CORONA THEOREM
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Abstract. For a wide class of domains $G \subset \mathbb{C}^d$ including balls and polydisks we prove the density of their canonical image in the spectrum of $H^\infty(G)$. This Corona Theorem is proved first in its abstract version for certain uniform algebras. We use properties of bands of measures and idempotents corresponding to Gleason parts. The essential tools are properties of hyper-Stonean spaces, normal and Henkin measures and some ideas based on Hoffman - Rossi theorem. We also use our previous results on weak-star closures of reducing bands of measures. Uniform bounds for operators used to solve the Gleason problem concerning ideals of analytic functions vanishing at given points are applied for bidual algebras.

Contents

1. Introduction 1
  1.1. Formulation of the problem 1
  1.2. Main ideas of the proof 2
  1.3. Preliminary notions 3
  1.4. Structure of the paper 3
2. Reducing bands 4
3. Embedding measures and spectra 8
4. Gleason parts and idempotents 12
5. Quotient algebras (Algebras of $H^\infty$-type) 14
6. Hoffman – Rossi type theorems 16
7. Abstract Theorems on Henkin Measures 19
8. Abstract Corona Theorem 21
9. Stability of $G$ 21
10. Classical setting 23
Additional Remarks 25
References 25

1. Introduction

1.1. Formulation of the problem. Corona problem is the question whether $\Omega$ is dense in the spectrum $\text{Sp}(H^\infty(\Omega))$ of the Banach algebra $H^\infty(\Omega)$ of bounded analytic functions on a domain $\Omega$ in the space $\mathbb{C}^d$, where $d \in \mathbb{N}$.

By a domain we mean an open, connected and bounded set, $\mathbb{C}$ denotes the complex plane and the natural embedding of $\Omega$ in $\text{Sp}(H^\infty(\Omega))$ is through the evaluation
functions $\delta_z : f \mapsto f(z)$ at the points $z \in \Omega$. For a uniform algebra $A$ we consider $\text{Sp}(A)$ as the set consisting of all nonzero linear, multiplicative functionals on $A$, with Gelfand (i.e. weak-star) topology. History of the research of corona problem is outlined in [10].

An equivalent formulation (see [15, Chapter X], [12, Thm. V.1.8]) asserts the existence of solutions $(g_1, \ldots, g_n) \in (H^\infty(\Omega))^n$ of the Bézout equation

$$f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) = 1, \quad z \in \Omega$$

for any finite corona data, which by definition is an $n$-tuple $(f_1, \ldots, f_n) \in H^\infty(\Omega)^n$ satisfying the following uniform estimate with some $c > 0$:

$$|f_1(z)| + \cdots + |f_n(z)| \geq c, \quad z \in \Omega.$$

In the original statement $\Omega$ was the open unit disc $\mathbb{D}$ on the complex plane. Using hard analysis methods Lennart Carleson has solved the problem for this case in [3]. His solution and most of the subsequent attempts to extend it for more general domains $\Omega$ were aimed at solving (1.1). That approach met serious difficulties in the case when $d > 1$ and only negative examples were known for some weakly pseudoconvex domains. The first positive higher-dimensional result seems to be the following statement:

**Theorem 1.1** (Corona Theorem). If $G \subset \mathbb{C}^d$ is either a domain strictly pseudoconvex or a strongly starlike polydomain then Corona Theorem holds true in $H^\infty(G)$: its spectrum is the Gelfand closure of the canonical image of $G$.

This theorem will be obtained in the last section, as a consequence of its abstract counterpart from Section 8. Our approach is based on theories of uniform algebras, bands of normal measures and properties of Henkin measures stated in Section 7.

1.2. **Main ideas of the proof.** The analogous result is easily available for the subalgebra $A(\Omega)$ in $H^\infty(\Omega)$ consisting of those elements that extend by continuity to the closure of $\Omega$. On the other hand, $H^\infty(\Omega)$ can be identified with a quotient algebra of the second dual of $A(\Omega)$. This allows one to apply duality methods.

We were trying to formulate an abstract analogue of corona theorem valid for uniform algebras $A$ satisfying certain general assumptions. Our aim was to find general assumptions implying the density of the canonical image of a Gleason part $G$ of $\text{Sp}(A)$ in the spectrum of an abstract counterpart of $H^\infty(\Omega)$. In the main application this part $G$ corresponds to the domain $\Omega$. The needed hypotheses on $G$ and $A$ correspond to some properties of bounded analytic functions that are well known to hold in many cases.

Based on ideas of Hoffman-Rossi Theorem and on lattice properties of real-valued continuous functions, first we show that for each representing measure $\mu$ at a point of $G$ its image $j(\mu)$ is concentrated on the Gelfand closure of $j(G)$. In this part we are also making essential use of properties of normal measures on hyper-Stonean spaces.

Next, we prove some abstract versions of Henkin measures Theorem in Section 7. This combined with description of the spectra of quotient algebras (Section 5) implies Abstract Corona Theorem stated in Section 8.
When we pass from $A(G)$ to $H^\infty(G)$, the fiber structure of the spectra changes: fibers over points $\lambda \in \partial G$ grow from singletons to much larger sets including (at least when $G = \mathbb{D}$) analytic discs $[15]$. However, for $\lambda \in G$ they remain singletons. The same is true in more general domains and for the spectrum of $A(G)^{**} = A_1$. This, as we call it stability of the main Gleason part is another key property.

The assumptions of Abstract Corona Theorem from Section 8 are verified in Sections 9 and 10 for a large class of domains $\Omega$. This implies the main "classical" result including the case of balls and polydisks in $\mathbb{C}^d$.

1.3. Preliminary notions. A concept that sheds some light on the geometry of the maximal ideal space of a uniform algebra $A$ (denoted here as $\text{Sp}(A)$) is that of Gleason parts, defined as equivalence classes under the relation

$$\|\phi - \psi\| < 2$$

given by the norm of the corresponding functionals $\phi, \psi \in \text{Sp}(A)$ (see [11]). In the most important case $A$ will be the algebra $A(G)$ of analytic functions on a domain $G \subset \mathbb{C}^d$, which extend to continuous functions on $\overline{G}$ -the Euclidean closure of $G$. Then by connectedness and by Harnack’s inequalities, the entire set $G$ corresponds to a single non-trivial Gleason part. Moreover $\|f\| = \sup_{z \in G}|f(z)|$ for all $f \in A(G)$. Trivial (i.e. one-point) parts correspond to certain boundary points $x \in \partial G$.

Given a non-empty, compact Hausdorff space $X$, we consider a closed, unital subalgebra $A$ of $C(X)$.

The second dual: $A^{**}$ of $A$, endowed with Arens multiplication $\cdot$ is then a uniform algebra and its spectrum will be denoted by $\text{Sp}(A^{**})$, while

$$Y := \text{Sp}(C(X)^{**})$$

is a hyper-Stonean envelope of $X$. Here one should bear in mind the Arens-regularity of closed subalgebras of $C^*$ algebras, so that the right and left Arens products coincide and are commutative, separately w-* continuous extensions to $A^{**}$ of the product from $A$.

1.4. Structure of the paper. The first five sections contain some preliminary results mostly known in different formulations, stated in the form needed in the sequel.

To a Gleason part of $A$ there corresponds an idempotent $g$ in the second dual algebra $A^{**}$. Its construction can be found in papers published nearly 50 years ago (like [23]), but Theorem 4.17 of the recent monograph [9] contains a version more convenient to apply in the present work.

Sections 6 and 7 contain key tools allowing us to obtain in Section 8 Abstract Corona Theorem. In Section 6 we prove results related to Hoffman-Rossi Theorem, which are exploited essentially in the next section, where new properties of Henkin measures are established.

In Section 9 we verify the general assumptions used in abstract setup – now for the concrete algebras over strictly pseudoconvex domains and over certain poly-domains in $\mathbb{C}^d$. This leads to the final result in Section 10.
2. Reducing bands

By $M(X)$ we denote the space of complex, regular Borel measures on $X$ and by $M_+(X)$ — its nonnegative part, with total variation norm. We use the standard notation $E^+ \cap M(X)$ (or $E^+$) for the annihilator of a set $E \subset C(X)$. $E^+$ consists of all $\mu \in M(X)$ satisfying $\int f \, d\mu = 0$ if $f \in E$.

A closed subspace $\mathcal{M}$ of $M(X)$ is called a band of measures if along with $\nu \in \mathcal{M}$ it contains any measure absolutely continuous with respect to $|\nu|$. Then the set $\mathcal{M}^s$ of all measures singular to all $\mu \in \mathcal{M}$ is also a band, forming a direct-$\ell^1$-sum decomposition: $M(X) = \mathcal{M} \oplus \mathcal{M}^s$. We denote the corresponding Lebesgue–type decomposition summands of $\mu$ by $\mu^{\mathcal{M}}$ and $\mu^s$, so that

$$\mu = \mu^{\mathcal{M}} + \mu^s \quad \text{with} \quad \mu^{\mathcal{M}} \in \mathcal{M}, \; \mu^s \in \mathcal{M}^s.$$ 

Also if $\mu \geq 0$, then both $\mu^{\mathcal{M}} \geq 0$ and $\mu^s \geq 0$.

**Definition 2.1.** A reducing band on $X$ for the subalgebra $A$ of $C(X)$ is the one satisfying $\mu^{\mathcal{M}} \in A^+ \cap M(X)$ for any $\mu \in A^+ \cap M(X)$. Here we even admit situation where $A$ is not separating the points of $X$, but we shall always assume that $1 \in A$ and that for any $f \in A$ we have $\|f\| = \sup\{|f(x)| : x \in X\}$.

A closed set $E \subset X$ is a reducing set for $A$ on $X$ if $M(E) := \{\chi_E \cdot \mu : \mu \in M(X)\}$ is a reducing band on $X$ for $A$. Here $(\chi_E \cdot \mu)(\Delta) = \mu(\Delta \cap E)$ and generally, we write $\chi \cdot \mu$ to indicate a measure $\mu$ multiplied by some ”density” function $\chi$.

For example, $E$ is a reducing set for $A$ (in a particular case) if $\chi_E \in A$, since $\nu \in A^+ \cap M(X)$ implies $\int_E d\nu = 0$. For clopen sets the converse is true:

**Lemma 2.2.** If $E$ is a clopen reducing set for $A$, then $\chi_E \in A$.

*Proof.* Now $\chi_E \in C(X)$ and no $\mu \in C(X)^*$ can separate $\chi_E$ from $A$. Indeed, if $\mu \in A^+$, then $\langle \mu, \chi_E \rangle = \langle \chi_E \cdot \mu, 1 \rangle = 0$, since $\mu = \chi_E \cdot \mu + (1 - \chi_E) \cdot \mu$ is a band decomposition of $M(X)$ with respect to $M(E)$. \hfill $\Box$

By a representing (respectively complex representing) measure for $\phi \in A^*$ (or at the point $\phi \in \text{Sp}(A)$) on $X$ we mean a nonnegative (respectively -complex) measure $\mu \in M(X)$ such that

$$\phi(f) = \int_X f \, d\mu \quad \text{for any} \quad f \in A.$$ 

Let us denote by $M_{\phi}(X)$ the set of all representing measures for $\phi$ on $X$.

In some places we assume that $A$ is a uniform algebra on $X$, which means that it separates the points of $X$ and $\|f\| = \sup_{x \in X} |f(x)|$. Sometimes we will suppose additionally that $A$ is a natural algebra in the sense that $\text{Sp}(A) = X$, so that any nonzero multiplicative and linear functional on $A$ is of the form $\delta_x : A \ni f \mapsto f(x) \in \mathbb{C}$ for some $x \in X$.

If $G$ is a non-trivial Gleason part, denote by $\mathcal{M}_G$ the smallest band in $M(\text{Sp}(A))$ containing all representing measures at a point $\phi \in G$. It does not depend on the choice of $\phi \in G$. 
Lemma 2.3. If $\mathcal{M} \subset M(X)$ is a reducing band on $X$ for $A$ which contains a measure $\mu$ representing a point $x \in \text{Sp}(A)$ then it contains all representing measures on $X$ for $x$. If $\mathcal{M} = M(H)$ for some closed reducing set $H \subset X$, then for every point $x \in X$ all its representing measures $\mu$ for $A$ either satisfy $\mu \in \mathcal{M}$ or all belong to $\mathcal{M}^s$. If moreover $H$ is clopen, then the same dichotomy appears for measures representing any point $x \in \text{Sp}(A)$.

