Generalized quasidisks and conformality: progress and challenges

Chang-Yu Guo1 · Haiqing Xu1,2

Received: 27 May 2020 / Accepted: 22 January 2021 / Published online: 18 February 2021
© The Author(s), under exclusive licence to Springer Nature Switzerland AG part of Springer Nature 2021

Abstract
In this note, we survey the recent developments on theory of generalized quasidisks. Based on standard techniques used earlier, we also provide some minor improvements on the recorded results. A few natural questions are posed.

Keywords Homeomorphism of finite distortion · Generalized quasidisk · Local connectivity · Three point property · Cusps

Mathematics Subject Classification 30C62 · 30C65

1 Introduction

1.1 Quasidisks and conformality

One calls a Jordan domain \( \Omega \subset \mathbb{R}^2 \) a quasidisk if it is the image of the unit disk \( \mathbb{D} \) under a quasiconformal mapping \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) of the entire plane. If \( f \) is \( K \)-quasiconformal, we say that \( \Omega \) is a \( K \)-quasidisk. Another possibility is to require that \( f \) is additionally conformal in the unit disk \( \mathbb{D} \).

The following characterization, which shows that there are no real differences between these two definitions, is essentially due to Kühnau.

**Theorem 1.1** ([5, 11]) A Jordan domain \( \Omega \subset \mathbb{R}^2 \) is a \( K \)-quasidisk if and only if \( \Omega \) is the image of \( \mathbb{D} \) under a \( K^2 \)-quasiconformal mapping \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that is conformal in \( \mathbb{D} \).

The concept of a quasidisk is central in the theory of planar quasiconformal mappings; see, for example, [2, 3, 5, 16]. There are two well-known simple geometric characterizations of quasidisks. The first one was given by Ahlfors.

**Theorem 1.2** ([1]) A Jordan domain \( \Omega \subset \mathbb{R}^2 \) is a quasidisk if and only if it satisfies the **three point property**:

\[
\min_{i=1,2} \text{diam}(\gamma_i) \leq C |P_1 - P_2|
\]

for any distinct pair of points \( P_1, P_2 \in \partial \Omega \), where \( \gamma_1 \) and \( \gamma_2 \) are the components of \( \partial \Omega \setminus \{P_1, P_2\} \) and \( C \) is a constant that depends on \( \Omega \).

Another one is due to Gehring.

**Theorem 1.3** ([4, 5]) A Jordan domain \( \Omega \subset \mathbb{R}^2 \) is a quasidisk if and only if it is linearly locally connected.

Recall that a domain \( \Omega \subset \mathbb{R}^2 \) is **linearly locally connected** (LLC) if there is a constant \( C \geq 1 \) so that

- (LLC-1) each pair of points in \( B(x, r) \cap \Omega \) can be joined by an arc in \( B(x, Cr) \cap \Omega \), and
- (LLC-2) each pair of points in \( \Omega \setminus B(x, r) \) can be joined by an arc in \( \Omega \setminus B(x, C^{-1}r) \).

For more on the function theoretic properties of a quasidisk, see the monograph [5].

1.2 Generalized quasidisks

A substantial part of the theory of quasiconformal mappings has recently been extended in a natural form to the setting...
of mappings of finite distortion with suitable integrability restrictions on the distortion function—particularly with locally exponentially integrable distortion—see monographs [8, 9] for a comprehensive overview. Here, we only briefly recall the basic definitions.

**Definition 1.4** We call a homeomorphism \( f : \Omega \rightarrow f(\Omega) \subset \mathbb{R}^2 \) a *homeomorphism of finite distortion* if \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \) and

\[
\|Df(x)\|^2 \leq K(x)J_f(x) \quad \text{almost everywhere in } \Omega, \tag{1.2}
\]

for some measurable function \( K(x) \geq 1 \) that is finite almost everywhere.

Recall here that \( J_f \in L^1_{\text{loc}}(\Omega) \) for each homeomorphism \( f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \) (see for instance [3]). In the distortion inequality (1.2), \( Df(x) \) is the formal differential of \( f \) at the point \( x \) and \( J_f(x) := \text{det } Df(x) \) is the Jacobian. The norm of \( Df(x) \) is defined as

\[
\|Df(x)\| \coloneqq \max_{e \in S^1} |Df(x)e|.
\]

For a homeomorphism of finite distortion, it is convenient to write \( K_f \) for the optimal distortion function. This is obtained by setting \( K_f(x) = \|Df(x)\|^2/J_f(x) \) when \( Df(x) \) exists and \( J_f(x) > 0 \), and \( K_f(x) = 1 \) otherwise. The distortion of \( f \) is said to be locally \( \lambda \)-exponentially integrable if \( \exp(\lambda K_f(x)) \in L^1_{\text{loc}}(\Omega) \) for some \( \lambda > 0 \). Note that if we assume \( K_f(x) \) to be bounded, \( K_f \leq K_f \), we recover the class of \( K \)-quasiconformal mappings. For this class, we have (see for instance [3]) that

\[
f \in W^{1,1}_{\text{loc}}(\Omega) \quad \forall p < 2K/(K - 1). \tag{1.3}
\]

Following [6, 7], we extend the definition of a quasidisk to the category of mappings of finite distortion, with an initial motivation to build a reasonable geometric counterpart for the theory of mappings with finite distortion.

**Definition 1.5** (Generalized quasidisk) A Jordan domain \( \Omega \subset \mathbb{R}^2 \) is a *generalized quasidisk* if it is the image of the unit disk \( \mathbb{D} \) under a homeomorphism \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that is conformal in \( \mathbb{D} \) and has locally exponentially integrable distortion in the entire plane.

Another possibility for the definition of a generalized quasidisk is to remove the extra conformality requirement for the global homeomorphism \( f \) in Definition 1.5—we shall refer to the latter case a *generalized quasidisk of second kind*. However, unlike the case of a quasidisk, this leads to different classes of domains. Before turning to more details, we introduce two model domains that play an important role in understanding the geometry of a generalized quasidisk.

**Example 1.6** (Outward-pointing cusps) For each \( s > 0 \), the model outward-pointing cusp domain is

\[
\Omega_s = \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, |x_2| < x_1^{1+s} \right\} \cup B(x_s, r_s), \tag{1.4}
\]

where \( x_s = (s + 2, 0) \) and \( r_s = \sqrt{(s + 1)^2 + 1} \).

