Convergent Iterative Solutions for a Sombrero-Shaped Potential in Any Space Dimension and Arbitrary Angular Momentum

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Abstract

We present an explicit convergent iterative solution for the lowest energy state of the Schrödinger equation with an $N$-dimensional radial potential $V = \frac{g^2}{2}(r^2 - 1)^2$ and an angular momentum $l$. For $g$ large, the rate of convergence is similar to a power series in $g^{-1}$.

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1. Introduction

The problem of a non-relativistic particle moving in an \( N \)-dimensional Sombrero-shaped potential provides a prototype example of the spontaneous symmetry breaking mechanism. Yet, even when \( N = 1 \), it is difficult to solve the corresponding Schrödinger equation with a quartic potential\[1-10]. In this paper, we shall give explicit convergent iterative solutions for the Schrödinger equation

\[
H \Psi = (-\frac{1}{2} \nabla^2 + V) \Psi = E \Psi
\]  

in \( N \)-dimension and with angular momentum \( l \). Let \( q \) be the Cartesian coordinates

\[
q = (q_1, q_2, \cdots, q_N)
\]

with \( N > 1 \) and \( \nabla^2 \) the Laplacian

\[
\nabla^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2}.
\]

The potential is

\[
V = \frac{g^2}{2} (r^2 - 1)^2,
\]

where

\[
r^2 = \sum_{i=1}^{N} q_i^2.
\]

To illustrate our approach, it may be useful to consider first the groundstate (\( s \)-state) of \( H \). By examining the shape of \( V \), we can guess a reasonable trial function \( \Phi \), which approximates \( \Psi \) when \( \Psi \) is large (i.e., near \( r = 1 \)). By taking the Laplacian of \( \Phi \), we can cast \( \Phi \) as the groundstate of a different Schrödinger equation.

Define

\[
U(r) - E_0 \equiv \Phi^{-1}(\frac{1}{2} \nabla^2 \Phi)
\]

and

\[
H_0 \equiv -\frac{1}{2} \nabla^2 + U(r),
\]

then

\[
H_0 \Phi = E_0 \Phi.
\]
Introducing
\[ w \equiv H_0 - H = U - V \]  
(1.9)
and
\[ \mathcal{E} = E_0 - E \]  
(1.10)
the original Schrödinger equation (1.1) can be written as
\[ (H_0 - E_0)\Psi = (w - \mathcal{E})\Psi. \]  
(1.11)

In (1.6), only the difference \( U - E_0 \) is defined. The constant \( E_0 \) may be chosen by requiring
\[ w(\infty) = 0. \]  
(1.12)

Following Refs.[11-14], the original Schrödinger equation (1.1) will be solved through an iterative sequence
\[ (H_0 - E_0)\Psi(m) = (w - \mathcal{E}_m)\Psi(m - 1) \]  
(1.13)
with \( m = 1, 2, \ldots \) and when \( m = 0 \)
\[ \Psi(0) = \Phi. \]  
(1.14)
The following simple discussion provides the motivation of this approach. Multiplying (1.13) by \( \Phi \) and (1.8) by \( \Psi(m) \), we find their difference given by the familiar Wronskian-type expression
\[ -\frac{1}{2} \nabla \cdot (\Phi \nabla \Psi(m) - \Psi(m) \nabla \Phi) = (w - \mathcal{E}_m)\Phi \Psi(m - 1). \]  
(1.15)
Integrating (1.15) over all space, we have
\[ \mathcal{E}_m = \int w\Phi \Psi(m - 1)d^Nq / \int \Phi \Psi(m - 1)d^Nq. \]  
(1.16)
Note that if \( \Psi^0(m) \) is a solution of (1.13), so is
\[ \Psi(m) = \Psi^0(m) + c \Phi, \]  
(1.17)
where \( c \) is a constant. Since \( \Phi \) is the groundstate of \( H_0 \), it is positive everywhere. By adjusting the constant \( c \), \( \Psi(m) \) can also be made positive everywhere. Thus, if \( w \) is bounded, so is \( \mathcal{E}_m \). This approach prevents the kind
of divergence encountered by the usual perturbative series, and enables us to
derive explicit convergent iterative solutions, as we shall see.

In Section 2, we show that when $V$ is a radial potential, the successive iter-
ative solutions $\Psi(m)$ can be solved by simple quadratures. For the Sombrero-
shaped potential (1.4) the low-lying wave function $\Psi$ is known reasonably well
when it is in the asymptotic region when $r$ is large, and also when its ampli-
tude is large (i.e., $r$ near 1). In Section 3, a reasonable trial function $\Phi$ is
constructed whose fractional deviation from $\Psi$ is large only for $r$ small (i.e.,
when the absolute magnitude of $\Psi$ is also very small); the same should also
apply to $w(r)$, the difference between $H_0$ and $H$, given by (1.9). As will be
proved in Section 4, $w(r)$ is positive everywhere and its radial derivative $w'(r)$
always negative, making $w$ maximal at $r = 0$. These conditions enable us to
apply the Hierarchy Theorem[12-14] which guarantees the convergence, as will
be discussed in Section 5. In Appendix B, we show that the rate of convergence
for large $g$ is similar to a power series in $g^{-1}$. 

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2. The Angular Momentum Operator and the Radial Equation

Express the Cartesian coordinates $q_1$, $q_2$, $\cdots$, $q_N$ of (1.2) in terms of the radial variable $r$ and the $(N - 1)$ angular variables

$$\theta_1, \theta_2, \cdots, \theta_{N-2} \text{ and } \theta_{N-1}$$

through

$$q_1 = r \cos \theta_1, \quad q_2 = r \sin \theta_1 \cos \theta_2,$$
$$q_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3, \cdots,$$
$$q_{N-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{N-2} \cos \theta_{N-1}$$

and

$$q_N = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{N-2} \sin \theta_{N-1}$$

with

$$0 \leq \theta_i < \pi \text{ for } i = 1, 2, \cdots, N - 2$$

and

$$0 \leq \theta_{N-1} \leq 2\pi.$$ 

Correspondingly, the line elements are

$$dr, \quad rd\theta_1, \quad r \sin \theta_1 d\theta_2, \quad r \sin \theta_1 \sin \theta_2 d\theta_3,$$
$$r \sin \theta_1 \sin \theta_2 \sin \theta_3 d\theta_4, \quad \cdots, \quad r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{N-2} d\theta_{N-1}$$

and the Laplacian operator is

$$\nabla^2 = \frac{1}{r^{2K}} \frac{\partial}{\partial r} \left(r^{2K} \frac{\partial}{\partial r}\right) - \frac{1}{r^2} L^2(N - 1)$$

where

$$K = \frac{1}{2}(N - 1),$$
\[ \mathcal{L}^2(N - 1) = -\frac{1}{\sin^{N-2} \theta_1 \partial \theta_1} \sin^{N-2} \theta_1 \partial \theta_1 \mathcal{L}^2(N - 2) + \frac{1}{\sin^2 \theta_1} \mathcal{L}^2(N - 2) \]

\[ \mathcal{L}^2(N - 2) = -\frac{1}{\sin^{N-3} \theta_2 \partial \theta_2} \sin^{N-3} \theta_2 \partial \theta_2 \mathcal{L}^2(N - 3) + \frac{1}{\sin^2 \theta_2} \mathcal{L}^2(N - 3) \]

\[ \cdots \cdots \]

\[ \mathcal{L}^2(2) = -\frac{1}{\sin \theta_{N-2} \partial \theta_{N-2}} \sin \theta_{N-2} \partial \theta_{N-2} \mathcal{L}^2(1) + \frac{1}{\sin^2 \theta_{N-2}} \mathcal{L}^2(1) \]

\[ \mathcal{L}^2(1) = -\frac{\partial^2}{\partial \theta_{N-1}^2}. \]

The square of the angular momentum operator on an \( n \)-sphere is \( \mathcal{L}^2(n) \). From (2.6), one sees readily that the commutator between any two \( \mathcal{L}^2(n) \) and \( \mathcal{L}^2(m) \) is zero; i.e.,

\[ [\mathcal{L}^2(n), \mathcal{L}^2(m)] = 0. \]  
(2.7)

As we shall show in Appendix A, the eigenvalues of each \( \mathcal{L}^2(n) \) are

\[ l(l + n - 1) \]  
(2.8)

with \( l = 0, 1, 2, \cdots \). Thus, for a radially symmetric potential, the wave function can be written as

\[ \Psi(r, \theta_1, \theta_2, \cdots, \theta_{N-1}) = R(r) \Theta(\theta_1, \theta_2, \cdots, \theta_{N-1}) \]  
(2.9)

with

\[ \mathcal{L}^2(N - 1) \Theta = l_1(l_1 + N - 2) \Theta, \]
\[ \mathcal{L}^2(N - 2) \Theta = l_2(l_2 + N - 3) \Theta, \]
\[ \cdots \]
\[ \mathcal{L}^2(2) \Theta = l_{N-2}(l_{N-2} + 1) \Theta \]  
(2.10)

and

\[ \mathcal{L}^2(1) \Theta = l_{N-1}^2 \Theta. \]
Correspondingly, by using (2.5) we find the radial part of the Schrödinger equation (1.1) for the lowest eigenstate of angular momentum \( l \) to be

\[
\left[ -\frac{1}{2} \nabla_r^2 + \frac{1}{2r^2} l(l + N - 2) + V(r) - E \right] R(r) = 0 \tag{2.11}
\]

with

\[
\nabla_r^2 = \frac{1}{r^2 K} \frac{d}{dr} \left( r^2 K \frac{d}{dr} \right) \tag{2.12}
\]

and \( l = l_1 \) given by the first equation of (2.10); i.e.,

\[
\mathcal{L}^2(N - 1) \Theta = l(l + N - 2) \Theta. \tag{2.13}
\]

Define

\[
R(r) = r^{-K} \psi(r). \tag{2.14}
\]

Eq. (2.11) becomes

\[
\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} k(k - 1) \right] V(r) - E \psi(r) = 0. \tag{2.15}
\]

where

\[
k = l + K = l + \frac{1}{2}(N - 1) \tag{2.16}
\]

and therefore \( k(k - 1) = K(K - 1) + l(l + N - 2) \), on account of \( K = \frac{1}{2}(N - 1) \).