Proof. Let $\nu$ be another representing measure at $x$ for $A$. Then the real measure $\mu - \nu$ annihilates $A$ and it decomposes into two parts $\mu - \nu^M \in \mathcal{M}$ and $\nu^s \in \mathcal{M}^s$, both annihilating $A$ by the reducing property. But $\nu^s$ must be equal 0 as a nonannihilating annihilating measure, which proves the first claim.

In the case when $\mathcal{M} = M(H)$, if $x \in H$ then $\delta_x \in M(H)$ is a representing measure and all representing measures must be in $M(H)$. In the second case -when $x \in X \setminus H$, we have $\delta_x \in M(X \setminus H)$, and $M(H)^*$ is a reducing band for $A$ on $X$, hence it contains all representing measures for $x$.

If $H$ is clopen and reducing, then by Lemma 2.2 we have $\chi_H \in A$. The integral $\int \chi_H \, d\mu$ equals $\chi_H(x)$. If this equals 0, then $\mu \in \mathcal{M}^s$. Otherwise $\int \chi_H \, d\mu = 1$ and then $\mu \in \mathcal{M}$. □

Remark 2.4. The Arens product in $A^{**}$ is separately continuous in $w^*$-topology and the $w^*$-density of the canonical image in the bidual space can be used to determine such product by iteration. The bidual $A^{**}$ of $A$ with the Arens multiplication can be considered as a closed subalgebra of $C(X)^{**}$ (closed both in norm and in $w^*$-topologies). The canonical map $\iota : A \rightarrow A^{**}$ is a counterpart of the corresponding

$$
\iota_C : C(X) \rightarrow C(X)^{**} \simeq C(Y)
$$

and $\iota$ is both a linear and multiplicative homomorphism. Let us denote by $J$ the isomorphism $J : C(Y) \rightarrow M(X)^*$. Here $Y$ is the hyper-Stonean envelope of $X$. The property of strongly unique preduals and the corresponding results collected in [8, Theorem 6.5.1] allow us to treat the action of the isomorphism $J^{-1}$ on $\iota_C(C(X))$ as identity, hence we shall omit $J$ in all formulae (except of the next one).

To any $\mu \in M(X)$ we can assign $k(\mu) \in M(Y)$, a unique measure given by

$$
(2.1) \quad \langle k(\mu), f \rangle = \langle J(f), \mu \rangle \quad f \in C(Y), \ \mu \in M(X).
$$

Note that (after the mentioned identification) for $f \in C(X)$ we also have

$$
(2.2) \quad \langle k(\mu), \iota_C(f) \rangle = \langle \mu, f \rangle, \quad \text{i.e.} \quad \mu = k(\mu) \circ \iota_C, \quad \text{or} \quad \iota_C^* \circ k = \text{id}.
$$

Here $id$ is the identity map. Then we obtain an embedding $k : M(X) \rightarrow M(Y)$. As a weak-star continuous functional on $C(Y)$, $k(\mu)$ is a normal measure on $Y$. (However one has to bear in mind that $k$ is usually not a $w^*$-continuous mapping.) The normal measures form the (strongly unique) predual of the space $C(Y)$ [8, Theorem 6.4.2] and are $w^*$-continuous. By the $w^*$-density of the unit ball of $C(X)$ in that of $C(Y)$ we have

Lemma 2.5. The mapping $k$ is an isometry.
Now given a bounded linear functional \( \phi \) on \( A \) and its representing measure \( \nu \in M_\phi(X) \), we apply the mapping \( k \) to obtain an “extension” of \( \phi \) to \( A^{**} \) denoted also by \( k(\phi) \). More precisely, for \( f \in A^{**} \subset C(Y) \) we have

\[
(2.3) \quad k(\phi)(f) = \int f(dk(\nu)), \quad \text{where} \quad \nu \in M_\phi(X).
\]

It is easy to note the independence of the choice of \( \nu \in M_\phi \), applying equality \((2.2)\) to the difference of any two such measures.

If \( f \in A \) is identified with its canonical image \( \iota_C(f) \), then from equality \((2.2)\) we deduce that

\[
k(\phi)(\iota_C(f)) = \phi(f),
\]

justifying the name “extension”. The w*-density of \( \iota(A) \) in \( A^{**} \) implies the uniqueness of \( k(\phi) \) among w*-continuous extensions of \( \phi \).

These facts including the following Proposition are consequences of Sections 5.4 and 6.5 in [8]:

**Proposition 2.6.** The measure \( k(\mu) \) is a normal measure on \( Y = \text{Sp}(C(X)^{**}) \) for any \( \mu \in M(X) \). Also any multiplicative linear functional \( \phi \) on \( A \) has a unique w*-continuous extension acting on \( A^{**} \). This extension \( k(\phi) \) is also multiplicative.

We also have the following property of \( k \) that results from the w*-density of \( \iota(A) \) in \( A^{**} \):

**Lemma 2.7.** If \( \mu \in A^\perp \), then \( k(\mu) \in (A^*)^\perp \). Also if \( \mu \) is representing at \( x \) for \( A \) then \( k(\mu) \) represents evaluation at \( k(x) \) for \( A^{**} \).

Note that for any \( h \in C(X), \mu \in M(X) \) we have

\[
(2.4) \quad \iota_C(h) \cdot k(\mu) = k(h \cdot \mu).
\]

Indeed, for any \( f \in C(Y) \) we have

\[
(\iota_C(h) \cdot k(\mu), f) = (k(\mu), \iota_C(h)f) = (k(h \cdot \mu), f) = (f, h \cdot \mu) = (k(h \cdot \mu), f).
\]

The last but one equality comes from the definition of the Arens product.

This allows us to define the product \( g \odot \mu \) of a measure \( \mu \in M(X) \) by \( g \in C(Y) \) as a measure on \( X \) given by the formula

\[
(2.5) \quad (g \odot \mu, f) := (g \cdot k(\mu), \iota_C(f)), \quad f \in C(X) \quad \text{i.e.} \quad g \odot \mu = (g \cdot k(\mu)) \circ \iota_C.
\]

In particular, for the constant function \( f = 1 \), since \( \iota_C(1) = 1 \), we have

\[
(2.6) \quad (g \odot \mu, 1) = (g \cdot k(\mu), 1) = (k(\mu), g) = (g, \mu).
\]

Here we treat \( g \in C(Y) \) as an element of \( M(X)^* \).

**Lemma 2.8.** The mapping \( M(Y) \ni \nu \mapsto g \odot \nu \in M(Y) \) is weak-star continuous.

**Proof.** Let \( \nu_\alpha \to \nu \) in w*-topology, which means that for any \( f \in C(Y) \) we have

\[
\int f dv_\alpha \to \int f dv. \quad \text{Consequently,}
\]

\[
\int f dg \cdot v_\alpha) = \int fg dv_\alpha \to \int fg dv = \int f dg \cdot v.
\]

\[\square\]
Lemma 2.9. The mapping $M(X) \ni \mu \mapsto g \circ \mu \in M(X)$ is norm continuous.

Proof. Since $k$ and $\iota_C$ are isometries, $\|g \circ \mu\| = \|(g \cdot k(\mu)) \circ \iota_C\| \leq \|g\|\|\mu\|$ and the result follows since $g \circ \mu$ depends linearly on $\mu$. \qed

If we put $g \circ M(X) := \{g \circ \mu : \mu \in M(X)\}$, we have the following

Remark 2.10. For $\mu \in M(X)$ and $g \in C(Y)$ we have $k(g \circ \mu) = g \cdot k(\mu)$. Consequently $k(g \circ M(X)) \subset g \cdot M(Y)$.

Proof. By (2.2) and (2.5), for $f \in C(X)$ we have $\langle k(g \circ \mu), \iota_C(f) \rangle = \langle g \cdot k(\mu), \iota_C(f) \rangle$. Now the claim follows from the weak-star density of $\iota_C(C(X))$ in $C(X)^{\ast\ast}$, and from the separate weak-star continuity of Arens product. \qed

Theorem 2.11. For any idempotent $g \in A^{\ast\ast}$ the subspace $g \circ M(X)$ of $M(X)$ is a reducing band on $X$ for $A$. Also $g \cdot M(Y) = M(\{y \in Y : g(y) = 1\})$ is a weak-star closed band in $M(Y)$ reducing for $A^{\ast\ast}$ on $Y$.

Proof. By Remark 2.10 and by Lemma 2.5, the subspace $g \circ M(X)$ is norm-closed in $M(X)$. To show that $g \circ M(X)$ is a band, take $\mu \in M(X)$ and $h \in L^1(\mu)$. We need to check whether $h \cdot (g \circ \mu)$ is in $g \circ M(X)$. By the density of $C(X)$ in $L^1(\mu)$, we may even assume that $h \in C(X)$. Then for $f \in C(X)$ we have

$$ (2.7) \quad \langle h \cdot (g \circ \mu), f \rangle = \langle g \circ \mu, h \cdot f \rangle = \langle g \cdot k(\mu), \iota_C(f) \cdot \iota_C(h) \rangle = \langle g \cdot \iota_C(h \cdot k(\mu), \iota_C(f) \rangle $$

By (2.4) $g \cdot \iota_C(h \cdot k(\mu)$ has the needed form $g \cdot k(h \cdot \mu)$ with $h \cdot \mu \in M(X)$. Hence by (2.5) $h \cdot (g \circ \mu) = g \circ (h \cdot \mu) \in g \circ M(X)$.

Next we consider the band decomposition $\eta = \eta_1 + \eta_2$ with respect to $g \circ M(X)$ of a measure $\eta \in A^\perp \cap M(X)$ (annihilating $A$) with $\eta_1 \in g \circ M(X)$, $\eta_2 \in (g \circ M(X))^\ast$. We claim that $\eta_1 = g \circ \eta$, $\eta_2 = (1 - g) \circ \eta$. To see this it suffices to show that $\mu_1 := g \circ \mu \perp \nu_2 := (1 - g) \circ \nu$ for any $\mu, \nu \in M(X)$.

Here we use the following metric characterisation (in the total variation norms) of singularity (see [3] Prop. 4.2.5):

$$ (2.8) \quad \mu_1 \perp \nu_2 \iff \|\mu_1 \pm \nu_2\| = \|\mu_1\| + \|\nu_2\|. $$

By (2.8) and Remark 2.10 using the isometry of $k$ we obtain

$$\|\mu_1 \pm \nu_2\| = \|g \circ k(\mu_1 \pm (1-g) \circ \nu)\| = \|k(g \circ \mu_1 \pm (1-g) \circ \nu)\| = \|k(g \circ \mu_1) \pm k((1-g) \circ \nu)\|
= \|g \cdot k(\mu_1) \pm (1-g) \cdot k(\nu)\| = \|g \cdot k(\mu_1)\| + \|(1-g) \cdot k(\nu)\|
= \|g \circ \mu_1\| + \|k((1-g) \circ \nu)\| = \|g \circ \mu_1\| + \|(1-g) \circ \nu\| = \|\mu_1\| + \|\nu_2\|,$$

which gives the desired singularity.

If $f \in A, \eta \in A^\perp$, then using the fact that (by Lemma (2.7) $k(\eta)$ annihilates $A^{\ast\ast}$ and (2.5), we obtain $\langle g \circ \eta, f \rangle = 0$, since $g \cdot \iota(f) \in A^{\ast\ast}$. Hence our band is reducing for $A^{\ast\ast}$. The last claim follows directly from the remark preceding Lemma 2.2 for $\chi_E = g$. \qed

Throughout the present paper we denote by $\overline{E}^\sigma$ the closure of a set $E$ in the corresponding $w^* -$ topology of a dual space. If $E \subset F$, where $F$ is a $w^* -$ closed subset of a dual space, then this $\overline{E}^\sigma$ is the closure of $E$ in the induced $w^*-$ topology on $F$. For example, if $E \subset \text{Sp}(A)$, then $\overline{E}^\sigma$ is the Gelfand topology closure of $E$. 

