A STRONG COLLAPSE INCREASING THE GEOMETRIC SIMPLICIAL LUSTERNIK-SCHNIRELMANN CATEGORY

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Abstract. In [3], after defining notions of LS category in the simplicial context, the authors show that the geometric simplicial LS category is non-decreasing under strong collapses. However, they do not give examples where it increases strictly, but they conjecture that such an example should exist, and thus that the geometric simplicial LS category is not strong homotopy invariant. The purpose of this note is to provide with such an example. We construct a simplicial complex whose simplicial and geometric simplicial LS categories are different, and using this, we provide an example of a strong collapse that increases the geometric simplicial LS category, thus settling the geometric simplicial LS category not being strong homotopy invariant.

1. Introduction

The Lusternik-Schnirelmann category (for short, LS category) for a topological space $X$ is defined as the smallest integer $k$ such that there is an open covering $(U_j)_{j \leq k+1}$ of cardinality $k + 1$ of $X$ such that the inclusion maps $i_j : U_j \hookrightarrow X$ are nullhomotopic. It is an important homotopy invariant, providing an upper bound for the critical points of a manifold, among several other applications.

In [3], the authors introduced the simplicial LS category, i.e. a notion of LS category for simplicial complexes. The advantage of their simplicial version is that it is a strongly homotopy invariant and it depends only in the simplicial structure, not on the chosen geometric realisation. For further information on the simplicial LS category, see also [4]. A nice introduction, in relation with finite topological spaces and using category theoretic language, may also be found in [5].

Let $K, L$ be simplicial complexes. Two simplicial maps $\varphi, \psi : K \to L$ are said to be contiguous if for every simplex $\sigma \in K$, $\varphi(\sigma) \cap \psi(\sigma)$ is a simplex in $L$. The contiguity relation is denoted by $\varphi \sim_c \psi$. The relation $\sim_c$ is symmetric and reflexive, but in general it is not transitive. Thus, as a simplicial equivalent to homotopy, the notion of the contiguity class needs to be introduced:

Definition 1. Let $K, L$ be simplicial complexes. Two simplicial maps $\varphi, \psi : K \to L$ belong to the same contiguity class ($\varphi \sim \psi$) if there is a sequence $\{\varphi_i\}_{i \leq n}$ of maps from $K$ to $L$ such that $\varphi = \varphi_0 \sim_c \varphi_1 \sim_c \varphi_2 \sim_c \cdots \sim_c \varphi_n = \psi$.

The role of sets whose inclusion is nullhomotopic is to be played by categorical subcomplexes.

Definition 2. Let $K$ be a simplicial complex. We say that a subcomplex $U \subset K$ is categorical if there exists a vertex $v \in K$ such that the inclusion map $i_U : U \hookrightarrow K$ and the constant map $c_v$ are in the same contiguity class, i.e. $i_U \sim_c c_v$.

Definition 3. The simplicial LS category of a simplicial complex $K$, denoted by $\text{scat}K$, is the least integer $k$ such that $K$ can be covered by $k + 1$ categorical subcomplexes. Such a cover is called categorical.

Let $K$ be a simplicial complex and $u, v \in K$ be two vertices. If for every maximal simplex $\tau \in K$ such that $u \in \tau$, we have that $v \in K$, we say that $u$ is dominated...
by $v$. Deleting such a vertex and removing all simplices that contain it is called \textit{elementary strong collapse} and results in the simplicial complex $K\setminus u$. We say that $K$ \textit{strong collapses} to $L$ if there is a sequence of elementary strong collapses from $K$ to $L$, and we denote it by $K \searrow L$. The inverse procedure, going from $L$ back to $K$ by adding dominated vertices, is called a \textit{strong expansion}. We say that $K$ and $L$ have the same \textit{strong homotopy type} if there is a sequence of strong collapses and expansions from $K$ to $L$. A well known result (see \cite[Corollary 2.12]{2}) is that $K$ and $L$ have the same strong homotopy type if and only if there are maps $\varphi : K \to L$ and $\psi : L \to K$ such that $\varphi \circ \psi \sim \text{Id}_L$ and $\psi \circ \varphi \sim \text{Id}_K$. Then, we denote $K \sim L$.

