Abstract This paper is devoted to studying the representation of measures of non-generalized compactness, in particular, measures of noncompactness, of non-weak compactness and of non-super weak compactness, defined on Banach spaces and its applications. With the aid of a three-time order-preserving embedding theorem, we show that for every Banach space $X$, there exist a Banach function space $C(K)$ for some compact Hausdorff space $K$ and an order-preserving affine mapping $T$ from the super space $\mathcal{B}$ of all the nonempty bounded subsets of $X$ endowed with the Hausdorff metric to the positive cone $C(K)^+$ of $C(K)$, such that for every convex measure, in particular, the regular measure, the homogeneous measure and the sublinear measure of non-generalized compactness $\mu$ on $X$, there is a convex function $\mathcal{F}$ on the cone $V = T(\mathcal{B})$ which is Lipschitzian on each bounded set of $V$ such that

$$\mathcal{F}(T(B)) = \mu(B), \quad \forall B \in \mathcal{B}.$$  

As its applications, we show a class of basic integral inequalities related to an initial value problem in Banach spaces, and prove a solvability result of the initial value problem, which is an extension of some classical results due to Banas and Goebel (1980), Goebel and Rzymowski (1970) and Rzymowski (1971).

Keywords representation of measures of noncompactness, convex analysis, Lebesgue-Bochner measurability, integral inequality, initial value problem in Banach spaces

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1 Introduction

There are three goals of this paper: (1) to establish the representation theorem of convex measures of noncompactness (convex MNCs, for simplicity), in particular, regular MNCs, homogeneous MNCs, sublinear MNCs and of their generalizations, including convex measures of non-weak compactness (convex MNWCs), convex measures of non-super weak compactness (convex MNSWCs), convex measures of
the non-Radon-Nikodým property (convex MNRNPs) and convex measures of non-Asplundness (convex MNAs); (2) to establish a class of basic integral inequalities related to an initial value problem in Banach spaces; (3) as their applications, to discuss the solvability of the initial value problem.

The letter $X$ will always be an infinite-dimensional real Banach space, and $X^*$ is its dual. $B_X$ stands for the closed unit ball of $X$, and $B(x, r)$ stands for the closed ball centered at $x$ with radius $r$. $\mathcal{B}(X)$ denotes the collection of all the nonempty bounded subsets of $X$ endowed with the Hausdorff metric. $\Omega$ stands for the closed unit ball $B_{X^*}$ of $X^*$ endowed with the norm topology, and $C_b(\Omega)$ is the Banach space of all the continuous bounded functions on $\Omega$ endowed with the sup-norm. For a subset $A \subset X$, $\overline{A}$ stands for the norm closure of $A$, and $\text{co}(A)$ stands for the convex hull of $A$.

The main results of this paper consist of the following three parts.

**Part I.** A representation theorem of convex measures of non-generalized compactness (convex MNGCs). For every Banach space $X$, there exist a Banach function space $C(K)$ for some compact Hausdorff space $K$ and an order-preserving affine 1-Lipschitz mapping $\mathbb{T} : \mathcal{B}(X) \rightarrow C(K)^+$ such that for every convex MNGC (in particular, convex MNC, convex MNWC, convex MNSWC, convex MNRNP and convex MNA) $\mu$ defined on $X$, there is a continuous convex function $F$ on the positive cone $V \equiv \mathbb{T}(\mathcal{B}(X))$, which is monotone increasing in the order of set inclusion and Lipschitzian on each bounded subset of $V$, satisfying

$$F(\mathbb{T}(B)) = \mu(B), \quad \forall B \in \mathcal{B}(X).$$

If, in addition, $\mu$ is a sublinear measure of non-generalized compactness, then $F$ is a $\mu(B_X)$-Lipschitzian sublinear functional on $V$.

**Part II.** A class of basic integral inequalities. For every nonempty subset $G \subset L_1([0, a], X)$ of integrable $X$-valued functions with $\psi(t) \equiv \sup_{g \in G} \| g(t) \|$ integrable on $[0, a]$ such that the mapping $$\| G : [0, a] \rightarrow C_b(\Omega)$$
defined for $t \in [0, a]$ by

$$\| G(t)(\omega) = \sup_{g \in G} \langle \omega, g(t) \rangle \equiv \sigma_{G(t)}(\omega), \quad \omega \in \Omega \equiv B_{X^*},$$
is strongly (Lebesgue-Bochner) measurable, then for every convex MNGC $\mu$ defined on $X$, we have

$$\mu\left\{ \int_0^\tau G(s)ds \right\} \leq \frac{1}{\tau} \int_0^\tau \mu\{\tau G(s)\}ds, \quad \forall 0 < \tau \leq a;$$

in particular, if $\mu$ is a sublinear measure of non-generalized compactness (sublinear MNC) or $\tau \leq 1$, then

$$\mu\left\{ \int_0^\tau G(s)ds \right\} \leq \int_0^\tau \mu\{G(s)\}ds.$$  

**Part III.** The solvability of a Cauchy problem. As an application of the results mentioned above, we consider the solvability of the following initial value problem:

$$\begin{cases} x'(t) = f(t, x), \quad a \geq t > 0, \\ x(0) = x_0 \end{cases}$$

and give an extension of some classical solvability results of the problem due to Goebel and Rzymowski [22] (see also [35]) and due to Banaś and Goebel [6].

The study of MNCs and of their applications has continued for more than 90 years since the first MNC (Kuratowski’s MNC in the present terminology) was introduced by Kuratowski [28] in 1930. It has been shown that the theory of MNCs is used in a wide variety of topics in nonlinear analysis. Roughly speaking, an MNC $\mu$ is a non-negative function defined on the family $\mathcal{B}(M)$ consisting of all the nonempty bounded subsets of a complete metric space $M$, in particular, a Banach space $X$ and satisfies some specific
properties such as non-decreasing monotonicity in the order of the set inclusion, and the (most important) noncompactness that \( \mu(B) = 0 \) if and only if \( B \) is relatively compact in \( X \).

The first MNC (denoted by \( \alpha \)) was introduced by Kuratowski [28] for every \( B \in \mathcal{B}(X) \) of a complete metric space \( X \):

\[
\alpha(B) = \inf \left\{ d > 0 : B \subset \bigcup_{j \in F} E_j \subset X, \ d(E_j) \leq d, \ F \subset \mathbb{N}, \ F^c < \infty \right\},
\]

where \( d(E_j) \) denotes the diameter of \( E_j \), and \( F^c \) denotes the cardinality of the set \( F \subset \mathbb{N} \). The earliest successful application of Kuratowski’s MNC was in the fixed point theory. In 1955, Darbo [18] extended the Schauder fixed point theorem to noncompact mappings, named set-contractive operators. Since then, the study of MNCs and of their applications has become an active research area, and various MNCs have appeared. Among many other MNCs, the Hausdorff MNC \( \beta \) is another widely used MNC, which was introduced by Gohberg et al. [23] in 1957. It is defined for \( B \in \mathcal{B}(X) \) by

\[
\beta(B) = \inf \left\{ r > 0 : B \subset \bigcup_{x \in F} B(x, r), \ F \subset X, \ F^c < \infty \right\},
\]

where \( B(x, r) \) denotes the closed ball centered at \( x \in X \) with radius \( r \).

It is easy to observe that if \( \mu \) is either Kuratowski’s MNC \( \alpha \) or the Hausdorff MNC \( \beta \), then it satisfies the following three conditions:

1. **Noncompactness** \( B \in \mathcal{B}(X) \), \( \mu(B) = 0 \) if \( B \) is relatively compact.
2. **Monotonicity** \( A, B \in \mathcal{B}(X) \) with \( A \supset B \Rightarrow \mu(A) \geq \mu(B) \).
3. **Order preserving** \( A, B \in \mathcal{B}(X) \Rightarrow \mu(A \cup B) = \mu(A) \lor \mu(B) \).

In particular, if \( X \) is a Banach space over the scalar field \( F \), then we have the following conditions:

4. **Convexification invariance** \( B \in \mathcal{B}(X) \Rightarrow \mu(\text{co}(B)) = \mu(B) \).
5. **Absolute homogeneity** \( B \in \mathcal{B}(X), k \in F \Rightarrow \mu(kB) = |k|\mu(B) \).
6. **Subadditivity** \( A, B \in \mathcal{B}(X) \Rightarrow \mu(A + B) \leq \mu(A) + \mu(B) \).

**Definition 1.1.** Let \( X \) be a Banach space, \( \mathcal{B}(X) \) be the family of all the nonempty bounded subsets of \( X \), and \( \mu : \mathcal{B}(X) \to \mathbb{R}^+ \) be a real-valued function.

(i) (See [7]) \( \mu \) is said to be a **regular MNC** on \( X \) provided that it satisfies all the six properties (1)–(6) above.

(ii) (See [30]) \( \mu \) is called a **homogeneous MNC** on \( X \) if it satisfies the properties (1), (2) and (4)–(6).

(iii) We say that \( \mu \) is a **sublinear MNC** on \( X \) if it satisfies the properties (1), (2), (4), (6) and the following property:

7. **Positive homogeneity** \( B \in \mathcal{B}(X), k \geq 0 \Rightarrow \mu(kB) = k\mu(B) \).

(iv) \( \mu \) is a **convex MNC** on \( X \) if it satisfies the properties (1), (2), (4) and the following property:

8. **Convexity** \( \forall A, B \in \mathcal{B}(X) \) and \( 0 \leq \lambda \leq 1 \),

\[
\mu(\lambda A + (1 - \lambda)B) \leq \lambda \mu(A) + (1 - \lambda)\mu(B).
\]

**Remark 1.2.** Clearly, a regular MNC is a homogeneous MNC, a homogeneous MNC is a sublinear MNC, and a sublinear MNC is a convex MNC. The notion of the homogeneous MNC is called the “sublinear full MNC” by Banaś and Geobel [6].

In this paper, we always assume that the space \( X \) in question is an infinite-dimensional real Banach space. We should mention here that a Banach space setting for the study of MNGCs would lose no generality, because every metric space is isometric to a subset of a Banach space [10, Lemma 1.1]. Besides, the concept is easy to be generalized by Banach space theory; for example, it has been generalized in various ways such as the measure of non-weak compactness (see, for example, [4, 12, 19, 24, 25]), the measure of the non-Radon-Nikodym property [37], the measure of non-super weak compactness [14, 36], and more generally, the measure of the non-property \( \mathcal{D} \) [14].
Recall that $\mathcal{P}(X)$ denotes the collection of all the nonempty bounded subsets of a Banach space $X$ endowed with the Hausdorff metric $d_H$ defined for $A, B \in \mathcal{P}(X)$ by

$$d_H(A, B) = \inf\{d > 0 : A \subset B + dB_X, B \subset A + dB_X\},$$

where $B_X$ is the closed unit ball of $X$.

We use $\mathcal{C}(X)$ (⊂ $\mathcal{P}(X)$) to denote the collection of all the nonempty bounded closed convex subsets of a Banach space $X$ (endowed with the Hausdorff metric) with the following set addition $\oplus$:

$$A \oplus B = A + B, \quad \forall A, B \in \mathcal{C}(X)$$

and the usual scalar multiplication of sets:

$$\lambda A = \{ka : a \in A\}, \quad \forall A \in \mathcal{C}(X) \quad \text{and} \quad k \in \mathbb{R}.$$ If we put

$$\|\|A\|\| = \sup_{a \in A} ||a||, \quad \forall A \in \mathcal{C}(X),$$

then it becomes a complete normed semigroup with respect to the Hausdorff metric [14] (see also Section 2 in detail).

**Definition 1.3** (See [14]). Let $X$ be a Banach space and $\mathcal{D} \equiv \mathcal{P}(X)$ be a closed subsemigroup of $\mathcal{C}(X)$. It is said to be fundamental provided that it satisfies

1. $\bigcup \{D \in \mathcal{D} : X; (2) A, B \in \mathcal{D}$ entails $\overline{\mathcal{W}}(\pm A \cup \pm B) \in \mathcal{D}$;
2. $\emptyset \neq B \subset D \in \mathcal{D}$ implies $\overline{\mathcal{W}}(B) \in \mathcal{D}$.

There are many possibilities. But the following are most interesting to us: (i) $\mathcal{D} = \mathcal{K}(X)$, the subsemigroup of all the nonempty compact convex subsets of $X$; (ii) $\mathcal{D} = \mathcal{W}(X)$, all the nonempty weakly compact convex subsets of $X$; (iii) $\mathcal{D} = \mathcal{R}(X)$, all the nonempty bounded closed convex subsets $D$ of $X$ admitting the Radon-Nikodým property; (iv) $\mathcal{D} = \mathcal{M}(X)$, all the nonempty bounded closed convex Asplund subsets $D$ of $X$; (v) $\mathcal{D} = \text{sup-}\mathcal{W}(X)$, all the nonempty convex super weakly compact subsets $D$ of $X$ (see [13–17]).

**Definition 1.4.** Let $X$ be a Banach space, $\mathcal{D} \equiv \mathcal{P}(X)$ be a fundamental subsemigroup of $\mathcal{C}(X)$, and $\mu : \mathcal{P}(X) \to \mathbb{R}^+$ be a real-valued function. $\mu$ is said to be a regular measure of the non-property $\mathcal{D}$ (regular MNP$\mathcal{D}$) provided that

1. (non-property $\mathcal{D}$) $B \in \mathcal{P}(X)$, $\mu(B) = 0 \Leftrightarrow \overline{\mathcal{W}}(B) \in \mathcal{D}$;
2. (monotonicity) $A, B \in \mathcal{P}(X)$ with $A \supset B \Rightarrow \mu(A) \geq \mu(B)$;
3. (order preserving) $A, B \in \mathcal{P}(X) \Rightarrow \mu(A \cup B) = \mu(A) \vee \mu(B)$;
4. (convexification invariance) $B \in \mathcal{P}(X) \Rightarrow \mu(\text{co}(B)) = \mu(B)$;
5. (absolutes homogeneity) $B \in \mathcal{P}(X), k \in \mathbb{F} \Rightarrow \mu(kB) = |k|\mu(B)$;
6. (subadditivity) $A, B \in \mathcal{P}(X) \Rightarrow \mu(A + B) \leq \mu(A) + \mu(B)$.

In particular, a regular MNP$\mathcal{D}$ is just a regular MNC if $\mathcal{D} = \mathcal{K}$, and a regular MNC$\mathcal{D}$ if $\mathcal{D} = \mathcal{W}$.

**Definition 1.5.** Let $X$ be a Banach space, $\mathcal{D} \equiv \mathcal{P}(X)$ be a fundamental subsemigroup of $\mathcal{C}(X)$ and $\mu : \mathcal{P}(X) \to \mathbb{R}^+$ be a real-valued function.

1. $\mu$ is called a homogeneous measure of the non-property $\mathcal{D}$ (homogeneous MNP$\mathcal{D}$) on $X$ if it satisfies the properties (1), (2) and (4)–(6) in Definition 1.4.
2. $\mu$ is called a sublinear measure of the non-property $\mathcal{D}$ (sublinear MNP$\mathcal{D}$) on $X$ if it satisfies the properties (1), (2), (4) and (6) in Definition 1.4 and the following property:
3. (Positive homogeneity) $B \in \mathcal{P}(X), k \geq 0 \Rightarrow \mu(kB) = k\mu(B)$.
4. We say that $\mu$ is a convex measure of the non-property $\mathcal{D}$ (convex MNP$\mathcal{D}$) on $X$ if it satisfies the properties (1), (2) and (4) in Definition 1.4 and the following property:
5. (Convexity) $\mu(\lambda A + (1 - \lambda)B) \leq \lambda\mu(A) + (1 - \lambda)\mu(B)$, $\forall A, B \in \mathcal{P}(X)$ and $0 \leq \lambda \leq 1$. Therefore, a sublinear MNP$\mathcal{D}$ is a convex one.
In particular, a convex (resp. homogeneous, sublinear) MNP is just a convex (resp. homogeneous, sublinear) MNC if \( \mathcal{D} = \mathcal{K} \), and a convex (resp. homogeneous, sublinear) MNWC if \( \mathcal{D} = \mathcal{W} \).

We have already known that each MNC is a non-negative real-valued function defined on the super space \( \mathcal{D}(X) \) of \( X \). The problem related to representation theory of MNCs is a fundamental and difficult question. The main obstacle to establishing the theory is that we do not know how to turn such functions defined on the super spaces into functions defined on usual metric spaces. Therefore, a great deal of effort has been made in studying the representation problem of some concrete MNCs, especially, the Hausdorff MNC on typical Banach spaces such as \( C(K) \) and \( L_\mu \) (see, for example, [3, 5, 6, 8]). Nevertheless, the representation problem of abstract MNCs on a general Banach space has never been studied before. With the help of a three-time order-preserving embedding theorem [14] established in 2018 (which will be introduced in Section 2) and by arguments in convex analysis, especially, in subdifferentiability of convex functions defined on convex sets with empty interiors, in Section 3 of this paper, we show the representation theorem previously mentioned in Section 1 for convex MNCs.

