Degrees of freedom and Hamiltonian formalism for $f(T)$ gravity

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The existence of an extra degree of freedom (d.o.f.) in $f(T)$ gravity has been recently proved by means of the Dirac formalism for constrained Hamiltonian systems. We will show a toy model displaying the essential feature of $f(T)$ gravity, which is the pseudo-invariance of $T$ under a local symmetry, to understand the nature of the extra d.o.f.

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1. $f(T)$ Gravity

The teleparallel equivalent of general relativity (TEGR) is a reformulation of general relativity (GR) in terms of a field of tetrads. It encompasses the vector basis $e_a = e^a_\mu \partial_\mu$ and its co-basis $E^a = E^a_\mu dx^\mu$, which are mutually dual: $E^a_\mu e^\mu_b = \delta^a_b$. Tetrads are related to the spacetime metric through the orthonormality condition

$$\eta_{ab} = g_{\mu\nu} e^a_\mu e^b_\nu, \quad g_{\mu\nu} = \eta_{ab} E^a_\mu E^b_\nu.$$ (1)

The spacetime underlying TEGR is endowed with a curvatureless, metric-compatible spin connection. Usually the Weitzenböck connection $\omega^a_{\mu\nu} = 0$ is chosen, which in coordinate bases means $T^a_{\mu\nu} = e^a_\rho \partial_\mu E^\rho_{\nu}$. TEGR Lagrangian is built from the torsion $T^a_{\mu\nu} = e^a_\rho (\partial_\mu E^\rho_{\nu} - \partial_\nu E^\rho_{\mu})$ through the torsion scalar $T$ defined as

$$T = -\frac{1}{4} T_{\mu\nu\sigma} T^{\mu\nu\sigma} - \frac{1}{2} T_{\mu\nu} T^{\mu\nu} + T^a_{\mu\rho} T^{\sigma\mu}_{\rho}.$$ (2)

TEGR Lagrangian $L = ET$ ($E$ stands for $\det(E^a_\mu) = |g|^{1/2}$) and GR Lagrangian $L = -ER$ ($R$ being the Levi-Civita scalar curvature) are dynamically equivalent
since they differ in a boundary term: \( E(R + T) = \partial_\mu (E T^\mu) \). So, both TEGR and GR govern the same d.o.f., which are associated with the metric tensor. The metric tensor is invariant under local Lorentz transformations of the tetrad, \( E^a \rightarrow E^{a'} = \Lambda^a_b(x) E^b \), which is thus a gauge symmetry of TEGR. The TEGR Lagrangian is used as a starting point to describe generalizations to GR inspired in \( f(R) \) theories; the so called \( f(T) \) gravity is governed by the action

\[
S = \frac{1}{2\kappa} \int d^4x \ E f(T). \tag{3}
\]

2. A Toy Model with Rotational Pseudo-Invariance

TEGR Lagrangian \( L = ET \) is not gauge invariant but pseudo-invariant, because \( T^\nu\mu \) in the above mentioned boundary term is not invariant under local Lorentz transformations of the tetrad. Therefore, a general function \( f \) will not allow the boundary term to be integrated out in the \( f(T) \) action \( E \); as a consequence, the theory will suffer a partial loss of the local Lorentz symmetry \( \mathbb{E} \), so an extra d.o.f. not related to the metric could appear. We will analyze this issue by resorting to a simple toy model with rotational pseudo-invariance (a similar one was introduced in a previous work, \( \mathbb{F} \), but the boundary term was simpler). Let be the Lagrangian

\[
L = 2 \left( \frac{d}{dt} \sqrt{zz} \right)^2 - U(z\bar{z}) + \bar{z} \frac{\partial g}{\partial z} + z \frac{\partial g}{\partial \bar{z}} \tag{4}
\]

The two first terms are invariant under local rotations \( z \rightarrow e^{i\alpha(t)} z \). The rest of \( L \) is a total derivative; it does not take part in the dynamics but can be affected by the local rotation. So, the Lagrangian \( L \) is just pseudo-invariant under a local rotation. As any gauge invariance the local pseudo-invariance implies the existence of constraints among the canonical momenta; a unique primary constraint is obtained in this case:

\[
G^{(1)} \equiv z \left( \frac{p_z - \partial g}{\partial z} \right) - \bar{z} \left( \frac{p_{\bar{z}} - \partial g}{\partial \bar{z}} \right) \approx 0. \tag{5}
\]

\( G^{(1)} \) is an angular momentum; it generates rotations. In fact, it is \( \{ G^{(1)}, z\bar{z} \} = 0 \), which means that the dynamical variable \( |z| \) is gauge invariant. As can be seen, the angular momentum not only is conserved in this case; since the symmetry is local (time-dependent), the conserved value is constrained to be zero.

