The Chiral Ring and the Periods of the Resolvent

Frank FERRARI

Service de Physique Théorique et Mathématique
Université Libre de Bruxelles and International Solvay Institutes
Campus de la Plaine, CP 231, B-1050 Bruxelles, Belgique
frank.ferrari@ulb.ac.be

The strongly coupled vacua of an $\mathcal{N} = 1$ supersymmetric gauge theory can be described by imposing quantization conditions on the periods of the gauge theory resolvent, or equivalently by imposing factorization conditions on the associated $\mathcal{N} = 2$ Seiberg-Witten curve (the so-called strong-coupling approach). We show that these conditions are equivalent to the existence of certain relations in the chiral ring, which themselves follow from the fact that the gauge group has a finite rank. This provides a conceptually very simple explanation of why and how the strongly coupled physics of $\mathcal{N} = 1$ theories, including fractional instanton effects, chiral symmetry breaking and confinement, can be derived from purely semi-classical calculations involving instantons only.

November 1, 2018
1 Introduction

When a four dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theory is deformed into an \( \mathcal{N} = 1 \) theory, many interesting strong coupling effects are expected to occur, like confinement, chiral symmetry breaking and the creation of a mass gap. Whereas the solution of the parent \( \mathcal{N} = 2 \) theory is governed by semi-classical instanton effects \([1]\), most vacua of the corresponding \( \mathcal{N} = 1 \) theory are strongly coupled and cannot be described in semi-classical terms. For example, chiral observables vacuum expectation values are typically given by fractional instanton series (that is to say, series for which the expansion parameter is a fractional power of the usual instanton factor).

Yet, it has been known for a long time that a very simple and consistent description of the strongly coupled \( \mathcal{N} = 1 \) vacua could be given in terms of the underlying \( \mathcal{N} = 2 \) theory \([1]\). This so-called “strong coupling” approach is based on the fact that at low energy, the parent \( \mathcal{N} = 2 \) theory is governed by a free abelian gauge theory. Assuming that the \( \mathcal{N} = 1 \) theory creates a mass gap, the \( \mathcal{N} = 2 \) moduli must be frozen at the singularities of the \( \mathcal{N} = 2 \) moduli space when the \( \mathcal{N} = 1 \) deformation is turned on. This is so because the only way a free abelian gauge theory can have a mass gap is through the usual Higgs mechanism, and this mechanism can only occur when charged fields, that are only present at the singularities, condense. This strong coupling procedure then implies confinement and chiral symmetry breaking \([1]\), and, when combined with the generalized Konishi anomaly equations \([2]\), essentially fixes all the correlators of chiral operators in the theory.

A second, equivalent description of \( \mathcal{N} = 1 \) theories can be given in the context of the gauge theory/matrix model correspondence \([3, 4]\). This is an elegant approach that allows to derive most of the known exact results in the field, including the Seiberg-Witten solution of \( \mathcal{N} = 2 \) super Yang-Mills \([1, 5]\), and many non-perturbative effects on the space of vacua of \( \mathcal{N} = 1 \) theories \([6, 7, 8, 9]\). On the matrix model side of the correspondence, the most general solution depends on a set of parameters, the filling fractions, that can be chosen arbitrarily. On the gauge theory side, however, these filling fractions correspond to the gluino condensates \( S_I \), and thus must be fixed, non-perturbative functions of the parameters (couplings in the tree-level superpotential and dynamically generated scale or gauge coupling constant).

Understanding the basic principles that fix the filling fractions \( S_I \) in the gauge theory is a major challenge. The original conjecture was that the filling fractions are determined by extremizing a certain superpotential, the so-called Dijkgraaf-Vafa
This proposal is well motivated by using the gauge/string correspondence (in the dual string formulation, the Dijkgraaf-Vafa superpotential is a flux superpotential [10]), but is difficult to understand from the field theory point of view. The equations (1.1) look like complicated dynamical constraints, consistently with the idea that a field theoretic proof would involve a deep understanding of the non-perturbative gauge dynamics [2]. Actually, we are going to show that a conceptually simple justification can be found.

A very nice property of the equations (1.1), pointed out in [9], is that they are mathematically equivalent to a set of quantization conditions for the periods of the one-form $\mathcal{R} dz$, where $\mathcal{R}$ is the gauge theory resolvent defined by

$$\mathcal{R}(z) = \text{tr} \frac{1}{z - X}. \quad (1.2)$$

The adjoint chiral superfield $X$ in the above formula is the superpartner of the vector superfield in the $\mathcal{N} = 2$ theory. A short proof of this statement is given for example in [11]. More precisely, the generalized Konishi anomaly equations [2] imply that $\mathcal{R} dz$ is a meromorphic differential on a certain hyperelliptic curve $\mathcal{C}$. If we write down the equation for $\mathcal{C}$ in the form

$$\mathcal{C} : \ y^2 = \prod_{I=1}^{d} (z - a_I^-)(z - a_I^+), \quad (1.3)$$

![Figure 1: The non-compact two-sheeted Riemann surface $\mathcal{C}$, with the contours $\alpha_I$ and $\gamma_{IJ}$ used in the main text.](image)
and define the contours as in Figure 1, the quantization conditions read¹

\[
\oint_{\gamma_{\ell,j}} \mathcal{R} \, dz \in 2i\pi \mathbb{Z}, \quad (1.4)
\]

\[
\oint_{\alpha_{\ell}} \mathcal{R} \, dz \in 2i\pi \mathbb{Z}. \quad (1.5)
\]

