On the oscillations of dissipative superfluid neutron stars

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We investigate the oscillations of slowly rotating superfluid stars, taking into account the vortex mediated mutual friction force that is expected to be the main damping mechanism in mature neutron star cores. Working to linear order in the rotation of the star, we consider both the fundamental f-modes and the inertial r-modes. In the case of the (polar) f-modes, we work out an analytic approximation of the mode which allows us to write down a closed expression for the mutual friction damping timescale. The analytic result is in good agreement with previous numerical results obtained using an energy integral argument. We extend previous work by considering the full range of permissible values for the vortex drag, e.g. the friction between each individual vortex and the electron fluid. This leads to the first ever results for the f-mode in the strong drag regime. Our estimates provide useful insight into the interdependence of, and relevance of, various equation of state parameters. In the case of the (axial) r-modes, we confirm the existence of two classes of modes. However, we demonstrate that only one of these sets remains purely axial in more realistic neutron star models. Our analysis lays the foundation for companion studies of the mutual friction damping of the r-modes at second order in the slow-rotation approximation, the first time evolutions for superfluid neutron star perturbations and also the first detailed attempt at studying the dynamics of superfluid neutron stars with both a relative rotation between the components and mutual friction.

I. INTRODUCTION

Neutron stars have a complex interior structure. With core densities reaching several times the nuclear saturation density, these objects require an understanding of physics that cannot be gained from laboratory experiments. This makes the modelling of neutron star dynamics an interesting challenge. On the one hand, one has to consider exotic physics that is, at best, poorly constrained. On the other hand, one may ask to what extent observations can distinguish between different possible models. An excellent example of this interplay concerns the possibility that the quarks may deconfine in the high density region. If this is the case, it will have a considerable effect on transport properties associated with viscosity and heat conductivity. In fact, such a quark core is expected to be a colour superconductor \cite{1}. The dynamics of this exotic state of matter, and the relevance of its different possible phases, is not yet certain. In order to improve our understanding of this problem, we need to build more precise stellar models and study, for example, their oscillation properties in detail. In this context, considerable attention has been focused on the inertial r-modes of a rotating star. The r-modes are interesting because they can be driven unstable by the emission of gravitational radiation, see \cite{2, 3} for literature reviews. The r-mode instability window is, however, sensitive to the physics of the neutron star interior. Since the bulk and shear viscosities are quite different in a quark core, compared to “normal” npe matter, one may hope to use observations to constrain the theory, see \cite{4} for a discussion of the relevant literature. In absence of a direct detection of an r-mode gravitational-wave signal, this analysis would have to be based on the nature of the instability window. The idea would be that, if an observed neutron star spins at a rate that would place it inside a predicted instability region, one may be able to rule out this particular theoretical model. Of course, this argument comes with a number of caveats. It could, for example, be that additional physics places a stronger constraint on the r-modes than the considered mechanisms. Inevitably, this becomes a “work in progress” where improved theoretical models are tested against better observational data.

In order to consider “realistic” neutron stars, it is important to appreciate the relevance of superfluidity. A neutron star is expected to contain a number of superfluid/superconducting components \cite{5}, and it is crucial to understand to what extent this affects the stars oscillation properties. It is well established that the behaviour of a superfluid system can differ significantly from standard hydrodynamics. The most familiar low-temperature system is, perhaps, He\textsubscript{3}, which exhibits superfluidity below a critical temperature near 2 K. Experimentally, it has been demonstrated that this system is very well described by the Navier-Stokes equations above the critical temperature. Below the critical temperature the behaviour is very different, and a “two-fluid” model is generally required (see \cite{6} for a very recent discussion). Superfluid neutron stars are, to some extent, similar. The second sound in Helium is analogous to a set of, more or less distinct, “superfluid” oscillation modes \cite{6, 7, 8} in a neutron star. These additional modes arise because the different components of a superfluid system are allowed to move “through” each other. The dissipation channels in a superfluid star are also quite different. Basically, the superfluid flows without friction. In the outer core of a neutron star, which is dominated by npe matter, one expects the neutrons to be superfluid while the protons form a superconductor. As a result, the shear viscosity is dominated by e-e scattering \cite{6, 10}. The bulk viscosity, which is due to the fluid motion driving the system away from chemical equilibrium and the resultant energy loss due to nuclear reactions, is also expected to be (exponentially) suppressed in a superfluid \cite{5}. These effects have direct
implications for the damping of neutron star oscillations, and play a key role in determining the r-mode instability window for a mature neutron star. This is, however, not the end of the story. A superfluid exhibits an additional dissipation mechanism, usually referred to as “mutual friction”. The mutual friction is due the presence of vortices in a rotating superfluid. In a neutron star core, the electrons can scatter dissipatively off of the (local) magnetic field of each vortex (see [11, 12, 13] for discussions and references). This effect may dominate the damping of realistic neutron star oscillation modes.

The basic requirements of a rudimentary model for superfluid neutron star oscillations should now be clear. One must account for the additional dynamical degree(s) of freedom, and also account for the mutual friction damping. This is obviously only a starting point, but the problem is sufficiently complicated that one may want to proceed with care. There has already been a number of studies of dissipative superfluid oscillations. The area was pioneered by Lindblom and Mendell who, in particular, demonstrated that the gravitational-wave instability of the fundamental f-modes would be suppressed in a superfluid star [14]. Following the discovery of the r-mode instability, they also provided the first accurate estimates of the relevance of the mutual friction for these modes [13]. Similar results were subsequently obtained by Lee and Yoshida [10]. These studies provide important assessments of the relevance of the mutual friction damping. There are, however, a number of reasons why we need to return to this problem. Most importantly, we want to consider more realistic neutron star models, including finite temperature effects, magnetic fields and the possible presence of exotic (hyperon and/or quark) cores. The additional physics brings additional complications, like additional fluid degrees of freedom, boundary layers at phase-transition interfaces and fundamental issues concerning dissipative multifluid systems [5, 17]. We also need to move away from the assumption that the vortex drag, which leads to the mutual friction, is weak. Strong arguments suggest that this is not going to be the case when the protons form a type II superconductor and there are magnetic fluxtubes present in the system [18, 19, 20]. The neutron vortices may be “pinned” to the fluxtubes leading to a strong drag regime. The strong drag problem has only been considered recently [21, 22], and the first results demonstrate the presence of a new instability in systems where the two components rotate at different rates. This instability, which may be relevant for the understanding of pulsar glitches [22], provides a direct demonstration that the dynamics in the strong drag regime may be both complicated and interesting. The present investigation lays the foundation for future work in this direction by allowing for strong drag. In particular, we retain the dynamic contribution to the mutual “friction” that has previously been neglected as a matter of course.

II. THE TWO-FLUID EQUATIONS

Our discussion is based on the standard two-fluid model for neutron star cores [17, 23]. That is, we consider two dynamical degrees of freedom loosely speaking representing the superfluid neutrons (labeled n) and a charge-neutral conglomerate of protons and electrons (labeled p). Assuming that the individual species are conserved, we have the usual conservation laws for the mass densities $\rho_x$,

$$\partial_t \rho_x + \nabla_i (\rho_x v^i_x) = 0$$

where the constituent index $x$ may be either p or n. Meanwhile, the equations of momentum balance can be written

$$\left(\partial_t + v^j_x \nabla_j\right)(\rho_n v^k_n + \varepsilon_x w^k_{yx}) + \nabla_i (\mu_x + \Phi) + \varepsilon_x w^j_{yx} \nabla v^k_j = f^k_x / \rho_x$$

where the velocities are $v^i_x$, the relative velocity is defined as $w^i_{xy} = v^i_x - v^i_y$ and $\mu_x = \mu_n / m_x$ represents the chemical potential (we will assume that $m_p = m_n$ throughout this paper). $\Phi$ represents the gravitational potential, and the parameter $\varepsilon_x$ encodes the non-dissipative entrainment coupling between the fluids [17, 23]. The force on the right-hand side of (2) can be used to represent various other interactions, including dissipative terms [17].

In the following we will focus on the vortex-mediated mutual friction. Assuming that the two fluids exhibit solid body rotation we have a force of form [12] (see also [11, 13])

$$f^k_x = 2\rho_n B' \epsilon_{ijk} \Omega^j u^k_{xy} + 2\rho_n B \epsilon_{ijk} \hat{\Omega}^j \epsilon^{klm} \Omega_l u^m_{xy}$$

(3)

Here, $\Omega^j$ is the angular frequency of the neutron fluid (a hat represents a unit vector). The mutual friction parameters are intimately related to the induced friction on the vortex. The latter is often described in terms of a dimensionless “drag” parameter $R$ such that

$$B' = RB = \frac{R^2}{1 + R^2}$$

(4)

In the standard picture, the mutual friction is due to the scattering of electrons off of the array of neutron vortices. This leads to $R \ll 1$, i.e., $B' \ll B$, and hence the first term in the mutual friction force can be ignored. There are,
however, good arguments for why the problem may be in the opposite regime. In particular if one considers the interaction between the fluxtubes in a type II proton superconductor and the neutron vortices \cite{18,19,20}. Then one would expect to be in the strong drag regime where $\mathcal{R} \gg 1$, i.e., $\mathcal{B}' \approx 1$ while $\mathcal{B}$ remains small. Superfluid oscillations in this regime have not previously (with the exception of \cite{22}) been considered.

Anyway, from (3) we see that the mutual friction will not be present in a non-rotating star. This is obvious since there would then be no vortices in the first place. Of course, any non-trivial motion of the superfluid neutrons leads to vortex generation. This means that a generic perturbation of a non-rotating star will be associated with a local vorticity which could lead to mutual friction. However, in this context the resulting mutual friction interaction would require a perturbative calculation to be carried out to second order. As far as we are aware, such calculations have not yet been attempted. It may be an interesting problem for the future.

III. THE PERTURBATION EQUATIONS

A. Decoupling the degrees of freedom

If we want to consider the effects of mutual friction we need to consider rotating stars. To keep the problem tractable (at least initially) we assume that the background configuration is such that the two fluids rotate together. Perturbing the equations of motion and working in a frame rotating with $\Omega^j$ we then have

$$\partial_t (\delta v_i^x + \varepsilon_x \delta w_i^x) + \nabla_i (\delta \mu_x + \delta \Phi) + 2\varepsilon_{ijk} \Omega^j \delta v^k_x = \delta (f_i^x / \rho_x)$$

and

$$\partial_t \delta \rho_x + \nabla_j (\rho_x \delta v^j_x) = 0$$

where $\delta$ represents an Eulerian variation.

