On Cobweb Posets and Discrete F-Boxes Tilings

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Abstract

F-boxes defined in [6] as hyper-boxes in $\mathbb{N}^\infty$ discrete space were applied here for the geometric description of the cobweb posets Hasse diagrams tilings. The F-boxes edges sizes are taken to be values of terms of natural numbers’ valued sequence $F$. The problem of partitions of hyper-boxes represented by graphs into blocks of special form is considered and these are to be called $F$-tilings.

The proof of such tilings’ existence for certain sub-family of admissible sequences $F$ is delivered. The family of $F$-tilings which we consider here includes among others $F = \text{Natural numbers, Fibonacci numbers, Gaussian integers with their corresponding } F\text{-nomial (Binomial, Fibonomial, Gaussian) coefficients as it is persistent typical for combinatorial interpretation of such tilings originated from Kwaśniewski cobweb posets tiling problem}.$

Extension of this tiling problem onto the general case multi $F$-nomial coefficients is here proposed. Reformulation of the present cobweb tiling problem into a clique problem of a graph specially invented for that purpose - is proposed here too. To this end we illustrate the area of our reconnaissance by means of the Venn type map of various cobweb sequences families.

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1 Introduction

The Kwaśniewski upside-down notation from [4] (see also [1, 2]) is being here taken for granted. For example $n$-th element of sequence $F$ is $F_n \equiv n_F$, consequently $n_F! = n_F \cdot (n-1)_F \cdot \ldots \cdot 1_F$ and a set $[n_F] = \{1, 2, \ldots, n_F\}$ however $[n]_F = \{1_F, 2_F, \ldots, n_F\}$. More about effectiveness of this notation see references in [4] and Appendix “On upside-down notation” in [6].
Throughout this paper we shall consequently use $F$ letter for a sequence of positive integers i.e. $F \equiv \{n_F\}_{n \geq 0}$ such that $n_F \in \mathbb{N}$ for any $n \in \mathbb{N} \cup \{0\}$.

1.1 Discrete $m$-dimensional $F$-Box

Let us define discrete $m$-dimensional $F$-box with edges sizes designated by natural numbers’ valued sequence $F$ as described below. These $F$-boxes from [6] where invented as a response to Kwaśniewski cobweb tiling problem posed in [1] (Problem 2 therein) and his question about visualization of this phenomenon.

**Definition 1** Let $F$ be a natural numbers’ valued sequence $\{n_F\}_{n \geq 0}$ and $m, n \in \mathbb{N}$ such that $n \geq m$. Then a set $V_{m,n}$ of points $v = (v_1, ..., v_m)$ of discrete $m$-dimensional space $\mathbb{N}^m$ given as follows

$$ V_{m,n} = [k_F] \times [(k + 1)_F] \times ... \times [n_F] \tag{1} $$

where $k = n - m + 1$ and $[s_F] = \{1, 2, ..., s_F\}$ is called $m$-dimensional $F$-box.

![Figure 1: F-Boxes $V_{2,3}$ and $V_{3,4}$ with sub-boxes.](image)

In the case of $n = m$ we write for short $V_{m,m} \equiv V_m$. Assume that we have a $m$-dimensional box $V_{m,n} = W_1 \times W_2 \times ... \times W_m$. Then a set $A = A_1 \times A_2 \times ... \times A_m$ such that

$$ A_s \subset W_s, \quad |A_s| > 0, \quad s = 1, 2, ..., m; $$

is called $m$-dimensional sub-box of $V_{m,n}$. Moreover, if for $s = 1, 2, ..., m$ these sets $A_s$ satisfy the following

$$ |A_s| = (\sigma \cdot s)_F $$

for any permutation $\sigma$ of set $\{1_F, 2_F, ..., m_F\}$ then $A$ is called $m$-dimensional sub-box of the form $\sigma V_m$. Compare with Figure 6.

Note, that the permutation $\sigma$ might be understood here as an orientation of sub-box’s position in the box $V_{m,n}$. Any two sub-boxes $A$ and $B$ are disjoint if its sets of points are disjoint i.e. $A \cap B = \emptyset$. 

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The number of points \( v = (v_1, ..., v_m) \) of \( m \)-dimensional box \( V_{m,n} \) is called \textit{volume}. It it easy to see that the \textit{volume} of \( V_{m,n} \) is equal to

\[
|V_{m,n}| = n_F \cdot (n - 1)_F \cdot ... \cdot (n - m + 1)_F = n_F^m
\]  

while for \( m = n \)

\[
|V_m| = |\sigma V_m| = m_F \cdot (m - 1)_F \cdot ... \cdot 1_F = m_F!
\]

### 1.2 Partition of discrete \( F \)-boxes

Let us consider \( m \)-dimensional \( F \)-box \( V_{m,n} \). A finite collection of \( \lambda \) pairwise disjoint sub-boxes \( B_1, B_2, ..., B_\lambda \) of the volume equal to \( \kappa \) is called \( \kappa \)-\textit{partition} of \( V_{m,n} \) if their set union of gives the whole box \( V_{m,n} \) i.e.

\[
\bigcup_{1 \leq j \leq \lambda} B_j = V_{m,n}, \quad |B_i| = \kappa, \quad i = 1, 2, ..., \lambda.
\]

\textbf{Convention.} In the following, we shall deal only with these \( \kappa \)-partition of \( m \)-dimensional boxes \( V_{m,n} \), which volume \( \kappa \) of sub-boxes is equal to the \textit{volume} of box \( V_m \) i.e. \( \kappa = |V_m| \).

Of course the box \( V_{m,n} \) has \( \kappa \)-partition not for all \( F \)-sequences \[8\]. Therefore we introduce the name: \( F \)-\textit{admissible} sequence which means that \( F \) satisfies the necessary and sufficient conditions for the box \( V_{m,n} \) to have \( \kappa \)-partitions. In order to proceed let us recall first what follows.

\textbf{Definition 2 (\[1, 2\])} Let \( F \) be a natural numbers' valued sequence \( F = \{n_F\}_{n \geq 0} \). Then \( F \)-nomial coefficient is identified with the symbol

\[
\binom{n}{m}_F = \frac{n_F!}{m_F!(n-m)_F!} = \frac{n_F^m}{m_F!}
\]

where \( n_F^0 = 0_F! = 1 \).

\textbf{Definition 3 (\[1, 2\])} A sequence \( F \) is called admissible if, and only if for any \( n, m \in \mathbb{N} \cup \{0\} \) the value of \( F \)-nomial coefficient is natural number or zero i.e.

\[
\binom{n}{m}_F \in \mathbb{N} \cup \{0\}
\]

while \( n \geq m \) else is zero.