Primary constraints have to be consistent with the evolution, as controlled by the primary Hamiltonian \( H_p = H + u(t) G^{(1)} \). In the case \( \mathbb{E}, \mathbb{F} \) it results that the consistency is fulfilled without specifying the Lagrange multiplier \( u(t) \). Thus, the evolution of any variable that does not commute with \( G^{(1)} \) is affected by an undetermined function \( u(t) \); this is the case of the phase of \( z \), which become a “pure gauge” variable, but not the case of \( |z| \), which is a genuine d.o.f. or observable. \( G^{(1)} \) is called first-class, since it commutes with all the constraints (it is the only constraint in this example). As it is well known, each first class constraint removes one d.o.f. from a Hamiltonian constrained system. In this toy model, one d.o.f. is removed from the pair \( (z, \bar{z}) \), showing that \( |z| \) is the only d.o.f. of the theory.
3. Modified toy model

We will deform the toy model of the previous section by introducing the Lagrangian $f(L)$. Let us show that this can be done by means of the Lagrangian

$$L = \phi L - V(\phi),$$

where $\phi$ is an auxiliary canonical variable. Equation (6) resembles the Jordan-frame representation of $f(R)$ gravity. From $L$ one gets the equation of motion for $\phi$:

$$L = V'(\phi).$$

Thus, $L$ is (on-shell) equal to the Legendre transform of $V(\phi)$; therefore it depends only on $L$, i.e. $L = f(L)$ (from the inverse Legendre transform we also know that $\phi = f'(L)$). Thus the Lagrangian $L$ is dynamically equivalent to a $f(L)$ theory. As expected for a $f(L)$ theory, $L$ is not pseudo-invariant under local rotations. This is because the total derivative coming with $L$ is now multiplied by $\phi$ in (6). We will present the main outcomes of the Hamiltonian formalism for this $f(L)$ model and see the implicancies of the lost pseudo-invariance.

By computing the canonical momenta for $L$ one gets two primary constraints: the angular momentum and the momentum conjugated to $\phi$,

$$G^{(1)} = z \left( p_z - \frac{\partial g}{\partial z} \right) - \overline{z} \left( \overline{p}_\overline{z} - \phi \frac{\partial g}{\partial \overline{z}} \right) \approx 0, \quad G^{(1)}_{\pi} = \pi = \frac{\partial L}{\partial \dot{\phi}} \approx 0. \quad (7)$$

The Poisson bracket between the constraints is

$$\{G^{(1)}, G^{(1)}_{\pi}\} = -\overline{z} \frac{\partial g}{\partial z} + z \frac{\partial g}{\partial \overline{z}}. \quad (8)$$

which depends on the function $g(z, \overline{z})$ appearing in the boundary term of $L$. Depending on $g$, the Poisson bracket could be zero or not, which would drastically affect the counting of d.o.f. So, we will separate two cases:

- **Case (i):** $g(z, \overline{z}) \neq v(z \overline{z})$. In this case it is $\{G^{(1)}, G^{(1)}_{\pi}\} \neq 0$, so the constraints are second class. The consistency is guaranteed by choosing the Lagrange multipliers $u^\pi(t)$ and $u(t)$ associated with $G_{\pi}$ and $G^{(1)}$, respectively. In particular, it results $u^\pi = 0$ which implies that $\phi$ does not evolve but is a constant. The constancy of $\phi$ also implies that $|z|$ evolves like in the undeformed theory governed by $L$. But now the evolution of the phase of $z$ is determined too, because the Lagrange multiplier $u(t)$ is no longer left free. Since the evolution is already consistent at this step, then the algorithm is over. The counting of d.o.f. goes like this: from the set of three canonical variables $(\phi, z, \overline{z})$, just one d.o.f. is removed due to the appearance of one pair of second class constraints. We are left with two d.o.f., which can be represented by the variables $(z, \overline{z})$. The Lagrangian $f(L)$ determines not only the modulus of $z$ but its phase as well.

- **Case (ii):** $g(z, \overline{z}) = v(z \overline{z})$. In this case it is $\{G^{(1)}, G^{(1)}_{\pi}\} = 0$. This case is trivial because if $g(z, \overline{z}) = v(z \overline{z})$ the entire Lagrangian $L$ will depend exclusively on $|z|$, so being locally invariant. Thus we do not expect an
extra d.o.f. in the deformed $f(L)$ theory. So, let us check that Dirac’s algorithm yields the right answer. The consistency of the constraints with the evolution leads to a new secondary constraint $G^{(2)} = L - V'(\phi) \approx 0$. Since $\{G^{(1)}, G^{(2)}\} = 0$, and $\{G_x^{(1)}, G^{(2)}\} = V''(\phi)$, then $G^{(1)}$ is first-class, while $G_x^{(1)}$, $G^{(2)}$ are second-class. The Lagrange multiplier $u^\pi(t)$ is fixed by the consistency equations. Instead $u(t)$ (associated with $G^{(1)}$ in $H_p$) is not fixed by the algorithm, so meaning that the variables that are sensitive to rotations, like the phase of $z$, will remain as pure gauge variables. The counting of d.o.f. goes like this: from the three canonical variables $(\phi, z, \bar{z})$ we remove two d.o.f., one coming from $G^{(1)}$ being first-class, and the other one because the pair $G_x^{(1)}$, $G^{(2)}$ is second-class, leaving us with the genuine d.o.f. $|z|$. Remarkably, $u^\pi(t)$ results in a non zero function; therefore $\phi$ is not a constant and affects the evolution of $|z|$, that departs from the evolution it had in the original undeformed theory $L$.

3.1. Conclusions

In principle $f(T)$ gravity is case-(i), since TEGR Lagrangian is pseudo-invariant under local Lorentz transformations of the tetrad. This means that $f(T)$ gravity entails an extra d.o.f. associated with the orientation of the tetrad. However we could wonder whether $f(T)$ gravity can be case-(ii) on-shell. This is an interesting point because, even though $f(T)$ gravity is case-(i), there could exist particular solutions to the equations of motion making zero the value of the Poisson bracket $\{\delta, \Delta\}$. For such solutions, $\phi$ (and so $T$ too) would be an evolving field, and no extra d.o.f. would manifest. Remarkably, flat FRW spacetime seems to be a good arena to test this conjecture, because it contains both solutions with $T$ equal to a constant and $T = -6H^2(t)$ an evolving function.

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