Theoretical studies of plain hinged linkage mechanisms movement in food processing robots-manipulators

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Abstract. Significance of the studied problem is conducted due to the lack of simple and accessible engineering methods that allow predicting kinematic characteristics and selecting parameters of the main elements of modern robotic manipulators that include mechanisms of parallel structure in their design. The solution of such tasks allows determining the ranges of operational indicators of the designed technological equipment in advance, as well as to develop criteria for their analysis. An algorithm for applying the Lagrange II equations to determine the equations of plain hinged linkage mechanisms movement with four degrees of freedom is proposed.

1. Introduction

The modular method design of the basic mechanisms of technological equipment, such as robot manipulators used in food production, based on groups of normalized joint units has unmatched advantages in comparison with traditional design methods. The main of which are:

- the possibility to design the specialized standard units without redundant functions, i.e. designed to solve a specific technological problem;
- reducing of the design process complexity by predicting of kinematic characteristics and selecting needed parameters of the main elements of mechanisms;
- increasing of the unit reliability due to the complete compliance of the mechanism design with the function that is performed for;
- intensification of processes for the finish product obtaining [1-5, 7].

The main functional indicators of technological equipment, such as robot manipulators, are the load capacity, mobility, and, above all, its positioning error. One of the methods oriented for reducing the last indicator is the parallel structure mechanisms using.

However, the complexity of their design prevents the widespread use of such systems. In this regard, the development of an engineering method used for kinematic characteristics predicting and parameters of the main design elements of modern parallel structure mechanisms selecting at the early stages of their design is an urgent scientific and practical task [12,14,15].
2. Formulation of the problem

Figure 1 shows a general schematic diagram of a plain four-degree-of-freedom hinged linkage mechanism commonly used for the robotic arms development in food processing.

![General diagram of a plain hinged linkage mechanism](image)

**Figure 1.** General diagram of a plain hinged linkage mechanism

The model of a plain hinged linkage mechanism, which has four degrees of freedom, in general form can be represented as a mechanical system consisting of:

- link 1, performing rotary movement in the horizontal plane relative to the point A, moving in a straight line;
- link 2 pin-connected to the link 1 in the point B performing translational movement with it and rotation relative to the point B;
- link 3, pin-connected to the link 2 in the point C, and performing translational movement with it and rotation relative to the point C.

The problem formulation is to determine the algorithm for any link position finding for the plain hinged linkage mechanism with four degrees of freedom, depending on the time of its movement from mechanism rest.

3. The proposed algorithm

It is necessary to equation obtain that allow to unambiguously determine the position of the mechanism as a time function. The following algorithm is proposed for that purpose:

- the independent kinematic parameters determination for every links of the mechanism;
- to draw up a system of equations containing a unified characteristic of the links system in the form of the mechanism kinetic energy[6,8-11];
- to compose a mathematical expression for the mechanism kinetic energy in general form, as a function of the joint coordinate system;
- to develop a mathematical model in the form of a system of differential equations describing the laws of movement of plain hinged linkage mechanisms with four freedom degrees;
- to find a solution to the obtained system of differential equations that is sufficiently accurate for engineering problems for preliminary design, which makes it possible to establish the laws of motion for individual links of the same kinds of mechanisms.
4. General solution to the problem

The position of link 3 at any time is determined by four independent parameters - coordinates $x_A$, $\varphi_1$, $\varphi_2$, $\varphi_3$. Consequently, a mechanical system, for the first approximation subordinate to ideal, and confining and holonomic constraints, has four degrees of freedom and four Lagrange equations can be written for it in the following general form [3,11,16]:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial X_A} \right) - \left( \frac{\partial T}{\partial X_A} \right) = Q$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \varphi_1} \right) - \left( \frac{\partial T}{\partial \varphi_1} \right) = Q_1$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \varphi_2} \right) - \left( \frac{\partial T}{\partial \varphi_2} \right) = Q_2$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \varphi_3} \right) - \left( \frac{\partial T}{\partial \varphi_3} \right) = Q_3$$

where: $X_A$, $\varphi_{1-3}$ – joint system coordinates; $\dot{X}_A, \dot{\varphi}_{1-3}$ – time derivatives of joint coordinates (generalized velocities); $T$ – kinetic energy of the system, expressed in terms of joint coordinates and generalized velocities; $Q, Q_1, Q_2, Q_3$ – generalized forces.

Let’s write an expression for the system kinetic energy $T$, where all the variable quantities included in it are given in joint coordinates and generalized velocities. The kinetic energy of the entire system can be represented by the corresponding sum [1-5]:

$$T_{mech} = T_1 + T_2 + T_3$$

where: $T_1, T_2$ and $T_3$ – kinetic energies of the mechanical system links.