Recall now also a combinatorial interpretation of the \( F \)-nomial coefficients in \( F \)-box reformulated form (consult Remark 5 in \[4\] and \[6\]). And note: these coefficients encompass among others Binomial, Gaussian and Fibonomial coefficients.
Fact 1 (Kwaśniewski [1, 2]) Let $F$ be an admissible sequence. Take any $m, n \in \mathbb{N}$ such that $n \geq m$, then the value of $F$-nomial coefficient $\binom{n}{m}_F$ is equal to the number of sub-boxes that constitute a $\kappa$-partition of $m$-dimensional $F$-box $V_{m,n}$ where $\kappa = |V_m|$.

**Proof.** This proof comes from Observation 3 in [1, 2] and was adopted here to the language of discrete boxes. Let us consider $m$-dimensional box $V_{m,n}$ with $|V_{m,n}| = \frac{n^m}{m^F}$. The volume of sub-boxes is equal to $\kappa = |V_m| = m_F!$. Therefore the number of sub-boxes is equal to

$$\frac{n^m}{m^F} = \binom{n}{m}_F$$

From definition of $F$-admissible sequence we have that the above is natural number. Hence the thesis $\blacksquare$

While considering any $\kappa$-partition of certain $m$-dimensional box we only assume that sub-boxes have the same volume. In the next section we shall take into account these partitions which sub-boxes have additionally established structure.

1.3 Tiling problem

Now, special $\kappa$-partitions of discrete boxes are considered. Namely, we deal with only these partitions of $m$-dimensional box $V_{m,n}$ which all sub-boxes are of the form $V_m$.

**Definition 4** Let $V_{m,n}$ be a $m$-dimensional $F$-box. Then any $\kappa$-partition into sub-boxes of the form $V_m$ is called tiling of $V_{m,n}$.

It was shown in [8] that just the admissibility condition (6) is not sufficient for the existence a tiling for any given $m$-dimensional box $V_{m,n}$. Kwaśniewski in his papers [1, 2] posed the following problem called Cobweb Tiling Problem, which was a starting point of the research with results being reported in the presents note.

**Problem 1 (Tiling)** Suppose now that $F$ is an admissible sequence. Under which conditions any $F$-box $V_{m,n}$ designated by sequence $F$ has a tiling? Find effective characterizations and/or find an algorithm to produce these tilings.

In the next sections we propose certain family $T_\lambda$ of sequences $F$. Then we prove that any $F$-box $V_{m,n}$, where $m, n \in \mathbb{N}$ designated by $F \in T_\lambda$ has a tiling with giving a construction of it.
1.4 Cobweb representation

In this section we recall [6] that discrete $F$-boxes $V_{m,n}$ are unique codings representing Cobwebs, introduced by Kwaśniewski [1, 2] as a special graded posets. Any poset might be represented as a Hasse digraph and this approach to tiling problem will be used throughout the paper.

Next we shall consider partitions of $m$-dimensional boxes as a partitions of cobwebs with $m$ levels into sub-cobwebs called blocks. In the following we quote some necessary notation of Cobwebs adopted to the tiling problem. For more on Cobwebs see source papers [1, 2, 4] and references therein.

**Definition 5** Let $F$ be a natural numbers’ valued sequence. Then a simple graph $\langle V, E \rangle$, such that $V = \bigcup_{k < s \leq n} \Phi_s$ and

$$E = \left\{ \{u, v\} : u \in \Phi_s \land v \in \Phi_{s+1} \land k \geq s < n \right\}$$

(7)

where $\Phi_s = \{1, 2, ..., s_F\}$ is called cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$.

![Figure 3: Cobweb layer $\langle \Phi_2 \rightarrow \Phi_4 \rangle$ designated by $F$-Natural numbers.](image)

Suppose that we have a cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ of $m$ levels $\Phi_s$, where $m = n - k + 1$. Then any cobweb layer $\langle \phi_1 \rightarrow \phi_m \rangle$ of $m$ levels $\phi_s$ such that

$$\phi_s \subseteq \Phi_s, \quad |\phi_s| = s_F, \quad s = 1, 2, ..., m;$$

(8)

is called cobweb block $P_m$ of layer $\langle \Phi_k \rightarrow \Phi_n \rangle$.

Additionally, one considers cobweb blocks obtained via permutation $\sigma$ of theirs levels’ order as follows (Compare with Figure 4).
Definition 6 Let a cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ with $m$ levels $\Phi_s$ be given, where $m = n - k + 1$. Then a cobweb block $P_m$ with $m$ levels $\phi_s$ such that

$\phi_s \subseteq \Phi_s$, $|\phi_s| = (\sigma \cdot s)_F$, $s = 1, 2, ..., m$; 

(9)

where $\sigma$ is a permutation of the set $\{1_F, 2_F, ..., m_F\}$ is called cobweb block of the form $\sigma P_m$.

Figure 4: Example of cobweb blocks $P_3$ and $\sigma P_3$.

While saying “a block $\sigma P_m$ of layer $\langle \Phi_k \rightarrow \Phi_n \rangle$” we mean that the number of levels in block and layer is the same i.e. $m = n - k + 1$ and each of levels of block are non-empty subsets of corresponding levels in the layer.

Assume that we have a cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$. A path $\pi$ from any vertex at first level $\Phi_k$ to any vertex at the last level $\Phi_n$, such that

$\pi = \{v_k, v_{k+1}, ..., v_n\}$, $v_s \in \Phi_s$, $s = k, k+1, ..., n$;

is noted as a maximal-path $\pi$ of $\langle \Phi_k \rightarrow \Phi_n \rangle$. In the same way we nominate maximal-path of cobweb block $\sigma P_m$.

Let $C_{\text{max}}(A)$ denotes a set of maximal-paths $\pi$ of cobweb block $A$. (Compare with [4]). Two cobweb blocks $A, B$ of layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ are max-disjoint or disjoint for short $([1] [2])$ if, and only if its sets of maximal-paths are disjoint i.e. $C_{\text{max}}(A) \cap C_{\text{max}}(B) = \emptyset$. The cardinality of set $C_{\text{max}}(A)$ is called size of block $A$.

Observation 1 ([6]) Let $F$ be a natural numbers' valued sequence and $k, n \in \mathbb{N}$. Then any $F$-box $V_{m,n}$ is uniquely represented by cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ and vice versa i.e.,

$V_{m,n} \leftrightarrow \langle \Phi_k \rightarrow \Phi_n \rangle$. 

(10)

where $k = n - m + 1$. 

Figure 5: $F$-Boxes of the form $\sigma V_2$ and cobweb blocks $\sigma P_2$. 

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Figure 6: $F$-Boxes of the form $\sigma V_3$ and cobweb blocks $\sigma P_3$.

