Vanishing of the $\tilde{K}Nil$ groups: localization methods

Frank BIHLER

Abstract:

The aim of this article is to introduce Vogel’s localization theorem for classes of $\mathcal{D}$-complexes: this generalization of Waldhausen’s localization theorem is especially useful and powerful in that it gives an explicit and computable description of the local objects. Next we present an excision theorem for transverse classes. Finally, we apply these methods to deduce partial results on Vogel’s Conjecture on a regular ring $R$ (cf [Bih01]).

Keywords:

COMPLEX OF DIAGRAMS – LOCALIZATION – EXCISION
VANISHING OF THE $\tilde{K}Nil$ GROUPS – VOGEL’S CONJECTURE

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1 Main Theorem

We shall expose at first the localization theorem, powerful tool introduced by Vogel in [Vog90], as a motivation to develop further notations and constructions, and then apply it to our [Conjecture] on a regular ring $R$ : namely, give a description of the algebraic K-theory spectrum $\tilde{KNil}(R; S)$ for $S$ a $R$-bimodule, flat from the left, reduce it through explicit localization computations, and ideally prove the vanishing of these obstruction groups to excision. In a few words, the reader can think about $\mathcal{D}$-complexes as representations of a diagram $\mathcal{D}$ in the category of chain complexes : h-small and h-compact are finiteness hypothesis to insure the existence of algebraic K-theory spectra, and the 'exactness' condition reminds about Waldhausen's 'stable under extensions subcategory' or about Verdier's 'thick subcategory' needed for a good localization. These form the natural setting for a generalization of the different categories of $Nil$ introduced by Waldhausen in [Wal78]. Precise definitions will just follow.

Theorem 1 [Localization] [Vog90]
Let $\mathcal{A} \subset \mathcal{B}$ be two exact classes of $\mathcal{D}$-complexes. Suppose that the class $\mathcal{B}$ is h-small and h-compact, and that the class $\mathcal{A}$ is stable in $\mathcal{B}$ under direct summands. Let $\mathcal{L}$ be the class of all $\mathcal{A}$-local $\mathcal{D}$-complexes $L$, such that there exists $X \in \mathcal{B}$ and an $\mathcal{A}$-equivalence $X \to L$. Then $\mathcal{L}$ is an exact class, and we have an homotopy equivalence of spectra : $K(\mathcal{B}, \mathcal{A}) \simeq K(\mathcal{L})$.

2 $\mathcal{D}$-complexes

We shall now expose, by order of increasing abstraction, the numerous definitions required to understand the hypothesis of the above localization theorem.

Definition 1.

1. A "diagram of bimodules" $\mathcal{D}$ is given by :
   • an oriented graph $\Gamma$
   • for each vertex $a \in \Gamma_0$, an associated ring $A_a$
   • for each edge $(f : a \to b) \in \Gamma_1$, an associated bimodule $S_f$ with an action of $A_b$ on the left, and an action of $A_a$ on the right.

Consider now the diagram $\mathcal{D}$ fixed.

2. A "$\mathcal{D}$-module" is given by :
   • for each vertex $a$, an associated right $A_a$-module $M_a$
   • for each edge $(f : a \to b)$, an associated map of modules $M_a \to M_b \otimes S_f$

The $\mathcal{D}$-modules form an abelian category.

We can thus take its derived category and define :

3. A "$\mathcal{D}$-complex" is a complex of $\mathcal{D}$-modules.

4. A $\mathcal{D}$-module is called "nilpotent" if there exists a length $n$ such that every composition of more than $n$ linear maps vanishes. Similarly, a $\mathcal{D}$-complex is called "h-nilpotent" or homotopy-nilpotent, or nilpotent up to homotopy, if there exists a length $n$ such that every composition of more than $n$ chain maps is null-homotopic.

Convention :

As the aim of this article is to treat objects fit to algebraic K-theory, we shall
from now on take this convention: a "ring" A shall mean a unitary associative ring; a "module" M shall mean a right projective module (unless otherwise stated); a "bimodule" S shall mean a flat on the left bimodule; a "complex" shall mean a Z-graded module M* equipped with a differential ∂: Mn → Mn−1 such that ∂ ∘ ∂ = 0. With this convention, we only ask a ∂-complex Cα to be composed by complexes Ca on each vertex a of the graph Γ, projective in each degree. Now a "chain map" shall mean a morphism of complexes. Define a "finite complex" C* to be a complex such that ⊕Cn is finitely generated. Finally, a "h-finite complex" D* shall mean a complex equipped with a finite complex C* and two chain maps f : C* → D* and g : D* → C*, such that f ∘ g ∼ Id : D* → D* and g ∘ f ∼ Id : C* → C*. We shall denote ΞR the class of h-finite complexes on the base ring R.

Examples:
Friedhelm Waldhausen introduced three categories Nil(R; S), Nil(R; S, T) and Nil(R; S, T, U, V) to study the algebraic K-theory of a tensor algebra, a free product of rings, and respectively a Laurent extension (or HNN-extension) of rings; these categories can be naturally viewed as categories of nilpotent D-modules, on the diagrams D1, D2 and D3 hereafter. In each case, the involved category of D-modules gives the obstructions to excision detailed in [Wal78].

- The category Nil(R; S) defined for the tensor algebra R[S] can be represented by the following diagram D1 of bimodules; similarly, one can draw the 'picture' of a D1-module, a nilpotent D1-module, a D1-complex, and a homotopy-nilpotent D1-complex:

\[
\begin{align*}
D_1 & = \bullet_R \bigcirc S \\
D_1\text{-module} & = \bullet_{M_R} \bigcirc f \quad \text{(with f : } M_R \to M_R \otimes_R S \text{ linear)} \\
\text{nilpotent } D_1\text{-module} & = \bullet_{M_R} \bigcirc f \quad (\exists n \geq 0, f^n = 0 : M_R \to M_R \otimes_R S^n) \\
D_1\text{-complex} & = \bullet_{C_R} \bigcirc f \quad \text{(with f : } C_R \to C_R \otimes_R S \text{ chain map)} \\
\text{h-nilpotent } D_1\text{-complex} & = \bullet_{C_R} \bigcirc f \quad (\exists n \geq 0, f^n \sim 0 : C_R \to C_R \otimes_R S^n)
\end{align*}
\]

- Let's consider Nil(R; S, T) defined for the generalized free product of rings, where the two pure embeddings α, β admit splittings as R-bimodules.

\[
\begin{align*}
R & \xrightarrow{\alpha} R \oplus S \\
R \oplus T & \xrightarrow{\beta} \text{pushout}
\end{align*}
\]

Here the naturally associated diagram D2 of bimodules is of the shape:

\[
\begin{align*}
D_2 & = \bullet_R \bigcirc S \\
\text{h-nilpotent } D_2\text{-complex} & = \bullet_{C_R} \bigcirc f \quad (\exists n \geq 0, f^n \sim 0 : C_R \to C_R \otimes_R (S \otimes_R T)^n)
\end{align*}
\]
Let’s consider $\text{Nil}(R; S, T, U, V)$ defined for the Laurent extension of rings, where the two pure embeddings $\alpha, \beta$ admit splittings as $R$-bimodules.

\[
\begin{array}{c}
R \xrightarrow{\alpha} A \xrightarrow{} \text{HNN-ext} \xrightarrow{} \text{HNN-ext} \\
& \xrightarrow{\beta}
\end{array}
\]

with $A = S \oplus U = T \oplus V; S = \alpha R; T = \beta R$

Here the naturally associated diagram $D_3$ is of the shape:

\[
D_3 = \begin{array}{c}
\bullet_R \xrightarrow{U} \bullet_R \\
\downarrow \downarrow \\
\bullet_R \xrightarrow{V} \bullet_R
\end{array}
\]

$h$-nilpotent $D_3$-complex = $h\begin{array}{c}
\bullet_C \xrightarrow{f} \bullet_R \\
\downarrow \downarrow \\
\bullet_R \xrightarrow{g} \bullet_D
\end{array}$

(with $f : C_R \rightarrow D_R \otimes_R U$ and $g : D_R \rightarrow C_R \otimes_R V$ and $h : C_R \rightarrow C_R \otimes_R S$ and $i : D_R \rightarrow D_R \otimes_R T$ chain maps and $\exists n \geq 0$, such that every composed map of length $n$ is null-homotopic; in particular this composed map $h^{n-4} \circ (g \circ f)^2 \sim 0 : C_R \rightarrow C_R \otimes_R (U \otimes_R V)^2 \otimes_R S^{n-4}$)

In terms of algebraic K-theory, we have the following decomposition:

\[
\text{KNil}(R; S) = K(R) \oplus \tilde{\text{KNil}}(R; S)
\]

More generally, for each diagram of bimodules $\mathcal{D}$, we can use the notion of $h$-nilpotent $\mathcal{D}$-complexes (here we use $h$-finite complexes) to define a K-theory spectrum $\tilde{\text{KNil}}(\mathcal{D})$ that will extend the usual notions presented above, and first introduced by Waldhausen in [Wal78].

3 Exact Classes of $\mathcal{D}$-complexes

Definition 2 .

Let $\mathcal{D}$ be a fixed diagram of bimodules, and $\mathfrak{A}$ a class of $\mathcal{D}$-complexes.

- The class $\mathfrak{A}$ is called “exact” if:
  - $\mathfrak{A}$ contains all finite acyclic $\mathcal{D}$-complexes (ie acyclic on each vertex of $\Gamma$)
  - $\mathfrak{A}$ verifies the ‘$2/3$ axiom’ : let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of $\mathcal{D}$-complexes, if two of them are in $\mathfrak{A}$, then so is the third.

- The “completion” of $\mathfrak{A}$, denoted by $\tilde{\mathfrak{A}}$, is the smallest exact class containing $\mathfrak{A}$ and stable under direct sum : $\forall i \in I, A_i \in \mathfrak{A} \Rightarrow \oplus A_i \in \tilde{\mathfrak{A}}$.

- A morphism $f$ of $\mathcal{D}$-complexes is a “$\mathfrak{A}$-equivalence” if the usual mapping cone $C(f) \in \mathfrak{A}$.

Recall the classical construction issued from Waldhausen’s cylinder-functor $T$:

\[
\begin{array}{c}
A \xrightarrow{j_1} T(f) \xrightarrow{j_2} B \\
\downarrow \downarrow \\
\xrightarrow{f} \downarrow \downarrow \\
\xrightarrow{p} B
\end{array}
\]

$0 \rightarrow A \xrightarrow{j_1} T(f) \xrightarrow{i_1} C(f) \rightarrow 0$

So $C(f) \in \mathfrak{A}$ by the ‘$2/3$ axiom’. 

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4 Finiteness Properties

Definition 3. Let $\mathcal{D}$ be a fixed diagram of bimodules, and $\mathfrak{A}$ a class of $\mathcal{D}$-complexes.

$\mathfrak{A}$ The class $\mathfrak{A}$ is “$h$-small” or homotopy-small, or small up to homotopy if

$$\exists (X_i)_{i \in I}, \forall Y \in \mathfrak{A}, \exists i, \exists X_i \xrightarrow{\sim} Y$$

Here $(X_i)_{i \in I}$ denotes a set of $\mathcal{D}$-complexes, and $\sim$ a homology equivalence.

