MULTIPLIER IDEALS IN ALGEBRAIC GEOMETRY

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INTRODUCTION

In this introductory survey text we introduce multiplier ideal sheaves in the context of general vanishing theorems and log-resolution of singularities. After discussing some basic properties of multiplier ideals, we then follow [EiLa] to obtain Kollár’s bound from [Kol] on the multiplicity of theta divisors on abelian varieties. We then develop the theory of asymptotic multiplier ideals and expose some of the ideas involved in the algebraic interpretation of Siu’s proof of deformation invariance of plurigenera for varieties of general type [Siu1]. We also define Nadel multiplier ideals in the analytic setting, and explain some ideas behind the analytic proofs of the vanishing theorems and of the deformation invariance of plurigenera, to give the reader an idea of the analytic side of the theory.

This text is by no means meant to be a complete self-contained introduction to the broad and rich field of multiplier ideals, and the reader is encouraged to look at the excellent rigorous in-depth treatments of the subject like the ones in [Laz], [Siu2], [Dem2] and references therein. Thus the purpose of this article is to explain some of the ideas and techniques in multiplier ideals, and to encourage the reader to learn this exciting field in more detail. Our exposition mostly follows [Laz] for the algebraic and [Siu2] for the analytic story.

Acknowledgements. I owe a debt of gratitude to Yum-Tong Siu, from whose classes (including the one the notes for which constitute [Siu2]), talks, and most importantly many insightful conversations with whom I first learned the subject. I have greatly benefited from detailed discussions with Robert Lazarsfeld, and learned the algebraic side of the theory from the preliminary versions of his book [Laz]. Professor Lazarsfeld has also made numerous valuable comments and suggested many clarifications for a preliminary version of this text. I am

Date: December 19, 2004.
2000 Mathematics Subject Classification. Primary: 14J17; Secondary: 14F17, 32L10.
Partially supported by the NSF Mathematical Sciences Postdoctoral Research Fellowship.
grateful to Carolina Araujo and Jordan Ellenberg, who attended and commented on a preliminary version of the talk, which led to further improvements in the talk and in this paper. I would also like to thank Richard Thomas for reading a draft of this text very closely and making many useful suggestions.

1. Classical theory and Kodaira’s vanishing

Convention. We will work over the field of complex numbers, and for simplicity will assume all the varieties to be smooth, though most of the methods have been generalized to deal with the singular case as well.

In this text we will be concerned with the study of line bundles on a projective variety \( X \) of (complex) dimension \( n \) and their cohomology. We do not make any distinction between a line bundle and the corresponding divisor, and use additive notations for line bundles, i.e. denote \( L \otimes L \) by \( 2L \). We denote the space of sections of a bundle \( L \) over \( X \) by \( \Gamma(X, \mathcal{O}_X(L)) \); its dimension is denoted \( h^0(X, \mathcal{O}_X(L)) \). The complete linear system \( |L| \) is the space of all one-dimensional linear subspaces of \( \Gamma(X, \mathcal{O}_X(L)) \) — the elements of \( |L| \) are divisors linearly equivalent to \( L \); the canonical bundle of \( X \) is denoted by \( K_X \). Let us now start with some classical definitions and results.

Definition 1.1. A line bundle \( L \) on \( X \) is called very ample if its sections embed \( X \) into a projective space, i.e. if for any basis \( s_0 \ldots s_N \in \Gamma(X, \mathcal{O}_X(L)) \) the map \( X \to \mathbb{P}^N \) obtained by sending a point \( x \in X \) to \( (s_0(x) : s_1(x) : \ldots : s_N(x)) \) is a well-defined embedding. A line bundle is called ample if there exists some number \( m \in \mathbb{N} \) such that \( mL \) is very ample.

The following numerical criterion of ampleness is classical:

Theorem 1.2 (Nakai-Moishezon-Kleiman criterion). A line bundle \( L \) on \( X \) is ample if and only if for any subvariety \( Y \subset X \) of any dimension \( d \) the intersection \( Y \cdot L^d > 0 \).

Ample line bundles are very special from the point of view of the cohomology theory in view of the following

Theorem 1.3 (Kodaira’s vanishing). If \( L \) is ample, then

\[
H^i(X, \mathcal{O}_X(K_X + L)) = H^{n-i}(X, \mathcal{O}_X(-L)) = 0
\]

for all \( i > 0 \) (the two cohomology groups are dual by Serre duality).

One may wonder whether this cohomological vanishing property characterizes ample line bundles. The answer is no, and the right
question to ask instead is for which bundles the vanishing holds. It seems the vanishing should hold at least for limits of ample bundles, which may not necessarily be ample themselves. These limits are well understood, and they are defined by relaxing both conditions of ampleness.

**Definition 1.4.** A line bundle $L$ is called numerically effective, or nef, if for any curve $C \subset X$ one has $L \cdot C \geq 0$. A line bundle is called big if for some $m \in \mathbb{Z}$ the rational map from $X$ to the projective space $\mathbb{P}^{h^0(X, \mathcal{O}_X(mL))-1}$ given by sections is birational. A variety $X$ is said to be of general type if its canonical bundle $K_X$ is big.

The reason one uses only intersections with curves, and not with higher-dimensional subvarieties, in the definition of nefness is the following

**Theorem 1.5** (Kleiman’s theorem). If a line bundle $L$ is nef, then its degree on any $d$-dimensional subvariety $Y \subset X$ is non-negative, i.e. $L^d \cdot Y \geq 0$.

This theorem means that for testing whether a divisor is nef intersecting only with curves suffices. One may thus wonder whether for testing ampleness checking the fact that $L \cdot C > 0$ for all curves suffices, and the answer is no, with the counterexamples already present for some smooth projective surfaces — see [Laz], 1.2-1.5 for a more detailed discussion.

**Theorem 1.6** (Kawamata-Viehweg). The vanishing theorem holds for nef and big bundles, i.e. if $L$ is both big and nef, then $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ for $i > 0$.

Kodaira’s vanishing theorem has found innumerable applications and has been used to obtain results in a variety of settings in algebraic geometry. The culmination of our discussion here will be an outline of the ideas behind the proof of the deformation invariance of plurigenera, which we now state.

**Theorem 1.7** (Siu, [Siu1]). The plurigenera of a variety of general type are deformation invariant, i.e. if we have a family $X \to \Delta$ of $n$-dimensional varieties over the unit complex disk $\Delta$, with all fibers $X_t$ being of general type, then for any integer $m$ the $m$th plurigenus $h^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$ is independent of $t$ for $t$ small enough.

It is quite easy to see that plurigenera are upper semicontinuous, because any family of sections of $mK_X$, that exists for all $t \neq 0$ can be extended to give an element of $\Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0}))$. Thus the hard
part of the proof is to show that the plurigenus cannot accidentally increase for \( X_0 \), i.e. essentially that any section of \( mK_{X_0} \) gives rise to a section in \( \Gamma(X_t, \mathcal{O}_{X_t}(mK_{X_t})) \) for all \( t \) sufficiently small. By gluing these together for all \( t \), this is the same as asking whether an element of \( \Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0})) \) extends to a section in \( \Gamma(X, \mathcal{O}_X(mK_X)) \).

At first sight it does not seem that the vanishing theorems are related to the invariance of plurigenera. However, vanishing theorems can in fact yield the invariance of plurigenera directly in some cases:

**Proposition 1.8.** If \( K_{X_0} \) is big and nef, the invariance of plurigenera holds.

**Proof.** Indeed, if \( K_{X_0} \) is big and nef, all its multiples \( mK_{X_0} \) are also big and nef, and so all the higher cohomologies \( H^i(X_0, \mathcal{O}_{X_0}(mK_{X_0})) \) are zero. By semicontinuity it follows that \( H^i(X_t, \mathcal{O}_{X_t}(mK_{X_t})) = 0 \) for all \( t \) small enough. Thus we see that \( h^0(X_t, \mathcal{O}_{X_t}(mK_{X_t})) \) is equal to the Euler characteristic \( \chi(mK_{X_t}) \). Since the Euler characteristic is a topological and thus a deformation invariant, it follows that \( h^0(X_t, \mathcal{O}_{X_t}(mK_{X_t})) \) is independent of \( t \), for \( t \) small enough. \( \square \)

This result is not general enough — for a variety of general type we know that \( K_{X_0} \) is big, but not necessarily nef. To generalize further, one is tempted to consider bundles that are big and “almost” nef. However, if a bundle is not nef, it means that there exists a curve \( C \subset X \) such that \( L \cdot C \leq -1 \). Thus there are no integral divisors that are “almost” nef, and we are led to consider divisors with rational coefficients.

