( \delta = 0 \)) formulas with the same stencil (\( \delta = 1 \)) formulas. We focus on this finite-difference discretization issue by keeping the setup of this numerical test as simple as possible.

Specifically, we consider solving (1) with parameters \( \rho = 1, H = 1, K = 0, T = 0 \) and \( D = 1 \) for the following manufactured solution on a unit square domain,

\[
u_e(x, y, t) = \cos(\pi x) \cos(\pi t).
\]

(35)

To avoid distractions from other factors involved in our algorithms such as interpolation on overlapping grids and artificial dissipation, this simple manufactured solution problem is solved using the C2 scheme without any artificial dissipation on a single grid as is shown in the left image of Figure 7.

| grid | \( ||E(w)|| \) ratio | \( ||E(v)|| \) ratio | \( ||E(a)|| \) ratio |
|------|----------------|----------------|----------------|
| \( G_1 \) | 4.7e-3 | 1.7e-1 | 17.2e0 |
| \( G_2 \) | 1.3e-3 | 3.58 | 3.9e-2 | 4.34 | 11.8e0 | 1.46 |
| \( G_4 \) | 3.1e-4 | 4.23 | 8.0e-3 | 4.84 | 11.6e0 | 1.02 |
| \( G_8 \) | 7.9e-5 | 3.93 | 2.5e-3 | 3.22 | 8.6e0 | 1.35 |
| \( G_{16} \) | 1.9e-5 | 4.07 | 6.0e-4 | 4.12 | 5.0e0 | 1.70 |

| rate | 1.99 | 2.02 | 0.40 |

| grid | \( ||E(w)|| \) ratio | \( ||E(v)|| \) ratio | \( ||E(a)|| \) ratio |
|------|----------------|----------------|----------------|
| \( G_1 \) | 4.7e-3 | 1.7e-1 | 17.2e0 |
| \( G_2 \) | 1.3e-3 | 3.58 | 3.9e-2 | 4.34 | 11.8e0 | 1.46 |
| \( G_4 \) | 3.1e-4 | 4.23 | 8.0e-3 | 4.84 | 11.6e0 | 1.02 |
| \( G_8 \) | 7.9e-5 | 3.93 | 2.5e-3 | 3.22 | 8.6e0 | 1.35 |
| \( G_{16} \) | 1.9e-5 | 4.07 | 6.0e-4 | 4.12 | 5.0e0 | 1.70 |

| rate | 2.00 | 0.40 |

Table 2: Maximum-norm errors at \( t = 0.1 \) and estimated convergence rates using the manufactured solution (35). The numbers in the ratio columns provide the ratio of the errors at the current grid to that on the next coarser grid. Results shown here are obtained using the C2 scheme for the plate subject to free boundary conditions.
Mesh refinement studies for the plate with free boundaries are performed using both the same order and the same stencil formulas; maximum-norm errors for all the solution components \((w, v, a)\) at \(t = 0.1\) are collected in Table 2. For second-order-accurate schemes, we expect the ratio to be around 4, and the estimated rates to be around 2. From this simple numerical example, we observe in Table 2 that the accuracies for \(w\) and \(v\) meet the expectation regardless of the \(\delta\) values; however, the accuracy for \(a\) is greatly affected by the choice of \(\delta\). To further examine how \(\delta\) influences the acceleration solution, we plot the contours of \(a(x, y, 0.1)\) for \(\delta = 0\) and \(\delta = 1\) in the middle and right images of Figure 7, respectively. As is seen in those plots, the acceleration of the \(\delta = 0\) case appears to be very noisy and inaccurate, while that of the \(\delta = 1\) case is smooth and accurate as expected.
The exact solution (35) is intentionally specified to be independent of $y$ variable, so that we can focus on understanding the discretization issue in $x$–direction only, and treat the problem as if it were in a 1D domain. Therefore, in Figure 8, we plot all the numerical solutions along the horizontal center line of the unit square domain; i.e., $w(x,0.5,0.1)$, $v(x,0.5,0.1)$ and $a(x,0.5,0.1)$. From these plots, we can see that using the same order formulas for computing lower-order derivatives causes degradation of the acceleration accuracy for plates with supported and free boundaries. This is because, if $\delta = 0$, the second-order derivative $\partial x^2 w$ that appears in both free and supported boundary conditions are approximated to 2nd-order accuracy with a 3-point stencil in the $x$–direction; this condition helps determine the $w$ solution on the first layer of ghost points. The acceleration is related to the fourth-order derivative $\partial x^4 w$ whose accuracy relies on the smoothness of $w$ solution across a 5-point stencil. Near the boundaries, values of $w$ on the ghost points are involved in the calculation of the acceleration. If derived using the 3-point stencil, the $w$ solution on the ghost points is less smooth than the other grid points that causes a boundary-layer error in the acceleration solution. The boundary-layer error then propagates through the entire domain that deteriorates the overall accuracy of the acceleration solution. However, if the same stencil formulas ($\delta = 1$) are used instead, the smoothness of the solution is consistent across the ghost and interior points. Therefore, the acceleration accuracy remains 2nd-order that is consistent with the truncation error of the spatial discretization.

