Quantum Memory Effects in the Measurement of Observables with a Continuous Spectrum

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Abstract
In the measurement of a continuous observable $Q$, the pure components of the reduced state do, in general, depend on the initial state. For measurements which attempt to localize the measured system in a certain region $R$, the localized wave functions are proportional to the original wave function outside of $R$. This “quantum memory” effect shows that it is not possible to perfectly localize a quantum particle.

In this paper we address the question of how to measure a quantum-mechanical observable with a continuous spectrum. Ever since the pioneering work of von Neumann [1], it has been known that for the measurement of an observable $Q$, the interaction energy between the observed system $S$ and the measuring apparatus $A$ has to be a function of $Q$ [2,3]. Von Neumann considered an interaction term which is linear in a position-like variable $Q$ [1]. More recently Zurek studied interactions between $S$ and $A$ in terms of a preferred “pointer basis” of $A$ [2]. Haake and Walls studied the case of an oscillator coupled to an infinite reservoir [3], following the earlier work of Ullersma [4]. Unruh and Zurek recently studied state reduction in a similar model, including the case of a free particle, and presented explicit calculations of reduced density matrices for states with Gaussian characteristic functions [5].

These studies show that the variance of $Q$ in the pure components of the reduced density matrix $\rho_S$ is much less than in the initial state of $S$, in accord with the idea
that an approximate measurement of $Q$ has been performed by $A$. However, the explicit calculations of $\rho_S$ also show that in general the set of eigenfunctions of $\rho_S$ depends quite strongly on the initial state. For a measurement, on the other hand, we require that the set of eigenstates of $\rho_S$ be independent of the initial state, and be identical to the set of eigenstates of the observable to be measured. In this sense, the linear models studied so far do not provide models for a measurement of $Q$.

In the following, we show that measurements which determine the value of a continuous observable $Q$ can be approximately realized by couplings of the form $\lambda_S(Q) \cdot V_A$, where $\lambda_S$ is a strongly localized function which vanishes rapidly outside a finite interval $I$, and $V_A$ is an apparatus operator. However, in this case the localized wave functions do not vanish, but are proportional to the original wave function outside of $I$.

Following von Neumann, we assume that the Hamiltonian of system $S$ plus apparatus $A$ takes the form

$$H = \lambda_S(Q) \cdot V_A + H_A$$

and that the initial state $\rho(0)$ of system $S$ plus apparatus $A$ at time $t = 0$ is the product of a pure state $|\psi_S\rangle\langle\psi_S|$ of $S$ with a state $\rho_A$ of the apparatus. Due to the $S$-$A$ interaction term, at later times $t > 0$ the state $\rho(t)$ is no longer a product, and the reduced state $\rho_S(t) = \text{trace}_A \rho(t)$ of $S$ is no longer a pure state. This state reduction models the measuring process.

The Hamiltonian (1) does not contain a part for the free motion of $S$. This corresponds to the assumption that the system $S$ does not change significantly over a time period $t$ required for the measurement.

We further assume that $[V_A, H_A] = 0$, and that the initial state of $A$ is invariant under the free motion of $A$. The matrix elements of $\rho_S(t)$ in the basis $\{|q\rangle\}$ of eigenstates of $Q$ are then (we choose units such that $\hbar = 1$)

$$\langle q' | \rho_S(t) | q'' \rangle = \text{trace}_A \langle q'| e^{-itH} (|\psi_S\rangle\langle\psi_S| \otimes \rho_A) e^{itH} | q'' \rangle = \langle q'| \psi_S \rangle \langle \psi_S | q'' \rangle \text{trace}_A (e^{-it\lambda_S(q')} V_A \rho_A e^{it\lambda_S(q'')} V_A) = \psi_S(q') \psi_S(q'')^* f_A \left(t (\lambda_S(q') - \lambda_S(q''))\right).$$

(2)
The overlap function $f_A(t) = \text{trace}_A (\rho_A e^{-i t V_A})$ has the property $f_A(0) = 1$, and we assume that $\lim_{t \to \infty} f_A(t) = 0$. We now apply equation (2) to three different cases of interest.

