LP-rounding algorithms for facility-location problems

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Abstract

We study LP-rounding approximation algorithms for metric uncapacitated facility-location problems. We first give a new analysis for the algorithm of Chudak and Shmoys, which differs from the analysis of Byrka and Aardal in that now we do not need any bound based on the solution to the dual LP program. Besides obtaining the optimal bifactor approximation as do Byrka and Aardal, we can now also show that the algorithm with scaling parameter equaling 1.58 is, in fact, an 1.58-approximation algorithm. More importantly, we suggest an approach based on additional randomization and analyses such as ours, which could achieve or approach the conjectured optimal 1.467-approximation for this basic problem.

Next, using essentially the same techniques, we obtain improved approximation algorithms in the 2-stage stochastic variant of the problem, where we must open a subset of facilities having only stochastic information about the future demand from the clients. For this problem we obtain a 2.2975-approximation algorithm in the standard setting, and a 2.4957-approximation in the more restricted, per-scenario setting.

We then study robust fault-tolerant facility location, introduced by Chechik and Peleg: solutions here are designed to provide low connection cost in case of failure of up to \( k \) facilities. Chechik and Peleg gave a 6.5-approximation algorithm for \( k = 1 \) and a \( (7.5k + 1.5) \)-approximation algorithm for general \( k \). We improve this to an LP-rounding \((k + 5 + 4/k)\)-approximation algorithm. We also observe that in case of oblivious failures the expected approximation ratio can be reduced to \( k + 1.5 \), and that the integrality gap of the natural LP-relaxation of the problem is at least \( k + 1 \).
1 Introduction

In facility location problems, we seek a subset of given locations where to build facilities, in order to service a given set of clients. The goal is to minimize the total cost of constructing facilities and the clients’ service cost, which in the metric setting is a function of distances between clients and the facilities that they are assigned to. In this paper we will only consider uncapacitated problems, where there is no restriction on the number of clients connected to a single facility. (We sometimes use the terminology of opening a subset of the existing facilities, rather then constructing them.) We present improved approximation algorithms for a variety of such problems using LP-rounding; we also sketch a possible approach to achieving or approaching the optimal approximation for the basic and perhaps most-studied variant, the Uncapacitated Facility Location problem (UFL).

The UFL is defined as follows. Given a set $F$ of facilities and a set $C$ of clients, we aim to open a subset of facilities and connect every client to an open facility. The cost of opening facility $i$ is $f_i$, and the cost of connecting client $j$ to facility $i$ is $c_{ij}$; the connection costs are assumed to define a symmetric metric. The goal is to choose the facilities and connections that minimize the sum of facility-opening costs and client-connection costs.

The UFL problem is NP-hard and hard to approximate better than the positive solution $s_0 \sim 1.46$ to the equation $s = 1 + 2e^{-s}$ [6], where $e$ denotes the base of the natural logarithm. Jain et al. [7] generalized this result to the following bifactor lower bound. Let $F_{OPT}$ and $C_{OPT}$ denote the facility-opening cost and client-connection cost of an optimal solution. They proved that it is unlikely, for any $\lambda \geq 1$, that there exists a polynomial-time algorithm that finds a solution with facility-opening cost at most $\lambda \cdot F_{OPT}$ and connection cost at most $(1 + 2e^{-\lambda}) \cdot C_{OPT}$. On the positive side, Shmoys, Tardos and Aardal [10] provided a constant factor LP-rounding approximation algorithm that exploits the assumption on the connection costs being metric. A series of algorithms improving the approximation ratio then followed, borrowing and contributing to essentially all known major approaches in approximation algorithms. The currently best-known approximation ratio is reached by the 1.5-approximation algorithm of Byrka and Aardal [2]; this is obtained by a combination of a greedy algorithm analyzed by a dual fitting technique [9] and a novel analysis of the LP-rounding algorithm of Chudak and Shmoys [4].

The first result of this paper is to simplify the analysis of Byrka and Aardal. In our case, the expected connection cost of a single client gets bounded with respect to the fractional connection cost of the client, rather than w.r.t. a combination of its fractional connection and a dual budget as in [5]. [3]. The main result of [2] remains unaffected: we still obtain that the expected cost of the solution is at most $\gamma \cdot F_{OPT} + (1 + 2e^{-\gamma}) \cdot C_{OPT}$ if the scaling parameter is $\gamma \geq 1.678$. However, our analysis is purely primal-based, and thus works if we have an approximately-optimal LP solution as well. (That is, our approximation bounds will be scaled by $(1 + \epsilon)$ if we have an $(1 + \epsilon)$-approximate solution to the LP – obtained, for instance, by some fast algorithm.) Furthermore, for smaller values of the parameter $\gamma$, we obtain bounds that are stronger than in [2]. In particular, for $\gamma = 1.575$ we obtain 1.575-approximate solutions which was not known before. Interestingly, the same ratio was previously obtained by yet another analysis, namely an analysis of Sviridenko [12], who considered essentially the same algorithm, but with the scaling parameter $\gamma$ drawn randomly from a certain nontrivial distribution. Perhaps most importantly, we suggest a new type of approach for the UFL based on our analysis, which appears promising in terms of approaching the optimal approximation of $s_0 \sim 1.46 \cdots$.

Next we consider the setting of uncertain, stochastic demand modeled as a 2-stage stochastic optimization problem. In the first stage, given stochastic information about the set of clients that needs to be served we decide to open a subset of facilities. Next, in the second stage, the actual set of clients is revealed to us and we can open additional facilities. Finally we connect each client to a facility opened in any of the stages. The essence of the problem is that facility-opening costs change over time, i.e., it is cheaper to open a facility earlier. We make the standard assumption that the stochastic demand is presented to us in the form of a polynomial number of possible scenarios, each scenario to be realized with a certain probability.
The goal is to minimize the total expected cost. Certain algorithms deliver slightly stronger, per-scenario bounds, i.e., the cost in each scenario is compared to the fractional cost in this scenario. The 2-stage stochastic facility location was introduced by Swamy and Shmoys [13]. The approximation ratio was then improved by Srinivasan [11], who obtained a 2.369-approximation in the general expectation setting and a 3.095-approximation in the per-scenario model. We use the techniques we develop for UFL to improve these ratios to 2.2975 and 2.4957 respectively.

Finally, we consider the Robust Fault-Tolerant Facility Location (RFTFL) problem introduced recently by Chechik and Peleg [4], and apply some insights from stochastic facility location. In RFTFL, one has to choose a set of facilities that are in a sense robust: i.e., in case of failure of up to $k$ of the opened facilities, where $k$ is viewed as a constant, the cost of connecting clients to the facilities that did not fail should be small. More precisely, we bound the total facility-opening cost plus a worst case client-connection cost. We start by observing that this problem can be modeled by an IP similar to the one used for the 2-stage stochastic problem. Now we say that facilities are opened only in the first stage and there are $\binom{|F|}{k}$ scenarios, each of them excluding the use of a certain subset of $k$ facilities. We present an LP-rounding $(k + 5 + 4/k)$-approximation algorithm, which improves (for $k > 1$) upon the bound of $7.5k + 1.5$ from [4]. We also show that if the scenario is chosen by an oblivious adversary, the bound can be improved to $k + 1.5$. Finally, we show a natural limit of this LP-rounding method by constructing simple instances for which the integrality gap of such an LP-relaxation of the problem is at least $k + 1$.

2 Uncapacitated Facility Location problem

We start with a sketch of the algorithm, and then discuss the crucial steps in more detail. Following this, we sketch our ideas for approaching the optimal $s_0 \approx 1.46\cdots$-approximation.