Link 1 moves at speed (portable) $V_A$ and performs rotational (relative) movement in the horizontal plane (with an angular velocity $\omega_{BA}$) relative to the point A. Consequently, the AB link makes a plane-parallel movement. Then the kinetic energy of link 1 in general form can be represented as:

$$T_1 = \frac{m_1 V_A^2}{2} + I_1 \frac{\omega_{BA}^2}{2}$$

where $V_A$ is the time derivative of the joint coordinate $X_A$ and it is the generalized speed $\dot{X}_A = \frac{dX_A}{dt}$.

$\omega_{BA}$ - respectively, the time derivative of the joint coordinate $\varphi_1$, therefore, is the generalized velocity $\dot{\varphi}_1 = \frac{d\varphi_1}{dt}$ (the angular velocity of the link 1). Thus:

$$T_1 = \frac{m_1 \dot{X}_A^2}{2} + I_1 \frac{\dot{\varphi}_1^2}{2}$$

For the purposes of the mechanism kinematic study, the AB link can be considered as a homogeneous rod, then taking into account its moment of inertia, and the expression (6) can be rewritten in the following form:

$$T_1 = \frac{m_1}{2} \left( \dot{X}_A^2 + \frac{AB^2 \cdot \dot{\varphi}_1^2}{3} \right)$$

Since the plane-parallel movement of a rigid body can be considered as the sum of its two simple motions: translational movement together with the pole and rotation around the pole [1,2,17,18], then,
based on this, link 2 moves together with link AB at a speed (portable) $\vec{V}_0^B$ and at the same time performs rotational (relative) motion in the horizontal plane (with an angular velocity $\omega_{BC}$) relative to the point B. Consequently, the link BC makes a plane-parallel movement. Then the kinetic energy of link 2 in general form can be represented as:

$$T_2 = \frac{m_2 \vec{V}_B^2}{2} + I_2 \cdot \frac{\omega_{BC}^2}{2}; \quad (9)$$

Point B, which belongs to the AB link, makes a complex movement (Figure 2). Its absolute speed will be determined by the vector expression:

$$\vec{V}_B = \vec{V}_A + \vec{V}_{BA} \quad (10)$$

The velocity $V_B$ is found from the algebraic equation:

$$V_B^2 = V_A^2 + V_{BA}^2 + 2 \cdot V_A \cdot V_{BA} \cos(90 - \phi_1) \quad (11)$$

since $\cos(90° - \varphi) = \sin \varphi$, then

$$V_B^2 = V_A^2 + V_{BA}^2 + 2 \cdot V_A \cdot V_{BA} \sin \phi_1 \quad (12)$$

where: $\phi_1$ - the value of the joint coordinate at a given time.

![Figure 2. Diagram for the speed of point B determination](image)

$\omega_{BC}$ is a time derivative of the joint coordinate $\phi_2$ and accordingly, it is the generalized speed $\dot{\phi}_2 = \frac{d\phi_2}{dt}$ (angular velocity of the link 2).

Thus:

$$T_2 = \frac{m_2 \dot{V}_B^2}{2} + I_2 \cdot \frac{\dot{\phi}_2^2}{2} \quad (13)$$

where it can be accepted that $I_2 = \frac{m_2 l_B^2}{12}$.

For the purposes of the kinematic study of the mechanism, the BC link can be considered a homogeneous rod, then taking into account the moment of inertia, the expression (13) can be rewritten in the following form:

$$T_2 = \frac{m_2}{2} (V_B^2 + \frac{BC^2 \dot{\phi}_2^2}{12}); \quad (14)$$

By delivering (10) to (13) and taking into account that $V_{BA} = \omega_{BA} \cdot BA = \phi_1 \cdot BA$, it gets to the next form:

$$T_2 = \frac{m_2}{2} \left[ \dot{X}_A^2 + (\phi_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \phi_1 + \frac{BC^2 \dot{\phi}_2^2}{12} \right] \quad (15)$$

By the similar way, considering the movement of link 3 in the general case as complex, in accordance with [1,2,5,14,15] we can assume that the link 3 moves along with the aircraft link at a speed of
(portable) $\overrightarrow{V_C}$ and simultaneously performs a rotational (relative) movement in a horizontal plane (with angular velocity $\omega_{CD}$) relative to the point C. Therefore, the CD link performs a plane-parallel movement. Then the kinetic energy of link 3 can be represented as:
\[
T_3 = m_3 \frac{V_C^2}{2} + I_3 \frac{\omega_{CD}^2}{2} \tag{16}
\]
assuming that $I_3 = \frac{m_3 l_3^2}{12}$; then
\[
T_3 = m_3 \frac{V_C^2}{2} + \frac{m_3 l_3^2}{12} \frac{\omega_{CD}^2}{2}. \tag{17}
\]

