On the Multiplicative Structure of Topological Hochschild Homology

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Abstract We show that the topological Hochschild homology $\text{THH}(R)$ of an $E_n$-ring spectrum $R$ is an $E_{n-1}$-ring spectrum. The proof is based on the fact that the tensor product of the operad $\text{Ass}$ for monoid structures and the the little $n$-cubes operad $C_n$ is an $E_{n+1}$-operad, a result which is of independent interest.

In 1993 Deligne asked whether the Hochschild cochain complex of an associative ring has a canonical action by the singular chains of the little 2-cubes operad. Affirmative answers for differential graded algebras in characteristic 0 have been found by Kontsevich and Soibelman [11], Tamarkin [15] and [16], and Voronov [18]. A more general proof, which also applies to associative ring spectra is due to McClure and Smith [14]. In [10] Kontsevich extended Deligne’s question: Does the Hochschild cochain complex of an $E_n$ differential graded algebra carry a canonical $E_{n+1}$-structure?

We consider the dual problem: Given a ring $R$ with additional structure, how much structure does the topological Hochschild homology $\text{THH}(R)$ of $R$ inherit from $R$? The close connection of $\text{THH}$ with algebraic $K$-theory and with structural questions in the category of spectra make multiplicative structures on $\text{THH}$ desirable.

In his early work on topological Hochschild homology of functors with smash product Bökstedt proved that $\text{THH}$ of a commutative such functor is a commutative ring spectrum (unpublished). The discovery of associative, commutative and unital smash product functors of spectra simplified the definition of $\text{THH}$ and the proof of the corresponding result for $E_\infty$-ring spectra considerably (e.g. see [13]).
In this paper we morally prove

**Theorem A:** For \( n \geq 2 \), if \( R \) is an \( E_n \)-ring spectrum then \( \text{T HH}(R) \) is an \( E_{n-1} \)-ring spectrum.

The same result has been obtained independently by Basterra and Mandell using different techniques [2].

Why “morally”? To define \( \text{T HH}(R) \) we need \( R \) to be a strictly associative spectrum. In general, \( E_n \)-structures do not have a strictly associative substructure. So we have to replace \( R \) by an equivalent strictly associative ring spectrum \( Y \), whose multiplication extends to an \( E_n \)-structure. Then the statement makes sense for \( Y \). Here is a more precise reformulation of Theorem A.

**Theorem B:** Let \( R \) be an \( E_n \)-ring spectrum. Then there are \( E_n \)-ring spectra \( X \) and \( Y \) and maps of \( E_n \)-ring spectra

\[
Y \longrightarrow X \longrightarrow R
\]

which are homotopy equivalences of underlying spectra such that the \( E_n \)-structure on \( Y \) extends a strictly associative ring structure and the topological Hochschild homology \( \text{T HH}(Y) \) inherits an \( E_{n-1} \)-ring structure from \( Y \).

Theorem B is an easy consequence of the universality of the \( W \)-construction of Boardman-Vogt [5], [17] and an interchange result involving the operad structures of the operad \( \mathcal{A ss} \) codifying monoid structures and the little \( n \)-cubes operad \( \mathcal{C}_n \). The interchange is codified by the tensor product of operads (for terminology see [16]). Our key result will be

**Theorem C:** \( \mathcal{A ss} \otimes \mathcal{C}_n \) is an \( E_{n+1} \)-operad satisfying Condition 1.2 below.

In [9] this theorem has been announced and the main ideas for a proof have been sketched. Here we will include a detailed proof by depicting the spaces \( (\mathcal{A ss} \otimes \mathcal{C}_n)(k) \) as iterated colimits of diagrams of contractible spaces over posets. The diagrams of this iterated colimit combine to give a diagram over a Grothendieck construction, whose realization will turn out to be an \( E_{n+1} \)-operad.

Since this iterated colimit construction might be of use in other cases we give a formal definition in an appendix.

In this paper we will be working in the categories \( \mathcal{T op}, \mathcal{T op}^*, \) and \( \mathcal{S p} \) of
1 \textbf{E}_n\text{-operads}

In this section we recall the basic definitions underlying the statement of Theorem C.

\textbf{1.1 Definition:} An operad is a topologically enriched strict symmetric monoidal category \((B, \oplus, 0)\) such that

\begin{itemize}
  \item \(\text{ob } B = \mathbb{N}\) and \(m \oplus n = m + n\)
  \item \(B(m, n) = \coprod_{r_1 + \ldots + r_n = m} B(r_1, 1) \times \ldots \times B(r_n, 1) \times \Sigma_{r_1} \times \ldots \times \Sigma_{r_n} \Sigma_m\)
\end{itemize}

Since the morphism spaces \(B(m, n)\) are determined by the spaces \(B(r, 1)\) we usually write \(B(r)\) for \(B(r, 1)\) and work with them. A map of operads \(f : B \to C\) is a continuous strict symmetric monoidal functor such that \(f(n) = n\) for all \(n \in \mathbb{N}\). It is called a weak equivalence if \(f : B(n) \to C(n)\) is a homotopy equivalence of spaces for all \(n\). We call it a \(\Sigma\)-equivalence if these maps are \(\Sigma_n\)-equivariant homotopy equivalences.

An operad \(B\) is called \(\Sigma\)-free if \(B(r) \to B(r)/\Sigma_r\) is a numerable principal \(\Sigma_r\)-bundle for all \(r\).

For technical reasons we sometimes require

\textbf{1.2 Condition:} \(\{id\} \subset B(1)\) is a closed cofibration.

