REMARKS ON MINIMAL MASS BLOW UP SOLUTIONS FOR A DOUBLE POWER
NONLINEAR SCHRÖDINGER EQUATION

NAOKI MATSUI

Abstract. We consider the following nonlinear Schrödinger equation with double power nonlinearity
\[ i\frac{\partial u}{\partial t} + \Delta u + |u|^\frac{4}{N} u \pm |u|^{p-1}u = 0, \quad 1 < p < 1 + \frac{4}{N} \]
in $\mathbb{R}^N$. For $N = 1, 2, 3$, Le Coz-Martel-Raphaël (2016) construct a minimal-mass blow-up solution. Moreover, the previous study derives blow-up rate of the blow-up solution. In this paper, we extend this result to the general dimension. Furthermore, we investigate the behaviour of the critical mass blow-up solution near the blow-up time.

1. Introduction

We consider the following nonlinear Schrödinger equation with double power nonlinearity
\[
(NLS^\pm) \begin{cases} 
    i\frac{\partial u}{\partial t} + \Delta u + |u|^\frac{4}{N} u \pm |u|^{p-1}u = 0, \\
    u(t_0) = u_0
\end{cases}
\]
in $\mathbb{R}^N$, where
\[ 1 < p < 1 + \frac{4}{N} \]
Then, $(NLS^\pm)$ is locally well-posed in $H^1(\mathbb{R}^N)$ ($\mathbb{R}$). This means that for any $u_0 \in H^1(\mathbb{R}^N)$, there exists a unique maximal solution $u \in C^1((-T_*, T^*), H^{-1}(\mathbb{R}^N)) \cap C((-T_*, T^*), H^1(\mathbb{R}^N))$. Moreover, the mass (i.e., $L^2$-norm) and energy $E$ of the solution are conserved by the flow, where
\[
E(u) := \frac{1}{2} \|\nabla u\|^2_2 - \frac{1}{2 + \frac{4}{N}} \|u\|^{2 + \frac{4}{N}} + \frac{1}{p + 1} \|u\|^{p+1}.
\]
Furthermore, there is a blow-up alternative
\[ T^* < \infty \Rightarrow \lim_{t \uparrow T^*} \|\nabla u(t)\|_2^2 = \infty. \]

1.1. Main results. In this paper, for $(NLS^\pm)$, we prove the following result, which is stronger than Le Coz, Martel, and Raphaël [7].

Theorem 1.1 (Existence of a minimal mass blow-up solution). For any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a critical mass radial initial value $u(t_0) \in \Sigma^1(\mathbb{R}^N)$ with $E(u_0) = E_0$ such that the corresponding solution $u$ for $(NLS^+)\,$ blows up at $T^* = 0$. Moreover,
\[
\left\| u(t) - \frac{1}{\lambda(t)^{\frac{4}{N}}} P \left(t, \frac{x}{\lambda(t)}\right) e^{-i\frac{\|\nabla u(t)\|^2_2 + 1}{\lambda(t)^2} + i\gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
\]
holds for some blow-up profile $P$, positive constants $C_1(p)$ and $C_2(p)$, positive-valued $C^1$ function $\lambda$, and real-valued $C^1$ functions $b$ and $\gamma$ such that
\[
P(t) \to Q \quad \text{in} \quad H^1(\mathbb{R}^N), \quad \lambda(t) = C_1(p)|t|^{-1-N(p-1)} (1 + o(1)),
\]
\[
b(t) = C_2(p)|t|^{-\frac{4-N(p-1)}{1-N(p-1)}} (1 + o(1)), \quad \gamma(t)^{-1} = O \left(|t|^{-\frac{4-N(p-1)}{1-N(p-1)}}\right)
\]
as $t \nearrow 0$.

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Here, $\Sigma^1$ is defined as
\[ \Sigma^1 := \{ u \in H^1(\mathbb{R}^N) \mid xu \in L^2(\mathbb{R}^N) \}. \]

**Theorem 1.2** (Non-existence of a minimal mass blow-up solution ([7])). For any critical-mass initial value $u(t_0) \in H^1(\mathbb{R}^N)$, the corresponding solution for \((\text{NLS}^-)\) is global and bounded in $H^1(\mathbb{R}^N)$.

**Theorem 1.3** (Existence of a supercritical mass blow-up solution ([7])). For any $\delta$, there is $u(t_0) \in H^1(\mathbb{R}^N)$ such that $\|u_0\|_2 = \|Q\|_2 + \delta$ and the corresponding solution for \((\text{NLS}^-)\) blows up at finite time.

Proofs of Theorem 1.2 and Theorem 1.3 in [7] is dimension-independent. In this paper, we prove only Theorem 1.1.

1.2. **Notations.** In this section, we introduce the notation used in this paper.

Let
\[ N := \mathbb{Z}_{\geq 1}, \quad N_0 := \mathbb{Z}_{\geq 0}. \]

Unless otherwise noted, we define
\[ (u, v)_2 := \Re \int_{\mathbb{R}^N} u(x) \overline{v}(x) dx, \quad \|u\|_q := \left( \int_{\mathbb{R}^N} |u(x)|^q dx \right)^{\frac{1}{q}}, \]
\[ f(z) := |z|^\frac{p+1}{2}, \quad F(z) := \frac{1}{2^\frac{p+1}{2} - 1} |z|^{\frac{p+1}{2}}, \quad g(z) := |z|^{p-1}z, \quad G(z) := \frac{1}{p+1} |z|^{p+1}. \]

By identifying $\mathbb{C}$ with $\mathbb{R}^2$, we denote the differentials of the functions $df$, $dg$, $dF$, and $dG$. We define
\[ \Lambda := \frac{N}{2} + x \cdot \nabla, \quad L_+ := -\Delta + 1 - \left(1 + \frac{4}{N}\right)Q^\frac{N}{2}, \quad L_- := -\Delta + 1 - Q^\frac{N}{2}. \]

