ON SEMI-ARMENDARIZ MATRIX RINGS

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Abstract. Given a positive integer $n$, a ring $R$ is said to be $n$-semi-Armendariz if whenever $f^n = 0$ for a polynomial $f$ in one indeterminate over $R$, then the product (possibly with repetitions) of any $n$ coefficients of $f$ is equal to zero. A ring $R$ is said to be semi-Armendariz if $R$ is $n$-semi-Armendariz for every positive integer $n$. Semi-Armendariz rings are a generalization of Armendariz rings. We characterize when certain important matrix rings are $n$-semi-Armendariz, generalizing some results of Jeon, Lee and Ryu from their paper (J. Korean Math. Soc. 47 (2010), 719–733), and we answer a problem left open in that paper.

1. Introduction

Throughout this paper, all rings are associative, and all rings have an identity except where explicitly indicated. For a ring $R$, the ring of polynomials in the indeterminate $x$ over $R$ is denoted by $R[x]$, and if $A \subseteq R$, then $A[x]$ stands for the set of polynomials in $R[x]$ whose all coefficients belong to $A$.

Recall that a ring $R$ is said to be an Armendariz ring if whenever the product of two polynomials over $R$ is zero, then the products of their coefficients are all zero, that is, in the polynomial ring $R[x]$ the following holds:

\[ \text{for any } f = \sum_{i=0}^{k} a_i x^i, \quad g = \sum_{j=0}^{m} b_j x^j \in R[x], \]

\[ \text{if } fg = 0, \text{ then } a_i b_j = 0 \text{ for all } i, j. \]

The name for such rings was chosen to honor E. P. Armendariz, who noted in [2] that all reduced rings (i.e., rings containing no nonzero nilpotent elements) satisfy condition (1). Various interesting properties and constructions of Armendariz rings can be found, e.g., in [1], [4], [6], [8], [11], [13], [14] and [15].

Armendariz rings, as well as many other classes of Armendariz-like rings, have recently been objects of intensive investigation (see [13]). These new
classes of Armendariz-like rings were defined using generalizations or modifications of condition (1). For example, by replacing in (1) the polynomial ring \( R[x] \) with the power series ring \( R[[x]] \), one obtains the definition of a power-serieswise Armendariz ring, introduced by N. K. Kim, K. H. Lee, and Y. Lee in [9]. By replacing in (1) the requirement that the products \( a_i b_j \) are all zero with the condition that the products \( a_i b_j \) are all nilpotent, we obtain the definition of a weak Armendariz ring, introduced by Z. Liu and R. Zhao in [12]. By considering in (1) the square of a single polynomial instead of the product of two polynomials, we obtain the definition of a 2-semi-Armendariz ring, introduced by Y. C. Jeon, Y. Lee and S. J. Ryu in [7], according to which a ring \( R \) is said to be 2-semi-Armendariz provided for any polynomial \( f = \sum_{i=0}^{m} a_i x^i \in R[x] \), if \( f^2 = 0 \), then \( a_i a_j = 0 \) for all \( i, j \).

More generally, for a positive integer \( n \), in [7] Jeon, Lee and Ryu define a ring \( R \) to be \( n \)-semi-Armendariz if for any polynomial \( f = \sum_{i=0}^{m} a_i x^i \in R[x] \), \( f^n = 0 \) implies \( a_{i_1} a_{i_2} \cdots a_{i_n} = 0 \) for any subset \( \{i_1, i_2, \ldots, i_n\} \subseteq \{0, 1, \ldots, m\} \), and they call a ring \( R \) a semi-Armendariz ring if \( R \) is \( n \)-semi-Armendariz for every positive integer \( n \). The following well-known result of D. D. Anderson and V. Camillo shows that all Armendariz rings are semi-Armendariz.

**Proposition 1.1** ([1, Proposition 1]). Suppose \( R \) is an Armendariz ring. If \( f_1, f_2, \ldots, f_n \in R[x] \) are such that \( f_1 f_2 \cdots f_n = 0 \), then \( a_1 a_2 \cdots a_n = 0 \), where \( a_i \) is a coefficient of \( f_i \).

The following proposition summarizes basic properties of \( n \)-semi-Armendariz rings and semi-Armendariz rings.

**Proposition 1.2** (see [7]).

(a) Every subring of an \( n \)-semi-Armendariz ring is \( n \)-semi-Armendariz.

(b) Direct sum of \( n \)-semi-Armendariz rings is \( n \)-semi-Armendariz.

(c) A ring \( R \) is \( n \)-semi-Armendariz if and only if the ring \( R[x] \) is \( n \)-semi-Armendariz.

(d) If a ring \( R \) is semi-Armendariz, then \( \text{nil}(R[x]) \subseteq \text{nil}(R)[x] \), where \( \text{nil}(A) \) denotes the set of nilpotent elements of a ring \( A \).

The aim of this paper is to characterize when some important matrix rings are \( n \)-semi-Armendariz. The motivation for this work were results of Jeon, Lee and Ryu from their paper [7] and a problem left open in [7].

In [7, Theorem 1.2] it was proved that for any \( n \geq 2 \) the \( n \times n \) upper triangular matrix ring \( U_n(R) \) over a ring \( R \) is \( n \)-semi-Armendariz if and only if \( R \) is reduced. In Section 2 we extend this and some other results of [7] to the ring of upper triangular \( n \times n \) matrices over a ring \( R \) whose diagonal entries belong to a given subring \( S \) of \( R \) (see Proposition 2.4 and Theorem 2.5).

For a ring \( R \) and an integer \( n \geq 2 \), the ring \( D_n(R) \) of upper triangular \( n \times n \) matrices over \( R \) whose diagonal entries are equal is a subring of \( U_n(R) \). Hence it follows from the aforementioned result [7, Theorem 1.2] and Proposition 1.2(a)
that if $R$ is a reduced ring, then the ring $D_n(R)$ is $n$-semi-Armendariz. It is natural to ask, whether the implication can be reversed, that is whether a ring $R$ has to be reduced if the ring $D_n(R)$ is $n$-semi-Armendariz. In [7] the problem was left unsolved. In Section 3 we show that for any integer $n \geq 2$ the answer to the problem is negative (see Example 3.2). The answer follows easily from a general result (Theorem 3.1), which also allows to construct further examples important for the theory of $n$-semi-Armendariz rings (see Examples 3.4 and 3.6).

