Fermions in gravity and the skyrmion backgrounds in six dimensional brane-worlds

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We construct brane solutions in six dimensional Einstein-Skyrme systems. A class of baby skyrmion solutions realize warped compactification of the extra dimensions and gravity localization on the brane for negative bulk cosmological constant. Coupling of the fermions with the brane skyrmions successfully lead to the brane localized fermions. The standard representation of the gamma matrices is used to obtain massive localized modes as well as the massless one. Nonlinear nature of the skyrmions brings richer information for the fermions level structure. In terms of the level crossing picture, emergence of the massive localized modes as well as the zero mode are observed.

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I. INTRODUCTION

Theories with extradimensions have been expected to solve the hierarchy problem and cosmological constant problem. Experimentally unobserved extradimensions indicate that the standard model particles and forces are confined to a 3-brane [1, 2, 3]. Intensive study has been performed for the RS brane model in five space-time dimensions [2, 3]. In this framework, the exponential warp factor in the metric can generate a large hierarchy of scales. This model, however, requires unstable negative tension branes and the fine-tuning between brane tensions and bulk cosmological constant.

There is hope that higher dimensional brane models than five could evade those problems appeared in five dimensions. In fact brane theories in six dimensions show a very distinct feature towards the fine-tuning and negative tension brane problems. In Refs. [2, 3], it was shown that the brane tension merely produces deficit angles in the bulk and hence it can take an arbitrary value without affecting the brane geometry. The model is based on the spontaneous compactification by the bulk magnetic flux. If the compactification manifold is a sphere, two branes have to be introduced with equal tensions. If it is a disk, no second 3-brane is needed. But still the fine-tuning between magnetic flux and the bulk cosmological constant can not be avoided although non-static solutions could be free of any fine-tuning [6].

Alternatively to the flux compactification in 6 dim., the nonlinear sigma model has been used for compactifications of the extra space dimensions [7, 8, 9, 10]. As in the flux compactification, no second 3-brane is needed if the parameters in the sigma model and bulk space-time are tuned.

Warped compactifications are also possible in six space-time dimensions in the model of topological objects such as defects and solitons. In this context strings [11, 12, 13, 14, 15] were investigated, showing that they can realize localization of gravity. Interestingly, if the brane is modeled in such a field theory language, the fine-tuning between bulk and brane parameters required in the case of delta-like branes turns to a tuning of the model parameters [10].

The Skyrme model is known to possess soliton solutions called baby skyrmions in two dimensional space [17, 18]. In this paper we therefore consider the warped compactification of the two dimensional extra space by the baby skyrmions. We find that in the 6 dim. Einstein-Skyrme systems, static solutions which realize warped compactification exist for negative bulk cosmological constant. Since the solution is regular except at the conical singularity, it has only single 3-brane. Thus no fine-tuning between brane tensions is required. The Skyrme model possess a rich class of stable multi soliton solutions. We find various brane solutions by such multi-solitons.

It should be noted that general considerations in the 6 dim. brane model with bulk scalar fields suggest that the mechanism of regular warped compactification with single positive tension brane is not possible [19]. However, the model under consideration is restricted to the bulk scalar field depending only on the radial coordinate in the extra space. The scalar field in the Skyrme model depend not only the radial coordinate but also the angular coordinate to exhibit nontrivial topological structure, which makes possible to realize regular warped compactification.

Study of localization of fermions and gauge fields on topological defects have been extensively studied with co-dimension one [20, 21, 22, 23] and two [24, 25, 26, 27, 28, 29]. In many years before, particle localization on a domain wall in higher dimensional space time was discussed [30, 31]. The authors suggested the possibility of localized massless fermions on the 1 dim. kink background in 4+1 space-time with Yukawa-type coupling manner. Localization of chiral fermions on RS scenario is in Ref. [20]. Analysis for the massive fermionic modes was done by Ringeval et al. in Ref. [21]. For co-dimension two, the localization on higher dimensional generalizations of the RS model was studied by the coupling of real scalar fields [24]. Many studies have been followed and most of

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them are based on the Abelian Higgs model coupled with the chiral fermions.

Problem of fermion mass hierarchy was discussed in Ref. [21, 22, 32, 33] within different mechanisms. Especially, in Ref. [23] the hierarchy between the fermionic generations are explained in terms of multi-winding number solutions of the complex scalar (Higgs) fields. They observed three chiral fermionic zero modes on a topological defect with winding number three and finite masses appear the mixing of those zero modes. Although any brane localization mechanism is absent in their discussion, the idea is promising.

In this article, we employ somewhat different set up: we consider the localization of the fermions on the skyrmion branes and, to treat the massive fermionic modes directly, we use the standard representation of the higher dimensional curvature instead of the chiral one. The fermion localization and the existence of the zero modes are confirmed through the analysis of spectral flow of the one particle state. Conclusion and discussion are given in Sec.IV.

This paper is organized as follows. In the next section we describe the Einstein-Skyrme system in six dimensions and derives the coupled equations for the Skyrme and gravitational fields. We derive a class of multi-winding number solutions. We will show some typical numerical solutions. We consider the model of the 6 dim. Einstein-Skyrme gravity system with a bulk cosmological constant coupled to the chiral fermions. The action comprises

\[ S = S_{\text{gravity}} + S_{\text{brane}} + S_{\text{fermion}}. \] (1)

Here \( S_{\text{gravity}} \) is the six dimensional Einstein-Hilbert action

\[ S_{\text{gravity}} = \int d^6x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \Lambda_b \right] \] (2)

In the parameter \( \kappa^2 = 1/M_b^6 \), \( M_b \) is the six-dimensional Planck mass, denoted the fundamental gravity scale, and \( \Lambda_b \) is the bulk cosmological constant.

