On Locality of Quantum Information in the Heisenberg Picture for Arbitrary States

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Abstract

The locality issue of quantum mechanics is a key issue to a proper understanding of quantum physics and beyond. What has been commonly emphasized as quantum nonlocality has received an inspiring examination through the notion of the Heisenberg picture of quantum information. Deutsch and Hayden established a local description of quantum information in a setting of quantum information flow in a system of qubits. With the introduction of a slightly modified version of what we call the Deutsch-Hayden matrix values of observables, together with our recently introduced parallel notion of the noncommutative values from a more fundamental perspective, we clarify all the locality issues based on such values as quantum information carried by local observables in any given arbitrary state of a generic composite system. Quantum information as the ‘quantum’ values of observables gives a transparent conceptual picture of all that. Spatial locality for a projective measurement is also discussed. The pressing question is if and how such information for an entangled system can be retrieved through local processes which can only be addressed with new experimental thinking.

Keywords : Quantum Information; Quantum Locality; Deutsch-Hayden Descriptors; Noncommutative Values of Observables

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I. INTRODUCTION

The question of quantum locality, or the lack of it, has been one of great concern for physicists working on fundamental theories since the Bohr-Einstein dialogue. With the modern appreciation of the great potential in the applications of quantum information science, a full understanding of the question becomes also a practical task. In 2000, Deutsch and Hayden, looking at quantum information flow for systems of qubits, established ‘the locality of quantum information from the Heisenberg picture’ \cite{1}. The local Heisenberg picture description, \textit{i.e.} description of the subsystems is, of course, a complete one. That “a complete description of a composite system can always be deduced from complete descriptions of its subsystems” \cite{1}. Otherwise, reduced density matrices certainly give a local description of the subsystems at the expense of the loss of some information about the full system.

The important result of the locality of quantum information \textit{in} the Heisenberg picture has not been as well-appreciated as it should be in the over twenty years after the publishing of Ref.\cite{1}, and “Today, most researchers in quantum foundations are still convinced not only that a local description of quantum systems has not yet been provided, but that it cannot exist.” \cite{2}. Many simply believe the lack of locality in quantum mechanics has been firmly established. However, the standard understanding of the (non)locality issue is based heavily on keeping the Newtonian space-time picture. Our recent work on an alternative picture of quantum mechanics as a theory of particle dynamics on a noncommutative geometric (phase-)space with the position and momentum observables as coordinates \cite{3,4}, we believe, provide a different conceptual perspective on the locality issue. The perspective is exactly one in the Heisenberg picture, \textit{i.e.} seeing the theory as, carrying information, about the values of the observables. Here, the values are ‘quantum’ in nature, like a kind of noncommutative number. As such observable has a definite state-dependent (noncommutative) value that is dictated by the theory of quantum mechanics, as explicitly presented in Ref.\cite{5}. Looking at the notion of ‘the locality of quantum information \textit{from} the Heisenberg picture’ as presented in Refs.\cite{1,2,6} from the point of view of a fundamental theorist, we also see a few conceptual difficulties causing confusion. Clarifying the issues is clearly very important, and our fundamental notion the values of physical quantities \cite{7} can only be better described in terms of mathematical systems, numbers, with a noncommutative rather than commutative product is exactly in line with the notion of quantum information.
presented in the paper by Deutsch and Hayden [1]. It is about the proper way to look at quantum information, at least from the theoretical point of view. We say a piece of quantum information is a (noncommutative) ‘value’ of a quantum observable. We make an effort to clarify all the issues completely here, based on the statement.

It is important to note that a local description of the full quantum information is one thing, being able to retrieve all that information locally is another. We do not have, at this point, an idea about how to directly obtain a noncommutative value experimentally. Nor has anyone exactly demonstrated that the corresponding full information, of a noncommutative value or its equivalence, for a local observable in an entangled system can indeed be retrieved locally. What is important though is that there is, apparently, no good reason to see that as impossible. As a theorist, we are focusing here first only on the question of how the perspective allows us to see a different side of the story and to see quantum information in a different light which is at least theoretically valid. The experimental challenge in relation is wide open. The usual projective measurements certainly have hardly anything to do with retrieving the noncommutative values. The current study is certainly not about criticizing the latter and all the important results we have learned from Bell and others concerning that. It is more complementary to all that, providing alternative thinking about the nature of quantum information which has important implications about its handling theoretically, and very probably also experimentally and technologically. The actual understanding of projective measurements as a physical process is mostly beyond the scope of our analysis. We make no commitment here on the latter subject matter and believe our results have no necessary conflict with any particular theory of the kind. We also address somewhat the question of spatial locality at the end, which is usually phrased in terms of projective measurement. We illustrate explicitly that what has been talked about as a ‘local’ projective measurement when it is seen as a quantum dynamical, *i.e.* unitary, process is certainly not a local one, hence the noncommutative values of the local observables for the other parts of the system would generally change. The same applies to its generalization as any ‘local’ POVM, as state projections are still involved.

Let us first clarify a background perspective. A physical theory describes the dynamical behavior of a definite system, say a particle, or two. The dynamical theory of quantum mechanics as such is about the kinematic set-up and the equation of motion, as given in the Schrödinger or the physically equivalent Heisenberg, picture. On the practical side, many
physicists may see the corresponding theory of measurement, as given in the projection postulate, as an integral part of the theory. However, how we can describe the dynamical behavior of a physical system is one thing, how we may practically obtain information about it is another. It has already been appreciated that projective measurement, as described by the projection postulate, is not the only possible way to extract information from a quantum system. Besides, unlike the unitary quantum evolution, the projection postulate in itself is not a dynamical theory, though there are available in the literature various proposals of such a dynamical theory behind, within the framework of quantum dynamics or otherwise. There are also physicists seeing projective measurement not as a physical process. In this article, we focus only on the dynamical theory of quantum (unitary) evolution, mostly independent of the projective measurement. We are interested firstly in the best way one can describe the quantum information, for an observable at a fixed state, as given in the mathematical dynamical quantum theory, especially the notion of locality in relation. Some of the questions on the practical side, in relation to projective measurements or otherwise, will be addressed at the end. Note that the direct results of projective measurements are really pieces of classical information. The complete quantum information the dynamical theory of quantum mechanics offers is really beyond even the full statistical distribution of results of the corresponding projective measurement on an ensemble of the same states. We want to emphasize though that a single projective measurement does not provide any useful information about the quantum system, only a statistic of such a measurement does. The latter as information for the quantum system is certainly beyond the notion of a single commutative, real number, quantity. Even checking the correlation of results of projective measurements of different parts of an entangled system has to rely on the kind of statistics. Those statistics are predicted by the quantum theory and independent of, any theory on, how the actual projection in each projective measurement happens. However, those statistics cannot be used to provide complete information about the system.

The Deutsch-Hayden descriptors as complete quantum information about a composite system of qubits are given as time-evolved observables, basic observables in terms of which all other single-qubit observables can be expressed in terms of. The idea that “the ‘values’ are matrices” is an idea of fundamental importance, though the phrase and its implications may easily be missed by readers. It shows up only once without much elaboration. The usual notion that the value of an observable has to be a real number is no more than a
model assumption that enjoys great success in classical physics. When it is unquestionably
adopted in quantum physics, we have to give up the idea of observables having definite
values. More importantly, we cannot recover the value of an observable as a product of
two observables from the product of values. That makes such values useless for checking
the validity of algebraic relations among observables experimentally. The descriptors as
matrix values can overcome the problem. It is not just about the basic observables in a
descriptor. The approach to quantum information should be seen as assigning a matrix
value to every observable when a state is taken/given, as the original authors kind of hinted
at. For any state, the matrix values of the observables preserve among themselves all the
algebraic relations among the observables as variables. The version of Deutsch-Hayden
matrix value we present below makes that plainly explicit. Independently, we have argued,
from a different line of work, that the latter properties are what a proper model of values
for observables should have. Unaware of the Deutsch-Hayden descriptors, we sought a
workable mathematical formulation of such a notion of noncommutative values of observables
and got the results first presented in Ref. [5]. The Deutsch-Hayden matrix value is essentially
a different mathematical form of the notion.

