Strategic Exploration for Innovation

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Abstract

This paper introduces a framework to study innovation in a strategic setting, in which innovators allocate their resources between exploration and exploitation in continuous time. Exploration creates public knowledge, while exploitation delivers private benefits. Through the analysis of a class of Markov equilibria, we demonstrate that knowledge spillovers accelerate knowledge creation and expedite its availability, thereby encouraging innovators to increase exploration. The prospect of the ensuing superior long-term innovations further motivates exploration, giving rise to a positive feedback loop. This novel feedback loop can substantially mitigate the free-riding problem arising from knowledge spillovers.

Keywords: Strategic experimentation, Encouragement effect, Innovation, Multi-armed bandit

JEL classification: C73; D83; O3.

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1 Introduction

Without innovation, we would still be living in caves and foraging for food in the wild. Although people celebrate disruptive inventions like penicillin and the Internet, innovations are often the result of small improvements through incremental experimentation. This approach can be observed in various fields and leads to significant advancements and innovations. In agriculture, breeding and selection of crops have led to new varieties with improved yields and disease resistance. In architecture, iconic structures that balance form and function have emerged through refinement of styles over time. And the gradual improvement of digital electronics has resulted in powerful, portable, and user-friendly devices.

The pursuit of innovation through experimentation typically entails opportunity costs and uncertainties, resulting in ubiquitous trade-offs between creating value through innovation (exploration) and capturing value through established operations (exploitation). As innovation is not guaranteed, innovators must carefully weigh the potential benefits of exploring new alternatives against the costs of diverting resources from their proven operations. As a result, incentives for experimentation can be undermined by free-riding when its outcomes are publicly observable, or when technologies can be reverse-engineered. Such information and knowledge spillovers lead to learning inefficiencies and stifle technological progress. Given the prevalence of free-riding opportunities, one might ask why anyone would pursue exploration. What are the trade-offs that individuals and organizations face when developing technologies with strategic considerations in mind? How does this strategic effect impact technological progress in the long run? We explore these questions in a model intended to capture the dynamics of collective exploration in a strategic setting, often seen in contexts such as research joint ventures, open-source software development, and academic collaborations.

This paper analyzes a game of strategic exploration in which a finite number of forward-looking players jointly search for innovative technologies. Given a set of publicly available technologies, at each point in time, each player decides how to allocate a perfectly divisible unit of resource between exploration, which expands the set of feasible technologies at a rate proportional to the resource allocated, and exploitation, which yields a flow payoff from the adoption of one of the feasible technologies. The qualities of technologies, which determine the flow payoff from exploitation, are represented by an (initially unknown) realized path of Brownian motion, referred to as the technological landscape. Not all technologies are readily available, and only the qualities associated with the explored technologies are known. The outcomes of exploration are transparent, and the technologies developed are treated as public goods. Consequently, perfect infor-
information and knowledge spillovers occur among players, offering abundant opportunities for free-riding, where some players benefit from the positive externalities generated by the experimentation of others.

One of the major contributions of this paper is to extend the classic game of strategic experimentation to a setting with an unboundedly expandable set of arms while allowing for a certain degree of correlation among them. When modeling technologies as arms, correlation between arms and discoveries of new arms are key features of technology development and innovation. However, most of the literature on strategic experimentation assumes a fixed set of independent arms. Indeed, the analysis in the previously studied models would be substantially complicated by correlation. By contrast, our proposed model demonstrates that under a form of correlation structure that is simple yet appropriate in the context of technology development, such complexities can be circumvented by simply focusing on incremental experimentation. This approach offers an intuitive way to capture the richness of the dynamic and path-dependent nature of technology development.

We first consider the efficient benchmark in which the players work jointly to maximize the average payoff. The solution to this cooperative problem takes a simple cutoff form: All players allocate their resource exclusively to exploration if the quality difference between the best available technology and the latest outcome of the experiment is below a time-invariant cutoff, and exclusively to exploitation otherwise. Players in a larger team are more ambitious, manifested by their persistence in exploration even after experiencing a prolonged series of setbacks. As team size grows, in the limit, exploration persists even when the quality difference becomes arbitrarily large, indefinitely broadening the set of long-run feasible technologies.

In the strategic problem, we restrict attention to Markov perfect equilibria (MPE) with the difference between the qualities of the best available technology and the latest technology under development as the state variable. Despite the considerable disparities between our setting and the two-armed bandit models in the literature on strategic experimentation, all MPE in our model exhibit a similar encouragement effect: The future information and knowledge spillovers from other players encourage at least one of them to continue exploration at states where a single agent would already have given up. The players thus act more ambitiously in the hope that successful outcomes will bring the other players back to exploration in the future, promoting innovation and sharing its burden. In addition, exactly because of such an incentive, any player who never explores would strictly prefer the deviation that resumes the exploration process as a volunteer at the state where all exploration stops. Therefore, no player always free-rides in equilibrium even though exploration per se never produces any payoffs.
We further show that there is no MPE in which all players use simple cutoff strategies as in the cooperative solution. This result suggests that in any symmetric equilibrium, each player chooses an interior allocation of their resource at some states. We establish the existence and uniqueness of the symmetric equilibrium, and provide closed-form representations for the equilibrium strategies and the associated payoff functions. In the symmetric equilibrium, for a set of parameters in which free-riding incentives are relatively weak, all players allocate their resource exclusively to exploration when the latest outcome is sufficiently close to the highest-known quality. However, for some other parameters, the incentives to free-ride can also be so strong that, right after the best technology to date has just been discovered, players, expecting incremental improvements to be achieved in no time, still allocate a positive fraction of their resource to exploitation. Within the region of interior allocation, the players gradually reduce the resource allocated to exploration as the latest outcomes deteriorate, regardless of the parameters.

Above all, we identify a novel innovation effect that is essential for understanding the incentives for technology development in dynamic and strategic environments. To explain this effect, we introduce the concept of returns to cooperation as follows. We say a technological landscape yields decreasing returns to cooperation if players’ payoff functions in the cooperative solution remain bounded as the number of players increases to infinity, reminiscent of the existing bandit models with a fixed set of arms. By contrast, exploration cooperatively over landscapes with nondecreasing returns to cooperation gives rise to unbounded payoffs in the limit. This asymptotic feature sets our model apart from the existing ones, as it demonstrates that introducing innovation into multi-armed bandits models allows us to identify a novel incentive for exploration: the prospect of technological advancement. We then examine the interplay between this innovation effect and the encouragement effect through the comparative statics of the unique symmetric equilibrium with respect to the number of players. Our analysis shows that the encouragement effect and the innovation effect reinforce each other. Moreover, the innovation effect boosts the encouragement effect to overcome the free-rider effect as the team gets large, if and only if the players are sufficiently patient and the underlying landscape yields increasing returns to cooperation. In such a case, even though the rate of exploration is suboptimal, the set of long-run feasible technologies expands indefinitely as the team size grows, resembling the cooperative solution. The prevalence of the encouragement effect in our model stands in marked contrast to the existing literature on strategic experimentation, in which the free-rider effect always prevails due to the lack of innovation possibilities.

In the symmetric equilibrium, exploration slows down as the latest outcomes deteri-
orate, but never fully stops. The reason is that as the latest outcomes keep deteriorating, exploration would slow down so severely that technologies never progress to the point at which all players prefer to allocate the resource exclusively to exploitation. This observation strongly suggests that asymmetric equilibria with the rate of exploration bounded away from zero in the region of interior allocation could improve welfare and long-run outcomes over the symmetric equilibrium. To investigate such a possibility, we construct a class of asymmetric MPE in which the players take turns performing exploration at unpromising states, so that each player achieves a higher payoff than in the symmetric equilibrium. It turns out that these asymmetric MPE are the best MPE in two-player games in terms of average payoffs. Unlike in the symmetric equilibrium, the players in these asymmetric MPE become more ambitious as team size grows, irrespective of the other parameters. The intuition for this result is that when alternation is allowed, the burden of keeping the exploration process active can be shared among more players in larger teams, and thus the players are willing to explore at less promising states. As a consequence, the set of long-run feasible technologies expands indefinitely as team size grows, irrespective of the patience of the players or the returns to cooperation. Nevertheless, similar to the symmetric equilibrium with impatient players, in these asymmetric equilibria innovations might arrive at a much slower rate than in the cooperative solution because of the low proportion of the overall resource allocated to exploration. As a result, the welfare loss might still be significant in large teams because of the strong free-riding incentives.

