\section*{A-2-Frames in A-2-Inner Product Spaces}

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\textbf{Abstract.} Certain results about frames are extended for the new frames in Hilbert $C^*$-modules. In this paper, we introduce the notion of $A$-2-frames in $A$-2-inner product spaces and give some characterizations for these frames. Then we define the tensor product of $A$-2-frames and prove some results for it.

\section{Introduction and Preliminaries}

Frames were first introduced in 1952 by Duffin and Schaeffer \cite{3}. In 2000, Frank-Larson \cite{4} introduced the notion of frames in Hilbert $C^*$-modules as a generalization of frames in Hilbert spaces and Jing \cite{6} continued to consider them. It is well known that Hilbert $C^*$-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a $C^*$-algebra rather than in the field of complex numbers. The theory of 2-inner product spaces as well as an extensive list of related references can be found in \cite{5,2}.

The concept of 2-frames for 2-inner product spaces was introduced by A. Arefijamal and Ghadir Sadeghi \cite{1} and described some fundamental properties of them. Recently T. Mehdiaibad and A. Nazari \cite{7} introduced the $A$-2-inner product space and investigate some inequalities in these spaces. The authors \cite{8} defined a 2-inner product that takes values in a locally $C^*$-algebra and studied some properties of it.

In this paper, we introduce an $A$-2-frame in the $A$-2-inner product space and describe some fundamental properties of them. The tensor product of $A$-2-frames in the $A$-2-inner product space is introduced. It is shown that the tensor product of two $A$-2-frames is an $A$-2-frame for the tensor product of $A$-2-inner product space. Also, we investigate tensor products of $A$-2-frames. From now, $A$ denotes a $C^*$-algebra.

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Definition 1.1. A pre-Hilbert \( \mathcal{A} \)-module is a complex vector space \( E \) which is also a left \( \mathcal{A} \)-module, compatible with the Complex algebra structure, equipped with an \( \mathcal{A} \)-valued inner product \( \langle ., . \rangle : E \times E \to \mathcal{A} \) which is \( \mathbb{C} \)-linear and \( \mathcal{A} \)-linear in its second variable and satisfies the following relations

\begin{align*}
(I_1) \quad & \langle x, x \rangle \geq 0 \text{ for every } x \in E, \\
(I_2) \quad & \langle x, y \rangle = \langle y, x \rangle^* \text{ for every } x, y \in E, \\
(I_3) \quad & \langle x, x \rangle = 0 \text{ if and only if } x = 0, \\
(I_4) \quad & \langle ax, by \rangle = a^* \langle x, y \rangle b \text{ for every } x, y \in E \text{ and } a, b \in \mathcal{A}, \\
(I_5) \quad & \langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle \text{ for every } x, y, z \in E \text{ and } \alpha, \beta \in \mathbb{C}.
\end{align*}

Example 1.2. Let \( l^2(\mathcal{A}) \) be the set of all sequences \( \{a_n\}_{n \in \mathbb{N}} \) of elements of a \( C^* \)-algebra \( \mathcal{A} \) such that the series \( \sum_{n \in \mathbb{N}} a_n a_n^* \) is convergent in \( \mathcal{A} \). Then \( l^2(\mathcal{A}) \) is a Hilbert \( \mathcal{A} \)-module with respect to the pointwise operations and inner product defined

\[
\langle \{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \rangle = \sum_{n \in \mathbb{N}} a_n b_n^*.
\]

Definition 1.3. Let \( E \) be a left \( \mathcal{A} \)-module, an \( \mathcal{A} \)-combination of \( x_1, x_2, ..., x_n \) in \( E \) is written as follows

\[
\sum_{i=1}^n a_i x_i = a_1 x_1 + a_2 x_2 + ... + a_n x_n \quad (a_i \in \mathcal{A}).
\]

\( x_1, x_2, ..., x_n \) are called \( \mathcal{A} \)-independent if the equation \( a_1 x_1 + a_2 x_2 + ... + a_n x_n = 0 \) has exactly one solution, namely \( a_1 = a_2 = ... = a_n = 0 \), otherwise, we say that \( x_1, x_2, ..., x_n \) are \( \mathcal{A} \)-dependent.

The maximum number of elements in \( E \) that are \( \mathcal{A} \)-independent is called the \( \mathcal{A} \)-rank of \( E \).

