Boundedness of commutators generated by $m$-th Calderón-Zygmund type singular integrals and local Campanato functions on generalized local Morrey spaces

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Abstract Let $T_m$ be the $m$-th Calderón-Zygmund type singular integral. In the paper, we consider the boundedness of $T_m$ on the generalized product local Morrey spaces $LM_{p_1,\varphi_1}^{(x_0)} \times LM_{p_2,\varphi_2}^{(x_0)} \times \cdots \times LM_{p_m,\varphi_m}^{(x_0)}$. And, the boundedness of the commutators of $T_m$ with local Campanato functions is obtained, also.

Key words $m$-th Calderón-Zygmund type singular integral, commutator, local Campanato function, generalized local Morrey space

1 Introduction

In recent years, the multilinear singular integrals have been attracting attention and great developments have been achieved (see [1-11]). The study for the multilinear singular integrals is motivated not only by a mere quest to generalize the theory of linear operators but also by their natural appearance in analysis.

Meanwhile, the commutators generated by the multilinear singular integral and BMO functions or Lipschitz functions also attract much attention, since the commutator is more singular than the singular integral operator itself.

Moreover, the classical Morrey space $M_{p,\lambda}$ were first introduced by Morrey in [11] to study the local behavior of solutions to second order elliptic partial differential equations. In [12], the authors studied the boundedess of the multilinear Calderón-Zygmund singular integral on the classical Morrey space $M_{p,\lambda}$. And, in [13], the authors introduced the local generalized Morrey space $LM_{p,\varphi}^{(x_0)}$, and they also studied the boundedness of the homogeneous singular integrals with rough kernel on these spaces.

Motivated by the works of [12,13], we are going to consider the boundedness of the multilinear Calderón-Zygmund singular integral and its commutator on the local generalized Morrey space $LM_{p,\varphi}^{(x_0)}$.

Now, let us give some related notations.

We are going to be working in $\mathbb{R}^n$. Let $m \in \mathbb{N}$ and $K(y_0, y_1, \ldots, y_m)$ be a function defined away from the diagonal $y_0 = y_1 = \ldots = y_m$ in $(\mathbb{R}^n)^{m+1}$. Let $T_m$ be a multilinear

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operator which was initially defined on the m-fold product of Schwartz space $S(\mathbb{R}^n)$ and take its values in the space of tempered distributions $S'(\mathbb{R}^n)$ and such that for $K$, the integral representation below is valid:

$$T_m(\tilde{f})(x) = T_m(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m)f_1(y_1) \cdots f_m(y_m)dy_1 \ldots dy_m,$$

whenever $f_i, i = 1, \ldots, m$, are smooth functions with compact support and $x \notin \cap_{i=1}^m \text{supp} f_i$.

Moreover, if the kernel $K$ satisfies the following size and smoothness estimates:

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{C}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}},$$

for all $(y_0, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ away from the diagonal;

$$|K(y_0, \ldots, y_i, \ldots, y_m) - K(y_0, \ldots, y'_i, \ldots, y_m)| \leq \frac{C|y_i - y'_i|^{\epsilon}}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn+\epsilon}},$$

for some $C > 0$ and $\epsilon > 0$, whenever $0 \leq j \leq m$ and $|y_i - y'_i| \leq 1/2 \max_{0 \leq k \leq m} |y_i - y_k|$, then the kernel is called a $m$-th Calderón-Zygmund kernel and the collection of such functions is denoted by $m - CZK(C, \epsilon)$. Let $T_m$ be as in (1.1) with a $m - CZK(C, \epsilon)$ kernel, then $T_m$ is called a $m$-th Calderón-Zygmund type singular integral and the collection of these operators is denoted by $m - CZO$.

Now, we define the commutators generated by the $m$-th multilinear Calderón-Zygmund type singular integral as follows.

Let $\tilde{b} = (b_1, \ldots, b_m)$ be a finite family of locally integrable functions, then the commutators generated by the $m$-th Calderón-Zygmund type singular integral and $\tilde{b}$ is defined by:

$$T_m^\tilde{b}(\tilde{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^m (b_i(x) - b_i(y_i))f_i(y_i)dy_1 \ldots dy_m.$$

In the following, we will establish the boundedness of $T_m$ on generalized product local Morrey spaces. And, we also consider the boundedness of the commutators generated by the $m$-th Calderón-Zygmund type singular integral $T_m$ and the local Campanato function on generalized product local Morrey spaces.

### 2 Some notations and lemmas

**Definition 2.1** Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p \leq \infty$. For any fixed $x_0 \in \mathbb{R}^n$, a function $f \in L_{loc}^q$ is said to belong to the local Morrey space, if

$$\|f\|_{LM_{p, \varphi}(x_0)} = \sup_{r > 0} \varphi^{-1}(x_0, r)|B(x_0, r)|^{-\frac{1}{p}}\|f\|_{L_p(B(x_0, r))} < \infty.$$
And, we denote

$$\text{LM}^{(x_0)}_{p,\varphi} = \text{LM}^{(x_0)}(\mathbb{R}^n) = \{ f \in L^0_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\text{LM}^{(x_0)}_{p,\varphi}} < \infty \}. $$

According to this definition, we recover the local Morrey space \(\text{LM}^{(x_0)}_{p,\varphi}\) under the choice \(\varphi(x_0, r) = r^{\frac{n}{p}}\).

