A FRACTIONAL DIFFERENTIAL EQUATION MODELING OF SARS-CoV-2 (COVID-19) DISEASE IN GHANA

Samuel Okyere\textsuperscript{a}, Ebenezer Bonyah\textsuperscript{b} and Joseph Ackora Prah\textsuperscript{c}

\textsuperscript{a,c}Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

\textsuperscript{b}Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Enterpreneurial Development, Kumasi, Ghana

Corresponding Author: Samuel Okyere; okyere2015@gmail.com

Abstract
The coronavirus (COVID -19) has spread through almost 224 countries and has caused over 5 million deaths. In this paper, we propose a model to study the transmission dynamics of SARS-CoV-2 in Ghana using fractional derivatives. The fractional derivative is defined in the Atangana – Beleanu – Caputo (ABC) sense. This model considers seven (7) classes, namely: Susceptible individuals (S), Exposed (E), Asymptomatic population (I\textsubscript{A}), Symptomatic (I\textsubscript{S}), Vaccinated (V), Quarantined (Q) and Recovered population (R). The equilibrium points, stability analysis and the basic reproduction number of the model have been determined. The existence and uniqueness of the solution and Ulam – Hyers stability are established. The model is tested using Ghana’s demographical and COVID-19 data. Further, two preventive control measures are incorporated into the model. The numerical analysis reveals the impact of the fractional-order derivative on the various classes of the disease model as one can get reliable information at any integer or non-integer value of the fractional operator. The results of the simulation depict the behaviour of Ghana’s COVID-19 infection cases. Analysis of the optimal control reveals social distancing leads to an increase in the susceptible population, whereas vaccination reduces the number of susceptible individuals. Both vaccination and social distancing leads to a decline in COVID-19 infections. It was established that the fractional-order derivatives could influence the behaviour of all classes in the proposed COVID -19 disease model.

Keywords: Fractional derivative, COVID-19, Atangana – Beleanu – Caputo, Reproductive Number, Stability Analysis, Hyers –Ulam Stability
1.0 Introduction

Coronaviruses are a large family of viruses that cause several illnesses such as cold, Middle East Respiratory Syndrome (MERS) and Severe Acute Respiratory Syndrome (SARS-CoV-2) [25]. This novel coronavirus is a new strain that has not been previously identified in humans [25]. The disease is transmitted through air, primarily via small droplets or particles such as aerosols, produced after an infected person breathes, coughs, sneezes, and talks [10, 11]. Common symptoms of the disease include fever, cough, fatigue, shortness of breath or breathing difficulties, and loss of smell and taste [1, 19, 26].

The disease has already spread across 224 countries and total confirmed cases stand at 276,707,027 as of December 22, 2021. The total reported deaths associated with COVID-19 are 5,388,419 and the recoveries stand at 248,241,353 [27]. Total confirmed cases for Africa is 9,347,424 with 226,825 deaths and 8,439,886 recoveries [27]. Ghana is not left out of the pandemic, the country had his first case of COVID-19 in March 2020, and since then there has been massive surge in the virus. Total confirmed cases as of 22nd December, 2021 is 133,046 with 1,263 COVID-19 deaths and recoveries at 129,829 [27]. Ghana’s cumulative case is shown in Fig.1.

Mathematical models have been proposed to study the transmission patterns of COVID-19 [3, 5, 13, 18]. Authors in [18], used SIR model to study the epidemic patterns of the COVID-19 disease taking into account some social measures such as distancing, lockdown, and quarantine. In recent years, fractional calculus has gained much attention and has been used to model diseases such as Tuberculosis and Syphilis [7, 22, 23] and recently COVID-19 [16, 20]. The
fractional – order can potentially describe more complex dynamics than the integer model and easily include memory effects present in many real – world phenomena. In this work, we propose fractional derivative to study the COVID-19 transmission in Ghana.

![Graph of Ghana’s cumulative COVID-19 cases from March, 2020 – November, 2021](image)

Fig.1: Ghana’s cumulative COVID-19 cases from March, 2020 – November, 2021 [27]

### 2 Model Formulations

In this section, we study the dynamic transmission of SARS-CoV-2 using fractional derivatives. The fractional derivative is defined in the Atangana – Beleanu – Caputo (ABC) sense. We define the fractional order operator as $\psi$, where $0 < \psi \leq 1$. The model considers seven (7) classes, namely: Susceptible individuals ($S$), Exposed ($E$), Asymptomatic ($I_A$), Symptomatic ($I_S$),
Vaccinated (V), Quarantined (Q) and Recovered population (R). The population is homogeneously mixing, with no restriction on age, mobility or other social factors. The susceptibles are recruited into the population at a rate $\Omega$. These individuals get exposed to the disease when they come in contact with the asymptomatic and symptomatic at a rate $\beta$. After being exposed to the covid-19 disease, they either progress to the asymptomatic class at the rate $(1-\alpha)\phi$ or the symptomatic class at the rate $\alpha\phi$. Both asymptomatic and symptomatic get quarantined at the rate $\rho$ and $\tau$ respectively. Those vaccinated according to this model, don’t get infected but may join the susceptible class at a rate $\Gamma$. The parameter $\mu$ and $\delta$, are the natural and the disease-induced death rate respectively. The parameters $\sigma, \theta$ and $\gamma$ are the rate of recovery for the asymptomatic, symptomatic, and quarantined class, respectively. All newborns are susceptible (no inherited immunity. The model diagram is displayed in Fig. 2.

![Flowchart of the COVID-19 fractional model](image)

Fig.2: Flowchart of the COVID-19 fractional model

The following fractional derivatives describe the model.
\[ \begin{align*}
^{ABC}D_{0,t}^\psi [S(t)] &= (1-\eta^\psi)\Omega^\psi + \Gamma^\psi V - \beta^\psi S \left( \frac{I_A + I_S}{N} \right) - \mu^\psi S \\
^{ABC}D_{0,t}^\psi [E(t)] &= \beta^\psi S \left( \frac{I_A + I_S}{N} \right) - (\phi^\psi + \mu^\psi)E \\
^{ABC}D_{0,t}^\psi [I_A(t)] &= (1-\alpha^\psi)\phi^\psi E - (\rho^\psi + \sigma^\psi + \mu^\psi + \delta^\psi)I_A \\
^{ABC}D_{0,t}^\psi [I_S(t)] &= \alpha^\psi \phi^\psi E - (\theta^\psi + \tau^\psi + \mu^\psi + \delta^\psi)I_S \\
^{ABC}D_{0,t}^\psi [Q(t)] &= \rho^\psi I_A + \tau^\psi I_S - (\gamma^\psi + \mu^\psi + \delta^\psi)Q \\
^{ABC}D_{0,t}^\psi [V(t)] &= \eta^\psi \Omega^\psi - (\Gamma^\psi + \mu^\psi)\nu \\
^{ABC}D_{0,t}^\psi [R(t)] &= \theta^\psi I_S + \sigma^\psi I_A + \gamma^\psi Q - \mu^\psi R \\
S(t) &\geq 0, E(t) \geq 0, I_A(t) \geq 0, I_S(t) \geq 0, Q(t) \geq 0, R(t) \geq 0
\end{align*} \]

2.1 Preliminaries

This section consists of the relevant definitions of Atanga – Baleanu derivative and integration in Caputo sense taken from [6]

**Definition 1**

The Liouville- Caputo (LC) of fractional derivative of order \( \psi \) as defined in [14] is

\[
^{AB_c}D_{0,t}^\psi r(t) = \frac{1}{\Gamma(1-\psi)} \int_0^t (t - p)^{-\psi} r(p) dp, \quad 0 < \psi \leq 1
\]

**Definition 2**

The Atangana – Baleanu definition in Liouville- Caputo sense as in [6] is

\[
^{AB_c}D_{0,t}^\psi r(t) = \frac{B(\psi)}{\Gamma(1-\psi)} \int_0^t (t - p)^{-\psi} r(p) dp
\]

Where \( B(\psi) = 1 - \psi + \frac{\psi}{\Gamma(\psi)} \) is the normalized function.

**Definition 3**

The corresponding fractional integral Atangana – Baleanu – Caputo derivative is defined as in [6] is

\[
^{AB_c}I_{0,t}^\psi r(t) = \frac{(1-\psi)}{B(\psi)} r(t) + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t - p)^{\psi-1} r(p) dp
\]
They found that when $\psi = 0$, they recovered the initial function and when $\psi = 1$, the ordinary integral is obtained. The Laplace transform of equation (4) gives

$$ABC D^\psi_{0,t} r(t) = \frac{(1-\psi)}{B(\psi)} g(t) + \frac{B(\psi)G(q)q^\psi - q^{\psi-1}r(0)}{(1-\psi)\left(q^\psi + \frac{\psi}{1-\psi}\right)}$$

(5)

**Theorem 1**

For a function $r \in C[a,b]$, the following results holds [9]:

$$\left\| ABC D^\psi_{0,t} r(t) \right\| < \frac{B(\psi)}{(1-\psi)} \left\| r(t) \right\|, \text{ where } \left\| r(t) \right\| = \max_{a \leq t \leq b} \left| r(t) \right|$$

(6)

Furthermore, the ABC derivatives fulfill the Lipschitz condition in [9]:

$$\left\| ABC D^\psi_{0,t} r_1(t) - ABC D^\psi_{0,t} r_2(t) \right\| < \alpha \left\| r_1(t) - r_2(t) \right\|$$

(7)