The point C belonging to the CB link performs a complex movement (figure 3). Its absolute speed will be determined by the vector expression:
\[
\overrightarrow{V}_C = \overrightarrow{V}_B + \overrightarrow{V}_{CB} \tag{18}
\]

The value of the velocity $V_C$ is found from the algebraic equation:
\[
V_C^2 = V_B^2 + V_{CB}^2 + 2 \cdot V_B \cdot V_{CB} \cdot \cos \varphi_2 \tag{19}
\]

where: $\varphi_3$ - is the value of the joint coordinate at a given time.

\[\begin{align*}
V_B^2 &= V_A^2 + V_{BA}^2 + 2 \cdot V_A \cdot V_{BA} \sin \varphi_1; \\
V_C^2 &= V_B^2 + V_{CB}^2 + 2 \cdot V_B \cdot V_{CB} \cdot \cos \varphi_2
\end{align*}\]

**Figure 3.** Diagram for the point C velocity determination

Taking into account (11) and taking into account that $V_{CB} = \omega_{BC} \cdot BC = \varphi_2 \cdot BC$, gets an expression for determining the point C velocity:
\[
V_C^2 = \dot{X}_A^2 + \dot{\phi}_1^2 \cdot BA^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \varphi_1 + \dot{\phi}_2^2 \cdot BC^2 + 2 \cdot \left(\sqrt{\dot{X}_A^2 + \dot{\phi}_1^2 \cdot BA^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \varphi_1}\right) \cdot \phi_2 \cdot BC \cdot \cos \varphi_2 \tag{20}
\]

For the purposes of the mechanism kinematic study, the CD link can be considered as a homogeneous rod, then taking into account its moment of inertia, equation (20) can be rewritten as follows:
\[
T_3 = \frac{m_3}{2} (V_C^2 + \frac{CD^2 \phi_2^2}{12}) \tag{21}
\]

By delivering (20) to (21) and taking into account that $V_{CB} = \omega_{BC} \cdot BC = \varphi_2 \cdot BC$, it gets to the next form:
\[
T_3 = \frac{m_3}{2} \left[\dot{X}_A^2 + \dot{\phi}_1^2 \cdot BA^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \varphi_1 + \dot{\phi}_2^2 \cdot BC^2 + 2 \cdot \left(\sqrt{\dot{X}_A^2 + \dot{\phi}_1^2 \cdot BA^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \varphi_1}\right) \cdot \phi_2 \cdot BC \cdot \cos \varphi_2 + \frac{CD^2 \phi_2^2}{12}\right] \tag{22}
\]
Taking into account the obtained results, the expression (5) for determining the mechanism kinetic energy takes the following form:

\[
T_{\text{mech}} = \frac{m_2}{2} \left( \dot{X}_A^2 + \frac{AB^2 \dot{\phi}_2^2}{3} \right) + \frac{m_2}{2} \left( \ddot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 + \frac{BC^2 \cdot \dot{\phi}_3^2}{12} \right) + \frac{m_3}{2} \left( \ddot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 + \dot{\phi}_2^2 \cdot BC^2 + 2 \cdot \sqrt{\left( \dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 \right) \cdot \dot{\phi}_2 \cdot BC \cdot \cos \phi_2 + \frac{CD^2 \cdot \dot{\phi}_3^2}{12}} \right)
\]

Find expressions for partial derivatives in equations (1-4):

\[
\frac{\partial T}{\partial \dot{X}_A} = \left( m_1 + m_2 + m_3 \right) \cdot \dot{X}_A + (m_2 + m_3) \cdot BA \cdot \dot{\phi}_1 \cdot \sin \phi_1 + m_3 \cdot \left( \frac{(\dot{\phi}_1 \cdot BA \cdot \sin \phi_1) \dot{\phi}_2 \cdot BC \cdot \cos \phi_2}{\left( \dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 \right)} \right)
\]

\[
\frac{\partial T}{\partial \dot{\phi}_1} = \left( m_3 + m_2 + m_3 \right) \cdot \dot{\phi}_1 \cdot BA^2 + (m_2 + m_3) \cdot BA \cdot \dot{X}_A \cdot \sin \phi_1 + m_3 \cdot \left( \frac{(\dot{\phi}_1 \cdot BA^2 + \dot{X}_A \cdot \sin \phi_1) \dot{\phi}_2 \cdot BC \cdot \cos \phi_2}{\left( \dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 \right)} \right)
\]

\[
\frac{\partial T}{\partial \dot{\phi}_2} = \left( \frac{m_2}{12} + m_3 \right) \cdot \dot{\phi}_2 \cdot BC^2 + m_3 \cdot \left( \frac{\dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1}{\left( \dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 \right)} \right) \cdot BC \cdot \cos \phi_2
\]

\[
\frac{\partial T}{\partial \dot{\phi}_3} = \frac{m_3 \cdot CD^2 \cdot \dot{\phi}_3}{12}
\]

\[
\frac{\partial T}{\partial X_A} = 0
\]

\[
\frac{\partial T}{\partial \phi_1} = (m_2 + m_3) \cdot BA \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot \cos \phi_1 + m_3 \cdot \left( \frac{\dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \cos \phi_1}{\left( \dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 \right)} \right)
\]

\[
\frac{\partial T}{\partial \phi_2} = -m_3 \cdot \left( \frac{\dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1}{\left( \dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 \right)} \right) \cdot BC \cdot \sin \phi_2
\]

\[
\frac{\partial T}{\partial \phi_3} = 0
\]

