MODELS OF CURVES OVER DVRS

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ABSTRACT. Let \( C \) be a smooth projective curve over a discretely valued
field \( K \), defined by an affine equation \( f(x, y) = 0 \). We construct a model
of \( C \) over the ring of integers of \( K \) using a toroidal embedding associated
to the Newton polygon of \( f \). We show that under ‘generic’ conditions it
is regular with normal crossings, and determine when it is minimal, the
global sections of its relative dualising sheaf, and the tame part of the
first étale cohomology of \( C \).

1. Introduction

The purpose of this paper is to construct regular models and study invari-
ants of arithmetic surfaces using a toric resolution derived from the defining
equation. Let \( K \) be a field with a discrete valuation \( v_K \) and residue
field \( k \), and \( C/K \) a smooth projective curve specified by an affine equation
\( f(x, y) = 0 \). We show that the Newton polytope \( \Delta \) of \( f \) with respect to \( v_K \),
under a ‘generic’ condition called \( \Delta \) -regularity, determines explicitly
— the minimal regular normal crossings model of \( C \) over \( O_K \);
— whether \( C \) and \( \text{Jac} C \) have good, semistable and tame reduction;
— the action \( \text{Gal}(\bar{K}/K) \supset H^1_{\text{ ét}}(C, \mathbb{Q}_l) \) when \( C \) is tamely ramified, and
the action on its wild inertia invariants in general, for \( l \neq \text{char} k \);
— a basis of global sections of the relative dualising sheaf;
— the reduction map on points from the generic to the special fibre.
A regular model is usually constructed by starting with any model over \( O_K \),
and repeatedly blowing up at points and components of the special fibre and
taking normalisations. In effect, the toric resolution replaces repeated blow-
ups along coordinate axes, and for \( \Delta \) -regular curves one such resolution is
enough to get a good model.

1.1. Regular model. Our main objects are the Newton polytopes of the
defining equation \( f = \sum a_{ij}x^iy^j \) of the curve \( C \) over \( K \) and over \( O_K \),
\[
\begin{align*}
\Delta &= \text{convex hull}( (i, j) \mid a_{ij} \neq 0 ) \subset \mathbb{R}^2, \\
\Delta_v &= \text{lower convex hull}( (i, j, v_K(a_{ij})) \mid a_{ij} \neq 0 ) \subset \mathbb{R}^2 \times \mathbb{R}.
\end{align*}
\]
(The lower convex hull is those points \( Q \) in the convex hull such that \( Q - (0, 0, \epsilon) \) is not in it for any \( \epsilon > 0 \).) Thus, \( \Delta \) is a 2-dimensional convex polygon,
and above every point \( P \in \Delta \) there is a unique point \( (P, v(P)) \in \Delta_v \). The
We construct a proper flat model of the absolute Galois group of \( C/K \) if all \( \bar{\Delta} \)-faces (4 outer, 1 inner). Under these conditions, \( C \) is the completion of \( \{ f = 0 \} \subset \mathbb{G}_m \) in the toric variety with fan \( \Delta \), see \( \S 2 \). In particular, the genus of \( C \) is \( |\Delta(\mathbb{Z})| \) and it has a basis of regular differentials \( x^i y^j \frac{dx}{y^{m_i}} \) indexed by \((i,j) \in \Delta(\mathbb{Z})\).

As an example, let \( \pi \in K \) be a uniformiser, and take a curve (left)

\[
C: y^2 + \pi x^3 y + x^3 + \pi^5 = 0
\]

Then \( \Delta \) is a quadrangle, with values of \( v \) on \( \Delta \cap \mathbb{Z}^2 \) as above (middle). It has \( v \)-faces \( F_1, F_2 \) with \( \delta_{F_1} = 6, \delta_{F_2} = 3 \), and five \( v \)-edges (4 outer, 1 inner). Here \( C \) has genus 2 and is \( \Delta_v \)-regular, for any ground field \( K \).

In \( \S 4 \) we construct a proper flat model \( C_\Delta/O_K \) of \( C/K \), using a toroidal embedding associated to \( \Delta_v \). For \( \Delta_v \)-regular curves the main result is

**Theorem 1.1** (3.13+8.12). If \( f \) is \( \Delta_v \)-regular, then \( C_\Delta/O_K \) is a regular model of \( C/K \) with strict normal crossings. Its special fibre \( C_k/k \) consists of:

1. A complete smooth curve \( \bar{X}_F/k \) of multiplicity \( \delta_F \), for every \( v \)-face \( F \).
2. For every \( v \)-edge \( L \) with slopes \( [s^L_1, s^L_2] \) pick \( \frac{m_i}{d_i} \in \mathbb{Q} \) so that

\[
s^L_1 = \frac{m_0}{d_0}, \frac{m_1}{d_1}, \ldots, \frac{m_r}{d_r} = s^L_2 \quad \text{with} \quad \left| \frac{m_i}{d_i} \right| = 1.
\]

Then \( L \) gives \( |X_L(\bar{k})| \) chains of \( \mathbb{P}^1 \)s of length \( r \) from \( \bar{X}_{F_1} \) to \( \bar{X}_{F_2} \) and multiplicities \( \delta_{L,d_1}, \ldots, \delta_{L,d_r} \).\(^3\) The absolute Galois group \( \text{Gal}(k^s/k) \) permutes the chains through its action on \( X_L(\bar{k}) \).

If \( f_y' \) is not identically zero, then the differentials

\[
\pi^{\nu(i,j)} x^{i-1} y^{j-1} \frac{dx}{f_y'}, \quad (i,j) \in \Delta(\mathbb{Z})
\]

form an \( O_K \)-basis of global sections of the relative dualising sheaf \( \omega_{C_\Delta/O_K} \).

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1The condition for outer \( v \)-faces can be made slightly weaker, see 3.9.
2If \( L \) is at the boundary of \( \Delta \), it has only one adjacent \( v \)-face and there is no \( \bar{X}_{F_2} \).
3If \( r = 0 \), this is interpreted as \( X_{F_1} \) meeting \( X_{F_2} \) transversally at \( |X_L(\bar{k})| \) points in the inner case, and no contribution from \( L \) in the outer case. In the outer case, this happens precisely when \( \delta_L = \delta_{F_1} \).
In the example above, the special fibre is shown on the right. See Theorem 3.14 for a slightly more general statement, for arbitrary curves.

1.2. Reduction. There is a natural description, in terms of \( \Delta_v \), of the dual graph of the special fibre (6.1), of the minimal regular model with normal crossings (5.7) and of the reduction map of points to the special fibre (3.19). From these one can also deduce the following criterion for good, semistable, and tame\(^4\) reduction for \( \Delta_v \)-regular curves and their Jacobians.

Write \( \Delta(\mathbb{Z})^F \subseteq \Delta(\mathbb{Z}) \) for points that are in the interiors of \( v \)-faces, and \( \Delta(\mathbb{Z})^L \) for the others (lying on \( v \)-edges). We let subscripts, such as \( \Delta(\mathbb{Z})_Z \) or \( \Delta(\mathbb{Z})_{Z_v}^F \), indicate that we further restrict to points with \( v(P) \in \mathbb{Z} \) or \( v(P) \in \mathbb{Z}_p \), respectively. (When \( p = 0 \), the latter is interpreted as an empty condition). See §1.6 for other notation.

**Theorem 1.2** (=6.7). Suppose \( C/K \) is \( \Delta_v \)-regular, of genus \( \geq 1 \). Then

1. \( C \) is \( \Delta_v \)-regular over every finite tame extension \( K'/K \).
2. \( C \) has good reduction \( \iff \Delta(\mathbb{Z}) = F(\mathbb{Z}) \) for some \( v \)-face \( F \) with \( \delta_F = 1 \).
3. \( C \) is semistable \( \iff \) every principal \( v \)-face \( F \) has \( \delta_F = 1 \).
4. \( C \) is tame \( \iff \) every principal \( v \)-face \( F \) has \( k \mid \delta_F \).

In this case, \( C \) is \( \Delta_v \)-regular over every finite extension \( K'/K \).

Suppose \( k \) is perfect, and let \( J \) be the Jacobian of \( C \). Then

5. \( J \) has good reduction \( \iff \Delta(\mathbb{Z}) = \Delta(\mathbb{Z})_Z^F \).
6. \( J \) is semistable \( \iff \Delta(\mathbb{Z}) = \Delta(\mathbb{Z})_Z \).
7. \( J \) is tame \( \iff \Delta(\mathbb{Z}) = \Delta(\mathbb{Z})_{Z_v}^\Z, \) where \( p = \text{char } k \). In this case,
   - \( J \) has potentially good reduction \( \iff \Delta(\mathbb{Z})^L = \emptyset \), and
   - \( J \) has potentially totally toric reduction \( \iff \Delta(\mathbb{Z})^F = \emptyset \).

1.3. Étale cohomology. Now let \( K \) be complete with perfect residue field \( k \) of characteristic \( p \geq 0 \). Assume that \( C/K \) is \( \Delta_v \)-regular, fix \( l \neq p \) and consider the étale cohomology representation of the absolute Galois group

\[
G_K = \text{Gal}(K^s/K) \supset H^1(C) = H^1_{\text{ét}}(C_K, \mathbb{Q}_l). 
\]

Let \( \mathcal{I}_\text{wild} \vartriangleleft \mathcal{I}_K \vartriangleleft G_K \) be the wild inertia and the inertia subgroups. It turns out that the action of \( \mathcal{I}_K \) on the subspace of wild inertia invariants \( H^1(C)^{\text{I}_{\text{wild}}=1} \) (which is all of \( H^1(C) \) if \( C \) is tame) depends only on \( \Delta_v \), and has the following elementary description.

Each point \( P \in \Delta(\mathbb{Z})_{Z_p} \) defines a tame character \( \chi_P : \mathcal{I}_K \rightarrow \{ \text{roots of unity} \} \) by \( \sigma \mapsto \sigma(\pi^{v(P)})/\pi^{v(P)} \). Let \( V^\text{ab}_{\text{tame}}, V^\text{toric}_{\text{tame}} \) be the unique continuous representations of \( \mathcal{I}_K \) over \( \mathbb{Q}_l \) that decompose over \( \overline{\mathbb{Q}}_l \) as

\[
V^\text{ab}_{\text{tame}} \cong \bigoplus_{P \in \Delta(\mathbb{Z})_{Z_p}^F} (\chi_P \oplus \chi_P^{-1}), \quad V^\text{toric}_{\text{tame}} \cong \bigoplus_{P \in \Delta(\mathbb{Z})_{Z_p}^L} \chi_P.
\]

\(^4\)By ‘tame’ we mean semistable over some tamely ramified extension of \( K \); this is automatically the case if \( \text{char } k = 0 \) or \( \text{char } k > 2 \text{genus}(C) + 1 \).
Then there is an isomorphism of $I_K$-modules (Theorem 6.4),

$$H^1(C)^{I_{\text{wild}}=1} \cong V_{\text{tame}}^{\text{ab}} \oplus V_{\text{tame}}^{\text{toric}} \otimes \text{Sp}_2.$$ 

In particular, the dimension of the wild inertia invariants is $2|\Delta(\mathbb{Z})_{\mathbb{Z}_p}|$.

When $k$ is finite, we describe $H^1(C)^{I_{\text{wild}}=1}$ as a full $G_K$-representation. To a $v$-face $F$ of $\Delta$ we associate a scheme $\overline{X}^{\text{tame}}_{\text{F}}/\mathcal{O}_K$, that depends only on the coefficients $a_P$ for $P$ along $F$. It is tame, defined by an equation that has only one $v$-face, and its $H^1$ admits an explicit description in terms of point counting (Theorem 7.3). Similarly we get schemes $X^{\text{tame}}_L/\mathcal{O}_K$ for $v$-edges $L$. Then there is in isomorphism of $G_K$-representations (see Theorem 7.2)

$$H^1(C)^{I_{\text{wild}}=1} \oplus 1 \cong \bigoplus_F (H^1(X^{\text{tame}}_F) \oplus 1) \oplus \bigoplus_{L \text{ inner \ v-edges}} (H^0(X^{\text{tame}}_L) \otimes \text{Sp}_2).$$

1.4. Applications. As an illustration, we

- recover Tate’s algorithm for elliptic curves (§9),
- determine regular models of Fermat curves $x^p + y^p = 1$ over $\mathbb{Z}_p$ and tame extensions of $\mathbb{Z}_p$, that seem to be unavailable (§10),
- compute regular models and their differentials for some curves from the literature, in an elementary way (3.16, 3.18, 8.15).

Generally, on the computational side our motivation came from determining arithmetic invariants of curves that enter the Birch–Swinnerton-Dyer conjecture, such as the differentials, conductor and Tamagawa numbers. On the theoretical side, for $\Delta_v$-regular curves we also recover explicit versions of various classical results: existence of (minimal) model with normal crossings, Saito’s criterion for wild reduction ([Sai, Thm 3]; see [Hal] as well), and Kisin’s theorem on the continuity of $l$-adic representations for $H^1$ of curves in the tame case (see 7.6). It also extends some computations of regular models as tame quotients [Vie, LoD, CES] to the wild case.

1.5. Remarks. (a) Note that Theorem 1.1 applies to curves with wild reduction, because the construction is ‘from the bottom’ (toroidal embedding) rather than ‘from the top’ (taking a quotient of a semistable model over some larger field). In fact, $\Delta_v$-regularity is automatic for the most ‘irregular’ reduction types, the ones where the only points in $\Delta_v \cap \mathbb{Z}^2$ are vertices of $\Delta_v$. (This is the case with the example above.) Then the shape of the special fibre of their regular models does not depend on the residue characteristic at all. This explains why for example elliptic curves $y^2 = x^3 + \pi$, $y^2 = x^3 + \pi^5$, $y^2 = x^3 + \pi x$, $y^2 = x^3 + \pi^3 x$ are always of Kodaira type II, II*, III, III*, irrespectively of whether $\text{char } k$ is 2, 3 or $\geq 5$.

(b) Also, the theorem produces a regular model with normal crossings, which is somewhat more pleasant than the minimal regular model for classification and computational purposes (as it can be encoded combinatorially in the dual graph of the special fibre), and sometimes for theoretical purposes as well; see e.g. [Sai, Hal].
(c) The technique appears to be useful for arithmetic schemes of higher
dimension as well, and over more general local rings. However, we only treat
the case of curves over DVRs here, as it seems sufficiently interesting and
already sufficiently involved.

(d) Although we have chosen to formulate the results for global equations,
the construction of the regular model is really a local process (in effect a
version of a repeated blow up).

(e) In a way, our results for arithmetic surfaces are a natural extension of
two classical lower-dimensional analogues — 1-dimensional schemes over a
field, and over a DVR. For a plane curve over a field, Baker’s theory com-
putes the genus, points at infinity and differentials from a Newton polygon
of the defining equation; we review this in §2. And for univariate \( f(x) \) with
coefficients in a complete DVR, the Newton polygon of \( f \) reflects how \( f \)
factors, in effect forcing a decomposition of \( H^0_{\text{ét}} \) of \( \text{Spec} \ f \); the results from
§1.3 simply extend this to bivariate \( f(x, y) \) and their \( H^1_{\text{ét}} \).

(f) Finally, there is an obvious question to which extent the theory extends
to arbitrary curves, not just the \( \Delta_v \)-regular ones. It appears that it does,
but this requires additional work. We postpone this to a future paper.

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helpful discussions.
1.6. **Notation.** Throughout the paper we use the following notation:

- $K$, $v_K$: field with a discrete valuation; in §2, $K$ is any field.
- $O_K$, $k$, $p$: ring of integers, residue field, char $k$.
- $\pi$: a fixed choice of a uniformiser of $K$.
- $f$: an equation in $K[x, y]$, not 0 or a monomial.
- $C$: smooth projective curve over $K$ birational to $f = 0$.
- $\Delta$: Newton polygon of $f$ over $K$ in $\mathbb{R}^2$, as in §1.1.
- $\Delta_v$: Newton polytope of $f$ over $O_K$ in $\mathbb{R}^3$, as in §1.1.
- $v$: piecewise affine function $v : \Delta \to \mathbb{R}$ induced by $v_K$, as in §1.1.
- $\partial$: boundary (of a convex polygon in $\mathbb{R}^2$).
- $L$, $F$: $v$-edges and $v$-faces of $\Delta$, the images of 1- and 2-dimensional faces of $\Delta_v$ under the homeomorphic projection $\Delta_v \to \Delta$.
- $\Delta(Z)$: interior$(\Delta) \cap \mathbb{Z}^2$; similarly $L(Z)$, $F(Z)$.
- $\overline{\Delta}(Z)$: closure$(\Delta) \cap \mathbb{Z}^2$; similarly $\overline{L}(Z)$, $\overline{F}(Z)$.
- $\delta_P, \delta_L, \delta_F$: the denominator of $v(P)$ for $P \in \Delta(Z)$, and the common denominators of $v(P)$ for $P \in L(Z)$ or $F(Z)$.
- $f|_L, f|_F$: restriction of $f$ to a $v$-edge or a $v$-face.
- $f_L, f_F$: reduction to $k[t], k[x, y]$ of the restrictions.
- $X_L, X_F$: affine schemes $\{f|_L = 0\} \subset \mathbb{G}_{m,k}$ and $\{f|_F = 0\} \subset \mathbb{G}_{m,k}$.
- $\overline{X}_F$: completion of $X_F$ with respect to its Newton polygon.
- r.n.c.: flat proper regular $O_K$-model with normal crossings of a curve over $K$ (exists in genus $\geq 1$, see [Liu, 9/1.7]).
- m.r.n.c.: minimal r.n.c. model (unique, see [Liu, 9/3.36]).
- $\mathcal{C}_\Delta/O_K$: flat proper model of $C/K$ constructed in §4.
- $\mathcal{C}_\Delta^{\text{min}}/O_K$: r.n.c. if $C$ is $\Delta_v$-regular (cf. 3.13, 3.14).
- $K^s$, $\overline{K}$: separable and algebraic closure of $K$.
- $G_K$: $\text{Gal}(K^s/K)$, the absolute Galois group.
- $I_K < G_K$: the inertia group.
- $I_{\text{wild}} < I_K$: the wild inertia subgroup.
- $\text{Jac} C$: Jacobian of $C$.
- $l$: prime different from $p = \text{char} k$.
- $\text{Sp}_2$: standard representation $I_K \to \text{GL}_2(\mathbb{Q}_l)$, $\sigma \mapsto \left(\begin{smallmatrix} t^{(\sigma)} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $t : I_K \to \mathbb{Z}_l$ is the $l$-adic tame character (see [TaN, 4.1.4]).
- ⊕, ⊙: direct sum and difference of representations.
- $I$: trivial representation.
- $\mathbb{G}_{m,K}$: Spec $K[t, t^{-1}]$, affine line minus the origin.
- $m_{ij}$, $m_{is}$: $j$th column and $i$th row of a matrix $M = (m_{ij})$.

All representations are finite-dimensional, models are flat and proper, lattices are affine, and polygons are convex lattice polygons (vertices in $\mathbb{Z}^2$). See Table 1 for examples of curves and the special fibres of their m.r.n.c. models, with a glossary of notation used in the pictures.
$\Delta_x$-regular

(i) 
\[ \begin{array}{c}
\text{Sample equations (right column)} \supset \text{Char } k = 0, \text{ say.}
\end{array} \]

$xy^2 = x^4 + x^2 + \pi^3$ (good elliptic curve meeting $\text{III}^*$).
On the left is $\Delta$ broken into $v$-edges and $v$-faces, with the values of $v$ on $\Delta \cap \mathbb{Z}^2$, and the special fibre of the m.r.n.c. model $C_{\Delta}^{\text{min}}$ of 3.13, 5.7.
Components of positive genus are marked g1, g2 etc.

