Analytic Continuation of $q$-Euler numbers and polynomials

Taekyun Kim

1 School of Electrical Engineering and Computer Science, Kyungpook National University, Taegu 702-701, S. Korea

Abstract In this paper we study that the $q$-Euler numbers and polynomials are analytically continued to $E_q(s)$. A new formula for the Euler’s $q$-Zeta function $\zeta_{E,q}(s)$ in terms of nested series of $\zeta_{E,q}(n)$ is derived. Finally we introduce the new concept of the dynamics of analytically continued $q$-Euler numbers and polynomials.

2000 Mathematics Subject Classification - 11B68, 11S40
Key words - $q$-Bernoulli polynomial, $q$-Riemann Zeta function

1 Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the ring of integers, the field of real numbers and the complex numbers, respectively.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex numbers or $p$-adic numbers. Throughout this paper, we will assume that $q \in \mathbb{C}$ with $|q| < 1$. The $q$-symbol $[x]_q$ denotes $[x]_q = \frac{1-q^x}{1-q}$. (see [1-16]).

In this paper we study that the $q$-Euler numbers and polynomials are analytically continued to $E_q(s)$. A new formula for the Euler’s $q$-Zeta function $\zeta_{E,q}(s)$ in terms of nested series of $\zeta_{E,q}(n)$ is derived. Finally we introduce the new concept of the dynamics of analytically continued $q$-Euler numbers and polynomials.

2 Generating $q$-Euler polynomials and numbers

For $h \in \mathbb{Z}$, the $q$-Euler polynomials were defined as

$$\sum_{n=0}^{\infty} \frac{E_n(x, h|q)}{n!} t^n = [2|q] \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t},$$ (2.1)
for $x, q \in \mathbb{C}$, cf. [1,7]. In the special case $x = 0$, $E_n(0, h|q) = E_n(h|q)$ are called $q$-Euler numbers, cf. [1,2,3,4]. By (2.1), we easily see that

$$E_n(x, h|q) = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^l + h q^{lx}}, \text{ cf.}[7, 8],$$

where $\binom{n}{l}$ is binomial coefficient. From (2.1), we derive

$$E_n(x, h|q) = (q^x E(h|q) + [x]_q)^n$$

with the usual convention of replacing $E^n(h|q)$ by $E_n(h|q)$. In the case $h = 0$, $E_n(x, 0|q)$ will be symbolically written as $E_n,q(x)$. Let $G_q(x, t)$ be generating function of $q$-Euler polynomials as follows:

$$G_q(x, t) = \sum_{n=0}^{\infty} E_n,q(x) t^n \frac{n!}{n!},$$

(2.3)

Then we easily see that

$$G_q(x, t) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t}.$$ 

(2.4)

For $x = 0$, $E_n,q = E_n,q(0)$ will be called $q$-Euler numbers.

From (2.3), (2.4), we easily derive the following: For $k (= \text{even})$ and $n \in \mathbb{Z}_+$, we have

$$E_{n,q}(k) - E_{n,q} = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n.$$ 

(2.5)

For $k (= \text{odd})$ and $n \in \mathbb{Z}_+$, we have

$$E_{n,q}(k) + E_{n,q} = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n.$$

(2.6)

By (2.4), we easily see that

$$E_{m,q}(x) = \sum_{l=0}^{m} \binom{m}{l} q^{xl} E_{l,q}[x]^m_l.$$ 

(2.7)

From (2.5), (2.6), and (2.7), we derive

$$[2]_q \sum_{l=0}^{k-1} (-1)^l[l]_q^n = (q^{kn} - 1) E_{n,q} + \sum_{l=0}^{k-1} \binom{n}{l} q^{kl} E_{l,q}[k]_q^{n-l},$$

(2.8)

where $k (= \text{even}) \in \mathbb{N}$. For $k (= \text{odd})$ and $n \in \mathbb{Z}_+$, we have

$$[2]_q \sum_{l=0}^{k-1} (-1)^l[l]_q^n = (q^{kn} + 1) E_{n,q} + \sum_{l=0}^{k-1} \binom{n}{l} q^{kl} E_{l,q}[k]_q^{n-l}.$$ 

