A (1+3) Relativistic Harmonic Oscillator Simulated by an Anti-de Sitter Background†

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Abstract

It is shown that a static (1 + 3) anti-de Sitter metric defines, in a natural way, a relativistic harmonic oscillator in Minkowski space. The quantum theory can be solved exactly and leads to wave functions having a significantly different behaviour with respect to the non-relativistic ones. The energy spectrum coincides, up to the ground state energy, with that of the non-relativistic oscillator.

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1 Introduction

The non-relativistic oscillator is one of the simplest and most useful systems in physics. However, despite of its simplicity, there is not a well-established relativistic generalization in the literature. The first proposal for a relativistic harmonic oscillator was given by Yukawa [1], followed by the work of Feynman et al. [2] and further developed in [3]. These works are all based in the naive covariant generalisation $x^\mu x_\mu$ of the non-relativistic potential thus leading to quantum timelike excitations, the interpretation of which presents some difficulties. On the other hand, Itô et al. [4] (see also [5, 6]), introduced a Dirac equation which is linear in both coordinates and momenta. In the non-relativistic limit, the equation satisfied by the large components is that of ordinary oscillator with a spin-orbit coupling term.

Recently a new proposal for the relativistic harmonic oscillator was outlined in Ref. [7] (see also [8]). It is based in the natural generalization of the symmetry algebra of quantum operators of a relativistic free system (i.e., the Poincaré algebra)

$$[\hat{E}, \hat{x}] = -i \frac{\hbar}{m} \hat{p}, \quad [\hat{E}, \hat{p}] = 0, \quad [\hat{x}, \hat{p}] = i\hbar(1 + \frac{1}{mc^2} \hat{E}),$$

(1)

and that of a non-relativistic harmonic oscillator (i.e., the Lie algebra of the Newton group)

$$[\hat{E}, \hat{x}] = -i \frac{\hbar}{m} \hat{p}, \quad [\hat{E}, \hat{p}] = im\omega^2 \hbar \hat{x}, \quad [\hat{x}, \hat{p}] = i\hbar \hat{E}.$$

(2)

$\hat{E}$, $\hat{p}$ and $\hat{x}$ are the energy (with the rest-mass energy subtracted), momentum and boost operators in the center of momentum frame. The proposed symmetry for the (1+1) relativistic oscillator was defined in terms of the unique Lie algebra which allows to be contracted to the above algebras

$$[\hat{E}, \hat{x}] = -i \frac{\hbar}{m} \hat{p}, \quad [\hat{E}, \hat{p}] = im\omega^2 \hbar \hat{x}, \quad [\hat{x}, \hat{p}] = i\hbar(1 + \frac{1}{mc^2} \hat{E}).$$

(3)

This Lie algebra corresponds to that of the $SO(1,2)$ group. Related approaches can be seen in ([9, 10, 11, 12])

The aim of this paper is to further elaborate on this proposal. In Sect.2 we shall interpret it geometrically showing that a static (1 + 1) anti-de Sitter metric can be used to simulate a one-dimensional relativistic oscillator. The results obtained in this way can be seen as complementary to those found by group theoretical methods ([14, 15, 16]) for the one-dimensional oscillator [7, 13]. The main goal of this paper is to extend to three spatial dimensions the proposal (3) for a (1 + 1) relativistic oscillator. This will be done in Sect.3 making use
of the geometrical interpretation developed in Sect.2. The quantum theory will be obtained by solving the corresponding Klein-Gordon equation and leads to wave functions having an appreciably different behaviour with respect to the non-relativistic ones. In Sect.4 we shall state our conclusions.