1.1 Related Literature

This paper combines two distinct strands of literature on learning and experimentation. The first strand studies experimentation in rich and complex environments, drawing on the seminal work by Callander (2011). He proposed modeling the correlation between technologies by Brownian path and studies experimentation conducted by a sequence of myopic agents. In his model, experiment outcomes closer to zero are preferable to the agents. In a similar setting, Callander and Matouschek (2019) consider agents with insatiable preferences and examine the impact of risk aversion on their search performance. Garfagnini and Strulovici (2016) extend Callander’s model to a setting with overlapping generations, in which short-lived yet forward-looking players search on a Brownian path for technologies with higher qualities. They mainly focus on the search patterns and the long-run dynamics, and establish the stagnation of search and the emergence of a tech-

\[1\] It is not clear whether these asymmetric MPE attain the highest average payoffs for games with more than two players.
nological standard in finite time. The qualities of technologies contribute exponentially to the payoffs in our model instead of linearly as in theirs. As a result, stagnation can be avoided in our model even under a negative drift of the Brownian path. All these models focus on non-strategic environments and preclude long-lived forward-looking agents, as their discrete-time frameworks create unexplored gaps between explored technologies, posing challenges for further analysis. To overcome this difficulty, we forgo the fine details of the learning dynamics for analytical tractability by imposing continuity on the experimentation process. This simplification can be interpreted as the qualities of the neighboring technology being revealed during experimentation, so that no unexplored territory remains between the explored technologies. Moreover, we impose a hard constraint on the scope of exploration to capture the scenario that technologies far ahead of their time are infeasible to be explored today. These abstractions allow us to derive explicit expressions for the equilibrium payoffs and strategies, perform comparative statics analysis, and construct asymmetric equilibria.

The second strand of literature, often referred to as strategic experimentation, originated from the Brownian model introduced by Bolton and Harris (1999), and further enriched by the exponential model in Keller, Rady, and Cripps (2005) and the Poisson model in Keller and Rady (2010) and Keller and Rady (2015). In all of these models, players face identical two-armed bandit machines, which consist of a risky arm with an unknown quality and a safe arm with a known quality. At each point in time, each player decides how to split one unit of a perfectly divisible resource between these arms, so that learning occurs gradually by observing other players’ actions and outcomes. These models differ in the assumptions on the probability distribution of the flow payoffs that each type of arm generates. By contrast, players in our model face a continuum of correlated arms. Local learning—learning the quality of a particular arm—occurs instantaneously. However, as the set of arms is unbounded, global learning—learning the qualities of all arms—occurs gradually.

The encouragement effect was first identified by Bolton and Harris (1999) in the symmetric equilibrium in their Brownian model. This effect is then established by Keller and Rady (2010) for all MPE in the Poisson model with inconclusive good news, and by Keller and Rady (2015) for the symmetric MPE in the Poisson model with bad news. Due to the absence of technological advancements, the encouragement effects in all these papers are not strong enough to dominate the free-rider effect.\(^\text{2}\) We demonstrate \(^\text{2}\)Bolton and Harris (1999) demonstrate the prevalence of the free-rider effect in their comparative statics analysis. They show that in the symmetric equilibrium of their model, the individual resource allocated to experimentation at beliefs below the myopic cutoff converge to zero as the number of players increases. The same feature can be observed in the symmetric MPE of the Poisson model in Keller and Rady (2010) and Keller and Rady (2015), despite the absence of a formal comparative statics analysis in

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not only the presence of the encouragement effect in all MPE of our model, but also perform comparative statics for the unique symmetric MPE to further investigate the strength of the encouragement effect. In contrast to the encouragement effect in their models, we find that the prospect of innovation and technological advancements enables the encouragement effect to overcome the free-rider effect, and provide the conditions for this to occur.

More broadly, this paper contributes to the literature on dynamic public-good games. Admati and Perry (1991), Marx and Matthews (2000), Yildirim (2006), and Georgiadis (2014) study voluntary contributions to a joint project in dynamic settings. While the public good in these papers is the progress toward the completion of a project, the public good in our model is the knowledge—the feasible technologies—built over time, which can be exploited once developed.

From a modeling perspective, continuous exploration on a Brownian sample path is independently studied by Wong (2022) and Urgun and Yariv (2023) in non-strategic environments. To the best of our knowledge, the only other study of collective exploration on a Brownian path in a strategic setting is an independent work by Cetemen, Urgun, and Yariv (2023). They focus on the exit patterns during a search process conducted jointly by heterogeneous players. In their model, exploitation is only possible after an irreversible exit chosen endogenously by each player. The players in our model, however, are not faced with stopping problems and thus are free to choose between exploration and exploitation, or even both simultaneously, at all times.

2 The Exploration Game

Time $t \in [0, \infty)$ is continuous, and the discount rate is $r > 0$. There are $N \geq 1$ players, each endowed with one unit of perfectly divisible resource per unit of time. Each player has to independently allocate her resource between exploration, which expands the feasible technology domain, and exploitation, which allows her to adopt one of the explored technologies. The feasible technology domain, which is common to all players and contains all the explored technologies at time $t$, is modeled as an interval $[0, X_t]$ with $X_0 = 0$. If a player allocates the fraction $k_t \in [0, 1]$ to exploration over an interval of time $[t, t + dt)$, the boundary $X_t$ is pushed to the right by an amount of $k_t \, dt$. With the fraction $1 - k_t$ allocated to exploitation, by adopting technology $x_t \in [0, X_t]$, the player receives a deterministic flow payoff $(1 - k_t) \exp(W(x_t)) \, dt$, where $W(x_t)$ denotes the quality of the adopted technology.
The technological landscape \( W : \mathbb{R}_+ \to \mathbb{R} \), which maps technologies to their qualities, is common to all players. Nevertheless, only the qualities of the feasible technologies in \([0, X_t]\) are known to each player at time \( t \). The status quo technology \( X_0 = 0 \) has a quality \( W(0) = s_0 \), whereas the qualities of the initially unexplored technologies on \( \mathbb{R}_{++} \) are specified by a realized path of Brownian motion in \( C(\mathbb{R}_+, \mathbb{R}) \) starting at \( w_0 = W(0^+) \leq s_0 \), with drift \( \mu \in \mathbb{R} \) and volatility \( \sigma > 0 \).\(^3\)

At the outset of the game, all players know the parameters of the landscape \( \mu, \sigma, w_0 \) and \( s_0 \), but not the realized Brownian path. Therefore, the process of exploration described above captures the dynamics of research—experimenting with unknown technologies—and development—expanding the set of feasible technologies. Collective exploration as such thus features both information and knowledge spillovers.\(^4\)

Given a player’s actions \( \{(k_t, x_t)\}_{t \geq 0} \), with \( k_t \in [0, 1] \) and \( x_t \in [0, X_t] \) measurable with respect to the information available at time \( t \), her total expected discounted payoff, expressed in per-period units, is

\[
E \left[ \int_0^\infty e^{-rt} (1 - k_t) e^{W(x_t)} \, dt \right].
\]

Note that whenever a player chooses exploitation, she always adopts one of the best feasible technologies \( x_t \in \arg \max_{x \in [0, X_t]} W(x) \) to maximize her total expected discounted payoff. Therefore, we can focus on such an exploitation strategy without loss and rewrite the above total payoff as

\[
E \left[ \int_0^\infty e^{-rt} (1 - k_t) e^{S_t} \, dt \right],
\]

where \( S_t = \max_{x \in [0, X_t]} W(x) \).