The quantization conditions (1.5) might seem obvious, because \(\frac{1}{2i\pi} \oint_{\alpha_{\ell}} \mathcal{R} \, dz\) may be interpreted as giving the number of eigenvalues of the matrix \(X\) in the cut \([a^{-1}_{\ell}, a^{+}_{\ell}]\). However, as we explain in the next Section, (1.5) is non-trivial from a fully non-perturbative point of view. Actually, both the quantization conditions (1.4) and (1.5) appear on an equal footing in the arguments that we present in this paper.

So we have two simple, elegant and physically well-motivated procedures to derive the solution of \(\mathcal{N} = 1\) gauge theories that are deformations of parent \(\mathcal{N} = 2\) gauge theories. It is known that these procedures are mathematically equivalent. Naively, it is a priori very difficult to justify these approaches from first principles. This might seem inevitable, since they are at the basis of the derivation of strongly coupled effects that cannot be described in semi-classical terms.

The main result of the present work is to provide an extremely simple argument, from first principles, proving directly the quantization conditions (1.4) and (1.5), or equivalently the validity of the strong coupling approach. The main idea of the proof is to concentrate on relations in the chiral ring that must exist because the gauge group is of finite rank. The existence of these relations is a basic difference with the associated matrix model, for which the size of the matrix is infinite. The main point is that the conditions (1.4) and (1.5) are simply equivalent to a particular form of the constraints. Since the constraints are operator relations, they must remain true in all the vacua of the deformed \(\mathcal{N} = 1\) theory if they are established in the \(\mathcal{N} = 2\) limit. This is possible if and only if (1.4) and (1.5) are satisfied in all the \(\mathcal{N} = 1\) vacua, or equivalently if and only if the \(\mathcal{N} = 1\) vacua are described by the usual factorized Seiberg-Witten curves.

We give full details in the case of the \(\mathcal{N} = 2\) \(U(N)\) theory deformed by a superpotential term \(\text{tr} W_{\text{tree}}(X)\), including when \(N_f \leq 2N\) flavors of quarks are present. However, our arguments are completely general. The same ideas actually apply as well to \(\mathcal{N} = 1\) theories that are not necessarily deformations of \(\mathcal{N} = 2\) theories.

The paper is organized as follows. In Section 2, we discuss the constraints in the chiral ring that follow from the finiteness of the number of colors \(N\). These constraints

¹The conditions (1.4) are equivalent to \(\partial W_{\text{DV}}/\partial S_I = \partial W_{\text{DV}}/\partial S_J\). As explained in the Section 4 of [11], the missing equation can then be derived from a standard Ward identity.
are at the basis of the main argument that is explained in Section 3. Finally, in Section 4 we present some open problems and discuss the relations of the present work with [11].

A note on notations: in the following, we consider expectation values of chiral operators $\mathcal{O}$ in various vacua. The symbol $\langle \mathcal{O} \rangle$ is used when we do not need to specify a particular vacuum, typically in expressions that are valid in all the vacua. Relations valid in all the vacua are also often noted as operator relations without brackets. On the other hand, in vacuum-dependent equations, we always specify explicitly in which vacuum $|0\rangle$ we are working, by using the symbol $\langle 0 | \mathcal{O} | 0 \rangle$.

2 On the relations in the chiral ring

Let us consider the U($N$) theory with $\mathcal{N} = 1$ supersymmetry and one adjoint chiral superfield $X$. The generalization to the theory with flavors is discussed in 3.4.

The chiral ring of the theory is generated by the operators [2]

$$u_k = \text{tr} X^k, \quad u^\alpha_k = \frac{1}{4\pi} \text{tr} W^\alpha X^k, \quad v_k = -\frac{1}{16\pi^2} \text{tr} W^\alpha W_\alpha X^k,$$

where $W^\alpha$ is the $\mathcal{N} = 1$ super field strength. It is well-known that for any $N \times N$ matrix $X$, the traces $\text{tr} X^k$ for $k > N$ can be expressed in terms of the $\text{tr} X^k$ for $1 \leq k \leq N$. These relations take the form

$$u_{N+p} = \mathcal{P}_{cl,p}(u_1, \ldots, u_N), \quad p \geq 1,$$

where the $\mathcal{P}_{cl,p}$ are homogeneous polynomials of degree $p + N$ in the $u_1, \ldots, u_N$ ($u_k$ being of degree $k$). Let us emphasize that these relations are simply identities, that follow from the finiteness of the rank of the gauge group. There are also similar relations relating the $u^\alpha_{N+p}$ and the $v_{N+p}$ for $p \geq 0$ to the $u^\alpha_0, \ldots, u^\alpha_{N-1}, u_1, \ldots, u_N$ and $v_0, \ldots, v_{N-1}, u_1, \ldots, u_N$ respectively, but we don’t need them for our analysis.