2 Anti-de Sitter space and the one-dimensional relativistic oscillator

The basic ingredient in the proposal of Ref.[4] for a relativistic oscillator is the \( SO(1, 2) \) group symmetry. The Lie algebra commutators of this group can be thought of as the natural generalization of those of the relativistic free particle and the non-relativistic oscillator. Due to anti-de Sitter (AdS) space is a homogeneous space of the \( SO(1, 2) \) group it is therefore natural to regard the harmonic oscillator interaction in \((1 + 1)\) Minkowski space as equivalent to a free system in AdS space (more precisely, in its universal covering space, which has the topology of \( \mathbb{R}^2 \)).[14] However, this definition for a relativistic oscillator is incomplete. Owing to general covariance we must specify which particular AdS metric properly simulates the relativistic oscillator interaction in Minkowski space. To solve this ambiguity we can resort to the non-relativistic limit. To adjust the non-relativistic limit we have to choose \( g_{00} \) as

\[
g_{00} = 1 + \frac{\omega^2}{c^2} x^2. \tag{4}\]

The requirement of having an AdS geometry, in particular the condition of constant scalar curvature, determines the remaining components of the metric

\[
ds^2 = (1 + \frac{\omega^2}{c^2} x^2) c^2 dt^2 - \frac{1}{1 + \frac{\omega^2}{c^2} x^2} dx^2. \tag{5}\]

The geodesic trajectories of motion can be obtained from the lagrangian

\[
\mathcal{L} = -mc \sqrt{1 - \frac{1}{1 + \frac{\omega^2}{c^2} x^2} v^2 + \frac{\omega^2}{c^2} x^2}, \tag{6}\]

where \( m \) is the (reduced) mass of the system. The underlying \( SO(1, 2) \) symmetry is realized by Poisson brackets between the three constants of motion

\[
m g_{\mu\nu} f_{(a)}^\mu \frac{dx^\nu}{d\tau}, \tag{7}\]
where $f_\mu^a$, $a = 1, 2, 3$ are the Killing vectors of (3)

$$f_1^\mu = (1, 0)$$

$$f_2^\mu = \left( \frac{\omega}{c} x, \sqrt{1 + \frac{\omega^2}{c^2} x^2} \cos \omega t, \sqrt{1 + \frac{\omega^2}{c^2} x^2} \sin \omega t \right)$$

$$f_3^\mu = \left( -\frac{\omega}{c} x, \sqrt{1 + \frac{\omega^2}{c^2} x^2} \sin \omega t, \sqrt{1 + \frac{\omega^2}{c^2} x^2} \cos \omega t \right).$$

The Killing vectors realize the algebra (3). $f_1^\mu$, $f_2^\mu$ and $f_3^\mu$ lead to the energy, boost and momentum generators respectively. It is also interesting to note that the periodic character of the trajectories of motion of the oscillator can be traced back to the existence of closed time-like lines in AdS space.

From (3) it is straightforward to compute the hamiltonian

$$H^2 = m^2 c^4 + p^2 c^2 + m^2 \omega^2 c^2 x^2 + 2\omega^2 x^2 p^2 + \frac{\omega^4}{c^2} x^4 p^2. \quad (11)$$

The quantum wavefunctions can be obtained from the Schrödinger-type equation associated with (11). With the standard substitutions $H \rightarrow i\hbar \frac{\partial}{\partial t}$, $x \rightarrow x$, $p \rightarrow -i\hbar \frac{\partial}{\partial x}$ and introducing the parameters $\alpha$, $\beta$ to account for the normal ordering ambiguities of the classical function (11)

$$x^4 p^2 \rightarrow -\hbar^2 \left( x^4 \frac{d^2}{dx^2} + 4x^3 \frac{d}{dx} + \alpha x^2 \right), \quad (12)$$

$$x^2 p^2 \rightarrow -\hbar^2 \left( x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + \beta \right), \quad (13)$$

the Schrödinger equation implies the Klein-Gordon equation in AdS space (14)

$$(\Box + \frac{m^2 c^2}{\hbar^2} + \xi R)\phi = 0,$$

where $\Box$ is the D’Alembertian operator for the AdS metric (3), $R = -2\frac{\omega^2}{c^2}$ the scalar curvature and $\xi$ a numerical factor. The equivalence is obtained through the transformation

$$\Psi = \frac{1}{\sqrt{1 + \frac{\omega^2}{c^2} x^2}} \phi,$$

and requires a restriction on the parameters $\alpha$ and $\beta$: $\alpha = \xi + \frac{1}{2}$, and $\beta = 2\xi + 2$. 