From previous work on superfluid neutron star perturbations (and indeed the large body of work on superfluid Helium) we know that the problem has two "natural" degrees of freedom, see for example \cite{6,7,8,24,25,26}. One of the degrees of freedom represents the total mass flux. Introducing

$$\rho \delta v^j = \rho_n \delta v^j_n + \rho_p \delta v^j_p$$

and combining the two Euler equations we find that

$$\partial_t \delta v_i + \nabla_i \delta \Phi + \frac{1}{\rho} \nabla_i \delta \rho - \frac{1}{\rho^2} \delta \rho \nabla_i p + 2\varepsilon_{ijk} \Omega^j \delta v^k = 0$$

where $\rho = \rho_n + \rho_p$ and the pressure is obtained from

$$\nabla_i p = \rho_n \nabla_i \mu_n + \rho_p \nabla_i \mu_p$$

In deriving this relation we have used

$$\rho_n \nabla_i \delta \mu_n + \rho_p \nabla_i \delta \mu_p = \nabla_i \delta p - \delta \rho \nabla_i \mu = \nabla_i \delta p - \frac{1}{\rho} \delta \rho \nabla_i p$$

where it has been assumed that the two fluids are in chemical equilibrium in the background. That is, we have $\mu_n = \mu_p = \mu$. We also have the usual continuity equation

$$\partial_t \delta \rho + \nabla_j (\rho \delta v^j) = 0$$

At this point we have two equations which are identical to the perturbation equations for a single fluid system. It is particularly notable that (8) does not have a force term. This follows immediately from the fact that we are only considering the mutual friction interaction. In other situations, say including shear viscosity, we would no longer have a homogeneous equation.

Of course, we are considering a two-fluid problem and there is a second degree of freedom to take into account. To describe this, it is natural to consider the difference in velocity. Thus, we introduce

$$\delta w^j = \delta v^j_p - \delta v^j_n$$
Combining the two Euler equations in the relevant way we have
\[(1 - \bar{\varepsilon}) \partial_t \delta w_i + \nabla_i \delta \beta + 2 B' \Sigma_i \delta \omega^k - \bar{B} \sum_i \delta \omega^l \epsilon^{lkm} \Omega \delta w_m = 0\]
Here we have defined
\[\delta \beta = \delta \mu_p - \delta \mu_n\]
which represents the (local) deviation from chemical equilibrium induced by the perturbations. We have also introduced the simplifying notation
\[\bar{\varepsilon} = \varepsilon_p / x_p, \quad B' = 1 - B' / x_p, \quad \bar{B} = B / x_p\]
where \(x_p = \rho_p / \rho\) is the proton fraction. Again, equation (13) does not couple the different degrees of freedom.

The coupling is entirely due to the second continuity equation. It is natural to use the proton fraction to complement the total density \(\rho\). Then we find that
\[\partial_t \delta x_p + \frac{1}{\rho} \nabla_j \left[ x_p (1 - x_p / \rho) \delta w_j \right] + \delta v^j \nabla_j x_p = 0\]
This equation shows that the two dynamical degrees of freedom are explicitly coupled unless the proton fraction is constant. This fact has already been pointed out by Prix and Rieutord [24].

Before moving on, it is worth asking to what extent it is possible to find solutions that are purely co-moving, i.e. for which \(\delta w^j = \delta \beta = 0\). From the above equations it is easy to see that such a solution would have to satisfy
\[\partial_t \delta x_p + \delta v^j \nabla_j x_p = 0\]
This condition is trivially satisfied if the proton fraction is uniform. In addition, it will be satisfied for fluid motion that has (for a spherical background configuration) no radial component and also do not lead to variations in \(\delta x_p\). Are there oscillation modes with this character? Indeed, to leading order in the slow-rotation approximation the standard r-mode satisfies these criteria. It is purely axial and the associated density perturbations appear at order \(\Omega^2\). However, in general we do not expect to find any oscillations of a “realistic” neutron star model to be purely co-moving. This means that a generic neutron star oscillation mode will be affected by mutual friction.

### B. Boundary conditions

To completely specify the perturbation problem, we need boundary conditions. At the centre of the star we simply require that all variables are regular. The surface of the star is somewhat more complex. In reality one does not expect the superfluid region to extend all the way to the surface. A real neutron star will always be covered by a single fluid envelope (e.g. the outer parts of the elastic crust). However, for simplicity we do not want to deal with the various interfaces in the present analysis [13, 27, 28]. Instead we will consider stars with a two-fluid surface, which is obviously somewhat artificial.

A reasonable approach is to assume that the perturbed star has a unique surface. That is, let the two perturbed fluids move together (in the radial direction) at the surface. In the two-fluid problem we have two distinct Lagrangian displacements \(\xi^I_s\) [29]. These follow from
\[\partial_t \xi^I_s = \delta v^I_s + \xi^I_j \nabla_j v^I_s + v^I_j \nabla_j \xi^I_s\]
We have assumed that the two fluids corotate in the background, i.e. we have \(v^I_j = v^I_s\). If we also impose that there is a common surface, then we have \(\xi^I_n = \xi^I_p\) at \(r = R\) and it follows that we should require
\[\delta w^R = \delta v^R_p - \delta v^R_n = 0, \quad \text{at } r = R\]
From (13) we see that this implies that, for a non-rotating configuration we must also have
\[\partial_t \delta \beta = 0, \quad \text{at } r = R\]
When we determine the rotational corrections to the f-mode we will impose this condition also at first slow-rotation order. This is not entirely consistent, cf. [13], but it is straightforward to relax this condition should it be required. If the two fluids move together at the surface, it also follows that
\[\delta p + \rho \xi^I_j \nabla_j \hat{\mu} = \delta p + \xi^I \partial_r \hat{p} = \Delta p = 0, \quad \text{at } r = R\]
where \(\xi^r = \xi_n^r = \xi_p^r\) at the surface. This is the usual single fluid condition of a vanishing Lagrangian pressure variation \(\Delta p\).
C. A bit of chemistry

Consider the various equations that we have written down. At this point the two degrees of freedom \([\delta v_i, \delta p]\) and \([\delta w_i, \delta \beta]\) only couple explicitly through (10). In fact, if we assume that the two fluids are incompressible, then there is no coupling at all. Since the mutual friction only enters the problem via (13), it is thus the case that any incompressible dynamics in the \([\delta v_i, \delta p]\) sector will be unaffected by mutual friction. This shows that, if we are interested in the effect of mutual friction on (say) the f-mode oscillations of a star it is not meaningful to consider an incompressible model. We know already from the outset that we would only find the mutual friction effects on the counter-moving “superfluid” modes. This may be an interesting problem, but it is not our main motivation here.

For compressible models, the two degrees of freedom also couple indirectly. Basically, we need to use the equation of state to relate \([\delta p, \delta \beta]\) to \([\delta p, \delta x_p]\). For models where the two fluids co-rotate in the background we can use

\[
\delta p = \left( \frac{\partial p}{\partial \rho} \right)_\beta \delta \rho + \left( \frac{\partial p}{\partial \beta} \right)_p \delta \beta
\]

and

\[
\delta x_p = \left( \frac{\partial x_p}{\partial \rho} \right)_\beta \delta \rho + \left( \frac{\partial x_p}{\partial \beta} \right)_p \delta \beta
\]

Using these relations, or their “inverse”, we see that the two degrees of freedom couple in a more subtle way. If we choose to reduce the problem by eliminating \(\delta \rho\) and \(\delta \beta\) then the coupling arises through the boundary conditions and the Euler equations. If, on the other hand, we eliminate \(\delta \rho\) and \(\delta x_p\) then the coupling enters through the continuity equations.

IV. DISSIPATION INTEGRALS

In order to estimate the damping associated with various dissipation mechanisms one can either (i) account for the dissipative terms in the equations of motion and solve for the damped modes directly, or (ii) solve the non-dissipative problem and use an energy integral argument to estimate the damping rate. In the typical situation when the damping is very slow the second strategy should be reliable. Indeed, all studies of damped neutron star oscillations have used this approach (see [3] for a discussion). Given this, it is natural to pause and consider the energy integral approach to the mutual friction problem.

A. The conserved energy

To work out a suitable energy associated with a given perturbation, we first multiply (5) with \(\rho \delta v_i\) (where the bar represents complex conjugation). Then we add the result to its complex conjugate. Combining the individual contributions from the neutron and proton fluids and integrating over the star we find that, when \(f^i_x = 0\), the result is a total time derivative of two terms. The first term is the “kinetic energy”, which follows from

\[
E_k = \frac{1}{2} \int \left[ (\rho_n - 2\alpha)\delta \dot{v}_n|^2 + 4\alpha \text{Re}(\delta \dot{v}_n^2 \delta \dot{v}_p^p) + (\rho_p - 2\alpha)\delta \dot{v}_p|^2 \right] dV
\]

where \(2\alpha = \rho_\text{x}^\text{c} \rho_\text{x}^\text{n}\). Alternatively, expressing this in the co- and countermoving variables, we have

\[
E_k = \frac{1}{2} \int \rho \left[ |\delta v|^2 + (1 - \varepsilon)x_p(1 - x_p)|\delta w|^2 \right] dV
\]

The “potential” energy requires a bit more work. Using the divergence theorem and the continuity equation one can show that we need

\[
\int \rho [\delta \dot{v}_n^i \nabla_i (\delta \dot{\mu}_x + \delta \Phi) + c.c] dV = \int \rho \left[ \delta \dot{v}_n^i (\delta \dot{\mu}_x + \delta \Phi) + c.c \right] e_i^x dS + \int [\delta \dot{\mu}_x + \delta \Phi] \partial_i \delta \dot{\phi}_x + c.c] dV
\]

(c.c. represents the complex conjugate). The surface term vanishes if \(\rho_x \rightarrow 0\) at the surface of the star. It also vanishes for modes that have no radial component. Adding the contributions for the neutron and proton fluids we see that we need

\[
\int \delta \Phi \partial_i \delta \rho dV = \frac{1}{4\pi G} \int \delta \Phi \partial_i \nabla^2 \delta \Phi dV = \frac{1}{4\pi G} \int \delta \Phi \partial_i \nabla_i \delta \Phi dS - \frac{1}{4\pi G} \int (\nabla_i \delta \Phi) \partial_i (\nabla^i \delta \Phi) dV
\]
Again, one can argue that the surface term vanishes.

Finally, we have terms of form

\[ \int \delta \tilde{\mu}_x \partial_t \delta \tilde{\rho}_x dV \]

It is natural to express these in terms of the perturbed densities using (this is valid for co-rotating background models only)

\[ \delta \tilde{\mu}_x = \left( \frac{\partial \tilde{\mu}_x}{\partial \rho_x} \right)_{\rho_y} \delta \rho_x + \frac{\partial \tilde{\rho}_x}{\partial \rho_y} \delta \rho_y \]  \hspace{1cm} (28)

Adding all the terms together we find that the “potential energy” follows from

\[ E_p = \frac{1}{2} \int \left[ \left( \frac{\partial \tilde{\mu}_n}{\partial \rho_n} \right)_{\rho_p} |\delta \rho_n|^2 + 2 \left( \frac{\partial \tilde{\rho}_n}{\partial \rho_p} \right)_{\rho_n} \text{Re} (\delta \rho_p \delta \tilde{\rho}_n) + \left( \frac{\partial \tilde{\mu}_p}{\partial \rho_p} \right)_{\rho_n} |\delta \rho_p|^2 - \frac{1}{4 \pi G} |\nabla \delta \Phi|^2 \right] dV \]  \hspace{1cm} (29)

or

\[ E_p = \frac{1}{2} \int \left\{ \rho \left( \frac{\partial \rho}{\partial \beta} \right)_{\beta} |\delta h|^2 + \left( \frac{\partial \rho}{\partial \beta} \right)_{\beta} \right\} dV \]  \hspace{1cm} (30)

With these definitions it follows that the total “energy” is conserved, i.e.

\[ \partial_t E = \partial_t (E_k + E_p) = 0 \]  \hspace{1cm} (31)

when \( f_i^* = 0 \). These energy expressions are equivalent to those used by Lindblom and Mendell \[14\].