### 2. \( \mathbb{Q} \)-divisors and Kawamata-Viehweg’s vanishing

**Definition 2.1.** A (Weil) \( \mathbb{Q} \)-divisor on \( X \) is a formal linear combination \( \sum a_i D_i \), where \( D_i \) are codimension one subvarieties of \( X \), and \( a_i \in \mathbb{Q} \) are arbitrary coefficients. The rounding up and down of a \( \mathbb{Q} \)-divisor are defined as \( [D] := \sum [a_i] D_i \) and \( |D| := \sum |a_i| D_i \), by taking respectively the least integer not smaller than, or the largest integer not greater than \( a_i \) for each \( i \).

**Remark 2.2.** Notice that while all arithmetic operations with \( \mathbb{Q} \)-divisors are functorial, i.e. commute with taking pullbacks and restrictions, the rounding is not. Indeed, consider a line \( \mathbb{C} \) in the plane \( \mathbb{C}^2 \) and the parabola \( P \) touching it at the point \( p \). Let \( D = \frac{1}{2} P \), so \( [D] = 0 \). However, when we restrict \( D \) to the line and then round down, we get \( [D]_{\mathbb{C}} = [2(p/2)] = p \neq 0 = [D]_{\mathbb{C}} \).
Introducing $\mathbb{Q}$-divisors allows one to talk about $\mathbb{Q}$-divisors that are close to an ample integral divisor. Moreover, one can multiply $\mathbb{Q}$-divisors by a large number to clear all the denominators, prove something for the resulting integral divisor, and then take the appropriate root to recover properties of the original divisor. However, it will turn out even more useful to work with integral divisors that are close to an ample $\mathbb{Q}$-divisor. Doing this allows one to perform some asymptotic constructions for integral divisors, essentially by proving that any large enough multiple $mL$ of a given integral divisor is uniformly $\varepsilon$-close to some ample $\mathbb{Q}$-divisor, and then arguing that then the integral divisor behaves almost as if it were ample, since the error is essentially $\varepsilon/m$.

This idea is indeed used in Siu’s proof of plurigenera, and setting it up rigorously is one of our main goals. First, we will need to understand $\mathbb{Q}$-divisors better.

**Definition 2.3.** Similarly to integral divisors, the support of a $\mathbb{Q}$-divisor $\sum a_iD_i$ is the integral divisor $\sum_{\{i|a_i\neq0\}} D_i$. A $\mathbb{Q}$-divisor is said to have simple normal crossings (s.n.c.) if its support has simple normal crossings, i.e. if all $D_i$ are smooth, and whenever a number of $D_i$ intersect, their normal vectors are linearly independent.

Simple normal crossings is the mildest singularity a divisor may have. The pullback of a s.n.c. divisor under the blowup at a point is also a s.n.c. divisor, and moreover restricting a divisor $D$ to some $E$ such that the union of $D$ and $E$ is s.n.c. commutes with rounding. The appropriate vanishing theorem for $\mathbb{Q}$-divisors is

**Theorem 2.4** (Kawamata-Viehweg’s vanishing, [Kaw],[Vie]). Suppose $L$ is an integral divisor, numerically equivalent to (i.e. its intersections with all effective curves are the same as those of) $B+D$, where $B$ is a big and nef $\mathbb{Q}$-divisor, and $D$ is a s.n.c. $\mathbb{Q}$-divisor such that $\lfloor D \rfloor = 0$. Then the vanishing holds for $L$, i.e. $H^i(X,\mathcal{O}_X(K_X+L)) = 0$ for $i > 0$.

This theorem makes precise what it means for an integral divisor $L$ to be “close” to a big and nef $\mathbb{Q}$-divisor $B$. We will not prove this powerful result, but will rather give a different version of it that will turn out to be equally useful.

**Corollary 2.5.** Suppose $L$ is an integral divisor, and $D$ is a s.n.c. $\mathbb{Q}$-divisor such that $L−D$ is big and nef. Then $H^i(X,\mathcal{O}_X(K_X+L−\lfloor D \rfloor)) = 0$ for $i > 0$.

Notice that restricting the Kawamata-Viehweg vanishing theorem to the case of $D$ integral yields precisely the vanishing for big and nef integral divisors. However, the improvement for $\mathbb{Q}$-divisors is crucial.
for further applications. What happens is essentially the following: the ample (or big and nef) cone is indeed a cone, i.e. is invariant under scaling. Given a big integral divisor that is not nef, it has to lie outside this cone, and in fact has to be a fixed distance from it, as there is a curve it intersects negatively. However, it may then happen that its multiples will be sufficiently close to some ample $\mathbb{Q}$-divisors, so that the Kawamata-Viehweg may be applied. Notice that the set of divisors for which Kawamata-Viehweg’s vanishing holds is not invariant under scaling.

3. Multiplier Ideals and Nadel’s Vanishing

In the previous section we have discussed the appropriate vanishing condition for $\mathbb{Q}$-divisor with s.n.c. However, to be able to use the vanishing in a wide variety of settings, we need to relax the s.n.c. condition. The way to do this is to try to resolve a more complicated singularity, and carrying this out naturally leads to defining a multiplier ideal.

**Definition 3.1.** For a $\mathbb{Q}$-divisor $D$ on $X$, a log-resolution (also called an embedded resolution) of the pair $(X, D)$ is a birational map $\mu : X' \to X$ with $X'$ non-singular such that the divisor $\mu^*D + \text{except}(\mu)$ on $X'$ is s.n.c., where $\text{except}(\mu)$ denotes the exceptional locus of the birational map.

The log-resolutions are not unique, as we can blow up a s.n.c. divisor again so that the preimage will still be s.n.c., or can do a resolution in different ways. Hironaka proved that by doing an appropriate sequence of blowups with smooth centers one can always construct a log resolution. This is a hard result in resolution of singularities, and as it falls far from the positivity questions that we are interested in we don’t discuss the proof here.

To get a version of the vanishing theorem that would work for $\mathbb{Q}$-divisors with any kind of singularities, we would need to do a log-resolution and then apply Kawamata-Viehweg for the resulting s.n.c. divisor.

**Definition 3.2.** The relative canonical divisor of a rational map $\mu : X' \to X$ is defined to be $K_{X'/X} := K_{X'} - \mu^*K_X$. Though both $K_X$ and $K_{X'}$ are only defined as linear series, the relative canonical is in fact an effective divisor: it is the locus where the differential of the map $\mu$ is degenerate. The relative canonical divisor is thus supported on $\text{except}(\mu)$, and thus its pushforward $\mu_*K_{X'/X} = \mathcal{O}_X$. 
Suppose now that \( \mu : X' \to X \) is a log-resolution of \((X, D)\). The second form of Kawamata-Viehweg vanishing theorem on \(X'\) then gives

\[
0 = H^i(X', \mathcal{O}_{X'}(K_{X'} + [\mu^*(L - D)]))
\]

\[
= H^i(X', \mathcal{O}_{X'}(K_{X'/X} + \mu^*(K_X + L) - [\mu^*D])).
\]

Now let us project this sheaf down to \(X\) by \(\mu\). If the higher direct images \(R^i\) vanish (and they in fact do, but we omit the proof), by the projection formula we would also get the vanishing of cohomology there:

\[
0 = H^i(X, \mathcal{O}_X(\mu_*([K_{X'/X} + \mu^*(K_X + L) - [\mu^*D]])))
\]

\[
= H^i(X, \mathcal{O}_X(K_X + L) \otimes \mu_*([K_{X'/X} - [\mu^*D]])).
\]

This indicates what the appropriate correction for the vanishing theorem should be.