$\mathfrak{B}$ The class $\mathfrak{A}$ is “$h$-compact” or homotopy-compact, or compact up to homotopy if $\forall X \in \mathfrak{A}, \forall (Y_i)_{i \in I} \in \mathfrak{A}^I, \forall (f : X \to \bigoplus Y_i) \in Fl(\mathfrak{A}),$

$$\exists (g : X' \xrightarrow{\sim} X) \in Fl(\mathfrak{A}) \text{ such that } f \circ g \text{ factorizes through a finite sum}.$$  

$\mathfrak{C}$ A $\mathcal{D}$-complex $X$ is “$\mathfrak{A}$-local” if every $\mathcal{D}$-complex morphism $f : Y \to X$ with $Y \in \mathfrak{A}$ factorizes through an acyclic $\mathcal{D}$-complex.

5 Easy Applications

1st example:
Let $\mathfrak{B}$ be the class of usual $h$-finite $\mathbb{Z}$-complexes (here the diagram $\mathcal{D}_4$ is reduced to one vertex with the ring $\mathbb{Z}$). And consider the sub-class $\mathfrak{A} \subset \mathfrak{B}$ of $\mathcal{D}_4$-complexes that are $\mathbb{Q}$-acyclic. Then $\mathfrak{A}, \mathfrak{B}$ are exact classes of $\mathcal{D}_4$-complexes. The class $\mathfrak{B}$ is $h$-small and $h$-compact, and $\mathfrak{A}$ is stable in $\mathfrak{B}$ under direct summands. Now Theorem 1 gives us the class $\mathfrak{L}$ of all $\mathbb{Z}$-complexes with the rational homotopy type of a finite complex, and an homotopy equivalence of spectra $\tilde{K}(\mathfrak{B}, \mathfrak{A}) \simeq K(\mathfrak{L})$. Here we consider $\mathfrak{L}'$ the class of all finite $\mathbb{Q}$-complex. The natural inclusion $\mathfrak{L}' \subset \mathfrak{L}$ induces an equivalence on the algebraic K-theory spectra by Waldhausen’s approximation theorem, and $K(\mathfrak{L}) = K(\mathfrak{L}') = K(\mathbb{Q})$. We can also identify $K(\mathfrak{B}) = K(\mathbb{Z})$ and $K(\mathfrak{A}) = \oplus K(\mathbb{F}_p)$ (the direct sum here is over the prime numbers $p$) by Quillen’s Dévissage. Hence we get the classical commutative localization long exact sequence, usually obtained via the Ore condition.

$$\ldots \to \oplus K_i(\mathbb{F}_p) \to K_i(\mathbb{Z}) \to K_i(\mathbb{Q}) \to \oplus K_{i-1}(\mathbb{F}_p) \to \ldots$$

2nd example:
Consider now the following diagram of bimodules:

$$\mathcal{D}_5 = \begin{array}{ccc}
\bullet_A & \xrightarrow{S} & \bullet_B \\
\downarrow_T & & \\
\bullet_B & & \end{array}$$

We note $\mathfrak{B}$ the class of $h$-nilpotent $\mathcal{D}_5$-complexes, ie $\mathfrak{B} = \{(C_A, C_B; \alpha, \beta)\}$ with $C_A, C_B$ $h$-finite complexes, $\alpha : C_A \to C_B \otimes S$ and $\beta : C_B \to C_A \otimes T$ verifying a nilpotency condition: $\exists n \geq 0, (\alpha \beta)^n \sim 0$. Consider now the sub-class $\mathfrak{A} \subset \mathfrak{B}$ of $\mathcal{D}_5$-complexes such that $C_A$ is contractible. Then the class $\mathfrak{B}$ is $h$-small and $h$-compact, and the class $\mathfrak{A}$ is stable in $\mathfrak{B}$ under direct summands. Now look at the $\mathfrak{A}$-local objects: we test on $(0, C'_B; 0, 0) \in \mathfrak{A}$ to obtain the following data:

$$\begin{array}{ccc}
\xrightarrow{0} & \xrightarrow{C'_B} & \\
\downarrow & \downarrow & \\
C_A & \xrightarrow{\alpha} & C_B \\
\downarrow & \downarrow & \\
\downarrow & \downarrow & \\
\end{array}$$

$^\dagger$This condition avoids set-theoretic problems for the existence of the K-theory $K(\mathfrak{A})$. 

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Every vertical map of $\mathcal{D}_3$-complex is thus given by $f : C_B' \to Ker\beta$. We can suppose that $\beta$ is surjective, so $\mathfrak{A}$-local implies that $\beta$ is a homology equivalence. Conversely, every $\mathcal{D}_3$-complex with $\beta$ a homology equivalence is easily seen to be $\mathfrak{A}$-local. Here the condition of $\mathfrak{A}$-equivalence says that in fact $C_B$ is $h$-finite. Now, such a data is just $(C_B, \alpha\beta)$, that is $\S$ an element of $\mathrm{Nil}(B; T \otimes_A S)$, thus Waldhausen’s approximation theorem gives the equivalence of spectra : $K(\mathfrak{A}) \simeq K\mathrm{Nil}(B; T \otimes_A S)$. Now Theorem 1 gives this homotopy fibration : $K(A) \twoheadrightarrow K(B) \twoheadrightarrow K\mathrm{Nil}(B; T \otimes_A S)$ symmetrically, we have also this other homotopy fibration : $K(B) \twoheadrightarrow K(A) \twoheadrightarrow K\mathrm{Nil}(A; S \otimes_B T)$. Hence comparing the two lines above gives the identification : $K(B) \simeq K(A) \oplus K(B) \oplus “\text{defect}”$.

**Proposition 1**.

Let $A, B$ be two rings, and let $A_S B$ and $B_T A$ be two bimodules, flat on the left. Then we have a homotopy equivalence : $K\mathrm{Nil}(B; T \otimes_A S) \simeq K\mathrm{Nil}(A; S \otimes_B T)$.

**Use** :

If $S$ is flat from the left and $A$ Noetherian regular, then $[\text{Wal78}]$ implies that $K\mathrm{Nil}(A; S \otimes_B T)$ is contractible. Thus so is the spectrum $K\mathrm{Nil}(B; T \otimes_A S)$ ! Hence some tricky reductions of the Nil groups on diagrams ...

**3rd example** :

If we keep the notations of $[\text{Bih02}]$, for a regular ring $R$ and a flat from the left bimodule $S$, we have the following homotopy equivalences of non-connective spectra : $K\mathrm{Nil}(R; S) \simeq K(\mathfrak{B}, \mathfrak{A}) = \lim K(P_n, S_n)$. So an approach of Vogel’s [Conjecture] based on localization should study first the graded case for $\mathfrak{B} : (\mathfrak{B} = P_{n+1}, \mathfrak{A} = P_n)$, and then the graded case for $\mathfrak{A} : (\mathfrak{B} = P_{n+1}, \mathfrak{A} = S_n)$. It appears that the $\mathfrak{B}$ case breaks out easily, but the nilpotent case $\mathfrak{A}$ gives only partial information, for lack of a good interpretation of local objects. We postpone the complete description of these computations till section 8.

6 Proof of the Main Theorem

This rather technical proof shall be decomposed in eight elementary lemmas. Notations are those of Theorem 1.

**Lemma 1**.

A class $\mathfrak{A}$ of $\mathcal{D}$-complexes is exact, if and only if $\mathfrak{A}$ is stable under the Coker of a cofibration, $\mathfrak{A}$ is stable under desuspension, and $\mathfrak{A}$ contains the acyclics.

**Proof** :

Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of $\mathcal{D}$-complexes. Suppose that $Z$ is in $\mathfrak{A}$. There exists an epimorphism $E \to Y$ with $E$ acyclic. We can now “make the triangle turn” : if we note $s^{-1}Z$ the desuspension of $Z$ in $\mathfrak{A}$, then $0 \to s^{-1}Z \to X \oplus E \to Y \to 0$ is also a short exact sequence. Now, if $X$ is in $\mathfrak{A}$, this shows that $Y$ is also in $\mathfrak{A}$. Hence the class $\mathfrak{A}$ is stable under extensions. Conversely, if $Y$ is in $\mathfrak{A}$, then $X \oplus E$ is also, hence $X = \operatorname{Coker}[E \to X \oplus E]$ is in $\mathfrak{A}$ too. Thus the class $\mathfrak{A}$ verifies the ’2/3 axiom’, $\mathfrak{A}$ contains the acyclics, so $\mathfrak{A}$ is exact. $\blacksquare$

$\S$The equivalence of the different definitions of $\mathrm{Nil}(R; S)$ is proven in $[\text{Bih02}]$. 
Lemma 2 .
Let $\mathfrak{A}$ be an exact class of $\mathscr{D}$-complexes.
Let $\mathfrak{A}_\mathfrak{L}$ be the class of $\mathfrak{A}$-local $\mathscr{D}$-complexes. Then the class $\mathfrak{A}_\mathfrak{L}$ is also exact.

Proof :
The class $\mathfrak{A}_\mathfrak{L}$ contains the acyclics. Like $\mathfrak{A}$, the class $\mathfrak{A}_\mathfrak{L}$ is stable under desuspension. So we need to prove that $\mathfrak{A}_\mathfrak{L}$ is stable under the $\text{Coker}$ of a cofibration. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence, with $X$ and $Y$ in $\mathfrak{A}_\mathfrak{L}$. Consider a $\mathscr{D}$-complex $U$ in $\mathfrak{A}$ and a $\mathscr{D}$-complex morphism $f : U \to Z$. There exists an epimorphism $E \to U \coprod_{\mathfrak{A}} Y$ with $E$ acyclic and onto the pullback $V \coprod_{\mathfrak{A}} Y$. Note $V = \text{Ker}[E \to U]$ : the exact class $\mathfrak{A}$ contains $U$ and $E$, thus $V$ too. We obtain the following commutative diagram :

\[
\begin{array}{ccc}
0 & \to & X \\
 & \downarrow & \downarrow \\
0 & \to & V \\
& \downarrow & \downarrow \\
0 & \to & E
\end{array}
\]

The map $V \to X$ admits a factorization through an acyclic $\mathscr{D}$-complex $F$. Build the pushout $W = F \coprod_{\mathfrak{A}} E$ : it’s an object in $\mathfrak{A}$, thus the map $W \to Y$ admits a factorization through an acyclic $\mathscr{D}$-complex $G$. We can suppose that $F \to G$ is a cofibration (if need be, we can add an acyclic to $G$), and denote $H = \text{Coker}[F \to G]$. Then the map $f$ admits a factorization through $H$ acyclic (cf the following diagram), and the class $\mathfrak{A}_\mathfrak{L}$ is exact. \qed

Lemma 3 .
Let $\mathfrak{A}$ be an exact, h-small and h-compact class of $\mathscr{D}$-complexes. Let $X$ be any $\mathscr{D}$-complex. Then there exists an $\mathfrak{A}$-local $\mathscr{D}$-complex $Y$ and an $\mathfrak{A}$-equivalence $X \to Y$.