Under the above assumptions we have the following properties.

Lemma 3.1. \( \lim k(\mu) \) is the smallest weak-star closed band containing \( (3.1) \) since \( \mathcal{M} \) is hyper-Stonean. Hence \( k(\mu) \) is of the form \( \mathcal{M} \) and, as normal mutually singular measures on a hyper-Stonean space \( Y \), they have disjoint clopen supports. Denote

\[ V := \bigcup \{ \text{supp}(k(\mu)) : \mu \in \mathcal{M} \}, \quad W := \bigcup \{ \text{supp}(k(\nu)) : \nu \in \mathcal{M}^* \}. \]

The sets \( V, W \) are open and disjoint which gives \( V^\sigma \cap W = \emptyset \). But \( V^\sigma \) is also open since \( Y \) is hyper-Stonean. Hence \( V^\sigma \cap W^\sigma = \emptyset \). Every weak-star closed band in \( M(Y) \) is of the form \( M(Y') \) for some closed subset \( Y' \subset Y \) and \( k(\mathcal{M}) \) is a weak-star closed band in \( M(Y) \) for any band \( \mathcal{M} \subset M(X) \) (see [20, Proposition 3]). This implies \( k(\mathcal{M})^\sigma = M(V') \) and \( k(\mathcal{M}^*)^\sigma = M(W') \) since \( M(V') \) (respectively \( M(W') \)) is the smallest weak-star closed band containing \( k(\mathcal{M}) \) (respectively \( k(\mathcal{M}^*) \)). Also by [20, Proposition 3] we have \( V^\sigma \cup W^\sigma = Y \). Putting \( E := V^\sigma \) and \( F := W^\sigma \) we obtain the desired conditions.

Assume now \( \mu \in (A^{**})^\perp \cap M(Y) \). This measure is a limit of some net. Namely, \( \mu = \lim k(\mu_\alpha) \), where \( \mu_\alpha \in A^+ \cap M(X) \) and \( \|\mu_\alpha\| \leq \|\mu\| \). This inequality can be obtained by the identification \( (A^{**})^\perp \sim A^+ \perp A^+ \sim (A^+)^{**} \). For every \( \alpha \) we have a decomposition \( \mu_\alpha = \mu^a_\alpha + \mu^s_\alpha \), where \( \mu^a_\alpha \in \mathcal{M}, \mu^s_\alpha \in \mathcal{M}^s \) and \( \|\mu_\alpha\| = \|\mu^a_\alpha\| + \|\mu^s_\alpha\| \). By the latter equality the nets \( k(\mu^a_\alpha) \) and \( k(\mu^s_\alpha) \) are bounded and, by the weak-star density of \( \iota(A) \) in \( A^{**} \), their terms annihilate \( A^{**} \). Hence they have weak-star adherent points \( \mu^a \in k(\mathcal{M})^\sigma, \mu^s \in k(\mathcal{M}^*)^\sigma \) annihilating \( A^{**} \). Moreover, \( \mu = \mu^a + \mu^s \).

\[ \text{3. Embedding measures and spectra} \]

Let \( \pi : \mathcal{Y} \to X \) be a continuous surjection between two compact spaces \( X, \mathcal{Y} \). For a uniform algebra \( A \subset C(X) \) on \( X \) let us denote by \( L^\pi(A) \) the algebra

\[ L^\pi(A) := \{ f \circ \pi : f \in A \} \subset C(\mathcal{Y}). \]

For a measure \( \mu \in M(\mathcal{Y}) \) let us define \( \pi^M(\mu) \in M(X) \) as the transport ("push-forward") measure obtained from \( \mu \) by \( \pi \) so that

\[ (3.1) \quad \int f \; d\pi^M(\mu) = \int (f \circ \pi) \; d\mu, \quad f \in C(X). \]

Lemma 3.1. Under the above assumptions we have the following properties.

1. For \( \nu \in M_+(\mathcal{Y}) \) the closed support: \( \text{supp}(\pi^M(\nu)) \) of \( \pi^M(\nu) \) equals \( \pi(\text{supp}(\nu)) \).
2. If \( A \) is a uniform algebra on \( X \) and \( E \subset \mathcal{Y} \) is a closed reducing set for the algebra \( L^\pi(A) \) on \( \mathcal{Y} \), then \( \chi_E = \chi_{\pi(E)} \circ \pi \).
(3) If moreover $E$ is clopen and the topology on $X$ is the strongest topology $T_\pi$ making $\pi$ continuous, then $\pi(E)$ is clopen and

$$\pi^M(M(E)) = M(\pi(E)).$$

(4) If under assumptions of (3), $E$ is reducing for $L^\pi(A)$, then $\pi(E)$ is reducing for $A$.

Proof. (1) If $U \subset X$ is open, then since $\pi^M(\nu)(U) = \nu(\pi^{-1}(U))$, we have

$$U \cap \text{supp}(\pi^M(\nu)) = \emptyset \Leftrightarrow \pi^M(\nu)(U) = 0 \Leftrightarrow \pi^{-1}(U)) \cap \text{supp}(\nu) = \emptyset \Leftrightarrow U \cap \pi(\text{supp}(\nu)) = \emptyset.$$

The equality in (1) follows now by easy comparison of their complements since a point not belonging to either of these sets has a neighbourhood $U$ disjoint with it.

(2) Assume on the contrary that there is $x \in \pi^{-1}(\pi(E)) \setminus E$. Then we can find $y \in E$ such that $\pi(x) = \pi(y)$. Hence for point-mass measures $\delta_x, \delta_y$ we have $\delta_x - \delta_y \in (L^\pi(A))^\perp$. Since $E$ is a reducing set, $\delta_x$ and $\delta_y$ must also annihilate $L^\pi(A)$ which is impossible for probabilistic measures.

(3) Since by (2), $\chi_{\pi(E)}$ is the factorisation of $\chi_E$ by $\pi$, the set $\pi(E)$ is clopen by the continuity of $\pi$ in $T_\pi$ and by the properties of quotient topologies. To show (3.2) note that for $\mu \in M(E)$ the support of $\pi^M(\mu)$ is contained in $\pi(E)$ by (1), showing the $\subset$ relation. The opposite inclusion can be obtained using Hahn - Banach theorem as follows. Any measure $\nu$ on $\pi(E)$ is a continuous linear functional on $C(\pi(E))$, a subspace contained isometrically (via $f \mapsto f \circ \pi$) in $C(E)$. Norm-preserving extension yields a measure $\nu_1$ on $E$ satisfying $\pi^M(\nu_1) = \nu$.

(4) Note that $\mu$ is annihilating $L^\pi(A)$ iff $\pi^M(\mu)$ annihilates $A$. Since $\chi_E \in L^\pi(A)$ by Lemma [2.2] from (3.1) we deduce that $\chi_{\pi(E)} \in A$.

\[\square\]

In $Y = \text{Sp}(C(X)^{**})$ we introduce the equivalence relation defined for $y_1, y_2 \in Y$ by

$$y_1 \sim y_2 \iff f(y_1) = f(y_2) \text{ for all } f \in A^{**}.$$

We have the canonical surjection

$$\Pi_1 : Y \to Y/\sim \subset \text{Sp}(A^{**}).$$

This mapping $\Pi_1$ transports (by push-forward) measures, defining the map

$$\Pi^M_1 : M(Y) \ni \mu \mapsto \Pi^M_1(\mu) \in M(Y/\sim).$$

**Definition 3.2.** Define an embedding $j : M(\text{Sp}(A)) \to M(\Pi_1(Y)) \subset M(\text{Sp}(A^{**}))$ as a composition

$$j = \Pi^M_1 \circ k,$$

i.e. $j$ is given by the formula

$$\int f \, dj(\mu) = \int (f \circ \Pi_1) \, d\kappa(\mu), \quad f \in C(Y/\sim).$$
As a consequence of this definition, we have the inclusions:

(3.6) \[ j(\text{Sp}(A)) \subset Y/\sim \quad \text{and} \quad j(M(\text{Sp}(A))) \subset M(Y/\sim) \]

Note that (as in the case of \( k \)), the mapping \( j \) usually is discontinuous with respect to weak-star topologies. The considered dependencies are illustrated below by diagrams:

\[
\begin{array}{ccc}
X & \xrightarrow{k} & Y \\
& j & \downarrow \Pi_1 \\
Y/\sim \subset \text{Sp}(A^{**}) & \xrightarrow{k} & M(Y) \\
& j & \downarrow M(Y/\sim) \subset M(\text{Sp}(A^{**}))
\end{array}
\]

\[ \text{Lemma 3.3.} \quad \text{For } \mu \in M(X) \text{ the measure } j(\mu) \text{ is } w^*-\text{continuous on } A^{**}. \]

\[ \text{Proof.} \text{ If } f_\alpha \in A^{**} \text{ converge weak-star to } 0 \in A^{**}, \text{ then } \langle j(\mu), f_\alpha \rangle = \langle k(\mu), f_\alpha \circ \Pi_1 \rangle = \langle f_\alpha, \mu \rangle \to 0. \]

Here we use the identification \( C(Y/\sim) \supset A^{**} \ni f \mapsto f \circ \Pi_1 \in A^{**} \subset C(Y). \Box \]

From Lemma 2.7, by the definition of \( j \) and from \( w^* \)-density of \( \nu(A) \) in \( A^{**} \) we conclude the following:

\[ \text{Lemma 3.4.} \quad \text{If } \nu \in A^+ \text{, then } j(\nu) \in (A^{**})^+ \text{ and if } \mu \text{ is representing for } A \text{ at } x, \text{ then } j(\mu) \text{ has analogous property for } A^{**} \text{ at } j(x). \]

\[ \text{Lemma 3.5.} \quad \text{The mapping } \Pi_1^M \text{ preserves absolute continuity for } \mu, \nu \in M_+(Y) : \mu \ll \nu \Rightarrow \Pi_1^M(\mu) \ll \Pi_1^M(\nu). \]

\[ \text{Proof.} \text{ If } 0 = (\Pi_1^M(\nu))(E) = \nu(\Pi_1^{-1}(E)) \text{ for some Borel set } E, \text{ then } 0 = \mu(\Pi_1^{-1}(E)) = (\Pi_1^M(\mu))(E). \Box \]

We need the following counterpart of Theorem 2.12

\[ \text{Theorem 3.6.} \quad \text{If a band } \mathcal{M} \text{ in } M(X) \text{ is reducing for } A \text{ on } X \text{ then } j(\mathcal{M})' \text{ and } j(k(\mathcal{M}))' \text{ are reducing bands for } A^{**} \text{ on } Y/\sim \text{ and } M(Y/\sim) = j(\mathcal{M})' \oplus j(k(\mathcal{M}))'. \]

Moreover, there are clopen disjoint subsets \( E, F \) of \( Y/\sim \) such that \( E \cup F = Y/\sim \) and

(3.7) \[ j(\mathcal{M})' = M(E) = \Pi_1^M \left( k(\mathcal{M})' \right), \quad j(k(\mathcal{M}))' = M(F) = \Pi_1^M \left( k(\mathcal{M})' \right). \]

\[ \text{Proof.} \text{ By Theorem 2.12 there exist reducing clopen sets } E, F \subset Y \text{ such that } k(\mathcal{M})' = \chi_E \cdot M(Y) \text{ and } k(\mathcal{M})' = \chi_F \cdot M(Y). \text{ Moreover } E \cup F = Y \text{ and we have the equalities } k(\mathcal{M})' = M(E), k(\mathcal{M})' = M(F). \text{ Let us define } \]

\[ \tilde{E} = \Pi_1(E) \quad \text{and} \quad \tilde{F} = \Pi_1(F) \]

and apply Lemma 3.1 with \( \pi = \Pi_1 \). From (4) of this lemma, \( \tilde{E} \) and \( \tilde{F} \) are reducing sets. Since \( E, F \) are disjoint and “saturated w.r. to \( \sim \)” by (2) in Lemma 3.1 their images are disjoint: \( \tilde{E} \cap \tilde{F} = \emptyset \).