As \( r \to 0 \), \( R \propto r^l \) and therefore \( \psi \propto r^{l+K} = r^k \).

The radial trial function \( \phi(r) \) shall be constructed in the next section. Similar to (2.15) it satisfies

\[
(H_0 - E_0) \phi(r) = 0 \tag{2.17}
\]

with

\[
H_0 = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2r^2} k(k - 1) + U(r). \tag{2.18}
\]

The iterative equation (1.11) becomes

\[
(H_0 - E_0) \psi_m(r) = (w(r) - \mathcal{E}_m) \psi_{m-1}(r) \tag{2.19}
\]

where, as before,

\[
w(r) = U(r) - V(r). \tag{2.20}
\]
Define
\[ f_m(r) \equiv \frac{\psi_m(r)}{\phi(r)}. \]  
(2.21)

As in (1.15)-(1.16), we have
\[-\frac{1}{2} \frac{d}{dr} (\phi^2 \frac{df_m}{dr}) = (w - E_m) \phi^2 f_{m-1} \]  
(2.22)

and
\[ E_m = \int_0^\infty w \phi^2 f_{m-1} dr / \int_0^\infty \phi^2 f_{m-1} dr. \]  
(2.23)

Eq. (2.22) can be readily integrated
\[ f_m(r) = f_m(\infty) - 2 \int_r^\infty \frac{dy}{\phi^2(y)} \int_y^\infty (w(x) - E_m) \phi^2(x) f_{m-1}(x) dx \]  
(2.24)
or, equivalently
\[ f_m(r) = f_m(0) - 2 \int_0^r \frac{dy}{\phi^2(y)} \int_0^y (w(x) - E_m) \phi^2(x) f_{m-1}(x) dx. \]  
(2.25)

Thus, with
\[ f_0(r) = 1, \]
each \( f_m(r) \) can be derived by quadratures. At each iteration, there is an arbitrary constant \( f_m(\infty) \) or \( f_m(0) \) (similar to the constant \( c \) in (1.17)). As we shall see, different choices of this constant can lead to different convergent iterative solutions.

Remarks: We note that in (2.11), the centrifugal potential \( \frac{1}{2r^2} l(l + N - 2) \) becomes the familiar \( l^2/2r^2 \) in two-dimension and \( l(l + 1)/2r^2 \) in three-dimension. Correspondingly, the term \( k(k - 1)/2r^2 \) in (2.15) becomes \( (l^2 - \frac{1}{4})/2r^2 \) in two-dimension, but remains \( l(l + 1)/2r^2 \) in three-dimension.
3. The Trial Function

To construct a reasonable trial function \( \phi(r) \), we start with the following two functions \( \phi_+(r) \) and \( \phi_-(r) \).

\[
\phi_+(r) = \frac{2r^k}{r+1} \left( \frac{1+a}{r+a} \right)^k e^{-gS_0(r)} \tag{3.1}
\]

and

\[
\phi_-(r) = \frac{2r^k}{r+1} \left( \frac{1+a}{r+a} \right)^k e^{-\frac{4}{3}g+gS_0(r)} \tag{3.2}
\]

where

\[
S_0(r) = \frac{1}{3} (r - 1)^2 (r + 2) \tag{3.3}
\]

and, as before, \( k \) is related to the angular momentum \( l \) and the dimensionality \( N \) by

\[
k = l + \frac{1}{2} (N - 1). \tag{3.4}
\]

The parameter \( a \) is positive but free within a range, which will be specified below. The trial function \( \phi(r) \) is given by

\[
\phi(r) = \begin{cases} 
    g_+ \phi_+(r) + g_- \phi_-(r) & \text{for } 0 \leq r < 1 \\
    (g_+ + g_- e^{-\frac{4}{3}g}) \phi_+(r) & \text{for } r > 1
\end{cases} \tag{3.5}
\]

where \( g_+ \) and \( g_- \) are constants given by

\[
g_\pm = g \pm \left( \frac{k}{a} + 1 \right). \tag{3.6}
\]

Thus, by construction \( r^{-k} \phi \) is regular at \( r = 0 \), with its derivative

\[
(r^{-k} \phi)' \equiv \frac{d}{dr} (r^{-k} \phi) = 0 \quad \text{at } r = 0. \tag{3.7}
\]

Throughout, ' denotes \( d/dr \). By construction, \( \phi(r) \) and \( \phi'(r) \) are continuous everywhere, with \( r \) varying from 0 to \( \infty \).
By differentiation, $\phi_+(r)$ and $\phi(r)$ satisfy respectively
\[
\left(-\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r^2}k(k - 1) + V + h\right)\phi_+ = E_0\phi_+ \tag{3.8}
\]
and, as in (2.17)-(2.20),
\[
\left(-\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r^2}k(k - 1) + V + w\right)\phi = E_0\phi, \tag{3.9}
\]
where
\[
E_0 = g(1 + ka), \tag{3.10}
\]
\[
h(r) = \frac{kag_-(r + a)}{r(r + a)} + \frac{ka^2g}{r + a} + \frac{1}{(r + 1)^2} + \frac{k(k + 1)}{2(r + a)^2} + \frac{ka}{(r + a)(r + 1)} \tag{3.11}
\]
and
\[
w(r) = h(r) + \hat{g}(r) \tag{3.12}
\]
with
\[
\hat{g}(r) = \begin{cases} 
\frac{2gg_-(1-\frac{ka(1-r^2)}{r(r+a)})}{g_+e^{\Lambda}+g_-}, & \text{for } 0 \leq r < 1 \\
0, & \text{for } r > 1,
\end{cases} \tag{3.13}
\]
where
\[
\Lambda = 2g(r - \frac{r^3}{3}) \tag{3.14}
\]
and by construction $[g_-(g_+e^{\Lambda}+g_-)] = g_+\phi_-/\phi$.

**Remarks**

(i) The exponent $\mp gS_0(r)$ in (3.1) and (3.2) satisfies
\[
\frac{1}{2}[\pm gS_0'(r)]^2 = \frac{1}{2}g^2(r^2 - 1)^2 = V(r). \tag{3.15}
\]

(ii) For $l = 0$ and $N = 1$, we have $k = 0$. Correspondingly, $\phi_+(r)$ of (3.1) becomes
\[
\phi_0(r) = \frac{2}{r + 1} e^{-gS_0(r)} \tag{3.16}
\]
which satisfies

\[
(-\frac{1}{2} \frac{d^2}{dr^2} + V + h_0)\phi_0 = g\phi_0
\]  

(3.17)

with

\[
h_0(r) = \frac{1}{(r + 1)^2}.
\]  

(3.18)

These equations reduce to the same expressions used before for the corresponding one-dimensional trial function[12, 13].

(iii) From (3.11) and (3.13), we see that at \(r = 0\) both \(h\) and \(\hat{g}\) contain an \(r^{-1}\) pole term. This is because neither the derivative of \(r^{-k}\phi_+(r)\) nor that of \(r^{-k}\phi_-(r)\) is zero at \(r = 0\). The trial function \(\phi(r)\) does satisfy \((r^{-k}\phi)' = 0\) at \(r = 0\), in accordance with (3.5) and (3.7). Thus, the potential function \(w(r) = h(r) + \hat{g}(r)\) given by (3.12) is regular at \(r = 0\).

Throughout the paper, we assume

\[
g > \frac{k}{a} + 1,
\]  

(3.19)

so that \(g_+ > 0\). Correspondingly, as will be shown, \(w(r)\) is positive and has a discontinuity at \(r = 1\).
4. Properties of \( w(r) \)

In this section, we shall establish

\[
\begin{align*}
    w(r) &> 0 \\
    w'(r) &< 0 \quad \text{for } r \geq 0.
\end{align*}
\] (4.1)

It is convenient to write (3.11) as

\[
h(r) = \sum_{i=1}^{5} h_i(r) \tag{4.2}
\]

with

\[
\begin{align*}
    h_1(r) &= \frac{kag}{r(r+a)}, \quad h_2(r) = \frac{ka^2g}{r+a}, \quad h_3(r) = \frac{1}{(r+1)^2}, \\
    h_4(r) &= \frac{k(k+1)}{2(r+a)^2} \quad \text{and} \quad h_5(r) = \frac{ka}{(r+a)(r+1)}. \tag{4.3}
\end{align*}
\]

Likewise, we decompose (3.13) for \( r < 1 \) as

\[
\hat{g}(r) = \sum_{i=6}^{8} \hat{g}_i(r) \tag{4.4}
\]

with

\[
\begin{align*}
    \hat{g}_6(r) &= \frac{2gg_-}{g_{+e\Lambda} + g_-}, \quad \hat{g}_7(r) = \frac{2gg_+kar}{(g_{+e\Lambda} + g_-)(r+a)} \\
    \hat{g}_8(r) &= -\frac{2kgg_-a}{(g_{+e\Lambda} + g_-)(r+a)}. \tag{4.5}
\end{align*}
\]

In order to show that \( w = h + \hat{g} \) satisfies (4.1), we note that for \( r > 1 \), \( \hat{g} = 0 \) in accordance with (3.3), and therefore

\[
w(r) = h(r).
\]

Since each of the \( h_i(r) \) in (4.3) satisfies

\[
h_i(r) > 0 \quad \text{and} \quad h'_i(r) < 0,
\]
therefore

\[ w(r) > 0 \quad \text{and} \quad w'(r) < 0 \quad \text{for} \quad r > 1. \quad (4.6) \]

For \( r < 1 \), it is convenient to combine first some of the \( h_i \) with \( \hat{g}_j \) given by (4.3) and (4.5).

