On almost Riemann solitons

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Abstract

We consider almost Riemann solitons \((V, \lambda)\) in a Riemannian manifold and underline their relation to almost Ricci solitons. When \(V\) is of gradient type, using Bochner formula, we explicitly express the function \(\lambda\) by means of the gradient vector field \(V\). Moreover, we deduce some properties for the particular cases when the potential vector field of the soliton is solenoidal or torse-forming, with a special view towards curvature.

1 Introduction

Ricci solitons and Riemann solitons correspond to self-similar solutions of Ricci flow and Riemann flow, respectively, introduced in 1982 [27] by R. S. Hamilton. The intrinsic geometric flows, Ricci flow

\[
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t))
\]

and Riemann flow

\[
\frac{\partial}{\partial t} G(t) = -2 R(g(t)),
\]

where \(G := \frac{1}{2} g \odot g\), for \(\odot\) the Kulkarni-Nomizu product, Ric the Ricci curvature tensor and \(R\) the Riemann curvature tensor of \(g\), are two evolution equations formally similar.

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to the heat equation, with applications in various fields. It’s been pointed out that some properties of the Riemann flow are close to that of the Ricci flow [34].

On a given \( n \)-dimensional smooth manifold \( M \), a Riemannian metric \( g \) and a non-vanishing vector field \( V \) is said to define a Ricci soliton [28] if there exists a real constant \( \lambda \) such that

\[
\frac{1}{2} \mathcal{L}_V g + \text{Ric} = \lambda g,
\]

respectively, a Riemann soliton [29] if there exists a real constant \( \lambda \) such that

\[
\frac{1}{2} \mathcal{L}_V g \odot g + R = \lambda G,
\]

where \( G := \frac{1}{2} g \odot g \), \( \mathcal{L}_V \) denotes the Lie derivative operator in the direction of the vector field \( V \), \( R \), Ric and scal are the Riemann, the Ricci and the scalar curvature of \( g \), respectively. If \( \lambda \) is a smooth function on \( M \), we call \((V,\lambda)\) an almost Ricci or an almost Riemann soliton. Moreover, if \( V \) is a gradient vector field, we call \((V,\lambda)\) a gradient almost Ricci or a gradient almost Riemann soliton. A soliton defined by \((V,\lambda)\) is said to be shrinking, steady or expanding according as \( \lambda \) is positive, zero or negative, respectively.

Ricci solitons are natural generalizations of Einstein manifolds and Riemann solitons are natural generalizations of spaces of constant sectional curvature. It was proved that on a compact Riemannian manifold, Ricci and Riemann solitons are gradient solitons (see [31] and [29]).

Note that the relations between Riemann flow and Ricci flow have been studied in [34] and some geometric properties of Riemann solitons in almost contact geometry, precisely, in Kenmotsu, Sasakian and \( K \)-contact manifolds, have been given in ([26], [35]).

In the present paper, we study some properties of almost Riemann and almost Ricci solitons, providing some relations between them. We compute explicitly the function \( \lambda \) by means of \( V \), and when \( V \) is a solenoidal vector field of gradient type, we express the volume of a compact Riemannian manifold in terms of the Ricci tensor’s norm. Moreover, we prove that any gradient Riemann soliton with potential vector field \( V \) of constant length is steady and \( V \) is a solenoidal vector field. Remark that the solenoidal vector fields of gradient type are the gradients of harmonic functions, and it’s known that they are relevant in electrostatic. We also consider the case when the potential vector field of the soliton is torse-forming and, in particular, we characterize the Ricci symmetric, Ricci semisymmetric manifolds admitting almost Riemann solitons with concircular vector field. Different ideas concerning Riemann, Ricci, Einstein or Yamabe solitons have been treated by the author and her collaborators in ([11] – [23], [25]).
2 Almost Riemann and almost Ricci solitons

