\textbf{C*-EXTREME MAPS AND NESTS}

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ABSTRACT. The generalized state space $S_H(A)$ of all unital completely positive (UCP) maps on a unital C*-algebra $A$ taking values in the algebra $B(H)$ of all bounded operators on a Hilbert space $H$, is a C*-convex set. In this paper, we establish a connection between C*-extreme points of $S_H(A)$ and a factorization property of certain algebras associated to the UCP maps. In particular, the factorization property of some nest algebras is used to give a complete characterization of those C*-extreme maps which are direct sums of pure UCP maps. This significantly extends a result of Farenick and Zhou [Proc. Amer. Math. Soc. 126 (1998)] from finite to infinite dimensional Hilbert spaces. Also it is shown that normal C*-extreme maps on type I factors are direct sums of normal pure UCP maps if and only if an associated algebra is reflexive. Further, a Krein-Milman type theorem is established for C*-convexity of the set $S_H(A)$ equipped with bounded weak topology, whenever $A$ is a separable C*-algebra or it is a type I factor. As an application, we provide a new proof of a classical factorization result on operator-valued Hardy algebras.

1. INTRODUCTION

Quantization in functional analysis and search for the right noncommutative analogue of various classical concepts have attracted considerable amount of interest among operator algebraists. Several different notions of quantizations of convexity have appeared in the literature over the decades, among which we cite a few [9, 11, 12, 13, 15, 20, 22]. One such natural extension is C*-convexity, where the idea is to replace scalar valued convex coefficients by C*-algebra valued coefficients. This particular notion has been explored in different contexts. The initial definition was for subsets of C*-algebras [20]. Subsequently it has been extended on similar lines for subsets of bimodules over C*-algebras [22], for spaces of unital completely positive maps [12], and for positive operator valued measures [13]. In all these frameworks, one of the primary goals has been to identify C*-extreme points of the corresponding C*-convex sets and look for an analogue of Krein-Milman theorem. Our focus in this paper is the C*-convexity structure of the generalized state space $S_H(A)$ of all unital completely positive (UCP) linear maps from a unital C*-algebra $A$ to $B(H)$, the algebra of all bounded operators on a separable Hilbert space $H$. Generalized state spaces are thought of as quantizations of usual state spaces.

Motivated by the ideas of Loebl and Paulsen [20], the notion of C*-convexity and C*-extreme points of $S_H(A)$ was defined and studied by Farenick and Morenz [12]. They developed some general properties, however the main focus remained on the case when $H$ is a finite dimensional Hilbert space, that is the case, $H = \mathbb{C}^n$ for some $n \in \mathbb{N}$. They gave a complete description of C*-extreme points of $S_{\mathbb{C}^n}(A)$, whenever $A$ is a commutative C*-algebra or a finite dimensional matrix algebra. Following this work, Farenick and Zhou [14] came up with an abstract characterization of C*-extreme points via Stinespring decomposition, using which the structure of all C*-extreme points of $S_{\mathbb{C}^n}(A)$ was illustrated for an arbitrary C*-algebra $A$. It was shown that all such maps are direct sums of pure UCP maps satisfying some ‘nested’ properties.

In the case when the C*-algebra $A$ is commutative and the Hilbert space $H$ is arbitrary dimensional, the techniques of positive operator valued measures were exploited by Gregg [16] to study necessary conditions for C*-extreme points of $S_H(A)$. Banerjee et al. [4] have recently used this approach to show in particular that all C*-extreme points of $S_H(A)$ are *-homomorphisms whenever $A$ is a commutative C*-algebra with countable spectrum. The purpose of this article is to

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undertake a systematic investigation of the structure of $C^*$-extreme points in $S_H(A)$ for arbitrary (not-necessarily commutative) $C^*$-algebras $A$.

We begin with a discussion of some abstract characterizations of $C^*$-extreme points of $S_H(A)$ in Section 2. Making use of a result from [14], we present a connection between $C^*$-extreme points and factorization property of certain subalgebras in some von Neumann algebras (see Definition 2.8 and Corollary 2.10). Our main insight is that characterizations of $C^*$-extreme points naturally lead to some nests of subspaces and we can invoke some well-known factorization results of the associated nest algebras. In other words we develop a strong mathematical link between the theory of $C^*$-extreme points of UCP maps and that of nest algebras. The theory of nest algebras and their factorization property has a long history, for which we refer the readers to the beautiful book by Davidson [8].

One of our main results is Theorem 3.7 in Section 3, which generalizes significantly a result of [14] from finite dimensional Hilbert spaces to infinite dimensions. More precisely, a complete description is given for countable direct sums of pure UCP maps to be $C^*$-extreme points. This result pinpoints a refinement needed to a sufficiency condition suggested by [14] for $C^*$-extremity of direct sums of pure UCP maps.

Section 4 is devoted to the study of normal $C^*$-extreme maps on type I factors. The structure of normal UCP maps on these algebras is well-known. This knowledge helps us to arrive at necessary and sufficient conditions for normal $C^*$-extreme maps to be direct sums of normal pure UCP maps (Theorem 4.8). In the course of the proof, we apply a fact recently proved by the authors [5] that all reflexive algebras having factorization are nest algebras.

A fundamental result in classical convexity theory is Krein-Milman theorem for compact convex sets in locally convex topological vector spaces. Naturally, an analogue of Krein-Milman theorem is expected for quantized convexity under appropriate topology. Several researchers have been quite successful in reaching this goal under varying set-ups, particularly when the operator-valued coefficients are taken from finite dimensional $C^*$-algebras: see for example, for compact $C^*$-convex subsets of $M_n$ [23], for compact matrix convex sets in locally convex spaces [29], and for weak$^*$-compact $C^*$-convex sets in hyperfinite factors [21]. However there are instances where such theorems fail to hold. In fact Magajna [22] produced an example of a weak$^*$-compact $C^*$-convex subset of an operator $B$-bimodule over a commutative von Neumann algebra $B$ which does not even possess any $C^*$-extreme point. Nevertheless, for $C^*$-convex spaces of UCP maps equipped with bounded weak topology, some promising results have appeared in restricted cases. More specifically, Krein-Milman type theorems are known to be true for $C^*$-convexity of the space $S_H(A)$ in the following two cases: (1) when $A$ is an arbitrary $C^*$-algebra and $H$ a finite dimensional Hilbert space [12], (2) when $A$ is a commutative $C^*$-algebra and $H$ has arbitrary dimension [4]. We extend this line of research in Section 5, by showing a Krein-Milman type theorem for $C^*$-convexity of $S_H(A)$, whenever $H$ is infinite dimensional and separable, and $A$ is a separable $C^*$-algebra or a type I factor (Theorem 5.3). Whether the same holds for $S_H(A)$ in full generality remains as an open question.

Finally in Section 6, we produce a number of examples of $C^*$-extreme maps, and consider their applications. At first, behaviour of $C^*$-extreme points under minimal tensor product of UCP maps are examined to derive more $C^*$-extreme points. Further, examples of certain $C^*$-extreme points in $S_H(C(T))$ are seen (here $C(T)$ is the space of all continuous functions on the unit circle $T$), using which we provide a new proof of a known classical result of Szegö and its operator valued analogue about factorization property of operator valued Hardy algebras. Lastly factorization property of some well-known algebras are utilized to produce examples of UCP maps, some of which are $C^*$-extreme and some are not.

The following convention will be followed throughout the paper. All Hilbert spaces on which completely positive maps act are complex and separable, where the inner product is assumed to be linear in the second variable. For Hilbert spaces $H$ and $K$, $B(H,K)$ denotes the space of all bounded linear operators from $H$ to $K$. We denote by $B(H)$ the algebra of all bounded operators on $H$. By subspaces, projections and operators, we mean closed subspaces, orthogonal projections and bounded operators respectively. For any subset $E$ of $H$, $[E]$ denotes the closed subspace generated by $E$. The orthogonal complement of a subspace $F$ in a subspace $E$ will be denoted by $E \ominus F$. If
\( \{ E_i \}_{i \in \Lambda} \) is a collection of subspaces of \( \mathcal{H} \), then we write \( \bigwedge_{i \in \Lambda} E_i = \cap_{i \in \Lambda} E_i \) and \( \forall_{i \in \Lambda} E_i = \bigvee_{i \in \Lambda} E_i \). For any subspace \( E \), we denote by \( P_E \) the projection onto \( E \). All \( C^* \)-algebras considered will be assumed to contain identity, which we denote by 1 or by \( I_\mathcal{H} \) if the Hilbert space \( \mathcal{H} \) on which the algebra acts needs to be specified. For any self-adjoint subalgebra \( \mathcal{M} \) of \( \mathcal{B}(\mathcal{H}) \), we denote by \( \mathcal{M}' \) the commutant of \( \mathcal{M} \) in \( \mathcal{B}(\mathcal{H}) \).

2. General properties of \( C^* \)-extreme points

We begin with some preliminaries on the theory of completely positive maps and their dilations. Let \( \mathcal{A} \) be a unital \( C^* \)-algebra, and \( \mathcal{H} \) a separable Hilbert space. A linear map \( \phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) is called positive if \( \phi(a) \geq 0 \) in \( \mathcal{B}(\mathcal{H}) \) whenever \( a \geq 0 \) in \( \mathcal{A} \), and \( \phi \) is called completely positive (CP) if \( \phi \otimes \text{id}_n : \mathcal{A} \otimes M_n \to \mathcal{B}(\mathcal{H}) \otimes M_n \) is positive for all \( n \geq 1 \). Here \( \text{id}_n \) is the identity map on the \( C^* \)-algebra \( M_n \) of \( n \times n \) complex matrices. A unital *-homomorphism from \( \mathcal{A} \) to \( \mathcal{B}(\mathcal{H}) \) is called a representation. Note that a map of the form \( a \mapsto V^* \pi(a) V \), \( a \in \mathcal{A} \), for a representation \( \pi \) on \( \mathcal{A} \) and an appropriate operator \( V \), is a CP map on \( \mathcal{A} \).

Conversely, the well-known Stinespring dilation theorem says: if \( \phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) is a CP map, then there is a triple \( (\pi, V, \mathcal{H}_\pi) \) of a Hilbert space \( \mathcal{H}_\pi \), an operator \( V \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\pi) \), and a representation \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi) \) such that \( \phi(a) = V^* \pi(a) V \) for all \( a \in \mathcal{A} \), and satisfies the minimality condition that \( \mathcal{H}_\pi = \pi(\mathcal{A}) \mathcal{H} \). Moreover, any such triple is unique up to unitary equivalence. We call \( (\pi, V, \mathcal{H}_\pi) \) the minimal Stinespring triple for \( \phi \). Note that \( V \) is an isometry if and only if \( \phi \) is unital (i.e. \( \phi(1) = I_\mathcal{H} \)).

We remark here that although the Hilbert space on which a CP map acts is assumed to be separable, the Hilbert space \( \mathcal{H}_\pi \) in the minimal Stinespring triple \( (\pi, V, \mathcal{H}_\pi) \) may not be separable. However, when the \( C^* \)-algebra \( \mathcal{A} \) is also separable, \( \mathcal{H}_\pi \) is separable. See \([2, 24, 25]\) for more details on the theory of CP maps. We fix the following notation for the rest of the article.

**Notation.** We denote by \( S_\mathcal{H}(\mathcal{A}) \) the collection of all unital completely positive (UCP) maps from a unital \( C^* \)-algebra \( \mathcal{A} \) to \( \mathcal{B}(\mathcal{H}) \).

The set \( S_\mathcal{H}(\mathcal{A}) \) is called generalized state space on the \( C^* \)-algebra \( \mathcal{A} \). Note that \( S_\mathcal{C}(\mathcal{A}) \) is the usual state space of \( \mathcal{A} \). The set \( S_\mathcal{H}(\mathcal{A}) \) possesses both linear as well as other quantized convexity structure. In particular, the \( C^* \)-convexity structure of \( S_\mathcal{H}(\mathcal{A}) \) has played a very important role in understanding general theory of completely positive maps and various concepts associated with them, and this is the main theme of this paper.

We list two very important theorems on completely positive maps proved by Arveson \([2]\). For any two completely positive maps \( \phi, \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \), we say \( \psi \preceq \phi \), if \( \phi - \psi \) is completely positive. Below we state a Radon-Nikodym type theorem (Theorem 1.4.2, \([2]\)) for comparison of two completely positive maps.

**Theorem 2.1** (Radon-Nikodym type Theorem). Let \( \phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) be a completely positive map with minimal Stinespring triple \( (\pi, V, \mathcal{H}_\pi) \). Then a completely positive map \( \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) satisfies \( \psi \preceq \phi \) if and only if there is a positive contraction \( T \in \pi(\mathcal{A})' \) such that \( \psi(a) = V^* T \pi(a) V \) for all \( a \in \mathcal{A} \).

Although we are mainly concerned about \( C^* \)-extreme points of UCP maps, we sometimes mention results about (linear) extreme points as well, in order to make comparisons between the two situations. Clearly, the set \( S_\mathcal{H}(\mathcal{A}) \) is a convex set (i.e. \( \sum_{i=1}^n \lambda_i \phi_i \in S_\mathcal{H}(\mathcal{A}) \), whenever \( \phi_i \in S_\mathcal{H}(\mathcal{A}) \) and \( \lambda_i \in [0, 1], 1 \leq i \leq n \) with \( \sum_{i=1}^n \lambda_i = 1 \)). The following is an abstract characterization of extreme points of UCP maps due to Arveson (Theorem 1.4.6, \([2]\)).

**Theorem 2.2** (Extreme point condition). Let \( \phi \in S_\mathcal{H}(\mathcal{A}) \), and let \( (\pi, V, \mathcal{H}_\pi) \) be its minimal Stinespring triple. Then \( \phi \) is extreme in \( S_\mathcal{H}(\mathcal{A}) \) if and only if the map \( T \mapsto V^* TV \) from \( \pi(\mathcal{A})' \) to \( \mathcal{B}(\mathcal{H}) \) is injective.

We now turn our attention to the main topic of \( C^* \)-convexity of the generalized state space \( S_\mathcal{H}(\mathcal{A}) \). The space \( S_\mathcal{H}(\mathcal{A}) \) is a \( C^* \)-convex set in the following sense: If \( \phi_i \in S_\mathcal{H}(\mathcal{A}) \) and \( T_i \in \mathcal{B}(\mathcal{H}) \)
for $1 \leq i \leq n$ with $\sum_{i=1}^{n} T_i = I_{\mathcal{H}}$, then their $C^*$-convex combination

$$\phi(\cdot) := \sum_{i=1}^{n} T_i \phi_i(\cdot)T_i$$

is in $S_\mathcal{H}(\mathcal{A})$. The operators $T_i$'s are called $C^*$-coefficients. When $T_i$'s are invertible, the sum is called a proper $C^*$-convex combination of $\phi$. Following [12], we consider the following definition:

**Definition 2.3.** A UCP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called a $C^*$-extreme point of $S_\mathcal{H}(\mathcal{A})$ if whenever

$$\phi(\cdot) = \sum_{i=1}^{n} T_i \phi_i(\cdot)T_i,$$

is a proper $C^*$-convex combination of $\phi$, then $\phi_i$ is unitarily equivalent to $\phi$ for each $i$ i.e. there is a unitary $U_i \in \mathcal{B}(\mathcal{H})$ such that $\phi_i = U_i^* \phi(U_i)$.

It is clear that every map unitarily equivalent to a $C^*$-extreme point is also $C^*$-extreme. The structure of $C^*$-extreme points of $S_\mathcal{H}(\mathcal{A})$ has been studied extensively, see [4, 12, 13, 14, 16, 22, 30] among others. The aim of this article is to understand the behaviour of $C^*$-extreme points of $S_\mathcal{H}(\mathcal{A})$, up to unitary equivalence.

A key ingredient in our approach is a result by Farenick and Zhou [14], who taking cue from Arveson’s extreme point condition for UCP maps provided an abstract characterization of $C^*$-extreme points of $S_\mathcal{H}(\mathcal{A})$ by making use of Stinespring decomposition. We restate their result with minor modifications in our notation and give an outline of the proof. In what follows, $\mathcal{R}(T)$ denotes the range of an operator $T$.

**Theorem 2.4** (Theorem 3.1, [14]). Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a UCP map with minimal Stinespring triple $(\pi, \mathcal{V}, \mathcal{H}_\pi)$. Then $\phi$ is $C^*$-extreme in $S_\mathcal{H}(\mathcal{A})$ if and only if for any positive operator $D \in \pi(\mathcal{A})'$ with $V^*DV$ invertible, there exist a partial isometry $U \in \pi(\mathcal{A})'$ with $\mathcal{R}(U^*) = \mathcal{R}(U^*U) = \mathcal{R}(D^{1/2})$ and an invertible $Z \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VZ$.

**Proof.** $\implies$ Let $\phi$ be $C^*$-extreme in $S_\mathcal{H}(\mathcal{A})$, and let $D \in \pi(\mathcal{A})'$ be positive with $V^*DV$ invertible. Choose $\alpha > 0$ small enough so that $I_{\mathcal{H}_\pi} - \alpha D$ is positive and invertible. Set $T_1 = (\alpha V^*DV)^{1/2}$ and $T_2 = (V^*(I_{\mathcal{H}_\pi} - \alpha D)V)^{1/2}$. Then $T_1, T_2$ are invertible, and satisfy $T_1T_2 + T_2T_1 = V^*V = I_{\mathcal{H}}$. Now we define $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\phi_1(a) = T_1^{-1}(\alpha V^*D(a)V)T_1^{-1}, \quad \text{and} \quad \phi_2(a) = T_2^{-1}V^*(I_{\mathcal{H}_\pi} - \alpha D)\pi(a)V^{-1}T_2^{-1}$$

for all $a \in \mathcal{A}$. Clearly $\phi_1$ and $\phi_2$ are UCP maps such that $\phi(a) = T_1\phi_1(a)T_1 + T_2\phi_2(a)T_2$, $a \in \mathcal{A}$. Since $\phi$ is a $C^*$-extreme point in $S_\mathcal{H}(\mathcal{A})$, there exists a unitary $W \in \mathcal{B}(\mathcal{H})$ such that for all $a \in \mathcal{A}$, we have $\phi(a) = W^*\phi_1(a)W$, that is,

$$\phi(a) = (\sqrt{\alpha}D^{1/2}VT_1^{-1}W^*)\pi(a)(\sqrt{\alpha}D^{1/2}VT_1^{-1}W) = X^*\pi(a)X,$$

where $X = \sqrt{\alpha}D^{1/2}VT_1^{-1}W$. It is easy to verify that $[\pi(\mathcal{A})X(\mathcal{H})] = \mathcal{R}(D^{1/2})$ (call it $K$). Then the triple $(\pi(\cdot)_{K}, X, K)$ is another minimal Stinespring triple for $\phi$; hence by uniqueness, there exists a unitary operator $\tilde{U} : K \to \mathcal{H}_\pi$ such that

$$\tilde{U}X = V, \quad \text{and} \quad \pi(a)\tilde{U} = \tilde{U}\pi(a)|_K \quad \text{for all } a \in \mathcal{A}.$$

Extend $\tilde{U}$ to $\mathcal{H}_\pi$ by assigning 0 on the complement of $K$, and call this map $U$. Then $U$ is a partial isometry (in fact, a co-isometry) with $\mathcal{R}(U^*) = K$. We also note that $UX = V$ and $\pi(a)U = U\pi(a)$ for all $a \in \mathcal{A}$, so $U \in \pi(\mathcal{A})'$. Further, we have

$$V = UX = U\sqrt{\alpha}D^{1/2}VT_1^{-1}W = UD^{1/2}VZ^{-1}$$

where $Z = \frac{1}{\sqrt{\alpha}}W^*T_1 \in \mathcal{B}(\mathcal{H})$ is invertible; hence we get $UD^{1/2}V = VZ$.