By Theorem 2.12 we have \( k(\mathcal{M})' = M(E) \). Hence \( \Pi_1^M(k(\mathcal{M})') = \Pi_1^M(M(E)) \). The latter band, equal \( M(\tilde{E}) \) by (3.2), is \( w^* \)-closed, since \( \tilde{E} \) is closed. Finally
\( \overline{j(M)^\sigma} = \Pi_1^M \left( \overline{k(M)^\sigma} \right) \), by w*-continuity of \( \Pi_1^M \). If we treat \( A^{**} \) as a subalgebra of \( C(Y/\sim) \), then \( L^{\Pi_1}(A^{**}) \) will be a subalgebra in \( C(Y) \) isomorphic with \( A^{**} \). By Lemma 3.1 for \( A^{**} \) the bands \( j(M)^\sigma, \overline{j(M)^\sigma} \) are reducing.

From Lemma 2.3 and from the above theorem it follows that the characteristic function of \( \tilde{E} \), call it \( g \), belongs to \( A^{**} \). Hence also the characteristic function of \( \tilde{F} \) is equal \( 1 - g \). Hence we have the following:

**Corollary 3.7.** We have \( \overline{j(M)^\sigma} = g \cdot M(Y/\sim) \) and \( \overline{j(M^*)} = (1 - g) \cdot M(Y/\sim) \).

Using the canonical embedding \( \iota : A \to A^{**} \) and its adjoint \( \iota^* : A^{***} \to A^* \) we have for \( f \in A \) and \( \psi \in A^{***} \)
\[
\langle \psi, \iota(f) \rangle = \langle \iota^*(\psi), f \rangle.
\]

Applied to \( \psi = z \in Sp(A^{**}) \subset A^{***} \) this yields
\[
(3.8) \quad (\iota(f))(z) = f(\iota^*(z)) = (f \circ \iota^*)(z), \quad \text{i.e.} \quad \iota(f) = f \circ (\iota^*|_{Sp(A^{**})}).
\]

Let us now define the following extension \( \Lambda \) of \( \iota \):
\[
(3.9) \quad \Lambda : C(Sp(A)) \ni h \mapsto h \circ (\iota^*|_{Sp(A^{**})}) \in C(Sp(A^{**}))
\]
Hence for \( h \in C(X) \) we have
\[
(3.10) \quad \iota_C(h) = \Lambda(h) \circ \Pi_1.
\]

The adjoint of \( \Lambda \) acts on \( \mu \in M(Sp(A^{**})) \) by the formula
\[
(3.11) \quad \langle \Lambda^* \mu, h \rangle = \langle \mu, \Lambda h \rangle = \langle \mu, h \circ (\iota^*|_{Sp(A^{**})}) \rangle, \quad h \in C(Sp(A)).
\]

This map \( \Lambda \) can be used to express the adjoint of \( \iota_C : C(X) \to C(X)^{**} \cong C(Y) \) as follows:
\[
(3.12) \quad \iota_C^* = \Lambda^* \circ \Pi_1^M
\]

By (3.4) and from formula (2.4) for \( h \in C(X) \) we have
\[
\overline{j(h \cdot \mu)} = \Pi_1^M (\overline{j(h \cdot \mu)}) = \Pi_1^M (\iota_C(h) \cdot j(\mu)).
\]
Hence from (3.10), we obtain
\[
\overline{j(h \cdot \mu)} = \Pi_1^M ((\Lambda(h) \circ \Pi_1) \cdot k(\mu)) = \Lambda(h) \cdot \Pi_1^M (k(\mu)) = \Lambda(h) \cdot j(\mu).
\]
The second equality is verified using (3.5) by integrating \( f \in C(Sp(A^{**})) \) as follows:
\[
\int f d(\Pi_1^M((\Lambda(h) \circ \Pi_1) \cdot k(\mu))) = \int f \circ \Pi_1 d((\Lambda(h) \circ \Pi_1) \cdot k(\mu))
\]
\[
= \int (f \cdot \Lambda(h)) \circ \Pi_1 dk(\mu) = \int f \cdot \Lambda(h) d(\Pi_1^M(k(\mu))).
\]

Therefore for any \( h \in C(X) \) and \( \mu \in M(X) \) we obtain
\[
(3.13) \quad \Lambda(h) \cdot j(\mu) = j(h \cdot \mu).
\]
Lemma 3.8. The mapping $j$ defined in (3.5) is an isometry. It has its left inverse equal to $\Lambda^*$. Namely,

$$
\Lambda^*(j(\mu)) = \mu \text{ for } \mu \in M(\text{Sp}(A)).
$$

Since $\Lambda^*$ extends $\iota^*$, we also have $(\iota^* \circ j)(z) = z$ for any $z \in \text{Sp}(A)$.

Proof. The total variation norm $\|j(\mu)\|$ is the least upper bound of $|\langle j(\mu), h \rangle|$ with $h$ varying through $\{h \in C(Y/\sim) : \|h\| \leq 1\}$. This norm is not less than $\|\mu\| = \sup\{|\langle \mu, f \rangle| : f \in C(X), \|f\| \leq 1\}$, since $\Lambda$ embeds isometrically $C(X)$ into $C(Y/\sim)$ and $\langle \mu, f \rangle = \langle j(\mu), \Lambda(f) \rangle$. Because $\Pi_1^\mu$ does not increase the norms and $k$ is (by Lemma 2.5) an isometry, the opposite inequality ($\|\mu\| \leq \|\mu\|$) is also true.

For $f \in C(\text{Sp}(A))$ and for $\mu \in M(\text{Sp}(A))$ we have

$$
\langle \Lambda^*(j(\mu)), f \rangle = \langle j(\mu), \Lambda(f) \rangle = \langle j(\mu), f \circ \iota^* \rangle = \langle f \circ \iota^*, \mu \rangle = \langle \Lambda(f), \mu \rangle = \langle \mu, f \rangle.
$$

The second and the last but one equalities result from (3.11). To prove the last statement of the lemma we use the following identities valid for $f \in A, z \in \text{Sp}(A)$:

$$
\langle f, \iota^*(j(z)) \rangle = \langle \iota(f), j(z) \rangle = \langle f, z \rangle.
$$

The last equality corresponds to (2.2) applied for $f \in A$. □

Since the set $\overline{k(G)^*}$ is compact and $\Pi_1$ is $w^*$-continuous, in view of Definition 3.2 we have the following equality for $w^*$-closures:

Lemma 3.9. $\Pi_1(\overline{k(G)^*}) = \overline{j(G)^*}$.

If $E$ is a reducing set for $A$ on $X$, then $M(E)/A^\perp$ can be considered as a subspace of $A^*$ so we can apply $j$ to this subspace of $A^*$.

As in Lemma 2.7 we see that

$$
j(A^\perp) \subset (A^{**})^\perp.
$$

4. Gleason parts and idempotents

We denote by $\mathcal{M}_G$ the smallest band containing all representing measures at the points of a (non-trivial) Gleason part $G$. From Theorem 3.6 we deduce that the following decomposition holds.

Corollary 4.1.

$$
\overline{j(\mathcal{M}_G)^*} \oplus \overline{j(\mathcal{M}_G^0)^*} = M(Y/\sim).
$$

Let $G$ be an arbitrary Gleason part of the spectrum of $A$. $G$ will be referred to in what follows as "a Gleason part of $A$" for the sake of brevity. The idea of associating to $G$ an idempotent $g_G \in A^{**}$ has appeared in an unpublished work of Brian Cole and in [11 Section 20]. The last part of the following Corollary is formulated in [9 Theorem 4.17] for natural uniform algebras and it can be obtained in the more general case by taking restrictions to $X$. Namely:

Proposition 4.2. There exists an idempotent $g_G \in A^{**}$ vanishing on other Gleason parts $G_\alpha \not= G$ of $A$ and equal 1 on $G$. Moreover, $g_Gg_{G_\alpha} = 0$ and if $f \in A^{**}$ satisfies $f(j(x_0)) = 1$ for some $x_0 \in G$, while $\|f\| \leq 1$, then $fg_G = g_G$. 
Note that \( j(G) \) is contained in a single Gleason part, since \( j \) is an isometry by Lemma 3.8. Hence \( M_{j(G)} \) can be understood as a band generated by this part.

**Lemma 4.3.** The band \( M_G \) generated by a Gleason part \( G \) is mapped by \( j \) into the band generated by \( j(G) \): we have \( j(M_G) \subset M_{j(G)} \) and \( \overline{j(M_G)} \subset M_{j(G)} \).

**Proof.** Note that \( j(\mu) \) is representing the point \( j(z) \) by Lemma 3.4. Now the inclusion follows, since members of \( M_G \) are absolutely continuous w.r. to some representing measures. Indeed, formula (3.13) shows that \( j \) preserves the absolute continuity. Consequently, \( \overline{j(M_G)} \subset M_{j(G)} \).

**Corollary 4.4.** If the band \( M \) is equal \( M_G \), then the related idempotent \( g \) found in Corollary 3.7 is equal \( g_G \). Moreover \( g_G \cdot \mu = \mu \) for \( \mu \in j(M_G) \), \( g_G \cdot \nu = 0 \) for \( \nu \in j(M_G) \), and hence \( g_G \cdot j(M_G) = j(M_G) \), \( g_G \cdot j(M_G) = \{0\} \).

**Proof.** As an idempotent satisfying \( g_G(j(x_0)) = 1 \) for some \( x_0 \in G \), \( g_G \) is equal 1 a.e. with respect to any measure \( \mu \) representing \( j(x_0) \) and \( g_G \cdot \mu = \mu \). Hence the same holds with respect to all measures from \( M_{j(G)} \) by [11 Corollary VI.1.2] and by Lemma 3.4 - also with respect to all measures from \( j(M_G) \). Now by Lemma 2.8 applied for \( Y/\sim \) in place of \( Y \) we conclude that \( g_G \cdot \mu = \mu \) for all \( \mu \in \overline{j(M_G)} \). Consequently by Corollary 3.7, \( g_G \geq g \). The opposite inequality follows from Proposition 4.2. The remaining part follows from Corollary 3.7.

**Corollary 4.5.** For \( \mu \in M(X) \) we have the equivalences:

\[
g_G \cdot j(\mu) = j(\mu) \iff \mu \in M_G, \quad (1 - g_G) \cdot j(\mu) = j(\mu) \iff \mu \in M_G^s.
\]

**Corollary 4.6.** If \( z \in \text{Sp}(A^{**}) \) then all its representing measures on \( Y/\sim \) belong either to \( j(M_G) \) or to \( \overline{j(M_G)} \).

**Proof.** By (3.7), these bands are of the form \( M(\hat{E}), M(\hat{F}) \) for clopen reducing sets \( \hat{E}, \hat{F} \), hence the result follows from Lemma 2.3.

From Proposition 4.2 we obtain the following

**Corollary 4.7.** The \( w^*- \) closures of images of different Gleason parts \( G_1, G_2 \) are disjoint: \( \overline{j(G_1)} \cap \overline{j(G_2)} = \emptyset \).

**Proof.** The pre-images \( g_{G_j}^{-1}(\{1\}) \) of 1 are containing \( \overline{j(G_j)} \), \( j = 1, 2 \), hence the result follows from the equality \( g_{G_1} \circ g_{G_2} = 0 \).

We can also define the product \( h \odot \mu \) of a measure \( \mu \in M(X) \) by \( h \in C(Y/\sim) \) using the formula

\[
(1.1) \quad \langle h \odot \mu, f \rangle := \langle h \cdot j(\mu), \Lambda(f) \rangle, \quad f \in C(X) \text{ i.e. } h \odot \mu = (h \cdot j(\mu)) \circ \Lambda.
\]

The function \( g_G \in A^{**} \) (defined in Proposition 4.2) is an idempotent on \( Y/\sim \). Its Gelfand transform \( g_G \in C(\text{Sp}(A^{**})) \) is also an idempotent. Usually we omit the symbol \( ^* \) treating \( g_G \) as an element of \( C(\text{Sp}(A^{**})) \). By Proposition 4.2 we have

\[
(1.2) \quad g_G(j(x)) = 0 \quad \text{for } x \in \text{Sp}(A) \setminus G.
\]
By (3.13), (4.1) and by Corollary 4.4, \( g_G \odot \mu \) equals \( \mu \) (respectively 0) for all measures absolutely continuous with respect to (respectively singular to all) measures representing points in \( G \) and by Lemma 2.9 we obtain

**Lemma 4.8.** \( g_G \odot \mu = \mu \) for \( \mu \in \mathcal{M}_G \), \( g_G \odot \mu = 0 \) for \( \mu \in \mathcal{M}_s^G \).