Recall that the Kulkarni-Nomizu product for two $(0,2)$-tensor fields $T_1$ and $T_2$ is defined by:

$$(T_1 \odot T_2)(X,Y,Z,W) := T_1(X,W)T_2(Y,Z) + T_1(Y,Z)T_2(X,W) - T_1(X,Z)T_2(Y,W) - T_1(Y,W)T_2(X,Z),$$

for any $X,Y,Z,W \in \chi(M)$. Then the Riemann soliton equation (4) is explicitly expressed as

$$2R(X,Y,Z,W) + [g(X,W)(\mathcal{L}_V g)(Y,Z) + g(Y,Z)(\mathcal{L}_V g)(X,W) - g(X,Z)(\mathcal{L}_V g)(Y,W) - g(Y,W)(\mathcal{L}_V g)(X,Z)] = 2\lambda [g(X,W)g(Y,Z) - g(X,Z)g(Y,W)],$$

which by contraction over $X$ and $W$, gives

$$\frac{1}{2} \mathcal{L}_V g + \frac{1}{n-2} \text{Ric} = \frac{(n-1)\lambda - \text{div}(V)}{n-2} g$$

and further

$$\text{scal} = (n-1)[n\lambda - 2 \text{div}(V)].$$

Remark 2.1. From the decomposition of the Riemann curvature tensor with respect to the Weil curvature tensor $\mathcal{W}$,

$$R = \mathcal{W} - \frac{\text{scal}}{2(n-1)(n-2)} g \odot g + \frac{1}{n-2} \text{Ric} \odot g,$$

if $(V,\lambda)$ define an almost Riemann soliton on the $n$-dimensional Riemannian manifold $(M,g)$, $n \geq 3$, then the Weil curvature tensor vanishes. Note that properties of the Ricci solitons with vanishing Weil curvature have been given in [24].

Now, if we assume that $V$ is a solenoidal vector field, i.e. $\text{div}(V) = 0$, the equation (6) becomes

$$\mathcal{L}_V g + \frac{2}{n-2} \text{Ric} = 2 \frac{n-1}{n-2} \lambda \cdot g$$

and the scalar curvature equals to

$$\text{scal} = n(n-1)\lambda,$$

so we can state:

Proposition 2.2. Let $(V,\lambda)$ define an almost Riemann soliton on an $n$-dimensional Riemannian manifold $(M,g)$ with $V$ a solenoidal vector field. Then

i) $(V,\bar{\lambda})$, where $\bar{\lambda} = \frac{n-1}{n-2}\lambda$, define an almost $\alpha$-Ricci soliton, for $\alpha = \frac{1}{n-2}$;

ii) $(\bar{V},\bar{\lambda})$, where $\bar{V} = (n-2)V$ and $\bar{\lambda} = (n-1)\lambda$, define an almost Ricci soliton.
2.1 Gradient solitons

In [5], using the Bochner formula, we have proved that for any gradient vector field \( V \in \chi(M) \), we have:

\[
\Delta(|V|^2) - 2|\nabla V|^2 = 2 \text{Ric}(V, V) + 2 V(\text{div}(V))
\]

and

\[
(\text{div}(\mathcal{L}_V g))(V) = 2V(\text{div}(V)) + 2 \text{Ric}(V, V),
\]

where \( Q \) is the Ricci operator, \( g(QX,Y) := \text{Ric}(X,Y) \).

Taking the divergence of (6), we get:

\[
\frac{1}{2} \text{div}(\mathcal{L}_V g) + \frac{1}{n-2} \text{div}(\text{Ric}) = \frac{(n-1)d\lambda - d(\text{div}(V))}{n-2}.
\]

Differentiating (7), we have:

\[
d(\text{scal}) = (n-1)[n\lambda - 2d(\text{div}(V))].
\]

Taking into account that \( \text{div}(\text{Ric}) = \frac{1}{2}d(\text{scal}) \), from (10) and (11), we obtain:

\[
(\text{div}(\mathcal{L}_V g))(V) = 2V(\text{div}(V)) - (n-1)V(\lambda)
\]

and using (9), we get:

\[
\text{Ric}(V, V) = -\frac{n-1}{2}V(\lambda).
\]

Computing now (6) in \((V, V)\) and using (12), we obtain:

\[
\text{div}(V) = (n-1)\lambda + \frac{n-1}{2} \cdot \frac{V(\lambda)}{|V|^2} - \frac{n-2}{2} \cdot \frac{V(|V|^2)}{|V|^2}.
\]

From (13) and (8) we explicitly determine \( \lambda \) by means of \( V \):