$\iff$ Assume the ‘only if’ condition, and let $\phi(\cdot) = \sum_{i=1}^{n} T_i^* \phi_i(\cdot)T_i$ be a proper $C^*$-convex combination of $\phi$. Then $T_i^* \phi_i(\cdot)T_i \leq \phi(\cdot)$ for each $i$, so by Radon-Nikodym type theorem (Theorem 2.1) there exists $D_i \in \pi(\mathcal{A})'$ with $0 \leq D_i \leq I_{\mathcal{H}_\pi}$ such that $T_i^* \phi_i(\cdot)T_i = V^*D_i \pi(\cdot)V$. Note that $V^*D_i V = T_i^*T_i$, so $V^*D_i V$ is invertible; hence by hypothesis, there exist a partial isometry
and polar decomposition of operators. This powerful characterization is a consequence of Theorem 3.1.5, [\textit{Corollary 2.5}]. Let \( \phi \) be a positive operator such that \( \psi = \phi(1) = I_\mathcal{H} \), and since \( W_i \) is invertible, it follows that \( W_i \) is unitary. Thus \( \phi_i \) is unitarily equivalent to \( \phi \) for each \( i \), which concludes that \( \phi \) is a \( C^* \)-extreme point in \( S_\mathcal{H}(A) \).

It is claimed in [\textit{14}] that the operator \( U \) in the statement of Theorem 2.4 above is a unitary. At this point, we do not know whether \( U \) can be chosen to be a unitary.

The following corollary is a characterization of \( C^* \)-extreme maps provided by Zhou [\textit{30}]. The proof follows directly from Theorem 2.4 and Radon-Nikodym type theorem. However, the statement as written in [\textit{30}] has a minor error; see Example 3.7 in [\textit{4}] for a counterexample (which is stated there in the language of positive operator valued measures). Also see Example 2.7 below. The proof of the following proceeds on almost the same lines as in [\textit{30}], so it is left to the readers.

**Corollary 2.5 (Theorem 3.1.5, \textit{[30]})**. Let \( \phi \in S_\mathcal{H}(A) \). Then \( \phi \) is \( C^* \)-extreme in \( S_\mathcal{H}(A) \) if and only if for any completely positive map \( \psi \) satisfying \( \psi \leq \phi \) with \( \psi(1) \) invertible, there exists an invertible operator \( T \in B(\mathcal{H}) \) such that \( \psi(a) = T^* \phi(a) T \) for all \( a \in A \).

We now give another abstract characterization of \( C^* \)-extreme points, whose proof follows from a direct application of Theorem 2.4 and polar decomposition of operators. This powerful characterization turns out to be the most useful for our purpose.

**Corollary 2.6.** Let \( \phi : A \to B(\mathcal{H}) \) be a UCP map with minimal Stinespring triple \((\pi, V, \mathcal{H}_\pi)\). Then \( \phi \) is \( C^* \)-extreme in \( S_\mathcal{H}(A) \) if and only if for any positive operator \( D \in \pi(A)' \) with \( V^* D V \) invertible, there exists \( S \in \pi(A) \) such that \( D = S^* S \), \( SVV^* = VV^* SV^* \) and \( V^* SV \) is invertible (i.e. \( S(V\mathcal{H}) \subseteq V\mathcal{H} \) and \( S_{1\mathcal{V}V} \) is invertible).

**Proof.** \( \implies \) We use the equivalent conditions for \( C^* \)-extreme points as in Theorem 2.4. Assume first that \( \phi \) is a \( C^* \)-extreme point in \( S_\mathcal{H}(A) \). Let \( D \in \pi(A)' \) be a positive operator such that \( V^* DV \) is invertible. By Theorem 2.4, there exist a partial isometry \( U \in \pi(A)' \) with \( U^* UD^{1/2} = D^{1/2} \) and an invertible \( Z \in B(\mathcal{H}) \) such that \( UD^{1/2} V = VZ \). Set \( S = UD^{1/2} \). Then \( S^* S = D^{1/2} U^* UD^{1/2} = D \) and \( V^* SV = V^* UD^{1/2} V = V^* VZ = Z \). Thus \( V^* SV \) is invertible, and we get

\[
SVV^* = UD^{1/2}VV^* = (VZ)V^* = VV^*(VZ)V^* = VV^*(UD^{1/2}V)V^* = VV^* SVV^*.
\]

\( \iff \) Assume the ‘only if’ conditions. To show that \( \phi \) is \( C^* \)-extreme in \( S_\mathcal{H}(A) \), let \( D \in \pi(A) \) be positive with \( V^* DV \) invertible. By hypothesis, there exists \( S \in \pi(A) \) such that \( D = S^* S \), \( SVV^* = VV^* SV^* \) and \( V^* SV \) is invertible. Let \( U = UD^{1/2} \) be the polar decomposition of \( S \), where \( U \) is a partial isometry with initial space \( \mathcal{R}(D^{1/2}) \) i.e. \( \mathcal{R}(U^*) = \mathcal{R}(D^{1/2}) \). Since \( S \in \pi(A)' \), and \( \pi(A)' \) is a von Neumann algebra, it follows that \( U \in \pi(A)' \). Further, we have

\[
UD^{1/2} V = SV = (SVV^*) V = (VV^* SVV^*) V = VV^* SV = VZ,
\]

where \( Z = V^* SV \in B(\mathcal{H}) \), which is invertible. That \( \phi \) is \( C^* \)-extreme in \( S_\mathcal{H}(A) \) now follows from the equivalent criteria of Theorem 2.4. This completes the proof.

In the corollary above, we cannot drop the assumption that \( V^* DV \) is invertible as the following example shows. Below \( \mathbb{T} \) is the unit circle with one dimensional Lebesgue measure, \( C(\mathbb{T}) \) is the space of continuous functions on \( \mathbb{T} \), and \( H^2 = H^2(\mathbb{T}) \) is the Hardy space.

**Example 2.7.** Consider the UCP map \( \phi : C(\mathbb{T}) \to B(H^2) \) defined by

\[
\phi(f) = P_{H^2} M_f 1_{H^2} = T_f \quad \text{for all} \; f \in C(\mathbb{T}).
\]

Here \( M_f \) is the multiplication operator on \( L^2(\mathbb{T}) \) by the symbol \( f \). Then \( \phi \) is a \( C^* \)-extreme point in \( S_{H^2}(C(\mathbb{T})) \) (Example 2, [\textit{12}]). Note that \( \phi \) is already in minimal Stinespring form with the representation \( \pi : C(\mathbb{T}) \to B(L^2(\mathbb{T})) \) given by \( \pi(f) = M_f \). Then it is well-known that
and Observation

Then for any positive

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Then for any positive and invertible element

This contradicts the assumption that zero set of the function

Corollary 2.10. Let \( \phi \) be a \( C^* \)-extreme point in \( S_\pi(H) \) with minimal Stinespring triple \((\pi, V, \mathcal{H}_\pi)\). Then for any positive \( D \in \pi(A)' \) with \( V^*DV \) invertible, we observe the following from the proof of Theorem 2.4:

- There is a co-isometry \( U \) with \( \mathcal{R}(U^*) = \overline{\mathcal{R}(D^{1/2})} \) and an invertible \( Z \) such that \( UD^{1/2}V = VZ \). In particular if \( D \) is one-one (equivalently, \( D \) has dense range), then \( U \) is unitary.
- If \( S = UD^{1/2} \), then \( S^* \) is one-one.
- Also \( V^*SV \) is invertible such that \( \| (V^*SV)^{-1} \|^2 = \| (V^*DV)^{-1} \| \).

The following proposition follows directly from the definition of factorization and \(*\)-closed property of \( C^* \)-algebras, and so we omit the proof. Here and elsewhere, \( S^* \) denotes the set \{\( S^*: S \in S \)\} for any subset \( S \) of a \( C^* \)-algebra \( A \).

Proposition 2.9. If a subalgebra \( \mathcal{M} \) has factorization in a \( C^* \)-algebra \( A \), then \( \mathcal{M}^* \) also has factorization in \( A \) i.e. for any positive and invertible element \( D \in A \), there is an invertible element \( S \in \mathcal{M} \) with \( S, S^{-1} \in \mathcal{M} \) such that \( D = SS^* \).

The factorization property of several non-selfadjoint algebras has widely been studied, of which we mention a few. The Cholesky factorization theorem talks about factorization property of the algebra of upper triangular matrices in \( M_n \), the algebra of all \( n \times n \) matrices. A result of Szeg"{o} says that the Hardy algebra \( H^\infty(T) \) on the unit circle has factorization in \( L^\infty(T) \) (see Corollary 6.7 below). Many other algebras like nest algebras, subdiagonal algebras etc. and their factorization property have attracted very deep study (see [1, 5, 8, 19]).

The next corollary provides a bridge between the theory of \( C^* \)-extreme maps and factorization property of certain algebras.

Corollary 2.10. Let \( \phi \) be a \( C^* \)-extreme point in \( S_\pi(H) \), and let \((\pi, V, \mathcal{H}_\pi)\) be its minimal Stinespring triple. If \( D \) is any positive and invertible operator in \( \pi(A)' \), then there exists an invertible operator \( S \in \pi(A)' \) such that \( D = SS^* \), \( SVV^* = VV^*SVV^* \), and \( V^*SV \) is invertible with inverse \( V^*S^{-1}V \). In particular, the algebra

has factorization in \( \pi(A)' \).

Proof. Let \( D \) be a positive and invertible operator in \( \pi(A)' \). Clearly \( V^*DV \) is invertible; hence by Theorem 2.4 and Observation 1, we get a co-isometry \( U \in \pi(A)' \) with initial space \( \overline{\mathcal{R}(D^{1/2})} \) and an invertible \( Z \in \mathcal{B}(\mathcal{H}) \) such that \( UD^{1/2}V = VZ \). Note that \( \mathcal{R}(D^{1/2}) = \mathcal{H}_\pi \) as \( D \) is invertible; so \( U \) is unitary. Set \( S = UD^{1/2} \). Then \( S \in \pi(A)' \) and \( S \) is invertible. Also \( D = SS^*S^* \) and \( SVV^* = VV^*SVV^* \) with \( V^*SV \) invertible. Note that

and since \( V^*SV \) is invertible, it follows that \( (V^*SV)^{-1} = V^*S^{-1}V \). Further

Further

and

Finally
Since $V^*SV$ is invertible, it follows that $(I_{H_\pi} - VV^*)S^{-1}V = 0$; hence $S^{-1}VV^* = VV^*S^{-1}VV^*$. In particular, $S, S^{-1} \in \mathcal{M}$, so we conclude that $\mathcal{M}$ has factorization in $\pi(\mathcal{A})'$. \hfill \Box

We end this section by considering the question of when a $C^*$-extreme point is also extreme, and vice versa. If $\mathcal{H}$ is a finite dimensional Hilbert space, then it was shown in [12] that every $C^*$-extreme point of $S_\mathcal{H}(\mathcal{A})$ is extreme as well. Whether this is true for infinite dimensional Hilbert spaces is not known. Conversely, there are examples where an extreme point in $S_\mathcal{H}(\mathcal{A})$ is not $C^*$-extreme (see pg. 1470 in [14]). We discuss some sufficient criteria under which condition of $C^*$-extremity automatically implies extremity. Also see Corollary 3.15 below.

**Proposition 2.11.** Let $\phi \in S_\mathcal{H}(\mathcal{A})$ with minimal Stinespring triple $(\pi, V, H_\pi)$ such that $\pi$ is multiplicity-free (i.e. $\pi(\mathcal{A})'$ is commutative). If $\phi$ is $C^*$-extreme in $S_\mathcal{H}(\mathcal{A})$, then $\phi$ is extreme in $S_\mathcal{H}(\mathcal{A})$.

**Proof.** To show $\phi$ is extreme in $S_\mathcal{H}(\mathcal{A})$, we use Arveson’s extreme point condition (Theorem 2.2). Let $D$ be a self-adjoint operator in $\pi(\mathcal{A})'$ such that $V^*DV = 0$. By multiplying by a small enough scalar, we assume without loss of generality that $-\frac{1}{2}I_{H_\pi} \leq D \leq \frac{1}{2}I_{H_\pi}$. Then $D + I_{H_\pi}$ is positive and invertible. By Corollary 2.10, there exists an invertible $S \in \pi(\mathcal{A})'$ satisfying $SVV^* = VV^*SV^*$ with $V^*SV$ invertible such that $D + I_{H_\pi} = S^*S$. Thus we have

$$(V^*SV)^*(V^*SV) = V^*S^*(VV^*SV^*)V = V^*S^*(SVV^*)V = V^*S^*SV = V^*DV + V^*V = I_{H_\pi},$$

and since $V^*SV$ is invertible, it follows that $V^*SV$ is unitary, that is, $V^*SV^*SV = I_{H_\pi}$. Further as $\pi(\mathcal{A})'$ is commutative by hypothesis, we have $SS^* = S^*S = D + I_{H_\pi}$; hence $V^*SS^*V = V^*(D + I_{H_\pi})V = I_{H_\pi}$. Therefore we get

$$[V^*S(I_{H_\pi} - VV^*)][V^*S(I_{H_\pi} - VV^*)]^* = V^*S(I_{H_\pi} - VV^*)S^*V = V^*SS^*V - V^*SVV^*S^*V = 0.$$ 

This implies $V^*S(I_{H_\pi} - VV^*) = 0$, which further yields

$$V^*S = VV^*SV^* = SVV^*.$$ 

In other words, $S$ commutes with $VV^*$ which also implies that $S^*$ commutes with $VV^*$; hence $D$ commutes with $VV^*$. Therefore, we have $DV = DVV^*V = VV^*DV = 0$. But then $D\pi(\mathcal{A})V = \pi(\mathcal{A})DV = 0$ and since $\pi(\mathcal{A})V \mathcal{H}$ is dense in $\mathcal{H}_\pi$, we conclude that $D = 0$. Since $D$ is arbitrary, this proves that $\phi$ is extreme in $S_\mathcal{H}(\mathcal{A})$. \hfill \Box

3. Direct sums of pure UCP maps

The question of whether the direct sum of two $C^*$-extreme points is also $C^*$-extreme is very natural. For the case when the Hilbert space is finite dimensional, a necessary and sufficient criterion for the validity of the assertion is known due to Farenick-Zhou [14]. In fact if $\mathcal{A}$ is a unital $C^*$-algebra and $n \in \mathbb{N}$, then every $C^*$-extreme point in $S_{\mathcal{C}^n}(\mathcal{A})$ is a direct sum of pure UCP maps (Theorem 2.1, [12]), so the question reduces to finding conditions under which direct sums of pure UCP maps are $C^*$-extreme (which was exploited in [14]). But it is no longer the case in infinite dimensional Hilbert space settings that a $C^*$-extreme point is a direct sum of pure UCP maps (see Example 2, [12]). Nevertheless, finding criteria for a direct sum of pure UCP maps in $S_\mathcal{H}(\mathcal{A})$ (for $\mathcal{H}$ infinite dimensional) to be $C^*$-extreme is interesting in its own right. In this section, we provide a complete characterization for such maps to be $C^*$-extreme. One of the main applications of this description would be in proving Krein-Milman type theorem in Section 5.

We begin with some general properties of $C^*$-extremity under direct sums. In the rest of the article, $\Lambda$ will usually be a countable indexing set for a family of maps or subspaces. For any family $\{\phi_i : \mathcal{A} \to B(H_i)\}_{i \in \Lambda}$ of UCP maps, their **direct sum** $\oplus_{i \in \Lambda} \phi_i$ is the UCP map from $\mathcal{A}$ to $B(\oplus_{i \in \Lambda} H_i)$ defined by $(\oplus_{i \in \Lambda} \phi_i)(a) = \oplus_{i \in \Lambda} \phi_i(a)$, for all $a \in \mathcal{A}$. The following remark records the minimal Stinespring triple for a direct sum of UCP maps, which is easy to verify.

**Remark 3.1.** Let $\phi_i : \mathcal{A} \to B(H_i)$, $i \in \Lambda$, be a collection of UCP maps with respective minimal Stinespring triple $(\pi_i, V_i, K_i)$. Then the minimal Stinespring triple for $\oplus_{i \in \Lambda} \phi_i$ is given by $(\pi, V, K)$, where $K = \oplus_{i \in \Lambda} K_i$, $V = \oplus_{i \in \Lambda} V_i$ and $\pi = \oplus_{i \in \Lambda} \pi_i$. 
We now recall some notions relevant to our results. If $\pi : A \to B(\mathcal{H}_a)$ is a representation, and $K \subseteq \mathcal{H}_a$ is a subspace invariant (and hence reducing) under $\pi(a)$ for all $a \in A$, then the map $a \mapsto \pi(a)|_K$ is a representation from $A$ to $B(K)$, called sub-representation of $\pi$. Two representations $\pi_i : A \to B(\mathcal{H}_{\pi_i})$, $i = 1, 2$, are said to be disjoint if no non-zero sub-representation of $\pi_1$ is unitarily equivalent to any sub-representation of $\pi_2$. We shall use the following fact about disjoint representations (see Proposition 2.1.4, [3]): If $\pi_1$ and $\pi_2$ are disjoint representations and $S\pi_1(a) = \pi_2(a)S$, for all $a \in A$, then $S = 0$.

A representation $\pi$ is called irreducible if it has no non-zero sub-representation (equivalently, $\pi(A)' = C \cdot I_{\mathcal{H}_a}$). Note that if $\pi_1$ and $\pi_2$ are two non-unitarily equivalent irreducible representations, then $\pi_1(\cdot) \otimes I_{K_1}$ and $\pi_2(\cdot) \otimes I_{K_2}$ are disjoint representations (for any Hilbert spaces $K_1$ and $K_2$).

We recall some more terminologies from CP map theory. A completely positive map $\phi$ is called pure if whenever $\psi$ is a completely positive map with $\psi \leq \phi$, then $\psi = \lambda \phi$ for some $\lambda \in [0, 1]$. It is easy to verify that if $(\pi, V, \mathcal{H}_\pi)$ is the minimal Stinespring triple of a completely positive map $\phi$, then $\phi$ is pure if and only if $\pi$ is irreducible (Corollary 1.4.3, [2]). All pure UCP maps are known to be $C^*$-extreme as well as extreme points of $S_H(A)$ (Proposition 1.2, [12]).

