ON CATEGORICAL APPROACH TO DERIVED PREFERENCE RELATIONS IN SOME DECISION MAKING PROBLEMS

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ABSTRACT. A structure called a decision making problem is considered. The set of outcomes (consequences) is partially ordered according to the decision maker’s preferences. The problem is how these preferences affect a decision maker to prefer one of his strategies (or acts) to another, i.e. it is to describe so called derived preference relations. This problem is formalized by using category theory approach and reduced to a pure algebraical question. An effective method is suggested to build all reasonable derived preferences relations and to compare them with each other.

Key words: Decision making problem, games with ordered outcomes, preference relations, preference functor, monoids of relations.

0. INTRODUCTION

It is known that at first a decision making problem (DMP) was built as a structure with a real-valued payoff function. Later on DMP appeared with a vector-valued payoff function. We consider a decision making problem with goal structure given by a partial ordering relation (preferences of a decision maker) on the set of outcomes. At the present moment such decision making problems have become of great interest. By the time the article had been written and tested, the interesting paper of Dubois, Fargier, Perny and Prade (2002) appeared devoted to the same subject and contained an extensive bibliography. It gives us an opportunity

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to refer a reader to this paper for the history of the problem. We only mention such papers that concern our results immediately. In spite of the common subjects the approaches of the present paper and the one cited above are different and results have no intersections. Below, we describe more exactly relations between the present paper and Dubua et al. (2002) that we name for short QDT.

The main problem considered in the present paper is the ”transference” of preferences from the set of outcomes (consequences) to the set of decision maker’s strategies (or acts), i.e. ranking the strategies knowing that some of consequences are better than others. Such ranking is called a derived preference relation. The point is that such transference of notions and principles used for classical DMPs to DMPs with preferences is based usually on intuition and leads frequently to anomalies. Further, in the next section, we explain what we mean. The similar problem is the subject of QDT, but there are some important differences.

The set of outcomes (consequences) is assumed in our paper to be partially ordered, while in QDT it is completely preordered (weakly ordered). A derived preference relation is regarded as a preorder, while in QDT it is assumed to be a partial order. But it is not essential because our approach works in all cases. It is essential that the authors of QDT formulate that “the real decision problem is to build this relation (ranking of acts) from information regarding the likelihood of states and the decision maker preference on consequences”, while we take in account only last one. They do not explain what ”to build” means but use some axioms motivated somehow or other. On the other hand, we concentrate our attention on giving a suitable mathematical formalization of the mentioned problem and on building all derived preference relations applying obtained mathematical results. Thus we think that the present paper is not a rival but an addition to QDT.

DMP considered in the paper is a structure, which components are a nonempty set $X$ of alternatives (strategies) of a decision maker, a nonempty set of states $A$, a partially ordered set $(A, \omega)$ of outcomes and a map $F$ that assigns an outcome $F(x, y)$ to every strategy $x$ if the state of the world is $y$. It is to mention that the
map $F$ determines a map $F^*$ from $X$ to the set $A^Y$. The set $A^Y$ is called the set of acts of the decision maker. Usually, the set of acts is considered instead of the set of strategies. But we do not suppose that every act can be realized by the considered decision maker.

A decision maker has to prefer one of his strategy to another according to his preferences in the set of outcomes which are given by the partial order $\omega$, that is he has to "build" a relation $\rho$ on the set of his strategies. We think that this relation has to be a preorder relation and has to be consistent with the preference relation $\omega$ and with the realization function $F$. We shall call every such relation $\rho$ a derivative preference relation. We look for a general principle to build the derivative preference relations for all DMPs like for example the maxmin principle that is the main optimality principle in the classical decision making theory. But absence of any one-valued (functional) connection between alternatives and outcomes makes the problem non-trivial.

Observe that a decision making problem can be considered as a game such that a player knows only his own preferences but does not know the preferences of other players. Namely this approach is used in the monograph Moulin (1981).

A new approach in the present paper is the use of category theory as a "methodological base" for formalizing the problem. This approach leads to a method of constructing of derived preference relations based on algebraic theory of monoids of binary relations. The main steps of solving the set up problem are the following ones. At first (section 2), the conditions (axioms) are formulated for a derivative preference relation. We proceed from minimal assumptions mentioned above and the assumption that so called Pareto-domination is a part of every derived preference relation. From the point of view of category theory, these conditions mean the existence of special functor (preference functor) from the category of DMPs to the category of preordered sets. In $\textbf{QDT}$, the so called Ordinal Invariance Axiom is assumed. May be it is at the bottom of some pessimistic conclusion in the mentioned paper. Instead of this axiom, we assume much weaker one that is fulfilled
for all suggested in literature preference relations.

Further (section 3), we show that constructing of the preference functor can be reduced to the constructing of some special functor from the category of ordered sets to the category of preordered sets. The main results are formulated in Theorem 1 and Theorem 2, that give an effective method to construct the required functors. Our method leads to anomalies-free preference relations, and it turns out that practically all known ways to specification of derivative preference relations are some concrete cases of the given construction (examples in Section 4). Using this new method we check some properties of derived preference relations. The last section 5 shows how to use the obtained results. Particularly, all derived preference relations form a complete lattice. In general case, some maximal and minimal preference relations are described. In all finite cases, this lattice can be effectively constructed.

In the first section, we give all necessary definitions and some motivations for the present research.

1 Preliminaries and motivations

A reader can find the necessary algebraic notions and facts in the book Grätzer (1979), so as some notions and facts concerning binary relations and ordered sets. But for the convenience of a reader, we recall some fundamental notions of relation theory.

A binary relation $\rho$ is a set of ordered pairs, usually it is a subset of the Cartesian product $A \times B$ of two sets, in this case it is called a binary relation between elements of these sets. The relation $\rho^{-1} = \{(b,a) | (a,b) \in \rho\}$ is called the inverse relation to $\rho$. The well known product of binary relations $\rho$ and $\sigma$ is denoted by $\sigma \circ \rho$, that is $\sigma \circ \rho = \{(a,c) | (\exists b)((a,b) \in \rho \ \& \ (b,c) \in \sigma)\}$. A relation $\rho \subseteq A \times A$ is called reflexive if $(\forall a \in A)(a,a) \in \rho$, it is called transitive if $\rho \circ \rho \subseteq \rho$, that is $(a,b) \in \rho$ and $(b,c) \in \rho$ implies $(a,c) \in \rho$. A transitive and reflexive relation is called a preorder (or quasiorder) relation, it is called a partial ordering relation if in addition it is antisymmetric, that is $(a_1,a_2) \in \rho \cap \rho^{-1} \Rightarrow a_1 = a_2$. Since we
consider in the present paper only partial ordering relations, further we often omit the word "partial".

For notational convenience we denote preorder and order relations by means of usual signs $\leq, \geq$ with a symbol of the corresponding relation on top. The greatest element $a$ with respect to a preorder relation $\preceq$ on a set $A$ is defined as usual, that is for all $x \in A$ we have $x \preceq a$. Of course, if it exists it is not unique in general, but with respect to an order relation it is unique if it exists. An element of an ordered set is called minimal or an atom if between it and the least element (if it exists) there are no elements. Dually, a maximal or a dual atom is defined. An ordered set is called a chain if every two elements of it are comparable ($a_1 \leq a_2$ or $a_2 \leq a_1$). An ordered set is called a lattice (complete lattice) if every its two elements have (every subset has) the least upper bound (supremum) and the greatest lower bound (infimum).

A map $f : A \rightarrow B$ from one preorder $(A, \preceq)$ to another $(B, \sigma)$ is called isotonic if it takes pairs from $\preceq$ to pairs from $\sigma$, that is $a_1 \preceq a_2 \Rightarrow f(a_1) \sigma f(a_2)$.

