Twisted Fourier-Mukai transforms for holomorphic symplectic fourfolds

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Abstract

We apply the methods of Căldăruşu to construct a twisted Fourier-Mukai transform between a pair of holomorphic symplectic four-folds. More precisely, we obtain an equivalence between the derived category of coherent sheaves on a certain four-fold and the derived category of twisted sheaves on its ‘mirror’ partner. As corollaries, we show that the two spaces are connected by a one-parameter family of deformations through Lagrangian fibrations, and we extend the original Fourier-Mukai transform to degenerations of abelian surfaces.

1 Introduction

Matsushita [32] proved that a projective holomorphic symplectic manifold can only be fibred by (holomorphic) Lagrangian abelian varieties; his results in [32, 33], together with the results of Cho, Miyaoka, and Shepherd-Barron [12], also strongly suggest that the base of the fibration must be projective space. In [40] the author reviewed what is known about such fibrations, and speculated on what may be true. In particular, we hope to obtain a classification (up to deformation) of holomorphic symplectic manifolds via this approach.

A central problem is: can a Lagrangian fibration which does not admit a global section be deformed to one that does? This is the motivation behind the present paper, which answers the question affirmatively for a particular example (Theorem 23) while introducing ideas which should have wider applications.

We investigate holomorphic symplectic four-folds fibred by abelian surfaces. Following Altman and Kleiman [2], we associate to such a fibration $X \to B$ its compactified relative Picard scheme $P := \widehat{\text{Pic}}^0(X/B)$, which is fibred over $B$ and admits a section. We regard $P$ as the dual fibration, and the double-dual fibration is $X^0 := \widehat{\text{Pic}}^0(P/B)$. If the singular fibres of $X$ are not too bad, both $P$ and $X^0$ are well-defined and smooth. It can also happen that $X$ and $X^0$ are locally isomorphic as fibrations, and then $X$ is a torsor over $X^0$. Our goal is to

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study the relation between \( X \) and \( X^0 \), and to show that \( X \) can be deformed to \( X^0 \) via Lagrangian fibrations.

The analogous situation for elliptic surfaces was studied by Kodaira \[27, 28\], for higher dimensional elliptic fibred varieties it is known as Ogg-Shafarevich theory (see \[14\] for instance). In those cases we use the compactified relative Jacobian \( J := \text{Jac}(X/B) \) of an elliptic fibration \( X \to B \) (since elliptic curves are self-dual, we don’t need to take the double-dual fibration). The group classifying all torsors \( X \) over \( J \) is the (cohomological) analytic Brauer group \( H^2(J, \mathcal{O}^*) \) of \( J \).

Recently Căldăraru \[10\] gave a more conceptual explanation for the appearance of the Brauer group in this context: \( J \) can be interpreted as a moduli space of sheaves on \( X \), and there is a holomorphic gerbe \( \beta \in H^2(J, \mathcal{O}^*) \) obstructing the existence of a universal sheaf for the moduli problem. Then \( X \) may be regarded as \( J \) ‘twisted’ by \( \beta \). Căldăraru also defined a twisted Fourier-Mukai transform, an equivalence between the derived category of coherent sheaves on \( X \) and the derived category of sheaves twisted by \( \beta^{-1} \) on \( J \). The existence of such an equivalence is indicative of the close geometric relation between the spaces \( X \) and \( J \).

Among the examples considered by Căldăraru are elliptic K3 surfaces and Calabi-Yau three-folds. In this paper we construct a twisted Fourier-Mukai transform for holomorphic symplectic four-folds fibred by Lagrangian abelian surfaces (Theorem \[24\]). The compactified relative Picard scheme \( P \) will be, a priori, a moduli space of sheaves on the four-fold \( X \). We will show that it is smooth and holomorphic symplectic, by identifying it with another well-known holomorphic symplectic four-fold. The double-dual fibration \( X^0 \) will also be a smooth holomorphic symplectic four-fold. Moreover, \( X \) will be a torsor over \( X^0 \), and will therefore correspond to a gerbe in \( H^2(P, \mathcal{O}^*) \). By analyzing the space \( H^2(P, \mathcal{O}^*) \) of gerbes on \( P \), we will show that there is a one-parameter family of Lagrangian fibrations connecting \( X \), which does not admit a global section, to \( X^0 \), which does (Theorem \[25\]). It should be stressed that our main interest is in obtaining geometric results such as this, and the application of the more abstract theory is just a tool to help understand the structure of the fibrations.

The twisted Fourier-Mukai transform will be an equivalence between the derived category of \( X \) and the twisted derived category of \( P \). It can be regarded as a family version of the Fourier-Mukai transform between an abelian surface and its dual. The twist arises when one tries to ‘assemble’ the fibrewise transforms into a global transform. As a corollary, we extend the Fourier-Mukai transform of abelian surfaces to degenerations of abelian surfaces (Corollary \[19\]).

On hyperkähler manifolds, a holomorphic Lagrangian fibration becomes a special Lagrangian fibration after a rotation of complex structures. We expect that our equivalence is a manifestation of homological mirror symmetry in the presence of a B-field, as applied to SYZ fibrations \[44\], although the precise relation between Fourier-Mukai transforms and homological mirror symmetry is not yet understood. Similar examples of mirror pairs of hyperkähler manifolds involving B-fields were constructed by Hausel and Thaddeus \[22\], though their examples are non-compact. We expect that our methods will produce more compact examples in higher dimensions, and that these will be related to Hausel
and Thaddeus’ examples via the construction of Donagi, Ein, and Lazarsfeld (namely, the compactified Hitchin system is a degeneration of the Beauville-Mukai system).

The paper is organized as follows. In Section 2 we review results of Mukai, Bridgeland, Maciocia, and Căldăraru on Fourier-Mukai and twisted Fourier-Mukai transforms. In Section 3 we introduce a pair of holomorphic symplectic four-folds which are fibred by abelian varieties and collect together some results about them. In Section 4 we construct a twisted Fourier-Mukai transform relating the derived category and twisted derived category of the pair of four-folds from Section 3. This is followed by our applications.

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2 FM and twisted FM transforms

We begin by reviewing Mukai’s work on integral transforms between derived categories. We state Bridgeland’s criterion for when we get an equivalence of categories, and the removable singularities result of Bridgeland and Maciocia. Then we review twisted Fourier-Mukai transforms, as they appear in Căldăraru’s thesis.

2.1 Fourier-Mukai transforms

Suppose $M$ is a moduli space of semi-stable sheaves on some given space $X$. The moduli space is fine if there exists a universal sheaf $U$ on $X \times M$

\[
\begin{array}{c}
\downarrow \\
\pi_X \downarrow \\
\pi_M
\end{array}
\]

\[U \quad X \quad X \times M \quad M.
\]

Given a sheaf $E$ on $M$, we can define a sheaf on $X$ by pulling $E$ back to $X \times M$, tensoring with $U$, and pushing down to $X$. This map extends to a functor, known as an integral transform, between the bounded derived categories of coherent sheaves on $M$ and $X$, which we denote

\[\Phi^U_{M \to X} : D^b_{coh}(M) \to D^b_{coh}(X)
\]

\[E^\bullet \mapsto R\pi_{X*}(U \otimes_{M} \pi_{M*} E^\bullet).
\]
Likewise, using the dual sheaf $U^\vee$ we obtain a functor

$$\Phi_{M\rightarrow X} : D^b_{coh}(X) \rightarrow D^b_{coh}(M).$$

These functors were investigated by Mukai [34, 36] in various cases: for example, when $X$ and $M$ are dual elliptic curves, dual abelian varieties, or K3 surfaces. In certain situations we get an equivalence of triangulated categories, and the functors are then known as Fourier-Mukai transforms. This phenomenon is closely related to dualities in physics, such as mirror symmetry, and may be regarded as some kind of quantum symmetry between spaces.

The following criterion, developed by Mukai, Bondal-Orlov, and Bridgeland, tells us precisely when an integral transform is a Fourier-Mukai transform. Let $O_m$ denote the skyscraper sheaf supported at $m \in M$. Then $U_m := \Phi_{M\rightarrow X} O_m$ is the sheaf on $X$ which the point $m \in M$ parametrizes.

**Theorem 1 (Bridgeland [7])** Suppose that $X$ and $M$ are smooth and of the same dimension. The functor $\Phi_{M\rightarrow X}$ is an equivalence of triangulated categories if and only if

1. for all $m \in M$, $U_m \otimes K_X = U_m$ and $U_m$ is simple, i.e.
   $$\text{Hom}_X(U_m, U_m) = \mathbb{C},$$
2. and for all integers $i$ and for all $m_1 \neq m_2 \in M$,
   $$\text{Ext}^i_X(U_{m_1}, U_{m_2}) = 0.$$

**Remark** The conditions in the theorem are roughly akin to the requirement that $\{U_m\}_{m \in M}$ behave like an orthonormal basis with respect to the Ext•-pairing on $D^b_{coh}(X)$.

**Example** We can take $X$ to be a smooth elliptic curve $E$, and $M$ to be its dual $\hat{E}$ (the Jacobian of $E$), regarded as the moduli space of degree zero line bundles on $E$. The Poincaré line bundle provides a universal bundle. In higher dimensions, we can take an abelian variety $A$, its dual Pic$^0 A$, and the Poincaré line bundle. These are the original examples of Mukai [34].

If $X$ is a curve or surface, techniques have been developed to determine whether a particular moduli space of sheaves $M$ on $X$ is smooth. In higher dimensions, however, much less is known. A priori we may have no way of knowing whether $M$ is smooth, which is why the following result of Bridgeland and Maciocia is particularly useful. More significantly for us, it weakens slightly the conditions we need to check in order to show that an integral transform is a Fourier-Mukai transform.

**Theorem 2 (Bridgeland-Maciocia [8], Proposition 6.1)** Suppose $X$ is a smooth projective variety of dimension $n$. Let $M$ be a fine moduli space of sheaves on $X$, with $M$ an irreducible projective scheme of dimension $n$. Let $U$ be a universal sheaf, with $U_m$ defined as before. Suppose that
1. for all \( m \in M \), \( U_m \otimes K_X = U_m \) and \( U_m \) is simple,

2. for all \( m_1 \neq m_2 \in M \),

\[
\text{Hom}_X(U_{m_1}, U_{m_2}) = 0,
\]

and the closed subscheme

\[
\Gamma(U) := \{(m_1, m_2) \in M \times M | \text{Ext}^i_X(U_{m_1}, U_{m_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\}
\]

of \( M \times M \) has dimension at most \( n + 1 \).

Then \( M \) is smooth and

\[
\Phi_{M \to X} : D_{\text{coh}}^b(M) \to D_{\text{coh}}^b(X)
\]

is an equivalence of categories. In particular, \( \Gamma(U) \) must be the diagonal in \( M \times M \).

### 2.2 Twisted Fourier-Mukai transforms

So far we have considered only fine moduli spaces. There are two kinds of obstructions to the existence of a global universal sheaf: firstly, universal sheaves may not exist locally, and secondly, the local universal sheaves may not patch together into a global sheaf. We will assume that all semi-stable sheaves are actually stable, in which case the first of these obstructions can be avoided. The second obstruction was studied by Căldăraru [10]; the main results of this subsection are quoted from his thesis.

Choose an open cover \( \{M_i\} \) of \( M \) such that there exists a local universal sheaf \( U_i \) over \( X \times M_i \) for all \( i \). Since \( M \) parametrizes stable sheaves, local universal sheaves always exist on small enough open sets; indeed it suffices to take a Stein covering of \( M \). (If we include strictly semi-stable sheaves this is no longer the case, as a point in the moduli space can represent a whole \( S \)-equivalence class of sheaves.)

Consider the restrictions of the sheaves \( U_i \) and \( U_j \) to the overlap \( X \times M_{ij} = X \times (M_i \cap M_j) \).

\[
\begin{array}{ccc}
U_i & \stackrel{\pi_X}{\rightarrow} & X \times M_{ij} \\
\downarrow & & \downarrow \\
X & \stackrel{\pi_M}{\rightarrow} & M_{ij}
\end{array}
\]

Let \( m \in M_{ij} \) and let \( U_m \) be the sheaf on \( X \) represented by \( m \). Since both \( U_i \) and \( U_j \) are universal sheaves over \( X \times M_{ij} \), it follows that their restrictions to \( X \times m \) are isomorphic, as

\[
U_i|_{X \times m} \cong U_m \cong U_j|_{X \times m}.
\]

Since \( M_{ij} \) is a Stein open set, these combine to give an isomorphism

\[
\phi_{ij} : U_i|_{X \times M_{ij}} \rightarrow U_j|_{X \times M_{ij}}.
\]
Since $U_m$ is stable and hence simple, the isomorphism $U_i|_{X \times M} \cong U_j|_{X \times M}$ is given by multiplication by a non-zero number. Then an alternate way of formulating the above statement is to say that there is a line bundle $L_{ij}$ on $M_{ij}$ such that $U_i|_{X \times M_{ij}}$ is equal (not just isomorphic) to $\pi_M^* L_{ij} \otimes U_j|_{X \times M_{ij}}$. (It is important that we have equality rather than isomorphism, as $L_{ij}$ is of course isomorphic to the trivial bundle on $M_{ij}$.)

On a triple intersection $X \times M_{ijk}$ composition gives

$$
\phi_{ki} \circ \phi_{jk} \circ \phi_{ij} : U_i|_{X \times M_{ijk}} \rightarrow U_i|_{X \times M_{ijk}}.
$$

On each $X \times m$ this is once again multiplication by a non-zero number, which we denote $\beta_{ijk}(m)$. Thus the composition is determined by a section $\beta_{ijk} \in \Gamma(M_{ijk}, O^*)$. It can be shown that these sections give a 2-cocycle representing a cohomology class $\beta \in H^2(M, O^*)$.

**Definition** A (holomorphic) gerbe on $M$ up to isomorphism is an element of the (cohomological) analytic Brauer group $H^2(M, O^*)$.

**Remark** Up to isomorphism, a line bundle corresponds to an element of $H^1(M, O^*)$, and hence a gerbe up to isomorphism may be regarded as a higher dimensional analogue of a line bundle.

Gerbes themselves can be defined in various ways (see Hitchin [23] for instance). One description is simply as a 2-cocycle representing the element of $H^2(M, O^*)$. Another description is as a collection of line bundles on two-fold intersections $M_{ij}$, such as our $L_{ij}$, satisfying various properties. One of these properties is that $L_{ij} \otimes L_{jk} \otimes L_{ki}$ should be trivial, which in our case follows from

$$
U_i|_{X \times M_{ijk}} = \pi_M^* (L_{ij} \otimes L_{jk} \otimes L_{ki}) \otimes U_i|_{X \times M_{ijk}}.
$$

Choosing trivializations of each $L_{ij}$ then gives another trivialization of $L_{ij} \otimes L_{jk} \otimes L_{ki}$ which is given by the section $\beta_{ijk} \in \Gamma(M_{ijk}, O^*)$.

Note also that if we began with different local universal bundles

$$
U_i' = \pi_M^* M_i \otimes U_i,
$$

where $M_i$ are line bundles on $M_i$, then we would have obtained different line bundles

$$
L_{ij}' = M_i \otimes M_j^{-1} \otimes L_{ij}.
$$

In this case $\{L_{ij}\}$ and $\{L_{ij}'\}$ define isomorphic gerbes (this is the definition of isomorphism, and is the same as saying that two 2-cocycles represent the same class in $H^2(M, O^*)$). Now if some choice leads to $L_{ij} = O_{M_{ij}}$ for all $i$ and $j$, then the local universal sheaves agree on overlaps and can be patched together to give a global universal sheaf.

