NUMERICAL PRIMARY DECOMPOSITION

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Abstract. Consider an ideal \( I \subset R = \mathbb{C}[x_1, \ldots, x_n] \) defining a complex affine variety \( X \subset \mathbb{C}^n \). We describe the components associated to \( I \) by means of numerical primary decomposition (NPD).

The method is based on the construction of deflation ideal \( I^{(d)} \) that defines the deflated variety \( X^{(d)} \) in a complex space of higher dimension. For every embedded component there exists \( d \) and an isolated component \( Y^{(d)} \) of \( I^{(d)} \) projecting onto \( Y \). In turn, \( Y^{(d)} \) can be discovered by existing methods for prime decomposition, in particular, the numerical irreducible decomposition, applied to \( X^{(d)} \).

The concept of NPD gives a full description of the scheme \( \text{Spec}(R/I) \) by representing each component with a witness set. We propose an algorithm to produce a collection of witness sets that contains a NPD and that can be used to solve the ideal membership problem for \( I \).

1. Introduction

Throughout the paper we use the following notation. Let \( I \) be an ideal in the polynomial ring \( R = \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n] \) generated by polynomials \( f_1, \ldots, f_N \in R \). The ideal \( I \) defines the variety \( X = V(I) \), which, set-theoretically, is the set of points in \( \mathbb{C}^n \) annihilated by all polynomials in ideal \( I \).

The basic description of the affine scheme \( \text{Spec}(R/I) \) is given by the set of associated prime ideals \( \text{Ass}(I) \) consisting of all ideals that appear as annihilators of elements of the \( R \)-module \( R/I \).

There exist symbolic algorithms (see, e.g., \([6, 7, 3, 5, 19, 16]\)) to compute an irredundant primary decomposition of \( I \), which is by definition a decomposition
\[
I = J_1 \cap \cdots \cap J_r
\]
such that all ideals \( J_i \) are primary and their radicals \( I_i = \sqrt{J_i} \in \text{Ass}(I) \) are pairwise distinct.

The set of subvarieties defined by the prime ideals
\[
\text{VAss}(I) = \{ V(P) \mid P \in \text{Ass}(I) \}
\]
is called the components associated to \( I \). A component is said to be isolated if it is maximal with respect to inclusion and embedded if it is not. The existing methods of numerical algebraic geometry can “see” only the isolated components.

The concept of numerical primary decomposition (NPD) that we propose describes all components in terms of the (generalized) witness sets (see Definition \([4,11]\)). Such a witness set for a component \( Y \in \text{VAss}(I) \) describes \( Y \) completely: one can sample the component, determine its degree, determine whether a given point belongs to \( Y \), etc. Moreover, with a
NPD in hand one can solve the ideal membership problem, i.e., given a polynomial \( f \in R \) decide if \( f \in I \). This is discussed in Subsection 4.2.

The idea of our method lies in the construction of the deflation ideal \( I^{(d)} \) (see Definition 2.3) that defines the deflated variety \( X^{(d)} = V(I^{(d)}) \) of order \( d \) in a higher-dimensional space. The latter is a stratified vector bundle over \( X \) that comes with the natural projection \( \pi_d : X^{(d)} \to X \) onto the base. This construction is related to a deflation technique for 0-dimensional isolated components [12, 13, 9].

The main theoretical result, Theorem 3.8, shows that an embedded component \( Y \in \text{VAss}(I) \) becomes visible for some deflation order \( d \), i.e., there is an isolated component \( Z \in \text{VAss}(I^{(d)}) \) such that \( \pi_d(Z) = Y \).

Based on this theorem, we outline a straightforward algorithm, Algorithm 3.11, to compute all visible components up to the given order. The advantage of this algorithm over other known techniques is that the problem of finding all components is reduced to that of finding only isolated components without performing any saturation steps.

Algorithm 5.3 specializes this general algorithm to a numerical method for prime decomposition, namely, the numerical irreducible decomposition (NID) [17].

The area of numerical algebraic geometry comprises novel approaches to computational algebraic geometry based on the numerical polynomial homotopy continuation methods. The recent book [18] may serve as a good introduction to the area.

While these methods involve computations that are approximate, a typical output of the algorithms consists of not only approximations of exact solutions, which are algebraic numbers, but also exact discrete information about the input. For instance, the main concept introduced in this paper, a numerical primary decomposition of an ideal provides exact data such as the dimensions and the degrees of the associated components. Another example is a solution to the ideal membership problem given via NPD. Although we do not provide a certification procedure for this method, such can be developed in theory, provided that results of our numerical computation can be refined to an arbitrary precision.

One advantage of hybrid numerical homotopy continuation techniques over purely symbolic methods, such as Gröbner bases, is that the former is easily parallelizable and the algorithms for computing the latter are intrinsically serial.

Due to a very small amount of data being stored on or transferred between the computational nodes and embarrassing parallelism, one can easily achieve linear speedups for numerical homotopy continuation on a parallel computing system with any architecture. This comes in contrast with the high interdependency of tasks performed in Gröbner bases computation and the phenomena of intermediate expression swell. (Although there is no doubt that limited speedups are possible in parallel Gröbner computation, the claims of good scalability are substantiated with experiments on preselected classes of problems and methods based on non-optimal serial algorithms. See, e.g., [1] for critique of such claims.)

We live in the age when the clock speed of processors has stopped growing fast and the computational capacity of computers increases mainly through building either multicore or distributed systems. This tendency implies both great present and even better future for the algorithms of numerical algebraic geometry.
The paper is structured as follows. After the introduction (Section 1), we make the main definitions of the paper – those of deflated ideal and variety – in Section 2. Next, we overview the dual space approach to looking at the structure of a polynomial ideal; Section 3 culminates in the proof of Theorem 3.8, the main theoretical result of the paper. Section 4 proposes a new numerical representation of an ideal called numerical primary decomposition (NPD) via (generalized) witness sets. In Section 5, we give a skeleton of an NPD algorithm and examples. We discuss the future of growing meat on the skeleton in conclusion, Section 6.