Definition 1.4. Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( E \) be a linear space by \( \mathcal{A} \)-rank greater than 1, which is also a left \( \mathcal{A} \)-module. We define a function \( \langle ., |. \rangle : E \times E \times E \to \mathcal{A} \) satisfies the following properties

\begin{align*}
(T_1) \quad & \langle x, x \mid y \rangle = 0, \text{ if and only if } x = ay \text{ for } a \in \mathcal{A} \\
(T_2) \quad & \langle x, x \mid y \rangle \geq 0 \text{ for all } x, y \in E \\
(T_3) \quad & \langle x, x \mid y \rangle = \langle y, x \mid x \rangle \text{ for all } x, y \in E \\
(T_4) \quad & \langle x, y \mid z \rangle = \langle y, x \mid z \rangle^* \text{ for all } x, y, z \in E \\
(T_5) \quad & \langle ax, by \mid z \rangle = a \langle x, y \mid z \rangle b^* \text{ for all } x, y, z \in E \text{ and } a, b \in \mathcal{A}.
\end{align*}
Example 1.5. [8] Let $\mathcal{A}$ be a commutative $C^*$-algebra and $E$ be a pre-Hilbert $\mathcal{A}$-module with inner product $\langle \cdot, \cdot \rangle$, define $\langle \cdot, \cdot \rangle : E \times E \times E \to \mathcal{A}$ by

$$
\langle x, y | z \rangle \mapsto \langle x, y | z \rangle = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle
$$

Then $(E, \langle \cdot, \cdot \rangle)$ is a $\mathcal{A}$-2-inner product space.

Theorem 1.6. Let $(E, \langle \cdot, \cdot \rangle)$ be an $\mathcal{A}$-2-inner product space on a commutative $C^*$-algebra $\mathcal{A}$. Then the following inequality holds,

$$
|\langle x, y | z \rangle|^2 = \langle x, y | z \rangle \langle x, y | z \rangle^* \leq \langle x, x | z \rangle \langle y, y | z \rangle \quad (x, y, z \in E).
$$

Proof. For $\lambda \in \mathcal{A}$ we have

$$
0 \leq \langle \lambda x - y, \lambda x - y | z \rangle = \langle \lambda x, \lambda x | z \rangle - \langle \lambda x, y | z \rangle - \langle y, \lambda x | z \rangle + \langle y, y | z \rangle
$$

$$
= \lambda^* \langle x, x | z \rangle \lambda - \lambda^* \langle x, y | z \rangle - \langle y, x | z \rangle \lambda + \langle y, y | z \rangle.
$$

Take $\lambda = \langle x, y | z \rangle (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1}$ then

$$
0 \leq \langle y, x | z \rangle (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1} \langle x, x | z \rangle \langle y, y | z \rangle (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1} \langle x, y | z \rangle - \langle y, x | z \rangle \langle y, y | z \rangle (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1} + \langle y, y | z \rangle,
$$

hence, $2\langle y, x | z \rangle \langle x, y | z \rangle \leq (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1} (\langle x, x | z \rangle \langle y, y | z \rangle \langle x, y | z \rangle$ $+ \langle y, y | z \rangle (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1} (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1} (\langle x, x | z \rangle + \varepsilon \mathbb{I})^{-1} + \langle y, y | z \rangle)$ then by $\varepsilon \to 0$ inequality holds. \hfill \Box

Definition 1.7. [7] Let $E$ be a real vector space that $\mathcal{A}$-rank is greater than 1 and $p : E \times E \to \mathbb{R}$ be a function such that

(1) $p(x, y) = 0$ if and only if $x, y \in E$ are linearly $\mathcal{A}$-dependent,
(2) $p(x, y) = p(y, x)$ for every $x, y \in E$,
(3) $p(\alpha x, y) = |\alpha| p(x, y)$, for every $x, y \in E$ and for every $\alpha \in \mathbb{C}$,
(4) $p(x + y, z) \leq p(x, z) + p(y, z)$, for every $x, y, z \in E$.
(5) $P(ax, y) \leq ||a||p(x, y)$, for every $x, y \in E$ and $a \in A$, The function $p$ is called an $A$-2-norm.

It follows from theorem 1.6 that

**Corollary 1.8.** [7] Let $E$ be an $A$-inner product space, For $x, z \in E$ we define $p(x, z) = \sqrt{\|\langle x, x \rangle \|}$. Then $||\langle x, y | z \rangle|| \leq p(x, y)p(y, z)$.

In the following theorem, we investigate some properties of an $A$-2-norm.

**Theorem 1.9.** Let $(E, \langle ., . | . \rangle)$ be an $A$-2-inner product space and $p$ be an $A$-2-norm, then

1. $p(x, y) = \sup \{\|\langle x, z | y \rangle\|; p(z, y) = 1\}$.
2. $p(x, y + ax) = p(x, y)$ for $a \in A$.

**Proof.** (1) By the Cauchy-schwarz inequality we observe that $\|\langle x, z | y \rangle\| \leq p(x, y)p(z, y) \leq p(x, y)$, for every $z \in E$ such that $p(z, y) \leq 1$. Moreover if $z = \frac{x}{p(x, y)}$ then $p(z, y) = 1$ and therefore $\|\langle x, z | y \rangle\| = p(x, y)$. □

Let $E$ be an $A$-inner product space. A sequence $\{a_n\}_{n \in \mathbb{N}}$ of $E$ is said to be convergent if there exists an element $a \in E$ such that $\lim_{n \to \infty} p(a_n - a, x) = 0$, for all $x \in E$. Similarly, we can define a Cauchy sequence in $E$. An $A$-inner product space $E$ is called an $A$-Hilbert space if it is complete.

