Regularity of the singular set for Mumford-Shah minimizers in \( \mathbb{R}^3 \) near a minimal cone.

Antoine Lemenant

Université Paris XI
antoine.lemenant@math.u-psud.fr

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Abstract. We show that if \((u, K)\) is a minimizer of the Mumford-Shah functional in an open set \( \Omega \) of \( \mathbb{R}^3 \), and if \( x \in K \) and \( r > 0 \) are such that \( K \) is close enough to a minimal cone of type \( \mathbb{P} \) (a plane), \( \mathbb{Y} \) (three half planes meeting with 120° angles) or \( \mathbb{T} \) (cone over a regular tetrahedron centered at the origin) in terms of Hausdorff distance in \( B(x, r) \), then \( K \) is \( C^{1,\alpha} \) equivalent to the minimal cone in \( B(x, cr) \) where \( c < 1 \) is an universal constant.

Introduction

The Mumford-Shah functional comes from an image segmentation problem. If \( \Omega \) is an open subset of \( \mathbb{R}^2 \), for example a rectangle, and \( g \in L^\infty(\Omega) \) is an image, D. Mumford and J. Shah [MS89] proposed to define

\[
J(K, u) := \int_{\Omega \setminus K} |\nabla u|^2\,dx + \int_{\Omega \setminus K} (u - g)^2\,dx + H^1(K)
\]

and, to get a good segmentation of the image \( g \), minimize the functional \( J \) over all the admissible pairs \((u, K) \in A\) (see definitions after). Any solution \((u, K)\) that minimizes \( J \) represents a “smoother” version of the image and the set \( K \) represents the edges of the image.

Existence of minimizers is a well known result (see for instance [GCL89]) using \( SBV \) theory.

The question of regularity for the singular set \( K \) of a minimizer is more difficult. The following conjecture from D. Mumford and J. Shah is currently still open.
Conjecture 1 (Mumford-Shah). \[\text{MS89}\] Let \((u, K)\) be a reduced minimizer for the functional \(J\). Then \(K\) is the finite union of \(C^1\) arcs.

Some partial results are true for this conjecture. For instance it is known that \(K\) is \(C^1\) almost everywhere (see [DAV96], [BON96] and [AFP97]).

Furthermore it is known that if \(B\) is a ball such that \(K \cap B\) is a \(C^{1,\alpha}\) graph, and if in addition \(g\) is of regularity \(C^k\), then \(K \cap B\) is \(C^k\) (cf Theorem 7.42 in [AFP00]) and even that if \(g\) is an analytic function, then \(K\) is also analytic (see [KLM05]).

Many results about the Mumford-Shah functional are about \(\mathbb{R}^2\). In dimension 3, lots of proprieties are still unknown. The theorem of L. Ambrosio, N. Fusco and D. Pallara [AFP97] about regularity of minimizers is one of the rare result valid in any dimension. It says in particular that if \(K\) is flat enough in a ball \(B\), and if the energy there is not too big, then \(K\) is a \(C^1\) hypersurface in a slightly smaller ball. The proof of this result is based on a “tilt-estimate” and does not seem to generalize to other geometric situations different than a hyperplane.

It is natural to think about situation in dimension 3. Some works on minimal surfaces and soap bubbles in dimension 3 tell us what can be the singularities of a Mumford-Shah minimizer, at least when the energy is small. In particular in Jean Taylor’s work [TAY76] we can find the description of the three minimal cones in \(\mathbb{R}^3\). Jean Taylor also proves that any minimal surface is locally \(C^1\) equivalent to one of those cones. So we can think that for Mumford-Shah minimizers a similar description is true.

What we prove here is that if in a ball, the singular set of a Mumford-Shah minimizer is close enough to a minimal cone, then it is \(C^{1,\alpha}\) equivalent to this cone. It is a generalization to cones \(\mathbb{Y}\) and \(\mathbb{T}\) of what L. Ambrosio, N. Fusco et D. Pallara have done with hyperplanes in [AFP97]. It is also a generalization in higher dimension of what G. David [DAV96] did in \(\mathbb{R}^2\) about the regularity near lines and propellers.

We start with a few definitions. Let \(\Omega\) be an open set of \(\mathbb{R}^N\). We consider the set of admissible pairs

\[\mathcal{A} := \{(u, K); \ K \text{ closed}, \ u \in W^{1,2}_{\text{loc}}(\Omega \setminus K)\}.\]

Definition 2. Let \((u, K) \in \mathcal{A}\) and \(B\) a ball such that \(\bar{B} \subset \Omega\). A competitor for the pair \((u, K)\) in the ball \(B\) is a pair \((v, L) \in \mathcal{A}\) such that

\[
\begin{align*}
    u &= v \\
    K &= L
\end{align*}
\]

in \(\Omega \setminus B\)

and in addition such that if \(x\) and \(y\) are two points in \(\Omega \setminus (B \cup K)\) that are separated by \(K\) then they are also separated by \(L\).

The expression “be separated by \(K\)” means that \(x\) and \(y\) lie in different connected components of \(\Omega \setminus K\).
Definition 3. A gauge function $h$ is a non negative and non decreasing function on $\mathbb{R}^+$ such that $\lim_{t \to 0} h(t) = 0$.

Definition 4. Let $\Omega$ be an open set of $\mathbb{R}^N$. A Mumford-Shah minimizer with gauge function $h$ is a pair $(u, K) \in \mathcal{A}$ such that for every ball $B \subset \Omega$ and every competitor $(v, L)$ in $B$ we have

$$
\int_{B \setminus K} |\nabla u|^2 dx + H^{N-1}(K \cap B) \leq \int_{B \setminus L} |\nabla v|^2 dx + H^{N-1}(L \cap B) + r^{N-1}h(r)
$$

with $r$ the radius of the ball $B$ and where $H^{N-1}$ denotes the Hausdorff measure of dimension $N - 1$.

It is not difficult to prove that a minimizer for the functional $J$ of the beginning of the introduction is a minimizer in the sense of Definition 4 with $h(r) = C_N \|g\|_2^2 r$ as gauge function where $C_N$ is a dimensional constant (see proposition 7.8 p. 46 of [DAV05]).

Definition 5. A global minimizer in $\mathbb{R}^N$ is a Mumford-Shah minimizer in the sense of Definition 4 with $\Omega = \mathbb{R}^N$ and $h = 0$.

We will not work on global minimizers in this paper but they take an important place in the study of the Mumford-Shah functional and that is why we introduced the definition. In dimension 2, only three types of connected sets can give a global minimizer: $K$ is a line and $u$ is locally constant, $K$ is a propeller (a union of three half-lines meeting with 120 degree angles) and $u$ is locally constant as well, and finally when $K$ is a half line and $u$ is a cracktip. Knowing whether there is another global minimizer would give a positive answer to the Mumford-Shah conjecture. The main fact is that every blow up limit of $(u, K)$ is a global minimizer. In [LEM08], one can find some informations about global minimizers in $\mathbb{R}^3$.

If $(u, K)$ is a Mumford-Shah minimizer and if we add to $K$ a small closed set of Hausdorff measure zero, then this new set is also a Mumford-Shah minimizer. That is why in the following we will always suppose that the minimizer is “reduced”. This means that a pair $(\tilde{u}, \tilde{K}) \in \mathcal{A}$ such that $\tilde{K} \subsetneq K$ and $\tilde{u}$ is an extension of $u$ in $W^{1,2}_{loc}(\Omega \setminus K)$ doesn’t exist. Given a pair $(u, K) \in \mathcal{A}$, one can always find a reduced pair $(\tilde{u}, \tilde{K}) \in \mathcal{A}$ such that $\tilde{K} \subset K$ and $\tilde{u}$ is an extension of $u$ (see Proposition 8.2 of [DAV05]).

Let us now define the minimal cones that will be used in the next sections. We define three types of cones. Cones of type 1 are planes in $\mathbb{R}^3$, also called $\mathbb{P}$. Cones of types 2 and 3 and their spines are defined as in [DPT] by the following way.

Definition 6. Define $\text{Prop} \subset \mathbb{R}^2$ by

$$
\text{Prop} = \{(x_1, x_2); x_1 \geq 0, x_2 = 0\}
$$
\[
\bigcup \{ (x_1, x_2); x_1 \leq 0, x_2 = -\sqrt{3} x_1 \}
\bigcup \{ (x_1, x_2); x_1 \leq 0, x_2 = \sqrt{3} x_1 \}.
\]

Then set \( Y_0 = \text{Prop} \times \mathbb{R} \subset \mathbb{R}^3 \). The spine of \( Y_0 \) is the line \( L_0 = \{ x_1 = x_2 = 0 \} \). A cone of type 2 (or of type \( Y \)) is a set \( Y = R(Y_0) \) where \( R \) is the composition of a translation and a rotation. The spine of \( Y \) is then the line \( R(L_0) \). We denote by \( \mathcal{Y} \) the set of all the cones of type 2. Sometimes we also may use the expression “of type \( \mathcal{Y} \)”.

**Definition 7.** Let \( A_1 = (1, 0, 0), A_2 = (-\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0), A_3 = (-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}), \) and \( A_4 = (-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}) \) the four vertices of a regular tetrahedron centered at 0. Let \( T_0 \) be the cone over the union of the 6 edges \([A_i, A_j]\) \( i \neq j \). The spine of \( T_0 \) is the union of the four half lines \([0, A_i]\). A cone of type 3 (or of type \( \mathcal{T} \)) is a set \( T = R(T_0) \) where \( R \) is the composition of a translation and a rotation. The spine of \( T \) is the image by \( R \) of the spine of \( T_0 \). We denote by \( \mathcal{T} \) the set of all the cones of type 3.

Cones of type \( \mathcal{P}, \mathcal{Y} \) and \( \mathcal{T} \) are the only sets (except the empty set) in \( \mathbb{R}^3 \) that locally minimizes the Hausdorff measure of dimension 2 under topological conditions (i.e. every competitor keep the same connected components outside the competitor ball). This fact is proved in [DAV\( a \)]. That is why in the following we will say “minimal cones” to design cones of type \( \mathcal{P}, \mathcal{Y} \) and \( \mathcal{T} \).

We denote by \( D_{x,r} \) the normalized Hausdorff distance between two closed sets \( E \) and \( F \) in \( B(x, r) \) defined by

\[
D_{x,r}(E, F) := \frac{1}{r} \left\{ \max \left\{ \sup_{y \in E \cap B(x, r)} d(y, F), \sup_{y \in F \cap B(x, r)} d(y, E) \right\} \right\}.
\]

We now come to the main result of the paper.

\(^1\)Thanks to Ken Brakke for those pictures.
Theorem 8. We can find some absolute positive constants $\varepsilon > 0$ and $c < 1$ such that all the following is true. Let $(u, K)$ be a reduced Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$, with gauge function $h$. Let $x \in K$ and $r > 0$ be such that $B(x, r) \subset \Omega$. Assume in addition that there is a minimal cone $Z$ of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ centered at $x$ such that
\[
D_{x,r}(K, Z) + h(r) \leq \varepsilon.
\]
Then there is a diffeomorphism $\phi$ of class $C^{1, \alpha}$ from $B(x, cr)$ to its image such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$.

When $(u, K)$ is a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^N$, and if $B(x, r)$ is a ball such that $\bar{B}(x, r) \subset \Omega$, we denote by $\omega_2(x, r)$ the normalized energy of $u$ in $B(x, r)$ defined by
\[
\omega_2(x, r) := \frac{1}{r^{N-1}} \int_{B(x,r) \setminus K} |\nabla u|^2 \, dx.
\]
We also have a version of Theorem 8 with only a condition on the normalized energy instead of the geometric condition.

Theorem 9. We can find some absolute positive constants $\varepsilon > 0$ and $c < 1$ such that the following is true. Let $(u, K)$ be a reduced Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$, with gauge function $h$. Let $x \in K$ and $r > 0$ be such that $B(x, r) \subset \Omega$ and
\[
\omega_2(x, r) + h(r) \leq \varepsilon.
\]
Then there is a diffeomorphism $\phi$ of class $C^{1, \alpha}$ from $B(x, cr)$ to its image, and there is a minimal cone $Z$ such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$.

In all the following we will work in $\mathbb{R}^3$. However, the proof of Theorem 8 still works in higher dimension for the case of hyperplanes so that we could have a new proof of L. Ambrosio, N. Fusco, D. Pallara’s entire result. With the same proof we could also imagine to have other results in $\mathbb{R}^N$, but the analogue of Jean Taylor’s Theorem in higher dimension is missing at the time when this paper is written.

Indeed, one of the ingredients of the proof of Theorem 8 is to apply some results about minimal sets. In particular we will use the paper of G. David [DAVa] following J. Taylor [TAY76], that is the analogue of Theorem 8 but for almost minimal sets. Let $E$ be a closed set in $\mathbb{R}^N$.

Definition 10. A MS-competitor for the closed set $E$ in $\Omega \subset \mathbb{R}^N$ is a closed set $F$ such that there is a ball $B \subset \Omega$ of radius $r$ with
\[
F \setminus B = E \setminus B
\]
and if $x, y \in \Omega \setminus (B \cup E)$ are separated by $E$ then they are also separated by $F$. 


Definition 11. A set $E \subset \Omega$ is MS-almost minimal with gauge function $h$ if

$$H^{N-1}(E \cap B) \leq H^{N-1}(F \cap B) + r^{N-1}h(r)$$

for all MS-competitor $F$ for $E$ in the ball $B$ of radius $r$. 

If $E$ is a MS-almost minimal set, we denote

$$\theta(x, r) = r^{-2}H^2(E \cap B(x, r)).$$

The limit at 0 of $\theta$ exists because $E$ is almost minimal so one can prove that $\theta$ is almost non decreasing (see 2.3 of [DAVa]). The limit is called “density” of $E$ at point $x$ and will be denoted by $d(x)$. Then we introduce the excess of density defined by

$$f(x, r) = \theta(x, r) - \lim_{t \to 0} \theta(x, t) = \theta(x, r) - d(x).$$

Now Proposition 12.28 of [DAVa] says the following.

**Theorem 12.** For each choice of $b \in (0, 1]$, $\bar{c} > 0$ and $C_0 > 0$ we can find $\varepsilon_1 > 0$ and $C \geq 0$ such that the following holds. Let $E$ be a reduced MS-almost minimal set in $\Omega \subset \mathbb{R}^3$ with gauge function $h$. Suppose that $0 \in E$, $r_0 > 0$ be such that $B(0, 110r_0) \subset \Omega$ and $h$ is satisfying

$$h(r) \leq C_0 r^b \quad \text{for } 0 < r < 220r_0.$$

Assume in addition that

$$f(0, 110r_0) + C_0 r_0^b \leq \varepsilon_1$$

and

$$D_{0,100r_0}(E, Z) \leq \varepsilon_1$$

where $Z$ is a minimal cone centered at the origin such that

$$H^2(Z \cap B(0, 1)) \leq d(0).$$

Then for all $x \in E$ and $r > 0$ such that $x \in E \cap B(0, 10r_0)$ and $0 < r < 10r_0$, we can find a minimal cone $Z(x, r)$, not necessarily centered at $x$ or at the origin, such that

$$D_{x,r}(E, Z(x, r)) \leq \bar{c} \left( \frac{r}{r_0} \right)^\alpha$$

The constant $\alpha$ is a universal constant depending on dimension and other geometric facts.

We also will need this result (Corollary 12.25 of [DAVa]).
Corollary 13. For each choice of $b \in (0,1]$, and $C_0 > 0$ we can find $\alpha > 0$ and $\varepsilon_1 > 0$ such that the following holds. Let $E$ be a reduced MS-almost minimal set in $\Omega \subset \mathbb{R}^3$ with gauge function $h$. Suppose that $0 \in E$, $r_0 > 0$ is such that $B(0, 110r_0) \subset \Omega$ and $h$ is satisfying

$$h(r) \leq C_0 r^b \quad \text{for } 0 < r < 220r_0.$$ 

Assume in addition that

$$f(0, 110r_0) + C_0 r_0^b \leq \varepsilon_1 \quad (1)$$

and

$$D_{0, 100r_0}(E, Z) \leq \varepsilon_1$$

where $Z$ is a minimal cone centered at the origin such that

$$H^2(Z \cap B(0, 1)) \leq d(0).$$

Then for $x \in E \cap B(0, r_0)$ and $0 < r \leq r_0$ there is a $C^{1, \alpha}$ diffeomorphism $\Phi : B(0, 2r) \rightarrow \Phi(B(x, 2r))$, such that $\Phi(0) = x$, $|\Phi(y) - y - x| \leq 10^{-2} r$ for $y \in B(0, 2r)$ and $E \cap B(x, r) = \Phi(Z) \cap B(x, r)$.