Defining

\[ w_I \equiv h_1 + \hat{g}_8, \quad (4.7) \]

we find

\[ w_I = \frac{ka}{r(r+a)}(g_- - \frac{2gg_-}{g_+e^\Lambda + g_-}) = \frac{kag_+g_-(e^\Lambda - 1)}{r(r+a)(g_+e^\Lambda + g_-)}. \quad (4.8) \]

Likewise, write

\[ h_2 + \hat{g}_7 = gka\left[ \frac{a}{r+a} + \frac{2g_-r}{(g_+e^\Lambda + g_-)(r+a)} \right] \]

\[ = gka\left\{ \frac{a}{r+a} + \frac{2g_-[r+a] - a}{(g_+e^\Lambda + g_-)(r+a)} \right\} \]

\[ = gka\left\{ \frac{a}{r+a}[1 - \frac{2g_-}{g_+e^\Lambda + g_-}] + \frac{2g_-}{g_+e^\Lambda + g_-} \right\}. \quad (4.9) \]

Separate the two terms inside the curly brackets and write

\[ h_2 + \hat{g}_7 = w_{II} + w_0 \quad (4.10) \]

with

\[ w_{II} = \frac{gka^2}{r+a}\left[ g_+e^\Lambda - g_- \right] \quad (4.11) \]

in which the factor inside the square brackets is identical to the corresponding one in (4.9), and

\[ w_0 = \frac{2gg_-ka}{g_+e^\Lambda + g_-}. \quad (4.12) \]

Next, combine the above \( w_0 \) with \( \hat{g}_6 \) of (4.5) into a single term as follows:

\[ w_{VI} \equiv w_0 + \hat{g}_6 = \frac{2gg_-(ka + 1)}{g_+e^\Lambda + g_-}. \quad (4.13) \]
We now express \( w = h + \hat{g} \), for \( r < 1 \), in terms of a new sum of six terms:

\[
w = \sum_{m=I}^{VI} w_m
\]  

(4.14)

with \( w_I, w_{II} \) and \( w_{VI} \) given by (4.9), (4.11) and (4.13) respectively; the rest \( w_{III} = h_3, w_{IV} = h_4 \) and \( w_V = h_5 \). In explicit forms

\[
w_I = \frac{Zka}{r(r + a)}, \quad w_{II} = \frac{Ygka^2}{r + a},
\]

\[
w_{III} = \frac{1}{(r + 1)^2}, \quad w_{IV} = \frac{k(k + 1)}{2(r + a)^2},
\]

\[
w_V = \frac{ka}{(r + a)(r + 1)}, \quad \text{and} \quad w_{VI} = X2g(ka + 1)
\]

(4.15)

where

\[
X = \frac{g_-}{g_+e^\Lambda + g_-},
\]

(4.16)

\[
Y = \frac{g_+e^\Lambda - g_-}{g_+e^\Lambda + g_-},
\]

(4.17)

and

\[
Z = \frac{g_+g_-(e^\Lambda - 1)}{g_+e^\Lambda + g_-}.
\]

(4.18)

For the \( w_m \) with \( m = III, IV \) and \( V \), it is clear that

\[
w_m > 0 \quad \text{and} \quad w_m' < 0.
\]

(4.19)

For \( m = VI \), since in accordance with (3.14), for \( r < 1 \), \( \Lambda > 0 \) and

\[
\Lambda' = 2g(1 - r^2) > 0,
\]

(4.20)

\( w_{VI} \) also satisfies (4.19).

Introduce

\[
\xi = \frac{1}{2}\Lambda = g(r - \frac{r^3}{3})
\]

(4.21)
and
\[ e^{2b} = \frac{g^+}{g^-}. \] (4.22)

Recall
\[ g^+ + g^- = 2g, \] (4.23)
on account of (3.6), we can express \( g_\pm \) in terms of \( g \) and \( b \):
\[ g^+ = \frac{ge^b}{\cosh b} \quad \text{and} \quad g^- = \frac{ge^{-b}}{\cosh b}. \] (4.24)

Thus, we can rewrite (4.18) as
\[ Z = \frac{g}{\cosh b} \left( \frac{\sinh \xi}{\cosh(\xi + b)} \right), \] (4.25)
and correspondingly
\[ \frac{Z}{r} = \frac{g}{\cosh b} \left( \frac{\xi}{r} \right) \left( \frac{\tanh \xi}{\xi} \right) \left( \frac{\cosh \xi}{\cosh(\xi + b)} \right). \] (4.26)

We observe that
\[ \frac{d}{dr} \left( \frac{\xi}{r} \right) = -g \frac{2r}{3} < 0 \] (4.27)
and for \( r < 1 \),
\[ \frac{d\xi}{dr} = g(1 - r^2) > 0. \] (4.28)

Since for \( \xi \) positive
\[ \frac{d}{d\xi} \left( \frac{\tanh \xi}{\xi} \right) = \frac{1}{2\xi^2 \cosh^2 \xi} (2\xi - \sinh 2\xi) < 0 \] (4.29)
and
\[ \frac{d}{d\xi} \ln \left( \frac{\cosh \xi}{\cosh(\xi + b)} \right) = \tanh \xi - \tanh(\xi + b) < 0, \] (4.30)
we have for $r < 1$

$$\frac{d}{dr} \left( \frac{Z}{r} \right) < 0. \quad (4.31)$$

Thus from (4.15),

$$w_I > 0$$

and

$$w'_I < 0 \quad \text{for} \quad r < 1. \quad (4.32)$$

Lastly, we examine $w_{II}$. From (4.17) we see that

$$Y > 0, \quad (4.33)$$

and therefore, in accordance with (4.15)

$$w_{II} = \frac{Y g k a^2}{r + a} > 0. \quad (4.34)$$

By using (4.17) and (4.20), we also have

$$\left( \ln Y \right)' = \left( \frac{1}{g_+ e^\Lambda - g_-} - \frac{1}{g_+ e^\Lambda + g_-} \right) g_+ e^\Lambda 2g(1 - r^2)$$

$$= \left( \frac{2}{e^{2(\Lambda + 2b)} - 1} \right)e^{\Lambda + 2b} 2g(1 - r^2)$$

$$= \frac{2g(1 - r^2)}{\sinh(\Lambda + 2b)} > 0 \quad (4.35)$$

for $r < 1$. Therefore

$$\left( \ln w_{II} \right)' = \left( \ln Y \right)' - \frac{1}{r + a}$$

$$= \frac{2g(1 - r^2)}{\sinh(\Lambda + 2b)} - \frac{1}{r + a}$$

$$= \frac{1}{(r + a) \sinh(\Lambda + 2b)} [T - S] \quad (4.36)$$

with

$$S \equiv \sinh(\Lambda + 2b) \quad (4.37)$$
and
\[
T \equiv 2g(1 - r^2)(r + a). \quad (4.38)
\]
We seek to show that \( T - S < 0 \) for \( 0 < r < 1 \). On account of (4.20),
\[
(T - S)' = -4gr(r + a) + 2g(1 - r^2) - \Lambda' \cosh(\Lambda + 2b)
\]
\[
= -4gr(r + a) + 2g(1 - r^2)[1 - \cosh(\Lambda + 2b)].
\]
At any \( r < 1 \), \( (T - S)' < 0 \) and therefore
\[
T(r) - S(r) < T(0) - S(0). \quad (4.39)
\]
When \( r = 0 \), \( \Lambda(0) = 0 \),
\[
S(0) = \sinh 2b = \frac{1}{2} \left( \frac{g_+ - g_-}{g_+} \right) = \frac{g_+^2 - g_-^2}{2g_+g_-}
\]
\[
= \frac{1}{2} \left[ \frac{(g + \frac{k}{a} + 1)^2 - (g - \frac{k}{a} - 1)^2}{g^2 - (\frac{k}{a} + 1)^2} \right]
\]
\[
= \frac{2g(k + 1)}{g^2 - (\frac{k}{a} + 1)^2} \quad (4.40)
\]
and
\[
T(0) = 2ga. \quad (4.41)
\]
Setting the parameter \( a \) to be within an upper bound defined by
\[
a < \frac{k + 1}{g^2 - (\frac{k}{a} + 1)^2}, \quad (4.42)
\]
we have \( T(0) - S(0) < 0 \), and therefore
\[
w'_{11}(0) < 0 \quad \text{for} \quad r < 1. \quad (4.43)
\]
The inequality (4.42) can also be written as
\[
g^2 < \left( \frac{k}{a} + 1 \right) \left( \frac{k}{a} + \frac{1}{a} + 1 \right). \quad (4.44)
\]
Combining with the inequality (3.19), we require
\[ 1 < \frac{g}{k + a} + 1 < \sqrt{1 + \frac{1}{k + a}}. \] (4.45)

This is the sufficient condition for \( w'_{II} < 0 \), and therefore also
\[ w' < 0 \quad \text{for} \quad r < 1. \] (4.46)

At \( r = 1 \), in accordance with (4.4)-(4.5)
\[ \hat{g}_7(1) + \hat{g}_8(1) = 0 \] (4.47)
and therefore
\[ \hat{g}(1) = \hat{g}_6(1) = \frac{2gg_+}{g_+e_3^3g + g_-} > 0. \] (4.48)

Thus, \( w(r) \) has a discontinuity at \( r = 1 \), with
\[ w(1-) - w(1+) = \hat{g}(1) > 0 \] (4.49)

Together with (4.6) and (4.46), this proves that for the parameter \( a \) within the range (4.45),
\[ w(r) > 0 \quad \text{and} \quad w'(r) < 0 \]
over the entire range of \( r \).

Remarks
(i) The first inequality in (4.45)
\[ \frac{k}{a} + 1 < g \]
prevents the limit \( a \to 0 \); otherwise \( h(r) \) would have a double pole \( r^{-2} \), in accordance with (4.2)-(4.3).

