VANISHING THEOREMS AND THE MULTIGRADED REGULARITY
OF NONSINGULAR SUBVARIETIES

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ABSTRACT. Given scheme-theoretic equations for a nonsingular subvariety, we prove that the
higher cohomology groups for suitable twists of the corresponding ideal sheaf vanish. From this
result, we obtain linear bounds on the multigraded Castelnuovo-Mumford regularity of a nonsingu-
lar subvariety, and new criteria for the embeddings by adjoint line bundles to be projectively normal.
A special case of our work recovers the vanishing theorem of Bertram, Ein, and Lazarsfeld.

1. INTRODUCTION

In higher dimensional geometry, vanishing theorems are indispensable for uncovering the deeper
relations between the geometry of a subvariety and its defining equations. The main result of this
paper, a new vanishing theorem for certain twists of the ideal sheaf of a nonsingular subvariety,
adds to this framework. Indeed, our vanishing theorem leads to linear bounds on the multigraded
Castelnuovo-Mumford regularity of nonsingular subvarieties and new conditions for the surjectiv-
ity of the multiplication maps between global sections of adjoint line bundles. Our techniques also
yield a new Griffiths-type vanishing theorem for vector bundles.

Let $X$ be a nonsingular complex projective variety with canonical line bundle $K_X$. For a nonsin-
gular subvariety $Y \subseteq X$ of codimension $e$ with ideal sheaf $\mathcal{I}_Y$, the following is our main theorem.

**Theorem 1.1.** Let $L$ be a line bundle on $X$ and let $m$ be a nonnegative integer. If $Y$ is defined
scheme-theoretically by the nef divisors $D_1, \ldots, D_r$ and $L \otimes \mathcal{O}_X(-(m+1)D_s_1 - D_s_2 - \cdots - D_s_e)$
is a big, nef line bundle for all subsets $\{s_1, s_2, \ldots, s_e\} \subseteq \{1, \ldots, r\}$, then we have
$$H^i(X, \mathcal{I}_Y^{m+1} \otimes K_X \otimes L) = 0, \quad \text{for all } i > 0.$$
Although a scheme $Y$ should be studied from within its ‘natural’ ambient space $X$, the advantages of our main theorem are most pronounced when the nef cone of $X$ has dimension at least two. For example, it is tailor-made to handle the ambient toric varieties that appear in mirror symmetry [CK], embeddings of Mori dream spaces [HK], and tropical compactifications [Tev]. As Remark 3.6 explains, the main theorem is a higher-codimensional version of the Kawamata-Viehweg vanishing theorem. Proposition 3.7 and Example 3.8 demonstrate that it is optimal from this perspective.

Our first primary application concerns the multigraded Castelnuovo-Mumford regularity of $Y$. Let $L$ be a line bundle on $X$ and fix free divisors $P_1, \ldots, P_\ell$ on $X$ such that some positive linear combination is very ample. Following [MS] or [HSS], a sheaf $\mathcal{F}$ is $L$-regular with respect to $P_1, \ldots, P_\ell$ if $H^i(X, \mathcal{F} \otimes L \otimes \mathcal{O}_X(-u_1 P_1 - \cdots - u_\ell P_\ell)) = 0$ for all $i > 0$ and all $(u_1, \ldots, u_\ell) \in \mathbb{N}^\ell$ satisfying $u_1 + \cdots + u_\ell = i$. This is the standard form of Castelnuovo-Mumford regularity when $\ell = 1$ (e.g., see Section 1.8 of [PAG1]). Since the globally generated line bundles $\mathcal{O}_X(P_1), \ldots, \mathcal{O}_X(P_\ell)$ do not have a natural total order, one cannot choose a smallest regularity. Bounding the regularity of $\mathcal{F}$ is, therefore, equivalent to describing a subset of $\{ L \in \text{Pic}(X) : \mathcal{F} \text{ is } L\text{-regular} \}$. By taking $m = 0$ in Theorem 1.1, we obtain the following bounds on the multigraded regularity of $\mathcal{I}_Y$.

**Corollary 1.2.** Let $N$ be a nef line bundle on $X$ and let $Y \subseteq X$ be defined scheme-theoretically by nef divisors $D_1, \ldots, D_r$. Assume that each $D_j$ can be expressed as a positive linear combination of the $P_1, \ldots, P_\ell$. If $L \otimes \mathcal{O}_X(-D_{s_1} - \cdots - D_{s_\ell} - u_1 P_1 - \cdots - u_\ell P_\ell)$ is a big, nef line bundle for all subsets $\{s_1, \ldots, s_\ell\} \subseteq \{1, \ldots, r\}$ and all $(u_1, \ldots, u_\ell) \in \mathbb{N}^\ell$ satisfying $u_1 + \cdots + u_\ell = \dim(Y) + 1$, then the ideal sheaf $\mathcal{I}_Y$ is $K_X \otimes L \otimes N$-regular with respect to $P_1, \ldots, P_\ell$.

As an immediate consequence, we see that the multigraded regularity of $Y$ grows at most linearly in terms of its defining equations (see Remark 4.1). In other words, even when embedded into an ambient space other than $\mathbb{P}^d$, the ‘algebraic complexity’ of a nonsingular variety is essentially as small as possible. In the special case that $X = \mathbb{P}^d$ and $\mathcal{O}_X(P_1) = \mathcal{O}_X(1)$, Example 4.3 shows that Corollary 1.2 becomes the sharp bound in Corollary 4 (i) in [BEL]. Since Theorem 4.7 in [CU] and Corollary 1.2 in [dFE] both generalize Corollary 4 in [BEL] by allowing $X$ and $Y$ to have mild singularities, one certainly hopes that Corollary 1.2 can be strengthened in a similar way, but our techniques do not currently accommodate singularities.

Our second application involves adjoint line bundles, that is line bundles of the form $K_X \otimes P$ where $P$ is a suitably positive line bundle. Set $d := \dim(X)$ and let $A_1, \ldots, A_d$ be very ample line bundles on $X$. Applying Theorem 3.4 to the diagonal $\Delta(X) \subseteq X \times X$ yields the following criterion.

**Corollary 1.3.** If $L_1$ and $L_2$ are line bundles on $X$ such that $L_j \otimes A_j$ is big and nef for all $1 \leq j \leq 2$, then the multiplication map

$$H^0(X, K_X \otimes L_1) \otimes H^0(X, K_X \otimes L_2) \to H^0(X, K_X^2 \otimes L_1 \otimes L_2)$$

is surjective. In particular, the adjoint bundle $K_X \otimes L_1$ defines a projectively normal embedding of $X$ provided it is very ample.
We recover Variant 3.2 in [BEL] when $A_1 = \cdots = A_d$ and Example 5.7 highlights the comparative power of Corollary 1.3. By replacing the very ample line bundles with globally generated line bundles that define a closed embedding into a product of projective spaces, Proposition 5.3 provides an analogous criteria for the multiplication map to be surjective. Example 5.6 illustrates the utility of these alternative hypotheses. Finally, by employing Theorem 1.1 with $m > 0$, Proposition 5.9 gives similar surjectivity statements for the $m$-th Wahl map.

In contrast with the multiplier ideal methods in [Ber] and [dFE], the key tool for proving Theorem 1.1 is an asymptotic multiplier ideal. Although the basic outline of our proof parallels [Ber], the asymptotic variant delivers three pivotal features. First, there is a slightly stronger variant of Nadel vanishing involving only a big line bundle (e.g., see Theorem 11.2.12 (ii) in [PAG1]). Second, asymptotic multiplier ideals depend only on the numerical class of a line bundle (see Example 11.3.12 in [PAG1]) and this provides some leeway in establishing that a given ideal is trivial. Third, we can use asymptotic constructions such as a generalization of Wilson’s theorem (see Lemma 3.2). Our approach, especially Lemma 3.3 and Theorem 3.4, underscores an intriguing connection between local properties of the divisors $D_1, \ldots, D_r$ and vanishing statements. As an added benefit, these ideas also produce the following vanishing theorem for vector bundles.

