Modeling ideas for ruled surfaces and their implementation in applied geometry

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Abstract. The study provides theoretical and applied issues related to the establishment of interdisciplinary relations between different disciplines. An algorithm for geometric modeling of ruled surfaces is presented. Its distinctive feature is the fact that in geometric modeling the model is always a geometric figure. New geometric configurations have been obtained, due to the mutual introduction of the foundations of technology and geometry. As a result of the study, it was found that the interdisciplinary connections of geometry and technology make it possible to obtain new geometric configurations, on the basis of which it is possible to achieve new results in aviation technology and in other industries.

1. Introduction

The foundations of descriptive geometry date back to the XVIII-XIX centuries. Previously, it was differentiated scientific discipline, the elements of which and, at the same time, its main scientific content were developed and systematized by the French geometer, thinker and philosopher Gaspard Monge. This scientist gave a great importance to descriptive geometry as a deep scientific and practical discipline. In this regard, it is enough to cite his vivid pedagogical commentary: "If I had to start this work again (we are talking about writing the content of a course in descriptive geometry), I would print it in two columns: in the first I would put the solutions of geometric problems by calculation, and in the second, I would put the solutions to the same problems, but performed by geometric constructions. Readers, perhaps, would be very surprised to see that the second column would almost always deserve preference, both in terms of clarity and simplicity of proofs". All of this made it possible to establish the dual nature of descriptive geometry, ranking it in the class of applied sciences and assigning it a second name: applied geometry. This, in particular, is explained by the fact that descriptive geometry plays certain and, as practice shows, a significant role in mathematics, being also its component in physics, theoretical mechanics, painting, construction of a wide variety of objects in the design system, aircraft construction, etc. [1, 2]. This is also seriously and thoroughly mentioned in John Horgan's book "The End of Science: Facing the Limits of Science in the Twilight of the Scientific Age" [3]: "Applied science will live for a long time, as scientists continue to develop new universal structures and materials, for example, fast and complex computers, new and more efficient genetic engineering technologies, ... more and more fully based on the achievements of mathematics, including higher applied geometry".
2. Methodology

In complex geometric structures, modeling and its varieties are usually present as a constituent element. So, for example, the mathematical model of an object is an abstract or sign model formed by means of the chosen mathematical apparatus, namely: system of equations, graph, logical structure, verbal description, etc. It should be noted that the mathematical model of a process or phenomenon may, in particular, be not quite adequate to the process, since the level of identity simply must be sufficient to implement the task.

Geometric modeling is a special case of such modeling. Its distinctive feature is the fact that in geometric modeling the model is always a geometric figure. However, the properties and characteristics of the model, of course, can have not only a geometric nature, they can have other functional properties, but always have the same mathematical or constructive-geometric description as the corresponding geometric prototype.

We introduce a geometric modeling algorithm:

- Analysis of the original (identification of the number of conditions: parameters, their definitions and existing functional and logical connections between them).
- Interpretation of information in geometric form.
- Selecting in the most optimal way one figure from a set of figures with similar characteristics.
- Construction of a geometric figure taking into account the principles of parametric geometry. It should be noted that geometric design is understood as the process of identifying and defining totality, mutual arrangement, and interrelations of the figure elements that meet certain predetermined (technological) requirements, both to the local characteristics of the original and to the characteristics in general.
- Real representation of the object (drawings, analytical dependencies, etc.).
- Research and analysis of the properties of the figure, possible optimization in a certain direction, etc.
- Reconstruction of the received information into the corresponding information about the object.
- The logical completion of the algorithm can be a smooth transition to the computer implementation of the tasks.

We introduce into consideration ruled surfaces, which are successfully used in various fields of science and technology, including in the design of various types of aircraft.

In the common case, a general ruled surface is a surface formed by some regular movement of a straight (generatrix) line along three guides $F (a, b, c)$. Since in the definition of these surfaces the leading role belongs to the arrangement of the generators, it is expedient here to consider some of them, which are necessary for the construction of the properties. In this case, congruences of lines are used, by which we mean a family of lines depending on two parameters.

Geometrically, it looks like this. Let some surface $F$, having a tangent plane $T$ at each point, be given. Then through each point of contact we can draw a normal $n$ (perpendicular to the surface $F$).
The set of all surface normals is congruence. It can be set differently. Two curves $m$ and $n$ are chosen, and there is a point $S$. Now an auxiliary conical surface is introduced into consideration $F^\text{con}(S, n)$ with apex at point $S$ (figure 2).

Moving point $S$ along the curve $m$, we can get a set of conical surfaces of general appearance with a common guide $n$. The generators of these surfaces fill a certain area (compartment) of space. The collection of these lines is a two-parameter set. An arbitrary line in this congruence is defined by the curves $m$ and $n$.

This implies another definition: set of all straight lines intersecting two predetermined curves is a two-parameter set, i.e. congruence.

Such constructions are successfully used in the practice of designing specific technical surfaces.

If we add to the already known curves $AB$ and $CD$ some additional curve $EF$, the proposed algorithm covers three possible cases:

- $EF$ lies inside the congruence $[AB, CD]$;
- $EF$ is partially included in the congruence;
- $EF$ is located outside the congruence $[AB, CD]$.

