Necessary conditions for ternary algebras

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Abstract
Ternary algebras, constructed from ternary commutators, or as we call them, ternutators, defined as the alternating sum of products of three operators, have been shown to satisfy cubic identities as necessary conditions for their existence. Here we examine the situation where we permit identities not solely constructed from ternutators or nested ternutators and we find that in general, these impose additional restrictions; for example, the anti-commutators or commutators of the operators must obey some linear relations among themselves.

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1. Introduction

The subject of ternary algebras and n-Lie algebras (as a particular case of n-ary algebras) is generally attributed to Filippov [1], but Filippov was following up on earlier studies which had appeared in the mathematics literature, primarily by Kurosh [2] (as remarked in [3]). The appearance of ternary algebras in physics was due to the pioneering work of Nambu [4], and more recently, the work of Bagger and Lambert [5] renewed interest in the theoretical physics community (see also [6]). In general, a ternary bracket is a composition law for three operators, which is completely antisymmetric in the three operators, as for example Nambu brackets, which are an extension of the idea of Poisson brackets to three functions. In this paper we study the purely algebraic structure of the algebra, with an associative product structure for the operators clearly in mind, and we refer to such brackets as ternutators. Ternary brackets satisfy identities of degree 7 in the operators which were first discovered by Bremner [7] 2 years ago, and re-discovered by one of us (JN) [8] more recently. These identities have been extended to n-ary algebras by Curtright et al [9]. The question arises as to whether such identities comprise all the necessary identities for ternary algebras, or whether some simpler
constraints, possibly of a different type exist. We shall demonstrate the existence of such constraints in this paper.

2. Ternutator basics

The ternutator bracket is a completely antisymmetrized trilinear composition law for three associative operators, just as the commutator is for two operators:

\[ [A, B, C] \equiv ABC + BCA + CAB - ACB - CBA - BAC \]  
\[ \equiv \frac{1}{2}([[A, B], C] + [[B, C], A] + [[C, A], B]). \]  

Note the appearance of anti-commutators; if instead all the brackets in (2) are commutators, the right-hand side is the Jacobi identity for Lie Brackets, and is zero. Corresponding operator algebras would read

\[ [A_i, A_j, A_k] = f_{ijk}^{\ m} A_m, \]  

where the structure constants \( f_{ijk}^{\ m} \) are completely antisymmetric in \( i, j, k \).

For a simplified notation, let us write

\[ (i j k l \ldots) \equiv A_i A_j A_k A_l \ldots \]  
\[ [i, j]_\pm \equiv [A_i, A_j]_\pm \]  
\[ [i, j, k] \equiv [A_i, A_j, A_k] \]  

respectively for the product of \( n \) arbitrary operators (4), for the anti-commutator or commutator (5), and for the ternutator (6).

3. Normal and non-normal order

We associate with each operator a label \( L \) equal to its index \( L(A_{j_i}) = j_i \). For a product of three operators \( A_{j_1} A_{j_2} A_{j_3} \), we say that they are in normal order if at least one of the set of indices \{\( j_1, j_2 \)\} and \{\( j_2, j_3 \)\} is in increasing order. For three given operators, five of their products are in ‘normal’ order and one is in “non-normal” order, for example,

\[ (321) \quad \text{non-normal order} \]  
\[ (123), (132), (213), (231), (312) \quad \text{normal order}. \]  

One clearly has

\[ (321) \equiv (321) + (123) + (132) + (213) + (231) + (312). \]  

In other words, the triple non-normal product (7) can be written as a sum of normal triple products up to a ternutator which is, through (3), of degree 1 in the operators (a decrease by 2°). More generally, any product of three operators in non-normal order can be written as the sum of operators in normal order up to a ternutator.

For \( i_3 > i_2 > i_1 \),

\[ (i_3 i_2 i_1) \equiv -[i_1, i_2, i_3] + (i_1 i_2 i_3) + (i_2 i_3 i_1) + (i_3 i_1 i_2) - (i_1 i_3 i_2) - (i_2 i_1 i_3). \]  

To simplify, we consider products which involve only operators with different indices

\[ A_{j_1} A_{j_2} A_{j_3} \ldots \quad \text{with} \quad A_{j_i} \neq A_{j_k} \quad \text{for} \quad k \neq m. \]
At the next ternutator level, we have to define the non-normal product of one ternutator and two operators or of two ternutators and one operator leading to the appearance of a ternutator of ternutators. At higher levels, one obtains nested ternutators of ternutators.