**Proposition 1.4.** Let $E$ be a vector bundle of rank $e$ on $X$ that is a quotient of a direct sum of line bundles $\mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_r) \rightarrow E$ where each divisor $D_j$ is nef. If $m$ is a positive integer and the line bundle $L \otimes \mathcal{O}_X((m+1)D_{s_1} + D_{s_2} + \cdots + D_{s_e})$ is big, nef for all subsets $\{s_1, \ldots, s_e\} \subseteq \{1, \ldots, r\}$, then we have $H^i(X, K_X \otimes \det(E) \otimes \text{Sym}^m(E) \otimes L) = 0$ for all $i > 0$.

Just as with the main theorem, we recover Proposition 1.12 in [BEL] under the assumption that each $D_j$ belongs to the linear system for some power of a single globally generated line bundle. Assuming $E$ is a quotient of a direct sum of nef line bundles, Proposition 1.4 also improves on Example 7.3.3 in [PAG2].

2. Notation and conventions

- We write $\mathbb{N}$ for the set of nonnegative integers.
- We work throughout over the complex numbers $\mathbb{C}$.
- Let $X$ be a nonsingular projective variety of dimension $d$ and let $K_X$ be its canonical line bundle.
- A subvariety $Y \subseteq X$ is defined scheme-theoretically by the divisors $D_1, \ldots, D_r$ on $X$ if we have $Y = D_1 \cap \cdots \cap D_r$ and the map $\bigoplus_{j=1}^r \mathcal{O}_X(-D_j) \rightarrow \mathcal{I}_Y$ determined by the $D_j$ is surjective.
- On a product $Z_1 \times \cdots \times Z_\ell$, set $E_1 \boxtimes \cdots \boxtimes E_\ell := p_1^*(E_1) \otimes \cdots \otimes p_\ell^*(E_\ell)$ where $E_j$ is a vector bundle on $j$-th factor $Z_j$ and $p_j : Z_1 \times \cdots \times Z_\ell \rightarrow Z_j$ is the projection.

3. The main vanishing theorem

In this section, we prove Theorem 3.4; Theorem 1.1 follows as a special case. We begin by presenting three parts of the argument as independent lemmas. After completing the proof, we show that the main theorem is optimal in some situations.
The first lemma is inspired by Proposition 2.1 in [Ber]. Following Definition 11.1.2 in [PAG2], the asymptotic multiplier ideal \( \mathcal{J}(\|B\|) \) associated to the line bundle \( B \) on \( X \) is the unique maximal member among the family of multiplier ideals \( \{ \mathcal{J}(\frac{1}{k} \cdot B^{\otimes k}) : k \in \mathbb{N} \} \).

**Lemma 3.1.** Let \( B \) be a big line bundle and let \( E \) be an effective divisor on \( X \) such that the line bundle \( B \otimes \mathcal{O}_X(E) \) is big and nef. If the closed subschemes in \( X \) defined by the ideal sheaves \( \mathcal{J}(\|B\|) \) and \( \mathcal{O}_X(-E) \) are disjoint, then we have \( H^i(X, K_X \otimes B) = 0 \) for all \( i > 0 \).

**Proof.** For notational brevity, set \( \mathcal{J} := \mathcal{J}(\|B\|) \) and \( \mathcal{I} := \mathcal{O}_X(-E) \). Tensoring the short exact sequence of the quotient \( \mathcal{O}_X/\mathcal{J} \) with the ideal sheaf \( \mathcal{J} \) yields the exact sequence

\[
(3.1.1) \quad 0 \longrightarrow \mathcal{J} \otimes \mathcal{J} \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_X/\mathcal{J} \longrightarrow 0.
\]

Since the subschemes defined by \( \mathcal{J} \) and \( \mathcal{I} \) are disjoint, the localization of \( \mathcal{J} \) establishes that

\[
0 = \mathcal{J} \otimes \mathcal{J} \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_X/\mathcal{J} \longrightarrow 0.
\]

Tensoring this second sequence with \( A := K_X \otimes B \otimes \mathcal{O}_X(E) \) and taking cohomology produces

\[
\cdots \longrightarrow H^i(X, A) \longrightarrow H^i \left( X, A \otimes \frac{\mathcal{O}_X}{\mathcal{J}} \right) \longrightarrow H^{i+1}(X, K_X \otimes B \otimes \mathcal{J}) \longrightarrow \cdots.
\]

The line bundle \( B \otimes \mathcal{O}_X(E) \) is big and nef, so the Kawamata-Viehweg vanishing theorem (e.g., see Theorem 4.3.1 in [PAG1]) implies that \( H^i(X, A) = 0 \) for all \( i > 0 \). Similarly, the line bundle \( B \) is big, so Theorem 11.2.12 (ii) in [PAG2] asserts that \( H^{i+1}(X, K_X \otimes B \otimes \mathcal{J}) = 0 \) for all \( i \geq 0 \). It follows that \( H^i(X, A \otimes (\mathcal{O}_X/\mathcal{J}) = 0 \) for all \( i > 0 \). Since \( \mathcal{J} \) and \( \mathcal{I} \) have disjoint cosupport, we also have \( (\mathcal{O}_X/\mathcal{J}) \oplus (\mathcal{O}_X/\mathcal{I}) = \mathcal{O}_X/\mathcal{J} \cdot \mathcal{I} \), so \( H^i(X, A \otimes (\mathcal{O}_X/\mathcal{I}) = 0 \) for all \( i > 0 \). Moreover, the canonical inclusion of (3.1.1) into (3.1.2) proves that \( \mathcal{J}/\mathcal{J} \cdot \mathcal{I} \cong \mathcal{O}_X/\mathcal{J} \). Thus, tensoring the exact sequence in (3.1.1) with \( A \) gives the short exact sequence

\[
0 \longrightarrow K_X \otimes B \otimes \mathcal{J} \longrightarrow K_X \otimes B \longrightarrow A \otimes \frac{\mathcal{O}_X}{\mathcal{J}} \longrightarrow 0.
\]

Having already established that the higher cohomology groups of the left and right terms in this sequence vanish, we conclude that the higher cohomology groups of \( K_X \otimes B \) also vanish. \( \square \)

Our second lemma is a minor variant of Wilson’s theorem (see Theorem 2.3.9 in [PAG1]).

**Lemma 3.2.** Let \( B \) be a big, nef line bundle on \( X \). Given a coherent sheaf \( \mathcal{F} \) on \( X \), there exists a positive integer \( k_0 \) and an effective divisor \( E \) such that \( \mathcal{F} \otimes B^{\otimes k} \otimes \mathcal{O}_X(-E) \) is globally generated for all \( k \geq k_0 \).

**Proof.** Fujita’s vanishing theorem (e.g., see Theorem 1.4.35 in [PAG1]) shows that there is a ample divisor \( D \) on \( X \) such that \( H^i(X, \mathcal{F} \otimes \mathcal{O}_X(D) \otimes N) = 0 \) for all \( i > 0 \) and any nef line bundle \( N \).
By replacing \( O_X(\mathcal{D}) \) with a sufficiently large power, we may also assume that \( O_X(\mathcal{D}) \) is globally generated. Since \( B \) is big, there exists a positive integer \( k_0 \) and an effective divisor \( E \) on \( X \) such that \( B^{\otimes k_0} \cong O_X((d + 1)\mathcal{D} + E) \) where \( d := \dim(X) \). It follows that \( H^i(X, \mathcal{F} \otimes B^{\otimes k} \otimes O_X(−E − iD)) = 0 \) for all \( i > 0 \) and all \( k \geq k_0 \). Therefore, the sheaf \( \mathcal{F} \otimes B^{\otimes k} \otimes O_X(−E) \) is \( O_X \)-regular with respect to \( O_X(\mathcal{D}) \) and globally generated (e.g., see Theorem 1.8.5 in [PAG1]).

