A SINGULAR LIMIT PROBLEM FOR CONSERVATION LAWS RELATED TO THE ROSENAU-KORTEWEG-DE VRIES EQUATION

GIUSEPPE MARIA COCLITE AND LORENZO DI RUVO

Abstract. We consider the Rosenau-Korteweg-de Vries equation, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to discontinuous weak solutions of the Burgers equation. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the $L^p$ setting.

1. Introduction

Dynamics of shallow water waves that is observed along lake shores and beaches has been a research area for the past few decades in oceanography (see [1, 39]). There are several models proposed in this context: Boussinesq equation, Peregrine equation, regularized long wave (RLW) equation, Kawahara equation, Benjamin-Bona-Mahoney equation, Bona-Chen equation etc. These models are derived from first principles under various different hypothesis and approximations. They are all well studied and very well understood.

In this context, there is also the Korteweg-de Vries equation

\[(1.1) \quad \partial_t u + \partial_x u^2 + \beta \partial_{xxx}^3 u = 0.\]

Observe that, if we send $\beta \to 0$ in (1.1), we pass from (1.1) to the Burgers equation

\[(1.2) \quad \partial_t u + \partial_x u^2 = 0.\]

In cite [24, 35], the convergence of the solution of (1.1) to the unique entropy solution of (1.2) is proven, under the assumption

\[(1.3) \quad u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta = o(\epsilon^2).\]

Appendixes A and B show that it is possible to obtain the same result of convergence, under the following assumptions

\[(1.4) \quad u_0 \in L^2(\mathbb{R}), \quad -\infty < \int_{\mathbb{R}} u_0(x)dx < \infty, \quad \beta = o(\epsilon^3), \quad u_0 \in L^2(\mathbb{R}), \quad \beta = o(\epsilon^4).\]

One generalization of (1.1) is the Ostrovsky equation (see [29]):

\[(1.5) \quad \partial_x(\partial_t u + \partial_x u^2 - \beta \partial_{xxx}^3 u) = \gamma u, \quad \beta, \gamma \in \mathbb{R}.\]

(1.5) describes small-amplitude long waves in a rotating fluid of a finite depth by the additional term induced by the Coriolis force. If we send $\beta \to 0$ in (1.5), we pass from (1.5) to the Ostrovsky-Hunter equation (see [4]).

\[(1.6) \quad \partial_x(\partial_t u + \partial_x u^2) = \gamma u, \quad t > 0, \quad x \in \mathbb{R}.\]

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The authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
In [11, 13, 18], the well-posedness of the entropy solutions of (1.6) is proven, in the sense of the following definition:

**Definition 1.1.** We say that \( u \in L_\infty((0, T) \times \mathbb{R}) \), \( T > 0 \), is an entropy solution of the initial value problem (1.6) if

1. \( u \) is a distributional solution of (1.6);
2. for every convex function \( \eta \in C^2(\mathbb{R}) \) the entropy inequality

\[
\partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u) P \leq 0, \quad q(u) = \int u f'(\xi) \eta'(\xi) d\xi,
\]

holds in the sense of distributions in \((0, \infty) \times \mathbb{R}\).

Under the assumption (1.3), in [12], the convergence of the solutions of (1.5) to the unique entropy solution of (1.6) is proven.

The dynamics of dispersive shallow water waves, on the other hand, is captured with slightly different models, like the Rosenau-Kawahara equation and the Rosenau-KdV-RLW equation [3, 20, 21, 23, 31].

The Rosenau-Korteweg-de Vries-RLW equation is following one:

(1.8) \[
\partial_t u + a \partial_x u + k \partial_x u^n + b_1 \partial^3_{xxx} u + b_2 \partial^3_{txx} u + c \partial^5_{txxxx} u = 0,
\]

Here \( u(t, x) \) is the nonlinear wave profile. The first term is the linear evolution one, while \( a \) is the advection or drifting coefficient. The two dispersion coefficients are \( b_1 \) and \( b_2 \). The higher order dispersion coefficient is \( c \), while the coefficient of nonlinearity is \( k \) where \( n \) is nonlinearity parameter. These are all known and given parameters.

In [31], the authors analyzed (1.8). They got solitary waves, shock waves and singular solitons along with conservation laws.

Considering the \( n = 2, a = 0, k = 1, b_1 = 1, b_2 = -1, c = 1 \):

(1.9) \[
\partial_t u + \partial_x u^2 + \partial^3_{xxx} u - \partial^3_{txx} u + \partial^5_{txxxx} u = 0.
\]

If \( n = 2, a = 0, k = 1, b_1 = 0, b_2 = -1, c = 1 \), (1.8) reads

(1.10) \[
\partial_t u + \partial_x u^2 - \partial^3_{txx} u + \partial^5_{txxxx} u = 0,
\]

which is known as Rosenau-RLW equation.

Arguing in [14], we re-scale the equations as follows

(1.11) \[
\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u - \beta \partial^3_{txx} u + \beta^2 \partial^5_{txxxx} u = 0,
\]

(1.12) \[
\partial_t u + \partial_x u^2 - \beta \partial^3_{txx} u + \beta^2 \partial^5_{txxxx} u = 0,
\]

where \( \beta \) is the diffusion parameter.

In [8], the authors proved that the solutions of (1.11) and (1.12) converge to the unique entropy solution of (1.2), under the assumptions

(1.13) \[
u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta = O(\varepsilon^4).
\]

(1.1) has also been used in very wide applications and undergone research which can be used to describe wave propagation and spread interaction (see [2, 17, 28, 37]).

In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using (1.1). To overcome this shortcoming of (1.1), Rosenau proposed the following equation (see [33, 34]):

(1.14) \[
\partial_t u + \partial_x u^2 + \partial^5_{txxxx} u = 0,
\]

which is also obtained by (1.8), taking \( n = 2, a = 0, k = 1, b_1 = 0, b_2 = 0, c = 1 \).
The existence and the uniqueness of the solution for (1.14) is proved in [30], but it is difficult to find the analytical solution for (1.14). Therefore, much work has been done on the numerical methods for (1.14) (see [5, 6, 7, 22, 25, 27]).

On the other hand, for the further consideration of the nonlinear wave, the viscous term \( \partial^3_{xxx} u \) needs to be included (see [38]). In this case, (1.14) reads

\[
\partial_t u + \partial_x u^2 + \partial^3_{xxx} u + \partial^5_{txxxx} u = 0,
\]

which is known as the Rosenau-Korteweg-de Vries (KdV) equation, and is also obtained by (1.8), taking \( n = 2, a = 0, k = 1, b_1 = 1, b_2 = 0, c = 1. \)

In [38], the author discussed the solitary wave solutions and (1.15). In [21], a conservative linear finite difference scheme for the numerical solution for an initial-boundary value problem of Rosenau-KdV equation is considered. In [19, 32], authors discussed the solitary solutions for (1.15) with usual solitary ansatz method. The authors also gave the two invariants for (1.15). In particular, in [32], the authors not only studied the two types of soliton solution, one is solitary wave solution and the other is singular soliton. In [36], the authors proposed an average linear finite difference scheme for the numerical solution of the initial-boundary value problem for (1.15).

Consider (1.14). Arguing as [14], we re-scale the equations as follows

\[
\partial_t u + \partial_x u^2 + \beta^2 \partial^5_{txxxx} u_{\epsilon, \beta} = 0.
\]

In [9], the authors proved that the solutions of (1.16) converge to the unique entropy solution of (1.2), choosing the initial datum in two different ways. The first one is:

\[
u_0 \in L^2(\mathbb{R}), \quad \beta = o(\epsilon^4).
\]

The second choice is given by (1.13).

In this paper, we analyze (1.15). Arguing as [14], we re-scale the equations as follows

\[
\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u + \beta^2 \partial^5_{txxxx} u = 0.
\]

We are interested in the no high frequency limit, we send \( \beta \to 0 \) in (1.18). In this way we pass from (1.18) to (1.2). We prove that, as \( \beta \to 0 \), the solutions of (1.18) to the unique entropy solution of (1.2). In order to do this, we can choose the initial datum and \( \beta \) in two different ways. Following [16, Theorem 7.1], the first choice is given by (1.17) (see Theorem 2.1). Since \( \|\cdot\|_{L^4} \) is a conserved quantity for (1.18), the second choice is given by (1.13) (see Theorem 3.1). It is interesting to observe that, while the summability on the initial datum in (1.13) is greater than the one of (1.17), the assumption on \( \beta \) in (1.13) is weaker than the one in (1.17).