**Example 1.7** (Inward-pointing cusps) For each \( s > 0 \), the model inward-pointing cusp domain is

\[
\Delta_s := B(x_s', r_s') \setminus \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, |x_2| \leq x_1^{1+s} \right\}, \tag{1.5}
\]

where \( x_s' = (-s, 0) \) and \( r_s' = \sqrt{(s + 1)^2 + 1} \).

These cusp domains do not satisfy the Ahlfors three point property, and thus, they are not quasidisks. But they are generalized quasidisks according to Definition 1.5. Indeed, we have the following result regarding the outward-pointing cusps.

**Theorem 1.8** ([13, 14]) Let \( s > 0 \). Then, for \( \lambda < \frac{1}{s} \), there is a homeomorphism \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with locally \( \lambda \)-exponentially integrable distortion such that \( \Omega_s = f(\mathbb{D}) \), while there is no such \( f \) if \( \lambda > \frac{1}{s} \). It is not clear what happens when \( \lambda = \frac{1}{s} \). Furthermore, if we require \( \Omega_s \) to be a generalized quasidisk, then the above critical bound for \( \lambda \) is \( \frac{1}{s} \).

Notice the difference to the setting of quasiconformal mappings: instead of the switch from \( K \) to \( K^2 \) under the additional conformality condition, one essentially switches from \( \lambda \) to \( \lambda /2 \). This type of conformality behavior disappears when we consider the inward-pointing cusps.

**Theorem 1.9** ([7]) Let \( s > 0 \). Then, for \( \lambda < \frac{1}{s} \), there is a homeomorphism \( f_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) of locally \( \lambda \)-exponentially integrable distortion so that

\[
f_s(\mathbb{D}) = \Delta_s.
\]

On the other hand, there is no homeomorphism \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) of locally exponentially integrable distortion such that \( g \) is quasiconformal in \( \mathbb{D} \) and \( g(\mathbb{D}) = \Delta_s \).

In fact, if \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a homeomorphism of finite distortion \( K_g \) such that \( g \) is \( K \)-quasiconformal in \( \mathbb{D} \) with \( g(\mathbb{D}) = \Delta_s \), then it was proved in [7] that \( K_g \notin L^p_{\text{loc}}(\mathbb{R}^2) \) if \( p > K/s \).

Thus an inward-pointing polynomial cusp rules out the extendability of a Riemann mapping function to a homeomorphism of locally exponentially integrable distortion, but such exterior cusps are not that dangerous.
1.3 Structure and notation

We sometimes associate the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ for convenience and denote by $\hat{\mathbb{C}}$ the extended complex plane. The closure of a set $U \subset \mathbb{R}^2$ is denoted $\overline{U}$ and the boundary $\partial U$. The open disk of radius $r > 0$ centered at $x \in \mathbb{R}^2$ is denoted by $B(x, r)$ and we simply write $\mathbb{D}$ for the unit disk. The boundary of $B(x, r)$ will be denoted by $\partial B(x, r)$ and the boundary of the unit disk $\mathbb{D}$ is written as $\partial \mathbb{D}$. The symbol $\Omega$ always refers to a domain, i.e., a connected and open subset of $\mathbb{R}^2$.

When we write $f(x) \leq g(x)$, we mean that $f(x) \leq Cg(x)$ is satisfied for all $x$ with some fixed constant $C \geq 1$. Similarly, the expression $f(x) \geq g(x)$ means that $f(x) \geq C^{-1}g(x)$ is satisfied for all $x$ with some fixed constant $C \geq 1$. We write $f(x) \approx g(x)$ whenever $f(x) \leq g(x)$ and $f(x) \geq g(x)$.

This paper is structured as follows. In Sect. 2 we introduce a standard way to extend a conformal mapping to the whole plane. Section 3 is about sufficient geometric conditions for generalized quasidisks. Section 4 is devoted to extension results of a particular class of quasiconformal mappings.

2 Extension of a conformal welding

In this section, we briefly describe the standard way of extending a conformal map $f : \mathbb{D} \to \Omega$, where $\Omega$ is a Jordan domain, to a mapping of the entire plane. First of all, by Caratheodory extension theorem, $f$ can be extended to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$. For simplicity, we denote this extended homeomorphism also by $f$. It follows from the Riemann Mapping Theorem that there exists a conformal mapping $g : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$ such that the complement of the closed unit disk gets mapped to the complement of $\overline{\Omega}$. In this correspondence the boundary curve $\Gamma = \partial \Omega$ is mapped homeomorphically onto the boundary circle $\partial D$ and hence the composed mapping $G = g^{-1} \circ f : \partial \mathbb{D} \to \partial \mathbb{D}$ is a well-defined circle homeomorphism, called conformal welding.

Suppose we are able to extend $G$ to the exterior of the unit disk, with the extension still denoted by $G$. Then the mapping $H = g \circ G$ will be well-defined outside the unit disk and it coincides with $f$ on the boundary circle $\partial \mathbb{D}$. Finally, if we define

$$F(x) = \begin{cases} H(x) & \text{if } |x| \geq 1 \\ f(x) & \text{if } |x| \leq 1. \end{cases}$$

then we obtain an extension of $f$ to the entire plane. In the case of a quasidisk [1], the extension $G$ can be chosen to be quasiconformal and hence the obtained map $F$ is also quasiconformal.

On the other hand, the extendability of a conformal mapping $f : \mathbb{D} \to \Omega$ to a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ of locally integrable distortion is essentially equivalent to being able to extend the conformal welding $G$ above to this class. Indeed, if $\hat{f}$ extends $f$, then $g^{-1} \hat{f}$ extends $G$ to the exterior of $\mathbb{D}$ and has the same distortion as $\hat{f}$. Reflecting (twice) with respect to the unit circle, one then further obtains an extension to $\mathbb{D} \setminus \{0\}$. Hence, one obtains an extension $\hat{G}$ of $G$ to $\mathbb{R}^2 \setminus \{0\}$ with distortion that has the same local integrability degree as the distortion of $\hat{f}$. If the latter distortion is sufficiently nice in a neighborhood of infinity (e.g., bounded), then this holds in all of $\mathbb{R}^2$ as well.

