RANDOM COUNTABLE ITERATED FUNCTION SYSTEMS WITH OVERLAPS AND APPLICATIONS

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ABSTRACT. We study invariant measures for random countable (finite or infinite) conformal iterated function systems (IFS) with arbitrary overlaps. We do not assume any type of separation condition. We prove, under a mild assumption of finite entropy, the dimensional exactness of the projections of invariant measures from the shift space, and we give a formula for their dimension, in the context of random infinite conformal iterated function systems with overlaps. There exist many differences between our case and the finite deterministic case studied in [7], and we introduce new methods specific to the infinite and random case. We apply our results towards a problem related to a conjecture of Lyons about random continued fractions ([11]), and show that for all parameters $\lambda > 0$, the invariant measure $\nu_\lambda$ is exact dimensional; and in addition, we give estimates for the pointwise and Hausdorff dimension of $\nu_\lambda$, for $\lambda$ in a certain interval. The finite IFS determining these continued fractions is not hyperbolic, but we can associate to it a random infinite IFS of contractions which have overlaps. We study then also other large classes of random countable iterated function systems with overlaps, namely: a) several types of random iterated function systems related to Kahane-Salem sets; and b) randomized infinite IFS in the plane which have uniformly bounded number of disc overlaps. For all the above classes, we prove dimensional exactness, and we find lower and upper estimates for the pointwise (and Hausdorff, packing) dimensions of the projection measures.

1. Introduction

Let $(X, \rho)$ be a metric space. A finite Borel measure $\mu$ on $X$ is called exact dimensional if

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}$$

exists for $\mu$-a.e. $x \in X$ and is equal to a common value denoted by $d_\mu$. Exact dimensionality of the measure $\mu$ has profound geometric consequences (for eg [6], [7], [13], [22], [25]). The question of which measures are exact dimensional attracted the attention at least since the seminal paper of L.S Young [31], where it was proved a formula for the Hausdorff dimension of a hyperbolic measure invariant under a surface diffeomorphism, formula involving the Lyapunov exponents of the measure. As a consequence of that proof, she established what (now) is called the dimensional exactness of such measures. The topic of dimensional exactness was then pursued by the breakthrough result of Barreira, Pesin, and Schmeling who proved in [1] the Eckmann–Ruelle conjecture asserting that any hyperbolic measure invariant under smooth diffeomorphisms is exact dimensional ([1]). Dimensional exactness,
without using these words, was also established in the book [14] for all projected invariant measures with finite entropy, in the setting of conformal iterated function systems with countable alphabet which satisfy the Open Set Condition (OSC); in particular for all projected invariant measures if the alphabet is finite and we have OSC. The next difficult task was the case of a conformal iterated function system with overlaps, i.e. without assuming the Open Set Condition. For the case of iterated function systems with finite alphabet and having overlaps, this was done by Feng and Hu in [7]. Overlaps in iterated function systems (IFS) are challenging. Our goal in the present paper is to extend the above mentioned paper of Feng and Hu, in two directions. Firstly, by allowing the alphabet of the system to be countable infinite; and secondly, to consider random iterated function systems rather than deterministic IFS. Random IFS’s contain deterministic IFS as a special case.

We prove under a mild assumption of finite conditional entropy, the dimensional exactness of the projections of invariant measures from the shift space, in the context of random conformal iterated function systems with countable alphabet and arbitrary overlaps. We thus deal simultaneously with two new, and qualitatively different issues: infinite alphabet rather than finite, and random, rather than deterministic choice of contractions. We thus need new ideas and techniques appropriate to the context of infinite alphabet and randomness. Randomization allows to have a unitary setting to study limit sets and measures, in a family of systems for generic parameter values, which proves useful in cases when studying individual systems is difficult. Moreover, randomization allows us to obtain new types of fractal sets defined by series of random variables.

Our main results are the following: in Theorem 3.13 we prove dimensional exactness, and provide a formula for the dimension of typical projection measures, by employing a random projectional entropy and the Lyapunov exponents of the measure with respect to the random countable IFS with overlaps. Thus, we show that the pointwise, Hausdorff and packing dimensions of such a typical projection measure, coincide.

Then, in Theorem 2.5 we give lower and upper bounds for the random projectional entropy of a measure. This allows us consequently, to obtain estimates for the pointwise dimension (and thus Hausdorff dimension, and packing dimension) of projection measures.

In Section 3, we introduce and investigate several concrete classes of random countable iterated systems with overlaps. Firstly, we will give several ways to randomize countable IFS related to generalizations of Kahane-Salem sets ([9]), and infinite convolutions of Bernoulli distributions. Some of these are fractal sets obtained from series of random variables, namely sets \( \left\{ \pm 1 + \sum_{i \geq 1} \sum_{(j,k) \in Z_i} \pm \rho_j^k \rho_2^k \right\} \), where for any pair of positive integers \((j,k) \in Z_i\) we have \(j + k = i, \ i \geq 1\), and where the sets \(Z_i\) are prescribed by the parameter \(\lambda \in \{1, 2\}^\mathbb{Z}\), and the signs \(\pm\) are arbitrary. We obtain also another type of random fractal sets, defined in the
following way: $J_{\lambda} = \{ \pm 1 \pm \lambda \rho_i \pm \lambda^2 \rho_i \rho_{i+1} \pm \ldots \}$, for all sequences of positive integers $\omega = (i_1, i_2, \ldots) \in E^\infty$. We provide estimates for the pointwise dimensions of invariant measures on these random fractals.

We will then study IFS related to random continued fractions, and will shed new light on a problem of Lyons (see \cite{11, 30}). These are continued fractions of type $[1, X_1, 1, X_2, \ldots] = \frac{1}{1 + \frac{1}{X_1 + \frac{1}{1 + \frac{1}{X_2 + \ldots}}}}$, where the random variables $X_i, i \geq 1$ take two values $0, \alpha > 0$, each with equal probability; the distribution of this random continued fraction is denoted by $\nu_\alpha$. The above random continued fractions correspond to a parabolic IFS with overlaps, whose limit set contains an interval in certain cases. In \cite{30}, by using a transversality condition, it was shown that for a certain interval of parameters $\alpha$, the invariant measure $\nu_\alpha$ is absolutely continuous; also for other parameters $\nu_\alpha$ is singular. In our current paper, we do not use transversality. One can associate to the above finite parabolic IFS, a random infinite hyperbolic IFS with overlaps. In Theorem 4.4 we will prove exact dimensionality of the measure $\nu_\alpha$, for all values of the parameters $\alpha$. And moreover, we will give lower numerical bounds for the pointwise dimension (and Hausdorff dimension, packing dimension) of $\nu_\alpha$, when $\alpha \in (\frac{\sqrt{5} - 1}{2}, 0.5)$. Our method can be extended also to other types of random continued fractions and associated infinite IFS with overlaps.

Then in Section 3, we provide also examples of random infinite conformal IFS with overlaps in the plane, with uniformly bounded preimage counting function. We study the projection measures on their limit sets, finding lower and upper bounds for their pointwise (and Hausdorff) dimension.

In general, infinite IFS with overlaps behave differently than finite IFS with overlaps (for eg \cite{14}, \cite{19}, etc). In the infinite case, the limit set is not necessarily compact (by contrast to the finite IFS case), also the diameters of the sets $\phi_i(X)$ converge to 0, etc. In addition, for an infinite IFS $\mathcal{S}$, the boundary at infinity $\mathcal{S}(\infty)$ plays an important role, and we have to take into consideration whether an invariant probability gives measure zero (or not) to $\mathcal{S}(\infty)$ (for eg \cite{14}). Even when OSC is satisfied, the Hausdorff dimension of the limit set is not always given as the zero of the pressure of a certain potential. However, a version of Bowen’s formula for the Hausdorff dimension still exists; see \cite{14}. For example even when assuming OSC, and unlike in the finite alphabet case, the Hausdorff measure can vanish and the packing measure may become locally infinite at every point. In addition for infinite systems with overlaps we may have infinitely many basic sets overlapping at points in the limit set $J$, or the number of overlaps may be unbounded over $J$. Also, in \cite{27} there was studied the thermodynamic formalism for random IFS which satisfy the Open Set Condition. In \cite{18}, we obtained lower estimates for the Hausdorff dimension of the limit set $J$ of a deterministic infinite IFS with overlaps, by using a pressure function and a preimage counting function for the overlaps at various points of $J$. 

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By extension, the case of random infinite IFS with overlaps presents even more differences and new phenomena, when compared to the case of finite IFS with overlaps. For instance several proofs that used compactness type arguments cannot be applied to random infinite IFS with overlaps. We also have to impose certain conditions on the randomization process $\theta : \Lambda \to \Lambda$ and on the invariant measure $\mu$ on $\Lambda \times E^\infty$, etc. Therefore, we develop several new methods in our current paper.

We mention that several authors investigated the question of dimension for invariant sets and measures in the context of random dynamical systems or finite iterated function systems, or for smooth endomorphisms, although in different settings than us; for example [2], [8], [15], [16], [17], [21], [23], [22], [31].

2. **Pointwise dimensions for self-measures of random countable IFS with overlaps.**

First let us recall some well-known geometric concepts, see for eg [6], [22], [25]). For a finite Borel measure $\mu$ on a metric space $(X, \rho)$, we denote by $d_\mu^-(x)$ and $d_\mu^+(x)$ respectively the lower and upper limit when $r \to 0$, of the ratio:

$$\frac{\log \mu(B(x, r))}{\log r}$$

These limits are called respectively, the **lower, and upper pointwise dimensions** of $\mu$ at $x$, and are guaranteed to exist at every $x \in X$, in contrast to the limit in (1.1). If $d_\mu^-(x) = d_\mu^+(x)$, then the common value is called the **pointwise dimension** of $\mu$ at $x$, and is denoted by $d_\mu(x)$.

Now one can define the following dimensions:

$$\text{HD}_*(\mu) := \inf \{\text{HD}(Y) : \mu(Y) > 0\} \quad \text{and} \quad \text{HD}^*(\mu) = \inf \{\text{HD}(Y) : \mu(X \setminus Y) = 0\}.$$

In the case when $\text{HD}_*(\mu) = \text{HD}^*(\mu)$, this common value is called the **Hausdorff dimension of the measure** $\mu$ and is denoted by $\text{HD}(\mu)$.

Analogous concepts can be formulated for packing dimension, with respective notation $\text{PD}_*(\mu), \text{PD}^*(\mu)$. If $\text{PD}(\mu)$ exists, it is called the **packing dimension of the measure** $\mu$; in this case it can be proved that $\text{PD}(\mu) = \sup \{s, d_\mu(x) \geq s, \mu - \text{almost all } x\}$.

The first relations between these concepts are given in the following theorem (for example [6], [25]):

**Theorem 2.1** (General properties of dimensions of measures on metric spaces).

(i) If $\mu$ is a finite Borel measure on a metric space $(X, \rho)$, then

$$\text{HD}_*(\mu) = \text{ess inf} \ d_\mu, \ \text{HD}^*(\mu) = \text{ess sup} \ d_\mu, \ \text{and} \ \text{PD}_*(\mu) = \text{ess inf} \ d_\mu, \ \text{PD}^*(\mu) = \text{ess sup} \ d_\mu$$
(ii) If \( \mu \) is an exact dimensional finite Borel measure on a metric space \( (X, \rho) \), then both its Hausdorff dimension and packing dimension are well-defined and
\[
\text{HD}(\mu) = \text{PD}(\mu) = d_{\mu}.
\]

Let now \( X \) be a compact connected subset of \( \mathbb{R}^q \), \( q \geq 1 \) with \( X = \text{Int}(X) \). Consider also \( E \) to be a countable set (either finite or infinite), called an alphabet.

**Definition 2.2.** A random countable conformal iterated function system
\[
S = (\theta : \Lambda \to \Lambda, \{\lambda \mapsto \phi^\lambda_e\}_{e \in E})
\]
is defined by an invertible ergodic measure-preserving transformation of a complete probability space \( (\Lambda, \mathcal{F}, m) \), namely
\[
\theta : (\Lambda, \mathcal{F}, m) \to (\Lambda, \mathcal{F}, m),
\]
and by a family of conformal injective maps \( \{\phi^\lambda_e\}_{e \in E, \lambda \in \Lambda} \),
\[
\phi^\lambda_e : X \to X,
\]
such that there exists a bounded open connected set \( W \subset \mathbb{R}^q \) with \( X \subset W \) so that all the maps \( \phi^\lambda_e : X \to X \) extend to conformal injective maps from \( W \) to \( W \), and such that the Lipschitz constants of all the maps \( \phi^\lambda_e \) do not exceed a common value \( 0 < s < 1 \).

From the definition of conformality it follows that \( \phi^\lambda_e \) is in \( C^1(W) \), for all \( e \in E, \lambda \in \Lambda \).

We will denote in the sequel by \( E^\infty \) the space of one-sided infinite sequences \( \omega = (\omega_1, \omega_2, \ldots) , \omega_i \in E, i \geq 1 \); and by \( E^* \) the set of all finite sequences \( \tau = (\tau_1, \tau_2, \ldots, \tau_k), \tau_i \in E, 1 \leq i \leq k, k \geq 1 \). We also have the usual shift map \( \sigma : E^\infty \to E^\infty \).

