1. Introduction

In this paper we consider the connection between the Szegö kernel of certain unbounded domains of \( \mathbb{C}^2 \) and the Bergman kernels of weighted spaces of entire functions of one complex variable.

Let \( p : \mathbb{C} \longrightarrow \mathbb{R}_+ \) denote a \( C^1 \)-function and define \( \Omega_p \subseteq \mathbb{C}^2 \) by

\[
\Omega_p = \{(z_1, z_2) \in \mathbb{C}^2 : \Re(z_2) > p(z_1)\}.
\]

Weakly pseudoconvex domains of this kind were investigated by Nagel, Rosay, Stein and Wainger [10,11], where estimates for the Szegö and the Bergman kernel of the domain were made in terms of the nonisotropic pseudometric defined in [12,13]. For the case where \( p(z) = |z|^k, k \in \mathbb{N} \), Greiner and Stein [5] found an explicit expression for the Szegö kernel of \( \Omega_p \), in which one can recognize the form of the pseudometric used for the nonisotropic estimates (see [2,8]). If \( p \) is a subharmonic function, which depends only on the real or only on the imaginary part of \( z \), then one can find analogous expressions and estimates in [9].

Let \( H^2(\partial \Omega_p) \) denote the space of all functions \( f \in L^2(\partial \Omega_p) \), which are holomorphic in \( \Omega_p \) and such that

\[
\sup_{y > 0} \int_{\mathbb{C}} \int_{\mathbb{R}} |f(z, t + iy) + ip(z) + iy)|^2 d\lambda(z) dt < \infty,
\]

where \( d\lambda \) denotes the Lebesgue measure on \( \mathbb{C} \). We identify \( \partial \Omega_p \) with \( \mathbb{C} \times \mathbb{R} \), and denote by \( S((z, t), (w, s)) \), \( z, w \in \mathbb{C} \), \( s, t \in \mathbb{R} \), the Szegö kernel of \( H^2(\partial \Omega_p) \).

We use the tangential Cauchy–Riemann operator on \( \partial \Omega_p \) to get an expression for the Bergman kernel \( K_\tau(z, w) \) in the space \( H_\tau \) of all entire functions \( f \) such that

\[
\int |f(z)|^2 \exp(-2\tau p(z)) d\lambda(z) < \infty,
\]
where $\tau > 0$; in this connection we suppose that the weight functions $p$ have a reasonable growth behavior so that the corresponding spaces of entire functions are nontrivial, for example if $p(z)$ is a polynomial in $\Re z$ and $\Im z$.

On the other hand, if one integrates the Bergman kernels with respect to the parameter $\tau$, one obtains a formula for the Szegö kernel of $H^2(\partial \Omega_p)$.

We apply the main result for special functions $p$ to get generalizations of results in [5,8,9]. In [7] one can find another approach to get explicit expressions for the Szegö kernel. Finally the Bergman kernels for the spaces $H_{\tau}$, where $p$ is a function of $\Re z$, are investigated, especially their asymptotic behavior, which leads to sharp estimates and applications to problems considered in [7] concerning a duality problem in functional analysis.

**Proposition 1.** Let $\tau > 0$. Then

\[(1) \quad K_{\tau}(z,w) = e^{\tau(p(z)+p(w))} \int_{\Re} \int_{\Re} S((z,t),(w,s)) \frac{e^{i\tau(s-t)}}{p(w) - is} ds dt,\]

where the integrals are to be understood in the sense of the Plancherel theorem, i.e. in general one has only $L^2$–convergence of the integrals.

The fact that the above formula (1) is not symmetric in $z$ and $w$ is due to the $L^2$–convergence of the integrals.

