Fractons from Partons

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Fracton topological phases host fractionalized excitations that are either completely immobile or only mobile along certain lines or planes. We demonstrate how such phases can be understood in terms of two fundamentally different types of parton constructions, in which physical degrees of freedom are decomposed into clusters of “parton” degrees of freedom subject to emergent gauge constraints. First, we employ non-interacting partons subject to multiple overlapping constraints to describe a fermionic fracton model. Second, we demonstrate how interacting partons can be used to develop new models of bosonic fracton phases, both with string and membrane logical operators (type-I fracton phases) and with fractal logical operators (type-II fracton phases). In particular, we find a new type-II model which saturates a bound on its information storage capacity. Our parton approach is generic beyond exactly solvable models and provides a variational route to realizing fracton phases in more physically realistic systems.

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Topologically ordered phases in three dimensions have presented a wealth of surprising phenomena, challenging conventional paradigms such as effective field theory and the notion of the thermodynamic limit itself. In particular, “fracton phases”\textsuperscript{[1–6]} are characterized by fractionalized excitations that are either completely immobile or mobile only along lines or planes, despite translation symmetry. More precisely, moving such excitations requires creating even more such excitations, and this energetic barrier against motion presents not only an exciting alternative to (disorder-driven) many-body localization\textsuperscript{[7, 8]} but also a marginally stable quantum memory at finite temperature\textsuperscript{[9, 10]}.

Much progress has been made in both understanding such fracton phases and developing new models. For example, fracton phases have been related to gauged classical systems with subsystem symmetry\textsuperscript{[11, 12]} and coupled-layer constructions\textsuperscript{[13, 14]}. Nonetheless, there remain many open questions involving possible field-theoretical descriptions and phase transitions out of fracton phases. Moreover, while there have been many new fracton models with string and membrane logical operators (“type-I” fracton models), the number of models with fractal logical operators (“type-II” fracton models) has thus far been limited\textsuperscript{[3, 4]}.

The goal of our work is to construct and understand fracton phases by decomposing physical degrees of freedom into clusters of “partons”. The parton approach has proven to be extremely valuable in illuminating the physics of interacting topological phases. In some remarkable cases, partons furnish exact solutions of spin-liquid models\textsuperscript{[15, 16]}, and they correspond directly to the deconfined excitations of the physical system. More generally, partons have provided useful variational wavefunctions for otherwise intractable spin systems.

In this work, we demonstrate how parton constructions can be used to describe, develop, and analyze fracton models. We first provide an explicit parton construction of a fermionic fracton model, in which the partons are non-interacting but are subject to multiple overlapping gauge constraints. We also construct a new framework involving interacting partons to develop new models of bosonic fracton phases. We illustrate this construction with exact parton descriptions of two new fracton models, one with string and membrane and one with fractal logical operators. The model with fractal logical operators is beyond the original Haah codes and notably saturates a bound on the number of encoded qubits\textsuperscript{[3]}. Our parton language provides a new perspective on fracton phases, potentially furnishes a route to even more exotic phases, and may suggest more physically realistic fracton models via a variational approach.

GENERAL CONSIDERATIONS

We begin by reviewing Kitaev’s parton construction of spins in terms of Majorana fermions\textsuperscript{[17]}, which serves as a basis for our more involved parton constructions. For a system of spin-half degrees of freedom, one can represent each spin $\sigma$ by four Majorana fermions (partons) $\gamma_{1,2,3,4}$ as $\sigma^x = i\gamma_1 \gamma_4$, $\sigma^y = i\gamma_2 \gamma_4$, and $\sigma^z = i\gamma_3 \gamma_4$. One can then guess an approximate ground state (i.e., a variational state) for the spin system by first considering the ground state of a non-interacting (quadratic) Hamiltonian for the partons. The key assumption behind such a guess is that the partons are emergent quasiparticles whose behavior is approximately governed by a non-interacting Hamiltonian. Importantly, however, there are specific models where the parton construction provides an exact solution to the spin system and therefore no assumptions are necessary. Notable examples include the Kitaev honeycomb model\textsuperscript{[17]} and the Wen plaquette model\textsuperscript{[18]}.