Then,
\[ L_- Q = 0, \quad L_+ (\Lambda Q) = -2Q, \quad L_- (|x|^2 Q) = -4\Lambda Q, \quad L_+ \rho = |x|^2 Q \]
holds, where $\rho$ is the unique radial Schwartz solution of $L_+ \rho = |x|^2 Q$. Furthermore, there is a $\mu > 0$ such that
\[ \forall u \in H^1_{\text{rad}}(\mathbb{R}^N), \quad (L_+ \Re u, \Re u)_2 + (L_- \Im u, \Im u)_2 \geq \mu \|u\|^2_{H^1} - \frac{1}{\mu} \left( (\Re u, Q)_2 + (\Re u, |x|^2 Q)_2 + (3 \Re u, \rho)_2 \right) \]
(e.g., see [11] [12] [14] [15]). We introduce
\[ \Sigma^m := \{ u \in H^m(\mathbb{R}^N) \mid |x|^m u \in L^2(\mathbb{R}^N) \}. \]

and denote by $\mathcal{Y}$ the set of functions $h \in C^\infty_{\text{rad}}(\mathbb{R}^N)$ such that
\[ \forall \alpha \in \mathbb{N}^N_0 \exists C_\alpha, \kappa_\alpha > 0, \quad \left| \frac{\partial^\alpha}{\partial x^\alpha} h(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x). \]

Finally, we use $\lesssim$ and $\gtrsim$ when the inequalities hold except for non-essential positive constant differences and $\approx$ when $\lesssim$ and $\gtrsim$ hold.

2. **Preliminaries**

We provide the following statements regarding notations.

**Proposition 2.1.** For any $\alpha \in \mathbb{N}^N_0$, there is a constant $C_\alpha > 0$ such that $\left| \left( \frac{\partial}{\partial x} \right)^\alpha \rho(x) \right| \leq C_\alpha Q(x)$. Similarly, $\left| \left( \frac{\partial}{\partial x} \right)^\alpha \rho(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x)$ holds (e.g., [7] [9]).

**Lemma 2.2.** For the ground state $Q$,
\[ (Q, \rho)_2 = \frac{1}{2} \| |x|^2 Q \|_2^2 \]
holds.
Lemma 2.3. For an appropriate function $w$,
\[
(|x|^{2p}w, \Lambda w)_2 = -p|||x|^p w||^2_{L^2}, \quad (-\Delta w, \Lambda w)_2 = ||\nabla w||^2_{L^2}, \quad (|w|^q w, \Lambda w)_2 = \frac{Nq}{2(q+2)} ||w||^{q+2}_{L^{q+2}}
\]
holds.

Lemma 2.4 (Properties of $F$ and $f$). For $F$ and $f$,
\[
\frac{\partial F}{\partial \Re} = \Re f, \quad \frac{\partial F}{\partial \Im} = \Im f, \quad \frac{\partial \Re f}{\partial \Im} = \frac{\partial \Im f}{\partial \Re},
\]
\[
\frac{\partial}{\partial s} F(z(s)) = f(z(s)) \cdot \frac{\partial z}{\partial s} = \Re \left( f(z(s)) \frac{\partial z}{\partial s} \right),
\]
d\[
dF(z)(w) = f(z) \cdot w = \Re (f(z)w),
\]
d\[
df(z)(w_1) \cdot w_2 = df(z)(w_2) \cdot w_1,
\]
\[
\frac{\partial}{\partial w} \int_{\mathbb{R}^N} (F(z(x) + w(x)) - F(z(x)) - dF(z(x))(w(x))) \, dx = f(z + w) - f(z),
\]
\[
L_+(\mathcal{R}Z) + iL_- (\mathcal{R}Z) = -\Delta Z + Z - df(Q)(Z)
\]
holds. When identifying $\mathbb{C}$ with $\mathbb{R}^2$, is the inner product of $\mathbb{R}^2$.

3. CONSTRUCTION OF A BLOW-UP PROFILE

In this section, we construct a blow-up profile $P$.

For $K \in \mathbb{N}_0$, let
\[
\Sigma_K = \{ (j,k) \in \mathbb{N}_0^2 \mid j + k \leq K \}.
\]

Proposition 3.1. Let $K \in \mathbb{N}$ be sufficiently large. Let $\lambda(s) > 0$ and $b(s) \in \mathbb{R}$ be $C^1$ function of $s$ such that $\lambda(s) + |b(s)| \ll 1$.

(i) Existence of blow-up profile. For any $(j,k) \in \Sigma_K$, there exist real-valued $P_{j,k}^+, P_{j,k}^- \in \mathcal{Y}$ and $\beta_{j,k} \in \mathbb{R}$ such that $P$ satisfies
\[
i \frac{\partial P}{\partial s} + \Delta P - P + f(P) + \lambda^\alpha g(P) + \theta \frac{|y|^2}{4} P = \Psi,
\]
where $\alpha = 2 - \frac{N(p-1)}{2}$, and $P$ and $\theta$ are defined by
\[
P(s, y) := Q(y) + \sum_{(j,k) \in \Sigma_K} \left( b(s)^{2j} \lambda(s)^{(k+1)\alpha} P_{j,k}^+(y) + ib(s)^{2j+1} \lambda(s)^{(k+1)\alpha} P_{j,k}^-(y) \right),
\]
\[
\theta(s) := \sum_{(j,k) \in \Sigma_K} b(s)^{2j} \lambda(s)^{(k+1)\alpha} \beta_{j,k}.
\]

Moreover, for some $\epsilon' > 0$ that is sufficiently small,
\[
\left\| e^{\epsilon'|y|} \Psi \right\|_{H^1} \lesssim \lambda^\alpha \left( \left| b + \frac{1}{\lambda} \frac{\partial b}{\partial s} \right| + |b^2 - \theta| \right) + (b^2 + \lambda^\alpha)^{K+2}.
\]