The Heisenberg picture focuses on using observables to describe (the evolution of) a
quantum system. The particular dynamical evolution as quantum information flows in
systems of qubits is of great practical importance. However, we expect our physical theories
to give a clear picture of the complete information, as values of the basic observables at
any fixed instance of time, which can always be identified as time $t = 0$ independent of
any particular evolution dependent on a physical Hamiltonian. Quantum information in a
system is first about information stored in the system at a fixed instance of time, potentially
available for future manipulations. Moreover, observables are dynamic variables and can
have fixed values only when the state is specified. And we would want to be able to describe
simultaneously such values for any number of states. Of course, we want to be able to do
that for arbitrary states of any quantum system. Naively, the descriptor seems to have
difficulty with all the issues highlighted here. Conceptually, as we will discuss in some detail
in the next section, the observation that the Heisenberg picture gives a local description is an
ingenious one. From the perspective of noncommutative values of observables, presented in
any form, that is indeed not difficult to appreciate for a generic composite quantum system.

Bédard looked at “the cost” of such a complete local description and got the answer that
“the descriptor of a single qubit has larger dimensionality than the Schrödinger state of the whole network—or of the Universe!” [6]. That dimensionality is certainly a lot larger. While Bédard’s analyses certainly help to clarify a few things about the descriptors, and some essentially equivalent notions of the local quantum information [8], it is easy to appreciate that many physicists may feel uncomfortable with the latter conclusion. We will clarify all that with a more direct and economic version of the Deutsch-Hayden matrix values as values of observables. The parallel locality property of our noncommutative values will also be presented.

After the more general and conceptual discussions in the next section, we present explicit illustrations of the Deutsch-Hayden matrix value of an observable in a language emphasizing its validity for a general quantum system independent of any notion of time evolution in section III. Of course, demonstrating its locality is a key point. The parallel illustrations will then be performed for our noncommutative value in section IV as inspired by the work of Deutsch and Hayden. The Deutsch-Hayden approach may be particularly well-adapted to qubit systems, especially in a practical setting of quantum information flow. Our noncommutative value would likely be more applicable to a general system. As the locality of the latter is completely transparent, the illustrations also aim at helping to make the notion of locality of quantum information in the Heisenberg picture easier to appreciate by a general reader. Some concluding remarks are put in the last section, including a discussion about the relevant issues of projective measurement with an explicit illustration in the case of two entangled qubits. The analysis is somewhat complementary to that given in Ref.[9]. Examples of explicit results for the Deutsch-Hayden matrix values and our noncommutative values of the local basic observables for the two-qubit system and that of our noncommutative values for a two-particle system in the Schrödinger wavefunction representation are given in the appendices.

Some readers may want to see a summary of the relevant background from the work of Deutsch and Hayden. We see our analysis presented as readable and understandable independent of that. Moreover, there are differences between the scope of our analysis and that of the original paper of Deutsch and Hayden [1], and we do not agree with some of the opinions presented there. Having stated that we summarize the relevant background from the paper here. Their focus was on a matrix triple of time-dependent local observables for each qubit under a (Heisenberg picture) time evolution of the quantum computation network.
That is, in our notation here, \((\sigma_1(t) \otimes I_r, \sigma_2(t) \otimes I_r, \sigma_3(t) \otimes I_r)\), where \(I_r\) is an identity matrix of the Hilbert space of the rest of the system, i.e. all the other qubits. The basis for the full Hilbert space and each of its subspaces for the individual qubits has been chosen such that all \(\sigma_i(0)\) are exactly the standard Pauli matrices with the state fixed as \(|0 \ldots 0\rangle\). Of course, any observable for the system at any later time, including the nonlocal ones, can be expressed in terms of all the \(\sigma_i(t)\) for the qubits. The key idea is that at time \(t\) these matrices can be seen as the values of the corresponding observables. The state description being formally fixed cannot carry any information for the system. All information for each such observable can be seen as encoded in the matrix representing it. The matrix itself hence provides a notion of value for the observable. As the value of all observables can be obtained from those of the basic triples, the authors called descriptors, the latter contains complete information about the system. Each descriptor is only about local observables, hence the locality of information about the quantum system obtained from the Heisenberg picture of the time evolution.

II. LOCALITY OF QUANTUM INFORMATION IN THE HEISENBERG PICTURE

The Heisenberg picture provides a local description of a quantum system. That is actually easy to appreciate, especially when one draws the parallel of the classical case from the right perspective that focuses on the similarity between the two theories. The quantum theory is not as different from the classical one as one may naively think. The Schrödinger picture is about the description of the state, which is well-appreciated to be nonlocal. The state of a composite system of parts \(A\) and \(B\) may be taken as a vector in the product Hilbert space \(\mathcal{H}_A \otimes \mathcal{H}_B\). While a set of basis vectors for the product space can be obtained as tensor products among a basis vector of \(\mathcal{H}_A\) and one of \(\mathcal{H}_B\), a generic vector may not be decomposed into a simple product of a vector of \(\mathcal{H}_A\) and one of \(\mathcal{H}_B\). Hence, one does not, in general, have a notion of the exact state for \(A\) or \(B\) for the composite, beyond the partial description in terms of reduced density matrices which do not carry the full information. Such a feature of entanglement does not exist in any classical theory. Yet, when the state is given, values for all observables are always determined. For the quantum case, the statement is still correct when the ‘value’ or ‘values’ of an observable is taken as the full statistical distribution of
eigenvalue outcomes for projective measurements of the same state. The Heisenberg picture is, of course, about the observables. A state is a point in the phase space, the description of which depends on a chosen system of coordinates as a frame of reference. In the classical case, those coordinates, the position and momentum variables for a general particle, for example, are the basic observables. Though the usual, real or complex number, coordinates of the quantum phase space are not taken as (quantum) observables, the latter can still be considered as generic Hamiltonian functions of such coordinates [10], as in the classical case. An operator $\beta$ on the Hilbert space, as an observable, corresponds to the generic Hamiltonian function

$$f_\beta(z_n, \bar{z}_n) = \frac{\langle \phi | \beta | \phi \rangle}{\langle \phi | \phi \rangle},$$  

with $z_n$ as complex coordinates of normalized state vectors $|\phi\rangle = \sum_n z_n |n\rangle$, $\{|n\rangle\}$ an orthonormal basis. The Hilbert space, or the projective Hilbert space are Kähler manifolds, with matching symplectic structures that give essentially the same Hamiltonian flows [4]. The set of $z_n$ has an ambiguity of a common factor as an overall complex phase for the vector $|\phi\rangle$, otherwise uniquely determine a point in the projective Hilbert space or a fixed projection operator as a pure state density matrix. The set $z_n$ also serves as its homogeneous coordinates. The noncommutative product among observables can be obtained as [5, 10]

$$f_{\beta\gamma} = f_\beta \star f_\gamma = f_\beta f_\gamma + \sum_n V_{\beta n} V_{\gamma n},$$

where $V_{\beta n} = \frac{\partial}{\partial z_n} f_\beta$ and $V_{\gamma n} = \frac{\partial}{\partial \bar{z}_n} f_\gamma$.

Exactly like the classical case, there is a set of basic observables all other observables can be expressed in terms of. For a generic particle, the set can be given by the position and momentum observables. We can even think about them as noncommutative, operator, coordinates of the phase space [3, 4]. For a qubit system, the basic observables may be conveniently taken as $\sigma_3$ and $\sigma_1$, or $\sigma_3$ and a linear combination of $\sigma_1$ and $\sigma_2$. For a composite system $AB$, an observable for $A$ is more than an operator $\beta_A$ on $\mathcal{H}_A$. It is an operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ in the form $\beta_A \otimes I_B$ and as such one can say that it knows about the presence of $B$. There are nonlocal observables of $AB$ that cannot be written as a simple tensor product of a $\beta_A$ and a $\gamma_B$. The latter feature is shared by the classical theory. Though the product for the observables for the system $AB$, as operators $\beta_{AB}$ or as generic Hamiltonian functions $f_{\beta_{AB}}$, is not a commutative one, local observables of $A$ or of $B$ are well defined as $\beta_A \otimes I_B$ and $I_A \otimes \beta_B$. 

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and as such, they form two independent sets. The full set of local basic observables for either part put together gives a complete set of generators for the observable algebra of $AB$ that includes the nonlocal observables. That can be called the locality of observables and is what is behind the locality of quantum information in the Heisenberg picture, as information contained in the values of the local basic observables. The values here, of course, have to be representations of the complete, quantum, information, contained. In particular, each descriptor is essentially two matrices each can be taken as a value for one of the two local basic observables for a qubit [6]. We call the value a Deutsch-Hayden matrix value, which can be defined for all observables in any system. Both the Deutsch-Hayden matrix values and our noncommutative values for the full set of local basic observables of a system give a complete local description of quantum information in the system, though the Deutsch-Hayden line of analyses is typically restricted to a system of qubits. We will show, in the analysis below, the Deutsch-Hayden matrix values for the local basic observables, however, contain redundant information tied to the picture of the latter as time-evolved observables.