Many mathematicians have made great effort to expect the following type of integral inequalities with various assumptions:

\[
\mu \left\{ \int_0^a x_n(\omega) d\omega : n \in \mathbb{N} \right\} \leq \int_0^a \mu \{ x_n(\omega) : n \in \mathbb{N} \} d\omega, \quad (1.8)
\]

where \( \{ x_n \} \) is a sequence of continuous \( X \)-valued functions in \( C([0, a], X) \), and \( \mu \) is the Hausdorff MNC or Kuratowski’s MNC. Such a type of integral inequalities arises from the initial value problem (1.5) in Banach spaces. Among many extra conditions which are sufficient for the solvability of the equation, one of the most important types is under hypotheses in terms of MNCs. Indeed, a standard proof for the existence of solutions employs the Darbo fixed point theorem in connection with the following Picard-Lindelöf operator:

\[
Ax(t) = x_0 + \int_0^t f(s, x(s)) ds, \quad x \in B, \quad (1.9)
\]

where \( B \) is a bounded subset of \( C([0, a], X) \), and \( f \) is called the Carathéodory function. One wants to obtain that if the Carathéodory function satisfies that \( f(t, \cdot) \) is condensing for all \( t \in [0, a] \), then the operator

\[
A : C([0, a], X) \to C([0, a], X)
\]

defined by (1.9) is also condensing. More precisely, one may expect that for \( \{ x_n \} \subset B \),

\[
\mu \left\{ \int_0^t f(s, x_n(s)) ds : n \geq 1 \right\} \leq \int_0^t \mu \{ f(s, x_n(s)) : n \geq 1 \} ds. \quad (1.10)
\]

Once we have established an estimate of the form (1.10), there is not much left to do to prove the existence of solutions to (1.5).

Some remarkable contributions have been made by Goebel and Rzymowdski [22], Banaś and Goebel [6], Mönch [31], Mönch and von Harten [32], Heinz [26] and Kunze and Schlüchtermann [27]. Nevertheless, it still comes as a surprise that the estimate (1.8) has been proven only under some very restrictive assumptions in the existing literature. A number of counterexamples constructed by Heinz [26] show that it is quite complicated and poses significant difficulties to present appropriate hypotheses to guarantee (1.8). In 1970, Goebel and Rzymowdski [22] first showed that (1.8) holds for the Hausdorff MNC \( \beta \) with the assumptions that \( \{ x_n \} \subset C([0, a], X) \) is bounded and equi-continuous. In 1980, Banaś and Goebel [6] further proved that (1.8) holds again under the same assumptions but one can substitute a sublinear MNC \( \mu \) for \( \beta \). Mönch and von Harten [31, 32] proved (1.8) with the assumptions that (a) \( \mu = \beta \), (b) \( X \) is a weakly compactly generated space (in particular, a separable Banach space) and (c) \( \{ x_n \} \subset C([0, a]; X) \) and there is \( \psi \in L_1([0, a]) \) such that \( \sup_n \| x_n(t) \| \leq \psi(t) \) a.e. In 1998, Kunze and Schlüchtermann [27] showed that (1.8) holds for a Grothendieck measure \( \mu \) assuming that \( X \) is strongly generated by a Grothendieck class. But for an arbitrary Banach space, one has to insert the factor 2 on the right-hand side of (1.10), i.e.,

\[
\alpha \left\{ \int_0^t x_n(s) ds : n \geq 1 \right\} \leq 2 \int_0^t \alpha \{ x_n(s) : n \geq 1 \} ds
\]
some classical results due to Goebel and Rzymowski \[22\], Banaś and Goebel \[6\] and Deimling \[20\].

In Section 4 of this paper, applying the representation theorem established in Section 3, we show the result presented in Part II.

We also show that the mapping $J_G$ is always weakly measurable if $G \subset L_1(I,X)$ is separable and the sup-envelop of $\{\|g\| : g \in G\}$ is again integrable. It is strongly measurable if, in addition, $G$ satisfies one of the following three conditions: (i) $G \subset C([0,a],X)$ is equi-continuous, (ii) $G$ is separable and equi-regulated, and (iii) $G$ is uniformly measurable.

The solvability question of the initial value problem (1.5) in infinite-dimensional Banach spaces is important because Peano’s existence theorem of the problem (1.5) in finite-dimensional spaces is not valid in infinite-dimensional Banach spaces. A number of mathematicians have made a series of contributions to this problem. In 1970, Goebel and Rzymowski \[22\], using the fixed point theorem, proved that (1.5) has a solution assuming that (a) $f$ is bounded and uniformly continuous on $[0,a] \times B(x_0,r)$, and (b) for the Hausdorff MNC $\beta$ defined on $X$, $\beta(f(t,B)) \leq w(t,\beta(B))$ for almost all $t \in [0,a]$, where $w$ is a Kamke function. In 1977, Deimling \[20\] by means of approximate solutions proved that (1.5) has a solution under the following assumptions: (a) $f$ is bounded and uniformly continuous on $[0,a] \times B(x_0,r)$, and (c) for Kuratowski’s MNC $\alpha$ defined on $X$, $\alpha(f(t,B)) \leq \alpha(t)A(B)$ for some $L > 0$ and for all $t \in [0,a]$. In 1980, Banaś and Goebel \[6\] showed that (1.5) has a solution assuming that (a) $f$ is bounded and uniformly continuous on $[0,a] \times B(x_0,r)$, and (d) $\mu(f(t,B)) \leq w(t,\mu(B))$ for almost all $t \in [0,a]$, where $w$ is a Kamke function and $\mu$ is a (symmetric) sublinear MNC. Motivated by Deimling \[20\], Mönch and von Harten \[32\] proved that (1.5) has a solution under the assumptions that $X$ is a weakly compactly generated space, $f$ is continuous, and there is a Kamke function $w$ such that $\beta(f(t,B)) \leq w(t,\beta(B))$ for almost all $t \in [0,a]$.

As an application of the representation theorem and the integral inequalities presented in Parts I and II, in Section 7 of this paper, we consider solvability of the initial value problem (1.5), and extend some classical results due to Goebel and Rzymowski \[22\], Banaś and Goebel \[6\] and Deimling \[20\].

2 Preliminaries

In this section, we introduce the three-time order-preserving embedding theorem established in \[14\]. This theorem plays an essential role in the study of the representation problem of measures of non-generalized compactness of this paper. It was proven by this theorem that every infinite-dimensional Banach space admits inequivalent regular measures of noncompactness, which is an affirmative answer to the question proposed by Goebel in 1978 (see \[1,2\]), also by Mallet-Paret and Nussbaum \[30\] in 2011. All the notions and results presented in this section can be found in \[14\].

**Definition 2.1.** Let $G$ be an abelian semigroup and $F \in \{\mathbb{R},C\}$. $G$ is said to be a module if there are two operations $(x,y) \in G \times G \to x + y \in G$ and $(\alpha,x) \in (F \times G) \to \alpha x \in G$ satisfying

$$(\lambda \mu)g = \lambda(\mu g), \quad \forall \lambda, \mu \in \mathbb{R} \quad \text{and} \quad g \in G,$$

$$\lambda(g_1 + g_2) = \lambda g_1 + \lambda g_2, \quad \forall \lambda \in F \quad \text{and} \quad g_1, g_2 \in G$$

and

$$1g = g \quad \text{and} \quad 0g = 0, \quad \forall g \in G.$$

A module $G$ endowed with a norm is called a normed semigroup.

For a (real) Banach space $X$, we denote by $\mathcal{C}(X)$ (or, simply, $\mathcal{C}$) the collection of all the nonempty bounded closed convex sets of $X$ with respect to the following linear operations:

(a) $A \oplus B = \{a + b : a \in A, b \in B\}$ for all $A, B \in \mathcal{C}$;

(b) $kC = \{ke : e \in C\}$ for all $k \in \mathbb{F}$ and $C \in \mathcal{C}$.

We further define the following function $||| \cdot |||$ for $C \in \mathcal{C}$ by

$$|||C||| = \sup_{c \in C} ||c||.$$  \hspace{1cm} (2.1)
Then it is easy to check that $|||\cdot|||$ is a norm on $\mathcal{C}$, i.e., it satisfies

(i) $|||C||| \geq 0$ for all $C \in \mathcal{C}$, and $|||C|||=0 \iff C=\{0\}$;

(ii) $|||kC|||=|k|\cdot|||C|||$ for all $C \in \mathcal{C}$ and $k \in \mathbb{R}$;

(iii) $|||A \oplus B||| \leq |||A||| + |||B|||$ for all $A,B \in \mathcal{C}$.

Therefore, $(\mathcal{C},|||\cdot|||)$ is a normed (real) semigroup.

If we equip the normed semigroup $\mathcal{C}$ with the Hausdorff metric $d_H$:

$$d_H(A,B) = \inf\{r>0 : A \subset B + rB_X, B \subset A + rB_X\}, \quad A,B \in \mathcal{C},$$

then

$$|||C||| = d_H(\{0\},C), \quad \forall C \in \mathcal{C}. \quad (2.3)$$

We also use $\mathcal{K}(X)$ to denote the (complete) subsemigroup of $\mathcal{C}(X)$ consisting of all the nonempty compact convex subsets.

**Definition 2.2.** Let $\Gamma_1$ and $\Gamma_2$ be two partially ordered sets, and $f : \Gamma_1 \to \Gamma_2$ be a mapping.

(i) Then $f$ is said to be (resp. fully) order-preserving provided that it is bijective and satisfies $f(x) \geq f(y)$ in $\Gamma_2$ if (resp. and only if) $x \geq y$ in $\Gamma_1$.

(ii) If both $\Gamma_1$ and $\Gamma_2$ are modules, then we call the mapping $f$ affine if it satisfies

$$f(ax+by) = af(x) + bf(y)$$

for all $a,b \geq 0$ and $x,y \in \Gamma_1$.

For each nonempty subset $A \subset X$, we denote by $\sigma_A$ the support function of $A$ restricted to $\Omega \equiv B_{X^*}$, i.e.,

$$\sigma_A(x^*) = \sup_{a \in A} \langle x^*, a \rangle \quad \text{for } x^* \in \Omega. \quad (2.4)$$

Let $\Omega$ be endowed with the norm topology of $X^*$, and $\mathcal{S}(\Omega)$ be the wedge of all the continuous and $w^*$-lower semicontinuous sublinear functions on $X^*$ but restricted to $\Omega$. Then it is a closed cone of $C_b(\Omega)$, the Banach space of all the continuous functions bounded on $\Omega$ endowed with the sup-norm. Let $J : \mathcal{C}(X) \to \mathcal{S}(\Omega)$ be defined by

$$J(C) = \sigma_C, \quad \forall C \in \mathcal{C}(X). \quad (2.5)$$

If we order the normed semigroup $\mathcal{C}(X)$ by set inclusion, i.e., $A \succeq B$ if and only if $A \supseteq B$, then we have the following theorem.

**Theorem 2.3** (See [14, Theorem 2.3]). Let $X$ be a Banach space. Then

(i) $\mathcal{C}(X)$ is complete with respect to the Hausdorff metric;

(ii) $J : \mathcal{C}(X) \to \mathcal{S}(\Omega)$ is a fully order-preserving affine surjective isometry, i.e., for all $A,B \in \mathcal{C}(X)$,

$$d_H(A,B) = |||J(A) - J(B)||| \equiv \sup_{x^* \in \Omega} |\sigma_A(x^*) - \sigma_B(x^*)|. \quad (2.6)$$

**Lemma 2.4** (See [14, Lemma 2.5]). Suppose that $X$ is a Banach space, and $\mathcal{D} = \mathcal{D}(X)$ is a subsemigroup of $\mathcal{C} = \mathcal{C}(X)$. Then $\mathcal{D}$ is closed if and only if every $C \in \mathcal{C}$ satisfies the condition that for every $\varepsilon > 0$ there exists a $D \in \mathcal{D}(X)$ so that $C \subset D + \varepsilon B_X$ is in $\mathcal{D}$.

All the notions related to Banach lattices and abstract $M$ spaces are the same as in [29]. Recall that a partially ordered real Banach space $Z$ is called a Banach lattice provided that

(i) $x \leq y$ implies $x+z \leq y+z$ for all $x,y,z \in Z$;

(ii) $ax \geq 0$ for all $a \geq 0$ in $Z$ and $a \in \mathbb{R}^+$;

(iii) both $x \lor y$ and $x \land y$ exist for all $x,y \in Z$;

(iv) $||x|| \leq ||y||$ whenever $|x| \equiv x \lor -x \leq y \lor -y \equiv |y|$.

It follows from (iv) and

$$|x-y| = |x \lor z - y \lor z| + |x \land z - y \land z| \quad \text{for all } x,y,z \in Z \quad (2.7)$$
that the lattice operations are norm continuous.

By a sublattice of a Banach lattice $Z$, we mean a linear subspace $Y$ of $Z$ so that $x \lor y$ (and also $x \land y = x + y - x \lor y$) belongs to $Y$ whenever $x, y \in Y$. A lattice ideal $Y$ in $Z$ is a sublattice of $Z$ satisfying that $|z| \leq |y|$ for $z \in Z$ and for some $y \in Y$ implies $z \in Y$. We say that an operator $U$ from a Banach lattice $X$ to another Banach lattice $Y$ is an order isometry provided that it is an isometry and satisfies that $x \geq y \in X$ entails $Ux \geq Uy$.

In the following discussion, unless stated explicitly, we always assume that the lattice operations $\lor$ and $\land$ in a real-valued function space $C(K)$ are defined for $x, y \in C(K)$ and $k \in K$ by

$$x(k) \lor y(k) = \max\{x(k), y(k)\}, \quad x(k) \land y(k) = \min\{x(k), y(k)\}. \quad (2.8)$$

Given a fundamental subsemigroup $\mathcal{D}$ of $\mathcal{C}(X)$ (see Definition 1.3), let $J$ be defined as (2.5), and put $E_\mathcal{D} = J\mathcal{D} - J\mathcal{D}$.

**Theorem 2.5** (See [14, Theorem 3.1]). Let $X$ be a Banach space. Then for any fundamental subsemigroup $\mathcal{D}$ of $\mathcal{C}(X)$, the space $E_\mathcal{D}$ is a Banach lattice.

**Theorem 2.6** (See [14, Theorem 3.2]). Suppose that $X$ is a Banach space. Then

(i) $E_{\mathcal{X}} = J\mathcal{X} - J\mathcal{X}$ is a lattice ideal of the Banach lattice $E_{\mathcal{X}} = J\mathcal{X} - J\mathcal{X}$;

(ii) the quotient space $E_{\mathcal{X}}/E_{\mathcal{X}}$ is an abstract $M$-space; therefore, there exist a Banach function space $C(K)$ for some compact Hausdorff space $K$ and an order isometry $T$ from $E_{\mathcal{X}}/E_{\mathcal{X}}$ onto a sublattice of $C(K)$;

(iii) $TQ\mathcal{C}$ is a closed cone contained in the positive cone $C(K)^+$ of $C(K)$, where $Q : E_{\mathcal{X}} \to E_{\mathcal{X}}/E_{\mathcal{X}}$ is the quotient mapping and $T : E_{\mathcal{X}}/E_{\mathcal{X}} \to C(K)$ is the corresponding order isometry.

Note that for a nonempty bounded subset $A \subset X^*$, $\| \cdot \|_A = \sigma_{A,u-A}$ defines a continuous semi-norm on $X$.

**Theorem 2.7** (See [14, Theorem 5.5]). Let all the notions be the same as in Theorem 2.6. Then for every nonempty bounded set $F \subset C(K)^*$ of positive functionals satisfying that for each $0 \neq u \in TQ\mathcal{C}$, there exists a $\varphi \in F$ so that $\langle \varphi, u \rangle > 0$, the following formula defines a homogeneous MNC $\mu$ on $X$:

$$\mu(B) = \| T(Q\mathcal{C}B) \|_F \quad \text{for all } B \in \mathcal{D}, \quad (2.9)$$

where $\| u \|_F = \sup_{\varphi \in F, u} \langle \varphi, u \rangle$ for all $u \in C(K)$.

In particular, the Hausdorff MNC $\beta$ on $X$ satisfies

$$\beta(B) = \| T\mathcal{C}B \| \quad \text{for all } B \in \mathcal{D}. \quad (2.10)$$

Therefore, $TQ\mathcal{C}B = 0$ if and only if $B$ is relatively compact.