The simplicial LS category is a strong homotopy invariant.

Another related notion is that of the geometric simplicial LS category, based on the notion of strong collapsibility. As conjectured in \cite{3} and proven in the present note, this is not a strong homotopy invariant. There are similar notions based on simple collapsibility, see \cite{1}.

\textbf{Definition 4.} Let $K$ be a simplicial complex. We say that $K$ is strongly collapsible if it strongly collapses to a point, i.e. if $\text{Id}_K \sim c_v$, where $c_v$ is the constant map to a vertex $v \in K$.

\textbf{Definition 5.} The geometric simplicial category of a simplicial complex $K$, denoted by $\text{gscat}K$, is the least integer $k$ such that $K$ can be covered by $k + 1$ strongly collapsible subcomplexes. Such a cover is called geometric.

By the definitions above, a strongly collapsible subcomplex is also categorical, but the opposite does not always hold. In fact, a categorical subcomplex need not even be connected, while a strongly collapsible one is necessarily connected.

The following results relate these two different notions of LS category for simplicial complexes.

\textbf{Proposition 6.} \cite[Proposition 4.2]{3} Let $K$ be a simplicial complex. Then, $\text{scat}K \leq \text{gscat}K$.

\textbf{Theorem 7.} \cite[Theorem 4.3]{3} Let $M, K$ be simplicial complexes such that $K$ is a strong collapse of $M$. Then, $\text{gscat}M \leq \text{gscat}K$.

The authors in \cite{3} remark that they did not have an example that satisfies the strict inequality in Theorem 7 but that, based on similar results regarding partially ordered sets and beat points, such an example should exist. In this short note, our purpose is to construct examples that satisfy these inequalities strictly, i.e. showing that the inequalities above do not degenerate to equalities. First, in section 2, we construct an example that satisfies the strict inequality in Proposition 6 with the further property that it has minimal categorical cover which consists of connected subcomplexes. This example is important in constructing a second one satisfying the strict inequality in Theorem 7 as the simplicial LS category is strong homotopy invariant, hinting that this second example should be searched among cases where simplicial and geometric simplicial LS categories are not equal.

The main contribution of this note is the next proposition, proven in sections 2 and 3, and the follow-up corollary, derived directly from it.

\textbf{Proposition 8.} There is a simplicial complex $K$ such that $\text{scat}K < \text{gscat}K$. Moreover, there is a simplicial complex $M$ such that $M \searrow K$ and $\text{gscat}M < \text{gscat}K$.

\textbf{Corollary 9.} The geometric simplicial LS category is not strong homotopy invariant.

\textbf{Remark 10.} The respective geometric realisations provide examples for the topological analogues of the strict inequalities. Of course, examples to these have already been known.
2. Example of strictly bigger geometric simplicial than simplicial LS category

In this section, we shall construct a simplicial complex $K$ such that $\text{gscat} K > \text{scat} K$. Let $K$ be the simplicial complex with set of vertices

$$V_K = \{(k, l) | -2 \leq k \leq 2, 0 \leq l \leq 2\}$$

and simplicial structure as shown in the figure below, where the rightmost and leftmost vertices, the rightmost and leftmost edges, and the vertices with coordinates $(0,0)$ (noted by the bulk points) are identified:

![Figure 1. The simplicial complex $K$. The rightmost and leftmost edges and vertices, as well as the 3 bulk points corresponding to the vertex $(0,0)$, are identified.](image)

It has degree 2 and 15 vertices, 45 edges and 30 2-simplices.

![Figure 2. A geometric realisation of $K$. It is a sphere in which we have created 2 handles by identifying 3 points together.](image)
It is clear that $K$ is not strongly collapsible (its geometric realisation is not even contractible), hence $\text{scat}K > 0$. The following categorical cover has cardinality 2, hence $\text{scat}K = 1$.