In the first case, $EF$ curve defines congruence, and a single ruled surface is selected within it. In the second case, accordingly, a certain area (compartment) of the ruled surface is defined. In the third case, the ruled surface is not defined.
Thus, in the general case, ruled surfaces are determined by specifying three guides, provided that the third guide is inside the congruence of straight lines determined by the first two (figure 3). In the second case, two curves are set, on which a one-to-one correspondence of their points is established (figure 4).

![Figure 3. Ruled surface design.](image1)

The figure 3 shows two guide curves $m$ and $n$. On the chords $AB$ and $CD$, proportional division was made and a one-to-one correspondence of the dot series was established. With the help of auxiliary normals on the curves $m$ and $n$, it is possible to construct points 1, 2, 3, connecting which it is possible to obtain a section of the ruled surface $F$. This method is used in the design of aviation equipment.

More visual and convenient in the practice of geometric constructions are ruled surfaces with a plane of parallelism or surfaces of Catalan, who first studied and applied these surfaces. Here the third parameter is the plane of parallelism, and all rectilinear generators are parallel to it, but not parallel to each other.

An example of such a surface is a ruled (hyperbolic) paraboloid (technical name is an oblique plane). In this case, the guides are intersecting straight lines, the plane of parallelism is usually the plane of a particular position $\Gamma$, and all $l'$ are $\parallel$ to $\Gamma$, but $l'^{+1}$ is not $\parallel$ to $l$. It is a second-order algebraic surface defined by the canonical equation

$$ \frac{x^2}{p} - \frac{y^2}{q} = 2z, \text{ where } (p > 0, q > 0) \quad (1) $$

carries a family of parabolas in planes parallel to the plane $xOz$.

In sections parallel to the plane $yOz$, where $x=h$, also parabolas defined by the equations

$$ \begin{cases} y^2 = -2q(z - \frac{h^2}{2p}) \\ x = h. \end{cases} \quad (2) $$

A visual representation of the surface is given in the figure 4. On this surface there are two series of rectilinear generators, the equations of which have the following form:

$$ \begin{cases} \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 2kz \\ \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = \frac{1}{k}. \end{cases} \quad (3) $$

$$ \begin{cases} \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = \frac{1}{l} \\ \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 2lz. \end{cases} \quad (4) $$

where (3) is the first series, (4) is the second series and numbers $k$ and $l$ are arbitrary parameters.
The rectilinear generatrices of the surfaces of a one-sheet hyperboloid of revolution and a linear paraboloid were used in his geometric designs by engineer V.G. Shukhov [4-10]. The area (compartment) of the oblique plane is shown in the figure 5.

Figure 5. Tapered wedge surface.

It should be noted that the entire set of ruled surfaces with a plane of parallelism can be represented by three classes, depending on the parametric characteristics:

- $F(a, b, \Gamma)$, where $a, b$ are the curved guides, ensuring the presence of congruences of straight lines (generators) inside the surface compartment, $\Gamma$ is the plane of parallelism, $l_{i+1}$ is not $\parallel$ to $l_i$. Such a surface is a cylindroid.
- $F(a, b, \Gamma)$, where $a$ is the straight guide, $b$ is the curve, $\Gamma$ is the plane of parallelism. Such a surface is a conoid. Moreover, if $a \perp \Pi_1$, this is a straight conoid.
- $F(a, b, \Gamma)$, where $a, b$ are the crossed straight, $\Gamma$ is the plane of parallelism. Such a surface is a ruled paraboloid or an oblique plane, considered earlier.

In technical applications, a straight conoid has another name: straight wedge. In the figure 6 (a, b, where a is a visual representation) shows an image of a straight wedge on the Monge’s epure.

Figure 6. Image of the surface of Catalan (straight conoid - "wedge").
If we cut the surface $F$ with a plane parallel to the base, in the section we get a curve similar to this base. In the figure 6 (b) there is an ellipse.

Examination of a more complex surface leads to the so-called oblique conoids or wedge-shaped surfaces with a twist. Surfaces of this kind are widely used in aviation technology. A wedge-shaped surface with a twist is formed if the $AB$ guide is not perpendicular to the parallelism plane $I$. This case is shown in the figure 5 (a) as a visual representation of the conoid surface.

The scheme for solving the problem from the field of aircraft construction is shown in the figure 5 (b). Here $\Sigma$ and $\Sigma'$ simulate two wings of an aircraft (these are the curves $a$, $b$) with vertical tangents $t_A$ || $t_B \perp \Pi_1$. The chords $AB$ and $CD$ are taken as oblique plane guides, which are then divided proportionally. Surfaces $AC$ and $BD$ that belong to planes $\Delta$ and $\Delta'$, cut on a straight line $m$ segment. The rest of the intermediate generators will intersect the segment inside it. The proportional division of the chords $AB$ and $CD$ ensures that the relation

$$\frac{AM}{MB} = \frac{CN}{ND}$$

Thus, as a result of this construction, it is possible to obtain a ruled surface, the generators of which will be based on a priori given arcs $a$ and $b$. This surface belongs to the class of wedge-shaped surfaces with a twist. From the drawing it follows that line $m$ is a representative of the second series of generators of the base oblique plane.

3. Conclusions

Thus, the study shows that interdisciplinary connections between geometry and technology allow obtaining new geometric configurations, on the basis of which it is possible to achieve new results in aviation technology.

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