Let $\phi_i : A \to B(\mathcal{H}_i)$, $i = 1, 2$, be two UCP maps. We say $\phi_2$ is a compression of $\phi_1$ if there exists an isometry $W : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\phi_2(a) = W^*\phi_1(a)W$, for all $a \in A$. If $\phi$ is a pure UCP map with the minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$, and $\psi = W^*\phi W$ is a compression of $\phi$ for some isometry $W$, then $(\pi, WV, \mathcal{H}_\pi)$ is the minimal Stinespring triple for $\psi$, and so $\psi$ is pure. This follows from the fact that $\pi(A)' = C \cdot I_{\mathcal{H}_a}$, so that $\pi(A)'' = B(\mathcal{H}_a)$, which further yields

$$[\pi(A)VW\mathcal{H}] = [\pi(A)''VW\mathcal{H}] = [B(\mathcal{H}_a)VW\mathcal{H}] = H_\pi.$$ 

Moreover, if $(\pi, V_i, \mathcal{H}_\pi)$ is the minimal Stinespring triple of UCP maps $\phi_i$, $i = 1, 2$ (i.e. both $\phi_1, \phi_2$ are compression of the same representation $\pi$), then one can easily show that $\phi_2$ is a compression of $\phi_1$ if and only if $V_i V_j^* \leq V_i V_j^*$ i.e. $\mathcal{R}(V_i) \subseteq \mathcal{R}(V_j)$.

Inspired from the notion of disjointness of representations, we define the same for UCP maps as follows. One can see this notion being considered for pure maps in [12].

**Definition 3.2.** For any two UCP maps $\phi_i : A \to B(\mathcal{H}_i)$, $i = 1, 2$ with respective minimal Stinespring triple $(\pi_i, V_i, \mathcal{H}_{\pi_i})$, we say $\phi_1$ is disjoint to $\phi_2$ if $\pi_1$ and $\pi_2$ are disjoint representations.

The major results of this paper deal with finding conditions under which direct sums of mutually disjoint UCP maps (especially, pure maps) are $C^*$-extreme. The next lemma and proposition are the first step in this direction.

It should be remarked that for a family of Hilbert spaces $\{\mathcal{H}_i\}_{i \in A}$, an operator $T$ in $B(\oplus_{i \in A} \mathcal{H}_i)$ will also be written in the matrix form $[T_{ij}]$, for some $T_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i)$.

**Lemma 3.3.** Let $\pi_i : A \to B(\mathcal{H}_i)$, $i \in \Lambda$, be a collection of mutually disjoint representations. If $\pi = \oplus_{i \in A} \pi_i$, then $\pi(A)' = \{\oplus_{i \in A} T_i; T_i \in \pi_i(A)\}'$.

**Proof.** Let $S \in \pi(A)' \subseteq B(\oplus_{i \in A} \mathcal{K}_i)$. Then $S = [S_{ij}]$ for some $S_{ij} \in B(K_j, K_i)$, such that for all $a \in A$, we have $[S_{ij}]\{(\oplus_{i \in A} \pi_i(a)) = [\oplus_{i \in A} \pi_i(a)](S_{ij});$ hence $S_{ij}(a) = \pi(a)S_{ij}$ for all $i, j$. For $i \neq j$, since $\pi_i$ is disjoint to $\pi_j$, it follows (from above mentioned result) that $S_{ij} = 0$. Also for each $i$, $S_{ii}(a) = \pi_i(a)S_{ii}$ for $a \in A$, implies that $S_{ii} \in \pi_i(A)'$. Thus $S = \oplus_{i \in A} S_{ii}$, where $S_{ii} \in \pi_i(A)'$. This shows that $\pi(A)' \subseteq \{\oplus_{i \in A} T_i; T_i \in \pi_i(A)'\}$.

The other inclusion is obvious. \(\Box\)

**Proposition 3.4.** Let $\{\phi_i : A \to B(\mathcal{H}_i)\}_{i \in A}$ be a collection of mutually disjoint UCP maps. Then $\phi = \oplus_{i \in A} \phi_i$ is $C^*$-extreme (resp. extreme) in $S_{\oplus_{i \in A} \mathcal{H}_i}(A)$ if and only if each $\phi_i$ is $C^*$-extreme (resp. extreme) in $S_{\mathcal{H}_i}(A)$.

**Proof.** Let $(\pi_i, V_i, K_i)$ be the minimal Stinespring triple for $\phi_i$, $i \in \Lambda$. Then as noted in Remark 3.1, $(\pi, K, V)$ is the minimal Stinespring triple for $\phi$, where $K = \oplus_{i \in A} K_i$, $\pi = \oplus_{i \in A} \pi_i$, and $V = \oplus_{i \in A} V_i$. Since $\pi_i$ is disjoint to $\pi_j$ for $i \neq j$, it follows from Lemma 3.3 that

$$\pi(A)' = \{\oplus_{i \in A} T_i; T_i \in \pi_i(A)\}' \subseteq B(\oplus_{i \in A} \mathcal{K}_i). \tag{3.1}$$

To prove the equivalent criteria for $C^*$-extremity, we shall use Corollary 2.6. Assume first that each $\phi_i$ is $C^*$-extreme in $S_{\mathcal{H}_i}(A)$. Let $D \in \pi(A)'$ be positive such that $V^*DV$ is invertible. Then it follows from (3.1) that $D = \oplus_{i \in A} D_i$ for some $D_i \in \pi_i(A)'$, and hence $V^*DV = \oplus_{i \in A} V_i^*D_iV_i$. Clearly
each $D_i$ is positive such that $V_j^* D_i V_i$ is invertible satisfying $\sup_{i \in \Lambda} \| (V_j^* D_i V_i)^{-1} \| = \| (V^* D V)^{-1} \|$. Since each $\phi_i$ is $C^*$-extreme, there exists an operator $\hat{S}_i \in \pi_i(A)'$ such that $D_i = S_i^* S_i V_i^* = V_i V_i^* S_i V_i^*$ and $V_i^* S_i V_i$ is invertible. Set $S = \oplus_{i \in \Lambda} S_i$. It is then immediate that $S \in \pi(A)'$, $D = S^* S$ and $SV V^* = V^* SV V^*$. Also from Observation 1, it follows that

$$\sup_{i \in \Lambda} \| (S_i V_i V_i^*)^{-1} \| = \sup_{i \in \Lambda} \| (V_i D_i V_i)^{-1} \| = \| (V^* D V)^{-1} \| < \infty,$$

which implies that $V^* S V = \oplus_{i \in \Lambda} V_i^* S_i V_i$ is invertible. Since $D$ is arbitrary, it follows that $\oplus_{i \in \Lambda} \phi_i$ is $C^*$-extreme.

Conversely, let $\oplus_{i \in \Lambda} \phi_i$ be $C^*$-extreme. Fix $j \in \Lambda$, and let $D_j \in \pi_j(A)'$ be a positive operator such that $V_j^* D_j V_i$ is invertible. For $i \neq j$, let $D_i = I_{K_i}$ and set $D = \oplus_{i \in \Lambda} D_i$. It is clear that $D \in \pi(A)'$. Also $D$ is positive and $V^* D V$ is invertible, as each $V_i^* D_i V_i$ is invertible whose inverse is uniformly bounded. Since $\oplus_{i \in \Lambda} \phi_i$ is $C^*$-extreme, there is an operator $S \in \pi(A)'$ such that $D = S^* S$, $SVV^* = V^* SVV^*$ and $V^* S V^*$ is invertible. Again from (3.1), we have $S = \oplus_{i \in \Lambda} S_i$ for some $S_i \in \pi_i(A)'$. Then the expressions $D = S^* S$ and $SVV^* = V^* SVV^*$ imply respectively that $D_j = S_j^* S_j$ and $S_j V^* V_j^* = V_j V_j^* S_j V_j^*$. Also invertibility of $V^* S V^*$ implies that $V_j^* S_j V_j$ is invertible. Since $D_j$ is arbitrary, we conclude that $\phi_j$ is $C^*$-extreme in $S_{H_j}(A)$. The case of equivalence of extreme points can be proved in a similar fashion using Arveson’s extreme point criterion (Theorem 2.2).

Some of the subsequent results about direct sums of pure maps involve a strong connection of their $C^*$-extremity conditions with the theory of nests of subspaces and corresponding nest algebras. To this end, we recall the basics of nest algebra theory. A family $E$ of subspaces of a Hilbert space $H$ is called a nest if $E$ is totally ordered by inclusion (i.e. $E \subseteq F$ or $F \subseteq E$ for any $E, F \in E$). The nest $E$ is complete if $0, H \in E$, and $\bigcup_{F \in \mathcal{F}} F \in E$ and $\bigvee_{F \in \mathcal{F}} F \in E$ for any subnest $\mathcal{F}$ of $E$. Note that any nest $E$ can be extended by adjoining $\{0\}, H$, and $\wedge$ and $\vee$ of arbitrary subfamily to get the smallest complete nest containing $E$, which we call the completion of $E$ (see Lemma 2.2, [26]). For any nest $E$, let Alg$E$ denote the collection of all operators in $B(H)$ which leave subspaces of $E$ invariant i.e.

$$\text{Alg} E = \{ T \in B(H); T(E) \subseteq E \text{ for all } E \in E \}.$$  

Clearly Alg$E$ is a unital closed algebra, called the nest algebra associated with $E$. If $\overline{E}$ denotes the completion of a nest $E$, then one can readily verify that Alg$\overline{E}$ = Alg$E$. The following theorem of Larson [19] about factorization property of certain nest algebras is very crucial to our results on $C^*$-extreme points. Recall Definition 2.8 for algebras having factorization.

**Theorem 3.5** (Theorem 4.7, [19]). Let $E$ be a nest in a separable Hilbert space $H$. Then Alg$E$ has factorization in $B(H)$ if and only if the completion $\overline{E}$ of the nest $E$ is countable.

We are now ready to prove the main result of this section, which gives necessary and sufficient criteria for direct sums of UCP maps to be $C^*$-extreme. This generalizes a result of Farenick and Zhou (Theorem 2.1, [14]) from finite to infinite dimensional Hilbert spaces.

Note that if $\phi, \psi : A \to B(H)$ are two pure UCP maps, then either $\phi$ and $\psi$ are mutually disjoint, or they are compression of the same irreducible representation. Therefore in view of Proposition 3.4, in order to give criteria of $C^*$-extremity of direct sums of pure UCP maps, it suffices to consider direct sum of only those pure UCP maps which are compression of the same irreducible representation (i.e. those pure maps which are not mutually disjoint), as done in the following theorem.

**Theorem 3.6.** Let $\psi_i : A \to B(H_i)$, $i \in \Lambda$, be a countable family of non-unitarily equivalent pure UCP maps with respective minimal Stinespring triple $(\pi_i, V_i, H_i)$, where $\pi$ is a fixed representation of $A$, and let $\phi_i = \psi_i(\cdot) \otimes I_{K_i}$, for some Hilbert space $K_i$. Set $H = \oplus_{i \in \Lambda} (H_i \otimes K_i)$, and $\phi = \oplus_{i \in \Lambda} \phi_i \in S_{H}(A)$. Then $\phi$ is $C^*$-extreme in $S_{H}(A)$ if and only if the following holds:

1. the family $\{ R(V_i^*) \}_{i \in \Lambda}$ of subspaces forms a nest in $H_i$, which induces an order on $\Lambda$ and
2. if $L_i = \oplus_{1 \leq j \leq i} K_j$ for $i \in \Lambda$, then completion of the nest $\{ L_i \}_{i \in \Lambda}$ in $\oplus_{i \in \Lambda} K_i$ is countable.
Proof. We know that each \( \psi_i \) is unitarily equivalent to the UCP map \( a \mapsto P_{R(V_i)}\pi(a)|_{R(V_i)} \), \( a \in \mathcal{A} \). So the fact from the hypothesis that \( \psi_i \) and \( \psi_j \) are not unitarily equivalent for \( i \neq j \) then implies that \( \mathcal{R}(V_i) \neq \mathcal{R}(V_j) \), that is,

\[
V_i V_i^* \neq V_j V_j^*, \quad \text{for all } i \neq j. \tag{3.2}
\]

Now set \( \mathcal{H}_\rho = \oplus_{i \in \Lambda} (\mathcal{H}_\pi \otimes \mathcal{K}_i) \), and consider the representation \( \rho : \mathcal{A} \to B(\mathcal{H}_\rho) \) defined by

\[
\rho(a) = \oplus_{i \in \Lambda} (\pi(a) \otimes I_{\mathcal{K}_i}) \quad \text{for all } a \in \mathcal{A},
\]

and the isometry \( V \in B(\mathcal{H}_\rho) \) given by

\[
V = \oplus_{i \in \Lambda} (V_i \otimes I_{\mathcal{K}_i}).
\]

It is clear that \( (\rho, V, \mathcal{H}_\rho) \) is the minimal Stinespring triple for \( \phi \). We identify the Hilbert space \( \mathcal{H}_\rho = \oplus_{i \in \Lambda} (\mathcal{H}_\pi \otimes \mathcal{K}_i) \) with the Hilbert space \( \mathcal{H}_\pi \otimes (\oplus_{i \in \Lambda} \mathcal{K}_i) \); so the representation \( \rho \) is given by

\[
\rho(a) = \pi(a) \otimes (\oplus_{i \in \Lambda} I_{\mathcal{K}_i}). \quad \text{Since} \quad \pi \text{ is irreducible}, \pi(A)' = \mathbb{C} \cdot I_{\mathcal{H}_\pi}; \quad \text{hence if we consider the operators on the Hilbert space} \mathcal{K} = \oplus_{i \in \Lambda} \mathcal{K}_i \text{ in matrix form, then} \rho(A)' \text{ is given by}
\]

\[
\rho(A)' = (\pi(A) \otimes I_{\mathcal{K}})' = I_{\mathcal{H}_\pi} \otimes B(\mathcal{K}) = \{ I_{\mathcal{H}_\pi} \otimes [T_{ij}]; \ T_{ij} \in B(\mathcal{K}_j, \mathcal{K}_i) \} \subseteq B(\mathcal{H}_\pi \otimes (\oplus_{i \in \Lambda} \mathcal{K}_i)).
\]

\[\implies \text{Assume now that } \oplus_{i \in \Lambda} \phi_i \text{ is a } C^*\text{-extreme point in } S_\mathcal{H}(\mathcal{A}). \text{ First we show that } \{ \mathcal{R}(V_i) \}_{i \in \Lambda} \text{ is a nest in } \mathcal{H}_\pi. \text{ Consider the subalgebra } \mathcal{M} \text{ of } B(\oplus_{i \in \Lambda} \mathcal{K}_i) \text{ given by}
\]

\[
\mathcal{M} = \{ [T_{ij}] \in B(\oplus_{i \in \Lambda} \mathcal{K}_i); \ (I_{\mathcal{H}_\pi} \otimes [T_{ij}])VV^* = VV^*(I_{\mathcal{H}_\pi} \otimes [T_{ij}])VV^* \}
\]

\[
= \{ [T_{ij}] \in B(\oplus_{i \in \Lambda} \mathcal{K}_i); \ V_i V_i^* \otimes T_{ij} = V_i V_i^* V_j V_j^* \otimes T_{ij} \ \forall \ i, j \in \Lambda \}. \tag{3.3}
\]

Since \( \oplus_{i \in \Lambda} \phi_i \) is \( C^*\)-extreme, it follows from Corollary 2.10 that \( I_{\mathcal{H}_\pi} \otimes \mathcal{M} \) has factorization in \( \rho(A)' = I_{\mathcal{H}_\pi} \otimes B(\mathcal{K}) \), which is to say that \( \mathcal{M} \) has factorization in \( B(\mathcal{K}) \).

Note that if there is an operator \( [T_{ij}] \in \mathcal{M} \) such that \( T_{mn} \neq 0 \) for some \( m, n \in \Lambda \), then since \( V_n V_n^* \otimes T_{mn} = V_m V_m^* \otimes T_{mn} \), it will follow that \( V_n V_n^* = V_m V_m^* V_n V_n^* \), which further implies \( V_n V_n^* \geq V_m V_m^* \). In other words, we have the following:

\[\text{If } V_m V_m^* \not\geq V_n V_n^* \text{ for some } m, n \in \Lambda, \text{ then } T_{mn} = 0 \text{ for all } [T_{ij}] \in \mathcal{M}. \tag{3.4}\]

For the remainder of this implication, we fix \( m, n \in \Lambda \) with \( m \neq n \). We shall prove that \( V_m V_m^* \geq V_n V_n^* \) or \( V_n V_n^* \geq V_m V_m^* \). Assume to the contrary that this is not the case. Then it follows from (3.4) that

\[T_{mn} = 0 \text{ and } T_{nm} = 0, \quad \text{for all } [T_{ij}] \in \mathcal{M}. \tag{3.5}\]

If \( \Lambda \) is a two point set, that is, \( \Lambda = \{ m, n \} \), then \( \mathcal{K} = \mathcal{K}_m \oplus \mathcal{K}_n \), and with respect to this decomposition, (3.5) implies that each element \( T \) in \( \mathcal{M} \) has the form

\[
\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \quad \text{for } T_1 \in B(\mathcal{K}_m) \text{ and } T_2 \in B(\mathcal{K}_n).
\]

But if we choose a positive and invertible operator \( D \) in \( B(\mathcal{K}) \) of the form \[
\begin{bmatrix} I_{\mathcal{K}_m} & D_1 \\ D_1^* & I_{\mathcal{K}_n} \end{bmatrix}
\]
with \( D_1 \in B(\mathcal{K}_m, \mathcal{K}_m) \) non-zero, then we cannot find any operator \( T \) in \( \mathcal{M} \) such that \( D = T^* T \). This will contradict the fact that \( \mathcal{M} \) has factorization in \( B(\mathcal{K}) \).

Therefore we assume for the rest of the implication that \( \Lambda \neq \{ m, n \} \). Now consider the sets

\[
\Lambda_1 = \{ l \in \Lambda \setminus \{ m, n \}; \ T_{lm} = 0 \text{ and } T_{nm} = 0 \text{ for all } [T_{ij}] \in \mathcal{M} \}, \tag{3.6}
\]

and

\[
\Lambda_2 = \Lambda \setminus (\Lambda_1 \cup \{ m, n \}).
\]

Note that \( \Lambda_1 \cap \Lambda_2 = \emptyset \) and \( \Lambda_1 \cup \Lambda_2 \cup \{ m, n \} = \Lambda \).