Based on the conclusion of this test, we proceed with the same stencil formulas by setting $\delta = 1$ in (9) and (10) for the rest of the numerical tests.

6.1.2. Manufactured solutions

Here we perform an exhaustive convergence study for all the numerical algorithms proposed in Section 3.3 subject to all the possible boundary conditions (2) – (4) using the method of manufactured solutions. To examine the stabilizing effects of these algorithms, we solve this problem using composite overlapping grid and artificial dissipation.

Specifically, we consider the following exact solution

$$u_e(x,y,t) = \cos(\pi x) \cos(\pi y) \cos(\pi t),$$

and solve the manufactured solution problem for a unit square plate that is discretized by a sequence of refined composite overlapping grids. The computational grid $G_2$ for this problem is shown in Figure 9. The physical parameters of the plate equation (1) are $\rho = H = K = T = D = 1$, and the Poisson’s ratio that is needed by the supported (3) and free (4) boundary conditions is $\nu = 0.3$.

In Figure 9, results of the plate with free boundaries are provided. Particularly, the errors shown in the figure are computed at $t = 0.1$ using the numerical solutions of the NB2 algorithm 4. Errors of the other three algorithms are similar, so their plots are omitted here to save space. We observe that the errors of all the solution components ($w$, $v$, $a$) are well behaved in that the magnitudes are small and they are smooth throughout the domain.

![Figure 9: Computational grids $G_2$, and the errors at $t = 0.1$ for the plate with free boundary conditions. Results shown here are computed using NB2 algorithm on the $G_{16}$ grid.](image_url)

Convergence studies of this test are presented in Figure 10. Here the manufactured solution problem is solved comprehensively with all of the proposed algorithms (C2, UPC2, PC22, NB2) subject to each of...
the boundary conditions (clamped, supported and free) on a sequence of refined meshes. The time step for each calculation is determined according to the formulas summarized in Table 1, where the stability factor is chosen to be $C_{sf} = 0.9$ for the explicit schemes (C2, UPC2, PC22) and $C_{sf} = 5$ for the implicit NB2 scheme. Since this problem is solved on composite overlapping grids, the artificial dissipation (22) with $C_{df} = 0.1$ is needed in the algorithms for stabilizing the simulations.

In Figure 10, the maximum-norm errors of all the numerical solutions are plotted against the target grid spacings in log-log scale, together with a reference line indicating 2nd-order accuracy. As are observed from these plots, 2nd-order accuracies are achieved in the solutions of $w, v$ and $a$ by all the proposed algorithms subject to any of the considered boundary conditions. The comprehensive convergence study validates the accuracy and stability properties of our approach for solving Kirchhoff-Love plate model on composite overlapping grid.

### 6.1.3. Analytical solutions for circular plates

Analytical solutions to (1) are available for circular plates with clamped (2) or supported (3) boundaries

![Figure 11: Plots of the first three transcendental functions and their smallest roots $\lambda_n$ with $n = 0, 1, 2$.](image)
We use these exact solutions to further verify the numerical properties of our algorithms for solving real plate problems as opposed to the manufactured ones in the previous tests. Given the circular domain \( \Omega = \{ x \in \mathbb{R}^2 : |x| \leq a \} \) and the parameters \( K = T = f = 0 \), analytical solutions for the Kirchhoff-Love model are derived in polar coordinates by separation of variables \( w(r, \theta, t) = R(r)\Theta(\theta)T(t) \). The general solution satisfying the boundary condition \( w = 0 \) reads,

\[
w(r, \theta, t) = A_n \left( J_n \left( \frac{\lambda r}{a} \right) - \frac{J_n(\lambda)}{I_n(\lambda)} \right) \left( \cos(n\theta) + \gamma_n \sin(n\theta) \right) \left( \sin(\omega t) + C_n \cos(\omega t) \right),
\]

where the natural frequency is \( \omega = \lambda^2 \sqrt{D/(\rho H)} \), and \( \lambda \) is related to the eigenvalue of \( \Delta^2 \), i.e., \( \Delta^2 w = \lambda^4 w \).

Figure 12: Computational grids \( G_4 \) and the initial displacements for \( n = 0,1,2 \) subject to the clamped boundary conditions. Initial conditions for the supported boundary condition are similar with slightly different scales (omitted here to save space)

The values of \( \lambda \) depend on the boundary conditions. Now, we consider deriving its values for the clamped (2) and supported (3) boundary conditions, respectively.