Non-perturbatively, the relations (2.2) may be modified. The new relations must be consistent with the U(1)$_A$ and U(1)$_R$ symmetries of the theory. These symmetries act on the operators, the parameters $g_k$ (defined in (3.9)) and the instanton factor $\Lambda^{2N}$ as

$$\begin{array}{cccccc}
U(1)_A & u_k & u^\alpha_k & v_k & \Lambda^{2N} & g_k \\
U(1)_R & 0 & 1 & 2 & 0 & -k - 1
\end{array}$$

(2.3)

\[2\text{By operator relations, we always mean operator relations in the chiral ring.}\]
The $U(1)_R$ symmetry implies that the $u_k$ cannot mix with the other operators $u^a_k$ or $v_k$, and that the quantum version of the relations (2.2) cannot depend on the couplings $g_k$. In other words, we can restrict ourselves (and this will turn out to be sufficient for our purposes) to the sector of the chiral ring with zero $U(1)_R$ charge. This sector is itself a ring $A$, generated by the $u_k$. The relations in $A$ can take the general form

$$u_{N+p} = \mathcal{P}_p(u_1, \ldots, u_N; \Lambda^{2N}), \quad p \geq 1,$$

(2.4)

where the $\mathcal{P}_p$ are polynomials of $U(1)_A$ charge $p+N$ that go to the classical polynomials $\mathcal{P}_{cl,p}$ when $\Lambda^{2N}$ goes to zero.\(^3\)

A fact we would like to emphasize is that the quantum corrections in (2.4) do not represent a “deformation,” in any sensible mathematical sense, of the classical chiral ring. The classical chiral ring $A_{cl}$ (in the sector of zero R-charge we’re interested in) is simply the polynomial algebra generated by the $u_k$ for $1 \leq k \leq N$. The quantum version $A$ of this ring must be commutative (since only bosonic variables are present) and generated freely by the same elements $u_1, \ldots, u_N$. This implies that

$$A = A_{cl} = \mathbb{C}[u_1, \ldots, u_N].$$

(2.5)

A more abstract way to understand this is to note that because the quantum theory can be seen as a smooth deformation of the classical theory obtained by turning on the instanton factor $\Lambda^{2N}$, the possible deformations of $A_{cl}$ can be studied using the standard deformation theory based on Hochschild cohomology. It is an elementary result that there is no possible non-trivial deformation of a polynomial algebra that preserves commutativity (see for example [12] for an elementary exposition). The conclusion is that the zero R-charge sector of the chiral ring is not quantum corrected.\(^4\)

So what is the interpretation of the relations (2.4)? They actually represent the definitions of what we call the $u_{N+p}$ for $p \geq 1$. These definitions are non-dynamical, and a priori can be completely arbitrary (as long as they are consistent with symmetries and the classical limit). This freedom has actually been used in some instances in the literature (for example to match results obtained by different methods [13, 14]). In particular, it is perfectly consistent to define the $u_k$ by the “classical” relations (2.2), even in the full quantum theory. However, depending on

\(^3\)Note that only the instanton factor $\Lambda^{2N}$ can enter, by $2\pi$ periodicity in the $\theta$ angle, because (2.4) is an operator relation and is thus valid in all the vacua. Fractional instanton effects do occur in $\mathcal{N} = 1$ theories, but only in relations that are valid in a particular vacuum (or a particular set of vacua).

\(^4\)The full chiral ring can be deformed, because for example elements that are nilpotent classically, like the glueball operator, are not in the quantum theory [2]. However these deformations are not related to the existence of the quantum-corrected relations (2.4).
the context, a natural non-perturbative definition of what is called \( u_{N+p} \) for \( p \geq 1 \) may involve quantum-corrected relations of the form (2.4).

For the purpose of our investigations, the natural variables are such that the anomaly equations, that were derived in perturbation theory in [2], take the same form in the full quantum theory: for all \( n \geq -1 \),

\[
-N \sum_{k \geq 0} g_k u_{n+k+1} + 2 \sum_{q_1+q_2=n} (v_{q_1} u_{q_2} + u_{q_1}^\alpha u_{q_2} \alpha) = 0, \tag{2.6}
\]

\[
-N \sum_{k \geq 0} g_k u_{n+k+1}^\alpha + 2 \sum_{q_1+q_2=n} v_{q_1} u_{q_2}^\alpha = 0, \tag{2.7}
\]

\[
-N \sum_{k \geq 0} g_k v_{n+k+1} + \sum_{q_1+q_2=n} v_{q_1} v_{q_2} = 0. \tag{2.8}
\]

Note that this assumption does not mean that the equations do not get non-perturbative corrections, but rather that the non-perturbative corrections can be absorbed in a proper definition of the variables. We also expect that this definition of the variables is the same as the one that enters naturally in the context of instanton calculus [14].

