LOCAL EXTREMA IN QUANTUM CHAOS

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Abstract. We numerically investigate both the number and the spatial distribution of local extrema of 'chaotic' Laplacian eigenfunctions on two-dimensional manifolds and demonstrate two new universality phenomena. Blum, Gnutzmann & Smilansky have numerically demonstrated that the \( k \)-th eigenfunction has typically \( \sim 0.06k \) nodal domains – we give numerical evidence that it typically has \( \sim \sigma \cdot k \) local extrema, where \( \sigma = 0.58 \pm 0.02 \) is a universal constant. Using the discrepancy as a measure of quality of distribution, we show that the local extrema are more regularly spread than a regular grid.

1. Introduction

1.1. Quantum Chaos. Quantum Chaos is concerned with the behavior of high-frequency Laplacian eigenfunctions
\[-\Delta u = \lambda u \quad \text{on compact manifolds } (M, g)\]
and their seemingly chaotic properties. Apart from highly particular cases which are usually characterized by completely integrable behavior of the geodesic flow, these eigenfunctions will appear to be somewhat 'random'. Indeed, should the behavior be not chaotic, then usually any small perturbation of the geometry of the domain will induce chaotic behavior: randomness is the generic case. It is of great interest to try to understand this randomness by specifying arising invariants.

![Figure 1](image-url)

An eigenfunction of the Laplacian with Neumann conditions once on \([0,1]^2\) (left) and once on a small perturbation of \([0,1]^2\) (right, the perturbation is not visible).

Some central questions of quantum chaos are

1. whether (and under which conditions on the geometry of the manifold) the \( L^2 \)-mass of the eigenfunctions tends towards uniform distribution – recent spectacular breakthroughs due to Anantharaman [1] and Lindenstrauss [20].
2. whether most eigenfunctions behave like 'random waves', i.e. whether for example
\[
\frac{\|u_k\|_{L^\infty}}{\|u_k\|_{L^2}} \lesssim (\log k)^{1/2} \quad \text{with high probability} \ [2]
\]
how many nodal domains there are (see [4, 5, 7] for the random wave model and [6, 24] for deterministic bounds) and how their volume is distributed (see e.g. [25]).

The number of nodal domains has received particular interest: in a highly influential paper by Blum, Gnutzmann & Smilansky [4], a universality statement for the number of nodal domains has been conjectured and numerically investigated: a generic Laplacian eigenfunction associated to the $k$-th eigenvalue seems to have $\sim 0.06k$ nodal domains. Bogomolny & Schmit [5] have worked out a percolation model simulating eigenfunctions in which the observation of Blum, Gnutzmann & Smilansky is confirmed: their model predicts that the number of nodal domains of the $k$-th eigenfunction is distributed with

$$\frac{3\sqrt{3} - 5}{\pi} k \sim 0.06k$$

mean and a variance of

$$\left( \frac{18}{\pi^2} + \frac{4\sqrt{3}}{\pi} - \frac{25}{2\pi} \right) k \sim 0.05k.$$

It is not yet understood to what extent these numbers are precise outside the model (recent numerical work of Konrad [18] suggests the mean to be $\sim 4\%$ smaller), however, they are certainly very good approximations. Our main contribution will show that a similar universal phenomenon seems to exist for the number of local extrema (a quantity that is trivially always at least as large as the number of nodal domains).

1.2. Local extrema. We will numerically investigate two new possible invariants: the number of local extrema as well as their spatial distribution. To the best of our knowledge, no research has been carried out in this direction. Our main observations are as follows.

1. If we denote the number of local extrema of the $k$-th Laplacian eigenfunction $u_k$ by $N(u_k)$, then for $k$ large this number seems to be centered around $N(u_k) \sim \sigma k$ where $\sigma \sim 0.58 \pm 0.02$ is a universal constant.

2. The spatial distribution of local extrema is highly regular: using the discrepancy, to be introduced further below, as a quantitative measure, we see that the points seem to tend towards uniform distribution rather quickly (in a way that will be made precise below).

Both statements seem to be completely new. Nodal domains are usually very non-local objects (as can be seen, for example, in the figure above or the fact that the Bogomolny-Schmit model is using percolation), one would perhaps not expected many universal local properties to hold: in particular, our observation suggests that highly elongated nodal domains are comprised of several somewhat 'independent' bumps each of which contains a local maximum. We regard this as further evidence that the underlying philosophy behind the random wave model is sound.
Indeed, combining our first observation with the observation of Blum, Gnutzmann & Smilansky (using the recent estimate of Konrad [18]), we can deduce that

a typical nodal domain contains on average \( \sim 9.7 \) local extrema.