The key property of a good mathematical model of the values for observables is that the evaluation map, given by any fixed state, which assigns each observable its value has to be a homomorphism from the observable algebra to the (noncommutative) algebra of the values of the observables. That is to say, any algebra relation among observables has to be preserved by their values [5]. Then, the value of any observable can be retrieved from the values of the local basic observables. That is actually beyond “the distributions of any measurement performed on the whole system”. Being able to predict that has been taken by many as the criteria for completeness [6].

Exactly as in the classical theory, an observable as a generic Hamiltonian function is a variable. The only information it carries is the algebraic relation between itself and the basic observables, which defines its explicit functional form. The variables cannot have fixed values until the state is fixed. Yet, even for the quantum case, once the state is fixed, the noncommutative values of all observables are fixed. However, one does not have enough information to fix the descriptors. Again, the latter contained extra information which is irrelevant to the notion of the quantum information in the system at a particular instance of time. The extra information is tied to how the system evolved from the Heisenberg reference state in the past. Hence, it is really local quantum information from the Heisenberg picture, instead of local quantum information in the Heisenberg picture. We will illustrate how the
Deutsch-Hayden descriptor, or our noncommutative values for the local basic observables, incorporated all information about the state into the values of the observables. Our noncommutative values do that directly, while the descriptor does that through a particular given time evolution.

One may call our locality description above a weak locality. There is a stronger requirement for a fully local description. “Descriptions of dynamically isolated – but possibly entangled – systems $A$ and $B$ are local if that of $A$ is unaffected by any process system $B$ may undergo, and vice versa.” [6]. Such a process is represented by a unitary transformation in the form $I_A \otimes U_B$. In the Heisenberg picture description of the process, the state is kept unchanged while observables are modified by the transformation. A $I_A \otimes U_B$, however, commutes with any $\beta_A \otimes I_B$ as a local observable of $A$ and is naturally unaffected by any process system $B$ may undergo. Explicitly,

$$(I_A \otimes U_B^\dagger)(\beta_A \otimes I_B)(I_A \otimes U_B) = \beta_A \otimes I_B \, .$$

(3)

Then, any properly defined values for the local observables should respect the same relation. We will present the explicit Deutsch-Hayden matrix value and our noncommutative values in the next section, illustrating their satisfying the strong locality requirement too.

III. EXPLICIT ILLUSTRATIONS ON THE LOCAL DEUTSCH-HAYDEN MATRIX VALUES

The Deutsch-Hayden matrix value for an observable $\beta$ is given by

$$\beta(t) = U^\dagger(t)\beta U(t) \, ,$$

where $U(t)$ is supposed to be a fixed unitary matrix giving a particular time evolution of the system, i.e. it is a fixed matrix of complex numbers without any parameter dependence. The corresponding Schrödinger picture of the particular time evolution gives $|\phi(t)\rangle = U(t)|\phi(0)\rangle$. We are interested in $\beta$ as one of the local basic observables such as $\tilde{\beta}_A \equiv \beta_A \otimes I_B$ of a composite system $AB$. Note that $U(t)$ as a matrix on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is in general not decomposable into a tensor product, hence neither is any matrix $\tilde{\beta}_A(t)$. It is easy to see that any $\tilde{\beta}_A(t)$ is local even in the strong sense.

When a process of the subsystem $B$ acts further on the system, the Schrödinger state would change to $\tilde{V}_B U(t)|\phi(0)\rangle$ where $\tilde{V}_B \equiv I_A \otimes V_B$ with $V_B$ being the unitary matrix on
\( \mathcal{H}_b \) which represents the action of the process. Switching back to the Heisenberg picture, we have \( \tilde{\beta}_A(t) \) taken to \( U^\dagger(t) \tilde{V}_b \tilde{\beta}_A \tilde{V}_b U(t) \) hence obviously unchanged. Note that the matrix \( \tilde{V}_b \) representing the action of the process of the subsystem \( B \) has the representation of \( \tilde{V}_b(t) = U^\dagger(t) \tilde{V}_b U(t) \) in the Heisenberg picture for the time evolution of the \( U(t) \) action. And, of course

\[
\tilde{V}_b(t) \tilde{\beta}_A(t) \tilde{V}_b(t) = U^\dagger(t) \tilde{V}_b(t) \tilde{\beta}_A(t) \tilde{V}_b(t) U(t) = U^\dagger(t) \tilde{V}_b \tilde{\beta}_A \tilde{V}_b U(t) = \tilde{\beta}_A(t).
\]

Up to here, we only presented what has been given in Ref.\[6\] in a more general language, mostly for completeness. Next, we illustrate how a \( \beta(t) \) naively as a time-evolved observable serves as a value for the observable. As said, a value of an observable must have information about the state incorporated into it. It is the state that gives observables their values. Given a fixed basis, the Hermitian matrix \( \beta \) gives the representation of the observable as a physical quantity. It is, of course, state-independent. It is a variable that carries no information about the system. For the \( \beta(t) \) then, the information can only be in the \( U(t) \) matrix. Actually, in the name of information flows as presented in the Heisenberg picture, the initial state \( \langle \phi(0) \rangle \) is generally taken as a fixed Heisenberg reference state, say the first basis vector here denoted as \( \vert 0 \rangle \). Then all information in the state \( \langle \phi(t) \rangle \) the \( \beta(t) \) represents the value of is completely inside the \( U(t) \) matrix. Instead of seeing \( \vert \phi(t) \rangle \) as the time-evolved state, one can take it simply as the particular fixed state the system is in at that instance of time. Any \( U(t) \) simply as a fixed matrix satisfying the equation \( \langle \phi(t) \rangle = U(t) \langle 0 \rangle \) then can be used to put the information in the state into \( \beta(t) \) as a matrix value of the observable \( \beta \), at the time instance \( t \). In that way, one frees \( U(t) \) and hence \( \beta(t) \) from any time evolution from the past and has direct matrix values for all observables. More explicitly, we can simply say that with the \( \vert \phi \rangle \), we pick a fixed unitary matrix \( U_\phi \) satisfying \( \langle \phi \rangle = U_\phi \langle 0 \rangle \) and use it to map each observable \( \beta \) to its matrix value \( [\beta]_\phi^{\mathcal{D}_H} = U_\phi^\dagger \beta U_\phi \). The new notation emphasizes its nature as a value determined by the state \( \langle \phi \rangle \), up to the somewhat arbitrary but fixed choice of \( U_\phi \) among solutions to the equation, without any nontrivial dependence on time. For any chosen \( U_\phi \), the map is an algebraic homomorphism. In particular \( [\beta\gamma]_\phi^{\mathcal{D}_H} = [\beta]_\phi^{\mathcal{D}_H}[\gamma]_\phi^{\mathcal{D}_H} \) as simply the matrix product.

From the above, it is easy to see the redundancy in the description. The equation \( \langle \phi \rangle = U_\phi \langle 0 \rangle \) fixes only the first column of the matrix \( U_\phi \). The latter is exactly the vector \( \langle \phi \rangle \).

Any \( U_\phi' = U_\phi U_{-1} \) where \( U_{-1} \) is any unitary matrix in the subspace of the full Hilbert space
complementary to the one-dimensional subspace spanned by the vector \(|0\rangle\) serves the purpose equally well. One may keep \(|\phi\rangle\), and hence also \(U_\phi\), for a generic state parametrized by as many independent real variables as the state admits. Say, one can that as the first column of the \(U_\phi\) matrix the complex coordinates \(z_n\) of \(|\phi\rangle = \sum_n z_n |n\rangle\) over the full orthonormal basis, and the rest of the columns as a fixed choice of state vectors that, together with \(|\phi\rangle\), make up a complete orthonormal set. That can, of course, be done with all the vectors expressed in terms of \(z_n\). It can then be easily seen that \(U_\phi\), or the matrix value \([\beta]^{DH}_{\phi}\), does not involve more parameters, or more degrees of freedom, than the state vector \(|\phi\rangle\).

If one counts also the degrees of freedom allowed in the choice of \(U_{-1}\), one would give it a much larger number of ‘dimensionality’ \([6]\). Of course, in an explicit setting of quantum information flow where one is interested in \(U(t)\) as the result of some practical processes, the story is different. In that case \(U(t)\) is given by those processes and contains information about them. It is always possible, and in our opinion desirable, to separate the description of physical processes from the notion of the values of observables for a system at an instance of time and have a description of the latter independent of the history of the system.

