Length Uniformity in Legal Dominating Sequences

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Abstract

A sequence of vertices \((v_1, \ldots, v_k)\) of a graph \(G\) is called a legal dominating sequence if \(\{v_1, \ldots, v_k\}\) is a dominating set of \(G\) and \(N[v_i] \not\subseteq \bigcup_{j=1}^{i-1} N[v_j]\) for every \(i\). A graph \(G\) is said to be \(k\)-uniform if all legal dominating sequences have equal length \(k\). Brešar et al. [4] characterized \(k\)-uniform graphs with \(k \leq 3\). In this article we extend their work by giving a complete characterization of all \(k\)-uniform graphs with \(k \geq 4\). We also discuss a variant of this problem for another type of sequence where open neighborhoods are considered instead of closed neighborhoods.

Keywords: domination, legal sequence

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1 Introduction

All graphs considered in this article are finite, simple, loopless and undirected. Given a graph \(G\), let \(\overline{G}\) be the complement of \(G\), and let \(V(G)\) and \(E(G)\) be the vertex set and the edge set of \(G\) respectively. A vertex \(u\) is a neighbor of vertex \(v\) if they are adjacent. The open neighborhood of \(v\), \(N(v)\), consists of all neighbors of \(v\), and the closed neighborhood of \(v\), \(N[v]\), is equal to \(N(v) \cup \{v\}\). A vertex \(v\) of \(G\) is called an isolated vertex of \(G\) if \(N(v) = \emptyset\) and it is called a dominating vertex of \(G\) if \(N[v] = V(G)\). Two distinct vertices \(u\) and \(v\) are called true twins if \(N[u] = N[v]\) and they are called false twins if \(N(u) = N(v)\) and \(uv \notin E(G)\). For a subset \(S\) of vertices of \(G\), let \(N(S) = \bigcup_{v \in S} N(v)\) and let \(G \setminus S\) denote the subgraph induced by the vertices of \(V(G) \setminus S\) (if \(S = \{u\}\) is a singleton, we simply write \(G \setminus u\)). Also, \(S\) is called a dominating set of \(G\) if \(S \cup N(S) = V(G)\) and it is called an independent set of \(G\) if every two vertices in \(S\) are nonadjacent. The join of two graphs \(G\) and \(H\), denoted by \(G \vee H\), has vertex set \(V(G \vee H) = V(G) \cup V(H)\) and edge set \(E(G \vee H) = \{uv \mid u \in V(G) \text{ and } v \in V(H)\}\). Let \(\vee tH\) denote the join of \(t\) copies of the graph \(H\) and \(tH\) denote disjoint union of \(t\) copies of \(H\). The complete graph, path graph
and cycle graph on \( n \) vertices are denoted by \( K_n, P_n \) and \( C_n \) respectively. Lastly, let \( K_{p_1,p_2} \) denote the complete bipartite graph with partition sizes \( p_1 \) and \( p_2 \), and let \( K_{p_1,\ldots,p_t} \) denote the complete multipartite graph with \( t \) parts of sizes \( p_1,\ldots,p_t \).

A sequence of \( k \) distinct vertices \((v_1, \ldots, v_k)\) in \( G \) is said to have length \( k \) and it is called a dominating sequence of \( G \) if the corresponding set \( \{v_1, \ldots, v_k\} \) is a dominating set of \( G \). A sequence \((v_1, \ldots, v_k)\) is called a legal sequence if, for each \( i \) with \( 2 \leq i \leq k \), we have

\[
N[v_i] \nsubseteq \bigcup_{j=1}^{i-1} N[v_j],
\]

or equivalently, \( N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset \). Each vertex in a legal sequence dominates a new vertex which is not dominated by any of the preceding vertices, so vertices of a legal sequence are called legal choices. We say that a graph \( G \) is uniform length dominating sequence graph if all legal dominating sequences have the same length. A graph \( G \) is called \( k \)-uniform if all legal dominating sequences of \( G \) have equal length \( k \). A sequence of vertices \((v_1, \ldots, v_k)\) of a graph \( G \) is called an open legal sequence of \( G \) if \( N(v_1) \neq \emptyset \) and

\[
N(v_i) \nsubseteq \bigcup_{j=1}^{i-1} N(v_j)
\]

for each \( i \) with \( 2 \leq i \leq k \). We say that a graph \( G \) is open \( k \)-uniform if \( G \) has no isolated vertices and all open legal dominating sequences have equal length \( k \).

