A novel approach to the spectral problem in the two photon Rabi model

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Received 4 November 2016, revised 4 March 2017
Accepted for publication 27 April 2017
Published 22 May 2017

Abstract
We explore the spectral problem of the two photon Rabi model from the point of view of complex differential equations in the Bargmann representation. The wave-functions are automatically entire but to ensure finite norm one has to effectively construct asymptotic expansions. This is achieved by means of the Mellin transformation and convergent factorial series, which allow direct computation of the spectral determinant. By further analysing the differential equation satisfied by the Mellin transform, we obtain a new form of the spectral conditions—in terms of holonomy matrices and contour integrals.

Keywords: two photon Rabi model, Mellin transformation, Bargmann representation

(Some figures may appear in colour only in the online journal)

1. Introduction

Although the Bargmann–Fock representation is commonly applied in quantum optics, its usage is often reduced to algebraic manipulation of infinite matrices and Fock basis, while its analytical aspects remain unexplored. In this article, we study the two photon Rabi model, which is a particularly challenging example when formulated in the language of complex differential equations. The main difficulty is that instead of standard boundary conditions, we have constraints on the asymptotic behaviour of wave-functions in all directions of the complex plane.

The main goal is to give an effective method of calculating spectra, which is directly derived from the properties of the Bargmann space of entire functions. The condition that the
norm of a state be finite is not straightforward to use in practical calculations. Our main result is to recast that requirement into constraints on the holonomy group of the differential equation obtained by the Mellin transform of the original system. This approach provides a novel theoretical framework, and additionally has a simple numerical implementation.

Behind this mathematical description the idea is very simple. The physical condition that the norm of an eigenstate is finite requires that solutions of differential equations with 'good' asymptotics glue nicely. Application of the Mellin transformation gives rise to a new system of differential equations for which the physical condition is now—the system has a solution which glue local solutions with 'good exponents'. That is exactly the same form as the quantisation problem formulated e.g. for the Rabi model [1]. At this point one can apply a variant of arbitrary methods used for studying such models. In this paper we propose a seemingly new one based on holonomy group of the system.

Stated another way, we wish to show how mathematical objects such as the analytical continuation, Mellin transform and asymptotic series can be effectively used to obtain the spectrum of physical problems whose Schrödinger equation can be considered on the complex plane. We will demonstrate all the relevant steps using an example from quantum optics.

The Hamiltonian of the two photon Rabi model has the form

\[ H = \omega a^\dagger a + \frac{\omega_0}{2} \sigma_z + 2g \left( (a^\dagger)^2 + (a)^2 \right) \sigma_x, \]

where \( \sigma_x, \sigma_y, \) and \( \sigma_z \) are the Pauli matrices; \( a^\dagger \) and \( a \) are photon creation and annihilation operators. It is also known as the two-photon Jaynes–Cummings model investigated initially in [2] and with a view to population inversion in [5]. For very recent studies of the two-photon Rabi model we refer the reader to [15, 16] where a detailed references for this subject can be found.

This system and the Rabi model were also extensively investigated in the PhD thesis of Emary [3]. In the subsequent paper [4] the authors show that the system is quasi-exactly-solvable, i.e. a finite number of eigen-states is known explicitly. However, for generic values of the physical parameters the spectrum is determined by a dedicated numerical methods based on diagonalisation. As noticed in the above references without the rotating wave approximation the system is not integrable.

On the other hand, in [13] the author claims that the system is solvable and that the eigen-energies are zeros of a certain expression called \( G \) function. Yet another scheme of a 'simple' spectra calculation for the Rabi two-photon model was proposed in [14] and according to the authors it gives a much simpler expression for the \( G \) function. However, as we have shown [6], the \( G \) function used in [13] is identically zero, so it cannot be reliably used to obtain the spectrum. Note that the Frobenius method, or its variants, utilised in these papers can be applied to obtain a solution or \( G \) function for any such system and hence says nothing about solvability.

This motivates the second goal of this article, which is to use the method introduced here to derive, as explicit as possible, a spectral determinant whose zeros are the eigen-energies of the two photon Rabi model. At the same time, we want the derivation to follow from the basic formulation of a quantum eigen-problem, without heuristics or conjectures.

The complete analysis consists of several crucial steps. First, the physical system has to be represented in purely mathematical terms, which in this case are: a system of ordinary differential equations on a complex plane and initial or boundary conditions. As it turns out, the boundary conditions here are strictly speaking the asymptotic behaviour at infinity. This initial translation into mathematical language is given in section 2.

The problem of asymptotic behaviour of the solutions will then be treated by means of the Mellin transform in section 3. As the growth of an entire function can be given in terms of the coefficients of its series expansion, the natural language here is that of recurrence relations and
the Mellin transform is very useful in solving them. This will lead to direct spectral conditions in terms of sequences instead of complex functions.

Although ready for applications, the recurrence approach has some drawbacks and the sections 4 and 5 are devoted to overcoming them. We will try to reformulate the spectral condition in analytical, rather than discrete language, giving it as a contour integral and discussing how the analytical continuation leads to the consideration of the holonomy group. Its action is the most important part of our study.

Finally, we will bring the elements of the analysis together formulating a complete criterion in section 6 and giving a possible implementation in section 7.

2. Formulation of the problem

In the Bargmann–Fock representation, see [7], the wave function of a two level system \( \psi = (\psi_1, \psi_2) \) is an element of Hilbert space \( \mathcal{H}^2 = \mathcal{H} \times \mathcal{H} \), where \( \mathcal{H} \) is the Hilbert space of entire functions of one variable \( z \in \mathbb{C} \). The elegant connection with the standard picture is that the annihilation and creation operators \( a, a^\dagger \) become \( \partial_z \) and multiplication by \( z \), respectively, for clearly \( [\partial_z, z] = 1 \). The scalar product in \( \mathcal{H} \) is given by

\[
\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)} g(z) e^{-|z|^2} d(\Re(z)) d(\Im(z)).
\]

It is worth mentioning that this space was also introduced, independently of Bargmann, by Newman and Shapiro [8, 9].

The Hilbert space \( \mathcal{H} \) has several peculiar properties. Let us mention two of them:

1. \( f(z) \in \mathcal{H} \) does not imply that \( f'(z) \in \mathcal{H} \).
2. \( f(z) \in \mathcal{H} \) does not imply that \( z f(z) \in \mathcal{H} \).

To understand these rather strange properties we have to recall some definitions and facts from the theory of entire functions, see [10, 11]. If \( f(z) \) is an entire function, then to characterise its growth, the following function is used:

\[
M_f(r) := \max_{|z|=r} |f(z)|.
\] (2)

We omit the subscript \( f \) later on, because the investigated function is known from the context. If for an entire function \( f(z) \) we have

\[
\lim_{r \to \infty} \sup_{|z|=r} \frac{\ln \ln M(r)}{\ln r} = \varrho, \quad \text{with} \quad 0 \leq \varrho \leq \infty,
\] (3)

then \( \varrho \) is called the order (or growth order) of \( f(z) \). If, further, the function has positive order \( \varrho < \infty \) and satisfies

\[
\lim_{r \to \infty} \sup_{|z|=r} \frac{\ln M(r)}{r^\sigma} = \sigma,
\] (4)

then we say that \( f(z) \) is of order \( \varrho \) and of type \( \sigma \).

Assume that \( f(z) \) belongs to \( \mathcal{H} \), then one can prove the following facts [7]:

1. \( f(z) \) is of order \( \varrho \leq 2 \).
2. If \( \varrho = 2 \), then \( f(z) \) is of type \( \sigma \leq \frac{1}{2} \).