**Proof.** Consider a cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ of $m$ levels $\Phi$ and $m$-dimensional box $V_{k,n}$. Observe that any maximal-path $\pi = (v_1, v_2, ..., v_m)$ of the layer corresponds to only one point $x = (x_1, x_2, ..., x_m)$ of $m$-dimensional box $V_{m,n}$, and vice versa, i.e.

$$\lfloor s_F \rfloor \ni x_s \iff v_s \in \lfloor s_F \rfloor, \quad s = 1, 2, ..., m;$$

and the number of these maximal-paths and points is the same (Compare with [4] and [6]) i.e.

$$|C_{\text{max}}(\langle \Phi_k \rightarrow \Phi_n \rangle)| = |V_{m,n}|$$

where $m = n - k + 1$. ■

Figure 7: Correspondence between tiling of $F$-box $V_{3,4}$ and $\langle \Phi_3 \rightarrow \Phi_4 \rangle$.

Next, we draw terminology of $F$-boxes’ partitions back to cobweb’s language, used in the next part of this note.

Take any cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ with $m$ levels. Then a set of $\lambda$ pairwise disjoint cobweb blocks $A_1, A_2, ..., A_\lambda$ of $m$ levels such that its size is equal to $\kappa$ and the union of $C_{\text{max}}(A_1), C_{\text{max}}(A_2), ..., C_{\text{max}}(A_\lambda)$ is equal to the set
$C_{\text{max}}(\langle \Phi_k \to \Phi_n \rangle)$ is called cobweb $\kappa$-partition. Finally, a $\kappa$-partition of layer $\langle \Phi_k \to \Phi_n \rangle$ with $m$ levels into cobweb blocks of the form $\sigma P_m$ is called cobweb tiling.

Let us sum it up with the following Table 1.

| Cobwebs | $F$-boxes |
|----------|------------|
| 1. Maximal-path $(v_1, ..., v_m) \in \langle \Phi_k \to \Phi_n \rangle$ | Point $(x_1, ..., x_m) \in V_{m,n}$ |
| 2. Cobweb layer $\langle \Phi_k \to \Phi_n \rangle$ | $F$-box $V_{m,n}$ |
| 3. Cobweb block $\sigma P_m \subset \langle \Phi_k \to \Phi_n \rangle$ | Sub-box $\sigma V_m \subset V_{m,n}$ |
| 4. Tiling of cobweb layer where $k = n - m + 1$. | Tiling of $F$-box |

## 2 Cobweb tiling sequences

Recall that for some $F$-admissible sequences there is no method to tile certain $F$-boxes $V_{m,n}$ or accordingly cobweb layers $\langle \Phi_k \to \Phi_n \rangle$ (no tiling property). For example see Figure 8 that comes from \[8\]. In the next part of this note, we define and consider only sequences with tiling property.

![Figure 8: Layer $\langle \Phi_5 \to \Phi_7 \rangle$ that does not have tiling with blocks $\sigma P_3$.](image)

**Definition 7** A cobweb admissible sequence $F$ such that for any $m, n \in \mathbb{N}$ the cobweb layer $\langle \Phi_k \to \Phi_n \rangle$ has a tiling is called cobweb tiling sequence.

Let $\mathcal{T}$ denotes the family of all cobweb tiling sequences. Characterization of whole family $\mathcal{T}$ is still open problem. Nevertheless we define certain subfamily $\mathcal{T}_\lambda \subset \mathcal{T}$ of non-trivial cobweb tiling sequences. This family contains among others Natural and Fibonacci numbers, Gaussian integers and others.
Notation 1 Let $T_\lambda$ denotes the family of natural number’s valued sequences $F \equiv \{n_F\}_{n \geq 1}$ such that for any $n$-th term of $F$ satisfies the following holds
\[ \forall m, k \in \mathbb{N}, \quad n_F = (m + k)_F = \lambda_K \cdot k_F + \lambda_M \cdot m_F \quad (11) \]
while $1_F \in \mathbb{N}$ and for certain coefficients $\lambda_K \equiv \lambda_K(k, m) \in \mathbb{N} \cup \{0\}$ and $\lambda_M \equiv \lambda_M(k, m) \in \mathbb{N} \cup \{0\}$.

Note, coefficients $\lambda_K$ and $\lambda_M$ might be considered as a natural numbers’ with zero valued infinite matrixes $\lambda_K \equiv [k_{ij}]_{i,j \geq 1}$ and $\lambda_M \equiv [m_{ij}]_{i,j \geq 1}$. Moreover the sequence $F \equiv \{n_F\}_{n \geq 0}$ is uniquely designated by these matrixes $\lambda_K, \lambda_M$ and first element $1_F \in \mathbb{N}$.

Corollary 1 Let a sequence $F \in T_\lambda$ with its coefficients’ matrixes $\lambda_K, \lambda_M$ and a composition $\vec{\beta} = \langle b_1, b_2, ..., b_k \rangle$ of number $n$ into $k$ nonzero parts be given. Then the following takes place
\[ n_F = 1_F \sum_{s=1}^{n} \lambda_s(\vec{\beta}) \cdot (b_s)_F \quad (12) \]
where
\[ \lambda_s(\vec{\beta}) = \lambda_K(b_s, b_{s+1} + ... + b_k) \prod_{i=1}^{s-1} \lambda_M(b_i, b_{i+1} + ... + b_k) \quad (13) \]
or equivalent
\[ \lambda_s(\vec{\beta}) = \lambda_M(b_{s+1} + ... + b_k, b_s) \prod_{i=1}^{s-1} \lambda_K(b_{i+1} + ... + b_k, b_i). \quad (14) \]

Proof. It is a straightforward algebraic induction exercise using property $(11)$ of the sequence $T_\lambda$. The first form $(13)$ of the coefficients $\lambda_s(\vec{\beta})$ comes from the following
\[ \left( b_1 + (n - b_1) \right)_F \Rightarrow \left( b_1 + b_2 + (n - b_1 - b_2) \right)_F \]
while the second one $(14)$ from
\[ \left( (n - b_k) + b_k \right)_F \Rightarrow \left( (n - b_k - b_{k-1}) + b_{k-1} + b_k \right)_F \]

If we take a vector $(1, 1, ..., 1)$ of $n$ ones i.e. $b_s = 1$ for any $s = 1, 2, ..., n$; then we obtain alternative formula to compute elements of the sequence $F$. 

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Corollary 2 Let $F \in T_\lambda$ be given. Then $n$-th element of the sequence $F$ satisfies
\[ n_F = 1_F \cdot \sum_{s=1}^{n} \lambda_K(1, n-s) \prod_{i=1}^{s-1} \lambda_M(1, n-i) \tag{15} \]
for any $n \in \mathbb{N}$.

Corollary 3 Let any sequence $F \in T_\lambda$ be given. Then for any $n, k \in \mathbb{N} \cup \{0\}$ such that $n \geq k$, the $F$-nomial coefficients satisfy below recurrence identity
\[ \binom{n}{k}_F = \lambda_K \binom{n-1}{k-1}_F + \lambda_M \binom{n-1}{k}_F \tag{16} \]
where $\binom{n}{0}_F = 1$.

