Boosted Simon-Wolff Spectral Criterion and Resonant Delocalization

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Abstract
Discussed here are criteria for the existence of continuous components in the spectra of operators with random potential. First, the essential condition for the Simon-Wolff criterion is shown to be measurable at infinity. By implication, for the i.i.d. case and more generally potentials with the $K$-property, the criterion is boosted by a zero-one law. The boosted criterion, combined with tunneling estimates, is then applied for sufficiency conditions for the presence of continuous spectrum for random Schrödinger operators. The general proof strategy that this yields is modeled on the resonant delocalization arguments by which continuous spectrum in the presence of disorder was previously established for random operators on tree graphs. In another application of the Simon-Wolff rank-one analysis we prove the almost sure simplicity of the pure point spectrum for operators with random potentials of conditionally continuous distribution. © 2015 Wiley Periodicals, Inc.

1 Introduction

Studies of the spectral effects of disorder often deal with self-adjoint operators of the form

\[(1.1) \quad H(\omega) = A + V(\omega)\]

acting in the $l^2$-space of functions over an infinite graph $G$, where $A$ is a self-adjoint bounded operator and $V(\omega)$ is a multiplication operator by a random function (the random potential) $[V(\omega)\psi](x) = V(x; \omega)\psi(x)$ with $\{V(x; \omega)\}_{x \in G}$ a collection of random variables with a specified joint distribution. The parameter $\omega$, which represents the disorder, ranges over a standard probability space $(\Omega, \mathcal{B}, \mathbb{P})$. In this setup $H(\omega)$ forms a weakly measurable, self-adjoint, operator-valued function (cf. [8]). The strength of the disorder is expressed here in the width of the distribution of $V(x; \omega)$, loosely speaking.

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The spectral measure associated with a specified realization of $H(\omega)$ and a vector $\psi \in \ell^2(\mathbb{G})$ is defined by the property
\begin{equation}
\langle \psi, F(H(\omega))\psi \rangle = \int F(E)\mu_{\psi}(dE; \omega)
\end{equation}
for all $F \in C_0(\mathbb{R})$ (continuous function that vanishes at infinity). Each measure can be decomposed into its pure-point component and a continuous one:
\begin{equation}
\mu_{\psi} = \mu_{\psi}^{pp} + \mu_{\psi}^c
\end{equation}
where $\mu_{\psi}^{pp}$ is a countable sum of point measures and $\mu_{\psi}^c$ is a continuous remainder. The distinction between the two spectra is reflected in the nature of the (possibly generalized) eigenfunctions: in the pure-point case these are proper elements of the $\ell^2$-space, whereas for the continuous spectrum the eigenfunctions are not square summable. The difference also carries significant implications for the recurrence properties of the unitary evolution generated by $H(\omega)$ (cf. [9]) and the conductive properties of particle systems with such one-particle Hamiltonians.

In the well-known Anderson localization phenomenon [6], at sufficiently high disorder as well as at extremal energies (with some exceptions [3]), the spectrum of $H(\omega)$ is almost surely of pure-point type, consisting of dense (random) collections of proper eigenvalues associated with square integrable eigenfunctions. There remains, however, a dearth of methods for establishing regimes of delocalization in the presence of disorder. On the short list of such are arguments based on resonant delocalization. This approach has been especially effective for random Schrödinger operators on tree graphs [3,4], but it was also shown to guide one to correct conclusions in other contexts [2]. Our main goal here is to advance this method, combining it with an improved version of the Simon-Wolff criterion for a related sufficiency criterion under which one may conclude the existence of continuous spectrum, and in some situations an absolutely continuous one.

In a related application of the Simon-Wolff criterion for point spectrum, in the Appendix (p. 2215) we present an improved result on the simplicity of the point spectrum, proving it for a naturally broad class of random potentials.

2 The Simon-Wolff Spectral Criterion and Its Boost

2.1 The Simon-Wolff Sufficiency Condition for Continuous Spectrum

Analysis of the spectral measures associated with the canonical basis elements $\delta_x \in \ell^2(\mathbb{G})$ is facilitated by considerations of the Green function
\[ G(x, y; z; \omega) := \langle \delta_x, (H(\omega) - z)^{-1}\delta_y \rangle. \]
When the potential of $H(\omega)$ is changed at a site $x \in G$ by $\delta V(x)$, an eigenfunction that solves
\[ (H - E)\varphi(y) = 0 \]
turns into a solution of the Green function equation
\[
([H + \delta V(x) P_x] - E)\psi(y) = [\delta V(x)\psi(x)]\delta_{x,y}.
\]
This elementary observation underlines a number of results concerning the structural similarity of the eigenfunctions to the kernel of the Green function with one of its arguments fixed. In particular, starting from Aronszajn’s analysis of rank-1 perturbations [7], B. Simon and T. Wolff noted the following [17]:

**Proposition 2.1.** Let \( G \) be a countable set, \( H_0 \) a bounded self-adjoint operator in \( \ell^2(G) \), and \( H_v \) the one-parameter family of operators defined by
\[
H_v = H_0 + vP_{\psi}
\]
with \( P_{\psi} \) a rank-1 orthonormal projection on a space spanned by a normalized function \( \psi \). Then for any \( v \neq 0 \) and \( E \in \mathbb{R} \) the following statements are equivalent:

i. \( E \) is a proper eigenvalue of \( H_v \), i.e., \( \mu_{v,\psi}(\{E\}) > 0 \).

ii. The following quantity is finite:
\[
\gamma_{0,\psi}(E) := \lim_{\eta \downarrow 0 \eta} \sum_y |(\delta_y, (H_0 - E - i\eta)^{-1}\psi)|^2 = \int \frac{\mu_{0,\psi}(dt; \omega)}{(t - E)^2} < \infty
\]
and
\[
(\psi, (H_0 - E - i0)^{-1}\psi) = -v^{-1}.
\]
Moreover, if the condition is satisfied, then \( \mu_{v,\psi}(\{E\}) = v^{-2}/\gamma_{\psi}(E) \).

Combining the above with a spectral averaging principle, Simon and Wolff presented a useful criterion in which reference is made to
\[
\gamma_x(E; \omega) := \lim_{\eta \downarrow 0 \eta} \sum_y |G(x, y; E + i\eta; \omega)|^2 = \int \frac{\mu_{\delta_x}(dt; \omega)}{(t - E)^2} \in (0, \infty)
\]
(the limit existing by monotonicity).

**Proposition 2.2 (Simon-Wolff [16,17]).** Let \( H(\omega) = A + V(\omega) \) be a self-adjoint operator on \( \ell^2(G) \) such that for each \( x \in G \) the random variable \( V(x) \) is of conditionally absolutely continuous distribution, conditioned on \( \{V(x), x \in G\} \). If, for a Borel subset \( I \subset \mathbb{R} \),
\[
\gamma_x(E; \omega) = \infty
\]
for Lebesgue almost every \( E \in I \) and \( \mathbb{P}\)-almost all \( \omega \), then almost surely \( H(\omega) \) has only continuous spectrum (if any) in \( I \).

For the proof, and hence also applications of this criterion, it is essential that the event [2.4] be of probability 1. Yet the second-moment analysis that has been employed in resonant delocalization arguments yields (initially) only a weaker result, that [2.4] holds with nonzero probability. The purpose of the following boost is to bridge this gap.
2.2 The Boost: A Zero-One Law

Let $V$ be a random potential, specified through the collection of random variables $\{V(x; \omega)\}_{x \in G}$. For each $\Lambda \subset G$, we denote by $\mathcal{B}_\Lambda$ the minimal $\sigma$-algebra of subsets $A \subset \Omega$ for which $\omega \mapsto 1_A(\omega)$ is a measurable function of $\{V(y)\}_{y \in \Lambda}$.

**Definition 2.3.**

1. A random variable $F : \Omega \mapsto \mathbb{R}$ is measurable at infinity if for each finite $\Lambda \subset G$, $F$ is measurable with respect to $\mathcal{B}_{\Lambda^c}$.
2. A stochastic process over a graph $G$ is said to have the $K$-property if any random variable that is measurable at infinity is constant almost surely.

The simplest example of processes with the $K$-property are those for which $\{V(y)\}_{y \in G}$ are independent random variables. For random potentials with this property the applicability of the Simon-Wolff criteria is hereby boosted by the following zero-one law.