B. Mutual friction

Even though we will include the mutual friction terms in the equations of motion, it is useful to work out the corresponding dissipation integrals. After all, this is the way that the mutual friction damping has traditionally been evaluated \[14, 15, 16\] and we want to be able to compare the two approaches.

First consider the \( B' \) terms. It is easy to show that these terms are not dissipative. We find that

\[ 2 \partial_t E_{B'} = 2 \int B' \epsilon_{ijk} \Omega_l \left[ \delta \tilde{\nu}^i \delta w^{jk}_n + \delta \tilde{v}^i \delta w^{jk}_p + \text{c.c.} \right] dV = 0 \]  \hspace{1cm} (32)

by symmetry. This result is not surprising. In fact, we see from \[13\] that the \( B' \) terms enter the equations of motion in the same way as the Coriolis force. Since the Coriolis terms vanish identically when we multiply each Euler equation with \( \delta v^*_i \), this should be true also for the non-dissipative part of the mutual friction.

Finally, it is straightforward to show that the dissipative terms lead to

\[ \partial_t E_B = \int \rho_n B [\delta \tilde{\nu}^i \delta w^{jk}_n + \delta \tilde{v}^i \delta w^{jk}_m + \text{c.c.}] dV = -2 \int \rho_n B \Omega_l (\delta \tilde{\nu}^i \delta \tilde{\Omega}^l_\Omega \delta \tilde{v}^i \delta w^{jk}_m + \text{c.c.}) dV \]  \hspace{1cm} (33)

Let us now ask how we can use these results to estimate the mutual friction damping timescale. Let us assume that we have a mode solution to the full dissipative problem. That is, we have a solution with time dependence \( e^{i \omega t} \) where \( \omega = \omega_r + i / \tau \) such that \( \tau \) is the damping timescale. From the fact that the energy is quadratic in the perturbations it follows that \[35\]

\[ \tau = \left| \frac{2E}{\partial_t E} \right| \]  \hspace{1cm} (34)

Moreover, since the solution satisfies the dissipative equations of motion we also know that

\[ \partial_t E = \partial_t E_B \]  \hspace{1cm} (35)

Hence, we can equally well use

\[ \tau = \left| \frac{2E}{\partial_t E_B} \right| \]  \hspace{1cm} (36)
As long as we are using the complete solution to evaluate this expression, it is an identity. However, in many cases we do not have access to the solution to the dissipative problem. (If we did, we would not need the energy integrals in the first place.) In these cases we can still estimate the damping timescale by evaluating the right-hand side of using the non-dissipative mode solution. When the damping is sufficiently slow, in the sense that the dissipative terms have a small effect on the eigenfunctions, this estimate should be reliable. Of course, one should not expect it to yield exactly the same result as the solution to the full dissipative problem.

C. Gravitational-wave emission

Finally, let us work out the multipole formulas for gravitational-wave emission from a two-fluid star. This exercise is particularly relevant if we are interested in oscillations that may be driven unstable by gravitational-wave emission \[3\]. The main motivation for including it here is that it demonstrates the intuitive result that gravitational waves are only generated by the co-moving degree of freedom.

Following \[30\] we need the mass multipoles

\[
\delta D_{lm} = \int \tau_{00} \bar{Y}_{lm} r^l dV \approx \int \delta T_{00} \bar{Y}_{lm} r^l dV
\]

and the current multipoles

\[
\delta J_{lm} = \int (-\tau_{0j}) \bar{Y}_{j,lm} r^j dV \approx \int (-\delta T_{0j}) \bar{Y}_{j,lm} dV
\]

In these expressions \(Y_{lm}\) are the standard spherical harmonics and \(Y_{Bj,lm}\) are the magnetic multipoles \[30\].

To work these out we start with the usual expression for the two-fluid stress-energy tensor in general relativity \[28\]

\[
T_{\mu\nu} = \Psi g_{\mu\nu} + n^a n^a_{\mu} n^a_{\nu} + n^p n^p_{\mu} n^p_{\nu}
\]

In the relativistic formulation, see \[31\] for a review and a survey of the literature, the central variables are the fluxes \(n^\mu_{x}\). The associated momenta follows from

\[
\mu^\nu_{x} = B^x n^\nu_{x} + A^{xy} n^y_{x}
\]

Hence, the \(A^{xy}\) coefficients encode the entrainment effect. Let us now work in the frame of an observer moving with four-velocity \(u^\mu\) such that

\[
u^\rho = [\gamma_x, \gamma_x v^\rho_x], \quad \text{with} \quad \gamma_x = (1 - v^2_x)^{-1/2}
\]

where \(v^\rho_x\) is the associated three-velocity. Then it follows that

\[
T_{00} = \Psi g_{00} + n^2_a B^a_{\gamma} \gamma^2_a + 2 n_a n_p A^{np} \gamma_n \gamma_p + n^2_p B^p_{\gamma} \gamma^2_p
\]

We want the Newtonian (low-velocity) limit of this expression. Thus we let

\[
g_{00} \rightarrow -1, \quad \text{and} \quad \gamma_x \rightarrow 1
\]

and we get

\[
T_{00} \approx -\Psi + n^2_a B^a + 2 n_a n_p A^{np} + n^2_p B^p
\]

Finally use the definition

\[
\Psi = \Lambda - n^a_{\mu} \mu^\mu_{a} - n^p_{\mu} \mu^\mu_{p} \approx \Lambda + n^2_a B^a + 2 n_a n_p A^{np} + n^2_p B^p
\]

and \(\Lambda = -\rho\) to arrive at

\[
T_{00} = \rho \rightarrow \tau_{00} \approx \delta \rho
\]

as one would have expected.

For the current multipoles we need

\[
T_{0j} = n^2_a B^a_{\gamma} \gamma^2_j + n_a n_p A^{np} \gamma_n \gamma_p (v_{j}^p + v_{j}^n) + n^2_p B^p_{\gamma} \gamma^2_j
\]
In the low-velocity limit, this leads to

\[ T_{0j} \approx n_n (n_n B^n + n_p A^{np}) v_j^n + n_p (n_p B^p v_j^p + n_n A^{np}) v_j^p \]  

Finally, we need to write this expression in terms of the Newtonian variables. This can be done by comparing the momenta,

\[ \frac{\mu_j^j}{m} \approx \frac{B^n n_n}{m} v_j^n + \frac{A^{np} n_p}{m} v_j^p = (1 - \varepsilon_n) v_j^n + \varepsilon_n v_j^p \]  

This suggests that we should identify

\[ \frac{B^n n_n}{m} = 1 - \varepsilon_n \quad \text{and} \quad \frac{A^{np} n_p}{m} = \varepsilon_n \]  

Using analogous expressions for the protons we see that

\[ T_{0j} \approx \rho_n v_j^n + \rho_p v_j^p \]  

which leads to (for a co-rotating background)

\[ \tau_{0j} \approx \rho_n \delta v_j^n + \rho_p \delta v_j^p + \delta \rho v_j = \rho \delta v_j + \delta \rho v_j \]  

These results show that it is only the co-moving degree of freedom that radiates gravitationally.

V. SLOW ROTATION PERTURBATION EQUATIONS

Let us now return to the problem of oscillating superfluid neutron stars. We will first derive the general perturbation equations for a slowly rotating superfluid star. To do this we expand all variables in spherical harmonics. Since we expect rotation to couple the various multipoles, we represent the velocity perturbations by the general expressions

\[ \delta v^j = \sum_l \left[ \frac{1}{r} W_l Y_l^m \dot{e}_r^j + \left( \frac{1}{r^2} \dot{V}_l \partial_\theta Y_l^m + \frac{m}{r^2 \sin \theta} U_l Y_l^m \right) \dot{e}_\theta^j + \frac{i}{r^2 \sin \theta} \left( m V_l Y_l^m + U_l \partial_\theta Y_l^m \right) \dot{e}_\phi^j \right] \]  

and

\[ \delta w^j = \sum_l \left[ \frac{1}{r} u_l Y_l^m \dot{e}_r^j + \left( \frac{1}{r^2} \ddot{V}_l \partial_\theta Y_l^m + \frac{m}{r^2 \sin \theta} \dot{u}_l Y_l^m \right) \dot{e}_\theta^j + \frac{i}{r^2 \sin \theta} \left( m \ddot{V}_l Y_l^m + \dot{u}_l \partial_\theta Y_l^m \right) \dot{e}_\phi^j \right] \]  

Note that we represent the “co-moving” degree of freedom by the uppercase amplitudes \([W_l, V_l, U_l]\) while the “counter-moving” degree of freedom corresponds to the lowercase quantities \([w_l, v_l, u_l]\). All scalar perturbations are expanded in spherical harmonics, i.e. we have \(\delta p = \sum_l \delta p_l Y_l^m\) et cetera. From now on the sum over \(l\) will be implied.

One can use a number of different strategies in writing down the perturbation equations. To some extent this is a matter of taste. However, in the slow-rotation problem it can be advantageous to work with a set of equations where the coupling between different multipoles is minimal. The set of equations that we use was chosen using this criterion. We also decided to use the velocity perturbations as our main variables. This approach is analogous to that used by Lindblom [31] in their analysis of inertial modes of single fluid stars. It is notably different from the two-potential formalism pioneered by Ipser and Lindblom [33], which was extended to superfluid stars by Lindblom and Mendell [13].