The strategy to obtain our main result is to control the normalized energy of $u$ (that is the quantity $\omega_2$). While the energy is under control with a decay as a power of radius $r$, we can say that our singular set is a MS-almost minimal with a gauge function that depends on the decay of $\omega_2$, thus we can apply Corollary 13.

We claim that if we had some similar statements as Theorem 12 and Corollary 13 in higher dimension, then the work in this paper should give a analogous result for the singular set of a minimizer for the Mumford-Shah functional in dimension $N > 3$. Unfortunately, if Guy David is quite able to give similar results for sets of dimension 2 in $\mathbb{R}^N$, the technics used to prove Theorem 12 and Corollary 13 seem not to work for lower co-dimensions.

The paper is organized as follow. In a first part we explain a method to construct a good competitor using a stoping time argument. This construction will use some preliminary work like the Whitney extension and geometric lemmas that are also used in [LEM] and which statements are recalled here. We begin by a good control of the normalized Jump in order to avoid some topological and geometric problems.

In the second Section we will use the competitor described in Section 1 in order to get some estimates about the two main quantities that will appear: normalized energy and bad mass. We also prove that the minimality defect depends on those quantities.

Finally in last section we prove the decay estimate that leads to regularity. At the end we state a few different versions of the main theorem.

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1 Construction of a competitor

It will be convenient to work with a set that is separating. That is why in a first part we have to control the jump of function $u$, that will be useful to estimate the size of holes in $K$. Before that, let us recall some definitions and geometric results from [LEM].

**Definition 14** (Almost Centered). Let $Z$ be a minimal cone and $B$ a ball that meets $Z$. We say that $Z$ is almost centered with constant $V$ if the center of $Z$ lies in $\frac{1}{V}B$. If $V = 2$ we just say that $Z$ is almost centered in $B$.

This lemma will be useful to deal with almost centered cones.

**Lemma 15.** [LEM] Let $Z$ be a minimal cone in $\mathbb{R}^3$ that contains 0 (but is not necessarily centered at 0). Then for all $r_0 > 0$ and for all constant $V \geq 1$ there is a $r_1$ such that

$$r_1 \in \{r_0, Vr_0, V^2r_0\}$$

and such that we can find a cone $Z'$, containing 0 and centered in $B(0, \frac{1}{V}r_1)$ with $Z \cap B(0, r_1) = Z' \cap B(0, r_1)$.

**Definition 16** (Separating). Let $Z$ be a minimal cone in $\mathbb{R}^3$ and $B$ a ball of radius $r$ such that $B \cap Z \neq \emptyset$. For all $a > 0$ we define $Z_a$ by

$$Z_a := \{y \in B; d(y, Z) \leq a\}.$$

Let $E$ be a closed set in $B$ such that $E$ is contained in $Z_{\varepsilon_0}$ for some $\varepsilon_0 < 10^{-5}$. We say that “$E$ is separating in $B$” if the connected components of $B \setminus Z_{\varepsilon_0}$ are contained in different connected components of $B \setminus E$. We denote by $k^B$ the number of connected component of $B \setminus Z_{2\varepsilon_0}$ (thus $k^B$ is equal to type$(Z) + 1$ if $Z$ is not centered too close to $\partial B$).

1.1 Separation and control of the Jump

So let $(u, K)$ be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$ (see Definition 4) with gauge function $h$ and let $\varepsilon$ be fixed. Suppose that there is a ball $B(x_0, r_0)$ such that in this ball, $K$ is $\varepsilon$-close to a minimal cone $Z^0$ of type $P, Y,$ or $T$ (see Definition 6 and 7), in other words there is a minimal cone $Z^0$ such that

$$K \cap B(x_0, r_0) \subset Z^0_{\varepsilon r_0} := \{y; d(y, Z^0) \leq \varepsilon r_0\}.$$

Equivalently we have

$$\beta(x_0, r_0) \leq \varepsilon.$$
where $\beta$ is the “generalized Peter Jones unilateral quantity” defined by

$$\beta(x, r) := \frac{1}{r} \inf_{Z} \{ \sup\{d(y, Z); y \in K \cap B(x, r)\} \}.$$ 

The infimum is taken over all the cones of type $\mathbb{P}$, $\mathbb{Y}$, or $\mathbb{T}$ that contain $x_0$ (but are not necessarily centered at $x_0$). Sometimes we will use the notation $\beta_K(x, r)$ to precise that the quantity is associated to the set $K$.

Moreover we suppose that $Z^0$ is centered at $x_0$. Throughout all this part of the paper, we will always work under these above hypothesis. We introduce now some additional notations. We denote by $k_0$ the number of connected components of $B(x_0, r_0) \setminus Z^0$ and for all $k \in \mathbb{N} \cap [1, k_0]$ we consider a ball $D_k$ of radius $\frac{1}{10}r_0$ such that each $D_k$ are situated in one of the connected components of $B(x_0, r_0) \setminus Z^0$, the farthest as possible from $Z^0$. We also denote by $m_k$ the mean value of $u$ on $D_k$. Then we introduce

$$\delta_{k,l}(x_0, r_0) = |m_k - m_l|$$

and finally, the normalized jump is defined by

$$J(x_0, r_0) := r_0^{-\frac{1}{2}} \min\{\delta_{k,l}; 1 < k, l < k_0^0 \text{ and } k \neq l\}.$$ 

In general, for all $x \in K$ and $r > 0$ such that $B(x, r)$ is included in $\Omega$ and such that there is a cone $Z$ almost centered in $B(x, t)$ and $10^{-5}$ close to $K$ in $B(x, r)$, we can define the normalized jump by the same way

$$J(x, r) := r^{-\frac{1}{2}} \min\{\delta_{k,l}; 1 < k, l < \kappa^{B(x,r)} \text{ and } k \neq l\}.$$ 

Here the $\delta_{k,l}$ are again defined as differences between mean values of $u$ on balls of radius equivalent to $r$ in each connected components of $B(x, r) \setminus Z$ far from $Z$.

If a ball $B(x, r)$ is such that $\beta(x, r) \leq 10^{-5}$ but with minimal cone that realize the infimum not almost centered, we can also define the normalized jump. Indeed, we know by the recentering Lemma [15] that $B(x, 2r)$ or $B(x, 4r)$ is associated to an almost centered cone. Then we define the normalized jump $J(x, r)$ as being equal to the jump of the first ball between $B(x, 2r)$ or $B(x, 4r)$ for which the cone is almost centered.

All the parameters that define the jump (choice of cone $Z^0$, constant 4 to have the almost centering property, diameter and position of the $D_k$) are not so important since the difference is just multiplying the jump by a constant.

First of all, we want to work with a new set $F$ that contains $K$ and such that $F$ is separating in $B(x_0, r_0)$ (see Definition [16]). The result is the same as Proposition 1 p. 303 of [DAV05] but generalized to the case of $\mathbb{Y}$ and $\mathbb{T}$. We also use the opportunity here to prove an
additional fact about the set $F$ (called Property $\star$) that will be used later. Recall that the normalized energy in the ball $B$ is denoted by
\[
\omega_2(x, r) := \frac{1}{r^2} \int_{B(x, r) \setminus K} |\nabla u|^2 dx.
\]

**Proposition 17.** Let $(u, K)$ be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$. Suppose that there is an $x \in \Omega$, a $r > 0$ and a positive constant $\varepsilon < 10^{-10}$ such that $B(x, r) \subset \Omega$ and suppose in addition that there is a minimal cone $Z$ almost centered in $B(x, r)$ such that
\[
\sup_{y \in K \cap B(x, r)} \frac{1}{r} d(y, Z) \leq \varepsilon.
\]
Moreover, assume that $J(x, r) \neq 0$,
\[
\omega_2(x, r) J^{-1}(x, r) \leq \varepsilon \tag{2}
\]
and that
\[
\omega_2(x, r)^{\frac{1}{2}} \leq C J(x, r) \tag{3}
\]
with $C$ a positive universal constant given by the demonstration. We call $D_k$ for $k \in \mathbb{N} \cap [1, K^{B(x, r)}]$ the domains in the definition of $J(x, r)$. Then there is a compact set $F(x, r) \subset B(x, r)$ such that
\[
K \cap B(x, r) \subset F(x, r) \subset \{ x \in B; d(x, Z) \leq C r \sqrt{\varepsilon} \} \tag{4}
\]
$F$ is separating each $D_k$ from $D_l$ for $k \neq l$ in $B(x, r)$
\[
H^2(F(x, r) \cap B(x, r) \setminus K) \leq C r^2 \omega_2(x, r)^{\frac{1}{2}} J(x, r)^{-1} \tag{5}
\]
Moreover $F$ is satisfying Property $\star$ (defined just after).

Property $\star$ shows that we control the geometry of $F$ at small scales when the geometry of $K$ is controlled. This is the definition.

**Definition 18 (Property $\star$).** $F$ satisfy Property $\star$ if, for all $\varepsilon_0 < 10^{-5}$, $y \in K \cap B(x, r)$ and $s > 0$ such that
\[
\inf \{ t; \forall t' \geq t, \beta_K(y, t') \leq \varepsilon_0 \} \leq s \leq d(y, \partial B(x, r))
\]
we have
\[
\beta_F(y, s) \leq \varepsilon_0.
\]

**Remark 19.** Condition (3) allows us to have Property $\star$ and Condition (2) is here to prove the last inclusion of (4). Proposition 17 is still true without Property $\star$ and without Conditions (2) and (3). In this case, (4) is proved by the use of a retraction as in 44.1 of [DAV05].
Proof: The first step is the same as Proposition 1 p. 303 of [DAV05] but applied to $Y$ and $T$ as well. However we will write the entire proof here because it will be easier next to show Property ⋆.

For all $\lambda$ we call
\[
S(\lambda) := \{ y \in B(x, r); d(y, Z) \leq \lambda r \}
\]
and denote by $A_k(\lambda)$ for $k \in \mathbb{N} \cap [1, \text{type}(Z) + 1]$ the connected component of $B(x, r) \setminus S(\lambda)$ which meets $D_k$. We set $V = B(x, r) \setminus K$. Let us find a function $v$ such that
\[
v(y) = m_k \quad \text{for } y \in A_k(1/10) \quad (6)
\]
and
\[
\int_V |\nabla v| \leq C \int_V |\nabla u|. \quad (7)
\]
To do this we consider for all $k$ a function $\varphi_k$ such that $0 \leq \varphi_k \leq 1$ and $\varphi_k = 1$ on $A_k(1/10)$, $\varphi_k = 0$ on $V \setminus A_k(1/100)$ and $|\nabla \varphi_k| \leq Cr^{-1}$. Then we set
\[
\varphi = 1 - \sum_k \varphi_k
\]
and
\[
v = \varphi u + \sum_k \varphi_k m_k
\]
while $m_k$ is the average of $u$ on $D_k$. We have (6) trivially. Concerning (7) we have
\[
\nabla v(y) = \varphi(y) \nabla u(y) - \sum_k 1_{A_k(1/100)}(y) \nabla \varphi_k(y)[u(y) - m_k]
\]
and since $\varepsilon < 10^{-5}$, the $A_k(1/100)$ do not meet $K$ and then applying Poincaré inequality in $A_k(1/100)$ gives
\[
\int_{A_k(1/100)} |\nabla \varphi_k(y)||u(y) - m_k|dy \leq C r^{-1} \int_{A_k(1/100)} |u(y) - m_k|dy
\]
\[
\leq C \int_{A_k(1/100)} |\nabla u(y)|dy
\]
and (7) is verified.
Now we want to replace $v$ with a smooth function $w$ in $V$ such that
\[
w(y) = m_k \quad \text{for } y \in A_k(1/10) \quad (8)
\]
and
\[
\int_V |\nabla w| \leq C \int_V |\nabla u|. \quad (9)
\]
We are going to use a Whitney extension. For all \( z \in V \) we denote by \( B(z) \) the ball \( B(z, 10^{-2} d(z, \partial V)) \), and let \( X \subset V \) be a maximal set such that for all \( z \in X \), the \( B(z) \) are disjoint. Note that by maximality, if \( y \in V \), then \( B(y) \) meets some \( B(z) \) for a certain \( z \in X \) hence \( y \in 4B(z) \) thus the \( 4B(z) \) cover \( V \).

For all \( z \in X \) we choose a function \( \varphi_z \) which support is included in \( 5B(z) \) such that \( \varphi_z(y) = 1 \) for all \( y \in 4B(z) \), \( 0 \leq \varphi_z(y) \leq 1 \) and \( |\nabla \varphi_z(y)| \leq C d(z, \partial V)^{-1} \) everywhere. Set \( \Phi(y) = \sum_{z \in X} \varphi_z(y) \) on \( V \). We have \( \Phi(y) \geq 1 \) because the \( 4B(z) \) cover \( V \) and the sum is locally finite (because all the \( B(z) \) are disjoint and because the \( 5B(z) \) that contain a fixed point \( y \) have a radius equivalent to \( d(y, \partial V) \)). Then we set \( \psi_z(y) = \varphi_z(y)/\Phi(y) \) such that \( \sum_{z \in X} \psi_z(y) = 1 \) on \( V \). Finally, if \( m_z \) is the mean value of \( v \) on \( B(z) \) we set for all \( y \in V \)

\[
    w(y) = \sum_{z \in X} m_z \psi_z(y).
\]

If \( y \in A_k(1/10) \), \( m_z = m_k \) for all \( z \in X \) such that \( y \in B(z) \) thus (8) is verified. In addition,

\[
    \nabla w(y) = \sum_{z \in X} m_z \nabla \psi_z(z) = \sum_{z \in X} [m_z - m(y)] [\nabla \psi_z(y)]
\]

where \( m(y) \) is the mean value of \( v \) on \( B(y) = B(y, 10^{-2} d(y, \partial V)) \). The sum at the point \( y \) has at most \( C \) terms, and all of these terms is less than

\[
    C d(y, \partial V)^{-1} |m_z - m(y)| \leq C d(y, \partial V)^{-3} \int_{10B(y)} |\nabla v|
\]

with using Poincaré inequality and because all the \( 5B(z) \) that contain \( y \) are included in \( 10B(y) \subset V \). Thus \( |\nabla w(y)| \leq C d(y, \partial V)^{-3} \int_{10B(y)} |\nabla v| \), and to obtain (9) it suffice to integrate on \( V \), apply Fubini and use (7).