The study of legal dominating sequences was initially motivated by some domination games in \cite{1,6,7} and variants of such sequences have connections to the minimum rank problem and the so called zero forcing number of the graph \cite{2,8}. The lengths of these sequences are also related to some other important graph parameters which have been extensively studied in the literature. For example, the minimum length of a legal sequence of a graph \( G \) is the well known domination number \( \gamma(G) \) of \( G \) and the maximum length of a legal dominating sequence of \( G \) is called the Grundy domination number \( \gamma_{gr}(G) \) of \( G \). For a graph \( G \) with no isolated vertices, the maximum length of an open legal dominating sequence is known as the Grundy total domination number \( \gamma_{gr}^t(G) \) of \( G \).

Brešar et al. \cite{4} gave a characterization of \( k \)-uniform graphs for \( k = 1, 2, 3 \) (see Theorem 3.6 in \cite{4}). In this article, we first complete the characterization of \( k \)-uniform graphs by finding all \( k \)-uniform graphs with \( k \geq 4 \) (Corollary 2.6) and then characterize open \( k \)-uniform graphs with \( k \leq 3 \) (Theorem 3.5).

2 Characterization of \( k \)-uniform graphs for \( k \geq 4 \)

Brešar et al. \cite{4} gave the following characterization of \( k \)-uniform graphs for \( k \leq 3 \).

**Theorem 2.1.** \cite{4} If \( G \) is a graph, then

- \( G \) is 1-uniform if and only if \( G \) is a complete graph;
- \( G \) is 2-uniform if and only if its complement \( \overline{G} \) is the disjoint union of one or more complete bipartite graphs;
• $G$ is 3-uniform if and only if $G$ is the disjoint union of a 1-uniform and a 2-uniform graph.

We use this characterization as the basis step of our induction and extend it to $k \geq 4$ by showing that every $k$-uniform graph is indeed a disjoint union of 1-uniform and 2-uniform graphs. To prove our result we make use of the following three observations.

**Lemma 2.2.** [3] Let $G$ be a uniform length dominating sequence graph with no true twins and let $x, y \in V(G)$. If $N[x] \subseteq N[y]$, then $x = y$.

**Remark 2.3.** If $G$ is a $k$-uniform graph then the subgraph $G \setminus N[v]$ is $(k-1)$-uniform for every vertex $v$ of $G$.

**Proof.** Let $v$ be a vertex of a $k$-uniform graph $G$ and $(v_1, \ldots, v_r)$ be a legal dominating sequence of the subgraph $G \setminus N[v]$. It is clear that $(v_1, \ldots, v_r, v)$ is a legal dominating sequence of $G$, as $v \notin \bigcup_{i=1}^r N[v_i]$. Hence $r = k - 1$ and the result follows. $\square$

**Remark 2.4.** Let $G$ be a graph with connected components $G_1, \ldots, G_c$. Then, $G$ is $k$-uniform if and only if each $G_i$ is $k_i$-uniform where $k = k_1 + \cdots + k_c$ and $k_i \geq 1$.

**Theorem 2.5.** If $G$ is a $k$-uniform graph with $k \geq 3$ and $G$ has no true twins, then $G$ is a disjoint union of 1-uniform and 2-uniform graphs.

**Proof.** We proceed by strong induction on $k$. For $k = 3$, the result follows from Theorem 2.1. Let $G$ be a $k$-uniform graph with $k \geq 4$ and $v$ be a vertex of $G$. By Remark 2.3, the subgraph $G \setminus N[v]$ is $(k-1)$-uniform. By the induction hypothesis, $G \setminus N[v]$ is a disjoint union of 1-uniform and 2-uniform graphs.