Given a sense-preserving homeomorphism $f : \partial \mathbb{D} \to \partial \mathbb{D}$ and $0 < t < \frac{1}{2}$, set

$$\delta_f(\theta, t) = \max \left\{ \frac{|f(e^{i(\theta+t)}) - f(e^{i\theta})|}{|f(e^{i\theta}) - f(e^{i(\theta-t)})|}, \frac{|f(e^{i\theta}) - f(e^{i(\theta+t)})|}{|f(e^{i\theta}) - f(e^{i(\theta+t)})|} \right\}.$$

(2.1)

Clearly $\delta_f$ is continuous in both variables, $\delta_f \geq 1$ and $\delta_f(\theta + 2\pi, t) = \delta_f(\theta, t)$. The scale-wise distortion of $f$ is defined as $\rho_f(t) = \sup_{\theta} \delta_f(\theta, t)$.

A well-known fact is that the extendability of a conformal welding $G : \partial \mathbb{D} \to \partial \mathbb{D}$ to a global homeomorphism of the entire plane with controlled distortion is related to the integrability of $\rho_f$; see for instance [6, Sect. 4] for more information on this. For our purpose, we recall the following result, which is essentially due to Zakeri [20].

**Proposition 2.1 ([6, 20])** Let $G : \partial \mathbb{D} \to \partial \mathbb{D}$ be a conformal welding. If

$$\rho_f(t) = O\left(\log \frac{1}{t}\right) \quad \text{as} \quad t \to 0,$$

then $G$ extends to a homeomorphism of the entire plane of locally exponentially integrable distortion. Furthermore, if

$$\rho_f(t) = O\left(t^{-a}\right) \quad \text{as} \quad t \to 0$$

for some $a > 0$, then $G$ extends to a homeomorphism of the entire plane of locally $p$-integrable distortion with any $p \in (0, \frac{1}{a})$.

3 Geometric criteria for generalized quasidisks

In this section, we review a known geometric criteria for a Jordan domain $\Omega$ to be a generalized quasidisk and present some improvements via basically the same techniques.
3.1 Relaxing the three point property and linear local connectivity

As observed from Theorems 1.8 and 1.9, we have to relax the Ahlfors three point property or linear local connectivity to include cusp domains. The extensions for these two concepts are straightforward.

Definition 3.1 (Generalized three point property) We say that a Jordan domain $\Omega \subset \mathbb{R}^2$ satisfies the three point property with a control function $\psi : [0, \infty) \to [0, \infty)$ if there exists a constant $C \geq 1$ and an increasing function $\psi : [0, \infty) \to [0, \infty)$ such that for each pair of distinct points $P_1, P_2 \in \partial \Omega$,

$$\min_{i=1,2} \text{diam} (\gamma_i) \leq \psi \left( C |P_1 - P_2| \right), \quad (3.1)$$

where $\gamma_1, \gamma_2$ are the components of $\partial \Omega \setminus \{P_1, P_2\}$.

Definition 3.2 (Generalized local connectivity) A domain $\Omega \subset \mathbb{R}^2$ is called $(\varphi, \psi)$-locally connected ($\varphi, \psi$-LC) if

- $(\varphi$-LC-1) each pair of points in $B(x, r) \cap \Omega$ can be joined by an arc in $B(x, \varphi(r)) \cap \Omega$, and
- $(\psi$-LC-2) each pair of points in $\Omega \setminus B(x, \varphi(r))$ can be joined by an arc in $\Omega \setminus B(x, \varphi(r))$,

where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are smooth increasing functions such that $\varphi(0) = \psi(0) = 0$, $\varphi(r) \geq r$ and $\varphi(r) \leq r$ for all $r > 0$.

For technical reasons, we assume that the function $t \mapsto \frac{1}{(\varphi^{-1}(\varphi(t)))^2}$ is decreasing and that there exist constants $C_1, C_2$ so that $C_1 \varphi(t) \leq \varphi(2t) \leq C_2 \varphi(t)$ and $C_1 \psi(t) \leq \psi(2t) \leq C_2 \psi(t)$ for all $t > 0$. If $\varphi^{-1} = \psi$ above, the domain $\Omega$ is called $\psi$-LC. By [6, Lemma 3.1], a Jordan domain $\Omega \subset \mathbb{R}^2$ has the three point property with control function $\psi$ if and only if $\psi$ is $1$-locally connected.

Using the generalized three point property, the following result was proved in [6].

Theorem 3.3 ([6]) If a Jordan domain $\Omega \subset \mathbb{R}^2$ has the three point property with control function $\varphi(t) = C t \log^s \log(e + \frac{1}{t})$ for some positive constant $C$ and $s \in (0, \frac{1}{2})$, then $\Omega$ is a generalized quasidisk.

Remark 3.4 In [6, Theorem 1.1], it was stated Theorem 3.3 holds for $\varphi(t) = \log^s \left( \frac{1}{t} \right)$ and this is not correct according to the proofs given there. We shall present a more general result with corrected proofs.

Via the nonlinear local connectivity, the following result was proved in [7].

Theorem 3.5 ([7]) Let $\Omega \subset \mathbb{R}^2$ be a $(\varphi, \psi)$-locally connected Jordan domain with

$$\lim_{r \to 0} \frac{r \cdot \varphi^{-1}\circ \psi(r)}{(\varphi^{-1}\circ \psi^{-1})(\varphi(r)) \cdot \log \log \frac{1}{r}} = 0, \quad (3.2)$$

where $\varphi, \psi$ satisfy the technical conditions

- $t \mapsto \frac{1}{(\varphi^{-1}(\varphi(t)))^2}$ is decreasing;
- there exist constants $C_1, C_2$ so that $C_1 \varphi(t) \leq \varphi(2t) \leq C_2 \varphi(t)$ and $C_1 \psi(t) \leq \psi(2t) \leq C_2 \psi(t)$ for all $t > 0$.

Then any conformal mapping $f : \Omega \to \Omega$ can be extended to the entire plane as a homeomorphism of locally exponentially integrable distortion.

Theorem 3.5 implies in particular that

Corollary 3.6 If a Jordan domain $\Omega \subset \mathbb{R}^2$ is $\psi$-locally connected with $\varphi^{-1}(t) = C t \log^s \log(e + \frac{1}{t})$ for some positive constant $C$ and some $s \in (0, \frac{1}{4})$, then $\Omega$ is a generalized quasidisk.