Then we apply the co-area formula (see [FED69] p.248, and also [DAV05] chapter 28) to the function \( w \) on \( V \). We obtain

\[
    \int_{\mathbb{R}} H^2(\Gamma_t) dt = \int_{V} |\nabla w| \leq C \int_{V} |\nabla u|
\]

where \( \Gamma_t := \{ y \in V; w(y) = t \} \) is the set of level \( t \) of the function \( w \). Recall that

\[
    J(x, r) := r^{-\frac{1}{2}} \min \{ \delta_{k,l}; k \neq l \}
\]

and

\[
    \delta_{k,l} = |m_k - m_l|
\]
where $m_k$ is the mean value of $u$ on $D_k$. For all $k_0 \neq k_1$ we know that $\delta_{k_0, k_1} \geq \sqrt{r} J(x, r)$. Using Tchebychev inequality we can choose $t_1 \in \mathbb{R}$ such that $t_1$ lies in $\frac{1}{10}[m_{k_0}, m_{k_1}]$ and such that

$$H^2(\Gamma_{t_1}) \leq C|m_{k_0} - m_{k_1}|^{-1} \int_V |\nabla u| \leq Cr^{-\frac{1}{2}} J(x, r)^{-1} \int_V |\nabla u| \leq Cr^2 J(x, r)^{-1} \omega_2(x, r)^{\frac{1}{2}} \quad (10)$$

For every pair $k_0 \neq k_1$ we do the same and choose $t_2$ etc, as many as required by the number of connected components of $B(x, r) \setminus Z$ (one if $Z$ is a plane, two if $Z$ is a $\mathbb{V}$ and three if $Z$ is a $\mathbb{T}$). Then we set

$$F = \bigcup_i \Gamma_{t_i} \cup [K \cap B(x, r)] \subset B(x, r).$$

$F$ is a closed set in $B(x, r)$ because each $\Gamma_{t_i}$ is closed in $V = B(x, r) \setminus K$ and $K$ is also a closed set. Since we have choosing some level sets, $F$ separates the $A_k(1/10)$ to each other in $B(x, r)$. Indeed, if it is not the case then there is $k, l$ and a continuous path $\gamma$ that join $A_k(1/10)$ to $A_l(1/0)$ and that is not meeting $K$ (because $K \subset F$). Then $\gamma \subset V$, thus $w$ is well defined and continuous on $\gamma$, it follows that there is a point $y \in \gamma$ such that $w(y) = t_i$. Then, $y \in F$, and this is a contradiction.

Now we have to prove the $\star$ property. Let $B(\bar{y}, s)$ be a ball centered on $K$ such that $\beta(\bar{y}, 2^l s) \leq \varepsilon_0$ for all $0 \leq l \leq L$ while $L$ is the first integer such that $B(\bar{y}, 2^L s)$ is not included in $B(x, r)$. Set $B_l := B(\bar{y}, 2^l s)$ and possibly by extracting a subsequence we may suppose using Lemma 15 that in each $B_l$ the minimal cone associated is almost centered. The radius of $B_l$ is not as before exactly $2^l s$ but is equivalent with a factor 4. Thus the balls $B_l$ form a sequence of balls centered at $\bar{y}$ such that $B_l \subset B_{l+1}$ and $B_0 = B(\bar{y}, s)$. Denote by $Z_l$ the cone associated to $B_l$. We want to show that $F \cap B(y, s) \subset Z_0(\varepsilon_0) := \{z; d(z, Z_0) \leq \varepsilon_0 s\}$. By definition of $F$, it suffice to show that for all $i$

$$w(y) \neq t_i \text{ in } B(\bar{y}, s) \setminus Z_0(\varepsilon_0). \quad (11)$$

So let $y \in B(\bar{y}, s) \setminus Z_0(\varepsilon_0)$ and recall that

$$w(y) = \sum_{z \in X} m_z \varphi_z(y).$$

Let $X(y) \subset X$ be the finite set of $z$ such that $\varphi_z(y) \neq 0$. We claim that

$$\forall z \in X(y), \quad |m_z - m_{D_k}| \leq Cr^\frac{3}{2} \omega_2(x, r)^{\frac{3}{2}} \quad (12)$$
where $m_{D_k}$ is the mean value of $u$ in the appropriate domain $D_k$ and $m_z$ is the mean value of $v$ on $B_z := B(z, 10^{-2}d(z, \partial V))$. First of all, we can use the proof of Lemma 15 in [LEM] to associate to each connected component of $B_l \setminus Z_l(\varepsilon_0)$, a component of $B_{l+1} \cap \{ y ; d(y, Z_{l+1}) \geq 10\varepsilon_0 r_l \}$, and by this way we can rely each component of $B_l \setminus Z_l(\varepsilon_0)$ to a certain $A_k$ (that contain a $D_k$) (the argument is just to do an iteration on the scale since we know that the set $K$ is close the a minimal cone at each scale that we look at). We denote by $O_0$ the component of $B_s \cap \{ y; d(y, Z_0) \geq \varepsilon_0 s \}$ that contains $y$ and by induction we denote by $O_l$ the component of $B_l \setminus Z_l(\varepsilon)$ that is relied to $O_0$. With help of the particulary geometrical configuration in each $B_l$ we can choose a domain $G_l$ included at the same time in $O_l$ and in $O_{l+1}$, and of diameter equivalent to the diameter of $B_l$. We denote by $m_l$ the mean value of $v$ on $G_l$. We are now ready to estimate

\[ |m_0 - m_L| \leq \sum_{l=0}^{L} |m_l - m_{l+1}| \leq \sum_{l=0}^{L} \frac{1}{|O_l|} \int_{O_l} |v - m_{l+1}| \]

\[ \leq \sum_{l=0}^{L} C(2^l s)^{-\frac{1}{2}} \left( \int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{1}{2}} \]

\[ \leq \sum_{l=0}^{L} C(2^l s)^{-\frac{1}{2}} \left( \int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{3}{8}} \left( \int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{1}{8}} \]

\[ \leq \sum_{l=0}^{L} C(2^l s)^{\frac{1}{2}} \left( \int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{1}{8}} \]  

\[ \leq C \left( \int_{V} |\nabla v|^2 \right)^{\frac{1}{8}} \sum_{l=0}^{L} (2^l s)^{\frac{1}{4}} \leq C \left( \int_{V} |\nabla v|^2 \right)^{\frac{1}{8}} \sum_{l=0}^{L} (2^{-l} r)^{\frac{1}{4}} \]

\[ \leq C \left( \int_{V} |\nabla v|^2 \right)^{\frac{1}{8}} \sum_{l=0}^{+\infty} (2^{-l} r)^{\frac{1}{4}} \leq C r^{\frac{1}{4}} \left( \int_{V} |\nabla v|^2 \right)^{\frac{1}{8}} \leq C r^{\frac{1}{4}} \left( \int_{V} |\nabla u|^2 \right)^{\frac{1}{8}} \]

\[ \leq C r^{\frac{3}{4}} \omega_2(x, r)^{\frac{1}{8}} \]  

(14)

for (13) we used the classical estimate on the gradient of a Mumford-Shah minimizer that is

\[ \int_{B(0,R) \setminus K} |\nabla u|^2 dx \leq C_N (1 + h(R)) R^{N-1} \]  

(15)
obtained by comparing \((u, K)\) and \((v, K')\) where \(v\) is equal to 0 in \(B(0, R)\) and \(K' = (K \setminus B(0, R)) \cup \partial B(0, R)\). With the same proof of (14) we get

\[
|m_L - m_{D_k}| \leq Cr^{\frac{1}{2}}\omega_2(x, r)^{\frac{1}{8}}.
\]

On the other hand, since \(z \in X(y)\), then \(\varphi_z(y)\) is not equal to zero. This implies that \(d(z, \partial V) \geq 2d(y, \partial V) \geq 2\varepsilon_0\) thus \(B_z := B(z, 10^{-2}d(z, \partial V)) \subset Z_0(\varepsilon_0)^c\). Since by hypothesis \(K\) does not meet this region, we can apply Poincaré inequality to prove that

\[
|m_z - m_0| \leq Cr^{\frac{1}{2}}\omega_2(x, r)^{\frac{1}{8}}.
\]

Finally

\[
|m_z - m_{D_k}| \leq |m_z - m_0| + |m_0 - m_L| + |m_L - m_{D_k}| \leq Cr^{\frac{1}{2}}\omega_2(x, r)^{\frac{1}{8}}
\]

and this completes the proof of (12).

Now since \(\sum z \varphi_z(y) = 1\) we deduce that

\[
|w(y) - m_k| = |w(y) - \sum z \varphi_z(y)m_k| \leq \sum |m_z - m_{D_k}| \leq Cr^{\frac{1}{2}}\omega_2(x, r)^{\frac{1}{8}}. \tag{16}
\]

Now if we return to the choice of the \(t_i\) (see near (10)) we have taken \(t_i \in \frac{1}{10}[m_{k_0}, m_{k_1}]\) for some \(k_0\) and \(k_1\).

So thanks to (16), if \(\omega_2(x, r)^{\frac{1}{8}}\) is small enough with respect to \(J(x, r)\) then we are sure that \(w(y) \neq t_i\) thus \(F\) does not meet the region \(Z_s(\varepsilon_0)\).

We have now to prove (4). With use of (2) and (10) we can find a cover of \(F\) by a family of balls \(B\{(x_j, r_j)\}\) centered on \(K\) and such that \(r_j = C\sqrt{\varepsilon}r\), otherwise we would have a hole in \(K\) of size greater than \(C\varepsilon r^2\) which is in contradiction with (10). On the other hand, since \(\beta_K(x, r) \leq \varepsilon\), we have \(\beta_K(x_j, r_j) \leq C\sqrt{\varepsilon}\). Now, for every \(y \in F \cap B(x_j, r_j)\) we have

\[
d(y, Z) \leq d(y, x_j) + d(x_j, Z) \leq C\sqrt{\varepsilon}r + \varepsilon r \leq C\sqrt{\varepsilon}r
\]

and the conclusion follows.

Lemma 7 on page 301 of [DAV05] shows how the normalized jump decreases. So we want to generalize this result to the cones of type \(T\) and \(Y\) as well. There is no difficulty to do that. We just have to be careful with the generalized definition of the jump that depends on the existence of almost centered cones, but this is not so troublesome. So if the lector already knows how to control the jump in dimension 2, and if he is convinced that it is also true for cones of type \(Y\) and \(T\) in \(\mathbb{R}^3\), he could just skip the proofs of the two following lemmas.
Lemma 20. Let $(u, K)$ be a Mumford-Shah minimizer in $\Omega$. Let $x \in K$, $r$ and $r_1$ being such that $B(x, r) \subset \Omega$ and $0 < r_1 \leq r \leq \frac{4}{3}r_1$. Suppose in addition that $\beta(x, r) \leq 100^{-1}$. Then

$$\left| \left( \frac{r_1}{r} \right)^{\frac{1}{2}} J(x, r_1) - J(x, r) \right| \leq C\omega_2(x, r)^{\frac{1}{2}} \leq C(1 + h(r))$$

(17)

with a constant $C$ that depends only on $N$.

Proof: For all $r_1 \leq t \leq 2r_1$ we denote by $Z_t$ a minimal cone such that

$$\forall y \in K \cap B(x, t), d(y, Z_t) \leq t\beta(x, t)$$

and for all $\lambda$ we also set

$$A_t(\lambda) := \{ y \in B(x, t), d(y, Z_t) \geq \lambda t \}.$$ 

Finally we denote by $A^k_t$ for $k \in \mathbb{N} \cap [1, k_0]$ the different connected components of $A_t$.

To begin, suppose that $Z_r$ is almost centered. Recall that in this case

$$J(x, r) = r^{-\frac{1}{2}} \min \{\delta_{k,l}\}$$

where $\delta_{k,l} = |m_k(r) - m_l(r)|$ and $m_k$ is the mean value of $u$ on a domain $D_k(x, r)$ in $A^k_r(\frac{1}{100})$.

Since $r_1 \geq \frac{3}{4}r$ and since $Z_r$ is almost centered and that $\beta(x, r) \leq \frac{1}{100}$, we may consider some balls $\tilde{D}_k$ in each connected components of $B(x, r_1) \setminus Z_r(\frac{1}{100})$ such that the radii of $\tilde{D}_k$ are equivalent to $r$ (and thus equivalent to $r_1$) and such that the $\tilde{D}_k$ are included in $A^k_r(\frac{1}{100})$.

By Poincaré inequality we have

$$|m_{\tilde{D}_k} - m_{A^k_t}| \leq Cr^2 \int_{A^k_t} |\nabla u|$$

and also

$$|m_{D_k(x, r)} - m_{A^k_t}| \leq Cr^2 \int_{A^k_t} |\nabla u|$$

where $m_{D_k(x, r)}$, $m_{A^k_t}$, $m_{\tilde{D}_k(x, r)}$ are the mean values of $u$ on $D_k(x, r)$, $A^k_r$, $\tilde{D}_k(x, r)$. We deduce that

$$|m_{\tilde{D}_k} - m_{D_k(x, r)}| \leq Cr^2 \int_{B(x, r)，{\setminus} K} |\nabla u| \leq C r^\frac{1}{2} \omega_2(x, r) \leq C(1 + h(r))r^{\frac{1}{2}}.$$

The last inequality comes from (15). By the same way we obtain

$$|m_{\tilde{D}_k} - m_{D_k(x, r_1)}| \leq Cr^2 \int_{B(x, r_1),{\setminus} K} |\nabla u| \leq C r^\frac{1}{2} \omega_2(x, r) \leq C(1 + h(r))r^{\frac{1}{2}}.$$
where the $D_k(x, r_1)$ are the domains in the definition of $J(x, t_1)$. This then gives the estimation of $r^\frac{1}{2} J(x, r) - r^\frac{1}{4} J(x, r_1)$ to prove \[17\].

Finally if $Z_r$ is not almost centered then we have two cases. The first one is when $Z_{r_1}$ is neither almost centered and then we can use $2r_1$ and $2r$ and that is the same as the above argument. The second case is when $Z_{r_1}$ is almost centered and then this implies that $Z_{r_1}$ is a cone of minor type than the type of $Z_r$ thus it suffice to control the mean values only in connected components $A_r$ that meets the $A_{r_1}$, and the difference between those mean values are always bounded by the jump $J(x, r)$.

**Lemma 21.** Let $(u, K)$ be a Mumford-Shah minimizer in $\Omega$. Then if $x \in K$ and $r$ are such that $B(x, r) \subset \Omega$ and for all $r_1 < t < r$, $\beta(x, t) \leq 10^{-1}$, then

$$J(x, r_1) \geq \left( \frac{r}{r_1} \right)^\frac{1}{2} [J(x, r) - C']$$

(18)

where $C' := C(1 + h(r))$ and $C$ depends only on $N$.

**Proof:** If $r_1 \leq r \leq \frac{4}{3}r_1$ then (18) is a consequence of Lemma 20. Otherwise we use a sequence of radii $r_k$ such that $r_k = \frac{4}{3}r_{k-1}$ and we apply Lemma 20 a number of time until $r_k$ is greater than $r$. We obtain

$$J(x, r_1) \geq \sqrt{4/3}^k J(x, (4/3)^k r_1) - C' \sqrt{4/3}^k (1 + \sqrt{4/3}^{-1} + \sqrt{4/3}^{-2} + \ldots)$$

$$\geq \sqrt{4/3}^k [J(x, (4/3)^k r_1) - \frac{C' \sqrt{4/3}}{1 - \sqrt{4/3}}].$$

(19)

and we conclude by using Lemma 20 a last time.

In the following we will sometimes use the notations $F$ and $B$ instead of $F(x_0, r_0)$ and $B(x_0, r_0)$. In addition, without loss of generality we may suppose by now that $x_0 = 0$.

### 1.2 Stopping times balls and bad mass

Our goal in this section is to construct a family of balls $S$ by a stopping time argument, with the condition that in all balls of $S$, the singular set $K$ will always looks like a minimal cone.

We suppose that $B(0, 4r_0) \subset \Omega$. For all $x \in K \cap B(0, r_0)$ and $r > 0$, we say that $B(x, r)$ is a good ball (and then denote $B(x, r) \in \mathcal{G}$) if

$$H^2(F \cap B(x, r)) - H^2(K \cap B(x, r)) \leq \varepsilon_0' r^2$$

(20)

and also if there is a minimal cone $Z$ such that

$$\forall y \in K \cap B(x, r), \ d(y, Z) \leq \varepsilon_0 r.$$  

(21)
Here, $\varepsilon'_0$ and $\varepsilon_0$ are such that $\varepsilon < \varepsilon'_0 < \varepsilon_0 < 10^{-5}$. Note that since $\beta(0, 4r_0) \leq \varepsilon$ the radii of balls that don’t verify (21) is bounded by $\frac{\varepsilon}{\varepsilon_0}r_0$ and if $C\omega_2(0, r_0)^{\frac{1}{2}}J(0, r_0)^{-1} \leq \varepsilon$, the radii of balls that don’t verify (20) is bounded by $\sqrt{\varepsilon}r_0$.