2. Deflation

A variant of the matrix defined below appears in the deflation method for regularizing singular isolated solutions of a polynomial system described in [13], as well as the computation of the multiplicity of an isolated point [4].

Definition 2.1. The deflation matrix of order $d$ of an ideal $I$ generated by polynomials $f_1, \ldots, f_N \in R$ is a matrix $A_I^{(d)}$ with

- entries in $R$;
- rows indexed by $x^\alpha f_j$, where $|\alpha| < d$ and $j = 1, 2, \ldots, N$;
- columns indexed by partial differential operators $\partial^{\beta} = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$, where $|\beta| \leq d$;

and with the entry at row $x^\alpha f_j$ and column $\partial^{\beta}$ set to be

$$(2.1) \quad \partial^{\beta} \cdot (x^\alpha f_j) = \frac{\partial^{|\beta|}(x^\alpha f_j)}{\partial x^{\beta}}.$$ 

If a point $x$ is an isolated solution, in other words, $\{x\}$ is 0-dimensional irreducible component, then its multiplicity – i.e., the vector space dimension of the ring $R/I$ localized at $x$ – equals corank $A_I^{(d)}(x)$ for a large enough $d$. This fact is shown, for example, in [4] where a method for computing the multiplicity using $A_I^{(d)}(x)$ is introduced.

Remark 2.2. A variant of the deflation matrix – the column of $A_I^{(d)}$ labeled with $\partial^0$ is dropped – is used in the (higher-order) deflation in [13] to construct an augmented system of equations for which the multiplicity of a given isolation solution $x$ of the original system drops. For computational purposes there is no need to keep the aforementioned column, however, this paper’s definition of deflation matrix makes our argument in Section 3 more compact.

Definition 2.3. Let $I = (f_1, \ldots, f_N) \subset R$ and let $a = (a_\beta)$, $|\beta| \leq d$, be a vector of indeterminates.

The ideal generated by $f_1, \ldots, f_N$ and the entries of the vector $A(x^a)^T$ in the ring $\mathbb{C}[x, a]$ is called the deflation ideal of $I$ of order $d$ and denoted by $I^{(d)}$.

The deflated variety of order $d$ is defined as

$$(d) \quad X^{(d)} = V(I^{(d)}) \subset \mathbb{C}^{B(n,d)},$$

where $B(n,d) = \dim \mathbb{C}[x, a] = n + \binom{n+d-1}{d}$.
Given an ideal \( I \), the variety \( X(d) \) is well defined in view of the following proposition, the proof of which is self-contained.

**Lemma 2.4.** For every \( g \in I \), the deflated variety \( Y(d) \) of its hypersurface \( Y = V(g) \) contains \( X(d) = V(I(d)) \).

**Proof.** Let \( Q_\alpha = \sum a_\beta \partial^\beta \in \mathbb{C}[x, \partial] \) be the linear differential operator corresponding to the vector \( \alpha = (a_\beta) \) of indeterminates. It is enough to show that for every point \( x \in X \) and every vector \( \alpha \) in the kernel of \( A_f(\partial)(x) \) the expression \( Q_\alpha g \) vanishes at \( x \). This would imply that the fiber of \( Y(d) \to Y \) over \( x \) contains the fiber of \( X(d) \to X \) over \( x \), thus proving the statement.

Let us write \( g = \sum_{i \leq N} c_{i, \alpha} x^\alpha f_i \), where \( c_{i, \alpha} \in \mathbb{C} \). Given a point \( x \in \mathbb{C}^n \) we can rewrite this as two sums:

\[
g = \sum_{|\alpha| \geq d, i \leq N} c'_{i, \alpha} (x - x^\alpha f_i + \sum_{|\alpha| < d, i \leq N} c'_{i, \alpha} x^\alpha f_i.
\]

Now, apply the operator \( Q_\alpha \) to \( g \). The first sum above vanishes at \( x \), since the order of vanishing of every summand is at least \( d + 1 \). The second sum vanishes, since \( \alpha \in \ker A_f(\partial)(x) \).

**Proposition 2.5.** If \( I_1 \subset I_2 \subset R \), then \( V(I_2(d)) \subset V(I_1(d)) \) for all \( d \).

In particular, the deflation variety \( X(d) \) does not depend on the set of generators \( f_1, \ldots, f_N \) of \( I \) chosen to construct \( A_f(d)(x) \).

**Proof.** This follows from Lemma 2.4 since for a fixed set of generators \( f_1, \ldots, f_N \) of \( I \), by definition,

\[
V(I(d)) = V(I) \cap V((f_1)^{(d)}) \cap \cdots \cap V((f_N)^{(d)}).
\]

Moreover, a stronger Proposition 2.7 holds; its proof requires a global argument less transparent than the local argument of Lemma 2.4 and 2.5.

**Lemma 2.6.** Let \( Q_\alpha = \sum_{|\beta| \leq d} a_\beta \partial^\beta \) and let \( \chi^\alpha Q_\alpha \) is a formal derivative of \( Q_\alpha \) with respect to \( \partial_i \) for \( 1 \leq i \leq n 

Then \( \chi^\alpha Q_\alpha \cdot f_j \in I(d) \) for every \( j \) and any \( \alpha \), where the deflation ideal \( I(d) \) is defined using \( f_1, \ldots, f_N \).

**Proof.** In case \( |\alpha| \geq d \), the expression \( \chi^\alpha Q_\alpha \) is a constant in \( \partial \), hence, \( \chi^\alpha Q_\alpha \cdot f_j \in I(d) \), since \( f_j \in I(d) \).