**Theorem 2.4.** Let $H(\omega) = H_0 + V(\omega)$ be a random self-adjoint operator on $\ell^2(G)$ with $V(\omega)$ a random potential with the $K$-property, and such that for each vertex $x \in G$ the conditional single-site distribution of $V(x)$ conditioned on $V \neq x$ is continuous. Then for Lebesgue almost every $E \in \mathbb{R}$,

\begin{equation}
\mathbb{P}(\gamma_x(E) < \infty) \text{ equals either 0 or 1.}
\end{equation}

As will be explained in the proof, the condition $\{\gamma_x(E) \leq \infty\}$ is essentially equivalent to

\begin{equation}
\kappa_x(E, \omega) := \lim_{\eta \downarrow 0} \frac{-1}{\eta} \text{Im} \left( \frac{1}{H(\omega) - E - i\eta} \delta_x \right)^{-1} < \infty.
\end{equation}

In effect, the proof of (2.5) proceeds by showing that for fixed $E \in \mathbb{R}$ the set $\{\omega \in \Omega \mid \kappa_x(E, \omega) < \infty\}$ is measurable at infinity—in the Lebesgue sense, that is up to corrections by sets of measure zero.

The rest of this section is devoted to the details of the argument outlined above. Other than the conclusion summarized in Theorem 2.4, the proof does not play a role in our discussion of resonant delocalization, which is the subject of Section 3.

2.3 Measurability at Infinity

In the proof of Theorem 2.4 we shall make use of rank-1 and rank-2 perturbation formulae, which express the dependence of $G(x, x, z; \omega)$ on $V(x)$, and its joint dependence on $V(x)$ and any other $V(y)$.

1. The dependence on the potential at $x$ is particularly simple:

\begin{equation}
\langle \delta_x, (H - z)^{-1} \delta_x \rangle =: [V(x) - \Sigma(x; z)]^{-1}
\end{equation}

with $\Sigma(x; z)$, which is referred to as the self-energy, a function of $V \neq x$ only.
2. For any pair of distinct sites, $x \neq y$:

$$\begin{pmatrix} G(x, x; z) & G(x, y; z) \\ G(y, x; z) & G(y, y; z) \end{pmatrix} = \begin{pmatrix} V(x) - \sigma(x; z) & -\tau(x, y; z) \\ -\tau(y, x; z) & V(y) - \sigma(y; z) \end{pmatrix}^{-1},$$

with $\sigma(x; z)$ and $\tau(x, y; z)$, which do not involve $V(x)$ and $V(y)$. Following [2] we refer to $\tau(x, y; E + i0)$ as the (pairwise) tunneling amplitude between the two sites, at energy $E$.

These expressions form two special cases of the Schur complement, or Krein-Feshbach formula. In the discussion of their implication on the properties of $\kappa_x(E)$, the following statement will be of relevance.

**Lemma 2.5.** Let

$$F_n(V) := \frac{a_n V + b_n}{c_n V + d_n}$$

stand for a sequence of Möbius functions with the property that for all $V \in \mathbb{R}$

1. $\text{Im} F_n(V) \geq 0$, and
2. $\text{Im} F_n(V)$ converges to a limit within $[0, \infty]$ (allowing the value $+\infty$).

Then, $\lim_{n \to \infty} \text{Im} F_n(V)$ is finite or infinite simultaneously for all except at most one value of $V \in \mathbb{R}$.

**Proof.** The fractional linear mapping $F_n : \mathbb{R} \to \{ z \in \mathbb{C} \mid \text{Im} z \geq 0 \} =: \mathbb{C}_0^+$ takes $\mathbb{R}$ (or rather its one-point compactification $\mathbb{R} \cup \{ \infty \}$) onto a generalized circle (possibly a line) in $\mathbb{C}_0^+$, preserving the canonical orientation. We denote the circle’s radius by $R_n \in [0, \infty]$ and its lowest point by $U_n \in \mathbb{C}_0^+$. In the degenerate case, $R_n = \infty$, we set $\text{Re} U_n = 0$. The asserted dichotomy holds trivially true (and without exceptional points) if either

1. $\limsup_{n \to \infty} \text{Im} U_n = \infty$ (in which case $\limsup_{n \to \infty} \text{Im} F_n(V) = \infty$ for all $V \in \mathbb{R}$), or
2. $(\text{Im} U_n)$ and $(R_n)$ are bounded sequences (in which case, for all $V \in \mathbb{R}$, $\limsup_{n \to \infty} \text{Im} F_n(V) < \infty$).

Hence it suffices to establish the claim for the case that $(\text{Im} U_n)$ is bounded and $\limsup_{n \to \infty} R_n = \infty$. Furthermore, since when $\lim_{n \to \infty} \text{Im} F_n(V)$ exists, it can be computed over any subsequence, it suffices to prove the assertion under the additional assumption that these limits exist, i.e.,

$$\lim_{n \to \infty} R_n = \infty, \quad \lim_{n \to \infty} \text{Im} U_n < \infty.$$

As a final simplification we note that the assertion holds for $F_n$ if and only if it holds for the sequence of shifted functions $F_n(V) - U_n$. Based on these considerations, we add without loss of generality the assumption that $U_n = 0$ for all $n \in \mathbb{N}$.

An equivalent form of the statement to be proven is the following: if

$$\lim_{n \to \infty} \text{Im} F_n(V) < \infty$$

(2.10)
for two distinct values $V_1 < V_2$, then (2.10) holds for all but at most one value of $V \in \mathbb{R}$. We shall prove it in this form.

Let $V_j$ with $j = 1, 2$ be two points for which (2.10) holds, and let $Y_j := \lim_{n \to \infty} \Im F_n(V_j)$. The convergence of the imaginary part does not ensure that of the real part, which may still oscillate between the region $\Re F_n(V) \geq 0$ and $\Re F_n(V) < 0$. As explained above, it suffices to restrict attention to a subsequence, and we select one for which the signs of the real part take consistently fixed $\pm 1$ values, i.e., for all $n$ and both $j = 1, 2$:

\begin{equation}
(2.11) \quad \text{sign} \, \Re F_n(V_j) = \sigma_j
\end{equation}

declaring sign $0 = +1$.

For the following argument it is convenient to have one more point with the properties of $V_j$, and we start by showing that such a point exists.

The removal from $\mathbb{R}$ of $V_1$ and $V_2$ splits the “generalized circle,” which is the image of $\mathbb{R}$ under $F_n$ into two arcs. We shall refer to the one that includes the point that minimizes $\Im F_n(V)$ as the “lower arc.” Along this arc the value of $\Im F_n$ is everywhere bounded by $\max\{\Im F_n(V_1), \Im F_n(V_2)\}$.

There are now two possibilities: the lower arc will contain the image of either the midpoint $(V_1 + V_2)/2$ or else the image of $V = \infty$, in which case it will also contain the image of $V_1 - (V_2 - V_1)/2$ (i.e., the midpoint’s reflection about $V_1$). Restricting to a subsequence, we may assume without loss of generality that which of the options applies does not change with $n$, and we shall pick the point $V_3 \in \mathbb{R}$ as either $(V_1 + V_2)/2$ or $V_1 - (V_2 - V_1)/2$, corresponding to this alternative. By this construction, $\Im F_n(V_3) \leq \max\{\Im F_n(V_1), \Im F_n(V_2)\}$, so that (2.10) also holds for $j = 3$. Possibly passing to a subsequence, it may be assumed that for one of the choices of $\sigma_3 \in \{-1, 1\}$, (2.11) also holds consistently for $j = 3$.

By the invariance of the cross ratio under fractional linear transformations, for any $V \in \mathbb{R}$ and the above $V_j$,

\begin{equation}
(2.12) \quad g(V; V_1, V_2, V_3) := \frac{(V - V_1)(V_2 - V_3)}{(V - V_2)(V_1 - V_3)} = \frac{[F_n(V) - F_n(V_1)][F_n(V_2) - F_n(V_3)]}{[F_n(V) - F_n(V_2)][F_n(V_1) - F_n(V_3)]}.
\end{equation}

We now will show that under the above assumptions, keeping the above three points fixed as $n \to \infty$,

\begin{equation}
(2.13) \quad \lim_{n \to \infty} \frac{F_n(V_2) - F_n(V_3)}{F_n(V_1) - F_n(V_3)} = \frac{\sigma_2 \sqrt{V_2} - \sigma_3 \sqrt{V_3}}{\sigma_1 \sqrt{V_1} - \sigma_3 \sqrt{V_3}},
\end{equation}

whereas for all $V \in \mathbb{R}$ with $\Im F_n(V) \geq T \geq 2W$, with $W := \max\{Y_1, Y_2\}$,

\begin{equation}
(2.14) \quad \limsup_{n \to \infty} \left| \frac{F_n(V) - F_n(V_1)}{F_n(V) - F_n(V_2)} - 1 \right| \leq C \sqrt{W/T}
\end{equation}

with a fixed constant.
The relation (2.13) is based on the observation that the coordinates of \( F_n(V_j) = X_{n,j} + i Y_{n,j} \) satisfy
\[
X_{n,j}^2 + (R_n - Y_{n,j})^2 = R_n^2 \quad \text{and} \quad \text{sign} \ X_{n,j} = \sigma_j
\]
and hence
\[
X_{n,j} = \sigma_j \sqrt{2R_n Y_{n,j} \sqrt{1 - Y_{n,j}^2}} / (2R_n).
\]
Since \( R_n \to \infty \), equation (2.13) easily follows. In essence, these estimates reflect the flatness and “horizontality” of the curve near its bottom due to the asymptotical vanishing of the circle’s curvature. Related considerations yield (2.14).