**Definition 2.2** [13] Let \(1 \leq q < \infty\) and \(0 \leq \lambda < 1/n\). A function \(f \in L^q_{\text{loc}}(\mathbb{R}^n)\) is said to belong to the space \(LC^{(x_0)}_{q,\lambda}\) (local Campanato space), if

$$\|f\|_{LC^{(x_0)}_{q,\lambda}} = \sup_{r>0} \left( \frac{1}{|B(x_0, r)|^{1+\lambda q}} \int_{B(x, r)} |f(y) - f_{B(x_0, r)}|^q dy \right)^{1/q} < \infty,$$

where

$$f_{B(x_0, r)} = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} f(y) dy.$$

Define

$$LC^{(x_0)}_{q,\lambda}(\mathbb{R}^n) = \{ f \in L^q_{\text{loc}}(\mathbb{R}^n) : \|f\|_{LC^{(x_0)}_{q,\lambda}} < \infty \}.$$

**Remark.** [13] Note that, the central BMO space \(CBMO_q(\mathbb{R}^n) = LC^{(0)}_{q,0}(\mathbb{R}^n)\), \(CBMO^{(x_0)}_q(\mathbb{R}^n) = LC^{(x_0)}_{q,0}(\mathbb{R}^n)\), and \(BMO_q(\mathbb{R}^n) \subset \bigcap_{q > 1} CBMO^{(x_0)}_q(\mathbb{R}^n)\). Moreover, one can imagine that the behavior of \(CBMO^{(x_0)}_q(\mathbb{R}^n)\) may be quite different from that of \(BMO(\mathbb{R}^n)\), since there is no analogy of the John-Nirenberg inequality of \(BMO\) for the space \(CBMO^{(x_0)}_q(\mathbb{R}^n)\).

**Lemma 2.1** Let \(1 < q < \infty\), \(0 < r_2 < r_1\) and \(b \in LC^{(x_0)}_{q,\lambda}\), then

$$\left( \frac{1}{|B(x_0, r_1)|^{1+\lambda q}} \int_{B(x_0, r_1)} |b(x) - b_{B(x_0, r_2)}|^q dx \right)^{1/q} \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) \|b\|_{LC^{(x_0)}_{q,\lambda}}. \quad (2.1)$$

And, from this inequality, we have

$$|b_{B(x_0, r_1)} - b_{B(x_0, r_2)}| \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) |B(x_0, r_1)|^{\lambda} \|b\|_{LC^{(x_0)}_{q,\lambda}}. \quad (2.2)$$

In this section, we are going to use the following statement on the boundedness of the weighted Hardy operator:

$$H_w g(t) := \int_t^\infty g(s) w(s) ds, \ 0 < t < \infty,$$

where \(w\) is a fixed function non-negative and measurable on \((0,\infty)\).
Moreover, if \( \varphi_1 \cdot \cdot \cdot + \varphi_n \) functions on \((0, \infty)\). The inequality
\[
\text{ess sup}_{t > 0} v(2t)H_w g(t) \leq \text{Cess sup}_{t > 0} v_1(t)g(t)
\] (2.3)
holds for some \( C > 0 \) and all non-negative and non-decreasing \( g \) on \((0, \infty)\) if and only if
\[
B : \text{ess sup}_{t > 0} v_2(t) \int_t^\infty \frac{w(s)ds}{\text{ess sup}_{s < \tau < \infty} v_1(\tau)}ds < \infty.
\]
Moreover, if \( \tilde{C} \) is the minimum value of \( C \) in (2.3), then \( \tilde{C} = B \).

**Lemma 2.2** [14] [15] Let \( v_1, v_2 \) and \( w \) be positive almost everywhere and measurable functions on \((0, \infty)\). The inequality
\[
\text{ess sup}_{t > 0} v_1(2t)H_w g(t) \leq C \text{ess sup}_{t > 0} v_1(t)g(t)
\]
holds for some \( C > 0 \) and all non-negative and non-decreasing \( g \) on \((0, \infty)\) if and only if
\[
B : \text{ess sup}_{t > 0} v_2(t) \int_t^\infty \frac{w(s)ds}{\text{ess sup}_{s < \tau < \infty} v_1(\tau)}ds < \infty.
\]
Moreover, if \( \tilde{C} \) is the minimum value of \( C \) in (2.3), then \( \tilde{C} = B \).

**Lemma 2.3** [2] Let \( T_m \) be a \( m - \text{CZO} \). Suppose that \( 1 \leq p_1, \cdots, p_m < \infty \) and \( 1/p = 1/p_1 + \cdots + 1/p_m \). If \( p_i > 1, i = 1, \cdots, m \), then there exists a constant \( C > 0 \), such that
\[
\|T_m \vec{f}\|_{L^p(B(x_0,r))} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}}.
\]

3 M-th Calderón-Zygmund type singular integral operator on generalized product local Morrey space

**Theorem 3.1** Let \( x_0 \in \mathbb{R}^n, 1 < p, p_1, p_2, \ldots, p_m < \infty \), such that \( 1/p = 1/p_1 + 1/p_2 + \cdots + p_m \). Then the inequality
\[
\|T_m \vec{f}\|_{L^p(B(x_0,r))} \lesssim r^{n/p} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))}t^{-n/p-1}dt
\]
holds for any ball \( B(x_0,r) \) and all \( f_i \in L^p_{\text{loc}}(\mathbb{R}^n), i = 1, 2, \ldots, m \).

**Proof.** Without loss of generality, it is suffice to show that the conclusion holds for \( T_2(f_1, f_2) \).

Let \( B = B(x_0, r) \). And, we write \( f_1 = f_1^0 + f_1^\infty \) and \( f_2 = f_2^0 + f_2^\infty \), where \( f_i^0 = f_i \chi_{2B}, f_i^\infty = f_i \chi_{(2B)^c} \), for \( i = 1, 2 \). Thus, we have
\[
\|T_2(f_1, f_2)\|_{L^p(B(x_0,r))} \leq \|T_2(f_1^0, f_2^0)\|_{L^p(B)} + \|T_2(f_1^0, f_2^\infty)\|_{L^p(B)} + \|T_2(f_1^\infty, f_2^0)\|_{L^p(B)} + \|T_2(f_1^\infty, f_2^\infty)\|_{L^p(B)} =: I + II + III + IV.
\]