In the sequel assume that the contraction maps \( \phi^\lambda_e : W \to W \) satisfy the following Bounded Distortion Property (BDP):

**Property 2.3 (BDP).** There exists a function \( K : [0, 1) \to [1, \infty) \) such that \( \lim_{t \to 0} K(t) = K(0) = 1 \), and
\[
\sup \left\{ \frac{|(\phi^\lambda_e)'(y)|}{|(\phi^\lambda_e)'(x)|} : e \in E, \lambda \in \Lambda, x \in X, ||y - x|| \leq t \cdot \text{dist}(x, \mathbb{R}^q \setminus W) \right\} \leq K(t).
\]

We also require some common measurability conditions. Precisely, we assume that for every \( e \in E \) and every \( x \in X \) the map
\[
\Lambda \ni \lambda \mapsto \phi^\lambda_e(x)
\]
is measurable. According to Lemma 1.1 in [3], this implies that, for all \( e \in E \), the maps
\[
\Lambda \times X \ni (\lambda, x) \mapsto \phi_e(x, \lambda) := \phi^\lambda_e(x)
\]
is measurable.
are (jointly) measurable. For every finite sequence $\omega \in E^*$, and every $\lambda \in \Lambda$, let us define also the (randomized) composition of contractions

$$\phi^\lambda_\omega := \phi^\lambda_{\omega_1} \circ \phi^\theta_{\omega_2} \circ \ldots \circ \phi^{|\omega|-1}_{\omega_l}(\lambda)$$

This formula exhibits the random aspect of our iterations: we choose consecutive generators $\phi_{\omega_1}, \phi_{\omega_2}, \ldots, \phi_{\omega_l}$ according to a random process governed by the ergodic map $\theta : \Lambda \to \Lambda$. This random aspect is particularly striking if $\theta$ is a Bernoulli shift when, in the random composition we choose $\phi^\lambda_\omega$ in an independent identically distributed way.

For $\omega \in E^\infty, \lambda \in \Lambda$, we define analogously to the deterministic case ([14], etc.), the point

$$\pi^\lambda_\omega := \bigcap_{n=1}^\infty \phi^\lambda_{\omega_n}(X),$$

and then the fractal limit set of the random countable IFS, corresponding to $\lambda \in \Lambda$ is:

$$J^\lambda := \pi^\lambda_{E^\infty}$$

Let us denote by $\pi^\lambda : \Lambda \times E^\infty \to \Lambda$ and $\pi_{E^\infty} : \Lambda \times E^\infty \to E^\infty$, the projections on the first, respectively the second coordinates. And by $\pi_{\mathbb{R}^q} : \Lambda \times E^\infty \to \mathbb{R}^q$ the projection defining the limit sets $J^\lambda$, $\lambda \in \Lambda$, namely $\pi_{\mathbb{R}^q}(\lambda, \omega) = \pi^\lambda_{\omega}(\omega)$, for $(\lambda, \omega) \in \Lambda \times E^\infty$.

Let us also denote by $\xi$ the partition of $E^\infty$ into initial cylinders of length 1; we will work in the sequel with conditional entropies of partitions and of probability measures (see for example [33], [12] for general definitions and properties).

Given a Lebesgue space $(Y, \mathcal{B}, \mu)$ and two measurable partitions of it, $\eta$ and $\zeta$, we will sometimes write $H_\mu(\eta|\zeta)$ without loss of generality, for the measure-theoretic conditional entropy $H_\mu(\eta|\zeta)$ of the partition $\eta$ with respect to the $\sigma$-algebra $\hat{\zeta}$ generated by $\zeta$. We will introduce now a notion of measure-theoretical projectional entropy for the random infinite system and for a projection measure.

**Definition 2.4.** Given the random countable iterated function system $\mathcal{S}$ as above, and a $\theta \times \sigma$–invariant probability measure $\mu$ on $\Lambda \times E^\infty$, define the random projectional entropy of the measure $\mu$ relative to the system $\mathcal{S}$, to be:

$$h_\mu(\mathcal{S}) := H_\mu(\pi_E^{-1}(\xi)|\pi_\Lambda^{-1}(\varepsilon_\Lambda) \vee (\theta \times \sigma)^{-1}(\pi_{\mathbb{R}^q}^{-1}(\varepsilon_{\mathbb{R}^q}))) - H_\mu(\pi_E^{-1}(\xi)|\pi_\Lambda^{-1}(\varepsilon_\Lambda) \vee \pi_{\mathbb{R}^q}^{-1}(\varepsilon_{\mathbb{R}^q})),$$

where $\varepsilon_\Lambda, \varepsilon_{\mathbb{R}^q}$ are the point partitions of $\Lambda$, respectively $\mathbb{R}^q$.

In the sequel we will consider only those $\theta \times \sigma$–invariant probability measures $\mu$ on $\Lambda \times E^\infty$ whose marginal measure on the parameter space $\Lambda$ is equal to $m$, i.e. such that

$$\mu \circ \pi_\Lambda^{-1} = m$$

We denote then by $(\mu_\lambda)_{\lambda \in \Lambda}$ the Rokhlin’s disintegration of the measure $\mu$ with respect to the fiber partition $(\pi_\Lambda^{-1})_{\lambda \in \Lambda}$. Its elements, $\{\lambda\} \times E^\infty, \lambda \in \Lambda$, will be frequently identified with the set $E^\infty$ and we will treat each probability measure $\mu_\lambda$ as defined on $E^\infty$. 


The desintegration \((\mu_\lambda)_{\lambda \in \Lambda}\) depending measurably on \(\lambda\), is uniquely determined by the property that for any \(\mu\)-integrable function \(g : \Lambda \times E^\infty \to \mathbb{R}\), we have
\[
\int_{\Lambda \times E^\infty} g d\mu = \int_{\Lambda} \int_{E^\infty} g \mu_\lambda dm(\lambda)
\]
Thus from the desintegration of conditional entropy in terms of a measurable partition (see \([26], [2]\)), we have the following equivalent desintegration formula for the random projectional entropy:
\[
(2.1) \quad h_\mu(S) = \int_\Lambda H_{\mu_\lambda}(\xi) \sigma^{-1}(\pi_\lambda^{-1}(\xi_{J_\theta(\lambda)})) dm(\lambda) - \int_\Lambda H_{\mu_\lambda}(\pi_\lambda^{-1}(\xi_{J_\lambda})) dm(\lambda)
\]
Using Definition 2.4 and the definitions of conditional entropy and conditional expectations (for eg from \([33]\), etc.), we can then further write:
\[
(2.2) \quad h_\mu(S) = \int_\Lambda \left[ \int_{E^\infty} \log E_{\mu_\lambda}(1_{[\omega]}|\pi_\lambda^{-1}(\xi_{J_\lambda}))](\omega) d\mu_\lambda(\omega) - \int_{E^\infty} \log E_{\mu_\lambda}(1_{[\omega]}|\pi_\lambda^{-1}(\xi_{J_\lambda}))](\omega) d\mu_\lambda(\omega) \right] dm(\lambda)
\]

We will see that there are important differences from the finite deterministic case, since here we have a family \((J_\lambda)_{\lambda \in \Lambda}\) of possibly non-compact limit sets, and a family of boundaries at infinity \(\{S_\lambda(\infty)\}_{\lambda \in \Lambda}\). The **boundary at infinity** of \(S_\lambda\), denoted by \(S_\lambda(\infty)\), is defined as the set of accumulation points of sequences \((\phi_{e_n}^\lambda(x_n))_n\), for arbitrary points \(x_n \in X\) and infinitely many different indices \(e_n \in E\). The family \(\{S_\lambda(\infty)\}_{\lambda \in \Lambda}\) is called also the **boundary at infinity** of the parametrized system \(S\). Similarly as in the deterministic case \([18]\), we can define for each \(\lambda \in \Lambda\) the set:
\[
S_\lambda^+(\infty) := \bigcup_{\omega \in E^\infty} \phi_\omega^{\theta(\lambda)}(S_\lambda(\infty))
\]

We give now some results about the relations between the random projectional entropy \(h_\mu(S)\) and the measure-theoretical entropy \(h(\mu)\) of the \(\theta \times \sigma\)-invariant measure \(\mu\) on \(\Lambda \times E^\infty\). In this way we obtain upper and lower bounds for the random projectional entropy \(h_\mu(S)\).

**Theorem 2.5.** In the above setting, if \(S\) is a random countable iterated function system and if \(\mu\) is a \((\theta \times \sigma)\)-invariant probability on \(\Lambda \times E^\infty\), we have the following inequalities:
(a) \[h_\mu(S) \leq h(\mu)\]
(b) Assume that there exists an integer \(k \geq 1\), such that for \(\mu\)-almost every \((\lambda, \omega) \in \Lambda \times E^\infty\) there exists \(r(\lambda, \omega) > 0\) and \(k\) indices \(e_1, \ldots, e_k \in E\), so that if the ball \(B(\pi_\lambda(\omega), r(\lambda, \omega)) \subset \mathbb{R}^q\) intersects a set of type \(\phi_e^n(J_\lambda)\), \(e \in E, \lambda \in \Lambda\), then \(e\) must belong to \(\{e_1, \ldots, e_k\}\). Then \[h_\mu(S) \geq h(\mu) - \log k\]
Proof. (a) Let us denote by $\mathcal{B}$ the $\sigma$-algebra of borelian sets in $\mathbb{R}^q$, and by $\hat{\xi}$ the $\sigma$-algebra generated by the partition $\hat{\xi} = \pi_{E^\infty}^{-1}\xi$ in $\Lambda \times E^\infty$. We want to prove first that

$$\hat{\xi} \lor (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}\mathcal{B} = \hat{\xi} \lor \pi_{R^q}^{-1}\mathcal{B}$$

But an element of the $\sigma$-algebra $\hat{\xi} \lor (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}\mathcal{B}$ is a set of type

$$\bigcup_{i \in E} (\Lambda \times [i]) \cap (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}A_i,$$

where $A_i \in \mathcal{B}, i \in E$. Let us take an element $(\lambda, \omega) \in \pi_{R^q}^{-1}(A_i)$, so $\pi_{R^q}(\lambda, \omega) \in A_i$, where $\omega = (\omega_1, \omega_2, \ldots)$. Then an element $\xi$ from the preimage set $(\theta^{-1} \times \sigma)^{-1}(\lambda, \omega)$, has the form $(\theta^{-1} \lambda, (\omega_0, \omega_1, \ldots)$, for arbitrary $\omega_0 \in E$; if this element belongs in addition to $\Lambda \times [i]$, then $\omega_0 = i$. Now $\pi_{R^q}(\xi) = \phi_i^{\theta^{-1}\lambda}(\pi_{R^q}(\lambda, \omega)) \in \phi_i^{\theta^{-1}\lambda}(A_i)$. Therefore we proved that

$$((\Lambda \times [i]) \cap (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}A_i = (\Lambda \times [i]) \cap \pi_{R^q}^{-1}(\phi_i^{\theta^{-1}\lambda}(A_i))$$

Thus $\hat{\xi} \lor (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}\mathcal{B} \subseteq \hat{\xi} \lor \pi_{R^q}^{-1}\mathcal{B}$, and after showing similarly the converse inclusion of $\sigma$-algebras, we obtain (2.3), i.e. that $\hat{\xi} \lor (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}\mathcal{B} = \hat{\xi} \lor \pi_{R^q}^{-1}\mathcal{B}$.

For an arbitrary integer $n \geq 1$, let us denote the measurable partition $\xi^{-1}_0 := \pi_{E^\infty}(\xi \lor \sigma^{-1}\xi \lor \ldots \lor \sigma^{-n}\xi)$. Using now the fact that the measure $\mu$ is $(\theta \times \sigma)$-invariant on $\Lambda \times E^\infty$, and the same type of argument as in Lemma 4.8 of [7], we obtain that for every integer $n \geq 1$,

$$H_\mu(\xi^{-1}_0 | (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}\mathcal{B}) - H_\mu(\xi^{-1}_0 | \pi_{R^q}^{-1}\mathcal{B}) = n \cdot [H_\mu(\xi | (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}\mathcal{B}) - H_\mu(\xi | \pi_{R^q}^{-1}\mathcal{B})]$$

Hence from formula (2.4) we obtain the following inequality:

$$nh_\mu(S) = H_\mu(\xi^{-1}_0 | (\theta \times \sigma)^{-1}\pi_{R^q}^{-1}\mathcal{B}) - H_\mu(\xi^{-1}_0 | \pi_{R^q}^{-1}\mathcal{B}) \leq H_\mu(\xi^{-1}_0)$$

Therefore, as $h(\mu)$ is the supremum of the limits of $\frac{1}{n}H_\mu\left(\bigvee_{0}^{n-1}(\theta \times \sigma)^{-1}\tau\right)$ when $n \to \infty$, over all partitions $\tau$ of $\Lambda \times E^\infty$, we obtain the upper bound $h_\mu(S) \leq h(\mu)$.