(ii) 
\[ \begin{array}{c}
\text{Removable 2-faces (not touching interior points $\Delta(\mathbb{Z})$ are grey; they do not affect the m.r.n.c. model; see §5. Here } y^2 = x^5 + x + \pi^6 \text{ has the same model as}
\end{array} \]

\[ y^2 = (x-1)(x-2)(x-3)(x-\pi)(x-2\pi)(x-3\pi)(x-4\pi) \]
Dashed blue lines mark empty chains of $P^1$s, that is main components meeting transversally. Here, the special fibre consists of two genus 1 curves meeting at two points.

(iii) 
\[ \begin{array}{c}
\text{Contractible 2-faces (meeting $\Delta(\mathbb{Z})$ in one point}
\end{array} \]

\[ P \text{ with } v(P) \in \mathbb{Z} \text{ are grey and marked with an arc; they give a chain of}
\]

\[ \mathbb{P}^1 \text{s between 2-faces at the end of the arc, here from the genus 1 component}
\]

\[ \tilde{X}_P \text{ to itself.} \]

(iv) 
\[ \begin{array}{c}
(2,1)
\end{array} \]

\[ y^2 = x^2(x-1)(x+2)(x+3) + \pi^4 \]
\[ \text{Contractible 2-faces (meeting $\Delta(\mathbb{Z})$ in one point}
\]

\[ P \text{ with } v(P) \in \mathbb{Z} \text{ are grey and marked with an arc; they give a chain of}
\]

\[ \mathbb{P}^1 \text{s between 2-faces at the end of the arc, here from the genus 1 component}
\]

\[ \tilde{X}_P \text{ to itself.} \]

\[ (y-1)^2 = (x-1)(x-2)(x-3)^2(x-4) + \pi^4: \text{this is}
\]

\[ \text{(iv) shifted by (3,1). It has a node at (3,1), and its depth cannot be determined from $\Delta_x$ alone.}
\]

(Here and below the conditions of Thm. 3.13 are not satisfied and, in fact, $C_{\Delta}$ is not regular.)

\[ (y-1)^2 = (x-1)^2(x-2)^2 + \pi^5: \]
\[ \text{as in (v), except the equation is reducible mod } \pi,
\]

\[ \text{and there are two components meeting at two nodes, again of undetermined depth.}
\]

\[ (y-1)^2 = (x-2)^5 + \pi^2: \]
\[ \text{singular point from a 2-face, which is not a node; here,}
\]

\[ \text{a cusp } y^2 = x^5 \text{ at (2,1).} \]

\[ y^2 = (x-1)(x-2)(x-3)^2(x-4) + \pi^4: \]
\[ \text{(iv) shifted by (3,0); singular point from an outer 1-face.}
\]

\[ x^4y^2 = x(y-x)^2 + \pi^3: \]
\[ \text{singular point from an inner 1-face.} \]

\[ \text{Table 1. Examples and notation in the pictures. For the}
\]

\[ \text{sample equations (right column) suppose char } k = 0, \text{ say.} \]
2. Baker’s theorem

By a classical result going back to Baker in 1893 [Bak], under generic conditions the genus of a plane curve is the number of interior integral points in its Newton polygon $\Delta$. Modern versions, over a general field, have been proved in [KWZ, Lemma 3.4], [BP, Thm 4.2] and [Bee, Thm 2.4]. There are extensions to higher-dimensional varieties and their invariants, notably over $\mathbb{C}$ [Kou, Kho, BKK],[Bat, §4]; see also [DL, Thm 1.3] over finite fields. The question which curves admit a ‘Baker model’ is addressed in detail in [CV]. Here we give a slightly different (and elementary) proof of Baker’s theorem. It describes regular differentials and points at infinity as well, and we will need the steps later.

Notation 2.1. Let $K$ be any field. Fix a polynomial, which is not zero or a monomial, 
$$f = \sum_{i,j} a_{ij} x^i y^j \in K[x,y].$$

Recall that the Newton polygon $\Delta$ of $f$ is 
$$\Delta = \text{convex hull}( (i, j) \mid a_{ij} \neq 0 ) \subset \mathbb{R}^2.$$ 
It is the smallest convex set containing the exponents of non-zero monomials, and by assumptions it is a convex polygon with at least two vertices. Write 
$$\Delta(Z) = Z^2 \cap \text{interior of } \Delta,$$
$$\bar{\Delta}(Z) = Z^2 \cap \Delta.$$
When $\Delta$ is a line segment ($\text{vol}(\Delta) = 0$), we view it as having two edges (1-dimensional faces) and $\Delta(Z) = \emptyset$. We will use ‘$L \in \partial \Delta$’ as a shorthand for ‘for $L$ an edge of $\Delta$’. For such an edge $L$, write 
$$L^* = L^*_{(\Delta)} = \text{unique affine function } Z^2 \to Z \text{ with } L^*|_L = 0 \text{ and } L^*|_\Delta \geq 0.$$ 
If $L \cap Z^2 = \{(i_0, j_0), \ldots, (i_r, j_r)\}$, ordered along $L$, set 
$$f_L(t) = \sum_{r=0}^t a_{i_n,j_n} t^n \in K[t].$$

Theorem 2.2. Let $f \in K[x,y]$, with Newton polygon $\Delta \subset \mathbb{R}^2$. If $C_0 : f = 0$ is a smooth curve in $\mathbb{G}^2_{m,K}$ and $f_L$ is square-free for every edge $L$, then

1. The non-singular complete curve $C$ birational to $C_0$ has genus $|\Delta(Z)|$.
2. $C$ has a basis of regular differentials, non-vanishing on $C_0$,
$$\omega_{ij} = x^{i-1} y^{j-1} \frac{dx}{f_y}, \quad (i, j) \in \Delta(Z).$$
(If $f_y = 0$, equivalently $K(C)/K(x)$ is inseparable, then swap $x \leftrightarrow y$.)
3. There is a natural bijection that preserves $\text{Gal}(K^s/K)$-action, 
$$C(\bar{K}) \setminus C_0(\bar{K}) \xleftarrow{1:1} \prod_{L \in \partial \Delta} \{ \text{roots of } f_L \text{ in } \bar{K}^x \}.  
$$

5The two possible orders give $f_L(t)$ vs $f_L(1/t)$; this choice does not affect anything.
6smooth=non-singular over $\bar{K}$, and square-freeness is also smoothness; recall that $\Delta=$line segment is allowed (non-connected curves) but not $\Delta=$point or $\emptyset$ (not a curve)
(4) If $P$ corresponds to a root of $f_L$ via (3), then, for any $(i, j) \in \mathbb{Z}^2$,

\begin{align*}
(4a) & \quad \text{ord}_P x^i y^j = L^*(i, j) - L^*(0, 0), \\
(4b) & \quad \text{ord}_P \omega_{ij} = L^*(i, j) - 1.
\end{align*}

Proof. Let $C$ be the non-singular complete curve birational to $C_0$, possibly non-connected. The statements are easy when $\Delta$ is a line segment $(C_0 = \text{union of } \mathbb{G}_m, C = \text{union of } \mathbb{P}^1 \mathbb{S})$, so assume that $\text{vol}(\Delta) > 0$. We will cover $C$ by charts $C_L, C_0 \hookrightarrow C_L \hookrightarrow C$, indexed by the edges $L$ of $\Delta$.

First, let $\phi : (i, j) \mapsto mj - ni + s$ be any affine function $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ which is non-negative on $\Delta$ and zero on some vertex; so $l = (m, n)$ is a primitive generator of the line $L = \phi^{-1}(0)$, going counterclockwise around $\Delta$.

Write $mn' - m'n = 1$ with $m', n' \in \mathbb{Z}$. In terms of $X = x^m y^n, Y = x^{n'} y^{n'}$,

$$f(x, y) = X^s Y^s (f_L(X) + Y h(X, Y)), \quad h \in K[X^\pm 1, Y].$$

The univariate polynomial $f_L$ is a non-zero constant when $L \cap \Delta = \text{vertex}$, and is separable (by assumption) of positive degree when $L \cap \Delta = \text{edge}$. In both cases,

$$C_L : f_L(X) + Y h(X, Y) = 0$$

is a smooth curve in $\mathbb{G}_m \times \mathbb{A}^1$ with $C_L \cap \mathbb{G}_m^2 = C_0$; so $C_0 \hookrightarrow C_L \hookrightarrow C$. The $K$-points in $C_L \setminus C_0$ have $Y = 0$ and correspond to the roots of $f_L$ in $(K^*)^s$, with the same Galois action. At these points,

- $(X =) x^m y^n$ is regular non-vanishing,
- $(Y =) x^{n'} y^{n'}$ is regular vanishing.

Conversely, pick $P \in C \setminus C_0$. Either $x$ or $y$ has a zero or a pole at $P$, so

$$(i, j) \mapsto \text{ord}_P x^i y^j$$

is a non-trivial linear map $\mathbb{Z}^2 \rightarrow \mathbb{Z}$. It has a rank 1 kernel spanned by some primitive vector $(b, -a)$. Changing its sign if necessary, it follows from the above construction that $P$ is ‘visible’ on $C_L$ obtained from $\phi : (i, j) \mapsto ai + bj + c$, for a unique $c$. Finally,

$$L^*(i, j) = ai + bj + c = \text{ord}_P x^i y^j + c, \quad L^*(0, 0) = c.$$

This proves (3) and (4a).

(4b) Suppose $P$ corresponds to a root of $f_L$ via (3).

It suffices to prove $\text{ord}_P \omega_{ij} = L^*(i, j) - 1$ for one $(i, j) \in \mathbb{Z}^2$ and use (4a) to get all $(i, j)$. By Lemma A.3, this claim is invariant under $\text{SL}_2(\mathbb{Z}) \subset \text{Aut } \mathbb{G}_m^2$, reducing to the case

$$L \subset \text{y-axis, } \quad C_0 : f_L(y) + x G(x, y) = 0, \quad \omega_{11} = \frac{dx}{f_L(y) + x G(x, y)}.$$

Since $f_L(y)$ is assumed to be separable and $P \in \text{y-axis}$, the denominator of $\omega_{11}$ does not vanish at $P$. Because $dx$ is regular non-vanishing on $\mathbb{A}^1 \times \mathbb{G}_m$, we get $\text{ord}_P \omega_{11} = 0$, as required.

(1), (2) The total degree of any $\omega_{ij}$ is the sum of $\text{ord}_P \omega_{ij} = L^*(i, j) - 1$ over all $L \subset \partial \Delta$, taken with multiplicity $\deg f_L$. An elementary calculation

\footnotesize

7Thus $\phi = L^*$ in the case $L \cap \Delta = \text{edge}$.
(Lemma A.2) shows that this sum is $2|\Delta(Z)| - 2$. This proves (1), as the degree of a differential form on a complete genus $g$ curve is $2g - 2$. Finally, the $\omega_{ij}$ with $(i, j) \in \Delta(Z)$ form a basis of regular differentials, as there are $g$ of them and there cannot be a linear relation between monomials $x^iy^j$ of degree $< \deg f$. (See Remark 2.3 for a slightly stronger statement.)

**Remark 2.3.** If $f = \sum a_{ij}x^iy^j$ satisfies the conditions of the theorem, the differentials $\omega_{ij}$ with $(i, j) \in \Delta(Z)$ have exactly one linear relation between them up to scalars, namely $\sum a_{ij}\omega_{ij} = 0$. This is clear from the minimality (=square-freeness) of the defining equation $f = 0$.