\(^3\)In other words, the mapping \( W \) is a realized Brownian path on \( \mathbb{R}_+ \), except possibly for a discontinuity at the origin with \( W(0) \geq W(0^+) \).

\(^4\)Dynamic games featuring pure information spillover include bandits-based games such as in Bolton and Harris (1999) and Keller, Rady, and Cripps (2005), in which all technologies are feasible at the outset, but their qualities are uncertain and to be learned. By contrast, dynamic games featuring pure knowledge spillover can be thought of as games of public goods provision such as in Admati and Perry (1991), Marx and Matthews (2000), Yildirim (2006), and Georgiadis (2014), in which players contribute to a joint project (e.g., developing a technology in the public domain) with its value known in advance. Neither of these two types of model fully captures the progressive and uncertain nature of the underlying process of technology development and innovation.
2.1 Reformulated Game

The environment above can be equivalently reformulated as follows. Players have prior beliefs represented by a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\), where \(\Omega = C(\mathbb{R}_+, \mathbb{R})\) is the space of Brownian paths, \(\mathbb{P}\) is the law of standard Brownian motion \(B = \{B_t\}_{t \geq 0}\), and \(\mathcal{F}_t\) is the canonical filtration of \(B\). Each player chooses her strategy from the space of admissible control processes \(\mathcal{A}\), which consists of all processes \(\{k_t\}_{t \geq 0}\) adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\) with \(k_t \in [0, 1]\). The public history of technology development is represented by the process \(\{W(X_t)\}_{t \geq 0}\), which is the original Brownian motion under the time change controlled by the players’ strategies. This process satisfies the stochastic differential equation

\[
dW(X_t) = \mu K_t \, dt + \sigma \sqrt{K_t} \, dB_t, \quad W(0+) = w_0,
\]

where \(K_t = \sum_{1 \leq n \leq N} k_{n,t}\) measures how much of the overall resource is allocated to exploration, and will be referred to as the intensity of exploration at time \(t\).

Given a strategy profile \(k = \{(k_{1,t}, \ldots, k_{N,t})\}_{t \geq 0}\), player \(n\)’s total expected discounted payoff can be written as

\[
E \left[ \int_0^\infty r e^{-rt} (1 - k_{n,t}) e^{S_t} \, dt \right],
\]

where

\[
S_t = \max_{0 \leq \tau \leq t} W(X_\tau)
\]

denotes the quality of the best feasible technology at time \(t\).

In addition, we use the term “gap”, denoted by \(A_t := S_t - W(X_t)\) for \(t > 0\), whereas by \(A_0 := s_0 - w_0\) for \(t = 0\), to refer to the quality difference between the best feasible technology and the latest technology under development. Henceforth, we shall use \(a\) and \(s\) when referring to the state variables as opposed to the stochastic processes \(\{A_t\}_{t \geq 0}\) and \(\{S_t\}_{t \geq 0}\) (i.e., if \(A_t = a\), then “the game is in state \(a\) at time \(t\”)).

A Markov strategy \(k_n : \mathbb{R}_+ \times \mathbb{R} \to [0, 1]\) with \((a, s)\) as the state variable specifies the action player \(n\) takes at time \(t\) to be \(k_n(A_t, S_t)\). A Markov strategy is called \(s\)-invariant if it depends on \((a, s)\) only through \(a\). Thus an \(s\)-invariant Markov strategy \(k_n : \mathbb{R}_+ \to [0, 1]\) takes the gap \(a\) as the state variable. Finally, an \(s\)-invariant Markov strategy \(k_n\) is a cutoff strategy if there is a cutoff \(\bar{a} \geq 0\) such that \(k_n(a) = 1\) for all \(a \in [0, \bar{a})\) and \(k_n(a) = 0\) otherwise.

Given an \(s\)-invariant Markov strategy profile \(k\), the homogeneity of the payoff functions enables us to write player \(n\)’s associated payoff at state \((a, s)\) as \(v_n(a, s \mid k) = \ldots\)
It is thus convenient to define $u_n(a | k) := v_n(a, 0 | k)$, which equals player $n$’s payoff at state $(a, s)$ normalized by $e^x$, the opportunity cost of exploration. We refer to $u_n : \mathbb{R}_+ \rightarrow \mathbb{R}$ as player $n$’s normalized payoff function, or simply as payoff function when it is clear from the context.

### 2.2 Exploration under Complete Information

To study the value of information, here we consider an alternative setting under complete information. More specifically, how would $N$ players allocate their resource cooperatively over time if the entire technological landscape $W$ is publicly known at the outset of the game?

Formally, in this subsection we replace $\mathcal{F}_t$ with $\mathcal{F}$ for each $t \geq 0$ while maintaining the assumption that the feasible technology domain $[0, X_t]$ can only be expanded by continuously pushing forward the boundary at the rate of $dX_t/dt = K_t$. For a given technological landscape $W$, denote the average (ex-post) value under complete information by

$$\hat{v}(W) := \sup \int_0^\infty r e^{-rt} (1 - K_t/N) e^{S_t} dt,$$

where $S_t = \max_{x \in [0, X_t]} W(x)$, and the supremum is taken over all measurable functions $t \mapsto K_t \in [0, N]$.

**Lemma 1** (Complete-information Payoff). Denote the average (ex-ante) value under complete information at state $(a, s)$ by $\hat{V}(a, s) := E_x[\hat{v}(W)]$. We have $\hat{V}(a, s) = e^x \hat{U}(a)$, where

$$\hat{U}(a) = \begin{cases} 1 + \exp(-\lambda a)/(\lambda - 1), & \text{if } \lambda > 1, \\ +\infty, & \text{otherwise}, \end{cases}$$

with $\lambda := (r/N - \mu)/(\sigma^2/2)$.

When $\lambda > 1$, there almost surely exists a first-best technology $\hat{x}_N \in \mathbb{R}_+$ so that the value $\hat{v}(W) < +\infty$ is achieved by exploring with full intensity up to the point when $\hat{x}_N$ is developed, and thereafter exploiting $\hat{x}_N$. On the contrary, if $\lambda \leq 1$, then with probability 1, the payoff can be improved indefinitely by delaying exploitation, and thus

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5See Lemma 5 in the Appendix.

6One can also examine the value of the landscape directly. More specifically, how much is a player willing to pay for the revelation of an unknown technological landscape, on which all technologies become feasible without any further development? The answer is simply $V := \lim_{x \rightarrow 0} \hat{V}$. One can then interpret the difference $V - \hat{V}$ as the value of knowledge (i.e., the saved opportunity costs from obviating the need for technology development), and the difference $V - V^*$ as the value of information, where $V^*$ is the value under incomplete information in Section 3. However, we do not find these objects relevant to our analysis.

7See the proof of Lemma 13 in the Online Appendix for the precise definition of $\hat{x}_N$. 

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\(\hat{b}(W) = +\infty\) almost surely. As a consequence, the value under incomplete information becomes infinite as well.

Let \(\theta := 2\mu/\sigma^2\) and \(\rho := \sigma^2/(2r)\). The condition \(\lambda > 1\) can be equivalently written in the following way.

**Assumption 1.** \(N\rho(1 + \theta) < 1\).

Moreover, for a reason that will become clear shortly, we introduce the concept of returns to cooperation associated with a technological landscape as follows.

**Definition 1.** We say (the prior distribution of) a technological landscape \(W\) yields

- decreasing returns to cooperation (DRC) if \(\theta < -1\),
- constant returns to cooperation (CRC) if \(\theta = -1\),
- and increasing returns to cooperation (IRC) if \(\theta > -1\).