**Proof.** Take any $F \in T_\lambda$ and $n \in \mathbb{N} \cup \{0\}$. Then from (11) of $T_\lambda$ and for any $m, k \in \mathbb{N} \cup \{0\}$ such that $m + k = n$ we have that $n$-th element of the sequence $F$ satisfies following recurrence
\[ n_F = (k + m)_F = \lambda_K \cdot k_F + \lambda_M \cdot m_F \]
Multiply both sides of above equation by $\frac{(n-1)_F!}{k_F! \cdot m_F!}$ to get
\[ \frac{n_F!}{k_F! \cdot m_F!} = \frac{(n-1)_F!}{(k-1)_F! \cdot m_F!} + \frac{(n-1)_F!}{k_F! \cdot (m-1)_F!} \]
And from Definition 2 of $F$-nomial coefficients we have
\[ \binom{n}{k}_F = \lambda_K \binom{n-1}{k-1}_F + \lambda_M \binom{n-1}{k}_F \square \]

It turns out that the recurrence formula (16) gives us a method to generating tilings of any layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ designated by sequence $F \in T_\lambda$.

**Theorem 1** Let $F$ be a sequence of $T_\lambda$ family. Then $F$ is cobweb tiling.

![Figure 9: Picture of Theorem 1 proof’s idea.](image-url)

**Proof.** Suppose that we have a cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ with $m$ levels designated by sequence $F$ from $T_\lambda$ family and $m = n - k$. Consider $\Phi_n$ level
with \( n_F \) vertices. From (11) we have that the number of vertices at this level is the sum of \( \lambda_M \cdot m_F \) and \( \lambda_K \cdot k_F \). Therefore we separate them by cutting into two disjoint subsets as illustrated by Figure 9 and cope at first \( \lambda_M \cdot m_F \) vertices in Step 1. Then we shall cope the rest \( \lambda_K \cdot k_F \) ones in Step 2.

**Figure 10: Picture of Theorem 1 proof’s Step 1.**

**Step 1.** Temporarily we have \( \lambda_M \cdot m_F \) fixed vertices on \( \Phi_n \) level to consider (Figure 10). Let us cover them \( \lambda_M \) times by \( m_F \) level of block \( \sigma P_m \), which has exactly \( m_F \) vertices. If \( \lambda_M = 0 \) we skip this step. What was left is the layer \( \langle \Phi_{k+1} \rightarrow \Phi_{n-1} \rangle \) and we might eventually partition it with smaller disjoint blocks \( \sigma P_{m-1} \) in the next induction step.

**Figure 11: Picture of Theorem 1 proof’s Step 2.**

**Step 2.** Consider now the second complementary situation, where we have \( \lambda_K \cdot k_F \) vertices on \( \Phi_n \) level being fixed (Figure 11). If \( \lambda_K = 0 \) we skip this step. Observe that if we move this level lower than \( \Phi_{k+1} \) level, we obtain exactly \( \lambda_K \) the same layers \( \langle \Phi_k \rightarrow \Phi_{n-1} \rangle \) to be partitioned with disjoint blocks of the form \( \sigma P_m \). This “move” operation is just permutation \( \sigma \) of levels’ order.

**Recapitulation.** The layer \( \langle \Phi_{k+1} \rightarrow \Phi_n \rangle \) might be partitioned into \( \sigma P_m \) blocks if \( \langle \Phi_{k+1} \rightarrow \Phi_{n-1} \rangle \) might be partitioned into \( \sigma P_{m-1} \) and \( \langle \Phi_k \rightarrow \Phi_{n-1} \rangle \) into \( \sigma P_m \) again. Continuing these steps by induction, we are left to prove that \( \langle \Phi_k \rightarrow \Phi_k \rangle \) might be partitioned into \( \sigma P_1 \) blocks and \( \langle \Phi_1 \rightarrow \Phi_m \rangle \) into \( \sigma P_m \) ones, what is trivial.

**Observation 2** Let \( F \) be a cobweb tiling sequence from the family \( T_\lambda \). Then
the number \( \{ {n \atop k} \}_F \) of different tilings of layer \( \langle \Phi_k \to \Phi_n \rangle \) where \( n, k \in \mathbb{N} \), \( n, k \geq 1 \) is equal to:

\[
\{ {n \atop k} \}_F = \frac{n_F!}{(m_F!)^{\lambda_M} \cdot ((k-1)_F)!^{\lambda_K}} \cdot \left( \{ {n-1 \atop k} \}_F \right)^{\lambda_M} \cdot \left( \{ {n-1 \atop k-1} \}_F \right)^{\lambda_K}
\]

(17)

where \( \{ {n \atop 1} \}_F = 1 \) and \( \{ {n \atop 1} \}_F = 1 \).

**Proof.** According to steps of the proof of Theorem 1 we might choose \( m_F \) vertices \( \lambda_M \) times at \( n \)-th level and next \( (k-1)_F \) vertices \( \lambda_K \) times out of \( n_F \) ones in \( \frac{n_F!}{(m_F!)^{\lambda_M} \cdot ((k-1)_F)!^{\lambda_K}} \) ways. Next recurrent steps of the proof of Theorem 1 result in formula (17) via product rule of counting.

\[ \square \]

Note that \( \{ {n \atop k} \}_F \) is not the number of all different tilings of the layer \( \langle \Phi_k \to \Phi_n \rangle \) i.e. \( \{ {n \atop k} \}_F \leq \{ {n \atop k} \}_F \) as computer experiments show [8]. There are much more other tilings with blocks \( \sigma F_m \).

### 3 Cobweb multi tiling

In this section, more general case of the tiling problem is considered. For that to do we introduce the so-called multi \( F \)-nomial coefficients that counts blocks of multi-block partitions.

**Definition 8** Let natural numbers’ valued sequence \( F \equiv \{ n_F \}_{n \geq 0} \) and a composition \( \langle b_1, b_2, ..., b_k \rangle \) of the number \( n \) be given. Then the multi \( F \)-nomial coefficient is identified with the symbol

\[
\left( \begin{array}{c} n \\ b_1, b_2, ..., b_k \end{array} \right) \equiv \frac{n_F!}{(b_1)_F! \cdot \cdot \cdot (b_k)_F!}
\]

(18)

while \( n = b_1 + b_2 + ... + b_k \).

**Corollary 4** Let \( F \) be any \( F \)-cobweb admissible sequence. Then value of the multi \( F \)-nomial coefficient is natural number or zero i.e.

\[
\left( \begin{array}{c} n \\ b_1, b_2, ..., b_k \end{array} \right) \in \mathbb{N} \cup \{0\}
\]

(19)

for any \( n, b_1, b_2, ..., b_k \in \mathbb{N} \) such that \( n = b_1 + b_2 + ... + b_k \).

