On weak solutions to a fractional Hardy–Hénon equation:  
Part I: Nonexistence

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Abstract  
This paper and [20] treat the existence and nonexistence of stable (resp. outside stable) weak solutions to a fractional Hardy–Hénon equation \((-\Delta)^s u = |x|^\ell |u|^{p-1}u\) in \(\mathbb{R}^N\), where \(0 < s < 1, \ell > -2s, p > 1, N \geq 1\) and \(N > 2s\). In this paper, the nonexistence part is proved for the Joseph–Lundgren subcritical case.

Keywords: fractional Hardy–Hénon equation, stable (stable outside a compact set) solutions, Liouville type theorem.

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1 Introduction

In this paper and [20], we consider a fractional Hardy–Hénon equation

\[ (-\Delta)^s u = |x|^\ell |u|^{p-1}u \quad \text{in } \mathbb{R}^N \]  

and throughout this paper, we always assume the following condition on \(s, \ell, p, N\):

\[ 0 < s < 1, \quad \ell > -2s, \quad p > 1, \quad N \geq 1, \quad N > 2s. \]  

In (1.1), \((-\Delta)^s\) is the fractional Laplacian, which is defined for any \(\varphi \in C^\infty_c(\mathbb{R}^N)\) by

\[ (-\Delta)^s \varphi(x) := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \varphi(x) - \varphi(y) \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy = C_{N,s} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy \]
for \( x \in \mathbb{R}^N \), where P.V. stands for the Cauchy principal value integral,

\[
C_{N,s} := 2^{2s} s (1 - s) \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)}
\]

(1.3)

and \( \Gamma(z) \) is the gamma function.

This paper and [20] are motivated by previous work in [7, 9, 16, 19] and we shall study the existence and nonexistence of stable solutions to (1.1). Farina [16] studied (1.1) in the case \( s = 1, \ell = 0 \) and \( N \geq 2 \) and he showed the existence (\( p \geq p_c(N) \)) and nonexistence (\( 1 < p < p_c(N) \)) of stable (resp. stable outside a compact set) solutions where the Joseph–Lundgren exponent \( p_c(N) \) is defined by

\[
p_c(N) := \begin{cases} 
   (N - 2)^2 - 4N + 8\sqrt{N - 1} 
   & \text{if } N \geq 11, \\
   \infty 
   & \text{if } 1 \leq N \leq 10.
\end{cases}
\]

Next, Dancer, Du and Guo [7, Theorem 1.2] and Wang and Ye [25, Theorem 1.7] studied the case \( \ell > -2 \) and \( N \geq 2 \) and showed the existence and nonexistence of stable and finite Morse index solutions in \( H^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \). We remark that in [25], they treated the weaker class of solutions than those in \( H^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \). The threshold on \( p \) is given by

\[
p_+(N, \ell) := \begin{cases} 
   \frac{(N - 2)^2 - 2(\ell + 2)(\ell + N) + 2\sqrt{(\ell + 2)^3(\ell + 2N - 2)}}{(N - 2)(N - 4\ell - 10)} 
   & \text{if } N > 10 + 4\ell, \\
   \infty 
   & \text{if } 2 \leq N \leq 10 + 4\ell.
\end{cases}
\]

On the other hand, the case \( s = 1/2, \ell = 0 \) and \( N \geq 2 \) was treated in Chipot, Chlebík, Fila and Shafrir [3] as an extension problem and it is shown that there exists a positive radial solution to (1.1) for \( p \geq \frac{N+1}{N+\frac{3}{2}} = p_S(N, 0) \) where \( p_S(N, \ell) \) is defined in (1.6). Harada [19] considered the same case \( s = 1/2, \ell = 0 \) and \( N \geq 2 \), introduced the notion corresponding to the Joseph–Lundgren exponent \( p_c(N) \) and proved the existence of a family of layered positive radial solutions when \( p \) is the Joseph–Lundgren supercritical or critical. In [19], the subcritical case is also treated. Dávila, Dupaigne and Wei [9] dealt with the case \( \ell = 0 \) and \( 0 < s < 1 \), and proved the existence and nonexistence of stable (resp. stable outside a compact set) solutions of (1.1). We remark that in [9], they treated solutions \( u \in C^{2\sigma}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \) (the sign of the weight in [9, Theorem 1.1] might be a misprint) where \( \sigma > s \) and

\[
L^q(\mathbb{R}^N, w(x)dx) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid \| u \|_{L^q(\mathbb{R}^N, w(x)dx)} := \left( \int_{\mathbb{R}^N} |u(x)|^q w(x) dx \right)^{\frac{1}{q}} < \infty \right\}.
\]

However, in order to make their argument work, it seems appropriate to assume \( u \in C^{2\sigma}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \). For this point, see Remark 2.4 and [9, Lemmata 2.1–2.4]. Notice that \( L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \subset L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \) since \( (1 + |x|)^{-N-2s} \in L^1(\mathbb{R}^N) \).

We also refer to Li and Bao [22] for the study of positive solutions of (1.1) with singularity at \( x = 0 \) in the case \( -2s < \ell \leq 0 \) and \( 1 < p \leq p_S(N, \ell) \).

The aim of this paper and [20] is to extend the results of [9, 19] into the case \( \ell \neq 0 \) and established the result which is a fractional counterpart of [7]. In this paper, we establish the
nonexistence result. On the other hand, in [20], we will consider the existence result and study properties of solutions.

After submitting this paper, we learned Barrios and Quaas [1] and the references therein from Alexander Quaas. In [1] and Dai and Qin [6], they studied the nonexistence of positive solution (and nonnegative nontrivial solutions) of (1.1) for \( \ell \in (-2s, \infty) \) and \( 0 < p < p_S(N, \ell) \). On the other hand, Yang [26] considered the existence of positive solution of (1.1) via the minimizing problem for \( p = p_S(N, \ell) \). Finally, Fazly–Wei [17] considered (1.1) for the case \( 0 < \ell \) and \( 1 < p \leq p_S(N, \ell) \). They proved the nonexistence of stable solutions for \( 1 < p < p_S(N, \ell) \), and for \( p = p_S(N, \ell) \), they obtained the same result under the finite energy condition for solutions. See also the comments after Theorem 1.1.

We first introduce the notation of solutions of (1.1).

**Definition 1.1.** Suppose (1.2). We say that \( u \) is a solution of (1.1) if \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \) and

\[
\langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^\ell |u|^{p-1}u \varphi \, dx \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^N) \tag{1.4}
\]

where

\[
\langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy \tag{1.5}
\]

and \( C_{N,s} \) is the constant defined by (1.3). Remark that \(|x|^\ell|u(x)|^{p-1}u(x) \in L^1_{\text{loc}}(\mathbb{R}^N)\) due to \( u \in L^\infty_{\text{loc}}(\mathbb{R}^N) \) and (1.2). For \( \Omega \subset \mathbb{R}^N \), we also set

\[
\|u\|_{H^s(\Omega)} := \left( \frac{C_{N,s}}{2} \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

**Remark 1.1.** In Section 2, we will see that

1. \( \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} \in \mathbb{R} \) for any \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx) \) and \( \varphi \in C^\infty_c(\mathbb{R}^N) \).

2. we may replace \( C^\infty_c(\mathbb{R}^N) \) in Definition 1.1 by \( C^1_c(\mathbb{R}^N) \).

3. our solution \( u \) satisfies (1.1) in the distribution sense, that is,

\[
\int_{\mathbb{R}^N} u (-\Delta)^s \varphi \, dx = \langle u, \varphi \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^\ell |u|^{p-1}u \varphi \, dx \quad \text{for every } \varphi \in C^\infty_c(\mathbb{R}^N).
\]

For the details, see Lemma 2.1.

In order to state our main result and for later use, following [9], we introduce some notation. We put

\[
B_R := \{ x \in \mathbb{R}^N \mid |x| < R \}, \quad S_R := \partial B_R = \{ x \in \mathbb{R}^N \mid |x| = R \},
\]

\[
B^+_R := \{ (x, t) \in \mathbb{R}^{N+1}_+ \mid |(x, t)| < R \}, \quad S^+_R := \partial B^+_R = \{ (x, t) \in \mathbb{R}^{N+1}_+ \mid |(x, t)| = R \},
\]

and \( B'_R := \mathbb{R}^N \setminus B_R \). For \( N \geq 1 \), \( s \in (0, 1) \) and \( \ell > -2s \), we write

\[
p_S(N, \ell) := \frac{N + 2s + 2\ell}{N - 2s} \in (1, \infty). \tag{1.6}
\]
Note that \( p_S(N,0) \) corresponds to the critical exponent of the fractional Sobolev inequality \( H^s(\mathbb{R}^N) \subset L^{p_S(N,0)}(\mathbb{R}^N) \). Next, for \( \alpha \in [0,(N-2s)/2) \), we set
\[
\lambda(\alpha) := 2^{2s} \frac{\Gamma\left(\frac{N+2s-2\alpha}{4}\right) \Gamma\left(\frac{N+2s-2\alpha}{4}\right)}{\Gamma\left(\frac{N-2s-2\alpha}{4}\right) \Gamma\left(\frac{N-2s-2\alpha}{4}\right)}.
\]  
(1.7)

It is known that
\[
\text{the function } \alpha \mapsto \lambda(\alpha) \text{ is strictly decreasing}
\] and \( \lambda(\alpha) \to 0 \) as \( \alpha \nearrow (N-2s)/2 \) (see, e.g. Frank, Lieb and Seiringer [18, Lemma 3.2] and Dávila, Dupaigne and Montenegro [8, Appendix]).

**Remark 1.2.** Let \( v_\alpha(x) := |x|^{-\left(\frac{N-2s}{2} - \alpha\right)} \) for \( 0 \leq \alpha < \frac{N-2s}{2} \). According to Fall [12, Lemma 4.1], the constant \( \lambda(\alpha) \) appears in the equation
\[
(-\Delta)^s v_\alpha = \lambda(\alpha)|x|^{-2s} v_\alpha \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]

Finally, we introduce the notation of stable, stable outside a compact set and Morse index equal to \( K \):

**Definition 1.2.** Let \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{-N-2s} dx) \) be a solution of (1.1). We say that \( u \) is stable if \( u \) satisfies
\[
p \int_{\mathbb{R}^N} |x|^{\ell} |u|^{p-1} \varphi^2 \, dx \leq \|\varphi\|^2_{H^s(\mathbb{R}^N)} \quad \text{for every } \varphi \in C^\infty_\text{c}(\mathbb{R}^N).
\]
(1.9)

On the other hand, \( u \) is called stable outside a compact set if there exists an \( R_0 \geq 0 \) such that
\[
p \int_{\mathbb{R}^N} |x|^{\ell} |u|^{p-1} \varphi^2 \, dx \leq \|\varphi\|^2_{H^s(\mathbb{R}^N)} \quad \text{for every } \varphi \in C^\infty_\text{c}(\mathbb{R}^N \setminus \overline{B_{R_0}}).
\]
(1.10)

Finally, a solution \( u \) is said to have a Morse index equal to \( K \) provided \( K \) is the maximal dimension of subspaces \( Z \subset C^\infty_\text{c}(\mathbb{R}^N) \) with
\[
\|\varphi\|^2_{H^s(\mathbb{R}^N)} - p \int_{\mathbb{R}^N} |x|^{\ell} |u|^{p-1} \varphi^2 \, dx < 0 \quad \text{for each } \varphi \in Z \setminus \{0\}.
\]

**Remark 1.3.**

1. By a density argument, in Definition 1.2, we may replace \( C^\infty_\text{c}(\mathbb{R}^N) \) and \( C^\infty_\text{c}(\mathbb{R}^N \setminus \overline{B_{R_0}}) \) by \( C^1_\text{c}(\mathbb{R}^N) \) and \( C^1_\text{c}(\mathbb{R}^N \setminus \overline{B_{R_0}}) \). In addition, (1.10) remains true for \( \varphi \in C^1_\text{c}(\mathbb{R}^N) \) with \( \varphi \equiv 0 \) on \( B_{R_0} \).

2. As in [16, Remark 1], we may check that if a solution \( u \) has a finite Morse index, then \( u \) is stable outside a compact set.

The following is the main result of this paper:

**Theorem 1.1.** Suppose (1.2) and let \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1+|x|)^{-N-2s} dx) \) be a solution of (1.1) which is stable outside a compact set.

(i) If \( 1 < p < p_S(N,\ell) \), then \( u \equiv 0 \).
(ii) If \( p = p_S(N, \ell) \), then \( u \) has finite energy, that is
\[
\|u\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |x|^{\frac{N-2s}{2}} |u|^{p+1} \, dx < +\infty.
\]
Furthermore, if \( u \) is stable, then \( u \equiv 0 \).

(iii) If \( p_S(N, \ell) < p \) with
\[
p \lambda \left( \frac{N-2s}{2} - \frac{2s + \ell}{p-1} \right) > \lambda(0),
\]
then \( u \equiv 0 \), where \( \lambda(\alpha) \) is the function given by (1.7).

As mentioned in the above, it might be necessary to suppose \( u \in L^2(\mathbb{R}^N, (1+|x|)^{-N-2s} \, dx) \) in [9] and also in [17] since a similar argument was used in [17]. Taking this point into consideration, we succeed to extend the nonexistence part of [9] and [17] into the case \( \ell \in (-2s, 0) \). Furthermore, Theorem 1.1 may be regarded as a fractional version of a part of [7, Theorem 1.2]. We remark that we deal with weak solutions and in [8, 17], classical solutions were studied. On the other hand, when \( p_S(N, \ell) < p \) holds and (1.11) fails to hold, then we will show the existence of stable solutions in [20] and observe the properties of those solutions. By Remark 1.3, Theorem 1.1 asserts also that there is no solution of (1.1) with finite Morse index when \( 1 < p < p_S(N, \ell) \) or \( p_S(N, \ell) < p \) with (1.11). When \( p = p_S(N, \ell) \), we find that any solution of \( u \) of (1.1) with finite Morse index satisfies \( u \in \dot{H}^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N, |x|^\ell \, dx) \).

The proof of Theorem 1.1 is basically similar to [9]. However, in [9], they use the fact that solutions are of class \( C^1 \) or smooth, for instance, see [9, the proofs of Theorem 1.1 for \( 1 < p \leq p_S(n) \) and Theorem 1.4, and at the end of proof of Theorem 1.1 for \( p_S(n) < p \)]. On the other hand, (1.1) contains the term \( |x|^\ell \) and especially in the case \( \ell < 0 \), solutions of (1.1) are not of class \( C^1 \) at the origin. Therefore, we need some modifications in the argument. In this paper, we first prove a local Pohozaev type identity as in Fall and Felli [13] and exploit it to show Theorem 1.1 for \( 1 < p \leq p_S(N, \ell) \). In addition, to show the monotonicity formula (Lemma 4.2), we use the idea in [13, section 3] where they studied the Almgren type frequency.

This paper is organized as follows. In subsection 2.1, we investigate the properties of the \( s \)-harmonic extension of functions in \( H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{-N-2s} \, dx) \), that is, functions satisfying the extension problem. Subsection 2.2 is devoted to the proof of local Pohozaev identity and the energy estimate is done in subsection 2.3. Section 3 contains the proof of Theorem 1.1 for \( 1 < p \leq p_S(N, \ell) \) and in section 4, we deal with the case \( p_S(N, \ell) < p \).

## 2 Preliminaries

This section is divided into three subsections. In subsection 2.1, we show properties of functions which belong to \( H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{-N-2s} \, dx) \), and give a relationship between a solution \( u \) of (1.1) and \( s \)-harmonic functions. In subsection 2.2, we recall local regularity estimates for the extension problem. This estimate is useful to establish the Pohozaev identity in Proposition 2.2. Furthermore, applying an argument similar to the one in [9], we also give energy estimates for solutions of (1.1) in subsection 2.3.

Throughout this paper, by the letter \( C \) we denote generic positive constants and they may have different values also within the same line. Furthermore, we write \( X := (x, t) \in \mathbb{R}_+^{N+1} \).
2.1 Remark on notion of weak solutions

We first prove properties of functions which belong to $H^s_{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx)$.

**Lemma 2.1.** Let $u \in H^s_{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx)$. Then the following hold.

(i) For any $\psi \in C_c^\infty(\mathbb{R}^N)$, $\langle u, \psi \rangle_{\dot{H}^s(\mathbb{R}^N)} \in \mathbb{R}$ and

$$
\left| \langle u, \psi \rangle_{\dot{H}^s(\mathbb{R}^N)} \right| \leq C(N, s) \left\{ \|u\|_{H^s(B_{2R})} \|\psi\|_{H^s(B_{2R})} + \|u\|_{L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx)} \|\psi\|_{L^1(B_{2R})} \right\}
$$

(2.1)

where $\text{supp} \, \psi \subset B_R$ with $R \geq 1$ and $C(N, s)$ is a constant depending on $N$ and $s$. In addition, let $\varphi_1 \in C_c^\infty(\mathbb{R}^N)$ with $\varphi_1(x) \equiv 1$ in $B_1$ and $\varphi_1(x) \equiv 0$ in $B_2^c$, and set $\varphi_n(x) := \varphi_1(n^{-1}x)$. Then for any $\psi \in C_c^\infty(\mathbb{R}^N),$

$$
\langle \varphi_n u, \psi \rangle_{\dot{H}^s(\mathbb{R}^N)} \rightarrow \langle u, \psi \rangle_{\dot{H}^s(\mathbb{R}^N)} \quad \text{as} \quad n \rightarrow \infty.
$$

In particular,

$$
\langle u, \psi \rangle_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u(-\Delta)^s \psi \, dx \quad \text{for each} \quad \psi \in C_c^\infty(\mathbb{R}^N).
$$

(2.2)

(ii) Put

$$
P_s(x, t) := p_{N,s} \frac{t^{2s}}{(|x|^2 + t^2)^{\frac{N+2s}{2}}}, \quad U(x, t) := (P_s(\cdot, t) * u)(x) \quad (2.3)
$$

where $p_{N,s} > 0$ is chosen so that $\|P_s(\cdot, t)\|_{L^1(\mathbb{R}^N)} = 1$. Then

$$
- \text{div} \left( t^{1-2s} \nabla U \right) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+, \quad U(x, 0) = u(x), \quad U \in H^1_{loc} \left( \mathbb{R}^{N+1}_+, t^{1-2s} dX \right) \quad (2.4)
$$

and for each $\psi \in C_c^\infty(\mathbb{R}^{N+1}_+)$ with $\partial_t \psi(x, 0) = 0$,

$$
- \lim_{t \rightarrow 0} \int_{\mathbb{R}^N} t^{1-2s} \partial_t U(x, t) \psi(x, t) \, dx = \kappa_s \int_{\mathbb{R}^N} u(x)(-\Delta)^s \psi(x, 0) \, dx
$$

(2.5)

Here

$$
H^1_{loc} \left( \mathbb{R}^{N+1}_+, t^{1-2s} dX \right)
$$

:= \left\{ V : \mathbb{R}^{N+1}_+ \rightarrow \mathbb{R} \left| \int_{B_R^+} t^{1-2s} \{ |\nabla V|^2 + V^2 \} dX < \infty \quad \text{for all} \ R > 0 \right\}
$$

and

$$
\kappa_s := \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}. \quad (2.6)
$$

Remark 2.1.
1. In Lemma 2.2 and Remark 2.2, we will see that (2.5) holds for every $\psi \in C^\infty_c(\mathbb{R}^{N+1}_+)$ if we assume that $u$ is a solution of (1.1).