(2.9)
3 $q$-Euler zeta function

It was known that the Euler polynomials are defined as

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!}t^n, \quad |t| < \pi, \text{ cf. [1-16]}.$$

(3.1)

For $s \in \mathbb{C}, x \in \mathbb{R}$ with $0 \leq x < 1$, define

$$\zeta_E(s, x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)s}, \quad \text{and} \quad \zeta_E(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

(3.2)

By (3.1) and (3.2) we see that Euler numbers are related to the Euler zeta function as

$$\zeta_E(-n) = E_n, \quad \zeta_E(-n, x) = E_n(x).$$

For $s, q, h \in \mathbb{C}$ with $|q| < 1$, we define $q$-Euler zeta function as follows:

$$\zeta_{E,q}(s, x|h) = [2]^q \sum_{n=0}^{\infty} \frac{(-1)^n q^{nh}}{[n + x]_q^s}, \quad \text{and} \quad \zeta_{E,q}(s|h) = [2]^q \sum_{n=1}^{\infty} \frac{(-1)^n q^{nh}}{[n]_q^s}.$$

(3.3)

For $k \in \mathbb{N}, h \in \mathbb{Z}$, we have

$$\zeta_{E,q}(-n|h) = E_n(h|q).$$

In the special case $h = 0$, $\zeta_{E,q}(s|0)$ will be written as $\zeta_{E,q}(s)$. For $s \in \mathbb{C}$, we note that

$$\zeta_{E,q}(s) = [2]^q \sum_{n=1}^{\infty} \frac{(-1)^n}{[n]_q^s}.$$

We now consider the function $E_q(s)$ as the analytic continuation of Euler numbers. All the $q$-Euler numbers $E_{n,q}$ agree with $E_q(n)$, the analytic continuation of Euler numbers evaluated at $n$,

$$E_q(n) = E_{n,q} \text{ for } n \geq 0.$$

Ordinary Euler numbers are defined by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n t^n}{n!}, \quad |t| < \pi.$$

(3.4)

By (3.4), it is easy to see that

$$E_0 = 1, \quad \text{and} \quad E_n = -\frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l, \quad n = 0, 1, 2, \ldots.$$

From (2.9) and (3.3), we can consider the $q$-extension of Euler numbers $E_n$ as follows:

$$E_{0,q} = \frac{[2]^q}{2}, \quad \text{and} \quad E_{n,q} = -\frac{1}{[2]^q} \sum_{l=0}^{n-1} \binom{n}{l} q^l E_{l,q}, \quad n = 1, 2, 3, \ldots.$$

(3.5)
In fact, we can express \( E'_q(s) \) in terms of \( \zeta'_{E,q}(s) \), the derivative of \( \zeta_{E,q}(s) \).

\[
E_q(s) = \zeta_{E,q}(-s), \quad E'_q(s) = \zeta'_{E,q}(-s), \quad E'_q(2n + 1) = \zeta'_{E,q}(-2n - 1),
\]

for \( n \in \mathbb{N} \cup \{0\} \). This is just the differential of the functional equation and so verifies the consistency of \( E_q(s) \) and \( E'_q(s) \) with \( E_{n,q} \) and \( \zeta(s) \).

From the above analytic continuation of \( q \)-Euler numbers, we derive

\[
E_q(s) = \zeta_{E,q}(-s), \quad E_q(-s) = \zeta_{E,q}(s) \Rightarrow E_{-n,q} = E_q(-n) = \zeta_{E,q}(n), n \in \mathbb{Z}._{+}.
\]

The curve \( E_q(s) \) runs through the points \( E_{-n,q} \) and grows \( \sim n \) asymptotically as \(-n \to -\infty\). The curve \( E_q(s) \) runs through the point \( E_q(-n) \) and

\[
\lim_{n \to -\infty} E_q(-n) = \lim_{n \to -\infty} \zeta_{E,q}(n) = -2.
\]

From (3.5), (3.6) and (3.7), we note that

\[
\zeta_{E,q}(-n) = E_q(n) \mapsto \zeta_{E,q}(-s) = E_q(s).
\]

