A PHRAGMÉN-LINDELOF PROPERTY OF VISCOSITY SOLUTIONS TO A CLASS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS: BOUNDED CASE

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Abstract. We study Phragmén-Lindelöf properties for viscosity solutions to a class of nonlinear parabolic equations of the type

\[ H(Du, D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^\sigma - u_t = 0 \]

under a certain boundedness condition on \( H \). We also state results for positive solutions to a class of doubly nonlinear equation

\[ H(Du, D^2u) - f(u)u_t = 0. \]

1. Introduction

In this work we study the Phragmén-Lindelöf property of viscosity solutions \( u(x,t) \) for a class of nonlinear parabolic equations on the infinite strip \( \mathbb{R}^n_T = \mathbb{R}^n \times (0,T) \), where \( n \geq 2 \) and \( 0 < T < \infty \). The current work may be viewed as partly complementing the work [7]. See also, [4].

Set \( \mathbb{R}^n_T = \mathbb{R}^n \times (0,T) \) and let \( g : \mathbb{R}^n \to (0,\infty) \) be continuous and \( f : [0,\infty) \to [0,\infty) \) be an increasing continuous function. As described in [7], the motivation for this work arises from the study of doubly nonlinear equations of the kind

\[ H(Du, D^2u) - f(u)u_t = 0, \text{ in } \mathbb{R}^n_T, \text{ with } u(x,0) = g(x), \forall x \text{ in } \mathbb{R}^n, \]

where \( H \) satisfies certain homogeneity conditions and \( u \in C(\mathbb{R}^n \times [0,T)) \) is a viscosity solution. See Section 2 for more details.

As noted in [6, 7], if \( f \) satisfies certain conditions then there is an increasing function \( \phi \) and a non-increasing function \( Z \geq 0 \) such that the change of variable \( u = \phi(v) \) transforms the differential equation in (1.1) to

\[ H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t = 0, \text{ in } \mathbb{R}^n_T, \text{ with } v(x,0) = \phi^{-1}(g(x)), \forall x \text{ in } \mathbb{R}^n. \]

It follows that the solutions of (1.2) and hence, the solutions of (1.1), satisfy a comparison principle, see [2, 3, 6]. Incidentally, we do not require that \( Z \) be defined in all of \( \mathbb{R} \), a matter that will be discussed later. For purposes of the current discussion, we will overlook this issue.
As done in [7], we consider a somewhat more general setting and study Phragmén-Lindelöf type results for equations of the kind

\[ H(Dv, D^2v + Z(v)Dv \otimes Dv) + \chi(t)|Dv|^\sigma - v_t = 0, \text{ in } \mathbb{R}^n_+, \]

(1.3)

\[ v(x, 0) = h(x), \forall x \in \mathbb{R}^n, \]

where \( \sigma \geq 0 \) and \( \chi : (0, T) \to \mathbb{R} \) and \( h : \mathbb{R}^n \to \mathbb{R} \) are both continuous and bounded.

In [7], we assumed that \( \sup_{\lambda} \left[ \min_{|e|=1} H(e, \lambda e \otimes e + I) \right] = \infty \), where \( e \) is a unit vector, \( I \) is the \( n \times n \) identity matrix and \( \lambda \) is a real valued parameter. We showed that the maximum principle was valid for solutions that satisfied certain growth rates for large \( x \). The class of operators, we considered, included, among others, the \( p \)-Laplacian \( (p \geq 2) \), the infinity-Laplacian and the Pucci operators. The current work addresses the case \( \sup_{\lambda} \left[ \max_{|e|=1} |H(e, \lambda e \otimes e \pm I)| \right] < \infty \) and, in a sense, complements [7]. In Section 2, we have listed some examples of operators that satisfy this condition.

We remark also that, much like [7], the imposed growth rates are influenced by the dueling terms \( Z(v)DV \otimes DV \) and \( \chi(t)|DV|^\sigma \) and the power \( \sigma \). Since \( Z \geq 0 \), by ellipticity, \( H(Du, D^2u) \leq H(Du, D^2u + Z(u)Du \otimes Du) \). Our work will show that, unlike [7], \( Z(s) \) can be allowed to vanish, i.e, \( Z(s) = 0, \forall s \geq s_0 \), for some \( s_0 \). The value of \( Z \) does not influence the bound on \( H(e, \lambda e \otimes e \pm I) \).

We have divided our work as follows. In Section 2, we introduce more notation and state the main results. Section 3 contains preliminary calculations and previously proven lemmas, useful for the current work, In Sections 4 and 5, we present the constructions of super-solutions and sub-solutions respectively. Section 6 addresses some special situations. The proofs of the main results appear in Section 7.

As a final note, we do not address questions of existence and uniqueness and nor do we address optimality of the growth rates stated in the theorems. Also, we direct the reader to [1, 9, 10, 11, 12] for related questions and discussion.

2. Notation and main results

In this work, sub-solutions, super-solutions and solutions are meant in the sense of viscosity. For definitions, we direct the reader to [6, 8].

We introduce notation that are used throughout this work. We address the problems in (1.1) and (1.3) on infinite strips in \( \mathbb{R}^{n+1} \) where \( n \geq 2 \). The letter \( o \) denotes the origin in \( \mathbb{R}^n \) and \( e \) denotes a unit vector in \( \mathbb{R}^n \). Let \( S^{n \times n} \) be the set of all symmetric \( n \times n \) real matrices. Let \( I \) be the identity matrix and \( O \) the \( n \times n \) zero matrix. The expressions \( \text{usc} \) and \( \text{lsc} \) stand for \textit{upper semi-continuous} and \textit{lower semi-continuous} respectively.
Through out this work, we assume that $H$ satisfies the following conditions.

**Condition A (Monotonicity):** Let $H : \mathbb{R}^n \times S^{n \times n} \to \mathbb{R}$ is continuous for any $(q, X) \in \mathbb{R}^n \times S^{n \times n}$. We require that

\begin{align}
(\text{i}) \ H(q, X) & \leq H(q, Y), \quad \forall \ q \in \mathbb{R}^n \text{ and } \forall \ X, Y \in S^{n \times n}, \text{ with } X \leq Y, \\
(\text{ii}) \ H(q, O) & = 0, \quad \forall \ q \in \mathbb{R}^n. \\
\end{align}

(2.1)

Clearly, for any $q \in \mathbb{R}^n$ and $X \in S^{n \times n}$, $H(q, X) \geq 0$ if $X \geq O$.

**Condition B (Homogeneity):** There is a constant $k_1 \geq 0$, such that for any $(q, X) \in \mathbb{R}^n \times S^{n \times n}$,

\begin{align}
(\text{i}) \ H(\theta q, X) & = |\theta|^{k_1} H(q, X), \quad \forall \ \theta \in \mathbb{R}, \text{ and} \\
(\text{ii}) \ H(q, \theta X) & = \theta H(q, X), \quad \forall \ \theta > 0. \\
\end{align}

(2.2)

We introduce two quantities before stating the next condition. For any unit vector $e \in \mathbb{R}^n$, we recall that $(e \otimes e)_{ij} = e_i e_j$, for any $i, j, = 1, 2, \cdots, n$. Moreover, $e \otimes e \geq O$. For $\lambda \in \mathbb{R}$, set

\begin{equation}
\Lambda_{\min}(\lambda) = \min_{|e|=1} H(e, \lambda e \otimes e - I) \quad \text{and} \quad \Lambda_{\max}(\lambda) = \max_{|e|=1} H(e, \lambda e \otimes e + I).
\end{equation}

(2.3)

By Condition A, both $\Lambda_{\min}(\lambda)$ and $\Lambda_{\max}(\lambda)$ are non decreasing functions of $\lambda$.

**Condition C (Growth at Infinity):** We impose that

$$
\max_{|e|=1} H(e, -I) < 0 < \min_{|e|=1} H(e, I).
$$

Set $\Lambda^{\sup} = \sup_{\lambda} \Lambda_{\max}(\lambda)$ and $\Lambda^{\inf} = \inf_{\lambda} \Lambda_{\min}(\lambda)$. Assume further that

\begin{equation}
\Lambda^{\sup} < \infty.
\end{equation}

(2.4)

It follows easily from (2.4), Condition A and Condition B (ii) that $H(e, e \otimes e) = 0$.

In this work, the requirement (2.4) will apply through out. For some of the results, we will require additionally that

$$
\Lambda^{\inf} > -\infty. \quad \Box
$$

We now present examples of operators that satisfy Conditions A, B and C, and include some observations.
Remark 2.1. (i) An example of an operator that satisfies Conditions A, B and C is

\[ H_p(q, X) = |q|^p \{ |q|^2 Tr(X) - q_i q_j X_{ij} \}, \quad p \geq 0, \quad \forall (q, X) \in \mathbb{R}^n \times S^{n \times n}, \]

where \( Tr(X) \) is the trace of \( X \). Clearly,

\[ H_p(Du, D^2 u) = |Du|^p (|Du|^2 \Delta u - \Delta_\infty u). \]

Thus, for any \( c \in \mathbb{R} \),

\[ H_p(q, X + cq \otimes q) = |q|^p \left[ |q|^2 Tr(X) + c|q|^4 - q_i q_j X_{ij} - c|q|^4 \right] = H_p(q, X). \]

In particular,

\[ H_p(e, \lambda e \otimes e \pm I) = H_p(e, \pm I) = \pm (n-1), \quad \text{for any } \lambda \in \mathbb{R}. \]

Note that \( k = k_1 + 1 \geq 1 \), see (2.6) below. A closely allied example is \( H(Du, D^2 u) = |Du|^4 \Delta_p u - (p-1)|Du|^p \Delta_\infty u. \)

(ii) A second example can be constructed as follows. Let \( \mu_i = \mu_i(X), \quad i = 1, 2, \ldots, n \) be the eigenvalues of any \( X \in S^{n \times n} \). We order these as \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \). Define

\[ H^m_p(q, X) = |q|^p \left( \sum_{i=m}^{n} \mu_i(X) \right), \quad p \geq 0 \text{ and } 2 \leq m < n. \]

Clearly, \( H \) satisfies Conditions A and B, \( H^m_p(e, \pm I) = \pm (n - m + 1) \).

Observe that \( \det(e \otimes e) = 0 \) and \( (e \otimes e)^2 = e \otimes e \) and \( (e \otimes e - \mu I)x = 0 \) \((x \perp e)\) if and only if \( \mu = 0 \) or \( \mu = 1 \) \((x \parallel e)\) implying that the eigenvalues of \( e \otimes e \) are 0 (multiplicity \( n-1 \)) and 1. Thus, the eigenvalues of \( \lambda e \otimes e + I \) are 1 (multiplicity \( n-1 \)) and \( \lambda + 1 \). Similarly, the eigenvalues of \( \lambda e \otimes e - I \) are \(-1 \) (multiplicity \( n-1 \)) and \( \lambda - 1 \). Thus,

\[ \begin{cases} 
H^m_p(e, \lambda e \otimes e + I) = n + 1 - m, & \lambda \geq 0, \\
H^m_p(e, \lambda e \otimes e - I) = \lambda - (n - m + 1), & \lambda \leq 0.
\end{cases} \]

Some of our results, in particular, the maximum principle in Theorem 2.2 given below, hold for this operator. The case \( m = 1 \) (Laplacian) is included in [7]. Observe that \( k = p + 1 \geq 1 \) in this case.

(iii) If \( H \) is odd in \( X \) i.e., \( H(q, -X) = -H(q, X) \) then (2.4) shows that \( H(e, \lambda e \otimes e + I) = -H(e, -\lambda e \otimes e - I) \) and \( \Lambda^{sup} = -\Lambda^{inf} < \infty \). Clearly, \( H(e, \pm e \otimes e) = 0 \).

(iv) We record a simple observation. If \( k_1 = 0 \) i.e., \( k = 1 \), then \( H(e, X) = H(e/s, X) \), for any \( s > 0 \). Thus, \( H(q, X) = H(0, X) = H(X). \) \( \square\)
We introduce some further notation. Set \( \mathbb{R}_T^n = \mathbb{R}^N \times (0, T) \). Let \( \chi : (0, T) \to \mathbb{R} \) be a bounded continuous function and, for some \( m \in \mathbb{R} \) (to be specified later) \( Z : [m, \infty) \to [0, \infty) \) be a non-increasing continuous function. For \( \sigma \geq 0 \), set
\[
(2.5) \quad P_\sigma(t, u, u_t, Du, D^2u) = H(Du, D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^\sigma - u_t.
\]

We assume throughout that \( H \) satisfies Conditions A, B and C. Define
\[
(2.6) \quad k = k_1 + 1 \quad \text{and} \quad \gamma = k + 1 = k_1 + 2.
\]
Clearly, \( \gamma \geq 2 \) and if \( k = 1 \) then \( k_1 = 0 \) and \( \gamma = 2 \). Next, define
\[
(2.7) \quad \forall \sigma > 1, \sigma^* = \frac{\sigma}{\sigma - 1}, \quad \text{and} \quad \forall k > 1, \gamma^* = \frac{\gamma}{k - 1} = \frac{\gamma}{\gamma - 2}.
\]

For a fixed \( z \in \mathbb{R}^n \) and \( \forall x \in \mathbb{R}^n \), set \( r = |x - z| \). Also, define \( B^R_T = \{(x, t) : |x - z| \leq R, \ 0 < t < T \} \). Let \( P_\sigma \) be as defined in (2.5).