By $CS(\gamma)$, we denote the algorithm of Chudak and Shmoys [5] with the scaling parameter equaling $\gamma$. A sketch of the $CS(\gamma)$ algorithm is as follows:

1. Solve the standard LP-relaxation (see below) of UFL.

2. Modify the fractional solution by:
   - scaling up the facility-opening variables by $\gamma$,
   - modifying the connection variables to completely use the “closest” fractionally open facilities,
   - splitting facilities, if necessary, such that there is no slack between the amount that a client is assigned to a facility, and the amount by which this facility is opened.

3. Divide clients into clusters based on the current fractional solution. In each cluster a specific client is assigned to be a “cluster center”. (This is a key step.)

4. For every cluster, open one of the “nearby” facilities of the cluster center.

5. For each facility not considered above, open it independently with probability equal to its (scaled) fractional opening value.

6. Connect each client to an open facility that is closest to it.

**IP formulation and relaxation.** UFL has a natural formulation as the following integer program.

$$\begin{align*}
\min \quad & \sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i \\
\text{s.t.} \quad & \sum_{i \in F} x_{ij} = 1 \quad \text{for all } j \in C, \\
& x_{ij} - y_i \leq 0 \quad \text{for all } i \in F, j \in C, \\
& x_{ij}, y_i \in \{0, 1\} \quad \text{for all } i \in F, j \in C.
\end{align*}$$ (1)
A linear relaxation of this IP formulation is obtained by replacing the integrality constraints by the constraint $x_{ij} \geq 0$, $y_i \geq 0$ for all $i \in F$, $j \in C$. We use this LP relaxation as a lower bound for the cost of the optimal integral solution.

**Scaling and clustering.** Inspired by a filtering technique of Lin and Vitter, the following scaling procedure has been successfully applied to facility location problems. Suppose that we have solved the LP relaxation, and that the optimal primal solution is $(x^*, y^*)$. We will start by modifying $(x^*, y^*)$ by scaling the $y$-variables by a constant $\gamma > 1$ to obtain a fractional solution $(x^*, \tilde{y})$, where $\tilde{y} = \gamma \cdot y^*$. Note that by scaling we might set some $\tilde{y}_i > 1$. In the filtering of Shmoys et al. such a variable would instantly be rounded to 1. However, for the compactness of a later part of our analysis it is useful not to round these variables, but rather to split facilities.

Before we discuss splitting, let us first modify the connection variables. Suppose that the values of the $y$-variables are scaled and fixed, but that we now have the freedom to change the values of the $x$-variables in order to minimize the connection cost. For each client $j$ we compute the values of the corresponding $\tilde{x}$-variables in the following way. We choose an ordering of facilities with non-decreasing distances to client $j$. We connect client $j$ to the first facilities in the ordering so that among the facilities fractionally serving $j$, only the last one can be opened by more than that it serves $j$ (i.e., for any facilities $i$ and $i'$ such that $i'$ is later in the ordering, if $\tilde{x}_{ij} < \tilde{y}_i$ then $\tilde{x}_{i'j} = 0$). In the next step, we eliminate the occurrences of situations where $0 < \tilde{x}_{ij} < \tilde{y}_i$. We do so by creating an equivalent instance of the UFL problem, where facility $i$ is split into two identical facilities $i'$ and $i''$. In the new setting, the opening of facility $i'$ is $\tilde{x}_{ij}$ and the opening of facility $i''$ is $\tilde{y}_i - \tilde{x}_{ij}$. The values of the $\tilde{x}$-variables are updated accordingly. By repeatedly applying this procedure we obtain a so-called complete solution $(\tilde{x}, \tilde{y})$, i.e., a solution in which no pair $i \in F, j \in C$ exists such that $0 < \tilde{x}_{ij} < \tilde{y}_i$ (see [12] Lemma 1 for a more detailed argument).

Based on the complete fractional solution $(\tilde{x}, \tilde{y})$, some of the facilities in our instance of UFL are grouped into clusters. (It is sometimes intuitive to view clusters as sets of clients instead.) Each cluster of facilities is created by picking a client as cluster center and creating a cluster from the facilities serving it in $(\tilde{x}, \tilde{y})$. We require that no facility belong to more than one cluster. Therefore when a cluster center is picked, any client that shares a facility with the cluster center can no longer be picked as the center of a new cluster. The algorithm of [5] uses the following procedure to obtain the clustering: while not all the clients are processed, greedily choose (in the manner described next) a new cluster center $j$, and build a cluster from $j$ and facilities serving $j$ in $(\tilde{x}, \tilde{y})$. Remove $j$ and any client that shares a facility with $j$. The greedy choice of the next cluster center depends on the distances between the clients and facilities serving them. For each client $j$, compute: (i) $d_j^{(c)}$, the average distance from $j$ to facilities serving it in $(\tilde{x}, \tilde{y})$, and (ii) $d_j^{(max)}$, the maximum distance to any facility serving it in $(\tilde{x}, \tilde{y})$. (For more formal definitions see the analysis of the algorithm.) The next cluster center is the remaining client with smallest $d_j^{(c)} + d_j^{(max)}$.

To obtain the integral solution, we round the fractional variables as follows. For each cluster the complete fractional opening variables $\tilde{y}_i$ sum to 1. We open exactly one facility within each cluster, with probabilities equal to the $\tilde{y}_i$’s. Any facility that is not in a cluster is opened independently with probability $\tilde{y}_i$. Each client is connected to the closest open facility.

### 2.1 Analysis

We will use the average distances between single clients and groups of facilities defined as follows. For any client $j \in C$, and for any subset of facilities $F' \subset F$ such that $\sum_{i \in F'} \tilde{y}_i > 0$, let $d(j, F') = \frac{\sum_{i \in F'} c_{ij} \tilde{y}_i}{\sum_{i \in F'} \tilde{y}_i}$. We will call the set of facilities $i \in F$ such that $\tilde{x}_{ij} > 0$ the set of close facilities of client $j$ and we denote it

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1Our algorithm is entirely primal and therefore may start with an arbitrary feasible fractional solution. We only start with an optimal fractional solution to be sure that its cost is no more then the cost of the optimal integral one.
by $C_j$. By analogy, we will call the set of facilities $i \in \mathcal{F}$ such that $x_{ij}^* > 0$ and $\overline{x}_{ij} = 0$ the set of \textit{distant facilities} of client $j$ and denote it $D_j$. Observe that $C_j \cap D_j = \emptyset$ for each client $j$.

We are interested in average distances from a client $j$ to sets of facilities fractionally serving it. Let $d_j$ be the average connection cost in $x_{ij}^*$, defined as

$$d_j = d(j, \mathcal{F}) = \frac{\sum_{i \in \mathcal{F}} c_{ij} \cdot x_{ij}^*}{\sum_{i \in \mathcal{F}} x_{ij}^*} = \sum_{i \in \mathcal{F}} c_{ij} \cdot x_{ij}^*.$$ 

Let $d_j^{(c)}$, $d_j^{(d)}$ be the average distances to close and distant facilities defined as

$$d_j^{(c)} = d(j, C_j) = \frac{\sum_{i \in \mathcal{F}} c_{ij} \cdot \overline{x}_{ij}}{\sum_{i \in \mathcal{F}} \overline{x}_{ij}} = \sum_{i \in C_j} c_{ij} \cdot \overline{y}_i,$$

$$d_j^{(d)} = d(j, D_j) = \frac{\sum_{i \in D_j} c_{ij} \cdot x_{ij}^*}{\sum_{i \in D_j} x_{ij}^*} = \sum_{i \in D_j} c_{ij} \cdot \overline{y}_i = \frac{\sum_{i \in \mathcal{F}} c_{ij} \cdot \overline{y}_i}{\gamma - 1}.$$

Let $d_j^{(\text{max})} = \max_{i \in C_j} c_{ij}$ be the maximum distance of $j$ to its close facilities, as mentioned above.