We replace each of the perturbed Euler equations with three equations. The first is the radial component of the vorticity equation that follows if we take the curl of Eq. (53) or (54). Assuming that the perturbations have a harmonic dependence on time, \(\exp(i \omega t)\), we get

\[ \{(l+1) \omega - 2m \Omega \} U_l Y_l^m + 2 \Omega (l+2)\{(W_l - iV_l)Q_{l+1} Y_{l+1}^m - 2 \Omega (l-1)\} W_l + (l+1)V_l \} Q_l Y^m_{l-1} = 0 \]  

and

\[ \{l(l+1) \omega (1 - \varepsilon) - 2m \Omega B^s - 2i \Omega |l(l+1) - m^2| B \} u_l Y_l^m + 2 \Omega [(l+2)B^s - imB] \{w_l - lv_l\} Q_{l+1} Y_{l+1}^m - 2 \Omega [(l-1)B^s + imB] \{(l+1)v_l + w_l\} Q_l Y^m_{l-1} = 0 \]
In deriving these equation we have made use of the standard recurrence relations
\[ \cos \theta Y_l^m = Q_{l+1} Y_{l+1}^m + Q_l Y_{l-1}^m \] (57)
and
\[ \sin \theta \partial_y Y_l^m = iQ_{l+1} Y_{l+1}^m - (l+1)Q_l Y_{l-1}^m \] (58)
where
\[ Q_l = \left[ \frac{(l-m)(l+m)}{(2l+1)(2l+3)} \right]^{1/2} \] (59)

For future reference, note that \( Q_m = 0 \) and \( Q_{m+1}^2 = 1/(2m+3) \).

Next we could use also the \( \theta \) (or \( \phi \)) components of the vorticity equation. However, as discussed in [34] there is a slightly simpler alternative. We first of all use a pair of equations analogous to the “divergence” equation in [34]. These can be written
\[ \{[(l+1)\omega - 2m\Omega]V_l - 2m\Omega W_l - il(l+1)[(1 - x_p)\delta \hat{\mu}_l^n + x_p\delta \hat{\mu}_{p1}^n]\} Y_l^m \]
\[ -2\Omega(l+2)U_lQ_{l+1}Y_{l+1}^m - 2\Omega(l-1)U_lQ_{l-1}Y_{l-1}^m = 0 \] (60)
and
\[ -\{il(l+1)\omega_l + 2m\Omega B^2l\} + [(l+1)\omega - 2m\Omega B^2]v_l \]
\[ + 2i\Omega B \{ [m^2 + (l+3)Q_{l+1}^2 + (l^2 - l - 2)Q_l^2] v_l \} Y_l^m \]
\[ -2\Omega(l+2)(lB' + imB)u_lQ_{l+1}Y_{l+1}^m - 2\Omega(l-1)[(l+1)B' - imB]u_lQ_{l-1}Y_{l-1}^m \]
\[ -2\Omega(l^2 + 3)(l^2 - l)v_lQ_{l+1}Q_{l+2}Y_{l+2}^m - 2i\Omega B(l-2)v_l + (l+1)v_lQ_{l-1}Y_{l-2}^m = 0 \] (61)

Meanwhile the radial components of the Euler equations lead to
\[ \{i[(1 - x_p)\rho \partial_r \delta \hat{\mu}_l^n + x_p\rho \partial_r \delta \hat{\mu}_{p1}^n] + 2m\Omega V_l - \omega W_l \} Y_l^m \]
\[ + 2\Omega U_lQ_{l+1}Y_{l+1}^m - 2\Omega(l+1)U_lQ_{l-1}Y_{l-1}^m = 0 \] (62)
and
\[ \{ i\rho \partial_r \delta \hat{\mu}_l - [\omega(1 - \varepsilon) - 2\Omega(1 - Q_l^2 - Q_{l+1}^2)]B v_l + 2\Omega(mB' - i(l+1)Q_l^2 - lQ_{l+1}^2)B v_l \} Y_l^m \]
\[ + 2\Omega(lB' + imB)u_lQ_{l+1}Y_{l+1}^m - 2\Omega(l+1)B' - imB]u_lQ_{l-1}Y_{l-1}^m \]
\[ + 2i\Omega(lv_l - w_l)BQ_{l+1}Q_{l+2}Y_{l+2}^m - 2i\Omega(l+1)v_l + w_lBQ_{l-1}Y_{l-2}^m = 0 \] (63)

Finally, the continuity equations become
\[ i\omega r^2 \delta \rho + \partial_r (\rho W_l) - l(l+1)\rho V_l = 0 \] (64)
and
\[ i\omega \rho r^2 \delta x_l + \partial_r [x_p(1 - x_p)\rho w_l] - x_p(1 - x_p)(l+1)\rho v_l + \rho W_l \partial_r x_p = 0 \] (65)

This completes the description of the general first order slow-rotation problem.

In order to deduce the relevant recurrence relations from the above equations we need to recall that we have been implying summation over \( l \). That is, we are considering relations of form
\[ \sum_l \{ a_l Q_{l-1} Y_{l-2}^m + b_l Q_{l+1} Y_{l+1}^m + c_l Y_l^m + d_l Q_{l+1} Y_{l+1}^m + e_l Q_{l+1} Q_{l+2} Y_{l+2}^m \} = 0 \] (66)

Using orthogonality of the spherical harmonics, i.e. multiplying by \( \bar{Y}_n^m \) and integrating over the sphere, we obtain the recurrence relation
\[ a_{n+2} Q_{n+1} Q_{n+2} + b_{n+1} Q_{n+1} + c_n + d_{n-1} Q_{n-1} + e_{n-2} Q_{n-1} Q_n = 0 \] (67)

Given this result, it is straightforward to write down recurrence relations for the various classes of oscillation modes of a rotating superfluid star. However, since the level of rotational coupling is different for different kinds of modes, it is not particularly useful to write down the general relations. Instead, we focus on two specific examples.
VI. THE F-MODES

Let us begin by considering modes that are non-trivial already in a non-rotating star. Then we first need to solve the non-rotating (and non-dissipative since the mutual friction damping requires rotation) problem. Simply setting $\Omega = 0$ in our perturbation equations we see that the polar and axial degrees of freedom decouple (as they should). It is also clear, cf. (55) and (56), that there will not exist any purely axial modes in the non-rotating case. This means that we can make the Ansatz

$$\omega = \omega_0 + \omega_1 \Omega$$

(68)

together with

$$W_l = W_l^0 + \Omega W_l^1, \quad V_l = V_l^0 + \Omega V_l^1, \quad U_l = \Omega U_l^1$$

(69)

and

$$w_l = w_l^0 + \Omega w_l^1, \quad v_l = v_l^0 + \Omega v_l^1, \quad u_l = \Omega u_l^1$$

(70)

and similarly for the various scalar perturbation quantities. For example, in the case of the proton fraction we have

$$\delta x_p = \sum_l \delta x_l Y_m^l$$

with

$$\delta x_l = \delta x_l^0 + \Omega \delta x_l^1$$

(71)

A. The non-rotating problem

At the non-rotating level the equations in Section IV provide the following relations

$$\rho_n \delta \ddot{\rho}_{n,l} + \rho_p \delta \ddot{\rho}_{p,l} = -i \omega_0 \rho V_l^0$$

(72)

$$\rho_n \delta \ddot{\beta}_{n,l} + \rho_p \delta \ddot{\beta}_{p,l} = -i \omega_0 \rho W_l^0$$

(73)

$$\delta \beta_l^0 = -i \omega_0 (1 - \bar{\varepsilon}) v_l^0$$

(74)

$$r \partial_r \delta \beta_l^0 = -i \omega_0 (1 - \bar{\varepsilon}) w_l^0$$

(75)

Meanwhile the continuity equations lead to

$$i \omega_0 r^2 \delta \rho_l^0 + \partial_r (r \rho W_l^0) - l(l+1) \rho V_l^0 = 0$$

(76)

and

$$i \omega_0 \rho r^2 \delta x_l^0 + \partial_r [x_p (1-x_p) \rho w_l^0] - x_p (1-x_p) l(l+1) \rho v_l^0 + \rho W_l^0 r \partial_r x_p = 0$$

(77)

Before we proceed, we will simplify the problem. Our aim is to determine analytic approximations for the fundamental modes of the system, including the mutual friction damping. Solving the problem numerically is, of course, straightforward but does not lead to the same level of insight into the dependence on the various parameters. To facilitate an analytic solution, we will combine an incompressible background model with compressible perturbations. This simplifies the calculations considerably. In addition, since this is the same model that was considered by Lindblom and Mendell [14] we can compare our final results directly to the available literature. We thus assume that $\rho_n$ and $\rho_p$ are both constant, while $\delta \rho_n$ and $\delta \rho_p$ are not.

It is also useful to introduce a new variable for the co-moving degree of freedom. Let us define

$$\delta h_l = \frac{1}{\rho} \delta \rho_l = \frac{1}{\rho} (\rho_n \delta \ddot{\rho}_n^l + \rho_p \delta \ddot{\rho}_p^l)$$

(78)

For a single barotropic fluid, $\delta h_l$ corresponds to the perturbed enthalpy. For a compressible background model we would have

$$\rho_n r \partial_r \delta \ddot{\rho}_n^l + \rho_p r \partial_r \delta \ddot{\beta}_p^l = \rho r \partial_r \delta h_l - \rho \delta \beta_l r \partial_r x_p$$

(79)
However, for the uniform density model the gradient of the proton fraction vanishes so we simply have
\[
\rho_r \rho \partial_r \delta \tilde{\mu}_n + \rho_p \rho \partial_r \delta \tilde{\mu}_p = \rho \rho_r \partial_r \delta h_l \tag{80}
\]
It is also worth noting that \( \delta h_l \) has the same dimension as \( \delta \beta_l \).

We now find that \(72\) and \(73\) can be written
\[
\delta h^0_l = -i \omega_0 V^0_l \tag{81}
\]
and
\[
r \partial_r \delta h^0_l = -i \omega_0 W^0_l \tag{82}
\]
Before we proceed, we need to decide what variables we want to work with. We can either remove \( \delta h_l, \delta \beta_l \) or \( \delta \rho_l, \delta x_p^l \) (or some other combination of these variables) from the problem using thermodynamic identities. Opting for the latter possibility, we use
\[
\delta \rho_l = \frac{\rho}{c_s^2} \delta h_l + \frac{\rho \alpha_1}{c_s^2} \delta \beta_l \tag{83}
\]
and
\[
\delta x_p^l = \frac{\alpha_1}{c_s^2} \delta h_l + \frac{\alpha_2 x_p}{c_s^2} \delta \beta_l \tag{84}
\]
In these relations we have defined, first of all, the speed of sound as
\[
c_s^2 = \left( \frac{\partial p}{\partial \rho} \right)_\beta = \rho \left( \frac{\partial h}{\partial \rho} \right)_\beta \tag{85}
\]
We have also introduced
\[
\alpha_1 = \frac{c_s^2}{\rho} \left( \frac{\partial \rho}{\partial \beta} \right)_h \tag{86}
\]
and
\[
\alpha_2 = \frac{c_s^2}{x_p} \left( \frac{\partial x_p}{\partial \beta} \right)_h \tag{87}
\]
and made use of the identity \(39\)
\[
\rho \left( \frac{\partial x_p}{\partial h} \right)_\beta = \rho^2 \left( \frac{\partial x_p}{\partial p} \right)_\beta = \left( \frac{\partial \rho}{\partial \beta} \right)_p \tag{88}
\]
This reduces the number of required “thermodynamic” quantities to three; \(c_s^2, \alpha_1 \) and \( \alpha_2 \).