Lacking a detailed non-perturbative analysis of the anomaly equations, we shall allow the relations (2.4) to take the most general possible form a priori. This implies an interesting subtlety that has been overlooked in previous works. It is clear that the gauge theory resolvent (1.2) does depend explicitly on the particular definitions of the higher moments \( u_{N+p} \). In particular, there are many consistent definitions, with relations of the form (2.4), that violate the quantization conditions (1.5). A nice feature of our approach is that we don’t need to assume (1.5), and both quantization conditions (1.4) and (1.5) will be derived at the same time.

3 Relations and the quantization of the periods

3.1 Picking a suitable vacuum

An important point is that the precise form of the constraints (2.4) can be derived from a purely semi-classical analysis. This is possible because we can always find a vacuum that is both arbitrarily weakly coupled and suitable to fix the relations (2.4) unambiguously.

For example, the R-symmetry implies that the polynomials \( \mathcal{P}_p \) in (2.4) cannot depend on the couplings in the tree-level superpotential. We can thus choose the
latter at our convenience. We pick a degree $N + 1$ superpotential such that

$$W'_{\text{tree}}(x) = \sum_{k=0}^{N} g_k x^k = g_N \prod_{I=1}^{N} (x - a_I) = g_N P_N(x).$$

(3.9)

Classically, the gauge theory has several vacua, depending on the numbers $N_I \geq 0$ of eigenvalues of the adjoint field $X$ that are taken to be equal to $a_I$. These classical vacua are denoted by $|N_1, \ldots, N_N\rangle$ and correspond to a pattern of gauge symmetry breaking $U(N) \to U(N_1) \times \cdots \times U(N_N)$. Let us focus on the Coulomb vacuum

$$|C\rangle = |1, \ldots, 1\rangle,$$

(3.10)

in which the low energy gauge group is $U(1)^N$. This vacuum can be made arbitrarily weakly coupled by going to the region $|a_I - a_J| \gg |\Lambda|$ in the space of parameters. Moreover, in this vacuum, the expectation values $\langle C|u_1|C\rangle, \ldots, \langle C|u_N|C\rangle$ are independent, unconstrained variables.\(^5\) Equivalently, we can take the $g_N \to 0$ limit of the $\mathcal{N} = 2$ theory, in which case the $u_1, \ldots, u_N$ are moduli. This means that if we can find polynomials $\hat{P}_p$ such that

$$\langle C|u_{N+p}|C\rangle = \hat{P}_p(\langle C|u_1|C\rangle, \ldots, \langle C|u_N|C\rangle; \Lambda^{2N}), \quad p \geq 1,$$

(3.11)

for arbitrary values of the $\langle C|u_k|C\rangle$, then we know automatically that

$$\mathcal{P}_p = \hat{P}_p.$$

(3.12)

The above reasoning is quite powerful: we are able to derive operator relations by studying the theory in a particular vacuum. This is a basic feature of our method. It is made possible by the fact that we know a priori that operator equations of the form (2.4) must exist.

### 3.2 Example

Let us use the above idea to compute the polynomials $\mathcal{P}_p$ in the theory with no flavor. Equations (2.6) and (2.8) can be easily solved (using in particular $\langle u_0^k \rangle = 0$ by Lorentz invariance) to yield a general formula for the gauge theory resolvent expectation value, valid in any vacuum [2]. For the degree $N + 1$ tree level superpotential (3.9), and

\(^5\)This is not true in general. For example, for a vacuum $|0\rangle$ with an unbroken gauge group and for which all the eigenvalues of $X$ are equal classically, we have automatically $\langle 0|u_k|0\rangle = N^{1-k}\langle 0|u_1|0\rangle^k$ at the perturbative level. This shows that there is only one independent variable. In the non-perturbative theory, the relations are modified by fractional instanton effects, but the number of independent variables do not change.
introducing degree \( N - 1 \) polynomials \( Q_{N-1} \) and \( R_{N-1} \) (whose precise forms depend on the particular vacuum under consideration), we have

\[
\langle \mathcal{R}(z) \rangle = \frac{Q_{N-1}(z)}{\sqrt{P_N(z)^2 - R_{N-1}(z)}}. \tag{3.13}
\]

Note that since

\[
\langle \mathcal{R}(z) \rangle \sim \frac{N}{z}, \quad z \to \infty, \tag{3.14}
\]

we know that \( Q_{N-1}(z) = N z^{N-1} + \cdots \) in all the vacua.