Since the Hilbert space is enlarged in the parton representation, the four partons of a given spin are subject to the constraint $G = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$. The physical varia-
tional state is then obtained from the ground state of the parton Hamiltonian by a projection imposing this constraint for each spin. From the partons’ point of view, the constraint operators $G$ are local gauge transformations (LGTs), and the resulting parton gauge theory is crucial for understanding the physical state. Indeed, one can classify spin-liquid phases by their parton variational states \[17\] via the invariant gauge group (IGG): the subgroup of the gauge group that leaves the parton Hamiltonian (and hence the parton state) invariant. For example, the Kitaev honeycomb model and the Wen plaquette model are both $Z_2$ spin liquids ($Z_2$ gauge theories) because their parton states are invariant under a $Z_2$ gauge group whose only non-trivial element is the global gauge transformation (the product of all LGTs).

Based on their ground-state degeneracies \[3\] and their logical operators \[4\] and their classical systems with gauged subsystem symmetries \[11, 12\], one expects the IGGs of the corresponding parton states to contain elements that are products of LGTs. Alternatively, if one understands fracton phases as classical systems with gauged subsystem symmetries \[11, 12\], one expects the IGGs of the corresponding parton states to contain elements that are products of LGTs along the appropriate subsystems, i.e., planes for type-I fracton phases and spikes at particular values of $L$ for type-II fracton phases. Alternatively, if one understands fracton phases as classical systems with gauged subsystem symmetries \[11, 12\], one expects the IGGs of the corresponding parton states to contain elements that are products of LGTs along the appropriate subsystems, i.e., planes for type-I fracton phases and spikes for type-II fracton phases. Based on this understanding, Kitaev’s parton construction with a non-interacting Majorana Hamiltonian is insufficient to describe a fracton phase because it necessarily gives rise to a simple $Z_2$ IGG. Since a two-Majorana term can only connect two different spins via one Majorana from each spin, the Majorana Hamiltonian can only be invariant under the product of the two corresponding LGTs. For a connected system, it immediately follows that the only non-trivial element of the IGG is then the product of all LGTs. This obstacle motivates us to change Kitaev’s parton construction in two different ways. First, we consider a parton construction where each constraint (i.e., each LGT) is substituted with several overlapping constraints. Second, we consider interacting parton Hamiltonians with four-Majorana terms that can each connect four different spins.

**FERMIonic FRACtON MODEL**

We first consider a modified version of Kitaev’s parton construction where the physical degrees of freedom are not spins but two Majorana fermions $\gamma_{A,B}$ at each site of a cubic lattice. These two physical Majorana fermions are represented by six Majorana partons $\gamma_{1,2,3,4,5,6}$ on the links adjacent to the site [see Fig. 1(a)]. For example, we may take $\gamma_A = i\gamma_2\gamma_4\gamma_6$ and $\gamma_B = i\gamma_1\gamma_3\gamma_5$. To account for the enlarged Hilbert space, we also impose two independent constraints $G_{xy} = \gamma_1\gamma_2\gamma_3\gamma_4 = 1$ and $G_{xz} = \gamma_1\gamma_2\gamma_5\gamma_6 = 1$, whose product is a dependent third constraint $G_{yz} = \gamma_3\gamma_4\gamma_5\gamma_6 = 1$. Note that these three gauge constraints are directional in the sense that they only act (respectively) on partons in the $xy$, $xz$, and $yz$ planes containing the given site [see Fig. 1(a)].

The parton state is constructed by imposing the constraint $i\gamma_j\gamma_k = 1$ for each pair of partons on the same link [see Fig. 1(b)]. Since these constraints commute, the corresponding parton Hamiltonian is simply the commuting-projector Hamiltonian $-\sum_{(j,k)} i\gamma_j\gamma_k$. For such a parton state, the IGG has several non-trivial elements due to the directionality of the gauge constraints. In particular, the product of all $G_{xy}$ in any $xy$ plane, the product of all $G_{xz}$ in any $xz$ plane, and the product of all $G_{yz}$ in any $yz$ plane each leave the parton state invariant. However, the product of all such “planar” IGG elements is trivial. For an $L \times L \times L$ system, the IGG is then $Z_2^{3L-1}$, which is indicative of a type-I fracton phase.

