RIGIDITY OF THE $L^p$-NORM OF THE POISSON BRACKET ON SURFACES

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ABSTRACT. For a symplectic manifold $(M,\omega)$, let $\{\cdot,\cdot\}$ be the corresponding Poisson bracket. In this note we prove that the functional $(F,G) \mapsto \|\{F,G\}\|_{L^p(M)}$ is lower-semicontinuous with respect to the $C^0$-norm on $C^\infty_c(M)$ when $\dim M = 2$ and $p < \infty$, extending previous rigidity results for $p = \infty$ in arbitrary dimension.

1. INTRODUCTION AND MAIN RESULT

One of the fascinating manifestations of rigidity in symplectic topology is the unexpected robust behavior of the Poisson bracket with respect to the $C^0$-norm on the space of smooth functions, discovered by Cardin–Viterbo [3]. To state their seminal result, let $(M,\omega)$ be a symplectic manifold without boundary, and let us endow the space $C^\infty_c(M)$ of smooth compactly supported functions on $M$ with the topology induced by the supremum norm $\|\cdot\|_{C^0}$. We write $C^0 \xrightarrow{\cdot} \text{C}^0 \xrightarrow{\cdot}$ to indicate convergence with respect to this topology.

The Poisson bracket of $F,G \in C^\infty(M)$ is the function
$$\{F,G\} = -\omega(X_F,X_G) = dF(X_G),$$
where for $H \in C^\infty(M)$ its Hamiltonian vector field $X_H$ is defined by $\omega(X_H,\cdot) = -dH$.

**Theorem 1.1** (Cardin–Viterbo [3]). Let $N$ be $\mathbb{R}^n$ or a closed manifold, and assume that $M = T^*N$ and $\omega$ is the canonical symplectic form. Let $F,G \in C^\infty_c(M)$ be such that $\{F,G\} \neq 0$. Then
$$\lim \inf_{\substack{F \xrightarrow{C^0} F,\ G \xrightarrow{C^0} G}} \|\{F,G\}\|_{C^0} > 0.$$
This means that if two functions do not Poisson commute, it is impossible to approximate them, in the $C^0$ sense, by Poisson commuting, or even asymptotically commuting, functions. This behavior is surprising because the Poisson bracket is defined in terms of the first derivatives of the functions and thus a priori it is unknown how it changes under $C^0$ perturbations. For surfaces, a stronger form of this statement was proved in [8]:

**Theorem 1.2.** Assume $\dim M = 2$. Then for $F, G \in C^\infty_c(M)$ the functional $\|\{\cdot, \cdot\}\|_{C^0}$ is lower-semicontinuous with respect to $C^0$-norm, meaning

$$\liminf_{\mathcal{F} \xrightarrow{C^0} F, \mathcal{G} \xrightarrow{C^0} G} \|\{\mathcal{F}, \mathcal{G}\}\|_{C^0} = \|\{F, G\}\|_{C^0}.$$ 

This result was proved using methods of classical analysis in dimension two. It was later generalized, using methods of “hard” symplectic topology, including the Hofer metric and the energy-capacity inequality, to arbitrary dimension:

**Theorem 1.3 ([4], [1]).** For $M$ of arbitrary dimension and any $F, G \in C^\infty_c(M)$ we have

$$\liminf_{\mathcal{F} \xrightarrow{C^0} F, \mathcal{G} \xrightarrow{C^0} G} \|\{\mathcal{F}, \mathcal{G}\}\|_{C^0} = \|\{F, G\}\|_{C^0}.$$ 

Our main result in this note is the following rigidity phenomenon in dimension two, proved using a refinement of the technique from [8]:

**Theorem 1.4.** Assume $\dim M = 2$ and $p \in [1, \infty)$. Then for $F, G \in C^\infty_c(M)$ we have

$$\liminf_{\mathcal{F} \xrightarrow{C^0} F, \mathcal{G} \xrightarrow{C^0} G} \|\{\mathcal{F}, \mathcal{G}\}\|_{L^p(M)} = \|\{F, G\}\|_{L^p(M)}.$$ 

Here and in the rest of the note we denote by $\|H\|_{L^p(X)}$ the $L^p$-norm, with respect to the measure induced by $\omega$, of a function $H$ defined on a measurable subset $X \subset M$.

Whether this behavior persists in higher dimension is currently unknown. Therefore we ask the following question.