If $\lambda$ is a binary relation then its first projection, $pr_1 \lambda$, is the set of elements $a$ such that there exists an element $b$ such that $(a, b) \in \lambda$, its second projection, $pr_2 \lambda$, is the first projection of the inverse relation $\lambda^{-1}$ and its reflexive projection, $pr_\Delta \lambda$, is the set of all elements $a$ such that $(a, a) \in \lambda$. The equality relation on a set $A$ we denote by $\Delta_A$. Thus $pr_\Delta \lambda = pr_1(\lambda \cap \Delta_A) = pr_2(\lambda \cap \Delta_A)$ for $\lambda \subset A \times A$.

As for the theory of categories and functors, we can refer a reader to the book Mac Lane (1971). But we use only basic notions as the most convenient language for our aim, and it is necessary to know only basic definitions. A category $\mathcal{C}$ consists of two collections, $\text{Ob}(\mathcal{C})$, whose elements are the so called objects of $\mathcal{C}$, and $\text{Hom}(\mathcal{C})$, the collection of morphisms (or arrows, homomorphisms, maps) of $\mathcal{C}$. To each morphism is assigned a pair of objects, called the domain and codomain of the morphism. The notation $f : A \rightarrow B$ means that $f$ is a morphism with the domain $A$ and the codomain $B$. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two morphisms, there is a morphism $g \circ f : A \rightarrow C$ called the composite of $g$ and $f$. For every object $A$
there is a morphism $id_A : A \to A$, called the identity of $A$. These data are subject to the following axioms:

(1) For $f : A \to B$, $g : B \to C$, $h : C \to D$,

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

(2) For $f : A \to B$,

$$f \circ id_A = id_B \circ f = f.$$

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is a map for which

(1) If $f : A \to B$ is a morphism of $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ is a morphism of $\mathcal{D}$;

(2) $F(id_A) = id_{F(A)}$;

(3) If $f : A \to B$, $g : B \to C$ , then $F(g \circ f) = F(g) \circ F(f)$.

All other definitions will be given when it will be necessary.

It is known that the maximin principle is the main optimality principle in the classical decision making theory. Let $X$ be a set of alternatives (strategies) of a decision maker, call $I$, $Y$ be a set of medium states and $f(x,y)$ be the payoff of $I$ if he chooses the strategy $x$ and the medium is situated in the state $y$. The alternative $x_0$ is maximin if $\inf_x f(x,y) = \inf_y f(x_0,y)$, i.e. if it maximizes the estimate $x \to \inf_y f(x,y)$.

If one considers the triplet $\mathcal{G} = (X,Y,f)$ as a two-player zero sum game, it appears along with the maximin principle for the player 1 the dual one for the player 2, i.e. the minimax principle: the alternative $y_0$ is minimax if it minimizes the estimate $y \to \sup_x f(x,y)$, i.e. $\inf_y \sup_x f(x,y) = \sup_x f(x,y_0)$. The values $v_1 = \sup_x \inf_y f(x,y)$ and $v_2 = \inf_y \sup_x f(x,y)$ are called the lower value and the upper value of the game respectively. By the way, $v_1 \leq v_2$ holds always. If $v_1 = v_2$ holds it is said that the game $\mathcal{G}$ has a value.

In two-player zero sum game theory, the saddle point principle is the most important one. We recall that a point $(x_0,y_0)$ is called a saddle point if $f(x,y_0) \leq f(x_0,y_0) \leq f(x_0,y)$ for all $x \in X, y \in Y$. The existence of a saddle point implies
the existence of the game value. Conversely, if a game has the value then every pair \((x_0, y_0)\), where \(x_0\) is a maximin strategy of the player 1 and \(y_0\) is a minimax strategy of the player 2, forms a saddle point, and every saddle point can be obtained by this way.

A game with ordered outcomes is a game in which preferences of players are given by partial ordering relations on the set of outcomes, that is the goal structure of a such game is a set of outcomes that every player orders according to his own preferences. In the case of an antagonistic game the goal structure is given by means of one partial ordering relation that expresses the preferences of the player 1, and the preferences of his opponent (the player 2) are given by the inverse partial ordering relation.

We consider an antagonistic game as an algebraic structure \(G = (X, Y, A, \omega, F)\), where \(X\) and \(Y\) are the sets of strategies of players 1 and 2, \((A, \omega)\) is a ordered set, \(F\) is a map \(F : X \times Y \rightarrow A\) that is called a realization function. It is assumed only that the sets \(X, Y\) and \(A\) contain more than one element. The fact \(a_1 \geq^\omega a_2\) is interpreted in the sense that the outcome \(a_1\) is more preferable for the player 1 than the outcome \(a_2\). If the player 1 chooses a strategy \(x\) and the player 2 chooses a strategy \(y\) it comes out a situation \((x, y)\) that leads to an outcome \(F(x, y)\). So, the problem arises for every player how to prefer one of his strategy to another.

The most natural point of view is the following one: a strategy \(x_1\) is more preferable for the player 1 than a strategy \(x_2\) if \(F(x_1, y) \geq^\omega F(x_2, y)\) takes place for every strategy \(y\) of the player 2. It is so called Pareto-domination. The strong Pareto-domination means that all inequalities in the definition of Pareto-domination are strong. The maximal according to Pareto-domination strategies are called Pareto-optimal. But this way has some shortcomings because the problem is reduced to the problem of choosing one of the Pareto-optimal strategies. Nevertheless it is obvious that if a strategy \(x_1\) Pareto-dominates a strategy \(x_2\) it is more preferable than last one in any other kind of domination.

While searching for other more suitable principals of domination, the notions are
introduced, that are analogies to the notions of maximin (minimax) strategy and the value of a game in the classic case [see Jentzsch (1964), Podinovski (1979)]. The saddle point is defined as usual (see above). Since it can be that an infimum of a subset of the ordered set \((A, \omega)\) does not exist, the following changing is made.

Denote by \(V_x\) the set of all outcomes that are "guaranteed" for the player 1 by the strategy \(x\), that is: \(V_x = \{a \in A | (\forall y \in Y) F(x, y) \geq \omega a\}\). The estimate \(x \to \inf_y f(x, y)\) is replaced by the estimate \(x \to V_x\). If the set \(V_x\) is assumed as a characteristic of the strategy \(x\) the relation of so called \(\alpha\)-dominating for the player 1 arises:

\[ x_1 \leq^\alpha x_2 \iff V_{x_1} \subseteq V_{x_2}. \]

The defined relation \(\alpha\) is a preorder relation on the set \(X\) of strategies of the player 1. The strategy \(x_0 \in X\) is called an \(\alpha\)-greatest one if it is the greatest element with respect to the preorder relation \(\alpha\). Dually, i.e. by means of rearrangement of players and replacing \(\omega\) by \(\omega^{-1}\), the definition of an \(\alpha\)-greatest strategy of the player 2 is obtained.

Denote \(U_y = \{a \in A | (\exists x \in X) F(x, y) \geq \omega a\}\) and set:

\[ V = \bigcup_x V_x \quad \text{(the lower characteristic set)}; \]
\[ U = \bigcap_y U_y \quad \text{(the upper characteristic set)}. \]

The inclusion \(V \subseteq U\) holds always. One can regard it as an analogy to the inequality \(v_1 \leq v_2\). The equality \(V = U\) is considered in Jentzsch (1964) as an analogy to the existence of a value of the game \(G\). In this case, we call the coincident lower and upper characteristic sets the generalized value of the game \(G\).

A game \(G\) with payoff function can be considered formally as a game with ordered outcomes (in this case, the set of outcomes is the set \(\mathbb{R}\) of real numbers with respect to usual order, and the function \(f\) becomes a payoff function). Then the \(\alpha\)-greatest strategy of the player 1 coincides with his maximin strategy, and the \(\alpha\)-greatest strategy of the player 2 coincides with his minimax strategy, and the equality \(V = U\) implies the equality \(v_1 = v_2\). Thus the introduced analogies seem to be
quite natural. But they have some anomalies (some of them were mentioned in Jentzsch (1964), Podinovski (1981), Rozen (2001)).