What parent Hamiltonian in terms of the physical Majorana fermions could have the projected parton state as its ground state? The topological bootstrap introduced by one of us \[18\] provides one route for deriving such a parent Hamiltonian. One seeks the minimal Hamiltonian in terms of the physical degrees of freedom, which, when written in terms of the partons, commutes with the parton Hamiltonian. Using this technique, we find that the parent Hamiltonian involves a product of eight physical Majorana fermions for each cube [see Fig. 1(c)]. In the
parton representation, the eight-Majorana term for each cube is then the product of the twelve terms $i\gamma_j\gamma_k$ on the twelve links surrounding the cube [19].

This model is equivalent to two copies of the Majorana cube model (MCM), which was introduced in Ref. [5] as a type-I fracton model. The MCM has only one flavor of Majorana fermion $\gamma_A$ at each site and involves interactions for only half of the cubes (i.e., a subset of cubes which are either disjoint or intersect at an edge). Our model can then be reproduced by taking one more copy of the MCM on the complementary subset of cubes involving a different flavor of Majorana fermion $\gamma_B$, and using a unitary transformation $\gamma_A \leftrightarrow \gamma_B$ on one sublattice of the (bipartite) cubic lattice.

Since our model is two copies of the MCM, it clearly captures a type-I fracton phase as well. Indeed, one can enumerate all characteristic fractional excitations of such a type-I fracton phase in our model [5]. First, a string of alternating Majorana flavors $\ldots \gamma_A \gamma_B \ldots$ along the $x$ direction creates two pairs of excitations at its endpoints that are mobile along the $x$ direction only. Second, a double string where two complementary strings $\ldots \gamma_A \gamma_B \ldots$ and $\ldots \gamma_B \gamma_A \ldots$ are displaced in the $z$ direction creates two pairs of excitations at its endpoints that are mobile in the $xy$ plane. Third, a rectangular checkerboard of $\gamma_A$ and $\gamma_B$ operators creates four excitations at its corners that are completely immobile.

Our parton construction is suggestive of coupling two-dimensional topologically ordered stacks [13, 14]. Indeed, if one imposed only one gauge constraint $G_{xy}$, $G_{xz}$, or $G_{yz}$ at each site, the model would consist of decoupled stacks of Wen plaquette models [16] with decoupled Majorana dimers in between. It is the presence of all three directional gauge constraints that produces a type-I fracton model. However, these constraints remarkably conspire to produce a fermionic fracton model.

**BOSONIC FRACTONS FROM INTERACTING PARTONS**

While it is possible to describe a bosonic fracton phase by non-interacting partons, a more natural choice for us, especially for describing type-II phases with fractal structures, is to consider a different construction involving interacting partons. Naively, this may not seem useful because interacting partons are in general as difficult to analyze as the (interacting) physical degrees of freedom. However, we focus on parton variational states that are ground states of interacting yet exactly solvable commuting-projector Hamiltonians.

![Bosonic type-I fracton model from interacting partons.](image)

**FIG. 2:** Bosonic type-I fracton model from interacting partons. (a) Two spin-one-halves at each site (blue sphere) are decomposed into eight Majorana partons (green spheres) subject to two constraints. (b) Parton state specified by four constraints for each parton cube. (c) Independent eight-spin interactions of the parent Hamiltonian.

**Type-I Fracton Model**

We first present an explicit example of such a construction that yields a new type-I fracton model with string and membrane logical operators. The physical degrees of freedom are two spin-one-halves $\sigma$ and $\tau$ at each site of a cubic lattice. Each spin is represented by four Majorana partons and is subject to a single constraint, as in Kitaev’s original construction [15]. In particular, the two spins at any given site are associated with the two tetrahedra formed by the eight Majorana partons $\gamma_1,...,\gamma_8$ surrounding the site [see Fig. 2(a)]. The two constraints are then $G_\sigma = \gamma_1 \gamma_4 \gamma_6 \gamma_7$ and $G_\tau = \gamma_2 \gamma_5 \gamma_7 \gamma_8$, while the spin components are $\sigma^x = i \gamma_1 \gamma_4, \sigma^y = i \gamma_1 \gamma_6, \sigma^z = i \gamma_1 \gamma_7$, $\tau^x = i \gamma_5 \gamma_8, \tau^y = i \gamma_3 \gamma_8$, and $\tau^z = i \gamma_2 \gamma_8$.