The obtained facts lead to the following result.

**Theorem 4.9.** The band \( \mathcal{M}_G = g_G \odot M(X) \) is a reducing band for \( A \). Its singular complement, \( \mathcal{M}_s^G \) equals \( (1 - g_G) \odot M(X) \).

The bands

\[
(4.3) \quad g_G \cdot M(Y/\sim) = M(g_G^{-1}(\{1\})) = j(\mathcal{M}_G)'
\]

and

\[
(1 - g_G) \cdot M(Y/\sim) = M(g_G^{-1}(\{0\})) = j(\mathcal{M}_s^G)'
\]

are also reducing on \( Y/\sim \) for \( A^{**} \).

**Proof.** The equality (4.3) follows from Corollary 3.7 and the bands appearing in (4.3) are reducing since \( g_G \in A^{**} \). This implies also the first claim by applying formula (4.1). \( \square \)

5. **Quotient Algebras (Algebras of \( H^\infty \)-Type)**

One of the approaches to the algebra \( H^\infty(G) \) of bounded analytic functions on a given domain \( G \subset \mathbb{C}^N \), is to consider its abstract counterpart, the algebra \( H^\infty(\mathcal{M}_G) \) corresponding to a nontrivial Gleason part \( G \) for a uniform algebra \( A \).

A band \( \mathcal{M} \) of measures on \( X \) is now considered as a Banach space, its dual space \( \mathcal{M}^* \) carries the weak-star topology. Moreover \( \mathcal{M}^* \) is a uniform algebra, since it is represented as an inductive limit of \( C^\ast \)-algebras \( L^\infty(\mu) \) taken over \( \mu \in \mathcal{M} \), where the index set \( \mathcal{M} \) is directed by the absolute continuity relation [7]. Elements \( f \in A \) act as bounded linear functionals on \( \mathcal{M} \), hence \( A \) may be treated as a subspace of \( \mathcal{M}^* \).

**Definition 5.1.** By \( H^\infty(A, \mathcal{M}) \) we denote the weak-star closure of \( A \) in \( \mathcal{M}^* \). We also put \( H^\infty(\mathcal{M}_G) := H^\infty(A, \mathcal{M}_G) \), if the meaning for \( A \) is clear.

**Remark 5.2.** From the above definition it follows that the pre-annihilator in \( \mathcal{M} \) of \( H^\infty(A, \mathcal{M}) \) is equal to the annihilator of \( A \) in \( \mathcal{M} \).

**Definition 5.3.** We say that a Borel set \( W \subset X \) is dominating for \( A \), if for any \( f \in A \) we have \( \|f\| = \sup \{|f(x)| : x \in W\} \). A subset \( E \) of the closed unit ball \( B^{[1]} \) of the Banach space \( B \) will be called maximising for a subspace \( \mathcal{F} \) of \( B^\ast \) if \( \|\varphi\| = \sup \{|\langle \varphi, u \rangle| : u \in E\} \) for all \( \varphi \in \mathcal{F} \).

The following consequence of Hahn-Banach theorem contained in [2, Proposition 2.8] (or [18, §20.8.(5)]) will be very useful:

**Lemma 5.4.** If a set \( W \subset B^{[1]} \) is maximising for \( B^\ast \), then the norm-closed absolutely convex hull \( \overline{\text{aco}}(W) \) contains \( B^{[1]} \).
We shall consider quotients of bidual algebras by ideals \( \mathcal{M}^I_G \) which are the annihilators of \( \mathcal{M}_G \) in \( A^{**} \). One can think of such annihilators as \( \mathcal{M}^I_G \cap A^{**} \), but we use the shorter notation \( \mathcal{M}^I_G \), which is unambiguous in expressions like \( A^{**}/\mathcal{M}^I_G \).

To establish isomorphism of Banach algebras it is useful to invoke the following result due to Nagasawa [25, V Theorem 31.1].

**Theorem 5.5.** If a unit-preserving mapping between two uniform algebras is a linear isometry, then it is also an isomorphism of their multiplicative structures.

**Proposition 5.6.** The algebra \( H^\infty(\mathcal{M}_G) \) is isometrically and algebraically isomorphic to \( A^{**}/\mathcal{M}^I_G \) and also it is isomorphic to \( g_G \cdot A^{**} \), where \( g_G \in A^{**} \) is the idempotent defined in Proposition 4.2. For \( h \in A^{**}, x \in \text{Sp}(A) \) we have

\[
(g_G h)(j(x)) = \langle g_G h, \delta_x \rangle
\]

**Proof.** Let \( h \in A^{**} \). Then there is a net \( \{h_\alpha\} \subset A \) such that \( \langle \iota(h_\alpha), \mu \rangle \to \langle h, \mu \rangle \) for any \( \mu \in M(X) \). Denote by \( f \) the restriction of \( h \) to \( \mathcal{M}_G \) (note that \( \mathcal{M}_G \) is a subspace of \( M(X) \) and \( A^{**} \subset C(X)^{**} = M(X)^\ast \), cf. Remark 2.1). Then \( f \) depends only on the equivalence class \( [h] \in A^{**}/\mathcal{M}^I_G \) of \( h \) and also \( \langle \iota(h_\alpha), \mu \rangle \to \langle f, \mu \rangle \) for \( \mu \in \mathcal{M}_G \), hence \( f \in H^\infty(\mathcal{M}_G) \). This means that the mapping

\[
A^{**}/\mathcal{M}^I_G \ni [h] \mapsto h|_{\mathcal{M}_G} \in H^\infty(\mathcal{M}_G)
\]

is well defined. The equality (5.1) follows from analogous relation valid for \( h_\alpha \) by passing to w∗-limit.

From the obvious relations \( \|h\| \geq \|h|_{\mathcal{M}_G}\| \) for any \( h' \in [h] \), we have \( \|[h]\| \geq \|h|_{\mathcal{M}_G}\| \).

On the other hand, let \( f \in H^\infty(\mathcal{M}_G) \). By Lemma 2.2 of [19], we can find a net \( \{f_\alpha\} \subset A \) such that \( f_\alpha \to f \) and \( \|f_\alpha\| \leq \|f\| \) for each \( \alpha \). Hence, by Banach-Alaoglu Theorem, the net \( \{f_\alpha\} \) has an adherent point \( h \in A^{**} \). So \( \langle f, \mu \rangle = \langle h, \mu \rangle \) for \( \mu \in \mathcal{M}_G \) and \( \|h\| \leq \|f\| \). Consequently the norm of \( [h] \) in \( A^{**}/\mathcal{M}^I_G \) is less than or equal to \( \|f\| \). This means that mapping (5.2) is isometric and surjective. To verify that \( A^{**}/\mathcal{M}^I_G \) is a uniform algebra it suffices to check that \( \|[h]\| = \|[h]\|^{**} \) for \( [h] \in A^{**}/\mathcal{M}^I_G \), which is easy.

By the Nagasawa’s Theorem 5.6 this mapping is also isomorphic.

The second isomorphism is established in the proof of [19, Lemma 20.4] \( \Box \)

**Corollary 5.7.** \( G \) can be identified as a subset of the spectrum of \( H^\infty(\mathcal{M}_G) \).

By Proposition 5.6 we identify the algebra \( H^\infty(\mathcal{M}_G) \) with the quotient algebra \( A^{**}/\mathcal{M}^I_G \). Note that \( \mathcal{M}^I_G \) is a weak-star closed ideal in the algebra \( A^{**} \) (see [20, Proposition 3, (4)]).

As in the remarks preceding Theorem 4.9 the function \( g_G \in A^{**} \) (defined in Proposition 4.2) will be treated as a function on \( Y/\sim \subset \text{Sp}(A^{**}) \). The fact that \( g_G \) is also an idempotent on \( Y/\sim \) is useful in proving the following result:

**Proposition 5.8.** The spectrum of \( H^\infty(\mathcal{M}_G) \) can be identified with the set of those elements of \( \text{Sp}(A^{**}) \) for which all representing measures concentrated on \( Y/\sim \) belong to \( \hat{\mathcal{M}_G} \).
Proof. If \( \nu \in j(M_G)^{\prime} \) for all \( \nu \) representing for \( z \in \text{Sp}(A^{**}) \) then
\[
(A^{**} \cap M_G^{\perp})(z) \equiv 0, \text{ in other words } z(A^{**} \cap M_G^{\perp}) \equiv 0
\]
where \( z \) is considered as a functional on \( A^{**} \). This means that \( z \) is also a well defined element of the spectrum of \( A^{**}/M_G^{\perp} \cap A^{**} \).

On the other hand assume now that \( z \) is a well defined functional on \( A^{**}/M_G^{\perp} \cap A^{**} \). It corresponds to a functional \( \tilde{z} \) on \( A^{**} \) obtained by composing \( z \) with the canonical surjection. Then we must have \( \tilde{z} \equiv 0 \) on \( A^{**} \cap M_G^{\perp} \). If \( \tilde{z} \) would have a representing measure \( \nu \not\in j(M_G)^{\prime} \) then by Corollary 4.6 it would have a representing measure \( \nu \in j(M_G)^{\prime} \). Now by Theorem 4.9 we have \( g_G = 1 \) a.e. with respect to every measure in \( j(M_G)^{\prime} \), and consequently \( 1 - g_G \in A^{**} \cap M_G^{\perp} \) which means that \( 1 - g_G \) is equivalent to zero in the quotient algebra \( A^{**}/M_G^{\perp} \cap A^{**} \). Since we have assumed that \( z \) is a well defined functional on \( A^{**}/M_G^{\perp} \cap A^{**} \) we conclude that \( 1 - g_G(\tilde{z}) = 0 \). This leads to a contradiction since \( g_G \equiv 0 \) on \( j(M_G)^{\prime} \) and consequently
\[
g_G(\tilde{z}) = \int g_G \, d\nu = 0.
\]

6. Hoffman – Rossi type theorems

Let \( G \) be a Gleason part of a uniform algebra \( \mathcal{A} \subset C(X) \). (In our setting either \( \mathcal{A} = A \) or \( \mathcal{A} = A^{**} \).

We say that \( \mu \) is a Henkin measure (or an \( A \)-measure) for \( \mathcal{A} \) with respect to \( G \) if any uniformly bounded, point-wise convergent to zero on \( G \) sequence \( u_n \in \mathcal{A} \) satisfies \( \int u_n \, d\mu \to 0 \). The following assumption will be later verified for a wide class of algebras \( H^\infty(G) \), \( G \subset \mathbb{C}^d \) and we assume it throughout this section:

**Assumption 6.1.** The fibers over points of \( G \) are singletons, i.e.
\[
(6.1) \quad j(G) = (\iota^*)^{-1}(G).
\]

**Proposition 6.2.** Under the above assumption \( j(G) \) is a Gleason part of \( A^{**} \).

Proof. Let \( \varphi \) be a linear multiplicative functional on \( A^{**} \) lying in its Gleason part containing \( j(G) \) and fix some \( \psi_0 \in G \). Then the ”restriction of \( \varphi \) to \( A \)”, namely the functional \( \iota^*(\varphi) \) given by \( A \ni f \mapsto \varphi(\iota(f)) \) belongs to \( G \). Indeed, for \( \psi = j(\psi_0) \in j(G) \) we have the norm estimate \( \|\iota^*(\psi) - \iota^*(\varphi)\| \leq \|\psi - \varphi\| < 2 \). Hence \( \iota^*(\varphi) \in G \). By (6.1), \( \varphi = j(\iota^*(\varphi)) \in j(G) \). Since \( \|j\| = 1 \), analogous estimate shows that for \( \psi_0, \psi_1 \in G \) their images by \( j \) are in the same Gleason part.