![Figure 3. A categorical cover of $K$](image)

However, this cover is not geometric. The green subcomplex is not strongly collapsible, as its core is not trivial. It can be strongly collapsed into the loop $\{(0, 2), (0, 0)\}, \{(0, 0), (0, 1)\}, \{(0, 2), (0, 1)\}$ that cannot be further strongly collapsed as it contains no dominated points.

The green subcomplex is categorical, but no strongly collapsible subcomplex contains it. This is an important fact as, otherwise, just by extending it we would have gotten a geometric cover, and thus the simplicial and the geometric simplicial LS categories would be equal. Hence, the existence of a categorical subcomplex which cannot be contained in a bigger strongly collapsible subcomplex is necessary for the simplicial and geometric simplicial LS categories to be different.

It is not difficult to see through combinatorial arguments that it is not possible to cover $K$ with 2 strongly collapsible subcomplexes. Heuristically, assume that $K$ is covered by strongly collapsible subcomplexes $A$ and $B$, and wlog that one of the triangles on the top of the figure is solely contained in $A$. For the moment, let’s treat the vertices $(0, 0)$ in the 3 different places in Figure 1 as not being identified. As no sequence of edges in $A$ may connect the top $(0, 0)$ with the middle or the bottom ones (as these loops cannot be in any strongly collapsible subcomplex in $K$), there must be a maximal connected subcomplex in $A$ that contains this triangle, and no triangles in the middle or bottom that contain $(0, 0)$. The boundary of this must then reside in $A \cap B$, and it must contain a loop that winds around one of the $(0, 0)$’s. But, for such a loop, there can only be a single minimal strongly collapsible subcomplex containing it. This subcomplex cannot be in both $A$ and $B$ as it must either contain the middle or bottom $(0, 0)$, thus it cannot be contained in $A$, or the top $(0, 0)$, thus it cannot contain $B$ by our initial assumptions (that there is a triangle in the top not contained in $B$). Thus $K$ cannot be covered by 2 strongly collapsible subcomplexes. For a more detailed proof, see Appendix. Hence, $\text{gscat}K > 1$.

The following figure shows a geometric cover of cardinality 3, which implies that $\text{gscat}K = 2$. Thus, the first part of Proposition 8 is proven.
3. A strong collapse that increases the geometric simplicial LS category

Based on the previous example, we consider $K$ as before. Let $M$ be a simplicial complex with vertices

$$V_M = V_K \cup \{a\}$$

and

$$M = K \cup \{\{a\}\} \cup \{\{a, (0,0), \{a, (2,0), \{a, (2,1), \{a, (2,2)\}\}$$

$$\cup \{\{a, (0,0), (2,0), \{a, (0,0), (2,1), \{a, (0,0), (2,2)\}\}$$

$$\cup \{\{a, (2,0), (2,1), \{a, (2,1), (2,2), \{a, (2,2), (2,0)\}\}$$

What we have essentially constructed is a simplicial complex in which we have added a new vertex $a$ and filled in the 3-simplices that contain $a$, $(0,0)$ and two of the $(2,0), (2,1), (2,2)$. We see that the new vertex $a$ is dominated by $(0,0)$, so $M \succ K$. 

![Diagram](https://via.placeholder.com/150)
Let $A \subset M$ be the subcomplex whose set of maximal simplices is

$$\{\{a,(0,0),(2,0),(2,1)\},\{a,(0,0),(2,1),(2,2)\},\{a,(0,0),(2,2),(2,0)\}\} \cup$$

$$\{\{(0,0),(-2,0),(-2,1)\},\{(0,0),(-2,1),(-2,2)\},\{(0,0),(-2,2),(-2,0)\}\}$$

We can easily see that $A$ strongly collapses to $(0,0)$, by first deleting the dominated vertex $a$, and then the other vertices. Let $B_0$ be the subcomplex consisting of all the maximal simplices of $M$ that are not in $A$, $B_1$ the one consisting of the 2-simplices $\{\{a,(2,0),(2,1)\},\{a,(2,1),(2,2)\},\{a,(2,2),(2,0)\}\}$, and $B = B_0 \cup B_1$. Again, one can show that $B$ strongly collapses to $a$: $B_0$ strongly collapses to the simplicial complex consisting of three 1-simplices $\{\{(2,0),(2,1)\},\{(2,1),(2,2)\},\{(2,2),(2,0)\}\}$, and then $(2,0),(2,1)$ and $(0,1)$ are dominated by $a$, hence $B_0 \cup B_1$ strongly collapses to $a$.