Note that the range for $s$ is smaller than the one obtained in Theorem 3.3.

If $\Omega$ does not contain inward-pointing cusps, then we have the following extension result. Comparing with Theorem 3.5, the extended map from $\mathbb{R}^2 \to \mathbb{R}^2$ only has local $L^p$-integrability with $p > 0$.

Theorem 3.7 ([6]) Let $\Omega \subset \mathbb{R}^2$ be an LLC-1 Jordan domain. Then any conformal mapping $f : \Omega \to \Omega$ can be extended to the entire plane as a homeomorphism of locally $p$-integrable distortion for some $p > 0$.

As the following example indicates, similar result fails if we only assume $\Omega$ has no outward-pointing cusps.

Example 3.8 Given any $\varepsilon > 0$, consider the domain

$$\Omega = B((-1, 0), \sqrt{\varepsilon}) \setminus \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, |x_2| < x_1 (\log e/x_1)^{1/(1+\varepsilon)}\}. \quad (3.3)$$

Then it is LLC-2 and $\varphi$-LC-1 with $\varphi^{-1}(t) = C t (\log^- \frac{1}{t})^{1/(1+\varepsilon)}$. However, by [7, Theorem 6.2], it fails to be a generalized quasidisk.

Note that there is an extra logarithm gap between Theorem 3.5 and Example 3.8.

We remark here that the generalized three point property is “symmetric” in the sense that both inward-pointing
and outward-pointing cusps are simultaneously allowed to have the same degree. While in [7, Theorems 5.1 and 6.1], the degree of an inward-pointing cusp and an outward-pointing cusp plays a different role in the extension result. This phenomenon is natural since polynomial interior cusps rule out the possibility of a locally exponentially integrable distortion extension, but polynomial exterior cusps do not.

On the other hand, from the technical view of point, doing modulus of curve family estimates does not distinguish different types of cusps. Thus one expects a similar kind of “symmetric” formulation which appears as in [6]. This is indeed the case; see Theorem 3.9.

### 3.2 A more general result via nonlinear local connectivity

Our aim of this section is to prove the following result, which extends Theorem 3.5.

**Theorem 3.9** Let $\Omega \subset \mathbb{R}^2$ be a $(\varphi, \psi)$-locally connected Jordan domain with$$\lim_{r \to 0} \frac{r}{\varphi^{-1}(\varphi(r) \log \log \frac{1}{r})} = 0.$$Then any conformal mapping $f : \mathbb{D} \to \Omega$ can be extended to the entire plane as a homeomorphism of locally exponentially integrable distortion. From our terminology, the relative distance of $E$ and $F$.

**Lemma 3.10** ([18]) Let $E, F$ be disjoint nondegenerate continua in a ball $B \subset \mathbb{R}^2$. Then there exists an absolute constant $C_0 > 0$ such that

$$\left( C_0 \log \left(1 + \delta(E, F)\right) \right)^{-1} \geq \text{Mod}(E, F, B) \geq C_0 \log \left(1 + \frac{1}{\delta(E, F)}\right)$$

(3.5)

when $\delta(E, F)$ is sufficiently small.

We would like to point out that the lower bound in (3.5) was incorrectly cited in [6, Lemma 3.3] and thus it leads to the incorrectly stated Theorem 1.1 there.

The second lemma gives uniform continuity of quasiconformal mappings from locally connected domains onto the unit disk.

**Lemma 3.11** ([12]) Suppose $g : \Omega \to \mathbb{D}$ is a $K$-quasiconformal mapping from a simply connected domain $\Omega$ onto the unit disk. Then there exists a positive constant $C = C(g) > 0$ such that for any $\omega, \zeta \in \Omega$,

$$|g(\omega) - g(\zeta)| \leq C d_\gamma(\omega, \zeta)^{\frac{1}{K}},$$

(3.6)

where $d_\gamma(\omega, \zeta)$ is defined as $\inf_{\gamma(\omega, \zeta) \subset \mathbb{D}} \text{diam}(\gamma(\omega, \zeta))$. In particular, if $\Omega$ above is $\varphi$-LC-1, then

$$|g(\omega) - g(\zeta)| \leq C \varphi(\omega - \zeta)^{\frac{1}{K}}.$$

(3.7)

**Proof of Theorem 3.9** The proof is a combination of [7, Proof of Theorem 5.1] and [6, Proof of Theorem 5.1]. Since $\Omega$ is a Jordan domain, $f$ extends to a homeomorphism between $\mathbb{D}$ and $\overline{\Omega}$ and we denote this extension by $f$. Let $e^{i(\theta - t)}$, $e^{i\theta}$ and $e^{i(\theta + t)}$ be three points on $S$. Since $f$ is a sense-preserving homeomorphism, $f(e^{i(\theta - t)})$, $f(e^{i\theta})$ and $f(e^{i(\theta + t)})$ will be on the boundary of $\Omega$ in order. Let $g : \mathbb{R}^2 \setminus \overline{\mathbb{D}} \setminus \mathbb{R}^2 \setminus \overline{\Omega}$ be a conformal mapping from the Riemann Mapping Theorem. Then $g$ extends to a homeomorphism between $\mathbb{R}^2 \setminus \mathbb{D}$ and $\mathbb{R}^2 \setminus \Omega$. As before, we still denote this extension by $g$. Based on the discussion in the previous section, we only need to estimate the scale-wise distortion of the conformal welding $G := g^{-1} \circ f$.

Let $P = e^{i(\theta + \varepsilon)}$ be the antipodal point of $e^{i\theta}$ on $\partial \mathbb{D}$ and let $\gamma_f(P, \theta - t)$ denote the arc from $f(P)$ to $f(e^{i(\theta - t)})$ on $\partial \mathbb{D}$. There exists a $t_0$ small enough such that $\text{diam}(\gamma_f(P, \theta - t)) \geq \text{diam}(\gamma_f(\theta, \theta + t))$ when $t \in [0, t_0]$. Set $\alpha_1 = \gamma_f(\theta, \theta + t)$, $\alpha_2 = \gamma_f(P, \theta - t)$ and $d = d(\alpha_1, \alpha_2)$.