Assume \( |\alpha| < d \). By generalized Leibnitz rule

\[
(2.2) \quad Q_\alpha \cdot (x^\alpha f_j) = \sum_{\gamma} \frac{1}{\gamma!} \frac{\partial^{\gamma}}{\partial x^\gamma} \chi^\alpha Q_\alpha \cdot f_j.
\]

The expression \( \frac{\partial^{\gamma}}{\partial x^\gamma} \) is zero if not \( \gamma = \alpha \) or \( |\gamma| < |\alpha| \). Assuming that the claim is established for degrees lower than \( |\alpha| \), it follows that \( \chi^\alpha Q_\alpha \cdot f_j \in I(d) \) for \( |\gamma| < |\alpha| \). Since
Observe that $V$ only isolated component is therefore, projecting isolated components of $a$ by the column vector $(\alpha, \beta, \gamma)$. Denote by $\gamma Q_\alpha \cdot f_j$, the summand corresponding to $\gamma = \alpha$, belongs to $I^{(d)}$ as well.

**Proposition 2.7.** The deflation ideal $I^{(d)}$ does not depend on the set of generators $I$ chosen to construct it.

**Proof.** By Lemma 2.6, we have $Q_\alpha \cdot (x^\alpha f_j) \in I^{(d)}$, therefore, $Q_\alpha \cdot h \in I^{(d)}$ for every $h \in I$, since $h$ is a linear combination of terms $x^\alpha f_j$ and the action of $Q_\alpha$ is linear.

The conclusion of the proposition follows immediately. □

**Example 2.8.** Let $I = (x_1^2, x_1 x_2 x_3) \subset \mathbb{C}[x_1, x_2, x_3]$. The radical $\sqrt{I} = (x_1)$, hence, the only isolated component is $V(x_1)$.

Compute the first order $(d = 1)$ deflation: multiply the deflation matrix

$$A^{(1)}_I(x) = \left[ \begin{array}{ccc} x_1^2 & \partial_1 & 0 \\ x_1 x_2 x_3 & 2x_1 & 0 \\ x_1 x_2 x_3 & x_2 x_3 & x_1 x_3 \\ x_1 x_2 x_3 & x_1 x_3 & x_1 x_2 \end{array} \right]$$

by the column vector $(a_0, a_1, a_2, a_3)^T$. The deflation ideal then is

$I^{(1)} = (x_1^2, x_1 x_2 x_3, a_1 x_1, a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2)$.

Observe that

$$\sqrt{I^{(1)}} = (x_1, a_1 x_2 x_3) = (x_1, a_1) \cap (x_1, x_2) \cap (x_1, x_3),$$

therefore, projecting isolated components of $X^{(1)}$ to $X$ gives two more (embedded) components: $V(x_1, x_2)$ and $V(x_1, x_3)$.

**Remark 2.9.** Denote by $\pi_d : X^{(d)} \rightarrow X$ the natural projection induced by the projection $\pi_d : \mathbb{C}^{B(n,d)} \rightarrow \mathbb{C}^n$, which maps $(x, a) \mapsto x$.

The deflated variety $X^{(d)}$ is a stratified vector bundle over the variety $X$ with the projection $\pi_d$ onto the base. The strata of $X$ over which $X^{(d)}$ is locally trivial are constructible algebraic subsets of $X$.

**Example 2.10.** Let $L$ be a plane in $\mathbb{C}^n$ of dimension $n - k$ defined by the vanishing of $I = (x_{j_1})$ for some $1 \leq j_1 \leq \cdots \leq j_k \leq n$.

Then it is not hard to establish that

$I^{(d)} = I + \{(a_\beta : |\beta| \leq d \text{ and } \beta_{j_i} \neq 0 \text{ for all } i\}$.

Therefore, $L^{(d)}$ is a plane in $\mathbb{C}^{B(n,k)}$ of dimension $n - k + \binom{n-k+d-1}{d}$.

Every plane in $\mathbb{C}^n$ can be brought to the above form with an affine change of coordinates. This change would result in a linear transformation on the derivations $\partial^\beta$; hence, a deflated variety of order $d$ of a plane of codimension $k$ is always a plane of dimension $n - k + \binom{n-k+d-1}{d}$.

### 3. Visible components

In this section we describe briefly the dual space (a.k.a. inverse system) approach to the local description of a point scheme. While from the computational point of view this approach has not been exploited yet beyond the 0-dimensional case, it comes handy in the proof of the main theorem of the section and the paper.
3.1. Dual spaces.

**Definition 3.1.** For $x \in \mathbb{C}^n$, let $\Delta^\beta_x : R \to \mathbb{C}$ be a linear functional defined by

$$\Delta^\beta_x(f) = (\partial^\beta \cdot f)(x) = \frac{\partial^{[\beta]} f}{\partial^\beta}(x), \quad f \in R.$$

The dual space $D_x[I]$ of ideal $I$ at a point $x$ is the subspace of the $\mathbb{C}$-span of functionals $\Delta^\beta_x$ consisting of the ones that annihilate all polynomials in $I$.

The dual space defined as above has been introduced under other names in literature; notably it is the local inverse system, the term going back to Macaulay [14]. The dual space $D_x[I]$ has a filtration

$$D_x^{(0)}[I] \subset D_x^{(1)}[I] \subset D_x^{(2)}[I] \subset \ldots$$

where $D_x^{(d)}[I]$ is the set of functionals of order at most $d$. For computational purposes, it is convenient to think of $D_x^{(d)}[I]$ as ker $A_I^{(d)}(x)$: indeed, given a vector $a \in \ker A_I^{(d)}(x)$, one can view it naturally as the functional $\sum_{|\beta| \leq d} a_\beta \Delta^\beta_x$. The natural isomorphism $\ker A_I^{(d)}(x) \simeq D_x^{(d)}[I]$ could be shown then by inspection of the definition of the deflation matrix. In particular, for the case of an isolated point $x$, the whole dual space $D_x[I]$ is isomorphic to $\ker A_I^{(d)}(x)$ for $d > 0$, which is shown, for example, in [11 Theorem 1].