A useful property of the Coulomb vacuum for the theory with no flavor is that the \( U(1)_A \) symmetry (2.3) implies that the \( \langle C|u_k|C \rangle \) cannot get quantum corrections for \( k \leq 2N - 1 \). This is so because in the Coulomb vacuum the quantum corrections are entirely generated by instantons, and the instanton factor \( \Lambda^{2N} \) has \( U(1)_A \) charge \( 2N \). We thus obtain

\[
\langle C|u_k|C \rangle = \sum_{I=1}^{N} a^k_I, \quad 1 \leq k \leq 2N - 1, \tag{3.15}
\]

which is equivalent to the following asymptotic condition,

\[
\langle C|\mathcal{R}(z)|C \rangle = \frac{P'_N(z)}{P_N(z)} + O(1/z^{2N+1}). \tag{3.16}
\]

Plugging (3.13) into (3.16), multiplying by \( \sqrt{P_N^2 - R_{N-1}} \) and expanding at large \( z \) immediately yield

\[
Q_{N-1}(z) = P'_N(z) + O(1/z^2). \tag{3.17}
\]

Since both \( Q_{N-1} \) and \( P'_N \) are polynomials, we must have

\[
Q_{N-1} = P'_N. \tag{3.18}
\]

Taking this result into account, (3.16) implies that

\[
\frac{1}{\sqrt{P_N(z)^2 - R_{N-1}(z)}} = \frac{1}{P_N(z)} + O(1/z^{3N}). \tag{3.19}
\]

Inverting this relation, taking the square and expanding at large \( z \) then yields

\[
P_N(z)^2 - R_{N-1}(z) = P_N(z)^2 + O(1), \tag{3.20}
\]

or equivalently that \( R_{N-1} \) must be a constant \( r \),

\[
R_{N-1}(z) = r. \tag{3.21}
\]
The relation (3.16), and thus (3.18) and (3.21), are of course valid only in the Coulomb vacuum.

We have thus achieved our goal: all the polynomials $P_p$ in (2.4) can be expressed in terms of the constant $r$, by expanding $\langle C | R(z) | C \rangle$ at large $z$. For example, the first non-trivial correction is obtained for $P_N$ and reads

$$P_N = P_{cl,N} + \frac{Nr}{2}.$$  \hspace{1cm} (3.22)

Clearly $r$ must be proportional to $\Lambda^{2N}$ and, at the expense of rescaling $\Lambda$, we can always choose $r = 4\Lambda^{2N}$, which is the standard convention.\footnote{The identification of the constant $r$ with $4\Lambda^{2N}$ is straightforward in the present case, but it plays no rôle in the following, and in particular is not needed to prove the quantization conditions (1.4) and (1.5).}

### 3.3 Encoding the form of the relations

In the theory with no flavor, we have been able to compute the polynomials $P_p$ by using a simple symmetry argument. This would not be the case for more general theories, for example when a large number of flavors are present. However, the only important point for us is that it is always possible to do this calculation in a purely semi-classical context (see also Section 4 for another possible way to derive the relations).

This being said, let us now show that these relations are equivalent to a simple algebraic equation satisfied by the quantum characteristic function

$$\mathcal{F}(z) = \det(z - X).$$  \hspace{1cm} (3.23)

This algebraic relation follow from a simple procedure\footnote{This trick has appeared several times in the literature, for example in [15] and [11].} to compute the polynomials $u_{N+p} = P_p$ recursively. The characteristic function $\mathcal{F}$ admits a simple expansion in terms of the $u_k$ of the form

$$\mathcal{F}(z) = z^N - \sum_{k \geq 1} F_k z^{N-k},$$  \hspace{1cm} (3.24)

where the $F_k = u_k/k + \cdots$ are polynomials in the $u_q$s of $U(1)_A$ charge $k$. The $F_k$s can be computed explicitly by writing

$$\mathcal{F}(z) = \det(z - X) = z^N e^{tr \ln(1 - X/z)} = z^N e^{-\sum_{k \geq 1} u_k/(kz^k)}$$  \hspace{1cm} (3.25)
and expanding at large $z$. At the classical level, $\mathcal{F}$ is a polynomial of degree $N$. We thus have $F_k = 0$ for all $k > N$. Since $F_k$ is the sum of $u_k/k$ plus terms that depend only on the $u_q$ for $q < k$, this yields convenient recursion relations that determine all the polynomials $\mathcal{P}_{cl,p}$. Quantum mechanically, we can find $\langle F \rangle$ in the Coulomb vacuum from the formula

$$\langle C | \mathcal{F}(z) | C \rangle = \frac{P'_N(z)}{\sqrt{P_N(z)^2 - 4\Lambda^2 N}}$$

(3.26)

derived in the previous subsection, by integrating the relation

$$\frac{d}{dz} \ln \langle F(z) \rangle = \langle \mathcal{F}(z) \rangle$$

(3.27)

and using

$$\langle F(z) \rangle \sim z^N.$$  

(3.28)

This yields

$$\langle C | \mathcal{F}(z) | C \rangle = \frac{1}{2} \left( P_N(z) + \sqrt{P_N(z)^2 - 4\Lambda^2 N} \right).$$

(3.29)

In particular, $\langle C | \mathcal{F}(z) | C \rangle$ is not a polynomial anymore, but it satisfies a simple quadratic equation

$$\langle C | \mathcal{F}(z) | C \rangle + \frac{\Lambda^2 N}{\langle C | \mathcal{F}(z) | C \rangle} = P_N(z).$$

(3.30)