For the sake of forthcoming combinatorial interpretation of multi \( F \)-nomial coefficients we introduce the following notation.
Definition 9  Let a cobweb layer \( \langle \Phi_1 \rightarrow \Phi_n \rangle \) of \( n \) levels \( \Phi_s \) and a composition \( \langle b_1, b_2, ..., b_k \rangle \) of number \( n \) into \( k \) non-zero parts be given. Then any cobweb layer \( \langle \phi_1 \rightarrow \phi_n \rangle \) of \( n \) levels \( \phi_s \) such that
\[
\phi_s \subseteq \Phi_s, \quad s = 1, 2, ..., n;
\]
where the cardinality of \( \phi_s \) is equal to \( s \)-th element of the vector \( L \) given as follows
\[
L = \sigma \cdot \langle 1, 2, ..., b_1, 1, 2, ..., b_2, 1, 2, ..., b_k \rangle
\]
for any permutation \( \sigma \) of a set \([n]\) is called cobweb multi-block of the form \( \sigma P_{b_1, b_2, ..., b_k} \).

Example 1
Take a sequence \( F \) of next natural numbers i.e. \( n_F = n \) and cobweb layer \( \langle \Phi_1 \rightarrow \Phi_4 \rangle \) designated by \( F \). A sample multi tiling of the layer \( \langle \Phi_1 \rightarrow \Phi_4 \rangle \) with the help of \( \left( \begin{array}{c} 4 \\ 2 \end{array} \right)_F = 6 \) disjoint multi blocks of the form \( \sigma P_{2,2} \) is in Figure 13.

Observation 3  Let \( \langle \Phi_1 \rightarrow \Phi_n \rangle \) be a cobweb layer and \( \langle b_1, ..., b_k \rangle \) be a composition of the number \( n \) into \( k \) nonzero parts. Then the value of multi \( F \)-nomial coefficient \( \left( \begin{array}{c} n \\ b_1, b_2, ..., b_k \end{array} \right)_F \) is equal to the number of blocks that form the cobweb \( \kappa \)-partition, where \( \kappa = |C_{\text{max}}(P_{b_1, ..., b_k})| \).

Proof. The proof is natural extension of Observation 3 in [1, 2]. The number of maximal paths in layer \( \langle \Phi_1 \rightarrow \Phi_n \rangle \) is equal to \( n_F! \). However the number of maximal paths in any multi block \( \sigma P_{b_1, b_2, ..., b_k} \) is \( (b_1)_F! \cdot (b_2)_F! \cdot ... \cdot (b_k)_F! \). Thus the number of such blocks is equal to
Figure 13: Sample multi tiling of layer $\langle \Phi_1 \rightarrow \Phi_4 \rangle$ from Example 2.

$$\frac{n_F!}{(b_1)_F! \cdot (b_2)_F! \cdot \ldots \cdot (b_k)_F!}$$

where $n = b_1 + b_2 + \ldots + b_k$ for any $n, k \in \mathbb{N}$

Of course for $k = 2$ we have

$$\left( \begin{array}{c} n \\ b, n-b \end{array} \right)_F = \left( \begin{array}{c} n \\ b \end{array} \right)_F = \left( \begin{array}{c} n \\ n-b \end{array} \right)_F$$

Note. For any permutation $\sigma$ of the set $[k]$ the following holds

$$\left( \begin{array}{c} n \\ b_1, b_2, \ldots, b_k \end{array} \right)_F = \left( \begin{array}{c} n \\ b_{\sigma 1}, b_{\sigma 2}, \ldots, b_{\sigma k} \end{array} \right)_F$$

as is obvious from Definition \[\text{of the multi F-nomial symbol. i.e.} \]

$$\frac{n_F!}{(b_1)_F! \cdot (b_2)_F! \cdot \ldots \cdot (b_{\sigma k})_F!}$$

Let us observe also that for any natural $n, k$ and $b_1 + \ldots + b_m = n - k$ the following holds

$$\left( \begin{array}{c} n \\ k \end{array} \right)_F \cdot \left( \begin{array}{c} n-k \\ b_1, b_2, \ldots, b_m \end{array} \right)_F = \left( \begin{array}{c} n \\ k, b_1, \ldots, b_m \end{array} \right)_F$$

Corollary 5 Let $F \in T_\lambda$ and a composition $\vec{\beta} = \langle b_1, \ldots, b_k \rangle$ of number $n$ into $k$ parts be given. Then the multi $F$-nomial coefficients satisfy the following recurrence relation

$$\left( \begin{array}{c} n \\ b_1, b_2, \ldots, b_k \end{array} \right)_F = \sum_{s=1}^{k} \lambda_s(\vec{\beta}) \cdot \left( \begin{array}{c} n-1 \\ b_1, \ldots, b_{s-1}, b_s-1, b_{s+1}, \ldots, b_k \end{array} \right)_F$$
for coefficients $\lambda_s(\vec{\beta})$ from (13) and for any $n = b_1 + \ldots + b_k$ and $(\binom{n}{0}, \ldots, 0)_F = 1$.

**Proof.** Take any $F \in T_\lambda$ and a composition $\vec{\beta} = (b_1, \ldots, b_k)$ of the number $n$. Then from Corollary 1 we have that for certain coefficients $\lambda_s(\vec{\beta})$ any $n$-th element of the sequence $F$ satisfies

$$n_F = \sum_{s=1}^{k} \lambda_s(\vec{\beta}) \cdot (b_s)_F$$

If we multiply both sides by $\frac{(n-1)_F!}{(b_1)_F! \cdot \ldots \cdot (b_k)_F!}$ then we obtain

$$\left( \begin{array}{c} n \\ b_1, \ldots, b_k \end{array} \right)_F = \sum_{s=1}^{k} \lambda_s(\vec{\beta}) \cdot \frac{(n-1)_F!}{(b_1)_F! \cdot \ldots \cdot (b_s-1)_F! \cdot (b_{s+1})_F! \cdot \ldots \cdot (b_k)_F!}$$

Hence the thesis □

**Theorem 2** Let any sequence $F \in T_\lambda$ be given. Then the sequence $F$ is cobweb multi tiling i.e. any layer $\{\Phi_1 \rightarrow \Phi_n\}$ might be partitioned into multi-blocks of the form $\sigma P_{b_1, b_2, \ldots, b_k}$ such that $b_1 + \ldots + b_k = n$.

**Proof.** Take any cobweb layer $\{\Phi_1 \rightarrow \Phi_n\}$ designated by sequence $F \in T_\lambda$ and a number $k \in \mathbb{N}$. We need to partition the layer into disjoint multi-blocks of the form $\sigma P_{b_1, b_2, \ldots, b_k}$.

Consider level $\Phi_n$ with $n_F$ vertices. From Corollary 1 we have that the number of vertices at this level is the following sum

$$n_F = \sum_{s=1}^{k} \lambda_s(\vec{\beta}) \cdot (b_s)_F$$

for certain coefficients $\lambda_s(\vec{\beta})$ where $1 \leq s \leq k$ and $\vec{\beta} = (b_1, b_2, \ldots, b_k)$.