**Definition 3.3** (Esnault-Viehweg). For a \(\mathbb{Q}\)-divisor \(D\) on \(X\) the multiplier ideal sheaf is defined to be

\[\mathcal{J}(X, D) := \mu_*(K_{X'/X} - [\mu^*D])\]

Since \(\mu_*(K_{X'/X}) = \mathcal{O}_X\), the sheaf \(\mathcal{J}(X, D)\) is an ideal subsheaf of \(\mathcal{O}_X\).

The discussion above indicates the direction one follows to prove the appropriate vanishing theorem for \(\mathbb{Q}\)-divisors. The result is as expected, but finishing the proof requires more work that we do not show here — see [Laz], 9.4.B.

**Theorem 3.4** (Nadel’s vanishing). If \(L\) is an integral divisor and \(D\) is a \(\mathbb{Q}\)-divisor such that \(L - D\) is big and nef, then

\[H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0\] for \(i > 0\).

**Remark 3.5.**

a) From the definition it is not a priori clear that \(\mathcal{J}(X, D)\) does not depend on the choice of the log-resolution. However, it can be proven that this is indeed the case.

b) If \(D\) is a \(\mathbb{Q}\)-divisor with s.n.c, the log-resolution is the identity, the relative canonical bundle is trivial, and thus \(\mathcal{J}(X, D) = \mathcal{O}_X(-[D])\)

c) If \(D\) is an integral divisor, then there is no rounding, and the definition commutes with pullback, so we get \(\mathcal{J}(X, D) = \mathcal{O}_X(-D)\).

d) In view of the above, for the study of the multiplier ideal all that matters is the fractional part of the \(\mathbb{Q}\)-divisor: the integral part contributes a factor of minus itself.
Definition 3.6. Since only the fractional part of the divisor is interesting for multiplier ideal considerations, given any integral divisor $D$, it is interesting to study the multiplier ideals $J(X, cD)$ as $c$ increases from 0 — then the multiplier ideal is trivially $\mathcal{O}_X$ — to 1, when the multiplier ideal is $\mathcal{O}_X(-D)$ and thus non-trivial. The log-canonical threshold of the pair $(X, D)$ is defined to be the infinum of $c$ such that $J(X, cD)$ is non-trivial. If this $c$ is equal to one, i.e. if we have $J(X, cD) = 0$ for all $0 < c < 1$, then the pair $(X, D)$ is called log-canonical.

Example 3.7. To demonstrate how the multiplier ideals are computed, let us do perhaps the simplest non-trivial example: $D$ is the sum of three lines $\ell_1, \ell_2$ and $\ell_3$ intersecting at the point $p$ in the plane $\mathbb{P}^2$.

Let $\mu : X' \to \mathbb{P}^2$ be the blow up at $p$, and let $E$ be the exceptional divisor on $X'$. The pullback of the divisor $cD$

$$\mu^*(cD) = c\mu^{-1}(\ell_1 + \ell_2 + \ell_3) + 3cE$$

is s.n.c.; thus $\mu$ is a log-resolution of $(\mathbb{P}^2, D)$. The relative canonical divisor $K_{X'/\mathbb{P}^2}$ is equal to $E$. Thus for $c < 1/3$ we have $\lfloor \mu^*(cD) \rfloor = 0$, while for $1/3 \leq c < 2/3$ we have $K_{X'/\mathbb{P}^2} - \lfloor \mu^*(cD) \rfloor = E - E = 0$, so that in both cases $J(\mathbb{P}^2, cD) = \mathcal{O}_{\mathbb{P}^2}$. However, for $2/3 \leq c < 1$ we have

$$J(\mathbb{P}^2, cD) = \mu_*(K_{X'/\mathbb{P}^2} - \lfloor \mu^*(cD) \rfloor) = \mu_*(-E) = m_p$$

is the maximal ideal of the point. Thus the log-canonical threshold for $D$ is equal to $2/3$.

Doing the log-resolution explicitly (as three consecutive blowups), one can show that for the case of $\ell = (\text{cusp } x^3 = y^2)$ in the plane we have $J(\mathbb{P}^2, c\ell) = \mathcal{O}_{\mathbb{P}^2}$ for $c < 5/6$, while $J(\mathbb{P}^2, c\ell) = m_p$ for $5/6 \leq c$. It can be shown that in general for the divisor of the curve $\{x^a = y^b\} \subset \mathbb{P}^2$ the log-canonical threshold is equal to $1/a + 1/b$ — this also works for three intersecting lines, which are $\{x^3 = y^3\}$.

4. Analytic approach to multiplier ideals and Nadel’s vanishing

In this section we will take a step back and indicate the analytic construction of Nadel multiplier ideal sheaves in full generality, and explain some of the analytic ideas that have been motivating the work in the subject. This is a very rich analytic field, to which we only present a simplified naïve introduction. The reader is encouraged to consult [Siu2], [Dem1], [Dem2] and references therein for a coherent rigorous exposition.

In the analytic setting one starts with a line bundle $L$ on a complex manifold $X$ of dimension $n$, and a Hermitian metric on it. A Hermitian
metric on a line bundle means simply a Hermitian scalar product on all fibers. If we locally trivialize the bundle, by choosing a basis vector $e(z)$ in each fiber (where $z = (z_1, \ldots, z_n)$ is the local coordinate system on $X$), then the Hermitian metric is determined by the value of the scalar product $h(z) := (e(z), \overline{e}(z))$.

**Definition 4.1.** The curvature of a Hermitian metric is the curvature of the unique complex connection on the tangent bundle compatible with this metric. It is a two-form on $X$ of type $(1,1)$, and is given in coordinates as

$$
\omega_h := \frac{\sqrt{-1}}{2} \partial \overline{\partial} \log h := \frac{\sqrt{-1}}{2} \sum_{i,j} \frac{\partial^2 \log h(z)}{\partial z_i \partial \overline{z}_j} dz_i d\overline{z}_j.
$$

The curvature form is closed, and thus defines a class in $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$. By Poincaré duality (if we think of $L$ as a linear combination of codimension one subvarieties) the line bundle $L$ itself also defines a class in that cohomology group, and the class of $\omega_h$ is linearly equivalent to $L$. In the algebraic setting we are interested in computing the degrees of the restriction of a line bundle to curves, or of its powers — to higher-dimensional subvarieties. Analytically the intersection number with a subvariety $Y \subset X$ of dimension $m$ is

$$
L^m \cdot Y = \int_Y (\omega_h)^m;
$$

notice that these integrals are independent of the choice of the Hermitian metric $h$ on the line bundle $L$.

There is a natural invariant metric, the Fubini-Study metric, on the tangent bundle to the projective space, which has constant positive curvature. Thus if we use a very ample line bundle to embed some $X$ in a projective space, the restriction of the Fubini-Study metric to the image will give a metric with positive curvature. This will also hold for roots of very ample bundles, i.e. for ample bundles, and thus the ampleness condition can be formulated analytically.

**Proposition 4.2.** A line bundle $L$ is ample if and only if it admits such a Hermitian metric $h$ that the corresponding curvature form is positive-definite, i.e. $\omega_h(v, \overline{v}) > 0$ for any non-zero holomorphic tangent vector $v \in T^{1,0}(X)$.

**Theorem 4.3** (Kodaira’s vanishing, analytic setting). If a line bundle $L$ on $X$ admits a Hermitian metric of positive curvature, then the cohomology groups $H^i(X, \mathcal{O}_X(K_X + L))$ are zero for all $i > 0$. 