\textbf{1.3 Definition:} Let \(C_n\) denote the little \(n\)-cubes operad [4, Chap. 2, Expl. 5]. An \textit{E}_n\text{-operad} is an operad \(B\) for which there exists a sequence of \(\Sigma\)-equivalences of operads

\[
B = B_0 \xrightarrow{f_0} B_1 \xleftarrow{f_1} \ldots \xrightarrow{f_r} B_r \xleftarrow{f_r} C_n
\]

\textbf{1.4} Let \(Opr\) denote the category of operads. In [5, Chap. III] Boardman-Vogt constructed as continuous functor

\[
W : Opr \longrightarrow Opr
\]
together with a natural transformation

\[ \varepsilon : W \longrightarrow \text{Id} \]

taking values in \( \Sigma \)-equivalences. It can be interpreted as a cofibrant replacement of \( B \) \[17\]. In particular, given a diagram of maps of operads

\[
\begin{array}{ccc}
C & \xrightarrow{g} & D \\
W B & \xleftarrow{f} & D
\end{array}
\]

such that \( B \) satisfies Condition \[1.2\] and \( g \) is a \( \Sigma \)-equivalence, then there exists a lift \( h : WB \rightarrow C \) up to homotopy through operad maps, and \( h \) is unique up to homotopy through operad maps. If \( B \) is also \( \Sigma \)-free, the same holds if \( g \) is only a weak equivalence \[5, \text{3.17}\].

This implies

1.5 Proposition: Let \( B \) and \( C \) be \( E_n \)-operads such that \( B \) satisfies Condition \[1.2\]. Let

\[ B = B_0 \xrightarrow{f_0} B_1 \xrightarrow{f_1} \cdots \xrightarrow{f_r} B_r \xrightarrow{f_r} C \]

be a sequence of weak equivalences connecting them. Then there is a diagram of weak equivalences

commuting up to homotopy through operad maps. The \( g_i \) are unique up to homotopy through operad maps. In particular, there exists a weak equivalence \( WC_n \rightarrow C \).

\( E_n \)-structures are closely related to \( n \) interchanging (\( E_1 = A_\infty \))-structures. Let \( C \) and \( D \) be two operads and \( X \) be an object having a \( C \)- and a \( D \)-structure. These structured are said to interchange if for each \( c \in C(n) \) the operation

\[ c : X^n \longrightarrow X \]
is a $\mathcal{D}$-homomorphism, or equivalently, for each $d \in \mathcal{D}(m)$ the operation
\[ d : X^m \longrightarrow X \]
is a $\mathcal{C}$-homomorphism, i.e. the diagram
\[
\begin{array}{ccc}
(X^m)^n & \cong & (X^n)^m \\
\downarrow c^n & & \downarrow d \\
X^n & \underset{c}{\longrightarrow} & X
\end{array}
\]
commutes for all $c \in \mathcal{C}(n)$ and all $d \in \mathcal{D}(m)$.

The resulting structure on $X$ is codified by an operad $\mathcal{C} \otimes \mathcal{D}$ called the tensor product of $\mathcal{C}$ and $\mathcal{D}$. Formally, $\mathcal{C} \otimes \mathcal{D}$ is the quotient of the categorical sum $\mathcal{C} \oplus \mathcal{D}$ in $\mathcal{Opr}$ by factoring out relation (1.6). For more details see [5, p. 40ff].

Theorem C will be proved in Section 4. In [4, 15] we will give an explicit chain of $\Sigma$-equivalences of operads connecting $(\text{Ass} \otimes \mathcal{C}_n)$ with $\mathcal{C}_{n+1}$.

## 2 Algebraic structures on spectra

### 2.1
The category $\mathcal{S}p$ of spectra is enriched over $\mathcal{T}op^*$ and complete and cocomplete in the enriched sense (for details see [8, Chap. VII]). If $K$, $L$ are based spaces and $M, N$ are spectra we have a natural isomorphism
\[ M \wedge (K \wedge L) \cong (M \wedge K) \wedge L \]
and natural homeomorphisms
\[ \mathcal{S}p(M \wedge K, N) \cong \mathcal{T}op^*(K, \mathcal{S}p(M, N)) \cong \mathcal{S}p(M, F(K, N)) \]
where $F(K, N)$ is the function spectrum. In particular,
\[ - \wedge K : \mathcal{S}p \longrightarrow \mathcal{S}p \]
preserves colimits.

### 2.2
We can form the based topological endomorphism operad $\mathcal{E}nd_M$, given by
\[ \mathcal{E}nd_M(n) = \mathcal{S}p(M^\wedge n, M) \]
with the 0-map as base point, where $M^\wedge 0 = S$ is the sphere spectrum.

If $\mathcal{C}$ is any operad in $\mathcal{T}op$, a $\mathcal{C}$-structure on $M$ is an operad map

$$\mathcal{C}_+ \longrightarrow \mathcal{E}nd_M$$

where $\mathcal{C}_+(n) = \mathcal{C}(n)_+ = \mathcal{C}(n) \sqcup \{\ast\}$ with basepoint $\ast$. This transforms the topological operad $\mathcal{C}$ into a based topological operad $\mathcal{C}_+$; the monoidal structure in $\mathcal{T}op^*$ is given by the smash product. Passing to adjoints a $\mathcal{C}$-structure on $M$ is given by a sequence of maps

$$\mathcal{C}(n)_+ \wedge_{\Sigma_n} M^\wedge n \longrightarrow M, \quad n \in \mathbb{N}$$

satisfying certain conditions due to the fact that $\mathcal{C}_+ \rightarrow \mathcal{E}nd_M$ is a symmetric monoidal functor.

$M$ together with a given $\mathcal{C}$-structure is called a $\mathcal{C}$-algebra or $\mathcal{C}$-ring spectrum.