(ii) Choose the parameter \( a \) within the range (4.45). As we will discuss in the next section, because of (4.1), the application of the Hierarchy Theorem leads to a convergent iterative solution in terms of quadratures for the lowest eigenstate Schroedinger wave function (1.1)-(1.5) in any dimension \( N \) and with arbitrary angular momentum \( l \).
5. The Convergent Iterative Solution

5.1 The Hierarchy Theorem

We begin with the iterative equations (2.22)-(2.25). Recalling that $f_m(r)$ and $\mathcal{E}_m$ are the $m^{th}$ order solutions for $f = \psi/\phi$ and $\mathcal{E} = E_0 - E$, we derive the corresponding $m^{th}$ order solutions for $\psi$ and $E$ to be

$$
\psi_m = \phi f_m \quad \text{and} \quad E_m = E_0 - \mathcal{E}_m \quad (5.1)
$$

Throughout this section, we assume the parameter $a$ to be restricted by the two inequalities given by (4.45), Therefore, for large $g$, $a$ is small. We distinguish two different conditions:

(A) $f_m(\infty) = 1$ for all $m$ \hspace{1cm} (5.2)

or

(B) $f_m(0) = 1$ for all $m$. \hspace{1cm} (5.3)

Correspondingly in (A), (2.24) becomes

$$
f_m(r) = 1 - 2 \int_0^{\infty} \frac{dy}{\phi^2(y)} \int_0^{\infty} (w(x) - \mathcal{E}_m)\phi^2(x) f_{m-1}(x)dx \hspace{1cm} (5.4)
$$

and in (B), (2.25) becomes

$$
f_m(r) = 1 - 2 \int_0^{r} \frac{dy}{\phi^2(y)} \int_0^{y} (w(x) - \mathcal{E}_m)\phi^2(x) f_{m-1}(x)dx. \hspace{1cm} (5.5)
$$

In both cases, we assume that the two inequalities of (4.45) hold. Thus $w(r)$ is positive and $w'(r)$ negative at all $r$, in accordance with (4.1). In case (A), apart from these two inequalities of (4.45) there is no other restriction on the magnitude of $w(r)$. In case (B), we assume $w(r)$ to be not too large so that for all $m$

$$
f_m(r) > 0 \quad \text{at all} \quad r. \hspace{1cm} (5.6)
$$

As we shall see, this is equivalent to requiring for all $m$

$$
f_m(\infty) > 0. \hspace{1cm} (5.7)
$$
Hierarchy Theorem[12-14]

(A) With the boundary condition \( f_m(\infty) = 1 \), we have for all \( m \)
\[
\mathcal{E}_{m+1} > \mathcal{E}_m
\]  
(5.8)
and
\[
\frac{d}{dr} \left( \frac{f_{m+1}(r)}{f_m(r)} \right) < 0 \quad \text{at any } r > 0.
\]  
(5.9)
Thus, the sequences \( \{\mathcal{E}_m\} \) and \( \{f_m(r)\} \) are all monotonic, with
\[
\mathcal{E}_1 < \mathcal{E}_2 < \mathcal{E}_3 < \cdots
\]  
(5.10)
and
\[
1 < f_1(r) < f_2(r) < f_3(r) < \cdots
\]  
(5.11)
at all finite \( r \).

(B) With the boundary condition \( f_m(0) = 1 \), we have for all odd \( m = 2n+1 \)
an ascending sequence
\[
\mathcal{E}_1 < \mathcal{E}_3 < \mathcal{E}_5 < \cdots,
\]  
(5.12)
but for all even \( m = 2n \), a descending sequence
\[
\mathcal{E}_2 > \mathcal{E}_4 > \mathcal{E}_6 > \cdots.
\]  
(5.13)
In addition, between any even \( m = 2n \) and any odd \( m = 2l+1 \)
\[
\mathcal{E}_{2n} > \mathcal{E}_{2l+1}.
\]  
(5.14)
Likewise, at any \( r \), for any even \( m = 2n \)
\[
\frac{d}{dr} \left( \frac{f_{2n+1}(r)}{f_{2n}(r)} \right) < 0,
\]  
(5.15)
whereas for any odd \( m = 2l+1 \)
\[
\frac{d}{dr} \left( \frac{f_{2l+2}(r)}{f_{2l+1}(r)} \right) > 0.
\]  
(5.16)
Furthermore,
\[
\lim_{m \to \infty} E_m = E
\]  
(5.17)
and
\[ \lim_{m \to \infty} f_m(r) = f(r) = \psi(r)/\phi(r). \] (5.18)

Thus, the boundary condition \( f_m(\infty) = 1 \) yields a sequence, in accordance with (5.10),
\[ E_1 > E_2 > E_3 > \cdots > E, \] (5.19)
with each member \( E_m \) an upper bound of \( E \), similar to the usual variational method.

On the other hand, with the boundary condition \( f_m(0) = 1 \), while the sequence of its odd members \( m = 2l + 1 \) yields a similar one, like (5.19), with
\[ E_1 > E_3 > E_5 > \cdots > E, \] (5.20)
its even members \( m = 2n \) satisfy
\[ E_2 < E_4 < E_6 < \cdots < E. \] (5.21)

It is unusual to have an iterative sequence of lower bounds of the eigenvalue \( E \). Together, these sequences may be quite efficient to pinpoint the limiting \( E \).

In Appendix B, by examining a simple prototype example in this class of problems, we shall show that for the Sombrero-shaped potential when \( g \) is large, the rate of convergence is similar to a power series in \( g^{-1} \).

### 5.2 Numerical Results

From (4.45), we see that for a given pair \((k, g)\), in order to apply the Hierarchy Theorem, the parameter \( a \) should be within a range
\[ a_{\text{min}} < a < a_{\text{max}} \] (5.22)
with the limits \( a_{\text{min}} \) and \( a_{\text{max}} \) determined by
\[ g = 1 + \frac{k}{a_{\text{min}}} \] (5.23)
and
\[ g^2 = (1 + \frac{k}{a_{\text{max}}})(1 + \frac{k + 1}{a_{\text{max}}}). \] (5.24)
Alternatively, for a given pair \((k, a)\), \(g\) should be within

\[ g_{\text{min}}^2 < g^2 < g_{\text{max}}^2 \]  

with

\[ g_{\text{min}} = 1 + \frac{k}{a} \]  

and

\[ g_{\text{max}}^2 = \left(1 + \frac{k}{a}\right)\left(1 + \frac{k + 1}{a}\right). \]  

Examples of these limiting values are given in Table 1.

Figure 1 gives examples of the iterative solutions of \(\psi_n\) and \(E_n\) with different \(n\) for the parameters \(g = 3\), \(k = 2\), and \(a = 1.2\). One sees that for both boundary conditions \(f_n(0) = 1\) and \(f_n(\infty) = 1\), the convergence sets in rapidly in accordance with the Hierarchy Theorem. In the case of the boundary condition \(f(0) = 1\), there is a limit to the range of these parameters, in order that \(f_n(\infty) > 0\), in accordance with (5.7). For example for \(g = 3\), the boundary condition \(f(0) = 1\) can be applied only for \(k \leq 2.5\), while the boundary condition \(f(\infty) = 1\) can be applied to any values of \(k\).

In Figures 2 and 3 we give the final radial wave function \(R(r)\) and energy \(E\) that satisfy (2.11) for

\( g = 3, \ l = 0 \) and \( N = 3, 4, 5, 6, \)

and also for

\( g = 3, \ N = 3 \) and \( l = 0, 1, 2, 3. \)  

Recalling (2.14) and (2.16), we have the curves in Fig.2 for \(l = 0\) and \(N = 3, 4, 5, 6\) corresponding to \(k = K = \frac{N-1}{2} = 1, 1.5, 2\) and 2.5; \(R(r) = \frac{1}{r^K}\psi(r)\). In Fig.3 the curves for \(N = 3\) and \(l = 0, 1, 2, 3\) correspond to \(K = 1\) and \(k = K + l = 1, 2, 3, 4\); \(R(r) = \frac{1}{r^K}\psi(r) = \frac{1}{r}\psi(r).\)
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Appendix A

Denote the solution $\Theta$ of (2.10) as

$$\Theta = \Theta_{l_1 l_2 \ldots l_{N-1}}^{(N-1)}(\theta_1, \theta_2, \ldots, \theta_{N-1}). \quad (A.1)$$

In this Appendix, we shall derive its explicit form inductively.

Write

$$\Theta_{l_1 l_2 \ldots l_{N-1}}^{(N-1)}(\theta_1, \theta_2, \ldots, \theta_{N-1}) = Z_{l_1 l_2 \ldots l_{N-1}}^{(N)}(\theta_2, \theta_3, \ldots, \theta_{N-1}) \quad (A.2)$$

with $Z_{l_1 l_2}^{(N)}$ depending only on $\theta_1$. By using (2.6) and (2.10), we see that $Z_{l_1 l_2}^{(N)}$ satisfies

$$\left[ \frac{1}{\sin^{N-2} \theta_1} \frac{\partial}{\partial \theta_1} \left( \frac{\sin^{N-2} \theta_1 \partial}{\partial \theta_1} \right) + \frac{l_2(l_2 + N - 3)}{\sin^2 \theta_1} - l_1(N - 2) \right] Z_{l_1 l_2}^{(N)} = 0. \quad (A.3)$$

For $N > 2$, it is convenient to denote

$$z = \cos \theta_1, \quad l_1 = l, \quad l_2 = m \quad (A.4)$$

and express (A.3) as

$$\left[ (1 - z^2) \frac{d^2}{dz^2} - (N - 1)z \frac{dz}{dz} + l(l + N - 2) - \frac{m(m + N - 3)}{1 - z^2} \right] Z_{l,m}^{(N)}(z) = 0. \quad (A.5)$$