**Clamped.** Clamped boundary conditions in polar coordinates are

\[
w = \frac{\partial w}{\partial r} = 0.
\]

Enforcing them to the general solution (36) leads to \( J_n(\lambda)I_{n+1}(\lambda) + I_n(\lambda)J_{n+1}(\lambda) = 0 \). The values of \( \lambda \) are determined by finding the roots of the following transcendental function,

\[
\phi_n^c(\lambda) = J_n(\lambda)I_{n+1}(\lambda) + I_n(\lambda)J_{n+1}(\lambda) - \frac{2\lambda}{1-\nu} J_n(\lambda)I_n(\lambda).
\]

For each \( n \), the transcendental function (37) has infinite number of roots. For the purpose of designing numerical tests, we take the smallest root of (37) for \( n = 0,1 \) and 2 (c.f., Figure 11a.), which are

\[
\lambda_0 \approx 3.196220616582554, \quad \lambda_1 \approx 4.610899879386510, \quad \lambda_2 \approx 5.905678237243653.
\]

**Supported.** The supported boundary conditions in polar coordinates are

\[
w = \frac{\partial^2 w}{\partial r^2} + \nu \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) = 0.
\]

Similarly, enforcing them to the general solution leads to the definition of the following transcendental function,

\[
\phi_n^s(\lambda) = J_n(\lambda)I_{n+1}(\lambda) + I_n(\lambda)J_{n+1}(\lambda) - \frac{2\lambda}{1-\nu} J_n(\lambda)I_n(\lambda).
\]
Figure 13: Convergence rates for the cases with the Bessel’s functions of order $n = 0$ (top row), $n = 1$ (middle row) and $n = 2$ (bottom row).
Assuming $\nu = 0.3$, we plot the first three $\phi_n^s(\lambda)$ in Figure 11b, whose smallest roots are

$$
\lambda_0 \approx 2.221519534965056, \quad \lambda_1 \approx 3.728024285469852, \quad \lambda_2 \approx 5.060958083288190
$$

(39)

We note that the general solution (36) consists of two sets of modal functions: one with $\cos(n\theta)$ and the other with $\sin(n\theta)$. To construct a standing wave solution, we specify the following initial conditions using the modal function with $\cos(n\theta)$ only; that is,

$$
\begin{align*}
& w_0(r, \theta) = \left[J_n\left(\frac{\lambda_n r}{a}\right) - J_n(\lambda_n) I_n\left(\frac{\lambda_n r}{a}\right)\right] \cos(n\theta) \quad \text{and} \quad v_0(r, \theta) = 0.
\end{align*}
$$

(40)

The exact solution to the plate equation (1) with this choice of initial conditions is

$$
\begin{align*}
& w_e(r, \theta, t) = \left[J_n\left(\frac{\lambda_n r}{a}\right) - J_n(\lambda_n) I_n\left(\frac{\lambda_n r}{a}\right)\right] \cos(n\theta) \cos\left(\lambda_n^2 \sqrt{\frac{D}{\rho H t}}\right).
\end{align*}
$$

(41)

Prescribing the initial conditions (40) for $n = 0, 1, 2$, we solve the circular plate with clamped and supported boundaries numerically, and then compare the numerical results with the corresponding exact solution (41) to verify the accuracy and stability of our schemes. The plate domain is assumed to be a unit circle (i.e., $a = 1$); and the complete parameter values used for this test are $\rho = H = D = 1$, $K = T = f = 0$, and $\nu = 0.3$. The values of $\lambda_n$ needed by the exact solution (41) and the initial conditions (40) are given in (38) for clamped and in (39) for supported boundary conditions, respectively. In Figure 12, we show the computational grids $G_4$ and the initial displacements for $n = 0, 1, 2$ subject to the clamped boundary conditions. The initial conditions for the supported boundary conditions are similar with slightly different scales, so their plots are omitted here to save space.
Convergence rates for all the cases are collected in Figure 13. It is clearly shown in this numerical experiment that the expected 2nd-order accuracy for all the solution components \((w, v, a)\) is consistently achieved by our methods proposed in this paper. To illustrate the error behavior over the entire domain, we plot the errors of the NB2 scheme on grid \(G_{16}\) at \(t = 0.2\) for the case \(n = 2\) in Figure 14. Again, we see that the errors of all the solution components \((w, v, a)\) subject to both clamped and supported boundary conditions are well behaved; that is, the errors are smooth and their magnitudes are small throughout the domain. The errors of all the other cases, which are not plotted here, behave similarly; their magnitudes can be read off the corresponding log-log plots collected in Figure 13.

6.2. Plate with numerous holes

Now we solve a plate with very complicated geometrical setting to showcase the capability and efficiency of our approach for solving the Kirchhoff-Love plate equation (1) using composite overlapping grids. We consider a circular plate with radius 4 and two layers of holes sitting on two rings inside of the circular domain. On the outer ring whose radius is 3.5, there are twenty-four small holes of radius 0.3 located on equally spaced angles; the angle of the \(k^{th}\) hole is \(\theta_k = (15k)^\circ\). On the inner ring whose radius is 2.25, there sit another twelve larger holes of radius 0.4 on equally spaced angles; the angle of the \(k^{th}\) hole is given by \(\theta_k = (30k + 15)^\circ\) so that it sits in between the holes on the outer ring.

