Noncommutative geometry in quantum field theory and the cosmogenic neutrino physics at the extreme energies

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Analysis of the covariant $\theta$-exact noncommutative (NC) gauge field theory (GFT), inspired by high energy cosmic rays experiments, is performed in the framework of the inelastic neutrino-nucleon scatterings. Next we have found neutrino two-point function and shows a closed form decoupled from the hard ultraviolet (UV) divergent term, from softened ultraviolet/infrared (UV/IR) mixing term, and from the finite terms as well. For a certain choice of the noncommutative parameter $\theta$ which preserves unitarity, problematic UV divergent and UV/IR mixing terms vanish. Non-perturbative modifications of the neutrino dispersion relations are asymptotically independent of the scale of noncommutativity in both, the low and high energy limits and may allow superluminal propagation.

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INTRODUCTION

The idea of noncommutative coordinates is not new and it was for the first time proposed by Heisenberg in 1930 in his letters to Ehrenfest [1] and Peierls [2]. At that time Heisenberg could not formulate this idea mathematically. Still, there was a hope that uncertainty relations for coordinates might provide a natural cut-off for divergent integrals of Quantum Field Theory (QFT). The idea was propagated and in 1943 Snyder published a paper on "Quantized Space Time" [3]. Pauli in a letter to Bohr mentioned this work to be mathematically ingenious but he rejected it for physical reasons. Later, due to the extreme success of renormalization of Quantum Electrodynamics (QED), Quantum Chromodynamics (QCD) and/or Standard Model (SM), the idea of possible Noncommutative Field Theory (NCFT) was ignored for a long time. However at the beginning of 90’s Noncommutative Geometry was developed, see Connes [4] and Madore [5] and references there. Research on divergences in a field theory on quantum spaces started by pioneering work of Filk [6]. Lately noncommutative coordinates appears in string theory indicating that Noncommutative Gauge Field Theory (NCGFT) could be one of its low-energy effective theories, as proposed by Seiberg and Witten in [7].

The noncommutativity of spacetime, for the following definition of Moyal-Weyl star($\star$)-product

\[
(f \star g)(x) = \left. e^{\frac{i}{\hbar} \partial_x \theta_{\mu \nu} \partial_y} f(x)g(y) \right|_{y \to x}; \quad \theta_{\mu \nu} \text{ constant, real, antisymmetric matrix},
\]

and for local coordinates $x^\mu$ promoted to hermitian operators $\hat{x}^\mu$ satisfying spacetime noncommutativity, is realized by the so-called $\star$-commutator and imply uncertainty relations

\[
[\hat{x}^\mu, \hat{x}^\nu] = [x^\mu \star x^\nu] = i\theta^{\mu \nu} \implies |\Delta x^\mu \Delta x^\nu| \geq \frac{1}{2} |\theta^{\mu \nu}|.
\]

The so-called Seiberg-Witten (SW) maps [7] and enveloping algebra based models, where one could deform commutative gauge theories with arbitrary gauge group and representation, having commutative instead of the noncommutative gauge symmetry preserved as the fundamental symmetry of the theory, shows a significant progress in last decade. Those are known as the Wess type of models, see [8–14].

Studies on noncommutative particle phenomenology [15] was motivated to find possible experimental signatures and/or predict/estimate bounds on space-time noncommutativity from collider physics experimental data: for example from the Standard Model (SM) forbidden $Z \to \gamma \gamma$ decays, or from the SM invisible part of $Z \to \tau \nu$ decays, and more important from the ultra high energy (UHE) processes occurring in the framework of the cosmogenic neutrino physics. Constraint on the scale of the NCGFT, $\Lambda_{NC}$, is possible due to a direct couplings of $Z\gamma\gamma$ and $Z\tau\nu$.