### 2.2 Existence and Uniqueness

We denote a banach space by $D(W)$ with $W = [0,b]$, containing real valued continuous function with sup norm and $T = D(W) \times D(W) \times D(W) \times D(W) \times D(W) \times D(W) \times D(W)$ and the given norm $\|(S,E,I_A,I_S,Q,V,R)\| = \|S\| + \|E\| + \|I_A\| + \|I_S\| + \|Q\| + \|V\| + \|R\|$ , where $\|S\| = \sup_{t \in J} |S|, \|E\| = \sup_{t \in J} |E|, \|I_A\| = \sup_{t \in J} |I_A|, \|I_S\| = \sup_{t \in J} |I_S|, \|Q\| = \sup_{t \in J} |Q|, \|V\| = \sup_{t \in J} |V|, \|R\| = \sup_{t \in J} |R|$. Using the ABC integral operator on the system (1), we have

$$S(t) - S(0) = ABC D^\psi_{0,t}[S(t)]\{(1-\eta^\psi)\Omega^\psi + \Gamma^\psi V(t) - \beta^\psi S(t)\left(\frac{I_A(t) + I_S(t)}{N}\right) - \mu^\psi S(t)\}$$

$$E(t) - E(0) = ABC D^\psi_{0,t}[E(t)]\{\beta^\psi S(t)\left(\frac{I_A(t) + I_S(t)}{N}\right) - (\phi^\psi + \mu^\psi)E(t)\}$$

$$I_A(t) - I_A(0) = ABC D^\psi_{0,t}[I_A(t)]\{(1-\alpha^\psi)\phi^\psi E(t) - (\rho^\psi + \sigma^\psi + \mu^\psi + \delta^\psi)I_A(t)\}$$

$$I_S(t) - I_S(0) = ABC D^\psi_{0,t}[I_S(t)]\{\alpha^\psi \phi^\psi E(t) - (\theta^\psi + \tau^\psi + \mu^\psi + \delta^\psi)I_S(t)\}$$

$$Q(t) - Q(0) = ABC D^\psi_{0,t}[Q(t)]\{\rho^\psi I_A(t) + \tau^\psi I_S(t) - (\gamma^\psi + \mu^\psi + \delta^\psi)Q(t)\}$$

$$V(t) - V(0) = ABC D^\psi_{0,t}[V(t)]\{\eta^\psi \Omega^\psi - (\Gamma^\psi + \mu^\psi)V(t)\}$$

$$R(t) - R(0) = ABC D^\psi_{0,t}[R(t)]\{\theta^\psi I_S(t) + \sigma^\psi I_A(t) + \gamma^\psi Q(t) - \mu^\psi R(t)\}$$

(8)
Now from Definition 1, we have

\[
S(t) - S(0) = \frac{1 - \psi}{B(\psi)} \Phi_1(\psi, t, S(t)) + \frac{\psi}{B(\psi)\Gamma(\psi)} \times \int_0^t (t - \tau)^{\gamma - 1} \Phi_1(\psi, \tau, S(\tau))d\tau,
\]

\[
E(t) - E(0) = \frac{1 - \psi}{B(\psi)} \Phi_2(\psi, t, E(t)) + \frac{\psi}{B(\psi)\Gamma(\psi)} \times \int_0^t (t - \tau)^{\gamma - 1} \Phi_2(\psi, \tau, E(\tau))d\tau,
\]

\[
I_A(t) - I_A(0) = \frac{1 - \psi}{B(\psi)} \Phi_3(\psi, t, I_A(t)) + \frac{\psi}{B(\psi)\Gamma(\psi)} \times \int_0^t (t - \tau)^{\gamma - 1} \Phi_3(\psi, \tau, I_A(\tau))d\tau,
\]

\[
I_S(t) - I_S(0) = \frac{1 - \psi}{B(\psi)} \Phi_4(\psi, t, I_S(t)) + \frac{\psi}{B(\psi)\Gamma(\psi)} \times \int_0^t (t - \tau)^{\gamma - 1} \Phi_4(\psi, \tau, I_S(\tau))d\tau,
\]

\[
Q(t) - Q(0) = \frac{1 - \psi}{B(\psi)} \Phi_5(\psi, t, Q(t)) + \frac{\psi}{B(\psi)\Gamma(\psi)} \times \int_0^t (t - \tau)^{\gamma - 1} \Phi_5(\psi, \tau, Q(\tau))d\tau,
\]

\[
V(t) - V(0) = \frac{1 - \psi}{B(\psi)} \Phi_6(\psi, t, V(t)) + \frac{\psi}{B(\psi)\Gamma(\psi)} \times \int_0^t (t - \tau)^{\gamma - 1} \Phi_6(\psi, \tau, V(\tau))d\tau,
\]

\[
R(t) - R(0) = \frac{1 - \psi}{B(\psi)} \Phi_7(\psi, t, R(t)) + \frac{\psi}{B(\psi)\Gamma(\psi)} \times \int_0^t (t - \tau)^{\gamma - 1} \Phi_7(\psi, \tau, R(\tau))d\tau,
\]

Where

\[
\Phi_1(\psi, \tau, S(t)) = (1 - \eta^\psi)\Omega^\psi + \Gamma^\psi V(t) - \beta^\psi S(t) \left( \frac{I_A(t) + I_S(t)}{N} \right) - \mu^\psi S(t),
\]

\[
\Phi_2(\psi, \tau, E(t)) = \beta^\psi S(t) \left( \frac{I_A(t) + I_S(t)}{N} \right) - (\phi^\psi + \mu^\psi)E(t),
\]

\[
\Phi_3(\psi, \tau, I_A(t)) = (1 - \alpha^\psi)\phi^\psi E(t) - (\rho^\psi + \sigma^\psi + \mu^\psi + \delta^\psi)I_A(t),
\]

\[
\Phi_4(\psi, \tau, I_S(t)) = \alpha^\psi \phi^\psi E(t) - (\theta^\psi + \tau^\psi + \mu^\psi + \delta^\psi)I_S(t),
\]

\[
\Phi_5(\psi, \tau, Q(t)) = \rho^\psi I_A(t) + \tau^\psi I_S(t) - (\gamma^\psi + \mu^\psi + \delta^\psi)Q(t),
\]

\[
\Phi_6(\psi, \tau, V(t)) = \eta^\psi \Omega^\psi - (\Gamma^\psi + \mu^\psi)V(t),
\]

\[
\Phi_7(\psi, \tau, R(t)) = \theta^\psi I_S(t) + \sigma^\psi I_A(t) + \gamma^\psi Q - \mu^\psi R(t)
\]

The function \( \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6 \) and \( \Phi_7 \) satisfies Lipschitz condition only if \( S(t), E(t), I_A(t), I_S(t), Q(t), V(t) \) and \( R(t) \) possess an upper bound. Suppose \( S(t) \) and \( S^*(t) \) are couple functions, then

\[
\| \Phi_1(\psi, t, S(t)) - \Phi_1(\psi, t, S^*(t)) \| = \| -[\beta^\psi (I_A(t) + I_S(t)) + \mu^\psi ](S(t) - S^*(t)) \|.
\]

Considering
\[ d_1 = \| (\beta^\nu (I_A(t) + I_5(t)) - \mu^\nu) \| \]

Equation (11) simplifies to
\[ \| \Phi_1(\psi, t, S(t)) - \Phi_1(\psi, t, S^*(t)) \| \leq d_1 \| (S(t) - S^*(t)) \| \quad (12) \]

Similarly,
\[ \| \Phi_2(\psi, t, E(t)) - \Phi_2(\psi, t, E^*(t)) \| \leq d_2 \| (E(t) - E^*(t)) \| \]
\[ \| \Phi_3(\psi, t, I_A(t)) - \Phi_3(\psi, t, I_A^*(t)) \| \leq d_3 \| (I_A(t) - I_A^*(t)) \| \]
\[ \| \Phi_4(\psi, t, I_5(t)) - \Phi_4(\psi, t, I_5^*(t)) \| \leq d_4 \| (I_5(t) - I_5^*(t)) \| \]
\[ \| \Phi_5(\psi, t, Q(t)) - \Phi_5(\psi, t, Q^*(t)) \| \leq d_5 \| (Q(t) - Q^*(t)) \| \]
\[ \| \Phi_6(\psi, t, V(t)) - \Phi_6(\psi, t, V^*(t)) \| \leq d_6 \| (V(t) - V^*(t)) \| \]
\[ \| \Phi_7(\psi, t, R(t)) - \Phi_7(\psi, t, R^*(t)) \| \leq d_7 \| (R(t) - R^*(t)) \| \quad (13) \]

where
\[ d_2 = \| (\varphi^\nu + \mu^\nu) \| \]
\[ d_3 = \| (\rho^\nu + \sigma^\nu + \mu^\nu + \delta^\nu) \| \]
\[ d_4 = \| (\theta^\nu + \tau^\nu + \mu^\nu + \delta^\nu) \| \]
\[ d_5 = \| (\gamma^\nu + \mu^\nu + \delta^\nu) \| \]
\[ d_6 = \| (\Gamma^\nu + \mu^\nu) \| \]
\[ d_7 = \| \mu^\nu \| \]

Hence Lipschitz condition holds. Now taking system (9) in a reiterative manner gives
\[ S_n(t) - S(0) = \frac{1 - \psi}{B(\psi)} \Phi_1(\psi, t, S_{n-1}(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t-\tau)^{\nu-1} \Phi_1(\psi, \tau, S_{n-1}(\tau)) \, d\tau, \]

\[ E_n(t) - E(0) = \frac{1 - \psi}{B(\psi)} \Phi_2(\psi, t, E_{n-1}(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t-\tau)^{\nu-1} \Phi_2(\psi, \tau, E_{n-1}(\tau)) \, d\tau, \]

\[ I_{A_n}(t) - I_{A}(0) = \frac{1 - \psi}{B(\psi)} \Phi_3(\psi, t, I_{A_{n-1}}(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t-\tau)^{\nu-1} \Phi_3(\psi, \tau, I_{A_{n-1}}(\tau)) \, d\tau, \]

\[ I_{S_n}(t) - I_{S}(0) = \frac{1 - \psi}{B(\psi)} \Phi_4(\psi, t, I_{S_{n-1}}(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t-\tau)^{\nu-1} \Phi_4(\psi, \tau, I_{S_{n-1}}(\tau)) \, d\tau, \]

\[ Q_n(t) - Q(0) = \frac{1 - \psi}{B(\psi)} \Phi_5(\psi, t, Q_{n-1}(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t-\tau)^{\nu-1} \Phi_5(\psi, \tau, Q_{n-1}(\tau)) \, d\tau, \]

\[ V_n(t) - V(0) = \frac{1 - \psi}{B(\psi)} \Phi_6(\psi, t, V_{n-1}(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t-\tau)^{\nu-1} \Phi_6(\psi, \tau, V_{n-1}(\tau)) \, d\tau, \]

\[ R_n(t) - R(0) = \frac{1 - \psi}{B(\psi)} \Phi_7(\psi, t, R_{n-1}(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t-\tau)^{\nu-1} \Phi_7(\psi, \tau, R_{n-1}(\tau)) \, d\tau, \]

(14)