We first state the results for \( k > 1 \) or equivalently for \( \gamma > 2 \).

**Theorem 2.2. (Maximum Principle)** Let \( 0 < T < \infty, h : \mathbb{R}^n \to \mathbb{R} \) be continuous with \( \sup_{\mathbb{R}^n} h(x) < \infty \) and, for some \( m, Z : [m, \infty) \to [0, \infty) \) be non-increasing and continuous. Suppose that (2.4) holds, i.e., \( \Lambda^{\sup} < \infty \). Let \( u \in \text{usc}(\mathbb{R}_T^n), \inf u > m \), solve
\[
P_\sigma(t, u, u_t, Du, D^2u) \geq 0 \quad \text{in} \quad \mathbb{R}_T^n,
\]
and \( u(x) \leq h(x), \forall x \in \mathbb{R}^n \).

Let \( \gamma^* \) and \( \sigma^* \) be as in (2.7). Suppose that \( \sup_{\mathbb{R}^n_T} u(x, t) = o(R^{\beta}) \), as \( R \to \infty \). Then the following hold.

(a) If \( 0 \leq \sigma \leq \gamma/2 \) and \( \beta = \gamma^* \) then
\[
\sup_{\mathbb{R}^n_T} u(x, t) \leq \begin{cases} \sup_{\mathbb{R}^n} h(x) + t(\sup_{[0, T]} |\chi(t)|), & \sigma = 0, \\ \sup_{\mathbb{R}^n} h(x), & 0 < \sigma \leq \gamma. \end{cases}
\]

(b) If \( \sigma > \gamma/2 \) and \( \beta = \sigma^* \) then
\[
\sup_{\mathbb{R}^n_T} u(x, t) \leq \sup_{\mathbb{R}^n} h(x). \quad \Box
\]

Observe that if \( m = -\infty \) then the restriction \( \inf u > m \) may be dropped. Also, note that if \( \sigma = \gamma/2 \) we get \( \sigma^* = \gamma/(\gamma - 2) = \gamma^* \).

**Theorem 2.3. (Minimum Principle)** Let \( 0 < T < \infty, h : \mathbb{R}^n \to \mathbb{R} \) be a continuous function, with \( \inf_{\mathbb{R}^n} h(x) > -\infty \), and \( Z : (-\infty, \infty) \to [0, \infty) \) be a non-increasing continuous function. We assume that \( \Lambda^{\sup} < \infty \).
Let $u \in \text{lsc}(\mathbb{R}^n_T)$ solve
\[
P_\sigma(t, u, u_t, Du, D^2u) \leq 0, \text{ in } \mathbb{R}^n_T \text{ and } u(x) \geq h(x), \forall x \in \mathbb{R}^n.
\]

Let $\gamma^*$ and $\sigma^*$ be as in (2.7). Suppose that $\sup_{B_R^T}(-u(x,t)) = o(R^\beta)$ as $R \to \infty$.

Then the following hold.

(a) If $0 \leq \sigma \leq \gamma/2$ and $\beta = \gamma^*$ then
\[
\inf_{\mathbb{R}^n_T} u(x, t) \geq \left\{ \begin{array}{ll}
\inf_{\mathbb{R}^n} h(x) - t \left( \sup_{[0,t]} |\chi(t)| \right), & \sigma = 0, \\
\inf_{\mathbb{R}^n} h(x), & 0 < \sigma \leq \gamma.
\end{array} \right.
\]

(b) If $\sigma > \gamma/2$ and $\beta = \sigma^*$ then
\[
\inf_{\mathbb{R}^n_T} u(x, t) \geq \inf_{\mathbb{R}^n} h(x).
\]

\[\Box\]

We impose no restrictions on $\Lambda^{\text{inf}}$ for Theorem 2.3.

We now state analogous results for $k = 1$, i.e., $\gamma = 2$. See Remark 2.1 (iv).

The statement that, for some $s > 0$, $w(r) = e^{o(r^s)}$ as $r \to \infty$, will mean that $\log v^+ = o(r^s)$ as $r \to \infty$, where $v^+ = \max(v, 0)$.

**Theorem 2.4. (Maximum Principle)** Let $0 < T < \infty$, $h : \mathbb{R}^n \to \mathbb{R}$ be continuous with $\sup_{\mathbb{R}^n} h(x) < \infty$. For some $m$, let $Z : [m, \infty) \to [0, \infty)$ be non-increasing and continuous. Suppose that (2.4) holds, i.e., $\Lambda^{\text{sup}} < \infty$.

Let $u \in \text{usc}(\mathbb{R}^n_T)$, $\inf u > m$, solve
\[
H(D^2u + Z(u)Du \otimes Du) - u_t \geq 0 \text{ in } \mathbb{R}^n_T, \text{ and } u(x) \leq h(x), \forall x \in \mathbb{R}^n.
\]

Let $\sigma^*$ be as in (2.7). Then the following hold

(a) Suppose that $\sigma = 0$. If $\sup_{B_R^T} u(x,t) = e^{o(R^2)}$, as $R \to \infty$, then
\[
u(x,t) \leq \sup_{\mathbb{R}^n} h(x) + \left( \sup_{(0,T)} \chi(t) \right) t, \forall (x,t) \in \mathbb{R}^n_T.
\]

(b) Let $0 < \sigma \leq 1$. If $\sup_{B_R^T} u(x,t) = e^{o(R)}$, as $R \to \infty$ then
\[
u(x,t) \leq \sup_{\mathbb{R}^n} h(x) + K(1 - \sigma) \left( \sup_{(0,T)} \chi(t) \right),
\]
where $K = K(\alpha, \Lambda^{\text{sup}}, \sigma, T)$. 
(c) Let \( 1 < \sigma < \infty \) and assume that \( \sup_{B_R^x} u(x, t) = o(R^{\sigma^*}) \), as \( R \to \infty \). Then
\[
\begin{align*}
  u(x, t) \leq \sup_{\mathbb{R}^n} h(x). \\
  \square
\end{align*}
\]

We now present a minimum principle. Note that the condition \( \Lambda^{\inf} > -\infty \) is needed only for parts (a) and (b) of the theorem. Part (c) of the theorem holds without this restriction.

**Theorem 2.5.** (Minimum Principle) Let \( 0 < T < \infty \), \( h : \mathbb{R}^n \to \mathbb{R} \) be continuous, with \( \sup_{\mathbb{R}^n} h(x) < \infty \), and \( Z : (-\infty, \infty) \to [0, \infty) \) be non-increasing and continuous. We assume that \( \Lambda^{\sup} < \infty \).

Let \( u \in \text{usc}(\mathbb{R}^n_T) \) solve
\[
H(D^2u + Z(u)Du \otimes Du) - u_t \leq 0 \text{ in } \mathbb{R}^n_T, \text{ and } u(x) \geq h(x), \forall x \in \mathbb{R}^n.
\]

Assume for parts (a) and (b) that \( \Lambda^{\inf} > -\infty \). Let \( \sigma^* \) be as in (2.7). Then the following hold.

(a) Suppose that \( \sigma = 0 \). If \( \sup_{B_R^x} (-u(x, t)) = e^{o(R^2)} \), as \( R \to \infty \), then
\[
\begin{align*}
  u(x, t) \geq \inf_{\mathbb{R}^n} h(x) - t \left( \sup_{(0, T)} \chi(t) \right), \forall (x, t) \in \mathbb{R}^n_T.
\end{align*}
\]

(b) Let \( 0 < \sigma \leq 1 \). If \( \sup_{B_R^x} (-u(x, t)) = e^{o(R)} \), as \( R \to \infty \) then
\[
\begin{align*}
  u(x, t) \geq \inf_{\mathbb{R}^n} h(x).
\end{align*}
\]

(c) Let \( 1 < \sigma < \infty \) and assume that \( \sup_{B_R^x} (-u(x, t)) = o(R^{\sigma^*}) \), as \( R \to \infty \). Then
\[
\begin{align*}
  u(x, t) \geq \inf_{\mathbb{R}^n} h(x). \quad \square
\end{align*}
\]

Finally, we present similar results for a class of doubly nonlinear equations of the type
\[
H(Du, D^2u) - f(u)u_t = 0, \text{ in } \mathbb{R}^n_T, \text{ with } u(x, 0) = g(x), \forall x \in \mathbb{R}^n.
\]

If \( k = 1 \), we assume that \( f \equiv 1 \) and the differential equation then reads
\[
(2.8) \quad H(D^2u) - u_t = 0, \text{ in } \mathbb{R}^n_T \text{ with } u(x, 0) = g(x), \forall x \in \mathbb{R}^n.
\]

The above is not doubly nonlinear but is contained in our work.
It is to be noted that the afore stated theorems are used to obtain a maximum principle for these equations. The minimum principle, however, requires a different treatment.

If \( k > 1 \) we take \( f : [0, \infty) \to [0, \infty) \) to be an increasing \( C^1 \) function such that \( f^{1/(k-1)} \) is concave and consider equations of the type

\[
(2.9) \quad H(Du, D^2u) - f(u)u_t = 0, \text{ in } \mathbb{R}_n^+, \text{ with } u(x, 0) = g(x), \forall x \in \mathbb{R}^n,
\]

where \( u > 0 \).

For \( k > 1 \), let \( F \) be a primitive of \( f^{-1/(k-1)} \). Since \( f(s) > f(0) \geq 0, \forall s > 0 \), we consider the following two situations:

\[
(2.10) \quad (i) \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) < \infty, \text{ and } (ii) \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) = \infty.
\]

We set \( \chi(t) \equiv 0 \) in Theorems 2.2 and 2.5.

\textbf{Theorem 2.6.} Let \( f : [0, \infty) \to [0, \infty) \) be a \( C^1 \) increasing function and \( g : \mathbb{R}^n \to (0, \infty) \), continuous, be such that \( 0 < \inf_x g(x) \leq \sup_x g(x) < \infty \). Assume that \( \Lambda^{\sup} < \infty \).

(a) Maximum Principle: Let \( k > 1 \) and \( f^{1/(k-1)} \) be a concave function. Suppose that \( \phi : \mathbb{R} \to [0, \infty) \) is a \( C^2 \) increasing function such that \( \phi'(\tau) = f(\phi(\tau))^{1/(k-1)} \). Recall \( \gamma^* \) from (2.7).

If \( u \in \text{usc}(\mathbb{R}^n_T) \), \( u > 0 \), solves

\[
H(Du, D^2u) - f(u)u_t \geq 0, \text{ in } \mathbb{R}_n^+, \text{ and } u(x, 0) \leq g(x), \forall x \in \mathbb{R}^n,
\]

and \( \sup_{B_R} u(x, t) \leq \phi(o(R^\gamma)) \), as \( R \to \infty \), then

\[
\sup_{\mathbb{R}_n^T} u(x, t) \leq \sup_{\mathbb{R}^n} g(x).
\]

Let \( k = 1 \) and \( f \equiv 1 \), i.e, \( H(D^2u) - u_t \geq 0 \). If \( \sup_{B_R} u(x, t) \leq e^{o(R^\gamma)} \), as \( R \to \infty \), then \( \sup_{\mathbb{R}_n^T} u(x, t) \leq \sup_{\mathbb{R}^n} g(x) \).

(b) Minimum Principle: Let \( k > 1 \), \( f \) and \( \phi \) be as in part (a).