2. Here we collect properties on the Poisson kernel $P_s(x,t)$. For the properties below, see, for instance, [4, 11, 13, 14, 21]. First of all, it is known that the Fourier transform of $P_s(x,t)$ is given by $\hat{P}_s(\xi,t) = \theta_0(2\pi|\xi|t)$ where

$$
\hat{v}(\xi) := \int_{\mathbb{R}^N} v(x)e^{-2\pi i x \cdot \xi} \, dx \quad \text{for } v \in L^1(\mathbb{R}^N),
$$

$$
\theta_0(t) := \frac{2}{\Gamma(s)} \left( \frac{t}{2} \right)^s K_s(t), \quad \theta_0' + \frac{1-2s}{t} \theta_0 - \theta_0 = 0 \quad \text{in } (0, \infty), \quad \theta_0(0) = 1,
$$

$K_s(t)$ is the modified Bessel function of the second kind with order $s$ and

$$
\kappa_s = \int_0^\infty t^{1-2s} \left\{ (\theta'_0(t))^2 + \theta_0^2(t) \right\} \, dt = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}.
$$

By these properties, it is possible to prove that for each $v \in H^s(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N+1} t^{1-2s} |\nabla(P_s(\cdot,t) * v)(x)|^2 \, dX = \kappa_s \int_{\mathbb{R}^N} (4\pi^2|\xi|^2)^s |\hat{v}(\xi)|^2 \, d\xi = \kappa_s \|v\|_{H^s(\mathbb{R}^N)}^2
$$

and for $U(x,t) = (P_s(\cdot,t) * u)(x)$ and $u \in \dot{H}^s(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N+1} t^{1-2s} |\nabla U|^2 \, dX = \kappa_s \|u\|_{\dot{H}^s(\mathbb{R}^N)}^2. \quad (2.7)
$$

Furthermore, for each $\zeta \in C^\infty_c(\mathbb{R}^{N+1}_+)$,

$$
\kappa_s \|\zeta(\cdot,0)\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N+1} t^{1-2s} |\nabla(P_s(\cdot,t) * \zeta(\cdot,0))(x)|^2 \, dX \leq \int_{\mathbb{R}^N+1} t^{1-2s} |\nabla \zeta|^2 \, dX. \quad (2.8)
$$

Finally, for $\varphi \in C^\infty_c(\mathbb{R}^N)$,

$$
- \lim_{t \to 0} \frac{1}{t^{1-2s}} \partial_t (P_s(\cdot,t) * \varphi)(x) = \kappa_s (-\Delta)^s \varphi(x) \quad \text{for any } x \in \mathbb{R}^N.
$$

Proof of Lemma 2.1. (i) We first show $\langle u, \psi \rangle_{\dot{H}^s(\mathbb{R}^N)} \in \mathbb{R}$ and (2.1). Let $\psi \in C^\infty_c(\mathbb{R}^N)$ with $\text{supp } \psi \subset B_R$ and $R \geq 1$. Since $\psi \equiv 0$ on $B_R^c$ and $|x-y| \geq |y|/2$ for $x \in B_R$ and $y \in B_R^c$, we see from $R \geq 1$ that

$$
\frac{1 + |y|}{|x-y|} \leq 2 \frac{1 + |y|}{|y|} \leq 4 \quad \text{for each } x \in B_R \text{ and } y \in B_R^c.
$$
Therefore, nated convergence theorem, observe that
\[
\langle u(x) - u(y), \psi(x) - \psi(y) \rangle dx dy
\]
\[
= \left( \int_{B_2 \times B_2} + 2 \int_{B_2 \times B_2^c} \frac{|u(x) - u(y)||\psi(x) - \psi(y)|}{|x - y|^{N+2s}} dx dy \right)
\]
\[
\leq \frac{2}{C_{N,s}} \|u\|_{H^s(B_2)} \|\psi\|_{H^s(B_2)}
\]
\[
+ 2 \int_{B_2} dx \int_{|y| \geq 2R} (|u(x)| + |u(y)|) |\psi(x)| (1 + |y|)^{-N-2s} \left( \frac{1 + |y|}{|x - y|} \right)^{N+2s} dy
\]
\[
\leq \frac{2}{C_{N,s}} \|u\|_{H^s(B_2)} \|\psi\|_{H^s(B_2)}
\]
\[
+ C(N,s) \int_{B_2} dx \int_{|y| \geq 2R} (|u(x)| + |u(y)|) |\psi(x)| (1 + |y|)^{-N-2s} dy
\]
\[
\leq \frac{2}{C_{N,s}} \|u\|_{H^s(B_2)} \|\psi\|_{H^s(B_2)}
\]
\[
+ C(N,s) \left\{ \|u\|_{L^2(B_2)} \|\psi\|_{L^2(B_2)} + \|\psi\|_{L^1(B_2)} \|u\|_{L^1(R^N, (1 + |x|)^{-N-2s} dx)} \right\}
\]
where $C_{N,s}$ is the constant given by (1.3). Since $\|u\|_{H^s(B_2)} < \infty$ due to $u \in H^s_{\text{loc}}(\mathbb{R}^N)$, we observe that $\langle u, \psi \rangle_{H^s(R^N)} \in \mathbb{R}$ and (2.1) holds.

The assertion $\langle \varphi_n u, \psi \rangle_{H^s(R^N)} \to \langle u, \psi \rangle_{H^s(R^N)}$ as $n \to \infty$ follows from (2.1) and $\|\varphi_n u - u\|_{L^1(R^N, (1 + |x|)^{-N-2s} dx)} \to 0$.

Finally we prove (2.2). We remark that due to Fall and Weth [15, Lemma 2.1] and supp $\psi \subset B_R$, there exists a $C_R > 0$ such that
\[
\left| \frac{\psi(x) - \psi(y)}{|x - y|^{N+2s}} dy \right| \leq C_R \|\psi\|_{C^2(R^N)} (1 + |x|)^{-N-2s} \quad \text{for all } x \in \mathbb{R}^N \text{ and } \varepsilon \in (0, 1).
\]
Therefore, $\int_{\mathbb{R}^N} u(-\Delta)^s \psi dx \in \mathbb{R}$ by $u \in L^1(R^N, (1 + |x|)^{-N-2s} dx)$. Moreover, by the dominated convergence theorem,
\[
\langle u, \psi \rangle = \frac{C_{N,s}}{2} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{(u(x) - u(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}} dx dy
\]
\[
= \lim_{\varepsilon \to 0} C_{N,s} \int_{\mathbb{R}^N} \left[ u(x) \int_{|x-y|>\varepsilon} \frac{\psi(x) - \psi(y)}{|x-y|^{N+2s}} dy \right] dx = \int_{\mathbb{R}^N} u(x) (-\Delta)^s \psi(x) dx.
\]
Hence, (i) holds.

(ii) Notice that $U$ is well-defined thanks to $u \in L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx)$. We prove (2.4). The assertion $-\text{div}(t^{1-2s} \nabla U) = 0$ in $\mathbb{R}^N_{++}$ follows from the fact $-\text{div}(t^{1-2s} \nabla P_s) = 0$ in $\mathbb{R}^N_{++}$. It is also easily seen that $P_s(x, t) \to \delta_0$ as $t \to +0$, hence, $U(x, 0) = u(x)$ for $u \in C_c^\infty(\mathbb{R}^N)$. For general $u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx)$, the assertion follows from $U \in H^1_{\text{loc}}(\mathbb{R}^N_{++}, t^{1-2s} dX)$ (this will be proved below) and the existence of the trace operator $H^s_{\text{loc}}(\mathbb{R}^N_{++}, t^{1-2s} dX) \to H^s_{\text{loc}}(\mathbb{R}^N)$ (see, for instance, Lions [23], and Demengel and Demengel [10]).
In order to prove $U \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s}dX)$, we will show $U \in H^1(B_R \times (0, R), t^{1-2s}dX)$ for each $R > 0$. To this end, let $\varphi_1$ be as in (i), $n_0 > 2R$ and write
\[ U(x, t) = (P_s(\cdot, t) * u)(x) = (P_s(\cdot, t) * (\varphi_{n_0}u + (1 - \varphi_{n_0})u))(x) =: U_1(x, t) + U_2(x, t). \]

For $U_1$, we have $\nabla U_1 \in L^2(\mathbb{R}^{N+1}, t^{1-2s}dX)$ due to $\varphi_{n_0}u \in H^s(\mathbb{R}^N)$ and Remark 2.1. In addition, Young’s inequality and the fact $\|P_s(\cdot, t)\|_{L^1(\mathbb{R}^N)} = 1$ imply
\begin{align*}
\int_0^R dt \int_{\mathbb{R}^N} t^{1-2s} |(P_s(\cdot, t) * (\varphi_{n_0}u))(x)|^2 dx &\leq \int_0^R t^{1-2s} \|P_s(\cdot, t)\|_{L^1(\mathbb{R}^N)}^2 \|\varphi_{n_0}u\|^2_{L^2(\mathbb{R}^N)} dt \\
&= C_{R, s} \|\varphi_{n_0}u\|^2_{L^2(\mathbb{R}^N)}.
\end{align*}

Hence, $U_1 \in H^1(B_R \times (0, R), t^{1-2s}dX)$ for every $R > 0$.

On the other hand, for $U_2$, thanks to $n_0 > 2R$, we may write as
\[ U_2(x, t) = p_{N, s} \int_{|y| \geq n_0} \frac{t^{2s}}{|y|^2 + t^2} \left(1 - \varphi_{n_0}(y)\right) u(y) dy. \]

Noting $|x - y| \geq |y|/2$ for all $|y| \geq n_0$ and $|x| \leq R$, we see that
\[ |U_2(x, t)| \leq C t^{2s} \int_{|y| \geq n_0} |u(y)|(1 + |y|)^{-N-2s} \left(\frac{1 + |y|}{|x|}ight)^{N+2s} dy \leq C t^{2s} \|u\|^2_{L^1(\mathbb{R}^N,(1+|x|)^{-N-2s}dx)} \]
and that
\[ |\nabla U_2(x, t)| \leq C \int_{|y| \geq n_0} \left(\frac{t^{2s-1}}{|x|+|y|} + \frac{t^{2s}}{|x|+|y|^{N+1+2s}}\right) |u(y)| dy \leq C \|u\|_{L^1(\mathbb{R}^N,(1+|x|)^{-N-2s}dx)} (t^{2s-1} + t^{2s}) \]
for all $(x, t) \in B_R \times (0, R)$. This yields $U_2 \in H^1(B_R \times (0, R), t^{1-2s}dX)$, which implies $U = U_1 + U_2 \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s}dX)$.

Next we prove (2.5). Let $\psi \in C^\infty_c(\mathbb{R}^{N+1}_+)$ with $\text{supp} \psi \subset B_{R/2} \times [0, R]$ and $\partial_t \psi(x, 0) = 0$. Then, by (2.3) and Fubini’s theorem, we have
\begin{align*}
- \int_{\mathbb{R}^N} t^{1-2s} \partial_t U(x, t) \psi(x, t) dx &= -t^{1-2s} \int_{\mathbb{R}^N} \partial_t \left(\int_{\mathbb{R}^N} p_{N, s} t^{2s} \psi(x, t) \frac{dy}{(|x - y|^2 + t^2)^{N+2s}}\right) \psi(x, t) dx \\
&= -t^{1-2s} \int_{\mathbb{R}^N} \partial_t \left[ \int_{\mathbb{R}^N} p_{N, s} t^{2s} \psi(x, t) \frac{dy}{(|x - y|^2 + t^2)^{N+2s}} \right] u(y) dy \\
&\quad + t^{1-2s} \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} p_{N, s} t^{2s} \partial_t \psi(x, t) \frac{dy}{(|x - y|^2 + t^2)^{N+2s}} \right] u(y) dy \\
&= \int_{\mathbb{R}^N} \left( I_1(y, t) + I_2(y, t) \right) u(y) dy \quad \text{(2.9)}
\end{align*}
where
\[
I_1(y, t) := -t^{1-2s} \partial_t \left[ \int_{\mathbb{R}^N} \frac{p_{N,s} t^{2s} \psi(x, t)}{|x-y|^2 + t^2} \ dx \right],
\]
\[
I_2(y, t) := t^{1-2s} \int_{\mathbb{R}^N} \frac{p_{N,s} t^{2s} \partial_x \psi(x, t)}{|x-y|^2 + t^2} \ dx.
\]

From $\psi(x, 0) \in C_c^\infty(\mathbb{R}^N)$ and Remark 2.1, we observe that
\[
- \lim_{t \to +0} t^{1-2s} \partial_t (P_s(\cdot, t) * \psi(\cdot, 0))(x) = \kappa_s(-\Delta)^s \psi(x, 0) \quad \text{for } x \in \mathbb{R}^N. \tag{2.10}
\]

Notice also that
\[
I_1(y, t) = -t^{1-2s} \left[ \partial_t \left( \int_{\mathbb{R}^N} \frac{p_{N,s} t^{2s} (\psi(x, t) - \psi(x, 0))}{|x-y|^2 + t^2} \ dx \right) \right. \nonumber
\]

\[
\left. + \partial_t \left( \int_{\mathbb{R}^N} \frac{p_{N,s} t^{2s} \psi(x, 0)}{|x-y|^2 + t^2} \ dx \right) \right] \tag{2.11}
\]

where
\[
I_2(y, t) = -t^{1-2s} \partial_t (P_s(\cdot, t) * \psi(\cdot, 0))(y)
\]

Thus, if we may prove that for all $y \in \mathbb{R}^N$
\[
\lim_{t \to +0} \left( |I_2(y, t)| + |I_3(y, t)| \right) = 0 \quad \text{strongly in } L^\infty(\mathbb{R}^N), \tag{2.12}
\]

\[
|I_2(y, t)| + |I_3(y, t)| + |t^{1-2s} \partial_t (P_s(\cdot, t) * \psi(\cdot, 0))(y)| \leq C|y|^{-N-2s} \quad \text{for each } |y| \geq 2R \text{ and } t \in (0, 1], \tag{2.13}
\]

then we have (2.5) by applying the dominated convergence theorem with $u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx)$, and (2.10)–(2.13) to (2.9).

We first deal with (2.12). Recall that $\text{supp } \psi \subset B_R$ and $\partial_\nu \psi(x, 0) \equiv 0$. Then,
\[
\frac{\psi(x, t) - \psi(x, 0)}{t} = \int_0^1 \partial_t \psi(x, t \theta) d\theta = \int_0^1 \partial_t \psi(x, t \theta) - \partial_t \psi(x, 0) d\theta =: \Psi(x, t) \in C^\infty(\overline{\mathbb{R}^N+1}),
\]
\[
\Psi(x, 0) = 0, \quad |\Psi(x, t)| \leq \varphi_R(x) t, \quad |\partial_t \Psi(x, t)| \leq \varphi_R(x) \quad \text{for each } (x, t) \in \mathbb{R}^N \times [0, 1]
\]
where \( \varphi_R \subset C_c(B_{3R}/2) \). Hence, we observe from \((|x-y|^2+t^2)^{1/2} \geq t\)

\[
|I_3(y,t)| = \left| t^{1-2s} \partial_t \left[ \int_{\mathbb{R}^N} \frac{p_{N,s}t^{1+2s}\Psi(x,t)}{|x-y|^2+t^2} \frac{dx}{N+2s} \right] \right| \\
\leq C \int_{\mathbb{R}^N} t \frac{\Psi(x,t)}{|x-y|^2+t^2} \frac{dx}{N+2s} + \int_{\mathbb{R}^N} t^2 \frac{p_{N,s}\partial_t \Psi(x,t)}{|x-y|^2+t^2} \frac{dx}{N+2s} + C t^2 \int_{\mathbb{R}^N} \Psi(x,t) \frac{dx}{|x-y|^2+t^2} \frac{dx}{N+2s} \\
\leq C \int_{\mathbb{R}^N} t^2 \frac{\varphi_R(x)}{|x-y|^2+t^2} N+2s \frac{dx}{N+2s} = t^{2-2s} \int_{\mathbb{R}^N} \frac{\varphi_R(y-tz)}{|z|^2+1} N+2s \frac{dx}{N+2s} \leq C \varphi_R \|L^\infty(\mathbb{R}^N)\| t^{2-2s} 
\]

where \( C > 0 \) is independent of \( y \in \mathbb{R}^N \). Hence, \( I_3(y,t) \rightarrow 0 \) in \( L^\infty(\mathbb{R}^N) \) as \( t \rightarrow 0 \).

Since \( \partial_t \psi(x,t) \leq t \varphi_R(x) \) for all \((x,t) \in \mathbb{R}^N \times [0,1]\) due to \( \partial_t \psi(x,0) = 0 \), in a similar way, we can check that \( I_2(y,t) \rightarrow 0 \) in \( L^\infty(\mathbb{R}^N) \) as \( t \rightarrow 0 \) and (2.12) holds.

Next, we treat (2.13). Let \(|y| \geq 2R\) and consider \( I_3 \). Noting \(|x-y| \geq \frac{|y|-|x|}{4} \geq |y|/4\) for all \(|y| \geq 2R\) and \(|x| \leq 3R/2\), by (2.14) we see that

\[
|I_3(y,t)| \leq C \int_{\mathbb{R}^N} t^2 \frac{\varphi_R(x)}{|x-y|^2+t^2} N+2s \frac{dx}{N+2s} \\
\leq C \int_{|x| \leq 3R/2} t^2 \|\varphi_R\|_{L^\infty(\mathbb{R}^N)} |x-y|^{-N-2s} \frac{dx}{N+2s} \leq Ct^2 |y|^{-N-2s}.
\]

In a similar way, we may prove

\[
|I_2(y,t)| \leq C t^2 |y|^{-N-2s}. 
\]

Furthermore, applying the change of variables and the integration by parts and noting \( \text{supp} \psi \subset B_R \), we obtain

\[
t^{1-2s} \partial_t(P_s(\cdot,t) * \psi(\cdot,0))(y) = t^{1-2s} \partial_t \left[ \int_{\mathbb{R}^N} \frac{p_{N,s}\psi(y-tz,0)}{|z|^2+1} N+2s \frac{dz}{N+2s} \right] \\
= -t^{1-2s} \int_{\mathbb{R}^N} \frac{p_{N,s}\nabla y \psi(y-tz,0) \cdot z}{|z|^2+1} N+2s \frac{dz}{N+2s} \\
= \int_{|x| \leq R} \frac{p_{N,s}\nabla x \psi(x,0) \cdot (x-y)}{|x-y|^2+t^2} N+2s \frac{dx}{N+2s} \\
= -\int_{|x| \leq R} p_{N,s} \psi(x,0) \text{div}_x \left[ \frac{x-y}{(|x-y|^2+t^2)^{N+2s}} \right] dx.
\]

Since it is easily seen that for \(|y| \geq 2R\)

\[
\int_{|x| \leq R} |\psi(x,0)| \text{div}_x \left[ \frac{x-y}{(|x-y|^2+t^2)^{N+2s}} \right] dx \leq C \int_{|x| \leq R} |x-y|^{-N-2s} dx \leq C |y|^{-N-2s},
\]

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by (2.17), we have
\[ |t^{1-2s}\partial_t (P_s(\cdot,t) \ast \psi(\cdot,0)) (y)| \leq C|y|^{-N-2s}. \]
This together with (2.15) and (2.16) implies (2.13), which completes the proof of Lemma 2.1.

Applying Lemma 2.1, we have the following.

Lemma 2.2. Let \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx) \) be a solution of (1.1) under (1.2), and U be the function given in (2.3). Then
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s}U \cdot \nabla \psi \, dX = \kappa_s \int_{\mathbb{R}^N} |x|^{\frac{N}{2}}|u|^{p-1}u\psi(x,0) \, dx = \kappa_s \langle u, \psi(\cdot,0) \rangle_{H^s(\mathbb{R}^N)} \quad (2.18)
\]
for every \( \psi \in C^1_c(\overline{\mathbb{R}^N}) \) where \( \kappa_s \) is the constant given in (2.6).

Remark 2.2. Since \( U \in C^\infty(\mathbb{R}^N) \), by (2.4), (2.18) and the integration by parts, for every \( \psi \in C^\infty(\mathbb{R}^N) \), we have
\[
- \lim_{t \to +0} \int_{\mathbb{R}^N} \tau^{1-2s} \partial_t U(x,\tau) \psi(x,\tau) \, dx = - \lim_{t \to +0} \int_{\mathbb{R}^N \times (\tau,\infty)} t^{1-2s}U \cdot \nabla \psi \, dX = \kappa_s \int_{\mathbb{R}^N} |x|^{\frac{N}{2}}|u|^{p-1}u\psi(x,0) \, dx = \kappa_s \langle u, \psi(\cdot,0) \rangle_{H^s(\mathbb{R}^N)}.
\]
Hence, for solutions \( u \in \tilde{H}^s(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx) \) of (1.1), the corresponding extension problem is
\[
\begin{aligned}
- \text{div} \left( t^{1-2s} \nabla U \right) &= 0 & & \text{in } \mathbb{R}^{N+1}_+, \\
U(x,0) &= u(x) & & \text{on } \mathbb{R}^N, \\
- \lim_{t \to +0} t^{1-2s} \partial_t U(x,t) &= \kappa_s |x|^{\frac{N}{2}}|u|^{p-1}u & & \text{on } \mathbb{R}^N.
\end{aligned}
\]

Proof of Lemma 2.2. We first prove (2.18) for \( \psi \in C^\infty_c(\overline{\mathbb{R}^N}) \) with \( \partial_t \psi(x,0) \equiv 0 \) on \( \mathbb{R}^N \). Let \( \varepsilon > 0, u \) be a solution of (1.1) and \( \psi \in C^\infty_c(\overline{\mathbb{R}^N}) \) with \( \partial_t \psi(x,0) = 0 \). Then we have
\[
0 = \int_{\mathbb{R}^N \times (\varepsilon,\infty)} - \text{div} \left( t^{1-2s} \nabla U \right) \psi(x,t) \, dX
= \int_{\mathbb{R}^N \times (\varepsilon,\infty)} t^{1-2s} \nabla U \cdot \nabla \psi \, dX + \int_{\mathbb{R}^N} \varepsilon^{1-2s} \partial_t U(\varepsilon,x) \psi(x,\varepsilon) \, dx.
\]
By letting \( \varepsilon \to 0 \) and (2.5), it follows that
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \cdot \nabla \psi \, dX = \kappa_s \int_{\mathbb{R}^N} u(x)(-\Delta)^s \psi(x,0) \, dx.
\]
Since \( u \) is a solution of (1.1), Lemma 2.1 yields
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \cdot \nabla \psi \, dX = \kappa_s \langle u, \psi(\cdot,0) \rangle_{H^s(\mathbb{R}^N)} = \kappa_s \int_{\mathbb{R}^N} |x|^{\frac{N}{2}}|u|^{p-1}u\psi(x,0) \, dx
\]
for every $\psi \in C^\infty_c(\mathbb{R}^N_{+1})$ with $\partial_t \psi(x, 0) \equiv 0$.

