New Approach for Solving Two Dimensional Spaces PDE

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Abstract. In this paper, new approach based on coupled Laplace transformation with decomposition method is proposed to solve type of partial differential equation. Then it's used to find the accurate solution for heat equation with initial conditions. Four examples introduced to illustrate the accuracy, efficiency of suggested method. The practical results show the importance of suggested method for solve differential equations with high accuracy and easy implemented.

1. Introduction
The differential model is a mathematical model that relates to unknown functions with its derivatives. In applications, functions usually describe physical quantities, derivatives describe their rates of change [1], and the differential model defines the relationship between the two. Since these relationships are so common, differential models play a prominent role in many disciplines including physics, engineering, biology, and economics. Many physical and chemistry phenomena can be modeled using the language of calculus. For example, compute the pollution of heavy metals in soil by using the mathematical model for that phenomena and when we solve that model we find the valued of pollution in soil by heavy metals and can be risk for it for more details see [2-6]. In that paper we explain the physical phenomena that considers how the distribution of some quantity (such as heat) develop over time in a solid medium, as it spontaneously pour from sites where it is higher towards sites where it is lower.

There are many researchers explain how to solve the PDEs by using coupled method such as, Salih et al. [7], used this method to solve modified regularized long wave equation. Tawfiq et al., [8, 9] used LA-transform homotopy perturbation method for solving autonomous equations, and 3-Dimensional Space PDEs. Suheil [10] used Laplace decomposition algorithm to solve a class of nonlinear ODEs. The combined Laplace transformation-differential transform method for solving linear non- homogeneous PDFs is suggested by Alquran etc. in 2012 [11]. Ali, et al. [12], used coupled method in parallel processing technique Recently many researchers used other coupled method based on decomposition method such [13].

Here, we solved non-linear 2-Dimentional space differential equation by using coupled Laplace transformation with Adomian decomposition method to find the accurate solution.
2. Two Dimensional Heat Flow equation

The distribution of heat flow in a 2-dimensional space governs the problem of the following initial-boundary value problem (14).

\[ \frac{\partial \mathcal{V}}{\partial t} = \mathcal{R}(\mathcal{V}_{xx} + \mathcal{V}_{yy}), \quad 0 < x < a, 0 < y < b, t > 0, \]  

(1)

BC \quad \mathcal{V}(0, y, t) = \mathcal{V}(a, y, t) = 0, \quad \mathcal{V}(x, 0, t) = \mathcal{V}(x, b, t) = 0,

(2)

IC \quad \mathcal{V}(x, y, 0) = f(x, y),

Where \( \mathcal{V} = \mathcal{V}(x, y, t) \) is the temperature of any point located at the \((x, y)\) of a rectangular plate at any time \( t \), and \( \mathcal{R} \) is the thermal diffusivity.

To solve that equation by using coupled Laplace transformation with decomposition method must tracing the following steps:

First we take the Laplace transformation for equation 1 we get

\[ s\mathcal{L}\{\mathcal{V}\} - \mathcal{V}(x, y, 0) = \mathcal{R} \left( \frac{d^2}{dx^2} \mathcal{L}\{\mathcal{V}\} + \frac{d^2}{dy^2} \mathcal{L}\{\mathcal{V}\} \right) \]  

(4)

put the initial conditions (2) in equation (4) we get

\[ \mathcal{L}\{\mathcal{V}\} = \frac{1}{s} f(x, y) + \frac{\mathcal{R}}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{\mathcal{V}\} + \frac{d^2}{dy^2} \mathcal{L}\{\mathcal{V}\} \right) \]  

(5)

Now, we using the decomposition representation for

\[ \mathcal{V} = \sum_{n=0}^{\infty} \mathcal{V}_n \]  

(6)

\[ \mathcal{L}\{\mathcal{V}_0\} = \frac{1}{s} f(x, y) \]  

(7)

Now, we applied the invers Laplace transformation to get \( \mathcal{V}_0 = f(x, y) \)

\[ \mathcal{L}\{\mathcal{V}_1\} = \frac{\mathcal{R}}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{\mathcal{V}_0\} + \frac{d^2}{dy^2} \mathcal{L}\{\mathcal{V}_0\} \right) \]  

(9)

\[ \mathcal{L}\{\mathcal{V}_{k+1}\} = \frac{\mathcal{R}}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{\mathcal{V}_k\} + \frac{d^2}{dy^2} \mathcal{L}\{\mathcal{V}_k\} \right) \]  

(10)

Apply the invers Laplace transformation to find the \( \mathcal{V}_1, \mathcal{V}_2, \ldots \)

And \( \mathcal{V} = \sum_{n=0}^{\infty} \mathcal{V}_n \) represent the accurate solution for the eq.1.