**Claim 1:** The subgraph $G \setminus N[v]$ has no true twins.

Let $G_1, \ldots, G_r$ be the 1-uniform connected components and $H_1, \ldots, H_t$ be the 2-uniform connected components of $G \setminus N[v]$, if any. So, $k-1 = r + 2t$ where $r, t \geq 0$. Suppose on the contrary that there is a 1-uniform graph, without loss, say $G_1$, which contains at least two vertices, say $v_1$ and $v'_1$. By Lemma 2.2 we have $N[v'_1] \setminus N[v_1] \neq \emptyset$ since $G$ has no true twins. Let $v_1$ be a vertex of $G_i$ and $u_i, u'_i$ be two nonadjacent vertices $H_i$ for each $i$. Now we can extend $(v_1, v'_1)$ to the legal dominating sequence $(v_1, v'_1, v_2, \ldots, v_r, u_1, u'_1, \ldots, u_i, u'_i, v)$ of $G$ which has length $r + 2t + 2 = k + 1$. The latter contradicts with $G$ being $k$-uniform and hence all of $G_1, \ldots, G_r$ must be equal to $K_1$. Now suppose on the contrary that there is a 2-uniform graph, without loss, say $H_1$, which contains a pair of true twins, say $u_1$ and $w_1$. By Lemma 2.2, $N[w_1] \setminus N[u_1] \neq \emptyset$ as $G$ has no true twins. Hence, we get a legal dominating sequence $(u_1, w_1, u'_1, u_2, w_2, \ldots, u_i, u'_i, v_1, \ldots, v_r, v)$ of $G$ which has size $r + 2t + 2 = k + 1$. Again, the latter contradicts with $G$ being $k$-uniform. Therefore, the subgraph $H_i$ must be equal to $\overline{K_2}$ for some integer $t_i \geq 2$ for each $i$ (note that $t_i \geq 2$ as $H_i$ is connected).

**Claim 2:** $G$ is disconnected.

Suppose on the contrary that $G$ is connected. Let $A_i = N(V(G_i)) \cap N(v)$ and $B_i = N(V(H_i)) \cap N(v)$ for each $i$. Since $G$ is connected, $A_i$ and $B_i$ are nonempty for each $i$. Let us show that $A_1, \ldots, A_r, B_1, \ldots, B_t$ are mutually disjoint.
A_i \cap A_j = \emptyset \text{ whenever } i \neq j:

Without loss, suppose on the contrary that there exists a vertex w in N(v) such that w is adjacent to both v_1 and v_2. Then, (v, w, v_3, \ldots, v_r, u_1, u'_1, \ldots, u_t, u'_t) is a legal dominating sequence of G which has length r + 2t = k - 1 which contradicts with G being k-uniform.

B_i \cap B_j = \emptyset \text{ whenever } i \neq j:

Without loss, suppose on the contrary that there exists a vertex w in N(v) such that w is adjacent to both u_1 and u_2. Then, (v, w, u'_1, u'_2, u_3, u'_3, \ldots, u_t, u'_t, v_1, \ldots, v_r) is a legal dominating sequence of G which has length r + 2t = k - 1. Again, a contradiction.

A_i \cap B_j = \emptyset \text{ for every } i \text{ and } j:

Without loss, suppose on the contrary that there exists a vertex w in N(v) such that w is adjacent to both v_1 and u_1. In this case we obtain a contradiction by finding the legal dominating sequence (v, w, v_2, \ldots, v_r, u'_1, u_2, u'_2, \ldots, u_t, u'_t) of G with length k - 1.

If r \geq 1, remove the vertex v_1 from G. The vertex v_1 has no neighbor in A_i or B_j for each i \neq 1 and j = 1, \ldots, t and A_i’s and B_j’s are nonempty. So, there is a path between every pair of vertices in G \ N[v_1] via w which makes the subgraph G \ N[v_1] connected. But this a contradiction because G \ N[v_1] is (k-1)-uniform by Remark 2.3 and hence must be disconnected by the induction hypothesis.