With the initial conditions

\[ S_o(t) = S(0), \quad E_o(t) = E(0), \quad I_{A_0}(t) = I_{A}(0), \quad I_{S_0}(t) = I_{S}(0), \quad Q_o(t) = Q(0), \quad V_o(t) = V(0), \quad R_o(t) = R(0), \]

difference of consecutive terms yields
\[ \Pi_{S_n}(t) = S_n(t) - S_{n-1}(t) = \frac{1-\psi}{B(\psi)}(\Phi_1(\psi, t, S_{n-1}(t)) - \Phi_1(\psi, t, S_{n-2}(t))) \]

\[ + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t-\tau)^{\psi-1}(\Phi_1(\psi, \tau, S_{n-1}(\tau)) - \Phi_1(\psi, \tau, S_{n-2}(\tau)))d\tau \]

\[ \Pi_{E_n}(t) = E_n(t) - E_{n-1}(t) = \frac{1-\psi}{B(\psi)}(\Phi_2(\psi, t, E_{n-1}(t)) - \Phi_2(\psi, t, E_{n-2}(t))) \]

\[ + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t-\tau)^{\psi-1}(\Phi_2(\psi, \tau, E_{n-1}(\tau)) - \Phi_2(\psi, \tau, E_{n-2}(\tau)))d\tau \]

\[ \Pi_{I_{A_n}}(t) = I_{A_n}(t) - I_{A_{n-1}}(t) = \frac{1-\psi}{B(\psi)}(\Phi_3(\psi, t, I_{A_{n-1}}(t)) - \Phi_3(\psi, t, I_{A_{n-2}}(t))) \]

\[ + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t-\tau)^{\psi-1}(\Phi_3(\psi, \tau, I_{A_{n-1}}(\tau)) - \Phi_3(\psi, \tau, I_{A_{n-2}}(\tau)))d\tau \]

\[ \Pi_{I_{S_n}}(t) = I_{S_n}(t) - I_{S_{n-1}}(t) = \frac{1-\psi}{B(\psi)}(\Phi_4(\psi, t, I_{S_{n-1}}(t)) - \Phi_4(\psi, t, I_{S_{n-2}}(t))) \]

\[ + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t-\tau)^{\psi-1}(\Phi_4(\psi, \tau, I_{S_{n-1}}(\tau)) - \Phi_4(\psi, \tau, I_{S_{n-2}}(\tau)))d\tau \]

\[ \Pi_{Q_n}(t) = Q_n(t) - Q_{n-1}(t) = \frac{1-\psi}{B(\psi)}(\Phi_5(\psi, t, Q_{n-1}(t)) - \Phi_5(\psi, t, Q_{n-2}(t))) \]

\[ + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t-\tau)^{\psi-1}(\Phi_5(\psi, \tau, Q_{n-1}(\tau)) - \Phi_5(\psi, \tau, Q_{n-2}(\tau)))d\tau \]

\[ \Pi_{V_n}(t) = V_n(t) - V_{n-1}(t) = \frac{1-\psi}{B(\psi)}(\Phi_6(\psi, t, V_{n-1}(t)) - \Phi_6(\psi, t, V_{n-2}(t))) \]

\[ + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t-\tau)^{\psi-1}(\Phi_6(\psi, \tau, V_{n-1}(\tau)) - \Phi_6(\psi, \tau, V_{n-2}(\tau)))d\tau \]

\[ \Pi_{R_n}(t) = R_n(t) - R_{n-1}(t) = \frac{1-\psi}{B(\psi)}(\Phi_7(\psi, t, R_{n-1}(t)) - \Phi_7(\psi, t, R_{n-2}(t))) \]

\[ + \frac{\psi}{B(\psi)\Gamma(\psi)} \int_0^t (t-\tau)^{\psi-1}(\Phi_7(\psi, \tau, R_{n-1}(\tau)) - \Phi_7(\psi, \tau, R_{n-2}(\tau)))d\tau \]

Where \( S_n(t) = \sum_{i=0}^{n} \Pi_{S_i}(t), E_n(t) = \sum_{i=0}^{n} \Pi_{E_i}(t), I_{A_n}(t) = \sum_{i=0}^{n} \Pi_{I_{A_i}}(t), I_{S_n}(t) = \sum_{i=0}^{n} \Pi_{I_{S_i}}(t), \)

\( Q_n(t) = \sum_{i=0}^{n} \Pi_{Q_i}(t), V_n(t) = \sum_{i=0}^{n} \Pi_{V_i}(t), R_n(t) = \sum_{i=0}^{n} \Pi_{R_i}(t). \) Taking into consideration equations (12) – (13) and considering

\[ \Pi_{A_{n-1}}(t) = S_{n-1}(t) - S_{n-2}(t), \Pi_{E_{n-1}}(t) = E_{n-1}(t) - E_{n-2}(t), \Pi_{I_{A_{n-1}}}(t) = I_{A_{n-1}}(t) - I_{A_{n-2}}(t), \]

10
\[ \Pi_{I_{s_{n-1}}}(t) = I_{s_{n-1}}(t) - I_{s_{n-2}}(t), \Pi_{Q_{n-1}}(t) = Q_{n-1}(t) - Q_{n-2}(t), \Pi_{V_{n-1}}(t) = V_{n-1}(t) - V_{n-2}(t), \Pi_{R_{n-1}}(t) = R_{n-1}(t) - R_{n-2}(t), \]

\[ \|\Pi_{S_n}(t)\| \leq \frac{1 - \psi}{B(\psi)} d_1 \|\Pi_{S_{n-1}}(t)\| \frac{\psi}{B(\psi)\Gamma(\psi)} d_1 \times \int_0^t (t - \tau) \|\Pi_{S_{n-1}}(\tau)\| d\tau \]

\[ \|\Pi_{E_n}(t)\| \leq \frac{1 - \psi}{B(\psi)} d_2 \|\Pi_{E_{n-1}}(t)\| \frac{\psi}{B(\psi)\Gamma(\psi)} d_2 \times \int_0^t (t - \tau) \|\Pi_{E_{n-1}}(\tau)\| d\tau \]

\[ \|\Pi_{I_{a_n}}(t)\| \leq \frac{1 - \psi}{B(\psi)} d_3 \|\Pi_{I_{a_{n-1}}}(t)\| \frac{\psi}{B(\psi)\Gamma(\psi)} d_3 \times \int_0^t (t - \tau) \|\Pi_{I_{a_{n-1}}}(\tau)\| d\tau \]

\[ \|\Pi_{I_{s_n}}(t)\| \leq \frac{1 - \psi}{B(\psi)} d_4 \|\Pi_{I_{s_{n-1}}}(t)\| \frac{\psi}{B(\psi)\Gamma(\psi)} d_4 \times \int_0^t (t - \tau) \|\Pi_{I_{s_{n-1}}}(\tau)\| d\tau \]

\[ \|\Pi_{Q_n}(t)\| \leq \frac{1 - \psi}{B(\psi)} d_5 \|\Pi_{Q_{n-1}}(t)\| \frac{\psi}{B(\psi)\Gamma(\psi)} d_5 \times \int_0^t (t - \tau) \|\Pi_{Q_{n-1}}(\tau)\| d\tau \]

\[ \|\Pi_{V_n}(t)\| \leq \frac{1 - \psi}{B(\psi)} d_6 \|\Pi_{V_{n-1}}(t)\| \frac{\psi}{B(\psi)\Gamma(\psi)} d_6 \times \int_0^t (t - \tau) \|\Pi_{V_{n-1}}(\tau)\| d\tau \]

\[ \|\Pi_{R_n}(t)\| \leq \frac{1 - \psi}{B(\psi)} d_7 \|\Pi_{R_{n-1}}(t)\| \frac{\psi}{B(\psi)\Gamma(\psi)} d_7 \times \int_0^t (t - \tau) \|\Pi_{R_{n-1}}(\tau)\| d\tau \]

**Theorem 2:** The system (1) has a unique solution for \( t \in [0, b] \) subject to the condition

\[ \frac{1 - \psi}{B(\psi)} d_i + \frac{\psi}{B(\psi)\Gamma(\psi)} b^\psi n_i < 1, i = 1, 2, 3, \ldots, 7 \]

holds.

**Proof:**

Since \( S(t), E(t), I_{A}(t), I_{S}(t), Q(t), V(t) \) and \( R(t) \) are bounded functions and Equation (12) – (13) holds. In a recurring manner, (16) reaches
\[
\begin{align*}
\|S_n(t)\| & \leq \left\| S_n(t) \right\| \left( \frac{1 - \psi}{B(\psi)} d_1 + \frac{\psi b^\psi}{B(\psi) \Gamma(\psi)} d_1 \right)^n \\
\|E_n(t)\| & \leq \left\| E_n(t) \right\| \left( \frac{1 - \psi}{B(\psi)} d_2 + \frac{\psi b^\psi}{B(\psi) \Gamma(\psi)} d_2 \right)^n \\
\|I_{n_s}(t)\| & \leq \left\| I_{n_s}(t) \right\| \left( \frac{1 - \psi}{B(\psi)} d_3 + \frac{\psi b^\psi}{B(\psi) \Gamma(\psi)} d_3 \right)^n \\
\|I_{n_s}(t)\| & \leq \left\| I_{n_s}(t) \right\| \left( \frac{1 - \psi}{B(\psi)} d_4 + \frac{\psi b^\psi}{B(\psi) \Gamma(\psi)} d_4 \right)^n \\
\|Q_n(t)\| & \leq \left\| Q_n(t) \right\| \left( \frac{1 - \psi}{B(\psi)} d_5 + \frac{\psi b^\psi}{B(\psi) \Gamma(\psi)} d_5 \right)^n \\
\|V_n(t)\| & \leq \left\| V_n(t) \right\| \left( \frac{1 - \psi}{B(\psi)} d_6 + \frac{\psi b^\psi}{B(\psi) \Gamma(\psi)} d_6 \right)^n \\
\|R_n(t)\| & \leq \left\| R_n(t) \right\| \left( \frac{1 - \psi}{B(\psi)} d_7 + \frac{\psi b^\psi}{B(\psi) \Gamma(\psi)} d_7 \right)^n \\
\end{align*}
\]