We discretize this complicated domain using a composite overlapping grid with 38 component grids, where a coarsened version is presented in Figure 15. The bulk of the domain is covered by a Cartesian grid, and the outer boundary of domain is descritized by an annulus boundary-fitted grid. The holes in the plate are represented by annulus boundary-fitted grids of smaller sizes that account for the remaining 36 component grids. The actual computational grid is \(G_{16}\) so that the target grid spacing is \(1/160\). The computational grid has 1.8 million grid points, of which 76 thousand are interpolation points. We note that a plate with similar configuration but modeled with 3D linear elasticity was studied in [44]; the computational grid with the same target grid spacing \((h = 1/160)\) for the 3D plate possess 42 million grid points. In this regard, the 2D Kirchhoff-Love plate model is much cheaper to solve numerically than a 3D plate model. Since our goal here is not to demonstrate the validity of the Kirchhoff-Love plate model for simplifying the 3D linear elasticity model, we do not compare our results with [44], which would require tuning the parameters, initial and boundary conditions of the Kirchhoff-Love model to match that with the 3D linear elasticity model.

For this numerical test, we solve (1) with physical parameters being \(\rho = 1, H = 1, K = 1, T = 1, D = 1\) and \(\nu = 0.3\) to investigate the vibrations of the plate in response to the following initial displacement

```
Figure 15: Computational setup for the plate with 24 outer and 12 inner holes. Left: coarsened version of the computational grid. Right: initial displacement.
```
Figure 16: Plate with holes simulation at \( t = 3.5 \) subject to: clamped (top row), supported (middle row), free (bottom row) boundary conditions. Results shown here are obtained using the NB2 scheme with \( C_{sf} = 50 \) and \( C_{df} = 0.1 \).

disturbance,

\[
\begin{align*}
    w(x, y, 0) &= \begin{cases} 
    0.25 \left( \cos(\sqrt{x^2 + y^2}/0.5) + 1 \right), & \sqrt{x^2 + y^2} \leq 0.5, \\
    0, & \text{otherwise}, 
\end{cases} \\
    v(x, y, 0) &= 0.
\end{align*}
\]

The contour of the initial displacement is plotted in Figure 15. The governing equation is solved using all the proposed numerical methods subject to all the boundary conditions (2)–(4). The time step and artificial dissipation for each scheme is determined according to formulas summarized in Table 1. The stability factor is \( C_{sf} = 0.9 \) for the explicit schemes, and \( C_{sf} = 50 \) for the NB2 scheme. And it suffices to stabilize all the schemes for this problem by setting the dissipation factor as \( C_{df} = 0.1 \).
Results obtained using all the numerical methods are similar, so only those of the NB2 scheme are shown here in Figure 16. The figure shows the evolution of the displacement, velocity and acceleration at time $t = 3.5$ for the plate with clamped (top row), supported (middle row), and free (bottom row) boundary conditions. We observe that the vibrations caused by the initial disturbance propagate outward from the center of the plate towards its perimeter. As the disturbance reaches the holes, the behavior of vibrations begin to differ for different boundary conditions. For the clamped and supported boundary conditions, we see that the displacement disturbance generally stops at the inner ring of holes because both boundary conditions prescribe zero displacement at the edges of the holes that serves to suppress the vibrations. For the solutions of the free boundary conditions, vibrations are able to propagate through both the inner and outer rings of holes since the edges of the holes are allowed to vibrate freely. All the plots shown here reflect the rich and complex dynamics of the Kirchhoff-Love plate model.

6.3. Traveling pulse on a guitar soundboard

As an illustration of the ability of our approach for solving realistic applications, we consider the problem of smoothed pulse traveling through a guitar soundboard. See Figure 17 for the geometric configuration of the soundboard and a coarsened version of the computational grid. Note that we do not intend to make a comparison for a particular guitar design or material; the focus of this paper is to demonstrate the numerical properties of the aforementioned algorithms. Therefore, in this test, we explore how an initial pulse travel through the guitar soundboard with hypothetical physical parameters.

Let us define a smoothed pulse,

$$ w_p(x, y, t) = \begin{cases} 
0.1 \left[ \cos \left( \pi \sqrt{\frac{(x - x_c - ct)^2}{r_a^2} + \frac{(y - y_c)^2}{r_b^2}} \right) + 1 \right] , & \text{if } \frac{(x - x_c - ct)^2}{r_a^2} + \frac{(y - y_c)^2}{r_b^2} \leq 1, \\
0, & \text{otherwise}.
\end{cases} $$

where $x_c = -1, y_c = 0, r_a = 0.1, r_b = 2, c = 1$. Then we specify the initial conditions accordingly,

$$ w(x, y, 0) = w_p(x, y, 0), \quad v(x, y, 0) = \frac{\partial w_p}{\partial t}(x, y, 0) \approx \frac{w_p(x, y, 0) - w_p(x, y, -\Delta t)}{\Delta t}. $$

The initial conditions are plotted in Figure 18. Clamped boundary conditions are enforced for the outer edge of the guitar domain, while the inner circular edge is allowed to move freely.

To investigate how the initial pulse travels through the domain, we consider two sets of physical parameters: (I) wave dominant case ($\rho = 1, H = 1, K = 1, T = 1, D = 1 \times 10^{-5}, \nu = 0.3$) and (II) bend dominant case ($\rho = 1, H = 1, K = 1, T = 1, D = 0.1, \nu = 0.3$). For the wave dominant case, the bending stiffness is set to be very small so that the initial pulse is expected to travel like a wave, while for the bend dominant
case, the bending effect dominates the vibrations. Computations are carried out for both cases using grid $G_{16}$ with UPC2 and NB2 schemes so that we can compare their respective performances.