For the remainder of this section, \( m \) is a positive integer and \( Y \subseteq X \) is a nonsingular subvariety of codimension \( e \) defined scheme-theoretically by the nef divisors \( D_1, \ldots, D_r \). Following [BEL] and [Ber], consider the blow-up \( \pi : X' := Bl_Y(X) \rightarrow X \) of \( X \) along \( Y \) and let \( E := \pi^{-1}(Y) \) be the exceptional divisor. Fix \( F_j := \pi^*(D_j) - E \) for \( 1 \leq j \leq r \). Since \( Y \) lies in \( D_j \), we have \( \text{mult}_Y(D_j) \geq 1 \). If \( D'_j \) denotes the proper transform of \( D_j \) in \( X' \), then each \( F_j = D'_j + (\text{mult}_Y(D_j) - 1)E \) is an effective divisor. Our third lemma constructs normal crossing divisors from certain subsets of \( F_1, \ldots, F_r \).

**Lemma 3.3.** If \( z \in E \) and \( y := \pi(z) \), then there exists a subset \( \{s_1, \ldots, s_e\} \subseteq \{1, \ldots, r\} \) such that the induced map \( \bigoplus_{j=1}^{e} O_{X,Y}(-D_{s_j}) \rightarrow \mathcal{J}_{Y,Y} \) is surjective. Moreover, by reindexing such a subset \( \{s_1, \ldots, s_e\} \), if necessary, we also have \( z \notin F_{s_1} \) and \( F_{s_2} + \cdots + F_{s_e} \) has normal crossings at \( z \).

**Proof.** Both \( X \) and \( Y \) are nonsingular, so \( Y \) is a local complete intersection in \( X \). Since \( Y \) is defined scheme-theoretically by \( D_1, \ldots, D_r \), it follows that, for every point \( y \in Y \), there is a subset \( \{s_1, \ldots, s_e\} \subseteq \{1, \ldots, r\} \) so that the induced map \( \bigoplus_{j=1}^{e} O_{X,Y}(-D_{s_j}) \rightarrow \mathcal{J}_{Y,Y} \) is surjective.

For the second part, choose an affine neighbourhood \( U \subseteq X \) of \( y \) such that \( R := O_X(U) \) is a regular ring, the local equation of \( D_{s_j} \) is \( f_j \in R \) for \( 1 \leq j \leq e \), and the prime ideal \( \mathcal{J}_Y(U) \subseteq R \) is generated by \( f_1, \ldots, f_e \). In other words, the elements \( f_1, \ldots, f_e \) form a regular system of parameters at the generic point of \( Y \cap U \). It follows that \( \pi^{-1}(U) \) is the union of the open subschemes

\[
W_j := \text{Spec} \left( \frac{R[t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_e]}{(f_1 - f_j t_1, \ldots, f_{j-1} - f_j t_{j-1}, f_{j+1} - f_j t_{j+1}, \ldots, f_e - f_j t_e)} \right)
\]

for \( 1 \leq j \leq e \),

and the ideal \( O_{W_j}(-E|_{W_j}) \) is generated by \( f_j \). Hence, we see that \( F_{s_j} \cap W_j = \emptyset \) and, for \( k \neq j \), the local equation of \( F_{s_k} \cap W_j \) is \( \frac{t_k}{f_j} = t_k \in O_{X}(W_j) \). By reindexing the subset \( \{s_1, \ldots, s_e\} \) if necessary, we may assume that \( z \) lies in \( W_1 \), so \( z \notin F_{s_1} \). Since the quotient \( O_{X}(W_1)/(f_{j}) \) is the polynomial ring \( k(y)[t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_e] \), we see that the divisor \( F_{s_2} + \cdots + F_{s_e} \) has normal crossings at \( z \).

With these three preliminary results, we can now prove our refinement of Theorem 1.1.

**Theorem 3.4.** Fix a collection \( \Phi \) of \( e \)-element subsets of \( \{1, \ldots, r\} \) such that, for each point \( y \in Y \), there exists \( \{s_1, \ldots, s_e\} \in \Phi \) for which the induced map \( \bigoplus_{j=1}^{e} O_{X,Y}(-D_{s_j}) \rightarrow \mathcal{J}_{Y,Y} \) is surjective. If the line bundle \( L \otimes O_X(-(m+1)D_{s_1} - D_{s_2} - \cdots - D_{s_e}) \) is a big and nef for all subsets \( \{s_1, s_2, \ldots, s_e\} \in \Phi \), then we have \( H^i(X, \mathcal{J}_{Y}^{m+1} \otimes K_X \otimes L) = 0 \) for all \( i > 0 \).

**Remark 3.5.** When \( m > 0 \), the line bundle \( L \otimes O_X(-(m+1)D_{s_1} - D_{s_2} - \cdots - D_{s_e}) \) depends on the choice of the first element \( s_1 \) not just the underlying set. In particular, the hypothesis “for all subsets \( \{s_1, \ldots, s_e\} \in \Phi \)” means all subsets together with all choices of a first element \( s_1 \).
Proof. Since $X$ and $Y$ are nonsingular, Lemma 4.3.16 in [PAG1] shows that
\[ H^i(X', \mathcal{I}'^{m+1} \otimes K_X \otimes L) = H^i(X', \pi^*(K_X) \otimes \pi^*(L) \otimes \mathcal{O}_{X'}(-(m+1)E)) \] for all $i > 0$.

We also have $K_{X'} \cong \pi^*(K_X) \otimes \mathcal{O}_{X'}((-e-1)E)$ because $Y$ is nonsingular of codimension $e$. Therefore, it is enough to show that $H^i(X', K_{X'} \otimes \pi^*(L) \otimes \mathcal{O}_{X'}(-(e+m)E)) = 0$ for all $i > 0$. By setting $B := \pi^*(L) \otimes \mathcal{O}_{X'}(-(e+m)E)$, we need to show that $H^i(X', K_{X'} \otimes B) = 0$ for all $i > 0$, so it suffices to prove that $B$ and $(e+m)E$ satisfy the three conditions in Lemma 3.1.

The first two conditions assert that $B$ is big and that $B \otimes \mathcal{O}_{X'}((e+m)E)$ is big and nef. By hypothesis, the line bundle $L \otimes \mathcal{O}_{X'}(-mD_{s_1} - \sum_{j=1}^e D_{s_j})$ is big and nef for any $\{s_1, \ldots, s_e\} \in \Phi$. Hence, by decomposing $B$ as the product of a big line bundle and an effective line bundle
\[ B = \pi^*(L \otimes \mathcal{O}_X(-mD_{s_1} - \sum_{j=1}^e D_{s_j})) \otimes \left( \mathcal{O}_{X'}(mF_{s_1} + \sum_{j=1}^e F_{s_j}) \right), \]
we see that $B$ is a big line bundle. Similarly, $B \otimes \mathcal{O}_{X'}((e+m)E)$ is the product of a big, nef line bundle and a nef line bundle
\[ B \otimes \mathcal{O}_{X'}((e+m)E) = \pi^*(L) \otimes \pi^*(L \otimes \mathcal{O}_X(-mD_{s_1} - \sum_{j=1}^e D_{s_j})) \otimes \pi^*(\mathcal{O}_X(mD_{s_1} + \sum_{j=1}^e D_{s_j})), \]
so the line bundle $B \otimes \mathcal{O}_{X'}((e+m)E)$ is big and nef.

