KOSZUL DUALITY FOR FILTERED $E_n$-ALGEBRAS

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Abstract. We study the Koszul duality between augmented $E_n$-algebras and augmented $E_n$-coalgebras in a symmetric monoidal stable infinity category equipped with a filtration in a suitable sense. We obtain that the Koszul duality constructions restrict to an equivalence between augmented algebras and coalgebras which have some positivity and completeness with respect to the filtration. We also obtain that the Koszul duality construction is functorial between carefully constructed generalised Morita categories consisting of those algebras/coalgebras in each dimension.

Contents

0. Introduction 1
Outline
Acknowledgment
1. Terminology and notations 3
2. Filtered stable category 4
2.0. Localisation of a stable category 5
2.1. Filtration of a stable category 7
2.2. Completion 9
2.3. The completion as a complete category 13
2.4. Totalisation 15
3. Monoidal filtered stable category 16
3.0. Pairing in filtered stable categories 16
3.1. Monoidal structure on a filtered category 17
3.2. The filtered category of filtered objects 18
3.3. Modules over an algebra in filtered stable category 20
4. Koszul duality for complete algebras 24
4.0. Koszul completeness of a positive algebra 24
4.1. Koszul completeness of a coalgebra 26
4.2. Koszul duality for $E_n$-algebras 27
4.3. Morita structure of the Koszul duality 28
References 33

0. Introduction

Let $n$ be a finite non-negative integer. The notion of an $E_n$-algebra was first introduced in iterated loop space theory, in the work of Boardman and Vogt [0]. $E_1$-algebra is an associative algebra, and an $E_n$-algebra can be inductively defined as an $E_{n-1}$-algebra with an additional structure of an associative algebra commuting

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with the $E_{n-1}$-structure. In other words, a structure of an $E_n$-algebra consists of $n$-fold associative structures and data for compatibility among them.

There is an issue that the notion of an $E_n$-algebra degenerates (unless $n \leq 1$) to that of a commutative algebra in a category whose higher homotopical structure is degenerate. Moreover, the kind of theory we aim to establish (the theory of the Koszul duality) fails in such a setting even for (the case $n = 1$ of) associative algebras. These issues force us to work in a homotopical setting. In order to work in such a setting, we use the convenient language of higher category theory. (For the main body, note our conventions stated in Section 1, which do not apply in this introduction.) We just remark here that associativity of an algebra in such a setting means a data for homotopy coherent associativity (which in particular is a structure rather than a property).

In this paper, we study the Koszul duality between $E_n$-algebras and $E_n$-coalgebras. By the Koszul dual of an augmented associative algebra, we mean the augmented associative coalgebra obtained as the bar construction or a suitable derived tensor product (see Section 4.0). For $E_n$-algebras, we simply consider the $n$-fold iteration of this construction. Coalgebras are simply algebras in the opposite category, so we obtain an augmented algebra from an augmented coalgebra by the same construction.

In some cases, this correspondence between algebras and coalgebras is an equivalence or close to it. Results of this form include the iterated loop space theory and the Verdier duality. (See Lurie’s book [9, Section 5.3] for the relation between these.)

In other contexts, the correspondence is far from an equivalence. In contexts which are closer to ours in this paper, it instead happens often that the category of algebras compares better with the category of suitable class of infinitesimal stacks around a point, in a derived version of suitable geometry. (See for instance, Francis [3], Lurie [10], Hirsch [6] for some precise such results.)

However, in this work, we find that with the help of some additional structure on the symmetric monoidal category in which we consider algebras and coalgebras, we can obtain simple classes of $E_n$-algebras and coalgebras between which the Koszul duality behaves nicely. The structure we consider is a filtration with respect to which the category becomes complete.

To get into more details, our setting is as follows. Let $\mathcal{A}$ be a symmetric monoidal stable infinity category. We assume that it has a filtration (Definition 2.11, note the conventions stated in Section 1) which is compatible with the symmetric monoidal structure in a suitable way.

Primary examples are the category of filtered objects in a reasonable symmetric monoidal stable infinity category (Section 3.2), and a symmetric monoidal stable infinity category with a compatible t-structure [9] (satisfying a mild technical condition, see Definition 2.30, Remark 2.31). Another family of examples is given by functor categories admitting the Goodwillie calculus [5], where the filtration is given by the degree of excisiveness (Example 2.13).

We further assume that $\mathcal{A}$ is complete with respect to the filtration in a suitable sense. The mentioned examples admit completion, and in these examples, the category $\mathcal{A}$ we indeed work in is the category of complete objects in any of the mentioned categories, with completed symmetric monoidal structure.

These categories satisfy a few further technical assumptions we need, which we shall not state here. (The theorems we state in this introduction shall be given references to their precise formulation in the main body. In order to understand
the formulation correctly, the reader should note our conventions stated in Section 1.)

In such a complete filtered infinity category \( \mathcal{A} \), any algebra comes with a natural filtration with respect to which it is complete. In the mentioned examples, the towers associated to the filtration are the canonical (or “defining”) tower, the Postnikov tower, and the Taylor tower, and the objects we deal with are the limits of the towers. We have established the Koszul duality for \( E_n \)-algebras in \( \mathcal{A} \) which is positively filtered. The corresponding restriction on the filtration of coalgebras is given by the condition we call copositivity (Definition 4.16). Our first main theorem is as follows.

**Theorem 0.0** (Theorem 4.18). Let \( \mathcal{A} \) be as above. Then the constructions of Koszul duals give inverse equivalences

\[
\text{Alg}_{E_n}(\mathcal{A})_+ \cong \text{Coalg}_{E_n}(\mathcal{A})_+
\]

between the infinity category of positive augmented \( E_n \)-algebras and copositive augmented \( E_n \)-coalgebras in \( \mathcal{A} \).

We have also shown that the Koszul duality further has a Morita theoretic functoriality.

To explain what this is, in [8], Lurie has outlined a generalisation of the Morita category for \( E_n \)-algebras. By collecting suitable versions of bimodules, one obtains an infinity \( (n+1) \)-category \( \text{Alg}_n(\mathcal{A}) \) of \( E_n \)-algebras, bimodules, etc. in a symmetric monoidal infinity category \( \mathcal{A} \), generalising the 2-category of associative algebras and bimodules.

In order to make the construction of this work, one usually assumes that the monoidal multiplication functors preserve geometric realisations variablewise. However, unless the monoidal multiplication also preserve totalisations, one cannot have both algebraic and coalgebraic versions of this in the same way. We have shown that in the kind of complete filtered category we work in, the construction works for both positive augmented algebras and copositive augmented coalgebras at the same time.

Let us denote the infinity \( (n+1) \)-categories we obtain by \( \text{Alg}_{n+}^+(\mathcal{A}) \) and \( \text{Coalg}_{n+}^+(\mathcal{A}) \) respectively. We have shown the following.

**Theorem 0.1** (Theorem 4.22). Let \( \mathcal{A} \) be as above. Then for every \( n \), the construction of the Koszul dual define a symmetric monoidal functor

\[
(\cdot)^+ : \text{Alg}_n^+(\mathcal{A}) \rightarrow \text{Coalg}_n^+(\mathcal{A})
\]

It is an equivalence with inverse given by the Koszul duality construction.

This paper is based on, and slightly revises, part of the author’s thesis [12].

**Outline.** Section 1 is for introducing conventions which are used throughout the main body.

In Sections 2 and 3 we establish basic notions and facts on symmetric monoidal filtered stable categories.

In Section 4 we develop the theory of Koszul duality for complete \( E_n \)-algebras.

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1. Terminology and notations

By a 1-category, we always mean an infinity 1-category. We often call a 1-category (namely an infinity 1-category) simply a category. A category with discrete sets of morphisms (namely, a “category” in the more traditional sense) will be called (1, 1)-category, or a discrete category.

In fact, all categorical and algebraic terms will be used in infinity (1-) categorical sense without further notice. Namely, categorical terms are used in the sense enriched in the infinity 1-category of spaces, or equivalently, of infinity groupoids, and algebraic terms are used freely in the sense generalised in accordance with the enriched categorical structures.

For example, for an integer $n$, by an $n$-category (resp. infinity category), we mean an infinity $n$-category (resp. infinity infinity category). We also consider multicategories. By default, multimaps in our multicategories will form a space with all higher homotopies allowed. Namely, our “multicategories” are “infinity operads” in the terminology of Lurie’s book [9].

Remark 1.0. We usually treat a space relatively to the structure of the standard (infinity) 1-category of spaces. Namely, a “space” for us is usually no more than an object of this category. Without loss of information, we shall freely identify a space in this sense with its fundamental infinity groupoid, and call it also a “groupoid”. Exceptions in which the term “space” means not necessarily this, include a “Euclidean space”, the “total space” of a fibre bundle, etc., in accordance with the common customs.

We use the following notations for over and under categories. Namely, if $C$ is a category and $x$ is an object of $C$, then we denote the category of objects $C$ lying over $x$, i.e., equipped with a map to $x$, by $C/x$. We denote the under category for $x$, in other words, $((C^{op})/x)^{op}$, by $C_x$.

More generally, if a category $D$ is equipped with a functor to $C$, then we define $D_x := D \times_C C/x$, and similarly for $D_x$. Note here that $C/x$ is mapping to $C$ by the functor which forgets the structure map to $x$. Note that the notation is abusive in that the name of the functor $D \to C$ that we are considering is clear from the context.

By the lax colimit of a diagram of categories indexed by a category $C$, we mean the Grothendieck construction. We choose the variance of the laxness so the lax colimit projects to $C$, to make it an op-fibration over $C$, rather than a fibration over $C^{op}$. (In particular, if $C = D^{op}$, so the functor is contravariant on $D$, then the familiar fibred category over $D$ is the op-lax colimit over $C$ for us.) Of course, we can choose the variance for lax limits compatibly with this, so our lax colimit generalises to that in any 2-category.

2. Filtered stable category

In this paper, we consider the Koszul duality in a symmetric monoidal stable category $A$, equipped with a “filtration” with respect to which $A$ becomes complete. The primary example will be given by the category of complete filtered objects, which shall be reviewed in Section 3.2. In fact, the influence to the present work comes from the use of complete filtered objects in a related context in Costello’s [11].
(see also the appendix of Costello–Gwilliam [2]). Filtration and completeness are also used in the work of Positselski on the Koszul duality [13].

Our approach, despite its slight abstractness, has the advantage of including a few more examples such as the filtration given by a t-structure, and hopefully of clarifying some logic. We shall develop these notions in this and the next sections, and then develop the Koszul duality theory in such a category, in Section 4.

2.0. Localisation of a stable category. We review some facts we need.

Definition 2.0. Let $C$ be a category.

A functor $C \to D$ is a left localisation if it has a fully faithful functor as a right adjoint.

A full subcategory $D$ of $C$ is a left localisation of $C$ if the inclusion functor $D \hookrightarrow C$ has a left adjoint.

Right localisation is defined similarly, so it is just left localisation in the opposite variance.

We consider the following situation. Let $A$ be a stable category, and let $A_\ell \subset A$ be a full subcategory which is a left localisation of $A$. Denote by $(\ )_\ell$ the localisation functor $A \to A_\ell$. By abuse of notation, we also denote by $(\ )_\ell$ the composite

$$A \xrightarrow{(\ )_\ell} A_\ell \hookrightarrow A.$$ 

Definition 2.1. A right localisation $A_r$ of $A$ is complementary to the left localisation $A_\ell$ of $A$ as above if for every $X \in A_r$ and $Y \in A_\ell$, the space $\text{Map}(X,Y)$ is contractible, and the sequence

$$(\ )_r \xrightarrow{\varepsilon} \text{id} \xrightarrow{\eta} (\ )_\ell : A \longrightarrow A,$$

where $(\ )_r$ is the right localisation functor considered as $A \to A$, and the maps are the counit and the unit maps for the respective adjunctions, is a fibre sequence (by the unique null homotopy of the composite $\eta \varepsilon$).

As a full subcategory of $A$, $A_r$ consists of objects $X \in A$ for which the counit $\varepsilon : X_r \to X$ is an equivalence, or equivalently, $X_r \simeq 0$. It follows that given any left localisation $A_\ell$ of $A$, if it has a complementary right localisation, then the right localisation is characterised as the right localisation to the full subcategory of $A$ consisting of objects $X \in A$ for which $X_\ell \simeq 0$.

Given any right localisation, its complementary left localisation is defined in the opposite way. It is immediate that if a left localisation has a complementary right localisation, then this left localisation is left complementary to its right complement.

Lemma 2.2. Let $A$ be a stable category, and let $A_\ell, A_r$ be left and right localisations of $A$ respectively which are complementary to each other. Then $A_\ell$ has a zero object, and dually for $A_r$.

Remark 2.3. All inclusion and localisation functors then preserves the zero objects.

Proof. $0_{\ell r} \simeq 0$ implies $0 \in A_\ell$, which is then a zero object of $A_\ell$. \qed

Proposition 2.4. A left localisation $(\ )_\ell : A \to A_\ell$ has a complementary right localisation if and only if

$$(\text{Fibre}[\eta ; \text{id} \to (\ )_\ell])_r \simeq 0.$$ 

Example 2.5. This condition is satisfied if the left localisation is exact in the sense that the functor $(\ )_\ell : A \to A_\ell$ (and equivalently, $(\ )_r : A \to A$) preserves finite limits.
Proof of Proposition 2.4. Necessity follows from the remark for Definition 2.1.

For sufficiency, define $A_r$ as the full subcategory of $A$ consisting of objects $X \in A$ for which $X \simeq 0$. Then the functor $(\ )_r := \text{Fibre}(\eta; \text{id} \rightarrow (\ )_r): A \rightarrow A$ lands in $A_r$. Denote the resulting functor $A \rightarrow A_r$ also by $(\ )_r$.

It will then follow that $(\ )_r$ is a right adjoint of the inclusion $A_r \hookrightarrow A$, with counit the canonical map $(\ )_r \rightarrow \text{id}$. Indeed, the defining fibre sequence for $X \in A_r$ gives for any $Y$, the fibre sequence
$$
\text{Map}(Y, X) \rightarrow \text{Map}(Y, X) \rightarrow \text{Map}(Y, X),
$$
but since $X \in A_r$, we have $\text{Map}(Y, X) = \text{Map}(Y, X)$. Now, if $Y \in A_r$, then this space is contractible, so we obtain that the map $\text{Map}(Y, X) \rightarrow \text{Map}(Y, X)$ is an equivalence, as was to be shown.

Thus we have obtained a right localisation $(\ )_r: A \rightarrow A_r$, and this came as complementary to the left localisation we started with.

Lemma 2.6. Let $A$ be a stable category with complementary left and right localisations $(\ )_\ell: A \rightarrow A_\ell$ and $(\ )_r: A \rightarrow A_r$, respectively. Then, for a cofibre sequence
$$W \rightarrow X \rightarrow Y
$$
in $A$, if $W$ belongs to the full subcategory $A_r$ of $A$, then the localised map $X_\ell \rightarrow Y_\ell$ is an equivalence.

Proof. $W$ belongs to $A_r$ if and only if $W \simeq 0$.

By applying the localisation functor $(\ )_r: A \rightarrow A_r$ to the given cofibre sequence, we obtain a cofibre sequence in $A_r$. If $W \simeq 0$, then the map $X_\ell \rightarrow Y_\ell$ in the sequence is an equivalence. □

Corollary 2.7. In the situation of Lemma 2.6, if $Y$ also belongs to $A_r$, then $X$ belongs to $A_r$.