At a key point of further considerations we need an extension of a Hoffman-Rossi’s result, Theorem 6.1 in [3].

For \( E \subset \text{Sp}(\mathcal{A}) \) -a Borel subset of \( X \) let \( \mathcal{A}_E := \{ f \in \mathcal{A} : f|_E = 0 \} \). If \( E = \{ \varphi \} \), we write \( \mathcal{A}_{\varphi} \) in place of \( \mathcal{A}_{\{\varphi\}} \) to denote the kernel of \( \varphi \). Since \( \varphi \) has real representing measures, the formula
\[
\varphi(\text{Re}(u)) := \text{Re}(\varphi(u)), \quad u \in \mathcal{A}
\]
defines $\varphi$ also on the real space $\text{Re}\, \mathcal{A} := \{\text{Re}(u) : u \in \mathcal{A}\}$.

Since for linear subspaces their annihilators are equal to their polar sets, by [18 §20.8.(7)] we have

\[(6.2) \quad \mathcal{A}_E^\perp = \left( \bigcap_{x \in E} \mathcal{A}_x \right) = \text{span}^\prime \{ \mathcal{A}_x^\perp : x \in E \}.\]

Let us begin with one of the implications appearing in Hoffman-Rossi Theorem. In our situation it is relating some positivity condition with approximation by elements of the sum $P + \text{Re}\, \mathcal{A}_\varphi$. Here $P$ denotes the cone of nonnegative functions in $C(X)$ and

$$\text{Re}\, \mathcal{A}_\varphi = \{\text{Re} u : u \in \mathcal{A}_\varphi\}.$$  

**Lemma 6.3.** Let $E$ be a Borel subset of $X$. If $u \in \text{Re} C(X)$ has non-negative integrals $\int u \, d\lambda \geq 0$ against all non-negative measures $\lambda \in (\mathcal{A}_E)^\perp$, then $u$ lies in the $\text{norm-closure of the cone } P + \text{Re}\, \mathcal{A}_\varphi.$

**Proof.** We use similar arguments as in the proof of the version of Hoffman-Rossi Theorem given in [23]. Assume $u$ is not in this closure, $u \notin P + \text{Re} \mathcal{A}_E$. Then there is a measure $\mu \in M_\mathbb{R}(X)$ such that $\mu(u) < \inf \{\mu(v) : v \in P + \text{Re} \mathcal{A}_E\}$. Since $P + \text{Re} \mathcal{A}_E$ is a cone – the infimum must be 0.

In particular, for $v \in P$ we have $\mu(v) \geq 0$, hence $\mu \geq 0$, $\mu \neq 0$. Since $\text{Re} \mathcal{A}_E$ is a linear subspace of $\text{Re} C(X)$, and since $\mu(v) \geq 0$ for $v \in \text{Re} \mathcal{A}_E$, $\mu$ must vanish on $\text{Re} \mathcal{A}_E$. Since $\mu$ is real, $\mu$ vanishes on $\mathcal{A}_E$, so by our hypotheses, $\mu(u) \geq 0$. This leads to a contradiction, since $\mu(u) < 0$. □

**Corollary 6.4.** If $u \in \text{Re} C(X)$ has non-negative integrals $\int u \, d\lambda \geq 0$ against all measures $\lambda$ representing $\varphi \in \text{Sp}(\mathcal{A})$, then $u$ lies in the $\text{norm-closure of the cone } P + \text{Re}\, \mathcal{A}_\varphi.$

**Proof.** If $\mu$ is probabilistic and annihilates $\mathcal{A}_\varphi$, then it is representing at $\varphi$. □

**Lemma 6.5.** Assume that $\mathcal{A}$ is a uniform algebra on a compact space $X$ and $G \subset X$ is an open Gleason part. Let $\mu$ be a probabilistic measure on $X$ annihilating $\mathcal{A}_E$, where $E \subset X$ is a Borel set. We assume that $\overline{E} \cap \overline{G} = \emptyset$. Then $\mu$ vanishes on $G$.

**Proof.** Any probabilistic measure from $\mathcal{A}_E^\perp$ is a representing measure for $x$, hence it belongs to $\mathcal{M}_G$, where $G_x$ is the Gleason part containing $x$. The reducing bands $\mathcal{M}_G$ are of the form $g_x \circ M(X)$ for idempotents $g_x \in C(Y/\sim)$. For $x \in E$ each idempotent $g_x$ is disjoint with idempotent $g$ corresponding to $G$. The idempotents $g_x \circ \Pi_1$ and $g \circ \Pi_1$ belong to $C(Y)$ and are also disjoint. Since $C(Y)$ is Dedekind-complete, we have the lattice supremum $g_E \circ \Pi_1$ of $\{g_x \circ \Pi_1 : x \in E\}$, also disjoint with $g \circ \Pi_1$, i.e. $(g_E \circ \Pi_1) \cdot (g \circ \Pi_1) = 0$ by the Distributive Law. Consequently, $g_E \cdot g = 0$. Hence the bands $\mathcal{M}_G$ and $g_E \circ M(X)$ are mutually singular. In particular, all measures $\mu \in g_E \circ M(X)$ satisfy $|\mu|(G) = 0$, since $M(G) \subset \mathcal{M}_G$ by [4 Lemma 20.5]. Consequently, since $G$ is open, all measures from $w^*$ closure of $g_E \circ M(X)$ vanish on $G$ and by (6.2) – all measures from $\mathcal{A}_E^\perp$ belong to $g_E \circ M(X)^\prime \prime$, and consequently – vanish on $G$. □
The main result of the present section concerns a nontrivial Gleason part \( G \) of the spectrum of our uniform algebra \( \mathcal{A} \), satisfying the following condition:

**Assumption 6.6.** There is a sequence of compact subsets \( K_m \subset G \) with the following properties: \( K_m \subset \text{int}(K_{m+1}), \bigcup K_m = G \).

**Remark 6.7.** Observe that then \( G = \bigcup \text{int}(K_n) \) is open and we have \( \bigcup (i^*)^{-1}(K_m) = (i^*)^{-1}(G) \). Indeed, since \( i^* \) is \( w^* \)-continuous, \( (i^*)^{-1}(K_m) \) are closed, hence compact and analogous pre-images of \( \text{int}(K_m) \) are open and fill \( j(G) \), hence \( j(G) \) is open. Moreover,

\[
(i^*)^{-1}(K_m) \subset (i^*)^{-1}(\text{int}(K_{m+1})) \subset \text{int}((i^*)^{-1}(K_{m+1})).
\]

**Proposition 6.8.** Let \( \mathcal{A} \) be a uniform algebra on \( X \), whose spectrum has a nontrivial Gleason part \( G \subset X \) satisfying Assumption 6.6 and let \( E \subset X \setminus G \) be a compact set. Then for any \( n \in \mathbb{N} \) there exist \( h_n \in \mathcal{A} \), with \( h_n \equiv 1 \) on \( E \) such that \( h_n \) converge to \( 0 \) on \( G \) and \( \|h_n\| \) converge to \( 1 \).

**Proof.** By Lemma 6.5 all the measures from \( \mathcal{A}_1^E \) are carried by \( X \setminus G \). Let \( u_n \in C(X), u_n \leq 0, u_n < -n \) on \( K_n \) and \( u_n \equiv 0 \) on \( X \setminus G \).

Now by Lemma 6.3 we have \( u_n \in \overline{P + \text{Re}(\mathcal{A}_E)} \) for any \( n \). Consequently, there exists a sequence \( \{g_n\} \subset \mathcal{A}_E \) and functions \( v_n \in C(X) \), \( v_n \geq 0 \) such that \( \|v_n + \text{Re}g_n - u_n\| < \frac{1}{n} \). Then for any \( n \) on \( K_n \) we have \( e^{\text{Re}g_n} \leq e^{v_n + \text{Re}g_n} < e^{-n+\frac{1}{n}} \). On the other hand, from \( g_n \in \mathcal{A}_E \), it follows that \( e^{g_n} = 1 \) on \( E \). Hence \( e^{g_n} \to 0 \) on \( G \), so we may take \( h_n = e^{g_n} \).

**Theorem 6.9.** Let \( \mathcal{A} \) be a uniform algebra, whose spectrum has a nontrivial Gleason part \( G \subset X \) satisfying Assumptions 6.1, 6.6. Then for any \( \mu \in \mathcal{M}_G \) we have \( \text{supp}(j(\mu)) \subset \overline{j(G)}^\sigma \). In other words, \( j(\mathcal{M}_G) \subset M(\overline{j(G)}^\sigma) \).

**Proof.** Here we use the notation \( Y/\sim \) and \( \Pi_1 \) introduced in Section 3. Let \( \mu \in M_\varphi \), where \( \varphi \in G \). Assume, that for some compact set \( E \) such that \( E \subset Y/\sim \setminus \overline{j(G)}^\sigma \) we have \( j(\mu)(E) > 0 \). By inner regularity of \( j(\mu) \) we can even enlarge \( E \) to obtain the estimate

\[
(6.3) \quad j(\mu)(E) > j(\mu) \left( (Y/\sim \setminus \overline{j(G)}^\sigma) \setminus E \right).
\]

Now we are applying Proposition 6.8 for \( j(G) \) in place of \( G \), \( X \) replaced by \( Y/\sim \). The assumptions of this Proposition are then satisfied by Proposition 6.2 and by Remark 6.7. In the Assumption 6.6 we replace \( K_n \) with \( (i^*)^{-1}(K_n) \). Hence we obtain a sequence \( h_n \in \mathcal{A}^* \) equal 1 on \( E \) and converging to 0 on \( j(G) \).

By Banach-Alaoglu Theorem, the bounded sequence \( h_n \) has a \( w^* \)-cluster point in \( \mathcal{A}^* \), let us call it \( h \). Since the points from \( j(G) \) act as \( w^* \)-continuous functionals on \( \mathcal{A}^* \), we have \( h = 0 \) on \( j(G) \) and by continuity, also on \( \overline{j(G)}^\sigma \). Moreover \( h \circ \Pi_1 \) is a \( w^* \)-accumulation point of \( h_n \circ \Pi_1 \).

The measure \( k(\mu) \) (defined by formula (2.11)) is normal, hence by [8, Theorem 4.7.8 (i)], its restriction to \( \Pi_1^{-1}(E) \) is also normal. Since \( M(X) \) is a predual of \( C(Y) \), which has a strongly unique isometric predual by [8, Theorem 6.4.2], the space \( \{k(\nu) : \nu \in M(X)\} \) can be identified with the predual of \( C(Y) \) in a way compatible...
with canonical embedding. The predual of $C(Y)$ is the set of all normal measures on $Y$. Hence the functional $C(Y) \ni f \mapsto \int_{\Pi_1^{-1}(E)} f \, d\kappa(\mu)$ is weak-star continuous. Consequently,

$$
\int_E h \, d\tilde{j}(\mu) = \int_{\Pi_1^{-1}(E)} h \circ \Pi_1 \, d\kappa(\mu) = \lim_n \int_{\Pi_1^{-1}(E)} h_n \circ \Pi_1 \, d\kappa(\mu) = \tilde{j}(\mu)(E),
$$

since $h_n \circ \Pi_1 = 1$ on $\Pi_1^{-1}(E)$. We have

$$
\int_{Y/\sim} h \, dj(\mu) = \int_{j(G)} h \, dj(\mu) + \int_{Y/\sim \setminus j(G)} h \, dj(\mu) = 0 + \int_{Y/\sim \setminus j(G)} h \, dj(\mu)
$$

However $j(\mu)$ is a representing measure for $j(\varphi)$, hence

$$
0 = h(j(\varphi)) = \text{Re} \int_{Y/\sim} h \, dj(\mu) = \text{Re} \int_{Y/\sim \setminus E} h \, dj(\mu) + \int_E h \, dj(\mu) > 0,
$$

where the last relation follows from inequality (6.3), since the integral over $E$ here is equal to $j(\mu)(E)$.

**Theorem 6.10.** Let $A$ be a uniform algebra, whose spectrum has a nontrivial Gleason part $G \subset X = \text{Sp}(A)$ satisfying Assumption 6.7 and any measure from $\mathcal{M}_G$ is a Henkin measure for $A$ with respect to $G$. Then for any point $\varphi \in G$ all its representing measures $\mu$ have supports contained in $G^\sigma$.