Next, we show (2.18) for $\psi \in C^\infty_c(\mathbb{R}^N_{+1})$. Let $\psi \in C^\infty_c(\mathbb{R}^N_{+1})$ and $(\rho_\varepsilon(t))_\varepsilon$ be a mollifier in $t$ with $\rho_\varepsilon(-t) = \rho_\varepsilon(t)$. Set

$$
\Psi(x, t) := \begin{cases} 
\psi(x, t) & \text{if } t \geq 0, \\
\psi(x, -t) & \text{if } t < 0.
\end{cases}
$$

Then $\Psi \in C^\infty(\mathbb{R}^N_{+1} \setminus \{t = 0\}) \cap W^{1,\infty}(\mathbb{R}^N_{+1})$ and $\Psi(\cdot, t) \in C^\infty_c(\mathbb{R}^N)$ for every $t \in \mathbb{R}$. Define $\Psi_\varepsilon$ by

$$
\Psi_\varepsilon(x, t) := \int_\mathbb{R} \rho_\varepsilon(t-\tau) \Psi(x, \tau) d\tau = \int_\mathbb{R} \rho_\varepsilon(\tau) \Psi(x, t-\tau) d\tau.
$$

It is easily seen that $\Psi_\varepsilon \in C^\infty_c(\mathbb{R}^N_{+1})$ with $\partial_t \Psi_\varepsilon(x, 0) = 0$ thanks to the symmetry of $\Psi$ and $\rho_\varepsilon$ in $t$. Since it holds that for any $k \in \mathbb{N}$ and $t_1 > 0$

$$
\lim_{\varepsilon \to 0} \left( \|\Psi_\varepsilon(\cdot, 0) - \psi(\cdot, 0)\|_{C^k(\mathbb{R}^N)} + \sup_{t \geq t_1, \tau > 0} \|\Psi_\varepsilon(\cdot, t) - \psi(\cdot, t)\|_{C^k(\mathbb{R}^N)} \right) = 0 \tag{2.19}
$$

and $\|\Psi_\varepsilon\|_{W^{1,\infty}(\mathbb{R}^N_{+1})} \leq M_0$ for some $M_0 > 0$, we deduce that

$$
\Psi_\varepsilon \to \psi \quad \text{strongly in } H^1(\mathbb{R}^N_{+1}, t^{1-2s} dX). \tag{2.20}
$$

Therefore, from

$$
\int_{\mathbb{R}^N_{+1}} t_1^{1-2s} \nabla U \cdot \nabla \Psi_\varepsilon \, dX = \kappa_s \int_{\mathbb{R}^N} |x|^s |u|^{p-1} u \Psi_\varepsilon(x, 0) \, dx
$$

with (2.19), (2.20) and $u \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, as $\varepsilon \to 0$, we have (2.18) for all $\psi \in C^\infty_c(\mathbb{R}^N_{+1})$. Finally, since we may approximate functions in $C^\infty_c(\mathbb{R}^N_{+1})$ by functions in $C^1_c(\mathbb{R}^N_{+1})$ in the $C^1(\mathbb{R}^N_{+1})$ sense, (2.18) holds for every $C^1_c(\mathbb{R}^N_{+1})$ and we complete the proof. \hfill \Box

\section{2.2 Local Regularity and the Pohozaev identity}

In this subsection we recall local regularity estimates for the extension problem in Remark 2.2, which are taken from [13, Section 3.1] (see also [5, 21]). Furthermore, we prove the Pohozaev identity for solutions to the extension problem.

We first recall local regularity estimates.

**Proposition 2.1.** (Fall and Felli [13, Proposition 3.2, Lemma 3.3], Jin, Li and Xiong [21, Proposition 2.6], Cabré and Sire [5, Lemma 4.5]) Let $R_0 > 0$, $x_0 \in \mathbb{R}^N$, $g(x, u) : B_{4R_0}(x_0) \times \mathbb{R} \to \mathbb{R}$ and $W \in H^1(B_{4R_0}(x_0), t^{1-2s} dX)$ be a weak solution to

$$
\begin{cases}
-\div(t^{1-2s} \nabla W) = 0 & \text{in } B_{4R_0}(x_0, 0), \\
-\lim_{t \to 0} t^{1-2s} \partial_t W(x, t) = \kappa_s g(x, W(x, 0)) & \text{on } B_{4R_0}(x_0),
\end{cases}
$$

that is, for all $\varphi \in C^\infty_c(B_{4R_0}(x_0, 0) \cup B_{4R_0}(x_0))$,

$$
\int_{B_{4R_0}(x_0)} t^{1-2s} \nabla W \cdot \nabla \varphi \, dX = \kappa_s \int_{B_{4R_0}(x_0)} g(x, W(x, 0)) \varphi(x, 0) \, dx.
$$

\section{References}

[13, 21, 5]
(i) Suppose that $g(x, u) := a(x)u + b(x)$ with $a, b \in L^q(B_{4R_0}(x_0))$ for some $q > N/(2s)$. For any $\mu > 0$ there exists a $C = C(N, s, q, \mu, \|a\|_{L^q(B_{4R_0}(x_0))})$ such that

$$\|W\|_{L^\infty(B_{2R_0}(x_0, 0))} \leq C \left[ \|W\|_{L^\mu(B_{2R_0}(x_0, 0))} + \|b\|_{L^q(B_{4R_0}(x_0))} \right].$$

In addition, there exists an $\alpha \in (0, 1)$ such that $W \in C^\alpha(B_{R_0}(x_0, 0))$ and

$$\|W\|_{C^\alpha(B_{R_0}(x_0, 0))} \leq C \left[ \|W\|_{L^\infty(B_{2R_0}(x_0, 0))} + \|b\|_{L^q(B_{4R_0}(x_0))} \right].$$

(ii) Suppose that $W \in C^\alpha(B_{2R_0}(x_0, 0))$ and $g(x, u) \in C^1(B_{4R_0}(x_0) \times \mathbb{R})$ for some $\alpha \in (0, 1)$. Then there exist $\beta \in (0, 1)$ and $C = C(N, s, \|g\|_{C^1(B_{2R_0}(x_0, 0) \times [-A, A])})$ where $A := \|W\|_{C^\alpha(B_{2R_0}(x_0, 0))}$ such that $\nabla_x W \in C^\beta(B_{R_0}(x_0, 0))$ and $t^{1-2s} \partial_t W \in C^\beta(B_{R_0}(x_0, 0))$ with

$$\|\nabla_x W\|_{C^\beta(B_{R_0}(x_0, 0))} + \|t^{1-2s} \partial_t W\|_{C^\beta(B_{R_0}(x_0, 0))} \leq C.$$

For solutions of (1.1), we have

**Lemma 2.3.** Let $u \in H^s_\text{loc}(\mathbb{R}^N) \cap L^\infty_\text{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N - 2s}dx)$ be a solution of (1.1) under (1.2) and $U$ be the function given in (2.3). Then for each $R > 1$ there exists an $\alpha_R \in (0, 1)$ such that $U \in C^{\alpha_R}(B_{R}^\mathbb{R})$ and $\nabla_x U, t^{1-2s} \partial_t U \in C^{\alpha_R}((B_{R} \setminus B_{1/R}) \times (0, R))$. As a consequence with Remark 2.2,

$$- \lim_{t \to +0} t^{1-2s} \partial_t U(x, t) = \kappa_s |x|^\ell |u(x)|^{p-1}u(x) \quad \text{in} \quad C_\text{loc}(\mathbb{R}^N \setminus \{0\}).$$

**Proof.** By $u \in L^\infty_\text{loc}(\mathbb{R}^N)$ and (1.2), we find some $q > N/(2s)$ such that $a(x) := |x|^\ell |u(x)|^{p-1} \in L^q(B_{4R})$ for each $R > 0$. Thus, we may apply Proposition 2.1 (i) for $U$ with $a(x)$ and there exists an $\alpha_R \in (0, 1)$ so that $U \in C^\alpha(B_{R}^\mathbb{R})$.

Next, notice that $g(x, u) := |x|^\ell |u|^{p-1}u \in C^1(B_{R} \setminus B_{1/R} \times \mathbb{R})$. Therefore, we may apply Proposition 2.1 (ii) and obtain the desired result. \hfill \Box

Next we prove the following Pohozaev identity.

**Proposition 2.2.** Let $u \in H^s_\text{loc}(\mathbb{R}^N) \cap L^\infty_\text{loc}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{-N - 2s}dx)$ be a solution of (1.1) with (1.2), and $U$ be the function given in (2.3). Then for all $R > 0$, there holds

$$- \frac{N - 2s}{2} \left[ \int_{B^+_{2R}} t^{1-2s} |\nabla U|^2 \, dX - \frac{2\kappa_s}{N-2s} \frac{N + \ell}{p + 1} \int_{B^+_{2R}} |x|^\ell |u|^{p+1} \, dx \right]$$

$$+ \frac{R}{2} \left[ \int_{S^+_{2R}} t^{1-2s} |\nabla U|^2 \, dS - \frac{2\kappa_s}{p + 1} \int_{S^+_{2R}} |x|^\ell |u|^{p+1} \, d\omega \right] = R \int_{S^+_{2R}} t^{1-2s} \frac{\partial U}{\partial \nu}^2 \, dS \quad \text{(2.21)}$$

and

$$\int_{B^+_{2R}} t^{1-2s} |\nabla U|^2 \, dx - \kappa_s \int_{B^+_{2R}} |x|^\ell |u|^{p+1} \, dx = \int_{S^+_{2R}} t^{1-2s} \frac{\partial U}{\partial \nu} \, dS. \quad \text{(2.22)}$$

Here $\nu = X/|X|$ is the unit outer normal vector of $S^+_{2R}$ at $X$ and $\kappa_s$ the constant given in (2.6).
Proof. We follow the argument in [13, Proof of Theorem 3.7]. Let \( u \) be a solution of (1.1) and \( U \) the function given in (2.3), and we take any \( R > 0 \). Then, by (2.4) we have

\[
\frac{N - 2s}{2} t^{1-2s} |\nabla U|^2 = \text{div} \left( \frac{1}{2} t^{1-2s} |\nabla U|^2 X - t^{1-2s} (X \cdot \nabla U) \nabla U \right)
\]

for \( X \in B_R^+ \). Let \( \rho < R \). Then, integrating (2.23) over the set

\[
\mathcal{O}_\delta := (B_R^+ \setminus \overline{B_\rho^+}) \cap \left\{ X = (x, t) \in \mathbb{R}^{N+1}_+ \mid t > \delta \right\}
\]

with \( \delta \in (0, \rho) \) and writing \( B_{R, \rho, \delta} := B_{\sqrt{R^2 - \rho^2}} \setminus B_{\sqrt{\rho^2 - \delta^2}} \subset \mathbb{R}^N \), we have

\[
\frac{N - 2s}{2} \int_{\mathcal{O}_\delta} t^{1-2s} |\nabla U|^2 \, dX = \int_{\mathcal{O}_\delta} \text{div} \left( \frac{1}{2} t^{1-2s} |\nabla U|^2 X - t^{1-2s} (X \cdot \nabla U) \nabla U \right) \, dX
\]

\[
= -\frac{1}{2} \delta^{2-2s} \int_{B_{R, \rho, \delta}} |\nabla U(x, \delta)|^2 \, dx + \delta^{2-2s} \int_{B_{R, \rho, \delta}} |U_t(x, \delta)|^2 \, dx
\]

\[
+ \delta^{1-2s} \int_{B_{R, \rho, \delta}} (x \cdot \nabla x U(x, \delta)) U_t(x, \delta) \, dx
\]

\[
+ \frac{R}{2} \int_{S^+_{R, t>\delta}} t^{1-2s} |\nabla U|^2 \, dS - R \int_{S^+_{R, t>\delta}} t^{1-2s} \frac{\partial U}{\partial \nu}^2 \, dS
\]

\[
- \frac{\rho}{2} \int_{S^+_{\rho, t>\delta}} t^{1-2s} |\nabla U|^2 \, dS + \rho \int_{S^+_{\rho, t>\delta}} t^{1-2s} \frac{\partial U}{\partial \nu}^2 \, dS.
\]

Now we claim that there exists a sequence \( \delta_n \to 0 \) such that

\[
\lim_{n \to \infty} \left[ \frac{1}{2} \delta^{2-2s} \int_{B_R} |\nabla U(x, \delta_n)|^2 \, dx + \delta^{2-2s} \int_{B_R} |U_t(x, \delta_n)|^2 \, dx \right] = 0.
\]

In fact, if there is no such sequence, then there exists a \( C > 0 \) such that

\[
\liminf_{\delta \to 0} \left[ \frac{1}{2} \delta^{2-2s} \int_{B_R} |\nabla U(x, \delta)|^2 \, dx + \delta^{2-2s} \int_{B_R} |U_t(x, \delta)|^2 \, dx \right] \geq C
\]

and thus there exists a \( \delta_0 > 0 \) such that for all \( \delta \in (0, \delta_0) \),

\[
\frac{1}{2} \delta^{1-2s} \int_{B_R} |\nabla U(x, \delta)|^2 \, dx + \delta^{1-2s} \int_{B_R} |U_t(x, \delta)|^2 \, dx \geq \frac{C}{2\delta}.
\]

Since \( U \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s} dX) \), integrating the above inequality in \( \delta \) over \( (0, \delta_0) \), we have a contradiction and (2.25) holds.

On the other hand, by Lemma 2.3, we see that

\[
\lim_{\delta \to 0} \delta^{1-2s} \int_{B_{R, \rho, \delta}} (x \cdot \nabla x U(x, \delta)) U_t(x, \delta) \, dx = -\kappa_s \int_{B_R \setminus B_\rho} (x \cdot \nabla u) |x|^{p-1} u \, dx.
\]
By (2.24)–(2.26) and replacing $\mathcal{O}_\delta$ with $\mathcal{O}_{\delta_n}$ for a sequence $\delta_n \to 0$, we conclude that
\[
\lim_{n \to \infty} \left[ \rho_n \int_{S_{p_n}^+} t^{1-2s} |\nabla U|^2 dS + \rho_n \int_{S_{p_n}^+} t^{1-2s} \left| \frac{\partial U}{\partial \nu} \right|^2 dS + \int_{S_{p_n}^+} |x|^{\ell+1} |u|^{p+1} d\omega \right] = 0.
\]

Therefore, taking $\rho = \rho_n$ and letting $n \to \infty$ in (2.27) and (2.28), we have (2.21).

On the other hand, since
\[
- \text{div} (t^{1-2s} \nabla U) = 0 \quad \text{in} \quad \bar{\mathcal{O}}_{\delta} := B_R^+ \cap \{ X \in \mathbb{R}^{N+1}_+ : t \geq \delta \}
\]
for any $\varphi \in C^\infty(B_R^+)$, integration by parts gives
\[
\int_{\bar{\mathcal{O}}_{\delta}} \int_{S_R^+ \cap [t > \delta]} t^{1-2s} \nabla U \cdot \nabla \varphi + \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} |x|^{\ell+1} |u|^{p+1} w \varphi (x, 0) dS - \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} \frac{\partial U}{\partial t} \varphi (x, \delta) dS.
\]

For the last term, by decomposing $\varphi$ into $\varphi (x, t) = \zeta (x, t) \varphi (x, t) + (1 - \zeta (x, t)) \varphi (x, t)$ where $\zeta \in C^\infty (B_{R/4}^+)$ with $\zeta \equiv 1$ on $B_{R/4}^+$ and noting $\zeta (x, t) \varphi (x, t) \in C^\infty (\mathbb{R}^{N+1}_+)$, Remark 2.2 and Lemma 2.3 yield
\[
- \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} \frac{\partial U}{\partial t} \varphi (x, \delta) dx \to \kappa_s \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} |x|^{\ell+1} |u|^{p-1} u \varphi (x, 0) dx \quad \text{as} \quad \delta \to 0.
\]

Therefore, for every $\varphi \in C^\infty (B_R^+)$,
\[
\int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} \int_{S_R^+ \cap [t > \delta]} t^{1-2s} \nabla U \cdot \nabla \varphi dS + \kappa_s \int_{B_R^+ \setminus \bar{\mathcal{O}}_{\delta}} |x|^{\ell+1} |u|^{p-1} u \varphi (x, 0) dx.
\]
From the fact that $U \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s}dX)$ can be approximated by functions of $C^\infty(\mathbb{B}^+_R)$ in the $H^1(\mathbb{B}^+_R, t^{1-2s}dX)$ sense, by setting $\varphi = U$ in the above, we obtain (2.22) and Proposition 2.2 follows.

2.3 Energy estimates

We first show several lemmata by following [9, Lemma 2.2 and Corollary 2.3].

**Lemma 2.4.** For $\zeta \in W^{1,\infty}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, define

$$
\rho(\zeta; x) := \int_{\mathbb{R}^N} \frac{(\zeta(x) - \zeta(y))^2}{|x - y|^{N+2s}} dy.
$$

Then there exists a $C = C(N, s) > 0$ such that

$$
\rho(\zeta; x) \leq C \left\{ (1 + |x|^2) \| \nabla \zeta \|_{L^\infty(\Omega(|x|))}^2 + \| \zeta \|_{L^\infty(\Omega(|x|))}^2 ight\} (1 + |x|)^{-2s} \quad \text{for all } |x| \geq 1,
$$

and

$$
\rho(\zeta; x) \leq C \left( 1 + \| \zeta \|_{W^{1,\infty}(\mathbb{R}^N)}^2 \right) \quad \text{for all } |x| \leq 1
$$

where

$$
\Omega(|x|) := \left\{ y \in \mathbb{R}^N \mid |y| \geq \frac{|x|}{2} \right\}.
$$

**Proof.** The proof is basically the same as the one in [9, Lemma 2.2]. We treat the case $|x| \geq 1$ and put

$$
D_1 := \left\{ y \in \mathbb{R}^N \mid |y - x| \leq \frac{|x|}{2} \right\}, \quad D_2 := \left\{ y \in \mathbb{R}^N \mid \frac{|x|}{2} \leq |y - x| \leq 2|x| \right\},
$$

$$
D_3 := \left\{ y \in \mathbb{R}^N \mid 2|x| \leq |y - x| \right\}.
$$

For $D_1$, notice that $D_1$ is convex and $D_1 \subset \Omega(|x|)$. Since it follows from $\zeta \in W^{1,\infty}(\mathbb{R}^N)$ that

$$
|\zeta(x) - \zeta(y)| \leq \| \nabla \zeta \|_{L^\infty(D_1)} |x - y| \leq \| \nabla \zeta \|_{L^\infty(\Omega(|x|))} |x - y|,
$$

we have

$$
\int_{D_1} \frac{(\zeta(x) - \zeta(y))^2}{|x - y|^{N+2s}} dy \leq C \| \nabla \zeta \|_{L^\infty(\Omega(|x|))}^2 \int_{D_1} |x - y|^{-N-2s} dy
$$

$$
\leq C \| \nabla \zeta \|_{L^\infty(\Omega(|x|))}^2 (1 + |x|)^{2-2s}.
$$

For $y \in D_2$, by $|x| \geq 1$, it holds that

$$
\int_{D_2} \frac{(\zeta(x) - \zeta(y))^2}{|x - y|^{N+2s}} dy \leq C |x|^{-N-2s} \int_{|y| \leq |x|} (\zeta(x)^2 + \zeta(y)^2) dy
$$

$$
\leq C \zeta(x)^2 |x|^{-2s} + C |x|^{-N-2s} \| \zeta \|_{L^2(\mathbb{R}^N)}^2
$$

$$
\leq C \| \zeta \|_{L^\infty(\Omega(|x|))}^2 (1 + |x|)^{-2s} + C \| \zeta \|_{L^2(\mathbb{R}^N)}^2 (1 + |x|)^{-N-2s}.
$$

$$
(2.33)
$$
For $y \in D_3$, since $|y| \geq |x|$, we have
\[
\int_{D_3} \frac{(\zeta(x) - \zeta(y))^2}{|x - y|^{N+2s}} \, dy \leq C\|\zeta\|^2_{L^\infty(\Omega(|x|))} \int_{D_3} |x - y|^{-N-2s} \, dy
\leq C\|\zeta\|^2_{L^\infty(\Omega(|x|))}(1 + |x|)^{-2s}.
\]

Putting (2.32)–(2.34) together, we have (2.30) for $|x| \geq 1$.

On the other hand, for $|x| \leq 1$, by dividing $\mathbb{R}^N$ into $B_2$ and $B_2^c$, and arguing as in (2.32) and (2.34), we obtain (2.31). We omit the details and Lemma 2.4 holds.

\[ \square \]

**Lemma 2.5.** For $m > N/2$, set
\[ \eta(x) := (1 + |x|^2)^{-\frac{m}{2}}. \]  

Let $R \geq R_0 \geq 1$ and $\psi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 0$ on $B_1$ and $\psi \equiv 1$ on $B_2^c$. Define $\eta_R$ and $\rho_R$ by
\[
\eta_R(x) := \eta\left(\frac{x}{R}\right)\psi\left(\frac{x}{R_0}\right), \quad \rho_R(x) := \rho(\eta_R; x).
\]

Then there exists a constant $C = C(N, s, m, R_0) > 0$ such that
\[
\rho_R(x) \leq \begin{cases} 
C\eta\left(\frac{x}{R}\right)^2 |x|^{-N-2s} + 2R^{-2s}\rho\left(\eta; \frac{x}{R}\right) & \text{if } |x| \geq 3R_0, \\
C + 2R^{-2s}\rho\left(\eta; \frac{x}{R}\right) & \text{if } |x| \leq 3R_0.
\end{cases} 
\]

**Remark 2.3.** By Lemma 2.4 and [9, Lemma 2.2], there exist two constants $c, C > 0$ such that
\[
c(1 + |x|)^{-N-2s} \leq \rho(\eta; x) \leq C(1 + |x|)^{-N-2s} \quad \text{for all } x \in \mathbb{R}^N.
\]

In addition, later (see (2.58)) we shall also prove that for all sufficiently large $R > 0$ and $x \in \mathbb{R}^N$,
\[
0 < c_R(1 + |x|)^{-N-2s} \leq \rho_R(x).
\]

**Proof of Lemma 2.5.** Applying Young’s inequality with the definition of $\eta_R$, we have
\[
\rho_R(x) \leq 2\eta\left(\frac{x}{R}\right)^2 \int_{\mathbb{R}^N} \left(\frac{\psi\left(\frac{x}{R_0}\right) - \psi\left(\frac{y}{R_0}\right)}{|x - y|^{N+2s}}\right)^2 \, dy
+ 2\int_{\mathbb{R}^N} \psi\left(\frac{y}{R_0}\right)^2 \frac{(\eta\left(\frac{x}{R}\right) - \eta\left(\frac{y}{R}\right))^2}{|x - y|^{N+2s}} \, dy.
\]

For the first term, if $|x| \geq 3R_0$, then $|x - y| \geq |x|/3$ for any $y \in B_{2R_0}$ and
\[
\int_{\mathbb{R}^N} \left(\frac{\psi\left(\frac{x}{R_0}\right) - \psi\left(\frac{y}{R_0}\right)}{|x - y|^{N+2s}}\right)^2 \, dy \leq \int_{B_{2R_0}} |x - y|^{-N-2s} \, dy \leq C_{R_0}|x|^{-N-2s}
\]

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and if $|x| \leq 3R_0$, then we see
\[
\int_{\mathbb{R}^N} \left( \frac{\psi \left( \frac{x}{R_0} \right) - \psi \left( \frac{y}{R_0} \right)}{|x-y|^{N+2s}} \right)^2 \, dy \leq \left( \int_{B_{R_0}(x)} + \int_{B_{R_0}^c(x)} \right) \left( \frac{\psi \left( \frac{x}{R_0} \right) - \psi \left( \frac{y}{R_0} \right)}{|x-y|^{N+2s}} \right)^2 \, dy
\leq R_0^{-2} \|\psi\|_{C^1_c(\mathbb{R}^N)}^2 \int_{|z| \leq R_0} |z|^{-N-2s+2} \, dz + \int_{|z| \geq R_0} |z|^{-N-2s} \, dz.
\]

Since
\[
\int_{\mathbb{R}^N} \psi \left( \frac{y}{R_0} \right) \left( \frac{\eta \left( \frac{y}{R} \right) - \eta \left( \frac{z}{R} \right)}{|x-y|^{N+2s}} \right)^2 \, dy \leq \int_{\mathbb{R}^N} \left( \frac{\eta \left( \frac{y}{R} \right) - \eta \left( \frac{z}{R} \right)}{|x-y|^{N+2s}} \right)^2 \, dy = R^{-2s} \rho \left( \frac{y}{R} \right),
\]
by (2.38), we have (2.37).