The parton state is constructed as follows. Each cube of the original cubic lattice contains eight Majorana partons from spins at eight different sites [see Fig. 2(b)]. For each such parton cube consisting of eight Majorana partons, we impose a Hamiltonian involving a four-Majorana term at each face of the cube. Since it has four independent (and two dependent) commuting terms, this parton Hamiltonian gives rise to a unique local ground state for each parton cube [20]. The parton state is then simply the direct product of these local ground states.

For such a parton state, the IGG has several non-trivial elements. First, the product of all $G_\sigma G_\tau$ in any $xy$, $xz$, or $yz$ plane leaves the parton state invariant. Second, if such a checkerboard pattern is consistent with the system size,
the product of all $G_y$ in one sublattice and all $G_z$ in the other sublattice of any $xy, xz,$ or $yz$ plane also leaves the parton state invariant. However, these elements of the IGG are not all independent. In fact, for an $L \times L \times L$ system, a detailed analysis shows that the IGG is $\mathbb{Z}_2^{3L-1}$ if $L$ is odd and $\mathbb{Z}_2^{L-4}$ if $L$ is even [19].

Once again, the topological bootstrap can be used to obtain a parent spin Hamiltonian whose ground state is the projected parton state. Using this technique, we find that the parent spin Hamiltonian has two independent eight-spin interactions for each cube [see Fig. 2(c)]. In the parton representation, each eight-spin term is then the product of four four-Majorana terms corresponding to four faces of neighboring parton cubes [19].

The system-size dependence of the IGG indicates that this model captures a type-I fracton phase. Indeed, the fractional excitations of this bosonic model are supported by string and membrane logical operators, which can be obtained from those of our fermionic model via the substitutions (i) $\gamma_A \rightarrow \sigma^y$ and $\gamma_B \rightarrow \tau^z$ and (ii) $\gamma_A \rightarrow \sigma^z$ and $\gamma_B \rightarrow \tau^y$. For example, a string of alternating spin types $\ldots \sigma^x \tau^x \ldots$ or $\ldots \sigma^y \tau^y \ldots$ along the $x, y,$ or $z$ direction creates two pairs of excitations at its endpoints that are mobile along the string direction only. Also, as expected for a type-I fracton model, the number of encoded qubits scales with the linear dimension of the system.

Type-II Fracton Model

We now demonstrate that our interacting parton construction can also yield a type-II fracton model with fractal logical operators. In fact, we derive a spin model beyond the original Haah codes and find that it saturates a bound on the number of encoded qubits [3]. Remarkably, our type-II construction is obtained from the type-I construction above by simply changing which set of eight Majorana partons interact with each other. Indeed, we use the same representation of spins in terms of Majorana partons [see Fig. 3(a)], but we choose a different unit cell of eight Majorana partons for the four-Majorana terms in our parton Hamiltonian [see Fig. 3(b)]. The parton state is again the direct product of the unique ground states for these eight-Majorana unit cells.

For such a parton state, the IGG is $\mathbb{Z}_2^N$, where $N$ has a peculiar dependence on the system size. For an $L \times L \times L$ system, $N$ is only 2 if $L$ is a generic odd number, while it reaches $2L$ if $L = 2^n$ [19]. In the former case, the only non-trivial elements of the IGG are global ones: the products of all $G_x$ and of all $G_y$. In the latter case, however, there are further non-trivial elements corresponding to products of $G_x$ and $G_z$ along fractal structures. Since the scaling ratio of each fractal structure is 2, it only fits into the system if $L = 2^n \ell$, where $\ell$ is the size of its base unit. For our fractal structures, the two smallest base units correspond to $\ell_1 = 1$ and $\ell_2 = 15$.

The topological bootstrap can again be used to obtain a parent spin Hamiltonian whose ground state is the projected parton state. Once again, we find that the parent spin Hamiltonian has two independent eight-spin interactions for each cube [see Fig. 3(c)] and that each eight-spin term is then the product of four four-Majorana terms in the parton representation [21].