To restrict to the particular line of analyses as presented in Refs.\([1, 6]\), the full system is considered to be one of many qubits and we can regard \(A\) as a particular qubit and \(B\) a composite of the rest of the qubits. A descriptor for qubit \(A\) may then be taken, under the new notation introduced, as \([[\tilde{\sigma}_1]^{DH}_{\phi}, [\tilde{\sigma}_3]^{DH}_{\phi}]\), where \(\sigma_{1,\phi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(\sigma_{3,\phi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) taken as the basic observables of qubit \(A\). An even more explicit illustration of what we discussed above in the example of a two-qubit system, namely having \(B\) as another qubit is presented in Appendix A.

IV. EXPLICIT ILLUSTRATIONS ON THE LOCAL NONCOMMUTATIVE VALUES

Like a Deutsch-Hayden matrix value, a noncommutative value of an observable is an element of a noncommutative algebra as the state-dependent homomorphic image of the observable algebra. The local basic observables form a set of generators for the observable algebra. In accordance, their values form a set of generators for the algebra of the values of observables for the fixed state, or the system at the instance. A particular representation
of elements in such an algebra of values can be given in terms of a set of real numbers with the set for a product observable obtainable from a noncommutative product. For our noncommutative value, the formalism is based on the Hamiltonian function of Eq. (1) and the product given in Eq. (2). Given the matrix elements \( \langle m|\beta|n \rangle \), over the basis, for a Hermitian \( \beta \), the noncommutative value of \( \beta \) for the state \( |\phi\rangle \) can be taken as \( [\beta]_\phi = \{ f_\beta |\phi, V_\beta |n \} \), i.e. the real value of \( f_\beta \) and all the complex values of \( V_\beta |n \), one for each basis vector \( |n\rangle \), all evaluated at the state \( [5, 11] \). We have

\[
V_{\beta n} = \frac{\partial f_\beta}{\partial z_n} = -f_\beta \bar{z}_n + \sum_m \bar{z}_m \langle m|\beta|n \rangle .
\] (4)

The product is then given by \( [\beta\gamma]_\phi = [\beta]_\phi \ast_n [\gamma]_\phi \) with

\[
f_{\beta\gamma} = f_\beta f_\gamma + \sum_n V_{\beta n} V_{\gamma n} ,
V_{\beta\gamma n} = -f_{\beta\gamma} \bar{z}_n + \sum_m \bar{z}_m \langle m|\beta\gamma|n \rangle = -f_{\beta\gamma} \bar{z}_n + \sum_{m,l} \bar{z}_m \langle m|\beta|l \rangle \langle l|\gamma|n \rangle ,
\] (5)

where \( V_{\gamma n} = \partial_n f_\gamma = \nabla_n \) is just the complex conjugate of \( V_{\gamma n} \). The \( V_{\beta n} \) has only the ambiguity of an overall phase factor, from the coordinate expressions of the state. As such, it is a common factor for all \( V_{\beta n} \) of all \( \beta \) in the observable algebra. All \( V_{\beta n} |\phi \) have the value zero for \( |\phi\rangle \) being an eigenstate of the observable \( \beta \). In that particular case, the noncommutative value becomes essentially a commutative value, i.e. its product with another noncommutative value is commutative, namely equals to multiplying the latter for an overall factor of the \( f_\beta \) value, which is exactly the eigenvalue in the case. We have also written down an expression for noncommutative value with the state given as a Schrödinger wavefunction explicitly. As the value of a wavefunction \( \phi(x_i) \) at a point of fixed values for the three \( x_i \) is really a coordinate for the phase space, like a \( z_n \), one has in the case the sequence of \( V_{\beta n} \) simply replaced by the functional derivative \( \delta_{\phi f_\beta} \).

Locality of the noncommutative value of local observables in a composite system is obvious when one checks. As the noncommutative value is based on the expectation value function \( f_\beta \), the value for a local observable \( [\tilde{\beta}_A]_\phi \) clearly does not change under a process on subsystem \( B \) of the composite system \( AB \), given by a unitary matrix \( I_A \otimes U_B \). In the Heisenberg picture description of the process, the state is untouched, and the commutativity of the observable \( \tilde{\beta}_A \equiv \beta_A \otimes I_B \) and \( I_A \otimes U_B \) says it is not changed by the transformation. The expectation value function \( f_{\tilde{\beta}_A} \) is invariant. And again, the map \( \beta \to [\beta]_\phi \) as an algebraic
homomorphism guarantees that the values of all observables, local or otherwise, are obtained from those of the set of all local basic observables. Note that a naive Schrödinger picture thinking about the situation would seem to give a different conclusion. As the state is transformed, the values of the $z_n$ coordinates change, and so do the $V_{\beta_n}$ values. However, that is really a change in the representation of the noncommutative values rather than changes in the values themselves. The feature has been analyzed under quantum reference frame transformations in Ref. [11]. An explicit representation of the noncommutative values, just like that of the Deutsch-Hayden matrix value, is simply dependent on the choice of basis for the Hilbert space. One needs always to be sure to check changes in such values under the same representation. After all, invariance of the expectation value function, instead of only its value at a single point, i.e. for a particular state, implies the invariance of all $V_{\beta_n}$ as its coordinate derivatives.

Taking subsystem $A$ as a qubit, we have the very simple results for the values of its local basic observables $[\tilde{\sigma}_{1A}]_\phi$ and $[\tilde{\sigma}_{3A}]_\phi$. We have

\begin{align*}
    f_{\tilde{\sigma}_{1A}} &= \sum_n (\bar{z}_{on}z_{1n} + \bar{z}_{1n}z_{on}) , \\
    V_{\tilde{\sigma}_{1A}}^{on} &= \bar{z}_{1n} - f_{\tilde{\sigma}_{1A}}z_{on} , \quad V_{\tilde{\sigma}_{1A}}^{1n} = \bar{z}_{on} - f_{\tilde{\sigma}_{1A}}\bar{z}_{1n} , \\
    f_{\tilde{\sigma}_{3A}} &= \sum_n (|z_{on}|^2 - |z_{1n}|^2) , \\
    V_{\tilde{\sigma}_{3A}}^{on} &= \bar{z}_{on}(1 - f_{\tilde{\sigma}_{3A}}) , \quad V_{\tilde{\sigma}_{3A}}^{1n} = -\bar{z}_{1n}(1 + f_{\tilde{\sigma}_{3A}}) ,
\end{align*}

(6)

where we have used $|\phi\rangle = \sum_n (z_{on}|0n\rangle + z_{1n}|1n\rangle)$ and put the coordinate indices for the derivatives of $f_{\tilde{\sigma}_{1A}}$ and $f_{\tilde{\sigma}_{3A}}$ as superscripts for convenience. Again, full explicit results are given for the examples of an entangled state in Appendix A.

V. CONCLUDING REMARKS

Unlike the notion of a local state for a subsystem in a composite system, local observables are always well-defined for a quantum system, with or without entanglement. Heisenberg picture is firstly about describing things with observables. To have a definite description, we need a state-specific notion of its value, Deutsch and Hayden introduced a notion of matrix values for the observables, though originally cast in the language of time-evolved observables. We provide here a more direct description of them simply as values for the
observables at a fixed instance of time. From the perspective of fundamental theories, we have recently introduced a more direct notion of noncommutative values for the observables. And, naturally, local observables for a subsystem are not affected by any process other subsystems may be subjected to. The same holds for their values, as our detailed analyses clearly show. The long ‘established’ emphasis of the quantum theory being nonlocal and the persistence of sticking to the idea that a value has to be a real number make it difficult to realize the kind of local description. Despite that, Deutsch and Hayden, in the study of quantum information flow, saw the light. After all, the information contained in or carried by a quantum system is quantum information and the information is not quantum if each piece of it can be described by a real number. The Deutsch-Hayden matrix values or our form of the noncommutative values presented are each simply a description of the full quantum information contained in an observable as the mathematical model of a physical quantity.

From the introduction of our notion of noncommutative values, we have emphasized that they are elements of a noncommutative algebra, serving as a homomorphic image of the observable algebra under an evaluation map given by the state. That is the proper picture of what the values of observables should be in general. In the classical case, the observable algebra is a commutative one, and so is any of its algebra of values, giving the latter as essentially a subalgebra of the algebra of real numbers. Early in his study of quantum mechanics, Dirac talked about classical observables as c-number quantities and quantum observables as q-number ones. The observables themselves are c-number, namely real number, and q-number variables, respectively. By definition, c-number variables take c-number values. In exact analog, q-number variables should take q-number values. What has been missing is exactly the q-numbers or a kind of noncommutative numbers. Deutsch and Hayden gave the first description of that, though not explicitly for all the observables, and without addressing the full algebraic properties of the set of such q-numbers. The noncommutative algebra of their matrix values for a fixed state is, when stripped of the irrelevant parts, simply a different mathematical presentation of the same algebra of q-numbers as our noncommutative values. In a definite practical setting of quantum information flow, however, the original Deutsch-Hayden formulation as time-evolved observables carrying extra information about the past time evolution may be of other interest in applications. That extra information is an independent part, which is about how those q-number values have evolved dynamically.