Furthermore, if we use the discrepancy to measure the regularity of their distribution in space, it seems that the distribution of local extrema of a chaotic eigenfunction is even more regular than that of the grid-structure arising from the extrema of a (non-chaotic) eigenfunction on the unperturbed domain \([0, 1]^2\).

The rest of the paper is structured as follows: in Section 2 we motivate our approach and describe the two methods we will employ to create quantum chaos (one is fairly standard, the other seems to be new), Section 3 describes our result on the number of local extrema, Section 4 describes our results about their spatial distribution; technical comments are given in the final section.

2. Generating quantum chaos

2.1. Motivation. Our approach was motivated by the following basic observation: the set of extrema of the function \( \sin (n\pi x) \), which is a solution to the eigenvalue problem

\[-\Delta u = n^2 \pi^2 u \quad \text{on } [0, 1]^2,\]

is extremely regular.

![Figure 3. The location of the extrema of \( \sin (5\pi x) \) on the unit interval.](image)

This is not altogether surprising: these eigenfunctions of the Laplacians may be alternatively described as minima of the Rayleigh quotient

\[\inf_{f \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} \quad \text{with } f \text{ being orthogonal on previous eigenfunctions.}\]

It is therefore not strange that two different extrema should be at a certain distance from each other: a maximum and a minimum too close together would induce a large gradient between them. Let us now consider the obvious generalization (with Dirichlet boundary conditions)

\[-\Delta u = \lambda u \quad \text{on the domain } [0, 1]^2,\]

it is well-known that the solutions are of the form

\[\{ \sin (n\pi x) \sin (m\pi x) : m, n \in \mathbb{N} \land \pi^2 (m^2 + n^2) = \lambda \}.\]

The multiplicative splitting of the eigenfunctions implies that the extrema will necessarily have a grid structure – there is no chaos. The splitting of the eigenfunction is induced by the geometry of the domain and a highly unstable phenomenon: any generic small perturbation of the domain will destroy that property.

2.2. Generating Chaos. There is a very natural way to generate objects that behave like chaotic Laplacian eigenfunctions. Take the torus \( T^2 \) or the sphere \( S^2 \) for which explicit representations of (non-chaotic) eigenfunctions exist; take now an eigenvalue with a large multiplicity and consider a random combination of the explicit eigenfunctions; heuristically, these objects should then behave in the same way as chaotic eigenfunctions while still being explicitly computable. This idea is, of course, directly inspired from the random wave model (see e.g. [2]). However, while these two cases are well suited for computational checking, we decided to complement this by computing
Laplacian eigenfunctions on random geometries. Our approach is based on the idea of identifying pairs of points on the unit square $[0, 1]^2$: this corresponds to introducing a small 'wormhole' connecting two points that were initially far away.

![Figure 4](image_url)

**Figure 4.** The flat square and the flat square with two points connected by a small wormhole.

To simplify matters, we will not give a geometrically precise definition of the nature of the identification but instead approximate the unit square $[0, 1]^2$ by a finite graph and then simply describe the modification being carried out on the finite graph. It follows immediately from work of Colin de Verdiere [11] that any such graph can conversely be approximated by a manifold which implies that performing the surgery directly on the graph is not a restriction to the discrete setting. We will be working on the grid-graph $G_n$ defined on the vertex set

$$V(G_n) = \{1, \ldots, n\}^2,$$

where two vertices are connected by an edge if and only if

$$(a, b) \sim_E (c, d) :\Leftrightarrow |a - c| + |b - d| = 1.$$

We will then induce a slight perturbation of the geometry by adding a few extra edges (which correspond to the 'wormholes'): interestingly, the results we obtain depend slightly on the number of added edges (where a rather small number, say 5 extra edges for a $n = 100$, suffices) but not really where the edges are added - we will actually add them at random. Once we have fixed our geometry, we are able to introduce the discrete Graph-Laplacian.