**Lemma 2.6.** Let $u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)$ be a solution of (1.1) under (1.2). Assume that $u$ is stable outside $B_{R_0}$. Let $\zeta \in C^1(\mathbb{R}^N)$ satisfy $\zeta \equiv 0$ on $B_{R_0}$ and
\[
|x| |\nabla \zeta(x)| + |\zeta(x)| \leq C (1 + |x|)^{-m} \quad \text{for all } x \in \mathbb{R}^N
\]
for some $m > N/2$. Then
\[
\int_{\mathbb{R}^N} |x|^t |u|^{p+1} \zeta^2 \, dx + \frac{1}{p} \|u\zeta\|_{H^s(\mathbb{R}^N)}^2 \leq C_{N,s} \frac{1}{p-1} \int_{\mathbb{R}^N} u(x)^2 \rho(\zeta; x) \, dx
\]
where $C_{N,s}$ is the constant given by (1.3).

**Remark 2.4.** Later, we shall use Lemma 2.6 for $\zeta(x) = \eta_R(x)$ and we require the right-hand side of (2.40) to be finite. For example, see Lemma 2.7 and the end of proof of it. Therefore, we need the condition $u \in L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)$ by Remark 2.3.

**Proof.** We follow the argument in [9, Lemma 2.1]. Remark that the right-hand side in (2.40) is finite due to (2.30), (2.31), (2.39) and $u \in L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)$. We first treat the case $\zeta \in C^1_c(\mathbb{R}^N)$ where $\zeta \equiv 0$ on $B_{R_0}$. Since $u$ is a solution of (1.1), by Lemma 2.3, we see that $u \in C^1(\mathbb{R}^N \setminus \{0\})$. Since $u \zeta \in C^1_c(\mathbb{R}^N)$, by Remark 1.1 we can take $\varphi = u \zeta^2$ as a test function in (1.4), and by (1.5) we have
\[
\int_{\mathbb{R}^N} |x|^t |u|^{p+1} \zeta^2 \, dx = \langle u, u \zeta^2 \rangle_{H^s(\mathbb{R}^N)}
= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(u(x)\zeta(x)^2 - u(y)\zeta(y)^2)}{|x-y|^{N+2s}} \, dx \, dy
= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x)^2 \zeta(x)^2 - u(x)u(y)(\zeta(x)^2 + \zeta(y)^2) + u(y)^2 \zeta(y)^2}{|x-y|^{N+2s}} \, dx \, dy
= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{(u(x)\zeta(x) - u(y)\zeta(y))^2 - (\zeta(x) - \zeta(y))^2 u(x)u(y)}{|x-y|^{N+2s}} \, dx \, dy
= \|u\zeta\|_{H^s(\mathbb{R}^N)}^2 - \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{(\zeta(x) - \zeta(y))^2 u(x)u(y)}{|x-y|^{N+2s}} \, dx \, dy.
\]
Applying the fundamental inequality $2ab \leq a^2 + b^2$ with (2.29), we deduce that
\[
\|u\|_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} |x|^\ell |u|^{p+1}\zeta^2\,dx \\
\leq \frac{C_{N,s}}{4} \left( \int_{\mathbb{R}^N} (\zeta(x) - \zeta(y))^2 \,dx\,dy + \int_{\mathbb{R}^N} (\zeta(x) - \zeta(y))^2 \,dx\,dy \right) \tag{2.41}
\]
\[
= \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} u(x)^2 \rho(\zeta; x)\,dx.
\]
Since $u$ is stable outside $B_{R_0}$, by (1.10) with $\varphi = u\zeta$ (see Remark 1.3) and (2.41), we have
\[
(p - 1) \int_{\mathbb{R}^N} |x|^\ell |u|^{p+1}\zeta^2\,dx \leq \|u\|_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} |x|^\ell |u|^{p+1}\zeta^2\,dx \leq \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} u(x)^2 \rho(\zeta; x)\,dx. \tag{2.42}
\]
Hence, by (2.41) and (2.42),
\[
\frac{1}{p} \|u\|_{H^s(\mathbb{R}^N)}^2 \leq \frac{1}{p} \int_{\mathbb{R}^N} |x|^\ell |u|^{p+1}\zeta^2\,dx + \frac{C_{N,s}}{2p} \int_{\mathbb{R}^N} u(x)^2 \rho(\zeta; x)\,dx
\]
\[
\leq \frac{C_{N,s}}{2(p-1)} \int_{\mathbb{R}^N} u^2 \rho(\zeta; x)\,dx.
\]
This together with (2.42) implies (2.40) for $\zeta \in C^1_c(\mathbb{R}^N)$ with $\zeta \equiv 0$ on $B_{R_0}$.

Next, let $\zeta \in C^1(\mathbb{R}^N)$ satisfy $\zeta \equiv 0$ on $B_{R_0}$ and (2.39). Let $(\varphi_n)_n$ be a sequence of cut-off functions in Lemma 2.1 and set $\zeta_n := \varphi_n\zeta \in C^1_c(\mathbb{R}^N)$. It is easily seen that $(\zeta_n)_n$ satisfies (2.39) uniformly with respect to $n$, namely, the constant $C$ in (2.39) is independent of $n$. Exploiting this fact with (2.30) and (2.31), we observe that there exists a $C > 0$ such that for all $x \in \mathbb{R}^N$ and $n$,
\[
\rho(\zeta_n; x) \to \rho(\zeta; x), \quad |\rho(\zeta_n; x)| \leq C (1 + |x|)^{-N-2s}.
\]
Therefore, from (2.40) with $\zeta_n$, we find that $(\zeta_n u)_n$ is bounded in $H^s(\mathbb{R}^N)$ and it is not difficult to see $|u(x)|^{p+1}\zeta_n(x)^2 \geq |u(x)|^{p+1}\zeta(x)^2$ for each $x \in \mathbb{R}^N$ and $\zeta_n u \rightharpoonup \zeta u$ weakly in $H^s(\mathbb{R}^N)$. Hence, from the monotone convergence theorem, the weak lower semicontinuity of norm, the fact $u \in L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx)$, (2.40) with $\zeta_n$ and the dominated convergence theorem, it follows that (2.40) holds for each $\zeta \in C^1(\mathbb{R}^N)$ with (2.39) and $\zeta \equiv 0$ on $B_{R_0}$. This completes the proof.

By using Lemmata 2.5 and 2.6, we have the following.

**Lemma 2.7.** Let $u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s}dx)$ be a solution of (1.1) with (1.2), which is stable outside $B_{R_0}$, and let $\rho_R$ be the function given in Lemma 2.5 with
\[
m \in \left(\frac{N}{2}, \frac{N + s(p+1) + \ell}{2}\right). \tag{2.43}
\]
Then there exists a constant $C = C(N, s, \ell, m, p, R_0) > 0$ such that
\[
\int_{\mathbb{R}^N} u^2 \rho_R \,dx \leq C \left( \int_{B_{3R_0}} u^2 \rho_R \,dx + R^{N-\frac{s}{p-1}(p+1)+\ell} + 1 \right) \tag{2.44}
\]
for all $R \geq 3R_0$. 
Remark 2.5. Due to (1.2), $0 < s(p+1) + \ell$ holds and we may choose an $m$ satisfying (2.43).

Proof of Lemma 2.7. We basically follow the idea in [9, Lemma 2.4] and let $R \geq 3R_0$. First, by Hölder’s inequality,

$$\int_{\mathbb{R}^N} u^2 \rho_R \, dx \leq \int_{B_{3R_0}} u^2 \rho_R \, dx + \int_{B_{3R_0}^c} u^2 \rho_R \left( |x|^{\frac{2}{p} } \eta_R \right) \frac{p+1}{p-1} \left( |x|^{\frac{2}{p} } \eta_R \right)^{-\frac{1}{p+1}} \, dx \quad (2.45)$$

For the case $3R_0 \leq |x| \leq R$, by Lemma 2.4 with (2.35), (2.36) and Remark 2.3, we have

$$2^{-\frac{m}{2}} = \eta(1) \leq \eta_R(x) \leq \eta(\frac{3R_0}{R}) \leq 1 \quad \text{and} \quad \rho \left( \eta, \frac{x}{R} \right) \leq C. \quad (2.46)$$

Then, by (2.37) we obtain

$$\rho_R(x) \leq C \left( |x|^{-N-2s} + R^{-2s} \right) \quad \text{for all} \ 3R_0 \leq |x| \leq R.$$

This together with (2.46) yields

$$\int_{B_{R} \setminus B_{3R_0}} |x| \frac{2\ell}{p-1} \rho_R^{\frac{p+1}{p-1}} \eta_R^{-\frac{4}{p-1}} \, dx \leq C \int_{3R_0}^{R} r^{N-1-\frac{2\ell}{p-1}} (r^{-(N+2s)} + R^{-2s}) \frac{p+1}{p-1} r^{N-1} \, dr \leq C \int_{3R_0}^{R} r^{N-1-\frac{2\ell}{p-1}} (1 + \log R) \frac{p+1}{p-1} r^{N-1} \, dr + CR^{-2s} \int_{3R_0}^{R} r^{N-1-\frac{2\ell}{p-1}} \, dr \quad (2.47)$$

Since

$$R^{-2s} \int_{3R_0}^{R} r^{N-1-\frac{2\ell}{p-1}} \, dr \leq \begin{cases} \left( CR^{-2s} \frac{p+1}{p-1} \right)^{\frac{p-1}{2}} & \text{if} \ N - \frac{2\ell}{p-1} < 0, \\ CR^{-2s} \frac{p+1}{p-1} (1 + \log R) & \text{if} \ N - \frac{2\ell}{p-1} = 0, \\ CR^{N-\frac{2}{p-1} (s(p+1)+\ell)} & \text{if} \ N - \frac{2\ell}{p-1} > 0, \end{cases}$$

we have

$$R^{-2s} \int_{3R_0}^{R} r^{N-1-\frac{2\ell}{p-1}} \, dr \leq \begin{cases} C & \text{if} \ N - \frac{2\ell}{p-1} \leq 0, \\ CR^{N-\frac{2}{p-1} (s(p+1)+\ell)} & \text{if} \ N - \frac{2\ell}{p-1} > 0, \end{cases}$$

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Combining (2.48), we see that
\[
\int_{B_R \setminus B_{3R_0}} |x|^{- \frac{2\ell}{p-1}} \rho_R^{\frac{p+1}{p-1}} \eta_R^{\frac{4}{p-1}} \, dx \leq C + CR^{N - \frac{2}{p-1}(s(p+1) + \ell)}. \tag{2.48}
\]

On the other hand, for the case $|x| \geq R \geq 3R_0$, by (2.35) and $R^2 + |x|^2 \leq 2|x|^2$, we obtain
\[
\eta \left(\frac{x}{R}\right)^2 |x|^{-N - 2s} \leq C \eta(1)^2 (R^2 + |x|^2)^{-\frac{N}{2} - s} \leq CR^{-N - 2s} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{N}{2} - s} \leq CR^{-2s} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{N}{2} - s}.
\]

This together with (2.30), (2.37) and Remark 2.3 implies that
\[
\rho_R(x) \leq CR^{-2s} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{N}{2} - s} \quad \text{for each } |x| \geq R.
\]

Thus, (2.35), (2.36) and the fact $|x| \geq R$ give
\[
|x|^{- \frac{2\ell}{p-1}} \rho_R(x)^{\frac{p+1}{p-1}} \eta_R(x)^{- \frac{4}{p-1}} \leq CR^{-2s} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{N+2s}{2}} \left(1 + \frac{|x|^2}{R^2}\right)^{\frac{2m}{p-1} + \frac{2\ell}{p-1}} \leq CR^{-2s} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{N+2s}{2}} \left(1 + \frac{|x|^2}{R^2}\right)^{\frac{2m}{p-1} + \frac{2\ell}{p-1}}.
\]

Since it follows from (2.43) that
\[
\alpha_{N, s, p, m, \ell} := -\frac{N + 2s}{2} + \frac{2m}{p-1} + \frac{\ell}{p-1} < -\frac{N}{2},
\]
by (2.49) we have
\[
\int_{B_R} |x|^{- \frac{2\ell}{p-1}} \rho_R^{\frac{p+1}{p-1}} \eta_R^{\frac{4}{p-1}} \, dx \leq CR^{-2s} \left(1 + \frac{r^2}{R^2}\right)^{\alpha_{N, s, p, m, \ell} - \frac{N-1}{2}} \int_{R}^{\infty} \frac{r^{N-1}}{r^2 + N-1} \, dr \leq CR^{-2s} \left(1 + \frac{r^2}{R^2}\right)^{\alpha_{N, s, p, m, \ell} - \frac{N-1}{2}} \int_{R}^{\infty} \frac{r^{N-1}}{r^2 + N-1} \, dr \leq CR^{N - \frac{2}{p-1}(s(p+1) + \ell)}. \tag{2.50}
\]

Combining (2.48) and (2.50), we obtain
\[
\int_{B_{3R_0}} |x|^{- \frac{2\ell}{p-1}} \rho_R^{\frac{p+1}{p-1}} \eta_R^{\frac{4}{p-1}} \, dx \leq C \left(R^{N - \frac{2}{p-1}(s(p+1) + \ell)} + 1\right). \tag{2.51}
\]
Now we substitute (2.51) into (2.45) and infer from (2.40) with $\zeta = \eta_R$ that
\[
\int_{\mathbb{R}^N} u^2 \rho_R \, dx
\leq \int_{B_{3R_0}} u^2 \rho_R \, dx + C \left( \int_{B_{3R_0}^c} |x|^{\ell} |u|^{p+1} \eta_R \, dx \right)^{\frac{2}{p+1}} \left( R^{N-\frac{2}{p-1}(s(p+1)+\ell)} + 1 \right)^{\frac{p-1}{p+1}}
\leq \int_{B_{3R_0}} u^2 \rho_R \, dx + C \left( \int_{\mathbb{R}^N} u^2 \rho_R \, dx \right)^{\frac{2}{p+1}} \left( R^{N-\frac{2}{p-1}(s(p+1)+\ell)} + 1 \right)^{\frac{p-1}{p+1}}
\]
for $R \geq 3R_0$. Dividing the both sides by $\left( \int_{\mathbb{R}^N} u^2 \rho_R \, dx \right)^{\frac{2}{p+1}} < \infty$ and noting
\[
\left( \int_{B_{3R_0}} u^2 \rho_R \, dx \right) \left( \int_{\mathbb{R}^N} u^2 \rho_R \, dx \right)^{-\frac{2}{p+1}} \leq \left( \int_{B_{3R_0}} u^2 \rho_R \, dx \right)^{\frac{p-1}{p+1}},
\]
we have (2.44) and Lemma 2.7 follows.

For the supercritical case, we have the following energy estimates for the function $U$, which is given in (2.3).

**Lemma 2.8.** Assume $p_S(N, \ell) < p$ and (1.2). Let $u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1 + |x|)^{-N-2s} \, dx)$ be a solution of (1.1) which is stable outside $B_{R_0}$ and $U$ be the function given in (2.3). Then there exists a $C = C(N, p, s, \ell, R_0, u) > 0$ such that
\[
\int_{B^+_R} t^{1-2s} U^2 \, dX \leq CR^{N+2(1-s) - \frac{2(2s+\ell)}{p-1}}
\]  
(2.52)
for all $R \geq 3R_0$.

**Remark 2.6.** If $p_S(N, \ell) < p$, then
\[
N - \frac{2}{p-1} (s(p+1) + \ell) > 0.
\]  
(2.53)

**Proof of Lemma 2.8.** Let $\zeta_{R_0} \in C^\infty(\mathbb{R}^N)$ with $0 \leq \zeta_{R_0} \leq 1$, $\zeta_{R_0} \equiv 1$ on $B_{3R_0}$ and $\zeta_{R_0} \equiv 0$ on $B^c_{4R_0}$. We decompose $u$ as
\[
u(x) = \zeta_{R_0}(x) u(x) + (1 - \zeta_{R_0}(x)) u(x) =: v(x) + w(x).
\]
Notice that $v \in H^s(\mathbb{R}^N)$ with supp $v \subset \overline{B_{4R_0}}$ and $w \in H^s_{\text{loc}}(\mathbb{R}^N)$. Recalling (2.3), we also decompose $U$ as
\[U(x, t) = (P_s(\cdot, t) * u)(x) = (P_s(\cdot, t) * v)(x) + (P_s(\cdot, t) * w)(x) =: V(x, t) + W(x, t).
\]
We first estimate $V(x, t)$. By Young’s inequality and $\|P_s(\cdot, t)\|_{L^1(\mathbb{R}^N)} = 1$, it follows that
\[
\|V(\cdot, t)\|_{L^2(\mathbb{R}^N)} \leq \|P_s(\cdot, t)\|_{L^1(\mathbb{R}^N)} \|v\|_{L^2(\mathbb{R}^N)} = \|v\|_{L^2(\mathbb{R}^N)} \quad \text{for each } t \in (0, \infty).
\]
Therefore,
\[ \int_{B_R^+} t^{1-2s} |V|^2 \, dX \leq \int_{0}^{R} dt \int_{\mathbb{R}^N} t^{1-2s} |V(x,t)|^2 \, dx \leq \int_{0}^{R} t^{1-2s} \|v\|_{L^2(\mathbb{R}^N)}^2 dt = CR^{2-2s}. \]
From
\[ 2 - 2s \leq N + 2(1-s) - \frac{2(2s + \ell)}{p-1}, \]
we infer that
\[ \int_{B_R^+} t^{1-2s} |V|^2 \, dX \leq CR^{N+2(1-s) - \frac{2(2s + \ell)}{p-1}}. \] (2.54)
Next, we consider \( W(x,t) \). By Hölder’s inequality,
\[ \int_{B_R^+} t^{1-2s} |W|^2 \, dX \]
\[ = \int_{B_R^+} t^{1-2s} \left( \int_{\mathbb{R}^N} (P_s(x-y,t))^{1/2}(P_s(x-y,t))^{1/2} w(y) \, dy \right)^2 \, dX \]
\[ \leq C \int_{B_R^+} dX t^{1-2s} \int_{\mathbb{R}^N} w(y)^2 \frac{t^{2s}}{(|x - y|^2 + t^2)^{N+2s/2}} \, dy \]
\[ \leq C \int_{B_R^+} dX t^{1-2s} \left( \int_{|x-y| \leq 3R} + \int_{|x-y| > 3R} \right) w(y)^2 \frac{t^{2s}}{(|x - y|^2 + t^2)^{N+2s/2}} \, dy. \] (2.55)
For \( y \in \mathbb{R}^N \) with \(|x - y| \leq 3R\), since it follows from (1.2) that
\[ -\frac{N + 2s}{2} + 1 < 0 \quad \text{if } N \geq 2, \quad -\frac{N + 2s}{2} + 1 = \frac{1 - 2s}{2} > 0 \quad \text{if } N = 1, \]
we see that
\[ \int_{B_R^+} \int_{|x-y| \leq 3R} w(y)^2 \frac{t^{2s}}{(|x - y|^2 + t^2)^{N+2s/2}} \, dy \]
\[ \leq \int_{0}^{R} dt \int_{B_R} dx \int_{|x-y| \leq 3R} w(y)^2 \frac{t^{2}}{(|x - y|^2 + t^2)^{N+2s/2}} \, dy \]
\[ = \int_{B_R} dx \int_{|x-y| \leq 3R} \left. w(y)^2 \frac{1}{2 - N - 2s} \frac{\partial}{\partial t} (|x - y|^2 + t^2) \right|_{t=0}^{t=2} \, dy \]
\[ = \frac{1}{2 - N - 2s} \int_{B_R} dx \int_{|x-y| \leq 3R} w(y)^2 \left( |x - y|^2 + R^2 \right)^{\frac{2-N-2s}{2}} - |x - y|^{2-N-2s} \, dy \]
\[ \leq \begin{cases} 
\frac{1}{2} & \text{if } N = 1, \\
\frac{1}{N + 2s - 2} & \text{if } N \geq 2.
\end{cases} \]
When \( N = 1 \), by \{ \{ y \in \mathbb{R}^N \mid |x - y| \leq 3R \} \subset B_{4R} \) for each \( x \in B_R \), we have
\[ \int_{B_R} dx \int_{|x-y| \leq 3R} w(y)^2 \left( |x - y|^2 + R^2 \right)^{\frac{1-2s}{2}} \, dy \leq CR^{1-2s} \int_{B_R} dx \int_{|x-y| \leq 3R} w(y)^2 \, dy \]
\[ \leq CR^{2-2s} \int_{B_{4R}} w(y)^2 \, dy. \]
On the other hand, when \( N \geq 2 \), since \( B_R(y) \subset B_{\ell R} \) for each \( y \in B_{\ell R} \), we have

\[
\int_{B_R} dx \int_{|x-y| \leq \ell R} w(y)^2 |x-y|^{2-N-2s} dy \leq C \int_{B_R} dx \int_{B_{\ell R}} w(y)^2 |x-y|^{2-N-2s} dy \\
= C \int_{B_{\ell R}} dy \int_{B_R(y)} w(y)^2 |z|^{2-N-2s} dz \\
\leq C \int_{B_{\ell R}} dy \int_{B_{\ell R}(y)} w(y)^2 |z|^{2-N-2s} dz \\
= CR^{2-2s} \int_{B_{\ell R}} w(y)^2 dy,
\]

hence, for \( N \geq 1 \) and \( R \geq 3R_0 \),

\[
\int_{B_{\ell R}}^2 dX^{1-2s} \int_{|x-y| \leq 3 R} w(y)^2 \frac{t^{2s}}{(|x-y|^2 + \ell^2)^{\frac{N+2s}{2}}} y dx \leq CR^{2-2s} \int_{B_{\ell R}} w(y)^2 y dy.
\]