For later convenience, it is useful to pause and consider the relative magnitude of the thermodynamic derivatives. To do this, take as an example an overall \( n = 1 \) polytrope with a proton fraction that is linear in the total density. This simple model is not completely unrealistic, and moreover it is straightforward to work out all the quantities we need. Assuming that
\[
p = K \rho^2 \tag{89}
\]
we find that
\[
\left( \frac{\partial p}{\partial \rho} \right)_\beta = 2 K \rho = c_s^2 \rightarrow \left( \frac{\partial h}{\partial \rho} \right)_\beta = \frac{c_s^2}{\rho} \tag{90}
\]
Combine this with the assumption that the proton fraction (in equilibrium) is linear in the density. That is, take
\[
x_p = \alpha \left( \frac{\rho}{\rho_c} \right) \tag{91}
\]
where $\alpha \sim 10^{-1}$ and $\rho_c$ is the central density of the star. This leads to

$$
\left( \frac{\partial \beta}{\partial \rho} \right)_h = \frac{1}{\rho} \left( \frac{\partial h}{\partial \rho} \right) = \frac{2K\rho}{x_p} = \frac{c_s^2}{x_p} \tag{92}
$$

and

$$
\left( \frac{\partial \beta}{\partial x_p} \right)_h = \frac{2K\rho}{x_p^2} = \frac{c_s^2}{x_p^2} \tag{93}
$$

These estimates suggest that $\alpha_1 \sim \alpha_2 \sim x_p$. Since we expect to have $x_p \ll 1$ it should be the case that

$$
\left( \frac{\partial \rho}{\partial h} \right)_\beta \gg \frac{1}{\rho} \left( \frac{\partial \rho}{\partial \beta} \right) \gg \left( \frac{\partial x_p}{\partial \beta} \right)_h \tag{94}
$$

This agrees with the more realistic equation of state considered by Lindblom and Mendell [14]. We will make explicit use of this ordering later.

Returning to the coupled system of equations, and combining the various relations we easily arrive at the two differential equations

$$
\partial_r (r^2 \partial_r \delta h_l^0) - l(l+1) \left[ 1 - \frac{\omega_0^2 r^2}{l(l+1)c_s^2} \right] \delta h_l^0 + \frac{\omega_0^2 \alpha_1 r^2}{c_s^2} \delta \beta_l^0 = 0 \tag{95}
$$

and

$$
\partial_r (r^2 \partial_r \delta \beta_l^0) - l(l+1) \left[ 1 - \frac{(1-\bar{\varepsilon})\omega_0^2 \alpha_1 r^2}{l(l+1)(1-x_p)c_s^2} \right] \delta \beta_l^0 + \frac{(1-\bar{\varepsilon})\omega_0^2 \alpha_1 r^2}{x_p(1-x_p)c_s^2} \delta h_l^0 = 0 \tag{96}
$$

These are the equations to be solved.

Before we proceed, it is useful to introduce dimensionless variables. First we introduce $\omega_0 = \sigma_0 \bar{\omega}$ where $\bar{\omega}^2 = GM/R^3$. Then we consider a new radial variable

$$
s = \frac{\bar{\omega} r}{c_s} \tag{97}
$$

This means that have

$$
\partial_s (s^2 \partial_s \delta h_l^0) - l(l+1) \left[ 1 - \frac{\sigma_0^2 s^2}{l(l+1)} \right] \delta h_l^0 + \alpha_1 \sigma_0^2 s^2 \delta \beta_l^0 = 0 \tag{98}
$$

together with

$$
\partial_s (s^2 \partial_s \delta \beta_l^0) - l(l+1) \left[ 1 - \frac{(1-\bar{\varepsilon})\alpha_2 \sigma_0^2 s^2}{l(l+1)} \right] \delta \beta_l^0 + \frac{(1-\bar{\varepsilon})\alpha_1 \sigma_0^2 s^2}{x_p} \delta h_l^0 = 0 \tag{99}
$$

For simplicity, we have assumed that $x_p$ is small ($\ll 1$). This should always be the case. When the equations are written in this form it becomes apparent that the coupling term in (99) is more important than that in (98). From this one can deduce that there should exist solutions to the problem such that $\delta h_l^0 \gg \delta \beta_l^0$. These are the modes that we will focus on. This is natural, if our main focus is on oscillations that radiate gravitational waves at a significant level, e.g. by being driven unstable [3].

As already mentioned, since we have assumed that the background configuration is uniform, our model is identical to the incompressible/compressible model considered by Lindblom and Mendell [14]. From their work we know that we can write down the solution to the coupled equations in closed form using (spherical) Bessel functions. This solution would contain all the modes of the system, fundamental modes and pressure modes with varying degree of co- and counter-moving character. However, this solution is not very practical for our present purposes. If we want to solve the order $\Omega$ problem explicitly, rather than estimate the mutual friction damping via the energy integral approach (as Lindblom and Mendell did), we need to be able to solve another system of equations where the leading order mode-solution acts as source. Expressed in terms of the Bessel-function solutions, the order $\Omega$ problem is very messy. Hence, we opt for a different strategy and introduce yet another simplifying approximation.
In order to proceed analytically, let us assume that $s^2 \ll 1$, in which case we can attempt to solve the problem using a power series. Is this reasonable? Well, let us again consider the case of an $n = 1$ polytrope. In that case

$$K = 2\pi G \left( \frac{R}{\pi} \right)^2$$

and it follows that

$$s^2 \leq \frac{\pi^2}{3} \left( \frac{\bar{\rho}}{\rho} \right)$$

where $\bar{\rho}$ is the average density of the star. This shows that, in our uniform parameter model, the power series Ansatz makes sense as long as we assume $\rho \gg 3\bar{\rho}$. That is, the calculation should be relevant for the conditions in a neutron star core. However, it is obviously not completely consistent. The assumptions will not hold near the surface of the star, since one tends to have $c_s^2 \to 0$ as $r \to R$. However, since the surface region is already dealt with in a rather ad hoc way this does not concern us too much.

Now that we have a small parameter in the problem, we can try to find a power series solution. It is natural to first rewrite the coupled problem as a single fourth-order equation for (say) $\delta h^0_l$. This is easily done by combining (98) and (99). Making the Ansatz

$$\delta h^0_l = s l N \sum_{n=0}^{\infty} a_n s^n$$

we find that the first few coefficients are determined by

$$a_1 = a_3 = a_5 = 0$$

and

$$\sigma_0^2(\alpha_1 - \alpha_2 x_p) a_0 - 2(1 + \alpha_2)(2l + 3) a_2 - \frac{8(2l + 3)(2l + 5)}{\sigma_0^2} a_4 = 0$$

We now insert this solution into (98). If we write

$$\delta \beta^0_l = s l \sum_{n=0}^{\infty} b_n s^n$$

then we must have

$$b_0 = -\frac{\sigma_0^2 a_0 + 2(2l + 3) a_2}{\alpha_1 \sigma_0^2}$$

and

$$b_2 = -\frac{\sigma_0^2 a_2 + 4(2l + 5) a_4}{\alpha_1 \sigma_0^2}$$

To complete the solution, we need to satisfy the boundary conditions. We want

$$\partial_s \delta \beta^0_l = 0 \quad \text{at} \quad s = \bar{\omega} R$$

Keeping the first two terms in the series for $\delta \beta^0_l$, this condition leads to another relation between the three coefficients $a_0$, $a_2$ and $a_4$. Combining this relation with (104) we arrive at an expansion for $\delta h^0_l$ where the overall scaling is given by $a_0$, and the only other unknown parameter is the frequency $\sigma_0$. To fix the frequency, we impose the remaining boundary condition. That is, we require

$$\partial_r \delta h^0_l + \left( \frac{\omega_0^2}{\partial_r \bar{\mu}} \right) \delta h^0_l = 0 \quad \text{at} \quad r = R$$
where (for a uniform background model)

$$\partial_t \tilde{\mu} \big|_{r=R} = -\frac{4\pi G \rho R}{3} = -\tilde{\omega}^2 R$$  \hspace{1cm} (110)$$

This leads to the condition

$$\partial_s \delta h^0_l - \frac{c_s}{\tilde{\omega} R} \sigma^2 \delta h^0_l = 0 \quad \text{at} \quad s = \frac{\tilde{\omega} R}{c_s}$$  \hspace{1cm} (111)$$

Some algebra now leads to a solution with frequency

$$\sigma^2 \approx l \left[ 1 - \frac{1}{2l+3} \left( \frac{\tilde{\omega} R}{c_s} \right)^2 \right]$$  \hspace{1cm} (112)$$

That is, we have

$$\omega^2_0 \approx \frac{4GM}{R^3} \left[ 1 - \frac{1}{2l+3} \left( \frac{\tilde{\omega} R}{c_s} \right)^2 \right]$$  \hspace{1cm} (113)$$

The leading order result is exactly what one would expect for an incompressible fluid ball in the Cowling approximation. The first correction to this is directly associated with the compressibility. The presence of the second fluid degree of freedom, e.g. the link to $\delta \beta$, appears at the next order of approximation. It is also worth pointing out that the solution is such that (omitting geometric factors of $l$)

$$b_0 \approx \frac{\alpha}{x_p} \left( \frac{\tilde{\omega} R}{c_s} \right)^2 a_0$$  \hspace{1cm} (114)$$

This demonstrates that the mode we have determined is such that $\delta h^0_l \gg \delta \beta^0_l$. The associated fluid motion is, indeed, predominantly co-moving. The conclusion that the co-moving f-mode is hardly at all affected by the two-fluid nature of the system accords well with the results of Lindblom and Mendell [14]

\section*{B. The slow-rotation corrections}

Having approximated the f-mode solution to the non-rotating problem, we will now work out the first order slow-rotation corrections. This will include the mutual friction damping.

The equations that need to be solved at order $\Omega$ are, first of all

$$2(l+2)(W_l^0 - lV_l^0)Q_{l+1}Y_{l+1}^m - 2(l-1)[W_l^0 + (l+1)V_l^0]Q_lY_{l-1}^m + l(l+1)\omega_0 U_l^0 Y_l^m = 0$$  \hspace{1cm} (115)$$

which determines the axial rotational correction $U_l^0$ to the f-mode. We are not going to determine this quantity here because it does not affect the mode damping, which is our main concern. A similar equation for the counter-moving degree of freedom determines the axial correction, $u_l^1$. This is also not of immediate relevance for our discussion, so we do not consider it.