Proof :
We shall build by induction a series $X = Z_0 \to Z_1 \to Z_2 \to \ldots$ of $\mathscr{D}$-complexes, where all maps $Z_i \to Z_{i+1}$ are cofibrations and $\mathfrak{A}$-equivalences, and where the colimit $Z$ shall be an $\mathfrak{A}$-local $\mathscr{D}$-complex. As the class $\mathfrak{A}$ is h-small, denote $\{X_{\lambda}\}_{\lambda \in \Lambda}$ the set of $\mathscr{D}$-complexes that generate $\mathfrak{A}$. Suppose that $Z_i$ is built for every $0 \leq i \leq n$. Let $T_n$ be the set of pairs $(\lambda, u)$ with $\lambda \in \Lambda$ and $u : X_{\lambda} \to Z_n$. Denote $U_n$ the direct sum of all $X_{\lambda}$ for $(\lambda, u) \in T_n$. We have a canonical map from $U_n$ to $Z_n$. Let $U_n \to E_n$ be a cofibration into an acyclic $E_n$, and build the pushout $Z_n+1 = Z_n \coprod_{U_n} E_n$. Then the map $Z_n \to Z_{n+1}$ is a cofibration; it’s an $\mathfrak{A}$-equivalence since $X_{\lambda} \in \mathfrak{A}$ implies $U_n \in \mathfrak{A}$. Let now $Z$ be the colimit of this series, and $f : Y \to Z$ any map from any $Y$ in $\mathfrak{A}$. Let $\alpha : \oplus Z_n \to \oplus Z_n$ be the difference between the identity and the stabilization map $Z_n \to Z_{n+1}$. There exists an epimorphism $E \to \oplus Z_n \coprod_{\mathfrak{A}} Y$ from an acyclic $E$ onto the pullback $\oplus Z_n \coprod_{\mathfrak{A}} Y$. We obtain the following commutative diagram :

\[
\begin{array}{ccc}
0 & \to & \oplus Z_n \\
 & \downarrow & \downarrow \\
0 & \to & \oplus Z_n \\
& \downarrow & \downarrow \\
0 & \to & K \\
& \downarrow & \downarrow \\
0 & \to & E
\end{array}
\]

Now we shall use twice the hypothesis of h-compactness on $\mathfrak{A}$ to reduce (modulo homology equivalence) the two infinite sums above to finite sums : there exists a homology equivalence $K' \to K$ such that the composed map $K' \to K \to \oplus Z_n$ admits a factorization through $\oplus Z_n$.

We can suppose that the map $K' \to K$ is a cofibration (if need be, we can add an acyclic to $K$ and $E$). Now the map $\alpha$ sends $\oplus Z_n$ to $\oplus Z_n$; by quotient, the middle vertical map sends $E/K'$ to $\oplus_{n+1}^{\infty} Z_n$. By h-compactness, there exists a homology equivalence $V \to E/K'$ that
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sends $V$ on a finite sum. Build now the pullback $E' = E \coprod_{E/K'} V$. We have thus built a finite factorization of the diagram above, for a certain finite integer $q$ (the cokernel $Y'$ is homologically equivalent to $Y$)

\[
\begin{array}{c}
  0 \to Z' \oplus Z_n \xrightarrow{\alpha} Z_n \to Z_q \to 0 \\
  0 \to E' \to Y' \to 0 \\
  0 \to Y \to Z
\end{array}
\]

Now $Y'$ being in $\mathcal{A}$ is equivalent to a certain $X_\lambda$, then the map $u : X_\lambda \to Y'$ is in $T_q$, thus the composed map $X_\lambda \to T_{q+1}$ admits by construction a factorization through an acyclic $F$. We now build the pushout $G = F \coprod_{X_\lambda} Y$, that is acyclic : $X_\lambda \to Y' \to Y$ is a homology equivalence, thus $F \to G$ is one also. By construction, the map $f$ factorizes through $G$, hence the colimit-object $Z$ in $\mathcal{A}$-local. Finally, consider $0 \to \bigoplus_0^\infty Z_n/X \to \bigoplus_0^\infty Z_n/X \to Z/X \to 0$ this short exact sequence shows that the colimit-map $X \to Z$ is a $\mathcal{A}$-equivalence.

**Lemma 4.**
Let $X$ be an $\mathcal{A}$-local $\mathcal{D}$-complex. Then $X$ is also $\tilde{\mathcal{A}}$-local.

**Proof:**
Let $\mathcal{G}$ be the class of all $\mathcal{D}$-complexes $Y$, such that, for all $L$ in $\mathcal{L}\mathcal{A}$, every map $Y \to L$ factorizes through an acyclic. This class contains $\mathcal{A}$. Like $\mathcal{L}\mathcal{A}$, it’s stable under desuspension. It’s stable under direct sum : actually, let $Y = \oplus Y_i$ and $f = \oplus f_i : Y_i \to L$ with $L$ in $\mathcal{L}\mathcal{A}$, each $f_i$ factorizes through an acyclic $E_i$, so $f$ factorizes through $E = \oplus E_i$ acyclic. Let’s prove that $\mathcal{G}$ is stable under the Coker of a cofibration : let $0 \to X \to Y \to Z \to 0$ be a short exact sequence, with $X$ and $Y$ in $\mathcal{G}$ : let $L$ be in $\mathcal{L}\mathcal{A}$ and consider $f : Z \to L$. The composed map $Y \to Z \to L$ factorizes through $E$ acyclic. We can choose $E$ such that $E \to L$ is surjective, and denote $L'$ the Kernel. As $\mathcal{L}\mathcal{A}$ is an exact class, $L'$ is in $\mathcal{L}\mathcal{A}$, thus the map at the level of the Kernels $X \to L'$ factorizes through $F$ acyclic. Let $E'$ be the direct sum of $E$ and an acyclic containing $F$ by a cofibration. Denoting $G = \text{Coker}[F \to E']$, that is also acyclic, we can complete the diagram of compatible short exact sequences :

\[
\begin{array}{c}
  0 \to L' \to E \to L \to 0 \\
  0 \to F \to E' \to G \to 0 \\
  0 \to X \to Y \to Z \to 0
\end{array}
\]

The class $\mathcal{G}$ is exact, stable under direct sum, and contains $\mathcal{A}$. Thus $\mathcal{G}$ contains $\tilde{\mathcal{A}}$ too.

**Lemma 5.**
Let $f : X \to Y$ be a morphism of $\mathcal{D}$-complexes, with $X \in \mathcal{B}$ and $Y \in \tilde{\mathcal{A}}$. Then $f$ admits a factorization through a $\mathcal{D}$-complex $Z \in \mathcal{A}$.

**Proof:**
Let $\mathcal{G}$ be the class of all $\mathcal{D}$-complexes $Y$ in $\tilde{\mathcal{A}}$ such that, for all $X$ in $\mathcal{B}$, every map $X \to Y$ factorizes through an object in $\mathcal{A}$. This class contains $\mathcal{A}$. Like $\mathcal{B}$, it’s stable under desuspension. Let’s prove that $\mathcal{G}$ is stable under direct sum : let $Y_i$ be objects in $\mathcal{G}$ and consider a map $f = \oplus f_i : X \to Y = \oplus Y_i$. As $\mathcal{B}$ is h-compact, there exists a homology equivalence $X' \xrightarrow{\cong} X$ such that $X' \to X \to Y$ factorizes through a finite sum $Y' = \oplus Y_i$.

The map $X' \to Y'$ factorizes through an object $Z$ in $\mathcal{A}$, because the class $\mathcal{G}$ is obviously stable under finite direct sum (in the finite case, we have “product=smash”). The map $f$ factorizes through the pushout $U = X \coprod_{X'} Z$ which is also in $\mathcal{A}$ : actually, $X' \xrightarrow{\cong} X$
is a homology equivalence, so \( Z \xrightarrow{\sim} U \) is one too; then \( Z \) is in \( \mathfrak{A} \) implies that \( U \) is also!
We can sum up the situation in the following diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{(Z \in \mathfrak{A})} & Z \\
\downarrow \cong & & \downarrow \\
X & \xrightarrow{f} & U \\
\downarrow \cong & & \downarrow \cong \\
Y & \xrightarrow{= Y_1} & Y_1 \\
\end{array}
\]

Let’s now prove that the class \( \mathcal{C} \) is stable under the Coker of a cofibration. Let \( 0 \to X \to Y \to Z \to 0 \) be a short exact sequence, with \( X \) and \( Y \) in \( \mathcal{C} \). Let \( U \) be in \( \mathfrak{A} \) and consider a map \( f : U \to Z \). Let \( E \) be an acyclic that maps onto the pullback \( Y \coprod_Z U \). Denote \( K = \text{Ker}[E \to U] \). As \( X \) is in \( \mathcal{C} \), the map \( K \to X \) factorizes through an object \( X_1 \) in \( \mathfrak{A} \). As \( Y \) is in \( \mathcal{C} \), the map from the pushout \( X_1 \coprod_K E \) to \( Y \) factorizes through an object \( Y_1 \) in \( \mathfrak{A} \). We can suppose that the map \( X_1 \to Y_1 \) is a cofibration, and denote \( Z_1 \) the Cokernel (if need be, we can add acyclics). By factorization of the Cokernels, the map \( U \to Z \) factorizes through \( Z_1 \), as we can see on the following diagram:

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\end{array}
\begin{array}{cccccc}
X & \rightarrow & Y & \rightarrow & Z & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
X_1 & \rightarrow & Y_1 & \rightarrow & Z_1 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K & \rightarrow & E & \rightarrow & U & \rightarrow & 0 \\
\end{array}
\]

The class \( \mathcal{C} \) is exact, stable under direct sum, and contains \( \mathfrak{A} \); thus \( \mathcal{C} \) contains \( \mathfrak{A} \). ■

**Lemma 6.**

Let \( \mathfrak{B}' \) be the class of \( \mathcal{D} \)-complexes \( X \), such that there exists a \( \mathcal{D} \)-complex \( U \in \mathfrak{B} \) and a \( \mathfrak{A} \)-equivalence \( U \to X \). Then the canonical inclusion \( \mathfrak{B} \subset \mathfrak{B}' \) induces a homotopy equivalence of spectra:

\( K(\mathfrak{B}, \mathfrak{A}) \simeq K(\mathfrak{B}', \mathfrak{A}) \).