Let us associate as follows a label \( L \) with a nested ternutator. Take all the indices \( i_1, \ldots, i_n \) of the operators which enter in the nested ternutator and define

\[
L = \min\{i_1, \ldots, i_n\}.
\]

(12)

It is then easy to define the non-normal product of three nested ternutators \( T_3, T_2, T_1 \) with label \( L_3, L_2, L_1 \), respectively. They are those with \( L_3 > L_2 > L_1 \). They are transformed into normal products by the obvious

\[
T_3 T_2 T_1 = -[T_1, T_2, T_3] + T_1 T_2 T_3 + T_1 T_3 T_2 + T_2 T_1 T_3 - T_2 T_3 T_1 - T_3 T_1 T_2.
\]

(13)

The ternutator \( [T_1, T_2, T_3] \) of higher nesting has label \( L = L_1 \).

Using these definitions, any product of different operators can be transformed into a sum of normal products.

Starting from a given product of operators, there are often many different paths which can be followed to transform them into normal order. The difference between two results when they are different leads to what we call ternutator identities.

4. General considerations about identities

It is known that ternutators enjoy the seven Bremner–Nuyts identities among seven operators. These identities, [7, 8], play the role of the Jacobi identity for ternary algebras and generate cubic necessary conditions on the structure constants of these algebras. These identities are also satisfied by Nambu brackets [4],[3], a trilinear antisymmetric composition law for three operators, which associates with three functions \( f(x, y, z), g(x, y, z), h(x, y, z) \) a ternary bracket of the form

\[
[f, g, h] = \det \begin{vmatrix}
\frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\
\frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \\
\frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z}
\end{vmatrix},
\]

(14)

just as the Poisson bracket of two functions obeys the Jacobi identity of Lie brackets. This is discussed further in [10]. They are also satisfied by other trilinear composition laws, such as that of Awata \textit{et al} [11]:

\[
[A_i, A_j, A_k]_{Aw} = [A_i, A_j] < A_k > + [A_j, A_k] < A_i > + [A_k, A_i] < A_j >.
\]

(15)

where \( < A_k > \) denotes the trace of the operator \( A_k \). An important question is related to the sufficiency of these conditions. For ternary algebras depending upon a composition law which intrinsically requires the composition of three operators, only ternutators of ternutators are allowed in the search for identities. However, since we also have a product at our disposal in terms of our definition (2), we can search for more general identical relations among the operators of the algebra. In this paper we show that, much to our surprise, there exist identities involving four and six operators which lead also to necessary requirements. If we assume the linear independence of the anti-commutators (or commutators) of the operators of the algebra, the identities for four (or six) operators severely restrict the allowable form of the structure constants. Conversely, the four-operator and six-operator identities may be interpreted as linear relations among the anti-commutators or commutators of operators of the algebra.
5. Products of operators. Identities

In this section, we discuss the identities which can be obtained starting with products of a certain number of operators.

Quite generally, we suppose that the basic operators conveniently labeled as \( A_1, A_2, \ldots \) (in some fixed order of the indices) are defined to be linearly independent.

The identities among the products of operators are of two kinds.

(i) First, we have the sui generis identities which involve nested products of ternutators where all the operators in the products appear in the form of ternutators only. It implies in particular that the products must involve an odd number of generators. In earlier papers [7, 8], it was demonstrated that there are no sui generis identities for products of five operators and seven identities for products of seven operators. For ternutator algebras (3), since the operators themselves are independent, they lead to cubic necessary conditions for the structure constants.

(ii) Second, we have the identities where some operators may appear in an unnested positions. In this section, we concentrate on such identities for the products of four, five and six operators. There are obviously no such identities for the products of two or three operators.

5.1. Products of four operators

The non-normal products of four operators are of three forms.

- The exceptional product

\[
(4321) \tag{16}
\]

where both the three first operators and the three last operators are not in normal form.

- The products

\[
(4312), \ (4213), \ (3214) \tag{17}
\]

\[
(1432), \ (2431), \ (3421). \tag{18}
\]

In the sets, we have underlined the operators which are not in the normal form. They are the three first operators in (17) and the three last operators in (18).

