ASYMPTOTIC FREEDOM AT SMALL $x$

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Abstract
We describe how perturbative QCD may be applied to inelastic $e-p$ scattering at high center-of-mass energies, i.e. at small $x$ and fixed $Q^2$.

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The structure functions for inclusive inelastic $e$-$p$ scattering depend on two scales: the virtuality $Q^2$ of the photon and the center-of-mass energy $W^2$ of the photon-proton collision. The Bjorken variable is defined as essentially the ratio of these two:

$$x = \frac{Q^2}{W^2 + Q^2 - m^2_p}.$$

If neither scale is large, it is generally agreed that a perturbative computation of the cross-section is not possible. If $Q^2$ becomes very large with $x$ fixed, mass factorization (or the OPE) and renormalization group invariance together imply that the appropriate scale for the running of the coupling is $Q^2$, and asymptotic freedom then justifies perturbative evolution in $Q^2$ of moments with respect to $x$ (see fig. 1).

At large $Q^2$ structure functions display Bjorken scaling, i.e. they are essentially flat, with logarithmic scaling violations. Structure functions are also essentially flat at fixed $Q^2$ but large $W^2$ (and thus small $x$), but as yet no entirely satisfactory explanation exists. The gentle growth of all physical cross-sections appears to be universal, and is often parameterized as $x^{\alpha - 1}$, where $\alpha$ is the intercept of the ‘pomeron’ singularity, and consequently close to unity. Logarithmic parameterizations are also acceptable, however. Finally, when both $Q^2$ and $W^2$ are large, but $x$ is small, structure functions rise quite steeply with both $x$ and $Q^2$, in accordance with the double scaling prediction of perturbative QCD, and consistent with the flat behaviour on either side.

The previous paragraph summarises the conventional theory of structure functions within perturbative QCD at high virtuality and Regge theory at high energy. To improve on this simple picture it is tempting to try to somehow derive the pomeron directly from QCD. At small $x$ the usual perturbative expansion is dominated by terms of the form $\alpha_s^n \ln^n 1/x$, and if these are summed at fixed $Q^2$ and fixed coupling they indeed generate a powerlike rise. However the relatively large value of the intercept $\alpha$ found in these calculations not only disagrees with the much softer rise in the data at low $Q^2$, but also spoils double scaling. Furthermore, if the extra logarithms are included in the perturbative evolution to higher $Q^2$, they result in a strong factorization scheme dependence, and in general generate too steep a rise in $x$ and $Q^2$. It thus seems that at best the higher order logarithms provide a poor approximation to the higher order terms in the conventional perturbation series: this may perhaps have been expected, since the series of logarithms converges, while the complete perturbative expansion is at best asymptotic.

The breakdown of the conventional formulation of perturbative QCD (based on the resummation of logarithms of $Q^2$) in the small $x$ limit thus cannot be cured by summing the large logarithms of $1/x$. However since the Regge limit involves a single large scale, $W^2 \gg Q^2$, there remains the possibility that perturbative QCD might still work provided it is reformulated in such a way that the evolution is with respect to $W^2$ (or $x$) and applies to moments with respect to $Q^2$. This will require a new factorization theorem, giving rise to novel evolution equations and parton distributions.

In a physical gauge a two particle reducible contribution to a physical cross-section may be factorized into an infrared finite coefficient function, depending on the renormalization scale $\mu$ only through the renormalised coupling $\alpha_s(\mu^2)$, and an ‘unintegrated’ parton distribution function $f(l^2; k^2; \mu^2)$ which gives the distribution of partons in the hadron according to their longitudinal and transverse momenta in the infinite momentum frame:

$$\sigma^{(2)}(S^2; \frac{Q^2}{\mu^2}; \mu^2) = \int_0^\infty \frac{dl^2}{l^2} \int_0^\infty \frac{dk^2}{k^2} C\left(\frac{S^2}{l^2}; \frac{Q^2}{k^2}; \alpha_s(\mu^2)\right) f\left(\frac{l^2}{\mu^2}; \frac{k^2}{\mu^2}; \mu^2\right),$$