Notice that \( w \equiv 0 \) on \( B_{3R_0} \), \(|w| \leq |w| \) and \( 0 < c \leq \eta_R(x) \) for any \( 3R_0 \leq |x| \leq 4R \). By Lemma 2.6 and \( N - 2\ell/(p - 1) > 0 \) due to \( p > p_S(N, \ell) \), we may argue as in (2.45) and (2.47) to obtain

\[
\int_{B_{\ell R}}^2 dX^{1-2s} \int_{|x-y| \leq 3 R} w(y)^2 \frac{t^{2s}}{(|x-y|^2 + \ell^2)^{\frac{N+2s}{2}}} y dx \\
\leq CR^{2-2s} \int_{3R_0 \leq |x| \leq 4R} u^2 \left( |x|^{\frac{4}{2}} \eta_R \right)^\frac{4}{p+1} \left( |x|^{\frac{4}{2}} \eta_R \right)^{-\frac{4}{p+1}} dx \\
\leq CR^{2-2s} \left( \int_{\mathbb{R}^N} |x|^{\frac{p}{2}} u^{p+1} \eta_R^2 dx \right)^\frac{2}{p+1} \left( \int_{3R_0 \leq |x| \leq 4R} |x|^{\frac{2N}{p-1}} dx \right)^\frac{p-1}{p+1} \\
\leq CR^{2(1-s)+(N-2\ell/(p-1))} \left( \int_{\mathbb{R}^N} u^2 \rho_R dx \right)^\frac{2}{p+1}.
\]

Furthermore, by (2.53) and Lemma 2.7, enlarging \( C \) if necessary, we obtain

\[
\int_{\mathbb{R}^N} u^2 \rho_R dx \leq CR^{N-2\ell/(s(p+1)+\ell)}, \tag{2.56}
\]

which yields

\[
\int_{B_{\ell R}}^2 dX^{1-2s} \int_{|x-y| \leq 3 R} w(y)^2 \frac{t^{2s}}{(|x-y|^2 + \ell^2)^{\frac{N+2s}{2}}} y dx \leq CR^{N+2(1-s)-\frac{2(2s+\ell)}{p-1}}. \tag{2.57}
\]

Next, we consider the second term in (2.55), namely the case \(|x-y| > 3R \). Since \(|x-y| \geq

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\[|y| - |x| \geq |y| - R \geq |y|/2\] and \(B_{3R}^c(x) \subset B_R^c\) for \(x \in B_R\) and \(y \in B_{2R}^c\), we have
\[
\int_{B_R^c} dX t^{1-2s} \int_{|x-y|>3R} w(y)^2 \frac{t^{2s}}{(|x-y|^2 + t^2)^{N+2s/2}} dy
\leq \int_{B_R^c} dx \int_{|x-y|>3R} dy \int_0^R w(y)^2 t|x-y|^{-N-2s} dt
\leq R^2 \int_{B_R^c} dx \int_{B_{5R}(x)} w(y)^2 |x-y|^{-N-2s} dy
\leq CR^2 \int_{B_R^c} dx \int_{B_{5R}(x)} w(y)^2 |y|^{-N-2s} dy
\leq CR^{2+N} \int_{|z|\geq 2R} w(z)^2 |z|^{-N-2s} dy.
\]
On the other hand, from the definition of \(\eta_R\) and \(\rho_R\), it follows that for \(|x| \geq 2R \geq 6R_0\) and \(e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N\),
\[
\rho_R(x) = \int_{\mathbb{R}^N} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{N+2s}} dy \geq \int_{|y|\geq 2R_0} \frac{(\eta(\frac{x}{R}) - \eta(\frac{y}{R}))^2}{|x-y|^{N+2s}} dy
= R^{-2s} \int_{|z|\geq 2R^{-1}R_0} \frac{(\eta(\frac{x}{R}) - \eta(z))^2}{|R^{-1}x-z|^{N+2s}} dz
\geq R^{-2s} \int_{|z-e_1|<\frac{3}{2}} \frac{(\eta(\frac{x}{R}) - \eta(z))^2}{|R^{-1}x-z|^{N+2s}} dz
\geq C_0 R^{-2s} |R^{-1}x|^{-N-2s} = C_0 R^N |x|^{-N-2s}
\]
for some \(C_0 > 0\). Thus, noting \(u \equiv w\) on \(|y| \geq 2R\) and (2.56), we obtain
\[
R^{2+N} \int_{|y|\geq 2R} w(y)^2 |y|^{-N-2s} dy \leq CR^2 \int_{|y|\geq 2R} w^2 \rho_R dy \leq CR^{N+2(1-s)-\frac{2(2s+\ell)}{p-1}},
\]
which implies
\[
\int_{B_R^+} dX t^{1-2s} \int_{|x-y|>3R} w(y)^2 \frac{t^{2s}}{(|x-y|^2 + t^2)^{N+2s/2}} dy \leq CR^{N+2(1-s)-\frac{2(2s+\ell)}{p-1}}. \tag{2.59}
\]
Substituting (2.57) and (2.59) into (2.55), we see that
\[
\int_{B_R^+} t^{1-2s}W^2 dX \leq CR^{N+2(1-s)-\frac{2(2s+\ell)}{p-1}}.
\]
This with (2.54) and \(U = V + W\) completes the proof of Lemma 2.8. \(\square\)

**Lemma 2.9.** Assume the same conditions as in Lemma 2.8. Then there exists a \(C = C(N, p, s, \ell, R_0, u) > 0\) such that
\[
\int_{B_R^+} t^{1-2s} |\nabla U|^2 dX + \int_{B_R} |x|^{\ell} |u|^{p+1} dx \leq CR^{N-\frac{2s}{p-1}(s(p+1)+\ell)} \tag{2.60}
\]
for all \(R \geq 3R_0\).
Proof. We first prove the weighted $L^{p+1}$ estimate for $u$. Let $\eta_R$ and $\rho_R$ be the functions given in Lemma 2.5. Then, since $u \in L^\infty_{loc}(\mathbb{R}^N)$ with (1.2), it holds that

$$\int_{B_{2R_0}} |x|^\ell |u|^{p+1} \, dx \leq C. \quad (2.61)$$

Furthermore, since $p_S(N, \ell) < p$ and (2.53), applying Lemmata 2.6 and 2.7, and noting $\eta_R \geq c_0 > 0$ on $B_R \setminus B_{2R_0}$, we see that for all $R \geq 3R_0$,

$$\int_{B_R \setminus B_{2R_0}} |x|^\ell |u|^{p+1} \, dx \leq C \int_{\mathbb{R}^N} |x|^\ell |u|^{p+1} \eta_R^2 \, dx \leq C \int_{\mathbb{R}^N} u^2 \rho_R \, dx \leq C \left(1 + R^{N - \frac{2}{p-1}((p+1)\ell)}\right) \leq CR^{N - \frac{2}{p-1}((p+1)\ell)}.$$

This together with (2.61) yields

$$\int_{B_R} |x|^\ell |u|^{p+1} \, dx \leq CR^{N - \frac{2}{p-1}((p+1)\ell)} \quad \text{for each } R \geq 3R_0. \quad (2.62)$$

Next we take a cut-off function $\zeta \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+ \setminus B_{R_0}^+})$ such that

$$\zeta \equiv \begin{cases} 
1 & \text{on } B_R^+ \setminus B_{2R_0}^+, \\
0 & \text{on } B_{R_0}^+ \cup (\mathbb{R}^{N+1}_+ \setminus B_{2R}^+),
\end{cases} \quad |\nabla \zeta| \leq CR^{-1} \quad \text{on } B_{2R}^+ \setminus B_R^+. \quad (2.63)$$

Then, taking $\psi = U\zeta^2 \in C^1_c(\overline{\mathbb{R}^{N+1}_+ \setminus B_{R_0}^+})$ as a test function in (2.18), we obtain

$$\kappa_s \int_{\mathbb{R}^N} |x|^\ell |u|^{p+1} \zeta(x,0)^2 \, dx = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \cdot \nabla (U\zeta^2) \, dX \quad (2.64)$$

$$= \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \{||\nabla (U\zeta)|^2 - U^2||\nabla \zeta|^2\} \, dX.$$

Since $u$ is stable outside $B_{R_0}$ and $U = u$ on $\partial \mathbb{R}^{N+1}_+$, we see from (2.8) that $U$ is stable outside $B_{R_0}^+$, that is, for any $\psi \in C^1_c(\overline{\mathbb{R}^{N+1}_+ \setminus B_{R_0}^+})$,

$$p\kappa_s \int_{\mathbb{R}^N} |x|^\ell |U(x,0)|^{p-1}\psi(x,0)^2 \, dx = p\kappa_s \int_{\mathbb{R}^N} |x|^\ell |u|^{p-1}\psi(x,0)^2 \, dx \leq \kappa_s ||\psi(\cdot,0)||^2_{H^s(\mathbb{R}^N)} \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} ||\nabla \psi||^2 \, dX. \quad (2.65)$$

By (2.64) and (2.65) with $\psi = U\zeta \in C^1_c(\overline{\mathbb{R}^{N+1}_+ \setminus B_{R_0}^+})$, we have

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \{||\nabla (U\zeta)|^2 - U^2||\nabla \zeta|^2\} \, dX \leq \frac{1}{p} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} ||\nabla (U\zeta)||^2 \, dX,$$

which implies

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} ||\nabla (U\zeta)||^2 \, dX \leq \frac{p}{p-1} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} U^2 ||\nabla \zeta||^2 \, dX. \quad (2.66)$$
By (2.66), (2.63) and (2.52), we see that
\[
\int_{B_R^+ \setminus B_R^{2R_0}} t^{1-2s} |\nabla U|^2 \, dX \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla (U \zeta)|^2 \, dX
\]
\[
\leq C \int_{\mathbb{R}^{N+1}_+} t^{1-2s} U^2 |\nabla \zeta|^2 \, dX
\]
\[
\leq C \left( \int_{B_{2R_0}^- \setminus B_R^+} t^{1-2s} U^2 \, dX + R^{-2} \int_{B_{2R}^- \setminus B_R^+} t^{1-2s} U^2 \, dX \right)
\]
\[
\leq C \int_{B_{2R_0}^- \setminus B_R^+} t^{1-2s} U^2 \, dX + CR^{N-\frac{2}{p-1}(s(p+1)+\ell)}
\]
for all $R \geq 3R_0$.

On the other hand, it follows from $U \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s} \, dX)$ due to Lemma 2.1 that
\[
\int_{B_{2R_0}^-} t^{1-2s} (|\nabla U|^2 + U^2) \, dX \leq C.
\]
This together with (2.62) and (2.67) yields (2.60), thus Lemma 2.9 follows.

\[\]

### 3 The subcritical and critical case

In this section, we prove Theorem 1.1 for the subcritical and critical case, that is, $1 < p \leq p_S(N, \ell)$.

**Proof of Theorem 1.1 for $1 < p \leq p_S(N, \ell)$**. Let $u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, (1+|x|)^{-N-2s} \, dx)$ be a solution of (1.1) which is stable outside $B_{R_0}^-$. As $R \to \infty$, $\eta_R(x) \to \psi(R^{-1}_0 x)$ for each $x \in \mathbb{R}^N$. Then by Lemma 2.5, $(\rho_R)_{R \geq R_0}$ is bounded in $L^\infty(\mathbb{R}^N)$ and we may check
\[
\rho_R(x) = \int_{\mathbb{R}^N} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{N+2s}} \, dy \to \int_{\mathbb{R}^N} \frac{(\psi(R^{-1}_0 x) - \psi(R^{-1}_0 y))^2}{|x-y|^{N+2s}} \, dy =: \rho_\infty(x).
\]

Next, from Lemmata 2.6 and 2.7 and the assumption $1 < p \leq p_S(N, \ell)$, it follows that $(\eta_R^2 R^{2(p+1)})_{R \geq 3R_0}$ is bounded in $L^{p+1}(\mathbb{R}^N, |x|^p \, dx)$ and $(\eta_R^2 R^{-1})_{R \geq 3R_0}$ is bounded in $\dot{H}^s(\mathbb{R}^N)$. Since
\[
u(x)\eta_R(x)^{\frac{2}{p+1}} \to u(x)\psi(R^{-1}_0 x)^{\frac{2}{p+1}}, \quad \nu(x)\eta_R(x) \to u(x)\psi(R^{-1}_0 x)
\]
for each $x \in \mathbb{R}^N$,
we infer that
\[
u_R^2 R^{2(p+1)} \to \psi(R^{-1}_0)^{\frac{2}{p+1}} \text{ weakly in } L^{p+1}(\mathbb{R}^N, |x|^p \, dx),
\]
\[
u_R \to \psi(R^{-1}_0) \text{ weakly in } \dot{H}^s(\mathbb{R}^N).
\]
In particular, we deduce that $u \in \dot{H}^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N, |x|^p \, dx)$.

Since $\varphi_n u \to u$ strongly in $\dot{H}^s(\mathbb{R}^N)$ where $(\varphi_n)_n$ appears in Lemma 2.1 and $u \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, we may use $\varphi_n u$ as a test function in (1.4):
\[
\int_{\mathbb{R}^N} |x|^p |u|^{p+1} \varphi_n \, dx = \langle u, \varphi_n u \rangle_{\dot{H}^s(\mathbb{R}^N)}.
\]
Letting $n \to \infty$, we obtain
\[
\int_{\mathbb{R}^N} |x|^{\ell}|u|^{p+1} \, dx = \|u\|^2_{H^s(\mathbb{R}^N)}.
\] (3.1)
Thus, the former assertion of Theorem 1.1(ii) is proved.

For the latter assertion of Theorem 1.1 (ii), assume that $p = p_S(N, \ell)$ and $u$ is stable. By the same argument as above, we can apply the stability inequality (1.9) with the test function $\varphi = u$:
\[
p \int_{\mathbb{R}^N} |x|^{\ell}|u|^{p+1} \, dx \leq \|u\|^2_{H^s(\mathbb{R}^N)}.
\]
This contradicts (3.1) unless $u \equiv 0$. So it remains to prove the subcritical case.

Since $u \in H^s(\mathbb{R}^N)$, notice that $\nabla U \in L^2(\mathbb{R}^N, p+1 \, dX)$ thanks to Remark 2.1. Then, similarly to (2.25) with $u \in L^{p+1}(\mathbb{R}^N, |x|^{\ell} \, dx)$, we claim that there exists a sequence $R_n \to \infty$ such that
\[
\lim_{n \to \infty} R_n \left[ \int_{S^N_{R_n}} t^{1-2s} |\nabla U|^2 \, dS + \int_{S_{R_n}} |x|^{\ell} |u|^{p+1} \, d\omega + \int_{S_{R_n}^t} t^{1-2s} \left| \frac{\partial U}{\partial \nu} \right|^2 \, dS \right] = 0. \tag{3.2}
\]
By (2.21), (3.2) and replacing $R$ with $R_n$ for a sequence $R_n \to \infty$, we conclude that
\[
\int_{\mathbb{R}^N} t^{1-2s} |\nabla U|^2 \, dX = \frac{2\kappa_s}{N-2s} \frac{N+\ell}{p+1} \int_{\mathbb{R}^N} |x|^{\ell} |u|^{p+1} \, dx.
\]
This together with (2.7) yields the following Pohozaev identity
\[
\frac{N+\ell}{p+1} \int_{\mathbb{R}^N} |x|^{\ell} |u|^{p+1} \, dx = \frac{N-2s}{2} \|u\|^2_{H^s(\mathbb{R}^N)}.
\]
Combining this identity with (3.1), we observe that $u \equiv 0$ for $p < p_S(N, \ell)$, and the proof of Theorem 1.1 is completed for $1 < p \leq p_S(N, \ell)$. \hfill \Box

4 The supercritical case

In this section, we follow the argument in [9] basically. However, due to the regularity issue of $U$ around 0 in $\mathbb{R}^{N+1}_+$, we prove the monotonicity formula (Lemma 4.2) via the argument in [13, section 3] and prove Theorem 1.1 for $p_S(N, \ell) < p$.

For $X \in \mathbb{R}^{N+1}_+$, we use the following notation:
\[
r := |X|, \quad \sigma := \frac{X}{|X|} \in S^+_1, \quad \sigma_{N+1} := \frac{t}{|X|}.
\]
Let $u \in H^s_{\text{loc}}(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{-N-2s} \, dx)$ be a solution of (1.1) and $U$ be the function given in (2.3). For every $\lambda > 0$, we define
\[
D(U; \lambda) := \lambda^{-(N-2s)} \left[ \frac{1}{2} \int_{B^+_\lambda} t^{1-2s} |\nabla U|^2 \, dX - \frac{\kappa_s}{p+1} \int_{B^+_\lambda} |x|^{\ell} |u|^{p+1} \, dx \right] \tag{4.1}
\]
and
\[
H(U; \lambda) := \lambda^{-(N+1-2s)} \int_{S^+_{\lambda}} t^{1-2s} U^2 \, dS = \int_{S^+_{\lambda}} \sigma^{1-2s}_{N+1} U(\lambda \sigma)^2 \, dS. \tag{4.2}
\]
Lemma 4.1. As a function of \( \lambda \), \( D, H \in C^1((0, \infty)) \) and

\[
\partial_\lambda D(U; \lambda) = \lambda^{-(N-2s)} \int_{S^+_\lambda} t^{1-2s} \left| \frac{\partial U}{\partial \nu} \right|^2 dS - \lambda^{-(N+1-2s)} \kappa_s \frac{2s + \ell}{p + 1} \int_{B_\lambda} |x|^{\ell} |u|^{p+1} dx
\]

and

\[
\partial_\lambda H(U; \lambda) = 2 \lambda^{-(N+1-2s)} \int_{S^+_\lambda} t^{1-2s} U \frac{\partial U}{\partial \nu} dS
\]

\[= 2 \lambda^{-(N+1-2s)} \left[ \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 dX - \kappa_s \int_{B_\lambda} |x|^{\ell} |u|^{p+1} dx \right].
\]

Proof. We first remark that by (2.3) and Lemma 2.3, \( U \in C(R_{N+1}^+) \), \( \nabla_x U \in C(R_{N+1}^+ \setminus \{0\}) \) and \( V := t^{1-2s} \partial_t U \in C(R_{N+1}^+ \setminus \{0\}) \). Hence, as a function of \( \lambda \),

\[
\lambda \mapsto \int_{B_\lambda} |x|^{\ell} |u|^{p+1} dx \in C^1((0, \infty)).
\]

On the other hand, since

\[
t^{1-2s} |\nabla U|^2 = t^{1-2s} |\nabla_x U|^2 + t^{2s-1} V^2
\]

and \( t^{1-2s} \in L^1_{\text{loc}}(R_{N+1}^+) \), it is easy to see that

\[
\lambda \mapsto \int_{S^+_\lambda} t^{1-2s} |\nabla U|^2 dS \in C((0, \infty)), \quad \lambda \mapsto \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 dX \in C^1((0, \infty)),
\]

which yields \( D(U; \lambda) \in C^1((0, \infty)) \).