**Remark 2.8.** Since the three mappings $J : \mathcal{C}(X) \to C_b(\Omega)$, $Q : E_{\mathcal{X}} \to E_{\mathcal{X}}/E_{\mathcal{X}}$ and $T : E_{\mathcal{X}}/E_{\mathcal{X}} \to C(K)$ are order-preserving, $T \equiv TQJ$ is an order-preserving 1-Lipschitzian mapping from $\mathcal{C}(X)$ to $C(K)^+$, which is called the three-time order-preserving affine mapping.

### 3 Representation of convex measures of noncompactness

In this section, we are devoted to the representation of convex MNCs. We denote again by $\mathcal{B}(X)$ the set of all the nonempty bounded subsets of a Banach space $X$, and by $\mathcal{C}(X)$ (resp. $\mathcal{K}(X)$) the cone of all the nonempty closed (resp. compact) convex subsets of $X$ endowed with the operations addition $\oplus$ and scalar multiplication of sets, and endowed with the Hausdorff metric $d_H$. For a Banach space $X$, the Banach spaces $C_b(\Omega)$, $E_{\mathcal{X}}$, $E_{\mathcal{X}}$ and $C(K)$ and the mappings $J$, $Q$, $T$ and $\mathcal{T} = TQJ$ are the same as in Section 2. Let $V = \mathcal{T}\mathcal{C}(X)$. Then $V$ is a closed subcone of the positive cone $C(K)^+$ of $C(K)$.

**Lemma 3.1.** Let $X$ be a Banach space, and $f, g \in V \equiv \mathcal{T}\mathcal{C}(X)$. Then $f \leq g$ if and only if there exist $A, B \in \mathcal{C}(X)$ with $f = T\mathcal{A}$ and $g = T\mathcal{B}$ such that for all $\varepsilon > 0$, there is $K_\varepsilon \in \mathcal{K}(X)$ satisfying

$$A \subset B + K_\varepsilon + \varepsilon B_X. \quad (3.1)$$
Proof. Sufficiency. Suppose that \( A, B \in \mathcal{C}(X) \) such that \( f = TA \) and \( g = TB \). Let

\[
B_\varepsilon = B + K_\varepsilon + \varepsilon B_X \quad (\varepsilon > 0).
\]

Since \( \mathbb{T} \) is affine and order-preserving with \( \mathbb{T} C = 0 \) for all \( C \in \mathcal{H}(X) \), we obtain

\[
\mathbb{T} A \leq \mathbb{T} B_\varepsilon = \mathbb{T} B + \mathbb{T} K_\varepsilon + \varepsilon \mathbb{T} B_X.
\]

Since \( \varepsilon \) is arbitrary, we obtain

\[
f = \mathbb{T} A \leq \mathbb{T} B = g.
\]

Necessity. Note that \( \mathbb{T} = TQJ \). Suppose \( f \leq g \in V \). By the definition of \( V \), there exist \( A, B \in \mathcal{C}(X) \) such that

\[
TQJ(A) = \mathbb{T} A = f \leq g = \mathbb{T} B = TQJ(B).
\]

Since \( T : E_{\mathcal{C}} / E_{\mathcal{X}} \to T(E_{\mathcal{C}} / E_{\mathcal{X}}) \subset C(K) \) is fully order-preserving, we get

\[
QJ(A) = (T^{-1} T) A \leq (T^{-1} T) B = QJ(B).
\]

Therefore,

\[
QJ(B) - QJ(A) \geq 0.
\]

By the definition of the order-preserving quotient mapping \( Q \), for every \( \varepsilon > 0 \), there exist \( C, D \in \mathcal{H}(X) \) such that

\[
J(B) - J(A) - (J(C) - J(D)) \geq -\varepsilon.
\]

Positive homogeneity of \( J(A), J(B), J(C) \) and \( J(D) \) entails

\[
J(B) - J(A) - (J(C) - J(D)) \geq -\varepsilon \cdot \|X\|
\]

Note that

\[
-(J(C) - J(D)) = J(D) - J(C) \leq J(D - C).
\]

Then

\[
J(B) + J(D - C) + \varepsilon \cdot \|X\| \geq J(A).
\]

Let \( K_\varepsilon = D - C \). Then the inequality above is equivalent to

\[
A \subset B + K_\varepsilon + \varepsilon B_X.
\]

This completes the proof. \( \square \)

Corollary 3.2. Let \( X \) be a Banach space and \( f, g \in V = \mathbb{T} \mathcal{C}(X) \). Then \( f = g \) if and only if there exist \( A, B \in \mathcal{C}(X) \) with \( f = TA \) and \( g = TB \) such that for all \( \varepsilon > 0 \), there are \( K_1, K_2 \in \mathcal{H}(X) \) satisfying

\[
B \in A + K_1 + \varepsilon B_X \quad \text{and} \quad A \in B + K_2 + \varepsilon B_X.
\]

Lemma 3.3. Let \( f, g \in V \) with \( \|f - g\| = r \). Then \( |f - g| \leq r \mathbb{T} (B_X) \).

Proof. Let \( f_C = \mathbb{T} C = TQJ(C) \) for all \( C \in \mathcal{C}(X) \). Since \( f, g \in V \), there exist \( A, B \in \mathcal{C}(X) \) such that

\[
f = \mathbb{T} A = f_A, \quad g = \mathbb{T} B = f_B.
\]

Since \( T : Q(J(\mathcal{C}(X))) \to V \) is a surjective order-preserving isometry,

\[
T^{-1}(f_A) = QJ(A) \quad \text{and} \quad T^{-1}(f_B) = QJ(B)
\]

satisfy

\[
\|Q(J(A)) - Q(J(B))\| = \|Q(J(A) - J(B))\| = r.
\]
By the definition of the quotient mapping \( Q : E_\varepsilon \rightarrow E_\varepsilon / E_\varepsilon \) for all \( \varepsilon > 0 \), there exist \( K_1, K_2 \subset X \) such that
\[
\|(J(A) - J(B)) - (J(K_1) - J(K_2))\|_{C_b(\Omega)} < r + \varepsilon.
\]
Positive homogeneity of the functions \( J(A), J(B), J(K_1) \) and \( J(K_2) \) entails
\[
|(J(A) - J(B)) - (J(K_1) - J(K_2))| < (r + \varepsilon)\|\cdot\|_{X^*}.
\]
Note that \( J(B_X) = \|\cdot\|_{X^*} \). Then
\[
J(A) + J(K_2) < (r + \varepsilon)J(B_X) + J(B) + J(K_1)
\]
and
\[
J(B) + J(K_1) < (r + \varepsilon)J(B_X) + J(A) + J(K_2).
\]
Consequently,
\[
\mathbb{T}(A) \leq \mathbb{T}(B) + (r + \varepsilon)\mathbb{T}(B_X), \quad \mathbb{T}(B) \leq \mathbb{T}(A) + (r + \varepsilon)\mathbb{T}(B_X).
\]
Therefore,
\[
|\mathbb{T}(A) - \mathbb{T}(B)| \leq (r + \varepsilon)\mathbb{T}(B_X).
\]
Since \( \varepsilon \) is arbitrary,
\[
|f - g| = |f_A - f_B| \leq r\mathbb{T}(B_X).
\]
This completes the proof. \( \square \)

Recall (see Definition 1.1(iv)) that for a Banach space \( X, \mu : \mathcal{B}(X) \rightarrow \mathbb{R}^+ \) is said to be a convex MNC provided that it satisfies the following four properties:

(P1) **(Noncompactness)** \( B \in \mathcal{B}(X), \mu(B) = 0 \iff B \) is relatively compact.

(P2) **(Monotonicity)** \( A, B \in \mathcal{B}(X) \) with \( A \supset B \Rightarrow \mu(A) \geq \mu(B) \).

(P3) **(Convexification invariance)** \( B \in \mathcal{B}(X) \Rightarrow \mu(co(B)) = \mu(B) \).

(P4) **(Convexity)** \( \mu(\lambda A + (1 - \lambda)B) \leq \lambda \mu(A) + (1 - \lambda)\mu(B), \forall A, B \in \mathcal{B}(X) \) and \( 0 \leq \lambda \leq 1 \).

Every convex MNC admits the following basic properties.

**Theorem 3.4.** Let \( X \) be a Banach space, and \( \mu : \mathcal{B}(X) \rightarrow \mathbb{R}^+ \) be a convex MNC. Then we have the following conditions:

(i) **(Density determination)** \( \mu(B) = \mu(B), \forall B \in \mathcal{B}(X) \).

(ii) **(Translation invariance)** \( \mu(B + C) = \mu(B), \forall B \in \mathcal{B}(X), C \in \mathcal{X} \).

(iii) **(Negligibility)** \( \mu(B \cup C) = \mu(B), \forall B \in \mathcal{B}(X), C \in \mathcal{X} \).

(iv) **(Continuity)** \( \emptyset \neq B_{n+1} \subset B_n \in \mathcal{B}(X), n \in \mathbb{N}; \mu(B_n) \rightarrow 0 \Rightarrow \bigcap_n \overline{B_n} \neq \emptyset \).

**Proof.**

(i) Monotonicity of \( \mu \) entails
\[
\mu(A) \leq \mu(\overline{A}), \quad \forall A \in \mathcal{B}(X).
\]
On the other hand, let
\[
M = \|\|A\|\| = \sup_{a \in A} \|a\| > 0.
\]
Then \( \forall M \geq \varepsilon > 0 \),
\[
\overline{A} \subset A + \frac{\varepsilon}{M}B_X \subset \left(1 - \frac{\varepsilon}{M}\right)A + \frac{\varepsilon}{M}(M + 1)B_X.
\]
Convexity and monotonicity of \( \mu \) imply
\[
\mu(\overline{A}) \leq \left(1 - \frac{\varepsilon}{M}\right)\mu(A) + \frac{\varepsilon}{M}\mu((M + 1)B_X).
\]
Since \( \varepsilon \) is arbitrary, we obtain
\[
\mu(\overline{A}) \leq \mu(A) \leq \mu(\overline{A}).
\]
(ii) It is easy to check that for each $B \in \mathcal{B}(X)$, $f(t) \equiv \mu(tB)$, $t \in \mathbb{R}$ defines a continuous convex function on $\mathbb{R}$. Indeed, since $\mu$ is a convex MNC, $f$ is convex on $\mathbb{R}$. Consequently, $f$ is continuous on $\mathbb{R}$.

For all $B \in \mathcal{B}(X)$, $C \in \mathcal{X}(X)$ and $1 \geq \lambda > 0$,

$$
\mu(B + C) = \mu \left[ \lambda \left( \frac{1}{\lambda} C \right) + (1 - \lambda) \left( \frac{1}{1 - \lambda} B \right) \right] \\
\leq \lambda \mu \left( \frac{1}{\lambda} C \right) + (1 - \lambda) \mu \left( \frac{1}{1 - \lambda} B \right) \\
= (1 - \lambda) \mu \left( \frac{1}{1 - \lambda} B \right) \rightarrow \mu(B),
$$

as $\lambda \to 0^+$. Thus,

$$
\mu(B + C) \leq \mu(B), \quad \forall x_0 \in X, \quad B \in \mathcal{B}(X).
$$

We substitute $B + C$ for $B$ in the inequality above. Then

$$
\mu(B) \leq \mu[-C + (B + C)] \leq \mu(B + C).
$$

Therefore,

$$
\mu(B + C) = \mu(B), \quad \forall B \in \mathcal{B}(X), \quad C \in \mathcal{X}(X).
$$

(iii) Given $B \in \mathcal{B}(X)$, $x_0 \in X$, choose any $b_0 \in B$ and let $B_1 = B - x_0$ and

$$
C = [0, x_0 - b_0] \equiv \{ \lambda(x_0 - b_0) : 0 \leq \lambda \leq 1 \}.
$$

Then by (ii), we have just proven

$$
\mu(\{x_0\} \cup B) = \mu(\{0\} \cup B_1).
$$

Note that $\{0\} \cup B_1 \subset B_1 + C$ and $C \in \mathcal{X}(X)$. Then again by (ii),

$$
\mu(\{0\} \cup B_1) \leq \mu(B_1 + C) = \mu(B_1) = \mu(B).
$$

Thus,

$$
\mu(\{x_0\} \cup B) = \mu(B).
$$

It follows that $\mu(B \cup F) = \mu(B)$ for any nonempty finite set $F \subset X$. For any $C \in \mathcal{X}(X)$ and for any $1 > \epsilon > 0$, let $F_\epsilon$ be a finite set such that $C \subset F_\epsilon + \epsilon B_X$. Then

$$
\mu(B \cup C) \leq \mu(B \cup (F_\epsilon + \epsilon B_X)) = \mu(B \cup \epsilon B_X) \\
\leq \mu \left( \frac{1}{1 + \epsilon} B + \frac{\epsilon}{1 + \epsilon} (B \cup (1 + \epsilon)B_X) \right) \\
\leq \frac{1}{1 + \epsilon} \mu(B) + \frac{\epsilon}{1 + \epsilon} \mu(B \cup (1 + \epsilon)B_X) \\
\rightarrow \mu(B), \quad \text{as } \epsilon \to 0^+.
$$

It follows that

$$
\mu(B \cup C) = \mu(B), \quad \forall B \in \mathcal{B}(X), \quad C \in \mathcal{X}(X).
$$

(iv) Let $\{B_n\} \subset \mathcal{B}(X)$ be a sequence of nonempty subsets with $B_{n+1} \subset B_n \in \mathcal{B}(X)$ and $n \in \mathbb{N}$, and $\mu(B_n) \to 0$. For each $n \in \mathbb{N}$, choose any $x_n \in B_n$. Then by (iii),

$$
\mu(\{x_n\}) = \mu(\{x_n\}_{n \geq k}) \leq \mu(B_k) \to 0, \quad \text{as } k \to \infty.
$$

It follows that $\{x_n\}$ is relatively compact. Let $x_0$ be a cluster point of $\{x_n\}$. Then $x_0 \in \bigcap_n \overline{B_n}$. \hfill \square

For $B \in \mathcal{B}(X)$, we simply denote $T(B)$ by $f_B$ in the sequel.
Lemma 3.5. Let \( \mu \) be a convex MNC on a Banach space \( X \). Then

(i) \( F(f_B) = \mu(B) \), \( B \in \mathcal{B}(X) \) defines a monotone convex function on \( V \);

(ii) for each \( r > 0 \), \( F \) is bounded by \( b_r \equiv \mu(rB_X) \) on \( V \cap (rB_{C(K)}) \);

(iii) for each \( f \in V \),
\[
p(g) = \lim_{t \to 0^+} \frac{F(f + tg) - F(f)}{t}, \quad g \in V
\]
defines a non-negative sublinear functional \( p \) on \( V \) satisfying
\[
p(g) \leq F(f + g) - F(f), \quad \forall g \in V.
\]

Proof. (i) We first show that \( F \) is well defined on \( V \). Indeed, it follows from Corollary 3.2 that for all \( A, B \in \mathcal{B}(X) \), \( f_A = f_B \) if and only if \( \forall \varepsilon > 0, \exists K_1, K_2 \in \mathcal{K}(X) \) such that
\[
B \subset K_1 + A + \varepsilon B_X \quad \text{and} \quad A \subset K_2 + B + \varepsilon B_X.
\]
Thus,
\[
F(f_A) = F(f_B) \iff \mu(A) = \mu(B).
\]

Let \( f_A, f_B \in V \) and \( \lambda \in [0, 1] \). Then
\[
F[\lambda f_A + (1 - \lambda)f_B] = F[\mathbb{T}(\lambda A + (1 - \lambda)B)]
\]
\[
= \mu(\lambda A + (1 - \lambda)B) \leq \lambda \mu(A) + (1 - \lambda)\mu(B)
\]
\[
= \lambda F(f_A) + (1 - \lambda)F(f_B).
\]
Therefore, the convexity of \( F \) is shown.

