Fractional branes on ALE orbifolds

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Abstract: We derive the classical type IIB supergravity solution describing fractional D3-branes transverse to a $\mathbb{C}^2/\Gamma$ orbifold singularity, for $\Gamma$ any Kleinian ADE subgroup. This solution fully describes the $\mathcal{N} = 2$ gauge theory with appropriate gauge groups and matter living on the branes, up to non-perturbative instanton contributions.

1 Introduction

One of the present major challenges in string theory is that of extending the AdS/CFT duality to more general gravity/gauge theory relations, valid in non-conformal situations. A possibility is that of considering fractional branes in singular spaces such as conifolds (corresponding to $\mathcal{N} = 1$ gauge theory) [1] or orbifolds (corresponding to $\mathcal{N} = 2$ gauge theory).

Supergravity solutions dual to non-conformal theories are generically singular. Some stringy mechanism (like the conifold deformation [2] or the enhàncèn mechanism in the orbifold cases) is expected to repair the singularities so as to match the true IR behaviour of the field theory.

In this communication, based on [4], we deal with $\mathcal{N} = 2$ gauge theories arising from fractional brane configurations, where one has good control on the field theory side (through the Seiberg-Witten [3] exact low energy effective action) and on the string theory side (strings on an orbifold). This allows for detailed comparisons at the perturbative level and (hopefully) at the level of instanton corrections.

Fractional branes on the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold have been thoroughly investigated in the literature [3, 4]. By considering the more general case of fractional brane configurations on $\mathbb{C}^2/\Gamma$, we find a very elegant agreement of the gravitational solutions, regulated by appro-
appropriate enhancements, with the perturbative treatment of the corresponding gauge theories. We also get more insight on the role of the flux of the RR five-form.

2 Closed and open strings on $\mathbb{C}^2/\Gamma$

We consider type IIB string theory on a space-time of the form $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma$, in which fractional D3-branes, transverse to the orbifold, are located at the orbifold singularity $z^1 = z^2 = 0$ (having indicated as $z^{1,2} \equiv x^{6,8} + ix^{7,9}$ the coordinates of $\mathbb{C}^2$). $\Gamma$ is a discrete subgroup of SU(2), acting on $\mathbb{C}^2$ by

$$g \in \Gamma : \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \mapsto Q(g) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix},$$

with $Q(g) \in$ SU(2) the defining 2-dim representation; the origin is clearly a fixed point. The Kleinian subgroups $\Gamma$ are ADE-classified (for instance, the $\mathbb{Z}_n$ cyclic groups correspond to $A_{n-1}$ algebras) through the McKay correspondence \[8\]: in the Clebsh-Gordan series

$$Q \otimes D_I = \oplus_J \tilde{A}_{IJ} D_J,$$

$\tilde{A}_{IJ}$ turns out to be the adjacency matrix of an extended Dynkin diagram of a simply-laced Lie algebra $G_\Gamma$. So, decomposing $I = (0,i)$, the irreducible representations $D_i$ correspond to the nodes of the diagram, i.e., to the simple roots $\alpha_i$; the trivial irrep $D_0$ corresponds to the extra node in the extended diagram, i.e., to $\alpha_0 \equiv - \sum_i d_i \alpha_i$. The dimensions $d_i$ of the irreps $D_i$ correspond to the Dynkin labels (while of course $d_0 = 1$). It is important in the following the fact that the extended Cartan matrix $\tilde{C}_{IJ} \equiv \alpha_I \cdot \alpha_J = 2 \delta_{IJ} - \tilde{A}_{IJ}$ is semidefinite positive, and admits the null eigenvector $d_I$: $\tilde{C}_{IJ} d_J = 0$.

The orbifold $\mathbb{C}^2/\Gamma$ is singular at the origin. Resolving this singularity in a canonical way produces \[9\] an Asymptotically Locally Euclidean (ALE) space $M_\Gamma$. This space has a non-trivial middle homology $H_2(M_\Gamma)$, generated by two-spheres $e_i$ arising in the resolution. The intersection form of these “exceptional” cycles is the (non-extended) Cartan matrix of $G_\Gamma$: $e_i \cdot e_j = -C_{ij}$. By Poincaré duality, the ALE space possesses a set of anti-self-dual two-forms $\omega^i$ such that

$$\int_{e_i} \omega^j = \delta^j_i \leftrightarrow \int_{\text{ALE}} \omega^j \wedge \omega^j = -(C^{-1})^{ij}.$$