(ii) Mass and energy properties of blow-up profile. Let define
\[
P_{\lambda,b,\gamma}(s, x) = \frac{1}{\lambda(s)^{\frac{N}{2}}} P \left( s, \frac{x}{\lambda(s)} \right) e^{-\frac{\epsilon |x|^2}{\lambda(s)^{N/2}}} e^{i\gamma(s)}.
\]

Then,
\[
\frac{d}{ds} \left\| P_{\lambda,b,\gamma} \right\|_{L^2} \lesssim \lambda^\alpha \left( \left| b + \frac{1}{\lambda} \frac{\partial b}{\partial s} \right| + |b^2 - \theta| \right) + (b^2 + \lambda^\alpha)^{K+2},
\]
\[
\frac{d}{ds} E(P_{\lambda,b,\gamma}) \lesssim \lambda^\alpha \left( \left| b + \frac{1}{\lambda} \frac{\partial b}{\partial s} \right| + |b^2 - \theta| + (b^2 + \lambda^\alpha)^{K+2} \right).
\]
hold. Moreover,
\[
8E(P_{\lambda,\gamma}) - \frac{||y||}{2} \left( \frac{b^2}{\lambda^2} - \frac{2\beta}{2 - \alpha} \lambda^{\alpha - 2} \right) \leq \frac{\lambda^\alpha (b^2 + \lambda^\alpha)}{\lambda^2},
\]
where
\[
\beta := \beta_{0,0} = \frac{2N(p - 1) ||Q||^{p+1}}{p + 1} \frac{||y||}{2}.
\]

proof. See [7, 9] for the proofs. The proofs are dimension-independent. \qed

Lemma 3.2 (Decomposition). There exist constants $I, \bar{I}, b, \bar{\gamma} > 0$ such that the following logic holds.

Let $I$ be an interval, let $\delta > 0$ be sufficiently small, and let $u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))$ satisfy that there exist functions $\lambda \in \text{Map}(I, (0, \bar{I}))$ and $\gamma \in \text{Map}(I, \mathbb{R})$ such that
\[
\forall t \in I, \quad \left\| \lambda(t) v(t, \lambda(t)) e^{i\gamma(t)} - Q \right\|_{H^s} < \delta.
\]
Then, (given $\lambda(0)$) there exist unique functions $\tilde{\lambda} \in C^1(I, (0, \infty))$ and $\tilde{b}, \tilde{\gamma} \in C^1(I, \mathbb{R})$ that are independent of $\lambda$ and $\gamma$ such that
\[
u(t, x) = \frac{1}{\lambda(t)^{\alpha/2}} (P + \varepsilon) \left( t, \frac{x}{\lambda(t)} \right) e^{-ib\tilde{\lambda}(t)|x|^2/4\tilde{\lambda}(t)^2 + i\tilde{\gamma}(t)}
\]
\[
\tilde{\lambda}(t) \in (\lambda(t)(1 - \bar{I}), \lambda(t)(1 + \bar{I}))
\]
\[
\tilde{b}(t) \in (-\bar{b}, \bar{b})
\]
\[
\tilde{\gamma}(t) \in \bigcup_{m \in \mathbb{Z}} (-\gamma - \gamma(t) + 2m\pi, \gamma - \gamma(t) + 2m\pi)
\]
holds and $\varepsilon$ satisfies the orthogonal conditions
\[
(\varepsilon, iLP)_2 = (\varepsilon, |\cdot|^2 P)_2 = (\varepsilon, i\rho)_2 = 0
\]
in $I$. In particular, $\tilde{\lambda}$ and $\tilde{b}$ are unique within functions and $\tilde{\gamma}$ is unique within continuous functions (and is unique within functions under modulo $2\pi$).

See [8, 9] for the proof.

4. Approximate blow-up law

In this section, we describe the initial values and the approximation functions of the parameters $\lambda$ and $b$ in the decomposition.

Lemma 4.1. Let
\[
\lambda_{\text{app}}(s) = \left( \frac{\alpha}{2} \right) \sqrt{\frac{2\beta}{2 - \alpha}} s^{-\frac{\alpha}{2}}, \quad b_{\text{app}}(s) = \frac{2}{\alpha s},
\]
Then, $(\lambda_{\text{app}}, b_{\text{app}})$ is solutions of
\[
\frac{\partial b}{\partial s} + b^2 - \beta \lambda^\alpha = 0, \quad b + \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} = 0
\]
in $s > 0$.

Lemma 4.2 (7, 9). Let define $C_0 := \frac{8E_0}{||y||^2}$ and $0 < \lambda_0 \ll 1$ such that $\frac{2\beta}{2 - \alpha} + C_0 \lambda_0^{2-\alpha} > 0$. For $\lambda \in (0, \lambda_0]$, we set
\[
\mathcal{F}(\lambda) := \int_0^\lambda \frac{1}{\mu^{\frac{2}{2-\alpha}} C_0 \mu^{2-\alpha}} d\mu.
\]
Then, for any $s_1 \gg 1$, there exist $b_1, \lambda_1 > 0$ such that
\[
\left| \frac{\lambda_1^{\frac{2}{2-\alpha}}}{\lambda_{\text{app}}(s_1)^2} - 1 \right| + \left| \frac{b_1}{b_{\text{app}}(s_1)} - 1 \right| \lesssim s_1^{\frac{\alpha}{2}} + s_1^{2-\frac{\alpha}{2}}, \quad \mathcal{F}(\lambda_1) = s_1, \quad E(P_{\lambda_1, b_1, \gamma}) = E_0.$
Moreover,
\[
\left| \mathcal{F}(\lambda) - \frac{2}{\alpha \lambda^{\frac{2}{2-\alpha}}} \sqrt{\frac{2\beta}{2-\alpha}} \right| \lesssim \lambda^{-\frac{2}{\alpha}} + \lambda^{2-\frac{4}{\alpha}}
\]
holds.