By expanding at large $z$, this equation yields the $\langle C | u_k | C \rangle$ for $k > N$ as a function of the $\langle C | u_1 | C \rangle, \ldots, \langle C | u_N | C \rangle$. To see how this works in details, let us write

$$\langle \mathcal{F}(z) \rangle = z^N - \sum_{k \geq 1} \langle F_k \rangle z^{N-k}, \quad \frac{1}{\langle \mathcal{F}(z) \rangle} = z^{-N} + \sum_{k \geq 1} \langle \tilde{F}_k \rangle z^{-N-k},$$

(3.31)

and plug these expansions into (3.30). Since the right hand side is a polynomial, all the terms with negative powers of $z$ must cancel in the left hand side, yielding

$$\langle C | F_{p+N} | C \rangle = \Lambda^{2N} \langle C | \tilde{F}_{p-N} | C \rangle$$

for all $p \geq 1$,  

(3.32)

with the convention that $\tilde{F}_0 = 1$ and $\tilde{F}_k = 0$ if $k < 0$. Since $\tilde{F}_k = u_k/k + \cdots$ depends only on the $u_q$ for $q \leq k$, (3.32) generates recursively all the relations (2.4).

Now comes the main point of our argument. The equations (3.32) not only provide a simple way to fix unambiguously the relations (2.4), they are actually equivalent

---

8Here we use the fact, already emphasized in Section 3.1, that the variables $\langle C | u_1 | C \rangle, \ldots, \langle C | u_N | C \rangle$ are independent in the Coulomb vacuum.
to them. Since (2.4) is valid in all the vacua, it must be so for (3.32). In other words, we have shown that the equations (3.32) are operator relations valid in all vacua,

\[ F_{p+N} = \Lambda^{2N} \tilde{F}_{p-N}, \quad p \geq 1, \]

because they simply correspond to a convenient rewriting of the operator relations (2.4). Moreover, since the couplings \( g_k \) cannot appear in (2.4), we know that (3.33) must be true for any tree-level superpotential \( W_{\text{tree}}(X) \), not necessarily of the form (3.9).

Using the operator relations (3.33), we deduce that \( \langle \mathcal{F}(z) \rangle + \Lambda^{2N}/\langle \mathcal{F}(z) \rangle \) has no negative powers of \( z \) in its large \( N \) expansion, not only in the Coulomb vacuum but also in all the other vacua of the \( \mathcal{N} = 1 \) theory. Equivalently, this implies that

\[ \langle \mathcal{F}(z) \rangle + \frac{\Lambda^{2N}}{\langle \mathcal{F}(z) \rangle} = P(z) \quad (3.34) \]

is a polynomial in all vacua. The precise form of \( P(z) = z^N + \cdots \) depends on the particular vacuum because the operator relations (3.32) do not constrain the positive powers in \( z \) in the left hand side of (3.34).

The fundamental point in our argument is that the algebraic equation (3.34) is not dynamical, but rather acts as a generating equation for the relations (2.4). It is very important that the algebraic equation satisfied by \( \mathcal{F} \) contains only this purely “kinematical” information. Again, this is why we can derive that the equation must be valid in all the vacua of the gauge theory.

### 3.4 The quantization conditions

We now have all the necessary ingredients to show that the quantization conditions (1.4) and (1.5) must always be valid. First of all, from (3.27) and (3.34) we find

\[ \langle \mathcal{F}(z) \rangle = \frac{1}{2} \left( P(z) + \sqrt{P(z)^2 - 4\Lambda^{2N}} \right), \quad (3.35) \]
\[ \langle \mathcal{R}(z) \rangle = \frac{P'(z)}{\sqrt{P(z)^2 - 4\Lambda^{2N}}}. \quad (3.36) \]

Both \( \langle \mathcal{F} \rangle \) and \( \langle \mathcal{R} \rangle \) are thus meromorphic functions defined on the same Riemann surface \( \mathcal{C} \), independently of the vacuum under consideration. The equations (3.35) and (3.36) have a form similar to (3.29) and (3.26), but the degree \( N \) polynomial \( P \) is vacuum-dependent and is not equal to \( P_N \) in general. In particular \( W'_{\text{tree}} \) can be any polynomial, not necessarily of the form (3.9). Consistency with the anomaly
equations (2.6) and (2.8) for arbitrary $W'_{\text{tree}}$ of degree $d$ actually immediately implies the existence degrees $d - 1$, $N - \tilde{d}$ and $d - \tilde{d}$ polynomials $\Delta_{d-1}$, $F_{N-\tilde{d}}$ and $H_{d-\tilde{d}}$ respectively such that

$$W'_{\text{tree}}(z)^2 - \Delta(z) = H_{d-\tilde{d}}(z)^2 y^2, \quad P(z)^2 - 4\Lambda^2 = F_{N-\tilde{d}}(z)^2 y^2. \quad (3.37)$$

These are the factorization equations (3.37) at the basis of the strong coupling approach to $\mathcal{N} = 1$ gauge theories! In particular, the curve

$$Y^2 = P(z)^2 - 4\Lambda^2$$

is the Seiberg-Witten curve of the $\mathcal{N} = 2$ theory obtained in the $W_{\text{tree}} \to 0$ limit.