To show the monotonicity of \( F \), let \( f_A \geq f_B \in V \). Then it follows from Lemma 3.1 that for all \( \varepsilon > 0 \), there is \( K_\varepsilon \in \mathcal{K} \) satisfying
\[
B \subset A + K_\varepsilon + \varepsilon B_X.
\]

It follows from Theorem 3.4(ii) that
\[
F(f_B) = \mu(B) \leq \mu(A + \varepsilon B_X) \leq (1 - \varepsilon)\mu\left(\frac{1}{1 - \varepsilon}A\right) + \varepsilon \mu(B_X)
\]
\[
= (1 - \varepsilon)F\left(\frac{1}{1 - \varepsilon}f_A\right) + \varepsilon F(f_{B_X}).
\]
Note that \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) defined for \( t \in \mathbb{R}^+ \) by \( g(t) = F(tf_A) \) is a convex function. It is continuous at \( t = 1 \). Thus, let \( \varepsilon \to 0^+ \) in the inequalities above. Then
\[
F(f_B) \leq F(f_A).
\]

(ii) Let \( r > 0 \) and \( f = f_B \in V \cap (rB_{C(K)}) \). Then \( ||f|| \leq r \). By Lemma 3.3, \( f \leq rf_{B_X} = f_{rB_X} \). Due to the monotonicity of \( F \),
\[
0 \leq F(f_B) \leq F(f_{rB_X}) = \mu(rB_X) = b_r.
\]

(iii) Let \( f, g \in V \). Since \( V \) is a cone, \( F(f + tg) \) is clearly well defined for all \( t \geq 0 \). By (i), we have just proven that \( F(f + tg) \) is convex in \( t \in \mathbb{R}^+ \). For all \( 0 < t < s \leq 1 \),
\[
0 \leq G(t) \equiv \frac{F(f + tg) - F(f)}{t}
\]
\[
= \frac{F[\frac{1}{t}(f + sg) + \frac{s-t}{s}f] - F(f)}{t}
\]
\[
\leq \frac{1}{t}F(f + sg) + \frac{s-t}{s}F(f) - F(f)
\]
\[
= \frac{1}{s}F(f + sg) - \frac{t}{s}F(f)
\]
\[
= \frac{F(f + sg) - F(f)}{s}
\]
\[
= G(s).
\]
Therefore, $G(t)$ is increasing and $t \in \mathbb{R}^+$. Consequently,

$$0 \leq p(g) = \lim_{t \to 0^+} G(t) \leq G(1) \leq F(f + g) - F(f).$$

Since $F$ is convex on $V$, $p$ is a non-negative sublinear functional on $V$ and satisfies

$$p(g) \leq F(f + g) - F(f), \quad \forall g \in V.$$

This completes the proof. \qed

**Lemma 3.6.** Let $\mu$ be a convex measure of noncompactness on a Banach space $X$, and $F$ be defined as Lemma 3.5. Then $F$ is continuous on $V$.

*Proof.* We first show that $F$ is continuous at 0. Let $\{u_n\} \subset V$ be a null sequence with $r_n = \|u_n\|$, $n \in \mathbb{N}$. By Lemma 3.3, for all sufficiently large $n \in \mathbb{N}$,

$$0 \leq F(u_n) \leq F(\|u_n\| f_B) \leq \|u_n\| F(f_B) \to 0.$$

Therefore, $F$ is continuous at 0.

To show that $F$ is continuous at each point $v_0 \in V$, let $\{v_n\} \subset V$ such that $v_n \to v_0$, and $r_n = \|v_n - v_0\|$. Again by Lemma 3.3,

$$|v_n - v_0| \leq r_n f_B, \quad n = 0, 1, \ldots$$

This implies that

$$v_n \leq v_0 + r_n f_B, \quad v_0 \leq v_n + r_n f_B, \quad n = 1, 2, \ldots \quad (3.5)$$

It follows from the first inequality of (3.5) that

$$v_n \leq \frac{1}{1 + r_n} [(1 + r_n)v_0] + \frac{r_n}{1 + r_n} [(1 + r_n)f_B].$$

Since $F$ is convex,

$$F(v_n) \leq \frac{1}{1 + r_n} F((1 + r_n)v_0) + \frac{r_n}{1 + r_n} F((1 + r_n)f_B). \quad (3.6)$$

Since $\xi$ (defined for $t \in \mathbb{R}^+$ by $\xi(t) \equiv F((1 + t)v_0)$) is a convex function, it is necessarily continuous in $t \in \mathbb{R}^+$. Therefore,

$$\frac{1}{1 + r_n} F((1 + r_n)v_0) \to F(v_0) \quad \text{and} \quad \frac{r_n}{1 + r_n} F((1 + r_n)f_B) \to 0, \quad (3.7)$$

and (3.6) and (3.7) together imply

$$\limsup_n F(v_n) \leq F(v_0). \quad (3.8)$$

It follows from the second inequality of (3.5) that

$$\frac{1}{1 + r_n} v_0 \leq \frac{1}{1 + r_n} v_n + \frac{r_n}{1 + r_n} f_B.$$

Convexity of $F$ implies that

$$F\left(\frac{1}{1 + r_n} v_0\right) \leq \frac{1}{1 + r_n} F(v_n) + \frac{r_n}{1 + r_n} F(f_B). \quad (3.9)$$

Therefore,

$$\liminf_n F(v_n) \geq F(v_0). \quad (3.10)$$

Continuity of $F$ at $v_0$ follows from (3.8) and (3.10). \qed

For a cone $C$ of a Banach space $Z$, we say that a linear functional $z^* \in Z^*$ is a positive functional on $C$ if it is non-negative valued on $C$ or equivalently, $z^*|_{X_C}$ is a positive functional of the subspace $X_C \equiv C - C$ with respect to the “positive” cone $C$ in the usual sense.
Lemma 3.7. Let $V_0 \subset V$ be a subcone of $V$ with $f_{B_X} \in V_0$, and $E_{V_0} = V_0 - V_0$. Then for every functional $x^* \in E_{V_0}^*$ which is positive on $V_0$, we have
\[
\|x^*\|_{E_{V_0}} = \langle x^*, f_{B_X} \rangle.
\] (3.11)

Proof. Let $u, v \in V_0$ and $h = u - v \in E_{V_0}$ with $\|h\| \leq 1$. Then by Lemma 3.3,
\[
u u \leq v + f_{B_X} \quad \text{and} \quad v \leq u + f_{B_X}.
\]
Since $x^*$ is positive on $V_0$, we obtain
\[
\langle x^*, u \rangle \leq \langle x^*, v \rangle + \langle x^*, f_{B_X} \rangle
\]
and
\[
\langle x^*, v \rangle \leq \langle x^*, u \rangle + \langle x^*, f_{B_X} \rangle.
\]
Therefore,
\[
|\langle x^*, h \rangle| = |\langle x^*, u - v \rangle| \leq \langle x^*, f_{B_X} \rangle.
\]
The inequality above is clearly equivalent to (3.11). \qed

Lemma 3.8. Suppose that $V_0 \subset V$ is a subcone of $V$, and $u \in V_0$ is in the relative interior $\text{int}(V_0)$ of $V_0$. Let the function $F$ be defined on $V$ as Lemma 3.5. If $u$ is a Gâteaux differentiability point of $F |_{V_0}$ (the restriction of $F$ to $V_0$), then its relative Gâteaux derivative $x^* = d_{GF} |_{V_0}(u)$ is a positive functional on $V_0$.

Proof. This is an immediate consequence of the definition of the Gâteaux derivative and the monotonicity of $F$ on $V$. \qed

Lemma 3.9. Suppose that $g$ is a continuous convex function on a closed convex subset $D$ of a Banach space $X$ with $\text{int}(D) \neq \emptyset$. Then the following statements of the mean-value theorem hold:

(i) $\forall x, y \in \text{int}(D)$, there exist $\xi \in [x, y] \equiv \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$ and $x^*_\xi \in \partial g(\xi)$ such that
\[
g(y) - g(x) = \langle x^*_\xi, y - x \rangle.
\] (3.12)

(ii) $\forall x^* \in \partial g(x)$ and $y^* \in \partial g(y)$, we have
\[
\langle y^*, y - x \rangle \geq g(y) - g(x) \geq \langle x^*, y - x \rangle.
\] (3.13)

Proof. Since $g$ is a continuous convex function on $\text{int}(D)$, $\partial g : \text{int}(D) \to 2^{X^*}$ is nonempty $w^*$-compact valued and norm-to-$w^*$ upper semicontinuous at each point of $\text{int}(D)$, and $\partial g$ is maximal monotone on $\text{int}(D)$ (see, for example, [34]).

(i) For each fixed $z \in [x, y]$, since $\partial g(z)$ is nonempty $w^*$-compact and convex,
\[
\{(z^*, y - x) : z^* \in \partial g(z)\}
\]
is an interval, i.e., $[a_z, b_z] \subset \mathbb{R}$, where
\[
a_z = \min\{\langle z^*, y - x \rangle : z^* \in \partial g(z)\} \quad \text{and} \quad b_z = \max\{\langle z^*, y - x \rangle : z^* \in \partial g(z)\}.
\]
Since $\partial g$ is monotone, for $0 \leq \alpha < \beta \leq 1$, $z_1 = \alpha x + (1 - \alpha)y$, $z_2 = \beta x + (1 - \beta)y$, and for all $z^*_1 \in \partial g(z_1)$ and $z^*_2 \in \partial g(z_2)$, we have
\[
\langle z^*_1, y - x \rangle \geq \langle z^*_2, y - x \rangle.
\]
Since $\partial g : \text{int}(D) \to 2^{X^*}$ is norm-to-$w^*$ upper semicontinuous, for every selection $\varphi$ of $\partial g$ on $[x, y]$ and for every $w^*$-cluster point $z^*_0$ of $\{\varphi(\beta x + (1 - \beta)y) : \beta \to \alpha^+\}$, we have $z^*_0 \in \partial g(z_1)$. Thus,
\[
\cup\{[a_z, b_z] : z \in [x, y]\}
\]
is again an interval. Consequently, (3.12) holds.

(ii) It follows from the definition of the subdifferential mapping $\partial g$. \qed
Lemma 3.10. Suppose that $X$ is a Banach space, $\mu$ is a convex measure of noncompactness on $X$, and $V_0 \subset V$ is a closed subcone of $V$ with $f_{B_X} \in V_0$ such that $E_{V_0} = V_0 - V_0$ is a finite-dimensional subspace. Let the function $f$ on $V$ be defined as Lemma 3.5. Then

(i) for each $r > 0$, $F$ is $c_r$-Lipschitzian on $V_0, r \equiv V_0 \cap rB_{C(K)}$, where $c_r = f((1+r)f_{B_X}) = \mu((1+r)B_X)$;

(ii) if, in addition, $\mu$ is a homogeneous measure of noncompactness, then $c_r = f(f_{B_X}) = \mu(B_X)$.

Proof. (i) Since $V_0$ is a finite-dimensional subcone of $V$, the relative interior $\text{int}_{\text{rel}}(V_0, r)$ is not empty. By Lemma 3.6, $F$ is convex and continuous on $V$. It follows that the restriction $F|_{V_0}$ is locally Lipschitzian on $\text{int}_{\text{rel}}(V_0)$. Therefore, $F|_{V_0}$ is Gâteaux differentiable at each point of a dense $G_\delta$-subset $M$ of $\text{int}_{\text{rel}}(V_0)$. Due to Lemma 3.8, for each $u \in M$, $\varphi \equiv d_{G}F|_{V_0}(u)$ is a positive functional on $V_0$ with $\|\varphi\|_{E_{V_0}} = \langle \varphi, f_{B_X} \rangle$.

On the other hand, by Lemma 3.5,

$$\langle \varphi, f_{B_X} \rangle \leq F(u + f_{B_X}) - F(u) \leq F(u + f_{B_X}).$$

(3.14)

Since $u \in V_0, r$, by Lemma 3.3 we obtain $u \leq f_{B_X} = rf_{B_X}$. Consequently,

$$\|\varphi\|_{E_{V_0}} = \langle \varphi, f_{B_X} \rangle \leq F((1+r)f_{B_X}).$$

(3.15)

Let $N = \{d_{G}F|_{V_0}(u) : u \in M\}$. Since the subdifferentiable mapping $\partial F|_{V_0} : \text{int}_{\text{rel}}(V_0, r) \to 2^{E_{V_0}}$ is norm-to-norm upper semicontinuous, the norm closure $N$ in $E_{V_0}$ contains a selection of $\partial F|_{V_0}$ on $V_0, r$. Thus, for all $u, v \in V_0, r$, there exist $\varphi_1 \in \partial F|_{V_0}(u) \cap N$ and $\varphi_2 \in \partial F|_{V_0}(v) \cap N$. It follows from Lemma 3.9 that

$$\langle \varphi_1, v - u \rangle \leq F(v) - F(u) \leq \langle \varphi_2, v - u \rangle.$$

(3.16)

Note that $\|\varphi\|_{E_{V_0}} \leq F((1+r)f_{B_X})$ for all $\varphi \in N$. Then it follows from (3.16) that

$$|F(v) - F(u)| \leq c_r \|v - u\|,$$

and this says that $F$ is $c_r$-Lipschitzian on $\text{int}_{\text{rel}}(V_0, r)$.

(ii) If, in addition, $\mu$ is a homogeneous measure of noncompactness, then $F$ is a non-negative sublinear functional on $V$. It follows from (3.14) that $c_r = F(f_{B_X})$.

Theorem 3.11. Suppose that $X$ is a Banach space. Then there is a Banach space $C(K)$ for some compact Hausdorff space $K$ such that for every convex MNC $\mu$ on $X$, there is a function $F$ on the cone $V \equiv \mathcal{T}(\mathcal{B}(X)) \subset C(K)^+$ satisfying

(i) $\mu(B) = F(\mathcal{T}B)$ for all $B \in \mathcal{B}(X)$;

(ii) $F$ is non-negative valued convex and monotone on $V$;

(iii) $F$ is bounded by $b_r = f(rB_{X})$ on $V \cap (rB_{C(K)})$ for all $r \geq 0$;

(iv) $F$ is $c_r$-Lipschitzian on $V \cap (rB_{C(K)})$ for all $r \geq 0$, where $c_r = f((1+r)B_{X}) = \mu((1+r)B_X)$;

(v) in particular, if $\mu$ is a sublinear MNC, then we can take $c_r = \mu(B_X)$ in (iv).

Proof. Given a convex measure of noncompactness $\mu$ on $X$, let

$$F(\mathcal{T}B) = \mu(B), \quad \forall B \in \mathcal{B}(X).$$

Then by Lemma 3.5, (i)–(iii) follow. Next, we show (iv). Given $r \geq 0$ and $f, g \in V$ with $\|f\|, \|g\| \leq r$, let $V_0 \subset V$ be the subcone generated by $\{f, g, \mathcal{T}(B_X)\}$. By Lemma 3.10(i), $F$ is $c_r$-Lipschitzian on $V_0 \cap rB_{C(K)}$. Therefore,

$$|F(f) - F(g)| \leq c_r \|f - g\|.$$

Consequently, (iv) follows.

(v) follows from Lemma 3.10(ii).

The following result is a converse version of Theorem 3.11 for sublinear MNCs.
Theorem 3.12. Suppose that $μ$ is a sublinear MNC defined on a Banach space $X$, and $C(K)$ is the Banach function space with respect to $X$ defined as in Theorem 3.11. Then there is a bounded subset $M \subset C(K)^+\cap$ of positive functionals such that

$$μ(B) = \sup_{ϕ \in M} \langle ϕ, T(B) \rangle, \quad ∀B \in \mathcal{B}(X).$$

Proof. Since $μ$ is a sublinear MNC, by Theorem 3.11, the corresponding convex function $F$ is $c (= μ(B_X))$-Lipschitzian on the positive cone $V = T(\mathcal{B}(X))$. Let

$$\mathfrak{I}(f) = \inf \{f(g) + c\|f - g\|_{C(K)} : g \in V\}, \quad ∀f \in C(K).$$

Then by a standard argument, we can see that $\mathfrak{I}$ is $c$-Lipschitzian on the whole space $C(K)$ with $\mathfrak{I}|_V = F$ on $V$. Let

$$M' = \partial \mathfrak{I}(V) \equiv \bigcup_{v \in V} \partial \mathfrak{I}(v).$$

Then $F(v) = \sup_{ϕ \in M'} \langle ϕ, v \rangle$. Next, we show that each $ϕ \in M'$ is a positive functional on $V$. Let $F \subset C(K)$ be a finite-dimensional subspace so that $F = V_F - V_F$, where $V_F \equiv V \cap F$. Then

$$\partial \mathfrak{I}(v)|_F \subset \partial \mathfrak{I}|_F(v).$$

Since $F$ is finite-dimensional, $\mathfrak{I}|_F$ is densely (Gâteaux) differentiable in $F$, and hence, densely differentiable in $V_F$. Let $D$ be the dense subset of $V_F$ such that at each point of $D$, $\mathfrak{I}|_F$ is differentiable. By Lemma 3.8, for each point $v \in D$ of $\mathfrak{I}|_F$, the Gâteaux derivative $d_G\mathfrak{I}|_F(v)$ is a positive functional on the space $F$. Thus, all the elements in $\overline{co}(d_G\mathfrak{I}|_F(v) : v \in D)$ are positive functionals on the space $F$. Note that

$$\partial \mathfrak{I}(V)|_F \subset \overline{co}(d_G\mathfrak{I}|_F(v) : v \in D).$$

Then we see that $\partial \mathfrak{I}(V)|_F$ is consisting of positive functionals on $F$. Since $F$ is arbitrary, each element in $\partial \mathfrak{I}(V)$ is a positive functional on the subspace $V - V$ of $C(K)$.