Assumption 1 is always satisfied when \(W\) yields DRC or CRC. Unless otherwise stated, we impose Assumption 1 for the remainder of the paper to ensure well-defined payoffs and deviations.

## 3 Joint Maximization of Average Payoffs

Suppose that \(N \geq 1\) players work cooperatively to maximize the average expected payoff. Denote by \(\mathcal{A}_N\) the space of all adapted processes \(\{K_t\}_{t \geq 0}\) with \(K_t \in [0,N]\). Formally, we are looking for the value function

\[
v(a,s) := \sup_{K \in \mathcal{A}_N} v(a, s \mid K),
\]

where

\[
v(a, s \mid K) := \mathbb{E}_{\text{as}} \left[ \int_0^\infty re^{-rt} (1 - K_t/N)e^{S_t} \, dt \right]
\]

is the average payoff function associated with the control process \(K = \{K_t\}_{t \geq 0}\), and an optimal control \(K^* \in \mathcal{A}_N\) such that \(v(a,s) = v(a,s \mid K^*)\). The structure of the problem allows us to focus on Markov strategies \(K : \mathbb{R}_+ \times \mathbb{R} \to [0,N]\) with \((a,s)\) as the state variable, so that the intensity of exploration at time \(t\) is specified by \(K_t = K(A_t, S_t)\).

\(^8\)For now, the existence of a Markovian optimal strategy is still a conjecture, which will be confirmed later by Proposition 1.
According to the dynamic programming principle, we have

\[ v(a, s) = \max_{K \in [0,N]} \left\{ r \left( 1 - \frac{K}{N} \right) e^{a} dt + E_{as} \left[ e^{-r dt} v(x + da, s + ds) \right] \right\}. \]

First, note that \( S \) can only change when \( A \), and thus \( dS = 0 \) for all positive gaps. Hence, for each \( a > 0 \) at which \( \partial^2 v / \partial a^2 \) is continuous, the value function \( v \) satisfies the Hamilton-Jacobi-Bellman (HJB) equation

\[ v(a, s) = \max_{K \in [0,N]} \left\{ \left( 1 - \frac{K}{N} \right) e^{a} + K \mathcal{P} \left( \frac{\partial^2 v(a, s)}{\partial a^2} - \theta \frac{\partial v(a, s)}{\partial a} \right) \right\}. \] (1)

Assume, as will be verified, that the optimal strategy is \( s \)-invariant. Then by the homogeneity of the value function we can replace \( v(a, s) \) with \( e^{a} u(a) \), and divide both sides of equation (1) by the opportunity cost \( e^{a} \), to obtain the normalized HJB equation

\[ u(a) = 1 + \max_{K \in [0,N]} K \{ \beta(a, u) - 1/N \}, \] (2)

where \( \beta(a, u) := \rho(u''(a) - \theta u'(a)) \) is the ratio of the expected benefit of exploration \( \rho(\partial^2 v / \partial a^2 - \theta \partial v / \partial a) \) to its opportunity cost \( e^{a} \). It is then straightforward to see that the optimal action takes the following “bang-bang” form. If the shared opportunity cost of exploration, \( 1/N \), exceeds the full expected benefit, the optimal choice is \( K(a) = 0 \) (all agents choose exploitation exclusively), which gives \( u(a) = 1 \). Otherwise, \( K(a) = N \) is optimal (all agents choose exploration exclusively), and \( u \) satisfies the second-order ordinary differential equation (henceforth ODE),

\[ \beta(a, u) = u(a)/N. \] (3)

The optimal strategy could presumably depend on both \( a \) and \( s \) and hence might not be \( s \)-invariant. Indeed, both the benefit and the opportunity cost of exploration increase as innovation occurs. Nevertheless, due to our specific form of flow payoff where the qualities of technologies contribute exponentially to the payoffs, the increased benefit of exploration exactly offsets the increased opportunity cost. As a result, the incentives for exploration at any fixed gap do not depend on the highest-known quality, which leads to an \( s \)-invariant optimal strategy. This conjecture is confirmed by the following proposition.

**Proposition 1 (Cooperative Solution).** Suppose Assumption 1 holds. In the \( N \)-agent
cooperative problem, there is a cutoff $a^* > 0$ given by

$$a^* = \frac{1}{\gamma_2 - \gamma_1} \left( \ln \left( 1 + \frac{1}{\gamma_2} \right) - \ln \left( 1 + \frac{1}{\gamma_1} \right) \right)$$

with $\gamma_1 < \gamma_2$ being the roots of $\gamma(\gamma - \theta) = 1/(N\rho)$, such that it is optimal for all players to choose exploitation exclusively when the gap is above the cutoff $a^*$ and it is optimal for all players to choose exploration exclusively when the gap is below the cutoff $a^*$.

The associated payoff at state $(a, s)$ can be written as $V^*(a, s) = e^s U^*(a)$, where the normalized payoff function $U^* : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$U^*(a) = \frac{1}{\gamma_2 - \gamma_1} \left( \gamma_2 e^{-\gamma_1(a^*-a)} - \gamma_1 e^{-\gamma_2(a^*-a)} \right)$$

when $a \in [0, a^*)$, and by $U^*(a) = 1$ otherwise.

If Assumption 1 is violated, then $U^*(a) = +\infty$ for all $a \geq 0$.

The cooperative solution is pinned down by the standard smooth pasting condition $u'(a^*) = 0$, and the normal reflection condition $(\partial v/\partial a + \partial v/\partial s)(0+, s) = 0$, which takes the form of $u(0) + u'(0+) = 0$ for $s$-invariant strategies.\(^9\)

Because of the lack of information on the qualities of technologies, the players might stop too early, giving up exploration before developing the first-best technology and thus ultimately adopting a suboptimal technology, or might stop too late, wasting too many resources for marginal improvement while the first-best technology has already been developed. The cooperative solution optimally balances these trade-offs between early and late stopping and therefore determines the efficient strategies under incomplete information.

**Corollary 1** (Comparative Statics of the Cooperative Solution). Suppose Assumption 1 holds for $N = 1$. The cooperative cutoff $a^*$ is strictly increasing in $N$ and strictly decreasing in $r$. For all $a \geq 0$, the cooperative payoff $U^*(a)$, if it is finite, is strictly below the complete-information payoff $\widetilde{U}(a)$. For each $r > 0$,

- if $W$ yields DRC, then $U_N^* \to \widetilde{U}_N < +\infty$ pointwise as $N \to +\infty$;
- if $W$ yields CRC, then $U_N^* \to +\infty$ as $N \to +\infty$;
- if $W$ yields IRC, then $U_N^* \to +\infty$ as $N \to 1/(\rho(1+\theta))$.\(^{10}\)

\(^9\)The normal reflection condition is not an optimality condition. It ensures that the infinitesimal change of the payoff at a zero gap has a zero $dS$ term, which is necessary for the continuation value process to be a martingale. See Peskir and Shiryaev (2006) for an introduction to the normal reflection condition in the context of optimal stopping problems, and the proof of Lemma 7 for more details.

\(^{10}\)Here we allow $N$ to be non-integral values for convenience. Also note that when $W$ yields IRC, Assumption 1 is violated for $N \geq 1/(\rho(1+\theta))$, in which case $U_N^* = +\infty$. 
In all cases \( a^*_N \to +\infty \).

A larger stopping cutoff \( a^* \) represents greater ambition among the players. The benefit of exploration decreases with \( r \), and thus more patient players are more ambitious and willing to explore at less promising states. Likewise, because the players work cooperatively, as the team size increases, extra resources brought by additional players enable a higher rate of exploration. Consequently, for each fixed level of ambition, resources are wasted for a shorter period of time before the exploration fully stops, which motivates the players to become more ambitious. This leads to the emergence of more advanced technologies in the long run and further drives the players to act more ambitiously, and so forth.