On the other hand, for any \( 0 < \lambda_1 < \lambda_2 < \infty \), there exists a \( C_{\lambda_1, \lambda_2} > 0 \) such that for every \( \lambda \in [\lambda_1, \lambda_2] \) and \( \sigma \in \mathbb{S}_1^+ \)

\[
\sigma_{N+1}^{1-2s} |\partial_\lambda (U(\lambda \sigma))|^2 \leq 2 \sigma_{N+1}^{1-2s} |U(\lambda \sigma)| |\nabla U(\lambda \sigma)| \leq 2 |U(\lambda \sigma)| (\sigma_{N+1}^{1-2s} |\nabla_x U(\lambda \sigma)| + |V(\lambda \sigma)|)
\]

\[\leq C_{\lambda_1, \lambda_2} (1 + \sigma_{N+1}^{1-2s}).
\]

Hence, the dominated convergence theorem gives \( H(U; \lambda) \in C^1((0, \infty)) \).
Next we compute the derivative of \( D \) and \( H \). Direct computations and (2.21) give

\[
\begin{align*}
\partial_\lambda D(U; \lambda) &= - (N - 2s) \lambda^{-(N-2s)-1} \left[ \frac{1}{2} \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 \, dX - \frac{\kappa_s}{p+1} \int_{B_\lambda} |x|^\ell \, dx \right] \\
&\quad + \lambda^{-(N-2s)-1} \lambda \left[ \frac{1}{2} \int_{S_{\lambda}^+} t^{1-2s} |\nabla U|^2 \, dS - \frac{\kappa_s}{p+1} \int_{S_\lambda} |x|^\ell \, d\omega \right] \\
&= - (N - 2s) \lambda^{-(N-2s)-1} \left[ \frac{1}{2} \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 \, dX - \frac{\kappa_s}{p+1} \int_{B_\lambda} |x|^\ell \, dx \right] \\
&\quad + \lambda^{-(N-2s)-1} \lambda \left[ \frac{N - 2s}{2} \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 \, dX - \frac{\kappa_s}{p+1} \int_{B_\lambda} |x|^\ell \, dx \right].
\end{align*}
\]

For \( H \), we compute similarly by using (2.22) and \( \nabla U(X) \cdot (X/|X|) = \partial U/\partial \nu \):

\[
\begin{align*}
\partial_\lambda H(U; \lambda) &= \int_{S_{\lambda}^+} \sigma^{1-2s}_N 2U(\lambda \sigma) \nabla U(\lambda \sigma) \cdot \sigma \, dS \\
&= 2\lambda^{N-1+2s} \int_{S_{\lambda}^+} t^{1-2s} U \frac{\partial U}{\partial \nu} \, dS \\
&= 2\lambda^{N-1+2s} \left[ \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 \, dX - \kappa_s \int_{B_\lambda} |x|^\ell \, dx \right].
\end{align*}
\]

Hence, we complete the proof. \( \square \)

Applying Lemma 4.1, we prove the following monotonicity formula (cf. [9, Theorem 1.4]).

**Lemma 4.2.** For \( \lambda > 0 \), define \( E(U; \lambda) \) by

\[
E(U; \lambda) := \lambda^{\frac{(2s+\ell)}{p-1}} \left( D(U; \lambda) + \frac{2s + \ell}{2(p-1)} H(U; \lambda) \right).
\]

Then it holds that

\[
\begin{align*}
\partial_\lambda E(U; \lambda) &= \lambda^{\frac{2(2s+\ell)}{p-1}} (s(p+1)+\ell-N) \int_{S_{\lambda}^+} t^{1-2s} \left| \frac{2s + \ell}{p-1} \frac{U}{\lambda} + \frac{\partial U}{\partial r} \right|^2 \, dS.
\end{align*}
\]

**Proof.** Put

\[
\gamma := \frac{2(2s + \ell)}{p-1}.
\]
By (4.3) and (4.5), we have
\[
\partial_\xi E(U; \lambda) = \gamma \lambda^{\gamma - 1} \left( D(U; \lambda) + \frac{\gamma}{4} H(U; \lambda) \right) + \lambda^{\gamma} \left( \partial_\xi D(U; \lambda) + \frac{\gamma}{4} \partial_\xi H(U; \lambda) \right) \\
= \lambda^{\gamma - 1} \left( \gamma D(U; \lambda) + \frac{\gamma^2}{4} H(U; \lambda) + \lambda \partial_\xi D(U; \lambda) + \frac{\gamma \lambda^2}{4} \partial_\xi H(U; \lambda) \right). \tag{4.6}
\]
Since it follows from (4.5) that
\[
\left( \frac{1}{2} - \frac{1}{p+1} \right) \gamma - 2s + \ell = \frac{p - 1}{2(p + 1)} \frac{2(2s + \ell)}{p - 1} - \frac{2s + \ell}{p + 1} = 0,
\]
by Lemma 4.1 and (2.22), we see that
\[
\lambda^{N - 2s} \left( \gamma D(U; \lambda) + \frac{\gamma^2}{4} H(U; \lambda) + \lambda \partial_\xi D(U; \lambda) + \frac{\gamma \lambda^2}{4} \partial_\xi H(U; \lambda) \right) \\
= \gamma \left[ \frac{1}{2} \int_{S^+\lambda} t^{1 - 2s} |\nabla U|^2 dX - \frac{\kappa_s}{p + 1} \int_{S^+\lambda} |x|^\ell |u|^{p + 1} dx \right] + \frac{\gamma^2}{4} \lambda^{\gamma - 1} \int_{S^+\lambda} t^{1 - 2s} U^2 dS \\
+ \lambda \int_{S^+\lambda} t^{1 - 2s} \left\{ \frac{\gamma^2}{4} \left( \frac{U}{\lambda} \right)^2 + \frac{U \partial U}{\lambda \partial \nu} + \left( \frac{\partial U}{\partial \nu} \right)^2 \right\} dS \\
+ \kappa_s \left\{ \left( \frac{1}{2} - \frac{1}{p + 1} \right) \gamma - \frac{2s + \ell}{p + 1} \right\} \int_{B^+\lambda} |x|^\ell |u|^{p + 1} dx \\
= \lambda \int_{S^+\lambda} t^{1 - 2s} \left\{ \frac{\gamma U}{2 \lambda} + \frac{\partial U}{\partial \nu} \right\}^2 dS.
\]
This together with (4.6) and \( \partial U/\partial \nu = \partial U/\partial r \) on \( S^+_{\lambda} \) implies (4.4).

Similar to [9, Theorem 5.1], we prove the nonexistence result of solutions which have a special form and are stable outside \( B^+_{R_0} \).

**Lemma 4.3.** Let \( R_0 > 0 \) and \( p_S(N, \ell) < p \). Suppose (1.11) and that \( W \) satisfies the following:
\[
\begin{cases}
W(X) = r^{-\frac{2s + \ell}{p - 1}} \psi(\sigma) \in H^{1}_{\text{loc}}(\mathbb{R}^{N+1}_{+}, t^{1 - 2s} dX), \\
\psi(\omega, 0) := \psi|_{\partial S^+_1} \in L^{p + 1}(\partial S^+_1), \\
\int_{\mathbb{R}^{N+1}} t^{1 - 2s} \nabla W \cdot \nabla \Phi dX = \kappa_s \int_{\mathbb{R}^{N}} |x|^\ell |W(x, 0)|^{p - 1} W(x, 0) \Phi(x, 0) dx \\
\text{for each } \Phi \in C^1_{c}(\mathbb{R}^{N+1}_{+}), \\
\kappa_s p \int_{\mathbb{R}^{N}} |x|^\ell |W(x, 0)|^{p - 1} \Phi(x, 0)^2 dx \leq \int_{\mathbb{R}^{N+1}} t^{1 - 2s} \left| \nabla \Phi \right|^2 dX \\
\text{for each } \Phi \in C^1_{c}(\mathbb{R}^{N+1}_{+} \setminus B^+_{R_0}).
\end{cases} \tag{4.7}
\]
Then \( W \equiv 0. \)
Remark 4.1.

1. The function \( W \) in Lemma 4.4 is not necessarily defined through the form \( W = P_s(\cdot, t)u \) where \( u \) is a solution of (1.1).

2. By \( p_S(N, \ell) < p \), we have

\[
\ell - \frac{p}{p-1}(2s + \ell) = - \frac{2sp + \ell}{p-1} > -N, \quad |x|^{\ell} |W(x, 0)|^{p-1} W(x, 0) \in L^1_{\text{loc}}(\mathbb{R}^N).
\]

3. Set

\[
H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) := \overline{C^1(S_1^+, \sigma_{N+1}^{1-2s} dS)},
\]

\[
||u||^2_{H^1(S_1^+, \sigma_{N+1}^{1-2s} dS)} := \int_{S_1^+} \sigma_{N+1}^{1-2s} \left( |\nabla S_1^+ u|^2 + u^2 \right) dS
\]

where \( \nabla S_1^+ \) stands for the standard gradient on the unit sphere in \( \mathbb{R}^{N+1} \). From [13, Lemma 2.2], there exists the trace operator \( H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \rightarrow L^2(\partial S_1^+) \).

4. Since

\[
-\text{div}(t^{1-2s} \nabla W) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+, \quad W \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s} dX),
\]

elliptic regularity yields \( W = r \frac{2s+\ell}{p-2s} \psi(\sigma) \in C^\infty(\mathbb{R}^{N+1}_+) \). In addition, from \( W \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s} dX) \), we see \( \psi \in H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \). Next, for \( k \geq 1 \), consider

\[
W_k(X) := \max \left\{ -k, \min \left\{ |X|^{\frac{2s+\ell}{p-2s}} W(X), k \right\} \right\}.
\]

Then

\[
W_k \in H^1(B_2^+ \setminus B_1^{1/2}, t^{1-2s} dX) \cap L^\infty(B_2^+ \setminus B_1^{1/2}), \quad |W_k(X)| \leq |X|^{\frac{2s+\ell}{p-2s}} |W(X)|.
\]

From this fact, we may find \((\psi_k)_k\) satisfying

\[
\psi_k \in H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \cap L^\infty(S_1^+), \quad |\psi_k(\sigma)| \leq |\psi(\sigma)| \quad \text{for any} \quad \sigma \in S_1^+,
\]

\[
\|\psi_k - \psi\|_{H^1(S_1^+, \sigma_{N+1}^{1-2s} dS)} \rightarrow 0, \quad \psi_k(\omega, 0) \rightarrow \psi(\omega, 0) \quad \text{strongly in} \quad L^{p+1}(\partial S_1^+). \tag{4.8}
\]

5. If \( \varphi \in H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \cap L^\infty(S_1^+) \), then we may find \((\varphi_k)_k \subset C^1(S_1^+)\) such that

\[
\sup_{k \geq 1} \|\varphi_k\|_{L^\infty(S_1^+)} < \infty, \quad \|\varphi_k - \varphi\|_{H^1(S_1^+, \sigma_{N+1}^{1-2s} dS)} \rightarrow 0. \tag{4.9}
\]

From the trace operator, we also have \( \varphi_k(\omega, 0) \rightarrow \varphi(\omega, 0) \) in \( L^2(\partial S_1^+) \).

Even though a proof of Lemma 4.3 is similar to the proof of [9, Theorem 5.1], for the sake of completeness, we give the proof here. Before a proof of Lemma 4.3, we recall [13, Lemma 2.1]:
Lemma 4.4. For $v(X) = f(r)\psi(\sigma) \in C^\infty(\mathbb{R}^{N+1})$,
\[- \text{div} \left( t^{1-2s} \nabla v \right) = - r^{-N} \left( r^{N+1-2s} f_r(r) \right) \sigma^{1-2s} \psi(\sigma) - r^{-1-2s} f(r) \text{div}_{S_1^+} \left( \sigma^{1-2s} \nabla_{S_1^+} \psi \right)\]
where $\text{div}_{S_1^+}$ is the standard divergence operator on the unit sphere in $\mathbb{R}^{N+1}$.

Proof of Lemma 4.3. Let $W = r^{-2s\ell/p+1} \psi(\sigma)$ be as in the statement. We divide our arguments into several steps.

Step 1: $\psi$ satisfies
\[
\begin{cases}
- \text{div}_{S_1^+} \left( \sigma^{1-2s} \nabla_{S_1^+} \psi \right) + \beta \sigma^{1-2s} \psi = 0 & \text{in } S_1^+, \\
- \lim_{\sigma \to 0} \sigma^{1-2s} \partial \sigma \psi = \kappa_s |\psi|^{p-1} \psi & \text{on } \partial S_1^+ = S_1
\end{cases}
\]
where $\beta := \frac{2s + \ell}{p-1} \left( N - \frac{2s p + \ell}{p-1} \right)$, namely, for each $\varphi \in H^1(S_1^+, \sigma^{1-2s} dS) \cap L^\infty(S_1^+)$,
\[
\int_{S_1^+} \sigma^{1-2s} \nabla_{S_1^+} \psi \cdot \nabla_{S_1^+} \varphi + \beta \sigma^{1-2s} \psi \varphi \, dS = \kappa_s \int_{\partial S_1^+} |\psi|^{p-1} \psi \varphi \, d\omega. \tag{4.10}
\]
Furthermore, we may also choose $\varphi = \psi$ in (4.10) and obtain
\[
\int_{S_1^+} \sigma^{1-2s} \nabla_{S_1^+} \psi^2 + \beta \sigma^{1-2s} \psi^2 \, dS = \kappa_s \int_{\partial S_1^+} |\psi|^{p+1} \, d\omega. \tag{4.11}
\]

Proof. For (4.10), by (4.9), $\psi(\omega, 0) \in L^{p+1}(\partial S_1^+)$ and the dominated convergence theorem, it is enough to prove it for $\varphi \in C^\infty(S_1^+)$.

For $V(X) = V(r \sigma) \in C_c^\infty(\mathbb{R}^{N+1})$, notice that
\[
\nabla V(X) = \partial_r V(r \sigma) \sigma + r^{-1} \nabla_{S_1^+} V, \\
\nabla W(X) = r^{-2s\ell/p-1} \left[ - \frac{2s + \ell}{p-1} \psi(\sigma) \sigma + \nabla_{S_1^+} \psi(\sigma) \right]. \tag{4.12}
\]

We see from (4.7), (4.12) and $\sigma \cdot \nabla_{S_1^+} h(\sigma) = 0$ for functions $h$ on $S_1^+$ that
\[
\kappa_s \int_{\mathbb{R}^N} |x|^{-2s\ell/p-1} |\psi(x/|x|, 0)|^{p-1} \psi(x/|x|, 0) V(x, 0) \, dx = \int_{\mathbb{R}^{N+1}} t^{1-2s} \nabla W \cdot \nabla V \, dX \tag{4.13}
\]
\[
= \int_{\mathbb{R}^{N+1}} t^{1-2s} r^{-2s\ell/p-1} \left[ - \frac{2s + \ell}{p-1} \partial_r V(r \sigma) \psi(\sigma) + r^{-1} \nabla_{S_1^+} V \cdot \nabla_{S_1^+} \psi \right] \, dX.
\]
If we choose \( V(X) = \eta(r)\varphi(\sigma) \) where \( \eta \in C^1_c([0, \infty)) \) and \( \varphi \in C^1(\overline{S_1^+}) \), then (4.13) is rewritten as

\[
\kappa_s \int_0^\infty r^{2p+\ell-\frac{2p+\ell}{p-1}+N-1} \eta(r) \, dr \left( \int_{\partial S_1^+} |\psi(\omega, 0)|^{p-1} \psi(\omega, 0) \varphi(\omega, 0) \, d\omega \right) = \frac{2s + \ell}{p-1} \int_0^\infty r^{N-\frac{2p+\ell}{p-1}} \eta'(r) \, dr \left( \int_{S_1^+} \sigma_{N+1}^{-1} \varphi(\sigma) \psi(\sigma) \, dS \right) + \left( \int_0^\infty r^{2p+\ell-\frac{2p+\ell}{p-1}+N-1} \eta(r) \, dr \right) \left( \int_{S_1^+} \sigma_{N+1}^{-1} \nabla S_1^+ \varphi \cdot \nabla S_1^+ \psi \, dS \right) .
\]

(4.14)

Since \( p_S(N, \ell) < p \) yields \( -\frac{2p+\ell}{p-1} + N > 0 \), it follows from the integration by parts that

\[
-\frac{2s + \ell}{p-1} \int_0^\infty r^{N-\frac{2p+\ell}{p-1}} \eta'(r) \, dr = \beta \int_0^\infty r^{N-\frac{2p+\ell}{p-1}-1} \eta(r) \, dr.
\]

Thus, by choosing \( \eta \geq 0 \) with \( \eta \neq 0 \), (4.14) implies

\[
\kappa_s \int_{\partial S_1^+} |\psi(\omega, 0)|^{p-1} \psi(\omega, 0) \varphi(\omega, 0) \, d\omega = \int_{S_1^+} \sigma_{N+1}^{-1} \nabla S_1^+ \psi \cdot \nabla S_1^+ \varphi \beta \sigma_{N+1}^{-1} \psi \varphi \, dS
\]

for every \( \varphi \in C^1(\overline{S_1^+}) \). Hence, (4.10) holds.

For (4.11), take \( (\psi_k)_k \) satisfying (4.8). Then (4.10) holds for \( \varphi = \psi_k \). Thanks to \( \psi(\omega, 0) \in L^p(\partial S_1^+) \) and the dominated convergence theorem, letting \( k \to \infty \), we obtain (4.11).

**Step 2:** For every \( \varphi \in H^1(S_1^+, \sigma_{N+1}^{-1}dS) \cap L^\infty(S_1^+) \),

\[
\kappa_s p \int_{\partial S_1^+} |\psi(\omega, 0)|^{p-1} \psi^2 \, d\omega \leq \int_{S_1^+} \sigma_{N+1}^{-1} |\nabla S_1^+ \varphi|^2 \, dS + \left( \frac{N-2s}{2} \right)^2 \int_{S_1^+} \sigma_{N+1}^{-2s} \varphi^2 \, dS.
\]

(4.15)

**Proof.** It is enough to treat the case \( \varphi \in C^1(\overline{S_1^+}) \) due to (4.9) as in Step 1. We recall the stability in (4.7): for any \( \phi \in C^1_c(\mathbb{R}^{N+1}_+ \setminus B^+_R) \),

\[
\kappa_s p \int_{\partial B^+_R} |x|^\ell |W|^{p-1} \phi(x, 0)^2 \, dx \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \phi|^2 \, dX.
\]

(4.16)

For \( 0 < \varepsilon \ll 1 \), we choose \( \tau_\varepsilon \) and a standard cutoff function \( \eta_\varepsilon \in C^1_c((0, \infty)) \) such that

\[
\tau_\varepsilon := \frac{1}{\sqrt{-\log \varepsilon}} \to 0, \quad \chi_{(R_0+2\tau_\varepsilon, R_0+\varepsilon^{-1})}(r) \leq \chi_{(R_0+\tau_\varepsilon, R_0+2\varepsilon^{-1})}(r), \quad |\eta_\varepsilon'(r)| \leq C \tau_\varepsilon^{-1} \quad \text{for} \quad r \in (R_0+\tau_\varepsilon, R_0+2\tau_\varepsilon), \quad |\eta_\varepsilon'(r)| \leq C \varepsilon \quad \text{for} \quad r \in (R_0+\varepsilon^{-1}, R_0+2\varepsilon^{-1})
\]

(4.17)

where \( \chi_A(\cdot) \) is the characteristic function of \( A \subset (0, \infty) \). For \( \varphi \in C^1(\overline{S_1^+}) \), we put

\[
\phi(X) = r^{-\frac{N-2s}{2}} \eta_\varepsilon(r) \varphi(\sigma) \quad \text{for} \quad X = r\sigma \in \mathbb{R}^{N+1}_+.
\]

(4.18)
Since $W = r^{-\frac{2s+\ell}{p+1}} \psi(\sigma)$, we have

$$
\int_{\partial \mathbb{R}^{N+1}_+} |x|^\ell |W|^{p-1} \phi^2 \, dx = \left( \int_0^{\infty} r^{-1}\eta_\varepsilon^2 \, dr \right) \left( \int_{\partial S^+_1} |\psi|^{p-1} \varphi^2 \, d\omega \right), \quad (4.19)
$$

On the other hand, by (4.18), we see that

$$
|\nabla \phi(X)|^2 = \left( \left( \frac{r - \frac{N-2s}{2} \eta_\varepsilon}{2} \right)^2 \varphi^2 + r^{-2} \left( \frac{r - \frac{N-2s}{2} \eta_\varepsilon}{2} \right)^2 |\nabla S^+_1 \varphi|^2 \right)
+ r^{-(N-2s)}(\eta'_\varepsilon)^2 \varphi^2 - (N-2s)r^{-(N-2s)} \eta_\varepsilon \eta'_\varepsilon \varphi^2. \quad (4.20)
$$

Since it follows from (4.17) that

$$
\int_0^{\infty} r^{N+1-2s} r^{-(N-2s)}(\eta'_\varepsilon)^2 \, dr \leq C \tau_{\varepsilon}^{-2} \int_{(R_0 + \tau_\varepsilon)} r \, dr + C \varepsilon \int_{R_0 + 1/\varepsilon} r \, dr \leq C \left( \tau_{\varepsilon}^{-1} + 1 \right),
$$

$$
\int_0^{\infty} r^{N+1-2s} r^{-(N-2s)} \eta_\varepsilon \eta'_\varepsilon \, dr \leq C \tau_{\varepsilon}^{-1} \int_{(R_0 + \tau_\varepsilon)} dr + C \varepsilon \int_{R_0 + 1/\varepsilon} dr \leq C,
$$

by (4.20) we have

$$
\int_{\mathbb{R}^{N+1}_+} r^{1-2s} |\nabla \phi|^2 \, dX
= \int_0^{\infty} dr \int_{S^+_1} r^N (r \sigma_{N+1})^{1-2s} \left\{ \left[ \left( \frac{N - 2s}{2} \right)^2 \varphi^2 + |\nabla S^+_1 \varphi|^2 \right] r^{-(N-2s)} \eta_\varepsilon^2 \right. \\
+ \left. r^{-(N-2s)}(\eta'_\varepsilon)^2 \varphi^2 - (N-2s)r^{-(N-2s)} \eta_\varepsilon \eta'_\varepsilon \varphi^2 \right\} \, dS
\leq \left( \int_0^{\infty} r^{-1}\eta_\varepsilon^2 \, dr \right) \left( \int_{S^+_1} \sigma_{N+1}^{1-2s} \left[ \left( \frac{N - 2s}{2} \right)^2 \varphi^2 + |\nabla S^+_1 \varphi|^2 \right] \, dS \right) + C \left( \tau_{\varepsilon}^{-1} + 1 \right). \quad (4.21)
$$