The constraints on the U(1) charges, known as the "no-go theorem" [16], are rescinded in our approach [17], and the noncommutative extensions of particle physics covariant SM (NCSM) and the noncommutative grand unified theories (NCGUT) models [11–13, 17–22] were constructed. The method known as SW map and/or enveloping algebra (Wess approach) avoids both the gauge group and the U(1) charge issues. It was shown mathematically rigorously that any U(1) gauge theory on an arbitrary Poisson manifold can be deformation-quantized to a noncommutative gauge theory via the the enveloping algebra approach [13] and later extended to the non-Abelian gauge groups [23, 24]. The
important step that has been missed in a paper [16] opposing above conclusions, is the use of reducible representations [17]. All these allow a minimal deformation with no new particle content and with the sacrifice that interactions include infinitely many terms defined through recursion (due to expansions) over the NC parameter $\theta^{\mu\nu}$.

The perturbative quantization of noncommutative field theories was first proposed by Filk [6], while other famous examples are the running of the coupling constant of NC QED [25] and the UV/IR mixing [26, 27]. Later well behaving one-loop quantum corrections to noncommutative scalar $\phi^4$ theories [28, 30] and the NC QED [31] have been found. Also the SW expanded NCSM [11, 18, 20, 22] at first order in $\theta$, albeit breaking Lorentz symmetry is anomaly free [32, 33], and has well-behaved one-loop quantum corrections [25, 27, 34, 42]. However, despite of some significant progress in the models [28–42], a better understanding of various models quantum loop corrections still remains in general a challenging open question. This fact is particularly true for the models constructed by using SW map expansion in the NC parameter $\theta$, [8, 11, 21, 43, 44]. Resulting models are very useful as effective field theories including their one-loop quantum properties [32, 42] and relevant phenomenology [45–51].

Quite recently restrictions due to expansions over parameter has been overcome by constructions of the $\theta$-exact SW map and enveloping algebra based theoretical models, in the framework of covariant noncommutative quantum gauge field theory [14], and applied in loop computation [52, 54] and to the phenomenology, as well [52, 57]. Namely, an expansion and cut-off in powers of the NC parameters $\theta^{\mu\nu}$ corresponds to an expansion in momenta and restrict the range of validity to energies well below the NC scale $\Lambda_{NC}$. Usually, this is no problem for experimental predictions because the lower bound on the NC parameters $\theta^{\mu\nu}$ runs higher than typical momenta involved in a particular process. However there are exotic processes, in the early universe as well as those involving ultra high energy cosmic rays [51, 53, 58, 55, 58], in which the typical energy involved is higher than the current experimental bound on the NC scale $\Lambda_{NC}$. Thus, the previous $\theta$-cut-off approximate results are inapplicable. To cure the cut-off approximation, we are using $\theta$-exact expressions, inspired by exact formulas for the SW map [55, 60], and expand in powers of gauge fields, as we did in [54]. In $\theta$-exact models we have studied the UV/IR mixing [52, 53], the neutrino propagation [54] and also some NC photon-neutrino phenomenology [51, 55, 56, 58], respectively. Due to the presence of the UV/IR mixing the $\theta$-exact model is not perturbatively renormalizable, thus the relations of quantum corrections to the observations [61] are not entirely clear.

In this work we present NCSM extended neutrino gauge bosons actions to all orders of $\theta$ and discuss their quantum and phenomenological properties for light-like noncommutativity which are allowed by unitarity condition [62, 63].

**COSMOGENIC NEUTRINO PHYSICS MOTIVATION**

Gauge boson direct coupling to neutral and “chiral” fermion particles [43, 55, 56], allow us via ultra-high energy cosmogenic neutrino experiments, to estimate a constraint on the scale of the NCGFT, $\Lambda_{NC}$, see i.e. Fig. **1**.

The observation of ultra-high energy (UHE) $\nu$’s from extraterrestrial sources would open a new window to look to the cosmos, as such $\nu$’s may easily escape very dense material backgrounds around local astrophysical objects, giving thereby information on regions that are otherwise hidden to any other means of exploration. In addition, $\nu$’s are not deflected on their way to the earth by various magnetic fields, pointing thus back to the direction of distant UHE cosmic-ray source candidates. This could also help resolving the underlying acceleration in astrophysical sources.
FIG. 2: $\nu N \rightarrow \nu + \text{anything}$ cross sections vs. $\Lambda_{NC}$ for $E_\nu = 10^{10}$ GeV (thick lines) and $E_\nu = 10^{11}$ GeV (thin lines). FKRT and PJ lines are the upper bounds on the $\nu$-nucleon inelastic cross section, denoting different estimates for the cosmogenic $\nu$-flux. SM denotes the SM total (charged current plus neutral current) $\nu$-nucleon inelastic cross section. The vertical lines denote the intersections of our curves with the RICE results.

