Outer billiards on the manifolds of oriented geodesics of the three dimensional space forms

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Abstract

For \( \kappa = 0, 1, -1 \), let \( M_\kappa \) be the three dimensional space form of curvature \( \kappa \), that is, \( \mathbb{R}^3 \), \( S^3 \) and hyperbolic 3-space \( H^3 \). Let \( G_\kappa \) be the manifold of all oriented (unparametrized) complete geodesics of \( M_\kappa \), i.e., \( G_0 \) and \( G_{-1} \) consist of oriented lines and \( G_1 \) of oriented great circles.

Given a strictly convex surface \( S \) of \( M_\kappa \), we define an outer billiard map \( B_\kappa \) on \( G_\kappa \). The billiard table is the set of all oriented geodesics not intersecting \( S \), whose boundary can be naturally identified with the unit tangent bundle of \( S \). We show that \( B_\kappa \) is a diffeomorphism under the stronger condition that \( S \) is quadratically convex.

We prove that \( B_1 \) and \( B_{-1} \) arise in the same manner as Tabachnikov’s original construction of the higher dimensional outer billiard on standard symplectic space \( (\mathbb{R}^{2n}, \omega) \). For that, of the two canonical Kähler structures that each of the manifolds \( G_1 \) and \( G_{-1} \) admits, we consider the one induced by the Killing form of \( \text{Iso}(M_\kappa) \). We prove that \( B_1 \) and \( B_{-1} \) are symplectomorphisms with respect to the corresponding fundamental symplectic forms. Also, we discuss a notion of holonomy for periodic points of \( B_{-1} \).

Key words and phrases: outer billiards, space of oriented geodesics, symplectomorphism, Jacobi field, holonomy

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1 Introduction

The dual or outer billiard map \( B \) is defined in the plane as a counterpart to the usual inner billiards. Let \( \gamma \subset \mathbb{R}^2 \) be a smooth, closed, strictly convex curve, and let \( p \) be a point outside of \( \gamma \). There are two tangent lines to \( \gamma \) through \( p \); choose one of them consistently, say, the right one from the viewpoint of \( p \), and define \( B(p) \) as the reflection of \( p \) in the point of tangency (see Figure 1).

The study of the dual billiard was originally popularized by Moser [15, 16], who considered the dual billiard map as a crude model for planetary motion and showed

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that orbits of the map cannot escape to infinity. The outer billiard map has since been studied in a number of settings; see [4, 20, 21, 23] for surveys.

The goal of the present article is to define and study an outer billiard system in another setting: on the space of oriented geodesics of the three-dimensional space form $M_\kappa$ of constant curvature $\kappa = 0, -1$ or $1$; that is, Euclidean space $\mathbb{R}^3$, hyperbolic space $H^3$, or the sphere $S^3$, respectively.

Let $G_\kappa$ be the space of all oriented geodesics of $M_\kappa$. This is a four dimensional manifold whose elements are the oriented trajectories of complete geodesics in $M_\kappa$, which can also be described as equivalence classes of unit speed geodesics $[\gamma]$, where $\gamma \sim \sigma$ if $\sigma(t) = \gamma(t + t_0)$ for some $t_0 \in \mathbb{R}$.

For convenience, if $\kappa = 0, -1$, we sometimes call $G_\kappa = L_\kappa$, the space of oriented lines of Euclidean or hyperbolic space (which are diffeomorphic to $TS^2$) and $G_1 = \mathcal{C}$, the space of oriented great circles of the three sphere (diffeomorphic to $S^2 \times S^2$).

Suppose first that $\kappa = 0$ or $-1$. Let $S$ be a strictly convex compact surface in $M_\kappa$, that is, for each $p \in S$, the complete totally geodesic surface tangent to $p$ intersects $S$ only at $p$. Let

$$U = L_\kappa - \{\text{oriented lines intersecting } S\}.$$

The space $U$ will serve as the outer billiard table; note that it is an open submanifold of $L_\kappa$ with boundary equal to the three-dimensional space of lines which are tangent to $S$.

The outer billiard map on $L_\kappa$ associated with $S$ is the map $B : U \to U$ defined as follows: Suppose that $\ell \in L_\kappa$ does not intersect $S$. There exist exactly two complete totally geodesic surfaces $P_+$ and $P_-$ tangent to $S$ containing $\ell$. The orientation of $\ell$ allows us to choose the surface on the right, say $P_+$. The orientation of $\ell$ allows us to choose the surface on the right, say $P_+$.
Now, let $\gamma_+ \subset \mathcal{L}_\kappa$ be the geodesic ray joining $q_+$ with $p_+$, with $\gamma_+(0) = q_+$ and $\gamma_+(d) = p_+$. Then $B(\ell)$ is the oriented line obtained by parallel transporting $\ell$ along $\gamma_+$ between 0 and $2d$. Observe that $B(\ell)$ is also in $P_+$.

Now we explain the choice between the two surfaces $P_-$ and $P_+$. Define $p_-$, $q_-$ and $\gamma_-$ analogously. Suppose that $\ell = [\gamma]$ with $\gamma (t_j) = q_j$ ($j = \pm$) and let $W_j$ be the parallel vector field along $\gamma$ with $W_j (t_j) = \gamma_j' (0)$. Then choose the sign $j$ in such a manner that $\{W_+, W_-, \gamma'\}$ is a positively oriented frame along $\gamma$.

![Figure 3: The outer billiard map on $\mathcal{L}_\kappa$ associated with $S$](image)

Notice that in the hyperbolic case (in contrast with the Euclidean), parallel translating a line $\ell$ a distance $d$ along unit speed geodesic rays $\gamma_1$ and $\gamma_2$ orthogonal to $\ell$ depends on the initial points $\gamma_1 (0)$ and $\gamma_2 (0)$ in $\ell = [\gamma]$, even if $\gamma_2' (0)$ is the parallel transport of $\gamma_1' (0)$ and along $\gamma$.

Given a strictly convex compact surface $S$ in the three sphere one can define the **outer billiard map on $\mathcal{C}$ associated with $S$** in a similar way as for the Euclidean and hyperbolic cases. Only the domain of definition must be further reduced. We call

$$\mathcal{C}_{\text{cut}}(S) = \{\text{oriented great circles intersecting } S\}$$

and $\mathcal{C}_{\text{Gauss}}(S)$ the Gauss image of the surface $S$, that is,

$$\mathcal{C}_{\text{Gauss}}(S) = \{\text{oriented great circles contained in the tangent spaces to } S\}$$

(recall that the Gauss map of $S$, taking values in the Grassmannian of oriented subspaces of dimension two of $\mathbb{R}^4$, assigns the subspace $T_p S \subset \mathbb{R}^4$ to each point $p \in S$).

**Proposition 1** Let $S$ be a strictly convex compact surface in $S^3$. The analogue of the outer billiard map associated with $S$ above is well defined on $(\mathcal{C} - \mathcal{C}_{\text{Gauss}}(S)) - \mathcal{C}_{\text{cut}}(S)$ and is a bijection onto this set.

As a first step towards the study of the billiards above, it would be convenient to put them in a more general setting: We present a general definition of outer billiard map which includes all the billiard maps we know where the boundary of the billiard table is smooth.
Definition 2 Let \( L \) be a smooth manifold and let \( U \) be a connected open set in \( L \) whose boundary is an embedded hypersurface \( M \) of \( L \).

Let \( T \in (0, \infty] \) and let \( F : M \times (-T, T) \rightarrow L \) be a smooth function such that:

a) For each \( p \in M \), the curve \( \alpha_p : (-T, T) \rightarrow L \) defined by \( \alpha_p(t) = F(p, t) \) satisfies that \( \alpha_p(0) = p \) and \( \alpha'_p(0) \in T_pM \).

b) The restrictions \( F_+, F_- \) of \( F \) to \( M \times (0, T) \) and \( M \times (-T, 0) \), respectively, are both diffeomorphisms onto \( U \).

Then we say that the triple \((F, U, M)\) determines the outer billiard map \( B : U \rightarrow U \) defined by \( B(q) = F_+(p, t) \), provided that \( F_-(p, -t) = q \ (t > 0) \).

Although for most examples \( L \) is actually a (pseudo-)Riemannian manifold and the curves \( \alpha_p \) are geodesics, we prefer to give the above more general definition of outer billiard, since for instance in the main example of this note it is more direct to give the tangent curves (which are defined naturally) rather than their initial velocities, with the additional advantage that we do not need to consider the (pseudo-)Riemannian structure on \( L \) from the beginning.

Examples. 1) The simplest outer billiard map, that is, the one induced by a smooth closed embedded plane curve \( \gamma \) with nowhere vanishing curvature, satisfies the above definition with \( L = \mathbb{R}^2 \), \( M \) the image of \( \gamma \) and \( U \) its exterior, \( T = \infty \) and \( \alpha_{\gamma(s)}(t) = \gamma(s) + t\gamma'(s) \) for all \( s, t \).

Higher dimensional outer billiards as in [22] also can be set in this framework, see equation (3) below.