Consider the following decomposition:

\[
\mathcal{K} = \oplus_{i \in \Lambda} \mathcal{K}_i = \mathcal{K}_m \oplus \mathcal{K}_n \oplus (\oplus_{i \in \Lambda, i \neq m} \mathcal{K}_i) \oplus (\oplus_{i \in \Lambda, i \neq n, i \neq m} \mathcal{K}_i) = \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \mathcal{Q}_3 \oplus \mathcal{Q}_4, \tag{3.7}
\]

where

\[
\mathcal{Q}_1 = \mathcal{K}_m, \quad \mathcal{Q}_2 = \mathcal{K}_n, \quad \mathcal{Q}_3 = \oplus_{i \in \Lambda_1} \mathcal{K}_i, \quad \text{and} \quad \mathcal{Q}_4 = \oplus_{i \in \Lambda_2} \mathcal{K}_i.
\]
We shall show that $Q_3 \neq \{0\}$ and $Q_4 \neq \{0\}$ (that is, $A_1$ and $A_2$ are non-empty), and that with respect to decomposition in (3.7), each $T$ in $\mathcal{M}$ has the following form:

$$T = \begin{bmatrix} T_1 & 0 & A_1 & 0 \\ 0 & T_2 & A_2 & 0 \\ 0 & 0 & X_1 & X_2 \\ B_1 & B_2 & X_3 & X_4 \end{bmatrix},$$  \hspace{1cm} (3.8)

for appropriate operators $T_1, T_2, \ldots$ etc. For that, we first claim the following: If for some $l \neq m, n$, there exists an operator $[S_{ij}] \in \mathcal{M}$ such that $S_{lm} \neq 0$ or $S_{ln} \neq 0$, then

$$T_{ml} = 0 \text{ and } T_{nl} = 0, \ \forall \ [T_{ij}] \in \mathcal{M}. \hspace{1cm} (3.9)$$

To prove the claim in (3.9), assume that $S_{lm} \neq 0$, and let $[T_{ij}]$ be an arbitrary operator in $\mathcal{M}$. Then it follows from (3.4) that $V_l V^*_l \geq V_m V^*_m$. Since $V_l V^*_l \neq V_m V^*_m$ from (3.2), it follows that $V_m V^*_m \ngeq V_l V^*_l$; again from (3.4), we get $T_{ml} = 0$. Further, we note that $V_n V^*_n \ngeq V_l V^*_l$ (otherwise we would have $V_n V^*_n \geq V_l V^*_l \geq V_m V^*_m$, and so $V_n V^*_n \geq V_m V^*_m$ which is against our assumption). This in turn implies by (3.4) that $T_{nl} = 0$. Similarly or by symmetry, the condition $S_{ln} \neq 0$ will imply the required claim in (3.9).

We now show that $A_1$ is a non-empty set. Assume otherwise that $A_1 = \emptyset$. Then for each $l \in \Lambda \setminus \{m, n\}$, we have $l \notin A_1$, so there exists $[S_{ij}] \in \mathcal{M}$ such that either $S_{lm} \neq 0$ or $S_{ln} \neq 0$. In either case, (3.9) implies that for all $T = [T_{ij}] \in \mathcal{M}$, we have $T_{ml} = 0$ and $T_{nl} = 0$; hence the $(m, n)$ entry of the matrix $TT^*$ satisfies

$$\sum_{l \in \Lambda} T_{ml} T_{nl}^* = T_{mn} T_{nm}^* + T_{mn} T_{nn}^* + \sum_{l \neq m, n} T_{ml} T_{nl}^* = 0,$$

as $T_{nn} = 0$ and $T_{nm}^* = 0$ from (3.5). Thus for any positive and invertible $D = [D_{ij}] \in \mathcal{B}(\mathcal{K})$ with $D_{mn} \neq 0$, we cannot find $T \in \mathcal{M}$ such that $D = TT^*$. We can always get such positive and invertible operator $D$ (see the operator in (3.11) below). This violates the fact that $\mathcal{M}^*$ and hence $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$. Thus our claim that $A_1 \neq \emptyset$ is true.

We next show that $A_2$ is non-empty. Let if possible, $A_2 = \emptyset$. Then for each $l \in \Lambda$ with $l \neq m, n$, it follows that $l \in A_1$; hence for all $T = [T_{ij}] \in \mathcal{M}$, we have $T_{lm} = 0$ and $T_{ln} = 0$, so that $(m, n)$ entry of $T^*T$ satisfies

$$\sum_{l \in \Lambda} T_{lm}^* T_{ln} = 0,$$

as $T_{mn} = 0$ and $T_{nm}^* = 0$. Again for a positive and invertible operator $D = [D_{ij}] \in \mathcal{B}(\mathcal{K})$ with $D_{mn} \neq 0$, we can’t find any $T \in \mathcal{M}$ such that $D = T^*T$, violating the fact that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$. This shows our claim that $A_2 \neq \emptyset$.

Further we note that if $l \notin A_2$, then $l \notin A_1$, so $S_{lm} \neq 0$ or $S_{ln} \neq 0$ for some $[S_{ij}] \in \mathcal{M}$; hence it follows from (3.9) that $T_{ml} = 0$ and $T_{nl} = 0$ for all $[T_{ij}] \in \mathcal{M}$. Thus we have

$$A_2 \subseteq \{ l \in \Lambda \setminus \{m, n\}; \ T_{ml} = 0 \ \text{and} \ \ T_{nl} = 0 \ \forall \ [T_{ij}] \in \mathcal{M} \}. \hspace{1cm} (3.10)$$

Now let $T = [T_{ij}] \in \mathcal{M}$, then since $T_{lm} = 0$ and $T_{ln} = 0$ for all $l \in A_1$, it follows that

$$P_{Q_1} T_{|Q_1} = \sum_{l \in A_1} P_{K_l} T_{|K_l} = \sum_{l \in A_1} T_{lm} = 0, \ \text{and} \ \ P_{Q_2} T_{|Q_2} = \sum_{l \in A_1} P_{K_l} T_{|K_l} = \sum_{l \in A_1} T_{ln} = 0.$$

The sum above is in strong operator topology. Similarly from (3.10), since $T_{ml} = 0$ and $T_{nl} = 0$ for all $l \in A_2$, it follows that $P_{Q_1} T_{|Q_1} = 0$ and $P_{Q_2} T_{|Q_2} = 0$. These observations along with (3.5) prove our claim that every operator $T \in \mathcal{M}$ has the form as in (3.8).

Now with respect to the decomposition in (3.7), consider the operator $D$ in $\mathcal{B}(\mathcal{K})$ given by

$$D = \begin{bmatrix} I_{Q_1} & D_1 & 0 & 0 \\ D_1^* & I_{Q_2} & 0 & 0 \\ 0 & 0 & I_{Q_3} & 0 \\ 0 & 0 & 0 & I_{Q_4} \end{bmatrix}, \hspace{1cm} (3.11)$$
where $D_1 \in \mathcal{B}(Q_3, Q_1)$ satisfies $0 < \|D_1\| < 1$. It is then clear that $D$ is a positive and invertible operator in $\mathcal{B}(\Lambda)$. Since $\mathcal{M}$ has factorization in $\mathcal{B}(\Lambda)$, there is an invertible operator $S \in \mathcal{M}$ with $S^{-1} \in \mathcal{M}$ such that $D = S^* S$. Then from (3.8), $S$ and $S^{-1}$ look like

$$
S = \begin{bmatrix} S_1 & 0 & A_1 & 0 \\ 0 & S_2 & A_2 & 0 \\ 0 & 0 & X_1 & X_2 \\ B_1 & B_2 & X_3 & X_4 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} T_1 & 0 & C_1 & 0 \\ 0 & T_2 & C_2 & 0 \\ 0 & 0 & Y_1 & Y_2 \\ E_1 & E_2 & Y_3 & Y_4 \end{bmatrix}.
$$

Now

$$
I_{\mathcal{K}} = SS^{-1} = \begin{bmatrix} S_1T_1 & 0 & S_1C_1 + A_1Y_1 & A_1Y_2 \\ 0 & S_2T_2 & S_2C_2 + A_2Y_1 & A_2Y_2 \\ X_1E_1 & X_2E_2 & X_1Y_1 + X_2Y_3 & X_1Y_2 + X_2Y_4 \\ B_1T_1 + X_4E_1 & B_2T_2 + X_4E_2 & B_1C_1 + B_2C_2 + X_3Y_1 + X_4Y_3 & X_3Y_2 + X_4Y_4 \end{bmatrix}.
$$

Thus we get $S_1T_1 = I_{Q_1}$ and $S_2T_2 = I_{Q_2}$. Similarly from the expression $S^{-1}S = I_{\mathcal{K}}$, we get $T_1S_1 = I_{Q_1}$ and $T_2S_2 = I_{Q_2}$. This shows that $T_1$ and $T_2$ are invertible. Further, from (4.1) entry of $SS^{-1}$, we have $B_1T_1 + X_4E_1 = 0$, which yields

$$
B_1 = -X_4E_1T_1^{-1} = X_4F_1,
$$

where $F_1 = -E_1T_1^{-1}$. Also, from (4.2) entry of $SS^{-1}$, we have $B_2T_2 + X_4E_2 = 0$, that is,

$$
B_2 = -X_4E_2T_2^{-1} = X_4F_2,
$$

where $F_2 = -E_2T_2^{-1}$. Next we note that (1,2) entry of $S^* S$ is $B_1^* B_2$, and (1,4) entry of $S^* S$ is $B_1^* X_4$. By substituting $B_1 = X_4F_1$ and $B_2 = X_4F_2$, and equating the corresponding entries of $D$, we get $F_1^* X_1 F_1 F_2 = B_1^* B_2 = D_1$ and $F_1^* X_1 F_2 = B_1^* X_4 = 0$. This implies that $D_1 = 0$, which is a contradiction. This again violates the fact that $\mathcal{M}$ has factorization in $\mathcal{B}(\Lambda)$. Thus we have shown our claim that $V_i^* V_n^* \geq V_m^* V_m^*$ or $V_m^* V_n^* \geq V_i^* V_n^*$, which is to say that $\mathcal{R}(V_m) \geq \mathcal{R}(V_n)$ or $\mathcal{R}(V_m) \geq \mathcal{R}(V_n)$. Since $m, n \in \Lambda$ are arbitrary, we conclude that $\mathcal{E} = \{\mathcal{R}(V_i)\}_{i \in \Lambda}$ is a nest.

Now we define an order on $\Lambda$ by assigning

$$
i \leq j \quad \text{if and only if} \quad V_i^* V_i^* \leq V_j^* V_j^*, \tag{3.12}\$$

for any $i, j \in \Lambda$. Since $V_i^* V_j^* \neq V_j^* V_i^*$ whenever $i \neq j$, the order on $\Lambda$ is well-defined. Also $\Lambda$ is a totally-ordered set, as $\{\mathcal{R}(V_i)\}_{i \in \Lambda}$ forms a nest of subspaces. For each $i \in \Lambda$, consider the subspace $\mathcal{L}_i$ of $\mathcal{K} = \oplus_{i \in \Lambda} \mathcal{K}_i$ given by

$$
\mathcal{L}_i = \bigoplus_{j \leq i} \mathcal{K}_j. \tag{3.13}
$$

Then it is clear that the collection $\mathcal{L} = \{\mathcal{L}_i; i \in \Lambda\}$ forms a nest in $\mathcal{K}$ such that $\mathcal{L}_i \subseteq \mathcal{L}_j$ if and only if $i \leq j$. We have to show that the completion $\overline{\mathcal{L}}$ of the nest $\mathcal{L}$ is countable. We claim that

$$\mathcal{M} = (\text{Alg} \mathcal{L})^* \tag{3.14}$$

Since $\mathcal{M}$ has factorization in $\mathcal{B}(\Lambda)$, it will then follow from the claim and Proposition 2.9 that $\text{Alg} \mathcal{L}$ has factorization in $\mathcal{B}(\Lambda)$, which further will imply our requirement using Theorem 3.5 that $\overline{\mathcal{L}}$ is countable (as $\mathcal{K}$ is separable).

To show the claim in (3.14), we first note that if an operator $S = [S_{ij}]$ in $\mathcal{B}(\Lambda)$ leaves all subspaces $\{\mathcal{L}_i\}$ invariant, then $S_{ij} = 0$ for all $i > j$; hence $\text{Alg} \mathcal{L} = \{[S_{ij}] \in \mathcal{B}(\Lambda); S_{ij} = 0$ for $i > j\}$, that is,

$$\text{Alg} \mathcal{L}^* = \{[S_{ij}] \in \mathcal{B}(\Lambda); S_{ij} = 0$ for $i < j\}. \tag{3.15}$$

Now let $[S_{ij}] \in \mathcal{M}$. Then $V_i^* V_j^* \otimes S_{ij} = V_i^* V_j^* V_i^* V_j^* \otimes S_{ij}$ for all $i, j \in \Lambda$. For $i < j$, since $V_i^* V_j^* V_j^* V_i^* = V_i^* V_j^*$ and $V_i^* V_j^* \neq V_j^* V_i^*$, it forces that $S_{ij} = 0$. This shows that $[S_{ij}] \in \{\text{Alg} \mathcal{L}^*\}$. Thus $\mathcal{M} \subseteq \{\text{Alg} \mathcal{L}^*\}$. Conversely, if $[S_{ij}] \in \{\text{Alg} \mathcal{L}^*\}$, then $S_{ij} = 0$ for $i < j$; hence $V_i^* V_j^* \otimes S_{ij} = 0 = V_i^* V_j^* V_i^* V_j^* \otimes S_{ij}$ for $i < j$. On the other hand, for $i \geq j$, we have $V_i^* V_j^* \geq V_j^* V_i^*$, so that $V_i^* V_j^* V_j^* \otimes S_{ij} = V_i^* V_j^* \otimes S_{ij}$. This shows that $V_i^* V_j^* V_j^* \otimes S_{ij} = V_j^* V_i^* \otimes S_{ij}$ for all $i, j \in \Lambda$, which is to say that $[S_{ij}] \in \mathcal{M}$. Thus we have shown our claim that $\mathcal{M} = \{\text{Alg} \mathcal{L}^*\}$.

$\iff$ To prove the converse implication, assume that the collection $\{\mathcal{R}(V_i)\}_{i \in \Lambda}$ is a nest (hence
A is a totally ordered set) such that completion \( \overline{\mathcal{L}} \) of the nest \( \mathcal{L} = \{ L_i : i \in \Lambda \} \) as in (3.13) is countable. Similar to the claim in (3.14), we note that \( \mathcal{M} = (\text{Alg } \mathcal{L})^* \). Since \( \overline{\mathcal{L}} \) is countable, it follows from Theorem 3.5 that \( \text{Alg } \mathcal{L} \) has factorization in \( \mathcal{B}(\mathcal{K}) \), which is to say that \( \mathcal{M} \) has factorization in \( \mathcal{B}(\mathcal{K}) \).

Now to show that \( \oplus_{i \in \Lambda} \phi_i \) is \( C^* \)-extreme, we use Corollary 2.6. Let \( \tilde{D} = I_{H \delta} \otimes [D_{ij}] \) be a positive operator in \( \rho(\mathcal{A})' \) such that \( V^* \tilde{D} V \) is invertible. We claim that \( [D_{ij}] \) is invertible. Since \( V^* \tilde{D} V \) is invertible, there exists \( \beta > 0 \) such that \( V^* \tilde{D} V \geq \beta V^* V \). Then we have

\[
0 \leq V^* \tilde{D} V - \beta V^* V = [V^*_i V_j \otimes D_{ij}] - \beta [V^*_i V_j \otimes \delta_{ij} I_{K_i}] = [V^*_i V_j \otimes (D_{ij} - \delta_{ij} \beta I_{K_i})],
\]

where \( \delta_{ij} \) denotes the Kronecker delta. In particular, for every finite subset \( \Lambda_0 \subseteq \Lambda \), we have

\[
[V^*_i V_j \otimes (D_{ij} - \delta_{ij} \beta I_{K_i})]_{i,j \in \Lambda_0} \geq 0.
\]

Now fix a finite subset \( \Lambda_0 \subseteq \Lambda \), and let \( h_{\Lambda_0} \in \cap_{i \in \Lambda_0} \mathcal{R}(V_i) \) be a unit vector (which exists because the set \( \{ \mathcal{R}(V_i) \}_{i \in \Lambda_0} \) is finite and is totally ordered). Then there exist unit vectors \( h_i \in H_i \) such that \( V_i h_i = h_{\Lambda_0} \) for each \( i \in \Lambda_0 \). So for any vector \( k_i \in K_i \), \( i \in \Lambda_0 \), it follows from (3.16) that

\[
0 \leq \sum_{i,j \in \Lambda_0} \langle (V^*_i V_j \otimes (D_{ij} - \delta_{ij} \beta I_{K_i}))(h_i \otimes k_j), (h_i \otimes k_i) \rangle
\]

\[
= \sum_{i,j \in \Lambda_0} \langle (V^*_i V_j h_j, h_i) \rangle \langle (D_{ij} - \delta_{ij} \beta I_{K_i})k_j, k_i \rangle = \sum_{i,j \in \Lambda_0} \langle V_j h_j, V_i h_i \rangle \langle (D_{ij} - \delta_{ij} \beta I_{K_i})k_j, k_i \rangle
\]

\[
= \sum_{i,j \in \Lambda_0} \langle h_{\Lambda_0}, h_{\Lambda_0} \rangle \langle (D_{ij} - \delta_{ij} \beta I_{K_i})k_j, k_i \rangle = \sum_{i,j \in \Lambda_0} \langle (D_{ij} - \delta_{ij} \beta I_{K_i})k_j, k_i \rangle.
\]

Since \( k_i \in K_i \) for \( i \in \Lambda_0 \), \( \beta I_{K_i} \) is arbitrary, we conclude that \( \langle (D_{ij} - \delta_{ij} \beta I_{K_i})k_j, k_i \rangle \geq 0 \) for all \( i,j \in \Lambda_0 \). Also since \( \Lambda_0 \) is an arbitrary finite subset of \( \Lambda \), it follows that \( \langle (D_{ij} - \delta_{ij} \beta I_{K_i})k_j, k_i \rangle \geq 0 \) for all \( i,j \in \Lambda \), proving our claim that \( D = [D_{ij}] \) is invertible.

Therefore, as \( \mathcal{M} \) has factorization in \( \mathcal{B}(\mathcal{K}) \), there is an invertible operator \( \tilde{S} \in \mathcal{B}(\mathcal{K}) \) such that \( S, S^{-1} \in \mathcal{M} \) and \( D = S^* S \). Set \( \tilde{S} = I_{H \delta} \otimes S \). Clearly \( \tilde{S} \in \rho(\mathcal{A})' \) and \( \tilde{D} = \tilde{S}^* \tilde{S} \). Since \( S^{-1} \in \mathcal{M} \), it follows that \( \tilde{S}^* V^* V = V^* \tilde{S}^* \tilde{S}^* V^* \); hence we have

\[
(V^* \tilde{S} V)(V^* \tilde{S}^* V) = V^* \tilde{S}(V^* V \tilde{S}^* V^* V) V = V^* \tilde{S} \tilde{S}^{-1} V^* V = V^* \tilde{S} \tilde{S}^{-1} V V^* V = V^* V = I_{H \delta}.
\]

Likewise we get \((V^* \tilde{S}^{-1} V)(V^* \tilde{S} V) = V^* V = I_{H \delta} \). This shows that \( V^* \tilde{S} \) is invertible. Thus for a given \( \tilde{D} \in \rho(\mathcal{A})' \) with \( V^* \tilde{D} \) invertible, we have got \( \tilde{S} \in \rho(\mathcal{A})' \) such that \( \tilde{D} = \tilde{S} \tilde{S}^* \). \( \tilde{S} V^* V = V^* \tilde{S} V V^* \) and \( V^* \tilde{S} \) is invertible. We now conclude from Corollary 2.6 that \( \phi = \oplus_{i \in \Lambda} \phi_i \) is a \( C^* \)-extreme point in \( S_H(\mathcal{A}) \).

\[\square\]

Combining Theorem 3.6 and Proposition 3.4 we have the following complete characterization of those \( C^* \)-extreme points which are direct sums of pure UCP maps.