Suppose that \( u \in \text{lsc}(\mathbb{R}^n_T) \), \( u > 0 \), solves

\[
H(Du, D^2u) - f(u)u_t \leq 0, \text{ in } \mathbb{R}_n^+, \text{ and } u(x, 0) \geq g(x), \forall x \in \mathbb{R}^n.
\]

If condition (2.10)(i) holds, i.e, \( \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) < \infty \) then

\[
u(x, t) \geq \inf_{\mathbb{R}^n} g(x), \forall (x, t) \in \mathbb{R}_n^T.
\]
If condition (2.10)(ii) holds, i.e., \( \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) = \infty \), and \( \inf_{B_R^\varepsilon} u(x,t) \geq \phi(-o(R^n)) \) as \( R \to \infty \) then
\[
\inf_{\bar{B}_R} g(x), \forall (x,t) \in \mathbb{R}_T^n.
\]

Suppose \( k = 1 \) and \( f \equiv 1 \), i.e., \( H(D^2u) - u_t \leq 0 \). If \( \inf_{B_R^\varepsilon} u(x,t) \geq -e^o(R^2) \), as \( R \to \infty \), then
\[
\inf_{\bar{B}_R} g(x), \forall (x,t) \in \mathbb{R}_T^n.
\]

3. Preliminaries

In this section, we present some definitions, lemmas and remarks we will use to prove the main results. Fix \( z \in \mathbb{R}^n \) and set \( r = |x - z|, \forall x \in \mathbb{R}^n \). A unit vector in \( \mathbb{R}^n \) is denoted by \( e = (e_1, e_2, \ldots, e_n) \).

We begin with an elementary remark that will be used frequently in our work.

**Remark 3.1.** Assume that \( w: \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) is a \( C^1 \) function in \( x \) and \( t \) and \( C^2 \) in \( x \) except, perhaps, at \( x \neq z \). We get, for \( r \neq 0 \),
\[
H(Dw, D^2w + Z(w) Dw \otimes Dw)
= \left\{ \begin{array}{ll}
H \left( w_r e, \left( \frac{w_r}{r} \right) I + \left( w_{rr} - \frac{w_r}{r} + \left( \frac{w_r}{r} \right)^2 Z(w) \right) e \otimes e \right), & \text{if } w_r \geq 0, \\
& \text{otherwise}
\end{array} \right.
\]
where \( e = (e_1, e_2, \ldots, e_n) \) with \( e_i = (x - z)_i/r, \forall i = 1, 2, \ldots, n. \) Let \( \kappa: (0, T) \to [0, \infty) \) be a \( C^1 \) function.

**Case (a) \( w_r \geq 0 \):** We apply Condition B, in (2.2), to (3.1). Factor \( w_r \) from the first entry, \( w_r/r \) from the second and use \( k = k_1 + 1 \) to get
\[
H(Dw, D^2w + Z(w) Dw \otimes Dw)
= \left( \frac{w^k_r}{r} \right) H \left( e, I + \left( \frac{rw_{rr}}{w_r} - 1 + rw_r Z(w) \right) e \otimes e \right), \forall r > 0.
\]

If \( w(x,t) = \kappa(t)v(r), \) with \( v'(r) \geq 0 \), then (3.2) implies that, in \( r > 0 \),
\[
H(Dw, D^2w + Z(w) Dw \otimes Dw)
= \frac{(\kappa(t)v'(r))^k}{r} H \left( e, I + \left( \frac{rv''(r)}{v'(r)} - 1 + r\kappa(t)v'(r) Z(w) \right) e \otimes e \right).
\]
Case (b) \((w_r \leq 0)\): Clearly, (3.1) leads to

\[
H(Dw, D^2w + Z(w) Dw \otimes Dw)
\]

\[
= \frac{|w_r|^k}{r} H\left(e, \left(1 - \frac{r w_{rr}}{w_r} + r|w_r|Z(w)\right) e \otimes e - I\right), \quad \forall r > 0.
\]

If \(w(x, t) = \kappa(t)v(r)\) and \(v'(r) \leq 0\) then (3.4) leads to the following analogue of (3.3):

\[
H(Dw, D^2w + Z(w) Dw \otimes Dw)
\]

\[
= \left(\kappa(t)|v'(r)|\right)^k r H\left(e, \left(r|v'(r)|\kappa(t)Z(w) + 1 - \frac{rv''(r)}{v'(r)}\right) e \otimes e - I\right). \quad \square
\]

The following lemma was proven in [7].

**Lemma 3.2.** Let \(\beta, \bar{\beta}\) be such that \(1 < \bar{\beta} < \beta\) and \(R > 0\). Fix \(z \in \mathbb{R}^n\), set \(r = |x - z|\) and define

\[
v(r) = \int_0^{r^\beta} \frac{1}{1 + \tau^p} d\tau, \quad \text{where} \quad p = \frac{\beta - \bar{\beta}}{\beta}.
\]

Then (i) \(0 < p < 1\), (ii) \((1 - p)\beta = \bar{\beta}\), and

\[
i(iii) \quad \forall r \geq 0, \quad \frac{r^\beta}{1 + r^{\beta p}} \leq v(r) \leq \min\left(r^\beta, \frac{r^\beta}{1 - p}\right).
\]

If \(R > 1\) then

\[
i(iv) \quad \frac{\beta}{2\beta} = \frac{1}{2(1 - p)} \leq \frac{v(r) - v(R)}{r^\beta - R^\beta} \leq \frac{1}{1 - p} = \frac{\beta}{\beta}, \quad \forall r \geq R.
\]

Moreover, \(v'(r) = \beta r^{\beta - 1} / (1 + r^{p\beta})\) implying that

\[
i(v) \quad v'(r) \leq \beta \min\left(r^{\beta - 1}, r^{\beta - 1}\right),
\]

\[
i(vi) \quad \left(\frac{(v'(r))^k}{r}\right) \leq \beta^k \min\left(r^{k\beta - \gamma}, r^{k\beta - \gamma}\right),
\]

and (vii) \(v''(r) = \beta r^{\beta - 2} \left(\frac{(\beta - 1) + (\bar{\beta} - 1)r^{p\beta}}{(1 + r^{p\beta})^2}\right)\).

**Comment:** Parts (iii) and (iv) of Lemma 3.2 show that \(v(r)\) grows like \(r^\beta\) near \(r = 0\) and like \(r^{\bar{\beta}}\) for large values of \(r\). Since \(\beta \geq \bar{\beta}\), one can design the function to decay fast enough at \(r = 0\) so as to be differentiable while its growth rate for large values of \(r\) may be slower.

**Proof.** Parts (i)-(iii) follow quite readily. For part (iv), we take \(R > 1\) and write

\[
v(r) = \int_0^{r^\beta} (1 + \tau^p)^{-1} d\tau = v(R) + \int_{R^\beta}^{r^\beta} (1 + \tau^p)^{-1} d\tau
\]
We estimate $(2\tau_p)^{-1} \leq (1 + \tau p)^{-1} \leq \tau^{-p}$, for $\tau \geq 1$, and use this in the second integral to obtain part (iv). For part (v), note that $1 + r^{p\beta} \geq \min(1, r^{p\beta})$. Using part (ii) yields the claim. Part (vi) follows by recalling that $\gamma = k + 1 = k_1 + 2$.

Next,
\[ v''(r) = \beta \left[ \frac{(\beta - 1)r^{\beta-2}}{1 + r^{p\beta}} - \frac{p\beta r^{p\beta+\beta-2}}{(1 + r^{p\beta})^2} \right]. \]

A simple calculation leads to part (vii). □

The following remark is useful for the construction of the auxiliary function. The values of $\bar{\beta}$ and $\beta$, used in the remark, are motivated by the work in Sections 4 and 5.

**Remark 3.3.** For Sub-Part (iv) of Part I in Section 4, we take $k > 1$ (i.e, $\gamma > 2$) and $\sigma > \gamma/2$. We set
\[ \beta = \gamma^* = \gamma/(\gamma - 2) \quad \text{and} \quad \bar{\beta} = \sigma^* = \sigma/(\sigma - 1). \]

Then $p = (\gamma^* - \sigma^*)/\gamma^* = (2\sigma - \gamma)/\gamma(\sigma - 1) > 0$. Clearly, $0 < p < 1$.

We take
\[ v(r) = \int_0^r \frac{1}{1 + \tau p} d\tau, \quad \text{where} \quad p = 1 - \frac{\sigma^*}{\gamma^*} = \frac{2\sigma - \gamma}{\gamma(\sigma - 1)}. \]

From Lemma 3.2 (i) $0 < p < 1$, (ii) $(1 - p)\gamma^* = \sigma^*$,

(iii) for $r \geq 0$, $\frac{r^{\gamma^*}}{1 + r^{\gamma^*}p} \leq v(r) \leq \min \left( r^{\gamma^*}, \frac{\gamma^* r^{\sigma^*}}{\sigma^*} \right),$

(iv) for any $R > 1$, $\frac{\gamma^*}{2\sigma^*} = \frac{1}{2(1 - p)} \leq \frac{v(r) - v(R)}{r^{\sigma^*} - R^{\sigma^*}} \leq \frac{1}{1 - p} = \frac{\gamma^*}{\sigma^*}, \forall r \geq R.$

Moreover, $v'(r) = \gamma^* r^{\gamma^* - 1}/(1 + r^{p\gamma^*}),$

(v) $v'(r) \leq \gamma^* \min \left( r^{\sigma^* - 1}, r^{\gamma^* - 1} \right),$ (vi) $\frac{(v'(r))^k}{r} \leq \gamma^k \min \left( r^{k\gamma^* - \gamma}, r^{k\sigma^* - \gamma} \right),$

and (vii) $v''(r) = \gamma^* r^{\gamma^* - 2} \left( \frac{\gamma^* - 1}{(1 + r^{\gamma^* p})^2} \right).$ □

**Remark 3.4.** The super-solutions and sub-solutions make use of functions that involve a $C^1$ function of $t$ and a $C^{1,\alpha}$ (for some $\alpha > 0$) function of $v(r)$. See the functions discussed in Remark 3.3. The calculations done in the remark hold in the sense of viscosity at $r = 0$. The verification can be found in [7]. □

We recall a comparison principle needed for our work, see [8]. See also [9] and [7].
Let $F : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ be continuous. Suppose that $F$ satisfies $\forall X, Y \in S^n$, with $X \leq Y$, that
\begin{equation}
F(t, r_1, q, X) \leq F(t, r_2, q, Y), \forall (t, q) \in \mathbb{R}^+ \times \mathbb{R}^n \text{ and } r_1 \geq r_2.
\end{equation}

In this work, $F(t, r, q, X) = H(q, X + Z(r)q \otimes q) + \chi(t)|q|^{\sigma}$, where $Z$ is a non-increasing continuous function, $\sigma \geq 0$ and $H$ satisfies Conditions A, B and C.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_T = \Omega \times (0, T)$ and $P_T$ be the parabolic boundary of $\Omega_T$.

**Lemma 3.5. (Comparison principle)** Let $F$ satisfy (3.6) and $\hat{f} : \mathbb{R}^+ \to \mathbb{R}^+$ be a bounded continuous function. For some $m$, let $Z : [m, \infty) \to \mathbb{R}$ be a non-increasing continuous function.

Let $u \in \text{usc}(\Omega_T \cup P_T)$ and $v \in \text{lsc}(\Omega_T \cup P_T)$ such that $\inf(\inf u, \inf v) > m$. Suppose that $u$ and $v$ solve
\[
F(t, u, Du, D^2u + Z(u)Du \otimes Du) - \hat{f}(t)u_t \geq 0, \quad \text{and} \quad F(t, v, Dv, D^2v + Z(v)Dv \otimes Dv) - \hat{f}(t)v_t \leq 0, \quad \text{in } \Omega_T.
\]
If $\sup_{P_T} v < \infty$ and $u \leq v$ on $P_T$ then $u \leq v$ in $\Omega_T$. □

Next, we discuss a change of variables that is used in the proof of Theorem 2.6 for doubly nonlinear equations of the kind
\begin{equation}
H(Du, D^2u) - f(u)u_t = 0, \quad \text{in } \mathbb{R}^n, u > 0, \text{ with } u(x, 0) = g(x), \forall x \in \mathbb{R}^n.
\end{equation}

**Remark 3.6.** Let $f : [0, \infty) \to [0, \infty)$ be an increasing $C^1$ function. Suppose that $k > 1$ and $f^{1/(k-1)}$ is concave.

Let $I$ be either $[0, \infty)$ or $(-\infty, \infty)$, see (b) and (c) below. We select $\phi : I \to [0, \infty)$, an increasing $C^2$ function, such that
\[
\phi'(\tau) = f(\phi(\tau))^{1/(k-1)}, \quad \forall \tau \in I, \quad \text{or} \quad \int_{\phi(\tau_0)}^{\phi(\tau)} f^{-1/(k-1)}(\theta) \, d\theta = \tau - \tau_0.
\]
We define the change of variable $u = \phi(v)$ by
\begin{equation}
v(u) - v(u_0) = \phi^{-1}(u) = \int_{u_0}^{u} f^{-1/(k-1)}(\theta) \, d\theta, \quad u \geq u_0,
\end{equation}
for some $u_0 \geq 0$.