In this paper, the full ring of $n \times n$ matrices over a ring $R$ is denoted by $M_n(R)$, and the ring of upper triangular $n \times n$ matrices over $R$ is denoted by $U_n(R)$. For a matrix $A \in M_n(R)$ and any $i, j \in \{1, 2, \ldots, n\}$ the $(i, j)$ entry of $A$ is denoted by $A(ij)$. The symbol $E_{ij}$ stands for the matrix with $(i, j)$ entry equal to 1 and all other entries equal to 0 (dimensions of the matrix $E_{ij}$ will be clear from the context). The canonical ring isomorphism of $M_n(R)[x]$ onto $M_n(R[x])$ is denoted by $\Phi$. Recall that the isomorphism $\Phi : M_n(R)[x] \rightarrow M_n(R[x])$ maps a polynomial

$$f = A_0 + A_1 x + A_2 x^2 + \cdots + A_k x^k \in M_n(R)[x]$$

to the $n \times n$ matrix $\Phi(f)$ over $R[x]$ whose $(i, j)$ entry is the polynomial

$$A_0^{(ij)} + A_1^{(ij)} x + A_2^{(ij)} x^2 + \cdots + A_k^{(ij)} x^k \in R[x]$$

for all $i, j \in \{1, 2, \ldots, n\}$. We will usually consider the isomorphism $\Phi$ restricted to a concrete subring of the ring $M_n(R)$; such a restriction will still be denoted by $\Phi$.

2. $n$-semi-Armendariz matrix rings

The aim of this section is to identify $n$-semi-Armendariz subrings of the full matrix ring $M_m(R)$, where $R$ is a ring and $m \geq 2$. We start by showing that the ring $M_m(R)$ is never $n$-semi-Armendariz for $n \geq 2$.

**Proposition 2.1.** Let $R$ be a ring and let $m, n \geq 2$ be integers. Then the ring $M_m(R)$ is not $n$-semi-Armendariz.

**Proof.** Let $f = A_0 + A_1 x + A_2 x^2 \in M_m(R)[x]$, where

$$A_0 = E_{1m}, \quad A_1 = E_{11} - E_{mm}, \quad A_2 = -E_{m1}.$$

Then $f^2 = 0$ and thus $f^n = 0$. Since $A_1^n = E_{11} + (-1)^n E_{mm} \neq 0$, the ring $M_m(R)$ is not $n$-semi-Armendariz.

It is well known that for every $n \geq 2$ and arbitrary ring $R$, the upper triangular matrix ring $U_n(R)$ is not Armendariz (see [8, Example 1]). However, for any reduced ring $R$ the ring $U_n(R)$ is $n$-semi-Armendariz, which was proved in [7, Theorem 1.2] (and which shows that the class of $n$-semi-Armendariz rings is indeed wider than the class of Armendariz rings). In Theorem 2.5 below, we generalize this result by showing that for any reduced subring $S$ of an arbitrary ring $R$, the upper triangular $n \times n$ matrices over $R$ whose diagonal entries
belong to $S$ form an $n$-semi-Armendariz ring. The following observation will be useful in our proofs.

**Lemma 2.2.** Let $R$ be a ring, let $I$ be an ideal of $R$ such that the factor ring $R/I$ is reduced, and let $m$ be a positive integer such that $I^m = 0$. Then $R$ is $n$-semi-Armendariz for every $n \geq m$.

**Proof.** Clearly, the set $I[x]$ of polynomials from $R[x]$ with all coefficients in $I$ is an ideal of $R[x]$. Since the ring $R/I$ is reduced, so is the ring $(R/I)[x]$. Therefore, if a polynomial $f = a_0 + a_1x + \cdots + a_kx^k \in R[x]$ satisfies $f^n = 0$, then $f \in I[x]$ and thus $a_i \in I$ for any $i$. Hence, if $n \geq m$, then for any $a_1, a_2, \ldots, a_n \in \{0, 1, \ldots, k\}$ we have $a_1a_2 \cdots a_n \in I^n \subseteq I^m = 0$, which proves that the ring $R$ is $n$-semi-Armendariz.

□

An immediate consequence of Lemma 2.2 is [7, Proposition 2.6]. Example 3.4 shows that the condition that $R/I$ is reduced in Lemma 2.2 is not superfluous.

**Remark 2.3.** Let $R$ be a ring. Recall that an ideal $J$ of $R$ is said to be completely prime if the factor ring $R/J$ is a domain. The intersection of all completely prime ideals of $R$ is called the generalized nil radical of $R$ and denoted by $N_g(R)$ (see [3, Example 3.8.16]). Note that if $I$ is an ideal of $R$ such that the ring $R/I$ is reduced and $I^m = 0$ for some positive integer $m$, then $I = N_g(R)$. Indeed, since $I^m = 0$, we deduce that $I$ is contained in every completely prime ideal of $R$ and $I \subseteq N_g(R)$ follows. The opposite inclusion is an immediate consequence of the fact that any reduced ring is a subdirect product of domains (see [10, Theorem 12.7]). Thus $I = N_g(R)$, as desired. Therefore, Lemma 2.2 can alternatively be formulated as follows: If $R$ is a ring such that $N_g(R)^n = 0$ for some positive integer $m$, then $R$ is $n$-semi-Armendariz for every $n \geq m$.

Let $S$ be a subring of a ring $R$. For any positive integer $n$ we set

$$U_n(S, R) = \{ A \in U_n(R) \mid A^{(i)} \in S \text{ for every } i \in \{1, 2, \ldots, n\} \},$$

i.e., $U_n(S, R)$ consists of all upper triangular $n \times n$ matrices over $R$ whose diagonal entries belong to $S$. Clearly, $U_n(S, R)$ is a subring of $U_n(R)$. Moreover, we set

$$N_n(R) = \{ A \in U_n(R) \mid A^{(i)} = 0 \text{ for every } i \in \{1, 2, \ldots, n\} \},$$

i.e., $N_n(R)$ is the set of upper triangular $n \times n$ matrices over $R$ with all diagonal entries equal to zero. Obviously, $N_n(R)$ is an ideal of both $U_n(R)$ and $U_n(S, R)$.

