Extension factor: definition, properties and problems. Part 1

Krassimir T. Atanassov¹ and József Sándor²

¹ Department of Bioinformatics and Mathematical Modelling
IBPhBME – Bulgarian Academy of Sciences,
Acad. G. Bonchev Str. Bl. 105, Sofia-1113, Bulgaria
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8010, Bulgaria
e-mail: krat@bas.bg

² Babes-Bolyai University of Cluj, Romania
e-mail: jjsandor@hotmail.com, jsandor@math.ubbcluj.ro

Received: 12 March 2019
Accepted: 30 June 2019

Abstract. A new arithmetic function, called “Extension Factor” is introduced and some of its properties are studied.

Keywords: Arithmetic function, Extension factor.

2010 Mathematics Subject Classification: 11A25

1 Introduction

In a series of papers, published during the last 35 years, the authors introduced some new arithmetic functions. One of them was called “Restrictive Factor” (see, [2, 3]). For each natural number \( n = \prod_{i=1}^{k} p_i^{\alpha_i} \), where \( k, \alpha_1, \alpha_2, ..., \alpha_k \geq 1 \) are natural numbers and \( p_1, p_2, ..., p_k \) are different prime numbers,

\[
RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1},
\]

\[
RF(1) = 1.
\]
In the present paper, for each natural number \( n \), having the above form, we will introduce a new arithmetic function, in some sense, opposite to the restrictive factor.

In the text, we will use also the definitions of the following three well-known arithmetic functions:

\[
\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1), \quad \varphi(1) = 1 - \text{Euler’s totient function},
\]

\[
\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i + 1), \quad \psi(1) = 1 - \text{Dedekind’s function},
\]

\[
\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}, \quad \sigma(1) = 1
\]

(see [4, 7]).

We will use also the arithmetic functions

\[
mult(n) = \prod_{i=1}^{k} p_i, \quad mult(1) = 1,
\]

\[
B(n) = \sum_{i=1}^{k} \alpha_i p_i, \quad B(1) = 1,
\]

(see [1, 7]), and

\[
\delta(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i-1} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k}, \quad \delta(1) = 1.
\]

(see [1]).

2 Main results

Here, we juxtapose to the natural number \( n \) the (natural) number

\[
EF(n) = \prod_{i=1}^{k} p_i^{\alpha_i+1}, \quad EF(1) = 1
\]

that we call Extension Factor.

Hence,

\[
EF(n) = n.multip(n).
\]

The first 40 values of \( EF \) are given in Table 1.

If \((m, n) = 1\), where for the natural numbers \( m, n \), \((m, n)\) is the Greatest Common Divisor (GCD), then

\[
EF(m.n) = EF(m).EF(n),
\]

i.e., \( EF \) is a multiplicative function,

\[
EF(n) = \prod_{i=1}^{k} EF(p_i^{\alpha_i}),
\]
\[
EF(n) = EF\left( \prod_{i=1}^{k} p_i^{\alpha_i} \right) = \prod_{i=1}^{k} p_i^{\alpha_i+1} \leq \prod_{i=1}^{k} p_i^{2\alpha_i} = n^2.
\]

On the other hand, it can be seen that if for every \( i \) (\( 1 \leq i \leq k \)) \( \alpha_i = 1 \), then
\[
EF(n) = n^2.
\]

Therefore, for each prime number \( p \):
\[
EF(p) = p^2.
\]

Moreover, for every natural number \( n \):
\[
\text{mult}(n^2) \leq EF(n) \leq n^2.
\]

From the definitions of functions \( RF \) and \( EF \) it follows the basic identity
\[
EF(n).RF(n) = n^2. \tag{1}
\]

Therefore, \( EF(n) = n^2 \) if and only if \( RF(n) = 1 \), i.e. when \( n = \text{mult}(n) \), so when \( n \) is a squarefree number.