By the proof of (5.2) in [7, Theorem 5.1], there exists a constant $C_0 > 0$ such that

$$\text{diam}(\gamma_f(\theta, \theta + t)) \leq C_0 \varphi^{-1} \varphi(d).$$

Thus, it follows from Lemma 3.10 that

$$\lim_{r \to 0} \varphi^{-1}(\varphi(r) \log \log \frac{1}{r}) = 0.$$
Mod $(\Gamma') \leq C \log^{-1} \left( 1 + \frac{\varphi^{-1} \psi \left( \text{diam} \left( a_1 \right) \right)}{\text{diam} \left( a_1 \right)} \right) + C. \quad (3.8)$

where $\Gamma'$ is the family of curves joining $a_1$ and $a_2$ in $\mathbb{R}^2 \setminus \Omega$. Indeed, we may choose $R >> 1$ such that $\Omega \subset B(0, R)$. We then divide the curves in $\Gamma$ into two subfamilies $\Gamma_1'$ and $\Gamma_2'$: $\Gamma_1'$ contains curves that completely stays inside $B(0, 2R)$ and $\Gamma_2'$ contains curves that leave $B(0, 2R)$. Note that each curve in $\Gamma_2'$ has a subcurve connecting $\partial B(0, R)$ and $\partial B(0, 2R)$, and thus $\text{Mod} (\Gamma_2')$ is bounded from above. We may apply Lemma 3.10 to $\Gamma'$ to conclude that $(3.8)$ holds with $\Gamma'$ replaced by $\Gamma'$. Then $(3.8)$ follows from the sub-additivity of modulus.

Again by conformal invariance of modulus, when $t$ is sufficiently small, we may apply Lemma 3.10 to deduce

$$\log(1 + \delta_C (\theta, t)) \leq \log \left( 1 + \frac{\min \{ \text{diam} \left( g^{-1}(a_1) \right), \text{diam} \left( g^{-1}(a_2) \right) \}}{d(g^{-1}(a_1), g^{-1}(a_2))} \right) \leq C_0^{-1} \text{Mod} (\Gamma'), \quad (3.9)$$

where $C_0$ is the constant from Lemma 3.10. Combining $(3.8)$ with $(3.9)$ gives us the estimate

$$\delta_C (\theta, t) \leq \exp \left( \frac{C_0 \text{diam} \left( a_1 \right)}{\varphi^{-1} \psi (\text{diam} \left( a_1 \right))} \right) + C_1.$$ 

On the other hand, by applying Lemma 3.11 and noticing that our technical assumptions on $\varphi$ imply that $\varphi^{-1}(t) \geq C t^a$ for some $a > 0$, we obtain that

$$\text{diam} \left( a_1 \right) \geq C \varphi^{-1}(t^2) \geq C t^{2a}.$$ 

Since $\frac{t}{\varphi^{-1}(\psi(t))}$ is nonincreasing, we conclude that

$$\delta_C (\theta, t) \leq \exp \left( \frac{C t^{2a}}{\varphi^{-1} \psi (t^{2a})} \right) + C_1. \quad (3.10)$$

Theorem 3.9 follows immediately from $(3.4)$, $(3.10)$ and Proposition 2.1. Indeed, $(3.4)$ implies

$$C \frac{t^{2a}}{\varphi^{-1} \psi (t^{2a})} \leq \log \log \frac{1}{t}$$

for $t$ sufficiently small and so

$$\delta_C (\theta, t) \leq \exp \left( \frac{C t^{2a}}{\varphi^{-1} \psi (t^{2a})} \right) \leq \log \frac{1}{t}$$

and thus Theorem 3.9 follows from Proposition 2.1. \(\Box\)

Comparing Theorem 3.9 with Example 3.8, it is clear that there is an extra logarithm gap for the control function $\varphi$, even when the domain does not have inward-pointing cusps. We thus pose the following question for further research.

**Question 1** Is Example 3.8 sharp for domains without inward-pointing cusps? In other words, suppose $\Omega$ is $\varphi$-LC-1 and LLC-2, with $\varphi^{-1}(t) = \frac{Ct}{\log 1/t}$. Is $\Omega$ then a generalized quasidisk?

## 4 Further analytic aspects of the extension

In this section, we study further analytic aspects of the extension problem for quasiconformal mappings $g : \mathbb{D} \to \Delta_s$, where $\Delta_s$ is defined in (1.5). We start with an extension result for the case when $g$ is conformal.

**Theorem 4.1** [(19)] Let $g$ be a conformal map from $\mathbb{D}$ onto $\Delta_s$. Then there is a homeomorphic extension $f : \mathbb{R}^2 \to \mathbb{R}^2$ of $g$ with finite distortion. Moreover we have that

$$f \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \quad \text{for all } p < \infty, \quad (4.1)$$

$$f^{-1} \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \quad \text{for all } p < \frac{2(s+2)}{2s+1}, \quad (4.2)$$

$$K_{f^{-1}} \in L^q_{\text{loc}}(\mathbb{R}^2) \quad \text{for all } q < \frac{s+2}{s}. \quad (4.3)$$

Note that the above $\Delta_s$ is slightly different from that defined in [19]. However, the proof of Theorem 1.2 from [19] can be modified in a straightforward manner to obtain Theorem 4.1. For each conformal map $g : \mathbb{D} \to \Delta_s$, we set $\mathcal{F}_s(g) := \{ f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a homeomorphic extension of } g \text{ with finite distortion } \}$. \(\Box\)

For the convenience of the readers, we briefly sketch the proof of $\mathcal{F}_s(g) \neq \emptyset$ in Theorem 4.1. After some simple reduction (composing with additional Möbius transformations), it suffices to prove that $\mathcal{F}_s(g_0) \neq \emptyset$ for a fixed conformal mapping $g_0 : \mathbb{D} \to \Delta_s$ via the bi-Lipschitz characterization of chord-arc domains [17], it is easy to construct an extension of $g_0$ on any region that is strictly away from the cusp point. Thus the essential task is to construct an extension of $g$ in a small neighborhood containing the cusp point. In this step, one can write down the extension by hand using the explicit geometry of $\Delta_s$ (see Step 1 in [19, Proof of Theorem 1.2]). Combining these two extensions leads to an element in the set $\mathcal{F}_s(g_0)$. 