The following result will be helpful in proving that some matrix rings of the form $U_n(S, R)$ are $k$-semi-Armendariz.

**Proposition 2.4.** Let $S$ be a subring of a ring $R$ and let $n$ be a positive integer. If there exists an ideal $I$ of $R$ such that $I \subseteq S$, and the ring $S/I$ is reduced, and $I^m = 0$ for some positive integer $m$, then the ring $U_n(S, R)$ is $k$-semi-Armendariz for every $k \geq n + m - 1$.  


Proof. Since $I$ is an ideal of $R$, the set
\[ J = \{ A \in U_n(S, R) \mid A^{(i)} \in I \text{ for every } i \in \{1, 2, \ldots, n\} \} \]

is an ideal of $U_n(S, R)$. Furthermore, the factor ring $U_n(S, R)/J$ is isomorphic to the direct sum of $n$ copies of the reduced ring $S/I$ and thus also $U_n(S, R)/J$ is reduced. Hence by Lemma 2.2, to complete the proof it suffices to show that $J^{n+m-1} = 0$. For this, it is enough to show that

for any $A_1, A_2, \ldots, A_{n+m-1} \in J$ we have $A_1A_2 \cdots A_{n+m-1} = 0$.

Note that for any $i, j \in \{1, 2, \ldots, n\}$ the $(i, j)$ entry of the matrix $A_1A_2 \cdots A_{n+m-1}$ is a sum of products of the form
\[
A_1^{(k_1k_2)}A_2^{(k_2k_3)}A_3^{(k_3k_4)} \cdots A_{n+m-1}^{(k_{n+m-1}k_{n+m})}, \quad \text{where } k_1 = i \text{ and } k_{n+m} = j.
\]

If $k_l > k_{l+1}$ for some $l \in \{1, 2, \ldots, n+m-1\}$, then $A_l^{(k_lik_{l+1})} = 0$ and thus the product (2) is equal to 0 in this case. Otherwise we have
\[
1 \leq i = k_1 \leq k_2 \leq k_3 \leq \cdots \leq k_{n+m-1} \leq k_{n+m} = j \leq n.
\]

Hence in this case there must exist at least $m$ pairs $(k_l, k_{l+1})$ with $k_l = k_{l+1}$, and since $A_1 \in J$, for any such a pair $(k_l, k_{l+1})$ we have $A_l^{(k_lik_{l+1})} \in I$. Consequently, the product (2) belongs to $I^m$ and thus it is equal to zero. Therefore, $A_1A_2 \cdots A_{n+m-1} = 0$. \qed

The following result extends [7, Theorem 1.2] to matrix rings of the form $U_n(S, R)$, with a different proof than that given in [7].

**Theorem 2.5.** Let $S$ be a subring of a ring $R$ and let $n \geq 2$ be an integer. Then

(a) The following conditions are equivalent:

(i) $U_n(S, R)$ is $n$-semi-Armendariz;

(ii) $U_n(S, R)$ is $k$-semi-Armendariz for every integer $k \geq n$;

(iii) $S$ is reduced.

(b) $U_{n+1}(S, R)$ is $n$-semi-Armendariz if and only if $S$ is reduced and $R$ is Armendariz.

*Proof. (a) (i) $\Rightarrow$ (iii): Assume $U_n(S, R)$ is $n$-semi-Armendariz. Then by Proposition 1.2(a), the ring $U_n(S)$ is $n$-semi-Armendariz. Hence by [7, Theorem 1.2], the ring $S$ is reduced.

(iii) $\Rightarrow$ (ii): Apply Proposition 2.4 with $I = 0$ and $m = 1$.

(ii) $\Rightarrow$ (i): Obvious.

(b) We will use the following observation:

For any ring $T$ and matrices $B_1, B_2, \ldots, B_n \in N_{n+1}(T)$ we have
\[
B_1B_2 \cdots B_n = B_1^{(1, 2)}B_2^{(2, 3)} \cdots B_n^{(n, n+1)}E_{1, n+1}.
\]
To establish (3), note that for any \( i, j \in \{1, 2, \ldots, n+1\} \) the \((i, j)\) entry of \( B_1B_2\cdots B_n \) is a sum of products of the form

\[
B_1^{(k_1,k_2)}B_2^{(k_2,k_3)}B_3^{(k_3,k_4)}\cdots B_n^{(k_n,k_{n+1})},
\]

where \( k_1 = i \) and \( k_{n+1} = j \).

For any \( l \in \{1, 2, \ldots, n\} \) we have \( B_l \in N_{n+1}(T) \) and thus if \( k_l \geq k_{l+1} \), then \( B_l^{(k_l,k_{l+1})} = 0 \). Hence the product (4) can be nonzero only in the case when

\[
1 \leq i = k_1 < k_2 < \cdots < k_n < k_{n+1} = j \leq n+1,
\]

that is, when \( k_l = l \) for every \( l \in \{1, 2, \ldots, n+1\} \). Now (3) follows.