Corollary 2.8. In the situation of Lemma 2.6, if $Y \in A_r$, then the canonical map $W \rightarrow X_r$ and $X_\ell \rightarrow Y$ are equivalences, so the fibre sequence is canonically equivalent to the canonical fibre sequence
$$X_r \rightarrow X \rightarrow X_\ell.
$$

Proof. The equivalences of objects is immediate from Lemma 2.6. The fibre sequences will then be canonically the same since the null-homotopy of the composite is unique. □

In a situation where we have left and right localisations complementary to each other, we will be particularly interested in how the localisations interact with limits (and colimits) in our stable category. We have seen in Lemma 2.2 that localisations contain $0 \in A$. More generally, we have the following.

Lemma 2.9. If a left localisation $A_\ell$ has a complementary right localisation, then $A_\ell$ is closed in $A$ under any limit which exists in $A$.

Proof. This follows since $A_r$ is the full subcategory of $A$ consisting of $X \in A$ for which $X \simeq 0$, and since the functor $(\ )_r: A \rightarrow A_r$ is a right adjoint, and hence preserves any limit. □

Note also that the limit taken in $A$ of a diagram lying in the full subcategory $A_\ell$ (which in fact belongs to $A_r$, according to the above) will be a limit in $A_r$ of the diagram. On the other hand, since the inclusion $A_r \hookrightarrow A$ preserves limits, if a limit of a diagram $A_r$ exists in the category $A_r$, then it also will be a limit in $A$.
In the next proposition, we assume given complementary left and right localisations \( (\_): \mathcal{A} \to \mathcal{A}_\ell \) and \( (\_): \mathcal{A} \to \mathcal{A}_r \) respectively, of a stable category \( \mathcal{A} \).

In this situation, we assume given classes of diagrams \( \mathcal{D}, \mathcal{D}_\ell, \mathcal{D}_r \), in \( \mathcal{A} \), \( \mathcal{A}_\ell \), and \( \mathcal{A}_r \) respectively, and consider limits of diagrams belonging to any of these classes. For example, we may be considering all finite limits in \( \mathcal{A}, \mathcal{A}_\ell, \) or \( \mathcal{A}_r \). Alternatively, we may be considering sequential limits.

We require that all inclusion and localisation functors between these categories take a diagram in the specified class in the source to one in the specified class in the target. In fact, from this requirement, it is immediate that the class \( \mathcal{D} \) determines the other classes. Namely, \( \mathcal{D}_\ell \) is the class of diagrams which belong to \( \mathcal{D} \) when considered as diagrams in \( \mathcal{A} \), and similarly for \( \mathcal{D}_r \).

One can start from any class of diagrams \( \mathcal{D} \) in \( \mathcal{A} \) which is closed under application of endofunctors \( (\_): \mathcal{A} \to \mathcal{A}_\ell \) and \( (\_): \mathcal{A} \to \mathcal{A}_r \) to have all three classes satisfying our requirements.

In this situation, if \( \mathcal{A} \) has limits of all diagrams in the class \( \mathcal{D} \) (see above), then \( \mathcal{A}_r \) has limits of all diagrams in the class \( \mathcal{D}_r \), and by Lemma 2.9, \( \mathcal{A}_\ell \) has limits of all diagrams in the class \( \mathcal{D}_\ell \).

**Proposition 2.10.** Let \( \mathcal{A}, \mathcal{A}_\ell, \mathcal{A}_r, \) and classes of diagrams \( \mathcal{D} \) in \( \mathcal{A}, \mathcal{D}_\ell, \mathcal{D}_r \) in \( \mathcal{A}_\ell, \mathcal{A}_r \) be as above. Assume that \( \mathcal{A} \) has limits of all diagrams in the class \( \mathcal{D} \) (see above).

Then the following are equivalent.

1. The left localisation functor \( (\_): \mathcal{A} \to \mathcal{A}_\ell \) takes limits of diagrams belonging to \( \mathcal{D} \), to corresponding limits in \( \mathcal{A}_\ell \).
2. The functor \( (\_): \mathcal{A} \to \mathcal{A} \) considered as \( \mathcal{A}_\ell \), takes limits of diagrams belonging to \( \mathcal{D} \), to corresponding limits in \( \mathcal{A} \).
3. The right localisation functor \( (\_): \mathcal{A}\to \mathcal{A} \) takes limits of diagrams belonging to \( \mathcal{D} \), to corresponding limits in \( \mathcal{A} \).
4. Given a diagram in \( \mathcal{A}_r \), belonging to \( \mathcal{D}_r \) (equivalently, a diagram in \( \mathcal{A} \) which belongs to \( \mathcal{D} \), and lands in \( \mathcal{A}_r \)), its limit taken in \( \mathcal{A} \), belongs to \( \mathcal{A}_r \).
5. The inclusion \( \mathcal{A}_r \hookrightarrow \mathcal{A} \) takes limits of diagrams belonging to \( \mathcal{D}_r \), to corresponding limits in \( \mathcal{A} \).

**Proof.** It is relatively simple to see that the first three are equivalent to each other. It is also easy to see that (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1) \( \Rightarrow \) (2). \( \square \)

2.1. **Filtration of a stable category.**

**Definition 2.11.** A filtration of a stable category \( \mathcal{A} \) is a sequence of full subcategories

\[
\mathcal{A} \supset \cdots \supset \mathcal{A}_{\geq r} \supset \mathcal{A}_{\geq r+1} \supset \cdots
\]

indexed by integers, each of which is the inclusion of a right localisation which has a complementary left localisation, denoted by \( (\_): \mathcal{A} \to \mathcal{A}^{<r} \).

A filtered stable category is a stable category which has all sequential limits, and is equipped with a filtration.

In particular, associated to a filtered stable category \( \mathcal{A} \), we have a sequence

\[
\mathcal{A} \to \mathcal{A}^{<r+1} \to \mathcal{A}^{<r} \to \cdots
\]

of left localisation functors. We would like to think of this as the tower associated to the filtration.

\( \mathcal{A}_{\geq r} \) can be considered as the pieces for the filtration. \( \mathcal{A}_{\geq r} \) is the full subcategory of \( \mathcal{A} \) formed by objects \( X \in \mathcal{A} \) for which \( X^{<r} \cong 0 \). We denote the right localisation by \( (\_): \mathcal{A} \to \mathcal{A}_{\geq r} \). Then the sequence \( (\_): \mathcal{A} \to \text{id} \to (\_): \mathcal{A} \to \mathcal{A}, \) equipped with the unique null homotopy of the composite \( (\_): \mathcal{A} \to (\_): \mathcal{A} \), is a fibre sequence.
An important example will be discussed in Section 3.2. Here are a few examples.

**Example 2.12.** If $\mathcal{A}$ is a stable category which has all sequential limits, then any t-structure \cite{[9]} on $\mathcal{A}$ gives a filtration.

**Example 2.13.** Let $\mathcal{A}$ be the functor category into a stable category, and assume it admits some version of the Goodwillie calculus \cite{[5]}. Then it has a filtration in which $\mathcal{A}^{\leq r}$ is the full subcategory consisting of $(r-1)$-excisive functors. The left localisation $\mathcal{A} \to \mathcal{A}^{\leq r}$ is given by the universal $(r-1)$-excisive approximation of functors.

The notion of a filtration on a stable category is self-dual in the following sense. Namely, if a stable category $\mathcal{A}$ is given a filtration, then $\mathcal{B} := \mathcal{A}^{op}$ has a filtration given by $\mathcal{B}_{\geq r} := (\mathcal{A}^{\leq r})^{op}$, where $\mathcal{A}_{\leq s} := \mathcal{A}^{s+1}$. Therefore, all notions and statements we formulate will have dual versions, which we shall speak about freely without further notices.

Let $r, s$ be integers such that $r \leq s$. Then $(X_{\geq r})^{<s}$ belongs to $\mathcal{A}_{\leq s}^{<r} := A_{\geq r} \cap A^{<s}$ since

$((X_{\geq r})^{<s})^{<r} = (X_{\geq r})^{<r} \simeq 0$,

and so does $(X^{<s})_{\geq r}$.

We would like to compare these objects.

We have a commutative diagram

\[ (2.14) \]

\[
\begin{array}{ccc}
X_{\geq r} & \xrightarrow{(X_{\geq r})^{<s}} & X \\
\downarrow & & \downarrow \\
X^{<s} & \xleftarrow{(X^{<s})_{\geq r}} & (X^{<s})_{\geq r}
\end{array}
\]

so the universal property of the map $X_{\geq r} \to (X_{\geq r})^{<s}$ implies that there is a unique pair consisting of a map $(X_{\geq r})^{<s} \to (X^{<s})_{\geq r}$ and a homotopy making the upper triangle of

\[ (2.15) \]

\[
\begin{array}{ccc}
X_{\geq r} & \xrightarrow{(X_{\geq r})^{<s}} & (X^{<s})_{\geq r} \\
\downarrow & & \downarrow \\
X^{<s} & \xleftarrow{(X^{<s})_{\geq r}} & X_{\geq r}
\end{array}
\]

commute.

Moreover, again by the universal property of the map $X_{\geq r} \to (X_{\geq r})^{<s}$, there is a unique pair consisting of

- a homotopy filling the lower triangle, and
- a higher homotopy between the homotopy filling the diamond in the diagram (2.14), and the homotopy obtained by pasting the homotopies in the diagram (2.15).

In other words, there is a unique quadruple consisting of

(0) a map $(X_{\geq r})^{<s} \to (X^{<s})_{\geq r}$
(1) a homotopy filling the upper triangle of (2.15)
(2) a homotopy filling the lower triangle of (2.15)
(3) a higher homotopy between the homotopy filling the diamond in the diagram (2.14), and the homotopy obtained by pasting the homotopies in the diagram (2.15).

Moreover, by the universal property of the map \((X^{<s})_{\geq r} \to X^{<s}\), the pair given by (1) and (2) above, must be the unique pair of this form.

It follows that for a map \((X_{\geq r})^{<s} \to (X^{<s})_{\geq r}\) the following data (in particular, existence of the data) are equivalent to each other.

- (1) above
- (2) above
- Extension to a quadruple above.

Lemma 2.16. Let \(r, s\) be integers such that \(r \leq s\). Then a map \((X_{\geq r})^{<s} \to (X^{<s})_{\geq r}\) which can be equipped with the equivalent data above, is an equivalence.

Proof. By looking at the cofibre of the map (drawn vertically) of fibre sequences

\[
\begin{array}{ccc}
X_{\geq s} & \to & X_{\geq s} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
X_{\geq r} & \to & X & \to & X^{<r},
\end{array}
\]

we obtain a fibre sequence

\[(X_{\geq r})^{<s} \to X^{<s} \to X^{<r}.\]

The map \(X^{<s} \to X^{<r}\) here is a map under \(X\), so can be identified with the canonical map \(X^{<s} \to (X^{<s})^{<r}\). Therefore, its fibre \((X_{\geq r})^{<s}\) is equivalent to \((X^{<s})_{\geq r}\) by a map over \(X^{<s}\). □

Definition 2.17. Let \(r, s\) be integers. Then we denote the canonically equivalent objects \((X_{\geq r})^{<s} = (X^{<s})_{\geq r}\) by \(X^{<s}_{\geq r}\). This belongs to \(A^{<s}_{\geq r}\).

2.2. Completion. Let \(A\) be a filtered stable category. Then define

\[A_{\geq r} := \lim_r A_{\geq r} = \bigcap_r A_{\geq r}.\]

We would like to investigate the sequence.

\[A_{\geq r} \to A \to \lim_r A^{<r}\]

obtained as the limit of the sequence

\[A_{\geq r} \to A \to A^{<r}.\]

Let us denote by \(\tau\) the functor \(A \to \lim_r A^{<r}\) here. This has a right adjoint which we shall denote by \(\lim\). For an object \(X = (X_r)_r\) of \(\lim_r A^{<r}\), it is given by

\[\lim X = \lim_r X_r,
\]

where the limit on the right hand side is taken in \(A\).

Definition 2.18. Let \(A\) be a filtered stable category. Then we denote \(\lim \tau X\) by \(\tilde{X}\). We say that \(X\) is complete if the unit map \(\eta: X \to \lim \tau X = \tilde{X}\) for the adjunction is an equivalence.

We denote by \(\tilde{A}\) the full subcategory of \(A\) consisting of complete objects.

Example 2.19. For every \(r\), \(A^{<r} \subset \tilde{A}\) in \(A\).

We compromise with the following definition, which may be more restrictive than it should be.
Definition 2.20. Let $\mathcal{A}$ be a filtered stable category. Then $\hat{\mathcal{A}}$ is said to be the completion of $\mathcal{A}$ if the following conditions are satisfied.

(0) The functor $\hat{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$ preserves sequential limits.
(1) $\hat{\mathcal{A}}$ lands in $\hat{\mathcal{A}}$.
(2) The map $\eta: \text{id} \to \hat{\mathcal{A}}$ makes $\hat{\mathcal{A}}$ a left localisation for the full subcategory $\hat{\mathcal{A}}$.

If $\mathcal{A}$ has $\hat{\mathcal{A}}$ as its completion in this sense, then we call the localisation functor the completion functor. In this case, we call $\eta$ the completion map.

We say that $\mathcal{A}$ is complete if $\hat{\mathcal{A}}$ is the whole of $\mathcal{A}$, namely, if every object of $\mathcal{A}$ is complete.

Remark 2.21. The conditions (1) and (2) follows if $\tau \lim \simeq \tau$ by the canonical map(s). This is also necessary since for every $r$, Example 2.19 will imply that the map $\eta^r: X^r \to \hat{X}^r$ is an equivalence for every $X$.

The following is a part of the motivation for Definition 2.20.

Lemma 2.22. If $\hat{\mathcal{A}}$ is the completion of $\mathcal{A}$, then the sequential limits exists in $\hat{\mathcal{A}}$, and the completion functor preserves sequential limits.

The following gives a sufficient condition for $\hat{\mathcal{A}}$ to be the completion of $\mathcal{A}$.

Lemma 2.23. If $\tau$ preserves sequential limits, then $\hat{\mathcal{A}}$ is the completion of $\mathcal{A}$.

Proof. The condition (0) of Definition 2.20 is automatic.

To prove the other conditions, it suffices to prove that $\tau \lim \simeq \tau$ by the canonical map(s). Let $X$ be an object of $\mathcal{A}$. Then it suffices to prove that for the unit map $\eta: X \to \lim \tau X$, the map $\eta^r$ is an equivalence for every $r$. By Lemma 2.24 it suffices to prove that the fibre $\lim_{r} X^r = \lim \tau X^r$ of $\eta$ belongs to $A_{\geq \infty}$.

We have $\tau \lim_{s} X_{s} = \lim_{s} \tau X_{s}$, so it suffices to show that this limit is 0. However, the limit over $s$ of the $r$-th object of $\tau X_{s}$ is $\lim_{s} \tau X_{s}^r = 0$ in $A_{< r}$, and coincides with the $r$-th object of $0 \in \lim_{r} A_{< r}$. It follows that this 0 is indeed the limit $\lim_{s} \tau X_{s}^r$.

□

Lemma 2.24. Let $\mathcal{A}$ be a filtered stable category. If the functor $\hat{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$ preserves sequential limits, then $\lim \tau \lim \simeq \lim \tau$ by the canonical map(s). In particular, if $\hat{\mathcal{A}}$ is the completion of $\mathcal{A}$, then $\lim \tau \lim$ lands in $\hat{\mathcal{A}}$, and will make $\hat{\mathcal{A}}$ a right localisation of $\lim_{r} A_{< r}$.