To make sense of the interchange diagram (1.6) we have to give $M^\wedge n$ a $\mathcal{C}$-structure: If $M$ and $N$ are $\mathcal{C}$-algebras, then the canonical $\mathcal{C}$-algebra structure on $M \wedge N$ is given by the maps

$$\mathcal{C}(n)_+ \rightarrow (\mathcal{C}(n) \times \mathcal{C}(n))_+ = \mathcal{C}(n)_+ \wedge \mathcal{C}(n)_+ \rightarrow \mathcal{E}nd_M(n) \wedge \mathcal{E}nd_N(n) \rightarrow \mathcal{E}nd_{M \wedge N}(n)$$

where the first map is induced by the diagonal and the last by the smash product.

Finally we will need

2.3 Proposition: If $M_\ast$ is a simplicial $\mathcal{C}$-algebra, then the realization $\left| M_\ast \right|$ inherits a $\mathcal{C}$-algebra structure.

This follows from the fact that $- \wedge \mathcal{C}(n)_+$ preserves colimits. For details see [8, X. 1.3, X. 1.4].

3 THH of $E_n$-ring spectra

3.1 Definition: Let $\mathcal{B}$ and $\mathcal{C}$ be $E_n$-operads, let $R$ be a $\mathcal{B}$-algebra and $M$ be a $\mathcal{C}$-algebra. An $E_n$-ring map from $R$ to $M$ is a pair $(\alpha, f)$ consisting of
an operad map $\alpha : B \to C$ and a homomorphism $f : R \to M$ of $B$-algebras, where the $B$-structure on $M$ is the pulled back $C$-structure.

3.2 Let $R \to M$ be an $E_n$-ring map between $E_n$-ring spectra. In view of (1.5) and Theorem C we may assume that it is a homomorphism of $W(\text{Ass} \otimes C_{n-1})$-algebras.

For any operad $B$ we have the free algebra functor $\mathbb{B}$ from spectra to $B$-algebras defined by

$$\mathbb{B}(X) = \bigvee_{n \geq 0} B(n) \wedge_{\Sigma_n} X^\wedge n.$$  

We now form the monadic bar constructions [8, Chap. XII] to obtain a diagram of $E_n$-ring spectra (here $W(\mathcal{C})$ stands for the free algebra functor associated with the operad $W$)

$$
\begin{array}{cccc}
B(\text{Ass} \otimes C_{n-1}, W(\text{Ass} \otimes C_{n-1}), R) & \leftarrow & B(W(\text{Ass} \otimes C_{n-1}), W(\text{Ass} \otimes C_{n-1}), R) & \rightarrow & R \\
\downarrow & & \downarrow & & \\
B(\text{Ass} \otimes C_{n-1}, W(\text{Ass} \otimes C_{n-1}), M) & \leftarrow & B(W(\text{Ass} \otimes C_{n-1}), W(\text{Ass} \otimes C_{n-1}), M) & \rightarrow & M
\end{array}
$$

and $E_n$-ring maps by (2.3). Since $\varepsilon : W(\text{Ass} \otimes C_{n-1}) \to \text{Ass} \otimes C_{n-1}$ is a weak $\Sigma$-equivalence the horizontal maps are homotopy equivalences of spectra [8, X. 2.4]. Let

$$f : Y_R \longrightarrow Y_M$$

denote the left vertical $(\text{Ass} \otimes C_{n-1})$-algebra homomorphism. In particular, $f$ is a homomorphism of strictly associative, unital ring spectra, so that $Y_M$ is a $Y_R$-bimodule. We can form the topological Hochschild homology of $Y_R$ with coefficients in $Y_M$:

3.3 Let $Q$ be a monoid in $\mathcal{S}p$ and $N$ a $Q$-bimodule. Then $THH(Q; N)$ is defined to be the realization of the simplicial spectrum

$$[n] \longrightarrow THH(Q; N)_n = Q^{\wedge n} \wedge N$$

with the well-known Hochschild boundary and degeneracy maps.

The inclusions of the 0-skeleton defines a natural map

$$\eta : N \longrightarrow THH(Q; N).$$

In our situation $THH(Y_R; Y_M)_*$ is a simplicial $C_{n-1}$-algebra by the interchange relation (1.6). Hence $THH(Y_R; Y_M)$ is a $C_{n-1}$-algebra and

$$\eta : Y_M \longrightarrow THH(Y_R; Y_M)$$
is a homomorphism of $C_{n-1}$-algebras. We obtain the following generalization of Theorem B.

3.4 Theorem: Let $f : R \to M$ be an $E_n$-ring map between $E_n$-ring spectra. Then there is a commutative diagram of $E_n$-ring spectra and $E_n$-ring maps

$$
\begin{array}{ccc}
Y_R & \longrightarrow & X_R \\
\downarrow f_Y & & \downarrow f_X \\
Y_M & \longleftarrow & X_M
\end{array}
$$

with the following properties:

1. The horizontal maps are homotopy equivalences of spectra.

2. $Y_R$ and $Y_M$ are $(\text{Ass} \otimes C_{n-1})$-algebras and $f_Y$ is an $(\text{Ass} \otimes C_{n-1})$-algebra homomorphism.

The second property implies that $THH(Y_R; Y_M)$ is a $C_{n-1}$-ring spectrum, and the natural map $\eta : Y_M \to THH(Y_R; Y_M)$ is a $C_{n-1}$-algebra homomorphism.