**Proposition 2.**

\[(2) \quad S((z,t),(w,s)) = \int_0^\infty K_{\tau}(z,w)e^{-\tau(p(z)+p(w))}e^{-i\tau(s-t)} d\tau.\]

2. **Proofs of Proposition 1. and 2.**

For the proof we consider the tangential Cauchy–Riemann operator

\[L = \frac{\partial}{\partial z_1} - 2i \frac{\partial p}{\partial z_1}(z_1) \frac{\partial}{\partial z_2}\]

on $\partial \Omega_p$. Then (see [8]) $L$ is a global tangential antiholomorphic vector field, and

\[H^2(\partial \Omega_p) = \{ f \in L^2(\partial \Omega_p) : L(f) = 0 \text{ as distribution} \}.\]

After the usual identification of $\partial \Omega_p$ with $\mathbb{C} \times \Re$ the tangential Cauchy–Riemann operator has the form

\[L = \frac{\partial}{\partial z} - \frac{\partial p}{\partial z} \frac{\partial}{\partial t}.\]

For a function $f \in L^2(d\lambda(z)dt)$ let $\mathcal{F}$ denote the Fourier transform with respect to the variable $t \in \Re$ :

\[(\mathcal{F}f)(z,\tau) = \int f(z,t)e^{-it\tau} dt.\]
Then
\[ \mathcal{F}L\mathcal{F}^{-1} = \frac{\partial}{\partial z} + \tau \frac{\partial p}{\partial z}. \]

\( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are to be taken in the sense of the Plancherel theorem.

Now let \( M \) denote the multiplication operator
\[ M : L^2(d\lambda(z)dt) \rightarrow L^2(e^{-2\tau p(z)}d\lambda(z)dt) \]
defined by
\[ (Mf)(z, \tau) = e^{\tau p(z)}f(z, \tau), \]
for \( f \in L^2(d\lambda(z)dt) \). Then one has
\[ (3) \quad \mathcal{F}L\mathcal{F}^{-1} = M^{-1} \frac{\partial}{\partial z} M. \]

Let \( \mathcal{P} \) denote the orthogonal projection
\[ \mathcal{P} : L^2(d\lambda(z)dt) \rightarrow \text{Ker}L, \]
and let \( P \) be the orthogonal projection
\[ P : L^2(e^{-2\tau p(z)}d\lambda(z)dt) \rightarrow \text{Ker} \frac{\partial}{\partial z}. \]

For fixed \( \tau > 0 \), let \( P_\tau \) be the orthogonal projection
\[ P_\tau : L^2(e^{-2\tau p(z)}d\lambda(z)) \rightarrow \text{Ker} \frac{\partial}{\partial z}. \]

Now we claim that
\[ (Pf)(z, \tau) = \begin{cases} (P_\tau f_\tau)(z), & \tau > 0 \\ 0, & \tau \leq 0 \end{cases}, \]
where \( f_\tau(z) = f(z, \tau) \), for \( f \in L^2(e^{-2\tau p(z)}d\lambda(z)dt) \). In order to see this it is enough to observe that a function \( f \in L^2(e^{-2\tau p(z)}d\lambda(z)dt) \) holomorphic with respect to the variable \( z \) has the property \( f(z, t) = 0 \), for almost all \( t \leq 0 \), which is a consequence of our assumption on the weight function \( p \).

The next step is to show that
\[ (4) \quad P = M\mathcal{F}\mathcal{P}\mathcal{F}^{-1}M^{-1}. \]

Denote the right side of (4) by \( Q \). We have to show that \( Q^2 = Q \) and that
\[ \text{Ker} \frac{\partial}{\partial z} \subseteq L^2(e^{-2\tau p(z)}d\lambda(z)dt) \]
coincides with the image of \( Q \). The first assertion follows directly from the definition of \( Q \). For the second assertion take a function \( f \in L^2(e^{-2\tau p(z)}d\lambda(z)dt) \) and use (3) to prove that
\[ \frac{\partial}{\partial z} Qf = M\mathcal{F}\mathcal{P}\mathcal{F}^{-1}M^{-1}f. \]
the last expression is zero, since $\mathcal{P}_f^{-1}M^{-1}f \in \text{Ker} L$, which implies that the image of $Q$ is contained in $\text{Ker} \frac{\partial}{\partial z}$. To prove the opposite inclusion set $g = Qf$ for $f \in \text{Ker} \frac{\partial}{\partial z}$. We are finish, if we can show that $Qg = f$. From (3) we get now