**Proof:**

We verify the hypothesis of Waldhausen’s approximation theorem for the inclusion functor \( F' : \mathfrak{B} \subset \mathfrak{B}' \). [App1]: Let \( X \) be an object in \( \mathfrak{B} \) such that \( F(\mathfrak{B}) \) is in \( \mathfrak{A} \). The identity map \( \text{Id} : X \to X \) factorizes through an object in \( \mathfrak{A} \). Thus there exists \( X' \) in \( \mathfrak{B} \) such that \( X \oplus X' \) is in \( \mathfrak{A} \). As \( \mathfrak{A} \) is stable under direct summand, \( X \) is already in \( \mathfrak{A} \). [App2]: Let \( X \) be an object in \( \mathfrak{B} \) and \( f : X \to Y \) any map in \( \mathfrak{B}' \). There exists \( Z \) in \( \mathfrak{B} \) and an \( \mathfrak{A} \)-equivalence \( Z \to Y \). Let \( E \) be an acyclic that contains \( Z \) by a cofibration, and denote \( U = \text{Coker}[Z \to Y \oplus E] \). The \( \mathcal{D} \)-complex \( U \) is in \( \mathfrak{A} \), and the map \( X \to Y \oplus E \) factorizes through an object \( V \) in \( \mathfrak{B} \). We build the pullback \( X' = V \coprod_Y (Y \oplus E) \): the map \( V \to U \) is an \( \mathfrak{A} \)-equivalence, thus \( X' \to Y \oplus E \) is one also; finally, \( X' \) is the extension of \( Z \) by \( V \) (two objects in \( \mathfrak{B} \)), thus \( X' \) is in \( \mathfrak{B} \) also. The following diagram proves the surjectivity hypothesis, and then Waldhausen’s approximation theorem gives the equivalence of K-theory spectra:

\[
\begin{array}{ccc}
Z & \xrightarrow{\sim} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{X'} & Y \oplus E \\
\downarrow & & \downarrow \\
V & \xrightarrow{\sim} & U
\end{array}
\]

Remark that the class \( \mathfrak{B}' \) contains the acyclics. It’s stable under desuspension (because \( \mathfrak{B} \) and \( \mathfrak{A} \) are). Finally, it’s stable under the Coker of a cofibration: let \( 0 \to X_1 \to X_2 \to X_3 \to 0 \) be a short exact sequence, with \( X_1 \) and \( X_2 \) in \( \mathfrak{B}' \). There exists an \( \mathfrak{A} \)-equivalence \( U_1 \to X_1 \) with \( U_1 \) in \( \mathfrak{B} \). We apply the axiom [App2] to the map \( U_1 \to X_1 \to X_2 \) to find a map \( u : U_1 \to U_2 \) in \( \mathfrak{B} \) and an \( \mathfrak{A} \)-equivalence \( U_2 \to X_2 \) such that the square commutes. Denote \( U_3 = \text{Coker}^b(u) \) in \( \mathfrak{B} \), we obtain by passing to the Cokernel the \( \mathfrak{A} \)-equivalence \( U_3 \to X_3 \) wanted: actually, \( \mathfrak{B} \) and \( \mathfrak{A} \) are stable under ‘2/3’ axiom. Hence the class \( \mathfrak{B}' \) is exact. ■
Lemma 7.
Let $\mathcal{E}$ be the category of short exact sequences of $\mathcal{D}$-complexes $X \to Y \to Z$ with $X \in \mathfrak{A}$ and $Z \in \mathcal{L}$. If we choose the homology equivalences at the level of the quotient term $Z$ to be the class $w\mathcal{E}$ of weak equivalences, then $\mathcal{E}$ has a structure of Waldhausen category. Let $F$ be the exact functor that sends each short exact sequence $X \to Y \to Z$ to its middle term $Y$. By Waldhausen’s approximation theorem, the functor $F$ induces a homotopy equivalence of spectra: $K(\mathcal{E}) \simeq K(\mathfrak{B}', \mathfrak{A})$.

Proof:
Let $F$ be the functor that sends each short exact sequence to its middle term. We shall verify the hypothesis of Waldhausen’s approximation theorem for $F$. 

$\text{[App1]}$: Let $X \to Y \to Z$ be a short exact sequence in $\mathcal{E}$. If $Y$ is in $\mathfrak{A}$, then $Z$ is in $\mathcal{L}$ and in $\mathfrak{A}$. By lemma 4, $Z$ is $\mathfrak{A}$-local, and the identity $\text{Id}: Z \to Z$ factorizes through an acyclic. Thus $Z$ is acyclic. We apply this argument to $\text{Coker}(f)$ for any map $f$ in $\mathcal{E}$ to deduce: $F(f)$ is in $\mathfrak{A}$ if and only if $f$ is a weak equivalence in $\mathcal{E}$. 

$\text{[App2]}$: Let $X \to Y \to Z = [S]$ be a short exact sequence in $\mathcal{E}$, and $f: Y \to Y_0$ any map in $\mathfrak{B}'$. We build the pushout $Z_0 = Z \coprod_{Y_0} Y_0$. As $\mathfrak{B}$ is h-small and h-compact, so is $\mathfrak{A}$; now by lemma 4, there exists an $\mathfrak{A}$-equivalence $Z_0 \to Z'$ with $Z'$ an $\mathfrak{A}$-local object. We add an acyclic $E$ that surjects onto $Z'$ to obtain a surjective map $(Y' = Y_0 \oplus E) \to Z'$, denote $X'$ its Kernel, we get a new short exact sequence in $\mathcal{E}$: $(X' \to Y' \to Z') = [S']$, dotted with a map of short exact sequences $\Phi : S \to S'$, and the map $f$ factorizes as $F(\Phi)$ followed by a homology equivalence (cf the following diagram).

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (S) {$[S]$};
  \node (S') at (1,0) {$[S']$};
  \node (Y) at (1,1) {$Y \in \mathfrak{B}'$};
  \node (Z) at (2,1) {$Z \in \mathcal{L}$};
  \node (Y0) at (1,2) {$Y_0 \in \mathfrak{B}'$};
  \node (Z0) at (2,2) {$Z_0$};
  \node (Y') at (1,3) {$Y' = Y_0 \oplus E$};
  \node (Z') at (2,3) {$Z'$};
  \draw[->] (S) -- (S');
  \draw[->] (S) -- (Y); \node at (1.5,1) {$\Phi$};
  \draw[->] (Y) -- (Z); \node at (1.5,1.5) {$f$};
  \draw[->] (Y') -- (Z'); \node at (1.5,2.5) {$\Phi$};
  \draw[->] (Y) -- (Y0); \node at (1.5,2.25) {(Id \oplus 0)};
  \draw[->] (Z) -- (Z0); \node at (2.25,2) {$\Lambda - eq$};
  \draw[->] (Y') -- (Y0); \node at (1.5,3) {(Id \oplus 0)};
\end{tikzpicture}
\end{figure}

Lemma 8.
Pose $\mathcal{L} = \mathfrak{B}' \cap \mathfrak{A}$. Then $\mathcal{L}$ is an exact class of $\mathcal{D}$-complexes. Let $G$ be the exact functor that sends an exact sequence $X \to Y \to Z$ to its quotient term $Z$. By Waldhausen’s additivity theorem, the functor $G$ induces a homotopy equivalence of spectra: $K(\mathcal{E}) \simeq K(\mathcal{L})$.

Proof:
The class $\mathcal{L}$ is the intersection $\mathfrak{B}' \cap \mathfrak{A}$, hence $\mathcal{L}$ is exact. By the additivity theorem, the K-theory spectrum $K(\mathcal{E})$ is equivalent to the product $K(\mathfrak{A}, \mathfrak{A}) \times K(\mathcal{L})$ and thus to $K(\mathcal{L})$.

This ends the proof of the Main Theorem.

7 Excision Theorem for Transverse Classes

Definition 4.
Let $\mathfrak{A}, \mathfrak{B}$ be two exact classes of $\mathcal{D}$-complexes.

We shall say that “$\mathfrak{A}$ is transverse to $\mathfrak{B}$”, denoted by $\mathfrak{A} \cap \mathfrak{B}$, if every map $f : A \to B$ with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ admits a factorization through a common
term $C \in \mathfrak{A} \cap \mathfrak{B}$. In that case, we shall note $\mathfrak{A} + \mathfrak{B}$ the smallest exact class of $\mathcal{D}$-complexes containing $\mathfrak{A}$ and $\mathfrak{B}$.

**Proposition 2.**

Let $\mathfrak{A}, \mathfrak{B}$ be two exact classes of $\mathcal{D}$-complexes. Suppose that $\mathfrak{A} \cap \mathfrak{B}$. Let $\mathfrak{C}$ be the class of $\mathcal{D}$-complexes $X$ (non-necessarily h-finite), such that there exists $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, and a short exact sequence: $\mathfrak{B} \rightarrow A \rightarrow X$. This class $\mathfrak{C}$ has a natural cylinder-functor, and verifies the '2/3 axiom'. As $\mathfrak{C}$ evidently contains $\mathfrak{A}$ and $\mathfrak{B}$, is exact, and is the smallest for these properties, thus we have: $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$.

**Proof:**

• With the definition above, we show the natural inclusions $\mathfrak{A} \subset \mathfrak{A} + \mathfrak{B}$ by the short exact sequence $0 \rightarrow A \rightarrow A$; and $\mathfrak{B} \subset \mathfrak{A} + \mathfrak{B}$ by the short exact sequence $s^{-1}B \rightarrow 0 \rightarrow B$. The first reflex is to verify if a short exact sequence 'in the other way' $A \rightarrow B \rightarrow X$ gives also an object in $\mathfrak{A} + \mathfrak{B}$. To this aim, we factorize $A \rightarrow B$ by $C$ in $\mathfrak{A} \cap \mathfrak{B}$, and write the diagram:

\[
\begin{array}{ccc}
A & \longrightarrow & \mathfrak{A} \\
\downarrow & & \downarrow \\
B & \longrightarrow & 0 \\
\end{array}
\]

• Let's look now at the four elementary cases of extension: we shall take the following notations $(X \in \mathfrak{A} + \mathfrak{B}), (A, A', A'' \in \mathfrak{A}), (B, B', B'' \in \mathfrak{B}),$ and $(C \in \mathfrak{A} \cap \mathfrak{B})$ obtained by factorization.

\[
\begin{array}{cccc}
A' & \longrightarrow & C & \longrightarrow & \mathfrak{B} \\
B & \longrightarrow & 0 & \longrightarrow & 0 \\
A'' & \longrightarrow & A & \longrightarrow & \mathfrak{A} \\
\end{array}
\]

These four diagram must be read vertically, and show that the class $\mathfrak{A} + \mathfrak{B}$ is stable under:

• the Kernel of a map going to $\mathfrak{A}$, the Cokernel of a map coming from $\mathfrak{B}$; the Cokernel of a map coming from $\mathfrak{A}$; finally, the Kernel of a map going to $\mathfrak{B}$. By composing these four elementary operations, we shall treat the '2/3' axiom: consider $(A \rightarrow X \rightarrow B), (A' \rightarrow Y \rightarrow B') \in \mathfrak{A} + \mathfrak{B}.

\[
\begin{array}{ccc}
A & \longrightarrow & X & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & K \\
\end{array}
\]

We consider the pushout $Z$ here. The third case above shows that $A \in \mathfrak{A}$ and $Y \in \mathfrak{A} + \mathfrak{B}$ imply $Z \in \mathfrak{A} + \mathfrak{B}$. Then the second case above shows that $Z \in \mathfrak{A} + \mathfrak{B}$ and $B \in \mathfrak{B}$ imply $K \in \mathfrak{A} + \mathfrak{B}$. Thus the '2/3' axiom holds.