**Example 2.4.** The following are genus 0 curves, with $\Delta(Z) = \emptyset$:

```

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Example: Newton polygons of genus 0 curves (say char $K \nmid 2mn$)

$y^n = 1, \quad y = x^n + 1, \quad x^2 + y^2 = 1, \quad xy + x + y = 1, \quad x^n - 2 = y(x^m - 1)$

Conversely, Rabinowitz [Rab, Thm 1], extending Arkin[l] [Ar, Lemma 1] shows that a convex lattice polygon $\Delta \subset \mathbb{R}^2$ with $\Delta(Z) = \emptyset$ is $GL_2(\mathbb{Z})$-equivalent to one of the pictures above: a line segment, a right triangle with legs $1, n$ or $2, 2$, or a trapezium of height 1 (right-angled as above, if desired).

**Example 2.5.** Curves of genus $\leq 4$ always have an equation satisfying Baker’s theorem if $K$ is large enough, but not in general in higher genus [CV]. For instance, in genus 3, every smooth projective curve $C/K$ is either hyperelliptic or a plane quartic (but not both). If, say, $K = \bar{K}$, then $C$ has an equation $f = 0$ with $\Delta$ either

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A basis of regular differentials is $\partial y, x\partial y, y\partial y$ and $\partial y, x\partial y, x^2\partial y$, respectively. Similarly, in genus $\leq 2$ all curves have hyperelliptic models when $K = \bar{K}$, and there are seven (disjoint) types in genus 4 [CV, §7].

**Remark 2.6.** Let $C_0 : f = 0$ be an equation in $\mathbb{G}_{m,K}^2$, with $\text{vol}(\Delta) > 0$.

(a) If $C_0$ satisfies the conditions of Theorem 2.2, the proof shows that the associated smooth complete curve $C$ is covered by charts

$C = \bigcup_{L \subset \partial \Delta} C_L, \quad C_L \hookrightarrow \mathbb{G}_m \times \mathbb{A}^1$ closed.

---

It is not hard to see that this is a basis of differentials that are regular on $C_0$ and have at most simple poles on $C \setminus C_0$, though we will not need this.
These $\mathbb{G}_m \times \mathbb{A}^1$ cover, up to a finite set, a toric variety $T_\Delta$ with polytope $\Delta$, and $C$ is simply the closure of $C_0$ in $T_\Delta$. It is often called the toric resolution of $C_0$ on $T_\Delta$ (see [CDV, §2.1], [Koe], [Cas]). Two standard examples are

- triangle on $(0,0), (n,0), (0,n)$ $\supset \mathbb{P}^2$ $\supset$ curve of degree $n$,
- rectangle $[n,0] \times [0,m] \mathbb{P}^1 \times \mathbb{P}^1$ $\supset$ curve of degree $(m,n)$.

(b) The theorem constructs $C_L$ as in (2.7) and the closure $C \subset T_\Delta$ of $C_0$ for any $f$ with Newton polygon $\Delta$, whether $C_0$ is smooth or not. We call this scheme the completion $C_0^{\Delta}$ of $C_0$ with respect to its Newton polygon. Since $T_\Delta$ is a complete variety, $C_0^{\Delta}$ is a proper scheme over $K$.

(c) When $C_0$ is smooth, the squarefreeness condition in Baker’s theorem to ensure smoothness of $C$ is sufficient but not quite necessary. We say that $C_0$ is outer regular at an edge $L$ of $\Delta$ if $C_L$ is smooth. When $C_0$ is outer regular at all $L$, in other words when $C_0^{\Delta}$ is smooth, the curve $C_0$ is often called nondegenerate with respect to $\Delta$ (CDV, §2, Def. 1). For example, $(x-1)^2 + (y-1)^2 = 1$ is smooth in $\mathbb{G}_m^2$, and its closure in $T_\Delta = \mathbb{P}^2$ is smooth as well. The restriction of $f$ to the two edges of $\Delta$ on the coordinate axes is not squarefree, but $C_0$ is outer regular there.

(d) Fix $\Delta$ with $\text{vol}(\Delta) > 0$ and let $V = \{\text{vertices of } \Delta\}$. All $f \in \bar{K}[x,y]$ with Newton polygon $\Delta$ form a family $C \subset \mathcal{P} \times T_\Delta$ with a parameter variety $\mathcal{P} \cong \mathbb{A}^{\Delta(\mathbb{Z}) \setminus V} \times \mathbb{G}_m^V$, flat over $\mathcal{P}$. Its generic member is smooth [CDV, §2, Prop 1], irreducible (dimension count), and hence connected. Consequently, any $C_0$ with Newton polygon $\Delta$ is connected (Zariski connectedness), and

$$C_0^{\Delta} \text{ has arithmetic genus } |\Delta(\mathbb{Z})|,$$

since arithmetic genus is constant in flat families [Har, III.9.13]. In particular, when $C_0$ is an integral scheme and so defines a curve, we have

$$\text{geometric genus of } C_0 \leq |\Delta(\mathbb{Z})| \quad \text{(Baker’s inequality)}.$$
Definition 3.1 (Faces). The images of the 0-, 1- and 2-dimensional (open) faces of the polytope $\Delta_v$ under the homeomorphic projection $\Delta_v \to \Delta$ are called $v$-vertices, $v$-edges and $v$-faces of $\Delta$. Thus, $v$-vertices are points in $\mathbb{Z}^2$, $v$-edges (denoted $L$) are homeomorphic to an open interval, and $v$-faces (denoted $F$) to an open disk.

Notation 3.2 (Integer points, denominators). For $v$-edges and $v$-faces write $L(\mathbb{Z}) = L \cap \mathbb{Z}^2$, $F(\mathbb{Z}) = F \cap \mathbb{Z}^2$. As before, $\Delta(\mathbb{Z}) = (\text{interior of } \Delta) \cap \mathbb{Z}^2$, and we write $\bar{L}(\mathbb{Z})$, $\bar{F}(\mathbb{Z})$, $\bar{\Delta}(\mathbb{Z})$ to include points on the boundary. We decompose $\Delta(\mathbb{Z})$ into $\Delta(\mathbb{Z})_F = \{P \in \Delta(\mathbb{Z}) \mid P \in F(\mathbb{Z}) \text{ for some } v\text{-face } F\}$, $\Delta(\mathbb{Z})_L = \{P \in \Delta(\mathbb{Z}) \mid P \notin F(\mathbb{Z}) \text{ for any } v\text{-face } F\}$.

We also use subscripts to restrict to points with $v(P)$ in a given set, such as $F(\mathbb{Z})_{Z_p} = \{P \in F(\mathbb{Z}) \mid p \nmid \text{den}(v(P))\} = \{P \in F(\mathbb{Z}) \mid v(P) \in \mathbb{Z}_p\}$.

(When $p = 0$, the above is interpreted as an empty condition.) We write $\Delta(\mathbb{Z})_Z^F$ etc. for multiple constraints. The denominator $\delta_\lambda$ of a $v$-edge or a $v$-face $\lambda$ is, equivalently (see Lemma A.1 for (1)$\iff$(3)),

1. common denominator of $v(P)$ for $P \in \lambda(\mathbb{Z})$;
2. smallest $m \geq 1$ for which $\lambda(\mathbb{Z})_Z^m = \lambda(\mathbb{Z})$;
3. index of the affine lattice spanned by $\lambda(\mathbb{Z})$ in its saturation in $\mathbb{Z}^2$.

Example 3.3. Let $C : f = 0$ be the genus 4 curve defined by

$$f = y^3 + y^2 + \pi^3 x^6 + \pi x^3 + \pi^3.$$ 

Here $\Delta_v$ has $v$-faces $F_1, F_2, F_3$ with $\delta_F=3,3,6$, $v$-edges with $\delta_L=1,1,1,2,3,3,3$ and five $v$-vertices (here from all of the coefficients of $f$). Below (left) is a picture of $\Delta$ broken into $v$-faces, and the values of $v$ on $\Delta(\mathbb{Z})$. ‘Capital’ ones indicate that the corresponding coefficient of $f$ has exactly that valuation (and not larger):

In this example, $|\Delta(\mathbb{Z})| = 4$, with $|\Delta(\mathbb{Z})_F| = |\Delta(\mathbb{Z})_L| = 2$.

To produce the regular model and describe the equations for the components of the special fibre (picture on the right), we need to extract monomials from $f$ that come from a given face, restrict them to a sublattice, and reduce the resulting equation:
Definition 3.4 (Restriction). Let \( g = \sum_{i \in \mathbb{Z}^n} c_i x^i \in K[x] \) be a polynomial in \( n \) variables \( x = (x_1, \ldots, x_n) \), and \( \emptyset \neq S \subset \mathbb{Z}^n \). Take \( \Lambda \) to be the smallest affine lattice with \( S \subset \Lambda \subset \mathbb{Z}^n \), say of rank \( r \). Choose an isomorphism \( \phi : \mathbb{Z}^r \to \Lambda \), write \( y = (y_1, \ldots, y_r) \) and define the restriction
\[
g|_S = \sum_{i \in \phi^{-1}(S)} c_{\phi(i)} y^i.
\]
Different choices of \( \phi \) change \( g|_S \) by a \( \text{GL}_r(\mathbb{Z}) \)-transformations of the variables and multiplication by monomials. For a \( v \)-vertex/edge/face \( \lambda \), set
\[
f|_\lambda = f|_{\chi(\mathbb{Z})\Lambda}.
\]

Definition 3.5 (Reduction). Suppose \( h \in K[x, y] \), and there are \( c, m, n \in \mathbb{Z} \) such that \( \hat{h}(x, y) = \pi^n h(\pi^m x, \pi^n y) \) has coefficients in \( O_K \), and \( h \in K[x, y] \) and \( \hat{h} \mod \pi \in k[x, y] \) have the same Newton polygon. Then we say that \( \hat{h} \mod \pi \) is the reduction of \( h \), and denote it \( \bar{h} \).

Example 3.6. Let \( f = y^2 - x^3 - \pi \), and \( F \) the unique \( v \)-face of \( \Delta \). In the notation of 3.4 and 3.5, the lattice \( \Lambda = \mathbb{Z}(3,0) + \mathbb{Z}(0,2) \) has index \( \delta_F = 6 \) in \( \mathbb{Z}^2 \), and \( f|_F = f|_{F(\mathbb{Z})\Lambda} = f|_{\Lambda} \) and its reduction \( \bar{f}|_F \) are as follows:

\[
\begin{align*}
\text{f} = y^2 - x^3 - \pi & \in K[x, y] \\
\text{f}|_F = y - x - \pi & \in K[x, y] \\
\bar{\text{f}}|_F = y - x - 1 & \in k[x, y]
\end{align*}
\]

Definition 3.7 (Components \( X_F, X_L \)). Define \( k \)-schemes
\begin{enumerate}
\item \( X_L : \{ \bar{f}|_L = 0 \} \subset \mathbb{G}_{m,k} \) for each \( v \)-edge \( L \) of \( \Delta \),
\item \( X_F : \{ \bar{f}|_F = 0 \} \subset \mathbb{G}_{m,k}^2 \) for each \( v \)-face \( F \) of \( \Delta \),
\item \( \bar{X}_F = \text{completion of } X_F \) with respect to its Newton polygon \( (2.6b) \);
\end{enumerate}

it is a geometrically connected scheme, proper over \( k \) \((2.6d)\).

The vertices and edges of the Newton polygon of \( \bar{f}|_F \) correspond to those of \( F \) (cf. example above), and so there is a bijection from Baker’s Theorem,
\[
(3.8) \quad \bar{X}_F(\bar{k}) \setminus X_F(\bar{k}) \longleftrightarrow \prod_{L \subset \partial F} X_L(\bar{k}).
\]

We can now define \( \Delta_v \)-regularity. Modulo one allowed exception, it is simply saying that all \( X_F \) need to satisfy Baker’s Theorem assumptions:

Definition 3.9 (\( \Delta_v \)-regularity). We say that \( C \) (or \( f \)) is \( \Delta_v \)-regular if all \( X_F, X_L \) are smooth over \( k \), except that for outer \( v \)-edges \( L \subset \partial F \) with \( \delta_L = \delta_F \), we drop the assumption that \( X_L \) is smooth but require \( X_F \) to be outer regular at \( L \), i.e. smooth at the points that correspond to \( L \) via \((3.8)\).

It is not hard to see that regularity does not depend on the choices made.
Example 3.10. The curve $C/K$ of Example 3.3 is $\Delta_\nu$-regular iff $\text{char } k \neq 3$.
(There is only one $\nu$-edge to check, all other restrictions are linear).

Remark 3.11 (Base change). Suppose $K'/K$ is an extension of finite ramification degree $e$. For a $\nu$-face $F$, write $X'_F$, for the scheme $X_F$ computed for $f$ as an element of $K'[x,y]$. Let $\Lambda_F$ be the affine lattice spanned by $F(\mathbb{Z})_2$, and $\Lambda_F \subseteq \Lambda'_F \subseteq \mathbb{Z}^2$ the unique intermediate lattice with $(\Lambda'_F : \Lambda_F) = \gcd(\delta_F, e)$. Let $\phi : \mathbb{G}_m^2 \rightarrow \text{ Spec } k[\Lambda'_F] \rightarrow \text{ Spec } k[\Lambda_F] = \mathbb{G}_m^2$ be the corresponding degree $(\Lambda'_F : \Lambda_F)$ cover. Then

$$X'_F \cong X_F \times_{\mathbb{G}_m^2, \phi} \mathbb{G}_m^2,$$

and a similar statement holds for $\nu$-edges. Thus,

1. If $\gcd(\delta_F, e) = 1$, then $X_F/k$ smooth $\iff X'_F$ is.
2. If $p \nmid \gcd(\delta_F, e)$, then $\phi$ is étale, so $X_F/k$ smooth $\Rightarrow X'_F$ is.
3. If $p \nmid \gcd(\delta_F, e)$ for every $\nu$-face $F$ of $\Delta$, e.g. $K'/K$ tame, then $C/K$ is $\Delta_\nu$-regular $\Rightarrow C/K'$ is $\Delta_\nu$-regular.

Finally, we need slopes of two $\nu$-faces $F_1, F_2$ meeting at a $\nu$-edge $L$. They will control the chains of $\mathbb{P}^1$s between $\bar{X}_{F_1}$ and $\bar{X}_{F_2}$:

Definition 3.12 (Slopes). Every $\nu$-edge $L$ is either inner, bounding two $\nu$-faces, say $F_1$ and $F_2$, or outer, bounding $F_1$ only. Choose $P_0, P_1 \in \mathbb{Z}^2$ with $L^*(p_0) = 0, L^*(P_1) = 1$ (see 2.1). The slopes $[s^L_1, s^L_2]$ at $L$ are

$$s^L_1 = \delta_L (v_1(P_1) - v_1(P_0)), \quad s^L_2 = \begin{cases} \delta_L (v_2(P_1) - v_2(P_0)) & \text{ for } L \text{ inner,} \\ \lfloor s^L_1 - 1 \rfloor & \text{ for } L \text{ outer,} \end{cases}$$

where $v_i$ is the unique affine function $\mathbb{Z}^2 \rightarrow \mathbb{Q}$ that agrees with $v$ on $F_i$.

In §4 from $\Delta_\nu$ we construct a scheme $\mathcal{C}_\Delta/O_K$. The reader is invited to skip the construction completely since regularity and the special fibre of $\mathcal{C}_\Delta$ can be described solely in terms of the above data. The results are:

Theorem 3.13 (Model in the $\Delta_\nu$-regular case). Suppose $C : f = 0$ is $\Delta_\nu$-regular. Then $\mathcal{C}_\Delta/O_K$ is a regular model of $C/K$ with strict normal crossings. Its special fibre $\mathcal{C}_k/k$ is as follows:

1. Every $\nu$-face $F$ of $\Delta$ gives a complete smooth curve $\bar{X}_F/k$ of multiplicity $\delta_F$ and genus $|F(\mathbb{Z})_2|$.
2. For every $\nu$-edge $L$ with slopes $[s^L_1, s^L_2]$ pick $\frac{m_1}{d_1} \in \mathbb{Q}$ so that

$$s^L_1 = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \ldots > \frac{m_r}{d_r} > \frac{m_{r+1}}{d_{r+1}} = s^L_2 \quad \text{ with } \frac{m_i m_{i+1}}{d_i d_{i+1}} = 1.$$

Then $L$ gives $|X_L(\bar{k})|$ chains of $\mathbb{P}^1$s of length $r$ from $\bar{X}_{F_1}$ to $\bar{X}_{F_2}$ (and open-ended in the outer case) and multiplicities $\delta_i d_{i-1}, \ldots, \delta_i d_r$.\footnote{If $r = 0$, this is interpreted as $X_{F_1}$ meeting $X_{F_2}$ transversally at $|X_L(\bar{k})|$ points in the inner case, and no contribution from $L$ in the outer case. In the outer case, this happens precisely when $\delta_L = \delta_{F_1}$.} The group $G_k$ permutes the chains by its natural action on $X_L(\bar{k})$.\footnote{If $r = 0$, this is interpreted as $X_{F_1}$ meeting $X_{F_2}$ transversally at $|X_L(\bar{k})|$ points in the inner case, and no contribution from $L$ in the outer case. In the outer case, this happens precisely when $\delta_L = \delta_{F_1}$.}
Contributions to $C_k$ from an inner $v$-edge $L_1$ and an outer $v$-edge $L_2$

This is a special case of the following more general version:

**Theorem 3.14 (General case).** Suppose $f(x, y) = 0$ defines a 1-dimensional scheme $C_0 \subset \mathbb{P}^2_{m,K}$. Then $C_\Delta/O_K$ is a proper flat model of the completion $C = C_0^\Delta$. Its special fibre $C_k$ is a union of closed subschemes $\bar{X}_F$ indexed by $v$-faces $F$ and $X_L \times_k \Gamma_L$ indexed by $v$-edges $L$, all 1-dimensional over $k$:

1. The $X_L, X_F, \bar{X}_F$ are as in 3.7. The $\bar{X}_F$ are geometrically connected, have arithmetic genus $|F(\mathbb{Z})|$, and come with multiplicity $\delta_F$ in the special fibre (i.e. defined by $f^{\Delta'} = 0$ in $C_k$).
2. For every $v$-edge $L$ with slopes $[s_1^L, s_2^L]$ pick $\frac{d_s}{d_t} \in \mathbb{Q}$ so that

$$s_1^L = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = s_2^L \quad \text{with} \quad \frac{n_i n_{i+1}}{d_i d_{i+1}} = 1.$$

Let $\Gamma_L = \Gamma_1^L \cup \ldots \cup \Gamma_r^L$ be a chain of $\mathbb{P}_k^1$'s, with multiplicities $\delta_i d_i$, meeting transversely: $\infty \in \Gamma_i^L$ is identified with $0 \in \Gamma_{i+1}^L$; when $r = 0$, let $\Gamma_L = \text{Spec } k$, viewing it as $0 \in \Gamma_1^L$ and $\infty \in \Gamma_r^L$.

3. The subscheme $X_L \times \{0\} \subset X_L \times \Gamma_L^i$ is identified with the subscheme of $\bar{X}_{F_1} \setminus X_{F_1}$ that corresponds to $L$ via (3.8); similarly for $X_L \times \{\infty\} \subset X_L \times \Gamma_L^i$ and $\bar{X}_{F_2} \setminus X_{F_2}$ when $L$ is inner.

4. The intersections of $\bar{X}_{F_1}$ $(i = 1, 2)$ with $X_L \times \Gamma_L$ and, when $r = 0$, of $\bar{X}_{F_1}$ with $\bar{X}_{F_2}$ at $X_L$ are transversal. In other words, the intersection, as a scheme, is given by $f_{\Delta'}^{\Delta'} = 0$.

The model $C_\Delta$ is given by explicit charts in §4. It is geometrically regular at

- the smooth locus of $X_F$, for every $v$-face $F$,
- (smooth locus of $X_L$) $\times \Gamma_L$, for every $v$-edge $L$,
- the smooth points of $\bar{X}_F \setminus X_F$ that correspond to $L$ via (3.8), when $L \subset \partial F$ is outer with $\delta_L = \delta_F$, and $r = 0$ in (2).

If $C_0$ is $\Delta_v$-regular, then $C$ is smooth, and $C_\Delta/O_K$ is its r.n.c. model.

The theorems are proved in §4. See Table 1 for some examples, both $\Delta_v$-regular and not. We end this section with a few comments:

**Remark 3.15 (Hirzebruch-Jung).** To see that sequences as in 3.14(2) exist, take all numbers in $[s_2^L, s_1^L] \cap \mathbb{Q}$ of denominator $\leq \max(\text{den } s_1^L, \text{den } s_2^L)$ in decreasing order. This is essentially a Haros (=Farey) series, and so satisfies the determinant condition [HW, Ch. III, Thm. 28]. Now repeatedly remove, in any order, terms of the form (loc. cit., Thm. 29)

$$\ldots > \frac{a}{b} > \frac{a+c}{b+d} > \frac{c}{d} > \ldots \quad \rightarrow \quad \ldots > \frac{a}{b} > \frac{c}{d} > \ldots,$$
until this is no longer possible; this corresponds to blowing down \( \mathbb{P}^1 \)'s of self-intersection \(-1\), see (3.17) below. The resulting minimal sequence is unique\(^{11}\). If \((s^L_2, s^L_1) \cap \mathbb{Z} = \{N, \ldots, N + a\}\) is non-empty, it has the form

\[
s^L_1 = \frac{n_0}{d_0} > \cdots > \frac{n_k}{d_k} > N + a > \cdots > N + 1 > \frac{n_l}{d_l} > \cdots > \frac{n_{r+1}}{d_{r+1}} = s^L_2,
\]

with \(d_0, \ldots, d_k\) decreasing and \(d_l, \ldots, d_{r+1}\) increasing. If \(s^L_2 > 0\), say (else shift by an integer), the numbers \(N > \frac{n_l}{d_l} > \cdots > \frac{n_{r+1}}{d_{r+1}}\) are the approximants of the Hirzebruch-Jung continued fraction expansion of \(s^L_2\), see [Ful, §2.6]. (Similarly, for the first terms consider the expansion of \(1 - s^L_1\).) These are designed to resolve toric singularities, which is essentially what the proof of 3.14 does, so their appearance is expected.

**Remark 3.16** (Minimal vs normal crossings). If \(C/K\) is \(\Delta_v\)-regular, the theorem constructs a r.n.c. model, in fact m.r.n.c. after an easy modification (see §5). Recall that this is not the same as the minimal regular model: the latter has generally fewer components but more complicated singularities in the special fibre. A r.n.c. model is encoded by the dual graph and the component multiplicities, so it is somewhat easier to work with (and to draw, for the matter). Note that if a component \(X\) of multiplicity \(m\) meets \(Y_1, \ldots, Y_n\) of multiplicities \(m_1, \ldots, m_n\) transversally, then the self-intersection

\[
(X \cdot X) = -\frac{m_1 + \cdots + m_n}{m} \quad (\in -\mathbb{N}).
\]

If \(X\) has genus 0 and \(X \cdot X = -1\), then \(X\) may be blown down. However, when \(n \geq 3\), the result is no longer r.n.c. For example, take the genus 12 curve \(C/\mathbb{Q}_7\) from [GRS2],

\[
x^7 = \frac{y^3 - 2y^2 - y + 1}{y^3 - y^2 - 2y + 1}.
\]

Letting \(y \mapsto \frac{2y - 3}{3y - 2}\) and clearing the denominator, we get a model (tree-like with no positive genus components, in agreement with [GRS2, App. A])

The components \(X_{F_1}, X_{F_2}\) (multiplicity 21) have self-intersection \(-1\). If they are blown down, we find components of multiplicity 14 meeting ‘7,6 and 1’, so they get self-intersection \(-1\) and can be blown down again. Ditto for ‘6’ and then ‘3’, resulting in a minimal regular model which has one multiplicity 1 component with two (bad) singularities on the special fibre.

**Example 3.18.** Here is an example for Theorem 3.14 (and not just 3.13). Consider the curve \(C_5/\mathbb{Q}_2\) from [PSS, §13.3]. Letting \(x \mapsto x + 1, y \mapsto xy + 1\) we find an equation

\(^{11}\)E.g., m.r.n.c. models would not be unique otherwise
Here $\Delta_v$ and the special fibre of the model $C_\Delta$ are as follows:

At the $v$-face $F_1$ (in red), the reduced equation is $\overline{f_{F_1}} = XY + X + Y + 1$ and it gives two irreducible components $X_{F_1^v} : X = 1$ and $X_{F_1^b} : Y = 1$ meeting transversally at $P = (1,1)$; both have multiplicity $\delta_{F_1} = 4$ on the special fibre. The point $P$ is not regular, and the regular model has a chain of $\mathbb{P}^1$s with multiplicity 4 from $X_{F_1^a}$ to $X_{F_1^b}$ (zigzag on the right picture). Its length is not determined by $\Delta_v$, and requires a computation in the local ring at $P$. (It is actually 5, see [PSS, §13.3].)

**Remark 3.19** (Reduction of points). Let $C/O_K$ be any regular model of a curve $C/K$. Write $N$ for the set of multiplicity 1 components of the special fibre $C_k$ of $C$, and $X^{ns} \subseteq X$ for the smooth locus of $X$ for $X \in N$. There is a natural reduction map

$$\text{red}: C(K) \to C_k(k),$$

landing in $\prod_{X \in N} X^{ns}(k)$ (see [Liu, 9/1.32]), and onto it if $K$ is Henselian. (In particular, if $N = \emptyset$, then $C(K) = \emptyset$.)

Now suppose $C/K$ is $\Delta_v$-regular and $C = C_\Delta$. The components $X \in N$ come from three sources, and are indexed by:

1. triples $(L, r, n)$ where $L$ is an *inner* $v$-edge with $\delta_L = 1$, $r$ a root of $\overline{f_{|L}}$ in $k^x$, and $n \in (s_L^b, s_L^s) \cap \mathbb{Z}$, plus

2. *$v$-faces* $F$ with $\delta_F = 1$.

Write $C_0$ for the curve $f = 0$ in $\mathbb{G}_m^2$, and let $P \in C(K)$ be a point.

(a) Suppose $P \in C(K) \setminus C_0(K)$. By Baker’s theorem 2.2(3) it corresponds to a root $r$ of $f_L$ for an edge $L$ of $\Delta$. The edge breaks into $v$-edges that correspond to factors of $f_L$ by its Newton polygon over the completion of $K$. The factor that has $r$ as a root gives the $v$-edge indexing the component to which $P$ reduces (Case $(1_o)$).

(b) Suppose $P \in C_0(K)$, so $x(P), y(P) \in K^x$. The linear form

$$L_P(i, j, z) = v(x(P))i + v(y(P))j + z$$

on $\Delta_v \subset \mathbb{R}^3$ is minimised on some face. It is easy to see that this face must project down either onto a $v$-edge of $\Delta$ (Case $(1_i)$ or $(1_o)$) or a $v$-face (Case (2)). It corresponds to the component to which $P$ reduces.

---

12as clear from the picture of $\Delta_v$ on the left, thanks to the scarcity of units in $F_2$
Example 3.20. The curve in Table 1(i) has $|N| = 2$, with one component from $F_2$ (of genus 1) and one from the leftmost $v$-edge (of genus 0). Here
\[
\begin{align*}
    v(x(P)) < 0 & \implies \text{red}(P) \in \text{genus 1 component}, \\
    v(x(P)) > 4 & \implies \text{red}(P) \in \text{genus 0 component}.
\end{align*}
\]
There are no points with $v(x(P)) \in \{0, 1, 2, 3, 4\}$, as all the corresponding $L_P$ are minimised on a vertex of $\Delta_v$; the $v$-edge $L$ between $F_1$ and $F_2$ gives no components since $(s_{F_2}^1, s_{F_2}^4) \cap \mathbb{Z} = \emptyset$. Replacing $\pi^4$ by $\pi^n$ with $n \geq 5$ in the equation of $C$ would introduce $\left\lfloor \frac{n-1}{4} \right\rfloor$ components of multiplicity 1 on the chain of $\mathbb{P}^1$s from $L$, to which points can reduce as well.

4. Construction of the model $C_\Delta$

We now construct the model $C_\Delta$ and prove Theorem 3.14.

If $p$ is harmless, $k = k$ and $C$ is $\Delta_v$-regular, one can get $C_\Delta$ by passing to $K_e = K(\sqrt{\pi})$ with $e = \text{lcm}_F \delta_F$, showing that $C/K_e$ is semistable (easy), and taking the quotient of a semistable model $\tilde{C}/O_K$ by the action of $\text{Gal}(K_e/K) \cong \mathbb{Z}/e\mathbb{Z}$; say $\varpi \in K$. This approach is well studied [Vie, LoD, CES, Hal], and the theorem follows from here for $\Delta_v$-regular curves, but only in the tame case $p || e$. In the wild case $p|e$, however, $C/K_e$ has no reason to be semistable. To deal with that, and with non-$\Delta_v$-regular curves, we take another approach and brutally construct $C_\Delta$ with a toroidal embedding over $O_K$. The numerology of the fan is, however, exactly the same as from the quotient construction. In fact, the charts are explicit, and toric geometry is only needed to get a free proof that $C_\Delta$ is separated.

4.1. Planes. Take a $v$-edge, say inner, $L \subset \partial F_1, \partial F_2$. Let $\tilde{F}_1, \tilde{F}_2, \tilde{L} \subset \mathbb{R}^3$ be the planes/line through the faces of $\Delta_v \subset \mathbb{R}^3$ above them. Pick an ‘origin’ $\tilde{P}_0 \in \tilde{L} \cap \mathbb{Z}^3$. We construct vectors $\nu, \omega_0, \ldots, \omega_r+1 \in \mathbb{Q}^3$ with $\omega_i$ lying above one another (same $x, y$ coordinates and $v$-coordinate decreasing with $i$), with
\[
\begin{align*}
    \tilde{L} &= \tilde{P}_0 + \mathbb{R}\nu, \quad \tilde{F}_1 = \tilde{P}_0 + \mathbb{R}\nu + \mathbb{R}\omega_0, \quad \tilde{F}_2 = \tilde{P}_0 + \mathbb{R}\nu + \mathbb{R}\omega_r+1, \\
    \text{det}(\nu, \omega_{r,i}) &= 1, \quad \text{det}(\nu, \omega_i, -\omega_i+1) = \frac{1}{\delta_L d_{i, i+1}}.
\end{align*}
\]

To be precise, let $L$ be any $v$-edge and $\tilde{L}$ as above. Write $\delta = \delta_L$. Fix $P_0, P_1 \in \mathbb{Z}^2$ with $L^*(P_0) = 0$, $L_{(P_0)}(P_1) = 1$ as in 3.12 and fractions $\frac{n_i}{d_i}$.
(i = 0, ..., r + 1) as in 3.14. Thus,

\[ v F_1 (P_1) - v F_1 (P_0) = \frac{n_0}{\delta d_0} > \frac{n_1}{\delta d_1} > \ldots > \frac{n_r}{\delta d_r} > \frac{n_{r+1}}{\delta d_{r+1}}. \]

Write \( \bar{L} = P_0 + R \nu \) with \( \nu = (v_x, v_y, v_z) \in \mathbb{Z}^3 \) primitive and \( (v_x, v_y) \) counterclockwise along \( \partial F_1 \). Write \( \bar{P}_1 - P_0 = (w_x, w_y) \) and \( \omega_i = (w_x, w_y, \frac{d_i}{d_i}) \); the choices force \( v_x w_y - v_y w_x = 1 \). We also make one modification:

If \( L \) is outer, redefine \( d_{r+1} = 0, n_{r+1} = -1, \omega_{r+1} = (0, 0, -\frac{1}{2}) \). (†)

These modified charts will be responsible for the ‘most outer’ points of \( C_\Delta \).

In all cases, define planes in \( \mathbb{R}^3 \) (see above picture, when \( L \) is inner),

\[ P_{L,i} = R \nu + R \omega_i \quad i = 0, \ldots, r + 1. \]

The planes \( P_0 + P_{L,i} \) rotate around \( \bar{L} \); the first one contains \( \bar{F}_1 \), and the last either contains \( \bar{F}_2 \) (\( L \) inner) or is vertical (\( L \) outer). Let

\[ P_{L,i}^{\perp} \subset P_{L,i} \]

be the ray \( (\equiv \mathbb{R}_+) \) for which the half-space \( P_0 + P_{L,i} + P_{L,i}^{\perp} \) contains \( \Delta_\nu \).

4.2. Toroidal embedding. For toroidal embeddings over a DVR, we follow [K2MS, §IV.3]. For a \( v \)-edge \( L \), define polyhedral cones in \( \mathbb{R}^2 \times \mathbb{R}_+ \),

- 0-dimensional cone \( \sigma_0 = \{ 0 \} \)
- 1-dimensional cones \( \sigma_{L,i} = P_{L,i}^{\perp} \quad (0 \leq i \leq r + 1) \)
- 2-dimensional cones \( \sigma_{L,i,i+1} = P_{L,i}^{\perp} + P_{L,i+1}^{\perp} \quad (0 \leq i \leq r) \)

The set \( \Sigma \) of all such cones from all \( L \) is a fan. The associated toric scheme

\[ X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma, \quad X_\sigma = \text{Spec} O_K [\sigma^\vee \cap \mathbb{Z}^3] \]

in the notation of [K2MS] is flat and separated over \( O_K \) (though not proper, as there are no 3-dimensional cones). Its generic fibre naturally contains \( \mathbb{G}^2_{m,K} \), and we define

\[ C_\Delta = \text{ closure of } \{ f = 0 \} \subset \mathbb{G}^2_{m,K} \text{ in } X_\Sigma. \]

4.3. Charts. Let us describe the charts \( X_\sigma \) explicitly. Fix \( L \) as in 4.1, and tweak \( \nu, \omega_i, -\omega_{i+1} \) into a basis of \( \mathbb{Z}^3 \); take \( \delta \nu \) (which is primitive) for the first basis vector. For the other two, shift \( d_i \omega_j \) by an appropriate multiple \( k_i \nu \) to get rid of the denominator in the last coordinate. In other words, pick \( k_i \in \mathbb{Z} \) with

\[ k_i \equiv -n_i \cdot (\delta v_z)^{-1} \mod \delta \quad (0 \leq i \leq r + 1). \]

As \( \delta \nu \) is primitive, \( \delta v_z \) is invertible mod \( \delta \), so this is possible; when \( \delta = 1 \), set \( k_i = 0 \), say. For \( 0 \leq i \leq r \) define

\[ M_{L,i} = \begin{pmatrix} \delta v_x & d_i w_y + k_i v_x & -d_{i+1} w_y - k_{i+1} v_x \\ \delta v_y & d_i w_z + k_i v_y & -d_{i+1} w_z - k_{i+1} v_y \\ \delta v_z & d_i w_z + k_i v_z & -d_{i+1} w_z - k_{i+1} v_z \end{pmatrix} = \begin{pmatrix} d_i & d_i w_y & -d_{i+1} w_y \\ d_i & d_i w_z & -d_{i+1} w_z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_i & k_{i+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tilde{N}. \]
Our choices guarantee that \( M = M_{L_1} \) has integer entries. Moreover,
\[
\det M = \det \tilde{M} = \delta v_y^{\frac{d_0 w_i z_n + d_1 v_z w_x}{\delta}} - \delta v_y^{\frac{d_0 w_i z_{n+1} + d_1 v_z w_x}{\delta}} + \delta v_z \cdot 0 \\
= \delta v_y^{\frac{d_0 w_i z_n + d_1 v_z w_x}{\delta}} - \delta v_y^{\frac{d_0 w_i z_{n+1} + d_1 v_z w_x}{\delta}} = 1.
\]

Thus \( M \in \text{SL}_3(\mathbb{Z}) \), with inverse \( M^{-1} = T^{-1} \tilde{M}^{-1} \),
\[
(4.1) \quad M_{L_1}^{-1} = \begin{pmatrix} 1 & -k_i/\delta & k_{i+1}/\delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_y/\delta & -w_x/\delta & 0 \\ n_{i+1}v_y - d_{i+1}v_zw_y & -n_{i+1}v_x + d_{i+1}v_zw_x & \delta_{i+1} \\ n_iv_y - d_iw_zw_y & -n_iv_x + d_iw_zw_x & \delta_i \end{pmatrix}.
\]

Because of the choices of \( \nu, \omega_i \) and their orientation, we have
\[
P_{L,i} = \mathbb{R} \nu + \mathbb{R} \omega_i = \mathbb{R} m_{i+1} + \mathbb{R} m_{i+2}, \quad \sigma_{L,i} = P_{L,i}^\perp = \mathbb{R}_+ \tilde{m}_{i+3},
\]
\[
P_{L,i+1} = \mathbb{R} \nu + \mathbb{R} \omega_{i+1} = \mathbb{R} m_{i+1} + \mathbb{R} m_{i+3}, \quad \sigma_{L,i+1} = P_{L,i+1}^\perp = \mathbb{R}_+ \tilde{m}_{i+2},
\]
\[
P_{L,i} \cap P_{L,i+1} = \mathbb{R} \nu = \mathbb{R} m_{i+1}, \quad \sigma_{L,i,i+1} = (P_{L,i}^\perp, P_{L,i+1}^\perp) = \mathbb{R}_+ \tilde{m}_{i+2} + \mathbb{R}_+ \tilde{m}_{i+3},
\]
and monomial exponents from the dual cones are
\[
\sigma_{L,i}^\perp \cap \mathbb{Z}^3 = \mathbb{Z} m_{i+1} + \mathbb{Z} m_{i+2} + \mathbb{Z}_+ m_{i+3} \\
\sigma_{L,i+1}^\perp \cap \mathbb{Z}^3 = \mathbb{Z} m_{i+1} + \mathbb{Z}_+ m_{i+2} + \mathbb{Z} m_{i+3} \\
\sigma_{L,i,i+1}^\perp \cap \mathbb{Z}^3 = \mathbb{Z} m_{i+1} + \mathbb{Z}_+ m_{i+2} + \mathbb{Z}_+ m_{i+3}.
\]

In other words, the coordinate transformation
\[
(X, Y, Z) = \left( \frac{x^{m_{i+1}}y^{m_{i+2}}z^{m_{i+3}}}{x^{m_{i+1}}y^{m_{i+2}}z^{m_{i+3}}} \right) = (x, y, \pi) \bullet M, \quad (x, y, \pi) = (X, Y, Z) \bullet M^{-1}
\]
gives ring homomorphisms (as \( \tilde{m}_{i+2}, \tilde{m}_{i+3} \geq 0 \), the denominator makes sense)
\[
K[x^{\pm 1}, y^{\pm 1}] \xrightarrow{M} \frac{O_K[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}]}{(\pi - X^{m_{i+3}}Y^{\tilde{m}_{i+3}}Z^{\tilde{m}_{i+3}})} \leftrightarrow \frac{O_K[X^{\pm 1}, Y, Z]}{(\pi - X^{m_{i+3}}Y^{\tilde{m}_{i+3}}Z^{\tilde{m}_{i+3}})} = R,
\]
and this is the inclusion \( O_K[\sigma_0^\perp] \leftrightarrow O_K[\sigma_{L,i,i+1}^\perp] \) of \([K^2MS]\). Thus,
\[
X_{\sigma_{L,i+1}} = \text{Spec } R, \quad X_{\sigma_{L,i}} = \text{Spec } R[Y^{-1}], \quad X_{\sigma_0} = \text{Spec } R[Y^{-1}, Z^{-1}],
\]
and these are all flat \( O_K \)-schemes of relative dimension 2. Glueing the \( X_{\sigma_{L,i,i+1}} \) for varying \( L \) and \( i \) along their common opens gives \( X_{\Sigma} \).

4.4. **The closure of** \( f = 0 \) **in** \( X_{\Sigma} \). The model \( C_\Delta \) is covered by schematic closures of \( \{ f = 0 \} \subset \mathbb{G}^2_{m,K} \) in the various \( X_{\sigma_{L,i,i+1}} \). Explicitly, view \( f \) as a polynomial in \( x, y, \pi \) whose non-zero coefficients \( u_{ij} \) are in \( O_K^\times \),
\[
(4.2) \quad f = \sum_{i,j} u_{ij} \pi^{v(a_{ij})} x^i y^j, \quad u_{ij} = \frac{a_{ij}}{\pi^{v(a_{ij})}}.
\]

Let \( M = (m_{ij}) \in \text{SL}_3(\mathbb{Z}) \), for the moment arbitrary. Write \( M^{-1} = (\tilde{m}_{ij}) \), suppose \( \tilde{m}_{23} \geq 0, \tilde{m}_{33} > 0 \), and change variables \( x = X^{\tilde{m}_{11}}Y^{\tilde{m}_{21}}Z^{\tilde{m}_{31}}, y = \ldots, \pi = \ldots \) according to \( M^{-1} \). For some unique \( n_Y, n_Z \in \mathbb{Z} \), we have
\[
f = Y^{n_Y}Z^{n_Z}F_M(X, Y, Z), \quad F_M \in O_K[X^{\pm 1}, Y, Z], \quad Y \uparrow F_M, \quad Z \uparrow F_M.
\]
Thus $\mathcal{F}_M$ and $f$ have the same coefficients $u_{ij}$, just different monomials, and $\mathcal{F}_M = 0$ defines a complete intersection

$$\mathcal{F}_M(X, Y, Z) = \pi - X^{\tilde{m}_{13}}Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}} = 0 \quad (\ast)$$

in Spec $O_K[X^{\pm 1}, Y, Z]$. Now let $M = M_{L,i}$ from 4.3. This Spec is $X_{\sigma_{L,i+1}}$, and $(\ast)$ defines the open set $\mathcal{C}_{\Delta} \cap X_{\sigma_{L,i+1}} \subseteq \mathcal{C}_{\Delta}$. These cover all of $\mathcal{C}_{\Delta}$.

4.5. Special fibre. The special fibre $\pi = 0$ of $X_{\Sigma}$ meets $(\ast)$ in $Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}} = 0$. Its underlying reduced subscheme is $Z = 0$ in the case $(\dagger)$, and $YZ = 0$ otherwise. To find $(\ast) \cap \{Z = 0\}$, write $f = Y^{n_Z}Z^{n_Z}F_M$ as above. Noting that $Z = 0 \Rightarrow \pi = 0$, we obtain a $k$-scheme