The system-size dependence of the IGG indicates that this model captures a type-II fracton phase. Indeed, the fractional excitations of this model are supported by fractal logical operators. For example, one such fractal operator is constructed iteratively as follows. First, the local operator $\sigma^x \tau^y$ anticommutes with six interaction terms and thus creates six excitations on the dual lattice [see Fig. 4(a)]. Next, by taking the product of six $\sigma^x \tau^y$ operators at the six points shown in Fig. 4(b), the resulting set of excitations is identical in shape but is rescaled by a factor of 2 with respect to the original one. This iterative procedure generates a fractal operator of size $2^n$ with six immobile excitations at its corners.

The model presented in Fig. 3(c) has several interesting features with respect to previously known type-II fracton models [3, 4]. As expected for any such model, the number of encoded qubits for an $L \times L \times L$ system has large spikes for $L = 2^n$. However, unlike any of the original Haah codes, our model can encode $4L$ qubits for $L = 2^n$, thereby saturating the upper bound for the number of encoded qubits in a type-II fracton system with two qubits per site and interactions supported on single cubes [3]. Furthermore, our model is a non-CSS code as each inter-
FIG. 4: Fractal structure in our type-II fracton model. (a) For the model in Fig. 3(c), the single-site operator $\sigma^x \tau^y$ creates six excitations (red dots). (b) Six such operators in the given configuration create six defects with the same shape as in (a) but with doubled linear dimension.

action involves both $x$-type and $z$-type spin operators, and it is therefore not clear how to realize it by gauging a classical spin model [11, 12]. Finally, to our knowledge, there are no string logical operators in this model, although rigorously proving this claim is challenging as techniques used in Refs. [3] and [4] are not directly applicable.

SUMMARY AND DISCUSSION

We have provided two different ways of describing and developing new fracton models by means of parton constructions. The first method uses non-interacting partons with multiple overlapping gauge constraints at each site and was used to obtain a fermionic type-I model, while the second method uses interacting partons governed by a commuting-projector Hamiltonian and was used to obtain bosonic type-I and type-II models. The new type-II model is particularly interesting because (i) it is the first non-CSS code involving qubits that captures a type-II fracton phase [22] and (ii) it saturates a bound on the maximal number of encoded qubits for particular system sizes.

In addition to providing exactly solvable fracton models, our parton approach may also enable a variational treatment of more realistic models that are not exactly solvable but are suspected to capture fracton phases. In fact, there are many variants of our interacting parton constructions that do not give rise to exactly solvable models. Nevertheless, based on the system-size dependence of their IGGs and the structures of their IGG elements, one can immediately deduce whether they correspond to an ordinary topological phase, a type-I fracton phase, or a type-II fracton phase. Furthermore, in the case of type-II fracton phases, one expects a direct correspondence between the fractal structures of the IGG elements and those of the logical operators.

There are many directions for extending and utilizing this parton approach, for example, in investigating phase transitions between fracton and other phases. Moreover, it would be interesting to apply the formalism of projective symmetry group, which has been useful for classifying conventional topological order, to fracton phases in the presence of symmetries.

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[21] Note that each interaction term is inversion symmetric while the two independent terms (and their dependent product) are transformed into each other by a simultaneous three-fold rotation around the [111] diagonal in both real space and spin space.
[22] See I. Kim, arXiv:1202.0052 (2012) for non-CSS qudit codes that are type-II fracton models.
Supplementary Material

Invariant gauge groups and ground-state degeneracies

For our interacting parton constructions and the corresponding exactly solvable spin models, one is interested in the invariant gauge group (IGG) of the parton construction and the ground-state degeneracy of the spin model as a function of the system size. In this section, we demonstrate that these quantities can be evaluated straightforwardly by means of standard $\mathbb{Z}_2$ linear algebra.