Let $e_i$ be an $n$-vector with one in the $i$-th position and zeros in the rest. On the space $D_x[0]$ of all differential functionals at a point $x$ we define

1. operators of integration $\delta_i : D_x^{(d)}[0] \to D_x^{(d+1)}[0]$, $i = 1, \ldots, n$, induced by the multiplication by $\partial_i$ in the Definition 3.1 i.e.:

$$\delta_i(\Delta^\beta_x)(f) = (\partial_i \partial^\beta \cdot f), \quad f \in R,$$

in other words,

$$\delta_i(\Delta^\beta_x) = \Delta^\beta_{x+e_i};$$

2. operators of differentiation $\chi_i : D_x^{(d+1)}[0] \to D_x^{(d)}[0]$, $i = 1, \ldots, n$, induced by the multiplication by $(x_i - x_i)$ in the Definition 3.1 i.e.:

$$\chi_i(\Delta^\beta_x)(f) = ((x_i - x_i) \partial^\beta \cdot f)$$

$$= ((\partial^\beta(x_i - x_i) + \beta_i \partial^{\beta-e_i} \cdot f),$$

therefore, after specializing $x = x$,

$$\chi_i(\Delta^\beta_x) = \beta_i \Delta^\beta_{x-e_i}.$$

Since $\Delta^\beta_x = \delta^\beta \Delta^0_x$, for a fixed $x$ we may look at the space of linear functionals $D_x[0]$ as $\mathbb{C}[\delta] = \mathbb{C}[\delta_1, \ldots, \delta_n]$ with differentiation operators $\chi_i$ acting as formal derivations $\frac{\partial}{\partial \delta_i}$. Note that $D_x[I] \subset \mathbb{C}[\delta_1, \ldots, \delta_n]$ is stable under this action of $\chi_i$ for every ideal $I \subset R$.

In fact, by (naturally) extending the space of linear functionals to the formal power series $\mathbb{C}[[\delta]]$ one may obtain the following fact (see, e.g., [15 Proposition 2.6]).

**Proposition 3.2.** The ideals of $R$ are in one-to-one correspondence with the vector subspaces of $\mathbb{C}[[\delta]]$ stable under differentiation and closed in $(\delta)$-adic topology.
For our purposes, given an ideal \( I \), we need just a local statement regarding \((R/I)_x = (R/I)_{m_x}\), the quotient local ring at a point \( x \), where \( m_x = (x_1 - x_1, \ldots, x_n - x_n) \subset R \).

One may show that the subspaces of \( C[\delta] \) stable under the differentiation are in one-to-one correspondence with the ideals of the local ring \( R_x \). The following lemma proves this statement in one direction.

**Lemma 3.3.** The image of \( f \in R \) is zero

1. in \((R/I + m_x^{d+1})_x\) iff \( Q \cdot f = 0 \) for all \( Q \in D_x^{(d)}[I] \);
2. in \((R/I)_x \) iff \( Q \cdot f = 0 \) for all \( Q \in D_x[I] \).

**Proof.** Statement (2) follows from (1).

To prove (1), notice that \( D_x[I + m_x^{d+1}] \) is exactly \( D_x^{(d)}[I] \). Indeed, every functional of the latter annihilates not only \( I \), but also all elements of \( m_x^{d+1} \); on the other hand, a functional of order larger than \( d \) does not kill the entire \( m_x^{d+1} \). \( \square \)

**Remark 3.4.** In case of an isolated point \( x \in X \), it follows from the lemma that the dimensions of the \( C \)-spaces \( D_x[I] \) and \((R/I)_x \) are equal. Stronger statements are available: for example, see [13, Theorem 3.1] and [15, Theorem 3.2]. The latter provides a recipe for constructing the \( m_x \)-primary component of the ideal \( I \) from a basis of \( D_x[I] \).

### 3.2. A visible deflation of a component

Recall the deflated variety \( X^{(d)} = V(I^{(d)}) \subset C^{B(n,d)} \) and the projection \( \pi_d : X^{(d)} \to X \) in the Definition 2.3.

**Definition 3.5.** We define the deflation of order \( d \) of a component \( Y \in \text{VAss}(I) \) as \( Y^{(d)} = \pi_d^{-1}Y^o \subset X^{(d)} \), where \( Y^o \) is the subset of generic points, that is all smooth points that do not belong to other components that do not contain \( Y \).

**Proposition 3.6.** A deflation of a component is an irreducible subvariety.

**Proof.** As was mentioned in Remark 2.9, \( X^{(d)} \) is a stratified bundle. The generic locus \( Y^o \) is a stratum in the stratification; indeed, the bundle is locally trivial at every generic point. Since \( \pi_d^{-1}Y^o \) is an open subset of \( Y^{(d)} \), the latter is irreducible. \( \square \)

**Definition 3.7.** We say that \( Y \in \text{VAss}(I) \) is visible at order \( d \), if \( Y^{(d)} \) is an isolated component of \( X^{(d)} \).

**Theorem 3.8.** Every component is visible at some order.

**Proof.** Note that an isolated component \( Y \) is visible for any \( d \geq 0 \), since \( Y^{(d)} = \pi_d^{-1}Y^o \) can not be contained in any other component in \( \text{VAss}(I^{(d)}) \).