When $N = 2$, the operator $L_2^{(N)} = L_2^{(1)}$ in the last equation of (2.6) is given by

$$L_2^{(1)} = \frac{\partial}{\partial \theta^2} \quad (A.6)$$

with $\theta$ denoting the corresponding $\theta_{N-1}$. Likewise, the last equation of (3.10) can be written as

$$-\frac{\partial^2}{\partial \theta^2} \Theta = l^2 \Theta \quad (A.7)$$

with $l^2$ denoting the corresponding $l_{N-1}^2$; its solutions will be designated as

$$\Theta = Z_{l,0}^{(2)} = \cos l \theta \quad (A.8)$$

for the functions even in $\theta$, and

$$\Theta = Z_{l,0}^{(2)} = \sin l \theta \quad (A.9)$$
for the functions odd in $\theta$, with

$$l = 0, 1, 2, \cdots$$  \hfill (A.10)

For $N = 3$, the eigenfunction of (A.5) are the Legendre polynomial when $m = 0$; i.e.,

$$Z_{l,0}^{(3)}(z) = P_l(z) = (-1)^l \frac{1}{2^l l!} \frac{d^l}{dz^l} (1 - z^2)^l$$  \hfill (A.11)

where $l = 0, 1, 2, \cdots$, as before. The corresponding $Z_{l,m}^{(N)}(z)$ for $m > 0$ is given by the associated Legendre function

$$Z_{l,m}^{(3)}(z) = P_l^m(z) = (1 - z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} P_l(z)$$  \hfill (A.12)

with

$$m \leq l.$$  \hfill (A.13)

In order to derive the functions $Z_{l,m}^{(N)}(z)$ for $N > 3$, it is useful to establish the following properties:

(i) Differentiate $m$ times the following equation for $u(z)$

$$(1 - z^2)\frac{d^2u}{dz^2} - az\frac{du}{dz} + bu = 0,$$  \hfill (A.14)

where $a$ and $b$ are constants, and denote

$$v(z) = \frac{d^m}{dz^m} u(z).$$  \hfill (A.15)

We obtain

$$(1 - z^2)\frac{d^2v}{dz^2} - \alpha_m z\frac{dv}{dz} + \beta_m v = 0$$  \hfill (A.16)

with

$$\alpha_m = a + 2m$$  \hfill (A.17)

and

$$\beta_m = b - ma - m(m - 1).$$

(ii) Instead of (A.14), $u(z)$ now satisfies

$$(1 - z^2)\frac{d^2u}{dz^2} - az\frac{du}{dz} + bu - \frac{cu}{1 - z^2} = 0$$  \hfill (A.18)
where $a$, $b$ and $c$ are again all constants. Write

$$w(z) = (1 - z^2)^{\frac{a}{2}} w(z). \quad (A.19)$$

we find

$$(1 - z^2)\frac{d^2w}{dz^2} - A_n z \frac{dw}{dz} + B_n w - \frac{C_n w}{1 - z^2} = 0 \quad (A.20)$$

with

$$A_n = a - 2n$$

$$B_n = b + n(a - n - 1)$$

and

$$C_n = c + n(a - n - 2); \quad (A.21)$$

therefore,

$$B_n - C_n = b - c + n. \quad (A.22)$$

We note that from (A.11) the Legendre polynomial $P_l(z) = Z_{l,0}^{(3)}(z)$ satisfies (A.14) with

$$a = 2, \quad \text{and} \quad b = l(l + 1). \quad (A.23)$$

Thus, for $N = \text{odd} = 2k + 1$, we have

$$Z_{l,0}^{(N)}(z) = \frac{d^{k-1}}{dz^{k-1}} P_{l+k-1}(z). \quad (A.24)$$

It can also be readily verified that for $m \geq 0$,

$$Z_{l,m}^{(N)}(z) = (1 - z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} Z_{l,0}^{(N)}(z). \quad (A.25)$$

For $N = \text{even} = 2k$, we start from $Z_{l,0}^{(2)}(z) = \cos l \theta$ with $z = \cos \theta$, and write

$$Z_{l,0}^{(N)}(z) = \frac{d^{k-1}}{dz^{k-1}} Z_{l+k-1,0}^{(2)}(z); \quad (A.26)$$

for $m > 0$, the corresponding $Z_{l,m}^{(N)}(z)$ is given by the same (A.25).
Appendix B

Because of (4.1), \( w > 0 \) and \( w' < 0 \), and the hierarchy theorem, the iterative solution with the boundary condition \( f(\infty) = 1 \) is convergent for any \( g > 0 \). By examining a simple prototype example in this class of problems, we shall show that for \( g \) large, the rate of convergence is similar to a power series in \( g^{-1} \).

Consider the Schroedinger equation

\[
-\frac{1}{2} \psi'' + (V(x) - E)\psi = 0 \tag{B.1}
\]

in one space dimension with

\[
V(x) = \frac{g^2}{2}(x^2 - 1)^2 \tag{B.2}
\]

and

\[
x \geq 0. \tag{B.3}
\]

Throughout this appendix, \( \cdot' \) denotes \( \frac{d}{dx} \). To simplify the analysis, we impose the boundary conditions

\[
\psi(\infty) = 0 \tag{B.4}
\]

and at the origin

\[
\left( \frac{\psi'}{\psi} \right)_{x=0} = g - 1.
\]

The trial function for the groundstate wave function is chosen to be

\[
\phi(x) = \frac{2}{x + 1} e^{-gS_0} \tag{B.5}
\]

with

\[
S_0(x) = \frac{1}{3}(x - 1)^2(x + 2); \tag{B.6}
\]

it satisfies

\[
-\frac{1}{2} \phi'' + (U(x) - g)\phi = 0 \tag{B.7}
\]

with

\[
U(x) = V(x) + u(x), \tag{B.8}
\]

27
\[ u(x) = \frac{1}{(1 + x)^2} \quad (B.9) \]

and the same boundary conditions \( \phi(\infty) = 0 \) and

\[ \left( \frac{\phi'}{\phi} \right)_{x=0} = g - 1. \quad (B.10) \]

Rewrite (B.1) as

\[- \frac{1}{2} \psi'' + (U(x) - g)\psi = (u(x) - E)\psi \quad (B.11)\]

with

\[ E = g - \mathcal{E}. \quad (B.12)\]

Introducing

\[ f \equiv \frac{\psi}{\phi}, \quad (B.13) \]

we find

\[- \frac{1}{2} (\phi^2 f')' = (u(x) - \mathcal{E})\phi^2 f; \quad (B.14)\]

at \( x = 0 \)

\[ f'(0) = 0. \quad (B.15) \]

To fix the relative normalization factor between \( \psi \) and \( \phi \), we impose at \( x = \infty \),

\[ f(\infty) = 1. \quad (B.16) \]

The groundstate wave function \( \psi(x) \) of the Schroedinger equation will be solved by introducing the iterative sequences

\[ \psi_1(x), \psi_2(x), \ldots, \psi_n(x), \ldots \]

and

\[ \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n, \ldots \]

with

\[- \frac{1}{2} \psi''_n + (U(x) - g)\psi_n = (u(x) - \mathcal{E}_n)\psi_{n-1} \quad (B.18)\]

and

\[ \psi_0 = \phi. \quad (B.19) \]
Define
\[ f_n(x) = \frac{\psi_n(x)}{\phi(x)}. \] (B.20)

From the above equations, it follows that
\[ -\frac{1}{2}(\phi^2 f_n')' = (u - \mathcal{E}_n)\phi^2 f_{n-1}; \] (B.21)

furthermore, similar to (B.15)-(B.16),
\[ f_n'(0) = 0 \] (B.22)

and
\[ f_n(\infty) = 1. \] (B.23)

From (B.21)-(B.22), we have
\[ \int_0^\infty (u(x) - \mathcal{E}_n)\phi^2(x)f_{n-1}(x)dx = 0. \] (B.24)

For any function \( F(x) \), define
\[ [F] = \int_0^\infty F(x)\phi^2(x)dx. \] (B.25)

From (B.24), we have
\[ \mathcal{E}_n = \frac{[uf_{n-1}]}{[f_{n-1}]} . \] (B.26)

It is convenient to regard (B.21) as an electrostatic analog problem with \(-\frac{1}{2}f_n'\)
as the electrostatic field,
\[ \kappa = \phi^2 \] (B.27)
as the dielectric constant,
\[ D_n = -\frac{1}{2}\kappa f_n' \] (B.28)
the displacement field and
\[ \sigma_n = (u - \mathcal{E}_n)\phi^2 f_{n-1} \] (B.29)
the electrostatic charge density. The electrostatic field equation is

\[ D'_n = \sigma_n, \quad (B.30) \]

and (B.24) gives the condition for zero total charge,

\[ \int_0^\infty \sigma_n(x) dx = 0. \quad (B.31) \]

Consequently,

\[ D_n(0) = 0. \quad (B.32) \]

At \( x = \infty \), because \( \phi(\infty) = 0 \), we also have

\[ D_n(\infty) = 0. \quad (B.33) \]

Furthermore, \( D_n(x) \) is positive and therefore

\[ f'_n(x) < 0. \quad (B.34) \]

From (B.22), (B.28) and (B.30), we find

\[ D_n(x) = \int_0^x \sigma_n(z) dz = -\int_x^\infty \sigma_n(z) dz \quad (B.35) \]

and, on account of (B.23),

\[ f_n(x) = 1 - 2 \int_x^\infty \phi^{-2}(y) dy \int_0^\infty \sigma_n(z) dz \quad (B.36) \]

which is equivalent to

\[ f_n(x) = f_n(0) - 2 \int_0^x \phi^{-2}(y) dy \int_0^y \sigma_n(z) dz. \]

Because \( u(x) \) satisfies \( u(x) > 0 \) and \( u'(x) < 0 \), the hierarchy theorem applies. Therefore

\[ E_{n-1} < E_n \quad (B.37) \]

\[ \left( \frac{f_n}{f_{n-1}} \right)' < 0 \quad (B.38) \]
and

\[
\left( \frac{f_n'}{f_{n-1}'} \right)' < 0. \tag{B.39}
\]

Define

\[ g_n(x) \equiv f_n(x) - f_{n-1}(x); \tag{B.40} \]

for \( n \geq 2 \)

\[ e_n \equiv \mathcal{E}_n - \mathcal{E}_{n-1}, \tag{B.41} \]

and for \( n = 1 \),

\[ e_1 \equiv \mathcal{E}_1 - \frac{1}{4}. \tag{B.42} \]

For clarity, we will present our analysis in the form of several simple theorems.