For each $ϕ \in M'$, it has a unique decomposition $ϕ = ϕ^+ - ϕ^-$ with $\|ϕ\| = \|ϕ^+\| + \|ϕ^-\|$, where $ϕ^\pm \in C^+(K)^\pm$, the cone of all the positive functionals of $C(K)$. Since $ϕ|_{V - V}$ is a positive functional of $V - V, ϕ|_{V - V} = ϕ^+|_{V - V}$. Thus, we finish the proof by letting $M = \{ϕ^+ : ϕ \in M'\}$. □

4 Applications to a class of basic integral inequalities

In this section, we use the representation theorem of the convex MNC (see Theorem 3.11) to establish the basic integral inequalities (1.8) and (1.10) with respect to the initial problem (1.5) presented in Section 1, which can be understood as an extension, improvement and unification of some classical results related to the problem.

The letter $X$ still denotes a real Banach space, and $I$ is the interval $[0, a] \subset \mathbb{R}^+$ with $a > 0$. We use $(I, Σ, m)$ to denote the Lebesgue measure space. For a set $A$, $χ_A$ stands for the characteristic function of $A$. Unless stated otherwise, all the notions and symbols will be the same as previously defined. We use $L_p(I, X)$ $(0 < p \leq \infty)$ to denote the space of all the $X$-valued measurable functions defined on $I$ such that $|f|^p$ is Lebesgue-Bochner integrable. $L_0(I, X)$ stands for the space of all the $X$-valued strongly measurable functions.

We are going to convert whether the integral equality is true to whether the mapping $J_G : I \rightarrow C_b(Ω)$ defined below is (Lebesgue-Bochner) measurable, where $Ω = B_X$, and $C_b(Ω)$ denotes again the Banach space of all the continuous bounded functions on $Ω$ endowed with the sup-norm. For a Banach space $X$, the Banach spaces $E_ε, E_ε^*$ and $C(K)$, the mappings $J, Q, T$ and $T = TQJ$ and the positive cone $V = T\mathcal{E}$ are the same as in Section 2.
**Definition 4.1.** Let $G \subset L_0(I, X)$ be a nonempty subset satisfying $\sup \|G(t)\| \equiv \sup_{g \in G} \|g(t)\| < \infty$ a.e. The mapping $\mathbb{J}_G : I \to C_b(\Omega)$ is defined for $t \in I$ by

$$\mathbb{J}_G(t)(x^*) = \sup_{g \in G} \langle x^*, g(t) \rangle, \quad x^* \in \Omega. \quad (4.1)$$

Note that $\mathbb{J}_G(t) = J(G(t))$, where $G(t) = \{g(t) : g \in G\}$. If no confusion arises, we simply write $J$ for $\mathbb{J}_G$.

We recall some notions and basic properties related to $X$-valued functions defined on the interval $I$ (from Definition 4.2 to Lemma 4.6), which can be found in [21].

**Definition 4.2.** Let $f : I \to X$ be a function.

(i) $f$ is said to be a simple function provided that there exist a finite $\Sigma$-partition $\{E_j\}_{j=1}^n$ and $n$ vectors $\{x_j\}_{j=1}^n \subset X$ such that $f = \sum_{j=1}^n x_j \chi_{E_j}$. The integral of $f$ is defined by $\int_I f dm = \sum_{j=1}^n x_j m(E_j)$.

(ii) $f$ is called (strongly) measurable if there is a sequence $\{f_n\}$ of simple functions such that $f_n(s) \to f(s)$ for almost all $s \in I$. If, in addition, the limit $\lim_n \int_I f_n dm$ exists, we then say that $f$ is (Lebesgue-Bochner) integrable and

$$\int_I f dm = \lim_n \int_I f_n dm$$

is called the integral of $f$ on $I$.

(iii) We say that $f$ is weakly measurable if for each $x^* \in X^*$, the numerical function $\langle x^*, f \rangle$ is ($\Sigma$-) measurable.

**Lemma 4.3.** A function $f : I \to X$ is (Lebesgue-Bochner) integrable if and only if $f$ is strongly measurable and $\int_I \|f\| dm < \infty$.

**Lemma 4.4** (Pettis’s measurability theorem). A function $f : I \to X$ is strongly measurable if and only if

1. $f(I) \subset X$ is essentially separable, i.e., there exists a null set $I_0 \subset I$ such that $f(I \setminus I_0)$ is separable;
2. $f$ is weakly measurable.

**Lemma 4.5.** Let $F$ be a closed linear operator defined inside $X$ having values in a Banach space $Y$. If both $f : I \to X$ and $Ff$ are Bochner integrable, then

$$F\left(\int_I f ds\right) = \int_I Ff ds.$$ 

**Lemma 4.6** (Jensen’s inequality). Assume that $f : [0, 1] \to X$ is integrable, and $p : X \to \mathbb{R}$ is a continuous convex function. If $p \circ f$ is integrable, then

$$p\left(\int_0^1 f ds\right) \leq \int_0^1 (p \circ f) ds.$$ 

**Lemma 4.7.** Let $X$ be a Banach space, $I = [0, a], G \subseteq L_1(I, X)$ be a nonempty set, and $\psi \in L_1(I, \mathbb{R}_+)$ such that $\sup_{g \in G} \|g(t)\| \leq \psi(t)$ a.e. $t \in I$. Assume that $\mathbb{J}_G : I \to C_b(\Omega)$ is strongly measurable. Then $TQ\mathbb{J}_G : I \to C(K)$ defined for $t \in I$ by $TQ\mathbb{J}_G(t) = TQJ(G(t))$ is integrable on $I$ and satisfies

$$0 \leq TQJ\left(\int_0^t G(s) ds\right) \leq \int_0^t TQJ(G(s)) ds. \quad (4.2)$$

**Proof.** Since $\mathbb{J}_G(t) = J(G(t))$ is strongly measurable with respect to $t$, and $\psi \in L_1(I, \mathbb{R}_+)$, it follows from

$$\|J(G(t))\|_{C_b(\Omega)} = \sup_{x^* \in \Omega} \sup_{g \in G} |\langle x^*, g(t) \rangle|$$

$$\leq \sup_{x^* \in \Omega} \sup_{g \in G} \|x^*\| \cdot \|g(t)\|$$

$$\leq \psi(t)$$

...
that \( \|J(G(t))\|_{C_b(\Omega)} \in L_1(I, \mathbb{R}^+) \). By Lemma 4.3, \( J(G(t)) \) is integrable on \( I \). Consequently, \( TQJ(G(t)) \) is integrable.

Note that for every \( x^* \in \Omega = B_{X^*} \), \( x^* \) can be regarded as the evaluation function \( \delta_{x^*} \subset C_b^+(\Omega) \). By Lemma 4.5,

\[
J \left( \int_0^t G(s)ds \right) (x^*) = \sup_{g \in G} \left\{ x^*, \int_0^t g(s)ds \right\} 
\leq \sup_{g \in G} \int_0^t (x^*, g(s))ds 
\leq \int_0^t \sup_{g \in G} (x^*, g(s))ds 
= \int_0^t J(G(s))(x^*)ds 
= \left( \int_0^t J(G(s))ds \right) (x^*).
\]

Thus, in the order induced by the positive cone of \( C_b(\Omega) \), we have

\[
J \left( \int_0^t G(s)ds \right) \leq \int_0^t J(G(s))ds.
\]

Since the quotient mapping \( Q \) is order-preserving, it follows from Lemma 4.5 that

\[
QJ \left( \int_0^t G(s)ds \right) \leq Q \int_0^t J(G(s))ds 
\leq \int_0^t QJ(G(s))ds.
\]

Since \( T \) is an order isometry,

\[
TQJ \left( \int_0^t G(s)ds \right) \leq T \int_0^t QJ(G(s))ds 
\leq \int_0^t TQJ(G(s))ds.
\]

Since \( TQJ(\mathcal{C}(X)) \subset C(K)^+ \) (see Theorem 2.6(iii)) and \( TQJ(G(\mathcal{B}(X)) = TQJ(G(\mathcal{B}(X))) \), we have

\[
0 \leq TQJ \left( \int_0^t G(s)ds \right) \leq \int_0^t TQJ(G(s))ds.
\]

This completes the proof. \( \square \)

**Theorem 4.8.** Let \( \mu \) be a convex \( MNC \) on a Banach space \( X \), \( I = [0, a] \), \( G \subseteq L_1(I, X) \) be a nonempty bounded subset, and \( \psi \in L_1(I, \mathbb{R}^+) \) such that \( \sup_{g \in G} \|g(t)\| \leq \psi(t) \) a.e. \( t \in I \). Assume that \( \mu_G : I \to C_b(\Omega) \) is strongly measurable. Then \( \mu(G(t)) \) is measurable and

\[
\mu \left( \int_0^t G(s)ds \right) \leq \frac{1}{t} \int_0^t \mu(tG(s))ds, \quad \forall 0 < t \leq a.
\]  \hspace{1cm} (4.3)

**Proof.** By Lemma 4.7, \( T(G) = TQJ(G) : I \to V \subset C(K)^+ \) defined for \( t \in I \) by \( T(G(t)) = TQJ(G(t)) \) is integrable on \( I \). On the other hand, it follows from Theorem 3.11 that there is a non-negative-valued continuous convex function \( F \) on \( V = TQJ(\mathcal{C}(X)) = T(\mathcal{B}(X)) \subset C(K)^+ \), which is Lipschitzian on each bounded subset of \( V \) such that

\[
F(T(B)) = \mu(B), \quad \forall B \in \mathcal{B}(X).
\]

Therefore,

\[
F(T(G)) = F(TQJ(G)) = \mu(G) : I \to \mathbb{R}^+ \quad \hspace{1cm} (4.4)
\]
is measurable on \( I \). Note that \( J_{\lambda G} = \lambda J_G \) for all fixed \( \lambda > 0 \). \( J_{tG} : I \to C_b(\Omega) \) is again strongly measurable for all fixed \( 0 < t \leq a \). Consequently, for all fixed \( 0 < t \leq a \),

\[
F(T(tG)) = F(tTQJ(G)) = \mu(tG) : I \to \mathbb{R}^+
\]

is again measurable on \( I \). Now, given \( 0 < t \leq a \), \( \int_0^t \mu(tG(s))ds = \infty \), then we finish the proof. If \( \int_0^t \mu(tG(s))ds < \infty \), then \( \mu(tG(s)) = t \mu(G(s)) \) is integrable with respect to \( s \in [0, t] \) with

\[
\frac{1}{t} \int_0^t \mu(tG(s))ds = \int_0^1 \mu(tG(ts))ds.
\]

Since \( \psi \in L_1(I, \mathbb{R}^+) \) with \( \sup_{g \in G} \|g(t)\| \leq \psi(t) \) a.e.,

\[
\int_0^1 tG(ts)ds = \int_0^t G(s)ds = \left\{ \int_0^1 g(s)ds : g \in G \right\} \subset C(I, X)
\]

is uniformly bounded by \( \int_0^t \psi(s)ds \) for all \( t \in I \).

Since \( F \) is a monotone continuous convex function on \( V \), it follows from (4.5), (4.6), Lemma 4.7 and Jensen’s inequality (see Lemma 4.6) that

\[
\mu\left( \int_0^t G(s)ds \right) = F\left( TQJ\left( \int_0^t G(s)ds \right) \right) \leq F\left( \int_0^t TQJ(G(s))ds \right) = F\left( \int_0^1 TQJ(tG(ts))ds \right) \leq \int_0^1 F(TQJ(tG(ts)))ds
\]

\[
= \int_0^1 \mu(tG(ts))ds = \frac{1}{t} \int_0^t \mu(tG(s))ds.
\]

Therefore, (4.3) holds.

**Corollary 4.9.** Let \( X \) be a Banach space, \( I = [0, a] \), \( G \subseteq L_1(I, X) \) be a nonempty bounded subset, and \( \psi \in L_1(I, \mathbb{R}^+) \) such that \( \sup_{g \in G} \|g(t)\| \leq \psi(t) \) a.e. \( t \in I \). Assume that \( J_G : I \to C_b(\Omega) \) is strongly measurable. Then the following inequality holds:

\[
\mu\left( \int_0^t G(s)ds \right) \leq \int_0^t \mu(G(s))ds, \quad \forall 0 < t \leq a,
\]

(4.7)

if one of the following conditions is satisfied:

(i) \( 0 < t \leq \min\{1, a\} \);

(ii) \( \mu \) is a sublinear measure of noncompactness.

**Proof.** By Theorem 4.8, it suffices to note that

\[
\mu(tG(s)) = F(T(tG(s))) \leq tF(T(G(s))), \quad \text{if } 0 \leq t \leq 1,
\]

and

\[
\mu(tG(s)) = t\mu(G(s)), \quad \forall t \geq 0,
\]

if \( \mu \) is a sublinear MNC.
5 Strong measurability and weak measurability of $\mathbb{J}_G$

In this section, we discuss measurability and weak measurability of the mapping $I \rightarrow C_b(\Omega)$ for a bounded set $G \subseteq L_1(I, X)$. As a result, we show that if $G$ is separable in $L_1(I, X)$ and there is $u \in L_p(I, X)$ for some $0 < p$ such that $\sup_{g \in G} \|g(t)\| < u(t)$ a.e., then $G$ is weakly measurable. Using this result, we further prove that $\mathbb{J}_G$ is strongly measurable if $G$ is one of the following classical classes:

(a) $G \subseteq C(I, X)$ is an equi-continuous subset;
(b) $G$ is an equi-regular subset of $R(I, X)$ (the Banach space of bounded functions on $I$ satisfying that $\lim_{t \rightarrow t_0 \pm 0} u(t)$ exist for all $u \in R(I, X)$ and $t_0 \in I$) endowed with the sup-norm [9];
(c) $G \subseteq L_1(I, X)$ is uniformly measurable.

Lemma 5.1. Let $T$ be a Hausdorff topological space, and $C_b(T)$ be the Banach space of all the bounded continuous functions on $T$ endowed with the sup-norm $\|f\| = \sup_{t \in T} |f(t)|$ for $f \in C_b(T)$. Then the closed unit ball $B_{C_b(T)}$ of the dual $C_b(T)^*$ satisfies

$$B_{C_b(T)}^* = w^* - \overline{co}\{\pm \delta_t : t \in T\},$$

where $\delta_t \in C_b(T)^*$ is the evaluation functional defined for $f \in C_b(T)$ by $(\delta_t, f) = f(t)$, and $w^* - \overline{co}(A)$ denotes the $w^*$-closed convex hull of $A$ in $C_b(T)^*$.

**Proof.** Since for each $f \in C_b(T)$,

$$\|f\| = \sup_{t \in T} |f(t)| = \sup_{t \in T} |(\delta_t, f)|,$$

$\{\pm \delta_t : t \in T\}$ is a norming set of $C_b(T)$. It follows from the separation theorem of convex sets in locally convex spaces. Indeed, suppose to the contrary, that there is

$$\varphi \in B_{C_b(T)} \setminus w^* - \overline{co}\{\pm \delta_t : t \in T\}.$$  

By the separation theorem in the locally convex space $(C_b(T)^*, w^*)$, there exists an $f \in C_b(T) = (C_b(T)^*, w^*)^*$ such that

$$\|f\| \geq \langle \varphi, f \rangle > \sup_{\psi \in w^* - \overline{co}(\pm \delta_t, t \in T)} \langle \psi, f \rangle = \sup_{t \in T} \langle \pm \delta_t, f \rangle = \|f\|,$$

which leads to a contradiction. \[]

Theorem 5.2. Let $X$ be a Banach space, $I = [0, a]$, and $G \subseteq L_1(I, X)$ be a separable subset satisfying that there is $u \in L_p(I, \mathbb{R}^+) \setminus p > 0$ such that $\sup_{g \in G} \|g(t)\| \leq u(t)$ for almost all $t \in I$. Then $\mathbb{J}_G : [0, a] \rightarrow C_b(\Omega)$ is weakly measurable.

**Proof.** We want to prove that for each $\varphi \in C_b(\Omega)^*$, the real-valued function $\langle \varphi, \mathbb{J}_G(\cdot) \rangle$ is measurable on $I$.