Now, we give the notion of a frame on a Hilbert $A$-module which is defined in [6, definition 3.1].

**Definition 1.10.** [6] Let $A$ be an unital $C^*$-algebra and $E$ be a Hilbert $A$-module. The sequence $\{x_j\}_{j \in J \subseteq \mathbb{N}}$ is called a frame for $E$ if there exist two positive elements $A$ and $B$ in real numbers such that

$$A\langle x, x \rangle \leq \sum_{j \in J} \langle x, x_j \rangle \langle x_j, x \rangle \leq B\langle x, x \rangle \quad (x \in E).$$

The frame $\{x_j\}$ is said to be tight frame if $A = B$, and said to be Parseval if $A = B = 1$. The operator $T : E \to l^2(A)$ defined by

$$Tx = \{\langle x, x_j \rangle\}_{j \in J}$$

is called the analysis operator. The adjoint operator $T^* : l^2(A) \to E$ is given by

$$T^*\{c_j\}_{j \in J} = \sum_{j \in J} c_j x_j$$
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is called the pre-frame operator or the synthesis operator. By composing $T$ and $T^*$, we obtain the frame operator $S : E \to E$, by

$$S = T^*Tx = \sum_{j \in J} \langle x, x_j \rangle x_j.$$ 

Also from this equation, we have

$$x = \sum_{j \in J} \langle x, S^{-1}x_j \rangle x_j \quad (x \in E).$$

Now we are ready to define an $A$-2-frame on an $A$-2-Hilbert space.

2. $A$-2-frames

In this section we define $A$-2-frames on $A$-2-Hilbert spaces, and we give some results about them.

**Definition 2.1.** Let $(E, \langle \cdot, \cdot \rangle)$ be an $A$-2-Hilbert space and $\xi \in E$. A sequence $\{a_i\}_{i \in \mathbb{N}}$ of $E$ is called an $A$-2-frame (associated to $\xi$) if there exist positive real numbers $A$ and $B$ such that

$$A\langle x, x|\xi \rangle \leq \sum_{i \in \mathbb{N}} \langle x, x_i\xi \rangle \langle x_i, x|\xi \rangle \leq B\langle x, x|\xi \rangle \quad (x \in E). \quad (2.1)$$

A sequence satisfying the upper $A$-2-frame condition is called an $A$-2-Bessel sequence, and every $x_i$ is $A$-independent to $\xi$.

**Proposition 2.2.** Let $A$ be a commutative and $(E, \langle \cdot, \cdot \rangle)$ be a Hilbert $A$-module and $\{x_i\}_{i \in \mathbb{N}}$ be a frame for $E$. Then for invertible element $\langle \xi, \xi \rangle$, it is an $A$-2-frame with the standard $A$-2-inner product.

**Proof.**

$$\sum_{i \in \mathbb{N}} \langle x, x_i\xi \rangle \langle x_i, x|\xi \rangle = \sum_{i \in \mathbb{N}} \langle x\xi, x|\xi \rangle - \langle \xi, x\xi \rangle \langle x_i, x|\xi \rangle - \langle \xi, x\xi \rangle \langle x_i, x|\xi \rangle$$

$$\leq B\langle x\xi, x|\xi \rangle - \langle \xi, x\xi \rangle x\xi - \langle \xi, x\xi \rangle$$

$$\leq B\langle \xi, \xi \rangle \left( \langle x, x\xi \rangle - \langle \xi, x\xi \rangle \right)$$

$$= B\langle \xi, \xi \rangle \left( \langle x, x\xi \rangle \right) \leq ||B\langle \xi, \xi \rangle|| \left( \langle x, x|\xi \rangle \right)$$

Take $D = ||B\langle \xi, \xi \rangle||$, the argument for lower bound is similar. \qed
In the following proposition, $E$ is a Hilbert $A$-module in which every closed submodule is orthogonally complemented and $\langle \xi, \xi \rangle$ is invertible and $L_\xi$ is the subspace generated with $\xi$.

**Proposition 2.3.** Let $A$ be a commutative and $(E, \langle ., . \rangle)$ be a Hilbert $A$-module and $\xi \in E$. Every $A$-2-frame associated with $\xi$ is a frame for $L_\xi^\perp$.