And
\[
\|S_n(t)\| \to 0, \|E_n(t)\| \to 0, \|I_{n_s}(t)\| \to 0, \|Q_n(t)\| \to 0, \|V_n(t)\| \to 0, \|R_n(t)\| \to 0
\]
as \(n \to \infty\). Incorporating the triangular inequality and for any \(j\), system (17) yields
\[
\begin{align*}
\|S_{n+j}(t) - S_n(t)\| & \leq \sum_{i=n+1}^{n+j} T_i^j = \frac{T_{n+1}^n - T_{n+k+1}^n}{1 - T_1} \\
\|E_{n+j}(t) - E_n(t)\| & \leq \sum_{i=n+1}^{n+j} T_i^j = \frac{T_{n+1}^n - T_{n+k+1}^n}{1 - T_2} \\
\|I_{n_s}(t) - S_n(t)\| & \leq \sum_{i=n+1}^{n+j} T_i^j = \frac{T_{n+1}^n - T_{n+k+1}^n}{1 - T_3} \\
\|I_{n_s}(t) - S_n(t)\| & \leq \sum_{i=n+1}^{n+j} T_i^j = \frac{T_{n+1}^n - T_{n+k+1}^n}{1 - T_4} \\
\|Q_{n+j}(t) - Q_n(t)\| & \leq \sum_{i=n+1}^{n+j} T_i^j = \frac{T_{n+1}^n - T_{n+k+1}^n}{1 - T_5} \\
\|V_{n+j}(t) - V_n(t)\| & \leq \sum_{i=n+1}^{n+j} T_i^j = \frac{T_{n+1}^n - T_{n+k+1}^n}{1 - T_6} \\
\|R_{n+j}(t) - R_n(t)\| & \leq \sum_{i=n+1}^{n+j} T_i^j = \frac{T_{n+1}^n - T_{n+k+1}^n}{1 - T_7} \\
\end{align*}
\]
Where \( T_i = \frac{1 - \psi}{B(\psi)} d_i + \frac{\psi}{\Gamma(\psi)} b^\psi d_i \leq 1 \). Hence there exists unique solution for system (1)

### 2.3 Hyers–Ulam Stability

**Definition 4**

The ABC fractional system given by equation (9) is said to be Hyers Ulam stable if for every \( \lambda_i > 0, i \in \mathbb{N} \) there exists constants \( h_i > 0, i \in \mathbb{N} \) satisfying:

\[
\begin{align*}
S(t) &- \frac{1 - \psi}{B(\psi)} \Phi_1(\psi, t, S(t)) + \frac{\psi}{\Gamma(\psi)} \times \int_0^t (t - \tau)^{\psi-1} \Phi_1(\psi, \tau, S(\tau)) d\tau \leq \lambda_1, \\
E(t) &- \frac{1 - \psi}{B(\psi)} \Phi_2(\psi, t, E(t)) + \frac{\psi}{\Gamma(\psi)} \times \int_0^t (t - \tau)^{\psi-1} \Phi_2(\psi, \tau, E(\tau)) d\tau \leq \lambda_2, \\
I_A(t) &- \frac{1 - \psi}{B(\psi)} \Phi_3(\psi, t, I_A(t)) + \frac{\psi}{\Gamma(\psi)} \times \int_0^t (t - \tau)^{\psi-1} \Phi_3(\psi, \tau, I_A(\tau)) d\tau \leq \lambda_3, \\
I_S(t) &- \frac{1 - \psi}{B(\psi)} \Phi_4(\psi, t, I_S(t)) + \frac{\psi}{\Gamma(\psi)} \times \int_0^t (t - \tau)^{\psi-1} \Phi_4(\psi, \tau, I_S(\tau)) d\tau \leq \lambda_4, \\
Q(t) &- \frac{1 - \psi}{B(\psi)} \Phi_5(\psi, t, Q(t)) + \frac{\psi}{\Gamma(\psi)} \times \int_0^t (t - \tau)^{\psi-1} \Phi_5(\psi, \tau, Q(\tau)) d\tau \leq \lambda_5, \\
V(t) &- \frac{1 - \psi}{B(\psi)} \Phi_6(\psi, t, V(t)) + \frac{\psi}{\Gamma(\psi)} \times \int_0^t (t - \tau)^{\psi-1} \Phi_6(\psi, \tau, V(\tau)) d\tau \leq \lambda_6, \\
R(t) &- \frac{1 - \psi}{B(\psi)} \Phi_7(\psi, t, R(t)) + \frac{\psi}{\Gamma(\psi)} \times \int_0^t (t - \tau)^{\psi-1} \Phi_7(\psi, \tau, R(\tau)) d\tau \leq \lambda_7,
\end{align*}
\]

And there exist \( \{S(t), E(t), S(t), I_S(t), I_A(t), Q(t), R(t)\} \) where

\[13\]
\[
\begin{align*}
\dot{S}(t) &= \frac{1 - \psi}{B(\psi)} \Phi_1(\psi, t, S(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t - \tau) \Phi_1(\psi, \tau, \dot{S}(\tau)) d\tau \\
\dot{E}(t) &= \frac{1 - \psi}{B(\psi)} \Phi_2(\psi, t, E(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t - \tau) \Phi_2(\psi, \tau, \dot{E}(\tau)) d\tau \\
\dot{I}_A(t) &= \frac{1 - \psi}{B(\psi)} \Phi_3(\psi, t, I_A(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t - \tau) \Phi_3(\psi, \tau, \dot{I}_A(\tau)) d\tau \\
\dot{I}_S(t) &= \frac{1 - \psi}{B(\psi)} \Phi_4(\psi, t, I_S(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t - \tau) \Phi_4(\psi, \tau, \dot{I}_S(\tau)) d\tau \\
\dot{Q}(t) &= \frac{1 - \psi}{B(\psi)} \Phi_5(\psi, t, Q(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t - \tau) \Phi_5(\psi, \tau, \dot{Q}(\tau)) d\tau \\
\dot{V}(t) &= \frac{1 - \psi}{B(\psi)} \Phi_6(\psi, t, V(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t - \tau) \Phi_6(\psi, \tau, \dot{V}(\tau)) d\tau \\
\dot{R}(t) &= \frac{1 - \psi}{B(\psi)} \Phi_7(\psi, t, R(t)) + \frac{\psi}{B(\psi) \Gamma(\psi)} \int_0^t (t - \tau) \Phi_7(\psi, \tau, \dot{R}(\tau)) d\tau
\end{align*}
\]

Such that
\[
\begin{align*}
\left| S(t) - \dot{S}(t) \right| &\leq \xi_1 \kappa_1, \left| E(t) - \dot{E}(t) \right| \leq \xi_2 \kappa_2, \left| I_A(t) - \dot{I}_A(t) \right| \leq \xi_3 \kappa_3, \left| I_S(t) - \dot{I}_S(t) \right| \leq \xi_4 \kappa_4, \\
\left| Q(t) - \dot{Q}(t) \right| &\leq \xi_5 \kappa_5, \left| V(t) - \dot{V}(t) \right| \leq \xi_6 \kappa_6, \left| R(t) - \dot{R}(t) \right| \leq \xi_7 \kappa_7
\end{align*}
\]

3 Model Analyses

This section looks at the steady state of system (1). We have the disease –free steady state and the endemic steady state. The disease – free equilibrium is the steady state solution where there is no infection in the population. The disease – free equilibrium \( (E_0) \) is given as

\[
E_0 = (S^0, E^0, I_A^0, I_S^0, Q^0, V^0, R^0) = \left( \frac{\Omega^\psi (\Gamma^\psi + \mu^\psi (1 - \eta^\psi))}{\mu^\psi (\mu^\psi + \Gamma^\psi)} , 0, 0, 0, 0, 0, 0 \right)
\]
3.1 Basic Reproductive Number

We now calculate the basic reproductive number \((R_0)\) of system (1). The basic reproductive number is the number of secondary cases produced, in a totally susceptible population, by a single infective individual during the time span of the infection [24]. Using the next generation operator method [24], denote \(F\) and \(V\), respectively, as matrices for the new infections generated and the transition terms we obtain

\[
F = \begin{bmatrix}
0 & \beta^v S^0 & \beta^v S^0 & 0 \\
(1-\alpha^v)\phi^v & 0 & 0 & 0 \\
\alpha^v \phi^v & 0 & 0 & 0 \\
0 & \rho^v & \tau^v & 0 
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
\phi^v + \mu^v & 0 & 0 & 0 \\
0 & \rho^v + \sigma^v + \mu^v + \delta^v & 0 & 0 \\
0 & 0 & \theta^v + \tau^v + \mu^v + \delta^v & 0 \\
0 & 0 & 0 & \gamma^v + \mu^v + \delta^v 
\end{bmatrix}
\]

\[
V^{-1} = \begin{bmatrix}
\frac{1}{(\phi^v + \mu^v)} & 0 & 0 & 0 \\
0 & \frac{1}{(\rho^v + \sigma^v + \mu^v + \delta^v)} & 0 & 0 \\
0 & 0 & \frac{1}{(\theta^v + \tau^v + \mu^v + \delta^v)} & 0 \\
0 & 0 & 0 & \frac{1}{(\gamma^v + \mu^v + \delta^v)} 
\end{bmatrix}
\]

Now the basic reproductive number is given as the spectra radius of the matrix \(FV^{-1}\).

\(R_0 = R_1 + R_2\), where

\[
R_1 = \frac{\beta^v \Omega^v (\Gamma^v + \mu^v (1-\eta^v))}{\mu^v (\mu^v + \Gamma^v)(\rho^v + \sigma^v + \mu^v + \delta^v)} \quad \text{and} \quad R_2 = \frac{\beta^v \Omega^v (\Gamma^v + \mu^v (1-\eta^v))}{\mu^v (\mu^v + \Gamma^v)(\theta^v + \tau^v + \mu^v + \delta^v)}
\]
Represents the reproduction number for system (1)

3.2 Endemic Steady States

The endemic equilibrium of system (1) is represented by $E^* = (S^*, E^*, V^*, I_A^*, I_S^*, Q^*, R^*)$

Where

$$\begin{align*}
S^* &= \frac{(1 - \eta^w) \Omega^v + \Gamma^v V^*}{\beta^w (I_A^* + I_S^*) + \mu^v}, \quad E^* = \frac{\beta^w S^* (I_A^* + I_S^*)}{\phi^v + \mu^v}, \quad I_A^* = \frac{(1 - \alpha^v) \phi E^*}{\rho^v + \sigma^v + \mu^v + \delta^v}, \\
I_S^* &= \frac{\alpha^v \phi^v E^*}{\rho^v + \sigma^v + \mu^v + \delta^v}, \quad Q^* = \frac{\eta^w \Omega^v}{\gamma^v + \mu^v + \delta^v}, \quad V^* = \frac{\eta^w \Omega^v}{\Gamma^v + \mu^v}, \quad R^* = \frac{\theta^v I_S^* + \sigma^v I_A^* + \gamma^v Q^*}{\mu^v}.
\end{align*}$$

4 Stability Analysis of the model

The necessary conditions for the local stability of the disease –free and endemic equilibrium are established in this section.