• For the cylinder-functor $T(f : X \rightarrow Y)$, we shall consider the short exact sequence $(Y \rightarrow T(f) \rightarrow sX)$; and then apply the stability under suspension and '2/3' axiom.

**Theorem 2 [Excision] [Vogel,Bihler]**

Let $\mathfrak{A}, \mathfrak{B}$ be two exact h-small classes of $\mathcal{D}$-complexes. Suppose that $\mathfrak{A} \cap \mathfrak{B}$. Then we have the homotopy equivalence of spectra: $K(\mathfrak{A} + \mathfrak{B}, \mathfrak{A}) \simeq K(\mathfrak{B}, \mathfrak{A} \cap \mathfrak{B})$.

**Proof:**

We apply Waldhausen's approximation theorem to the inclusion $(\mathfrak{B}, \mathfrak{A} \cap \mathfrak{B}) \subset (\mathfrak{A} + \mathfrak{B}, \mathfrak{A})$. We must therefore define the weak equivalences $(f : X \rightarrow Y) \in v(\mathfrak{A} + \mathfrak{B}) \Rightarrow \text{Coker}^h(f) \in \mathfrak{A}$. Similarly, $(g : B \rightarrow B') \in w(\mathfrak{B}) \Rightarrow \text{Coker}^h(g) \in \mathfrak{A} \cap \mathfrak{B}$. The only non-trivial hypothesis that we must verify is the surjectivity axiom [App2]. Consider on the left-side $B' \in \mathfrak{B}$ and on the right-side $(A \rightarrow X \rightarrow B) \in \mathfrak{A} + \mathfrak{B}$ dotted with a map $f : X \rightarrow B'$. We want to factorize $f$ through a map in $\mathfrak{B}$, modulo a v-equivalence. The composed map $A \rightarrow X \rightarrow B'$ factorizes
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through an object $C \in \mathfrak{A} \cap \mathfrak{B}$. We build the pushout $Y = X \amalg_a C$. The map $X \to Y$ is a $v$-equivalence; actually, $\text{Coker}^h(X \to Y) = \text{Coker}^h(A \to C) \in \mathfrak{A}$. Moreover, the map $f$ factorizes as $X \to Y \to B'$. Finally, the object $Y$ is in $\mathfrak{B}$ (it’s the extension of $C$ by $B$, both in $\mathfrak{B}$, which is stable under ’2/3’). The axiom [App2] is hence verified, and Waldhausen’s approximation theorem gives: $K(\mathfrak{B}, \mathfrak{A} \cap \mathfrak{B}) \simeq K(\mathfrak{A} + \mathfrak{B}, \mathfrak{A})$. This situation is summed up in the following diagram:

$$
\begin{array}{ccc}
A & \to & X \\
\downarrow v & & \downarrow v \\
C & \to & Y \\
& & \downarrow \quad \downarrow \\
& & \quad B \\
& & \quad B' \\
\end{array}
$$

Remark:
Even if the transversality hypothesis is not symmetric, the conclusion of our excision theorem is! Actually, consider the following diagram where the lines & columns are homotopy fibrations:

$$
\begin{array}{cccc}
K(\mathfrak{B}, \mathfrak{A} \cap \mathfrak{B}) & \to & K(\mathfrak{A} + \mathfrak{B}, \mathfrak{A}) \\
\downarrow & & \downarrow \\
K(\mathfrak{B}) & \to & K(\mathfrak{A} + \mathfrak{B}) & \to & K(\mathfrak{A} + \mathfrak{B}, \mathfrak{B}) \\
\downarrow & & \downarrow & & \downarrow \\
K(\mathfrak{A} \cap \mathfrak{B}) & \to & K(\mathfrak{A}) & \to & K(\mathfrak{A}, \mathfrak{A} \cap \mathfrak{B}) \\
\end{array}
$$

If one of the two inclusions (above or on the right) gives an equivalence in K-theory, then the square $\square$ is ’exact’ (ie Cartesian & Cocartesian), and this implies that the other map gives a homotopy equivalence between the K-theory spectra. Anyway, we can deduce a Mayer-Vietoris long exact sequence ($\forall i \in \mathbb{Z}$):

$$
\ldots \to K_i(\mathfrak{A} \cap \mathfrak{B}) \to K_i(\mathfrak{A}) \oplus K_i(\mathfrak{B}) \to K_i(\mathfrak{A} + \mathfrak{B}) \to K_{i-1}(\mathfrak{A} \cap \mathfrak{B}) \ldots
$$

Example:
Let $R$ be a noetherian regular ring. For any ideal, we have a notion of order defined via the Krull-dimension. For each complex $C_*$, consider the annihilator ideal $\mathcal{J} = \{ a \in R \mid C_* \twoheadrightarrow C_* \sim 0 \}$: we get a notion of order for complexes. Now take two prime integers $p \wedge q = 1$; define $\mathfrak{A} = \{ C_* \mid \exists m \geq 1, \text{order}(H_*(C)) = p^m \}$ and $\mathfrak{B} = \{ C_* \mid \exists n \geq 1, \text{order}(H_*(C)) = q^n \}$. Then we can verify that $\mathfrak{A} \pitchfork \mathfrak{B}$, and thus we can apply the excision theorem, knowing that $\mathfrak{A} \cap \mathfrak{B}$ contains only the acyclics: thus the equivalence $K_i(\mathfrak{A} + \mathfrak{B}) \simeq K_i(\mathfrak{A}) \oplus K_i(\mathfrak{B})$, $\forall i \in \mathbb{Z}$.

8 Application to the $\tilde{K}\text{Nil}$ groups

We shall introduce the structures we need to study Waldhausen’s $\tilde{K}\text{Nil}$ groups in the setting of Vogel’s [Conjecture]. Hereafter, $R$ will be a fixed regular ring in the sense of Vogel and $S$ a fixed $R$-bimodule, flat on the left (cf [Bih01] for precisions). After a few technical definitions, we shall apply our localization and excision theorems on the class $\mathcal{B}_n$ and $\mathcal{A}_n$ of nilpotent $\mathcal{D}_n$-complexes, and gives examples of concrete calculus of local objects.
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Definition 5 .
Let \( D_0 \) be the diagram of bimodules given by :

- The oriented graph \( \Gamma \) is the set \( \mathbb{N} \) of natural integers, doted with maps \( (\lambda : n \to n+1) \) and \( (\alpha : n+1 \to n) \) such that \( \alpha \circ \lambda = \lambda \circ \alpha \).
- To each vertex \( n \in \mathbb{N} \) we associate the ring \( R \).
- To each map \( \lambda \) we associate the bimodule \( R \), to each map \( \alpha \) the bimodule \( S \).

Definition 6 .

\[
\begin{align*}
B_* &= 0 \\
B_* &= 0 \quad B_1 \xrightarrow{\lambda_1} B_2 \xrightarrow{\lambda_2} B_3 \xrightarrow{\lambda_3} \cdots \xrightarrow{\lambda_n} B_n \xrightarrow{\lambda_{n+1}} B_{n+1} \xrightarrow{\lambda_{n+2}} \cdots
\end{align*}
\]

where \( B_0 = 0 \) and each \( B_i \) is a h-finite complex

( of projective right \( R \)-modules )

each \( \lambda_i : B_i \to B_{i+1} \) is a chain map, and a cofibration

( We suppose furthermore that \( \lambda_i \) is a weak equivalence for all \( i \geq n \) )

each \( \alpha_i : B_i \to B_{i-1} \otimes_R S \) is a chain map

( If all \( \alpha_i \) are surjective, the \( D_0 \)-complex \( B_* \) will be called reduced )

finally, we ask that the two composed maps \( B_i \to B_{i+1} \to B_i \otimes S \)

and \( B_i \to B_{i-1} \otimes S \) verify \( \alpha \circ \lambda = \lambda \circ \alpha \).

Definition 7 .

- For all \( n \geq 0 \), let \( \mathcal{A}_n \) be the class of \( B_* \in \mathcal{B}_n \) such that \( B_n \) is contractible.
- The inclusion functors \( \mathcal{B}_n \subset \mathcal{B}_{n+1} \) for all \( n \geq 0 \) define \( \mathcal{B} = \lim \mathcal{B}_n \).
- The inclusion functors \( \mathcal{A}_n \subset \mathcal{A}_{n+1} \) for all \( n \geq 0 \) define \( \mathcal{A} = \lim \mathcal{A}_n \).

Theorem 3 .
Let \( R \) be a right regular ring (in the sense of Vogel ), and let \( S \) be any \( R \)-bimodule, flat on the left. Then we have the following homotopy equivalence of non-connective K-theory spectra : \( Knil(R; S) \simeq K(\mathcal{B}, \mathcal{A}) \simeq \lim K(\mathcal{B}_n, \mathcal{A}_n) \).

We refer the interested reader to [Bih02] for a complete proof.

Calculus of local objects :
1. We shall first treat the easy graded case \( \mathcal{B}_n \subset \mathcal{B}_{n+1} \).

Proposition 3 .
Let \( C_* \) be a reduced \( D_0 \)-complex in \( \mathcal{B} \).
Then \( C_* \) is \( \mathcal{B}_n \)-local if and only if \( C_* \) is contractible \( \forall 0 \leq i \leq n \).

Proof :
- (i) \( \Rightarrow \) (ii) : We consider the test-objects \( D_* = g_m(D) = (0 \to 0 \to \cdots 0 \to D = D = \cdots) \)
beginning with \( m \) zeros, where the complex \( D = (\cdots \to 0 \to R \to 0 \to \cdots) \) is the base ring
concentrated in degree 0, and every index \( 1 \leq m \leq n \). Let \( \text{Hom}(D_*, C_i) \) denote the set of maps \( f_* : D_* \to C_* \) respecting the gradations \( f_k : D_k \to C_k \), the structural maps \( (\lambda, \alpha) \) of \( D_0 \)-complexes, but a priori not the degrees, and not the differentials ( we call these maps ‘algebraic morphisms’ ). It’s a graded differential \( R \)-module : \( \text{Hom}(D_*, C_*) \) of \( f : D_k \to C_k \) and following Leibniz rule : \( (df) = d(f) + (-1)^{deg(f)} f \circ d \) By \( R \)-linearity, a morphism \( f_* : D_* \to C_* \) is given by the image \( x = f(1_R) \in C_m \). To respect the \( \lambda \) maps, we pose \( f_k = 0 \) for \( k < m \)
and \( f_{m+k+1} = \lambda_{m+k} \circ \alpha_{m+k} \). To respect the \( \alpha \) maps, we have the condition \( \alpha_m(0) = 0 \).
Then we write \( \alpha_{m+k} \circ f_{m+k}(1_R) = \lambda_{m+k} \circ \cdots \lambda_m \circ \alpha_m(0) = 0 \) due to the relation \( \alpha \circ \lambda = \lambda \circ \alpha \) available in any \( D_0 \)-complex. Thus we have an isomorphism of graded \( R \)-modules :
Conversely, we shall now verify that every object of this type is \( \mathcal{B}_n \)-local. Let \( \mathcal{B}_n \)-local and \( D_n \in \mathcal{B}_n \), the map factors through an acyclic \( E_n \). Thus \( f = dy \) is a boundary. Conclusion: the complex \( \text{Ker}(\alpha_m) \) is acyclic. A finite induction on \( i \) shows that each complex \( C_i \) is acyclic for \( i \leq n \); actually, it's true for \( i = 0 \); now look at the short exact sequence \( 0 \to \text{Ker}(\alpha_1) \to C_i^* \to C_{i-1}^* \otimes S \to 0 \) (here the object \( C_* \) is reduced). Then \( C_i^* \) is acyclic by induction hypothesis, thus \( C_{i-1}^* \otimes S \) is acyclic too, because \( S \) is flat on the left; \( \text{Ker}(\alpha_1) \) is acyclic due to the condition imposed by the test-object \( \eta_i(D) \); hence \( C_i^* \) is acyclic also. We have proven condition (ii).