*Proof.*

\[
\sum_{i \in \mathbb{N}} \langle x \xi, x \xi \rangle - \langle \xi, x \rangle \langle x, x \rangle - \langle x, x \rangle \langle x \xi, x \xi \rangle - \langle \xi, x \rangle \langle x, x \rangle \xi, x \rangle \leq B \langle x \xi, x \xi \rangle \langle x, x \rangle - \langle x, x \rangle \langle x \xi, x \xi \rangle - \langle \xi, x \rangle \langle x, x \rangle \xi, x \rangle \leq B \langle x \xi, x \xi \rangle \langle x, x \rangle - \langle x, x \rangle \langle x \xi, x \xi \rangle - \langle \xi, x \rangle \langle x, x \rangle \xi \rangle \]

Then

\[
A \langle x, \xi \rangle \langle x, \xi \rangle^2 \leq \langle x, x \rangle \langle x, x \rangle \langle x, x \rangle \langle x, x \rangle \leq B \langle x, \xi \rangle ^2 \langle x, x \rangle \quad (x \in L_\xi^\perp)
\]

Since $\langle \xi, \xi \rangle$ is invertible, the proof is completed. \qed

Let $(E, \langle ., . \rangle)$ be an $A$-2-Hilbert space and $L_\xi$ be the subspace generated with $\xi$ for a fix element $\xi$ in $E$. Denote by $M_\xi$ the algebraic complement of $L_\xi$ in $E$. So $L_\xi \oplus M_\xi = E$. We define the semi-inner product $\langle ., . \rangle_\xi$ on $E$ as following

\[
\langle x, z \rangle_\xi = \langle x, z \rangle |\xi\rangle.
\]

This semi-inner product induces an inner product on the quotient space $E/L_\xi$ as

\[
\langle x + L_\xi, z + L_\xi \rangle_\xi = \langle x, z \rangle_\xi, \quad (z, x \in E).
\]

By identifying $E/L_\xi$ with $M_\xi$ in an obvious way, we obtain an inner product on $M_\xi$.

Now if $\{x_i\}_{i \in \mathbb{N}} \subseteq E$ is an $A$-2- frame associated with $\xi$ with bounds $A$ and $B$, we can rewrite(2.1) as

\[
A \langle x, x \rangle_\xi \leq \sum_{i \in \mathbb{N}} \langle x, x_i \rangle \langle x, x \rangle \langle x, x \rangle \langle x, x \rangle \leq B \langle x, x \rangle_\xi \quad (x \in M_\xi).
\]

That is, $\{x_i\}_{i \in \mathbb{N}}$ is a frame for $M_\xi$. Let $E_\xi$ be the completion of the inner product space $E_\xi$, then the sequence $\{x_i\}_{i \in \mathbb{N}}$ is also a frame for $E_\xi$. To summarize, we have the following theorem.
Theorem 2.4. Let \((E, \langle \cdot, \cdot \rangle)\) be an \(A\)-2-Hilbert space. Then \(\{x_i\}_{i \in \mathbb{N}} \subseteq E\) is an \(A\)-2-frame associated with \(\xi\) if and only if it is a frame for the Hilbert space \(E_\xi\).

Lemma 2.5. Let \(\{x_i\}_{i \in \mathbb{N}}\) be an \(A\)-2-Bessel sequence in \(E\). Then the \(A\)-2-pre frame operator \(T : l^2(A) \to E_\xi\) defined by

\[ T_\xi \{c_i\} = \sum_{i \in \mathbb{N}} c_i x_i \]

is well-defined and bounded.

Proof.

\[
\left\| \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{m} c_i x_i, \xi \right\|^2 = \sup \{ \left\| \sum_{i=m+1}^{n} c_i y, \xi \right\|, y \in E_\xi, \|y, \xi\| = 1 \}
\]

\[
\leq \sup \{ \left\| \sum_{i=m+1}^{n} \langle x, y \rangle, \xi \right\|, y \in E_\xi, \|y, \xi\| = 1 \} \left\| \sum_{i=m+1}^{n} c_i c_i^* \right\|
\]

\[
\leq \left\| \sum_{i=m+1}^{n} c_i c_i^* \right\| \|B\|.
\]

Since \(B\) is the upper bound of \(\{x_i\}\) this implies that \(\sum_{i=1}^{\infty} c_i x_i\) is well defined as an element of \(E_\xi\). Moreover if \(\{c_i\}\) is a sequence in \(l^2(A)\), then an argument as above shows that

\[
\left\| T_\xi \{c_i\} \right\| \leq \sqrt{\|B\|} \left\| \sum_{i \in \mathbb{N}} c_i c_i^* \right\|
\]

In particular, \(\|T_\xi\| \leq \sqrt{\|B\|}\).

We have \(\langle x, T(c_j) \rangle = \langle x, T(c_j) \rangle_\xi = \langle x, \sum c_j x_j \rangle_\xi = \sum \langle x, x_j \rangle_\xi c_j^* = \langle \langle x, x_j \rangle_\xi, c_j \rangle\) Next, we can compute \(T_\xi^*\), the adjoint of \(T_\xi\) as

\[ T_\xi^* : E_\xi \to l^2(A); \quad T_\xi^* x = \{ \langle x, x_i \rangle_\xi \}_{i \in \mathbb{N}}. \]

\(T_\xi^*\) is well-defined and bounded, because

\[
\left\| T_\xi^* (x) \right\|^2 = \left\| \{ \langle x, x_i, \xi \rangle \}_{i \in \mathbb{N}} \right\|^2 = \left\| \sum_{i \in \mathbb{N}} \langle x, x_i, \xi \rangle \langle x, x_i \rangle_\xi \right\| \leq \|B\| \|x, \xi\|
\]