4.1 Local Stability of the Disease – free Equilibrium

**Theorem 3:** The disease-free equilibrium is locally asymptotically stable if $R_o < 1$.

Proof:

The Jacobian matrix of system (1) is given as

$$J = \begin{pmatrix}
-\mu^v & 0 & -\beta^w S & -\beta^v S & 0 & \Gamma^v & 0 \\
\beta^w (I_A^* + I_S^*) - (\phi^v + \mu^v) & \beta^w S & \beta^v S & 0 & 0 & 0 \\
0 & (1 - \alpha^v) \phi^v & -(\rho^v + \sigma^v + \mu^v + \delta^v) & 0 & 0 & 0 \\
0 & \alpha^v \phi^v & 0 & -(\theta^v + \tau^v + \mu^v + \delta^v) & 0 & 0 & 0 \\
0 & 0 & \rho^v & \tau^v & -(\gamma^v + \mu^v + \delta^v) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -(\Gamma^v + \mu^v) & 0 \\
0 & 0 & \sigma^v & \theta^v & \gamma^v & 0 & -\mu^v
\end{pmatrix}$$

The Jacobian matrix evaluated at the disease-free equilibrium point is given as
Theorem 4: The disease-free equilibrium of system (1) is globally asymptotically stable if \( R_o < 1 \) and the following conditions are satisfied.

4.2 Global Stability of the Disease – free Equilibrium Point
1. \( \frac{dV}{dt} = P(V,0) = 0 \)

2. \( P = (V,F) = AV - T(V,F) \geq 0 \)

**Proof:**

We apply the theorem in [8]. We start by dividing system (1) into two submodels, namely, the infectious class and noninfectious class. We denote the infectious class by \( V \) and the noninfectious class by \( F \). The variable \( V \) and \( F \) are written as \( V = (E,I_A,I_S,Q) \in R_+^4 \) and \( F = (S,V,R) \in R_+^3 \). The system (1) can now be written as

\[
\frac{dV}{dt} = P(V,F), \quad \frac{dF}{dt} = T(V,F) \tag{31}
\]

The two valued function \( P(V,F) \) and \( T(V,F) \) are given by

\[
P(V,F) = \begin{cases} 
\beta^v (I_A + I_S)S - (\phi^v + \mu^v)E \\
(1-\alpha^v)\phi^v E - (\rho^v + \sigma^v + \mu^v + \delta^v)I_A \\
\alpha^v \phi^v E - (\theta^v + \tau^v + \mu^v + \delta^v)I_S \\
\rho^v I_A + \tau^v I_S - (\gamma^v + \mu^v + \delta^v)Q 
\end{cases} \quad \text{and} \quad
T(V,F) = \begin{cases} 
(1-\eta^v)\Omega^v + \Gamma^vV - \beta^v (I_A + I_S)S - \mu^v S \\
\eta^v \Omega^v - (\Gamma^v + \mu^v) V \\
\theta^v I_S + \sigma^v I_A + \gamma^v Q - \mu^v R 
\end{cases} \tag{32}
\]

The reduce form of the system \( \frac{dF}{dt} = T(F,0) \) is given as

\[
\frac{dS}{dt} = (1-\mu^v)\Omega^v + \Gamma^vV - \mu^v S, \\
\frac{dV}{dt} = \eta^v \Omega^v - (\Gamma^v + \mu^v) V, \tag{33} \\
\frac{dR}{dt} = -\mu^v R,
\]

\( F^* = (S^*,V^*,R^*) = \left( \frac{(1-\eta^v)\Omega^v + \Gamma^vV^*}{\mu^v}, \frac{\eta^v \Omega^v}{\Gamma^v + \mu^v}, 0 \right) \) is a globally asymptotically stable equilibrium point for the reduce system \( \frac{dF}{dt} = T(F,0) \).
Solving the third equation of system (33) gives \( R(t) = R(0)e^{-\mu^v(t)} \to 0 \) as \( t \to \infty \). Solving the second equation of system (33) gives \( V(t) = \frac{\eta^v \Omega^v}{\Gamma^v + \mu^v} + V(0)e^{-(\Gamma^v + \mu^v)t} \to \frac{\eta^v \Omega^v}{\Gamma^v + \mu^v} \) as \( t \to \infty \).

Solving the first equation of system (33) gives

\[
S(t) = \frac{1}{\mu^v} \left(1 - \mu^v \right) \Omega^v + \Gamma \left( \frac{\eta^v \Omega^v}{\Gamma^v + \mu^v} + V(0)e^{-(\Gamma^v + \mu^v)t} \right) \] 

\[
= S(0) e^{-\mu^v(t)} \to \frac{1}{\mu^v} \left(1 - \mu^v \right) \Omega^v + \Gamma \left( \frac{\eta^v \Omega^v}{\Gamma^v + \mu^v} \right)
\]

as \( t \to \infty \).

Hence, the convergence of system (1) is global in \( \Psi \). \( P = (V, F) \), satisfies the following conditions given in [8].

3. \( \frac{dV}{dt} = P(V, 0) = 0 \)

4. \( P = (V, F) = AV - T(V, F) \geq 0 \)

Where \( A = \Phi_v P(V, 0) = \)

\[
\begin{pmatrix}
- (\varphi^v + \mu^v) E \\
(1 - \alpha^v) \varphi^v E - (\rho^v + \sigma^v + \mu^v + \delta^v) I_A \\
\alpha^v \varphi^v E - (\theta^v + \tau^v + \mu^v + \delta^v) I_S \\
\rho^v I_A + \tau^v I_S - (\gamma^v + \mu^v + \delta^v) Q
\end{pmatrix}
\]
satisfy the conditions above.

### 4.3 Local Stability of the Endemic Equilibrium

**Theorem 5.** If \( R_o > 1 \), then the pandemic equilibrium \( (E^*) \) is locally asymptotically stable.

**Proof:**

The Jacobian matrix of system (1) evaluated at the endemic equilibrium point is given as
The first three eigenvalues can be obtained from the fifth, sixth, and seventh columns which are; 
\(- (\gamma + \mu + \delta), \quad -(\Gamma + \mu) \quad \text{and} \quad -\mu \). The rest of the eigenvalues are obtained from system (35) by excluding the fifth, sixth and seventh rows and columns of system (34). Hence we have

\[
J_{\xi} = \begin{pmatrix}
\begin{array}{cccccc}
-\mu^\nu & 0 & -\beta^\nu S^\nu & -\beta^\nu S^\nu & 0 & \Gamma & 0 \\
\beta^\nu \left( I_1 + I_3 \right) & -(\phi^\nu + \mu^\nu) & \beta^\nu S^\nu & \beta^\nu S^\nu & 0 & 0 & 0 \\
0 & (1 - \alpha^\nu) \phi^\nu & -(\rho^\nu + \sigma^\nu + \mu^\nu + \delta^\nu) & 0 & 0 & 0 & 0 \\
0 & \alpha^\nu \phi^\nu & 0 & -(\theta^\nu + \tau^\nu + \mu^\nu + \delta^\nu) & 0 & 0 & 0 \\
0 & 0 & \rho & \tau^\nu & -\left( \gamma^\nu + \mu^\nu + \delta^\nu \right) & 0 & 0 \\
0 & 0 & 0 & \theta & \gamma & 0 & -\mu^\nu \\
0 & 0 & \sigma & \theta & \gamma & 0 & \mu^\nu \\
\end{array}
\end{pmatrix}
\]

The characteristic equation of system (35) is given as

\[
\lambda^4 + B_1 \lambda^3 + B_2 \lambda^2 + B_3 \lambda + B_4 = 0 
\]

Where

\[
B_1 = \mu^\nu + J_{22} + J_{33} + J_{44}
\]

\[
B_2 = \mu^\nu \left[ J_{22} + J_{33} + J_{44} \right] + J_{22} \left[ J_{33} + J_{44} \right] - \beta^\nu S^\nu \left( J_{32} + \alpha^\nu \phi^\nu \right) + J_{33} J_{44}
\]

\[
B_3 = \mu^\nu J_{33} \left( J_{44} + J_{22} \right) - \mu^\nu J_{32} \beta^\nu S^\nu + \mu^\nu J_{22} J_{44} - \mu^\nu \alpha^\nu \phi^\nu \beta^\nu S^\nu + \beta^\nu \left( I_1^* + I_3^* \right) S^\nu \left( J_{32} + \alpha^\nu \phi^\nu \right) + A
\]

\[
B_4 = \mu^\nu J_{44} \left( J_{22} J_{33} - J_{32} \beta^\nu S^\nu \right) + \alpha^\nu \phi^\nu \beta^\nu S^\nu \left( J_{33} \left( \beta^\nu - \mu^\nu \right) + \beta^\nu \left( I_1^* + I_3^* \right) S^\nu \right) J_{32} J_{44}
\]

\[
J_{22} = \phi^\nu + \mu^\nu, \quad J_{33} = \rho^\nu + \sigma^\nu + \mu^\nu + \delta^\nu, \quad J_{44} = \theta^\nu + \tau^\nu + \mu^\nu + \delta^\nu, \quad J_{32} = \left( 1 - \alpha^\nu \right) \phi^\nu
\]

\[
A = J_{44} \left( J_{22} J_{33} - J_{32} \beta^\nu S^\nu - \alpha^\nu \phi^\nu J_{33} \beta^\nu S^\nu \right)
\]

From Routh- Hurwitz stability criterion, if the conditions \( B_1 > 0, B_3 > 0, B_4 > 0 \) and \( B_1 B_2 B_3 > B_2^2 + B_4^2 \) are satisfied, then the characteristic equation above has negative real parts and hence a stable equilibrium.
5 Numerical Scheme of the Fractional Derivative

Let us consider the first equation of system (1)

\[ ^{ABC}D_{0,t}^\alpha [S(t)] = r(t, S(t)), \quad S(0) = S_o \quad (37) \]