We first consider the parton constructions. For a given system size, we assume that there are $N_\gamma$ Majorana partons, $N_H$ interaction terms in the parton Hamiltonian, and $N_G$ independent gauge constraints. Each interaction term $H$ (gauge constraint $G$) is a product of Majorana partons and it can thus be represented by an $N_\gamma$-component vector $h$ ($g$) of $\mathbb{Z}_2$ elements such that each element is 1 if the product contains the corresponding Majorana parton and 0 if it does not. Furthermore, we may include all interaction terms in the $N_H \times N_\gamma$ matrix $H$ and all gauge constraints in the $N_G \times N_\gamma$ matrix $G$. In general, an interaction term and a gauge constraint either commute or anticommute:

$$[H, G] = 0 \iff h \cdot g = 0 \pmod{2},$$

$$\{H, G\} = 0 \iff h \cdot g = 1 \pmod{2}.$$  

By definition, each element of the IGG is an appropriate product of gauge constraints $G$ that commutes with all the interaction terms $H$. The number of independent IGG elements is then

$$N = N_G - \text{rank}[H \cdot G^T],$$

while the IGG elements themselves are contained in $\text{ker}[H \cdot G^T]$. Note that the rank and the kernel (null space) both must be taken modulo 2. Since all elements are $\mathbb{Z}_2$, the IGG is given by $\mathbb{Z}_2^N$.

We next consider the corresponding spin models. For a given system size, we assume that there are $N_\sigma$ spin-one-half degrees of freedom and there are $N_\tilde{H}$ commuting interaction terms in the spin Hamiltonian. Each interaction term $\tilde{H}$ is a product of spin operators $\sigma^x, \sigma^y, \text{and } \sigma^z \propto \sigma^x \sigma^y$ and it can thus be represented by a $2N_\sigma$-component vector $\tilde{h}$ of $\mathbb{Z}_2$ elements such that each pair of elements is $\{0, 0\}$ if the interaction term does not act at the corresponding spin, while it is $\{0, 1\}, \{1, 0\}$, and $\{1, 1\}$ if the interaction term acts at the corresponding spin by $\sigma^x, \sigma^y, \text{and } \sigma^z$ operators, respectively. Furthermore, we may include all interaction terms in the $N_\tilde{H} \times 2N_\sigma$ matrix $\tilde{H}$. In general, the interaction terms are not all independent and a subset $\{k\}$ of them may satisfy

$$\prod_{\{k\}} \tilde{H}_k \propto 1 \iff \sum_{\{k\}} \tilde{h}_k = 0 \pmod{2}.$$  

Since the Hilbert space contains $N_\sigma$ effective qubits, and each independent (commuting) interaction term specifies one effective qubit, the actual number of qubits encoded in global degrees of freedom is

$$\tilde{N} = N_\sigma - \text{rank}[\tilde{H}].$$

Once again, the rank must be taken modulo 2. For $\tilde{N}$ such qubits encoded in global degrees of freedom, the ground-state degeneracy of the spin model is then given by $2^{\tilde{N}}$.

For our interacting parton constructions, the number $N$ of $\mathbb{Z}_2$ factors in the IGG is given in Table I, while for the corresponding spin models, the number $\tilde{N}$ of encoded qubits is given in Table II. For the purpose of benchmarking, we also include the Wen plaquette model and its parton construction. While $N$ and $\tilde{N}$ might not have identical system-size dependence and, in particular, they might have different even-odd oscillations, they follow the same qualitative behavior: they are both approximately constant for the Wen plaquette model, scale with the linear system dimension for our type-I fracton model, and spike at particular system sizes for our type-II fracton model.

Parent Hamiltonians for Projected Parton States

The topological bootstrap [13] provides a simple heuristic to obtain a parent Hamiltonian for a projected parton state. We briefly review the bootstrap construction, but ultimately only the heuristic is necessary. In this section, we first review how both the bootstrap and the heuristic apply to the Wen plaquette model, and then we apply the heuristic to the parton constructions of the fracton models considered in the main text.
TABLE I: Number $N$ of $\mathbb{Z}_2$ factors in the IGG for an $L \times L$ system in the case of the Wen plaquette model and for an $L \times L \times L$ system in the case of our bosonic fracton models.

| System size $(L)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| Wen plaquette model | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Type-I fracton model | 2 | 8 | 8 | 20 | 14 | 32 | 20 | 44 | 26 | 56 | 32 | 68 | 38 | 80 | 44 | 92 |
| Type-II fracton model | 2 | 4 | 2 | 8 | 2 | 4 | 2 | 16 | 2 | 4 | 2 | 8 | 2 | 4 | 26 | 32 |