To prove (b), set \( V = U_{n+1}(S, R) \) and \( N = N_{n+1}(R) \). Assume \( V \) is \( n\)-semi-Armendariz. Since \( U_n(S, R) \) is isomorphic to a subring of \( V \), the ring \( U_n(S, R) \) is \( n\)-semi-Armendariz by Proposition 1.2(a), and thus by the already proved part (a), the ring \( S \) is reduced. To show that \( R \) is Armendariz, consider any polynomials \( f = a_0 + a_1x + \cdots + a_kx^k \), \( g = b_0 + b_1x + \cdots + b_lx^l \in R[x] \) such that \( fg = 0 \). Without loss of generality we can assume that \( k = l \). For any \( m \in \{0, 1, \ldots, k\} \) we set

\[
A_m = a_mE_{12} + b_mE_{23} + E_{34} + \cdots + E_{n,n+1} \in N,
\]

and we put

\[
h = A_0 + A_1x + \cdots + A_kx^k \in N[x].
\]

Let \( \Phi : U_{n+1}(R)[x] \to U_{n+1}(R[x]) \) be the canonical isomorphism, and let \( H = \Phi(h) \). Since \( h \in N[x] \), it follows that \( H \in N_{n+1}(R[x]) \). Hence (3) implies

\[
H^n = H^{(1,2)}H^{(2,3)}H^{(3,4)}\cdots H^{(n,n+1)}E_{1,n+1} = fg(1+x+\cdots+x^k)^{n-2}E_{1,n+1} = 0,
\]

and thus \( h^n = 0 \). Since \( V \) is \( n\)-semi-Armendariz and \( h^n = 0 \), it follows that \( A_iA_jA_i^{n-2} = 0 \) for any \( i, j \). Since \( A_i, A_j, A_i \in N \), (3) implies

\[
a_iA_j = A_j^{(1,2)}A_j^{(2,3)}A_j^{(3,4)}\cdots A_j^{(n,n+1)} = 0,
\]

which proves that \( R \) is Armendariz.

To prove the converse, assume \( S \) is reduced and \( R \) is Armendariz, and consider any polynomial \( q = a_0 + a_1x + \cdots + a_kx^k \in V[x] \) with \( q^n = 0 \). We claim that \( A_i \in N \) for each \( i \). To see this, note that since \( S \) is reduced and \( N \) is an ideal of \( V \) such that \( V/N \) is isomorphic to the direct sum of \( n+1 \) copies of \( S \), the ring \( V/N \) is reduced, and thus so is the ring \( V[x]/N[x] \cong (V/N)[x] \). Hence \( q^n = 0 \) yields \( q \in N[x] \), which proves our claim.

Let \( \Phi : U_{n+1}(R)[x] \to U_{n+1}(R[x]) \) be the canonical isomorphism, and let \( Q = \Phi(q) \). Since \( q \in N[x] \), it follows that \( Q \in N_{n+1}(R[x]) \). Furthermore, \( Q^n = \Phi(q^n) = \Phi(0) = 0 \), and thus (3) implies

\[
Q^{(1,2)}Q^{(2,3)}\cdots Q^{(n,n+1)} = 0.
\]

Hence, if for any \( i \in \{1, 2, \ldots, n\} \) we set

\[
f_i = A_0^{(i,i+1)} + A_1^{(i,i+1)}x + \cdots + A_k^{(i,i+1)}x^k \in R[x],
\]
then \( f_1f_2 \cdots f_n = 0 \). Since \( R \) is Armendariz, by Proposition 1.1 we have
\[ A_{s_1}^{(1,2)}A_{s_2}^{(2,3)} \cdots A_{s_n}^{(n,n+1)} = 0 \]
for any \( n \)-tuple \((s_1, s_2, \ldots, s_n)\), and (3) implies \( A_{s_1}A_{s_2} \cdots A_{s_n} = 0 \). Thus \( V \) is \( n \)-semi-Armendariz. \( \square \)

As an immediate consequence of the above theorem we obtain [7, Theorem 1.2], which is just the first part of the following corollary. The second part says that if \( R \) is a reduced ring, then \( U_n(R) \) is a maximal \( n \)-semi-Armendariz subring of the full matrix ring \( M_n(R) \).

**Corollary 2.6.** For any ring \( R \) and integer \( n \geq 2 \), the following conditions are equivalent:

1. \( U_n(R) \) is \( n \)-semi-Armendariz;
2. \( U_n(R) \) is \( k \)-semi-Armendariz for every \( k \geq n \);
3. \( U_{n+1}(R) \) is \( n \)-semi-Armendariz;
4. \( R \) is reduced.

If any of these equivalent conditions holds, then \( U_n(R) \) is a maximal \( n \)-semi-Armendariz subring of \( M_n(R) \).

**Proof.** Since reduced rings are Armendariz, Theorem 2.5 implies that (i), (ii), (iii), and (iv) are equivalent. To prove the second part of the corollary, assume \( R \) is reduced and \( T \) is an \( n \)-semi-Armendariz subring of \( M_n(R) \) such that \( U_n(R) \subsetneq T \). Then there exists a matrix \( A \in T \) such that for some \( i > j \) the \((i,j)\) entry of \( A \), say \( a \), is non-zero. Since \( E_{ii}, E_{jj} \in T \), also \( E_{ii}AE_{jj} = aE_{ij} \in T \). Thus for the matrices \( A_0 = aE_{ij} \), \( A_1 = aE_{ii} - aE_{jj} \) and \( A_2 = -aE_{ji} \) we have
\[ f = A_0 + A_1x + A_2x^2 \in T[x] \]. It is easy to see that \( f^2 = 0 \), and thus \( f^n = 0 \). Since \( T \) is \( n \)-semi-Armendariz, \( 0 = A_0^n = a^nE_{ii} - a^nE_{jj} \). Hence \( a^n = 0 \), and since \( R \) is reduced, we obtain \( a = 0 \), a contradiction. \( \square \)

The following result is an immediate consequence of Corollary 2.6.

**Corollary 2.7** (cf. [7, Corollary 1.3]). For any ring \( R \) the following conditions are equivalent:

1. \( R \) is reduced;
2. \( U_2(R) \) is semi-Armendariz;
3. \( U_3(R) \) is semi-Armendariz.

Given integers \( n \geq 2 \) and \( m \geq n + 2 \), in the context of the first part of Corollary 2.6 it is natural to ask, when for a ring \( R \) the ring \( U_m(R) \) is \( n \)-semi-Armendariz. The following result shows that the answer is “never” (cf. [7, Example 1.6]).

**Proposition 2.8.** Let \( R \) be a ring and let \( n \geq 2 \) be an integer. Then for any integer \( m \geq n + 2 \) the ring \( U_m(R) \) is not \( n \)-semi-Armendariz.