(1)
where $S^2 \equiv \Lambda^2/x$, $l^2 \equiv \Lambda^2/y$, and $\Lambda$ is some fixed scale typical of the strong interaction. Since all the $\mu$ dependence is contained in the coupling, the separation of the integral over the momenta of the intermediate parton into longitudinal and transverse convolutions is entirely kinematic. The convolutions can be undone by Mellin transforms, to give

$$
\sigma_{NM}^{(2)} = C_{NM}(\alpha_s(\mu^2))f_{NM}(\mu^2).
$$

At large $Q^2$, the double factorization eq.(1) may be used to derive the more familiar mass factorization

$$
\sigma_N(Q^2/\mu^2; \mu^2) = C_N(Q^2/\mu^2; \alpha_s(\mu^2))F_N(\mu^2) + O(1/Q^2),
$$

where now $F_N(\mu^2)$ is the Mellin transform of a parton distribution $F(y, \mu^2)$, which can be expressed in terms of the logarithmic moments with respect to $k^2$ of the original unintegrated distribution. Multiparton (i.e. two particle irreducible) contributions to the cross-section are suppressed by powers of $Q^2$ (higher twist). Renormalization group invariance of the right hand side of eq.(3) then gives the usual evolution equation for the parton distribution and renormalization group equation for the coefficient function, whose solution leads to the choice of $Q^2$ as an appropriate factorization scale.

At large $S^2$ (i.e. small $x$), because the double factorization eq.(1) is essentially symmetrical in transverse and longitudinal scales, a similar argument gives us instead the energy factorization

$$
\sigma_M(S^2/\mu^2; \mu^2) = C_M(S^2/\mu^2; \alpha_s(\mu^2))F_M(\mu^2) + \sigma_M^q(S^2) + O(1/S^2).
$$

Here $F_M(\mu^2)$ is the Mellin transform of a different integrated parton distribution function $F(k^2, \mu^2)$, which gives the distribution of partons in transverse momentum, and can be expressed in terms of logarithmic moments with respect to $l^2$ of the unintegrated distribution. Multiparton contributions are however no longer suppressed by powers of $S^2$, though they cannot grow with $S^2$ because collinear and soft singularities only arise from emissions from a single parton. They may thus be ignored asymptotically, even though at practical energy scales they may constitute an important background to the leading single parton contribution.

Renormalization group invariance of the right hand side of eq.(4) now gives an evolution equation

$$
\mu^2 \frac{\partial}{\partial \mu^2} F_M(\mu^2) = \gamma_M(\alpha_s(\mu^2))F_M(\mu^2),
$$

for the integrated parton distribution $F_M(\mu^2)$, and a renormalization group equation for the coefficient function, which may be solved in the usual way to yield

$$
\sigma_M(S^2/\mu^2; \mu^2) = C_M(1; \alpha_s(S^2))F_M(S^2) + \sigma_M^q(S^2) + O(1/S^2).
$$

The appropriate choice of factorization scale at small $x$ is thus $S^2$: QCD becomes asymptotically free as $x \to 0$, and perturbative calculations of the coefficient function and evolution of the parton distribution may be performed self-consistently. This is in contrast to the more usual approach in which the logarithms of $x$ are added to the usual evolution in $Q^2$: there the coupling is either fixed or runs with $Q^2$, and the calculation becomes inconsistent in the high energy limit since the coupling is no longer falling to compensate for the large logarithms.
The anomalous dimension $\gamma_M(\alpha_s)$ in the longitudinal evolution equation (5) may be thought of as the Mellin transform of a longitudinal splitting function. This may be calculated in a very similar way to the Altarelli–Parisi calculation of the usual (transverse) splitting function, by considering parton emission in the Weizsäcker–Williams approximation, but now with strong ordering of longitudinal rather than transverse momenta. The computation of LO graphs is then sufficient to determine the cross section at the leading logarithmic level to all orders in the coupling. Since large logarithms are now generated by both $s$ and $t$ channel emissions, it is necessary to calculate the leading log contribution from the LO graphs in fig. 2a. The virtual graph cancels the infrared singularity in the real emissions, and the final result is \[ 7 \]

$$\gamma_M(\alpha_s) = C_A \frac{\alpha_s}{\pi} \left[ 2\psi(1) - \psi(M) - \psi(1 - M) \right] + O(\alpha_s^2).$$

It follows that if the coupling were fixed, at LO the longitudinal evolution equation (5) would reduce to the BFKL equation if in addition distinctions between integrated and unintegrated distributions were ignored.