Let \( I = \bigcap_{Z \in \text{VAss}(I)} J_Z \) be an irredundant primary decomposition of \( I \), where \( V(J_Z) = Z \) for all \( Z \). Fix an embedded component \( Y \) with a generic point \( y \in Y^o \) and consider \( I_{>Y} = \bigcap_{Y \subset Z \in \text{VAss}(I)} J_Z \). Then \( I_{>Y} \supseteq I \), moreover, the local quotient ring \((R/I_{>Y})_y \) is a proper quotient of \((R/I)_y \). Hence, according to Lemma 3.3 there is a proper inclusion \( D_y[I] \supset I_{>Y} \).

For any \( Z \in \text{VAss}(I) \) containing \( Y \) the fiber of its deflation \( \pi_d^{-1}(y) \cap Z^{(d)} \) is contained in \( D_y^{(d)}[I_{>Y}] \), since only the fibers over the generic locus \( Z^o \) define \( Z^{(d)} \). Now, there exists \( d \) such that

\[
\pi_d^{-1}(y) \cap Y^{(d)} = D_y^{(d)}[I] \supset D_y^{(d)}[I_{>Y}] \supset \pi_d^{-1}(y) \cap Z^{(d)}
\]
that implies immediately that $Z^{(d)}$ does not contain $Y^{(d)}$. \hfill \Box

**Remark 3.9.** If $Y$ is visible at order $d$ then it is visible at any order $d' \geq d$. Indeed, the component
\[ Y^{(d')} = \pi_{d'}^{-1}Y = \pi_{d'}^{-1}(Y^{(d)})^0, \]
where $\pi_{d'\rightarrow d} : X^{(d')} \rightarrow X^{(d)}$ is the natural projection, has to be isolated if $Y^{(d)}$ is isolated.

Since there is a finite number of components there is an order, at which all components are visible.

**Example 3.10.** Consider two ideals of $\mathbb{C}[x_1, x_2, x_3]$:
\[ I = (x_1^2, x_1x_2x_3), \]
\[ J = (x_1^2, x_1x_2^2x_3, x_1x_2x_3^2). \]

The first order deflation of $I$ has been computed in Example 2.8. For $J$ we have
\[
A_J^{(1)} = \begin{bmatrix} x_1^2 & x_1\partial_1 & x_1\partial_2 & x_1\partial_3 \\
2x_1 & x_1x_2x_3 & x_1x_2x_3^2 & x_1x_2x_3^2 \\
x_1x_2x_3 & x_1x_2x_3 & x_1x_2x_3^2 & x_1x_2x_3^2 \\
x_1x_2x_3^2 & x_1x_2x_3^2 & x_1x_2x_3^2 & 2x_1x_2x_3 \end{bmatrix}
\]
The entries of $A_J^{(1)}(a_0, a_1, a_2, a_3)^T$ together with the original generators of $J$ generate
\[
J^{(1)} = (x_1^2, x_1x_2^2x_3, x_1x_2x_3^2, a_1x_1, a_1x_2x_3 + 2a_2x_1x_2x_3 + a_3x_1x_2^2),
\]
\[
a_1x_2x_3^2 + a_2x_1x_2^2 + 2a_3x_1x_2x_3 \). \]

It is easy to see that
\[
\sqrt{J^{(1)}} = \sqrt{I^{(1)}} = (x_1, a_1x_2x_3) = (x_1, a_1) \cap (x_1, x_2) \cap (x_1, x_3),
\]
therefore, we need a higher order deflation to distinguish $J$ from $I$.

One may check that the second order deflation uncovers another embedded component in VAss($J$), the origin, which is not a component in VAss($J$). However, still $\sqrt{J^{(2)}} = \sqrt{I^{(2)}}$, for $I$ the origin is a pseudo-component!

It is not until the third deflation that we see the difference and here is why. The difference may be seen by comparing the dual spaces at the origin: $D_0[I]$ and $D_0[J]$. First of all, $D_0[I] \subset D_0[J]$, since $J \subset I$. This inclusion is proper as $\Delta_0^{(1,1,1)} \in D_0[J] \setminus D_0[I]$, a functional of order 3. On the other hand, $D_0^{(2)}[I] = D_0^{(2)}[J]$.

Here we would like to propose the most general algorithm for primary decomposition, which take the order of deflation $d$ as a parameter and returns all components for $d \gg 0$.

**Algorithm 3.11.** $\mathcal{N} = \text{VisibleComponents}(I, d)$

**Require:** $I$, ideal of $R = \mathbb{K}[x]$ where $\mathbb{K}$ is a field of characteristic 0; $d > 0$.

**Ensure:** $C$, the set components visible at order $d$.

\[ C = \pi_d \{ \text{isolated components of } I^{(d)} \}. \]
The correctness of the algorithm is assured by Theorem 3.8; although we use $\mathbb{K} = \mathbb{C}$, the argument holds for an arbitrary algebraically closed field of characteristic 0.

In fact, by passing to the algebraic closure, one can make the same conclusion for an arbitrary $\mathbb{K}$ of characteristic 0. The only serious issue in this case is that a component that is irreducible over $\mathbb{K}$ may become reducible over its algebraic closure. However, the deflation procedure preserves irreducibility and the property of algebraic closedness is exploited only locally in the proof of Theorem 3.8.

While any routine for prime decomposition can be used to compute the isolated components of the deflated variety, in what follows we concentrate on a numerical approach, which is applicable only for $\mathbb{K} = \mathbb{C}$.

4. Witness sets and numerical primary decomposition

4.1. Generalized and classical witness sets. All numerical algorithms based on homotopy continuation boil down to the computation of approximations to points of a 0-dimensional variety. That is why for every component $Y$ we need to invent a presentation that would consist of a finite number of points and, perhaps, some additional (finite) information.

