On the potential energy in an electrostatically bound two-body system

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Abstract The potential energy problem in an electrostatically bound two-body system is studied in the framework of a recently proposed impact model of the electrostatic force and in analogy to the potential energy in a gravitationally bound system. The physical processes are described that result in the variation of the potential energy as a function of the distance between the charged bodies. The energy is extracted from distributions of hypothetical interaction entities modified by the charged bodies.

Keywords Potential energy, electrostatics, closed systems, impact model

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1 Introduction

In analogy to the gravitational potential treated in Wilhelm and Dwivedi (2015, Paper 1), we can—according to Landau and Lifshitz (1976)—write the Lagrangian of a closed system consisting of two bodies A and B in motion with masses \( m_A \) and \( m_B \), respectively, as

\[
L = \frac{1}{2} \left( m_A V_A^2 + m_B V_B^2 \right) - U(r_A, r_B) = T - U ,
\]

where \( r_A, r_B \) are the radius vectors of the bodies and \( V_A = \frac{d r_A}{dt}, V_B = \frac{d r_B}{dt} \) their (non-relativistic) velocities. The sum \( T \) is the kinetic energy and the function \( U \) here designates the electrostatic potential energy of the system. The conservation law of energy can be derived from the homogeneity of time: The energy \( E = T + U \) of the closed system remains constant during the motion, because \( L \) does not explicitly depend on time.

The external electrostatic potential of a spherically symmetric body A with charge \( Q \) is

\[
\phi_A(r) = \frac{Q}{4\pi \varepsilon_0 r} ,
\]

where \( r \) is the distance from the centre of the body (cf., e.g. Jackson 1999). The electric constant is \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F m}^{-1} \) (exact).

Although the introductory statements on the potential energy are textbook knowledge, Carlip (1998) wrote: “—after all, potential energy is a rather mysterious quantity to begin with—”. This remark motivated us to think about the gravitational potential energy (Paper 1). Here we will discuss the electrostatic aspects of the “mystery”.

In order to have a well-defined configuration for our discussion, we will assume that body A has a positive charge \(|Q|\) and is positioned beneath body B with either a charge \(|q|\) in Fig. 1 or \(-|q|\) in Fig. 2. Only the processes near the body B are shown in detail. Since the electrostatic forces between charged particles A and B are typically many orders of magnitude larger than the gravitational forces, we only take the electrostatic effects into account and neglect the gravitational interaction.

1 Follows from the definition of \( \mu_0 = 4\pi \times 10^7 \text{ H m}^{-1} \), the magnetic constant (Bureau International des Poids et Mesures, BIPM, 2006), and \( \varepsilon_0 = (\mu_0 c_0^2)^{-1} \) with the speed of light in vacuum \( c_0 = 299792458 \text{ m s}^{-1} \) (exact) — according to the definition of the SI base unit “metre”.

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2 Free particles

The energy $E_m$ and momentum $p$ of a free particle with mass $m$ moving with a velocity $V$ relative to an inertial reference system are related by

$$E_m^2 - p^2 c_0^2 = m^2 c_0^4,$$

where the momentum $p$ is

$$p = V E_m c_0^2$$  \[\text{(4)}\]

\text{(Einstein 1905a\textsuperscript{a,b}).}

For an entity in vacuum with $m = 0$, such as a photon (cf. Einstein 1905c; Lewis 1926; Okun 2009), the energy-momentum relation in Eq. (3) reduces to

$$E = p c_0 = p c_0,$$

\[\text{(5)}\]

3 Electrostatic impact model and dipoles

In analogy to Eq. (3), we assumed for hypothetical massless entities (named “dipoles”)

$$E_D = |p_D| c_0 = p_D c_0,$$

\[\text{(6)}\]

where $p_D$ is the momentum vector of the dipoles, and constructed an electrostatic impact model (Wilhelm, Dwivedi and Wilhelm 2014, Paper 2). The interaction rates of dipoles with bodies A and B

$$\frac{\Delta N_{Q,q}}{\Delta t} = \frac{\Delta N_{q,Q}}{\Delta t} \quad \text{(the same for both bodies even for } |Q| \neq |q|),$$

required to emulate Coulomb’s law in the static case and Newton’s third law can be obtained from Eqs. (31) and (32) of Paper 2:

$$\left| \frac{\Delta P_E(r)}{\Delta t} \right| = p_D \frac{\Delta N_{Q,q}(r)}{\Delta t} = p_D \eta E \frac{\kappa_E}{c_0} \frac{|Q| |q|}{4 \pi r^2},$$

\[\text{(8)}\]

where $r$ is the separation distance between both bodies and $|\Delta P_E/\Delta t|$ is the norm of the momentum change rate for A and B leading together with

$$p_D \eta E \kappa_E = c_0 / \varepsilon_0$$

\[\text{(9)}\]

to an attractive or repulsive electrostatic force of

$$F_E(r) = \pm |q| \frac{|Q|}{4 \pi \varepsilon_0 r^2}.$$  \[\text{(10)}\]
The quantities $\eta_E$ and $\kappa_E$ are the electrostatic emission and absorption coefficients with the following definitions: The absorption coefficient is the dipole absorption rate of a charge from the background

$$\frac{\Delta N_Q}{\Delta t} = \kappa_E \rho_E |Q| = \eta_E |Q|, \quad (11)$$

where $\rho_E = \Delta N_E/\Delta V$ is the spatial background number density of dipoles in the volume element $\Delta V$, and $\eta_E = \kappa_E \rho_E$ is the emission coefficient leading to the same emission rate $\Delta N_Q/\Delta t$.