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It is worthwhile to remark that the transformation (13) can be understood in terms of the Klein-Gordon scalar product in curved space (see, for instance [15])

\[
\langle \phi_1 | \phi_2 \rangle = i \int_{\Sigma} d\sigma \sqrt{g} g^{\mu \nu} (\phi_1 \partial_\nu \phi_2^*),
\]

where \( \Sigma \) is the initial value hypersurface. For the line element (5) and choosing \( \Sigma \) as \( t = 0 \), (16) becomes

\[
\langle \phi_1 | \phi_2 \rangle = -i \int dx \frac{1}{1 + \frac{\omega^2}{c^2} x^2} (\phi_1 \partial_0 \phi_2^*).
\]

For stationary states the scalar product (17) is proportional to the standard scalar product of Schrödinger wave functions

\[
\langle \Psi_1 | \Psi_2 \rangle = \int dx \Psi_1 \Psi_2^*.
\]

From now on we shall be mainly concerned with the Klein-Gordon equation to study the quantum theory of the relativistic oscillator. We shall also keep free the parameter \( \xi \).

The D’Alembertian operator for the AdS metric (3) is

\[
\Box = \frac{1}{1 + \frac{\omega^2}{c^2} x^2} \frac{1}{c^2} \partial_t^2 - 2 \frac{\omega^2}{c^2} \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} x \partial_x - (1 + \frac{\omega^2}{c^2} x^2) \frac{\partial^2}{\partial x^2}.
\]

To solve the Klein-Gordon equation we shall first look for positive frequency states. To find the spatial dependence of the wave functions we shall further propose the ansatz

\[
\phi(x, t) = e^{-i \omega t} (1 + \frac{\omega^2}{c^2} x^2)^{-\frac{\lambda}{2}} \phi^\lambda(x),
\]

where \( \lambda \) is an arbitrary positive parameter. The wave function (20) verifies the Klein-Gordon equation (14) if the function \( \phi^\lambda(x) \) satisfies the following differential equation

\[
\left\{ (1 + \frac{\omega^2}{c^2} x^2) \frac{d^2}{dx^2} - 2 \frac{\omega^2}{c^2} (\lambda - 1) x \frac{d}{dx} + \frac{\omega^2}{c^2} \left( \lambda(\lambda - 1) - N^2 + 2 \xi \right) \right\} \phi^\lambda(x) = 0,
\]

where \( N = \frac{mc^2}{\hbar \omega} \). In terms of the variable \( w = -\frac{\omega^2}{c^2} x^2 \), (21) is a standard hypergeometric equation

\[
\left\{ (1 - w) \frac{d^2}{dw^2} + \left( \frac{1}{2} - (\frac{3}{2} - \lambda) w \right) \frac{d}{dw} - \frac{1}{4} \left( \lambda(\lambda - 1) - N^2 + 2 \xi \right) \right\} \phi^\lambda(w) = 0.
\]
The regularity condition at infinity restricts the allowed values of the \( \lambda \) parameter. We obtain
\[
\lambda = \frac{1}{2} + n + \gamma ,
\]
where \( n = 0, 1, 2, \ldots \) and
\[
\gamma = \frac{1}{2} \sqrt{1 + 4N^2 - 8\xi}.
\]
The energy spectrum is then
\[
E = \left( \frac{1}{2} + n + \frac{1}{2} \sqrt{1 + 4\frac{m^2c^4}{\hbar^2\omega^2} - 8\xi} \right) \hbar \omega .
\]
We have chosen the positive sign of the square root of (24) to fit the non-relativistic approximation of (25). The above equation makes clear the physical meaning of the \( \xi \) parameter. It is related with the zero-point energy of system.