Now suppose that r = 0 and t \geq 2. If k = 4, then (G \ N[u_1]) \ N[u'_1] is 2-uniform by Remark 2.3. But this is not possible because \{v, u_2, u'_2\} is an independent set of size 3 which cannot be contained in a 2-uniform graph. Now we may assume assume that k \geq 5. First let us show that u_1 and u'_1 have the same neighbors in N(v). Suppose on the contrary that there is a vertex w in N(v) which is adjacent to exactly one of u_1 and u'_1. Without loss, assume thatwu_3 \in E(G) and wu'_1 \notin E(G). So, G \ N[u'_1] is (k-1)-uniform and connected. The latter contradicts with the induction hypothesis. Hence, the vertex u_1 is an isolated vertex of G \ N[u'_1]. The subgraph (G \ N[u'_1]) \ u_1 must be (k-2)-uniform by Remark 2.3. By the induction hypothesis, (G \ N[u'_1]) \ u_1 is a disjoint union of 1-uniform and 2-uniform graphs. This is again a contradiction, as (G \ N[u'_1]) \ u_1 is connected.

Thus, G is disconnected and the result follows by induction and Remark 2.4.

Let G' be a graph obtained from another graph G by adding a new true twin vertex and k be any positive integer. Then, observe that G' is k-uniform if and only if G is k-uniform. Thus, we obtain a characterization of all k-uniform graphs as an immediate consequence of Theorem 2.5.

**Corollary 2.6.** Every k-uniform graph is a disjoint union of 1-uniform and 2-uniform graphs.
3 Open $k$-uniform graphs

If $G$ is a graph with no isolated vertices then every open legal sequence in $G$ can be extended to an open legal dominating sequence of $G$. Also, a graph $G$ contains an open legal dominating sequence if and only if $G$ has no isolated vertices. We shall implicitly make use these observations in the sequel. We begin with showing some properties of open 3-uniform graphs.

Lemma 3.1. If $G$ is an open 3-uniform graph, then

(i) $G$ is $P_4$-free;
(ii) $G$ is $2K_2$-free;
(iii) If $G$ has no false twins, then $G$ is $(P_3 \cup K_1)$-free.

Proof. (i) Suppose that $G$ contains an induced $P_4$ with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4\}$. Now $(v_1, v_4)$ is an open legal sequence because $v_3 \in N(v_4) \setminus N(v_1)$, $v_1 \in N(v_2) \setminus (N(v_1) \cup N(v_4))$ and $v_4 \in N(v_3) \setminus (N(v_1) \cup N(v_2) \cup N(v_2))$. One can extend $(v_1, v_4, v_2, v_3)$ to an open legal dominating sequence of $G$ with length at least four.

(ii) Suppose that $G$ contains an induced $2K_2$ whose edges are $uv$ and $u'v'$. Now $(u, v, u', v')$ is an open legal sequence because $u \in N(v) \setminus N(u)$, $v' \in N(u') \setminus (N(u) \cup N(v))$ and $u' \in N(v') \setminus (N(u) \cup N(v) \cup N(u'))$. One can extend $(u, v, u', v')$ to an open legal dominating sequence of $G$ with length at least four.

(iii) Suppose that $G$ contains an induced $P_3 \cup K_1$ whose vertex set is $\{u, v, w, x\}$ and edge set is $\{uw, vw\}$. Since $u$ and $w$ are nonadjacent and $G$ has no false twins, there exists a vertex $u'$ which is adjacent to exactly one of $u$ and $w$. Without loss, suppose that $u'u \in E(G)$ and $u'w \notin E(G)$. Since $G$ has no isolated vertices, the vertex $x$ has at least one neighbor $x'$ (it is possible that $x' = u'$). Now, $(w, u, v, x')$ is an open legal sequence of $G$ because $u' \in N(u) \setminus N(w)$, $u \in N(v) \setminus (N(w) \cup N(u))$ and $x \in N(x') \setminus (N(w) \cup N(u) \cup N(v))$. One can extend $(w, u, v, x')$ to an open legal dominating sequence of $G$ with length at least four.