Now, for all $x \in K$ we define the stopping time function

$$d(x) := \inf\{r; \forall t \geq r, B(x, t) \in \mathcal{G}\}.$$

Then with help of the Vitali covering lemma, from the collection of balls

$$\{B(x, Ad(x))\}_{x \in K \cap B(0, r_0)}$$

with $A$ a constant that will be chosen later, we get a disjoint subfamily $\{B_i\}_{i \in I}$ such that $\{5B_i\}_{i \in I}$ is covering. Denote $S := \{B_i\}_{i \in I}$ the “Bad balls”. For all $r \leq r_0$ we set

$$I_r := \{i \in I; B_i \cap B(0, r) \neq \emptyset\}$$

and we introduce a new quantity called “Bad mass” defined by

$$m(0, r) := \frac{1}{r^2} \sum_{i \in I_r} r_i^2.$$

By convention, a single point $\{x\}$ with $d(x) = 0$ will be identified with the ball $B(x, d(x))$.

### 1.3 Whitney extension

Here we have to recall some definitions and a result from [LEM] in a little weaker form.

Let $K$ be a closed set in $\bar{B}(x_0, r_0)$ such that $H^2(K \cap \bar{B}(x_0, r_0)) < +\infty$. Suppose that there is a positive constant $\varepsilon_0 < 10^{-5}$ and a minimal cone $Z$, centered at $x_0$, such that

$$\sup\{d(x, Z); x \in K \cap B(x_0, r_0)\} \leq r_0 \varepsilon_0$$

and that $K$ is separating in $B(x_0, r_0)$. For all $x \in K \cap B(x_0, r_0)$ and $r > 0$ such that $B(x, r) \subset B(0, r_0)$ recall that

$$\beta(x, r) = \inf_{Z \ni x} \frac{1}{r} \sup_{r \leq Z} \{d(x, Z); x \in K \cap B(x, r)\}.$$

Let $\rho \in [\frac{1}{4}r_0, \frac{3}{4}r_0]$ and assume that we have an application

$$\delta : B(x_0, \rho) \rightarrow [0, \frac{1}{4}r_0].$$
with the property that

\[ \beta(x, r) \leq \varepsilon_0, \text{ for all } x \in K \cap B(x_0, \rho) \text{ and } r \text{ such that } \delta(x) \leq r \leq \frac{1}{4}r_0. \]  

(24)

In addition we suppose that

\[ \delta \text{ is } C_0 - \text{Lipschitz}. \]  

(25)

The application \( \delta \) will be called the “geometric function”.

**Definition 22** (Hypothesis \( \mathcal{H} \)). We will say that a closed set \( K \subset B(x_0, r_0) \) with finite \( H^2 \) measure is satisfying hypothesis \( \mathcal{H} \) if

i) There is a minimal cone \( Z \) that verify (22) for a “geometric constant” \( \varepsilon_0 < 10^{-5} \) and a “Lipschitz constant” \( C_0 \).

ii) \( K \) is separating in \( B(x_0, r_0) \).

iii) There is a geometric function \( \delta \) satisfying (23), (24) and (25) for a radius \( \rho \in [\frac{1}{2}r_0, \frac{3}{4}r_0] \).

Let \( U > 1 \) be a constant that will be fixed later, depending on \( C_0 \) and a dimensional constant. In addition we assume that \( \varepsilon_0 \) is very small compared to \( U^{-1} \). For all \( t > 0 \) we define

\[ v := \bigcup_{x \in K \cap B(0, \rho)} B(x, \frac{10}{U} \delta(x)). \]  

(26)

We also set

\[ v' := \bigcup_{x: B(x, \frac{10}{U} \delta(x)) \cap \partial B(x_0, \rho) \neq \emptyset} B(x, \frac{10}{U} \delta(x)). \]  

(27)

Recall that by hypothesis, \( K \) is separating in \( B(x_0, r_0) \) and that for all \( k \in [1, \kappa_B^{B(x_0, r_0)}] \) we denote by \( A_k(x_0, r_0) \) the connected components of \( B(x_0, r_0) \setminus Z_{\varepsilon_0 r_0} \) and by \( \Omega_k(x_0, r_0) \) the connected component of \( B(x_0, r_0) \setminus K \) that contains \( A_k(x_0, r_0) \). We also set

\[ \Delta_k := B(x_0, \rho) \cap (\Omega_k(x_0, r_0) \cup v'). \]  

(28)

Then we have the following lemma.

**Lemma 23.** (Whitney Extension) Let \( K \) be a closed set in \( B(x_0, r_0) \) satisfying Hypothesis \( \mathcal{H} \) with a geometric function \( \delta \), a minimal cone \( Z \), a constant \( \varepsilon_0 < 10^{-5} \) and a radius \( \rho \in [\frac{1}{2}r_0, \frac{3}{4}r_0] \). Then for all function \( u \in W^{1,2}(B(0, r_0) \setminus K) \), and for all \( k \in [1, \kappa_B^{B(x_0, r_0)}] \), there is a function

\[ v_k \in W^{1,2}(\Delta_k \setminus v'_k) \]
such that

\[ v_k = u \text{ in } B(x_0, \rho) \setminus \mathcal{V} \]

and

\[ \int_{\Delta_k \setminus \mathcal{V}_{\rho}} |\nabla v_k|^2 \, dx \leq +C \int_{B(0, r_0) \setminus K} |\nabla u|^2 \, dx \] (29)

where \( C \) is a constant depending only on dimension and where \( \mathcal{V}, \mathcal{V}_{\rho}, \) and \( \Delta_k \) are defined in (26), (27), and (28) with constant \( U > 30C_0 \) depending also on dimension.

From balls of \( S \), we want to apply Lemma 23 to get a good extension of \( u \) near the bad balls. This extension will allow us replace in each bad ball the set \( K \) by a new set in order to get some estimates. So we begin by introducing a geometric function associated to the balls of \( S \). We define

\[ \forall x \in \mathbb{R}^3; \quad \delta(x) := \inf_{B \in S} \{d(x, B) + r_B\} \] (30)

where \( r_B \) is the radius of the ball \( B \) (that could be equal to 0).

**Proposition 24.** Application \( \delta \) is a geometric function associated to \( F \) in \( B(0, r_0) \) for all \( \rho \in \left[ \frac{1}{2} r_0, \frac{3}{4} r_0 \right] \) with Lipschitz constant \( C_0 = 1 \) and geometric constant \( 10\varepsilon_0 \). In addition, we have Hypothesis \( \mathcal{H} \) on \( F \) in \( B(x_0, r_0) \) and

\[ \bigcup_{i \in I} \frac{10}{U} B_i \subset \mathcal{V} \]

where \( \mathcal{V} \) is defined in (26).

**Remark 25.** Note that since \( C_0 = 1 \), \( U \) is depending only on dimension.

**Proof:** We have to verify (23), (24) and (25). Let \( \rho \in \left[ \frac{1}{2} r_0, \frac{3}{4} r_0 \right] \). Recall that \( F \) is separating in \( B(0, r_0) \) and

\[ F \cap B(0, r_0) \subset \{y \in B(x_0, r_0); d(y, Z^0) \leq r_0 C \sqrt{\varepsilon}\} \]

Then if \( \varepsilon \) is small enough with respect to \( \varepsilon_0 \), for all \( x \in F \cap B(0, \rho) \) and for all ball \( B(y, r) \)

with \( r > \frac{1}{100} r_0 \) that contain \( x \) we have \( \beta(y, r) \leq \varepsilon_0 \) thus for all \( x \in F \cap B(0, \rho) \) we easily have

\[ \delta(x) \leq \frac{1}{4} r_0 \]

and (23) is proved.

Now let \( x \in F \cap B(0, \rho) \) and let \( r \) be a radius such that \( \delta(x) \leq r \leq \frac{1}{4} r_0 \). Let \( B \) be a ball of \( S \) such that

\[ d(x, B) + r_B \leq 2r \]
(we always have one $B$ like that by definition of $\delta$). Let $x_B$ be the center of $B$. Then we have
\[
d(x, x_B) \leq 2r
\]
thus $x \in B(x_B, 2r)$ and $B(x, r) \subset B(x_B, 3r)$. Since $r \geq r_B \geq d(x)$, we know by definition of $d(x)$ that $\beta(x_B, 3r) \leq \varepsilon_0$. Moreover, for all $t > 3r$ we have
\[
\beta(x_B, t) \leq \varepsilon_0.
\]
Then we can apply Property $\star$ in $B(x_B, 3r)$ in order to get a cone $Z$ containing $x_B$ such that for all $y \in F \cap B(x_B, 3r)$, $d(y, Z) \leq \varepsilon_0 3r$. Since $B(x, r) \subset B(x_B, 3r)$ and $x \in F$ we deduce that $\beta_F(x, r) \leq 10 \varepsilon_0$ and (24) is proved with $10 \varepsilon_0$ instead of $\varepsilon_0$.

Finally, if $B$ is a ball of $S$ then for all $x$ and $y$ we have
\[
\begin{align*}
d(x, B) & \leq d(x, y) + d(y, B) \\
d(x, B) + r_B & \leq d(x, y) + d(y, B) + r_B \\
\delta(x) & \leq d(x, B) + r_B \\
\delta(x) & \leq d(x, y) + d(y, B) + r_B
\end{align*}
\]
then passing to the infimum we deduce
\[
|\delta(x) - \delta(y)| \leq d(x, y)
\]
and that shows that the application $x \mapsto \delta(x)$ is 1-Lipschitz.

So we deduce that we have Hypothesis $\mathcal{H}$ on $F$ in $B(x_0, r_0)$ with application $\delta$ defined in (30). Let us show that
\[
\bigcup_{i \in I} \frac{10}{U} B_i \subset \mathcal{V}
\]
(31)

Let $B_i = B(x_i, r_i) \in S$ be a bad ball. We claim that
\[
\delta(x_i) \geq r_i.
\]
Indeed, recall that the balls $B \in S$ are disjoint. If we take $B_i$ in the infimum of the definition of $\delta$ we get $d(x_i, B_i) + r_i = r_i$ and if we take a ball $\tilde{B}$ out of $B$ we get again $d(x_i, \tilde{B}) + r_B \geq r_i$. Thus
\[
B(x_i, \frac{10}{U} r_i) \subset B(x_i, \frac{10}{U} \delta(x_i)) \subset \mathcal{V}.
\]

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2 Useful estimates

We are now ready to compute some estimates about different quantities that will lead to regularity. The main point is to show some decay estimates on the normalized energy $\omega_2(x, r)$. This decay will come from the same sort of argument as in [DAV96]. In dimension 2, the intersection between $\partial B(x, r)$ and $K$ is mainly constituted of single points. Here in dimension 3, $\partial B(x, r) \cap K$ is more complicated and this will lead some problems. We start by finding a judicious radius $\rho$ to begin the estimates.

2.1 Choice of the radius

For the choice of the radius we select a $\rho \in R := [\frac{3}{4}r_0, \frac{3}{4}r_0]$, such that the mass of the bad balls $\{B_i\}_{i \in I}$ that are meeting $\partial B(0, \rho)$ is less than average. Recall that the $B_i$ are the bad balls $B(x_i, r_i) \in S$. Set $I(\rho) := \{i \in I; B_i \cap \partial B(0, \rho) \neq \emptyset\}$ and let $r_i$ be the radius of $B_i$. By such a choice of $\rho$ we have

$$\sum_{i \in I(\rho)} r_i^2 \leq \frac{1}{|R|} \int_R \sum_{i \in I(t)} r_i^2 dt \leq \frac{1}{|R|} \sum_{i \in I} \int_{t_i \in I(t)} r_i^2 \leq C \frac{1}{|R|} \sum_{i \in I} r_i^3. $$

Finally we have found a $\rho$ that verify

$$\sum_{i \in I(\rho)} r_i^2 \leq C \frac{1}{r_0} \sum_{i \in I} r_i^3 \leq C \sup_i \{r_i\} \sum_{i \in I} r_i^2 \leq C \sqrt{\varepsilon} r_0^2 m(r). \tag{32}$$

2.2 Comparison with an energy minimizing function

Since $\rho$ is now chosen, we are ready to compare with an energy minimizing function and use the decay result of [LEM]. By construction of $S$, the set $F$ is $(\varepsilon_0, \sqrt{\varepsilon})$-minimal in sense of Definition 8 of [LEM]. In fact, we know that $F$ is $\varepsilon_0$-minimal in the complement of the $\{B_i\}_{i \in I}$, and for all $i$, we have that $r_i \leq \sqrt{\varepsilon} r_0$. Set

$$G := F^p = (F \setminus \bigcup_{i \in I(\rho)} B_i) \cup \bigcup_{i \in I(\rho)} \partial B_i.$$

Then if $\varepsilon$ is small enough with respect to $\varepsilon_0$ and $\varepsilon_2$ (the constant of [LEM]) we can apply Theorem 9 of [LEM]. Thus we know that the normalized energy decreases for all energy minimizer in $B(0, r_0) \setminus G$. In particular if $w$ is the energy minimizer in $B \setminus G$ that is equal to $u$ on $\partial B \setminus G = \partial B \setminus F$ (for the existence of such a minimizer, one can see for example [DAV05 page 97]), applying Theorem 9 of [LEM] with $0 < \gamma < 0.8$, we have that for all
\( a < \frac{1}{2} \), there is a \( \varepsilon_2 \) (that depends on \( a \) and \( \varepsilon_0 \)), such that if \( \varepsilon \) is small enough (depending on \( \varepsilon_0 \) and \( a \)),

\[
\frac{1}{(ar_0)^2} \int_{B(0,ar_0) \setminus G} |\nabla w|^2 \leq a^\gamma \frac{1}{r_0^2} \int_{B(0,r_0) \setminus G} |\nabla w|^2. \tag{33}
\]

The second useful fact is the following. Since \((u, K)\) is a Mumford-Shah minimizer and \((w, G)\) is a competitor we have

\[
\int_{B(0,\rho) \setminus K} |\nabla u|^2 + H^2(K \cap B(0, \rho)) \leq \int_{B(0,\rho) \setminus G} |\nabla w|^2 + H^2(G \cap B(0, \rho)) + \rho^2 h(\rho).
\]

Hence

\[
\int_{B(0,\rho) \setminus K} |\nabla u|^2 - \int_{B(0,\rho) \setminus G} |\nabla w|^2 \\
\leq H^2(G \cap B(0, \rho)) - H^2(K \cap B(0, \rho)) + \rho^2 h(\rho) \\
\leq Cr_0^2 \omega_2(0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + C \sum_{i \in I(\rho)} r_i^2 + \rho^2 h(\rho) \\
\leq Cr_0^2 \omega_2(0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + C \sqrt{\varepsilon} r_0^2 m(0, r_0) + \rho^2 h(\rho). \tag{34}
\]

The third point is that \( \nabla w \) and \( \nabla (w - u) \) are orthogonal in \( L^2(B(0, r_0)) \). That comes from the fact that \( w \) is an energy minimizer in \( B(0, r_0) \setminus G \) and \( u \) is a competitor for \( w \). Thus

\[
\int_{B(0, r_0) \setminus G} |\nabla u - \nabla w|^2 = \int_{B(0, r_0) \setminus G} |\nabla u|^2 - \int_{B(0, r_0) \setminus G} |\nabla w|^2.
\]

We can now estimate the energy of \( u \). Let \( 0 < a < \frac{1}{2} \), then

\[
\int_{B(0,ar_0) \setminus G} |\nabla u|^2 \leq 2 \int_{B(0,ar_0) \setminus G} |\nabla w|^2 + 2 \int_{B(0,ar_0) \setminus G} |\nabla w - \nabla u|^2 \\
\leq 2a^{2+\gamma} \int_{B(0, r_0) \setminus G} |\nabla w|^2 + 2 \int_{B(0, r_0) \setminus G} |\nabla w - \nabla u|^2 \\
\leq 2a^{2+\gamma} \int_{B(0, r_0) \setminus G} |\nabla u|^2 + 2 \int_{B(0, r_0) \setminus G} |\nabla u|^2 - 2 \int_{B(0, r_0) \setminus G} |\nabla w|^2.
\]

Hence,

\[
\omega_2(0, ar_0) \leq 2a^{\gamma} \omega_2(0, r_0) + \frac{1}{a^2} \omega_2(0, r_0)^{\frac{1}{2}} J(0, r_0)^{-1} + C \sqrt{\varepsilon} \frac{1}{a^2} m(0, r_0) + \frac{1}{a^2} \rho^2 h(\rho). \tag{35}
\]

Inequality (33) is the fundamental estimate that will be used to control the energy.
Compactness lemmas for almost minimal sets

The purpose of this section is to show some geometrical results about almost minimal sets (see definition [11]). In the future estimates, we will use an argument which allows us to win something in each bad ball, in order to prove that there are not so many. The main lemma says the following. If \( B(x, r) \) is a ball such that \( x \in K \) and \( \beta(x, r) \leq \varepsilon_0 \) but \( \beta(x, ar) > \varepsilon_0 \), then there is a set that does better than \( K \) in \( B(x, r) \) in terms of \( H^2 \)-measure.