**Question 1.5.** Is the functional $(F, G) \mapsto \|\{F, G\}\|_{L^p(M)}$ lower semi-continuous for $M$ of arbitrary dimension and finite $p$?

In view of the results in [1], it is also natural to ask the following.

**Question 1.6.** What is the modulus of semi-continuity of the functional $(F, G) \mapsto \|\{F, G\}\|_{L^p(M)}$? Is there a constant $\kappa > 0$ such that

$$\inf_{\|\mathcal{F} - F\|_{C^0}, \|\mathcal{G} - G\|_{C^0} \leq \delta} \|\{\mathcal{F}, \mathcal{G}\}\|_{L^p(M)} \geq \|\{F, G\}\|_{L^p(M)} - \text{const}(F, G) \cdot \delta^\kappa?$$

The rigid behavior with respect to the $C^0$-norm should be contrasted with the following result.

**Theorem 1.7 ([7]).** Let $M$ have arbitrary dimension $2n$, let $q \in [1, \infty)$, and let $F, G \in C^\infty_c(M)$. Then for any $\varepsilon > 0$ and a compact submanifold with boundary

\[\]
$C \subset M$ of dimension $2n$, whose interior contains $\text{supp} \, F \cup \text{supp} \, G$, there exist $\widetilde{F}, \widetilde{G} \in C^\infty_c(M)$ supported in $C$ such that

$$\|\widetilde{F} - F\|_{C^0} < \varepsilon, \quad \|\widetilde{G} - G\|_{L^p(M)} < \varepsilon^{1/q}, \quad \text{and} \quad \{\widetilde{F}, \widetilde{G}\} \equiv 0.$$  

In particular, for any $p \in [1, \infty]$,

$$\liminf_{\widetilde{F} \to F, \widetilde{G} \to G} \|\{\widetilde{F}, \widetilde{G}\}\|_{L^p(M)} = \liminf_{\widetilde{F} \to F, \widetilde{G} \to G} \|\{\widetilde{F}, \widetilde{G}\}\|_{L^p(M)} = 0.$$

Here we use the fact that $\|H\|_{L^p(M)} \leq (\int_C \omega^n)^{1/q} \cdot \|H\|_{C^0}$ for $H \in C^\infty(M)$ with $\text{supp} \, H \subset C$. This means that the $L^p$-norm of the Poisson bracket becomes flexible if we take the $L^q$-topology on $C^\infty_c(M)$ for finite $q$.

For the sake of completeness, we provide a proof of the theorem in the next section.

**Remark 1.8.** Note that for continuous functions the $L^\infty$-norm and the $C^0$-norm coincide.

2. Proofs

**Proof** (of Theorem 1.4). Let us give an overview of the proof before passing to the details. The actual logical order of the proof is somewhat different from this summary.

We define the map $\Phi : M \to \mathbb{R}^2$ by

$$\Phi(z) = (F(z), G(z)).$$

The main point is that since $\dim M = 2$, the Poisson bracket $\{F, G\}$ is related to $\Phi$ via

$$\Phi^*(dx \wedge dy) = dF \wedge dG = -\{F, G\} \omega,$$

where $(x, y)$ are the coordinate functions on $\mathbb{R}^2$. We see that a point $z \in M$ is regular for $\Phi$ if and only if $\{F, G\}(z) \neq 0$. We let $U \subset \mathbb{R}^2$ be the set of regular values of $\Phi$ in $\text{im} \, \Phi$.

Consider now the subset $K_n \subset U$ comprised of squares of size $\frac{1}{n}$ with vertices in the grid $\frac{1}{n} \mathbb{Z} \times \frac{1}{n} \mathbb{Z}$ with $n \in \mathbb{N}$ large so that $\|\{F, G\}\|_{L^p(\Phi^{-1}(K_n))}$ is close to $\|\{F, G\}\|_{L^p(M)}$. Next we subdivide each square in $K_n$ into squares of size $\frac{1}{kn}$, $k \in \mathbb{N}$. For such a square $Q$, a connected component $Q' \subset \Phi^{-1}(Q)$, and $k$ large, the oscillation of $\{F, G\}$ over $Q'$ can be made arbitrarily small for all such $Q'$, which allows us to relate the $L^1$- and the $L^p$-norms of $\{F, G\}$ over $Q'$. Then we use the *lower semi-continuity of the $L^1$-norm* to pass to $\|\{\widetilde{F}, \widetilde{G}\}\|_{L^1(Q')}$. Finally the H"older inequality brings us back to $\|\{F, G\}\|_{L^p(Q')}$.  