![Figure 5](image_url)

**Figure 5.** The domains and the relevant graph approximations.

Given a finite, simple, connected Graph $G = (V, E)$ the natural analogue of the Laplacian is the discrete Graph-Laplacian given by a $|V| \times |V|$-matrix $L$ with entries

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -(d_id_j)^{-\frac{1}{2}} & \text{otherwise,} \end{cases}$$

where $d_i$ is the degree of the vertex $i$. The first few eigenvalues/eigenvectors of the matrix will then approximate the first few eigenvalues/eigenfunctions of the Laplacian with Neumann boundary condition (for details we refer to [23]). The standard reference on the Graph Laplacian is the book of Chung [8].
3. The number of local extrema

Our approach to testing the universality hypothesis is to look at several structurally very different examples and compare results; natural examples are given by taking random linear combinations of eigenfunctions on $T$ and $S^2$ belonging to the same eigenspace. Our second approach is outlined above and consisted of studying eigenfunctions on grid graphs with an additional number of random edges; we are not aware of any results in the literature taking this approach to generating functions that should essentially behave like chaotic eigenfunctions, however, it is certainly very natural.

3.1. The torus $T^2$. Our approach was to use a random linear combination (with coefficients uniformly chosen from $[0, 1]$) of explicit eigenfunctions on the torus. It would certainly be desirable to have more large-scale computations with a larger $n$, unfortunately the reliable detection of local extrema is computationally costly; we refer to the last section for details. Our results were as follows.

| eigenvalue | dim | $n$ | mean | variance |
|------------|-----|-----|------|----------|
| $1105\pi^2$ | 8   | 100 | 0.581k | 0.137k  |
| $1325\pi^2$ | 6   | 100 | 0.574k | 0.404k  |
| $2125\pi^2$ | 8   | 100 | 0.582k | 0.487k  |

Table 1. For each eigenvalue, the dimension of the eigenspaces and the mean and variance of our normalized number of extrema.

3.2. The sphere $S^2$. We can use the explicit representation of the spherical harmonics for additional testing: the multiplicity of the $\ell$-th eigenvalues is $2\ell - 1$. We encounter two competing factors:

- the randomly generated functions should be even closer to chaotic eigenfunctions as the eigenspace is very large but
- the large eigenspace means that the $k$-th eigenvalue is going to be piecewise constant and then jump for small $k$; the error term in the Weyl asymptotic is sharp in the case.

This has two effects, the first of which is helping our efforts because things get very chaotic very quickly while the second effect is perturbing things for small $k$: there is no natural $k$-th eigenvalue because the $i$-th eigenfunctions with

$$i = \ell^2, \ell^2 + 1, \ldots, \ell^2 + 2\ell + 1$$

all have the same eigenvalue. This data seems to suggest that the universal constant should be $\sim 0.6$, however, this is misleading: the data was computed based on the assumption that $\ell^2$ is the $k$-th eigenvalue. A natural renormalization is to interpret the $2\ell - 1$ dimensional eigenspace associated to the eigenvalue $\ell^2$ in such a way that the `average’ value of $k$ should not be $k = \ell^2$ but rather

$$k \sim \ell^2 + \ell + \frac{1}{2}.$$ 

Note that the correction tends to 1 as $k$ becomes large, i.e. we expect both the normalized und unnormalized mean to be slightly smaller than suggested by the data here.

| $\ell^2$ | 100 | 196 | 289 |
|---------|-----|-----|-----|
| mean    | 0.604k | 0.606k | 0.609k |
| variance| 0.608k | 0.540k | 0.368k |
| renormalized mean | 0.548k | 0.564k | 0.574k |

Table 2. Mean and variance for the number of local extrema of a linear combination of the $k$-th eigenfunctions for $k = \ell^2$ and $n = 50$. 
3.3. Random graphs. Here we computed eigenfunctions on 100 random graphs in each case. A minor difficulty occurs: our phrasing in terms of the Graph Laplacian uses implicitly Neumann conditions whereas our calibration used Dirichlet conditions; this leads to certain boundary effects. We circumvented the problem by only looking at the number of extrema in the fixed box $[0.05, 0.95]^2$ and rescaling (thereby exploiting another universality phenomenon: quantum chaos in a subregion should not behave any differently than in the whole space). The results are summarized in the following table.

| $k$ | 100     | 200     | 300     |
|-----|---------|---------|---------|
| 5 random edges | (0.69$k$, 0.06$k$) | (0.52$k$, 0.07$k$) | (0.59$k$, 0.71$k$) |
| 10 random edges | (0.71$k$, 0.06$k$) | (0.53$k$, 0.22$k$) | (0.59$k$, 0.37$k$) |
| 20 random edges | (0.75$k$, 0.16$k$) | (0.56$k$, 0.13$k$) | (0.60$k$, 0.28$k$) |

Table 3. Mean and variance for the number of local extrema of the $k$-th eigenfunction on a $150 \times 150$ grid graph for a different number of random edges added. For each case mean/variance was taken over a random sample with $n = 100$.