**Theorem 1.** For every two natural numbers \( m \) and \( n \):
\[
EF(m).EF(n) = EF(m.n).\text{mult}(m.n)).
\]

**Proof.** Let \((m, n) = r \geq 1\) and let \( m = s.r, n = t.r \). Then
\[
EF(m).EF(n) = EF(s.r).EF(t.r) = (s.r.\text{mult}(s.r)).(t.r.\text{mult}(t.r))
\]
\[
= s.r^2.t.\text{mult}(s).\text{mult}(r)^2.\text{mult}(t) = (s.r^2.t.\text{mult}(s).\text{mult}(r).\text{mult}(t)).\text{mult}(r)
\]
\[
= EF(m.n).\text{mult}(r) = EF(m.n).\text{mult}(m.n)). \qed
\]
Theorem 1 follows also from the definitions, and the following property of the function \texttt{mult}:

\[
\text{mult}(n) \cdot \text{mult}(m) = \text{mult}(mn) \cdot \text{mult}((m,n)).
\]

**Theorem 2.** For every natural number \(n\):

\[
RF(EF(n)) = n \geq EF(RF(n)).
\]

**Proof.** For \(n = 1\), the statement is true. Let \(n = \prod_{i=1}^{k} p_i^{\alpha_i}\) and let for each real number \(x\)

\[
\text{sg}(x) = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x \leq 0
\end{cases}
\]

Then

\[
RF(EF(n)) = RF\left(\prod_{i=1}^{k} p_i^{\alpha_i+1}\right)
\]

\[
= \prod_{i=1}^{k} p_i^{\alpha_i} = n \geq \prod_{i=1}^{k} p_i^{\alpha_i \cdot \text{sg}(\alpha_i-1)}
\]

(so, we eliminate the prime numbers with power 1)

\[
= EF\left(\prod_{i=1}^{k} p_i^{\alpha_i-1}\right) = EF(RF(n)).
\]

Another proof of Theorem 2 follows from:

\[
\text{mult}(n \cdot \text{mult}(n)) = n
\]

and

\[
\text{mult}\left(\frac{n}{\text{mult}(n)}\right) \leq n.
\]

(2) follows from the fact that \(n\) and \(\text{mult}(n)\) have the same prime factors, while (3) from the fact that the prime factors of \(\frac{n}{\text{mult}(n)}\) are among the prime factors of \(n\).

There is equality in (3) only when \(n > 1\) is squarefull number (i.e. when from each prime power divisor \(p^a\) of \(n\) one has \(a \geq 2\)). Thus one has

\[
\text{mult}(RF(n)) \leq \text{mult}(n)
\]

and

\[
\text{mult}(EF(n)) = n
\]

and the result follows.

It could be mentioned that there is equality in Theorem 2 only when \(n\) is squarefull.

**Theorem 3.** For every natural number \(n\):

(a) \(\varphi(EF(n)) = \varphi(n) \cdot \text{mult}(n)\),
(b) $\psi(EF(n)) = \psi(n).\text{mult}(n)$.

(c) $\sigma(EF(n)) \geq \sigma(n).\text{mult}(n)$.

**Proof.** The statement is obviously true for $n = 1$. Let $n > 1$ be a natural number. Then

$$\varphi(EF(n)) = \varphi \left( \prod_{i=1}^{k} p_i^{\alpha_i+1} \right) = \prod_{i=1}^{k} p_i^{\alpha_i}(p_i - 1) = \varphi(n).\text{mult}(n),$$

i.e., (a) is valid. (b) is proved analogously, while the proof of (c) is the following.

$$\sigma(EF(n)) = \sigma \left( \prod_{i=1}^{k} p_i^{\alpha_i+1} \right) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+2} - 1}{p_i - 1} = \sigma(n).\prod_{i=1}^{k} \frac{p_i^{\alpha_i+2} - 1}{p_i^{\alpha_i+1} - 1} \geq \sigma(n).\text{mult}(n). \text{ □}$$

Another proof of inequality (c) of Theorem 3 is based on the known inequality $\sigma(ab) \geq a.\sigma(b)$, with equality only for $a = 1$. Let $a = \text{mult}(n)$, $b = n$, and the result follows.