**proof.** See [4, 9] for the proof. The proof is dimension-independent. \(\square\)

5. Uniformity estimates for decomposition

In this section, we estimate modulation terms.

Let define
\[
\mathcal{C} := \frac{\alpha}{4-\alpha} \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2-\alpha}} \right)^{-\frac{4}{\alpha}}.
\]
For \(t_1 < 0\) that is sufficiently close to 0, we define
\[
s_1 := |C^{-1}t_1|^{-\frac{1}{4-\alpha}}.
\]
Additionally, let \(\lambda_1\) and \(b_1\) be given in Lemma 4.2 for \(s_1\) and \(\gamma_1 = 0\). Let \(u\) be the solution for \((\text{NLS}+)\) with an initial value
\[
(4) \quad u(t_1, x) := P_{\lambda_1, b_1, 0}(x).
\]
Then, since \(u\) satisfies the assumption of Lemma 3.2 in a neighbourhood of \(t_1\), there exists a decomposition \((\tilde{\lambda}_t, \tilde{b}_t, \tilde{\gamma}_t, \tilde{\varepsilon}_t)\) such that \(3\) in a neighbourhood I of \(t_1\). The rescaled time \(s_{t_1}\) is defined as
\[
s_{t_1}(t) := s_1 - \int_t^{I_{t_1}} \frac{1}{\lambda_{t_1}(\tau)^{\frac{2}{\alpha}}} d\tau.
\]
Therefore, we define an inverse function \(s_{t_1}^{-1} : s_{t_1}(I) \rightarrow I\). Therefore, we define
\[
t_{t_1} := s_{t_1}^{-1}, \quad \lambda_{t_1}(s) := \tilde{\lambda}(t_{t_1}(s)), \quad b_{t_1}(s) := \tilde{b}(t_{t_1}(s)), \quad \gamma_{t_1}(s) := \tilde{\gamma}(t_{t_1}(s)), \quad \varepsilon_{t_1}(s, y) := \tilde{\varepsilon}(t_{t_1}(s), y).
\]
If there is no risk of confusion, the subscript \(t_1\) is omitted. In particular, it should be noted that \(u \in C((-T_*, T^*), \Sigma^2(\mathbb{R}^N))\) and \(|x|\nabla u \in C((-T_*, T^*), L^2(\mathbb{R}^N))\). Furthermore, let \(I_{t_1}\) be the maximal interval such that a decomposition as \(3\) is obtained and \(J_{s_1} := s(I_{t_1})\). Additionally, let \(s_0 (\leq s_1)\) be sufficiently large and let \(s' := \max\{s_0, \inf J_{s_1}\}\).

Let \(0 < M < \min\{\frac{1}{2}, \frac{1}{\alpha} - 2\}\) and \(s_*\) be defined as
\[
s_* := \inf \{\sigma \in (s', s_1) \mid 4\) holds on \([\sigma, s_1]\}.
\]
where
\[
(5) \quad \|\varepsilon(s)\|_{H^1}^2 + b(s)^2 \cdot |\varepsilon(s)|^{\frac{4}{\alpha}} < s^{-2K}, \quad \left| \frac{\lambda(s)^{\frac{4}{\alpha}}}{\lambda_{\text{app}}(s)^{\frac{4}{\alpha}}} - 1 \right| + \left| \frac{b(s)}{b_{\text{app}}(s)} - 1 \right| < s^{-M}.
\]
\[
(6) \quad \text{Mod} := \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2 - \theta, 1 - \frac{\partial \gamma}{\partial s} \right).
\]

Finally, we define
\[
\text{Mod} := \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2 - \theta, 1 - \frac{\partial \gamma}{\partial s} \right).
\]

In the following discussion, the constant \(\epsilon > 0\) is a sufficiently small constant. If necessary, \(s_0\) and \(s_1\) are recalculated in response to \(\epsilon > 0\).
Moreover, we have
\[ \frac{\partial \varepsilon}{\partial s} + \Delta \varepsilon - \varepsilon + f(P(\varepsilon) - f(P) - \lambda^\alpha (g(P + \varepsilon) - g(P)) \]
\[ = -i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \Lambda(\varepsilon) + \left( 1 - \frac{\partial \gamma}{\partial s} \right) (P + \varepsilon) + \left( \frac{\partial b}{\partial s} + b^2 - \theta \right) \frac{|y|^2}{4} (P + \varepsilon) - \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) b \frac{|y|^2}{2} (P + \varepsilon) \]
(7)
\[ \varepsilon = \Psi \]
holds.

**Lemma 5.2.** For \( s \in (s_*, s_1] \),
\[ |(\varepsilon(s), Q)| \lesssim s^{-(K+2)}, \quad |\text{Mod}(s)| \lesssim s^{-(K+4)} \]
hold.

**proof.** We outline the proof. See [1, 2] for detail of the proof.

Let \( s_* := \inf \left\{ s \in [s_*, s_1] \mid |(\varepsilon(\tau), P)| < \tau^{-(K+2)} \right\} \) holds on \([s, s_1]\).

We work on the interval \([s_*, s_1]\).