$$(\ast) \cap \{Z = 0\} = \text{Spec } k[X^{\pm 1}, Y]/(F_M(X, Y, 0)).$$

The non-zero terms in $F(X, Y, 0) \in k[X, Y]$ come from monomials $\pi^{n(a_{ij})}x^iy^j$ in (4.2) that, when rewritten in $X, Y, Z$, have minimal exponent $(=n_Z)$ of $Z$. In other words, they minimise the following linear form $Z^3 \to Z$ on $\Delta_v$,

$$\phi: (\alpha, \beta, \gamma) \mapsto \text{ord}_Z(x^\alpha y^\beta z^\gamma) = \tilde{m}_{31}\alpha + \tilde{m}_{32}\beta + \tilde{m}_{33}\gamma.$$  

Geometrically, $\mathcal{P} = \ker \phi$ is the plane spanned by the two columns of $M$, and $\phi$ is minimised on some face $\lambda \subset \Delta_v$ parallel to $\mathcal{P}$. Let $\lambda$ be the projection of $\Delta$ onto $\Delta_{\nu}$; it is a $v$-vertex/edge/face. With an appropriate choice of the basis of the lattice in 3.4, 3.5, we find that

$$\mathcal{F}_0(X, Y) = \overline{f_0}.$$  

This, and an identical computation for $(\ast) \cap \{Y = 0\}$ to $M = M_{L,i}$ gives

$$\mathcal{F}_M(X, 0, 0) = \overline{f_L}(X)$$

$$\mathcal{F}_M(X, 0, Z) = \begin{cases} \overline{f_F}(X, Y) & i = 0 \\ \overline{f_L}(X) & i > 0 \end{cases} \quad (\ast\ast)$$

For a fixed $v$-face $F$ and varying $L \subset \partial F$, the subsets $\overline{f_F} = 0$ of $\mathbb{C}_{m,k} \times \mathbb{A}_k^1$ are precisely the charts that define the completion $\tilde{X}_F$. Glueing the pieces together, we find that $\mathcal{C}_k$ is the union of $\tilde{X}_F$ over all $v$-faces $F$, and $X_L \times \Gamma_L$ over $v$-edges $L$, as claimed in Theorem 3.14 (and correct multiplicities). Since $X_L$ are finite and so proper over $k$, and the $\tilde{X}_F$ and $\Gamma_L$ are proper as well, $\mathcal{C}_k/k$ is proper. By [Liu, 3/3.28] (or EGA IV.15.7.10), $\mathcal{C}_{\Delta}/O_K$ is proper.

4.6. Regularity. Let $P \in \mathcal{C}_k(\overline{k})$ be a point. It corresponds to an ideal

$$m_P = (\pi, X - x_0, Y - y_0, Z - z_0) \subset O_K[X^{\pm 1}, Y, Z]/(\ast).$$

To prove the claims in 3.14 when $P$ is regular, we compute $\dim m_P/m_P^2$. The equation $\pi = X^{\tilde{m}_{13}}Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}}$ of $(\ast)$ eliminates $\pi$ (recall that $\tilde{m}_{23}, \tilde{m}_{33} \geq 0$), and the assumptions of 3.14 on $\overline{f_L}, \overline{f_F}$ eliminate either $X$ or $Y$ from $m_P/m_P^2$; so $\mathcal{C}$ is regular at $P$ under those assumptions.

Finally, suppose $C$ is $\Delta_v$-regular. Then $\mathcal{C}$ is geometrically regular at every point of $\mathcal{C}_k$, as above. As geometric regularity is an open condition,
this implies that the generic fibre, which is \( \tilde{C} \) by construction, is smooth. So \( C \) is a regular model of \( \tilde{C} \) in this case.

This proves Theorems 3.14 and 3.13. \( \square \)

5. Minimal model

From now on, assume that \( C \) has genus \( \geq 1 \).

Here, in Theorem 5.7, we describe the \textit{minimal regular normal crossings} (m.r.n.c.) model, a canonical regular model obtained from any normal crossings model by ‘removing unnecessary stuff’, that is repeatedly blowing down genus 0 components of self-intersection \(-1\). In our case, ‘unnecessary stuff’ comes from \( v \)-faces that avoid \( \Delta(Z) \). We will see later that \( \Delta(Z) \) also plays a crucial role in the description of \( \acute{e} \)tale cohomology and differentials.

\textbf{Definition 5.1.} Let \( F \) be a \( v \)-face of \( \Delta \). We say that

- \( F \) is \textit{removable} if \( F \cap \Delta(Z) = \emptyset \).
- \( F \) is \textit{contractible} if \( F \cap \Delta(Z) \) has a unique point \( P \), all \( v \)-edges \( L \subset \partial F \) with \( P \notin L(Z) \) are outer, and \( v(P) \in Z \).
- \( F \) is \textit{principal} in all other cases.

We will refer to chains of \( \mathbb{P}^1 \)'s from Theorem 3.13(2) as \textit{inner chains} or \textit{outer chains}, depending on whether they come from an inner or outer \( v \)-edge.

\textbf{Proposition 5.2.} Let \( C/K \) be a \( \Delta_v \)-regular curve, and \( X_F \) a component of \( C_k \) from a \( v \)-face \( F \), as in 3.13. The following conditions are equivalent:

\begin{enumerate}[(a)]  
  \item \( X_F \) has genus 0 and meets the rest of the special fibre either at one point or at two points one of which lies on an outer chain.
  \item \( F \) is \textit{GL}_2(\mathbb{Z})\text{-equivalent to an} \( n \times 1 \) right-angled triangle whose short sides are either outer \( v \)-edges (as in Table 2, top left) or one outer \( v \)-edge and one inner, of length 1, bounding a removable face.
  \item \( F \) is removable.
\end{enumerate}

Similarly, the following conditions are equivalent:

\begin{enumerate}[(a)]  
  \item \( X_F \) has genus 0 and meets the rest of the special fibre at two points, that are not outer chains.
  \item \( F \) is \textit{GL}_2(\mathbb{Z})\text{-equivalent to a} \text{ polygon in Table 2, bottom row (an} \( n \times 1 \) \text{ or} \( 2 \times 2 \) right-angled triangle, a chopped version of it, or a right-angled trapezoid with parallel sides} \( 1,n \), \text{ with inner} \( v \)-edges as shaded and of denominator 1.
  \item \( F \) is contractible.
\end{enumerate}

\textit{Proof.} \( ^{13} \) Denote the four shapes in (b) by \( S_1, S_2, S_3, S_4 \) (cf. Table 2). The edges of \( S_3 \) and \( S_4 \) span \( \mathbb{Z}^2 \), so such faces have \( \delta_F = 1 \); similarly \( \delta_F|n \) for \( S_1 \) and \( \delta_F|2 \) for \( S_2 \).

\( (b_i) \Rightarrow (c_i) \) Clear.

\( ^{13} \) A direct classification-free proof of \( (a_i) \Leftrightarrow (c_i) \) would be very much appreciated
MODELS OF CURVES OVER DVRS

Removable $v$-face $F$: \[ \begin{array}{c}
\includegraphics{removable_face}
\end{array} \]

\[ \Leftrightarrow \begin{array}{c}
\includegraphics{removable_contractible}
\end{array} \]

Contractible $v$-face $F$: \[ \begin{array}{c}
\includegraphics{contractible_face}
\end{array} \]

\[ \Leftrightarrow \begin{array}{c}
\includegraphics{contractible_contractible}
\end{array} \]

Table 2. Removable and contractible faces. Shading indicates the direction towards other parts of $\Delta$ (inner $v$-edges)

(b$_1$)$\Rightarrow$(a$_i$) By Theorem 3.13, outer $v$-edges contribute nothing, and inner ones give one chain ($S_1, S_4$) or two ($S_2, S_3$).

(a$_i$)$\Rightarrow$(b$_i$) Depending on the case, we have

\[(5.3)\quad 1 \text{ or } 2 = \# \text{ chains from } X_F = \sum_L |X_L(\bar{k})| = \sum_L |L(\mathbb{Z})_{vL}|,\]

summing over inner $v$-edges $L$, and outer $v$-edges $L \subset \partial F$ with $\delta_L < \delta_F$ (see Theorem 3.13).

Write $\tilde{F}$ for $F$ restricted to the sublattice $\Lambda$ spanned by $\bar{F}(\mathbb{Z})$. The genus of $X_F$ is $|F(\mathbb{Z})_{vL}|$, assumed to be 0. Therefore, by the Arkinstall–Rabinowicz classification of convex polygons with no interior points (Example 2.4), $\tilde{F}$ is one of $S_1, \ldots, S_4$. As $\mathbb{Z}^2/\Lambda$ is cyclic of order $\delta_F$,

\[
\begin{align*}
\delta_F = 1 & \Rightarrow \text{ all } \delta_L = 1 \\
\delta_F > 1 & \Rightarrow \delta_L < \delta_F \text{ for all } L \text{ except possibly those parallel to one fixed line in } \mathbb{Z}^2.
\end{align*}
\]

Suppose LHS of (5.3) is 1. Then $F$ has one inner face $L$ with $L(\mathbb{Z})_{vL} = \emptyset$, and all the others are outer with $\delta_L = \delta_F$. This eliminates $\tilde{F} = S_2$. In the $S_3$ and $S_4$-case, $\delta_F = 1$ by ($\ast$), but then one inner face with $L(\mathbb{Z})_{vL} = \emptyset$ is impossible by convexity, as there is no space behind $L$ to contain interior integral points (while genus$(C) > 0$). So $\tilde{F}$ is $S_1$, the inner face gives the unique chain of $\mathbb{P}^1$s, and two outer ones have $\delta_L = \delta_F$ which forces $\delta_F = 1$ by ($\ast$). Thus, $F$ is $S_1$ (and, moreover, $n = 1$).

If the LHS of (5.3) is 2, then the same (but more tedious) considerations give the list of possible cases.

(c$_i$)$\Rightarrow$(b$_i$) Similar: as $F$ has no interior integer points, it is one of the $S_i$ by the Arkinstall–Rabinowicz classification; the rest is by inspection. $\square$

Remark 5.4. Let $C/K$ be $\Delta_v$-regular curve of genus $g > 0$.

- If $g = 1$ with $|\Delta(\mathbb{Z})_{PQ}| = 1$, there are no principal $v$-faces.
- In all other cases, e.g. if $g > 1$, there is always a principal $v$-face.

Indeed, in the first case, it follows from Theorem 3.13, after contracting the chains, that the special fibre is a reduced $n$-gon of $\mathbb{P}^1$s (Kodaira type $I_n$). In the second case, if $\Delta(\mathbb{Z})^F \neq \emptyset$ or if $g = 1$, $\Delta(\mathbb{Z})_{PQ}^F = \emptyset$, then there is a principal face by definition. Otherwise for distinct $P, Q \in \Delta(\mathbb{Z})$, every point in the interior of the line segment $PQ$ belongs to principal $v$-face(s).
Proposition 5.5. Let $C$ be a $\Delta_v$-regular curve of genus $g > 0$. If $v(P) \in \mathbb{Z}$ for every $P \in \Delta(\mathbb{Z})$, then exactly one of the following holds:

- $g > 1$ and every principal $v$-face $F$ has $\delta_F = 1$.
- $g = 1$ and $C$ is a semistable elliptic curve (type $1_n$ with $n \geq 0$).
- $g = 1$, $C(K) = \emptyset$, and up to an isomorphism of the form $x \mapsto x^a y^b \pi^c$, $y \mapsto x^h y^i \pi^j$, $\Delta_v$ has one of the three exceptional shapes

$$
\begin{align*}
&1 \quad 1/2 \\
&1/2 \\
&2 -\sqrt{2} -1 -\sqrt{2} = 0
\end{align*}
$$

(5.6)

The special fibre $C_k$ consists of a unique genus 1 component $X_F$, of multiplicity 2,2,3, respectively.

Proof. Case 2. $g = 1$, $|\Delta(\mathbb{Z})|_Z^f = 1$. See Remark 5.4.

Case 3. $g = 1$, $|\Delta(\mathbb{Z})|_Z^f = 1$. There is a unique principal $v$-face $F$. Up to $\text{GL}_2(\mathbb{Z})$-equivalence, there are 15 possible convex lattice polygons with one interior point [Rab, Thm. 5]. Of these, in 12 cases the coordinates of the vertices and the interior point generate $\mathbb{Z}^2$ as an affine lattice, so $\delta_F = 1$. The other three shapes (Cases 1,4,9 of loc.cit.) are as in (5.6), with lattice index 2,2,3. Assume we are in one of these, so $\delta_F = 2,2,3$, respectively. After rescaling $x$, $y$ by powers of $\pi$, the valuations can be made as in (5.6).

Case 1. $g > 2$. Suppose $F$ is a principal $v$-face with $\delta_F > 1$. If $|F(\mathbb{Z})| < 2$, from 2.4 and the proof of Case 3, the only possibilities for $F$ are

(a) an $n \times 1$ right triangle; here $|F(\mathbb{Z})| = 0$, $1 < \delta_F | n$,
(b) a $2 \times 2$ right triangle; here $|F(\mathbb{Z})| = 0$, $\delta_F = 2$,
(c) the three cases in (5.6); here $|F(\mathbb{Z})| = 1$, $\delta_F = 2,2,3$ respectively.

In (b) and (c), as $F$ is not the whole of $\Delta$, at least one $v$-edge $L \subset \partial F$ must be inner with $\delta_L > 1$, contradicting $\Delta(\mathbb{Z}) = \Delta(\mathbb{Z})_Z$. Similarly, in (a), as $F$ is not removable or contractible, one of the corners of the $v$-edge $L \subset \partial F$ of length $n$ must in $\Delta(\mathbb{Z})$, and this again makes $L$ inner with $\delta_L > 1$.

Finally, if $|F(\mathbb{Z})| \geq 2$, subdivide $F$ into convex lattice polygons one of which has exactly one interior integer point, and apply the same argument to it to show that it is in (c), to get the same contradiction.

\[\square\]

Theorem 5.7. Let $C : f = 0$ be a $\Delta_v$-regular curve, with $f = \sum a_{ij} x^i y^j$. Its m.r.n.c. model $C_{k}^\text{min}/O_K$ has the following special fibre $C_k^{\text{min}}/k$.

1. Let $f_{nr} = \sum' a_{ij} x^i y^j$, the sum taken over those $(i, j)$ that lie in the closure of some non-removable $v$-face of $\Delta$. Then $f_{nr}$ is $\Delta_v$-regular.
2. Let $C$ be the model of $f_{nr} = 0$ of §4 and Theorem 3.13, and $C_k$ its special fibre. The principal $v$-faces of $\Delta$ are in 1-1 correspondence with principal components of $C_k$ in the sense of Xiao [Xi] (positive genus or meeting the rest of $C_k$ in $\geq 3$ points).
MODELS OF CURVES OVER DVRTS

(3) Starting with \( C_k \), repeatedly contract \( \mathbb{P}^1 \)'s of self-intersection \(-1\) in the chains between two principal v-faces\(^{14}\). This gives \( C_k^{\min} / k \).

Proof. Follows from Proposition 5.2 and Remark 3.15. \( \square \)

6. REDUCTION AND INERTIA

Throughout this section, \( C \) is a \( \Delta_v \)-regular curve, and \( C_k \) the special fibre of the r.n.c. model \( C_\Delta \) of \( \S 4 \) and Theorem 3.13. Write \( \Gamma(C_k) \) for its dual graph: it has geometric components as vertices, with an edge for every double point. Viewing it as a topological space, we compute its homology:

**Theorem 6.1** (Homology of the dual graph). Suppose \( C \) is \( \Delta_v \)-regular. Then \( H_1(\Gamma(C_k), \mathbb{C}) \) has dimension \( |\Delta(Z)_{L,\delta}| \), and

\[
H_1(\Gamma(C_k), \mathbb{C}) \cong \ker \left( \bigoplus_{\text{v-edges } L} \mathbb{Z}[X_L(\bar{k})] \xrightarrow{\phi} \mathbb{Z}[v\text{-faces}] \right)
\]
as \( G_k \)-modules. Here \( G_k \) acts naturally on \( \mathbb{Z}[X_L(\bar{k})] \), trivially on \( \mathbb{Z}[v\text{-faces}] \), and for \( L \subset \partial F_1, \partial F_2 \) and every \( r \in X_L(\bar{k}) \), we set \( \phi(r) = F_1 - F_2 \).

Proof. For every connected directed graph \( \Gamma \), there is an \( \text{Aut } \Gamma \)-equivariant isomorphism \( H_1(\Gamma, \mathbb{Z}) \cong \ker \phi \), where

\[
\phi: \mathbb{Z}^\text{edges} \to \mathbb{Z}^\text{vertices}, \quad \phi(v_1 \to v_2) = v_1 - v_2
\]
is as above. Now take \( \Gamma = \Gamma(C_k) \), obtained from Theorem 3.13; note that it is connected. The choice of the ordering \( F_1, F_2 \) of v-faces whose boundary is a given v-edge \( L \) (already implicit in the definition of \( \phi \) in the statement) makes \( \Gamma \) directed. Now modify \( \Gamma \), by viewing every chain of \( \mathbb{P}^1 \)'s as one component (i.e. edge), and contracting all outer chains. This produces a new graph \( \Gamma' \) with \( H_1(\Gamma, \mathbb{Z}) = H_1(\Gamma', \mathbb{Z}) \).

We find a planar realisation of \( \Gamma' \) with 2-dimensional faces in 1-1 correspondence with elements of \( |\Delta(Z)_{L,\delta}| \); these give a \( \mathbb{Z} \)-basis of homology. The statement on the \( G_k \)-action follows from Theorem 3.13. \( \square \)

Next, recall that \( \text{Pic}^0 C_k \) is an algebraic group over \( k \), an extension of an abelian variety (the abelian part) by a linear group; the latter is an extension of a unipotent group by a torus (the toric part) [BLR, §9.2].

\(^{14}\)if there are no contractible faces, and all chains in 3.13(2) are chosen to be minimal (cf. 3.15), then \( C_k \) is already minimal and there is nothing to contract
Corollary 6.2 (Special fibre). Suppose $C/K$ is $\Delta_v$-regular, and $k$ is perfect.

1. The abelian part of $\text{Pic}^0 C_k$ is isomorphic to $\bigoplus_{v \text{-edges} \subset \Delta} \text{Pic}^0 \tilde{X}_F$, and has dimension $|\Delta(Z)_{\mathbb{F}_\ell}|$.
2. The toric part of $\text{Pic}^0 C_k$ has dimension $|\Delta(Z)_{\mathbb{F}_\ell}|$ and character group $H_1(\Gamma(C_k), \mathbb{Z})$ given by Theorem 6.1.
3. Let $l \neq \text{char } k$, and denote $H^1(-) = H^1(-, \mathbb{Q}_l)$. We have

$$H^1(C_k) \otimes \mathbb{Q}_l \cong \bigoplus_{v \text{-edges} \subset \Delta} (H^1(\tilde{X}_F) \otimes 1) \oplus \bigoplus_{v \text{-edges} \subset \Delta} H^0(\tilde{X}_L).$$

Proof. (1), (2) Combine Theorems 3.13, 6.1 with [BLR, 9.2.5/8]. (3) The image of $\phi$ of Theorem 6.1 is the degree 0 part of $\mathbb{Z}[v \text{-faces}]$, as $\Gamma = \Gamma(C_k)$ is connected. Tensoring with $\mathbb{Q}_l$, we find that the toric part is

$$H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{Q}_l \cong H_1(\Gamma, \mathbb{Z}) \otimes \mathbb{Q}_l \cong \bigoplus_{L} H^0(\tilde{X}_L) \otimes \mathbb{Z}[v \text{-faces}] \otimes 1.$$

Moving $\mathbb{Z}[v \text{-faces}]$ into the abelian part gives the claim. \hfill \Box

Remark 6.3. When $K$ is local and $C/K$ is semistable, this also determines

- the conductor exponent $f_{C/K} = |\Delta(Z)_{\mathbb{F}_\ell}|$ (the toric dimension), and