If \( \varrho = 2 \) and \( \sigma = \frac{1}{2} \), then the question whether \( f(z) \in \mathcal{H} \) requires a separate investigation. Exactly in the mentioned case when \( f(z) \in \mathcal{H} \) but \( f'(z) \notin \mathcal{H} \) the function is of order \( \varrho = 2 \) and type \( \sigma = \frac{1}{2} \). For additional details see [12].
The usefulness of this representation can immediately be seen with the harmonic oscillator, which represents the radiation. The time-independent Schrödinger equation for energy \( E \) is simply \( H \psi(z) = z \psi'(z) = E \psi(z) \) and one immediately recovers the orthonormal eigenbasis as \( \{ z^n/\sqrt{n!} \}_{n \in \mathbb{N}} \). The connection with the usual space of square-integrable functions of \( q \) is given by the integral transformation
\[
\psi(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -(z^2 + q^2) / 2 + \sqrt{2} q z \right] \phi(q) \, dq.
\]
and the kernel is one of the forms of the generating function for the Hermite polynomials. Each \( z^n \) thus corresponds to the appropriately normalised wavefunction \( e^{-q^2/2} H_n(q) \). In this basis, the annihilation operator \( a \) is just an infinite matrix with entries on the superdiagonal, so the Hamiltonian can be constructed as tensor products of such matrices with the sigma matrices, giving a simple band structure. This allows for direct numerical diagonalization. However, the open question that we wish to tackle is how to determine the spectrum with as explicit exact formulas as possible while avoiding heuristic reasoning.

Now we want to write down the Schrödinger equation for the two photon Rabi model. First, we apply a unitary transformation given by \( U = (\sigma_x + \sigma_z) / \sqrt{2} \) to the Hamiltonian (1), which yields
\[
\tilde{H} = U^\dagger H U = \omega a^\dagger a + \frac{\omega_0}{2} \sigma_x + 2g \left[ (a^\dagger)^2 + a^2 \right] \sigma_z = 2g \left\{ 2x a^\dagger a + \mu \sigma_x + \left[ (a^\dagger)^2 + a^2 \right] \sigma_z \right\},
\]
where we set \( \omega = 4 \chi \phi \) and \( \omega_0 = 4 \mu g \). We rewrite the rescaled Hamiltonian in matrix form
\[
K := \frac{1}{2g} \tilde{H} = 2xa^\dagger a + \mu \sigma_x + \left[ (a^\dagger)^2 + a^2 \right] \sigma_z = \begin{bmatrix} 2xa^\dagger a + \left[ (a^\dagger)^2 + a^2 \right] & \mu \\ \mu & 2xa^\dagger a - \left[ (a^\dagger)^2 + a^2 \right] \end{bmatrix}.
\]

The operator \( K \) can be decomposed as
\[
K = 2x A^\dagger A + \mu \sigma_z - 2x \sin^2(\eta),
\]
where
\[
A = \cos(\eta) a + \sin(\eta) a^\dagger \sigma_z, \quad \sin(2\eta) = \frac{1}{x}.
\]
For a normalized eigen-state \( \psi = (\psi_1, \psi_2) \) we then have
\[
\langle \psi | K | \psi \rangle = 2x \| A \psi \|^2 + \mu \langle \psi | \sigma_z | \psi \rangle - 2x \sin^2(\eta) = E,
\]
which gives the constraint
\[
E \geq -\mu - \tan(\eta).
\]
In the Bargmann representation, the stationary Schrödinger equation \( K \psi = E \psi \), has the form of the following system of differential equations
\[
\begin{align*}
\psi_1''(z) + 2xz \psi_1'(z) + (z^2 - E) \psi_1(z) + \mu \psi_2(z) &= 0, \\
\psi_2''(z) - 2xz \psi_2'(z) + (z^2 + E) \psi_2(z) - \mu \psi_1(z) &= 0.
\end{align*}
\]
In some calculations it is more convenient not to take the above system, but rather the corresponding fourth order equation, obtained by elimination of \( \psi_2 \).
\[ \psi_0'' + ((2 - 4\xi^2)\psi'' + 4\xi\psi''') + 4(1 + E\xi - \xi^2)\psi' + (2 - E^2 + \mu^2 - 4\xi^2 + \xi^4)\psi_0 = 0. \]  
(13)

All solutions of this equation are entire, so the other component \( \psi_2 \) obtained from the first equation of (12) is also entire. This way, the first requirement of the Bargmann picture is identically satisfied. What remains to be checked is the finiteness of the norm.

Essentially all the subsequent work is devoted to the analysis of the behaviour of the solutions at infinity and the structure of the holonomy group of the relevant Mellin transform (introduced in the next section).

There are four basis solutions, and they can be numbered according to their parity, which is generated by the transformation \( \tau \) of the state

\[ \tau \psi(z) = \sigma \psi(iz). \]  
(14)

The Hamiltonian commutes with \( \tau \) and the associated symmetry group is \( \mathbb{Z}_4 \) isomorphic to \( \{1, \tau, \tau^2, \tau^3\} \). For a solution with parity \( s \) one has

\[
\begin{align*}
\psi_1(iz) &= s \psi_2(z), \quad \\
\psi_2(iz) &= s \psi_1(z),
\end{align*}
\]  
(15)

where \( s \in \{+1, -1, i, -i\} \). Because \( \tau^2 \psi(z) = \psi(-z) \), the distinction between even and odd solutions represents the \( \mathbb{Z}_2 \) subgroup of the symmetry and can be used to simplify the calculation.

In what follows we consider the even case, which includes parities \( s = \pm 1 \); calculations for the odd case \( s = \pm i \) are completely analogous and the main stages are given in Appendix A.

Let us define the function \( f(\xi) \) such that \( f(z^2) := \psi(z) \), for which system (12) becomes

\[
\begin{align*}
4\xi f''(\xi) + 2(2\xi + 1)f'((\xi - E)f_1(\xi) + \mu f_2(\xi) &= 0, \\
4\xi f''(\xi) - 2(2\xi - 1)f_2(\xi) + (\xi + E)f_2(\xi) - \mu f_1(\xi) &= 0.
\end{align*}
\]  
(16)

This system has a regular singular point at \( \xi = 0 \) but we are only interested in its entire solutions, given by series convergent in the whole complex plane

\[ f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \]  
(17)

whose vector coefficients \( a_n = [a_n^1, a_n^2]^T \) satisfy the matrix recurrence relation

\[ 2n(2n - 1)a_n = \begin{bmatrix} E - 4\xi(n - 1) & -\mu \\ \mu & -E + 4\xi(n - 1) \end{bmatrix} a_{n-1} - a_{n-2}. \]  
(18)

The initial conditions of the two solutions in question are fixed, save for a multiplicative constant, by the choice of parities. The symmetry action is \( \tau f(\xi) = \sigma \sigma f(-\xi) \), or, in terms of the series coefficients,

\[
\begin{align*}
a_0 &= [1, 1]^T, \quad \text{for } s = +1, \\
a_0 &= [1, -1]^T, \quad \text{for } s = -1.
\end{align*}
\]  
(19)

The only problem left is if the state has finite norm, which comes down to the aforementioned growth order and type. Given the series expansion \( \psi_1(z) = \sum_{k=0}^{\infty} c_k z^k \) of a solution of (13), these quantities can be calculated as the following limits

\[ \varrho = \limsup_{k \to \infty} \frac{k \ln k}{k^{1/\varrho}}, \quad (\varrho \varrho)^{1/\varrho} = \limsup_{k \to \infty} k^{1/\varrho} |c_k|^{1/k}. \]  
(20)
Note that because $\xi = z^2$, the coefficients satisfy $c_{2k} = a_k^1$ for even functions, and $c_{2k+1} = a_k^1$ for odd ones. The order of $\psi_1(z)$ is thus double that of $f_1(\xi)$, while the types are the same.

All the possible orders and types of $\psi(z)$ can be checked quickly by substituting a formal series of the form

$$\psi_1(z) = \exp(\sigma z^\varrho) z^{\alpha n + \rho A (n^{-1/\varrho})},$$

(23)

where $A$ is an asymptotic formal power series with some suitable integer $p$. For a general method of finding the asymptotic solutions of linear recurrence relations see [18].

Using methods of this reference, it can be shown that such a basis can be taken to be asymptotically simple in the sense that its members exhaust all possible behaviors at infinity, i.e. the sets $\{\alpha, \rho, \beta, A\}$. Consequently, each $a_n$ can be represented as a linear combination of such asymptotically simple $b_n$ and this decomposition will provide information about asymptotics of $a_n$, $f(\xi)$ given by (17) and hence also about $\psi(z)$.