It is easy to see that, using (9), the products in (17) and in (18) can be brought to the normal form by following a path which is unique. This is not the case for the product (4321) in (16) where two different paths can be followed depending on which set of three operators is used first the set (432) or the set (321). Let us follow the two paths. For path 1, one finds

\[
(4321) = ([4, 3, 2]1) + (4231) + (3421) + (3241) - (2341) - (2431)
\]

\[
= \ldots
\]

\[
= ([4, 3, 2]1) + ([3, 4, 2, 1]) - (2[4, 3, 1]) - ([3, 2, 1]4)
+ (4231) + (3412) - (3142) - (2413)
+ (2143) - (1234) + (1324). \tag{19}
\]

Following the same pattern, one finds for path 2,

\[
(4321) = ([4, 3, 2, 1]) - ([4, 2, 1]3) + ([4, 3, 1]2) - ([4, 2, 1]3)
+ (4231) + (3412) - (3142) - (2413)
+ (2143) - (1234) + (1324). \tag{20}
\]
Subtracting the results of (19) and (20), one finds the identity
\[ I_4(4, 3, 2, 1) = [[4, 3, 2], 1] + [[4, 3, 1], 2] + [[4, 2, 1], 3] - [[3, 2, 1], 4] = 0. \] (21)

Note the appearance of anti-commutators. This identity has also appeared in Curtright and Zachos [10], in equation (84) of that paper, but these authors did not pursue its implications.

The identity can be written as
\[ I_4(\{4, 3, 2, 1\}) = \sum_{\mu_1 < \mu_2 < \mu_3 < \mu_4} \text{sign}(\sigma) \{A_{\mu_1}, A_{\mu_2}, A_{\mu_3}, A_{\mu_4}\} = 0, \] (22)

where \(S_4\) are the permutations of \(\{i_1, i_2, i_3, i_4\}\) and \(\text{sign}(\sigma)\) is their sign.

These relations imply new necessary conditions on ternutator algebras (see below).

Indeed there also exist analogs to the identity of degree 4 for Nambu brackets, three of them to be precise. These follow simply from the following argument. Consider the expansion of the determinant
\[
\begin{vmatrix}
\frac{\partial f}{\partial \alpha} & \frac{\partial g}{\partial \alpha} & \frac{\partial h}{\partial \alpha} & \frac{\partial k}{\partial \alpha} \\
\frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} & \frac{\partial k}{\partial x} \\
\frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial k}{\partial y} \\
\frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial k}{\partial z}
\end{vmatrix} = 0,
\] (23)

for \(\alpha = x, y\) or \(z\). Expanding the determinant on the first row, we have
\[
\frac{\partial f}{\partial \alpha} [g, h, k]_{NB} - \frac{\partial g}{\partial \alpha} [h, k, f]_{NB} + \frac{\partial h}{\partial \alpha} [k, j, g]_{NB} - \frac{\partial k}{\partial \alpha} [f, g, h]_{NB} = 0.
\] (24)

These equations, for the various choices of \(\alpha\), are the analogs of identity (21), for Nambu brackets. A referee has pointed out the existence of another type of 4-identity which we write in the form
\[ [[4, 3, 2], 1] = [[4, 1], 3, 2] + [4, [3, 1], 2] + [[4, 3, 2], 1]. \] (25)

This identity is of a different nature from (21) as it is in the nature of a derivation, and applies not only to the ternutator, but to any product of operators. Indeed, the Jacobi identity itself is of the form
\[ [a, [b, c]] = [[a, b], c] + [[b, a], c]. \] (26)

While we can use (21) to obtain restrictions upon the structure functions of the ternutators involved, we cannot do this with (25) without assuming that the commutators themselves close upon operators in the ternutator algebra; in other words, the operators themselves must satisfy a Lie algebra. We shall avoid making this assumption, and have a reason to believe that there exist ternary algebras which are not equivalent to Lie algebras.