From the mathematical point of view, the two assumptions require two different arguments for the \( L^\infty \) estimate (see Lemmas 2.2 and 3.1). Indeed, the proof of Lemma 2.2 under the assumption (1.17), is more technical than the one of Lemma 3.1. Moreover, due to the presence of the third order term, Lemmas 2.2 and 3.2 is finer than [9] Lemmas 2.2 and 3.2. Indeed, with respect to [9, Lemma 2.2], in Lemma 2.2 we need to prove the existence of two positive constants, while, with respect to [9, Lemma 3.2], in Lemma 3.2 we need to prove the existence of four positive constants.

The paper is organized in four sections. In Section 2, we prove the convergence of (1.18) to (1.2) in \( L^p \) setting, with \( 1 \leq p < 2 \). In Section 3, we prove the convergence of (1.18) to (1.2) in \( L^p \) setting, with \( 1 \leq p < 4 \). The Section A is an appendix where we prove that the solutions of the the Benjamin-Bona-Mahony equation converge to discontinuous weak solutions of (1.2) in \( L^p \) setting, with \( 1 \leq p < 2 \).
2. The Rosenau-KdV-equation: \( u_0 \in L^2(\mathbb{R}) \).

In this section, we consider \((1.18)\), and assume \((1.17)\) on the initial datum. We study the dispersion-diffusion limit for \((1.18)\). Therefore, we fix two small numbers \(0 < \varepsilon, \beta < 1\) and consider the following fifth order approximation

\[
\begin{align*}
\partial_t u_{\varepsilon, \beta} + \partial_x u_{\varepsilon, \beta}^2 + \beta \partial_{xxx}^2 u_{\varepsilon, \beta} + \beta^2 \partial_{xxxxx}^2 u_{\varepsilon, \beta} = \varepsilon \partial_{xx}^2 u_{\varepsilon, \beta}, & \quad t > 0, \ x \in \mathbb{R}, \\
\left. u_{\varepsilon, \beta}(0, x) = u_{\varepsilon, \beta}(0, x), \right|_{x \in \mathbb{R}},
\end{align*}
\]

where \(u_{\varepsilon, \beta, 0}\) is a \(C^\infty\) approximation of \(u_0\) such that

\[
\left. u_{\varepsilon, \beta, 0} \to u_0 \right|_{L^p(\mathbb{R})}, \ 1 \leq p < 2, \ \text{as} \ \varepsilon, \beta \to 0,
\]

and \(C_0\) is a constant independent on \(\varepsilon\) and \(\beta\).

The main result of this section is the following theorem.

**Theorem 2.1.** Assume that \((1.17)\) and \((2.2)\) hold. Fix \(T > 0\), if

\[
\beta = O\left(\varepsilon^4\right),
\]

then, there exist two sequences \(\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}\), with \(\varepsilon_n, \beta_n \to 0\), and a limit function

\[
u \in L^\infty((0,T); L^2(\mathbb{R})),
\]

such that

i) \(u_{\varepsilon_n, \beta_n} \to u\) strongly in \(L^p(\mathbb{R} \times \mathbb{R})\), for each \(1 \leq p < 2\),

ii) \(u\) is a distributional solution of \((1.2)\).

Moreover, if

\[
\beta = o\left(\varepsilon^4\right),
\]

iii) \(u\) is the unique entropy solution of \((1.2)\).

Let us prove some a priori estimates on \(u_{\varepsilon, \beta}\), denoting with \(C_0\) the constants which depend only on the initial data.

**Lemma 2.1.** For each \(t > 0\),

\[
\|u_{\varepsilon, \beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \leq C_0.
\]

**Proof.** We begin by observing that

\[
\int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{xxx}^3 u_{\varepsilon, \beta} \, dx = 0.
\]

Therefore, arguing as [9 Lemma 2.1], we have \((2.5)\). \(\square\)

**Lemma 2.2.** Fix \(T > 0\). Assume \((2.3)\) holds. There exists \(C_0 > 0\), independent on \(\varepsilon, \beta\) such that

\[
\|u_{\varepsilon, \beta}\|_{L^\infty((0,T) \times \mathbb{R})} \leq C_0 \beta^{-\frac{1}{4}}.
\]

Moreover,

i) the families \(\{\beta^\frac{1}{4} \partial_x u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{1}{4} \varepsilon \partial_x u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{3}{4} \varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{3}{4} \varepsilon^2 \partial_{xxx}^2 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \)

are bounded in \(L^\infty((0,T); L^2(\mathbb{R}))\);

ii) the families \(\{\beta^\frac{1}{4} \varepsilon \partial_{xx}^3 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{1}{4} \varepsilon^2 \partial_{xxx}^3 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{1}{4} \varepsilon^2 \partial_{xxxx}^3 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{3}{4} \varepsilon^2 \partial_{xx}^3 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{3}{4} \varepsilon^2 \partial_{xxx}^3 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{3}{4} \varepsilon^2 \partial_{xxxx}^3 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \)

are bounded in \(L^2((0,T) \times \mathbb{R})\).
Proof. Let $0 < t < T$. Let $A, B$ be some positive constants which will be specified later.

Multiplying (2.1) by $-\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta}$, we have

\[
\left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial_t u_{\varepsilon, \beta} \\
+ 2 \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \\
+ \beta \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial^3_{txx} u_{\varepsilon, \beta} \\
+ \beta^2 \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial^5_{txxxx} u_{\varepsilon, \beta} \\
= \varepsilon \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial^2_{xx} u_{\varepsilon, \beta}.
\]

(2.7)

We observe that

\[
\int_{\mathbb{R}} \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial_t u_{\varepsilon, \beta} dx \\
= \frac{\beta^\frac{1}{2}}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})} + A\beta \varepsilon \left( \int_{\mathbb{R}} \partial^3_{txx} u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})} \\
+ B \varepsilon \left( \int_{\mathbb{R}} \partial_t u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})}.
\]

(2.8)