For \( S_{\text{brane}} \) we use the action of baby-Skyrme model [17, 18]

\[ S_{\text{brane}} = \int d^6x L_{\text{brane}} \] (3)

with

\[ L_{\text{brane}} = \sqrt{-g} \left[ \frac{F^2}{2} \partial_M \vec{\phi} \cdot \partial^M \vec{\phi} + \frac{1}{4e^2} (\partial_M \vec{\phi} \times \partial_N \vec{\phi})^2 + \mu^2 (1 + \vec{n} \cdot \vec{\phi}) \right]. \] (4)

where \( M, N \) run over \( 0, \cdots, 5 \) and \( \vec{n} = (0, 0, 1) \). \( \vec{\phi} = (\phi^1, \phi^2, \phi^3) \) denotes a triplet of scalar real fields with the constraint \( \phi^1 \cdot \phi^2 = 1 \). The \( F, e, \mu \) are the Skyrme model parameters with the dimension of \( M^2, M^{-1}, M^3 \) respectively. The first term in (4) is familiar from the \( \sigma \)-model. The second term is the analogue of the Skyrme fourth order term in the usual Skyrme model which works as a stabilizer for obtaining the soliton solution. The last term is referred to as a potential term which guarantees the stability of a baby-skyrmion.

The solutions of the model would be characterized by following topological charge in curved space-time

\[ Q = \frac{1}{4\pi} \int d^2x \vec{\phi} \cdot (\nabla_1 \vec{\phi} \times \nabla_2 \vec{\phi}) \] (5)

where \( \nabla_\mu \) means the space-time covariant derivative. Let us assume that the matter Skyrme fields depend only on the extra coordinates and impose the hedgehog ansatz

\[ \vec{\phi} = (\sin f(r) \cos n\theta, \sin f(r) \sin n\theta, \cos f(r)) \] (6)

The function \( f(r) \) which is often called profile function, has following boundary condition

\[ f(0) = -(m-1)\pi, \lim_{r \to \infty} f(r) = \pi \] (7)

where \( (m, n) \) is arbitrary integer. This ansatz ensures the topological charge

\[ Q = n(1 - (-1)^n)/2. \] (8)

We consider the maximally symmetric metric with vanishing 4D cosmological constant,

\[ ds^2 = B^2(r)\eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + C^2(r) d\theta^2 \] (9)

where \( \eta_{\mu\nu} \) is the Minkowski metric with the signature \( (-, +, +, +) \) in our convention and \( 0 \leq r < \infty \) and \( 0 \leq \theta \leq 2\pi \). This ansatz has been proved to realize warped compactification of the extra dimension in models where branes are represented by global defects [34].

\( S_{\text{fermion}} \) is the action of a fermions coupled with the skyrmions and the warp factors; that would be described in Sec. III.

The general forms of the coupled system of Einstein equations and the equation of motion of the Skyrme model are

\[ G_{MN} = \kappa^2 (-\Lambda_b g_{MN} + T_{MN}), \] (10)

\[ \frac{1}{\sqrt{-g}} \partial_N \left( \sqrt{-g} F^2 \vec{\phi} \times \partial^N \vec{\phi} \right) \]

\[ + \frac{1}{e^2} \partial_M \vec{\phi} \left( (\partial^M \vec{\phi} \cdot (\vec{\phi} \times \partial^N \vec{\phi})) \right) + \mu^2 \vec{\phi} \times \vec{n} = 0, \] (11)
where the stress-energy tensor $T_{MN}$ is given by
\[ T_{MN} = \frac{\delta L_{\text{brane}}}{\delta g_{MN}} + g_{MN} L_{\text{brane}} \]
\[ = F^2 \partial_M \phi \cdot \partial_N \phi + \frac{1}{e^2} g^{AB} (\partial_A \phi \times \partial_M \phi) \cdot (\partial_B \phi \times \partial_N \phi) \]
\[ + g_{MN} L_{\text{brane}}. \] (12)

Inserting Eq. (1) into Eq. (4), one obtains the Lagrangian
\[ L_{\text{brane}} = -B^4 \tilde{C} F^4 e^2 \left[ u f'^2 + \frac{n^2 \sin^2 f}{C^2} + 2\tilde{\mu}(1 + \cos f) \right] \] (13)
where we have introduced the dimensionless quantities
\[ \tilde{x}_\mu = e F x_\mu, \quad y = e F r, \quad \tilde{C} = e F C, \quad \tilde{\mu} = \frac{1}{e F^2 \mu} \] (14)
and
\[ u = 1 + \frac{n^2 \sin^2 f}{C^2}. \] (15)
The prime denotes derivative with respect to the radial component $y$ of the two extra space. The Skyrme field equation is thus
\[ f'' + \left( \frac{4B'}{B} + \frac{\tilde{C}'}{C} + \frac{u'}{u} \right) f' \]
\[ - \frac{1}{2u} \left[ \frac{n^2 \sin^2 f}{C^2} (1 + f'^2) + 2\tilde{\mu}^2 \sin f \right] = 0 \] (16)
where
\[ \frac{u'}{u} = \frac{n^2}{C^2 + 2n^2 \sin^2 f} \left[ f' \sin 2f - 2\tilde{C}' \sin^2 f \right] \] (17).