According to the orthogonality properties, we have
\[ 0 = \frac{d}{ds} (i \varepsilon, \Lambda P)_2 \]
\[ = \frac{d}{ds} (i \varepsilon, i \cdot |^2 P)_2 \]
\[ = \frac{d}{ds} (i \varepsilon, \rho)_2 \]
\[ = (i \varepsilon, \frac{\partial (\Lambda P)}{\partial s})_2 = O(s^{-(K+3)}) + O(s^{-1}|\text{Mod}(s)|). \]
Moreover, we have
\[ (i \varepsilon, \frac{\partial \Lambda P}{\partial s})_2 \]
\[ = (L+ \Re \varepsilon + iL- \Im \varepsilon - f(P) + f(P) - df(Q)\varepsilon) + \lambda^\alpha (g(P + \varepsilon) - g(P)) \]
\[ + i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \Lambda(\varepsilon) - \left( 1 - \frac{\partial \gamma}{\partial s} \right) (P + \varepsilon) - \left( \frac{\partial b}{\partial s} + b^2 - \theta \right) \frac{|y|^2}{4} (P + \varepsilon) + \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) b \frac{|y|^2}{2} (P + \varepsilon) + \Psi, \Lambda P)_2 \]
Here, we have
\[ |(\varepsilon| \cdot |^2 P, \Lambda P)_2| = \frac{1}{\lambda} |\cdot |^2 P, \Lambda P)_2| \]
\[ = O(s^{-(K+2)} + s^{-1}|\text{Mod}(s)|) \]
and
\[ (| \cdot |^2 P, \Lambda P)_2 = -|| \cdot |^2 P, \Lambda P)_2| + O(s^{-2}). \]
Moreover, we have
\[ f(P + \varepsilon) - f(P) - df(Q)\varepsilon = f(P + \varepsilon) - f(P) - df(P)(\varepsilon) + df(P)(\varepsilon) - df(Q)\varepsilon. \]
We prove only the case \( N \geq 4 \). If \( Q < 3|\lambda^\alpha Z| \), then we obtain
\[ |(f(P + \varepsilon) - f(P) - df(P)(\varepsilon)) \Lambda P| \lesssim \lambda^\alpha (1 + |\cdot |^2)(Q_{\Delta}^\pm + |\varepsilon|^\pm)|\varepsilon| Q \]
since \( 1 \lesssim \lambda^\alpha (1 + |\cdot |) \). If \( 3|\lambda^\alpha Z| \leq Q \) and \( Q < 3|\varepsilon| \), then we obtain
\[ |(f(P + \varepsilon) - f(P) - df(P)(\varepsilon)) \Lambda P| \lesssim (1 + |\cdot |^\alpha)Q_{\Delta}^\pm |\varepsilon|^2. \]
If $|\varepsilon| \leq Q$, then $P - |\varepsilon| > \frac{1}{3}Q > 0$. We have
\[
| (f(P + \varepsilon) - f(P) - df(P)(\varepsilon)) \Lambda^T | \lesssim (1 + |\varepsilon|) \frac{Q}{\varepsilon} |\varepsilon|^2.
\]
Therefore, we have
\[
(f(P + \varepsilon) - f(P) - df(P)(\varepsilon), \Lambda P)_2 = O(s^{-(K+2)}).
\]
Similarly, for $(df(P)(\varepsilon) - df(Q)(\varepsilon)) \Lambda P$, we have
\[
(df(P)(\varepsilon) - df(Q)(\varepsilon), \Lambda P)_2 = O(s^{-(K+2)}).
\]
Accordingly, we have
\[
\left( i \frac{\partial \varepsilon}{\partial s}, \Lambda P \right)_2 = -\frac{1}{4} ||Q|| \left( \frac{\partial b}{\partial s} + b^2 - \theta \right) + O(s^{-(K+2)}) + O(s^{-1} |\text{Mod}(s)|)
\]
and
\[
\frac{\partial b}{\partial s} + b^2 - \theta = O(s^{-(K+2)}) + O(s^{-1} |\text{Mod}(s)|).
\]
The same calculations for $\|Q\|$ and $\|\theta\|$ yield
\[
\frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b = O(s^{-(K+2)}) + O(s^{-1} |\text{Mod}(s)|), \quad 1 - \frac{\partial \gamma}{\partial s} = O(s^{-(K+2)}) + O(s^{-1} |\text{Mod}(s)|).
\]
Consequently, we have
\[
|\text{Mod}(s)| \lesssim s^{-(K+2)}, \quad \left\| \varepsilon \gamma(y) \Psi \right\|_{H^1} \lesssim s^{-(K+4)}.
\]
The rest of the proof is the same as the proof in [7, 9].

6. Modified energy function

In this section, we proceed with a modified version [8, 9] of the technique presented in Le Coz, Martel, Raphaëlt
and Martel and Szeftel [14]. Let $m > 0$ be sufficiently large and define
\[
H(s, \varepsilon) := \frac{1}{2} ||\varepsilon||^2_{H^1} + b^2 ||y||_2^2 - \int_{\mathbb{R}^N} (F(P + \varepsilon) - F(P) - dF(P)(\varepsilon)) dy
\]
\[
- \lambda^\alpha \int_{\mathbb{R}^N} (G(P + \varepsilon) - G(P) - dG(P)(\varepsilon)) dy,
\]
\[
S(s, \varepsilon) := \frac{1}{\lambda^m} H(s, \varepsilon).
\]

Lemma 6.1 (Coercivity of $H$). For $s \in (s^*, s_1]$, \[
||\varepsilon||_{H^1}^2 + b^2 ||y||_2^2 + O(s^{-2(K+2)}) \lesssim H(s, \varepsilon) \lesssim ||\varepsilon||_{H^1}^2 + b^2 ||y||_2^2
\]
hold.

**proof.** We prove only the case $N \geq 4$.

If $2|\varepsilon| \geq |P|$, then we have
\[
\left| \int_{\mathbb{R}^N} (F(P + \varepsilon) - F(P) - dF(P)(\varepsilon, \varepsilon)) \frac{1}{2} d^2 F(P)(\varepsilon, \varepsilon) \right| \lesssim |\varepsilon|^{\frac{3}{2} + 2}.
\]

If $2|\varepsilon| < |P|$, then we have
\[
\left| \int_{\mathbb{R}^N} (F(P + \varepsilon) - F(P) - dF(P)(\varepsilon, \varepsilon)) \frac{1}{2} d^2 F(P)(\varepsilon, \varepsilon) \right| \lesssim (|P| - |\varepsilon|)^{\frac{3}{2} - 1} |\varepsilon|^3 \lesssim |\varepsilon|^{\frac{3}{2} + 2}.
\]
Therefore, we obtain
\[
\int_{\mathbb{R}^N} \left( F(P(y) + \varepsilon(y)) - F(P(y)) - dF(P(y))(\varepsilon(y), \varepsilon(y)) \right) dy = o(||\varepsilon||^2_{H^1}).
\]