Applying the fundamental theorem of fractional calculus to equation (37), we obtain

\[ S(t) - S(0) = \frac{1-\psi}{B(\psi)} r(t, S(t)) + \frac{\psi}{\Gamma(\psi)B(\psi)} \int_0^t g(\tau, S(\tau))(t - \tau)^{\psi - 1} d\tau \quad (38) \]

Where \( B(\psi) = 1 - \psi + \frac{\psi}{\Gamma(\psi)} \) is a normalised function and at \( t_{n+1} \) we have,

\[ S_{n+1} = S_o + \frac{(1-\psi)\Gamma(\psi)}{(1-\psi)\Gamma(\psi) + \psi} r(t_n, S(t_n)) + \frac{\psi}{\Gamma(\psi) + \psi(1-\Gamma(\psi))} \sum_{m=0}^{n} h \times (t_{n+1} - \tau)^{\psi - 1}. \quad (39) \]

Implementing two-step Lagrange’s interpolation polynomial on the interval \([t_n, t_{n+1}]\) [21], we have

\[ Y = \frac{g(t_m, S_m)}{h} (\tau - t_{m-1}) - \frac{r(t_{m-1}, S_{m-1})}{h} (\tau - t_m) \quad (40) \]

Equation (40) is replaced with equation (39) and by performing the steps given in [21], we obtain

\[ S(t_{n+1}) = S(t_n) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, S(t_n)) + \frac{1}{(\psi + 1)\Gamma(\psi) + \psi} \sum_{m=0}^{n} h^\psi r(t_m, S(t_m))(n+1-m)^\psi \]
\[ \times (n+m+2+p) - (n+m)^\psi (n+m+2+2p) - h^\psi r(t_{m-1}, S(t_{m-1}))(n+1-m)^{\psi+1}(n+m+2+p) \]
\[ - (n+m)^\psi (n+m+1+p) \quad (41) \]

To obtain high stability, we replace the step size \( h \) in equation (41) with \( \phi(h) \) such that \( \phi(h) = h + O(h^2), \quad 0 < \phi(h) \leq 1 \) [17].

The new scheme which is called the nonstandard two –step Lagrange interpolation method (NS2LIM) is given as:
\[ S(t_{n+1}) = S(t_0) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, S(t_n)) + \frac{1}{(\psi + 1)(1-\psi)\Gamma(\psi) + \psi} \sum_{m=0}^{n} \phi(m)\psi r(t_n, S(t_m))(n+1-m)^\psi \\
\times (n-m+2+\psi) - (n-m)^\psi (n-m+2+2\psi) - \phi(h)^\psi r(t_{m-1}, S(t_{m-1}))(n+1-m)^{\psi+1} (n-m+2+\psi) \\
- (n-m)^\psi (n-m+1+\psi) \tag{42} \]

Similarly,

\[ E(t_{n+1}) = E(t_0) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, E(t_n)) + \frac{1}{(\psi + 1)(1-\psi)\Gamma(\psi) + \psi} \sum_{m=0}^{n} \phi(m)\psi r(t_n, E(t_m))(n+1-m)^\psi \\
\times (n-m+2+\psi) - (n-m)^\psi (n-m+2+2\psi) - \phi(h)^\psi r(t_{m-1}, E(t_{m-1}))(n+1-m)^{\psi+1} (n-m+2+\psi) \\
- (n-m)^\psi (n-m+1+\psi) \tag{43} \]

\[ I_A(t_{n+1}) = I_A(t_0) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, I_A(t_n)) + \frac{1}{(\psi + 1)(1-\psi)\Gamma(\psi) + \psi} \sum_{m=0}^{n} \phi(m)\psi r(t_n, I_A(t_m))(n+1-m)^\psi \\
\times (n-m+2+\psi) - (n-m)^\psi (n-m+2+2\psi) - \phi(h)^\psi r(t_{m-1}, I_A(t_{m-1}))(n+1-m)^{\psi+1} (n-m+2+\psi) \\
- (n-m)^\psi (n-m+1+\psi) \]

\[ I_S(t_{n+1}) = I_S(t_0) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, I_S(t_n)) + \frac{1}{(\psi + 1)(1-\psi)\Gamma(\psi) + \psi} \sum_{m=0}^{n} \phi(m)\psi r(t_n, I_S(t_m))(n+1-m)^\psi \\
\times (n-m+2+\psi) - (n-m)^\psi (n-m+2+2\psi) - \phi(h)^\psi r(t_{m-1}, I_S(t_{m-1}))(n+1-m)^{\psi+1} (n-m+2+\psi) \\
- (n-m)^\psi (n-m+1+\psi) \]

\[ Q(t_{n+1}) = Q(t_0) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, Q(t_n)) + \frac{1}{(\psi + 1)(1-\psi)\Gamma(\psi) + \psi} \sum_{m=0}^{n} \phi(m)\psi r(t_n, Q(t_m))(n+1-m)^\psi \\
\times (n-m+2+\psi) - (n-m)^\psi (n-m+2+2\psi) - \phi(h)^\psi r(t_{m-1}, Q(t_{m-1}))(n+1-m)^{\psi+1} (n-m+2+\psi) \\
- (n-m)^\psi (n-m+1+\psi) \]

\[ V(t_{n+1}) = V(t_0) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, V(t_n)) + \frac{1}{(\psi + 1)(1-\psi)\Gamma(\psi) + \psi} \sum_{m=0}^{n} \phi(m)\psi r(t_n, V(t_m))(n+1-m)^\psi \\
\times (n-m+2+\psi) - (n-m)^\psi (n-m+2+2\psi) - \phi(h)^\psi r(t_{m-1}, V(t_{m-1}))(n+1-m)^{\psi+1} (n-m+2+\psi) \\
- (n-m)^\psi (n-m+1+\psi) \]

\[ R(t_{n+1}) = R(t_0) + \frac{\Gamma(\psi)(1-\psi)}{\Gamma(\psi)(1-\psi) + \psi} r(t_n, R(t_n)) + \frac{1}{(\psi + 1)(1-\psi)\Gamma(\psi) + \psi} \sum_{m=0}^{n} \phi(m)\psi r(t_n, R(t_m))(n+1-m)^\psi \\
\times (n-m+2+\psi) - (n-m)^\psi (n-m+2+2\psi) - \phi(h)^\psi r(t_{m-1}, R(t_{m-1}))(n+1-m)^{\psi+1} (n-m+2+\psi) \\
- (n-m)^\psi (n-m+1+\psi) \]

6 Numerical Simulation

In this section, we validate the COVID-19 model by using COVID-19 confirmed cases data from Ghana Health Service for the period March – September, 2020 [12]. We also estimate
the parameters of the COVID-19 model and test the effect of the fractional order derivative on the various classes of the model. After formulating a model, one important thing is to validate the model to see if it will stand the test of time. Model validation is the process of determining the degree to which a mathematical model is an accurate representation of the available data. Using matlab gaussfit, the cumulative data of confirmed COVID-19 cases for the period March – September, 2020 is depicted in Fig. 3. And Fig.4 shows the residuals of the best fitted curve. The parameter values is given in Table 1

| Parameter | Description | Value | Source |
|-----------|-------------|-------|--------|
| $\Omega^\nu$ | Recruitment rate | 29.08 | [15] |
| $\beta^\nu$ | Transmission rate | 0.9 | Estimated |
| $\phi^\nu$ | The rate at which exposed individuals become infectious | 0.021199 | Assumed |
| $\mu$ | Natural death rate | 0.4252912 × 10^{-4} | Estimated |
| $\delta^\nu$ | Disease-induced death rate | 1.6728 × 10^{-5} | [2] |
| $\theta^\nu$ | Recovery rate of symptomatic individuals | 1/14 | [4] |
| $\Gamma^\nu$ | Rate at which vaccinated individuals lose their immunity | 1.52 × 10^{-7} | Assumed |
| $\sigma^\nu$ | Recovery rate of asymptomatic individuals | 0.25 | Assumed |
| $\gamma^\nu$ | Recovery rate of quarantine individuals | 1/14 | Assumed |
| $\tau^\nu$ | Rate at which symptomatic individuals move to the quarantine class | 0.01 | [4] |
| $\rho^\nu$ | Rate at which asymptomatic individuals move to the quarantine class | 1.026 × 10^{-7} | Assumed |
| $\eta^\nu$ | Rate at which susceptible individuals are vaccinated | 0.0269 | [25] |
Fig. 3: Cumulative cases of Ghana’s COVID-19 from March to September, 2020 with the best fitted curve.

Fig. 4: Residuals for the best fitted curve.
Given the initial conditions \( S(0) = 30800000, E(0) = 100, I_A(0) = 100, I_S(0) = 100 \), \( Q(0) = 0, V(t) = 0 \) and \( R(t) = 0 \). The simulation is displayed in Figures 5 – 11, where Fig.5 -11 depicts the behaviour of the susceptible, exposed, asymptomatic, symptomatic, quarantine, vaccinated and recovered individuals respectively.

![Graph showing behaviour of susceptible individuals](image)

Fig. 5: Behaviour of the susceptible individuals at different values of \( \psi \).
Fig. 6: Behaviour of the exposed individuals at different values of $\psi$.

Fig. 7: Behaviour of the asymptomatic at different values of $\psi$. 
Fig. 8: Behaviour of the symptomatic at different values of $\psi$.

Fig. 9: Behaviour of the quarantine at different values of $\psi$. 
**Fig. 10**: Behaviour of the vaccinated individuals at different values of $\psi$.

**Fig. 11**: Behaviour of the recovered individuals at different values of $\psi$. 
Fig. 5 depicts the behavior of the susceptible individuals for integer and non-integer values of $\nu$. The number of the susceptible individuals reduces as the fractional order derivative $(\nu)$ decreases from 1.0 to 0.55 within 200 days. Figs. 6 – 11 depict the behavior of the exposed, asymptomatic, symptomatic, quarantined and recovered individuals for integer and non-integer values of $\nu$. The number of exposed individuals increases as the fractional order derivative $\nu$ increases from 0.55 to 1.0 within 200 days. The number of asymptomatic, symptomatic, and quarantine individuals increases as the fractional order derivative increases from 0.55 to 1.0 within the 200-day period.