All observables, local or nonlocal, can be expressed in terms of a number of local basic ob-
servables. For a generic particle without spin, they are given by the position and momentum observables. The q-number values for the observables, as the homomorphic image, of course, have all algebraic relations among the observables preserved among them. Hence, the local quantum information in the full set of q-number values for the local basic observables is complete. The q-number values of all observables can be retrieved from them.

The popular version of the story of quantum nonlocality is described with a projective measurement of an entangled state of a two-qubit system $AB$. For example, we take the state $|\psi\rangle = e^{-i\frac{\pi}{2}} \sqrt{\frac{1+r}{2}} |00\rangle + e^{i\frac{\pi}{2}} \sqrt{\frac{1-r}{2}} |11\rangle$ used in the analysis presented in Appendix A, which has an entanglement given by $\sqrt{1-r^2}$ for $0 \leq r < 1$. A projective measurement on $\hat{\sigma}_3^B$ yielding an eigenvalue of 1 would have the state collapsed to $|00\rangle$. Certainly, no local observable for the qubit $A$ can have a sensible identical value for the states $|\psi\rangle$ and $|00\rangle$. However, one needs to look at the situation more carefully. Some physicists do not consider the ‘wavefunction collapse’ resulting from the measurement as a physical process within the scope of the description of quantum mechanics itself. Otherwise, we believe the decoherence theory [12, 13] provides quite a successful description of that, at least in principle. The description involves, actually unavoidable, couplings of the system to the measuring apparatus and the environment. One would expect the latter to couple to both $A$ and $B$. The results accessible are only given in terms of reduced density matrices. Hewitt-Horsman and Vedral, in their interesting effort to develop the “Deutsch-Hayden approach” [9] have addressed the projective measurement. We present a somewhat complementary analysis here for further clarification of the, arguably, locality issue receiving the most attention. Hewitt-Horsman and Vedral consider a CNOT gate as a model for the “measurement-type interactions” on a qubit. More exactly, the qubit to be measured is taken as the control and the target qubit serves as the ‘apparatus/pointer’. Starting from an initial pointer state $|0\rangle$ coupling to a qubit to be measured, say the $B$ qubit of our system $AB$ in the above state, the CNOT gate turns the state for, here, the system $ABC$ into one with a perfect correlation between qubits $B$ and $C$. Note however that a fixed pointer output of either $|0\rangle$ and $|1\rangle$, the so-called wavefunction collapse still has not happened. Explicitly, we have

$$|\psi\rangle_{AB} \otimes |0\rangle_C \rightarrow e^{-i\frac{\pi}{2}} \sqrt{\frac{1+r}{2}} |000\rangle_{ABC} + e^{i\frac{\pi}{2}} \sqrt{\frac{1-r}{2}} |111\rangle_{ABC} .$$

The decoherence theory further invokes coupling to the many more degrees of freedom of the ‘environment’, the exact initial state of which cannot be practically determined. The
system then evolves into a macro state that has the subsystem $ABC$ ‘collapsed’ to either $|000\rangle$ and $|111\rangle$, or more exactly the reduced density matrix of $|000\rangle\langle 000|$ or $|111\rangle\langle 111|$. The undetermined exact microstate generally does not stabilize as the macrostate. The macrostate outcome is supposed to be quantum mechanically determined, depending on the practically inaccessible initial (micro)state of the ‘environment’ in line with the Born probability expectations. Even the exact microstate of the complete system at any instance is supposed to be quantum mechanically determined. Short of going through the exact description of all such details, we want to point out a few things that are important to the understanding of the locality issues involved. First of all, concerning the strong notion of locality discussed, the CNOT gate as a quantum process for two qubits is not a local one. Similarly, the unitary evolution given by (7) is not local. Explicitly, it cannot be written in the form $I_A \otimes U_{BC}$ not to say $I_A \otimes U_B \otimes U_C$. The further part of quantum evolution from a complete decoherence theory picture would very unlikely be in the form of $I_A \otimes U$ either. One way or another, a projective measurement cannot be a local process on the subsystem being measured. More directly, the ‘collapse’ taking the state $|\psi\rangle$ to $|00\rangle$ can only be implemented within the system by the unitary transformation $U^{-1}_q$ of the $U_q$ given in the appendix, i.e. $|00\rangle = U^{-1}_q |\psi\rangle$. But $U_q$, or $U^{-1}$, is not a local transformation of the form $I_A \otimes U_B$. We have been naively thinking about such a projective measurement to be a local process. That actually cannot be justified. Then, the process of measuring subsystem $B$ causing changes to the values of local observables for subsystem $A$ does not violate the locality of those observables or the locality of quantum information in the Heisenberg picture.

There is another notion of locality that seems to dictate the projective measurement to be local, namely spatial locality. One considers the case that $A$ and $B$ are ‘spatially well separated’, say with spacelike separation in the Minkowski spacetime picture. Then measuring $B$ cannot result in changes in $A$ without having action-at-a-distance violating special relativity. The q-number value picture, however, has a deep implication that suggests a completely different way to look at the issue. The idea of observables taking values in a noncommutative algebra applying to the position observables gives us a new perspective on the notion of a point in space or an event in spacetime. Such points each described by such the q-number position (coordinate) values cannot be a point in the c-number space. The classical Newtonian space and Minkowski spacetime models should be replaced by some noncommutative geometric models [3, 4]. Questions about spatial locality have to be
reformulated accordingly. At this point, it is still an open question.

We want to emphasize that the description concerning projective measurements as given above is only an attempt to clarify some of the related questions. Our main analysis and results are independent of that. Again, how we are to understand the physics of projective measurements is one thing, a good theoretical picture of the quantum information given by the dynamical theory of quantum mechanics, independent of the latter, is another. We cannot rule out the possibility of having some other dynamical theory, local or nonlocal in their appropriate senses, giving a better description of the physics of projective measurements. Nor should we rule out the possibility of having some other ingenious experimental designs to retrieve or manipulate the quantum information in ways beyond the limit of projective measurements.

The whole dynamical theory of quantum mechanics can be described in the Heisenberg picture, that is to say, by using only the observables. With the notion of noncommutative values for the observables, the full set of local basic observables can be seen as a system of noncommutative coordinates for the quantum phase space \([3, 4]\). From that perspective, the Schrödinger picture is the c-number (real/complex number) coordinate picture while the Heisenberg picture is the q-number (noncommutative) coordinate picture. As Timpson \([14]\) discussed in the language of the Deutsch-Hayden descriptors, such noncommutative values of the local noncommutative coordinates can be seen to give descriptions of the exact local states, through values of their q-number coordinates, even for a composite system with entanglement. The full picture describes the theory as simply a noncommutative version of its classical analog. Quantum reality is then simply a noncommutative reality \([7]\), not necessarily practically inaccessible or having unpalatable features as Timpson stated. It has been discussed in Ref. \([7]\) how the real number readings we get out of our measuring equipment are only the results of the real number we put in to calibrate the scales for displaying the outputs. Otherwise, nothing in Nature says physical quantities have real number values. Nothing says that the information we have obtained, say what the pointer on the scale indicates, is a piece of classical information, beyond the case that the idea may work well enough for the familiar particular settings. In the age of quantum information science, it is time we think about dealing with the full information in a quantum observable and general quantum information directly. In the future, we may even find ways to experimentally determine the noncommutative value or q-number value of an observable directly in one
Finally, let us further clarify a few points against plausible confusion for the benefit of the readers. Firstly, we are not proposing any new theory or new model to describe physics. The work is theoretical, analyzing exactly the logical and mathematical implications of the dynamical theory of quantum mechanics. We are not challenging the correctness of Bell’s theorem so long as the results of projective measurements and theories of local hidden variables, as c-number valued variables, are concerned. Our analysis has hardly touched on the kind of measurements or much of any issues about practical measurements. Neither are we questioning the nonlocal character of an entangled state within the Schrödinger picture, or more precisely so long as the usual description of the state in terms of a commutative/c-number geometric picture of the phase space is concerned. We have demonstrated that the dynamical theory of quantum mechanics, independent of any theory of projective measurements, essentially predicts theoretical information about an observable for a definitive state richer than what is encoded in the full statistical distribution of the corresponding projective measurement. That full theoretical information is an element of a noncommutative algebra, of the noncommutative values. One can think about them as exact q-numbers. We see that as what quantum information is about. That quantum information for a full set of basic observables, which can be naturally identified among the local observables for a composite system, is complete in the sense that the corresponding information for all observables can be mathematically determined from it. Such local quantum information for a part of a composite system is not affected by any physical process local to other parts of the system, even for entangled states. That is our clarification of the full local picture based on describing everything in terms of (values of) observables, which is simply an exact analog of the classical case. We call that the Heisenberg picture, generalizing from its usual discussion involving a time evolution.