The main tool to achieve this will be a modified Mellin transform which gives convergent expressions for $a_n$ in the form of factorial series. Such expressions are both valid for finite $n$ and have prescribed asymptotic behaviour, so they can be compared with $a_n$ obtained recursively from (19). As the solutions of a linear recursion relation form a vector space, the problem will come down to checking dependence of finite-dimensional vectors.

3. The Mellin transform

Following Okubo [17], to determine solutions of the above difference equation with prescribed asymptotic behaviour, we will use the integral representation

$$b_n = \mathcal{M}[\nu]n := \frac{1}{\Gamma(1 + n/\varrho)} \int_C u^n \nu(u) du,$$

(24)

where $\nu = [v_1(u), v_2(u)]^T$, and the contour $C$ will be chosen such that the integrand resumes its initial value after $u$ has described $C$. When compared to [17], the index $n$ is shifted by 1 and we have modified the argument of the $\Gamma$ function to reflect the behaviour of coefficients of an entire function of order $\varrho$. This can be easily seen from the asymptotics
\[
\frac{1}{\Gamma(n/\varrho)} \sim \sqrt{n \frac{e^{\varrho n}}{2\pi\varrho}}^{n/\varrho},
\]  
(25)

while the formulae (20) gives a simple example of coefficients of an entire function of order \(\varrho\):

\[
c_n = \left(\frac{\sigma e^{\varrho n}}{n}\right)^{n/\varrho}.
\]  
(26)

In other words, the \(\Gamma\) factor ensures the order of \(\varrho\), whereas the \(\sigma^n\) factor, which specifies the type, will have to be recovered from the integral, by using a suitable \(v(u)\).

In our case, the order of \(\psi(z)\) is 2, so we should use \(\Gamma(n/2)\) to analyse (12) or (13) directly, but thanks to the separation of even and odd solutions, we can instead deal with \(f(\xi)\), which has order 1.

Thus, substituting the Mellin transform with \(\varrho = 1\) into relation (18), we can transform it into a differential equation with integration by parts of the form

\[
\int_C n^j u^{n-1} v(u) du = [n^{j-1} u^n v(u)]_C - \int_C n^{j-1} u^n v'(u) du.
\]  
(27)

This allows to factor the integrand so that in the end the difference equation becomes

\[
\frac{1}{\Gamma(n)} \int_C n^{j-1} (4u^2 + 4x\sigma z, u + 1)v' + (6u + (E + 4x)\sigma z - i\mu\sigma_y) v) \, du
\]

\[-\frac{1}{\Gamma(n)} [n^{j-1} (4u^2 + 4x\sigma z, u + 1)v]_C = 0,
\]  
(28)

so the system to solve for \(v\) is

\[
\frac{dv}{du} = M(u)v, \quad M(u) := -\begin{bmatrix} \frac{6u + 4x + E}{4u^2 + 4xu + 1} & \frac{-\mu}{4u^2 + 4xu + 1} \\ \frac{\mu}{4u^2 + 4xu + 1} & \frac{6u - 4x - E}{4u^2 + 4xu + 1} \end{bmatrix}.
\]  
(29)

This system has five regular singular points \(u_0\)

\[
u_0 \in \Omega = \left\{ \pm \frac{\kappa}{2}, \pm \frac{1}{2\kappa}, \infty \right\}, \quad \text{where } x = \frac{\kappa}{2} + \frac{1}{2\kappa}.
\]  
(30)

and by the restrictions on \(x\) we choose \(0 < \kappa < 1\). The characteristic exponents at these points are the following

\[
\left\{ 0, -\chi \right\}, \quad \text{for } u_0 = \pm \frac{\kappa}{2},
\]

\[
\left\{ 0, \chi = \frac{3}{2} \right\}, \quad \text{for } u_0 = \pm \frac{1}{2\kappa},
\]

\[
\left\{ -\frac{3}{2}, \chi = \frac{3}{2} \right\}, \quad \text{for } u_0 = \infty,
\]  
(31)

where we introduced a natural spectral parameter

\[
\chi = \frac{\kappa(E + \kappa)}{2(1 - \kappa^2)} + 1.
\]  
(32)

It is not a coincidence that the positions of these points are precisely the growth types of the entire functions \(f(\xi)\) and \(\psi(z)\), as will soon become apparent. We also note that there is a bound by (11), \(\chi \geq \frac{\mu\kappa}{2(1 - \kappa^2)} + 1\), because \(\kappa = \tan(\eta)\).
Now the question is how to distinguish local solutions appropriate for the Mellin formula. On the one hand, the integral cannot vanish identically, so the solution cannot be single valued if the contour is a loop. On the other, the integrand must resume the same value on both ends of C. Depending on χ then, we are lead to several possibilities.

If χ is not a negative integer, then one local solution around \( u = \pm \kappa/2 \) always has some kind of singularity: either there is a branch point, a pole or a logarithmic term. This allows for choosing the contour C simply as a loop starting at \( u = 0 \) encircling a specific singular point \( \pm \kappa/2 \) in the positive direction and going back to zero. Choosing \( v(u) \) to be a solution which is multivalued in a disk centred at \( u_0 \) will ensure that the integral (24) does not vanish identically, while the boundary term in integration by parts will vanish at both ends, i.e. \( u = 0 \), provided that \( u \) enters with positive power. The lowest power boundary term in (28) is \( u^{n-1}v; \) consequently, such contour will work for at least \( n \geq 2 \).

If \( \chi \in \mathbb{Z}^- \), the contour can be a line from the origin to the singular point because the solution with the higher exponent has a zero of order at least 1 at \( u = \pm \kappa/2 \), and both boundary terms vanish.

In a neighbourhood of each singular point \( u_0 \) there exists a solution with exponent \( \nu \) given by the convergent series

\[
v(u) = (u - u_0)^\nu \sum_{j=0}^{\infty} h_j(u - u_0)^j.
\]

If the contour \( C \) lies entirely in this neighbourhood, and the series converges uniformly, the summation and integration can be exchanged, and the Mellin transform yields the factorial series representation

\[
\mathcal{M}[v]_n = \frac{1}{\Gamma(n + 1)} \int_C u^n \sum_{j=0}^{\infty} h_j(u - u_0)^j+\nu du
\]

\[
= \frac{2i \sin(\pi \nu) u_0^{n+1}}{\Gamma(n + 1)} \sum_{j=0}^{\infty} \frac{(-u_0)^{\nu+j} \Gamma(n+1) \Gamma(1+j+\nu)}{\Gamma(2+j+n+\nu)} h_j,
\]

which follows from the integral representation of the Beta function

\[
\frac{1}{2i \sin(\pi \nu)} \int_C u^{\alpha-1}(u - 1)^{\beta-1} du = B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}.
\]

If \( C \) is a line, the \( 2i \sin(\pi \nu) \) factor is absent.

The regular point used, \( u_0 \), determines the crucial asymptotic behaviour and this is all one needs for practical purpose of finding the ‘good’ recurrence solutions \( b_n \) and gluing it with \( a_n \). This first method of obtaining the spectrum is described in appendix C.

Although \( a_n \) can be calculated explicitly for any \( n \), and the expressions for \( b_n \) are given by the factorial series, there are problems with direct computational implementation. First, the complexity of the terms grows quickly with \( n \), and this makes the error estimates for \( b_n \) cumbersome. Second, because the singular points depend on the parameters, the region \( \kappa > 1/\sqrt{2} \) requires separate procedures of appendix D, for which the involved expressions are even longer. Finally, we have to find a common zero of 4 functions \( (3 \times 3 \text{ minors}) \) to locate the spectrum.

An alternative approach is based on the observation that \( b_n \) are given by contour integrals, so they can be calculated by numerical integration of \( v \) instead of numerical summation of series. As it turns out, we can do even more than that by noticing how the \( \mathbb{Z}_4 \) symmetry is
reflected in the holonomy group. In the process of finding the appropriate solution \( v \) for the integral, we discover a very concise form of the spectral conditions.

