Bell’s theorem for trajectories

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In classical theory, the trajectory of a particle is entirely predetermined by the complete set of initial conditions via dynamical laws. Based on this, we formulate a no-go theorem for the dynamics of classical particles, i.e. a Bell’s inequality for trajectories, and discuss its possible violation in a quantum scenario. A trajectory, however, is not an outcome of a quantum measurement, in the sense that there is no observable associated to it, and thus there is no “direct” experimental test of the Bell’s inequality for trajectories. Nevertheless, we show how to overcome this problem by considering a special case of our generic inequality that can be experimentally tested point-by-point in time. Such inequality is indeed violated by quantum mechanics and the violation pertains during an entire interval of time and not just at a particular singular instant. We interpret the violation to imply that trajectories (or at least pieces thereof) cannot exist predetermined, within a local-realistic theory.

I. INTRODUCTION

The essential feature of classical mechanics is that successive positions of a point-like particle constitute a continuous trajectory that is uniquely defined by dynamical equations together with the appropriate set of initial conditions. Bell’s theorem [1], on the other hand, demonstrates that quantum theory is incompatible with the outcomes of measurements being predetermined. In fact, the violation of Bell’s inequalities guarantees that there cannot exist any local and deterministic classical model that accounts for the observed statistics. A typical Bell’s scenario features two distant (space-like separated) observers, Alice and Bob, who each performs local measurements on their respective system (these two systems may have interacted in the past). It is assumed that each of them freely and independently picks measurement settings (or inputs), a and b for Alice and Bob, and obtain outcomes α and β, respectively. The assumption of “local realism” means that the probability distribution of the respective local outcomes are conditionally independent (i.e. locally factorable), given that one takes into account all the possible “hidden variables” λ, i.e.

\[ p(\alpha, \beta | a, b) = \int_\Lambda d\lambda \, q(\lambda) p(\alpha | a, \lambda) p(\beta | b, \lambda). \] (1)

The mutual dependence between measurement outcomes is solely due to the lack of experimental control (lack of knowledge) of the full set of parameters λ ∈ Λ, which are distributed according to some distribution q(λ). This form of factorization can be used to derive no-go theorems in the form of inequalities (known as Bell’s inequalities) which put strict bound on possible statistics of measurable quantities. As such, within a theory that predicts a violation of Bell’s inequalities, local realism cannot be upheld. Quantum mechanics is indeed an example of such a theory and it is by now a corroborated experimental result that Bell’s inequalities can be violated using entangled quantum states [2-4].

As a concrete example, consider the simplest Bell’s inequality (known as Clauser-Horne-Shimony-Holt inequality, after the physicists who put it forward [5]), where both inputs and outputs are binary variables, i.e. \( a, b \in \{0,1\} \) and \( \alpha, \beta \in \{-1,1\} \). Then the condition of local realism [1] implies the inequality

\[ S := |\langle \alpha \beta \rangle_{00} + \langle \alpha \beta \rangle_{01} + \langle \alpha \beta \rangle_{10} - \langle \alpha \beta \rangle_{11} | \leq 2, \] (2)

where \( \langle \alpha \beta \rangle_{ab} = \sum_{\alpha,\beta} p(\alpha,\beta | a, b) \alpha \beta \) are the correlations between measurement outcomes, given the inputs. Quantum physics allows to violate this inequality, reaching a maximal value of \( S = 2\sqrt{2} \) (known as Tsirelson’s bound [6]).

Based on the same argument of local realism, we put forward a no-go theorem aimed at ruling out the existence of local classical dynamics, i.e. a Bell’s inequality for trajectories. A violation of such an inequality would preclude the possibility of accommodating predefined trajectories of particles in any empirically adequate theory (possibly beyond quantum mechanics).

However, although the formulation of such a Bell’s theorem is relatively simple, its empirical confirmation is more challenging. The main problem is that in the quantum theory there is no observable associated to the trajectory of a particle. Hence, there is no straightforward means on how to directly measure it. Similar problems

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have been studied in context of consistent histories \[7\,8\] and, more specifically, of entangled histories \[9\,10\]. It seems that the only experimentally accessible object is a single point of the trajectory, obtained by measuring particle’s position at a given instant of time. From this perspective, the trajectory can be seen as a sequence of such points during an interval of time. This observation will serve us to derive a whole class of experimentally accessible Bell’s inequalities for trajectories, where the actual test is done in a point-by-point fashion. This “local in time” violation is indeed obtained in quantum mechanics. In this case, the local settings are, for each party, encoded in choices (\(x_0, p_0\)), thus disproving their local-realistic description.