**Proposition 3 (Căldăraru [10])** Suppose that $M$ be a moduli space of stable sheaves on $X$. There is a gerbe $\beta \in H^2(M, O^*)$ (defined up to isomorphism) representing the obstruction to the existence of a global universal sheaf on $X \times M$. This is the sole obstruction: $\beta$ vanishes if and only if there exists a global universal sheaf.
Remark Henceforth everything we will say can be made to depend only on the isomorphism class of the gerbe, and therefore we will refer to gerbes up to isomorphism simply as gerbes.

Since the obstruction is completely encoded in the gerbe \( \beta \), we can construct Fourier-Mukai transforms by incorporating \( \beta \) into the construction. Specifically, this means working with twisted sheaves.

**Definition** Let \( \beta \) be a gerbe on \( M \). A \( \beta \)-twisted sheaf on \( M \) is a collection of sheaves \( F_i \) on \( M_i \) and isomorphisms \( \psi_{ij} : F_i|_{M_{ij}} \to F_j|_{M_{ij}} \) on the overlaps \( M_{ij} \) such that

1. for all \( i \) and \( j \), \( \psi_{ji} = \psi_{ij}^{-1} \),
2. and for all \( i, j, \) and \( k \), the composition
   \[ \psi_{ki} \circ \psi_{jk} \circ \psi_{ij} : F_i|_{M_{ijk}} \to F_M|_{M_{ijk}} \]
   is given by \( \beta_{ijk} \text{Id} \).

**Example** The gerbe \( \beta \in H^2(M, \mathcal{O}^*) \) can be pulled back by \( \pi_M \) to give a gerbe \( \pi_M^* \beta \) on \( X \times M \). The collection \( \{U_i\} \) of local universal sheaves with isomorphisms \( \phi_{ij} \) then gives a \( \pi_M^* \beta \)-twisted sheaf on \( X \times M \). Let us denote this twisted universal sheaf simply by \( U \), as in the untwisted case.

The category of \( \beta \)-twisted sheaves over \( M \) is an abelian category, and one can construct its derived category \( D_{\text{coh}}^b(M, \beta) \). As in the untwisted case, we can construct a functor

\[
\Phi_{M \to X}^L : D_{\text{coh}}^b(M, \beta^{-1}) \to D_{\text{coh}}^b(X)
\]

\[ \mathcal{E}^* \mapsto R\pi_X^* (U \otimes \pi_M^* \mathcal{E}^*) \).

The inverse of \( \beta \) is defined in the obvious way

\[ (\beta^{-1})_{ijk} = (\beta_{ijk})^{-1} \]

Note that \( \mathcal{E}^* \) is a complex of \( \beta^{-1} \)-twisted sheaves, so \( \pi_M^* \mathcal{E}^* \) is a complex of \( \pi_M^* \beta^{-1} \)-twisted sheaves; when we tensor it with \( U \), which is a \( \pi_M^* \beta \)-twisted sheaf, the twistings cancel each other, and hence

\[ U \otimes \pi_M^* \mathcal{E}^* \]

is a (untwisted) sheaf on \( X \times M \).

If \( \Phi_{M \to X}^L \) is an equivalence of triangulated categories it is called a twisted Fourier-Mukai transform. Căldăraru generalized Bridgeland’s criterion for when this happens. First observe that the skyscraper sheaf \( \mathcal{O}_m \) on \( M \) can be regarded as a twisted sheaf; simply choose the cover \( \{M_i\} \) so that \( m \) lies in precisely one open set, then just one local sheaf is non-vanishing and all isomorphisms are zero. As in the untwisted case, \( U_m := \Phi_{M \to X}^L \mathcal{O}_m \) is the sheaf parametrized by the point \( m \in M \).
Theorem 4 (Căldăruș [10], Theorem 3.2.1) Suppose that \( X \) and \( M \) are smooth and of the same dimension, and \( \mathcal{U} \) is a \( \pi_M^* \beta \)-twisted universal sheaf on \( X \times M \). The functor \( \Phi^M_{M \to X} \) is an equivalence of triangulated categories if and only if

1. for all \( m \in M \), \( \mathcal{U}_m \otimes K_X = \mathcal{U}_m \) and \( \mathcal{U}_m \) is simple, i.e.
   \[ \text{Hom}_X(\mathcal{U}_m, \mathcal{U}_m) = \mathbb{C}, \]

2. and for all integers \( i \) and for all \( m_1 \neq m_2 \in M \)
   \[ \text{Ext}^i_X(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0. \]

2.3 Elliptic fibrations

Căldăruș [10] discusses two main examples of twisted Fourier-Mukai transforms: when \( X \) and \( M \) are K3 surfaces, and when \( X \) and \( M \) are elliptic Calabi-Yau three-folds. We will focus on the second case, and then generalize some of the results to fibrations by abelian varieties.

Let \( p_X : X \to B \) be an elliptic fibration of arbitrary dimension. Assume for the moment that all the fibres are reduced and irreducible (i.e. either they are smooth elliptic curves, or they contain a single node or cusp).

**Definition** [D’Souza [16]] The compactified relative Jacobian \( J := \overline{\text{Jac}}(X/B) \) of \( X \) is the moduli space parametrizing families of torsion-free rank one sheaves of degree zero on the fibres of \( X \). The degree of a sheaf \( \mathcal{E} \) on a fibre \( X_t \) is defined by \( \chi(\mathcal{E}) - \chi(\mathcal{O}_{X_t}) \).

**Remark** Given a degree zero torsion-free rank one sheaf on a fibre, we can push-forward by the inclusion of the fibre in \( X \), to obtain a torsion sheaf on \( X \) itself. According to Simpson’s terminology [43], this sheaf is pure-dimensional. Since it is rank one on a fibre of \( X \), and the fibres are irreducible, a destabilizing sheaf cannot exist. A little more work shows that this actually gives an isomorphism between \( J \) and an irreducible component of the Simpson moduli space of stable sheaves on \( X \), and we will henceforth use these two descriptions interchangeably.

There is the obvious projection \( p_J : J \to B \), and \( J \) is locally isomorphic to \( X \) as a fibration: it is clear that corresponding smooth fibres of \( X \) and \( J \) are isomorphic, and in fact this is true also for singular fibres (see Section 6.3 of Căldăruș [10]). Smoothness of \( J \) now follows. Note that if we allowed \( X \) to have fibres with worse singularities then \( J \) need not be smooth. The case of a type \( I_2 \) singular fibre is studied in Section 6 of Căldăruș [10]: since it is not irreducible, the elements of \( J \) supported on such a fibre need not be stable. This leads to singularities in \( J \).

Now \( J \) can be regarded as the family of elliptic curves dual to the family \( X \to B \). The fact that \( X \) and \( J \) are locally isomorphic is then a consequence of the self-duality of elliptic curves (in higher dimensions it won’t always be true,
as abelian varieties are not always self-dual). So we have a family version of an elliptic curve and its dual. Locally we can give an explicit construction of universal sheaves by extending the Poincaré line bundle to a family over a small open set $B_i \subset B$. Note that this actually gives a local universal bundle on the fibre product $X \times B_i$, $J_i$, where $J_i := p^{-1}_i(B_i)$, but it can be push forward to $X \times J_i$ by the natural embedding. As usual, these local universal sheaves need not patch together to give a global universal sheaf: the obstruction is a gerbe $\beta \in H^2(J, \mathcal{O}^*)$.

In this example $J$ admits a canonical section (given by the trivial line bundle on each fibre of $X$) and $X$ is a torsor over $J$. Thus $X$ is isomorphic to $J$ if and only if it admits a global section. Moreover, to extend the Poincaré line bundle to a local family is essentially the same as choosing a local section of $X$; if we have a global section it can be extended globally. Such arguments lead to the next result.

**Proposition 5 (Căldăraru [10])** The following are equivalent

1. $X$ is isomorphic to $J$,
2. $X$ admits a section,
3. there is a global universal sheaf on $X \times J$,
4. $\beta$ vanishes.

Moreover, when $\beta$ does not vanish, the fibration $X$ can be reconstructed from $J$ and $\beta$ (this construction will be described in the next subsection). If $J$ is projective, then $X$ is also projective if and only if the corresponding gerbe $\beta$ is a torsion element in $H^2(M, \mathcal{O}^*)$, or equivalently, an element of the étale cohomology group $H^2_{\text{ét}}(J, \mathcal{O}^*)$ known as the (cohomological) Brauer group. Thus Căldăraru’s approach gives a conceptual explanation for the appearance of the Brauer group in Ogg-Shafarevich theory [14], where it is essentially the group classifying all (minimal, projective) elliptic fibrations with a given relative Jacobian $J$.

There are of course some subtleties: in general elliptic fibrations will have fibres which are not irreducible, nor reduced. For instance, it is not currently known whether there exists an elliptic Calabi-Yau three-fold with only irreducible fibres. Also, the following remark shows that not any element of the (cohomological) Brauer group can be used to construct a fibration $X$ from $J$.

**Remark** Gerbes on $B$ can be pulled back to $J$ by the projection $p_J$, giving an inclusion

$$H^2(B, \mathcal{O}^*) \hookrightarrow H^2(J, \mathcal{O}^*).$$

Moreover $J \to B$ admits a section so the inclusion splits. A refinement of the theory shows that the obstruction $\beta$ really lies in $H^2(J, \mathcal{O}^*)/H^2(B, \mathcal{O}^*)$. 
2.4 Abelian fibrations

The main purpose of this article is to describe twisted Fourier-Mukai transforms for fibrations by abelian surfaces. We described an elliptic fibration and its relative Jacobian as a family version of an elliptic curve and its dual. We now want to discuss the family version of a higher dimensional abelian variety and its dual.

Let $p_X : X \to B$ be a fibration by abelian varieties. Once again, difficulties arise for fibres which are ‘too’ singular, so we will assume that all fibres are reduced and irreducible (this assumption will be satisfied by our main example later on).

**Definition** [Altman-Kleiman 2] The compactified relative Picard scheme $P := \text{Pic}^0(X/B)$ of $X$ is the moduli space parametrizing families of torsion-free rank one sheaves of degree zero on fibres of $X$. We say a sheaf $\mathcal{E}$ on a fibre $X_t$ has degree zero if it is in the same connected component of $\text{Pic}(X_t)$ as $\mathcal{O}_{X_t}$.

**Remark** As with the compactified relative Jacobian of an elliptic fibration, we can take the push-forward of these sheaves under the inclusion of the fibre, and hence regard them as torsion sheaves on $X$ itself. Then they are sheaves of pure-dimension according to Simpson [43]. Since we have assumed the fibres of $X$ are reduced and irreducible, there can be no destabilizing sheaves. Thus $P$ parametrizes a family of stable sheaves on $X$.

The typical element in this family is the push-forward of a degree zero line bundle supported on a fibre of $X$. Let $Q$ be the irreducible component of the Simpson moduli space of stable sheaves on $X$ containing this typical element. One can show that we have a (holomorphic) embedding $P \hookrightarrow Q$. The following lemma implies that $P$ is actually isomorphic to $Q$, an important fact for our later applications. Thus we can (and will) identify the compactified relative Picard scheme $P$ with a component of the Simpson moduli space of stable sheaves on $X$.

Note that every sheaf in $Q$ is supported on a fibre, and thus we have a projection $p_Q : Q \to B$.

**Lemma 6** Every point $m$ of $Q$ parametrizes a sheaf $\mathcal{U}_m$ on $X$ of the form $\iota_* \mathcal{E}$, where $\iota : X_t \hookrightarrow X$ is the inclusion of a fibre ($t := p_Q(m) \in B$) and $\mathcal{E}$ is a stable rank one degree zero sheaf on $X_t$.

**Proof** This essentially follows from the assumption that the fibres of $X$ are reduced and irreducible. As above, this implies that all sheaves in $Q$ are stable, and hence local universal sheaves exist. Let $\mathcal{U}_t$ be a local universal sheaf over $X \times Q_t$, where $Q_t \subset Q$ is some open subset. Let $\Gamma$ be the graph of the projection $p_X : X \to B$ in $X \times B$. We claim that $\mathcal{U}_t$ is supported on the inverse image of
2.4 Abelian fibrations

\[ \Gamma \underbrace{\text{under the map given by the identity on } X \text{ and projection on } Q_i.} \]

\[ \downarrow \]

\[ \begin{array}{ccc}
X \times Q_i & \xrightarrow{\text{Id}_X \times p_Q} & X \times B \\
\cup & & \cup \\
(Id_X \times p_Q)^{-1}(\Gamma) & & \Gamma
\end{array} \]

The inverse image of \( \Gamma \) and the support of \( U_i \) are closed subsets cut out locally by some set of algebraic equations, and hence it suffices to check that they agree near smooth fibres. But this is obvious, as the sheaves \( U_m \) supported on a smooth fibre \( X_t \) (i.e. a smooth abelian variety) are just push-forwards of line bundles on the fibre, and hence the claim follows.

Now if \( Q_t \) is a fibre of \( Q \subset P \) (where \( t \in B \)), then it follows from the above claim that the restriction of \( U_t \) to \( X \times Q_t \) must be supported on \( X_t \times Q_t \). Since \( X_t \subset X \) is irreducible and reduced, it follows that \( U_t \mid_{X \times Q_t} \) is the push-forward of a sheaf on \( X_t \times Q_t \). The lemma now follows. \( \square \)

**Remark** Although the proof relied on the existence of a local universal sheaf, which does not exist when \( X \) contains reducible fibres (and hence \( Q \) contains strictly semi-stable sheaves), one expects the result to remain true provided no fibres of \( X \) contain non-reduced components. On the other hand, if \( X \) contained non-reduced fibres, we expect there could exist ‘fat’ sheaves supported on those fibres, i.e. sheaves which are not the push-forward of a sheaf on the reduction of the fibre. The point of the lemma is to avoid such behaviour.

Unlike in the case of an elliptic fibration, \( P \) need not be locally isomorphic to \( X \) as a fibration, for two reasons. Firstly, if the fibres of \( X \) are not principally polarized then even a smooth fibre of \( P \) need not be isomorphic to the corresponding smooth fibre of \( X \) (they will only be isogenous). Secondly, even for principal polarizations the corresponding singular fibres of \( P \) and \( X \) may not be isomorphic.

Regarding the second of these problems, let \( X_t \) be a singular fibre of \( X \). We have a description of the singular fibre \( P_t \) of \( P \) as the moduli space of stable rank one degree zero sheaves on \( X_t \). Further analysis needs to be done on a case-by-case basis, depending on the structure of \( X_t \).

The first problem can be resolved by taking the double dual of \( X \), namely \( X^0 := \text{Pic}^d(P/B) \). Clearly this is locally isomorphic to \( X \) as a fibration away from singular fibres. We will postpone further discussion of singular fibres to specific examples; at this stage let us just say that the local isomorphisms can sometimes be extended over singular fibres, and we will assume this in the following discussion.