Finally, remark that

$$
\int_0^{\infty} r^{-1}\eta_\varepsilon^2 \, dr \geq \int_{R_0 + 2\tau_\varepsilon} r^{-1} \, dr = \log \left( R_0 + \varepsilon^{-1} \right) - \log \left( R_0 + 2\tau_\varepsilon \right) \geq \frac{\tau_{\varepsilon}^{-2}}{2}
$$

holds for sufficiently small $\varepsilon$. Therefore, substituting (4.19) and (4.21) to (4.16), dividing by $\int_0^{\infty} r^{-1}\eta_\varepsilon^2 \, dr$ and taking $\varepsilon \to 0$, we obtain (4.15).

\[ \square \]

**Step 3:** For $\alpha \in [0, \frac{N-2s}{2})$, set

$$
v_\alpha(x) := |x|^{-\left( \frac{N-2s}{2} - \alpha \right)}, \quad V_\alpha(X) := (P_s(\cdot; t) * v_\alpha)(x).
$$

Then for each $X \in \mathbb{R}^{N+1}_+$ and $\lambda > 0$,

$$
V_\alpha(\lambda X) = \lambda^{-\frac{N-2s}{2} + \alpha} V_\alpha(X). \quad (4.22)
$$
and \( \phi_\alpha(\sigma) := V_\alpha(\sigma) = V_\alpha(r^{-1}X) \in C(S_1^n) \cap C^1(S_1^+) \cap H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \). Moreover, \( \phi_\alpha \) also satisfies \( \phi_\alpha > 0 \) in \( S_1^+ \), for any \( \varphi \in H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \cap L^\infty(S_1^+) \),

\[
\int_{S_1^+} \sigma_{N+1}^{1-2s} |\nabla S_1^+ \varphi|^2 \, dS + \left[ \frac{(N-2s)}{2} - \alpha^2 \right] \int_{S_1^+} \sigma_{N+1}^{1-2s} \varphi^2 \, dS = \kappa_s \lambda(\alpha) \int_{\partial S_1^+} \varphi^2 \, d\omega + \int_{S_1^+} \sigma_{N+1}^{1-2s} \alpha \left| \nabla S_1^+ \left( \frac{\varphi}{\alpha} \right) \right|^2 \, dS
\]

(4.23)

and

\[
0 \leq \alpha_1 \leq \alpha_2 < \frac{N - 2s}{2} \quad \Rightarrow \quad \phi_{\alpha_1} \leq \phi_{\alpha_2} \quad \text{in} \quad S_1^+.
\]

(4.24)

**Proof.** By direct computation, we may check (4.22). For the assertion \( \phi_\alpha \in C(S_1^n) \cap C^1(S_1^+) \cap H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \), we remark \( V_\alpha(x) = (P_\alpha(t) \ast \nu_\alpha)(x) \in C^\infty(\mathbb{R}_+^{N+1}) \). By \( \phi_\alpha = V_\alpha|_{S_1^+} \), to show \( \phi_\alpha \in C(S_1^n) \cap C^1(S_1^+) \cap H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \), it suffices to prove

\[
V_\alpha, \ t^{1-2s} \partial_t V_\alpha, \ \nabla_x V_\alpha \in C \left( \frac{N+1}{4}(\partial S_1^+) \right), \quad N_1^+(A) := \left\{ X \in \mathbb{R}_+^{N+1} \mid \operatorname{dist}(X, A) < r \right\}.
\]

(4.25)

To this end, decompose \( v_\alpha = v_{\alpha,1} + v_{\alpha,2} \) where \( \operatorname{supp} v_{\alpha,1} \subset B_2 \setminus B_1/4 \) and \( v_{\alpha,1} \equiv v_\alpha \) on \( B_{3/2} \setminus B_{1/2} \), and set \( V_{\alpha,i}(X) := (P_\alpha(t) \ast v_{\alpha,i})(x) \). Since \( v_{\alpha,1} \in C^\infty(\mathbb{R}^N) \) and \( N_{1/4}(\partial S_1^+) \cap \operatorname{supp} v_{\alpha,2} = 0 \) where \( N_r(A) := \{ x \in \mathbb{R}^N \mid \operatorname{dist}(x, A) < r \} \), it is not difficult to show (4.25) and \( \phi_\alpha \in C(S_1^n) \cap C^1(S_1^+) \cap H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \).

The assertion \( \phi_\alpha > 0 \) in \( S_1^+ \) follows from \( v_\alpha > 0 \) in \( \mathbb{R}^N \setminus \{0\} \) and the definition of \( V_\alpha \).

For (4.23) and (4.24), remark that in [12, Lemma 4.1], it is proved that \( (-\Delta)^s v_\alpha = \lambda(\alpha)|x|^{-2s} v_\alpha \) in \( \mathbb{R}^N \), where \( \lambda(\alpha) \) appears in (1.7). Hence, by the property of \( P_\alpha(x, t) \) and \( v_\alpha \), we may check that for each \( \varphi \in C^\infty_c(\mathbb{R}^N \setminus \{0\}) \),

\[
- \lim_{t \to 0} \int_{\mathbb{R}^N} t^{1-2s} \partial_t V_\alpha(X) \varphi(x) \, dx = \kappa_s \int_{\mathbb{R}^N} v_\alpha(-\Delta)^s \varphi \, dx = \kappa_s \int_{\mathbb{R}^N} \lambda(\alpha)|x|^{-2s} v_\alpha \varphi \, dx.
\]

(4.26)

For any fixed \( \omega \in \partial S_1^+ \), we consider a curve

\[
\gamma_\omega(\tau) := \left( \frac{\sqrt{1 - \tau^2} \, \omega}{\tau} \right) \in S_1^+.
\]

Then \( \phi_\alpha(\gamma_\omega(\tau)) = V_\alpha(\gamma_\omega(\tau)) \) and

\[
\frac{d}{d\tau} V_\alpha(\gamma_\omega(\tau)) = \nabla V_\alpha(\gamma_\omega(\tau)) \cdot \left( - \frac{\tau}{\sqrt{1 - \tau^2}} \omega \right) = - \frac{\tau}{\sqrt{1 - \tau^2}} \nabla_x V_\alpha(\gamma_\omega(\tau)) \cdot \omega + \partial_t V_\alpha(\gamma_\omega(\tau)).
\]

Combining this fact with (4.25), \( v_\alpha(1) = 1 \) and (4.26), we deduce that

\[
- \lim_{\sigma_{N+1} \to 0} \sigma_{N+1}^{1-2s} \partial_{\sigma_{N+1}} \phi_\alpha(\sigma) = \kappa_s \lambda(\alpha) \quad \text{on} \quad \partial S_1^+.
\]

(4.27)
Due to (4.22), we notice that

\[ V_\alpha(X) = r^{-\frac{N-2s}{2} + \alpha} \phi_\alpha(\sigma). \]

Furthermore, since \( 0 = -\text{div}(t^{1-2s} \nabla V_\alpha) \) in \( \mathbb{R}^{N+1} \) and \( V_\alpha = v_\alpha \) on \( \partial \mathbb{R}^{N+1} \setminus \{0\} \), we have \( \phi_\alpha = v_\alpha = 1 \) on \( \partial S_1^+ \), and by Lemma 4.4 with \( f(r) = r^{-\frac{N-2s}{2} + \alpha} \) and \( \psi(\sigma) = \phi_\alpha(\sigma) \), \( \phi_\alpha \) is a solution of

\[
\begin{aligned}
-\text{div}_{S_1^+} \left( \sigma_{N+1}^{1-2s} \nabla_{S_1^+} \phi_\alpha \right) + \left[ \left( \frac{N-2s}{2} \right)^2 - \alpha^2 \right] \sigma_{N+1}^{1-2s} \phi_\alpha &= 0 \quad \text{in} \quad S_1^+, \\
\phi_\alpha &= 1 \quad \text{on} \quad \partial S_1^+.
\end{aligned}
\]

(4.28)

Now we prove (4.23). Since \( \phi_\alpha \in C(\overline{S_1^+}) \) and \( \phi_\alpha > 0 \) in \( \overline{S_1^+} \), for every \( \varphi \in H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \cap L^\infty(S_1^+) \) and \( (\varphi_k)_k \) with (4.9), it follows that

\[
\frac{\varphi_k}{\phi_\alpha} \rightarrow \frac{\varphi}{\phi_\alpha} \quad \text{strongly in} \quad H^1(S_1^+, \sigma_{N+1}^{1-2s} dS), \quad \frac{\varphi}{\phi_\alpha} \in H^1(S_1^+, \sigma_{N+1}^{1-2s} dS).
\]

Hence, it suffices to show (4.23) for \( \varphi \in C^1(\overline{S_1^+}) \). For \( \varphi \in C^1(\overline{S_1^+}) \), notice that

\[
\nabla_{S_1^+} \phi_\alpha \cdot \nabla_{S_1^+} \left( \frac{\varphi^2}{\phi_\alpha} \right) = \nabla_{S_1^+} \phi_\alpha \cdot \left[ \frac{2 \varphi \nabla_{S_1^+} \varphi}{\phi_\alpha} - \frac{\varphi^2 \nabla_{S_1^+} \phi_\alpha}{\phi_\alpha^2} \right] = \left| \nabla_{S_1^+} \varphi \right|^2 - \left| \nabla_{S_1^+} \left( \frac{\varphi}{\phi_\alpha} \right) \right|^2 \phi_\alpha^2.
\]

Thus, multiplying (4.28) by \( \varphi^2/\phi_\alpha \), we see from (4.27) that (4.23) holds.

Finally, for \( 0 \leq \alpha_1 \leq \alpha_2 < \frac{N-2s}{2} \), we infer from \( 0 < \phi_\alpha \) that

\[
-\text{div}_{S_1^+} \left( \sigma_{N+1}^{1-2s} \nabla_{S_1^+} \phi_\alpha \right) = - \left[ \left( \frac{N-2s}{2} \right)^2 - \alpha_1^2 \right] \sigma_{N+1}^{1-2s} \phi_\alpha \leq - \left[ \left( \frac{N-2s}{2} \right)^2 - \alpha_2^2 \right] \sigma_{N+1}^{1-2s} \phi_\alpha \quad \text{on} \quad S_1^+,
\]

which yields

\[
-\text{div}_{S_1^+} \left( \sigma_{N+1}^{1-2s} \nabla_{S_1^+} (\phi_{\alpha_2} - \phi_{\alpha_1}) \right) + \left[ \left( \frac{N-2s}{2} \right)^2 - \alpha_2^2 \right] \sigma_{N+1}^{1-2s} (\phi_{\alpha_2} - \phi_{\alpha_1}) \geq 0.
\]

Multiplying this inequality by \( (\phi_{\alpha_2} - \phi_{\alpha_1})_- := \max\{0, - (\phi_{\alpha_2} - \phi_{\alpha_1})\} \in H^1(S_1^+, \sigma_{N+1}^{1-2s} dS) \) and integrating it over \( S_1^+ \), by \( \phi_{\alpha_1} = 1 = \phi_{\alpha_2} \) on \( \partial S_1^+ \) and \( (\frac{N-2s}{2})^2 - \alpha_2^2 > 0 \), we deduce that \( (\phi_{\alpha_2} - \phi_{\alpha_1})_- \equiv 0 \), hence, (4.24) holds.

\( \square \)

**Step 4: Conclusion**

Now we are ready to prove the assertion of Lemma 4.3. By \( p_S(N, \ell) < p \) and \( \ell > -2s \) thanks to (1.2), we set

\[
\tilde{\alpha} = \frac{N-2s}{2} - \frac{2s + \ell}{p-1} \in \left( 0, \frac{N-2s}{2} \right).
\]

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By this choice of $\tilde{\alpha}$, we see that
\[
\left(\frac{N - 2s}{2}\right)^2 - \tilde{\alpha}^2 = \frac{2s + \ell}{p - 1} \left(\frac{N - 2s}{2} - \frac{2s + \ell}{p - 1}\right) = \beta
\] (4.29)

where $\beta$ appears in Step 1. Let $(\psi_k)_k$ be the functions in (4.8) and notice that $\phi_0/\phi_\tilde{\alpha} \in C(S^+_1) \cap H^1(S^+_1, \sigma^0_{N+1})$, $\phi_0 = \phi_\tilde{\alpha} = 1$ on $\partial S^+_1$ and $\psi_k \phi_0 / \phi_\tilde{\alpha} \in H^1(S^+_1, \sigma^0_{N+1}) \cap L^\infty(S^+_1)$. Hence, (4.15) and (4.23) with $\varphi = \psi_k \phi_0 / \phi_\tilde{\alpha}$ and $\alpha = 0$ give
\[
\kappa_s \int_{\partial S^+_1} |\psi|^{p-1} \psi_k^2 d\omega
\leq \int_{S^+_1} \sigma^0_{N+1} \left| \nabla S^+_1 \left( \frac{\psi_k \phi_0}{\phi_\tilde{\alpha}} \right) \right|^2 dS + \left(\frac{N - 2s}{2}\right)^2 \int_{S^+_1} \sigma^0_{N+1} \left( \frac{\psi_k \phi_0}{\phi_\tilde{\alpha}} \right)^2 dS
\]

This together with (4.24) implies
\[
\kappa_s \int_{\partial S^+_1} |\psi|^{p-1} \psi_k^2 d\omega \leq \kappa_s \lambda(0) \int_{\partial S^+_1} \psi_k^2 d\omega + \int_{S^+_1} \sigma^0_{N+1} \phi_\tilde{\alpha}^2 \left| \nabla S^+_1 \left( \frac{\psi_k}{\phi_\tilde{\alpha}} \right) \right|^2 dS.
\] (4.30)

Substituting (4.23) with $\varphi = \psi_k$ and $\alpha = \tilde{\alpha}$ into (4.30), we observe from (4.29) that
\[
\kappa_s \int_{\partial S^+_1} |\psi|^{p-1} \psi_k^2 d\omega
\leq \kappa_s \lambda(0) \int_{\partial S^+_1} \psi_k^2 d\omega + \int_{S^+_1} \sigma^0_{N+1} \left| \nabla S^+_1 \psi_k \right|^2 dS
\]
\[
+ \left[\left(\frac{N - 2s}{2}\right)^2 - \tilde{\alpha}^2\right] \int_{S^+_1} \sigma^0_{N+1} \psi_k^2 dS - \kappa_s \lambda(\tilde{\alpha}) \int_{\partial S^+_1} \psi_k^2 d\omega
\]
\[
= \kappa_s (\lambda(0) - \lambda(\tilde{\alpha})) \int_{\partial S^+_1} \psi_k^2 d\omega + \int_{S^+_1} \sigma^0_{N+1} \left| \nabla S^+_1 \psi_k \right|^2 dS + \beta \int_{S^+_1} \sigma^0_{N+1} \psi_k^2 dS.
\] (4.31)

On the other hand, by (4.23) with $\varphi = \psi_k$ and $\alpha = \tilde{\alpha}$, we have
\[
\int_{S^+_1} \sigma^0_{N+1} \left| \nabla S^+_1 \psi_k \right|^2 dS + \beta \int_{S^+_1} \sigma^0_{N+1} \psi_k^2 dS \geq \kappa_s \lambda(\tilde{\alpha}) \int_{\partial S^+_1} \psi_k^2 d\omega.
\]

From (4.31) and the fact $\lambda(0) > \lambda(\tilde{\alpha})$ due to (1.8), it follows that
\[
\kappa_s \int_{\partial S^+_1} |\psi|^{p-1} \psi_k^2 d\omega \leq \frac{\lambda(0)}{\lambda(\tilde{\alpha})} \left\{ \int_{S^+_1} \sigma^0_{N+1} \left| \nabla S^+_1 \psi_k \right|^2 dS + \beta \int_{S^+_1} \sigma^0_{N+1} \psi_k^2 dS \right\}.
\]

Letting $k \to \infty$ and noting (4.8) and (4.11), we obtain
\[
\kappa_s \int_{\partial S^+_1} |\psi|^{p+1} d\omega \leq \frac{\lambda(0)}{\lambda(\tilde{\alpha})} \kappa_s \int_{\partial S^+_1} |\psi|^{p+1} d\omega.
\]

Thus, we obtain $\lambda(\tilde{\alpha}) p \leq \lambda(0)$ unless $\psi \equiv 0$. Therefore, if (1.11) holds, namely $\lambda(\tilde{\alpha}) p > \lambda(0)$, then $\psi \equiv 0$, and by $W = r^{-s(\frac{2s}{p-1} + 1)} \psi$, we have $W \equiv 0$. Hence, Lemma 4.3 follows. 
\[\square\]
Now we are ready to prove Theorem 1.1 for $p_S(N, \ell) < p$. Following [9], we use the blow-down analysis.

**Proof of Theorem 1.1 for $p_S(N, \ell) < p$.** Assume (1.2) and (1.11). Let $u \in H_{\text{loc}}^s(\mathbb{R}^N) \cap L_{\text{loc}}^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be a solution of (1.1) which is stable outside $B_{R_0}$ and let $U$ be the function given in (2.3). Recall $D$, $H$ and $E$ in (4.1), (4.2) and (4.3), respectively. Then, by Lemma 2.8 we see that

$$\lambda^{\frac{2(2s+\ell)}{p-1}} D(U; \lambda) \leq C \lambda^{\frac{2}{p-1}((p+1)+\ell)-N} \left( \int_{B^+_\lambda} t^{1-2s} |\nabla U|^2 dX + \int_{B^+_\lambda} |x|^\ell |u|^{p+1} dx \right) \leq C \quad (4.32)$$

for $\lambda \geq 3R_0$. Since $E$ is nondecreasing with respect to $\lambda$ due to Lemma 4.2, by (4.32) and Lemma 2.8, for $\lambda \geq 3R_0$, we have

$$E(U; \lambda) \leq \lambda^{-1} \int_\lambda^{2\lambda} E(U; \xi) d\xi$$

$$= \lambda^{-1} \int_\lambda^{2\lambda} \xi^{\frac{2(2s+\ell)}{p-1}} \left[ D(U; \xi) + \frac{2s+\ell}{2(p-1)} H(U; \xi) \right] d\xi$$

$$\leq C + \lambda \int_\lambda^{2\lambda} d\xi \xi^{\frac{2(2s+\ell)}{p-1}-N-1+2s} \int_{S^+_t} t^{1-2s} U^2 dS$$

$$\leq C + \lambda \int_\lambda^{2\lambda} d\xi \xi^{\frac{2(2s+\ell)}{p-1}-N-1+2s} \int_{B^+_\lambda} t^{1-2s} U^2 dX$$

$$\leq C + \lambda \left( \frac{\sqrt{p-1}}{p-1} \right)^{2s+\ell+2s-N-2} \lambda^{N+2(1-s)-2(2s+\ell)/p-1} \leq C.$$  

This implies that

$$\lim_{\lambda \to \infty} E(U; \lambda) < +\infty. \quad (4.33)$$

On the other hand, for $X \in \mathbb{R}^{N+1}_+$, let

$$V_\lambda(X) := \lambda^{\frac{2s+\ell}{p-1}} U(\lambda X).$$

Then it is easy to check that

$$V_\lambda(X) = \left( P_s(\cdot, t) * \left( \lambda^{\frac{2s+\ell}{p-1}} u(\lambda \cdot) \right) \right)(x),$$

$$\lim_{t \to 0^+} t^{1-2s} \partial_t V_\lambda(x, t) = \kappa_s |x|^\ell \left| V_\lambda(x, 0) \right|^{p-1} V_\lambda(x, 0), \quad (4.34)$$

$$\lambda^{\frac{2(2s+\ell)}{p-1}} D(U; \lambda R) = D(V_\lambda; R), \quad \lambda^{\frac{2(2s+\ell)}{p-1}} H(U; \lambda R) = H(V_\lambda; R), \quad E(U; \lambda R) = E(V_\lambda; R)$$

for each $\lambda \geq 3R_0$. Since $u$ is stable outside $B_{R_0}$, as in the proof of Lemma 2.9 (see (2.65),

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by (2.8), $U$ is stable outside $B_{R_0}^+$. Therefore, for every $\psi \in C_c^1(\mathbb{R}^{N+1}_+ \setminus B_{\lambda-1}^+)$,

$$p\kappa_s \int_{\mathbb{R}^N} |x|^\ell |V_\lambda(x,0)|^{p-1} \psi(x,0)^2 \, dx = p\kappa_s \lambda^{2s-N} \int_{\mathbb{R}^N} |x|^\ell |u(x)|^{p-1} \psi(\lambda^{-1}x,0)^2 \, dx$$

$$\leq \lambda^{2s-N} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla(\psi(\lambda^{-1}X))|^2 \, dX$$

$$= \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \psi|^2 \, dX,$$

which implies that $V_\lambda$ is stable outside $B_{\lambda-1}^+$. Furthermore, by (2.52) and (2.60), $(V_\lambda)_{\lambda \geq 3R_0}$ is bounded in $H^1_{loc}(\mathbb{R}^{N+1}_+, t^{1-2s}dX)$ and $(V_\lambda(x,0))_{\lambda \geq 3R_0}$ is bounded in $L_{loc}^{p+1}(\mathbb{R}^N, |x|^{\ell}dX)$.