The relations (2.12), (2.13), and (2.14) imply that for each fixed \( T > 2W \), all \( V \in \mathbb{R} \) with \( \text{Im} \ F_n(V) > T \) at \( n \) large enough,
\[
\tag{2.15}
g(V; V_1, V_2, V_3) = \frac{\sigma_2 \sqrt{Y_2} - \sigma_3 \sqrt{Y_3}}{\sigma_1 \sqrt{Y_1} - \sigma_3 \sqrt{Y_3}} \leq C \sqrt{\frac{W}{T}} \frac{\sigma_2 \sqrt{Y_2} - \sigma_3 \sqrt{Y_3}}{\sigma_1 \sqrt{Y_1} - \sigma_3 \sqrt{Y_3}}.
\]
The cross ratio of \( V \) with fixed \( \{V_j\}_{j=1,2,3} \) forms a continuous \( \mathbb{R} \)-valued function over \( \mathbb{R} \) that determines \( V \) uniquely. The intersection of the bound (2.15) over a sequence of values of \( T \to \infty \) implies that the set of points for which \( \limsup_{n \to \infty} \text{Im} \ F_n(V) = \infty \) consists of only the one point for which
\[
g(V; V_1, V_2, V_3) = \frac{\sigma_2 \sqrt{Y_2} - \sigma_3 \sqrt{Y_3}}{\sigma_1 \sqrt{Y_1} - \sigma_3 \sqrt{Y_3}} = \frac{2}{3} \frac{1}{\sqrt{3}}. \quad \square
\]

Using Lemma 2.5 we now turn to prove the zero-one law that was introduced above.

**Proof of Theorem 2.4** For a convenient reformulation of the condition in (2.5), let
\[
\mathcal{F}_x(z, \omega) := -\frac{1}{\text{Im} z} \left[ \frac{1}{\delta_x} \frac{1}{H(\omega) - z} \delta_x \right]^{-1}.
\]
It satisfies
\[
\tag{2.17}
\text{Im} \mathcal{F}_x(E + i \eta, \omega) = |G(x, x; E + i \eta; \omega)|^{-2} \left[ \frac{1}{(H(\omega) - E)^2 + \eta^2 \delta_x} \right].
\]
For a full measure of energies \( E \in \mathbb{R} \), the limit
\[
G(x, x; E + i \eta; \omega) := \lim_{\eta \downarrow 0} G(x, x; E + i \eta; \omega)
\]
exists and is finite nonzero for \( \mathbb{P} \)-almost every \( \omega \). (By Fubini’s theorem the statement is equivalent to its reversed-order form, and that is implied by the de la Vallée-Poussin theorem.) It follows that for \( E \) in this full measure set
\[
\tag{2.18}
\mathbb{P} (\gamma_x(E) < \infty) = \mathbb{P} (\kappa_x(E) < \infty)
\]
(for the quantities defined in (2.5) and (2.6)). We now proceed, restricting our attention to this regular set of energies \( E \).
Let us denote the event:
\[ K_x(E) := \{ \omega \in \Omega \mid \kappa_x(E; \omega) < \infty \}. \]

Using Lemma 2.5 we shall now prove that for every finite set \( \Lambda \subset G \) with \( \{ x \} \subset \Lambda \), the regular conditional probability of \( K_x \) conditioned on \( B_{\Lambda^c} \) is either 0 or 1, or more explicitly
\[ \mathbb{E}[1_{K_x} \mid B_{\Lambda^c}](\omega) \overset{a.s.}{=} 1_{K_x}(\omega). \]

where the quantity on the left is the conditional probability \( \mathbb{P}(K_x|B_{\Lambda^c})(\omega) \) expressed as an average in which the variables \( \{ V(u) \}_{u \in \Lambda} \) are integrated out with their appropriate conditional distribution.

Lemma 2.5 is applicable here due to (2.8). It tells us that for any site \( u \in G \) the indicator function \( 1_{K_x} \) is almost surely constant as \( V(u) \) is varied at fixed values of \( V \neq u \). Due to the continuity of the conditional distribution, the set of \( \omega \) for which \( V(u; \omega) \) takes one of the exceptional values (of which there are at most two) is of zero probability. It follows that for any \( u \in G \)
\[ \mathbb{E}[1_{K_x} \mid B_{\Lambda^c}](\omega) \overset{a.s.}{=} 1_{K_x}(\omega). \]

Denoting by \( P_{\Lambda} \) the orthogonal projections in \( L^2(\Omega, \mathbb{P}) \) corresponding to the conditional expectations, \( \psi \mapsto \mathbb{E}[\psi \mid B_{\Lambda^c}] \), equation (2.20) can be equivalently stated in the form
\[ \mathbb{E}[|1_{K_x} - P_{\{u\}} 1_{K_x}|^2] = 0. \]

By an elementary orthogonality bound, for any finite (and by implication also infinite) \( \Lambda \subset G \),
\[ \mathbb{E}[|1_{K_x} - P_{\Lambda} 1_{K_x}|^2] \leq \sum_{u \in \Lambda} \mathbb{E}[|1_{K_x} - P_{\{u\}} 1_{K_x}|^2]. \]

Therefore, (2.21) implies that the above quantity also vanishes for all finite \( \Lambda \). This proves (2.19) for finite \( \Lambda \subset G \). Through the martingale convergence theorem (or through just an extension of the above variance bounds) we conclude that for any \( \Lambda \subset G \)
\[ \mathbb{E}[1_{K_x} \mid B_{\Lambda^c}](\omega) \overset{a.s.}{=} 1_{K_x}(\omega), \]

and thus \( 1_{K_x} \) is measurable at infinity. Under the assumption that the joint distribution of the potential has the \( K \)-property, it follows that \( \mathbb{P}(\kappa_x(E) < \infty) \) can equal only 0 or 1, and through (2.18) this implies the claim (2.5). \( \square \)

### 2.4 A Weak Form of Spectral Dichotomy

Theorem 2.4 along with with the Simon-Wolff criterion, Theorem 2.2 imply:

1. The real line is covered up to a zero measure subset by the disjoint union \( \mathcal{C}_x \cup \mathcal{P}_x \) of the nonrandom sets
\[ \mathcal{C}_x := \{ E \in \mathbb{R} \mid \mathbb{P}(\gamma_x(E) = \infty) = 1 \}, \]
\[ \mathcal{P}_x := \{ E \in \mathbb{R} \mid \mathbb{P}(\gamma_x(E) < \infty) = 1 \}. \]
2. With probability one $C_x$ serves as a support for the continuous spectrum of $H(\omega)$ in the sense that
   i. $\mu_{\delta_x}^{PP}(C_x; \omega) = 0$ for $\mathbb{P}$-almost all $\omega$,
   ii. for any $\varepsilon > 0$ and Lebesgue almost every $E \in C_x$:
   \begin{equation}
   \mu_{\delta_x}^c((E - \varepsilon, E + \varepsilon); \omega) > 0 \quad \text{for } \mathbb{P} \text{-almost every } \omega.
   \end{equation}

3. $\mathcal{P}_x$ supports the pure-point spectrum of $H(\omega)$ together with the real part of the resolvent set. In particular,
   \begin{equation}
   \mu_{\delta_x}^c(\mathcal{P}_x; \omega) = 0 \quad \text{for } \mathbb{P} \text{-almost all } \omega.
   \end{equation}

One may add to it that the condition $\lim_{\eta \downarrow 0} \langle \delta_x, \frac{1}{H(\omega) - E - i\eta} \delta_x \rangle > 0$, in which existence of the limit is part of the statement, is also measurable at infinity—a fact that has already been noted before [12, cor. 1.1.3]. Denoting
\[ \mathcal{A}_x := \left\{ E \in \mathbb{R} \mid \mathbb{P}\left( \text{Im} \left( \frac{1}{H - E - i0} \delta_x \right) > 0 \right) = 1 \right\}, \]
and observing that $\mathcal{A}_x \subset C_x$ (which follows from the spectral representation), one may add to the above:

4. the support of the continuous spectrum also admits a nonrandom disjoint decomposition, with
   i. $\mathcal{A}_x$ providing an almost sure support of the absolutely continuous spectrum.
   ii. $C_x \setminus \mathcal{A}_x$ serving as an almost sure support of the singular continuous spectrum.