**Remark** To construct \( X^0 \) from \( X \) in one step, we can take the relative Albanese scheme of \( X \), following Markushevich [30]. However, use of the relative Picard scheme is essential for our moduli space interpretation.
Take a (Stein) open cover \( \{ B_i \} \) of the base \( B \), such that we have (local) isomorphisms
\[
f_i : X_i \to X_i^0
\]
where \( X_i := p_X^{-1}(B_i) \) and \( X_i^0 := p_0^{-1}(B_i) \). On the overlap \( X_{ij}^0 := p_0^{-1}(B_{ij}) \), define
\[
\alpha_{ij} := f_j \circ f_i^{-1}.
\]
Then
\[
\alpha_{ij} : X_{ij}^0 \to X_{ij}^0
\]
is given by a translation in each fibre over \( B_{ij} \). Since \( X^0 \) has a canonical section, a translation is equivalent to a local section of \( p_0 : X^0 \to B \). These local sections form a 1-cocycle \( \alpha \in H^1(B, X^0) \), where we have implicitly identified \( X^0 \) with its sheaf of local sections. Moreover \( X \) can be completely recovered from \( \alpha \), so there is a bijection between torsors \( X \) over \( X^0 \) and elements of \( H^1(B, X^0) \).

By definition each point of \( X^0 \) represents a stable rank one degree zero sheaf (generically a line bundle) supported on a fibre of \( P \to B \). Therefore each local section \( \alpha_{ij} \) of \( p_0 : X^0 \to B \) determines a line bundle \( L_{ij} \) on \( P_{ij} := p_P^{-1}(B_{ij}) \), and the collection of these line bundles determines the gerbe \( \beta \in H^2(P, O^*) \). Thus we have shown the next result.

**Proposition 7** The following are equivalent

1. \( X \) is isomorphic to \( X^0 \),
2. \( X \) admits a section,
3. there is a global universal sheaf on \( X \times P \),
4. \( \beta \) vanishes.

Conversely, if the gerbe \( \beta \in H^2(P, O^*) \) can be represented by a collection of line bundles \( L_{ij} \) on \( P_{ij} \) which have degree zero when restricted to each fibre of \( P \), then the construction can be reversed. Thus in this case, there exists a torsor \( X \) over \( X^0 \) corresponding to the gerbe \( \beta \).

### 3 Holomorphic symplectic manifolds

We review some examples of holomorphic symplectic manifolds. In particular, we describe some holomorphic symplectic four-folds which are fibred by abelian surfaces, and collect together some facts about these spaces that will be used in the next section.

#### 3.1 Definition and examples

**Definition** Let \( X \) be a compact Kähler manifold. We call \( X \) a holomorphic symplectic manifold if it admits a closed non-degenerate two-form \( \sigma \) of type \((2, 0)\), i.e.
\[
\sigma \in H^0(X, \Lambda^2 T^*) = H^{2,0}(X),
\]
which we call a holomorphic symplectic form. If \( X \) is simply-connected and \( \sigma \) generates \( H^2,0(X) \cong \mathbb{C} \) then we say that \( X \) is irreducible.

If \( X \) has (complex) dimension \( 2n \) then \( \sigma^n \) trivializes the canonical bundle \( K_X \). By Yau’s theorem, \( X \) admits a hyperkähler metric; conversely, a hyperkähler manifold is holomorphic symplectic for each choice of complex structure compatible with the hyperkähler metric. By the Bogomolov decomposition theorem, a holomorphic symplectic manifold has a finite cover which is the cartesian product of a complex tori and irreducible holomorphic symplectic manifolds. In this sense, all holomorphic symplectic manifolds can be built out of irreducible ones.

In dimension two, K3 surfaces are the only irreducible examples, and they form a single family up to deformation. In dimension four, there are just two currently known examples, up to deformation.

**Example** The first higher dimensional example was discovered by Fujiki [19]. Let \( S \) be a K3 surface and \( \text{Blow}_\Delta(S \times S) \) the blow up of the diagonal. Quotienting by the involution which exchanges the two copies of \( S \) gives a smooth four-fold

\[
\text{Hilb}^2S := \text{Blow}_\Delta(S \times S)/\mathbb{Z}_2.
\]

Fujiki showed that \( \text{Hilb}^2S \) is an irreducible holomorphic symplectic four-fold. Beauville [4] generalized this example to produce an irreducible holomorphic symplectic manifold in each even dimension \( 2n \). These are the Hilbert schemes \( \text{Hilb}^nS \), which parametrizes length \( n \) zero-dimensional subschemes of \( S \), and are smooth resolutions of the symmetric products \( \text{Sym}^nS \).

By beginning with an abelian surface, instead of a K3 surface, Beauville [4] also constructed another family of examples, one in each even dimension, known as the generalized Kummer varieties. The following examples will also be important, though up to deformation they do not give us new spaces.

**Example** Let \( S \) be a K3 surface with ample divisor \( H \). The Mukai lattice is

\[
H^\bullet(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})
\]

endowed with the bilinear form

\[
((v_0,v_2,v_4),(w_0,w_2,w_4)) := \int_S -v_0w_4 + v_2w_2 - v_4w_0.
\]

The Mukai vector of a sheaf \( \mathcal{E} \), defined by \( v(\mathcal{E}) := \text{ch}(\mathcal{E})td^{1/2} \in H^\bullet(S,\mathbb{Z}) \), is a convenient way to encode the topological type of the sheaf. For example, if \( \mathcal{E} \) is a rank \( r \) vector bundle with Chern classes \( c_1 \) and \( c_2 \) then

\[
v(\mathcal{E}) = (r, c_1, r + c_1^2/2 - c_2).
\]

For fixed \( v \) in the Mukai lattice, the Mukai moduli space \( \mathcal{M}^v_H(v) \) is the moduli space of stable (with respect to \( H \)) sheaves \( \mathcal{E} \) on \( S \) with fixed Mukai vector \( v(\mathcal{E}) = v \).
Mukai [35] showed that, for general $H$, $\mathcal{M}_H(v)$ is smooth, quasi-projective, and holomorphic symplectic of dimension $2n := (v,v) + 2$. If $v$ is primitive and $v_0 > 0$ then $\mathcal{M}_H(v)$ is also compact: in fact it is an irreducible holomorphic symplectic manifold. It is also a deformation of $\text{Hilb}^n S$ (these results were proved by Göttche, Huybrechts, O’Grady, and Yoshioka; see [45]). For other choices of $v$, we can compactify to the moduli space of semi-stable sheaves $\mathcal{M}^{ss}_H(v)$, but this introduces singularities.

### 3.2 Abelian fibrations

Elliptic K3 surfaces are dense and of codimension one in the moduli space of all K3 surfaces. In higher dimensions we have the following result.

**Theorem 8 (Matsushita [32, 33])** Let $X^{2n}$ be a projective irreducible holomorphic symplectic manifold. Suppose $p_X : X \to B$ is a proper surjective morphism, whose generic fibre is connected, and with projective base $B$ of dimension strictly between $0$ and $2n$. Then

1. the generic fibre is a (holomorphic) Lagrangian abelian variety of dimension $n$,

2. the base is Fano with the same Hodge numbers as $\mathbb{P}^n$.

In particular, when $n = 2$ the base is $\mathbb{P}^2$.

It is currently an open problem whether an arbitrary holomorphic symplectic manifold can be deformed to a fibration by abelian varieties. This is possible for all the known irreducible holomorphic symplectic manifolds. In particular, for $\text{Hilb}^n S$ we have the following example.

**Example** Let $S$ be a K3 surface which contains a smooth genus $g \geq 2$ curve $C$, but is otherwise generic. Then $C$ moves in a $g$-dimensional linear system $|C| \cong \mathbb{P}^g$, and taking $H = C$ as an ample divisor gives us an embedding $S \hookrightarrow (\mathbb{P}^g)^\vee$ (unless $g = 2$, in which case we instead get a double cover of the plane). Let $Z$ be the Mukai moduli space $\mathcal{M}_H((0,[C],1))$, where $[C]$ denotes the class of $C$ in $H^2(S,\mathbb{Z})$. Then $Z$ is smooth and compact. The typical element is the push-forward of a degree $g$ line bundle on a smooth curve $D \subset [C]$. Thus $Z$ is fibred over $|C| \cong \mathbb{P}^g$, and the generic fibre is a smooth abelian variety of dimension $g$, namely the degree $g$ Picard group $\text{Pic}^g D$ of a smooth genus $g$ curve.

$$\text{Pic}^g \hookrightarrow Z \downarrow \quad |C| \cong \mathbb{P}^g$$

Let us argue that $Z$ is birational to $\text{Hilb}^g S$. A generic element of $Z$ gives a generic degree $g$ line bundle on a smooth genus $g$ curve $D \subset S$. This line bundle will have a unique section, up to scale, which will vanish at precisely $g$ distinct points. This gives us a rational map $Z \dasharrow \text{Hilb}^g S$. 
In the other direction, a generic element of $\text{Hilb}^g S$ consisting of $g$ distinct points determines a hyperplane in $(\mathbb{P}^g)^\vee$. For $g > 2$ this hyperplane cuts $S \subset (\mathbb{P}^g)^\vee$ in a smooth curve of the linear system $|C|$; for $g = 2$ it can be pulled back from $(\mathbb{P}^2)^\vee$ to give such a curve. Moreover, the $g$ points lie on the curve and determine a degree $g$ line bundle. Thus we obtain a rational map $\text{Hilb}^g S \rightarrow Z$.

This example was used by Beauville \cite{Beauville} to count the number of rational curves (with nodes) in each linear system $|C|$. Note that as in the first remark in Section 2.3, $Z$ can be identified with the compactified relative Jacobian $\text{Pic}^G(C/|C|)$ (see D’Souza \cite{DSouza}, Altman, Iarrobino, and Kleiman \cite{AIK}) of the family of curves $C \rightarrow |C|$ in the linear system. Huybrechts \cite{Huybrechts} showed that birational holomorphic symplectic manifolds have the same periods and represent non-separated points in the moduli space, which is non-Hausdorff. It follows that $Z$ and $\text{Hilb}^g S$ are also deformation equivalent.

Debarre \cite{Debarre} used similar methods to Beauville to count the number of genus two curves (with nodes) in a linear system on an abelian surface. This included showing that the generalized Kummer varieties can be deformed to fibrations by abelian varieties (see also Example 3.8 in Sawon \cite{Sawon}).

### 3.3 More about $Z$

In this subsection we concentrate on the fibrations by abelian surfaces which are deformations of $\text{Hilb}^2(S)$. We collect together some facts that will be of use in the next section.

Let $S \rightarrow (\mathbb{P}^2)^\vee$ be a hyperelliptic K3, i.e. a double cover of the plane ramified over a sextic $\delta$. We will assume that $\delta$ is generic; in particular it does not admit a tritangent (this is a codimension one condition on the space of plane sextics). The pull-back of a generic line in $(\mathbb{P}^2)^\vee$ gives a smooth genus two curve $C \subset S$, whose linear system is the $\mathbb{P}^2$ dual to $(\mathbb{P}^2)^\vee$. We also take $H = C$ as an ample divisor on $S$. Let us use $Z^2$ to denote the moduli space $Z = \mathcal{M}_H^2((0, [C], 1))$ described in the previous subsection. Thus $Z^2$ is a four-fold fibred by abelian surfaces.

$$
\text{Pic}^2 \hookrightarrow Z^2 \\
\downarrow \\
|C| \cong \mathbb{P}^2
$$

For all $d \in \mathbb{Z}$, we can construct a similar fibration $Z^d$, whose generic fibre is the degree $d$ Picard group $\text{Pic}^d D$ of a smooth genus two curve $D \subset |C|$.

$$
\text{Pic}^d \hookrightarrow Z^d \\
\downarrow \\
|C| \cong \mathbb{P}^2
$$

Indeed, $Z^d$ is just the Mukai moduli space $\mathcal{M}_H^d((0, [C], k - 1))$. Since the Mukai vectors are primitive, $Z^d$ are smooth compact irreducible holomorphic symplectic four-folds, deformation equivalent to $\text{Hilb}^2(S)$. 
As in the first remark in Section 2.3, $Z^d$ can be identified with the compactified relative Jacobian $\text{Pic}^d(C\setminus|C|)$ of the family of curves $C \to |C|$, and Markushevich [29, 30] gave explicit constructions of $Z^0$ and $Z^1$ via this approach (we will see shortly that the other spaces $Z^d$ are all isomorphic to one of these two). Note that the curves in $|C|$ are

1. smooth genus two curves, generically; pull-backs of lines in $(\mathbb{P}^2)^\vee$ meeting $\delta$ transversely,

2. genus one curves with one node, in codimension one; pull-backs of lines tangent to $\delta$ at precisely one point,

3. genus one curves with one cusp, in codimension two; pull-backs of flex lines of $\delta$,

4. and rational curves with two nodes, in codimension two; pull-backs of bitangents to $\delta$.

The singular curves sit above the curve $\Delta \subset \mathbb{P}^2$ dual to $\delta$. By the Plücker formulae [20] $\Delta$ is a degree 30 curve with 72 cusps and 324 nodes. The two kinds of most singular fibres sit above the cusps and nodes respectively.

We see in particular that all the curves are integral (i.e. reduced and irreducible). Altman, Iarrobino, and Kleiman [1] proved that for families of integral curves embedded in surfaces, the fibres of $\text{Pic}^d(C\setminus|C|)$ are always irreducible. In fact, this follows from the weaker result of D’Souza [16] since our curves have at worst nodes or cusps as singularities; D’Souza also shows that the fibres are reduced and equidimensional in this case. In fact we will give an explicit description of all the fibres.

Clearly for a smooth genus two curve $D$, of type (1), the Jacobian $\text{Pic}^dD$ is already compact, and is a smooth abelian surface. For type (2) we have the following description.

**Lemma 9** Let $D$ be a genus one curve with one node $r$, i.e. of arithmetic genus two. Let $\pi : \tilde{D} \to D$ be the normalization of $D$, and let $p$ and $q$ be the two points of $\tilde{D}$ which are identified by $\pi$. Let $\mathcal{L}$ be the Poincaré line bundle on $\tilde{D} \times \text{Pic}^0\tilde{D}$, and let $\mathcal{L}_p$ and $\mathcal{L}_q$ be the restrictions to $\{p\} \times \text{Pic}^0\tilde{D}$ and $\{q\} \times \text{Pic}^0\tilde{D}$ respectively. Then $\text{Pic}^0\tilde{D}$ is given by taking the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{L}_p \oplus \mathcal{L}_q)$ over $\text{Pic}^0\tilde{D}$ and identifying $s_0 := \mathbb{P}(\mathcal{L}_p) \cong \text{Pic}^0\tilde{D}$ and $s_\infty := \mathbb{P}(\mathcal{L}_q) \cong \text{Pic}^0\tilde{D}$ with a translation by $\mathcal{O}(p-q)$. (Note that $\mathcal{O}(p-q)$ is a point on $\text{Pic}^0\tilde{D}$, by which we can translate.)

**Proof** This is Example (1) on page 83 of [37]. First let us make one general comment: a torsion-free rank one sheaf on a curve must be locally free at all smooth points, and at a singular point $r$ it is either locally free or isomorphic to the maximal ideal $\mathfrak{m}_r$.

Consider first the locally free case. A line bundle on $D$ is given by a line bundle on $\tilde{D}$ plus ‘gluing data’: an element of $\mathbb{C}^*$ describing how the fibres at $p$
and $q$ are to be identified. Thus we have the extension

$$0 \rightarrow G_m \rightarrow \text{Pic}^0 D \rightarrow \text{Pic}^0 \tilde{D} \rightarrow 0$$

as in pages 247-253 of [21].