As explained before, the solution in the \( t \) space, the \( x \) space, or the \( y \) space will output the same chain solution. However, the solution in the \( t \) space decreases the size of calculations compared with the other space solutions.

We take four examples, two examples are homogeneous and two examples are inhomogeneous.

3. Homogeneous Heat Equations

The equation that contains in all terms the dependent variable \( \mathcal{V}(x, y, t) \) that equation was said the homogeneous heat equation. We applied the modify Laplace transformation with decomposition method to solve that equation by given some of examples to explain that case.

Example 1:-

Consider a linear homogeneous heat equation

PDE \[ \mathcal{V}_t = \mathcal{V}_{xx} + \mathcal{V}_{yy}, \quad 0 < x, y < \pi \ & t > 0, \]  

(11)

BC \quad \mathcal{V}(0, y, t) = \mathcal{V}(\pi, y, t) = 0, \quad \mathcal{V}(x, 0, t) = \mathcal{V}(x, \pi, t) = 0,

(12)

IC \quad \mathcal{V}(x, y, 0) = \sin x \sin y

To solve that equation by using modify Laplace transformation with decomposition method:
First we take the Laplace transformation for equation 1 we get
\[ \mathcal{L}(\mathcal{V}) = \mathcal{L}(\mathcal{V}_{xx}) + \mathcal{L}(\mathcal{V}_{yy}) \]
\[ s\mathcal{L}(\mathcal{V}) - \mathcal{V}(x, y, 0) = \frac{d^2}{dx^2}\mathcal{L}(\mathcal{V}) + \frac{d^2}{dy^2}\mathcal{L}(\mathcal{V}) \]
\[ \text{(13)} \]
put the initial conditions (13) in equation (14) we get
\[ \mathcal{L}(\mathcal{V}) = \frac{1}{s} \sin x \sin y + \frac{1}{s} \left( \frac{d^2}{dx^2}\mathcal{L}(\mathcal{V}) + \frac{d^2}{dy^2}\mathcal{L}(\mathcal{V}) \right) \]
\[ \text{Now, we using the decomposition representation} \]
\[ \mathcal{V} = \sum_{n=0}^{\infty} \mathcal{V}_n \]
\[ \mathcal{L}(\mathcal{V}_0) = \frac{1}{s} \sin x \sin y \]
\[ \text{(14)} \]
we take the invers Laplace transformation for eq. (16) to get \( \mathcal{V}_0 \)
\[ \mathcal{V}_0 = \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_1) = \frac{1}{s} \left( \frac{d^2}{dx^2}\mathcal{L}(\mathcal{V}_0) + \frac{d^2}{dy^2}\mathcal{L}(\mathcal{V}_0) \right) \]
\[ \mathcal{L}(\mathcal{V}_1) = -\frac{1}{s^2} \sin x \sin y - \frac{1}{s^2} \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_1) = -\frac{1}{s^2} \sin x \sin y \]
\[ \mathcal{V}_1 = -2t \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_2) = \frac{1}{s} \left( \frac{d^2}{dx^2}\mathcal{L}(\mathcal{V}_1) + \frac{d^2}{dy^2}\mathcal{L}(\mathcal{V}_1) \right) \]
\[ \mathcal{L}(\mathcal{V}_2) = \frac{2}{s^4} \sin x \sin y + \frac{2}{s^4} \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_2) = \frac{2}{s^4} \sin x \sin y \]
\[ \mathcal{V}_2 = \frac{(2t)^4}{2} \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_3) = \frac{1}{s} \left( \frac{d^2}{dx^2}\mathcal{L}(\mathcal{V}_2) + \frac{d^2}{dy^2}\mathcal{L}(\mathcal{V}_2) \right) \]
\[ \mathcal{L}(\mathcal{V}_3) = -\frac{4}{s^4} \sin x \sin y - \frac{2}{s^4} \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_3) = -\frac{1}{s^4} \sin x \sin y \]
\[ \mathcal{V}_3 = -\frac{(2t)^3}{3!} \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_4) = \frac{1}{s} \left( \frac{d^2}{dx^2}\mathcal{L}(\mathcal{V}_3) + \frac{d^2}{dy^2}\mathcal{L}(\mathcal{V}_3) \right) \]
\[ \mathcal{L}(\mathcal{V}_4) = \frac{8}{s^4} \sin x \sin y + \frac{8}{s^4} \sin x \sin y \]
\[ \mathcal{L}(\mathcal{V}_4) = \frac{16}{s^4} \sin x \sin y \]
\[ \mathcal{V}_4 = \frac{(2t)^4}{4!} \sin x \sin y \]
\[ \vdots \]
\[ \mathcal{L}(\mathcal{V}_{k+1}) = \frac{1}{s} \left( \frac{d^2}{dx^2}\mathcal{L}(\mathcal{V}_k) + \frac{d^2}{dy^2}\mathcal{L}(\mathcal{V}_k) \right) \]
\[ \text{(16)} \]
\[ V = \sum_{n=0}^{\infty} V_n \]  
\[ V = V_0 + V_1 + V_2 + V_3 + V_4 \ldots \]  
\[ V = \sin x \sin y - 2t \sin x \sin y + \frac{(2t)^2}{2!} \sin x \sin y - \frac{(2t)^3}{3!} \sin x \sin y + \frac{(2t)^4}{4!} \sin x \sin y \]  
\[ - \ldots \]
\[ V = \sin x \sin y \left(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \ldots\right) \]