When $a = n$, $b = \text{mult}(n)$, one obtains another inequality:

$$\sigma(EF(n)) \geq n.\sigma(\text{mult}(n)) = n.(p_1 + 1)...(p_k + 1),$$

where $p_1, ..., p_k$ are the distinct prime factors of $n$. Since

$$(p_1 + 1)...(p_k + 1) = \frac{\psi(n)}{RF(n)},$$

we get the inequality:

$$\sigma(EF(n)) \geq \frac{n\psi(n)}{RF(n)}.$$  

Another result of this type is the following

**Theorem 4.** For every natural number $n$:

$$\sigma(EF(n)) \leq \frac{\sigma(n).\psi(n)}{RF(n)}.$$  

**Proof.** For $n = 1$, the statement is obviously true. Applying the known inequality $\sigma(ab) \leq \sigma(a).\sigma(b)$ for $a = n$ and $b = \text{mult}(n)$, and assuming the distinct prime factors of $n$ to be $p_1, ..., p_k$, note that one has

$$\sigma(\text{mult}(n)) = (p_1 + 1)...(p_k + 1) = \frac{\psi(n)}{RF(n)}.$$  

The result follows by the definitions. \text{ □}

**Theorem 5.** For every natural number $n > 1$:

$$EF(n) > \sigma(n).$$  

**Proof.** Let $n > 1$ be a natural number. Then

$$EF(n) = \prod_{i=1}^{k} p_i^{\alpha_i+1} > \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = \sigma(n). \text{ □}$$
The inequality of Theorem 5 can be improved when \( n \) is odd.

**Theorem 6.** When \( n > 1 \) is odd number, then

\[
EF(n) > \sigma(n) + n.
\]

**Proof.** Apply the known inequality \( \sigma(n).\phi(n) < n^2 \) (see e.g. [4, 7]). Thus, \( \sigma(n) < \frac{n^2}{\phi(n)} \). Since

\[
\frac{n}{\phi(n)} = \frac{p_1 \cdots p_k}{(p_1 - 1)(p_2 - 1)}, \quad \text{and } EF(n) - n = n(p_1 \cdots p_k - 1),
\]

it will be sufficient to prove that:

\[
\frac{p_1 \cdots p_k}{(p_1 - 1)(p_2 - 1)} \leq p_1 \cdots p_k - 1.
\]

Put \( p_i - 1 = x_i \). Since \( n \) is odd, one has \( x_i \geq 2 \) for all \( i = 1, 2, ..., k \). We have to prove the inequality

\[
x_1 \cdots x_k \leq (x_1 + 1)(x_k + 1)(x_1 \cdots x_k - 1),
\]
or

\[
x_1 \cdots x_k + (x_1 + 1)(x_k + 1) \leq x_1 \cdots x_k(x_1 + 1)(x_k + 1).
\]

Put \( x_1 \cdots x_k = a, (x_1 + 1)(x_k + 1) = b \). Then we have to prove that \( a + b \leq a.b \), or, this can be written also as \( (a - 1)(b - 1) \geq 1 \). This is true, as \( a - 1 \geq x_k - 1 \geq 1 \), and \( b \geq x_1 + 1 \geq 3 \). The inequality is strict. \( \square \)

Now, we will formulate and prove the following common refinement of the last two theorems.

**Theorem 7.**

a) For any natural \( n > 1 \) one has

\[
\sigma(n) < n(\omega(n) + 1) \leq EF(n)
\]

(4)

b) For any odd \( n > 1 \) one has

\[
\sigma(n) < n(\omega(n) + 1) \leq EF(n) - n,
\]

(5)

where \( \omega(n) \) denotes the number of distinct prime factors of \( n \).

**Proof.** The first inequalities of both a) and b), namely

\[
\sigma(n) < (\omega(n) + 1)
\]

appeared for the first time in paper [5] from 1989. A proof is included also in paper [6] from 2010.

Now, to prove the second inequality of (4), note that

\[
\text{mult}(n) = p_1 \cdots p_k \geq 2^k,
\]

where \( p_1, ..., p_k \) are the prime divisors of \( n \), and \( k = \omega(n) \). Now, \( 2^k \geq k + 1 \) holds true for any \( k \geq 1 \). Thus (4) follows, as \( EF(n) = n.\text{mult}(n) \).

For the proof of second inequality of b), note that when \( n > 1 \) is odd, then \( \text{mult}(n) \geq 3^k \), as \( p_1, ..., p_k \geq 3 \). Now, the inequality \( 3^k \geq k + 2 \) for \( k \geq 1 \) follows at once, e.g., by mathematical induction. This proves \( \text{mult}(n) \geq k + 2 \), so (5) follows. \( \square \)
Theorem 8. For every natural number $n$:
\[ EF(n) + RF(n) \geq 2n, \]
with equality only for $n = 1$.