Since $G \subseteq L_1(I, X)$ is separable, for every $x^* \in \Omega = B_{X^*}$, the real-valued function $\mathbb{J}_G(\cdot)(x^*)$ defined for $t \in I$ by

$$\mathbb{J}_G(t)(x^*) = \sup_{g \in G} \langle x^*, g(t) \rangle$$

is measurable. Indeed, let $\{g_n\}$ be a dense sequence of $G$. Then

$$\mathbb{J}_{\{g_n\}}(t)(x^*) = \sup_n \langle x^*, g_n(t) \rangle = \mathbb{J}_G(t)(x^*) \quad \text{for almost all } t \in I.$$

Thus, the measurability of $\mathbb{J}_G(\cdot)(x^*)$ follows from the measurability of $\mathbb{J}_{\{g_n\}}(\cdot)(x^*)$.

Assume that $\varphi \in C_b(\Omega)^*$ with $\|\varphi\| = 1$. By Lemma 5.1, there is a net

$$\{\varphi_i\} \subset \overline{co}\{\pm \delta_{x^*} : x^* \in \Omega\}$$

such that $\varphi_i \rightarrow \varphi$ in the $w^*$-topology of $C_b(\Omega)^*$. Therefore,

$$\langle \varphi_i, \mathbb{J}_G(t) \rangle \rightarrow \langle \varphi, \mathbb{J}_G(t) \rangle, \quad t \in I. \quad (5.1)$$
Since for each $x^* \in \Omega \equiv B_{X^*}$,
\[
\langle \delta_{x^*}, J_G(t) \rangle = \langle J_G(t), x^* \rangle = \sup_{g \in G} \langle x^*, g(t) \rangle
\]
is measurable, $\langle \varphi_i, J_G(t) \rangle$ and $\text{sgn}(\langle \varphi_i, J_G(t) \rangle)$ are measurable for each $\varphi_i$. Let
\[
E = \{ t \in I : \langle \varphi, J_G(t) \rangle \neq 0 \}.
\]
Then
\[
\text{sgn}(\langle \varphi_i, J_G(t) \rangle) \cdot \chi_E \to \text{sgn}(\langle \varphi, J_G(t) \rangle), \quad t \in I.
\]
On the other hand, since $\{ \text{sgn}(\langle \varphi_i, J_G(t) \rangle) \} \subset L_2(I, \mathbb{R})$ is a bounded net, there is a subnet of it weakly convergent to an element $v \in L_2(I, \mathbb{R})$. Clearly,
\[
v(t)\chi_E(t) = \text{sgn}(\langle \varphi, J_G(t) \rangle) \quad \text{for almost all } t \in I.
\] (5.2)

Let $u_i = |\langle \varphi_i, J_G(t) \rangle|$. Since
\[
|\langle \varphi_i, J_G(t) \rangle| = \| J_G(t) \| = \sup_{x^* \in \Omega, g \in G} |\langle x^*, g(t) \rangle| = \sup_{g \in G} \| g(t) \| \leq u(t) \quad \text{a.e.}
\]
and $u \in L_p(I, \mathbb{R}), \{ u_i^I \} \subset L_2(I, \mathbb{R})$ is a bounded net which is pointwise convergent to $|\langle \varphi, J_G(t) \rangle|^I$ for $t \in I$. This entails that $\{ u_i^I \}$ admits a subnet weakly convergent to some element $w \in L_2(I, \mathbb{R})$. Clearly, $w(t) = |\langle \varphi, J_G(t) \rangle|^I$ for almost all $t \in I$. Therefore, $|\langle \varphi, J_G(t) \rangle|^I \in L_2(I, \mathbb{R})$. Consequently, $|\langle \varphi, J_G(t) \rangle|$ is measurable. Hence, $\chi_E$ is measurable. This and (5.2) entail that $\text{sgn}(\langle \varphi, J_G(t) \rangle)$ is measurable. Since
\[
\langle \varphi, J_G(t) \rangle = |\langle \varphi, J_G(t) \rangle|\langle \varphi, J_G(t) \rangle,
\]
$\langle \varphi, J_G(t) \rangle$ is measurable. \hfill \Box

Theorem 5.3. Let $G \subset C(I, X)$ be a nonempty equi-continuous subset. Then $J_G : I \to C_b(\Omega)$ is continuous, and hence, strongly measurable.

Proof. It follows from the equi-continuity of $G$ on $I$ that for all $\varepsilon > 0$ and $t_0 \in I$, there is $\delta > 0$ such that
\[
t \in I, \quad |t - t_0| < \delta \Rightarrow \| g(t) - g(t_0) \| < \varepsilon, \quad \forall g \in G.
\]
Thus, $t \in I$ and $|t - t_0| < \delta$ imply
\[
\| J_G(t) - J_G(t_0) \| \leq \left| \sup_{x^* \in \Omega, g \in G} \langle x^*, g(t) \rangle - \sup_{x^* \in \Omega, g \in G} \langle x^*, g(t_0) \rangle \right| \\
\leq \left| \sup_{x^* \in \Omega, g \in G} \langle x^*, g(t) - g(t_0) \rangle \right| = \sup_{g \in G} \| g(t) - g(t_0) \| \leq \varepsilon.
\]
This completes the proof. \hfill \Box

The next result follows from the theorem above and Corollary 4.9.

Corollary 5.4. Let $X$ be a Banach space, $\mu$ be a convex MNC on $X, I = [0, a]$, and $G \subset C(I, X)$ be a nonempty equi-continuous subset. Then the following inequality holds:
\[
\mu \left( \int_0^t G(s)ds \right) \leq \frac{1}{t} \int_0^t \mu(tG(s))ds, \quad \forall 0 < t \leq a.
\] (5.3)

In particular,
\[
\mu \left( \int_0^t G(s)ds \right) \leq \int_0^t \mu(G(s))ds, \quad \forall 0 < t \leq a,
\] (5.4)

if one of the following conditions is satisfied:

(i) $0 < t \leq \min\{1, a\}$;

(ii) $\mu$ is a sublinear MNC.
Remark 5.5. Corollary 5.4 is a generalization of Goebel et al.’s results. In 1970, Goebel and Rzymowski [22] showed that the inequality (5.4) holds for equi-continuous subsets \(G \subset C(I, X)\) and for the Hausdorff MNC \(\beta\), i.e., \(\mu = \beta\) (see also [35]). In 1980, Banaś and Goebel [6] showed the inequality (5.4) holds for equi-continuous subsets \(G \subset C(I, X)\) and for the homogeneous MNC \(\mu\).

Before stating the next result, we recall the notions of regulated functions and of equi-regulated sets of such functions (see, for example, [33]).

**Definition 5.6.** (i) A function \(f : [a, b] \to X\) is said to be regulated provided that for every \(t \in [a, b]\), the right-hand side limit \(\lim_{s \to t^+} f(s) \equiv f(t^+)\) exists, and for every \(t \in (a, b]\), the left-hand side limit \(\lim_{s \to t^-} f(s) \equiv f(t^-)\) exists.

We denote by \(R([a, b], X)\) the Banach space of all the regulated functions defined on the interval \([a, b]\) endowed with the sup-norm.

(ii) A nonempty subset \(G \subset R(I, X)\) is called equi-regulated if

\[
\forall t \in (a, b), \quad \forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall g \in G, \quad \forall t_1, t_2 \in (t - \delta, t) \cap [a, b], \quad \|g(t_2) - g(t_1)\| < \varepsilon,
\]

\[
\forall t \in [a, b), \quad \forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall g \in G, \quad \forall t_1, t_2 \in (t, t + \delta) \cap [a, b], \quad \|g(t_2) - g(t_1)\| < \varepsilon.
\]

**Theorem 5.7.** Let \(X\) be a Banach space and \(I = [0, a]\). Assume that \(G \subset R(I, X)\) is a separable equi-regulated set. Then the mapping \(\mathbb{J}_G : I \to C_b(\Omega)\) is strongly measurable.

**Proof.** By the definition of regulated functions and equi-regulated sets, it is easy to see that \(G\) is separable in \(L_1(I, X)\) and bounded in \(R(I, R)\), i.e.,

\[
\sup_{g \in G} \|g\|_{R(I, R)} = \sup_{g \in G, t \in I} \|g(t)\|_X = \varepsilon < \infty.
\]

By Theorem 5.2, \(\mathbb{J}_G\) is weakly measurable. To prove that \(\mathbb{J}_G\) is strongly measurable, it suffices to show that \(\mathbb{J}_G(I)\) is separable in \(C_b(\Omega)\).

Through a routine argument of compactness of \(I\) similar to Olszowy and Zajac [33, Theorem 3.1], we obtain that for all \(\varepsilon > 0\), there is an open cover of \(I\) consisting of finitely many open intervals of the form

\[
[0, \delta_0), (s_1 - \delta_1, s_1 + \delta_1), \ldots, (s_{n-1} - \delta_{n-1}, s_{n-1} + \delta_{n-1}), (s_n, a]\}
\]

such that

\[
\sup_{g \in G} \|g(t_2) - g(t_1)\| < \varepsilon, \quad \forall t_1, t_2 \in [0, \delta_0) \text{ or } (s_n, a],
\]

\[
\sup_{g \in G} \|g(t_2) - g(t_1)\| < \varepsilon, \quad \forall t_1, t_2 \in (s_1 - \delta_j, s_j) \text{ or } (s_j, s_j + \delta_j),
\]

where \(j = 1, 2, \ldots, n - 1\) for some \(n \in \mathbb{N}\), \(\delta_j > 0\), \(j = 0, 1, \ldots, n - 1\), and \(\delta_0, s_n, s_j, s_j - \delta_j, s_j + \delta_j \in (0, a)\), \(j = 1, \ldots, n - 1\). Put

\[
I_0^+ = [0, \delta_0), \quad I_n^+ = (s_n, a],
\]

\[
I_j^+ = (s_j, s_j + \delta_j), \quad I_j^- = (s_1 - \delta_j, s_j), \quad j = 1, 2, \ldots, n - 1.
\]

Therefore,

\[
\|J_G(t_2) - J_G(t_1)\| \leq \sup_{g \in G, x^* \in \Omega} \|x^*, g(t_2) - g(t_1)\| \leq \sup_{g \in G} \|g(t_2) - g(t_1)\| < \varepsilon,
\]

whenever \(t_1, t_2 \in I_j^+, j = 0, 1, \ldots, n\) or \(t_1, t_2 \in I_j^-\), \(j = 1, 2, \ldots, n - 1\). Thus, each element of \(J_G(I_j^+)\) \((j = 0, 1, \ldots, n)\) and \(J_G(I_j^-)\) \((j = 1, 2, \ldots, n - 1)\) is contained in a ball of \(C_b(\Omega)\) with radius \(\varepsilon\). Consequently, \(J_G(I)\) is contained in the union of \(3n\) balls of \(C_b(\Omega)\) with radius \(\varepsilon\). This entails that \(J_G(I)\) is a relatively compact subset of \(C_b(\Omega)\), and hence, it is separable. \(\Box\)
The result below follows from the theorem above and Corollary 4.9.

**Corollary 5.8.** Let $X$ be a Banach space, $\mu$ be a convex MNC on $X$, $I = [0, a]$, and $G \subseteq R(I, X)$ be a nonempty separable equi-regulated subset. Then the following inequality holds:

$$\mu\left(\int_0^t G(s) ds\right) \leq \frac{1}{t} \int_0^t \mu(tG(s)) ds, \quad \forall 0 < t \leq a.$$  \hspace{1cm} (5.5)

In particular,

$$\mu\left(\int_0^t G(s) ds\right) \leq \int_0^t \mu(G(s)) ds, \quad \forall 0 < t \leq a,$$  \hspace{1cm} (5.6)

if one of the following conditions is satisfied:

(i) $0 < t \leq \min\{1, a\}$;

(ii) $\mu$ is a sublinear MNC.

**Remark 5.9.** Corollary 5.8 is a generalization of Olszowy and Zajac [33, Theorem 3.1] in 2020, where they showed that the inequality (5.6) holds for the Hausdorff MNC $\beta$, i.e., $\mu = \beta$.

For a finite measure space $(\Gamma, \Sigma, \eta)$ and a Banach space $X$, we denote by $L_1(\Gamma, \eta, X)$ the Banach space of all the $X$-valued $\eta$-measurable functions $f$ endowed with the $L_1$-norm $\|f\| = \int_{\Gamma} \|f(\gamma)\| d\eta$. In particular, if $\Gamma = I = [a, b] \subset \mathbb{R}$ and $\eta$ is the Lebesgue measure on $\mathbb{R}$, then we simply denote it by $L_1(I, X)$.

The following useful notion of uniform measurability of functions was introduced by Kunze and Schliectermann [27, Definition 3.16].

**Definition 5.10.** A bounded set $G \subset L_1(\Gamma, \eta, X)$ is said to be uniformly $\eta$-measurable provided that for all $\varepsilon > 0$ and $A \in \Sigma$, there exist $A_1, \ldots, A_n \in \Sigma$ with $\bigcup_{j=1}^n A_j = A$ and $\eta(A \setminus A_j) < \varepsilon$ such that for $j = 1, 2, \ldots, n$, we can choose $\gamma_j \in A_j$ satisfying

$$\sup_{g \in G} \left\| g(\cdot) \chi_{A_j} - \sum_{j=1}^n g(\gamma_j) \chi_{A_j} \right\|_{L_1} < \varepsilon.$$  \hspace{1cm} (5.7)

Given a subset $G \subset L_0(\Gamma, \eta, X)$ of $X$-valued $\eta$-measurable functions, we define

$$G(\gamma) = \{g(\cdot) : g \in G\}, \quad \gamma \in \Gamma,$$

and let $\mathbb{J}_G : \Gamma \to C_b(\Omega)$ be again defined for $\gamma \in \Gamma$ by

$$\mathbb{J}_G(\gamma)(x^*) = J(G(\gamma))(x^*) = \sup_{g \in G} (x^*, g(\gamma)) = \sigma_{G(\gamma)}(x^*), \quad x^* \in \Omega \equiv B_{X^*}.$$  \hspace{1cm} (5.8)

**Theorem 5.11.** Let $G \subset L_0(\Gamma, \eta, X)$ be a subset with

$$\sup_{g \in G} \|g(\gamma)\| < \infty \quad \text{for almost all } \gamma \in \Gamma.$$  \hspace{1cm} (5.9)

If $G$ is uniformly $\eta$-measurable, then $\mathbb{J}_G : \Gamma \to C_b(\Omega)$ is strongly measurable.

**Proof.** Without loss of generality, we can assume that (5.9) holds for all $\gamma \in \Gamma$.

Given $m \in \mathbb{N}$, let $A_1 = \Gamma$ and $\varepsilon = 1/m$. It follows from Definition 5.10 that there exist $\Gamma_{m,1} \subset \Sigma$, mutually disjoint $A_{m,1}^1, \ldots, A_{m,n_{m,1}}^1 \subset \Sigma$, and $\gamma_{j}^1 \in \Gamma$ for $j = 1, 2, \ldots, n_{m,1}$ with $\bigcup_{j=1}^{n_{m,1}} A_{m,j}^1 = \Gamma_1$ and $\eta(\Gamma \setminus \Gamma_{m,1}^1) < 1/m$ such that

$$\sup_{g \in G} \left\| g(\cdot) \chi_{A_{m,j}^1} - \sum_{j=1}^{n_{m,1}} g(\gamma_{j}^1) \chi_{A_{m,j}^1} \right\| < \frac{1}{m}. $$  \hspace{1cm} (5.10)

Next, we substitute $A_2 \equiv \Gamma \setminus \Gamma_{m,1}^1$ for $A_1$ (= $\Gamma$) in the procedure above. Then there exist $\Gamma_{m,2} \subset \Sigma$, mutually disjoint $A_{m,1}^2, \ldots, A_{m,n_{m,2}}^2 \subset \Sigma$, and $\gamma_{j}^2 \in \Gamma$ for $j = 1, 2, \ldots, n_{m,2}$ with $\bigcup_{j=1}^{n_{m,2}} A_{m,j}^2 = \Gamma_2$ and $\eta(\Gamma \setminus \Gamma_{m,2}^2) < 1/m$ such that

$$\sup_{g \in G} \left\| g(\cdot) \chi_{A_{m,j}^2} - \sum_{j=1}^{n_{m,2}} g(\gamma_{j}^2) \chi_{A_{m,j}^2} \right\| < \frac{1}{m}. $$  \hspace{1cm} (5.11)
We can claim that $\Gamma_m^2 \subset A^2$; otherwise, we can replace $\Gamma_m^2$ by $\Gamma_m^2 \cap A^2$.