2) Theorem 3 below states that the outer billiard map on \( \mathcal{L}_0, \mathcal{L}_{-1} \) or \( \mathcal{C} \) associated with a strictly convex compact surface \( S \) presented above satisfies the conditions of Definition 2, provided that \( S \) is quadratically convex, that is, the shape operator at any point of \( S \) is definite. For instance, for the outer billiard map on \( \mathcal{C} \) associated with a surface \( S \), the manifold \( L \) is \( \mathcal{C} - \mathcal{C}_{\text{Gauss}(S)} \), \( U = L - \mathcal{C}_{\text{cut}(S)} \) and \( T = \pi/2 \).

3) Following Tabachnikov’s article [22, page 235], the definition covers also the case of inner billiards for a compact quadratically convex hypersurface \( S \) in \( \mathbb{R}^m \): Let \( n : S \rightarrow S^{m-1} \) denote the inward pointing unit normal vector field. Let

\[
\mathcal{L}_{\text{normal}(S)} = \{ [t \mapsto p + \delta t n(p)] \mid p \in S, \delta = \pm 1 \}
\]

and \( L = \mathcal{L}_0 - \mathcal{L}_{\text{normal}(S)} \). Let \( U = \{ \ell \in L \mid \ell \text{ intersects the interior of } S \} \), whose boundary in \( L \) is the set \( M \) of oriented lines tangent to \( S \), which may be identified with \( T^1S \).

For \( v \in T_pS \) define the curve \( c_v : (-\pi/2, \pi/2) \rightarrow T^1\mathbb{R}^m \) by

\[
c_v(t) = (\cos t)v + (\sin t)n(p).
\]

Let \( F : T^1S \times (-\pi/2, \pi/2) \rightarrow L \) be given by \( F(v, t) = [\gamma_{c_v(t)}] \). Then the triple \((F, U, T^1S)\) determines an outer billiard map which coincides with the inner billiard map with table the interior of \( S \) (except on \( \mathcal{L}_{\text{normal}(S)} \), on which the latter acts simply by changing the orientation). Observe that \( (\alpha_v)'(0) \in T_{[\gamma_v]}M \) by Lemma 14 below, since the corresponding Jacobi field along \( \gamma_v \) vanishes at \( s = 0 \).

We comment that by means of the free \( \mathbb{Z}_2 \)-action \(* \) on \( T^1S \times (0, \pi) \) defined by \( \varepsilon * (v, t) = (-v, \pi - t) \ (\varepsilon = \pm 1) \) and a similar one on \( T^1S \times (-\pi, 0) \) one can make the
inner billiard almost fit into Definition 2, without having to omit the normal lines to S.

The next theorem states that, under the stronger condition that S is quadratically convex, the outer billiard maps on G_κ associated with S may also be written in the language of Definition 2. This encompasses the result that they are smooth (and in particular, diffeomorphisms).

On a strictly convex surface S in M_κ we consider the complex structure i defined by \( i_p(u) = n(p) \times u \) for \( p \in S \), where \( n : S \to TM_κ \) is the unit normal vector field pointing inwards and \( \times \) is the cross product associated with the canonical orientation of M_κ. By abuse of notation we will write \( iu \) omitting the foot point.

Given \( v \in TM_κ \), we denote by \( \gamma_v \) the unique geodesic in M_κ with initial velocity \( v \).

**Theorem 3** Suppose that \( \kappa = 0 \) or \(-1\), that is, \( M_κ \) is Euclidean or hyperbolic space, and let \( S \) be a quadratically convex compact surface of \( M_κ \). Let

\[
M = \{ \text{oriented lines tangent to } S \} \cong T^1S
\]

and

\[
U = \{ \text{oriented lines which do not intersect } S \} .
\]

For each \( u \in T^1S \) and \( t \in \mathbb{R} \), let \( u_t \in T^1M_κ \) be the parallel transport of \( u \) between 0 and \( t \) along \( \gamma_{iu} \). Let

\[
F : M \times \mathbb{R} \cong T^1S \times \mathbb{R} \to L_κ, \quad F(u, t) = [\gamma_{iu}].
\]

Then \((F,U,M)\) determines an outer billiard map that coincides with the outer billiard map on \( L_κ \) associated with \( S \).

The analogous statement for the outer billiard map on \( C \) associated with a surface is true.

The choice of \( M \) as the boundary of the billiard table was inspired by [7].

In [22] (see also [5, 23]), Tabachnikov generalized planar outer billiards (Figure 1) to even-dimensional standard symplectic space \((\mathbb{R}^{2n}, \omega)\) as follows: given a smooth quadratically convex compact hypersurface \( M \) in \( \mathbb{R}^{2n} \), the restriction of \( \omega \) to each tangent space \( T_qM \) has a 1-dimensional kernel, called the characteristic line. Tabachnikov showed that for each point \( p \) outside of \( M \), exactly two characteristic lines of \( M \) pass through \( p \), so that a choice of orientation induces a well-defined outer billiard map. Moreover, he proved that the map is symplectic.

In the phrasing of Definition 2, the billiard above is the one determined by \((F,U,M)\), where \( U \) is the exterior of \( M \) in \( \mathbb{R}^{2n} \cong \mathbb{C}^n \), and

\[
F : M \times \mathbb{R} \to \mathbb{C}^n, \quad F(q, t) = q + \sin(t)q,
\]

with \( n \) the outward pointing unit normal vector field of \( M \).

We will show that our outer billiard can also be defined in a similar manner. For that, we recall the canonical Kähler or Poisson structures on \( G_κ \). See the details in Subsection 2.2.

For \( \kappa = 1, 0, -1 \), given \( \ell \in G_κ \), the rotation in \( M_κ \) through an angle of \( \pi/2 \) which fixes \( \ell \) induces a diffeomorphism of \( G_κ \) which fixes \( \ell \), whose differential at \( \ell \) is a linear transformation of \( T_\ell G_κ \) squaring to \(-\text{id}\). This yields a **complex structure** \( J \) on \( G_κ \).
For \( \kappa = 1, -1 \), the manifold \( \mathcal{G}_\kappa \) has two canonical pseudo-Riemannian metrics \( g_\times \) and \( g_K \), induced by the cross product on \( M_\kappa \) and the Killing form on \( \text{Iso}(M_\kappa) \), respectively. The metric \( g_\times \) is also well defined on \( \mathcal{G}_0 \). Moreover, in any case, \( (g_K, J) \) and \( (g_\times, J) \) are Kähler structures on \( \mathcal{G}_\kappa \). In the Euclidean case, the Killing form \( g_K \) degenerates, however, on \( \mathcal{G}_0 \) there is a weaker structure also compatible with \( J \), namely, a Poisson bivector field \( \mathcal{P} \).

The next two theorems assert that for \( \kappa = \pm 1 \), the outer billiard on \( \mathcal{G}_\kappa \) is obtained by an analogue of Tabachnikov’s construction, provided \( \mathcal{G}_\kappa \) is endowed with the Kähler structure \( (g_K, J) \), and moreover it is a symplectomorphism with respect to the associated fundamental form. Afterwards we show that this is not the case for \( (g_\times, J) \) and that \( \mathcal{P} \) is preserved by the outer billiard map on \( \mathcal{G}_0 \).

**Theorem 4** For \( \kappa = \pm 1 \), let \( S \) be a quadratically convex compact surface in \( M_\kappa \). Let \( \mathcal{G}_\kappa \) be endowed with the Kähler structure \( (g_K, J) \). Then the outer billiard on \( \mathcal{G}_\kappa \) determined by \( (F, U, \mathcal{M}) \) satisfies

\[
F(u, t) = \Gamma_{J(N(u))}(t),
\]

for all \( u \in T^1S \cong \mathcal{M} \) and all \( t \), where \( N \) is the unit normal vector field on \( \mathcal{M} \) pointing outwards and \( \Gamma_\xi \) denotes the geodesic in \( (\mathcal{G}_\kappa, g_K) \) with initial velocity \( \xi \).

**Theorem 5** For \( \kappa = \pm 1 \), the outer billiard map associated with a quadratically convex compact surface in \( M_\kappa \) is a symplectomorphism with respect to the fundamental symplectic form of \( (\mathcal{G}_\kappa, g_K, J) \).

**Proposition 6** For \( \kappa = 0, 1, -1 \), the billiard map does not preserve the fundamental symplectic form of the Kähler structure \( (g_\times, J) \) on \( \mathcal{G}_\kappa \).

In the Euclidean case, the outer billiard map associated with a quadratically convex compact surface \( S \) preserves parallelism, yielding an \( S^2 \)-worth of planar outer billiards as in Figure 1. The complements of the shadows of \( S \) provide billiard tables with shapes changing smoothly (for each \( v \) in \( \mathbb{R}^3 \), the orthogonal projection of \( S \) onto any plane orthogonal to \( v \) determines a closed convex curve in the plane called the shadow of \( S \) along \( v \)). For each fixed direction the outer billiard map on the plane is well-understood, in particular, it preserves the area. For the more general system on \( \mathcal{G}_0 \) this translates into the assertion in the following proposition.

**Proposition 7** The outer billiard map associated with a quadratically convex compact surface in \( \mathbb{R}^3 \) preserves the canonical Poisson structure \( \mathcal{P} \) on \( \mathcal{G}_0 \), and its symplectic leaves, which are the submanifolds of parallel lines. In particular, the restrictions to them are symplectomorphisms.