Closed type IIB strings in this orbifold background admit an exact SCFT description \[10\]; in the untwisted sectors only $\Gamma$-invariant states are retained, and there are additional states from the twisted sectors. The twisted sectors are in correspondence with the conjugacy classes of $\Gamma$; in these sectors the momentum has no components along the directions in which $\Gamma$ acts, so the twisted fields have a 6-dimensional dynamics and are stuck at the fixed point of the orbifold. Viewing “geometrically” the orbifold background as the singular limit of the ALE space for vanishing two-cycles, the low-energy supergravity theory can be obtained à la Kaluza-Klein: in particular, decomposing the closed string massless fields on the two-forms $\omega^i$ dual to the cycles reproduces the spectrum of twisted fields. In our solution, the following twisted scalar fields \[^{1}\]will be crucial:

$$b_i \equiv \frac{1}{2\pi} \int_{e_i} B_2, \quad c_i \equiv \frac{1}{2\pi} \int_{e_i} C_2.$$

\[^{1}\]In the closed string theory, the $b_i$ fields should be periodic of period 1, as world-sheet instantons on the vanishing cycle $e_i$ contribute $\exp(2\pi ib_i)$ to the partition function. At the supergravity level, this symmetry is not explicit.
Reducing type IIB sugra on the singular orbifold, one arrives to the bulk supergravity action (restricted to the fields that will be turned on in the fractional brane solutions)

\[
S_b = \frac{1}{2\kappa^2} \left\{ \int d^{10}x \sqrt{-\text{det}G} \, R - \frac{1}{2} \int \frac{1}{4\pi^2} d\gamma_i \wedge *d\bar{\gamma}_j \wedge \omega^i \wedge *\omega^j \\
- \frac{i}{8\pi^2} C_4 \wedge d\gamma_i \wedge d\bar{\gamma}_j \wedge \omega^i \wedge \omega^j + \frac{1}{2} \bar{F}_5 \wedge *\bar{F}_5 \right\},
\]

where \( \bar{F}_5 \equiv dC_4 + C_2 \wedge dB_2 \) is self-dual, and \( \gamma_i \equiv c_i - ib_i \). Also, one has \( \kappa = 8\pi^{7/2}g_s(\alpha')^2 = 2\pi^{3/2}g_s \) (we use \( 2\pi\alpha' = 1 \)).

When we introduce also D3 branes transverse to the orbifold space the resulting open strings carry Chan-Paton factors that specify to which D3-branes they are attached. The group \( \Gamma \) may act on the Chan-Paton factors as well as on the string fields along the \( \mathbb{C}^2 \) directions. If we want a D3 to be located at a generic point \( p \) inside the orbifold, then the image branes at \( gp, g \in \Gamma \), fill a complete orbit. The Chan-Paton indices of open strings attached to such branes transform in the regular representation of \( \Gamma \); such configurations are called regular or bulk branes. However, the regular representation decomposes as \( \mathcal{R} = \oplus I d_I D_I \), so a regular brane at the origin of the orbifold (which is fixed under \( \Gamma \)) it can decompose into fractional branes. A fractional brane of type \( I \) corresponds to Chan-Paton indices in the irrep \( D_I \), and is stuck at the fixed point \( \Gamma I \). In relation with the gauge theory living on the branes, the position of a regular D3-brane inside \( \mathbb{C}^2/\Gamma \) corresponds to the Higgs branch, while the positions of its fractional constituents in the \( x^4, x^5 \) plane correspond, as we will see, to the Coulomb branch.

The fractional branes can be described as boundary states in the orbifold SCFT but admit also a (quasi)-geometrical interpretation as D5 branes wrapped on the vanishing cycles \( e_i \). By either methods, it is possible to infer the world-volume action for a collection of \( \{m^I\} \) fractional branes of the various types:

\[
S_{\text{wv}} = - \frac{T_3}{\kappa} \left\{ \int_{D_3} \sqrt{-\text{det}G} m^J b_J + \int_{D_3} C_4 m^J b_J + \int_{D_3} m^J A_{4,J} \right\}.
\]

Here the tension has the usual expression \( T_p = \sqrt{\pi(2\pi\sqrt{\alpha'})^{3-p}} \), and the twisted RR 4-forms \( A_{4,J} \), arising from \( C_6 = A_{4,J} \wedge \omega^J \), are dual in 6 dimensions to the scalars \( c_I \). Notice from Eq. (4) that the tension and charge of a brane of type \( I \) are proportional to the twisted field \( b_I \).