**Remark 2.1.** We wish to note here that the use of two scales, $\frac{1}{n}$ and $\frac{1}{kn}$, seems to stem from convenience rather than being a reflection of something deeper. We must simultaneously approximate the $L^p$-norm of $\{F, G\}$ and control its oscillation, and this double subdivision is a way to do it.

We now give the details of the proof.

**Remark 2.2.** Since $M$ is assumed to have no boundary, and $\{F, G\}$ has compact support, $\Phi$ is a covering map over $U$. In particular, the lifting property of a covering implies that if $Y \subset U$ is a path-connected simply connected subset, then $\Phi|_{\Phi^{-1}(Y)} : \Phi^{-1}(Y) \to Y$ is a trivial covering, that is, $\Phi^{-1}(Y)$ is a disjoint union of
path components, each one projected homeomorphically onto $Y$ by $\Phi$. If $Y$ is in addition a submanifold with corners, then, since $\Phi$ is smooth, these components are themselves submanifolds with corners, projected in fact diffeomorphically onto $Y$.

For $n \in \mathbb{N}$ let $K_n \subset \mathbb{R}^2$ be the union of squares of the form $\left[ \frac{i}{n}, \frac{i+1}{n} \right] \times \left[ \frac{j}{n}, \frac{j+1}{n} \right]$, where $i, j \in \mathbb{Z}$, contained in $U$. For $k \in \mathbb{N}$ consider a square $Q = \left[ \frac{i}{kn}, \frac{i+1}{kn} \right] \times \left[ \frac{j}{kn}, \frac{j+1}{kn} \right]$, where $i, j \in \mathbb{Z}$, and assume it is contained in $K_n$. By Remark 2.2, $\Phi^{-1}(Q)$ is a disjoint union of connected components, each of which is mapped by $\Phi$ diffeomorphically onto $Q$. See Figure 1. Let $Q_{n,k}$ be the collection of all such connected components for all such $Q$. The next lemma states that the oscillation of $|\{F, G\}|_p$ over the sets in $Q_{n,k}$ can be made arbitrarily small as $k \to \infty$.

**Figure 1.** Illustrating $K_n \subseteq U$ and an element $\Phi(Q') = Q$ in its subdivision.

**Lemma 2.3.** $\lim_{k \to \infty} \max_{Q' \in Q_{n,k}} \text{osc}_{Q'} |\{F, G\}|_p = 0$.

Fix $\varepsilon > 0$ and let $k \in \mathbb{N}$ be such that $\max_{Q' \in Q_{n,k}} \text{osc}_{Q'} |\{F, G\}|_p \leq \varepsilon$. Pick $Q' \in Q_{n,k}$, let $Q = \Phi(Q') \subset \mathbb{R}^2$, and let $i, j \in \mathbb{Z}$ be such that $Q = \left[ \frac{i}{kn}, \frac{i+1}{kn} \right] \times \left[ \frac{j}{kn}, \frac{j+1}{kn} \right]$. For $\delta \in (0, \frac{1}{2kn})$ denote $Q_\delta = \left[ \frac{i}{kn} + \delta, \frac{i+1}{kn} - \delta \right] \times \left[ \frac{j}{kn} + \delta, \frac{j+1}{kn} - \delta \right].$ The following lemma is a quantitative local surjectivity result for $C^0$-perturbations of $\Phi$. Its proof is an almost verbatim repetition of the one of [8, Lemma 3.1] and is omitted.

**Lemma 2.4.** Let $\delta \in (0, \frac{1}{2kn})$ and let $\overline{F}, \overline{G} \in C_c^\infty(M)$ be such that

$$
\|\overline{F} - F\|_{C^0} \leq \delta, \quad \|\overline{G} - G\|_{C^0} \leq \delta.
$$

Define $\overline{\Phi} : M \to \mathbb{R}^2$ by

$$
\overline{\Phi}(z) = (\overline{F}(z), \overline{G}(z)).
$$

Then we have

$$
\overline{\Phi}(Q') \supseteq Q_\delta.
$$

Fix $\delta \in (0, \frac{1}{2kn})$ and $\overline{F}, \overline{G} \in C_c^\infty(M)$ with $\|\overline{F} - F\|_{C^0} \leq \delta, \|\overline{G} - G\|_{C^0} \leq \delta$. Let $q$ be such that $1/p + 1/q = 1$. The Hölder inequality allows us to relate the $L^p$- and the $L^1$-norms of $\{\overline{F}, \overline{G}\}$:

$$
\|\{\overline{F}, \overline{G}\}\|_{L^p(Q')}^p \geq \|\{F, G\}\|_{L^1(Q')}^p \|1\|_{L^q(Q')}^{-q}.
$$