Regarding the dynamics of the outer billiard map on \( \mathcal{L}_{-1} \) associated with a surface in \( H^3 \), most of the natural questions remain unanswered. In Propositions 8 and 9 below we present the first steps into this subject, that hint at its complexity. The hyperbolic case is more interesting than the Euclidean one, because the lines \( \ell, B(\ell), B^2(\ell), \ldots \) in \( H^3 \) are in general not parallel, in the sense that they are not orthogonal.
to a fixed totally geodesic surface, as in the Euclidean case. This comes from the fact that if you parallel translate a line \( \ell \) in \( H^3 \) along a line \( \ell' \) orthogonal to it, then \( \ell' \) is the unique line preserved by the motion, in contrast with the Euclidean case. The following proposition illustrates this phenomenon.

**Proposition 8** Let \( 0 < \theta < \pi/2 \). There exist a quadratically convex compact surface \( S \) in \( H^3 \) and an oriented line \( \ell \) not intersecting \( S \) such that \( B^3(\ell) \) and \( \ell \) intersect at a point forming the angle \( \theta \), where \( B \) is the outer billiard map on \( \mathcal{L}_{-1} \) associated with \( S \).

Similar arguments show the existence of \( \ell \) as above such that \( B^3(\ell) \) is different from \( \ell \) and asymptotic to it.

For \( \kappa = 0, -1 \), let \( \varpi : P = T^1 M_\kappa \to \mathcal{L}_\kappa \) be the tautological line bundle, which is an \((\mathbb{R}, +)\)-principal bundle. The right action \( \rho : P \times \mathbb{R} \to P \) is given by \( \rho(u, t) = \gamma'_\omega(t) \).

Let \( S \) be a strictly convex compact surface in \( M_\kappa \) and let \( U \) as before the set of oriented lines in \( M_\kappa \) which do not intersect \( S \).

The line bundle \( \mathcal{P}_U = \varpi^{-1}(U) \to U \) has two distinguished sections \( \sigma_+ \) and \( \sigma_- \): For \( j = \pm \), \( \sigma_j([\gamma]) = \gamma'(0) \) if \( \gamma(0) = q_j(\ell) \), where \( q_+ \) and \( q_- \) are as in the definition of the outer billiard map on \( \mathcal{L}_\kappa \) associated with \( S \) in the introduction. They allow us to define the notion of holonomy of a periodic point of the corresponding billiard map.

Let \( \ell_0 \in U \) be a periodic point of \( B \), say \( B^n(\ell_0) = \ell_0 \) for some \( n > 1 \) (in particular, \( n \geq 3 \)) and suppose that \( B^m(\ell_0) \neq \ell_0 \) for \( 0 < m < n \). For \( 0 \leq k \leq n \) call \( \ell_k = B^k(\ell_0) \) and let \( d_k \) be the signed distance between \( q_+(\ell_k) \) and \( q_-(\ell_k) \); more precisely, if \( \ell_k = [\gamma_k] \) with \( \gamma_k(0) = q_-(\ell_k) \), then \( d_k \) is defined by \( \gamma_k(d_k) = q_+(\ell_k) \). Then the holonomy of \( B \) at \( \ell \) is the number \( d = \sum_{k=0}^{n-1} d_k \). We use the word holonomy in a wider sense: It makes sense only for periodic points and it is not associated with a particular connection, but rather with a combination of the flat connections induced by \( \sigma_\pm \) and the Levi-Civita connection on \( M_\kappa \) along the shortest segments joining \( \ell_k \) with \( \ell_{k+1} \).

It is easy to see that the holonomy of a periodic point of the billiard map is zero in the Euclidean case. We will give an example of a surface in hyperbolic space whose associated billiard map has a periodic point with nontrivial holonomy.

**Proposition 9** There exist a quadratically convex compact surface \( S \) in \( H^3 \) and an oriented line \( \ell \) not intersecting \( S \) which is a periodic point of the associated outer billiard map and whose holonomy is not zero.

Apart from the usual questions on the dynamics of a general system, the following is specific of the outer billiard map \( B \) on \( \mathcal{L}_{-1} \) associated with a surface \( S \) in \( H^3 \): For \( S \) the round sphere in hyperbolic space, \( B \) preserves parallelism (in the sense that for any \( \ell \in \mathcal{L}_{-1} \), \( B^n(\ell) \) remains orthogonal to a certain totally geodesic surface). We wonder whether this property characterizes the sphere. This is related to the issue of defining shadows of convex surfaces in \( H^3 \).

## 2 Preliminaries

### 2.1 The Jacobi fields of the three dimensional space forms

We recall that \((M_\kappa, \langle \cdot, \cdot \rangle_\kappa)\) denotes the three-dimensional complete simply connected manifold of constant sectional curvature \( \kappa = 0, 1, -1 \), that is, \( \mathbb{R}^3, S^3 \) and hyperbolic
space $H^3$. The curvature tensor of $M_\kappa$ is given by
\[ R_\kappa (x, y) z = \kappa (\langle z, x \rangle_\kappa y - \langle z, y \rangle_\kappa x), \]
for vector fields $x, y, z$ on $M_\kappa$.

Let $M$ be a complete Riemannian manifold and let $\gamma$ be a unit speed geodesic of $M$. A Jacobi field $J$ along $\gamma$ is by definition a vector field along $\gamma$ arising via a variation of geodesics as follows: Let $\delta > 0$ and $\phi : \mathbb{R} \times (-\delta, \delta) \to M$ be a smooth map such that $r \mapsto \phi (r, s)$ is a geodesic for each $s \in (-\delta, \delta)$ and $\phi (r, 0) = \gamma (r)$ for all $r$. Then
\[ J (r) = \frac{d}{ds} |_0 \phi (r, s). \]

It is well known that Jacobi fields along a geodesic $\gamma$ and orthogonal to $\gamma'$ of $M_\kappa$ are exactly the vector fields $J$ along $\gamma$ with $\langle J, \gamma' \rangle_\kappa = 0$ satisfying the equation
\[ \frac{d^2 J}{ds^2} + \kappa J = 0. \] (5)

In the rest of the article, following a common abuse of notation, given a smooth vector field $J$ along a curve $\gamma$, we write $J' = \frac{dJ}{dr}$ if there is no danger of confusion.

Recall that a Jacobi field $J$ along $\gamma$ is determined by the values $J (0)$ and $J' (0)$. Suppose that a Jacobi field $J$ along $\gamma$ satisfies $J (0) = u + a \gamma' (0)$ and $J' (0) = v + b \gamma' (0)$ where $a, b \in \mathbb{R}$ and $u, v \in \gamma' (0)$\!. Let $U$ and $V$ be the parallel vector fields along $\gamma$ with $U (0) = u$ and $V (0) = v$. Then
\[ J (r) = c_\kappa (r) \ U (r) + s_\kappa (r) \ V (r) + (a + rb) \gamma' (r), \] (6)
where
\[ c_1 (r) = \cos r, \quad c_0 (r) = 1, \quad c_{-1} (r) = \cosh r, \]
\[ s_1 (r) = \sin r, \quad s_0 (r) = r, \quad s_{-1} (r) = \sinh r \]
(in particular, $s'_\kappa = c_\kappa$ and $c'_\kappa = -\kappa s_\kappa$). This expression will allow us to perform most computations without having to resort to coordinates of $M_\kappa$ or a particular model of it.

For $\kappa = 0, 1, -1$, we presented in the introduction the space $\mathcal{G}_\kappa$ of all the oriented geodesics of $M_\kappa$. Next we see that tangent vectors to $\mathcal{G}_\kappa$ at an oriented geodesic $[\gamma]$ may be identified with Jacobi fields along $\gamma$.

Let $\gamma$ be a complete unit speed geodesic of $M_\kappa$ and let $\mathfrak{J}_\gamma$ be the space of all Jacobi fields along $\gamma$ which are orthogonal to $\gamma'$. There is a canonical isomorphism
\[ T_\gamma : \mathfrak{J}_\gamma \to T_{[\gamma]} \mathcal{G}_\kappa, \quad T_\gamma (J) = \frac{d}{ds} |_0 [\gamma_s], \] (7)
where $\gamma_s$ is any variation of $\gamma$ by unit speed geodesics associated with $J$. Moreover, if $J$ is the Jacobi field associated to a variation $\phi : \mathbb{R} \times (-\delta, \delta) \to M_\kappa$ of $\gamma$ by unit speed geodesics ($\gamma$ is not necessarily orthogonal to $\gamma'$), then
\[ T_\gamma (J^N) = \frac{d}{ds} |_0 [\phi_s], \] (8)
where $J^N (r) = J (r) - \langle J (r), \gamma' (r) \rangle_\kappa \gamma' (r)$ (see Section 2 in [13] and [18]).
2.2 Kähler structures on the manifolds of oriented geodesics

In the introduction we mentioned the two canonical Kähler structures \((g_K, J)\) and \((g_\kappa, J)\) on the space of oriented geodesics \(G_\kappa\) for \(\kappa = 1, -1\), and the Kähler structure \((g_K, J)\) and the Poisson bivector field \(P\) on \(G_0\). Next we present the precise definitions, in terms of the isomorphism \([7]\). We also include the expressions of the associated fundamental forms (see \([12, 17, 18, 6, 1, 9]\) (Theorem 1 in \([18]\) is valid also for \(\kappa = 0, 1\)). The geometry of these spaces of geodesics has been studied for instance in \([10, 11, 19]\). See for instance in \([8]\) or \([14]\) the proof that \(C\) is diffeomorphic to \(S^2 \times S^2\).