Within this ansatz, the components of the stress-energy tensor (12) becomes
\[ T_{\mu\nu} = -F^4 e^2 \tilde{B}^2 \eta_{\mu\nu} \tau_0(y), \]
\[ \tau_0(y) = \frac{u}{2} f'^2 + \frac{n^2 \sin^2 f}{2C^2} + \tilde{\mu}^2(1 + \cos f) \] (18)
\[ T_{rr} = -F^4 e^2 \tau_r(y), \]
\[ \tau_r(y) = -\frac{u}{2} f'^2 + \frac{n^2 \sin^2 f}{2C^2} + \tilde{\mu}^2(1 + \cos f) \] (19)
\[ T_{\theta\theta} = -F^4 \tilde{C} \tau_\theta(y), \]
\[ \tau_\theta(y) = \frac{\tilde{u}}{2} f'^2 - \frac{n^2 \sin^2 f}{2C^2} + \tilde{\mu}^2(1 + \cos f) \] (20)
where
\[ \tilde{u} = 1 - \frac{n^2 \sin^2 f}{C^2}. \] (21)
The Einstein equations with bulk cosmological constant are written down in the following form
\[ 3\tilde{b}' + 6\tilde{b}^2 + 3\tilde{b} \dot{c} + \dot{c}' + c^2 = -\alpha(\tilde{\Lambda}_b + \tau_0(y)) \] (22)
\[ 6\tilde{b}^2 + 4\tilde{b} \dot{c} = -\alpha(\tilde{\Lambda}_b + \tau_r(y)) \] (23)
\[ 4\tilde{b}' + 10\tilde{b}^2 = -\alpha(\tilde{\Lambda}_b + \tau_\theta(y)) \] (24)
where $\alpha = \kappa^2 F^2$ is a dimensionless coupling constant and $\tilde{\Lambda}_b = \Lambda_b/e^2 F^4$ is a dimensionless bulk cosmological constant. Also, we introduce $\tilde{b} \equiv B'/B, \dot{c} \equiv C'/C$ for convenience.

**B. Boundary conditions**

At infinity, all components of the energy-momentum tensor vanishes and the Einstein equations (22) are then reduced to
\[ 3\tilde{b}' + 6\tilde{b}^2 + 3\tilde{b} \dot{c} + \dot{c}' + c^2 = -\alpha\tilde{\Lambda}_b \] (25)
\[ 6\tilde{b}^2 + 4\tilde{b} \dot{c} = -\alpha\tilde{\Lambda}_b \] (26)
\[ 4\tilde{b}' + 10\tilde{b}^2 = -\alpha\tilde{\Lambda}_b. \] (27)
The general solution has been obtained in Ref. [14, 15] which is given by
\[ \tilde{b} = p \frac{A e^{\frac{5}{2} y} - e^{-\frac{5}{2} y} B}{A e^{\frac{5}{2} y} + e^{-\frac{5}{2} y}}, \quad \dot{c} = \frac{5p^2}{2b} - \frac{3}{2} \tilde{b} \] (28)
where $A$ is an arbitrary constant and
\[ p = \sqrt{-\frac{\alpha\tilde{\Lambda}_b}{10}}. \] (29)

Since we are interested in regular solutions with warped compactification of the extra-space, the functions $B$ and $C$ must converge at infinity. This is achieved only when $\tilde{\Lambda}_b < 0$ and $A = 0$ with the solution
\[ B \to e_1 e^{-pr}, \quad C \to e_2 e^{-pr} \] (30)
where $e_1$ and $e_2$ are arbitrary constants. Then, the asymptotic form of the metric which realizes warped compactification is given by
\[ ds^2_{\infty} = e_1 e^{-\sqrt{-\frac{\alpha\tilde{\Lambda}_b}{10}} y} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 + e_2 e^{-2\sqrt{-\frac{\alpha\tilde{\Lambda}_b}{10}} y} d\theta^2. \] (31)

The four-dimensional reduced Planck mass $M_{pl}$ is derived by the coefficient of the four-dimensional Ricci scalar, which can be calculated inserting the metric (22) into the action (2),
\[ \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g} R^{(4)} = \frac{M_6^4}{2} \int d^3x \sqrt{-B^2} (r) R^{(4)} \]
\[ = \frac{M_6^4}{2} \int d^4x \sqrt{-g} R^{(4)} \int dr d\theta B^2 (r) C (r) \]
\[ = 2\pi M_6^4 \int dr B^2 (r) C (r) \int d^4x \sqrt{-g} R^{(4)} \]
where the superscript (4) represents a tensor defined on the four-dimensional submanifold. Thus, we find the relation between $M_{pl}$ and $M_6$ as
\[ M_{pl}^2 = 2\pi M_6^4 \int_0^\infty dr B^2 (r) C (r). \] (32)
we can write
\[ f(y) = \tilde{f} + \delta f(y), \tag{33} \]
where for \( y \gg 1 \), \( \tilde{f} \sim 0 \). The linearized field equations are given by
\[ \delta f'' - 5p\delta f' - \hat{\mu}\delta f = 0. \tag{34} \]
Assuming that \( f \) falls off exponentially, one obtains for \( x \gg 1 \)
\[ \delta f(y) \sim f_c e^{-qy} \quad \text{with} \quad q = \sqrt{25\hat{p}^2 + 4\hat{\mu}} - 5\hat{p} \tag{35} \]
where \( f_c \) is an arbitrary constant.