For the last point (4) we recall that the spectral measure’s absolutely continuous component is
\[ \mu_{\delta_x}^{ac}(dE) = \pi^{-1} \text{Im} \left( \frac{1}{H - E - i0} \delta_x \right) dE. \]

Thus, the boosted Simon-Wolff criterion implies a measure-theoretic form of spectral dichotomy. It should, however, be appreciated that $C_x$ and $\mathcal{P}_x$ generally do not coincide with the topological definition of the continuous and pure-point spectra since these sets may in general not be closed. In this context, let us recall that the nonrandomness of the topological supports of the different spectra is also known quite generally for ergodic operators [8, 9, 14].

3 Resonant Delocalization

Delocalization in the presence of extensive disorder turned out to be more elusive than Anderson localization. Challenges remain open at both the level of compelling physical argument and of mathematical existence proofs of spectral regimes with extended states for random Schrödinger operators in finite dimensions. In particular, the Bloch-Floquet mechanism for the formation of bands of ac spectrum
for periodic operators is unstable with respect to even weak homogeneous disorder. That is drastically demonstrated by the one-dimensional example, where at arbitrarily weak disorder the entire spectrum changes to pure point \[9,11\].

In the following, we build on one of the very few effective arguments for the formation of continuous spectrum through local resonances to have emerged in mathematical works on delocalization. For simplicity, we focus on operators of the form (1.1) with independent and identically distributed (i.i.d.) potentials, whose single-site distribution is absolutely continuous with bounded density \( \rho \), and assume homogeneity in the following sense.

**Definition 3.1.** A self-adjoint random operator on a transitive graph \( G \) is said to be _homogeneous_ if for each \( T \) in a transitive collection of graph homomorphisms of \( G \) the action of \( T \) on \( G \) lifts to measure-preserving transformation on \( \Omega \) under which \( H(T\omega) \) is unitarily equivalent to \( H(\omega) \) and satisfies
\[
\langle \delta_{T(x)}, F(H(\omega))\delta_{T(x)} \rangle \overset{a.s.}{=} \langle \delta_x, F(H(T\omega))\delta_x \rangle
given \text{for all bounded continuous } F : \mathbb{R} \to \mathbb{R} \text{ and all } x \in G.
\]

In this setup, the mean density of states measure, which is generally defined by \( \nu(I) = \mathbb{E}[\mu_{\delta_x}(I)] \) for Borel \( I \subset \mathbb{R} \), does not depend on \( x \in G \), and is known to be absolutely continuous, with a bounded derivative satisfying
\[
n(E) := \frac{\nu(dE)}{dE} \leq \|\rho\|_{\infty}.
\]
(The function \( n(E) \) is referred to as the density of states; the estimate is known as the Wegner bound, and its proof can be based on the spectral averaging principle \[16,17\].)

By the zero-one law of Theorem \[2,4\] and the Simon-Wolff criterion, in order to establish the presence of continuous spectrum within an interval \( I \), it suffices to prove that for a positive measure set of energies \( E \in I \),
\[
\mathbb{P}\left\{ \lim_{\eta \downarrow 0} \sum_x |G(0,x;E+i\eta;\omega)|^2 = \infty \right\} > 0,
\]
where the sum can also be written as \( \langle \delta_0, (H(\omega) - E)^2 + \eta^2 \rangle^{-1} \delta_0 \). The representation provided in \[2,17\] makes it clear that, with the possible exception of a zero measure set of energies, the above limit diverges for each \( E \) in the set
\[
\mathcal{A}_0 = \{ E \in \mathbb{R} \mid \mathbb{P}(\text{Im } G(0,0;E+i0) > 0) \neq 0 \}.
\]

Our discussion will therefore focus on conditions implying the divergence under the assumption that \( G(0,0;E+i0,\omega) \) is real for \( \mathbb{P} \)-almost all \( \omega \). (It is not difficult to see that this event also satisfies a zero-one law for almost every \( E \in \mathbb{R} \), but this observation will not be needed here.)

For brevity of notation, from this point on we shall omit the explicit reference to the limit, denoting
\[
G(x,y;E) := \lim_{\eta \downarrow 0} G(x,y;E+i\eta).
\]
The limit exists for almost all \((E, \omega)\) simultaneously for all \(x, y \in \mathcal{G}\) (as follows from the de la Vallée-Poussin theorem).

### 3.1 Rare But Destabilizing Resonances

A simple lower bound for the quantity of interest is, for any \(R \in \mathbb{N}\),

\[
\gamma_0(E; \omega) = \lim_{\eta \downarrow 0} \sum_{x \in \mathcal{G}} |G(0, x; E + i \eta, \omega)|^2 \\
\geq \sum_{x : d(0, x) = R} |G(0, x; E)|^2 \\
= |G(0, 0; E)|^2 \sum_{x : d(0, x) = R} |g(x; E)|^2,
\]

with (in terms introduced in (2.8))

\[
g(x; E) := \frac{G(0, x; E)}{G(0, 0; E)} = \frac{\tau(0, x; E)}{V(x) - \sigma(x; E)}. \tag{3.4}
\]

We now present a scenario under which a given site 0 is resonant at energy \(E\) with a random collection of many other sites \(x \in \mathcal{G}\) for which \(|g(x; E)| \approx 1\). The resonances on which we shall focus are expressed in the joint occurrences of the following three events:

\[
\mathcal{T}_x := \{ |\tau(0, x; E)| \geq t(0, x; E) \}, \\
\mathcal{E}_x := \{ |V(x) - \Sigma(x; E)| \leq t(0, x; E) \}, \\
\mathcal{N}_x := \{ |V(0) - \sigma(0; E)| \geq |\tau(x, 0; E)| \},
\]

with \(\Sigma(x; E)\) defined by the rank-1 relation (2.7) and \(t(x, y; E) \in (0, 1]\) selected so that

\[
\lim_{R \to \infty} \min_{d(x, y) = R} \mathbb{P}(|\tau(x, y; E)| \geq t(x, y; E)) = 1. \tag{3.6}
\]

A related quantifier of the tunneling amplitude’s distribution that will play a role is its truncated average

\[
T(x, y; E) := \mathbb{E}[\min(|\tau(x, y; E)|, 1)] \geq ct(x, y; E), \tag{3.7}
\]

which for any \(\varepsilon > 0\) satisfies \(T(x, y; E) \geq (1 - \varepsilon)t(x, y; E)\) at sufficiently large distances, \(d(x, y) \geq R_x\).

The events \(\mathcal{T}_x\) and \(\mathcal{N}_x\) do not depend on \(V(x)\), and in the situation discussed below, they will be found to occur with asymptotically full probability as \(d(x, 0) \to \infty\). In contrast, \(\mathcal{E}_x\) depends on the value of \(V(x)\), and requires it to fall within an extremely narrow range of values (near \(\Sigma(x; E)\), which depends on \(V_{\neq x}\)). A key fact is that at any site \(x \in \mathcal{G}\) for which the three conditions are met for a given potential, i.e., under the event \(\mathcal{T}_x \cap \mathcal{E}_x \cap \mathcal{N}_x\), the ratio that appears in (3.3) is bounded below:

\[
|g(x, E)| \geq \frac{1}{2}. \tag{3.8}
\]
For a proof let us note that (2.7) with (2.8) yields
\[ \Sigma(x; z) = \sigma(x; z) + \frac{\tau(x, 0; z) \tau(0, x; z)}{V(0) - \sigma(0; z)}, \]
which shows that under the event \( N_x \),
\[ |\Sigma(x; E) - \sigma(x; E)| \leq |\tau(0, x; E)|. \]
The lower bound (3.8) follows by combining (3.10) with (3.4) and the conditions defining \( E_x \) and \( T_x \).