The simulation results of the wave dominant case are shown in Figure 19. In the figure, the evolution of all the solution components (i.e., $w,v$ and $a$) are presented at two selected times ($t = 3.5$ and $t = 5$). For this case, we observe that the initial pulse travels towards the hole. When the leading edge of the pulse hits the hole, more waves are generated by diffractions and reflections. By the end of the simulation ($t = 5$), a complex distribution of elastic waves are observed over the entire soundboard. The velocity and acceleration evolve similarly as the displacement.

![Figure 18: Initial displacement (left) and velocity (right).](image)

![Figure 19: Simulation results for the wave dominant case ($\rho = 1, H = 1, K = 1, T = 1, D = 1 \times 10^{-5}, \nu = 0.3$).](image)
The simulation results of the bend dominant case are shown in Figure 20, where the displacement, velocity and acceleration of the plate are plotted at $t = 0.2$ and $t = 1$, respectively. For this case, we observe that the initial pulse induced vibrations radiate rapidly through the entire domain that exhibits a much richer dynamics than the wave dominant case. The patterns for the velocity and acceleration are also observed to quickly travel towards the outer edge of the guitar and display a very complex dynamics by the time $t = 1$.

It is worth noting that, even though the above simulations are done for hypothetical physical parameters, interesting guitar designers or artists could exploit our approach and code to improve their guitar designs. For example, they may modify the shape of the guitar and specify physical parameters for specific materials in the above simulations to investigate the acoustic quality of their intended designs.
We also make a rough comparison between the run-time CPU costs for the simulations shown in Figure 19 and 20. The problems are solved until $t = 5$ using a single processor of a MacBook Pro equipped with 16GB memory. The performance results are summarized in Figure 21, in which we compare the step size $\Delta t$, the CPU time in seconds per step (time/step), the total number of steps and the total computational times. We see that $\Delta t$ of the implicit NB2 scheme is generally 10 times bigger than that of the explicit UPC2 scheme, so it takes 10 times more time steps for the UPC2 scheme to reach the final simulation time $t = 5$. For each time step, the UPC2 scheme is more efficient and takes much less time than the NB2 scheme since the former scheme does not require solving a linear system. In terms of the overall simulation, the total computation times for both schemes are more or less on the same order. However, for harder (stiffer) problems with larger values of $D$, the time step of the explicit schemes can be significantly smaller. Under this circumstance, the implicit NB2 scheme can be more efficient for the overall simulation. The performance comparison presented here can serve as a guidance for us to choose the appropriate algorithm for a particular problem.

7. Conclusions

One novelty of this work lies in the manifestation that finite difference methods are well-suited for solving plate models with realistic and complex geometries. In fact, our finite-difference-based approaches can be more advantageous and efficient for solving plate problems because it solves the strong formulation of the governing equation (1) directly using structured overlapping grids without the need of any reformulation or extra complexity of non-standard finite elements. In this work, four novel numerical algorithms, referred to as the C2, UPC2, PC22 and NB2 schemes, are developed for the efficient and accurate solution of the Kirchhoff-Love plate model in complex domains. The proposed schemes are based on the standard spatial discretization on composite overlapping grid which involves curvilinear finite-difference approximations for spatial derivatives on each component grid and interpolating formulas for coupling together solutions on the different component grids. To the best of our knowledge, the methods presented here are the first finite-difference based algorithms for solving Kirchhoff-Love model with complex geometries that are suitable for realistic applications.

Our approaches resolve the numerical challenge due to the weak instability excited by the presence of interpolating equations in the discretization formulas on overlapping grids. The time-stepping schemes are stabilized by including a novel artificial hyper-dissipation term to the governing equations. Analysis on the frequency domain of the problem is performed to illustrate the stability of our methods. The analysis also leads to the derivation of explicit formulas (another novelty of the current work) for determining stable time steps and sufficient artificial dissipations that are applicable in real computations. Moreover, unlike many existing algorithms which solve the plate equation for the displacement only, we have seamlessly incorporated the solution of velocity and acceleration into our algorithms. This contribution can be useful for multi-physics coupling such as FSI, since the plate velocity and acceleration are important information that needs to be accurately transferred across the multi-physical interfaces.

Quadratic eigenvalue problems for a simplified model plate on 1D overlapping grids are derived for all the proposed algorithms to reveal the weak instability associated with the interpolation among component grids. Using this model problem, we also investigate how the artificial hyper-dissipation stabilizes the unstable modes for each of the algorithms. Carefully designed test problems are solved to demonstrate the properties and applications of our numerical approaches. In particular, the stability and accuracy of the schemes are verified by mesh refinement studies using method of manufactured solutions and using the analytical solutions of a circular plate. For the demonstration of applying our approach for plates with complex geometries arising from realistic applications, two benchmark problems are considered. In the first problem, we solve a plate with numerous holes subject to an initial disturbance of the displacement. In the second problem, we look at the traveling pulse on a guitar soundboard with two sets of hypothetical physical parameters. The future direction of this research is to deploy these novel algorithms for FSI applications involving plates with complex geometries in conjunction with our previously developed FSI scheme [15].