Now let $(\lambda_i)_{i=1}^\infty$ satisfy $\lambda_i \to \infty$ and $V_{\lambda_i} \rightharpoonup U_\infty$ weakly in $H^1_{loc}(\mathbb{R}^{N+1}_+, t^{1-2s}dX)$. Thanks to the above fact, without loss of generality, we may also assume that

$$U_\infty(x,0) \in L_{loc}^{p+1}(\mathbb{R}^N, |x|^{\ell}dX).$$

We shall claim that

$$V_{\lambda_i}(x,0) \to U_\infty(x,0) \quad \text{strongly in } L^q_{loc}(\mathbb{R}^N) \quad \text{for } 1 \leq q < \frac{2N}{N-2s},$$

$$V_{\lambda_i}(X) \to U_\infty(X) \quad \text{strongly in } L^2_{loc}(\mathbb{R}^{N+1}_+, t^{1-2s}dX).$$

Due to the boundedness of the trace operator from $H^1(B_R \times (0, R), t^{1-2s}dX)$ to $H^s(B_R)$ for each $R$ (see [10]), we also have $V_{\lambda_i}(x,0) \rightharpoonup U_\infty(x,0)$ weakly in $H^s(B_R)$. By the compactness of embedding $H^s(B_R) \subset L^q(B_R)$ where $1 \leq q < 2N/(N-2s)$, we get (4.37). For (4.38), since $H^1(B_R \times (R^{-1}, R), t^{1-2s}dX) = H^1(B_R \times (R^{-1}, R))$, we first remark that

$$V_{\lambda_i} \rightharpoonup U_\infty \quad \text{strongly in } L^2_{loc}(\mathbb{R}^{N+1}_+, t^{1-2s}dX).$$

Around $t = 0$, we notice that for $\psi \in C^1(\mathbb{R}^{N+1}_+)$,

\[
|\psi(x,t)| \leq |\psi(x,0)| + \int_0^t \tau^{\frac{2s-1}{2}} |\partial_\tau \psi(x,\tau)| \, d\tau
\]

$$\leq |\psi(x,0)| + \left[ \frac{1}{2s} t^{2s} \left( \int_0^t \tau^{1-2s} |\partial_\tau \psi(x,\tau)|^2 \, d\tau \right)^{1/2} \right].$$

Therefore,

$$\int_{B_R \times (0,T)} t^{1-2s} |\psi(X)|^2 \, dX$$

$$\leq 2 \int_{B_R \times (0,T)} t^{1-2s} \left[ |\psi(x,0)|^2 + \frac{t^{2s}}{s} \int_0^t \tau^{1-2s} |\partial_\tau \psi(x,\tau)|^2 \, d\tau \right] \, dX$$

$$\leq C_s \left[ T^{2-2s} ||\psi(\cdot,0)||^2_{L^2(B_R)} + T^2 \int_{B_R \times (0,T)} t^{1-2s} |\partial_\tau \psi(X)|^2 \, dX \right].$$
By the density argument, (4.40) holds for every \( W \in H^1_\text{loc}(\mathbb{R}^{N+1}_+, t^{1-2s} dX) \). Thus, by (4.39),

\[
\limsup_{i \to \infty} \int_{B_R \times (0, R)} t^{1-2s}|V_{\lambda_i} - U_\infty|^2 \, dX \\
= \limsup_{i \to \infty} \left( \int_{B_R \times (0,T)} + \int_{B_R \times (T, R)} \right) t^{1-2s}|V_{\lambda_i} - U_\infty|^2 \, dX \leq C(T^{2-2s} + T^2).
\]

Since \( T \in (0, R) \) is arbitrary and \( C \) is independent of \( T \), (4.38) holds.

Next, we shall prove that \( U_\infty \) satisfies (4.7). For the third property in (4.7), we observe from (2.18) and (4.34) that for each \( \varphi \in C^1_c(\mathbb{R}^{N+1}_+) \),

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla V_{\lambda_i} \cdot \nabla \varphi \, dX = \kappa_s \int_{\mathbb{R}^N} |x|^{t_1} |V_{\lambda_i}(x,0)|^{p-1} V_{\lambda_i}(x,0) \varphi(x,0) \, dx.
\]

By (4.37), we may also suppose that \( V_{\lambda_i}(x,0) \to U_\infty(x,0) \) for a.a. \( x \in \mathbb{R}^N \). Since \( (V_{\lambda_i}(x,0))_i \) is bounded in \( L^{p+1}_\text{loc}(\mathbb{R}^N, |x|^t \, dx) \), a variant of Strauss’ lemma (see Strauss [24, Compactness Lemma 2] and Berestycki and Lions [2, Theorem A.I]) and the fact \( V_{\lambda_i} \rightharpoonup U_\infty \) weakly in \( H^1_\text{loc}(\mathbb{R}^{N+1}_+, t^{1-2s} dX) \) give

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U_\infty \cdot \nabla \varphi \, dX = \kappa_s \int_{\mathbb{R}^N} |x|^{t_1} |U_\infty(x,0)|^{p-1} U_\infty(x,0) \varphi(x,0) \, dx.
\]

In a similar way, by (4.35), we also observe that \( U_\infty \) is stable outside \( B^+_\varepsilon \) for any \( \varepsilon > 0 \), that is, for each \( \psi \in C^1_c(\mathbb{R}^{N+1}_+ \setminus B^+_\varepsilon) \),

\[
\kappa_s p \int_{\mathbb{R}^N} |x|^{t_1} |U_\infty(x,0)|^{p-1} \psi(x,0)^2 \, dx \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \psi|^2 \, dX.
\]

Finally, we prove \( U_\infty(X) = r^{-2s+\ell} U_\infty(r^{-1} X) \). If this is true, then (4.36) gives \( \psi(\omega,0) = U_\infty(x/r,0) \in L^{p+1}(\partial S^+_1) \) and Lemma 4.3 is applicable for \( U_\infty \). Remark that (4.34) implies

\[
(-\Delta)^{s} V_{\lambda}(x,0) = |x|^{t_1} |V_{\lambda}(x,0)|^{p-1} V_{\lambda}(x,0) \quad \text{in} \quad \mathbb{R}^N
\]

and Proposition 2.2 and Lemma 4.2 hold for \( V_{\lambda} \). Hence, for \( R_2 > R_1 > 0 \), by (4.33), (4.34) and Lemma 4.2, we have

\[
0 = \lim_{i \to \infty} \left\{ E(U; \lambda_i R_2) - E(U; \lambda_i R_1) \right\} = \lim_{i \to \infty} \left\{ E(V_{\lambda_i}; R_2) - E(V_{\lambda_i}; R_1) \right\}
\geq \liminf_{i \to \infty} \int_{R_1}^{R_2} \frac{\partial}{\partial r} E(V_{\lambda_i}; r) \, dr.
\]

This together with (4.4), (4.38) and the weak lower semicontinuity of norms yield

\[
0 \geq \liminf_{i \to \infty} \int_{R_1}^{R_2} r^{-2s(p+1)\ell - N} \left( \int_{S^+_r} t^{1-2s} \left( \frac{2s + \ell V_{\lambda_i}}{p-1} \frac{\partial V_{\lambda_i}}{\partial r} \right)^2 \, dS \right) \, dr
\]

\[
= \liminf_{i \to \infty} \int_{B^+_{R_2} \setminus B^+_{R_1}} t^{1-2s} r^{-2s(p+1)\ell - N} \left( \frac{2s + \ell U_\infty}{p-1} \frac{\partial U_\infty}{\partial r} \right)^2 \, dX
\geq \int_{B^+_{R_2} \setminus B^+_{R_1}} t^{1-2s} r^{-2s(p+1)\ell - N} \left( \frac{2s + \ell U_\infty}{p-1} \frac{\partial U_\infty}{\partial r} \right)^2 \, dX.
\]
Noting that $U_\infty \in C^\infty(\mathbb{R}^{N+1}_+)$ thanks to $\text{div}(t^{1-2s}\nabla U_\infty) = 0$ and elliptic regularity, by the arbitrariness of $R_1$ and $R_2$, we have

$$0 = \frac{\partial U_\infty}{\partial r} + \frac{2s + \ell}{p - 1} \frac{U_\infty}{r} = r^{-\frac{2s + \ell}{p - 1}} \frac{\partial}{\partial r} \left( r^{\frac{2s + \ell}{p - 1}} U_\infty \right) \text{ in } \mathbb{R}^{N+1}_+.$$ Integrating this equality with respect to $r$, we obtain

$$U_\infty(X) = r^{-\frac{2s + \ell}{p - 1}} U_\infty(r^{-1} X)$$ and hence, $U_\infty$ satisfies (4.7).

It follows from Lemma 4.3 that $U_\infty \equiv 0$. Since the weak limit does not depend on choices of subsequences, we infer that $V_\lambda \rightharpoonup 0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^{N+1}_+, t^{1-2s}dx)$ and from (4.38) that

$$\int_{B^+_R} t^{1-2s}|V_\lambda|^2 dX \to 0 \quad \text{as} \quad \lambda \to \infty.$$ (4.44)

Recalling (4.34), (4.35), (4.41) and the proof of (2.66) in Lemma 2.9, we see

$$\int_{\mathbb{R}^N \setminus B^+_R(t)} t^{1-2s} |\nabla (V_\lambda \zeta)|^2 dX \leq \frac{p}{p - 1} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |V_\lambda|^2 |\nabla \zeta|^2 dX$$ (4.45)

where $\zeta \in C^1_c(\mathbb{R}^{N+1}_+)$ satisfying

$$\zeta \equiv 1 \quad \text{in } B^+_R \setminus B^+_t, \quad \zeta \equiv 0 \quad \text{in } B^+_R \setminus (\mathbb{R}^{N+1}_+ \setminus B^+_R) \quad \text{for } \lambda^{-1} R_0 < r < R.$$ From (4.44), (4.45) and the property of $\zeta$, we observe that for any $0 < r < R$,

$$\lim_{\lambda \to \infty} \int_{B^+_R \setminus B^+_t} t^{1-2s} |\nabla V_\lambda|^2 dX = 0.$$ (4.46)

Furthermore, by (2.64) with $V_\lambda$, (4.44) and (4.45), for each $0 < r < R$,

$$\lim_{\lambda \to \infty} \int_{B_R \setminus B_r} |x|^{\ell} |V_\lambda(x, 0)|^{p+1} dx = 0.$$ (4.47)

Next, we shall prove $E(U; \lambda) \to 0$ as $\lambda \to \infty$. In view of (4.34), for each $\varepsilon \in (0, 1)$, we have

$$\lambda^{\frac{2(2s + \ell)}{p - 1}} D(U; \lambda) = D(V_\lambda; 1)$$

$$= \frac{1}{2} \int_{B^+_t} t^{1-2s} |\nabla V_\lambda|^2 dX - \frac{\kappa_s}{p + 1} \int_{B^+_t} |x|^{\ell} |V_\lambda(x, 0)|^{p+1} dx$$

$$+ \frac{1}{2} \int_{B^+_1 \setminus B^+_t} t^{1-2s} |\nabla V_\lambda|^2 dX - \frac{\kappa_s}{p + 1} \int_{B^+_1 \setminus B^+_t} |x|^{\ell} |V_\lambda(x, 0)|^{p+1} dx$$

$$= \varepsilon^{N-2s} D(V_\lambda; \varepsilon) + \frac{1}{2} \int_{B^+_1 \setminus B^+_t} t^{1-2s} |\nabla V_\lambda|^2 dX$$

$$- \frac{\kappa_s}{p + 1} \int_{B^+_1 \setminus B^+_t} |x|^{\ell} |V_\lambda(x, 0)|^{p+1} dx$$

$$= \varepsilon^{N-2s} D(V_\lambda; \varepsilon) + \left[ \frac{\varepsilon}{\lambda^{\frac{2(2s + \ell)}{p - 1}}} D(U; \lambda \varepsilon) \right]$$

$$+ \frac{1}{2} \int_{B^+_1 \setminus B^+_t} t^{1-2s} |\nabla V_\lambda|^2 dX - \frac{\kappa_s}{p + 1} \int_{B^+_1 \setminus B^+_t} |x|^{\ell} |V_\lambda(x, 0)|^{p+1} dx.$$ (4.48)
By \((4.32)\) and \((4.46)-(4.48)\), we see that

\[
\limsup_{\lambda \to \infty} \left| \lambda^{\frac{2(2s+\ell)}{p-1}} D(U; \lambda) \right|
\]

\[
= \limsup_{\lambda \to \infty} \left| \varepsilon^{N-\frac{2}{p-1}(s(p+1)+\ell)} \left( \lambda \varepsilon \right)^{\frac{2(2s+\ell)}{p-1}} D(U; \lambda \varepsilon) \right|
\]

\[
+ \frac{1}{2} \int_{B_1^+ \setminus B^\varepsilon_2} t^{1-2s} |\nabla V\lambda|^2 \, dX - \frac{\kappa_s}{p+1} \int_{B_1 \setminus B^\varepsilon} |x|^\ell |V_\lambda(x,0)|^{p+1} \, dx
\]

\[
\leq C_0 \varepsilon^{N-\frac{2}{p-1}(s(p+1)+\ell)}
\]

for some \(C_0 > 0\). Since \(\varepsilon \in (0,1)\) is arbitrary and \(N - \frac{2}{p-1}((p+1)s + \ell) > 0\), we obtain

\[
\lim_{\lambda \to \infty} \lambda^{\frac{2(2s+\ell)}{p-1}} D(U; \lambda) = 0. \quad (4.49)
\]

On the other hand, from \((4.44)\) it follows that

\[
0 = \lim_{\lambda \to \infty} \int_{B^+_2} t^{1-2s} |V\lambda|^2 \, dX = \lim_{\lambda \to \infty} \int_0^2 dr \int_{S^+_r} t^{1-2s} |V\lambda|^2 \, dS.
\]

By choosing a subsequence \((\lambda_i)\),

\[
\int_{S^+_r} t^{1-2s} |V\lambda_i|^2 \, dS \to 0 \quad \text{a.a. } r \in (0,2).
\]

Therefore, there exists an \(r_0 \in (0,2)\) such that \((4.34)\) gives

\[
\lambda_i^{\frac{2(2s+\ell)}{p-1}} H(U; r_0 \lambda_i) = H(V\lambda_i; r_0) = r_0^{-(N+1-2s)} \int_{S^+_{r_0}} t^{1-2s} |V\lambda_i|^2 \, dS \to 0.
\]

With \((4.34)\), \((4.49)\) and the monotonicity of \(E(U; \lambda)\), we have

\[
\lim_{\lambda \to \infty} E(U; \lambda) = \lim_{i \to \infty} E(U; \lambda_i r_0) = 0. \quad (4.50)
\]

We shall prove \(U \equiv 0\). To this end, we shall prove \(E(U; \lambda) \to 0\) as \(\lambda \to 0\). Since \(U \in C(\mathbb{R}^{N+1}_+)\) holds by Lemma 2.3, as \(\lambda \to 0\), we have

\[
0 \leq \lambda^{\frac{2(2s+\ell)}{p-1}} H(U; \lambda) = \lambda^{\frac{2(2s+\ell)}{p-1}} \int_{S^+_N} \sigma_{N+1}^{-2s} U(\lambda \sigma)^2 \, dS \leq C\|U\|^2_{L^\infty(B^+_1)} \lambda^{\frac{2(2s+\ell)}{p-1}} \to 0 \quad (4.51)
\]

and by \((1.2)\),

\[
\lambda^{\frac{2(2s+\ell)}{p-1}-(N-2s)} \int_{B^+_1} |x|^\ell |u|^{p+1} \, dx \leq C\|u\|^{p+1}_{L^\infty(B^+_1)} \lambda^{\frac{2(2s+\ell)}{p-1}+2s+\ell} \to 0. \quad (4.52)
\]
By (4.50) and the monotonicity of $E(U; \lambda)$, we have $E(U; \lambda) \leq 0$ for all $\lambda \in (0, \infty)$. In view of this fact with (4.51) and (4.52), it follows that
\[
\limsup_{\lambda \to +0} \frac{\lambda^{2(2s+\ell) - (N-2s)}}{2} \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 dX
= \limsup_{\lambda \to +0} \left[ E(U; \lambda) + \frac{\lambda^{2(2s+\ell) - (N-2s)}}{p+1} \int_{B_\lambda} |x|\ell |u|^{p+1} dx - \lambda \frac{2(2s+\ell) - (N-2s)}{2(p-1)} H(U; \lambda) \right]
\leq 0,
\]
which yields
\[
\lim_{\lambda \to +0} \lambda^{2(2s+\ell) - (N-2s)} \int_{B_\lambda^+} t^{1-2s} |\nabla U|^2 dX = 0. \tag{4.53}
\]
Now, (4.51)–(4.53) yields $E(U; \lambda) \to 0$ ($\lambda \to 0$). Therefore, $E(U; \lambda) \equiv 0$ thanks to (4.50) and the monotonicity of $E$. Applying the argument similar to (4.42) and (4.43), we see that $U = r^{\frac{2s+\ell}{p-1}} U(r^{-1}X)$. In addition, since $U \in C(\mathbb{R}^{N+1})$, $\psi(\omega, 0) = U(\omega, 0) \in L^\infty(\partial S_1^+) \subset L^\infty(\partial \Sigma_1^+)$ and $u$ (respectively $U$) is stable outside $B_{R_0}$ (respectively $B_{R_0}^+$), (4.7) is satisfied and it follows from Lemma 4.3 that $U \equiv 0$. This completes the proof. \qed

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**References**

[1] B. Barrios and A. Quaas, The sharp exponent in the study of the nonlocal Hénon equation in $\mathbb{R}^N$: a Liouville theorem and an existence result. Calc. Var. Partial Differential Equations **59** (2020), no. 4, Paper No. 114, 22 pp. 3, 45

[2] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I Existence of a ground state, Arch. Ration. Mech. Anal. **82** (4) (1983) 313–345. 42

[3] M. Chipot, M. Chlebík, M. Fila and I. Shafrir, Existence of positive solutions of a semilinear elliptic equation, J. Math. Anal. Appl. **223** (1998), 429–471. 2

[4] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations **32** (2007), 1245–1260. 7

[5] X. Cabrè and Y. Sire, Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates, Ann. Inst. H. Poincaré **31** (2014), 23–53. 13

[6] W. Dai and G. Qin, Liouville type theorems for fractional and higher order Hénon-Hardy type equations via the method of scaling spheres, arXiv:1810.02752 [math.AP]. 3

45
[7] E.N. Dancer, Y. Du and Z. Guo, Finite Morse index solutions of an elliptic equation with supercritical exponent, J. Differential Equations, 250 (2011), 3281–3310. 2, 5

[8] J. Dávila, L. Dupaigne and M. Montenegro, The extremal solution of a boundary reaction problem, Commun. Pure Appl. Anal. 7 (2008), 795–817. 4, 5

[9] J. Dávila, L. Dupaigne and J. Wei, On the fractional Lane–Emden equation, Trans. Amer. Math. Soc. 369 (2017), 6087–6104. 2, 3, 5, 17, 18, 19, 21, 29, 31, 32, 33, 40

[10] F. Demengel and G. Demengel, Functional spaces for the theory of elliptic partial differential equations. Translated from the 2007 French original by Reinie Erné. Universitext. Springer, London; EDP Sciences, Les Ulis, 2012. 8, 41

[11] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math. 136 (2012), no. 5, 521–573. 7

[12] M.M. Fall, Semilinear elliptic equations for the fractional Laplacian with Hardy potential. Nonlinear Anal. 193 (2020), 111311, 29 pp. 4, 37

[13] M.M. Fall and V. Felli, Unique continuation property and local asymptotics of solutions to fractional elliptic equations, Comm. Partial Differential Equations 39 (2014), 354–397. 5, 7, 13, 15, 29, 33

[14] M.M. Fall and V. Felli, Unique continuation properties for relativistic Schrödinger operators with a singular potential. Discrete Contin. Dyn. Syst. 35 (2015), no. 12, 5827–5867. 7

[15] M.M. Fall and T. Weth, Nonexistence results for a class of fractional elliptic boundary value problems, J. Funct. Anal. 263 (2012), 2205–2227. 8

[16] A. Farina, On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$, J. Math. Pures Appl. (9) 87 (2007), 537–561. 2, 4

[17] M. Fazly and J. Wei, On stable solutions of the fractional Hénon-Lane-Emden equation. Commun. Contemp. Math. 18 (2016), no. 5, 1650005, 24 pp. 3, 5

[18] R.L. Frank, E.H. Lieb and R. Seiringer, Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators. J. Amer. Math. Soc. 21 (2008), no. 4, 925–950. 4

[19] J. Harada, Positive solutions to the Laplace equation with nonlinear boundary conditions on the half space, Calc. Var. Partial Differential Equations 50 (2014), 399–435. 2

[20] S. Hasegawa, N. Ikoma and T. Kawakami, On weak solutions to a fractional Hardy–Hénon equation: Part 2: Existence, in preparation. 1, 2, 3, 5

[21] T. Jin, Y.Y. Li and J. Xiong, On a fractional Nirenberg problem, part I: Blow up analysis and compactness of solutions, J. Eur. Math. Soc. 16 (2014), 1111-1171. 7, 13

[22] Y. Li and J. Bao, Fractional Hardy–Hénon equations on exterior domains. J. Differential Equations 266 (2019), no. 2-3, 1153–1175. 2
[23] J.-L. Lions, Théorèmes de trace et d’interpolation. I. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 13 (1959), 389–403. 8

[24] W.A. Strauss, Existence of solitary waves in higher dimensions. Comm. Math. Phys. 55 (1977), no. 2, 149–162. 42

[25] C. Wang and D. Ye, Some Liouville theorems for Hénon type elliptic equations. J. Funct. Anal. 262 (2012), no. 4, 1705–1727 and Corrigendum to ”Some Liouville theorems for Hénon type elliptic equations” [J. Funct. Anal. 262 (4) (2012) 1705–1727] [MR2873856]. J. Funct. Anal. 263 (2012), no. 6, 1766–1768. 2

[26] J. Yang, Fractional Sobolev-Hardy inequality in $\mathbb{R}^N$. Nonlinear Anal. 119 (2015), 179–185. 3