4 Proof of Theorem C

In general analyzing the homotopy type of the tensor product of operads is an intractable problem. Our strategy is to represent $(\text{Ass} \otimes C_{n})(k)$ as the colimit of a diagram

$$
F_k : \mathcal{K}_{n+1}(k) \longrightarrow \mathcal{T op}
$$

of contractible spaces, indexed by the $k$-th space of a modification of Berger’s complete graphs operad, a poset operad defined below, such that

- the diagrams are compatible with the operad structures of $\mathcal{K}_{n+1}$ and $\text{Ass} \otimes C_{n}$

- the canonical map $\text{hocolim } F_k \to \text{colim } F_k = (\text{Ass} \otimes C_{n})(k)$ is a homotopy equivalence

Then the collection of the hocolim $F_k$ forms an operad hocolim $F$ and we have a chain of weak equivalences of operads

$$
|\mathcal{K}_{n+1}| = \text{hocolim}_{\mathcal{K}_{n+1}} \ast \text{hocolim } F \longrightarrow \text{Ass} \otimes C_{n}
$$
Since the topological realization $|K_{n+1}|$ of $K_{n+1}$ is a $\Sigma$-free topological operad, so is hocolim $F$. We will show that $(\text{Ass} \otimes C_n)$ is $\Sigma$-free, hence both weak equivalences are $\Sigma$-equivalences. Moreover, using a corrected version of Berger’s argument in [3] we will prove that $|K_{n+1}|$ is an $E_{n+1}$-operad. Hence $\text{Ass} \otimes C_n$ is an $E_{n+1}$-operad, too.

The modified complete graphs operad $K$: A coloring of the complete graph on the set of vertices $\{1, 2, 3, \ldots, k\}$ is an assignment of colors to each edge of the graph from the countable set of colors $\{1, 2, 3, \ldots\}$. A monochrome acyclic orientation of a colored complete graph on $k$ vertices is an assignment of direction to each edge of the graph such that no directed cycles of edges of the same color occur. The poset $K(k)$ has as elements pairs $(\mu, \sigma)$, where $\mu$ is a coloring and $\sigma$ is a monochrome acyclic orientation of the complete graph on $k$ vertices. The order relation on $K(k)$ is determined as follows: we say that $(\mu_1, \sigma_1) \leq (\mu_2, \sigma_2)$ if for any colored oriented edge $a \xrightarrow{i} b$ in $(\mu_1, \sigma_1)$ the corresponding edge in $(\mu_2, \sigma_2)$ has orientation and coloring $a \xrightarrow{j} b$ with $j \geq i$ or $b \xrightarrow{j} a$ with $j > i$. The $n$-th filtration $K_n(k)$ is the subposet of $K(k)$ where the colorings are restricted to take values in the subset $\{1, 2, 3, \ldots, n\}$.

The action of the symmetric group $\Sigma_k$ on $K(k)$ is via permutation of the vertices. The composition

$$K(k) \times K(m_1) \times K(m_2) \times \cdots \times K(m_k) \longrightarrow K(m_1 + m_2 + \cdots + m_k)$$

assigns to a tuple of orientations and colorings in $K(k) \times K(m_1) \times K(m_2) \times \cdots \times K(m_k)$ the orientation and coloring obtained by subdividing the set of $m_1 + m_2 + \cdots + m_k$ vertices into $k$ adjacent blocks containing $m_1$, $m_2$, $\ldots$, $m_k$ vertices respectively. The edges connecting vertices within the $i$-th block are oriented and colored according to the given element in $K(m_i)$. The edges connecting vertices between blocks $i$ and $j$ are all oriented and colored according to the corresponding edge in the given element of $K(k)$.

Berger’s complete graphs operad $K_n^B$ is the suboperad of $K_n$ consisting of those oriented colored graphs which do not have any cycles, i.e., polychromatic cycles are also disallowed for elements in $K_n^B(k)$

4.1 Analysis of $\text{Ass} \otimes C_n$: By [3] Thm. 5.5] the space $(\text{Ass} \otimes C_n)(k)$ is the quotient of $\Sigma_k \times C_n(1)^k$ by the relation

$$(\pi; c_1, \ldots, c_k) \sim (\rho; c_1, \ldots, c_k)$$

iff $\pi^{-1}(i) < \pi^{-1}(j)$ and $\rho^{-1}(i) > \rho^{-1}(j)$ imply that $(c_i, c_j) \in C_n(2)$. The element $(\sigma, c_1, \ldots, c_k) \in \Sigma_k \times C_n(1)^k$ represents the operation $(x_1, \ldots, x_k) \mapsto$
\(c_{\sigma_1}(x_{\sigma_1}) \cdot \ldots \cdot c_{\sigma_k}(x_{\sigma_k})\), where \(\cdot\) stands for the monoid multiplication. This also specifies the operad structure.

**Observation:** Since \((\text{Ass} \otimes C_n)(1) = C_n(1)\) and \(C_n\) satisfies Condition \(1.2\), so does \(\text{Ass} \otimes C_n\).

Since there is only one color for edges of elements in \(K_1(k)\), the operads \(K_1\) and \(K_1^B\) coincide. In particular, the elements in \(K_1(k)\) do not contain any cycles. An orientation with no cycles of the complete graph on the set of vertices \(\{1, 2, \ldots, k\}\) is a total ordering of \(\{1, 2, \ldots, k\}\), which in turn can be identified with a permutation of \(\{1, 2, \ldots, k\}\). Hence a representative of an element in \((\text{Ass} \otimes C_n)(k)\) can be identified with an oriented graph \(\lambda \in K_1(k)\) together with a labelling of the vertices by elements of \(C_n(1)\).

To take care of the relation (4.1) we enlarge the modified complete graphs operad: we allow complete graphs with partial monochrome acyclic orientations and partial colorings. Such graphs \(\lambda'\) are obtained from oriented colored graphs \(\lambda \in K(k)\) by choosing a subset \(S\) of the set \(E(k)\) of edges of \(\lambda\) and forgetting the orientations and colors of all edges in \(S\). The graph \(\lambda''\) obtained from \(\lambda\) by forgetting the orientations and colors of all edges in \(E(k) \setminus S\) is called a complementary graph of \(\lambda\). Let \(\widehat{K}(k)\) denote the poset of all pairs \((\mu, \sigma)\), where \(\mu\) is a partial coloring and \(\sigma\) is a partial monochrome acyclic orientation of the complete graph on \(k\) vertices obtained from some element in \(K(k)\). The order relation is defined as follows: \((\mu_1, \sigma_1) \leq (\mu_2, \sigma_2)\) if every uncolored unoriented edge in \((\mu_1, \sigma_1)\) is also uncolored unoriented in \((\mu_2, \sigma_2)\), and for any colored oriented edge \(a \xrightarrow{i} b\) in \((\mu_1, \sigma_1)\) the corresponding edge in \((\mu_2, \sigma_2)\) is either uncolored unoriented or has orientation and coloring \(a \xrightarrow{j} b\) with \(j \geq i\) or \(b \xrightarrow{j} a\) with \(j > i\).