Following regularity of the geometry at the center of the defect are imposed
\[ B'(0) = 0, \quad C(0) = 0, \quad C'(0) = 1 \tag{36} \]
and we can arbitrarily fix \( B(0) = 1 \). Boundary conditions for the warp factors and the profile function at the origin are determined by expanding them around the origin. For the different topological sectors, the first few terms are schematically written down as
\[ f(y) = -(m - 1)\pi + f^{(n)}(0)y^n + O(y^{n+1}) \tag{37} \]
\[ b(y) = B y + O(y^3) \tag{38} \]
\[ \hat{C}(y) = y + C y^3 + O(y^5) \tag{39} \]

Thus one finds that the only \( f'(0) \) or \( f''(0) \) is the free parameter vicinity of the origin.

Consider linear combinations of Eqs. (22)-(24), we obtain
\[ \hat{b}' + 4\hat{b}^2 + \hat{b}\hat{c} = -\frac{1}{2} \alpha \tilde{\Lambda}_{\hat{b}} + \frac{\alpha}{4} (\tau_\tau + \tau_\theta), \tag{44} \]
\[ 4\hat{b}\hat{c} + \hat{c}' + c^2 = -\frac{1}{2} \alpha \tilde{\Lambda}_{\hat{c}} + \frac{\alpha}{4} (4\tau_0 + \tau_\tau - 3\tau_\theta). \tag{45} \]
Integrating Eqs. (44), (45) from zero to $x_c$, we get
\[ B^3(x_c)B'(x_c)C(x_c) = -\frac{\alpha}{2} \tilde{\Lambda}_0 \int_0^{x_c} B^4 C dx - \frac{\alpha}{4}(\mu_r + \mu_0), \] (46)
\[ B^4(x_c)C'(x_c) = 1 - \frac{\alpha}{2} \tilde{\Lambda}_0 \int_0^{x_c} B^4 C dx - \frac{\alpha}{4}(4\mu_0 + \mu_r - \mu_0). \] (47)
\[ \tilde{\Lambda}_0 \] is the six-dimensional analogue of the relation determining the Tolman mass whereas Eq. (47) is the generalization of the relation giving the angular deficit. Combining these the following relations are obtained in the
\[ x \rightarrow \infty \]
\[ \alpha \int_0^\infty B^4 \frac{n^2 \sin^2 f}{C}(1 + f^2) dx = 1 \] (48)
or
\[ \alpha \int_0^\infty B^4 \left[ \frac{n^2 \sin^2 f}{C} + 2\tilde{\Lambda}_0 C + 2\tilde{\mu} C(1 - \cos f) \right] dx = 1. \] (49)
These conditions are used for checking the numerical accuracy of our calculations.

C. Numerical solutions

The equations (10), (22), (23) should be solved numerically since they are highly nonlinear. The simple technique to solve the Einstein-Skryme equations are the shooting method combined with the 4th order Runge-Kutta forward integration \[35\]. However, a unique set of boundary conditions at $x = 0$ produces 2 distinct solutions, one of which grows exponentially and the other decays exponentially as $x \rightarrow \infty$. This causes instability of solutions when the forward integration is performed. Instead, we use a backward integration following Refs. [14] where the 6 dim. vortex-like regular brane solutions were constructed. The backward integration method requires a set of boundary conditions at infinity. We, however, truncate and take the distance $x_{\text{max}}$ far enough from the origin so that the Skryme profile would fall off before it reaches $x_{\text{max}}$. The set of boundary conditions at $x = x_{\text{max}}$ produces a unique solution which satisfies the boundary conditions at $x = 0$ and hence it is numerically stable. We present our typical numerical results in Figs. [12].

III. FERMIONS

A. Basic formalism

The action of the fermions coupled with the Skryme field in a Yukawa coupling manner can be written as
\[ S_{\text{fermion}} = \int d^6x \mathcal{L}_{\text{fermion}} \] (50)
with
\[ \mathcal{L}_{\text{fermion}} = \sqrt{-g} \left[ \bar{\Psi}(i\Gamma^A D_A - M \tau \cdot \phi)\Psi \right]. \] (51)

The six-dimensional gamma matrices $\Gamma^A$ are defined with the help of the vielbein $e^A_{\hat{a}}$ and those of the flat-space $\gamma^a$, i.e., $\Gamma^A := e^A_{\hat{a}} \gamma^a$. The covariant derivative is defined as
\[ D_A := \frac{1}{2} \nabla_A + \frac{1}{2} \omega_{\hat{a}\hat{b}}^A \sigma_{\hat{a}\hat{b}} \] (52)
where $\omega_{\hat{a}\hat{b}}^A := \frac{1}{4} e^{\hat{a}B} \nabla_A e^{\hat{b}B}$ are the spin connection with generators $\sigma_{\hat{a}\hat{b}} := \frac{1}{4} [\gamma_{\hat{a}}, \gamma_{\hat{b}}]$. The symbol $\nabla$ implies that $\bar{\psi} \nabla \psi = \bar{\psi} \partial \psi - (\partial \bar{\psi}) \psi$. Here $A, B = 0, \cdots, 5$ are the six-dimensional space-time index and $\hat{a}, \hat{b} = 0, \cdots, 5$ corresponds to the flat tangent six-dimensional Minkowski space. The vielbein is defined through $\hat{e}_A := e^A_{\hat{a}} e_{\hat{a}B} = \hat{e}_A e_{\hat{a}B}$. The definition which was introduced in Refs. [26, 24] is simply defined as
\[ e^{\hat{a}}_{\mu} = B(r) \delta^{\hat{a}}_{\mu}, \quad \mu = 0, \cdots, 3, \]
\[ e^{\hat{a}}_{r} = 1, \quad e^{\hat{a}}_{\theta} = C(r). \] (53)