(b) We remind that $\xi$ is the partition of $E^\infty$ into the 1-cylinders $[i] := \{\omega \in E^\infty, \omega = (\omega_1, \omega_2, \ldots), \omega_1 = i\}$, for $i \in E$; we also recall that, for simplicity of notation, given in general 2 measurable partitions $\eta, \zeta$ of a Lebesgue space $(Y, \nu)$, we may write $H_\nu(\eta | \zeta)$ instead of $H_\nu(\eta | \zeta)$, where $\zeta$ is the $\sigma$-algebra generated by $\zeta$. We now assume that for $\mu$-almost every $(\lambda, \omega) \in \Lambda \times E^\infty$, there are at most $k$ indices $e \in E$ so that sets of type $\phi_e^{\lambda'}(J_{\lambda'})$, $\lambda' \in \Lambda$ intersect the ball $B(\pi_\lambda(\omega), r(\lambda, \omega))$. Let us consider next the partition $\mathcal{P}_n$ of $\mathbb{R}^q$ with sets of type $I_{i_1, \ldots, i_q} := \left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right] \times \ldots \times \left[\frac{i_q}{2^n}, \frac{i_q+1}{2^n}\right]$, for all multi-indices $\left(i_1, \ldots, i_q\right) \in \mathbb{Z}^q$.

For $m$-almost every $\lambda \in \Lambda$ we will now construct the subpartition $\mathcal{R}_n(\lambda) \subseteq \mathcal{P}_n$, which uses only those sets $I_{i_1, \ldots, i_q} \in \mathcal{P}_n$ that contain points $\pi_\lambda(\omega) \in J_{\lambda}$, $\omega \in E^\infty$, with $r(\lambda, \omega) > q/2^n$, and where the union of all the remaining cubes $I_{i_1, \ldots, i_q}$ of $\mathcal{P}_n$ represents just one element of $\mathcal{R}_n(\lambda)$. But we assumed that for $\mu$-almost all $(\lambda, \omega) \in \Lambda \times E^\infty$, there exists a radius $r(\lambda, \omega) > 0$, such that:

$$\text{Card}\{i \in E, \exists \lambda' \in \Lambda \text{ s.t } B(\pi_\lambda(\omega), r(\lambda, \omega)) \cap \phi_i^{\lambda'}(J_{\lambda'}) \neq \emptyset\} \leq k$$
So using the fact that \( n \) was chosen so that any cube \( I_{i_1, \ldots, i_q} \in \mathcal{R}_n(\lambda) \) contains at least a point of type \( \pi_\lambda(\omega), \omega \in E^\infty \) with \( r(\lambda, \omega) > \frac{4}{\pi} \), we obtain that any fixed set \( A \) from the partition \( \pi_\lambda^{-1}(\mathcal{R}_n(\lambda)) \) of \( E^\infty \), intersects at most \( k \) elements of the partition \( \xi \vee \pi_\lambda^{-1}(\mathcal{R}_n(\lambda)) \) of \( E^\infty \). From general properties of conditional entropy for partitions (see [33]), we have:

\[
(2.6) \quad H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))) = H_{\mu_\lambda}(\xi \vee \pi_\lambda^{-1}(\mathcal{R}_n(\lambda)) - H_{\mu_\lambda}(\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))) \leq \log k
\]

But now, since we known that for \( \mu \)-almost all \((\lambda, \omega) \in \Lambda \times E^\infty \) there exists a radius \( r(\lambda, \omega) > 0 \) satisfying condition (2.5), we infer that \( \pi_\lambda^{-1}(\mathcal{R}_n(\lambda)) \) is not \( \mathcal{R}_n(\lambda) \)-almost every \( \pi_\lambda^{-1}(\mathcal{R}_n(\lambda)) \), when \( n \to \infty \); and the same conclusion for the respective \( \sigma \)-algebras generated by these partitions in \( E^\infty \). Therefore from (2.6) and [20], and since \( \mu \circ \pi_\lambda^{-1} = m \), it follows that for \( m \)-almost every \( \lambda \in \Lambda \), the conditional entropy \( H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}\mathcal{B}) \) satisfies the inequality

\[
(2.7) \quad H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}\mathcal{B}) = H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}(\mathcal{B})) = \lim_{n \to \infty} H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))) \leq \log k
\]

In addition, if \( \mathcal{B}(E^\infty) \) represents the \( \sigma \)-algebra of subsets of \( E^\infty \) generated by cylinders, and by using the expression for conditional entropy with respect to the \( \sigma \)-algebra associated to a partition (see [20], [26]), we have that for \( m \)-almost all parameters \( \lambda \in \Lambda \),

\[
(2.8) \quad H_{\mu_\lambda}(\xi|\sigma^{-1}(\pi_{\theta(\lambda)}^{-1}\mathcal{B})) \geq H_{\mu_\lambda}(\xi|\sigma^{-1}(\mathcal{B}(E^\infty)))
\]

Consider now the measurable partition of \( \Lambda \times E^\infty \) defined by \( \zeta := \{\{\lambda\} \times [\omega_1], \lambda \in \Lambda, \omega_1 \in E\} = \epsilon_\Lambda \times \xi \). Then, since \( \xi \) is a \( \sigma \)-generator for \( \mathcal{B}(E^\infty) \) and since \( \theta \) is invertible, it follows that the partition \( \zeta \) is a \((\theta \times \sigma)\)-generator for the Lebesgue space \((\Lambda \times E^\infty, \mathcal{B}(\Lambda \times E^\infty), \mu)\). Hence from [20], [26], or [33],

\[
h(\mu) = h(\mu, \zeta) = H_{\mu}(\zeta|((\theta \times \sigma)^{-1}(\mathcal{B}(\Lambda \times E^\infty))
\]

But then, from the properties of conditional measures and of conditional entropies (see Rokhlin [26]), we obtain that:

\[
h(\mu) = H_{\mu}(\zeta|((\theta \times \sigma)^{-1}(\mathcal{B}(\Lambda \times E^\infty))) = \int_\Lambda H_{\mu_\lambda}(\zeta \cap \{\{\lambda\} \times E^\infty\}|((\theta \times \sigma)^{-1}(\mathcal{B}(\Lambda \times E^\infty))) \, dm(\lambda)
\]

Therefore from the definition of \( h_\mu(S) \), and from (2.7), (2.8) and the last displayed equality, we conclude that, in this case b), the random projectional entropy is bounded below by:

\[
h_\mu(S) \geq h(\mu) - \log k
\]

\[\square\]

**Remark 2.6.** We remark that the condition in Theorem [2.5] part (b), implies that there are no points from \( S_\lambda(\infty) \) in any of the limit sets \( J_\lambda \) for all \( \lambda, \lambda' \in \Lambda \). We shall give an example of such a random infinite system with overlaps in the last section. The difficulty without this condition is that, there may be a variable number of overlaps at points from the possibly non-compact fractal \( J_\lambda \), and that this number may tend to \( \infty \) even for a given \( \lambda \), or that it...
may tend to $\infty$ when $\lambda$ varies in $\Lambda$; in both of these cases, we cannot obtain however a lower estimate for $h_\mu(S)$ like the one in Theorem 2.5 (b).

3. Pointwise dimension for random projections of measures.

Given a metric space $(X, \rho)$ and a measurable map $H : E^\infty \to X$, then for every sequence $\omega \in E^\infty$ and every $r > 0$, we shall denote by

$$B_H(\omega, r) := H^{-1}(B_\rho(H(\omega), r)).$$

Our main result in this section is the exact dimensionality of random projections $\mu_\lambda$ on $J_\lambda$, of $(\theta \times \sigma)$-invariant probabilities $\mu$ from $\Lambda \times E^\infty$, for $m$-almost all parameters $\lambda \in \Lambda$. We start with the following:

**Lemma 3.1.** For all integers $k \geq 0$, every $e \in E$, $m$-a.e $\lambda \in \Lambda$, and $\mu_\lambda$-a.e. $\omega \in E^\infty$, we have

$$\lim_{r \to 0} \frac{\log \mu_\lambda(B_{\pi^{\theta_k(\lambda)} \circ \sigma_k}(\omega, r) \cap [e])}{\mu_\lambda(B_{\pi^{\theta_k(\lambda)} \circ \sigma_k}(\omega, r))} = \log E_{\mu_\lambda} \left( \frac{1}{1} \right)(\omega).$$

**Proof.** Fix $e \in E$ and define the following two Borel measures on $\mathbb{R}^q$:

(3.1) \quad \nu_\lambda := \mu_\lambda \circ (\pi^{\theta_k(\lambda)} \circ \sigma_k)^{-1}, \quad \text{and}\n
(3.2) \quad \nu_\lambda^e(D) := \mu_\lambda([e] \cap (\pi^{\theta_k(\lambda)} \circ \sigma_k)^{-1}(D)), \quad D \text{ Borel set in } \mathbb{R}^d.

Since $\nu_\lambda^e \leq \nu_\lambda$, the measure $\nu_\lambda^e$ is absolutely continuous with respect to $\nu_\lambda$. Let us then define the Radon-Nikodym derivative of $\nu_\lambda^e$ with respect to $\nu_\lambda$:

$$g_\lambda^e := \frac{d\nu_\lambda^e}{d\nu_\lambda}$$

Then, by Theorem 2.12 in [13], we have that:

(3.3) \quad g_\lambda^e(x) = \lim_{r \to 0} \frac{\nu_\lambda^e(B(x, r))}{\nu_\lambda(B(x, r))}.\n
for \( \nu_\lambda \)-a.e. \( x \in \mathbb{R}^q \). On the other hand, for every set \( F \in (\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}) \), say \( F = (\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\tilde{F}) \), \( \tilde{F} \in \mathcal{B}_{\mathbb{R}^q} \), we have

\[
\int_F E_{\mu_\lambda}(\mathbb{1}_{[e]}(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q})) d\mu_\lambda = \int_F \mathbb{1}_{[e]} d\mu_\lambda = \mu_\lambda(F \cap [e])
\]

\[
= \mu_\lambda((\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\tilde{F}) \cap [e]) = \nu_\lambda^e(\tilde{F}) = \int_{\tilde{F}} g_\lambda^e d\nu_\lambda
\]

\[
= \int_{\mathbb{R}^q} g_\lambda^e d(\mu_\lambda \circ (\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}) = \int_{\mathbb{R}^q} \mathbb{1}_F g_\lambda^e d(\mu_\lambda \circ (\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1})
\]

\[
= \int_{E^\infty} \mathbb{1}_F g_\lambda^e d(\mu_\lambda \circ (\pi_{\theta^k(\lambda)} \circ \sigma^k)) d\mu_\lambda
\]

\[
= \int_{E^\infty} g_\lambda^e d(\mu_\lambda \circ (\pi_{\theta^k(\lambda)} \circ \sigma^k)) d\mu_\lambda.
\]

Since, in addition, both functions \( E_{\mu_\lambda}(\mathbb{1}_{[e]}(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q})) \) and \( g_\lambda^e \circ (\pi_{\theta^k(\lambda)} \circ \sigma^k) \) are non-negative and measurable with respect to the \( \sigma \)-algebra \( (\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}) \), we conclude that:

\[
(g_\lambda^e \circ (\pi_{\theta^k(\lambda)} \circ \sigma^k))(\omega) = E_{\mu_\lambda}(\mathbb{1}_{[e]}(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}))(\omega)
\]

for \( \mu_\lambda \)-a.e. \( \omega \in E^\infty \). Along with (3.3) this means that

\[
\lim_{r \to 0} \frac{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r) \cap [e])}{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r))} = \log E_{\mu_\lambda}(\mathbb{1}_{[\omega]}(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}))(\omega)
\]

for \( \mu_\lambda \)-a.e. \( \omega \in E^\infty \). Taking logarithms the lemma follows.

\[\square\]

**Corollary 3.2.** For all integers \( k \geq 0 \), m-a.e. \( \lambda \in \Lambda \), and \( \mu_\lambda \)-a.e. \( \omega \in E^\infty \), we have

\[
\lim_{r \to 0} \log \frac{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r) \cap [\omega_1])}{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r))} = \log E_{\mu_\lambda}(\mathbb{1}_{[\omega_1]}(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}))(\omega).
\]

**Proof.** We have

\[
\lim_{r \to 0} \log \frac{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r) \cap [\omega_1])}{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r))} = 
\]

\[
= \sum_{e \in E} \mathbb{1}_{[e]}(\omega) \lim_{r \to 0} \frac{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r) \cap [e])}{\mu_\lambda(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r))}
\]

\[
= \sum_{e \in E} \mathbb{1}_{[e]}(\omega) \log E_{\mu_\lambda}(\mathbb{1}_{[e]}(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}))(\omega)
\]

\[
= \log E_{\mu_\lambda}(\mathbb{1}_{[\omega_1]}(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}))(\omega).
\]

\[\square\]

Now we shall prove the following.
Lemma 3.3. If $H_{\mu}(\pi^{-1}_{E\infty}(\xi)|\pi^{-1}_{\Lambda}(\varepsilon_{\Lambda})) < \infty$, then the function
\[
g(\lambda, \omega) := -\inf_{r>0} \log \frac{\mu_{\lambda}(\lfloor \omega \rfloor \cap B_{\pi_{\sigma_{\lambda}}(\omega, r)})}{\mu_{\lambda}(B_{\pi_{\sigma_{\lambda}}(\omega, r)})} \in \mathbb{R}
\]
is integrable with respect to the measure $\mu$, that is it belongs to $L^1(\mu)$.