The symmetric group actions and composition in \(\widehat{K}\), and the \(n\)-th filtration \(\widehat{K}_n\) are defined as in \(K\). We shall refer to \(\widehat{K}\) and its filtrations as the augmented complete graphs operad.

While the topological realization of \(K_n\) is an \(E_n\)-operad, this is not true for \(\widehat{K}_n\): \(\mid \widehat{K}_n(k)\mid\) is equivariantly contractible to the \(\Sigma_k\) fixed point specified by the complete graph on \(\{1, 2, \ldots, k\}\) with all its edges unoriented and uncolored.

If we now label the vertices of \(\lambda \in \widehat{K}_1(k)\) by elements in \(C_n(1)\) with the extra condition that the pair of labels \((c, c')\) of the end points of a non-oriented edge is an element of \(C_n(2)\), then \(\lambda\) with its vertex labels \((c_1, \ldots, c_k)\) represents the equivalence class in \((\text{Ass} \otimes C_n)(k)\) of all \((\lambda'; c_1, \ldots, c_k)\), where \(\lambda' \in K_1(k)\)
is an element from which \( \lambda \) can be obtained by forgetting orientations and colors. These labelled augmented complete graphs form an operad \( \hat{\mathcal{K}}_1 \# \mathcal{C}_n \). Its composition is induced by the composition in \( \hat{\mathcal{K}}_1 \) and the following labelling condition: if the \( i \)-th vertex of the element in \( \hat{\mathcal{K}}_1(k) \) has the label \( a \) we compose the labels of the vertices of the elements in \( \hat{\mathcal{K}}_1(m_i) \) from the left with \( a \). So \( (\hat{\mathcal{K}}_1 \# C_n)(k) \) is the disjoint union of all \( A(\lambda), \lambda \in \hat{\mathcal{K}}_1(k) \), where \( A(\lambda) \subset C_n(1)^k \) denotes the space of possible vertex labels of \( \lambda \). We obtain that \( (\mathcal{A}_s \otimes C_n)(k) \) is a quotient of \( (\hat{\mathcal{K}}_1 \# C_n)(k) \). More precisely, the analysis of \( \mathcal{A}_s \otimes C_n \) of 4.1 can be restated as

**4.2 Lemma:** \((\mathcal{A}_s \otimes C_n)(k)\) is the colimit of the diagram

\[ A : \hat{\mathcal{K}}_1(k)^{op} \to \mathcal{T}_{op}, \quad \lambda \mapsto A(\lambda) \]

where we consider each poset as a category with a morphism \( \lambda_1 \to \lambda_2 \) whenever \( \lambda_1 \leq \lambda_2 \).

Our next step is to depict \( A(\lambda) \) as a colimit of contractible subspaces. Here \( \mathcal{K}_{n+1} \) comes into the picture. We embed \( \hat{\mathcal{K}}_1 \) into \( \hat{\mathcal{K}}_{n+1} \) by changing the color 1 of the colored edges of its graphs to \( n + 1 \), and we define \( T_k(\lambda) \) be the subposet of \( \hat{\mathcal{K}}_{n}(k) \) of all \( \lambda' \) which are complementary graphs of \( \lambda \). More explicitly: let \( S \subset E(k) \) be the set of colored edges of \( \lambda \); then an element \( \lambda' \in \hat{\mathcal{K}}_{n}(k) \) lies in \( T_k(\lambda) \) if there is an element \( \overline{\lambda} \in \mathcal{K}_{n+1}(k) \) whose edges in \( S \) are oriented as in \( \lambda \) and colored \( n + 1 \) and \( \lambda' \) is obtained from \( \overline{\lambda} \) by forgetting orientations and colors of the edges in \( S \).

We define \( n \) strict order relations on \( \mathcal{C}_n(1) \) as follows: Let \( c_1, c_2 \in \mathcal{C}_n(1) \) and let \( (x_1, \ldots, x_n) \) be the highest corner of \( c_1 \) and \( (y_1, \ldots, y_n) \) the lowest corner of \( c_2 \). For \( 1 \leq i \leq n \) we define

\[ c_i <_i c_2 \quad \iff \quad x_i \leq y_i \]

For each \( \mu \in \hat{\mathcal{K}}_{n}(k) \) we define a closed subspace \( H(\mu) \subset \mathcal{C}_n(1)^k \) by

\[ H(\mu) = \{(c_1, \ldots, c_k) \in \mathcal{C}_n(1)^k; \quad c_p <_i c_q \quad \text{if} \quad p \xrightarrow{i} q \quad \text{in} \quad \mu \}; \]

and we have a functor

\[ F_k(\lambda) : T_k(\lambda) \to \mathcal{T}_{op}, \quad \lambda' \mapsto \bigcup \{ H(\mu); \mu \in T_k(\lambda) \mu \leq \lambda' \} \]

where the union is taken in \( \mathcal{C}_n(1)^k \).
4.3 Lemma: $A(\lambda) = \text{colim} F_k(\lambda)$. Moreover, for any element $\alpha \in T_k(\lambda)$ the restriction of this colimit to the subposet $\mathcal{P} = \{ \beta \in T_k(\lambda); \, \beta < \alpha \}$ is a subspace of $A(\lambda)$.