Since

\[
2 \int_{\mathbb{R}} \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial_x u_{\varepsilon, \beta} dx \\
= -2\beta^\frac{1}{2} \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} dx \partial^2_{xx} u_{\varepsilon, \beta} dx - 2A\beta \varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} \partial^3_{txx} u_{\varepsilon, \beta} dx \\
+ 2B \varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} \partial_t u_{\varepsilon, \beta} dx, \\
\beta \int_{\mathbb{R}} \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial^3_{txx} u_{\varepsilon, \beta} dx \\
=A\beta^2 \varepsilon \int_{\mathbb{R}} \partial^2_{xx} u_{\varepsilon, \beta} \partial^4_{txxx} u_{\varepsilon, \beta} dx + B \beta \varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon, \beta} \partial^3_{txx} u_{\varepsilon, \beta} dx, \\
\beta^2 \int_{\mathbb{R}} \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial^5_{txxxx} u_{\varepsilon, \beta} dx \\
= \frac{\beta^\frac{3}{2}}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} \partial^4_{txxx} u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})} + \frac{\beta^3}{2} \varepsilon \left( \int_{\mathbb{R}} \partial^5_{txxxx} u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})} \\
+ B \beta^2 \varepsilon \left( \int_{\mathbb{R}} \partial^5_{txxxx} u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})}, \\
\varepsilon \int_{\mathbb{R}} \left( -\beta^\frac{1}{2} \partial^2_{xx} u_{\varepsilon, \beta} - A\beta \varepsilon \partial^3_{txx} u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta} \right) \partial^2_{xx} u_{\varepsilon, \beta} dx \\
= -\beta^\frac{1}{2} \varepsilon \left( \int_{\mathbb{R}} \partial^2_{xx} u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})} - \frac{A\beta^2 \varepsilon}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} \partial^2_{xx} u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})} \\
- \frac{B \varepsilon}{2} \frac{d}{dt} \left( \int_{\mathbb{R}} \partial_t u_{\varepsilon, \beta}(t, \cdot) \right)^2_{L^2(\mathbb{R})}.
\]
an integration on $\mathbb{R}$ of (2.8) gives
\[
\frac{d}{dt} \left( \frac{\beta^2 t + B \varepsilon}{2} \right) \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{A \beta \varepsilon}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
+ \frac{\beta^2}{2} \frac{d}{dt} \| \partial_{xxx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta \varepsilon \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
+ B \varepsilon \| \partial_{x} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + A \beta^2 \varepsilon \| \partial_{xxxx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
- \frac{B \beta \varepsilon}{2} \| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right)
\]
\[
= 2\beta^2 \int_R u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} dx + 2A \beta \varepsilon \int_R u_{\varepsilon, \beta} \partial_x^3 u_{\varepsilon, \beta} \partial_{xxxx}^2 u_{\varepsilon, \beta} dx
\]
\[
- 2B \varepsilon \int_R u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \partial_{x} u_{\varepsilon, \beta} dx - A \beta^2 \varepsilon \int_R \partial_{xx}^2 u_{\varepsilon, \beta} \partial_{xxxx}^2 u_{\varepsilon, \beta} dx
\]
\[
- B \beta \varepsilon \int_R \partial_{xx} u_{\varepsilon, \beta} \partial_{xxxx}^2 u_{\varepsilon, \beta} dx.
\]
Using (2.8), $0 < \beta < 1$, and the Young inequality,
\[
2 \beta^2 \int_R \| u_{\varepsilon, \beta} \partial_x^2 u_{\varepsilon, \beta} \| \| \partial_{xx}^2 u_{\varepsilon, \beta} \| dx = \beta^2 \int_R \left( \frac{2u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}}{\varepsilon^2} \right) \left( \frac{\varepsilon^2}{\partial_x^2 u_{\varepsilon, \beta}} \right) dx
\]
\[
\leq C_0 \varepsilon \| u_{\varepsilon, \beta} \|_{L^2((0, T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{B \beta \varepsilon}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
2A \beta \varepsilon \int_R \| u_{\varepsilon, \beta} \partial_x^3 u_{\varepsilon, \beta} \| \| \partial_{xxxx}^2 u_{\varepsilon, \beta} \| dx = \varepsilon \int_R \left( \frac{2A u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}}{\sqrt{B}} \right) \left( \frac{\sqrt{B} \partial_x^3 u_{\varepsilon, \beta}}{\varepsilon} \right) dx
\]
\[
\leq \frac{2A^2 \varepsilon}{B} \int_R \| u_{\varepsilon, \beta} \|_{L^2((0, T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{B \beta \varepsilon}{2} \| \partial_{xxxx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
2B \varepsilon \int_R \| u_{\varepsilon, \beta} \partial_x^3 u_{\varepsilon, \beta} \| \| \partial_{xx}^2 u_{\varepsilon, \beta} \| dx = B \varepsilon \int_R \left( \frac{2u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}}{\varepsilon} \right) \left( \frac{\varepsilon}{\partial_x^3 u_{\varepsilon, \beta}} \right) dx
\]
\[
\leq \frac{2B \varepsilon}{2} \int_R \| u_{\varepsilon, \beta} \|_{L^2((0, T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{B \beta \varepsilon}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
A \beta^2 \varepsilon \int_R \| \partial_{xx}^2 u_{\varepsilon, \beta} \| \| \partial_{xxxx}^2 u_{\varepsilon, \beta} \| dx = A \varepsilon \int_R \left( \frac{\beta \varepsilon}{2} \partial_x u_{\varepsilon, \beta} \right) \left( \frac{\beta \varepsilon}{\partial_{xx}^2 u_{\varepsilon, \beta}} \right) dx
\]
\[
\leq \frac{A \beta \varepsilon}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{A \beta^2 \varepsilon}{2} \| \partial_{xxxx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
\leq \frac{A \beta^2 \varepsilon}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{A \beta^2 \varepsilon}{2} \| \partial_{xxxx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
\]
\[
B \beta \varepsilon \int_R \| \partial_x u_{\varepsilon, \beta} \partial_{xxx}^2 u_{\varepsilon, \beta} dx = \varepsilon \int_R \| \partial_x u_{\varepsilon, \beta} \| \| B \beta \partial_{xxx}^2 u_{\varepsilon, \beta} \| dx
\]
\[
\leq \frac{\varepsilon}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{B \beta^2 \varepsilon}{2} \| \partial_{xxx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2
Therefore, (2.9) gives
\[ \frac{d}{dt} \left( \frac{\beta^2}{2} + B \varepsilon^2 \right) + \frac{\beta^2}{2} \frac{d}{dt} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{2} \frac{d}{dt} \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{4}{5} B \varepsilon (1 - A) \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ + \frac{4}{5} \varepsilon \left(1 - A\right) \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \varepsilon \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})}^2 \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ + 2 A^2 \varepsilon \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 B \varepsilon \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \cdot \]
Choosing \( A = \frac{1}{2}, \ B = \frac{1}{2} \), from (2.10), we have
\[ \frac{d}{dt} \left( \frac{2 \beta^2}{4} + \frac{\varepsilon^2}{4} \right) \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ + \frac{\varepsilon}{4} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ \leq C_0 \varepsilon \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})}^2 \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ + 2 A^2 \varepsilon \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 B \varepsilon \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \cdot \]
(2.2), (2.5), and an integration on \((0, t)\) give
\[ \frac{2 \beta^2}{4} + \frac{\varepsilon^2}{4} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2}{4} \frac{d}{dt} \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ \leq C_0 + C_0 \varepsilon \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[ + \frac{\varepsilon}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_0 \left(1 + \left\| u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \right) . \]
We prove (2.6). Due to (2.5), (2.11), and the Hölder inequality,
\[ u_{\varepsilon, \beta}(t, x) = 2 \int_{-\infty}^{x} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} dx \leq 2 \int_{\mathbb{R}} |u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}| dx \]
that is
\[\| u_{\varepsilon, \beta} \|^2_{L^2((0,T) \times \mathbb{R})} \leq L \left( 1 + \| u_{\varepsilon, \beta} \|^2_{L^2((0,T) \times \mathbb{R})} \right),\]

Arguing as \cite{9} Lemma 2.2, we have (2.6).

It follows from (2.6) and (2.11) that
\[
\frac{2\beta \frac{3}{2} + \frac{\varepsilon^2}{4}}{4} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\beta \varepsilon^2}{4} \| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\beta}{2} \int_0^t \| \partial_{xxx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds
\]
\[+ \varepsilon \int_0^t \| \partial_t u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds + \frac{\beta \varepsilon}{4} \int_0^t \| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds + \frac{\beta \varepsilon}{8} \int_0^t \| \partial_{xxx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C_0 \beta^{-\frac{1}{2}},\]

that is,
\[
\frac{2\beta + \beta \frac{3}{2} \varepsilon^2}{4} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\beta \varepsilon^2}{4} \| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{\beta}{2} \int_0^t \| \partial_{xxx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds
\]
\[+ \beta \frac{3}{2} \varepsilon \int_0^t \| \partial_t u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds + \beta \varepsilon \int_0^t \| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds + \beta \varepsilon \int_0^t \| \partial_{xxx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C_0.
\]

Hence,
\[
\beta \frac{3}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0,
\]
\[
\beta \frac{3}{2} \varepsilon \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0,
\]
\[
\beta \frac{3}{2} \varepsilon \| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0,
\]
\[
\beta \frac{3}{2} \| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0,
\]
\[
\beta \frac{3}{2} \varepsilon \int_0^t \| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C_0,
\]
\[
\beta \frac{3}{2} \varepsilon \int_0^t \| \partial_t u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C_0,
\]
\[
\beta \frac{3}{2} \varepsilon \int_0^t \| \partial_{xxx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C_0,
\]
\[
\beta \frac{3}{2} \varepsilon \int_0^t \| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C_0.
\]
Lemma 2.3. Let $t < T$ for every $0 < W$.

Definition 2.1. $(\mathcal{L}, \eta)$, assume that

Lemma 2.4. is convex/compactly supported.

Moreover, we consider the following definition.

Definition 2.1. A pair of functions $(\eta, q)$ is called an entropy–entropy flux pair if $\eta : \mathbb{R} \to \mathbb{R}$ is a $C^2$ function and $q : \mathbb{R} \to \mathbb{R}$ is defined by

An entropy-entropy flux pair $(\eta, q)$ is called convex/compactly supported if, in addition, $\eta$ is convex/compactly supported.

We begin by proving the following result

Lemma 2.4. Assume that (1.17), (2.2) and (2.3) hold. Then for any compactly supported entropy–entropy flux pair $(\eta, q)$, there exist two sequences $\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

such that

Proof. Let us consider a compactly supported entropy–entropy flux pair $(\eta, q)$. Multiplying (2.1) by $\eta'(u_{e, \beta})$, we have

where

Fix $T > 0$. Arguing in [12, Lemma 3.2], we have that $I_1, e, \beta \to 0$ in $H^{-1}((0, T) \times \mathbb{R})$, and $\{I_2, e, \beta\}_{\varepsilon, \beta \to 0}$ is bounded in $L^1((0, T) \times \mathbb{R})$. Arguing in [9, Theorem B.1], $I_3, e, \beta \to 0$ in $H^{-1}((0, T) \times \mathbb{R})$, and $I_4, e, \beta \to 0$ in $L^1((0, T) \times \mathbb{R})$, while arguing in [9, Lemma 2.4], $I_5, e, \beta \to 0$ in $H^{-1}((0, T) \times \mathbb{R})$, and $\{I_6, e, \beta\}_{e, \beta \to 0}$ is bounded in $L^1((0, T) \times \mathbb{R})$. 

\[
\beta\varepsilon \int_0^t \|\partial_{xx}^2 u_{e, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0,
\]

for every $0 < t < T$. \hfill \Box
Therefore, (2.12) follows from Lemmas 2.2, 2.3 and the $L^p$ compensated compactness of [35]. Arguing in [8, Theorem 2.1], we have (2.13).