- the local root number $w_{C/K} = \text{multiplicity of 1 in } H_1(\Gamma, \mathbb{C}), \text{say e.g.}$ [D2r, Prop. 3.23].

Theorem 6.4 (Inertia action). Suppose $k$ is perfect, $C/K$ is a $\Delta_v$-regular curve, and $l \neq p$. For $P \in \Delta(Z)_{\mathbb{Z}_p}$ define a tame character

$$\chi_P : I_K \longrightarrow \{ \text{roots of unity} \}, \quad \sigma \longmapsto \sigma(\pi^{v(P)})/\pi^{v(P)}.$$

Let $V^\text{tame}_{\text{ab}}$, $V^\text{tame}_{\text{toric}}$ be the unique continuous $l$-adic representations of $I_K$ that decompose over $\mathbb{Q}_l$ as

$$V^\text{tame}_{\text{ab}} \cong_{\mathbb{Q}_l} \bigoplus_{P \in \Delta(Z)_{\mathbb{F}_p}} (\chi_P \oplus \chi_P^{-1}), \quad V^\text{tame}_{\text{toric}} \cong_{\mathbb{Q}_l} \bigoplus_{P \in \Delta(Z)_{\mathbb{F}_p}} \chi_P.$$

Then there are isomorphisms of $I_K$-modules,

$$H^1_{\text{et}}(C_K, \mathbb{Q}_l)^{\text{wild}} \cong (V^\text{tame}_{\text{ab}} \cdot \text{Jac } C)^{\text{wild}} \cong V^\text{tame}_{\text{ab}} \oplus V^\text{toric}_{\text{tame}} \otimes \text{Sp}_2.$$

In particular, $\text{Jac } C$ is wildly ramified $\iff \Delta(Z)_{\mathbb{Z}_p} \subsetneq \Delta(Z)$.

Proof. Write $J = \text{Jac } C$. The statement only concerns inertia, so we may assume $k = \bar{k}$ and that $K$ is complete. The first isomorphism is standard.

Now $V^\text{tame}_{\text{ab}}$, $V^\text{toric}_{\text{tame}}$ are $I_K/I^{\text{wild}}$-representations that factor through some cyclic group $C_d$, $p \nmid d$. They are rational (realisable over $\mathbb{Q}$) by Lemma A.1 (2), and uniquely determined by the dimensions of their invariants under subgroups: for $H < C_d$ of index $e$,

$$\dim(V^\text{tame}_{\text{tame}})^H = 2|\Delta(Z)_{\mathbb{F}_p}|, \quad \dim(V^\text{toric}_{\text{tame}})^H = |\Delta(Z)_{\mathbb{F}_p}|.$$

On the other side, there is a canonical filtration of $I_K$-modules [SGA7, IX]

$$(V^J)^{\text{toric}} \subseteq (V^J)^{I_K} \subseteq V^J$$.
that gives a decomposition

\[ V_l J \cong (V_l J)^{ab} \oplus (V_l J)^{toric} \otimes \text{Sp}_2. \]

Decompose the wild inertia invariants accordingly,

\[ (V_l J)^{l, \text{wild}} \cong (V_l J)^{l, \text{tame}} \oplus (V_l J)^{l, \text{toric}} \otimes \text{Sp}_2. \]

The \( I_K/I_{\text{wild}} \)-representations \( (V_l J)^{l, \text{tame}} \), \( (V_l J)^{l, \text{toric}} \) also factor through finite (cyclic) groups by the Grothendieck monodromy theorem. They are rational, as their characters are independent of \( l \), and a representation of cyclic group with a rational character is rational. We need to show that they are \( \cong V_l^{ab} \) and \( \cong V_l^{toric} \). First note that the inertia invariants match:

\[ \dim(V_l J)^{l, K} = \dim V_l \text{Pic}^0 C_k = 2|\Delta(Z)^F|_Z + |\Delta(Z)^F|_Z, \]

with the first equality by Theorem B.1, and the second by Corollary 6.2 on the unipotent part of \( \text{Pic}^0 C_k \) has no \( l \)-power torsion. By Deligne’s argument on Frobenius weights [SGA7, I, §6], “the abelian and toric parts do not mix”, so \( (V_l J)^{l, \text{tame}} \) reduces to the abelian part, and \( (V_l J)^{l, \text{toric}} \) to the toric part of \( \text{Pic}^0 C_k \). In other words,

\[ \dim(V_l J)^{l, \text{tame}} = 2|\Delta(Z)^F|_Z, \quad \dim(V_l J)^{l, \text{toric}} = |\Delta(Z)^F|_Z. \]

The same argument applies over the (unique) tame extension of \( K \) of degree \( e \) for every \( e \geq 1 \), because \( C \) stays \( \Delta_v \)-regular in finite tame extensions by Remark 3.11, and the function \( v \) on \( \Delta \) gets multiplied by \( e \). This proves the claimed isomorphism. Finally,

\[ \dim V_l J = 2 \dim J = 2g(C) = 2|\Delta(Z)| = 2|\Delta(Z)^F|_Z + 2|\Delta(Z)^F|_Z + 2|\Delta(Z)^F|_Z. \]

Comparing the dimensions with (6.5) for \( H = \{1\} \), we see that the last term is non-zero if and only if \( V_l J \neq (V_l J)^{l, \text{wild}} \). \( \square \)

Example 6.6. The curve from Example 3.3 has wild reduction when \( \text{char } k = 2 \) or \( 3 \), and tame otherwise. In the tame case, \( C \) becomes semistable over any extension \( K'/K \) of ramification degree 6. Let \( \chi : I_K/I_{\text{wild}} \to \overline{\mathbb{Q}}_l^\times \) be an order 6 character. By Theorem 6.4, as an \( I_K \)-module (\( \text{char } k \neq 2, 3 \)),

\[ H^1_{\text{ét}}(C_K, \mathbb{Q}_l) \cong (\chi \oplus \chi^{-1} \oplus \chi^2 \oplus \chi^{-2}) \oplus (\chi^2 \oplus \chi^{-2}) \otimes \text{Sp}_2. \]

Theorem 6.7 (Reduction). Suppose \( C/K \) is \( \Delta_v \)-regular, of genus \( \geq 1 \). Then

1. \( C \) is \( \Delta_v \)-regular over every finite tame extension \( K'/K \).
2. \( C \) has good reduction \( \iff \Delta(Z) = F(Z) \) for some \( v \)-face \( F \) with \( \delta_F = 1 \).
3. \( C \) is semistable \( \iff \text{every principal } v \)-face \( F \) has \( \delta_F = 1 \).
4. \( C \) is tame \( \iff \text{every principal } v \)-face \( F \) has \( p \nmid \delta_F \) (\( p = \text{char } k \)).

In this case, \( C \) is \( \Delta_v \)-regular over every finite extension \( K'/K \).

Suppose \( k \) is perfect, and let \( J \) be the Jacobian of \( C \). Then

5. \( J \) has good reduction \( \iff \Delta(Z) = \Delta(Z)^F \).
6. \( J \) is semistable \( \iff \Delta(Z) = \Delta(Z)_Z \).
(7) \( J \) is tame \( \iff \Delta(Z) = \Delta(Z)_{Z_p} \). In this case, 
\( J \) has potentially good reduction \( \iff \Delta(Z)^L = \emptyset \), and 
\( J \) has potentially totally toric reduction \( \iff \Delta(Z)^F = \emptyset \).

**Proof.** Let \( C/O_K \) be the m.r.n.c. model of \( C/K \) from Theorems 3.13, 5.7, with special fibre \( \mathcal{C}_k \).

(1), (4, second claim) This is proved in Remark 3.11.

(3, \( \Leftarrow \)) For every principal \( v \)-face \( F \), we have \( \delta_L = 1 \) for all \( L \subset \partial F \) (inner or outer) since \( \delta_F = 1 \). It follows from 3.13, 5.7 and 3.15 that \( C \) is a semistable model of \( C \).

(3, \( \Rightarrow \)) If \( \delta_F > 1 \) for some principal \( v \)-face, then \( C \) is not semistable as it has a multiple fibre. By the uniqueness of the m.r.n.c. model, \( C \) is not semistable.

(2, \( \Leftarrow \)) All faces but \( F \) are removable and \( \mathcal{C}_k \) has one smooth component, of multiplicity 1. So \( C \) has good reduction.

(2, \( \Rightarrow \)) Conversely, the m.r.n.c. model has one component, and so (i) there is only one principal face \( F \), (ii) \( \delta_F = 1 \) as \( C \) is semistable, and (iii) the dual graph \( \Gamma \) of \( \mathcal{C}_k \) is a tree by 6.1, so \( \Delta(Z)^L = \emptyset \).

(6) For an abelian variety \( A/K \) semistable reduction is equivalent to inertia \( I_K \) acting unipotently on \( V_lA \) [SGA7, IX, 3.5/3.8], [BLR, 7.4.6], so the claim follows from 6.4.

(5) Semistable reduction is good if and only if the toric part of \( \text{Pic}^0 \mathcal{C}_k \) is trivial, so this follows from 6.2.

(4, first claim), (7). By (1), \( C/K \) remains \( \Delta_v \)-regular in tame extensions \( K'/K \), and the function \( v : \Delta \to \mathbb{R} \) gets multiplied by \( e_{K'/K} \). So (3), (6) and 6.2(1,2) give the claims. \( \square \)

**Remark 6.8.** Let \( C \) be a \( \Delta_v \)-regular curve, and suppose \( k \) is perfect. Then 6.7 and 5.5 show that the following conditions are equivalent:

(1a) Jac \( C \) is semistable (equivalently, \( I_K \) acts unipotently on \( \text{Jac} C \)),

(1b) \( C \) is either semistable of genus \( \geq 1 \), or a genus 1 curve with \( C(K) \neq \emptyset \) and special fibre consisting of one non-reduced genus 1 component, and the following are equivalent as well:

(2a) \( C \) has tame reduction,

(2b) Principal components of the special fibre of the some r.n.c. model have multiplicities prime to \( p \).

Thus, 6.4 extends theorems of Saito [Sai, 3.8] and Halle [Hal, 7.1] for \( \Delta_v \)-regular curves, describing explicitly the wild inertia invariants.

7. Étale cohomology over local fields

Suppose \( K \) is complete and \( k \) is finite, so \( K \) is a local field. We write \( I_{\text{wild}} \triangleleft I_K \triangleleft G_K \) for the wild inertia and inertia. Recall that a choice of a Frobenius element \( \Phi \in G_K \) (acting as \( x \mapsto x^{[k]} \) on \( \bar{k} \)) identifies \( G_K = I_K \rtimes \hat{\mathbb{Z}} \), and the Weil group \( W_K = I_K \rtimes \mathbb{Z} \), with \( \hat{\mathbb{Z}}, \mathbb{Z} \) generated by \( \Phi \). Fix a prime \( l \neq \text{char} k \); for a scheme \( Y/F \) (\( F \in \{ k, K \} \)) write \( H^i(Y) = H^i_{\text{ét}}(Y_F, \mathbb{Q}_l) \).
We note that 6.2 gives the local polynomial \( \det(1 - \Phi^{-1}T \mid H^1(C)^I_K) \),
\begin{equation}
(7.1) \quad L(T) = \det(1 - \Phi^{-1}T \mid H^1(C_k, \mathbb{Z})) \prod_{v\text{-faces } F} \det(1 - \Phi^{-1}T \mid \text{Pic}^0 \bar{X}_F).
\end{equation}
The first factor has roots of absolute value 1. The second is a product of \( \zeta \)-functions of \( \bar{X}_F \) with trivial factors \( (1-T)(1-|k|T) \) removed, and has roots of absolute value \(|k|^{-1/2}\).

**Theorem 7.2** (Tame decomposition). Let \( C/K \) be a \( \Delta_v \)-regular curve. For \( v\text{-edges/faces } \lambda \) of \( \Delta \), define \( O_K \)-schemes \( X^\text{tame}_\lambda : f|_{\lambda(Z)_p} = 0 \). Then
\[ H^1(C)^{I_{\text{wild}}} \oplus 1 \cong \bigoplus_{v\text{-faces } F} (H^1(X^\text{tame}_F) \oplus 1) \oplus \bigoplus_{L\text{ inner } v\text{-edges}} (H^0(X^\text{tame}_L) \otimes \text{Sp}_2). \]

**Proof.** Choose an embedding \( \bar{Q}_l \hookrightarrow \mathbb{C} \) and view LHS and RHS as complex Weil-Deligne representations of \( W_K \). From (7.1), 6.2 (3) and (B.2), we find that \( \text{LHS}^I \cong \text{RHS}^I \) as \( G_K \)-modules. Generally, if \( K'/K \) is a finite tame extension, then \( \text{LHS}^{I_{K'}} \cong \text{RHS}^{I_{K'}} \) as \( G_{K'/I_{K'}} \)-modules, using the same argument as in Theorem 6.4 and that \( \Delta_v \)-regularity is stable in tame extensions. As LHS and RHS are tame Frobenius-semisimple Weil representations that have the same local polynomial in all finite tame extensions of \( K \), they are isomorphic, by [D2w, Thm 1.1]. (This is shown in [D2w] without the two words ‘tame’, but the tame case is an easy consequence.) Finally, LHS and RHS are Weil-Deligne representations of the form \( \text{Weil} \oplus \text{Weil} \otimes \text{Sp}_2 \), so their semisimplifications determine the representations themselves, since Frobenius has different weights on the two constituents. \( \square \)

**Theorem 7.3** (One tame face). Suppose \( C : \sum a_{ij} x^i y^j = 0 \) is \( \Delta_v \)-regular, \( K \) is local with \( k = \mathbb{F}_q \), \( \Delta_v \) has only one \( v \)-face \( F \), and \( p \nmid \delta_F = e \). Then

(1) \( C \) has good reduction over \( K(\sqrt[3]{\pi}) \).

(2) The action \( G_K \subset H^1(C) \) factors through \( G = \text{Gal}(K^{nr}(\sqrt[3]{\pi})/K) \). It is faithful on \( G \), and is uniquely determined by (3) and (4) below.

For \( u \in \bar{F}_q^\times \) define a complete smooth curve \( C_u/\mathbb{F}_q(u) \) by an affine equation
\[ C_u : \sum_{i,j} a_{ij} u_{v_F(i,j)} x^i y^j = 0, \quad \text{where} \quad a_p = \begin{cases} 0 & \text{if } v(P) \notin \mathbb{Z} \\ \bar{a}_p & \text{mod } \pi & \text{if } v(P) \in \mathbb{Z}. \end{cases} \]

Write \( I \cong \frac{\mathbb{Z}}{e \mathbb{Z}} < G \) for the inertia group, and \( \Phi \in G \) for the Frobenius element of \( K(\sqrt[3]{\pi}) \).

(3) For every 1-dimensional character \( \chi \) of \( I \) of order \( n \), the multiplicity of \( \chi \) in \( H^1(C) \) viewed as an \( I \)-module is
\[ \frac{1}{\phi(n)} \left| \{ P \in \Delta(\mathbb{Z}) \mid v(P) \text{ has denominator } n \} \right|. \]

(4) Every conjugacy class of the Weil group \( W = I \rtimes \mathbb{Z} < G \) which is not in \( I \) contains either an element \( \sigma \Phi^d \) or \( (\sigma \Phi^d)^{-1} \) with \( d > 0, \sigma \in I \).
and $\frac{\sigma(\sqrt[p]{\pi})}{\sqrt[p]{\pi}}$ reducing to $u^{\frac{d-1}{3}}$ for some $u \in \mathbb{F}_q^\times$. We have
\[
\text{Tr}\left(\left(\sigma \Phi^d\right)^{-1} \mid H^1(C)\right) = q^d + 1 - |\tilde{C}_d(\mathbb{F}_q^\times)|.
\]

**Proof.** (1) Recall from Remark 3.11(3) that $C$ stays $\Delta_v$-regular in finite tame extensions of $K$. This applies to $K' = K(\sqrt[p]{\pi})$, where $\delta_F$ becomes 1, and then $C/K'$ has good reduction by Theorem 3.13.

(2) Theorem 3.13 also shows that the reduction is bad over $K(\sqrt[p]{\pi})$ for $d < e$. By the Néron-Ogg-Shafarevich criterion for $\text{Jac} C$ (see [ST, §2]), $G_K$ acts on $H^1(C)$ through $G$, and faithfully on $I$; the determinant of this action is a power of the cyclotomic character, so it is faithful on all of $G$. Finally, the traces of $g \circ H^1(C)$ for $g \in G$ determine $H^1(C)$ (as it is semisimple), so it is enough to compute them for $g \in G \setminus I$ and to describe $H^1(C)|_I$.

(3) This is a special case of Theorem 6.4 and the fact that $H^1(C)|_I$ is a rational representation.

(4) Let $\Psi \in W \setminus I$, and let $U = \langle \Psi \rangle < W$ be the subgroup it generates. Then $\Psi$ has infinite order, $U \cap I = \{1\}$, and the image of $U$ in $W/I \cong \mathbb{Z}$ is of some finite index $d$. The closure of $U$ in $G_K$ cuts out a finite extension $K''/K$, of ramification degree $e$, residue degree $d$, and $\Psi$ is a Frobenius element of $K''$ or its inverse. Inverting $\Psi$ if necessary, suppose from now on that it is the former. So $\Psi = \sigma \Phi^d$, for some $\sigma \in I$.

Let $K \subset K' \subset K''$ be the degree $d$ unramified extension of $K$. Pick uniformisers $\pi''$ of $K''$ and $\pi'$ of $K'$ with $(\pi'')^e = \pi'$. Then $(\pi'')^e = \tilde{u} \pi$ with $\tilde{u} \in K'$ a unit, whose reduction we call $u \in \mathbb{F}_q^\times$. Thus, $K'' = K(\zeta_{q^e}, \sqrt[d]{u\pi})$.

Now, $\Psi$ fixes $\sqrt[d]{\pi}$, and sends $\sqrt[d]{u}$ to $(\sqrt[d]{u})^{q^d}$ modulo higher order terms, whence $\frac{\Psi(\sqrt[p]{\pi})}{\sqrt[p]{\pi}}$ reduces to $u^{\frac{d-1}{3}}$. Therefore $\sigma = \Psi \Phi^{-d}$ acts on $\sqrt[p]{\pi}$ in the same way (as $\Phi$ fixes it). This proves the first assertion.

Finally, as in (1), $C/K''$ has good reduction and the reduced curve is exactly $\tilde{C}_u/\mathbb{F}_q$. As $\Psi \circ H^1(C)$ and $\text{Frob} \circ H^1(\tilde{C}_u)$ act in the same way, the Lefschetz trace formula gives the second claim in (4).

**Example 7.4.** Consider the two curves
\[
\begin{align*}
y^2 &= x^3 + 7^2 \quad \text{over } K = \mathbb{Q}_7, \quad \pi = 7 \\
y^2 &= x^3 + 5^2 \quad \text{over } K = \mathbb{Q}_5, \quad \pi = 5.
\end{align*}
\]
Both are $\Delta_v$-regular, with one $v$-face $F$ that has $e = \delta_F = 3$.

Tables 3 and 4 list, for various $u$ and $d$ the reduced curve $\tilde{C}_u/\mathbb{F}_q$, its number of points and the conclusion of Theorem 7.3 (4) in that case. These determine (with an overkill) the action of $G = \frac{\mathbb{Z}}{3\mathbb{Z}} \rtimes \hat{\mathbb{Z}}$ on $H^1(C)$, shown at the bottom of the tables. In the first case, $I = \frac{\mathbb{Z}}{6\mathbb{Z}}$ commutes with $\hat{\mathbb{Z}}$ and the action is diagonal; here $\tau \in I$ is chosen to be the generator that multiplies $\sqrt[7]{7}$ by a 3rd root of unity congruent to 2 mod 7. In the second case, $\hat{\mathbb{Z}}$ acts on $I$ by inversion and $G$ is non-abelian; here $\tau \in I$ is any generator (the two are conjugate in $G$). In both tables, $\Phi$ is the Frobenius element of $K(\sqrt[p]{\pi})$, as in the theorem.
Table 3. \( G = \langle \tau, \Phi \rangle = C_3 \times \hat{\mathbb{Z}} \subset H^1(C) \) for \( C/\mathbb{Q}_7 : y^2 = x^3 + 7^2 \)

| \( u \) | \( d \) | \( C_u \) | \( |C_u(\mathbb{F}_y)| \) | \( \text{Tr} \) |
|---|---|---|---|---|
| ±1 | 1 | \( y^2 = x^3 + 1 \) | 12 | \( \text{Tr}(\Phi^{-1}) = 7 + 1 - 12 = -4 \) |
| 2 | | 48 | \( \text{Tr}(\Phi^{-2}) = 49 + 1 - 48 = 2 \) |
| ±2 | 1 | \( y^2 = x^3 + 4 \) | 3 | \( \text{Tr}(\Phi^{-1}\tau) = 7 + 1 - 3 = 5 \) |
| 2 | | 39 | \( \text{Tr}(\Phi^{-2}\tau) = 49 + 1 - 39 = 11 \) |
| ±3 | 1 | \( y^2 = x^3 + 2 \) | 9 | \( \text{Tr}(\Phi^{-1}\tau^2) = 7 + 1 - 9 = -1 \) |
| 2 | | 63 | \( \text{Tr}(\Phi^{-2}\tau^2) = 49 + 1 - 63 = -13 \) |