$$L\mathcal{F}^{-1}M^{-1}f = \mathcal{F}^{-1}M^{-1}\frac{\partial}{\partial z}f,$$

which is zero by the assumption on $f$, hence $\mathcal{F}^{-1}M^{-1}f \in \text{Ker} L$ and therefore

$$\mathcal{P}\mathcal{F}^{-1}M^{-1}f = \mathcal{F}^{-1}M^{-1}f.$$

The last equality yields

$$Qg = M\mathcal{F}\mathcal{P}\mathcal{F}^{-1}M^{-1}f = M\mathcal{F}\mathcal{F}^{-1}M^{-1}f = f,$$

which proves formula (4).

For a fixed $\tau > 0$ take a function $F \in L^2(e^{-2\tau p(z)}d\lambda(z))$ and define

$$f(z,t) = \begin{cases} \chi(z)F(z), & t \geq \tau \\ 0, & t < \tau \end{cases},$$

where $\chi$ is a nonnegative, smooth function with the properties $(\chi(z))^2 = p(z)$, for $|z| \leq 1$ and $\chi(z) = 1$, for $|z| \geq 2$.

Since

$$\int_{C} \int_{\mathbb{R}} |f(z,t)|^2e^{-2tp(z)} dtd\lambda(z) = \int_{C} \int_{\tau}^{\infty} |\chi(z)F(z)|^2e^{-2tp(z)} dtd\lambda(z)$$

$$= \int_{C} \frac{1}{2p(z)}|\chi(z)F(z)|^2e^{-2\tau p(z)} d\lambda(z) \leq \text{Const.} \int_{C} |F(z)|^2e^{-2\tau p(z)} d\lambda(z),$$

it follows that

$$f \in L^2(e^{-2\tau p(z)}d\lambda(z)dt).$$

Now we use formula (4) to obtain (1): application of the operators $M^{-1}$ and $\mathcal{F}^{-1}$ to the function $f$ from above yields

$$\mathcal{F}^{-1}M^{-1}f(w,t) = \int_{\tau}^{\infty} \chi(w)F(w)e^{t(i\sigma - p(w))} dt$$

$$= \frac{\chi(w)F(w)e^{-\tau(p(w)-i\sigma)}}{p(w) - i\sigma},$$

which is a function in $L^2(d\lambda(w)d\sigma)$, by the properties of the function $\chi$.

The next operator in (4) is now $\mathcal{P}$, which is the Szegö projection, hence an application of this operator can be expressed by integration over the Szegö kernel $S((z,t),(w,\sigma))$. Finally we carry out the action of the operators $\mathcal{F}$ and $M$ and recall the properties of the operator $P$ on the left side of (4), which imply that this operator is for a fixed $\tau$ the Bergman projection in a weighted space of entire functions in one variable. The function $\chi$ appears on both sides and hence cancels out. In this way we get formula (1). In order to prove (2) one writes (4) in the form

$$\mathcal{P} = \mathcal{F}^{-1}M^{-1}PMF,$$

and applies an analogous procedure as above.
3. Examples

(a) Let $\alpha \in \mathbb{R}$, $\alpha > 0$. We consider the function $p(z) = |z|^\alpha$ and get from [6] the following expression for the Bergman kernel $K_\tau(z, w)$ in the space $H_\tau$:

$$K_\tau(z, w) = \frac{2\pi}{\alpha} \sum_{k=0}^{\infty} (2\tau)^{2(k+1)/\alpha} \left( \Gamma\left(\frac{2(k+1)}{\alpha}\right)\right)^{-1} z^k w^k.$$