8. Acknowledgment

L. Li is grateful to Professor W.D. Henshaw of Rensselaer Polytechnic Institute (RPI) for helpful conversations.
Appendix A.

Appendix A.1. Transformed PDE on reference domain

The specific formula for Kirchhoff-Love shell model (1) transformed into the reference domain is given by

\[
\rho H \frac{\partial^2 W}{\partial t^2} = \mathcal{L}W + F,
\]

where

\[
\mathcal{L}W = (-K) W
\]

\[
+ (T (r_{xx} + r_{yy}) - D (r_{xxxx} + 2 r_{xxyy} + r_{yyyy})) W_r
+ (T (s_{xx} + s_{yy}) - D (s_{xxxx} + 2 s_{xxyy} + s_{yyyy})) W_s
+ (T (r_{x}^2 + r_{y}^2) - D (3 r_{xx}^2 + 2 r_{x} r_{yy} + 4 r_{xy}^2 + 3 r_{yy}^2 + 4 r_{x} r_{xx} + 4 r_{xy} r_{yy} + 4 r_{xy} r_{yy})) W_{rr}
+ (T (2 s_{x} + 2 r_{y} s_{y}) - D (4 r_{x} s_{xxxx} + 4 r_{x} s_{xxyy} + 4 r_{xy} s_{x} + 6 r_{xx} s_{xx})
+ 8 r_{xy} s_{xy} + 4 r_{x} s_{xy} + 4 r_{y} s_{xy} + 2 r_{x} s_{xy} + 2 r_{y} s_{xy} + 4 r_{y} s_{xy} + 4 r_{yyyy} s_{yy} + 6 r_{yy} s_{yy})) W_{ss}
+ (T (s_{y}^2 + s_{x}^2) - D (3 s_{xx}^2 + 2 s_{xx} s_{yy} + 4 s_{xy}^2 + 3 s_{yy}^2 + 4 s_{xx} s_{xxx} + 4 s_{xy} s_{xyy} + 4 s_{xy} s_{xyy} + 4 s_{y} s_{xyy})) W_{ss}
+ (-D (6 r_{x} r_{xx} + 2 r_{xx} r_{yy} + 6 r_{xy}^2 r_{yy} + 2 r_{x} r_{xy} r_{yy} + 2 r_{xy} r_{xy} r_{yy} + 4 r_{x} r_{xy} r_{xy})) W_{xxx}
+ (-D (2 s_{x} r_{yy} + 2 r_{xy} s_{xy} + 3 r_{xx} s_{x} + 2 r_{y} s_{xy} + 3 r_{yy} s_{xy})) W_{yyy}
+ 2 s_{x} (r_{x} r_{xy} + 2 r_{xy} r_{yy}) + r_{x} (3 r_{x} s_{xxx} + 3 r_{xx} s_{x}) + 4 r_{xy} (r_{x} s_{xy} + r_{y} s_{x}) + r_{y} (3 r_{y} s_{yyy} + 3 r_{yy} s_{y})
+ 9 r_{x} r_{xx} s_{x} + 4 r_{x} r_{y} s_{xy} + 4 r_{xx} r_{x} s_{x} + 2 r_{y} s_{xy} + 3 r_{y} s_{xy} s_{xy} + 9 r_{y} r_{yy} s_{yy}) W_{rrssss}
+ (-D (6 s_{x}^2 s_{xx} + 2 s_{xx} s_{yy} + 6 s_{y}^2 s_{yy} + 2 s_{x} s_{xy} + 2 s_{x} s_{yy} + 4 s_{xx} s_{xy} + 4 s_{y} s_{xy})) W_{ass}
+ (-D (r_{x}^4 + 2 r_{x}^2 r_{y}^2 + r_{y}^4)) W_{rrrr}
+ (-D (r_{x}^2 + 2 r_{x} r_{y} r_{xy} + 2 r_{xy} s_{xy} r_{y}) + 2 + r_{x}^2 s_{x} + 4 r_{y}^2 s_{y} + 2 r_{x} r_{yy} s_{xy})) W_{rrrs}
+ (-D (6 r_{x}^2 s_{x}^2 + 6 r_{y}^2 s_{y}^2 + r_{x} (r_{x} s_{xy} + 2 r_{y} s_{xy} + 2 + s_{x} (r_{x} s_{xy} + 2 r_{x} r_{xy} s_{xy}))) W_{rrss}
+ (-D (s_{x} (r_{x} s_{xx} + 2 r_{y} s_{xy} + 2 + 4 r_{x} s_{xx} + 4 r_{y} s_{xy} + 2 + 4 r_{x} s_{xx} + 4 r_{y} s_{xy}))) W_{assss}
+ (-D (s_{y}^4 + 2 s_{y}^2 s_{x} + s_{x}^4)) W_{ssssss}.
\]

The definitions of coefficients \(a_i(r), b_i(r), c_i(r), d_i(r)\) (coefficients in front of the red terms) can be readily read off the above formula for \(\mathcal{L}W\).