Following [24], we prove the following result

**Lemma 2.5.** Assume that (1.17), (2.2) and (2.4) hold. Then for any compactly supported entropy-entropy flux pair $(\eta, q)$, there exist two sequences $\{\epsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}$, with $\epsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0, T); L^2(\mathbb{R})),$$

such that (2.12) holds and

$$u \text{ is the unique entropy solution of } (1.2).$$

**Proof.** Let us consider a compactly supported entropy-entropy flux pair $(\eta, q)$. Multiplying (2.1) by $\eta'(u_{\epsilon, \beta})$, we have

$$\partial_t \eta(u_{\epsilon, \beta}) + \partial_x q(u_{\epsilon, \beta}) = \epsilon \eta'(u_{\epsilon, \beta}) \partial^2_{xx} u_{\epsilon, \beta} - \beta \partial^3_{xxx} u_{\epsilon, \beta} = I_{1, \epsilon, \beta} + I_{2, \epsilon, \beta} + I_{3, \epsilon, \beta} + I_{4, \epsilon, \beta} + I_{5, \epsilon, \beta} + I_{6, \epsilon, \beta}$$

where $I_{1, \epsilon, \beta}, I_{2, \epsilon, \beta}, I_{3, \epsilon, \beta}, I_{4, \epsilon, \beta}, I_{5, \epsilon, \beta}, I_{6, \epsilon, \beta}$ are defined in (2.14).

As in Lemma 2.4, we obtain that $I_{1, \epsilon, \beta} \to 0$ in $H^{-1}((0, T) \times \mathbb{R}), \{I_{2, \epsilon, \beta}\}_{\epsilon, \beta > 0}$ is bounded in $L^1((0, T) \times \mathbb{R}), I_{3, \epsilon, \beta} \to 0$ in $H^{-1}((0, T) \times \mathbb{R}), I_{4, \epsilon, \beta} \to 0$ in $L^1((0, T) \times \mathbb{R}), I_{5, \epsilon, \beta} \to 0$ in $H^{-1}((0, T) \times \mathbb{R})$, while arguing in [9, Lemma 2.4], $I_{6, \epsilon, \beta} \to 0$ in $L^1((0, T) \times \mathbb{R})$

Arguing in [8, Theorem 2.1], we have (2.13).

**Proof of Theorem 2.7.** Theorem 2.1 follows from Lemmas 2.4 and 2.5

3. The Rosenau-KdV-equation. $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$.

In this section, we consider (1.18), and assume (1.13) on the initial datum. We consider the approximation (2.1), where $u_{\epsilon, \beta, 0}$ is a $C^\infty$ approximation of $u_0$ such that

$$u_{\epsilon, \beta, 0} \to u_0 \text{ in } L^p_{\text{loc}}(\mathbb{R}), 1 \leq p < 2, \text{ as } \epsilon, \beta \to 0,$$

$$\|u_{\epsilon, \beta, 0}\|_{L^2_p(\mathbb{R})} + \|\partial_x u_{\epsilon, \beta, 0}\|^2_{L^2(\mathbb{R})} + \left(\beta \frac{\epsilon^2}{\beta^2} + \epsilon^2\right) \|\partial^3_{xxx} u_{\epsilon, \beta, 0}\|^2_{L^2(\mathbb{R})} \leq C_0, \epsilon, \beta > 0,$$

$$\beta^2 \|\partial^4_{xxxx} u_{\epsilon, \beta, 0}\|^2_{L^2(\mathbb{R})} \leq C_0, \epsilon, \beta > 0,$$

and $C_0$ is a constant independent on $\epsilon$ and $\beta$.

The main result of this section is the following theorem.

**Theorem 3.1.** Assume that (1.13) and (3.1) hold. Fix $T > 0$, if (2.3) holds, there exist two sequences $\{\epsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}$, with $\epsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0, T); L^2(\mathbb{R}) \cap L^4(\mathbb{R})),$$

such that

i) $u_{\epsilon_n, \beta_n} \to u$ strongly in $L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$, for each $1 \leq p < 4$,

ii) $u$ is the unique entropy solution of (1.2).

Let us prove some a priori estimates on $u_{\epsilon, \beta}$, denoting with $C_0$ the constants which depend only on the initial data.
Lemma 3.1. Fix $T > 0$. Assume (2.3) holds. There exists $C_0 > 0$, independent on $\varepsilon, \beta$ such that (2.6) holds. In particular, we have
\[
\beta \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2([0, T])}^2 + \beta^2 \|\partial_{xxx} u_{\varepsilon, \beta}(t, \cdot)\|_{L^2([0, T])}^2
\]
(3.2)
\[
+ \frac{3\beta\varepsilon}{2} \int_0^t \|\partial_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2([0, T])}^2 \, ds \leq C_0,
\]
for every $0 < t < T$. Moreover,
\[
\|\partial_x u_{\varepsilon, \beta}\|_{L^\infty((0, T) \times \mathbb{R})} \leq C_0 \beta^{-\frac{2}{3}}.
\]
(3.3)

Remark 3.1. Observe that the proof of Lemma 3.1 is simpler than one of the Lemma 2.2. Indeed, we only need to prove (2.6).

Proof of Lemma 3.1. Let $0 < t < T$. Multiplying (2.1) by $-\beta^\frac{2}{3}\partial_{xx} u_{\varepsilon, \beta}$, we have
\[
-\beta^\frac{2}{3} \partial_{xx} u_{\varepsilon, \beta} \partial_t u_{\varepsilon, \beta} - 2 \beta^\frac{4}{3} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta}
\]
\[
+ \beta^\frac{8}{3} \partial_{xx} u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta} - \beta^\frac{10}{3} \partial_{xxxx} u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta} = -\beta^\frac{1}{3} \varepsilon (\partial_{xx} u_{\varepsilon, \beta})^2.
\]
We note that
\[
\beta^\frac{2}{3} \int_\mathbb{R} \partial_{xx} u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta} dx = 0.
\]
Therefore, arguing as [9] Lemma 3.1], we have (2.6), (3.2) and (3.3).

Following [10] Lemma 2.2], or [11] Lemma 4.2], we prove the following result.

Lemma 3.2. Fix $T > 0$. Assume (2.3) holds. Then:

i) the family \{u_{\varepsilon, \beta}\}_{\varepsilon, \beta} is bounded in $L^\infty((0, T); L^4(\mathbb{R}))$;

ii) the families \{\partial_x u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{2}{3} \varepsilon \partial_{xx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta \partial_{xx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^2 \partial_{xxx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$ are bounded in $L^\infty((0, T); L^2(\mathbb{R}))$;

iii) the families \{\beta^\frac{1}{3} \varepsilon \partial_{xx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\varepsilon \partial_t u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{2}{3} \varepsilon \partial_{xxx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^\frac{1}{3} \varepsilon \partial_{xx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta \partial_{xx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}, \{\beta^2 \partial_{xxx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, are bounded in $L^2((0, T) \times \mathbb{R})$;