5.2. Products of five operators

Take five independent operators \(A_1, \ldots, A_5\), a REDUCE computation shows explicitly that there are ten independent identities involving the product of two of the operators and one ternutator (made of the three remaining operators), namely \([A_i, A_j, A_k]A_l A_m\), \([A_i, A_j, A_k]A_m A_l\), \([A_i, A_j, A_k]A_m A_l\) and \([A_i, A_j, A_k][A_l, A_m][A_j, A_k]\). There are a priori 60 such products and hence 60 arbitrary coefficients; (five choices for \(l\) and four for \(m\) in each...
of the three classes). Grouping these terms as $A_iA_m[A_i, A_j, A_k] + A_i[A_i, A_j, A_k]A_m$ and $[A_i, A_j, A_k]A_iA_m + A_i[A_i, A_j, A_k]A_m$, we see that they fall into ten sets which are identities in consequence of the four-operator identity.

There are 50 independent relations among these coefficients. Through REDUCE, it has been checked explicitly that a basis for the ten identities is given by the ten products $A_3 I_4(4, 3, 2, 1), A_4 I_4(5, 3, 2, 1), A_5 I_4(5, 4, 2, 1), A_6 I_4(5, 4, 3, 1)$.

$I_4(5, 4, 3, 2) A_1$.  

(27)

They all involve four-operator identities. Hence, no new identity exists at this degree. This has also been checked explicitly by hand. Remember that there are no sui generis identities for five operators.

5.3. Products of six operators

There are 21 identities involving the product of six operators $A_1, \ldots, A_6$ and which are quadratic in ternutators.

- There are $C_6^3 = 20$ identities which are simple consequences of the four identity. They are indexed by the choice of separating the six operators in two non-overlapping sets $\{i_1 < i_2 < i_3\}$ and $\{i_4 < i_5 < i_6\}$. They are conveniently written as

$$\sum_{\{i_1 < i_2 < i_3\}} \text{sign}(P)[[A_{i_1}, A_{i_2}, A_{i_3}], A_{i_4}, A_{i_5}, A_{i_6}]_+$$

$$- \sum_{\{i_4 < i_5 < i_6\}} \text{sign}(P)[[A_{i_4}, A_{i_5}, A_{i_6}], A_{i_1}, A_{i_2}, A_{i_3}]_+ = 0. \quad (28)$$

- The remaining identity can be written democratically as

$$6 \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6} \epsilon_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6} [[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], [A_{\mu_4}, A_{\mu_5}, A_{\mu_6}]]_+$$

$$- \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6} \epsilon_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6} [[A_{\mu_1}, A_{\mu_2}, A_{\mu_3}], [A_{\mu_4}, A_{\mu_5}, A_{\mu_6}]]_+ = 0. \quad (29)$$

This identity can be written in many apparently different forms adding terms which are zero when using the four-identity.

We have first obtained the exceptional identity (29) by using REDUCE to build all the possible identities. Another approach to this identity is to use the Bremner–Nuyts [7], [8] seven-identity in the form of equation (13) of [8] singling out an $A_7$ operator on the right. Then the coefficient of $A_7$ constitutes the identity at level 6. Another way is to use the path analysis for six operators; one obtains an identity by comparing for the initial configuration (654321), the path starting with (6 5 4 3 2 1) with the path starting for example with (6 5 4 3 2 1). Any other path starting with (6 5 4 3 2 1) or (6 5 4 3 2 1) and where 6 is a spectator will give the same identity, as all three paths are equivalent as they involve five operators known to be equivalent through $I_4$. This is another proof that the 6-identity is unique.

There seem to be no identities for six operators linear in the structure constants, i.e. of first degree in the ternutators of the basic operators, except those which follow from the 4-identity. This was essentially proved using REDUCE. The same referee who pointed out the 4-identity (25) also drew our attention to the existence of the similar identities of degree 6. Similar conclusions exist as to the utility of such identities unless closure of the operators under commutation is assumed.
6. Conditions on the structure constants from the identities of degree 4

Identity (21) implies the following type of conditions on the structure constants:

\[ f_{m}^{432}[A_m, A_1] - f_{m}^{431}[A_m, A_2] + f_{m}^{421}[A_m, A_3] - f_{m}^{321}[A_m, A_4] = 0. \]  

(30)

If we assume that the anti-commutators are linearly independent, we find the conditions

- from \( A_1^2 \rightarrow f_{1}^{432} = 0 \),
- from \( A_1A_2 \rightarrow f_{432}^2 - f_{431}^1 = 0 \),
- using \( m \notin \{1, 2, 3, 4\} \), we recover a condition of the form (31),
- from \( A_1A_5 \rightarrow f_{5}^{432} = 0 \).