Example 1

Consider the game $G = (X, Y, A, \omega, F)$ with ordered outcomes, where the ordered set $(A, \omega)$ is the following 5-elements lattice: $\{0 < a, b, c < 1\}$, and with realization function $F$ given by the table:

|   | $F$ | $y_1$ | $y_2$ | $y_3$ |
|---|-----|-------|-------|-------|
| $x_1$ |       |       |       |       |
| $x_2$ |       |       |       |       |

Here $x_1$ is the $\alpha$-greatest strategy of the player 1, $y_1$ is the $\alpha$-greatest strategy of the player 2 and $V = U$. However, the pair $(x_1, y_1)$ does not form a saddle point, and this game has no saddle points at all. Besides this, the dual condition $V^* = U^*$ is not satisfied, i.e. the condition of the existence of a generalized value is not invariant with respect to duality, when the players change places.

Example 2

Consider the antagonistic game $G = (X, Y, \mathbb{R}^2, \omega, F)$ with vectorial payoffs (i.e. with respect to component-wise ordering), where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the function $F$ is given by the following table:

|   | $F$ | $y_1$ | $y_2$ |
|---|-----|-------|-------|
| $x_1$ |       |       |       |
| $x_2$ |       |       |       |

We see that

$V_{x_1} = \{(p, q) \in \mathbb{R}^2| (p, q) \leq (2, 1) \& (p, q) \leq (1, 3)\} = \{(p, q) \in \mathbb{R}^2| (p, q) \leq (1, 1)\}$;

$V_{x_2} = \{(p, q) \in \mathbb{R}^2| (p, q) \leq (4, 0) \& (p, q) \leq (0, 5)\} = \{(p, q) \in \mathbb{R}^2| (p, q) \leq (0, 0)\}$.

Hence $V_{x_2} \subseteq V_{x_1}$, that is $x_1 \geq^\alpha x_2$ and therefore $x_1$ is the $\alpha$-greatest strategy. However, if we ”convolve” the vectorial payoffs according the formula $(p, q) \rightarrow p + q$ we receive the following game:

|   | $F$ | $y_1$ | $y_2$ |
|---|-----|-------|-------|
| $x_1$ |       |       |       |
| $x_2$ |       |       |       |
In this game, $x_2$ is the $\alpha$-greatest strategy but not $x_1$. The new game is an homomorphic image of the previous one. Thus for the games with ordered outcomes, an $\alpha$-optimal strategy can be changed by passing on to homomorphic images.

**Example 3**

With the preceding example notation, let the antagonistic game $G$ have the following function $F$:

|    | $y_1$ | $y_2$ |
|----|-------|-------|
| $x_1$ | (2,1) | (1,2) |
| $x_2$ | (3,1) | (1,3) |

Here the strategy $x_2$ Pareto-dominates the strategy $x_1$ but it holds $V_{x_1} = V_{x_2}$. Thus it may be that two strategies are equivalent with respect to the $\alpha$-domination and one of them strongly Pareto-dominates the other.

**Example 4**

Consider the game $G = (X, Y, A, \omega, F)$ with ordered outcomes, where the ordered set $(A, \omega)$ is the following 5-elements lattice: $\{0 < a, b, c < 1\}$, and with realization function $F$ given by the table:

|    | $y_1$ | $y_2$ |
|----|-------|-------|
| $x_1$ | b     | 1     |
| $x_2$ | c     | a     |
| $x_3$ | 1     | 0     |

Here the strategy $x_1$ is the $\alpha$-greatest strategy. Extend the set $A$ up to $\bar{A}$ by adding new elements $f, g$ and $h$ and extend the order on $\bar{A}$ by setting $0 < f, g, h; \ g < a, c; \ h < b, c$. In the obtained game $\bar{G}$, we have $\bar{V}_{x_2} = \{0, g\}, \bar{V}_{x_1} = \{0, b\}, \bar{V}_{x_3} = \{0\}$, hence the strategies $x_1$ and $x_2$ are incomparable with respect to the $\alpha$-domination. Thus the condition to be an $\alpha$-greatest strategy is not preserved by adding some non-realized outcomes.

On the base of shortcomings mentioned above, we can make a conclusion that the set of all outcomes that are "guaranteed" for the player 1 by a strategy is not a good characteristic of it. There is more fine characteristic based on another approach. Let $(A, \omega)$ be an ordered set. It is possible to extend the relation $\omega$ up to set of all subsets of $A$ by the following two ways:

$$A_1 \leq^{\omega} A_2 \iff (\forall a_1 \in A_1)(\exists a_2 \in A_2)a_1 \leq^{\omega} a_2.$$
$A_1 \leq \tilde{\omega} A_2 \Leftrightarrow (\forall a_2 \in A_2)(\exists a_1 \in A_1)a_1 \leq \omega a_2.$

Applying these relations to the set of the kind $A_x = \{F(x, y) | y \in Y\}$ we obtain notions of so called $\beta$ – domination and dual $\beta$ – domination. The explicit forms of them are the following ones:

$$x_1 \leq^\beta x_2 \Leftrightarrow (\forall y_1 \in Y)(\exists y_2 \in Y)F(x_1, y_2) \leq^\omega F(x_2, y_1);$$

$$x_1 \leq_{\tilde{\beta}} x_2 \Leftrightarrow (\forall y_1 \in Y)(\exists y_2 \in Y)F(x_1, y_1) \leq^\omega F(x_2, y_2);$$

These relations agree with the approach of Berge (1957) to sort out the players into active and passive ones. An active player wants to have at least one good outcomes and a passive one does not want to have a bad outcome.

$\beta$ – domination is interrogated in Rozen (2001), it coincides with relation $R_{g2}$ introduced by Podinivski (1979).

2 Preference functor

The main model under consideration is DMP of the following kind

$$G = (X, Y, A, \omega, F), \quad (1)$$

where $X, Y, A$ are arbitrary (nonempty) sets, $\omega \subseteq A \times A$ is a (partial) order relation, $F : X \times Y \to A$ is a map. This model is interpreted in the following way. The set $X$ is a set of alternatives (strategies) of a decision maker, the set $Y$ is a set of states and $A$ is a set of outcomes (results). The order relation $\omega$ expresses the preferences of the decision maker. And the map $F$ is a realization function (although its values are not assumed to be real numbers). We assume the following underlying conditions (axioms) for any derived preference relation $\rho \subseteq X^2$:

(A1) the relation $\rho$ is a preorder relation on $X$;

(A2) the relation $\rho$ contains the Pareto-domination relation;

(A3) a ”strengthening” of the relation $\omega$ implies a ”strengthening” of the relation $\rho$, in other words, by adding new outcomes or new comparable pairs to $\omega$, the existing preferences of alternatives have to be saved.
We note that the axioms (A1)-(A3) are from the valuable point of view the minimal system of natural requirements. The mathematical sense of axioms A1 and A2 is clear, and an exact form of the axiom A3 will be given below on the base of category theory.

Fix sets $X$ and $Y$ and consider the category $\mathcal{G}(X,Y)$ which has as objects the DMPs $G = (X,Y,A,\omega,F)$, and a morphism from $G = (X,Y,A,\omega,F)$ to $G^1 = (X,Y,B,\delta,H)$ is a map $f : A \to B$ satisfying the following two conditions:

$$a_1 \leq^\omega a_2 \Rightarrow f(a_1) \leq^\delta f(a_2); (2)$$

$$f \circ F = H. (3)$$

The condition (2) means that $f$ is an isotonic map of the ordered set $(A,\omega)$ to the ordered set $(B,\delta)$ and the condition (3) means that $f$ transfers the function $F$ to the function $H$. In other words, we consider a DMP (1) as a structure with base set $A$, one binary relation $\omega$ and a family of 0-ary operations $\{F(x,y)\}_{(x,y) \in X \times Y}$, and the defined morphisms are usual homomorphisms of structures (see Grätzer (1979)).