We discuss some examples. Let $\alpha > 0$, $a \geq 0$ and $f(s) = (s + a)^{\alpha}$, $\forall s \geq 0$. Then $f(s)^{1/(k-1)}$ is concave if $\alpha \leq k - 1$. Set $c_k = (k - 1 - \alpha)/(k - 1)$. We may take $u_0 = 0$.
in (3.8), we get that
\[ u = \phi(v) = \begin{cases} 
[c_kv + a^{c_k}]^{1/c_k} - a, & 0 < \alpha < k - 1, \ a \geq 0, \\
ae^v - a, & \alpha = k - 1, \ a > 0.
\end{cases} \]

See also part (b) below.

If \( a = 0 \), take \( f(s) = s^{k-1} \) then \( u = be^v \) for any \( b > 0 \). But, \( u_0 \neq 0 \), see part (c).

We make some observations about (3.8).

(a) It is clear that \( v \) is an increasing concave function of \( u \). The concavity follows since \( f \) is non-decreasing. Since \( v \) is increasing, \( u \) is a convex function of \( v \).

(b) If the integral in (3.8) is convergent for \( u_0 = 0 \) we then define
\[ v = \phi^{-1}(u) = \int_0^u f^{-1/(k-1)}(\theta) \, d\theta. \]

Thus, \( v(0) = 0 \) and \( v > 0 \).

We choose \( I = [0, \infty) \) and \( \phi : [0, \infty) \rightarrow [0, \infty) \). This applies to examples like
\[ f(s) = \begin{cases} 
 s^\alpha, & 0 \leq \alpha < k - 1, \\
 (s + a)^\alpha, & 0 \leq \alpha \leq k - 1,
\end{cases} \]

where \( a > 0 \).

(c) If the integral in (3.8) is divergent for \( u_0 = 0 \) then \( v(u_0) \rightarrow -\infty \) as \( u_0 \rightarrow 0^+ \). In this case, we select a primitive
\[ v = \phi^{-1}(u) = \int_0^u f^{-1/(k-1)}(\theta) \, d\theta. \]

We choose \( I = (-\infty, \infty) \) and \( \phi : (-\infty, \infty) \rightarrow (0, \infty) \). This includes examples such as \( f(s) = s^{k-1}, \ (s + \log(s + 1))^{k-1} \) etc.

(d) We show that in parts (b) and (c), \( v \rightarrow \infty \) if \( u \rightarrow \infty \). Set \( \nu(s) = f^{1/(k-1)}(s) \).

Since \( \nu(s) \) is concave in \( (0, \infty) \), it is clear that, for a fixed \( \varepsilon > 0 \),
\[ \nu(s) \leq \nu(\varepsilon) + (s - \varepsilon)\nu'(\varepsilon), \ s \geq \varepsilon. \]

Using (3.8), we get that
\[ v(u) = v(\varepsilon) + \int_\varepsilon^u \frac{1}{\nu(s)} \, ds \geq v(\varepsilon) + \int_\varepsilon^u \frac{1}{\nu(\varepsilon) + (s - \varepsilon)\nu'(\varepsilon)} \, ds. \]

The claim holds.

(e) It is clear from (3.8) that
\[ \frac{\phi''(v)}{\phi'(v)} = \left( \frac{d}{ds} f^{1/(k-1)}(s) \right) \bigg|_{\phi(\varepsilon)}, \]
and $\phi''(v)/\phi'(v)$ is non-increasing in $v$ since $f^{1/(k-1)}$ is concave and $\phi(v)$ is increasing in $v$.

Suppose that there are constants $0 < \omega_1 \leq \omega_2 < \infty$ such that

$$\omega_1 \leq \phi''(v)/\phi'(v) \leq \omega_2.$$  \hfill (3.9)

Integrating from $s = 0$ to any $s > 0$, we get that,

$$\omega_1 s \leq f^{1/(k-1)}(s) - f^{1/(k-1)}(0) \leq \omega_2 s, \quad \forall s \geq 0.$$  

Since $f(0) \geq 0$, we get that, for some $\omega > 0$, $(\omega_1 s + \omega)^{k-1} \leq f(s) \leq (\omega_2 s + \omega)^{k-1}$, $\forall s \geq 0$.

If $\omega > 0$ then we use $v$ as in part (b). If $\omega = 0$ then we use part (c).

(f) The change of variable $u = \phi(v)$, as given by (3.8), transforms (3.7) into

$$H(Dv,D^2v + Z(v)Dv \otimes Dv) - v_t = 0 \quad \text{in } \Omega_T,$$

where $Z(v) = \phi''(v)/\phi'(v)$, see Lemma 2.3 in [6]. By part (e), $Z(v)$ is non-increasing in $v$ and the domain of $Z$ contains either $(0, \infty)$ or $(-\infty, \infty)$. \hfill \square

We now state a comparison principle for doubly nonlinear equations.

**Lemma 3.7.** Let $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $k > 1$ and $f : [0, \infty) \to [0, \infty)$ is a non-decreasing $C^1$ function such that $f^{1/(k-1)}$ is concave. Set $\Omega_T = \Omega \times (0,T)$ and $P_T$ to be the parabolic boundary of $\Omega_T$.

Let $u \in \text{usc}(\Omega_T)$, $v \in \text{lsc}(\Omega_T)$ and $u > 0$ and $v > 0$. Suppose that

$$H(Du,D^2u) - f(u)u_t \geq 0, \quad \text{in } \Omega_T,$$

$$H(Dv,D^2v) - f(v)v_t \leq 0, \quad \text{in } \Omega_T,$$

where $H$ satisfies conditions $A$, $B$ and $C$.

If $u \leq v$ on $P_T$ then $u \leq v$ in $\Omega_T$.

**Proof.** We employ Lemma 3.5 and Remark 3.6. Let $u$ and $v$ be as in the statement of the theorem. Set

$$F(s) = \int_{s}^{\hat{s}} f^{-1/(k-1)}(\theta)d\theta, \quad \forall \hat{s} \geq s \geq 0.$$  

We define $F(\hat{u},0) = \lim_{s \to 0^+} F(\hat{u},s)$, if it exists.

(i) Suppose that $F(1,0) < \infty$ then we define

$$\bar{u} = \phi^{-1}(u) = F(u,0) \quad \text{and} \quad \bar{v} = \phi^{-1}(v) = F(v,0).$$

By parts (a) and (b) of Remark 3.6 $\bar{u} > 0$ and $\bar{v} > 0$. Also, by part (f) of Remark 3.6

$$H(D\bar{u},D^2\bar{u} + Z(\bar{u})D\bar{u} \otimes D\bar{u}) - \bar{u}_t \geq 0$$

and

$$H(D\bar{v},D^2\bar{v} + Z(\bar{v})D\bar{v} \otimes D\bar{v}) - \bar{v}_t \leq 0,$$
in $\Omega_T$, where $Z(s) = \phi''(s)/\phi'(s)$ is non-increasing in $s$. Note that the domain of $Z$ contains $(0, \infty)$. Using Lemma 3.5 $\bar{u} \leq \bar{v}$ in $\Omega_T$ thus implying that $u \leq v$ in $\Omega_T$.

(ii) Suppose now that $F(1,0)$ is divergent, see part (c) of Remark 3.6. Fix a primitive

$$F(s) = \int^s f^{-1/(k-1)}(\theta)d\theta, \quad s > 0.$$  

Define $\bar{u} = \phi^{-1}(u) = F(u)$ and $\bar{v} = \phi^{-1}(v) = F(v)$. Then $-\infty < \bar{u}, \bar{v} < \infty$ and by parts (e) and (f) of Remark 3.6 we get in $\Omega_T$,

$$H(D\bar{u}, D^2\bar{u} + Z(\bar{u})D\bar{u} \otimes D\bar{u}) - \bar{u}_t \geq 0$$

and

$$H(D\bar{v}, D^2\bar{v} + Z(\bar{v})D\bar{v} \otimes D\bar{v}) - \bar{v}_t \leq 0,$$

where the domain of $Z$ is $(-\infty, \infty)$. Using Lemma 3.5 $\bar{u} \leq \bar{v}$ in $\Omega_T$ thus implying that $u \leq v$ in $\Omega_T$. □

4. Super-solutions

In this section, we construct super-solutions of (1.3) and these are used to prove Theorems 2.2, 2.4 and 2.6. We have divided our work into two parts. Part I addresses the case $k > 1$ (or $\gamma > 2$) and Part II discusses the case $k = 1$ or $\gamma = 2$. In each part, the work is further sub-divided to address various situations based on the values of $\sigma$. Since the auxiliary functions are non-negative, we assume that the domain of $Z$ is at least $(0, \infty)$, see discussion below.

Part I has four sub-parts: (i) $\sigma = 0$, (ii) $0 < \sigma < \gamma/2$, (iii) $\sigma = \gamma/2$ and (iv) $\sigma > \gamma/2$, and Part II has three sub-parts: (i) $0 \leq \sigma \leq 1$, (ii) $1 < \sigma \leq 2$, and (iii) $\sigma > 2$.

We recall from (2.5) that

$$(4.1) \quad \mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) := H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t,$$

where $\sigma \geq 0$, and $Z(s) \geq 0$ and is a non-increasing continuous function of $s$.

Let $m < \min(0, \inf_{\mathbb{R}^n} h, \inf_{\mathbb{R}^n} u)$, where $h$ is the initial data in (1.3) and $u$ is the given sub-solution. We assume that the domain of $Z$ is at least $[m, \infty)$.

Recall from (2.3) and (2.4) that $\Lambda_{\text{sup}} = \sup_{\lambda} \left( \max_{|e|=1} H(e, I + \lambda e \otimes e) \right)$. We set

$$(4.2) \quad \alpha = \sup_{[0,T]} |\chi(t)| \quad \text{and} \quad M = \max(\Lambda_{\text{sup}}, 1).$$

We also recall from (2.6) and (2.7) that

$$k = k_1 + 1, \quad \gamma = k_1 + 2 = k + 1, \quad \gamma \geq 2 \quad \text{and} \quad \gamma^* = \frac{\gamma}{\gamma - 2} \quad \text{if} \ \gamma > 2.$$  

Moreover, $\gamma = 2$ if and only if $k = 1(k_1 = 0)$.

Super-solutions:
**Part I** \((k > 1)\): Since \(\gamma > 2\), we see that

\[
\gamma^* - 1 = \frac{2}{\gamma - 2}, \quad \gamma^* - 2 = \frac{4 - \gamma}{\gamma - 2} > -1 \quad \text{and} \quad k\gamma^* - \gamma = \frac{\gamma}{\gamma - 2} = \gamma^*.
\]

We start with the case \(0 \leq \sigma \leq \gamma/2\) and first carry out some calculations that will hold for the entire interval \([0, \gamma/2]\). We will then discuss the cases \(\sigma = 0, 0 < \sigma < \gamma/2\) and \(\sigma = \gamma/2\) separately.

Let \(z \in \mathbb{R}^n\) be fixed, set \(r = |x - z|, \forall x \in \mathbb{R}^n\), and define

\[
w(x,t) = at + b(1+t)v(r), \quad v'(r) \geq 0, \quad \forall (r,t) \in \mathbb{R}_T^n,
\]

where \(a \geq 0\) and \(0 < b \leq 1\) are to be determined. We do this in each of the three cases listed above and also calculate \(\lim_{b \to 0} a\), wherever it is meaningful.

Using (3.3), (4.1) and (4.2), we get

\[
P_\sigma(t,w,w_t,Dw,D^2w) = \left[ b(1+t)v'(r) \right]^k_H \left( e, I + \left( \frac{rv''(r)}{v'(r)} - 1 + b(1+t)rv'(r)Z(w) \right) e \otimes e \right) + \chi(t)[b(1+t)v'(r)]^\sigma - a - bv(r)
\]

\[
\leq \frac{M[b(1+T)]^k v'(r)^k}{r} + \alpha [b(1+T)]^\sigma (v'(r))^\sigma - a - bv(r).
\]

We use the above inequality in both Parts I and II.