Corollary 1 underlines a key component absent in the literature on bandits-based games: innovation. The set of feasible technologies (arms) is predetermined and fixed in the Brownian model in Bolton and Harris (1999) and the Poisson model in Keller and Rady (2010). Even though additional players expedite learning, the discounted payoff stream derived from technology adoption is bounded. As a result, with a bounded technology space, each player’s payoff eventually maxes out as the team size grows. In our model, technology adoption still offers a bounded payoff stream. However, as more players join exploration, they become more ambitious and thus more advanced technologies emerge in the long run. Whether this prospect of superior innovations qualitatively alters the asymptotic behavior of players’ payoffs depends on the returns to cooperation associated with the technological landscape. As characterized in Corollary 1, collective exploration over a DRC landscape entails diminishing marginal welfare improvement with respect to team size, and thus resembles the bandits-based games with a fixed set of arms, whereas a CRC or IRC landscape unleashes the power of innovation as team gets large.\(^{11}\) This innovation effect has important implications in the upcoming analysis of the strategic problem.

### 3.1 Long-run Outcomes

Consider an \( s \)-invariant strategy profile \( k \) such that the set of states at which the intensity of exploration is bounded away from zero takes the form of a half-open interval \([0, \bar{a})\). We denote by \( \bar{x}(\bar{a}) := \lim_{t \to \infty} X_t = \int_0^\infty K_t \, dt \) the amount of exploration under \( k \). In addition, we denote by \( \bar{s}(\bar{a}) := \lim_{t \to \infty} S_t \) the long-run technological standard, which is defined as the quality of the best technology available as time approaches infinity. For

\(^{11}\)As shown in Corollary 1, \( U^*_N \to +\infty \) under both CRC and IRC landscapes. What differentiates these two cases is that \( U^*_N < +\infty \) for any finite \( N \) under a CRC landscape, whereas \( U^*_N \to +\infty \) as \( N \) approaches some finite number under an IRC landscape.
a given Brownian path \( W \), it is straightforward to see that both \( \bar{s}(\bar{a}) \) and \( \bar{x}(\bar{a}) \) depend only on the initial state \((a, s)\) and the stopping threshold \( \bar{a} \), and are independent of the intensity of exploration. Moreover, they are clearly nondecreasing in \( \bar{a} \). We can explicitly express the prior belief on the distribution of \( \bar{s}(\bar{a}) \) as follows.

**Lemma 2.** At state \((a, s)\), for a strategy profile with stopping threshold \( \bar{a} \), the long-run technological standard \( \bar{s}(\bar{a}) \) has the same distribution as \( \max\{s, M - a\} \), where the random variable \( M \) has an exponential distribution with mean \( (e^{\theta \bar{a}} - 1)/\theta \).\(^{12}\)

Recall from Section 2.2 that the first-best technology, denoted by \( \hat{x}_N \), is the technology \( x \geq 0 \) that yields the highest payoff for \( N \) cooperative agents under complete information, taking the opportunity cost of its development into account. Denote by \( q(\bar{a}) := \Pr_{as}(\hat{x}_N \in [0, \bar{x}(\bar{a})]) \) the probability that the first-best technology will be explored in the long run under a strategy profile with stopping threshold \( \bar{a} \).

**Lemma 3.** At state \((a, s)\), we have \( q(\bar{a}) \rightarrow 1 \) as \( \bar{a} \rightarrow +\infty \).

Therefore, the comparative statics in Corollary 1 imply that \( \hat{x}_N \) will be developed under the cooperative solution with probability \( q_N(a_N^*) \rightarrow 1 \) as the team size grows.\(^{13}\)

### 4 The Strategic Problem

From now on, we assume that there are \( N > 1 \) players acting noncooperatively. We study equilibria in the class of \( s \)-invariant Markov strategies, which are the Markov strategies with the gap as the state variable and will hereafter be referred to as Markov strategies. In this section, we provide characterizations of the best responses and the associated payoff functions, which establish useful properties of the equilibria for further analysis.

#### 4.1 Best Responses and Equilibria

We denote by \( \mathcal{K} \) the set of Markov strategies that are right-continuous and piecewise Lipschitz-continuous, and denote by \( \mathcal{A} \) the space of admissible control processes as in Section 2.1.\(^{14}\) A strategy \( k_n^* \in \mathcal{K} \) for player \( n \) is a best response against her opponents’

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\(^{12}\)For \( \theta = 0 \), we take \( \lim_{\theta \rightarrow 0} (e^{\theta \bar{a}} - 1)/\theta = \bar{a} \) for the mean of \( M \).

\(^{13}\)This result is not as apparent as it seems, because not only \( a_N^* \), but the first-best technology \( \hat{x}_N \) could depend on the parameter \( N \) as well. However, the convergence in Lemma 3 does not require the parameters to stay the same, provided that the parameters satisfy Assumption 1 along the sequence of the stopping thresholds.

\(^{14}\)Piecewise Lipschitz-continuity means that \( \mathbb{R}_+ \) can be partitioned into a finite number of intervals such that the strategy is Lipschitz-continuous on each of them. This requirement rules out the infinite-switching strategies considered in Section 6.2 of Keller, Rady, and Cripps (2005).
strategies \( k_{-n} = (k_1, \ldots, k_{n-1}, k_{n+1}, k_N) \in \mathcal{K}^{N-1} \) if

\[
v_n(a, s \mid k_n^*, k_{-n}) = \sup_{k_n \in \mathcal{K}} v_n(a, s \mid k_n, k_{-n})
\]

at each state \((a, s) \in \mathbb{R}_+ \times \mathbb{R}\). This definition turns out to be equivalent to

\[
u_n(a \mid k_n^*, k_{-n}) = \sup_{k_n \in \mathcal{K}} u_n(a \mid k_n, k_{-n})
\]

for each gap \(a \geq 0\), with the normalized payoff function \(u_n(a \mid k)\) defined as in Section 2.1. A Markov perfect equilibrium is a profile of Markov strategies that are mutually best responses.

Denote the intensity of exploration carried out by player \(n\)’s opponents by \(\alpha(n, a) = \sum_{i \neq n} k_i(a)\), and the benefit-cost ratio of exploration by \(\beta(a, u_n)\) as in Section 3. The following lemma characterizes all MPE in the exploration game.

**Lemma 4** (Equilibrium Characterization). A strategy profile \(k = (k_1^*, \ldots, k_N^*) \in \mathcal{K}^N\) is a Markov perfect equilibrium with \(u_n(a \mid k)\) being the corresponding payoff function of player \(n\) for each \(n \in \{1, \ldots, N\}\), if and only if for each \(n\), function \(u_n\)

1. is continuous on \(\mathbb{R}_+\) and once continuously differentiable on \(\mathbb{R}_+;\)
2. is piecewise twice continuously differentiable on \(\mathbb{R}_+;\)
3. satisfies the normal reflection condition

\[
u_n(0) + u_n'(0+) = 0;
\]

4. satisfies, at each continuity point of \(u_n''\), the HJB equation

\[
u_n(a) = 1 + K_{-n}(a)\beta(a, u_n) + \max_{k_n \in [0,1]} k_n\{\beta(a, u_n) - 1\},
\]

with \(k_n^*(a)\) achieving the maximum on the right-hand side, i.e.,

\[
k_n^*(a) \in \arg \max_{k_n \in [0,1]} k_n\{\beta(a, u_n) - 1\}.
\]

These conditions are standard in optimal control problems. Condition 4 and the smooth pasting condition, which is implicitly stated in Condition 1, are the optimality conditions. The rest are properties for general payoff functions.