Further, a "strengthening" of preferences on the set of outcomes of the given DMP $G = (X,Y,A,\omega,F)$ can be formalized as an isotonic map $f : A \to B$ of the ordered set $(A,\omega)$ to an ordered set $(B,\delta)$, and as a result a new DMP $G^1 = (X,Y,B,\delta,H)$ appears, where $H = f \circ F$. Thus a "strengthening" is a morphism of the category $\mathcal{G}(X,Y)$.

Denote by $Q(X)$ the category which objects are preorder relations on $X$, and for $\rho \subseteq X \times X$ and $\sigma \subseteq X \times X$ the set $Hom(\rho,\sigma)$ contains only one morphism if $\rho \subseteq \sigma$ and is empty otherwise. Since our aim is to associate with every DMP a preorder relation $\rho \subseteq X \times X$ (a derived preference relation) in such a way that the condition (A3) is satisfied, we have to consider a map $\Psi$ which assigns to every $\mathcal{G}(X,Y)$-object $G$ a preorder relation $\Psi(G)$ on $X$ in such way that for every morphism $G \to G^1$ the inclusion $\Psi(G) \subseteq \Psi(G^1)$ is satisfied. Thus we have to construct functors from the category $\mathcal{G}(X,Y)$ to the category $Q(X)$. This is the mentioned category-theoretical formalization of the axiom A3.
Consider the set $A^Y$ of all maps from $Y$ to $A$. For every order relation $\omega$ on the set $A$, we have the order relation $\hat{\omega}$ on $A^Y : (\alpha, \beta) \in \hat{\omega} \Leftrightarrow (\forall y \in Y) \alpha(y) \leq^\omega \beta(y)$.

As it is known, we can associate with every map $F : X \times Y \rightarrow A$ the map $F^* : X \rightarrow A^Y$, that assigns to every element $x \in X$ the map $F^*_x : Y \rightarrow A$ by the rule: $F^*_x(y) = F(x, y)$ for all $y \in Y$.

Let $P$ be a map that assigns to every DMP $G = (X, Y, A, \omega, F)$ the relation $F^{-1}(\hat{\omega}) = F^{-1} \circ \hat{\omega} \circ F^*$. We have

$$P(G) = \{(x_1, x_2) | (\forall y \in Y) F(x_1, y) \leq^\omega F(x_2, y)\}. \quad (4)$$

It is exactly the Pareto-domination for $G$. Let $f : A \rightarrow B$ be a morphism from $G = (X, Y, A, \omega, F)$ to $G^1 = (X, Y, B, \delta, H)$. Then according to definition of morphisms, we have:

$$(x_1, x_2) \in P(G) \Rightarrow (\forall y \in Y) (f \circ F)(x_1, y) \leq^\delta (f \circ F)(x_2, y) \Rightarrow$$

$$(\forall y \in Y) H(x_1, y) \leq^\delta H(x_2, y) \Rightarrow (x_1, x_2) \in P(G^1),$$

that is $P$ is a functor from the category $G(X, Y)$ to the category $Q(X)$. We call it the Pareto functor.

**Definition 1.** A functor $\Psi : G(X, Y) \rightarrow Q(X)$ is called a preference functor if $P(G) \subseteq \Psi(G)$ for every DMP $G = (X, Y, A, \omega, F)$.

From the point of view of this definition, the conditions (A1)-(A3) mean that we deal with preference functors from the category $G(X, Y)$ to the category $Q(X)$.

Our definition of a preference functor is very general. All preference relations that are considered in the literature indeed satisfy more one condition which seems to be natural, but we have never seen it in an explicit form. Roughly speaking this condition means that we choose the same preferences being in similar situations. Consider two situations $(x_1, y_1)$ and $(x_2, y_2)$ that lead to outcomes $F(x_1, y_1)$ and $F(x_2, y_2)$ respectively. We can compare these situations by means of order in $A$, namely the second situation is preferable than the first one if $F(x_1, y_1) \leq^\omega F(x_2, y_2)$. This
approach leads to a relation in the set of states for every pair of strategies \((x_1, x_2)\).

It seems to be naturally to take into account this relation. Below we give the exact definition.

**Definition 2.** The state-preference for the pair \((x_1, x_2)\) with respect to DMP \(G\) is the following relation on the set of states \(Y\):

\[
\rho_G(x_1, x_2) = \{(y_1, y_2)| F(x_1, y_1) \leq^\omega F(x_2, y_2)\}.
\]  

(5)

**Definition.** A preference functor \(\Psi : G(X,Y) \rightarrow Q(X)\) is called regular if it satisfies the following condition:

\[(A4)\]

if \(\rho_G(x_1, x_2) = \rho_G'(x_3, x_4)\) then

\[(x_1, x_2) \in \Psi(G) \Leftrightarrow (x_3, x_4) \in \Psi(G')\]  

(6)

Now we show that so called Ordinal Invariance Axiom (OIA) (see \(\text{QDT}\)) is more stronger that A4. OIA means in our notations that for \(x_1, x_2, x_3, x_4 \in X\), if

\[
\{y \in Y| F(x_1, y) \leq^\omega F(x_2, y)\} = \{y \in Y| F(x_3, y) \leq^\delta F(x_4, y)\}
\]

then \((x_1, x_2) \in \Psi(G) \Leftrightarrow (x_3, x_4) \in \Psi(G')\).

Let \(\rho_G(x_1, x_2) = \rho_G'(x_3, x_4)\). The fact \(F(x_1, y) \leq^\omega F(x_2, y)\) means that \((y, y) \in \rho_G(x_1, x_2)\). The last one is equal to \((y, y) \in \rho_G(x_3, x_4)\), that is equal for one’s turn to \(F(x_1, y) \leq^\delta F(x_2, y)\). Hence \(\{y \in Y| F(x_1, y) \leq^\omega F(x_2, y)\} = \{y \in Y| F(x_3, y) \leq^\delta F(x_4, y)\}\) and OIA implies that \((x_1, x_2) \in \Psi(G) \Leftrightarrow (x_3, x_4) \in \Psi(G')\). Thus OIA implies A4.

3. **Main results**

In this section, we give a method of constructing of all regular preference functors \(\Psi : G(X,Y) \rightarrow Q(X)\) and consider connections between them. It is obvious that the required construction is not to depend on sets of strategies \(X\) and realization functions \(F\) because the last ones only choose and rename some maps from \(Y\) to \(A\). We should mention that some authors consider from the very beginning preorder
relations on $A^Y$, the sets of acts of a decision maker (see for example Barthélemy et al. (1982), Dubois et al. (2002)). But we do not assume in contrast to this point of view that all mappings are possible acts. However by constructing of preference functors, we consider preorder relations on the set $A^Y$ without connections with functions $F$. The following fact gives a reason for such approach.

Recall that every map $f : A \rightarrow B$ determines a map $\tilde{f} : A^Y \rightarrow B^Y$ under formula: $\tilde{f}(\varphi) = f \circ \varphi$ for all $\varphi : Y \rightarrow A$. Denote by $Q$ the category of all preordered sets and isotonic maps between them.

**Proposition 1.**

(1) Let $\Psi : \mathcal{G}(X,Y) \rightarrow Q(X)$ be a preference functor. Consider a map $\Xi$ that assigns 1) to every DMP $G = (X,Y,A,\omega,F)$ the preorder relation on the set $A^Y$ generated by $F^* \circ \Psi(G) \circ F^{*-1}$ and $\hat{\omega}$ and 2) to every morphism of DMPs $f$ the map $\tilde{f}$. Then $\Xi$ is a functor from the category $\mathcal{G}(X,Y)$ to the category $Q$.

2) Assume that we have a functor $\Xi$ from the category of all ordered sets to the category $Q$ which assigns to every ordered set $(A,\omega)$ a preorder relation $\Xi(\omega)$ on the set $A^Y$ and such that the following two conditions are satisfied:

(i) $\hat{\omega} \subseteq \Xi(\omega)$;

(ii) if $f : A \rightarrow B$ is an isotonic map of the ordered set $(A,\omega)$ into the ordered set $(B,\delta)$ then $\Xi(f) = \tilde{f} : A^Y \rightarrow B^Y$.