Definition 4.1. A witness set $W = W_Y$ of a component $Y \in VAss(I)$ is a triple $(d, L, w) = (d_Y, L_Y, w_Y)$ consisting of

1. an order $d$, such that $Y$ is visible at order $d$;
2. a generic $(\text{codim} Y^{(d)})$-plane $L \subset \mathbb{C}^{B(n,d)}$;
3. the (finite) set of witness points $w = Y^{(d)} \cap L$;

All items can be presented with finite data: in particular, $L$ can be represented by a linear basis. We do not include as elements of the witness set generators of $I$, we assume that those are fixed and available.

We also assume that there is a procedure $\mathcal{H}_{L,L'}$ that for another generic $(\text{codim} Y^{(d)})$-plane $L' \subset \mathbb{C}^{B(n,d)}$ takes the witness points $w_Y$ as input and produces a new set of witness points $w'_Y$ forming a witness set $(d, L', w'_Y)$. In numerical algebraic geometry such a procedure is provided by a sufficiently randomized homotopy continuation that deforms $L$ into $L'$ without encountering an intermediate plane that is singular with respect to $Y^{(d)}$ and a numerical routine that tracks the paths starting at the witness points $w_Y$.

We would like to remark that, in principle, it is enough to store only one witness point, since the rest can be obtained due to the action of the monodromy group, which is transitive on $w_Y$. In practice, this can be done by following a random homotopy cycle $\mathcal{H}_{L,L}$ a finite number of times. However, it is, of course, more practical to store the whole set $w_Y$.

Remark 4.2. An isolated component is visible at order $d = 0$. In this case, the plane $L$ and the set $w$ give what we would call a classical witness set, a concept which is used, for example, throughout [15] where a generalization of it is made (page 237) in relation to the deflation of an isolated (but multiple) component.
Example 4.3. The components of $I = (x_3^2, x_3(x_2 + x_1^2)) \subset \mathbb{C}[x_1, x_2, x_3]$ are $Z = V(x_3)$ and $Y = V(x_3, x_2 + x_1^2)$.

The isolated component $Z$ can be presented by the witness set using any line $L$ that is not parallel to $Z$ and such that $Z \cap L \not\in Y$.

To represent $Y$, we have to look at the first order deflation:

\[
I^{(1)} = (x_3^2, x_3(x_2 + x_1^2), a_3x_3, 2a_1x_1x_3 + a_2x_3 + a_3(x_2 + x_1^2)) \subset \mathbb{C}[x_1, x_2, x_3, a_0, a_1, a_2, a_3]
\]

\[
X^{(1)} = V(x_3, a_3(x_2 + x_1^2)) \subset \mathbb{C}^7
\]

The deflated variety $X^{(1)}$ is 5-dimensional; we take the following 5 equations for $L$:

\[
\begin{align*}
x_2 &= -3x_1 + 2; \\
a_3 &= x_1 - 3; \\
a_i &= c_i x_1 + d_i, \quad c_i, d_i \in \mathbb{C}, \quad i = 0, 1, 2.
\end{align*}
\]

The first two equations together with the second defining equation of $X^{(1)}$ give

\[
(x_1 - 3)(x_1 - 2)(x_1 - 1) = 0.
\]

The set $\pi_1(X^{(1)} \cap L) = \{(3, -7, 0), (2, -4, 0), (1, -1, 0)\}$ contains projections of two subsets of witness points,

\[
\begin{align*}
\pi_1(w_Z) &= \{(3, -7, 0)\}, \\
\pi_1(w_Y) &= \{(2, -4, 0), (1, -1, 0)\},
\end{align*}
\]

of the witness sets of the first order $W_Z = (1, L, w_Z)$ and $W_Y = (1, L, w_Y)$, respectively.

Remark 4.4. For a witness point $(y, a) \in w_Y$, the vector $a$ translates into a functional $Q_a \in D_Y^{(d)}[I]$. For a fixed generic $y \in Y$ the set $\{Q_a \mid (y, a) \in Y^{(d)}\}$ equals the dual space $D_Y^{(d)}[I]$. Therefore, in practice, we can compute $D_Y^{(d)}[I]$ from the witness set $(d, L, w)$ of $Y$ by tracking a homotopy that creates another witness set $(d, L_i, w_i)$, where the plane $L_i$ is a random plane such that $\pi_d(L) \cap Y \ni y$. If the procedure is carried out for sufficiently many $L_i$, then the functionals $\{Q_a \mid (y, a) \in w_i\}$ span $D_Y^{(d)}[I]$.

4.2. Numerical primary decomposition and the ideal membership problem.

Definition 4.5. A collection of witness sets is called a numerical primary decomposition (NPD) of $I$ if it contains precisely one witness set for each component in $\text{VAss}(I)$.

NPD contains exhaustive information about the ideal $I$ and, in particular, the scheme $\text{Spec}(R/I)$ due to the possibility of solving the ideal membership problem. This is so, since for every $Y$ the projection $\pi_{d_Y}(z)$ of a witness point $z \in w_Y$ gives a generic point of $Y$ and in the view of the following:

Theorem 4.6. A polynomial $g \in R$ is contained in the ideal $I$ iff for all $Y \in \text{VAss}(I)$ and every (any) generic point $y \in Y$, all functionals in the dual space $D_Y^{(\deg g)}[I]$ annihilate $g$. 
Proof. A polynomial \( g \in I \) iff its image in \( R/I \) is zero or, equivalently, for all \( x \in X = V(I) \) its image in \( (R/I)_x \) vanishes. It suffices to check the latter statement for one generic point per component for all components.

It follows that, in practice, we can solve the ideal membership problem by checking the condition in the theorem at a finite number of points. Therefore, we have the following algorithm.