To establish the third condition from Lemma 3.1, we must show that the closed subschemes of $X'$ defined by $\mathcal{J}(|B|)$ and $\mathcal{O}_{X'}((-e+m)E)$ are disjoint. Fix a point $z \in E = \text{Supp}((e+m)E)$. By Lemma 3.3, there exists a subset $\{s_1, \ldots, s_e\} \subseteq \Phi$ such that $z \not\in F_{s_1}$ and $F_{s_2} + \cdots + F_{s_e}$ has normal crossings at $z$. Since $\pi^*(L \otimes \mathcal{O}_X(-mD_{s_1} - \sum_{j=1}^e D_{s_j}))$ is a big, nef line bundle, Lemma 3.2 provides a positive integer $k_0$ and an effective divisor $D$ such that the linear series
\[ \mathcal{O}_{X'}(mF_{s_1} + \sum_{j=1}^e F_{s_j}) \otimes \pi^*(L^{\otimes k}) \otimes \mathcal{O}_{X'}(-km \pi^*(D_{s_1}) - k\sum_{j=1}^e \pi^*(D_{s_j}) - \tilde{D}) \]
has no base points for all $k \geq k_0$. Choose an effective divisor $F'$ in this linear series that does not pass through $z$ and, for each $k \geq k_0$, consider the effective divisor
\[ G := \frac{k+1}{k}(mF_{s_1} + \sum_{j=1}^e F_{s_j}) + \frac{1}{k}F'. \]
Since $G$ has normal crossing support at $z$, the multiplier ideal sheaf $\mathcal{J}(G)$ is trivial at $z$ (cf. Example 9.2.13 in [PAG2]).

To complete the proof, we relate the multiplier ideal sheaf $\mathcal{J}(G)$ to the asymptotic multiplier ideal sheaf $\mathcal{J}(|B|)$. Since $kG \in |B^{\otimes k} \otimes \mathcal{O}_{X'}(-\tilde{D})|$ and $|B^{\otimes k} \otimes \mathcal{O}_{X'}(-\tilde{D})| \subseteq |B^{\otimes k}|$, Proposition 9.2.32 in [PAG2] yields
\[ \mathcal{J}(G) \subseteq \mathcal{J}(\frac{1}{k}B^{\otimes k} \otimes \mathcal{O}_{X'}(-\tilde{D})) \subseteq \mathcal{J}(\frac{1}{k}B^{\otimes k}), \]
for all $k \geq k_0$.

The definition of the asymptotic multiplier ideal implies that $\mathcal{J}(\frac{1}{k}B^{\otimes k}) \subseteq \mathcal{J}(|B|)$. Hence, for sufficiently large integers $k$, we have $\mathcal{J}(G) \subseteq \mathcal{J}(|B|)$, so the asymptotic multiplier ideal sheaf $\mathcal{J}(|B|)$ must also be trivial at the point $z \in E$. In other words, the closed subschemes of $X'$ defined by $\mathcal{J}(|B|)$ and $\mathcal{O}_{X'}(-(e+m)E)$ are disjoint. \[ \square \]
Proof of Theorem 1.1. Lemma 3.3 shows that, for any \( y \in Y \), there exist a subset \( \{ s_1, \ldots, s_e \} \subseteq \{ 1, \ldots, r \} \) for which the induced map \( \bigoplus_{j=1}^e \mathcal{O}_{X,Y}(-D_{s_j}) \to \mathcal{I}_{Y,Y} \) is surjective. Thus, Theorem 1.1 is simply Theorem 3.4 when \( \Phi \) consists of all \( e \)-element subsets of \( \{ 1, \ldots, r \} \).

\[ \square \]

Remark 3.6. When \( m = 0 \) and \( r = 1 \), Theorem 1.1 specializes to the Kawamata-Viehweg vanishing theorem. Indeed, if \( Y \subseteq X \) is an effective Cartier divisor, then we have \( \mathcal{I}_Y = \mathcal{O}_X(-Y) \). Given any big and nef line bundle \( B \) on \( X \), set \( L := B \otimes \mathcal{O}_X(Y) \). In this case, Theorem 1.1 implies that \( H^i(X, K_X \otimes B) = H^i(X, \mathcal{I}_Y \otimes K_X \otimes L) = 0 \) for all \( i > 0 \). Consequently, one can view our main theorem as a higher-codimensional analogue of the Kawamata-Viehweg vanishing theorem.

Although Theorem 1.1 generalizes the Kawamata-Viehweg vanishing theorem, the next result shows that it is essentially as sharp. More precisely, given a witness for the optimality of the Kawamata-Viehweg vanishing theorem, we obtain a witness for the optimality of Theorem 1.1.

Proposition 3.7. Let \( i \) and \( e \) be positive integers. If \( N \) is a nef line bundle on \( X \) such that \( H^{i+e-1}(X, K_X \otimes N) \neq 0 \), then every complete intersection \( Y = D_1 \cap \cdots \cap D_e \) in \( X \), where each divisor \( D_i \) is big and nef, satisfies \( H^i(X, \mathcal{I}_Y \otimes K_X \otimes \mathcal{O}_X(D_1 + \cdots + D_e) \otimes N) = 0 \).

Proof. Fix a complete intersection \( Y = D_1 \cap \cdots \cap D_e \) in \( X \) such that each divisor \( D_i \) is big and nef. Let \( \binom{[e]}{k} \) denote the set of all \( k \)-element subsets of \( \{ e \} := \{ 1, \ldots, e \} \). Tensoring the Koszul complex associated to \( Y \) with the line bundle \( K_X \otimes \mathcal{O}_X(D_1 + \cdots + D_e) \otimes N \) yields the exact sequence

\[
0 \to K_X \otimes N \to \bigoplus_{\{ s_1 \} \in \binom{[e]}{1}} K_X \otimes \mathcal{O}_X(D_{s_1}) \otimes N \to \bigoplus_{\{ s_1, s_2 \} \in \binom{[e]}{2}} K_X \otimes \mathcal{O}_X(D_{s_1} + D_{s_2}) \otimes N \to \cdots
\]

\[
\to K_X \otimes \mathcal{O}_X \left( \sum_{j=1}^{e-1} D_{s_j} \right) \otimes N \to \mathcal{I}_Y \otimes K_X \otimes \mathcal{O}_X \left( \sum_{j=1}^{e} D_j \right) \otimes N \to 0.
\]

For any \( 1 \leq k \leq e \) and any nonempty subset \( \{ s_1, \ldots, s_k \} \subseteq \{ 1, \ldots, e \} \), the Kawamata-Viehweg vanishing theorem shows that \( H^i(X, K_X \otimes \mathcal{O}_X(\sum_{j=1}^{k} D_{s_j}) \otimes N) = 0 \) for all \( i > 0 \). Chopping this exact sequence into short exact ones and taking the associated long exact sequences in cohomology (cf. Proposition B.1.2 in [PAG1]), we conclude that

\[
H^i(X, \mathcal{I}_Y \otimes K_X \otimes \mathcal{O}_X(D_1 + \cdots + D_e) \otimes N) = H^{i+e-1}(X, K_X \otimes N) \neq 0.
\]

\[ \square \]

The next example shows that Theorem 1.1 is optimal for a very general line bundle \( L \) on a product of projective spaces.

Example 3.8. Let \( X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell} \) and write \( \mathcal{O}_X(m_1, \ldots, m_\ell) := \mathcal{O}_{\mathbb{P}^{n_1}}(m_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_\ell}}(m_\ell) \).

With this notation, we have \( K_X = \mathcal{O}_X(-n_1 - 1, \ldots, -n_\ell - 1) \). Moreover, \( \mathcal{O}_X(m_1, \ldots, m_\ell) \) is nef (resp. big) if and only if \( m_j \geq 0 \) (resp. \( m_j > 0 \)) for all \( 1 \leq j \leq \ell \). Consider the line bundle \( N = \mathcal{O}_X(0, m_2, \ldots, m_\ell) \) where \( m_j \geq n_j + 1 \) for all \( 2 \leq j \leq \ell \). In particular, \( N \) is nef but not big. The Künneth formula gives \( H^{n_1}(X, K_X \otimes N) \neq 0 \). Thus, for any \( 1 \leq e \leq n_1 \) and any complete intersection \( Y = D_{1} \cap \cdots \cap D_{e} \) in \( X \) where each divisor \( D_{i} \) is big and nef, Proposition 3.7 establishes that \( H^{n_1 - e + 1}(X, \mathcal{I}_Y \otimes K_X \otimes \mathcal{O}_X(D_1 + \cdots + D_e) \otimes N) \neq 0 \). By permuting the factors of \( X \), it follows
that, for any line bundle $L$ sufficiently far into the relative interior of a facet of the nef cone, one cannot weaken the hypotheses on $L$ in Theorem 1.1 and continue to have all of the higher cohomology groups vanish.