**Theorem 1**

\[
\left( \frac{g_n}{f_{n-1}} \right)' < 0, \quad \left( \frac{g_n'}{f_{n-1}'} \right)' < 0, \tag{B.43}
\]

and

\[ g_n' < 0 \tag{B.44} \]

**Proof** From (B.40),

\[
\frac{g_n}{f_{n-1}} = \frac{f_n}{f_{n-1}} - 1 \quad \text{and} \quad \frac{g_n'}{f_{n-1}'} = \frac{f_n'}{f_{n-1}'} - 1. \tag{B.45}
\]

Therefore, (B.38) and (B.39) lead to (B.43) and therefore

\[
\left( \frac{g_n}{f_{n-1}} \right)' = \frac{g_n'}{f_{n-1}} - \frac{g_n f_{n-1}'}{f_{n-1}^2} < 0.
\]

Because \( f_{n-1}' < 0 \) and \( g_n, f_{n-1} \) both positive, it follows then

\[ g_n' < \frac{g_n f_{n-1}'}{f_{n-1}} < 0. \tag{B.46} \]

**Theorem 2**

\[ e_n > 0. \tag{B.47} \]

**Proof** For \( n \geq 2 \), (B.37) gives \( e_n > 0 \). For \( n = 1 \), (B.26) and \( f_0 = 1 \) lead to

\[ \mathcal{E}_1 = \left[ \frac{u}{[1]} \right]. \tag{B.48} \]
As we shall prove

\[ [e_1] = ([\mathcal{E}_1 - \frac{1}{4}] [1] = [u - \frac{1}{4}] > \frac{1}{3} e^{-\frac{4}{3}g}. \quad (B.49) \]

By using (B.9), we obtain

\[ u - \frac{1}{4} = \frac{3 + x}{4(1 + x)^2} (1 - x). \quad (B.50) \]

Introduce

\[ \xi(x) = e^{-2gS_0} = \frac{\xi'(x)}{2g(1 - x^2)}. \quad (B.51) \]

As \( x \) varies from 0 to \( \infty \), \( \xi \) follows a path \( \mathcal{P} \), starting from \( \xi = e^{-\frac{4}{3}g} \), increasing to \( \xi = 1 \) when \( x = 1 \), and then decreasing to \( \xi = 0 \). Thus,

\[ [u - \frac{1}{4}] = \int_0^\infty \phi^2(u - \frac{1}{4})dx = \int_\mathcal{P} \frac{3 + x}{2g(1 + x)^5} d\xi. \quad (B.52) \]

Divide the positive \( x \)-axis into three sections:

1. \( x \) from 0 to 1; (B.53)
   correspondingly \( \xi \) from \( e^{-\frac{4}{3}g} \) to 1, and \( S_0 \) from \( \frac{2}{3} \) to 0.

2. \( x \) from 1 to \( \sqrt{3} \); (B.54)
   \( \xi \) from 1 to \( e^{-\frac{4}{3}g} \) and \( S_0 \) from 0 to \( \frac{2}{3} \).

3. \( x \) from \( \sqrt{3} \) to \( \infty \); (B.55)
   \( \xi \) from \( e^{-\frac{4}{3}g} \) to 0 and \( S_0 \) from \( \frac{2}{3} \) to \( \infty \).

Define

\[ F(x) = \frac{3 + x}{(1 + x)^5}. \quad (B.56) \]

Eq. (B.52) can be written as

\[ [u - \frac{1}{4}] = I(g) - II(g) - III(g) \quad (B.57) \]
where

\begin{align*}
I(g) &= \int_{0}^{\frac{2}{3}} F(x_{-}(S_{0})) e^{-2gS_{0}} dS_{0} > 0 \\
II(g) &= \int_{0}^{\frac{2}{3}} F(x_{+}(S_{0})) e^{-2gS_{0}} dS_{0} > 0 \quad (B.58)
\end{align*}

and

\[ III(g) = \int_{\frac{2}{3}}^{\infty} F(x_{+}(S_{0})) e^{-2gS_{0}} dS_{0} > 0, \]

with \( x_{-}(S_{0}) \) and \( x_{+}(S_{0}) \) given below.

In accordance with (B.6), when \( x \) varies from \(-2\) to \(2\), \( S_{0} \) ranges from \(0\) to \(\frac{4}{3}\). In this range, for each \( S_{0} \) there are three real roots of \( x \) that satisfy

\[ x^3 - 3x = 3S_{0} - 2. \quad (B.59) \]

Let

\[ \cos \theta \equiv 1 - \frac{3}{2}S_{0}. \quad (B.60) \]

With \( \theta = 0 \) at \( S_{0} = 0 \) and \( x = 1 \). For each \( S_{0} \) within \( 0 \) and \( \frac{4}{3} \) designate these three roots as

\[ x_{+}(S_{0}) = 2 \cos \alpha_{+}(S_{0}), \quad x_{-}(S_{0}) = 2 \cos \alpha_{-}(S_{0}) \]

and

\[ x_{0}(S_{0}) = 2 \cos \alpha_{0}(S_{0}), \quad (B.61) \]

with

\[ \alpha_{+}(S_{0}) = -60^0 + \frac{\theta}{3} \]
\[ \alpha_{-}(S_{0}) = 60^0 + \frac{\theta}{3} \quad (B.62) \]
\[ \alpha_{0}(S_{0}) = 180^0 + \frac{\theta}{3} \]

The \( x \) referred to in (B.53) and (B.54)-(B.55) are respectively the above \( x_{-}(S_{0}) \) and \( x_{+}(S_{0}) \) together with its analytical extension to \( x_{+} > 2 \).
When \( g = 0 \), by using
\[-S'_0 = 1 - x^2\]
and
\[(3 + x)(1 - x^2) = (1 + x)(4 - (1 + x)^2),\]
we have from (B.52) and (B.57)
\[\left[u - \frac{1}{4}\right]_{g=0} = \int_{0}^{\infty} \frac{1}{(1 + x)^4} \left(4 - (1 + x)^2\right) dx = I(0) - II(0) - III(0) = \frac{1}{3}.\] (B.64)

Take the expression for III\((g)\) in (B.58). Since in its integrand \( e^{-2gS_0} < e^{-\frac{4}{3}g} \), we have
\[\text{III}(g) < e^{-\frac{4}{3}g}\text{III}(0).\] (B.65)

Next consider
\[I(g) - II(g) = \int_{0}^{2\pi} \left( F(x-(S_0)) - F(x+(S_0)) \right) e^{-2gS_0} dS_0\] (B.66)
in which, since \( x+(S_0) > x-(S_0) \),
\[F(x-(S_0)) - F(x+(S_0)) > 0.\] (B.67)

In addition, because in its integrand
\[e^{-2gS_0} > e^{-\frac{4}{3}g},\] (B.68)
we find
\[I(g) - II(g) > e^{-\frac{4}{3}g}\left(I(0) - II(0)\right).\] (B.69)

Combining with (B.64)-(B.65), we derive
\[I(g) - II(g) - III(g) > e^{-\frac{4}{3}g}\left(I(0) - II(0) - III(0)\right) > \frac{1}{3}e^{-\frac{4}{3}g}\] (B.70)
which leads to (B.49) and \( e_1 > 0 \), and thereby completes the proof of Theorem 2.
Theorem 3
\[ e_1 = \frac{[u - \frac{1}{4}]}{[1]} \]
and for \( n \geq 2 \)
\[ e_n = \frac{[(u - \mathcal{E}_{n-1})g_{n-1}]}{[f_{n-1}]} < \frac{[(u - \frac{1}{4})g_{n-1}]}{[1]} \]  \hspace{1cm} (B.71)

Proof  The first equation for \( n = 1 \) follows from (B.42). For \( n \geq 2 \),
\[ e_n = \mathcal{E}_n - \mathcal{E}_{n-1} = \frac{[(u - \mathcal{E}_{n-1})f_{n-1}]}{[f_{n-1}]} \]  \hspace{1cm} (B.72)

In accordance with (B.40) and (B.26),
\[ f_{n-1} = f_{n-2} + g_{n-1} \]  \hspace{1cm} (B.73)
and
\[ [(u - \mathcal{E}_{n-1})f_{n-2}] = 0. \]  \hspace{1cm} (B.74)
Thus, the equality in (B.71) follows. Since for all \( n \),
\[ \mathcal{E}_n \geq \mathcal{E}_1 > \frac{1}{4}, \]  \hspace{1cm} (B.75)
we establish also the inequality in (B.71).

Theorem 4
\[ \frac{[(u - \frac{1}{4})g_n]}{[1]} < \frac{1}{8g}(14g_n(0) + 3G_n), \]  \hspace{1cm} (B.76)
where
\[ G_n \equiv \text{maximum of} -g'_n(x). \]  \hspace{1cm} (B.77)

Proof  As in (B.52), we write
\[ [(u - \frac{1}{4})g_n] = \int_0^\infty \frac{3 + x}{2g(1 + x)^5}g_n(x) \frac{d}{dx}(e^{-2gS_0}) dx \]
\[ = -\frac{3}{2g}g_n(0)e^{-\frac{3}{4}g} + \frac{1}{8g} \int_0^\infty \phi^2(Ag_n - Bg'_n) dx \]  \hspace{1cm} (B.78)
with
\[ A(x) = \frac{3 + x}{(1 + x)^3} \left( \frac{5}{1 + x} - \frac{1}{3 + x} \right) = \frac{2(7 + 2x)}{(1 + x)^4} \leq 14 \]  \hspace{1cm} (B.79)
and
\[ B(x) = \frac{3 + x}{(1 + x)^3} \leq 3. \] (B.80)
Since the first term on the right hand side of (B.78) is negative, Theorem 4 is proved.