Using the \( L^p \) boundedness of \( T_2 \) (Lemma 2.3), we have
\[
I \lesssim \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B)} \int_{2r}^\infty \frac{dt}{t^{n/p+1}}
\]
\[
\lesssim \frac{1}{2^p} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \int_{2r}^\infty \frac{dt}{t^{n/p+1}}.
\]

(3.1)
Moreover, when $x \in B(x_0, r)$ and $y \in (2B)^c$, we have

$$
\frac{1}{2} |x_0 - y| \leq |x - y| \leq \frac{3}{2} |x_0 - y|.
$$

(3.2)

Then, it follows from (1.2) that

$$
|T_2(f_1^0, f_2^\infty)(x)| \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1(y_1)||f_2^\infty(y_2)|}{|x - y_1| + |x - y_2|^{2n}} dy_1 dy_2
\lesssim \int_{2B} |f_1(y_1)| dy_1 \int_{(2B)^c} \frac{|f_2(y_2)|}{|x_0 - y_2|^{2n}} dy_2
\lesssim \int_{2B} |f_1(y_1)| dy_1 \int_{(2B)^c} |f_2(y_2)| \left[ \int_0^\infty \frac{dt}{t^{2n+1}} \right] dy_2
\lesssim \|f_1\|_{L^{p_1}(2B)} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}.
$$

(3.3)

where $1/p = 1/p_1 + 1/p_2$.

Thus,

$$
II = \|T_2(f_1^0, f_2^\infty)\|_{L^p(B)} \lesssim r^{n/p} \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}.
$$

(3.4)

Similarly, we have

$$
III = \|T_2(f_1^\infty, f_2^0)\|_{L^p(B)} \lesssim r^{n/p} \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}.
$$

Moreover, similar to the estimate of (3.3), we have

$$
|T_2(f_1^\infty, f_2^\infty)(x)| \lesssim \int_{(2B)^c} \int_{(2B)^c} \frac{|f_1(y_1)||f_2(y_2)|}{|x_0 - y_1| + |x_0 - y_2|^{2n}} dy_1 dy_2
\lesssim \int_{2B} |f_1(y_1)| dy_1 \int_{(2B)^c} |f_2(y_2)| \left[ \int_0^\infty \frac{dt}{t^{2n+1}} \right] dy_2
\lesssim \int_{2B} |f_1(y_1)| dy_1 \int_{B(x_0,t)} |f_2(y_2)| \left[ \int_{B(x_0,t)} \frac{dt}{t^{2n+1}} \right] dy_2
\lesssim \int_{2B} \|f\|_{L^{p_1}(B(x_0,t))} \|f\|_{L^{p_1}(B(x_0,t))} \|f\|_{L^{p_1}(B(x_0,t))} \frac{dt}{t^{n/p+1}}.
$$

Thus,

$$
IV = \|T_2(f_1^\infty, f_2^\infty)\|_{L^p(B)} \lesssim r^{n/p} \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} \frac{dt}{t^{n/p+1}}.
$$

(3.5)
Combining the above estimates, we obtain
\[
\|T_2(f_1, f_2)\|_{L^p(B)} \lesssim r^{n/p} \int_{2r}^{\infty} \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} \frac{dt}{t^{n/p+1}}.
\]

**Theorem 3.2** Let \(x_0 \in \mathbb{R}^n\), \(1 < p, p_1, p_2, \ldots, p_m < \infty\) such that \(1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m\). If functions \(\varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, +\infty), (i = 1, 2, \cdots, m)\) satisfy the condition
\[
\int_r^{\infty} \inf_{t < s < \infty} \frac{\prod_{i=1}^m \varphi_i(x_0, s) s^{n/p}}{t^{n/p+1}} dt \leq C \psi(x_0, r),
\]
where constant \(C > 0\) doesn’t depend on \(r\). Then the operator \(T_m\) is bounded from the product space \(LM^{p_1}_{x_1} \times LM^{p_2}_{x_2} \times \cdots \times LM^{p_m}_{x_m}\) to \(LM_{x_0}\). Moreover, the following inequality
\[
\|T_m(\vec{f})\|_{LM^{p_1}_{x_1} \times \cdots \times LM^{p_m}_{x_m}} \lesssim \prod_{i=1}^m \|f_i\|_{LM^{p_i}_{x_i, \varphi_i}}.
\]
holds.

**Proof.** Taking \(v_1(r) = \prod_{i=1}^m \varphi_i^{-1}(x_0, r)r^{-n/p}, v_2(r) = \psi^{-1}(x_0, r), g(r) = \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0, r))}\) and \(w(r) = r^{-n/p-1}\), then we have
\[
\text{ess sup}_{t > 0} v_2(t) \int_t^{\infty} \frac{w(s)ds}{\text{ess sup}_{s < \tau < \infty} v_1(\tau)} < \infty.
\]
Thus, by Lemma 2.2, we have
\[
\text{ess sup}_{t > 0} v_2(t) H_{\psi} g(t) \leq C \text{ess sup}_{t > 0} v_1(t) g(t). \tag{3.7}
\]
Therefore, from Theorem 3.1 and (3.7), it follows that
\[
\|T_m(\vec{f})\|_{LM^{p_1}_{x_1} \times \cdots \times LM^{p_m}_{x_m}} \\
= \sup_{r > 0} \psi^{-1}(x_0, r)|B(x_0, r)|^{-1/p} \|T_m(\vec{f})\|_{L^{p}(B(x_0, r))} \\
\lesssim \sup_{r > 0} \psi^{-1}(x_0, r)|B(x_0, r)|^{-1/p} r^{-n/p} \int_{2r}^{\infty} \|f_i\|_{L^{p_i}(B(x_0, t))} t^{-n/p-1} dt \\
\lesssim \sup_{r > 0} \prod_{i=1}^m \varphi_i^{-1}(x_0, r)^{-n/p} \|f_i\|_{L^{p_i}(B(x_0, r))} \\
\lesssim \sup_{r > 0} \prod_{i=1}^m \varphi_i^{-1}(x_0, r)^{-n/p} \|f_i\|_{L^{p_i}(B(x_0, r))} \\
= \prod_{i=1}^m \|f_i\|_{LM^{p_i}_{x_i, \varphi_i}}.
\]
4 Commutators generated by m-th Calderón Zygmund type
singular integral operators and local Campanato functions