We now introduce the operation of averaging over the hidden parameters \(a\) and \(b\) of the two trajectories, given the inputs \(a\) and \(b\), by averaging over the distribution of trajectories and consider subadditive functionals, not necessarily symmetric ones; in that case it reads

\[
S := \sum_{a, b=0}^1 (-1)^{ab} (F)_{ab} \geq 0, \quad (5)
\]

We can generalize this inequality to include arbitrary subadditive functionals, not necessarily symmetric ones; in that case it reads

\[
S = \sum_{a, b=0}^1 (1-a)(1-b)(X - 0) \geq 0.
\]

II. BELL’S INEQUALITY FOR TRAJECTORIES

In order to derive Bell’s inequality for trajectories, we consider the standard Bell’s scenario. Two parties, Alice (A) and Bob (B), reside in two distant laboratories and each has a particle that they can manipulate locally. Alice and Bob can freely and independently choose binary settings (inputs), respectively labelled by \(a\) and \(b\). They then locally encode their choices by specifying a set of dynamical parameters to govern the dynamics of their respective particles. For example, A and B can encode their choices into potentials, \(V^{(a)}(x)\) and \(V^{(b)}(y)\), as shown in Figure 1. For simplicity, we shall derive our Bell’s inequality for 1D trajectories, however a generalization to higher dimensions is straightforward.

Figure 1. Setting up local potentials. Alice (A) and Bob (B) locally encode their freely chosen binary inputs \((a,b)\) in potentials \(V^{(a)}(x)\) and \(V^{(b)}(y)\) that will govern the dynamics of their particles. Each particle is constrained to move along a line, parameterized by the \(x\)-coordinate for Alice’s particle and the \(y\)-coordinate for Bob’s.

Let us start by describing the evolution of one particle only, say Alice’s. As already recalled, in classical physics the trajectory of a particle is entirely defined by the complete set of initial conditions and dynamical laws. Suppose now that Alice, in her laboratory, has a control over a certain set of parameters \(a\) (which play the role of measurement settings in the standard Bell’s experiments), and conducts an experiment to determine the trajectory of the particle. In a realistic scenario, however, Alice will in general lack control over some other relevant “hidden” parameters \(\lambda_A\), e.g. controllable parameters \(a\) could specify the Hamiltonian of the system and \(\lambda_A\) could refer to the uncontrollable initial conditions \((x_0, p_0)\). Yet, it is necessary to specify the full set of parameters \((a, \lambda_A)\) in order to deterministically characterize the unique trajectory \(X^{(a)}\) of the particle. Therefore, the probability (density) to get a particular trajectory \(X\) in the time interval \([0, \tau]\) given the setting \(a\) reads

\[
p_{A}[X|a] = \int d\lambda_A \mu_A(\lambda_A) \prod_{t=0}^{\tau} \delta[(X(t) - X^{(a)}_{A}(t))],
\]

with some probability distribution \(\mu_A\) over all possible values of the hidden variables \(\lambda_A\); and we have a similar expression in Bob’s case.

Coming back to a bipartite scenario, one can, in a similar fashion, construct the joint probability distribution of the two trajectories, given the inputs \(a, b\), by averaging over the hidden parameters \(\lambda_A\) and \(\lambda_B\) for A and B respectively. If the evolutions of the two particles are to be governed by their respective local Hamiltonians only (assumption of local realism), the joint conditional distribution is of the form

\[
p[X, Y|a, b] = \int d\lambda_A d\lambda_B \mu(\lambda_A, \lambda_B)
\times \prod_{t=0}^{\tau} \delta[(X(t) - X^{(a)}_{A}(t))\delta[Y(t) - Y^{(b)}_{B}(t)]], \quad (3)
\]

where \(\mu(\lambda_A, \lambda_B)\) is the joint distribution of the hidden parameters. We now introduce the operation of averaging over the distribution of trajectories and consider functionals that take trajectories as inputs. For some functional of the difference of two trajectories, \(F[X - Y]\), its mean value is given by

\[
\langle F \rangle_{ab} = \int DXDY \ p[X, Y|a, b]F[X - Y]
= \int d\lambda_A d\lambda_B \mu(\lambda_A, \lambda_B)F[X^{(a)}_{A} - Y^{(b)}_{B}], \quad (4)
\]

which directly follows from local form of the conditional probability distribution provided in \(3\).