The accurate solution is
\[ V = e^{-2t} \sin x \sin y \]  
\[ \therefore V = e^{-2t} \sin x \sin y \]  
\[ (18) \]

**Example 2:**

Consider a linear homogeneous heat equation

\[ PDE \quad V_t = V_{xx} + V_{yy} - V, \quad 0 < x, y < \pi \text{ and } t > 0, \]  
\[ (19) \]

\[ BC \quad V(0, y, t) = V(\pi, y, t) = 0, \]
\[ V(x, 0, t) = -V(x, \pi, t) = e^{-3t} \sin x, \]  
\[ (20) \]

\[ IC \quad V(x, y, 0) = \sin x \cos y. \]

To solve that equation by using modified Laplace transformation with decomposition method:

\[ sL(V) - V(x, y, 0) = \frac{d^2}{dx^2}L(V) + \frac{d^2}{dy^2}L(V) - L(V) \]  
\[ (21) \]

Put the initial conditions (22) in equation (23) we get

\[ L(V) = \frac{1}{s} \sin x \cos y + \frac{1}{s} \left(\frac{d^2}{dx^2}L(V) + \frac{d^2}{dy^2}L(V) - L(V)\right) \]

Now, we using the decomposition representation

\[ V = \sum_{n=0}^{\infty} V_n \]  
\[ (22) \]

\[ L(V_0) = \frac{1}{s} \sin x \cos y \]  
\[ (23) \]

We take the inverse Laplace transformation for eq. (25) to get \( V_0 \)

\[ V_0 = \sin x \cos y. \]

\[ L(V_1) = \frac{1}{s} \left(\frac{d^2}{dx^2}L(V_0) + \frac{d^2}{dy^2}L(V_0) - L(V_0)\right) \]

\[ L(V_1) = -\frac{1}{s^3} \sin x \cos y - \frac{1}{s^3} \sin x \cos y - \frac{1}{s^3} \sin x \cos y \]

\[ L(V_1) = -\frac{1}{s^3} \sin x \cos y \]

\[ V_1 = -3t \sin x \cos y \]

\[ L(V_2) = \frac{1}{s} \left(\frac{d^2}{dx^2}L(V_1) + \frac{d^2}{dy^2}L(V_1) - L(V_1)\right) \]

\[ L(V_2) = \frac{3}{s^3} \sin x \cos y + \frac{3}{s^3} \sin x \cos y + \frac{3}{s^3} \sin x \cos y \]

\[ L(V_2) = \frac{3}{s^3} \sin x \cos y \]

\[ V_2 = \frac{(3t)^2}{2!} \sin x \cos y \]

\[ L(V_3) = \frac{1}{s} \left(\frac{d^2}{dx^2}L(V_2) + \frac{d^2}{dy^2}L(V_2) - L(V_2)\right) \]

\[ L(V_3) = \frac{3}{s^3} \sin x \cos y \]

\[ V_3 = \frac{(3t)^3}{6} \sin x \cos y \]
\[ \mathcal{L}\{V_3\} = - \frac{9}{s^4} \sin x \cos y - \frac{9}{s^4} \sin x \cos y - \frac{9}{s^4} \sin x \cos y \]

\[ \mathcal{L}\{V_3\} = - \frac{3t}{3!} \sin x \cos y \]

\[ \mathcal{L}\{V_4\} = \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V_3\} + \frac{d^2}{dy^2} \mathcal{L}\{V_3\} - \mathcal{L}\{V_3\}\right) \]

\[ \mathcal{L}\{V_4\} = \frac{27}{s^5} \sin x \cos y + \frac{27}{s^5} \sin x \cos y + \frac{27}{s^5} \sin x \cos y \]