They are the definitions which produce the gamma matrices parallel to the polar coordinate $(r, \theta)$. Thus, special care is needed to explore the conserved quantities like angular momenta from the corresponding hamiltonian. Non-vanishing components of the corresponding spin connection are found to be
\[ \omega_{\hat{a}\hat{b}}^A = \delta^A_{\hat{a}} B'(r), \quad \omega_{\hat{a}\hat{b}}^{35} = -C'(r) \] (54)
where $'$ is $\frac{d}{dr}$. The Dirac equation takes the form
\[ \left[ i \gamma_{\hat{a}}^A \frac{1}{B} \gamma^a \partial_{\mu} + i \gamma^3 \left( \partial_\mu + \frac{2B'}{B} + \frac{C'}{2C} \right) \right] \Psi = 0. \] (55)
The corresponding hamiltonian reads
\[ H = -i \gamma^0 \gamma^3 (\partial_\mu + \frac{2B'}{B} + \frac{C'}{2C}) \gamma^0 \partial_\mu \gamma^0 M \tau \cdot \phi. \] (56)
The equation (55) can be solved numerically in terms of, for example, the shooting method. At present study, however, we will treat the problem by somewhat different way. According to the Rayleigh-Ritz variational method \[30\], the upper bound of the spectrum can be obtained from the secular equation:
\[ \det (A - \epsilon B) = 0 \] (57)
where
\[ A_{ij} = \int d^3x \phi_i^{\dagger} H \phi_j, \quad B_{ij} = \int d^3x \phi_i^{\dagger} \phi_j \]
where \( \{ \varphi_i \} \ (i = 1, \ldots, N) \) is some complete set of the plane-wave spinor basis. For \( N \to \infty \), the spectrum \( \epsilon \) becomes exact. Eq. (57) can be solved numerically. For simplicity, we are to construct plane-wave basis in the flat space-time, \( i.e., B = 1, B' = 0, C = r \), as \( \{ \varphi_i \} \). However, no corresponding flat Hamiltonian subject to Eq. (56) exists; thus it is not easy task to construct the plane-wave basis.

We employ a different form of the vielbein which was used at, \( e.g., \) Ref. [28], that is

\[
\begin{align*}
\epsilon^\mu_\rho &= B(r)\delta^\mu_\rho, \quad \mu = 0, \ldots, 3, \\
\epsilon^4_r &= \cos \theta, \quad \epsilon^5_r = \sin \theta, \\
\epsilon^\theta_\theta &= C(r) \sin \theta, \quad \epsilon^\phi_\theta = C(r) \cos \theta.
\end{align*}
\]

The new definition produces the gamma matrices parallel to the Cartesian coordinate \((x_4, x_5)\). Non-vanishing components of the corresponding spin connections are

\[
\begin{align*}
\Gamma^4_\mu &= 0 \quad (\text{decoupled}), \quad \Gamma^5_\mu = \frac{\partial \theta}{\partial x_5},
\end{align*}
\]

with \( (\mathbf{FIG. 3}) \). Fermion number density for the background skyrmion with \((m, n) = (1, 1)\). The results for the coupling constant \( m = 0 \) (decoupled) and \( m = 0.86 \) are plotted. If fermions couple to the skyrmions, only the state \( K_3 = 0 \) is localized on the brane core. The all other states are not observed because they are strongly delocalized.

The Dirac equation is now

\[
\Psi = \delta^\mu_\nu B'(r) \cos \theta, \quad \omega^\mu_\nu = \delta^\mu_\nu B'(r) \sin \theta,
\]

\[
\tilde{\omega}^\theta_\theta = 1 - C'(r).
\]

The Dirac equation is now

\[
\begin{align*}
\left[ i \frac{1}{B} \delta^\mu_\nu \partial_\mu + i (\cos \theta \gamma^4 + \sin \theta \gamma^5) (\partial_r + \frac{2B'}{B} - \frac{1}{2C}) - i (\sin \theta \gamma^4 - \cos \theta \gamma^5) \frac{1}{C} \partial_\theta - M \vec{r} \cdot \vec{\phi} \right] \Psi = 0.
\end{align*}
\]

The Dirac gamma matrices should satisfy the anti-commutation relations \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \) and there are

**FIG. 5:** Spectral flow of the fermion energy of \((m, n) = (1, 1)\), with some excited energy spectra. The zero crossing spectrum has the quantum number \( K_3 = 0 \), which corresponds to the localizing zero mode.
FIG. 6: The typical brane solution with \((m, n) = (1, 3)\); the profile function and the warp factors (left), and the stress energy tensors and the topological charge (right).