Proof. Fix $\lambda \in \Lambda$ such that $\mu_{\lambda}$ is defined. Fix also $e \in E$. As in the proof of Lemma 3.1 consider measures $\nu_{\lambda}$ and $\nu_{\lambda}^e$ defined by (3.1) and (3.2) respectively. By Theorem 2.19 in [13] we have that
\[
\begin{align*}
\nu_{\lambda}^e\left( \{ x \in \mathbb{R}^q : \inf_{r>0} \left\{ \frac{\nu_{\lambda}^e(B(x, r))}{\nu_{\lambda}(B(x, r))} \right\} < t \} \right) &= \\
&= \nu_{\lambda}^e\left( \{ x \in \mathbb{R}^q : \sup_{r>0} \left\{ \frac{\nu_{\lambda}(B(x, r))}{\nu_{\lambda}^e(B(x, r))} \right\} > 1/t \} \right) \\
&\leq C_q t \nu_{\lambda}(\mathbb{R}^q) = C_q t,
\end{align*}
\]
where $1 \leq C_q < \infty$ is a constant depending only on $q$. What we obtained means that
\[
\mu_{\lambda}\left( \{ \omega \in E^\infty : \inf_{r>0} \left\{ \frac{\mu_{\lambda}(\lfloor e \rfloor \cap B_{\pi_{\sigma_{\lambda}}(\omega, r)})}{\mu_{\lambda}(B_{\pi_{\sigma_{\lambda}}(\omega, r)})} \right\} < t \} \right) \leq C_q t.
\]
Let us define also the function:
\[
G_{\lambda}^e(\omega) := \inf_{r>0} \left\{ \frac{\mu_{\lambda}(\lfloor e \rfloor \cap B_{\pi_{\sigma_{\lambda}}(\omega, r)})}{\mu_{\lambda}(B_{\pi_{\sigma_{\lambda}}(\omega, r)})} \right\}.
\]
Then the previous inequality can be rewritten as:
\[
\mu_{\lambda}(\{ G_{\lambda}^e \}^{-1}([0, t])) \leq C_q t.
\]
Define now the function $g_{\lambda} : E^\infty \to \mathbb{R}$ by $g_{\lambda}(\omega) = g(\lambda, \omega)$. Thus the following equility holds:
\[
g_{\lambda} = \sum_{e \in E} -\lfloor e \rfloor \log G_{\lambda}^e.
\]
Noting also that \(g_{\lambda} \geq 0\), we obtain therefore:

\[
\int_{E^\infty} g_{\lambda} d\mu_{\lambda} = \sum_{e \in E} - \int_{[e]} \log G_{\lambda}^e d\mu_{\lambda} = \sum_{e \in E} - \int_{0}^{\infty} \mu_{\lambda}(\{\omega \in [e] : -\log G_{\lambda}^e(\omega) > s\}) ds
\]

\[
= \sum_{e \in E} - \int_{0}^{\infty} \mu_{\lambda}(\{\omega \in [e] : G_{\lambda}^e(\omega) < e^{-s}\}) ds
\]

\[
= \sum_{e \in E} - \int_{0}^{\infty} \mu_{\lambda}(\{\omega \in E^\infty : G_{\lambda}^e(\omega) < e^{-s}\} \cap [e]) ds
\]

\[
\leq \sum_{e \in E} - \int_{0}^{\infty} \min\{\mu_{\lambda}([e]), Cq e^{-s}\} ds
\]

\[
= \sum_{e \in E} \left(\int_{0}^{\mu_{\lambda}([e]) + \log Cq} \mu_{\lambda}([e]) ds + \int_{\mu_{\lambda}([e]) + \log Cq}^{\infty} Cq e^{-s} ds\right)
\]

\[
= \sum_{e \in E} \left(-\mu_{\lambda}([e]) \log \mu_{\lambda}([e]) + \log(Cq) \mu_{\lambda}([e])\right)
\]

\[
= 1 + \log(Cq) + \sum_{e \in E} \left(-\mu_{\lambda}([e]) \log \mu_{\lambda}([e])\right)
\]

\[
= 1 + \log(Cq) + H_{\mu_{\lambda}}(\xi)
\]

Since \(H_{\mu}(\pi_{E^\infty}^{-1}(\xi)|\pi_{\Lambda}^{-1}(\epsilon_{\Lambda})) < \infty\), it therefore follows from the desintegration formula for entropy with respect to conditional measures \([26], [2]\), that

\[
\int_{\Lambda \times E^\infty} gd\mu = \int_{\Lambda} \int_{E^\infty} g_{\lambda} d\mu_{\lambda} dm(\lambda) \leq 1 + \log(Cq) + \int_{\Lambda} H_{\mu_{\lambda}}(\xi) \ dm(\lambda)
\]

\[
= 1 + \log(Cq) + H_{\mu}(\pi_{E^\infty}^{-1}(\xi)|\pi_{\Lambda}^{-1}(\epsilon_{\Lambda})) < \infty
\]

The proof is thus finished.

\[\square\]

**Remark 3.4.** We assumed above the finite entropy condition \(H_{\mu}(\pi_{E^\infty}^{-1}(\xi)|\pi_{\Lambda}^{-1}(\epsilon_{\Lambda})) < \infty\). This is not a restrictive condition, and it is satisfied by many measures and systems. For example, it is clearly satisfied if the alphabet \(E\) is finite. More interestingly, it is also satisfied when \(E\) is infinite and \(\mu = m \times \nu\), where \(m\) is an arbitrary \(\theta\)-invariant probability on \(\Lambda\), and \(\nu\) is a \(\sigma\)-invariant probability on \(E^\infty\) satisfying \(\nu([i]) = \nu_i, i \in E\) and

\[
h(\nu) = - \sum_{i \in E} \nu_i \log \nu_i < \infty
\]

Indeed, if \(A\) is the \(\sigma\)-algebra generated in \(\Lambda \times E^\infty\) by the partition \(\pi_{\Lambda}^{-1}(\epsilon_{\Lambda})\), and if \(\xi := \pi_{E^\infty}^{-1}\xi\), then \(H_{\mu}(\xi|A) = \int I_{\mu}(\xi|A)\), where \(I_{\mu}(\xi|A)\) is the information function

\[
I_{\mu}(\xi|A) := - \sum_{\Lambda \in \xi} \chi_{A} \cdot \log E_{\mu}(\chi_{A}|A)
\]

Now, the conditional expectation \(E_{\mu}(\chi_{A}|A) := g_{A}\) is \(A\)-measurable, and \(\int_{B \times E^\infty} g_{A} \ dm = \int_{B \times E^\infty} \chi_{A} d\mu\), for all sets \(B\) measurable in \(\Lambda\). Hence if \(A = \Lambda \times \{i\}\), then \(\int g_{A} d\mu = \mu(A \cap (B \times \{i\}))\).
Now we shall prove the following: 

\[ E^\infty) = m(B) \cdot \nu_i, \text{ so } g_A = \nu_i \text{ and } H_\mu(\xi|A) = -\sum_{i \in E} \nu_i \log \nu_i. \text{ Therefore, if } h(\nu) < \infty, \text{ then } H_\mu(\xi|A) < \infty \]

\[ \square \]

As an immediate consequence of Lemma 3.3, Corollary 3.2 and Lebesgue’s Dominated Convergence Theorem, we get the following:

**Lemma 3.5.** If \( H_\mu(\pi_{E^\infty}(\xi)|\pi^{-1}_\Lambda(\xi)) < \infty \), then

\[
\lim_{r \to 0} \frac{\mu_\lambda([\omega_1] \cap B_{\pi_\lambda}(\omega, K)(\phi_{\omega_1}^1(\pi_{\theta(\lambda)}(\sigma(\omega))))|r)}{\mu_\lambda(B_{\pi_\lambda}(\omega, r))} = \log E_\mu(\mathbb{1}_{[\omega_1]}(\pi_{\theta(\lambda)} \circ \sigma^k)^{-1}(B_r^\Lambda_\mu(\omega)))
\]

for \( \mu \text{-a.e. } (\lambda, \omega) \in \Lambda \times E^\infty, \) and the convergence holds also in \( L^1(\mu) \).

Now we shall prove the following:

**Lemma 3.6.** For every \( K \geq 1 \) there exists \( R_1 > 0 \) such that

\[ [\omega_1] \cap B_{\pi_\lambda}(\omega, K)(\phi_{\omega_1}^1(\pi_{\theta(\lambda)}(\sigma(\omega))))|r) \supset [\omega_1] \cap B_{\pi_\lambda}(\omega, r) \]

for all \( \lambda \in \Lambda \), all \( \omega \in E^\infty \), and all \( r \in [0, R_1] \).

**Proof.** Let \( \tau \in B_{\pi_\lambda}(\omega, r) \subset [\omega_1] \). Then \( \tau_1 = \omega_1 \) and \( \pi_{\theta(\lambda)}(\sigma(\tau)) \in B(\pi_{\theta(\lambda)}(\sigma(\omega)), r) \). Hence,

\[
\pi_\lambda(\tau) = \phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\tau))) \in \phi_{\omega_1}^\lambda(B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)) \]

\[
\subset B(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))), K(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))))|r)
\]

\[
= B(\pi_\lambda(\omega), K(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))))|r),
\]

where, because of the Bounded Distortion Property (BDP), the inclusion sign ”\( \subset \)" holds assuming \( r > 0 \) to be small enough. This means that

\[
\tau \in \pi_\lambda^{-1}(B(\pi_\lambda(\omega), K(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))))|r)) = B_{\pi_\lambda}(\omega, K(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))))|r)
\]

Since also already know that \( \tau_1 = \omega_1 \), we are thus done.

\[ \square \]

**Lemma 3.7.** For every \( K \geq 1 \) there exists \( R_2 > 0 \) such that

\[ [\omega_1] \cap B_{\pi_\lambda}(\omega, K^{-1}(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))))|r) \subset [\omega_1] \cap B_{\pi_\lambda}(\omega, r) \]

for all \( \lambda \in \Lambda \), all \( \omega \in E^\infty \), and all \( r \in [0, R_2] \).

**Proof.** Because of the Bounded Distortion Property (BDP), we have for all \( r \geq 0 \) small enough, say \( 0 \leq r \leq R_2 \), that

\[
B_{\pi_\lambda}(\omega, K^{-1}(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))))|r) = \pi_\lambda^{-1}(B(\pi_\lambda(\omega), K^{-1}(\phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\omega))))|r))
\]

\[
\subset \pi_\lambda^{-1}(\phi_{\omega_1}^\lambda(B(\pi_\lambda(\sigma(\omega)), r)))
\]

\[ 14 \]
So, fixing $\tau \in [\omega_1] \cap B_{\pi \omega}(\omega, K^{-1}(\phi_{\omega_1}^\lambda)'(\pi_{\theta(\lambda)}(\sigma(\omega)))|r)$, we have $\tau_1 = \omega_1$ and

$$\pi_\lambda(\tau) = \phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}(\sigma(\tau))) \in \phi_{\omega_1}^\lambda(B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)).$$

This means that $\pi_{\theta(\lambda)}(\sigma(\tau)) \in B(\pi_{\theta(\lambda)}(\sigma(\omega)), r))$, or equivalently, $\tau \in B_{\pi_{\theta(\lambda)}\circ \sigma}(\omega, r)$. The required inclusion is thus proved and the proof is complete.

Since the measure $\mu$ is fiberwise invariant, we have for all $\omega \in E^\infty$, all $r > 0$, and $m$-a.e. $\lambda \in \Lambda$ that

$$\mu_\lambda(\pi_{\theta(\lambda)}\circ \sigma(\omega), r)) = \mu_\lambda((\pi_{\theta(\lambda)} \circ \sigma)^{-1}(B(\pi_{\theta(\lambda)} \circ \sigma(\omega), r))$$

(3.4)

$$= \mu_\lambda(\pi_{\theta(\lambda)}^{-1}(B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)))$$

$$= \mu_{\theta(\lambda)}(B_{\pi_{\theta(\lambda)}(\sigma(\omega)), r}))$$

As an immediate consequence of this formula along with Lemma 3.6 and Lemma 3.7, we get the following:

**Lemma 3.8.** For every $K > 1$ there exists $R_K > 0$ such that

$$\frac{\mu_\lambda([\omega_1] \cap B_{\pi \omega}(\omega, K^\lambda(\phi_{\omega_1}^\lambda)'(\pi_{\theta(\lambda)}(\sigma(\omega)))|r))}{\mu_{\theta(\lambda)}(B_{\pi_{\theta(\lambda)}(\sigma(\omega)), r}))} \leq \frac{\mu_\lambda([\omega_1] \cap B_{\pi_{\theta(\lambda)}\circ \sigma}(\omega, r))}{\mu_{\lambda}(B_{\pi_{\theta(\lambda)}\circ \sigma}(\omega, r))}$$

and

$$\frac{\mu_\lambda([\omega_1] \cap B_{\pi \omega}(\omega, K^{\lambda-1}(\phi_{\omega_1}^\lambda)'(\pi_{\theta(\lambda)}(\sigma(\omega)))|r))}{\mu_{\theta(\lambda)}(B_{\pi_{\theta(\lambda)}(\sigma(\omega)), r}))} \geq \frac{\mu_\lambda([\omega_1] \cap B_{\pi_{\theta(\lambda)}\circ \sigma}(\omega, r))}{\mu_{\lambda}(B_{\pi_{\theta(\lambda)}\circ \sigma}(\omega, r))}$$

for all $\omega \in E^\infty$, all $r \in (0, R_K]$, and $m$-a.e. $\lambda \in \Lambda$.