Let $\rho_j$ be defined as $\rho_j = \frac{d_j - d_j^{(c)}}{d_j}$ if $d_j > 0$, and define $\rho_j = 0$ otherwise. Observe that $\rho_j$ takes value between 0 and 1. $\rho_j = 0$ implies $d_j^{(c)} = d_j = d_j^{(d)}$, and $\rho_j = 1$ occurs only when $d_j^{(c)} = 0$. Large values of the parameter $\rho_j$ are desirable for one part of the analysis, and small values are good for another part.

\textbf{Lemma 2.1} $d_j^{(d)} = d_j(1 + \frac{\rho_j}{\gamma - 1})$.

Observe that $d_j^{(\text{max})} \leq d_j^{(d)}$. We will also use the following lemmas from [2]:

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{figure1.png}
  \caption{Distances to facilities serving client $j$ in $(x^*, y^*)$. The width of a rectangle corresponding to facility $i$ is equal to $\gamma \cdot \overline{x}_{ij}^* = \overline{y}_i$. This figure helps us understand the meaning of $\rho_j$.}
\end{figure}
Lemma 2.2 Suppose $\gamma < 2$ and that clients $j, j' \in C$ share a facility in $(\overline{x}, \overline{y})$, i.e., $\exists i \in F$ s.t. $\overline{x}_{ij} > 0$ and $\overline{x}_{ij'} > 0$. Then, either $C_{j'} \setminus (C_j \cup D_j) = \emptyset$ or

$$d(j, C_{j'} \setminus (C_j \cup D_j)) \leq d_{j'}^{(d)} + d_{j'}^{(max)} + d_{j'}^{(c)}.$$  

Lemma 2.3 Given are a random vector $y \in \{0, 1\}^{|F|}$ produced by CS Algorithm $CS(\gamma)$, a subset $A \subseteq F$ of facilities such that $\sum_{i \in A} \overline{y}_i > 0$, and a client $j \in C$. Then, the following holds:

$$E \left[ \min_{i \in A, y_i = 1} c_{ij} \mid \sum_{i \in A} y_i \geq 1 \right] \leq d(j, A)$$

In the analysis of our algorithm we will also use the following result:

Lemma 2.4 Suppose we are given $n$ independent events that occur with probabilities $p_1, p_2, \ldots, p_n$ respectively. The probability that at least one of these events occurs is lower-bounded by $1 - e^{-\sum_{i=1}^{n} p_i}$.

Let $\gamma_0$ be defined as the only positive solution to the following equation.

$$\left( e^{-\gamma} + e^{-\gamma} \right) - (1 + 2e^{-\gamma}) = 0 \tag{2}$$

An approximate value of this constant is $\gamma_0 \approx 1.67736$. As we will observe in the proof of Theorem 2.5, equation (2) appears naturally in the analysis of algorithm $CS(\gamma)$.

Theorem 2.5 For $1 \leq \gamma < 2$, Algorithm $CS(\gamma)$ produces a solution with expected costs

$$E[\text{cost}(\text{OPEN}_i)] = \gamma \cdot F_i^s,$$

$$E[\text{cost}(\text{CONN}_j)] \leq \max \left\{ 1 + 2e^{-\gamma}, \frac{e^{-1} + \gamma}{1 - \frac{1}{\gamma}} \right\} \cdot C_j^s$$

where $F_i^s = f_i y_i^s$, $C_j^s = \sum_{i \in F} c_{ij} x_{ij}^s$, $F^s = \sum_{i \in F} F_i^s$ and $C^s = \sum_{j \in C} C_j^s$.

Proof: The expected facility-opening cost is $E[\text{cost}(\text{OPEN}_i)] = f_i y_i^s = \gamma f_i y_i^s = \gamma \cdot F_i^s$.

To bound the expected connection cost, we show that for each client $j$ there is an open facility within a certain distance with a certain probability. If $j$ is a cluster center, one of its close facilities is open and the expected distance to this open facility is $d_j^{(c)}$.

If $j$ is not a cluster center, it first considers its close facilities (see Figure 2). If any of them is open, by Lemma 2.3 the expected distance to the closest open facility is at most $d_j^{(c)}$. From Lemma 2.4 at least one close facility is open with probability $p_c \geq (1 - \frac{1}{\gamma})$. Suppose none of the close facilities of $j$ is open, but at least one of its distant facilities is open. Let $p_d$ denote the probability of this event. Again by Lemma 2.3 the expected distance to the closest facility is then at most $d_j^{(d)}$. If neither any close nor any distant facility of client $j$ is open, then $j$ connects itself to the facility serving its cluster center $j'$. Again from Lemma 2.4 such an event happens with probability $p_s \leq \frac{1}{\gamma}$. We will now use the fact that if $\gamma < 2$ then, by Lemma 2.2 and Lemma 2.3 the expected distance from $j$ to the facility opened around $j'$ is at most $d_j^{(d)} + d_{j'}^{(max)} + d_{j'}^{(c)}$.

Finally, we combine the probabilities of particular cases with the bounds on the expected connection for each of the cases, to obtain the following upper bound on the expected connection cost.

$$E[\text{cost}(\text{CONN}_j)] \leq p_c \cdot d_j^{(c)} + p_d \cdot d_j^{(d)} + p_s \cdot (d_j^{(d)} + d_{j'}^{(max)} + d_{j'}^{(c)})$$
Figure 2: Facilities that $j$ considers: close and distant, as well as close facilities of its cluster center $j'$.

\[
\begin{align*}
\text{The penultimate line above is due to the fact that } &0 \leq \rho_j \leq 1. \text{ The total cost follows easily. Therefore, } \\
CS(\gamma) \text{ is a } &\left(\gamma, \max \left\{1 + 2e^{-\gamma}, \frac{e^{-1} + e^{-\gamma}}{1 - \frac{1}{\gamma}}\right\}\right) \text{ bi-factor approximation for UFL (}1 \leq \gamma \leq 2). \text{ Note that } \\
(\gamma, 1 + 2e^{-\gamma}) \text{ is the bi-factor approximation lower bound [7, 6].} \\
\end{align*}
\]

**Corollary 2.6** The $CS(1.575)$ algorithm is an 1.575-approximation algorithm for the UFL problem. Also, $CS(\gamma)$ is an optimal bi-factor approximation for UFL for $\gamma_0 < \gamma < 2$.

### 2.2 Approaching an optimal approximation

Our purely primal-based analysis of $CS(\gamma)$ suggests a way to approach the optimal $s_0$-approximation. Our analysis shows that if the adversary knows the value of $\gamma$, then the adversary’s optimal strategy – in
selecting a “bad” instance and corresponding LP solution \((x^*, y^*)\) – is just to make one of two choices, as follows. For each \(j\),

- either all facilities \(i\) with \(x^*_{ij} > 0\) are at the same distance from \(j\),
- or there exist \(0 < a_j < b_j\) such that all facilities \(i\) with \(x^*_{ij} > 0\) are at distance either \(a_j\) or \(b_j\) from \(j\), such that \(\sum_{i: c_{ij}=a_j} x^*_ij\) is infinitesimally smaller than \(1/\gamma\).

Given \(\gamma\), the adversary will make the above choice in order to maximize the expected objective-function value after running \(CS(\gamma)\). (This optimization, as well as the corresponding choices for \(a_j\) and \(b_j\), can be carried out explicitly in a straightforward manner.)