Springer
Analogously, we next explore extension results in [19] when \( g : \mathbb{D} \to \Delta_s \) is more generally a quasiconformal mapping. To this end, for \( s \in (0, \infty) \) and \( K \in (1, \infty) \), we set

\[
G_s(K) = \{ g : g : \mathbb{D} \to \Delta_s \text{ is a } K\text{-quasiconformal mapping from } \mathbb{D} \text{ onto } \Delta_s \}.
\]

Obviously \( G_s(K) \neq \emptyset \), since there are conformal mappings in \( G_s(K) \). Given any \( g \in G_s(K) \), we set \( \mathcal{F}_s(g) \) as in (4.4). For \( a \in \mathbb{R} \) we denote \( a_+ = (a + |a|)/2 \). We have the following result that partially extends [19, Theorem 1.2].

**Theorem 4.2** Given any \( g \in G_s(K) \), we have that \( \mathcal{F}_s(g) \neq \emptyset \) and

\[
\sup \{ p \in [1, \infty) : f \in \mathcal{F}_s(g) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \} = \frac{2K}{K-1}.
\]

Moreover, we have

\[
\inf_{g \in G_s(K)} \sup \{ p \in [1, \infty) : f \in \mathcal{F}_s(g), f^{-1} \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \} = \frac{2(2+s)}{2s + 2 - K^{-1}}.
\]

We need a couple of auxiliary results for the proof of Theorem 4.2. In the first lemma, we provide a standard method to extend mappings in \( G_s(K) \).

**Lemma 4.3** For any \( g \in G_s(K) \), there exists an \( f \in \mathcal{F}_s(g) \) such that

\[
f \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \text{ for all } p < \frac{2K}{K-1},
\]

and

\[
f^{-1} \in W^{1,q}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \text{ for all } q < \frac{2(2+s)}{2s + 2 - K^{-1}}.
\]

**Proof** Analogously to the construction of \( g_s \) as in [19, (2.3.3)], there is a conformal mapping \( \varphi : \mathbb{D} \to \Delta_s \) satisfying \( \varphi(\mathbb{D}) = \varphi(\mathbb{D}) \) for all \( z \in \mathbb{D} \). By Theorem 4.1 there is a homeomorphic extension \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) of \( \varphi \) with finite distortion. For \( g \in G_s(K) \), we set \( \psi = \varphi^{-1} \circ g \). Then \( \psi : \mathbb{D} \to \mathbb{D} \) is a \( K \)-quasiconformal mapping. Via reflection, we obtain a \( K \)-quasiconformal extension \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) of \( \psi \). Set

\[
f = \Phi \circ \Psi.
\]

To show that \( f \in \mathcal{F}_s(g) \), it suffices to check that \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \). Alternatively, this will be done if we can prove (4.9).

To this end, we let \( p_1 \in (1, \infty) \) and \( p_2 \in \left( 0, \frac{2p_K}{(p_1-1)(K-1)} \right) \). Set \( p \) by \( p^{-1} = (2p_1)^{-1} + p_2^{-1} \). By monotonicity we have that

\[
p < \frac{1}{2p_1} + \frac{1}{2p_2} < \frac{2K}{K-1}.
\]

From the chain rule and the Lusin \( (N^{-1}) \) property of \( \Psi \), it follows that \( Df(z) \) exists and

\[
Df(z) = D\Phi(\Psi(z))D\Psi(z)
\]

for almost every \( z \in \mathbb{R}^2 \). Fix an arbitrary compact set \( M \subset \mathbb{R}^2 \). By Hölder’s inequality, we have

\[
\int_M |Df|^p \leq \left( \int_M |D\Phi(\Psi)|^{2p_1} |D\Psi|^2 \right)^{\frac{p}{2}} \left( \int_M |D\Psi|^{\frac{2p_2}{p_2}} \right)^{\frac{p_2}{p}}.
\]

On the one hand, by the area formula and (4.1) we obtain that

\[
\int_M |D\Phi(\Psi)|^{2p_1} |D\Psi|^2 \approx \int_M |D\Phi(\Psi)|^{2p_1} |D\Phi|^{2p_1} \leq \int_M |D\Phi|^{2p_1} < \infty.
\]

On the other hand, note that \( (p_1-1)p_2/p_1 < 2K/(K-1) \). Hence via (1.3) we have that \( \int_M |D\Psi|^{(p_1-1)p_2/p_1} < \infty \). Therefore from (4.12) and (4.11) we obtain that \( f \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) for all \( p < 2K/(K-1) \).

It remains to show (4.10) and the proof is analogous to that of (4.9). Let \( q_1 \in \left( 1, \frac{K}{K-1} \right) \), \( q_2 \in \left( 0, \frac{p_K+2+2}{s+1} \right) \), \( q_3 \in \left( 0, \frac{2p_2(s+2)}{q_1(s+2)+1} \right) \). Define \( q \) by \( q^{-1} = q_1^{-1} + q_3^{-1} \). By monotonicity, we have that

\[
q < \frac{q_1(s+2)}{q_1(s+2)+1} + \frac{2(q_1(s+2)+1)}{q_1(s+2)} < \frac{2(s+2)}{2s + 2 - K^{-1}}.
\]

Fix a compact set \( M \subset \mathbb{R}^2 \). Hölder’s inequality implies that

\[
\int_M \left| Df^{-1} \right|^q \leq \left( \int_M |D\Phi^{-1}(\Phi^{-1})^{q_1} |D\Phi^{-1}|^{\frac{q_2}{q_1}} \right)^{\frac{q_1}{q_2}} \left( \int_M |D\Phi^{-1}|^{(q_1-1)\frac{q_2}{q_1}} \right)^{\frac{q_2}{q_1}}
\]

(4.14)

On the one hand, by (4.2) and the fact that \( (1-\frac{1}{q_1})q_3 < 2(s+2)/(2s+1) \) we conclude that \( J < \infty \). On the other hand, by Hölder’s inequality, we have that
\[ I \leq \left( \int_M |D\Psi^{-1}(\Phi^{-1})|^{2q_1} J_{\Phi^{-1}} \right)^{\frac{q_1}{2}} \left( \int_M K_{\Phi^{-1}}^{2q_1-2q_1} \right)^{1-\frac{q_1}{2}}. \]  

(4.15)

Note that \( 2q_1 < 2K/(K-1) \). Hence by the area formula and (1.3) we infer that

\[ \int_M |D\Psi^{-1}(\Phi^{-1})|^{2q_1} J_{\Phi^{-1}} < \infty. \]

Since \( q_2/2q_1 < (s+2)/(s+1) \), we obtain that

\[ \int_M K_{\Phi^{-1}}^{2q_1/(2q_1-q_2)} < \infty. \]

Hence (4.3) implies that \( \int_M K_{\Phi^{-1}}^{q_1} < \infty \). Therefore \( I \) as in (4.15) is finite. From (4.13) and (4.14) we obtain (4.10). \( \square \)

In the next step, we construct two quasiconformal mappings from \( \mathcal{D} \) onto \( \Delta_{x} \).