Inductively, we obtain a sequence $\{\Gamma_m^n\}_{n=1}^\infty$ of mutually disjoint measurable sets with $\eta(\Gamma \setminus \bigcup_{n=1}^\infty \Gamma_m^n) = 0$ and a family of strongly measurable functions $\{g_{f,m} : f \in M\}$: $f_{g,m} : \Gamma_m \equiv \bigcup_{n=1}^\infty \Gamma_m^n \to X$ defined for $\gamma$

\[
f_{g,m}(\gamma) = \sum_{j=1}^{n_{g,m}} g_{\gamma,j} \chi_{A^{g,m}_j}, \quad \gamma \in \Gamma_m^l, \quad l \in \mathbb{N},
\]

which satisfies

\[
\sup_{g \in G, \gamma \in \Gamma_m} ||g(\gamma)\chi_{\Gamma_m}(\gamma) - f_{g,m}(\gamma)|| < \frac{1}{m}
\]

and $\eta(\Gamma \setminus \Gamma_m) = 0$. Clearly, for each $g \in G$ and $l \in \mathbb{N}$, $f_{g,m}$ is a simple function on $\Gamma_m^l$. Let $G_{m,l} = \{f_{g,m} | \Gamma_m \}$.

By taking $\lim$ we obtain the following two sequences:

\[
J_{m,1} : \Gamma_m \to C_b(\Omega)
\]

defines a simple function $J_{m,l} : \Gamma_m^l \to C_b(\Omega)$. Note that for $\gamma \in \Gamma_m^l$,

\[
||J_G(\gamma) - J_{m,l}(\gamma)||_{C_b(\Omega)} \equiv \sup_{x^* \in \Omega} (J_G(\gamma)(x^*) - J_{m,l}(\gamma)(x^*)).
\]

For all $\varepsilon > 0$ and $\gamma \in \Gamma_m^l$, let $x^*_\varepsilon \in \Omega$ be such that

\[
||J_G(\gamma) - J_{m,l}(\gamma)||_{C_b(\Omega)} < J_G(\gamma)(x^*_\varepsilon) - J_{m,l}(\gamma)(x^*_\varepsilon) + \varepsilon.
\]

Then there is $g_\varepsilon \in G$ such that $J_G(\gamma)(x^*_\varepsilon) < \langle x^*_\varepsilon, g_\varepsilon(\gamma) \rangle + \varepsilon$. Therefore,

\[
J_G(\gamma)(x^*_\varepsilon) - J_{m,1}(\gamma)(x^*_\varepsilon) + \varepsilon < \langle x^*_\varepsilon, g_\varepsilon(\gamma) \rangle - \langle x^*_\varepsilon, f_{g,m}(\gamma) \rangle \leq ||g_\varepsilon(\gamma) - f_{g,m}(\gamma)|| + \varepsilon < \frac{1}{m} + \varepsilon.
\]

Since $\varepsilon$ is arbitrary, the inequalities above and (5.14) imply that for all $\gamma \in \Gamma_m^l$,

\[
||J_G(\gamma) - J_{m,l}(\gamma)||_{C_b(\Omega)} \leq \frac{1}{m}.
\]

Now, we define $J_m : \Gamma_m \to C_b(\Omega)$ by $J_m(\gamma) = J_{m,n}(\gamma)$, $\gamma \in \Gamma_m^n$, $n = 1, 2, \ldots$. Clearly, $J_m$ is strongly measurable and it satisfies

\[
||J_G(\gamma) - J_m(\gamma)||_{C_b(\Omega)} \leq \frac{1}{m}, \quad \gamma \in \Gamma_m^n,
\]

and $J_m(\Gamma_m) = \bigcup_{n=1}^\infty J_m(\Gamma_m^n)$ is a countable subset of $C_b(\Omega)$.

By taking $m=1, 2, \ldots$, we can obtain the following two sequences:

(a) a sequence $\{\Gamma_m\}$ of measurable sets satisfying $\eta(\Gamma \setminus \Gamma_m) = 0$ for all $m \in \mathbb{N}$, and

(b) a sequence $\{J_m\}$ of strongly measurable functions $J_m$ defined on $\Gamma_m$ with countable ranges satisfying

\[
||J_G(\gamma) - J_m(\gamma)||_{C_b(\Omega)} \leq \frac{1}{m}, \quad \gamma \in \Gamma_m^n, \quad m \in \mathbb{N}.
\]

Finally, let $\Gamma_0 = \bigcap_{m=1}^\infty \Gamma_m$.

By taking $m \to \infty$, we obtain the following two sequences:

(a) a sequence $\{\Gamma_n\}$ of strongly measurable sets satisfying $\eta(\Gamma \setminus \Gamma_n) = 0$ for all $n \in \mathbb{N}$, and

(b) a sequence $\{J_n\}$ of strongly measurable functions $J_n$ defined on $\Gamma_n$ with countable ranges satisfying

\[
||J_G(\gamma) - J_n(\gamma)||_{C_b(\Omega)} \leq \frac{1}{n}, \quad \gamma \in \Gamma_n, \quad n \in \mathbb{N}.
\]

Consequently, $J_G : \Gamma \to C_b(\Omega)$ is strongly measurable.

The following result is a consequence of Theorem 5.11 and Corollary 4.9.
Corollary 5.12. Let \( X \) be a Banach space, \( \mu \) be a convex MNC on \( X \), \( I = [0, a] \), and \( G \subseteq L_1(I, X) \) be a nonempty separable and uniformly measurable subset. Then the following inequality holds:

\[
\mu \left( \int_0^t G(s)ds \right) \leq \frac{1}{t} \int_0^t \mu(tG(s))ds, \quad \forall 0 < t \leq a.
\]

(5.18)

In particular,

\[
\mu \left( \int_0^t G(s)ds \right) \leq \int_0^t \mu(G(s))ds, \quad \forall 0 < t \leq a,
\]

(5.19)

if one of the following conditions is satisfied:

(i) \( 0 < t \leq \min\{1, a\} \);

(ii) \( \mu \) is a sublinear MNC.

Remark 5.13. Corollary 5.12 is a generalization of Kunze and Schlüchtermann [27, Corollary 3.19], where they showed that the inequality (5.19) holds for the Hausdorff MNC \( \beta \), i.e., \( \mu = \beta \).

6 On representation of measures of the non-property \( \mathcal{D} \)

In this section, we discuss a more general class of measures on Banach spaces, namely, the convex MNP\( \mathcal{D} \) (defined in Definition 1.5), which contains convex MNCs, convex MNWCs, convex MNSWCs, convex MNRNPs and convex MNAs as its special cases. We show that the results presented in Sections 3–5 for MNCs are still true for these measures of non-generalized compactness.

In this section, we again use \( \mathcal{D}(X) \) (or, simply, \( \mathcal{D} \)) to denote a fundamental subsemigroup of \( \mathcal{E}(X) \) (see Definition 1.3).

Proposition 6.1. Every fundamental closed subsemigroup \( \mathcal{D} \) of \( \mathcal{E}(X) \) contains all the nonempty compact convex subsets of \( X \), i.e., \( \mathcal{D}(X) \supseteq \mathcal{K}(X) \).

Proof. Since \( \mathcal{E}(X) \) is complete, and \( \mathcal{D} \) is closed in \( \mathcal{E}(X) \), \( \mathcal{D} \) is necessarily complete. On the other hand, it follows from Definition 1.3 that every fundamental closed subsemigroup \( \mathcal{D} \) of \( \mathcal{E}(X) \) contains all the nonempty convex polyhedrons of \( X \), i.e., for any finite subset \( F \subset X \), \( \text{co}(F) \in \mathcal{D} \). Note that the set \( \mathcal{D}(X) \) of all the nonempty convex polyhedrons of \( X \) is dense in \( \mathcal{K}(X) \). Completeness of \( \mathcal{D} \) entails \( \mathcal{D}(X) \supseteq \mathcal{K}(X) \). \( \square \)

Definition 6.2. A convex measure of the non-property \( \mathcal{D} \) on \( X \) is said to be, successively, a convex measure of noncompactness, of non-weak compactness, of non-super weak compactness, of the non-Radon-Nikodým property and of non-Asplundness if we take, successively,

\begin{align*}
\mathcal{D} &= \mathcal{K}(X), \text{ the fundamental semigroup consisting of all the nonempty convex compact subsets of } X; \\
\mathcal{D} &= \mathcal{W}(X), \text{ the fundamental semigroup consisting of all the nonempty convex weakly compact subsets of } X; \\
\mathcal{D} &= \sup\mathcal{W}(X), \text{ the fundamental semigroup consisting of all the nonempty convex super weakly compact subsets of } X; \\
\mathcal{D} &= \mathcal{R}(X), \text{ the fundamental semigroup consisting of all the nonempty closed bounded convex subsets of } X \text{ with the Radon-Nikodým property}; \\
\mathcal{D} &= \mathcal{A}(X), \text{ the fundamental semigroup consisting of all the nonempty closed bounded convex Asplund subsets of } X.
\end{align*}

Definition 6.3. (i) Kuratowski’s measure \( \alpha_{\mathcal{D}} \) of the non-property \( \mathcal{D} \) on \( X \) is defined for \( B \in \mathcal{D}(X) \) by

\[
\alpha_{\mathcal{D}}(B) = \inf \left\{ r > 0 : \exists D \in \mathcal{D}, \{E_j\}_{j \in F} \subset \mathcal{D}(X) \text{ s.t. } B \subset D + \bigcup_{j \in F} E_j \right\},
\]

where \( F \subset \mathbb{N} \) is a finite set with \( \text{diam}(E_j) \leq r \) for all \( j \in F \).

(ii) (See [14]) The Hausdorff measure \( \beta_{\mathcal{D}} \) of the non-property \( \mathcal{D} \) on \( X \) is defined for \( B \in \mathcal{D}(X) \) by

\[
\beta_{\mathcal{D}}(B) = \inf \{ r > 0 : \exists D \in \mathcal{D} \text{ s.t. } B \subset D + rB_X \}.
\]
Theorem 6.4. Let \( X \) be a Banach space, and \( \mu : \mathcal{B}(X) \to \mathbb{R}^+ \) be a convex measure of the non-property \( \mathcal{D} \). Then

(i) (density determination) \( \mu(\mathcal{B}) = \mu(B) \), \( \forall B \in \mathcal{B}(X) \);

(ii) (translation invariance) \( \mu(B + D) = \mu(B) \), \( \forall B \in \mathcal{B}(X) \), \( D \in \mathcal{P}(X) \);

(iii) (negligibility) \( \mu(B \cup D) = \mu(B) \), \( \forall B \in \mathcal{B}(X) \), \( D \in \mathcal{P}(X) \).

If, in addition, one of the following four conditions is satisfied:

(a) \( \mathcal{D} = \mathcal{X} \);

(b) \( \mathcal{D} = \mathcal{W} \);

(c) \( \mathcal{D} = \sup \mathcal{W} \);

(d) \( X \) is reflexive,

then

(iv) (continuity) \( \emptyset \neq C_{n+1} \subseteq C_n \in \mathcal{C}(X) \), \( n \in \mathbb{N} \); \( \mu(C_n) \to 0 \Rightarrow \bigcap_n C_n \neq \emptyset \).

Proof. By Proposition 6.1, \( \mathcal{D} \supset \mathcal{X} \). Therefore, (i)–(iii) follow from an argument which is the same as the proof of Theorem 3.4 but substituting the “convex MNPD” for the “convex MNC”. It remains to show (iv).

If (a) \( \mathcal{D} = \mathcal{X} \), then \( \mu \) is a convex MNC. Thus, (iv) follows from Theorem 3.4.

If (b) \( \mathcal{D} = \mathcal{W} \), then \( \mu \) is a convex MNWC. For each \( n \in \mathbb{N} \), choose any \( x_n \in C_n \) and let \( W_n = \mathcal{W}(x_k)_{k \geq n} \).

Then

\[
\mu(W_1) = \mu(W_n) \leq \mu(C_n) \to 0, \quad n \to \infty.
\]

Thus, \( \mu(W_1) = 0 \), which entails that \( W_1 \) is weakly compact. Weak compactness and non-increasing monotonicity of \( \{W_n\} \) imply

\[
\emptyset \neq \bigcap_n W_n \subset \bigcap_n C_n.
\]

If (c) \( \mathcal{D} = \sup \mathcal{W} \), then by (b), it suffices to note that every super weakly compact set is weakly compact.

If (d) \( X \) is reflexive, then every \( C_n \) is nonempty weakly compact. Consequently, \( \bigcap_n C_n \neq \emptyset \).

The following representation theorem of the convex MNPD is an analogy of Theorem 3.11.

Theorem 6.5. Suppose that \( X \) is a Banach space and \( \mathcal{D} \) is a fundamental subsemigroup of \( \mathcal{C}(X) \). Then there is a Banach space \( C(K) \) for some compact Hausdorff space \( K \) such that for every \( \mu \) which is a convex measure of the non-property \( \mathcal{D} \) with respect to \( \mathcal{D} \) on \( X \), there is a function \( F = F_\mathcal{D} \) on the cone \( V \equiv \mathcal{T}(\mathcal{B}(X)) \subset C(K)^+ \) satisfying

(i) \( \mu(B) = F(\mathcal{T}B) \) for all \( B \in \mathcal{B}(X) \);

(ii) \( F \) is non-negative convex and monotone on \( V \);

(iii) \( F \) is bounded by \( b_r = F(r\mathcal{T}B_X) \) on \( V \cap (rB_C(K)) \) for all \( r \geq 0 \);

(iv) \( F \) is \( c_r \)-Lipschitzian on \( V \cap (rB_C(K)) \) for all \( r \geq 0 \), where \( c_r = F((1+r)\mathcal{T}B_X) = \mu((1+r)B_X) \);

(v) in particular, if \( \mu \) is a sublinear of the non-property \( \mathcal{D} \), then we can take \( c_r = \mu(B_X) \) in (iv).

Proof. Let \( \mu \) be a convex measure of the non-property \( \mathcal{D} \) with respect to \( \mathcal{D} \) on \( X \). Let

\[
F_\mathcal{D}(\mathcal{T}(B)) = \mu(B), \quad \forall B \in \mathcal{B}(X).
\]

By Proposition 6.1, we know that all the propositions, lemmas and theorems in Section 3 hold again if we substitute \( F_\mathcal{D} \) for the convex function \( F \). In particular, Theorem 3.11 is still true for \( F_\mathcal{D} \).

The following theorem is an analogy of Theorem 4.8.

Theorem 6.6. Let \( \mu \) be a convex measure of the non-property \( \mathcal{D} \) on a Banach space \( X \), \( I = [0,a] \), \( G \subseteq L^1(I,X) \) be a nonempty bounded subset, and \( \varphi \in L^1(I,\mathbb{R}^+) \) such that \( \sup_{\varphi \in \mathcal{G}} \|g(t)\| \leq \varphi(t) \) a.e. \( t \in I \).

Assume that \( \mathcal{I}_G : I \to C_0(\Omega) \) is strongly measurable. Then \( \mu(G(t)) \) is measurable and

\[
\mu\left(\int_0^t G(s)ds\right) \leq \frac{1}{t} \int_0^t \mu(tG(s))ds, \quad \forall 0 < t \leq a.
\]
Corollary 6.7. Let $X$ be a Banach space, $\mu$ be a convex measure of the non-property $\mathcal{D}$ on $X$, $I = [0,a]$, $G \subseteq L_1(I,X)$ be a nonempty bounded subset, and $\psi \in L_1(I,\mathbb{R}^+)$ such that $\sup_{g \in G} \|g(t)\| \leq \psi(t)$ a.e. $t \in I$. Assume that $\mathcal{G}_G : I \to C_b(\Omega)$ is strongly measurable. Then the following inequality holds:
\[
\mu\left(\int_0^t G(s)ds\right) \leq \int_0^t \mu(G(s))ds, \quad \forall 0 < t \leq a,
\]
if one of the following conditions is satisfied:
(i) $0 < t \leq \min\{1,a\}$;
(ii) $\mu$ is a sublinear measure of the non-property $\mathcal{D}$.

The next result follows from Corollary 6.7 and Theorems 5.3, 5.7 and 5.11.

Corollary 6.8. Let $X$ be a Banach space, $\mu$ be a convex measure of the non-property $\mathcal{D}$ on $X$, and $I = [0,a]$. If $G$ satisfies one of the following conditions:
(i) $G \subseteq C(I,X)$ is a nonempty equi-continuous subset;
(ii) $G \subseteq R(I,X)$ is a nonempty separable equi-regulated subset;
(iii) $G \subseteq L_1(I,X)$ is a nonempty separable uniformly measurable set,
then the following inequality holds:
\[
\mu\left(\int_0^t G(s)ds\right) \leq \frac{1}{t} \int_0^t \mu(tG(s))ds, \quad \forall 0 < t \leq a.
\]
In particular,
\[
\mu\left(\int_0^t G(s)ds\right) \leq \int_0^t \mu(G(s))ds, \quad \forall 0 < t \leq a,
\]
if one of the following conditions is satisfied:
(a) $0 < t \leq \min\{1,a\}$;
(b) $\mu$ is a sublinear measure of the non-property $\mathcal{D}$.