Another description, which makes it clearer how to compactify, is to begin with a degree zero line bundle $L$ on $\tilde{D}$ and push-forward by $\pi$. The resulting sheaf $\pi_\ast L$ has fibre $L_p \oplus L_q$ at the node $r$. Let $L_0$ be a one-dimensional subspace of $L_p \oplus L_q$ and let $\xi$ be the composition

$$\pi_\ast L \rightarrow L_p \oplus L_q \rightarrow L_p \oplus L_q / L_0$$

where here we regard $L_p$, $L_q$, and $L_0$ as skyscraper sheaves supported at $r$. Then $\ker\xi$ is a torsion-free rank one degree zero sheaf on $D$.

The line bundle on $\text{Pic}^0 \tilde{D}$ whose fibre over $L$ is $L_p \otimes L_q^\vee$ is precisely $L_p \otimes L_q$, and hence the one-dimensional subspace $L_0$ is really a point in the fibre of $\mathbb{P}(L_p \oplus L_q)$ over $L$. The above description therefore leads to the $\mathbb{P}^1$-bundle $\mathbb{P}(L_p \oplus L_q)$ over $\text{Pic}^0 \tilde{D}$. This is not quite the compactified Jacobian of $D$, but rather the scheme representing the presentation functor $[3]$; in fact, it is the normalization $\tilde{\text{Pic}}^0 D := \tilde{\text{Pic}}^0 D$ of the compactified Jacobian of $D$, as we now explain.

If $L_0 = L_p$, then the above composition becomes

$$\pi_\ast L \rightarrow L_q$$

and $\ker\xi = \pi_\ast (L(-q))$. For another line bundle $L'$ on $\tilde{D}$, and $L'_0 = L'_q$, we get $\pi_\ast (L'(-p))$; this is clearly isomorphic to $\pi_\ast (L(-q))$ when $L' \cong L \otimes \mathcal{O}(p - q)$. These sheaves represent the zero and infinity sections $s_0$ and $s_\infty$, and we see that they are glued with a translation by $\mathcal{O}(p - q)$.

Next consider type (3) curves.

**Lemma 10** Let $D$ be a genus one curve with one cusp $r$, i.e. of arithmetic genus two. Let $\pi : \tilde{D} \rightarrow D$ be the normalization of $D$, and let $p \in \tilde{D}$ be the preimage of $r$. Let $\mathcal{L}$ be the Poincaré line bundle on $D \times \text{Pic}^0 \tilde{D}$ and $\mathcal{L}_p$ the restrictions to $\{p\} \times \text{Pic}^0 \tilde{D}$. Let $J^1 \mathcal{L}_p$ denote the first jet bundle of $\mathcal{L}_p$; there is an exact sequence

$$0 \rightarrow \Omega^1_{\text{Pic}^0 \tilde{D}} \otimes \mathcal{L}_p \rightarrow J^1 \mathcal{L}_p \rightarrow \mathcal{L}_p \rightarrow 0.$$

Then $\tilde{\text{Pic}}^0 D$ is given by taking the $\mathbb{P}^1$-bundle $\mathbb{P}(J^1 \mathcal{L}_p)$ over $\text{Pic}^0 \tilde{D}$ and contracting along $s_\infty := \mathbb{P}(\Omega^1 \otimes \mathcal{L}_p) \cong \text{Pic}^0 \tilde{D}$ in a certain direction, to produce a locus of cusps. Note that the contraction is not pure in the direction of the fibres.

**Proof** This is Theorem 10 of [20] (see also [3]). In the locally free case, we have an extension [21]

$$0 \rightarrow G_m \rightarrow \text{Pic}^0 D \rightarrow \text{Pic}^0 \tilde{D} \rightarrow 0$$
which once again splits as a sequence of abelian groups \[5\].

We can also get a description of the compactified Jacobian by beginning with a degree zero line bundle \(L\) on \(D\) and pushing forward by \(\pi\). The resulting sheaf \(\pi_* L\) has fibre \(J^1 L_p\) at the node \(r\). Let \(L_0\) be a one-dimensional subspace of \(J^1 L_p\) and let \(\xi\) be the composition

\[
\pi_* L \xrightarrow{\ev} J^1 L_p \to J^1 L_p / L_0.
\]

Then \(\ker \xi\) is a torsion-free rank one degree zero sheaf on \(D\). This leads to the description of the normalization \(\Pic D\) of the compactified Jacobian of \(D\) as the \(\mathbb{P}^1\)-bundle \(\mathbb{P}(J^1 L_p)\) over \(\Pic^0 D\). As with the previous lemma, some further identifications need to be made, and these amount to contracting along the locus \(s_\infty\) in a certain direction. The precise direction is described in Kleiman’s paper \[26\].

Finally, consider type (4) curves.

Lemma 11 Let \(D\) be a rational curve with two nodes \(r_1\) and \(r_2\), i.e. of arithmetic genus two. Let \(\pi : \tilde{D} \to D\) be the normalization of \(D\), and let \(\{p_1, q_1\}\) and \(\{p_2, q_2\}\) be the pairs of points of \(\tilde{D}\) which are identified by \(\pi\). Since \(\tilde{D} \cong \mathbb{P}^1\), we can define \(\lambda\) to be the cross-ratio of the four points \(\{p_1, q_1, p_2, q_2\}\). Note that multiplication by \(\lambda\) gives an isomorphism \(\mathbb{P}^1 \to \mathbb{P}^1\) which fixes 0 and \(\infty\).

Then \(\Pic^0 D\) is given by taking \(\mathbb{P}^1 \times \mathbb{P}^1\) and identifying \(s_0' := \{0\} \times \mathbb{P}^1\) and \(s'_\infty := \{\infty\} \times \mathbb{P}^1\) via multiplication by \(\lambda\), and identifying \(s_0 := \mathbb{P}^1 \times \{0\}\) and \(s'_\infty := \mathbb{P}^1 \times \{\infty\}\) via multiplication by \(\lambda\).

Proof This is Example (2) on pages 83-84 of \[27\]. A degree zero line bundle on \(\tilde{D} \cong \mathbb{P}^1\) is necessarily isomorphic to the trivial bundle. In the locally free case, the gluing data is a point in \(\mathbb{C}^* \times \mathbb{C}^*\), which tells us how to identify the fibres at \(p_1\) and \(q_1\), and those at \(p_2\) and \(q_2\), to get a line bundle on \(D\). Thus

\[
\Pic^0 D \cong \mathbb{G}_m \times \mathbb{G}_m.
\]

To see how to compactify this, begin with a degree zero line bundle \(L\) on \(\tilde{D}\) (isomorphic to the trivial bundle) and push-forward by \(\pi\). The resulting sheaf \(\pi_* L\) has fibres \(L_{p_1} \oplus L_{q_1}\) and \(L_{p_2} \oplus L_{q_2}\) at the nodes \(r_1\) and \(r_2\), respectively. Let \(L_{01}\) and \(L_{02}\) be one-dimensional subspaces of \(L_{p_1} \oplus L_{q_1}\) and \(L_{p_2} \oplus L_{q_2}\), respectively, and let \(\xi\) be the composition

\[
\pi_* L \xrightarrow{\ev_{r_1} \oplus \ev_{r_2}} (L_{p_1} \oplus L_{q_1}) \oplus (L_{p_2} \oplus L_{q_2}) \to (L_{p_1} \oplus L_{q_1} / L_{01}) \oplus (L_{p_2} \oplus L_{q_2} / L_{02}).
\]

Then \(\ker \xi\) is a torsion-free rank one degree zero sheaf on \(D\). Thus we see that the normalization \(\tilde{\Pic}^0 D\) of the compactified Jacobian of \(D\) is

\[
\Pic(L_{p_1} \oplus L_{q_1}) \oplus \Pic(L_{p_2} \oplus L_{q_2}) \cong \mathbb{P}^1 \times \mathbb{P}^1.
\]

Now if \(L_{01} = L_{p_1}\), we find \(\ker \xi = \pi_* (L(-q_1))\), and similarly if \(L'_{01} = L'_{q_1}\), then \(\ker \xi = \pi_* (L'(-p_1))\). Clearly \(L'\) is always isomorphic to \(L \otimes \mathcal{O}(p_1 - q_1)\), as both
are degree one line bundles on $\tilde{D} \cong \mathbb{P}^1$. However, tensoring with $\mathcal{O}(p_1 - q_1)$ changes the other $\mathbb{P}^1 \cong \mathbb{P}(L_{p_2} \oplus L_{q_2})$ factor. More specifically, if we let $f$ be a meromorphic function on $\mathbb{P}^1$ vanishing at $p_1$ and with a simple pole at $q_1$, so that $p_1 - q_1 = (f)$, then

$$L_{02}' \subset L_{p_2}' \oplus L_{q_2}' = (L \otimes \mathcal{O}((f)))_{p_2} \oplus (L \otimes \mathcal{O}((f)))_{q_2}.$$ 

Hence if $L_{02}'$ corresponds to $w \in \mathbb{P}^1$, then $L_{02}$ corresponds to $\lambda w \in \mathbb{P}^1$, where $\lambda = f(p_2)/f(q_2)$ is the cross-ratio of the four points $\{p_1, q_1, p_2, q_2\}$. The same factor arises when we glue $s_0^2$ to $s_{\infty}^2$. This completes the proof. □

**Remark** All of the compactified Jacobians described above are irreducible (as required by [1]). Therefore the isomorphism

$$\text{Pic}^0 D \to \text{Pic}^d D$$

given by tensoring with some fixed degree $d$ line bundle extends to an isomorphism

$$\overline{\text{Pic}}^0 D \to \overline{\text{Pic}}^d D$$

and we have a description of the compactified Picard schemes of all degrees.

**Remark** Only the compactified Jacobian of type (4) has non-zero Euler characteristic, equal to one. Therefore only these fibres make a non-trivial contribution to the Euler characteristic of $Z^d$, which is therefore 324 (see Beauville [5] for how this method can be used to calculate the Euler characteristic of $\text{Hilb}^n S$).

**Remark** A fact that we will use later is that $Z^d \to \mathbb{P}^2$ is a flat fibration. This follows from the corollary after Theorem 23.1 in Matsumura’s book [31], since $Z^d$ is smooth and the fibres are equidimensional.

We can use the first of the above remarks to prove the next lemma.

**Lemma 12** All the spaces $Z^{2m}$ (for $m \in \mathbb{Z}$) are isomorphic and all the spaces $Z^{2m+1}$ (for $m \in \mathbb{Z}$) are isomorphic, so we essentially just have $Z^0$ and $Z^1$. Moreover, $Z^1$ is a torsor over $Z^0$.

**Proof** First note that $Z^0$ admits a global section, given by taking the trivial (degree zero) line bundle $\mathcal{O}_D$ on each curve $D \in |C|$. Since the fibres of $Z^d$ are reduced and irreducible, choosing a local section of $\mathcal{C}$ over $U \subset |C|$ will give a local isomorphism $Z^d/U \to Z^{d+1}/U$, and hence all the spaces $Z^d$ are locally isomorphic as fibrations. In particular, this means $Z^d$ is a torsor over $Z^0$.

Now for all $d \in \mathbb{Z}$ there is a global isomorphism

$$Z^d \cong Z^{d+2}$$

which over $D \in |C|$ is given by tensoring stable sheaves with the canonical bundle $K_D$ (of degree two). This completes the proof. □
**Proposition 13**  For a generic hyperelliptic K3 surface $S$ (i.e. the sextic $\delta$ is generic), the spaces $Z^0$ and $Z^1$ are not isomorphic. Indeed they have different periods, and hence are not even birational.

**Proof**  We will use O’Grady’s description [38] of the weight two Hodge structure of the Mukai moduli space $\mathcal{M}_H^s(v)$ to show that $Z^0$ and $Z^1$ have non-isomorphic Picard lattices, and hence different periods. The Mukai lattice $H^\bullet(S,\mathbb{Z})$ can be given the Hodge structure whose $(2,0)$, $(1,1)$, and $(0,2)$ components are

$$H^{2,0}(S), \quad H^{0,0}(S) \oplus H^{1,1}(S) \oplus H^{2,2}(S), \quad \text{and} \quad H^{0,2}(S)$$

respectively. Then O’Grady proved that for $(v,v) > 0$, the weight two Hodge structure of $\mathcal{M}_H^s(v)$ is isomorphic to $v^\perp$ (and the induced quadratic form agrees with the Beauville-Bogomolov quadratic form on $H^2(\mathcal{M}_H^s(v),\mathbb{Z})$). For generic $S$ we can assume that the Picard lattice $H^{1,1}(S) \cap H^2(S,\mathbb{Z})$ is generated by $[C]$. Then the Picard lattice of $\mathcal{M}_H^s(v)$ is isomorphic to

$$\{ (a,b,c) \in \mathbb{Z}^3 | (a,b[C],c) \in v^\perp \}.$$

For the particular spaces we are interested in, we find the Picard lattices of $Z^0 = \mathcal{M}_H^s((0,\lfloor C \rfloor, -1))$ and $Z^1 = \mathcal{M}_H^s((0,\lfloor C \rfloor, 0))$ are

$$\mathbb{Z}(-2,\lfloor C \rfloor, 0) \oplus \mathbb{Z}(0,0,1)$$

and

$$\mathbb{Z}(-1,0,0) \oplus \mathbb{Z}(0,0,1)$$

respectively, and the induced quadratic forms

$$\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are easily seen to be non-isomorphic (only the second is unimodular). □

**Corollary 14**  Generically, $Z^1$ does not admit a section. It does, however, admit a rational 2-valued section.

**Proof**  If $Z^1$ admitted a section it would be isomorphic to $Z^0$. A 2-valued section is a pair of degree one line bundles on each curve $D \in |C|$ (since we will construct a rational 2-valued section, we can avoid the singular fibres, and therefore talk about genuine line bundles rather than stable rank one sheaves). Note that a degree two line bundle (such as $K_D$) on each curve does not canonically give a pair of degree one line bundles. However, by adjunction

$$K_D = \mathcal{O}(D)|_D = \mathcal{O}(C)|_D$$

as $K_S$ is trivial. In other words, $C|_D$ is just a pair of points on each curve $D$, giving us a pair of degree one line bundles as required. □
Remark This is only a rational 2-valued section as obviously \( C \) does not intersect itself transversely. A genuine 2-valued section appears not to exist, though this will not present any difficulties. From the discussion in Section 2, we know that there should exist an element \( \alpha \in H^1(\mathbb{P}^2, Z^0) \) classifying the torsor \( Z^1 \). The fact that \( Z^1 \) admits a rational 2-valued section means that \( \alpha \) should be 2-torsion, at least in étale cohomology \( H^1_{\text{\acute{e}t}}(\mathbb{P}^2, Z^0) \). This element was constructed explicitly by Markushevich in [30].

Remark An example of a non-generic hyperelliptic K3 surface \( S \) arises when the sextic admits a tritangent: this condition is codimension one on the space of sextics. In this case the pull-back of the tritangent to \( S \) gives a reducible curve, consisting of two rational curves \( C_1 \) and \( C_2 \) which intersect transversely at three points. Let us assume that \( S \) is otherwise as generic as possible, so that its Picard lattice is generated by \([C_1]\) and \([C_2]\) (since \([C] = [C_1] + [C_2]\) and \(C.C = 2\), \([C_1]\) and \([C_2]\) cannot be proportional). The Picard lattice of \( \mathcal{M}_H(v) \) is therefore isomorphic to

\[
\{(a, b, c, d) \in \mathbb{Z}^4 | (a, b[C_1] + c[C_2], d) \in v^+ \}\.
\]

A bit of work shows that now \( Z^0 \) and \( Z^1 \) have isomorphic Picard lattices, with quadratic form

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -10
\end{pmatrix}
\]

with respect to the bases

\[
\mathbb{Z}(-1, [C_1], 1) \oplus \mathbb{Z}(-5, 4[C_1] + [C_2], 1) \oplus \mathbb{Z}(-10, 9[C_1] + [C_2], 5)
\]

and

\[
\mathbb{Z}(-1, 0, 0) \oplus \mathbb{Z}(0, 0, 1) \oplus \mathbb{Z}(0, [C_1] - [C_2], 0)
\]

respectively.