## 4 Analytic continuation of \( q \)-Euler polynomials

For consistency with the redefinition of \( E_{n,q} = E_q(n) \) in (4.5) and (4.6), we have

\[
E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q}q^{ks}[x]_q^{n-k}.
\]

Let \( \Gamma(s) \) be the gamma function. Then the analytic continuation can be obtained as

\[
n \mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C}, \\
E_{k,q} \mapsto E_q(k + s - [s]) = \zeta_{E,q}(-(k + (s - [s]))), \\
\binom{n}{k} \mapsto \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\
\Rightarrow E_{n,q}(s) \mapsto E_q(s, w) = \sum_{k=1}^{[s]} \frac{\Gamma(1+s)E_q(k + s - [s])q^{k+s-[s]}w^{[s]-k}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\
= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s)E_q((k - 1) + s - [s])q^{([k-1]+s-[s])w^{[s]+1-k}}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)},
\]

where \([s]\) gives the integer part of \( s \), and so \( s-[s] \) gives the fractional part.

Deformation of the curve \( E_q(2, w) \) into the curve of \( E_q(3, w) \) via the real analytic continuation \( E_q(s, w), 2 \leq s \leq 3, -0.5 \leq w \leq 0.5 \).

**ACKNOWLEDGEMENTS.** This paper is supported by Jangjeon Mathematical Society and Jangjeon Research Institute for Mathematical Science and Physics( 2007-001-JRIMS 1234567)
References

[1] M. Cenkci, M. Can, Some results on q-analogue of the Lerch zeta function, Advan. Stud. Contemp. Math., Vol 12(2006), 213-223.

[2] M. Cenkci, The p-adic generalized twisted (h,q)-Euler-l-function and its applications, Advan. Stud. Contemp. Math., Vol 15(2007), 37-47

[3] T. Kim, q-Euler numbers and polynomials associated with p-adic q-integrals, J. Nonlinear Math. Phys., Vol 14(2007), 15-27.

[4] T. Kim, On p-adic interpolating function for q-Euler numbers and its derivatives, J. Math. Anal. Appl., Vol 339(2008), 598-608.

[5] T. Kim, A Note on p-Adic q-integral on \( \mathbb{Z}_p \) Associated with q-Euler Numbers, Advan. Stud. Contemp. Math., Vol 15(2007), 133-137.

[6] T. Kim, On p-adic q-L-functions and sums of powers, J. Math. Anal. Appl., Vol 329(2007), 1472-1481.

[7] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys., Vol 9 (2002), 288-299.

[8] T. Kim, A note on some formulas for the q-Euler numbers and polynomials, Proc. Jangjeon Math. Soc., Vol 9(2006), 227-232.

[9] T. Kim, Multiple p-adic L-function, Russ. J. Math. Phys., Vol 13(2006), 151-157.

[10] T. Kim, On explicit formulas of p-adic q-L-functions, Kyushu J. Math., Vol 43(1994), 73-86.

[11] A. Kudo, A congruence of generalized Bernoulli number for the character of the first kind, Advan. Stud. Contemp. Math., Vol 2(2000), 1-8.

[12] Q.-M. Luo, F. Qi, Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials, Advan. Stud. Contemp. Math., Vol 7(2003), 11-18.

[13] Q.-M. Luo, Some recursion formulae and relations for Bernoulli numbers and Euler numbers of higher order, Advan. Stud. Contemp. Math., Vol 10 (2005), 63-70.

[14] H. Ozden, Y. Simsek, S.H. Rim, I. Cangul, A note on p-adic q-Euler measure, Advan. Stud. Contemp. Math., Vol 14(2007), 233-239.

[15] Y. Simsek, Theorem on twisted L-function and twisted Bernoulli numbers, Advan. Stud. Contemp. Math., Vol 12(2006), 237-246.

[16] C. S. Ryoo, T. Kim, R. P. Agarwal, Exploring the multiple Changhee q-Bernoulli polynomials, Inter. J. Comput. Math., Vol 82(2005), 483-493.