The structure of CR manifolds.

Abstract.

In this paper we study the topology of CR pseudoconvex manifo lds whose Reeb flow preserve the Levi metric.

Definition 1.
A CR-manifold of dimension $2n + 1$ is a manifold $N$ of dimension $2n + 1$ endowed with the followings properties:

Let $TN$ be the tangent bundle of $N$, there exists a subbundle $V$ of $TN \otimes \mathbb{C}$, of complex dimension $n$, such that $V \cap \overline{V} = 0$, where $\overline{V}$ is the complex conjugate of $V$, and $[V, V] \subset V$.

The bundle $\overline{V} = H$ is the kernel of a real 1-form $\theta \in T^*N$, where $T^*N$ is the cotangent bundle of $N$.

Let $(U_p)_{p \in I}$ be an open contractible covering of $N$ by open subsets. We denote by $\theta_p$ the restriction of $\theta$ to $U_p$, and $(u^p_1, \ldots, u^p_n)$ a basis of the restriction of the dual of $V$ to $U$, we have:

$$d(\theta_p) = \sum i h_{cd} u_c \wedge \overline{u_d}$$

The coefficients $h_{cd}$, define a pseudo-Hermitian metric on $V$. We can extend this metric to $H$ by supposing that $V$ is orthogonal to $\overline{V}$, and the complex conjugate is an isometry. This metric is called the Levi metric.

Definition 2.
The CR-manifold is pseudo convex if the Levi metric is positive definite. In this case there exists a vector field $X$ of $TN$ such that $L_X \theta = 0$, where $L_X$ is the Lie derivative. The vector field $X$ is called the Reeb vector field.

The purpose of this paper is to show the following result:

Theorem 1.
Let $N$ be a compact pseudo-convex CR manifold, $X$ a flow transverse to $H$ which preserves the Levi metric then the closure of the orbits of $X$ are torus, if these closure have the same dimension, then $N$ is the total space of a bundle whose typical fiber is the closure of an orbit of $X$ flow over a Satake manifold.

The complex transverse geometry of the Reeb flow.

Let $N$ be a CR pseudo-convex manifold, and $X$ a flow transverse to $H$ which preserves the Levi metric. The complex transverse bundle $TX$ of $X$ is
the quotient of $TN \otimes I_C$ by $X \otimes I_C$. Let $U$ be a contractible open subset of $N$, and $T$ a submanifold of $U$ transverse to $X$. We denote by $L_X N_U$, the restriction of the bundle of complex frames $L_X N$ of $TX N$ to $U$, we have a projection $p_U : L_X N \to LT \otimes I_C$, where $LT$ is the bundle of transverse complex frames of $T$. The kernel of the differential $dp_U$ of $dP$ define on $L_X N_U$ a distribution tangent to a flow. This distribution is independent of the choice of the local transversal $T$. We have thus defined on $L_X N$ a flow $\hat{X}$, which is called the lift of $X$.

The Levi metric $<,>$, defines an Hermitian reduction $H_X$ of $L_X N$ (which is invariant by the orbit of $\hat{X}$) of $L_X N$, since the manifold $N$ is compact, $H_X$ is also a compact submanifold. To show the theorem 1, we shall prove that the restriction $X'$ of $\hat{X}$ to $H_X$ is a riemannian flow, (in fact, we are going to construct a transverse parallelism to $X'$) and apply a well-known result of Yves Carriere on the structure of Riemannian flows.

**The transverse parallelism of $X'$.**

The manifold $H_X$ is a principal bundle over $N$ whose typical fiber is $U(2n)$, the group of Hermitian matrices. Each element $c$ of the Lie algebra $u(2n)$ of $U(2n)$ defines on $H_X$ a vector field $c_N$ defined by $c_N(x) = \frac{d}{dt=0} \exp(tc_N)(x)$. The vector $c_N$ are called the fundamental vector fields of $H_X$.

Let $\alpha$ be the fundamental form of $H_X$. Recall that for every element $u$ of the tangent space $TH_x N_x$ of $x \in TH_x$ (x is a linear map $L^{2n} \to T_{p(x)} N \otimes I_C / I_C \otimes X$, where $p : H_X \to N$ is the bundle projection map, $\alpha(u) = x^{-1} (dp_x (u))$. For each element $y$ of $L^{2n}$, we can define the vector field $\hat{y}$ of $H_X$ by setting $\alpha(\hat{y}) = y$.

The vector $\hat{y}$, $y \in L^{2n}$ and the fundamental vector fields define the transverse parallelism to $X'$.

**Proof of theorem 1.**

Let $X$ be an isometry transverse to the Levi metric, then the lift $X'$ of $X$ to $H_X$ is a riemannian flow, we can apply the result of Carriere. The closure of the orbits of $X$ are the projections of the closure of the orbits of $X'$ by the bundle map $H_X \to N$.

We can obtain this most general result:

**Theorem 2.**

Let $N$ be a pseudo convex compact CR manifold, endowed with a transverse flow $X$ which is a conformal flow in respect to the Levi metric, then the closure of the orbits of $X$ are torus.

**Proof.**

Consider the group generated by $U(n)$ and the complex homothetic maps, and we denote $L(n)$, its quotient by an homothetic map $h_{\lambda_0}$ such that the norm of $\lambda_0$ is strictly superior to 1. The transverse bundle of $X$ can be reduced to
The lifts $X'$ of $X$ to $L(n)$ is a transversely conformal analytic flow of codimension greater than 3. We can apply the theorem of Tarquini which asserts that in this situation that the flow of $X'$ is riemannian or Moebius. Then we apply the structure theorem for riemannian flows and Moebius flows.

The previous result suggests the study of transversely Hermitian foliations:

**Definition 3.**
Let $N$ be a manifold, and $\mathcal{H}$ a foliation defined on $N$, the foliation $\mathcal{H}$ is a transversely Hermitian foliation, if there exists a symmetric Hermitian basic 2-tensor $\langle , \rangle$ defined on $TN \otimes IC'$ such that:

- For each $u \in N$, $Tu\mathcal{H} \otimes IC'$ the tensor product of the subspace $T_x\mathcal{H}$ of $TuN$ tangent to $\mathcal{H}$ and $IC'$ is the kernel of $\langle , \rangle$.
- The projection of $\langle , \rangle$ to the transverse complex bundle of $\mathcal{H}$ is a positive definite Hermitian metric. We have the following nice result:

**Theorem 3.**
Let $N$ be a compact manifold endowed with a transversely Hermitian foliation $\mathcal{H}$, then the closure of the leaves of $\mathcal{H}$ are submanifolds.

**References.**
1. Yves Carriere, Flots riemanniens Asterisque 116
2. Tarquini, Feuilletages conformes, Annales Institut Fourier 2004