However, what if we select \(\gamma\) randomly from some appropriate distribution? If the adversary selects the second choice above, then a suitably “large” value of \(\gamma\) will defeat the adversary’s goal of making \(\sum_{i: c_{ij}=a_j} x^*_ij\) (infinitesimally) smaller than \(1/\gamma\), leading to a significantly-improved approximation. On the other hand, if the adversary selects the first choice above, then a choice of \(\gamma\) close to \(s_0\) will lead to an approximation close to \(s_0\).

We are unable at the moment to carry out the above idea in an optimal manner: choosing the optimal distribution for \(\gamma\), for instance. However, we view this as a potentially-fruitful approach since it starts with the knowledge of what exactly are the limits of deterministic strategies for choosing \(\gamma\).

3 Two-stage stochastic facility location

We now consider two-stage stochastic facility location in the “explicitly-given polynomially-many scenarios” model presented in the introduction. The LP-rounding problem is thus to find integral solutions “close” to the optimal solution of the following LP, where the scenarios are indexed by the symbol \(A\) and their respective probabilities are given by the values \(p_A\):

\[
\text{minimize } \sum_{i \in F} f_i y_i + \sum_A p_A \left( \sum_i f^A_i y_{A,i} + \sum_j \sum_i c_{ij} x_{A,ij} \right) \text{ subject to:}
\]

\[
\sum_i x_{A,ij} \geq 1 \forall A \forall j \in A;
\]

\[
x_{A,ij} \leq y_i + y_{A,i} \forall i \forall A \forall j \in A;
\]

\[
x_{A,ij}, y_i, y_{A,i} \geq 0 \forall i \forall A \forall j \in A.
\]

First, in Section 3.1 we give an algorithm for the standard setting, then in Section 3.2 we consider the per-scenario version of the problem.

3.1 General expected cost: analysis with a dual bound

Consider the following dual formulation of the 2-stage stochastic facility location problem:

\[
\text{maximize } \sum_A p_A (\sum_{j \in A} v_{j,A}) \text{ subject to:}
\]

\[
w_{ij,A} \geq v_{j,A} - c_{ij}
\]

\[
f^A_i \leq \sum_A p_A (\sum_{j \in A} w_{ij,A})
\]

\[
f_i^A \leq \sum_{j \in A} w_{ij,A}
\]

\[
w_{ij,A}, v_{j,A} \geq 0.
\]
Let \((x^*, y^*)\) and \((v^*, w^*)\) be optimal solutions to the primal and the dual programs, respectively. Note that by complementary slackness, we have \(c_{ij} \leq v_{j,A}\) if \(x_{A,ij} > 0\).

**Algorithm.** We now describe a randomized LP-rounding algorithm that transforms the fractional solution \((x^*, y^*)\) into an integral solution \((\hat{x}, \hat{y})\) with bounded expected cost. The expectation is over the random choices of the algorithm, but not over the random choice of the scenario. Note that we need to decide the first stage entries of \(\hat{y}\) not knowing \(A\). W.l.o.g. we assume that no facility is fractionally opened in \((x^*, y^*)\) in both stages, i.e., for all \(i\) we have \(y^*_i = 0\) or for all \(A\) \(y^*_{A,i} = 0\). To obtain this property it suffices to have two identical copies of each facility, one for Stage I and one for Stage II.

We start by scaling the fractional solution \((x^*, y^*)\) by a factor of 2. As a result, we obtain a fractional solution \((\overline{x}, \overline{y})\) with \(\overline{x}_{A,ij} = 2 \cdot x^*_{A,ij}, \overline{y}_i = 2 \cdot y^*_i\), and \(\overline{y}_{A,i} = 2 \cdot y^*_{A,i}\). Note that the scaled fractional solution \((\overline{x}, \overline{y})\) can have facilities with fractional opening of more than 1. For simplicity of the analysis, we do not round these facility-opening values to 1, but rather split such facilities. More precisely, we split each facility \(i\) with fractional opening \(\overline{y}_i > \overline{x}_{A,ij} > 0\) (or \(\overline{y}_{A,i} > \overline{x}_{A,ij} > 0\)) for some \((A, j)\) into \(i\) and \(i'\), such that \(\overline{y}_{i'} = \overline{x}_{A,ij}\) and \(\overline{y}_{i''} = \overline{y}_i - \overline{x}_{A,ij}\). We also split facilities whose fractional opening exceeds one. By splitting facilities we create another instance of the problem, then we solve this modified instance and interpret the solution as a solution to the original problem in the natural way. The technique of splitting facilities is precisely described in [12].

Define \(\overline{x}_{A,ij}^{(I)} = \min\{\overline{x}_{A,ij}, \overline{y}_i\}\), and \(\overline{x}_{A,ij}^{(II)} = \overline{x}_{A,ij} - \overline{x}_{A,ij}^{(I)}\). Observe, that for a client-scenario pair \((j, A)\) either \(\sum_{i \in F} \overline{x}_{A,ij}^{(I)} \geq 1\), or \(\sum_{i \in F} \overline{x}_{A,ij}^{(II)} \geq 1\). In the former case, we call such a pair a first stage served, and we denote the set of the first stage served pairs by \(S\).

Since we can split facilities, for each \((j, A) \in S\) we can assume that there exists a subset of facilities \(F_{(j,A)} \subseteq F\), such that \(\sum_{i \in F_{(j,A)}} \overline{x}_{A,ij}^{(I)} = 1\), and for each \(i \in F_{(j,A)}\) we have \(\overline{x}_{A,ij}^{(I)} = \overline{y}_i\). Also for each \((j, A) \notin S\) we can assume that there exists a subset of facilities \(F_{(j,A)} \subseteq F\), such that \(\sum_{i \in F_{(j,A)}} \overline{x}_{A,ij}^{(II)} = 1\), and for each \(i \in F_{(j,A)}\) we have \(\overline{x}_{A,ij}^{(II)} = \overline{y}_{A,i}\). Let \(R_{(j,A)} = \max_{i \in F_{(j,A)}} c_{ij}\) be a maximal distance from \(j\) to an \(i \in F_{(j,A)}\). Recall that, by complementary slackness, we have \(R_{(j,A)} \leq v_{j,A}\).

The algorithm opens facilities randomly in each of the stages with the probability of opening facility \(i\) equal to \(\overline{y}_i\) in Stage I, and \(\overline{y}_{A,i}\) in Stage II of scenario \(A\). Some facilities are grouped in disjoint clusters in order to correlate the opening of facilities from a single cluster. The clusters are formed in each stage by the following procedure. Let all facilities be initially unclustered. In Stage I, consider all client-scenario pairs \((j, A) \in S\) (in Stage II of scenario \(A\), consider all clients \(j\) such that \((j, A) \notin S\) in the order of non-decreasing values \(R_{(j,A)}\). If the set of facilities \(F_{(j,A)}\) contains no facility from the previously formed clusters, then form a new cluster containing facilities from \(F_{(j,A)}\), otherwise do nothing. In each stage, open exactly one facility in each cluster. Recall that the total fractional opening of facilities in each cluster equals 1. Within each cluster choose the facility randomly with the probability of opening facility \(i\) equal to the fractional opening \(\overline{y}_i\) in Stage I, or \(\overline{y}_{A,i}\) in Stage II of scenario \(A\). For each unclustered facility \(i\) open it independently with probability \(\overline{y}_i\) in Stage I, and with probability \(\overline{y}_{A,i}\) in Stage II of scenario \(A\). Finally, at the end of Stage II of scenario \(A\), connect each client \(i \in A\) to the closest open facility.