Considering the possible effects of resonant delocalization, we obtain the following result concerning conditions inducing continuous spectrum in specified energy regimes. The essential role in the proof is played by the Simon-Wolff criterion of Theorem 2.2 boosted by the zero-one law of Theorem 2.4. The assumed regularity assumption will be expressed invoking the convolution
\[ (1_{\varepsilon} \ast \varrho)(v) := \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \varrho(v + w)dw. \]

**Theorem 3.2.** Let \( G \) be an infinite transitive graph and \( H(\omega) = A + V(\omega) \) a homogeneous random operator on \( \ell^2(\mathbb{Z}) \) with \( V \) an i.i.d. potential whose single-site distribution is absolutely continuous with density satisfying
\[ \varrho \leq c \inf_{\varepsilon \in (0, \delta)} (1_{\varepsilon} \ast \varrho) \]
for some \( \delta > 0 \) and \( c < \infty \). Then for any Borel set \( I \subset \mathbb{R} \) and vertex \( 0 \in G \) a sufficient condition for
\[ \mu^{(c)}_{\delta_0}(I; \omega) > 0 \]
is that the following conditions (A1)-(A3) hold for a positive Lebesgue measure subset of values \( E \in I \):

(A1) The functions \( T \) and \( t \), of which the first is defined by (3.7) and \( t(u, v; E) \) is taken to depend only on \( d(u, v) \), satisfy
\[ \lim_{d(x, 0) \to \infty} T(x, 0; E) = 0 \quad \text{and} \quad \lim_{R \to \infty} \sum_{x : d(0, x) = R} t(0, x; E) = \infty. \]

(A2) For some finite \( C_T < \infty \), at all sufficiently large distances \( R < \infty \) for all \( x \) with \( d(0, x) = R \),
\[ \sum_{y : d(0, y) = R} T(x, y; E) \leq C_T \sum_{x : d(0, x) = R} t(0, x; E). \]

(A3) The operator has a well-defined and strictly positive density of states at energy \( E \):
\[ n(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \overline{\text{Im} \left( \sum_{x} \frac{1}{H - E - i\eta \delta_x} \delta_x \right)} \in (0, \infty). \]
Before we turn to the proof, let us comment on the feasibility of the assumed
conditions \( (A1) \) and \( (A2) \), as it is illustrated on the case where \( G \) is a regular tree
graph of constant degree \( K + 1 \) and \( A \) is the graph’s adjacency operator.

On tree graphs the tunneling amplitude \( \tau(x, y; E) \) factorizes into a product of
similar but shifted random variables, which can be associated with the decomposi-
tion of arbitrary paths from \( x \) to \( y \) into a sequence of “no-return” steps. One then
finds \cite[theorem 3.2]{4} that both \( t(x, y; E) \) and \( T(x, y; E) \) decay exponentially:
\[
(3.17) \quad t(x, y; E) \leq C_0 e^{-L_0(E) \text{dist}(x, y)}, \quad T(x, y; E) \leq C_1 e^{-L_1(E) \text{dist}(x, y)},
\]
with \( L_0(E) \), which can be identified as the Lyapunov exponent of a transfer-matrix-
driven dynamics, and \( L_1(E) \leq L_0(E) \) by \cite[(3.7)]{4}. In that situation, condition \((A1)\)
requires
\[
(3.18) \quad L_0(E) < \log K,
\]
which is satisfied when the rate of typical tunneling decay is below the rate of the
(geometric) surface growth. On regular tree graphs assumption \((A2)\) is also valid
when \( (3.18) \) holds. This follows from the inequality \cite[theorem 3.2]{4}
\[
(3.19) \quad L_1(E) \geq \log \sqrt{K}
\]
and the hyperbolic geometry of the tree: for each \( x \) with \( \text{dist}(x, 0) = R \), most of
the \( R \)-sphere is asymptotically further from \( x \) than from the center 0 by a factor
that asymptotically can be chosen arbitrarily close to 2. Due to this, the surface
average (over \( y \)) of \( T(x, y; E) \) decays at asymptotically twice the decay rate of the
surface average of \( T(0, y; E) \) (the exact calculation is elementary).

The zero-disorder values of the above pair of exponents are, for regular tree
graphs,
\[
(3.20) \quad L_0(E) = L_1(E) = \begin{cases} 
\log \sqrt{K} & \text{for } E \in [-2\sqrt{K}, 2\sqrt{K}] = \sigma(A), \\
\log K & \text{for } |E| = [-(K + 1), K + 1].
\end{cases}
\]
This expresses the facts that: (i) there are no fluctuations, (ii) over the spectrum
of \( A \) the \( \ell^2 \)-sum \( \sum_{x: \text{dist}(x, 0) = R} |G_0(0, x; E)|^2 \) tends to a finite constant as \( R \rightarrow \infty \), and (iii) the Lyaponov exponent grows as the distance of \( E \) from the spectrum
increases. Thus, in this case sufficient Lyaponov exponent bounds are in place,
and partial continuity arguments for the exponents at positive disorder can be de-
veloped, making Theorem \cite[3.2]{4} applicable at weak disorder throughout—and even
beyond—the spectrum of \( A \) \cite[4]{4}.

To establish Theorem \cite[3.2]{4} it suffices to prove the following estimate for
\[
(3.21) \quad N_R(E) := \sum_{x: |d(x, 0)| = R} \mathbb{1}_{T_x \cap \mathcal{E}_x \cap \mathcal{N}_x},
\]
which counts the number of sites \( x \in G \) at which in a given realization there is a
strong enough resonance at energy \( E \) to be noted at 0 in the sense defined by the
conditions stated in \cite[(3.5)]{4}.
LEMMA 3.3. Let $H(\omega)$ be a random operator satisfying the assumptions of Theorem 3.2 and $I \subset \mathbb{R}$ a Borel set such that for almost every $E \in I$ the conditions (A1)–(A3) are satisfied and also

$$\mathbb{P}(\text{Im} \, \Sigma(0; E) = 0) = 1. \tag{3.22}$$

Then for any $M < \infty$ and almost every $E \in I$:

$$\liminf_{R \to \infty} \mathbb{P}(N_R(E) \geq M) \geq p_0(E). \tag{3.23}$$

with some $p_0(E) > 0$ that does not depend on $M$.

Once this is proven, Theorem 3.2 readily follows: it is already known that the $\ell^2$-sum in (3.2) diverges at energies at which $\text{Im} \, \Sigma(0; E) \neq 0$, and the lemma allows to conclude positive probability of the sum’s divergence (under the assumptions (A1)–(A3) regardless of (3.22). The zero-one law of Theorem 2.4 then allows raising the probability in (3.2) to 1, and the rest follows by the Simon-Wolff criterion (Theorem 2.2).

Let us now turn to the proof of Lemma 3.3.

### 3.2 The Second-Moment Proof

Lemma 3.3 is proved below through a two-step argument: the first is to show that the mean (i.e., first moment of $N_R$) diverges (as expressed in (3.25)). Then, the second-moment test will be used to establish a uniformly positive lower bound on the probability the random variables $N_R$ assume values comparable with their mean. The alternative that needs to be ruled out here is that the mean diverges only due to very large contributions of very rare events, while the typical range of values (e.g., the median) remains finite. A convenient tool for such a purpose is the Paley-Zygmund inequality, which states that

$$\mathbb{P}(N \geq \theta \mathbb{E}[N]) \geq (1 - \theta)^2 \frac{\mathbb{E}[N^2]}{\mathbb{E}[N]^2} \tag{3.24}$$

for any random variable $N$ and any $\theta \in (0, 1)$. To employ it, one needs to derive a lower bound on $\mathbb{E}[N_R]$ and an upper bound on $\mathbb{E}[N_R^2]$.

### The Lower Bound

The first of the two steps outlined above is the following:

**Lemma 3.4.** Under the assumptions of Theorem 3.2 for Lebesgue-almost all $E \in \mathbb{R}$ at which (3.22) and (A1)–(A3) hold,

$$\lim_{R \to \infty} \mathbb{E}[N_R] = \infty. \tag{3.25}$$

**Proof.** Let us first consider the probability of the rare event $\mathbb{P}(E_x)$. Applying the shift covariance, one gets

$$\liminf_{d(x, \theta) \to \infty} \frac{\mathbb{P}(E_x)}{2\pi(0, x; E)} = \liminf_{\epsilon \searrow 0} \frac{1}{2\epsilon} \mathbb{E} \left[ \int \mathbb{1}[|v - \Sigma(x; E)| \leq \epsilon] \phi(v) dv \right].\tag{3.26}$$
On the other hand, by the rank-1 perturbation formula (2.7), the density of states function that is given by (3.16) can be presented as

\[
n(E) = \lim_{\eta \to 0} \frac{1}{\eta \pi} \mathbb{E} \left[ \frac{1}{\Im \left( V(x) - \Sigma(x; E + i \eta) \right)} \right].
\]