Firstly, assume that $\lambda_S(Q) \equiv Q$, and that $Q$ has a discrete spectrum $\{q_\alpha\}$, such that $|q_\alpha - q_\beta| \geq \Delta q > 0$ for $\alpha \neq \beta$. Then with $\theta_{\alpha \beta} = t(q_\alpha - q_\beta)$ and $|\theta_{\alpha \beta}| < t \Delta q$ we have for the matrix elements of $\rho_S$ in the basis $\{|q_\alpha\}\$:

$$\langle q_\alpha | \rho_S(t) | q_\alpha \rangle = |\langle q_\alpha | \psi_S \rangle|^2$$

$$\langle q_\alpha | \rho_S(t) | q_\beta \rangle = \langle q_\alpha | \psi_S \rangle \langle \psi_S | q_\beta \rangle f(\theta_{\alpha \beta})$$

$$\to 0 \text{ for } t \to \infty, \alpha \neq \beta. \quad (3)$$

The off-diagonal elements go to zero uniformly in $\alpha$ and $\beta$ for $t \to \infty$. This shows that $\rho_S(t)$ converges to its diagonal part in the limit $t \to \infty$.

Secondly, we consider $\lambda_S(Q) \equiv Q$ with a continuous observable $Q$, and a Gaussian overlap function.

$$f_A(t(q' - q'')) = \exp\left(-\frac{1}{4} \kappa_A^2(t)(q' - q'')^2\right)$$

If the initial state $|\psi_S\rangle$ is the vacuum state of the shifted number operator $N = \frac{1}{2} \left(\kappa_S^{-2} P^2 + \kappa_S^2 (Q - q_0)^2\right) - \frac{1}{2}$, then

$$\psi_S(q) = \langle q | \psi_S \rangle = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2} \kappa_S^2 (q - q_0)^2\right), \text{ and }$$

$$\langle q' | \rho_S | q'' \rangle = \frac{1}{\pi} \exp\left(-\frac{1}{2} \kappa_S^2 \left((q' - q_0)^2 + (q'' - q_0)^2 - \frac{1}{4} \kappa_A^2 (q' - q'')^2\right)\right) \quad (4)$$

The state $\rho_S$ is a canonical ensemble $\rho_S = (1 - e^{-\beta}) e^{-\beta N_R}$ whose pure components are the eigenstates of the squeezed, shifted number operator $N_R$

$$N_R = \frac{1}{2} \left(\kappa_R^{-2} P^2 + \kappa_R^2 (Q - q_0)^2\right) - \frac{1}{2}, \text{ with }$$

$$\kappa_R^2 = \kappa_S \sqrt{\kappa_S^2 + \kappa_A^2} \text{ and } e^{-\beta} = \frac{(\kappa_R^2 - \kappa_S^2)}{(\kappa_R^2 + \kappa_S^2)}. \quad (5)$$

For significant state reduction, $\kappa_A^2 \gg \kappa_S^2$, and $\kappa_R^2 \approx \kappa_S \kappa_A$ still depends strongly on the initial state through $\kappa_S$ and $q_0$. The eigenvalues of $\rho_S$ are independent of $q_0$, the location of the maximum of $\psi_S$. 

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Finally, we consider the case of continuous $Q$ with strongly localized coupling function $\lambda_S(q)$. In the physically unrealistic case where $\lambda_S = 1$ inside and $\lambda_S = 0$ outside of a finite interval $I$, the eigenstates of $\rho_S$ are wave functions with support inside or outside of $I$. In this situation, the observable being measured is the spectral projection operator associated with eigenvalues of $Q$ in $I$.

For physically realizable interactions, the function $\lambda_S(q)$ will not be a step function, but will go to zero for $|q| \to \infty$. An example is the coupling between two levels of an atom through the electric field of an external electron, which depends on the distance between electron and atom.

In such a situation, if the wave function $\psi_S(q)$ does not vary strongly in the region, where $\lambda_S(q)$ is appreciably different from zero, we expect the pure components of the reduced state to be independent of the initial state vector $|\psi_S\rangle$. More specifically, we assume that at sufficiently large time $t$, there exists a value $q_0$, such that

$$f_A(t\lambda_S(q)) \approx 1 \text{ for } |q| > q_0, \quad f_A(t\lambda_S(0)) \approx 0 \quad \text{and} \quad \psi_S(q) \approx \psi_S(0) \text{ for } |q| < q_0. \quad (6)$$

From Eq. (2) we find that the eigenfunctions of $\rho_S$ with non-zero eigenvalues are proportional to $\psi_S(q)$ for $|q| > q_0$. In this subspace of wave functions, the density matrix $\rho_S$ still depends on $\psi_S(0)$ or on the probability $p_0 = 2q_0|\psi_S(0)|^2$ of finding the system in the interval $I = [-q_0, q_0]$. For small values of $p_0$, a perturbation expansion in $\psi_0(0) \equiv \psi$ gives the following results. The eigenfunctions of $\rho_S$ take the form:

$$\psi_N(q) = \psi_S(q)f_A(t\lambda_S(q)) + O(\psi^2),$$
$$\psi_i(q) = \psi_0(q) + \lambda_iM\psi_M(q) + O(\psi^2), \quad (7)$$

where $\psi_0(q) \approx 0$ and $\psi_M(q) \approx \psi_N(q) \approx \psi_S(q)$ for $|q| > q_0$.