4. Discrepancy of Local Extrema

4.1. Discrepancy. This section is devoted to the introduction of the notion of discrepancy; it is a widely used and foundational concept in the theory of Quasi-Monte Carlo methods, where the distribution properties of a set of points directly imply error bounds for numerical integration schemes using these set of points. We refer to the classical books of Niederreiter [21] and Drmota & Tichy [14] as well as the recent monograph of Dick & Pillichshammer [12] for more detailed information about discrepancy. Formally, let $\mathcal{P} = \{p_1, \ldots, p_N\}$ be a set of points in the unit square, i.e. $\mathcal{P} \subset [0,1]^2$ and write $p_i = (x_i, y_i)$. We define the star-discrepancy $D_N^*$ via

$$D_N^*(\mathcal{P}) := \sup_{0 \leq x,y \leq 1} \left| \frac{\#\{1 \leq i \leq N : 0 \leq x_i \leq x \land 0 \leq y_i \leq y\}}{|\mathcal{P}|} - xy \right|.$$ 

A graphical description is as follows: given the set of all axis-parallel rectangles anchored in $(0,0)$, we measure the maximal deviation between the area of the rectangle and the relative size of the set $\mathcal{P}$ contained in the rectangle. If one wishes to replace the set of rectangles anchored in the origin by the set of all axis-parallel rectangles, this gives rise to the notion of discrepancy. We will only work with star-discrepancy since it is technically easier and $D_N^* \leq D_N \leq 2D_N^*$. A celebrated result of Schmidt [22] states that

$$D_N(\mathcal{P}) \geq \frac{1}{100} \log \frac{N}{N}$$

for any two-dimensional set $\mathcal{P}$ with $\#\mathcal{P} = N$ and this is asymptotically best possible.

Figure 6. Set of points and a rectangle containing three points.
A simple calculation shows that a regular grid always satisfies $D_N^*(\text{grid}) \gtrapprox N^{-1/2}$. Using the definition, it follows that for a set of $N$ points $\mathcal{P}$ with discrepancy $D_N^*(\mathcal{P})$ and any axis-parallel rectangle $R \subset [0, 1]^2$, we have that

$$|R| - 2D_N^*(\mathcal{P}) \leq \frac{\#(\mathcal{P} \cap R)}{\#\mathcal{P}} \leq |R| + 2D_N^*(\mathcal{P}).$$

### 4.2. Sets with small discrepancy

In this section we briefly describe the different constructions of deterministic point sets, which serve as benchmark with which we will compare the regularity of the local extrema of Laplacian eigenfunctions – we will create quantum chaos using again random linear combinations of eigenfunctions on the torus $\mathbb{T}^2$ as well as eigenfunctions on random graphs. The sphere $S^2$ is not as suited because no isometric embedding $\phi : S^2 \to [0, 1]^2$ exists (one could study other notions of discrepancy on $S^2$ but this outside of the scope of this paper).

#### 4.2.1. Random points

The star discrepancy of a point set with $N$ random elements in $[0, 1]^2$ is a random variable with expectation $D_N^* \sim N^{-1/2}$. We will always include discrepancy results for random point as a natural benchmark of the simplest possible method.

#### 4.2.2. Regular grid

In contrast to random point sets, the regular grid $G_m = \left\{ \left( \frac{i}{m}, \frac{j}{m} \right) : 0 \leq i, j \leq m-1 \right\}$ of $N = m^2$ points is a highly structured point set with star discrepancy $D_N(G_m) = \frac{2m-1}{m^2} \sim \frac{2}{\sqrt{N}}$.

It is easy to see that shifting the grid by a random vector $x \in [0, 1/m]^2$ can only decrease the value of the star discrepancy; however, at most by a constant factor of $1/2$.