3 The gauge theory on the brane

The gauge theory living on a configuration of fractional D3 branes is determined by the spectrum (and interactions) of the massless open string fields stretching between such branes and surviving the orbifold projection, taking into account also the action of \( \Gamma \) on the Chan-Paton factors. The result is that there is a \( U(m^I) \) \( \mathcal{N} = 2 \) gauge multiplet from the strings stretching between the \( m^I \) branes of type \( I \). Moreover, there are \( \tilde{A}_{IJ} \) hypermultiplets in the \( (m^I, \tilde{m}^J) \) representation of \( U(m^I) \) and \( U(m^J) \) respectively, from strings stretched between branes of type \( I \) and \( J \); namely, the number of hypermultiplets from such strings is the number of links between the nodes \( I \) and \( J \) in the extended Dynkin

\footnote{The brane of type 0 is a D5 wrapped on \( e_0 = -\sum_i d_i e_i \) with a background gauge flux \( f_{e_0} \mathcal{F} = e_\pi \).}
The $\beta$-functions of the various $U(m^I)$ gauge factors have only one-loop perturbative contributions. The coefficients are given by

$$2m^I - \hat{A}_{IJ} m^J = \hat{C}_{IJ} m^J,$$

the positive contributions being from the gauge multiplet, the negative ones from the matter hypermultiplets. Thus, if we define a bare coupling $\bar{g}_I$ at an UV scale $\rho_0$, it dimensionally transmutes into a RG invariant scale $\Lambda_I$ and the (complexified) coupling constants run as described by

$$\tau_I(\mu) \equiv \frac{4\pi i}{g_I^2} + \frac{\theta_I}{2\pi} = \frac{i}{2\pi} \hat{C}_{IJ} m^J \ln \frac{\mu}{\Lambda_I}, \quad \Lambda_I = \rho_0 e^{-\frac{8\pi^2}{\hat{g}_I^2 \hat{C}_{IJ} m^J}}.$$

This $\mathcal{N} = 2$ theory has a moduli space, parametrized by the v.e.v.’s of the adjoint complex scalars in the vector multiplets, $\langle \phi \rangle = \text{diag}(a_1, \ldots, a_{m^I})$, and by the masses $M_k$ ($k = 1, \ldots, N_h = \hat{A}_{IJ} m^J$) of the hypers. A Seiberg-Witten description of the low energy effective action is possible. The effective couplings have a perturbative part, which is determined by one-loop diagrams with massive vector- and hyper-multiplets running in the loop, plus non-perturbative corrections due to instantonic effects. The analysis of [7] in the $\mathbb{C}^2/\mathbb{Z}_2$ case indicates that in the limit of many branes the instantonic effects arise very suddenly near the scales $\Lambda_I$ and, below $\Lambda_I$, the couplings $\tau_I$ tend to stop running.

## 4 The supergravity solution vs the gauge theory

The equations of motion following from the bulk and boundary actions Eq. (5) and Eq. (6) for the metric and the RR five-form (for the explicit expressions see [4]) can be solved with the usual D3 brane ansatz in terms of a function $H$ which depends upon all the transverse coordinates, $z$ and $x^6, \ldots, x^9$. The other untwisted fields, in particular the dilaton $\varphi$ and the RR scalar $C_0$ are trivial. The only excited twisted fields are the scalars $\gamma_i = c_i - ib_i$; they depend only on $z$. One finds that the choice of constant dilaton and axion is consistent provided $\gamma_i(z)$ are analytic functions. This being the case, one can also prove that half susies are preserved in the bulk (see [15] in the case of a non-collapsing ALE).

Solving the eq. of motion for the twisted scalars we find in fact the harmonic functions:

$$\gamma_I(z) = -i \frac{g_s}{2\pi} \hat{C}_{IJ} m^J \ln \frac{z}{\Lambda_I},$$

where $\Lambda_I$ are introduced as IR (small $z$) regulators. We see that these fields have exactly the same logarithmic running as the gauge couplings, suggesting the correspondence

$$\tau_I(z) \leftrightarrow -\gamma_I(z)/g_s.$$

In this perspective, $z$ corresponds to the complexified energy scale, $\Lambda_I$ to the dynamically generated scales; moreover, the distance $|z| = \rho_0$ where the twisted fields attain their perturbative orbifold value $b_I = d_I/|\Gamma|$ corresponds to the UV scale where the bare theory is defined, with bare coupling $g_I^2 = 4\pi g_s |\Gamma|/d_I$.