Appendix A.2. Formula of the Discrete transformation

The Fourier transform (symbol) of the discrete operator \(\mathcal{L}_h\) is

\[
\hat{Q}(\xi, \xi_r) = -K - b_1 \frac{4 \sin^2(\xi_r / 2)}{h_2^2} - b_2 \frac{\sin(\xi_r) \sin(\xi_r)}{h_1 h_2} - b_3 \frac{4 \sin^2(\xi_r / 2)}{h_2^2} + d_{1111} \frac{16 \sin^4(\xi_r / 2)}{h_1^2 h_2^4}
+ d_{1112} \frac{4 \sin^2(\xi_r / 2) \sin(\xi_r) \sin(\xi_r)}{h_1^2 h_2^4} + d_{1122} \frac{16 \sin^2(\xi_r / 2) \sin^2(\xi_r / 2)}{h_1^2 h_2^4} + d_{1222} \frac{4 \sin(\xi_r) \sin^2(\xi_r / 2) \sin(\xi_r)}{h_1^2 h_2^4} + d_{2222} \frac{16 \sin^4(\xi_r / 2)}{h_1^2 h_2^4}.
\]
References

[1] J. N. Reddy, Theory and analysis of elastic plates and shells, CRC press, 2006.

[2] L. W. Ehrlich, Solving the biharmonic equation as coupled finite difference equations, SIAM J. Numer. Anal. 8 (2) (1971) 278–287.

[3] G. Chen, Z. Li, P. Lin, A fast finite difference method for biharmonic equations on irregular domains and its application to an incompressible stokes flow, Adv. Comput. Math. 29 (2008) 113–133.

[4] E. Bécache, G. Derveaux, P. Joly, An efficient numerical method for the resolution of the Kirchhoff-Love dynamic plate equation, Numer. Methods Partial Differential Equations 21 (2) (2005) 323–348.

[5] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, Philadelphia, 2002.

[6] T. Belytschko, W. K. Liu, B. Moran, Nonlinear Finite Elements for Continua and Structures, John Wiley and Sons, New York, 2005.

[7] L. S. D. Morley, The triangular equilibrium element in the solution of plate bending problems, Aero. Quart. 19 (1968) 149–169.

[8] S. C. Brenner, Two-level additive Schwarz preconditioners for nonconforming finite element methods, Math. Comput. 65 (215) (1996) 897–921.

[9] S. C. Brenner, L. yeng Sung, Balancing domain decomposition for nonconforming plate elements, Numer. Math. 83 (1999) 25–52.

[10] W. Ming, J. Xu, The Morley element for fourth order elliptic equations in any dimensions, Numer. Math. 103 (2006) 155–169.

[11] M. Li, X. Guan, S. Mao, New error estimates of the Morley element for the plate bending problems, J. Comput. Appl. Math 263 (2014) 405–416.

[12] L. Noels, R. Radovitzky, A new discontinuous galerkin method for kirchhoff–love shells, Computer Methods in Applied Mechanics and Engineering 197 (33-40) (2008) 2901–2929.

[13] D. T. A. Nguyen, L. Li, H. Ji, Stable and accurate numerical methods for generalized Kirchhoff-Love plates, arXiv:2008.01693 (2020) 1–26.

[14] H. Ji, L. Li, Numerical methods for thermally stressed shallow shell equations, J. Comput. Appl. Math. 362 (2019) 626–652.

[15] L. Li, W. D. Henshaw, J. W. Banks, D. W. Schwendeman, G. A. Main, A stable partitioned FSI algorithm for incompressible flow and deforming beams, J. Comput. Phys. 312 (2016) 272–306.

[16] J. W. Banks, W. D. Henshaw, D. W. Schwendeman, Q. Tang, A stable partitioned FSI algorithm for rigid bodies and incompressible flow. Part I: Model problem analysis, J. Comput. Phys. 343 (2017) 432–468.

[17] J. W. Banks, W. D. Henshaw, D. W. Schwendeman, Q. Tang, A stable partitioned FSI algorithm for rigid bodies and incompressible flow. Part II: General formulation, J. Comput. Phys. 343 (2017) 469–500.

[18] J. W. Banks, W. D. Henshaw, D. W. Schwendeman, Q. Tang, A stable partitioned FSI algorithm for rigid bodies and incompressible flow in three dimensions, J. Comput. Phys. 373 (2018) 455–492.

[19] D. A. Serino, J. W. Banks, W. D. Henshaw, D. W. Schwendeman, A stable added-mass partitioned (AMP) algorithm for elastic solids and incompressible flow: Model problem analysis, SIAM J. Sci. Comput. 41 (4) (2019) A2464–A2484.
[20] D. A. Serino, J. W. Banks, W. D. Henshaw, D. W. Schwendeman, A stable added-mass partitioned (AMP) algorithm for elastic solids and incompressible flow, J. Comput. Phys. 399 (2019) 1–30.