That implies \(\|T_\xi^*\| \leq \sqrt{\|B\|}\).
Definition 2.6. Let \( \{x_i\}_{i \in \mathbb{N}} \) be an \( \mathcal{A} \)-2-frame associated to \( \xi \) with bounds \( A \) and \( B \) in an \( \mathcal{A} \)-2-Hilbert space \( E \). The operator \( S_\xi : E_\xi \to E_\xi \) defined by
\[
S_\xi x = \sum_{i \in \mathbb{N}} \langle x, x_i | \xi \rangle x_i.
\]
is called the \( \mathcal{A} \)-2-frame operator for \( \{x_i\}_{i \in \mathbb{N}} \).

In the next theorem, we investigate some properties of \( S_\xi \).

Theorem 2.7. Let \( \{x_i\}_{i \in \mathbb{N}} \) be an \( \mathcal{A} \)-2-frame associated to \( \xi \) for an \( \mathcal{A} \)-2-Hilbert space \( (E, \langle .,. \rangle) \) with \( \mathcal{A} \)-2-frame operator \( S_\xi \). Then \( S_\xi \) is bounded, invertible, self-adjoint, and positive.

Proof. It is clear that \( S_\xi = T_\xi T_\xi^* \) is self adjoint and
\[
\|S_\xi\| = \|T_\xi T_\xi^*\| = \|T_\xi\|^2 \leq \|B\|.
\]
We can conclude the boundedness of \( S_\xi \) directly
\[
\|S_\xi(x), \xi\|^2 = \sup\{\|\langle S_\xi(x), y | \xi \rangle\|^2, y \in E_\xi, \|y, \xi\| = 1\}
\]
\[
= \sup\{\|\sum_{i=1}^{\infty} \langle x, x_i | \xi \rangle x_i, y | \xi \rangle\|^2, y \in E_\xi, \|y, \xi\| = 1\}
\]
\[
\leq \sup\{\|\sum_{i=1}^{\infty} \langle x, x_i | \xi \rangle \langle x, x_i | \xi \rangle\| \| \sum_{i=1}^{\infty} \langle y, x_i | \xi \rangle \langle x, x_i | \xi \rangle\|, \|y, \xi\| = 1\}
\]
\[
\leq \|B\|^2 \|\langle x, x | \xi \rangle\|
\]
The inequality (2.1) means that
\[
A\langle x, x \rangle_\xi \leq \langle S_\xi(x), x \rangle_\xi \leq B\langle x, x \rangle_\xi
\]
\( S_\xi \) is a positive element in the set of all bounded operators on the Hilbert space \( E_\xi \). □

By the definition of \( S_\xi \) we get the following results.

Corollary 2.8. Let \( \{x_i\}_{i \in \mathbb{N}} \) be an \( \mathcal{A} \)-2-frame in an \( \mathcal{A} \)-2-Hilbert space \( (E, \langle .,. \rangle) \) with frame operator \( S_\xi \). Then each \( x \in E_\xi \) has an expansion of the following
\[
x = SS_\xi^{-1}x = \sum_{i \in \mathbb{N}} \langle S_\xi^{-1}x, x_i | \xi \rangle x_i.
\]
Corollary 2.9. Let $\xi$ and $\eta$ be $\mathcal{A}$-independent and $\{x_i\}_{i \in \mathbb{N}}$ be an $\mathcal{A}$-2-frame associated with $\xi$ and $\eta$, and for $x \in E_\xi \cap E_\eta$, the operators $S_\xi, E_\eta : E_\xi \cap E_\eta \rightarrow E_\xi \cap E_\eta$ defined by,
\[ S_\xi x = \sum_{i \in \mathbb{N}} \langle x, x_i \mid \xi \rangle x_i, \]
\[ S_\eta x = \sum_{i \in \mathbb{N}} \langle x, x_i \mid \eta \rangle x_i. \]

Then we have $\langle S_\eta x, x \mid \xi \rangle = \langle S_\xi x, x \mid \eta \rangle^*$. 

Proof.
\[ \langle S_\eta x, x \mid \xi \rangle = \left( \sum_{i \in \mathbb{N}} \langle x, x_i \mid \eta \rangle x_i, x \mid \xi \rangle \right) \]
\[ = \sum_{i \in \mathbb{N}} \langle x, x_i \mid \eta \rangle \langle x_i, x \mid \xi \rangle \]
\[ = \left( \sum_{i \in \mathbb{N}} \langle x, x_i \mid \xi \rangle \langle x_i, x \mid \eta \rangle \right)^* \]
\[ = \left( \sum_{i \in \mathbb{N}} \langle x, x_i \mid \xi \rangle x_i, x \mid \eta \rangle \right)^* \]
\[ = \langle S_\xi x, x \mid \eta \rangle^*. \]

\[ \square \]