(33)

Summarizing the results, we find

\[ f_{i}^{m} = 0 \quad \text{for} \quad m \notin \{i, j, k\} \]

\[ f_{ij}^{m} = f_{ij}^{k} \quad \text{for} \quad m \notin \{i, j\} \]

(34)

These results severely restrict the possible ternary algebras unless we drop the assumption of independence, and re-interpret (34) as a set of linear relations among the anti-commutators of the operators. This is a surprising result.

7. Alternative iterative approach

There is another way of looking at the identities which we have found. Consider the following iterative situation:

\[ [[A_1, A_2], A_3] + [[A_2, A_3], A_1] + [[A_3, A_1], A_2] = 0, \]

(35)

\[ [[A_1, A_2], A_3] + [[A_2, A_3], A_1] + [[A_3, A_2], A_1] + 2[A_1, A_2, A_3]. \]

(36)

The first is the Jacobi identity and the second is twice the ternutator \([A_1, A_2, A_3]\). Now add one more operator,

\[ [[A_1, A_2, A_3], A_4] + [[A_2, A_3, A_4], A_1] + [[A_3, A_1, A_2], A_4] = 0. \]

(37)

\[ [[A_1, A_2, A_3], A_4] + [[A_2, A_3, A_4], A_1] + [[A_3, A_4, A_1], A_2] + [[A_4, A_1, A_2], A_3] = 2[A_1, A_2, A_3, A_4]. \]

(38)

The first equation is the 4-identity \( I_4(4321) = 0 \) and the second equation is the definition of the antisymmetric 4-bracket \([A_1, A_2, A_3, A_4]\) with an additional factor of 2.

This pattern of repeated nested alternating commutators and anti-commutators persists. Writing the antisymmetric \( n \)-bracket in a generalized notation as

\[ [i_1, i_2, \ldots, i_n] \equiv [A_{i_1}, A_{i_2}, \ldots, A_{i_n}], \]

(39)

we find

- for an even number of operators

\[ [i_1, i_2, \ldots, i_{2n}] = \frac{1}{2} \sum_{\text{cyclic}} \text{sign}(C)[[i_1, i_2, \ldots, i_{2n-1}], i_{2n}]. \]

\[ 0 = \sum_{\text{cyclic}} \text{sign}(C)[[i_1, i_2, \ldots, i_{2n-1}], i_{2n}], \]

(40)

where the summation is over the cyclic permutation of \{i_1, i_2, \ldots, i_{2n}\} and \text{sign}(C) is the sign of the permutation,
for an odd number of operators

\[ 0 = \sum_{\text{cyclic}} \text{sign}(C)[[i_1, i_2, \ldots, i_{2n}], i_{2n+1}] \]

\[ [i_1, i_2, \ldots, i_{2n+1}] = \frac{1}{2} \sum_{\text{cyclic}} \text{sign}(C)[[i_1, i_2, \ldots, i_{2n}], i_{2n+1}], \quad (41) \]

with summation over the cyclic permutations of \{i_1, i_2, \ldots, i_{2n+1}\}.

As has been demonstrated, the identity at level 5 is not new. The level 6 identities contain the 20 identities (28) in a democratic fashion. There are no new identities at level 9 [12].

8. Discussion and conclusion

Usually, the only relevant identities are those composed of only ternary operations, i.e. operations sui generis, of nested ternutators, just as for Lie algebras, the relevant identities are those composed of iterated or nested commutators. It is known that in this case the Jacobi identity is both necessary and sufficient. In the case of ternary algebras, we have demonstrated an identity at the level of four operators (and one identity at the level of six operators), which involves anti-commutators (or commutators) and which can be interpreted in various ways. Either we assume that anti-commutators (or commutators) of all the operators involved are independent of one another, in which case there are very few viable ternary algebras, or we assert that having found a representation of the ternutator algebra, linear relations amongst the anti-commutators (or commutators) must be automatically satisfied. Perhaps we may be able to exploit these relations in the search for the representations of ternutators. In addition to the structures considered here, there are other binary–ternary algebraic structures to which our attention has been drawn, specifically Lie–Yamaguti algebras [13], Bol algebras [14] and Akivis algebras [15].

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