Theorem 4.1 Let \( x_0 \in \mathbb{R}^n, 1 < p, p_i, q_i < \infty (i = 1, 2, \ldots, m) \) such that \( 1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m + 1/q_1 + 1/q_2 + \cdots + 1/q_m \) and \( b_i \in LC_{q_i, \lambda_i}^{q(x_0)} \) for \( 0 < \lambda_i < 1/n, i = 1, 2, \ldots, m. \)
Then the inequality
\[
\|T_m^\circ (\tilde{f})\|_{L^p(B(x_0,r))} \lesssim \prod_{i=1}^m \|b_i\|_{LC_{q_i, \lambda_i}}^{r^n/p} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^m t^n \sum_{i=1}^m \lambda_i - \sum_{i=1}^m 1/p_i - 1 \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))} dt
\]
holds for any ball \( B(x_0, r) \) and all \( f_i \in L^{p_i}_{loc}(\mathbb{R}^n), i = 1, 2, \ldots, m. \)

Proof. Without loss of generality, it is suffice for us to show that the conclusion holds for \( m = 2. \)

Let \( \mathcal{B} = B(x_0, r), f_1 = f_1^0 + f_1^\infty \) and \( f_2 = f_2^0 + f_2^\infty, \) where \( f_i^0 \) and \( f_i^\infty \) are as in the proof of Theorem 3.1, for \( i = 1, 2. \) Thus, we have
\[
T_2^{(b_1, b_2)}(f_1, f_2)(x) = T_2^{(b_1, b_2)}(f_1^0, f_2^\infty)(x) + T_2^{(b_1, b_2)}(f_1^\infty, f_2^0)(x) + T_2^{(b_1, b_2)}(f_1^\infty, f_2^\infty)(x).
\]

So,
\[
\|T_2^{(b_1, b_2)}(f_1, f_2)\|_{L^p(\mathcal{B})} \leq \|T_2^{(b_1, b_2)}(f_1^0, f_2^\infty)\|_{L^p(\mathcal{B})} + \|T_2^{(b_1, b_2)}(f_1^\infty, f_2^0)\|_{L^p(\mathcal{B})} + \|T_2^{(b_1, b_2)}(f_1^\infty, f_2^\infty)\|_{L^p(\mathcal{B})} =: I + II + III + IV.
\]

Let us estimate \( I, II, III \) and \( IV, \) respectively.

Since,
\[
(b_1(x) - b_1(y))(b_2(x) - b_2(y)) = (b_1(x) - (b_1)_B)(b_2(x) - (b_2)_B) - (b_1(x) - (b_1)_B)(b_2(y) - (b_2)_B) - (b_1(y) - (b_1)_B)(b_2(x) - (b_2)_B) + (b_1(y) - (b_1)_B)(b_2(y) - (b_2)_B).
\]

Then,
\[
\|T_2^{(b_1, b_2)}(f_1^0, f_2^0)\|_{L^p(\mathcal{B})} = \|(b_1 - (b_1)_B)(b_2 - (b_2)_B)T_2(f_1^0, f_2^0)\|_{L^p(\mathcal{B})} + \|(b_1 - (b_1)_B)T_2(f_1^0, f_2^0)(b_2 - (b_2)_B)\|_{L^p(\mathcal{B})} + \|(b_2 - (b_2)_B)T_2((b_1 - (b_1)_B)f_1^0, f_2^0)\|_{L^p(\mathcal{B})} + \|T_2((b_1 - (b_1)_B)(b_2 - (b_2)_B))f_1^0, f_2^0)\|_{L^p(\mathcal{B})} =: I_1 + I_2 + I_3 + I_4.
\]
Let $1 < \tilde{p}, \tilde{q} < \infty$, such that $1/\tilde{p} = 1/p_1 + 1/p_2$ and $1/\tilde{q} = 1/q_1 + 1/q_2$. Then, using the Hölder’s inequality and Lemma 2.3, we have

$$I_1 \lesssim \|(b_1 - (b_1)_B)(b_2 - (b_2)_B)\|_{L^\tilde{p}(B)}\|T_2(f_1^0, f_2^0)\|_{L^\tilde{p}(B)}$$

$$\lesssim \|b_1 - (b_1)_B\|_{L^q(B)}\|b_2 - (b_2)_B\|_{L^r(B)}\|f_1\|_{L^{p_1}(2B)}\|f_2\|_{L^{p_2}(2B)}$$

$$\lesssim \|b_1 - (b_1)_B\|_{L^{q_1}(B)}\|b_2 - (b_2)_B\|_{L^{r_2}(B)r^{(1/p_1 + 1/p_2)n}}$$

$$\times \int_{2r}^\infty \|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}\frac{dt}{t^{(1/p_1 + 1/p_2)n+1}}\text{ (4.3)}$$

Similarly,

$$I_2 \lesssim \|b_1 - (b_1)_B\|_{L^{q_1}(B)}\|b_2\|_{L^{r_2}(B)}\|f_1\|_{L^{p_1}(B)}\|f_2\|_{L^{p_2}(B)}$$

$$\lesssim \|b_1 - (b_1)_B\|_{L^{q_1}(B(2B))}\|f_1\|_{L^{p_1}(B(2B))}\|f_2\|_{L^{p_2}(B(2B))}\text{ (4.4)}$$

where $1 < s < \infty$, such that $1/s = 1/p_2 + 1/q_2 = 1/\tau - 1/p_1$.