Proof. Let $0 < t < T$. Let $A, B, C, E$ be some positive constants which will be specified later. Multiplying (2.1) by
\[
u_{\varepsilon, \beta}^3 - A \varepsilon^2 \partial_{xx} u_{\varepsilon, \beta} - B \beta \varepsilon \partial_{xxx} u_{\varepsilon, \beta} + C \varepsilon \partial_t u_{\varepsilon, \beta} + E \beta^2 \partial_{xxxx} u_{\varepsilon, \beta},
\]
we have
\[
\begin{align*}
(u_{\varepsilon, \beta}^3 & - A \varepsilon^2 \partial_{xx} u_{\varepsilon, \beta} - B \beta \varepsilon \partial_{xxx} u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} \\
& + (C \varepsilon \partial_t u_{\varepsilon, \beta} + E \beta^2 \partial_{xxxx} u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \\
& + 2 \left(\partial_{xx} u_{\varepsilon, \beta} - B \beta \varepsilon \partial_{xxx} u_{\varepsilon, \beta}\right) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \\
& + 2 \left(C \varepsilon \partial_t u_{\varepsilon, \beta} + E \beta^2 \partial_{xxxx} u_{\varepsilon, \beta}\right) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \\
& + \beta \left(u_{\varepsilon, \beta}^3 - A \varepsilon^2 \partial_{xx} u_{\varepsilon, \beta} - B \beta \varepsilon \partial_{xxx} u_{\varepsilon, \beta}\right) \partial_{xx} u_{\varepsilon, \beta} \\
& + \beta \left(C \varepsilon \partial_t u_{\varepsilon, \beta} + E \beta^2 \partial_{xxxx} u_{\varepsilon, \beta}\right) \partial_{xx} u_{\varepsilon, \beta} \\
& + \beta^2 \left(u_{\varepsilon, \beta}^3 - A \varepsilon^2 \partial_{xx} u_{\varepsilon, \beta} - B \beta \varepsilon \partial_{xxx} u_{\varepsilon, \beta}\right) \partial_{xxx} u_{\varepsilon, \beta} \\
& + \beta^2 \left(C \varepsilon \partial_t u_{\varepsilon, \beta} + E \beta^2 \partial_{xxxx} u_{\varepsilon, \beta}\right) \partial_{xxx} u_{\varepsilon, \beta} \\
& = \varepsilon \left(u_{\varepsilon, \beta}^3 - A \varepsilon^2 \partial_{xx} u_{\varepsilon, \beta} - B \beta \varepsilon \partial_{xxx} u_{\varepsilon, \beta}\right) \partial_{xx} u_{\varepsilon, \beta} \\
& + \varepsilon \left(C \varepsilon \partial_t u_{\varepsilon, \beta} + E \beta^2 \partial_{xxxx} u_{\varepsilon, \beta}\right) \partial_{xx} u_{\varepsilon, \beta}.
\end{align*}
\]
Since

\[ \int_{\mathbb{R}} \left( u_{\varepsilon, \beta}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \beta} - B\beta \varepsilon \partial_{xxxx}^3 u_{\varepsilon, \beta} \right) \, \partial_{t} u_{\varepsilon, \beta} \, dx \]

\[ = \frac{1}{4} \left. \frac{d}{dt} \right|_{t=0} \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^1(\mathbb{R})}^4 + \frac{A\varepsilon^2}{2} \left. \frac{d}{dt} \right|_{t=0} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + B\beta \varepsilon \left. \frac{d}{dt} \right|_{t=0} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2, \]

\[ \int_{\mathbb{R}} \left( C\varepsilon \partial_t u_{\varepsilon, \beta} + E\beta^2 \partial_{xxxx}^4 u_{\varepsilon, \beta} \right) \, \partial_{t} u_{\varepsilon, \beta} \, dx \]

\[ = C\varepsilon \left. \left\| \partial_t u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{E\beta^2}{2} \left. \frac{d}{dt} \right|_{t=0} \left\| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \]

\[ 2 \int_{\mathbb{R}} \left( u_{\varepsilon, \beta}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \beta} - B\beta \varepsilon \partial_{xxxx}^3 u_{\varepsilon, \beta} \right) \, \partial_{x} u_{\varepsilon, \beta} \, dx \]

\[ = -2A\varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta} \partial_{x} u_{\varepsilon, \beta} \, dx - 2B\beta \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta} \partial_{txx}^3 u_{\varepsilon, \beta} \, dx, \]

\[ 2 \int_{\mathbb{R}} \left( C\varepsilon \partial_t u_{\varepsilon, \beta} + E\beta^2 \partial_{xxxx}^4 u_{\varepsilon, \beta} \right) u_{\varepsilon, \beta} \, \partial_{x} u_{\varepsilon, \beta} \, dx \]

\[ = 2C \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{x} u_{\varepsilon, \beta} \partial_{t} u_{\varepsilon, \beta} \, dx - 2E\beta^2 \int_{\mathbb{R}} \left( \partial_{x} u_{\varepsilon, \beta} \right)^2 \partial_{xxxx}^3 u_{\varepsilon, \beta} \, dx \]

\[ - 2E\beta^2 \left. \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} \partial_{x} u_{\varepsilon, \beta} \partial_{txx}^3 u_{\varepsilon, \beta} \, dx \right), \]

\[ -2E\beta^2 \int_{\mathbb{R}} \left( \partial_{x} u_{\varepsilon, \beta} \right)^2 \partial_{xxxx}^3 u_{\varepsilon, \beta} \, dx - 2E\beta^2 \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} \partial_{txx}^3 u_{\varepsilon, \beta} \, dx \]

\[ = 2 \int_{\mathbb{R}} \left( C\varepsilon \partial_t u_{\varepsilon, \beta} + E\beta^2 \partial_{xxxx}^4 u_{\varepsilon, \beta} \right) u_{\varepsilon, \beta} \, \partial_{x} u_{\varepsilon, \beta} \, dx \]

\[ = 2C \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{x} u_{\varepsilon, \beta} \partial_{t} u_{\varepsilon, \beta} \, dx - \frac{5E\beta^2}{2} \int_{\mathbb{R}} \left( \partial_{x} u_{\varepsilon, \beta} \right)^2 \partial_{xxxx}^3 u_{\varepsilon, \beta} \, dx, \]

\[ \beta \int_{\mathbb{R}} \left( u_{\varepsilon, \beta}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \beta} - B\beta \varepsilon \partial_{xxxx}^3 u_{\varepsilon, \beta} \right) \, \partial_{xxxx} u_{\varepsilon, \beta} \, dx \]

\[ = -3\beta \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_{x} u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} \, dx - B\beta^2 \varepsilon \int_{\mathbb{R}} \partial_{txx}^3 u_{\varepsilon, \beta} \partial_{xxxx}^3 u_{\varepsilon, \beta} \, dx, \]

\[ \beta \int_{\mathbb{R}} \left( C\varepsilon \partial_t u_{\varepsilon, \beta} + E\beta^2 \partial_{xxxx}^4 u_{\varepsilon, \beta} \right) \, \partial_{xxxx}^3 u_{\varepsilon, \beta} \, dx \]

\[ = C\beta \varepsilon \int_{\mathbb{R}} \partial_{txx}^3 u_{\varepsilon, \beta} \partial_{txx}^3 u_{\varepsilon, \beta} \, dx, \]

\[ \beta^2 \int_{\mathbb{R}} \left( C\varepsilon \partial_t u_{\varepsilon, \beta} + E\beta^2 \partial_{xxxx}^4 u_{\varepsilon, \beta} \right) \, \partial_{txx}^3 u_{\varepsilon, \beta} \, dx \]

\[ = C\beta^2 \varepsilon \left. \left\| \partial_{txx}^3 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{E\beta^4}{2} \left. \frac{d}{dt} \right|_{t=0} \left\| \partial_{xxxx}^3 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \]
\[ \varepsilon \int_{\mathbb{R}} (u_{e,\beta}^3 - A\varepsilon^2 \partial_{xx}^2 u_{e,\beta} - B\beta\varepsilon \partial_\beta^4 u_{e,\beta}) \partial_{xx}^2 u_{e,\beta} \, dx \]
\[ = -3\varepsilon \| u_{e,\beta}(t, \cdot) \partial_x u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} - A\varepsilon^3 \| \partial_{xx}^2 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \]
\[ - \frac{B\beta^2}{2} \frac{d}{dt} \| \partial_{xx}^2 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})}, \]
\[ \varepsilon \int_{\mathbb{R}} (C\varepsilon \partial_t u_{e,\beta} + E\beta^2 \partial_\beta^4 u_{e,\beta}) \partial_{xx}^2 u_{e,\beta} \, dx \]
\[ = -\frac{C\varepsilon^2}{2} \frac{d}{dt} \| \partial_x u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} - E\beta^2 \varepsilon \| \partial_{xxx}^3 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})}. \]

an integration on \( \mathbb{R} \) of (3.5) gives

\[
\frac{d}{dt} \left( \frac{1}{4} \| u_{e,\beta}(t, \cdot) \|^2_{L^4(\mathbb{R})} + \frac{(A + C)\varepsilon^2}{2} \| \partial_x u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ \frac{d}{dt} \left( \frac{A\beta^2}{2} \| \partial_{xxx}^2 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{E\beta^4}{2} \| \partial_{xxxx}^4 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ \frac{B\beta^2 + E\beta^3}{2} \frac{d}{dt} \| \partial_{xx}^2 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + B\beta\varepsilon \| \partial_x u_{e,\beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
+ C\varepsilon \| \partial_x u_{e,\beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + B\beta^3 \varepsilon \| \partial_{xxx}^3 u_{e,\beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
+ C\beta^2 \varepsilon \| \partial_{xxx}^3 u_{e,\beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 3\varepsilon \| u_{e,\beta}(t, \cdot) \partial_x u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ A\varepsilon^3 \| \partial_{xx}^2 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + E\beta^2 \varepsilon \| \partial_{xxx}^4 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
= 2A\varepsilon^2 \int_{\mathbb{R}} u_{e,\beta} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx + 2B\beta\varepsilon \int_{\mathbb{R}} \partial_x u_{e,\beta} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx \\
+ 2C\varepsilon \int_{\mathbb{R}} \partial_x u_{e,\beta} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx + \frac{5E\beta^2}{2} \int_{\mathbb{R}} (\partial_x u_{e,\beta})^2 \partial_{xxx}^3 u_{e,\beta} \, dx \\
+ 3\beta \int_{\mathbb{R}} u_{e,\beta}^2 \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx + B\beta^2 \varepsilon \int_{\mathbb{R}} \partial_{xxx}^3 u_{e,\beta} \partial_{xx}^2 u_{e,\beta} \, dx \\
- C\beta \varepsilon \int_{\mathbb{R}} \partial_{xxx}^3 u_{e,\beta} \partial_x u_{e,\beta} \, dx + 3\beta^2 \int_{\mathbb{R}} u_{e,\beta}^2 \partial_x u_{e,\beta} \partial_{xxx}^4 u_{e,\beta} \, dx.
\]