**Analysis.** Consider the solution \((\hat{x}, \hat{y})\) constructed by our LP-rounding algorithm. We fix scenario \(A\) and bound the expectation of \(\text{COST}(A) = \sum_{i \in F} (f^i_1 \hat{y}_i + f^i_A \hat{y}_{A,i}) + \sum_{j \in A} \sum_{i \in F} c_{ij} \hat{x}_{A,ij}\). Define \(C_A = \sum_{j \in A} C_{(j,A)} = \sum_{j \in A} \sum_{i \in F} c_{ij} x^*_{A,ij}\), \(F_A = \sum_{j \in A} (f^j_1 y^*_j + f^j_A y^*_{A,j})\), \(V_A = \sum_{j \in A} v^*_j\).

**Lemma 3.1** \(E[\text{COST}(A)] \leq e^{-2} \cdot 3 \cdot V_A + (1 - e^{-2}) \cdot C_A + 2 \cdot F_A\) in each scenario \(A\).

**Proof:** Since the probability of opening a facility is equal to its fractional opening in \((\overline{x}, \overline{y})\), the expected facility-opening cost of \((\hat{x}, \hat{y})\) equals facility-opening cost of \((\overline{x}, \overline{y})\), which is exactly twice the facility-opening cost of \((x^*, y^*)\).
Fix a client \( j \in A \). The total (from both stages) fractional opening in \( \overline{y} \) of facilities serving \( j \) in \((\overline{x}, \overline{y})\) is exactly 2, hence the probability that at least one of these facilities is open in \((\hat{x}, \hat{y})\) is at least \( 1 - e^{-2} \).

Observe that, on the condition that at least one such facility is open, by an analogous to the one from Lemma 2.3 the expected distance to the closest of the open facilities is at most \( C_{(j,A)} \).

With probability at most \( e^{-2} \), none of the facilities fractionally serving \( j \) in \((\overline{x}, \overline{y})\) is open. In such a case we need to find a different facility to serve \( j \). We will now prove that for each client \( j \in A \) there exists a facility \( i \) which is open in \((\hat{x}, \hat{y})\), such that \( c_{ij} \leq 3 \cdot v_{j,A} \).

Assume \( (j, A) \in S \) (for \( (j, A) \notin S \) the argument is analogous). If \( F_{(j,A)} \) is a cluster, then at least one \( i \in F_{(j,A)} \) is open and \( c_{ij} \leq v_{j,A} \). Suppose \( F_{(j,A)} \) is not a cluster, then by the construction of clusters, it intersects a cluster \( F_{(j',A')} \) with \( R_{(j',A')} \leq R_{(j,A)} \leq v_{j,A} \). Let \( i \) be the facility opened in cluster \( F_{(j',A')} \) and let \( i' \in F_{(j',A')} \cap F_{(j,A)} \). Since \( i' \) is in \( F_{(j,A)} \), \( c_{i'j} \leq R_{(j,A)} \). Since both \( i \) and \( i' \) are in \( F_{(j',A')} \), both \( c_{ij'} \leq R_{(j',A')} \) and \( c_{i'j} \leq R_{(j',A')} \). Hence, by triangle inequality, \( c_{ij} \leq R_{(j,A)} + 2 \cdot R_{(j',A')} \leq 3 \cdot R_{(j,A)} \leq 3 \cdot v_{j,A} \).

Thus, the expected cost of the solution in scenario \( A \) is:

\[
E[COST(A)] \leq e^{-2} \cdot 3 \cdot \sum_{j \in A} v_{j,A} + (1 - e^{-2}) \left( \sum_{j \in A} \sum_{i \in F} c_{ij}^* x_{A,ij}^* \right) + 2 \cdot \left( \sum_{i \in F} f_i^* y_{i}^* + f_i^A y_{iA,i}^* \right) \\
\leq e^{-2} \cdot 3 \cdot V_A + (1 - e^{-2}) \cdot C_A + 2 \cdot F_A.
\]

Define \( F^* = \sum_{i \in F} f_i^* y_i + \sum_A p_A \sum_{i \in F_i} f_i^A y_{iA,i} \) and \( C^* = \sum_A p_A \sum_{i \in F} c_{ij}^* x_{A,ij} \). Note that we have \( F^* = \sum_A p_A F_A \), \( C^* = \sum_A p_A C_A \), and \( F^* + C^* = \sum_A p_A V_A \). Summing up the expected cost over scenarios we obtain the following estimate on the general expected cost, where the expectation is both on the choice of the scenario and on the random choices of our algorithm.

**Corollary 3.2** \( E[COST(\hat{x}, \hat{y})] \leq 2.4061 \cdot F^* + 1.2707 \cdot C^* \).

**Proof:**

\[
E[COST(\hat{x}, \hat{y})] = \sum_A p_A E[COST(A)] \\
\leq \sum_A p_A \left( e^{-2} \cdot 3 \cdot V_A + (1 - e^{-2}) \cdot C_A + 2 \cdot F_A \right) \\
= (1 - e^{-2}) \cdot C^* + 2 \cdot F^* + 3e^{-2}(\sum_A p_A V_A) \\
= (1 - e^{-2}) \cdot C^* + 2 \cdot F^* + 3e^{-2}(F^* + C^*) \\
= (2 + e^{-2} \cdot 3)F^* + (1 + e^{-2} \cdot 2)C^* \\
\leq 2.4061 \cdot F^* + 1.2707 \cdot C^*.
\]

Combining two algorithms. The above described algorithm can be combined with an algorithm from [11] to obtain a 2.2975-approximation algorithm. See Appendix A for details.

### 3.2 Per-scenario bounds: primal analysis

Consider again the 2-stage facility location problem, and a corresponding optimal fractional solution. We now describe a randomized rounding scheme so that for each scenario \( A \), its expected final (rounded)
cost is at most $2.4061$ times its fractional counterpart $Val_A = \sum_{i \in F}(f_i^y y_i + f_i^A y_i^*) + \sum_{j \in A} \sum_{i \in F} c_{ij} x_{A,ij}^*$, improving on the $3.095 \cdot Val_A$ bound of [1].

Note that we cannot use dual bounds in this setting, as the dual budgets $V_A$ do not have to equal $Val_A$ in each scenario (see Appendix B for an example). Instead, we scale the facility-opening values a little more and show that scaling by a factor of $2.4061$ is sufficient to bound the expected connection cost in each scenario by $2.4061$ times the fractional connection cost in this scenario. Details are given in Appendix B.

4 Robust fault-tolerant UFL

In recent work, Chechik and Peleg have introduced a new variant of facility location problems [4]. They study a setting that can be described as follows. Once we choose the facilities to open, an adversary closes up to $k$ of them (which models possible failures of facilities), and then clients are connected to the closest of the remaining open facilities. The goal is to minimize the facility-opening cost plus the worst-case (over the choice of facilities to close) connection cost. Observe that integral solutions $(x, y)$ of the following linear program are exactly the feasible solutions to the problem we study.

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in F} f_i y_i + \max_A \sum_{j} \sum_{i} c_{ij} x_{A,j} \\
\text{subject to} & \quad \sum_{i} x_{A,j} \geq 1 \quad \forall A \forall j; \quad (4) \\
& \quad x_{A,j} \leq y_i \quad \forall i \forall A \forall j; \quad (5) \\
& \quad x_{A,j} = 0 \quad \forall A \forall i \in A \forall j; \quad (6) \\
& \quad x_{A,j}, y_i, y_{A,i} \geq 0 \quad \forall i \forall A \forall j. \end{align*}
\]

The “scenarios” $A$ in the above program are all the subsets of facilities of cardinality $k$, and they encode the facilities closed by the adversary. Note that the connection cost is calculated as a maximum over the scenarios. The above program is of polynomial size only for fixed $k$ and we will only study settings with such small $k$.