Proof: Let $R(\lambda) = \text{colim} F_k(\lambda)$. By construction, $A(\lambda) = \bigcup_{\lambda' \in T(\lambda)} H(\lambda')$. Since the $H(\lambda')$ are closed subspaces of $A(\lambda)$, it suffices to show that the canonical map $p: R(\lambda) \to A(\lambda)$ is bijective. It is clearly surjective. So let $x \in H(\lambda_1) \cap H(\lambda_2) \subset A(\lambda)$. We need to show that $x \in H(\lambda_1)$ is related to $x \in H(\lambda_2)$ in the colimit $R(\lambda)$. Now $x = (c_1, \ldots, c_k) \in C_n(1)^k$, and the little cubes $c_1, \ldots, c_k$ satisfy ordering conditions specified by $\lambda_1$ and $\lambda_2$. Define $\lambda_3 \in \hat{K}_n(k)$ as follows: the edge between $p$ and $q$ obtains no color or orientation if the corresponding edges in $\lambda_1$ and $\lambda_2$ are not colored (note: by definition of $T_k(\lambda)$ an edge in $\lambda_1$ is not colored iff the corresponding edge in $\lambda_2$ is not colored). If both are colored, the corresponding edge in $\lambda_3$ obtains the color and orientation of the edge with the smaller color (if the colors agree, so do the orientations; this is forced by the ordering conditions for $c_1, \ldots, c_k$). The ordering conditions for $c_1, \ldots, c_k$ also imply that $\lambda_3$ does not have monochrome cycles. By construction, $\lambda_3 \in T_k(\lambda)$ and $\lambda_3 \leq \lambda_1$ and $\lambda_3 \leq \lambda_2$, and $x \in H(\lambda_3)$. Hence $x \in H(\lambda_1)$ and $x \in H(\lambda_2)$ represent the same element in the colimit.

This argument also proves the second statement. $\square$

4.4 Remark: If $\lambda$ is the complete graph with no colors, then $T_k(\lambda) = K_n(k)$, and Lemma 4.3 gives, in the terminology of Berger [3], a “cellular decomposition” of $C_n(k)$ over $K_n(k)$. In [3] Berger claimed that the same construction gives a cellular decomposition of $C_n(k)$ over $K^B_n(k)$ and used this to show that $|K^B_n|$ is an $E_n$-operad. The following example illustrates that this construction does not give such a cellular decomposition over $K^B_n$.

Let $(c_1, c_2, c_3) \in C_3(3)$ be the configuration with $c_1 = [0, \frac{1}{3}] \times [\frac{2}{3}, 1] \times [0, \frac{1}{3}]$, $c_2 = [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \times [\frac{1}{2}, \frac{2}{3}]$, and $c_3 = [\frac{1}{3}, 1] \times [0, \frac{1}{3}] \times [\frac{2}{3}, 1]$. Over $K^B_3$ this configuration lies in the interior of the cells $C_\alpha$ and $C_\beta$ where
So the cells $C_\alpha$ and $C_\beta$ do not have disjoint interiors, which violates Berger’s notion of a cellular decomposition. In contrast to $\mathcal{K}_3^B$, over $\mathcal{K}_3$, this configuration lies in $C_\gamma$ with

$$
\gamma = 3
\quad 1
\quad 2
\quad 2
$$

which is in the boundary of $C_\alpha$ and $C_\beta$. For definitions and terminology consult [3].

We want to point out that results from [3] and [1] imply that $|\mathcal{K}_n^B|$ is an $E_n$-operad and that the inclusion $|\mathcal{K}_n^B| \subset |\mathcal{K}_n|$ is a $\Sigma$-equivalence.

The colimit decompositions of the $A(\lambda)$ are functorial with respect to the colimit decomposition of $(\text{Ass} \otimes C_n)(k)$ of Lemma 4.2 in the following sense: $T_k$ defines a functor

$$
T_k : \hat{\mathcal{K}}_1(k)^\text{op} \longrightarrow \text{Posets}, \quad \lambda \mapsto T_k(\lambda)
$$

For a morphism $f : \lambda_1 \rightarrow \lambda_2$ in $\hat{\mathcal{K}}_1(k)^\text{op}$, i.e. $\lambda_2 \leq \lambda_1$, we have a map of posets $T_k(f) : T_k(\lambda_1) \longrightarrow T_k(\lambda_2)$ defined as follows: let $S_1$ be the set of uncolored unoriented edges of $\lambda_i$. Then $S_2 \subset S_1$, and $E(k) \setminus S_1$ is the set of uncolored edges of any $\mu \in T(\lambda_1)$. The map $T_k(f)$ sends $\mu$ to $\overline{\mu}$ obtained from $\mu$ by forgetting the orientations and colors of all edges in $E(k) \setminus S_2$. Moreover, $F_k(\lambda_1)(\mu) \subset F_k(\lambda_2)(\overline{\mu})$ because we have less order conditions on the cubes in $F_k(\lambda_2)(\overline{\mu})$. Hence the collection of functors $\{F_k(\lambda) ; \lambda \in \hat{\mathcal{K}}_1(k)^\text{op}\}$ is a $T_k$-indexed family of functors in the sense of 5.1 below and we can combine the diagrams to a diagram

$$
F_k : \hat{\mathcal{K}}_1(k)^\text{op} \int T_k \longrightarrow T_{\text{op}}, \quad (\lambda, \mu) \mapsto F_k(\lambda)(\mu)
$$

where $\hat{\mathcal{K}}_1(k)^\text{op} \int T_k$ is the Grothendieck construction. Its objects are pairs $(\lambda, \lambda')$ with $\lambda \in \hat{\mathcal{K}}_1(k)^\text{op}$ and $\lambda' \in T_k(\lambda)$, and morphisms $(f, g) : (\lambda_1, \lambda_1') \rightarrow (\lambda_2, \lambda_2')$ with $f : \lambda_1 \rightarrow \lambda_2$ in $\hat{\mathcal{K}}_1(k)^\text{op}$, i.e. $\lambda_2 \leq \lambda_1$ in $\hat{\mathcal{K}}_1(k)$, and $g : T_k(f)(\lambda_1') = \overline{\lambda_1'} \rightarrow \lambda_2'$, i.e. $\overline{\lambda_1'} \leq \lambda_2'$, in $\hat{\mathcal{K}}_n(k)$. 