Given \(\ell = [\gamma] \in G_\kappa\), the linear complex structure \(J_{\ell}\) on \(\mathfrak{J}_\gamma = T_0G_\kappa\) is defined by

\[
J_{\ell}(J) = \gamma' \times J, \quad \text{for all } J \in \mathfrak{J}_\gamma \cong T_0G_\kappa.
\]

The square norms of the metrics \(g_K\) and \(g_\kappa\) may be written as follows (we put \(\|Z\| = (Z, Z)\)):

\[
\|J\|_x = \langle \gamma' \times J, J' \rangle_\kappa \quad \text{and} \quad \|J\|_K = |J|_\kappa^2 + \kappa |J'|_\kappa^2.
\]

Notice that by \([5]\) the right hand sides are constant functions, so the left hand sides are well defined. Proposition 2 in \([18]\) provides a geometric interpretation for the metrics \(g_\kappa\) and \(g_K\) in the case \(\kappa = -1\): whether a curve in \(G_{-1}\) is space-like or time-like is related to the dichotomies positive versus negative screw for the former and translation versus rotation for the latter. By polarization we have

\[
\begin{align*}
2 g_\kappa (I, J) &= \langle I \times J' + J \times I', \gamma' \rangle_\kappa \quad \text{for } \kappa = -1, 0, 1; \\
g_K (I, J) &= \langle I, J \rangle_\kappa + \kappa \langle I', J' \rangle_\kappa \quad \text{for } \kappa = \pm 1.
\end{align*}
\]

The second one is the push down onto \(G_\kappa\) from the left invariant pseudo-Riemannian metric on the Lie group \(\text{Iso}_0(M_\kappa)\) given at the identity by the Killing form of its Lie algebra. It is Riemannian for \(\kappa = 1\) and split for \(\kappa = -1\). It is well known that \((G_\kappa, g_\kappa, J)\) is Kähler for \(\kappa = 0, 1, -1\) (see, for example, page 753 of \([9]\)).

The associated fundamental forms are given by

\[
\begin{align*}
\omega_\kappa (I, J) &= g_\kappa (J(I), J) = \frac{1}{2} (\langle I'I, J \rangle_\kappa - \langle I, J' \rangle_\kappa), \\
\omega_K (I, J) &= g_K (J(I), J) = \langle I \times J + \kappa I' \times J', \gamma' \rangle_\kappa.
\end{align*}
\]

The bilinear form \(g_K\) degenerates for \(\kappa = 0\). On \(G_0\) we have the canonical Poisson structure, given by

\[
P(\ell) = J_\ell \wedge J_{\ell}(J)
\]

where \(J\) is any parallel Jacobi field along \(\ell\) with \(\|J\| \equiv 1\). Although no such a section \(\ell \mapsto J_\ell \in T_0G_0\) exists globally (otherwise, it would induce a unit vector field on the 2-sphere), \(P\) is easily seen to be well defined and smooth; the Schouten bracket \([P, P]\) vanishes, since the distribution on \(G_0\) induced by \(P\) is integrable. In fact, the symplectic leaves are the submanifolds of parallel lines. The proof of Proposition \([7]\) is straightforward from the well-known fact that the planar outer billiard map preserves the area and we omit it.
3 Proofs of the results

Before proving Proposition 1, we comment on the set $C_{\text{Gauss}}(S)$ defined in (2). We define the map $c : S \to C$ as follows: Given $p \in S$, let $c(p)$ be the oriented great circle obtained by intersecting $S^3$ with the subspace $T_p S \subset \mathbb{R}^4$, endowed with the orientation induced by that of $T_p S$. Equivalently, and without using the immersion of the sphere in $\mathbb{R}^4$, for any positively oriented orthonormal basis $\{u,v\}$ of $T_p S$, $c(p) = [C_p]$, where

$$C_p(s) = \text{Exp}_p \left( \frac{s}{2} (\cos s u + \sin s v) \right) \cong \cos s u + \sin s v.$$  

With this notation, $C_{\text{Gauss}}(S)$ is the image of $c$. Notice that the fact that $S$ is compact and strictly convex implies that if an oriented plane is in $C_{\text{Gauss}}(S)$, then the same plane with the opposite orientation is also in $C_{\text{Gauss}}(S)$.

Proof of Proposition 1. We verify that the same procedure as for the Euclidean and hyperbolic spaces applies here. By (4), $S$ is contained in a hemisphere, say $S^3_+ = \{ x \in S^3 | x_0 > 0 \}$, where $x_0, \ldots, x_3$ are the usual coordinates in $\mathbb{R}^4$. Consider the central projection $\Pi$ from $\{1\} \times \mathbb{R}^3 \cong \mathbb{R}^3$ to $S^3_+$ (called the Beltrami map), which preserves both strict and quadratical convexity, since planes correspond with half great spheres and the order of contact is preserved by diffeomorphisms.

Let $c \in C - C_{\text{cut}}(S)$. If $c$ is not contained in the equator $x_0 = 0$, then $\Pi^{-1}(c)$ is an oriented line $\ell$ in $\mathbb{R}^3$. Exactly two planes in $\mathbb{R}^3$ containing $\ell$ will intersect $\Pi^{-1}(S)$ at, say, the tangent points $\bar{p}_1$ and $\bar{p}_2$. They project to two half great spheres containing $c$, which are tangent to $S$ at $p_j = \Pi(\bar{p}_j), j = 1, 2$. These points play the roles of the corresponding points in the definition of the outer billiard maps on $\mathcal{L}_0$ and $\mathcal{L}_{-1}$ in the introduction. If $c$ is in the equator $x_0 = 0$ and $u \in e^+_0$ is a unit vector orthogonal to $c$, then the planes orthogonal to $u$ project to half great spheres containing $c$. Among these parallel planes, exactly two of them intersect $\Pi^{-1}(S)$ and one obtains as before two great spheres containing $c$, which are tangent to $S$ at some points $p_1$ and $p_2$.

Now, let us suppose additionally that $c \notin C_{\text{Gauss}}(S)$. In the following argument the Beltrami map is not needed any more. Let $d$ be the distance between $p_1$ and $c$. If $d < \pi/2$, there exists exactly one point $q_1$ on $c$ realizing the distance. If $d = \pi/2$, since $c$ is contained in the great sphere $\text{Exp}_{p_1}(T_{p_1}S)$, which is tangent to $p_1$, then $c \in C_{\text{Gauss}}(S)$, which is a contradiction. The proof continues as in the Euclidean and hyperbolic cases.

Next we prove some technical lemmas, working towards the proof of Theorem 3.

Before we state the next lemma we give some geometric intuition for it. Consider the situation in which $S$ is a strictly convex compact surface in $M_\kappa$, $U$ is the collection of oriented geodesics not intersecting $S$, and its boundary $\mathcal{M}$ is the collection of geodesics tangent to $S$. Given $\ell \in \mathcal{M}$, one may consider three perturbations of $\ell$ which stay in $\mathcal{M}$: one which skates along $S$ in the direction of $\ell$, one which parallel translates $\ell$ along the surface in the direction orthogonal to $\ell$, and one which rotates $\ell$, maintaining the point of tangency.

Lemma 10 Let $S$ be a strictly convex compact surface in $M_\kappa$, let $u \in T^1_p S$ and call $v = iu$. For $m = 1, 2, 3$, let $w_m : \mathbb{R} \to T^1 S$ be the curve defined by

$$w_1(s) = \sigma'_u(s), \quad w_2(s) = P^{\sigma_v}_{0,s}(u) \quad \text{and} \quad w_3(s) = \cos s u + \sin s v,$$
where \( \sigma_v \) is the geodesic on \( S \) with initial velocity \( v \) and \( P^s_{0,v} \) denotes the parallel transport on \( S \) along \( \sigma \) between 0 and \( s \). Then \( \mathcal{B} = \{ w'_1(0), w'_2(0), w'_3(0) \} \) is a basis of \( T_u T^1 S \).

**Proof.** Let \( \pi : TS \to S \) be the canonical projection and let \( \mathcal{K}_u : T_u TS \to T_p S \) be the connection operator. We claim that under the linear isomorphism

\[
\varphi_u : T_u TS \to T_p S \times T_p S,
\]

(see for instance [2]) the elements of \( \mathcal{B} \) are mapped to \((u, 0), (v, 0) \) and \((0, v)\), respectively, which are linearly independent. We compute

\[
d\pi_u(w'_1(0)) = \frac{d}{ds}\big|_0 \pi(w_1(s)) = \frac{d}{ds}\big|_0 \sigma_u(s) = u,
\]

and, by definition of \( \mathcal{K}_u \),

\[
\mathcal{K}_u(w'_1(0)) = \frac{D}{ds}\big|_0 w_1(s) = \frac{D}{ds}\big|_0 \sigma'_u(s) = 0.
\]

Hence, \( \varphi_u(w'_1(0)) = (u, 0) \). The other cases are similar. \( \blacksquare \)

We recall some notation. We denote by \( \gamma_w \) the unique geodesic in \( M_\kappa \) with initial velocity \( w \in T^1 M_\kappa \). Given a strictly convex compact surface \( S \) in \( M_\kappa \), we call \( n \) the unit normal vector field on \( S \) pointing inward. The complex structure \( i \) on \( S \) is defined by \( i w = n(p) \times w \), for \( w \in T_p S \).