From Lemma 2.1, it is easy to see that

$$\|b_i - (b_i)_B\|_{L^{q_i}(B)} \leq C_T^{n/q_i + n\lambda_i}\|b_i\|_{L^{q_i}(x_0)},$$

and

$$\|b_i - (b_i)_B\|_{L^{q_i}(2B)} \leq \|b_i - (b_i)_B\|_{L^{q_i}(B)} + \|(b_i)_B - (b_i)_B\|_{L^{q_i}(2B)} \leq C_T^{n/q_i + n\lambda_i}\|b_i\|_{L^{q_i}(x_0)},$$

for $i = 1, 2$.

Then,

$$I_2 \lesssim \|b_1\|_{L^{q_1}(x_0)}\|b_2\|_{L^{q_2}(x_0)}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}\frac{dt}{t^{(1/p_1 + 1/p_2)n+1}}\text{ (4.5)}$$

Similarly,

$$I_3 \lesssim \|b_1\|_{L^{q_1}(x_0)}\|b_2\|_{L^{q_2}(x_0)}\|f_1\|_{L^{p_1}(B(x_0,t))}\|f_2\|_{L^{p_2}(B(x_0,t))}\frac{dt}{t^{(1/p_1 + 1/p_2)n+1}}$$

Moreover, let $1 < \tau_1, \tau_2 < \infty$, such that $1/\tau_1 = 1/p_1 + 1/q_1$ and $1/\tau_2 = 1/p_2 + 1/q_2$. It is easy to see that $1/p = 1/\tau_1 + 1/\tau_2$. Then by Lemma 2.3, Hölder’s inequality and (4.5),
we obtain
\[ I_4 \lesssim \| (b_1 - (b_1)_B) f_2^0 \| L^1(\mathbb{R}^n) \| (b_2 - (b_2)_B) f_2^0 \| L^2(\mathbb{R}^n) \]
\[ \lesssim \| b_1 - (b_1)_B \| L^q_1(2\mathbb{B}) \| b_2 - (b_2)_B \| L^{r_2}(2\mathbb{B}) \| f_1 \| L^{p_1}(2\mathbb{B}) \| f_2 \| L^{p_2}(2\mathbb{B}) \]
\[ \lesssim \| b_1 \|_{L^{c_0}(\mathbb{R}^n)} \| b_2 \|_{L^{c_0}(\mathbb{R}^n)} \| \| (\phi_1(\mathbb{R}^n)) \|^{r/n} \times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \left( (\lambda_1 + \lambda_2)^n - (\frac{1}{p_1} + \frac{1}{p_2}) n - 1 \right) \| f_1 \|_{L^{p_1}(B(x_0,t))} \| f_2 \|_{L^{p_2}(B(x_0,t))} dt. \]

Therefore, combining the estimates of $I_1, I_2, I_3$ and $I_4$, we have
\[
I \lesssim \| b_1 \|_{L^{c_0}(\mathbb{R}^n)} \| b_2 \|_{L^{c_0}(\mathbb{R}^n)} \| \| (\phi_1(\mathbb{R}^n)) \|^{r/n} \times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \left( (\lambda_1 + \lambda_2)^n - (\frac{1}{p_1} + \frac{1}{p_2}) n - 1 \right) \| f_1 \|_{L^{p_1}(B(x_0,t))} \| f_2 \|_{L^{p_2}(B(x_0,t))} dt. \]

Let us estimate $II$.

It’s analogous to (4.2), we have
\[
\| \mathbb{T}_2(b_1, b_2) (f_1^0, f_2^\infty) \|_{L^p(B)} = \| (b_1 - (b_1)_B) (b_2 - (b_2)_B) T_2(f_1^0, f_2^\infty) \|_{L^p(B)} + \| (b_1 - (b_1)_B) T_2(f_1^0, (b_2 - (b_2)_B) f_2^\infty) \|_{L^p(B)} + \| (b_2 - (b_2)_B) T_2((b_1 - (b_1)_B) f_1^0, (b_2 - (b_2)_B) f_2^\infty) \|_{L^p(B)} \]
\[=: II_1 + II_2 + I_3 + I_4. \]

Let $1 < p, q < \infty$, such that $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$. Then, using the Hölder’s inequality and (3.4), we have
\[
II_1 \lesssim \| (b_1 - (b_1)_B) (b_2 - (b_2)_B) \|_{L^q_1(2\mathbb{B})} \| T_2(f_1^0, f_2^\infty) \|_{L^p(B)} \]
\[\lesssim \| b_1 \|_{L^{c_0}(\mathbb{R}^n)} \| b_2 \|_{L^{c_0}(\mathbb{R}^n)} \| (\phi_1(\mathbb{R}^n)) \|^{r/n} \times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \left( (\lambda_1 + \lambda_2)^n - (\frac{1}{p_1} + \frac{1}{p_2}) n - 1 \right) \| f_1 \|_{L^{p_1}(B(x_0,t))} \| f_2 \|_{L^{p_2}(B(x_0,t))} dt. \]

Moreover, using (1.2) and (3.2), we have
\[
| T_2(f_1^0, (b_2 - (b_2)_B) f_2^\infty)(x) | \lesssim \int_{2\mathbb{B}} | f_1(y_1) | dy_1 \int_{2\mathbb{B}} \frac{| b_2(y_2) - (b_2)_B | | f_2(y_2) |}{| x_0 - y_2 |^{2n}} dy_2. \]

It’s obvious that
\[
\int_{2\mathbb{B}} | f_1(y_1) | dy_1 \lesssim \| f_1 \|_{L^p_1(2\mathbb{B})} |2B|^{1-1/p_1}, \]