This RG/twisted field duality can be justified in the brane-probe approach. Indeed, a single test brane of type $I$ placed at a point $z$ on the fixed plane in the background generated by a configuration of $\{m^I\}$ branes is BPS. The effective two-derivative action for the $U(1)$ living on the test brane is found to be an $\mathcal{N} = 2$ effective theory with coupling
\( \tau_I(z) \), \( z \) being the position of the probe, where \( \tau_I(z) \), see Eqs. (9,10) is the twisted scalar generated by the background. This is the same answer given by the perturbative part of the Seiberg-Witten prepotential for this particular breaking of \( U(m_I + 1) \) to \( U(1) \times U(m_I) \). These perturbative contributions arise from 1-loop diagrams in which the massive vector-and hyper-multiplet fields run in the loop and correspond to loops of open strings stretched between the test brane and the background branes. Upon open/closed duality, these diagrams correspond to the tree-level propagation of massless closed strings, the twisted scalars, from the background branes to the probe, that is, to the evaluation of the probe action in the classical background. The situation is similar to the relations between RG flows and NS-NS tadpoles found in different models [10,17].

Besides the twisted fields, the fractional brane solution displays a non-trivial metric and RR 4-form, parametrized by the function \( H(z, x^i) \). The effective tension and RR charge of a type \( I \) fractional branes, proportional to \( b_I(z) \), attains its orbifold value \( d_I/|\Gamma| \) at the UV scale \( \rho_0 \), while it vanishes at the enhançon radius \( \Lambda_I \). The construction of the boundary action is justified at the orbifold point, but we assume we can trust it down to \( \Lambda_I \), where instanton effects become very suddenly important. So we consider the fractional branes of each type \( i \) to be located on an enhançon shell of radius \( \Lambda_i \) in the \( z \)-plane (and those of type 0 at a point where \( b_0 = 1 \)). The solution of the RR 4-form and Einstein equation involves IR divergent integrations, which are regulated via the enhançon (recall that by Seiberg-Witten analysis [7] the twisted fields \( \gamma_i \) are expected to basically stop running below their enhançon); see also [18]. Above the highest enhançon, the function \( H \) can be expressed in terms of the UV cutoff \( \rho_0 \) as

\[
H = 1 + \frac{g_s m^I d_I}{\pi |\Gamma|} \frac{1}{r^4} + \frac{g_s^2 m^I \tilde{C}_{I,J} m^J}{4\pi^2} \left[ \ln \frac{r^4}{\rho_0^2 \sigma^2} - 1 + \frac{\rho^2}{\sigma^2} \right],
\]

where \( \rho^2 = (x^4)^2 + (x^5)^2 \), \( \sigma^2 = \sum_{i=0}^9 (x^i)^2 \) and \( r^2 = \rho^2 + \sigma^2 \). At lower scales, the expression is modified as the various types of twisted fields contribute only above the corresponding enhançon. Without these modifications, the metric exhibits singularities where \( H = 0 \).

The flux \( \Phi_5(\rho) \) that measures the RR D3-charge contained within a surface \( \Sigma \) comprising a disk of radius \( \rho \) in fixed \( z \)-plane is found to be, for \( \rho \) above the highest enhançon,

\[
\Phi_5(\rho) = 4\pi^2 g_s \left( \frac{m^I d_I}{|\Gamma|} + \frac{g_s m^I \tilde{C}_{I,J} m^J \ln \frac{\rho}{\rho_0}}{2\pi} \right) = 4\pi^2 g_s Q(\rho),
\]

where \( Q(\rho) = m^I b_I(\rho) \) is the D3 charge encoded in the boundary action. The logarithmic running of the five-form flux is quite general for fractional brane solutions, and is expected to be related to a measure of the degrees of freedom of the system. In our case, notice that for any configuration, we always have \( \tilde{C}_{I,J} m^I m^J \geq 0 \), so that the flux, i.e. the untwisted charge \( Q(\rho) \) is always decreasing towards the IR. It satisfies the differential equation

\[
\frac{dQ}{d\ln \rho} = \frac{g_s}{2\pi} \tilde{C}_{I,J} m^I m^J \propto \beta_i \beta_j G^{ij},
\]

where we introduced the logarithmic derivatives of the twisted scalars, \( \beta_i \equiv \frac{d\tau_i}{d\ln \rho} = i\tilde{C}_{I,J} m^I/(2\pi) \), that is the \( \beta \)-functions for the gauge couplings \( \tau_i \), and the metric \( G^{ij} \propto (C^{-1})^{ij} \) appearing in the kinetic terms for the twisted scalars themselves, as it can be seen from Eq. (6) and Eq. (9). The behaviour of the five-form flux has thus a very

\[\footnote{We consider configurations such that all but one twisted field run to UV asymptotic freedom; then the remaining one, say \( b_0 \), which is not independent, has the opposite running.}\]
suggestive formal analogy with that of the holographic $c$-function of [19], which satisfies $dc/d\ln \rho = 2\beta_i\bar{\beta}_j G^{ij}$. Though the latter is defined in the different context of 5-dimensional gauged supergravity, still we think this analogy, to be better investigated, supports the possible relation of the flux to a sensible measure of the degrees of freedom of the system.

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