13
4.5 Lemma: \((\mathcal{A}s \otimes C_n)(k) = \text{colim } F_k\)

Proof: For \(\lambda \in \hat{\mathcal{K}}_1(k)^{\text{op}}\) let

\[ i_\lambda : T_k(\lambda) \to \hat{\mathcal{K}}_1(k)^{\text{op}} \int T_k \]

denote the inclusion. Then

\[ \text{colim} \hat{\mathcal{K}}_1(k)^{\text{op}} \int T_k F_k = \text{colim}(\hat{\mathcal{K}}_1(k)^{\text{op}} \to \mathcal{T} \text{op}), \lambda \mapsto \text{colim}_{T_k(\lambda)} F_k \circ i_\lambda \]

Since \(F_k \circ i_\lambda = F_k(\lambda)\) Lemmas 4.2 and 4.3 imply the statement. \(\square\)

4.6 Lemma: There is an isomorphism of categories

\[ \varphi : \mathcal{K}_{n+1}(k) \to \hat{\mathcal{K}}_1(k)^{\text{op}} \int T_k \]

Proof: The isomorphism is defined by sending \(\lambda \in \mathcal{K}_{n+1}(k)\) to \((\varphi_1(\lambda), \varphi_2(\lambda))\)

where \(\varphi_1(\lambda)\) is obtained from \(\lambda\) by replacing the color \(n+1\) by the color 1 and by deleting colors and orientations of edges colored by any \(i \leq n\), and \(\varphi_2(\lambda)\) by forgetting colors and orientations of all edges colored by \(n+1\). Observe that \(\varphi_2(\lambda)\) is a complementary graph of \(\varphi_1(\lambda)\). \(\square\)

4.7 Lemma: For each \(\lambda \in \mathcal{K}_{n+1}(k)\) the space \((F_k \circ \varphi)(\lambda)\) is contractible.

Proof: \((F_k \circ \varphi)(\lambda) = \bigcup\{H(\mu); \mu \in T_k(\varphi_1(\lambda)), \mu \leq \varphi_2(\lambda)\}\). Berger has proved the contractability of \(\bigcup\{H(\mu); \mu \in \mathcal{K}_n(k), \mu \leq \lambda\}\) with \(\lambda \in \mathcal{K}_n^p(k)\) [3, Thm. 5.5]. The same argument applies to our situation. \(\square\)

4.8 Lemma: The diagram \(F_k \circ \varphi\) is Reedy-cofibrant, i.e. for all \(\lambda \in \mathcal{K}_{n+1}(k)\)

\[ \text{colim}_{\mu < \lambda}(F_k \circ \varphi)(\mu) \to (F_k \circ \varphi)(\lambda) \]

is a closed cofibration.

Proof: By 4.3 \(\text{colim}_{\mu < \lambda}(F_k \circ \varphi)(\mu) = \bigcup_{\mu < \lambda} H(\varphi_2(\mu))\), so that we have to show that

\[ \bigcup_{\mu < \lambda} H(\varphi_2(\mu)) \subset \bigcup_{\mu \leq \lambda} H(\varphi_2(\mu)) \]
is a closed cofibration. Let $\mu_1, \ldots, \mu_n$ be the predecessors of $\lambda$ and let $\lambda_i = \varphi_2(\mu_i), \lambda' = \varphi_2(\lambda)$. Since

$$
\bigcup_{i=1}^n H(\lambda_i) \cap H(\lambda') \longrightarrow H(\lambda')
$$

is a pushout, it suffices to show that

$$
\bigcup_{i=1}^n H(\lambda_i) \longrightarrow F_k(\lambda)
$$

is a closed cofibration. By Lillig’s union theorem [12, Cor. 3] this holds if for any choice of objects $\mu_1, \ldots, \mu_r \in \mathcal{K}_n(k)$ the map

$$
H(\mu_1) \cap \ldots \cap H(\mu_r) \longrightarrow H(\mu_r)
$$

is a closed cofibration. A little cube $c \in C_n(1)$ is determined by its lowest vertex $x = (x_1, \ldots, x_n)$ and its highest vertex $y = (y_1, \ldots, y_n)$. So $C_n(1)^k \subset \mathbb{R}^{2nk}$ is given by inequalities

$$
0 \leq x_{ij} < y_{ij} \leq 1 \quad i = 1, \ldots, k, j = 1, \ldots, n.
$$

The subspace $H(\mu) \subset C_n(1)^k$ consists of elements satisfying additional non-strict inequalities given by the ordering conditions.

Let $A \subset \mathbb{R}^{2nk}$ be the subspace given by all inequalities determining $H(\mu_1) \cap \ldots \cap H(\mu_r)$ made non-strict, and $X$ the corresponding space obtained from $H(\mu_r)$. Then $A \subset X$ clearly is a closed cofibration. Define $\tau : X \rightarrow [0, 1]$ to be the product of all $(y_{ij} - x_{ij})$ for which we have strict inequalities in $H(\mu_1) \cap \ldots \cap H(\mu_r)$ (they are the same as the ones in the list of $H(\mu_r)$), and let $V = \tau^{-1}([0, 1])$. Then by a result of Dold [7, Satz 1]

$$
V \cap A = H(\mu_1) \cap \ldots \cap H(\mu_r) \subset V = H(\mu_r)
$$

is a closed cofibration.