Since \( \lim_{\eta \to 0} \Sigma(x; E + i \eta) = \Sigma(x; E) \in \mathbb{R} \) (in the distributional sense), the delta function principle that is stated here more explicitly in Lemma A.1 is applicable, and it implies a relation between the last expressions in (3.26) and (3.27), with the consequence that

\[
\liminf_{d(x,0) \to \infty} \frac{\mathbb{P}(E_x)}{2t(0, x; E)} \geq n(E).
\]

Let now \( \{Z_x\}_{x \in \mathbb{G}} \) be a family of events for which

1. \( Z_x \) is independent of \( V(x) \) for all \( x \in \mathbb{G} \), and
2. \( \lim_{R \to \infty} \max_{x: d(x,0) = R} \mathbb{P}(Z_x^c) = 0 \),

and \( E \in \mathbb{R} \) is an energy at which (3.22) and (A1)–(A3) hold. One may evaluate the joint probability by first conditioning on \( V \preceq x \):

\[
\mathbb{P}(E_x \cap \mathcal{Z}_x^c) = \mathbb{E} \left[ 1_{\mathcal{Z}_x^c} \mathbb{P}(E_x | V \preceq x) \right]
\]

\[
= \mathbb{E} \left[ 1_{\mathcal{Z}_x^c} \int 1[v - \Sigma(x; E)] \leq t(0, x; E)] \phi(v) dv \right]
\]

\[
\leq 2t(0, x; E) \|\phi\|_{\infty} \mathbb{P}(Z_x^c).
\]

Hence \( \lim_{d(x,0) \to \infty} \mathbb{P}(E_x \cap \mathcal{Z}_x^c) / 2t(0, x; E) = 0 \). Since \( \mathbb{P}(E_x \cap \mathcal{Z}_x) = \mathbb{P}(E_x) - \mathbb{P}(E_x \cap \mathcal{Z}_x^c) \), we thus get the following extension of (3.28):

\[
\liminf_{d(x,0) \to \infty} \frac{\mathbb{P}(E_x \cap \mathcal{Z}_x)}{2t(0, x; E)} = \liminf_{d(x,0) \to \infty} \frac{\mathbb{P}(E_x)}{2t(0, x; E)} \geq n(E).
\]

To verify that the above applies to \( \mathcal{Z}_x = \mathcal{T}_x \cap \mathcal{N}_x \), we note that by its definition this event is independent of \( V(x) \). To check the other assumption, let us note the following:

1. By our selection of the function \( t(0, x) \), for all \( x \in \mathbb{G} \),

\[
\lim_{R \to \infty} \min_{d(x,0) = R} \mathbb{P}(\mathcal{T}_x^c) = 1.
\]

2. Since the random variable \( V(0) \) is independent of \( \sigma(0; E) \) and its distribution is absolutely continuous with density \( \phi \in L^\infty(\mathbb{R}) \), we also have

\[
\mathbb{P} \left( \mathcal{N}_x^c \right) \leq \mathbb{E} \left[ \min\{2\|\phi\|_{\infty} |\tau(x, 0; E)|, 1\} \right].
\]

The right-side turns to 0 uniformly for all \( x \) with dist\((0,x) = R\) in the limit \( R \to \infty \) by assumption (A2).

By (3.28) we may now conclude that for all \( R \in \mathbb{N} \) sufficiently large and all \( x \) with \( d(x,0) = R \),

\[
\mathbb{P}(\mathcal{T}_x \cap E_x \cap \mathcal{N}_x) \geq \frac{n(E)}{2} t(0, x; E),
\]
and hence
\[ \mathbb{E}[N_R] = \sum_{x : d(x,0) = R} \mathbb{P}(T_x \cap \mathcal{E}_x \cap N_x) \geq \frac{n(E)}{2} \sum_{x : d(x,0) = R} t(0, x). \]
The claimed divergence (3.25) for \( R \to \infty \) is then implied by (3.14) of assumption (A1).

The Upper Bound

For the second moment (upper) bound we start from
\[ \mathbb{E}[N_R(N_R - 1)] = \sum_{x \neq y}^{(R)} \mathbb{P}(T_x \cap \mathcal{E}_x \cap N_x \cap T_y \cap \mathcal{E}_y \cap N_y), \]
where the sums in \( \sum^{(R)} \) are over sites of \( \mathbb{G} \) at distance \( R \) from 0.

By (3.9) the event \( \mathcal{E}_x \) corresponds to \( \{|G(x,x; E + i0)| \geq 1/t(0,x; E)\} \), and likewise for \( y \). The challenge is to bound the effects of correlations between such rare events. Considering the restriction of the resolvent kernel to the two-dimensional space spanned by \( \mathbf{y}_x \) and \( \mathbf{y}_y \), we have the following estimate (which forms a slight variant of [4, theorem A.2]).

**Lemma 3.5.** Let \( H_0 \) be a self-adjoint operator in \( \ell^2(\mathbb{G}) \) and
\[ \rho(du dv) = \varrho_1(u)\varrho_2(v)du dv \]
an absolutely continuous probability measure on \( \mathbb{R}^2 \) with bounded densities \( \varrho_j \in L^\infty(\mathbb{R}), j = 1, 2 \). Then there is some \( C < \infty \) such that the Green function of
\[ H_{u,v} := H_0 + u \mathbb{1}_{\{y\}} + v \mathbb{1}_{\{x\}} \]
satisfies for any \( z \in C \setminus \mathbb{R} \) and any \( a, b > 0 \)
\[ \rho(\{ (u, v) \in \mathbb{R}^2 : |G_{u,v}(x, x; z)| > a^{-1} \text{ and } |G_{u,v}(y, y; z)| > b^{-1} \}) \]
\[ \leq 4\|\varrho\|^2_\infty \sqrt{ab} \min \{2(\sqrt{ab} + \sqrt{|\tau(x, y; z)|}||\tau(x, y; z)|), \]
\[ \max \{\sqrt{a/b}, \sqrt{b/a}\} \} \]
with \( \tau(x, y; z) \) the tunneling amplitude associated with \( (\delta_x, \delta_y) \) at \( z \) and \( \|\varrho\|_\infty := \max\{\|\varrho_1\|_\infty, \|\varrho_2\|_\infty\} \).

**Proof.** The rank-2 Schur complement formula (2.8) reveals the dependence of the diagonal Green functions on \( (u, v) \). Abbreviating
\[ U := \sqrt{\frac{b}{a}}(u - \sigma(x; z)), \quad V := \sqrt{\frac{a}{b}}(v - \sigma(y; z)), \]

and \( \gamma := \tau(x, y; z)\tau(y, x; z) \), we can translate the lower bounds on \( |G_{u,v}(x, x; z)| \) and \( |G_{u,v}(y, y; z)| \) to

\[
U - \frac{\gamma}{V} \leq \sqrt{ab}, \quad V - \frac{\gamma}{U} \leq \sqrt{ab}.
\]

The claim can be proven through the following two observations about the set in the \((U, V)\)-plane over which the conditions (3.35) are met (the set’s shape is indicated in Figure 3.1):

i. For any solution

\[
\min\{|U|, |V|\} \leq \sqrt{|\gamma|} + \sqrt{ab}.
\]

ii. For specified \( v \), the set of \( U \) for which (3.35) holds is an interval of length at most \( 2\sqrt{ab} \), and a similar statement holds for \( V \) and \( U \) interchanged.

The area bound that these yield upon integration translates directly into (3.33). \( \square \)

We now return to the main result, which as was explained above hinges on Lemma 3.3.

**Proof of Lemma 3.3.** Lemma 3.5 yields

\[
\mathbb{P}(\mathcal{E}_x \cap \mathcal{E}_y) \leq 8\left( t(0, x)t(0, y) + \sqrt{t(0, x)t(0, y)}\mathbb{E}\left[ \min\{ \sqrt{|\tau(x, y; E)\tau(y, x; E)|}, 1 \} \right] \right).
\]

The assumption (3.22) allows us to conclude that \( \text{Im}(\psi, (H - E - i0)^{-1}\psi) = 0 \) for all \( \psi \in \ell^2(\mathbb{G}) \), and thus \( |\tau(x, y; E)| = |\tau(y, x; E)| \). Thus the expectation

![Figure 3.1. The solution set of (3.33) in the (U, V)-plane for real \( \gamma \) and \( \sigma \).](image)
value in \((3.37)\) coincides with \(T(x, y)\) of \((3.7)\). Applying the Cauchy-Schwarz inequality, we get

\[
(3.38) \quad \mathbb{E}[N_R(N_R - 1)] \leq 8\|\varrho\|_{\infty}^2 \mathbb{E}[N_R]^2 + 8\|\varrho\|_{\infty}^2 \sum_{x:|d(x,0)|=R} t(0, x) \sum_{y:|d(y,0)|=R} T(x, y).
\]

Assumption (A2) then allows us to conclude the second-moment bound:

\[
(3.39) \quad \limsup_{L \to \infty} \frac{\mathbb{E}[N_R(N_R - 1)]}{\mathbb{E}[N_R]^2} \leq 8\|\varrho\|_{\infty}^2 (1 + C_T).
\]

Through the Paley-Zygmund criterion \((3.24)\), the pair of moment bounds \((3.25)\) and \((3.39)\) yield Lemma \(3.3\) \(\Box\)

As was noted below, the statement of the just proven lemma, it readily implies Theorem \(3.2\)

4 Discussion: Exploring the Argument’s Limits

For a sense of how far the above analysis can be extended, let us consider the following, admittedly “optimistic” picture of the possible reach of the resonant delocalization argument.