Given \( p \in S \), the shape operator \( A_p : T_p S \to T_p S \) is defined by

\[
A_p(x) = -\nabla_x n, \tag{14}
\]

where \( \nabla \) denotes the Levi-Civita connection of \( M_\kappa \). We assume that \( A_p \) is positive definite at each \( p \in S \) (that is, \( S \) is quadratically convex).

Let \( u \in T^1_p S \) and call \( v = iu \in T^1_p S \). We consider the matrix of \( A_p \) with respect to the orthonormal basis \( \{u, v\} \) and call \( b_{ij} \) its entries, that is, \( [A_p]_{\{u,v\}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \). In particular, we have that \( b_{12} = b_{21} \).

We call \( u_r, v_r \) and \( n_r \) the vectors in \( T_{\gamma_w(r)} M_\kappa \) obtained by parallel transporting \( u, v \) and \( n(p) \), respectively, along \( \gamma_v \), between 0 and \( r \).

For \( m = 1, 2, 3 \), let \( w_m \) be the curve in \( T^1 S \) defined in Lemma [10] and let \( f_m : \mathbb{R}^2 \to M_\kappa \) be the parametrized surface in \( M_\kappa \) defined by

\[
f_m(r, s) = \gamma_{iw_m(s)}(r).
\]

In particular \( f_m(r, 0) = \gamma_v(r) \). We call \( \sigma_m = \pi \circ w_m = f_m(0, \cdot) \); we have that \( \sigma_1 = \sigma_u, \sigma_2 = \sigma_v \) and \( \sigma_3 \equiv p \).

**Lemma 11** For \( m = 1, 2, 3 \), the Jacobi vector field \( K_m \) along \( \gamma_v \) associated with the geodesic variation \( f_m \), that is,

\[
K_m(r) = \frac{d}{ds}\big|_0 f_m(r, s), \tag{15}
\]

is given by

\[
\begin{align*}
K_1(r) &= c_\kappa(r) \ u_v + b_{21}s_\kappa(r) \ n_r, \\
K_2(r) &= v_r + b_{22} s_\kappa(r) \ n_r, \\
K_3(r) &= -s_\kappa(r) \ u_r.
\end{align*}
\tag{16}
\]
Proof. We compute the initial values of $K_m$ and $K'_m$ and use (6). We write down the details for $m = 1$. The other cases are similar. We have that

$$K_1(0) = \frac{d}{ds}f_1(0,s) = \frac{d}{ds}\sigma_1(s) = u$$

and

$$K'_1(0) = \frac{\partial}{\partial s}f_1(r,s) = \frac{\partial}{\partial s}f_1(r,s) = \frac{\partial}{\partial s}0 = i w_1(s).$$

We compute the coordinates of $K'_1(0)$ with respect to the orthonormal basis $\{u, v, n(p)\}$ of $T_pM$. To obtain $\langle K'_1(0), n(p) \rangle_k$, we observe that $\langle iw_1(s), n(\sigma_1(s)) \rangle_k = 0$ for all $s$. Hence,

$$\langle \frac{\partial}{\partial s}0 iw_1(s), n(p) \rangle_k = -\langle iu, \frac{\partial}{\partial s}n(\sigma_1(s)) \rangle_k = -\langle v, \nabla u n \rangle_k = \langle v, A_p u \rangle_k = b_{21}.$$

In the same way, $\langle K'_1(0), u \rangle_k = 0 = \langle K'_1(0), v \rangle_k$. Therefore, $K'_1(0) = b_{21}n(p)$. Notice that $K_m$ is not necessarily orthogonal to $\gamma'$. ■

Lemma 12 For $m = 1, 2, 3$, let $N_m$ be the vector field along $f_m$ defined as follows: $N_m(r, s)$ is the parallel transport of $n(f_m(0, s)) = n(\sigma_m(s))$ between 0 and $r$ along the geodesic $\gamma_{i w_m(s)} = f_m(\cdot, s)$. Let

$$\zeta_m(r) = \langle \frac{d}{ds}0 N_m(r, s), u_r \rangle_k.$$

Then

$$\zeta_1 \equiv -b_{11}, \quad \zeta_2 \equiv -b_{12} \quad \text{and} \quad \zeta_3 \equiv 0.$$

Proof. We compute

$$\zeta'_m(r) = \frac{d}{dr}\langle \frac{d}{ds}0 N_m(r, s), u_r \rangle_k = \langle \frac{d}{dr}\frac{d}{ds}0 N_m(r, s), u_r \rangle_k$$

$$= \langle \frac{d}{ds}0 \frac{d}{dr}N_m(r, s) + R_k \langle \frac{d f_m}{d r}(r, 0), \frac{d f_m}{d s}(r, 0) \rangle N_m(r, 0), u_r \rangle_k$$

$$= \langle R_k \langle v_r, K_m(r) \rangle n_r, u_r \rangle_k = \kappa \langle \langle n_r, v_r \rangle_k, K_m(r) - \langle n_r, K_m(r) \rangle_k, v_r, u_r \rangle_k = 0,$$

by (4). Hence, $\zeta_m$ is constant, equal to

$$\zeta_m(0) = \langle \frac{d}{ds}0 N_m(0, s), u_0 \rangle_k = \langle \nabla \sigma'_m(0) n, u \rangle_k = -\langle A_p(\sigma'_m(0)), u \rangle_k.$$

Now, the assertions follow from the definition of $\sigma_m$ and the values of the entries of the matrix $[A_p]_{u,v}$. ■

Lemma 13 For $m = 1, 2, 3$, let $W_m$ be the vector field along $f_m$ defined analogously as $N_m$, with $w_m(s)$ instead of $n(\sigma_m(s))$. Let $Y_m$ be the vector field along $\gamma_v = f_m(\cdot, 0)$ given by

$$Y_m(r) = \frac{d}{dr}0 W_m(r, s)$$

Then

$$Y_1(r) = \kappa s_r(r) v_r + b_{11} n_r, \quad Y_2(r) = b_{12} n_r \quad \text{and} \quad Y_3(r) = c_r(r) v_r.$$
Proof. Observe that \( \langle W_m, W_m \rangle_\kappa, \langle W_m, N_m \rangle_\kappa \) and \( \langle W_m, \frac{\partial f_m}{\partial r} \rangle_\kappa \) are constant functions of the second variable \( s \). Hence we can compute the components \( Y_m(r) \) with respect to the basis \( \{u_r, v_r, n_r\} \) of \( T_{\gamma_r}(M_\kappa) \) as follows:

\[
\langle Y_m(r), u_r \rangle_\kappa = \langle \frac{\partial}{\partial s}\big|_0 W_m(r, s), W_m(r, 0) \rangle_\kappa = 0.
\]

Also

\[
\langle Y_m(r), v_r \rangle_\kappa = \langle \frac{\partial}{\partial s}\big|_0 W_m(r, s), v_r \rangle_\kappa = -\langle W_m(r, 0), \frac{\partial}{\partial s}\big|_0 \gamma_{iw_0}(s)(r) \rangle_\kappa
\]

\[
= -\langle u_r, \frac{\partial}{\partial s}\big|_0 \gamma_{iw_0}(s)(r) \rangle_\kappa = -\langle u_r, \frac{\partial}{\partial r} K_m(r) \rangle_\kappa.
\]

where \( K_m \) is the vector field along \( \gamma_v \) defined in (15). By (16), we have

\[-\langle u_r, K_m(r) \rangle_\kappa = \begin{cases} 
\kappa s_\kappa(r), & \text{if } m = 1 \\
0, & \text{if } m = 2 \\
c_\kappa(r), & \text{if } m = 3.
\end{cases}\]

In the same way,

\[
\langle Y_m(r), n_r \rangle_\kappa = \langle \frac{\partial}{\partial s}\big|_0 W_m(r, s), N_m(r, 0) \rangle_\kappa = -\langle W_m(r, 0), \frac{\partial}{\partial s}\big|_0 N_m(r, s) \rangle_\kappa = -\zeta_m(r),
\]

with \( \zeta_m \) as in Lemma 12.

For \( \ell \in M \), unless otherwise stated, we consider the parametrization \( \gamma \) with \( \gamma(0) \in S \) and \( [\gamma] = \ell \). The next lemma refers to the isomorphism defined in (7).

Lemma 14 The tangent space \( T_{[\gamma]}M \) identifies with \( \{K \in \mathfrak{X}_\gamma \mid K(0) \in T_{\gamma(0)}S\} \) via the isomorphism \( T_{\gamma} \).