4. The spectral condition as contour integral

Given an appropriate solution \( v \), the contour integrals provide successive values of \( b_n \) for all \( n \geq 2 \); alternatively, the whole sequence can equally well be generated by (18) from just two consecutive elements \( b_{n_0} \) and \( b_{n_0+1} \) obtained from the contour integrals. The asymptotic growth of \( b_n \) is guaranteed by the above, but the representation fails at \( b_0 \), and in particular it is not always the case that \( b_n \equiv 0 \) for \( n < 0 \) contradicting the initial conditions (19). However, only when they hold can we say that we have found the coefficients of a function \( f(\xi) \) that is both entire and of proper asymptotic type.

In order to express the initial conditions in terms of the function \( v \), we can integrate the whole system (29) over the contour in question. To this end we rewrite it as

\[
(4u^2 + 1)v' + 4xu\sigma_3v' = -6uv - (4x + E)\sigma_3v + i\mu\sigma_3v, \quad (36)
\]

and use integration by parts to get

\[
-\int_C (8u + 4x\sigma_3)v\,du + \int_C v'\,du = -\int_C [6u + (4x + E)\sigma_3 - i\mu\sigma_3] v\,du, \quad (37)
\]

\[
2\int_C uv\,du = \int_C (E\sigma_3 - i\mu\sigma_3)v\,du + [v]_C, \quad (38)
\]

where \([v]_C := v(\gamma(1)) - v(\gamma(0)),\) and \([0, 1] \ni t \mapsto \gamma(t)\) is a parametrisation of the contour \( C \). As we are in fact dealing with several singular points, we have several pairs of contours and solutions \( \{C, v\}_{u_0} \), in this particular case only two points \( u_0 = \pm \kappa/2 \) matter, so we will simply index the pairs with \( + \) and \( - \). As explained in the previous section, the choice at each point depends on the characteristic exponent, which also means that each \( v \) is determined up to a multiplicative constant.

A general solution \( b_n \) of the recurrence (18) can be any linear combination of particular solutions, each of which corresponds to certain \( \{C, v\}_{u_0} \). Its asymptotic type is determined by \( u_0 \) and to ensure that such \( b_n \) defines an entire function, it must coincide with \( a_n \) so the linear dependence

\[
a_n = \alpha_+ b_n^+ + \alpha_- b_n^-, \quad (39)
\]

must hold. If so, then by the definition of \( b_n^\pm \)

\[
a_n = \frac{1}{n!} \int_{C_+} u^n \alpha_+ v_+ \,du + \frac{1}{n!} \int_{C_-} u^n \alpha_- v_- \,du, \quad (40)
\]

and thanks to the aforementioned freedom of rescaling, we will write \( v_\pm = \alpha_\pm \tilde{v}_\pm \). For \( n = 1 \) recurrence relation (18) reads \( 2a_1 = (E\sigma_3 - i\mu\sigma_3)a_0 \), which, by substituting \( a_0 \) and \( a_1 \) as expressed in (40), becomes

\[
2\sum_i \int_{C_i} uv_i \,du = (E\sigma_3 - i\mu\sigma_3) \sum_i \int_{C_i} v_i \,du. \quad (41)
\]
Combining this condition with the integrated system (38) yields
\[
\sum_i \int_{C_i} (E \sigma_z - i \mu \sigma_y) v_i du + \sum_i [v_i]_{C_i} = \sum_i \int_{C_i} (E \sigma_z - i \mu \sigma_y) v_i du, \tag{42}
\]
or
\[
\sum_i [v_i]_{C_i} = 0. \tag{43}
\]
In the generic case, when the contour is a loop, this can be written in terms of the holonomies
\[
\sum_i (F_i - \mathbb{I}) v_i(0) = 0, \tag{44}
\]
where, by definition, the holonomy matrix \( F_i \) is the value of the fundamental matrix \( V(u) \) of the system (29) analytically continued over a closed loop \( C_i \), starting with the initial condition \( V(0) = \mathbb{I} \).

At this point we have replaced the need of constructing local series representations of \( v \) around each singular point with just obtaining the holonomy matrices for the system (29). The holonomy (or, equivalently, monodromy) naturally incorporates the information about characteristic exponents, which was also necessary before to chose the right series and the contour. Moreover, we no longer need to calculate several consecutive elements \( b_n \) or even integrals \( \int u^n v du \), because to obtain \( F_i \), and hence condition (43), only one integration over \( C_i \) is sufficient.

5. The impact of symmetry on the Holonomy

We notice first, that the matrix \( M(u) \) defining the right hand sides of the the Mellin system (29) satisfies
\[
M(u) \sigma_x + \sigma_x M(-u) = 0. \tag{45}
\]
Thus, if \( v(u) \) is a solution, so is \( \tau_M(v(u)) := \sigma_x v(-u) \). We show that this \( \mathbb{Z}_2 \) symmetry holds for any fundamental matrix, and that its columns can be chosen as the eigenvectors of this symmetry.

Let \( T(u) = V(u)^{-1} \sigma_x V(-u) \), where \( V(u) \) is an arbitrary fundamental matrix of the Mellin system (29), then \( T'(u) = 0 \). Direct differentiation gives
\[
T'(u) = \left[ V(u)^{-1} \right]' \sigma_x V(-u) + V(u)^{-1} \sigma_x [V(-u)]'
= -V(u)^{-1} M(u) \sigma_x V(-u) + V(u)^{-1} \sigma_x [-M(-u) V(-u)]
= -V(u)^{-1} [M(u) \sigma_x + \sigma_x M(-u)] V(-u) = 0. \tag{46}
\]
Hence \( T = T(u) \) is a constant matrix.

Now, assume that \( V(0) = \mathbb{I} \), then \( T = \sigma_x \) and hence
\[
\sigma_x V(-u) = V(u) \sigma_x. \tag{47}
\]

Next, we investigate the holonomy matrices of the system. Let \( a \in \mathbb{C} \) be a non-singular point of the system and \( V(u) \) its local fundamental matrix defined in a neighbourhood of \( a \). We consider analytic continuation of initial state \( Y_0 = V(a) \) of the system along a loop \( C \) given parametrically by the map
\[
[0, 1] \ni t \mapsto \gamma(t) \in \mathbb{C},
\]
}\]
with $\gamma(0) = \gamma(1) = a$. The result of this continuation is a matrix $Y_1 = Y(1)$ given by the solution of initial value problem

$$\frac{d}{dt}Y(t) = \gamma(t)M(\gamma(t))Y(t), \quad Y(0) = Y_0.$$  \hfill (48)

This is a change of independent variable such that $V(\gamma(t)) = Y(t)$, and the fact that $t = 0$ and $t = 1$ both correspond to $u = a$, while $Y_0 \neq Y_1$ in general, reflects the fact that $V(u)$ is not necessarily single-valued.

The holonomy matrix $F_\gamma$ is then defined via the linear map

$$\Delta_\gamma : Y_0 \mapsto Y_1 = F_\gamma Y_0,$$  \hfill (49)

and it does not depend on the particular initial condition chosen, but only on the homotopy class of $\gamma$. This can be seen by noticing that any fundamental matrix of (48) is a constant matrix $A$, so that $Y_0A \mapsto Y_1A = F_\gamma Y_0A$.

For further purposes we consider two parametrised loops $C_+$ and $C_-$ encircling counterclockwise the singular point $u = \pm \kappa/2$, respectively. They have one common point $u = 0$.