**Theorem 3.7.** Let \( \psi \) be a direct sum of pure UCP maps in \( S_H(\mathcal{A}) \), so that \( \psi \) is unitarily equivalent to \( \bigoplus_{\alpha \in \Gamma} \bigoplus_{i \in \Lambda_\alpha} \psi^i_\alpha(\cdot) \otimes I_{K^\alpha_i} \), where \( \psi^i_\alpha \) is a pure UCP map with minimal Stinespring triple \( (\pi_\alpha, V_\alpha^i, H_{\pi_\alpha}) \) such that \( \psi^i_\alpha \) is non-unitarily equivalent to \( \psi^j_\beta \) for each \( i \neq j \) in \( \Lambda_\alpha \), \( \alpha \in \Gamma \), and \( \pi_\alpha \) is disjoint to \( \pi_\beta \) for \( \alpha \neq \beta \). Then \( \psi \) is \( C^* \)-extreme in \( S_H(\mathcal{A}) \) if and only if the following holds for each \( \alpha \in \Gamma \):

1. \( \{ \mathcal{R}(V^i_\alpha) \}_{i \in \Lambda_\alpha} \) is a nest in \( \mathcal{H}_{\pi_\alpha} \), which makes \( \Lambda_\alpha \) a totally ordered set, and
2. if \( \mathcal{L}_\alpha = \bigoplus_{i \leq j} K^\alpha_{ij} \) for \( i \in \Lambda_\alpha \), then the completion of the nest \( \{ \mathcal{L}^\alpha_i \}_{i \in \Lambda_\alpha} \) in \( \bigoplus_{i \in \Lambda_\alpha} K^\alpha_i \) is countable.

**Remark 3.8.** Based on their results for finite dimensions, Farenick and Zhou in their remarks towards the end of [14] suggest that Condition (1) in Theorem 3.7 is perhaps sufficient, even in infinite dimensions, for a direct sum of pure UCP maps to be \( C^* \)-extreme. Here in this Theorem we observe that Condition (1) is to be supplemented with Condition (2), which is a somewhat more delicate restriction and is a purely infinite dimensional phenomenon. It has no role to play in finite dimensions (see Example 3.9 below).
Example 3.9. Let $\mathcal{G}$ be a separable Hilbert space and let $\{\mathcal{G}_q\}_{q \in \mathbb{Q}}$ be a collection of subspaces of $\mathcal{G}$ indexed by rationals $\mathbb{Q}$ such that $\mathcal{G}_q \subseteq \mathcal{G}_{q'}$ for $q < p$. Let $\mathcal{K}$ be another Hilbert space with an orthonormal basis $\{e_q\}_{q \in \mathbb{Q}}$ and let $P_q$ denote the projection onto the one dimensional subspace $C e_q$. Consider the space $\mathcal{H} = \bigoplus_{q \in \mathbb{Q}} (\mathcal{G}_q \otimes C e_q) \subseteq \mathcal{G} \otimes \mathcal{K}$, and define the UCP map $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ by

$$\phi(X) = \bigoplus_{q \in \mathbb{Q}} P_{\phi q} X_{\phi q} \otimes P_q, \quad X \in \mathcal{B}(\mathcal{G}).$$

Then it is clear that $\phi$ is a direct sum of pure UCP maps. It is immediate to see that Condition (1) of Theorem 3.7 is satisfied (since $\{\mathcal{G}_q\}_{q \in \mathbb{Q}}$ forms a nest). On the other hand, if $L_p = \bigoplus_{q \leq p} C e_q$, then $\{L_p\}_{p \in \mathbb{Q}}$ is a nest whose completion is uncountable (indeed, indexed by reals $\mathbb{R}$). Thus Condition (2) of Theorem 3.7 fails to hold (compare this example with Example 6.10).

Some straightforward corollaries of Theorems 3.6 and Theorem 3.7 are immediate as given below.

Corollary 3.10. Let $\phi = \oplus_{i \in \Lambda} \phi_i$ be a direct sum of pure UCP maps $\phi_i$. If $\phi$ is $C^*$-extreme, then for each $i, j \in \Lambda$, either $\phi_i$ and $\phi_j$ are disjoint, or one of $\{\phi_i, \phi_j\}$ is a compression of the other.

Corollary 3.11. Let $\phi : A \to \mathcal{B}(\mathcal{H})$ be a direct sum of pure UCP maps. Then $\phi \oplus \phi$ is a $C^*$-extreme point in $S_{\mathcal{H} \otimes \mathcal{H}}(A)$ if and only if $\phi$ is a $C^*$-extreme point in $S_{\mathcal{H}}(A)$.

Since a finite nest containing $\{0\}$, $\mathcal{H}$ is always complete, the following corollary is immediate from Theorem 3.6. This result along with Proposition 3.4 also recover Theorem 2.1 in [14].

Corollary 3.12. Let $\{\phi_i : A \to \mathcal{B}(\mathcal{H}_i)\}_{i=1}^n$ be a finite collection of pure UCP maps with respective minimal Stinespring triple $(\pi_i, V_i, \mathcal{H}_i)$ (so that each $\phi_i$ is compression of the same irreducible representation $\pi$). Then $\phi = \oplus_{i=1}^n \phi_i$ is $C^*$-extreme in $S_{\bigoplus_{i=1}^n \mathcal{H}_i}(A)$ if and only if the family $\{\mathcal{H}_i\}_{i=1}^n$ is a nest.

If $\Lambda$ is a subset of the set of integers $\mathbb{Z}$, and if $\mathcal{E} = \{E_n\}_{n \in \Lambda}$ is a nest in a Hilbert space $\mathcal{K}$ with the property that $E_n \subseteq E_m$ for $n < m$, then the completion of $\mathcal{E}$ is given by the nest $\mathcal{E} = \{E_n\}_{n \in \Lambda}$, which is already countable. Thus the following corollary is immediate from Theorem 3.6.

Corollary 3.13. Let $\Lambda = \mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Z}_-$, or $\{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$, and let $\phi_n : A \to \mathcal{B}(\mathcal{H}_n)$ be a pure UCP map for $n \in \Lambda$. If $\phi_n$ is a compression of $\phi_{n+1}$ for each $n$ with $n, n+1 \in \Lambda$, then the direct sum $\phi = \oplus_{n \in \Lambda} \phi_n$ is a $C^*$-extreme point in $S_{\bigoplus_{n \in \Lambda} \mathcal{H}_n}(A)$, where $\mathcal{H} = \bigoplus_{n \in \Lambda} \mathcal{H}_n$.

We end this section by giving a necessary and sufficient criterion for a direct sum of pure UCP maps to be extreme. Note that in view of Proposition 3.4, it is enough to consider direct sums of only those pure UCP maps which are compression of the same irreducible representation.

Proposition 3.14. Let $\phi_i : A \to \mathcal{B}(\mathcal{H}_i)$, $i \in \Lambda$, be a family of pure UCP maps with respective minimal Stinespring triple $(\pi_i, V_i, \mathcal{H}_i)$. Then $\phi = \oplus_{i \in \Lambda} \phi_i$ is extreme in $S_{\bigoplus_{i \in \Lambda} \mathcal{H}_i}(A)$ if and only if $V_i^* V_j \neq 0$ for all $i, j \in \Lambda$.

Proof. Set $\mathcal{H} = \bigoplus_{i \in \Lambda} \mathcal{H}_i$. Note that $(\rho, V, \mathcal{H}_\rho)$ is the minimal Stinespring triple for $\phi$, where $\mathcal{H}_\rho = \bigoplus_{i \in \Lambda} \mathcal{H}_i$, $\rho = \bigoplus_{i \in \Lambda} \pi$ and $V = \bigoplus_{i \in \Lambda} V_i$. Since $\pi$ is irreducible, $\pi(A)' = \mathcal{C} \cdot I_{\mathcal{H}_\rho}$; so it follows that

$$\rho(A)' = \{[\lambda_{ij} I_{\mathcal{H}_\rho}] ; \lambda_{ij} \in \mathbb{C}\} \subseteq \mathcal{B}(\bigoplus_{i \in \Lambda} \mathcal{H}_i).$$

First assume that $\phi$ is extreme in $S_{\mathcal{H}}(A)$, and fix $m, n \in \Lambda$. Let $\lambda \neq 0$ in $\mathbb{C}$. Consider the operator $T = [\lambda_{ij} I_{\mathcal{H}_i}] \in \rho(A)'$, where $\lambda_{mn} = \lambda$ and $\lambda_{ij} = 0$ otherwise. Then $T \neq 0$. Since $\phi$ is extreme, it follow from Arveson’s extreme point condition (Theorem 2.2) that $V^* T V \neq 0$. But $V^* T V = [\lambda_{ij} V_i^* V_j]$, and since $\lambda_{ij} V_i^* V_j = 0$ for all $(i, j) \neq (m, n)$, it follows that $\lambda_{mn} V_m^* V_n \neq 0$, showing that $V_m^* V_n \neq 0$.

Conversely, let $V_i^* V_j \neq 0$ for all $i, j \in \Lambda$. Let $T = [\lambda_{ij} I_{\mathcal{H}_i}] \in \rho(A)'$, $\lambda_{ij} \in \mathbb{C}$, be such that $V^* T V = 0$. Then for each $i, j \in \Lambda$, we have $\lambda_{ij} V_i^* V_j = 0$, which yields $\lambda_{ij} = 0$; hence $T = 0$. Again by extreme point condition of Arveson, we conclude that $\phi$ is extreme in $S_{\mathcal{H}}(A)$. □
The following corollary is another condition (along which Proposition 2.11) under which a C*-extreme map is also extreme.

Corollary 3.15. Let φ ∈ SH(A) decompose as a direct sum of pure UCP maps. If φ is a C*-extreme point in SH(A), then φ is also an extreme point in SH(A).

Proof. Let φ = ⊕i∈Λi φi for some pure UCP maps φi, i ∈ Λ. By separating out disjoint UCP maps and then invoking Proposition 3.4 if needed, we assume without loss of generality that each φi is a compression of the same irreducible representation, say π. Let (π, V ⊗ K) be the minimal Stinespring triple for φi. Since φ is C*-extreme, it follows from Theorem 3.6 that either V∗i Vj ≥ Vi V∗j or V∗i Vj ≥ Vi V∗j for all i, j ∈ Λ. In either case, it is immediate that V∗i Vj ̸= 0 for i, j ∈ Λ. The required assertion now follows from Proposition 3.14.

4. Normal C*-extreme maps

Our attention now shifts towards the study of structure of normal C*-extreme maps on von Neumann algebras, specifically on type I factors (i.e. B(G) for some Hilbert space G). First, we see some basic properties and examples of such maps. The main result of this section (Theorem 4.8) provides necessary and sufficient criteria for normal C*-extreme UCP maps to be direct sums of normal pure UCP maps.

Let B ⊆ B(G) be a von Neumann algebra. Recall that a positive linear map φ : B → B(H) is called normal if whenever {Xi} is a net of increasing (or decreasing) self-adjoint operators converging to X in strong operator topology (SOT), then φ(Xi) → φ(X) in SOT.

Let NSH(B) denote the collection of all normal UCP maps from B to B(H). It is clear that NSH(B) itself is a C*-convex set. Hence one can define and study C*-extreme points of NSH(B) on the lines of Definition 2.3, and look into its structure. However we see below (Proposition 4.2) that any normal UCP map on B is C*-extreme in NSH(B) if and only if it is C*-extreme in SH(B).

Therefore it does not matter whether we explore C*-extremity conditions in the set NSH(B) or the set SH(B).

Lemma 4.1. Let φ, ψ : B → B(H) be two completely positive maps such that ψ ≤ φ. If φ is normal, then ψ is normal.

Proof. Let {Xi} be a net of decreasing positive elements in B such that Xi ↓ 0 in SOT. Then φ(Xi) → 0 in SOT, as φ is normal. As ψ is positive, we note that {ψ(Xi)} is a decreasing net of positive elements; hence ψ(Xi) → Y in SOT for some positive operator Y ∈ B(H). But since ψ(Xi) ≤ φ(Xi) for all i, it follows by taking limit in SOT that Y ≤ 0; hence Y = 0.

Proposition 4.2. A normal UCP map φ : B → B(H) is C*-extreme in NSH(B) if and only if it is C*-extreme in SH(B).

Proof. Since NSH(B) ⊆ SH(B), it is immediate that every normal C*-extreme point of SH(B) is also a C*-extreme point of NSH(B). Conversely, let φ be a C*-extreme point of NSH(B). Let φ = ∑n i=1 T∗i φi(·)Ti be a proper C*-convex combination in SH(B) for some φi ∈ SH(B). Then for each i, we have T∗i φi(·)Ti ≤ φ(·), so it follows from Lemma 4.1 that T∗i φi(·)Ti is normal; hence φi is normal. Since φ is C*-extreme in SH(B), there is a unitary U i ∈ B(H) such that φi(·) = U∗i φ(·)Ui, as required to prove that φ is C*-extreme in SH(B).

For the rest of this section, we assume that all von Neumann algebras are of the form B(G) for some separable Hilbert space G. We now recall the well-known structure of normal representations and the Stinespring dilation of normal UCP maps (see Theorem 1.41, [25]).

Theorem 4.3. Let φ : B(G) → B(H) be a normal UCP map. Then there exist a separable Hilbert space K and an isometry V : H → G ⊗ K such that

φ(X) = V∗(X ⊗ I K)V for all X ∈ B,

and satisfies the minimality condition: G ⊗ K = span{(X ⊗ I K)V h; h ∈ H, X ∈ B}.

In Theorem 4.3, if we recognize the Hilbert space G ⊗ K as direct sum of dim K copies of G, we get the following structure theorem for normal UCP maps (see Theorem 2.3, [10]).
Corollary 4.4. Let \( \phi : B(\mathcal{G}) \rightarrow B(\mathcal{H}) \) be a normal UCP map. Then there exists a finite or countable sequence \( \{V_n\}_{n \geq 1} \) of operators \( \phi(B(\mathcal{H},\mathcal{G})) \) such that
\[
\phi(X) = \sum_{n \geq 1} V_n^* XV_n \quad \text{in SOT, (4.1)}
\]
for all \( X \in B \).

Note that the commutator of the set \( \{X \otimes I_K; X \in B(\mathcal{G})\} \) in \( B(\mathcal{G} \otimes \mathcal{K}) \) is the algebra \( \{I_G \otimes T; T \in B(\mathcal{K})\} \). So a normal UCP map \( \phi : B(\mathcal{G}) \rightarrow B(\mathcal{H}) \) is pure if and only if \( \dim K = 1 \). Let \( \phi(X) = V^* XV \) for some isometry \( V \) from \( \mathcal{H} \) to \( \mathcal{G} \).

The \( C^* \)-extreme condition (Corollary 2.6) for normal \( C^* \)-extreme points of \( S_d(\mathcal{G}) \) translates as follows:

Theorem 4.5. Let \( \phi : B(\mathcal{G}) \rightarrow B(\mathcal{H}) \) be a normal UCP map with minimal Stinespring form \( \phi(X) = V^*(X \otimes I_K)V \) for some Hilbert space \( K \). Then \( \phi \) is \( C^* \)-extreme in \( S_d(\mathcal{G}) \) if and only if for any positive operator \( D \in B(K) \) with \( V^*(I_G \otimes D)V \) invertible, there exists \( S \in B(K) \) such that \( D = S^* S \), \( (I_G \otimes S)VV^* = VV^*(I_G \otimes S)VV^* \) and \( V^*(I_G \otimes S)VV^* \) is invertible.

Remark 4.6. Let \( \phi : B(\mathcal{G}) \rightarrow B(\mathcal{H}) \) be a normal UCP map with minimal Stinespring form \( \phi(X) = V^*(X \otimes I_K)V \). We identify the subspace \( V \mathcal{H} \) with \( \mathcal{H} \), so that \( \mathcal{H} \) is a subspace of \( \mathcal{G} \otimes \mathcal{K} \). It then follows from Theorem 4.5 that \( \phi \) is a \( C^* \)-extreme point in \( S_d(\mathcal{G}) \) if and only if the subspace \( \mathcal{H} \) of \( \mathcal{G} \otimes \mathcal{K} \) satisfies the following factorization property:

\( (1) \) for any positive operator \( D \in B(K) \) with \( P_H(I_G \otimes D)_{\mathcal{H}} \) invertible, there exists \( S \in B(K) \) satisfying \( D = S^* S \), \( (I_G \otimes S)(\mathcal{H}) \subseteq \mathcal{H} \) and \( (I_G \otimes S)_{\mathcal{H}} \) is invertible.

Therefore, in order to understand the structure of normal \( C^* \)-extreme maps, one can characterize subspaces of \( \mathcal{G} \otimes \mathcal{K} \) with factorization property \( (1) \).

The following proposition provides a family of examples of subspaces in \( \mathcal{G} \otimes \mathcal{K} \) satisfying factorization property \( (1) \).

Proposition 4.7. Let \( \mathcal{H} = \bigvee_{i \in \Lambda} \mathcal{G}_i \otimes \mathcal{K}_i \) be a subspace of \( \mathcal{G} \otimes \mathcal{K} \), for some family \( \{\mathcal{G}_i\}_{i \in \Lambda} \) and \( \{\mathcal{K}_i\}_{i \in \Lambda} \) of subspaces of \( \mathcal{G} \) and \( \mathcal{K} \) respectively, such that \( \mathcal{G} \otimes \mathcal{K} = \bigvee_{i \in \Lambda} (\mathcal{G}_i \otimes \mathcal{K}_i) \). If either of the following is true:

1. \( \mathcal{G}_i \perp \mathcal{G}_j \) for all \( i \neq j \) and \( \{\mathcal{K}_i\} \) is a nest whose completion is countable,
2. \( \mathcal{G}_i \) is a nest and \( \mathcal{K}_i \perp \mathcal{K}_j \) for \( i \neq j \) such that the completion of the nest \( \{\mathcal{G}_i\}_{i \in \Lambda} \) is countable,

then \( \mathcal{H} \) satisfies factorization property \( (1) \).

Proof. (1) Firstly it is easy to verify that \( \mathcal{K} = \bigvee_{i \in \Lambda} \mathcal{K}_i \) (indeed, if \( k \in \mathcal{K} \cap \bigvee_{i \in \Lambda} \mathcal{K}_i \), then for any non-zero \( g \in \mathcal{G} \), we will have \( g \otimes k \perp \{\mathcal{X} \otimes \mathcal{K}_i; h \in \mathcal{H}, X \in B(\mathcal{G})\} \), which will yield \( g \otimes k = 0 \).