In this case, there is a birational map \( Z^0 \dashrightarrow Z^1 \) inducing the isomorphism of periods: since \( C_1.C = 1 \) the restriction of the line bundle \( \mathcal{O}(C_1) \) on \( S \) to a generic curve \( D \in [C] \) is a degree one line bundle, and tensoring with this line bundle induces the birational map. This line bundle also provides a (birational) section of \( Z^1 \).

4 Derived equivalences

In Section 3 we described a certain four-fold \( Z^0 \) which is fibred by abelian surfaces, as well as another fibration \( Z^1 \) which is a torsor over \( Z^0 \). In this section we will reinterpret this torsor via moduli spaces and twisted Fourier-Mukai transforms. In particular, we will show that the derived category of sheaves on \( Z^1 \) is equivalent to the derived category of twisted sheaves on \( Z^0 \).
4.1 The dual fibration

Since $Z^1$ is fibred over $\mathbb{P}^2$, we can define its compactified relative Picard scheme, which we regard as a dual fibration. Our first goal is to understand this space. Firstly, observe that $Z^0$ and $Z^1$ are locally isomorphic as fibrations, and therefore their dual fibrations $\overline{\text{Pic}}^0(Z^0/\mathbb{P}^2)$ and $\overline{\text{Pic}}^0(Z^1/\mathbb{P}^2)$ must also be locally isomorphic. Since these dual fibrations both admit canonical global sections (given by the trivial bundle on each fibre of $Z^0$, respectively $Z^1$), we must have

$$\overline{\text{Pic}}^0(Z^0/\mathbb{P}^2) \cong \overline{\text{Pic}}^0(Z^1/\mathbb{P}^2).$$

In this section we will define $P$ to be $\overline{\text{Pic}}^0(Z^0/\mathbb{P}^2)$, and we will show that $P$ is also isomorphic to $Z^0$; thus $Z^0$ is self-dual. By the above observations, this also implies that $Z^0$ is the dual fibration of $Z^1$ (the operation of taking the dual fibration is only locally reflexive). Thus we will interpret $Z^0$, which is a priori a moduli space of sheaves on the K3 surface $S$, as also being a moduli space of stable sheaves on the four-fold $Z^1$.

It is clear that corresponding smooth fibres of $Z^0$ and $P$ are isomorphic: if $\text{Pic}^0 D$ is a smooth fibre of $Z^0$ (i.e. $D$ is of type (1), a smooth genus two curve), then the corresponding fibre of $P$ is

$$\text{Pic}^0((\text{Pic}^0 D) \cong \text{Pic}^0 D.$$ 

This result also extends to the singular fibres by the following autoduality result of Esteves and Kleiman. Firstly, since the fibres of $Z^0$ are reduced and irreducible, recall that by Lemma 6, the fibres of $P$ are just the compactified Picard schemes of the fibres of $Z^0$.

**Theorem 15 (Esteves and Kleiman [18])** Let $D$ be a surficial curve with at worst nodes and/or cusps as singularities. Then the compactified Picard scheme of the compactified Jacobian of $D$ is isomorphic to the compactified Jacobian of $D$, i.e.

$$\overline{\text{Pic}}^0(\text{Pic}^0 D) \cong \overline{\text{Pic}}^0 D.$$ 

**Remark** In an earlier paper Esteves, Gagné, and Kleiman [17] proved that if $D$ is a surficial curve with at worst double points as singularities (which includes nodes or cusps), then

$$\text{Pic}^0((\text{Pic}^0 D) \cong \text{Pic}^0 D.$$ 

Now it is not immediately true that the same result must hold for compactified Picard schemes. A priori, a compactified Picard scheme could consist of several irreducible components, with whole components parametrizing torsion-free rank one sheaves none of which are locally free. Such examples even exist (see Altman, Iarrobino, and Kleiman [1] for an example of a space curve whose compactified Jacobian has this property), but only in the presence of particularly bad kinds of singularities. It was also shown in [1] that the compactified Jacobian of a surficial curve must be irreducible. So when the singularities are mild, as in the theorem, one expects much better behaviour.
4.1 The dual fibration

In particular, Theorem 15 applies to our curves of type (2), (3), and (4). Let us take a closer look at the type (2) case.

**Lemma 16** If \( D \) is a curve of type (2), then the compactified Picard scheme of the compactified Jacobian of \( D \) is isomorphic to the compactified Jacobian of \( D \), i.e.

\[
\overline{\text{Pic}}^0 (\text{Pic}^0 D) \cong \text{Pic}^0 D.
\]

**Proof** Our argument will identify the two spaces as sets. More care needs to be taken to prove that their analytic structures agree (for instance, one could exhibit a universal sheaf for the moduli problem); for this we refer to Esteves and Kleiman’s proof of the more general Theorem 15.

We begin by simplifying the description of \( \overline{\text{Pic}}^0 D \) given in Lemma 9. Pick an arbitrary point \( p_0 \) in the normalization \( \tilde{D} \). Then we have an isomorphism \( \tilde{D} \cong \overline{\text{Pic}}^0 \tilde{D} \) given by taking the point \( p \in \tilde{D} \) to \( \mathcal{O}(p - p_0) \). Under this isomorphism the bundle \( L_{p_0} = L|_{\{p\} \times \overline{\text{Pic}}^0 \tilde{D}} \) on \( \overline{\text{Pic}}^0 \tilde{D} \) becomes the bundle \( L|_{\tilde{D} \times \{\mathcal{O}(p - p_0)\}} = \mathcal{O}(p - p_0) \) on \( \tilde{D} \). Translation by \( \mathcal{O}(p - q) = \mathcal{O}(p - p_0) \otimes \mathcal{O}(q - p_0) \) on \( \overline{\text{Pic}}^0 \tilde{D} \) becomes translation by \( p - q \) on \( \tilde{D} \). Thus the compactified Jacobian of \( D \) is isomorphic to \( \mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O}(q)) \) over \( \tilde{D} \) with \( s_0 := \mathbb{P}(\mathcal{O}(p)) \cong \tilde{D} \) identified with a translation by \( p - q \). The normalization \( \tilde{D} \) of the compactified Jacobian is isomorphic to \( \mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O}(q)) \) itself.

Now consider \( \text{Pic}^0(\overline{\text{Pic}}^0 D) \). We will begin with a line bundle \( \mathcal{E} \) on the normalization \( \tilde{\text{Pic}}^0 D \), and then glue \( \mathcal{E}|_{s_0} \) to \( \mathcal{E}|_{s_\infty} \) to obtain a sheaf on \( \overline{\text{Pic}}^0 D \). We choose \( \mathcal{E} \) to be the pull-back \( \gamma^* L \) of a degree zero line bundle \( L \) on \( \tilde{D} \), under the projection

\[
\gamma : \mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O}(q)) \to \tilde{D}.
\]

Both \( s_0 \) and \( s_\infty \) are identified with \( \tilde{D} \), by the map \( \gamma \); therefore

\[
\mathcal{E}|_{s_0} = (\gamma|_{s_0})^* L
\]

and

\[
\mathcal{E}|_{s_\infty} = (\gamma|_{s_\infty})^* L.
\]

However, \( s_0 \) and \( s_\infty \) are not glued by \( (\gamma|_{s_\infty})^{-1} \circ (\gamma|_{s_0}) \), but rather by

\[
\tau := (\gamma|_{s_\infty})^{-1} \circ \text{tr} \circ (\gamma|_{s_0}) : s_0 \to s_\infty
\]

where \( \text{tr} \) is translation by \( p - q \). Any degree zero line bundle on \( \tilde{D} \) can be written as \( \mathcal{O}(a - b) \) for some points \( a \) and \( b \in \tilde{D} \); therefore

\[
\text{tr}^* \mathcal{O}(a - b) \cong \mathcal{O}((a + p - q) - (b + p - q)) \cong \mathcal{O}(a - b).
\]
In particular, $\operatorname{tr}^*L \cong L$, and so we still have

$$\tau^*(\mathcal{E}|_{s_\infty}) \cong \mathcal{E}|_{s_0}.$$  

Choosing a non-zero element of

$$\operatorname{Hom}_{s_0}(\mathcal{E}|_{s_0}, \tau^*(\mathcal{E}|_{s_\infty})) \cong \operatorname{Hom}_D(L, \operatorname{tr}^*L) \cong \mathbb{C}$$

thus enables us to glue $\mathcal{E}|_{s_0}$ to $\mathcal{E}|_{s_\infty}$, giving a rank-one locally free sheaf on $\widetilde{\operatorname{Pic}}^0 D$. Thus we have described the Picard scheme of $\widetilde{\operatorname{Pic}}^0 D$ as a $\mathbb{C}^*$-bundle over $\operatorname{Pic}^0 \widetilde{D}$. It will become clear when we compactify that this is, in fact, the same as the $\mathbb{C}^*$-bundle arising in the description of $\operatorname{Pic}^0 D$, and thus

$$\operatorname{Pic}^0(\widetilde{\operatorname{Pic}}^0 D) \cong \operatorname{Pic}^0 D$$

as required by [17].

To compactify, we imitate the proofs of Lemmas 9, 10, and 11. First push $\mathcal{E}$ forward under the normalization map

$$\pi : \widetilde{\operatorname{Pic}}^0 D \to \operatorname{Pic}^0 D.$$  

Over the singular locus $s \cong \widetilde{D}$, $\mathcal{E}$ looks like

$$\mathcal{E}|_{s_0} \oplus \tau^*(\mathcal{E}|_{s_\infty})$$

on $s_0$, or equivalently, $L \oplus \operatorname{tr}^*L$ on $\widetilde{D}$. Choose a degree zero line subbundle $L_0$ of $L \oplus \operatorname{tr}^*L$. Let $\xi$ be the composition

$$\pi_*\mathcal{E} \to L \oplus \operatorname{tr}^*L \to L \oplus \operatorname{tr}^*L/L_0$$

where we regard $L \oplus \operatorname{tr}^*L$ and $L_0$ as torsion sheaves supported on $s$. Then ker$\xi$ is a torsion-free rank one sheaf on $\operatorname{Pic}^0 D$, which is clearly in the same connected component as the trivial sheaf.

Since $\operatorname{tr}^*L \cong L$, we must have $L_0 \cong L$. Thus the line subbundle $L_0$ is completely determined by examining the fibres over a single point in $\widetilde{D}$. Choosing this point to be $p$, we see that

$$(L_0)_p \subset (L \oplus \operatorname{tr}^*L)_p = L_p \oplus L_q$$

is given by a point of $\mathbb{P}(L_p \oplus L_q)$. Therefore we arrive at a description of the normalization

$$\widetilde{\operatorname{Pic}}^0(\operatorname{Pic}^0 D) := \widetilde{\operatorname{Pic}}^0(\operatorname{Pic}^0 D)$$

of the compactified Jacobian of $\operatorname{Pic}^0 D$, namely it is the total space of the $\mathbb{P}^1$-bundle $\mathbb{P}(L_p \oplus L_q)$ over $\operatorname{Pic}^0 \widetilde{D}$. In particular, comparing to Lemma 9 we see that

$$\operatorname{Pic}^0(\operatorname{Pic}^0 D) \cong \operatorname{Pic}^0 D.$$

Finally, we need to identify some points in \( \tilde{\text{Pic}}^0 (\text{Pic}^0 D) \). If we choose \((L_0)_p\) to be \(L_p\) then we find

\[
\ker \xi = \pi_\ast (\mathcal{E}(-s_\infty)) = \pi_\ast ((\gamma^\ast L)(-s_\infty)).
\]

For another sheaf \(\mathcal{E}' = \gamma^\ast L'\), with the choice \((L'_0)_p = L'_q\) we get

\[
\pi_\ast (\mathcal{E}'(-s_0)) = \pi_\ast ((\gamma^\ast L')(-s_0)).
\]

Let \(f_p\) and \(f_q\) be sections of \(\mathcal{O}(p)\) and \(\mathcal{O}(q)\), respectively, on \(\tilde{D}\). Define a map

\[
\mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O}(q)) \to \mathbb{P}(\mathcal{O}(p + q) \oplus \mathcal{O}(q + p)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O})
\]

by taking the point \([x, y]\) over \(t \in \tilde{D}\) to \([xf_q(t), yf_p(t)]\). Since \(\mathbb{P}(\mathcal{O} \oplus \mathcal{O})\) is the trivial \(\mathbb{P}^1\)-bundle on \(\tilde{D}\), this defines a meromorphic function on \(\mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O}(q))\). Moreover, it has a zero when \(x\) or \(f_q\) vanish (i.e. along \(s_0\) and \(\gamma^{-1}(q)\)) and a pole when \(y\) of \(f_p\) vanish (i.e. along \(s_\infty\) and \(\gamma^{-1}(p)\)). Therefore we have a linear equivalence of divisors

\[
s_0 + \gamma^{-1}(q) \sim s_\infty + \gamma^{-1}(p).
\]

It follows that the sheaves \((\gamma^\ast L')(-s_0)\) and \((\gamma^\ast L)(-s_\infty)\) are isomorphic when \(L' \cong L \otimes \mathcal{O}(p - q)\). This is precisely the same translation that occurred for the compactified Jacobian of \(D\).

We have made numerous choices, so it is not clear a priori that we have a complete description of the compactified Picard scheme of \(\text{Pic}^0 D\), which by definition is the connected component of the trivial sheaf in the moduli space of torsion-free rank one sheaves on \(\text{Pic}^0 D\). At a point which represents a locally free sheaf, one can compute the dimension of this moduli space and show that it is smooth. Therefore another irreducible component could only meet the one we have just described along its singular locus.

To construct other irreducible components, we could have started with a different torsion-free rank one sheaf \(\mathcal{E}\) on \(\text{Pic} D\), which in general would look like a line bundle tensored with the ideal sheaf of a zero-dimensional subscheme \(Z \subset \text{Pic} D\). Since \(s_0\) and \(s_\infty\) are smooth curves, the restrictions \(\mathcal{E}|_{s_0}\) and \(\mathcal{E}|_{s_\infty}\) would once again be line bundles, and so the rest of our construction would still make sense. We could also have chosen the line subbundle \(L_0\) differently; for instance, of degree less than zero. All of these variations produce other irreducible components of the moduli space of torsion-free rank one sheaves, but one can check directly that they don’t meet the irreducible component that we first described. For instance, the Euler characteristic of a sheaf must remain constant in a connected component, and therefore it can be used to distinguish many connected components (though this on its own is by no means sufficient).

Thus our description

\[
\text{Pic}^0 (\text{Pic}^0 D) \cong \text{Pic}^0 D
\]

for type (2) curves is complete. \(\square\)
Remark The above argument can probably be extended to type (3) and (4) curves: we leave this as an exercise for the (patient) reader.