The fact that $\mathcal{F}$ and $\mathcal{R}$ are defined on the same Riemann surface is also all we need to derive the quantization conditions (1.4) and (1.5). Indeed, in general, integrating (3.27) taking into account (3.28) yields

$$\langle \mathcal{F}(z) \rangle = \lim_{\mu_0 \to \infty} \left( \mu_0^N \exp \int_{\mu_0}^{\infty} \langle \mathcal{R}(z') \rangle dz' \right), \quad (3.39)$$

where $\mu_0$ is a point on the first sheet (the sheet for which (3.28) is valid) of the Riemann surface. This formula shows that $\langle \mathcal{F}(z) \rangle$ is generically a multivalued function on the curve $\mathcal{C}$ on which $\langle \mathcal{R}(z) \rangle$ is well-defined, because one must specify the contour from the point at infinity $\mu_0$ to $z$ to do the integral in (3.39). The integral representation also shows that $\langle \mathcal{F}(z) \rangle$ will be single valued if and only if the quantization conditions

$$\oint_{\gamma_{Ij}} \langle \mathcal{R} \rangle dz \in 2i\pi \mathbb{Z}, \quad \oint_{\alpha_I} \langle \mathcal{R} \rangle dz \in 2i\pi \mathbb{Z}, \quad (3.40)$$

are satisfied. We have thus completed the proof of (1.4) and (1.5).

### 3.5 Generalization to the case with flavors

Let us now add $N_f = 2N$ flavors of fundamental and antifundamental quarks and antiquarks $Q_q$ and $\tilde{Q}^q$. The other cases $N_f < 2N$ can be obtained by integrating out some of the flavors. The tree-level superpotential has the form

$$W = \text{tr} W_{\text{tree}}(X) + \sum_{q=1}^{2N} T \tilde{Q}^q (X - m_q) Q_q. \quad (3.41)$$

The most general classical vacuum $|N_I; \nu_q \rangle$ is specified by the numbers of eigenvalues of the matrix $X$, $N_I \geq 0$ and $\nu_Q = 0$ or 1, that are equal to the $I^{th}$ extremum of $W_{\text{tree}}$ and $m_q$ respectively [9]. In particular,

$$\sum_I N_I + \sum_q \nu_q = N. \quad (3.42)$$
The pattern of gauge symmetry breaking in a vacuum \(|N_f; \nu_q\rangle\) is \(U(N) \rightarrow \prod_I U(N_I)\).

The anomaly equations [16] and associated matrix model [17] for this theory are well-known, and we shall not repeat the details here (a very detailed discussion is included in [4]). The chiral ring relations are still of the general form (2.4), but the instanton factor is now

\[ h = e^{2i\pi \tau} \]  

(3.43)

and the polynomials \(\mathcal{P}_p\) can depend symmetrically on the masses \(m_q\) (but cannot depend on the couplings in \(W_{\text{tree}}\)).

We can compute the \(\mathcal{P}_p\) semi-classically by choosing \(W_{\text{tree}}\) of the form (3.9) and going to the Coulomb vacuum \(|C\rangle = |N_I = 1; \nu_q = 0\rangle\). Symmetries are no longer enough when \(N_f = 2N\) to fix completely the solution to the anomaly equations in this vacuum, but we can rely on explicit instanton calculations [18]. Introducing the polynomial

\[ U(z) = \prod_{q=1}^{2N} (z - m_q), \]  

(3.44)

it can be shown that the characteristic function (3.23) satisfies

\[ \langle C| \mathcal{F}(z)|C \rangle + \frac{h U(z)}{\langle C| \mathcal{F}(z)|C \rangle} = P_{|C\rangle}(z) \]  

(3.45)

for some degree \(N\) polynomial \(P_{|C\rangle}\). The conditions obtained by expanding at large \(z\) and writing that the negative powers in \(z\) in the left hand side of (3.45) must vanish determine recursively the polynomials \(\mathcal{P}_p\). Conversely, the operator relations (2.4) then implies that

\[ \langle \mathcal{F}(z) \rangle + \frac{h U(z)}{\langle \mathcal{F}(z) \rangle} = P(z) \]  

(3.46)

in all the vacua, for a certain vacuum-dependent polynomial \(P\). This in turn yields the quantization conditions (1.4) and (1.5) (or the appropriate factorization of the associated Seiberg-Witten curves) in full generality.

4 Conclusion and open problems

In this paper, we have obtained an extremely simple interpretation of the quantization conditions (1.4) and (1.5). These conditions simply encode the precise form of vacuum-independent operator relations between the chiral observables \(u_k = \text{tr} X^k\). Since the relations are completely fixed by studying a weakly coupled Coulomb vacuum, the validity of (1.4) and (1.5) in all the vacua of the \(\mathcal{N} = 1\) theory, including the
strongly coupled confining vacua, can be derived from a purely semi-classical analysis. It is particularly startling that such a conceptually simple understanding can be achieved. It yields in particular a straightforward justification from first principles of the well-known strong coupling approach.