Recall that for any almost minimal set \( E \) in \( B(x, r) \), we denote by \( f(r) \) the excess of density

\[
f(r) = \theta(x, r) - \lim_{t \to 0} \theta(x, t)
\]

with

\[
\theta(x, r) = r^{-2} H^2(E \cap B(x, r)).
\]

The limit at 0 of \( \theta \) exists because \( E \) is almost minimal (see 2.3. of [DAVb]). For \( x \in E \) we call \( d(x) \) the density at \( x \), that is \( d(x) = \lim_{t \to 0} \theta(x, t) \). The function \( d(x) \) can only take a finite number of values, more precisely \( d(x) \in \{0, \pi, \frac{3\pi}{2}, d_+\} \) that are (excepted 0) densities of the three minimal cones in \( \mathbb{R}^3 \).

For an almost minimal set \( E \), the function \( \theta(x, t) \) is non decreasing in \( t \) thus the limit when \( t \) tend to 0 exists and that allows us to define the function \( d(x) \). Unfortunately, if \( E \) is now the singular set of a Mumford-Shah minimizer, the monotonicity of \( \theta \) is not known. So we have some difficulties so define the analogue of \( f(r) \) for a Mumford-Shah minimizer.

In order to use Theorem 12, we want to control \( f(r) \). That will be the role of the following Lemmas. Our goal is to obtain a statement analogous to Theorem 12 but with only an hypothesis on \( \beta(0, r_0) \) instead of \( f(0, r_0) \).

First of all, an application of Proposition 16.24 of [DAVb] in \( B(x, r 10^{-3}) \) with \( \eta_1 = \varepsilon_2 10^3 \), mixed with Proposition 18.1 of [DAVb] in \( B(x, r 10^{-5}) \) and \( \eta_1 = \varepsilon_7 10^{-5} \) (where \( \varepsilon_7 \) and \( \varepsilon_3 \) are defined in [DAVb]) allows us to state the following lemma. Recall that \( D_{x,r} \) is the normalized bilateral Hausdorff distance.

**Lemma 26. [DAVb]** For each choice of \( b \in \mathbb{R}^3 \) there is a \( \eta_1 \geq 0 \) such that if \( E \) is an almost minimal set in an open set \( U \subset \mathbb{R}^3 \), with gauge function \( h(r) = C_0 r^b \), if \( x \in E \) and \( r > 0 \) are such that \( B(x, r) \subset U \), if there is \( Z \), centered at \( x \), of type \( \mathbb{P}, \mathbb{Y} \) or \( \mathbb{T} \) such that

\[
D_{x,r}(E, Z) \leq \eta_1, \quad h(2r) \leq \eta_1, \quad \int_0^{2r} h(t) \frac{dt}{t} \leq \eta_1
\]

and if \( E \) is separating in \( B(x, r) \), then there is a point \( x \in E \cap B(x, r 10^{-5}) \), of the same type of \( Z \).

We say that \( x \) has the same type as \( Z \) if \( d(x) \) is equal to the density of the cone \( Z \).
Remark 27. The hypothesis of separating are only useful for the case of $T$. See Propositions 16.24 and 18.1 of [DAVb] for more details.

Remark 28. Lemma 26 is not trivial because we can imagine that $E$ is very close to a cone of type $T$ in $B(x, r)$ but contains only $\mathbb{P}$ points and $\mathbb{Y}$-points (see [DAVb] Section 19). The lemma says that under separating conditions and if $h$ and $\beta$ are small enough, this is not possible.

Here is now the statement that will be useful for the next sections. The reader is invited to compare it with Theorem 12.

Lemma 29. For each choice of $b \in (0, 1], \bar{c} > 0$ and $C_0 > 0$ we can find $\eta_2 > 0$ and $C \geq 0$ such that the following holds. Let $E$ be a reduced MS-almost minimal set in $\Omega \subset \mathbb{R}^3$ with gauge function $h$. Suppose that $0 \in E$, $r_0 > 0$ such that $B(0, 110r_0) \subset \Omega$ and $h$ is satisfying

$$h(r) \leq C_0 r^b \quad \text{for } 0 < r < 220r_0$$

and

$$h(220r_0) \leq \eta_2, \quad \int_0^{220r_0} h(t) \frac{dt}{t} \leq \eta_2.$$  

Assume in addition that

$$D_{0,100r_0}(E, Z) \leq \eta_2$$

where $Z$ is a minimal cone centered at the origin such that

$$H^2(Z \cap B(0, 1)) \leq d(0).$$

Then for all $x \in E \cap B(0, 4r_0)$ and for all $0 < r < 5r_0$ there is a minimal cone $Z(x, r)$ such that

$$D_{x,r}(E, Z(x, r)) \leq \bar{c} \left( \frac{r}{r_0} \right)^{\alpha}.$$  

Proof: We take $\eta_2 < \varepsilon_1$ (the constant of Theorem 12). In order to apply Theorem 12 all we have to prove is that

$$f(0, 110r_0) \leq \varepsilon_1.$$  

If $\eta_2$ is smaller than $\eta_1$ we can apply Lemma 26 to $E$ in $B(x, 110r_0)$ thus there is a point $z$ in $B(x, 10^{-3}r_0)$ of same type of $Z$. In particular $d(z) = H^2(Z \cap B(z, 1)) = \frac{1}{4} H^2(Z \cap B(z, r))$ for all $r$. Hence we can compute the excess of density at $z$ in $B(z, 55r_0)$ by

$$f(z, 55r_0) = \frac{1}{(55r_0)^2} [H^2(E \cap B(z, 55r_0)) - H^2(Z \cap B(z, 55r_0))].$$

Now define a competitor $L$ by

$$L = \begin{cases} \ M \cup Z \cap B(z, 55r_0) & \text{in } \hat{B}(z, 55r_0) \\ E & \text{in } \Omega \setminus B(z, 55r_0) \end{cases}.$$
where $M$ is a little wall:

$$M := \{ x \in \partial B(z, 55r_0); d(x, Z) \leq 500\eta_2 r_0 \}.$$

The set $L$ is a MS-competitor for $E$ thus

$$H^2(E \cap B(z, 55r_0)) \leq H^2(L \cap B(z, 55r_0)) + (55r_0)^2 h(55r_0) \leq H^2(M) + H^2(Z \cap B(z, 55r_0)) + (55r_0)^2 h(55r_0).$$

Since $H^2(M) \leq Cr_0^2 \eta_2$ we deduce

$$f(z, 55r_0) \leq C\eta_2.$$

Now if $\eta_2$ is small enough compared to $\varepsilon_1$, we can apply Theorem 12 in $B(z, 55r_0)$ then for all $y \in E \cap B(z, 5r_0)$ and $0 < r < 10r_0$ we have

$$\beta(y, r) \leq \bar{c} \left( \frac{r}{r_0} \right)^\alpha. \quad (36)$$

In addition, since $d(x, z) \leq 10^{-3}r_0$ we deduce that (36) is true for all $y \in B(x, 4r_0)$ and $0 < r < 10r_0$. $\square$

**Definition 30.** By now we will call $\bar{\eta}_2$ the constant given by Lemma 29 with $\bar{c} = 1$, $r_0 = \frac{1}{200}$, $C_0 = 0$ and $b = 0$, and we call $\bar{r}$ the radius such that

$$\left( \frac{\bar{r}}{2} \right)^\alpha = \frac{1}{2}\varepsilon_0.$$

Now we are ready to prove our fundamental lemma that will be used later to count the mass of bad balls.

**Lemma 31.** For all $\varepsilon_0 > 0$, and for all $r < \bar{r}$, there is a constant $\eta_0$ such that if $E$ is a closed set of finite $H^2$ measure in $B(0, 1) \subset \mathbb{R}^3$ that contains the origin, with the uniform concentration Property (with constant $C_u$), and assume that

$$\beta(0, 1) \leq \bar{\eta}_2 \quad (37)$$

$$\beta(0, r) \geq \varepsilon_0 \quad (38)$$

such that the cone in $\beta(0, 1)$ is centered in $B(0, 10^{-5})$. If in addition we assume that there is a set $F$ that contains $E$, that is separating in $B(0, 1)$ (see Definition 40) and such that

$$H^2(F) - H^2(E) \leq \eta_0.$$ 

Then there is a MS-competitor $L$ for $E$ in $B(0, \frac{3}{4})$ such that

$$H^2(E) - H^2(L) \geq \eta_0.$$
Proof : The argument is by contradiction. If the lemma is not true, then there is a $r < \bar{r}$ and there is an $\varepsilon_0 < \frac{1}{100} \bar{\eta}_2$ such that for all $\eta$ there is a set $E_\eta$ that verify (37) and (38). In addition for all MS-competitor $L_\eta$ we have

$$H^2(E_\eta) - H^2(L_\eta) \leq \eta. \quad (39)$$

And for all $\eta$ there is a set $F_\eta$ that contain $E_\eta$, is separating in $B(0, 1)$, and such that

$$H^2(F_\eta) - H^2(E_\eta) \leq \eta. \quad (40)$$

Now let $\eta$ tend to 0. Passing if necessary to a subsequence, we may assume that the sequence of sets $E_\eta$ converges to a certain $E_0$ in sense of Hausdorff distance. Passing to the limit, we deduce that this set $E_0$ still verify (37) and (38).

We want to show that $E_0$ is a minimal set in $B(0, \frac{3}{4})$. Let $L$ be a MS-competitor for $E_0$ in $B(0, \frac{3}{4})$. Since $E_\eta$ tend to $E_0$ for the Hausdorff distance $D_H$, we know that for all $\tau$ there is a $\eta'$ such that for all $\eta < \eta'$, $D_H(E_0, E_\eta) \leq \tau$. Thus if $T_\tau := \{x \in \partial B(0, 1); d(x, L) \leq \tau\}$, we deduce that $E_\eta \cap \partial B(0, 1) \subset T_\tau$. Therefore, the set $L_\eta := L \cup (E_\eta \cap B(0, 1) \setminus B(0, \frac{3}{4})) \cup T_\tau$ is a MS-competitor for $E_\eta$. Then applying (39) we obtain

$$H^2(E_\eta \cap B(0, \frac{3}{4})) \leq H^2(L_\eta) - H^2(E_0) \leq H^2(L) + H^2(T_\tau) + \eta \leq H^2(L \cap B(0, \frac{3}{4})) + \eta + C\tau.$$  

In addition, by hypothesis the sets $E_\delta$ verify the uniform concentration property with same constant $C_u$. This allows us to say that (see [DAV05] section 35)

$$H^2(E_0) \leq \lim_{\eta \to 0} H^2(E_\eta).$$

Hence, letting $\eta$ tend to 0 we obtain

$$H^2(E_0) \leq H^2(L) + \tau$$

then letting $\tau$ tend to 0,

$$H^2(E_0) \leq H^2(L)$$

thus $E_0$ is a minimal set (i.e. almost minimal set with gauge function equal to zero).

On the other hand, $E_0$ is separating in $B(0, 1)$, because if it is not the case, we can find a continuous path $\gamma$ that join $A^+$ and $A^-$ (two points in different connected component of $B(0, 1) \setminus Z_{10-5}$) in $B(0, 1)$ and such that $\gamma$ does not meet $E_0$. Since $E_\eta$ converge to $E_0$ for the Hausdorff distance, for all $\tau$ there is a $\eta_\tau$ such that if $\eta < \eta_\tau$, all the $E_\eta$ are $\tau$ close to $E_0$. Let $x$ be the point of $\gamma$ that realize the infimum of $d(x, E_0)$. Since $\gamma$ is disjoint from $E_0$, there is a ball centered at $x$ with positive radius $r$ that is not meeting $E_0$. Thus if we
choose \( \eta \) smaller than \( r \) we get that all the \( E_\eta \) for \( \eta < \eta_r \) contain a hole of size \( r \), but this is not possible according to (40).

Thus finally \( E_0 \) is a minimal set in \( B(0, \frac{3}{4}) \), which is separating and verifies (37) and (38). We want now to apply Lemma 29 to obtain a contradiction. We know that

\[
\beta(0, 1) \leq \bar{\eta}_2
\]

and that the cone associated is centered in \( B(0, 10^{-5}) \). We claim that

\[
D_{\frac{1}{2}}(E_0, Z) \leq \bar{\eta}_2.
\]  

(41)

All we have to show is that for all \( x \in Z \), \( d(x, E_0) \leq \bar{\eta}_2 \). If it is not the case, then we can find \( x \in Z \) such that \( B(x, \bar{\eta}_2) \cap E_0 = \emptyset \). But then we can find a continuous path that join two different connected components of \( B(0, 1) \setminus Z \) without meeting \( E \), and that is not possible if \( E \) is separating. So we have shown (41) and then we can apply Lemma 29 in \( B(z, \frac{1}{2}) \) (i.e. \( r_0 = \frac{1}{200} \)), which implies that

\[
\beta(0, r) \leq \frac{1}{2} \varepsilon_0
\]

because of the definition of \( \bar{r} \), and this yields a contradiction with (38) so the proof is now complete.

Applying Lemma 31 we can deduce to following proposition.

**Proposition 32.** Let \( i \in I \) be an index such that \( \frac{1}{4} B_i := B(x_i, d(x_i)) \) do not verify (21). Then there is a MS-competitor \( L \) for \( K \) in

\[
\tilde{B}_i := B(x_i, \frac{M}{\bar{r}} d(x_i))
\]

such that

\[
H^2(K \cap \tilde{B}_i) - H^2(L \cap \tilde{B}_i) \geq \eta_0 \tilde{r}_i^2
\]

with \( \tilde{r}_i := \frac{M}{\bar{r}} d(x_i) \) and \( M \) is a constant equal to 1, \( 10^5 \) or \( 10^{10} \).

**Proof:** Since \( B_i \) do not verify (21), we know that

\[
\beta(x_i, d(x_i)) \geq \varepsilon_0
\]

and in addition

\[
\beta(x_i, \frac{1}{\bar{r}} d(x_i)) \leq \varepsilon_0
\]

Multiplying if necessary the radius by \( 10^5 \) or \( 10^{10} \), and by use of the re-centering Lemma 15 (with constant \( V = 10^5 \)), we can suppose that the center of the cone is in a ball of radius \( 10^{-5} \) times smaller in \( B(x_i, \frac{M}{\bar{r}} d(x_i)) \) (\( M \) is the constant equal to 1, \( 10^5 \) or \( 10^{10} \)). Set

\[
\tilde{r}_i := \frac{M}{\bar{r}} d(x_i)
\]

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Then if $\varepsilon_0$ is small enough compared to $\eta_2$ we have that
\[ \beta(x_i, \tilde{r}_i) \leq \varepsilon_0 \leq \eta_2 \]
with a cone centered in $B(x_i, 10^{-5}\tilde{r}_i)$. Moreover we have
\[ \beta(x_i, \tilde{r}_i) \geq \frac{1}{M}\varepsilon_0. \]
We also have $F \cap B(x_i, \tilde{r}_i)$, that is a separating set in $B(x_i, \tilde{r}_i)$ and such that
\[ H^2(F \cap B(x_i, \tilde{r}_i)) - H^2(K \cap B(x_i, \tilde{r}_i)) \leq \varepsilon_0'\tilde{r}_i. \]
Therefore, we can apply lemma 31 in $B(x_i, \tilde{r}_i)$ with $\frac{1}{M}\varepsilon_0$ instead of $\varepsilon_0$ that we may suppose smaller than $C\varepsilon_1$. We can also take $\varepsilon_0' << \eta_0$. Finally, Lemma 31 is stated in $B(0,1)$ but by translation and dilatation it stays true in every ball $B(x,r)$.