Therefore let us separate these $n_F$ vertices by cutting into $k$ disjoint subsets as illustrated by Fig. 14 and cope at first $\lambda_1 \cdot (b_1)_F$ vertices in Step 1, then
\( \lambda_2 \cdot (b_2)_F \) ones in Step 2 and so on up to the last \( \lambda_k \cdot (b_k)_F \) vertices to consider in the last \( k \)-th step. If any \( \lambda_i = 0 \) we skip the \( i \)-th step.

**Step 1.** Temporarily we have \( \lambda_1 \cdot (b_1)_F \) fixed vertices at level \( \Phi_n \) to consider. Let us cover them \( \lambda_1 \) times by \( (b_1)_F \)-th level of block \( P_{b_1,b_2,...,b_k} \), which has exactly \( (b_1)_F \) vertices. What was left is the layer \( \langle \Phi_1 \to \Phi_{n-1} \rangle \) and we might partition it with smaller disjoint blocks \( \sigma P_{1-1,b_2,...,b_k} \) in the next induction step.

Note. In the next induction steps we use smaller blocks \( \sigma P \) without levels which we have been already used in previous steps (disjoint of blocks condition).

**Step 2.** Consider now the second situation, where we have \( \lambda_2 \cdot (b_2)_F \) vertices at level \( \Phi_n \) being fixed. We cover them \( \lambda_2 \) times by \( (b_1+b_2)_F \)-th level of block \( P_{b_1,b_2,...,b_k} \), which has \( (b_2)_F \) vertices. Then we obtain smaller layer \( \langle \Phi_1 \to \Phi_{n-1} \rangle \) to be partitioned with blocks \( \sigma P_{b_1,b_2-1,b_3,...,b_k} \).

And so on up to ...

**Step \( k \).** Analogously to previous steps, we cover the last \( \lambda_{b_k} \) vertices by the last \( (b_1+b_2+...+b_k) = n \)-th level of block \( P_{b_1,b_2,...,b_k} \), obtaining smaller layer \( \langle \Phi_1 \to \Phi_{n-1} \rangle \) to be partitioned with blocks \( \sigma P_{b_1,...,b_{k-1},b_k-1} \).

**Conclusion.**
The layer \( \langle \Phi_1 \to \Phi_n \rangle \) might be partitioned into blocks \( \sigma P_{b_1,b_2,...,b_k} \) if \( \langle \Phi_1 \to \Phi_{n-1} \rangle \) might be partitioned into \( \sigma P_{b_1-1,b_2,...,b_k} \) and \( \langle \Phi_1 \to \Phi_{n-1} \rangle \) into \( \sigma P_{b_1,b_2-1,b_3,...,b_k} \) again and so on up to the layer \( \langle \Phi_1 \to \Phi_{n-1} \rangle \) which might be partitioned into \( \sigma P_{b_1,...,b_{k-1},b_k-1} \). Continuing these steps by induction, we are left to prove that \( \langle \Phi_1 \to \Phi_k \rangle \) might be partitioned into blocks \( \sigma P_{1,1,...,1} \) or \( \langle \Phi_1 \to \Phi_1 \rangle \) by \( \sigma P_{1,0,...,0} \) ones, which is trivial.

### 4 Family \( T_\lambda(\alpha, \beta) \) of cobweb tiling sequences

In this section a specific family of cobweb tiling sequences \( F \in T_\lambda \) is presented as an exemplification of a might be source method. We assume that coefficients \( \lambda_K \) and \( \lambda_M \) of \( F \in T_\lambda \) take a form

\[
\lambda_M(k,m) = \alpha^k \quad \lambda_K(k,m) = \beta^m
\]  

(25)

while \( \alpha, \beta \in \mathbb{N} \).

**Notation 2** Let \( T_\lambda(\alpha, \beta) \) denotes a family of natural numbers’ valued sequences \( F \equiv \{ n_F \}_{n \geq 0} \) constituted by \( n \)-th coefficients of the generating function \( \mathcal{F}(x) \) expansion i.e. \( n_F = [x^n] \mathcal{F}(x) \), where

\[
\mathcal{F}(x) = 1_F \cdot \frac{x}{(1-\alpha x)(1-\beta x)}
\]

(26)

for certain \( \alpha, \beta \in \mathbb{N} \cup \{ 0 \} \) and \( 1_F \in \mathbb{N} \).
1. If \((\alpha = \beta)\), then \(F(x) = 1_F \cdot \frac{x}{1-ax} + \alpha x F(x)\) which leads to
\[
n_F = 1_F \cdot n \cdot \alpha^{n-1} \quad n \geq 1 \tag{27}
\]

2. If \((\alpha \neq \beta)\), then \(F(x) = \frac{1_F}{\alpha - \beta} \left( \frac{1}{1-ax} - \frac{1}{1-bx} \right)\) gives us
\[
n_F = \frac{1_F}{\alpha - \beta} (\alpha^n - \beta^n) \quad n \geq 1 \tag{28}
\]

**Proposition 1** Let \(F \in T_k(\alpha, \beta)\) and composition \(\bar{b} = \langle b_1, b_2, \ldots, b_k \rangle\) of the number \(n\) into \(k\) non-zero parts be given. Then any \(n\)-th element of the sequence \(F\) satisfies the following recurrence identity
\[
n_F = \left( \sum_{s=1}^{k} b_s \right)_F = \sum_{s=1}^{k} \lambda_s(\bar{b}) \cdot (b_s)_F \tag{29}
\]
where
\[
\lambda_s(\bar{b}) = \alpha^{b_{s+1} + \ldots + b_k - b_s} \cdot \beta^{b_1 + \ldots + b_{s-1}}
\]
for any \(n = b_1 + \ldots + b_k\).

**Proof.** Take any composition \(\bar{b} = \langle b_1, b_2, \ldots, b_k \rangle\) of the number \(n \in \mathbb{N}\) into \(k\) nonzero parts i.e. \(b_1 + b_2 + \ldots + b_k = n\).