**Theorem 5** For \( 0 < x \leq 1 \),
\[ -g_n'(x) < \frac{3}{g_n-1(0)}. \] (B.81)

**Proof** Integrating (B.21) from 0 to \( x \), and using (B.22), we obtain
\[ -\frac{1}{2} \phi^2(x) f_n'(x) = \int_0^x \phi^2(z)(u(z) - E_n) f_{n-1}(z) dz \]
and
\[ -\frac{1}{2} \phi^2(x) f_{n-1}'(x) = \int_0^x \phi^2(z)(u(z) - E_{n-1}) f_{n-2}(z) dz. \] (B.82)
The difference between these two equations gives
\[ -\frac{1}{2} \phi^2(x) g_n'(x) = \int_0^x \phi^2(z)\left\{ (u(z) - E_n) g_{n-1}(z) - e_n f_{n-2}(z) \right\} dz \]
\[ < \int_0^x \phi^2(z)(u(z) - E_n) g_{n-1}(z) dz, \] (B.83)
since \(-e_n f_{n-2}(z) < 0\). From (B.44) and (B.49) it follows then
\[ g_{n-1}(z) < g_{n-1}(0), \] (B.84)
\[ E_n \geq E_1 > \frac{1}{4} \] (B.85)
and (B.83) becomes
\[ -\frac{1}{2} \phi^2(x) g_n'(x) < \int_0^x \phi^2(z)\left( u(z) - \frac{1}{4} \right) g_{n-1}(z) dz \]
\[< g_{n-1}(0) \int_0^x \phi^2(z)(u(z) - \frac{1}{4})dz = g_{n-1}(0) \int_0^x \frac{3 + x}{2g(1 + x)^5}(e^{-2gS_0})'dx. \] (B.86)

Because
\[
\left( \ln \frac{3 + x}{(1 + x)^5} \right)' = -\frac{14 + 4x}{(3 + x)(1 + x)} < 0,
\]
(B.86) leads to
\[\frac{-1}{2} \phi^2(x)g'_n(x) < \frac{3g_{n-1}(0)}{2g} \left( e^{-2gS_0(x)} - e^{-\frac{4}{3}g} \right) < \frac{3g_{n-1}(0)}{2g} e^{-2gS_0(x)}. \]

By using (B.5), we derive
\[-g'_n(x) < \frac{3}{g} \frac{(1 + x)^2}{4}g_{n-1}(0) \] (B.87)

which for \(0 < x < 1\) leads to Theorem 5.

**Lemma**  When \(n = 1\) and \(x > 1\),
\[- (1 + x)^2 f_1' < 0 \] (B.88)
provided
\[g > 2. \] (B.89)

The prove of the lemma is given at the end of this Appendix. We will now proceed assuming its validity.

**Theorem 6**  For \(x > 1\) and \(g > 2\),
\[- (1 + x)^2 f_n' < 0 \] (B.90)
and
\[- (1 + x)^2 g'_n < 0. \] (B.91)
Proof Since
\[-(1 + x)^2 f'_n = \frac{f'_n}{f'_{n-1}} \cdot \frac{f'_{n-1}}{f'_{n-2}} \cdots \frac{f'_{2}}{f'_{1}}\left(-(1 + x)^2 f'_1\right) \quad (B.92)\]
and
\[-(1 + x)^2 g'_n = \frac{g'_n}{g'_{n-1}} \cdot \frac{g'_{n-1}}{g'_{n-2}} \cdots \frac{g'_{2}}{g'_{1}}\left(-(1 + x)^2 f'_1\right). \quad (B.93)\]
Using (B.39), (B.43) and the lemma, we see that the derivatives of (B.92) and (B.93) are negative.

Theorem 7 For \(g > 2\) and \(x > 1\),
\[G_n = \max \left(-g'_n(x)\right) < \frac{3}{g} g_{n-1}(0). \quad (B.94)\]
Proof For \(x > 1\),
\[\left(-g'_n(x)\right)' < 0 \quad (B.95)\]
on account of (B.91); therefore
\[-g'_n(x) < -g'_n(1). \quad (B.96)\]
Together with Theorem 5, (B.94) is proved.

Theorem 8 For \(g > 2\) and for all \(x > 0\),
\[g_n(x) < g_n(0) < \frac{9}{g} g_{n-1}(0) < \cdots < \left(\frac{9}{g}\right)^n \quad (B.97)\]
Proof Since
\[g_n(0) = g_n(1) - \int_0^1 g'_n(x)dx < g_n(1) + G_n \quad (B.98)\]
and
\[g_n(1) = -\int_1^\infty g'_n(x)dx = \int_1^{\infty} \frac{dx}{(1 + x)^2} \cdot \quad (B.99)\]
On account of (B.91), for \(x > 1\)
\[-(1 + x)^2 g'_n(x) < -4g'_n(1); \quad (B.100)\]
therefore (B.99) yields
\[ g_n(1) < -4g'_n(1) \int_1^\infty \frac{dx}{(1 + x)^2} = -2g'_n(1) < 2G_n. \] (B.101)

Combining this result with (B.98), we derive
\[ g_n(0) < 3G_n < \frac{9}{g} g_{n-1}(0), \] (B.102)
on account of (B.94).

**Theorem 9**  For \( g > 2 \),
\[ e_n < \frac{5}{24} \left( \frac{9}{g} \right)^n \] (B.103)

**Proof**  Using (B.76), (B.94) and (B.97), we derive
\[
\frac{[(u - \frac{4}{3})g_{n-1}]}{[1]} < \frac{1}{8g} \left( 14g_{n-1}(0) + 3G_{n-1} \right)
< \frac{1}{8g} \left( 14 \left( \frac{9}{g} \right)^{n-1} + \left( \frac{9}{g} \right)^{n-1} \right) = \frac{15}{8g} \left( \frac{9}{g} \right)^{n-1}
= \frac{5}{24} \left( \frac{9}{g} \right)^n.
\]

Substituting this result into (B.71), we complete the proof of Theorem 9.

We will now turn to the proof of the lemma. For \( n = 1 \), (B.29) and (B.36) lead to
\[ f'_1(x) = 2\phi^{-2}(x) \int_x^\infty (u(z) - \mathcal{E}_1)\phi^2(z)dz. \] (B.104)

Define
\[ F(x) = -\frac{1}{2} (1 + x)^2 f'_1(x) \] (B.105)
\[ \xi(x) = \frac{\phi^2(x)}{4(1 + x)^2} = \frac{1}{(1 + x)^4} e^{-2gS_0(x)} \] (B.106)
and
\[ \eta(x) = -\frac{1}{4} \int_x^\infty (u(z) - \mathcal{E}_1)\phi^2(z)dz. \] (B.107)
We find
\[ F = \frac{\eta}{\xi}, \]  \hspace{1cm} (B.108)
with
\[ \eta' = -\xi p, \]  \hspace{1cm} (B.109)
\[ \xi' = -\xi q, \]  \hspace{1cm} (B.110)
\[ p = (1 + x)^2 \mathcal{E}_1 - 1 \]  \hspace{1cm} (B.111)
and
\[ q = \frac{4}{1 + x} + 2g(x^2 - 1). \]  \hspace{1cm} (B.112)

Since \( \mathcal{E}_1 > \frac{1}{4} \) in accordance with (B.49), for \( x > 1 \), we have
\[ u(x) = \frac{1}{(x + 1)^2} < \mathcal{E}_1. \]  \hspace{1cm} (B.113)

Therefore
\[ \eta(x) > 0; \]
in addition,
\[ p(x) > 0 \quad \text{and} \quad q(x) > 0. \]  \hspace{1cm} (B.114)

Regarding \( \eta = \eta(\xi) \), we define
\[ L(\xi) \equiv \xi \frac{d\eta}{d\xi} - \eta. \]  \hspace{1cm} (B.115)

Its derivative is
\[ \frac{dL}{d\xi} = \xi \frac{d^2\eta}{d\xi^2} = \frac{pq' - qp'}{q^3}. \]  \hspace{1cm} (B.116)

Likewise, by differentiating (B.108), we derive
\[ \xi^2 \frac{dF}{d\xi} = \xi \left( \frac{d\eta}{d\xi} - \frac{\eta}{\xi} \right) = L. \]  \hspace{1cm} (B.117)

At \( x = \infty \), (B.106) and (B.107) yield
\[ \xi(\infty) = 0 \quad \text{and} \quad \eta(\infty) = 0; \]  \hspace{1cm} (B.118)

their ratio is
\[ F(\infty) = \frac{\eta(\infty)}{\xi(\infty)} = \frac{\eta'(\infty)}{\xi'(\infty)} = \frac{p(\infty)}{q(\infty)} = \frac{\mathcal{E}_1}{2g} > 0. \]  \hspace{1cm} (B.119)
From (B.115), we see that at $x = \infty$

$$L(\xi)_{x=\infty} = L(0) = 0. \quad (B.120)$$

As $x$ decreases, since $\xi'$ and $\eta'$ both are negative, $\xi(x)$ and $\eta(x)$ increase and are both positive. Next, we shall prove that for $x > 1$ and $g > 2$

$$\lambda(x) \equiv \frac{1}{4}(pq' - qp') > 0; \quad (B.121)$$

i.e.,

$$\frac{dL}{d\xi} = \frac{4}{q^3} \lambda > 0, \quad (B.122)$$

and therefore, because of (B.117),

$$L > 0 \quad (B.123)$$

and therefore, because of (B.117),

$$\frac{dF}{d\xi} > 0. \quad (B.124)$$

In turn, because of (B.110), for $x > 1$

$$F' = -\xi q \frac{dF}{d\xi} < 0 \quad (B.125)$$

and that leads to, on account of (B.105)

$$-(1 + x)^2 f_1' < 0$$

which is the lemma (B.88)-(B.89).