Now we apply formula (2) to this sum and get

$$S((z, t), (w, s)) = \frac{2\pi}{\alpha} \sum_{k=0}^{\infty} \frac{(2\tau)^{2(k+1)/\alpha}}{\Gamma\left(\frac{2(k+1)}{\alpha}\right)} z^k w^k \int_0^{\infty} 2^{2(k+1)/\alpha} e^{-\tau(|z|^\alpha+|w|^\alpha)} e^{-i\tau(s-t)} d\tau,$$

evaluation of the last integral gives

$$\Gamma\left(\frac{2(k+1)}{\alpha} + 1\right) \left[|z|^\alpha + |w|^\alpha + i(s-t)\right]^{-2(k+1)/\alpha-1},$$

by the functional equation of the $\Gamma$-function we have

$$\Gamma\left(\frac{2(k+1)}{\alpha} + 1\right) = \frac{2(k+1)}{\alpha} \Gamma\left(\frac{2(k+1)}{\alpha}\right),$$

hence

$$S((z, t), (w, s)) = \frac{2\pi}{\alpha} \sum_{k=0}^{\infty} \frac{2(k+1)}{\alpha} 2^{2(k+1)/\alpha} z^k w^k \left[|z|^\alpha + |w|^\alpha + i(s-t)\right]^{-2(k+1)/\alpha-1}.$$ 

Now we set

$$A = \frac{1}{2}(|z|^\alpha + |w|^\alpha + i(s-t))$$

and carry out the summation over $k$ with the result

$$S((z, t), (w, s)) = \frac{2\pi}{\alpha^2} A^{-1-2/\alpha} \left(1 - \frac{z\overline{w}}{A^{2/\alpha}}\right)^{-2}.$$

This generalizes a result of Greiner and Stein [5], where the same formula appears for $\alpha \in \mathbb{N}$ (see also [2,8]).

(b) If the weight function $p$ depends only on the real part of $z$ and satisfies

$$\int_{\mathbb{R}} e^{-2p(x)+2yx} dx < \infty,$$

for each $y \in \mathbb{R}$, then the Bergman kernel of $H_\tau$ is given by

$$K_\tau(z, w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(\eta(z + \overline{w}))}{\exp(2\eta(x - \overline{x}(\alpha)))} d\eta,$$

(6)
This follows by a modification of methods developed in [9]. To show (6) we proceed in the following way:

In sake of simplicity we set $\tau = 1$. Similar to the proofs of Proposition 1 and 2 we consider the multiplication operator

$$M_p : L^2(d\lambda(z)) \longrightarrow L^2(e^{-2p(x)}d\lambda(z)),$$

defined by $(M_p f)(z) = e^{p(x)} f(z)$, $f \in L^2(d\lambda(z))$. Now a computation shows that

$$\frac{\partial}{\partial \bar{z}} \left( e^{p(x)} f(z) \right) = e^{p(x)} \left( \frac{1}{2} \frac{\partial p}{\partial x} f + \frac{\partial f}{\partial \bar{z}} \right),$$

which can be expressed by the operator identity

$$L(f) := \left( M_{-p} \frac{\partial}{\partial \bar{z}} M_p \right)(f) = \frac{1}{2} \frac{\partial p}{\partial x} f + \frac{\partial f}{\partial \bar{z}}.$$

Let $\mathcal{F}$ denote the Fourier transform with respect to $y$:

$$\mathcal{F} f(x, \eta) = \int_{-\infty}^{\infty} f(x, y)e^{-iy\eta} dy.$$ 

Then in the sense of distributions we have

$$\mathcal{F} L(f)(x, \eta) = \frac{1}{2} \left( e^{-p(x)+\eta x} \frac{\partial}{\partial x} \left( e^{p(x)-\eta x} \mathcal{F} f(x, \eta) \right) \right).$$

We set $\psi(x, \eta) = e^{p(x)-\eta x}$ and define the multiplication operator

$$\mathcal{M}_\psi : L^2(d\lambda(z)) \longrightarrow L^2(e^{-2p(x)+2yx}d\lambda(z))$$

by $(\mathcal{M}_\psi g)(x, \eta) = \psi(x, \eta) g(x, \eta)$, for $g \in L^2(d\lambda(z))$. Combining this with the last results we get

$$L = \frac{1}{2} \mathcal{F}^{-1} \mathcal{M}_{-\psi} \frac{\partial}{\partial x} \mathcal{M}_\psi \mathcal{F},$$

and finally

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} M_p \mathcal{F}^{-1} \mathcal{M}_{-\psi} \frac{\partial}{\partial x} \mathcal{M}_\psi \mathcal{F} M_p.$$ 

In this context we consider differentiation with respect to $x$ as an operator

$$\frac{\partial}{\partial x} : L^2(e^{-2p(x)+2yx}d\lambda(z)) \longrightarrow L^2(e^{-2p(x)+2yx}d\lambda(z)),$$

in the sense of distributions.