**Theorem 2.3.** Let \((V, \lambda)\) define a gradient almost Riemann soliton on the \(n\)-dimensional Riemannian manifold \((M, g)\). Then

\[
\lambda = \frac{1}{2(n-1)|V|^2}[\Delta(|V|^2) - 2|\nabla V|^2 + (n-2)V(|V|^2 - 2V(\text{div}(V))] + \frac{1}{n-1} \text{div}(V).
\]
Remark 2.4. Similarly, if \((V, \lambda)\) define a gradient almost Ricci soliton, then, in the same way, we can explicitly express the function \(\lambda\) in terms of \(V\), namely

\[
\lambda = \frac{1}{2|V|^2} [\Delta(|V|^2) - 2|\nabla V|^2 + V(|V|^2) - 2V(\text{div}(V))].
\]

If \(\lambda\) is a constant and \(V\) is a gradient vector field of constant length, from (12) we get \(\text{Ric}(V, V) = 0\) and from (8), we obtain \(|\nabla V|^2 + V(\text{div}(V)) = 0\). Then Theorem 2.3 implies \(\lambda = \frac{1}{n-1} \text{div}(V)\), which by integration, in the compact case, give \(\lambda = 0\), therefore, \(\text{div}(V) = 0\), and we can state:

**Theorem 2.5.** On a compact Riemannian manifold \((M, g)\), any gradient Riemann soliton \((V, \lambda)\) with \(V\) of constant length is steady and \(V\) is a solenoidal vector field (hence, \(f\) is a harmonic function).

Also we obtain:

**Proposition 2.6.** Let \((V, \lambda)\) define a gradient Riemann soliton on an \(n\)-dimensional compact Riemannian manifold \((M, g)\), \(n \geq 3\). Then

\[
\lambda \cdot \text{vol}(M) = \frac{n-2}{2(n-1)} \int_M \frac{(\text{grad}(f))(\text{grad}(f))^2}{|\text{grad}(f)|^2} d\mu_g.
\]

If \(V = \text{grad}(f)\), equation (6) becomes

\[
(14) \quad \text{Hess}(f) + \frac{1}{n-2} \text{Ric} = \frac{(n-1)\lambda - \Delta(f)}{n-2} g.
\]

By tracing (14), we get

\[
(15) \quad \text{scal} = n(n-1)\lambda - 2(n-1)\Delta(f)
\]

which by differentiating gives

\[
(16) \quad d(\Delta(f)) = \frac{n}{2} d\lambda - \frac{1}{2(n-1)} d(\text{scal})
\]

and by taking the gradient of (15)

\[
(17) \quad \text{grad}(\Delta(f)) = \frac{n}{2} \text{grad}(\lambda) - \frac{1}{2(n-1)} \text{grad}(\text{scal}).
\]

Applying the divergence operator to (14) and using \(\text{div}(\text{Ric}) = \frac{1}{2} d(\text{scal})\), we obtain

\[
(18) \quad \text{div}(\text{Hess}(f)) = \frac{n-1}{n-2} d\lambda - \frac{1}{n-2} d(\Delta(f)) - \frac{1}{2(n-2)} d(\text{scal}).
\]
Using the relation proved in [5], namely
\[
\text{div}(\text{Hess}(f)) = d(\Delta(f)) + i_{Q(\text{grad}(f))}g,
\]
we get
\[
d(\Delta(f)) = d\lambda - \frac{1}{2(n-1)} d(\text{scal}) - \frac{n-2}{n-1} i_{Q(\text{grad}(f))}g.
\]

Equating (16) and (20), we get:

**Proposition 2.7.** If \((V = \text{grad}(f), \lambda)\) define a gradient almost Riemann soliton on the \(n\)-dimensional Riemannian manifold \((M, g)\), then
\[
\text{grad}(\lambda) = -\frac{2}{n-1} Q(\text{grad}(f)).
\]

In equation (6), by taking the scalar product with \(\text{Ric}\), we get
\[
\text{div}^{(\text{Ric}, \nabla)}(V) = \frac{1}{n-2} \left\{ (n-1)[n(n-1)\lambda^2 - (3n-2)\lambda \cdot \text{div}(V) + 2(\text{div}(V))^2] - |\text{Ric}|^2 \right\}
\]
and by taking the scalar product with \(\mathcal{L}_V g\), we get
\[
\text{div}^{(\text{Ric}, \nabla)}(V) = \frac{n-2}{4} \left\{ \frac{4[(n-1)\lambda \cdot \text{div}(V) - (\text{div}(V))^2]}{n-2} - |\mathcal{L}_V g|^2 \right\}.
\]