Substituting (24-31) into equations (1-4) and notation making

\[
K_i := \left[ \frac{\dot{X}_A + \dot{\phi}_1 \cdot BA \cdot \cos \phi_1 - \phi_1 \cdot \dot{\phi}_1 \cdot BA \cdot \sin \phi_1 \cdot \dot{\phi}_2 \cdot BC \cdot \cos \phi_2}{\sqrt{(\dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \cos \phi_1)}} \right] + \left[ \frac{\dot{X}_A + \dot{\phi}_1 \cdot BA \cdot \cos \phi_1 \cdot \dot{\phi}_2 \cdot BC \cdot \cos \phi_2}{\sqrt{(\dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \cos \phi_1)}} \right] + \left[ \frac{\dot{X}_A + \dot{\phi}_1 \cdot BA \cdot \cos \phi_1 \cdot \dot{\phi}_2 \cdot BC \cdot \cos \phi_2}{\sqrt{(\dot{X}_A^2 + (\dot{\phi}_1 \cdot BA)^2 + 2 \cdot \dot{X}_A \cdot \dot{\phi}_1 \cdot BA \cdot \cos \phi_1)}} \right]
\]
\[ \begin{align*}
&+ \frac{\left( \dot{X}_A + \phi_1 \cdot BA \cdot \cos \phi_1 \right) \cdot \dot{\phi}_2 \cdot \phi_2 \cdot BC \cdot \sin \phi_2}{\sqrt{\left( X_A^2 + \left( \dot{X}_A + \phi_1 \cdot BA \right)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 \right)}} \times \\
&\times \frac{\left( \dot{X}_A \cdot \dot{X}_A + \dot{X}_A \cdot \phi_1 + \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 + \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \phi_1 - \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \phi_1 \right)}{\sqrt{\left( X_A^2 + \left( \phi_1 \cdot BA \right)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 \right)}} \times \\
&\times \frac{\left( \dot{X}_A \cdot \dot{X}_A + \phi_1 \cdot BA \cdot \cos \phi_1 \right) \cdot \phi_2 \cdot BC \cdot \cos \phi_2}{\sqrt{\left( X_A^2 + \left( \phi_1 \cdot BA \right)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 \right)}}.
\end{align*} \]

\[ K_2 = \left( \frac{\dot{X}_A + \dot{X}_A \cdot \phi_1 - \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \phi_1}{\sqrt{\left( \dot{X}_A^2 + \left( \dot{X}_A + \phi_1 \cdot BA \right)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 \right)}} \right) \times \\
\times \left( \frac{\phi_1 \cdot BA \cdot \cos \phi_1}{\sqrt{\left( \phi_1 \cdot BA \cdot \cos \phi_1 \right)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1}} \right) \times \\
\times \left( \frac{\phi_2 \cdot BC \cdot \cos \phi_2 + \left( \phi_1 \cdot BA^2 + \dot{X}_A \cdot BA \cdot \cos \phi_1 \right) \cdot \phi_2 \cdot BC \cdot \cos \phi_2}{\sqrt{\left( \phi_1 \cdot BA^2 + \dot{X}_A \cdot BA \cdot \cos \phi_1 \right)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1}} \right) \times \\
\times \left( \frac{\left( \dot{X}_A \cdot \dot{X}_A + \phi_1 \cdot \phi_1 + \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 + \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 - \dot{X}_A \cdot \phi_1 \cdot BA \cdot \sin \phi_1 \right)}{\sqrt{\left( X_A^2 + \left( \phi_1 \cdot BA \right)^2 + 2 \cdot \dot{X}_A \cdot \phi_1 \cdot BA \cdot \cos \phi_1 \right)}} \right). \]
The final form can be given in the next view:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_A} \right) - \frac{\partial T}{\partial x_A} = (m_1 + m_2 + m_3) \cdot \ddot{x}_A + \left( m_2 + m_3 \right) \cdot BA \cdot \left( \ddot{\phi}_1 \cdot \cos \varphi_1 - \ddot{\phi}_2 \cdot \sin \varphi_1 \right) + m_3 \cdot K_1 = Q \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}_1} \right) - \frac{\partial T}{\partial \phi_1} = \left( \frac{m_1}{12} + m_2 + m_3 \right) \cdot \ddot{\phi}_1 \cdot BA^2 + \left( m_2 + m_3 \right) \cdot BA \cdot \left( \ddot{x}_A \cdot \cos \varphi_1 - \dot{x}_A \cdot \dot{\phi}_1 \cdot \sin \varphi_1 \right) + m_3 \cdot K_2 = Q_1 \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}_2} \right) - \frac{\partial T}{\partial \phi_2} = (m_2 + m_3) \cdot \ddot{\phi}_2 \cdot BC^2 + m_3 \cdot K_3 = Q_2 \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}_3} \right) - \frac{\partial T}{\partial \phi_3} = \frac{m_3 \cdot CD^2 \cdot \ddot{\phi}_3}{3} = Q_3
\]