\( \Phi^{-1} : \begin{pmatrix} -2+\sqrt{-3} & 0 \\ 0 & -2-\sqrt{-3} \end{pmatrix} \quad \tau : \begin{pmatrix} -1+\sqrt{-3} & 0 \\ 0 & -1-\sqrt{-3} \end{pmatrix} \quad \Phi\tau = \tau\Phi \)

Table 4. \( G = \langle \tau, \Phi \rangle = C_3 \times \hat{\mathbb{Z}} \subset H^1(C) \) for \( C/\mathbb{Q}_5 : y^2 = x^3 + 5^2 \)

| \( u \) | \( d \) | \( C_u \) | \( |C_u(\mathbb{F}_y)| \) | \( \text{Tr} \) |
|---|---|---|---|---|
| \( \in F_5 \) | 1 | \( y^2 = x^3 + u^2 \) | 6 | \( \text{Tr}(\Phi^{-1}) = 5 + 1 - 6 = 0 \) |
| \( = \text{Tr}(\Phi^{-1}) = \text{Tr}(\Phi^{-1}\tau^2), \text{ same class} \) |
| 2 | 2 | \( y^2 = x^3 + \zeta_{12} \) | 21 | \( \text{Tr}(\Phi^{-1}\tau^2) = 25 + 1 - 21 = 5 \) |
| \( \zeta_{12} \) | 2 | \( y^2 = x^3 + \zeta_6 \) | 21 | \( \text{Tr}(\Phi^{-2}\tau^2) = 25 + 1 - 21 = 5 \) |

\( \Phi^{-1} : \begin{pmatrix} 0 & \sqrt{-5} \\ -\sqrt{-5} & 0 \end{pmatrix} \quad \tau : \begin{pmatrix} -1+\sqrt{-5} & 0 \\ 0 & -1-\sqrt{-5} \end{pmatrix} \quad \Phi\tau\Phi^{-1} = \tau^{-1} \)

Example 7.5. Take any tame ‘2-variable Eisenstein equation’ over \( \mathbb{Q}_p \),

\[ C : y^n + x^m + p \sum_{i,j} a_{ij}x^iy^j = 0, \quad p \nmid mn \ a_{00}, \]

with \((i, j)\) confined to the triangle of exponents of \( y^n, x^m \) and 1. Then \( C \) is \( \Delta_v \)-regular, with one \( v \)-face \( F \). We find that

- \( C \) has good reduction over \( K(\sqrt{p}) \), with \( e = \delta_F = \text{lcm}(m, n) \) (tame).
- \( H^1(C)|_{I_{\mathbb{Q}_p}} \) does not depend on \( C \). It factors through the unique \( C_e \)-quotient of \( I_{\mathbb{Q}_p} \), and is the permutation representation (see Thm. 6.4)

\[ H^1(C)|_{I_{\mathbb{Q}_p}} \cong \mathbb{Q}_{l}[C_e/C_e/m] \otimes \mathbb{Q}_{l}[C_e/C_e/n]. \]

- \( H^1(C) \) depends only on \( a_{00} \mod p \) (see Thm. 7.3).

Remark 7.6. Generally, by a theorem of Kisin [Kis] \( l \)-adic representations are locally constant in \( p \)-adic families of varieties. Thus, 6.4 and 7.3 may be viewed as an explicit version of this statement for tame \( \Delta_v \)-regular curves.

Remark 7.7. We end by noting that 7.2, 7.3 and 7.5 all fail without the tameness assumption. There is one Kodaira type for elliptic curves, namely \( \Gamma_{\ast} \), when the regular model does not determine whether the reduction is potentially good or potentially multiplicative (see [LoM]). So Theorem 6.4 (and its refinements 7.2, 7.3) do not have an obvious analogue for the description of \( I_K \subset H^1(C) \) in the wild case just in terms of \( \Delta_v \).
Similarly, the condition \( \text{char } k \nmid mn \) in Example 7.5 is necessary: for instance, the two elliptic curves over \( \overline{\mathbb{Q}} \)
\[
y^2 = x^3 + 3, \quad y^2 = x^3 + 3x + 3
\]
have conductors \( 2^13^5 \) and \( 2^33^13^3 \), so their Galois representations at 3 are certainly not the same (while the regular model is the same, of type II.)

8. Differentials

Let \( v : K^\times \to \mathbb{Z}, O_K, \pi, k, p = \text{char } k \) be, as usual, a discretely valued field and the associated invariants. Let \( C : f = 0 \) be a \( \Delta_v \)-regular curve; we choose variables as in §2 so that \( f'_y \neq 0 \). Recall that 2.2 (4) gives a basis of differentials for \( C/K \). Here we aim to modify it to give a basis of the global sections of the relative dualising sheaf for \( C_{\Delta}/O_K \) (Theorem 8.12).

We refer to §4 for the charts for the components \( \tilde{X}_F, X_L \times \Gamma_L \) of the special fibre of \( C_{\Delta} \), defined in 3.7.

**Proposition 8.1.** Let \( F \) be a \( v \)-face of \( \Delta \), and \( F^* : \mathbb{Z}^2 \to \mathbb{Z} \) be the unique affine function that equals \( -\delta_F v \) on \( F(\mathbb{Z}) \). Then
\[
\text{ord}_{\tilde{X}_F} x^i y^j = F^*(i, j) - F^*(0, 0), \\
\text{ord}_{\tilde{X}_F} x^i y^j \frac{dx}{\pi xy f'_y} = F^*(i, j) - 1.
\]

**Proof.** 8.2. Coordinate transformation. Write \( S = \text{Spec } O_K \), and let
\[
(X, Y, Z) = (x, y, \pi) \cdot M, \quad M = (m_{ij}), \quad M^{-1} = (\tilde{m}_{ij})
\]
be a coordinate transformation as in 4.1-4, associated to \( F \) and some \( v \)-edge \( L \) of \( F \). Recall from the formula (4.1) for \( M^{-1} \in \text{SL}_3(\mathbb{Z}) \) that
\[
\tilde{m}_{13} = k_{i+1}d_i - k_id_{i+1}, \quad \tilde{m}_{23} = \delta_L d_{i+1}, \quad \tilde{m}_{33} = \delta_L d_i.
\]
If \( \text{char } k = p > 0 \) and \( p|\delta_L \), then \( \gcd(\tilde{m}_{13}, \tilde{m}_{23}, \tilde{m}_{33}) = 1 \) forces \( p \nmid \tilde{m}_{13} \); if \( p \nmid \delta_L, p \nmid d_i, p|\tilde{m}_{23} \), we choose \( k_{i+1} \) (defined modulo \( \delta \)) so that \( p \nmid \tilde{m}_{13} \). Thus, \( \tilde{m}_{13}, \tilde{m}_{23} \neq (0, 0) \in K^2 \). Equivalently,
\[
(m_{13}, m_{23}) \neq (0, 0) \in K^2,
\]
since \( M \) preserves \((0, 0, 1) \in K^3 \) if and only if \( M^{-1} \) does.

Also, the first two columns of \( M \) span the plane orthogonal to \( \ker F^* \), so this plane is parallel to \((\tilde{m}_{31}, \tilde{m}_{32}, \tilde{m}_{33})^\perp \). As \( \delta_F = \tilde{m}_{33} \), we find
\[
F^*(i, j) = \tilde{m}_{31}i + \tilde{m}_{32}j + F^*(0, 0).
\]

8.5. Transformed equation. Write \( G(X, Y, Z) = f((X, Y, Z) \cdot M^{-1}) \),
\[
G_0(X, Y, Z) = Z^{F^*(0, 0)} G(X, Y, Z) \quad \text{and} \quad H(X, Y, Z) = X^{\tilde{m}_{13}} Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}},
\]
a transformed version of \( f \) and \( \pi \) to the \( X, Y, Z \)-chart. Then
\[
U : \quad G_0 = H = 0
\]
defines a complete intersection in \( \mathbb{A}^3_S \), and restricted to \( \mathbb{A}^3_S \setminus \{XY = 0\} \) it gives an open subset \( U : Z = 0 \) of \( \tilde{X}_F \subset \mathcal{C}^{\text{red}}_k \). By [Liu, 6/4.14], the sheaf
\(\omega_{C/S}\) is generated on \(\tilde{X}_F\) by \(dZ/|^{(G_\delta)_X}_{H_X'(\bar{H}_X')}|\) if this determinant is non-zero. We will show that it is indeed non-zero (8.6, 8.8), and that

\[
Z^{F^*(0,0)}dZ/\left|^{(G_\delta)_X}_{H_X'(\bar{H}_X')}\right| = \frac{\pi^{-1}XYdZ}{m_{23}XG_Y' - m_{13}YG_Y'} = \pi^{-1}XY \frac{dx}{xyf_y'}.
\]

As \(X,Y\) are units on \(U\), and \(Z\) vanishes to order 1, we get the claim for \((i,j) = (0,0)\). As \(\text{ord}_U x = \tilde{m}_{31}, \text{ord}_U y = \tilde{m}_{32}\), the theorem follows from (8.4).

8.6. Relation \(f' \leftrightarrow G'\). As in Lemma A.3, from the chain rule we get

\[
\begin{pmatrix}
xf_x'
yf_y'
\pi f_y'
\end{pmatrix} = \begin{pmatrix}XG_Y' \\ YG_Y' \\ ZG_Y'\end{pmatrix} M = \begin{cases} xf_x' = m_{11}XG_Y' + m_{12}YG_Y' + m_{13}ZG_Y', \\
yf_y' = m_{21}XG_Y' + m_{22}YG_Y' + m_{23}ZG_Y'.\end{cases}
\]

Therefore,

\[
m_{23}xf_x' - m_{13}yf_y' = (m_{23}m_{11} - m_{13}m_{21})XG_Y' + (m_{23}m_{12} - m_{13}m_{22})YG_Y' + 0 \cdot ZG_Y' = m_{23}XG_Y' - m_{13}YG_Y'.
\]

8.8. Non-vanishing. We claim that \(m_{23}xf_x' - m_{13}yf_y' \neq 0\) in \(K(C)\). If not, there is a linear relation in \(K[x,y]\), non-trivial by (8.3),

\[c_1xf_x' + c_2yf_y' + c_3f = 0, \quad c_i \in K.\]

(The coefficient \(c_3\) must be constant, and not a higher degree polynomial in \(x\) and \(y\) by degree considerations.) Then \(\sum a_{ij}(c_1i + c_2j + c_3)x^iy^j\) is identically zero, so all monomial exponents are in the kernel of a non-trivial linear form. But this contradicts \(\text{vol}(\Delta) > 0\).

8.9. \(dx/f_y' \leftrightarrow dZ/\det\). Recall that \(Z = x^{m_{13}}y^{m_{23}}\pi^{m_{33}}\). From the relations

\[
dZ = m_{13}\frac{dx}{x} + m_{23}\frac{dy}{y} \quad \text{and} \quad df = f_x'dx + f_y'dy = 0 \text{ in } \Omega_{C/K},
\]

we get

\[
\frac{dZ}{Z} = (\frac{m_{13}}{x} - \frac{m_{23}}{y} \frac{f_x'}{f_y'})dx = \frac{dx}{xyf_y'}(m_{13}f_y' - m_{23}xf_x').
\]

Combined with (8.7) this yields

\[
\frac{dZ}{m_{23}XG_Y' - m_{13}YG_Y'} = Z \frac{dx}{xyf_y'}.
\]

\(\square\)

**Remark 8.11.** In the notation of Theorem 2.2 and Proposition 8.1, \(\pi''\omega_F\) is regular on \(\tilde{X}_F\) if and only if \(n \geq \lfloor v_F(P) \rfloor\), and regular non-vanishing on \(\tilde{X}_F\) if and only if either \(\delta_F = 1, v_F(P) = n\) or \(\delta_F > 1, v_F(P) = n + \frac{1}{d_F}\).

**Theorem 8.12.** If \(C\) is \(\Delta_v\)-regular, then the differentials

\[\omega_{ij} = \pi^{[v(i,j)]} x^{-1} y^{-1} \frac{dx}{y_t'}\]

form an \(O_K\)-basis of global sections of the relative dualising sheaf \(\omega_{{C/\Omega_{C/K}}}\).
Proof. By Proposition 8.1, each $\pi^{[v(i,j)]}\omega_{ij}$ has order $\geq 0$ at every component of $C_k$. So they are global sections, linearly independent by Baker’s theorem, and it remains to prove that the lattice they span is saturated in the global sections. Suppose not. There there is a combination of the form

$$\left(8.13\right) \frac{1}{\pi} \sum_{(i,j) \in \Sigma} u_{ij} \omega_{ij}^v \quad (\emptyset \neq \Sigma \subset \Delta(Z), \ u_{ij} \in O_K^s),$$

which is regular along every component. Pick $P \in \Sigma$ with $\epsilon = v(P) - [v(P)]$ minimal, and a $v$-face $F$ with $P \in \bar{F}(Z)$. Write

$$m = \text{ord}_{\bar{X}_F} \frac{1}{\pi} \pi^{[v(P)]}\omega_P; \quad m \overset{8.1}{=} -\delta_F + \epsilon\delta_F < 0.$$

As (8.13) is supposed to be regular along $\bar{X}_F$, and all its terms have order $\geq m$ on $\bar{X}_F$ by minimality of $\epsilon$ and lower convexity of $\Delta$, those of order $m$ must cancel along $\bar{X}_F$. Let $\Sigma_m \subset \Sigma$ be their indices.

First, note that $\Sigma_m \subset \bar{F}(Z)$. Indeed, else there is $P' \in \Sigma_m$ on some $v$-face $F' \neq F$ of $\Delta$ for which $v(P') - [v(P')] < \epsilon$, contradiction. Now, from Remark 2.3 it follows that $\Sigma_m$ contains all vertices of $\Delta$. This allows us to replace $F$ by any of the neighbouring $v$-faces, changing $P$ if necessary but not changing $m$, and it also shows that $F \cap \partial \Delta = \emptyset$ (as $\Sigma \cap \partial \Delta = \emptyset$). Proceeding inductively, we find that (8.13) cannot exist at all (cf. [O’C]). \hfill \Box

Example 8.14. Let $C : y^2 = \pi x^4 + \pi^3$. If char $k \neq 2$, then $C$ is $\Delta_v$-regular, and its $\Delta_v$ and the special fibre of the m.r.n.c. model are as follows:

$$\begin{array}{cccc}
0 & 2 & g1 \\
3/2 & 1/2 & 1 \\
1 & 1/2 & -2 & 3/2 & 2 \end{array}$$

It is a genus 1 curve with $C(K) = \emptyset$ by 3.19, so it is not an elliptic curve; it has bad reduction but its Jacobian has good reduction, by 6.7(25). The differential $\pi dx/y$ spans the global sections of the relative dualising sheaf $\omega_{C_{\Delta}/O_K}$ by the above theorem, but it vanishes along the unique (reduced) component $\Gamma$ of the special fibre, by 8.1. Hence, $\omega_{C_{\Delta}/O_K} (=O(\Gamma))$ is not generated by global sections. In contrast, $\omega_{C_{\Delta}/O_K}$ is trivial for genus 1 curves with a rational point, see [Liu, 9.4, Exc. 4.16].

Example 8.15. Consider the genus 2 curve ‘188’ from [FLS3W] at $p = 2$,

$$C : y^2 = x^5 - x^4 + x^3 + x^2 - 2x + 1.$$ 

Letting $x \rightarrow \frac{2}{x-1}$, $y \rightarrow \frac{y+1}{(x-1)^2} + \frac{2}{x-1}$, the equation becomes

$$y^2 + 2(x^3 - 2x^2 + x + 1)y + x(2x^2 - 5x + 4)^2 = 0.$$ 

This is $\Delta_v$-regular, with special fibre of the m.r.n.c. model as follows:
The differentials \( dx/y, xdx/y \) form a basis of the relative dualising sheaf.

9. **Example: Elliptic curves**

As an application, we recover Tate’s algorithm for elliptic curves, and some immediate consequences for the Galois representation and the Néron component group. All of this is well-known: see [TaA, Si1, §IV.9]+[Di2] for Tate’s algorithm and the valuations of the coefficients of the minimal Weierstrass model, [Kra, Roh, ST] for the Galois representation, and [Liu, §10.2, Exc. 2.2] for r.n.c. models of elliptic curves.

As always, we have \( v : K^\times \to \mathbb{Z}, O_K, \pi, k \) and \( p = \text{char} \, k \). Recall that an elliptic curve \( E/K \) has a Weierstrass equation,

\[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in K, \Delta_E \neq 0. \tag{W}
\]

Here \( \Delta_E \) is the discriminant of \( E \), defined via

\[
\begin{align*}
b_2 &= a_1^2 + 4a_2, & b_4 &= 2a_4 + a_1a_3, & b_6 &= a_3^2 + 4a_6, \\
b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, & \Delta_E &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6.
\end{align*}
\]

We write \( f \) for LHS-RHS of \((W)\), so that \( f = 0 \) is the equation defining \( E \).

**Theorem 9.1.** Assume\(^{15}\) \( k \) is perfect or \( p > 3 \). An elliptic curve \( E/K \) has a \( \Delta_v \)-regular Weierstrass equation \((W)\) with \( \Delta_v \) as in Table 5, column 2, up to removable faces. The Kodaira type of \( E \), special fibre of the m.r.n.c. model, condition for \( E \) to have tame reduction, inertia action on \( H^1_{\text{ét}}(E/K, \mathbb{Q}_l) \) if \( E \) is tame and \( k \) is perfect, and the Néron component group \( E(K)/E_0(K) \) are as in columns 1, 3–6.

**Proof.** Start with a Weierstrass equation \((W)\). As \( x = \pi^{-2}x', y = \pi^{-3}y' \) changes \( a_i \mapsto \pi^i a_i \), we can assume that all \( v(a_i) \geq 0 \). Write \( P = (1,1) \in \mathbb{Z}^2 \).

A transformation \( x \mapsto x + r, y \mapsto y + sx + t \) with \( r, s, t \in O_K \) keeps \( \Delta_E \) unchanged, and \( \Delta \) confined to the triangle \((0,0)-(0,0)-(2,0) \). As long as \((W)\) is not \( \Delta_v \)-regular, we will see that there is such a transformation that increases \( v(P) \). Note that vertices of any \( v \)-face inside the triangle span an affine lattice of index \( 12 \) in \( \mathbb{Z}^2 \), so \( 12v(P) \in \mathbb{Z} \). But \( v(a_i) \geq i v(P) \) for all \( i \), so \( 12v(P) \leq v(\Delta_E) \) and the algorithm terminates.

There are four reasons why \( f \) could be non-\( \Delta_v \)-regular:

1. \( \Delta_v \) has \( v \)-edge \( \lambda \) (left, \((0,0)-(0,2)\)) with \( f_\lambda \) non-squarefree, and bounding a non-removable face.
2. \( \Delta_v \) has \( v \)-edge \( \sigma \) (skew, \((0,2)-(2,0)\)) with \( f_\sigma \) non-squarefree.
3. \( \Delta_v \) has \( v \)-edge \( \beta \) (bottom, \((0,0)-(2,0), (0,0)-(3,0) \) or \((1,0)-(3,0)\)) with \( f_\beta \) non-squarefree, and bounding a non-removable face.

\(^{15}\)Otherwise there are indeed more reduction types, classified in [Szy]; the proof of 9.1 uses that a multiple root of a polynomial of degree 2 or 3 over \( k \) is always \( k \)-rational.
| Type | $\Delta_v$ | Model | tame ↔ tame if tame | Inertia if tame | $[E(K):E_0(K)]$ |
|------|------------|--------|---------------------|----------------|----------------|
| I₀   | $0$        | $\frac{1}{2} \cdot \frac{1}{2}$ | $p \neq 2, 3$ | $C_1$ | $\{0\}$ |
|      | $1$        | $\frac{1}{3} \cdot \frac{1}{3}$ | $p \neq 2, 3$ | $C_6$ | $\{0\}$ |
|      | $2$        | $\frac{1}{4} \cdot \frac{1}{4}$ | $p \neq 2$ | $C_4$ | $\mathbb{Z}/2\mathbb{Z}$ |
|      | $3$        | $\frac{1}{5} \cdot \frac{1}{5}$ | $p \neq 3$ | $C_3$ | $\mathbb{Z}/3\mathbb{Z}$ if $l=2$ |
|      | $4$        | $\frac{1}{6} \cdot \frac{1}{6}$ | $p \neq 3$ | $C_3$ | $\{0\}$ if $l=0$ |
|      | $5$        | $\frac{1}{7} \cdot \frac{1}{7}$ | $p \neq 2$ | $C_2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ if $b=3$ |
|      | $6$        | $\frac{1}{8} \cdot \frac{1}{8}$ | $p \neq 2$ | $C_3$ | $\mathbb{Z}/2\mathbb{Z}$ if $b=1$ |
|      | $7$        | $\frac{1}{9} \cdot \frac{1}{9}$ | $p \neq 2$ | $C_4$ | $\mathbb{Z}/2\mathbb{Z}$ |
|      | $8$        | $\frac{1}{10} \cdot \frac{1}{10}$ | $p \neq 2, 3$ | $C_6$ | $\{0\}$ |
|      | $9$        | $\frac{1}{11} \cdot \frac{1}{11}$ | $p \neq 2$ | $C_2$ | $\mathbb{Z}/n\mathbb{Z}$ if $s=2$ |
|      | $10$       | $\frac{1}{12} \cdot \frac{1}{12}$ | $p \neq 2$ | $C_2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ if $b=3$ |
|      | $11$       | $\frac{1}{13} \cdot \frac{1}{13}$ | $p \neq 2$ | $C_2$ | $\mathbb{Z}/2\mathbb{Z}$ if $b=1$ |
|      | $12$       | $\frac{1}{14} \cdot \frac{1}{14}$ | $p \neq 2$ | $C_2$ | $\mathbb{Z}/4\mathbb{Z}$ if $l=2$ |

**Table 5.** Reduction types of elliptic curves (Theorem 9.1)
(4) $\Delta_v$ has a principal $v$-face $F$, with $P$ in its interior, and $\overline{f_F} = 0$ is a geometrically singular subscheme of $G_m^2$.