Loop $C_+$, parametrised by $\gamma_+(t)$, gives the holonomy matrix $F_+$, i.e. $\dot{Y} = \gamma_+M(\gamma_+(t))Y$ and $Y_1 = F_+Y_0$. Similarly, we have $\dot{Z} = \gamma_-M(\gamma_-(t))Z$ and $Z_1 = F_-Z_0$. In order to find a relation between $F_+$ and $F_-$ we assume the loop $C_-$ is obtained from $C_+$ by the reflection through the origin. Then $\gamma_-(t) = -\gamma_+(t)$ is a parametrisation of $C_-$, and by (45)

$$\frac{d}{dt}(\sigma, X) = \sigma \dot{X} = -\gamma_+\sigma M(-\gamma_+)Z = \gamma_+\sigma M(\gamma_+)\sigma Z,$$  \hfill (50)

which means that $\sigma, Z(t) = Y(t)A$, for some constant matrix $A$, and

$$\Delta_{\gamma_-} : Z_0 \mapsto Z_1 = \sigma_1^{-1}Y_1A = \sigma_1^{-1}F_+Y_0A = \sigma_1^{-1}F_+\sigma Z_0,$$  \hfill (51)

so we have obtained the fundamental formula

$$F_- = \sigma_1^{-1}F_+\sigma_1.$$  \hfill (52)

We also note that there is a direct link with the monodromy group, which is another representation of how the solutions change under analytic continuation along the contours. The monodromy matrices depend on the choice of the fundamental matrix but if the standard initial condition $V(0) = 1$ is used, they are numerically identical to the respective holonomies, so that an analogous formula $M_- = \sigma_1^{-1}M_+\sigma_1$ holds.

6. Holonomy criteria

We now bring together all the above elements to formulate criteria for determining the spectrum. Thanks to the symmetry of the holonomy, only analysis around one singular point is necessary; while the contour formulation allows us to work with initial conditions $v(0)$.

**Criterion 1.** If $\chi$ belongs to the spectrum, i.e. the function $\psi(z)$ is entire and normalizable, then there exists a common eigenvector $e$ of $F_+$, $F_-$ and $\sigma_e$.

The local holonomy group is thus seen to be solvable because the matrices are simultaneously triangularizable. The above is just a necessary condition, and we further have
Criterion 2. Depending on the value of the parameters, the sufficient conditions are

1. $\chi \notin \mathbb{Z}$: the eigenvector $\mathbf{e}$ has the eigenvalue $\exp(2\pi i \chi)$.
2. $\chi \in \mathbb{Z}$ and there are logarithmic solutions or, equivalently, the holonomy has a Jordan block, the necessary condition is also sufficient.
3. $\chi \in \mathbb{Z}$ and there are no logarithms or, equivalently, the holonomies are $F_\pm = 1$: the solution $\mathbf{v}_+$ corresponding to the eigenvector $\mathbf{e}$ has the characteristic exponent $-\chi$ at the regular point $\kappa/2$.

To demonstrate all the points in turn, we will consider the initial conditions of solutions $\mathbf{v}(0)$, denoted by $\mathbf{e}$, and the action of the holonomy.

If $\chi \notin \mathbb{Z}$, the eigenvalues of $F_\pm$ are $\lambda = \exp(2\pi i \chi)$ and $1$. Take the eigenvector $\mathbf{e}_\lambda^+$ and construct $\mathbf{e}_\lambda^- = -\alpha \sigma_x \mathbf{e}_\lambda^+$, $\alpha \in \mathbb{C}^*$, which must be an eigenvector of $F_-$ to the eigenvalue $\lambda$ by (52). If the spectral condition (43) is satisfied, then

$$
(F_+ - 1)\mathbf{e}_\lambda^+ + (F_- - 1)\mathbf{e}_\lambda^- = (\lambda - 1)\mathbf{e}_\lambda^+ - \alpha(\lambda - 1)\sigma_x \mathbf{e}_\lambda^+ = 0,
$$

but because $\lambda \neq 1$ for noninteger $\chi$, it follows that $\mathbf{e}_\lambda^+ = \alpha \sigma_x \mathbf{e}_\lambda^+$ and $\mathbf{e}_\lambda^- = -\mathbf{e}_\lambda^+$. Thus, there is a common eigenvector of $F_+, F_-$. In addition, it must be an eigenvector of $\sigma_x$, so one of $[1, \pm 1]^T$.

If $\chi \in \mathbb{Z}$ and $F_\pm$ has a Jordan block, there is a solution whose initial solutions at zero satisfy $F_+ \mathbf{e}_0^+ = \mathbf{e}_0^+$ and a logarithmic solution which corresponds to the generalized eigenvector, i.e. $F_-\mathbf{e}_0^- = \mathbf{e}_0^- + 2\pi i \mathbf{e}_0^+$. Taking $\mathbf{e}_i = -\alpha \sigma_x \mathbf{e}_i^+$, $i \in \{0, 1\}$, gives

$$
F_- \mathbf{e}_j^- = -\alpha \sigma_x \mathbf{e}_j^+ - 2\pi i \alpha \sigma_x \mathbf{e}_0^+ = \mathbf{e}_j^+ + 2\pi i \mathbf{e}_0^+.
$$

The spectral condition is then

$$
(F_+ - 1)\mathbf{e}_j^+ + (F_- - 1)\mathbf{e}_j^- = 2\pi i (\mathbf{e}_0^+ + \mathbf{e}_0^-) = 0,
$$

so these eigenvectors must be proportional and, like before, $\mathbf{e}_j^+ = \alpha \sigma_x \mathbf{e}_0^+ = -\mathbf{e}_0^-$ is the common eigenvector of the form $[1, \pm 1]^T$.

If $\chi \in \mathbb{Z}$ and $F_\pm$ is diagonalizable, it must be the identity matrix, so the necessary condition is trivial; additionally $\chi = 0$ is excluded as it always leads to logarithms. For the sufficient condition we notice, that for $\chi \in \mathbb{Z}$ and no logarithms, the solution $\mathbf{v}(u)$ has a pole at the regular point but it is not multivalued. The Mellin integral is thus not identically zero, but the contour condition $[\mathbf{v}]_C$ is identically satisfied around each point independently. As stated in appendix C, this leads to pairs of explicit solutions discovered by Emary and Bishop [4]. When $\chi$ is non-positive, $\mathbf{v}(u)$ has a zero at the regular point and the contour has to be the line from 0 to $\kappa/2$. The corresponding solution around $-\kappa/2$ is $\mathbf{v}_-(u) := -\alpha \sigma_x \mathbf{v}_+(u)$ and the contour condition (52) is

$$
\mathbf{v}_+(\kappa/2) - \mathbf{v}_+(0) + \mathbf{v}_-(\kappa/2) - \mathbf{v}_-(0) = -\mathbf{e}^+ - \mathbf{e}^- = 0,
$$

and by the symmetry (43), we must once again have $\mathbf{e}^- = -\alpha \sigma_x \mathbf{e}^+ = -\mathbf{e}^+$, so that an eigenvector of $\sigma_x$ must correspond to the solution with the positive exponent $-\chi$, i.e. vanishing at $\kappa/2$.

This completes the proof, and we also note that in the last case the matrix $F_\pm$ cannot be used to obtain the eigenvector $\mathbf{e}$; but to check which solution vanishes at $\kappa/2$ one can make use of Cauchy’s integral

$$
\mathbf{v}_h(\kappa/2) = \frac{1}{2\pi i} \oint \frac{\mathbf{v}(u)}{u - \frac{\kappa}{2}} \, du,
$$

which will be valid for the whole fundamental matrix, since both solutions are analytic.

In each of the above cases, the fundamental quantity is the determinant.
\[ \det[v(u), \sigma_x v(-u)], \]  
(58) 

taken at \( u = 0 \), where \( v \) is just \( e \), so that if \( \chi \) belongs to the spectrum

\[ W := \det[e, \sigma_x e] = 0. \]  
(59) 

This determinant arises in complete analogy with the Wronskian introduced by the authors in \[6\]. Although here we are dealing with a determinant of numeric quantities, these are the initial conditions of solutions, and the connection the Wronskian of \( v(u) \) is

\[ \text{Wr}[v_1, v_2] = W \frac{(1 - 4\kappa^2 u^2) \chi^{-3/2}}{(1 - 4u^2/\kappa^2) \chi}. \]  
(60)

7. Implementation

All the cases can now be gathered into a simple algorithm for computing the spectral determinant. The whole goal of finding the correct values of the energy comes down to verifying that the main equation has, for a given set of parameters \((E, x, \mu)\), entire normalisable solutions. We will work directly with the quantities \((\chi, \kappa, \mu)\), because they are more natural, e.g. the explicit Emary-Bishop solutions appear for (half)integer values of \( \chi \), and \( \kappa \) lies between 0 and 1.