For an infinite transitive metric graph, let

\[
\chi(R) := \log[\text{card}\{x \in G : \text{dist}(0, x) \in [R, R + 1]\}]
\]

so that by definition the number of sites at distance \(R\) from \(0 \in G\) grows as \(e^{\chi(R)}\). For example, on a regular tree of degree \(K + 1\), \(\chi(R)\) grows as \(R \log K\), while for the \(N\)-hypercube the analogous function grows much faster as long as \(R \leq N/2\).

Consider now the case where the tunneling amplitude is exponentially small in \(\chi(\text{dist}(0, x))\),

\[
\tau(0, x) \approx \exp(-[L + o(1)]\chi(\text{dist}(0, x))).
\]

but the effective decay rate assumes, in addition to its typical value \(L_0\), also a range of values \(L < L_0\), though at only a random and exponentially small fraction of sites on the \(R\)-sphere. To be more specific, assume it exhibits large-deviation behavior in the sense that for any fixed \(\delta > 0\):

\[
(4.1) \quad \Pr\left\{ \left| \frac{\log \tau(0, x)}{\chi(\text{dist}(0, x))} + L \right| < \delta \right\} \approx e^{-[\gamma(L)+o(1)]\chi(\text{dist}(0, x))}
\]

with a good rate function \(\gamma(L)\). (The standard vocabulary of the large-deviation theory and its relevant basic results may be found in, e.g., [10].)

To estimate the effect of possible resonances between 0 and points on the subset of the \(R\)-sphere to which the tunneling amplitude \(\tau(0, x)\) is larger than normal, let us consider the three events described by \((3.5)\) with the cutoff function modified to

\[
(4.2) \quad t(0, x) = e^{-L\chi(\text{dist}(0, x))}
\]
at a fixed \( L < L_0 \). Compared with the analysis in Section 3, \( \mathcal{T}_x \) is now made into a rare event, of probability \( \approx e^{-\gamma(L)} \). However, for the sites where it is realized, the probability of \( \mathcal{E}_x \) (which may still be expected to be of the order \( e^{-L \chi(\text{dist}(0,x))} \)), while still small, will be much larger than the previous \( e^{-L_0 \chi(\text{dist}(0,x))} \). Assuming this part of the argument could be carried through, instead of (3.32) we would get for the first moment a lower bound of the form

\[
\mathbb{E}(N_R) \geq C e^{-\delta R} e^{\chi(R)} e^{-[\gamma(L) + L \chi(R)]},
\]

where the two exponential factors are the surface area, and the fraction of sites at which the modified events \( \mathcal{T}_x \cap \mathcal{E}_x \) occur.

Thus, assuming the validity of the above sketched large-deviation structure and considerations, a sufficient condition for passing the first-moment test (3.25) may well be

\[
- \varphi(1) := \inf_{L \geq 0} [\gamma(L) + L] < 1,
\]

with \( \varphi(\cdot) \) the Legendre transform of the function \( \gamma \) (interpreted as \( \gamma(L) = +\infty \) for \( L < 0 \)):

\[
- \varphi(s) := \inf_{L \geq 0} [\gamma(L) + sL].
\]

(The regrettable negative sign is to keep notational consistency with [4].)

Under the above assumptions (in particular applicability of the large-deviation picture), for \( s < 1 \),

\[
\varphi(s) = \lim_{\text{dist}(0,x) \to \infty} \frac{1}{\chi(\text{dist}(0,x))} \log \mathbb{E}(|\tau(0,x;E)|^s)
\]

\[
= \lim_{\text{dist}(0,x) \to \infty} \frac{1}{\chi(\text{dist}(0,x))} \log \mathbb{E}(\min[|\tau(0,x;E)|,1]^s).
\]

The restriction to \( s < 1 \) is to avoid the spurious effects of the trivial divergence of the first moment. The truncation makes no difference for \( s < 1 \), since in that case the moments are bounded uniformly in \( x \); however, it allows extending the definition to \( s = 1 \).

Thus, the first-moment test for delocalization could conceivably be proven, under the above assumptions, for the regime in which

\[
\sum_x e^{\varepsilon d(x,0)} \mathbb{E}(\min[|\tau(0,x;E)|,1]) = \infty
\]

for some \( \varepsilon > 0 \).

However, to turn the above discussion into a proof, we would need to establish also the second-moment upper bound (3.24). That would be more involved than what was faced in Section 3, since it now requires bounding correlations between not just pairs of essentially local events, at \( (x,y) \) as above, but also between large deviations of the corresponding path-related tunneling amplitudes. For tree graphs such a program was carried out in [4].
It is of interest to note that (4.5) has the appearance of a complementary condition (except for transitional points) to the fractional moment localization criterion, by which pure-point spectrum in a Borel set of positive measure $I \subset \mathbb{R}$ can be concluded from the property

\[(4.6) \quad \text{for some} \ s < 1 \ \text{and all} \ E \in I \sum_x \mathbb{E}(|\tau(0, x; E)|^s) < \infty.\]

(Missing is a proof that the transition between convergence and divergence occurs along a reasonably regular boundary in the $(E, \lambda)$-plane).

Although the notion of continuous spectrum applies only to spectra of operators on infinite graphs, resonant delocalization can also play an essential role for finite graphs. In particular, that was shown to be the case for the complete graph on $1 \ll N < \infty$ points [2]. It would be of interest to see the present techniques explored further in the context of finite graphs.

**Appendix A A Stochastic Delta Function Principle**

Below is the delta function principle, invoked in the proof of Lemma 3.4.

**Lemma A.1.** Let $\Xi_n = X_n + iY_n$ be a sequence of random variables with values in $\mathbb{C}^+$ that converge in distribution to a real-valued random variable $X$, and let $V$ be an independent random variable of an absolutely continuous distribution which for some $\delta > 0$ and $c(\delta) < \infty$ satisfies the pointwise bound

\[(A.1) \quad \rho \leq c(\delta) \inf_{\varepsilon \in (0, \delta)} (1_\varepsilon \ast \rho).\]

Then

\[(A.2) \quad \lim_{n \to \infty} \mathbb{E} \left[\text{Im} \frac{\pi^{-1}}{V - \Xi_n}\right] \leq c(\delta) \inf_{\varepsilon \in (0, \delta)} \frac{1}{2\varepsilon} \mathbb{P}(|V - X| \leq \varepsilon).\]

**Proof.** We will make use of the fact that for any $\rho \in L^1 \cap L^\infty$ the family of functions $f_\rho : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ defined through

\[f_\rho(\alpha, \beta) := \begin{cases} \int (1_\varepsilon \ast \rho)(v) \frac{\beta}{(v - \alpha)^2 + \beta^2} \frac{dv}{\pi}, & \beta > 0, \\ (1_\varepsilon \ast \rho)(\alpha), & \beta = 0, \end{cases}\]

is bounded and continuous for any $\varepsilon > 0$. Its boundedness is evident. For a proof of continuity, we use its Fourier representation

\[(A.3) \quad f_\rho(\alpha, \beta) = \int \hat{\rho}(k) \text{sinc}(k\varepsilon) e^{i\alpha k - \beta |k|} \frac{dk}{2\pi},\]

where $\text{sinc}(\xi) := \sin(\xi)/\xi$ if $\xi \neq 0$ and $\text{sinc}(0) := 1$. The function $\hat{\rho}(k) := \int \rho(v) e^{ikv} \frac{dv}{2\pi}$ is square-integrable since $\rho \in L^1 \cap L^\infty$ (the latter by (A.1)). This renders the integrand in (A.3) absolutely integrable even in case $\beta = 0$. The claimed continuity then follows from (A.3) with the help of the dominated convergence theorem.
By assumption, the sequence of probability measures

$$\mu_n(d\alpha d\beta) = \mathbb{P}(\mathbb{Z}_n \in d\alpha + i d\beta)$$

on $\mathbb{R} \times [0, \infty)$ converges weakly to $\mu(d\alpha)\delta_0(d\beta)$, with $\mu$ the probability distribution of $X$ and $\delta_0$ Dirac’s measure at zero. By using the estimate

$$\mathbb{E} \left[ \text{Im} \left( \frac{\pi^{-1}}{V - \mathbb{E}_n} \right) \right] = \int \int \rho(v) \frac{\pi^{-1} \beta}{(v - \alpha)^2 + \beta^2} \, dv \, \mu_n(d\alpha d\beta) \leq c(\delta) \int f_\varepsilon(\alpha, \beta) \mu_\eta(d\alpha d\beta),$$

the claim follows, since by the convergence of $\mu_n$ and the continuity of $f_\varepsilon$,

$$\lim_{n \to \infty} \int f_\varepsilon(\alpha, \beta) \mu_n(d\alpha d\beta) = \int f_\varepsilon(\alpha, 0) \mu(d\alpha) = \frac{1}{2\varepsilon} \mathbb{P}(|V - X| \leq \varepsilon)$$

for any $\varepsilon \in (0, \delta)$.