For Part I, we take \(v(r) = r^{\gamma^*}\). Using (4.3) and \(k = \gamma - 1\) in (4.5), we find that

\[
P_\sigma(t,w,w_t,Dw,D^2w) \leq M[b\gamma^*(1+T)]^k (r^{\gamma^* - 1})^k + \alpha [b\gamma^*(1+T)]^\sigma (r^{\gamma^* - 1})^\sigma - a - br^{\gamma^*}
\]

\[
\leq M [\gamma^*(1+T)]^k (b^{\gamma^*}) + \alpha [\gamma^*(1+T)]^\sigma (b^{\gamma^*/(\gamma - 2)}) - a - (br^{\gamma^*}).
\]

In order to write more compactly, we set

\[
E = M [\gamma^*(1+T)]^k \quad \text{and} \quad F = [\gamma^*(1+T)]^\sigma.
\]

Thus, (4.6) reads

\[
P_\sigma(t,w,w_t,Dw,D^2w) \leq E(b^{\gamma^*}) + \alpha F(b^{\gamma^*/(\gamma - 2)}) - a - (br^{\gamma^*}).
\]

**Sub-Part (i)** \((\sigma = 0)\): Taking \(\sigma = 0\) in (4.7), we get that \(F = 1\) and

\[
P_\sigma(t,w,w_t,Dw,D^2w) \leq b(Eb^{k-1} - 1)r^{\gamma^*} + \alpha - a.
\]
Select $a = \alpha$ and $0 < b < \min(1, E^{1-k})$. Clearly, $w(x, t)$ is a super-solution in $\mathbb{R}^n_T$ and
\[(4.8) \quad w(x, t) = \alpha t + b(1 + t)r^{\gamma^*}. \quad \Box \]

**Sub-Part (ii) (0 < \sigma < \gamma/2):** Since $\gamma^* = \gamma/(\gamma - 2)$, (4.7) yields that
\begin{align*}
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) & \leq Eb^k r^{\gamma^*} - br^{\gamma^*} + \alpha F b^{\sigma - 1} r^{2\sigma/(\gamma - 2)} - a \\
(4.9) & = \frac{br^{\gamma^*}}{r^{(\gamma - 2\sigma)/(\gamma - 2)}} \left( Eb^{k-1} - 1 + \frac{\alpha F b^{\sigma - 1}}{R^{(\gamma - 2\sigma)/(\gamma - 2)}} \right) - a.
\end{align*}

We choose
\begin{align*}
(4.10) \quad \left\{ \begin{array}{ll}
0 < b^{k-1} & < \min(1, (4E)^{-1}), \quad R = (4\alpha F b^{\sigma - 1})^{(\gamma - 2)/(\gamma - 2\sigma)}, \\
\text{and} \quad a & = Eb^k R^{\gamma^*} + \alpha F b^{\sigma} R^{2\sigma/(\gamma - 2)}.
\end{array} \right.
\end{align*}

The choice for $a$ shows that $w$ is a super-solution in $B_R(z) \times (0, T)$. In $r \geq R$, using $0 < \sigma < \gamma/2$ and the selections for $b$ and $R$, stated in (4.10), in (4.9), we get
\[Eb^{k-1} - 1 + \frac{\alpha F b^{\sigma - 1}}{R^{(\gamma - 2\sigma)/(\gamma - 2)}} \leq -\frac{3}{4} \quad \frac{1}{4} \quad = \frac{1}{2}.\]
Thus, $w$ is a super-solution in $\mathbb{R}^n_T$ for any $a$ and $b > 0$ satisfying the requirement in (4.10).

We now evaluate $\lim_{b \to 0} a$. If $\sigma \geq 1$, it is clear from (4.11) that $\lim_{b \to 0} a = 0$. Let $0 < \sigma < 1$. Using (4.10), $\gamma^* = \gamma/(\gamma - 2)$ and $k = \gamma - 1$, we obtain that, for some $K_1$ and $K_2$, independent of $b$,
\[b^k R^{\gamma^*} = K_1 b^{\gamma - 1} \left(b^{(\sigma - 1)(\gamma - 2)/(\gamma - 2\sigma)}\right)^{(\gamma - 2)/(\gamma - 2)} = K_1 b^{\gamma - (\sigma - 1)(\gamma - 2)/(\gamma - 2\sigma)}, \]
and
\[b^\sigma R^{2\sigma/(\gamma - 2)} = K_2 b^{\sigma} \left(b^{(\sigma - 1)(\gamma - 2)/(\gamma - 2\sigma)}\right)^{2\sigma/(\gamma - 2)} = K_2 b^{\sigma - 1}(\gamma - 2)/(\gamma - 2\sigma).\]

It is clear that
\[(4.11) \quad \lim_{b \to 0} a = 0. \quad \Box \]

**Sub-Part (iii) ($\sigma = \gamma/2$):** We modify $w$ as follows. Take
\[(4.12) \quad w(x, t) = b(t + 1)r^{\gamma^*}, \]
where $b > 0$ is to be determined. Note that
\[\gamma^* = \frac{\gamma}{\gamma - 2} = \frac{2\sigma}{\gamma - 2}.\]
Taking $a = 0$ in (4.7) and observing that $k > 1$ and $\gamma > 2$, we get

\[
\mathcal{P}_\sigma(t,w,w_t,Dw,D^2w) \leq Eb^k r^{\gamma^*} + \alpha Fb^\sigma r^{2\sigma/(\gamma-2)} - br^{\gamma^*} = Eb^k r^{\gamma^*} + \alpha Fb^\gamma r^{\gamma^*} - br^{\gamma^*} = br^{\gamma^*} \left( Eb^{k-1} + \alpha Fb^{(\gamma-2)/2} - 1 \right) \leq 0,
\]

if $0 < b \leq b_0$, for some $b_0 = b_0(\alpha,k,\gamma,E,F)$ chosen small enough. Thus,

(4.13) \hspace{1cm} w(x,t) = b(1+t)r^{\gamma^*}, \hspace{0.5cm} \forall 0 < b \leq b_0,

is a super-solution in $\mathbb{R}_+^n$. \ \Box

**Sub-part (iv) ($\sigma > \gamma/2$):** We use Remark 3.3 and take

(4.14) \hspace{1cm} w(x,t) = at + b(1+t)v(r),

where

\[
v(r) = \int_0^{r^{\gamma^*}} \frac{1}{1 + r^p} \, dr, \quad p = \frac{\gamma^* - \sigma^*}{\gamma^*}, \quad \gamma^* = \frac{\gamma}{\gamma - 2} \quad \text{and} \quad \sigma^* = \frac{\sigma}{\sigma - 1}.
\]

Here $a > 0$ and $0 < b \leq 1$ are to be determined. Note that $v(r)$ grows like $r^{\gamma^*}$ near $r = 0$ and like $r^{\sigma^*}$ for large $r$.

Recall (4.5) i.e.,

(4.15) \hspace{1cm} \mathcal{P}_\sigma(t,w,w_t,Dw,D^2w) \leq \frac{M[b(1+T)]^k v'(r)^k}{r} + \alpha [b(1+T)]^\sigma v'(r)^\sigma - a - bv(r).

We use parts (ii)-(viii) of Remark 3.3 $k = \gamma - 1$ and $(\sigma^* - 1)\sigma = \sigma^*$. Note that

\[
(v'(r))^\sigma \leq (\gamma^*)^\sigma \min \left( r^{\sigma^* - 1}, \, r^{\gamma^* - 1} \right)^\sigma = (\gamma^*)^\sigma \min \left( r^{\sigma^*}, \, r^{2\sigma/(\gamma - 2)} \right),
\]

and

\[
\left( \frac{v'(r)}{r} \right)^k \leq \min(\gamma^*)^k \left( r^{k\sigma^* - \gamma}, \, r^{k\gamma^* - \gamma} \right) = (\gamma^*)^k \min \left( r^{(\gamma - \sigma)/(\sigma - 1)}, \, r^{\gamma^*} \right).
\]

Using the above in (4.15) and recalling the definitions of $E$, $F$ (see the line following (4.6)) we get that

(4.16) \hspace{1cm} \mathcal{P}_\sigma(t,w,w_t,Dw,D^2w) \leq Eb^k r^{(\gamma - \sigma)/(\sigma - 1)} + \alpha Fb^\sigma r^{\sigma^*} - a - bv(r).

A lower bound for $v(r)$ is obtained by setting $R = 1$ in Remark 3.3(iv) and ignoring $v(1)$. Taking $r \geq 1$, (4.16) yields that

(4.17) \hspace{1cm} \mathcal{P}_\sigma(t,w,w_t,Dw,D^2w) \leq Eb^k r^{(\gamma - \sigma)/(\sigma - 1)} + \alpha Fb^\sigma r^{\sigma^*} - a - \frac{b\gamma^* (r^{\sigma^* - 1})}{2\sigma^*} = Eb^k r^{(\gamma - \sigma)/(\sigma - 1)} + \alpha Fb^\sigma r^{\sigma^*} + \frac{b\gamma^*}{2\sigma^*} - a - \frac{b\gamma^* r^{\sigma^*}}{2\sigma^*},

where we have used that $1 - p = \sigma^*/\gamma^*$. 
We select

\begin{equation}
(4.18) \quad a = Eb^k + \alpha Fb^\sigma + \frac{b\gamma^*}{\sigma^*},
\end{equation}

From (4.17) and (4.18), it follows that \( w \) is a super-solution in \( B_1(o) \times [0, T] \).

Since \( r^{(\gamma-\sigma)/(\sigma-1)} \leq r^{\sigma^*} \), in \( r \geq 1 \), using (4.18) in (4.17) implies that

\[
\mathcal{P}_\sigma(t, w_t, Dw, D^2w) \leq Eb^k r^{\sigma^*} + \alpha Fb^\sigma r^{\sigma^*} + \frac{b\gamma^*}{2\sigma^*} - a - \frac{br^{\sigma^*}}{2\sigma^*} \leq b r^{\sigma^*} \left( Eb^{k-1} + \alpha Fb^{\sigma-1} - \frac{\gamma^*}{2\sigma^*} \right) \leq 0,
\]

if we select \( 0 \leq b \leq b_0 \), where \( b_0 \) depends only on \( \alpha, \gamma, \sigma, E \) and \( F \), and is chosen small enough.

Thus, \( w \) is super-solution in \( \mathbb{R}^n_T \) and

\begin{equation}
(4.19) \quad \lim_{b \to 0} a = 0. \quad \square
\end{equation}

**Part II** \((k = 1)\): In this case, \( \gamma = 2 \) and \( k_1 = 0 \).

By Remark 2.1(iv), \( H(q, X) = H(X), \forall (q, X) \in \mathbb{R}^n \times S^{n \times n} \). Thus, we work with

\[
H(D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^{\sigma} - u_t \geq 0, \quad \text{in} \quad \mathbb{R}^n_T \quad \text{with} \quad u(x, 0) \leq h(x), \forall x \in \mathbb{R}^n.
\]

We treat separately the three possibilities: (i) \( 0 \leq \sigma \leq 1 \), (ii) \( 1 < \sigma \leq 2 \) and (iii) \( 2 < \sigma < \infty \).

**Sub-Part (i) \((0 \leq \sigma \leq 1)\):** Take

\begin{equation}
(4.20) \quad w(x, t) = at + b(1 + t)v(r), \quad \forall (x, t) \in \mathbb{R}^n_T,
\end{equation}

where \( a \geq 0 \) and \( 0 < b \leq 1 \) are to be determined.

(a) \((\sigma = 0)\): We choose

\[
v(r) = e^{cr^2}.
\]

where \( c > 0 \) is to be determined. We note the following elementary facts.

\[
v'(r) = 2ce^{cr^2}, \quad \frac{v'(r)}{r} = 2ce^{cr^2}, \quad \text{and} \quad \frac{rv''(r)}{v'(r)} = 1 + 2cr^2.
\]

Using these in (4.5) and using \( \sigma = 0 \), we get

\[
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq b(1 + T)M \left( \frac{v'(r)}{r} \right) + \alpha - a - bv(r)
\]

\[
= 2bc(1 + T)M e^{cr^2} + \alpha - a - be^{cr^2}.
\]

Set \( a = \alpha, \bar{E} = 2(1 + T)M \) and \( c = 1/\bar{E} \) to obtain \( \mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq 0 \) in \( \mathbb{R}^n_T \).
Thus,
\[ w(x, t) = \alpha t + b(1 + t)e^{r^2/E}, \quad \forall (x, t) \in \mathbb{R}_T^n, \]
is a super-solution in $\mathbb{R}_T^n$ for any $b > 0$. Moreover,
\begin{equation}
\lim_{b \to 0} w(x, t) = \alpha t. \quad \Box
\end{equation}

(b) $(0 < \sigma \leq 1)$: For $a > 0$, $0 < b \leq 1$ and $c > 0$ (to be determined), we define
\begin{equation}
w(x, t) = at + b(1 + t)v(r), \quad \text{in } \mathbb{R}_T^n, \quad \text{where } v(r) = e^{cr} - (1 + cr).
\end{equation}

Thus,
\[ v'(r) = c(e^{cr} - 1), \quad c^2 \leq \frac{v'(r)}{r} \leq c^2e^{cr}, \quad \text{and} \quad 1 \leq \frac{rv''(r)}{v'(r)} \leq \frac{c\max(1, cr)}{e - 1}. \]
In the last estimate, for $0 < \theta < 1$ we used that $\theta e^\theta/(e^\theta - 1)$ is increasing and for $1 < \theta$, we used that $e^\theta/(e^\theta - 1)$ is decreasing.