Let \( D \in B(K) \) be a positive operator such that \( P_H(I_G \otimes D)_{\mathcal{H}} \) is invertible. We claim that \( D \) is invertible. Let \( \beta > 0 \) be such that \( P_H(I_G \otimes D)_{\mathcal{H}} \geq \beta I_{\mathcal{H}} \). Since \( g \otimes k \in \mathcal{H} \), for any \( 0 \neq g \in \mathcal{G}_i \) and \( k_i \in \mathcal{K}_i \), we get
\[
|g_i|^2 (Dk_i, k_i) = \langle (I_G \otimes D)(g_i \otimes k_i), g_i \otimes k_i \rangle \geq \beta \langle g_i \otimes k_i, g_i \otimes k_i \rangle = \beta |g_i|^2 \langle k_i, k_i \rangle,
\]
which implies that \( \langle Dk_i, k_i \rangle \geq \beta \langle k_i, k_i \rangle \). Since \( \bigcup_{i \in \Lambda} \mathcal{K}_i \) is dense in \( \mathcal{K} \), we conclude that \( \langle Dk, k \rangle \geq \beta \langle k, k \rangle \) for all \( k \in \mathcal{K} \); hence \( D \) is invertible.

Since the nest \( \{\mathcal{K}_i\}_{i \in \Lambda} \) has a countable completion, by Theorem 3.5 there exists an invertible operator \( S \in B(K) \) satisfying \( D = S^* S \) and \( S(\mathcal{K}_i) \subseteq \mathcal{K}_i \). Clearly then \( (I_G \otimes S)(\mathcal{H}) \subseteq \mathcal{H} \). Note that \( (S^{-1}_i)_{\mathcal{K}_i} = (S_{\mathcal{K}_i})^{-1} \in B(\mathcal{K}_i) \) for each \( i \in \Lambda \) and \( \sup_{i \in \Lambda} \| (S_{\mathcal{K}_i})^{-1} \| = \| S^{-1} \| < \infty \). Hence \( \bigoplus_{i \in \Lambda} I_{\mathcal{G}_i} \otimes (S_{\mathcal{K}_i})^{-1} \) is a bounded operator on \( \mathcal{H} \) and
\[
(I_G \otimes S)_{\mathcal{H}} \bigoplus_{i \in \Lambda} I_{\mathcal{G}_i} \otimes (S_{\mathcal{K}_i})^{-1} = \bigoplus_{i \in \Lambda} I_{\mathcal{G}_i} \otimes S_{\mathcal{K}_i} \bigoplus_{i \in \Lambda} I_{\mathcal{G}_i} \otimes I_{\mathcal{K}_i} = I_{\mathcal{H}}.
\]
Similarly, \( \bigoplus_{i \in \Lambda} I_{\mathcal{G}_i} \otimes (S_{\mathcal{K}_i})^{-1} \) is invertible. This proves that \( (I_G \otimes S)_{\mathcal{H}} \) is invertible. Since \( D \in B(K) \) is arbitrary, we have shown that \( \mathcal{H} \) satisfies factorization property \( (1) \).

(2) This assertion follows from Theorem 3.6, as the map \( \phi(X) = P_H(X \otimes I_K)_{\mathcal{H}} = \bigoplus_{i \in \Lambda} (P_{\mathcal{G}_i} X_{\mathcal{G}_i} \otimes I_{\mathcal{K}_i}) \) from \( B(\mathcal{G}) \) to \( B(\mathcal{H}) \) satisfies the equivalent criteria for it to be \( C^* \)-extreme in \( S_d(\mathcal{G}) \). \( \square \)
At this point, we are not sure if we can write subspaces of Part (1) in Proposition 4.7 in the form of subspaces in Part (2), and vice versa. However one can easily verify that if the concerned nests are already complete, then the two parts produce the same set of subspaces.

Before proving the main result of this section, we recall some terminologies for the purpose. Let $\mathcal{E}$ be a complete nest on a separable Hilbert space $\mathcal{K}$. For any $E \in \mathcal{E}$, define

$$E_- = \bigvee \{F \in \mathcal{E} : F \subseteq E\} \quad \text{and} \quad E_+ = \bigwedge \{F \in \mathcal{E} : E \subseteq F\}.$$  

An atom of $\mathcal{E}$ is a subspace of the form $E \ominus E_-$, for some $E \in \mathcal{E}$ with $E \neq E_-$. Clearly, any two atoms of $\mathcal{E}$ are orthogonal. The nest $\mathcal{E}$ is called atomic if there is a countable collection of atoms $\{K_n\}$ of $\mathcal{E}$ such that $\mathcal{K} = \bigoplus_n K_n$.

Now let $\mathcal{M}$ be a subalgebra of $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Then its lattice $\text{Lat}\mathcal{M}$ is defined by

$$\text{Lat}\mathcal{M} = \{E \subseteq \mathcal{K} : E \text{ is a subspace such that } T(E) \subseteq E \text{ for all } T \in \mathcal{M}\}.$$  

Dually, for any collection $\mathcal{E}$ of subspaces of $\mathcal{K}$, consider the unital closed algebra $\text{Alg}\mathcal{E}$ defined by

$$\text{Alg}\mathcal{E} = \{T \in \mathcal{B}(\mathcal{K}) : T(E) \subseteq E \text{ for all } E \in \mathcal{E}\}.$$  

It is clear that $\mathcal{M} \subseteq \text{Alg}\text{Lat}\mathcal{M}$. The subalgebra $\mathcal{M}$ is called reflexive if $\mathcal{M} = \text{Alg}\text{Lat}\mathcal{M}$. Any nest algebra is an example of reflexive algebra. More generally, any algebra of the form $\text{Alg}\mathcal{E}$ (for some collection $\mathcal{E}$ of subspaces) is reflexive. One can refer to [8] for more details.

We now mention a crucial result proved in [5] about reflexive algebras having factorization property to be applied below (see Corollary 2.11 and Lemma 4.3 in [5]): If $\mathcal{M}$ is a reflexive algebra having factorization in $\mathcal{B}(\mathcal{K})$ for some separable Hilbert space $\mathcal{K}$, then $\text{Lat}\mathcal{M}$ is a complete atomic nest, and hence $\mathcal{M}$ is a nest algebra.

The following theorem is another major result of the article, which provides a necessary and sufficient criteria for a normal $C^*$-extreme map to be direct sum of normal pure UCP maps.

**Theorem 4.8.** Let $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal $C^*$-extreme map with minimal Stinespring form $\phi(X) = V^*(X \otimes I_\mathcal{K})V$, for some Hilbert space $\mathcal{K}$. Then $\phi$ is unitarily equivalent to a direct sum of normal pure UCP maps if and only if the algebra $\mathcal{M} = \{T \in \mathcal{B}(\mathcal{K}) : (I_\mathcal{G} \otimes T)(V\mathcal{H}) \subseteq V\mathcal{H}\}$ is reflexive.

**Proof.** By identifying the Hilbert space $\mathcal{H}$ with $V\mathcal{H}$, we assume that $\mathcal{H}$ is a subspace of $\mathcal{G} \otimes \mathcal{K}$, so that $\phi(X) = P_H(X \otimes I_\mathcal{K})|_H$ for $X \in \mathcal{B}(\mathcal{G})$ and $\mathcal{M} = \{T \in \mathcal{B}(\mathcal{K}) : (I_\mathcal{G} \otimes T)\mathcal{H} \subseteq \mathcal{H}\}$.

First we assume that the algebra $\mathcal{M}$ is reflexive. Since $\phi$ is $C^*$-extreme in $S(H)(\mathcal{B}(\mathcal{G}))$, it follows from Corollary 2.10 that $I_\mathcal{G} \otimes \mathcal{M}$ has factorization in $I_\mathcal{G} \otimes \mathcal{B}(\mathcal{K})$, which is to say that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$. Since $\mathcal{K}$ is separable, it then follows from the result from Corollary 2.11 in [5] as mentioned above, that $\text{Lat}\mathcal{M}$ is an atomic nest. Therefore by definition of atomic nests, there exists an orthonormal basis $\{e_n\}_{n \geq 1}$ of $\mathcal{K}$ such that each $e_n$ is contained in one of the atoms of $\text{Lat}\mathcal{M}$. Now for all $n \geq 1$, consider the subspace $\mathcal{G}_n$ of $\mathcal{G}$ given by

$$\mathcal{G}_n = \{g \in \mathcal{G} : g \otimes e_n \in \mathcal{H}\}.$$  

We claim that

$$\mathcal{H} = \bigoplus_{n \geq 1} (\mathcal{G}_n \otimes e_n).$$  

Clearly, $\mathcal{G}_n \otimes e_n \subseteq \mathcal{H}$ for all $n \geq 1$; hence $\bigoplus_{n \geq 1} (\mathcal{G}_n \otimes e_n) \subseteq \mathcal{H}$. Conversely, let $h \in \mathcal{H}$. Then as $\{e_n\}_{n \geq 1}$ is an orthonormal basis of $\mathcal{K}$ and $h \in \mathcal{G} \otimes \mathcal{K}$, there exists a sequence $\{g_n\}_{n \geq 1}$ of vectors in $\mathcal{G}$ such that

$$h = \sum_{n \geq 1} g_n \otimes e_n.$$  

Now for any unit vector $e \in \mathcal{K}$, we denote by $|e\rangle\langle e|$ the rank one projection on $\mathcal{K}$ defined by

$$|e\rangle\langle e|(k) = e\langle e,k\rangle$$  

for all $k \in \mathcal{K}$.

Then we note for all $n \geq 1$ that $|e_n\rangle\langle e_n|$ is in $\text{Alg}\text{Lat}\mathcal{M}$ (indeed, if $E \ominus E_-$ is an atom of $\text{Lat}\mathcal{M}$ and $e \in E \ominus E_-$ is a unit vector, then $|e\rangle\langle e|(F) = 0 \subseteq F$ for $F \subseteq E_-$, and $|e\rangle\langle e|(F) = \mathbb{C} \cdot e \subseteq F$.
for $F \supseteq E$). Since $\mathcal{M}$ is reflexive, it then follows that $|e_n\rangle\langle e_n| \in \mathcal{M}$; hence $(I_G \otimes |e_n\rangle\langle e_n|)H \subseteq H$, which implies
\[(I_G \otimes |e_n\rangle\langle e_n|)h = g_n \otimes e_n \in H.\]
In particular, $g_n \in G_n$ and hence $g_n \otimes e_n \in G_n \otimes e_n$. This shows that $h = \sum_{n \geq 1} g_n \otimes e_n \in \oplus_{n \geq 1} G_n \otimes e_n$. Since $h \in H$ is arbitrary, we conclude our claim that $H = \oplus_{n \geq 1}(G_n \otimes e_n)$. Now for each $n \geq 1$, define the map $\phi_n : B(G) \to B(G_n)$ by\[\phi_n(X) = P_{G_n}X|_{\phi_n}, \quad \text{for all } X \in B(G).\]
Note that $G_n$ can be a zero subspace, in which case we ignore the map $\phi_n$. Then it is clear that $\phi_n$ is a normal pure UCP map, and for all $X \in B(G)$ we have\[\phi(X) = P_{H}(X \otimes I_K)|_{H} = \sum_{n \geq 1} P_{G_n}X|_{\phi_n} \otimes |e_n\rangle\langle e_n| = \bigoplus_{n \geq 1} \phi_n(X) \otimes |e_n\rangle\langle e_n|.
\]
This proves the required assertion that $\phi$ is unitarily equivalent to a direct sum of normal pure UCP maps $\phi_n$.

To prove the converse, let $\phi$ be a direct sum of normal pure UCP maps. Then for some countable indexing set $J$, there is a collection $\{G_i\}_{i \in J}$ of distinct subspaces of $G$ and a collection $\{K_i\}_{i \in J}$ of mutually orthogonal subspaces of $K$ such that $H = \oplus_{i \in J}(G_i \otimes K_i)$. Since $\phi$ is $C^*$-extreme in $S_{H}(B(G))$, the collection $\{G_i\}_{i \in J}$ is a nest by Theorem 3.6. This nest induces an order on $J$ making it a totally ordered set. If we set $L_i = \oplus_{j \geq i} K_j$ for $i \in J$, then $\{L_i\}_{i \in J}$ is a nest, and it is easy to verify that $M = \{T \in B(K): I_G \otimes T)(H) \subseteq H\} = \text{Alg}\{L_i; i \in J\}$ (to show this, one can imitate the same argument as in (3.14) in the proof of Theorem 3.6). Thus we conclude that $\mathcal{M}$ is reflexive.

It is a known fact due to Juschenko [18] that any subalgebra having factorization in the finite dimensional matrix algebra $M_n$ is a nest algebra, and hence is automatically reflexive (also see [5] for an alternative proof). Thus the following corollary is immediate from Theorem 4.8 and Theorem 3.6.

**Corollary 4.9.** Let $\mathcal{H}$ be a subspace of $G \otimes K$, where $K$ is a finite dimensional Hilbert space, such that the normal UCP map $\phi : B(G) \to B(\mathcal{H})$ given by $\phi(X) = P_{H}(X \otimes I_K)|_{H}$, for $X \in B(G)$, is in minimal Stinespring form. Then $\phi$ is $C^*$-extreme in $S_{H}(B(G))$ if and only if $\phi$ is unitarily equivalent to a direct sum of a finite sequence of normal pure UCP maps $\{\phi_i\}_{i=1}^n$ such that $\phi_i$ is a compression of $\phi_{i+1}$.\[\text{Corollary 4.9 was proved for the case when the UCP map is from } M_n \text{ to } M_r \text{ for some } n, r \in \mathbb{N} \text{ (Theorem 4.1, [12]) through rather tedious matrix computations. Here we have provided a more conceptual approach using algebra theory.}

This Corollary suggests that perhaps the algebra $\mathcal{M}$ in Theorem 4.8 is always reflexive when $\phi$ is $C^*$-extreme. But we are not able to prove it. If this turns out to be true, then Theorem 4.8 along with Theorem 3.6 would characterize all normal $C^*$-extreme maps on $B(G)$. Thus we propose the following conjecture:

**Conjecture 4.10.** Every normal $C^*$-extreme map on a type I factor is a direct sum of normal pure UCP maps.

5. **A Krein-Milman type theorem**

The Krein-Milman theorem is a very important result in classical functional analysis, which says that in a locally convex topological vector space, a convex compact subset is closure of the convex hull of its extreme points. So it is desired to have an analogue of Krein-Milman theorem for $C^*$-convexity in the space $S_{H}(A)$ equipped with an appropriate topology. We equip the set $S_{H}(A)$ with bounded weak (BW) topology. Convergence in BW-topology is given by: a net $\phi_{i}$ converges to $\phi$ in $S_{H}(A)$ if $\phi_{i}(a) \to \phi(a)$ in weak operator topology (WOT) for all $a \in A$. It is known that $S_{H}(A)$ is a compact set with respect to BW-topology. See [2, 24] for more details on this topology.
So a generalized Krein-Milman theorem for $S_H(A)$ would be to ask whether $S_H(A)$ is the closure of the $C^*$-convex hull of its $C^*$-extreme points in $BW$-topology. Here the $C^*$-convex hull of any subset $K$ of $S_H(A)$ is given by

$$\left\{ \sum_{i=1}^{n} T_i^* \phi_i(\cdot) T_i; \phi_i \in K, T_i \in \mathcal{B}(H) \text{ with } \sum_{i=1}^{n} T_i^* T_i = I_H \right\}. \quad (5.1)$$

The goal of this section is to prove a Krein-Milman type theorem for $S_H(A)$, whenever $A$ is a separable $C^*$-algebra or $A$ is of the form $B(G)$ for some Hilbert space $G$. The proof of these two cases are different. Note that $B(G)$ is not separable, when $G$ is infinite dimensional. As yet we do not know the result in full generality (i.e. for non-separable $C^*$-algebras). We recall here that as mentioned before such a theorem can be found in [12] for $S_H(A)$, when $H$ is a finite dimensional Hilbert space and $A$ is an arbitrary $C^*$-algebra, and in [4] for general $H$ and commutative $C^*$-algebra $A$. Thus our result provides an important development towards this theorem in infinite dimensional Hilbert space settings.

**Lemma 5.1.** Let $\phi \in S_H(A)$ be such that $\phi(a) = \sum_{n \geq 1} \phi_n(a)$ in WOT, for all $a \in A$, where $\{\phi_n : A \to B(H)\}_{n \geq 1}$ is a countable family of pure completely positive maps. Then $\phi$ is in the $BW$-closure of $C^*$-convex hull of $C^*$-extreme points of $S_H(A)$.

**Proof.** We assume that the collection $\{\phi_n\}_{n \geq 1}$ is the collection in the sum of $\phi$ is countably infinite. The finite case follows similarly and easily. For each $n \geq 1$, let $(\pi_n, V_n, K_n)$ be the minimal Stinespring triple for $\phi_n$. Then each $\pi_n$ is irreducible, as $\phi_n$ is pure by hypothesis. Note that

$$\sum_{n \geq 1} V_n^* V_n = \sum_{n \geq 1} \phi_n(1) = \phi(1) = I_H, \quad \text{in WOT.}$$

Set $A_n = V_n^* V_n \in B(H)$, and let $V_n = W_n A_n^{1/2}$ be the polar decomposition of $V_n$. Here $W_n \in B(H, K_n)$ is the partial isometry with initial space $\mathcal{R}(A_n^{1/2})$ and final space $\mathcal{R}(V_n)$. Define the map $\zeta_n : A \to B(H)$ by

$$\zeta_n(a) = W_n^* \pi_n(a) W_n \quad \text{for all } a \in A.$$

It is immediate to verify that $\zeta_n$ is a completely positive map with the minimal Stinespring triple $(\pi_n, V_n, K_n)$. Let $\theta_n : A \to \mathbb{C}$ be a pure state that is a compression of $\zeta_n$ (e.g. take a unit vector $e_n \in \mathcal{R}(W_n)$ and define $\theta_n(a) = \langle e_n, \pi_n(a) e_n \rangle$ for all $a \in A$). Now we define $\xi_n : A \to B(H)$ by

$$\xi_n = \zeta_n + (1 - P_n) \theta_n,$$

where $P_n = W_n^* W_n$ is the projection from $H$ onto $\mathcal{R}(A_n^{1/2})$. Note that $\xi_n$ is a UCP map from $A$ to $B(H)$. If we set $U_n = W_n|_{\mathcal{R}(P_n)}$ (so that $U_n$ is an isometry from $\mathcal{R}(P_n)$ to $K_n$), then it is straightforward to verify that $\xi_n$ is unitarily equivalent to the UCP map $\tilde{\xi}_n : A \to B(\mathcal{R}(P_n) \oplus \mathcal{R}(P_n^*))$ given by

$$\tilde{\xi}_n(a) = U_n^* \pi_n(a) U_n \oplus \theta_n(a) I_{\mathcal{R}(P_n^*)}, \quad \text{for all } a \in A.$$

Since $\theta_n$ is a compression of the map $a \mapsto U_n^* \pi_n(a) U_n$ (which is pure, as $\pi_n$ is irreducible), it follows from Theorem 3.6 that $\tilde{\xi}_n$ is $C^*$-extreme in $S_{\mathcal{R}(P_n) \oplus \mathcal{R}(P_n^*)}(A)$; hence $\xi_n$ is $C^*$-extreme in $S_H(A)$.