the candidates preserving such Clifford algebra. In most of previous studies in 6 dim., their analyses are based on the localization on the Abelian vortex and the chiral representation of gamma matrices. In this representation, the spinor can be expanded into the right and the left components. The zero mode appears as a eigenstate of right or left component. For the massive modes, they can be estimated from the mixing of both components. To treat the massive fermionic modes directly, we employ the standard representation of the higher dimensional gamma matrices instead of the chiral one. Since the eigenvalues depend on the properties of the background brane solutions, zero modes should appear as a special case of the massive modes. The standard representation of the gamma matrices in 6 dim. can be defined as

\[
\gamma^\mu := \left( \begin{array}{cc} \tilde{\gamma}^\mu & 0 \\ 0 & -\tilde{\gamma}^\mu \end{array} \right), \quad \gamma^\mu = \left\{ \begin{array}{c} I_2 \\ 0 \end{array} \right\}, \quad \left( \begin{array}{cc} 0 & 0 \\ 0 & -\sigma \end{array} \right) \right) \right\} \gamma^4 := \left( \begin{array}{cc} 0 & -iI_4 \\ -iI_4 & 0 \end{array} \right), \quad \gamma^5 := \left( \begin{array}{cc} 0 & -I_4 \\ I_4 & 0 \end{array} \right) \right\} \right) \right) \right)
\]

(61)

where \(I_n\) means the identity matrix of the dimension \(n\). The six dimensional spinor \(\Psi\) can be decomposed into the four dimensional and the extra space-time components

\[
\Psi(x^\mu, r, \theta) = \psi(x^\mu) \left( \begin{array}{c} U(r, \theta) \\ V(r, \theta) \end{array} \right).
\]

(62)

Here the four dimensional part \(\psi(x^\mu)\) is the solution of the corresponding Dirac equation on the brane

\[
i\tilde{\gamma}^\mu \partial_\mu \psi = w\psi
\]

(63)

in which the eigenvalues \(w\) indicate an 4 dim. effective mass of the fermions. Substituting the ansatz (62) and (63) into the Dirac equation (55) yields the equations for \((U, V)\)

\[
-Be^{-i\theta}(\partial_\theta - i\frac{\partial_0}{C}) + A_y)\psi + Bm\tau \cdot \phi U = \tilde{w}U
\]

\[
Be^{i\theta}(\partial_\theta + i\frac{\partial_0}{C}) + A_y)\psi - Bm\tau \cdot \phi V = \tilde{w}V
\]

(64)

where \(A_y := 2B'/B - (1 - \tilde{C}')/2\tilde{C}\). Here we have introduced the dimensionless coupling constant \(m := M/eF\) (and \(\tilde{\omega} := \omega/eF\)). In order to eliminate such induced potential \(A_y\) from the equations, it is convenient to replace the eigenfunctions \((U, V)\) into \((u, v)\) by

\[
\begin{pmatrix} U \\ V \end{pmatrix} = \exp \left[ -2\ln B(y) - \frac{1}{2} \ln \tilde{C}(y) + \frac{1}{2} \int^y dy' \frac{d\tilde{y}'}{\tilde{C}(y')} \right] \begin{pmatrix} u \\ v \end{pmatrix}.
\]

The eigenproblem thus reads

\[
H \begin{pmatrix} u \\ v \end{pmatrix} = \tilde{w} \begin{pmatrix} u \\ v \end{pmatrix}
\]

(65)

\[
H := B \left( \begin{array}{cc} \frac{m\tau \cdot \phi}{C} & -e^{i\theta}(\partial_\theta - i\frac{\partial_0}{C}) \\ e^{-i\theta}(\partial_\theta + i\frac{\partial_0}{C}) & -m\tau \cdot \phi \end{array} \right).
\]

(66)

\(H\) is regarded as a effective Hamiltonian of the model. If background is flat \((i.e., B = 1, B' = 0, C = r)\), the corresponding Hamiltonian easily reads

\[
H_{\text{flat}} = -\gamma^\delta\gamma^4\partial_4 - \gamma^\delta\gamma^5\partial_5 + \gamma^\delta m\tau \cdot \phi
\]

(67)

where the partial derivatives \(\partial_4, \partial_5\) in Cartesian coordinate are defined as \(\partial_4 \equiv i\partial_0 = e^{i2}(\partial_\phi \pm i\partial_\theta)/y\). For obtaining the concise form (67), we introduce and use the additional component of the gamma matrix \(\gamma^\delta := I \otimes \sigma^3\). One easily confirms that \(H_{\text{flat}}\) commutes with “grand-spin”

\[
K_3 := l_3 + \frac{n\tilde{\omega}}{2}, \quad l_3 := x_4p_5 - x_5p_4
\]

(68)

and also “time-reversal operator”

\[
T := i\gamma^3 \otimes \tau^2 C
\]

(69)

where \(C\) is the charge conjugation operator. We emphasize that these operators also commute with the Hamiltonian in the curved space-time. As a consequence,
eigenstates are specified by the magnitude of the grandspin, i.e.,
\[ K_3 = 0, \pm 1, \pm 2, \pm 3 \cdots \text{for odd } n \]
\[ K_3 = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2} \cdots \text{for even } n. \]  
(70)

Since the Hamiltonian is invariant under time reverse, the states of \( \pm K_3 \) are degenerate in energy.