Consider now any symmetric (i.e. \(F[X] = F[-X]\)) and subadditive (i.e. \(F[X + Y] \leq F[X] + F[Y]\)) functional. One can write the following general Bell’s inequality\[1\]

\[
S := \sum_{a, b=0}^1 (1-a)(1-b)\langle F \rangle_{ab} \geq 0,
\]

\[1\] We can generalize this inequality to include arbitrary subadditive functionals, not necessarily symmetric ones; in that case it reads

\[
S = \sum_{a, b=0}^1 (1-a)(1-b)(X - 0) \geq 0.
\]
which follows directly from the triangle inequality (see Appendix A). However, in order to evaluate the averages entering Eq. (5), we would need the entire trajectories $X$ and $Y$ as outcomes of measurements, which is problematic in quantum theory, because trajectories are not observables (in a strict mathematical sense). This is an obstacle, even in principle, to directly test our Bell’s inequality. However, it is reasonable to expect violation in quantum setting, at least in some form.

To make our Bell’s inequality testable, we provide an operational meaning to these trajectory measurements in the following sense. Suppose $F[X - Y] = \int_0^\tau dt f[X(t) - Y(t)]$, where $f$ is a symmetric and subadditive function, i.e. $f(x - y) = f(y - x)$ and $f(x + y) \leq f(x) + f(y)$ (such as the norm distance $f(x - y) = |x - y|$). Clearly, this property induces the subadditivity of $F$, and thus Eq. (5) holds. The expression for averages in Eq. (5) now reads

$$\langle F \rangle_{ab} = \int_0^\tau dt \langle f(t) \rangle_{ab}, \quad (6)$$

with $\langle f(t) \rangle_{ab} = \int d\lambda A \lambda B \mu(\lambda_A, \lambda_B)f[X_{\lambda_A}^{(a)}(t) - Y_{\lambda_B}^{(b)}(t)]$. Finally, the Bell’s inequality (5) becomes

$$S = \int_0^\tau dt S(t) \geq 0, \quad (7)$$

where we introduced time dependent Bell’s parameter $S(t) := \sum_{a,b=0}^1 (-1)^{ab} \langle f(t) \rangle_{ab}$, satisfying “local in time” inequality $S(t) \geq 0$ for every $t$ (assuming local realism). This is actually a continuous family of Bell-like inequalities for the coordinates $x$ and $y$ of the pair of particles for each particular instant of time. The quantity $S$ is now experimentally testable, for it can be evaluated from point-by-point measurements of the Bell’s parameter $S(t)$ in time. In this view, the expression (7) is understood as a Bell’s inequality for trajectories, because the violation of the inequality $S(t) \geq 0$ for some finite continuous interval of time $[t_i, t_f]$ during the evolution of the particles (not necessarily during the whole interval $[0, \tau]$) would rule out the possibility for the particles to have predetermined trajectories (more precisely, the pieces thereof that correspond to the interval $[t_i, t_f]$ during which $S(t) < 0$), within any local-realistic theory. In fact, if $S(t) < 0$ during some interval $[t_i, t_f] \subset [0, \tau]$, then we necessarily have $S < 0$ at least for that interval of time, and the corresponding pieces of trajectories cannot be accounted for by any local-realistic theory.

In what follows, we demonstrate, by using a simple dynamical model, that quantum mechanics can indeed allow this kind of violation.