If \(V\) is a solenoidal vector field, then
\[
\frac{1}{n-2} [n(n-1)^2\lambda^2 - |\text{Ric}|^2] = -\frac{n-2}{4} |\mathcal{L}_V g|^2,
\]
which imply
\[
|\text{Ric}|^2 \geq n(n-1)^2\lambda^2.
\]
But \(|\mathcal{L}_V g|^2 = 4|\nabla V|^2\), which gives
\[
|\text{Ric}|^2 = n(n-1)^2\lambda^2 + (n-2)^2 |\nabla V|^2.
\]

If we assume that \(V\) is a solenoidal vector field of gradient type, then (8) becomes
\[
|\nabla V|^2 = \frac{1}{2} \Delta(|V|^2) - \text{Ric}(V, V),
\]
therefore, using (12), we obtain:
Proposition 2.8. Let \((V, \lambda)\) define a gradient almost Riemann soliton on the \(n\)-dimensional Riemannian manifold \((M, g)\). If \(V\) is a solenoidal vector field, then

\[
|\text{Ric}|^2 = n(n-1)^2 \lambda^2 + \frac{(n-2)^2}{2} |\nabla V|^2 + (n-1)V(\lambda).
\]

From (13) and Proposition 2.8, we get:

Corollary 2.9. If \((V, \lambda)\) define a gradient almost Riemann soliton on the \(n\)-dimensional Riemannian manifold \((M, g)\) with \(V\) a solenoidal vector field, then

\[
|\text{Ric}|^2 = n(n-1)^2 \lambda^2 - (n-1)(n-2)^2 |V|^2 \lambda + \frac{(n-2)^2}{2} \left[ \Delta (|V|^2) + (n-2)V(|V|^2) \right].
\]

Moreover, if \(V\) is a unitary vector field, then \(M\) is a Ricci-flat manifold.

2.2 Solitons with torse-forming potential vector field

Assume next that \(V\) is a torse-forming vector field, i.e. \(\nabla V = aI + \psi \otimes V\), with \(a\) a smooth function and \(\psi\) a 1-form, where \(\nabla\) is the Levi-Civita connection of \(g\). Then the Lie derivative of \(g\) in the direction of \(V\) is

\[(21) \quad \mathcal{L}_V g = 2ag + \psi \otimes \theta + \theta \otimes \psi,\]

where \(\theta\) is the \(g\)-dual 1-form of \(V\). Also \(\text{div}(V) = na + \psi(V)\) and (6) becomes

\[
\frac{1}{2} \mathcal{L}_V g + \frac{1}{n-2} \text{Ric} = \frac{[(n-1)\lambda - na - \psi(V)]}{n-2} g
\]

and replacing \(\mathcal{L}_V g\) from (21), we get

\[(22) \quad \text{Ric} = [(n-1)(\lambda - 2a) - \psi(V)] g - \frac{n-2}{2} (\psi \otimes \theta + \theta \otimes \psi),\]

\[
Q = [(n-1)(\lambda - 2a) - \psi(V)] I - \frac{n-2}{2} (\psi \otimes V + \theta \otimes \zeta),
\]

hence

\[
\text{scal} = (n-1)[n(\lambda - 2a) - 2\psi(V)],
\]

where \(\zeta\) is the \(g\)-dual of \(\psi\).

From (5), for \(\nabla V = aI + \psi \otimes V\), we also obtain

\[
R(X,Y)V = J(X)Y - J(Y)X + [(\nabla_X \psi)Y - (\nabla_Y \psi)X]V,
\]

for any \(X,Y \in \chi(M)\), where \(J := da - a\psi\).
Moreover, if ψ is a Codazzi tensor field, i.e. \((\nabla_X \psi)Y = (\nabla_Y \psi)X\), for any \(X,Y \in \chi(M)\), then the Jacobi operator \(R(\cdot,V)V\) w.r.t. \(V\) is given by
\[
R(\cdot,V)V = da \otimes V - V(a)I - a[\psi \otimes V - \psi(V)I]
\]
and
\[
(23) \quad \text{Ric}(V,V) = (1 - n)[V(a) - a\psi(V)].
\]

But if \((V,\lambda)\) define an almost Riemann soliton with torse-forming potential vector field \(V\), then
\[
(24) \quad \text{Ric}(V,V) = (n - 1)[(\lambda - 2a) - \psi(V)]|V|^2,
\]
and we can state:

**Proposition 2.10.** If \((V,\lambda)\) define an almost Riemann soliton on the \(n\)-dimensional Riemannian manifold \((M,g)\), \(n \geq 3\), such that the potential vector field \(V\) is torse-forming and \(\psi\) is a Codazzi tensor field, then
\[
\lambda = 2a + \frac{1}{|V|^2}[(a + |V|^2)\psi(V) - V(a)].
\]

Moreover, if \(V\) is a concircular vector field, i.e. \(\nabla V = aI\), with \(a\) a non zero constant, then

i) \(\lambda = 2a\);

ii) \(M\) is a Ricci-flat manifold (i.e. Ric \(= 0\)).