For some special classes of nonsingular varieties, there are vanishing results that improve upon the Kawamata-Viehweg vanishing theorem. Given Remark 3.6, we cannot expect Theorem 1.1 to be uniformly optimal for all ambient varieties $X$.

**Question 3.9.** Is it possible to strengthen Theorem 1.1 by exploiting some properties of $X$?

### 4. Multigraded Castelnuovo-Mumford regularity

This section proves Corollary 1.2 which bounds the multigraded Castelnuovo-Mumford regularity of $\mathcal{I}_Y$. To assess this corollary, we include a couple of examples. Throughout the section, we measure multigraded Castelnuovo-Mumford regularity with respect to the free divisors $P_1, \ldots, P_\ell$ on $X$ such that some positive linear combination is very ample.

**Proof of Corollary 1.2.** We must verify that $H^i(\mathcal{I}_Y \otimes K_X \otimes L \otimes N \otimes \mathcal{O}_X(-u_1 P_1 - \cdots - u_\ell P_\ell)) = 0$ for all $i > 0$ and all $(u_1, \ldots, u_\ell) \in \mathbb{N}^\ell$ satisfying $u_1 + \cdots + u_\ell = i$. For $i > \dim(X)$, this is immediate and, for $i \leq \dim(Y) + 1$, it follows from Theorem 1.1. To establish the remaining vanishing, tensor the exact sequence $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ with $A := K_X \otimes L \otimes N \otimes \mathcal{O}_X(\sum_{j=1}^\ell -u_j P_j)$ and take cohomology to obtain the exact sequence

$$
\cdots \to H^{i-1}(X, \mathcal{O}_Y \otimes A) \to H^i(X, \mathcal{I}_Y \otimes A) \to H^i(X, A) \to \cdots.
$$

We have $H^{i-1}(X, \mathcal{O}_Y \otimes A) = 0$ for all $i > \dim(Y)$. Since each divisor $D_j$ is a positive linear combination of the $P_1, \ldots, P_\ell$, we have $-u_1 P_1 - \cdots - u_\ell P_\ell = -D_{s_1} - \cdots - D_{s_\ell} - u_1' P_1 - \cdots - u_\ell' P_\ell$ where $(u_1', \ldots, u_\ell') \in \mathbb{N}^\ell$ satisfies $u_1' + \cdots + u_\ell' \leq \dim(Y)$. Hence, the line bundle

$$
L \otimes \mathcal{O}_X(\sum_{j=1}^\ell -u_j P_j) = L \otimes \mathcal{O}_X(-D_{s_1} - \cdots - D_{s_\ell} - u_1' P_1 - \cdots - u_\ell' P_\ell)
$$

is big and nef, so the Kawamata-Viehweg vanishing theorem implies that $H^i(X, A) = 0$ for all $i > 0$. Therefore, we have $H^i(X, \mathcal{I}_Y \otimes A) = 0$ for all $\dim(Y) + 1 < i \leq \dim(X)$.

**Remark 4.1.** Fix a big, nef line bundle $B$ on $X$ and consider the line bundle

$$
L = B \otimes \mathcal{O}_X((d - e + 1) P_1 + \cdots + (d - e + 1) P_\ell + D_1 + \cdots + D_r),
$$

where $d := \dim(X)$ and $e := \text{codim}(Y)$. Corollary 1.2 implies that $\mathcal{I}_Y$ is $K_X \otimes L$-regular, so we have a linear bound on multigraded Castelnuovo-Mumford regularity for nonsingular varieties.

**Remark 4.2.** Assuming that $\mathcal{O}_X$ is $K_X \otimes L \otimes N$-regular with respect to $P_1, \ldots, P_\ell$, the cohomology group $H^i(X, A)$ appearing in (4.0.1) vanishes. Thus, with this alternative hypothesis, we do not need to assume that each nef divisor $D_j$ is a positive linear combination of the $P_1, \ldots, P_\ell$.

When the ambient space is simply projective space, the following example shows that we recover Corollary 4 in [BEL] (also see Remark 1.8.44 in [PAG1]).
Example 4.3. Let \( X = \mathbb{P}^d, \ell = 1, \) and \( \mathcal{O}_X(P) = \mathcal{O}_X(1). \) Since each \( D_j \) is nef, there is a positive integer \( m_j \) such that \( D_j \in [m_jP] \) for all \( 1 \leq j \leq r. \) We may assume that \( m_1 \geq m_2 \geq \cdots \geq m_r. \) Thus, the line bundle
\[
L = \mathcal{O}_X(P) \otimes \mathcal{O}_X((d - e + 1)P + D_1 + \cdots + D_e) = \mathcal{O}_X(d - e + 2 + m_1 + \cdots + m_e)
\]
satisfies the hypothesis of Corollary 1.2 and \( \mathcal{J}_Y \) is \( \mathcal{O}_X(m_1 + \cdots + m_e - e + 1) \)-regular.

The final example shows, perhaps unexpectedly, that the best bounds arising from Corollary 1.2 do not necessarily come from the ideal-theoretic equations for \( Y. \)

Example 4.4. Fix \( X = \mathbb{P}^2 \times \mathbb{P}^2 \) as the ambient variety and choose free divisors \( P_1 \) and \( P_2 \) on \( X \) such that \( \mathcal{O}_X(P_1) = \mathcal{O}_X(1, 0) \) and \( \mathcal{O}_X(P_2) = \mathcal{O}_X(0, 1). \) Proposition 6.10 in [MS] shows that \( \mathcal{O}_X \) is \( \mathcal{O}_X \)-regular with respect to \( P_1, P_2. \) Let \( Y \subset X \) be the image of the map from \( \mathbb{P}^1 \) to \( \mathbb{P}^2 \times \mathbb{P}^2 \) given by \([t_0 : t_1] \mapsto ([t_0^6 : t_0^3 t_1^3 : t_1^6], [t_0^2 : t_0 t_1 : t_1^2]),\) so \( Y \) is nonsingular of codimension 3. By composing this map with a Segre embedding of \( \mathbb{P}^2 \times \mathbb{P}^2 \) in \( \mathbb{P}^8, \) we obtain a Veronese embedding of \( \mathbb{P}^1 \) in \( \mathbb{P}^8. \) If \( S := \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2], \) where \( \deg(x_i) = [\frac{1}{2}] \in \mathbb{Z}^2 \) and \( \deg(y_j) = [\frac{1}{2}] \in \mathbb{Z}^2, \) is the Cox ring or total coordinate ring of \( X \) and \( J := (x_0, x_1, x_2) \cap (y_0, y_1, y_2) \) is the irrelevant ideal, then the \( J \)-saturated \( S \)-ideal associated to \( Y \) is \( I_Y = (f_0, \ldots, f_5) \) where
\[
\begin{align*}
f_0 &= x_1^2 - x_0 x_2, & f_2 &= x_2 y_0 y_1 - x_1 y_2^2, & f_4 &= x_2 y_0^2 - x_1 y_1 y_2, \\
f_1 &= y_1^2 - y_0 y_2, & f_3 &= x_1 y_0 y_1 - x_0 y_2^2, & f_5 &= x_1 y_0^2 - x_0 y_1 y_2.
\end{align*}
\]
Since \( \deg(f_0) = [\frac{5}{2}], \deg(f_1) = [\frac{9}{2}], \) and \( \deg(f_2) = \cdots = \deg(f_5) = [\frac{1}{2}], \) Corollary 1.2 together with Remark 4.2 imply that \( \mathcal{J}_Y = I_Y \) is \( \mathcal{O}_X(4, 6) \)-regular. On the other hand, the intersection of prime ideal \( I_Y \) and the \((y_0, y_1, y_2)\)-primary ideal \((y_2^2, y_1 y_2, y_1^2 - y_0 y_2, y_0 y_1, y_0^2)\) equals \((f_1, \ldots, f_5)\) which implies that \( Y \) is defined scheme-theoretically by the last five equations. Using this smaller collection of equations, we see that \( \mathcal{J}_Y \) is \( \mathcal{O}_X(3, 6) \)-regular. However, Theorem A in [Loz] establishes that \( \mathcal{J}_Y \) is also \( \mathcal{O}_X(1, 5) \)-regular because the curve \( Y \) has bidegree \((6, 2)\).