(4.9)
\begin{align*}
& \int_{(2B)^t} \frac{|b_2(y_2) - (b_2)_{B}| |f_2(y_2)|}{|x_0 - y_2|^{2n}} dy_2 \\
& \lesssim \int_{(2B)^t} |b_2(y_2) - (b_2)_{B}| |f_2(y_2)| \left[ \int_{|x_0 - y_2|}^{\infty} \frac{dt}{t^{2n+1}} \right] dy_2 \\
& \lesssim \int_{2r}^{\infty} \|b_2(y_2) - (b_2)_{B(x_0,t)}\| \|f_2\| \|L^p(B(x_0,t))\| \|B(x_0,t)^{1-(1/p_2+1/q_2)} \frac{dt}{t^{2n+1}} \\
& \quad + \int_{2r}^{\infty} \|b_2\|_{L^q_2} \frac{dt}{t^{2n+1}} + \|b_2\|_{L^q_{2,\alpha_2}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 |f_2| \|L^p_2(B(x_0,t))\| \|B(x_0,t)^{1-1/p_2} \frac{dt}{t^{2n+1}} \\
& \lesssim \|b_2\|_{L^q_{2,\alpha_2}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 t^{-n+n_2-n/p_2-1} |f_2| \|L^p_2(B(x_0,t))\| dt.
\end{align*}

Therefore, from (4.9) and (4.10), it follows that
\[
|T_2(f_1^0, (b_2 - (b_2)_{B})f_2^\infty)(x)| \\
\lesssim \|b_2\|_{L^q_{2,\alpha_2}} \|f_1\| \|L^p_1(2B)\| \|B\|^{1-1/p_1} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 t^{-n+n_2-n/p_2-1} |f_2| \|L^p_2(B(x_0,t))\| dt \\
\lesssim \|b_2\|_{L^q_{2,\alpha_2}} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \lambda_n \|L^p_1(2B)\| \|f_2\| \|L^p_2(B(x_0,t))\| dt.
\]

Thus, let $1 < \tau < \infty$, such that $1/p = 1/q_1 + 1/\tau$, then similarly to the estimate of (4.3), we have
\[
II_2 = \|(b_2 - (b_1)_{B})T_2(f_1^0, (b_2 - (b_2)_{B})f_2^\infty)\|_{L^p(B)} \\
\lesssim \|b_2 - (b_1)_{B}\| \|L^q_1(B)\| \|T_2(f_1^0, (b_2 - (b_2)_{B})f_2^\infty)\|_{L^p(B)} \\
\lesssim \|b_2\|_{L^q_{1,\alpha_1}} \|b_2\|_{L^q_{2,\alpha_2}} \|B\|^{1-1/q_1+1/\tau} \\
\times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \lambda_n \|f_1\| \|L^p_1(B(x_0,t))\| \|f_2\| \|L^p_2(B(x_0,t))\| dt \\
\lesssim \|b_2\|_{L^q_{1,\alpha_1}} \|b_2\|_{L^q_{2,\alpha_2}} \|f_2\| \|L^p_1(B(x_0,t))\| \|f_2\| \|L^p_2(B(x_0,t))\| dt.
\]

Similarly, we have
\[
II_3 \lesssim \|b_2\|_{L^q_{1,\alpha_1}} \|b_2\|_{L^q_{2,\alpha_2}} \left( \frac{t}{r} \right)^{n/p} \\
\times \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \lambda_n \|f_1\| \|L^p_1(B(x_0,t))\| \|f_2\| \|L^p_2(B(x_0,t))\| dt.
\]
Let us estimate $II_4$.

Since,

$$|T_2((b_1 - (b_1)_B)f_1^0, (b_2 - (b_2)_B)f_2^\infty)(x)| \leq \int_{2B} |b_1(y_1) - (b_1)_B||f_1(y_1)|dy_1 \int_{(2B)^c} \frac{|b_2(y_2) - (b_2)_B||f_2(y_2)|}{|x_0 - y_2|^{2n}}dy_2,$$

and

$$\int_{2B} |b_1(y_1) - (b_1)_B||f_1(y_1)|dy_1 \lesssim \|b_1\|_{LC^{(x_0)}_{q_1,\lambda_1}} \|B\|^\lambda_1 + 1 - p_1/2 \|f_1\|_{L^p(2B)} \|f_2\|_{L^{p_2}(B(x_0, t))}. \quad (4.12)$$

Then, by (4.10) and (4.12), we have

$$|T_2((b_1 - (b_1)_B)f_1^0, (b_2 - (b_2)_B)f_2^\infty)(x)| \lesssim \|b_1\|_{LC^{(x_0)}_{q_1,\lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2,\lambda_2}} \int_{2r} \left(1 + \ln \frac{t}{r}\right) t^{(\lambda_1 + \lambda_2) - n(1/p_1 + 1/p_2) - 1} \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} dt.$$

Therefore,

$$II_4 = |T_2((b_1 - (b_1)_B)f_1^0, (b_2 - (b_2)_B)f_2^\infty)(x)| \lesssim \|b_1\|_{LC^{(x_0)}_{q_1,\lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2,\lambda_2}} \int_{2r} \left(1 + \ln \frac{t}{r}\right) t^{(\lambda_1 + \lambda_2) - n(1/p_1 + 1/p_2) - 1} \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} dt.$$

Combining the estimates of $II_1 - II_4$, we have

$$II \lesssim \|b_1\|_{LC^{(x_0)}_{q_1,\lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2,\lambda_2}} \int_{2r} \left(1 + \ln \frac{t}{r}\right) t^{(\lambda_1 + \lambda_2) - n(1/p_1 + 1/p_2) - 1} \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} dt.$$

Similarly,

$$III \lesssim \|b_1\|_{LC^{(x_0)}_{q_1,\lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2,\lambda_2}} \int_{2r} \left(1 + \ln \frac{t}{r}\right) t^{(\lambda_1 + \lambda_2) - n(1/p_1 + 1/p_2) - 1} \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} dt.$$

For $IV$, we have

$$\|T_2((b_1, b_2)(f_1^\infty, f_2^\infty))\|_{L^p(B)} \leq \|(b_1 - (b_1)_B)(b_2 - (b_2)_B)T_2(f_1^\infty, f_2^\infty)\|_{L^p(B)} + \|(b_1 - (b_1)_B)T_2(f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)\|_{L^p(B)} + \|(b_2 - (b_2)_B)T_2((b_1 - (b_1)_B)f_1^\infty, f_2^\infty)\|_{L^p(B)} + \|T_2((b_1 - (b_1)_B)f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)\|_{L^p(B)} =: IV_1 + IV_2 + IV_3 + IV_4.$$
Let us estimate $IV_1$, $IV_2$, $IV_3$ and $IV_4$, respectively.