\[ \mathcal{L}\{V_4\} = \frac{81}{s^5} \sin x \cos y \]

\[ V_4 = \frac{(3t)^4}{4!} \sin x \cos y \]

\[ \mathcal{L}\{V_{k+1}\} = \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V_k\} + \frac{d^2}{dy^2} \mathcal{L}\{V_k\} - \mathcal{L}\{V_k\}\right) \]

\[ V = \sum_{n=0}^{\infty} V_n \]

\[ V = V_0 + V_1 + V_2 + V_3 + V_4 \ldots \]

\[ V = \sin x \cos y - 3t \sin x \cos y + \frac{(3t)^2}{2!} \sin x \cos y - \frac{(3t)^3}{3!} \sin x \cos y + \frac{(3t)^4}{4!} \sin x \cos y \]

\[ V = \sin x \cos y \left( 1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!} - \ldots \right) \]

The accurate solution is

\[ V = e^{-3t \sin x \cos y} \]

4. Inhomogeneous Heat Equations

The heat equation that not contains in one or more terms the dependent variable $V$ that equation was said the inhomogeneous heat equation. We applied the modify Laplace transformation with decomposition method to solve that equation by given some of examples to explain that case.

Example 3:

Consider a linear inhomogeneous heat equation

\[ \mathcal{L}\{V\} = \mathcal{L}\{V_{xx}\} + \mathcal{L}\{V_{yy}\} + \mathcal{L}\{\sin y\} \]

\[ s\mathcal{L}\{V\} - \mathcal{L}\{V(x, 0, t)\} = \frac{d^2}{dx^2} \mathcal{L}\{V\} + \frac{d^2}{dy^2} \mathcal{L}\{V\} + \mathcal{L}\{\sin y\} \]

Put the initial conditions (31) in equation (32) we get

\[ \mathcal{L}\{V\} = \frac{1}{s} \left( \sin x \sin y + \sin y \right) + \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V\} + \frac{d^2}{dy^2} \mathcal{L}\{V\} + \mathcal{L}\{\sin y\} \right) \]

Now, we using the decomposition representation
\[ V = \sum_{n=0}^{\infty} V_n \quad (28) \]

\[ \mathcal{L}\{V_0\} = \frac{1}{s} (\sin x \sin y + \sin y) + \frac{1}{s^2} \sin y \quad (29) \]

we take the inverse Laplace transformation for eq. (34) to get \( V_0 \)

\[ V_0 = \sin x \sin y + \sin y + t \sin y \]

\[ \mathcal{L}\{V_1\} = \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V_0\} + \frac{1}{s^2} \frac{d^2}{dy^2} \mathcal{L}\{V_0\} \right) \]

\[ \mathcal{L}\{V_1\} = -\frac{1}{s^3} \sin x \sin y - \frac{1}{s^4} \sin x \sin y - \frac{1}{s^3} \sin y - \frac{1}{s^4} \sin y \]

\[ \mathcal{L}\{V_1\} = -\frac{2}{s^2} \sin x \sin y - \frac{1}{s^2} \sin y - \frac{1}{s^3} \sin y \]

\[ V_1 = -2t \sin x \sin y - t \sin y - \frac{t^2}{2} \sin y \]

\[ \mathcal{L}\{V_2\} = \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V_1\} + \frac{1}{s^2} \frac{d^2}{dy^2} \mathcal{L}\{V_1\} \right) \]

\[ \mathcal{L}\{V_2\} = \left( \frac{2}{s^3} \right) \sin x \sin y + \left( \frac{2}{s^4} \right) \sin x \sin y + \left( \frac{1}{s^3} \right) \sin y + \left( \frac{1}{s^4} \right) \sin y \]

\[ \mathcal{L}\{V_2\} = \left( \frac{4}{s^3} \right) \sin x \sin y + \left( \frac{4}{s^4} \right) \sin x \sin y + \left( \frac{1}{s^3} \right) \sin y + \left( \frac{1}{s^4} \right) \sin y \]

\[ V_2 = \frac{2}{2!} \sin x \sin y + \left( \frac{t^2}{2!} \right) \sin y + \left( \frac{t^3}{3!} \right) \sin y \]

\[ \mathcal{L}\{V_3\} = \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V_2\} + \frac{1}{s^2} \frac{d^2}{dy^2} \mathcal{L}\{V_2\} \right) \]

\[ \mathcal{L}\{V_3\} = -\left( \frac{8}{s^3} \right) \sin x \sin y - \left( \frac{8}{s^4} \right) \sin x \sin y - \left( \frac{1}{s^3} \right) \sin y - \left( \frac{1}{s^4} \right) \sin y \]