Now set $B_n = I_H - \sum_{j=1}^{n} A_j$. Since $\sum_{n \geq 1} A_n = \sum_{n \geq 1} V_n^* V_n = I_H$ in WOT; it follows that $B_n \geq 0$, and $B_n \to 0$ in WOT as $n \to \infty$. Now fix a $C^*$-extreme point $\xi$ in $S_H(A)$ and define the map $\psi_n : A \to B(H)$ by

$$\psi_n(a) = B_n^{1/2} \xi(a) B_n^{1/2} + \sum_{j=1}^{n} A_j^{1/2} \xi_j(a) A_j^{1/2}, \quad \text{for all } a \in A.$$

It is clear that each $\psi_n$ is a UCP map such that $\psi_n$ is a $C^*$-convex combination of $C^*$-extreme points of $S_H(A)$. Since $B_n \to 0$ in WOT, it follows that $B_n^{1/2} \to 0$ in SOT; hence $B_n^{1/2} \xi(a) B_n^{1/2} \to 0$
in WOT for all \( a \in A \). This implies that
\[
\lim_{n \to \infty} \psi_n(a) = \sum_{j=1}^{\infty} A_j^{1/2} \xi_j(a) A_j^{1/2} \quad \text{in WOT, for all } a \in A.
\]

Note that \( A_j^{1/2}(I - P_j) = 0 \) for all \( j \). Hence for all \( a \in A \), we get \( A_j^{1/2} \xi_j(a) A_j^{1/2} = A_j^{1/2} \xi_j(a) A_j^{1/2} \), which further yields in WOT convergence
\[
\lim_{n \to \infty} \psi_n(a) = \sum_{j=1}^{\infty} A_j^{1/2} \xi_j(a) A_j^{1/2} = \sum_{j=1}^{\infty} A_j^{1/2} W_j^{*} \pi_j(a) W_j A_j^{1/2} = \sum_{j=1}^{\infty} V_j^{*} \pi_j(a) V_j = \sum_{j=1}^{\infty} \phi_j(a) = \phi(a).
\]

In other words, \( \psi_n \to \phi \) in BW-topology. Thus we have approximated \( \phi \) in BW-topology by a sequence \( \psi_n \), belonging to the \( C^{*} \)-convex hull of \( C^{*} \)-extreme points of \( S_{\mathcal{H}}(A) \).

The following proposition seems to be a well-known result. However, we could trace the proof only when \( \mathcal{H} \) is a finite dimensional Hilbert space. So we outline a proof for the sake of completeness.

**Proposition 5.2.** Let \( A \) be a von Neumann algebra, and let \( \phi: A \to B(\mathcal{H}) \) be a UCP map. Then there exists a sequence \( \phi_n: A \to B(\mathcal{H}) \) of normal UCP maps such that \( \phi_n(a) \to \phi(a) \) in SOT for all \( a \in A \). In particular, the set \( NS_{\mathcal{H}}(A) \) of normal generalized states is dense in the set \( S_{\mathcal{H}}(A) \) of all generalized states in BW-topology.

**Proof.** If \( \mathcal{H} \) is finite dimensional, then the assertion is proved in (Corollary 1.6.3, [6]). So assume that \( \mathcal{H} \) is infinite dimensional. Let \( \{P_n\}_{n \geq 1} \) be an increasing sequence of projections on \( \mathcal{H} \) with finite dimensional ranges such that \( P_n \to I_{\mathcal{H}} \) in SOT. Fix a normal UCP map \( \psi: A \to B(\mathcal{H}) \), and for each \( n \geq 1 \), consider the map \( \phi_n: A \to B(\mathcal{H}) \) given by
\[
\phi_n(a) = P_n \phi(a) P_n + (1 - P_n) \psi(a)(1 - P_n), \quad \text{for all } a \in A.
\]

Since \( P_n \to I_{\mathcal{H}} \) in SOT, we note that \( \phi_n(a) \to \phi(a) \) in SOT for all \( a \in A \). Also the second term in the above sum is normal, as \( \psi \) is normal. So it suffices to approximate the map \( P_n \phi(\cdot) P_n \) by normal completely positive maps. The problem now reduces to approximation of (unital) completely positive maps by normal (unital) completely positive maps acting on finite dimensional Hilbert spaces, which is possible as already noted.

We are now ready to show a Krein-Milman type theorem for \( S_{\mathcal{H}}(A) \) for the case when \( A \) is a separable \( C^{*} \)-algebra or a type I factor, and \( \mathcal{H} \) is an infinite dimensional Hilbert space.

**Theorem 5.3.** Let \( A \) be a separable \( C^{*} \)-algebra or a type I factor, and let \( \mathcal{H} \) be a separable Hilbert space. Then \( S_{\mathcal{H}}(A) \) is BW-closure of \( C^{*} \)-convex hull of its \( C^{*} \)-extreme points.

**Proof.** Case I: First assume that \( A \) is a separable \( C^{*} \)-algebra. Let \( \phi \in S_{\mathcal{H}}(A) \), and let \( (\pi,V,\mathcal{H}_\pi) \) be its minimal Stinespring triple. Since both \( A \) and \( \mathcal{H} \) are separable, the Hilbert space \( \mathcal{H}_\pi \) is also separable. By a corollary of Voiculescu’s theorem (see Theorem 42.1, [7]), there exists a sequence \( \{U_n\} \) of unitaries on \( \mathcal{H}_\pi \) and a representation \( \rho: A \to B(\mathcal{H}_\pi) \) such that \( \rho \) is a direct sum of irreducible representations and
\[
\pi(a) = \lim_{n \to \infty} U_n^* \rho(a) U_n \quad \text{in WOT},
\]
for all \( a \in A \). Therefore if we set \( W_n = U_n V \), then each \( W_n \) is an isometry, and \( \phi(a) = \lim_{n \to \infty} W_n^* \rho(a) W_n \) in WOT for all \( a \in A \). In other words, \( \phi \) is approximated in BW-topology by UCP maps, all of which are compression of the representation \( \rho \) that is a direct sum of irreducible representations. Thus, without loss of generality, we assume that \( \pi \) itself is a direct sum of a finite or countable irreducible representations, say,
\[
\pi = \oplus_{n \geq 1} \pi_n,
\]
where \( \pi_n: A \to B(K_n) \) is an irreducible representation on some Hilbert space \( K_n \). Now for each \( n \geq 1 \), let \( Q_n \) denote the projection of \( \mathcal{H}_\pi \) onto \( K_n \), and let \( V_n = Q_n V \in B(\mathcal{H}_\pi,K_n) \). Consider the completely positive map \( \phi_n: A \to B(\mathcal{H}) \) defined by \( \phi_n(a) = V_n^* \pi_n(a) V_n \) for all \( a \in A \). Since \( \pi_n \) is
irreducible, each \( \phi_n \) is a pure completely positive map. Also note that in WOT convergence, we have
\[
\sum_{n \geq 1} \phi_n(a) = \sum_{n \geq 1} V^*Q_n \pi_n(a)Q_n V = V^* \left( \sum_{n \geq 1} Q_n \pi_n(a)Q_n \right) V = V^*(\oplus_{n \geq 1} \pi_n(a))V = V^*\pi(a)V = \phi(a),
\]
for all \( a \in A \). The required assertion that \( \phi \) is in BW-closure of \( C^*\)-convex hull of \( C^*\)-extreme points of \( S_H(A) \) now follows from Lemma 5.1.

Case II: Let \( A \) be a type I factor, say \( A = B(G) \) for some Hilbert space \( G \). In view of Proposition 5.2, it suffices to approximate a normal UCP map \( \phi \) by \( C^*\)-convex combinations of \( C^*\)-extreme points of \( S_H(B(G)) \). Let \( \phi : B(G) \to B(H) \) be a normal UCP map. Then by Corollary 4.4, there exists a finite or countable sequence of contractions \( \{V_n\}_{n \geq 1} \) in \( B(H,G) \) such that
\[
\phi(X) = \sum_{n \geq 1} V_n^* XV_n \quad \text{for all } X \in B(G), \quad \text{(WOT Convergence).} \tag{5.3}
\]
Note that the maps \( X \mapsto V_n^* XV_n \) from \( B(G) \) to \( B(H) \) are pure maps. The claim now follows from Lemma 5.1. \( \square \)

**Remark 5.4.** In case I of Theorem 5.3 above, we have invoked a corollary of Voiculescu’s result for representations on separable \( C^*\)-algebras acting on separable Hilbert spaces. In recent years, there have been some study of Voiculescu’s theorems beyond separable case by applications coming from logic to operator algebras (see Vaccaro [28]). The results of [28] are for separably acting representations on certain non-separable \( C^*\)-algebras, and are not directly applicable in the current situation, as the Hilbert space \( H_n \) in Case I of Theorem 5.3 need not remain separable if \( A \) is not separable. Nevertheless, one can try to modify the proof above or look for possible variations in Vaccaro’s result to extend our work beyond separable case.

### 6. Examples and Applications

In the final section, we discuss a number of examples of UCP maps with their \( C^*\)-extremity properties. We shall also see an application to a well-known result from classical functional analysis about factorization property of Hardy algebras. We believe that the connection between \( C^*\)-extreme points and factorization property of the algebra \( M \) in Corollary 2.10 will produce many more examples and applications.

First we look into the question of when tensor products of two \( C^*\)-extreme points are \( C^*\)-extreme. This will help us in producing more \( C^*\)-extreme points out of the existing ones. The tensor product in question is minimal tensor product. See [24] for definitions and related properties.

For any two unital \( C^*\)-algebras \( A_1 \) and \( A_2 \), let \( A_1 \otimes A_2 \) denote their minimal (or spatial) tensor product (when \( A_1 \) and \( A_2 \) are von Neumann algebras, we denote by \( \overline{A_1 \otimes A_2} \) the von Neumann algebra generated by \( A_1 \otimes A_2 \)). Then for any two UCP maps \( \phi_i : A_i \to B(H_i), i = 1, 2 \), the assignment
\[
a_1 \otimes a_2 \mapsto \phi_1(a_1) \otimes \phi_2(a_2), \quad a_i \in A_i,
\]
extends to a UCP map from \( A_1 \otimes A_2 \) to \( B(H_1 \otimes H_2) \), which we denote by \( \phi_1 \otimes \phi_2 \) (see Theorem 12.3, [24]). The next proposition talks about \( C^*\)-extremity of tensor products, where one of the components is pure. We use the following well-known fact: if \( B_i \subseteq B(H_i), i = 1, 2 \), are two von Neumann algebras, then \( (B_1 \otimes B_2)' = B_1' \overline{\bigotimes} B_2' \) (Theorem IV.5.9, [27]).

**Proposition 6.1.** Let \( \phi_i : A_i \to B(H_i), i = 1, 2 \), be two UCP maps, and let \( \phi_2 \) be pure. Then \( \phi_1 \) is \( C^*\)-extreme (resp. extreme) in \( S_{H_1}(A_1) \) if and only if \( \phi_1 \otimes \phi_2 \) is \( C^*\)-extreme (resp. extreme) in \( S_{H_1 \otimes H_2}(A_1 \otimes A_2) \).

**Proof.** Let \( (\pi_1, V_1, K_1) \) be the minimal Stinespring triple of \( \phi_i \) for \( i = 1, 2 \). Then it is immediate that \( (\pi_1 \otimes \pi_2, V_1 \otimes V_2, K_1 \otimes K_2) \) is the minimal Stinespring triple for \( \phi_1 \otimes \phi_2 \). Set \( \pi = \pi_1 \otimes \pi_2 \). Note that since \( \pi_2(A_2)' = \mathbb{C} \cdot I_{K_2} \) (as \( \phi_2 \) is pure), it follows from above mentioned result that
\[
\pi(A)' = (\pi_1(A_1) \otimes I_{K_2})' = \pi_1(A_1)' \overline{\bigotimes} I_{K_2} = \pi_1(A_1)' \otimes I_{K_2}.
\]
Now for any operator \( D = D_1 \otimes I_{K_2} \in \pi(A)' \), we note that \( D_1 \) is positive and \( V_1^* D_1 V_1 \) is invertible if and only if \( D_1 \otimes I_{K_2} \) is positive and \( (V_1 \otimes V_2)^{*} (D_1 \otimes I_{K_2}) (V_1 \otimes V_2) \) is invertible. Also \( D_1 (V_1 H_1) \subseteq V_1 H_1 \) if and only if \( (D_1 \otimes I_{K_2}) (V_1 \otimes V_2) (H_1 \otimes H_2) \subseteq (V_1 \otimes V_2) (H_1 \otimes H_2) \). The assertion
about equivalence of $C^*$-extreme points now follows from equivalent criteria in Corollary 2.6. The assertions about extreme points follow similarly using Extreme point condition (Theorem 2.2). □

Since the identity representation $\text{id}_n : M_n \to M_n$ is pure, the following corollary about amalgamation of a $C^*$-extreme map is immediate.

**Corollary 6.2.** Let $\phi$ be a $C^*$-extreme point in $S_H(\mathcal{A})$. Then the map $\phi \otimes \text{id}_n : \mathcal{A} \otimes M_n \to B(\mathcal{H} \otimes C^n)$ is $C^*$-extreme in $S_{H\otimes C^n}(\mathcal{A} \otimes M_n)$, for each $n \in \mathbb{N}$.

For the next proposition, we set up some notations. Let $X$ be a countable set. For any Hilbert space $\mathcal{H}$ and a von Neumann algebra $\mathcal{B} \subseteq B(\mathcal{H})$, we consider the Hilbert space $\ell^2_X(\mathcal{H})$ and von Neumann algebra $\ell^\infty_B(X)$ given by

$$\ell^2_X(\mathcal{H}) = \{ f : X \to \mathcal{H} : \Sigma_{x \in X} |f(x)|^2 < \infty \}, \quad \ell^\infty_B(X) = \{ F : X \to \mathcal{B} : F \text{ is bounded} \}.$$  

Then $\ell^\infty_B(X)$ acts on the Hilbert space $\ell^2_X(\mathcal{H})$ via the operator $M_F, F \in \ell^\infty_B(X)$, defined by

$$M_F f(x) = F(x) f(x), \quad f \in \ell^2_X(\mathcal{H}) \quad \text{and} \quad x \in X.$$ 

We write $\ell^2_X(\mathcal{H})$ and $\ell^\infty_B(X)$ simply by $\ell^2(\mathcal{H})$ and $\ell^\infty(B)$ respectively. Also we identify the Hilbert space $\ell^2_X(\mathcal{H})$ with $\ell^2(\mathcal{H}) \otimes \mathcal{H}$, and the algebra $\ell^\infty_B(X)$ with $\ell^\infty(B) \otimes \mathcal{B}$, so that we shall use them interchangeably. If there is no possibility of confusion, we shall drop $X$ from $\ell^2_X(\mathcal{H}), \ell^\infty_B(X)$ etc.

**Proposition 6.3.** Let $\phi$ be a $C^*$-extreme point in $S_H(\mathcal{A})$, and let $i : \ell^\infty(\mathcal{X}) \to B(\ell^2(\mathcal{X}))$ be the natural inclusion map for some countable set $X$. Then $i \otimes \phi$ is $C^*$-extreme in $S_{\ell^\infty(\mathcal{X})}(\ell^\infty \otimes \mathcal{A})$.

**Proof.** Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Stinespring triple for $\phi$. Then $(\rho, U, \mathcal{H}_\rho)$ is the minimal Stinespring triple for $i \otimes \phi$, where $\mathcal{H}_\rho = \ell^2 \otimes \mathcal{H}_\pi = \ell^2_{\mathcal{H}_\rho}$, $U = i \otimes V : \ell^2 \otimes \mathcal{H} \to \ell^2 \otimes \mathcal{H}_\pi$, and $\rho = i \otimes \pi$. As mentioned above, we have

$$\rho(\ell^\infty \otimes \mathcal{A})' = (\ell^\infty \otimes \pi(\mathcal{A})')' = \ell^\infty_{\pi(\mathcal{A})}' = (\ell^\infty(\mathcal{A}))'.$$

Now let $M_D \in \ell^\infty_{\pi(\mathcal{A})}'$, be a positive operator such that $U^* M_D U$ is invertible. Then there exists $\alpha > 0$ such that $U^* M_D U \geq \alpha U^* U$. Note that for any $f \in \ell^2_{\mathcal{H}}$ and $x \in X$, we have

$$U^* M_D U f(x) = (V^* D(x)V) f(x).$$

Therefore for any unit vectors $g \in \ell^2$ and $h \in \mathcal{H}$, we have

$$\alpha \leq \langle U^* M_D U (g \otimes h), g \otimes h \rangle = \sum_{x \in X} \langle (V^* D(x)V) g(x) h, g(x) h \rangle = \sum_{x \in X} \langle (V^* D(x)V) h, h \rangle |g(x)|^2,$$

and since $g \in \ell^2$ varies over all unit vectors, it follows (by choosing $g$ to be the canonical basis elements of $\ell^2$) that $\langle (V^* D(x)V) h, h \rangle \geq \alpha$ for all $x \in X$. Again since $h \in \mathcal{H}$ is arbitrary, it follows that $V^* D(x)V \geq \alpha$ for all $x \in X$, i.e. $V^* D(x)V$ is invertible in $B(\mathcal{H})$. Since $\phi$ is $C^*$-extreme in $S_H(\mathcal{A})$, there exists an operator $S(x) \in \pi(\mathcal{A})'$ for each $x \in X$, such that $D(x) = S(x)^* S(x), S(x)V V^* = V V^* S(x)V V^*$ and $V S(x)V$ is invertible. Also note that

$$\| (V S(x)V)^{-1} \|^2 = \| (V^* D(x)V)^{-1} \|^2 \leq 1/\alpha.$$  

If $S$ denotes the map $x \mapsto S(x)$ from $X$ to $\pi(\mathcal{A})'$, then it is immediate to conclude that $S \in \ell^\infty_{\pi(\mathcal{A})}'$, such that $M_D = M_S^* M_S$ and $M_S U U^* = U U^* M_S U U^*$. Also since sup$_{x \in X} \| (V S(x)V)^{-1} \|^2 \leq 1/\alpha$, it follows that $U^* M_S U$ is invertible. Since $M_D$ is arbitrary, we conclude that $i \otimes \phi$ is $C^*$-extreme. □

If the set $X$ in Proposition 6.3 is a two point set, then we get the following:

**Corollary 6.4.** Let $\phi$ be a $C^*$-extreme point in $S_H(\mathcal{A})$. Then the map $\psi : \mathcal{A} \oplus \mathcal{A} \to B(\mathcal{H} \oplus \mathcal{H})$ defined by $\psi(a \oplus b) = \phi(a) \oplus \phi(b)$, for all $a, b \in \mathcal{A}$, is a $C^*$-extreme point in $S_{\mathcal{H} \oplus \mathcal{H}}(\mathcal{A} \oplus \mathcal{A})$.