**Proof.** i) and ii) follow from (23), (24) and (22).

If \(a\) is a non zero constant, then Ric\((V,V) = 0\) and we get \(\lambda = 2a\) and Ric \(= 0\).

**Remark 2.11.** i) If \(V\) is concircular, i.e. \(\nabla V = aI\), with \(a\) a smooth function, \(a(x) \neq 0\), for any \(x \in M\), then
\[
V = \frac{1}{2a} \text{grad}(|V|^2)
\]
whose divergence is div\((V) = na\).

ii) If the torse-forming vector field \(V\) and \(\zeta\) are \(g\)-orthogonal and \(\psi\) is a Codazzi tensor field, then \(M\) is an almost quasi-Einstein manifold with associated functions
\[
\left(- (n - 1) \frac{V(a)}{|V|^2}, - \frac{n - 2}{2}\right).
\]
Next we shall find a condition such that an almost Ricci soliton to provide an almost Riemann soliton. Recall that the conharmonic curvature tensor $\mathcal{H}$ was defined in (32)

$$\mathcal{H}(X,Y)Z := R(X,Y)Z$$

(25) \[ \frac{1}{\dim(M) - 2} [g(Z,X)QY - g(Y,Z)QX + \text{Ric}(Z,X)Y - \text{Ric}(Y,Z)X], \]

for any $X, Y, Z \in \chi(M)$.

**Proposition 2.12.** Let $(V, \lambda)$ define an almost Ricci soliton on the $n$-dimensional Riemannian manifold $(M, g)$, $n \geq 3$. Then $(V, 2\lambda)$ define an almost Riemann soliton if and only if the conharmonic curvature tensor field $\mathcal{H}$ of $g$ satisfies $\mathcal{H} = \frac{n-3}{n-2} R$.

**Proof.** Applying the Kulkarni-Nomizu product with $g$ to the Ricci soliton equation (3), we deduce that $\text{Ric} \circ g = R$ if and only if

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + \text{Ric}(Y,Z)X - \text{Ric}(X,Z)Y,$$

for any $X, Y, Z \in \chi(M)$, which gives the conclusion. \qed

**Corollary 2.13.** If $(V, \lambda)$ define an almost Ricci soliton on the $n$-dimensional Riemannian manifold $(M, g)$, $n \geq 3$, with $V$ a concircular vector field, then $(V, 2\lambda)$ define an almost Riemann soliton if and only if

$$R(X,Y)Z = 2(\lambda - a)[g(Y,Z)X - g(X,Z)Y],$$

for any $X, Y, Z \in \chi(M)$. Moreover, we get

i) $V(a)\theta = |V|^2 da$;

ii) $\text{grad}(a) = V(a)_{|V|^2}$;

iii) $M$ is of constant scalar curvature.

**Proof.** If $V$ is concircular, from the Ricci soliton equation (3), we get

$$\text{Ric} = (\lambda - a)g, \quad Q = (\lambda - a)I, \quad \text{scal} = n(\lambda - a).$$

Then from Proposition 2.12, we deduce that the Riemann soliton equation (5) is satisfied for $(V, 2\lambda)$ if and only if

$$R(X,Y)Z = 2(\lambda - a)[g(Y,Z)X - g(X,Z)Y],$$
for any $X, Y, Z \in \chi(M)$. Since

$$R(X, V)V = X(a)V - V(a)X,$$

for any $X \in \chi(M)$, we have

$$[2(\lambda - a)|V|^2 + V(a)]X = [2(\lambda - a)\theta(X) + X(a)]V,$$

where $\theta = i_V g$, and applying $\theta$, we get i) and ii). Also, taking into account that $\text{div}(\text{Ric}) = \frac{1}{2} d(\text{scal})$, we get $d\lambda = da$, hence $d(\text{scal}) = 0$ and we deduce iii).