In order to treat the eigenproblem \([65]\), we employ the method which was originally proposed by Kahana-Ripka \([37]\) for solving the Dirac equation with non-linear chiral background. We construct the plane-wave basis in large circular box with radius \( D \) as a set of eigenstates of the flat, unperturbed \((B = 1, B' = 0, C = r, f = \pi)\) Hamiltonian i.e., \( H_0 = -\gamma^6 \gamma^4 \partial_4 - \gamma^6 \gamma^5 \partial_5 - \gamma^6 m^3 \tau^3 \). The solutions are
\[
\begin{pmatrix}
    u \\
    v
\end{pmatrix}
0_{\text{up}} = \begin{pmatrix} M_{k_1} & \left( \omega_{+k_1} J_{p-1}(k_1 y) e^{i(p-1)\theta} \\
\omega_{-k_1} J_{p}(k_1 y) e^{ip\theta} \end{pmatrix} \otimes \begin{pmatrix} 1 \\
0
\end{pmatrix},
\begin{pmatrix}
    u \\
    v
\end{pmatrix}
0_{\text{down}} = \begin{pmatrix} N_{l_1} & \left( \omega_{+l_1} J_{q+1}(l_1 y) e^{i(q+1)\theta} \\
\omega_{-l_1} J_{q}(l_1 y) e^{iq\theta} \end{pmatrix} \otimes \begin{pmatrix} 0 \\
1
\end{pmatrix},
\end{pmatrix}
\]  
(71)

with
\[
M_{k_1} = \left[ \frac{2\pi D^2 \epsilon_{k_1}}{\epsilon_{k_1} + m} \left( J_{p-1}(k_1 y) \right) \right]^{2-1/2}, \\
N_{l_1} = \left[ \frac{2\pi D^2 \epsilon_{l_1}}{\epsilon_{l_1} + m} \left( J_{q+1}(l_1 y) \right) \right]^{2-1/2},
\]
and \( \omega_{+k_1,\omega_{-k_1}}>0 = \frac{1}{2}, \omega_{-k_1}>0 = -\text{sgn}(\epsilon_k)/(/\epsilon_k + m) \). The momenta \( k_1, l_1 \) are discretized. by the boundary conditions
\[
J_{p}(k_1 y) = 0, \quad J_{q}(l_1 y) = 0 \quad (72)
\]
\[ p := K_3 + \frac{1}{2} - \frac{n}{2}, \quad q := K_3 - \frac{1}{2} + \frac{n}{2}. \]

The orthogonality of the basis is then satisfied by
\[
\int_0^D dr r J_{\nu}(k_1 r) J_{\nu}(k_1 r) = \int_0^D dr r J_{\nu+1}(k_1 r) J_{\nu+1}(k_1 r) \\
= \delta_{ij} \frac{D^2}{2} \left[ J_{\nu+1}(k_1 D) \right]^2, \quad \nu := K_3 + \frac{1}{2} + \frac{n}{2}. \quad (73)
\]

Expanding the eigenstates of Eq.\([65]\) in terms of the plane-wave basis, the eigenproblem reduces to the symmetric matrix diagonalization problem. A special care is taken on the estimation of the matrix element from the Hamiltonian \([66]\). In order to hold the Hermiticity of the matrix, the following differential rule is imposed
\[
\langle \psi | \partial_y^\ast | \phi \rangle = \int dy \partial_y \bar{C}(y) \frac{1}{2} \left[ \psi^\ast \partial_y \phi - (\partial_y \psi^\ast) \phi \right], \quad (74)
\]

Once the desired eigenfunctions are obtained, angular averaged fermion densities on the brane can be estimated as follows
\[
R(y) = N(y) \rho(y) \quad (75)
\]
\[ \rho(y) := \int d\theta [u^\dagger(y, \theta) u(y, \theta) + v^\dagger(y, \theta) v(y, \theta)] \quad (76)
\]
\[ N(y) = \exp \left[-4 \ln B(y) + \int^y \frac{dy'}{C(y')} \right]. \quad (77)
\]
For a numerical convenience, we divide the extra spacetime into two domains. We introduce an effective size of the background $D^*$ that the stress energy tensors are finite inside. Outside this domain, no brane exists and only the warping of the geometry is effective. We assume that the fermions we should observe in our 4D space-time are perfectly trapped inside and never leak outside. We set the size as $D^* = D$ and tentatively choose the value $D = 10$, which is larger than the distribution of the stress energy tensors and the topological charge (see Fig.[2]). Of course outside of the domain is not Minkowski space so that the fermions at the boundary feel the effects. The numerical results necessarily depend on the choice of $D$. This crude approximation works well if the effects of the geometry do not affect to the localized modes of the fermions.

For the numerical analysis except for the results of Figs.[9,10], we use the brane solutions with following parameter set for the background fields

\[(m, n) = (1, 1):\]
\[\alpha = 0.5, \tilde{\Lambda}_b = -0.1, \tilde{\mu} = 0.220525915, \]
\[f'(0) = 0.7292182131078184; \quad (78)\]

\[(m, n) = (1, 3):\]
\[\alpha = 0.11, \tilde{\Lambda}_b = -0.1, \tilde{\mu} = 0.49588, \]
\[f^{(3)}(0) = 0.1913003760462098. \quad (79)\]

The numerical results are shown in Figs.[2,3].

Fig.[2] shows the density $\rho(y)$ for the background skyrmion of $(m, n) = (1, 1)$. We present a tower of the massive modes as well as the ground state. As one easily see that only the lowest mode is peaked on the brane while all other modes escape from the core thus we cannot observe them.