3. Tensor product of $\mathcal{A}$-2-Frames

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras, $E$ an $\mathcal{A}$-2-Hilbert space and $F$ be a $\mathcal{B}$-2-Hilbert space. We take $\mathcal{A} \otimes \mathcal{B}$ as the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with the spatial norm. Hence $\mathcal{A} \otimes \mathcal{B}$ is a $C^*$-algebra and for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have $\|a \otimes b\| = \|a\| \|b\|$. The algebraic tensor product $E \otimes_{alg} F$ is a pre-Hilbert $\mathcal{A} \otimes \mathcal{B}$-module with module action
\[ (a \otimes b)(x \otimes y) = ax \otimes by \quad (a \in \mathcal{A}, b \in \mathcal{B}, x \in E, y \in F) \]
and $\mathcal{A} \otimes \mathcal{B}$-valued 2-inner product
\[ \langle x_1 \otimes y_1, x_2 \otimes y_2 \mid \xi \otimes \eta \rangle = \langle x_1, x_2 \mid \xi \rangle \otimes \langle y_1, y_2 \mid \eta \rangle \quad (x_1, x_2, \xi \in E, y_1, y_2, \eta \in F). \]
The $\mathcal{A} \otimes \mathcal{B}$-2-norm on $E \otimes F$ is defined by
\[ \|x_1 \otimes x_2, y_1 \otimes y_2\|_{\mathcal{A} \otimes \mathcal{B}} = \|x_1, y_1\|_{\mathcal{A}} \|x_2, y_2\|_{\mathcal{B}} \quad (x_1, y_1 \in E, x_2, y_2 \in F). \]
where $\|., .\|_{\mathcal{A}}$ and $\|., .\|_{\mathcal{B}}$ are norms generated by $\langle ., . \rangle_{\mathcal{A}}$ and $\langle ., . \rangle_{\mathcal{B}}$ respectively. The space $E \otimes F$ is complete with the above 2-inner product. Therefore, the space $E \otimes F$ is
an \( A \otimes B \)-2-Hilbert space.
The following definition is the extension of (2.1) to the sequence \( \{ x_i \otimes y_i \}_{i \in \mathbb{N}} \).

**Definition 3.1.** Let \( \{ x_i \} \) and \( \{ y_i \} \) be two sequences in \( A \)-2-Hilbert space \( E \) and \( B \)-2-Hilbert space \( F \), respectively. Then, the sequence \( \{ x_i \otimes y_i \}_{i \in \mathbb{N}} \) is said to be a tensor product of \( A \otimes B \)-2-frame for the tensor product of \( A \otimes B \)-2-Hilbert space \( E \otimes F \) associated to \( \xi \otimes \eta \) if there exist two constants \( 0 < A \leq B < \infty \) such that

\[
A \langle x \otimes y, x \otimes y | \xi \otimes \eta \rangle \leq \sum_{i,j} \langle x \otimes y, x_i \otimes y_j | \xi \otimes \eta \rangle \langle x_i \otimes y_j, x \otimes y | \xi \otimes \eta \rangle \leq B \langle x \otimes y, x \otimes y | \xi \otimes \eta \rangle 
\]

for all \( x \in E, y \in F \).

The numbers \( A \) and \( B \) are called lower and upper frame bounds of the tensor product of \( A \otimes B \)-2-frame respectively.

**Lemma 3.2.** Let \( \{ x_i \}_{i \in I} \) be an \( A \)-2-frame for \( E \) with frame bounds \( A \) and \( B \) associated to \( \xi \) and let \( \{ y_j \}_{j \in J} \) be a \( B \)-2-frame for \( F \) with frame bounds \( C \) and \( D \) associated with \( \eta \). Then \( \{ x_i \otimes y_j \}_{i \in I, j \in J} \) is an \( A \otimes B \)-2-frame for \( E \otimes F \) with frame bounds \( AC \) and \( BD \), if \( \{ x_i \}_{i \in I} \) and \( \{ y_j \}_{j \in J} \) are tight or Parseval frames, then so is \( \{ x_i \otimes y_j \}_{i \in I, j \in J} \).

**Proof.** Let \( x \in E \) and \( y \in F \). Then we have

\[
A \langle x, x | \xi \rangle \leq \sum_{i \in I} \langle x, x_i | \xi \rangle \langle x_i, x | \xi \rangle \leq B \langle x, x | \xi \rangle \quad (3.1)
\]

\[
C \langle y, y | \eta \rangle \leq \sum_{j \in J} \langle y, y_i | \eta \rangle \langle y_i, y | \eta \rangle \leq B \langle y, y | \eta \rangle. \quad (3.2)
\]

We know if \( a, b \) are hermitian elements of \( A \) and \( a \leq b \), then for every positive element \( x \) of \( B \), we have \( a \otimes x \leq b \otimes x \). Therefore

\[
A \langle x, x | \xi \rangle \otimes \langle y, y | \eta \rangle \leq \sum_{i \in I} \langle x, x_i | \xi \rangle \langle x_i, x | \xi \rangle \otimes \langle y, y_i | \eta \rangle \langle y_i, y | \eta \rangle \leq B \langle x, x | \xi \rangle \otimes \langle y, y | \eta \rangle.
\]