Introducing the variable \( z = -i\omega c \) and for the discrete values (23) the equation (21) turns out to be the equation of Gegenbauer polynomials [16]
\[
\left\{ (1 - z^2) \frac{d^2}{dz^2} - (1 - 2(n + \gamma)) z \frac{d}{dz} - n(n + 2\gamma) \right\} \phi^\lambda_n(z) = 0.
\]
Therefore the Klein-Gordon energy-eigenstates are
\[
\phi^\gamma_n(x, t) = e^{-iEt} \left( 1 + \frac{\omega^2}{c^2} x^2 \right)^{-\frac{\lambda}{2}} C^{-(n+\gamma)} \left( -i\frac{\omega}{c} x \right),
\]
and, in terms of the Schrödinger wave functions (see (15)), they are
\[
\Psi^\gamma_n(x, t) = N^\gamma_n e^{-iE} \left( 1 + \frac{\omega^2}{c^2} x^2 \right)^{-\frac{\lambda+1}{2}} C^{-(n+\gamma)} \left( -i\frac{\omega}{c} x \right),
\]
where \( N^\gamma_n \) are normalization constants
\[
(N^\gamma_n)^{-2} = \sqrt{\frac{\hbar \pi}{m\omega}} \frac{4^n}{n!} \frac{(2\gamma)!}{(2\gamma+n)!} \left[ \frac{(\gamma+n)!}{(\gamma)!} \right]^2 \frac{1}{\gamma + n + \frac{1}{2}} \Gamma(\gamma + 1) \Gamma(\gamma + \frac{1}{2}).
\]
We must stress that the Hermite polynomials \( H_n(\zeta) \), where \( \zeta = \sqrt{\frac{m\omega}{\hbar}} x \), are naturally recovered in the non-relativistic limit
\[
\lim_{c \to \infty} \frac{i^n n!}{N^\gamma_n^{\frac{1}{2}}} C^{-(n+\gamma)} \left( -i\frac{\zeta}{\sqrt{N}} \right) = H_n(\zeta),
\]
we also have
\[
\lim_{c \to \infty} \frac{N^n}{(n!)^2} (N^\gamma_n)^2 = \sqrt{\frac{m\omega}{\hbar \pi}} \frac{1}{n!2^n}.
\]
where the r.h.s. of (31) are the normalization constants of the Hermite polynomials. Therefore, the l.h.s. of (30), without the limit, can be seen as a relativistic generalization of the Hermite polynomials. In fact, it is not difficult to check that they are proportional to the so-called relativistic Hermite polynomials of Ref. ([13, 7]).

To finish this section we would like to comment on the issue of the zero-point energy. From a group theoretical point of view the quantum wave functions should provide a carrier space for irreducible lowest weight representations of the symmetry group $SO(1, 2)$. The dimensionless parameter $E_0/\hbar \omega = \frac{1}{2} + \gamma$ is the lowest weight and characterizes the representation. For the (1 + 1) relativistic oscillator we have $E_0/\hbar \omega > \frac{1}{2}$. This means that not all the lowest weight representations ($E_0/\hbar \omega \in [0, +\infty[$) can be realized physically. The natural barrier $E_0/\hbar \omega = \frac{1}{2}$ corresponds to the so-called Mock representation [17].

3 The three-dimensional relativistic oscillator

In this section we shall extend our study of the one-dimensional relativistic oscillator to the three-dimensional case. To this end we shall first find out the appropriate form of the (1 + 3) metric. Imposing the non-relativistic limit and using the spherical symmetry of the (1 + 3) AdS space we can write

$$ds^2 = (1 + \frac{\omega^2}{c^2} r^2)c^2 dt^2 - F(t, r) dr^2 - G(t, r) (d\theta^2 + \sin^2 \theta d\phi^2),$$

where $0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$ are the usual spherical coordinates. Moreover, to recover the one-dimensional oscillator when the angular coordinates are frozen we have to choose the function $F$ as follows

$$F(t, r) = \frac{1}{1 + \frac{\omega^2}{c^2} r^2}. \quad (33)$$

Imposing now the anti-de Sitter geometry we find that the appropriate line element should read as

$$ds^2 = (1 + \frac{\omega^2}{c^2} r^2)c^2 dt^2 - \frac{dr^2}{1 + \frac{\omega^2}{c^2} r^2} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (34)$$