#### 4.2.3. Hammersley point sets

Hammersley point sets and their generalizations are classical examples of low discrepancy point sets created using number theory. Let $n = \sum_{i=0}^{\infty} n_i b^i$ be the $b$-adic expansion of the integer $n$ with $0 \leq n_i \leq b-1$. Then the radical-inverse function, $\phi_b : \mathbb{N} \to [0, 1]$, is defined as

$$\phi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \ldots = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$}

Furthermore, for $N = b^m$ the two-dimensional Hammersley point set in base $b$ is then defined by

$$\mathcal{H}_{b,m} = \left\{ \left( \phi_b(n), \frac{n}{b^m} \right) : 0 \leq n \leq b^m - 1 \right\}.$$}

We refer to [17] for an explicit formula for $D_N^*(\mathcal{H}_{b,m})$ and its generalization using permutations of the digits in the $b$-adic expansion of $n$.

### 4.3. Results

Our results are summarized in the following table. Each entry gives both the mean as well as the standard deviation of the discrepancy sampled over $n = 20$ random constructions (which is already sufficient for a conclusive result as the distribution is tightly concentrated around its mean, note the small variance). We refer to the appendix for additional technical details.

| # points | 50        | 100       | 200       | 500       |
|----------|-----------|-----------|-----------|-----------|
| random-graph | (0.110, 0.014) | (0.074, 0.012) | (0.058, 0.011) | (0.029, 0.003) |
| random-torus | (0.105, 0.005) | (0.078, 0.003) | (0.055, 0.004) | (0.033, 0.004) |
| random points | (0.202, 0.046) | (0.116, 0.019) | (0.088, 0.016) | (0.054, 0.011) |
| standard grid | 0.265     | 0.190     | 0.128     | 0.085     |
| shifted grid | 0.133     | 0.095     | 0.064     | 0.043     |
| Hammersley sets | 0.082     | 0.0544    | 0.035     | 0.022     |

Table 4. Mean and standard deviation of the discrepancy of various sets.
We note that the discrepancy is clearly decaying implying that the distribution of local extrema tends towards uniform distribution (as one would expect). However, it is certainly surprising that the discrepancy of the local extrema seems to be smaller than both that of the standard/shifted grid as well as that of fully random points. It seems natural to conjecture that asymptotically the discrepancy of the set $P_k$ of local extrema of the $k$–eigenfunction should scale as $\sim \frac{c}{\sqrt{k}}$.

5. Methodology and technical details

5.1. Detection of local extrema. Finding local extrema computationally is in general difficult: any function might have tiny oscillations at a small scale which could potentially never be discovered by a computer, or, reversely, inaccuracies of the computer might generate tiny artificial extrema depending on the geometry and parameters of the underlying grid. We start by giving some heuristics why this is not to be expected in the case of Laplacian eigenfunctions: basically, eigenfunctions of second order partial differential equations of elliptic type with constant coefficients come equipped with a natural length scale on which oscillations occur. This insight motivates to count extrema using a local, fast and easy-to-implement algorithm, which requires the evaluation of the function $f$ on a fine grid imposed on $[0,1]^2$. Despite our heuristics that ensure the accuracy of the local extrema count, we also sketch an alternative, topological algorithm. While this algorithm is computationally more demanding, it is provably reliable in much more general situations than the one we study here.

5.1.1. Stability of the problem. A standard estimate for a Laplacian eigenfunction satisfying $-\Delta u = \lambda u$ on some domain with Dirichlet conditions at the boundary is that $\|\nabla u\|_{L^\infty} \sim \lambda^{1/2} \|u\|_{L^\infty}$. This clarifies a natural heuristic saying that such an eigenfunction may be seen as a collection of waves with wavelength $\lambda^{-1/2}$. Furthermore, the elliptic equation $-\Delta u = \lambda u$ implies quantitative control on the concavity of $u$ around a local maximum (and, conversely, quantitative control on the convexity around a local minimum). This quantitative control weakens whenever $u$ has a local extremum with $|u|$ being very small – something that is not to be expected in the case of quantum chaos. This implies that, at least in the generic case, two extrema are at least distance $\sim \lambda^{-1/2}$ away from each other.

5.1.2. Local Method. The local method is straightforward and easy to implement as it simply checks all eight neighbors of a given grid point $(i,j)$. If $f(i,j)$ is greater (resp. smaller) than the value of every neighbor the algorithm reports a local extremum (resp. minimum).