Let us define the function

$$
n_{\overline{\Phi}} : \mathbb{R}^2 \to \mathbb{N} \cup \{0, \infty\},
$$
where \( n_\Phi(u) \) is the number of preimages of \( u \) by the restriction of \( \Phi \) to \( Q' \). Note that by Lemma 2.4 \( n_\Phi(u) \geq 1 \) for every \( u \in Q_s \). The so-called area formula from geometric measure theory [6, Theorem 3.2.3] implies in our case the following identity:

\[
\int_{Q'} |dF \wedge dG| = \int_{\mathbb{R}^2} n_\Phi \, dx \wedge dy,
\]
where for a two-form \( \beta \) on \( M \) we let \( |\beta| \) denote the corresponding density.\(^3\)

Next, we relate the \( L^1 \)-norms of \( \{F, G\} \) and \( \{F, G\} \) over \( Q' \):

\[
\|\{F, G\}\|_{L^1(Q')} = \int_{Q'} |dF \wedge dG| \quad \text{by the definition of \{\cdot, \cdot\} }
= \int_{\mathbb{R}^2} n_\Phi \, dx \wedge dy \quad \text{by the area formula }
\geq \int_{Q_\delta} dx \wedge dy \quad \text{since } n_\Phi|_{Q_\delta} \geq 1.
\]

The last integral is the area of \( Q_\delta \), which equals

\[
\left( \frac{1}{kn} - 2\delta \right)^2 = (1 - 2kn\delta)^2 \text{area}(Q) = (1 - 2kn\delta)^2 \int_Q dx \wedge dy.
\]

We continue:

\[
\|\{F, G\}\|_{L^1(Q')} \geq (1 - 2kn\delta)^2 \int_Q dx \wedge dy
= (1 - 2kn\delta)^2 \int_{\Phi(Q')} |dx \wedge dy|
= (1 - 2kn\delta)^2 \int_{Q'} |\Phi^*(dx \wedge dy)|
= (1 - 2kn\delta)^2 \int_{Q'} |dF \wedge dG|
= (1 - 2kn\delta)^2 \|\{F, G\}\|_{L^1(Q')},
\]
therefore

\[
\|\{F, G\}\|_{L^1(Q')} \geq (1 - 2kn\delta)^2 \|\{F, G\}\|_{L^1(Q')}^p.
\]

Now we relate the \( L^1 \)- and \( L^p \)-norms of \( \{F, G\} \) over \( Q' \). Since \( \text{osc}_{Q'} |\{F, G\}| \leq \epsilon \), we have

\[
\|\{F, G\}\|_{L^1(Q')}^p \geq (\min_{Q'} |\{F, G\}| \int_{Q'} \omega)^p
= \min_{Q'} |\{F, G\}|^p \left( \int_{Q'} \omega \right)^p
\geq (\max_{Q'} |\{F, G\}|^p - \epsilon) \left( \int_{Q'} \omega \right)^p,
\]
therefore, since \( \|1\|_{L^p(Q')} = \left( \int_{Q'} \omega \right)^{-p/q} \):

\[
\|\{F, G\}\|_{L^1(Q')}^p \|1\|_{L^p(Q')}^p \geq (\max_{Q'} |\{F, G\}|^p - \epsilon) \left( \int_{Q'} \omega \right)^{-p/q}.
\]

Since \( p - p/q = 1 \), we obtain

\[
\max_{Q'} |\{F, G\}|^p \left( \int_{Q'} \omega \right)^{p-p/q} = \max_{Q'} \|\{F, G\}\|_{L^p(Q')}^p \int_{Q'} \omega \geq \|\{F, G\}\|_{L^p(Q')}^p.
\]

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3 This can be thought of as the nonnegative measure induced by \( \beta \); if \( f \in C^\infty(M) \) is such that \( \beta = f \omega \), then \( f|\beta| \equiv f |f| \omega \).
thus in total
\[ \| \{F, G\} \|_{L^p}^p \geq 1 - \epsilon \int Q' \omega. \]