Then the map $\Psi$ which assigns to every DMP $G = (X,Y,A,\omega,F)$ the preorder relation $F^{*-1}(\Xi(\omega))$ determines a preference functor from category $\mathcal{G}(X,Y)$ to the category $Q(X)$.

**Proof.** (1) Given a DMP $G = (X,Y,A,\omega,F)$, consider the kernel of the map $F^*$: $\varepsilon_{F^*} = \{ (x_1, x_2) | F^*_{x_1} = F^*_{x_2} \}$. We see that $\varepsilon_{F^*}$ is included in the Pareto-domination relation. Since $P(G) \subseteq \Psi(G)$ and $\varepsilon_{F^*} \subseteq P(G)$, the inclusion $\varepsilon_{F^*} \subseteq \Psi(G)$ holds. Therefore the preorder relation generated by $F^* \circ \Psi(G) \circ F^{*-1}$ and $\hat{\omega}$ is equal to $\hat{\omega} \circ F^* \circ \Psi(G) \circ F^{*-1} \circ \hat{\omega} \cup \hat{\omega}$. Now we prove that the map $\tilde{f} : a^Y \rightarrow b^Y$ is isotonic. In fact, we have $\tilde{f} \circ F^* \circ \Psi(G) \circ F^{*-1} \circ \tilde{f}^{-1} = (f \circ F)^* \circ \Psi(G) \circ (f \circ F^*)^{-1} = H^* \circ \Psi(G) \circ H^{*-1}$.
It is well known (and quite obvious) that if \( f : A \to A \) is the identity map then
\[ \Xi(f) = \tilde{f} \] is also the identity map on \( A^Y \) and that for \( f : A \to B \), \( g : B \to C \) the equality \( \tilde{g} \circ \tilde{f} = \tilde{\tilde{g}} \circ \tilde{f} \). Thus \( \Xi \) is a functor.

(2) Let \( \Xi \) be a functor satisfying the hypotheses. Let \( G = (X,Y,A,\omega,F) \) and \( G^1 = (X,Y,B,\delta,H) \) and \( f : G \to G^1 \) be a morphism. Since \( f \) is an isotonic map, the map \( \tilde{f} \) also is isotonic and therefore \( (F^*_{x_1}, F^*_{x_2}) \in \Xi(\omega) \Rightarrow (\tilde{f}(F^*_{x_1}), \tilde{f}(F^*_{x_2})) \in \Xi(\delta) \). Under formula (3), \( \tilde{f}(F^*_{x}) = H^*_{x} \) holds for arbitrary \( x \in X \). Therefore the following implication is true: \( (F^*_{x_1}, F^*_{x_2}) \in \Xi(\omega) \Rightarrow (H^*_{x_1}, H^*_{x_2}) \in \Xi(\delta) \). Hence \( \tilde{\tilde{F}^*(-1)}(\Xi(\omega)) \subseteq \tilde{\tilde{H}^*(-1)}(\Xi(\delta)) \). In other words, we can define that \( \Psi(f) \) is an inclusion \( \Psi(G) \subseteq \Psi(G^1) \). It is enough for conclusion that \( \Psi \) is a functor. The condition (i) implies that \( P(G) \subseteq \Psi(G) \), i.e. the functor \( \Psi \) is a preference functor. \( \square \)

According to Proposition above, we begin to construct functors \( \Xi \) from the category of ordered sets to the category of preordered sets satisfying the conditions for a regular preference functor, that is

1) it assigns to every ordered set \((A,\omega)\) a preorder relation \( \Xi(\omega) \) on the set \( A^Y \) and to every isotonic map \( f : A \to B \) the map \( \tilde{f} : A^Y \to B^Y \);

2) it includes the Pareto functor: \( \hat{\omega} \subseteq \Xi(\omega) \) for all ordered sets \((A,\omega)\);

3) it is regular: if \((A,\omega)\) and \((B,\delta)\) are ordered sets and \( \psi^{-1} \circ \omega \circ \varphi = \psi_1^{-1} \circ \omega \circ \varphi_1 \) holds for \( \varphi, \psi : Y \to A \) and \( \varphi_1, \psi_1 : Y \to B \) then \( (\varphi, \psi) \in \Xi(\omega) \Leftrightarrow (\varphi_1, \psi_1) \in \Xi(\delta) \).

We call such a functor also as a regular preference functor.

We recall that a **monoid** is a nonempty set \( M \) together with an associative binary operation (multiplication) and an element \( 1 \) called an *identity*, such that \( m1 = 1m = m \) for every \( m \in M \). The set \( \mathcal{R}(Y) \) of all binary relations on \( Y \) is a monoid with respect to composition of binary relations and the identity \( \Delta_Y \) (see Introduction). We call a subset \( \mathcal{A} \) of \( \mathcal{R}(Y) \) a **closed submonoid** if it satisfies the following conditions:

1) \( \alpha, \beta \in \mathcal{A} \Rightarrow \beta \circ \alpha \in \mathcal{A} \);

2) \( \Delta_Y \in \mathcal{A} \);

3) \( \alpha \in \mathcal{A} \) \& \( \alpha \subseteq \beta \Rightarrow \beta \in \mathcal{A} \).
Consider an ordered set \((A, \omega)\) and choose some closed submonoid \(\mathfrak{A}\) of \(\mathfrak{R}(Y)\). Define a relation \(\mathfrak{A}(\omega)\) on the set \(A^Y\) setting for arbitrary \(\varphi_1, \varphi_2 \in A^Y:\)

\[
(\varphi_1, \varphi_2) \in \mathfrak{A}(\omega) \iff \{(y_1, y_2) \in Y^2 \mid \varphi_1(y_1) \leq \omega \varphi_2(y_2)\} \in \mathfrak{A}.
\]

(7)

It can be expressed also by the formula:

\[
(\varphi_1, \varphi_2) \in \mathfrak{A}(\omega) \iff \varphi_2^{-1} \circ \omega \circ \varphi_1 \in \mathfrak{A}.
\]

(8)

It is easy to see that \(\mathfrak{A}(\omega)\) is a preorder relation on \(A^Y\). Indeed, for every \(\varphi : Y \to A\) it holds \(\Delta_Y \subseteq \varphi^{-1} \circ \omega \circ \varphi\), thus we have \(\varphi^{-1} \circ \omega \circ \varphi \in \mathfrak{A}\), hence \((\varphi, \varphi) \in \mathfrak{A}(\omega)\). Further, if \((\varphi, \psi) \in \mathfrak{A}(\omega)\) and \((\psi, \gamma) \in \mathfrak{A}(\omega)\), then \((\varphi, \gamma) \in \mathfrak{A}(\omega)\), because \(\gamma^{-1} \circ \omega \circ \psi \circ \psi^{-1} \circ \omega \circ \varphi \subseteq \gamma^{-1} \circ \omega \circ \varphi\).

**Theorem 1.** Let \(\mathfrak{A}\) be a closed submonoid of \(\mathfrak{R}(Y)\). The map \(\Psi\) that assigns to every DMP \(G = (X, Y, A, \omega, F)\) the preorder relation \(F^{*-1}(\mathfrak{A}(\omega))\) determines a regular preference functor from the category \(\mathcal{G}(X, Y)\) to the category \(\mathcal{Q}(X)\).

**Proof.**

Let \((A, \omega)\) and \((B, \delta)\) be ordered sets. Let \(f : A \to B\) be an isotonic map. Let further \((\varphi, \psi) \in \mathfrak{A}(\omega)\). The last one means that \(\psi^{-1} \circ \omega \circ \varphi \in \mathfrak{A}\). Since \(\psi^{-1} \circ \omega \circ \varphi \subseteq \psi^{-1} \circ f^{-1} \circ f \circ \omega \circ f^{-1} \circ f \circ \varphi \subseteq (f \circ \psi)^{-1} \circ \delta \circ (f \circ \varphi)\), we have \((f \circ \varphi, f \circ \psi) \in \mathfrak{A}(\delta)\).