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This condition means that there is partition of \(\mathbb{R}_+\) into a finite number of intervals such that \(u_n''\) is continuous on the interior of each of them.
In any MPE, Lemma 4 provides the following characterization of best responses. If \( \beta(a, u_n) < 1 \), then \( k_n^*(a) = 0 \) is optimal and \( u_n(a) = 1 + K_n^{-1}(a) \beta(a, u_n) < 1 + K_n^{-1}(a) \). If \( \beta(a, u_n) = 1 \), then the optimal \( k_n^*(a) \) takes arbitrary values in \([0, 1]\) and \( u_n(a) = 1 + K_n^{-1}(a) \). Finally, if \( \beta(a, u_n) > 1 \), then \( k_n^*(a) = 1 \) is optimal and \( u_n(a) = (1 + K_n^{-1}(a)) \beta(a, u_n) > 1 + K_n^{-1}(a) \). In short, player \( n \)'s best response to a given intensity of exploration \( K_n^{-1} \) by the others depends on whether \( u_n \) is greater than, equal to, or less than \( 1 + K_n^{-1} \).

On the intervals where each \( k_n \) is continuous, HJB equation (5) gives rise to the ODE
\[
\dot{u}_n(a) = 1 - k_n(a) + K(a) \beta(a, u_n). \tag{6}
\]
In particular, on the intervals where each \( k_n \) is constant, the ODE above admits the explicit solution
\[
U(a) = 1 - k_n + C_1 e^{\gamma_1 a} + C_2 e^{\gamma_2 a}, \tag{7}
\]
where \( \gamma_1 \) and \( \gamma_2 \) are the roots of the equation \( \gamma(\gamma - \theta) = 1/(K \rho) \), and \( C_1, C_2 \) are constants to be determined.

Lastly, on the intervals where an interior allocation is chosen by player \( n \), the ODE from the indifference condition \( \beta(a, u_n) = 1 \) has the general solution
\[
U(a) = \begin{cases} 
C_1 + C_2 a + a^2/(2 \rho), & \text{if } \theta = 0, \\
C_1 + C_2 e^{\theta a} - a/(\rho \theta), & \text{if } \theta \neq 0,
\end{cases} \tag{8}
\]
where \( C_1, C_2 \) are constants to be determined.

### 4.2 Properties of MPE

First, note that in any MPE, the average payoff can never exceed the \( N \)-player cooperative payoff \( U_1^* \), and no individual payoff can fall below the single-agent payoff \( U_1^* \). The upper bound follows directly from the fact that the cooperative solution maximizes the average payoff. The lower bound \( U_1^* \) is guaranteed by playing the single-agent optimal strategy, as the players can only benefit from the exploration efforts of others.

Second, all Markov perfect equilibria are inefficient. Along the efficient exploration path, the benefit of exploration tends to \( 1/N \) of its opportunity cost as the gap \( A_t \) approaches the efficient stopping threshold. A self-interested player thus has an incentive to deviate to exploitation whenever the benefit of exploration drops below its full opportunity cost.

Note also that in any MPE, the set of states at which the intensity of exploration
is positive must be an interval $[0, \bar{a})$ with $a_1^* \leq \bar{a} \leq a_N^*$. The bounds on the stopping threshold follow directly from the bounds on the average payoffs and imply that the long-run outcomes in any MPE cannot outperform the cooperative solution.

**Corollary 2.** In any Markov perfect equilibrium with stopping threshold $\bar{a}$, at any state $(a, s) \in \mathbb{R} \times [0, 1]$ we have $\tilde{s}(\bar{a}) \leq \tilde{s}(a_N^*)$ almost surely, and $q(\bar{a}) \leq q(a_N^*)$.

Moreover, the intensity of exploration must be bounded away from zero on any compact subset of $[0, \bar{a})$. If this were not the case, there would exist some gap $a < \bar{a}$ such that the process $\{W(X_t)\}_{t \geq 0}$ starting from $w_0 = s_0 - a$ would never reach the best-known quality $s_0$ because of diminishing intensity, and therefore allocating a positive fraction of the resource to exploration at gap $a$ is clearly not optimal for any player.

In the two-armed bandit models reviewed in Section 1.1, the players always use the risky arm at beliefs higher than some myopic cutoff, above which the expected short-run payoff from the risky arm exceeds the deterministic flow payoff from the safe arm. Because our model lacks such a myopic cutoff, it might seem reasonable to conjecture that some player $\mathcal{B}$, with her payoff function $u_{\mathcal{B}}$ bounded from above by $1 + \hat{\nu}_{\mathcal{B}}$, never explores, and thus free-rides the technologies developed by the other players. Such a conjecture is refuted by the following proposition.

**Proposition 2** (No Player Always Free-rides). In any Markov perfect equilibrium, no player allocates her resource exclusively to exploitation for all gaps.

The intuition behind this result is that in equilibrium, the cost and benefit of exploration must be equalized at the stopping threshold for each player, whereas any player who never explores would find this benefit outweighs the cost, manifested by a kink in her payoff function at the stopping threshold. She would then have a strict incentive to resume exploration immediately after the other players give up, hoping to reduce the gap to bring the exploration process back alive. Therefore, in equilibrium, every player must perform exploration at some states, respecting the smooth pasting condition (Condition 1 in Lemma 4).

This result shows that each player strictly benefits from the presence of the other players in equilibrium through information and knowledge spillovers. As a result, the future exploration efforts of the others encourage some of the players to explore at gaps larger than their single-agent cutoffs. Such an encouragement effect is exhibited in all MPE of the exploration game.

**Proposition 3** (Encouragement Effect). In any Markov perfect equilibrium, at least one player explores at gaps above the single-agent cutoff $a_1^*$.
With the same intuition as the encouragement effect, our last general result on Markov perfect equilibria concerns the nonexistence of equilibria where all players use cutoff strategies.

**Proposition 4 (No MPE in Cutoff Strategies).** In any Markov perfect equilibrium, at least one player uses a strategy that is not of the cutoff type.

Next, we look into Markov perfect equilibria in greater depth.

## 5 Symmetric Equilibrium

Our characterization of best responses and the nonexistence of MPE in cutoff strategies suggest that in any symmetric equilibrium, the players choose an interior allocation at some states. At these states of interior allocation, the benefit of exploration must be equal to the opportunity cost, and therefore the common payoff function solves the ODE $\beta(a, u) = 1$. As a consequence of equation (6), the payoff of each player at the states of interior allocation in the symmetric equilibrium must also satisfy $u = 1 + K_{-n} \leq N$. Therefore, whenever the common payoff exceeds $N$, each player allocates her resource exclusively to exploration, and the payoff function satisfies the same ODE (3) as in the cooperative solution. However, it is worth pointing out that for some configurations of the parameters, because of the strength of free-riding incentives among the players, the common payoff could be below $N$ for all gaps, and accordingly, the resource constraint of each player is not necessarily binding even when the gap is zero, in marked contrast to the cooperative solution. Lastly, the common payoff satisfies $u = 1$ at the gaps for which the resource is exclusively allocated to exploitation.

The solutions to the corresponding ODEs provided in equations (7) and (8), together with the normal reflection condition (4) and the smoothness requirement on the equilibrium payoff functions, uniquely pin down the strategies and the associated payoff functions in the symmetric equilibrium, which can be expressed in closed form as follows.

**Proposition 5 (Symmetric Equilibrium).** The $N$-player exploration game has a unique symmetric Markov perfect equilibrium with the gap as the state variable. There exists a stopping threshold $\bar{a} \in (a_1^*, a_N^*)$ and a full-intensity threshold $a^\dagger \geq 0$ such that the fraction $k^\dagger(a)$ of the resource that each player allocates to exploration at gap $a$ is given
by
\[
k^\dagger(a) = \begin{cases} 
0, & \text{on } [\bar{a}, +\infty), \\
\frac{1}{(N-1)\rho} \int_{0}^{\bar{a}-a} \phi_\theta(z) \, dz \in (0, 1), & \text{on } [a^\dagger, \bar{a}), \\
1, & \text{on } [0, a^\dagger) \text{ if } a^\dagger > 0,
\end{cases}
\] (9)

with \( \phi_\theta(z) := (1 - e^{-\theta z}) / \theta. \)

The corresponding payoff function is the unique function \( U^\dagger : \mathbb{R}_+ \rightarrow [1, +\infty) \) of class \( C^1 \) with the following properties: \( U^\dagger(a) = 1 \) on \([\bar{a}, +\infty)\); \( U^\dagger(a) = 1 + (N - 1)k^\dagger(a) \in (1, N) \) and solves the ODE \( \beta(a, u) = 1 \) on \((a^\dagger, \bar{a})\); if \( a^\dagger > 0 \), then \( U^\dagger(a) > N \) and solves the ODE \( \beta(a, u) = u / N \) on \((0, a^\dagger)\).