Assume that the potential vector field $V$ of the Riemann soliton defined by $(V, \lambda)$ is a torse-forming vector field, i.e. $\nabla V = aI + \psi \otimes V$, with $a$ a smooth function and $\psi$ a 1-form, where $\nabla$ is the Levi-Civita connection of $g$. Let $\theta = i_V g$ and $i_\zeta g = \psi$. Notice that

$$\nabla \theta = ag + \psi \otimes \theta,$$

$$(\nabla_V \psi)V + (\nabla_V \theta)\zeta = V(\psi(V)), \quad (\nabla_\zeta \psi)V + (\nabla_\zeta \theta)\zeta = \zeta(\psi(V)),$$

$$(\nabla_V \theta)\zeta \cdot |\zeta|^2 = (\nabla_\zeta \theta)\zeta \cdot \psi(V).$$

In particular, if $V$ and $\zeta$ are $g$-orthogonal, then $a = -\frac{1}{2|V|^2} V(|V|^2)$.

But differentiating covariantly (22), we get

$$(26) \quad (\nabla_X \text{Ric})(Y, Z) = \{(n - 1)[X(\lambda) - 2X(a)] - X(\psi(V))\} g(Y, Z)$$

$$- \frac{n - 2}{2} [\psi(Y)(\nabla_X \theta)Z + \psi(Z)(\nabla_X \theta)Y + \theta(Y)(\nabla_X \psi)Z + \theta(Z)(\nabla_X \psi)Y],$$

for any $X, Y, Z \in \chi(M)$ and we can state:

**Proposition 2.14.** If $(V, \lambda)$ define an almost Riemann soliton with concircular potential vector field $V$, then

i) $M$ is Ricci symmetric (i.e. $\nabla \text{Ric} = 0$) if and only if $d\lambda = 2da$;

ii) the Ricci tensor field $\text{Ric}$ is $\theta$-recurrent (i.e. $\text{Ric} = \theta \otimes \text{Ric}$) if and only if

$$\text{grad}(\lambda - 2a) = (\lambda - 2a)V;$$

iii) $\text{Ric}$ is a Codazzi tensor field (i.e. $(\nabla_X \text{Ric})(Y, Z) = (\nabla_Y \text{Ric})(X, Z)$, for any $X, Y, Z \in \chi(M)$) if and only if

$$d(\lambda - 2a) \otimes I = I \otimes d(\lambda - 2a);$$
iv) $M$ has cyclic Ricci tensor (i.e. $(\nabla_X \text{Ric})(Y,Z)+(\nabla_Y \text{Ric})(Z,X)+(\nabla_Z \text{Ric})(X,Y) = 0,$ for any $X,Y,Z \in \chi(M)$) if and only if

$$\text{grad}(\lambda - 2a) = -2V(\lambda - 2a) \frac{V}{|V|^2}.$$ 

**Proof.** If $V$ is concircular, from (26) we get

$$(\nabla_X \text{Ric})(Y,Z) = (n - 1)[X(\lambda) - 2X(a)]g(Y,Z),$$

for any $X,Y,Z \in \chi(M)$ and we easily get the conclusions.

Next we shall assume the curvature condition $R(V, \cdot) \cdot \text{Ric} = 0,$ where by $\cdot$ we denote the derivation of the tensor algebra at each point of the tangent space.

**Proposition 2.15.** Let $(V, \lambda)$ define an almost Riemann soliton on the $n$-dimensional Riemannian manifold $(M, g),$ $n \geq 3,$ with torse-forming potential vector field $V.$ If $R(V, \cdot) \cdot \text{Ric} = 0,$ then $\lambda = \frac{1}{2} \cdot \frac{V(|V|^2)}{|V|^2} + a$ or $a = \frac{1}{2} \cdot \frac{V(|V|^2)}{|V|^2} \pm |V| \cdot |\zeta|.$

**Proof.** If $V$ is torse-forming, from (3) we get

$$2R(X,Y)Z = [2(\lambda - 2a)g(Y,Z) - \psi(Y)\theta(Z) - \theta(Y)\psi(Z)]X$$

$$- [2(\lambda - 2a)g(X,Z) - \psi(X)\theta(Z) - \theta(X)\psi(Z)]Y$$

$$- [g(Y,Z)\psi(X) - g(X,Z)\psi(Y)]V - [g(Y,Z)\theta(X) - g(X,Z)\theta(Y)]\zeta$$

and, in particular,

$$2R(V,Y)Z = [2(\lambda - 2a) - \psi(V)]g(Y,Z)V - \theta(Z)Y - \psi(Z)\theta(Y)V - |V|^2Y$$

$$- [|V|^2g(Y,Z) - \theta(Y)\theta(Z)]\zeta.$$ 

(27)