**Algorithm 4.7.** \( b = \text{IMP}(g, I, N) \)

**Require:** \( I \), ideal of \( R \) represented by a finite set of generators; \( N \), a NPD of \( I \).

**Ensure:** \( b = \text{“} g \in I \text{”} \), a boolean value.

Let \( d = \deg g \) and \( g = \sum_{|\beta|\leq d} c_\beta x^\beta \); for all \( (d', L, w) \in N \) do

Pick \( x \in \pi'_{d'}(w) \);

Compute a linear basis \( K \) of \( \ker A_{d'}(x) \);

if \( Q \cdot f \neq 0 \) (*) for some \( Q \in D_{x_{d'}(I)} \) corresponding to an element of \( K \) then

Return false;

end if

end for

Return true.

According to Theorem 4.6 \( f \in I \) iff there is no \( x \in X \), for which the condition (*) holds. In fact, it suffices to check (*) for a set of generic points (one per component).

**Remark 4.8.** In case the above algorithm is executed numerically, i.e., only approximations of points on the components are generated, we would like to point out several practical issues.

First, having the whole NPD it easy to generate other points on any given component and recheck the condition (*) at as many points as desired, therefore, lowering the probability of this algorithm returning an incorrect result due to picking a non-generic point.

Second, since the approximations of generic points can be refined to an arbitrary precision, the condition (*) can be effectively checked by “zooming in” on the exact point for which the computation is carried out. However, a rigorous certification procedure has to be developed in the future in the spirit of the alpha-test for an approximate zero of a univariate polynomial.

Algorithm 4.7 is practical only for polynomials of low degrees due to the high complexity of construction of the deflation matrix of order \( d \) and computing its kernel. However, we are confident that improvements can be made as this matrix is highly structured and the fact that it is enough to check the condition (*) for only one generic element in the kernel.

5. **Algorithm for Numerical Primary Decomposition**

In the description of our NPD algorithm, we assume the following subroutines are at our disposal:
Algorithm 5.1. $\mathcal{M} = \text{NID}(I)$

Require: I, ideal of $\mathbb{R}$.
Ensure: $\mathcal{M}$, the set of classical witness sets $(L_Y, w_Y)$ for all isolated components $Y \in \text{VAss}(I)$.

There are two approaches to NID in the numerical algebraic geometry: the “top-down” method described in detail in [18, Chapter 15] and “equation-by-equation” method the philosophy of which is outlined in [18, §16.2].

Algorithm 5.2. $b = \text{IsInComponent}(y, W_Z)$

Require: $y \in \mathbb{C}^n$; $W_Z = (d, L, w)$ a witness set for $Z \in \text{VAss}(I)$.
Ensure: $b = "y \in Z", a boolean value.$

Pick a generic $(\text{codim } Z)$-plane $M \subset \mathbb{C}^n$ and a $(\text{dim } L)$-plane $L' \subset \mathbb{C}^{B(n,d)}$ such that $\pi_d(L') = M \ni y$.
Use a procedure similar to the usual containment test routine [18, §15.1], i.e., track the points $w$ along a generic homotopy $\mathcal{H}_{L,L'}$.
Return “$y \in \pi_d(\mathcal{H}_{L,L'}(w))$”.

There are also two subroutines, for which finding efficient algorithms is an open problem:

- **StopCriterium**$(d, I, \mathcal{N})$ implements a termination criterion that guarantees that all components in $\text{VAss}(I)$ are visible at order less than $d$.
- **IsComponent**$((d, L, w), I, \mathcal{N})$ is used to filter out witness sets $(d, L, w)$ that represent *false components* that appear due to singularities. As a parameter it takes partial NPD $\mathcal{N}$ that includes the witness sets for all components visible at order $d$ of dimensions higher than the dimension of the alleged component.

Both routines will be discussed later in this section.

Now we are ready to outline the main algorithm of this paper.

Algorithm 5.3. $\mathcal{N} = \text{NPD}(I)$

Require: I, ideal of $\mathbb{R}$.
Ensure: $\mathcal{N}$, the set of witness sets for all $Y \in \text{VAss}(I)$ visible at order $d$.

\[
\mathcal{N} = \emptyset;
\text{repeat}
\quad \mathcal{N} = \{ (L, w) \in \text{NID}(I^{(d)}) : \neg \text{IsInComponent}(\pi_d(w_1), W) \text{ for all } W \in \mathcal{C} \};
\quad \text{for all } (L, w) \in \mathcal{N} \text{ in an order of decreasing } \dim Y_{(L,w)} \text{ do}
\quad \quad \text{if } \text{IsComponent}((d, L, w), I, \mathcal{N}) \text{ then}
\quad \quad \quad \mathcal{N} = \mathcal{N} \cup \{ (d, L, w) \};
\quad \quad \text{end if}
\quad \text{end for}
\quad d = d + 1;
\text{until } \text{StopCriterium}(d, I, \mathcal{N})
The order of deflation sufficient to discover all components can be bound by the maximum of the regularities of the (local) Hilbert functions at the points of \(X\). For the latter a crude bound exists, which is doubly exponential in the number of variables. Obviously, one can not use \texttt{StopCriterium} based on this bound for practical purposes.

In reality, for many nontrivial examples all embedded components are discovered by the deflation of order as low as 1. While finding a reasonable termination criterion is the matter of the future, currently it makes sense to run a \textit{truncated} computation with

\[
\text{StopCriterium}(d, I, \mathcal{N}) = "d > d_{\text{max}}",\]

where \(d_{\text{max}}\) is the maximal deflation order considered.