FIG. 3: The intersections of our curves with the RICE results (cf. Fig.2) as a function of the fraction of Fe nuclei in the UH cosmic rays. The terminal point on each curve represents the highest fraction of Fe nuclei above which no useful information on $\Lambda_{NC}$ can be inferred with our method.

In the energy spectrum of UHE cosmic rays at $\sim 4 \times 10^{19}$ eV the GZK-structure has been observed recently with high statistical accuracy [64]. Thus the flux of the so-called cosmogenic $\nu$’s, arising from photo-pion production on the cosmic microwave background $p\gamma_{CMB} \rightarrow \Delta^* \rightarrow N\pi$ and subsequent pion decay, is now guaranteed to exist. Possible ranges for the size of the flux of cosmogenic $\nu$’s can be obtained from separate analysis of the data [65, 66].

Using the upper bound on the $\nu N$ cross section derived from the RICE Collaboration search results [67] at $E_\nu = 10^{11}$ GeV ($4 \times 10^{-3}$ mb for the FKRT $\nu$-flux [65]), one can infer from $\theta$-truncated model on the NC scale $\Lambda_{NC}$ to be greater than 455 TeV, a really strong bound. Here we use $\theta^{n\nu} \equiv c^{n\nu}/\Lambda_{NC}^2$ such that the matrix elements of $c^{n\nu}$ are of order 1.

One should however be careful and suspect this result as it has been obtained from the conjecture that the $\theta$-expansion stays well-defined in the kinematical region of interest. Although a heuristic criterion for the validity of the perturbative $\theta$-expansion, $\sqrt{s}/\Lambda_{NC} \lesssim 1$, with $s = 2E_\nu M_N$, would underpin our result on $\Lambda_{NC}$, a more thorough inspection on the kinematics of the process does reveal a more stronger energy dependence $E_\nu^{1/2}s^{1/4}/\Lambda_{NC} \lesssim 1$. In spite of an additional phase-space suppression for small $x$’s in the $\theta^2$-contribution [45] of the cross section relative to the $\theta$-contribution, we find an unacceptably large ratio $\sigma(\theta^2)/\sigma(\theta) \simeq 10^4$, at $\Lambda_{NC} = 455$ TeV. Hence, the bound on $\Lambda_{NC}$ obtained this way is incorrect, and our last resort is to modify the model adequately to include the full-$\theta$ resummation, thereby allowing us to compute nonperturbatively in $\theta$. Total cross section, as a function of the NC scale at fixed $E_\nu = 10^{10}$ GeV and $E_\nu = 10^{11}$ GeV, together with the upper bounds depending on the actual size of the cosmogenic $\nu$-flux (FKRT [65] and PJ [66]) as well as the total SM cross sections at these energies, are depicted in our Figure 2.

Even if the future data confirm that UHE cosmic rays are composed mainly of Fe nuclei, as indicated by the PAO data, then still valuable information on $\Lambda_{NC}$ can be obtained with our method, as seen in Fig.3. Here we see the
intersections of our curves with the RICE results (cf. Fig. as a function of the fraction $\alpha$ of Fe nuclei in the UHE cosmic rays. Results depicted in Figs. shows convergent behavior. In our opinion those were the strong signs to continue research towards quantum properties and phenomenology of such $\theta$-exact NCGFT model.