We recall that the fundamental matrix \( V(u) \) has the initial condition

\[ V(0) = 1, \]  
and for numerical integration, the contour around \( \kappa/2 \) can be parametrised with the path

\[ \gamma_+(t) = \frac{1}{4} - \frac{1}{4} \exp[2\pi i t], \quad t \in [0, 1]. \]  
(61) 

The value that \( V \) attains at 0, having described the contour \( C \), will be the holonomy matrix \( F_+ \).

For the exceptional last case, we use Cauchy’s formula (57) for the whole matrix \( V \) to obtain \( V(\kappa^2/2) \), and its null eigenvector will be the desired eigenvector \( e \).

Algorithm 1. Spectral determinant \( W(\chi, \kappa, \mu) \)

Require: \( \chi, \kappa, \mu \)

Integrate system (29) to obtain \( F_+ = V(\gamma_+(1)) \).

if \( \chi \notin \mathbb{Z} \) then

Determine the eigenvector \( e \) of \( F_+ \) to the eigenvalue \( e^{2\pi i \chi} \).

else if \( \chi \in \mathbb{Z} \land F_+ \neq 1 \) then

Take the only eigenvector \( e \).

else if \( \chi \in \mathbb{Z}_+ \land F_+ = 1 \) then

Two Emary–Bishop states exist: \( e \) can be either of \([1, 1]\) and \([1, -1]\).

else if \( \chi \in \mathbb{Z}_- \land F_+ = 1 \) then

Integrate the fundamental matrix according to Cauchy’s formula (57).

Solve \( V(\kappa/2)e = 0 \) for \( e \).

end if

\[ W = \det[e, \sigma_x e]. \]

The odd parities are completely analogous, with their Mellin system:

\[ \frac{dv}{du} = -\begin{bmatrix} \frac{2\mu+2E}{4\mu^2+4\mu+1} & -\frac{\mu}{4\mu^2+4\mu+1} \\ \frac{\mu}{4\mu^2+4\mu+1} & \frac{2\mu-2E}{4\mu^2-4\mu+1} \end{bmatrix} v, \]  
(62) 

and with \( \chi \in \frac{1}{2}\mathbb{Z} \) for odd Emary–Bishop states.
A numerical example for a generic situation is presented in figure 1 and a spectrum with Emary-Bishop states is presented in figure 2. We notice in particular, that the function is smooth (or has a removable discontinuity in the degenerate case) which is not the case in other methods which introduce artificial singularities at integer values of the exponent.

8. Conclusions

The two photon Rabi model, as formulated in the Bargmann representation, is unusual in that the respective differential equation has only entire solutions. The condition that a function is an eigenstate is reduced only to the finiteness of its norm or, in other words, the proper
asymptotic behaviour at infinity, as specified by the growth order and type. Whereas in the standard Rabi model one has to ensure analyticity by gluing together solutions around different regular singular points, here the problem lies in gluing solutions with appropriate asymptotic growth.

As infinity is an irregular singular point, in theory such connection problem would require dealing with the Stokes phenomenon between formal solutions across the sectors at infinity. However, by using the Mellin transformation we have shown how to obtain solutions with prescribed global asymptotics. The intermediate step is the construction of entire power series, while the transformation is necessary to select appropriate solutions of the recurrence relation satisfied by the coefficient of such series.

We note that the starting point of this approach is just the requirement that the eigen-state be an element of the Bargmann–Fock space. As opposed to other ad hoc methods in the literature, we thus arrive at a practical method which is well founded.

The crucial element in the asymptotic analysis are the factorial series, which, unlike the standard asymptotic expansions, are convergent. They can be used both for functions of a complex variable and for solutions of recurrence relations, and they give a concise way to solve the connection problem or to determine the Stokes phenomenon as shown in [17].

Because the system is, in general, not solvable, there are no explicit elementary formulae for the Stokes multipliers or the connection coefficients. Thus, even using the factorial series means that eventually some numerical approximation has to be used. By noticing that this can be implemented already at the stage of the Mellin transformation, we further refine our results by investigating how to give the spectral conditions in terms of contour integrals. These can then be treated numerically much easier than the relevant infinite series.

It turns out that the existence of an eigen-state is directly connected with the properties of the holonomy group of a second order system of linear differential equations. Furthermore, the $\mathbb{Z}_4$ symmetry further simplifies the problem, because it provides a partial connection formula between the holonomy matrices.

Finally, despite the formal development, the holonomy for a linear system is very easy to compute and leads to a practical algorithm 1, whose precision is in essence limited only by the particular chosen scheme of numerical integration.

Acknowledgments

This work has been supported by the grant No. DEC-2011/02/A/ST1/00208 of National Science Centre of Poland.

Appendix A. Odd entire solutions

By defining a new function $f$ such that $zf(z^2) := \psi(z)$ and introducing again $\xi = \xi^2$ we have the series expansion

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n,$$

and the coefficients satisfy the matrix difference equation

$$2n(2n-1)c_n = \begin{bmatrix} E - 2x(2n - 1) & -\mu \\ \mu & -E + 2x(2n - 1) \end{bmatrix} c_n - c_{n-2}. \quad \text{(A.2)}$$
Through the same Mellin integral as for the even case we obtain the differential system
\[ \frac{dv}{du} = M(u)v, \quad M(u) := -\begin{bmatrix} \frac{2(u+x)+\rho}{4u^2+4ux+1} & \frac{-\mu}{2} \\
\frac{\mu}{4u^2-4ux+1} & \frac{2(u-x)+\rho}{4u^2-4ux+1} \end{bmatrix}, \] (A.3)
whose characteristic exponents are
\[ \left\{ 0, \frac{1}{2} - \chi \right\}, \quad \text{for } \ u_0 = \pm \kappa \]
\[ \{0, \chi - 1\}, \quad \text{for } \ u_0 = \pm \frac{1}{2\kappa} : \]
\[ \left\{ -\frac{1}{2}, -\frac{1}{2} \right\}, \quad \text{for } \ u_0 = \infty, \] (A.4)
and the spectral parameter \( \chi \) is the same as before.

We note that the logarithmic and Juddian solutions can now arise only for half integer values of \( \chi \), and this is the main difference between the even and odd cases.

Appendix B. The case \( |\sigma| = \frac{1}{2} \)

When \( x = 1 \), there are only two available exponential factors in the asymptotic expansion: \( \exp(\pm \frac{1}{2}z^2) \), and the convergence of the Bargmann norm has to be checked in each sector separately. E.g. the integral of \( \exp(\frac{1}{2}z^2) \) is finite over the region \(-\pi/4 \leq \arg(z) \leq \pi/4\), but not over \( \pi/4 \leq \arg(z) \leq 3\pi/4\). This means that a normalizable solution must change its (generalized) type as \( \arg(z) \) increases.

If one continues analytically a solution which behaves properly around the real axis, i.e. \( f \sim \exp(-\frac{1}{2}z^2) \), and there is no Stokes phenomenon, it will behave as \( \exp(\frac{1}{2}|z|^2) \) around the imaginary axis, and the Bargmann integral will be infinite. A proper eigenstate cannot have this behaviour.