From Section IV we see that the equations we actually need to solve are

$$[l(l+1)\omega_1 - 2m]V_l^0 + l(l+1)\omega_0 V_l^1 - 2mW_l^0 - il(l+1)\delta h_l^1 = 0$$  \hspace{1cm} (116)$$

and

$$i\nu \partial_t \delta h_l^1 + 2mV_l^0 - \omega_1 W_l^0 - \omega_0 W_l^1 = 0$$  \hspace{1cm} (117)$$

together with

$$- \left\{ il(l+1)\delta \beta^1_l + 2m\tilde{B}'w_0^0 - [l(l+1)\omega_1(1-\varepsilon) - 2m\tilde{B}']v_l^0 - l(l+1)\omega_0(1-\varepsilon)v_l^1 \right. \nonumber $$

$$+ 2i\tilde{B}[l(l+3)Q_{l+1}^2 + (l-2)Q_l^2]w_0^0 + 2i\tilde{B}[m + l(l+3)Q_{l+1}^2 + (l+1)(l-2)Q_l^2]v_l^0 \left. \right\} Y_l^m \nonumber $$

$$- 2i(l+3)(v_l^0 - w_l^0)\tilde{B}Q_{l+1}Q_{l+2}Y_{l+2}^m - 2i(l-2)[w_l^0 - (l+1)v_l^0]\tilde{B}Q_lQ_{l-1}Y_{l-2}^m = 0$$  \hspace{1cm} (118)$$
We want to find solutions that satisfy the boundary conditions together with the continuity equation (assuming a uniform background)

Note that there is no multipole coupling in these equations. For the other degree of freedom we have

\[ i\omega_0 r^2 \delta \rho^1 + \rho \partial_r (r W^1) - l(l + 1) \rho V_1^0 = -i\omega_1 r^2 \delta \rho^0 \]

(122)

If we want to determine the rotational correction to the \( f \)-mode, then we only need to consider the order \( \Omega \) terms

\[ l(l + 1)[i \delta h^1_1 - \omega_0 V^1_1] = [l(l + 1)\omega_1 - 2m] V^0_1 - 2m W^0_1 \]

(120)

and

\[ i r \partial_r \delta h^1_1 - \omega_0 W^1_1 = \omega_1 W^0_1 - 2m V^0_1 \]

(121)

together with the continuity equation (assuming a uniform background)

(122)

Note that there is no multipole coupling in these equations. For the other degree of freedom we have

\[ l(l + 1)[i \delta \beta^1 - \omega_0(1 - \bar{\varepsilon})v^1_1] = -2m \bar{B} w^1_0 + [l(l + 1)\omega_1(1 - \bar{\varepsilon}) - 2m \bar{B}] v^0_1 - 2i \bar{B}[1 - \bar{B}(1 - 2)Q^2_{l+1} + (l + 1)Q^2_{l+1} + (l + 1)(l - 2)Q^2_{l+1}] v^0_1 \]

(123)

and

\[ i r \partial_r \delta \beta^1_1 - \omega_0(1 - \bar{\varepsilon})w^1_1 = [\omega_1(1 - \bar{\varepsilon}) - 2i \bar{B}(1 - Q^2_l - Q^2_{l+1})] w^0_1 - 2\{m \bar{B} - i[(l + 1)Q^2_l - lQ^2_{l+1}] \bar{B} \} v^0_1 \]

(124)

together with

\[ i \omega_0 r^2 \delta x^1_1 + x_1(1 - x_p)[\partial_r (r w^1_1) - l(l + 1)v^1_1] = -i\omega_1 r^2 \delta x^0_1 \]

(125)

We want to find solutions that satisfy the boundary conditions

\[ i \omega_0 \delta h^1_1 + \frac{1}{r} W^1_1 \partial_r \mu = -i \omega_1 \delta h^0_1, \quad \text{at } r = R \]

(126)

and

\[ i \omega_0 \partial_r \delta \beta^1_1 = -i \omega_1 \partial_r \delta \beta^0_1 = 0, \quad \text{at } r = R \]

(127)

After some manipulations (making use of the leading order relations) we arrive at the two coupled equations

\[ \partial_r (r^2 \partial_r \delta h^1_1) - l(l + 1) \left[ 1 - \frac{\omega_0^2 r^2}{l(l + 1)c_s^2} \right] \delta h^1_1 + \frac{\omega_0^2 \alpha_1 r^2}{c_s^2} \delta \beta^1_1 = -\frac{2\omega_1 r^2}{c_s^2} \left[ \delta h^0_1 + \alpha_1 \delta \beta^0_1 \right] \]

(128)

and

\[ \partial_r (r^2 \partial_r \delta \beta^1_1) - l(l + 1) \left[ 1 - \frac{(1 - \bar{\varepsilon})\omega_0^2 \alpha_1 r^2}{l(l + 1)(1 - x_p)c_s^2} \right] \delta \beta^1_1 + \frac{\omega_0^2 \alpha_1 r^2}{x_p(1 - x_p)c_s^2} \delta h^1_1 = -\frac{2\omega_1 \alpha_1 r^2}{x_p(1 - x_p)c_s^2} \left[ (1 - \bar{\varepsilon})\omega_1 - i(1 + (2l - 1)Q^2_l - Q^2_{l+1}) \right] \delta h^0_1 \]

\[ + \frac{2i \bar{B}}{(1 - \bar{\varepsilon})\omega_0} \left[ 1 + (2l - 1)Q^2_l - (2l + 3)Q^2_{l+1} \right] r \partial_r \delta \beta^0_1 \]

\[ + \frac{2i \bar{B}}{(1 - \bar{\varepsilon})\omega_0} \left[ m^2 - l(l + 1) + (l + 1)(2l - 1)Q^2_l + l(2l + 3)Q^2_{l+1} \right] \delta \beta^0_1 \]

\[ - \frac{2\alpha_2 \omega_1 r^2}{(1 - x_p)c_s^2} \left[ (1 - \bar{\varepsilon})\omega_1 - i(1 - Q^2_l - Q^2_{l+1}) \bar{B} \right] \delta \beta^0_1 \]

(129)
The problem has the anticipated form, a coupled system of equations for $\delta h_l^1$ and $\delta \beta_l^1$ which differs from the non-rotating problem only by the presence of leading order source terms. To proceed, we will follow the same strategy as in the leading order calculation. However, it is beneficial to first note that the source term in the second equation simplifies considerably if we focus on the $l = m f$-modes. Since $Q_{m+1}^2 = 1/(2m + 3)$ we see that the factors in front of the first two $\delta \beta_l^0$ pieces in the right-hand side of (129) are then identically zero. Moreover, we know from the leading order calculation that $\delta \beta_l^0 \ll \delta h_l^0$, cf. (114). As long as we are only interested in the leading order rotational correction and the leading order mode damping, this allows us to neglect also the remaining $\delta \beta_l^0$ part of the source. Using $\omega_0/c_s = \sigma_0 s$ as well as $\sigma_1 = \omega_1/\omega_0$ and assuming that $x_p \ll 1$ we then arrive at the simplified equations

$$\partial_s (s^2 \partial_s \delta h_l^1) - l(l + 1) \left[ 1 - \frac{\sigma_0^2 s^2}{l(l + 1)} \right] \delta h_l^1 + \sigma_0^2 s^2 \delta \beta_l^1 = -2\sigma_0^2 \sigma_1 s^2 \delta h_l^0$$

(130)

and

$$\partial_s (s^2 \partial_s \delta \beta_l^1) - l(l + 1) \left[ 1 - \frac{(1 - \bar{\epsilon})\sigma_0^2 \alpha_2 s^2}{l(l + 1)} \right] \delta \beta_l^1 + \frac{\sigma_0^2 s^2 (1 - \bar{\epsilon}) \sigma_1}{x_p} \delta h_l^1 = -\frac{2\sigma_0^2 s^2 \sigma_1}{x_p} \delta h_l^0$$

(131)

where

$$D = \sigma_0^2 \left( (1 - \bar{\epsilon}) \sigma_1 - \frac{2i(l + 1)\bar{\beta}}{\sigma_0 \omega(2l + 3)} \right)$$

(132)

Combining the two equations, making the same Ansatz as in the non-rotating case

$$\delta h_l^1 = s^l \sum_{n=0}^{N} a_n s^n$$

(133)

and taking the source term to be $\delta h_l^0 = C s^l$ [41], we find that

$$a_1 = a_3 = a_5 = 0$$

(134)

Meanwhile, we have

$$\frac{\sigma_0^2 (\alpha_1^2 - \alpha_2 x_p)}{x_p} a_0 - 2(1 + \alpha_2)(2l + 3)a_2 - \frac{8(2l + 3)(2l + 5)}{\sigma_0^2} a_4 + 2 \left( \frac{\alpha_1^2 D}{x_p} - \sigma_0 \sigma_1 \alpha_2 \right) C = 0$$

(135)

The difference now is that we are only interested in the particular solution due to the presence of the source term. In order to remove the unwanted homogeneous solution we set $a_0 = 0$. Then (135) becomes a relation between the known mode amplitude $C$ and the two coefficients $a_2$ and $a_4$. As in the non-rotating problem, we get a second such relation from the boundary condition for $\delta \beta_l^1$. From (130) and

$$\delta \beta_l^1 = s^l \sum_{n=0}^{N} b_n s^n$$

(136)

we get

$$b_0 = -\frac{2}{\alpha_1} \left[ \frac{(2l + 3)}{\sigma_0^2} a_2 + \sigma_1 C \right]$$

(137)

and

$$b_2 = -\frac{1}{\alpha_1} \left[ a_2 + \frac{4(2l + 5)a_4}{\sigma_0^2} \right]$$

(138)

This solution has to satisfy the surface condition

$$\partial_s \delta \beta_l^1 = 0 \quad \text{at} \quad s = \frac{\bar{\omega} R}{c_s}$$

(139)
This leads to another relation between $C$, $\sigma_2$ and $a_4$. This means that we can write down the solution for $\delta h^1_l$ with an overall scale $C$ (as expected for the particular solution) and with the frequency correction $\sigma_1$ as the only remaining undetermined quantity.

The final condition to be satisfied can be written (again, for $l = m$)

$$\partial_s \delta h^1_l - \frac{c_s}{\omega R} \sigma_0^2 \delta h^1_l = \frac{c_s}{\omega R} \left[ \sigma_1 \left( \sigma_0^2 + l \right) - \frac{2l}{\omega \omega_0} \right] C s^l \quad \text{at} \quad s = \frac{\omega R}{c_s} \quad (140)$$

Inserting the power series solution, one can show that this condition leads to the leading order rotation correction of the f-mode frequency being

$$\Re \sigma_1 \approx \frac{1}{\omega^{1/2}} \quad (141)$$

Meanwhile the leading damping term (the imaginary part of $\sigma_1$) is

$$\Im \sigma_1 \approx \frac{2(l + 1)(3l + 5)}{\sqrt{l(2l + 3)^3(2l + 5)}} \frac{\alpha^2}{x_p} \left( \frac{\omega R}{c_s} \right)^2 \mathcal{B} = \frac{2(l + 1)(3l + 5)}{(2l + 3)^3(2l + 5)} \frac{1}{\rho^2_p} \left( \frac{\partial \rho}{\partial \beta} \right)_h^2 \left( \frac{GM}{R} \right)^2 \mathcal{B} \quad (142)$$

These are the final results of the f-mode analysis. After retracing our steps to recall the various definitions, we find that the mutual friction damping follows from

$$\Im \omega_1 \approx \frac{2(l + 1)(3l + 5)}{(2l + 3)^3(2l + 5)} \frac{\alpha^2}{x_p} \left( \frac{GM}{R c_s^2} \right)^2 \mathcal{B} = \frac{2(l + 1)(3l + 5)}{(2l + 3)^3(2l + 5)} \frac{1}{\rho^2_p} \left( \frac{\partial \rho}{\partial \beta} \right)_h^2 \left( \frac{GM}{R} \right)^2 \mathcal{B} \quad (143)$$

That is, the damping timescale is

$$\tau = \frac{1}{\Im \omega_1 \Omega} = \left( \frac{2(l + 3)^3(2l + 5)}{2(l + 1)(3l + 5)} \right) \left[ \frac{1}{\rho^2_p} \left( \frac{\partial \rho}{\partial \beta} \right)_h^2 \left( \frac{GM}{R} \right)^2 \mathcal{B} \right]^{-1} \frac{1}{\mathcal{B} \Omega} \quad (144)$$

This result completes our analysis of the (co-moving) f-mode in a superfluid neutron star.