4.9 Corollary: The canonical map $\text{hocolim}(F_k \circ \varphi) \rightarrow \text{colim}(F_k \circ \varphi) = (\text{Ass} \otimes C_n)(k)$ is a homotopy equivalence.

For a proof see [1, Prop. 6.9].

4.10 Lemma: The operad $\text{Ass} \otimes C_n$ is $\Sigma$-free.
Proof: By [6, Cor. 5.7] the $\Sigma_k$-action on $(\text{Ass} \otimes C_n)(k)$ is free. Since each space $(\text{Ass} \otimes C_n)(k)$ is Hausdorff and paracompact, the lemma follows. □

4.11 The maps

$$|K_{n+1}(k)| \xleftarrow{\text{hocolim}(F_k \circ \varphi)} (\text{Ass} \otimes C_n)(k)$$

assemble to $\Sigma$-equivalences of operads.

For each $k$ the maps are homotopy equivalences by Lemma 4.7 and Corollary 4.9. Both are equivariant homotopy equivalences since $|K_{n+1}(k)|$ and $(\text{Ass} \otimes C_n)(k)$ are free $\Sigma_k$-spaces. It remains to prove that the collection of these maps form maps of operads. For this it suffices to show that

4.12 $H(\varphi_2(\lambda)) \circ (H(\varphi_2(\lambda_1) \times \ldots \times H(\varphi_2(\lambda_k))) \subset H(\varphi_2(\lambda \circ (\lambda_1 \oplus \ldots \oplus \lambda_k))$

for $\lambda \in K_{n+1}(k)$ and $\lambda_i \in K_{n+1}(l_i)$, $i = 1, \ldots, k$. On the left side, composition is determined by the one in $\hat{K}_{1}[C_n]$, on the right side we have composition in $K_{n+1}$.

Condition (4.12) is a consequence of the following properties of our order relations on $C_n(1)$:

4.13 (i) $c_1 < c_2$ iff $c_3 \circ c_1 < c_3 \circ c_2$ for all $c_3 \in C_n(1)$

(ii) $c_1 < c_2 \Rightarrow c_1 \circ c_3 < c_2 \circ c_4$ for all $c_3, c_4 \in C_n(1)$

Finally we show

4.14 Lemma: $|K_n|$ is an $E_n$-operad for each $n$.

Proof: $\varphi(K_n(k)) \subset \hat{K}_{1}(k)^{op} \int T_k$ exactly consists of all those pairs $(\varphi_1(\lambda), \varphi_2(\lambda))$ for which $\varphi_1(\lambda)$ does nor have any colors. By our arguments of Remark 4.4, Diagram 4.11 restricts to a diagram of $\Sigma_k$-equivariant homotopy equivalences

$$|K_n(k)| \xleftarrow{\text{hocolim}(F_k \circ (\varphi|K_n(k))}} C_n(k)$$

To distinguish between the $F_k$ for the various $n$ we denote $F_k$ above by $F^{(n)}_k$ and similarly for $\varphi$. We summarize:
4.15 There is an explicit chain of $\Sigma$-equivalences

$$\mathcal{A}_{ss} \otimes C_n \leftarrow \text{hocolim}(F(n) \circ \varphi(n)) \rightarrow |\mathcal{K}_{n+1}| \leftarrow \text{hocolim}(F(n+1) \circ (\varphi(n+1)|\mathcal{K}_{n+1})) \rightarrow C_{n+1}$$

Together with the Observation 4.11 this completes the proof of Theorem C.

5 Appendix: Iterated colimits and the Grothendieck construction

Our description of $(\mathcal{A}_{ss} \otimes C_n)(k)$ as an iterated colimit is a special case of a more general situation, which may be of separate interest.

Let $F : A \to \text{Cat}$ be any functor. Recall that the Grothendieck construction $A \int F$ is the category whose objects are pairs $(A, B)$ with $A \in \text{obj}A$ and with $B \in \text{obj}F(A)$. A morphism $(A_1, B_1) \to (A_2, B_2)$ is a pair $(\alpha, \beta)$ where $\alpha : A_1 \to A_2$ and $\beta : F(\alpha)(B_1) \to B_2$.

5.1 Definition: An $F$-indexed family of functors into a category $C$ is a collection of functors

$$\{G_A : F(A) \to C : A \in \text{obj}A\}$$

and natural transformations

$$\{\eta_a : G_{A_1} \to G_{A_2} F(\alpha) : \alpha : A_1 \to A_2 \in \text{mor}A\}$$

satisfying $\eta_{id_A} = id_{G_A}$ for all $A \in \text{obj}A$ and satisfying the the following associativity conditions

$$G_{A_1} \xleftarrow{\eta_{a_2 a_1}} G_{A_3} F(\alpha_2 \alpha_1) = G_{A_3} F(\alpha_2) F(\alpha_1)$$

for any composable pair of morphisms

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3$$

in $\mathcal{A}$. 
An $F$-indexed family of functors determines a functor $G \int F : \mathcal{A} \int F \to \mathcal{C}$ given on objects by $G \int F(A, B) = G_A(B)$ and on morphisms $(\alpha, \beta) : (A_1, B_1) \to (A_2, B_2)$ by

$$G_{A_1}(B_1) \xrightarrow{\eta_\alpha} G_{A_2}F(\alpha)(B_1) \xrightarrow{G_{A_2}(\beta)} G_{A_2}(B_2).$$

Now suppose $\mathcal{C}$ is a category with small colimits. Then the natural transformations $\eta_\alpha$ induce a functor $\mathcal{A} \to \mathcal{C}$, which takes an object $A$ to $\text{colim}_{F(A)} G_A$. We then have

**5.2 Proposition:** $\text{colim}_{A \in \text{obj} \mathcal{A}} \left( \text{colim}_{F(A)} G_A \right) \cong \text{colim}_{\mathcal{A} \int F} G \int F$

The proof is straightforward.

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