**Corollary 17** The spaces $P$ and $Z^0$ are isomorphic. In particular, $P$ is a smooth holomorphic symplectic four-fold.

**Proof** Because of Theorem 15, we know that

$$\text{Pic}^0(\text{Pic}^0 D) \cong \text{Pic}^0 D$$

for all four kind of curves. Moreover, Esteves and Kleiman’s result (Theorem 4.1 in [18]) also holds in the relative case. So for the family of curves $C \to \mathbb{P}^2$, we have

$$\text{Pic}^0(\text{Pic}^0(C/\mathbb{P}^2)) \cong \text{Pic}^0(C/\mathbb{P}^2)$$

which is precisely $P \cong Z^0$. □

**Remark** Even if we didn’t have the relative version of Theorem 15, we could still conclude that $Z^0 \cong P$ by arguing that they are locally isomorphic as fibrations and then observing that they both admit global sections. To prove they are locally isomorphic we can take a (Stein) open cover $\{U_i\}$ of $\mathbb{P}^2$ and consider the induced maps of each open set $U_i$ into a compactification of the moduli space of principally polarized abelian surfaces. Then over $U_i$, both $Z^0$ and $P$ will be given by pulling back the same universal variety.

The difficulty with this argument is that there are different compactifications of the moduli space of abelian surfaces, not all of which admit a universal variety. If we use Mumford’s compactification, then there exist local universal varieties (see Hulek, Kahn, and Weintraub [24]), which suffice since the $U_i$ are Stein open sets. However, not all degenerations of abelian surfaces arise in Mumford’s compactification: we get compactified Jacobians of type (2) curves, but not of types (3) and (4). To proceed we first remove from $U_i$ the points corresponding to type (3) and (4) curves (fortunately $U_i$ remains simply-connected). We conclude then that $Z^0$ and $P$ are isomorphic as fibrations over the complement of a finite set of points in $\mathbb{P}^2$. This also gives a birational map between $Z^0$ and $P$. Since this map is an isomorphism in codimension one, the holomorphic symplectic form $\sigma$ on $Z^0$ induces a holomorphic two-form on $P$ (a priori, we don’t know whether it is non-degenerate).

Now the argument becomes a little delicate: we can show using Fourier-Mukai methods (see below) that $P$ is smooth and derived equivalent to $Z^0$. Then the fact that $Z^0$ is holomorphic symplectic implies the same is true of $P$ (the derived equivalence preserves the Serre functor, so $P$ certainly has trivial canonical bundle; this implies the holomorphic two-form on $P$ is indeed non-degenerate). Finally, a birational map between holomorphic symplectic four-folds must be a composition of Mukai flops of embedded $\mathbb{P}^2$s (see Burns, Hu, and Luo [19]), but the indeterminacy of our birational map is contained in the (disjoint) union of the set of singular fibres corresponding to curves of types (3) and (4). In particular, it contains no $\mathbb{P}^2$s and thus the birational map must extend to an isomorphism.
4.2 A Fourier-Mukai transform

We have seen that $P$ parametrizes stable sheaves on $Z^0$. Moreover, since $Z^0$ admits a global section, there exists a universal sheaf $\mathcal{U}$ on $Z^0 \times P$. We can therefore construct the functor

$$\Phi^U_{P \to Z^0} : D^b_{\text{coh}}(P) \to D^b_{\text{coh}}(Z^0).$$

**Theorem 18** The functor $\Phi^U_{P \to Z^0}$ is an equivalence of triangulated categories.

**Proof** We will apply Bridgeland and Maciocia’s Theorem 2. Let $\mathcal{U}_m := \Phi^U_{P \to Z^0} \mathcal{O}_m$ be the sheaf on $Z^0$ which the point $m \in P$ parametrizes. We must show

1. for all $m \in P$, $\mathcal{U}_m \otimes \mathcal{K}_{Z^0} = \mathcal{U}_m$ and $\mathcal{U}_m$ is simple,
2. for all $m_1 \neq m_2 \in P$,

$$\text{Hom}_{Z^0}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0,$$

and the closed subscheme

$$\Gamma(\mathcal{U}) := \{(m_1, m_2) \in P \times P | \text{Ext}^i_{Z^0}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\}$$

of $P \times P$ has dimension at most five.

Since $\mathcal{U}_m$ is stable, it is also simple. Since $Z^0$ is holomorphic symplectic, it has trivial canonical bundle $K_{Z^0}$. Thus condition (1) follows.

If $\mathcal{U}_{m_1} \to \mathcal{U}_{m_2}$ is a non-trivial morphism, then it must be an isomorphism since $\mathcal{U}_{m_2}$ is stable. Therefore $m_1 = m_2$, proving the first part of condition (2).

It remains to prove the bound on the dimension of $\Gamma(\mathcal{U})$.

Firstly, suppose that $m_1$ and $m_2$ lie in different fibres of $P$. Then $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are sheaves supported on different (disjoint) fibres of $Z^0$. Therefore

$$\text{Ext}^i_{Z^0}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0$$

for all $i$ because all local $\mathcal{E}xt$ sheaves vanish.

Next suppose that $m_1$ and $m_2$ lie in the same smooth fibre of $P$. Then $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are supported on the smooth fibre $Z^0_i = \text{Pic}^0D$ of $Z^0$; in fact they are of the form $\iota_* L_1$ and $\iota_* L_2$ respectively, where $\iota : Z^0_i \to Z^0$ is inclusion and $L_1$ and $L_2$ are degree zero line bundles on the (smooth) abelian surface $Z^0_i$. Now we have the following spectral sequence, taken from Section 7.2 of Bridgeland and Maciocia

$$E^{p,q}_2 := \text{Ext}^p_{Z^0_i}(L_1 \otimes \Lambda^q \mathcal{O}_{Z^0_i}^\oplus, L_2) \implies \text{Ext}^{p+q}_{Z^0}(\iota_* L_1, \iota_* L_2)$$

where $\mathcal{O}_{Z^0_i}^\oplus$ is really the conormal bundle of $Z^0_i$ in $Z^0$, which is trivial. If $m_1 \neq m_2$ then $L_1$ and $L_2$ are not isomorphic. The cohomology of the non-trivial
degree zero line bundle $L_1^\vee \otimes L_2$ on $Z_0^0$ therefore vanishes in all degrees (see Chapter 3 of Birkenhake and Lange [6]), i.e.

$$H^p(Z_0^0, L_1^\vee \otimes L_2) = 0 \text{ for all } p \in \mathbb{Z}. $$

It follows that the spectral sequence vanishes and we have proved

$$\text{Ext}^i_{Z_0^0}(\mathcal{U}_1, \mathcal{U}_2) = \text{Ext}^i_{Z_0^0}(\iota_* L_1, \iota_* L_2) = 0 \text{ for all } i \in \mathbb{Z}.$$ 

for all $i \in \mathbb{Z}$ in this case.

We have shown that $\Gamma(\mathcal{U})$ is a subset of

$$\text{Diag} \cup \{(m_1, m_2) \in P \times P|m_1 \text{ and } m_2 \text{ lie in the same singular fibre}\}$$

where Diag is the diagonal in $P \times P$. But the singular fibres of $P$ sit above the curve $\Delta \subset \mathbb{P}^2$ and have dimension two. Thus

$$\{(m_1, m_2) \in P \times P|m_1 \text{ and } m_2 \text{ lie in the same singular fibre}\}$$

has dimension five, and Diag clearly has dimension four. Therefore $\Gamma(\mathcal{U})$ has dimension at most five and the theorem is proved.

\[ \square \]

**Remark** Theorem 2 also implies that $P$ is smooth. We used this fact in the remark following Corollary 17 to give an alternate proof of the isomorphism $P \cong Z^0$.

**Remark** Since $P \cong Z^0$ by Corollary 17, the equivalence

$$\Phi_{P \rightarrow Z^0} : \mathcal{D}^b_coh(P) \rightarrow \mathcal{D}^b_coh(Z^0)$$

is really an auto-equivalence of the derived category of $Z^0$. It is non-trivial: for instance, on a smooth fibre it induces the non-trivial auto-equivalence of the derived category of a principally polarized abelian surface (which was first constructed by Mukai [34]).

In fact, we can extend Mukai’s results to degenerate abelian surfaces as follows. Let $D$ be an arbitrary curve of type (2), (3), or (4). We claim that $D$ occurs in some hyperelliptic K3 surface $S$. This is straightforward: we write $D$ as a double cover of $\mathbb{P}^1$ branched over six points, some of them coinciding. Then we embed the line $\mathbb{P}^1$ in $(\mathbb{P}^2)^{\vee}$ and find a sextic $\delta$ meeting the line in the given six points, with $\delta$ touching the line to order two or three at a point of multiplicity two, respectively three. The space of plane sextics moduli automorphisms of $\mathbb{P}^2$ has dimension 19, whereas the points impose just six conditions, so we clearly can find such a sextic, and moreover we can assume that it does not admit a tritangent, so that the corresponding K3 surface $S$ is generic.

Next we construct $Z^0$ and $P$ as before. The compactified Jacobian $J := \overline{\text{Jac}} D$ occurs as a fibre of $Z^0$, and its dual $\overline{\text{Pic}} J$ is the corresponding fibre of $P$. Use $\iota$ to denote the inclusion of $J$ in $Z^0$, $\overline{\text{Pic}} J$ in $P$, and $J \times \overline{\text{Pic}} J$ in $Z^0 \times P$ (the usage will always be clear from the context). The universal sheaf $\mathcal{U}$ on $Z^0 \times P$ restricts to a universal sheaf $\mathcal{U}_D := \iota^* \mathcal{U}$ on $J \times \overline{\text{Pic}} J$, which generalizes the Poincaré line bundle for a smooth abelian surface.
Corollary 19 Let $D$ be an arbitrary curve of type (2), (3), or (4), and let $J := \text{Jac}D$ be its compactified Jacobian, which is a degeneration of a principally polarized abelian surface. The universal sheaf $\mathcal{U}_D$ induces an integral transform

$$\Phi^{\mathcal{U}_D, \mathcal{P}ic_0} : \mathcal{D}^b_{\text{coh}}(\mathcal{P}ic^0_{\mathcal{P}ic_0}) \rightarrow \mathcal{D}^b_{\text{coh}}(J)$$

which is an equivalence of triangulated categories.

Proof Our proof is taken directly from Section 6 of Chen [11]. Write $\Phi$ for $\Phi^{\mathcal{U}_P, Z_0}$ and $\Phi_D$ for $\Phi^{\mathcal{U}_D, \mathcal{P}ic_0}$. Label the various projections as in the following diagram:

We first claim there is a natural isomorphism of functors

$$\Phi \circ \iota_* \cong \iota_* \circ \Phi_D$$

(this is Lemma 6.1 in [11]). Let $\mathcal{E} \in \mathcal{D}^b_{\text{coh}}(\mathcal{P}ic^0_{\mathcal{P}ic_0})$; using flat base change and the projection formula we have

$$\Phi(\iota_* \mathcal{E}) = R\pi_0* (\mathcal{U} \otimes L\pi_P^*(\iota_* \mathcal{E})) 
\cong R\pi_0* (\mathcal{U} \otimes R\iota_*(L(\pi_P^0)*) \mathcal{E})) 
\cong R\pi_0* (R\iota_*(\iota^* \mathcal{U} \otimes L(\pi_P^0)*) \mathcal{E}) 
\cong R\iota_*(R(\pi_0^0)*) (\mathcal{U}_D \otimes L(\pi_P^0)*) \mathcal{E}) 
= R\iota_*(\Phi_D \mathcal{E})$$

and the claim follows.

Now let us prove that $\Phi_D$ is an equivalence (this is Lemma 6.2 in [11]). Let $\mathcal{V}$ be the right adjoint of $\Phi$; then $\Psi$ is an integral transform given by some sheaf $\mathcal{V}$ on $P \times Z_0$. Let

$$\Psi_D : \mathcal{D}^b_{\text{coh}}(J) \rightarrow \mathcal{D}^b_{\text{coh}}(\mathcal{P}ic^0_{\mathcal{P}ic_0})$$

be the integral transform given by $\iota^* \mathcal{V}$. Then one can show there is a natural isomorphism of functors

$$\Psi \circ \iota_* \cong \iota_* \Psi_D$$

using the same argument as for the claim above. We therefore have a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^b_{\text{coh}}(P) & \xrightarrow{\Phi} & \mathcal{D}^b_{\text{coh}}(Z_0) \\
\uparrow \iota_* & & \uparrow \iota_* \\
\mathcal{D}^b_{\text{coh}}(\mathcal{P}ic^0_{\mathcal{P}ic_0}) & \xrightarrow{\Phi^0} & \mathcal{D}^b_{\text{coh}}(P) \\
\uparrow \iota_* & & \uparrow \iota_* \\
\mathcal{D}^b_{\text{coh}}(J) & \xrightarrow{\Psi^0} & \mathcal{D}^b_{\text{coh}}(\mathcal{P}ic^0_{\mathcal{P}ic_0}).
\end{array}$$
In particular
\[(\Psi \circ \Phi) \circ \iota_\ast \cong \iota_\ast \circ (\Psi_D \circ \Phi_D),\]
but \(\Psi \circ \Phi\) is equivalent to the identity functor on \(\mathcal{D}_{\text{coh}}^b(P)\). As \(\iota\) is a closed embedding, this implies \(\Psi_D \circ \Phi_D\) is equivalent to the identity functor on \(\mathcal{D}_{\text{coh}}^b(\text{Pic}^0 J)\). Similarly, using \(\Phi \circ \Psi \cong \text{Id}_{\mathcal{D}_{\text{coh}}^b(Z^0)}\) we can show that \(\Phi_D \circ \Psi_D\) is equivalent to the identity functor on \(\mathcal{D}_{\text{coh}}^b(J)\). It follows that \(\Phi_D\) is an equivalence. \(\square\)

**Corollary 20** Let \(D\) be an arbitrary curve of type (2), (3), or (4), and let \(J := \text{Jac}D\) be its compactified Jacobian. Let \(L_1\) and \(L_2\) be two non-isomorphic torsion-free rank one sheaves of degree zero on \(J\). Then
\[\text{Ext}^p_J(L_1, L_2) = 0\]
for all \(p\).

**Proof** This follows from Corollary 19, since \(L_1\) and \(L_2\) are the Fourier-Mukai transforms of two skyscraper sheaves supported at distinct points of \(\text{Pic}^0 J\), and \(\Phi_D\) preserves the \(\text{Ext}^\bullet\)-pairing. However, we can give a direct proof using Theorem 18, which says that \(\Phi^{U_P \to Z^0}_D\) is an equivalence of triangulated categories. By Bridgeland’s criterion, Theorem 1, it follows that we must have
\[\text{Ext}^i_{Z^0}(U_{m_1}, U_{m_2}) = 0\]
for all integers \(i\) and all \(m_1 \neq m_2 \in P\). In particular, suppose that \(m_1\) and \(m_2\) are points in the singular fibre \(\text{Pic}^0 J\) corresponding to \(L_1\) and \(L_2\), so that \(U_{m_1} = \iota_\ast L_1\) and \(U_{m_2} = \iota_\ast L_2\).