**Lemma 4.4** For any \( K \in (1, \infty) \), there is a \( K \)-quasiconformal mapping \( g : \mathcal{D} \to \Delta_{x} \) satisfying that \( g(z) = \overline{g(z)} \) for all \( z \in \mathcal{D} \) and

\[ \text{diam} (g(S^1 \cap B(e^{i\xi}, r))) \approx r^{2K} \]  

(4.16)

whenever \( r \ll 1 \).

**Proof** Let \( \varphi_1(z) = |z + 1|^{-1}(z + 1) \) for \( z \in \mathbb{D} \). Obviously \( \varphi_1 \) is a \( K \)-quasiconformal mapping. Denote \( D_K = \varphi_1(\mathbb{D}) \). Then \( D_K \) is convex and symmetric with respect to the real axis. Moreover \( \partial D_K \) is piecewise smooth without cusp points. Hence \( D_K \) is a chord-arc domain, and so is \( D_K \cap R_+^2 \). Define a parameterization \( \varphi_1 : \partial(D_K \cap R_+^2) \to \partial(\mathbb{D} \cap R_+^2) \) by letting it be constant speeds from \( D_K \cap R_+^2 \) onto \( \mathbb{D} \cap R_+^2 \) and from \( D_K \cap R_-^2 \) onto \( \mathbb{D} \cap R_-^2 \). Without loss of generality, we assume that \( \varphi_1(0) = e^{i\pi} \). Then \( \varphi_1 \) is a bi-Lipschitz mapping. By [17] there is a bi-Lipschitz extension \( \varphi_2 : D_K \cap R_+^2 \to \mathbb{D} \cap R_+^2 \) of \( \varphi_1 \). Afterwards by reflection there is a bi-Lipschitz mapping \( \varphi_3 : D_K \to \mathbb{D} \) satisfying that \( \varphi_2(z) = \varphi_2(z) \) for all \( z \in D_K \) and \( \varphi_2(0) = e^{i\pi} \). By the arguments in [19], Sect. 2.3, there exists a conformal mapping \( \varphi_3 : \mathcal{D} \to \Delta_x \) such that \( \varphi_3(z) = \overline{\varphi_3(z)} \) for all \( z \in \mathbb{D} \), \( \varphi_3(e^{i\pi}) = 0 \) and

\[ \text{diam} \left( \varphi_3(S^1 \cap B(e^{i\xi}, r)) \right) \approx r^2 \]  

(4.17)

whenever \( r \ll 1 \).

Set \( g = \varphi_3 \circ \varphi_2 \circ \varphi_1 \). Then \( g : \mathbb{D} \to \Delta_x \) is a \( K \)-quasiconformal mapping with \( g(z) = \overline{g(z)} \) for all \( z \in \mathbb{D} \). By the definition of \( \varphi_1 \), it easy to check that \( \text{diam} \left( \varphi_3(S^1 \cap B(e^{i\xi}, r)) \right) \approx r^2 \) whenever \( r \ll 1 \). Together with (4.17) and the bi-Lipschitz property of \( \varphi_2 \), this gives (4.16). \( \square \)

**Remark 4.5** Let \( \hat{\varphi}_1(z) = |z + 1|^{-1}(z + 1) \) for \( K \in (1, \infty) \). Analogously to Lemma 4.4, replacing \( \varphi_1 \) by \( \hat{\varphi}_1 \), one can show that there exists a \( K \)-quasiconformal mapping \( g : \mathbb{D} \to \Delta_x \) satisfying that \( g(z) = \overline{g(z)} \) for all \( z \in \mathbb{D} \) and \( \text{diam} \left( g(S^1 \cap B(e^{i\xi}, r)) \right) \approx r^{2/K} \) whenever \( r \ll 1 \).

Let \( g \) be the quasiconformal mapping from Lemma 4.4. In the following lemma, we give an upper bound for the Sobolev exponent of the inverse of extensions of \( g \). Analogous results are discussed in [10] for \( g \) without quasiconformality.

**Lemma 4.6** Let \( g \) be as in Lemma 4.4. For any \( f \in F_{s}(g) \), if \( f^{-1} \in W_{loc}^{1,p}(\mathbb{R}_+^2, \mathbb{R}_+^2) \) for some \( p \geq 1 \), then necessarily \( p < \frac{2(s+2)}{2s+2-K-1} \).

**Proof** Given a constant \( c > 0 \), we let \( I_x = \{(x,y) : y \in [-|x|^{s+1}, |x|^{s+1}]\} \) for \( x \in (0, c) \). Via the ACL property of Sobolev functions, we have that

\[ \frac{\text{osc}_{I_x} f^{-1}}{x^{(s+1)(p-1)}} \leq \int_{I_x} |Df^{-1}(x,y)|^p dy. \]  

(4.18)

Notice that by (4.16) we have \( \text{osc}_{I_x} f^{-1} \approx x^{1/(2K)} \). Hence via Fabini’s theorem we obtain from (4.18) that

\[ \int_0^c x^{\frac{2}{pK} - (s+1)(p-1)} dx \leq \int_{B(0,1)} |Df^{-1}|^p. \]

Therefore by the Sobolev assumption of \( f^{-1} \), we have that \( \frac{2}{pK} - (s+1)(p-1) > -1 \), that is, \( p < 2(s+2)/(2s+2-K-1) \). \( \square \)

**Remark 4.7** Take \( g \in G(K) \) and \( f \in F_s(g) \). Notice that by [7, Lemma 4.2]

\[ \text{osc}_{I_x} f^{-1} 

(4.19)

for all \( x \in (0, c) \). Analogously to Lemma 4.6, if \( f^{-1} \in W_{loc}^{1,p}(\mathbb{R}_+^2, \mathbb{R}_+^2) \) for \( p \geq 1 \), then

\[ p < 2(s+2)/(2s+2-K-1). \]  

**Proof of Theorem 4.2** First of all, Lemma 4.3 shows that \( F_s(g) \neq \emptyset \) for any \( g \in G(K) \). By (1.3), it is obvious that \( 2K/(K-1) \) is an upper bound of the supremum in (4.6). Moreover by (4.9) we obtain that this supremum equals \( 2K/(K-1) \). This proves (4.6). By (4.10) we see that \( 2(s+2)/(2s+2-K-1) \) is a lower bound for the infimum in (4.7). Together with Lemma 4.6 we may conclude (4.7).