To see how the the solution behaves with nontrivial Stokes phenomenon we can employ the Laplace representation again, which will be particularly simple for \( x = 1 \). The main equation (13) for \( z^2 = \zeta \), which amounts to taking parities \( \pm 1 \) (the \( \pm i \) case is analogous), is
\[ 16\zeta^2 f'''' + 48\zeta f''' + 4(3 + 4\zeta - 2\zeta^2) f'' + (8 - 12\zeta) f' + (1 + \mu^2 - 4\zeta + \zeta^2) f = 0. \] (B.1)

The integral representation of \( f(\zeta) = \int \exp\left(\frac{\zeta}{2}u\right) g(u)du \) (B.2)
gives the following differential equation for \( g \)
\[ 4(1-u^2)g'' + 4(5u-2)(u^2-1)g' + (\mu^2 - 3 - 12u + 15u^2) = 0, \] (B.3)
whose general solution, for \( \mu \neq 1 \) is
\[ g = c_1(1-u)^\rho(1+u)^{-\rho - 3/2} + c_2(1+u)^{\rho - 1}(1-u)^{-\rho - 1/2}, \quad \rho = -1 + \sqrt{1 - \frac{\mu^2}{4}}, \] (B.4)
or, for \( \mu = 1 \),
\begin{equation}
  g = (1 - u)^{-1/4}(1 + u)^{-5/4} \left( c_1 + c_2 \log \left( \frac{1 + u}{1 - u} \right) \right).
\end{equation}

Using the methods of [6], we obtain the positions of the Stokes lines to be \( \arg(z) \in \{ \pi/4, -\pi/4, 3\pi/4, -3\pi/4 \} \), and that the Stokes matrices are triangular. This means that even if one chooses a solution with finite partial norm in some sector \( S = \{ z : \alpha \leq z \leq \beta \} \)

\begin{equation}
  \| f(z) \|_{\alpha, \beta} = \frac{1}{\pi} \int_{S} e^{-|z|^2} |f(\Re(z))| d(\Im(z)),
\end{equation}

its continuation will contain both asymptotics in the next sector rendering the global integral infinite.

In the generic case, when \( x < 1 \) (\( \kappa \) is no longer real), there are again four exponential types (22), except this time they all lie on the circle \( |\sigma| = \frac{1}{2} \) and form a rectangle whose sides are parallel to the real and imaginary axes. Because the type alone will not be enough to check the norm, let us go back to equation (13) and write the solution in the even case (odd being completely analogous again)

\begin{equation}
  \psi_1(z) = \sum_{n=0}^{\infty} c_n z^{2n},
\end{equation}

whose coefficients correspond to \( a_n^1 \) of (17). The asymptotic form of these coefficients can be ascertained either by direct substitution into the recurrence relation or from the Mellin representation (34). This time a formal expression is all we need, because the Bargmann norm will be finite if

\begin{equation}
  \| \psi_1 \| = \sum_{n=0}^{\infty} (2n)! |c_n|^2 < \infty,
\end{equation}

and only the behaviour of \( c_n \) at infinity matters. Namely, we will use the Gauss test which states that when

\begin{equation}
  \left| \frac{u_n}{u_{n+1}} \right| = 1 + \frac{h}{n} + \mathcal{O}(n^{-r}), \quad r > 1,
\end{equation}

then the positive series given by \( u_n \) converges if and only if \( h > 1 \).

Since the coefficients in question behave as

\begin{equation}
  c_n \sim \frac{1}{n!} \sigma^n n^\beta,
\end{equation}

its absolute value behaves as

\begin{equation}
  |c_n| = \frac{1}{n!2^n} n^{\Re(\beta)} \left( 1 + \mathcal{O}(n^{-1}) \right),
\end{equation}

where

\begin{equation}
  \beta \in \left\{ \frac{1}{4} \pm \frac{E + x}{4\sqrt{x^2 - 1}}, -\frac{5}{4} \pm \frac{E + x}{4\sqrt{x^2 - 1}} \right\},
\end{equation}

in accordance with

\begin{equation}
  \sigma \in \left\{ \frac{1}{2} (-x \pm \sqrt{x^2 - 1}), \frac{1}{2} (x \pm \sqrt{x^2 - 1}) \right\}.
\end{equation}
The norm series to be analysed is given by $u_n = (2n)!|c_n|^2$ so

$$\left| \frac{u_n}{u_{n+1}} \right| = 1 + \frac{1 - 4\Re(\beta)}{2n} + O(n^{-2}), \quad (B.14)$$

and the deciding term is $1/n$ for the first two choices of $\beta$ and $3/n$ for the other two.

At this point we have to employ the residual $Z_2$ symmetry of the even solutions, because there are more solutions of the recurrence than of the differential equation. This happens because a series solution of the differential equation has imposed on it the additional conditions $c_n \equiv 0$ for $n < 0$. Specifically, we can only obtain two entire even functions, and there are four pairs of $(\sigma, \beta)$ specifying asymptotic solutions of the recurrence.

Fortunately it is not necessary to solve the full connection problem, i.e. decide which asymptotic expansion corresponds to which entire series. Instead, we recall that if a solution exists, it can be projected onto parity eigenstates, which satisfy

$$\psi''(z) + 2xz\psi'(z) + (z^2 - E)\psi(z) + \mu s \psi(iz) = 0, \quad s = \pm 1. \quad (B.15)$$

By direct substitution we find that the series coefficients of a solution of parity $s$ must be a combination of two solutions $c_n$ corresponding to $\sigma$ and $-\sigma$:

$$d_n = \frac{1}{n!} \sigma^\beta n^\beta (1 + O(n^{-1})) - \frac{s\mu}{8\pi n!} (-\sigma)^\beta n^{\beta-1} (1 + O(n^{-1})), \quad (B.16)$$

where now there are only two possibilities

$$\sigma = \frac{1}{2}(-x \pm \sqrt{x^2 - 1}), \quad \beta = \frac{1}{4} \pm \frac{E + x}{4\sqrt{x^2 - 1}}, \quad (B.17)$$

so that in effect the asymptotics of $d_n$ is dominated by the larger $\beta$ and

$$\left| \frac{u_n}{u_{n+1}} \right| = 1 + \frac{1}{n} + O(n^{-2}), \quad (B.18)$$

so by Gauss’s criterion the norm series is always divergent, proving no proper eigenstate exists in this case.

**Appendix C. Spectral conditions through factorial series**

The reason why the formula (34) can be readily put into practice is that the regular point used, $u_0$, determines the crucial asymptotic behaviour of the sequence $b_n$. Because the $\Gamma$ function factors in the sum are of the order $O(n^{-1-j-\nu})$, asymptotically one has

$$b_n \sim \frac{1}{\sqrt{2\pi}} \left( \frac{u_0(e/n)}{n} \right)^n n^{-\nu-3/2} \left( u_0 h_0 + O(n^{-1}) \right), \quad (C.1)$$

so, by (20), $u_0$ is the type of the associated entire function $f(\xi)$ and also of $\psi(z)$. It will thus suffice to consider only solutions and contours around the two points $u_0 = \pm \kappa/2$, which give the normalizable types. The other singular points influence the radius of convergence of (33), so for this series to be integrated term by term over the contour $C$, the point $u_0$ must lie closer to zero than to any other singular point, giving the condition $\kappa \leq 1/\sqrt{2}$. When this condition does not hold, a change of variable is required, which amounts to using a different series for $v$, as explained in detail in appendix D, but in the end $b_n$ is still represented by a factorial series.
When one exponent is \(-m \in \mathbb{Z}_-\), the logarithmic solution \(v = v_1 + \log(u - u_0)v_2\) has to be used, and we notice that continuation around the contour \(C\) acts on this solution as \(v \rightarrow v + 2\pi i v_2\). Both \(v_i\) are single-valued so the contour can be decomposed into two line segments and an arbitrarily small circle giving

\[
\int_C u^v du = \int_{0}^{u_0 - \varepsilon} u^v \log(u - u_0)v_2 du + \oint_{|u-u_0| = \varepsilon} u^v du + \int_{u_0 - \varepsilon}^{0} u^v (\log(u - u_0) + 2\pi i)v_2 du = I(\varepsilon),
\]

and because the exponent of \(v_2\) is zero, we can take the limit \(I(\varepsilon) \rightarrow \varepsilon \rightarrow 0\)

\[
\text{res}_{u(u_0)}(u^v v_1) - 2\pi i \int_{0}^{u_0} u^v v_2 du = 2\pi i \left( u_0 \sum_{j=0}^{m-1} \frac{h_{-j-1}n!}{j!(n-j)!} - \int_{0}^{u_0} u^v v_2 du \right),
\]

so the situation is the same as before, because the summand behaves as \(O(n^j)\), and the integral gives another factorial series as in (34).