1. If \((\alpha = \beta)\) then from (27)
\[
\left( \sum_{s=1}^{k} b_s \right)_F = 1_F \left( \sum_{s=1}^{k} b_s \right) \cdot \alpha^{n-1} = \sum_{s=1}^{k} 1_F b_s \alpha^{b_s-1} \alpha^{n-b_s} = \sum_{s=1}^{k} (b_s)_F \alpha^{n-b_s}
\]

2. If \((\alpha \neq \beta)\) then from (28)
\[
\left( \sum_{s=1}^{k} b_s \right)_F = \frac{1_F}{\alpha - \beta} \alpha^{b_1 + \sum_{s=2}^{k} b_s} - \frac{1_F}{\alpha - \beta} \beta^{b_k + \sum_{s=1}^{k-1} b_s} = A + B
\]
Next, denote \(S_\pm(m)\) for \(1 < m < k\) such that \(S_+(m) + S_-(m) = 0\) as follows \(S_\pm(m) = \pm \frac{1_F}{\alpha - \beta} \alpha^{\sum_{s=m+1}^{k} b_s} \cdot \beta^{\sum_{s=1}^{m-1} b_s}\). Then observe that if we add to the \(A + B\) the sum of \(S_\pm(m)\) where \(1 < m < k\) i.e.
\[
A + B = A + B + \sum_{1 < j < k} S_+(j) + S_-(j)
\]
then we obtain
\[
\begin{cases}
A + S_-(1) = (b_1)_F \cdot \alpha \sum_{s=2}^{k} b_s \beta^0 \\
S_+(1) + S_-(2) = (b_2)_F \cdot \alpha \sum_{s=3}^{k} b_s \cdot \beta b_1 \\
\ldots \\
S_+(k-1) + B = (b_k)_F \cdot \alpha^0 \cdot \beta^{b_{k-1}} b_s
\end{cases}
\]
And finally
\[
\left( \sum_{s=1}^{k} b_s \right)_F = A + B = \sum_{s=1}^{k} (b_s)_F \cdot \alpha^{b_{s+1} + \ldots + b_k} \beta^{b_1 + \ldots + b_{s-1}}
\]

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Note. If $k = 2$ then for any $m, b \in \mathbb{N} \cup \{0\}$ we have

\[(m + b)F = \lambda_mmF + \lambda_bbF = \alpha^mF + \beta^mbF \quad (30)\]

Let us compare above with condition (11) for sequences that are cobweb tiling from family $T_\lambda$ and let us sum up this with the following corollary.

**Corollary 6** Let family of sequences $T_\lambda(\alpha, \beta)$ and family $T_\lambda$ of cobweb tiling sequences be given. Then the following takes place

\[T_\lambda(\alpha, \beta) \subset T_\lambda \quad (31)\]

thus any sequence $F \in T_\lambda(\alpha, \beta)$ is cobweb tiling.

**Proof.** We only need to show that $T_\lambda(\alpha, \beta) \neq T_\lambda$. As an example we show that the sequence $F$ of Fibonacci numbers is cobweb tiling of the form $T_\lambda$ but does not belong to the family $T_\lambda(\alpha, \beta)$. Ones show that $n$-th element of the Fibonacci numbers satisfies

\[n_F = \frac{1}{\alpha - \beta} (\alpha^n - \beta^n) \quad (32)\]

but $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are not natural numbers - compare with (26). However its elements satisfy another equivalent relation for any $m, k \in \mathbb{N} \cup \{0\}$

\[(k + m)_F = (m - 1)_F \cdot kF + (k + 1)_F \cdot mF \quad (33)\]

Therefore $F \in T_\lambda$ and $F \notin T_\lambda(\alpha, \beta)$. Hence the thesis \[\blacksquare\]

**Corollary 7** Let $F \in T_\lambda$ be given. Then for any $n, k \in \mathbb{N} \cup \{0\}$ the following holds

\[(k \cdot n)_F = \left(\sum_{k}^{n} n + n + \ldots + n\right)_F = n_F \cdot \sum_{s=1}^{k} \alpha^{(k-s)n} \beta^{(s-1)n} \quad (34)\]

From Proposition 11 we obtain an another explicit formula for $n$-th element of the sequence $F \in T_\lambda$ i.e.

\[n_F = (n \cdot 1)_F = 1_F \cdot \sum_{s=1}^{n} \alpha^{(n-s)} \beta^{(s-1)}. \quad (35)\]

5 Examples of cobweb tiling sequences

In this section we are going to show a few examples of cobweb-tiling sequences. Throughout this part we shall consequently use the condition convention: $n = k + m.$
5.1 Examples of $T_\lambda(\alpha, \beta)$ family

1. Natural numbers

Putting $\alpha = \beta = 1$ gives us a sequence $n_F = 1_F \cdot n$ with the recurrence $(k + m)_F = k_F + m_F$. If $1_F = 1$ then we obtain Natural numbers with Binomial coefficients’ recurrence:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

2. Powers’ sequence

If $\alpha = 0$, $\beta = 1$ then $n_F = q^n$ and $(k + m)_F = q^m \cdot k_F$ with its $F$-nomial coefficients’ recurrence

$$\binom{n}{k}_F = q^m \binom{n-1}{k-1}_F + q^k \binom{n-1}{k}$$

3. Gaussian numbers

If $\alpha = 1$, $\beta = q$ then $n_F = \frac{1_F}{1-q} (1 - q^n)$ and $(k + m)_F = k_F + q^m m_F$ with the recurrence for Gaussian coefficients

$$\binom{n}{k}_q = \binom{n-1}{k-1}_F + q^k \binom{n-1}{k}$$

4. Modified Gaussian integers

For $\alpha = \beta = q \in \mathbb{N}$ we have $n_F = 1_F \cdot n \cdot q^{n-1}$ and $(k + m)_F = q^m k_F + q^k m_F$ with the recurrence

$$\binom{n}{k}_F = q^m \binom{n-1}{k-1}_F + q^k \binom{n-1}{k}$$

5.2 Fibonacci numbers

In the following, we prove that sequence of Fibonacci numbers is tiling sequence i.e. any cobweb layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ might be partitioned into blocks of the form $\sigma P_m$.

Definition 10 Let $F(p)$ be a natural numbers’ valued sequence such that for any $k, m \in \mathbb{N} \cup \{0\}$ its elements satisfy the following relation

$$(k + m)_F = (m - 1)_F \cdot k_F + (k + 1)_F \cdot m_F$$

while $1_F = 1$ and $2_F = p$.

From Theorem [11] and condition [11] on the sequence $T_\lambda$, we have that $F(p)$ is cobweb tiling. Moreover, it is easy to see, that explicit formula for $n$-th element of $F(p)$ is

$$n_F = \frac{1}{\sqrt{5F^2 + 4}} (\phi_1^n - \phi_2^n)$$
where $\phi_{1,2} = \frac{2F\pm\sqrt{2F^2+4}}{2}$ and $1_F = 1$ while $2_F = p$.

**Examples of** $F(p) = \{n_F\}_{n \geq 0}$

- $F(1) \equiv (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots) \equiv$ Fibonacci numbers
- $F(2) \equiv (0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, \ldots)$
- $F(3) \equiv (0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, \ldots)$
- $F(4) \equiv (0, 1, 4, 17, 72, 305, 1292, 5473, 23184, 98209, 416020, \ldots)$

**Corollary 8** *The sequence of Fibonacci numbers is cobweb tiling.*

**Proof.** If we put $1_F = 2_F = 1$ in (36) then we obtain Fibonacci numbers and well-known recurrence relation for Fibonomial coefficients [7]

$$\binom{n}{k}_F = (m - 1)_F \binom{n - 1}{k - 1}_F + (k + 1)_F \binom{n - 1}{k}_F \quad \blacksquare$$

**Observation 4** *Let $F$ be a sequence of the form $F(p)$. Take any composition $(b_1, b_2, \ldots, b_k)$ of a number $n$ into $k$ nonzero parts. Then $n$-th element of $F$ satisfies*

$$n_F = \sum_{s=1}^{k} (b_s)_F \cdot \prod_{i=1}^{s-1} (b_i + 1)_F \cdot (b_{s+1} + \ldots + b_k - 1)_F \quad (39)$$

while $n, k \in \mathbb{N}$.