**Remark 33. (Choice of $A$)** We can now fix our constant $A$. We want that for every bad ball $B_i := B(x_i, Ad(x_i))$ with $i \in I$, the ball
\[ B(x_i, \tilde{r}_i) := B(x_i, \frac{M}{\bar{r}}d(x_i)) \subset B(x_i, \frac{10A}{U}d(x_i)) \subset \mathcal{V} \]
in order to have that the extension of $u$ given by Lemma 23 is well defined in each $B(x_i, \tilde{r}_i)$. Thus it suffices to take for instance
\[ A = \frac{U10^{-20}}{\bar{r}}. \]

Before continuing, it is time now to recapitulate in which order the principal constants are introduced, to see who is controlled by who. Recall that at beginning we have a Mumford-Shah minimizer $K$ with $\beta(0, r_0)$ less than a certain $\varepsilon$. Then we use a stopping time argument about being close to cones at small scales with stopping constant $\varepsilon_0$ for the geometry and $\varepsilon_0'$ for the topology (separating condition). We obtain a collection of balls that we call “small scales” on which we do some manipulations.

At small scales : The regularity theorem of Guy David gives a $\varepsilon_1$ for which $\beta$ decays like a power of radius for a minimal set with excess density (function $f(0, r_0)$) smaller than $\varepsilon_1$. An other lemma controls $f(0, r_0)$ by $\beta(0, r_0)$ whenever $\beta(0, r_0)$ is smaller than a certain $\eta_1$. Thus we obtain $\bar{r}$, that depends on $\varepsilon_0$, for which $\beta(0, \bar{r}) < \frac{\varepsilon_0}{2}$ for all minimal set that is separating in $B(0,1)$ and such that $\beta(0, 1) < \eta_2$. In the proof of this compactness lemma we fix $\varepsilon_0$ small enough compared to $\bar{\eta}_2$. The lemma gives a $\eta_0$ that is the winning of surface in each bad ball, depending on $\varepsilon_0$ and $\bar{r}$. In the other hand we have to be sure that $\varepsilon_0'$ is smaller than $\eta_0$ to apply the Lemma in future. So at this stage we have (each quantity is depending on what is on the right of the symbol $\prec$):
\[ \varepsilon_0' \prec \eta_0 \prec \bar{r} \prec \varepsilon_0 \prec \bar{\eta}_2 \]
At big scale: In the big scale we want to show that some quantities in the ball of radius $ar_0$ are controlled by the same quantity in the ball of radius $r_0$, for a certain $a$ that is chosen later with some arithmetical conditions, in particular $a^3 < \frac{1}{8}$ where $\gamma$ is close as we want to 0.8. We apply Theorem 9 of [LEM] with $F$ a $(\varepsilon_0, \frac{\varepsilon}{r_0})$-minimal set and $\varepsilon_0$ is like in the above paragraph. Theorem 9 of [LEM] gives a $\varepsilon_2$ (depending on $\varepsilon_0$, $\gamma$ and $a$) and assure a decay of energy if $\varepsilon$ is small enough in respect with $\varepsilon_2$ and $\varepsilon_0$. Thus in addition of (42) we have

$$\varepsilon \prec \varepsilon_2 \prec \left\{ \varepsilon_0 \prec \varepsilon_1 \right\}$$

2.4 Bounds for the bad mass

The following proposition is an estimate about $m$. Recall that $\rho$ is the radius chosen in $[\frac{r_0}{2}, \frac{3}{4}r_0]$.

**Proposition 34.** If $m(0, \frac{\rho}{2}) > \frac{m(0, r_0)}{10}$ then

$$m(0, \frac{\rho}{2}) \leq \frac{C}{\eta_0} \left( \omega_2(0, r_0) + \omega_2(0, r_0)^{\frac{3}{2}} J(0, r_0)^{-1} + h(r_0) \right).$$

(43)

**Proof:** To prove Proposition 34 we will count the number of $B_i$ for $i \in I$ and use Proposition 32 to say that there are not so many. Recall that the $B_i$ are disjoints.

In order to estimate the bad mass we will take a good competitor for $(u, K)$ in $B(0, r_0)$. Set $I_1$ the indices of bad balls $B_i$ such that $B(x_i, d(x_i))$ don’t verify (21) and $I_2 := I \setminus I_1$. In particular, balls of $I_2$ don’t verify (20). Hence we know that if $i \in I_2$ and if $r_i := d(x_i)$ we have

$$r_i^2 \leq \frac{1}{\varepsilon_0} \left( H^2(F \cap B(x_i, r_i)) - H^2(K \cap B(x_i, r_i)) \right)$$

and since the $B_i$ are disjoints we deduce that

$$\sum_{i \in I_2} r_i^2 \leq C \frac{1}{\varepsilon_0} \left( H^2(F(0, r_0)) - H^2(K \cap B(0, r_0)) \right) \leq Cr_0^2 \omega_2(0, r_0)^{\frac{3}{2}} J(0, r_0)^{-1}.$$  

Now we have to count the contribution of $I_1$. We will modify each $B_i$ for $i \in I_1$ with the use of Proposition 32. Set

$$\tilde{G} := \left\{ \begin{array}{ll} F(0, r_0) & \text{in } B(0, r_0) \setminus \bigcup_{i \in I_1; B_i \cap B(0, r_0) \neq \emptyset} B_i \\ L_i & \text{in } B_i \text{ for all } i \in I_1; B_i \cap B(0, \rho) \neq \emptyset \end{array} \right.$$  

where $L_i$ is the set given by Proposition 32. Then set

$$G := \tilde{G} \cup \bigcup_{i \in I_1} \partial B_i,$$  

where $G$ is the set given by Proposition 32. Then set

$$G := \tilde{G} \cup \bigcup_{i \in I_1} \partial B_i.$$

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For the function we use the extension of Proposition 23 which can be applied in \( B(0, \rho) \) by Proposition 24. Thus we take 
\[
v = v^k \text{ in } \Omega^k.
\]
By choice of constant \( A \) we know that the function \( v \) is well defined in \( B(0, r_0) \setminus G \). Set 
\[
I'_1 := \{ i \in I_1; B_i \cap B(0, \rho) \neq \emptyset \text{ and } B_i \cap \partial B(0, \rho) = \emptyset \}
\]
and 
\[
I''_1 := \{ i \in I_1; B_i \cap \partial B(0, \rho) \neq \emptyset \}.
\]
Notice that 
\[
m(0, \frac{\rho}{2}) \leq C \frac{1}{r_0} \sum_{i \in I'_1} r_i^2 \quad \text{and} \quad \sum_{i \in I''_1} r_i^2 \leq \sqrt{\varepsilon} m(0, r_0). \]
In addition \( G \) is a competitor. To see this we can use the same argument as Remark 1.8. in \cite{DAVb}. We apply now the fact that \((u, K) \) is a Mumford-Shah minimizer and we obtain
\[
\int_{B(0, r_0) \setminus K} |\nabla u|^2 + H^2(K \cap B(0, r_0)) \leq \int_{B(0, r_0) \setminus G} |\nabla v|^2 + H^2(G \cap B(0, r_0)) + r_0^2 h(r_0)
\]
\[
\leq C \int_{B(0, r_0) \setminus K} |\nabla u|^2 + H^2(F(0, r_0)) - \eta_0 \sum_{i \in I'_1} r_i^2 + C \sum_{i \in I''_1} r_i^2 + C \sum_{i \in I_2} r_i^2 + r_0^2 h(r_0).
\]
Hence,
\[
\eta_0 C r_0 m(0, \frac{\rho}{2}) - C \sqrt{\varepsilon} r_0 m(0, r_0) \leq C \int_{B(0, r_0) \setminus K} |\nabla u|^2 + r_0^2 \omega_2(0, r_0) \frac{1}{2} J(0, r_0)^{-1} + r_0^2 h(2r_0).
\]
Therefore, if \( \varepsilon \) is small enough compared to \( \eta_0 \) and since \( m(0, \frac{\rho}{2}) \geq \frac{0.01 m(r_0)}{10} \) we deduce
\[
m(0, \frac{\rho}{2}) \leq \frac{C}{\eta_0} \left( \omega_2(0, r_0) + \omega_2(0, r_0) \frac{1}{2} J(0, r_0)^{-1} + \beta(0, r_0) + h(2r_0) \right)
\]
and the proposition follows.

Now by the same sort of argument as Proposition before, we have this second estimate about \( m \).

**Proposition 35.**
\[
m(0, r_0(1 - 5\sqrt{\varepsilon})) \leq \frac{C}{\eta_0} \left( \omega_2(0, r_0) + \omega_2(0, r_0) \frac{1}{2} J(0, r_0)^{-1} + \beta(0, r_0) + h(2r_0) \right).
\]
\[
(44)
\]

**Proof:** The proof is very similar to Proposition 34. We modify each \( B_i \) for \( i \in I_1 \) with the use of Proposition 32. Set
\[
\tilde{G} := \begin{cases} 
F(0, r_0) & \text{in } B(0, r_0) \setminus \bigcup_{i \in I_1} B_i \\
L_i & \text{in } B_i \text{ for all } i \in I_1
\end{cases}
\]

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where the $L_i$ are the sets given by Proposition 32. Our competitor is now

$$G := \tilde{G} \cup T_\beta$$

where $T_\beta$ is a little wall of size $\beta := 10\beta(0, r_0)$

$$T_\beta := \{ y \in \partial B(0, r_0); d(y, Z) \leq \beta r_0 \}$$

with $Z$ a minimal cone centered at the origin at distance less than $\beta(0, r_0)$ of $K$ in $B(0, r_0)$.

We keep the same notation $I_1, I_2, I'_1$ and $I''_1$ as before but now with $\rho = r_0$. As before we have

$$\sum_{i \in I_2} r_i^2 \leq C \frac{1}{\varepsilon_0} (H^2(F(0, r_0)) - H^2(K \cap B(0, r_0))) \leq C r_0^2 \omega_2(0, r_0)^{\frac{3}{2}} J(0, r_0)^{-1}.$$ 

For the function we use the extension of Proposition 23 in $B(0, 2r_0)$ with $\rho = r_0$ and with application $\delta$ defined in [30]. We set

$$v = v^k \text{ in } \Omega^k.$$ 

By choice of constant $A$ we know that the function $v$ is well defined in $B(0, r_0) \setminus G$ and since we added $T_\beta$ there is no boundary problem.

We apply now the fact that $(u, K)$ is a Mumford-Shah minimizer and we obtain with same notations as Proposition before,

$$\int_{B(0, r_0) \setminus K} |\nabla u|^2 + H^2(K \cap B(0, r_0)) \leq \int_{B(0, r_0) \setminus G} |\nabla v|^2 + H^2(G \cap B(0, r_0)) + r_0^2 h(r_0)$$

$$\leq C \int_{B(0, r_0) \setminus K} |\nabla u|^2 + H^2(F(0, r_0)) - \eta_0 \sum_{i \in I'_1} r_i^2 + C \sum_{i \in I_2} r_i^2 + H^2(T_\beta) + r_0^2 h(r_0)$$

Hence,

$$\eta_0 m(0, r_0(1 - 5\sqrt{\varepsilon})) \leq C \int_{B(0, r_0) \setminus K} |\nabla u|^2 + r_0^2 \omega_2(0, r_0)^{\frac{3}{2}} J(0, r_0)^{-1} + C r_0^2 \beta(0, r_0) + r_0^2 h(2r_0)$$

because all the $B_i$ have a radius less than $\sqrt{\varepsilon} r_0$ thus all the $B_i$ for $i \in I_1$ such that $5B_i \cap \partial B(0, r_0) = \emptyset$ are included in $B(0, r_0(1 - 5\sqrt{\varepsilon}))$, and the proposition follows. \qed
Control of the minimality defect

In this section we want to control the defect of minimality of $K$ in terms of energy and bad mass. For some topological reasons we are not going to work directly on $K$, but we will use the set $F$ to be sure that it is separating in $B$. We show in this section that for all MS-competitor $L$ for $F$ we can give a function $w$ such that $(L, w)$ is a Mumford-Shah competitor for $(u, K)$, and with good bounds on the energy of $w$. Here is a more precise statement:

**Proposition 36.** There is a positive constant $c_{10} < 1$ such that for all MS-competitor $L$ for the set $F$ (see Definition 10) in the ball $B(0, c_{10} r_0)$, we have:

$$\frac{1}{r_0^2} [H^2(F \cap B(0, c_{10} r_0)) - H^2(L \cap B(0, c_{10} r_0))] \leq C \left[ \omega_2(0, r_0) + \sqrt{\varepsilon m}(0, r_0) + h(r_0) \right]$$

**Proof:** Let $Z^0$ be the cone such that $d(x, Z^0) \leq \varepsilon r_0$ for all $x \in K \cap B(0, r_0)$. We call as usual $Z^0_\varepsilon$ the region

$$Z^0_\varepsilon := \{ x \in B(0, r_0); d(x, Z_0) \leq \varepsilon \}. \quad (45)$$

We consider our ball $\{B_i\}_{i \in I}$ obtained by the stopping time argument. We define the functions

$$\psi_i := \begin{cases} r_i & \text{on } B_i \\ 0 & \text{in the complement of } 2B_i \end{cases}$$

then for all $x$ we define

$$d_1(x) := \sum_{i \in I} \psi_i(x).$$

Finally, for all $x \in B(0, \rho)$ set

$$\delta(x) := \max(d(x, \partial B(0, \rho)), d_1(x)).$$

As usual, $\delta(x)$ is a geometric function associated to $F$ in $B(0, r_0)$. Thus applying Lemma 23 we get $\mathcal{H}^\rho$ functions $v^k$ such that $v^k \in W^{1,2}(\Omega^k \cup \mathcal{V})$ and such that

$$\int_{\Omega^k \cup \mathcal{V} \setminus \mathcal{V}_\rho} |\nabla v^k|^2 \leq C \int_{B(0, \rho) \setminus F} |\nabla u|^2$$

in addition, $v^k$ is equal to $u$ on $\partial B(0, \rho) \cap \Omega^k \setminus \mathcal{V}$. Moreover, since $\delta(x) \geq d(x, \partial B(x, \rho))$, if $\varepsilon$ is small enough we can easily deduce that there is a constant $c_{10} < \frac{1}{2}$ depending on constant $U$ such that $B(0, c_{10} r_0) \subset \mathcal{V}$. Set

$$G' = \begin{cases} F & \text{in } B(0, r_0) \setminus B(0, c_{10} r_0) \\ L & \text{in } B(0, c_{10} r_0) \end{cases}$$
If $L$ is a MS-competitor for $F$ in $B(0, c_{10}r_0)$, we know that $L$ in separating $B(0, c_{10}r_0)$ into $k^\rho$ big connected components (because $F$ is separating and $L$ is a topological competitor). Thus $G'$ is separating in $B(0, \rho)$ and we note $(B(0, \rho) \setminus G)^k$ the big connected components.

Then set

$$G := G' \cup \bigcup_{i \in I(\rho)} \partial B_i$$

and

$$v := \begin{cases} 
  u & \text{in} \quad B(0, r_0) \setminus B(0, \rho) \\
  v^k & \text{in} \quad (B(0, \rho) \setminus G)^k \\
  0 & \text{in other components of} \quad B(0, \rho) \setminus G
\end{cases}$$

Using that $(u, K)$ is a Mumford-Shah minimizer and that $(v, G)$ is a competitor we obtain

$$\int_{B(0, \rho) \setminus K} |\nabla u|^2 + H^2(K) \leq \int_{B(0, \rho) \setminus G} |\nabla v|^2 + H^2(G) + \rho^2 h(\rho)$$

thus

$$H^2(K \cap B(0, c_{10}r_0)) - H^2(L \cap B(0, c_{10}r_0))$$

$$\leq C \left[ \int_{B(0, r_0) \setminus K} |\nabla u|^2 + \sum_{i \in I'} r_i^2 + r_0^2 \omega(0, r)^\frac{1}{2} J(0, r)^{-1} + r_0^2 h(r_0) \right]$$

and the proposition follows.

\[\square\]

### 2.6 Conclusion about regularity

Now we are ready to use all the preceding estimates in order to prove some regularity. We begin with this proposition about self-improving estimates.