Further we remark that $\text{Ker} \frac{\partial}{\partial x}$ consists of all functions $g \in L^2(e^{-2p(x)+2yx}d\lambda(z))$, which are constant in $x$. 

By our assumption on the weight function $p$ the space $L^2(e^{-2p(x)+2yx}dx)$ contains the constants for each $y \in \mathbb{R}$. Let $P_y$ denote the orthogonal projection of $L^2(e^{-2p(x)+2yx}dx)$ onto the constants and $P$ the orthogonal projection of $L^2(e^{-2p(x)+2yx}d\lambda(z))$ onto $\text{Ker} \frac{\partial}{\partial x}$. Then it is easily seen that

$$(Pg)(x, y) = P_y g_y(x),$$

for $g \in L^2(e^{-2p(x)+2yx}d\lambda(z))$, where $g_y(x) = g(x, y)$.

For a fixed $y \in \mathbb{R}$ and a function $h \in L^2(e^{-2p(x)+2yx}dx)$ one has

$$P_y h = \frac{(h, 1)}{(1, 1)} \left( \int_{\mathbb{R}} e^{-2p(x)+2yx} dx \right)^{-1} \int_{\mathbb{R}} h(x)e^{-2p(x)+2yx} dx.$$ 

Finally let $P$ denote the orthogonal projection of $L^2(e^{-2p(x)}d\lambda(z))$ onto $H_1 = \text{Ker} \frac{\partial}{\partial z}$.

With the help of the above operator identities we readily establish now

$$P = M_pF^{-1}M_{-\psi}P_M\psi FM_p.$$ 

This identity, together with the above remarks on the orthogonal projection $P$, implies formula (6).

Using (2) one gets

$$S((z, t), (w, s)) = \frac{1}{2\pi} \int_{0}^{\infty} \int_{\mathbb{R}} \frac{\tau \exp(\tau(\eta(z + w) - p(z) - p(w) - i(s - t)))}{\int_{\mathbb{R}} \exp(2\tau(r\eta - p(r))) dr} d\eta d\tau,$$

which is similar to an expression in [9].

Now we investigate the asymptotic behavior of the integral

$$(7) \int_{\mathbb{R}} \exp(2\tau(r\eta - p(r))) dr,$$

which appears in formula (6), first as a function of $\eta$, for $|\eta| \to \infty$.

We restrict our attention to the case where the weight function $p$ is of the form

$$p(r) = \frac{|r|^\alpha}{\alpha}, \quad \alpha > 1, \quad r \in \mathbb{R}.$$

Let $p^*$ denote the Young conjugate of $p$ which is given by

$$(8) \quad p^*(\eta) = \sup_{x \geq 0} [x|\eta| - p(x)] = \frac{|\eta|^{\alpha'}}{\alpha'},$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Note that $p^{**} = p$. Now we can estimate the integral (7) from above.