**Proof.** It is a straightforward algebraic exercise using an idea from the proof of Corollary 1. If we use the substitutions $m = a + b$ in the formula (36) then we obtain the case of 3 terms

$$(k + m)_F = (k + a + b)_F = \lambda_K k_F + \lambda_a a_F + \lambda_b b_F$$

where $\lambda_K = (a + b - 1)_F$, $\lambda_a = (k + 1)_F \cdot (b - 1)_F$ and $\lambda_b = (k + 1)_F \cdot (a + 1)_F$. And so on by induction $\blacksquare$

### 6 Cobweb tiling problem as a particular case of clique problem

Recall that the clique problem is the problem of determining whether a graph contains a clique of at least a given size $d$. In this section, we show that the cobweb tiling problem might be considered as the clique problem in specific graph. Namely reformulation of the $F$-cobweb i.e. $F$-boxes tiling
problem into a clique problem of a graph specially invented for that purpose - is proposed.

Suppose that we have a cobweb layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \) designated by any sequence \( F \). Let \( B (\langle \Phi_k \rightarrow \Phi_n \rangle) \) denotes a family of all blocks of the form \( \sigma P_m \), where \( m = n - k + 1 \) of that layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \) and assume that \( b_{k,n} \) is a cardinality of that family i.e. \( b_{k,n} = |B (\langle \Phi_k \rightarrow \Phi_n \rangle)| \).

**Observation 5** The number \( b_{k,n} \) is given by the following formula

\[
b_{k,n} = \sum_{\sigma \in S_m} \prod_{s=1}^{m} \binom{(k + s - 1)F}{(\sigma \cdot s)F}
\]

where \( m = n - k + 1 \) and \( S_m \) is a set of permutations \( \sigma \) of the set \( \{k_F, (k + 1)_F, ..., n_F\} \).

**Proof.** Suppose that we have the layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \). Take any permutation \( \sigma \in S_m \) of \( m \) levels of the block \( \sigma P_m \). Let \( s \in [m] \); for such order of levels, cope \( (\sigma \cdot s)_F \) vertices by \( s \)-th element of the block \( \sigma P_m \) from all of vertices i.e. \( (k + s - 1)_F \) of the \( (k + s) \)-th level in the layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \). To the end sum the above after all of permutation \( \sigma \) ■

Let us define now a simple not directed graph \( G(\langle \Phi_k \rightarrow \Phi_n \rangle) = (V, E) \) such that set of vertices is \( V \equiv B (\langle \Phi_k \rightarrow \Phi_n \rangle) \) i.e. for any cobweb block \( \beta \) we have that

\( \beta \in B (\langle \Phi_k \rightarrow \Phi_n \rangle) \iff v_{\beta} \in V \)

while set of edges \( E \) is defined as follows

\[\{v_{\alpha}, v_{\beta}\} \in E \iff C_{max}(\alpha) \cap C_{max}(\beta) = \emptyset\]

for any two cobweb blocks \( \alpha, \beta \in B (\langle \Phi_k \rightarrow \Phi_n \rangle) \) where \( C_{max}(\gamma) \) is a set of maximal paths of block \( \gamma \).

**Corollary 9** Cobweb tiling problem of layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \) is the clique of size \( d \) in graph \( G(\langle \Phi_k \rightarrow \Phi_n \rangle) \) problem, where \( d = m_F! \).

**Proof.** Suppose that we have a cobweb layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \) and consider the family \( B (\langle \Phi_k \rightarrow \Phi_n \rangle) \) of all blocks of the form \( \sigma P_m \) of layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \), where \( m = n - k + 1 \).

Assume that a cobweb tiling of layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \) contains \( d \) pairwise disjoint blocks of the form \( \sigma P_m \), where \( m = n - k + 1 \). From combinatorial interpretation of \( F \)-nomial coefficients we have that \( d = \binom{n}{m_F} \). Thus if the family \( B (\langle \Phi_k \rightarrow \Phi_n \rangle) \) contains \( d \) blocks that are pairwise disjoint then the layer has tiling \( \pi \). In the other words, if a graph \( G \) has \( d \) vertices that are pairwise incidence then of course has a clique \( \chi \) of size \( d \). Moreover this clique \( \chi \) of graph \( G \) corresponds to the cobweb tiling \( \pi \) of layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \) and vice versa i.e. \( \pi \leftrightarrow \chi \) ■
Corollary 10 If a graph $G(\langle \Phi_k \rightarrow \Phi_n \rangle)$ has a clique $\chi$ of size $d = m_F!$ then $\chi$ is maximal clique of the graph.

Corollary 11 The number of all cobweb tilings of layer $\langle \Phi_k \rightarrow \Phi_n \rangle$ is equal to the number of all maximal cliques in graph $G(\langle \Phi_k \rightarrow \Phi_n \rangle)$.

7 Map of cobweb sequences

Here down in Figure 15 we present a Venn type diagram map of cobweb sequences. Note that the boundary of the whole family of Cobweb Tiling sequences is still not known (open problem).

Cobweb Admissible sequences family $\mathcal{A}$ is defined in [9], GCD-morphic sequences family in [8]. Subfamily $T_\lambda$ of cobweb tiling sequences $T$ is introduced in this note.

1. $A = (1, 3, 5, 7, 9, ...)$;
2. $B = (1, 2, 2, 2, 1, 4, 1, 2, ...) = B_{2,2} \cdot B_{2,3}$;
3. $C = (1, 2, 2, 1, 2, 2, 1, ...)$;
4. $E = (1, 2, 3, 2, 1, 6, 1, ...) = B_{2,2} \cdot B_{3,3}$;
5. $F = (1, 2, 1, 2, 1, 2, ...) = B_{2,2}$;
6. Natural numbers, Fibonacci numbers;
7. $G = 1, 4, 12, 32, 80, 192, 448, 1024, ...$ (Example 4 in Section 5);

Sequences $B_{c,M}$ and $A_{c,t}$ are defined in [8].
Additional information

In [12] we deliver some computer applications for generating tilings of any layer \( \langle \Phi_k \rightarrow \Phi_n \rangle \) based on an algorithm from the proof of Theorem [1]. There one may find also a visualization application for drawing all multi blocks of the form \( \sigma F_{k,n-k} \) of a layer \( \langle \Phi_1 \rightarrow \Phi_n \rangle \).

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\[ \langle \Phi_2 \rightarrow \Phi_5 \rangle = \]