The geodesics of (34) can be derived from the lagrangian

$$\mathcal{L} = -mc \sqrt{1 + \frac{\omega^2}{c^2} r^2 - \frac{v^2}{c^2} + \frac{\omega^2}{c^4} (\vec{x} \cdot \vec{v})^2} \quad (35)$$
and, according with our scheme, we can view the above lagrangian as defining a three-dimensional relativistic oscillator in Minkowski space. The $SO(3, 2)$ symmetry (i.e., the generalization of the three-dimensional Poincaré and Newton algebras)

$$[\hat{E}, \hat{x}^i] = -i \frac{\hbar}{m} \hat{p}^i, \quad [\hat{E}, \hat{p}^i] = i m \omega^2 \hbar \hat{x}^i, \quad [\hat{x}^i, \hat{p}^j] = i \hbar (1 + \frac{1}{mc^2} \hat{E}) \delta^{ij}. \quad (36)$$

(we have omitted the rotation generators) can be realized now by the Killing vectors of (34).

The next step now is to compute the hamiltonian. We obtain

$$H^2 = (1 + \frac{\omega^2}{c^2} r^2) (m^2 c^4 p^2 c^2 + \omega^2 (\vec{x} \cdot \vec{p})^2). \quad (37)$$

As in the one-dimensional oscillator it is not difficult to test that the Schrödinger equation associated with the hamiltonian (37) can be transformed, with a particular normal-ordering prescription, into a Klein-Gordon equation. Introducing the parameters $\alpha, \beta, \eta, \zeta$ for the operator ordering ambiguities of the classical function (37)

$$x_i^4 p_i^2 \rightarrow -\hbar^2 (x_i^4 \frac{\partial^2}{\partial x_i^2} + 4 x_i^3 \frac{\partial}{\partial x_i} + \alpha x_i^2), \quad (38)$$

$$x_i^2 p_i^2 \rightarrow -\hbar^2 (x_i^2 \frac{\partial^2}{\partial x_i^2} + 2 x_i \frac{\partial}{\partial x_i} + \beta), \quad (39)$$

$$x_i^3 p_i \rightarrow -i \hbar (x_i^3 \frac{\partial}{\partial x_i} + \eta), \quad (40)$$

$$x_i p_i \rightarrow -i \hbar (x_i \frac{\partial}{\partial x_i} + \zeta), \quad (41)$$

the Schrödinger equation leads to the Klein-Gordon equation in AdS space with metric (34) ($R = -12 \frac{\omega^2}{c^2}$). The Schrödinger and Klein-Gordon wave functions are related by

$$\Psi = \frac{1}{\sqrt{1 + \frac{\omega^2}{c^2} r^2}} \phi. \quad (42)$$

The normal ordering parameters are then fixed as

$$\alpha = \frac{11}{2} + 8 \xi, \quad \beta = \frac{1}{2} + 2 \xi, \quad \eta = \frac{3}{2}, \quad \zeta = \frac{1}{2}. \quad (43)$$

So, there is only one free parameter left, which is essentially the curvature factor $\xi$. 

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Now we want to solve the corresponding Klein-Gordon equation. The D’Alembertian operator is given by

\[ \Box = \frac{1}{1 + \frac{\omega^2}{c^2} r^2} \frac{\partial^2}{\partial t^2} - \frac{2 \omega^2}{c^2} r \frac{\partial}{\partial r} - \frac{1 + \omega^2 r^2}{c^2} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{\vec{L}^2}{r^2}, \tag{44} \]

where \( \vec{L}^2 \) is the orbital angular momentum operator

\[ \vec{L}^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \tag{45} \]

Separation of variables and an ansatz analogous to that of (20) leads to the following wave functions

\[ \phi(\vec{x}, t) = e^{-i \lambda \omega t} Y^l_m(\theta, \varphi) (1 + \frac{\omega^2}{c^2} r^2)^{-\frac{l}{2}} r^l \phi^\lambda_l(r), \tag{46} \]

where \( Y^l_m(\theta, \varphi) \) are the spherical harmonics and the functions \( \phi^\lambda_l(r) \) are required to verify the equation

\[ \left\{ (1 + \frac{\omega^2}{c^2} r^2) \frac{d^2}{dr^2} - \frac{2 \omega^2}{c^2} r (\lambda - l - 2) \frac{d}{dr} + 2 \frac{l + 1}{r} \frac{d}{dr} + \frac{\omega^2}{c^2} (\lambda(\lambda - 2l - 3) + l(l + 3) - N^2 + 12 \xi) \right\} \phi^\lambda_l(r) = 0. \tag{47} \]