In [4] Chechik and Peleg gave a $6.5$-approximation algorithm for $k = 1$ and a $(7.5k + 1.5)$-approximation algorithm for general $k$. We improve the latter to a $(k + 5 + 4/k)$-approximation algorithm for the studied problem. This we obtain by showing that scaling the facility opening variables by $(k + 5 + 4/k)$ is sufficient to provide enough fractional opening, which is then rounded by a dependent rounding method described in [3] (Sections 3 and 4). More details are given in Appendix C.

We also briefly discuss an oblivious version of the problem, where the adversary does not know the random choices of the algorithm when deciding the facilities to close. In this setting we provide a randomized LP-rounding algorithm that delivers integral solutions of expected cost at most $k + 1.5$ times the cost of the initial fractional solution. (see Appendix C)

We also show that these methods cannot be extended to obtain approximation ratios sublinear in $k$ by providing instances with integrality gap that are arbitrarily close to $k + 1$. (see Appendix C)

References

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[2] J. Byrka and K. Aardal. An optimal bifactor approximation algorithm for the metric uncapacitated facility location problem. SIAM Journal on Computing, 39(6):2212–2231, 2010.
Appendix

A Combining two algorithms for 2-stage stochastic facility location

We have described an algorithm that returns solutions of expected cost at most $2.4061 \cdot F^* + 1.2707 \cdot C^*$. Let us call this algorithm ALG1.

In [11], Srinivasan gave a different approximation algorithm for our 2-stage stochastic facility location problem. This algorithm also splits the client-scenario pairs into two groups, namely those that are left to be connected in the second stage, and those that are right to be connected in the second stage. The decision is made by comparing fractional “first stage” connection of each pair with a certain threshold. Once the split is made, the obtained instances of the standard Uncapacitated Facility Location problem are solved with the JMS algorithm [7]. The threshold is chosen randomly from a distribution parametrized by $\alpha$. For the choice of a parameter $\alpha = 0.2485$ the resulting algorithm is shown in [11] to be a 2.369-approximation algorithm. It is easy to show that by setting $\alpha = 0.37$ in the algorithm of [11], we obtain an algorithm that returns solutions of expected cost of at most $2.24152F^* + 2.8254C^*$. We will call this algorithm ALG2.

Consider the algorithm ALG3, which tosses a coin that comes heads with probability $p = 0.3396$. If the coin comes heads, then ALG1 is executed; if it comes tails ALG2 is used. The expected cost of the solution produced by ALG3 can be estimated as: $F^*(p \cdot 2.4061 + (1-p) \cdot 2.24152) + C^*(p \cdot 1.2707 + (1-p) \cdot 2.8254) \leq 2.2975(C^* + F^*)$. Therefore, ALG3 is a 2.2975-approximation algorithm for the 2-stage stochastic facility location problem.
B 2-stage stochastic facility location with per-scenario bounds

Let us first note that it is not possible to directly use the analysis from the previous setting in the per-scenario model. This is because the dual costs $V_A$ do not need to be equal $Val_A = F_A + C_A$ in each scenario $A$. It is possible, for instance, that the fractional opening of a facility in the first stage is entirely paid form the dual budget of a single scenario, despite the fact that clients not active in this scenario benefit from the facility being open. This can be observed, e.g., in the following simple example. Consider two clients $c^1$ and $c^2$, and two facilities $f^1$ and $f^2$. All client facility distances are 1, except $c_{1,2} = \text{dist}(c^1, f^2) = 3$. Scenarios are: $A^1 = \{c^1\}$ and $A^2 = \{c^2\}$, and they occur with probability 1/2 each. The facility-opening costs are: $f^1_1 = 2$, $f^1_2 = \epsilon$, $f^2_1 = f^2_2 = 4$ for both scenarios $A$. It is easy to see that the only optimal fractional solution is integral and it opens facility $f^1$ in the first stage, and opens no more facilities in the second stage. Therefore, $Val(A^1) = Val(A^2) = 3$. However, in the dual problem, client $c^2$ has an advantage over $c^1$ in the access to the cheaper facility $f^2$, and therefore in no optimal dual solution client $c^2$ will pay more then $\epsilon$ for the opening of facility $f^1$. In consequence, most of the cost of opening $f^1$ is paid by the dual budget of scenario $A^1$. Therefore, the dual budget $V_{A^1}$ is strictly greater then the primal bound $Val_{A^1}$ which we use as an estimate of the cost of the optimal solution in scenario $A^1$.

Bearing the above example in mind, we construct an LP-rounding algorithm that does not rely on the dual bound on the length of the created connections. We use a primal bound, which is obtained by scaling the opening variables a little more and using just a subset of fractionally connected facilities for each client in the process of creating clusters. Such a simple filtering technique, whose origins can be found in the work of Lin and Vitter [8], provides slightly weaker but entirely primal, per-scenario bounds.

Algorithm. As before, we describe a randomized LP-rounding algorithm that transforms the fractional solution $(x^*, y^*)$ into an integral solution $(\tilde{x}, \tilde{y})$ with bounded expected cost. The expectation is over the random choices of the algorithm, but not over the random choice of the scenario.

We start by scaling the fractional solution $(x^*, y^*)$ by a factor of $\gamma > 2$. As a result, we obtain a fractional solution $(\tilde{x}, \tilde{y})$ with $\tilde{x}_{A,ij} = \gamma \cdot x^*_{A,ij}$, $\tilde{y}_i = \gamma \cdot y^*_i$, and $\tilde{y}_{A,i} = \gamma \cdot y^*_A$. Note that the scaled fractional solution $(\tilde{x}, \tilde{y})$ may have facilities with fractional opening of more than 1. Again, for simplicity of the analysis, we do not round these facility-opening values to 1, but rather split such facilities. More precisely, we split each facility $i$ with fractional opening $\tilde{y}_i > \tilde{x}_{A,ij} > 0$ (or $\tilde{y}_{A,i} > \tilde{x}_{A,ij} > 0$) for some $(A, j)$ into $i'$ and $i''$, such that $\tilde{y}_{i'} = \tilde{x}_{A,ij}$ and $\tilde{y}_{i''} = \tilde{y}_i - \tilde{x}_{A,ij}$.

We also split facilities whose fractional opening exceeds one.

As before, define $\tilde{x}^{(I)}_{A,ij} = \min\{\tilde{x}_{A,ij}, \tilde{y}_i\}$, and $\tilde{x}^{(II)}_{A,ij} = \tilde{x}_{A,ij} - \tilde{x}^{(I)}_{A,ij}$. Define

$$
F^{I}_{(j,A)} = \begin{cases} 
\arg\min_{F' \subseteq F, \sum_{i \in F'} \tilde{x}^{(I)}_{A,ij} \geq 1} \max_{i \in F'} c_{ij} & \text{if } \sum_{i \in F} \tilde{x}^{(I)}_{A,ij} \geq 1 \\
\emptyset & \text{if } \sum_{i \in F} \tilde{x}^{(I)}_{A,ij} < 1
\end{cases}
$$

$$
F^{II}_{(j,A)} = \begin{cases} 
\arg\min_{F' \subseteq F, \sum_{i \in F'} \tilde{x}^{(II)}_{A,ij} \geq 1} \max_{i \in F'} c_{ij} & \text{if } \sum_{i \in F} \tilde{x}^{(II)}_{A,ij} \geq 1 \\
\emptyset & \text{if } \sum_{i \in F} \tilde{x}^{(II)}_{A,ij} < 1
\end{cases}
$$

Note that these sets can easily be computed by considering facilities in an order of non-decreasing distances $c_{ij}$ to the considered client $j$. Since we can split facilities, w.l.o.g., for all $C$ we assume that if $F^{I}_{(j,A)}$ is nonempty then $\sum_{i \in F^{I}_{(j,A)}} \tilde{x}^{(I)}_{A,ij} = 1$, and if $F^{II}_{(j,A)}$ is not empty then $\sum_{i \in F^{II}_{(j,A)}} \tilde{x}^{(II)}_{A,ij} = 1$. Define $d^{I}_{(j,A)} = \max_{i \in F^{I}_{(j,A)}} c_{ij}$ and $d^{II}_{(j,A)} = \max_{i \in F^{II}_{(j,A)}} c_{ij}$. Let $d_{(j,A)} = \min\{d^{I}_{(j,A)}, d^{II}_{(j,A)}\}$.