By the remark preceding Lemma 12, the fibration \(Z^0 \to \mathbb{P}^2\) is flat. Therefore we can pull-back the Koszul resolution of a point in \(\mathbb{P}^2\) (any point) to get a resolution of the structure sheaf of the corresponding fibre. It follows that the spectral sequence in Section 7.2 of Bridgeland and Maciocia [8] exists also for singular fibres in our case, and thus we have
\[E^{p,q}_2 := \text{Ext}^p_J(L_1 \otimes L^q \mathcal{O}^{\mathbb{P}^2}_J, L_2) \implies \text{Ext}^{p+q}_{Z^0}(\iota_\ast L_1, \iota_\ast L_2).\]
The vanishing of the right hand side allows us to conclude that
\[E^{0,0}_2 = \text{Ext}^0_J(L_1, L_2) = \text{Ext}^0_{Z^0}(\iota_\ast L_1, \iota_\ast L_2)\]
vanishes (which also follows since \(L_1\) and \(L_2\) are stable and not isomorphic). Suppose \(\text{Ext}^p_J(L_1, L_2)\) vanishes for all \(p \leq k\). Then \(E^{p,q}_2\) vanishes for \(p \leq k\) and for all \(q\). Therefore
\[E^{p+1,0}_2 = \text{Ext}^{p+1}_J(L_1, L_2) = \text{Ext}^{p+1}_{Z^0}(\iota_\ast L_1, \iota_\ast L_2)\]
also vanishes. By induction
\[\text{Ext}^p_J(L_1, L_2)\]
vanes for all \(p\). Note that since \(J\) is not smooth, \(L_1\) and \(L_2\) need not have finite projective resolutions. Thus we show the vanishing for all \(p\), rather than just \(p \leq 2 = \dim J\). \(\square\)
4.3 A twisted Fourier-Mukai transform

Remark  Note that $J$ has trivial normal bundle in $Z^0$, and $Z^0$ has trivial canonical bundle, so by adjunction the canonical bundle $K_J$ of $J$ is also trivial. Also, the sheaves on $J$ parametrized by $\text{Pic}^0 J$ are clearly simple. One might expect that combining these observations with Corollary 20 we could obtain another proof of Corollary 19 using Bridgeland’s criterion, Theorem 1. However, Bridgeland’s result is for smooth varieties and it is not immediately clear how to generalize his proofs to singular varieties.

Remark  Let us make one more remark about Theorem 18: although $P := \text{Pic}^0 (Z^0/\mathbb{P}^2) \cong \text{Pic}^0 (Z^1/\mathbb{P}^2)$, it is important to distinguish these spaces in the sense that $P$ should be regarded as the dual fibration of $Z^0$. In this paper, our four-folds $Z^0$ and $Z^1$ are fibred by principally polarized abelian surfaces, which are self-dual, which is why $Z^0$ happens to be isomorphic to its dual fibration.

However, there exist holomorphic symplectic manifolds which are fibred by non-principally polarized abelian varieties. The generalized Kummer varieties provide examples (see Debarre [13]); indeed Proposition 5.3 of [40] shows that generalized Kummer varieties cannot be fibred by principally polarized abelian varieties. The possibility of constructing a Fourier-Mukai transform for generalized Kummer four-folds was briefly discussed in Section 5.4 of [41]; there remain many technical details to be resolved.

4.3 A twisted Fourier-Mukai transform

At the beginning of this section we made the observation that

$$P := \text{Pic}^0 (Z^0/\mathbb{P}^2) \cong \text{Pic}^0 (Z^1/\mathbb{P}^2).$$

Thus $P$, which is by definition the dual fibration of $Z^0$, is also the dual fibration of $Z^1$. In particular, $P$ parametrizes stable sheaves on $Z^1$, so there exist local universal sheaves on $Z^1 \times P$ (for some cover $\{P_i\}$ of $P$). As in Section 2, there exists a gerbe $\beta \in H^2(P, \mathcal{O}^*)$ which is the obstruction to the existence of a global universal sheaf on $Z^1 \times P$. Since $Z^1$ does not admit a section, we know from Proposition 7 that $\beta$ is non-zero. The collection of local universal sheaves gives us a $\pi_P^*\beta$-twisted universal sheaf $\mathcal{U}$ on $Z^1 \times P$, where $\pi_P$ is projection to $P$. We can therefore construct the functor

$$\Phi_{P^* \to Z^1} : \mathcal{D}_{\text{coh}}^b(P, \beta^{-1}) \to \mathcal{D}_{\text{coh}}^b(Z^1).$$

Theorem 21  The functor $\Phi_{P^* \to Z^1}$ is an equivalence of triangulated categories.

Proof  We will apply Căldăraru’s Proposition 4 which is the twisted version of Bridgeland’s criterion, Theorem 1. Let $\mathcal{U}_m := \Phi_{P^* \to Z^1}^* \mathcal{O}_m$ be the sheaf on $Z^1$ which the point $m \in P$ parametrizes. We must show

1. for all $m \in P$, $\mathcal{U}_m \otimes K_{Z^1} = \mathcal{U}_m$ and $\mathcal{U}_m$ is simple,

2. for all integers $i$ and all $m_1 \neq m_2 \in P$,

$$\text{Ext}_{Z^1}^i(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0.$$
The proof of condition (1) is the same as in the proof of Theorem 18. Namely, since \( U_m \) is stable, it is also simple. Since \( Z^1 \) is holomorphic symplectic, it has trivial canonical bundle \( K_{Z^1} \).

Regarding condition (2), first suppose that \( m_1 \) and \( m_2 \) lie in different fibres of \( P \). Then as before \( U_{m_1} \) and \( U_{m_2} \) are sheaves supported on different (disjoint) fibres of \( Z^1 \). Therefore

\[
\text{Ext}^i_{Z^1}(U_{m_1}, U_{m_2}) = 0
\]

for all \( i \) because all local \( \text{Ext} \) sheaves vanish.

Next suppose that \( m_1 \) and \( m_2 \) lie in the same fibre of \( P \), which may be smooth or singular. Then \( U_{m_1} \) and \( U_{m_2} \) are of the form \( \iota_* L_1 \) and \( \iota_* L_2 \) respectively, where \( \iota : Z^1_t \to Z^1 \) is inclusion, and \( L_1 \) and \( L_2 \) are torsion-free rank one sheaves on \( Z^1_t \) of degree one. As before we have the spectral sequence

\[
E_2^{p,q} := \text{Ext}^p_{Z^1_t}(L_1 \otimes \Lambda^q \mathcal{C}_{Z^1_t}^{\otimes 2}, L_2) \Rightarrow \text{Ext}^{p+q}_{Z^1_t}(\iota_* L_1, \iota_* L_2)
\]

which exists for both smooth and singular fibres (see the comments in the proof of Corollary 20). If \( Z^1_t \) is smooth, and \( L_1 \) and \( L_2 \) are not isomorphic, then the spectral sequence vanishes as in the proof of Theorem 18. If \( Z^1_t \) is singular, then it is the compactified Jacobian of a curve of type (2), (3), or (4). Then Corollary 20 showed that \( \text{Ext}^p_{Z^1_t}(L_1, L_2) \) vanishes for all \( p \) if \( L_1 \) and \( L_2 \) are not isomorphic. So once again, all terms in the spectral sequence vanish. It follows that

\[
\text{Ext}^i_{Z^1_t}(U_{m_1}, U_{m_2}) = 0
\]

for all \( i \in \mathbb{Z} \) and for all \( m_1 \neq m_2 \in P \). This concludes the proof of condition (2), and of the theorem. \( \square \)

**Remark** Since the proof of Bridgeland and Maciocia’s Theorem 2 only relies on local arguments, it can be generalized to the twisted case. This would lead to a direct proof of Theorem 21 without the need to first prove Theorem 18 and thereby obtain Corollary 20.

**Remark** Recall that \( Z^1 \) is a torsor over \( Z^0 \cong P \). Since \( Z^0 \) admits a section and \( Z^1 \) does not, we could regard \( Z^1 \) as being a ‘twisted’ version of the space \( Z^0 \). Thus the theorem says that the derived category of twisted sheaves on the ‘untwisted’ space \( Z^0 \) is equivalent to the derived category of (untwisted) sheaves on the ‘twisted’ space \( Z^1 \).

### 4.4 Deformations of fibrations

In this final subsection we will show that \( Z^1 \) and \( Z^0 \) can be connected by a one parameter family in their space of deformations, which only passes through Lagrangian fibrations. This follows by considering the subspace of \( H^2(P, \mathcal{O}^*) \)
consisting of gerbes on $P$ which arise from torsors over $Z^0$, and showing that it is connected.

We start with the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$
on $P$. The cohomology of each of these sheaves on $P$ can be computed using the projection $p_P : P \to \mathbb{F}^2$ and the Leray spectral sequence. Thus we have

$$E_2^{i,j}(\mathbb{Z}) := H^i(\mathbb{F}^2, R^j p_P_* \mathbb{Z}) \Rightarrow H^{i+j}(P, \mathbb{Z})$$

and since $R^0 p_P_* \mathbb{Z}_P \cong \mathbb{Z}_{\mathbb{F}^2}$ we can compute the bottom row of the left hand side and obtain

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
H^0(\mathbb{F}^2, R^2 p_P_* \mathbb{Z}) & H^1(\mathbb{F}^2, R^2 p_P_* \mathbb{Z}) & H^2(\mathbb{F}^2, R^2 p_P_* \mathbb{Z}) & \ldots \\
H^0(\mathbb{F}^2, R^1 p_P_* \mathbb{Z}) & H^1(\mathbb{F}^2, R^1 p_P_* \mathbb{Z}) & H^2(\mathbb{F}^2, R^1 p_P_* \mathbb{Z}) & \ldots \\
Z & 0 & Z & 0 \\
\end{array}$$

Moreover, the right-most term $\mathbb{Z}$ must survive the higher derivations $d_2(\mathbb{Z})$, $d_3(\mathbb{Z})$, and $d_4(\mathbb{Z})$, since the class generating $H^4(\mathbb{F}^2, \mathbb{Z})$ can be pulled-back to give a non-trivial class in $H^4(P, \mathbb{Z})$ (the class of a fibre).

Next we have

$$E_2^{i,j}(\mathcal{O}) := H^i(\mathbb{F}^2, R^j p_P_* \mathcal{O}) \Rightarrow H^{i+j}(P, \mathcal{O}).$$

Matsushita \[33]\ proved that $R^j p_P_* \mathcal{O}_P \cong \Omega^j_{\mathbb{F}^2}$ and therefore we can compute the left hand side precisely, obtaining

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & C & \ldots \\
0 & C & 0 & \ldots \\
C & 0 & 0 & \ldots \\
\end{array}$$

The spectral sequence degenerates at the $E^2$ term and gives

$$H^k(P, \mathcal{O}) \cong \begin{cases} C & k \text{ even,} \\ 0 & k \text{ odd.} \end{cases}$$

Finally we have

$$E_2^{i,j}(\mathcal{O}^*) := H^i(\mathbb{F}^2, R^j p_P_* \mathcal{O}^*) \Rightarrow H^{i+j}(P, \mathcal{O}^*).$$

In this case we know that $R^0 p_P_* \mathcal{O}_P^* \cong \mathcal{O}_{\mathbb{F}^2}^*$, and thus we can compute the bottom row of the left hand side (using the exponential long exact sequence on $\mathbb{F}^2$), obtaining

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
H^0(\mathbb{F}^2, R^2 p_P_* \mathcal{O}^*) & H^1(\mathbb{F}^2, R^2 p_P_* \mathcal{O}^*) & H^2(\mathbb{F}^2, R^2 p_P_* \mathcal{O}^*) & \ldots \\
H^0(\mathbb{F}^2, R^1 p_P_* \mathcal{O}^*) & H^1(\mathbb{F}^2, R^1 p_P_* \mathcal{O}^*) & H^2(\mathbb{F}^2, R^1 p_P_* \mathcal{O}^*) & \ldots \\
\mathcal{O}^* & Z & 0 & Z \\
\end{array}$$
Now $Z^1$ is a torsor over $Z^0$, corresponding to a gerbe $\beta \in H^2(P, \mathcal{O}^*)$. Moreover, there is a cover $\{P_i\}$ of $P$ obtained by pulling back a cover of $\mathbb{P}^2$, and $\beta$ can be represented by line bundles $L_{ij}$ on pair-wise intersections $P_{ij}$. Moreover, these line bundles have degree zero on each fibre of $p_P : P \to \mathbb{P}^2$. Conversely, given a gerbe $\beta'$ with these properties, we can work backwards to construct the torsor $Z^{\beta'}$ over $Z^0$ (see Subsection 2.4). Now a family of line bundles on fibres of $P$ is a local section of $R^1 p_P^* \mathcal{O}^*$, and the degree is given by the coboundary map $\delta_1$ of the long exact sequence of direct image sheaves

$$\cdots \to R^1 p_P^* \mathcal{O} \to R^1 p_P^* \mathcal{O}^* \xrightarrow{\delta_1} R^2 p_P^* \mathcal{O} \to R^2 p_P^* \mathcal{O} \to \cdots$$

Thus $\beta$ corresponds to a cocycle in $H^1(\mathbb{P}^2, R^1 p_P^* \mathcal{O}^*)$ which moreover can be represented by local sections of

$$\ker \delta_1 \subset R^1 p_P^* \mathcal{O}^*.$$ 

Looking at the spectral sequence, we see that this cocycle must also lie in the kernel of

$$d_2(\mathcal{O}^*) : H^1(\mathbb{P}^2, R^1 p_P^* \mathcal{O}^*) \to H^3(\mathbb{P}^2, R^0 p_P^* \mathcal{O}^*) \cong \mathbb{Z}$$

since it survives to give a class in $H^2(P, \mathcal{O}^*)$. We claim that this makes one of the earlier conditions redundant.

**Lemma 22** Suppose that $\alpha$ lies in the kernel of

$$d_2(\mathcal{O}^*) : H^1(\mathbb{P}^2, R^1 p_P^* \mathcal{O}^*) \to H^3(\mathbb{P}^2, R^0 p_P^* \mathcal{O}^*) \cong \mathbb{Z}.$$

Then $\alpha$ can be represented by line bundles $L_{ij}$ on pair-wise intersections $P_{ij}$ which have degree zero on each fibre of $p_P : P \to \mathbb{P}^2$.

**Proof** By the functoriality of spectral sequences, the coboundary maps of the long exact sequence of direct image sheaves

$$\cdots \to R^0 p_P^* \mathcal{O}^* \xrightarrow{\delta_0} R^1 p_P^* \mathcal{O} \to R^1 p_P^* \mathcal{O} \to R^1 p_P^* \mathcal{O}^* \xrightarrow{\delta_1} R^2 p_P^* \mathcal{O} \to \cdots$$

induce maps which commute with the derivations of the spectral sequences. Thus we obtain a commutative diagram

$$\begin{array}{ccc}
H^1(\mathbb{P}^2, R^1 p_P^* \mathcal{O}^*) & \xrightarrow{d_2(\mathcal{O}^*)} & H^3(\mathbb{P}^2, R^0 p_P^* \mathcal{O}^*) \\
\downarrow \text{id} & & \downarrow \text{id} \\
H^1(\mathbb{P}^2, R^2 p_P^* \mathcal{O}) & \xrightarrow{d_2(\mathcal{O}^*)} & H^3(\mathbb{P}^2, R^1 p_P^* \mathcal{O}).
\end{array}$$

Therefore since $\alpha$ is in the kernel of $d_2(\mathcal{O}^*)$, $H^1(\delta_1)\alpha$ must lie in the kernel of $d_2(\mathcal{O}^*)$. Now as observed above, $H^4(\mathbb{P}^2, R^0 p_P^* \mathcal{O}^*) \cong \mathbb{Z}$ must survive to give a class in $H^4(P, \mathcal{O})$, and thus the map

$$d_3(\mathcal{O}) : \ker d_2(\mathcal{O}) \subset H^1(\mathbb{P}^2, R^2 p_P^* \mathcal{O}) \to H^4(\mathbb{P}^2, R^0 p_P^* \mathcal{O})$$
4.4 Deformations of fibrations

is trivial. This means that $H^1(\delta_1)\alpha$ also lies in the kernel of $d_3(Z)$, and thus it survives to give a class in $H^3(P, Z)$. However, $P$ is a deformation of the Hilbert scheme of two points on a K3 surface, and thus $H^3(P, Z) = 0$. We conclude that $H^1(\delta_1)\alpha = 0$.