Due to the Young inequality,

\[
2A\varepsilon^2 \int_{\mathbb{R}} \| u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \| dx = \int_{\mathbb{R}} \| u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \| dx \\
\leq \frac{\varepsilon}{2} \| u_{e,\beta}(t, \cdot) \partial_x u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2A\varepsilon^3 \| \partial_{xx}^2 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]

\[
2B\beta\varepsilon \int_{\mathbb{R}} \| u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_x u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \|_{L^2(\mathbb{R})} dx = \varepsilon \int_{\mathbb{R}} \| u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \|_{L^2(\mathbb{R})} dx \\
\frac{\varepsilon}{2} \| u_{e,\beta}(t, \cdot) \partial_x u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + 4B\beta^2 \varepsilon \| \partial_{xxx}^3 u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]

\[
2C\varepsilon \int_{\mathbb{R}} \| u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_x u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \|_{L^2(\mathbb{R})} dx = \varepsilon \int_{\mathbb{R}} \| u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \|_{L^2(\mathbb{R})} dx \\
\frac{\varepsilon}{2} \| u_{e,\beta}(t, \cdot) \partial_x u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + 2C\varepsilon \| \partial_t u_{e,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]

\[
B\beta^2 \varepsilon \int_{\mathbb{R}} \| \partial_{xxx}^3 u_{e,\beta} \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \|_{L^2(\mathbb{R})} dx = \beta^2 \varepsilon \int_{\mathbb{R}} \| 2B\partial_{xxx}^3 u_{e,\beta} \|_{L^2(\mathbb{R})} dx
\]
\[
\beta \leq D^2 \varepsilon^4,
\]
where \( D \) is a positive constant that will be specified later. It follows from (3.3), (3.8) and the Young inequality that

\[
\frac{5E\beta^2}{2}\int_R (\partial_x u_{\varepsilon, \beta})^2 |\partial_{xxx} u_{\varepsilon, \beta}| \, dx = E \beta^2 \int_R \frac{5}{2 \varepsilon^2} (\partial_x u_{\varepsilon, \beta})^2 |\varepsilon \frac{2}{\beta} \partial_{xxx} u_{\varepsilon, \beta}| \, dx \\
\leq \frac{25E\beta^2}{8\varepsilon} \int_R (\partial_x u_{\varepsilon, \beta})^2 \, dx + \frac{E\beta^2}{2 \varepsilon^2} |\partial_{xxx} u_{\varepsilon, \beta}(t, \cdot)|^2_{L^2(R)} \\
\leq \frac{25E\beta^2}{8\varepsilon} \|\partial_x u_{\varepsilon, \beta}\|_{L^\infty((0,T) \times R)}^2 \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(R)}^2 \\
+ \frac{E\beta^2}{2 \varepsilon^2} |\partial_{xxx} u_{\varepsilon, \beta}(t, \cdot)|^2_{L^2(R)} \\
\leq C_0 \beta^2 \frac{1}{\varepsilon} \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(R)}^2 + \frac{E\beta^2}{2 \varepsilon^2} |\partial_{xxx} u_{\varepsilon, \beta}(t, \cdot)|^2_{L^2(R)} \\
\leq C_0 D \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(R)}^2 + \frac{E\beta^2}{2 \varepsilon^2} |\partial_{xxx} u_{\varepsilon, \beta}(t, \cdot)|^2_{L^2(R)},
\]

\[
3\beta \int_R u_{\varepsilon, \beta}^2 |\partial_{xx} u_{\varepsilon, \beta}| \, dx \leq 3\beta \|u_{\varepsilon, \beta}\|_{L^\infty((0,T) \times R)}^2 \int_R |\partial_x u_{\varepsilon, \beta}| \|\partial_{xx} u_{\varepsilon, \beta}| \, dx \\
\leq 3C_0 D\varepsilon^2 \int_R |\partial_x u_{\varepsilon, \beta}|^2 u_{\varepsilon, \beta} \, dx = 3 \int_R \varepsilon \frac{2}{\beta} \partial_x u_{\varepsilon, \beta} |\partial_{xx} u_{\varepsilon, \beta}| \, dx
\]
\[
3\beta^2 \int_{\mathbb{R}} u_{\epsilon,\beta}^2 |\partial_x u_{\epsilon,\beta}|^4 \, dx 
\leq \frac{3\varepsilon}{2} \left( \| \partial_x u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + C_0^2 D^2 \varepsilon^3 \left( \| \partial_t^2 u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \right)
\]

Then, it follows from (3.7) that

\[
\frac{d}{dt} \left( \frac{1}{4} \left( A + C \right) \varepsilon^2 + \frac{(A + C) \varepsilon^2}{2} \| \partial_x u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \right)
+ \frac{d}{dt} \left( \frac{A^2 \varepsilon^2}{2} \| \partial_{x}^3 u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{E \beta^2}{2} \| \partial_{xxx}^2 u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \right)
+ \left( \frac{C}{2} - (1 - 2C) \right) \varepsilon^2 \| \partial_{xx}^2 u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{B \beta^2 \varepsilon^2}{2} \| \partial_{xxx}^2 u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})}
+ \left( \frac{3}{2} - \frac{C_0 D}{B} \right) \varepsilon \| u_{\epsilon,\beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C_0 \varepsilon \| \partial_x u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})}.
\]

We search \( A, B, C, E \) such that

\[
\begin{align*}
1 - 2C &> 0, \\
C &> 2 - 8B^2 > 0, \\
E - 1 &> 0, \\
A - 2A^2 - C_0^2 D^2 &> 0, \\
3 - \frac{C_0 D}{B} &> 0,
\end{align*}
\]

that is

\[
\begin{align*}
C &< \frac{1}{2}, \\
B^2 &< \frac{C}{16}, \\
E &> 1, \\
2A^2 - A + C_0^2 D^2 &< 0, \\
& \frac{3B}{2C_0}.
\end{align*}
\]
We choose

\[(3.11)\quad C = \frac{1}{4}, \quad E = 2.\]

It follows from the second inequality of \((3.10)\), and \((3.11)\) that

\[B < \frac{1}{8}.\]

Hence, we can choose

\[(3.12)\quad B = \frac{1}{9}.\]

Substituting \((3.12)\) in the fifth inequality of \((3.10)\), we have

\[(3.13)\quad D < \frac{1}{6C_0}\]

The fourth inequality admits solution when

\[(3.14)\quad D < \frac{2\sqrt{2}}{8C_0}.

It follows from \((3.13)\) and \((3.14)\) that

\[(3.15)\quad D < \min\left\{ \frac{1}{6C_0}, \frac{2\sqrt{2}}{8C_0} \right\} = \frac{1}{6C_0}.

Therefore, from \((3.10)\) and \((3.11)\), there exist \(0 < A_1 < A_2\) such that

\[(3.16)\quad 0 < A_1 < A < A_2.

Substituting \((3.11)\), \((3.12)\), and \((3.15)\) in \((3.9)\), from \((3.16)\), we get

\[
\frac{d}{dt} \left( \frac{1}{4} \| u_{\epsilon,\beta}(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{(4A + 1) \epsilon^2}{8} \| \partial_x u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ \frac{d}{dt} \left( \frac{A \beta^2 \epsilon^2}{2} \| \partial_{xxx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \beta^4 \| \partial_{xxxx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \right) \\
+ \frac{\beta \epsilon^2 + 18 \beta^2}{18} \frac{d}{dt} \| \partial_{xx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\beta \epsilon^2}{9} \| \partial_{x} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ \frac{\epsilon}{8} \| \partial_t u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\beta \epsilon^2}{18} \| \partial_{xxx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ \frac{73 \beta^2 \epsilon}{648} \| \partial_{xx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\beta \epsilon^2}{2} \| \partial_{xxx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ K_2 \epsilon^3 \| \partial_{xx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + K_2 \epsilon \| u_{\epsilon,\beta}(t, \cdot) \| \partial_{xx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
\leq C_0 \epsilon \| \partial_x u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})},
\]

for some \(K_1, K_2 > 0\).