Since quarks make no contribution to the LO splitting function, only the gluon distribution actually evolves: the quark contribution to the cross-section may thus be included in the second (background) term in eq.(1). Coefficients functions for $F_2$ and $F_L$ may be deduced from the quark box calculations in ref.[5]. Only ratios of coefficient functions have physical significance, since an arbitrary factor may be absorbed into the definition of the gluon distribution. In fig. 2 we also show some of the graphs which contribute to the emission amplitude at NLO. Real emissions fig. 2b contain both leading logarithms which are iterations of the LO real emissions in fig. 2a (and are thus reproduced when solving the LO evolution equations), but also next-to-leading logarithms, i.e. contributions to the NLO anomalous dimension. The one-loop corrections fig. 2c further contain leading logarithms which are not obtained by simple iteration of the LO graphs, but are instead proportional to $\beta_0$ and are thus generated by the running of the coupling with $S^2$ in the LO evolution. This follows as a direct consequence of the Callan-Symanzik equation satisfied by the gluon four-point function; notice that the virtuality of the $s$-channel gluon in the first two graphs is $k^2/y$. Finally the two-loop graphs fig. 2d are necessary to cancel infrared singularities. There is furthermore a set of graphs involving quark loops and production of $q\bar{q}$ pairs. \[ 2 \]

We may now consider some of the implications of the longitudinal evolution eq.(5). At high energies we will find logarithmic scaling violations, just as with the usual transverse evolution at high virtualities. But because the longitudinal anomalous anomalous dimension has a minimum at $M = \frac{1}{2}$, rather than decreasing monotonically, and because there is only one evolving parton, all (singlet) structure functions will exhibit the same universal behaviour at high energies:

$$F(x, Q^2) \sim N \left( \frac{Q^2}{\Lambda_0^2} \right)^{\frac{1}{2}} \left( \ln \ln \frac{1}{x} \right)^{-\frac{1}{2}} \left( \ln \frac{1}{x} \right)^{4 \ln 2 \gamma^2 - 1}.$$

$1$It is not clear to us in precisely which circumstances all these graphs might be equivalent to the ‘Reggeized’ graphs calculated in ref.[10, 11]. What is clear however is that a NLO calculation can only be useful when corrections to both splitting functions and coefficient functions have been computed in the same renormalization and factorization scheme.
In this way perturbative QCD reproduces the gentle growth in the high energy cross-section, with no need for a Regge trajectory with intercept greater than unity. Being logarithmic, the growth is sufficiently gentle to match smoothly onto the double scaling behaviour at higher $Q^2$, and all the difficulties found in [1, 6, 7] are resolved. Furthermore a careful examination of the $Q^2 \to 0$ limit shows that the photoproduction cross-section remains finite, and there is no violation of unitarity bounds. However in this limit our perturbative approach may break down due to infrared logarithms in the same way that the usual analysis at large $Q^2$ breaks down as $x \to 1$ due to Sudakov logarithms.

We are thus led naturally into a world without a pomeron. There, we can approach a variety of problems from the vantage point of perturbative QCD: besides the calculation of NLO corrections to structure functions, we might consider perturbative computations of diffractive processes, of high energy photon-photon cross-sections, and of high energy hadronic processes in which there is no large transverse scale. There remains much to be done.

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References

[1] For a recent review see R.D. Ball and A. DeRoeck, WG1 summary at ‘DIS96’, [hep-ph/9609309], and ref. therein.

[2] A. De Rújula et al., Phys. Rev. 10 (1974) 1649.

[3] V. Del Duca, these proceedings, and ref. therein.

[4] R.D. Ball and S. Forte, Phys. Lett. B335 (1994) 77.

[5] S. Catani and F. Hautmann, Nucl. Phys. B427 (1994) 475.

[6] S. Forte and R.D. Ball, talk at ‘DIS95’, [hep-ph/9507211].

[7] R.D. Ball and S. Forte, talk at ‘DIS96’, [hep-ph/9607291].

[8] R.D. Ball and S. Forte, Phys. Lett. (in press), [hep-ph/9703417].

[9] R.K. Ellis et al, Nucl. Phys. B152 (1979) 285.

[10] V.S. Fadin, these proceedings, [hep-ph/9704267], and ref. therein.

[11] G. Camici and M. Ciafaloni, Phys. Lett. B386 (1996) 341, [hep-ph/9701303].
Figure 1: Regions and limits in the \((Q^2, \frac{1}{x})\) plane. The shaded area denotes the region of nonperturbative dynamics where there is no large scale. To the right of an initial condition set at \(Q^2 = Q^2_0\) we may evolve using the transverse evolution equation, while above an initial condition set at \(x = x_0\) we may evolve using the longitudinal evolution equation.
Figure 2: Contributions to gluon emission amplitudes: a) LO contributions b)-d) NLO contributions described in the text. There are also many contributions due to self energy insertions, and NLO contributions from emission of $q\bar{q}$ pairs and quark loops.