[21] L. B. da Veiga, J. Niiranen, R. Stenberg, A family of $C^0$ finite elements for Kirchhoff plates I: Error analysis, SIAM J. Numer. Anal. 45 (2007) 2047–2071.

[22] L. B. da Veiga, J. Niiranen, R. Stenberg, A family of $C^0$ finite elements for kirchhoff plates II: Numerical results, Comput. Method. Appl. Mech. Engrg. 197 (21) (2008) 1850–1864.

[23] L. B. da Veiga, T. J. R. Hughes, J. Kiendl, C. Lovadina, J. Niiranen, A. Reali, H. Speleers, A locking-free model for Reissner-Mindlin plates: Analysis and isogeometric implementation via NURBS and triangular NURPS, Mathematical Models and Methods in Applied Sciences 25 (08) (2015) 1519–1551.

[24] S. GM, Continuum-based shell elements, Ph.D. thesis, Stanford University (1985).

[25] S. Klinkel, F. Gruttmann, W. Wagner, A continuum based three-dimensional shell element for laminated structures, Comput. Struct. 71 (1) (1999) 43–62.

[26] T. Hughes, J. Cottrell, Y. Bazilevs, Isogeometric analysis: Cad, finite elements, nurbs, exact geometry and mesh refinement, Comput. Method. Appl. Mech. Engrg. 194 (39) (2005) 4135–4195.

[27] Y. Bazilevs, L. B. da Veiga, J. A. Cottrell, T. J. R. Hughes, G. Sangalli, Isogeometric analysis: approximation, stability and error estimates for h-refined meshes, Mathematical Models and Methods in Applied Sciences 16 (07) (2006) 1031–1090.

[28] J. Kiendl, K.-U. Bletzinger, J. Linhard, R. Wüchner, Isogeometric shell analysis with kirchhoff-love elements, Comput. Method. Appl. Mech. Engrg. 198 (49) (2009) 3902–3914.

[29] J. Kiendl, Y. Bazilevs, M.-C. Hsu, R. Wüchner, K.-U. Bletzinger, The bending strip method for isogeometric analysis of kirchhoff-love shell structures comprised of multiple patches, Computer Methods in Applied Mechanics and Engineering 199 (37-40) (2010) 2403–2416.

[30] J. Kiendl, M.-C. Hsu, M. C. Wu, A. Reali, Isogeometric kirchhoff-love shell formulations for general hyperelastic materials, Comput. Method. Appl. Mech. Engrg. 291 (2015) 280–303.

[31] N. Nguyen-Thanh, N. Valizadeh, M. Nguyen, H. Nguyen-Xuan, X. Zhuang, P. Areias, G. Zi, Y. Bazilevs, L. De Lorenzis, T. Rabczuk, An extended isogeometric thin shell analysis based on kirchhoff-love theory, Computer Methods in Applied Mechanics and Engineering 284 (2015) 265–291.

[32] Z. Zou, T. Hughes, M. Scott, R. Sauer, E. Savitha, Galerkin formulations of isogeometric shell analysis: Alleviating locking with greville quadratures and higher-order elements, Comput. Method. Appl. Mech. Engrg. 380 (2021) 113757.

[33] D. Benson, Y. Bazilevs, M. Hsu, T. Hughes, Isogeometric shell analysis: The Reissner-Mindlin shell, Comput. Method. Appl. Mech. Engrg. 199 (5) (2010) 276 – 289, computational Geometry and Analysis.

[34] G. S. Chesshire, W. D. Henshaw, Composite overlapping meshes for the solution of partial differential equations, J. Comput. Phys. 90 (1) (1990) 1–64.

[35] J. W. Banks, W. D. Henshaw, Upwind schemes for the wave equation in second-order form, J. Comput. Phys. 231 (17) (2012) 5854–5889.

[36] J. Angel, J. W. Banks, W. D. Henshaw, High-order upwind schemes for the wave equation on overlapping grids: Maxwell’s equations in second-order form, J. Comput. Phys. 352 (2018) 534–567.

[37] V. Z. Vlasov, Beams, plates and shells on elastic foundation, Israel Program for Scientific Translation.

[38] W. Flügge, Viscoelasticity, Springer Science & Business Media, 2013.
[39] S. Bilbao, A family of conservative finite difference schemes for the dynamical von Karman plate equations, Numer. Methods Partial Differential Equations 24 (1) (2008) 193–216.

[40] J. Angel, J. W. Banks, W. D. Henshaw, Efficient high-order upwind difference schemes for the second-order wave equation on overlapping grids, Tech. rep., in preparation (2018).

[41] N. M. Newmark, A method of computation for structural dynamics, proceedings of the american society of civil engineers 85 (EM 3) (1959) 67–74.

[42] P. J. Roache, Verification and Validation in Computational Science and Engineering, Hermosa Publishers, New Mexico, 1998.

[43] T. Wah, Vibration of circular plates, The Journal of the Acoustical Society of America 34 (3) (1962) 275–281.

[44] D. Appelö, J. W. Banks, W. D. Henshaw, D. W. Schwendeman, Numerical methods for solid mechanics on overlapping grids: Linear elasticity, J. Comput. Phys. 231 (18) (2012) 6012–6050.