The condition that must be satisfied by $\text{Ric}$ is:

$$\text{Ric}(R(V,X)Y,Z) + \text{Ric}(Y, R(V,X)Z) = 0,$$

for any $X, Y, Z \in \chi(M).$

Replacing the expression of $R(V, \cdot) \cdot \text{Ric}$ from (2.2) in (28) we get:

$$[2(\lambda - 2a) - \psi(V)]g(X,Y)\text{Ric}(V,Z) - \theta(Y)\text{Ric}(X,Z) + g(X,Z)\text{Ric}(V,Y) - \theta(Z)\text{Ric}(Y,X)].$$

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\[-\psi(Y)\theta(X) \, \text{Ric}(V, Z) + \psi(Y)|V|^2 \, \text{Ric}(X, Z) - \psi(Z)\theta(X) \, \text{Ric}(V, Y) + \psi(Z)|V|^2 \, \text{Ric}(X, Y)\]

\[-|V|^2 g(X, Y) - \theta(X)\theta(Y)] \, \text{Ric}(\zeta, Z) - [|V|^2 g(X, Z) - \theta(X)\theta(Z)] \, \text{Ric}(\zeta, Y) = 0,\]

for any \(X, Y, Z \in \chi(M)\).

For \(Y = Z := V\) we have:

\[[\lambda - 2a - \psi(V)] [\theta(X) \, \text{Ric}(V, V) - |V|^2 \, \text{Ric}(X, V)] = 0.\]

But from [22]:

\[
\text{Ric}(X, V) = \left[ (n - 1)(\lambda - 2a) - \frac{n}{2} \psi(V) \right] \theta(X) - \frac{n - 2}{2} |V|^2 \psi(X),
\]

which implies

\[[\lambda - 2a - \psi(V)] [\psi(V)\theta(X) - |V|^2 \psi(X)] = 0.\]

Then either \(\psi(V) = \lambda - 2a\) or \(\psi(V)V = |V|^2 \zeta\). But \(\psi(V) = \frac{1}{2|V|^2} V(|V|^2) - a\), hence the conclusion.

\[
\square
\]

References

[1] Blaga, A. M.: A note on almost \(\eta\)-Ricci solitons in Euclidean hypersurfaces. Serdica Math. J. 43(3-4), 361-368 (2017).

[2] Blaga, A. M.: A note on warped product almost quasi-Yamabe solitons. Filomat 33(7), 2009-2016 (2019).

[3] Blaga, A. M.: Almost \(\eta\)-Ricci solitons in \((LCS)_n\)-manifolds. Bull. Belg. Math. Soc. Simon Stevin 25(5), 641-653 (2018).

[4] Blaga, A. M.: \(\eta\)-Ricci solitons on Lorentzian para-Sasakian manifolds. Filomat 30(2), 489-496 (2016).

[5] Blaga, A. M.: \(\eta\)-Ricci solitons on para-Kenmotsu manifolds. Balkan J. Geom. Appl. 20(1), 1-13 (2015).

[6] Blaga A. M.: Geometric solitons in a \(D\)-homothetically deformed Kenmotsu manifold, arXiv:2008.03502.

[7] Blaga A. M.: Gradient solitons in statistical manifolds, arXiv:2005.13470.

[8] Blaga A. M.: Harmonic aspects in an \(\eta\)-Ricci soliton, Int. El. J. Geom. 13(1), 41-49 (2020).
[9] Blaga, A. M.: *Last multipliers on \( \eta \)-Ricci solitons*. Matematichki Bilten 42(2), 85-90 (2018).

[10] Blaga, A. M.: *On gradient \( \eta \)-Einstein solitons*. Kragujevak J. Math. 42(2), 229-237 (2018).

[11] Blaga, A. M.: *On harmonicity and Miao-Tam critical metrics in a perfect fluid spacetime*. Bol. Soc. Mat. Mexicana, https://doi.org/10.1007/s40590-020-00281-4 (2020).

[12] Blaga, A. M.: *On solitons in statistical geometry*. Int. J. Appl. Math. Stat. 58(4), (2019).