**COVARIANT $\theta$-EXACT $U_\star(1)$ MODEL**

We start with the following SW type of NC $U_\star(1)$ gauge model:

$$S = \int -\frac{1}{4} F_{\mu\nu} \ast F_{\mu\nu} + i\Psi \ast \bar{D}\Psi,$$

with the NC definitions of the nonabelian field strength and the covariant derivative, respectively:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \ast A_\nu], \quad D_\mu \Psi = \partial_\mu \Psi - i[A_\mu \ast \Psi].$$

All noncommutative fields in this action $(A_\mu, \Psi)$ are images under (hybrid) Seiberg-Witten maps of the corresponding commutative fields $(a_\mu, \psi)$. Here we shall interpret the NC fields as valued in the enveloping algebra of the underlying gauge group. This naturally corresponds to an expansion in powers of the gauge field $a_\mu$ and hence in powers of the coupling constant $e$. At each order in $a_\mu$ we shall determine $\theta$-exact expressions.

In the next step we expand the action in terms of the commutative gauge fields $a_\mu$ (and/or coupling constant) and using the SW map solution up to the $O(a^3)$ order:

$$\Lambda = \lambda - \frac{1}{2} \theta^{ij} a_2 \partial_j \lambda, \quad A_\mu = a_\mu - \frac{1}{2} \theta^{\rho\mu} a_\nu \ast_2 (\partial_\rho a_\mu + f_{\rho\mu}),$$

$$\Psi = \psi - \theta^{\mu\nu} a_\mu \ast_2 \partial_\nu \psi + \frac{1}{2} \theta^{\mu\nu} \theta^{\rho\sigma} \left[ (a_\rho \ast_2 (\partial_\sigma a_\mu + f_{\sigma\mu})) \ast_2 \partial_\nu \psi + 2a_\mu \ast_2 (\partial_\rho a_\mu \ast_2 \partial_\nu \psi) - a_\mu \ast_2 (\partial_\rho a_\mu \partial_\nu \psi) - (a_\rho \partial_\mu \psi (\partial_\nu a_\sigma + f_{\nu\sigma}) - \partial_\mu \partial_\nu \psi a_\sigma a_\sigma) \ast_3 \right],$$

with $\Lambda$ being the NC gauge parameter and $f_{\mu\nu}$ the abelian commutative field strength $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$.

The generalized Mojal-Weyl star products $\ast_2$ and $\ast_3$, appearing in (5), are defined, respectively, as

$$f(x) \ast_2 g(x) = \left[ f(x) \ast g(x) \right] = \sin \left( \frac{\partial_\mu \theta \partial_\nu g_\mu}{\partial_\sigma \theta \partial_\delta g_{\sigma\delta}} f(x_1) g(x_2) \right)_{x_1 = x_2 = x},$$

$$f(x) g(x) h(x) \ast_3 = \left( \sin \left( \frac{\partial_\mu \theta \partial_\nu \partial_\lambda \theta}{\partial_\sigma \theta \partial_\delta \partial_\ell \theta} \right) \sin \left( \frac{\partial_\lambda \theta \partial_\ell \theta}{\partial_\sigma \theta \partial_\delta \partial_\ell \theta} \right) + \{1 \leftrightarrow 2\} \right) f(x_1) g(x_2) h(x_3) \left|_{x_1 = x} \right.$$

where $\ast$ is associative but noncommutative, while $\ast_2$ and $\ast_3$ are both commutative but nonassociative.

The resulting expansion defines $\theta$-exact neutrino-photon $U_\star(1)$ actions, for a gauge and a matter sectors respectively. Pure gauge field (3-photon) action reads:

$$S_g = \int i\partial_\mu a_\nu \ast [a^\mu \ast a^\nu] + \frac{1}{2} \theta_{\mu} \left( \theta^{\rho\sigma} a_\rho \ast_2 (\partial_\sigma a_\nu + f_{\sigma\nu}) \right) \ast f^{\mu\nu}.$$
Note that actions for gauge and matter fields obtained above, (8) and (9) respectively, are nonlocal objects due to the presence of the star products: $\star_1$, $\star_2$ and $\star_3$. Feynman rules from above actions (Fig.4), are given explicitly in [54].