$$\int_{\mathbb{R}} \exp(2\tau(r\eta - p(r))) dr = \int_{0}^{\infty} \exp(2\tau(r\eta - p(r))) dr + \int_{0}^{\infty} \exp(2\tau(r\eta - p(r))) dr.$$
Let \( \lambda > 1 \). Then we have for \( \eta \geq 1 \)

\[
\int_0^\infty \exp(2\tau (r\eta - p(r))) \, dr \leq \int_0^\infty \exp(2\tau (r\eta - \lambda \eta r + p^*(\lambda \eta))) \, dr
\]

\[
= \exp(2\tau (p^*(\lambda \eta))) \int_0^\infty \exp(-2\tau (\lambda - 1)\eta r) \, dr
\]

\[
= \frac{\exp(2\tau p^*(\lambda \eta))}{2\tau (\lambda - 1)\eta}
\]

and for the second part of the integral

\[
\int_{-\infty}^0 \exp(2\tau (r\eta - p(r))) \, dr = \int_0^\infty \exp(-2\tau r\eta) \, dr
\]

\[
\leq \int_0^\infty \exp(-2\tau r\eta) \, dr
\]

\[
= \frac{1}{2\tau \eta}.
\]

For \( \eta \leq -1 \) we estimate in the analogous way.

Finally for \( |\eta| < 1 \) we get

\[
\int_0^\infty \exp(2\tau (r\eta - p(r))) \, dr \leq \int_0^\infty \exp(2\tau (r - p(r))) \, dr,
\]

\[
\int_{-\infty}^0 \exp(2\tau (r\eta - p(r))) \, dr = \int_0^\infty \exp(-2\tau (r - p(r))) \, dr
\]

\[
\leq \int_0^\infty \exp(2\tau (r - p(r))) \, dr.
\]

Hence for each \( \eta \in \mathbb{R} \) we obtain

\[
\int_{\mathbb{R}} \exp(2\tau (r\eta - p(r))) \, dr \leq C(\lambda, \tau) \exp(2\tau p^*(\lambda \eta)),
\]

for each \( \lambda > 1 \), where \( C(\lambda, \tau) > 0 \) is a constant depending on \( \lambda \) and \( \tau \).

To estimate the integral in (7) from below we denote by \( \mu \) the inverse function of the derivative \( p' \)

\[
\mu(\eta) := (p')^{-1}(\eta) = |\eta|^{1/(\alpha - 1)}.
\]

First suppose that \( \eta \geq 0 \) and observe that \( p' \) is strictly increasing and that the supremum in formula (8) is attained in the point \( \mu(\eta) \), hence

\[
\int_{\mathbb{R}} \exp(2\tau (r\eta - p(r))) \, dr \geq \int_0^\infty \exp(2\tau (r\eta - p(r))) \, dr
\]

\[
\geq \exp(2\tau (\eta (\mu(\eta) + 1) - p(\mu(\eta) + 1))).
\]

Next we claim that for each \( \lambda, 0 < \lambda < 1 \), the following inequality holds

\[
2\tau (\eta (\mu(\eta) + 1) - p(\mu(\eta) + 1)) > 2\tau (\lambda \eta (\mu(\eta) + 1) - p(\mu(\lambda \eta))) = D(\tau, \lambda)
\]

where

\[
D(\tau, \lambda) := (\lambda^2 - \lambda + 1)\tau - \eta (\lambda^2 - \lambda + 1)\tau.
\]
for each $\eta \geq 0$, where $D(\tau, \lambda) > 0$ is a constant depending on $\tau$ and $\lambda$.

To see this we remark that

$$
\eta(\mu(\eta) + 1) - p(\mu(\eta) + 1) = \eta^{\alpha/(\alpha - 1)} + \eta - 1/\alpha \left( \eta^{1/(\alpha - 1)} + 1 \right)^\alpha,
$$

and

$$
\lambda \eta \mu(\lambda \eta) - p(\mu(\lambda \eta)) = (1 - 1/\alpha) \lambda^{\alpha/(\alpha - 1)} \eta^{\alpha/(\alpha - 1)}.
$$

It suffices to show that

$$
\left( 1 - (1 - 1/\alpha) \lambda^{\alpha/(\alpha - 1)} \right) \eta^{\alpha/(\alpha - 1)} + \eta \geq 1/\alpha \left( \eta^{1/(\alpha - 1)} + 1 \right)^\alpha - D(\lambda),
$$

for each $\eta \geq 0$, where $D(\lambda) > 0$ is a constant depending on $\lambda$. But this follows easily from the fact that

$$
1 - (1 - 1/\alpha) \lambda^{\alpha/(\alpha - 1)} > 1/\alpha.
$$

For $\eta < 0$ we argue in a similar way.