The fact above can be expressed in the following words: every closed submonoid \(\mathfrak{A}\) of \(\mathfrak{R}(Y)\) determines a functor \(\Xi\) from the category of ordered sets to the category \(\mathcal{Q}\) of preordered sets. This functor assigns to every ordered set \((A, \omega)\) the preordered set \((A^Y, \mathfrak{A}(\omega))\) and to every isotonic map \(f\) of \((A, \omega)\) in \((B, \delta)\) the isotonic map \(\tilde{f}\) of \((A^Y, \mathfrak{A}(\omega))\) into \((B^Y, \mathfrak{A}(\delta))\).

Now we apply the part (2) of Proposition. The condition (ii) is fulfilled. Verify the condition (i), that is for all \(\varphi_1, \varphi_2 \in A^Y:\)

\((\forall y \in Y) \varphi_1(y) \leq \omega \varphi_2(y) \Rightarrow (\varphi_1, \varphi_2) \in \mathfrak{A}(\omega)\) holds.

Indeed, if \((\forall y \in Y) \varphi_1(y) \leq \omega \varphi_2(y)\) then \(\Delta_Y \subseteq \varphi_2^{-1} \circ \omega \circ \varphi_1\) and therefore \(\varphi_2^{-1} \circ \omega \circ \varphi_1 \in \mathfrak{A}\).
Then the conclusion of Proposition means that the map \( \Psi \) which assigns to every DMP \( G = (X, Y, A, \omega, F) \) the preorder relation \( F_{x_2}^{-1}(\mathfrak{A}(\omega)) \) and to every morphism \( f : G = (X, Y, A, \omega, F) \to G^1 = (X, Y, B, \delta, H) \) the inclusion \( F_{x_2}^{-1}(\mathfrak{A}(\omega)) \subseteq H_{x_1}^{-1}(\mathfrak{A}(\delta)) \) is a preference functor from the category \( G(X, Y) \) to the category \( Q(X) \).

Further, we have for two DMPs \( G = (X, Y, A, \omega, F) \) and \( G^1 = (X, Y, B, \delta, H) : \)

\[
(x_1, x_2) \in \Psi(G) \Leftrightarrow (F_{x_1}^*, F_{x_2}^*) \in \mathfrak{A}(\omega),
\]

\[
(x_3, x_4) \in \Psi(G^1) \Leftrightarrow (H_{x_3}^*, H_{x_4}^*) \in \mathfrak{A}(\delta).
\]

If \( \rho_G(x_1, x_2) = \rho_{G^1}(x_3, x_4) \), then according to (5) right sides of these formulas are equivalent and hence \( (x_1, x_2) \in \Psi(G) \Leftrightarrow (x_3, x_4) \in \Psi(G^1) \), that is (6) is fulfilled. Thus the functor \( \Psi \) is regular. \( \Box \)

Theorem 1 shows that choosing some closed submonoid \( \mathfrak{A} \) of the monoid of all binary relations on the set \( Y \) we can construct derivative preference relations for every DMP from \( G(X, Y) \). The next theorem shows that this is a way to obtain all regular preference functors.

**Theorem 2.**

Let \( \Xi \) be a regular preference functor from the category ordered sets to the category \( Q \). Denote by \( \mathfrak{A} \) the set of all binary relations on the set \( Y \) of the form \( \psi^{-1} \circ \omega \circ \varphi \) where \( (\varphi, \psi) \in \Xi(\omega) \). Then \( \mathfrak{A} \) is a closed monoid of relations and \( \mathfrak{A}(\omega) = \Xi(\omega) \) for every ordered set \( (A, \omega) \).

**Proof.** Consider the trivially ordered set \( (Y, \Delta_Y) \). Since for every \( \varphi : Y \to Y \), \( (\varphi, \varphi) \in \Xi(\Delta_Y) \) holds, we have \( \varphi^{-1} \circ \Delta_Y \circ \varphi \in \mathfrak{A} \). For the case \( \varphi \) is the identity map, we obtain that \( \Delta_Y \in \mathfrak{A} \).

Let \( \rho, \sigma \in \mathfrak{A} \). Then there are ordered sets \( (A, \omega) \) and \( (B, \delta) \) and there are maps \( \varphi_1, \psi_1 : Y \to A \), \( \varphi_2, \psi_2 : Y \to B \) such that \( (\varphi_1, \psi_1) \in \Xi(\omega) \), \( (\varphi_2, \psi_2) \in \Xi(\delta) \) and \( \rho = \psi_1^{-1} \circ \omega \circ \varphi_1 \), \( \sigma = \psi_2^{-1} \circ \omega \circ \varphi_2 \). It can be assumed that sets \( A \) and \( B \) have not common elements. Consider the relation \( \tau = \omega \cup \delta \cup \delta \circ \varphi_2 \circ \psi_1^{-1} \circ \omega \) on the set
\[ A \cup B. \] It is easy to see that \( \tau \) is an order relation on this set. For each \( y \in Y \) we have
\[
(\psi_1(y), \varphi_2(y)) \in \delta \circ \varphi_2 \circ \psi_1^{-1} \circ \omega \subseteq \tau.
\]

Consider two inclusions \( f : A \to A \cup B \) and \( g : B \to A \cup B \). Under definition \( \tau \), they are isotonic. We conclude from the formula above, that \((f \circ \psi_1, g \circ \varphi_2) \in \Xi(\tau)\). But since the maps are isotonic, we have also \((f \circ \varphi_1, f \circ \psi_1) \in \Xi(\tau)\) and \((g \circ \varphi_2, g \circ \psi_2) \in \Xi(\tau)\). Therefore \((f \circ \varphi_1, g \circ \psi_2) \in \Xi(\tau)\). Further we have:
\[
\psi_2^{-1} \circ g^{-1} \circ \tau \circ f \circ \varphi_1 = \psi_2^{-1} \circ \delta \circ \varphi_2 \circ \psi_1^{-1} \circ \omega \circ \varphi_1 = \sigma \circ \rho
\]
and hence \( \sigma \circ \rho \in A \). Thus we have proved that \( A \) is a monoid.

Now we have to proof that this monoid is a closed submonoid of \( \mathfrak{A}(Y) \). Firstly observe a trivial fact that every relation \( \sigma \subseteq Y \times Y \) can be represented in the form:
\[
\sigma = \psi^{-1} \circ \omega \circ \varphi \text{ for some ordered set } (A, \omega) \text{ and maps } \varphi, \psi : Y \to A. \text{ Indeed, let } Y_1 \text{ and } Y_2 \text{ be two copies of } Y \text{ without common elements. Let } \varphi : Y \to Y_1 \text{ and } \psi : Y \to Y_2 \text{ be the corresponding identification maps. Define } \omega = \Delta_{Y_1} \cup \Delta_{Y_2} \cup \psi \circ \sigma \circ \varphi^{-1}.
\]
It is true that \( \omega \) is an order relation on \( A = Y_1 \cup Y_2 \) because a pair of different elements belongs to \( \omega \) only if the first of them belongs to \( Y_1 \) and the second one belongs to \( Y_2 \). It is easy to see that \( \sigma = \psi^{-1} \circ \omega \circ \varphi \).