4 The potential energy

We may now ask the question, whether the electrostatic impact model can provide an answer to the "mysterious" potential energy problem in a closed system, where dipoles are interacting with two charged bodies. The number of dipoles travelling at any instant of time from one charge to the other can be calculated from the interaction rate in Eq. (8) multiplied by the travel time $T = r/c_0$.

$$\Delta N_{Q,q}(r) = \frac{\eta_E \kappa_E}{c_0^3} \frac{|Q||q|}{4\pi r}. \quad (12)$$

The same number of dipoles is moving in the opposite direction. The energy of the dipoles interacting with the corresponding charge then is

$$\Delta E_E(r) = \Delta N_{Q,q}(r) \rho_D \kappa_D = \frac{\rho_D \eta_E \kappa_E}{c_0^3} \frac{|Q||q|}{4\pi r} = \frac{|Q||q|}{4\pi \varepsilon_0 r}. \quad (13)$$

The last term shows – with reference to Eqs. (2) and (11) – that the energy $\Delta E_E$ equals the absolute value of the electrostatic potential energy of body B

$$U_B(r) = \phi_A(r) q = \frac{|Q||q|}{4\pi \varepsilon_0 r}, \quad (14)$$

at a distance $r$ from body A. The symmetry in $Q$ and $q$ implies that the potential energy of body A at a distance $r$ from body B is the same. To simplify the following arguments, we will now assume that body A has a mass $m_A$ much larger than $m_B$ of body B and can be considered to be at rest in an inertial system. We then calculate the difference of the potential energies for a displacement of B from $r$ to $r + \Delta r$ as well as the difference of the energies of the interacting dipoles and get

$$U_B(r) - U_B(r + \Delta r) = \frac{|Q||q|}{4\pi \varepsilon_0} \left( \frac{1}{r} - \frac{1}{r + \Delta r} \right) \approx \frac{|Q||q| \Delta r}{4\pi \varepsilon_0 r^2}, \quad (15)$$

and

$$\Delta E_E(r) - \Delta E_E(r + \Delta r) = \{\Delta N_{Q,q}(r) - \Delta N_{Q,q}(r + \Delta r)\} \rho_D \kappa_D c_0 = \frac{|Q||q|}{4\pi \varepsilon_0} \left( \frac{1}{r} - \frac{1}{r + \Delta r} \right) \approx \frac{|Q||q| \Delta r}{4\pi \varepsilon_0 r^2}, \quad (16)$$

where the approximations are valid for $|\Delta r| \ll r$. For $+|q|$ and $\Delta r > 0$, Eqs. (15) and (16) correspond to the case of repulsion in Fig. 1. If, on the other hand, the charge of body B in Fig. 2 is $-|q|$ and $\Delta r < 0$, the equations describe attraction between the bodies. In both cases, the result of Eq. (15) is positive, i.e. $U_B(r) > U_B(r + \Delta r)$. The difference of the potential energies can be transformed into kinetic energy with respect to the inertial system defined.

The last term of Eq. (10) gives the variation of the energy $\Delta E_E$ between $r$ and $r + \Delta r$. It is positive for repulsion with $\Delta r > 0$ and equal to the result of Eq. (16). The source of the potential energy for this process (shown in Fig. 1) thus is the difference of the number of interacting dipoles on their way to body B and the corresponding difference in energy.

In the case of attraction, the result of Eq. (16) is negative, whereas the difference of the energies in Eq. (15) was positive. This can be understood by considering that the number of indirect interactions in Fig. 2 increases and the excess direct interactions from the background are needed to provide the negative force $F_E(r)$ in Eq. (10), which is, however, controlled by the number of indirect interactions.

A question remains concerning the dipoles travelling to body A. Eqs. (12) and (13) are symmetric in $q$ and $Q$ and, therefore, the difference in their number with a distance variation of $\Delta r$ must be the same as that of the dipoles on their way to B, i.e.,

$$\Delta N_{Q,q}(r) - \Delta N_{Q,q}(r + \Delta r) = \Delta N_{q,Q}(r) - \Delta N_{q,Q}(r + \Delta r). \quad (17)$$

What happens to the corresponding difference in energy, since body A in our approximation is basically at rest? The answer is that the change in potential energy of body A at the new relative position with respect to B is given by the absolute value of the results of Eq. (13) and thus accounts for the energy difference of Eq. (15).

5 Conclusion

In the framework of a recently proposed electrostatic impact model in Paper 2, the physical processes related to the variation of the electrostatic potential energy of two charged bodies have been described and the "source
region” of the potential energy in such a system could be identified. In a configuration with repulsion, the potential energy is directly related to the energy of the interacting dipoles on their way from body A to body B. For attraction, the negative force stems from the excess direct dipole interactions from the background distribution – in analogy to the gravitational attraction in Paper 1.
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