We shall also make use of the following result on $(C_4, P_4, 2K_2)$-free graphs which are also known as threshold graphs.

Lemma 3.2. [9] If $G$ is a $(C_4, P_4, 2K_2)$-free graph then $G$ has either a dominating vertex or an isolated vertex.

Lemma 3.3. Every open 3-uniform graph contains false twins.

Proof. Let $G$ be an open 3-uniform graph. Suppose on the contrary that $G$ has no false twins. By Lemma 3.2 and Lemma 3.1 (i,ii), the graph $G$ must contain an induced $C_4$. Let $\{a, b, c, d\}$ be the vertex set of the induced $C_4$ and $\{ab, bd, dc, ca\}$ be its edge set. There exists a vertex $w$ which is a nonneighbor of both $c$ and $d$ because otherwise $(c, d)$ would be an open legal dominating sequence of length 2. If $w$ is a neighbor of $a$, the vertices $w, a, c, d$
induce a $P_4$ which contradicts with Lemma 3.1 (i). If $w$ is not a neighbor of $a$ then the vertices $w, a, c, d$ induce a $P_3 \cup K_1$ which contradicts with Lemma 3.1 (iii). In each case we obtain a contradiction.

**Lemma 3.4.** If $G$ is an open 3-uniform graph with a false twin vertex $u$, then the subgraph $G \setminus u$ is also open 3-uniform.

**Proof.** Let $u$ and $u'$ be false twins of $G$. It is clear that the subgraph $G \setminus u$ has no isolated vertex as $N(u) = N(u')$. It is also easy to check that $G \setminus u$ has no dominating vertex. Suppose that $G \setminus u$ has an open legal dominating sequence of length two. Such sequence must contain the vertex $u'$ and a vertex from $V(G) \setminus (N[u] \cup N[u'])$, say vertex $v$, because otherwise the sequence would also be an open legal dominating sequence of $G$. The vertices in $V(G) \setminus (N[u] \cup N[u'])$ cannot be dominated by $u'$, so $v$ is adjacent to every other vertex in $V(G) \setminus (N[u] \cup N[u'])$. Now, the vertex $v$ has at least one nonneighbor in $N(u')$ because otherwise $v$ and a vertex from $N(u')$ would form an open legal dominating sequence of $G$. Let $w$ be a nonneighbor of $v$ in $N(u')$. Moreover, there is a vertex $v'$ in $V(G) \setminus (N[u] \cup N[u'])$ which is different from $v$ because otherwise $u'$ and a neighbor of $v$ in $N(u')$ would form an open legal dominating sequence of $G$. If $w$ is not adjacent to $v'$ then the vertices $u', w, v, v'$ induce a $2K_2$ in $G$ which contradicts with $G$ being open 3-uniform by Lemma 3.1 (ii). If $w$ is adjacent to $v'$ then the vertices $u', w, v, v'$ induce a $P_4$ in $G$ which contradicts with $G$ being open 3-uniform by Lemma 3.1 (i). Thus, every open legal dominating sequence of $G \setminus u$ has length three. □

We are now ready to give a characterization of open $k$-uniform graphs with $k \leq 3$.

**Theorem 3.5.** Let $G$ be a graph, then

(i) There are no open 1-uniform graphs;

(ii) $G$ is open 2-uniform if and only if $G$ is a complete multipartite graph $K_{p_1, \ldots, p_t}$ where $t \geq 2$ and $p_i \geq 2$ for each $i = 1, \ldots, t$;

(iii) There are no open 3-uniform graphs.