The established expressions (35-38) are the mathematical model that generally describes the laws of motion in the form of Lagrange equations of the second kind of plane hinged linkage mechanisms with four degrees of freedom. To simplify expressions (35-38), additional notation can be introduced:

\[
m_1 + m_2 + m_3 = a \\
(m_2 + m_3) \cdot BA = b \\
\left( \frac{m_1}{12} + m_2 + m_3 \right) \cdot BA^2 = c \\
(m_2 + m_3) \cdot BC^2 = d; \\
f = \frac{m_3 \cdot t^2}{3}
\]

Substituting (38-42) in (34-37), the following system of differential equations can be obtained:

\[
a \cdot \ddot{x}_A + b \cdot \left( \ddot{\phi}_1 \cdot \cos \varphi_1 - \ddot{\phi}_2 \cdot \sin \varphi_1 \right) + m_3 \cdot K_1 = Q \\
c \cdot \ddot{\phi}_1 + b \cdot \left( \ddot{x}_A \cdot \cos \varphi_1 - \dot{x}_A \cdot \dot{\phi}_1 \cdot \sin \varphi_1 \right) + m_3 \cdot K_2 = Q_1 \\
d \cdot \ddot{\phi}_2 + m_3 \cdot K_3 = Q_2 \\
f \cdot \ddot{\phi}_3 = Q_3
\]

The generalized forces \( Q \) must be defined at the next step. Doing that the possible and independent elementary displacements \( \delta q \), for every coordinates \( \delta x_A, \delta \phi_1, \delta \phi_2, \delta \phi_3 \) must be set. Let’s give the system
consecutive elementary moves $\delta x_A \neq 0$ by $\delta \phi_1 = 0$, and $\delta \phi_1 \neq 0$ by $\delta x_A = 0$ respectively. At that condition the following cases are possible when the forces acting on the mechanism will be reduced to one of its links (figure 4):

a) by $\delta x_A \neq 0$ and $\delta \phi_1 - 3 = 0$ forces are reduced to the resultant $R = N$, then $Q_1 = N$;

b) by $\delta \phi 1 \neq 0$, $\delta \phi_2,3=0$ and $\delta x_A = 0$ forces are also reduced to a single resultant, similarly $Q_2 = N$;

c) if there are elastic forces in the system ($F_{elas} = C \cdot \Delta S$) and $\delta \phi_2 \neq 0$, $\delta \phi_1,3=0$ by $\delta x_A = 0$ the value of the generalized force can be represented as $Q_2 = R + F_{elas}$;

d) by $\delta \phi_3 \neq 0$, $\delta \phi_1,2=0$ and $\delta x_A = 0$ forces are also reduced to a single resultant, which can be expressed as $Q_3 = M$ as a moment of forces that is caused by the action $F_{res.}$, attached at the point $D$.

![Figure 4. Scheme for generalized forces $Q_i$ determination](image)

The possible numerical values of the coefficients $K_{1-3}$ should be determined for the purposes of obtained system of differential equations (44-47) practical using. Let’s introduced the new notations that do not affect the results of the study, but would simplify mathematical expressions:

$$K_1 = \frac{P - Q - G}{W},$$

where:

$$P = 2 \cdot AB \cdot BC \cdot x^2 \cdot y \cdot \cos \phi_1 \cdot \cos \phi_2 \cdot (z^2 + (x \cdot BA)^2 + 2 \cdot z \cdot x \cdot BA \cdot \cos \phi_1)$$

$$Q = 2 \cdot (z + AB \cdot x \cdot \sin \phi_1) \cdot y^2 \cdot BC \cdot \sin \phi_2 \cdot (z^2 + (x \cdot BA)^2 + 2 \cdot z \cdot x \cdot BA \cdot \cos \phi_2)$$

$$G = (z + AB \cdot x \cdot \sin \phi_1) \cdot z \cdot y \cdot x^2 \cdot AB \cdot BC \cdot \cos \phi_1 \cdot \sin \phi_4$$

$$W = (z^2 + (x \cdot BA)^2 + 2 \cdot z \cdot x \cdot BA \cdot \cos \phi_1)^{1.5}$$