One is tempted to conjecture that a bundle is nef if and only if it admits a Hermitian metric with non-negative curvature form. The “if” part — if there is a Hermitian metric with non-negative curvature, then the bundle is nef — is indeed trivially true. However, not all nef bundles do in fact admit metrics of non-negative curvature. The correct criterion is

**Proposition 4.4.** Fix a Kähler form \( \nu \) on \( X \) (i.e. a closed positive \((1,1)\) form on \( X \), which does not need to be the curvature of some bundle); then a line bundle \( L \) on \( X \) is nef if and only if for any \( \varepsilon > 0 \) there exists a metric on \( L \), whose curvature \( \omega_\varepsilon \) is such that \( \omega_\varepsilon + \varepsilon \nu \) is strictly positive-definite.

If a line bundle \( L \) is not ample, then we cannot choose a metric on it with positive-definite curvature form, so we would like to work with a metric that is as close to being positive as possible. One way to do this is to somehow bound from below the negativity of the curvature form, but it turns out that the way that naturally leads to multiplier ideals is rather to allow the metric to singularize, while preserving the positivity.

Indeed, given a big line bundle (which is the case in our ultimate goal here — the invariance of plurigenera of general type, whence the canonical bundle is big), we can use a high power of this bundle defining a rational map to the projective space to pull back the Fubini-Study metric. In doing this, we end up with a metric that is well-defined, smooth, and has positive-definite curvature form away from the indeterminacy locus of the rational map. On the indeterminacy locus, however, the metric may singularize, i.e. acquire a singularity as to become meromorphic there (in the algebraic setting we don’t encounter essential singularities). Thus it will be natural in the following discussion to allow Hermitian metrics with singularities.

**Remark 4.5.** It is amusing to note that when working with Riemann surfaces, one also can either deal with a hyperbolic metric on a surface, i.e. a metric of constant negative curvature, or can instead introduce a flat Euclidean (zero-curvature) metric on a Riemann surface, which would then be singular at a finite number of points. In a way what we do now is a generalization and interpretation of this idea in terms of metrics on general complex manifolds.

Conceptually, we consider metrics that have zero curvature and singularize along divisors. Suppose we are given an effective irreducible \( \mathbb{Q} \)-divisor \( D \), equal to \( aH \) for some non-singular subvariety \( H \subset X \) of codimension one and some \( a \in \mathbb{Q}_+ \). Then in local coordinate \( z \) on
X near some point \( p \in H \) the subvariety \( H \) is the zero locus of some function \( f(z) \), the vanishing order of which along \( H \) is equal to one. Then locally we can consider the singular Hermitian metric given by \(|f(z)|^{-2a}\). The curvature form of this metric,

\[
\omega_f = \frac{\sqrt{-1}}{2} \partial \overline{\partial} (-2a \log |f|),
\]

is identically equal to zero away from \( H \), and gives \( a \) times the delta-function of the subvariety \( H \) (we recall that on \( \mathbb{C} \) we have \( \sqrt{-1} \partial \overline{\partial} \log |z| = 2\pi \delta(z) \)).

A bit more precisely, what this means is the following. The curvature “form” \( \omega_f \) is in fact not a smooth form on \( X \), but rather has a singularity. However, given a smooth \((n-1, n-1)\)-form \( \xi \) on \( X \), the integral \( \int_X \omega_f \wedge \xi \) can still be computed. This means that technically we should think of \( \omega_f \) as a current of type \((1,1)\) — an object dual to smooth \((n-1, n-1)\)-forms on \( X \). On the other hand, \( H \) can also be paired with \( \xi \) by taking \( \int_H \xi \). The delta-function statement means simply that we have \( \int_X \omega_f \wedge \xi = a \int_H \xi \) for all \( \xi \). Still a bit more technically, we should note that this should only work locally, as the defining function \( f \) for \( H \) can only be obtained locally, and thus the above should rather be considered for smooth forms \( \xi \) in a small analytic neighborhood of a point \( p \in H \). For the rigorous discussion and details on currents we refer, for example, to [Dem1].

The case of an irreducible divisor easily generalizes to the case of a simple normal crossing divisor that is equal to \( \sum a_i H_i \), with \( f_i \) being the (local) defining function for \( H_i \). In this case we consider the singular metric \( \prod |f_i(z)|^{-2a_i} \). Recall that for simple normal crossing divisors we know that the multiplier ideal is \( \mathcal{J}(X, D) = \mathcal{O}(-\lfloor D \rfloor) \). If we think of this analytically, it means that the germ of the multiplier ideal consists of locally defined holomorphic functions \( F \) with vanishing order along each of \( H_i \) equal to at least \( |a_i| \). This is equivalent to \( |F|^2 / \prod |f_i(z)|^{2a_i} \) being locally integrable near each of the \( H_i \) and their intersections.

Analytically it is natural to attach the multiplier ideal to any singular metric \( e^{-2\phi} \), not only to those \( \phi \) that come from divisors. The technical condition is that the real function \( \phi \) must be plurisubharmonic, i.e. that whenever \( \phi \) is finite and smooth, its Laplacian \( \sqrt{-1} \partial \overline{\partial} \phi \) is positive-definite, while \( \phi \) is everywhere upper-semi-continuous, and allowed to take the value of \(-\infty\) at some points.

**Definition 4.6.** For a plurisubharmonic function \( \phi \) on \( X \) we define its associated multiplier ideal \( \mathcal{J}(\phi) \subset \mathcal{O}_X \) to be the sheaf of germs of local holomorphic functions \( F \) such that \( |F|^2 e^{-2\phi} \) is locally integrable.
The discussion for effective normal crossing divisors can in fact be applied to any divisor $D$, by passing to a small neighborhood where all of its irreducible components become principal, and defining the metric $e^{-2\phi_D}$ as above, irrespective of whether the components are normal crossing or not.

**Proposition 4.7** (see [Laz], 9.3.D). In the algebraic setting, the analytic and the algebraic multiplier ideals agree: for any $\mathbb{Q}$-divisor $D$ we have $J(X, D) = J(\phi_D)$.

We note, however, that the analytic multiplier ideal is defined in a more general setting, and allows one to potentially tackle the non-algebraic situations as well.

Let us now discuss the analytic formulation and the intuition behind Nadel’s vanishing.

**Theorem 4.8** (Analytic Nadel’s vanishing, [Nad]). Let $L$ be a line bundle on a projective algebraic variety $X$, with singular metric $e^{-2\phi}$, such that its curvature current $\omega_{\phi}$ dominates some smooth positive $(1,1)$-form $\nu$ on $X$ as a current (i.e. for any positive $(n-1, n-1)$-form $\xi$ the integral $\int_X (\omega_{\phi} - \nu) \wedge \xi > 0$). Then the cohomology groups $H^i(X, \mathcal{O}_X(K_X + L) \otimes J(\phi))$ vanish for all $i > 0$.

*Analytic idea of the proof, following [Sin2].* The basic method is to try to smooth out the function $\phi$, while controlling the smoothing in such a way that we can see that the vanishing condition for the cohomology is preserved. The way to smooth a function is to translate it by the flow of a holomorphic vector field, and then average the translates (i.e. take the integral of all translates for translation times from $-t$ to $t$ for some small $t$, and then divide the result by $2t$).

Let us embed $X$ in a large projective space $\mathbb{P}^N$ and then take a generic projection $\pi : \mathbb{P}^N \to \mathbb{P}^n$. Let then $U := \pi^{-1}(\mathbb{C}^n) \subset X$, and let $Z := X - U$ be the preimage $\pi^{-1}(\mathbb{P}^{n-1})$. Let us choose a global trivialization for $L|_U$.

The map $\pi$ restricted to $U$ is a branched cover. Let $Y$ be its branch locus, so that the map $\pi : U - \pi^{-1}(Y) \to \mathbb{C}^n - Y$ is an unbranched cover. The vector fields $\frac{\partial}{\partial z_j}$ on $\mathbb{C}^n - Y$ lift to holomorphic (this is why we needed to eliminate the branching locus — otherwise a branching singularity could develop) vector fields on $U - \pi^{-1}(Y)$, which we denote by $v_j$.