On the whole we have now proved that

$$
(10) \quad D(\tau, \lambda) \exp(2\tau p^*(\eta/\lambda)) \leq \int_\mathbb{R} \exp(2\tau(r\eta - p(r))) \, dr \leq C(\lambda, \tau) \exp(2\tau p^*(\lambda \eta)),
$$

for each $\eta \in \mathbb{R}$ and $\lambda > 1$.

For the conjugate function $p^*$ one obtains by the same methods

$$
(11) \quad D_1(\tau, \lambda) \exp(2\tau p(r/\lambda)) \leq \int_\mathbb{R} \exp(2\tau(r\eta - p^*(\eta))) \, d\eta \leq C_1(\lambda, \tau) \exp(2\tau p(\lambda r)),
$$

for each $r \in \mathbb{R}$ and $\lambda > 1$.

The asymptotic behavior of (7) as a function of $\tau$, $\tau \to \infty$, can be derived from [1], pg. 65:

$$
\int_\mathbb{R} \exp(2\tau(r\eta - p(r))) \, dr \approx \left( \frac{\tau p''(\mu(\eta))}{2\pi} \right)^{1/2} \exp(2\tau p^*(\eta)).
$$

Let

$$
\exp(2\tau \varphi^*(\eta)) = \int_\mathbb{R} \exp(2\tau(r\eta - p(r))) \, dr.
$$

Then formula (6') can be written in the form

$$
(12) \quad K_\tau(z, w) = \frac{\tau}{2\pi} \int_\mathbb{R} \exp \left( 2\tau(\eta \left( \frac{z + \overline{w}}{2} \right) - \varphi^*(\eta)) \right) \, d\eta.
$$

In view of (10) and (11) this means that the Bergman kernel $K_\tau(z, w)$ is in a certain sense an analytical continuation of the original weight $\exp(2\tau p(r))$, namely in the form

$$
\exp \left( 2\tau \varphi \left( \frac{z + \overline{w}}{2} \right) \right).
$$
For \( p(z) = x^2/2 \) everything can be computed explicitly:

\[
\int_{\mathbb{R}} \exp(2\tau(r\eta - r^2/2)) \, dr = (\pi/\tau)^{1/2} \exp(\tau \eta^2),
\]

(13) \[ K_\tau(z, w) = \frac{\tau}{2\pi} \exp \left( \frac{\tau}{4} (z + \overline{w})^2 \right) \]

and

(14) \[ S((z, t), (w, s)) = \frac{1}{2\pi} \left( \frac{1}{4}(z + \overline{w})^2 - \frac{1}{8}(z + \overline{z})^2 - \frac{1}{8}(w + \overline{w})^2 - i(s - t) \right)^{2} \]

Applying formula (1) to the expression for the Szegö kernel in (14), we arrive again at (13), now the integral with respect to \( s \) converges only in \( L^2 \).

Results of this type have also been obtained by Gindikin (see [4] or [3]).

Finally we mention an estimate for the Bergman kernel, which plays an important role in the duality problem of [7] and which, in itself, seems to be interesting.

For the Bergman kernel in formula (13) the following condition is satisfied: for each \( \tau_1 > \tau \) there exists \( \tau_0, 0 < \tau_0 < \tau \), such that

\[
\int_{\mathbb{C}} \int_{\mathbb{C}} |K_{\tau}(z, w)|^2 \exp(-2\tau_1 p(z) - 2\tau_0 p(w)) \, d\lambda(z) \, d\lambda(w) < \infty.
\]

This follows by a direct computation using (13). In the general case the integration with respect to the variable \( z \) causes no problems, as the function \( z \mapsto K_{\tau}(z, w) \) belongs to the Hilbertspace \( H_{\tau_1} \), for each fixed \( w \). But, afterwards, the integration with respect to the variable \( w \) makes difficulties, because \( \tau_0 < \tau \).
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