An important question, which we have addressed before, is the practical accessibility of the full quantum information in a noncommutative value. In terms of its representation as a sequence of complex numbers, which is finite for a system with a finite-dimensional Hilbert space, the noncommutative value can be determined up to any required precision in principle. In the interest of the current analysis, it suffices to say that those involve information about the state description which is hence not local. Extracting quantum information is certainly not simply about making a few projective measurements. The no-
tion of noncommutative values is about treating quantum information directly as quantum information. It clearly indicates that there is information in the system beyond the access of projective measurements, and the complete information, including the part that is nonlocal in the conventional state picture, is encoded in those values of the basic local observables. How to practically retrieve that full information is a challenge to experimentalists and practical quantum information scientists to whom the author as a fundamental theorist can only appeal.

Another question some readers may have in mind is temporal locality. So long as 'nonrelativistic' quantum dynamics is concerned, time is not an observable and should even not be seen as a kinematic variable. It is only the (real) parameter describing the Hamiltonian evolution, mathematically as a one-parameter group of symmetry transformation on the phase space, which is exactly the unitary one-parameter group of unitary transformations on the Hilbert space [15]. As such, it does not bring in extra locality issues. If a unitary transformation may change the noncommutative values of a local variable, of course, depends on whether it is local to other parts of the system, as discussed above. The situation in 'relativistic' quantum dynamics is more of a tricky story. For the standard theory as available in textbooks, time is still just that evolution parameter, and nothing changes. However, we see the theory as not truly Lorentz covariant. A physical Lorentz covariant picture should have spacetime position observables as components of a Minkowski four-vector. We have formulated exactly such a theory [16] and addressed the corresponding picture of its Hamiltonian dynamics and the key differences with the standard theory [16, 17]. The theory is essentially an exact 1 + 3 analog of the standard 'nonrelativistic' theory with noncommuting four-vector position and momentum operators and a Minkowski metric operator on the vector space of states for a particle. Locality issues and noncommutative values concerning the time part are then simply an exact parallel of what we have here. It is important to note that the literature on the topic of temporal correlation and locality is (see for example Ref. [18]) mostly not about the basic quantum dynamics of (pseudo-)unitary evolution, but rather about some theories extending that basic dynamics which essentially include a theory about projective measurements. We have refrained from going in that direction and stayed from committing to any theory of the kind. Again, our philosophy is that no matter how important and useful projective measurements or POVM have been, it is about a kind of process the actual physics of which we do not quite have a complete understanding. How
the theory of quantum mechanics describes nature is one thing. How a certain kind of process extracts information the theory predicts is another, and the physical understanding of those processes still is somewhat something different. Quantum mechanics ascribes information to an observable part of which is not accessible to the kind of measurements, or any measurement with results depending only on those ‘probabilities’ concerning the eigenstates of the particular observable measured. The full information is experimentally accessible in principle, though obtaining any such noncommutative value directly is a huge challenge. We cannot prove that it would not turn out to be impossible, though we do not see any reason to be that pessimistic at this point. We are interested in that full information here, hence not on the projective measurements.

Our last word: Despite our personal inclination, logically, there are two ways to look at the locality question physically, depending on if the nonlocal information in the Schrödinger picture encoded in the noncommutative values of the local observables is indeed locally experimentally retrievable. If yes, complete quantum information could be retrieved locally and the nonlocal feature in the Schrödinger picture is simply an artifact of the description of states not in terms of physical values of observables. We can, and in that case definitely should, describe the states using the noncommutative values of their local basic observables. Otherwise, one would have to admit the strange conclusion that the noncommutative values of the local observables are nonlocal in themselves!

Appendix A: Deutsch-Hayden Matrix Values and Noncommutative Values of Local Basic Observables for the Two-qubit System

In the two-qubit system, a generic normalized state vector can be written, up to an overall phase factor, as

$$|\Psi\rangle = z_{00} |00\rangle + z_{01} |01\rangle + z_{10} |10\rangle + z_{11} |11\rangle ,$$  

(8)

with the complex coordinates $z_{jk}$ given by

$$z_{00} = e^{-i\frac{\zeta}{2}} q_+ c_A c_B + e^{i\frac{\zeta}{2}} q_- s_A s_B ,$$  

$$z_{01} = e^{-i\frac{\zeta}{2}} q_+ c_A s_B - e^{i\frac{\zeta}{2}} q_- s_A c_B ,$$  

$$z_{10} = e^{-i\frac{\zeta}{2}} q_+ s_A c_B - e^{i\frac{\zeta}{2}} q_- c_A s_B ,$$  

$$z_{11} = e^{-i\frac{\zeta}{2}} q_+ s_A s_B + e^{i\frac{\zeta}{2}} q_- c_A c_B ,$$  

(9)
where \( q_+ = \sqrt{\frac{1+r}{2}} \), \( q_- = \sqrt{\frac{1-r}{2}} \), \( c_A = \cos(\frac{\theta_A}{2})e^{-i\frac{\psi_A}{2}} \), \( s_A = \sin(\frac{\theta_A}{2})e^{i\frac{\psi_A}{2}} \), \( c_B = \cos(\frac{\theta_B}{2})e^{-i\frac{\psi_B}{2}} \), \( s_B = \sin(\frac{\theta_B}{2})e^{i\frac{\psi_B}{2}} \), \( 0 \leq r \leq 1 \), \( 0 \leq \theta_A, \theta_B \leq \pi \), \( 0 \leq \zeta, \psi_A, \psi_B < 2\pi \). The six real parameters \( \{r, \zeta, \theta_A, \theta_B, \psi_A, \psi_B\} \) uniquely specify a physical state as a pure state density matrix and a point in the projective Hilbert space. The value of \( r \) completely characterizes the entanglement as \( \sqrt{1-r^2} \). At \( r = 1 \), the \(|\Psi\rangle\) reduces the general product state \((c_A|0⟩+s_A|1⟩)⊗(c_B|0⟩+s_B|1⟩)\). For the descriptor as \( \{[\tilde{\sigma}_{1A}]_{DH}, [\tilde{\sigma}_{3A}]_{DH}\} \), the Heisenberg reference state is then taken as \(|00⟩\) with then \(|\Psi⟩ = U_\Psi |00⟩\). A generic \( U_\Psi \) is given by \((U_A \otimes U_B)U_qU_{i3}\) where

\[
U_{i3} = \begin{pmatrix} 1 & 0 \\ 0 & U_3 \end{pmatrix}, \quad U_A = \begin{pmatrix} c_A & -s_A \\ s_A & \bar{c}_A \end{pmatrix}, \quad U_B = \begin{pmatrix} c_B & -s_B \\ s_B & \bar{c}_B \end{pmatrix}, \quad U_q = \begin{pmatrix} e^{\frac{\zeta}{2}}q_+ & 0 & -e^{\frac{\zeta}{2}}q_- \\ 0 & I_2 & 0 \\ e^{\frac{\psi}{2}}q_- & 0 & e^{\frac{\psi}{2}}q_+ \end{pmatrix},
\]

and \( U_3 \) any SU(3) matrix. One can check that \( U_\Psi \), or simply \((U_A \otimes U_B)U_q\), gives exactly \(|\Psi⟩\) as the first column vector. Adding back an overall phase factor to \( U_\Psi \) by multiplying the factor to \( U_q \) and allowing a generic \( U_3 \) with eight real parameters, one gets back the fifteen-parameter SU(4) matrix. So long as the Deutsch-Hayden matrix values for observables at the particular instances of time are concerned, we can take \((U_A \otimes U_B)U_q\) as our choice of \( U_\Psi \). Anyway, we see a \( U_\Psi \) has exactly the information of the six real parameters that describe the state \(|\Psi⟩\). For this particular \( U_\Psi \), we do not have all the matrix elements expressed in terms of the four complex coordinates of \(|\Psi⟩\) though.