III. QUANTUM SCENARIO

Let us suppose that Alice and Bob share a pair of quantum particles, both of mass $M$, prepared in some pure initial state $|\Psi_0\rangle$. Alice and Bob then encode their freely-chosen inputs $a$ and $b$ in the potentials that will govern the dynamics of their particles. For a given pair of inputs $(a, b)$, we have a pair of Hamiltonians

$$\hat{H}_A^{(a)} = \frac{\hat{p}_A^2}{2M} + \hat{V}^{(a)}(\hat{x}), \quad \hat{H}_B^{(b)} = \frac{\hat{p}_B^2}{2M} + \hat{V}^{(b)}(\hat{y}). \quad (8)$$

Since there is no interaction between the particles during the evolution, their initial state evolves, in the Schrodinger picture, as $|\Psi^{(a,b)}(t)\rangle = U_A^{(a)}(t) \otimes U_B^{(b)}(t)|\Psi_0\rangle$, where $U_A^{(a)}(t) = \exp \left( -\frac{i}{\hbar} \hat{H}_A^{(a)} t \right)$. The “quantum” Bell’s parameter thus reads

$$\hat{S}_{QM}(t) = \sum_{a,b=0}^1 (-1)^{ab} \langle \Psi^{(a,b)}(t) | \hat{f}(\hat{x} - \hat{y}) | \Psi^{(a,b)}(t) \rangle. \quad (9)$$

Alternatively, and more appropriately for our purpose, we can switch to the Heisenberg picture. Therefore, we consider the time evolution of $\hat{f}(\hat{x} - \hat{y})$ for a given pair of inputs $(a, b)$,

$$\hat{f}_{ab}(t) = U_B^{(b)}(t) U_A^{(a)}(t) \hat{f}(\hat{x} - \hat{y}) U_A^{(a)}(t) U_B^{(b)}(t). \quad (10)$$

In this picture, the Bell’s parameter reads

$$\hat{S}_{QM}(t) = \langle \Psi_0 | \hat{S}_{QM}(t) | \Psi_0 \rangle = \sum_{a,b=0}^1 (-1)^{ab} \langle \Psi_0 | \hat{f}_{ab}(t) | \Psi_0 \rangle. \quad (11)$$

If for some particular instant of time $t = T$ there exists an eigenvalue of the operator $\hat{S}_{QM}(T)$, as defined in (11), that is smaller than zero, then we can choose the corresponding eigenfunction to be the initial state $|\Psi_0\rangle$ and thus assure that $\hat{S}_{QM}(T) < 0$. From the continuity of the Bell’s parameter as a function of time, we expect that the local-realistic inequality $S(t) \geq 0$ must also be violated by $\hat{S}_{QM}(t)$ in some neighborhood of $t = T$, i.e. for some finite, continuous interval of time around $T$.

The problem is thus reduced to finding an appropriate initial state $|\Psi_0\rangle$ together with the potentials $\hat{V}^{(a)}(x)$ and $\hat{V}^{(b)}(y)$ such that $\hat{S}_{QM}(t) < 0$ during some finite, continuous interval of time, thus leading to the violation of (7), at least during that particular interval.

IV. EXAMPLE OF VIOLATION

To set up a concrete dynamical model that allows a violation of the Bell’s inequality for trajectories, we use a pair of quantum harmonic oscillators. Alice and Bob locally encode their inputs $(a, b)$ by setting up harmonic potentials $\hat{V}_A^{(a)} = \frac{M\omega_A^2}{2} \hat{x}^2$ and $\hat{V}_B^{(b)} = \frac{M\omega_B^2}{2} \hat{y}^2$. More precisely, the inputs are encoded by tuning the frequency parameters $(\omega_A, \omega_B)$ of the potentials (see Figure 2).

As an ansatz for the initial state $|\Psi_0\rangle$, we consider a general state of two quantum harmonic oscillators,
both having frequency $\Omega$, that belongs to the subspace spanned by the basis \(\{|m\rangle_A |n\rangle_B \mid m, n = 0, 1, \ldots, 8\}\), i.e.

\[
|\Psi_0\rangle = \sum_{m,n=0}^{8} c_{mn} |m\rangle_A |n\rangle_B ,
\]

with some a priori undetermined amplitudes $c_{mn}$. We could, of course, take a more general ansatz, but in order to see the violation it turns out to be enough to consider only the first nine harmonics for each oscillator. Note that $|m\rangle_A$ and $|n\rangle_B$ are energy eigenstates of the respective oscillators for the frequency $\Omega$; they need not be eigenstates for the evolution operators $U_A^{(s)}$ and $U_B^{(s)}$ because these operators depend on the choice of the parameters $\omega_a$ and $\omega_b$. The relevant parameters of the system, $M$ and $\Omega$, provide the units of time and length. The unit of time is simply $\Omega^{-1}$, and the unit of length $(\hbar/M\Omega)^{1/2}$. Together, they set the scale of the problem. For example, if we take the mass of an electron $M \approx 10^{-30}$ kg and frequency $\Omega \approx 10^8$ rad/s, the relevant length scale is $(\hbar/M\Omega)^{1/2} \approx 1 \mu$m.