**QUANTUM PROPERTIES: NEUTRINO SELF ENERGY**

As depicted in Fig. 5 there are four Feynman diagrams contributing to the $\nu$-self-energy at one-loop. With the aid of (9), we have verified by explicit calculation that the 4-field tadpole ($\Sigma_2$) does vanish. The 3-fields tadpoles ($\Sigma_3$ and $\Sigma_4$) can be ruled out by invoking the NC charge conjugation symmetry [21]. Thus only the $\Sigma_1$ diagram needs to be evaluated. In spacetime of the dimensionality $D$ we obtain

$$
\Sigma_1 = \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \left( \frac{\sin \frac{2\theta p}{q \mu}}{\frac{q \mu}{2}} \right)^2 \frac{1}{q^2 (\mu + q)^2} \cdot \left[ (q \theta p)^2 (4 - D) (\dot{\phi} + \dot{\theta} p) + (q \theta p) \left( \dot{\theta} (2p^2 + 2p \cdot q) - \ddot{p} (2q^2 + 2p \cdot q) \right) + \dot{p} \left( \ddot{q} (2p^2 + 2p \cdot q) - q^2 (2q^2 + 2p \cdot q) \right) \right],
$$

where $\ddot{p}^\mu = (\theta p)^\mu = \theta^{\mu \nu} p_\nu$, and in addition $\ddot{p}^\mu = (\theta p)^\mu = \theta^{\mu \nu} \theta_{\nu \rho} p^\rho$. To perform computations of those integrals using the dimensional regularization method, we first use the Feynman parametrization on the quadratic denominators, then the Heavy Quark Effective theory (HQET) parametrization [68] is used to combine the quadratic and linear denominators. In the next stage we use the Schwinger parametrization to turn the denominators into Gaussian.
integrals. Evaluating the relevant integrals for $D = 4 - \epsilon$ in the limit $\epsilon \to 0$, we obtain the closed form self-energy

$$
\Sigma_1 = \gamma_\mu \left[ p^\mu A + \theta \theta p^\mu \frac{p^2}{(\theta p)^2} B \right], 
$$

(11)

$$
A = \frac{-1}{(4\pi)^2} \left[ p^2 \left( \frac{\theta \theta p}{(\theta p)^2} + 2 \frac{(\theta p p)^2}{(\theta p)^4} \right) A_1 + \left( 1 + p^2 \left( \frac{\theta \theta p}{(\theta p)^2} \right) \right) A_2 \right], 
$$

(12)

$$
A_1 = \frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) + \ln(\pi e^{\gamma_E}) + \sum_{k=1}^{\infty} \left( \frac{\mu^2(\theta p)^2/4}{\Gamma(2k+2)} \right) \left( \ln \frac{\mu^2(\theta p)^2}{4} - 2\psi_0(2k+2) - \frac{8(k+1)}{(2k+1)(2k+3)} \right), 
$$

(13)

$$
A_2 = -\frac{(4\pi)^2}{2} B = -2 
+ \sum_{k=0}^{\infty} \frac{(\mu^2(\theta p)^2/4)^{k+1}}{(2k+1)(2k+3)\Gamma(2k+2)} \left( \ln \frac{\mu^2(\theta p)^2}{4} - 2\psi_0(2k+2) - \frac{8(k+1)}{(2k+1)(2k+3)} \right),
$$

(14)

with $\gamma_E \simeq 0.577216$ being Euler’s constant.

The $1/\epsilon$ UV divergence could in principle be removed by a properly chosen counterterm. However due to the specific momentum-dependent coefficient in front of it, a nonlocal form for it is required.

Turning to the UV/IR mixing problem, we recognize a soft UV/IR mixing term represented by a logarithm,

$$
\Sigma_{UV/IR} = \hat{p} \frac{p^2}{(4\pi)^2} \ln \left( \frac{1}{|\mu(\theta p)|^2} \right) \left( \frac{\theta \theta p}{(\theta p)^2} + 2 \frac{(\theta p p)^2}{(\theta p)^4} \right),
$$

(15)

Instead of dealing with nonlocal counterterms, we take a different route here to cope with various divergences besetting (14). Since $\theta^0 \neq 0$ makes a NC theory nonunitary, we can without loss of generality chose $\theta$ to lie in the $(1, 2)$ plane

$$
\theta^{\mu\nu} = \frac{1}{A_{NC}^2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \rightarrow \frac{\theta \theta p}{(\theta p)^2} + 2 \frac{(\theta p p)^2}{(\theta p)^4} = 0, \forall p.
$$