The closed-form expressions for the common payoff function \( U^\dagger \) and the thresholds \( a^\dagger \) and \( \bar{a} \) are provided in the Appendix.

As we have already pointed out, depending on the parameters, it is possible that \( a^\dagger = 0 \), in which case \( k^\dagger(a) < 1 \) for all \( a > 0 \), and hence will be referred to as the non-binding case. The opposite case, where \( a^\dagger > 0 \), will be referred to as the binding case. Figure 1 illustrates the symmetric equilibrium for these two cases.

As the outcomes are publicly observed and newly developed technologies are freely available, the players have incentives to free-ride. Such a free-rider effect becomes more

\[1^\text{We define } \phi_0(z) := \lim_{\theta \to 0} \phi_\theta(z) = z.\]
pronounced through the comparison between the benefit-cost ratio of exploration at the states of interior allocation in the symmetric MPE

$$\beta(a, U^\dagger) = 1,$$

and the one in the cooperative solution

$$\beta(a, U^*) = U^*(a)/N.$$

Exploration in equilibrium thus requires the benefit of exploration to cover the cost, whereas the efficient strategy entails exploration at states where the cost exceeds the benefit, as $\beta(a, U^*) < 1$ whenever $U^*(a) < N$. Figure 2 illustrates the comparison between the common payoff function in the symmetric equilibrium and the cooperative solution in a two-player exploration game.

Figure 2. From top to bottom: average payoff $\hat{U}_N$ under complete information, average payoff $U^*_N$ in the cooperative solution, common payoff $U^\dagger_N$ in the symmetric equilibrium, and payoff $U^*_1$ in the single-agent optimum. Parameter values: $\rho = 2$, $\theta = -1$, $N = 2$. 
5.1 Comparative Statics

In this section, we examine the comparative statics of the symmetric equilibrium with respect to the discount rate \( r \) and the number of players \( N \).\(^{17}\)

**Corollary 3** (Effect of \( r \)). The stopping threshold \( \bar{a}_r \) is strictly decreasing in \( r \), and the full-intensity threshold \( a^\dagger \) is weakly decreasing in \( r \). For any gap \( a \geq 0 \), the equilibrium strategy \( k^\dagger(a) \) and the common payoff \( U^\dagger(a) \) are weakly decreasing in \( r \).

As \( r \) decreases, the players become more patient and have greater incentives for exploration. Moreover, the increased exploration efforts of others encourage each player to raise their own effort further.

The common payoff is decreasing in \( r \) for two reasons. First, higher patience increases players’ payoffs directly. Second, as in the cooperative solution, the increased patience raises the level of ambition \( \bar{a}_r \), and hence more advanced technologies will be developed and adopted in the long run.

**Corollary 4** (Effect of \( N \)). On \( \{ N \geq 1 \mid a^\dagger_N > 0 \} \), which is the range of \( N \) for which the players’ resource constraints are binding in the symmetric equilibrium, the stopping threshold \( \bar{a}_N \) is strictly increasing in \( N \) and the common payoff function \( U^\dagger_N \) is weakly increasing in \( N \). Whereas on \( \{ N \geq 1 \mid a^\dagger_N = 0 \} \), both \( \bar{a}_N \) and \( U^\dagger_N \) are constant over \( N \), and the equilibrium strategy \( k^\dagger_N \) is weakly decreasing in \( N \).

In the binding case, note that \( k^\dagger_N(a) \) is not monotone in \( N \) because the full-intensity threshold \( a^\dagger_N \) could be decreasing in \( N \). This situation occurs when the free-rider effect outweighs the encouragement effect. On the one hand, extra encouragement brought by additional players raises the stopping threshold \( \bar{a}_N \). On the other hand, the increased free-riding incentives due to extra players tighten the requirement \( u > N \) for binding resource constraints, which enlarges the region \( (a^\dagger_N, \bar{a}_N) \) of interior allocation. The total effect of increasing \( N \) on the intensity of exploration is determined by these two competing forces and hence is not monotone in \( N \).

In the non-binding case, as \( N \) increases, each player adjusts their individual intensity of exploration downward, maintaining the same equilibrium payoff. The incentive to free-ride in such a situation is so strong that it completely offsets further encouragement brought by additional players. Even the overall intensity of exploration \( Nk^\dagger \) is decreasing in \( N \) for the gaps in \( [0, \bar{a}_N) \). Thus, free-riding slows down exploration considerably. In the worst scenario, the overall intensity when the gap is zero could even be lower than that in the single-agent problem.

\(^{17}\)The effect of discount rate \( r \) on the payoff \( U_r \) carries over to \( U_r/r \). In other words, the following comparative statics with respect to \( r \) are not driven by the normalizing constant \( r \) in the flow payoff.
Also, note that whenever the resource constraints are not binding, the full-intensity threshold $a_N^+$ remains constant at zero for any further increase in $N$ because $k_N^+$ would be even lower. Therefore, if $a_N^+$ ever hits zero as $N$ goes up, the resource constraints in the symmetric equilibrium remain non-binding for any larger $N$.

As we have seen, depending on whether or not the resource constraints are binding in the symmetric equilibrium, a larger team size can have qualitatively different effects on the welfare and long-run outcomes. If the resource constraints are not binding in equilibrium, any extra resource brought by additional players translate entirely to free-riding, which results in a highly inefficient outcome in a large team in terms of average payoffs, the likelihood of developing the first-best technology, and the technological standard in the long run. The question then naturally arises of whether the resource constraints would ever fail to be binding in the symmetric equilibrium as $N$ goes up. Or, conversely, would the encouragement effect eventually overcome the free-rider effect?\footnote{The purpose of classifying the symmetric MPE into binding and non-binding cases is to help describe the comparative statics. We do not intend to suggest this classification of equilibria in a large team determines the prevalence of the encouragement or free-rider effect. On the very contrary, it is the consequence of the relative strength between these two forces.}

To investigate this question, we now examine the effect on the symmetric equilibrium as $N$ increases toward infinity, while keeping the other parameters fixed. For ease of exposition, we allow $N \geq 1$ to take non-integral values and drop Assumption 1 for the remainder of this section.

\textbf{Corollary 5} (Asymptotic Effect of $N$). Suppose Assumption 1 holds for $N = 1$. If $W$ yields IRC and $r < \hat{r} := (\sigma \theta)^2/(2(\theta - \ln(1 + \theta)))$,\footnote{Note that $\hat{r}$ is well defined only if $W$ yields IRC.} then we have $U^1_N \to +\infty$, $a_N^+ \to +\infty$, and $q_N(\tilde{a}_N) \to 1$ as $N \to 1/(\rho(1 + \theta))$; otherwise, we have $a_N^+ = 0$ for sufficiently large $N$, and we have $\lim U^1_N(a) < \lim U^*_N(a)$ for each $a \geq 0$, $\tilde{a}_N$ is bounded, and $q_N(\tilde{a}_N)$ is bounded away from 1 as $N \to +\infty$.

The free-rider effect and the encouragement effect in our model are two competing forces shared in several models in the literature on strategic experimentation (e.g., Bolton and Harris (1999), Keller and Rady (2010), and Keller and Rady (2015)). In the symmetric equilibrium of the Brownian model of Bolton and Harris (1999), the free-rider effect eventually dominates the encouragement effect as team size grows. This is not necessarily the case here.