In order to find a simple case of the violation of the Bell’s inequality for trajectories, we consider here, among all symmetric and subadditive functions, the absolute value (Euclidean distance) $d(x - y) = |x - y|$. In coordinate representation, the corresponding operator satisfies $\langle x, y | d(\hat{x} - \hat{y}) | \Psi \rangle = |x - y| \Psi(x, y)$. In this particular case, the time dependent Bell’s parameter reads

\[
S_{QM}(t) = \sum_{a,b=0}^{1} (-1)^{ab} \langle \Psi_0 | \hat{d}_{ab}(t) | \Psi_0 \rangle ,
\]

where $\hat{d}_{ab}(t)$ represents the specific choice of the operator $\hat{f}_{ab}(t)$ defined in Eq. \[10\]. The matrix elements of the Bell’s operator $S_{QM}(t)$, generally defined in Eq. \[11\], in the reduced basis \(\{|m\rangle_A |n\rangle_B \mid m, n = 0, 1, \ldots, 8\}\) are

\[
\sum_{a,b=0}^{1} (-1)^{ab} \langle m'|n\rangle | \hat{d}_{ab}(t) | m\rangle |n\rangle .
\]

One only has to find a particular instant of time $t = T$ for which there exists an eigenvalue of $S_{QM}(T)$ that is smaller than zero (classical bound). The eigenstate of $S_{QM}(T)$ for this particular eigenvalue will be our initial state $|\Psi_0\rangle$. From the continuity of $S_{QM}(t)$ follows that $S_{QM}(t) < 0$ also in some interval $[t_i, t_f]$ around $t = T$, implying the violation of the Bell’s inequality for trajectories at least during that particular interval of time.

To illustrate the above-described procedure, we provide, in Figure \[3\] a graphical representation of the evolution of Bell’s parameter $S_{QM}(t)$ in time. The example of violation that we provided is not particularly strong, but this is a proof of principle that quantum mechanics allows such a violation. Details of the calculation are given in Appendix B. It would be desirable for future work to find further physical examples that lead to stronger violations, perhaps by choosing a different ansatz for the initial state or considering some other functional of the trajectories.

**V. CONCLUSIONS AND OUTLOOK**

In this paper, we have derived a *Bell’s theorem for trajectories*. Were trajectories fully predetermined – as it is assumed in classical physics – their pieces would clearly be predetermined, too. We have shown that quantum mechanics precludes this possibility, at least for some finite, continuous interval of time during the evolution. As a matter of fact, quantum mechanics (by means of Bell’s theorem) gave us good reasons to question the existence of predetermined values of physical observables. We showed that this argument has even more severe consequences, for it can be extended to prima facie unobservable quantities: trajectories of particles in general do not exist predetermined.
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Appendix A

We prove that expression [5] holds for any symmetric, subadditive functional, i.e. $F[X] = F[-X]$ and $F[X + Y] \leq F[X] + F[Y]$. Let us start by explicitly writing the averages appearing in the inequality [5], by plugging Eq. (4) in it:

$$S = \int d\lambda_A d\lambda_B \mu(\lambda_A, \lambda_B) \sum_{a,b=0} \frac{1}{2} (-1)^{ab} F[X^{(a)}_{\lambda_A} - Y^{(b)}_{\lambda_B}].$$

(A1)

To prove that $S \geq 0$, it is sufficient to demonstrate that the sum over $a, b$ is non-negative (since $\mu \geq 0$, being a distribution), i.e.

$$\sum_{a,b=0} \frac{1}{2} (-1)^{ab} F[X^{(a)}_{\lambda_A} - Y^{(b)}_{\lambda_B}] = F[X^{(0)}_{\lambda_A} - Y^{(0)}_{\lambda_B}] + F[X^{(0)}_{\lambda_A} - Y^{(1)}_{\lambda_B}] + F[X^{(1)}_{\lambda_A} - Y^{(0)}_{\lambda_B}] - F[X^{(1)}_{\lambda_A} - Y^{(1)}_{\lambda_B}] \geq 0.$$  