**Lemma 3.9.** We have that

$$\int_{-\infty}^{\infty} \int_{E^\infty} \log \mu_\lambda(B_{\pi \omega}(\omega, r))d\mu_\lambda(\omega)d\mu_\lambda(\lambda) > -\infty$$

for all $r > 0$.

**Proof.** Since $X$ is compact there exist finitely many points $z_1, z_2, \ldots, z_l$ in $X$ such that

$$\bigcup_{j=1}^l B(z_j, r/2) \supset X.$$ 

For every $\lambda \in \Lambda$ and every integer $n \geq 0$ define the set of sequences:

$$A_n(\lambda) := \{\omega \in E^\infty : e^{-(n+1)} < \mu_{\lambda}(B_{\pi \omega}(\omega, r)) \leq e^{-n}\}.$$ 

Assume that

$$A_n(\lambda) \cap \pi_{\theta(\lambda)}^{-1}(B(z_j, r/2)) \neq \emptyset$$

for some $1 \leq j \leq l$. Fix $\omega \in A_n(\lambda) \cap \pi_{\theta(\lambda)}^{-1}(B(z_j, r/2))$ arbitrary. Then, because of the triangle inequality, $\pi_{\theta(\lambda)}^{-1}(B(z_j, r/2)) \subset B_{\pi \omega}(\omega, r)$. Therefore,

$$\mu_\lambda(A_n(\lambda) \cap \pi_{\theta(\lambda)}^{-1}(B(z_j, r/2))) \leq \mu_\lambda(\pi_{\theta(\lambda)}^{-1}(B(z_j, r/2))) \leq \mu_\lambda(B_{\pi \omega}(\omega, r)) \leq e^{-n}.$$
However, \( \mu_\lambda(A_n(\lambda)) \cap \pi_\lambda^{-1}(B(z_i, r/2)) = 0 \leq e^{-n} \) if \( A_n(\lambda) \cap \pi_\lambda^{-1}(B(z_i, r/2)) = \emptyset \), for some \( 1 \leq i \leq l \). Hence, since \( \{ \pi_\lambda^{-1}(B(z_j, r/2)) \}_{j=1}^l \) is a cover of \( E^\infty \), this implies that

\[
\mu_\lambda(A_n(\lambda)) \leq le^{-n}
\]

Therefore we obtain,

\[
\int_{E^\infty} - \log \mu_\lambda(B_{\pi_\lambda}(\omega, r)) d\mu_\lambda(\omega) = \sum_{n=0}^{\infty} \int_{A_n(\lambda)} - \log \mu_\lambda(B_{\pi_\lambda}(\omega, r)) d\mu_\lambda(\omega)
\]

\[
\leq \sum_{n=0}^{\infty} (n+1)le^{-n} = l \sum_{n=0}^{\infty} (n+1)e^{-n} < \infty.
\]

Hence, from the above, we can conclude that

\[
- \int_{\Lambda} \int_{E^\infty} \log \mu_\lambda(B_{\pi_\lambda}(\omega, r)) d\mu_\lambda(\omega) d\mu(\lambda) \leq l \sum_{n=0}^{\infty} (n+1)e^{-n} < \infty.
\]

\[\square\]

Then employing this lemma and Birkhoff’s Ergodic Theorem, we obtain the following:

**Lemma 3.10.** If \( \mu \) is ergodic, then for all \( r > 0 \) and \( \mu \)-a.e. pair \((\lambda, \omega) \in \Lambda \times E^\infty\), we have:

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu_{\theta^n(\lambda)}(B_{\pi_{\theta^n(\lambda)}}(\sigma^n(\omega), r)) = 0
\]

Now, we shall prove the following estimates for the measure \( \mu_\lambda \) of \( \pi_\lambda \)-preimages of balls:

**Lemma 3.11.** If the above \((\theta \times \sigma)\)-invariant measure \( \mu \) is ergodic on \( \Lambda \times E^\infty \) and if \( H_\mu(\pi_\lambda^{-1}(\xi)|\pi_\lambda^{-1}(\varepsilon_\Lambda)) < \infty \), then for every \( K > 1 \), all \( r \in (0, R_K) \) and \( \mu \)-a.e. \((\lambda, \omega) \in \Lambda \times E^\infty\), we have that

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu_\lambda(B_{\pi_\lambda}(\omega, K^{-n}|(\phi_{\omega|_n}^\lambda)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))))|r)) \leq -h_\mu(S),
\]

and moreover

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu_\lambda(B_{\pi_\lambda}(\omega, K^n|(\phi_{\omega|_n}^\lambda)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))))|r)) \geq -h_\mu(S).
\]
Proof. We prove the first inequality by relying on the second inequality of Lemma 3.8. The proof of the second inequality of the lemma is analogous and will be omitted. We have:

\[ T_{λ, n}^−(ω) = \]

\[ : = \log \mu_λ(\mathcal{B}_{λ}(ω, K_n)| (\phi_{ω|n})^i((πθ^n(λ)(σ^n(ω))))|r)) \]

\[ = \sum_{j=0}^{n-1} \log \frac{μ_θ(λ) \big( B_{λ}(σ^j(ω), K^{-(n-j)}) \big) \big( (\phi_{σ^j(ω)|n-j}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)}{μ_θ(λ) \big( B_{λ+1}(σ^{j+1}(ω), K^{-n-(j+1)}) \big) \big( (\phi_{σ^{j+1}(ω)|n-(j+1)}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)} + \]

\[ - \sum_{j=0}^{n-1} \log \frac{μ_θ(λ) \big( [σ^j(ω)]_1 \big) \big( B_{λ}(σ^j(ω), K^{-(n-j)}) \big) \big( (\phi_{σ^j(ω)|n-j}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)}{μ_θ(λ) \big( B_{λ+1}(σ^{j+1}(ω), K^{-n-(j+1)}) \big) \big( (\phi_{σ^{j+1}(ω)|n-(j+1)}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)} - \]

\[ - \sum_{j=0}^{n-1} \log \frac{μ_θ(λ) \big( [σ^j(ω)]_1 \big) \big( B_{λ}(σ^j(ω), K^{-(n-j)}) \big) \big( (\phi_{σ^j(ω)|n-j}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)}{μ_θ(λ) \big( B_{λ+1}(σ^{j+1}(ω), K^{-n-(j+1)}) \big) \big( (\phi_{σ^{j+1}(ω)|n-(j+1)}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)} + \]

\[ \sum_{j=0}^{n-1} W^-_{n-j}(θ × σ)^j(λ, ω) - \sum_{j=0}^{n-1} G^-_{n-j}(θ × σ)^j(λ, ω) + \log μ_θ^n(λ) \big( B_{θ^n(λ)}(σ^n(ω), r) \big), \]

where for all i ≥ 1,

\[ W^-_i(λ, ω) := \log \frac{μ_λ \big( [ω^i]_1 \big) \big( B_{λ}(σ^j(ω), K^{-i-1}) \big) \big( (φ_{σ^j(ω)|i}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)}{μ_λ \big( B_{λ}(σ^j(ω), K^{-i-1}) \big) \big( (φ_{σ^j(ω)|i}^θ)^i((πθ^n(λ)(σ^n(ω))))|r) \big)} \]

and where

\[ G^-_i(λ, ω) := \log \frac{μ_λ \big( [ω^i]_1 \big) \big( B_{λ}(σ^j(ω), K^{-(i-1)}) \big) \big( φ_{ω^i}^λ((πθ^n(λ)(σ^n(ω))))|r) \big)}{μ_λ \big( B_{λ}(σ^j(ω), K^{-(i-1)}) \big) \big( φ_{ω^i}^λ((πθ^n(λ)(σ^n(ω))))|r) \big)}. \]

Now, by virtue of Lemma 3.5, we see that Corollary 1.6, p. 96 in [12], applies to the sequences (W^-_i)_{i=1}^∞ and (G^-_i)_{i=1}^∞. This, in conjunction with Lemma 3.5, Lemma 3.10, the ergodicity of the measure μ with respect to the dynamical system θ × σ, and formula (2.2),
gives us the following inequalities:

\[
\lim_{n \to \infty} T_{\lambda,n}^- (\omega) \leq \int_{\Lambda \times E^\infty} \left( \log E_{\mu_\lambda} (\mathbb{1}_{[\omega_1]} (\pi_{\theta(\lambda)} \circ \sigma)^{-1}( \mathcal{B}_{\mathbb{R}^q})) (\omega) - \log E_{\mu_\lambda} (\mathbb{1}_{[\omega_1]} (\pi_{\lambda}^{-1}( \mathcal{B}_{\mathbb{R}^q})) (\omega) \right) d\mu_\lambda (\omega) d\mu (\lambda)
\]

\[= -h_\mu (S).\]

This finishes thus the proof.

\[\square\]

**Definition 3.12.** In the above setting, let us define the Lyapunov exponent of an ergodic measure \(\mu\) with respect to the endomorphism \(\theta \times \sigma : \Lambda \times E^\infty \to \Lambda \times E^\infty\) and the random countable iterated function system \(S\) by:

\[\chi_\mu := \int_{\Lambda \times E^\infty} -\log \left| \phi_{\omega_1}^\lambda (\pi_{\theta(\lambda)} (\sigma (\omega))) \right| d\mu (\lambda, \omega).\]

Since the above dynamical system is ergodic, then Birkhoff’s Ergodic Theorem yields that, for \(\mu\)-a.e. \((\lambda, \omega) \in \Lambda \times E^\infty\), we have

\[\lim_{n \to \infty} \frac{1}{n} \log \left| \phi_{\omega_1}^\lambda (\pi_{\theta(\lambda)} (\sigma^n (\omega))) \right| = \chi_\mu.\]

As a consequence of this lemma and Lemma 3.11 we now prove the main result of our paper:

**Theorem 3.13.** If \(\mu\) is ergodic and \(H_\mu (\pi_{E^\infty}^{-1} (\xi) | \pi_{\lambda}^{-1} (\varepsilon_\lambda)) < \infty\), then for \(\mu\)-a.e. \((\lambda, \omega) \in \Lambda \times E^\infty\), we have

\[\lim_{r \to 0} \frac{1}{\log r} \log \mu_\lambda (B_{\pi_\lambda} (\omega, r)) = \frac{h_\mu (S)}{\chi_\mu}.\]

**Proof.** What we want to prove is that:

\[\lim_{r \to 0} \frac{\log \mu_\lambda (B_{\pi_\lambda} (\omega, r))}{\log r} = \frac{h_\mu (S)}{\chi_\mu}.\]