Example 4.4 also shows that better bounds can be obtained by using additional numerical invariants. The following seems like an excellent place to start exploring this phenomena.

Question 4.5. Suppose that \( Y \) is a nonsingular curve embedded in a nonsingular toric variety \( X. \) What are the optimal bounds on the multigraded Castelnuovo-Mumford regularity of \( Y \) in terms of its multidegree?

5. Maps arising from the global sections of adjoint bundles

The goal of this section is to provide sufficient conditions for the surjectivity of both the canonical multiplication map between the global sections of adjoint bundles and the Wahl maps arising from adjoint bundles. Let \( A_1, \ldots, A_d \) denote very ample line bundles on \( X \) where \( d := \dim(X). \)

Proof of Corollary 1.3. Tensoring the short exact sequence for the diagonal \( \Delta(X) \subset X \times X \) with \((K_X \otimes L_1) \boxtimes (K_X \otimes L_2) = K_{X \times X} \otimes (L_1 \boxtimes L_2) \) yields the exact sequence
\[
0 \longrightarrow \mathcal{J}_{\Delta(X)} \otimes K_{X \times X} \otimes (L_1 \boxtimes L_2) \longrightarrow K_{X \times X} \otimes (L_1 \boxtimes L_2) \longrightarrow \mathcal{O}_{\Delta(X)} \otimes K_{X \times X} \otimes (L_1 \boxtimes L_2) \longrightarrow 0.
\]
By restricting to the diagonal $\Delta(X)$ and taking global sections, we obtain the multiplication map $H^0(X, K_X \otimes L_1) \otimes H^0(X, K_X \otimes L_2) \rightarrow H^0(X, K_X^2 \otimes L_1 \otimes L_2)$. Therefore, it suffices to show that $H^1(X \times X, \mathcal{I}_{\Delta(X)} \otimes K_{X \times X} \otimes (L_1 \otimes L_2)) = 0$.

Since each $A_k$ is very ample, the diagonal $\Delta(X) \subseteq X \times X$ is defined scheme-theoretically by the complete linear series $|A_k \boxtimes A_k|$ (cf. Lemma 5.4). As $\Delta(X)$ and $X \times X$ are both nonsingular, we can choose, for each point $x \in \Delta(X)$ and each $1 \leq k \leq d$, a single divisor $P_k \in |A_k \boxtimes A_k|$ such that the induced map $\bigoplus_{k=1}^{d} \mathcal{O}_{X \times X, x}(-P_k) \rightarrow \mathcal{I}_{\Delta(X), x}$ is surjective. Thus, the vanishing follows from Theorem 3.4 because

$$(L_1 \boxtimes L_2) \otimes (\mathcal{O}_X(-P_1) \boxtimes \mathcal{O}_X(-P_1)) \otimes \cdots \otimes (\mathcal{O}_X(-P_d) \boxtimes \mathcal{O}_X(-P_d))$$

$$= (L_1 \otimes A_1^{-1} \otimes \cdots \otimes A_d^{-1}) \boxtimes (L_2 \otimes A_1^{-1} \otimes \cdots \otimes A_d^{-1})$$

is big and nef. 

\[\square\]

\textbf{Remark 5.1.} Corollary 1.3 specializes to Variant 3.2 in [BEL] when $A_1 = \ldots = A_d$. On the other hand, Lazarsfeld points out that one can obtain another proof for Corollary 1.3 by adapting the argument for Variant 3.2 in [BEL].

To determine if an adjoint bundle is very ample, we have the following simple observation.

\textbf{Corollary 5.2.} If $A_1, \ldots, A_{d+2}$ are very ample line bundles on $X$ and $N$ is a nef line bundle on $X$, then the line bundle $K_X \otimes A_1 \otimes \cdots \otimes A_{d+2} \otimes N$ is very ample and defines a projectively normal embedding.

\textbf{Proof.} By applying Corollary 1.3, it is enough to show that $K_X \otimes A_1 \otimes \cdots \otimes A_{d+2} \otimes N$ is very ample. Since the tensor product of a globally generated line bundle and a very ample one is also very ample, it suffices to prove that $B := K_X \otimes A_1 \otimes \cdots \otimes A_{d+1} \otimes N$ is globally generated. To accomplish this, we proceed by induction on $d := \dim(X)$.

Suppose that $d = 1$. On a smooth curve $C$ of genus $g$, a divisor is ample (nef) if and only if it has positive (nonnegative) degree and a divisor of degree at least $2g$ has no base point. Since $\deg(K_C) = 2g - 2$, it follows that $K_C \otimes A_1 \otimes A_2 \otimes N$ is globally generated.

Now suppose that $d > 1$ and fix a point $x \in X$. The line bundle $A_{d+1}$ is very ample, so Bertini’s Theorem provides a smooth irreducible divisor $H \in |A_{d+1}|$ passing through $x$. Tensoring the short exact sequence $0 \rightarrow \mathcal{O}_X(-H) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_H \rightarrow 0$ with $B$ yields

$$0 \rightarrow B \otimes A_{d+1}^{-1} \rightarrow B \rightarrow K_H \otimes A_1 \otimes \cdots \otimes A_d \otimes N \rightarrow 0.$$ Since $B \otimes A_{d+1}^{-1} = K_X \otimes A_1 \otimes \cdots \otimes A_d \otimes N$, the Kawamata-Viehweg vanishing theorem shows that $H^1(X, B \otimes A_{d+1}^{-1}) = 0$. Hence, the map $H^0(X, B) \rightarrow H^0(H, K_H \otimes A_1 \otimes \cdots \otimes A_d \otimes N)$ is surjective. The induction hypothesis ensures that there exists a global section of $K_H \otimes A_1 \otimes \cdots \otimes A_d \otimes N$ that does not vanish at the point $x$. By choosing a preimage in $H^0(X, B)$, we obtain a global section of $B$ that does not vanish at $x$. 

\[\square\]
To give alternative conditions for the surjectivity of the multiplication map, fix globally generated line bundles \( \mathcal{O}_X(P_1), \ldots, \mathcal{O}_X(P_\ell) \) on \( X \) such that the associated complete linear series define a closed embedding \( X \hookrightarrow \mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_\ell \). Loosely speaking, we view the line bundles \( \mathcal{O}_X(P_1), \ldots, \mathcal{O}_X(P_\ell) \) as a reasonable factorization of the very ample line bundle \( \mathcal{O}_X(P_1 + \cdots + P_\ell) \).

**Proposition 5.3.** If \( L_1 \) and \( L_2 \) are line bundles on \( X \) such that \( L_j \otimes \mathcal{O}_X(-u_1 P_1 - \cdots - u_\ell P_\ell) \) is big and nef for each \( j \) and all \( (u_1, \ldots, u_\ell) \in \mathbb{N}^\ell \) such that \( u_1 + \cdots + u_\ell = d \) and \( 0 \leq u_k \leq (n_k+1)/2 \), then the multiplication map \( H^0(X, K_X \otimes L_1) \otimes H^0(X, K_X \otimes L_2) \to H^0(X, K_X^\otimes \otimes L_1 \otimes L_2) \) is surjective.

Before proving this proposition, we record a multigraded extension of Lemma 3.1 in [BEL].

**Lemma 5.4.** The diagonal \( \Delta(X) \subseteq X \times X \) is defined scheme-theoretically by divisors obtained from \( (n_k+1)/2 \) sections of \( \mathcal{O}_X(P_k) \boxtimes \mathcal{O}_X(P_k) \) for each \( 1 \leq k \leq \ell \).