(A2)

In order to show this, let us relabel the arguments of the functional in the previous expression as $Z_{00}$, $Z_{01}$, $Z_{10}$ and $Z_{11}$, respectively, from the left to the right. (For example, $Z_{00} := X^{(0)}_{\lambda_A} - Y^{(0)}_{\lambda_B}$.) In this way, the inequality (A2) reads

$$F[Z_{11}] \leq F[Z_{00}] + F[Z_{01}] + F[Z_{10}].$$

(A3)

The symmetry of $F$ ensures $F[Z_{00}] = F[-Z_{00}]$, which together with the subadditivity completes the proof:

$$F[Z_{11}] = F[-Z_{00} + Z_{01} + Z_{10}] \leq F[-Z_{00}] + F[Z_{01}] + F[Z_{10}] = F[Z_{00}] + F[Z_{01}] + F[Z_{10}].$$

(A4)

Appendix B

Here we present, in some more detail, the procedure of finding the initial state $|\Psi_0\rangle$ that ensures a violation of the inequality $S(t) > 0$ at a particular instant of time $t = T$, which immediately extends to some finite continuous interval $[t_i, t_f]$ such that $T \in [t_i, t_f]$, due to continuity of $S(t)$. For that, we make a transition to the Heisenberg picture.

Under the action of 1D harmonic oscillator Hamiltonian, with some generic frequency $\omega$, the coordinate and momentum operators for Alice’s particle, $\hat{x}(t)$ and $\hat{p}_x(t)$ (and likewise $\hat{y}(t)$ and $\hat{p}_y(t)$ for Bob’s particle), evolve in time
according to the Heisenberg’s equations:

\[
\begin{align*}
\dot{x}(t) &= x(0) \cos(\omega t) + \frac{\hat{p}_x(0)}{M\omega} \sin(\omega t), \\
\dot{\hat{p}}_x(t) &= \hat{p}_x(0) \cos(\omega t) - M\omega \hat{x}(0) \sin(\omega t).
\end{align*}
\] (B1) (B2)

For \(\omega t = 2\pi\), this “rotation” reduces to the identity transformation, whereas for \(\omega t = \pi/2\) we get the interchange of the operators – the coordinate operator becomes \(\hat{p}(0)/M\omega\) and the momentum operator becomes \(-M\omega \hat{x}(0)\), because the transformation is just an ordinary Fourier transform,

\[
\hat{F}_{A/B} = e^{-i\hat{N}_{A/B} \pi/2},
\] (B3)

where \(\hat{N}_{A/B}\) is the number operator.

The evolution of the operator \(\hat{d}(\hat{x} - \hat{y})\) for a given pair of inputs \((a, b)\) is

\[
\hat{d}_{ab}(t) = e^{i\hat{H}_a^{(a)}t} e^{i\hat{H}_b^{(b)}t} \hat{d}(\hat{x} - \hat{y}) e^{-i\hat{H}_a^{(a)}t} e^{-i\hat{H}_b^{(b)}t},
\] (B4)

with \(\hat{H}_a^{(a/b)} = \hbar \omega_{a/b}(\hat{N}_{a/b} + 1/2)\).

For a given frequency \(\Omega\), let us fix a particular instant of time, say \(t = T = (\pi/2)\Omega^{-1}\), at which we want to obtain the maximal violation. Alice and Bob agree to set their frequencies to \(\Omega\) for the input value 1, and to set them to \(4\Omega\) if for the input value 0. This leads to four possible cases:

1. For \((a, b) = (0, 0)\) we have \((\omega_a, \omega_b) = (4\Omega, 4\Omega)\) and hence \(\omega_aT = \omega_bT = 2\pi\). Therefore,

\[
\hat{d}_{00}(T) = \hat{d}(\hat{x} - \hat{y}).
\] (B5)

2. For \((a, b) = (0, 1)\) we have \((\omega_a, \omega_b) = (4\Omega, \Omega)\) and hence \(\omega_aT = 2\pi\) and \(\omega_bT = \pi/2\). Therefore,

\[
\hat{d}_{01}(T) = \hat{F}_B^\dagger \hat{d}(\hat{x} - \hat{y}) \hat{F}_B.
\] (B6)