Let \( \rho \subseteq \sigma \) and \( \rho \in \mathfrak{A} \). The last one means that there exist \( \varphi_1, \psi_1 : Y \to B \) and order relation \( \delta \) in \( B \) such that \( \rho = \psi_1^{-1} \circ \delta \circ \varphi_1 \) and \( (\varphi_1, \psi_1) \in \Xi(\delta) \). As it was mentioned above, \( \sigma = \psi^{-1} \circ \omega \circ \varphi \) for some maps \( \varphi, \psi : Y \to B \) and order relation \( \omega \) in \( B \). Consider the ordered set \( (A \times B, \omega \times \delta) \) where \(((a_1, b_1), (a_2, b_2)) \in \omega \times \delta \leftrightarrow (a_1, a_2) \in \omega \ \& \ (b_1, b_2) \in \delta \). We have two natural maps \( \alpha, \beta : Y \to A \times B \) where \( \alpha(y) = (\varphi(y), \varphi_1(y)) \) and \( \beta(y) = (\psi(y), \psi_1(y)) \). It is easy to see that \( \beta^{-1} \circ (\omega \times \delta) \circ \alpha = \rho \). Since the functor \( \Xi \) is regular, the last equality means that \( (\alpha, \beta) \in \Xi(\omega \times \delta) \). Using the isotonic map \( \pi : A \times B \to A \), projection on \( A \), we obtain that \( (\varphi, \psi) \in \Xi(\omega) \), and hence \( \sigma \in \mathfrak{A} \). Therefore the monoid \( \mathfrak{A} \) is closed.

Now let \((\varphi, \psi) \in \mathfrak{A}(\omega)\) for some ordered set \((A, \omega)\). It is equivalent according to (8) that \( \rho = \psi^{-1} \circ \omega \circ \varphi \in \mathfrak{A} \). The last fact means that there exist \( \varphi_1, \psi_1 : Y \to B \)
and order relation $\delta$ in $B$ such that $\rho = \psi_1^{-1} \circ \delta \circ \varphi_1$ and $(\varphi_1, \psi_1) \in \Xi(\delta)$. Since the functor $\Xi$ is regular, it is equivalent to $(\varphi, \psi) \in \Xi(\omega)$. Therefore $\mathfrak{A}(\omega) = \Xi(\omega). \quad \square$

3. Examples and applications.

Thus to observe all closed preference functors from the category $G(X, Y)$, we have to observe all closed monoids of relations on the set $Y$. In reality, it is sufficient to choose a submonoid of $\mathfrak{R}(Y)$ and then consider all relations which include its members. All closed submonoids of $\mathfrak{A}$ form a lattice with respect to inclusion. This lattice is complete. The least element of this lattice is submonoid of all reflexive relations and the greatest one is $\mathfrak{R}(Y)$ itself. We show below that the submonoid of all reflexive relations gives the Pareto-domination. Clearly, $\mathfrak{R}(Y)$ gives the greatest preference relation, i. e. the complete relation on $X$: $X \times X$. If $\mathfrak{A}$ is a closed submonoid of $\mathfrak{R}(Y)$, then the set of all relations $\varrho^{-1}$ for $\varrho \in \mathfrak{A}$ also is a submonoid of $\mathfrak{R}(Y)$. Thus to every preference functor, there is the dual one. If they coincide, we have a self-dual preference functor. Below we give some useful examples.

Examples

1. Consider the set of all reflexive relations on the set $Y$. Clearly, it is a submonoid of $\mathfrak{R}$. Let $\mathfrak{A}$ be this submonoid. The corresponding derivative preference relation is the Pareto-domination. Indeed,

$$(\alpha, \beta) \in \mathfrak{A}(\omega) \iff \Delta_Y \subseteq \beta^{-1} \circ \omega \circ \alpha \iff (\forall y \in Y) \alpha(y) \leq^\omega \beta(y) \iff (\alpha, \beta) \in \hat{\omega}. \quad (9)$$

2. We call a relation on $Y$ surjective if its second projection is equal to $Y$. Clearly, all such relations form a submonoid of $\mathfrak{A}$. Let $\mathfrak{A}$ be the submonoid of all surjective relations. The corresponding derivative preference relation is the $\beta$-domination defined in Section 1. Indeed,

$$(\alpha, \beta) \in \mathfrak{A}(\omega) \iff pr_2(\beta^{-1} \circ \omega \circ \alpha) = Y \iff (\forall y \in Y) (\exists y' \in Y) \alpha(y') \leq^\omega \beta(y). \quad (10)$$

Thus

$$(x_1, x_2) \in F^{*-1}(\mathfrak{A}(\omega)) \iff (\forall y)(\exists y')F(x_1, y') \leq^\omega F(x_2, y). \quad (11)$$
3. Dually to the previous example, let \( \mathfrak{A} \) be the submonoid of all everywhere defined relations (it means the first projection of a relation is equal to \( Y \)). The corresponding derivative preference relation is the dual \( \beta \)-domination. The proof is the same that above.

4. Recall that a filter on the set \( Y \) is a set \( \mathfrak{F} \) of non-empty subsets of \( Y \) satisfying the following conditions: 1) \( Y \in \mathfrak{F} \), 2) \( A, B \in \mathfrak{F} \) implies \( A \cap B \in \mathfrak{F} \) and 3) \( A \in \mathfrak{F} \) and \( A \subseteq B \) implies \( B \in \mathfrak{F} \).

Let \( \mathfrak{F} \) be a filter on the set \( Y \). Define \( \mathfrak{A} = \{ \lambda \subseteq Y \times Y \mid pr_\Delta \in \mathfrak{F} \} \). The corresponding derivative preference relation is the preference according to the filter. It can be expressed as follows: a strategy \( x_2 \) is more preferable than a strategy \( x_1 \) if the set of all \( y \) for which \( F(x_1, y) \leq_\omega F(x_2, y) \) belongs to the filter \( \mathfrak{F} \). The well known interpretation: "the majority vote for".

A filter \( \mathfrak{F} \) is called principal if it is of the following form: \( \mathfrak{F} = \{ P \subseteq Y \mid P_0 \subseteq P \} \), where \( P_0 \) is a fixed subset of \( Y \). One can consider the set \( P_0 \) as a system of dominators. If particularly \( P_0 \) consists of one state only, this state can be considered as an indicator (or a dictator): a strategy \( x_1 \) is preferable than \( x_2 \) if and only if the strategy \( x_1 \) gives for this state the result better than \( x_2 \). If the set \( Y \) is finite, every filter is principal.

5. Let \( \sigma \) be an idempotent relation on the set \( Y \), that is \( \sigma \circ \sigma = \sigma \). It means that a decision maker has some special preferences on the set of states. Denote by \( \mathfrak{A} \) the closed monoid generated by \( \sigma \). It consists of reflexive binary relations and all binary relations containing \( \sigma \). We have

\[(\alpha, \beta) \in \mathfrak{A}(\omega) \iff (\alpha, \beta) \in \hat{\omega} \lor (\forall y_1, y_2 \in Y)((y_1, y_2) \in \sigma \Rightarrow \alpha(y_1) \leq_\omega \beta(y_2)). \quad (12)\]

According to this we obtain a new derived preference relation.

We recall that Example 3 given in Section 1 presents two strategies equivalent with respect to \( \alpha \)-domination such that one of them strongly Pareto-dominates the other. Although \( \alpha \)-domination can not be realized by means of a preference functor, such undesirable event can appear, for example if one chooses the greatest monoid \( \mathfrak{A} \).
Definition 3. Let $G$ be a DMP. Let $P(G)_{str}$ denote the strong Pareto-domination for $G$. A preference functor $\Psi : \mathcal{G}(X,Y) \to \mathcal{Q}(X)$ is called suitable for the $G$ if the following condition is satisfied:

$$(A5)$$

$$\Psi(G) \cap P(G)_{str}^{-1} = \emptyset.$$ 

The condition above means that it is impossible that two strategies $x_1$ and $x_2$ are equivalent with respect to preference relation $\Psi(G)$ but $x_2$ strong Pareto-dominates $x_1$. It is obvious that Pareto preference functor is suitable for all DMPs. In general case a preference functor can be suitable for some DMPs and non-suitable for another ones.

Proposition 2. Let $\mathfrak{A}$ be a closed submonoid of $\mathcal{R}(Y)$. If every relation $\rho \in \mathfrak{A}$ has a fixed point $y$, i.e. $(y,y) \in \rho$, then the preference functor $\Psi$ determined by $\mathfrak{A}$ is suitable for every DMP.