If $2|\lambda^\alpha Z| \geq Q$, then we have
\[
\left| \frac{1}{2} d^2 F(P)(\varepsilon, \varepsilon) - \frac{1}{2} d^2 F(Q)(\varepsilon, \varepsilon) \right| \lesssim |\lambda^\alpha Z|^\frac{3}{2} |\varepsilon|^2.
\]
If $2|\lambda Z| < Q$, then we have

$$\left| \frac{1}{2} d^2 F(P) (\varepsilon, \varepsilon) - \frac{1}{2} d^2 F(Q) (\varepsilon, \varepsilon) \right| \lesssim \lambda^\alpha (Q - |\lambda^\alpha Z|)^{\frac{\alpha}{2} - 1} |\varepsilon|^2 |Z| \lesssim (1 + |\cdot|^\alpha) \lambda^\alpha |\varepsilon|^2 Q^\frac{\alpha}{2}. $$

Therefore, we obtain

$$\int_{\mathbb{R}^N} \left( \frac{1}{2} d^2 F(P(y)) (\varepsilon(y), \varepsilon(y)) - \frac{1}{2} d^2 F(Q) (\varepsilon(y), \varepsilon(y)) \right) dy = o(\|\varepsilon\|_{H^1}^2). $$

Moreover, we have

$$\int_{\mathbb{R}^N} (G(P(y) + \varepsilon(y)) - G(P(y)) - dG(P(y)) (\varepsilon(y))) dy = O (\|\varepsilon\|_{H^1}^2). $$

Finally, we have

$$\|\varepsilon\|_{H^1}^2 - \int_{\mathbb{R}^N} d^2 F(Q) (\varepsilon(y), \varepsilon(y)) dy = \langle L_+ \Re \varepsilon, \Re \varepsilon \rangle + \langle L_- \Im \varepsilon, \Im \varepsilon \rangle \geq \mu \|\varepsilon\|_{H^1}^2 \left( \frac{1}{\mu} \left( \|\Re \varepsilon, Q\|_2^2 + \|\Re \varepsilon, |z|^2 Q\|_2^2 + \|3 \varepsilon, \rho\|_2^2 \right) \right) = \mu \|\varepsilon\|_{H^1}^2 + O(s^{-2(K+2)}). $$

\[ \square \]

**Corollary 6.2** (Estimation of $S$). For $s \in (s_*, s_1)$,

$$\frac{1}{\lambda^m} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_{L^2}^2 + O(s^{-2(K+2)}) \right) \lesssim S(s, \varepsilon) \lesssim \frac{1}{\lambda^m} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\|_{L^2}^2 \right)$$

hold.

**Lemma 6.3.** For $s \in (s_*, s_1)$ and $0 \leq q \leq \frac{4}{N-4} (N \geq 5)$ and $0 \leq q < \infty (N \leq 4)$,

$$|(|P + \varepsilon|^q (P + \varepsilon) - |P|^q P, \Lambda \varepsilon)_2| \lesssim \|\varepsilon\|_{H^1}^2 + s^{-3K}$$

holds.

**proof.** If $q = 0$, then the lemma holds clearly. Therefore, we may $p \neq 0$.

Let

$$j(z) = |z|^q z, \quad J(z) = \frac{1}{q + 2} |z|^{q + 2}. $$

Calculated in the same way as in Section 5.4 in [7], we have

$$\nabla (J(P + \varepsilon) - J(P) - dJ(P)(\varepsilon)) = \Re \left( j(P + \varepsilon) \nabla (\overline{P} + \varepsilon) - j(P) \nabla \overline{P} - dj(P)(\varepsilon) \nabla \overline{P} - j(P) \nabla \varepsilon \right) = \Re \left( (j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) \nabla \overline{P} + (j(P + \varepsilon) - j(P)) \nabla \varepsilon \right). $$
Therefore, we have

\[
(j(P + \varepsilon) - j(P), \Lambda \varepsilon) = \Re \int_{\mathbb{R}^N} (j(P + \varepsilon) - j(P)) \Lambda \varepsilon \, dy
= \Re \int_{\mathbb{R}^N} (j(P + \varepsilon) - j(P)) \left( \frac{N}{2} \varepsilon + y \cdot \nabla \varepsilon \right) \, dy
= \Re \int_{\mathbb{R}^N} \left( \frac{N}{2} (j(P + \varepsilon) - j(P)) \varepsilon - y \cdot \left( (j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) \nabla P \right)
+ \nabla (j(P + \varepsilon) - J(P) - dj(J(P)(\varepsilon))) \right) \, dy
= \Re \int_{\mathbb{R}^N} \left( \frac{N}{2} (j(P + \varepsilon) - j(P)) \varepsilon - (j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) y \cdot \nabla P
- N (j(P + \varepsilon) - J(P) - dj(J(P)(\varepsilon))) \right) \, dy.
\]

Firstly,

\[
|(j(P + \varepsilon) - j(P)) \varepsilon| + |J(P + \varepsilon) - J(P) - dj(J(P)(\varepsilon))| \lesssim (1 + |\cdot|^\kappa) Q^q + |\varepsilon|^q |\varepsilon|^2
\]

holds.

Next, we consider \((j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) y \cdot \nabla P\). If \(q > 1\), then we have

\[
|(j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) y \cdot \nabla P| \lesssim (1 + |\cdot|^\kappa)(Q + |\varepsilon|)^{q-1} |\varepsilon|^2 Q.
\]

On the other hands, we assume \(q \leq 1\). If \(Q < 3|\lambda^\alpha Z|\), then we have

\[
|(j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) y \cdot \nabla P| \lesssim \lambda^{K\alpha} (1 + |\cdot|^\kappa)(Q^q + |\varepsilon|^q) |\varepsilon| Q
\]

since \(1 \lesssim \lambda^{\alpha}(1 + |\cdot|)\).