Let $1 < \tau < \infty$, such that $1/p = 1/q_1 + 1/q_2 + 1/\tau$. Then, from Hölder’s inequality and (3.5), we get

$$IV_1 \lesssim \|b_1 - (b_1)_B\|_{L^{p_1}(B)} \|b_2 - (b_2)_B\|_{L^{p_2}(B)} \|T_2(f_1^\infty, f_2^\infty)\|_{L^\tau(B)}$$

$$\lesssim \|b_1\|_{LC^{(x_0)}_{\epsilon_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{\epsilon_2, \lambda_2}} \|B\|_{(\lambda_1 + \lambda_2) + (1/q_1 + 1/q_2) + 1/\tau}$$

$$\times \int_0^\infty \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} t^{-(n/(p_1 + 1/p_2)) - 1 - 1} \, dt$$

$$\lesssim \|b_1\|_{LC^{(x_0)}_{\epsilon_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{\epsilon_2, \lambda_2}} \|B\|_{r^{n/p}} \times \int_0^\infty \left(1 + \ln \frac{t}{r}\right)^{(\lambda_1 + \lambda_2) - 1} t^{-(n/(p_1 + 1/p_2)) - 1} \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} \, dt.$$

Moreover, by (1.2) and (3.2), we have

$$|T_2(f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)(x)|$$

$$\lesssim \int_{(2B)^c} \int_{(2B)^c} \frac{|b_2(y_2) - (b_2)_B|}{(|x_0 - y_1| + |x_0 - y_2|)^{2n}} dy_1 dy_2$$

$$\lesssim \int_{(2B)^c} \int_{(2B)^c} |f_1(y_1)||b_2(y_2) - (b_2)_B||f_2(y_2)| \left[ \int_{|x_0 - y_1| + |x_0 - y_2|}^\infty \frac{dt}{t^{2n + 1}} \right] dy_1 dy_2$$

$$\lesssim \int_{2r} \left[ \int_{B(x_0, t)} |f_1(y_1)| dy_1 \right] \left[ \int_{B(x_0, t)} |b_2(y_2) - (b_2)_B||f_2(y_2)| dy_2 \right] \frac{dt}{t^{2n + 1}}.$$

Since,

$$\int_{B(x_0, t)} |f_1(y_1)| dy_1 \lesssim \|f_1\|_{L^{p_1}(B(x_0, t))} t^{n(1 - 1/p_1)},$$

and

$$\int_{B(x_0, t)} |b_2(y_2) - (b_2)_B||f_2(y_2)|$$

$$\lesssim \|b_2 - (b_2)_B\|_{L^{q_2}(B(x_0, t))} \|f_2\|_{L^{p_2}} \|B(x_0, t)\|_{1 - 1/p_2 + 1/q_2}$$

$$+ \|b_2\|_{L^{q_2}_2} \|B(x_0, t)\|^{|1/q_2 + \lambda_2|} \|f_2\|_{L^{p_2}} \|B(x_0, t)\|_{1 - 1/p_2 + 1/q_2}$$

$$+ \|b_2\|_{LC^{(x_0)}_{\epsilon_2, \lambda_2}} \left(1 + \ln \frac{t}{r}\right)^2 |B(x_0, t)|^{\lambda_2} \|f_2\|_{L^{p_2}} \|B(x_0, t)\|_{1 - 1/p_2}$$

$$\lesssim \|b_2\|_{LC^{(x_0)}_{\epsilon_2, \lambda_2}} \left(1 + \ln \frac{t}{r}\right)^2 t^{n\lambda_2 - n/p_2} \|f_2\|_{L^{p_2}(B(x_0, t))}.$$  

Then,

$$|T_2(f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)(x)|$$

$$\lesssim \|b_2\|_{LC^{(x_0)}_{\epsilon_2, \lambda_2}} \int_0^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{n\lambda_2 - (1/p_1 + 1/p_2) - 1} \|f_1\|_{L^{p_1}(B(x_0, t))} \|f_2\|_{L^{p_2}(B(x_0, t))} \, dt.$$

(4.13)
Let \( 1 < \tau < \infty \), such that \( 1/p = 1/q_1 + 1/\tau \). Then, from Hölder’s inequality and (4.13), we have

\[
IV_2 \lesssim \|b_1 - (b_1)_B\|_{L^{n_1}(B)} \|T_2(f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)\|_{L^r(B)} \\
\lesssim \|b_1\|_{LC^{(x_0)}_{q_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2, \lambda_2}} \frac{r^{n/p}}{2^\lambda_2} \\
\times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right) t^{(\lambda_1 + \lambda_2)n - (1/p_1 + 1/p_2)n - 1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\]

Similarly,

\[
IV_3 \lesssim \|b_1\|_{LC^{(x_0)}_{q_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2, \lambda_2}} \frac{r^{n/p}}{2^\lambda_2} \\
\times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right) t^{(\lambda_1 + \lambda_2)n - (1/p_1 + 1/p_2)n - 1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\]