7 Some applications to a Cauchy problem

In this section, as applications of the representation theorem of the convex MNC (see Theorem 3.11) and the inequalities in Section 4, we give two examples of solvability of the Cauchy problem (1.5) in Section 1, i.e.,
\[
\begin{cases}
  x'(t) = f(t,x), & a \geq t > 0, \\
  x(0) = x_0
\end{cases}
\]
in a Banach space $X$. This result can be regarded as an extension of the classical result of Goebel and Rzymowski [22] (see also [35]) and Banaš and Goebel [6]. Our proof is based on their nice and concise constructions but with slight modifications.

The first example (see Theorem 7.3) is to show that (7.1) has at least one solution on the interval $I = [0,a]$ under the assumptions (1) $f : I \times B(x_0,\tau) \to X$ is uniformly continuous and bounded, and (2) for every nonempty bounded set $B \in \mathcal{B}(X)$, we have $\mu(f(t,B)) \leq w(t,\mu(B))$, where $w(t,u)$ is a Kamke function and $\mu$ is a convex MNC. The same result has been proven by Banaš and Goebel [6] but under the assumption that $\mu$ is a “symmetric” sublinear MNC.

Let $w(t,u)$ be a real non-negative function defined on $(0, a] \times [0, +\infty)$ which is continuous with respect to $u$ for any fixed $t$ and measurable with respect to $t$ for each fixed $u$. We call $w(t,u)$ a Kamke comparison function if $w(t,0) = 0$ and the constant function $u = 0$ is the unique solution to the integral inequality
\[
u(t) \leq \int_0^t w(s,u(s))ds \quad \text{for} \quad t \in (0,a],
\]
which satisfies the condition
\[
\lim_{t \to a^+} \frac{u(t)}{t} = \lim_{t \to a^+} u(t) = 0.
\]
For $x \in C(I, X)$ and $B \subset C(I, X)$, where $I = [0, a]$ and $X$ is a Banach space, the modulus of continuity of $x$ (resp. $B$) is defined by
\[
\omega(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, a], |t - s| \leq \varepsilon\} 
\]
(resp. $\omega(B, \varepsilon) = \sup \{\omega(x, \varepsilon) : x \in B\}$).\hfill (7.3)\n
The following result is an extension of Banas and Goebel [6, Lemma 13.2.1].

**Lemma 7.1.** Let $X$ be a Banach space, $I = [0, a]$, $\mu$ be a convex MNC on $X$, and $B$ be a nonempty bounded subset of $C(I, X)$. Then
\[
|\mu(B(t)) - \mu(B(s))| \leq L_B \cdot \omega(B, |t - s|), \quad \forall t, s \in I, 
\]
where $L_B > 0$ is a constant. If $B$ is equi-continuous, then $\mu(B(t))$ is continuous with respect to $t$.\hfill (7.4)

**Proof.** Since $B \subset C(I, X)$ is bounded, $B_0 \equiv \{x(t) : x \in B, t \in I\}$ is a bounded subset of $X$. Let $r = \|f_{B_0}\|_{C(K)}$, where $f_{B_0} = T(B_0) = TQJ(B_0)$, and $C(K), T, Q$ and $J$ are defined as in Section 2. It follows from Theorem 3.11 that $f(B) = \mu(B)$ for $B \in \mathcal{B}(X)$ defines a monotone continuous convex function on $V$, and it is $L_B$-Lipschitzian on $V \cap r B C(K)$, where $L_B = f((1 + r)f_{B_X})$. Note that for all $t \in I$, $B(t) \subset B_0$. Since the three mappings $T$, $Q$ and $J$ are order-preserving, we conclude that $f_{B(t)} \leq f_{B_0}$, and hence, $f_{B(t)} \in V \cap r B C(K)$. We have
\[
|\mu(B(t)) - \mu(B(s))| = |f(f_{B(t)}) - f(f_{B(s)})| 
\]
\[
\leq L_B \cdot \|f_{B(t)} - f_{B(s)}\| 
\]
\[
= L_B \cdot \|TQJ(B(t)) - TQJ(B(s))\| 
\]
\[
= L_B \cdot \|Q[J(B(t)) - J(B(s))]\| 
\]
\[
= L_B \cdot \|J(B(t)) - J(B(s))\| 
\]
\[
= L_B \cdot d_\mu(B(t), B(s)) 
\]
\[
\leq L_B \cdot \omega(B, |t - s|), \quad \forall s, t \in I. 
\]

Thus, the proof is completed. \hfill \Box

**Lemma 7.2.** Suppose that $\mu$ is a convex measure of noncompactness on $X$, $B_0 \subset C(I, X)$ is a nonempty equi-continuous bounded set, and $B_{n+1} \subset B_n \neq \emptyset$, $n = 0, 1, 2, \ldots$. Then $\{\mu(B_n)\}_{n \geq 1} \subset C(I, R)$ is again an equi-continuous subset.\hfill (7.5)

**Proof.** Since $B_0$ is an equi-continuous bounded set, each $B_n \neq \emptyset$ is again an equi-continuous bounded subset. Let $C_1 := \{x(t) : x \in B_0, t \in I\}$ and $r = \|f_{C_1}\|$, where $f_{C_1} = TQJ(C_1)$. Then it follows from Lemma 7.1 that for all $n \in \mathbb{N}$ and $t, s \in I$,
\[
|\mu(B_n(t)) - \mu(B_n(s))| \leq L_n \cdot \omega(B_n, |t - s|), 
\]
where $L_n = f((1 + \|f_{B_n(t)}\|)f_{B_X})$ and $B_n(t) = \{x(t) : x \in B_n, t \in I\}$. Since $B_n(I) \subset C_1$, $f_{B_n(t)} \leq f_{C_1}$. According to the monotonicity of $f$, we have $L_n \leq f((1 + \|f_{C_1}\|)f_{B_X})$. It holds that
\[
|\mu(B_n(t)) - \mu(B_n(s))| \leq f((1 + \|f_{C_1}\|)f_{B_X}) \cdot \omega(B_n, |t - s|) 
\]
\[
\leq f((1 + r)f_{B_X}) \cdot \omega(B_0, |t - s|). 
\]

Since $B_0$ is equi-continuous, $\{\mu(B_n(t))\}_{n \geq 1}$ is equi-continuous with respect to $t \in I$. \hfill \Box

The following result is an extension of Banas and Goebel [6, Theorem 13.3.1].

**Theorem 7.3.** Let $X$ be a Banach space, $I = [0, a]$, $\mu$ be a convex MNC on $X$, and
\[
B(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}. 
\]
Suppose that \( f(\cdot, \cdot) : [0, a] \times B(x_0, r) \to X \) is uniformly continuous and \( w = w(\cdot, \cdot) \) is a Kambke function. If \( f \) is bounded on \([0, a] \times B(x_0, r)\) by \((0 \leq \xi \leq r \text{ and the following condition is satisfied: for all } B \in \mathcal{B}(X) \text{ and almost all } t \in [0, a], \)
\[
\mu(f(t, B)) \leq w(t, \mu(B)),
\]
then the Cauchy problem (7.1) has at least one solution \( x \in C([0, a_1], X) \), where \( a_1 = \min\{1, a\} \). In particular, if \( \mu \) is a sublinear MNC, then \( a_1 = a \), i.e., the Cauchy problem (7.1) has at least one solution \( x \in C([0, a], X) \).

**Proof.** Let \( I = [0, a] \) and
\[
B_0 = \{ x(t) \in C(I, X) : x(0) = x_0 \text{ with } \| x(t) - x(s) \| \leq \xi|t - s|, \forall s, t \in I \}.
\]
Then \( B_0 \) is an equi-continuous bounded closed convex set. The transformation \( A : B_0 \to C(I, X) \) defined for \( x \in B_0 \) and \( t \in I \) by
\[
(Ax)(t) = x_0 + \int_0^t f(s, x(s))ds
\]
is a continuous self-mapping on \( B_0 \). Let
\[
B_{n+1} = \text{co}(A(B_n)), \quad n = 0, 1, 2, \ldots
\]
It is easy to observe that \( \{B_n\} \) is a decreasing sequence of equi-continuous closed convex subsets. Let \( u_n(t) = \mu(B_n(t)) \). Obviously, \( 0 \leq u_{n+1}(t) \leq u_n(t) \) for \( n = 0, 1, \ldots \). By Lemma 7.2, \( \{u_n\}_{n \geq 1} \) is a bounded equi-continuous subset of \( C(I, X) \). Therefore, it converges uniformly to a function \( u_\infty \) defined by
\[
u_\infty(t) = \lim_{n \to \infty} u_n(t)
\]
for all \( t \in I \). Let \( g(t) = x_0 + f(0, x_0)t \) for all \( t \in I \). Then for every \( x \in A(B_0) \) and for all \( t \in I \), we have
\[
\| x(t) - g(t) \| \leq t \cdot \sup \{ \| f(0, x_0) - f(s, x) \| : s \leq t, \| x - x_0 \| \leq \xi s \} \equiv ta(t).
\]
Note that \( \lim_{t \to 0^+} a(t) = 0 \) and
\[
(A(B_0))(t) \subset B(g(t), ta(t)) = x_0 + tf(0, x_0) + ta(t)B(0, 1).
\]
Since \( \mu \) is a convex MNC,
\[
u_1(t) = \mu(B_1(t)) = \mu(\text{co}(A(B_0)(t)))
\leq \mu(x_0 + tf(0, x_0) + ta(t)B(0, 1))
= \mu(tf(0, x_0) + ta(t)B(0, 1))
= \mu(ta(t)B(0, 1)).
\]
Due to the convexity of \( \mu \), for all \( 0 \leq t \leq a \) with \( ta(t) \leq 1 \),
\[
\mu(ta(t)B(0, 1)) \leq ta(t)\mu(B(0, 1)).
\]
Thus, we conclude that
\[
\lim_{t \to 0^+} \frac{u_1(t)}{t} = \lim_{t \to 0^+} a(t)\mu(B(0, 1)) = 0.
\]
Since \( 0 \leq u_\infty(t) \leq u_n(t) \leq u_1(t) \),
\[
\lim_{t \to 0^+} \frac{u_\infty(t)}{t} = 0.
\]
Note that the uniform continuity of \( f(\cdot, \cdot) \) implies that the mapping \( F \) defined by \((Fx)(t) = f(t, x(t))\) maps every equi-continuous subset of \( B_0 \) into an equi-continuous subset of it.
If either \(0 \leq t \leq \min\{1, a\}\) or \(\mu\) is a sublinear MNC, then by Corollary 4.9,

\[
u_{n+1}(t) = \mu(\overline{\text{co}}(A(B_n)(t))) = \mu\left(\int_{0}^{t} f(s, B_n(s)) \, ds\right)
\leq \int_{0}^{t} \mu(f(s, B_n(s))) \, ds \leq \int_{0}^{t} w(s, u_n(s)) \, ds,
\]

which implies \(u_{\infty}(t) \leq \int_{0}^{t} w(s, u_{\infty}(s)) \, ds\). Consequently, \(u_{\infty}(t) = 0\) for all \(0 \leq t \leq \min\{1, a\}\). Since \(u_n\) converges to \(u_{\infty} = 0\) uniformly on \(I\),

\[
\lim_{n \to \infty} \max_{t \in [0, a]} u_n(t) = 0.
\]

It follows from the generalized Arzela-Ascoli theorem that the set \(B_{\infty} = \bigcap_{n=1}^{\infty} B_n\) is a nonempty convex compact set in \(C(I, X)\). Obviously, \(A\) is a self-mapping on \(B_{\infty}\). It follows from the Schauder theorem that \(A\) has a fixed point \(x \in B_{\infty}\).

The next result (i.e., the second example) of solvability of the problem (7.1) is a generalization of Theorem 7.3. It is also an extension of Banaš and Goebel [6, Theorem 13.3.1]. The space \(C_{b}(\Omega)\) and the mapping \(J_G\) are the same as in Definition 4.1.

**Theorem 7.4.** Let \(X\) be a Banach space, \(I = [0, a]\) and \(\mu\) be a convex MNC on \(X\). Suppose that

(i) \(f(\cdot, \cdot) : I \times X \to X\) is bounded and continuous, and \(w = w(\cdot, \cdot)\) is a Kamke function;

(ii) for any equi-continuous set \(G \subset C(I, X)\), \(J_G[0, a] \subset C_{b}(\Omega)\) is essentially separable in \(C_{b}(\Omega)\); in particular, \(\overline{F (\cdot, t)}\) satisfies one of the following three conditions:

(a) \(F\) maps every equi-continuous set into an equi-continuous set of \(C(I, X)\);

(b) \(F\) maps every equi-continuous set into an equi-regulated set of \(R(I, X)\) (see Definition 5.6(ii));

(c) \(F\) maps every equi-continuous set into a uniformly measurable set of \(L_1(I, X)\) (see Definition 5.10);

(iii) for all \(B \in \mathcal{B}(X)\) and almost all \(t \in [0, a]\),

\[
\mu(f(t, B)) \leq w(t, \mu(B)).
\]

Then the Cauchy problem (7.1) has at least one solution \(x \in C([0, a_1], X)\), where \(a_1 = \min\{1, a\}\). In particular, if \(\mu\) is a sublinear MNC, then \(a_1 = a\), i.e., the Cauchy problem (7.1) has at least one solution \(x \in C([0, a], X)\).

**Proof.** Assume that \(f(t, x) : [0, a] \times X \to X\) is bounded by \(\xi\), and let

\[
B_0 = \{x(t) \in C(I, X) : x(0) = x_0 \text{ with } \|x(t) - x(s)\| \leq \xi|t-s|, \forall s, t \in I\}.
\]

Then \(B_0\) is an equi-continuous bounded closed convex set. The transformation \(A : B_0 \to C(I, X)\) defined for \(x \in B_0\) and \(t \in I\) by

\[
(Ax)(t) = x_0 + \int_{0}^{t} f(s, x(s)) \, ds
\]

is a continuous self-mapping on \(B_0\), and it maps every nonempty subset of \(C(I, X)\) into an equi-continuous subset. Let

\[
B_{n+1} = \overline{\text{co}}(A(B_n)), \quad n = 0, 1, 2, \ldots
\]

Then we obtain a decreasing sequence \(\{B_n\}\) of equi-continuous closed convex subsets. If we repeat the procedure of the proof of Theorem 7.3, then we get a decreasing sequence \(\{u_n\}\) of continuous functions \(u_n(t) = \mu(B_n(t))\) and \(u_{\infty} \leq u_n\) for all \(n \in \mathbb{N}\) defined by

\[
u_{\infty}(t) = \lim_{n \to \infty} u_n(t)
\]

for all \(t \in I\) such that \(\lim_{t \to 0^+} \frac{u_{\infty}(t)}{t} = 0\).
Since for each \( n \in \mathbb{N} \), \( J_{B_n}(I) \) is essentially separable in \( C_b(\Omega) \), by Theorem 5.2, \( J_{B_n} : I \to C_b(\Omega) \) is strongly measurable. If either \( 0 < t \leq \min\{1, a\} \) or \( \mu \) is a sublinear MNC, then it follows from Corollary 4.9 that

\[
u_{n+1}(t) = \mu(\overline{co}(A(B_n)(t))) = \mu \left( \int_0^t f(s, B_n(s)) \, ds \right) \leq \int_0^t \mu \{ f(s, B_n(s)) \} \, ds \leq \int_0^t w(s, u_n(s)) \, ds,
\]

which implies \( \nu_{\infty}(t) \leq \int_0^t w(s, u_{\infty}(s)) \, ds \). Consequently, \( \nu_{\infty}(t) = 0 \) for all \( 0 \leq t \leq 1 \). Since \( u_n \) converges to \( u_{\infty} = 0 \) uniformly on \( I \),

\[
\lim_{n \to \infty} \max_{t \in [0, 1]} u_n(t) = 0.
\]

It follows from the generalized Arzela-Ascoli theorem that the set \( B_{\infty} = \bigcap_{n=1}^{\infty} B_n \) is a nonempty convex compact set in \( C(I, X) \). Obviously, \( A \) is a self-mapping on \( B_{\infty} \). It follows from the Schauder theorem that \( A \) has a fixed point \( x \in B_{\infty} \).

We finish the proof by noting that one of the three particular cases (a)–(c) can always guarantee that the mappings \( J_{B_n} : I \to C_b(\Omega) \) are strongly measurable (see Theorems 5.3, 5.7 and 5.11).

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