Proof. Let $X = (X_r)_r$ be an object of $\lim_{r} A_{< r}$. Then

$$\lim \tau \lim X = \hat{\lim}_{r} \hat{X}_r = \hat{\lim}_{r} X_r = \lim_{r} \hat{X}_r = \lim_{r} X_r = \lim X.$$

□

It would be natural to ask whether completion has a complementary right localisation. Let us first give a characterisation of objects with vanishing completion.

Lemma 2.25. Let $\mathcal{A}$ be a filtered stable category with $\hat{\mathcal{A}}$ its completion. Then the completion of an object $X$ of $\mathcal{A}$ vanishes if and only if $X$ belongs to $A_{\geq \infty}$.

Proof. $X$ belongs to $A_{\geq \infty}$ if and only if $\tau X \simeq 0$. The result then follows from Lemma 2.21. Indeed, $\tau X$ is contained in the full subcategory of $\lim_{r} A_{< r}$ which by the lemma, is a right localisation, and is identified with $\mathcal{A}$. Therefore, $\tau X \simeq 0$ in $\lim_{r} A_{< r}$ if and only if it is so in this full subcategory of $\lim_{r} A_{< r}$. However, the object of $\mathcal{A}$ corresponding to $\tau X$ under the identification, is $\hat{X} \in \hat{\mathcal{A}}$. □
Lemma 2.26. Let $\mathcal{A}$ be a filtered stable category with $\hat{\mathcal{A}}$ its completion. Suppose given an inverse system

$$\cdots \leftarrow X_i \leftarrow X_{i+1} \leftarrow \cdots$$

in $\mathcal{A}$, and suppose there is a sequence $(r_i)_i$ of integers, tending to $\infty$ as $i \to \infty$, such that $X_i$ belongs to $\mathcal{A}_{\geq r_i}$ for every $i$.

Then $\lim_i X_i$ belongs to $\mathcal{A}_{\geq \infty}$.

Proof. From the previous lemma, it suffices to prove that its completion vanishes. However,

$$\widehat{\lim_i X_i} = \lim_i \hat{X}_i = \lim_r \lim_i X_i^{< r} \simeq \lim_r 0 = 0.$$

\[\square\]

Corollary 2.27. Let $\mathcal{A}$ be a filtered stable category with $\hat{\mathcal{A}}$ its completion. Suppose given a map of inverse systems

$$\cdots \leftarrow X_i \leftarrow X_{i+1} \leftarrow \cdots$$

$$\cdots \leftarrow Y_i \leftarrow Y_{i+1} \leftarrow \cdots$$

in $\mathcal{A}$, and suppose there is a sequence $(r_i)_i$ of integers, tending to $\infty$ as $i \to \infty$, such that the fibre of $f_i$ belongs to $\mathcal{A}_{\geq r_i}$ for every $i$.

Then the map $\lim_i f_i : \lim_i X_i \to \lim_i Y_i$ is an equivalence after completion.

Proof. This follows from Lemma and Lemma 2.6. \[\square\]

Proposition 2.28. Let $\mathcal{A}$ be a filtered stable category with $\hat{\mathcal{A}}$ its completion. Then the full subcategory $\mathcal{A}_{\geq \infty}$ of $\mathcal{A}$ is a right localisation complementary to the left localisation $\hat{\mathcal{A}}$.

Proof. It suffices to show that completion has a complementary right localisation, since the right localisation will then be identified with $\mathcal{A}_{\geq \infty}$ by Lemma 2.25. Existence of the complement follows from Proposition 2.24 and Lemma 2.26 since the fibre of the completion map is $\lim_i X_{\geq r_i}$. \[\square\]

Corollary 2.29. Let $\mathcal{A}$ be a filtered stable category with $\hat{\mathcal{A}}$ its completion. Then a limit of complete objects is complete. In particular, a limit of bounded above objects is complete.

Proof. This follows from Lemma 2.9. \[\square\]

In practice, it may not be clear when $\tau$ preserves sequential limits, since limits in $\lim_i \mathcal{A}^{< r}$ is not always objectwise. The following condition will lead to the same conclusions on the completion, but involves only the sequential limits in $\mathcal{A}$.

Definition 2.30. Let $\mathcal{A}$ be a filtered stable category. Then we say that sequential limits are uniformly bounded in $\mathcal{A}$ if there exists a finite integer $d$ such that for every integer $r$, and for every inverse sequence in the full subcategory $\mathcal{A}_{\geq r}$ of $\mathcal{A}$, the limit of the sequence taken in $\mathcal{A}$, belongs to $\mathcal{A}_{\geq r+d}$.

Remark 2.31. $\mathcal{A}$ is assumed to have finite limits and sequential limits, so it has countable products at least, and if sequential limits are uniformly bounded, then so are countable products in the similar sense. In the case where the filtration is given by a t-structure, if countable products in $\mathcal{A}$ are uniformly bounded below by $b$, then the familiar computation of a sequential limit in terms of countable products by Milnor shows that sequential limits will be bounded by $b-1$. 


In the case of Goodwillie’s filtration (Example 2.13), sequential limits are bounded by 0.

However, it turns out that in order to prove that $\hat{A}$ is the completion of $A$ in this case, one necessarily proves that the functor $\tau$ preserves limits as well. Namely, we have the following two lemmas.

**Lemma 2.32.** Let $A$ be a filtered stable category with uniformly bounded sequential limits. Then $\tau$ is a left localisation. In other words, the functor $\lim: \lim_{<r} A \to A$ lands in $\hat{A}$, and induces an equivalence $\lim_{<r} A \sim \hat{A}$.

**Lemma 2.33.** In the case $\tau$ is a left localisation functor, $\tau$ preserves sequential limits if and only if $\hat{(\_): A \to A}$ preserves sequential limits.

**Proof assuming Lemma 2.32.** Through the identification of $\lim_{<r} A$ with $\hat{A}$ by the equivalence $\lim$, $\tau$ gets identified with $\hat{(\_): A \to A}$. □

**Proof of Lemma 2.32.** It suffices to prove that the counit $\varepsilon: \tau \lim \to \text{id}$ of the adjunction is an equivalence.

Let $X = (X_r)_r$ be an object of $\lim_{<r} A$. Then the counit for the adjunction is given by

$$(\lim_{<r} X_s) \to X_{r}$$

for each $r$. Let $d \leq 0$ be a uniform bound for sequential limits. We can apply Lemma 2.6 to the fibre sequence

$$\lim_{s}(X_{s})_{\geq r-d} \to \lim_{s} X_s \to \lim_{s}(X_{s})_{<r-d},$$

where the fibre belongs to $A_{\geq r}$, and the cofibre is $X_{r-d}$. We get that that the induced map $(\lim X)^{<r} \to X_{r-d}^{<r} = X_r$ is an equivalence. □

**Lemma 2.34.** Let $A$ be a filtered stable category with uniformly bounded sequential limits. Then $\hat{(\_): A \to A}$ preserves sequential limits.

**Proof.** Let

$$\cdots \leftarrow X_i \leftarrow X_{i+1} \leftarrow \cdots$$

be a sequence in $A$. Then

$$(\lim_{i} X_i)^{<r} = (\lim_{i} X_{i}^{<r-d})^{<r}.$$ $$\text{The limit of this as } r \to \infty \text{ can then be computed as } \lim_{r} \lim_{s} (\lim_{i} X_{i}^{<s})^{<r}, \text{ but } \lim_{i} X_{i}^{<s} \text{ belongs to } A^{<s} \text{ by Lemma 2.9 so}$$

$$\lim_{r} (\lim_{i} X_{i}^{<s})^{<r} = \lim_{i} X_{i}^{<s}.$$ $$\text{Now } \lim_{s} \lim_{i} X_{i}^{<s} = \lim_{i} \hat{X}_i, \text{ so we have proved that } \hat{\lim_{i} X_i} = \lim_{i} \hat{X}_i \text{ as desired.}$$ □

We have proved the following.

**Proposition 2.35.** Let $A$ be a filtered stable category with uniformly bounded sequential limits. Then $\hat{A}$ is the completion of $A$, with complementary right localisation $A_{\geq \infty}$ as a full subcategory of $A$.

**Corollary 2.36.** If sequential limits are uniformly bounded in $A$, then $A$ is complete if and only if $A_{\geq \infty} \simeq 0$. 
2.3. The completion as a complete category. When \( \hat{A} \) is the completion of a filtered stable category \( A \), then it will be useful if the completion is itself a complete filtered stable category. We would like to first consider a sufficient condition for the completion to be a stable category.

We start with the general situation of localisation.

**Definition 2.37.** Let \( A \) be a stable category, and \( A_\ell \) be its left localisation. Then we say that the localisation is **stable exact** if \( A_\ell \) is a stable category, and the localisation functor and the inclusion functor \( A_\ell \hookrightarrow A \) are both exact.

A **right** localisation of a stable category is **stable exact** if it is so as a left localisation of the opposite category.

For a left localisation \( A_\ell \) of a stable category \( A \), there is actually another set of conditions which is equivalent to stable exactness. First, note that \( A_\ell \) has finite limits if and only if as a full subcategory of \( A \), it is closed under finite limits (in \( A \)). On the other hand, \( A_\ell \) has all finite colimits, and preservation of finite colimits by the inclusion functor is equivalent to that \( A_\ell \) is closed as a full subcategory of \( A \) under finite colimits.

**Lemma 2.38.** Let \( A \) be a stable category, and let \( A_\ell \) be a left localisation of \( A \). If \( A_\ell \) is closed under both finite limits and finite colimits in \( A \), then \( A_\ell \) is stable, and the inclusion functor \( A_\ell \hookrightarrow A \) is exact.

**Proof.** From the discussion above, \( A_\ell \) has all finite limits and finite colimits, and the inclusion is exact. We need to show that the initial object and the terminal object coincide in \( A_\ell \), and that pushout squares and pullback squares coincide in \( A_\ell \). However, these are obvious since those limits and colimits are all those in \( A \), and the required coincidences hold in \( A \). \( \square \)

**Corollary 2.39.** A left localisation \( A_\ell \) of a stable category \( A \) is stable exact if (and only if) \( A_\ell \) as a full subcategory of \( A \), is closed under finite limits and finite colimits, and if the localisation functor \( A \to A_\ell \) is exact.

**Lemma 2.40.** Let \( A \) be a stable category, and let \( A_\ell \) and \( A_r \) be left and right localisations of \( A \) which are complementary to each other. Then the following are equivalent. (Recall from Lemma 2.9 that \( A_\ell \) and \( A_r \) both have finite limits and finite colimits.)

- \( A_\ell \) is a stable exact localisation.
- \( A_r \) is a stable exact localisation.
- The localisation functors are both exact.
- The inclusion functors are both exact.

**Proof.** Apply Proposition 2.10 to finite limits and finite colimits. \( \square \)

**Definition 2.41.** Let \( A \) be a filtered stable category with \( \hat{A} \) its completion. Then we say that the completion is **stable exact** if \( \hat{A} \) is a stable exact localisation of \( A \).

We shall look for a sufficient condition for the completion to be stable exact.

**Definition 2.42.** Let \( A \) be a filtered stable category. An integer \( \omega \) is said to be a **uniform bound for loops** in \( A \) if for every integer \( r \), and for every object of the full subcategory \( A_{\geq r} \) of \( A \), its loop in \( A \) belongs to \( A_{\geq r+\omega} \). We say that **loops are uniformly bounded** in \( A \) if loops in \( A \) have a uniform bound.

**Remark 2.43.** By Corollary 2.7, \( \omega \) is a uniform bound for loops if and only if it is a uniform bound for **fibres** in the similar sense. Indeed, if \( W \to X \to Y \) is a fibre sequence in \( A \), then there is a fibre sequence \( \Omega Y \to W \to X \).
Loops are uniformly bounded if the action of the category of (finite) spectra on $A$ by tensoring, is compatible with the filtrations (on the category of spectra and on $A$) in a way similar to (or slightly more general than) the way we consider in Definition 4.4. In this case, the suspension functor raises the filtration, as we shall consider in Definition 4.3.

**Example 2.44.** $\omega$ can be taken as $-1$ if the filtration is a t-structure on $A$.

$\omega$ can be taken as 0 for Goodwillie’s filtration. In fact, all localisations are stable exact in this filtration.

**Lemma 2.45.** Let $A$ be a filtered stable category. If loops are uniformly bounded in $A$, then for any finite category $K$, limits of $K$-shaped diagrams are uniformly bounded in $A$.

*Proof.* A uniform bound of fibres more generally bounds fibre products. Then the result follows from the arguments of the proof of Corollary 4.4.2.4 of [7]. □

**Lemma 2.46.** Let $A$ be a filtered stable category with $\hat{A}$ its completion. If loops are uniformly bounded in $A$, then the completion is an exact left localisation of $A$.

*Proof.* By Proposition 2.10 it suffices to prove that the full subcategory $A_{\geq \infty}$ of $A$ is closed under finite limits in $A$.

Let $K$ be a finite category, and let $X$ be a $K$-shaped diagram in the full subcategory $A_{\geq \infty}$ of $A$. Then we would like to prove that $\lim_K X$ belongs to $A_{\geq \infty}$.

However, for every $r$, $\lim_K X$ belongs to $A_{\geq r}$ since $X$ is in particular a diagram in $A_{\geq r-k}$ for a uniform bound $k$ of $K$-shaped limits. □

**Definition 2.47.** Let $A$ be a filtered stable category. Then we say that suspensions are uniformly bounded in $A$ if there exists a finite integer $\sigma$ such that for every integer $r$, and for every object of the full subcategory $A^{<r}$ of $A$, its suspension in $A$ belongs to $A^{<r+\sigma}$.

**Example 2.48.** $\sigma$ can be taken as 1 if the filtration is a t-structure on $A$. Goodwillie’s filtration is stable exact as mentioned in Example 2.44, so $\sigma$ can be taken as 0 in this case.

**Lemma 2.49.** Let $A$ be a filtered stable category with $\hat{A}$ its completion. If suspensions are uniformly bounded in $A$, then the right localisation $(\cdot)_{\geq \infty} : A \to \hat{A}_{\geq \infty}$ is exact.

*Proof.* By Proposition 2.10 it suffices to prove that the full subcategory $\hat{A}$ of $A$ is closed under finite colimits in $A$.

Let $K$ be a finite category, and let $X$ be a diagram in the full subcategory $\hat{A}$ of $A$. Then we would like to prove that $\colim_K X$ belongs to $\hat{A}_{\geq \infty}$.

However, $\colim_K X = \lim_r \colim_K X^{<r}$, and $\colim_K X^{<r}$ belongs to $A^{<r+k}$ for a uniform bound $k$ of $K$-shaped colimits. The result follows from Corollary 2.29. □

Combining the lemmas, we obtain the following.

**Proposition 2.50.** Let $A$ be a filtered stable category with $\hat{A}$ its completion. If loops and suspensions are uniformly bounded in $A$, then the completion is stable exact.

**Proposition 2.51.** Let $A$ be a filtered stable category with $\hat{A}$ its completion. If the completion is stable exact, then the canonical tower

$$\hat{A} \to \cdots \to A^{<r} \to A^{<r-1} \to \cdots$$

makes $\hat{A}$ into a complete filtered stable category.
Proof. As we have remarked in Example 2.13, for every \( r \), \( A^<r \subset \hat{A} \) as full subcategories of \( \hat{A} \). It follows that the restriction to \( \hat{A} \) of the localisation functor \( \hat{A} \to A^{<r} \) is a left localisation. A complementary right localisation to this is given by \( A_{\geq r} \cap \hat{A} \).