Proof. Let \( X \in T_{[\gamma]}M \) and let \( c \) be a smooth curve on \( M \) (defined on an interval \( I \) containing 0) such that \( c(0) = [\gamma] \) and \( c'(0) = X \). For each \( t \in I \), let \( [\gamma_t] \in M \) such that \( \gamma_t(0) \in S \) and \( [\gamma_t] = c(t) \). Hence, by (8), \( X = T_{\gamma}(J^N) \), where \( J \) is given by

\[ J(s) = \frac{d}{dt}\big|_0 \gamma_t(s). \]

Now, \( t \mapsto \gamma_t(0) \) is a smooth curve on \( S \). Then \( J(0) = \frac{d}{dt}\big|_0 \gamma_t(0) \in T_{\gamma(0)}S \) and so \( J^N(0) \in T_{\gamma(0)}S \), since \( \gamma'(0) \in T_{\gamma(0)}S \).

Thus, \( T_{[\gamma]}M \subset T_{\gamma}\{K \in \mathfrak{X}_\gamma \mid K(0) \in T_{\gamma(0)}S\} \). The proof concludes noting that both spaces have dimension 3.

Proof of Theorem 3. For \( \kappa = 0, -1 \), let \( T = \infty \) and for \( \kappa = 1 \), let \( T = \pi/2 \). We consider

\[ F : M \times (-T, T) \cong T^1S \times (-T, T) \to \mathcal{G}_\kappa \]

defined by \( F(u, t) = [\gamma_{ut}] \), where \( u_t \) is the parallel transport of \( u \) between 0 and \( t \) along \( \gamma_{ut} \). Clearly, \( F \) is a smooth function. Now, we will prove that condition a) of Definition 2 holds. Via the identification of \( M \) with \( T^1S \), given \( u \in T^1S \) we consider the curve \( t \mapsto \alpha_u(t) = F(u, t) \), which satisfies that \( \alpha_u(0) = [\gamma_u] \). Besides, the Jacobi field along \( \gamma_u \) given by

\[ J(s) = \frac{d}{dt}\big|_0 \gamma_u(s) \]
holds \( J(0) = iu \in T_{\gamma_u(0)}S \). Therefore, \( J \in \mathcal{J}_{\gamma_u} \) and by Lemma 14, \( \alpha_u'(0) \in T_{[\gamma_u]}\mathcal{M} \).

The fact that \( F_+ \) and \( F_- \) are bijections onto \( \mathcal{U} \) and that the billiard map determined by \((F,\mathcal{U},\mathcal{M})\) coincides with the outer billiard map on \( G_{\kappa} \) associated with \( S \) follow directly from the definitions. So, to verify condition b), it remains to prove that \( F_+ \) and \( F_- \) are diffeomorphisms. For this we have to check that

\[
dF_{(u,t)} : T_{(t,t)} (\mathcal{M} \times (-T, T)) \cong T_u T^1 S \times T_t \mathbb{R} \rightarrow T_{F(u,t)} \mathcal{G}_{\kappa}
\]

is non singular for all \( u \in T^1 S \) and \( 0 \neq |t| < T \), where \( \ell = [\gamma_u] \).

We consider the basis \( B \) of \( T_{(t,t)} (\mathcal{M} \times (-T, T)) \cong T_u T^1 S \times T_t \mathbb{R} \) given by

\[
B = \{(w_1'(0), 0), (w_2'(0), 0), (w_3'(0), 0), (0, \frac{d}{ds}|_t)\},
\]

where \( w_m \) are the curves in \( T^1 S \) as in Lemma 10. For \( m = 1, 2, 3 \) we have

\[
(dF)_{(u,t)}(w_m'(0), 0) = \frac{d}{ds}|_0 \alpha_{w_m(s)}(t)
\]

and

\[
(dF)_{(u,t)}(0, \frac{d}{ds}|_t) = \frac{d}{ds}|_0 \alpha_u(t + s).
\]

By (8), these vectors correspond to the Jacobi fields \( J_m \in \mathcal{J}_{\gamma_{ut}} \) whose initial conditions are (using the notation given immediately before Lemma 11)

\[
J_4(0) = v_t \quad \text{and} \quad J'_4(0) = 0,
\]

and for \( m = 1, 2, 3 \),

\[
J_m(0) = K_m(t) - \langle K_m(t), u_t \rangle u_t \quad \text{and} \quad J'_m(0) = \frac{d}{ds}|_0 W_m(t, s) = Y_m(t),
\]

where \( K_m \) is as in (15) and \( W_m \) is defined in Lemma 13.

Now, we consider the basis

\[
B_t = \{E_1^t, \cdots, E_4^t\}
\]

of \( \mathcal{J}_{\gamma_{ut}} \), where \( E_m^t \) are the Jacobi fields along \( \gamma_{ut} \) whose initial conditions are

\[
E_1^t(0) = 0, \quad E_2^t(0) = n_t, \quad E_3^t(0) = 0, \quad E_4^t(0) = v_t,
\]

\[
(E_1^t)'(0) = n_t, \quad (E_2^t)'(0) = 0, \quad (E_3^t)'(0) = v_t, \quad (E_4^t)'(0) = 0.
\]

By Lemmas 11 and 13 we have

\[
J_m(0) = \begin{cases}
  b_{21}s_\kappa(t)n_t, & m = 1 \\
  v_t + b_{22}s_\kappa(t)n_t, & m = 2 \\
  0, & m = 3 \\
  v_t, & m = 4
\end{cases}
\]  

and

\[
J'_m(0) = \begin{cases}
  \kappa s_\kappa(t)v_t + b_{11}n_t, & m = 1 \\
  b_{12}n_t, & m = 2 \\
  c_\kappa(t)v_t, & m = 3 \\
  0, & m = 4.
\end{cases}
\]

Hence, calling \( C \) the matrix of \( (dF)_{(u,t)} \) with respect to the bases \( B \) and \( B_t \), we obtain that

\[
C = \begin{pmatrix}
  b_{11} & b_{12} & 0 & 0 \\
  b_{21}s_\kappa(t) & b_{22}s_\kappa(t) & 0 & 0 \\
  \kappa s_\kappa(t) & 0 & c_\kappa(t) & 0 \\
  0 & 1 & 0 & 1
\end{pmatrix}.
\]

14
Therefore, det $C = c_\kappa(t)s_\kappa(t)b$, where $b = \det[A_p]_{\{u,v\}}$. Since, by hypothesis, $0 \neq |t| < T$ and $[A_p]_{\{u,v\}}$ is definite, we have that $C$ is non singular. ■

**Proof of Theorem 5** By the Definition 2 of the outer billiard $B : \mathcal{U} \to \mathcal{U}$ we have that $B = F_+ \circ g \circ F_-^{-1}$, where $g : T^1S \times (-T,0) \to T^1S \times (0,T)$ is defined by $g(u, t) = (u, -t)$.

Given $\ell \in \mathcal{U}$, suppose that $\ell = F_-(u, -t)$ for some $0 < t < T$. We compute the matrix of $dB_\ell$ with respect to the canonical bases $\mathcal{B}_{-t}$ and $\mathcal{B}_t$ of $\mathcal{J}_{\gamma_{-t}}$ and $\mathcal{J}_{\gamma_t}$ as in (18), respectively, obtaining

$$[dB_\ell]_{\mathcal{B}_{-t}, \mathcal{B}_t} = \left( \begin{array}{cc} R & 0_2 \\
D & R \end{array} \right),$$

(19)

where $R = \left( \begin{array}{cc} 1 & 0 \\
0 & -1 \end{array} \right)$ and $D = \frac{2}{b} \left( \begin{array}{cc} s_\kappa(t) \kappa b_{22} & \kappa b_{12} \\
b_{21} & -s_\kappa(t) \end{array} \right)$, with $b = \det [A_p]_{\{u,v\}}$. Thus,

$$dB_\ell \left( E_{1}^{-t} \right) = E_1^t + \frac{2}{b} s_\kappa(t) \kappa b_{22} E_3^t - \frac{2}{b} b_{21} E_4^t,$n$$

$$dB_\ell \left( E_{2}^{-t} \right) = -E_2^t + \frac{2}{b} \kappa b_{12} E_3^t - \frac{2}{b} b_{11} s_\kappa(t) E_4^t,$n$$

$$dB_\ell \left( E_{3}^{-t} \right) = E_3^t \text{ and } dB_\ell \left( E_{4}^{-t} \right) = -E_4^t.$$

Recall from (13) the definition of the symplectic form $\omega_K$. Straightforward computations yield that $\langle E_i^t(0) \times E_j^t(0), u_\ell \rangle_\kappa = \langle (E_i^t)'(0) \times (E_j^t)'(0), u_\ell \rangle_\kappa = 0$ for all $1 \leq i < j \leq 3$ except for

$$\langle E_2^t(0) \times E_4^t(0), u_\ell \rangle_\kappa = \langle (E_1^t)'(0) \times (E_3^t)'(0), u_\ell \rangle_\kappa = -1.$$