The next proposition provides a family of $C^*$-extreme points, which can be thought as a generalization of Example 2 in [12], and whose proof follows almost the same lines. We give the proof for the sake of completeness. For doing so, we need the following fact from $C^*$-convexity of unit ball of $B(\mathcal{H})$ (see [17] for definitions and Theorem 1.1 therein): all isometries and co-isometries are $C^*$-extreme points of closed unit ball of $B(\mathcal{H})$. 

□
We also use the following assertion which is easy to verify (also see Theorem 3.18, [24]): if \((\pi, V, \mathcal{H}_\pi)\) is the minimal Stinespring triple for a UCP map \(\phi \in S_\mathcal{H}(\mathcal{A})\), then for any \(a \in \mathcal{A}\), \(\phi(a^*a) = \phi(a^*a)\) and only if \(V\phi(a) = \pi(a)V\).

Below, \(C^*(T)\) denotes the unital \(C^*\)-algebra generated by an operator \(T\).

**Proposition 6.5.** Let \(S\) be a unitary, and let \(\phi : C^*(S) \to \mathcal{B}(\mathcal{H})\) be a UCP map such that \(\phi(S)\) is an isometry or a co-isometry. Then \(\phi\) is \(C^*\)-extreme as well as extreme in \(S_{\mathcal{H}}(C^*(S))\).

**Proof.** We assume that \(\phi(S)\) is an isometry. The case of \(\phi(S)\) a co-isometry follows similarly. Let \((\pi, V, \mathcal{H}_\pi)\) be the minimal Stinespring triple for \(\phi\). Since \(\phi(S)\) is an isometry, we have \(\phi(S)^*\phi(S) = I_\mathcal{H} = \phi(1) = \phi(S^*S)\), so it follows (as mentioned above) that \(V\phi(S) = \pi(S)V\). This in particular implies for each \(n \in \mathbb{N}\) that \(V\phi(S)^n = \pi(S)^nV\), which yields

\[
\phi(S)^n = V^*\pi(S)^nV = V^*\pi(S^n)V = \phi(S^n).
\]  

(6.1)

Now to prove that \(\phi\) is \(C^*\)-extreme in \(S_{\mathcal{H}}(C^*(S))\), let \(\phi = \sum_{i=1}^n T_i^*\phi_i(\cdot)T_i\) be a proper \(C^*\)-convex combination for some UCP maps \(\phi_i\) and invertible operators \(T_i \in \mathcal{B}(\mathcal{H})\). Then \(\phi_S = I_\mathcal{H} = \sum_{i=1}^n T_i^*T_i = I_\mathcal{H}\). Since \(\phi(S)\) is an isometry, it is a \(C^*\)-extreme point in the closed unit ball of \(\mathcal{B}(\mathcal{H})\) (Theorem 1.1, [17]); hence there exist unitaries \(U_i \in \mathcal{B}(\mathcal{H})\) satisfying

\[
\phi(S) = U_i^*\phi_i(S)U_i
\]

for each \(i\). This implies that each \(\phi_i(S)\) is an isometry, and in a similar fashion as in (6.1), we get

\[
\phi_i(S)^n = \phi_i(S^n) \quad \text{for all } n \in \mathbb{N}.
\]

(6.2)

Thus for each \(n \in \mathbb{N}\), we have

\[
\phi(S^n) = (U_i^*\phi_i(S)U_i)^n = U_i^*\phi(S^n)U_i = U_i^*\phi_i(S^n)U_i.
\]

By taking adjoint both the sides, we also get \(\phi(S^{*n}) = U_i^*\phi(S^{*n})U_i\). Since \(S\) is unitary, it follows that \(\mathbb{S}(S^n, S^{*n}; n, m \in \mathbb{N}) = C^*(S)\). Thus we conclude that \(\phi(T) = U_i^*\phi_i(T)U_i\) for every \(T \in C^*(S)\) i.e. \(\phi\) is unitarily equivalent to \(\phi_i\). The case of \(\phi\) being extreme follows on similar lines, as isometries and co-isometries are extreme points of the closed unit ball of \(\mathcal{B}(\mathcal{H})\). \(\square\)

As a special case of Proposition 6.5, we have the following result. Here \(z \in C(T)\) is the function on the unit circle \(T\) given by \(z(e^{i\theta}) = e^{i\theta}\) for \(\theta \in \mathbb{R}\).

**Corollary 6.6.** Let \(\phi : C(T) \to \mathcal{B}(\mathcal{H})\) be a UCP map such that \(\phi(z)\) is an isometry or a co-isometry. Then \(\phi\) is \(C^*\)-extreme as well as extreme in \(S_{\mathcal{H}}(C(T))\).

As an application of Corollary 6.6, we give a new and simplified proof of a classical result of Szegö and its operator valued analogue about factorization property of Hardy algebras. Let \(K\) be a Hilbert space (possibly infinite dimensional), and let \(L^2_K(T)\) denote the Hilbert space of \(K\)-valued square integrable functions on \(T\) with respect to one-dimensional Lebesgue measure (which is isomorphic to \(L^2(T) \otimes K\)). Let \(H^\infty_K(T)\) denote the subspace

\[
\{f \in L^2_K(T); \int_0^{2\pi} f(e^{i\theta})e^{-i\theta n} d\theta = 0 \text{ for all } n < 0\}
\]

of \(L^2_K(T)\) (called vector-valued Hardy space). Let \(L^\infty_{B(K)}(T)\) be the von Neumann algebra of all essentially bounded measurable functions from \(T\) to \(B(K)\), which acts on \(L^2_K(T)\) by left multiplication i.e. for \(F \in L^\infty_{B(K)}(T)\), the operator \(M_F : L^2_K(T) \to L^2_K(T)\) is defined by

\[
M_F f(x) = F(x)f(x) \quad \text{for all } f \in L^2_K(T), x \in T.
\]

Let \(H^\infty_{B(K)}(T)\) be its subalgebra defined by

\[
H^\infty_{B(K)}(T) = \{F \in L^\infty_{B(K)}(T); \int_0^{2\pi} F(e^{i\theta})e^{-i\theta n} d\theta = 0 \text{ for all } n < 0\}.
\]

The algebra \(H^\infty_{B(K)}(T)\) is called the operator-valued Hardy algebra. Note that \(C(T) \subseteq L^\infty(T) \subseteq L^\infty_{B(K)}(T)\). We have the following factorization property of \(H^\infty_{B(K)}(T)\) in \(L^\infty_{B(K)}(T)\).
Theorem 6.6. Let \( H \) be a Hilbert space, and let \( S \in B(H) \) be a UCP map. Let \( \phi : C(T) \to B(H) \) be a UCP map on \( C(T) \) (Theorem 2.6, [24]), and it follows from Corollary 6.6 that \( \phi \) is \( C^\ast\)-extreme as well as extreme in \( S_H(C(T)) \).

Following is an example \( \phi \) of a \( C^\ast\)-extreme map of \( S_H(C(T)) \) such that \( \phi(z) \) need not be an isometry or a co-isometry.

Example 6.9. Let \( g : T \to T \) be a homeomorphism, and let \( \phi : C(T) \to B(H) \) be a UCP map. Set \( \psi : C(T) \to B(H) \) by \( \psi(f) = \phi(f \circ g) \) for all \( f \in C(T) \). Then it is easy to verify that \( \phi \) is \( C^\ast\)-extreme in \( S_H(C(T)) \) if and only if \( \psi \) is \( C^\ast\)-extreme in \( S_H(C(T)) \). Moreover one can choose a homeomorphism \( f \) such that \( \phi(z) \) is an isometry but \( \psi(z) \) is neither an isometry nor a co-isometry.

The following are two examples of (normal) UCP maps which are not \( C^\ast\)-extreme points. In order to show this, we use the fact that nest algebras associated with uncountable complete nests do not have factorization.

Example 6.10. Let \( K \) be a Hilbert space, and let \( \{K_q\}_{q \in Q} \) be a nest of subspaces indexed by rationals \( Q \) such that \( K_{q'} \subset K_q \) if \( q < q' \), and \( K = \bigvee_{q \in Q} K_q \). Let \( G \) be a Hilbert space, and let \( \{G_q\}_{q \in Q} \) be any collection of mutually orthogonal subspaces of \( G \). Consider the subspace \( H = \bigoplus_{q \in Q} G_q \otimes K_q \) of \( G \otimes K \), and the map \( \phi : B(G) \to B(H) \) defined by
\[
\phi(X) = P_H(X \otimes I_K)_{|H}, \quad \text{for all } X \in B(G).
\]

Note that the algebra \( \mathcal{M} = \{T \in B(K); (I_G \otimes T)(H) \subseteq H\} \) is nothing but \( \text{Alg} \mathcal{E} \), where \( \mathcal{E} \) is the nest \( \mathcal{E} = \{K_q\}_{q \in Q} \). Even though the nest \( \mathcal{E} \) is countable, its completion is not a countable nest. Consequently, \( I_G \otimes \mathcal{M} \) does not have factorization in \( B(K) \). Consequently, \( I_G \otimes B(K) = \pi(\mathcal{A})' \), where \( \pi(X) = X \otimes I_K \) is the minimal Stinespring representation of \( \phi \). Thus we conclude from Corollary 2.10 that \( \phi \) is not a \( C^\ast\)-extreme point in \( S_H(B(G)) \).

Example 6.11. Let \( K = L^2([0,1]) \) with respect to Lebesgue measure, and let \( H = \{\chi \Delta f; f \in L^2([0,1] \times [0,1]) \} \subseteq K \otimes K \), where \( \Delta = \{(s,t); s,t \in [0,1], 0 \leq s < t \leq 1 \} \subseteq [0,1] \times [0,1] \). Here \( \chi \Delta \) denotes the characteristic function on the set \( \Delta \). Define \( \phi : B(K) \to B(H) \) by
\[
\phi(X) = P_H(X \otimes I_K)_{|H}, \quad \text{for all } X \in B(K).
\]

We claim that \( \phi \) is not a \( C^\ast\)-extreme point in \( S_H(B(K)) \). First consider the following observations, which are straightforward to verify:

- \( \mathcal{H} = \bigoplus\bigcap_{q \in Q}(\chi_{[0,1]}f \otimes \chi_{[t,1]}g; t \in [0,1], f,g \in K) \).
- \( \mathcal{H}^\perp = \bigoplus\bigcap_{q \in Q}(\chi_{[s,1]}f \otimes \chi_{[0,1]}g; s \in [0,1], f,g \in K) \).
- \( \mathcal{K} \otimes \mathcal{K} = \bigoplus (X \otimes I_K)_{|H}, h \in H, X \in B(K) \).
- \( \phi(X) = P_{\mathcal{H}'}(X)_{|H} \) is the minimal Stinespring dilation for \( \phi \) where \( \pi : B(K) \to B(K \otimes K) \) is defined by \( \pi(X) = X \otimes I_K, X \in B(K) \).
- \( \pi(B(K))' = \{I_K \otimes S; S \in B(K)\} \).
Let $\mathcal{M} = \{ S \in B(\mathcal{K}); (I_{\mathcal{K}} \otimes S)(\mathcal{H}) \subseteq \mathcal{H} \}$. We claim that $\mathcal{M} \subseteq \text{Alg } \mathcal{E}$, for the complete nest $\mathcal{E} = \{ E_t; t \in [0,1] \}$, where

$$E_t = \{ \chi_{[t,1]}f; f \in \mathcal{K} \}, \text{ for } t \in [0,1].$$

Since $\mathcal{E}$ is uncountable, it will follow from Theorem 3.5 that $\text{Alg } \mathcal{E}$ does not have factorization in $B(\mathcal{K})$; hence $\mathcal{M}$ does not have factorization in $B(\mathcal{K})$, that is, $I_{\mathcal{K}} \otimes \mathcal{M}$ does not have factorization in $I_{\mathcal{K}} \otimes B(\mathcal{K}) = \pi(B(\mathcal{K})).$ This will imply from Corollary 2.10 that $\phi$ is not $C^*$-extreme in $S_{\text{Hyp}}(B(\mathcal{K})).$

Now let $S \in \mathcal{M}$, so that $(I_{\mathcal{K}} \otimes S)(\mathcal{H}) \subseteq \mathcal{H}$. Fix $t \in (0,1]$, and let $0 < s < t$. Note that $E_s^x = \{ \chi_{[0,s]}f; f \in \mathcal{K} \}$. Now for any $f, g \in \mathcal{K}$, we note from above observations that $\chi_{[0,t]} \otimes \chi_{[t,1]}g \in \mathcal{H}$ (so that $(I_{\mathcal{K}} \otimes S)(\chi_{[0,t]} \otimes \chi_{[t,1]}g) \in \mathcal{H}$) and $\chi_{[0,s]} \otimes \chi_{[0,s]}f \in \mathcal{H}^x$; hence

$$0 = \langle (I_{\mathcal{K}} \otimes S)(\chi_{[0,t]} \otimes \chi_{[t,1]}g), \chi_{[0,s]} \otimes \chi_{[0,s]}f \rangle = \langle \chi_{[0,t]} \otimes S(\chi_{[t,1]}g), \chi_{[0,s]} \otimes \chi_{[0,s]}f \rangle = (t-s)\langle S(\chi_{[t,1]}g), \chi_{[0,s]}f \rangle.$$  

Since $t-s \neq 0$, it follows that $\langle S(\chi_{[t,1]}g), \chi_{[0,s]}f \rangle = 0$. This shows that $S(\chi_{[t,1]}g) \perp E_s^x$, which is to say $S(\chi_{[t,1]}g) \in E_s$. Since $g \in \mathcal{K}$ is arbitrary, it follows that $S(E_t) \subseteq E_s$. Since $s < t$ is arbitrary, we conclude that

$$S(E_t) \subseteq \bigcap_{0<s<t} E_s = E_t.$$  

This shows that $S \in \text{Alg } \mathcal{E}$; thus we conclude our claim that $\mathcal{M} \subseteq \text{Alg } \mathcal{E}$.

Inspired from the example of $C^*$-extreme point as in (2.1), we now consider its noncommutative analogue. For a $C^*$-subalgebra $\mathcal{A}$ of $B(\mathcal{K})$ and a subspace $\mathcal{H}$ of $\mathcal{K}$, consider the UCP map $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$ given by

$$\phi(X) = P_\mathcal{H}X|_\mathcal{H}, \text{ for } X \in \mathcal{A}.$$  

If $\mathcal{A} = B(\mathcal{K})$, then clearly $\phi$ is a pure map, so that $\phi$ is $C^*$-extreme in $S_{\text{Hyp}}(\mathcal{A})$. An example of $C^*$-extreme point of this form (when $\mathcal{A} \neq B(\mathcal{K})$) is the map in (6.3). But for arbitrary $\mathcal{A}$, we do not know if $\phi$ is always $C^*$-extreme in $S_{\text{Hyp}}(\mathcal{A})$.

Let $\mathcal{A}$ be a finite von Neumann algebra with a distinguished faithful trace $\tau$. Let $L^2(\tau)$ denote the Hilbert space induced by $\tau$, which is the closure of $\mathcal{A}$ with respect to the inner product on $\mathcal{A}$ defined by $\langle x, y \rangle = \tau(x^*y)$ for $x, y \in \mathcal{A}$. Then the left regular representation $\pi: \mathcal{A} \rightarrow B(L^2(\tau))$ defined by $\pi(x) = L_x$ for all $x \in \mathcal{A}$, is cyclic with cyclic vector $\delta = 1$, where $L_x: L^2(\tau) \rightarrow L^2(\tau)$ is given by

$$L_x(y) = xy, \text{ for all } y \in \mathcal{A}.$$  

Now let $\mathcal{M}$ be a subalgebra of $\mathcal{A}$ such that $\mathcal{M}$ has factorization in $\mathcal{A}$ (as defined in 2.8). Examples of such algebras are finite maximal subdiagonal algebras introduced by Arveson [1], which also include nest subalgebras. Consider the subspace $H^2 = [\mathcal{M}] \subseteq L^2(\tau)$ (called noncommutative Hardy space), and let $\phi: \mathcal{A} \rightarrow B(H^2)$ be the map defined by

$$\phi(x) = P_{H^2}L_x|_{H^2},$$  

for $x \in \mathcal{A}$. It is clear that $\phi$ is a UCP map. We have the following:

**Proposition 6.12.** For $\mathcal{A}, \mathcal{M}$ and $\phi$ as above, $\phi$ is a $C^*$-extreme point in $S_{H^2}(\mathcal{A})$.

**Proof.** Note that $(\pi, V, L^2(\tau))$ is the minimal Stinespring triple, where $V$ is the inclusion map from $H^2$ to $L^2(\tau)$. It is a well-known fact that $\pi(\mathcal{A}') = \{ R_x; x \in \mathcal{A} \}$ (see Proposition 11.16, [25]), where $R_x \in B(L^2(\tau))$ is the right multiplication operator defined by $R_x(y) = yx$ for all $y \in \mathcal{A}$.

Now to show that $\phi$ is $C^*$-extreme in $S_{H^2}(\mathcal{A})$, we let $R_x$ to be a positive operator in $\pi(\mathcal{A}')$ for some $x \in \mathcal{A}$ such that $P_{H^2}R_x|_{H^2}$ is invertible. Clearly $x \geq 0$ in $\mathcal{A}$. We claim that $x$ is invertible in $\mathcal{A}$. Since $P_{H^2}R_x|_{H^2}$ is invertible, there is an $\alpha > 0$ such that $P_{H^2}R_x|_{H^2} \geq \alpha I_{H^2}$. Hence for all $z \in \mathcal{M}$, we have $(zx, z) = \langle R_xz, z \rangle \geq \alpha(z, z)$, that is, $\tau((zx - \alpha)z^*z) = \tau((z - \alpha)z) \geq 0$. Since $\{ z^*z; z \in \mathcal{M} \}$ is dense in the set of all positive elements of $\mathcal{A}$ (as $\mathcal{M}$ has factorization in $\mathcal{A}$), it follows that $\tau((x - \alpha)y) \geq 0$, for all $y \geq 0$ in $\mathcal{A}$. Hence for all $a \in \mathcal{A}$, we get using the trace property of $\tau$ that

$$\langle (x - \alpha)a, a \rangle = \tau((x - \alpha)a^*a) = \tau((x - \alpha)aa^*) \geq 0,$$

which is to say that $x - \alpha \geq 0$ in $\mathcal{A}$. This shows that $x$ is invertible. Therefore by factorization of $\mathcal{M}$ in $\mathcal{A}$, there exists an invertible element $z$ with $z, z^{-1} \in \mathcal{M}$ such that $x = zz^*$; thus $R_x = R_{zz^*} = \ldots$
Further, since \( z \in \mathcal{M} \), it follows that \( R_z(\mathcal{M}) \subseteq \mathcal{M} \) and hence \( R_z(H^2) \subseteq H^2 \). Also since \( z^{-1} \in \mathcal{M} \), we have \( R_{z^{-1}}(H^2) = R_{z^{-1}}(H^2) \subseteq H^2 \), which in particular implies that \( R_{z^{-1}} \) is invertible. Since \( R_z \) is arbitrary in \( \pi(\mathcal{A})' \), we conclude that \( \phi \) is a \( C^* \)-extreme point in \( S_{H^2}(\mathcal{A}) \). □

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