So $\alpha$ is given by local sections $\alpha_{ij}$ of $R^1p_{*}\mathcal{O}^*$ (i.e. the line bundles $L_{ij}$) over the pair-wise intersections $P_{ij}$, such that the cocycle $\{\delta_1\alpha_{ij}\}$ is actually a coboundary. So there exist local sections $\gamma_i$ of $R^2p_{*}\mathbb{Z}$ over $P_i$ such that

$$\delta_1\alpha_{ij} = \gamma_i - \gamma_j.$$ 

Consider the image of $\gamma_i$ under the map

$$R^2p_{*}\iota : R^2p_{*}\mathbb{Z} \to R^2p_{*}\mathcal{O}$$

induced by the inclusion $\iota : \mathbb{Z} \to \mathcal{O}$. Observe that

$$R^2p_{*}\iota \gamma_i - R^2p_{*}\iota \gamma_j = R^2p_{*}\iota(\delta_1\alpha_{ij})$$

vanishes, by the exactness of

$$\ldots \to R^1p_{*}\mathcal{O}^* \xrightarrow{\delta_1} R^2p_{*}\mathbb{Z} \xrightarrow{R^2p_{*}\iota} R^2p_{*}\mathcal{O} \to \ldots$$

Therefore the $R^2p_{*}\iota \gamma_i$ agree on overlaps and can be patched together to give a global section of $R^2p_{*}\mathcal{O}$. However, Matsushita proved in [33] that $R^2p_{*}\mathcal{O} \cong \Omega^2_{\mathbb{P}^2}$, which has no global sections, and therefore $R^2p_{*}\iota \gamma_i$ vanishes for all $i$.

Using the exactness of the long exact sequence of direct image sheaves once again, we conclude that there exist local sections $\epsilon_i$ of $R^1p_{*}\mathcal{O}^*$ over $P_i$ such that

$$\gamma_i = \delta_1\epsilon_i.$$ 

Now define

$$\alpha'_{ij} := \alpha_{ij} - \epsilon_i + \epsilon_j.$$ 

These are local sections of $R^1p_{*}\mathcal{O}^*$ over $P_{ij}$ (we have written the group action additively, but if one wants to think of these as families of line bundles on fibres then the group action is just tensor product). Moreover the collection $\{\alpha'_{ij}\}$ represents the same cohomology class in $H^1(\mathbb{P}^2, R^1p_{*}\mathcal{O}^*)$ as $\alpha$, though now we have

$$\delta_1\alpha' = \delta_1\alpha_{ij} - \delta_1\epsilon_i + \delta_1\epsilon_j = \gamma_i - \gamma_j - \gamma_i + \gamma_j = 0.$$ 

So $\alpha'_{ij}$ corresponds to a line bundle $L'_{ij}$ on $P_{ij}$ which has degree zero on each fibre of $p_P : P \to \mathbb{P}^2$. This completes the proof. \hfill \Box

Finally we prove the following result.
Theorem 23 There is a one-parameter family of Lagrangian fibrations \(Z'\) over \(\mathbb{P}^2\), connecting \(Z^0\) and \(Z^1\) in their moduli space of deformations. Each fibration \(Z' \to \mathbb{P}^2\) is a torsor over \(Z^0\), and it corresponds to a gerbe \(\beta_t \in H^2(P, \mathcal{O}^*)\).

Proof Let \(\beta \in H^2(P, \mathcal{O}^*)\) be the gerbe corresponding to \(Z^1\), and \(\alpha\) the corresponding class in \(H^1(\mathbb{P}^2, R^1p_{pp,*}\mathcal{O}^*)\). Then \(\alpha\) survives the spectral sequence to give the class \(\beta\). Part of the exponential long exact sequence on \(P\) looks like

\[
\cdots \to \mathbb{C}^2 \xrightarrow{H^2(\exp)} H^2(P, \mathcal{O}^*) \to H^3(P, \mathbb{Z}) \to \cdots
\]

since \(H^2(P, \mathcal{O}) \cong \mathbb{C}\). We know that \(\beta\) can be represented by line bundles \(L_{ij}\) on \(P\) which have degree zero on each fibre of \(p_p: P \to \mathbb{P}^2\). This implies that \(H^1(\delta_1)\alpha = 0\), and hence \(\beta\) must map to zero in \(H^3(P, \mathbb{Z})\). Note that we have shown this without using the fact that \(H^3(P, \mathbb{Z})\) vanishes.

By exactness, \(\beta\) is the image under \(H^3(\exp)\) of an element in \(H^2(P, \mathcal{O}) \cong \mathbb{C}\). Likewise, \(\alpha\) must be the image of a class

\[
\kappa \in H^1(\mathbb{P}^2, R^1p_{pp,*}\mathcal{O}) \cong \mathbb{C}
\]

under the map \(H^1(R^1p_{pp,*}\exp)\) coming from \(\exp: \mathcal{O} \to \mathcal{O}^*\). Next let us define \(\alpha_t \in H^1(\mathbb{P}^2, R^1p_{pp,*}\mathcal{O}^*)\) to be the image of \(t\kappa \in H^1(\mathbb{P}^2, R^1p_{pp,*}\mathcal{O})\) under \(H^1(R^1p_{pp,*}\exp)\). Since

\[
\begin{array}{c c c c c c c}
H^1(\mathbb{P}^2, R^1p_{pp,*}\mathcal{O}) & d_2(\mathcal{O}^*) & H^3(\mathbb{P}^2, R^0p_{pp,*}\mathcal{O}) & = 0 \\
\downarrow H^1(R^1p_{pp,*}\exp) & & \downarrow H^3(R^0p_{pp,*}\exp) \\
H^1(\mathbb{P}^2, R^1p_{pp,*}\mathcal{O}^*) & d_2(\mathcal{O}^*) & H^3(\mathbb{P}^2, R^0p_{pp,*}\mathcal{O}^*)
\end{array}
\]

commutes, \(\alpha_t\) must lie in the kernel of \(d_2(\mathcal{O}^*)\). Thus not only does it survive to give a gerbe \(\beta_t \in H^2(P, \mathcal{O}^*)\), but by Lemma 22 it can be represented by line bundles \(L_{ij}\) on \(P\) which have degree zero on each fibre of \(p_p: P \to \mathbb{P}^2\). This is precisely what is required to construct the torsor \(Z'\) over \(Z^0\), concluding the proof. \(\square\)

Remark Note that in the proof of Lemma 22 the vanishing of \(H^3(P, \mathbb{Z})\) was used to show that \(H^1(\delta_1)\alpha = 0\). However, in the proof of Theorem 23 we already know that \(H^1(\delta)\alpha_t = 0\), so the argument would work even if \(H^3(P, \mathbb{Z})\) did not vanish. So even if \(H^2(P, \mathcal{O}^*)\) is not connected, the gerbes on \(P\) which arise from torsors over \(Z^0\) form a connected subspace. This is significant because for the generalized Kummer four-fold \(K_4\) we have \(H^3(K_4, \mathbb{Z}) \cong \mathbb{Z}^{2g}\). In Section 5.4 of [11] the author suggested one might be able to find new deformation classes of holomorphic symplectic four-folds by constructing Lagrangian fibrations from gerbes in different connected components of \(H^2(K_4, \mathcal{O}^*)\). Unfortunately the above argument implies that this will not work.

Remark Let \(\text{Def}(Z^0)\) be the Kuranishi space parametrizing deformations of \(Z^0\) as a complex manifold. In [12] the author proved that there is a subspace \(\Delta \subset \text{Def}(Z^0)\) of codimension one parametrizing deformations of \(Z^0\) which are
Lagrangian fibrations, and a subspace $\Delta' \subset \Delta \subset \text{Def}(Z^0)$ of codimension one in $\Delta$ (and hence codimension two in $\text{Def}(Z^0)$) parametrizing deformations of $Z^0$ which are Lagrangian fibrations with global sections. The one-parameter family described in Theorem 23 is consistent with these results; it can be regarded as a deformation inside $\Delta$ but transverse to $\Delta'$.

References

[1] A. Altman, A. Iarrobino, and S. Kleiman, *Irreducibility of the compactified Jacobian*, Real and complex singularities, Proc. Ninth Nordic Summer School/NAVF (Oslo, 1976), Sijthoff and Noordhoff, 1977, 1–12.

[2] A. Altman and S. Kleiman, *Compactifying the Picard scheme*, Adv. in Math. 35 (1980), no. 1, 50–112.

[3] A. Altman and S. Kleiman, *The presentation functor and the compactified Jacobian*, The Grothendieck Festschrift, Vol. I, 15–32, Progr. Math. 86, Birkhäuser, 1990.

[4] A. Beauville, *Variétés Kähleriennes dont le 1ère classe de Chern est nulle*, Jour. Diff. Geom. 18 (1983), 755–782.

[5] A. Beauville, *Counting rational curves on K3 surfaces*, Duke Math. J. 97 (1999), no. 1, 99–108.

[6] C. Birkenhake and H. Lange, *Complex abelian varieties*, Springer-Verlag, Berlin, 1992.

[7] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. 31 (1999), no. 1, 25–34.

[8] T. Bridgeland and A. Maciocia, *Fourier-Mukai transforms for K3 and elliptic fibrations*, J. Algebraic Geom. 11 (2002), no. 4, 629–657.

[9] D. Burns, Y. Hu, and T. Luo, *HyperKähler manifolds and birational transformations in dimension 4*, Vector bundles and representation theory (Columbia, MO, 2002), 141–149, Contemp. Math. 322, Amer. Math. Soc., 2003.

[10] A. Căldăruț, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, Cornell PhD thesis, May 2000 (available from www.math.upenn.edu/~andreic/).

[11] J-C. Chen, *Flops and equivalences of derived categories for three-folds with only terminal Gorenstein singularities*, J. Differential Geom. 61 (2002), no. 2, 227–261.
REFERENCES

[12] K. Cho, Y. Miyaoka, and N. Shepherd-Barron, *Characterizations of projective space and applications to complex symplectic manifolds*, in Higher Dimensional Birational Geometry, Advanced Studies in Pure Mathematics 35, 2002, 1–88.

[13] O. Debarre, *On the Euler characteristic of generalized Kummer varieties*, Amer. J. Math. 121 (1999), no. 3, 577–586.

[14] I. Dolgachev and M. Gross, *Elliptic threefolds. I. Ogg-Shafarevich theory*, J. Algebraic Geom. 3 (1994), no. 1, 39–80.

[15] R. Donagi, L. Ein, and R. Lazarsfeld, *Nilpotent cones and sheaves on K3 surfaces*, in Birational Algebraic Geometry (Baltimore 1996), Contemp. Math. 207, Amer. Math. Soc., 1997, 51–61.

[16] C. D’Souza, *Compactification of generalized Jacobians*, Proc. Indian Acad. Sci. A88 (1979), 419–457.

[17] E. Esteves, M. Gagné, and S. Kleiman, *Autoduality of the compactified Jacobian*, J. London Math. Soc. 65 (2002), no. 3, 591–610.

[18] E. Esteves and S. Kleiman, *The compactified Picard scheme of the compactified Jacobian*, arXiv preprint math.AG/0410537.

[19] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, in Classification of algebraic and analytic manifolds. Progr. Math. 39 (1983), 71–250.

[20] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley, New York, 1978.

[21] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics 187, Springer-Verlag, New York, 1998.

[22] T. Hausel and M. Thaddeus, *Examples of mirror partners arising from integrable systems*, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), No. 4, 313–318.

[23] N. Hitchin, *Lectures on special Lagrangian submanifolds*, in Proceedings of the Harvard Winter School on Mirror Symmetry January 1999, International Press, 2001.

[24] K. Hulek, C. Kahn, and S. Weintraub, *Moduli spaces of abelian surfaces: compactification, degeneration, and theta functions*, de Gruyter Expositions in Mathematics 12, Walter de Gruyter, 1993.

[25] D. Huybrechts, *Birational symplectic manifolds and their deformations*, J. Differential Geom. 45 (1997), no. 3, 488–513.
[26] S. Kleiman, *The structure of the compactified Jacobian: a review and announcement*, Geometry seminars (Bologna, 1982/1983), 81–92, Univ. Stud. Bologna, 1984.

[27] K. Kodaira, *On compact analytic surfaces II, III*, Ann. of Math. 77 (1963), 563–626, and 78 (1963), 1–40.

[28] K. Kodaira, *On the structure of compact complex analytic surfaces, I*, Am. J. Math. 86 (1964), 751–798.

[29] D. Markushevich, *Completely integrable projective symplectic 4-dimensional varieties*, Izvestiya: Mathematics 59 (1995), no. 1, 159–187.

[30] D. Markushevich, *Lagrangian families of Jacobians of genus 2 curves*, J. Math. Sci. 82 (1996), no. 1, 3268–3284.

[31] H. Matsumura, *Commutative ring theory*, Cambridge studied in advanced mathematics 8, Cambridge University Press, 1986.

[32] D. Matsushita, *On fibre space structures of a projective irreducible symplectic manifold*, Topology 38 (1999), No. 1, 79–83. Addendum, Topology 40 (2001), No. 2, 431–432.

[33] D. Matsushita, *Higher direct images of Lagrangian fibrations*, arXiv preprint math.AG/0010283.

[34] S. Mukai, *Duality between $\mathcal{D}(X)$ and $\mathcal{D}(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153–175.

[35] S. Mukai, *Symplectic structure of the moduli space of simple sheaves on an abelian or K3 surface*, Invent. Math. 77 (1984), 101–116.

[36] S. Mukai, *On the moduli space of bundles on K3 surfaces I*, in Vector Bundles on Algebraic Varieties, M. F. Atiyah et al., Oxford University Press (1987), 341–413.

[37] T. Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. 253 (1979), 1–90.

[38] K. O’Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface*, J. Algebraic Geom. 6 (1997), no. 4, 599–644.

[39] C. J. Rego, *The compactified Jacobian*, Ann. Sci. École Norm. Sup. 13 (1980), No. 4, 211-223.

[40] J. Sawon, *Abelian fibred holomorphic symplectic manifolds*, Turkish Jour. Math. 27 (2003), no. 1, 197–230.

[41] J. Sawon, *Derived equivalence of holomorphic symplectic manifolds*, Proceedings of the Workshop on algebraic structures and moduli spaces, CRM Montreal, July 2003, to appear.
[42] J. Sawon, *Deformations of holomorphic Lagrangian fibrations*, preprint (2004).

[43] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publ. Math. I.H.E.S 79 (1994), 47–129.

[44] A. Strominger, S-T. Yau, E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), 243–259.

[45] K. Yoshioka, *Irreducibility of moduli spaces of vector bundles on K3 surfaces*, arXiv preprint math.AG/9907001.