The function $\psi_N$ has zero amplitude at $q = 0$, and corresponds to a negative result for the attempt to localize the system $S$ in $I$, which this measurement represents. The functions $\psi_i$, on the other hand, represent different modes of localization of $S$ within $I$. The coefficients $\lambda_iM$ are non-zero and proportional to $\psi_S(0)$. Thus, the localized wave functions do not vanish outside of $I$, but are proportional to the original wave function there.
To illustrate these findings, we have diagonalized $\rho_S$ numerically by discretizing the matrix elements in the interval $I$. In the example shown here, the coupling function is $\lambda_S(q) = \exp(-q^2)$, overlap function $f(\theta) = \exp(-\theta^2)$ and time $t = 4$. The resulting eigenfunctions were scaled by the square root of their eigenvalues, and the scaled eigenfunctions with largest eigenvalues are shown in Figure 1 for values of $p_0 = 0.01$ and $p_0 = 0.36$. As can be seen, the eigenfunctions for both cases are almost identical. The non-localized wave function is very close to $\psi_N$ of Eq. (7) even for large values of $p_0$. In addition, there exist about nine localized wave functions with significant eigenvalues. For $p_0 = 0.36$, the asymptotic tail of the localized eigenfunctions for large $q$, resulting from the coefficient $\lambda_i M$, is clearly visible. This relatively large number of different modes of localization is due to the fact that we have treated the apparatus $A$ as completely unobservable. For a more realistic model, in which $A$ is partly observable, a smaller number of modes may result.

The results obtained here are interesting in that they show a fundamental difference in measurability between continuous observables with linear couplings and observables with nonlinear, localized couplings. Examples of the first category with linear couplings only are field variables, such as the electric and magnetic fields. Such variables cannot be measured in the strict sense of the word, since the pure components of their reduced state depend upon the initial state. In the second example above, the corresponding eigenfunctions are all centered at the same location $q_0$ as the original wave function $\psi_S$, while the eigenvalues are independent of $q_0$. The first example shows that this is due exclusively to the continuous nature of $Q$. While in the discrete case, the density matrix elements converge uniformly to diagonal form, in the continuous case convergence is non-uniform. For a continous variable, there does not exist a diagonal form of the density matrix, and example two shows that the reduced state converges to zero in the norm for large times. We also note that our result (4) for the reduced matrix elements, although obtained under somewhat restrictive conditions on the total Hamiltonian, is in agreement with the results obtained by Unruh and Zurek [5]. They find that the term involving $\kappa_A^2$ is dominant for times $t \sim \Gamma^{-1}$, where $\Gamma$ is the cut-off frequency of the infinite reservoir of oscillations representing $A$ in their work.
Finally, example three demonstrates the possibility of measuring continuous observables such as the position of a charged particle, which couple to a measuring apparatus through localized functions of position, e.g. their electric field. Such couplings allow one to decide the question whether or not the particle is approximately localized at a given point, with corresponding eigenfunctions that are almost independent of the initial state. In this sense, these observables can be measured indirectly. However, in this case, the asymptotic behavior of the localized eigenfunctions at large distances reveals a quantum-memory effect, or incomplete localization of the particle by this type of measurement. Thus, if the particle had been found localized at \( q = 0 \), an immediately following localization experiment of the same type, but at a different location \( q = q_1 \), would yield a positive result with a probability proportional to \( |\psi_S(0)|^2 \langle \psi_S(q_1) \rangle^2 \).
References

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Figure Caption

Figure 1: The most important localized eigenfunctions and the non-localized eigenfunction of the reduced density matrix are shown for two different values of the local probability $p_0$. Solid curves correspond to $p_0 = 0.01$, and dashed curves to $p_0 = 0.36$. The functions have been scaled by the square root of their eigenvalue.