**Proof.** Since the line bundles \( \mathcal{O}_X(P_1), \ldots, \mathcal{O}_X(P_\ell) \) give a closed embedding \( X \subseteq \mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_\ell \), we have \( X \times X \subseteq (\mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_\ell) \times (\mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_\ell) \) and \( \Delta(X) = \Delta(\mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_\ell) \cap (X \times X) \). Each \( \Delta(\mathbb{P}^n_k) \subseteq \mathbb{P}^{n_k} \times \mathbb{P}^{n_k} \) is defined ideal-theoretically by \( (n_k+1)/2 \) sections of \( \mathcal{O}_{\mathbb{P}^{n_k}}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(1) \) (e.g., see Exercise 13.15.b in [Eis]). Thus, the diagonal \( \Delta(\mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_\ell) \) is defined scheme-theoretically by entire collection of sections. The claim then follows by pulling-back to \( X \times X \).

**Proof of Proposition 5.3.** Just as in the proof of Corollary 1.3, it suffices to show that

\[
H^1(X \times X, \mathcal{J}_{\Delta(X)} \otimes K_{X \times X} \otimes (L_1 \boxtimes L_2)) = 0.
\]

Lemma 5.4 shows that \( \Delta(X) \) is defined scheme-theoretically by divisors obtained from \( (n_k+1)/2 \) sections of \( \mathcal{O}_X(P_k) \boxtimes \mathcal{O}_X(P_k) \) for \( 1 \leq k \leq \ell \). Moreover, the divisor

\[
(L_1 \boxtimes L_2) \otimes (\mathcal{O}_X(-P_1) \boxtimes \mathcal{O}_X(-P_2))^{\otimes u_1} \otimes \cdots \otimes (\mathcal{O}_X(-P_\ell) \boxtimes \mathcal{O}_X(-P_\ell))^{\otimes u_\ell} = (L_1 \boxtimes \mathcal{O}_X(-u_1 P_1 - \cdots - u_\ell P_\ell)) \boxtimes (L_2 \boxtimes \mathcal{O}_X(-u_1 P_1 - \cdots - u_\ell P_\ell))
\]

is big and nef for each \( (u_1, \ldots, u_\ell) \in \mathbb{N}^\ell \) such that \( u_1 + \cdots + u_\ell = d \) and \( 0 \leq u_k \leq (n_k+1)/2 \). Therefore, the required vanishing follows from Theorem 1.1.

**Remark 5.5.** Whenever \( \ell = 1 \), we again recover Variant 3.2 in [BEL] from Proposition 5.3.

The first example in this section demonstrates that, for some varieties, Proposition 5.3 can yield stronger results than Corollary 1.3 or Variant 3.2 in [BEL].

**Example 5.6.** Let \( X \) be the blow-up of \( \mathbb{P}^d \) along the intersection of two hyperplanes. In other words, \( X \) is the toric variety obtained by blowing-up \( \mathbb{P}^d \) along a codimension-two torus orbit closure (e.g., see Proposition 3.3.15 in [CLS]). If \( E \) is the exceptional divisor and \( H \) is the proper transform of a hyperplane, then \( \mathcal{O}_X(E) \) and \( \mathcal{O}_X(H) \) form a basis for \( \text{Pic}(X) \) with \( K_X = \mathcal{O}_X(-(d+1)H + E) \). The numerical classes of \( P_1 := H - E \) and \( P_2 := H \) generate the nef cone \( \text{Nef}(X) \), and the numerical classes \( [E] \) and \( [P_1] \) generated the pseudoeffective cone of \( X \). Moreover, the complete linear series associated to \( \mathcal{O}_X(P_1) \) and \( \mathcal{O}_X(P_2) \) define a closed embedding \( X \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^d \); in particular, we
have \( n_1 + 1 = \dim H^0(X, \mathcal{O}_X(P_1)) = 2 \) and \( n_2 + 1 = \dim(H^0(X, \mathcal{O}_X(P_2))) = d + 1 \). Hence, Proposition 5.3 shows that the multiplication map \( H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2) \) is surjective provided that each \( L_j \) has the form \( \mathcal{O}_X(P_2) \otimes N \) for some nef line bundle \( N \) (see the shaded-region in Figure 1a). In contrast, Corollary 1.3 or Variant 3.2 in [BEL] apply only to adjoint bundles of the form \( \mathcal{O}_X((d - 1)P_1) \otimes N \) where \( N \) is a nef line bundle. \hfill \bigcirc

Our second example shows the reverse: Corollary 1.3 can be stronger than Proposition 5.3.

**Example 5.7.** Let \( X \subseteq \mathbb{P}^3 \) be a nonsingular quartic surface that contains a rational quartic curve \( C \) (for existence, see Theorem 1 in [Mor]). Let \( H \) be a plane section of \( X \). Since \( X \) is a K3 surface, we have \( K_X = \mathcal{O}_X \). Remark 4 in [Mor] indicates that \( H^2 = 4, H.C = 4, C^2 = -2, \) and \( \text{Pic}(X) = \mathbb{Z}H + \mathbb{Z}C \). In addition, Theorem 2 in [Kov] implies that the cone of curves on \( X \) is generated by numerical classes of \( C \) and \( H + (2 - \sqrt{6})C \). Since \( (H + 2C).C = 0 \), we deduce that the nef cone \( \text{Nef}(X) \) is generated by numerical classes \( [H + 2C] \) and \( [H + (2 - \sqrt{6})C] \). Using results of Saint-Donat (e.g., see Theorem 5 in [Mor]), we also see that \( H, H + C \) are very ample and \( H + 2C \) has no base points. For \( H \) and \( H + C \), Corollary 1.3 shows that the multiplication map \( H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2) \) is surjective provided that both \( [L_j] \) lies in the cone \( Q = [2H + C] + \text{Nef}(X) \) (see the shaded-region in Figure 1b). In contrast, applying Proposition 5.3 to \( P_1 := H + 2C \) and \( P_2 := H + C \) only shows that the multiplication map is surjective when each \( [L_j] \) lies in the cone

\[
\left( [2P_1] + \text{Nef}(X) \right) \cap \left( [P_1 + P_2] + \text{Nef}(X) \right) \cap \left( [2P_2] + \text{Nef}(X) \right) = \left[ 2 + \frac{2}{\sqrt{6}} \right]H + \left[ 2 + \frac{4}{\sqrt{6}} \right]C + \text{Nef}(X) \subseteq Q.
\]

In fact, no matter how we choose of the divisors \( P_1, \ldots, P_t \), Proposition 5.3 will not improve upon Corollary 1.3 in this case. \hfill \bigcirc

By combining Example 5.6 and Example 5.7, we see that the relative strengths of Corollary 1.3 and Proposition 5.3 are incomparable. However, there is an obvious common refinement in which one factors each very ample line bundle into appropriate globally generated line bundles; we leave the details to the interested reader.
Although one can use the ideas from Corollary 1.3 or Proposition 5.3 together with the methods in [Ina] to say something about the higher syzygies of \(X\), we expect these results to be far from optimal. We suspect that following question points in a more fruitful direction.

**Question 5.8.** Is there an analogue of Theorem 1 in [EL] which mirrors either Corollary 1.3 or Proposition 5.3?

To exhibit the utility of the parameter \(m\) in our main theorem, we conclude this section with the following result.

**Proposition 5.9.** If \(L_1\) and \(L_2\) are line bundles on \(X\) such that each \(L_j \otimes \mathcal{O}_X(-u_1 P_1 - \cdots - u_\ell P_\ell)\) is big and nef for all \(j = 1, 2\) and all \((u_1, \ldots, u_\ell) \in \mathbb{N}^\ell\) with \(u_1 + \cdots + u_\ell = d + m\) and \(0 \leq u_k \leq \left(\frac{n_k+1}{2}\right)\), then the \(m\)-th Wahl map

\[
H^0(X \times \mathcal{O}_{\Delta(X)}^m \otimes K_X \times \mathcal{O}_X \otimes (L_1 \boxtimes L_2)) \longrightarrow H^0(X, \text{Sym}^m(\Omega^1_X) \otimes K_X^2 \otimes L_1 \otimes L_2)
\]

is surjective.