Applying the above to (4.5), we obtain
\begin{equation}
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w)
\end{equation}
\begin{equation}
\leq b(1 + T)M \left( \frac{v'(r)}{r} \right) + \alpha [b(1 + T)v'(r)]^\sigma - a - bv(r)
\end{equation}
\begin{equation}
\leq bc^2(1 + T)M e^{cr} + \alpha [bc(1 + T) (e^{cr} - 1)]^\sigma - a - b(e^{cr} - 1 - cr).
\end{equation}
Set $E = (1 + T)M$ and $F = \alpha(1 + T)^\sigma$. A rearrangement of the above leads to
\begin{equation}
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq b(1 + cr) + (c^2E) be^{cr} + (c^\sigma F) (be^{cr})^\sigma - be^{cr} - a.
\end{equation}

Applying Young’s inequality $(be^{cr})^\sigma \leq (1 - \sigma) + \sigma be^{cr}$, (4.24) implies that
\begin{equation}
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq b(1 + cr) + e^{cr} (c^2E + \sigma c^\sigma F - 1) + (1 - \sigma)c^\sigma F - a
\end{equation}
\begin{equation}
\leq [(1 - \sigma)c^\sigma F - a] + b [(1 + cr) + e^{cr} (c^2E + \sigma c^\sigma F - 1)].
\end{equation}

Select $c > 0$ such that $c^2E + \sigma c^\sigma F = 1 - \varepsilon$, for a fixed small $0 < \varepsilon < 1$. Hence,
\begin{equation}
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq [(1 - \sigma)c^\sigma F - a] + b [(1 + cr) - \varepsilon e^{cr}].
\end{equation}

The maximum of the function $1 + cr - \varepsilon e^{cr}$ occurs at $r_0 = c^{-1}\log(1/\varepsilon)$ and the maximum value is $\log(1/\varepsilon)$. Select
\[ a = b\log(1/\varepsilon) + (1 - \sigma)c^\sigma F. \]

Using the choice for $a$ in (4.25), we get that $\mathcal{P}_\sigma(t, w_t, Dw, D^2w) \leq 0$, in $\mathbb{R}_T^n$. Thus, $w$ is a super-solution in $\mathbb{R}_T^n$ and
\begin{equation}
\lim_{b \to 0} a = (1 - \sigma)c^\sigma F, \quad 0 < \sigma \leq 1. \quad \Box
\end{equation}
Observe that if \( \sigma = 0 \) then \( \lim_{b \to 0} a = \alpha \). While this agrees with part (a), the growth rate allowed in part (a) is greater. Also, if we take \( \sigma = 1 \), \( \lim_{b \to 0} a = 0 \).

**Sub-Part (iii) \( 1 < \sigma \leq 2 \):** For \( a > 0 \) and \( 0 < b \leq 1 \) (to be determined), we select

\[
(4.27) \quad w(x,t) = at + b(1+t)r^{\sigma^*}, \quad \forall (x,t) \in \mathbb{R}^n_T, \text{ where } \sigma^* = \frac{\sigma}{\sigma - 1}.
\]

Note that \( \sigma^* \geq 2 \). Setting \( v(r) = r^{\sigma^*} \), we find that

\[
\frac{v'(r)}{r} = \sigma^* r^{\sigma^*-2} = \sigma^* r^{2(\sigma-\sigma)/(\sigma-1)}, \quad v(r)^\sigma = (\sigma^*)^\sigma r^{\sigma^*} \quad \text{and} \quad \frac{rv''(r)}{v'(r)} = \sigma^* - 1.
\]

Using the above in (4.23) or (4.5) and recalling the definitions of \( \tilde{E} \) and \( \tilde{F} \) (see Sub-Part (ii)) we obtain that

\[
(4.28) \quad \mathcal{P}_\sigma(t,w,w_t,Dw,D^2w) \leq \tilde{E} \left( \frac{bv'(r)}{r} \right) + \tilde{F}(bv'(r))^\sigma - a - bv(r)
\]

Choose \[ R = \sqrt{4\sigma^*\tilde{E}}, \quad 0 < b < \left( \frac{1}{4\sigma^*\tilde{F}} \right)^{1/(\sigma-1)} \quad \text{and} \quad a = (\sigma^*\tilde{E})bR^{\sigma^*-2} + (\sigma^*\tilde{F})b^\sigma R^{\sigma^*}.
\]

Employing the above values in (4.28) and noting that \( \sigma^* \geq 2 \), we see that \( w \) is super-solution in \([0,R] \times [0,T]\). In \( r \geq R \),

\[
\mathcal{P}_\sigma(t,w,w_t,Dw,D^2w) \leq (\sigma^*\tilde{E})bR^{\sigma^*-2} + (\sigma^*\tilde{F})b^\sigma R^{\sigma^*} - a - bR^{\sigma^*}
\]

Using the values of \( R \) and \( b \), it is clear that \( w \) is super-solution in \( \mathbb{R}^n_T \). Moreover,

\[
(4.29) \quad \lim_{b \to 0} a = 0. \quad \square
\]

**Sub-Part (iv) \( 2 < \sigma < \infty \):** We choose

\[
(4.30) \quad w(x,t) = at + b(1+t)v(r), \quad \forall (x,t) \in \mathbb{R}^n_T,
\]

where

\[
v(r) = \int_0^{r^2} \frac{1}{1 + \tau^p} \, d\tau \quad \text{with} \quad p = 1 - \frac{\sigma^*}{2} = \frac{\sigma - 2}{2(\sigma - 1)}.
\]

Observe that \( 2(1 - p) = \sigma^* \) and also, that \( v(r) \) is like \( r^2 \) near \( r = 0 \) and like \( r^{\sigma^*} \) for large \( r \).
In Lemma 3.2 we set $\beta = 2$ and $\bar{\beta} = \sigma^*$. Thus, parts (iv), (v) and (vi) yield

\[
(iv) \quad \frac{1}{\sigma^*} \leq \frac{v(r) - v(1)}{r^{\sigma^*} - 1} \leq \frac{2}{\sigma^*}, \quad \forall r \geq 1,
\]

\[
(v) \quad v'(r) \leq 2 \min \left( r^{1/(\sigma - 1)}, r \right) \quad \text{and} \quad (vi) \quad \frac{v'(r)}{r} \leq 2.
\]

Using the above values and expressions in (4.23) or (4.5) and recalling $\bar{E}$ and $\bar{F}$, we get

\[
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq b(1 + T)M \left( \frac{v'(r)}{r} \right) + \left[ \alpha(b(1 + T))^\sigma \right] (v'(r))^\sigma - a - bv(r)
\]

\[
(4.31) \quad \leq 2b\bar{E} + (2b)^\sigma \bar{F} \min \left( r^{\sigma^*}, r^\sigma \right) - a - bv(r).
\]

We choose

\[
a = 2b\bar{E} + (2b)^\sigma \bar{F} + \frac{b}{\sigma^*} \quad \text{and} \quad 0 < b < \left( \frac{1}{2^{\sigma^*} F} \right)^{1/(\sigma - 1)}.
\]

Using the above, $w$ is a super-solution in $0 \leq r \leq 1$ and $0 \leq t \leq T$.

In $r \geq 1$, we employ values of $a$, $b$ and the bound $v(r) \geq (r^{\sigma^*} - 1)/\sigma^*$ in (4.31) to find that

\[
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq 2b\bar{E} + (2b)^\sigma \bar{F} \min \left( r^{\sigma^*}, r^\sigma \right) - a - b \left( \frac{r^{\sigma^*} - 1}{\sigma^*} \right)
\]

\[
\leq (2b)^\sigma \bar{F} r^{\sigma^*} - \frac{br^{\sigma^*}}{\sigma^*} \leq br^{\sigma^*} \left( 2^{\sigma^*} - 1 \right) \leq 0.
\]

Thus, $w$ is super-solution in $\mathbb{R}^p_T$. Moreover,

\[
(4.32) \quad \lim_{b \to 0} a = 0. \quad \Box
\]

We summarize: select $w(x, t) = at + b(1 + t)v(r)$ where $v(r)$ is as follows

\[
(4.33)
\]

(I) $k > 1$: $v(r) = \begin{cases} r^{\gamma^*}, & 0 \leq \sigma \leq \gamma/2, \\ \int_0^{r^{\gamma^*}} (1 + \tau^{p})^{-1}d\tau, & \sigma > \gamma/2, \end{cases}$

\[
\lim_{b \to 0} a = \begin{cases} \alpha, & \sigma = 0, \\ 0, & \sigma > 0, \end{cases}
\]

where $p = 1 - (\sigma^*/\gamma^*)$,

(II) $k = 1$: $v(r) = \begin{cases} e^{cr^2}, & \sigma = 0, \\ e^{cr} - 1 - cr, & 0 < \sigma \leq 1, \\ r^{\sigma^*}, & 1 < \sigma \leq 2, \end{cases}$

\[
\lim_{b \to 0} a = \begin{cases} \alpha, & \sigma = 0, \\ (1 - \sigma)e^{\sigma^* \bar{F}}, & 0 < \sigma \leq 1, \\ 0, & \sigma > 1. \end{cases}
\]

where $p = 1 - (\sigma^*/2)$.

See (4.8), (4.11) and (4.19), (4.21), (4.26), (4.29) and (4.32). Recall that $v(r)$ grows like $r^{\sigma^*}$ in (I) (for $\sigma > \gamma/2$) and in (II) (for $\sigma > 2$).
5. Sub-solutions

The work in this section is quite similar to that in Section 4. Although, $H$ is not assumed to be odd in $X$, the auxiliary functions used in Section 4 continue to apply here. We will not repeat the calculations done in Section 4, instead, provide an outline as to how to use them to obtain sub-solutions. We require that the domain for $Z$ be $(-\infty, \infty)$.

We use functions of the type $w(x,t) = -[at + b(1 + t)v(r)]$, where $a > 0$ and $b > 0$, small, $v(r) > 0$ and $v'(r) \geq 0$. Recalling (3.5), we see that

$$P_{\sigma}(t,w,w_t,Dw,D^2w)$$

$$= \frac{[b(1+t)v'(r)]^k}{r}H\left(e, \left(1 - \frac{rv''(r)}{v'(r)} + b(1 + t)rZ(w)v'(r)\right)e \otimes e - I\right)$$

$$+ \chi(t)[b(1+t)v'(r)]^\sigma + a + bv(r).$$

(5.1)

We set

$$\alpha = \sup_{[0,T]} |\chi(t)| \quad \text{and} \quad N = \inf_{\lambda} \left(\min_{|e|=1} H(e, \lambda e \otimes e - I)\right).$$

We note that $N \leq 0$ since $H(e,-I) \leq 0$, see Condition C in Section 2.

As done in Section 4, we take $v(r)$ to be either a power of $r$ (power greater than 1) or $e^{ar^2}$ or $e^{cr}$. For the exponential type functions, since $1 - (rv''(r))/v'(r)$ could become unbounded, a lower bound on $H$ is needed. However, $1 - (rv''(r))/v'(r)$ is bounded from below if $v(r)$ is a power of $r$ and the bound depends on the power. Since $H$ is continuous and non-decreasing in $X$, we get a natural lower bound depending on the power of $r$. We use $N$ to denote the lower bound in both situations.

With the above discussion in mind, (5.1) implies

$$P_{\sigma}(t,w,w_t,Dw,D^2w) \geq \frac{[b(1+T)v'(r)]^k}{r}N - \alpha[b(1+T)v'(r)]^\sigma + a + bv(r)$$

$$= -\left(\frac{[b(1+T)v'(r)]^k}{r}|N| + \alpha[b(1+T)v'(r)]^\sigma - a - bv(r)\right).$$

(5.2)

We now use auxiliary functions $v(r)$ that are similar to those in Section 4. The goal is to choose $a \geq 0$ and $0 < b < 1$ such that the expression in (5.2) is non-positive i.e,

$$\frac{[b(1+T)v'(r)]^k}{r}|N| + \alpha[b(1+T)v'(r)]^\sigma - a - bv(r) \leq 0.$$

The analysis is almost identical to Section 4. We list the choice for $w(x,t)$ for the various values of $\sigma$. 
**Part I** $k > 1$: Recall that $\gamma > 2$ and $\gamma^* = \gamma/(\gamma - 2)$. Set $r = |x - z|$, for some fixed $z \in \mathbb{R}^n$, and take

$$ w(x, t) = \begin{cases} -at - b(1 + t)r^\gamma^*, & 0 \leq \sigma < \gamma/2, \\ -b(1 + t)r^\gamma^*, & \sigma = \gamma/2, \\ -at - b(1 + t)v(r), & \sigma > \gamma/2, \end{cases} $$

where $v(r) = \int_0^{r^\gamma^*} \frac{1}{1 + \tau^p} \, d\tau$ with $\sigma^* = \frac{\sigma}{\sigma - 1}$, $p = 1 - \frac{\sigma^*}{\gamma^*} = \frac{2\sigma - \gamma}{\gamma(\sigma - 1)}$.