Assembling all of the above, we obtain the main estimate
\[ \| \{F, G\} \|_{L^p}^p \geq (1 - 2kn\delta)^{2p} \| \{F, G\} \|_{L^p}^p - \epsilon \int Q' \omega. \]

Note that \( \Phi^{-1}(K_n) \) is the essentially disjoint\(^4\) union of the sets \( Q' \in \mathcal{Q}_{n,k} \) and that \( \| \cdot \|_{L^p} \) is additive with respect to essentially disjoint unions. Thus we have

\[ \| \{F, G\} \|_{L^p(M)}^p \geq \| \{F, G\} \|_{L^p(\Phi^{-1}(K_n))}^p \]

\[ = \sum_{Q' \in \mathcal{Q}_{n,k}} \| \{F, G\} \|_{L^p(Q')}^p \]

\[ \geq (1 - 2kn\delta)^{2p} \sum_{Q' \in \mathcal{Q}_{n,k}} (\| \{F, G\} \|_{L^p(Q')}^p - \epsilon \int Q' \omega) \]

\[ = (1 - 2kn\delta)^{2p} \| \{F, G\} \|_{L^p(\Phi^{-1}(K_n))} - \epsilon \int \Phi^{-1}(K_n) \omega \]

\[ \geq (1 - 2kn\delta)^{2p} \| \{F, G\} \|_{L^p(\Phi^{-1}(K_n))} - \epsilon \cdot \text{area supp}\{F, G\}, \]

where for \( \geq \) we used the main estimate (1) and in the last inequality we used \( \Phi^{-1}(K_n) \subset \text{supp}\{F, G\} \). Taking \( \delta \to 0 \), we see that

\[ \liminf_{T \to \infty} \| \{F, G\} \|_{L^p(M)}^p \geq \| \{F, G\} \|_{L^p(\Phi^{-1}(K_n))} - \epsilon \cdot \text{area supp}\{F, G\}, \]

and since \( \epsilon \) was arbitrary, we have

\[ \liminf_{T \to \infty} \| \{F, G\} \|_{L^p(M)}^p \geq \| \{F, G\} \|_{L^p(\Phi^{-1}(K_n))}^p. \]

It remains to invoke the following lemma, which says that the \( L^p \)-norm of \( \{F, G\} \) can be approximated by looking at the sets \( \Phi^{-1}(K_n) \).

**Lemma 2.5.** \( \sup_{n \in \mathbb{N}} \| \{F, G\} \|_{L^p(\Phi^{-1}(K_n))}^p = \| \{F, G\} \|_{L^p(M)}^p. \)

The proof is thus finished, assuming Lemmas 2.3 and 2.5. \( \square \)

It remains to prove the lemmas. We keep the notations introduced during the proof of Theorem 1.4.

**Proof** (of Lemma 2.3). Let \( C \subset K_n \) be a square entering the definition of \( K_n \).

By Remark 2.2, \( \Phi^{-1}(C) \) is a disjoint union of a finite number of components, each projecting diffeomorphically onto \( C \) by \( \Phi \). Let \( \mathcal{C} \) be the collection of all such connected components for all the squares \( C \subset K_n \). Note that \( \mathcal{C} \) is finite. Pick \( C' \in \mathcal{C} \), let \( C = \Phi(C') \), and let \( P_{C'} : C \to \mathbb{R} \) be the function \( ||\{F, G\}||^p \circ (\Phi|_{C'})^{-1} \).

Since \( \Phi|_{C'} : C' \to C \) is a diffeomorphism, we have for any \( Z \subset C' \):

\[ \text{osc}_Z \| \{F, G\} \|^p = \text{osc}_{\Phi(Z)} P_{C'}. \]

It then follows that it is enough to prove the following for every \( C' \in \mathcal{C} \):

\[ \lim_{k \to \infty} \max_{k' \in \mathcal{Q}_{n,k}, Q' \subset C'} \| \text{osc}_{\Phi(Q')} P_{C'} \| = 0. \]

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\(^4\) A countable union of subsets is essentially disjoint if the intersection of every two subsets has measure zero.
This follows from the fact that $P_C'$ is a smooth function—in particular, it has bounded derivatives—and therefore its oscillation over $\Phi(Q')$ is bounded by a constant times the diameter of $\Phi(Q')$, which is $\frac{\sqrt{2}}{kn}$.

\[\square\]