Similar to the estimate of (4.13), we have

\[
|T_2((b_1 - (b_1)_B)f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)(x)| \\
\lesssim \int_{(2B)^c} \int_{(2B)^c} |b_1(y_1) - (b_1)_B| |b_2(y_2) - (b_2)_B| |f_1(y_1)| |f_2(y_2)| \left[ \int_{[x_0 - y_1|x_0 - y_2]} dt \right] \frac{dt}{t^{2n+1}} dy_1 dy_2 \\
\lesssim \int_{2r}^{\infty} \left[ \int_{B(x_0,t)} |b_1(y_1) - (b_1)_B| |f_1(y_1)| dy_1 \right] \left[ \int_{B(x_0,t)} |b_2(y_2) - (b_2)_B| |f_2(y_2)| dy_2 \right] \frac{dt}{t^{2n+1}} \\
\lesssim \|b_1\|_{LC^{(x_0)}_{q_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2, \lambda_2}} \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 t^{n(\lambda_1 + \lambda_2) - n(1/p_1 + 1/p_2) - 1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\]

Thus,

\[
IV_4 \lesssim \|b_1\|_{LC^{(x_0)}_{q_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2, \lambda_2}} \frac{r^{n/p}}{2^\lambda_2} \\
\times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 t^{(\lambda_1 + \lambda_2)n - (1/p_1 + 1/p_2)n - 1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\]

Then, from the estimates of \( IV_1 - IV_4 \), we deduce that

\[
IV \lesssim \|b_1\|_{LC^{(x_0)}_{q_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2, \lambda_2}} \frac{r^{n/p}}{2^\lambda_2} \\
\times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 t^{(\lambda_1 + \lambda_2)n - (1/p_1 + 1/p_2)n - 1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\]

So, combining the estimates for \( I, II, III \) and \( IV \), we have

\[
\|\mathcal{T}^{(b_1,b_2)}_2(f_1, f_2)\|_{L^p(B)} \\
\lesssim \|b_1\|_{LC^{(x_0)}_{q_1, \lambda_1}} \|b_2\|_{LC^{(x_0)}_{q_2, \lambda_2}} \frac{r^{n/p}}{2^\lambda_2} \\
\times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 t^{(\lambda_1 + \lambda_2)n - (1/p_1 + 1/p_2)n - 1} \|f_1\|_{L^{p_1}(B(x_0,t))} \|f_2\|_{L^{p_2}(B(x_0,t))} dt.
\]
Therefore, we complete the proof of Theorem 4.1.

**Theorem 4.2** Let $x_0 \in \mathbb{R}^n$, $1 < p_i, p_i, q_i < \infty$, for $i = 1, 2, \ldots, m$ such that $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_n + 1/q_1 + 1/q_2 + \cdots + 1/q_n$. Suppose that $0 < \lambda_i < 1/n$ such that $b_i \in LC_{q_i, \lambda_i}^{(x_0)}$, for $0 < \lambda_i < 1/n$, $i = 1, 2, \ldots, m$. If functions $\varphi, \varphi_i : \mathbb{R}^n \times (0, \infty) \to (0, +\infty)$, $(i = 1, 2, \ldots, m)$ satisfy the condition

$$
\int_r^\infty \left(1 + \ln \frac{t}{r}\right)^m \frac{n}{t} \sum_{i=1}^m \lambda_i - n \sum_{i=1}^m 1/p_i - 1 \operatorname{ess} \inf_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0, s)s^{n/p_i}dt \leq C\psi(x_0, r),
$$

where constant $C > 0$ doesn’t depend on $r$. Then the operator $T_m^\varphi$ is bounded from product space $LM_{p_1, \varphi_1}^{(x_0)} \times LM_{p_2, \varphi_2}^{(x_0)} \times \cdots \times LM_{p_m, \varphi_m}^{(x_0)}$ to $LM_{p, \varphi}^{(x_0)}$. Moreover, the inequality

$$
\|T_m^\varphi(f)\|_{LM_{p, \varphi}^{(x_0)}} \lesssim \prod_{i=1}^m \|b_i\|_{LC_{q_i, \lambda_i}^{(x_0)}} \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{(x_0)}}.
$$

holds.

**Proof.** Taking $v_1(t) = \prod_{i=1}^m \varphi_i^{-1}(x_0, t)t^{-n/p_i}$, $v_2(t) = \psi^{-1}(x_0, t)$, $g(t) = \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0, t))}$

and $w(t) = (1 + \ln t)^m n \sum_{i=1}^m \lambda_i - n \sum_{i=1}^m 1/p_i - 1$, then we have

$$
\operatorname{ess} \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess} \sup_{s<\tau<\infty} v_1(\tau)} < \infty.
$$

Thus, by Lemma 2.2, we have

$$
\operatorname{ess} \sup_{t>0} v_2(t)Hw g(t) \leq \operatorname{Cess} \sup_{t>0} v_1(t) g(t).
$$

So,

$$
\|T_m^\varphi(f)\|_{LM_{p, \varphi}^{(x_0)}} = \sup_{r>0} \psi^{-1}(x_0, r)|B(x_0, r)|^{-1/p}\|T_m^\varphi(f)\|_{L^p(B(x_0, r))}
\lesssim \prod_{i=1}^m \|b_i\|_{LC_{q_i, \lambda_i}^{(x_0)}} \sup_{r>0} \psi^{-1}(x_0, r) \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^m n \sum_{i=1}^m \lambda_i - n \sum_{i=1}^m 1/p_i - 1 \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0, t))} dt
\lesssim \prod_{i=1}^m \|b_i\|_{LC_{q_i, \lambda_i}^{(x_0)}} \sup_{r>0} \|f_i\|_{L^{p_i}(B(x_0, r))} \prod_{i=1}^m \|f_i\|_{LM_{p_i, \varphi_i}^{(x_0)}}.
$$

Thus we complete the proof of Theorem 4.2.

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