Hence,

$$[\omega_K]_{\mathcal{B}_t} = \left( \begin{array}{cc} 0_2 & \rho \\
-\rho & 0_2 \end{array} \right)$$

with $\rho = \left( \begin{array}{cc} \kappa & 0 \\
0 & 1 \end{array} \right)$. We observe that $[\omega_K]_{\mathcal{B}_{-t}} = [\omega_K]_{\mathcal{B}_t}$. Hence, calling $H$ the matrix in (19), we have to check that

$$H^T [\omega_K]_{\mathcal{B}_t} H = [\omega_K]_{\mathcal{B}_{-t}}.$$

The left hand side equals

$$\left( \begin{array}{cc} -D^T \rho R + R \rho D & \rho \\
-\rho & 0_2 \end{array} \right)$$

and

$$-D^T \rho R + R \rho D = \frac{2}{a} (1 - \kappa^2) \left( \begin{array}{cc} 0 & -b_{12} \\
b_{21} & 0 \end{array} \right),$$

which is the zero matrix since $\kappa = \pm 1$, as desired. ■
Proof of Proposition 6. Following the computations in the proof of Theorem 5 we have
\[ [\omega_x]_{B_t} = \begin{pmatrix} j & 0_2 \\ 0_2 & j \end{pmatrix}, \]
where \( j = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \([\omega_x]_{B_{-t}} = [\omega_x]_{B_t} \). Further computations yield
\[ H^T [\omega_x]_{B_t} H = \begin{pmatrix} j & -(RjD)^T \\ RjD & j \end{pmatrix}. \quad (20) \]
Now, \( RjD = \frac{1}{6} \begin{pmatrix} -b_{21} & -b_{11}/s_\kappa (t) \\ s_\kappa (t) \kappa b_{22} & \kappa b_{12} \end{pmatrix} \) and \( b_{11}/s_\kappa (t) \neq 0 \) since by the hypothesis \( S \) is quadratically convex. Therefore, \( RjD \neq 0_2 \) and the expression (20) is not equal to \([\omega_x]_{B_{-t}} \). \( \blacksquare \)

In order to prove Theorem 4 we need the presentation of \( G_\kappa \) as a symmetric homogeneous space. The details of the following description can be found for instance in [9]. For \( \kappa = \pm 1 \), we consider the standard presentation of \( M_\kappa \) as a submanifold of \( \mathbb{R}^4 \): If \( \{e_0, e_1, e_2, e_3\} \) is the canonical basis of \( \mathbb{R}^4 \), then \( M_k \) is the connected component of \( e_0 \) of the set
\[ \{x \in \mathbb{R}^4 | \kappa x_0^2 + x_1^2 + x_2^2 + x_3^2 = \kappa \} \].

Let \( G_\kappa \) be the identity component of the isometry group of \( M_\kappa \), that is, \( G_1 = SO_4 \) and \( G_{-1} = O_o(1,3) \). The group \( G_\kappa \) acts smoothly and transitively on \( G_\kappa \) as follows: \( g \cdot [\gamma] = [g \circ \gamma] \). Let \( \gamma_0 \) be the geodesic in \( M_\kappa \) with \( \gamma_0(0) = e_0 \) and initial velocity \( e_1 \in T_{e_0}M_\kappa \) and let \( H_\kappa \) be the isotropy subgroup of \( G_\kappa \) at \([\gamma_0]\). Then there exists a diffeomorphism \( \phi : G_\kappa / H_\kappa \to G_\kappa \), given by \( \phi(gH_\kappa) = g \cdot [\gamma_0] \).

The Killing form of \( \text{Lie} (G_\kappa) \) provides \( G_\kappa \) with a bi-invariant metric and thus there exists a unique pseudo-Riemannian metric \( \tilde{g}_K \) on \( G_\kappa / H_\kappa \) such that the canonical projection \( \pi : \tilde{G}_\kappa \to G_\kappa / H_\kappa \) is a pseudo-Riemannian submersion. The diffeomorphism \( \phi \) turns out to be an isometry onto \( G_\kappa \) endowed with a constant multiple of the metric \( g_K \) defined in (11).

Besides, it is well known that \((G_\kappa / H_\kappa, \tilde{g}_K)\) is a pseudo-Riemannian symmetric space. In particular, if \( \text{Lie} (G_\kappa) = \text{Lie} (H_\kappa) \oplus p_\kappa \) is the Cartan decomposition determined by \([\gamma_0]\), then for any \( Z \in p_\kappa \), the curve \( t \mapsto \exp (tZ) H_\kappa \) is a geodesic of \( G_\kappa / H_\kappa \).

Proof of Theorem 4. We fix \( u \in T_p^1 S \) and we want to see that the curve \( \Gamma (t) = [\gamma_{ut}] \) is the geodesic in \((G_\kappa, \tilde{g}_K)\) with initial velocity \( \mathcal{J} (N (u)) \). First we verify that \( \Gamma' (0) = \mathcal{J} (N (u)) \) and afterwards that \( \Gamma \) is a geodesic of \( \mathcal{G}_\kappa \).

The initial velocity of \( \Gamma \) corresponds, via the isomorphism \( T_{\gamma_u} \) of (7), with the Jacobi field along \( \gamma_u \) given by
\[ J(s) = \frac{d}{dt} \bigg|_0 \gamma_{ut} (s). \]
A straightforward computation shows that it is determined by the conditions \( J(0) = iu \) and \( J'(0) = 0 \).

On the other hand, if \( N \) is the unit normal vector field on \( \mathcal{M} \) pointing outwards, then, after the identification with Jacobi fields,
\[ N([\gamma_u]) = I \in \mathfrak{J}_{\gamma_u} \text{ satisfying } I(0) = -n(p) \text{ and } I'(0) = 0. \]
Indeed, by \((11)\),
\[
g_K(I, I) = \langle I, I \rangle_\kappa + \kappa \langle I', I' \rangle_\kappa = (-1)^2 |n(p)|^2 = 1
\]
and \(g_K(I, K) = -\langle n(p), K(0) \rangle_\kappa = 0\) for all \(K \in T_{[\gamma_u]} \mathcal{M}\), since \(K(0) \in T_{\gamma_u(0)} S\) by Lemma \([14]\).

Now, by the definition of the complex structure \(J\) in \([9]\), the identity \(J_{[\gamma_u]} \left( N_{[\gamma_u]} \right) = \Gamma'(0)\) translates into \(J = \gamma_u' \times I\), and this is true since \(J'(0) = 0 = \Gamma'(0)\) and
\[
J(0) = iu = n(p) \times u = \gamma_u'(0) \times I(0).
\]

Next we show that \(\Gamma\) is a geodesic. By homogeneity, we may suppose that \(p = e_0\), the inward pointing unit normal vector of \(S\) at \(e_0\) is \(e_3\) and \(u = e_1\). Hence \(iu = e_2\) and \(u_t = e_1 \in T_{\gamma(t)} M_\kappa\) where \(\gamma(t) = c_\kappa(t) = e_0 + s_\kappa(t) e_2\).

Let \(Z\) be the linear transformation of \(\mathbb{R}^4\) defined by \(Z(e_0) = e_2\), \(Z(e_2) = -\kappa e_0\) and \(Z(e_1) = Z(e_3) = 0\). It is easy to verify that \(Z \in \text{Lie} (G_\kappa)\) and \(\exp (tZ) [\gamma_\kappa] = [\gamma_\kappa]\) for all \(t\). Now, one can see in the preliminaries of \([9]\) (page 752) that \(Z \in \mathfrak{p}_\kappa\), and so \(\Gamma\) is a geodesic by the properties of symmetric spaces presented above. ■

Before proving Proposition \([8]\) we comment on the Klein model of hyperbolic space, that is, the open ball \(\mathcal{H}\) centered at the origin with radius 1, where the trajectories of geodesics are the intersections of Euclidean straight lines with the ball. The intersections of \(\mathcal{H}\) with Euclidean planes are totally geodesic hyperbolic planes.

We recall the following well-known constructions on \(\mathcal{H}\) (see for instance Chapter 6 of \([3]\)). For an oriented line \(\ell\) in \(\mathcal{H}\) we call \(\ell^+\) and \(\ell^-\) its forward and backward ideal end points in the two sphere \(\partial \mathcal{H}\).

Let \(\ell_1\) and \(\ell_2\) be two coplanar oriented lines in \(\mathcal{H}\) such that the corresponding extensions to Euclidean straight lines intersect in the complement of the closure of \(\mathcal{H}\). In particular, \(\ell_1\) and \(\ell_2\) do not intersect and are not asymptotic and hence there exists the shortest segment joining them with respect to the hyperbolic metric; we call it \(s(\ell_1, \ell_2)\).

The hyperbolic midpoint of \(s(\ell_1, \ell_2)\) is the intersection of the Euclidean segments joining \(\ell_1^+\) with \(\ell_2^-\) and \(\ell_1^-\) with \(\ell_2^+\), or joining \(\ell_1^+\) with \(\ell_2^+\) and \(\ell_1^-\) with \(\ell_2^-\), depending on the orientation of the lines.

Suppose that \(\ell_1, \ell_2\) lie in the plane \(P\) and call \(D\) the disc \(P \cap \mathcal{H}\) and \(C\) its boundary. Now we describe the construction of the segment \(s(\ell_1, \ell_2)\) itself. We will need it only in the case when one of the lines is a diameter in \(D\), say \(\ell_1\). Let \(p\) be the intersection of the tangent lines to \(C\) through the ideal end points \(\ell_2^+\) and \(\ell_2^-\). Then \(s(\ell_1, \ell_2)\) is the segment joining \(\ell_1\) and \(\ell_2\) contained in the straight line through \(p\) perpendicular to \(\ell_1\) (the point \(p\) is called the pole of \(\ell_2\) in the plane \(P\)).