Notice that $Y$ is of codimension one in $\mathbb{C}^n$, and is the zero set of some polynomial $F : \mathbb{C}^n \to \mathbb{C}$. Consider the family of compact sets $K_a \subset \mathbb{C}^n$ for $a \geq 0$ such that $K_a$ is essentially the set of points sufficiently far
away from infinity and from $Y$, and such that $\cup_{a \geq 0} K_a = \mathbb{C}^n - Y$. Technically, we can define the compact set $K_a := \{ z | a \geq |F(z)|^{-2} + |z|^2 \}$, and denote its preimage by $\Omega_a := \pi^{-1}(K_a)$.

Now on $\Omega_a$ the vector fields $v_j$ are smooth and bounded (in terms of $a$), and we can use them to smooth out $\phi$. In other words, we can get smooth plurisubharmonic functions $\phi_\varepsilon$ on $\Omega_a$ monotonically decreasing to $\phi$ as $\varepsilon \to 0$. The fact that the smoothings $\phi_\varepsilon$ are monotonically decreasing follows from the sub-mean-value property, i.e. from the maximum principle for plurisubharmonic functions.

Finally we are ready to think about the vanishing. We think of $H^i$ as the Dolbeault cohomology, i.e. an element of $H^i(X, \mathcal{O}_X(K_X + L) \otimes J(\phi))$ is a $\overline{\partial}$-closed $(0, i)$ form $g$ on $X$ such that $|g|^2 e^{-2\phi}$ is integrable. To show that the cohomology is in fact zero we need to show that there then exists some $(0, i - 1)$-form $u$ such that $\overline{\partial}u = g$. Then $g$ is exact and thus represents the zero cohomology class.

Since we know that the smooth functions $\phi_\varepsilon \geq 0$ are monotonically decreasing to $\phi$ and $|g|^2 e^{-2\phi}$ is integrable on $\Omega_a$, it means that $|g|^2 e^{-2\phi_\varepsilon}$ is also integrable on $\Omega_a$. Since everything is smooth and compact now, we can find $u_{a, \varepsilon}$ such that $\overline{\partial}u_{a, \varepsilon} = g$ on $\Omega_a$ (the cohomology of $\mathbb{C}^n$ is trivial). Moreover, we can choose $u_{a, \varepsilon}$ such that the integral $\int_{\Omega_a} |u_{a, \varepsilon}|^2 e^{-2\phi_\varepsilon}$, a.k.a. the $L^2$-norm of $u_{a, \varepsilon}$ on $\Omega_a$ with weight $\phi_\varepsilon$, is bounded by some constant independent of $a$ and $\varepsilon$.

The crucial step in the proof is the claim that this constant is independent of $a$ and $\varepsilon$, and to prove this we use the fact that $\omega_\phi$ dominates a smooth positive $(1, 1)$-form $\nu$. Indeed, using this fact we can bound both the growth of $u_{a, \varepsilon}$ near $X - \Omega_a$, since there $\nu$ is also smooth, and the approximation error in replacing $\phi$ by $\phi_\varepsilon$, as on compacts by choosing $\varepsilon$ small enough we can ensure that $\phi_\varepsilon$ still dominates $\nu$.

Thus finally we get a family of solutions $\overline{\partial}u_{a, \varepsilon} = g$ on compacts $\Omega_a$ with bounded $L^2$ norms with respect to weights $\phi_\varepsilon$. Taking first the limit as $\varepsilon \to 0$ and then the limit as $a \to \infty$, and using the fact that a bounded normal family must converge, we finally show the existence of the limit $u := \lim_{\varepsilon \to 0, a \to \infty} u_{a, \varepsilon}$ with bounded $L^2$ norm with respect to $\phi$, and it follows that $\overline{\partial}u = g$ on $X$. We refer the reader to [Nad] for the details of the original proof.

5. Multiplicity of multiplier ideals and Kollár’s theorem

In this section we will study the multiplicity of divisors and the non-triviality of the corresponding multiplier ideals, and will prove Kollár’s
Theorem on the multiplicity of the theta divisor of a principally polarized abelian variety.

**Proposition 5.1.** Let $D$ be a $\mathbb{Q}$-divisor on $X$, let $n = \dim X$, and let $x \in X$ be a point. If the multiplicity $\text{mult}_x D \geq n + p - 1$, then $\mathcal{J}(X, D) \subset \mathfrak{m}^p_x$.

**Proof.** Let us construct a log-resolution $\mu : X' \to X$ of $(X, D)$ by first blowing up the point $x$ and then doing whatever else is necessary. Let us denote by $E$ the preimage in $X'$ of the exceptional divisor of the first blow-up. We can compute $\text{ord}_E(K_{X'/X}) = n - 1$, since $X'$ is computed by first blowing up at $x$, i.e. by inserting a $\mathbb{P}^{n-1}$ instead of $x$.

The order to which the pullback of any divisor $F$ on $X$ contains $E$ is equal to the multiplicity of vanishing of $F$ at $x$. Thus $\text{ord}_E(\mu^*D) = \text{mult}_x D \geq n + p - 1$, and in general $\mu_*\mathcal{O}_{X'}(-aE) = \mathfrak{m}_x^a$.

Therefore

$$\text{ord}_E(K_{X'/X} - [\mu^*D]) \leq (n - 1) - (n + p - 1) = -p.$$ 

It follows that

$$\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D]) \subset \mathcal{O}_{X'}(-pE),$$

and thus finally

$$\mu_*\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D]) \subset \mu_*\mathcal{O}_{X'}(-pE) = \mathfrak{m}_x^p.$$

□

This proposition shows that if a divisor has very high multiplicity at some point, then the corresponding multiplier ideal sheaf is non-trivial (i.e. not equal to $\mathcal{O}_X$). This can be generalized to higher-dimensional subvarieties of $X$ — the proof is analogous.

**Proposition 5.2.** Suppose $Z \subset X$ is a subvariety of codimension $e$ such that $\text{mult}_Z D \geq e + p - 1$. Then $\mathcal{J}(X, D) \subset I^{(p)}_Z$, where $I^{(p)}_Z$ is the symbolic power — the ideal of functions vanishing on $Z$ to order at least $p$.

Now we will apply this to bound the multiplicity of theta divisors.

**Definition 5.3.** A principally polarized abelian variety $(A, \Theta)$ of dimension $g$ is a complex torus $A$, i.e. a $g$-dimensional projective variety with the structure of an abelian group on its points, together with the choice of a principal polarization, i.e. an ample line bundle $\Theta$ such that $h^0(A, \mathcal{O}_X(\Theta)) = 1$.

Principally polarized abelian varieties are a very classical object. Classically they can be thought of as $A = \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$, where $\tau$ is a
complex symmetric $g \times g$ matrix with positive-definite imaginary part, and the theta function (the unique up to a constant factor section of $\Theta$) is then given, for $z \in \mathbb{C}^g$, by

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i (n, \tau n) + 2\pi i (n, z)),$$

where the transformation rule for $\theta(z)$ as we add to $z$ a vector in the lattice $\mathbb{Z}^g + \tau \mathbb{Z}^g$ is what defines the bundle $\Theta$ on $A$. Theta functions were studied extensively at least ever since the works of Riemann. One natural question to ask is to describe the vanishing locus of the theta function and its order of vanishing. In particular one may wonder what is the maximal possible multiplicity the theta function may have at a point. Despite this being a question that already Riemann could ask, a classical analytical solution is, to the best of our knowledge, still not known. However, one can answer this question rather easily by using multiplier ideals.

**Theorem 5.4 (Kollár, [Kol]).** The theta divisor (or theta function) cannot have multiplicity greater than $g$ at any point of any principally polarized abelian variety. More generally, for any $g$-dimensional principally polarized abelian variety $A$ we have

$$\dim\{x \in A| \text{mult}_x \Theta \geq k\} \leq g - k.$$

*Proof.* We will use the above relation of the multiplicity of the divisor and the non-triviality of the corresponding multiplier ideal sheaf. In fact we will show below that the pair $(A, \Theta)$ is log-canonical. Then the theorem would follow: indeed if at some point $x \in A$ we had $\text{mult}_x \Theta > g$, then for some $\varepsilon$ sufficiently close to 1 we would also have $\text{mult}_x (\varepsilon \Theta) > g$ and thus by proposition 5.1 the ideal $\mathcal{J}(\varepsilon \Theta)$ would be non-trivial, which would be a contradiction. The bound for the dimension of the high-multiplicity set is obtained analogously.