It is easy to check that (see Remark 3.3) that

$$ 1 - \frac{rv''(r)}{v'(r)} = \frac{2 - \gamma^* + (2 - \sigma^*)r^\gamma^*}{1 + r^\gamma^*} \geq 2 - \sigma^* > -\infty. $$

We choose $N$ to be an appropriate lower bound for $H$, see the right hand side of (5.1). Thus, (5.2) holds without any restrictions on $\inf_\lambda [\min_{|e| = 1} H(e, \lambda e \otimes e - I)]$. Moreover, from (4.33),

$$ (5.3) \quad \lim_{b \to 0} a = \begin{cases} \alpha, & \sigma = 0, \\ 0, & \sigma > 0. \end{cases} $$

**Part II** $k = 1$: In this case, $\gamma = 2$ and $k_1 = 0$. Set $\sigma^* = \sigma/(\sigma - 1)$. We choose $a \geq 0$, $0 < b < 1$ and $c > 0$ such that (5.2) in non-positive. We select

$$ w(x, t) = \begin{cases} -a - b(1 + t)e^{cr^2}, & \sigma = 0, \\ -at - b(1 + t)(e^{cr} - 1 - cr), & 0 < \sigma \leq 1, \\ -at - b(1 + t)r^{\sigma^*}, & 1 < \sigma \leq 2, \\ -at - b(1 + t)v(r), & 2 < \sigma < \infty, \end{cases} $$

where $v(r) = \int_0^{r^2} \frac{1}{1 + \tau^p} \, d\tau$ with $p = 1 - \frac{\sigma^*}{2} = \frac{\sigma - 2}{2(\sigma - 1)}$.

If $0 \leq \sigma \leq 1$ then $1 - rv''(r)/v'(r) \leq 0$ becomes unbounded as $r \to \infty$. Thus, we impose that $|\inf_\lambda [\min_{|e| = 1} H(e, \lambda e \otimes e - I)]| < \infty$. For $\sigma > 1$, however, no such requirement is made.

Moreover, from (4.33),

$$ (5.4) \quad \lim_{b \to 0} a = \begin{cases} \alpha, & \sigma = 0, \\ (1 - \sigma)\alpha(c(1 + T))^{\sigma}, & 0 < \sigma \leq 1, \\ 0, & \sigma > 1. \end{cases} $$
6. SOME SPECIAL CASES

In this section we consider some special cases. Recall that

\[ (6.1) \quad \mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) = H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t. \]

As before, set

\[ N = \inf_\lambda \left( \min_{|e|=1} H(e, \lambda e \otimes e - I) \right). \]

We discuss the following three cases.

**Case (i):** \( k \geq 1 \) and \( \chi \equiv 0 \). The equations reads

\[ H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t = 0, \text{ in } \mathbb{R}^n_T, v > 0, \text{ with } v(x, 0) = h(x), \forall x \in \mathbb{R}^n. \]

As observed in \((1.1), (1.2)\) and part (f) of Remark 3.6, this applies to the doubly nonlinear case by employing a change of variables. Moreover, as noted in Remark 3.6 and Lemma 3.7, the convergence or the divergence of the integral

\[ I = \int_0^1 f^{-1/(k-1)}(\theta) \, d\theta, \quad k > 1, \]

determines the domain of \( Z \). In particular, if \( I < \infty \) then the domain of \( Z \) is \((0, \infty)\) or \([0, \infty)\), and if \( I = \infty \) then the domain is \((-\infty, \infty)\).

The super-solutions in Section 4, (in particular, the one in Sub-Part (i) or Part I) being positive, are also super-solutions of \((6.1)\) regardless the domain of \( Z \). However, the domain of \( Z \) needs to be stated more precisely for sub-solutions. If the integral \( I \) diverges then the work in Section 5, in particular, Part I applies since the domain of \( Z \) is \((-\infty, \infty)\). If \( I \) converges then the domain is \((0, \infty)\) or \([0, \infty)\) and a different sub-solution needs to be calculated. We do this in this section.

We also include here the case \( k = 1 \) where \( Z \) is defined on \((0, \infty)\) or \([0, \infty)\). The two Part II’s in Sections 4 and 5 address the case where the domain is \((-\infty, \infty)\).

The next two cases bring out the influence of the sign of \( \chi \).

**Case (ii):** We discuss super-solutions in the case \( \chi \leq 0 \) and we derive a maximum principle.

**Case (iii):** We study sub-solutions for \( \chi \geq 0 \) and this leads to a minimum principle. The cases (ii) and (iii) are related.

Let \( z \in \mathbb{R}^n \) be a fixed and set \( r = |x - z|, \forall x \in \mathbb{R}^n. \)

We begin with Case (i).
Case (i-1): We take \( k > 1, \chi \equiv 0, \sigma = 0 \) and assume that the domain of \( Z \) contains \((0, \infty)\). Thus, the equation reads

\[
P_0(t, w, w_t, Dw, D^2w) = H(Dw, D^2w + Z(w) Dw \otimes Dw) - w_t.
\]

Since our goal is to construct positive sub-solutions \( w \), it suffices to find a \( w \) such that \( H(Dw, D^2w) - w_t \geq 0 \) since ellipticity \((Z \geq 0)\) implies the desired conclusion.

Let \( R > 0 \) and set \( B^R_T = B_R(z) \times (0, T) \). We construct a sub-solution \( w \) for any large \( R \). More precisely, \( w \geq 0 \) solves

\[
H(Dw, D^2w) - w_t \geq 0, \text{ in } B^R_T \text{ and } w(x, 0) \leq g(x), \forall x \in B_R(z).
\]

We define

\[
w(x, t) = \psi(t)v(r) = \frac{D \left[ R^{(k+1)/k} - r^{(k+1)/k} \right]^{k/(k-1)}}{(E + t)^{1/(k-1)}}, \quad \forall (x, t) \in B^R_T,
\]

where \( D, E > 0 \) are to be determined. One recalls from (3.4) that if \( w = \psi(t)v(r) \), with \( w_r \leq 0 \), then

\[
H(Dw, D^2w) - w_t = \frac{(|\psi(t)v'(r)|)^k}{r}H \left( e, \left( 1 - \frac{rv''(r)}{v'(r)} \right) e \otimes e - I \right) - v(r)\psi'(t)
\]

(6.3)

\[
\geq - \frac{|N|(|\psi(t)v'(r)|)^k}{r} - v(r)\psi'(t).
\]

Using the expression for \( w \) and setting \( c_k = [(k + 1)/(k - 1)]^k \), we see that

\[
-v(r)\psi'(t) - \frac{|N|(|\psi(t)v'(r)|)^k}{r} = \frac{Dv(r)}{(k-1)(E + t)^{k/(k-1)}} - \frac{c_k|N|D^k v(r)}{(E + t)^{k/(k-1)}}
\]

\[
= \frac{Dv(r)}{(k-1)(E + t)^{k/(k-1)}} \left[ 1 - (k-1)c_k|N|D^{k-1} \right].
\]

Choosing

\[
D = \left( \frac{1}{c_k(k-1)|N|} \right)^{1/(k-1)},
\]

and using the above in (6.3), we get a sub-solution \( w \geq 0 \) in \( B^R_T \) such that \( w(R, t) = 0 \).

Next, we calculate \( E \) by requiring that

\[
w(z, 0) = w(0, 0) = \frac{DR^{(k+1)/(k-1)}}{E^{1/(k-1)}} = \inf_x h(x) = \mu.
\]

Thus,

\[
w = \frac{DR^{(k+1)/(k-1)}}{E^{1/(k-1)}} \left[ 1 - (r/R)^{(k+1)/k} \right]^{k/(k-1)} \frac{k}{1 + (t/E)^{1/(k-1)}} = \mu \left[ 1 - (r/R)^{(k+1)/k} \right]^{k/(k-1)}(1 + (t/E))^{1/(k-1)}.
\]

Note that \( E = O(R^{k+1}) \) and

\[
w(z, t) = w(0, t) = \frac{\mu}{(1 + (t/E))^{1/(k-1)}} \to \mu \text{ as } R \to \infty.
\]
We record that in $0 \leq r < R$, 

$$w(x, t) = \mu \left[ 1 - \frac{(r/R)^{k+1}}{(1 + (t/E))^{1/(k-1)}} \right]^{1/(k-1)}$$

where $E = \frac{R^{k+1}}{c_k R^{k-1} (k-1)|N|}$. 

**Case (i-2):** We now study $k = 1$. We take $w(x, t) = De^{-E^2 e^{-Ft}}$ and recall (6.3).

We get

$$-|N| \psi(t) \frac{|v'(r)|}{r} - v(r) \psi'(t)$$

$$= DF e^{-E^2 e^{-Ft}} - |N|2 DE e^{-E^2 e^{-Ft}} = De^{-E^2 e^{-Ft}} (F - 2|N|E)$$

We take $F = 2|N|E$ and $D = \mu$ and obtain a sub-solution

$$w(x, t) = \mu e^{-E^2 e^{-2|N|E t}}, \forall E > 0.$$

It is clear that $W \to \mu$ as $E \to 0$. □

**Case (ii):** We consider

$$P_\sigma(t, w, w_t, Dw, D^2 w) = H(Dw, D^2 w + Z(w) Dw \otimes Dw) + \chi(t)[Dw]^\sigma - w_t$$

where $\chi \leq 0$. We set

$$\hat{\alpha} = \sup_{(0, T)} \chi(t)$$

and assume that $\hat{\alpha} < 0$. We further assume that

$$k \geq 1 \text{ and } \sigma \geq k.$$ 

Our goal here is to construct super-solutions $w \geq 0$, i.e., $P_\sigma(t, w, w_t, Dw, D^2 w) \leq 0$ in cylinders $B^R_T$.

Selecting $w(x, t) = at + (1+t)v(r)$, $\nu' \geq 0$, setting $M = \sup_{\lambda} \left[ \max_{|\nu| = 1} H(e, I + \lambda e \otimes e) \right]$ and recalling (3.3) and (6.6) we find that

$$P_\sigma(t, w, w_t, Dw, D^2 w)$$

$$\leq \left[ \frac{(1 + t)\nu'(r)}{r} \right]^k \left[ M - |\hat{\alpha}|[(1 + t)\nu'(r)]^{\sigma-k} r \right] - a - v(r).$$

For $R > 0$, set

$$v(r) = (R^2 - r^2)^{-1}, \quad 0 \leq r < R.$$

Since

$$v'(r) = (2r)(R^2 - r^2)^{-2},$$
\[(6.8)\) yields that, in \(0 \leq r < R\),
\[P_{\sigma}(t, w, w_t, Dw, D^2w) \leq \frac{[2(1 + t)]^{k}r^{k-1}}{(R^2 - r^2)^{2k}} \left( M - |\hat{\alpha}| \left( \frac{2(1 + t)}{(R^2 - r^2)^2} \right)^{\sigma - k} r^{\sigma - k + 1} \right) - a \]
\[
(6.9)
\]

**Sub-Case (ii-1) \((\sigma = k):\)** Set \(r^* = M/\hat{\alpha}\) and take \(R > r^*\). Then \((6.9)\) yields that
\[(6.10)\]
\[P_{\sigma}(t, w, w_t, Dw, D^2w) \leq \left( \frac{2(1 + t)}{(R^2 - r^2)^{2}} \right)^{k} (M - |\hat{\alpha}| r^{k-1} - a - \frac{1}{R^2 - r^2}).
\]
Select \(a = M \left( \frac{2(1 + T)}{(R^2 - (r^*)^2)^{2}} \right)^{k} (r^*)^{k-1}\).

With this choice \((6.10)\) shows that \(w\) is a super-solution in \(B_R^T\). Thus,
\[(6.11)\]
\[w(x, t) = at + (1 + t)v(r) \text{ and } \lim_{R \to \infty} a = 0. \quad \Box\]

**Case (ii-2) \((\sigma > k):\)** From \((6.9)\) we have that
\[P_{\sigma}(t, w, w_t, Dw, D^2w)
\leq \left( \frac{2(1 + t)}{(R^2 - r^2)^{2}} \right)^{k} \left( M - |\hat{\alpha}| \left( \frac{2}{(R^2 - r^2)^2} \right)^{\sigma - k} r^{\sigma - k + 1} \right) r^{k-1} - a.
\]
Since the function \(f(r) = r^{\sigma - k + 1} (R^2 - r^2)^{2(\sigma - k)}\) is continuous and increasing in \(0 \leq r < R\), \(f(0) = 0\) and \(f(r) \to \infty\), as \(r \to R\), there is an \(r^* = r^*(R) < R\) such that \(2^{\sigma - k} |\hat{\alpha}| f(r^*) = M\). Choose
\[a = M \left( \frac{2(1 + T)}{(R^2 - (r^*)^2)^{2}} \right)^{k} (r^*)^{k-1}.
\]
Clearly, \(w\) is super-solution in \(0 \leq r < R\).