We recall the formula for the hyperbolic distance between a point in \(\mathcal{H}\) and the midpoint of any chord containing it: Let \(x, y\) be two distinct points in \(S^2 = \partial \mathcal{H}\) and let \(c\) be the midpoint of the segment joining \(x\) and \(y\), that is, \(c = \frac{1}{2} (x + y)\). Then, for any \(t \in (0, 1)\),
\[
d\left( c, c + t \frac{y-x}{2} \right) = \text{arctanh} \ t
\]
holds, where \(d\) is the hyperbolic distance in \(\mathcal{H}\).

Since quadratical contact is invariant by diffeomorphisms, a quadratically convex surface of \(\mathbb{R}^3\) contained in \(\mathcal{H}\) is also quadratically convex with the hyperbolic metric.
By abuse of notation, we describe an oriented geodesic \( \ell \) in \( \mathcal{H} \) by the straight Euclidean \( p + \mathbb{R} u \) line containing \( \ell \), with \( p \in \mathcal{H} \) and \( u \) a unit vector giving the orientation.

**Proof of Proposition 8.** We use the Klein model of hyperbolic space. Let \( \ell = \mathbb{R} e_3 \) and \( \ell_\theta = \mathbb{R} (\sin \theta e_2 + \cos \theta e_3) \). We construct a surface \( S \) contained in the region \( x \geq 0, y \geq 0 \) of \( \mathcal{H} \) whose associated outer billiard map \( B \) satisfies \( B^3 (\ell) = \ell_\theta \). We fix a real number \( r \) in the interval \((\sin \theta, 1)\). Call \( \ell_1 = re_1 + \mathbb{R} e_3, \ell_2 = re_2 + \mathbb{R} e_3 \) and let \( p = (x_o, 0, 0) \) and \( q = (0, y_o, z_o) \) be the hyperbolic midpoints between \( \ell \) and \( \ell_1 \) and between \( \ell_\theta \) and \( \ell_2 \), respectively.

![Figure 4: Elements for the construction of \( S \)](image)

There exists a quadratically convex compact surface \( S \) contained in \( \mathcal{H} \) tangent to the vertical planes \( y = 0, y + x = r \) and \( x = 0 \), at the points \( p, (r/2, r/2, 0) \) and \( q \), respectively. In fact, consider a quadratically convex compact surface \( S' \rangle \) tangent to those planes at the points \( p, (r/2, r/2, 0) \) and \( (0, y_o, 0) \), respectively, such that the absolute value of the height function \( |z|_{S'} \) is bounded by \( \varepsilon \) for some \( \varepsilon > 0 \). Let \( T \) be the unique affine transformation of \( \mathbb{R}^3 \) fixing the vertical planes through \( p \) and \( (r/2, r/2, 0) \) and sending \( (0, y_o, 0) \) to \( q \). Then \( S = T (S') \) satisfies the desired conditions, provided that \( \varepsilon \) is small enough (notice that affine transformations preserve quadratical contact). By the properties of \( S \) we have that \( B (\ell) = \ell_1, B^2 (\ell) = \ell_2 \) and \( B^3 (\ell) = \ell_\theta \).

**Proof of Proposition 9.** As in the proof of Proposition 8 we use the Klein model \( \mathcal{H} \) for hyperbolic space. We write \( \mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \). Given \( 0 < a < \frac{1}{2} < r_o < 1 \) and \( h_o = \sqrt{1 - r_o^2} \), we consider the straight lines

\[
\gamma_0 (t) = (t, 0), \quad \gamma_1 (t) = \left( \frac{t}{2} + t (1 - ai), 0 \right), \\
\gamma_2 (t) = \left( \frac{t}{2} + t (1 - ai), h_o \right), \quad \gamma_3 (t) = (t, h_o).
\]

For \( k = 0, 1, 2, 3 \), let \( \ell_k \) be the corresponding oriented geodesic in \( \mathcal{H} \) and set \( \ell_4 = \ell_0 \). They are pairwise coplanar. We call \( P_k \) the hyperbolic plane containing \( \ell_k \) and \( \ell_{k+1} \) and \( \sigma_k \) the shortest segment joining them (they are not asymptotic since \( 0 < 2a < 1 \)).

We will show the existence of a quadratically convex compact surface \( S \) in \( \mathcal{H} \) not intersecting \( \ell_0 \) such that the associated billiard map \( B \) satisfies \( B^k (\ell_0) = \ell_k \) for \( k = 0, \ldots, 4 \) and its holonomy at \( \ell_0 \) is not trivial.

We make computations for general values of \( r \) and \( h = \sqrt{1 - r^2} \), with \( 0 < a < \frac{1}{2} < r \leq 1 \), in order to deal simultaneously with \( \ell_0 \) and \( \ell_1 \) on the one hand (case \( r = 1 \) and
\( \ell_2 \) and \( \ell_3 \) on the other (case \( r = r_0 \)), since the former lie in a disc of radius 1 at height 0 and the latter in a disc of radius \( r_0 \) at height \( h_0 \).

The end points of \( \ell_1 \) and \( \ell_2 \) are given by

\[
\ell_1^\varepsilon = (z_\varepsilon (1), 0) \quad \text{and} \quad \ell_2^\varepsilon = (z_\varepsilon (r_0), h_0),
\]

for \( \varepsilon = \pm 1 \), where \( z_\varepsilon (r) = \frac{i}{2} + t_\varepsilon (r) (1 - ai) \), with \( t_- (r) < t_+ (r) \) being the solutions of the equation \( t^2 + (\frac{1}{2} - at)^2 = r^2 \). Since \( \ell_1 \) and \( \ell_2 \) are parallel, the end points of \( \sigma_1 \) are the respective midpoints, whose (common) component in \( \mathbb{C} \) is

\[
z_o = \frac{1}{2} (z_+ (1) + z_- (1)) = \frac{1}{2} (z_+ (r_0) + z_- (r_0)) = \frac{1}{2(a^2 + 1)} (a + i).
\]

Hence, \( q_+ (\ell_1) = (z_o, 0) \) and \( q_- (\ell_2) = (z_o, h_0) \). Similarly, \( q_- (\ell_0) = (0, 0) \) and \( q_+ (\ell_3) = (0, h_0) \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/figure5.png}
\caption{Elements for the construction of \( S \)}
\end{figure}

Let \( p_1 = (w (1), 0) \) and \( p_2 = (w (r_0), h_0) \) be the poles of the line \( \ell_1 \) in the plane \( \mathbb{C} \times \{0\} \) and of the line \( \ell_2 \) in the plane \( \mathbb{C} \times \{h_0\} \), respectively. That is, \( w (r) \) is the intersection of the lines tangent to the circle of radius \( r \) in \( \mathbb{C} \) at the points \( z_- (r) \) and \( z_+ (r) \). Using the construction of the shortest segment joining two oriented lines, the segment \( \sigma_2 \) is contained in the line perpendicular to \( \ell_3 \) passing through \( (w (r_0), h_0) \), that is, the line \( (\text{Re} \, w (r_0) + \Re i, h_0) \). Putting \( r = 1 \), we get that \( \sigma_0 \) is contained in the line \( (\text{Re} \, w (1) + \Re i, 0) \).

We have that \( w (r) = z_+ (r) + s_o (r) iz_+ (r) \), where \( s_o \) is the solution of the equation

\[
z_+ (r) + siz_+ (r) = z_- (r) - siz_- (r).
\]

A straightforward computation yields \( \text{Re} \, w (r) = 2ar^2 \).

Now, computing the intersections of the remaining \( \sigma_k \) with the lines \( \ell_j \), we obtain the rest of the \( q_\delta (\ell_j) \), \( \delta = \pm 1 \):

\[
q_+ (\ell_0) = (\text{Re} \, w (1), 0) = 2 (a, 0), \quad q_- (\ell_1) = 2 \left( a + i \left( \frac{1}{2} - a^2 \right), 0 \right)
q_+ (\ell_2) = 2 \left( ar_o + i \left( \frac{1}{4} - a^2 r_o^2 \right), h_0 \right), \quad q_- (\ell_3) = (\text{Re} \, w (r_0), h_0) = (2ar_o^2, h_0).
\]
As in Proposition 8, for each $a > 0$ there exists a quadratically convex surface $S_a$ in $\mathcal{H}$ tangent to the plane $P_k$ at the midpoint of $\sigma_k$ for any $k = 0, \ldots, 3$. The associated billiard map $B_a$ satisfies $(B_a)^4 (\ell_0) = \ell_0$ and its holonomy at $\ell_0$ turns out to be

$$H(a) = \sum_{k=0}^{3} (-1)^k d(q_+ (\ell_k), q_- (\ell_k)).$$

Particularizing $r_o = h_o = 1/\sqrt{2}$, using (21) we obtain that

$$H(a) = \operatorname{arctanh} (2a) - \operatorname{arctanh} \left( a\sqrt{4a^2 + 3} \right) + \operatorname{arctanh} \left( a\sqrt{2a^2 + 1} \right) - \operatorname{arctanh}(a).$$

We compute $H(0) = 0$ and $H'(0) \neq 0$. Therefore for a small enough positive $a$, the holonomy of $B_a$ at $\ell_0$ does not vanish.

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