Choosing the appropriate grid size is of course essential. Our rule of thumb was that the grid size $h$ should scale as $h \sim 0.2\lambda^{-1/2}$. This implies that the grid is really smaller than the wavelength, certainly a necessary condition for the approximation to work. The situation simplifies a bit since we have the explicit form of the eigenfunctions at our disposal. For example on $T$, the eigenfunctions associated to the eigenvalues $2210\pi^2$ with the highest oscillation frequency are $\sin(\pi x) \sin(47\pi y)$ and $\sin(\pi y) \sin(47\pi x)$. This implies that a $150 \times 150$ grid certainly satisfies the rule of thumb. We additionally checked the stability of the method by performing iterative refinements on larger grids and comparing the results to ensure sufficient stability.
5.1.3. Topological method. The topological method uses the concept of persistent homology and is a standard method in computational topology. We refer to the book of Edelsbrunner & Harer [16] for further details. We evaluate the function again on a finite number of points (not necessarily a regular grid), which we extend to a simplicial complex. We then compute the lower (upper) star filtration of this complex and use it to compute the 0-th extended persistence diagram of $f$. This diagram describes the evolution of the 0-dimensional homology of the sequence of sub- and superlevel sets of $f$ from which we can read off the number of extrema. While this method is more involved and computationally more expensive, its main advantage lies in the various stability results of persistent topology [9, 10, 16] which formally ensure that this method indeed detects all relevant extrema of $f$ on any scale. We used a standard implementation of the persistence algorithm to confirm the results of the local algorithm.

5.2. Discrepancy. The purpose of this section is to describe various technical details regarding the results presented in Section 4.3. One common necessity is that in our construction of random Laplacian eigenfunctions it is not a priori clear how many local extrema one can expect; however, due to the underlying universality phenomenon, it was rather easy to pick the right $k$ such that the $k$–th eigenfunction will have ‘on average’ the desired number of local extrema (indeed, this is how we discovered the universal behavior for the number of local extrema).

5.2.1. Computation. For the computation of the discrepancy we use a recent implementation of the Dobkin-Eppstein-Mitchell algorithm [15] by Magnus Wahlström [26], which is freely available online. This algorithm computes the star discrepancy exactly; for details on the implementation and a fast randomized algorithm for the approximation of the star discrepancy of large multi-dimensional point sets we refer to [13].

5.2.2. Random graphs. We used a $n \times n$ grid graph, where $n = 120$ for eigenfunctions containing up to 200 extrema and $n = 170$ for higher accuracy in the final case. Furthermore, we always added 10 edges (in the first three cases) and 50 random edges (in the final case), respectively, between edges chosen fully at random (the precise number of edges added seemed to have very little impact on the overall results). We then looked for the $k$ such that the the $k$th eigenfunction has, on average, the desired number of local extrema: on the $170 \times 170$ graph, this was, for example, the 1050th eigenfunction. However, the results seem very stable and it does not seem to have a big effect whether one chooses the 1050th eigenfunction or, say, the 1037th. Since we are again computing Neumann solutions with the usual boundary phenomena, we again did an intersection of the points with the fixed box $[0.05, 0.95]^2$ followed by the appropriate rescaling.

5.2.3. Random combinations on the torus. The problem that arose here is that, at least for small eigenvalues, not many of them have high multiplicity; it is thus not always possible to produce random combinations having on average the desired number of local extrema. Our choice was as follows.

| eigenvalue | dim(eigenspace) | expected number of local extrema |
|------------|-----------------|---------------------------------|
| $125\pi^2$ | 4               | $\sim 55$                       |
| $221\pi^2$ | 4               | $\sim 100$                      |
| $481\pi^2$ | 4               | $\sim 220$                      |
| $1105\pi^2$| 8               | $\sim 500$                      |

In particular, it should be noted that in the following comparison in the cases of 50 and 200 points, the point sets arising as local extrema will have 10% more points than the sets used for benchmark (which, however, will changes the resulting data by less than 10% due to superlinear scaling).

5.2.4. Deterministic sets. The standard grid, the randomly shifted grid as well as fully random points require no further explanation; regarding parameters, the grid was computed with $m = 7, 10, 15, 23$ to generate point sets with $m^2$ many points ($m = 15$ was chosen instead of $m = 14$ to compensate for the corresponding effect on the torus). Similarly, the same numbers were used as basis for the generation of corresponding Hammersley point sets of the same size.
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