(16)

With (16), $\Sigma_1$, in terms of Euclidean momenta, receives the following form:

$$
\Sigma_1 = \frac{-1}{(4\pi)^2} \gamma_\mu \left[ p^\mu \left( 1 + \frac{\theta \theta p}{2(\theta p)^2} \right) - 2(\theta p p)^2 \right] A_2.
$$

(17)

By inspecting (14) one can be easily convinced that $A_2$ is free from the $1/\epsilon$ divergence and the UV/IR mixing term, being also well-behaved in the infrared, in the $\theta \to 0$ as well as $\theta p \to 0$ limit. We see, however, that the two terms in (17), one being proportional to $\hat{p}$ and the other proportional to $\theta \theta p$, are still ill-behaved in the $\theta p \to 0$ limit. If, for the choice (16), $\hat{p}$ denotes the momentum in the $(1, 2)$ plane, then $\theta p = \theta P$. For instance, a particle moving inside the NC plane with momentum $P$ along the one axis, has a spatial extension of size $|\theta P|$ along the other. For the choice (16), $\theta p \to 0$ corresponds to a zero momentum projection onto the $(1, 2)$ plane. Thus, albeit in our approach the commutative limit ($\theta \to 0$) is smooth at the quantum level, the limit when an extended object (arising due to the fuzziness of space) shrinks to zero, is not. We could surely claim that in our approach the UV/IR mixing problem is considerably softened; on the other hand, we have witnessed how the problem strikes back in an unexpected way. This is, at the same time, the first example where this two limits are not degenerate.

Computing dispersion relations we probe physical consequence of the 1-loop quantum correction, with $\Sigma_{1-loop_{\lambda\mu}}$ from Eq. (3.25) in [54]. We have to modify the propagator

$$
\Sigma = \frac{1}{\Sigma_1 - \Sigma_{1-loop_{\lambda\mu}}} = \frac{\Sigma}{\Sigma^2},
$$

(18)

and further we choose the NC parameter to be (16) so that the denominator is finite and can be expressed explicitly:

$$
\Sigma^2 = p^2 \left[ \hat{A}_2 \left( \frac{p^4}{p^2} + 2 \frac{p^4}{p^2} + 5 \right) - \hat{A}_2 \left( 6 + 2 \frac{p^2}{p^2} \right) + 1 \right],
$$

(19)

where $p_r$ represents $r$-component of the momentum $p$ in a cylindrical spatial coordinate system and $\hat{A}_2 = e^2 A_2/(4\pi)^2 = -B/2$. 
From above one see that \( p^2 = 0 \) defines one set of the dispersion relation, corresponding to the dispersion for the massless neutrino mode, however the denominator \( \Sigma^2 \) has one more coefficient \( \Sigma' \) which could also induce certain zero-points. Since the \( A_2 \) is a function of a single variable \( p^2 \), with \( p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 \) and \( p_2^2 = p_1^2 + p_2^2 \), the condition \( \Sigma' = 0 \) can be expressed as a simple algebraic equation

\[
\hat{A}_2 z^2 - 2 \left( A_2 - \hat{A}_2 \right) z + \left( 1 - 6 \hat{A}_2 + 5 \hat{A}_2^2 \right) = 0,
\]

(20)

of new variables \( z := p^2 / p_r^2 \), in which the coefficients are all functions of \( y := p^2 / p_r^2 / \Lambda_{NC}^4 \). Formal solutions of (20) are

\[
z = \frac{1}{A_2} \left[ \left( 1 - \hat{A}_2 \right) \pm 2 \left( \hat{A}_2 - \hat{A}_2^2 \right)^{1/2} \right],
\]

(21)

are birefringent. The behavior of solutions (21), is next analyzed at two limits \( y \to 0 \), and \( y \to \infty \).