**Appendix B  Simplicity of the Pure-Point Spectrum**

The rank-1 (and more generally finite rank) perturbation method that underlines the above criteria for continuous spectrum also allows us to shed light on properties of the pure-point spectrum and in particular its almost sure simplicity. Results in this vein were initially presented by B. Simon and then V. Jakšić and Y. Last, who proved simplicity of the point spectrum [15] and more generally singular spectrum [13] for operators with random potential of absolutely continuous conditional distribution. The following streamlined statement shows that the absolute continuity of the distribution of $V(x)$ (which enables one to apply the spectral averaging principle) is an unnecessarily strong condition for the simplicity of the point spectrum.

**Theorem B.1.** Let $H(\omega) = A + V(\omega)$ be a random operator on $\ell^2(\mathbb{G})$ with $A$ bounded and self-adjoint and $V(\omega)$ a random potential such that for any $x \in \mathbb{G}$ the conditional distribution of $V(x)$ conditioned on $V \neq x := \{V(u)\}_{u \neq x}$ is continuous. Then the pure-point spectrum of $H(\omega)$ is almost surely simple.

Our proof proceeds through proposition 2.1 of Simon-Wolff [17], extended by the observation that under its condition the vector $(H_0 - E - i0)^{-1}\psi$ provides the unique (up to a multiplicative constant) proper eigenfunction of $H_v = H_0 + vP_\psi$ within its cyclic subspace

$$\mathcal{H}_\psi := \text{span}\{(H_v - \xi)^{-1}\psi : \xi \in \mathbb{C} \setminus \mathbb{R}\} \subset \ell^2(\mathbb{G}).$$

**Lemma B.2.** Let $\mathbb{G}$ be a countable set, $H_0$ a bounded self-adjoint operator in $\ell^2(\mathbb{G})$, and $H_v$ the one-parameter family of operators defined by (2.1) with $\psi \in \ell^2(\mathbb{G})$. Then for any countable subset $S \subset \mathbb{R}$ and any probability measure $\rho$ that is continuous

$$\int \mu_{v,\psi}(S) \rho(dv) = 0.$$
Proof. By the countable additivity of the measure $\int \mu_{v,\psi}\,\rho(dv)$ it suffices to prove (B.1) for one-point sets $S = \{E\}$ at arbitrary $E \in \mathbb{R}$. The contribution to the integral from the one-point set $v = 0$ vanishes since $\rho(\{0\}) = 0$. For $v \neq 0$, by Proposition 2.1 the integrand does not vanish only if

$$v = -(\psi, (H_0 - E - i0)^{-1}\psi)^{-1}. $$

However, this point is also not charged, since $\rho$ is a continuous measure. Hence the integral vanishes.  

**Proof of Theorem B.1.** For $x \in \mathbb{G}$ let

$$\Omega_x := \{\omega \in \Omega \mid \text{for some } E \in \mathbb{R} : \dim \text{range } P_{\{E\}}(H(\omega)) \geq 2$$

and $P_{\{E\}}(H(\omega))\delta_x \neq 0\}$. Our goal is to prove that this set is of vanishing probability. To highlight the dependence of the random operator $H = H(\omega)$ on $V(x)$ we write it in the form

$$H =: H_0 + V(x)\delta_{\{x\}}, $$

where $H_0$ is independent of $V(x)$ and defined through the above equation. The full Hilbert space may be presented as the direct sum of the cyclic subspace $\mathcal{H}_{H,x} = H_{H_0,x}$ and its orthogonal complement

$$\ell^2(\mathbb{G}) = H_{H_0,x} \oplus H_{H_0,x}^\perp. $$

If the vector $\delta_x$ is cyclic for $H$, then $H_{H_0,x}^\perp$ consists of just the zero vector and the pure-point spectrum of $H$ is simple. In the following we therefore concentrate on the case that $H_{H_0,x}^\perp \neq \{0\}$.

The operator $H$ leaves both $\mathcal{H}_{H_0,x}$ and its orthogonal complement invariant. Its point spectrum is therefore the union of point spectra the operator has in the two subspaces. Two notable features of this decomposition are: (i) the spectrum of $H$ in $\mathcal{H}_{H_0,x}$ is nondegenerate, and (ii) the spectrum in $H_{H_0,x}^\perp$ does not vary with $V(x)$.

Let $S_x$ denote the set of eigenvalues of the restriction of $H$ to $\mathcal{H}_{H_0,x}^\perp$. Its independence of $V(x)$ follows from the observation that the eigenfunctions $\psi_E$ in $\mathcal{H}_{H_0,x}^\perp$ have to vanish at $x$, since for any $\varphi = f(H)\delta_x \in \mathcal{H}_{H_0,x}$

$$0 = \langle \varphi, \psi_E \rangle = \langle \delta_x, f(H)\psi_E \rangle = f(E)\psi_E(x). $$

This implies that $\psi_E$ is also an eigenfunction of $H_0 + \tilde{V}(x)\delta_{\{x\}}$ for any other $\tilde{V}(x) \in \mathbb{R}$ with the same eigenvalue $E$.

Since the set $S_x$ is independent of $V(x)$, by Lemma B.2 the conditional expectation of $\mu_{\delta_x}(S_x)$, conditioned on $V_{\neq x}$, is 0 for each value of all the other parameters. (In this argument $\mu_{\delta_x}$ is the spectral measure associated with $H$ and the vector $\delta_x \in \ell^2(\mathbb{G})$, and use is made of the continuity of the conditional distribution of $V(x)$). Hence

$$\mathbb{E}[\mu_{\delta_x}(S_x)] = \mathbb{E}[\mathbb{E}[\mu_{\delta_x}(S_x) \mid V_{\neq x}]] = 0. $$
This means that the point spectrum of $H$ in $\mathcal{H}_{H_0,x}$, which supports $\mu_\delta$, almost surely does not intersect $S_x$, or
\begin{equation}
\text{Prob}(\Omega_x) = 0.
\end{equation}
Since countable unions of null sets carry zero probability, also $\Omega_0 := \bigcup_{x \in \mathbb{G}} \Omega_x$ has this property.
On the complement of $\Omega_0$ for each $E \in \mathbb{R}$ the vectors $P_{(E)}(H(\omega)) \delta_x$ and $P_{(E)}(H(\omega)) \delta_y$ at different $x, y \in \mathbb{G}$ are collinear when nonzero. Since their collection spans the full range of $P_{(E)}(H(\omega))$ in $\ell^2(\mathbb{G})$, the point spectrum is simple for any $\omega$ in the complement of the null set $\Omega_0$.
\[\square\]
In the simple case that $H_0 \equiv 0$ and the potential is given by i.i.d. random variables, the continuity of the distribution of $V$ is trivially both sufficient and necessary for the almost sure simplicity of the spectrum. Hence the sufficient condition of Theorem B.1 cannot be weakened for a statement that is valid regardless of $H_0$.

One may also note that Theorem B.1 may be extended to random potentials that behave as assumed there only along a subset $\mathbb{G}' \subset \mathbb{G}$ provided
\begin{equation}
\text{span}\{(H(\omega) - z)^{-1} \delta_x \mid z \in \mathbb{C} \setminus \mathbb{R}, x \in \mathbb{G}'\} = \ell^2(\mathbb{G}),
\end{equation}
i.e., the collection of vectors $(\delta_x)_{x \in \mathbb{G}'}$ is cyclic under the action of $H(\omega)$.

More extensive discussions of the behavior of spectra under independent rank-1 perturbations can be found in [5, 16].

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