The proof of Proposition 2.8 will be based on the following lemma. In the statement of the lemma, in an obvious way, we extend the notion of an \( n \)-semi-Armendariz ring to rings without unity. Recall that \( N_{n+2}(R) \) consists of upper
triangular \((n + 2) \times (n + 2)\) matrices over \(R\) with all diagonal entries equal to zero. Clearly, \(U_{n+2}(R)\) is a non-unital subring of \(U_{n+2}(R)\) (even more: it is an ideal of \(U_{n+2}(R)\)).

**Lemma 2.9.** For any ring \(R\) and integer \(n \geq 2\) the non-unital ring \(U_{n+2}(R)\) is not \(n\)-semi-Armendariz.

**Proof.** We start the proof with the case \(n = 2\). Set \(A = E_{12} + E_{34}\) and \(B = E_{12} + E_{13} - E_{24} + E_{34}\). Then \(f = A + Bx \in N_4(R)[x]\) and \(f^2 = 0\), but \(AB = -E_{14} \neq 0\) and thus \(N_4(R)\) is not \(2\)-semi-Armendariz.

Next assume that \(n \geq 3\). In this case we simply repeat arguments of [7, Example 1.6]. We set

\[
A = E_{12} + E_{23} + \cdots + E_{n-2,n-1} + E_{n-1,n+1} + E_{n,n+2} \in N_{n+2}(R),
B = E_{n-1,n} + E_{n-1,n+1} + E_{n,n+2} - E_{n+1,n+2} \in N_{n+2}(R).
\]

It is easy to verify that \(B^2 = BA^2 = BAB = 0\),

\[
A^k = E_{1,k+1} + E_{2,k+2} + \cdots + E_{n-k-1,n-1} + E_{n-k,n+1} \quad \text{for} \quad 2 \leq k \leq n - 2,
\]

and \(A^{n-1} = E_{1,n+1}\). Hence for \(f = f + Bx \in N_{n+2}(R)[x]\) we have

\[
f^n = A^n + (A^{n-2}BA + A^{n-1}B)x = (E_{1,n+2} - E_{1,n+2})x = 0.
\]

Since \(A^{n-2}BA = E_{1,n+2} \neq 0\), the ring \(N_{n+2}(R)\) is not \(n\)-semi-Armendariz. \(\square\)

Now we are ready to prove Proposition 2.8.

**Proof of Proposition 2.8.** From Proposition 1.2(a) and Lemma 2.9 we deduce that the ring \(U_{n+2}(R)\) is not \(n\)-semi-Armendariz. To complete the proof, consider any integer \(m \geq n + 2\). Then the ring \(U_{n+2}(R)\) is isomorphic to a subring of \(U_m(R)\). Since the ring \(U_{n+2}(R)\) is not \(n\)-semi-Armendariz, Proposition 1.2(a) implies that \(U_m(R)\) is not \(n\)-semi-Armendariz. \(\square\)

Let \(R\) be a ring and let \(n, k\) be integers such that \(2 \leq n < k\). By Corollary 2.6, \(R\) being reduced implies that the ring \(U_n(R)\) is \(k\)-semi-Armendariz. The following example shows that the opposite implication is not true.

**Example 2.10.** (For any \(2 \leq n < k\) there exists a nonreduced ring \(R\) such that the ring \(U_n(R)\) is \(k\)-semi-Armendariz.) Let \(p\) be a prime number, and let \(m = k - n + 1\). Let \(R = \mathbb{Z}_{p^m}\) be the ring of integers modulo \(p^m\) and let \(I = (p)\) be the ideal of \(R\) generated by \(p\). Then the factor ring \(R/I \cong \mathbb{Z}_p\) is reduced and \(I^n = 0\). Hence by Proposition 2.4 the ring \(U_n(R)\) is \(k\)-semi-Armendariz. Furthermore, since \(m \geq 2\), the ring \(R\) is not reduced.

Corollary 2.6 implies that the ring \(U_n(R)\) constructed in Example 2.10 is not \(n\)-semi-Armendariz. Hence, the same example shows also that for any \(2 \leq n < k\) there exists a \(k\)-semi-Armendariz ring that is not \(n\)-semi-Armendariz (cf. [7, p. 725]).
3. Answer to Jeon-Lee-Ryu’s problem

Given an integer $n \geq 2$, in [7, Theorem 1.2] Jeon, Lee and Ryu characterized rings $R$ for which the ring $U_n(R)$ is $n$-semi-Armendariz. Namely, they proved that $U_n(R)$ is $n$-semi-Armendariz if and only if $R$ is reduced (see also Corollary 2.6). Since the set

$$D_n(R) = \{ A \in U_n(R) \mid A^{(1,1)} = A^{(2,2)} = \ldots = A^{(n,n)} \}$$

(i.e., the set of all upper triangular $n \times n$ matrices over $R$ whose diagonal entries are equal) is a subring of $U_n(R)$, it follows from Proposition 1.2(a) that if $R$ is a reduced ring, then the ring $D_n(R)$ is $n$-semi-Armendariz. It is natural to ask, whether the implication can be reversed, which led the authors of [7] to the following problem (see [7, p. 724, the paragraph preceding Proposition 1.5]):

(5) Problem: If $n \geq 2$ and the ring $D_n(R)$ is $n$-semi-Armendariz, must $R$ be a reduced ring?

In [7] the problem was left unsolved. In Example 3.2 we will show that the answer to the problem is negative for any integer $n \geq 2$. The answer will easily follow from the general Theorem 3.1.

The main reason why in Theorem 3.1 we focus on rings $D_n(S)$ with $S$ of the form $S = D_2(R)$ is that the ring $S = D_2(R)$ is never reduced, so in the context of Jeon-Lee-Ryu’s problem (5) it is natural to ask, when the ring $D_n(S) = D_n(D_2(R))$ is $n$-semi-Armendariz. In the theorem below we answer this question in the case where the ring $R$ is reduced and commutative.

For a ring $R$, if $n$ is a positive integer and $r \in R$, then $nr$ denotes the sum $r + r + \cdots + r$ with $n$ summands.