Next, we recall that
\[f(r^*) = \frac{(r^*)^{\sigma - k + 1}}{|R^2 - (r^*)^2|^{2(\sigma - k)}} = \frac{M}{2^{\sigma - k} |\hat{\alpha}|}.
\]
Clearly, \(r^* \to \infty\), as \(R \to \infty\). For calculating \(\lim_{R \to \infty} a\), we use the formula for \(f(r^*)\) and observe that for an appropriate constant \(D\), we have
\[\frac{(r^*)^{k-1}}{|R^2 - (r^*)^2|^{2k}} = \frac{D(r^*)^{k-1}}{(r^*)^{k(\sigma - k + 1)/(\sigma - k)}} = \frac{D}{(r^*)^{1+k/(\sigma - k)}}.
\]
Thus,
\[(6.12)\]
\[\lim_{R \to \infty} a = 0. \quad \Box\]
Case (iii) Sub-solution: We construct a function \( w(x,t) \) such that
\[
P_\sigma(t,w,w_t,Dw,D^2w) = H(Dw,D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \geq 0, \quad \text{in } \mathbb{R}^n_T,
\]
where \( \chi \geq 0 \). Set
\[
N = \inf_{\lambda} \left[ \min_{|e|=1} H(e, \lambda e \otimes e - I) \right] \quad \text{and} \quad \hat{\alpha} = \inf \chi(t).
\]
Select \( w(x,t) = -at - (1 + t)v(r), \ v' \geq 0, \) and recall (3.5):
\[
P_\sigma(t,w,w_t,Dw,D^2w) \geq - \left[ \frac{[(1 + t)v'(r)]^k}{r} |N| - \chi(t)[(1 + t)v'(r)]^\sigma - a - v(r) \right].
\]

Defining
\[
v(r) = \frac{1}{R^2 - r^2}, \quad \forall \ 0 \leq r < R,
\]
and proceeding as in Case (ii), one can construct a sub-solution \( w \) with the same properties. \( \square \)

Remark 6.1. We point out that, except for Case (i) in this section all the auxiliary functions in this work are of the kind \( w(x,t) = at + b(1 + t)v(r) \), where \( v(r) \) is an appropriately chosen function, \( r = |x - z|, \forall x \in \mathbb{R}^n \) and \( z \in \mathbb{R}^n \) is fixed.

Case (i) is used in proving the minimum principle in Theorem. For \( k > 1 \) we utilize \( w \) in (6.4) and for \( k = 1 \) we use \( w \) in (6.5). Note that \( k > 1 \) requires no lower bound except \( u > 0 \), however, for \( k = 1 \) we assume a lower bound.

Case (ii) implies a maximum principle without any imposition of an upper bound. Case (iii) leads to a minimum principle without requiring any lower bound.

We provide details in Section 7. \( \square \)

7. Proofs of the main results

Assume that \( -\infty < \inf_{\mathbb{R}^n} g \leq \sup_{\mathbb{R}^n} g < \infty \) and set
\[
\mu = \inf_{\mathbb{R}^n} g \quad \text{and} \quad \nu = \sup_{\mathbb{R}^n} g.
\]

Proofs of Theorems 2.2 and 2.3: \((k > 1)\)
We first present the proof of Theorem 2.2. Select $\varepsilon > 0$ small and $R_0 > 0$ such that
\begin{equation}
(7.1) \quad \sup_{[0,R]\times[0,T]} u(x,t) \leq \varepsilon R^\beta, \quad \forall R \geq R_0.
\end{equation}
where $\beta$ is as described in the statements of the theorem.

Recall from (4.4) and (4.33) that the super-solution $w(x,t)$ can be written as
\[ w(x,t) = at + b(1+t)v(r), \]
for an appropriate $v(r) > 0$. Observe that $w$ is a super-solution for any small $b > 0$. Also, $v$ grows like $r^\beta$, see (4.33) and the constructions of the super-solutions in Section 4. Define
\[ W(x,t) = \nu + w(x,t). \]
Let $\hat{k} > 2$ be a constant so that $\hat{k}v \geq r^\beta$ for $r \geq R_1$, where $R_1$ is large. We take $b = \hat{k}\varepsilon$ in $W(x,t)$ and consider the cylinder $B_R(z) \times [0,T]$, where $R \geq \max(R_0,R_1)$.

At $t = 0$, $W(x,0) = \nu + \hat{k}\varepsilon v(r) \geq \nu \geq u(x,0)$. On $|x-z| = R$,
\[ W(x,t) \geq \hat{k}\varepsilon v(R) \geq \varepsilon R^\beta. \]
Thus, $W \geq u$ on the parabolic boundary of $B_R(z) \times (0,T)$. We apply Lemma 3.5 to conclude that $W \geq u$ in $B_R(z) \times (0,T)$ for any $R$, i.e.,
\[ u(x,t) \leq at + \hat{k}\varepsilon(1+t)v(r) + \nu, \quad \forall |x-z| \leq R. \]
Taking $x = z$, we get that $u(z,t) \leq at + \nu$. Letting $R \to \infty$ and then $\varepsilon \to 0$ (i.e. $b \to 0$) and using (4.33) (employ $\lim_{b \to 0} a$) we obtain the conclusion of the Theorem.

The proof of Theorem 2.3 can be obtained by using Part I of Section 5 and arguing analogously. We omit the details. \hfill $\Box$

Proofs of Theorems 2.4 and 2.5: ($k = 1$)

We first prove Theorem 2.4. We recall (II) in (4.33).

We take $\sigma = 0$. Let $0 < \varepsilon < c/10$ be small and fixed. Set
\[ W(x,t) = \nu + \alpha t + \varepsilon(1+t)e^{\sigma r^2}, \quad \forall (x,t) \in \mathbb{R}^n_T. \]
Then $W$ is super-solution for any small $\varepsilon > 0$.

Choose $R_0 > 0$ such that $\sup_{B_R(z)\times[0,T]} u(x,t) \leq e^{\varepsilon R^2}$ and $\varepsilon e^{\varepsilon R^2} > e^{\varepsilon R^2}, \forall R > R_0$.

We apply the comparison principle Lemma 3.5 to prove the claim in the theorem. Observe that $W(x,0) \geq \nu \geq u(x,0), \forall x \in \mathbb{R}^n_T$. On $|x-z| = R > R_1, W(x,t) \geq \varepsilon e^{\varepsilon R^2} \geq e^{\varepsilon R^2}$. By Lemma 3.5 $u(x,t) \leq W(x,t), \forall (x,t) \in B_R(z) \times (0,T)$, for any $R > R_0$. Hence,
\[ u(z,t) \leq W(z,t) = \nu + \alpha t + \varepsilon(1+t)e^{\sigma r^2}. \]
Since the above holds for any large \( R \), we let \( \varepsilon \to 0 \) to obtain the claim in part (a).

Part (b) may now be shown by arguing as above. Part (c) may be shown by following the ideas in the Proof of Theorem 2.2. Theorem 2.5 follows analogously, see Part II in Section 5. \( \square \)

We now present the proof of Theorem 2.6. We start with the maximum principle.

**Proof of Theorem 2.6(a): (Maximum principle)** We refer to Remark 3.6 and the comparison principle in Lemma 3.7. We set \( \alpha = 0 \) in part (a) of Theorem 2.2.

Suppose that \( u = \phi(v) \) where the change of variable is as in Remark 3.6. If

\[
H(Du, D^2u) - f(u)u_t \geq 0, \quad \text{in } \mathbb{R}_T^n, \quad u > 0, \quad \text{with } u(x, 0) \leq g(x), \quad \forall x \in \mathbb{R}^n,
\]

then

\[
H(Dv, D^2v + Z(v)DV \otimes Dv) - v_t \geq 0, \quad \text{in } \mathbb{R}_T^n, \quad \text{with } v(x, 0) \leq \phi^{-1}(g(x)), \quad \forall x \in \mathbb{R}^n,
\]

where \( Z(s) = \phi''(s)/\phi'(s) \) and the domain of \( Z \) contains \((0, \infty)\). See Remark 3.6.

The super-solution \( w \) used in the proof of Theorem 2.2 is positive. Clearly, \( Z(w) \) is well-defined. Using Lemma 3.5 (or Lemma 3.7) and arguing as in the proof of Theorem 2.2 we get that

\[
\sup_{B_R(z) \times (0, T)} v(x, t) = o(R^{r^*}) \quad \text{as } R \to \infty.
\]

Thus, the claim holds for \( u \).

**Proof of Theorem 2.6: (Minimum principle)**

(i) Suppose that

\[
\lim_{\delta \to 0^+} F(1) - F(\delta) < \infty.
\]

We choose

\[
v = \phi^{-1}(u) = \int_0^u f^{-1/(k-1)}(\theta) d\theta, \quad u > 0.
\]

Then

\[
H(Dv, D^2v + Z(v)DV \otimes Dv) - v_t \leq 0, \quad v > 0, \quad \text{in } \mathbb{R}_T^n, \quad \text{with } v(x, 0) \geq \phi^{-1}(g(x)), \quad \forall x \in \mathbb{R}^n,
\]

where the domain of \( Z \) contains \((0, \infty)\). We recall Case(i-1) from Section 6 and (6.4) i.e.,

\[
w(x, t) = \frac{\mu}{\hat{\mu}} \left[ 1 - \frac{(r/R)^{(k+1)/k}}{(1 + (t/E))^{1/(k-1)}} \right]^{k/(k-1)}, \quad \text{where } E = \frac{R^{k+1}}{c_k \hat{\mu}^{k-1}(k-1) |N|},
\]

for any large \( R > 0 \). Here \( \hat{\mu} = \phi^{-1}(\mu) \).

We use comparison in \( B_R(z) \times [0, T] \). It is clear that \( v(x, 0) \geq \phi^{-1}(g(x)) \geq w(x, 0), \quad \forall|x-z| < R \). Since \( v > 0 \) in \( \mathbb{R}_T^n \), working with \( R' < R \), close to \( R \), we see that
Applying Lemma 3.5 to the parabolic boundary of \( B_R(z) \times (0, T) \), we get that \( v(x, t) \geq w(x, t) \) in \( B_R(z) \times (0, T) \). Thus,
\[
v(z, t) \geq w(z, t) = \frac{\hat{\mu}}{(1 + (t/E))^{1/(k-1)}},
\]
Letting \( R \to \infty \) (i.e. \( E \to \infty \)), we get that \( v(z, t) \geq \hat{\mu} \) and the claim follows for \( u \). For \( k = 1 \) and \( f \equiv 1 \), we use Case (i-2) in Section 6 and (6.5) and assume that 
\[
\inf_{B_R(z) \times [0, T]} u(x, t) \geq \mu e^{-\varepsilon R^2}, \quad \forall \varepsilon > 0.
\]
Recall from (6.5) that 
\[
w(x, t) = \mu e^{-E r^2} e^{-2|N|^2 t}, \quad \forall E > 0,
\]
is a sub-solution in \( \mathbb{R}^n_T \). Working in cylinders \( B_R(z) \times (0, T) \), for large \( R \), we find that 
\[
u(x, 0) \geq \mu \geq w(x, 0), \quad \forall E > 0.
\]
Fix an \( E > \varepsilon \). On \(|x - z| = R\), \( w(x, t) \leq u(x, t) \) implying that \( w(x, t) \leq u(x, t) \) in \( B_R \times (0, T) \), for any large \( R \), and, hence, in \( \mathbb{R}^n_T \). Thus,
\[
w(z, t) = \mu e^{-2|N|^2 t} \leq u(z, t), \quad \forall E > \varepsilon.
\]
Since the above holds for any \( R \) and, hence, for any \( \varepsilon > 0 \), we get that the above estimate holds for any \( E > 0 \). Clearly, the claim holds.

(ii) Suppose that
\[
\int_0^1 f^{-1/(k-1)}(\theta)d\theta < \infty.
\]
We choose
\[
v = \phi^{-1}(u) = \int_0^u f^{-1/(k-1)}(\theta)d\theta, \quad u > 0.
\]
Then
\[
H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t \leq 0, \quad \text{in } \mathbb{R}^n_T \text{ with } v(x, 0) \geq \phi^{-1}(g(x)), \quad \forall x \in \mathbb{R}^n,
\]
where the domain of \( Z \) contains \((0, \infty)\). This is similar to the proof of Theorem 2.5.

The case \( k = 1 \) and \( f \equiv 1 \) also follows in an analogous way. \( \square \)

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