For a client-scenario pair $(j, A)$, if we have $d_{(j,A)} = d^{I}_{(j,A)}$, then we call such a pair first-stage clustered, and put its cluster candidate $F_{(j,A)} = F^{I}_{(j,A)}$. Otherwise, if $d_{(j,A)} = d^{II}_{(j,A)} < d^{I}_{(j,A)}$, we say that $(j, A)$ is second-stage clustered and put its cluster candidate $F_{(j,A)} = F^{II}_{(j,A)}$. 

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Recall that we use \( C_{(j,A)} = \sum_i c_{ij} x_{A,ij}^* \) to denote the fractional connection cost of client \( j \) in scenario \( A \). Let us now argue that distances to facilities in cluster candidates are not too large.

**Lemma B.1** \( d_{(j,A)} \leq \frac{\gamma}{\gamma - 2} C_{(j,A)} \) for all pairs \((j, A)\).

*Proof:* Fix a client-scenario pair \((j, A)\). Assume \( F_{(j,A)} = F^I_{(j,A)} \) (the other case is symmetric). Recall that in this case we have \( d_{(j,A)} = d^I_{(j,A)} \leq d^I_{(j,A)} \). Consider the following two subcases.

**Case 1.** \( \sum_{i \in F^I_{(j,A)}} \frac{d^I_{(j,A)}}{x_{A,ij}} = 1 \). Observe that we have \( c_{ij} \geq d_{(j,A)} \) for all \( i \in F' = F \setminus (F^I_{(j,A)} \cup F^II_{(j,A)}) \). Note also that \( \sum_{i \in F'} x_{A,ij} = \gamma - 2 \) and \( \sum_{i \in F} x_{A,ij} = \frac{\gamma - 2}{\gamma} \). Hence, \( C_{(j,A)} = \sum_{i \in F'} x_{A,ij} c_{ij} \geq \sum_{i \in F'} x_{A,ij} c_{ij} \geq \frac{\gamma - 2}{\gamma} \cdot d_{(j,A)} \).

**Case 2.** \( \sum_{i \in F^II_{(j,A)}} \frac{d^II_{(j,A)}}{x_{A,ij}} = 0 \), which implies that \( \sum_{i \in F} \frac{d^II_{(j,A)}}{x_{A,ij}} < 1 \). Observe that now we have \( \sum_{i \in F} x_{A,ij} > \gamma - 1 \), and therefore \( \sum_{i \in F \setminus F^I_{(j,A)}} \frac{d^II_{(j,A)}}{x_{A,ij}} > \gamma - 2 \). Recall that \( c_{ij} \geq d_{(j,A)} \) for all \( i \in (F \setminus F^I_{(j,A)}) \), hence \( C_{(j,A)} = \sum_{i \in F} x_{A,ij} c_{ij} \geq \sum_{i \in (F \setminus F^I_{(j,A)})} x_{A,ij} c_{ij} \geq \frac{\gamma - 2}{\gamma} \cdot d_{(j,A)} \). \( \square \)

Like in Section 3.1, the algorithm opens facilities randomly in each of the stages with the probability of opening facility \( i \) equal to \( \gamma_i \) in Stage I, and \( \gamma_{A,i} \) in Stage II of scenario \( A \). Some facilities are grouped in disjoint clusters in order to correlate the opening of facilities from a single cluster. The clusters are formed in each stage by the following procedure. Let all facilities be initially unclustered. In Stage I, consider all first-stage clustered client-scenario pairs, i.e., pairs \((j, A)\) such that \( d_{(j,A)} = d^I_{(j,A)} \). (in Stage II of scenario \( A \), consider all second-stage clustered client-scenario pairs) in the order of non-decreasing values \( d_{(j,A)} \). If the set of facilities \( F_{(j,A)} \) contains no facility from the previously formed clusters, then form a new cluster containing facilities from \( F_{(j,A)} \), otherwise do nothing. In each stage, open exactly one facility in each cluster. Recall that the total fractional opening of facilities in each cluster equals 1. Within each cluster choose the facility randomly with the probability of opening facility \( i \) equal to the fractional opening \( \gamma_i \) in Stage I, or \( \gamma_{A,i} \) in Stage II of scenario \( A \). For each unclustered facility \( i \) open it independently with probability \( \gamma_i \) in Stage I, and with probability \( \gamma_{A,i} \) in Stage II of scenario \( A \).

Finally, at the end of Stage II of scenario \( A \), connect each client \( i \in A \) to the closest open facility.

**Analysis.** The expected facility-opening cost is obviously \( \gamma \) times the fractional opening cost. More precisely, the expected facility-opening cost in scenario \( A \) equals \( \gamma \cdot F^A_\gamma = \gamma \cdot \sum_{i \in F} y^I_{A,i} + \sum_{i \in F} y^II_{A,i} \gamma_i \), it remains to bound the expected connection cost in scenario \( A \) in terms of \( C^A_\gamma = \sum_{j \in A} \sum_i c_{ij} x_{A,ij} \).

**Lemma B.2** The expected connection cost in scenario \( A \) is at most \((1 + \frac{2\gamma + 2}{\gamma - 2} \cdot e^{-\gamma}) \cdot C_{(j,A)} \).

*Proof:* Consider a single client-scenario pair \((j, A)\). Observe that the facilities fractionally connected to \( j \) in scenario \( A \) have the total fractional opening of \( \gamma \) in the scaled facility-opening vector \( \gamma \). Since there is no positive correlation (only negative correlation in the disjoint clusters formed by the algorithm), with probability at least \( 1 - e^{-\gamma} \) at least one such facility will be opened, moreover, by Lemma B.3 the expected distance to the closest open facilities from this set will be at most the fractional connection cost \( C_{(j,A)} \).

Just like in the proof of Lemma 3.1 from the greedy construction of the clusters in each phase of the algorithm, with probability 1, there exists facility \( i \) opened by the algorithm such that \( c_{ij} \leq 3 \cdot d_{(j,A)} \). We connect client \( j \) to facility \( i \) if no facility from facilities fractionally serving \((j, A)\) was opened. We obtain that the expected connection cost of client \( j \) is at most \((1 - e^{-\gamma}) \cdot C_{(j,A)} + e^{-\gamma} \cdot 3d_{(j,A)} \). By Lemma B.1 this can be bounded by \((1 - e^{-\gamma}) \cdot C_{(j,A)} + e^{-\gamma} \cdot 3 \cdot \frac{\gamma - 2}{\gamma} \cdot C_{(j,A)} = (1 + \frac{2\gamma + 2}{\gamma - 2} \cdot e^{-\gamma}) \cdot C_{(j,A)} \). \( \square \)

To equalize the opening and connection cost approximation ratios we solve \((1 + \frac{2\gamma + 2}{\gamma - 2} \cdot e^{-\gamma}) = \gamma \) and obtain the following.
Theorem B.3 The described algorithm with $\gamma = 2.4957$ delivers solutions such that the expected cost in each scenario $A$ is at most $2.4957$ times the fractional cost in scenario $A$.