TABLE II: Number $\tilde{N}$ of encoded qubits for an $L \times L$ system in the case of the Wen plaquette model and for an $L \times L \times L$ system in the case of our bosonic fracton models.

| System size $(L)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| Wen plaquette model | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| Type-I fracton model | 2 | 12 | 14 | 36 | 26 | 60 | 38 | 84 | 50 | 108 | 62 | 132 | 74 | 156 | 86 | 180 |
| Type-II fracton model | 2 | 8 | 2 | 16 | 2 | 8 | 2 | 32 | 2 | 8 | 2 | 16 | 2 | 8 | 50 | 64 |

The Wen plaquette model consists of spin-1/2s at the sites of a square lattice, with plaquette interactions given by

$$H = \sum_l \sigma_l^y \sigma_{l+x}^x \sigma_l^y \sigma_{l+y}^x.$$ 

Following Kitaev [15] and Wen [16], each spin is represented by four Majorana fermions $\gamma_1, 2, 3, 4$ on the links adjacent to the spin site [see Fig. 5(a)] that are subject to the constraint $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$. The parton state is defined as the ground state of the Hamiltonian $-\sum_{(j,k)} i \gamma_j^* \gamma_k$, where $(j,k)$ label Majoranas on the same link [see Fig. 5(b)].

Given this parton state, how would one obtain the parent spin Hamiltonian without knowing about the Wen plaquette model? The method of the topological bootstrap involves coupling two systems $A$ and $B$, where $A$ is the system of four Majoranas at each site, now treated as physical degrees of freedom without any constraints, and $B$ is a system of free spin-1/2s at each site. The full Hamiltonian we consider is a combination of the Majorana Hamiltonian specified above and a Kondo coupling between the Majoranas and the spins:

$$H = H_A + H_{AB},$$

$$H_A = -\sum_{(j,k)} i \gamma_j \gamma_k,$$

$$H_{AB} = g \sum_{l,\alpha,\beta} \bar{\sigma}_l \cdot (\gamma_l^\dagger \tau_{\alpha \beta} \gamma_l^\dagger),$$

where $\bar{\tau}_{\alpha \beta}$ is a vector of the Pauli matrices, and the four Majorana fermions $\gamma_l^{1,2,3,4}$ around any site $l$ form a spinful complex fermion given by

$$c_{l, \uparrow} = \frac{\gamma_l^1 + i \gamma_l^2}{2}, \quad c_{l, \downarrow} = \frac{\gamma_l^3 + i \gamma_l^4}{2}.$$

FIG. 5: (a) The physical spin-1/2 at each site (blue sphere) is decomposed into four Majorana partons $\gamma_1, 2, 3, 4$ (green spheres) subject to the constraint $\gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$. (b) The parton state for the Wen plaquette model is the product state of dimers formed by pairs of Majoranas on each link (green zigzag).
As noted in Ref. 18, the desired parent Hamiltonian is the lowest-order effective Hamiltonian for the $B$ system that can be obtained from degenerate perturbation theory. In this case, the lowest-order interaction preserving the Majorana integrals of motion is generated in perturbation theory by four applications of the Kondo interaction and it is precisely the four-spin interaction of the Wen plaquette model [see Fig. 6(a) for the Majoranas involved]. Thus, the heuristic is to find the minimal physical interaction, which, when written in terms of the partons, is a product of parton integrals of motion. One can then readily check that such physical interactions do indeed give rise to a suitable parent Hamiltonian. For example, the four-spin interaction of the Wen plaquette model is equivalent to the eight-Majorana interaction shown in Fig. 6(a). Since the link bilinears are each unity in the Majorana ground state, the product of all eight Majoranas is also unity, and thus the (projected) Majorana ground state is indeed the ground state of the Wen plaquette model.

In the same way, we can apply this heuristic to obtain the parent Hamiltonians described in the main text. In Fig. 6(b,c,d), we explicitly illustrate the physical interactions of our three fracton models in terms of the Majorana partons. In each case, one can check that the resulting product of Majoranas is unity for the Majorana ground state.