In terms of the variable \( \rho = -\frac{\omega^2}{c^2} r^2 \), the above equation turns out to be a hypergeometric differential equation

\[ \left\{ (1 - \rho) \rho \frac{d^2}{d\rho^2} + \left( l + \frac{3}{2} - (l + \frac{5}{2} - \lambda) \rho \right) \frac{d}{d\rho} - \frac{1}{4} \left( \lambda(\lambda - 2l - 3) + l(l + 3) - N^2 + 12 \xi \right) \right\} \phi^\lambda_l(\rho) = 0. \tag{48} \]

The regularity condition at the origin and the square integrability of the wave functions yield to the following energy spectrum

\[ E = \left( \frac{3}{2} + 2n + l + \frac{1}{2} \sqrt{9 + \frac{4m^2 c^4}{\hbar^2 \omega^2} - 48 \xi} \right) \hbar \omega, \tag{49} \]

where \( n, l = 0, 1, 2, 3, \ldots \) We observe again that the energy spectrum coincides, up to the zero-point energy, with that of the non-relativistic limit. In the non-relativistic limit \( c \to \infty \) the spectrum behaves as

\[ E \xrightarrow{c \to \infty} E^{NR} + mc^2, \tag{50} \]
where $E^{NR} = \left( \frac{3}{2} + 2n + l \right) \hbar \omega$ is the ordinary energy spectrum of three-dimensional non-relativistic oscillator.

For the values (49) the regular hypergeometric functions solving (48) are

$$\phi_{nl}^\lambda(r) = \, _2F_1(-n, n + l + \frac{3}{2} - \lambda, l + \frac{3}{2}; -\frac{\omega^2}{c^2} r^2),$$

(51)

where $\lambda = \frac{E}{\hbar \omega}$. Rewriting the dimensionless parameter $\frac{\omega^2}{c^2} r^2$ as $\frac{m \omega}{\hbar} N r^2$ it is easy to check that, in the limit $c \to \infty$, the functions (51) become the confluent hypergeometric ones appearing in the non-relativistic wave functions

$$\lim_{c \to \infty} \, _2F_1(-n, n + l + \frac{3}{2} - \lambda, l + \frac{3}{2}; -\frac{m \omega}{\hbar} N r^2) = \, _1F_1(-n, l + \frac{3}{2}; \frac{m \omega}{\hbar} r^2).$$

(52)

Taking into account the relation of Jacobi polynomials with the hypergeometric functions [16]

$$\, _2F_1(-n, n + \alpha + \beta + 1, \alpha + 1; z) = \frac{n!}{(\alpha + 1)_n} P_n^{(\alpha, \beta)}(1 - 2z),$$

(53)

a orthonormal basis for the Hilbert space is given by

$$\Psi_{nlm}(\vec{x}, t) = C_{nlm}^\gamma e^{-iE t} Y_l^m(\theta, \phi) (1 + \frac{\omega^2}{c^2} r^2)^{-\frac{l+1}{2}} r^l P_n^{(l+\frac{1}{2}, -\lambda)} (1 + 2\frac{\omega^2}{c^2} r^2)$$

(54)

where $C_{nlm}^\gamma$ are normalization constants. Observe that the polynomials $P_n^{(l+\frac{1}{2}, -\lambda)}$ have the appropriate $c \to \infty$ limit

$$\lim_{c \to \infty} P_n^{(l+\frac{1}{2}, -\lambda)} (1 + 2\frac{m \omega}{\hbar} N r^2) = L_n^{(l+\frac{1}{2})} (\frac{m \omega}{\hbar} r^2),$$

(55)

where $L_n^{(l+\frac{1}{2})}$ are the generalized Laguerre polynomials. To illustrate our relativistic generalization of them we give the first few polynomials

$$P_0^{[l+\frac{1}{2}, -(l+\frac{3}{2} + \gamma)]}(1 + 2\frac{\omega^2}{c^2} N) = 1$$