So let us prove that $\mathcal{J}(A, \varepsilon \Theta) = \mathcal{O}_A$ for all $0 < \varepsilon < 1$. Assume the contrary: that for some $\varepsilon$ this is not the case, and then denote by $Z$ the zero locus of $\mathcal{J}(A, \varepsilon \Theta)$. We must clearly have then $Z \subset \Theta$. Consider the exact sequence

$$0 \to \mathcal{O}_A(\Theta) \otimes \mathcal{J}(A, \varepsilon \Theta) \to \mathcal{O}_A(\Theta) \to \mathcal{O}_Z(\Theta) \to 0,$$

and look at the corresponding long exact sequence for cohomology. Let us apply Nadel’s vanishing to $A$ with the integral divisor $L := \Theta$ and the $\mathbb{Q}$-divisor $D := \varepsilon \Theta$, so that indeed $L - D = (1 - \varepsilon)\Theta$ is ample. Since the canonical bundle $K_A$ is trivial, we get for $i > 0$

$$H^i(A, \mathcal{O}_A(K_A + L) \otimes \mathcal{J}(A, D)) = H^i(A, \mathcal{O}_A(\Theta) \otimes \mathcal{J}(A, \varepsilon \Theta)) = 0.$$
Thus from the long exact sequence we get the piece
\[ H^0(A, \mathcal{O}_A(\Theta)) \to H^0(Z, \mathcal{O}_Z(\Theta)) \to 0, \]
which must be exact. The first term of this sequence is just \( \mathbb{C} \), as \( \Theta \) has a unique section, and thus the first map is zero, as we are restricting the theta function to \( Z \), which lies entirely in its zero locus. Thus we must have \( H^0(Z, \mathcal{O}_Z(\Theta)) = 0 \); however, this is impossible as we can always construct a section by differentiating \( \Theta \) in some direction sufficiently many times (and using the fact that the derivative of a section \( s \) of some line bundle is the section of the same bundle when restricted to the zero set of \( s \)). Alternatively we can prove it by translating the theta divisor by some small \( a \in A \), so that there would certainly be a section (since \( \Theta_a \) meets \( \Theta \), and thus \( Z \subset \Theta \), properly), and then using semicontinuity to show that there is a section of \( \mathcal{O}_Z(\Theta) \) as well.

So we have arrived at a contradiction, and thus we must have \( Z \) empty, so that \( J(A, \varepsilon \Theta) = \mathcal{O}_A \) for all \( 0 < \varepsilon < 1 \), the pair \( (A, \Theta) \) is log-canonical, and thus the multiplicity of the theta divisor is at most \( g \) at all points. \( \Box \)

6. Asymptotic methods and plurigenera

In this and the next section we develop the necessary techniques and explain the proof of Siu’s theorem on invariance of plurigenera for varieties of general type, following the algebraic exposition in [Laz], and referring to [Siu1] for the original analytic proof. Let us recall the statement:

**Theorem 6.1 (Invariance of plurigenera).** If \( X \to \Delta \) is a family of varieties of general type over the unit disk \( \Delta \), then for \( t \in \Delta \) small enough the dimension of \( \Gamma(X_t, mK_{X_t}) \) is independent of \( t \), for all integer \( m \).

As explained in proposition 1.8, if we know that the higher cohomologies of \( mK_{X_0} \) vanish (and thus by semicontinuity the higher cohomologies of \( mK_{X_t} \) also vanish), then the invariance of plurigenera follows from the invariance of Euler characteristics. One way to try to ensure vanishing is by applying Nadel’s vanishing theorem. However, in Nadel’s vanishing the multiplier ideal enters, and thus we need to be able to control it, which means controlling the singularities of divisors \( D_m \in |mK_{X_0}| \). If we try to do this directly for all \( m \), the log-resolutions get out of hand. This is when we are aided by the fact that we are now working with \( \mathbb{Q} \)-divisors: philosophically what we can try to do is resolve some \( D_m \) for \( m \) very large and then take “roots” to resolve lower multiples of the canonical bundle. To do this, let us introduce the asymptotic multiplier ideals properly.
Definition 6.2. The Iitaka dimension of a linear system $|L|$ on a projective variety $X$ of dimension $n$ is the number $\kappa(L) := \lim_{N \to \infty} \frac{\log h^0(X, O_X(NL))}{\log N}$.

Notice that by definition $L$ is big if and only if $\kappa(L) = n$, and $X$ is of general type if and only if $\kappa(K_X) = n$.

Definition 6.3. For a linear system $|L|$ on $X$ with $\kappa(L) \geq 0$ the asymptotic multiplier ideal is defined to be the direct limit

$$J(X, c ||L||) := \lim_{N \to \infty} J(X, \frac{c}{N} |NL|).$$

Here the multiplier ideal of a linear system means that we choose a general divisor $D_N \in |NL|$ and compute the multiplier ideal $J(X, \frac{c}{N} D_N)$, and thus the non-negativity of the Iitaka dimension $\kappa(L)$ is needed to ensure the existence of such a section $D_N$ for all $N$ sufficiently large. The existence of the limit, and the fact that the ideals in the sequence grow as $m$ increases follows from the inclusion $J(X, \frac{c}{N} |NL|) \subset J(X, \frac{c}{kN} |kNL|)$, which holds for all integers $k > 0$.

In general the asymptotic multiplier ideals differ from the usual multiplier ideals. However, the following proposition provides us with an example when they coincide.

Proposition 6.4. If the ring of sections $\bigoplus_{m=1}^{\infty} \Gamma(X, O_X(mL))$ is finitely generated, then for $k \gg 0$ the equality $J(X, ||mkL||) = J(X, |mkL|)$ holds for all $m \geq 1$.

Proof. Indeed, let us choose once and for all a log-resolution for all the generators of the ring of sections simultaneously, and use it independent of $m$ — everything will get resolved, and thus we are done. \qed

Remark 6.5. In view of this proposition, the asymptotic multiplier ideals for linear systems with finitely generated rings of sections are easier to deal with. However, for the case of the canonical linear system that we are primarily concerned with, the finite generation is a very hard open problem. Indeed, if finite generation were known, the projectivization of the ring of sections — the pluricanonical ring — would provide a canonical model for $X$ and thus prove the minimal model program. Thus instead of using finite generation to understand the multiplier ideals, one may instead try to use multiplier ideals to tackle finite generation. This has not been achieved yet, but the proof of the invariance of plurigenera can be viewed as an indication that there may be hope in this approach.

One way to understand a linear series is via its base locus. We have the following easy observation...
Proposition 6.6. The base ideal (the ideal dual to the base locus) of a linear system is contained in the asymptotic multiplier ideal, i.e. $b(|L|) \subset \mathcal{J}(X, ||L||)$.

Proof. The proposition is proven by fixing $k \gg 0$ large enough that it computes the asymptotic multiplier ideal, i.e. such that $\mathcal{J}(X, ||L||) = \mathcal{J}(X, \frac{1}{k}|kL|)$, considering a log-resolution $\mu : X' \to X$ resolving both $|L|$ and $|kL|$, noting that $b(|L|)^k \subset b(|kL|)$ and remembering that the relative canonical class in the definition of the multiplier ideal is an effective divisor. □

We now formulate the appropriate vanishing theorem for asymptotic multiplier ideals.