If \(3|\lambda^\alpha Z| \leq Q\) and \(Q < 3|\varepsilon|\), then we have

\[
|(j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) y \cdot \nabla P| \lesssim (1 + |\cdot|^\kappa) Q^q |\varepsilon|^2.
\]

If \(3|\varepsilon| \leq Q\), then we have

\[
|(j(P + \varepsilon) - j(P) - dj(P)(\varepsilon)) y \cdot \nabla P| \lesssim (1 + |\cdot|^\kappa) Q^q |\varepsilon|^2.
\]

\(\square\)

**Lemma 6.4** (Derivative of \(H\) in time). For \(s \in (s_*, s_1)\),

\[
\frac{d}{ds} H(s, \varepsilon(s)) \geq -C b \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\varepsilon\|_2^2 \right) + O(s^{-2(K+2)})
\]

holds.

**proof.** We prove the lemma by combining Lemma 6.3 and the proofs in 7 8 9. \(\square\)

**Lemma 6.5** (Derivative of \(S\) in time). Let \(m > 0\) be sufficiently large. Then,

\[
\frac{d}{ds} S(s, \varepsilon(s)) \geq \frac{b}{\lambda^m} \left( \|\varepsilon\|_{H^1}^2 + b^2 \|y\varepsilon\|_2^2 + O(s^{-(2K+3)}) \right)
\]

holds for \(s \in (s_*, s_1)\).

**proof.** See 7 9 for the proof. \(\square\)
7. Bootstrap

In this section, we use the estimates obtained in Section 6 and the bootstrap to establish the estimates of the parameters. However, we introduce the following lemmas without proofs. Regarding lemmas in this section, see [7, 9] for the proof.

Lemma 7.1 (Re-estimation). For \( s \in (s_*, s_1) \),
\[
\|e(s)\|_{H^1}^2 + b(s)^2 \|e(s)\|_{L^2}^2 \lesssim s^{-(2K+2)},
\]
(12)
\[
\left| \frac{\lambda(s)^{\frac{2}{
\lambda_{app}(s)}}}{1} + \frac{b(s)}{\lambda_{app}(s)} - 1 \right| \lesssim s^{-\frac{1}{2}} + s^{-\frac{1}{2}}
\]
holds.

Corollary 7.2. If \( s_0 \) is sufficiently large, then \( s_* = s' \).

Lemma 7.3. If \( s_0 \) is sufficiently large, then \( s' = s_0 \).

Lemma 7.4 (Interval). If \( s_0 \) is sufficiently large, then there is a \( t_0 < 0 \) that is sufficiently close to 0 such that for \( t_1 \in (t_0, 0) \),
\[
[t_0, t_1] \subset s_t^{-1}([s_0, s_1]), \quad \left| \mathcal{C} s_t(t) \frac{s_{\n\lambda(t)}}{H^1} - |t| \right| \lesssim |t|^{1 + \frac{\alpha}{4}} (t \in [t_0, t_1])
\]
holds.

Lemma 7.5 (Conversion of estimates). Let
\[
\mathcal{C}_\lambda := C^{-\frac{2}{4-n}} \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2 - \alpha}} \right)^{\frac{1}{2}} , \quad \mathcal{C}_b := \frac{2}{\alpha} C^{-\frac{2}{4-n}}.
\]
For \( t \in [t_0, t_1] \),
\[
\tilde{\lambda}_t(t) = \mathcal{C}_\lambda |t|^\frac{\alpha}{4-n} \left( 1 + \epsilon_{\tilde{\lambda},t_1}(t) \right) , \quad \tilde{b}_t(t) = \mathcal{C}_b |t|^\frac{\alpha}{4-n} \left( 1 + \epsilon_{\tilde{b},t_1}(t) \right) , \quad ||\tilde{\epsilon}_{t_1}(t)||_{H^1} \lesssim |t|^\frac{\alpha}{4-n}, \quad || \cdot |\tilde{\epsilon}_{t_1}(t)||_{L^2} \lesssim |t|^\frac{\alpha}{4-n-1}
\]
holds. Furthermore,
\[
\sup_{t_1 \in [t_0, t_1]} |\epsilon_{\tilde{\lambda},t_1}(t)| \lesssim |t|^\frac{\alpha}{4-n} , \quad \sup_{t_1 \in [t_0, t_1]} |\epsilon_{\tilde{b},t_1}(t)| \lesssim |t|^\frac{\alpha}{4-n}.
\]

8. Proof of the Main Result

proof. Let \((t_n)_{n \in \mathbb{N}} \subset (t_0, 0)\) be a monotonically increasing sequence such that \( \lim_{n \to \infty} t_n = 0 \). For each \( n \in \mathbb{N} \), \( u_n \) is the solution for (NLS+) with an initial value
\[
u_n(t_n, x) := P_{\lambda_{t_n}, b_{t_n}, 0}(x)
\]
at \( t_n \), where \( b_{t_n} \) and \( \lambda_{t_n} \) are given by Lemma 4.2 for \( t_n \).

According to Lemma 6.2 with an initial value \( \tilde{\eta}_n(t_n) = 0 \) on \([t_0, t_1]\), there exists a decomposition
\[
u_n(t, x) = \frac{1}{\lambda_n(t)} \left( P + \tilde{\eta}_n \right) \left( t, \frac{x}{\lambda_n(t)} \right) e^{-i \int_0^t \frac{b_{\lambda_n}(t) + \frac{\alpha}{2} \lambda_n(t)^2}{\lambda_n(t)^2} \lambda_n(t) dt} + \eta_n(t).
\]
Then, \((u_n(t_0))_{n \in \mathbb{N}}\) is bounded in \( \Sigma^1 \). Therefore, up to a subsequence, there exists \( u_\infty(t_0) \in \Sigma^1 \) such that
\[
u_n(t_0) \rightharpoonup u_\infty(t_0) \quad \text{in} \quad \Sigma^1 , \quad u_n(t_0) \to u_\infty(t_0) \quad \text{in} \quad \Sigma \quad (n \to \infty),
\]
see [7, 8] for details.