The explicit form of the full expression of \( U_\Psi \) as \((U_A \otimes U_B)U_q\) is still quite tedious. Besides, with the lastly mentioned undesirable properties, it is not particularly illustrative to look at. However, so long as the locality issue is concerned, it is illustrative enough to look at the special case of \( U_\Psi = U_q \), where \(|\psi⟩\) is the entangled state \( e^{\frac{i\psi}{2}}q_+ |00⟩ + e^{\frac{i\psi}{2}}q_- |11⟩ \) generalizing a Bell state. We have then the explicit Deutsch-Hayden matrix values of the local basic
observables obtained as

\[
[\tilde{\sigma}_{1A}]_{\psi}^{DH} = \begin{pmatrix}
0 & \bar{z}_{11} & \bar{z}_{00} & 0 \\
\bar{z}_{11} & 0 & 0 & \bar{z}_{00} \\
\bar{z}_{00} & 0 & 0 & -\bar{z}_{11} \\
0 & \bar{z}_{00} & -\bar{z}_{11} & 0
\end{pmatrix}
= \begin{pmatrix}
0 & e^{\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}} e^{\frac{\sqrt{1-r}}{2}} & 0 \\
e^{\frac{\sqrt{1-r}}{2}} & 0 & 0 & e^{\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}} \\
e^{-\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}} e^{-\frac{\sqrt{1-r}}{2}} & 0 & 0 & e^{-\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}} \\
0 & e^{-\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}} e^{-\frac{\sqrt{1-r}}{2}} & 0 & 0
\end{pmatrix},
\]

\[
[\tilde{\sigma}_{1B}]_{\psi}^{DH} = \begin{pmatrix}
0 & \bar{z}_{00} & \bar{z}_{11} & 0 \\
\bar{z}_{00} & 0 & 0 & -\bar{z}_{11} \\
\bar{z}_{11} & 0 & 0 & \bar{z}_{00} \\
0 & -\bar{z}_{11} & \bar{z}_{00} & 0
\end{pmatrix}
= \begin{pmatrix}
0 & e^{\frac{i\zeta}{2}} \sqrt{\frac{1-r}{2}} e^{\frac{\sqrt{1-r}}{2}} & 0 \\
e^{-\frac{i\zeta}{2}} \sqrt{\frac{1-r}{2}} e^{-\frac{\sqrt{1-r}}{2}} & 0 & 0 & e^{-\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}} \\
e^{-\frac{i\zeta}{2}} \sqrt{\frac{1-r}{2}} e^{-\frac{\sqrt{1-r}}{2}} & 0 & 0 & e^{\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}} \\
0 & e^{-\frac{i\zeta}{2}} \sqrt{\frac{1-r}{2}} e^{-\frac{\sqrt{1-r}}{2}} & 0 & 0
\end{pmatrix},
\]

\[
[\tilde{\sigma}_{3A}]_{\psi}^{DH} = [\tilde{\sigma}_{3B}]_{\psi}^{DH} = \begin{pmatrix}
|z_{00}|^2 - |z_{11}|^2 & 0 & 0 & -\bar{z}_{00} \bar{z}_{11} \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-\bar{z}_{00} \bar{z}_{11} & 0 & 0 & |z_{11}|^2 - |z_{00}|^2
\end{pmatrix}
= \begin{pmatrix}
r & 0 & 0 & -\sqrt{1-r^2} \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-\sqrt{1-r^2} & 0 & 0 & -r
\end{pmatrix},
\]

where we have used \(z_{01} = z_{10} = 0\). Note that the reduced density matrices for the qubits do not contain any information about \(\zeta\), though they do have \(r\) showing up as the purity parameter showing any value of \(r\) other than 1 they are not pure states.

For our noncommutative values, we have

\[
[\beta]_{\Psi} = \{f_{\beta}\psi; \psi, V_{\beta}^{00}\psi, V_{\beta}^{01}\psi, V_{\beta}^{10}\psi, V_{\beta}^{11}\psi\}.
\]

The simple results for the special case of \(|\Psi\rangle\) reduced to \(|\psi\rangle\) are given for comparison.

\[
f_{\tilde{\sigma}_{1A}} = 0, \quad V_{\tilde{\sigma}_{1A}}^{00} = -f_{\tilde{\sigma}_{1A}} \bar{z}_{00} = 0, \quad V_{\tilde{\sigma}_{1A}}^{01} = \bar{z}_{11} = 0 = e^{\frac{i\zeta}{2}} \sqrt{\frac{1-r}{2}},
\]
\[
V_{\tilde{\sigma}_{1A}}^{10} = \bar{z}_{00} = e^{\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}}, \quad V_{\tilde{\sigma}_{1A}}^{11} = -f_{\tilde{\sigma}_{1A}} \bar{z}_{11} = 0,
\]
\[
f_{\tilde{\sigma}_{1B}} = 0, \quad V_{\tilde{\sigma}_{1B}}^{00} = -f_{\tilde{\sigma}_{1B}} \bar{z}_{00} = 0, \quad V_{\tilde{\sigma}_{1B}}^{01} = \bar{z}_{00} = 0 = e^{\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}},
\]
\[
V_{\tilde{\sigma}_{1B}}^{10} = \bar{z}_{11} = 0 = e^{\frac{i\zeta}{2}} \sqrt{\frac{1-r}{2}}, \quad V_{\tilde{\sigma}_{1B}}^{11} = -f_{\tilde{\sigma}_{1B}} \bar{z}_{11} = 0,
\]
\[
f_{\tilde{\sigma}_{3A}} = |z_{00}|^2 - |z_{11}|^2 = r, \quad V_{\tilde{\sigma}_{3A}}^{00} = (1 - f_{\tilde{\sigma}_{3A}}) \bar{z}_{00} = (1 - r)e^{\frac{i\zeta}{2}} \sqrt{\frac{1+r}{2}},
\]
\[
V_{\tilde{\sigma}_{3A}}^{01} = 0, \quad V_{\tilde{\sigma}_{3A}}^{10} = 0, \quad V_{\tilde{\sigma}_{3A}}^{11} = -(1 + f_{\tilde{\sigma}_{3A}}) \bar{z}_{11} = -(1 + r)e^{\frac{i\zeta}{2}} \sqrt{\frac{1-r}{2}}.
\]
and again $[\tilde{\sigma}_{3A}]_\phi = [\tilde{\sigma}_{3B}]_\phi$. We note in particular that for $r = 1$, $|\psi\rangle$ reduces to the product state $e^{-i\xi} |00\rangle$ (as $q_+ = 1$ and $q_- = 0$) for which we have $[\tilde{\sigma}_{1A}]_{\phi(r=1)} = \{0; 0, 0, e^{i\xi}, 0\}$, $[\tilde{\sigma}_{1B}]_{\phi(r=1)} = \{0; 0, e^{i\xi}, 0, 0\}$, and $[\tilde{\sigma}_{3A}]_{\phi(r=1)} = [\tilde{\sigma}_{3B}]_{\phi(r=1)} = \{1; 0, 0, 0, 0\}$. The last result is the consequence that the state is an eigenstate of $\tilde{\sigma}_{3A}$ and $\tilde{\sigma}_{3B}$ of eigenvalue, and hence expectation value, 1. As mentioned in the main text, for an eigenstate of an observable, the noncommutative value becomes essentially a commutative value. $e^{i\xi}$, in the case, is just an overall phase factor, the conjugate of that for the state. We can drop it and take $[\tilde{\sigma}_{1A}]_{\phi(r=1)} = \{0; 0, 0, 1\}$ and $[\tilde{\sigma}_{1B}]_{\phi(r=1)} = \{0; 0, 1, 0\}$.

Appendix B: Noncommutative Values of Local Basic Observables for the Two-particle System (The exact EPR case)

Let us look at a different example of the noncommutative value picture for the local basic observables of a system of two quantum particles concerning the locality issue of the exact EPR state. Noncommutative values for the position and momentum operators under a Schrödinger wavefunction representation have been explicitly given in Ref.\[11\]. Here, we adapt that to the two-particle wavefunction $\phi(x_1, x_2)$. For convenience, we take the particles to be of equal mass $m$. Take the entangled state of fixed total momentum $p$ ($\hbar$ taken as unity), and a separation between the particles given by $r = vt$ increasing linearly with time. $p$ and $r$ should then by eigenvalues of the dynamics observables $\hat{P}$ and $\hat{R}$ to which the system maintains in simultaneous eigenstates. Note that the observables are parts of the canonical pairs of observables for the center of mass and relative motion degrees of freedom with

$$
\hat{X} = \frac{\hat{X}_1 + \hat{X}_2}{2}, \quad \hat{P} = \hat{P}_1 + \hat{P}_2, \\
\hat{R} = \hat{X}_1 - \hat{X}_2, \quad \hat{Q} = \frac{\hat{P}_1 - \hat{P}_2}{2},
$$

with $[\hat{X}, \hat{P}] = i = [\hat{R}, \hat{Q}]$ as the only nonzero commutators among them. The wavefunction of the EPR state is then written as

$$
= \delta(r - r_o)e^{ipx} = \delta(x_1 - x_2 - r_o) e^{i\frac{p(x_1 + x_2)}{2}},
$$

where $x = \frac{x_1 + x_2}{2}$, and $r = x_1 - x_2$. The state is, of course, not an eigenstate of any of the local basic observables as the position and momentum of the individual particles. The entanglement is seen in the fact that the wavefunction $\phi(x_1, x_2)$ cannot be written in terms...
of a product of a $\phi_1(x_1)$ and a $\phi_2(x_2)$ as wavefunctions for states of the individual particles.