The low-energy regime: \( p^2 p_r^2 \ll \Lambda_{NC}^4 \)

For \( y \ll 1 \) we set \( \hat{A}_2 \) to its zeroth order value \( e^2 / 8 \pi^2 \),

\[
p^2 \sim \left( \frac{8 \pi^2}{e^2} - 1 \right) \pm 2 \left( \frac{8 \pi^2}{e^2} - 1 \right)^{1/2} \cdot p_r^2 \simeq (859 \pm 59) \cdot p_r^2,
\]

(22)

obtaining two (approximate) zero points. From the definition of \( p^2 \) and \( p_r^2 \) we see that both solutions are real and positive. Taking into account the higher order (in \( y \)) correction these poles will locate nearby the real axis of the complex \( p_0 \) plane thus correspond to some metastable modes with the above defined dispersion relations. As we can see, the modified dispersion relation (22) does not depend on the noncommutative scale, therefore it introduces a discontinuity in the \( \Lambda_{NC} \to \infty \) limit, which is not unfamiliar in noncommutative theories.

The high-energy regime: \( p^2 p_r^2 \gg \Lambda_{NC}^4 \)

At \( y \gg 1 \) we analyze the asymptotic behavior of \( A_2 \), therefore (21) can be reduced

\[
A_2 \sim \frac{i \pi^2}{8 \sqrt{y}} (1 - \frac{16i}{\pi y} e^{-\frac{1}{2} \sqrt{y}}) + \mathcal{O}(y^{-1}), \quad \longrightarrow \quad z \sim -1 \pm 2i \quad \rightarrow \quad p_0^2 \sim p_3^2 \pm 2i p_r^2.
\]

(23)

We thus reach two unstable deformed modes besides the usual mode \( p^2 = 0 \) in the high energy regime. Here again the leading order deformed dispersion relation does not depend on the noncommutative scale \( \Lambda_{NC} \).

PHENOMENOLOGY: RATE OF \( Z \to \tau \nu \) DECAYS

To illustrate phenomenological effects of our \( \theta \)-exact construction, we present a computation the \( Z \to \tau \nu \) decay rate in the \( Z \)-boson rest frame, which is then readily to be compared with the precision \( Z \) resonance measurements, where \( Z \) is almost at rest. Since the complete \( Z\tau \nu \) interaction on noncommutative spaces was discussed in details in \([17, 53, 54, 56]\), we shall not repeat it here. Using the almost complete \( Z\tau \nu \) vertex from \([17]\) we have found the following \( Z \to \tau \nu \) partial width \([57]\)

\[
\Gamma(Z \to \tau \nu) = \Gamma_{SM}(Z \to \tau \nu)
\]

\[+ \frac{\alpha}{3M_Z |\vec{E}_{\theta}|} \left[ \kappa (1 - \kappa + \kappa \cos 2\theta_W) \sec^2 \theta_W \cos \left( \frac{M_Z |\vec{E}_{\theta}|}{4} \right) - 8 \csc^2 2\theta_W \right] \sin \left( \frac{M_Z |\vec{E}_{\theta}|}{4} \right)
\]

\[+ \frac{\alpha M_Z}{12} \left[ -2\kappa^2 + (\kappa (2\kappa - 1) + 2) \sec^2 \theta_W + 2 \csc^2 \theta_W \right],
\]

(24)

where \( \kappa \) is an arbitrary constant \([76]\). The NC part vanishes when \( \vec{E}_{\theta} \to 0 \), i.e. for vanishing \( \theta \) or space-like noncommutativity, but not light-like \([62, 63]\).

A comparison of the experimental \( Z \) decay width \( \Gamma_{invis} = (499.0 \pm 1.5) \text{ MeV} \) \([69]\) with its SM theoretical counterpart, allows us to set a constraint \( \Gamma(Z \to \tau \nu) - \Gamma_{SM}(Z \to \tau \nu) \lesssim 1 \text{ MeV} \), from where a bound on the scale of noncommutativity \( \Lambda_{NC} = |\vec{E}_{\theta}|^{-1/2} \gtrsim 140 \text{ GeV} \) is obtained (see Fig. (6), for the choice \( \kappa = 1 \).
CONCLUSIONS

In the energy range of interest, $10^{10}$ to $10^{11}$ GeV, where there is always energy of the system ($E$) larger than the NC scale ($E/\Lambda_{NC} > 1$), the perturbative expansion in terms of $\Lambda_{NC}$ retains no longer its meaningful character, thus it is forcing us to resort to those NC field-theoretical frameworks involving the full $\theta$-resummation. Our numerical estimates of the contribution to the processes coming from the photon exchange, pins impeccably down a lower bound on $\Lambda_{NC}$ to be as high as around up to $O(10^6)$ GeV, depending on the cosmogenic $\nu$-flux.