It is obviously relevant to compare the estimated f-mode damping timescale to previous work. To do this, we first need to recall that all previous work has focused on the case where electron scattering off the vortex array is the main cause of mutual friction. Then we have [12, 13]

$$\mathcal{B} \approx 4 \times 10^{-4} \left( \frac{m^*_{\rho}}{m_p} \right)^2 \left( \frac{m_p}{m^*_{\rho}} \right)^{1/2} \left( \frac{x_p}{0.05} \right)^{7/6} \left( \frac{\rho}{10^{14} \text{g/cm}^3} \right)^{1/6} \quad (145)$$

where we have used the relation between the entrainment and the effective proton mass:

$$\varepsilon_p = 1 - \frac{m^*_{\rho}}{m_p} \quad (146)$$

Taking $m^*_{\rho}/m_p = 0.3$ we have $\mathcal{B} \approx 5.5 \times 10^{-4}$ in good agreement with the result used by Lindblom and Mendell in their investigation of the f-mode problem [14]. The overall scaling with $l$ in [14] also appears to be similar to their result. This is evident from the results in Table 1 which compares our results to data from Table 1 in [14]. This comparison shows that, in the $m^*_{\rho}/m_p = 0.3$ case, our damping times are about a factor of 2 longer than those estimated by Lindblom and Mendell. The results also differ in the predicted dependence on the entrainment. In our calculation, the entrainment only enters [14] indirectly through its effect on $\mathcal{B}$. The data given by Lindblom and Mendell hints at a different behaviour. In the $m^*_{\rho}/m_p = 0.8$ case we find that the difference between our damping result and Table 1 in [14] is closer to a factor of 3.

Most likely the main difference originates from the use of the energy integral approach in one case and the direct determination of dissipative mode solutions in the other. In order to check this assumption, we have estimated the mutual friction damping using the energy integral approach (together with our leading order f-mode solution). That is, we evaluate [30] for the non-rotating f-mode solution. Then we find that

$$\tau = \frac{2(l + 3)^3(2l + 5)}{6(2l^2 + 6l + 5)} \left[ \frac{1}{\rho^2_p} \left( \frac{\partial \rho}{\partial \beta} \right)_h^2 \left( \frac{GM}{R} \right)^2 \right]^{-1} \frac{1}{\mathcal{B} \Omega} \quad (147)$$
This scaling with $l$ here differs somewhat from that in (144). This introduces a numerical factor of $\approx 2$ for small values of $l$. This factor brings our results very close to those of Lindblom and Mendell, cf. Table I. The main difference between the energy integral result and the full dissipative mode calculation is a geometric factor of order unity. This is what one would expect. At the end of the day the astrophysical implications of the results are the same.

Before we move on, it is worth making a comment on the apparent lack of entrainment scaling. The damping timescale in (144) does not depend directly on the entrainment parameter $\varepsilon$ due to a series of cancellations that occur when we impose the order $\Omega$ boundary condition $\nabla_i \delta \beta = 0$. If we were to impose the (slightly more realistic) condition $\delta w^i = 0$ at the surface (corresponding to a common surface), these cancellations would not occur and there would be an explicit dependence on $\varepsilon$ in the damping timescale. This does not, however, affect the numerical results significantly.

### C. The f-mode instability window

To conclude the discussion of the superfluid f-mode, it is worth considering the impact of the results on the gravitational-wave driven instability of this mode. The close agreement between our mutual friction damping rates and the results of Lindblom and Mendell (14) obviously means that their key conclusions stand. That is, the instability of the f-mode is likely to be completely suppressed in a superfluid neutron star. However, we think that this result has sometimes been misunderstood. The result does not show that the secular f-mode instability cannot play a role for astrophysical neutron stars. To show this, we have combined the different timescales for gravitational-wave growth of an unstable mode with the damping due to shear- and bulk viscosity from Ipser and Lindblom (32). The results, for the $l = m = 4$ f-mode that leads to the strongest instability in a Newtonian star, are shown in Figure 1. The data in the figure corresponds to an $n = 1$ polytrope with mass $1.5M_{\odot}$ and $12.533$ km (the average density is $3.6 \times 10^{14}$ g/cm$^3$). In order to connect this with our mutual friction approximation, we have used the model parameters from (14), i.e.,

\[ \rho = 4 \times 10^{14} \text{g/cm}^3 \]
\[ x_p = 0.06 \]
\[ \left( \frac{\partial \rho}{\partial \beta} \right)_p = 1.911 \times 10^{-7} \text{gs}^2/\text{cm}^5 \]

Combined with the canonical value for $B$ given in (145) these parameter values lead to the mutual friction damping completely overwhelming any gravitational-wave driving of the f-mode. Table I provides similar results for the model considered in (14), a star with radius $15$ km and average density $4 \times 10^{14}$ g/cm$^3$ (which means that the mass is quite large, $2.84M_{\odot}$). We have scaled the frequencies and the timescales using $\Omega_0 = \sqrt{\pi G \bar{\rho}}$ where $\bar{\rho}$ is the average density.

However, this result is only relevant below the critical temperature at which the stars core become superfluid. Suppose that we take the critical temperature to be $5 \times 10^9$ K, which is a typical value (10). Then the f-mode instability window remains unaltered in hotter stars. This is evident from Figure 1. Of course, as soon as a sizeable part of the core is superfluid, the instability is no longer present. However, it seems that there is still scope for the unstable f-modes to play a role in the evolution of a nascent neutron star born spinning near the breakup velocity. One

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TABLE I: Estimated mutual friction damping timescales for the (co-moving) f-mode of a superfluid neutron star. We compare our results to previous work by Lindblom and Mendell (14). The parameters are those discussed in the main text, and correspond to $m_p^* / m_p = 0.3$. The stellar model has radius $15$ km and average density $4 \times 10^{14}$ g/cm$^3$. Note that, in our case the non-rotating f-mode frequency is given by $\omega_0^2 = 4l \Omega_0^2 / 3$, where $\Omega_0 = \sqrt{\pi G \bar{\rho}}$ and $\bar{\rho}$ is the average density. The difference between the listed frequencies ($15\%$ for $l = 2$ and decreasing with increasing $l$) should mainly be due to our use of the Cowling approximation. The dissipative mode result (144) for the mutual friction timescales (which scales as $\Omega_0/\Omega$) differs by about a factor of $2$ from the tabulated Lindblom-Mendell results. By comparing to (147) we learn that this is mainly due to different geometric factors in the energy integral approach and the direct dissipative mode calculation.

| $l$ | $\omega(0)/\Omega_0$ | $\tau \Omega_0$ | $\omega(0)/\Omega_0$ from (144) | $\tau \Omega_0$ from (144) |
|-----|------------------|-----------------|-------------------------------|-----------------|
| 2   | 1.407            | 1.071 \times 10^3 | 1.63                         | 9.2 \times 10^3  |
| 3   | 1.809            | 1.638 \times 10^4 | 2                            | 1.5 \times 10^4  |
| 4   | 2.141            | 2.347 \times 10^4 | 2.31                         | 2.1 \times 10^4  |
| 5   | 2.430            | 3.195 \times 10^4 | 2.58                         | 2.9 \times 10^4  |
| 6   | 2.689            | 4.182 \times 10^4 | 2.83                         | 3.8 \times 10^4  |
FIG. 1: The f-mode instability window for the $l = m = 4$ f-mode. The data for gravitational radiation reaction, shear- and bulk viscosity are taken from [35]. The instability is active above a critical rotation rate (thick solid line) at each given temperature. The mutual friction, which acts only below the superfluid transition temperature (here taken to be $5 \times 10^9$ K, lower temperatures are indicated by the grey region in the figure) is estimated using (144) and the parameter values given in the main text. To illustrate the role of a weak mutual friction we show (as thin solid lines) the instability curves for $B$ in the range $10^{-10} - 10^{-7}$, more than three orders of magnitude weaker than the canonical value $10^{-5}$. The solid rotation rate of the star is given as a fraction of the breakup rate $\Omega_K \approx 0.639 \Omega_0$.

should also remember that the instability is stronger in a relativistic model. In fact, in this case the $l = m = 2$ mode may also become unstable. Based on the available evidence it would be premature to rule out the f-mode instability for realistic neutron star models. The problem requires further attention.

In Figure 1 we also show the effect of a weaker mutual friction. Suppose that (145) is, for some reason, not the typical value. Our understanding of neutron star core physics is not complete, so it is interesting to consider a range of possibilities. The different thin solid curves in the figure show the effect of mutual friction for the given values of $B$. For the considered $1.5M_\odot$ model the results correspond to (for $l = 4$)

$$\tau \Omega_0 \approx 2.5 \times 10^6 \left( \Omega / \Omega_0 \right)^{-1} \left( B / 10^{-4} \right)^{-1}$$

(148)

From these results we learn that the mutual friction must be at least three orders of magnitude weaker than the canonical value in order for the f-mode to be unstable below the superfluid transition temperature.

Finally, it is worth noting the following. The coefficient $B'$ was not present in our final f-mode equations. This is due to a series of, perhaps surprising, cancelations. The upshot of this is that our results are also valid in the strong drag regime. Expressed in terms of the drag parameter $R$, the data in Figure 1 shows that the mutual friction suppresses the f-mode instability in a superfluid neutron star with $10^{-7} < R < 10^1$. This conclusion is interesting since the strong drag regime has not been considered before. It also shows that the suppression of the f-mode takes place for much of the plausible parameter range.