**Proposition 37.** There is an $\varepsilon > 0$, some $\tau_4 < \tau_3 < \tau_2 < \tau_1 < \varepsilon$ and $a < 1$ such that if $x \in K$ and $r$ are such that $B(x, r) \subset \Omega$, and

$$h(r) + J(x, r)^{-1} \leq \tau_4, \quad \omega_2(x, r) \leq \tau_3, \quad m(x, r) \leq \tau_2, \quad \beta(x, r) \leq \tau_1$$

then (46) is still true with $ar$ instead of $r$. 

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Proof: We choose \( \varepsilon < \varepsilon_0 \) and \( \varepsilon_1 \) such that all the results of the preceding sections are true. We choose \( a < \frac{1}{16} \) such that applying (35) to \((u, K)\) gives

\[
\omega_2(x, ar) \leq \frac{1}{8} \omega_2(x, r) + C_2 \omega_2(x, r)^{\frac{1}{2}} J(x, r)^{-1} + C_2 \sqrt{\varepsilon} m(x, r) + C_2 h(r). \tag{47}
\]

Since \( a \) is chosen, we can fix \( \tau_1 \) small enough such that for all \( ar < t < r \) we have \( \beta(x, t) \leq 10^{-1} \). Hence by Lemma 21

\[
J(x, ar) \geq a^{-\frac{1}{2}}[J(x, r) - C'] \geq \frac{1}{2} a^{-\frac{1}{2}} J(x, r)
\]

if \( \tau_4 \) is small enough compared to \( C' \). Then we deduce

\[
J(x, ar)^{-1} \leq 2a^\frac{1}{2} J(x, r)^{-1} \leq \frac{J(x, r)^{-1}}{2}
\]

because \( a < \frac{1}{16} \). In addition if \( \tau_4 \) is small enough compared to \( \tau_3 \), we have

\[
C \tau_3^{-\frac{1}{2}} \tau_4 \leq \frac{1}{8} \tau_3. \tag{48}
\]

Therefore by (47),

\[
\omega_2(x, ar) \leq \frac{3}{8} \tau_3 + C_2 \sqrt{\varepsilon} m(x, r) \leq \frac{\tau_3}{2}
\]

under the condition that

\[
8C_2 \sqrt{\varepsilon} \tau_2 < \tau_3. \tag{49}
\]

Now for \( m(x, r) \) we have two cases. If \( m(x, ar) \leq \frac{m(x, r)}{10} \) then \( m(x, ar) \leq \frac{\tau_2}{10} \) and it is what we want. Otherwise, we have \( m(x, ar) > \frac{m(x, r)}{10} \) which implies \( m(x, \frac{\varepsilon}{2}) > \frac{2m(x, r)}{5} \) and then we can use the proof of Proposition 34 with a slightly different constant (depending on \( a \)) to obtain

\[
m(x, ar) \leq \frac{C(a)}{\varepsilon_1} (\tau_3 + \frac{1}{2} \tau_4 + \tau_4) \leq \frac{C(a)}{\varepsilon_1} \tau_3 \leq \frac{\tau_2}{2}
\]

if

\[
2 \frac{C(a)}{\varepsilon_1} \tau_3 \leq \tau_2. \tag{50}
\]

So it suffice to choose \( \varepsilon \) small enough compared to \( \varepsilon_0 \) and \( C \) in order to have the existence of \( \tau_3 < \tau_2 \) that verify simultaneously (49) and (50). Hence, we control \( \omega_2(x, ar) \) and \( m(x, ar) \).

To finish we have to control \( \beta(x, ar) \). For that we use the estimate in Proposition 36 and Lemma 31 that we apply in \( B(x, c_{10}r) \). Indeed, suppose that \( a << \bar{r}(\varepsilon_0) \) is such that

\[
\beta(x, ar) \geq \tau_1 \tag{51}
\]
Then applying Lemma 31 with $\varepsilon_0 = \tau_1$ gives a $\eta_0(\tau_1, a)$ and a competitor $L$ for $K$ in $B(x, c_{10}r)$ such that

$$H^2(K) - H^2(L) \geq \eta_0(\tau_1, a).$$

(52)

On the other hand, according to Proposition 36, if we choose $\tau_2$ and $\tau_3$ small enough compared to $\eta_0(\tau_1, a)$, the inequality (52) cannot hold. This shows that

$$\beta(x, ar) \leq \tau_1$$

and gives a contradiction with (51) which achieves the proof of the proposition.  

We keep the constants $a$ and $\tau_i$ given by the preceding proposition. Let $b$ be the positive power such that $a^b = \frac{1}{2}$. Set

$$\tilde{h}_r(t) = \sup \left\{ \left( \frac{t}{s} \right)^b h(s); t \leq s \leq r \right\}$$

for $t < r$ and $\tilde{h}_r(t) = h(t)$ for $t > r$. According to [DAV05] page 318, the function $\tilde{h}$ is still a gauge function (i.e. monotone and with limit equal to 0 at 0). We also trivially have that $h(t) \leq \tilde{h}_r(t)$ and one can prove that

$$\tilde{h}_r(t) \geq \left( \frac{t}{t'} \right)^b \tilde{h}_r(t') \quad \text{for } 0 < t < t' \leq r.$$  

(53)

Note that since $a^b = \frac{1}{2}$, we have

$$\tilde{h}_r(at) \geq \frac{1}{2} \tilde{h}_r(t) \quad \text{for } 0 < t \leq r.$$  

(54)

The purpose of Proposition 37 is just to have $\beta(x, r) \leq \tau_1$ at all scales in order to have more decay for the other quantities. Notice that at this step, we could prove that $K$ is the bi-hölderian image of a minimal cone using [DPT]. This will be done in Corollary 40 to prove that $K$ is a separating set. Before that we will prove some more decay estimates.

Proposition 38. We assume that we have the same hypothesis as in the proposition before. Then for all $0 < t < r$ we have

$$J(x, t)^{-1} \leq 2 \left( \frac{t}{r} \right)^b \tau_4$$

$$\omega_2(x, t) \leq C \left( \frac{t}{r} \right)^b \tau_3 + C\tilde{h}_r(t)$$

$$m(x, t) \leq C \left( \frac{t}{r} \right)^b \tau_2 + C\tilde{h}_r(t).$$
Proof: The first step is to control the jump. Since $\tau_1$ is small enough to have $\beta(x, t) \leq 10^{-1}$ for all $t < r$, then by Lemma 21

$$J(x, t) \geq \left(\frac{r}{t}\right)^{-\frac{1}{2}} [J(x, r) - C'] \geq \frac{1}{2} \left(\frac{r}{t}\right)^{-\frac{1}{2}} J(x, r)$$

if $\tau_1$ is small enough compared to $C'$. We deduce

$$J(x, t)^{-1} \leq \left(\frac{t}{r}\right)^{\frac{1}{2}} J(x, r)^{-1}.$$ And since $a < \frac{1}{4}$ we have

$$J(x, a^n r)^{-1} \leq 2 \left(\frac{1}{2}\right)^n \tau_4.$$ (55)

Now we want to show by induction that

$$\omega_2(x, a^n r_0) \leq 2^{-n} \tau_3 + C_3 \tilde{h}_r(a^n r) \quad \text{and} \quad m(x, a^n r) \leq 2^{-n} \tau_2 + C_3 \tilde{h}_r(a^n r) \quad (56)$$

For $n = 0$ we have (56) trivially. Suppose by now that (56) is true for $n$. Then applying inequality (35) in $B(x, a^n r)$

$$\omega_2(x, a^{n+1} r) \leq \frac{1}{8} \omega_2(x, a^n r) + C_2 \omega_2(x, a^n r)^{\frac{1}{2}} J(x, a^n r)^{-1} + C_2 \sqrt{\varepsilon} m(x, a^n r) + C_2 h(a^n r). \quad (57)$$

Now, using the inequality $2ab \leq a^2 + b^2$ we obtain

$$\omega_2(x, a^n r)^{\frac{1}{2}} \leq \frac{1}{20C_2} \omega_2(x, a^n r) J(x, a^n r) + 5C_2 J(x, a^n r)^{-1}.$$ Thus (57) yields

$$\omega_2(x, a^{n+1} r) \leq \frac{7}{40} \omega_2(x, a^n r) + 5C_2^2 J(x, a^n r)^{-2} + C_2 \sqrt{\varepsilon} m(x, a^n r) + C_2 h(a^n r).$$

Now using (55), and the induction hypothesis we obtain

$$\omega_2(x, a^{n+1} r) \leq \frac{7}{40} 2^{-n} \tau_3 + 5C_2^2 \varepsilon 2^{-n} \tau_2 + \left(\frac{7}{40} C_3 + C_2 \sqrt{\varepsilon} C_3 + C_2\right) \tilde{h}_r(a^n r).$$

Now, using that $\tau_4$ controlled by $\tau_3$, since $\varepsilon$ is small as we want compared to $C_2$, using also (49) and (54), and finally if we choose $C_3$ larger than $100C_2$ we deduce that

$$\omega_2(x, a^{n+1} r) \leq \left(\frac{8}{40} + \frac{1}{8}\right) 2^{-n} \tau_3 + C_3 \tilde{h}_r(a^n r) \leq 2^{-(n+1)} \tau_3 + C_3 \tilde{h}_r(a^{n+1} r).$$
Concerning \( m(x, r) \) it is a similar argument, suppose that \( m(x, a^{n+1}r) > 2^{-(n+1)}m(x, a^n r) \). Then we can apply Proposition 34 in the ball \( B(x, a^n r) \) thus

\[
m(x, a^{n+1}r) \leq \frac{C(a)}{\varepsilon_1} \left( \varepsilon_1 \omega_2(x, a^n r) + \omega_2(x, a^n r) \frac{1}{2} J(x, a^n r)^{-1} + h(a^n r) \right)
\]

\[
\leq \frac{C(a)}{\varepsilon_1} \left( \frac{3}{2} \omega_2(x, a^n r) + \frac{1}{2} J(x, a^n r)^{-2} + h(a^n r) \right).
\]

(58)

Setting \( C_4 = \frac{C(a)}{\varepsilon_1} \), using (55) and induction hypothesis we obtain

\[
m(x, a^{n+1}r) \leq C_4 2^{-n} \tau_3 + C_4 2^{-n} \tau_4 + 2C_4 \tilde{h}_r(a^n r)
\]

\[
\leq 2^{-n} \tau_2 + C_3 \tilde{h}_r(a^{n+1} r)
\]

because \( \tau_3 \) and \( \tau_4 \) are small as we want with respect to \( C_4 \) and \( \tau_2 \), and because we can chose \( C_3 \) bigger than \( 10C_4 \) and we have used (54).

To finish the proof let \( 0 < t < r \) and \( n \) such that \( a^{n+1} \leq t \leq a^n r \). Then we have

\[
\omega_2(x, t) = \frac{1}{t^2} \int_{B(0, t) \setminus K} |\nabla u|^2 \leq \left( \frac{a^n r}{t} \right)^2 \omega_2(x, a^n r)
\]

\[
\leq \frac{1}{a^2} 2^{-n} \tau_3 + C_3 \tilde{h}_r(a^n r)
\]

\[
\leq \frac{1}{a^2} a^{bn} \tau_3 + C_3 \tilde{h}_r(t)
\]

\[
\leq C \left( \frac{t}{r} \right)^b \tau_3 + C_3 \tilde{h}_r(t)
\]

and

\[
m(x, t) \leq \frac{a^{2n} r^2}{t^2} m(x, a^n r) \leq \frac{a^{2n} r^2}{t^2} 2^{-n} \tau_2 + C_3 \tilde{h}_r(a^n r)
\]

\[
\leq \frac{1}{a^2} a^{bn} \tau_2 + C_3 \tilde{h}_r(t) \leq C \left( \frac{t}{r} \right)^b \tau_2 + C_3 \tilde{h}_r(t).
\]

Proposition 39. There is a positive constant \( b \) such that the following is true. Let \((u, K)\) be a Mumford-Shah minimizer in \( \Omega \subset \mathbb{R}^3 \) with gauge function \( h \). Let \( x_0 \in K \) and \( r_0 \) be such that \( B(x_0, r_0) \subset \Omega \). Then there is \( \varepsilon > 0 \) and \( \tau_4' < \tau_2' < \tau_1' < \varepsilon \) such that if

\[
h(r_0) + J(x_0, r_0)^{-1} \leq \tau_4', \quad \omega_2(x_0, r_0) \leq \tau_3', \quad m(x_0, r_0) \leq \tau_2', \quad \beta(x_0, r_0) \leq \tau_1'
\]

then for all \( x \in B(x_0, \frac{1}{10} r_0) \) and for all \( 0 < t < \frac{1}{2} r_0 \) we have

\[
J(x, t)^{-1} \leq C \left( \frac{t}{r_0} \right)^b
\]

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\[
\omega_2(x, r) \leq C \left( \frac{t}{r_0} \right)^b + C\tilde{h}_r(t)
\]
\[
m(x, t) \leq C \left( \frac{t}{r_0} \right)^b + C\tilde{h}_r(t)
\]
\[
\beta(x, t) \leq \tau_1
\]

**Proof:** It suffice to show that there is \(\tau'_4 < \tau'_3 < \tau'_2 < \tau'_1 < \varepsilon\) such that if
\[
h(r_0) + J(x_0, r_0)^{-1} \leq \tau'_4, \quad \omega_2(x_0, r_0) \leq \tau'_3, \quad m(x_0, r_0) \leq \tau'_2, \quad \beta(x_0, r_0) \leq \tau'_1
\]
than for all \(x \in B(x_0, \frac{1}{10}r_0)\) we have
\[
h\left(\frac{1}{2}r_0\right) + J(x, \frac{1}{2}r_0)^{-1} \leq \tau_4, \quad \omega_2(x, \frac{1}{2}r_0) \leq \tau_3, \quad m(x, \frac{1}{2}r_0) \leq \tau_2, \quad \beta(x, \frac{1}{2}r_0) \leq \tau_1
\]

hence we could apply all the work of preceding sections in \(B(x, \frac{1}{10}r_0)\) and conclude.

Note that for all \(x \in K \cap B(x_0, \frac{1}{10}r_0)\) we have
\[
\omega_2(x, \frac{1}{2}r_0) \leq 4\omega_2(x_0, r_0)
\]
\[
m(x, \frac{1}{2}r_0) \leq 2m(x_0, r_0)
\]
\[
\beta(x, \frac{1}{2}r_0) \leq 2\beta(x_0, r_0)
\]

in addition if \(\beta(x_0, r_0)\) is small enough then
\[
J^{-1}(x, \frac{1}{2}r_0) \leq 2J^{-1}(x_0, r_0).
\]

Finally, since \(h\) is non decreasing
\[
h\left(\frac{1}{2}r_0\right) \leq h(r_0).
\]

We deduce that for \(i \in [1, 4]\) we can set
\[
\tau'_i := \frac{1}{4} \tau_i
\]

and the proposition follows.