We first discuss $\theta$-exact computation of the one-loop quantum correction to the $\nu$-propagator. General expression for the neutrino self-energy [11] contains in [12] both a hard $1/\epsilon$ UV term and the UV/IR mixing term with a logarithmic infrared singularity $\ln|\theta p|$. Results shows complete decoupling of the UV divergent term from softened UV/IR mixing term and from the finite terms as well. Our deformed dispersion relations at both the low and high energies and at the leading order do not depend on the noncommutative scale $\Lambda_{NC}$. The low energy dispersion [22] is capable of generating a direction dependent superluminal velocity. This is clear from the maximal attainable velocity of the neutrinos

$$\frac{v_{max}}{c} = \frac{dE}{d|\vec{p}|} \sim \sqrt{1 + (859 \pm 59) \sin^2 \vartheta},$$

where $\vartheta$ is the angle with respect to the direction perpendicular to the NC plane. This gives one more example how such spontaneous $\theta$-background breaking of Lorentz symmetry could affect the particle kinematics through quantum corrections, even without divergent behavior like UV/IR mixing. On the other hand one can also see that the magnitude of superluminosity is in general very large in our model as a quantum effect, thus seems contradicting various observations which suggests much smaller values [70–72]. On the other hand, note that the large superluminal velocity issue may also be reduced/removed by taking into account several considerations and/or properties:
- Selection of a constant nonzero $\theta$ background in this paper is due to the computational simplicity. The results will, however, still hold for a NC background that is varying sufficiently slowly with respect to the scale of noncommutativity. There is no physics reason to expect $\theta$ to be a globally constant background ether. In fact, if the $\theta$ background is only nonzero in tiny regions (NC bubbles) the effects of the modified dispersion relation will be suppressed macroscopically. Certainly a better understanding of possible sources of NC is needed.
- We have considered only the purely noncommutative neutrino-photon coupling. However, it has been pointed out that modified neutrino dispersion relation could open decay channels within the commutative standard model framework [23]. In our case this would further provide cascade decay channel(s) which can, via bremsstrahlung, bring superluminal neutrinos to normal ones.
- Our results differs with respect to non SW map models since in our case both terms are proportional to the spacetime noncommutativity dependent $\theta$-ratio (the scale-independent structure!) factor in [10], which arise from the natural non-locality of our actions. Besides the divergent terms, a new spinor structure ($\theta\theta p$) with finite coefficients emerges in our computation, see [11–14]. All these structures are proportional to $p^2$, therefore if appropriate renormalization conditions are imposed, the commutative dispersion relation $p_\mu^2 = 0$ can still hold. Also it is reasonable to conjecture that SW map freedom may also serve as one possible remedy to (23) issue.
- Finally, we mention that our approach to UV/IR mixing should not be confused with the one based on a theory with UV completion ($\Lambda_{UV} < \infty$), where a theory becomes an effective QFT, and the UV/IR mixing manifests itself via a
specific relationship between the UV and the IR cutoffs [74]. This is fortunate with regard to the use of low-energy NCQFT as an important window to holography [61] and quantum gravity [75].

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The constant $\kappa$ measures a correction from the $\star$-commutator coupling of the right handed neutrino $\nu_R$ to the noncommutative hypercharge $U(1)_Y$ gauge field $B_\mu^\nu[r]$. Coupling is chiral blind and it vanishes in the commutative limit. The non-$\kappa$-proportional term, on the other hand, is the noncommutative deformation of standard model $Z$-neutrino coupling, which involves the left handed neutrinos only. Details can be found in section four of [17].