- (ii) \( \Rightarrow \) (i): Conversely, we shall now verify that every object of this type is \( \mathcal{B}_n \)-local. Let \( f : D_* \to C_* \) be any map from any \( D_\bullet \in \mathcal{B}_n \) to \( C_* \) with \( C_i \) contractible \( \forall 0 \leq i \leq n \). We shall build by induction on \( i \) a contractible object \( E_* \) through which \( f \) admits a factorization. For \( i \leq n \), we pose \( E_i = C_i \) dotted with the same \((\lambda, \alpha)\) structural maps. For \( i \geq n \), we pose \( E_{i+1} = E_i \coprod_{D_i} D_{i+1} \). So now we must define the maps \( \lambda_i : E_i \to E_{i+1} \) and \( \alpha_{i+1} : E_{i+1} \to E_i \otimes S \) in a compatible way with the maps on \( D_* \) and \( C_* \). As \( D_i \to D_{i+1} \) is a cofibration and a homology equivalence (because \( D_* \) is in \( \mathcal{B}_n \)), the pushout gives \( \lambda_i : E_i \to E_{i+1} \) cofibration and homology equivalence, compatible with \( D_* \). Next consider the following pushout, that will define the map \( E_{i+1} \to C_{i+1} \) (by the universal property) compatible with the maps \( \lambda \):

\[
\begin{array}{ccc}
D_i & \xrightarrow{\sim} & D_{i+1} \\
\downarrow & \nearrow & \downarrow \\
E_i & \xrightarrow{\sim} & E_{i+1} \\
\downarrow & \nearrow & \downarrow \\
C_i & \xrightarrow{\sim} & C_{i+1}
\end{array}
\]

Actually, the exterior “square” commutes, because the initial map \( f \) commutes with \( \lambda \). We must now define \((\alpha_{i+1} : E_{i+1} \to E_i \otimes S)\) on \( D_{i+1} \), take \( D_{i+1} \to D_i \otimes S \to E_i \otimes S \); on \( E_i \), take the composition \( E_i \to E_{i-1} \otimes S \to E_i \otimes S \). These maps coincide on \( D_i \) as suggested below:

\[
\begin{bmatrix}
0 \\
2 \\
3
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} =
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots
\end{bmatrix}
\]

Thus we can define \( \alpha_{i+1} \) on \( E_{i+1} \) by the universal property of the pushout, and it will be compatible with that defined on \( D_* \). It remains to verify the compatibility with \( C_* \) on \( D_{i+1} \), it’s just the compatibility of the initial map \( f \) with \( \alpha_i \) on \( E_i \), we go back to the induction hypothesis, with exactly the same manipulation as above, but between \( E_i \) and \( C_{i+1} \). Hence we have built by induction on \( i \) a contractible \( \mathcal{B}_n \)-complex \( E_* \) through which \( f : D_* \to C_* \) admits a factorization. The objects verifying (ii) are thus well \( \mathcal{B}_n \)-local.

We can now identify the local objects, via Waldhausen’s approximation theorem, to obtain a substitute of Quillen’s dévissage for the class \( \mathcal{B} \) of \( \mathcal{D}_0 \)-complexes.

**Proposition 4.**

Let \( \mathcal{L} \) be the class of reduced \( \mathcal{B}_n \)-local \( \mathcal{D}_0 \)-complexes \( L \), such that there exists \( B \) in \( \mathcal{B}_{n+1} \) and a \( \mathcal{B}_n \)-equivalence from \( B \) to \( L \). By Waldhausen’s approximation theorem, the exact projection functor \( F : \mathcal{L} \to \mathcal{C}_R \) defined by \( L_* \to L_{n+1} \) induces a homotopy equivalence of \( K \)-theory spectra \( K(\mathcal{B}_{n+1}, \mathcal{B}_n) \simeq K(R) \).

**Proof:**

- By use of the main theorem, we know the equivalence \( K(\mathcal{B}_{n+1}, \mathcal{B}_n) \simeq K(\mathcal{L}) \). It
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Now remains to identify the objects $L$ in $\mathcal{L}$: by the proposition above, $L_i$ is contractible $\forall 0 \leq i \leq n$. Moreover, we have a short exact sequence $(A \rightarrow B \rightarrow L)$ with $A$ in $\mathcal{B}_R^n$ and $B$ in $\mathcal{B}_{n+1}$. In particular, the $\lambda_i$ are homotopy equivalence $\forall i > n$. Let’s prove that $L_{n+1} \in \mathcal{C}_R$, and thus that the projection functor $F$ is well defined: consider the exact sequence $(A_n \rightarrow B_n \rightarrow *)$. $B_n$ is in $\mathcal{C}$, thus $A_n$ too. Now look at the exact sequence $(A_{n+1} \rightarrow B_{n+1} \rightarrow L_{n+1})$: as $A_{n+1} = A_n$ is in $\mathcal{C}_R$ and $B_{n+1}$ too, hence $L_{n+1}$ is in $\mathcal{C}_R$ also!

- The projection functor $F$ sends a homotopy equivalence in the class of local objects to a homotopy equivalence in $\mathcal{C}_R$. Conversely, let $f: A_+ \rightarrow B_+$ between local objects, such that $f_{n+1}$ be a homotopy equivalence. As $A_i \simeq A_{i+1}$ and $B_i \simeq B_{i+1}$ for all index $i > n$ (these are homotopy equivalences because we work in the class $\mathcal{C}_R$), then $f_i \simeq f_{n+1}$ is a homotopy equivalence. For the index $i \leq n$, all complexes $A_i$ and $B_i$ are contractible, so $f_i$ is a homotopy equivalence! Thus $f_* = (f_i)$ is a homotopy equivalence, and the axiom $[\text{App1}]$ is verified.

- Let’s now study the surjectivity $[\text{App2}]$: let $C_*$ be a local object described in the proposition above, and let $f: C_{n+1} \rightarrow D$ be any map in $\mathcal{C}_R$. We want to build a $\mathcal{B}_0$-complex $D_*$ local and a map $f_*$ that lift $f$. By a factorization through the cylinder-functor on $\mathcal{C}_R$, we can suppose that $f$ is a cofibration. We shall build $D_*$ by induction on $i$. For the index $i < n$, pose $D_i = C_i$ with the same structural maps $(\lambda, \alpha)$. For $i = n$, pose $D_n = C_n$ with the same $\alpha_n$, but here $\lambda_n = f \circ \lambda_n$. Problem: can we define $\alpha_{n+1}: D \rightarrow C_{n+1} \otimes S$ that lift the map $\alpha_{n+1}$ already defined on $C_{n+1}$? As $C_n$ is acyclic, and $S$ flat on the left, then $C_n \otimes S$ is acyclic also: there is no obstruction to lift along the cofibration $f$. Then it remains all the right-side of $D_*$ to build. For this, we use the same argument as in the proof $(ii) \rightarrow (i)$ above: the pushout! For $i > n$, pose $D_{i+1} = D_i \bigsqcup C_i$, and now all structural maps $(\lambda, \alpha)$ are defined functorially! So, we have built a map of $\mathcal{B}_0$-complexes $f_*$ that lifts $f$ (followed eventually by a homotopy equivalence after the factorization through the cylinder . . .) and the multicomplex $D_*$ we’ve constructed has the same properties as $C_*$: contractible for the index $i \leq n$, then the $\lambda_i$ cofibrations and homology equivalences make their way in the pushout. So $D_*$ is local. Moreover $D_{n+1} = D_n \in \mathcal{C}_R$, thus the constructed object $D_*$ is in $\mathcal{L}$! Every hypothesis is verified for Waldhausen’s approximation theorem, thus $K(\mathcal{B}_{red}^{n+1}, \mathcal{B}_R^{red}) \simeq K(\mathcal{L}) \simeq K(\mathcal{C}_R, \text{homology equivalences}) \simeq K(R)$.

- Let’s now study some extra-precations when dealing with “reduced objects” first. remark that if a $\mathcal{B}_0$-complex $C_*$ is reduced, then testing if $C_*$ is $\mathcal{B}_n$-local or $\mathcal{B}_R^{red}$-local is the same; moreover, we know also $[\text{Hill02}]$ that $K(\mathcal{B}_{n+1}, \mathcal{B}_n) \simeq K(\mathcal{B}_{n+1}^{red}, \mathcal{B}_n^{red})$. Now by our main theorem, this is the $K$-theory of reduced objects $C_*$ for which $C_i$ is contractible $\forall 0 \leq i \leq n$, and the other $C_i$ equal $C_{n+1} \in \mathcal{C}_R$ via the homology equivalences $\lambda_i$. One could object that the contractible object $E_*$ that we built is not reduced: that’s true, but in $[\text{Hill02}]$ we have a functorial way to send the global diagram to reduced objects: one compose on the right to obtain the wanted $C_* \rightarrow D_* \rightarrow D_*^{red}$. We finish by remarking that $D^{red}_{n+1} \leftarrow D_{n+1} \rightarrow D$ is better than two homology equivalences: as “acyclic=contractible” in $\mathcal{C}_R$, here are two opposite homotopy equivalences (they have some inverse!), so $D^{red}_{n+1} \simeq D$.

Now, let’s exercise our excision theorem on this same class $\mathcal{B}$. Let $\mathcal{F} \subset \mathcal{B}_{n+1}$ be the class of $\mathcal{B}_0$-complexes $(0 \rightarrow B_1 = B_1 = \ldots)$. Let $\mathcal{G} \subset \mathcal{B}_{n+1}$ be the class of $\mathcal{B}_0$-complexes $(0 \rightarrow 0 \rightarrow B_2 \rightarrow B_3 \ldots)$. Then we have the following homotopy equivalences of Waldhausen categories: $\mathcal{F} \simeq \mathcal{C}_R$ via the projection functor and $\mathcal{G} \simeq \mathcal{B}_n$ via the translation functor. Moreover $\mathcal{F} \cap \mathcal{G} = 0$: there is no non-trivial map from $\mathcal{F}$ to $\mathcal{G}$. Hence the transversality: $\mathcal{F} \cap \mathcal{G}$, and we have $\mathcal{F} + \mathcal{G} = \mathcal{B}_{n+1}$. The excision theorem gives the equivalence of spectra: $K(\mathcal{B}_{n+1}, \mathcal{B}_n) \simeq K(R)$. 15
By an obvious induction on \( n \), we obtain the result:

\[
K(P_n) \simeq K(R)^n
\]

We shall now study the nilpotent case \( A_n \subset A_{n+1} \).