It is easy to see that \( \hat{A} \) is complete with respect to this filtration. \(\square\)

Lemma 2.52. Let \( A \) be a filtered stable category with \( \hat{A} \) its stable exact completion. Then any class of limits which exist in \( A \) (and therefore also in \( \hat{A} \) by Lemma 2.7) and are uniformly bounded, have the same uniform bound in \( \hat{A} \).

Proof. Lemma 2.51 in fact states that \( \hat{A} \) is closed under the limits which exists in \( A \). The result follows since the full subcategory \( A_{\geq r} \) in the filtration of \( \hat{A} \) is just \( A_{\geq r} \cap \hat{A} \) as a full subcategory of \( A \). \(\square\)

2.4. Totalisation. In this section, we shall prove a technical result which will be very useful for our study of the Koszul duality.

Definition 2.53. Let \( A, B \) be filtered stable categories, and let \( F: A \to B \) be an exact functor. Then we say that an integer \( b \) is a lower bound of \( F \) if for every \( r \), \( F \) takes the full subcategory \( A_{\geq r} \) of the source to the full subcategory \( B_{\geq r+b} \) of the target.

We say that \( F \) is bounded below if it has a lower bound.

Let \( \Delta_f \) denote the subcategory of the category \( \Delta \) of combinatorial simplices, where only face maps (maps strictly preserving the order of vertices) are included. A covariant functor \( X^\bullet: \Delta_f \to A \) is a cosimplicial object ‘without degeneracies’ of \( \hat{A} \). Its totalisation \( \text{tot} X^\bullet \) is by definition, the limit over \( \Delta_f \) of the diagram \( X^\bullet \).

Proposition 2.54. Let \( A, B \) be filtered stable categories, and let \( F: A \to B \) be an exact functor which is bounded below. Assume that loops and sequential limits are uniformly bounded in \( A \), and \( \hat{B} \) is the completion of \( B \).

Let \( X^\bullet: \Delta_f \to A \) be such that there exists a sequence \( r = (r_n)_n \) of integers, tending to \( \infty \) as \( n \to \infty \), such that for a uniform bound \( \omega \) for loops, and for every \( n \), \( X_n \) belongs to \( A_{\geq -\omega n + r_n} \). Then the canonical map

\[ F(\text{tot} X^\bullet) \longrightarrow \text{tot} FX^\bullet \]

is an equivalence after completion.

Proof. According to the sequence of full subcategories

\[ \Delta_f \supset \cdots \supset \Delta_{\leq n} \supset \Delta_{\leq n-1} \supset \cdots, \]

where objects of \( \Delta_{\leq n} \) are simplices of dimension at most \( n \), we have the sequence

\[ \text{tot} X^\bullet \longrightarrow \cdots \longrightarrow \text{sk}_n \text{tot} X^\bullet \longrightarrow \text{sk}_{n-1} \text{tot} X^\bullet \longrightarrow \cdots \]

such that \( \text{tot} X^\bullet = \lim_n \text{sk}_n \text{tot} X^\bullet \), where “sk \( n \) tot” is a single symbol representing the operation of taking the limit over \( \Delta_{\leq n} \).

It is standard that the fibre of the map \( \text{sk}_n \text{tot} X^\bullet \to \text{sk}_{n-1} \text{tot} X^\bullet \) is equivalent to \( \Omega^n X^\bullet \). It follows from our assumption that this belongs to \( A_{\geq r_{n-1}} \). It follows that the fibre of the map \( \text{tot} X^\bullet \to \text{sk}_n \text{tot} X^\bullet \) belongs to \( A_{\geq r_n + d} \) for \( d \) a uniform bound for sequential limits.

It follows that the fibre of the map \( F(\text{tot} X^\bullet) \to \text{sk}_n \text{tot} FX^\bullet \) belongs to \( B_{\geq r_{n+d}+b} \) for a bound \( b \) of \( F \). By taking the limit over \( n \), we obtain the result from Corollary 2.27. \(\square\)

Similarly, we can consider simplicial objects ‘without degeneracies’ and their geometric realisations.
Proposition 2.55. Let \( A, B \) be filtered stable categories, and let \( F : A \to B \) be an exact functor which is bounded below. Assume that \( \hat{B} \) is the completion of \( B \).

Let \( X^*: \Delta_0^{op} \to A \) be such that there exists a sequence \( r = (r_n)_n \) of integers, tending to \( \infty \) as \( n \to \infty \), such that for every \( n \), \( X^n \) belongs to \( A_{\geq r_n} \). Then the canonical map

\[
[F^X] \to F[X^*]
\]

is an equivalence after completion.

Proof. The proof of this is simpler. One just notes that the full subcategory \( B_{\geq r} \) of \( B \) is closed under any colimit by Lemma 2.9. \( \square \)

3. Monoidal filtered stable category

3.0. Pairing in filtered stable categories.

Definition 3.0. Let \( A, B, C \) be stable categories. Then, a pairing from \( A, B \) to \( C \) is a functor

\[
\langle , \rangle : A \times B \to C
\]

which is exact in each variable.

Definition 3.1. Let \( A, B, C \) be filtered stable categories. Then, a pairing \( \langle , \rangle : A \times B \to C \) is compatible with the filtrations if for any \( r, s \), it takes \( A_{\geq r} \times B_{\geq s} \) to \( C_{\geq r+s} \).

Example 3.2. Let \( S \) denote the stable category of finite spectra, filtered by connectivity. Let \( A \) be another stable category with a t-structure. Then the pairing \( S \times A \to A \) given by tensoring is compatible with the filtrations (ignoring that \( S \) is not closed under sequential limits).

Remark 3.3. More generally, we may say that the pairing is bounded below if there is a finite integer \( d \) such that the pairing takes \( A_{\geq r} \times B_{\geq s} \) to \( C_{\geq r+s+d} \). In this case, if \( d \) can be taken only as a negative number, then the pairing is not compatible with the filtrations in the above sense. However, this can be corrected by reindexing the filtrations in any suitable way.

It might seem natural to further require the pairing to preserve sequential limits (variable-wise). However, this condition is too strong to require in practice. We will see that the compatibility defined above ensures that certain sequential limits are preserved (up to completion). This will turn out to be useful for applications.

Definition 3.4. Let \( A \) be a filtered stable category. Then an object \( X \) of \( A \) is said to be bounded below in the filtration if there exists an integer \( r \) such that \( X \in A_{\geq r} \).

Let \( A, B, C \) be filtered stable categories, and consider a pairing \( \langle , \rangle : A \times B \to C \), compatible with the filtrations. The following is an immediate consequence of Lemma 2.26.

Lemma 3.5. Assume that \( \hat{C} \) is the completion of \( C \).

Suppose given an inverse system

\[
\cdots \leftarrow X_i \leftarrow X_{i+1} \cdots
\]

in \( A \), and suppose there is a sequence \( (r_i)_i \) of integers, tending to \( \infty \) as \( i \to \infty \), such that for every \( i \), \( X_i \) belongs to \( A_{\geq r_i} \). Then for every bounded below \( Y \in B \), \( \lim_i \langle X_i, Y \rangle \) belongs to \( C_{\geq \infty} \).
Proposition 3.6. Let \( \langle \cdot, \cdot \rangle : A \times B \to C \) be a pairing on filtered stable categories, compatible with the filtrations. Assume that \( \hat{C} \) is the completion of \( C \). Assume also that sequential limits are uniformly bounded in \( A \).

Suppose given an inverse system

\[
\cdots \leftarrow X_i \leftarrow X_{i+1} \leftarrow \cdots
\]

in \( A \), and suppose there is a sequence \((r_i)\)_i of integers, tending to \( \infty \) as \( i \to \infty \), such that for every \( i \), the fibre of the map \( X_{i+1} \to X_i \) belongs to \( A_{\geq r_i} \). Then for every \( Y \in B \), if \( Y \) is bounded below, then the map

\[
\lim_i \langle X_i, Y \rangle \longrightarrow \lim_i \langle X_i, Y \rangle
\]

is an equivalence after completion.

Proof. This follows from Corollary 2.27. \( \square \)

3.1. Monoidal structure on a filtered category. By a monoidal structure on a stable category \( A \), we mean a monoidal structure on the underlying category of \( A \) whose multiplication operations are exact in each variable.

Definition 3.7. Let \( A \) be a filtered stable category, and let \( \otimes \) be a monoidal structure on the stable category (underlying) \( A \). We say that the monoidal structure is compatible with the filtration on \( A \) if for every finite totally ordered set \( I \), and every sequence \( r = (r_i)_{i \in I} \) of integers, the functor \( \otimes_{i \in I} : A^I \to A \) takes the full subcategory \( \prod_{i \in I} A_{\geq r_i} \) of the source, to the full subcategory \( A_{\geq \sum r_i} \) of the target.

We call a filtered stable category \( A \) equipped with a compatible monoidal structure a monoidal filtered stable category. If the monoidal structure is symmetric, then it will just be a symmetric monoidal filtered category.

In other words, a symmetric monoidal filtered stable category is just a commutative monoid object of a suitable multicategory of filtered stable categories.

Example 3.8. In the case where \( A \) is a functor category with Goodwillie’s filtration, if the target category is a symmetric monoidal stable category, then the pointwise symmetric monoidal structure on \( A \) is compatible with the filtration.

Remark 3.9. More generally, we may say that the monoidal structure is bounded below if the unit \( 1 \) and the pairing \( A^2 \otimes A \) are bounded below. All the results we consider in the following on monoidal filtered categories will be valid for filtered stable categories with a bounded below monoidal structure, after making suitable (and straightforward) modifications.

We shall only state the results for monoidal filtered categories in our sense, in order to keep the exposition simple.

Remark 3.10. Even though both filtration and monoidal structure are self-dual notion on a stable category, the boundedness below of the monoidal structure is not self-dual. Namely, boundedness below in \( A^{op} \) means boundedness above in \( A \).

Corollary 3.11. Let \( A \) be a monoidal filtered stable category with uniformly bounded sequential limits.

Suppose given an inverse system

\[
\cdots \leftarrow X_i \leftarrow X_{i+1} \leftarrow \cdots
\]

in \( A \), and suppose there is a sequence \((r_i)\)_i of integers, tending to \( \infty \) as \( i \to \infty \), such that for every \( i \), the fibre of the map \( X_{i+1} \to X_i \) belongs to \( A_{\geq r_i} \). Then for every \( Y \in A \), if \( Y \) is bounded below, then the map

\[
(\lim_i X_i) \otimes Y \longrightarrow \lim_i (X_i \otimes Y)
\]

is an equivalence after completion.
Definition 3.12. Let $\mathcal{A}$ be a monoidal filtered stable category with $\hat{\mathcal{A}}$ completing the filtration. Then we say that the monoidal structure is **completable** if there is a monoidal structure on $\hat{\mathcal{A}}$ such that the completion functor $\mathcal{A} \to \hat{\mathcal{A}}$ is monoidal.

Remark 3.13. Together with a monoidal structure of the completion functor, the monoidal structure on $\hat{\mathcal{A}}$ will be uniquely determined.

Lemma 3.14. Let $\mathcal{A}$ be a monoidal filtered stable category with $\hat{\mathcal{A}}$ completing the filtration. Then the monoidal structure is completable if and only if the monoidal product functor lands in the full subcategory $\mathcal{A}_{\geq \infty}$ of $\mathcal{A}$ whenever one of the factors is in $\mathcal{A}_{\geq \infty}$.

Proof. Necessity is obvious.

Conversely, if the assumption is satisfied, then the lax monoidal structure whose operations are the composites

$$\hat{A}^n \longrightarrow A^n \otimes \hat{A} \longrightarrow \hat{A},$$

is in fact a genuinely associative structure. □

Proposition 3.15. Let $\mathcal{A}$ be a monoidal filtered stable category with $\hat{\mathcal{A}}$ its stable exact completion. If the monoidal structure on $\mathcal{A}$ is completable, then $\hat{\mathcal{A}}$ with the induced structures is a monoidal (complete) filtered stable category.

Proof. The variable-wise exactness of the induced monoidal structure on $\hat{\mathcal{A}}$ follows from the description of the monoidal product functor in the proof of Lemma 3.14 (see Remark 3.13).

We further need to prove that this monoidal structure is compatible with the induced filtration on $\hat{\mathcal{A}}$ (see Proposition 2.51). This follows since $\hat{A}_{\geq r} \subset A_{\geq r}$ as full subcategories of $\mathcal{A}$, and the completion functor $\mathcal{A} \to \hat{\mathcal{A}}$ takes the full subcategory $\mathcal{A}_{\geq r}$ of the source, to the full subcategory $\hat{\mathcal{A}}_{\geq r}$ of the target. □

Lemma 3.16. Let $\mathcal{A}$ be as in Proposition 3.15. If the monoidal multiplication functor on $\mathcal{A}$ preserves variable-wise, a certain class of colimits (specified as in Proposition 2.10, and assumed to exist), then so does the completed monoidal operation on $\hat{\mathcal{A}}$ if for every $r$, the full subcategory $\mathcal{A}^{< r}$ of $\mathcal{A}$ are closed under the class of colimits taken in $\mathcal{A}$.

Proof. In view of the description of the monoidal multiplication functor in the proof of Lemma 3.14 it suffices to prove under our assumption, that the inclusion functor $\hat{\mathcal{A}}$ preserves the class colimits in question. By Proposition 2.10 this condition is equivalent to that the localisation functor $(\ )_{\geq \infty}: \mathcal{A} \to \mathcal{A}_{\geq \infty}$ preserves the class of colimits in question.

Recall that $\mathcal{A}_{\geq \infty} = \lim_r \mathcal{A}_{\geq r}$. Since this limit is along colimit preserving functors, colimits in $\lim_r \mathcal{A}_{\geq r}$ is object-wise. Therefore, it suffices to show for every $r$, that $(\ )_{\geq r}: \mathcal{A} \to \mathcal{A}_{\geq r}$ preserves colimits.

We conclude by invoking Proposition 2.10 again. □

3.2. The filtered category of filtered objects. In this section, we shall give a simple example of a filtered stable category, for which limits of any kind are uniformly bounded by 0. We also show how this filtered stable category may have a completable symmetric monoidal structure.

Let us denote by $\text{Sta}$ the following symmetric (2-)multicategory. Its object is a stable category. Given a family $\mathcal{A} = (\mathcal{A}_s)_{s \in S}$ of stable categories indexed by a finite set $S$, and a stable category $\mathcal{B}$, we define a multimap $\mathcal{A} \to \mathcal{B}$ to be a functor
Let $Z$ be the category
defined by the poset of integers. This is a commutative monoid (in the category of poset), so the functor category $\text{Fun}(Z, \text{Sta})$ is a symmetric multicategory.

Let $B$ be an object of $\text{Fun}(Z, \text{Sta})$, and let $A$ be the category of lax morphisms $\ast \to B$ in $\text{Fun}(Z, \text{Cat})$. To be more precise about the variance in the definition of a lax functor here, we consider the functor $y: Z \to \text{Cat}$, $n \mapsto \mathbb{Z}/n$, and define $A$ to be the category of (genuine, rather than lax) morphisms $y \to B$.

**Definition 3.17.** In the case where the sequence $B$ is constant at a stable category $C$, we call an object of $A$ a filtered object of $C$, so $A$ will be the category of filtered objects of $C$.