Let \( u_\infty \) be the solution for (NLS+) with an initial value \( u_\infty(t_0) \) and \( T^* \) be the supremum of the maximal existence interval of \( u_\infty \). Moreover, we define \( T := \min \{ 0, T^* \} \). Then, for any \( T' \in [t_0, T) \), \([t_0, T'] \subset [t_0, t_n] \) if \( n \) is sufficiently large. Then, there exist \( n_0 \) and \( C(T', t_0) > 0 \) such that
\[
\sup_{n \geq n_0} \| u_n \|_{L^\infty([t_0, T'], \Sigma^1)} \leq C(T', t_0)
\]
holds. According to Lemma B.2 in [8],

\[ u_n \to u_\infty \quad \text{in} \quad C \left( [t_0, T], L^2(\mathbb{R}^N) \right) \quad (n \to \infty) \]

holds. In particular, \( u_n(t) \to u_\infty(t) \) in \( \Sigma^1 \) for any \( t \in [t_0, T] \). Furthermore, from the mass conservation, we have

\[ \| u_\infty(t) \|_2 = \| u_\infty(t_0) \|_2 = \lim_{n \to \infty} \| u_n(t_0) \|_2 = \lim_{n \to \infty} \| u_n(t_n) \|_2 = \lim_{n \to \infty} \| P(t_n) \|_2 = \| Q \|_2. \]

Based on weak convergence in \( H^1(\mathbb{R}^N) \) and Lemma 3.2, we decompose \( u_\infty \) to

\[ u_\infty(t, x) = \frac{1}{\lambda_\infty(t)} \left( P + \tilde{\varepsilon}_\infty \right) \left( t, \frac{x}{\lambda_\infty(t)} \right) e^{-\frac{\lambda_\infty(t)}{4} \frac{|x|^2}{\lambda_\infty(t)^2} + i \tilde{\varepsilon}_\infty(t)}, \]

where an initial value of \( \tilde{\varepsilon}_\infty \) is \( \gamma_\infty(t_0) \in \left( |t_0|^{-1} - \pi, |t_0|^{-1} + \pi \right) \cap \tilde{\gamma}(u_\infty(t_0)) \) (which is unique, see [8]). Furthermore, for any \( t \in [t_0, T) \), as \( n \to \infty \),

\[ \lambda_n(t) \to \lambda_\infty(t), \quad b_n(t) \to \tilde{b}_\infty(t), \quad e^{i \tilde{\varepsilon}_n(t)} \to e^{i \tilde{\varepsilon}_\infty(t)}, \quad \tilde{\varepsilon}_n(t) \to \tilde{\varepsilon}_\infty(t) \]

holds. Consequently, for a uniform estimate of Lemma 7.5 as \( n \to \infty \), we have

\[ \lambda_\infty(t) = C \lambda |t|^{\frac{1}{2n}} (1 + \epsilon_\lambda(t)), \quad \tilde{b}_\infty(t) = C \epsilon_\lambda,0(t), \quad \| \tilde{\varepsilon}_\infty(t) \|_{H^1} \lesssim |t|^{\frac{1}{2n}}, \quad \| |b| \|_{H^1} \lesssim |t|^{\frac{1}{2n}}, \quad \epsilon_\lambda,0(t) \lesssim |t|^{\frac{1}{2n}}. \]

Consequently, we obtain that \( u \) converge to the blow-up profile in \( \Sigma^1 \).

Finally, we check energy of \( u_\infty \). Since

\[ E (u_n) - E \left( P_{\lambda_n, b_n, \tilde{\varepsilon}_n} \right) = \int_0^1 \left\langle E' \left( P_{\lambda_n, b_n, \tilde{\varepsilon}_n} \right), \frac{d}{dt} \tilde{\varepsilon}_n \right\rangle \, dt \]

and \( E'(w) = -\Delta w - |w|^4 w - |y|^{-2\sigma w} \), we have

\[ E (u_n) - E \left( P_{\lambda_n, b_n, \tilde{\varepsilon}_n} \right) = O \left( \frac{1}{\lambda_n^2} \| \tilde{\varepsilon}_n \|_{H^1} \right) = O \left( |t|^{\frac{1}{2n}} \right). \]

Similarly, we have

\[ E (u_\infty) - E \left( P_{\lambda_\infty, \tilde{b}_\infty, \tilde{\varepsilon}_\infty} \right) = O \left( \frac{1}{\lambda_\infty^2} \| \tilde{\varepsilon}_\infty \|_{H^1} \right) = O \left( |t|^{\frac{1}{2n}} \right). \]

From continuity of \( E \), we have

\[ \lim_{n \to \infty} E \left( P_{\lambda_n, b_n, \tilde{\varepsilon}_n} \right) = E \left( P_{\lambda_\infty, \tilde{b}_\infty, \tilde{\varepsilon}_\infty} \right) \]

and from the conservation of energy,

\[ E (u_n) = E (u_n(t_n)) = E \left( P_{\lambda_1, b_1, \tilde{\varepsilon}_1} \right) = E_0. \]

Therefore, we have

\[ E (u_\infty) = E_0 + o_{n \to \infty} \]

and since \( E (u_\infty) \) is constant for \( t \), \( E (u_\infty) = E_0 \). \( \square \)

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(N. Matsui) DEPARTMENT OF MATHEMATICS, TOKYO UNIVERSITY OF SCIENCE, 1-3 KAGURAZAKA, SHINJUKU-KU, TOKYO 162-8601, JAPAN

*Email address*, N. Matsui: [1120703@ed.tus.ac.jp](mailto:1120703@ed.tus.ac.jp)