Thus, we have constructed a cover of $M$ consisting of 2 strongly collapsible subcomplexes, and hence $\text{gscat} M = 1$, while we have seen that $\text{gscat} K = 2$ and $M \setminus \mathcal{K}$, showing the second part of Proposition 8.

**Appendix**

**Lemma 1.** The simplicial complex $K$, as in section 2, cannot be covered by two strongly collapsible subcomplexes.

**Proof.** We will argue by contradiction. The idea is that a strongly collapsible subcomplex may not contain certain loops, e.g. a loop starting from $(0,0)$ at the top of the figure and ending at $(0,0)$ in the bottom or the middle. Assume that there exist two strongly collapsible subcomplexes $A, B \subset K$ such that $A \cup B = K$. Out of the three edges $\{(0,0),(2,0)\}, \{(2,0),(1,0)\}$ and $\{(1,0),(0,0)\}$ one has to belong solely to $A$ and one solely to $B$, as no strongly collapsible subcomplex may contain the loop $\{(0,0),(2,0)\}, \{(2,0),(1,0)\}, \{(1,0),(0,0)\}$. Let’s enumerate the triangles in the following way and assume that $\{(0,0),(2,0)\} \not\in B$ and $\{(1,0),(0,0)\} \not\in A$. The other cases can be argued in a similar way.

![Figure 6. An enumeration of the 2-simplices of $K$.](image)

We must have that $a_1, a_2 \in A$ and $c_2, c_3 \in B$. Then, $\{(1,1),(1,2)\} \not\in B$, as if the loop $\{(0,0),(1,2)\}, \{(1,1),(1,2)\}, \{(1,1),(0,0)\}$ is contained in $B$, $B$ cannot be strongly collapsed to a point. Hence $b_5, c_6 \in A$. As $A$ may not contain any loops in the contiguity class of $\{(0,0),(2,0)\}, \{(2,0),(1,0)\}, \{(1,0),(2,0)\}$, it must be that $c_1, c_4, d_1 \in B$. 
Similarly, as $B$ may not contain the loop $\{(0, 2), (0, 0)\}$, $\{(0, 0), (0, 1)\}$, $\{(0, 1), (0, 2)\}$, or any loop that can be deformed to it while fixing $(0, 0)$, $e_5, d_5 \in A$, and by the previous argument $d_3, d_4 \in B$, and then again $d_6 \in A$. This implies $d_2 \in B$, and then $e_5 \in A$.

Thus, assuming that $\{(0, 0), (2, 0)\} \notin B$ and $\{(1, 0), (0, 0)\} \notin A$, out of necessity we have shown:

The edge $\{(0, 0), (-2, 1)\}$ cannot be contained in $A$, as then it would not be strongly collapsible containing a loop from the top $(0, 0)$ to the bottom one, hence $f_2, f_3 \in B$. But then $B$ cannot contain $\{(-1, 0), (-2, 0)\}$, hence $e_2, e_3 \in A$. But then there is a loop $\{((-1, 2), (-1, 0)), ((-1, 0), (-1, 1)), ((-1, 1), (-1, 2))\} \subset A$, and $A$ cannot strongly collapse to a point.

In a similar way, one can reach a contradiction by assuming $\{(0, 0), (2, 0)\} \notin B$ and $\{(2, 0), (1, 0)\} \notin A$, or $\{(2, 0), (1, 0)\} \notin B$ and $\{(1, 0), (0, 0)\} \notin A$. Hence $K$ cannot be covered by 2 strongly collapsible subcomplexes.

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