(56)

$$P_1^{[l+\frac{1}{2}, -(l+\frac{3}{2} + \gamma)]}(1 + 2\frac{\omega^2}{c^2} N) = (l + \frac{3}{2}) - 2\frac{\gamma}{N} \zeta^2$$

(57)

$$P_2^{[l+\frac{1}{2}, -(l+\frac{3}{2} + \gamma)]}(1 + 2\frac{\omega^2}{c^2} N) = \frac{1}{2} (l + \frac{3}{2}) (l + \frac{5}{2}) - \frac{1}{2} \frac{\gamma (l + \frac{5}{2})}{2 N} \zeta^2 + \frac{1}{8} \frac{\gamma (\gamma + 1)}{N^2} \zeta^4,$$

(58)

where now $\gamma = \frac{1}{2} \sqrt{9 + 4N^2 - 48\xi}$. 


4 Conclusions and final comments

In this paper we have shown that a particular anti-de Sitter metric can be used to simulate, in a natural way, a relativistic harmonic oscillator. The system is exactly solvable and leads to radial energy eigen-functions composed of a weight-function

\[ \left(1 + \frac{\omega^2}{c^2 r^2}\right)\left(-\frac{1}{4}\sqrt{\frac{9 + 4N^2 - 48\xi}{9} - \frac{2n+1}{2}}\right), \tag{59} \]

reducing to the Gaussian one \( e^{-\frac{1}{2} m\omega^2 r^2} \) in the limit \( c \to \infty \), and a polynomial

\[ r^l P_n^{\left[\frac{l+1}{2},-\lambda\right]}(1 + 2\frac{\omega^2}{c^2 r^2}), \tag{60} \]

going to its non-relativistic counterpart.

We observe from the expression (59) that the probability density for the relativistic oscillator is less confined in the classical region than the corresponding one of the non-relativistic oscillator. It penetrates more appreciably in the classically forbidden region. This can be understood in terms of the behaviour of the null geodesics in AdS. They go to infinity in a finite lapse of coordinate time. So that, in the limit \( N = 0 \), the geodesics are not confined in a finite region of space and this fact is partially reflected by the asymptotic behaviour of the wave functions. Despite of this, the spacing of the energy levels is identical to the non-relativistic one. However for the ground state energy we have

\[ E_0 = \hbar \omega \left(\frac{3}{2} + \frac{1}{2}\sqrt{9 + 4N^2 - 48\xi}\right), \tag{61} \]

representing some sort of mixing between the non-relativistic zero-point energy \( \frac{3}{2} \hbar \omega \) and the relativistic rest mass energy. The mixing is just parametrized by the curvature factor \( \xi \).

Another point which merits some comments is the question of how to extend our approach to spinning particles. When dealing with spin \( \frac{1}{2} \) particles one could construct the corresponding wave equation by means of Dirac equation in the AdS background, i.e.

\[ (i\gamma^\mu(\partial_\mu - \Gamma_\mu) - \frac{mc}{\hbar})\Psi = 0, \tag{62} \]

where \( \gamma^\mu = e_\alpha^\mu \gamma^\alpha \) are the Dirac matrices in AdS space (\( e_\alpha^\mu \) are the vierbeins) and \( \Gamma_\mu \) is the spin connection. Using the identity

\[ R_{\mu\nu\sigma\rho} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho = -2R, \tag{63} \]
where $R_{\mu\nu\sigma\rho}$ is the Riemann curvature tensor, it is not difficult to see that the Dirac equation (62) implies a Klein-Gordon equation with $\xi = \frac{1}{4}$

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2 + \frac{1}{4}R\right)\Psi = 0.$$  \hspace{1cm} (64)

The term $\frac{1}{4}R$ plays the role of the standard spin-dependent term $\frac{2}{2}F_{\mu\nu}\sigma^{\mu\nu}$ which appears when coupling the Dirac field with a electromagnetic potential. Note that this term is now diagonal in the spin components and then does not yield to a spin-orbit coupling.

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