Fix \(K > 1\). Fix also \((\lambda, \omega) \in \Lambda \times E^\infty\). Consider any \(r \in \left(0, K^{-1} \left| \phi_{\omega_1}^\lambda (\pi_{\theta(\lambda)} (\sigma (\omega))) \right| \right)\). There then exists a largest integer \(n \geq 0\) such that

\[r \leq K^{-n} \left| \phi_{\omega_1}^\lambda (\pi_{\theta(\lambda)} (\sigma^n (\omega))) \right| R_K,\]

hence for this specific integer \(n\), we obtain:

\[B_{\pi_\lambda} (\omega, r) \subset B_{\pi_\lambda} (\omega, K^{-n} \left| \phi_{\omega_1}^\lambda (\pi_{\theta(\lambda)} (\sigma^n (\omega))) \right| R_K),\]

and

\[r \geq K^{-(n+1)} \left| \phi_{\omega_1}^\lambda (\pi_{\theta(n+1)(\lambda)} (\sigma^{n+1}(\omega))) \right| R_K.\]
Therefore,
\[
\frac{\log \mu_\lambda((B_{\pi_\lambda}(\omega, r))}{\log r} \geq \frac{\log \mu_\lambda(\omega, K^{-n})}{\log r} \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K) \right)
\]
\[
\geq \frac{\log \mu_\lambda(\omega, K^{-n})}{\log r} \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K) \right) - (n + 1) \log K + \log \left( (\phi_{\omega|n+1})'(\pi_{\theta^{n+1}(\lambda)}(\sigma^{n+1}(\omega))) \right) + \log R_K
\]
\[
= \frac{\frac{1}{n} \log \mu_\lambda(\omega, K^{-n})}{\log r} \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K) \right) - (1 + \frac{1}{n}) \log K + \frac{1}{n} \log \left( (\phi_{\omega|n+1})'(\pi_{\theta^{n+1}(\lambda)}(\sigma^{n+1}(\omega))) \right) + \frac{1}{n} \log R_K.
\]
Hence, applying formula \((3.5)\) from Lemma \((3.11)\) and also Lemma \((3.7)\) we get
\[
\lim_{r \to 0} \frac{\log \mu_\lambda((B_{\pi_\lambda}(\omega, r))}{\log r} \geq \frac{h_\mu(S)}{\log K + \chi_\mu}
\]
for all \((\lambda, \omega)\) in some measurable set \(\Omega^+_K \subset \Lambda \times E^\infty\) with \(\mu(\Omega^+_K) = 1\). Then
\[
\mu \left( \Omega^+ := \bigcap_{j=1}^{\infty} \Omega^+_{j+1} \right) = 1
\]
and
\[
(3.8) \lim_{r \to 0} \frac{\log \mu_\lambda((B_{\pi_\lambda}(\omega, r))}{\log r} \geq \frac{h_\mu(S)}{\chi_\mu}
\]
for all \((\lambda, \omega) \in \Omega^+.\) For the proof of the opposite direction fix any \( K > 1 \) so small that
\[
(3.9) K^{-1} > \text{ess sup } \{ \| (\phi_e^\lambda)'' \| : e \in E, \lambda \in \Lambda \}.
\]
Having \((\lambda, \omega) \in \Lambda \times E^\infty\) fix any \( r \in (0, KR_K \text{ ess sup } \{ \| (\phi_e^\lambda)'' \| : e \in E, \lambda \in \Lambda \} \). Because of \((3.9)\) there exists a least \( n \geq 1 \) such that
\[
K^n \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K \leq r.
\]
Then, because of our choice of \( r \), we have that \( n \geq 2 \),
\[
K^{n-1} \left( (\phi_{\omega|n-1})'(\pi_{\theta^{n-1}(\lambda)}(\sigma^{n-1}(\omega))) | R_K \leq r,
\]
and
\[
B_{\pi_\lambda}(\omega, r) \supset B_{\pi_\lambda}(\omega, K^n) \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K \right).
\]
Therefore,
\[
\frac{\log \mu_\lambda((B_{\pi_\lambda}(\omega, r))}{\log r} \leq \frac{\log \mu_\lambda(\omega, K^n) \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K \right)}{\log r}
\]
\[
\geq \frac{\log \mu_\lambda(\omega, K^n) \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K \right)}{\log r} - (n - 1) \log K + \log \left( (\phi_{\omega|n-1})'(\pi_{\theta^{n-1}(\lambda)}(\sigma^{n-1}(\omega))) \right) + \log R_K
\]
\[
= \frac{\frac{1}{n} \log \mu_\lambda(\omega, K^n) \left( (\phi_{\omega|n})'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) | R_K \right)}{\log r} - (1 - \frac{1}{n}) \log K + \frac{1}{n} \log \left( (\phi_{\omega|n-1})'(\pi_{\theta^{n-1}(\lambda)}(\sigma^{n-1}(\omega))) \right) + \frac{1}{n} \log R_K.
Hence, applying formula (3.5) from Lemma 3.11 and also Lemma 3.7, we get
\[
\lim_{r \to 0} \frac{\log \mu(\Lambda \times E^\infty)}{\log r} \leq \frac{h_\mu(S)}{\log(K + \kappa)}
\]
for all \((\lambda, \omega)\) in some measurable set \(\Omega_K \subset \Lambda \times E^\infty\) with \(\mu(\Omega_K) = 1\). Then we have:
\[
\mu (\Omega^- := \bigcap_{j=k}^\infty (\Omega_{1/k}^- \cup \ldots)) = 1,
\]
where \(k \geq 1\) is taken to be so large that \(\frac{k+1}{k} \sup \{||\phi^\prime_r|| : e \in E, \lambda \in \Lambda\} < 1\). Also,
\[
\lim_{r \to 0} \frac{\log \mu(\Lambda \times E^\infty)}{\log r} \leq \frac{h_\mu(S)}{\kappa}
\]
for all \((\lambda, \omega) \in \Omega^-\). Along with (3.8) this yields \(\mu(\Omega^+ \cap \Omega^-) = 1\) and moreover,
\[
\lim_{r \to 0} \frac{\log \mu(\Lambda \times E^\infty)}{\log r} = \frac{h_\mu(S)}{\kappa}
\]
for all \((\lambda, \omega) \in \Omega^+ \cap \Omega^-\), which gives therefore the required dimensional exactness. \(\square\)

From the above Theorem 3.13 and Theorem 2.1 we obtain the following result, giving the (common) Hausdorff dimension and packing dimension of the projections \(\mu_\lambda \circ \pi_{\lambda}^{-1}\) on the random limit sets \(J_\lambda\):

**Corollary 3.14.** In the above setting if \(\mu\) is a \(\theta \times \sigma\)-invariant ergodic probability on \(\Lambda \times E^\infty\) whose marginal on \(\Lambda\) is \(m\), and if \(H_\mu \left( \pi_{E^\infty}^{-1}(\xi) | \pi_{\lambda}^{-1}(\epsilon_\lambda) \right) < \infty\), then for \(m\)-a.e \(\lambda \in \Lambda\), we have
\[
\text{HD}(\mu_\lambda \circ \pi_{\lambda}^{-1}) = \text{PD}(\mu_\lambda \circ \pi_{\lambda}^{-1}) = \frac{h_\mu(S)}{\kappa}.
\]

4. Classes of random countable IFS with overlaps.

In this section we study several large classes of examples of random countable IFS \(S\) with overlaps, together with invariant measures \(\mu\) and projections \((\pi_{\lambda})_* \mu_\lambda\) of their conditional measures on fibers. In all these examples we give estimates for the pointwise (and Hausdorff, packing) dimensions of the projected measures \((\pi_{\lambda})_* \mu_\lambda = \mu_\lambda \circ \pi_{\lambda}^{-1}\).

4.1. Randomizations related to Kahane-Salem sets.

In [9] Kahane and Salem studied the convolution of infinitely many Bernoulli distributions, namely the measure \(\mu = B(r_0) * B(r_1) * B(r_2) * \ldots\), where \(B(x)\) denotes the Bernoulli probability supported at the points \(-x, x\) and giving measure \(\frac{1}{2}\) to each of them. The support of \(\mu\) is the set \(F\) of points of the form \(\epsilon_0 r_0 + \epsilon_1 r_1 + \ldots\), where \(\epsilon_k\) is equal to \(+1\) or \(-1\) with
equal probabilities. If we assume $\sum_{0}^{\infty} r_k = 1$, and if we introduce the sequence of positive numbers $(\rho_n)_{n \geq 0}$ defined by:

$$r_0 = 1 - \rho_0, r_1 = \rho_0 (1 - \rho_1), r_2 = \rho_0 \rho_1 (1 - \rho_2), \ldots,$$

then it can be seen that, if $\rho_k > \frac{1}{2}$ for all but finitely many $k$’s, then $F$ contains intervals. If, on the other hand, $\rho_k < \frac{1}{2}$ for all $k \geq 0$, then $F$ is a Cantor set. In addition to this, $\lim_{k \to \infty} 2^k \rho_0 \ldots \rho_{k-1} = 0$, then $F$ has zero Lebesgue measure and $\mu$ is singular.

A particular though interesting case is when $r_k = \rho^k, k \geq 0$, for some $\rho \in (0, 1)$. Then the corresponding set $F = F_\rho$ is the set of real numbers of type $\pm 1 \pm \rho \pm \rho^2 \pm \ldots$. If $\rho < \frac{1}{2}$, then $F_\rho$ contains intervals. The convolution $\mu^{(\rho)}$ is equal to the invariant probability of the IFS with two contractions

$$\phi_1(x) = \rho x + 1, \phi_2(x) = \rho x - 1,$$

taken with probabilities $1/2, 1/2$. This is a conformal system with overlaps, and $F_\rho$ is equal to the limit set $J_\rho$ of this IFS. The measure $\mu^{(\rho)}$ is the projection $\nu_{(1/2,1/2)} \circ \pi^{-1}$ of the probability $\nu_{(1/2,1/2)}$ from $\{1, 2\}^\mathbb{N}$, through the canonical projection $\pi : \{1, 2\}^\mathbb{N} \to J_\rho$. In [5] Erdös proved that when $1/\rho$ is a Pisot number (i.e a real algebraic integer greater than 1 so that all its conjugates are less than 1 in absolute value), then the measure $\mu^{(\rho)}$ is singular. In [24] it was shown that its Hausdorff dimension is strictly smaller than 1. In the other direction, B. Solomyak showed in [32] that $\mu^{(\rho)}$ is absolutely continuous for a.e $\rho \in [1/2, 1)$. Shmerkin showed in [29] that in fact there exists a set $E \subset [1/2, 1)$ such that $HD(E) = 0$, and $\mu^{\rho}$ is absolutely continuous for every $\rho \in [1/2, 1) \setminus E$.

Here we will give several ways to extend and randomize the idea of this construction, and will apply our results on pointwise dimensions of projection measures for random infinite IFS with overlaps:

**Random system 4.1.1**

A type of random IFS can be obtained by fixing numbers $r_1, r_2 \in (0, 1)$, letting $\Lambda = \{1, 2\}^\mathbb{Z}$, $\theta : \Lambda \to \Lambda$ be the shift homeomorphism, and setting $E = \{1, 2\}$ so the alphabet is finite in this case. For arbitrary $\lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots) \in \Lambda$ and $e \in E$, consider then the affine contractions $\phi^\lambda_e$ in one real variable, defined by:

$$\phi^\lambda_e(x) = r_{\lambda_0} x + 1, \quad \phi^\lambda_e(x) = r_{\lambda_0} x - 1 \quad (4.1)$$

Then, for arbitrary $\lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots) \in \{1, 2\}^\mathbb{Z}$, the corresponding fractal limit set is

$$J_\lambda := \pi_\lambda(E^\infty) = \{\phi^\lambda_{\omega_1} \circ \phi^\theta(\lambda) \circ \ldots, \omega = (\omega_1, \omega_2, \ldots) \in E^\infty\},$$
which can actually be described as a set of type
\[
\left\{ \pm 1 + \sum_{i \geq 1} \sum_{(j,k) \in Z_i} \pm r_1^k r_2^j \right\},
\]
where for any pair of positive integers \((j,k) \in Z_i\) we have \(j + k = i, \ i \geq 1\), and where the sets \(Z_i\) are prescribed by the parameter \(\lambda \in \{1, 2\}^\mathbb{Z}\), while the signs \(\pm\) are arbitrary.

We then consider the 1-sided shift space \(E^\infty\), and a Bernoulli measure \(\nu = \nu_Q\) on \(E^\infty\) given by a probability vector \(Q = (q_1,q_2)\). Let also a Bernoulli measure \(m = m_P\) on \(\Lambda\) associated to the probability vector \(P = (p_1,p_2)\), and the probability \(\mu = m \times \nu\) on \(\Lambda \times E^\infty\). The above random finite IFS is denoted by \(S\).

Next, by desintegrating \(\mu\) into conditional measures \(\mu_\lambda\), and projecting \(\mu_\lambda\) to the limit set \(J_\lambda\), we obtain the projection measure \(\mu_\lambda \circ \pi_\lambda\). \(\lambda \in \Lambda\). In this case the finiteness condition of entropy from the statement of Theorem 3.13 is clearly satisfied since \(E\) is finite, so we obtain the exact dimensionality of the measures \(\mu_\lambda \circ \pi_\lambda^{-1}\) on \(J_\lambda\) for \(m\)-almost all \(\lambda \in \Lambda\). And from Corollary 3.14 and Theorem 2.5 we obtain

**Corollary 4.1.** In the setting of 4.1.1, we obtain the following upper estimate for the pointwise (and Hausdorff, packing) dimensions of the projection measures, for \(\mu\)-almost all \((\lambda, \omega) \in \Lambda \times E^\infty\):

\[
d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) = \frac{h_\mu(S)}{h_\mu(S)} \leq \frac{h(m_P) + h(\nu_Q)}{-p_1 \log r_1 - p_2 \log r_2} = \frac{p_1 \log p_1 + p_2 \log p_2 + q_1 \log q_1 + q_2 \log q_2}{p_1 \log r_1 + p_2 \log r_2}
\]

Also, another possibility is to take \(\mu = m \times \nu\) on \(\Lambda \times E^\infty\), where \(m = m_P\) as before and \(\nu\) is an equilibrium measure of a Hölder continuous potential on the 1-sided shift space \(E^\infty\).