**Proposition 5.**

Let \( C_\ast \) be a reduced \( \mathcal{D} \)-complex in \( A \).

Then “\( C_\ast \) is \( A_n \)-local” if and only if “the map \( \lambda \) induces a homology equivalence between \( \text{Ker}(\alpha_m) \) and \( \text{Ker}(\alpha_{m+1}) \) for all index \( 1 \leq m \leq n \).” We can visualize more easily the \( A_n \)-local multicodimensional objects by writing horizontally the maps \( \lambda \) and vertically the maps \( \alpha \), the symbol \( \bullet \) designing the “exact” squares (ie squares that are both homotopy-pushout and homotopy-pullback):

\[
\begin{array}{ccccccccccc}
0 & \rightarrow & C_1 & \rightarrow & C_2 & \rightarrow & C_3 & \cdots & \rightarrow & C_{n-1} & \rightarrow & C_n & \rightarrow & C_{n+1} & \cdots \\
\downarrow & & \downarrow \bigcirc & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & C_1 \otimes S & \rightarrow & C_2 \otimes S & \cdots & \rightarrow & C_{n-2} \otimes S & \rightarrow & C_{n-1} \otimes S & \rightarrow & C_n \otimes S & \cdots
\end{array}
\]

**Proof:**

1. \( (i) \rightarrow (ii) \): Consider the test-objects \( D_\ast = g_{m+1}^{m+1}(D) = (0 \rightarrow \cdots \rightarrow 0 \rightarrow D \rightarrow CD) = \ldots \), beginning with \( m \) zeros, where the complex \( D = (\cdots \rightarrow 0 \rightarrow R \rightarrow 0 \rightarrow \cdots) \) is the base ring concentrated in degree 0, and \( CD \) is the mapping-cone of the identity on \( D \) (that’s contractible), and every index \( 1 \leq m < n \). Let \( \mathcal{H} \text{om}(D_\ast, C_\ast) \) denote the set of “algebraic morphisms” \( f : D_\ast \rightarrow C_\ast \). By \( \lambda \)-linearity, \( f \) is given by the image \( x = f(1_R) \in C_m \). To commute with \( \alpha \), we must have \( x \in \text{Ker}(\alpha_m) \). At the index \( m + 1 \), the cone \( CD \) is given by two copies of \( R \): one generated by 1 in degree 0, that is the image of \( D \) by \( \lambda \), and the other generated by \( e \) in degree \(-1\). The map \( f_{m+1} \) is thus given by the images \( \lambda(x) = f_{m+1}(1_R) \in \text{Ker}(\alpha_{m+1}) \) and \( y = f_{m+1}(e) \in \text{Ker}(\alpha_{m+1+1}) \). It remains to complete on the right side by the composed maps:

\[
\begin{align*}
& f_{m+1+k} = \lambda_{m+k} \circ \cdots \lambda_{m+1} \circ f_{m+1} \\
\text{to have the commutations with every structural map.}
\end{align*}
\]

Hence we can have an isomorphism of graded \( R \)-modules \( \mathcal{H} \text{om}(D_\ast, C_\ast) \simeq \text{Ker}(\alpha_m) \otimes \text{Ker}(\alpha_{m+1}) \) defined by \( f \leftrightarrow (x, y) \). Now describe the differential through this correspondence: Leibniz’ formula \( f \mapsto d \circ f - (-1)^{\deg(f)} f \circ df \) applied to \( D \) gives \( [x \mapsto dx] \); on the cone \( CD \), it gives \( [y \mapsto dy - (-1)^{\deg(f)} \lambda(x)] \); hence the correspondence \( [df \leftrightarrow (dx, dy - (-1)^{\deg(f)} \lambda(x))] \).

Next we consider the short exact sequence:

\[
\text{Ker}(\alpha_m+1)_{p+1} \xrightarrow{\lambda} \mathcal{H} \text{om}(D_\ast, C_\ast)_p \xrightarrow{\text{Ker}(\alpha_m)_p}
\]

where the injection is given by \( i(y) = (0, y) \) respecting \( i(dy) = (0, dy) \) and the projection is given by \( \pi(x, y) = x \) respecting \( \pi(dx, y) = dx \). The two maps \( i \) and \( \pi \) are thus chain morphisms and induce a long exact sequence in homology; moreover, \( C_\ast \) being \( A_n \)-local and \( D_\ast \in \mathcal{A}_n \), thus the complex \( \mathcal{H} \text{om}(D_\ast, C_\ast) \) is acyclic; hence we obtain the following isomorphism in homology \( H_p(\text{Ker}(\alpha_m)) \simeq H_p(\text{Ker}(\alpha_{m+1})) \). We must identify this connecting morphism: let \( x \in \text{Ker}(\alpha_m) \) be a cycle, such that \( dx = 0 \). We choose a section \( (x, 0) \in \mathcal{H} \text{om}(D_\ast, C_\ast) \), then its boundary \( d(x, 0) = (-1)^{\deg(f)} \lambda(x) \). The connecting morphism thus corresponds (up to a question of sign) to \( \lambda \). Conclusion: \( \lambda \) induces a homology equivalence between \( \text{Ker}(\alpha_m) \) and \( \text{Ker}(\alpha_{m+1}) \) for all index \( 1 \leq m < n \).

2. \( (ii) \rightarrow (i) \): The converse is much harder, and shall be proven directly for all index \( n \) by the construction of an explicit homotopy, that is also available for the case \( n = \infty \) (in other words, it means the case of \( \mathcal{A}_n \)-local objects). Let’s start by reviewing our notations. Let \( B_\ast \) be a reduced \( \mathcal{D} \)-complex in \( \mathcal{A} \) (that means \( B_\infty \) is contractible) of the type:

\[
\begin{array}{ccccccccccc}
\cdots & \rightarrow & B_{-1} & \rightarrow & B_{0} & \rightarrow & B_{1} & \rightarrow & B_{2} & \cdots \\
\downarrow & & & & \uparrow \bigcirc & & \downarrow & & \uparrow \bigcirc & \\
\cdots & \rightarrow & B_{-2} \otimes S & \rightarrow & B_{-1} \otimes S & \rightarrow & B_{0} \otimes S & \rightarrow & B_{1} \otimes S & \cdots
\end{array}
\]

where all squares for positive index are “exact”. Consider a test-object \( A_\ast = (\cdots 0 \rightarrow A_0 \rightarrow A_1 \rightarrow A_2 \ldots) \) in \( \mathcal{A} \), doted with a map of \( \mathcal{D} \)-complexes \( f_\ast : A_\ast \rightarrow B_\ast \) given by a family of
maps \( \hat{f}_i : A_i \to B_i \) for all \( i \geq 0 \). We want to prove that \( f \sim 0 \). Take a family of compatible algebraic splittings for the following short exact sequences: ( on the right side for \( n \geq 0 \) )

\[
\begin{array}{c}
\xymatrix{ A_n & A_{n+1} \ar[l]_{\lambda_n} \ar[r]^{\delta_n} & C_{n+1} \\
K & B_n \ar[l]_{j_n} \ar[r]_{\beta_n} & B_{n+1} \ar[r]_{\sigma_n} & B_{n-1} \otimes S }
\end{array}
\]

with the two structural maps :

\[
\theta_n : A_n \to A_{n-1} \otimes S, \quad \mu_n : B_n \to B_{n+1}
\]

and the compatibility relation :

\[
d\theta_n = \delta_n \theta_n, \quad d\mu_n = -\mu_n \delta_n + \alpha_n
\]

and the differentials:

\[
d\hat{u}_n = \phi_n + 1 \pi_n, \quad d\hat{v}_n = -\lambda_n \delta_n + 1
\]

We shall moreover need some specific notations ( independent from \( n \) ) :

\[\forall p \geq 0, \text{we shall note :}\]

\[
\begin{array}{c}
T_0 = \delta_n j_n : K \otimes S \to K \\
T_p = \delta \sigma \ldots \sigma j_n : K \otimes S^p \to K
\end{array}
\]

Lemma 9 .

These chain morphisms in degree \(-1\) verify the functional relation :

\[
dT_p = \sum_{i+j=p-1} T_i T_j
\]

Proof : We already know that \( d\hat{u}_n = d\hat{v}_n = 0 \) for all \( k \), thus the differential \( dT_p \) has only terms with \( d\sigma_k = -j_k \delta_k \). By enumerating them ( \( k^{th} \) between \( p \) choices, from the end ), and forgetting the \( (\otimes S) \) useless and cumbersome, we write :

\[
dT_p = \sum_{k=1}^{p} (-1)^{k} \delta_{n+p} \sigma_{n+p} \ldots \sigma_{n+k+1}(-j_{n+k} \delta_{n+k}) \sigma_{n+k} \ldots \sigma_{n+1} j_n
\]

\[
= \sum_{k=1}^{p} \sum_{k=1}^{p} (-1)^{k} \delta_{n+p} \sigma_{n+p} \ldots \sigma_{n+k+1} j_{n+k} \sigma_{n+k} \ldots \sigma_{n+1} j_n
\]

\[
= \sum_{k=1}^{p} T_{p-k} T_{k-1} = \sum_{i+j=p-1} T_i T_j
\]

Lemma 10 .

We proceed to different re-writings and re-interpretations of the maps coming from \( f_n \) : we begin from \( f_n : A_n \to B_n \) first, that we decompose on the maps \( f_n : A_n \to K \), and then in a more synthetic manner, we write them from a unique \( F : A \to K \). The important intermediary formulas are the following :

\[
\hat{f}_{n+1} = j_{n+1} \theta_n f_n \hat{u}_n + \sigma_{n+1} f_n \hat{v}_{n+1} + j_{n+1} f_{n+1} \pi_{n+1}
\]

\[
f_n = F \lambda^\infty \otimes \nu_n
\]

\[
\hat{f}_n = \sum_{k=0}^{n} \sigma_n \ldots \sigma_{k+1} j_k F \lambda^\infty \alpha_{k+1} \ldots \alpha_n
\]

Proof : • Compatibility of the maps of \( \theta_0 \)-complexes : from the following commutative diagram, we shall write the maps \( f_n : A_n \to B_n \) inductively on \( n \), from the maps \( f_n : A_n \to K \) ( we shall proceed by necessary, then sufficient condition ).

\[
\begin{array}{c}
\xymatrix{ A_n & A_{n+1} \ar[l]_{\lambda_n} \ar[r]^{\delta_n} & C_{n+1} \\
K & B_n \ar[l]_{j_n} \ar[r]_{\beta_n} & B_{n+1} \ar[r]_{\sigma_n} & B_{n-1} \otimes S }
\end{array}
\]

We must then verify the two conditions by induction :

\[
\hat{f}_{n+1} \lambda_n = \mu_n \hat{f}_n \quad \beta_{n+1} \hat{f}_{n+1} = \hat{f}_n \theta_{n+1}
\]
The naive approach uses the algebraic sections, then corrects the eventual defect: let's pose $f_{n+1} = \mu_n f_{n} n + \sigma_{n+1} F_{n+1} \delta$. We write the two conditions, using the induction hypothesis, and the two equalities: $Id_{A_{n+1}} = \lambda_{n} u_{n} + v_{n+1} \lambda_{n+1}$ and $Id_{B_{n+1}} = \lambda_{n+1} \delta + \sigma_{n+1} + \beta_{n+1}$.