7 Fractional Optimal Control Problem

We add two control functions, $u_1$ and $u_2$ into the system (1). Where control $u_1$ and $u_2$ are social distancing and vaccination respectively. We include the time-dependent controls into system (1) and we have

$$\begin{align*}
ABC D^\nu_{0,t} [S(t)] &= (1-\eta^\nu) \Omega^\nu + \Gamma^\nu V - (1-u_1) \beta^\nu \frac{S(I_A + I_S)}{N} - \mu^\nu S - u_2 S \\
ABC D^\nu_{0,t} [E(t)] &= (1-u_1) \beta^\nu \frac{S(I_A + I_S)}{N} - (\phi^\nu + \mu^\nu) E \\
ABC D^\nu_{0,t} [I_A(t)] &= (1-\alpha^\nu) \phi^\nu E - (\rho^\nu + \sigma^\nu + \mu^\nu + \delta^\nu) I_A \\
ABC D^\nu_{0,t} [I_S(t)] &= \alpha^\nu \phi^\nu E - (\theta^\nu + \tau^\nu + \mu^\nu + \delta^\nu) I_S \\
ABC D^\nu_{0,t} [Q(t)] &= \rho^\nu I_A + \tau^\nu I_S - (\gamma^\nu + \mu^\nu + \delta^\nu) Q \\
ABC D^\nu_{0,t} [V(t)] &= \eta^\nu \Omega^\nu - (\Gamma^\nu + \mu^\nu) V + u_2 S \\
ABC D^\nu_{0,t} [R(t)] &= \theta^\nu I_A + \sigma^\nu I_A + \gamma^\nu Q - \mu^\nu R \\
S(t) &\geq 0, E(t) \geq 0, I_A(t) \geq 0, I_S(t) \geq 0, Q(t) \geq 0, R(t) \geq 0
\end{align*}$$

The objective function for fixed time $t_f$ is given as

$$J(u_1, u_2) = \int_0^{t_f} \left[ G_1 S(t) + G_2 E(t) + G_3 I_A(t) + G_4 I_S(t) + G_5 Q(t) + \frac{1}{2} (T_1 u_1^2 + T_2 u_2^2) \right] dt$$

(44)
Where $T_1$ and $T_2$ are the measure of relative cost of interventions associated with the controls $u_1$ and $u_2$. We find optimal controls $u_1$ and $u_2$ that minimizes the cost function

$$J(u_1, u_2, u_3) = \int_0^{t_f} \xi(S, E, I_A, I_S, Q, V, R) dt$$

(45)

Subject to the constraint

$$D_{0+}^{\alpha_1}[S(t)] = \xi_1, D_{0+}^{\alpha_2}[E(t)] = \xi_2, D_{0+}^{\alpha_3}[I_A(t)] = \xi_3, D_{0+}^{\alpha_4}[I_S(t)] = \xi_4, D_{0+}^{\alpha_5}[Q(t)] = \xi_5, D_{0+}^{\alpha_6}[V(t)] = \xi_6, D_{0+}^{\alpha_7}[R(t)] = \xi_7$$

(46)

Where $\xi_i = \xi(S, E, I_A, I_S, Q, V, R), i = 1, 2, 3, \ldots, 7, \Phi = (u_1, u_2) | u_i$ is a Lebesgue measurable on $[0,1]$ such that $0 \leq (u_1, u_2) \leq 1, \forall t \in [0, t_f]$, where $t_f$ is the final time and with initial conditions

$$S(0) = S_0, E(0) = E_0, I_A(0) = I_{A0}, I_S(0) = I_{S0}, Q(0) = Q_0, V(0) = V_0, R(0) = R_0.$$

To define the fractional optimal control, we consider the following modified cost function [22]:

$$J = \int_0^{t_f} H_{\psi}(S, E, I_A, I_S, Q, V, R, u_j, t) - \sum_{i=1}^{7} \lambda_i \xi_i(S, E, I_A, I_S, Q, V, R, u_j, t) dt$$

(47)

Where $i = 1, \ldots, 7$ and $j = 1, 2, 3$.

For the fractional optimal control, the Hamiltonian is

$$H_{\psi}(S, E, I_A, I_S, Q, V, R, u_j, t) = v(S, E, I_A, I_S, Q, V, R, u_j, t) + \sum_{i=1}^{7} \lambda_i \xi_i(S, E, I_A, I_S, Q, V, R, u_j, t)$$

(48)

Where $i = 1, \ldots, 7$ and $j = 1, 2, 3$. The following are essential for the formulation of the fractional optimal control [22].
\[ \begin{align*}
ABC D_{0,j}^\nu \Lambda_S &= \frac{\partial H}{\partial \Lambda_S},
ABC D_{0,j}^\nu \Lambda_E &= \frac{\partial H}{\partial \Lambda_E},
ABC D_{0,j}^\nu I_A &= \frac{\partial H}{\partial I_A},
ABC D_{0,j}^\nu I_S &= \frac{\partial H}{\partial I_S},
ABC D_{0,j}^\nu \Lambda_Q &= \frac{\partial H}{\partial \Lambda_Q},
\end{align*} \]

(48)

Moreover,

\[ 0 = \frac{\partial H}{\partial u_i}, \]

(49)

Moreover,

\[ \Lambda_S(t_f) = \Lambda_E(t_f) = \Lambda_{I_A}(t_f) = \Lambda_{I_S}(t_f) = \Lambda_Q(t_f) = \Lambda_V(t_f) = \Lambda_R(t_f) = 0 \]

(50)

are the Lagrange multipliers. Equation (48) and (49) provides the necessary conditions for the fractional optimal control in terms of the Hamiltonian for the optimal control problem defined above. The Hamiltonian, H, defined by

\[ H = k_1S^* + k_2E^* + k_3I_A^* + k_4I_S^* + k_5Q^* + \frac{1}{2}(T_1u_1^2 + T_2u_2^2) \]

(51)

\[ + {\ ABC \ D}_{0,j}^\nu \Lambda_S + {\ ABC \ D}_{0,j}^\nu \Lambda_E + {\ ABC \ D}_{0,j}^\nu I_A + {\ ABC \ D}_{0,j}^\nu I_S + {\ ABC \ D}_{0,j}^\nu \Lambda_Q + {\ ABC \ D}_{0,j}^\nu \Lambda_V + {\ ABC \ D}_{0,j}^\nu \Lambda_R \]

Theorem 6: Given an optimal control \((u_1^*, u_2^*) \in U\) and corresponding solution \(S^*, E^*, I_A^*, I_S^*, Q^*, V^*, R^*\) that minimizes \(J(u_1, u_2)\) over U. Furthermore, there exist adjoint variables \(\Lambda_S, \Lambda_E, \Lambda_{I_A}, \Lambda_{I_S}, \Lambda_Q, \Lambda_V, \Lambda_R\), satisfying

\[ - \frac{d\Lambda_i}{dt} = \frac{\partial H}{\partial i} \]

(52)
Where $\Lambda_S, \Lambda_E, \Lambda_{I_A}, \Lambda_{I_S}, \Lambda_Q, \Lambda_V, \Lambda_R$, with the transversality conditions

$$\Lambda_S(t_f) = \Lambda_E(t_f) = \Lambda_{I_A}(t_f) = \Lambda_{I_S}(t_f) = \Lambda_Q(t_f) = \Lambda_V(t_f) = \Lambda_R(t_f) = 0$$

(53)

**Proof:**

The differential equation characterized by the adjoint variables are obtained by considering the right hand side differentiation of system (51) determined at the optimal control. The adjoint equations derived are given as

$$
\begin{align*}
\text{ABC} \frac{\partial \Lambda_S}{\partial t} &= \beta^v (I_A - I_S)(1-u_1)[\Lambda_S - \Lambda_E] + (\mu^v + u_2)\Lambda_S + u_2\Lambda_V, \\
\text{ABC} \frac{\partial \Lambda_E}{\partial t} &= (\phi^v + \mu^v)\Lambda_E - (1-\alpha^v)\phi^v \Lambda_{I_A} - \alpha^v \phi^v \Lambda_{I_S}, \\
\text{ABC} \frac{\partial \Lambda_{I_A}}{\partial t} &= (\rho^v + \sigma^v + \mu^v + \delta^v)\Lambda_{I_A} + (1-u_1)\beta^v S\Lambda_S - \rho^v \Lambda_Q - \sigma^v \Lambda_R, \\
\text{ABC} \frac{\partial \Lambda_{I_S}}{\partial t} &= (\theta^v + \tau^v + \mu^v + \delta^v)\Lambda_{I_S} + (1-u_1)\beta^v S\Lambda_S - \tau^v \Lambda_Q - \theta^v \Lambda_R, \\
\text{ABC} \frac{\partial \Lambda_Q}{\partial t} &= (\gamma^v + \mu^v + \delta^v)\Lambda_Q - \gamma^v \Lambda_R, \\
\text{ABC} \frac{\partial \Lambda_V}{\partial t} &= -\Gamma^v \Lambda_S + (\Gamma^v + \mu^v)\Lambda_V, \\
\text{ABC} \frac{\partial \Lambda_R}{\partial t} &= \mu^v \Lambda_R
\end{align*}
$$

By obtaining the solution for $u_1^*$ and $u_2^*$ subject to the constraints, we have

$$
\begin{align*}
0 &= \frac{\partial H}{\partial u_1} = -T_1 u_1 + \beta^v S(I_A + I_S)[\Lambda_E - \Lambda_S], \\
0 &= \frac{\partial H}{\partial u_2} = -T_2 u_2 + S[\Lambda_S - \Lambda_V]
\end{align*}
$$

(55)

This gives

$$
\begin{align*}
u_1^* &= \min \left( 1, \max \left( 0, \frac{\beta^v S(I_A + I_S)[\Lambda_E - \Lambda_S]}{T_1} \right) \right), \\
u_2^* &= \min \left( 1, \max \left( \frac{S[\Lambda_S - \Lambda_V]}{T_2} \right) \right)
\end{align*}
$$

(56)
8 Numerical Simulation of the Optimal Control Model

In this section, we analyze the numerical behavior of the optimal control model. Using the parameter values given in Table 1, and the same initial conditions $S(0)=30,800000$, $E(0)=100$, $A(0)=100$, $Q(0)=100$, $V(0)=0$, $R(0)=0$,

8.1 Prevention Control ($u_1$)

we simulate the behavior of the compartments. Setting $u_1 = 0.25$, the results are displayed in Fig. 12 - 16

![Graph showing the behavior of the susceptible class with and without control](image)

Fig. 12: Behaviour of the susceptible class with and without control ($\psi = 1, u_1 = 0.25$)
Fig. 13: Behaviour of the exposed class with and with control \((\psi = 1, u_i = 0.25)\)

Fig. 14: Behaviour of the asymptomatic class with and without control \((\psi = 1, u_i = 0.25)\)
Fig. 15: Behaviour of the symptomatic class with and without control ($\psi = 1, u_1 = 0.25$)

Fig. 16: Behaviour of the quarantine class with and without control ($\psi = 1, u_1 = 0.25$)
Figs. 12 – 16 depicts the behaviour of the susceptible, exposed, asymptomatic, symptomatic and quarantine when the fractional derivative $\psi = 1$ and the control $u_1 = 0.25$. Fig. 12 depicts an increase in the number of susceptible individuals whilst there is a decline in the number of exposed, asymptomatic, symptomatic and quarantined individuals. Hence social distancing measures taken by the government rather increased the number of susceptible and also reduces infection.