Theorem 6.7 (see [Laz], 11.2.12). Let $|L|$ be a linear system on $X$ with $\kappa(L) \geq 0$. Then
(a) For any big and nef integral divisor $A$ we have
$$H^i(X, \mathcal{O}_X(K_X + mL + A)) \otimes \mathcal{J}(X, ||mL||) = 0$$
for all $m \geq 1$ and all $i > 0$.
(b) If we additionally assume $\kappa(L) = n$, i.e. that $L$ is big, then it is enough to take $A$ nef, and not necessarily big, in the above. In particular we can take $A = 0$ to get in this case
$$H^i(X, \mathcal{O}_X(K_X + mL)) \otimes \mathcal{J}(X, ||mL||) = 0.$$

Part (a) of this vanishing theorem follows from the fact that the usual multiplier ideals stabilize to the asymptotic ideal, and then applying the usual Nadel’s vanishing. To prove part (b) more work is needed, and it is achieved by essentially adapting the proof of Nadel’s vanishing to this case.

Corollary 6.8. In the setting of the theorem above, if $B$ is any ample globally generated bundle, then
$$\mathcal{O}_X(K_X + mL + A + nB) \otimes \mathcal{J}(X, m||L||)$$
is globally generated for all $m \geq 1$. If we additionally assume $\kappa(L) = n$, then $A$ does not need to be big, and in particular $A = 0$ can be taken.

Proof. From Castelnuovo-Mumford regularity (see [Laz], 1.8) we know that if for some very ample divisor $B$ and some sheaf $F$ we have $H^i(X, F \otimes \mathcal{O}_X(-iB)) = 0$, then $F$ is globally generated. Thus the global generation follows from the above vanishing theorem for the asymptotic multiplier ideals. □

This in particular implies an interesting characterization of big and nef bundles among all big bundles.
Proposition 6.9. A big linear system \(|L|\) is nef if and only if we have \(J(X, ||mL||) = \mathcal{O}_X\) for all \(m \geq 1\).

Proof of the “if” part. Assume that the asymptotic multiplier ideals are trivial. Consider then some very ample bundle \(B\), and let \(A = B\). Then the assumptions of the above corollary are satisfied, and so \(\mathcal{O}_X(K_X + (n + 1)B + mL) \otimes J(m||L||) = \mathcal{O}_X(K_X + (n + 1)B + mL)\) is big globally generated, and therefore nef. So for any effective curve \(C \subset X\) we have \((K_X + (n + 1)B) \cdot C + mL \cdot C \geq 0\). By taking the limit as \(m \to \infty\) it then follows that \(L \cdot C \geq 0\), and thus \(L\) itself is nef. \(\square\)

The theorems above are refinements of Nadel’s vanishing, and thus one can try to apply them to obtain deformation invariance of plurigenera. The result one gets is the following

Theorem 6.10 (Siu’s generalization of Levine’s theorem). If for some \(k\) there exists a divisor \(D \in |K_{X_0}|\) such that the pair \((X_0, D)\) is log-canonical, then the invariance of plurigenera holds for all the plurigenera \(H^i(X_t, \mathcal{O}_{X_t}((m + 1)K_{X_t}))\) for \(m < k\).

Proof. Indeed, \(X_0\) is of general type, so \(K_{X_0}\) is big, and we can apply to it part (b) of theorem 6.7 to get

\[ H^i(X_0, \mathcal{O}_{X_0}((m + 1)K_{X_0})) \otimes J(X_0, ||mK_{X_0}||) = 0. \]

By the log-canonicity assumption \(J(X_0, mK_{X_0}) = J(X_0, ||mK_{X_0}||) = \mathcal{O}_{X_0}\) for any \(m < k\), and thus

\[ J(X_0, m||K_{X_0}||) = \mathcal{O}_{X_0} = J(X_0, ||mK_{X_0}||). \]

Therefore the vanishing above becomes \(H^i(X_0, \mathcal{O}_{X_0}((m + 1)K_{X_0})) = 0\), and the invariance of plurigenera \(H^i(\mathcal{O}_{X_t}((m + 1)K_{X_t}))\) for \(m < k\) follows as before in section 1.8 for the big and nef case. \(\square\)

7. Proof of Siu’s theorem on the invariance of plurigenera

We will now explain the proof of the invariance of plurigenera for arbitrary varieties of general type. This is more delicate and more technical than the considerations from the previous section, as the vanishing by itself does not suffice. The trick will be to compare the asymptotic multiplier ideals for multiples of \(K_{X_0}\) and for the multiples of \(K_X\), then restricted to \(X_0\). Indeed, given any line bundle \(L\) on the total space \(X\) of the family, we can consider the restriction maps

\[ \phi_k : \Gamma(X, \mathcal{O}_X(kL)) \to \Gamma(X_0, \mathcal{O}_{X_0}(kL_0)) \]
The images of these maps define a graded family of linear systems, and thus we can consider the associated asymptotic multiplier ideal as before, which we denote by

$$J(X_0, ||L||_0) := \lim_{k \to \infty} J(X_0, \frac{1}{k} \phi_k \left( |\Gamma(X, \mathcal{O}_X(kL))| \right)),$$

where the subscript of 0 after the notation of the asymptotic multiplier ideal signifies the fact that we take sections of $L$ first and then restrict to the fiber over zero.

By working carefully with the definition of $J(X_0, ||L||_0)$, we can see that ([Laz], 11.5.5)

$$\Gamma(X_0, \mathcal{O}_{X_0}(K_{X_0} + L_0) \otimes J(X_0, ||L||_0))$$

lies inside the image of the restriction map

$$\Gamma(X, \mathcal{O}_X(K_X + L)) \to \Gamma(X_0, \mathcal{O}_{X_0}(K_{X_0} + L_0)),$$

essentially since the asymptotic multiplier ideal comes from restrictions of sections over $X$.

Thus for the case of $L = (m - 1)K_X$ in the above, if we could show that the whole space of sections of $K_{X_0} + L_0$ comes from the sections in $(\ast)$, i.e. that

$$\Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes J(X_0, ||(m - 1)K_X||_0)) = \Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0})), \quad (\ast\ast)$$

the invariance of plurigenera would follow, as we would have all sections of $mK_{X_0}$ lying in the image of restricting from $mK_X$, i.e. we would then see that all sections of $mK_{X_0}$ can be extended to sections of $mK_X$, and thus we would be done.

Thus what we need to prove is $(\ast\ast)$, the fact that all the sections of $mK_{X_0}$ vanish along $J(X_0, ||(m - 1)K_X||_0))$. We now notice that there is another asymptotic multiplier ideal that can be naturally defined, which is just

$$J(X_0, ||mK_{X_0}||) = \lim_{k \to \infty} J \left( X_0, \frac{1}{k} \left| \Gamma(X_0, \mathcal{O}_{X_0}(kmK_{X_0})) \right| \right).$$

By proposition 6.6 (base locus is contained in the asymptotic multiplier ideal) we know that the sections of $mK_{X_0}$ indeed do vanish along this ideal, i.e. that

$$\Gamma \left( X_0, \mathcal{O}_{X_0}(mK_{X_0}) \otimes J(X_0, ||mK_{X_0}||) \right) = \Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0})).$$

Thus in particular if somehow we had

$$J(X_0, ||mK_{X_0}||) \subset J(X_0, ||(m - 1)K_X||_0),$$
the statement (**) would follow and we would be done — but of course there is no reason for this inclusion to hold. However, there is a weaker version that does hold, which still suffices to finish the proof.

**Claim 7.1.** There exists an integer $a$ independent of $m$, and a pluricanonical section $s$ of $aK_{X_0}$ with zero locus $D := \{s = 0\} \subset X_0$, such that

$$\mathcal{J}(X_0, ||mK_{X_0}||)(-D) \subset \mathcal{J}(X_0, ||(m + a - 1)K_X||_0).$$

The point of the claim is that though the two asymptotic multiplier ideals we have defined are not the same, they are “at most $a$ off from each other”, and, very importantly, that this $a$ is independent of the multiple $m$ of the canonical bundle that we take. Given the claim, the proof of the invariance of plurigenera is obtained in the following way.