Assume that each $B_n$ has all sequential limits. Then $A$ (in which sequential limits are given object-wise) is a filtered stable category as follows.

Concretely, $B$ is a sequence of stable categories
defined by the poset of integers. This is a commutative monoid (in the category of poset), so the functor category $\text{Fun}(Z, \text{Sta})$ is a symmetric multicategory.

Let $\prod_s A \to B$ which is exact in each variable. (Note that the condition is vacuous when there is no variable, i.e., when $S$ is empty and the product is one point.)

Let $A$ be the category
\[
\cdots \leftarrow n \leftarrow n + 1 \leftarrow \cdots
\]
defined by the poset of integers. This is a commutative monoid (in the category of poset), so the functor category $\text{Fun}(Z, \text{Sta})$ is a symmetric multicategory.

Let $B$ be an object of $\text{Fun}(Z, \text{Sta})$, and let $A$ be the category of lax morphisms $\ast \to B$ in $\text{Fun}(Z, \text{Cat})$. To be more precise about the variance in the definition of a lax functor here, we consider the functor $y: Z \to \text{Cat}$, $n \mapsto \mathbb{Z}/n$, and define $A$ to be the category of (genuine, rather than lax) morphisms $y \to B$.

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Let $\prod_s A \to B$ which is exact in each variable. (Note that the condition is vacuous when there is no variable, i.e., when $S$ is empty and the product is one point.)

Let $Z$ be the category
\[
\cdots \leftarrow n \leftarrow n + 1 \leftarrow \cdots
\]
defined by the poset of integers. This is a commutative monoid (in the category of poset), so the functor category $\text{Fun}(Z, \text{Sta})$ is a symmetric multicategory.

Let $B$ be an object of $\text{Fun}(Z, \text{Sta})$, and let $A$ be the category of lax morphisms $\ast \to B$ in $\text{Fun}(Z, \text{Cat})$. To be more precise about the variance in the definition of a lax functor here, we consider the functor $y: Z \to \text{Cat}$, $n \mapsto \mathbb{Z}/n$, and define $A$ to be the category of (genuine, rather than lax) morphisms $y \to B$.

**Definition 3.17.** In the case where the sequence $B$ is constant at a stable category $C$, we call an object of $A$ a filtered object of $C$, so $A$ will be the category of filtered objects of $C$.

Assume that each $B_n$ has all sequential limits. Then $A$ (in which sequential limits are given object-wise) is a filtered stable category as follows.

Concretely, $B$ is a sequence of stable categories
defined by the poset of integers. This is a commutative monoid (in the category of poset), so the functor category $\text{Fun}(Z, \text{Sta})$ is a symmetric multicategory.

Let $\prod_s A \to B$ which is exact in each variable. (Note that the condition is vacuous when there is no variable, i.e., when $S$ is empty and the product is one point.)

Let $Z$ be the category
\[
\cdots \leftarrow n \leftarrow n + 1 \leftarrow \cdots
\]
defined by the poset of integers. This is a commutative monoid (in the category of poset), so the functor category $\text{Fun}(Z, \text{Sta})$ is a symmetric multicategory.

Let $B$ be an object of $\text{Fun}(Z, \text{Sta})$, and let $A$ be the category of lax morphisms $\ast \to B$ in $\text{Fun}(Z, \text{Cat})$. To be more precise about the variance in the definition of a lax functor here, we consider the functor $y: Z \to \text{Cat}$, $n \mapsto \mathbb{Z}/n$, and define $A$ to be the category of (genuine, rather than lax) morphisms $y \to B$.

**Definition 3.17.** In the case where the sequence $B$ is constant at a stable category $C$, we call an object of $A$ a filtered object of $C$, so $A$ will be the category of filtered objects of $C$.

Assume that each $B_n$ has all sequential limits. Then $A$ (in which sequential limits are given object-wise) is a filtered stable category as follows.

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Let $Z$ be the category
\[
\cdots \leftarrow n \leftarrow n + 1 \leftarrow \cdots
\]
defined by the poset of integers. This is a commutative monoid (in the category of poset), so the functor category $\text{Fun}(Z, \text{Sta})$ is a symmetric multicategory.
In this case, $\mathcal{A}$ inherits a symmetric monoidal structure. Namely, if $X = (F_nX)_n$ and $Y = (F_nY)_n$ are objects of $\mathcal{A}$, then we have $X \otimes Y$ defined by
$$F_n(X \otimes Y) = \colim_{i+j \geq n} L^{i+j-n}(F_iX \otimes F_jY),$$
the colimit taken in $\mathcal{B}_n$. This monoidal multiplication preserves colimits variable-wise.

**Proposition 3.21.** The symmetric monoidal structure on $\mathcal{A}$ is compatible with the filtration on $\mathcal{A}$.

**Proof.** Let $r$, $s$ be integers, and let $X \in \mathcal{A}_{\geq r}$ and $Y \in \mathcal{A}_{\geq s}$. Then we would then like to prove $X \otimes Y \in \mathcal{A}_{\geq r+s}$.

In terms of the sequence defining $X$ and $Y$, the given conditions are that the map $F_iX \leftarrow L^{r-i}F_rX$ is an equivalence for $i \leq r$, and similarly for $Y$. Under these assumptions, we need to prove that the map $F_n(X \otimes Y) \leftarrow L^{r+s-n}F_{r+s}(X \otimes Y)$ is an equivalence for $n \leq r + s$.

By definition, $F_n(X \otimes Y)$ was the colimit over $i, j$ such that $i + j \geq n$, of $L^{i+j-n}(F_iX \otimes F_jY)$. It suffices to prove that, for $n \leq r + s$, this colimit is the same as the colimit of $L^{i+j-n}(F_iX \otimes F_jY)$ over $i, j$ such that $i \geq r$ and $j \geq s$. However, the given assumptions imply that the diagram over $i, j$ such that $i + j \geq n$, is the left Kan extension of its restriction to $i, j$ such that $i \geq r$ and $j \geq s$, since the assumptions imply that the map $F_iX \otimes F_jY \leftarrow F_{\max(i, r)}X \otimes F_{\max(j, s)}Y$ (in $\mathcal{B}_{i+j}$; we have omitted $L$ from the notation) will be an equivalence for all $i, j$. The result follows. \hfill $\square$

**Proposition 3.22.** The monoidal structure on $\mathcal{A}$ is completable.

**Proof.** We want to show that if $X \in \mathcal{A}_{\geq \infty}$ and $Y \in \mathcal{A}$, then $X \otimes Y \in \mathcal{A}_{\geq \infty}$.

The given condition is the same as that the map $F_nX \leftarrow F_{n+1}X$ is an equivalence for every $n$. We then want to prove that the map
$$F_n(X \otimes Y) = \colim_{i+j \geq n} F_iX \otimes F_jY \leftarrow \colim_{i+j \geq n+1} F_iX \otimes F_jY = F_{n+1}(X \otimes Y)$$
is an equivalence for every $n$.

However, an inverse to this map can be constructed as the colimit
$$\colim_{i+j \geq n} F_iX \otimes F_jY \longrightarrow \colim_{i+j \geq n} F_{i+1}X \otimes F_jY$$
of the maps induced from the inverses $F_iX \rightarrow F_{i+1}X$ to the given equivalences. \hfill $\square$

### 3.3. Modules over an algebra in filtered stable category.

Further examples of filtered stable categories will be found by considering modules over an algebra in a filtered stable category. We shall investigate them further.

Let us start from the situation of general localisation of a stable category.

Thus, let $\mathcal{A}$ be a stable category, and let a monoidal structure $\otimes$ on $\mathcal{A}$ be given. Recall that we assume by convention that the monoidal multiplication is exact in each variable.

Let us further assume given a left localisation $(\_)_L : \mathcal{A} \rightarrow \mathcal{A}_L$ of $\mathcal{A}$ with a complementary right localisation $(\_)_R : \mathcal{A} \rightarrow \mathcal{A}_r$.

We assume given an associative algebra $A$ in $\mathcal{A}$, and would like to have a corresponding localisation of the category $\operatorname{Mod}_A$ of (say, right) $A$-modules, in a natural way. A sufficient condition so one can do this is that the functor $\_ \otimes A : \mathcal{A} \rightarrow \mathcal{A}$ take $\mathcal{A}_L$ to $\mathcal{A}_R$. (There is no difference if $\mathcal{A}$ is not assumed to be monoidal, but it is given an action by any monad, in place of an action of an algebra object in $\mathcal{A}$. For our applications, we do not need to use this language.)
Indeed, if \( A \) satisfies this condition, then for any object \( X \) of \( \mathcal{A}_r \) and \( Y \) of \( \mathcal{A} \), and for any integer \( n \geq 0 \), we have that the map

\[
\text{Map}(X \otimes A^{\otimes n}, Y_r) \rightarrow \text{Map}(X \otimes A^{\otimes n}, Y)
\]

is an equivalence. It follows that the category \( \text{Mod}_{\mathcal{A}_r} := \text{Mod}_{\mathcal{A}}(\mathcal{A}_r) \) of \( \mathcal{A} \)-modules in \( \mathcal{A}_r \), is a full subcategory of \( \text{Mod}_{\mathcal{A}}(\mathcal{A}) = \text{Mod}_{\mathcal{A}} \) by the functor induced from \( \mathcal{A}_r \hookrightarrow \mathcal{A} \), and is a right localisation of \( \text{Mod}_{\mathcal{A}} \). It further follows that the square

\[
\begin{array}{ccc}
\text{Mod}_{\mathcal{A}_r} & \longrightarrow & \text{Mod}_{\mathcal{A}} \\
\downarrow & & \downarrow \\
\mathcal{A}_r & \longrightarrow & \mathcal{A},
\end{array}
\]

where the vertical arrows are the forgetful functors, is Cartesian, and the localisation functor \( \text{Mod}_{\mathcal{A}} \rightarrow \text{Mod}_{\mathcal{A}_r} \) is the functor induced from the localisation functor \((\_)_r: \mathcal{A} \rightarrow \mathcal{A}_r\), and its lax linearity over the action of \( \mathcal{A} \). In particular, the localisation functor lifts \((\_)_r\) canonically.

It follows that the complementary left localisation \( \text{Mod}_{\mathcal{A},\ell} \) of \( \text{Mod}_{\mathcal{A}} \) is given by the Cartesian square

\[
\begin{array}{ccc}
\text{Mod}_{\mathcal{A},\ell} & \longrightarrow & \text{Mod}_{\mathcal{A}} \\
\downarrow & & \downarrow \\
\mathcal{A}_\ell & \longrightarrow & \mathcal{A}.
\end{array}
\]

As a full subcategory of \( \text{Mod}_{\mathcal{A}} \), this can be also expressed as \( \text{Mod}_{\mathcal{A}}(\mathcal{A}_\ell) \), modules with respect to the op-lax action of powers of \( \mathcal{A} \) on \( \mathcal{A}_\ell \) by \( X \mapsto X \otimes \ell A^{\otimes n} := (X \otimes A^{\otimes n})_\ell \).

**Remark 3.23.** The action of powers of \( \mathcal{A} \) on \( \mathcal{A}_\ell \) is in fact genuinely associative. To see this, it suffices to show that for any object \( X \) of \( \mathcal{A} \), the map \( X \otimes \ell A^{\otimes n} \rightarrow X_\ell \otimes \ell A^{\otimes n} \) is an equivalence. This follows from the cofibre sequence

\[
X_r \otimes A^{\otimes n} \rightarrow X \otimes A^{\otimes n} \rightarrow X_\ell \otimes A^{\otimes n}
\]

and Lemma 2.6.

The left localisation \( \text{Mod}_{\mathcal{A}} \rightarrow \text{Mod}_{\mathcal{A},\ell} \) lifts \((\_)_\ell: \mathcal{A} \rightarrow \mathcal{A}_\ell\) canonically, since the left localisation functor is the cofibre of the right localisation map, and the forgetful functor \( \text{Mod}_{\mathcal{A}} \rightarrow \mathcal{A} \) preserves cofibre sequences.

In terms of objects, if \( K \) is an \( \mathcal{A} \)-module in \( \mathcal{A} \), then \( K_\ell \) has a canonical structure of an \( \mathcal{A} \)-module. This comes from the canonical structure of an \( \mathcal{A} \)-module on \( K_r \), and the canonical structure of an \( \mathcal{A} \)-module map on the right localisation map \( K_r \rightarrow K \) (which together exist uniquely).

**Remark 3.24.** In particular, \( \mathcal{A}_r \) is an \( \mathcal{A} \)-bimodule, and hence \( \mathcal{A}_\ell \) becomes an \( \mathcal{A} \)-algebra. However, the \( \mathcal{A} \)-module \( K_\ell \) does not in general come from an \( \mathcal{A}_r \)-module.

Let us now consider a filtered stable category \( \mathcal{A} \) with compatible monoidal structure, and let \( \mathcal{A} \) be an associative algebra in \( \mathcal{A} \). A sufficient condition so the constructions above can be applied to this context is that the underlying object of \( \mathcal{A} \) belong to \( \mathcal{A}_{\geq 0} \).

Thus, let \( \mathcal{A} \) be in fact, an associative algebra in \( \mathcal{A}_{\geq 0} \). Then we have a filtration on \( \text{Mod}_{\mathcal{A}} \), where \( \text{Mod}_{\mathcal{A}_{\geq r}} = \text{Mod}_{\mathcal{A}}(\mathcal{A}_{\geq r}) \subset \text{Mod}_{\mathcal{A}} \), and the localisation functor \( \text{Mod}_{\mathcal{A}} \rightarrow \text{Mod}_{\mathcal{A}_{\geq r}} \) is induced from \((\_)_r: \mathcal{A} \rightarrow \mathcal{A}_{\geq r} \).
The complementary left localisation also lifts that on $\mathcal{A}$, and the square

$$
\begin{array}{ccc}
\text{Mod}_{\mathcal{A}}^{<r} & \longrightarrow & \text{Mod}_{\mathcal{A}} \\
\downarrow & & \downarrow \\
\mathcal{A}^{<r} & \longrightarrow & \mathcal{A}
\end{array}
$$

is Cartesian for every $r$. As a full subcategory of $\text{Mod}_{\mathcal{A}}$, this can be also expressed as $\text{Mod}_{\mathcal{A}}(\mathcal{A}^{<r})$, modules with respect to the action of $\mathcal{A}$ on $\mathcal{A}^{<r}$ by $X \mapsto (X \otimes \mathcal{A})^{<r}$.

The localisation functor $\text{Mod}_{\mathcal{A}} \rightarrow \text{Mod}_{\mathcal{A}}^{<r}$ lifts $(\ )^{<r} : \mathcal{A} \rightarrow \mathcal{A}^{<r}$.

**Remark 3.25.** As noted in the previous remark, the $\mathcal{A}$-module $\mathcal{K}^{<r}$ does not in general come from an $\mathcal{A}^{<r}$-module. However, it is always true that $\mathcal{A}_{\leq 0}^{<r} := \mathcal{A}^{<r} \cap \mathcal{A}_{\geq 0}$ comes with a canonical monoidal structure, together with a canonical monoidal structure on the functor $\mathcal{A}_{\geq 0} \rightarrow \mathcal{A}_{\leq 0}^{<r}$. Note that the algebra $\mathcal{A}^{<r}$ is obtained in $\mathcal{A}_{\leq 0}^{<r}$ using this. If $\mathcal{A}$-module $\mathcal{K}$ is in $\mathcal{A}_{\geq 0}$, then $\mathcal{K}^{<r}$ can be obtained as an $\mathcal{A}^{<r}$-module in $\mathcal{A}_{\leq 0}^{<r}$ from which the structure of an $\mathcal{A}$-module on $\mathcal{K}^{<r}$ gets recovered.