**Proof.** Let $\mu \in M_\varphi$, where $\varphi \in G$. Assume on the contrary, that for some compact set $E \subset X \setminus G^\sigma$ we have $\mu(E) > 0$. By Proposition 6.8 we have uniformly bounded sequence of functions $h_n \in A$ converging to 0 on $G$ and such that $\int_E h_n \, d\mu = \mu(E) > 0$. Hence $\chi_E \cdot \mu$ is not a Henkin measure. This contradicts the assumption on $\mathcal{M}_G$, since $\chi_E \cdot \mu \in \mathcal{M}_G$.

7. **Abstract Theorems on Henkin Measures**

Let $A_0 := A$ and for $n = 0, 1, 2, \ldots$ let $A_{n+1} := (A_n)^{**}$ with $\tilde{\gamma} : A_{n-1} \to A_n$ denoting the canonical embeddings, their n-fold composition will give the embedding $\tilde{\gamma} : A_0 \to A_n$. With the Arens product, each $A_n$ is a uniform algebra with unit $\mathbf{1}_n = \tilde{\gamma}(1_0)$, where $1_0$ is the unit element of $A_0$. Extending Definition 3.2 to measures on $n$-th level spaces $\text{Sp}(A_n)$, we obtain embeddings $\tilde{j} : M(\text{Sp}(A_{n-1})) \to M(\text{Sp}(A_n))$, and their n-fold composition $j$ allowing us to embed $M(\text{Sp}(A))$ in $M(\text{Sp}(A_n))$.

We also use the notation $\tilde{j}(\lambda), j(\lambda)$ at the points $\lambda$ from the corresponding spectra defined as $\tilde{j}(\delta_\lambda), j(\delta_\lambda)$. (In the present section we are using the above notation only for $n = 1$ and $n = 2$.)

Assume that analogous to (6.11) condition holds for $j(G)$:

$$
(7.1) \quad j(G) = (\tilde{j}^*)^{-1}(G).
$$

**Theorem 7.1** (Abstract Henkin measures Theorem for $A^{**}$). If the Gleason part $G$ of $A$ satisfies Assumptions 6.1 and 6.6, then $\{\delta_x : x \in j(G)\}$ is maximising for the algebra $H^\infty(A^{**}, \mathcal{M}_{j(G)})$. The measures from $\mathcal{M}_{j(G)}$ are Henkin measures for $A^{**}$ w.r. to $j(G)$ and are concentrated on $\overline{j(G)}$. 

Proof. By Theorem 6.9 applied for $A^{**}$ we obtain

$$\tilde{j}(\mathcal{M}_{j(G)}) \subset M(\tilde{j}(j(G))).$$

Therefore $\tilde{j}(j(G))$ is (by Proposition 6.2) a Gleason part of $\text{Sp}(A^{****})$ and is dominating for $g_{j(G)} \cdot A^{****} \simeq H^\infty(A^{**}, \mathcal{M}_{j(G)})$. Indeed, for $\mu \in \mathcal{M}_{j(G)}, h \in A^{****}$ we have

$$\langle g_{j(G)}h, \mu \rangle = \tilde{j}(\mu), g_{j(G)}h \rangle = \int g_{j(G)}h \tilde{j}(\mu).$$

Hence using (5.1), we have

$$|\langle g_{j(G)}h, \mu \rangle| \leq \sup_{j(G)} |g_{j(G)}h| \cdot \|\tilde{j}(\mu)\| = \sup\{|\langle g_{j(G)}h, \delta_x \rangle| : x \in j(G)\} \cdot \|\tilde{j}(\mu)\|,$$

where the equality follows from the identity

$$\sup_{j(G)} |g_{j(G)}h| = \sup_{x \in j(G)} |(g_{j(G)}h)(\tilde{j}(x))|.$$

This means that $\{\delta_x : x \in j(G)\}$ is also maximising for $H^\infty(A^{**}, \mathcal{M}_{j(G)})$. Now Lemma 5.4 implies that any measure from $\mathcal{M}_{j(G)}$ belongs to the norm-closure (in the quotient norm) of $\text{span}(\{\delta_x : x \in j(G)\})$ and consequently, is a Henkin measure for $A^{**}$ on $j(G)$. The rest follows from Theorem 6.10.

As a direct consequence of this theorem and of the equality $\mathcal{M}_G = \Lambda^*(\mathcal{M}_{j(G)})$, we have the following abstract version of A-measures Theorem, giving a new proof in the case of $A = A(G)$ for strictly pseudoconvex domains and polydomains.

**Theorem 7.2** (Abstract Henkin measures Theorem). If $G$ a Gleason part of $\text{Sp}(A)$ satisfying Assumptions 6.1 and 6.6, then all measures from $\mathcal{M}_G$ are Henkin measures for $A$ with respect to $G$ and $\{\delta_x : x \in G\}$ is maximising for $H^\infty(A, \mathcal{M}_G)$.

**Proof.** From Theorem 6.9 we see that for $h \in H^\infty(A, \mathcal{M}_G)$

$$\sup_{x \in G} |h(j(x))| \geq \sup_{\mu \in \mathcal{M}_G, \|\mu\| \leq 1} |\langle j(\mu), h \rangle|.$$

Rewriting both sides of this inequality, we can put it in the form

$$\sup_{x \in G} |\langle h, \delta_x \rangle| \geq \sup_{\mu \in \mathcal{M}_G, \|\mu\| \leq 1} |\langle h, \mu \rangle|.$$

This is the maximisation of $G$ for $H^\infty(A, \mathcal{M}_G)$. Now by Lemma 5.4 the closed (in the quotient norm) absolutely convex hull of $G$ contains the unit ball of $\mathcal{M}_G$ and all measures from $\mathcal{M}_G$ are Henkin measures.

**Remark 7.3.** The uniform approximation resulting from Lemma 5.3 allows us to obtain even stronger version of Henkin measures property in Theorems 7.1 and 7.2. The bounded sequences $(u_n)$ converging point-wise to $0$ on $G$ (resp. on $j(G)$) can be replaced by bounded nets.
8. Abstract Corona Theorem

**Theorem 8.1.** Assume that $G$ satisfies Assumptions 6.1, 6.6 and that $\{\delta_x : x \in G\}$ is a maximising set for $H^\infty(A, \mathcal{M}_G)$, for which (7.1) holds true. Then $\overline{j(G)}^\sigma$ is a reducing set for $A^{**}$ on $\mathcal{Sp}(A^{**})$.

**Proof.** Assume towards a contradiction that there exists $\mu \in (A^{**})^\perp$ such that $\mu|_{\overline{j(G)}^\sigma} \notin (A^{**})^\perp$. Then

$$
(8.1) \quad \int_{\overline{j(G)}^\sigma} f \, d\mu \neq 0 \quad \text{for some } f \in A^{**}.
$$

There exists a sequence of compact sets $K_n$ disjoint with $\overline{j(G)}^\sigma$ such that the sequence $\mu(\mathcal{Sp}(A^{**}) \setminus (\overline{j(G)}^\sigma \cup K_n))$ converges to 0.

Then by formula (6.2) applied for $E = j(G)$ and by Theorem 7.1, all the measures from $(\mathcal{A}^{**})_{j(G)}$ are carried by $\overline{j(G)}^\sigma$. Let $u_n \in C(\mathcal{Sp}(A^{**}))$, $u_n \leq 0$, $u_n < -n$ on $K_n$ and $u_n \equiv 0$ on $\overline{j(G)}$.

Now by Lemma 6.3 we have $u_n \in \overline{P + \mathrm{Re}(\mathcal{A}^{**})_{j(G)}}$ for any $n$. Consequently, there exists a sequence $\{g_n\} \subset (\mathcal{A}^{**})_{j(G)}$ and functions $v_n \in C(\mathcal{Sp}(A^{**}))$, $v_n \geq 0$ such that $\|v_n + \mathrm{Re}g_n - u_n\| < \frac{1}{n^2}$. Then for any $n$ on $K_n$ we have $e^{\mathrm{Re}g_n} \leq e^{v_n + \mathrm{Re}g_n} < e^{-n + \frac{1}{n}}$. If we take $h_n = e^{v_n}$, then $|h_n| < e^{-n + \frac{1}{n}}$ on $K_n$. On the other hand, from $g_n \in (\mathcal{A}^{**})_{j(G)}$, it follows that $h_n = 1$ on $\overline{j(G)}^\sigma$. Hence $\int h_n f d\mu$ converge to $\int_{\overline{j(G)}^\sigma} f d\mu$, so by Formula (8.1), for $n$ sufficiently large $\int f h_n d\mu \neq 0$, contradicting the relation $\mu \in (A^{**})^\perp$. \qed

**Theorem 8.2** (Abstract Corona Theorem). Let $G$ be an open Gleason part in $\mathcal{Sp}(A)$ satisfying Assumption 6.1 and condition (7.1). Assume also that the norm topology of $A^*$ is equal to $G$ on the Gelfand topology. Then the canonical image of $G$ is dense in the spectrum of $H^\infty(A, \mathcal{M}_G)$ in its Gelfand topology.

**Proof.** We need to show that Assumption 6.6 holds in this setting. For a fixed $x_0 \in G$ we define $K_m := \{x \in \mathcal{Sp}(A) : \|x - x_0\| \leq 2 - \frac{1}{n}\}$, then the sets $K_m$ are compact in Gelfand topology, by Banach-Alaoglu Theorem. Their interiors are equal $\{x \in \mathcal{Sp}(A) : \|x - x_0\| < 2 - \frac{1}{n}\}$ also in Gelfand topology, by our assumption. Theorem 8.1 implies that $\overline{j(G)}$ is a reducing set for $A^{**}$ on $\mathcal{Sp}(A^{**})$, therefore all representing measures at points of $\overline{j(G)}$ for $A^{**}$ are concentrated on $\overline{j(G)}^\sigma$. Hence by Proposition 5.8 we have $\mathcal{Sp}(H^\infty(A, \mathcal{M}_G)) = \overline{j(G)}^\sigma$. \qed

9. Stability of $G$

From now on let $A = A(G)$, where $G \subset \mathbb{C}^d$ is a bounded domain and let us fix a point $\lambda = (\lambda_1, \ldots, \lambda_d) \in G$. By $z_k$ we denote the $k$-th coordinate function on $\overline{G}$, so that $z_k \in A$.

**Definition 9.1.** We say that the domain $G$ is stable, if for any fixed point $\lambda \in G$ there exist continuous linear operators

$$
T_{\{k\}} : A(G) \ni f \mapsto T_{\{k\}} f \in A(G), \quad k = 1, \ldots, d
$$
such that
\begin{equation}
  f(z) - f(\lambda) = \sum_{k=1}^{d} (z_k - \lambda_k)(T_{\{k\}}f)(z), \quad z \in G, \ f \in A(G).
\end{equation}

In other words, when \( f(\lambda) \) and \( \lambda_k \) are treated as constant functions in \( A(G) \) and \( z_k \in A(G) \) as the coordinate functions, we have
\begin{equation}
  f - f(\lambda) = \sum_{k=1}^{d} (z_k - \lambda_k)T_{\{k\}}f
\end{equation}
and for some constant \( C \) depending only on \( \lambda \) the norms satisfy \( \|T_{\{k\}}f\|_G \leq C\|f\|_G \).

Here the sup-norm over \( G \) is the same as the norm in \( A(G) \).

**Proposition 9.2.** If a domain \( G \subset \mathbb{C}^d \) is either strictly pseudoconvex with \( C^2 \)-boundary or a polydomain, then \( G \) is stable in the sense of Definition 9.1.

**Proof.** In the case of strictly pseudoconvex domains with smooth boundaries the solution of Gleason problem was published in [14] and the strengthening giving bounds \( \|T_{\{k\}}f\| \leq C\|f\| \) has appeared in [17]. For polydomains analogous decompositions are easily available by induction on \( d \) after fixing all but one variables. \( \Box \)

We are now going to show that the above result transfers to the second dual and even further. Here we use notations introduced at the beginning of Section 7. In particular, for \( f \in A_n, \lambda \in \text{Sp}(A) \) we simplify notation writing \( f(\lambda) := f(\hat{\delta}_\lambda) \).