Substituting the numerical values of the mechanism geometric dimensions into expressions (48-51) and setting its initial kinematic parameters in the practically used ranges, it is clear that the last terms of the equations (44-46), in comparison with the rest in absolute value, represent values of a lower order. That is the case when the resulting differential equations (48-51) can be written in a simplified form:
\[
a \cdot \dddot{X}_A + b \cdot (\ddot{\varphi}_1 \cdot \cos \varphi_1 - \dot{\varphi}_1^2 \cdot \sin \varphi_1) = N \tag{52}
\]
\[
c \cdot \dddot{\varphi}_1 + b \cdot (\dddot{X}_A \cdot \cos \varphi_1 - \dddot{X}_A \cdot \dot{\varphi}_1 \cdot \sin \varphi_1) = N \tag{53}
\]
\[
d \cdot \ddot{\varphi}_2 = N + C \cdot l \cdot \varphi_2 \tag{54}
\]
\[
f \cdot \dot{\varphi}_3 = F_{\text{res}} \tag{55}
\]

where: \( l \) - the reduced linear size of the link 2; \( C \) is a reduced elasticity coefficient of the elements installed on the mechanism.

5. Discussion

To solve the resulting system of differential equations (52-55), the followed additional notation can be introduced:

\[
\frac{b}{a} = A; \quad \frac{b}{c} = B; \quad \frac{N}{a} = D; \quad \frac{N}{c} = R; \tag{56}
\]

After substituting (56) in (52) and (53), the new view of equation system can be obtained:

\[
\begin{align*}
\dddot{X}_A + A \cdot \dddot{\varphi}_1 \cdot \cos \varphi_1 - A \cdot \dddot{\varphi}_1^2 \cdot \sin \varphi_1 &= D \\
\dddot{\varphi}_1 + B \cdot \dddot{X}_A \cdot \cos \varphi_1 - B \cdot \dddot{X}_A \cdot \dot{\varphi}_1 \cdot \sin \varphi_1 &= R
\end{align*} \tag{57}
\]

Take into account that:

\[
\frac{d(\ddot{\varphi}_1 \cdot \cos \varphi_1)}{dt} = \dddot{\varphi}_1 \cdot \cos \varphi_1 - \dddot{\varphi}_1^2 \cdot \sin \varphi_1 \tag{58}
\]

\[
\frac{d(\ddot{X}_A \cdot \cos \varphi_1)}{dt} = \dddot{X}_A \cdot \cos \varphi_1 - \dddot{X}_A \cdot \dot{\varphi}_1 \cdot \sin \varphi_1 \tag{59}
\]

Then the system of differential equations (57) can be rewritten in the following form:

\[
\dddot{X}_A + A \cdot \frac{d(\ddot{\varphi}_1 \cdot \cos \varphi_1)}{dt} = D \tag{60}
\]

\[
\dddot{\varphi}_1 + B \cdot \frac{d(\ddot{X}_A \cdot \cos \varphi_1)}{dt} = R \tag{61}
\]

After integration, it will get into the next form:

\[
\dddot{X}_A + A \cdot \dddot{\varphi}_1 \cdot \cos \varphi_1 = D \cdot t + C_1 \tag{62}
\]

\[
\dddot{\varphi}_1 + B \cdot \dddot{X}_A \cdot \cos \varphi_1 = R \cdot t + C_2 \tag{63}
\]

Substituting the given initial conditions:

\( C_1 = 0 \) и \( C_2 = 0 \).
and multiply (63) by $A \cdot \cos \phi_1$ and the result is subtracted from (62). By the way, the equation (62) multiplied by $B \cdot \cos \phi_1$ and accordingly subtract from (63), and finally the following views will be obtained:

$$X' - A \cdot B \cdot X' \cdot \cos^2 \phi_1 = D \cdot t - R \cdot t \cdot A \cdot \cos \phi_1$$  \hspace{1cm} (64)

$$\dot{\phi}_1 - A \cdot B \cdot \dot{\phi}_1 \cdot \cos^2 \phi_1 = R \cdot t - D \cdot t \cdot \cos \phi_1$$  \hspace{1cm} (65)

From (64, 65) it follows that:

$$\dot{X}_A = \frac{(D - R \cdot A \cdot \cos \phi_1) \cdot t}{1 - A \cdot B \cdot \cos^2 \phi_1}$$  \hspace{1cm} (66)

$$\dot{\phi}_1 = \frac{(R - D \cdot B \cdot \cos \phi_1) \cdot t}{1 - A \cdot B \cdot \cos^2 \phi_1}$$  \hspace{1cm} (67)

at the same time let’s assume that $\cos^2 \phi_1 \neq \frac{1}{A \cdot B}$ и $0 < |A \cdot B| < 1$