As to \texttt{IsComponent}, first of all, we remark that elimination of fake components from \(\mathcal{N}\) may also be done at the end of the algorithm. There is a way to do this by checking whether for large enough \(d\) all functionals in the \(D^{(d)}_y[I]\) for a generic \(y\) in an alleged component \(Y\) “come” from the components that contain \(Y\). We do not describe the procedure here as it is quite technical and not practical at the moment, since the question how large \(d\) should be relates to the question of finding a good \texttt{StopCriterium}.

\textbf{Remark 5.4.} Establishing an efficient \texttt{StopCriterium} has a higher priority (over \texttt{IsComponent}), since \(\mathcal{N}\) containing additional witness sets of fake components can be used instead of a true NPD for many tasks. In particular, our \texttt{IMP} routine (Algorithm 4.7) would still work.

Below is an example of a “truncated computation”, where only the first order deflation is computed.

\textbf{Example 5.5.} Consider the cyclic 4-roots problem:

\[
I = \langle x_1 + x_2 + x_3 + x_4, \ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1, \\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2, \ x_1x_2x_3x_4 - 1 \rangle.
\]

The calculation of associated primes via symbolic software (we used \textit{Macaulay} 2 [8]) gives:

\[
\text{Ass}(I) = \{ (x_2 + x_4, x_1 + x_3, x_3x_4 + 1), \\
(x_2 + x_4, x_1 + x_3, x_3x_4 - 1), \\
(x_4 - 1, x_3 + 1, x_2 + 1, x_1 - 1), \\
(x_4 - 1, x_3 - 1, x_2 + 1, x_1 + 1), \\
(x_4 + 1, x_3 + 1, x_2 - 1, x_1 - 1), \\
(x_4 + 1, x_3 - 1, x_2 - 1, x_1 + 1), \\
(x_3 + x_4, x_2 + x_4, x_1 - x_4, x_3^2 + 1), \\
(x_3 - x_4, x_2 + x_4, x_1 + x_4, x_3^2 + 1) \}
\]

The first two ideals correspond to the irreducible curves that are the two isolated components. The rest are embedded 0-dimensional components; note that the last two ideals are irreducible over the ground field \(\mathbb{Q}\), but not \(\mathbb{C}\).

Over complex numbers, there are 8 embedded components that are all visible at order 1; The numerical computation of the irreducible components of the first deflation finds all
components. An excerpt from the numerical output is given below: we list the projections of the witness points for the components $\text{VAss}(I^{(1)})$.

```python
>>> projections of witness points for
>>> component #1:
[x1 = -4.4882+2.0260*I, x2 = .18509+.83550e-1*I,  
x3 = 4.4882-2.0260*I, x4 = -.18509-.83550e-1*I]
[x1 = .52885e-1-.87608*I, x2 = -.68654e-1+1.1373*I,  
x3 = -.52885e-1+.87608*I,x4 = .68654e-1+1.1373*I]
[x1 = -.15083+.49191*I, x2 = .56975+1.8582*I,  
x3 = .15083-.49191*I, x4 = -.56975-1.8582*I]
[x1 = .41488+.24720*I, x2 = -1.7788+1.0599*I,  
x3 = -.41488-.24720*I, x4 = 1.7788-1.0599*I]
>>> component #2:
[x1 = -.95775+.36799*I, x2 = -.90980-.34957*I,  
x3 = .95775-.36799*I, x4 = .90980+.34957*I]
[x1 = .71538+.12328*I, x2 = 1.3576-.23395*I,  
x3 = -.71538-.12328*I, x4 = -1.3576+.23395*I]
[x1 = -3.7686+1.7072*I, x2 = -.22017-.99738e-1*I,  
x3 = 3.7686-1.7072*I, x4 = .22017+.99738e-1*I]
[x1 = -.16036-.30943*I, x2 = -1.3202+2.5476*I,  
x3 = .16036+.30943*I, x4 = 1.3202-2.5476*I]
>>> component #3:
[x1 = -1.0-.53734e-17*I, x2 = 1.0-.20045e-16*I,  
x3 = 1.0+.89149e-17*I, x4 = -1.0+.18026e-17*I]
...
>>> component #10:
[x1 = -.59351e-17+1.0*I, x2 = -.46995e-16+1.0*I,  
x3 = .16158e-16-1.0*I, x4 = .22439e-16-1.0*I]
```

Note that there are 2 witness sets of 4 points corresponding to the 2 isolated curves and 8 singletons for the embedded points.

See the webpage [10] for the scripts in Macaulay 2 and Maple (using PHCmaple package [11]) that perform prime decomposition and numerical irreducible decomposition, respectively, for the first deflation ideal in this example.

6. Discussion and conclusion

We consider this paper as one laying a theoretical foundation to the method that at this point works only on small examples. However, it is our believe that this technique would be able to solve problems unsolvable by purely symbolic methods in the future. The improvements are expected to be made both in the software and in the theory.

We remark that the software in the area of numerical algebraic geometry is as young as the area itself. For the purposes of numerical irreducible decomposition there exist only two software options: PHCpack [20] that we use via PCHmaple [11] and Bertini [2]. The practical computation using the ideas in this paper is limited by the capabilities of these
software systems; we expect the implementations of numerical irreducible decomposition algorithms to improve. Both PHCpack and Bertini move towards the throughout parallelization; as we argued in the introduction, easy parallelization is a crucial feature of numerical methods that distinguishes them from the symbolic ones.

The future theoretical work should, in particular, concentrate on the construction of special homotopy methods to tackle deflated ideals (systems), which possess an obvious multihomogeneous structure: they are linear in the additional variables. Also, while the global algorithms such as NID have been well established, there are still no efficient local procedures, e.g., for determining the local dimension at a given point on a variety. The same can be said about the local dual space computation: while it is possible to compute the truncation at some degree using the deflation matrix, an efficient description of the whole (possibly infinite-dimensional) dual space and ways to create such description are yet to be found.

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