Proof. This follows from the classical inequality $x + y \geq 2\sqrt{xy}$ applied for $x = EF(n), y = RF(n)$, and using the basic identity (1).

Another simple related inequality is the following.

Theorem 9. For every natural number $n > 1$:
\[ n^2 \geq \frac{EF(n)}{RF(n)} \geq 4\omega(n), \]
where $\omega(n)$ is the number of distinct prime factors of $n$.

Proof. There is equality on the right only when $n$ has a single prime factor, i.e., when $\omega(n) = 1$, and on the left, when $n$ is squarefree number. This follows from
\[ \frac{EF(n)}{RF(n)} = (\text{mult}(n))^2. \]
Now, from $\text{mult}(n) \leq n$, the left side inequality follows. For the right side, note that $\text{mult}(n) \geq 2^k$ as any prime divisor is greater or equal to 2.

Theorem 10. For every natural number $n > 1$:
\[ B(\text{EF}(n)) = B(n) + B(\text{mult}(n)). \]

Proof. Let $n > 1$ be a natural number. Then
\[ B(\text{EF}(n)) = B\left(\prod_{i=1}^{k} p_i^{\alpha_i+1}\right) = \sum_{i=1}^{k} (\alpha_i + 1).p_i = \sum_{i=1}^{k} (\alpha_i).p_i + \sum_{i=1}^{k} p_i \]
\[ = B(n) + B(\text{mult}(n)). \]

Theorem 11. For every natural number $n > 1$:
\[ \delta(\text{EF}(n)) = \delta(n)\text{mult}(n) + n\delta(\text{mult}(n)). \]

Proof. Let $n > 1$ be a natural number. Then
\[ \delta(\text{EF}(n)) = \delta\left(\prod_{i=1}^{k} p_i^{\alpha_i+1}\right) = \sum_{i=1}^{k} (\alpha_i + 1)p_1^{\alpha_i+1}...p_{i-1}^{\alpha_i+1}p_i^{\alpha_i}p_{i+i}^{\alpha_i+1}...p_k^{\alpha_k+1} \]
\[ = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_i+1}...p_{i-1}^{\alpha_i+1}p_i^{\alpha_i}p_{i+i}^{\alpha_i+1}...p_k^{\alpha_k+1} + \sum_{i=1}^{k} p_1^{\alpha_i+1}...p_{i-1}^{\alpha_i+1}p_i^{\alpha_i}p_{i+i}^{\alpha_i+1}...p_k^{\alpha_k+1} \]
\[ = \delta(n)\text{mult}(n) + n\delta(\text{mult}(n)). \]
3 Conclusion

In conclusion, we will mention, that in the second part we will study the following problems.

**Problem 1.** To find other equalities and inequalities related to function $EF$.

**Problem 2.** To generalize the function $EF$ to $EF_s$, so that for each natural number $n$:

$$EF_1(n) = EF(n).$$

**Problem 3.** To study the properties of $EF_s$.

References

[1] Atanassov, K. (1987). New integer functions, related to $\varphi$ and $\sigma$ functions, *Bulletin of Number Theory and Related Topics*, XI (1), 3-26.

[2] Atanassov, K. (2002). Restrictive factor: definition, properties and problems. *Notes on Number Theory and Discrete Mathematics*, 8(4), 117-119.

[3] Atanassov, K. (2016) On function “Restrictive factor”, *Notes on Number Theory and Discrete Mathematics*, 22(2), 1722.

[4] Mitrinović, D. S. & Sándor, J., & Crstici, B. (1995). *Handbook of number theory*, Kluwer Acad. Publ.

[5] Sándor, J. (1989). On some Diophantine equations for particular arithmetic functions, *Seminarul de Teoria Structurilor, Univ. Timisoara, Romania*, 53, 1-10.

[6] Sándor, J. (2010). Two arithmetic inequalities, *Advanced Studies in Contemporary Mathematics*, 20(2), 197-202.

[7] Sándor, J. et al., *Handbook of number theory I*, Springer Verlag, 2005 (First printing 1995 by Kluwer Acad. Publ.)