If further, $\hat{\mathcal{A}}$ is the completion of $\mathcal{A}$, then $\hat{\text{Mod}}_{\mathcal{A}}$ is the completion of $\text{Mod}_{\mathcal{A}}$, and this completion lifts the completion of $\mathcal{A}$. As a full subcategory of $\text{Mod}_{\mathcal{A}}$, $\hat{\text{Mod}}_{\mathcal{A}} = \text{Mod}_{\mathcal{A}} \times_{\mathcal{A}} \hat{\mathcal{A}}$.

**Corollary 3.26.** $\text{Mod}_{\mathcal{A}}$ is complete if $\mathcal{A}$ is complete.

The full subcategory $\mathcal{A}_{\geq \infty}$ of $\mathcal{A}$ is preserved by the action of $\mathcal{A}$, so the general argument can be applied to completion as well. In particular, $\hat{\text{Mod}}_{\mathcal{A}}$ can be identified with $\hat{\text{Mod}}_{\mathcal{A}}(\hat{\mathcal{A}})$, where $\mathcal{A}$ acts on $\hat{\mathcal{A}}$ by $X \mapsto X \hat{\otimes} \mathcal{A} := \hat{X} \otimes \mathcal{A}$. The inclusion $\hat{\text{Mod}}_{\mathcal{A}} \hookrightarrow \text{Mod}_{\mathcal{A}}$ then gets identified with the functor induced from the lax $\mathcal{A}$-linear functor $\hat{\mathcal{A}} \hookrightarrow \mathcal{A}$.

If further, the monoidal structure on $\mathcal{A}$ is completeable, then the action of $\mathcal{A}$ on $\hat{\mathcal{A}}$ is through the action of the algebra $\hat{\mathcal{A}}$ in $\hat{\mathcal{A}}$ (indeed we will have $X \otimes \mathcal{A} \rightarrow X \otimes \hat{\mathcal{A}}$) on $\hat{\mathcal{A}}$, and the completion functor

$$
\text{Mod}_{\mathcal{A}}(\mathcal{A}) \rightarrow \hat{\text{Mod}}_{\mathcal{A}}(\mathcal{A}) \simeq \hat{\text{Mod}}_{\mathcal{A}}(\hat{\mathcal{A}}) = \text{Mod}_{\hat{\mathcal{A}}}(\hat{\mathcal{A}})
$$

is just the functor induced from the monoidal functor $(\ ) : \mathcal{A} \rightarrow \hat{\mathcal{A}}$.

Let $\mathcal{A}$ be a monoidal filtered stable category, and let $\mathcal{A}$ be an associative algebra in $\mathcal{A}_{\geq 0}$. Then we consider the pairing

$$
- \otimes_{\mathcal{A} } - : \text{Mod}_{\mathcal{A}} \times_{\mathcal{A}} \text{Mod}_{\mathcal{A}} \rightarrow \mathcal{A},
$$

where $\text{Mod}_{\mathcal{A}}$ denotes the filtered stable category of left $\mathcal{A}$-modules. This pairing is compatible with the filtrations.

**Corollary 3.27.** Let $\mathcal{A}$ be a monoidal filtered stable category with uniformly bounded sequential limits.

Let $\mathcal{A}$ be an associative algebra in $\mathcal{A}_{\geq 0}$. Suppose given an inverse system

$$
\cdots \leftarrow K_i \leftarrow K_{i+1} \leftarrow \cdots
$$

in $\text{Mod}_{\mathcal{A}}$, and suppose there is a sequence $(r_i)_i$ of integers, tending to $\infty$ as $i \rightarrow \infty$, such that for every $i$, the fibre of the map $K_{i+1} \rightarrow K_i$ belongs to $\text{Mod}_{\mathcal{A}_{\geq r_i}}$ (namely, its underlying object belongs to $\mathcal{A}_{\leq r_i}$). Then for every left $\mathcal{A}$-module $L$, if (the underlying object of) $L$ is bounded below, then the map

$$
(\lim_i K_i) \otimes_{\mathcal{A}} L \longrightarrow \lim_i (K_i \otimes_{\mathcal{A}} L)
$$

is an equivalence after completion.
We discuss a few simple consequences. (More consequences will be discussed in the next section.)

Firstly, associativity of tensor product holds for bounded below modules over positive augmented algebras to be defined as follows.

**Definition 3.28.** Let $A$ be a monoidal filtered stable category. We say that an augmented algebra $A$ in $A$ is positive if the augmentation ideal $I$ of $A$ belongs to $A_{\geq 1}$.

**Lemma 3.29.** Let $A$ be a monoidal complete filtered stable category. Let $A_{+i}$, $i = 0, 1, 2, 3$, be positive augmented algebras in $A$, and let $K_{+i+1}$ be a left $A_{+i}$-right $A_{+i+1}$-bimodule for $i = 0, 1, 2$, whose underlying object is bounded below.

Then the resulting map

$$K_{01} \circ A_{1} \circ K_{12} \circ A_{2} \circ K_{23} \rightarrow (K_{01} \circ A_{1}, K_{12}) \circ A_{2} \circ A_{3} \circ K_{23}$$

is an equivalence, where the source denotes the realisation of the bisimplicial bar construction.

**Proof.** Denote the augmentation ideal of $A_{i}$ by $I_{i}$. We express the tensor product $K_{01} \circ A_{1}, K_{12}$ etc. as the geometric realisation of the simplicial bar construction $B_{\circ}(K_{01}, I_{1}, I_{12})$ etc. without degeneracies (in the sense that it is a diagram over $\Delta_j$), associated to the actions of the non-unital algebra $I_{1}$ etc. See Section 2.4

It is easy to check that the usual bar construction, with degeneracies, associated to the unital algebra $A_{1}$ etc., is the left Kan extension of the version here, so the geometric realisations are equivalent.

The target then can be written as $|B_{\circ}(K_{01} \circ A_{1}, K_{12}, I_{2}, K_{23})|$

For every $n$, the functor $\ldots \circ I_{2}^{\circ n} \circ K_{23}$ is bounded below, so Proposition 2.55 implies that

$$B_{\circ}(K_{01} \circ A_{1}, K_{12}, I_{2}, K_{23}) = |B_{\circ}(I_{01}, I_{1}, I_{12}, I_{2}, K_{23})|,$$

where the realisation is in the variable $*$.

However, the realisation of this is nothing but the source. \qed

Let $A$ be a monoidal complete filtered stable category, and let $A$ be an augmented associative algebra in $A$. Let $\varepsilon: A \rightarrow 1$ be the augmentation map, and $I := \text{Fibre}(\varepsilon)$ be the augmentation ideal of $A$.

Let us define the powers of $I$ by $I^{r} := I^{\circ r}$. Note that multiplication of $A$ gives an $A$-bimodule map $I^{r} \rightarrow I^{s}$ whenever $r \geq s$. Denote the cofibre of this map by $I^{s}/I^{r}$. When $s = 0$, this, $A/I^{r}$, is an $A$-algebra.

**Lemma 3.30.** Let $A$ be a monoidal filtered stable category, and let $A$ be a positive augmented associative algebra in $A$.

Let $K$ be a right $A$-module which is bounded below. Then the map $K \rightarrow \lim A/I^{r}$ is an equivalence after completion.

**Proof.** Since the fibre of the map $A \rightarrow A/I^{r}$ (namely $I^{r}$) belongs to $\text{Mod}_{A_{\geq r}}$, the result follows from Lemma 2.27 (Write $K$ as $K \circ A_{A}$.) \qed

**Corollary 3.31.** Let $A$ be a positive augmented associative algebra in a monoidal complete filtered stable category $A$. Then, the functor $- \circ A_{1}: \text{Mod}_{A_{\geq r}} \rightarrow A_{\geq r}$ reflects equivalences.

**Proof.** Suppose an $A$-module $K$ in $A_{\geq r}$ satisfies $K \circ A_{1} \simeq 0$. We want to show that $K \simeq 0$.

In order to do this, it suffices, from the previous lemma, to prove $K \circ (I^{s}/I^{s+1}) \simeq 0$ for all $s \geq 0$. However, $I^{s}/I^{s+1} \simeq 1 \circ A_{I^{s}}$ as a left $A$-module. \qed
4. Koszul duality for complete algebras

In this section, we shall obtain our main results on the Koszul duality using the basic results developed in the previous two sections.

4.0. Koszul completeness of a positive algebra. The Koszul duality we consider will be between augmented algebras and coalgebras. We first need to consider the condition on an augmented coalgebra, corresponding to the positivity of an algebra.

Definition 4.0. Let $\mathcal{A}$ be a monoidal filtered stable category with uniformly bounded loops and sequential limits. An augmented associative coalgebra $C$ in $\mathcal{A}$ is said to be copositive (with respect to the filtration) if the augmentation ideal $J$ belongs to $\mathcal{A}_{\geq 1 - \omega}$ for a uniform bound $\omega$ for loops in $\mathcal{A}$.

Example 4.1. If the filtration is a $t$-structure, then copositivity means that $\Omega J$ belongs to $\mathcal{A}_{\geq 1}$.

Let us now consider the Koszul duality. For an augmented coalgebra $C$, recall that its Koszul dual is an augmented associative algebra $C^!$ described as follows.

First of all, its underlying object is $1 \Box_C 1$, where $1$ is given the structure of a $C$-module coming from the module structure over the unit coalgebra, through the augmentation map $\varepsilon: 1 \rightarrow C$, and $\Box_C$ denotes the cotensor product operation over $C$.

In other words, it is an object representing the presheaf $\mathcal{A}^{op} \rightarrow \text{Space}$, $X \mapsto \text{Map}_{\text{Mod}}(X \otimes 1, 1)$, where $X \otimes 1 = X$ is made into a $C$-module by the action of $C$ on the factor $1$. The structure of an associative algebra of $C$ results from this, and we take as the augmentation the map $\eta: C \rightarrow 1$ for the unit $\eta: C \rightarrow 1$.

From this description, $C^!$ represents the presheaf on the category of augmented associative algebras which maps an object $A$ to the space of $A$-module structures on the $C$-module $1$, lifting the $A$-module structure on the underlying object given by the augmentation map of $A$. In particular, $\text{Map}_{\text{Alg}}(A, C^!) = \text{Map}_{\text{Coalg}}(A^!, C)$, where $A^! = 1 \otimes_A 1$ is the augmented associative coalgebra Koszul dual to $A$. The subscripts $*$ here indicates that the categories are those of augmented algebras and coalgebras. (For example, the map $A^! \xrightarrow{\varepsilon} 1 \xrightarrow{\eta} C$ corresponds to the map $A \xrightarrow{\varepsilon} 1 \xrightarrow{\eta} C$.)

Lemma 4.2. Let $\mathcal{A}$ be a monoidal filtered stable category with uniformly bounded loops and sequential limits. If $C$ is a copositive augmented associative coalgebra in $\mathcal{A}$, then its Koszul dual algebra is positive.

Proof. Let $J$ denote the augmentation ideal of $C$. Let us write the cotensor product in a way convenient for us. Namely, we write $1 \Box_C 1 = \text{tot} B^*(1, J, 1)$, where $B^*$ denotes the cosimplicial bar construction without degeneracies, for the actions of the non-unital coalgebra $J$.

The augmentation ideal of $C^!$ can then be written as

$$\lim_n \text{Fibre}[\text{sk}_n \text{tot} B^*(1, J, 1) \rightarrow \text{sk}_0 \text{tot} B^*(1, J, 1)].$$

Since for every $n$, $\Omega^n B^n(1, J, 1) = \Omega^n J \otimes^n$ belongs to the full subcategory $\mathcal{A}_{\geq n}$ of $\mathcal{A}$ from the assumptions, $\text{Fibre}[\text{sk}_n \text{tot} B^*(1, J, 1) \rightarrow 1] \in \mathcal{A}_{\geq 1}$ for every $n$.

Let $d \leq 0$ be a uniform bound for sequential limits in $\mathcal{A}$. Then for every $n \geq -d$,

$$\text{Fibre}[\text{sk}_n \text{tot} B^*(1, J, 1) \rightarrow \text{sk}_{-d} \text{tot} B^*(1, J, 1)]$$

belongs to $\mathcal{A}_{\geq -d + 1}$. It follows that $\text{Fibre}[C^! \rightarrow \text{sk}_{-d} \text{tot} B^*(1, J, 1)] \in \mathcal{A}_{\geq 1}$.

The result follows. $\square$
Definition 4.3. Let \( \mathcal{A} \) be a filtered stable category. Then we say that the filtration of \( \mathcal{A} \) is \textit{sound} if there is a uniform bound \( \omega \) for loops in \( \mathcal{A} \) such that for every \( r \), the suspension functor \( \Sigma: \mathcal{A} \rightarrow \mathcal{A} \) sends the full subcategory \( \mathcal{A}_{\geq r} \) to the full subcategory \( \mathcal{A}_{\geq r - \omega} \).

Remark 4.4. This happens if the tensoring of the finite spectra on \( \mathcal{A} \) is compatible with the filtrations. See Remark 2.43.

Examples include the category of filtered objects (Section 3.2), a stable category with a t-structure (Example 2.12), and a functor category with Goodwillie’s filtration (Example 2.13).

Remark 4.5. In general, if \( \mathcal{A} \) is soundly filtered, then a uniform bound \( \omega \) for loops satisfying the condition of Definition 4.3 must be the greatest among all the lower bounds for loops.

Lemma 4.6. Let \( \mathcal{A} \) be a monoidal soundly filtered stable category. If \( \mathcal{A} \) is a positive augmented associative algebra in \( \mathcal{A} \), then its Koszul dual coalgebra is copositive.

Proof. Similar to the proof of Lemma 4.2, but is simpler. \( \square \)

Proposition 4.7. Let \( \mathcal{A} \) be a monoidal complete filtered stable category with uniformly bounded loops and sequential limits.

Let \( A \) be a positive augmented associative algebra, and \( C \) a copositive augmented associative coalgebra, both in \( \mathcal{A} \). Let \( K \) be a right \( A \)-module, \( L \) an \( A \)-\( C \)-bimodule, and let \( X \) be a left \( C \)-module, all bounded below.

Then the canonical map
\[
K \otimes_A (L \square_C X) \rightarrow (K \otimes_A L) \square_C X
\]
is an equivalence (where the left \( A \)-module structure of \( L \square_C X \) and the right \( C \)-module structure of \( K \otimes_A L \) are induced from the \( A \)-\( C \)-bimodule structure of \( L \)).

Proof. Write
\[
L \square_C X = \text{tot} B^*(L, J, X).
\]
Since the functor \( K \otimes_A - \) is bounded below, we obtain from Proposition 2.54 that this functor sends this cotensor product to \( \text{tot} K \otimes_A B^*(L, J, X) \).

Since \( J \) and \( X \) are bounded below, it follows from Proposition 2.55 that
\[
K \otimes_A B^*(L, J, X) = B^*(K \otimes_A L, J, X).
\]
Therefore, we get the result by totalising this. \( \square \)

Let \( A \) be an augmented associative algebra. Then, for a right \( A \)-module \( K \), we define a right \( A' \)-module \( \mathbb{D}_A K \) as \( K \otimes_A 1 \). Dually, if \( C \) is an augmented associative coalgebra, then for a right \( C \)-module \( L \), we have a right \( C' \)-module \( \mathbb{D}_C L = L \square_C 1 \).