Now by (3.2), we have

\[
AC \langle x, x | \xi \rangle \otimes \langle y, y | \eta \rangle \leq \sum_{i \in I} \sum_{j \in J} \langle x, x_i | \xi \rangle \langle x_i, x | \xi \rangle \otimes \langle y, y_i | \eta \rangle \langle y_i, y | \eta \rangle \leq B \langle x, x | \xi \rangle \otimes \sum_{j \in J} \langle y, y_i | \eta \rangle \langle y_i, y | \eta \rangle \leq BD \langle x, x | \xi \rangle \otimes \langle y, y | \eta \rangle.
\]
Consequently, we have
\[
AC\langle x \otimes y, x \otimes y | \xi \otimes \eta \rangle \leq \sum_{i,j} \langle x \otimes y, x_i \otimes y_j | \xi \otimes \eta \rangle \langle x_i \otimes y_j, x \otimes y | \xi \otimes \eta \rangle \\
\leq BD\langle x \otimes y, x \otimes y | \xi \otimes \eta \rangle.
\]
From these inequalities, it follows that for all 
\[
z = \sum_{k=1}^n u_k \otimes v_k \in E \otimes_{alg} F
\]
\[
AC\langle z, z | \xi \otimes \eta \rangle \leq \sum_{i,j} \langle z, z | \xi \otimes \eta \rangle \langle x_i \otimes y_j, z | \xi \otimes \eta \rangle \\
\leq BD\langle z, z | \xi \otimes \eta \rangle.
\]
Hence it holds for all \(z \in E \otimes F\). \(\square\)

If take \(A = C\) and define \(C \otimes B\)-2-product by
\[
\langle x_1 \otimes y_1, x_2 \otimes y_2 | \xi \otimes \eta \rangle = \langle x_1, x_2 | \xi \rangle \langle y_1, y_2 | \eta \rangle \quad (x_1, x_2, \xi, y_1, y_2, \eta \in F).
\] (3.3)

We have the following proposition.

**Proposition 3.3.** Let \(\{x_i \otimes y_i\}_{i \in I, j \in J}\) be a \(C \otimes B\)-2-frame for \(E \otimes F\) associated with \(\xi \otimes \eta\), then \(\{y_i\}_{j \in J}\) is a \(B\)-2-frame for \(F\) associated with \(\eta\).

**Proof.** Suppose that \(\{x_i \otimes y_i\}_{i \in I, j \in J}\) is a \(C \otimes B\)-2-frame for \(E \otimes F\) associated to \(\xi \otimes \eta\). Then for \(x \otimes y \in E \otimes F - \{0 \otimes 0\}\), we have
\[
A\langle x \otimes y, x \otimes y | \xi \otimes \eta \rangle \leq \sum_{i,j} \langle x \otimes y, x_i \otimes y_j | \xi \otimes \eta \rangle \langle x_i \otimes y_j, x \otimes y | \xi \otimes \eta \rangle \leq B\langle x \otimes y, x \otimes y | \xi \otimes \eta \rangle.
\]
By using (3.3) the above equation becomes
\[
A\langle x, x | \xi \rangle \langle y, y | \eta \rangle \leq \sum_{i \in I} |\langle x, x_i | \xi \rangle|^2 \sum_{j \in J} \langle y, y_j | \eta \rangle \langle y_j, y | \eta \rangle \\
\leq B\langle x, x | \xi \rangle \langle y, y | \eta \rangle.
\]
This gives
\[
\frac{A\langle x, x | \xi \rangle}{\sum_{i \in I} |\langle x, x_i | \xi \rangle|^2} \langle y, y | \eta \rangle \leq \sum_{j \in J} \langle y, y_j | \eta \rangle \langle y_j, y | \eta \rangle \\
\leq \frac{B\langle x, x | \xi \rangle}{\sum_{i \in I} |\langle x, x_i | \xi \rangle|^2} \langle y, y | \eta \rangle.
\]
Therefore

\[ A_1(y, y|\eta) \leq \sum_{j \in J} \langle y, y_j|\eta \rangle \langle y_j, y|\eta \rangle \leq B_1(y, y|\eta). \quad (\forall y \in F) \]

Where \( A_1 = \frac{A(x,x|\xi)}{\sum_{i \in I} |(x,x_i|\xi)|^2} \) and \( B_1 = \frac{B(x,x|\xi)}{\sum_{i \in I} |(x,x_i|\xi)|^2} \).