In terms of the degrees of freedom for the canonical observables of Eq. (12), i.e. in terms of real variables $r$ and $x$, however, there is no entanglement.

The noncommutative values for $\hat{P}$ and $\hat{R}$, assuming the matrix elements, are given by $[\hat{P}]_\phi = \{p; V_\rho\}$ and $[\hat{R}]_\phi = \{vt; V_\rho\}$ where

$$V_\rho \equiv \delta_\phi f_\rho = 0,$$
$$V_\rho \equiv \delta_\phi f_\rho = 0.$$  \quad (14)

The last equations express the collection of ‘coordinates’ derivatives of the corresponding expectation value function(al)s as the functional derivatives. The vanishing results are a consequence of $\phi$ being an eigenstate of the observables. We further take the expectation values of all the position observables at $t = 0$ and that of the momentum observable $\hat{Q}$ to be zero. We have then $[\hat{X}]_\phi = \{\bar{x}; V_x\}$ and $[\hat{Q}]_\phi = \{\bar{q}; V_q\}$ where

$$V_x \equiv \delta_\phi f_x = e^{-ipx} \delta(r - r_0) (x - \bar{x}) ,$$
$$V_q \equiv \delta_\phi f_q = e^{-ipx} (i\partial_r - \bar{q}) \delta(r - r_0).$$  \quad (15)

Note that $\bar{x}$ and $\bar{q}$ denote the expectation values of the observables. We expect their values to be zero. Yet, it serves our purpose here better to simply leave them in the direct formal integrals as

$$\bar{x} = \int dx \ x |e^{ipx}|^2 ,$$
$$\bar{q} = \int dr \ \delta(r - r_0) (-i\partial_r) \delta(r - r_0) ,$$  \quad (16)

to be used below. All the expressions involving the noncommutative values we have given so far factorize into products of functions of $x$ and functions of $r$. When rewritten in terms of variables $x_1$ and $x_2$, $V_x$ and $V_q$ do not factorize. That indicates the nature of the state as being an entangled one in the corresponding degrees of freedom, i.e. those of the individual particles.

For the local basic observables, the noncommutative values can be obtained as $[\hat{X}_1]_\phi = \{\bar{x}_1; V_{x_1}\}$, $[\hat{X}_2]_\phi = \{\bar{x}_2; V_{x_2}\}$, $[\hat{P}_1]_\phi = \{\bar{x}_1; V_\rho\}$, and $[\hat{P}_2]_\phi = \{\bar{x}_2; V_\rho\}$, with the expectation
values formally as

\[ \bar{x}_1 = \int dx \, x_1 |e^{ipx}|^2 \delta^2 (r - r_o) = \bar{x} + \frac{r_o}{2}, \]

\[ \bar{x}_2 = \int dx \, x_2 |e^{ipx}|^2 \delta^2 (r - r_o) = \bar{x} - \frac{r_o}{2}, \]

\[ \bar{p}_\lambda = e^{-ipx} \delta (r - r_o) (-i \partial_{x_1}) e^{ipx} \delta (r - r_o) = \frac{p}{2} + \int dr \, \delta (r - r_o) (-i \partial_r) \delta (r - r_o), \]

\[ \bar{p}_\lambda = e^{-ipx} \delta (r - r_o) (-i \partial_{x_2}) e^{ipx} \delta (r - r_o) = \frac{p}{2} - \int dr \, \delta (r - r_o) (-i \partial_r) \delta (r - r_o), \quad (17) \]

and

\[ V_{\bar{x}_1} = e^{-ipx} \delta (r - r_o) (x_1 - \bar{x}_1), \]

\[ V_{\bar{x}_2} = e^{-ipx} \delta (r - r_o) (x_2 - \bar{x}_2) = (x_2 - \bar{x}_2) \delta (x_1 - x_2 - r_o) e^{-ip(x_1 + x_2)}, \]

\[ V_{\bar{p}_\lambda} = (i \partial_{x_1} - \bar{p}_1) e^{-ipx} \delta (r - r_o) = e^{-ipx} \left( \frac{p}{2} + i \partial_r - \bar{p}_1 \right) \delta (r - r_o), \]

\[ V_{\bar{p}_\lambda} = (i \partial_{x_2} - \bar{p}_2) e^{-ipx} \delta (r - r_o) = e^{-ipx} \left( \frac{p}{2} - i \partial_r - \bar{p}_2 \right) \delta (r - r_o). \quad (18) \]

The results above clearly demonstrate the linearity of the noncommutative values giving

\[ [\hat{X}]_\phi = \frac{[\hat{X}_1]_\phi + [\hat{X}_2]_\phi}{2}, \quad [\hat{P}]_\phi = \frac{[\hat{P}_1]_\phi + [\hat{P}_2]_\phi}{2}, \]

\[ [\hat{R}]_\phi = [\hat{X}]_\phi - [\hat{X}_2]_\phi, \quad \hat{Q}_\phi = \frac{[\hat{R}]_\phi - [\hat{P}]_\phi}{2}. \quad (19) \]

We can further express the noncommutative values for the local basic observables as

\[ [\hat{X}]_\phi = \left\{ \bar{x} + \frac{r_o}{2}; \left( x_1 - \bar{x} - \frac{r_o}{2} \right) \delta (x_1 - x_2 - r_o) e^{-ip(x_1 + x_2)} \right\}, \]

\[ [\hat{X}_2]_\phi = \left\{ \bar{x} - \frac{r_o}{2}; \left( x_2 - \bar{x} + \frac{r_o}{2} \right) \delta (x_1 - x_2 - r_o) e^{-ip(x_1 + x_2)} \right\}, \]

\[ [\hat{P}_1]_\phi = \left\{ \frac{p}{2} + \bar{q}; \frac{e^{-ip(x_1 + x_2)}}{2} \left( i \partial_{x_1} - \bar{q} \right) \delta (x_1 - x_2 - r_o) \right\}, \]

\[ [\hat{P}_2]_\phi = \left\{ \frac{p}{2} - \bar{q}; \frac{e^{-ip(x_1 + x_2)}}{2} \left( i \partial_{x_2} + \bar{q} \right) \delta (x_1 - x_2 - r_o) \right\}, \quad (20) \]

showing explicitly how they encode the information that would be seen as nonlocal in the language of the states. \( V_{\bar{x}_1} \) of \( [\hat{X}]_\phi \), for example, involves the \( x_1 \) values and the nonfactorizable nature of it indicates the entanglement. Note that the nonvanishing \( V_{\bar{x}_1}, V_{\bar{x}_2}, V_{\bar{p}_1}, \) and \( V_{\bar{p}_2} \) parts of the noncommutative values indicate that the state is not an eigenstate of the observables. The Heisenberg uncertainty of an observable \( \hat{A} \) is simply given by the integral of \( |V_{\hat{A}}|^2 \). The cancellation between \( V_{\bar{x}_1} \) and \( V_{\bar{x}_2} \) in \( V_{\bar{p}} \) and that of \( V_{\bar{p}_1} \) and \( -V_{\bar{p}_2} \) in \( V_{\bar{p}} \) is the
statement that the state is an eigenstate of $\hat{R}$ and $\hat{P}$ with the exact correlations between the pairs of observables involved. Explicitly, we have

$$V_{\hat{R}} = V_{\hat{x}_1} - V_{\hat{x}_2} = (x_1 - x_2 - r_o)\delta(x_1 - x_2 - r_o) e^{-ip(x_1+x_2)/2} = 0,$$

$$V_{\hat{P}} = V_{\hat{p}_1} + V_{\hat{p}_2} = e^{-ip(x_1+x_2)/2}i(\partial_{x_1} - \partial_{x_2})\delta(x_1 - x_2 - r_o) = 0.$$  \hspace{1cm} (21)

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