[13] Blaga, A. M.: *On warped product gradient \( \eta \)-Ricci solitons*. Filomat 31(18), 5791-5801 (2017).

[14] Blaga, A. M.: *Remarks on almost \( \eta \)-solitons*. Matematicki Vesnik 71(3), 244-249 (2019).

[15] Blaga, A. M.: *Solitons and geometrical structures in a perfect fluid spacetime*. Rocky Mountain J. Math. 50(1), 41-53 (2020).

[16] Blaga, A. M.: *Solutions of some types of soliton equations in \( \mathbb{R}^3 \).* Filomat 33(4), 1159-1162 (2019).

[17] Blaga, A. M.: *Some geometrical aspects of Einstein, Ricci and Yamabe solitons*. J. Geom. Sym. Phys. 52, 17-26 (2019).

[18] Blaga, A. M., Baishya, K. K., Sarkar, N.: *Ricci solitons in a generalized weakly (Ricci) symmetric D-homothetically deformed Kenmotsu manifold*. Ann. Univ. Paedagog. Crac. Stud. Math. 18, 123-136 (2019).

[19] Blaga, A. M., Crasmareanu, M. C.: *Inequalities for gradient Einstein and Ricci solitons*. Facta Univ. Math. Inform. 35(2), 351-356 (2020).

[20] Blaga, A. M., Crasmareanu, M. C.: *Torse-forming \( \eta \)-Ricci solitons in almost paracontact \( \eta \)-Einstein geometry*. Filomat 31(2), 499-504 (2017).

[21] Blaga A. M., Özgür, C.: *Almost \( \eta \)-Ricci and almost \( \eta \)-Yamabe solitons with torse-forming potential vector field*, arXiv:2003.12574.

[22] Blaga, A. M., Perkta¸s, S. Y.: *Remarks on almost \( \eta \)-Ricci solitons in \( \varepsilon \)-para Sasakian manifolds*. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 68(2), 1621-1628 (2019).

[23] Blaga, A. M., Perkta¸s, S. Y., Acet, B. E., Erdogan, F. E.: *\( \eta \)-Ricci solitons in \( \varepsilon \)-almost paracontact metric manifolds*. Glasnik Matematicki 53(1), 377-410 (2018).

[24] Catino G., Mastrolia P., Monticelli D. D.: *Gradient Ricci solitons with vanishing conditions on Weil*, J. Math. Pures et Appl. 108(1), 1-13 (2017).
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[25] De, K., Blaga, A. M., De, U. C.: \textit{*-Ricci solitons on (e)-Kenmotsu manifolds.} Palestine Math. J. \textbf{9}(2), 984-990 (2020).

[26] Devaraja M. N., Kumara H. A., Venkatesha V.: \textit{Riemann solitons within the framework of contact geometry}, Questiones Math., 1-15 (2020).

[27] Hamilton R. S.: \textit{Three-manifolds with positive Ricci curvature}, J. Diff. Geom. \textbf{17}(2), 255-306 (1982).

[28] Hamilton R. S.: \textit{The Ricci flow on surfaces}, Math. and general relativity (Santa Cruz, CA, 1986), Contemp. Math. \textbf{71}, 237-262 (1988), AMS.

[29] Hircă I. E., Udrîște C.: \textit{Ricci and Riemann solitons}, Balkan J. Geom. Appl. \textbf{21}, 35-44 (2016).

[30] Oubina J. A.: \textit{New classes of almost contact metric structures}, Publicationes Mathematicae Debrecen \textbf{32}(3-4), 187-193 (1985).

[31] Perelman G.: \textit{Ricci flow with surgery on three-manifolds}, arXiv:math/0303109

[32] Pokhariyal G. P., Mishra R. S.: \textit{Curvature tensors and their relativistic significance}, Yokohama Math. J. \textbf{18}, 105-108 (1970).

[33] Sasaki S.: \textit{On differentiable manifolds with certain structures which are closely related to almost contact structure I}, Tohoku Math. J. \textbf{12}(2), 459-476 (1960).

[34] Udrîște C.: \textit{Riemann flow and Riemann wave via bialternate product Riemannian metric}, arXiv:1112.4279.

[35] Venkatesha V., Kumara H. A., Devaraja M. N.: \textit{Riemann solitons and almost Riemann solitons on almost Kenmotsu manifolds}, Int. J. Geom. Meth. Modern Phys. \textbf{17}(7), (2020).

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