**Random system 4.1.2**

Consider now a fixed sequence \(\tilde{\rho} = (\rho_i)_{i \geq 1}\) of numbers in \((0,1)\) which are smaller than some fixed \(\rho \in (0,1)\), and let the parameter space \(\Lambda = \{1, 2, \ldots\}^\mathbb{Z}\) with the shift homeomorphism \(\theta : \Lambda \to \Lambda\). Let also an infinite probability vector \(P = (p_1,p_2,\ldots)\), and the \(\theta\)-invariant Bernoulli measure \(m_P\) on \(\Lambda\) satisfying \(m_P([i]) = p_i, \ i \geq 1\), where \([i] := \{\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots), \omega_0 = i\}, \ i \geq 1\), and \(h(m_P) < \infty\). Let us take then the set \(E := \{1, 2, \ldots\}^\mathbb{Z}\) and a \((\theta \times \sigma)\)-invariant probability measure \(\mu\) on \(\Lambda \times E^\infty\), having its marginal on \(\Lambda\) equal to \(m_P\). For example we can take \(\mu = m_P \times \nu_Q\), where \(Q = (q_1,q_2,\ldots)\) is a probability vector, and where \(\nu_Q([j]) = q_j, j \geq 1\) is a \(\sigma\)-invariant Bernoulli probability on \(E^\infty\); we assume in addition that the entropy of \(\nu_Q\) is finite, i.e that

\[-\sum_{j \geq 1} q_j \log q_j < \infty\]
We now define infinitely many contractions $\phi^\lambda_n$ on a fixed large enough compact interval $X$, for arbitrary $e \in E, \lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots) \in \Lambda, \; \lambda_i \in \{1, 2 \ldots\}, i \in \mathbb{Z}$, by:

$$\phi^\lambda_n(x) = \rho_{\lambda_n} \cdot x + (-1)^{\lambda_0}, \; n \geq 1$$

It is clear that $\phi^\lambda_n$ are conformal contractions and they satisfy Bounded Distortion Property. We construct thus a random infinite IFS denoted by $\mathcal{S}(\rho)$, which has overlaps.

For every $\lambda \in \Lambda$, we construct then the fractal limit set $J_\lambda := \pi_\lambda(E^\infty)$, which may be non-compact. The fractal $J_\lambda$ is the set of points given as $\phi^{\lambda_1}_n \circ \phi^{\lambda_2}_n \circ \ldots$, for all $\omega \in E^\infty$. The main difference from the previous example 4.1.1 is that now, the plus and minus signs in the series giving the points of $J_\lambda$ are not arbitrary, instead they are determined by $\lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots) \in \Lambda$. The randomness in the series comes now from the various possibilities to choose the sequences $\omega$ that the condition $\theta$ is satisfied. Moreover we have from Theorem 2.5 that the random projectional entropy of the projection measure $\mu$ with respect to the random infinite system $\mathcal{S}(\rho)$ is equal to:

$$\chi_\mu = - \int_{\Lambda \times E^\infty} \log \rho_{\lambda_1} \, d\mu(\lambda, \omega) = - \sum_{i \geq 1} q_i \int \log \rho_\lambda \, dm_P(\lambda)$$

Moreover we have from Theorem 2.5 that the random projectional entropy of $\mu$ satisfies

$$h_\mu(\mathcal{S}(\rho)) \leq h(\mu) = h(m_P) + h(\nu_Q) = - \sum_{i \geq 1} p_i \log p_i - \sum_{j \geq 1} q_j \log q_j.$$

**Corollary 4.2.** In the setting of 4.1.2, we obtain a concrete upper estimate for the pointwise (and Hausdorff, packing) dimension of $\mu_\lambda \circ \pi_{\lambda}^{-1}$, for $\mu$-almost all $(\lambda, \omega) \in \Lambda \times E^\infty$:

$$d_{\mu_\lambda \circ \pi_{\lambda}^{-1}}(\pi_\lambda(\omega)) \leq \frac{\sum_{i \geq 1} p_i \log p_i + \sum_{j \geq 1} q_j \log q_j}{\sum_{j \geq 1} p_j \log \rho_j}$$

**Random system 4.1.3**

Let us fix a sequence $\bar{\rho} = (\rho_0, \rho_1, \rho_2, \ldots)$ in $(0, 1)$, and $\Lambda = [1 - \varepsilon, 1 + \varepsilon]$ for some small $\varepsilon > 0$, together with a homeomorphism $\theta : \Lambda \to \Lambda$ which preserves an absolutely continuous and ergodic probability $m$ on $\Lambda$. Let us take also the set $E = \{1, 2, \ldots\}$ and the $\sigma$-invariant Bernoulli measure $\nu$ on $E^\infty$ given by $\nu([i]) = \nu_i, i \geq 1$, where $(\nu_1, \nu_2, \ldots)$ is a probability
vector. We assume also that $h(\nu) = -\sum_{i \geq 1} \nu_i \log \nu_i < \infty$. For arbitrary $e \in E$ and $\lambda \in \Lambda$, we now define the sequence of parametrized contractions:

$$
\rho_{2n+1}(x) = \lambda \rho_n x + 1, \quad \rho_{2n+2}(x) = \lambda \rho_n x - 1, \quad n \geq 0.
$$

By considering also the $(\theta \times \sigma)$-invariant probability $\mu = m \times \nu$ we obtain the random infinite IFS with overlaps $S(\bar{\rho})$. Notice that, since $m$ is ergodic with respect to $\theta$ and $\nu$ is mixing with respect to $\sigma$, it follows that $\mu$ is ergodic with respect to $\theta \times \sigma$ (for e.g. [10], [33]).

The corresponding limit set $J_\lambda := \pi_\lambda(E^\infty)$ can be thought of as the set determined, for $\lambda \in \Lambda$, in the following way:

$$
J_\lambda = \{ \pm 1 \pm \lambda \rho_i \pm \theta(\lambda) \rho_i \rho_i \pm \ldots, \text{for all sequences of positive integers } \omega = (i_1, i_2, \ldots) \}
$$

The projection $(\pi_\lambda)_* \mu_\lambda = \mu_\lambda \circ \pi_{\lambda,1}$ of the measure $\mu_\lambda$, is a probability measure on $J_\lambda$. We see that both (2.3) and the entropy condition $H_{\mu}(\pi_{\lambda,1}) < \infty$, are satisfied in this case.

Hence, we can apply Theorem 3.13 and Corollary 3.14, to obtain that for $m$-almost all parameters $\lambda \in [1 - \varepsilon, 1 + \varepsilon]$, the projection measure $\mu_\lambda \circ \pi_{\lambda,1}$ is exact dimensional, and that its Hausdorff dimension has a common value, which is equal to

$$
HD(\mu_\lambda \circ \pi_{\lambda,1}) = \frac{h_{\mu}(S(\bar{\rho}))}{\lambda_{\mu}},
$$

where the Lyapunov exponent of $\mu$ with respect to $S(\bar{\rho})$ is given by:

$$
\chi_{\mu} = -\int_{\Lambda \times E^\infty} \log(\lambda \rho_1 \omega_1 \omega_1) \, d\mu(\lambda, \omega) = -\int_{\Lambda} \log \lambda \, dm(\lambda) - \sum_{i \geq 0} (\nu_{2i+1} + \nu_{2i+2}) \log \rho_i
$$

From Theorem 2.5 we obtain an upper estimate for the random projectional entropy, $h_{\mu}(S) \leq h(m) - \sum_i \nu_i \log \nu_i$, and an upper estimate for the pointwise dimension and Hausdorff dimension of $\mu_\lambda \circ \pi_{\lambda,1}$.

**Corollary 4.3.** In the setting of 4.1.3, we obtain that for $\mu$-almost every $(\lambda, \omega) \in [1 - \varepsilon, 1 + \varepsilon] \times E^\infty$,

$$
d_{\mu_\lambda \circ \pi_{\lambda,1}}(\pi_\lambda(\omega)) = HD(\mu_\lambda \circ \pi_{\lambda,1}) \leq \frac{h(m) - \sum_{i \geq 1} \nu_i \log \nu_i}{-\int_{\Lambda} \log \lambda \, dm(\lambda) - \sum_{i \geq 1} (\nu_{2i+1} + \nu_{2i+2}) \log \rho_i}
$$

If all the contraction factors $\rho_i$ are equal to some fixed $\rho$, then $J_\lambda$ is a perturbation of the set from the beginning of 4.1.

4.2. Random continued fractions.

By the continued fraction $[a_1, a_2, \ldots]$ with digits $a_1, a_2, \ldots$, we understand the ratio

$$
\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}
$$
R. Lyons studied in [11] random continued fractions [1, X1, 1, X2, ...], where the random variables X_i, i ≥ 1 are i.i.d and take the values 0, α each with probability 1/2, and where α is a fixed number in (0, ∞). Let ν_α be the distribution of this random continued fraction. In fact the measure ν_α is the invariant measure of the iterated function system \( S_α = \{φ_1^α, φ_2^α\} = \{\frac{x+α}{x+α+1}, \frac{x}{x+1}\} \), where the two generator maps are applied with equal probabilities.

If \( P_α \) is the fixed point of \( φ_1^α \), then \( P_α = -\frac{α+\sqrt{α^2+4α}}{2} \), and it can be seen that \( φ_1^α(0) > φ_2(P_α) \) if and only if \( α > \frac{1}{2} \). Thus for \( α > \frac{1}{2} \), the support of \( ν_α \) is a Cantor set contained in \( [0, P_α] \cup [0, P_α] \). If \( α \in (0, \frac{1}{2}) \), then there are strict overlaps and the limit set of \( S_α \) is the interval \( [0, \frac{-α+\sqrt{α^2+4α}}{2}] \).

Lyons showed in [11] that the measure \( ν_α \) is singular for all \( α \in (α_c, \frac{1}{2}) \), where \( α_c \in (0.2688, 0.2689) \). Later, by employing a transversality condition, Simon, Solomyak and Urbanski [30] made progress and showed that \( ν_α \) is absolutely continuous for Lebesgue-almost all \( α \in (0.215, α_c) \). They left it open whether \( ν_α \) is absolutely continuous or singular for small values of \( α \), as transversality may fail in that case.

In our paper, we do not use transversality, but view the original system \( S_α \) as a random parabolic IFS with overlaps. The system \( S_α \) is not hyperbolic (the map \( φ_2 \) is not contracting everywhere), hence we cannot apply directly our results above. However we can associate to it an infinite random IFS with overlaps containing only contractions, using the jump transformation ([28]). In this way, infinite random IFS of contractions with overlaps, will appear naturally in this situation.

Our goal is to show that the invariant measure \( ν_α \) of the IFS \( S_α \) is exact dimensional, and to give estimates on its pointwise dimension. Once we have exact dimensionality, it means that all fractal invariants of the measure (pointwise dimension, Hausdorff dimension, packing dimension) are the same.

**Theorem 4.4.** Consider the IFS \( S_λ = \{φ_1^λ, φ_2^λ\} = \{\frac{x+λ}{x+λ+1}, \frac{x}{x+1}\} \), and let the invariant measure \( ν_λ \) obtained by applying the maps of \( S_λ \) each with probability \( \frac{1}{2} \).

a) Then for all \( λ \in (0, ∞) \), the measure \( ν_λ \) is exact dimensional, and its pointwise, Hausdorff, and packing dimensions are all equal to each other.

b) For every \( λ \in (\frac{1}{2}, 0.5) \), the pointwise (and Hausdorff) dimension of \( ν_λ \) is larger than 0.174.