If \( K \) is a left \( A \)-module, then we simply define a left \( A' \)-module \( \mathbb{D}_A K \) by \( 1 \otimes_A K \), and similarly for left \( C \)-modules.

The following is a special case of Proposition 4.7.

Corollary 4.8. Let \( \mathcal{A} \) be a monoidal complete soundly filtered stable category with uniformly bounded sequential limits. Let \( A \) be an augmented associative algebra in \( \mathcal{A} \), and assume it is positive. Let \( K \) be a right \( A \)-module, \( L \) a left \( A' \)-module, and assume both of these are bounded below.

Then the canonical map
\[
K \otimes_A \mathbb{D}_A L \rightarrow \mathbb{D}_A K \square_{A'} L
\]
is an equivalence.

Proof. The coalgebra \( A' \) is copositive by Lemma 4.6. \( \square \)
Corollary 4.9. In the situation of the previous corollary, the canonical map $D_A D_A L \to L$ is an equivalence.

Proof. Apply the previous corollary to $K = 1$. □

Theorem 4.10. Let $A$ be a monoidal complete soundly filtered stable category with uniformly bounded sequential limits. Let $A$ be a positive augmented associative algebra in $A$, and $K$ be a right $A$-module which is bounded below. Then the canonical map $K \to D_A D_A K$ is an equivalence (of $A$-modules). In particular, the canonical map $A \to A^!$ (of augmented associative algebras) is an equivalence.

Proof. By Corollary 3.31 it suffices to prove that the map is an equivalence after we apply the functor $- \otimes_A 1$ to it. However, this follows by applying Corollary 4.9 to the (right) $A^!$-module $D_A K$. □

4.1. Koszul completeness of a coalgebra. In order to complete our study of the Koszul duality for associative algebras, we shall establish the results similar to those established for positive augmented algebras, for copositive coalgebras.

Let us start with the following situation. Namely, let $C_i$, $i = 0, 1, 2, \ldots$, be coalgebras in $A$, and let $K_{i+1}$ for $i = 0, 1$ be a left $C_i$-right $C_{i+1}$-bimodule. Then we would like $K_0 \square C_1 K_2$ to be a $C_0 - C_2$-bimodule in a natural way.

We have this in the following case. Namely, assume $A$ to be a monoidal complete filtered stable category with uniformly bounded loops and sequential limits. Moreover, assume that $C_1$ is a copositive augmented coalgebra, and $K_{i+1}$ are bounded below. Then for any bounded below object $L$, the canonical map

$$(K_0 \square C_1 K_2) \otimes L \to K_0 \square C_1 (K_2 \otimes L)$$

is an equivalence by Proposition 2.54.

It follows that if $C_0$ and $C_2$ are bounded below, then the bimodule structures of $K_{i+1}$, $i = 0, 1$ induce a structure of a $C_0 - C_2$-bimodule on the cotensor product. In fact, the resulting bimodule has the universal property to be expected of the cotensor product.

Lemma 4.11. Let $A$ be a monoidal complete filtered stable category with uniformly bounded loops and sequential limits. Let $C_i$, $i = 0, 1, 2, 3$, be copositive augmented coalgebras in $A$, and let $K_{i+1}$ be a left $C_i$-right $C_{i+1}$-bimodule for $i = 0, 1, 2$, whose underlying object is bounded below.

Then the resulting map

$$(K_0 \square C_1 K_2) \square C_3 K_{23} \to K_0 \square C_1 K_2 \square C_3 K_{23}$$

is an equivalence of $C_0 - C_3$-modules, where the target denotes the totalisation of the bicosimplicial bar construction.

The proof is similar to the proof of Lemma 5.20. One uses Proposition 2.54 instead of Proposition 2.55.

Lemma 4.12. Let $A$ be a monoidal complete filtered stable category with uniformly bounded loops and sequential limits. Let $C$ be a copositive augmented coalgebra, $K$ a right $C$-module, and $L$ a left $C$-module, all in $A$. If for integers $r$ and $s$, (the underlying object of) $K$ belongs to $A_{\geq r}$, and $L$ belongs to $A_{\geq s}$, then $K \square C L$ belongs to $A_{\geq r+s}$.

Proof. Denote by $I$ the augmentation ideal of $C$, and write $K \square C L = \text{tot } B^*(K, I, L)$. For every integer $n \geq 0$, $\Omega^n B^*(K, I, L)$ belongs to $A_{\geq r+s+n}$. In particular, $sk_n \text{tot } B^*(K, I, L)$ belongs to $A_{r+s}$ for every $n$.

Let $d \leq 0$ be a uniform bound for sequential limits in $A$. Then for $n \geq -d$, the fibre of the canonical map $sk_n \text{tot } B^*(K, I, L) \to sk_{-d} \text{tot } B^*(K, I, L)$ belongs to $A_{r+s-d}$. It follows that the limit of this as $n \to \infty$ belongs to $A_{r+s}$.
However, the limit is the fibre of the canonical map $K \square_C L \to \text{sk}_- \text{tot} B^\bullet(K, I, L)$, so the result follows. □

Let $\text{Mod}_{C,>\infty}$ denote the category of bounded below $C$-modules.

**Lemma 4.13.** Let $A$ be a monoidal complete filtered stable category with uniformly bounded loops and sequential limits. Let $C$ be a copositive augmented coalgebra in $A$. Then the functor $-\square_C 1 : \text{Mod}_{C,>\infty} \to A$ reflects equivalences.

**Proof.** We would like to apply the arguments of the proof of Corollary 3.31. We simply need to establish the analogue of Lemma 3.30. This follows from Lemma 4.12. □

**Lemma 4.14.** Let $C$ be as in Lemma 4.13, and let $K$ be a right $C$-module which is bounded below. Then the canonical map $D_C \square C K \to K$ is an equivalence (of $C$-modules). In particular, the canonical map $C'' \to C$ (of augmented associative coalgebras) is an equivalence.

**Proof.** Similar to the proof of Theorem 4.10. Note that the assumptions imply that $C''$ is positive, so Proposition 4.7 can be applied. □

**Theorem 4.15.** Let $A$ be a monoidal complete filtered stable category with uniformly bounded loops and sequential limits. Let $C$ be a copositive augmented coalgebra in $A$. Then the functor $D_C : \text{Mod}_{C,>\infty} \to \text{Mod}_{C'',>\infty}$ is an equivalence with inverse $D_C'$.

From Lemma 4.13 and Theorem 4.10, we also obtain immediately the case $n = 1$ of Theorem 4.18 below.

4.2. **Koszul duality for $E_n$-algebras.** In this section, we would like to prove our first main theorem, which extracts an equivalence of categories from the Koszul duality. In this section, we assume that $A$ is a monoidal complete soundly filtered stable category with uniformly bounded sequential limits.

We define the Koszul duality functor for $E_n$-algebras inductively as the composite

$$\text{Alg}_{E_1}(\text{Alg}_{E_{n-1}}) \to \text{Coalg}_{E_1}(\text{Alg}_{E_{n-1}}) \to \text{Coalg}_{E_1}(\text{Coalg}_{E_{n-1}}),$$

where the first map is the associative Koszul duality construction, and the next map is induced from the inductively defined $E_{n-1}$-Koszul duality functor, which is canonically op-lax symmetric monoidal by induction.

We would like to analyse this for a suitable restricted classes of algebras and coalgebras (considered as algebras in the opposite category). The restriction will be given by some positivity conditions as below.

**Definition 4.16.** Let $A$ be a symmetric monoidal filtered stable category with uniformly bounded loops.

An augmented $E_n$-algebra $A$ is said to be **positive** if its augmentation ideal belongs to $A_{\geq 1}$.

An augmented $E_n$-coalgebra $C$ in $A$ is said to be **copositive** if there is a uniform bound $\omega$ for loops in $A$ such that the augmentation ideal $J$ of $C$ belongs to $A_{\geq 1 - \omega}$.

**Lemma 4.17.** Let $A$, $B$ be positive augmented associative algebras in $A$. Then the canonical map $(A \otimes B)^! \to A^! \otimes B^!$ is an equivalence.

**Proof.** This follows from Lemma 3.30. □
It follows that the functor $A \mapsto A'$ is symmetric monoidal, so in particular, if $A$ is an positive augmented $E_{n+1}$-algebra, then the Koszul dual $1 \otimes A$ of its underlying associative algebra becomes an $E_n$-algebra in the category of augmented associative coalgebras. Moreover, by Proposition 2.55, this $E_n$-algebra is equivalent to the tensor product $1 \otimes A$ taken in the category of $E_n$-algebras.

By Lemma 4.11, similar results hold for copositive $E_n$-coalgebras as well. It follows that Lemmas similar to Lemma 4.2 and 4.6 holds for $E_n$-algebras.

We shall now state our first main theorem. Let $\text{Alg}_{E_n}(A)_+$ denote the category of positive augmented $E_n$-algebras in $\mathcal{A}$, and similarly, $\text{Coalg}_{E_n}$ for copositive coalgebras.

**Theorem 4.18.** Let $\mathcal{A}$ be a monoidal complete soundly filtered stable category with uniformly bounded sequential limits. Then the constructions of Koszul duals give inverse equivalences

$$\text{Alg}_{E_n}(A)_+ \sim \text{Coalg}_{E_n}(A)_+. $$

*Proof.* The proof will be by induction. We claim the equivalence as well as the following. Namely, under the claimed equivalence, if $A \in \text{Alg}_{E_n}(A)_+$ and $C \in \text{Coalg}_{E_n}(A)_+$ correspond to each other, then we further claim that for every integer $r$, the condition that the augmentation ideal $I$ of $A$ belongs to $\mathcal{A}_{\geq r}$, is equivalent to that the augmentation ideal $J$ of $C$ belongs to $\mathcal{A}_{\geq r-\omega}$ for the uniform bound $\omega$ for loops in $\mathcal{A}$ satisfying the condition stated in Definition 4.3. We prove these claims by induction on $n$.

The case $n = 0$ is obvious, so assume the claims for an integer $n \geq 0$. Then we would like to prove the claims for $n + 1$.

With the preparation above, the arguments of the previous sections apply, under a modification, to augmented associative algebras and coalgebras in the category of $E_n$-algebras. The modification needed is as follows. Namely, the arguments refer to depth of objects in the filtration. Since we are here dealing not with objects of $\mathcal{A}$, but with $E_n$-algebras in $\mathcal{A}$, we should understood the depth of algebras as the depth of the underlying objects in the filtration of $\mathcal{A}$. From this, we obtain that the constructions of the Koszul duals restrict to an equivalence

$$\text{Alg}_{E_n}(\text{Alg}_{E_n}(A)_+) \sim \text{Coalg}_{E_n}(\text{Alg}_{E_n}(A)_+).$$

On the other hand, from the inductive hypothesis, we have an equivalence

$$\text{Alg}_{E_n}(A)_+ \sim \text{Coalg}_{E_n}(A)_+$$

(which is symmetric monoidal by iteration of Lemma 4.17), in which the condition $I \in \mathcal{A}_{\geq r}$ corresponds to $J \in \mathcal{A}_{\geq r-\omega}$.

We obtain the desired equivalence for $n + 1$ from these. Moreover, suppose $A \in \text{Alg}_{E_{n+1}}(A)$ and $C \in \text{Coalg}_{E_{n+1}}(C)$ correspond to each other in the equivalence. Then it follows from Lemmas 4.2 and 4.6 that the condition $I \in \mathcal{A}_{\geq r}$ is equivalent to that the augmentation ideal of the associative Koszul dual of $A$ belongs to $\mathcal{A}_{\geq r-\omega}$. Moreover, by the inductive hypothesis, this is equivalent to that $J \in \mathcal{A}_{\geq r-(n+1)\omega}$.

This completes the inductive step. $\square$

### 4.3. Morita structure of the Koszul duality.

Let $\mathcal{A}$ be a symmetric monoidal category whose monoidal multiplication functor preserves geometric realisations (variable-wise). Then there is an $(n+1)$-category of $E_n$-algebras, generalising the Morita 2-category of associative algebras.

If the condition of the preservation of geometric realisation is dropped, then one has to be cautious. Let us work in $\mathcal{A}^{op}$ instead, and see a case where a suitably restricted higher dimensional Morita category of coalgebras in $\mathcal{A}$ makes sense.

Let us get back to the construction of a version of the $E_n$-Morita category. For $\mathcal{A}$ a symmetric monoidal complete soundly filtered stable category with uniformly
bounded sequential limits, we would like to construct a version \( \text{Coalg}_n^+ (A) \) of a
Morita \((n+1)\)-category of augmented coalgebras, in which \( k \)-morphisms are copositive as an augmented \( E_{n−k} \)-coalgebra in \( A \). The construction will be indeed the same as in (a version where everything is augmented, of) the familiar case. We shall simply observe that the usual construction makes sense under the mentioned restriction on the objects to be morphisms in \( \text{Coalg}_n^+ (A) \).

Let us see why this is true. We shall follow the construction outlined by Lurie in [8]. Firstly, an object of \( \text{Coalg}_n^+ (A) \) will be a copositive augmented \( E_n \)-coalgebra in \( A \). Given objects \( C, D \) as such, then we would like to define the morphism \( n \)-category \( \text{Map}(C, D) \) in \( \text{Coalg}_n^+ (A) \) to be what we shall denote by \( \text{Coalg}_{n−1}^+ (\text{Bimod}_{C−D}(A)) \).

By this, we mean the Morita \( n \)-category to be seen to be well-defined, for the \( E_{n−1} \)-monoidal category of \( C−D \)-bimodules, in which \((k−1)\)-morphisms are copositive as an augmented \( E_{n−k} \)-coalgebra in \( A \). (We understand an augmentation of an coalgebra in \( \text{Bimod}_{C−D}(A)_{>−\infty} \) to be given by a map from 1, but not from the unit of \( \text{Bimod}_{C−D}(A)_{>−\infty} \).) For the moment, it will suffice to see that the cotensor product over \( C^{\text{op}} \otimes D \) makes the category \( \text{Bimod}_{C−D}(A)_{>−\infty} \) of bounded below bimodules into an \( E_{n−1} \)-monoidal category. The reason why this will suffice is that it will be clear as we proceed that the rest of the arguments for well-definedness of \( \text{Coalg}_{n−1}^+ (\text{Bimod}_{C−D}(A)) \) is similar (with obvious minor modifications) to the arguments for well-definedness of \( \text{Coalg}_n^+ \) we are discussing right now, so we can ignore the issue of well-definedness of \( \text{Coalg}_{n−1}^+ (\text{Bimod}_{C−D}(A)) \) for the moment by understanding that the whole argument will be inductive at the end (as in Lurie’s description of the construction in the more familiar case).

To investigate the cotensor product operation in \( \text{Bimod}_{C−D}(A)_{>−\infty} \), the associativity follows from Proposition 4.11. If \( n−1 \geq 2 \), we need to have compatibility of this operation with itself. This follows from the following general considerations.

**Lemma 4.19.** Let \( C \) be a copositive augmented \( E_2 \)-coalgebra. Let \( C_i, D_j \), \( i = 0, 1 \), be associative coalgebras in the category \( \text{Mod}_C \) of augmented \( C \)-modules in \( A \). Assume that these are copositive as an augmented associative coalgebra in \( A \).