An integration on \((0, t)\), \((2.5)\), and \((3.1)\) give

\[
\frac{1}{4} \| u_{\epsilon,\beta}(t, \cdot) \|^4_{L^4(\mathbb{R})} + \frac{(4A + 1) \epsilon^2}{8} \| \partial_x u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ \frac{A \beta^2 \epsilon^2}{2} \| \partial_{xxx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \beta^4 \| \partial_{xxxx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ \frac{\beta \epsilon^2 + 18 \beta^2}{18} \| \partial_{xx} u_{\epsilon,\beta}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\beta \epsilon^2}{9} \int_0^t \| \partial_{xxx} u_{\epsilon,\beta}(s, \cdot) \|^2_{L^2(\mathbb{R})} ds
\]

for some \(0 < A_1 < A_2\) such that
Let us consider a compactly supported entropy–entropy flux pair $(\eta, q)$. Multiplying \(2.14\) by $\eta'(u_{\varepsilon, \beta})$, we have

$$
\partial_t \eta(u_{\varepsilon, \beta}) + \partial_x q(u_{\varepsilon, \beta}) = \varepsilon \eta'(u_{\varepsilon, \beta}) \partial_x^2 u_{\varepsilon, \beta} - \beta^2 \partial_{xxx} u_{\varepsilon, \beta} - \beta^2 \varepsilon \eta'(u_{\varepsilon, \beta}) \partial_x \partial_{xxx} u_{\varepsilon, \beta}
$$

where $I_{1, \varepsilon, \beta}$, $I_{2, \varepsilon, \beta}$, $I_{3, \varepsilon, \beta}$, $I_{4, \varepsilon, \beta}$, $I_{5, \varepsilon, \beta}$, $I_{6, \varepsilon, \beta}$ are defined in \(2.14\).

As in \[9\] Theorem 3.1, we obtain that $I_{1, \varepsilon, \beta} \rightarrow 0$ in $H^{-1}((0, T) \times \mathbb{R})$, $\{I_{2, \varepsilon, \beta}\}_{\varepsilon, \beta>0}$ is bounded in $L^1((0, T) \times \mathbb{R})$, $I_{4, \varepsilon, \beta} \rightarrow 0$ in $H^{-1}((0, T) \times \mathbb{R})$, $I_{5, \varepsilon, \beta} \rightarrow 0$ in $L^1((0, T) \times \mathbb{R})$, while as in \[8\] Theorem 2.1 $I_{3, \varepsilon, \beta} \rightarrow 0$ in $H^{-1}((0, T) \times \mathbb{R})$, and, $I_{4, \varepsilon, \beta} \rightarrow 0$ in $L^1((0, T) \times \mathbb{R})$.

Arguing in \[8\] Theorem 2.1, we have \(2.13\).
APPENDIX A. THE BENJAMIN-BONA-MAHONY EQUATION

In this appendix, we consider the Benjamin-Bona-Mahony equation
\begin{equation}
\partial_t u + u \partial_x u - \beta \partial_x^3 u = 0.
\end{equation}
We augment (A.1) with the initial condition
\begin{equation}
u(0, x) = u_0(x),
\end{equation}
on which we assume (1.17). We study the dispersion-diffusion limit for (A.1). Therefore, we fix two small numbers \( \varepsilon, \beta \) and consider the following third order problem
\begin{equation}
\begin{aligned}
\partial_t u_{\varepsilon, \beta} + u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} - \beta \partial_x^3 u_{\varepsilon, \beta} &= \varepsilon \partial_x^2 u_{\varepsilon, \beta}, \quad t > 0, \ x \in \mathbb{R}, \\
u_{\varepsilon, \beta}(0, x) &= u_{\varepsilon, \beta, 0}(x), \quad x \in \mathbb{R},
\end{aligned}
\end{equation}
where \( u_{\varepsilon, \beta, 0} \) is a \( C^\infty \) approximation of \( u_0 \) such that
\begin{equation}
u_{\varepsilon, \beta, 0} \to u_0 \quad \text{in} \ L^p_{\text{loc}}(\mathbb{R}), \ 1 \leq p < 2, \ \text{as} \ \varepsilon, \ \beta \to 0,
\end{equation}
and \( C_0 \) is a constant independent on \( \varepsilon \) and \( \beta \).

The main result of this section is the following theorem.

Theorem A.1. Assume that (1.17) and (A.4) hold. If (2.3) holds, then, there exist two sequences \( \{\varepsilon_n\}_{n \in \mathbb{N}}, \ \{\beta_n\}_{n \in \mathbb{N}} \), with \( \varepsilon_n, \beta_n \to 0 \), and a limit function
\begin{equation}
u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R})),
\end{equation}
such that
\begin{enumerate}
\item \( u_{\varepsilon_n, \beta_n} \to u \) strongly in \( L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \), for each \( 1 \leq p < 2 \),
\item \( u \) a distributional solution of (1.2).
\end{enumerate}

Moreover, if (2.4) holds
\begin{enumerate}
\item \( u \) is the unique entropy solution of (1.2).
\end{enumerate}

Let us prove some a priori estimates on \( u_{\varepsilon, \beta} \), denoting with \( C_0 \) the constants which depend only on the initial data.

Arguing as [35], we have the following result

Lemma A.1. For each \( t > 0 \),
\begin{equation}
\|u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \varepsilon \int_0^t \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0.
\end{equation}

Moreover,
\begin{equation}
\|u_{\varepsilon, \beta}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_0 \beta^{-\frac{1}{4}}.
\end{equation}

Lemma A.2. Assume (2.3). For each \( t > 0 \),
\begin{equation}
\begin{aligned}
\beta \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 &+ \frac{2 \beta^2 + \beta^2 \varepsilon}{2} \|\partial_x^2 u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&+ \frac{3 \beta \varepsilon}{2} \int_0^t \|\partial_x^2 u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \frac{\beta^2 \varepsilon}{2} \int_0^t \|\partial_x^3 u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0.
\end{aligned}
\end{equation}
Proof. Let $t > 0$. Multiplying \((A.3)\) by $-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}$, we have

\[
\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta} + \left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} \leq -\beta \left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}\right)\partial^3_{lxx}u_{\varepsilon,\beta}
\]
\[
= \varepsilon \left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}.
\]

(A.8)

Since

\[
\int_{\mathbb{R}} \left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta} \, dx = \beta^{\frac{1}{2}} \frac{d}{dt} \|\partial_x u_{\varepsilon,\beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \beta\varepsilon \left\|\partial^2_{lxx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})},
\]

\[
- \beta \int_{\mathbb{R}} \left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}\right)\partial^3_{lxx}u_{\varepsilon,\beta} \, dx
\]

\[
= \beta^{\frac{1}{2}} \frac{d}{dt} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})} + \beta\varepsilon \left\|\partial^3_{lxx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})},
\]

\[
\varepsilon \int_{\mathbb{R}} \left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta} - \beta\varepsilon\partial^3_{lxx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta} \, dx = -2\beta^{\frac{1}{2}}\varepsilon \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})} - \beta\varepsilon \frac{d}{dt} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})},
\]

integrating (A.8) on $\mathbb{R}$, we get

\[
\frac{d}{dt} \left(\beta^{\frac{1}{2}} \|\partial_x u_{\varepsilon,\beta}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \frac{2\beta^{\frac{1}{2}} + \beta\varepsilon^2}{2} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})}\right)
\]

\[
+ 2\beta^{\frac{1}{2}}\varepsilon \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})} + \beta\varepsilon \left\|\partial^3_{lxx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})}
\]

\[
= 2\beta^{\frac{1}{2}} \int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta} \, dx - \beta\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^3_{lxx}u_{\varepsilon,\beta} \, dx.
\]

Due to (2.3), (A.6), and the Young inequality,

\[
2\beta^{\frac{1}{2}} \int_{\mathbb{R}} |u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}| \partial^2_{xx}u_{\varepsilon,\beta} \, dx = \beta^{\frac{1}{2}} \int_{\mathbb{R}} \left|\frac{2u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac{1}{2}}}\right| \left|\varepsilon^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}\right| \, dx
\]

\[
\leq \frac{2\beta^{\frac{1}{2}}}{\varepsilon} \int_{\mathbb{R}} u_{\varepsilon,\beta}^2 (\partial_x u_{\varepsilon,\beta})^2 \, dx + \frac{\beta^{\frac{1}{2}}\varepsilon}{2} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})}
\]