A natural question is whether the same ideas can be used to derive the quantization conditions in the Coulomb vacuum as well, in the most general cases, and independently of the semi-classical approximation. This seems plausible, because it is absolutely not obvious a priori that the solutions to the anomaly equations can be consistent with the existence of *vacuum-independent* relations between the variables. This consistency requirement does not arise in the similar-looking loop equations of the planar matrix model, because in this case all the variables are independent. In the gauge theory, it is natural to conjecture that consistency can be achieved if and only if the conditions (1.4) and (1.5) are satisfied in the Coulomb vacuum (and thus in all the other vacua by the arguments above).

An outstanding open problem is to provide a non-perturbative proof of the generalized Konishi anomalies. The existing derivations are made in perturbation theory, with a fixed classical background gauge field [2]. Contrary to a statement often made in the literature, the equations do get non-perturbative corrections, but it is believed that these corrections can be made implicit by suitably defining the variables, as explained in Section 2. Since the equations must be valid in all the vacua, it is enough to make a proof in the context of instanton calculus, and this is presently under investigation.

In [11], it was also shown, from another point of view, that the quantization conditions (1.4) and (1.5) are not dynamical but rather follow from general consistency conditions. In this respect, [11] and the present work share the same philosophy. The argument of [11] was based on the analysis of the gauge invariance of the operator $\mathcal{F}(z) = \det(z - X)$ for all values of $z$. Gauge invariance turns out to be consistent with the analytic continuation in $z$ if and only if the conditions (1.4) and (1.5) are satisfied. However, gauge invariance of $\det(z - X)$ can be achieved only for some particular definitions of the variables $u_{N+P}$, and thus is not trivial a priori (this is similar to the fact that the quantization conditions (1.5) are not trivial a priori). Again, an explicit non-perturbative definition of $\det(z - X)$ can certainly be given in the context of the instanton calculus. As suggested in [11], the standard representation of the determinant in terms of a fermionic integral is then likely to make gauge invariance manifest. Combined with the non-perturbative analysis of the anomaly equations, the arguments of [11] would then provide a new and elegant way to sum up instanton series explicitly, independently of the localization methods used in [18].
Acknowledgements

This work is supported in part by the belgian Fonds de la Recherche Fondamentale Collective (grant 2.4655.07), the belgian Institut Interuniversitaire des Sciences Nucléaires (grant 4.4505.86), the Interuniversity Attraction Poles Programme (Belgian Science Policy) and by the European Commission FP6 programme MRTN-CT-2004-005104 (in association with V. U. Brussels). The author is on leave of absence from Centre National de la Recherche Scientifique, Laboratoire de Physique Théorique de l’École Normale Supérieure, Paris, France.

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19, erratum B 430 (1994) 485, hep-th/9407087; Nucl. Phys. B 431 (1994) 484, hep-th/9408099.

[2] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, J. High Energy Phys. 12 (2002) 071, hep-th/0211170.

[3] R. Dijkgraaf and C. Vafa, Nucl. Phys. B 644 (2002) 3, hep-th/0206255; Nucl. Phys. B 644 (2002) 21, hep-th/0207106; A Perturbative Window into Non-Perturbative Physics, hep-th/0208048.

[4] F. Ferrari, Supersymmetric Gauge Theories, Matrix Models and Geometric Transitions, to appear in Phys. Rep.

[5] F. Cachazo and C. Vafa, $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Geometry from Fluxes, hep-th/0206017.

[6] F. Ferrari, Phys. Rev. D 67 (2003) 85013, hep-th/0211069.

[7] F. Ferrari, Phys. Lett. B 557 (2003) 290, hep-th/0301157.

[8] F. Cachazo, N. Seiberg and E. Witten, J. High Energy Phys. 02 (2003) 042, hep-th/0301006.

[9] F. Cachazo, N. Seiberg and E. Witten, J. High Energy Phys. 04 (2003) 018, hep-th/0303207.

[10] F. Cachazo, K. Intriligator and C. Vafa, Nucl. Phys. B 603 (2001) 3, hep-th/0103067.

[11] F. Ferrari, J. High Energy Phys. 06 (2006) 039, hep-th/0602249.
[12] M. Penkava and P. Vanhaecke, J. Algebra 227 (2000) 365; Comm. C. Math. 3 (2001) 393.

[13] N. Dorey, V.V. Khoze and M.P. Mattis, Phys. Lett. B 396 (1997) 141, hep-th/9612231.

[14] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, Phys. Rep. 371 (2002) 231, hep-th/0206063.

[15] P. Svrcek, J. High Energy Phys. 04 (2004) 036, hep-th/0308037.

[16] N. Seiberg, J. High Energy Phys. 01 (2003) 061, hep-th/0212225.

[17] R. Argurio, V.L. Campos, G. Ferretti and R. Heise, Phys. Rev. D 67 (2003) 65005, hep-th/0210291.

[18] N. Nekrasov, Adv. Theor. Math. Phys. 7 (2004) 831, hep-th/0206161; Seiberg-Witten Prepotential from Instanton Counting, Proceedings of the International Congress of Mathematicians (ICM 2002), hep-th/0306211, N. Nekrasov and A. Okounkov, Seiberg-Witten Theory and Random Partitions, hep-th/0306238.