Let \( D_{i,j} \), \( j = 0, 1 \), be a bounded below \( D_j−C_i \)-bimodule if \( i + j \) is even, and a bounded below \( C_i−D_j \)-bimodule if \( i + j \) is odd. Then the canonical map in \( A \) from \( (D_0 \boxtimes D_0^{\text{op}}(D_1) \boxtimes C_0 \boxtimes C_1^{\text{op}}(D_0 \boxtimes D_1, D_1)) \) to the totalisation of the corresponding bicosimplicial bar construction is an equivalence.

**Proof.** Let us denote the augmentation ideal of \( C \) and \( C_i \) by \( I \) and \( I_i \), respectively, and the augmentation ideal of \( D_j \) by \( J_j \). The bicosimplicial bar construction is

\[
B^*(B^*(D_0, J_0^{\text{op}}, D_0), B^*(I_0, I_1^{\text{op}}, B^*(D_10, J_1, D_11)))
\]

where the cosimplicial indices are \( \bullet \) and \( \ast \). The result follows since the totalisation of this in the index \( \ast \) is

\[
B^*(D_0 \boxtimes D_0^{\text{op}}D_0, C_0 \boxtimes C_1^{\text{op}}, D_0 \boxtimes D_1, D_1)
\]

by Proposition 2.54 and boundedness below of the monoidal operations. \( \square \)

**Lemma 4.20.** Let \( C \) be an augmented associative coalgebra in \( A \), and let \( K, L \) be a right and a left \( C \)-modules respectively, both of which are augmented. Let \( \varepsilon \) be the augmentation map of \( C, K, \) or \( L \), and assume that, for an integer \( r \) and a uniform bound \( \omega \) for loops in \( A \), the cofibre in \( A \) of \( \varepsilon \) belongs to \( A_{\geq r−\omega} \) for \( C \), and to \( A_{\geq r} \) for \( K \) and \( L \). Then the cofibre of the map

\[
1 = 1 \boxtimes 1 \longrightarrow K \boxtimes C L
\]

belongs to \( A_{\geq r} \).

**Proof.** Similar to Lemma 4.12. \( \square \)
Corollary 4.21. Let $k \geq 0$ and $m$ be integers such that $m \geq k + 1$. In Lemma, if $C$ is a copositive augmented $E_n$-coalgebra, and $K$ and $L$ are further $E_k$-coalgebras in $(\text{Mod}_C)_1$ which are copositive as augmented $E_k$-coalgebras in $\mathcal{A}$, then $K \subseteq C \subseteq L$ is copositive as an augmented $E_k$-coalgebra in $\mathcal{A}$.

It follows that $n - 1$ monoidal structures on $\text{Bimod}_{C \otimes D}(A)_{\geq -\infty}$, all of which are given by the cotensor product over $C^{op} \otimes D$, has the compatibility required for them to together define an $E_{n-1}$-monoidal structure on this category of bounded below bimodules.

Thus, we can try to see if the construction of the Morita category can be applied for restricted class of augmented coalgebras in $\text{Bimod}_{C \otimes D}(A)_{\geq -\infty}$, to give an $n$-category $\text{Coalg}^+_n(A)$ for copositive augmented $E_n$-coalgebras $C$, $D$, define composition in the desired category $\text{Coalg}^+_n(A)$ enriched in $n$-categories. Cotensor product of copositive objects remain copositive by Corollary 4.21. The functoriality of the cotensor operations follows from Lemma 4.19 (in the case $C_1$ are $C$). Finally, the associativity of the composition defined by cotensor product follows from Proposition 4.21.

To summarise, the usual construction of the Morita category (as outlined in [8]) works under our assumptions on $\mathcal{A}$ and the copositivity of the class of objects we include, since in construction of any composition of morphisms, application of the totalisation functor to any iterated multicosimplicial bar construction which appear can be always postponed to the last step.

In the next theorem, we shall see that the Koszul duality construction is functorial on the positive Morita category, and gives an equivalence of the algebraic and coalgebraic Morita categories.

Let us assume that the monoidal multiplication functor on $\mathcal{A}$ preserves geometric realisation, and denote by $\text{Alg}^+_n(A)$ the positive part of the augmented version of the Morita $(n+1)$-category of (augmented) $E_n$-algebras in $\mathcal{A}$.

Theorem 4.22. Let $\mathcal{A}$ be a symmetric monoidal complete soundly filtered stable category with uniformly bounded sequential limits. Assume that the monoidal multiplication functor on $\mathcal{A}$ preserves geometric realisation.

Then for every $n$, the construction of the Koszul dual define a symmetric monoidal functor

$$
\mathcal{A}^+ \colon \text{Alg}^+_n(A) \to \text{Coalg}^+_n(A).
$$

It is an equivalence with inverse given by the Koszul duality construction.

Proof. Let us first describe the functor underlying the claimed symmetric monoidal functor. In order to do this, it suffices to consider the following, more general case. Namely, let $A_i$, $i = 0, 1$, be positive augmented $E_{n+1}$-algebras. Then we would like to see that the Koszul duality constructions define a functor

$$
\text{Alg}^+_n(\text{Bimod}_{A_{n+1}}) \to \text{Coalg}^+_n(\text{Bimod}_{A'_n})
$$

Indeed, the original case is when $A_i$ are the unit algebra in $\mathcal{A}$.

Similarly to how it was in the construction of the higher Morita category, we need to consider here algebras $A_i$, possibly in the category of bimodules over some $E_{n+2}$-algebras. In order to understand (4.23) including this case, recall first that
in general, an $E_{k+1}$-algebra can be considered as an $E_k$-algebra in the category of $E_1$-algebras. Given an $E_{k+1}$-algebra $A$, let us denote by $A_{(1,1)}$ the $E_1$-algebra in $E_k$-coalgebras which is obtained as the $E_k$-Koszul dual of $A$. If $A_i$, $i = 0, 1$, are $E_{k+1}$-algebras (possibly again in a bimodule category, and inductively), and $B$ is an augmented $E_k$-algebra in $(\text{Bimod}_{A_0-\text{A}_1})_{>\infty}$, then by $B'$, we mean the canonical augmented $E_k$-coalgebra in $(\text{Bimod}_{A_0^{(1,1)}-A_1^{(1,1)}})_{>\infty}$ (bimodules with respect to the $E_1$-algebra structures of $A_{i}^{(1,1)}$) lifting the $E_k$-Koszul dual (in inductively the similar sense) of the augmented $E_k$-algebra underlying $B$ after forgetting the bimodule structure of $B$ over $A_0$ and $A_1$. Note that the $E_k$-monoidal structure of $(\text{Bimod}_{A_0^{(1,1)}-A_1^{(1,1)}})_{>\infty}$ is the ‘plain’ tensor product, lifting the $E_k$-monoidal structure (underlying the $E_{k+1}$-monoidal structure) of the underlying objects. Note that if $A_i$ here are again algebras in a bimodule category, then $A_i^{(1,1)}$ are interpreted in the similar way, and inductively. In particular, if one forgets all the way down to $A$, then as an augmented $E_k$-coalgebra in $A$, $B'$ is the Koszul dual of the augmented $E_k$-algebra in $A$ underlying $B$. We are just taking into account the natural algebraic structures carried by it.

Let us now describe the construction of \([1.23]\). Note that $A' = (A^{(1,1)})^{(1,1)}$, where $()^{(1,1)}$ is the Koszul duality construction with respect to the remaining $E_1$-algebra structure. Using this, \([1.23]\) will be constructed as the composition of two functors. Namely, it will be constructed as a functor factoring through $\text{Coalg}^+_{m} (\text{Bimod}_{A_0^{(1,1)}-A_1^{(1,1)}})$.

The functor

$$\text{Coalg}^+_{m} (\text{Bimod}_{A_0^{(1,1)}-A_1^{(1,1)}}) \rightarrow \text{Coalg}^+_{m} (\text{Bimod}_{A_{0}^{1,1}-A_{1}^{1,1}})$$



to be one of the factors, will be induced from an ox-lax $E_n$-monoidal functor $(\text{Bimod}_{A_0^{(1,1)}-A_1^{(1,1)}})_{>\infty} \rightarrow (\text{Bimod}_{A_{0}^{1,1}-A_{1}^{1,1}})_{>\infty}$ whose underlying functor is $D\Delta_{A_0^{(1,1)}-A_1^{(1,1)}} : K \mapsto 1 \otimes_{A_0^{1,1}} K \otimes_{A_1^{1,1}} 1$. Note that this functor will preserve copositivity of coalgebras once it is given an $E_n$-monoidal structure.

To see the ox-lax $E_n$-monoidal structure of $D := D\Delta_{A_0^{(1,1)}-A_1^{(1,1)}}$, let $S$ be a finite set, and let $m$ be an $S$-ary operation in the operad $E_n$. Then for a family $K = (K_s)_{s \in S}$ of objects of $(\text{Bimod}_{A_0^{(1,1)}-A_1^{(1,1)}})_{>\infty}$, we have the map

$$Dm_K \mapsto \Delta^*_m D\Delta_{A_0^{(1,1)}-A_{1}^{1,1}} \otimes_{A_0^{1,1}} m, K = m, DK$$



where

- $m : (\text{Bimod}_{A_0^{(1,1)}-A_{1}^{1,1}})^S \rightarrow \text{Bimod}_{A_0^{(1,1)}-A_{1}^{1,1}}$ is the monoidal multiplication along $m$

- $\Delta^*_m : \text{Bimod}_{A_0^{(1,1)}-A_{1}^{1,1}} \otimes_{A_0^{1,1}} m, A_0^{1,1} \rightarrow \text{Bimod}_{A_0^{1,1}}$ is the (“co”)extension of scalars along the comultiplication operations $\Delta_m$ along $m$ of $A_0^{1,1}$ and $A_1^{1,1}$.

- $m : (\text{Bimod}_{A_0^{(1,1)}-A_{1}^{1,1}})^S \rightarrow \text{Bimod}_{A_0^{(1,1)}-A_{1}^{1,1}}$ is the external monoidal multiplication along $m$ (so $m_l = \Delta^*_m m_l$).

- $m : (\text{Bimod}_{A_0^{(1,1)}-A_{1}^{1,1}})^S \rightarrow \text{Bimod}_{A_0^{1,1}-A_1^{1,1}}$ is the monoidal multiplication along $m$, and the map is the instance for $m, K$ of the extension of scalars of the $A_0^{(1,1)} \otimes_{A_1^{1,1}} m, A_1^{1,1}$-bimodule map $\Delta_m, D\Delta^*_m \rightarrow D\Delta_{A_0^{(1,1)}-A_{1}^{1,1}} \otimes m$ induced from $\Delta_m$ of $A_0^{(1,1)}$ and $A_1^{(1,1)}$.

Next, we would like to describe the other factor

\[(4.24) \quad \text{Alg}^+_{m} (\text{Bimod}_{A_0-\text{A}_1}) \rightarrow \text{Coalg}^+_{m} (\text{Bimod}_{A_0^{(1,1)}-A_{1}^{1,1}}).\]
If $B$ is an object of the source, then the object of $\text{Coalg}_{n+1}^+(\text{Bimod}_{A^i_0(A^i_{n-1}-A^i_{n-1}))}$ associated to it is the $E_n$-Koszul dual $B'$. To see the functoriality of this construction, let $B_i$, $i = 0, 1$, be objects of $\text{Alg}_{n}^+(\text{Bimod}_{A^i_n(A^1_n)})$. Then we first need a functor

$$\text{Alg}_{n-1}^+(\text{Bimod}_{B_0-B_1}) \longrightarrow \text{Coalg}_{n-1}^+(\text{Bimod}_{B_0'-B_1'}).$$

Note that this is the same form of functor as (4.28). Therefore, we may assume that we have this functor by assuming we have (4.24) for $n - 1$ by an inductive hypothesis, once we check the base case. However, the base case is the identity functor of $(\text{Bimod}_{B_0-B_1})_{\geq 1}$ for positive $E_1$-algebras $B_i$ (in the bimodule category in the bimodule category in $\ldots$).

Next, we would like to see the compatibility of the functors (4.25) with the compositions. Thus, let $B_2$ be another object, and let maps

$$B_0 \xrightarrow{K_{01}} B_1 \xrightarrow{K_{12}} B_2$$

be given in $\text{Alg}_{n}^+(\text{Bimod}_{A^i_n(A^i_1)})$. Then the version of Lemma 4.19 for positive algebras implies that the $E_{n-1}$-Koszul dual $(K_{01} \otimes B_1 K_{12})'$ is equivalent by the canonical map to the realisation of a bicosimplicial object which is also equivalent to $K_{01}^i \otimes B_1^i K_{12}^i$ by the canonical map, again by Lemma 4.19. Moreover, the canonical map

$$\mathbb{D}_{B_0^{(i)}-B_2^{(i)}} \left( K_{01}^i \otimes B_1^{(i)} K_{12}^i \right) \longrightarrow \left( \mathbb{D}_{B_1^{(i)}-B_2^{(i)}} K_{01}^i \right) \square_{(B_1^{(i)}-B_2^{(i)})} \left( \mathbb{D}_{B_0^{(i)}-B_2^{(i)}} K_{12}^i \right),$$

is an equivalence by Proposition 4.7 and Theorem 4.10.

This essentially completes the inductive step, so we have given a description of the underlying functor of the desired symmetric monoidal functor. Moreover, the symmetric monoidality of the functor is straightforward.

It follows in the same way that we also have a functor in the other direction, and it follows from Theorems 4.10 and Lemma 4.14 that these are inverse to each other. □

Remark 4.26. As the proof shows, the equivalence is in fact more than an equivalence of $(n + 1)$-categories. Namely, the equivalence $A \simeq A'$ for $A$ in any dimension is an honest equivalence of algebras, rather than merely an equivalence in the Morita category.

Remark 4.27. Theorem seems to be suggesting that $\text{Coalg}_{n}^+(A)$ is a meaningful thing at least in the case where the monoidal operation of $A$ preserves geometric realisations. However, the construction of $\text{Coalg}_{n}^+(A)$ was independent of this assumption, and a similar construction for $\text{Alg}_{n}^+(A)$ works without preservation of geometric realisations. Moreover, Theorem remains true in this generality.

Recall that any $E_n$-algebra $A$, as an object of the Morita $(n + 1)$-category $\text{Alg}_{n}(A)$, is $n$-dualisable. All dualisability data are in fact given by $A$, considered as suitable morphisms in $\text{Alg}_{n}(A)$.

It is then immediate to see that if $A$ is an augmented $E_n$-algebra, then the dualisability data (and the field theory) for $A$ in $\text{Alg}_{n}(A)$ can be lifted to those for $A$ in $\text{Alg}_{n}^+(A)$, the augmented version of $\text{Alg}_{n}(A)$. Moreover, if $A$ is positive, then those data belongs to $\text{Alg}_{n}^+(A)$. In particular, $A$ will be $n$-dualisable in $\text{Alg}_{n}^+(A)$.

Corollary 4.28. Let $A$ be a symmetric monoidal complete soundly filtered stable category with uniformly bounded sequential limits. Then any object of the symmetric monoidal category $\text{Coalg}_{n}^+(A)$ is $n$-dualisable.

There is a concrete description of the fully extended $n$-dimensional framed topological field theory associated to an object $A \in \text{Alg}_{n}(A)$, using the topological chiral...
homology. See Lurie [8]. In [11], we shall give a concrete description of the framed topological field theory associated to a copositive $E_n$-coalgebra, using compactly supported topological chiral homology. See also Francis [4].

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