**Theorem 3.1.** Let $R$ be a reduced commutative ring and let $n$ be a positive integer. Then the following conditions are equivalent:

(i) The ring $D_n(D_2(R))$ is $n$-semi-Armendariz

(ii) The ring $(D_2(R)[x])/(x^n)$ is $n$-semi-Armendariz, where $(x^n)$ denotes the ideal of $D_2(R)[x]$ generated by $x^n$;

(iii) $nr = 0 \Rightarrow r = 0$ for any $r \in R$.

**Proof.** The case $n = 1$ is trivial, because every ring is obviously 1-semi-Armendariz. Thus to the end of the proof we assume that $n \geq 2$.

(i) $\Rightarrow$ (ii): Assume the ring $D_n(D_2(R))$ is $n$-semi-Armendariz. The factor ring $(D_2(R)[x])/(x^n)$ is isomorphic to the matrix ring

$$\begin{bmatrix}
A_1 & A_2 & \cdots & A_{n-1} & A_n \\
0 & A_1 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & A_1 & A_2 \\
0 & 0 & \cdots & 0 & A_1
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_n
\end{bmatrix},$$

for $A_1, A_2, \ldots, A_n \in D_2(R)$.
which is a subring of $D_0(D_2(R))$. Hence Proposition 1.2(a) implies that the
ring $(D_2(R)[x])/(x^n)$ is $n$-semi-Armendariz.

(ii) $\Rightarrow$ (iii): Put $P = (D_2(R)[x])/(x^n)$ and assume $P$ is $n$-semi-Armendariz.
We will use the bar notation for elements of $P$, that is, for any $h \in D_2(R)[x]$ we
will write $\bar{h}$ to denote the coset $h + (x^n) \in P$. Let $r \in R$ be such that $nr = 0$. Set
\[
A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in D_2(R)
\]
and consider the polynomial $f = \bar{A}x + \bar{B}t \in P[t]$. Note that in $P$ we have
\[
\bar{A}\bar{B} = \bar{B}\bar{A}, \quad \bar{B}^2 = 0, \quad \bar{x}^n = 0, \quad \text{and} \quad n\bar{A} = 0.
\]
Hence, by the binomial theorem,
\[
f^n = (\bar{A}x + \bar{B}t)^n = \sum_{i=0}^n \binom{n}{i} (\bar{A}x)^{n-i}(\bar{B}t)^i = A^n\bar{x}^n + nA^{n-1}\bar{x}^{n-1}\bar{B}t = 0.
\]
Since the ring $P$ is $n$-semi-Armendariz, it follows that $(\bar{A}\bar{x})^{n-1}\bar{B} = 0$ and thus
$A^{n-1}Bx^{n-1} \in (x^n)$. Hence $A^{n-1}B = 0$ and consequently $r^{n-1} = 0$. Since $R$ is
reduced, we obtain $r = 0$, as desired.

(iii) $\Rightarrow$ (i): Set $D(R) = D_0(D_2(R))$. To the end of the proof the ring $D(R)$ is
considered as a subring of $U_{2n}(R)$. For any $B \in D(R)$ and $i, j \in \{1, 2, \ldots, 2n\}$,
$B^{(i)}$ denotes the $(i, j)$ entry of the matrix $B$.

Obviously $N(R) = \{B \in D(R) : B^{(i)} = 0 \text{ for any } i \in \{1, 2, \ldots, 2n\}\}$ is an
ideal of $D(R)$. We start with describing the form of any product of $n$ matrices from
$N(R)$, which will be crucial in the proof of the implication (iii) $\Rightarrow$ (i). The
reader should note that in this part of the proof we do not put any assumption
on the ring $R$.

Let $B_1, B_2, \ldots, B_n \in N(R)$ and let $B = B_1B_2 \cdots B_n$. Then for any $i, j \in \{1, 2, \ldots, 2n\}$, the $(i, j)$ entry of $B$ is a sum of products of the form
\[
6. \quad b = B_1^{(k_0k_1)}B_2^{(k_1k_2)} \cdots B_n^{(k_{n-1}k_n)}, \quad \text{where } k_0 = i \text{ and } k_n = j.
\]
Assume $b \neq 0$. Then $B_t^{(k_{t+1})} \neq 0$ for every $t \in \{0, 1, \ldots, n - 1\}$, and since
$B_t \in N(R)$, it follows that $k_t < k_{t+1}$. Thus
\[
1 \leq i = k_0 < k_1 < k_2 < \cdots < k_{n-1} < k_n = j < 2n.
\]
Notice that for any matrix $A \in D(R) = D_0(D_2(R))$ we have $A^{(pq)} = 0$ for
every pair $(p, q)$ with even $p$ and odd $q$. Hence, since $b \neq 0$, if $k_t$ is even for
some $t \in \{0, 1, \ldots, n - 1\}$, then so are $k_{t+1}, k_{t+2}, \ldots, k_n$. In particular, if $k_0$
would be even, then so would be $k_1, k_2, \ldots, k_n$, and we would obtain $n + 1$
even integers in the interval $(1, 2n)$, a contradiction. Thus $k_0$ is odd. If also
$k_1, k_2, \ldots, k_n$ would be odd, then there would be $n + 1$ odd integers between
1 and $2n$, a contradiction. Therefore, there exists $m \in \{1, 2, \ldots, n\}$ such that
$k_0, k_1, \ldots, k_{m-1}$ are odd and $k_m, k_{m+1}, \ldots, k_n$ are even. Hence
\[
7. \quad k_0 + 2(m - 1) \leq k_{m-1} \text{ and } k_m \leq k_n - 2(n - m).
\]
Since furthermore $k_{m-1} < k_m$, it follows from (7) that $2n - 1 \leq k_n - k_0$, and thus $k_0 = 1$ and $k_n = 2n$. Putting these values of $k_0$ and $k_n$ into (7), we obtain $k_{n-1} = 2m - 1$ and $k_m = 2m$.