**Proof.** Following Section 1 in [Wah], the \(m\)-th Wahl maps is defined as follows. For brevity, set \(\mathcal{F} := \mathcal{F}_{\Delta(X)}\). Since \(\mathcal{F}/\mathcal{F}^2\) is isomorphic to \(\Omega^1_X\) as an \(\mathcal{O}_{\Delta(X)}\)-module and \(\mathcal{F}^m/\mathcal{F}^{m+1} \cong \text{Sym}^m(\Omega^1_X)\), there is an exact sequence \(0 \longrightarrow \mathcal{F}^{m+1} \longrightarrow \mathcal{F}^m \longrightarrow \text{Sym}^m(\Omega^1_X) \otimes \mathcal{O}_{\Delta(X)} \longrightarrow 0\). Tensoring with \(K_X \otimes (L_1 \boxtimes L_2) = (K_X \otimes L_1) \boxtimes (K_X \otimes L_2)\) and taking global sections yields the \(m\)-th Wahl map

\[
H^0(X \times \mathcal{O}_{\Delta(X)}^m \otimes K_X \times \mathcal{O}_X \otimes (L_1 \boxtimes L_2)) \longrightarrow H^0(X, \text{Sym}^m(\Omega^1_X) \otimes K_X^2 \otimes L_1 \otimes L_2)
\]

To show that it is surjective, it suffices to prove that \(H^1(X \times \mathcal{O}_{\Delta(X)}^m \otimes K_X \times \mathcal{O}_X \otimes (L_1 \boxtimes L_2))\) vanishes which follows by combining Lemma 5.4 and Theorem 1.1. \(\square\)

**Remark 5.10.** We recover Corollary 3.4 in [BEL] from Proposition 5.9 when \(m = 1\), \(\ell = d + m\) and \(P_1 = \cdots = P_{d+m}\). The version of Proposition 5.9 that mimics Corollary 1.3 is left to the interested reader.

### 6. Vanishing for Vector Bundles

This section presents the proof of our Griffiths-type vanishing theorem for vector bundles. Throughout, \(E\) denotes a vector bundle of rank \(e\) on \(X\) which is a quotient of a direct sum of line bundles \(\mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_r) \longrightarrow E\) where each divisor \(D_i\) is nef.

**Proof of Proposition 1.4.** Let \(\pi : X' := \mathbb{P}(E) \longrightarrow X\) be the projective bundle of one-dimensional quotients of \(E\) and let \(\mathcal{O}_{X'}(1)\) be the tautological line bundle. From the cotangent bundle sequence for \(\pi\) and the relative Euler sequence, we obtain \(K_{X'} = \pi^*(K_X \otimes \det(E)) \otimes \mathcal{O}_{X'}(-e)\), and the projection formula gives \(\pi_*(K_{X'} \otimes \mathcal{O}_{X'}(m+e) \otimes \pi^*(L)) = K_X \otimes \det(E) \otimes \text{Sym}^m(E) \otimes L\). Since \(m \geq 0\), we have \(R^i \pi_*(\mathcal{O}_{X'}(m)) = 0\) for all \(i > 0\), so

\[
H^i(X', K_{X'} \otimes \mathcal{O}_{X'}(m+e) \otimes \pi^*(L)) = H^i(X, K_X \otimes \det(E) \otimes \text{Sym}^m(E) \otimes L), \quad \text{for all } i.
\]

Setting \(B := \pi^*(L) \otimes \mathcal{O}_{X'}(m+e)\), it suffices to show that \(H^i(X', K_{X'} \otimes B) = 0\) for all \(i > 0\).
Since $E$ is as a quotient of $\bigoplus_{j=1}^r \mathcal{O}_X(D_j)$, there is a map $\mathcal{O}_X(D_j) \rightarrow E$ for each $1 \leq j \leq r$. Let $F_j \in \mathcal{O}_X(1) \otimes \mathcal{O}_X(-\pi^*(D_j))$ be the effective divisor on $X'$ associated to the global section $\mathcal{O}_X \rightarrow \text{Sym}^1(E) \otimes \mathcal{O}_X(-D_i)$. We see that $B$ is big from the decomposition

$$B = \pi^*(L \otimes \mathcal{O}_X(mD_1 + \sum_{j=1}^e D_j)) \otimes \mathcal{O}_X(mF_1 + \sum_{j=1}^e F_j).$$

Hence, Theorem 11.2.12 (ii) in [PAG2] implies that $H^i(X', K_{X'} \otimes B \otimes \mathcal{J}(\|B\|)) = 0$ for all $i > 0$ and we need only show that $\mathcal{J}(\|B\|) = \mathcal{O}_{X'}$.

To achieve this, we claim that, for each $z \in X'$, there is a subset $\{s_1, \ldots, s_e\} \subseteq \{1, \ldots, r\}$ such that $z \notin F_{s_1}$ and $F_{s_2} + \cdots + F_{s_e}$ has normal crossings at $z$ (cf. Lemma 3.3). Indeed, there exists an affine neighbourhood $U$ of $\pi(z)$ such that $\pi^{-1}(U) \cong \text{Spec}(\mathcal{O}_X(U)[t_1, \ldots, t_e])$. By shrinking $U$ if necessary, we see that there is a subset $\{s_1, \ldots, s_e\} \subseteq \{1, \ldots, r\}$ such that the local equations corresponding to the sections $\mathcal{O}_X \rightarrow \text{Sym}^1(E) \otimes \mathcal{O}_X(-D_j)$ form a regular system of parameters in the ring $\mathcal{O}_X(U)[t_1, \ldots, t_e]$. Hence, we may assume that $z \notin F_{s_1}$. Since $\mathcal{O}_X(U)[t_1, \ldots, t_e]$ is a polynomial ring, we also deduce that $F_{s_2} + \cdots + F_{s_e}$ has normal crossings at $z$.

Now, the line bundle $\pi^*(L \otimes \mathcal{O}_X(mD_{s_1} + \sum_{j=1}^e D_j))$ is a big and nef, so Lemma 3.2 yields a positive integer $k_0$ and an effective divisor $D$ on $X'$ such that the linear series

$$\left| \mathcal{O}_{X'}(mF_{s_1} + \sum_{j=1}^e F_j) \otimes \pi^*(L^k) \otimes \mathcal{O}_{X'}(km \pi^*(D_{s_1}) + k \sum_{j=1}^e \pi^*(D_j) - D) \right|$$

has no base points for all $k \geq k_0$. Choose an effective divisor $F'$ in this linear series that does not pass through $z$, and for each $k \geq k_0$, consider the effective divisor $G := \frac{k}{k+1}(mF_{s_1} + \sum_{j=1}^e F_j) + \frac{1}{k}F'$. Since $G$ has normal crossing support at $z$, $\mathcal{J}(G)$ is trivial at $z$. Finally, we relate the multiplier ideal sheaf $\mathcal{J}(G)$ to the asymptotic multiplier ideal sheaf $\mathcal{J}(\|B\|)$ as in the proof of Theorem 1.1. \qed

We end with an uncomplicated example illustrating Proposition 1.4.

**Example 6.1.** Let $X = \mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_\ell$. If the divisor $D_k$ is the pull-back of the hyperplane from the $k$-factor $\mathbb{P}^n_k$, then the Euler sequence (e.g., see Theorem 8.1.6 in [CLS]) implies that the tangent bundle $T_X$ is a quotient of the direct sum $\bigoplus_{k=1}^\ell \bigoplus_{j=1}^{n_k+1} \mathcal{O}_X(D_k)$. Since each line bundle $\mathcal{O}_X(D_k)$ is nef and $\text{det}(T_X)$ is the anticanonical line bundle on $X$, Proposition 1.4 shows that $H^i(X, \text{Sym}^m(T_X) \otimes B) = 0$ for all $i > 0$, all $m \geq 0$, and all big, nef line bundles $B$. If $n_j \geq \ell$ for all $1 \leq j \leq \ell$, then $H^i(X, \text{Sym}^m(T_X) \otimes N) = 0$ for all $i > 0$, all $m \geq 0$, and all nef line bundles $N$.

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