Finally, we remark that the exceptional case when \(-m \in \mathbb{Z}\) and the logarithmic term vanishes, corresponds to the Juddian solutions discovered by Emary and Bishop. This can be verified by comparing the values of energy numbered by the integer \(m\) and the algebraic conditions on the other parameters which guarantee the absence of logarithms. The solutions are then of the form \(\exp(\sigma z^2)P(z)\), for a polynomial \(P\), so their expansions are still infinite series, but their Laplace transforms, hence the solutions \(v\), are rational.

Let now \(b^\pm_n\) denote the solutions of the recurrence equation (18), constructed by means of the Mellin transform around \(u_0 = \pm \kappa/2\), respectively. Their asymptotic growth is as required, and it remains to be checked whether the solution \(a_n\), obtained around \(\xi = 0\), is their linear combination. Because we are dealing with a linear recurrence it is enough to check the linear dependence for two consecutive elements, which means that if, for some \(n_0 \geq 2\), the rank of the \(4 \times 3\) matrix

\[
\begin{bmatrix}
  a_{n_0} & b^+_n & b^-_n \\
  a_{n_0+1} & b^+_n & b^-_n \\
\end{bmatrix},
\]

is less than 3, then \(a_n \in \text{Span}(b^+_n, b^-_{n})\). In practice one has to check that all the \(3 \times 3\) minors of the above matrix vanish. If that happens for some value of energy, there exists an entire solution of the desired asymptotics, i.e. with finite norm.

**Appendix D. Factorial series for general position of singular points**

Let us deal with the radius of convergence of the series (33). The crucial obstacle is that both the regular points \(\frac{\kappa}{2}\) and \(\frac{1}{\kappa}\) can be arbitrarily close to \(\frac{1}{2}\) as \(\kappa\) gets close to 1 (and likewise for their negative counterparts), so the radius of convergence gets smaller and smaller. To remedy this one can choose the following new independent variable

\[
w = \left( \frac{u}{u_0} \right)^p,
\]

with a sufficiently large, real \(p\). For clarity let us look at the positive regular point \(u_0 = \kappa/2\) as the negative case is analogous. The point of larger absolute value will be mapped into \((1/\kappa^2)^p\), which can be made larger than 2 by taking
$p > \log_{\kappa} (2) = -\frac{\ln 2}{2 \ln \kappa}, \quad \text{(D.2)}$

or, for computational purposes,

$$p = \max \left\{ 1, \frac{1}{2(1 - \kappa)} \right\}, \quad \text{(D.3)}$$

because for $\kappa < 2^{-1/2} \approx 0.7$ there radius of convergence is already large enough, and otherwise we have $-\ln \kappa > 1 - \kappa$.

With proper $p$, the above mapping will send a small connected region around $u = u_0$ into the disk of radius 1 centered at $w = 1$, so the Mellin integral will change to

$$\mathcal{M}[v]_n = \frac{u_0^n}{p \Gamma(n + 1)} \int_C w^{n-1} \tilde{v}(w) \, dw, \quad \text{(D.4)}$$

with $\tilde{v}(w) = v(u_0 w^{1/p}) = v(u)$ being holomorphic around $w = 1$. An example of such a disk map is shown in figure D1. One can then expand $\tilde{v}(w)$, and tie it with the expansion of $v(u)$

$$\tilde{v}(w) = \sum_{j=0}^{\infty} H_j (w - 1)^{\nu + j}$$

$$v(u_0 w^{1/p}) = \sum_{k=0}^{\infty} h_k u_0^{\nu+k} (w^{1/p} - 1)^{\nu+k} = \sum_{k=0}^{\infty} h_k u_0^{\nu+k} \left( \sum_{l=1}^{\infty} \left( \frac{1}{p} \right)^l (w - 1)^l \right)^{\nu+k}. \quad \text{(D.5)}$$

Comparing the two series the following relation between their coefficients can be found

$$H_j = u_0^{\nu} \sum_{k=0}^{j} B_{j-k} \sum_{m=0}^{k} A_{m,k} h_m u_0^m, \quad \text{(D.6)}$$
where $A_{m,j}$ are given recursively by
\[
A_{0,j} = \delta_{0,j},
A_{1,j} = \left(\frac{1}{p}\right)^{j},
A_{m,j} = \sum_{l=1}^{j-m+1} \left(\frac{1}{p}\right)^{l} A_{m-1,j-l}, \text{ for } j \geq m,
A_{m,j} = 0, \text{ for } j < m,
\]
(D.7)

and $B_{j}$ are the series coefficient in
\[
(w^{1/p} - 1)^{\nu} =: (w - 1)^{\nu} \sum_{j=0}^{\infty} B_{j}(w - 1)^{j}.
\]
(D.8)

Finally, the modified Mellin transform can be given as
\[
\mathcal{M}[g]_{n} = \frac{w^{n}}{p \Gamma(n + 1)} \sum_{j=0}^{\infty} (-1)^{\nu+j} \Gamma\left(\frac{n}{p} + 1\right) \Gamma\left(1 + j + \nu\right) \frac{H_{j}}{\Gamma\left(2 + j + \frac{n}{p} + \nu\right)}
\]
(D.9)

Alternatively, $H_{j}$ can be obtained directly, without the use of $h_{j}$, by writing the system in the variable $w$. Such differential equation has coefficients which are not rational but they admit power series expansion in the relevant region so the solution can be constructed by the Frobenius method around $w = -1$.

References

[1] Maciejewski A J, Przybylska M and Stachowiak T 2014 Analytical method of spectra calculations in the Bargmann representation Phys. Lett. A 378 3445–51
[2] Gerry C 1988 Two-photon Jaynes-Cummings model interacting with the squeezed vacuum Phys. Rev. A 37 2683
[3] Emery C 2001 PhD Thesis Manchester
[4] Emery C and Bishop R F 2002 Exact isolated solutions for the two-photon Rabi Hamiltonian J. Phys. A: Math. Gen. 35 8231–41
[5] Penna V and Raffa F A 2016 Off-resonance regimes in nonlinear quantum Rabi models Phys. Rev. A 93 043814
[6] Maciejewski A J, Przybylska M and Stachowiak T 2015 An exactly solvable system from quantum optics Phys. Lett. 379 1503–9
[7] Bargmann V 1961 On a Hilbert space of analytic functions and an associated integral transform Commun. Pure Appl. Math. 14 187–214
[8] Newman D J and Shapiro H S 1971 A Hilbert space of entire functions related to the operational calculus unpublished
[9] Newman D J, Shapiro H S 1966 Certain Hilbert spaces of entire functions Bull. Am. Math. Soc. 72 971–7
[10] Levin B Y 1996 Lectures on Entire Functions (Translations of Mathematical Monographs vol 150) (Providence, RI: American Mathematical Society) (In collaboration with and with a preface by Y Lyubarskii, M Sodin and V Tkachenko, Translated from the Russian manuscript by Tkachenko)
[11] Boas R P Jr 1954 Entire Functions (New York: Academic)
[12] Vourdas A 2006 Analytic representations in quantum mechanics J. Phys. A: Math. Gen. A 39 R65
[13] Travêncel I 2012 Solvability of the two-photon Rabi hamiltonian Phys. Rev. A 85 043805
[14] Chen Q H, Wang C, He S, Liu T and Wang K L 2012 Exact solvability of the quantum Rabi model using Bogoliubov operators Phys. Rev. A 86 023822
[15] Duan L, Xie Y-F, Braak D and Chen Q-H 2016 Two-photon Rabi model: analytic solutions and spectral collapse J. Phys. A: Math. Theor. 49 464002

[16] Lü Z, Zhao C and Zheng H 2017 Quantum dynamics of two-photon quantum Rabi model J. Phys. A: Math. Theor. 50 074002

[17] Okubo K 1963 A global representation of a fundamental set of solutions and a Stokes phenomenon for a system of linear ordinary differential equations J. Math. Soc. Japan 15 268–88

[18] Turrittin H L 1960 The formal theory of systems of irregular homogeneous linear difference and differential equations Bol. Soc. Mat. Mex. 5 255–64