By the above, if $b \neq 0$, then $i = 1, j = 2n$ and the sequence

$$(k_0, k_1, k_2, \ldots, k_{n-1}, k_n)$$

is of the following form:

$$(1) \quad (1, 3, \ldots, 2m-1, 2m, 2m+2, \ldots, 2n), \quad \text{for some } m \in \{1, 2, \ldots, n\}.$$  

The sequence (8) will be denoted by $\gamma_m$ and the set of all such sequences will be denoted by $C_n$, i.e., $C_n = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. For example, $C_4$ consists of the following four sequences:

$\gamma_1 = (1, 2, 4, 6, 8), \quad \gamma_2 = (1, 3, 4, 6, 8), \quad \gamma_3 = (1, 3, 5, 6, 8), \quad \gamma_4 = (1, 3, 5, 7, 8).$

To simplify notation, for any $\gamma = (k_0, k_1, \ldots, k_{n-1}, k_n) \in C_n$, the product (6) will be denoted by $\gamma(B_1, B_2, \ldots, B_n)$, i.e.,

$$\gamma(B_1, B_2, \ldots, B_n) = B_1^{(k_0 k_1)} B_2^{(k_1 k_2)} B_3^{(k_2 k_3)} \cdots B_n^{(k_{n-1} k_n)}.$$  

Summarizing, and using the just introduced notation, if $B_1, B_2, \ldots, B_n \in N(R)$, then the $(1, 2n)$ entry of the product $B_1 B_2 \cdots B_n$ is equal to the sum

$$\sum_{m=1}^n \gamma_m(B_1, B_2, \ldots, B_n),$$

and all other entries are equal to 0, i.e.,

$$(2) \quad B_1 B_2 \cdots B_n = \left( \sum_{m=1}^n \gamma_m(B_1, B_2, \ldots, B_n) \right) E_{1, 2n}.$$  

Now we are in a position to prove the implication (iii) $\Rightarrow$ (i). Assume $R$ is a reduced commutative ring and $n \geq 2$ is an integer satisfying (iii), i.e., we have $nr = 0 \Rightarrow r = 0$ for any $r \in R$. To prove (i), we consider an arbitrary polynomial

$$f = A_0 + A_1 x + \cdots + A_t x^t \in D(R)[x] \quad \text{such that } f^n = 0,$$

and we have to show that

$$(3) \quad A_{s_1} A_{s_2} \cdots A_{s_n} = 0 \quad \text{for any } n\text{-tuple } (s_1, s_2, \ldots, s_n).$$

Since $N(R)$ is an ideal of $D(R)$ and the ring $D(R)/N(R)$ is isomorphic to $R$, it follows that $D(R)/N(R)$ is reduced. Hence also the ring

$$D(R)[x]/N(R)[x] \cong (D(R)/N(R))[x]$$

is reduced and from $f^n = 0$ we deduce that $f \in N(R)[x]$. Thus $A_0, A_1, \ldots, A_t \in N(R)$, and (9) implies that to prove (10) it suffices to show that for any $m \in \{1, 2, \ldots, n\}$ and $n$-tuple $(s_1, s_2, \ldots, s_n)$ we have

$$\gamma_m(A_{s_1}, A_{s_2}, \ldots, A_{s_n}) = 0.$$  

To prove (11), let $\Phi : D(R)[x] \to D(R[x])$ be the canonical isomorphism and let $F = \Phi(f)$. Since $f \in N(R)[x]$ and $F^n = 0$, we have $F \in N(R[x])$ and $F^n = 0$. Hence (9) implies
\begin{equation}
\sum_{m=1}^{n} \gamma_m(F, F, \ldots, F) = 0.
\end{equation}

Recall that $\gamma_m$ denotes the sequence (8) and thus
\begin{equation}
\gamma_m(F, F, \ldots, F) = p^{(1,3)} p^{(3,5)} \ldots p^{(2m-3,2m-1)} p^{(2m-1,2m)} p^{(2m,2m+2)} \ldots p^{(2n-2,2n)}.
\end{equation}

Note that since $F \in D(R[x]) = D_n(D_2(R[x]))$, for any odd $i \in \{1, 2, \ldots, 2n\}$ we have $F^{(i)} = F^{(1,2)}$ and if furthermore $i \leq 2n - 3$, then $F^{(i+1, i+2)} = F^{(i+1, i+3)}$.

Moreover, since $R$ is commutative, the ring $R[x]$ is commutative. Combining all of these with (13), we obtain
\begin{align}
\gamma_m(F, F, \ldots, F) &= p^{(2,4)} p^{(4,6)} \ldots p^{(2m-2,2m)} p^{(2m,2m+2)} \ldots p^{(2n-2,2n)} = p^{(1,2)} p^{(2,4)} p^{(4,6)} \ldots p^{(2m-2,2m)} p^{(2m,2m+2)} \ldots p^{(2n-2,2n)} \\
&= \gamma_1(F, F, \ldots, F).
\end{align}

We have shown that
\begin{equation}
\gamma_m(F, F, \ldots, F) = \gamma_1(F, F, \ldots, F) \quad \text{for any } m \in \{1, 2, \ldots, n\},
\end{equation}

and thus (12) implies
\begin{equation}
n \cdot \gamma_1(F, F, \ldots, F) = 0.
\end{equation}

Since by (iii) we have $nr = 0 \Rightarrow r = 0$ for any $r \in R$, it follows from (15) that $\gamma_1(F, F, \ldots, F) = 0$, and thus, by (14), for any $m \in \{1, 2, \ldots, n\}$ we have
\begin{equation}
\gamma_m(F, F, \ldots, F) = 0.
\end{equation}

We will show that (16) implies (11), which is enough to complete the proof. Let $\gamma_m = (k_0, k_1, k_2, \ldots, k_{n-1}, k_n)$ (at this point, the exact form of $\gamma_m$ does not matter). From (16) we obtain
\begin{equation}
p^{(k_0k_1)} p^{(k_1k_2)} \ldots p^{(k_{n-1}k_n)} = 0.
\end{equation}

For any $i \in \{1, 2, \ldots, n\}$, set
\begin{equation}
f_i = A_i^{(k_0-1,k_1)} + A_i^{(k_1-1,k_2)} x + \ldots + A_i^{(k_{n-1}-1,k_n)} x^t \in R[x].
\end{equation}