**Proof (of Lemma 2.5).** Let $X = \text{supp } F \cap \text{supp } G$, $V = \Phi^{-1}(U)$, and $Z = X - V$, which is the subset of $X$ consisting of points lying over singular values of $\Phi$. Let $S, R \subset M$ be the sets of critical and regular points of $\Phi$, respectively. We have the disjoint union

\[Z = (Z \cap S) \cup (Z \cap R).\]

At the beginning of the proof of Theorem 1.4 we noted that $z \in S$ if and only if $\{F, G\}(z) = 0$, therefore

\[\int_{Z \cap S} |\{F, G\}|^p \omega = 0.\]

We claim that $Z \cap R$ has measure zero. Indeed, $Z \cap R = (\Phi|_R)^{-1}(\text{im } \Phi - U)$, and the claim follows from the fact that $R$ is an open subset of $M$ (therefore a submanifold), $\Phi|_R$ is a local diffeomorphism, the fact that $\text{im } \Phi - U$ has measure zero by Sard’s theorem, and the following lemma.

**Lemma 2.6.** Let $N, P$ be manifolds, let $f : N \to P$ be a local diffeomorphism, and let $Y \subset P$ a subset of measure zero. Then $f^{-1}(Y)$ has measure zero.

**Proof.** Since our manifolds are paracompact, they are second countable, and in particular $N$ can be covered with countably many charts, such that on each one of them $f$ is a diffeomorphism onto its image. Since diffeomorphisms preserve the property of having measure zero, it follows that $f^{-1}(Y)$ is covered by countably many measure zero sets, and thus it is itself such.

This implies

\[\int_{Z \cap R} |\{F, G\}|^p \omega = 0,\]

and therefore we have

\[\int_{M} |\{F, G\}|^p \omega = \int_{X} |\{F, G\}|^p \omega = \int_{V} |\{F, G\}|^p \omega.\]

From the regularity of the measure $|\{F, G\}|^p \omega$ we obtain

\[\int_{V} |\{F, G\}|^p \omega = \sup_{K \subset V \text{ compact}} \int_{K} |\{F, G\}|^p \omega.\]

It is therefore enough to show that for any compact $K \subset V$ there is $n \in \mathbb{N}$ such that $\Phi(K) \subset K_n$. This follows from the fact that $\Phi(K)$ is compact and contained in $U$, therefore $d(\Phi(K), \mathbb{R}^2 - U) > 0$ and

\[\lim_{n \to \infty} d(K_n, \mathbb{R}^2 - U) = 0,\]

where $d$ is the Euclidean distance between subsets of $\mathbb{R}^2$. This limit is indeed zero since $K_n$ contains all the points at a distance at least $\sqrt{2}/n$ from $\mathbb{R}^2 - U$.

We now prove the flexibility result, Theorem 1.7.

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5 We thank Lev Buhovsky for a suggestion that led to a simplification of the proof of the lemma.

6 Note that there may be regular points of $\Phi$ which are mapped to singular values.
Proof (of Theorem 1.7). Let \( C \subset M \) be as in the formulation of the theorem and fix \( \varepsilon > 0 \). We need to construct a Poisson commuting pair \( \tilde{F}, \tilde{G} \in C_c^\infty (M) \) supported in \( C \) and satisfying

\[
\| \tilde{F} - F \|_{C^0} < \varepsilon, \quad \| \tilde{G} - G \|_{L^p} < \varepsilon.
\]

Fix a Riemannian metric \( d \) on \( M \). By a simplex in \( M \) we mean the image of an embedding \( \Delta \to M \), where \( \Delta \) is a closed simplex in \( \mathbb{R}^{2n} \). A triangulation of \( C \) is a representation of \( C \) as a union of such simplices, where every two simplices intersect only in a common face (which is a simplex of lower dimension). A construction described in [2] produces a finite such triangulation; moreover given \( \delta > 0 \), every simplex in this triangulation may be assumed to have diameter \( < \delta \) with respect to \( d \).

Since \( C \) is compact, \( F \) is uniformly continuous on it, that is, there exists \( \delta \) such that if \( x, y \in C \) satisfy \( d(x, y) < \delta \) then \( |F(x) - F(y)| < \delta \). Fix such \( \delta \) and take a triangulation of \( C \) with all the simplices having diameter \( < \delta \).