It is obvious that the system (66,67) has a trivial solution when:

$$\left\{ \begin{array}{l} \Phi_1 = 0 \\ D \cdot B = R \end{array} \right.$$  \hspace{1cm} (68)

where from

$$\dot{X}_A = \frac{D - R \cdot A}{1 - A \cdot B} \cdot t$$

To find the exact solution of the system (66,67), it is assumed that:

$$R - D \cdot B \cdot \cos \phi_1 \neq 0$$

The system (57) for $t=0$ under initial conditions can be written as follows:

$$X'(0) + A \cdot \dot{\phi}_1(0) = D$$  \hspace{1cm} (69)

$$\dot{\phi}_1(0) + B \cdot \dot{X}_A(0) = R$$  \hspace{1cm} (70)

where from $|A \cdot B| < 1$:

$$\dot{X}_A(0) = \frac{D - R \cdot A}{1 - A \cdot B}$$  \hspace{1cm} (71)

$$\dot{\phi}_1(0) = \frac{R - D \cdot B}{1 - A \cdot B}$$  \hspace{1cm} (72)

Let’s integrate the equation (67) by separating of the variables:

$$\int \frac{d\phi_1(1 - A \cdot B \cdot \cos^2 \phi_1)}{R - D \cdot B \cdot \cos \phi_1} = \frac{t^2}{2}$$  \hspace{1cm} (73)

and so on:
\[
\varphi_1 \frac{d\varphi_1}{R - D \cdot B \cdot \cos \varphi_1} - A \cdot B \int_{\varphi_0}^{\varphi_1} \frac{(1 - \sin^2 \varphi_1)}{R - D \cdot B \cdot \cos \varphi_1} = \frac{t^2}{2}
\]

(73)

then:

\[
\left( \int_{\varphi_0}^{\varphi_1} \frac{d\varphi_1}{R - D \cdot B \cdot \cos \varphi_1} \right) (1 - A \cdot B) - A \cdot B \int_{\varphi_0}^{\varphi_1} \frac{\sqrt{(1 - \cos^2 \varphi_1)} \, d\cos \varphi_1_{11}}{R - D \cdot B \cdot \cos \varphi_1} = \frac{t^2}{2}
\]

(74)

The next notation for the integrals should be introduced:

\[
I_1(\varphi_1) = \int_{\varphi_0}^{\varphi_1} \frac{d\varphi_1}{R - D \cdot B \cdot \cos \varphi_1}
\]

(75)

\[
I_2(\varphi_1) = \int_{\varphi_0}^{\varphi_1} \frac{\sqrt{(1 - \cos^2 \varphi_1)} \, d\cos \varphi_1_{11}}{R - D \cdot B \cdot \cos \varphi_1}
\]

(76)

Thus, the solution of equation (67) can be represented in the following compact form:

\[
(1 - A \cdot B) \cdot I_1(\varphi_1) - A \cdot B \cdot I_2(\varphi_1)
\]

(77)

In accordance with the recommendations [4] it can be written:

\[
I_1(\varphi_1) = \frac{t^2}{2},
\]

(78)

\[
I_2(\varphi_1) = - \frac{1}{D \cdot B} \int_{1}^{\cos \varphi_1} \int_{1}^{\cos \varphi_1} \frac{\sqrt{(1 - X_A^2)} \, dX_A}{(X_A - \frac{R}{D \cdot B} \cdot \cos \varphi_1)} = - \frac{1}{D \cdot B} \int_{1}^{\cos \varphi_1} \sqrt{R} \, dX_A = 
\]

\[
= - \frac{1}{D \cdot B} \left[ \frac{X_A \cdot dX_A}{\sqrt{1 - X_A^2}} - \frac{R}{D \cdot B} \cdot \int_{1}^{\cos \varphi_1} \frac{dX_A}{\sqrt{1 - X_A^2}} \int_{1}^{\cos \varphi_1} \left( 1 - \frac{R^2}{D^2 \cdot B^2} \right) \right. 
\]

\[
\left. \left. + \frac{dX_A}{(X_A + p) \sqrt{1 - X_A^2}} \right] \right] 
\]