In Fig.[3] we plot the densities for some ranges of coupling constant $m$. As you see in the figure, only zero mode (or quasi zero modes) are localized and the others go far away from the brane. In Fig.[4] we plot some crucial eigenvalues for varying $m$. The result clearly exhibits so called the level crossing picture [37]. Spectral flow is defined as the number of eigenvalues of Dirac Hamiltonian that cross zero from below minus the number of eigenvalues that cross zero from above for varying the properties of the background fields. The spectral flow exactly coincides with the topological charge and the zero modes are emerged when they cross the zero. In this case, for varying coupling constant $m$, one zero mode is observed.

Libanov, Troitsky have discussed relation between the topological charge and the fermionic generation [25]. Four-dimensional fermions appear as zero-modes trapped in the core of the global vortex with winding number three. We shall investigate this speculation with our solution for $(m, n) = (1, 3)$. The background brane solution is given in Fig.[6] As is expected, we observe three localized solutions (Fig.[7] which can be regarded as the generations of the fermions. A main difference from the previous studies is that we use non-linear type of soliton solutions for constructing the branes. They have a richer information of the topology and the spectra exhibit doubly degenerate ground states and a higher excited state. On the other hand, the linear type of solitons like Abelian vortex have the spectra with double degeneracy only. Fermion generation of our universe comprises approximate doubly degenerate states and a very high energy state which suggests the underlying topology that we employs.

More generally, we explore the fermion level behavior for various brane solutions with large parameter space. In Fig.[9] we present the parameter dependence of the solutions with $(m, n) = (1, 1)$. By using these, we examine the fermion energies and show the results in Fig.[10] in which we present the level crossing for fixed value of $m, \tilde{\Lambda}_b$ (thus the each lines become the function of $\alpha, \tilde{\mu}$). The emergence of the zero mode is observed and it strongly depends on the property of the branes.

By using the asymptotics of the warp factors (80), we easily speculate the behaviour $N(y)$ at zero and infinity [27]

\[N(y \to 0) \to y \]
\[N(y \to \infty) \to \frac{1}{e^{4y}} \exp(4py + \frac{1}{p e^{2y}}). \quad (80)\]

At the far from the core, the densities of the excited modes are more enhanced than the localized mode.
The Dirac equation in 6 dim. curved space-time was constructed for the fermion localization on the skyrmion branes. The Dirac equation with inside and outside boundary conditions at the distance far from the brane and integrate in backward. For the numerical integration method is unstable as for the parameter ranges, which means the existence of the massive modes as well as the massless one. For \((m, n) = (1, 1)\), three solutions localized on the brane are found. They are the lowest modes of doubly degenerate and an excited state, which approximately agree with the experimental measurements of fermion masses. On the other hand, in the Abelian Higgs model one would expect the spectra include only double degeneracy.

In Ref. [10], the authors had an elaborate analysis for the massive modes in a 5 dim. anti-de Sitter space-time. They set a domain that is defined by the effective potential, and then solved the Dirac equation with inside and out. By imposing the boundary conditions at the surface, they obtained a number of massive modes. In our case, the setup is more crude; for the numerical reason, we tentatively assumed that the fermions are confined within a hypothetical domain. If the size goes to infinity, there is no effect to the zero mode; the others will be delocalized. Therefore the massive modes that we observe in Fig. 4 might be an artifact of our numerical setup. On the other hand, for \((m, n) = (1, 3)\) the lower three modes still localize on the vicinity at the origin; thus we successfully observe the massive localized mode.

IV. CONCLUSION

In this article, we have proposed new brane solutions in six dimensional space-time and have discussed about localization of the fermions on them. The brane have constructed by a baby-skyrmion, a generalization of a O(3) non-linear \(\sigma\) model, and realizes regular warped compactification in 6 dim. anti-de Sitter space-time. The metric is non-factorizable with the warp factors either exponentially diverging or decaying in static solutions. But only exponentially decaying warp factors can allow gravity localization near the brane in the sense that the 6 dim. Planck mass takes a finite value. Such solutions can be obtained numerically if we impose suitable boundary conditions at the distance far from the brane and integrate in backward. Forward integration method is unstable as a unique set of boundary conditions at the origin does not produce an unique solution.

Once the skyrmion branes were found, we studied the fermion localization on the skyrmion branes. The Dirac equation in 6 dim. curved space-time was constructed in terms of introducing the vielbein and six dimensional generalization of the gamma matrices. To treat the massive modes directly, we employ the standard representation of the gamma matrices. In order to solve the eigenproblem, we introduce the plane wave basis in a large circular box with radius \(D\). To study the peculation that the charge of the background skyrmions is identified as the generation of the fermions of our universe, we investigated the fermions with charge \(Q = 1, 3\). This conjecture may be confirmed in terms of the fermion spectral flow. They exactly coincide with the topological charge and practically embody the localized modes. We found localized fermion modes corresponding to the topological charge. For \((m, n) = (1, 1)\), we observed the solutions with some parameter ranges, which means the existence of the massive modes as well as the massless one. For \((m, n) = (1, 3)\), three solutions localized on the brane are found. They are the lowest modes of doubly degenerate and an excited state, which approximately agree with the experimental measurements of fermion masses. On the other hand, in the Abelian Higgs model one would expect the spectra include only double degeneracy.

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