VII. THE R-MODES

Having explored the f-mode in a superfluid star, revisiting the issue of the mutual friction damping, we will now consider the Coriolis driven r-modes in a slowly rotating star. The r-modes are interesting because they also suffer a gravitational-wave driven instability [3]. In contrast to the f-modes, which only become unstable at fast rotation rates, the r-mode instability may set in already at quite modest spins.
In a single fluid star, the r-modes are purely axial to leading order. Moreover, their frequency is linear in the rotation rate. Hence, it is natural to make the Ansatz

$$\omega = \omega_0 \Omega$$ (149)

together with

$$W_l = \Omega W_l^1, \quad V_l = \Omega V_l^1, \quad U_l = U_l^0$$ (150)

and

$$w_l = \Omega w_l^1, \quad v_l = \Omega v_l^1, \quad u_l = \Omega u_l^0$$ (151)

Note that, if we want to work out the order $\Omega$ corrections to the mode (with the above ordering) we will first need to account for the centrifugal force and the change in shape of the star. We will discuss this problem elsewhere [36]. Here we will focus on the problem at linear slow-rotation order.

From the general slow-rotation equations in Section 4, we immediately see that the r-mode assumption decouples the two degrees of freedom. First of all, the average vorticity equation leads to

$$[l(l+1)+2m]U_0^m Y_l^m = 0$$ (152)

This shows that we must have a single multipole solution, with frequency

$$\omega_0 = \frac{2m}{l(l+1)}$$ (153)

To determine the associated eigenfunction we consider the divergence equation and the radial Euler equation. These lead to the recurrence relations

$$-in(n+1)\delta h_n^1 - 2(n-1)(n+1)Q_n U_{n-1}^0 - 2[(n+1)^2 - 1]Q_{n+1} U_{n+1}^0 = 0$$ (154)

and

$$ir\partial_r \delta h_n^1 + 2(n-1)Q_n U_{n-1}^0 - 2(n + 2)Q_{n+1} U_{n+1}^0 = 0$$ (155)

For simplicity, we have assumed that the background is uniform (as in the f-mode analysis in the previous section). These equations show that the only way to have a single multipole axial solution is to have $U_m^0 \neq 0$. This follows since $Q_m = 0$. In other words, we will have non-trivial modes only for $l = m$. Inserting $n = m - 1$ in the two equations we have

$$-i(m + 1)\delta h_{m+1}^1 - 2mQ_{m+1} U_m^0 = 0$$ (156)

and

$$ir\partial_r \delta h_{m+1}^1 + 2mQ_{m+1} U_m^0 = 0$$ (157)

These combine to

$$r\partial_r U_m^0 - (m + 1)U_m^0 = 0$$ (158)

and the familiar solution

$$U_m^0 = Ar^{m+1}$$ (159)

This analysis shows that, to leading order, the standard r-mode remains unchanged in a superfluid star. We also see that we need to go to higher orders in rotation if we want to determine the mutual friction damping of these modes. Such calculations have been carried out by Lindblom and Mendell [15] and Lee and Yoshida [16]. Motivated by the recent evidence that the strong drag regime may be relevant, we are currently revisiting this problem [36].

Now consider the counter-moving degree of freedom. In that case, the difference vorticity equation leads to

$$\left\{l(l+1)(1-\bar{\varepsilon})\omega_0 - 2m\bar{B'} - 2[l(l+1) - m^2]\bar{B} \right\} u_l^0 Y_l^m = 0$$ (160)
That is, there should exist a single multipole solution with frequency
\[
\omega_0 = \frac{1}{1 - \bar{\varepsilon}} \left\{ \frac{2m}{l(l+1)} \bar{\mathcal{B}}' + \frac{2i}{l(l+1)} [(l+1) - m^2] \bar{\mathcal{B}} \right\}
\]  
(161)

As in the co-moving problem, the associated eigenfunctions follow from the divergence equation and the radial Euler equations. These lead to the recurrence relations;
\[
- i n(n+1) \delta \beta_n^1 - 2(n+1) [(n+1) \bar{\mathcal{B}}' + i m \bar{\mathcal{B}}] Q_n u_n^{0} - 2[(n+2) \bar{\mathcal{B}}' - im \bar{\mathcal{B}}] Q_{n+1} u_{n+1}^{0} = 0
\]  
(162)
and
\[
ir \partial_r \delta \beta_n^1 + 2[(n+1) \bar{\mathcal{B}}' + i m \bar{\mathcal{B}}] Q_n u_n^{0} - 2[(n+2) \bar{\mathcal{B}}' - im \bar{\mathcal{B}}] Q_{n+1} u_{n+1}^{0} = 0
\]  
(163)

Again, it is easy to see that the only way to have a single multipole solution is to have \( u_m^{0} \neq 0 \), i.e. we must have \( l = m \). This leads to
\[
- i (m+1) \delta \beta_{m+1}^1 - 2m [\bar{\mathcal{B}}' + i \bar{\mathcal{B}}] Q_{m+1} u_{m}^{0} = 0
\]  
(164)
and
\[
ir \partial_r \delta \beta_{m+1}^1 + 2m [\bar{\mathcal{B}}' + i \bar{\mathcal{B}}] Q_{m+1} u_{m}^{0} = 0
\]  
(165)
That is, we have
\[
r \partial_r u_m^{0} - (m+1) u_m^{0} = 0
\]  
(166)
which means that the counter-moving solution also takes the form
\[
u_m^{0} = Br^{m+1}
\]  
(167)

What do we learn from this exercise? First of all, we should recognize that we have been somewhat cavalier in the discussion. Since we have assumed that \( \bar{\varepsilon}, \bar{\mathcal{B}}' \) and \( \bar{\mathcal{B}} \) are all constant, the analysis leading to (161) is clearly only valid for uniform background models. This tells us that the purely axial counter-moving solution only exists for this simplified model. In a more general case, this mode will become an axial-led inertial mode. In the weak drag regime, these inertial modes have been determined numerically by Lee and Yoshida [16].

The counter-moving r-modes are nevertheless interesting. Two particular features are worth noting. Let us first consider the mode pattern speed
\[
\sigma_p = - \frac{\text{Re} \omega}{m}
\]  
(168)
In the case of the normal r-mode we see from (153) that the pattern speed is always negative. That is, these modes are retrograde with respect to the star’s rotation. This is the criterion that renders the mode unstable to gravitational-wave emission at all rotation rates (in an otherwise non-dissipative star). Meanwhile, from (161) we find that
\[
\sigma_p = - \frac{1}{1 - \bar{\varepsilon}} \frac{2m}{(m+1)} \bar{\mathcal{B}}' = - \frac{1}{1 - \bar{\varepsilon}/x_p} \frac{2m}{(m+1)} \left[ 1 - \frac{\bar{\mathcal{B}}'}{x_p} \right]
\]  
(169)
This relation shows that the second class of superfluid r-modes may, in fact, be prograde. For this to be the case we must have (assuming that \( 1 - \bar{\varepsilon} > 0 \), see below)
\[
\bar{\mathcal{B}}' > x_p
\]  
(170)
which may well happen. Recall that the mutual friction parameter is related to the induced friction on the vortex. From (4) we see that we need to have
\[
R^2 > \frac{1}{1 - x_p} > 1
\]  
(171)
in order for the mode to be prograde. Clearly, systems in the strong drag regime (where \( R \to \infty \)) will satisfy this condition. Alternatively, we may require
\[
\bar{\varepsilon} = \frac{\bar{\varepsilon}}{x_p} > 1
\]  
(172)
Recalling that $\varepsilon_n = (\rho_p/\rho_n)\varepsilon_p$ and using the relation between the entrainment and the effective proton mass $\rho_p$, we find that $\varepsilon_p \approx 0.1$ and $m^*_p \approx 0.5m_p$, but the possibility is nevertheless interesting. In particular since the mode will actually be unstable (due to the presence of the mutual friction) if the condition is met. Since $\bar{B} > 0$ it is clear from (161) that the imaginary part of the mode frequency is negative if (172) is satisfied. The existence of this unstable regime is interesting, at least conceptually.

As a final check let us compare the damping timescale calculated in (161) with that calculated using the integral approach, cf. (36). Using the definitions for the energy, (24), and the dissipation due to mutual friction, (33), one readily finds for the counter-moving r-mode solution:

$$\partial_t E_B = -2\rho(1-x_p)\bar{B}\Omega \int_0^R [l(l+1) - m^2]|u_l^0|^2r^4dr$$

Meanwhile

$$E = \frac{1}{2}\rho x_p(1-x_p)(1-\bar{\varepsilon}) \int_0^R l(l+1)|u_l^0|^2r^4dr$$

These lead to

$$\text{Im} \, \omega = \frac{1}{\tau} = \frac{2\Omega\bar{B}}{(1-\bar{\varepsilon})} \left[\frac{l(l+1) - m^2}{l(l+1)}\right]$$

which agrees perfectly with the damping timescale extracted from (161). Since we are using the full dissipative mode-solution in the energy integrals, this is as expected.

**VIII. CONCLUDING REMARKS**

The aim of this paper was to lay the foundation for a renewed assault on the problem of dissipative superfluid neutron star oscillations. We have discussed the oscillations of slowly rotating superfluid stars, taking into account the mutual friction force at linear order in the (presumed) slow rotation of the star. We have considered both the fundamental f-modes and the inertial r-modes. Our analysis differs from previous studies in that we do not assume weak mutual friction from the outset, the final results are also valid in the strong drag regime.

In the case of the f-modes, we worked out an analytic approximation for the mode which allowed us to write down a closed expression for the mutual friction damping timescale. This result, which is in good agreement with previous numerical results of Lindblom and Mendell [14], provides useful insight into the dependence on, and relevance of, various equation of state parameters. The scaling with the harmonic index $l$ is also obvious from our final formula. The analysis is readily extended to stars with superfluid cores that do not extend all the way to the surface (as assumed in our analysis), although the result is then less transparent.

In the case of the r-modes, we have confirmed the existence of two classes of modes. However, we demonstrated that only one of these sets will remain purely axial in more realistic situations. This agrees with previous results of Lee and Yoshida [16]. We discussed some peculiarities of the counter-moving r-modes. In particular, the fact that they may be unstable for some parameter values. Even though we do not expect this instability to be relevant for realistic superfluid stars, its existence is of conceptual interest.

Building on the formalism and the results presented in this paper, we are currently carrying out a detailed study of the mutual friction damping of the r-modes at second order in the slow-rotation approximation [36]. At the same time we are considering neutron stars with exotic hyperon and/or quark cores. Since the multifluid aspects of those problems have never been considered in detail, these are exciting developments. They are, in fact, necessary if we want to understand the dynamics of realistic models of mature neutron stars.
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[37] Caution: In the general case when the background fluids are not co-rotating, there will be additional contributions associated with the entrainment in this relation.
[38] Note that there will be entrainment contributions here in the general case when the two background fluids are not co-rotating.
[39] Ultimately, this relation follows from the fact that the partial derivatives with respect to the number densities commute, so the mixed second derivatives of the energy functional (the “equation of state”) must be equal.
[40] Note that we are neglecting the entrainment in these equations. This is, however, not important. By including the relevant entrainment factors one can show that they do not affect the f-mode result to the order of approximation at which our solution is valid.
[41] Since we are only interested in the leading corrections it is sufficient to use the leading term in the non-rotating solution as source.