(79)

\[
= - \frac{1}{D \cdot B} \left[ \sin \varphi_1 - \frac{R}{D \cdot B} \cdot \varphi_1 + \left( \frac{R^2}{D^2 \cdot B^2} - 1 \right) \cdot I_3 \right]
\]

where: \( p = \frac{R}{D \cdot B}, |p| > 1, R = 1 - X_A^2 \), \( t = \frac{1}{X_A + p} \)

\[
I_3 = \int_{(1 + p)^{-1}}^{(\cos \varphi_1 + p)^{-1}} \frac{dt}{\sqrt{-1 + 2 \cdot p \cdot t + (1 - p^2) \cdot t^2}} = 
\]

\[
= \frac{1}{\sqrt{p^2 - 1}} \cdot \arcsin \left( (1 - p^2) \cdot t + p \right) \left| (\cos \varphi_1 + p)^{-1} \right| 
\]

\[
= \frac{|D \cdot B|}{\sqrt{R^2 - D^2 \cdot B^2}} \cdot \left[ \frac{\pi}{2} - \arcsin \left( \frac{D^2 \cdot B^2 - R^2}{D \cdot B (D \cdot B \cdot \cos \varphi_1 - R)} - \frac{R}{D \cdot B} \right) \right]
\]

(80)
Thus, equation (68) under the next conditions: \( A \cdot B \neq 1, D \cdot B \neq 0, (cos\varphi_1)^2 \neq \frac{1}{A \cdot B} \) (by A-B≠0) and \( R > D \cdot B \) takes the new form:

\[
\frac{2}{\sqrt{R^2-D^2-B^2}} \cdot \frac{1}{|D \cdot B|} \cdot arct g \left( \frac{\sqrt{R^2-D^2-B^2} \cdot t g \left( \frac{\varphi_1}{2} \right)}{R-D-B} \right) + \frac{A}{D} \cdot \left\{ sin\varphi_1 \cdot \frac{R}{R-D-B} \cdot \varphi_1 + \frac{\sqrt{R^2-D^2-B^2}}{|D \cdot B|} \cdot \left( \frac{D^2-B^2-R^2}{D \cdot B(D \cdot B \cdot cos\varphi_1-R)} - \frac{R}{D \cdot B} \right) \right\} = \frac{t^2}{2}
\]

The resulting expression is the desired dependence \( \varphi_1(t) \), which describes the rotational movement of link 1 around point A.

Dividing (66) by (67) an equation of the form \( X_A(\varphi_1) \) can be obtained:

\[
X_A = \int_0^{\varphi_1} \frac{D \cdot R \cdot A \cdot cos\varphi_1}{R-D \cdot B \cdot cos\varphi_1} \cdot d\varphi_1 = \frac{R \cdot A}{D \cdot B} \cdot \varphi_1 + \frac{D^2-B^2-R^2 \cdot A}{D \cdot B} \cdot l_1
\]

Whence, taking into account (39-41), (56) and (78) it can be written as:

\[
X_A = \varphi_1 + \frac{N}{2} \cdot t^2 \cdot \frac{(m_1 + m_2 + m_3) \cdot BA^2 - m_1 + m_2 + m_3}{2 \cdot (m_1 + m_2 + m_3) \cdot (m_1 + m_2 + m_3) \cdot BA^2}
\]

which is the equation of the translational movement of link 1.

Then, let’s turn to (54) and rewrite this expression in the following form:

\[
\frac{d}{dt} \varphi_2 - \frac{C_1}{\alpha} \varphi_2 = \frac{N}{\alpha} \cdot t \cdot const \quad \text{или} \quad \frac{d}{dt} \varphi_2 - k \cdot \varphi_2 = U
\]

under the initial conditions \( \varphi_2(0) = 0; \varphi_2(0) = 0 \) the solution to the equation has the following form:

\[
\varphi_2 = \frac{U}{2k} \cdot e^{-\sqrt{k} \cdot t} \cdot (e^{\sqrt{k} \cdot t} - 1)^2 \quad \text{or} \quad \varphi_2 = \frac{N}{2 \cdot C \cdot l} \cdot e^{-\frac{C_1}{\alpha} \cdot t} \cdot (e^{\frac{C_1}{\alpha} \cdot t} - 1)^2
\]

which is the equation of link 2 movement around the point B.

The solution to equation (55) can be represented as:

\[
\varphi_3 = 0.5 \cdot \frac{M_{res}}{f} \cdot t^2 + C_1 \cdot \tau + C_2
\]

where \( C_1 \) and \( C_2 \) are integration constants determined from the initial conditions \( \varphi_3(0) = 0; \varphi_3(0) = 0 \)

\[ \varphi_3 = 0.5 \cdot \frac{F_{res}}{f} \cdot t^2 \] after substitution \( f = \frac{m_2 \cdot l_2^2}{3} \) the equation of the link 3 movement can be obtained:

\[
\varphi_3 = 1.5 \cdot \frac{F_{res}}{m_3 \cdot l_3^2} \cdot t^2
\]
6. Conclusion

As a result of theoretical research, it is established that the Lagrange equations of the second kind can be successfully used for a practical engineering method of the kinematic parameters predicting for designing of the three-link plain hinged linkage mechanisms of a parallel structure. That method does not require of cumbersome computing systems using, that are usually developed for every individual configuration of the technological machines components, including robots-manipulators used in food production.

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