Proof. Let $G = (X,Y,A,\omega,F)$ be a DMP. Suppose that $(\alpha, \beta) \in \mathfrak{A}(\omega)$. It means that $\beta^{-1} \circ \omega \circ \alpha \in \mathfrak{A}$. Under hypotheses, this relation has a fixed point $y_0$. For this point $\alpha(y_0) \leq^\omega \beta(y_0)$ holds. Hence it is impossible that $(\forall y)\beta(y) <^\omega \alpha(y)$. It means that $\Psi(G) \cap P(G)_{str}^{-1} = \emptyset$. $\blacksquare$

Corollary 1. The preference functor according to a filter on the set $Y$ (example 4) is suitable for every DMP.

Proposition 3. Let $\mathfrak{A}$ be a non-universal closed submonoid of $\mathcal{R}(Y)$. If $\mathfrak{A}(\omega) \cap \omega_{str}^{-1} \neq \emptyset$, then $(\mathfrak{A},\omega)$ contains a chain $a_1 < a_2 < \ldots < a_k$ for every positive integer $k$.

Proof. Suppose that there are $\varphi, \psi \in A^Y$ such that $(\varphi, \psi) \in \mathfrak{A}(\omega)$ but $\psi(y) <^\omega \varphi(y)$ for all $y \in Y$. Then the relation $\rho = \{(y_1,y_2) | \varphi(y_1) \leq^\omega \psi(y_2)\} = \psi^{-1} \circ \omega \circ \varphi$ belongs to $\mathfrak{A}$. Hence $\rho^k \in \mathfrak{A}$ for arbitrary natural number $k$. Since $\varphi \circ \psi^{-1} \subseteq \omega_{str}$, $\rho^2 = \psi^{-1} \circ \omega \circ \varphi \circ \psi^{-1} \circ \omega \circ \varphi \subseteq \psi^{-1} \circ \omega_{str} \circ \varphi$, and further $\rho^3 \subseteq \psi^{-1} \circ \omega_{str}^2 \circ \varphi$, and so long, ..., $\rho^k \subseteq \psi^{-1} \circ \omega_{str}^{k-1} \circ \varphi$. Since submonoid $\mathfrak{A}$ is closed $\psi^{-1} \circ \omega_{str}^k \circ \varphi \in \mathfrak{A}$ for all $k$. Under hypotheses, $\mathfrak{A}$ is not universal and therefore all these relations
are not empty. It gives a sequence $a_1, \ldots, a_k$ for every integer $k > 0$ such that $a_1 < a_2 < \ldots < a_k$. □

**Corollary 2.** If lengths of all strong chains in the set of outcomes of a DMP $G$ are bounded above, then every non trivial derivative preference relation is suitable for $G$, particularly for all DMPs with finite set of outcomes.

Proposition 3 generalizes the following result obtained in [Rozen (2001), Theorem 2].

**Corollary 3.** If the ordered set of outcomes for a DMP $G$ satisfies descending (increasing) chain condition then $\beta$-domination (inverse $\beta$-domination) preference functor is suitable for $G$.

**Proof.** The submonoid $\mathfrak{A}$ corresponding to $\beta$-domination preference functor consists of all surjective relations. Suppose that $\mathfrak{A}(\omega) \cap \omega_{\text{str}}^{-1} \neq \emptyset$. Then with the notation of the proof of Proposition 3, the relation $\psi^{-1} \circ \omega_{\text{str}}^k \circ \varphi$ is surjective. It means that for every $y \in Y$ there is $y_1 \in Y$ such that $(\varphi(y_1), \varphi(y)) \in \omega_{\text{str}}^k$. It leads to an infinite descending chain in $A$. This contradiction shows that $\beta$-domination preference functor is suitable for $G$.

## 5. Conclusion

The approach suggested in the paper and based on category theory leads to a strong definition of derived preferences of a decision maker and the obtained results give a method to observe all such preferences and compare them with each other. The most important point is that we start from minimal conditions and therefore include all reasonable other approaches but exclude all anomalies mentioned in Section 2. The second point is that we reduce the problem formulated in Introduction to pure algebraical one, namely to study closed monoids of binary relations. Indeed, using Theorems 1 and 2 it is possible to build (of course, if the set $Y$ of states is finite) all regular preference functors for given set $X$ of strategies. One must mention that these theorems establish connections between regular functors
(from the category of all ordered sets to the category of preordered sets) and closed monoids and are new.

It was said that all closed monoids of relations on the set $Y$ form a lattice. This lattice is a complete lattice. If a monoid $\mathcal{A}_1$ is a submonoid of a monoid $\mathcal{A}_2$ and $\xi_1$ and $\xi_2$ are the corresponding derived preferences relations on the set $X$ for a DMP $G$, then we have $\xi_1 \subseteq \xi_2$. It can be interpreted that preference $\xi_2$ is stronger than $\xi_1$: if $x_2$ is preferable than $x_1$ in the sense of $\xi_1$ it is preferable than $x_1$ in the sense of $\xi_2$.

In the simple cases, when the set $Y$ is not too large, one can describe explicitly all derivative preference relations. For example, when the set $Y$ has only two elements there are only 16 binary relations on $Y$, the derivative preference relations are: two extremal relations (the universal one and the Pareto-domination), two relations associated with two principal filters, the $\beta$-domination, the inverse $\beta$-domination (four maximal relations only) and their intersections. In the case when $Y$ contains more elements it is more complicated computational problem, but it seems that the most important thing is to give a qualitative description which does not depend on a set of states.

The Pareto-domination is the weakest derivative preference relation. The universal (all strategies are equivalent) relation is the strongest derivative preference relation. It is very easy to prove that the monoid of all surjective relations and the monoid of all everywhere defined relations are maximal closed submonoids (dual atoms). It means that $\beta$-domination and inverse $\beta$-domination are maximal preferences, i.e. there are no preference relations between each of them and the universal relation.

It may be that a decision maker has some preferences also in the set $Y$. One can take such preferences in consideration choosing a filter on the set $Y$ or some idempotent relation on it. This leads to the derived preference relations described in the examples 4 and 5 in the Section 4. The preference relations associated with principal filters (the case of dictators) are also maximal ones.
Taking intersections of these known monoids we obtain new preference relations that are weaker than their parents. For example, considering the intersection of \(\beta\)- and inverse \(\beta\)-domination (i.e. the monoid of all everywhere defined surjective relations) gives a new preference relation that can be expressed by the formula: \(x_2\) is preferable than \(x_1\) iff

\[
(\forall y_1)(\exists y_2) F(x_1, y_1) \leq F(x_2, y_2) \land (\forall y_2)(\exists y_1) F(x_1, y_1) \leq F(x_2, y_2).
\] (13)

On the other hand one can consider minimal preference relations (atoms). Let \(y_0 \in Y\) be a fixed element. Build the relation \(\rho = Y \times Y \setminus \{(y_0, y_0)\}\). It is a maximal relation and \(\rho \circ \rho = Y \times Y\). Thus all reflexive relations and \(\rho\) form a closed monoid that clearly is a minimal element in the lattice of all closed monoids. This is a way to build all minimal derived preference relations. The sense of them is obvious: \(x_2\) is preferable than \(x_1\) iff \(x_2\) Pareto-dominates \(x_1\) or \(F(x_1, y_1) \leq F(x_2, y_2)\) for all pairs of states \((y_1, y_2)\) different from \((y_0, y_0)\). It means that if a decision maker wants to ignore for some reason the correlation between \(F(x_1, y_0)\) and \(F(x_2, y_0)\) he has to check the inequality above for all pairs of states, with the exception of \((y_0, y_0)\).

We see that there are many different preference functors. But it is not a shortcoming, it is the nature of things. Practically, we choose one of these functors following supplementary data. For example, one can consider some structures on the set of states like mentioned above or a probability distribution and use corresponding preference functors.
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