**Theorem 9.3.** Assume that \( A = A(G) \), where \( G \subset \mathbb{C}^d \) is a stable domain. For \( f \in A_n, \ (0 \leq n < \infty) \) and for \( \lambda \in G \) there exist \( T_{\{n,k\}}f \in A_n \) such that
\begin{equation}
  f - f(\lambda)1_n = \sum_{k=1}^{d} (\tilde{\alpha}(z_k) - \lambda_k)T_{\{n,k\}}f, \\
\end{equation}
where \( f(\lambda) \) and \( \lambda_k \) are treated as constant functions and \( z_k \in A \) as the coordinate functions, and for some constant \( C = C_\lambda \) depending only on \( \lambda \) the norms satisfy \( \|T_{\{n,k\}}f\| \leq C\|f\| \).

**Proof.** We will prove the theorem by induction. The step 0 follows from \( \text{(9.2)} \) for \( A_0 = A(G) \). Assume that our claim holds for \( n \). Let \( f \in A_{n+1} \). Then there is a net \( \{f_\alpha\} \subset A_n \) such that (in \( \text{w}^* \) topology) \( \tilde{\alpha}(f_\alpha) \to f \) and \( \|f_\alpha\| \leq \|f\| \). By the step \( n \) we conclude for every \( \alpha \) the existence a \( d \)-tuple \( T_{\{n,k\}}f_\alpha \in A_n \) satisfying \( \text{(9.3)} \) and such that \( \|T_{\{n,k\}}f_\alpha\| \leq C\|f\| \). The directed families \( \tilde{\alpha}(T_{\{n,k\}}f_\alpha) \) have adherent points \( T_{\{n+1,k\}}f \in A_{n+1} \) with norms bounded by the same constant. Passing to suitable subnets, we can write \( \tilde{\alpha}(T_{\{n,k\}}f_\alpha) \to T_{\{n+1,k\}}f \). Consequently for their limits we have \( \text{(9.3)} \) with \( \|T_{\{n+1,k\}}f\| \leq C\|f\| \), where norms are those from \( A_{n+1} \). \( \Box \)

**Corollary 9.4.** Under the above assumptions, for \( \lambda, \tilde{\lambda} \in G, n \in \mathbb{N} \) we have
\[ \|\hat{j}(\lambda) - \hat{j}(\tilde{\lambda})\| \leq \sup_{\|f\| \leq 1} \sum_{k=1}^{d} |\lambda_k - \tilde{\lambda}_k| \|T_{\{n+1,k\}}f(\tilde{\lambda})\| \leq \sum_{k=1}^{d} |\lambda_k - \tilde{\lambda}_k| C_\lambda. \]
Gelfand, Euclidean and norm topologies coincide on $G$ and $G$ is open in the spectrum of $A(G)$.

Proof. The above inequalities follow directly from Theorem 9.3 and they imply the equalities between the considered topologies. Proposition 9.2 implies, in particular, that the part of spectrum of $A(G)$ in the fibers over $G$ is mapped in a bijective way to $G$ by the Gelfand transform $(\hat{\gamma}(z_k))$ of the coordinate functions. By the results of [13], the entire $\text{Sp}(A(G))$ is mapped bijectively onto $\overline{G}$. □

Theorem 9.5. Let $A = A(G)$, where $G \subset \mathbb{C}^d$ is a stable domain. Then for $0 \leq n < \infty$, we have

$$\mathfrak{j}(G) = (\mathfrak{r}^*)^{-1}(G)$$

and this set is a Gleason part of $\text{Sp}(A_n)$.

Proof. The equation (9.3) implies that the fiber over $\lambda$ of the spectrum of $A_n$ consists of one point - the evaluation functional at $\mathfrak{j}(\delta_\lambda)$. Indeed, applying a linear multiplicative functional $\varphi \in \text{Sp}(A_n)$ such that $\varphi(\hat{\gamma}(z_k)) = \lambda_k$ for any $k \leq d$ to both sides of this equation, we deduce that $\varphi(f) = f(\lambda)$.

Let $\varphi$ be a linear multiplicative functional on $A_n$ lying in its Gleason part containing the image $\mathfrak{j}(G)$ of $G$. Then its "restriction to $A_0$", namely the functional $\mathfrak{r}^*(\phi)$ given by $A_0 \ni f \mapsto \phi(\hat{\gamma}(f))$ is of the form $\delta_\lambda$ for some element $\lambda$ of $G$. Indeed, for $\psi = \mathfrak{j}(\psi_0) \in \mathfrak{j}(G)$ the norm $\|\mathfrak{r}^*(\psi) - \mathfrak{r}^*(\varphi)\| \leq \|\psi - \varphi\| < 2$. Hence $\mathfrak{r}^*(\varphi)$ is in the same Gleason part as $\psi_0$ and is an evaluation at some $\lambda \in G$. Taking values at $z_k \in A(G)$ we see that $\varphi$ is in the fiber over $\lambda \in G$, which is just shown to be a singleton $\mathfrak{j}(\delta_\lambda)$. □

10. Classical setting

Let $z \in G$ and let $\nu_z$ be its representing measure with respect the algebra $A$. For $f \in A$ we have $f(z) = \int f \, d\nu_z$. By the weak-star density of $A$ in $H^\infty(A, \mathcal{M}_G)$ we also have $f(z) = \langle f, \nu_z \rangle$ for $f \in H^\infty(A, \mathcal{M}_G)$.

Lemma 10.1. The value $f(z)$ does not depend on the choice of representing measure. It can be assigned to $f \in H^\infty(A, \mathcal{M}_G)$:

$$f(z) = \int f \, d\nu_z.$$

Proposition 10.2. If $G$ is a bounded domain in $\mathbb{C}^d$ and $f \in H^\infty(A, \mathcal{M}_G)$ then the defined above mapping $f|_G : G \ni z \mapsto f(z)$ is a bounded analytic function of $z \in G$.

Proof. Let $z_0 \in G$. The analyticity of $f|_G$ on $G$ can be established by routine methods, like [20, Proposition 16], or by normal family arguments. □

In the next theorem we consider a bounded domain of holomorphy $G \subset \mathbb{C}^n$ such that its closure $\overline{G}$ is the spectrum of $A(G)$. For this it suffices to assume either that $\overline{G}$ is the intersection of a sequence of domains of holomorphy, that is, $\overline{G}$ has a Stein neighbourhoods basis [22], or that it has a smooth boundary [13]. Moreover $G$ is a Gleason part for $A(G)$ (see remarks in the Introduction and [24, Section 16]). Here $A(G)$ plays the role of our initial uniform algebra $A$. 

For the purpose of the proof we formulate the following two assumptions

**Assumption 10.3.**

1. $G$ is a bounded domain of holomorphy $G \subset \mathbb{C}^n$ such that its closure $\overline{G}$ is the spectrum of $A(G)$.
2. Any $f \in H^\infty(G)$ is a pointwise limit of a uniformly bounded sequence of $f_n \in A(G)$.

In two most important cases these conditions will be verified. Let us say that $G$ is **strongly starlike**, if there exists a point $x_0 \in G$ such that for the translate $G_{x_0} := G - x_0$ and for any $\rho > 1$ we have $\overline{G_{x_0}} \subset \rho \cdot G_{x_0}$. It is easy to see that any convex domain is strongly starlike with respect to any of its points.

**Lemma 10.4.** For strictly pseudoconvex domains with $C^2$ boundaries and for polydomains $G$ which are strongly starlike Assumptions [10.3] are satisfied.

**Proof.** In the first case (1) follows from [13], (2) is proved in [6, Theorem 2.4]. For polydomains (i.e. cartesian products $G_1 \times \cdots \times G_n$ of plane domains) (1) is easy to see, since $(z_j - \lambda_j)^{-1} \in A(G_j)$ for $\lambda_j \notin \overline{G_j}$, $j = 1, \ldots, n$. To see (2) in this case, we may assume without loss of generality that $G$ is strongly starlike with respect to 0. For $f \in H^\infty(G)$ the sequence $f_n \in A(G)$ defined as $f_n(z) := f((1 - \frac{1}{n})z)$ ($z \in G$) does the job.

**Theorem 10.5.** If $G$ is a domain in $\mathbb{C}^d$ satisfying above Assumptions [10.3], then the algebras $H^\infty(G)$ and $H^\infty(A, \mathcal{M}_G)$ are isometrically isomorphic. The associated homeomorphism of spectra preserves the points of $G$.

**Proof.** By Proposition [10.2] we have $\{f|_G : f \in H^\infty(A, \mathcal{M}_G)\} \subset H^\infty(G)$ and the norm of every element $h \in H^\infty(A, \mathcal{M}_G)$ is greater or equal to the supremum of $|h(z)|$ on $G$.

We need to check whether $H^\infty(G) \subset H^\infty(A, \mathcal{M}_G)$ in the sense of isometric embedding. For $f \in H^\infty(G)$ let $f_n \in A(G)$ converge pointwise, boundedly to $f$. The family $\{f_n\}_{n \in \mathbb{N}}$ has an adherent point $h \in H^\infty(A, \mathcal{M}_G)$. Passing to a suitable subnet, we can write $\langle \mu, f_n \rangle \to \langle h, \mu \rangle$ for $\mu \in \mathcal{M}_G$. In particular, $\langle \nu_z, f_n \rangle \to \langle h, \nu_z \rangle$ as $n \to \infty$ for any measure $\nu_z$ representing any $z \in G$. On the other hand $\langle \nu_z, f_n \rangle = f_n(z) \to f(z)$ for $z \in G$. Hence $h$ and $f$ agree on $G$. The mapping $f \to h$ is isometric, since $\|h\| \leq \lim \inf \|f_n\|$ and for every $r < 1$ we have $\|f_n\| \leq \|f\|_r$.

By Theorem [7,2] the set $\{\delta_x : x \in G\}$ is maximising for $H^\infty(A, \mathcal{M}_G)$, hence the mapping $H^\infty(A, \mathcal{M}_G) \ni f \mapsto f|_G \in H^\infty(G)$ is injective.

Finally, we obtain our main result:

**Theorem 10.6** (Corona Theorem). If a stable domain $G \subset \mathbb{C}^d$ satisfies Assumptions [10.3], then Corona Theorem holds true for $H^\infty(G)$: its spectrum is the Gelfand closure of (the canonical image of) $G$.

**Proof.** By Theorem [10.5] the spectra of algebras $H^\infty(G)$ and $H^\infty(A, \mathcal{M}_G)$ are homeomorphic by a mapping preserving the points of $G$ (sending $x \in G$ to its canonical image in $\text{Sp}(H^\infty(A, \mathcal{M}_G))$ to be more precise). By Corollary [9.1] the norm- and Gelfand topologies are equal on $G$. Now it remains to apply Theorem [8.2].

Proof of Theorem 1.1. By Proposition 9.2 and by Lemma 10.4 the assumptions of Theorem 10.6 are satisfied for two important classes of domains: strictly pseudo-convex with $C^2$ boundaries and for strongly star-like cartesian products of plane domains. Hence corona problem is solved in these cases.

Additional Remarks

The results from complex analysis employed in the present work come mostly from the late 1970’s. General theory of uniform algebras has been developed around the same period, the main sources used here are [11], [8] and [4]. We are also using analysis of bands of measures related to Gleason parts and their weak star closures developed in the first part of [20].

In the earlier version of our paper (published in ArXiv we were making the assumption of strong star-likeness also for strictly pseudoconvex domains. Our student, Sebastian Gwizdek has noted that the results of [6] allow us to eliminate this assumption here.

Unfortunately, there was a gap in the proof of Theorem 6 in that paper – noticed and kindly communicated to us by Garth Dales. Theorem 6 in [20] and its corollaries do not hold in the claimed generality, in particular – for the algebras of all continuous functions. The present work is using only the results of [20] preceding Theorem 6 – with one exception: when in the proof of Proposition 10.2 we invoke a method used in the proof of [20, Proposition 16], which is however quite routine.

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