In all four cases, the offending 1- or 2-face has denominator 1, for otherwise the corresponding reduction is linear and cannot define a singular scheme; for the same reason, in (1) and (3) the face is assumed to be non-removable.

In case (1)-(3), the reduction $\overline{f_F} (\ast \in \{\lambda, \sigma, \beta\})$ has a unique multiple root $\alpha \in k$, and a shift $y \mapsto y + \alpha \pi^d$, $y \mapsto y + \alpha \pi^d x$ or $x \mapsto x + \alpha \pi^d$ with $d$ defined by the slope of $\ast$ shifts the root to 0, breaking the 1-face $\ast$. Let us call this change of variables the ‘shift along $\ast$’.

We refer to cases, i.e. table rows, by Kodaira types (first column).

Step 1. ($I_n$) If $P$ is a $v$-vertex of $\Delta$, then $v(1, 0) > v(P), v(0, 1) > v(P)$, and $x \mapsto x - \frac{a_1}{a_6}, y \mapsto y + \frac{a_2}{a_6}$ makes $a_3 = a_4 = 0$; as $\Delta_E \neq 0$, we have $a_6 \neq 0$. Now all $v$-faces are contractible, with linear reductions, and $f$ is always $\Delta_v$-regular. The reduction type is $I_n$ (cf. Remark 5.4), split in this case.

Step 2. If $\Delta_v$ has the $v$-edge $\sigma : (0,2)-(2,0)$, there are 3 possibilities.

If $\delta_\sigma = 1$ and $\overline{f_\sigma}$ is not-squarefree, then a shift along $\sigma$ increases $v(P)$, and we go back to Step 1.

($I_n^r$) If $\delta_\sigma = 1$ and $\overline{f_\sigma}$ is squarefree, we are again in the $I_n$-case, possibly non-split (with picture as in the table, up to possible removable faces).

($I_n^* r$) Otherwise $\delta_\sigma = 2$, and $v(P) = \frac{a_2}{a_6}, v(2,0) = m$ for some odd $m$. The only possibly singular faces are the $v$-edges $\lambda$ and $\beta$. If one of them is, shift along it. This increases the (smallest) slope of a face left of $P$. Since

$$v(a_3), v(a_4) \geq t; v(a_6) \geq 2t \Rightarrow v(b_4) \geq t; v(b_6), v(b_8) \geq 2t \Rightarrow v(\Delta_E) \geq 2t,$$

this process terminates in a $\Delta_v$-regular model, of Type $I_n^r$.

Step 3. Finally suppose $P$ is neither $v$-vertex nor lies on a $v$-edge; in other words, it lies inside a unique principal $v$-face, say $F$.

If $\{f_F = 0\} \subset G_m^2$ is singular (Case (4)), it has arithmetic genus 1, implying $\delta_F = 1$, and the singular point is unique and $k$-rational. Use a shift in $x$ and $y$ to translate it to $(0,0)$. If $v(P)$ increases or $P$ is not interior in a $v$-face any more, go back to Step 1. Otherwise, the only possibly singular faces now are the $v$-edges $\lambda$ and $\beta$. If one of them is, shift along it; this necessarily increases $v(P)$ and we go back to Step 1.

If we reached this stage, we now have a $\Delta_v$-regular model, with one non-removable face. Depending on $v(P) = \text{integer} + 0, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}$ (every face with vertices $(3,0),(0,2)$ and somewhere else in $\Delta$ has $\delta_F \mid 4$ or $|6|$, we get types $I_0, II, III, IV, I_n^*, \text{IV}^*, \text{III}^*, \text{II}^*$ after a rescaling.

Finally, the Galois action on the components of multiplicity 1 follows from Theorem 3.13(2), and the ‘tame’ and ‘inertia if tame’ columns in the table by Theorems 6.7(4) and 6.4. \qed

**Remark 9.2.** One also recovers the behaviour of minimal models of elliptic curves in tame extensions, see [D2t] Thm 3 (1)-(3).
10. Example: Fermat curves

Tate's algorithm relies on Weierstrass models having a unique singularity that can be shifted to the origin. In higher genus, there may be multiple ones, and the algorithm 'repeatedly shift to resolve a singularity' of 9.1 often works but only \textit{locally}. Thus, 3.14 may be applied for several choices of generators \(x, y \in K(C)\) and the resulting schemes merged to a r.n.c. model.

For example, consider a genus 2 hyperelliptic curve

\[ C/K: \pi Y^2 = X^3(X - 1)^2 + \pi \quad \text{(char } k > 3) \]

The polynomial on the right has a triple root \(X \equiv 0\) and a double root \(X \equiv 1\) mod \(\pi\), so the equation is a model \(C_{\text{sing}}/O_K\) with two singular points \(P_1 = (X, Y, \pi)\) and \(P_2 = (X - 1, Y, \pi)\).

Applying Theorem 3.14 to \((x_1, y_1) = (X, Y)\) and \((x_2, y_2) = (X - 1, Y)\) yields two schemes (with special fibres left and centre), both still singular:

\[\pi y^2 = x^3(x - 1)^2 + \pi \quad \quad x \mapsto x + 1 \quad \quad x \mapsto \frac{1}{x + 1}, \ y \mapsto \frac{y}{(x + 1)^3}\]

Shifts and glueing; red = singular point on the model

The left one successfully resolves \(P_1\) (dashed line and above) but not \(P_2\) (red dot) and the centre one \(P_2\) (dashed line and above) but not \(P_1\) (red dot).

They agree on two components — thick one of multiplicity 1 coming from the thick \(v\)-edge of \(\Delta\) above it, and a thick one of multiplicity 2 coming from the shaded \(v\)-face of \(\Delta\) above it, with \(P_1, P_2\) removed. Glueing them together gives a r.n.c. model, bottom right. Here, one \textit{could} resolve both \(P_1\) and \(P_2\) simultaneously by a Möbius transformation that sends their \(x\)-coordinates to 0 and \(\infty\) respectively (top right), but that is generally not possible.

As an application, we construct regular models of Fermat curves

\[x^p + y^p = 1, \quad p \geq 3\]

over \(\mathbb{Z}_p\) (and thus over its tame extensions as well, see Remark 3.11). These are known over \(\mathbb{Z}_p[\mu_p]\) [McC, CM], but seemingly not over \(\mathbb{Z}_p\). Let \(x \mapsto x - y + 1\), so that the new equation is

\[(x - y + 1)^p + y^p = 1.\]

The five monomials \(x^p, px, -py, -py^{p-1}, -pxy^{p-1}\) determine \(\Delta\) and \(\Delta_v\), as all the others have valuations \(\geq 1\) and lie inside the Newton polygon.
Example: $\Delta_v$ for $(x-y+1)^p + y^p = 1$ with $p = 7$.

The $\Delta_v$-regularity conditions are automatic except at the leftmost $v$-edge $L$, where the reduced equation is non-linear, namely

$$f_L = \frac{1}{p}((1-y)^p + y^p - 1) \in \mathbb{F}_p[y].$$

As shown in [CM, Lem. 3.1] (or [McC, p. 59]), considering the derivatives

$$f'_L = -(1-y)^{p-1} + y^{p-1}, \quad f''_L = (p-1)(1-y)^{p-2} + (p-1)y^{p-2},$$

and noting that the roots of $f'_L$ are 2, 3, ..., $p-1$, we see that all multiple roots of $f'_L$ are double and $\mathbb{F}_p$-rational. Let $\rho$ be their number, and lift them to $r_1, ..., r_\rho \in \mathbb{Z}$. Pick one, say $r_i$, move it to zero with a shift $y \mapsto y + r_i$, and apply Theorem 3.14 to the shifted equation

$$(x-y+1-r_i)^p + (y+r_i)^p = 1.$$
\[ p - 2\rho \] chains from \( X_p \) with multiplicities \( p - 1, p - 2, \ldots, 2, 1 \),

- Components \( Y_1, \ldots, Y_{\rho} \) of multiplicity \( 2p - 2 \), meeting \( X_p \),

- A chain with multiplicities \( 2p - 2, 2p - 1, \ldots, 2, 1 \) from each \( Y_i \),

- A component of multiplicity \( p - 1 \) from each \( Y_i \).

When \( p = 3 \), we have \( \rho = 0 \) and \( C_k \) is a type IV* elliptic curve, see Table 5.

The multiplities on the chains come from the slopes. For example, to see why \( Y_1 \) meets \( X_p \) transversally, consider \( C_1 \) and compute the slopes at the \( v \)-edge \( L' \) where \( F_1 \) meets \( F_2 \). In the notation of 3.12, let \( P = (\frac{p-1}{2}, 1) \). Then

\[
s_1' = \frac{p}{2p-2}, \quad s_2' = \frac{(p+1)/2}{p}, \quad \left| \frac{p}{2p-2} (p+1)/2 \right| = 1.
\]

The chain between \( Y_1 \) and \( X_p \) is therefore empty; it continues to be empty in tame extensions of \( \mathbb{Z}_p \) with \( v | (2p - 2) \).

**Remark 10.1.** Theorem 3.14 (and gluing as above) also gives an alternative approach to the construction of regular models for semistable hyperelliptic curves in odd residue characteristic \([M^2D^2] \).

**Appendix A. Elementary facts about lattice polygons in \( \mathbb{R}^2 \)**

We start with equivalent characterisations of \( \delta_F \) for a \( v \)-face \( F \), and verify a statement that reflects rationality of étale cohomology (Thm. 6.4).

**Lemma A.1.** Let \( F \subset \mathbb{R}^2 \) be a convex lattice polygon, and \( v : F \to \mathbb{R} \) an affine function, \( \mathbb{Z} \)-valued on the vertices of \( F \).

1. If \( F \) has positive volume, then the following three numbers are equal:
   - \( A = \text{index of the lattice } v^{-1}(\mathbb{Z}) \text{ in } \mathbb{Z}^2 \).
   - \( B = \text{common denominator of } v(P) \text{ for } P \in \mathbb{Z}^2 \).
   - \( C = \text{common denominator of } v(P) \text{ for } P \in F(\mathbb{Z}) \).

2. Fix \( d \in \mathbb{N}_{\geq 2} \). Define two counting functions: for \( n \in (\mathbb{Z}/d\mathbb{Z})^\times \) let
   \[
   N(\frac{n}{d}) = |(\Delta(\mathbb{Z})n/2 + \mathbb{Z})|, \quad \bar{N}(\frac{n}{d}) = |\bar{\Delta}(\mathbb{Z})n/2 + \mathbb{Z}|.
   \]

   (i) \( n \mapsto N(\frac{n}{d})+N(\frac{-n}{d}) \) and \( n \mapsto \bar{N}(\frac{n}{d})+\bar{N}(\frac{-n}{d}) \) are constant on \( (\mathbb{Z}/d\mathbb{Z})^\times \).

   (ii) If \( F \) is a line segment or a parallelogram, then \( N, \bar{N} \) are constant.

**Proof.** (1) \( A = B, C | B \): clear from definitions. \( B | C \): Replace \( F \) by any lattice triangle \( T \subset F \). Now reflect \( T \), completing it to a parallelogram, and tile the plane with it.

   (2i) Suppose \( F \) is a line segment. Translate an endpoints of \( F \) to \((0,0)\), and replace \( v \mapsto v - v(0,0) \). This does not affect the statement, but now \( v : V(\mathbb{Z}) \to \mathbb{Q} \) is linear, where \( V \) is the line generated by \( F \). It descends to \( v : F(\mathbb{Z}) \setminus \{ \text{endpoint} \} \to \mathbb{Q}/\mathbb{Z} \), and all reduced fractions \( \frac{n}{d} \) have preimage sizes independent of \( n \). The same argument works for a parallelogram.

   (2ii) If \( F \) is a triangle, reflect it in one of its edges \( L \), completing it to a parallelogram \( R = F \cup F' \cup L \). The values of \( v \) on \( F' \) are \( \text{minus} \) those on \( F \), so the claim follows from (2i) for \( L \) and for \( R \). In general, break \( F \) into triangles and use (2i) on the edges. \( \square \)
Lemma A.2. Let $\Delta \subset \mathbb{R}^2$ be a convex lattice polygon with positive volume, and $\mathcal{L} = \{1\text{-dim faces of } \Delta\}$. For $L \in \mathcal{L}$ let $L^* : \mathbb{Z}^2 \to \mathbb{Z}$ be the unique affine function vanishing on $L$ and non-negative on $\Delta$. Then for every $(i, j) \in \mathbb{Z}^2$,
\[ \sum_{L \in \mathcal{L}} (L(Z)| - 1) (L^*(i, j) - 1) = 2|\Delta(Z)| - 2. \] (\text{*}_\Delta)

Proof. Induction on $|\Delta(Z)|$. If $|\Delta(Z)|=3$, then LHS($\text{\text{*}}_\Delta$)=RHS($\text{\text{*}}_\Delta$)=0. Otherwise cut $\Delta$ into two lattice polygons $A, B$; say $L^*_0 = A \cap B$. Then (\text{*}_A), (\text{*}_B) hold by induction and we claim that adding them up gives $\text{\text{*}}_\Delta$ minus $2(L^*_0(Z)| - 1)$ in both sides. For the right-hand side,
\[ 2|A(Z)| - 2 + 2|B(Z)| - 2 = 2(|\Delta(Z)| - |L^*_0(Z)|) - 4. \]
For the left-hand side, the terms of (\text{*}_A), (\text{*}_B) add up to those of (\text{*}_\Delta) except the contribution from $L^*_0$. But $L^*_0$ with respect to $A$ is minus that for $B$, so the two contributions also add up to $-2$ times $|L^*_0(Z)| - 1$. \hfill $\square$

Finally, the proof of Baker’s theorem 2.2 used that the differentials $\omega_{ij}$ behave well under $\text{Aut} \mathbb{G}^2_m$:

Lemma A.3. Let $C : \{f(x, y) = 0\} \subset \mathbb{G}^2_{m,K}$ be a smooth curve. Consider a transformation of $\mathbb{G}^2_{m,K}$
\[ x = X^a Y^b, \quad X = x^d y^{-b}, \quad Y = x^{-c} y^a \quad \text{with} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1. \]
Let $F(X, Y) = f(X^a Y^b, X^c Y^d)$, and assume that $f'_y$ and $F'_Y$ are not identically zero. Then
\[ \frac{1}{xy f'_y} = \frac{1}{XY F'_Y} \frac{dx}{X} \frac{dX}{Y}. \]

Proof. From the chain rule for $f(x, y) = F(x^d y^{-b}, x^{-c} y^a)$, we find
\[ \begin{pmatrix} x f'_x \\ y f'_y \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} X F'_X \\ Y F'_Y \end{pmatrix}. \] (\text{*})
On the other hand, using \((gh)' = g' + h'/k\) and the relation $(dF =) F'_X dX + F'_Y dY = 0$ on $C$, we have
\[ \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} dX \\ dY \end{pmatrix} = \frac{1}{XY} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Y F'_Y \\ X F'_X \end{pmatrix} \frac{dX}{F'_Y}. \] (\text{**})
Combining (\text{*}) (2nd row) and (\text{**}) (first row), the formula follows. \hfill $\square$

Appendix B. Inertia invariants on étale cohomology

Let $O_K$ be a complete DVR, with field of fractions $K$ and perfect residue field $k$ of characteristic $p \geq 0$.

Theorem B.1. Let $C/O_K$ be a proper regular model of a smooth projective geometrically connected curve $C/K$. Write
• $\mathcal{A} = \text{Néron model of } \text{Jac } C \text{ over } O_K$.
• $\mathcal{C}_k, \mathcal{A}_k = \text{special fibres of } \mathcal{C} \text{ and } \mathcal{A} \text{ base changed to } \bar{k}$,
• $d = \gcd(\text{multiplicities of components of } \mathcal{C}_k)$,
• $\Phi = \text{component group } \mathcal{A}_k/\mathcal{A}_0^k$.

For every prime $l \neq \text{char } k$ and $n \geq 1$, there is a natural $G_k$-homomorphism

$$\text{Pic}(\mathcal{C}_k)[l^n] \longrightarrow \text{Pic}(C_K)[l^n]^I_K,$$

with kernel killed by $d$ and cokernel killed by $d|\Phi|$. In the limit,

$$V_l(A_0^\infty) \cong V_l \text{Pic}(\mathcal{C}_k) \cong V_l \text{Pic}(C_K)^I_K,$$

as $G_k$-modules; when $l \nmid d|\Phi|$, the same is true with $\mathbb{Z}_l$-coefficients.

Proof. This is essentially well-known, and the proof follows [BW, Prop 2.6] and [Liu, §10.4.2] closely. (The claim is proved in [BW] when $d = 1$.)

Write $S_0 = \text{Spec } O_K$, $S = \text{Spec } O_{K^w}$ and $X = C \times_{S_0} S$. Denote by $X_s$ and $X_\eta$ the special fibre and the generic fibre of $X$, and $X^\eta_\bar{K} = X_\eta/K^s$.

Let $\Gamma_1, \ldots, \Gamma_n$ be components of $X_s$, with multiplicities $d_1, \ldots, d_n$, so that $d = \gcd(d_i)$. Write $D$ for the free abelian group on the $D_i$ and $D^\vee$ for its dual, with dual basis $\Gamma^*_1, \ldots, \Gamma^*_n$. There are natural maps

$$D \xrightarrow{\lambda} \text{Pic } X \xrightarrow{\phi} D^\vee \xrightarrow{\alpha} D \xrightarrow{\alpha = \phi \lambda} D^\vee,$$

with $\ker \alpha = d^{-1} X_s \mathbb{Z}$, and a commutative diagram with exact rows

$$\begin{array}{ccc}
\ker s & \longrightarrow & \lambda(D) \\
\downarrow & & \downarrow \\
\ker \phi & \longrightarrow & \text{Pic } X \\
\downarrow s & & \downarrow \\
\ker \psi & \longrightarrow & \text{Pic } X_\eta \\
\end{array}$$

with $s = \ker(\phi : \lambda(D) \rightarrow \alpha(D))$ is killed by $d$ because $\ker \alpha = d^{-1} X_s \mathbb{Z}$ and $X_s \in \ker \lambda$. The multiplication by $l^n$ map applied to the left column of (B.3) gives a kernel-cokernel exact sequence

$$0 \longrightarrow \ker s[l^n] \longrightarrow \ker \phi[l^n] \longrightarrow \ker \psi[l^n] \longrightarrow \frac{\ker s}{l^n \ker s}.$$

Now $\ker \phi[l^n] \cong \text{Pic } X[l^n]$ from the middle row of (B.3), as $D^\vee$ is torsion-free, and $\text{Pic } X[l^n] \cong \text{Pic } X_\eta[l^n]$ by [Liu, 10.4.14-15]. As for $\ker \psi$, from the bottom row of (B.3)

$$0 \longrightarrow \ker \psi[l^n] \longrightarrow \text{Pic } X_\eta[l^n] \longrightarrow \frac{D^\vee}{\alpha(D)[l^n]} = \Phi[l^n],$$
the last equality by [Liu, 10/4.12]. Putting everything together, we get
\[
\text{(B.4) } \text{Pic} X_\mathbb{A}[m] \rightarrow \text{Pic} X[m] \rightarrow \ker \phi[m] \rightarrow \ker \psi[m] \rightarrow \text{Pic} X_\theta[m]
\]
with kernel and cokernel of (1) killed by $d$ and cokernel of (2) by $|\Phi|$.

Finally, Pic $X_\theta = (\text{Pic} X_\theta)^{1 \mathbb{K}}$ from the exact sequence (for any variety)
\[
H^1(G, K^s[X_\theta]^\times) \rightarrow \text{Pic} X_\theta \rightarrow (\text{Pic} X_\theta)^{1 \mathbb{K}} \rightarrow H^2(G, K^s[X_\theta]^\times),
\]
see [CTS, 1.5.0, p.386]. As $X_\theta$ is proper and geometrically integral, $K^s[X_\theta]^\times = (K^s)^\times$, so the first term is 0 by Hilbert 90, and the last one is the Brauer group, also 0 for a separably closed field.

This proves the first claim, and passing to the limit and tensoring with $\mathbb{Q}_l$ gives the other two; also, $A_\theta(m)^{1 \mathbb{K}} \cong A_s(m)$ by [ST, Lemma 2].

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