We obtain: $2\lambda = \sigma_{n+1} + \mu_{n} f_{n} u_{n} + (\lambda_{n+1} + \sigma_{n+1}) \delta F_{n+1} \tau_{n+1} \lambda_{n+1}$. The second mixed term can be interpreted as $j_{n+1} f_{n+1} \pi_{n+1} + 1$ with some $f_{n+1} : C_{n+1} \to K$ on which we have no condition.

Grouping the terms finishing by $f_{n+1} u_{n}$, we can write in a synthetic manner:

$$J_{n+1} = \lambda_{n+1} \delta + j_{n+1} \delta F_{n+1} \tau_{n+1} \lambda_{n+1} + j_{n+1} f_{n+1} \pi_{n+1}$$

- Now we shall think algebraically at the $f_{n} : C_{n} \to K$ as coordinates of a unique map $F : A \to K$ where the total space is $A = A_{\infty} = \oplus C_{n}$, and the maps $A_{n} : A_{n} \to A$ give the projections on each factor: $f_{n} = F_{\lambda_{n}} v_{n}$. We have a correspondence: $(f_{n}) \leftrightarrow (F : A \to K)$.

So for a homotopy we shall search a map $(g_{n} : C_{n} \to K) \leftrightarrow (g_{n} = dF_{n})$ rather in the form $(g_{n}) \leftrightarrow (G : A \to K)$.

- Let's precise the conventions for the third formula: if $k = n$, then the compositions with $\sigma$ on the left and $\alpha$ on the right are both empty: if $k = n - 1$, we have only one term $\sigma_{n}$ and $\alpha_{n}$; and if $k < n - 1$, then the index are going down on the left side, and up on the right side. Now, the method for finding this formula is 'magic': try and guess for small index, then prove it officially by induction. For $n = 0$, we get: $f_{0} = j_{0} f_{0} \pi_{0} = j_{0} F_{\lambda_{0}} \pi_{0} = j_{0} F_{\lambda_{0}}$ because $C_{0} = A_{0}$.

Suppose now the formula available for the index $n$: we want to calculate $f_{n+1}$. We replace $j_{n+1}$ by the first formula; $f_{n+1}$ by $F_{\lambda_{n+1}} v_{n+1}$; and $v_{n+1} \pi_{n+1}$ by $F_{\lambda_{n+1}}$. We then obtain the wanted result, plus the defect: $J_{n+1}(\phi_{n} \sum_{k=0}^{n} \sigma_{n-k} j_{k} F_{\lambda_{k+1}} \alpha_{n-k+1} + \pi_{n}) \lambda_{n} u_{n}$.

But the composition $\phi_{n} \lambda_{n}$ is trivial, hence every term vanishes except for $k = n$. It remains: $J_{n+1}(\phi_{n} \lambda_{n} F_{\lambda_{n+1}} - F_{\lambda_{n+1}} \lambda_{n})$. But here $\phi_{n} \lambda_{n} = Id$ and the parenthesis vanishes. Thus the third formula is proven by induction.

**Lemma 11**

Differentiating the induction formulae for $J_{n}$, we obtain the following formula, giving $G$ associated with $g_{n} = dJ_{n}$:

$$dF - G = \sum_{i \geq 0} T_{i} F_{\alpha}^{i+1}$$

Then we build a new differential $\delta$ defined by:

$$F \mapsto \delta F = dF - \sum_{i \geq 0} T_{i} F_{\alpha}^{i+1}$$

According to the principle for inverting formal series, we obtain for every cycle $\delta F = 0$, the general formula giving a homotopy $G = F$:

$$G = \sum_{p \geq 0} \left( \sum_{k_{1}, \ldots, k_{p}} (-1)^{p+1} \delta g(F) T_{1}, \ldots, T_{p}, T_{p+1} \ldots T_{n} F_{k_{1} \alpha_{p+1}^{i+1} + \ldots + k_{p} \alpha_{p+1}^{i+1}} \alpha_{p+1}^{i+1} \right)$$

**Proof:**

- We differentiate the formula giving $J_{n+1}$, and identify $g_{n} = df_{n}$. After simplification, we obtain: $g_{n+1} = df_{n+1} + (-1)^{p} \delta g(F) \phi_{n} \phi_{n+1} - \delta_{n} \phi_{n+1} \pi_{n+1}$. Then we differentiate the expression: $f_{n} = F_{\lambda_{n}} \pi_{n}$ and we find the value of $G$ by identification, from its coordinates $g_{n+1} = G_{\lambda_{n+1}} v_{n+1}$ projected on $C_{n+1}$: $g_{n+1} = dF_{n} \alpha_{n+1} = \delta T_{1} F_{\alpha}^{i+1}$.

In a more compact way, we proved by projection the following formula: $dF - G = \sum_{i \geq 0} T_{i} F_{\alpha}^{i+1}$.

- After re-interpretation, we have $\phi_{n} \phi_{n+1} (\phi_{n} \phi_{n+1} : \phi : F) \mapsto G$ where the new differential is given by $f \mapsto \delta F = dF - \sum_{i \geq 0} T_{i} F_{\alpha}^{i+1}$. Let's verify that $\delta$ is really a differential:

$$\delta^{2} F = d^{2} F - \sum_{i \geq 0} (T_{i} F_{\alpha}^{i+1}) + \sum_{j \geq 0} T_{i} (T_{j} F_{\alpha}^{j+1}) - \sum_{i \geq 0} T_{i} \left( (T_{j} F_{\alpha}^{j+1}) \alpha^{i+1} \right)$$

The double differential $d^{2} F$ vanishes, and the term $i = 0$ from the first sum also, because $f_{0} = 0$; the second sum simplifies with the beginning of the third sum, and it remains:

$$\delta^{2} F = \sum_{i \geq 1} \left( \sum_{k \geq i} T_{k} T_{i} F_{\alpha}^{i+1} \right) + \sum_{j \geq 0} \left( \sum_{k \geq j} T_{k} T_{j} F_{\alpha}^{j+1} \right)$$

But after re-indexation, these two sums both equal $\sum_{a \geq 0} \left( \sum_{k \geq a} T_{k} T_{a} F_{\alpha}^{a+2} \right)$, thus $\delta^{2} F = 0$ and $\delta$ is really a differential!
**VANISHING OF THE $\mathcal{Kn}il$ GROUPS : LOCALIZATION METHODS**

- We know that $A$ is a projective $\mathcal{A}$-module: hence there exists a homotopy $k : A \to A$ of degree $1$ such that $Id_A = dk$. Let $F$ be a cycle such that $dF = 0$. Then $d(Fk) = (-1)^{\deg(F)}F$ : it’s a solution to $dF = \delta G$ at the order $0$ in $T$ ! The problem is now, knowing that $d(Fk) = \alpha k - ka$, to guess a solution to find our homotopy, according to the (rather complicated) principle for inverting formal series. Here I prefer to parachute the result (found by P. Vogel) coldly, and we shall verify it formally.

$$G = \sum_{p \geq 0} \left( \sum_{i_1, \ldots, i_p} (-1)^{p+1} \cdot \deg(F) T_{i_p} T_{i_{p-1}} \cdots T_{i_1} F k a^{i_1+1} k a^{i_2+1} \cdots k a^{i_p+1} k \right)$$

Let’s write methodically its differential $\delta G$ and we shall obtain four triple sums: the first corresponds to juxtapose $T_i$ on the left, and $\alpha^{i+1}$ on the right; the second to derive one of the $T_j$; the third is the differential $dF$ (which we write as a sum, from the hypothesis that $F$ is a cycle: $\delta F = 0$); at last, the fourth corresponds to derive one of the $k$.

$$\delta G = \sum_i \left( \sum_p \left( \sum_{i_1, \ldots, i_p} (-1)^{p+1} \cdot \deg(F) T_i T_{i_p} \cdots T_{i_1} F k a^{i_1+1} k a^{i_2+1} \cdots k a^{i_p+1} k \right) \right)$$

Now re-interpret each of these sums: for the first, pose $i = i_{p+1}$, then it lacks $a$ on the right; for the second, we cut $\alpha^{i+1} = \alpha^{i+1} \alpha^{m+1}$, then it lacks a $k$ between the two $\alpha$; for the third, pose $i = i_0$, then it lacks a $k$ on the left. With these simplifications, the fourth (with the index $p + 1$) vanishes with the terms of the three others (with the index $p$). For this, we take an index $1 \leq w \leq p$, and we impose three conditions for the annulation of the signs:

\[
\begin{array}{l}
\epsilon_{p+1}(-1)^{\deg(F)} + \epsilon_p(1)^w + \epsilon_p(1)^w = 0 \\
\epsilon_{p+1}(-1)^{\deg(F)} + \epsilon_p(1)^w = 0 \\
\epsilon_{p+1}(-1)^{\deg(F)} p + dF = 0
\end{array}
\]

Remark: the notation $\epsilon$ allows us to make the theoretical verification of the signs in the sums indexed by $(i_1, \ldots, i_p)$. Actually, all three conditions are equivalent to: $\epsilon_{p+1} = \epsilon_p(1)^{\deg(F)}$. Hence the existence of a sign: $\epsilon = \epsilon_0(1)^{\deg(F)}$. But we have seen in our first development ‘by the hand’ that $\epsilon_0 = (-1)^{\deg(F)}$. Thus the formula vanishes well. It then remains two minor points of detail: first, the change of index in the “middle” term. We enumerate the integral points in the plane by their usual coordinates $(m, l)$ or by the diagonals $(l + m = i_w - 1, w \geq 1)$: actually, the case $i_w = 0$ has a trivial differential. Finally, the compensation lets untouched the term $p = 0$ in the fourth sum: here, we have $\epsilon_0(-1)^{\deg(F)} F = F$, so we have proved that $\delta G = F$. ■

This ends the proof of Proposition 5. ■

**Remark:**

We lack a good interpretation of these $\mathcal{A}$-local objects, to apply Waldhausen’s approximation theorem, and allow us an ersatz of Quillen’s dévissage for the difficult class $\mathcal{A}$ of nilpotent $\partial_n$-complexes. Actually, if one manages to prove $K(\mathcal{A}_{n+1}, \mathcal{A}_n) \simeq K(R)$, then Vogel’s excision [Conjecture] would be solved.
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Mr BIHLER Frank

U.F.R. de Mathématiques – Case 7012
Université de Paris 7 – Denis Diderot
2, Place Jussieu
75251 PARIS cedex 05
FRANCE

Mail : bihler@math.jussieu.fr
Webpage : [http://www.institut.math.jussieu.fr/~bihler](http://www.institut.math.jussieu.fr/~bihler)