### 8.2 Prevention Control $(u_2)$

We simulate the behaviour of the compartments. Setting $u_2 = 0.005$, the results are displayed in Figs. 17-21.

![Graph showing the behaviour of the susceptible with and without control](image)

**Fig. 17:** Behaviour of the susceptible with and without control ($\psi = 1, u_2 = 0.005$)
Fig. 18: Behaviour of the exposed with and without control ($\psi = 1, u_2 = 0.005$)

Fig. 19: Behaviour of the asymptomatic with and without control ($\psi = 1, u_2 = 0.005$)
Fig. 20: Behaviour of the symptomatic with and without control \((\psi=1, u_2 = 0.005)\)

Fig. 21: Behaviour of the quarantine with and without control \((\psi=1, u_2 = 0.005)\)
Figs. 17 – 21 depicts the behaviour of the the susceptible, exposed, asymptomatic, symptomatic and quarantine when the fractional derivative $\psi = 1$ and the control $u_2 = 0.005$. There is a decline in the number of susceptible, exposed, asymptomatic, symptomatic and quarantined individuals when there is a vaccination control.

9 Conclusion

In this study, a COVID-19 model has been examined using the fractional ABC operator in the Caputo sense. The basic properties of the model were examined. The equilibrium points of the model were found and stability analyses were carried out. The disease-free equilibrium was both locally and globally stable. The basic reproductive number of the model was determined. The existence and the uniqueness of solution are established along with Hyers –Ulam Stability. The numerical scheme for the operator was carried out to obtain a numerical simulation to support the analytical results. The proposed fractional – order can potentially describe more complex dynamics than the integer model and easily include memory effects present in many real – world phenomena. It was established that the fractional order derivatives could influence the behaviour of all classes in the COVID -19 disease model. The two preventive control measures suggest a reduction in total infections when there is social distancing and also vaccination, however, social distancing increases individuals’ susceptibility to COVID-19 disease whilst vaccination reduces the number of people susceptible to the disease.

Data Availability
The data/information supporting the formulation of the mathematical model in this paper are/is from Ghana health service website: https://www.ghs.gov.gh/covid19/ which has been cited in the manuscript.

Declaration of conflict of interest

No conflict of interest regarding the content of this article

Funding

The research did not receive funding from any sources.

Acknowledgement

This Manuscript was submitted as a pre-print in the link https://arxiv.org/ftp/arxiv/papers/2201/2201.08689.pdf and has been referenced.

Reference

[1] Agyemang AA, Chin KL, Landersdorfer CB, Liew D, Ofori-Asenso R. Smell and taste Dysfunction in Patients with COVID-19: A systematic Review and Meta – analysis. Mayo Clin. Proc., 2020; 95(8):1621-1631. Doi:10.1016/j.mayocp.2020.05.030 https://www.ncbi.nlm.nih.gov/pmc/articles/PMC7275152/

[2] Ahmed I, Modu GU, Yusuf A, Kumam P and Yusuf I. A mathematical model of coronavirus disease (COVID-19) containing asymptomatic and symptomatic classes. Elsevier public health emergency collection, Results Phys. 2021; 21:103776, doi: 10.1016/j.rinp.2020.103776. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC7787076/

[3] Ajbar AM, Ali E, Ajbar A. Modeling the evolution of the coronavirus disease (COVID-19) in Saudi Arabia. J infect Dev Ctries, 2021; 15(7):918-924, doi:10.3855/jidc.13568.PMID:34343116 https://jidc.org/index.php/journal/article/view/34343116

[4] Anggriani N, Ndii MZ, Amelia R, Suryaningrat W and Pratama MAA. A mathematical COVID-19 model considering asymptomatic and symptomatic classes with waning immunity. Alexandria engineering journal, 2022; vol. 61, issue 1, pg. 113 – 124. https://www.sciencedirect.com/science/article/pii/S1110016821003513

[5] Appiah-Otoo I, Kursah MB. Modelling spatial variations of novel coronavirus disease (COVID-19): evidence from a global perspective. GeoJournal 2021;;1-15. PMID: 33935350. https://dx.doi.org/10.1007/s10708-021-10427-0
[6] Atangana, A. and Baleanu, D. New Fractional Derivatives with Non-Local and Non-Singular Kernel: Theory and Application to Heat Transfer Model. Thermal Science, 2016; 20, 763-769. http://dx.doi.org/10.2298/TSCI160111018A

[7] Bonyah E, Chukwu W, Juga M, Fatmawati F. Modeling fractional-order dynamics of Syphilis via Mittag-Leffler law. AIMS Mathematics, 2021; 6(8):8367-8389 DOI:10.3934/math.2021485

[8] Castillo-Chavez C, and B. Song ‘Dynamical model of tuberculosis and their applications’, Math. Biosci. Eng., 2002; 1:361- 404. https://pubmed.ncbi.nlm.nih.gov/20369977/

[9] Coll C, Herrero A, Sanchez E, Thome N. A dynamic model for a study of diabetes. Math Comput Model, 50 (5–6)(2009), pp. 713-716. https://dl.acm.org/doi/abs/10.1016/j.mcm.2008.12.027

[10] Duncan J. Two cases of coronavirus confirmed in Ghana”. Citi Newsroom https://citinewsroom.com/2020/03/two-cases-of-coronavirus-confirmed-in-ghana/ (accessed 16 March2020)

[11] European Centre for Disease Prevention and Control, ‘Transmission of COVID-19’. https://www.ecdc.europa.eu/en/covid-19/latest-evidence/transmission, (Retrieved 12 September 2020)

[12] Ghana Health Service, ‘COVID-19 Updates(Ghana’, https://www.ghs.gov.gh/covid19/, (accessed 1st December, 2021)

[13] Ivorra B, Ferrández MR, Vela-Peréz M, Ramos AM, Mathematical modeling of the spread of the coronavirus disease 2019 (COVID-19) taking into account the undetected infections. The case of China. Communications in nonlinear science and numerical simulation. 2020: 105303. Doi: 10.1016/j.cnsns.2020.105303

[14] Moghaddam BP, Yaghoobi S, Machado JT. An extended predictor-corrector algorithm for variable-order fractional delay differential equations. J Comput Nonlinear Dyn, 2006; 1: 1-11

[15] Indexmundi, ‘Ghana Birth rate’, 2021, www.indexmundi.com (accessed 11th December, 2021)

[16] Ndaïrou F, Area I, Nieto JJ, Silva CJ, Torres DFM. Fractional model of COVID-19 applied to Galicia, Spain, and Portugal. Chaos Solitons & Fractals, 2021; 144: 110652, ISSN 0960-0779, https://doi.org/10.1016/j.chaos.2021.110652.

[17] Patidar KC. Nonstandard finite difference methods: recent trends and further developments. J Diff Equat Appl, 2016; 22:6, 817-849 doi:10.1080/10236198.2016.1144748,

[18] Sameni R. Mathematical modeling of epidemic diseases: A case study of the COVID-19 coronavirus. Quantitative biology, 2020; arXiv:2003.11371v4 [q-bio.PE]. https://arxiv.org/abs/2003.11371

[19] Saniasiaya J and Islam MA. Prevalence and characteristics of taste disorders in Cases of COVID-19: Systematic review and meta-analysis of 29349 patients. Otolaryngology – Head and neck Surgery, 2021; 165(1): 33-42. doi:10.1177/0194599820981018. https://journals.sagepub.com/doi/10.1177/0194599820981018

[20] Shaikh AS, Shaikh IN, Nisar KS. A mathematical model of COVID-19 using fractional derivative: outbreak in India with dynamics of transmission and control. Adv Differ Equ. 2020; 373. https://doi.org/10.1186/s13662-020-02834-3
[21] Solís-Pérez JE, Gómez-Aguilar JF, Atangana A. Novel numerical method for solving variable-order fractional differential equations with power, exponential and Mittag-Leffler laws. Chaos Solitons Fractals, 2018; 114:175-185. http://www.sciencedirect.com/science/article/pii/S096007791830350

[22] Sweilam NH, AL-Mekhlafi SM, Baleanu D. Optimal control for a fractional tuberculosis infection model including the impact of diabetes and resistant strains. Journal of Advanced Research, 2019; 17: 125-137, ISSN 2090-1232, https://doi.org/10.1016/j.jare.2019.01.007.

[23] Ullah S, Khan MA, Farooq M, Hammouch Z and Baleanu D. A fractional model for the dynamics of tuberculosis infection using Caputo-Fabrizio derivative. American institute of mathematical sciences, 2020; 13(3): 975-993. doi: 10.3934/dcdss.2020057

[24] van den Driessche P, Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math Biosci. 2002;180:29-48. doi: 10.1016/s0025-5564(02)00108-6. PMID: 12387915
https://www.sciencedirect.com/science/article/pii/S0025556402001086?via%3Dihub

[25] World Health Organization (WHO). Coronavirus disease (COVID-19). 2021
https://www.who.int/westernpacific/health-topics/detail/coronavirus

[26] World Health Organization, ‘Q&A on coronaviruses (COVID-19)’, 2020;
https://www.who.int/emergencies/diseases/novel-coronavirus-2019/question-and-answers-hub/q-a-detail/q-a-coronaviruses (accessed 14 May 2020)

[27] Worldometer, “COVID-19 Coronavirus Pandemic”, 2021,
http://www.worldometers.info/coronavirus

[28] S. Okyere, J.A. Prah and E. Bonyah, “A fractional differential equation modelling of SARS – CoV – 2 (COVID-19) disease: A case study of Ghana”, arXiv:2201.08689v1 [q-bio.PE],
https://arxiv.org/ftp/arxiv/papers/2201/2201.08689.pdf