For every simplex \( Q \) from the triangulation we fix open subsets with smooth boundary \( Q_3 \in Q_2 \in Q_1 \in Q \) satisfying

\[
\text{Vol}(Q \setminus Q_3) < \varepsilon \cdot \frac{\text{Vol}(Q)}{\|G\|_{C^0}^q \cdot \text{Vol}(C)}.
\]

Here \( A \Subset B \) means that the closure of \( A \) is contained in the interior of \( B \), and \( \text{Vol} \) here is the volume with respect to \( \omega^n \). The condition on the volumes is essential for constructing a suitable \( \tilde{G} \).

Construction of \( \tilde{F} \). Consider a simplex \( Q \) with open subsets \( Q_2 \Subset Q_1 \Subset Q \) as above. We take an auxiliary smooth function \( \varphi : Q \to [0, 1] \) such that \( \varphi|_{Q_2} \equiv 0 \) and \( \varphi|_{Q_1} \equiv 1 \). Fix a point \( x_0 \in Q_2 \). Define \( \tilde{F} \) on \( Q \) to be

\[
\tilde{F}(x) = \varphi(x)F(x) + (1 - \varphi(x))F(x_0).
\]

We see that on \( Q_2 \) we have \( \tilde{F} \equiv F(x_0) \), while outside \( Q_1 \) we have \( \tilde{F} \equiv F \). See Figure 2. Next, define \( \tilde{F} \) on \( C \) by gluing all these partially defined functions. Note that the resulting function is well-defined and smooth. Moreover, since \( F \) vanishes near \( \partial C \), it is also true for \( \tilde{F} \). Therefore we can extend \( \tilde{F} \) by zero to a smooth function on \( M \) with support in \( C \). For any \( x \in Q \) we have

\[
|\tilde{F}(x) - F(x)| = |\varphi(x)F(x) + (1 - \varphi(x))F(x_0) - F(x)|
\leq 1 \cdot |F(x) - F(x_0)| < \varepsilon,
\]

where the last inequality holds since \( \text{diam}(Q) < \delta \). Since \( Q \) is arbitrary, we obtain \( \|\tilde{F} - F\|_{C^0} < \varepsilon \).

Construction of \( \tilde{G} \). Consider again a simplex \( Q \) from our triangulation with the subsets \( Q_3 \Subset Q_2 \Subset Q_1 \Subset Q \), such that

\[
\text{Vol}(Q \setminus Q_3) \leq \varepsilon \cdot \frac{\text{Vol}(Q)}{\|G\|_{C^0}^q \cdot \text{Vol}(C)}.
\]

Take a smooth function \( \psi : Q \to [0, 1] \) satisfying \( \psi|_{Q_3} \equiv 1 \), \( \psi|_{Q \setminus Q_2} \equiv 0 \), and define \( \tilde{G} : Q \to \mathbb{R} \) by \( \tilde{G} = \psi G \). We have \( \tilde{G} \equiv G \) on \( Q_3 \) and \( \tilde{G}|_{Q \setminus Q_2} \equiv 0 \). Take \( \tilde{G} \) to be the function on \( M \) defined in this way on every simplex \( Q \), and extended by zero to \( M \setminus C \). This again is a well-defined smooth function with support in \( C \).
On a single simplex $Q$ we have
\[
\int_Q |\tilde{G} - G|^q \omega^n = \int_{Q \setminus Q_3} |\tilde{G} - G|^q \omega^n \\
= \int_{Q \setminus Q_3} (1 - \psi)^q |G|^q \omega^n \\
\leq \int_{Q \setminus Q_3} |G|^q \omega^n \\
\leq \|G\|_{C^0}^q \cdot \text{Vol}(Q \setminus Q_3) \\
< \|G\|_{C^0}^q \cdot \varepsilon \cdot \frac{\text{Vol}(Q)}{\|G\|_{L^\infty}^q \cdot \text{Vol}(C)} = \varepsilon \cdot \frac{\text{Vol}(Q)}{\text{Vol}(C)}.
\]

Therefore on the whole of $M$ we get the bound
\[
\|\tilde{G} - G\|^q_{L^q(M)} = \int_M |\tilde{G} - G|^q \omega^n < \varepsilon.
\]

It remains to note that for every simplex $Q$ in our triangulation and the associated subsets $Q_3 \subset Q_2 \subset Q_1 \subset Q$, $\tilde{F}$ is constant on $Q_2$, while $\tilde{G} \equiv 0$ outside $Q_2$, meaning $\{\tilde{F}, G\} \equiv 0$ as claimed. $\Box$
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