\[ \mathcal{L}\{V_3\} = -\left( \frac{16}{s^4} \right) \sin x \sin y - \left( \frac{1}{s^5} \right) \sin y - \left( \frac{1}{s^6} \right) \sin y \]

\[ V_3 = \frac{3}{2!} \sin x \sin y + \left( \frac{t^3}{3!} \right) \sin y - \left( \frac{t^5}{4!} \right) \sin y \]

\[ \mathcal{L}\{V_4\} = \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V_3\} + \frac{1}{s^2} \frac{d^2}{dy^2} \mathcal{L}\{V_3\} \right) \]

\[ \mathcal{L}\{V_4\} = \left( \frac{8}{s^5} \right) \sin x \sin y + \left( \frac{8}{s^6} \right) \sin x \sin y + \left( \frac{1}{s^5} \right) \sin y + \left( \frac{1}{s^6} \right) \sin y \]

\[ \mathcal{L}\{V_4\} = \left( \frac{16}{s^5} \right) \sin x \sin y + \left( \frac{1}{s^6} \right) \sin y + \left( \frac{1}{s^7} \right) \sin y \]

\[ V_4 = \frac{4}{4!} \sin x \sin y + \left( \frac{t^4}{4!} \right) \sin y + \left( \frac{t^5}{5!} \right) \sin y \]

\[ \vdots \]

\[ \mathcal{L}\{V_{k+1}\} = \frac{1}{s} \left( \frac{d^2}{dx^2} \mathcal{L}\{V_k\} + \frac{1}{s^2} \frac{d^2}{dy^2} \mathcal{L}\{V_k\} \right) \quad (30) \]

\[ V = \sum_{n=0}^{\infty} V_n \quad (31) \]

\[ V = V_0 + V_1 + V_2 + V_3 + V_4 \ldots \]
\[ V = \sin x \sin y \left( 1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} - \ldots \right) + \sin y \left( 1 - t + \frac{(t)^2}{2!} - \frac{(t)^3}{3!} + \frac{(t)^4}{4!} - \ldots \right) \]

There are noise terms in the \( V \) which are

\[ \sin y \left( t - \frac{(2)^2}{2!} + \frac{(t)^3}{3!} - \frac{(t)^4}{4!} + \frac{(t)^5}{5!} \ldots \right) \]

And to verify that the remaining canceled time for which the equation is achieved, then the accurate solution is

\[ V = e^{-2t} \sin x \sin y + \sin y \]

**Example 4:**

Consider a linear inhomogeneous heat equation

PDE \( V_t = V_{xx} + V_{yy} + 2 \cos x \cos y \), \( 0 < x, y < \pi \) \& \( t > 0 \),

BC \( V(0, y, t) = V(\pi, y, t) = (1 - e^{-2t}) \cos y \)

To solve that equation by using modify Laplace transformation with decomposition method:

\[ L(V_t) = L(V_{xx}) + L(V_{yy}) + L[2 \cos x \cos y] \]

Put the initial conditions (40) in equation (41) we get

\[ L[V] = \frac{1}{s} \left( \frac{d^2}{dx^2} L[V] + \frac{d^2}{dy^2} L[V] + \frac{2}{s} \cos x \cos y \right) \]

Now, we using the decomposition representation

\[ V = \sum_{n=0}^{\infty} V_n \]

We take the invers Laplace transformation for eq. (43) to get \( V_0 \)

\[ V_0 = 2t \cos x \cos y \]

We take the invers Laplace transformation for eq. (43) to get \( V_0 \)

\[ V_1 = \frac{1}{s} \left( \frac{d^2}{dx^2} L[V_0] + \frac{d^2}{dy^2} L[V_0] \right) \]

\[ L[V_2] = \frac{1}{s} \left( \frac{d^2}{dx^2} L[V_1] + \frac{d^2}{dy^2} L[V_1] \right) \]

\[ L[V_2] = \frac{8}{s^3} \cos x \cos y \]

\[ V_2 = \frac{(2t)^3}{3!} \cos x \cos y \]

\[ L[V_3] = \frac{1}{s} \left( \frac{d^2}{dx^2} L[V_2] + \frac{d^2}{dy^2} L[V_2] \right) \]

\[ L[V_3] = -\frac{8}{s^5} \cos x \cos y - \frac{8}{s^5} \cos x \cos y \]
The accurate solution is
\[ \mathcal{V}(t) = (1 - e^{-2t}) \cos x \cos y \]
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