Since $F = \Phi(f)$, it follows from (17) that
\begin{equation}
f_1 f_2 \cdots f_n = 0.
\end{equation}

By assumption the ring $R$ is reduced, so $R$ is Armendariz as well, and thus (18) and Proposition 1.1 imply that for any $n$-tuple $(s_1, s_2, \ldots, s_n)$ we have
\begin{equation}
A_{s_1}^{(k_0k_1)} A_{s_2}^{(k_1k_2)} \ldots A_{s_n}^{(k_{n-1}k_n)} = 0,
\end{equation}

where $A_i^{(k_0-1,k_1)} + A_i^{(k_1-1,k_2)} x + \ldots + A_i^{(k_{n-1}-1,k_n)} x^t \in R[x]$. \hfill \ensuremath{\square}
that is,
\[ \gamma_m(A_{s_1}, A_{s_2}, \ldots, A_{s_n}) = 0 \text{ for any } m \in \{1, 2, \ldots, n\}. \]
Since \( A_{s_1}, A_{s_2}, \ldots, A_{s_n} \in N(R) \), from (9) we obtain
\[ A_{s_1}A_{s_2} \cdots A_{s_n} = 0, \]
which is just condition (10). The proof is complete. \( \square \)

Now we are in a position to answer Jeon-Lee-Ryu’s problem (5).

Example 3.2. (There exists a nonreduced ring \( R \) such that for any \( n \geq 2 \) the ring \( D_n(R) \) is \( n \)-semi-Armendariz.) Let, as usually, \( \mathbb{Z} \) denote the ring of integers. The ring \( R = D_2(\mathbb{Z}) \) is not reduced but, by Theorem 3.1, for any \( n \geq 2 \) the ring \( D_n(R) \) is \( n \)-semi-Armendariz.

Remark 3.3. Using the same arguments as in the proof of Theorem 3.1 one can prove the following generalization of Theorem 3.1.

Let \( S \) be a reduced subring of a commutative Armendariz ring \( R \) and let \( n \geq 2 \) be an integer. Then the following conditions are equivalent:

(i) The ring \( D_n(D_2(S, R)) \) is \( n \)-semi-Armendariz;
(ii) The ring \( (D_2(S, R)[x])/(x^n) \) is \( n \)-semi-Armendariz, where \((x^n)\) denotes the ideal of \( D_2(S, R)[x] \) generated by \( x^n \);
(iii) \( \forall r \in R \) such that for any \( r \in R \).

Let \( R \) be a ring and let \( I \) be a proper ideal of \( R \). Assume \( R/I \) and \( I \) are \( n \)-semi-Armendariz, where \( I \) is considered as an \( n \)-semi-Armendariz ring without unity. One could conjecture that in this case \( R \) has to be \( n \)-semi-Armendariz. However, this is not the case as shown in [7, Example 2.2]. Below we use Theorem 3.1 to construct another example of a ring \( R \) with an ideal \( I \) such that \( R/I \) and \( I \) are \( n \)-semi-Armendariz but \( R \) is not \( n \)-semi-Armendariz.

Example 3.4. (For any \( n \geq 2 \) there exist a ring \( R \) and an ideal \( I \) of \( R \) such that \( I \) and \( R/I \) are \( n \)-semi-Armendariz, but \( R \) is not \( n \)-semi-Armendariz.) Let \( n = p_1^a_1p_2^a_2 \cdots p_k^a_k \) be the prime factorization of \( n \) and let \( m = p_1p_2 \cdots p_k \). Note that the ring \( \mathbb{Z}_m \) of integers modulo \( m \) is reduced. Set \( R = (D_2(\mathbb{Z}_m)[x])/(x^n) \) and \( I = (x^{n-1})/(x^n) \). Then \( I \) is an ideal of \( R \) with \( I^2 = 0 \) and thus \( I \) is \( n \)-semi-Armendariz. Furthermore, the ring \( R/I \cong (D_2(\mathbb{Z}_m)[x])/(x^n) \) is isomorphic to a subring of \( U_{n-1}(D_2(\mathbb{Z}_m)) \) (see the proof of the implication (i) \( \Rightarrow \) (ii) in Theorem 3.1). Since \( N = N_2(\mathbb{Z}_m) \) is an ideal of \( D_2(\mathbb{Z}_m) \) such that the ring \( D_2(\mathbb{Z}_m)/N \cong \mathbb{Z}_m \) is reduced and \( N^2 = 0 \), Proposition 2.4 implies that the ring \( U_{n-1}(D_2(\mathbb{Z}_m)) \) is \( n \)-semi-Armendariz, and thus the ring \( R/I \) is \( n \)-semi-Armendariz as well. Finally note that by Theorem 3.1 the ring \( R \) is not \( n \)-semi-Armendariz.

Recall that a ring \( R \) is said to be abelian if every idempotent of \( R \) is central. In [7, Example 2.4(1)] it was shown that an abelian prime ring need not be semi-Armendariz. Bellow we show the same, constructing a simpler example than this in [7].
Example 3.5. (An abelian prime ring need not be semi-Armendariz.) Let $S$ be a domain and let $n \geq 2$. Set $R = D_{n+2}(S)$. Then by [5, Lemma 2] the ring $R$ is abelian. To see that $R$ is prime, consider any non-zero matrices $A, B \in R$. Let $a_{ij}$ and $b_{kl}$ be any nonzero entries of $A$ and $B$, respectively. Then the $(i,l)$ entry of $AE_{jk}B$ is equal to $a_{ij}b_{kl} \neq 0$. Thus $ARB \neq 0$, proving that $R$ is prime. Finally, Lemma 2.9 implies that $R$ is not $n$-semi-Armendariz and thus $R$ is not semi-Armendariz.

We close the paper with an example of a commutative ring which is not semi-Armendariz (cf. [7, Example 2.4(2)]).

Example 3.6. (A commutative ring need not be semi-Armendariz.) Obviously, the ring $R = D_2(D_2(\mathbb{Z}_2))$ is commutative. However, $R$ is not 2-semi-Armendariz by Theorem 3.1 and thus $R$ is not semi-Armendariz.

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