**Corollary 40.** In the same situation as in proposition before, if \(\tau_1\) is small enough we can choose
\[
F(x_0, \frac{1}{10}r_0) = K \cap \overline{B(x_0, \frac{1}{10}r_0)}.
\]
Proof: The method is to prove that $K$ is separating in $B(x_0, \frac{1}{10} r_0)$. This will show that we can take $F = K$ in this ball. To show that $K$ is separating we will apply Theorem 1.1 of [DPT], even if we could prove the same result without using [DPT] but with a longer explication. The main point is to show that for all $x \in B(x_0, \frac{1}{10} r_0)$ and for all $r$ such that $B(x, r) \subset B(x_0, \frac{1}{5} r_0)$ there is a cone $Z(x, r)$ such that

$$D_{x,r}(K, P(x, r)) \leq \varepsilon'$$

with $\varepsilon'$ a certain constant given by Theorem 1.1. of [DPT]. Recall that according to the notations of [DPT], $D_{x,r}$ is the Hausdorff distance

$$D_{x,r}(E, F) := \frac{1}{r} \max \left\{ \sup_{z \in E \cap B(x, r)} \{d(z, F)\}, \sup_{z \in F \cap B(x, r)} \{d(z, E)\} \right\}. \quad (60)$$

If we choose $\tau_1$ small enough compared to $\varepsilon'$ we know that for all $x$ and for all $r$ we have $\beta(x, r) \leq \varepsilon'$ by the preceding proposition. Hence we can find a cone $Z(x, r)$ that satisfy the first half of $D(x, r)$. We have to show now that

$$\sup \{d(z, K), z \in Z(x, r)\} \leq r\varepsilon'.$$

We know that $J(x, r)^{-1} \leq \tau_4$ and $\omega_2(x, r) \leq \tau_3$. Thus there is a set $F(x, r)$ that is separating in $B(x, r)$ and such that

$$H^2(F(x, r) \cap K \cap B(0, r)) \leq C\omega_2(x, r)^{\frac{1}{2}} J(x, r)^{-1} \leq \frac{1}{8} \tau_3 r^2.$$

Then for all $z \in Z(x, r)$, we have

$$d(z, K) \leq d(z, y) + d(y, K)$$

with $y$ a point of $F(x, r)$ such that $d(z, F(x, r)) = d(z, y)$. If $\tau_1 \leq \frac{\varepsilon'}{2}$ we can suppose that $F(x, r) \subset \{y; d(y, Z) \leq r\varepsilon' \}$. Thus $d(z, y) \leq r\varepsilon'$. We claim that $d(y, K) \leq r\varepsilon'$. The argument is by contradiction. If it is not true, then $K \cap B(y, r\varepsilon') = \emptyset$. But $F(x, r)$ is included in $T := \{y; d(y, Z) \leq r\varepsilon' \}$. Let $A^k$ be the connected components of $B(y, r\varepsilon') \setminus T$. Then $F(x, r)$ separates the $A^k$ in $B(y, r\varepsilon')$, and the minimal set that have this property is a cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ of area greater than $C\varepsilon'^2 r^2$. On the other hand $H^2(F(x, r) \setminus K) \leq \tau_3 r^2$. Thus if $\tau_3$ is small enough compared to $\varepsilon'$ it is not possible, thus finally $d(y, K) \leq \frac{\varepsilon'}{2}$ and

$$D_{x,r}(K, P) \leq \varepsilon'.$$

Now Theorem 1.1 of [DPT] says that $K$ is containing the image of a minimal cone by a homeomorphism from $B(x_0, \frac{1}{10} r_0)$ to $B(x_0, \frac{1}{5} r_0)$. This proves that $K$ separates $D^+$ from $D^-$ in $B(x_0, \frac{1}{10} r_0)$. \hfill $\square$

**Theorem 41.** There is some absolute positive constants $\varepsilon$ and $c$ such that the following is true. Let $(u, K)$ be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$ with gauge function $h$, let $x \in K$ and $r$ be such that $B(x, r) \subset \Omega$ and

$$\omega_2(x, r) + \beta(x, r) + J(x, r)^{-1} + h(r) \leq \varepsilon$$

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where the best cone in $\beta(x,r)$ named $Z$ is of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ centered at $x$. Then there is a diffeomorphism $\phi$ of class $C^{1,\alpha}$ from $B(x,cr)$ to its image such that $K \cap B(x,cr) = \phi(Z) \cap B(x,cr)$.

**Proof**: We want to apply Corollary 12.25 of [DAVa] (or see Corollary 13).

Thus to prove Theorem 41, it suffice to show that $K \cap B(x,cr)$ is an almost minimal set that verify the hypothesis of Corollary 13. If $\varepsilon$ is small enough, all the quantities $\omega_2(x,r)$, $\beta(x,r)$, $J^{-1}(x,r)$ and $h(r)$ verify the hypothesis of Proposition 39. In addition, according to Proposition 35 (applied in $B(x,r(1 - \sqrt{\varepsilon})^{-1})$), $m(r)$ is also smaller that $\tau_2$. So we can apply the result of the preceding Propositions.

By Corollary 40, we know that $F = K$ in $B(x, \frac{1}{10}r)$. So we can apply Proposition 36 directly on $K$ (instead of $F$) and the monotonicity of $\omega$ and $m$ obtained in Proposition 39 shows that $K$ is an almost minimal set in $B(x, \frac{1}{10}c_{10}r)$ with gauge function

$$\hat{h}(t) := C \left( \frac{t}{r} \right)^b + \tilde{h}(t).$$

To conclude we have to verify (1). If $\varepsilon$ and $c$ are small enough we have that $\hat{h}(cr) \leq \varepsilon_1$ so we only have to control $f(x,r)$. To do this we can use the same argument as we used in Lemma 29. We use Lemma 26 to find a point $x$ of same type of cone $Z$ that define $f(x,r)$. Then we use the same competitor $L$ as in the proof of 29 that is $Z \cup M$ where $M$ is a small wall. We deduce a bound of $f$ by $\beta$. Thus if the $\tau_i$ are small enough compared to $\varepsilon_1$, (1) is verified hence the proof is achieved.

**Remark 42.** Constant $c$ in Theorem 41 is depending on $c_{10}$, $U$, $\alpha$, and other constants. Thus, constant $c$ is fairly small but one might give an explicit value by doing some long computations.

Now we want to prove that the conditions on $J$ and $\omega_2$ can be removed in Theorem 41 if we suppose that $c$ and $\varepsilon$ are a bit smaller. To begin, we have to use this following lemma.

**Lemma 43.** There is some absolute positive constants $\varepsilon_3$ and $\eta_1$ such that if $x \in K$, $B(x,r) \subset \Omega$,

$$\omega_2(x,r) + h(r) + \beta(x,r) \leq \varepsilon_3$$

then $J(x,r) \geq \eta_1$.

**Proof**: The proof is like Lemma 8 page 365 and Proposition 10 page 297 of [DAV05]. The generalization of these lemmas in higher dimension is not a problem by the same way as we have proved Lemma 17, Lemma 20 and Lemma 21.

About the normalized energy we also have this result that naturally comes from an argument with blow up limits. One can find a similar statement about dimension 2 in Lemma...
3 page 504 of [DAV05]. The proof is the same for the case of $Y$ and $T$ in $\mathbb{R}^3$ so it has been omitted here. Recall that $D_{x,r}$ is the normalized bilateral Hausdorff distance defined in (60).

**Lemma 44.** For each $\eta_2 > 0$ there is constants $\varepsilon_3$ and $a_0$ with the following property. Let $\Omega \subset \mathbb{R}^3$ and let $(u, K)$ be a Mumford-Shah minimizer in $\Omega$ with gauge function $h$. Let $x \in K$ and $r > 0$ be such that $B(x, r) \subset \Omega$. Suppose that $h(r) \leq \varepsilon_3$ and that we can find a cone $Z$ of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ centered at $x$ such that

$$D_{x,r}(K, Z) \leq \varepsilon_3.$$  

Then

$$\omega_2(x, a_0 r) \leq \eta_2.$$  

Now we can state the main theorem.

**Theorem 45.** There is some absolute positive constants $\varepsilon$ and $c$ such that the following is true. Let $(u, K)$ be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$ with gauge function $h$, let $x \in K$ and $r > 0$ be such that $B(x, r) \subset \Omega$ and $h(r) \leq \varepsilon$. Assume in addition that there is a cone $Z$ of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ centered at $x$ such that

$$D_{x,r}(K, Z) \leq \varepsilon.$$  

Then there is a diffeomorphism $\phi$ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image, such that $K \cap B(x, cr) = \phi(Z)$.

**Proof :** We have to control the normalized jump and then apply Theorem 41. Firstly, if $\varepsilon$ is small enough compared to $\varepsilon_3$ we can use Lemma 43 and obtain that

$$J(x, r) \geq \eta_1$$  

for a certain $\eta_1 > 0$. Then, by Lemma 21 we have, for $r' \leq r$,

$$J(x, r') \geq \left( \frac{r'}{r} \right)^{\frac{1}{2}} [J(x, r) - \omega_2(x, r)].$$  

If $\varepsilon$ is small enough compared to $\eta_1$, the quantity $J(x, r) - \omega_2(x, r)$ is positive. Then by a good choice of $r'$, and if $\varepsilon$ is small enough compared to $\frac{\eta_1}{r}$, we deduce that

$$J(x, r')^{-1} \leq \bar{\varepsilon}$$  

where $\bar{\varepsilon}$ is the constant of Theorem 41.

Now since $\varepsilon$ is still small as we want, we can assume that the cone in $\beta(x, r')$ is still centered near $x$ and in addition

$$\beta(x, r') + J(x, r')^{-1} + \omega_2(x, r') + h(r') \leq \bar{\varepsilon}.$$  

Then we apply Theorem 41 in $B(x, r')$ and the conclusion follows. $\square$

This is an example of statement in terms of functional $J$.  

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Corollary 46. There is some absolute positive constants \( \varepsilon \) and \( c \) such that the following is true. Let \( g \in L^\infty \) and \( \Omega \subset \mathbb{R}^3 \). There is a \( \tilde{r} \) that depends only on \( \|g\|_\infty \) such that for all pair \((u, K) \in A \) that minimize the functional

\[
J(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_{\Omega \setminus K} (u - g)^2 \, dx + H^1(K),
\]

for all \( x \in K \) and \( r < \tilde{r} \) such that there is a cone \( Z \) of type \( \mathbb{P}, \mathbb{Y} \) or \( \mathbb{T} \) centered at \( x \) with

\[
D_{x,r}(K, Z) \leq \varepsilon
\]

there is a diffeomorphism \( \phi \) of class \( C^{1,\alpha} \) from \( B(x, cr) \) to \( B(x, 10cr) \) such that \( K \cap B(x, cr) = \phi(Z) \cap B(x, cr) \).

Proof : We know by Proposition 7.8, p 46 of [DAV05] that \((u, K)\) is a Mumford-Shah minimizer with gauge function

\[
h(r) = C_N \|g\|_\infty^2 r
\]

where \( C_N \) depends only on dimension. The conclusion follows applying Theorem 45 in \( B(x, r) \) if we choose

\[
\tilde{r} = \frac{\tilde{\varepsilon}}{2C_N \|g\|_\infty^2}
\]

where \( \tilde{\varepsilon} \) is the constant of Theorem 45.

Now we want a statement with only a condition about energy. We begin by this following lemma (\( D_H \) denotes the Hausdorff distance).

Lemma 47. For every \( \eta_4 > 0 \) there exist a radius \( R > 1 \) and a \( \eta_3 > 0 \) such that for every Mumford-Shah minimizer \((u, K)\) in \( B(x, R) \subset \mathbb{R}^3 \) such that \( x \in K \) and

\[
\omega_2(x, R) + h(R) \leq \delta_3,
\]

there is a minimal cone \( Z \) of type \( \mathbb{P}, \mathbb{Y} \) or \( \mathbb{T} \) that contains \( x \) and such that

\[
D_H(K \cap B(0, 1), Z \cap B(0, 1)) \leq \delta_4.
\]

Proof : The argument is by compactness. If it is not true, then we can find a \( \eta_4 > 0 \) such that for all \( n > 0 \), there is a Mumford-Shah minimizer \((u_n, K_n)\) in \( B(x, n) \) such that

\[
\omega_2(x, n) + h(n) \leq \frac{1}{n^3}
\]

and

\[
\sup_Z D_H(K_n \cap B(0, 1), Z \cap B(0, 1)) \geq \eta_4
\]
where the supremum is taken over all minimal cones containing $x$. We let now tend $n$ to infinity. Since $(u_n, K_n)$ is a sequence of Mumford-Shah minimizers, with same gauge function $h_l(r) := \sup\{h(nr); n \geq l\},$ and such that

$$\int_{B(x,n)} |\nabla u|^2 \leq r \frac{1}{n} \leq C$$

by Proposition 37.8 of [DAV05] we can extract a subsequence such that $(u_{n_k}, K_{n_k})$ converges to $(u, K)$ in the following sense : $D_H(K_{n_k} \cap A, K \cap A)$ tends to 0 for every compact set $A$ in $\mathbb{R}^3$. Moreover for all connected component $\Omega$ of $\mathbb{R}^3 \setminus K$ and for all compact set $A$ of $\Omega$, there is a sequence $a_k$ such that $\{u_{n_k} - a_k\}_{k \in \mathbb{N}}$ converges to $u$ in $L^1(A)$. Then, using (61) and Proposition 37.18 of [DAV05], we know that for every ball $B \subset \mathbb{R}^3$,

$$\int_{B \setminus K} |\nabla u|^2 \leq \liminf_{k \to +\infty} \int_{B \setminus K_{n_k}} |\nabla u_n|^2 \leq \lim_{k \to +\infty} r \frac{1}{n_k} = 0.$$

Thus $\nabla u = 0$ and $u$ is locally constant. Finally, Theorem 38.3 of [DAV05] says that the limit $(u, K)$ is a Mumford-Shah minimizer with gauge function $h_l(4r)$. Since it is true for all $l$, and that $\sup h_l = 0$, we can suppose that $(u, K)$ is a Mumford-Shah minimizer with gauge function equal to zero, and $u$ is locally constant. But in this case we know by [DAVa] that $K$ is a minimal cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$, and since for all $n$, $K_n$ is containing $x$, it is still true for the limit $K$. In addition, there is a rank $L$ such that for all $k \geq L$ we have $D_H(K \cap B(0,1), K_{n_k} \cap B(0,1)) \leq \frac{\eta_4}{2}$ which is in contradiction with (62) and achieve the proof. 

Lemma 47 implies the following Theorem.

**Theorem 48.** There is some positive constants $\varepsilon$ and $c < 1$ such that the following is true. Let $(u, K)$ be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$ with gauge function $h$, let $x \in K$ and $r$ be such that $B(x, r) \subset \Omega$ and

$$\omega_2(x, r) + h(r) \leq \varepsilon.$$

Then there is a diffeomorphism $\phi$ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image, and there is a minimal cone $Z$ such that $K \cap B(x, Cr) = \phi(Z) \cap B(x, cr)$.

**Proof:** Denote by $\varepsilon$ the constant of Theorem 45. We apply Lemma 47 to $(u, K)$ with $\eta_4 = \varepsilon$. We know that there is a constant $c < 1$ and there is a cone $Z$ that contains $x$ such that

$$D_{x, cr}(Z, K) \leq \varepsilon.$$

Dividing if necessary $c$ by 16 we may assume that the center of the cone lies in $\frac{1}{2}B(x, cr)$. Thus there is an $y \in B(x, cr) \supset \frac{1}{2}B(x, cr)$ such that, possibly taking a smaller $\varepsilon$,

$$D_{y, cr}(Z, K) + \omega_2(y, cr) + h(r) \leq \varepsilon$$

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and then we can apply Theorem 45 in $B(y, c^2 r)$, and the conclusion follows.

By the same way of Corollary 46 in terms of functional $J$ we have the following statement.

**Corollary 49.** There exist some positive constants $\varepsilon$ and $c$ such that the following is true. Let $g \in L^\infty$ and $\Omega \subset \mathbb{R}^3$. There is a $\tilde{r}$ depending only on $\|g\|_\infty$, such that for all pair $(u, K) \in A$ that minimizes

$$ J(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_{\Omega \setminus K} (u - g)^2 \, dx + H^2(K), $$

for all $x \in K$ and $r < \tilde{r}$ such that

$$ \omega_2(x, r) \leq \varepsilon $$

there is a diffeomorphism $\phi$ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$.

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ADDRESS:
Antoine LEMENANT
e-mail : antoine.lemenant@math.u-psud.fr
Université Paris XI
Bureau 15 Bâtiment 430
ORSAY 91400 FRANCE
Tél: 00 33 169157951