**Remark 3.4.** If the sequences \( \{x_i\}_{i \in I}, \{y_j\}_{j \in J} \) and \( \{x_i \otimes y_j\}_{i \in I, j \in J} \) are 2-frames for Hilbert spaces \( E_\xi, F_\eta \) and \( (E \otimes F)_{\xi \otimes \eta} \) respectively and \( S_\xi, S_\eta \) and \( S_{x_i \otimes y_j} \) are frame operators of above frames respectively, then from (2.6), we have the following

\[ S_\xi x = \sum_{i \in I} \langle x, x_i|\xi \rangle x_i, S_\eta y = \sum_{j \in J} \langle y, y_j|\eta \rangle y_j. \]

\[ S_{\xi \otimes \eta}(x \otimes y) = \sum_{i \in I, j \in J} \langle x \otimes y, x_i \otimes y_j|\xi \otimes \eta \rangle (x_i \otimes y_j), \quad (x \in E, y \in F, x \otimes y \in E \otimes F) \]

**Theorem 3.5.** If the sequences \( \{x_i\}_{i \in I}, \{y_j\}_{j \in J} \) and \( \{x_i \otimes y_j\}_{i \in I, j \in J} \) are 2-frames for Hilbert spaces \( E_\xi, F_\eta \) and \( (E \otimes F)_{\xi \otimes \eta} \), respectively and \( S_\xi, S_\eta \) and \( S_{\xi \otimes \eta} \) are frame operators respectively, then \( S_{\xi \otimes \eta} = S_\xi \otimes S_\eta \).

**Proof.** For \( x \otimes y \in E \otimes F \), we have

\[
S_{\xi \otimes \eta}(x \otimes y) = \sum_{i \in I, j \in J} \langle x \otimes y, x_i \otimes y_j|\xi \otimes \eta \rangle (x_i \otimes y_j)
= \sum_{i \in I, j \in J} \langle x, x_i|\xi \rangle A(y, y_j|\eta) B(x_i \otimes y_j)
= \sum_{i \in I} \langle x, x_i|\xi \rangle A x_i \otimes \sum_{j \in J} \langle y, y_j|\eta \rangle B y_i
= S_\xi x \otimes S_\eta y = (S_\xi \otimes S_\eta)(x \otimes y).
\]

Hence \( S_{\xi \otimes \eta} = S_\xi \otimes S_\eta \). \qed

**Lemma 3.6.** Suppose that \( \{x_i\}_{i \in I} \) is a sequence in \( A \)-2-Hilbert space \( E \), with \( x = \sum_{i \in I} \langle x, x_i|\xi \rangle x_i \) holds for all \( x \in E \), then \( \{x_i\}_{i \in I} \) is an \( A \)-2-normalized tight frame for \( E \).

**Proof.** \( x = \sum_{i \in I} \langle x, x_i|\xi \rangle x_i \), so \( \langle x, x|\xi \rangle = \langle \sum_{i \in I} \langle x, x_i|\xi \rangle x_i, x|\xi \rangle \), hence \( \Rightarrow \langle x, x|\xi \rangle = \sum_{i \in I} \langle x, x_i|\xi \rangle \langle x_i, x|\xi \rangle \). \( \qed \)
Corollary 3.7. Assume that \( \{x_i \otimes y_j\}_{i \in I, j \in J} \) is a sequence in \( \mathcal{A} \otimes \mathcal{B} \)-2-Hilbert space \( E \otimes F \) and \( x \otimes y = \sum_{i \in I, j \in J} \langle x \otimes y, x_i \otimes y_j \rangle \xi \otimes \eta (x_i \otimes y_j) \), for all \( x \in E, y \in F \). Then \( \{x_i \otimes y_j\}_{i \in I, j \in J} \) is an \( \mathcal{A} \otimes \mathcal{B} \)-2-normalized tight frame for \( E \otimes F \).

Proof. By lemma (3.6) is clear. □

References

[1] A. Arefijamaal and G. Sadeghi, Frames in 2-Inner Product Spaces, Iranian Journal of Mathematical Sciences and Informatics, 8 (2013), 123-130.
[2] Y. J. Cho, Paul C. S. Lin, S. S. Kim and A. Misiak, Theory of 2-inner product spaces, Nova Science Publishers, Inc. New York, 2001.
[3] J. Duffin, A.C. Schaefer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., 72 (1952), 341-366.
[4] M. Frank and D. R. Larson, A module frame concept for Hilbert C*-modules, Functional and Harmonic Analysis of Wavelets (San Antonio, TX, Jan. 1999), Contemp. Math., 247 (2000), 207-233.
[5] S. Gahler, Lineare 2-normierte Räume, Math. Nachr., 28 (1965), 1-43.
[6] W. Jing, Frames in Hilbert C*-modules, Ph.D. Thesis, University of Central Florida, 2006.
[7] T. Mehdiabad Mahchari and A. Nazari, 2-Hilbert C*-modules and some Gruss type inequalities in \( \mathcal{A} \)-2-inner product space, Math. Inequal. Appl., 18 (2) (2015), 721-754.
[8] B. Mohebbi Najmabadi and T.L. Shateri, 2-inner product which takes values on a locally C*-algebra, Indian J. Math. Soc., 85(1–2) (2018), 217–225

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