To establish (B.121), we use the identity

$$\int_0^\infty \phi^2(x) \left( \frac{1}{(1 + x)^n} - \frac{1}{2^n} \right) dx = -\frac{2^n - 1}{2^{n-1}g} e^{-\frac{4}{g}}$$

$$+ \frac{1}{4g} \int_0^\infty \phi^2(x) \left( \frac{n + 3}{(1 + x)^{n+2}} + \frac{n + 2}{2(1 + x)^{n+1}} + \frac{n + 1}{2^2(1 + x)^n} + \cdots + \frac{4}{2^{n-1}(1 + x)^3} \right) dx. \quad (B.126)$$
It follows then
\[
\int_0^\infty \phi^2(x) \left( \frac{1}{(1 + x)^2} - \frac{1}{4} \right) dx = -\frac{3}{2g} e^{-\frac{4}{g}}
\]
\[
+ \frac{1}{4g} \int_0^\infty \phi^2(x) \left( \frac{5}{(1 + x)^4} + \frac{4}{2(1 + x)^3} \right) dx,
\]  \hspace{1cm} (B.127)

\[
\int_0^\infty \phi^2(x) \left( \frac{1}{(1 + x)^2} - \frac{1}{4} - \frac{9}{2^6 g} \right) dx = -\left( \frac{3}{2g} + \frac{103}{2^5 g^2} \right) e^{-\frac{4}{g}}
\]
\[
+ \frac{1}{2^4 g^2} \int_0^\infty \phi^2(x) \left( \frac{35}{(1 + x)^6} + \frac{54}{2(1 + x)^5} + \frac{45}{2^2(1 + x)^4} + \frac{36}{2^3(1 + x)^3} \right) dx.
\]  \hspace{1cm} (B.128)

and
\[
\int_0^\infty \phi^2(x) \left( \frac{1}{(1 + x)^2} - \frac{1}{4} - \frac{9}{2^6 g} - \frac{85}{2^9 g^2} \right) dx = \left[ \frac{\delta}{g^3} \right]
\]  \hspace{1cm} (B.129)

where
\[
\left[ \frac{\delta}{g^3} \right] = -\left( \frac{3}{2g} + \frac{103}{2^5 g^2} + \frac{2403}{2^8 g^3} \right) e^{-\frac{4}{g}}
\]
\[
+ \frac{1}{2^6 g^3} \int_0^\infty \phi^2(x) \left( \frac{315}{(1 + x)^8} + \frac{712}{2(1 + x)^7} + \frac{938}{2^2(1 + x)^6} + \frac{1020}{2^3(1 + x)^5} 
\]
\[
+ \frac{850}{2^4(1 + x)^4} + \frac{680}{2^5(1 + x)^3} \right) dx.
\]  \hspace{1cm} (B.130)

Next, using (B.48), we can write the above expression (B.129) as
\[
\mathcal{E}_1 = \frac{1}{4} + \frac{9}{2^6 g} + \frac{\gamma}{g^2}
\]  \hspace{1cm} (B.131)

where
\[
\gamma = \frac{85}{2^9} + \frac{\delta}{g}
\]  \hspace{1cm} (B.132)

with \(\delta\) defined by (B.130). From (B.111)-(B.112), we find that \(\lambda(x)\) defined by (B.121) is given by
\[
\lambda(x) = g\mathcal{E}_1(x + 1)^2 - gx + \frac{1}{(1 + x)^2} - 3\mathcal{E}_1.
\]  \hspace{1cm} (B.133)
Its derivatives are
\[ \lambda'(x) = 2gE_1(x + 1) - g - \frac{2}{(1 + x)^3} \]  \hspace{1cm} (B.134)
and
\[ \lambda''(x) = 2gE_1 + \frac{6}{(1 + x)^4} > 0. \]  \hspace{1cm} (B.135)

At \( x = 1 \),
\[ \lambda'(1) = 4gE_1 - g - \frac{1}{4}. \]  \hspace{1cm} (B.136)

Using (B.131), we find
\[ \lambda'(1) = \frac{5}{16} + \frac{4\gamma}{g}. \]  \hspace{1cm} (B.137)

Neglecting \( O(e^{-\frac{4}{3}g}) \), \( \delta \) is positive; therefore \( \gamma > 0 \) and \( \lambda'(1) > 0 \). Because \( \lambda''(x) > 0, \lambda'(x) > 0 \) for all \( x > 1 \), and the minimum of \( \lambda(x) \) is at \( x = 1 \) with
\[ \lambda(1) = \frac{1}{16} + \frac{4}{g} \left( \gamma - \frac{27}{256} \right) - \frac{3\gamma}{g^2}. \]  \hspace{1cm} (B.138)

In order that \( \lambda(1) > 0 \), we require
\[ 64\gamma > g \left( \frac{27 - 4g}{4g - 3} \right). \]  \hspace{1cm} (B.139)

Assuming \( g \) not too small so that we can neglect \( \delta/g \) in (B.132); hence
\[ \gamma \approx \frac{85}{2^9} \]  \hspace{1cm} (B.140)
and
\[ 64\gamma \approx \frac{85}{8} = 10.625. \]  \hspace{1cm} (B.141)

The righthand side of (B.139) is
\[ g \left( \frac{27 - 4g}{4g - 3} \right) = \frac{21}{2} \text{ when } g = 1.5 \]  \hspace{1cm} (B.142)
consistent with the inequality (B.139). If in (B.132), we take \( \delta/g \) into account, then a sufficient condition for \( \lambda(1) > 0 \) and therefore \( \lambda(x) > 0 \) for \( x \geq 1 \) is
\[ g > 2. \]  \hspace{1cm} (B.143)

This, together with (B.122)-(B.125) complete the proof of the lemma.
Table 1

| $g = 3$ | $a$ | $a_{max}$ | $a_{min}$ | $g^2_{max}$ | $g^2_{min}$ |
|---------|-----|-----------|-----------|-------------|-------------|
| $k$     |     |           |           |             |             |
| 0.5     | 0.4 | .46       | .25       | 10.69       | 5.06        |
| 1       | 0.6 | .72       | .5        | 11.56       | 7.13        |
| 1.5     | 0.8 | .98       | .75       | 11.86       | 8.29        |
| 2       | 1.2 | 1.23      | 1.0       | 9.33        | 7.13        |
| 2.5     | 1.3 | 1.49      | 1.25      | 10.79       | 8.52        |
| 3       | 1.6 | 1.74      | 1.5       | 10.06       | 8.29        |
| 3.5     | 1.8 | 1.97      | 1.75      | 10.31       | 8.64        |
| 4       | 2.1 | 2.24      | 2.0       | 9.82        | 8.41        |

Figure caption

Fig. 1. Iterative radial wave functions $\psi_n(r)$ and their corresponding energies $E_n$ for both the boundary conditions $f_n(0) = 1$ (upper curves) and $f_n(\infty) = 1$ (lower curves). The parameters are $g = 3$, $k = 2$ and $a = 1.2$.

Fig. 2. Final $R(r) = R_{N,l}(r)$ and $E = E_{N,l}$ that satisfy (2.11) for $g = 3$, $l = 0$ and $N = 3, 4, 5, 6$. The overall normalization factor for these curves are determined by requiring $f(\infty) = 1.1$, 1.1, 1.0 and 0.7 for the corresponding $N = 3, 4, 5$ and 6.

Fig. 3. Final $R(r) = R_{N,l}(r)$ and $E = E_{N,l}$ that satisfy (2.11) for $g = 3$, $N = 3$ and $l = 0, 1, 2, 3$. The overall normalization factor for these curves are all determined by requiring $f(\infty) = 1$.

Table caption

Table 1. List of parameter $a$ used in Figures 1-3 for different $k = l + \frac{1}{2}(N - 1)$, but with the same $g = 3$. The two last columns are $g^2_{max} = (\frac{k}{a} + 1)(\frac{k}{a} + \frac{1}{a} + 1)$ and $g^2_{min} = (\frac{k}{a} + 1)^2$ for different pairs $(k, a)$ used in these figures.
\[ g = 3, \quad k = l + \frac{1}{2}(N - 1) = 2, \quad a = 1.2 \]

\[ \psi_n(r) \]

\[ f_n(0) = 1 \]

\begin{tabular}{|c|c|}
  \hline
  \( n \) & \( E_n \) \\
  \hline
  0 & 10.2 \\
  1 & 4.58834 \\
  2 & 4.33415 \\
  3 & 4.49772 \\
  4 & 4.45779 \\
  5 & 4.46930 \\
  6 & 4.46620 \\
  \hline
\end{tabular}

\[ f_n(\infty) = 1 \]

\begin{tabular}{|c|c|}
  \hline
  \( n \) & \( E_n \) \\
  \hline
  0 & 10.2 \\
  1 & 4.58834 \\
  2 & 4.52449 \\
  3 & 4.48937 \\
  4 & 4.47446 \\
  5 & 4.46924 \\
  6 & 4.46757 \\
  \hline
\end{tabular}

Fig.1
\[
\mathcal{R}_{N,l}(r)
\]

Fig. 2

| $N$ | $l$ | $k$ | $E = E_{N,l}$ |
|-----|-----|-----|---------------|
| 3   | 0   | 1   | 2.897         |
| 4   | 0   | 1.5 | 3.578         |
| 5   | 0   | 2   | 4.467         |
| 6   | 0   | 2.5 | 5.526         |
Fig. 3

$\mathcal{R}_{3,l}(r)$

| $N$ | $l$ | $k$ | $E = E_{N,l}$ |
|-----|-----|-----|----------------|
| 3   | 0   | 1   | 2.897          |
| 3   | 1   | 2   | 4.467          |
| 3   | 2   | 3   | 6.732          |
| 3   | 3   | 4   | 9.517          |

$l = 0$