(A.10)

\[
\leq C_0\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\beta}^2 (\partial_x u_{\varepsilon,\beta})^2 \, dx + \frac{\beta^{\frac{1}{2}}\varepsilon}{2} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})}
\]

\[
\leq C_0\varepsilon \left\|u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^\infty(\mathbb{R})} \left\|\partial_x u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^{\frac{1}{2}}\varepsilon}{2} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})}
\]

\[
\leq C_0\varepsilon \left\|u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^{\frac{1}{2}}\varepsilon}{2} \left\|\partial^2_{xx}u_{\varepsilon,\beta}(t, \cdot)\right\|^2_{L^2(\mathbb{R})}.
\]
Thanks to (A.6), and the Young inequality,
\[
\beta \varepsilon \int_{\mathbb{R}} |u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}| \partial_{x x x}^2 u_{\varepsilon, \beta} |dx = \varepsilon \int_{\mathbb{R}} |u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}| |\beta \partial_{x x x}^2 u_{\varepsilon, \beta}| \, dx
\]
\[
\leq \frac{\varepsilon}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 (\partial_x u_{\varepsilon, \beta})^2 \, dx + \frac{\beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2
\]
\[
\leq \frac{\varepsilon}{2} \left\| u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^\infty}^2 \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 + \frac{\beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2
\]
\[
\leq \frac{\varepsilon}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 + \frac{\beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2.
\]
(A.11)

It follows from (A.9), (A.10), and (A.11) that
\[
\frac{d}{dt} \left( \beta \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 + \frac{2 \beta^2 + \beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 \right)
\]
\[
\leq \frac{3 \beta \varepsilon}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 + \frac{\beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2
\]
\[
+ \beta \varepsilon \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 \leq C_0 \varepsilon \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2.
\]

Hence,
\[
\frac{d}{dt} \left( \beta \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 + \frac{2 \beta^2 + \beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 \right)
\]
\[
\leq \frac{3 \beta \varepsilon}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 + \frac{\beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2
\]
\[
+ \beta \varepsilon \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 \leq C_0 \varepsilon \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2.
\]

An integration on (0, t) and (A.5) give
\[
\beta \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2 + \frac{2 \beta^2 + \beta \varepsilon^2}{2} \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2}^2
\]
\[
+ \frac{3 \beta \varepsilon}{2} \int_0^t \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(s, \cdot) \right\|_{L^2}^2 \, ds
\]
\[
+ \frac{\beta \varepsilon^2}{2} \int_0^t \left\| \partial_{x x x}^2 u_{\varepsilon, \beta}(s, \cdot) \right\|_{L^2}^2 \, ds \leq C_0 + C_0 \varepsilon \int_0^t \left\| \partial_x u_{\varepsilon, \beta}(s, \cdot) \right\|_{L^2}^2 \, ds \leq C_0,
\]
that is (A.11).

We continue by proving the following result

Lemma A.3. Assume that (1.17), (2.3), and (A.4) hold. Then, for any compactly supported entropy–entropy flux pair (\eta, q), there exist two sequences \{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}, with \varepsilon_n, \beta_n \to 0, and a limit function
\[
u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R})),
\]
such that (2.12) holds and
(A.12) \quad \nu is a distributional solution of (1.2).
Proof. Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\). Multiplying \((A.3)\) by \(\eta'(u,\beta)\), we have
\[
\partial_t \eta(u,\beta) + \partial_x q(u,\beta) = \varepsilon \eta'(u,\beta) \partial_{xx}^2 u_{e,\beta} + \beta \eta'(u,\beta) \partial_{xx}^3 u_{e,\beta} = I_{1,\varepsilon,\beta} + I_{2,\varepsilon,\beta} + I_{3,\varepsilon,\beta} + I_{4,\varepsilon,\beta},
\]
where
\[
I_{1,\varepsilon,\beta} = \partial_x(\varepsilon \eta'(u,\beta) \partial_x u_{e,\beta}),
I_{2,\varepsilon,\beta} = -\beta \eta''(u,\beta)(\partial_x u_{e,\beta})^2,
I_{3,\varepsilon,\beta} = \partial_x(\beta \eta'(u,\beta) \partial_{xx}^2 u_{e,\beta}),
I_{4,\varepsilon,\beta} = -\beta \eta''(u,\beta) \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta}.
\]
Fix \(T > 0\). Arguing in [12, Lemma 3.2], we have that \(I_{1,\varepsilon,\beta} \to 0\) in \(H^{-1}((0,T) \times \mathbb{R})\), and \(\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta > 0}\) is bounded in \(L^1((0,T) \times \mathbb{R})\).
We claim that
\[
I_{3,\varepsilon,\beta} \to 0 \quad \text{in} \quad H^{-1}((0,T) \times \mathbb{R}), \quad T > 0, \quad \text{as} \quad \varepsilon \to 0.
\]
By \((2.3)\) and \((A.7)\),
\[
\| \beta \eta'(u,\beta) \partial_{xx}^2 u_{e,\beta} \|^2_{L^2(0,T) \times \mathbb{R}}
\leq \beta^2 \| \eta' \|_{L^\infty(\mathbb{R})} \| \partial_{xx}^2 u_{e,\beta} \|^2_{L^2(0,T) \times \mathbb{R}}
= \| \eta' \|_{L^\infty(\mathbb{R})} \beta^2 \varepsilon \| \partial_{xx}^2 u_{e,\beta} \|^2_{L^2(0,T) \times \mathbb{R}}
\leq C_0 \| \eta' \|_{L^\infty(\mathbb{R})} \varepsilon \to 0.
\]
Let us show that
\(I_{4,\varepsilon,\beta}\) is bounded in \(L^1((0,T) \times \mathbb{R})\), \(T > 0\).
Thanks to \((2.3)\), \((A.5)\), \((A.7)\), and the Hölder inequality,
\[
\| \beta \eta''(u,\beta) \partial_{xx}^2 u_{e,\beta} \|_{L^1(0,T) \times \mathbb{R}}
\leq \beta \| \eta'' \|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} \partial_x u_{e,\beta} \partial_{xx}^2 u_{e,\beta} |dsdx|
= \| \eta'' \|_{L^\infty(\mathbb{R})} \frac{\beta^2 \varepsilon}{\beta^2 + \varepsilon} \| \partial_x u_{e,\beta} \|_{L^2(0,T) \times \mathbb{R}} \| \partial_{xx}^2 u_{e,\beta} \|_{L^2(0,T) \times \mathbb{R}}
\leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})}.
\]
Arguing as in [35], we have \((A.12)\). \(\square\)

**Lemma A.4.** Assume \((1.17)\), \((2.3)\), and \((A.3)\) hold. Then, for any compactly supported entropy–entropy flux pair \((\eta, q)\), there exist two sequences \(\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}\), with \(\varepsilon_n, \beta_n \to 0\), and a limit function
\[
u \in L^\infty(\mathbb{R}^+ ; L^2(\mathbb{R})),
\]
such that \((2.12)\) and \((2.15)\) hold.

**Proof.** Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\). Multiplying \((A.3)\) by \(\eta'(u,\beta)\), we have
\[
\partial_t \eta(u,\beta) + \partial_x q(u,\beta) = \varepsilon \eta'(u,\beta) \partial_{xx}^2 u_{e,\beta} + \beta \eta'(u,\beta) \partial_{xx}^3 u_{e,\beta} = I_{1,\varepsilon,\beta} + I_{2,\varepsilon,\beta} + I_{3,\varepsilon,\beta} + I_{4,\varepsilon,\beta},
\]
where $I_1, I_2, I_3, I_4$ are defined in (A.13).

As in Lemma 2.4 we have that $I_1, I_2, I_3, I_4$ \( \to 0 \) in $H^{-1}((0, T) \times \mathbb{R})$, \{I_{2, \beta} \}_{\beta > 0}$ is bounded in $L^1((0, T) \times \mathbb{R})$, while $I_{4, \beta} \to 0$ in $L^1((0, T) \times \mathbb{R})$.

Arguing as in [24], we have (2.15). \( \square \)

**Proof of Theorem A.1.** Theorem A.1 follows from Lemmas A.3 and A.4. \( \square \)

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(Giuseppe Maria Coclite and Lorenzo di Ruvo)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BARI, VIA E. ORABONA 4, 70125 BARI, ITALY

E-mail address: giuseppemaria.coclite@uniba.it, lorenzo.diruvo@uniba.it

URL: http://www.dm.uniba.it/Members/coclitegm/