C Robust fault-tolerant UFL

C.1 The $(k + 5 + 4/k)$-approximation rounding routine

Like in the algorithms in the previous sections we first scale up the fractional facility-opening costs, we then cluster certain facilities to correlate their opening, and then use a randomized rounding routine to decide the subset of facilities to open. Once we open facilities and the adversary chooses which $k$ of them to close, clients get connected to the closest of the remaining open facilities.

Let $(x^*, y^*)$ be an optimal solution to the above LP relaxation of the problem. We first scale up the opening of facilities by $\gamma = k + 5 + 4/k$, i.e., we set $\overline{y}_i = \min\{1, \gamma \cdot y_i^*\}$. We also set $\overline{x}_{Aj} = \min\{1, \gamma \cdot x_{Aj}^*\}$.

Consider a single client-scenario pair $(j, A)$. Consider facilities $i$ fractionally serving this pair in solution $(x^*, y^*)$ in an order $i^1, i^2, \ldots$ of non-decreasing distance to $c_{ij}$. Let $i'$ be the first facility in this order such that $x_{Aj}^* + x_{i^2Aj}^* + x_{i^3Aj}^* \geq \frac{k+1}{\gamma} = \frac{k}{k+4}$. Recall that $C_{(j,A)} = \sum_j c_{ij} x_{Aj}^*$ denotes the fractional connection cost of client $j$ in scenario $A$. By an argument analogous to the one in Lemma 2.3, we obtain that $c_{i'j} \leq \frac{(k+4)(k+1)}{3k} \cdot C_{(j,A)}$. We now distinguish two cases.

Case 1. There exists $i$ among $i^1, i^2, \ldots, i'$ such that $\overline{y}_i = 1$. Then facility $i$ will be deterministically opened by the algorithm. Note that since $x_{Aj}^* > 0$, we have $i \not\in A$ (i.e., facility $i$ is not closed by the adversary in scenario $A$); hence, we can connect $j$ to $i$ in scenario $A$ in our constructed integral solution. It remains to observe that $c_{ij} \leq \frac{(k+4)(k+1)}{3k} \cdot C_{(j,A)} \leq (k + 5 + 4/k) \cdot C_{(j,A)}$ is a distance that we can accept.

Case 2. There is no $i$ among $i^1, i^2, \ldots, i'$ such that $\overline{y}_i = 1$. Then we have $\overline{x}_{Aj} + \overline{x}_{i^2Aj} + \ldots + \overline{x}_{i' Aj} \geq k + 1$, which is the fractional connection to at least $k + 1$ facilities, each of them within the distance of $\frac{(k+4)(k+1)}{3k} \cdot C_{(j,A)}$. With a randomized rounding technique described below, they will be turned into $k + 1$ facilities opened within the distance of $3 \cdot \frac{(k+4)(k+1)}{3k} \cdot C_{(j,A)} = (k + 5 + 4/k) \cdot C_{(j,A)}$. Since at most $k$ of these facilities will be closed by the adversary, there remains an open facility for client $j$ in scenario $A$ at distance at most $(k + 5 + 4/k) \cdot C_{(j,A)}$.

It remains to argue that we can turn $k + 1$ fractional connections to facilities at distance at most $d$ into $k + 1$ integral connections to facilities at distance at most $3d$. This can be seen as a situation typical for LP-rounding algorithms for the standard fault-tolerant facility location problem. Indeed, exactly this property is associated with the rounding scheme in [3] (Sections 3 and 4). It is obtained by carefully constructing a laminar family of subsets of facilities and performing a dependent rounding procedure guided by the subsets. It can also be thought of as an application of the pipage-rounding technique [1].

C.2 Better bound in the oblivious setting

Let us now consider the oblivious setting where the $k$ facilities to close/fail are chosen without the knowledge of our opening of facilities. In this setting we give a bound on the expected connection cost, where the expectation is over the random choices of the algorithm. More precisely, we will argue that the expected connection cost of client $j$ in scenario $A$ is bounded with respect to the fractional connection cost of $j$ in scenario $A$.

The difference with the previous setting is that now we can use the argument that after scaling the facility-opening variables by a constant $\gamma$, for a client $j$ in scenario $A$, with probability at least $1 - e^{-\gamma}$, at least one facility from those fractionally serving $(j, A)$ will be opened. Moreover, we can bound the expected distance to such facility by the fractional connection cost of $(j, A)$. This allows us to use those facilities that get opened with certainty (as described in Section C.1) only with a certain small probability. In such a situation, it is beneficial to scale the facility-opening variables by a little less.

The algorithm is like in Section C.1, only the scaling parameter $\gamma$ is smaller (say $\gamma = 1.5 + k$), and
the analysis is different. We argue that for every client \( j \) in each single scenario (choice of the \( k \) facilities to close) \( A \) the expected connection cost is bounded. As before we distinguish two cases.

**Case 1.** (There exists \( i \) such that \( \bar{x}_{A,ij} = 1 \)) If there is such facility at distance at most \((1.5 + k) C_{j,A}\) we just connect to it, otherwise, the average connection cost in \( x^* \) is only smaller then the average connection cost in \( x^\star \), and we may use a version of the Lemma 2.3 to argue that the expected connection cost to the closest of the facilities randomly opened by the algorithm is at most \( C_{j,A} \).

**Case 2.** (There is no such facility, and therefore there is no \( i \) among \( i^1, i^2, \ldots, i' \) such that \( y_i = 1 \)) Then we have \( \bar{x}_{A,i^1} \bar{x}_{A,i^2} + \cdots + \bar{x}_{A,i'} \geq k + 1 \), which is the fractional connection to at least \( k + 1 \) facilities, each of them within the distance of \((3+2k) \cdot C_{j,A}\). Just like in Section C.1, we argue that as a result of dependent rounding we obtain \( k + 1 \) facilities deterministically opened within the distance \((3 + 2k) \cdot C_{j,A}\). And now we propose a suboptimal assignment procedure to bound the cost of the optimal one. In the suboptimal assignment, client first looks at facilities fractionally serving him. If one of them is opened then connect to the closest one, which would incur an expected cost of \( C_{j,A} \), and if non is open, then take a facility deterministically opened within distance \((3 + 2k) \cdot C_{j,A}\). Like for the other results in this paper, we then argue that the expected connection cost is at most \((1 - e^{-(1.5+k)}) \cdot C_{j,A} + e^{-(1.5+k)} \cdot (3 + 2k) \cdot C_{j,A}\), which is less then \((1.5 + k) \cdot C_{j,A}\) for \( k \geq 2 \).

### C.3 Integrality gap example

Let us now show that the program (8)-(11) has integrality gap at least \( k + 1 - \epsilon \). Consider the following instance. There is a single client and \( n \) identical facilities. All the facility-opening costs are 1, and all the connection costs are 0. The optimal fractional solution opens each facility to the extent of \( \frac{1}{n-k} \), incurring cost \( \frac{n}{n-k} \cdot \lim_{n \to \infty} \frac{n}{n-k} = 1 \). Any integral solution, however, needs to open at least \( k + 1 \) facilities and therefore has cost at least \( k + 1 \). Therefore, for any \( \alpha < k + 1 \) there exists an instance of the \( k \)-robust fault tolerant problem, for which the integrality gap of the program (8)-(11) at least \( \alpha \).