Fix some $f \in \Gamma(X_0, mK_{X_0})$; we want to show that $f$ vanishes along the ideal $\mathcal{J}(X_0, ||(m-1)K_X||_0)$. Consider the section $f^N \cdot s \in \Gamma(X_0, (mN + a)K_{X_0})$ for $N$ very large. It vanishes along $\mathcal{J}(X_0, ||mNK_{X_0}||)$ (since $f^N$ vanishes there), and also on $D$, since $s$ vanishes there. By the claim it then follows that $f^N \cdot s$ vanishes along $\mathcal{J}(X_0, ||(mN + a - 1)K_X||_0)$. We now use fact that for all multiplier ideals we have $\mathcal{J}(X, ||kL||) \subset \mathcal{J}(X, ||L||)^k$ (this is known as the subadditivity of the multiplier ideals, see [Laz] 9.5.B; notice that for base ideals, which are subideals of the multiplier ideals by proposition 6.6 the inclusions peculiarly goes the other way) this means that $f^N \cdot s$ vanishes along $\mathcal{J}(X_0, ||mK_X||_0)^N$. Since this is the case for all $N$, by the claim this should imply that $f$ vanishes along $\mathcal{J}(X_0, ||(m - 1)K_X||_0)$, once we choose $N \gg a$. Conceptually we could actually try to say that $f$ vanishes on $\mathcal{J}(X_0, ||(m - \varepsilon)K_X||_0)$, but we do not need this. Technically we need to use here the integral closure of ideals, and the details can be found in [Laz], 11.5.6. Thus the invariance of the plurigenera for varieties of general type is obtained once we prove the claim.

**Proof of claim 7.1.** The idea is that we want to add a very ample fixed piece to the canonical divisor, so that everything becomes positive enough so that we can handle the computation. In doing this we will be aided by the so-called Kodaira’s lemma: the statement that given any big divisor $L$ and any divisor $F$ there is always a multiple of $L$ “bigger” than $F$, i.e. that for some $a$ there exists an effective divisor in the linear system $|aL - F|$.

So choose a very ample bundle $B$ on $X$, positive enough so that the linear system $F := 2K_X + (n+1)B$, where $n = \dim X_0$, is basepoint free. Using Kodaira’s lemma, choose $a$ such that there exists an effective divisor $D$ in the linear series $|aK_X - F|$ — this $D$ is then the zero
set of some section \( s \in \Gamma(X, \mathcal{O}_X(aK_X)) \). We will now show inductively that the statement of the claim holds with these \( a, s|_{X_0}, \) and \( D_0 := D|_{X_0} \) (in general the subscript 0 will denote restricting divisors from \( X \) to \( X_0 \)).

To start the induction we need to show that

\[
\mathcal{J}(X_0, ||K_{X_0}||)(-D_0) \subset \mathcal{J}(X_0, ||aK_X||_0).
\]

First notice that since the linear system \(|2F|\) is basepoint free, a general divisor \( E \in |2F| \) is non-singular. Moreover, the intersection of a general such \( E \) with any subvariety is also non-singular, and a generic \( E \) intersects any smooth subvariety transversely. Thus if we do a log-resolution for \((X, D)\), it also gives a log-resolution for \((X, D + cE)\) for any constant \( c \). Therefore it follows from the definition of the multiplier ideal that \( \mathcal{J}(X, D) = \mathcal{J}(X, D + cE) \) for all \( 0 < c < 1 \) — this statement is sometimes known as Kollár-Bertini theorem. So finally the base of induction is obtained by observing that

\[
\mathcal{J}(X_0, ||K_{X_0}||)(-D_0) \subset \mathcal{O}_{X_0}(-D_0) = \mathcal{J}(X_0, D_0) = \mathcal{J}(X_0, ||aK_{X_0}||_0).
\]

Lastly, we need to prove the step of induction: that if the inclusion of the claim holds for \( k \), then it also holds for \( k + 1 \). Let us twist both sides of the claim by \( \mathcal{O}_{X_0}((k + a)K_{X_0}) \) — it is enough to prove that the inclusion holds then. Writing down what we get after this twist, on the left-hand-side of the claim for \( k + 1 \) we get

\[
\mathcal{O}_{X_0}(((k + a)K_{X_0} - D_0) \otimes \mathcal{J}(X_0, ||(k + 1)K_{X_0}||), \quad (LHS)
\]

which is globally generated, since \( aK_X - D \) is linearly equivalent to \( 2K_X + (n + 1)B \), and thus we can use corollary 6.8 with big \( L := K_X \), \( m := k + 1 \), and \( A := 0 \).

By the induction hypothesis we know that any section \( u \) of \((k + a)K_{X_0} \) that vanishes along \( \mathcal{J}(X_0, ||(k + 1)K_{X_0}||)(-D_0) \) — this is what the sections of \((LHS)\) are — must also vanish along \( \mathcal{J}(X_0, ||(k + a - 1)K_X||_0) \). Thus from the inclusion of the space of sections \((*)\) in the image of the restriction map it follows that this \( u \) extends from the zero fiber, to give a section \( \tilde{u} \) of \((k + a)K_X\), which thus has to vanish along \( \mathcal{J}(X_0, ||(k + a)K||_0) \). So we have shown that all the sections \( u \) of \((LHS)\) in fact lie in

\[
\mathcal{O}_{X_0}(((k + a)K_{X_0}) \otimes \mathcal{J}(X_0, ||(k + a)K_X||_0), \quad (RHS)
\]

But this is exactly the right-hand-side of the claim for \( k + 1 \), and since the sheaf \((LHS)\) is globally generated, it means it is a subsheaf of
(RHS), which is precisely the statement of claim 7.1 that we want, for \( k + 1 \).

\[ \square \]

Analytic viewpoint, following \[ \text{[Siu1]} \]. The analytic version of the asymptotic multiplier ideal is obtained as follows. For a bundle \( L \) on \( X \) with \( \kappa(L) \geq 0 \) one chooses a basis \( s^{(k)}_1 \ldots s^{(k)}_{N_k} \) for \( \Gamma(X, \mathcal{O}_X(kL)) \), takes the \( k^{th} \) roots of these and then arranges those, for all \( k \), into a power series defining the singular metric. Formally, one chooses a sequence \( \{ a_k \} \) such that the sum

\[ e^{-2\phi} := \sum_{k=1}^{\infty} a_k \sum_{i=1}^{N_k} |s^{(k)}_i|^{2/k} \]

converges uniformly. In the analytic setting the sequence \( a_k \) can be chosen arbitrarily, but should be the same for both \( \Gamma(X_0, \mathcal{O}_{X_0}(mK_{X_0})) \) and for \( \Gamma(X, \mathcal{O}_X(mK_X))|_{X_0} \), so that the analytic multiplier ideals corresponding to the two resulting metrics (denote them by \( e^{-2\phi} \) and by \( e^{-2\tilde{\phi}} \)) that are constructed by using these series can then be compared.

The analytic multiplier ideal is determined by the singular behavior of the metric. Thus to compare two multiplier ideals one only needs to compare the singularities of the metrics. Technically this means that if one can show that the singularity of \( e^{-2m\phi} \) is worse than that of \( e^{-2m\tilde{\phi}} \) by some fixed amount (i.e. the pole order of the ratio is bounded, or something like that) for all \( m \), then the multiplier ideals corresponding to \( e^{-2\phi} \) and \( e^{-2\tilde{\phi}} \) coincide. The technical result needed here to show integrability is a generalization of an extension theorem of Ohsawa-Takegoshi and Manivel.

The comparison of the singularities of \( e^{-2m\phi} \) and of \( e^{-2m\tilde{\phi}} \), which is the analytic analog of claim 7.1, can be proven essentially the same way as in the algebraic setting, using again the extension theorems, and also Skoda’s technique for generating elements of multiplier ideals. \[ \square \]

Remark 7.2. Siu later established in \[ \text{[Siu3]} \] the technical analytical result that was the sticking point in extending the analytic techniques of \[ \text{[Siu1]} \] to arbitrary varieties, and thus proved the deformation invariance of plurigenera for all varieties, not necessarily of general type.

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