It remains to prove (4.8). By Remark 4.7 and (1.3), we obtain that the minimum in (4.8) is an upper bound for the
supremum in (4.8). Let $g$ be as in Remark 4.5. By the analogous proof of $F_s(g) \neq \emptyset$ for Theorem 4.1, we construct by hand a mapping $f \in F_s(g)$ satisfying that $f^{-1} \in W^{1,p}_\text{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for all $p < \min \{2K/(K-1), 2(s+2)/(2s+2-K)\}$ We leave the details to interested readers. Hence the minimum in (4.8) is a lower bound for the supremum in (4.8). The proof is therefore complete. □

Comparing Theorem 4.2 with [19, Theorem 1.2], we miss the optimal regularity of distortions of extensions, and that of distortions of their inverse. We formulate the missing part of Theorem 4.2 as a conjecture below.

**Conjecture 4.8** Let $G_s(K)$ be as in (4.5) and $F_s(g)$ be as in (4.4). We conjecture that the following equations hold:

$$\inf_{g \in G_s(K)} \sup \{q \in (0, \infty) : f \in F_s(g), K_f \in L^q_{\text{loc}}(\mathbb{R}^2)\} = \max \left\{ \frac{1}{K_s}, 1 \right\},$$

$$\sup_{g \in G_s(K)} \sup \{q \in (0, \infty) : f \in F_s(g), K_f \in L^q_{\text{loc}}(\mathbb{R}^2)\} = \max \left\{ K, \frac{1}{s}, 1 \right\},$$

$$\sup \{q \in (0, \infty) : f \in F_s(g), K_{f^{-1}} \in L^q_{\text{loc}}(\mathbb{R}^2)\} = \frac{2 + s}{s}$$

for any $g \in G_s(K)$.

Iwaniec et al. in [10] discussed Conjecture 4.8 when $g$ above loses quasiconformality. In [15], via reflections Koskela and Zhu studied extensions of Sobolev functions on cusp domains. Note that Conjecture 4.8 is closely related to Theorem 1.9. As $\Delta_s$ from (1.5) is a special example of the more general class of John disks, it is natural to pose the following question.

**Question 4.9** What are analogous results in Theorem 4.2 when $\Delta_s$ is replaced by a John disk?

**Acknowledgements** This survey is dedicated to our former Ph.D supervisor Prof. Pekka Koskela for his excellent guidance during our postgraduate studies and for bringing us to the world of quasiconformal analysis.

**Funding** C.-Y. Guo is supported by the Qilu funding of Shandong University (No. 625008963197). H. Xu is supported by the Academy of Finland (No. 21000046081).

**References**

1. Ahlfors, L.V.: Quasiconformal reflections. Acta Math. 109, 291–301 (1963)
2. Ahlfors, L.V.: Lectures on Quasiconformal Mappings, Second edition, With Supplemental Chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. University Lecture Series, 38. American Mathematical Society, Providence, RI (2006)
3. Astala, K., Iwaniec, T., Martin, G.: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton University Press, Princeton (2009)
4. Gehring, F.W.: Characteristic Properties of Quasidisks, Séminaire de Mathématiques Supérieures, 84. Presses de l’Université de Montréal, Montreal (1982)
5. Gehring, F.W., Hag, K.: The Ubiquitous quasidis. With Contributions by Ole Jacob Broch. Mathematical Surveys and Monographs, vol. 184. American Mathematical Society, Providence (2012)
6. Guo, C.-Y.: Generalized quasidisks and conformality II. Proc. Am. Math. Soc. 143(8), 3505–3517 (2015)
7. Guo, C.-Y., Koskela, P., Takkinen, J.: Generalized quasidisks and conformality. Publ. Mat. 58(1), 193–212 (2014)
8. Hencl, S., Koskela, P.: Lecture Notes on Mappings of Finite Distortion. Lecture Notes in Mathematics, vol. 2096 (2014)
9. Iwaniec, T., Martin, G.: Geometric function theory and non-linear analysis. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (2001)
10. Iwaniec, T., Onninen, J., Zheng, Z.: Singularity in $L^p$-quasidisks, submitted preprint (2019)
11. Kühnau, R.: Möglichst konforme Spiegelung an einer Jordankurve. Jahresber. Deutsch. Math.-Verein 90 (1988), 90–109
12. Koskela, P., Onninen, J., Tyson, J.T.: Quasihyperbolic boundary conditions and capacity: Hölder continuity of quasiconformal mappings. Comment. Math. Helv. 76(3), 416–435 (2001)
13. Koskela, P., Takkinen, J.: Mappings of finite distortion: formation of cusps. Publ. Mat. 51(1), 223–242 (2007)
14. Koskela, P., Takkinen, J.: A note to mappings of finite distortion: formation of cusps II. Conform. Geom. Dyn. 14, 184–189 (2010)
15. Koskela, P., Zheng, Z.: Sobolev extension via reflections, submitted preprint (2018)
16. Lehto, O., Virtanen, K.I.: Quasiconformal Mappings in the Plane, 2nd edn. Springer, New York (1973)
17. Tukia, P.: The planar Schönhflies theorem for Lipschitz maps. Ann. Acad. Sci. Fenn. Ser. A I Math. 5(1), 49–72 (1980)
18. Vuorinen, M.: Conformal Geometry and Quasiregular Mappings. Lecture Notes in Mathematics, vol. 1319. Springer, Berlin (1988)
19. Xu, H.: Optimal extensions of conformal mappings from the unit disk to Cardioid-type domains, published in J. Geom, Anal (2020)
20. Zakeri, S.: On boundary homeomorphisms of trans-quasiconformal maps of the disk. Ann. Acad. Sci. Fenn. Math. 33(1), 241–260 (2008)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.