3. For \((a, b) = (1, 0)\) we have \((\omega_a, \omega_b) = (\Omega, 4\Omega)\) and hence \(\omega_aT = \pi/2\) and \(\omega_bT = 2\pi\). Therefore,

\[
\hat{d}_{10}(T) = \hat{F}_A \hat{d}(\hat{x} - \hat{y}) \hat{F}_A.
\] (B7)

4. For \((a, b) = (1, 1)\) we have \((\omega_a, \omega_b) = (\Omega, \Omega)\) and hence \(\omega_aT = \omega_bT = \pi/2\). Therefore,

\[
\hat{d}_{11}(T) = \hat{F}_B^\dagger \hat{F}_A \hat{d}(\hat{x} - \hat{y}) \hat{F}_A \hat{F}_B.
\] (B8)

We assume that the initial state \(|\Psi_0\rangle\) belongs to the subspace spanned by \(|m\rangle_A |n\rangle_B\) for \(m, n = 0, 1, \ldots, 8\), where \(|m\rangle_A\) and \(|n\rangle_B\) are energy eigenstates for frequency \(\Omega\). In this basis, operators \(\hat{F}_A\) and \(\hat{F}_B\) are represented by the following matrices:

\[
\begin{align*}
F_A &= \begin{bmatrix}
1 & -i \\
\vdots & \ddots \\
(-i)^8 & \ddots & 1
\end{bmatrix} \otimes \mathbb{I}_{9 \times 9}, & F_B &= \mathbb{I}_{9 \times 9} \otimes \begin{bmatrix}
1 & -i \\
\vdots & \ddots \\
(-i)^8 & \ddots & 1
\end{bmatrix}.
\end{align*}
\] (B9)

Finally, the Bell’s operator at \(t = T\) is

\[
\hat{S}_{QM}(T) = \sum_{a,b=0}^1 (-1)^{ab} \hat{d}_{ab}(T) = \hat{d}(\hat{x} - \hat{y}) + \hat{F}_B^\dagger \hat{d}(\hat{x} - \hat{y}) \hat{F}_B + \hat{F}_A \hat{d}(\hat{x} - \hat{y}) \hat{F}_A - \hat{F}_B^\dagger \hat{F}_A^\dagger \hat{d}(\hat{x} - \hat{y}) \hat{F}_A \hat{F}_B,
\] (B10)

and it is represented by a certain \(81 \times 81\) matrix. By solving the eigenvalue problem of this matrix (using Wolfram Mathematica) we find that its spectrum has a minimal negative eigenvalue \(\xi_- \approx -0.034\), the unit of length being \((\hbar/M\Omega)^{1/2}\), which depends on the parameters \(M\) and \(\Omega\).
If we now identify the initial state $|\Psi_0\rangle$ with the (entangled) eigenstate $|\xi_-\rangle$, our procedure ensures a violation at $t = T$, i.e.

$$S_{QM}(T) = \sum_{a,b=0}^{1} (-1)^{ab} \langle \Psi^{(a,b)}_{AB}(T) | \hat{d}(\hat{x} - \hat{y}) | \Psi^{(a,b)}_{AB}(T) \rangle < 0. \quad (B11)$$

Since $S_{QM}(t)$ is a continuous function of time, we expect also to have $S_{QM}(t) < 0$ in some finite continuous interval $[t_i, t_f]$ such that $T \in [t_i, t_f]$, hence $S = \int_{t_i}^{t_f} dt S_{QM}(t) < 0$. Having selected the appropriate initial state, we can propagate it from $t = 0$ to any $t > 0$, for all four instances of $(a, b)$, and calculate analytically the whole function $S_{QM}(t)$; in particular, we can find $t_i$ and $t_f$. We conclude that the pieces of the trajectories of the particle’s that correspond to the interval $[t_i, t_f]$ cannot be accounted for by any theory that assumes local realism. In our case, $T = \frac{\pi}{2} \Omega^{-1}$, $t_i \approx 1.485 \Omega^{-1}$ and $t_f \approx 1.665 \Omega^{-1}$, see Figure 3.