**Proof.** (i) Let $G$ be open 1-uniform. Then, by definition, $G$ has no isolated vertices. If $G$ contains two adjacent vertices $u$ and $v$, then $(u, v)$ is an open legal sequence of $G$, as $N(u) \neq \emptyset$ and $u \in N(v) \setminus N(u)$. Since $G$ has no isolated vertices, $(u, v)$ can be extended to an open legal dominating sequence of length at least 2 and this contradicts with $G$ being 1-uniform.

(ii) It is easy to check that a complete multipartite graph $K_{p_1, \ldots, p_t}$ with $t, p_1, \ldots, p_t \geq 2$ is open 2-uniform. So we shall only show that if $G$ is open 2-uniform, then $G$ is equal to $K_{p_1, \ldots, p_t}$ for some $t, p_1, \ldots, p_t \geq 2$. Let $A_1, \ldots, A_t$ be a partition of $V(G)$ into maximal vertex subsets consisting of false twins. For each $i$, $A_i$ is an independent set and $N(u_i) = N(v_i)$ for every pair of vertices $u_i, v_i$ in $A_i$. Also, if $i \neq j$, $u_i \in A_i$ and $u_j \in A_j$, then $u_i$ and $u_j$ are not false twins. It is clear that $t \geq 2$ since otherwise $G$ would consist of isolated vertices.

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Claim 1: For every pair \( i, j \) with \( i \neq j \), if \( u_i \in A_i \) and \( u_j \in A_j \), then \( u_i u_j \in E(G) \).

Suppose on the contrary that there exists a pair \( i, j \) with \( i \neq j \) where \( u_i \in A_i \) and \( u_j \in A_j \) but \( u_i u_j \notin E(G) \). Since \( u_i \) and \( u_j \) are not false twins, there exists a vertex which is adjacent to exactly one of them. Without loss, let \( u_k \) be a vertex such that \( u_k u_i \in E(G) \) and \( u_k u_j \notin E(G) \). All vertices in \( A_j \) have the same neighbors, so \( u_k \) must belong to some other subset \( A_k \) with \( k \notin \{i, j\} \). The vertex \( u_j \) has a neighbor, say \( u'_j \), as \( G \) has no isolated vertices. Now, \( (u_i, u_k, u'_j) \) is an open legal sequence because \( \{u_i\} \subseteq N(u_k) \setminus N(u_i) \) and \( \{u_j\} \subseteq N(u'_j) \setminus (N(u_i) \cup N(u_k)) \). One can extend the sequence \( (u_i, u_k, u'_j) \) to an open legal dominating sequence of \( G \) with length at least three and this contradicts with \( G \) being open 2-uniform.

Claim 2: \(|A_i| \geq 2\) for each \( i = 1, \ldots, t \).

If there is some subset \( A_i \) with exactly one vertex \( u_i \), then \( u_i \) would be adjacent to all other vertices in the graph \( G \) by the above argument. Hence, \( (u_i) \) would be an open legal dominating sequence of \( G \) with length one which contradicts with \( G \) being open 2-uniform.

Thus, \( G \) is a complete multipartite graph with \( A_1, \ldots, A_t \) being the independent sets of the partition.

(iii) Given an open 3-uniform graph, one can successively remove false twin vertices from the graph until there is no pair of false twins and the resulting new graph would still be open 3-uniform by Lemma 3.4. Therefore, if there exists an open 3-uniform graph with false twins then there must exist an open 3-uniform graph with no false twins as well. However, by Lemma 3.3 every open 3-uniform graph must contain false twins. Thus we conclude that there are no open 3-uniform graphs.

4 Concluding Remarks

Characterization of open \( k \)-uniform graphs for \( k \geq 4 \) remains unsolved. Unlike \( k \)-uniform graphs, we cannot proceed inductively to find open \( k \)-uniform graphs because removal of a subgraph may yield isolated vertices in the resulting graph and in that case the new graph does not have any open legal dominating sequences at all. So it seems that a different approach is necessary to characterize open \( k \)-uniform graphs. We believe that there are no open \( k \)-uniform graphs when \( k \) is odd and every open \( k \)-uniform graph with \( k \) even is a disjoint union of open 2-uniform graphs.

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