**Proof.** a) In order to keep our notation for random systems, replace \( α \) by \( λ \). We have the maps:

\[
φ_1^λ(x) = \frac{x + λ}{x + λ + 1}, \quad φ_2^λ(x) = \frac{x}{x + 1}
\]

Consider \( X = [0, 1] \), and for each \( λ \in (0, ∞) \), take the transformation \( θ(λ) = λ \) on \( Λ_λ = \{λ\} \), which invariates the Dirac distribution \( δ_λ \). We see that both \( φ_1^λ \) and \( φ_2 \) are increasing, and that \( φ_1^λ(0) = \frac{λ}{λ+1}, φ_1^λ(1) = \frac{λ+1}{λ+2}, \) and \( φ_2(0) = 0, φ_2(1) = \frac{1}{2} \). Also the map \( φ_1^λ \) is a contraction for \( λ > 0 \), but \( φ_2 \) is not contracting. According to the jump transformation ([28]), we can
associate to our parabolic IFS $\hat{S}_\lambda$, a hyperbolic IFS $S_\lambda = \{\psi_n^\lambda, n \geq 0\}$, formed by the transformations of $X$,

$$\psi_n^\lambda = \phi_2^n \circ \phi_1^\lambda, \ n \geq 0$$

From the definition, it follows that the maps $\psi_n^\lambda$ are all contractions, and by induction on $n$,

$$\psi_n^\lambda(x) = \frac{x + \lambda}{(n + 1)(x + \lambda) + 1}$$

We denote, for $\lambda \in (0, \infty)$, by $S_\lambda$ the random infinite IFS with overlaps obtained above. For $\lambda > 0$, the measure $\nu_\lambda$ is the projection onto the limit set $J_\lambda$ of the Bernoulli measure on $\{1, 2\}^\mathbb{N}$, associated to $(\frac{1}{2}, \frac{1}{2})$. If $\phi_* \mu$ denotes in general, the push forward of a measure $\mu$ through a map $\phi$, then $\nu_\lambda$ is the unique probability measure satisfying the condition:

$$h(\mu_\lambda) = h(\nu) = \sum_{n \geq 1} \log \frac{2^n}{2} = \log 2 \sum_{n \geq 1} \frac{n}{2^n} = 2 \log 2 < \infty$$

Then, Remark 3.4 applies, and we obtain from Theorem 3.13 that for all $\lambda \in (0, \infty)$ and $\nu$-almost all $\omega \in E^\infty$, the projection $\nu_\lambda = \nu \circ \pi_\lambda^{-1}$ is exact dimensional at the point $\pi_\lambda(\omega)$. The Lyapunov exponent of the measure $\mu_\lambda$ is defined by:

$$\chi_{\mu_\lambda} = -\int_{(\lambda) \times E^\infty} \log \| (\psi_n^\lambda)'(\pi_\theta(\lambda)(\sigma\omega)) \| \ d\mu_\lambda(\lambda, \omega)$$

From the above formula for $\psi_n^\lambda$, $(\psi_n^\lambda)'(x) = \frac{1}{[(n+1)(x+\lambda)+1]^2}$; thus, since $x \in [0, 1]$,

$$\frac{1}{[\lambda(n+1)+n+2]^2} \leq |(\psi_n^\lambda)'(x)| \leq \frac{1}{[\lambda(n+1)]^2}$$

Hence from above we obtain:

$$\chi_{\mu_\lambda} \leq \sum_{n \geq 1} \frac{1}{2n} \log[\lambda(n + 1) + n + 2] < \infty$$

In conclusion from Theorem 3.13 we obtain that for any $\lambda > 0$, $\nu_\lambda$ is exact dimensional, and

$$d_{\nu_\lambda}(\pi_\lambda(\omega)) = HD(\nu_\lambda) = \frac{h_{\mu_\lambda}(S_\lambda)}{\chi_{\mu_\lambda}}$$
b) We want now to estimate the random projectional entropy \( h_\mu(\mathcal{S}_\lambda) \) for certain values of \( \lambda \), which will give estimates also the pointwise dimension. Firstly, we know that \( \psi_n^\lambda(0) = \frac{\lambda}{\lambda(n+1)+1} = \frac{1}{n+1+\frac{\lambda}{n+1}} \), \( \psi_n^\lambda(1) = \frac{\lambda+1}{(n+1)(\lambda+1)+1} = \frac{1}{n+1+\frac{\lambda}{n+1}} \), and that clearly \( (\psi_n^\lambda(0))_n \) and \( (\psi_n^\lambda(1))_n \) are strictly decreasing sequences in \( n \), as \( \lambda > 0 \).

We want to see what is the maximum number of intervals \( I_j^\lambda := \psi_j^\lambda([0,1]) \) that any given interval \( I_n^\lambda \) intersects. Assuming that \( I_{n+k}^\lambda \) intersects \( I_n^\lambda \), it follows that \( \frac{1}{n+k+1+\frac{\lambda}{n+1}} > \frac{1}{n+1+\frac{\lambda}{n+1}} \), thus \( \lambda(\lambda+1) < \frac{1}{k} \). Looking next at intervals of type \( I_{n-k'}^\lambda \), we have an overlap between \( I_n^\lambda \) and \( I_{n-k'}^\lambda \) if \( \frac{1}{n-k'+1+\frac{\lambda}{n}} < \frac{1}{n+1+\frac{\lambda}{n+1}} \), which means again that \( \lambda(\lambda+1) < \frac{1}{k'} \). By combining, we obtain that \( I_n^\lambda \) intersects \( k + k' \) intervals \( I_j^\lambda, j \neq n \), if \( \lambda(\lambda+1) < \max\{\frac{1}{k}, \frac{1}{k'}\} \). In particular, if \( \lambda > \frac{-1+\sqrt{3}}{2} \), then each interval \( I_n^\lambda \) intersects strictly less than 4 other intervals \( I_j^\lambda \).

For \( \lambda > 0 \), let us take now again the transformation \( \theta \) as the identity \( \theta = \id : \{\lambda\} \to \{\lambda\} \), with its invariant ergodic Dirac measure \( \delta_\lambda \), also \( E = \mathbb{N} \), the measure \( \tilde{\nu} \) on \( E^\infty \) being the Bernoulli measure associated to the probability vector \( \left( \frac{1}{2}, \frac{1}{2}, \ldots \right) \), and \( \mu_\lambda = \delta_\lambda \times \tilde{\nu} \). As in a), for each \( \lambda > 0 \) they form the random system \( \mathcal{S}_\lambda \) with overlaps (and one fiber); recall also that we proved that the entropy \( h(\mu_\lambda) = h(\tilde{\nu}) = 2 \log 2 \).

Since for \( \lambda > \frac{-1+\sqrt{3}}{2} \), each image \( \psi_n^\lambda([0,1]) \) intersects at most 3 other image intervals of type \( \psi_j^\lambda([0,1]) \), it follows from Theorem 2.5 and the proof of part a) that, for any \( \lambda > 0 \), the random projectional entropy \( h_{\mu_\lambda}(\mathcal{S}_\lambda) \) can be estimated as:

\[
h_{\mu_\lambda}(\mathcal{S}_\lambda) \geq h(\mu_\lambda) - \log 3 = 2 \log 2 - \log 3 = \log \frac{4}{3}
\]

On the other hand, from the definition of the random Lyapunov exponent \( \chi_{\mu_\lambda} \), and (4.3),

\[
\chi_{\mu_\lambda} \leq \sum_{n \geq 1} \frac{1}{2^n} \log[\lambda(n+1) + n + 2] \leq \sum_{n \geq 1} \frac{1}{2^n} \log\left(\frac{3n}{2} + \frac{5}{2}\right)
\]

In order to estimate the last sum, notice that \( \log x \leq 1.2^x \), for \( x \geq 4 \) (where the logarithm is the natural one). Thus \( \log\left(\frac{3n+5}{2}\right) \leq (\sqrt{1.2})^{3n+5} \), for \( n \geq 1 \). Therefore by applying this inequality to the terms of our series, we obtain that for any \( n_1 \geq 1 \),

\[
(4.5) \quad \sum_{n \geq n_1} \frac{1}{2^n} \log\left(\frac{3n}{2} + \frac{5}{2}\right) \leq 1.577 \sum_{n \geq n_1} 0.65^n
\]

The terms \( \frac{1}{2^n} \log\left(\frac{3n+5}{2}\right) \), and \( 0.65^n \), of the two series above, converge rapidly to 0; so for instance it is enough to compute only the first 14 terms, and in (4.5) we can take \( n_1 = 15 \). Thus the Lyapunov exponent of \( \mu_\lambda \) satisfies \( \chi_{\mu_\lambda} \leq 1.65 \).

Then from (4.3) and the above estimates, we obtain that for every \( \lambda \in (-\frac{1+\sqrt{3}}{2}, 0.5) \) and for \( \tilde{\nu}\text{-a.e } \omega \in E^\infty \),

\[
d_{\mu_\lambda \circ \pi^{-1}}(\pi_\lambda(\omega)) = d_{\nu_\lambda}(\pi_\lambda(\omega)) = HD(\nu_\lambda) \geq \frac{\log \frac{4}{3}}{1.65} \geq 0.174
\]

\[
\square
\]
Another possibility is to take $\Lambda = [0,1]$, $\theta : \Lambda \to \Lambda$ to be an expanding smooth bijective map, and $m$ to be its absolutely continuous invariant and ergodic measure on $[0,1]$ (see [12]). We can then form the random system $S$ and the measure $\nu_\lambda$ as before. By applying our results, we again obtain that $\nu_\lambda$ is exact dimensional, and we can find another estimate for its Hausdorff (and packing) dimension.

We make the observation that our method can be applied also to other random continued fractions, whose digits take values in some fixed set.

4.3. Random infinite IFS with bounded number of overlaps in the plane.

In Example 5.11 of [18], we gave an example of a deterministic infinite IFS defined as follows: let $X = \bar{B}(0,1) \subset \mathbb{R}^2$ be the closed unit disk and for $n \geq 1$ take $C_n$ to be the circle centered at the origin and having radius $r_n \in (0,1)$, $r_n \nrightarrow 1$. For each $n \geq 1$ we cover the circle $C_n$ with closed disks $D_n(i), i \in K_n$, of the same radius $r_n$, where $K_n$ is a finite set and each disk $D_n(i)$ intersects only two other disks of the form $D_n(j), j \in K_n$, and where none of the disks $D_n(i)$ intersects $C_k, k \neq n$. Moreover, we assume that for any $m \neq n, m, n \geq 1$, the families $\{D_m(i)\}_{i \in K_m}$ and $\{D_n(i)\}_{i \in K_n}$ consist of mutually disjoint disks.

Consider now contraction similarities $\phi_{n,i} : X \rightarrow X, i \in K_n, n \geq 0$ whose respective images of $X$ are the above disks $D_n(i), i \in K_n, n \geq 0$. For this countable deterministic system, the boundary at infinity $S(\infty)$ is contained in $\partial X$.

Assume now in addition, that there exists $\varepsilon > 0$, such that for $m \neq n$, any disk $(1 + \varepsilon)D_n(i), i \in K_n$ does not intersect any disk of type $(1 + \varepsilon)D_m(j), j \in K_m$ (where in general for $\beta > 0$, $\beta D_n(i)$ denotes the disk of the same center as $D_n(i)$ and radius equal to $\beta r_n$), and that any disk $(1 + \varepsilon)D_n(i)$ intersects only two other disks $(1 + \varepsilon)D_n(j), j \in K_n$.

Let us take $\Lambda = [1-\varepsilon, 1+\varepsilon]$ and $\theta : \Lambda \rightarrow \Lambda$ a homeomorphism which preserves an absolutely continuous ergodic probability $m$ on $[1-\varepsilon, 1+\varepsilon]$. Consider the following countable alphabet

$$E = \{(n,i), i \in K_n, n \geq 0\}$$

Consider also a fixed probability vector $P = (\nu_e)_{e \in E}$, and the associated Bernoulli probability $\nu = \nu_P$ on $E^\infty$, and let us assume that $h(\nu) < \infty$.

We now define the conformal contraction $\phi_{(n,i)}^\lambda(x)$, as being a similarity with image $\phi_{(n,i)}^\lambda(X)$ equal to $\lambda D_n(i)$, for $i \in K_n, n \geq 0$ and $\lambda \in \Lambda$; its contraction factor is equal to $\lambda r_n$, $n \geq 0$.

Consider now the probability measure $\mu = m \times \nu$ defined on $\Lambda \times E^\infty$. Since $m$ is ergodic with respect to $\theta$ and $\nu$ is mixing with respect to $\sigma$, then $\mu$ is ergodic with respect to $\theta \times \sigma$ (see [10], [33]). We have thus constructed a random conformal infinite IFS with overlaps, denoted by $S$; and from Remark 3.14 and since $h(\nu) < \infty$, we obtain also the finite entropy condition $H_\mu(\pi_{E^\infty}^{-1}\xi | \pi_\Lambda^{-1}\epsilon_\Lambda) < \infty$.

The conditions in Theorem 3.13 and Corollary 3.14 are satisfied, and thus for Lebesgue-a.e $\Lambda \in \Lambda$ and for $\nu$-a.e $\omega \in E^\infty$, the projection measure $(\pi_\Lambda)_* \mu_\Lambda = \mu_\Lambda \circ \pi_\Lambda^{-1}$ on the non-compact
limit set $J_\lambda := \pi_\lambda(E^\infty)$ is exact dimensional, and its pointwise (and Hausdorff) dimension is given by:

$$d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) = \frac{h_\mu(S)}{\chi_\mu},$$

where the Lyapunov exponent of $\mu$ with respect to the random system $S$ is equal to:

$$\chi_\mu = -\log \lambda - \sum_{e=(n,i) \in E} \nu_e \log r'_n > 0$$

From the construction of the disks $\lambda D_n(i), i \in K_n, n \geq 0, \lambda \in \Lambda$ above, we notice that the condition in Theorem 2.5, part b) is satisfied with $k = 2$. Hence we can obtain a lower estimate for the random projectional entropy of $\mu$, namely

$$h_\mu(S) \geq h(\mu) - \log 2 = h(m) - \sum_{e \in E} \nu_e \log \nu_e - \log 2$$

Hence, by combining the last two displayed formulas and using Theorem 2.5 we obtain:

**Corollary 4.5.** In the setting of 4.3, for $\mu$-almost every pair $(\lambda, \omega) \in \Lambda \times E^\infty$, the pointwise dimension of $\mu_\lambda \circ \pi_\lambda^{-1}$ satisfies the following estimates:

$$\frac{h(m) - \sum_{e \in E} \nu_e \log \nu_e - \log 2}{-\log \lambda - \sum_{e=(n,i) \in E} \nu_e \log r'_n} \leq d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) \leq \frac{h(m) - \sum_{e \in E} \nu_e \log \nu_e}{-\log \lambda - \sum_{e=(n,i) \in E} \nu_e \log r'_n}$$

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