If the technological landscape yields DRC, then the marginal welfare improvement is certainly diminishing asymptotically with respect to the team size. Unsurprisingly, like the Brownian model, the marginal encouragement effect yields to the marginal free-rider effect, as the latter does not abate as the team gets larger.
However, when $W$ does not lead to DRC, the innovation effect introduced in Section 3 can make a difference. Stemming from the encouragement effect, the future exploration from an additional player encourages everyone to become more ambitious, which then leads to the emergence of more advanced technologies in the long run. This novel innovation effect in our model in turn motivate the players to explore, reinforcing the encouragement effect, and vice versa. Therefore, exploration over a non-DRC technological landscape allows the encouragement effect to prevail, which is exhibited by the unlimited expansion of the full-intensity region, as illustrated in Figure 3.

Even so, collective exploration over a CRC or IRC landscape is only necessary for the prevalence of the encouragement effect, but not sufficient. The returns to cooperation determine how likely or how advanced the technologies are expected to be developed, but the timing of their availability also matters. Naturally, players’ patience plays a role: The prevalence of the encouragement effect requires both an IRC technological landscape and sufficiently patient players, as stated in Corollary 5. The innovation effect from the exploration over a CRC landscape, or an IRC landscape with impatient players, fails to boost the encouragement effect up to the magnitude required for overcoming the free-rider effect. In such cases, as team size grows to infinity, the stopping threshold remains bounded, the full-intensity region vanishes, and the equilibrium payoff functions stay bounded, standing in marked contrast to the cooperative solution in which payoffs grow without bound.

Corollary 5 highlights the key role of innovation in strategic learning and experimentation, which has been largely overlooked in the literature despite its importance. The absence of technological advancements is partly responsible for the prevalence of free-riding in two-armed bandit models. Our result suggests that innovation is an essential element toward understanding the incentives for experimentation and technology development in dynamic and strategic environments.

6 Asymmetric Equilibria and Welfare Properties

Note that in the symmetric equilibrium, the intensity of exploration dwindles down to zero as the gap approaches the stopping threshold. As a result, the threshold is never reached and exploration never fully stops. This observation suggests that welfare can be improved if the players take turns between the roles of explorer and free-rider, keeping the intensity of exploration bounded away from zero until all exploration stops. In this section, we investigate this possibility by constructing a class of asymmetric Markov perfect equilibria.
Figure 3. Full-intensity thresholds and the stopping thresholds in the symmetric equilibrium ($\theta = -0.09, \sigma = \sqrt{2}$) for different discount rate $r$. If the players are sufficiently patient (solid curves), resource constraints are binding ($a_N^\dagger > 0$) for all $N$. Otherwise (dashed curve), resource constraints are not binding ($a_N^\dagger = 0$) for sufficiently large $N$.

6.1 Construction of Asymmetric Equilibria

Our construction of asymmetric MPE is based on the idea of the asymmetric MPE proposed in Keller and Rady (2010). We let the players adopt the common actions in the same way as in the symmetric equilibrium whenever the resulting average payoff is high enough to induce an overall intensity of exploration greater than one, and let the players take turns exploring at less promising states in order to maintain the overall intensity at one. Such alternation between the roles of explorer and free-rider leads to an overall intensity of exploration higher than in the symmetric equilibrium, yielding higher equilibrium payoffs.

In what follows, we briefly address the two main steps in our construction. In the first step, we construct the average payoff function $\bar{u}$. We let $\bar{u}$ solve the same ODE $\beta(a, u) = \max\{u(a)/N, 1\}$ as the common payoff function in the symmetric equilibrium whenever $u > 2 - 1/N$, which ensures the corresponding overall intensity is greater than one. Whenever $1 < u < 2 - 1/N$, we let $\bar{u}$ solve the ODE $u(a) = 1 - 1/N + \beta(a, u)$, which is the ODE for the average payoff function among $N$ players associated with an overall intensity $K = 1$. The boundary conditions for the average payoff function $\bar{u}$, namely the smooth pasting condition at the stopping threshold and the normal reflection condition (4), are identical to the conditions in Lemma 4, simply because those conditions remain unchanged after taking the average. The unique solution of class $C^1(\mathbb{R}_{++})$ to the ODE above serves as the average payoff function, which also gives thresholds $a^b > a^\dagger \geq 0$ such that $\bar{u} = 1$ on $[a^b, +\infty)$, $1 < \bar{u} < 2 - 1/N$ on $(a^\dagger, a^b)$ and $\bar{u} > 2 - 1/N$ on $[0, a^\dagger)$. In the second step, equilibrium-compatible actions are assigned to each player.
On \([0, a^b]\), if it is nonempty, we let the players adopt the common action \(k_n(a) = \min\{(\bar{u}(a) - 1)/(N - 1), 1\}\) in the same way as in the symmetric equilibrium. On \([a^b, a^b]\), players alternate between the roles of explorer and free-rider so as to keep the overall intensity at one. We first split \([a^b, a^b]\) into subintervals in an arbitrary way and then meticulously choose the switch points of their actions, so that all individual payoff functions have the same values and derivatives as the average payoff function at the endpoints of these subintervals. Lastly, our characterization of MPE in Lemma 4 confirms that the assigned action profile is compatible with equilibrium. We leave the method for choosing the switch points and further details to the Appendix.

For \(N = 2\), Figure 4 illustrates the intensity of exploration in the asymmetric MPE, compared with the symmetric equilibrium. The resource constraints are binding for small gaps in the depicted equilibria, but this may not be the case for different parameters. For example, if the players are too impatient, the average payoff function could be bounded by \(2 - 1/N\) from above, resulting in an intensity of exploration equal to 1 over the entire region \([0, a^b]\). The states \(\bar{a}_F\) and \(\bar{a}_E\) in the figure demarcate the switch points at which these two players swap roles when they take turns exploring on

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20In fact, this technique can be used to construct asymmetric equilibria with strategies that take values in \(\{0, 1\}\) only, which are referred to as simple equilibria in Keller, Rady, and Cripps (2005). See Proposition 10 in the Appendix. However, it is not clear whether such equilibria achieve higher average payoffs than the symmetric equilibrium.
Figure 5. Average payoff and possible individual payoffs in the best two-player asymmetric equilibria, compared to the common payoff in the symmetric equilibrium ($\rho = 2$, $\theta = -1$, $N = 2$).

$[a^a, a^b]$. The volunteer explores on $[a^a, \bar{a}_F] \cup [\bar{a}_E, a^b]$, whereas the free-rider explores on $[\bar{a}_F, \bar{a}_E]$. These switch points are chosen in a way that ensures their individual payoff functions are of class $C^1(\mathbb{R}^+)$ and coincide on $[0, a^a]$.

Figure 5 illustrates the associated average payoff function (dashed curve) and the individual payoff functions (solid curves) that can arise in the equilibria in a two-player game, compared with the common payoff function in the symmetric equilibrium (solid dotted curve). Note that the payoff function of the volunteer is strictly higher than that of the free-rider at the states immediately to the left of the stopping threshold $a^b$. In fact, the free-rider has a payoff equal to 1 on $[\bar{a}_E, a^b]$. This observation stands in marked contrast to the models in Keller, Rady, and Cripps (2005) and Keller and Rady (2010), where the volunteer is worse off in this region. The intuition behind this feature is similar to that in Proposition 2. A kink has to be created at $a^b$ in the free-rider’s payoff function in order to attain a higher payoff than the volunteer’s at states immediately to the left of $a^b$, because the free-rider’s ODE $u(a) = 1 + \beta(a, u)$ must be satisfied.\footnote{Even though the free-rider has a payoff equal to 1 around $a^b$, she still benefits from free-riding in this region. This benefit, however, is offset by the relatively high burden of exploration effort she must bear in equilibrium at more promising states to reward the volunteer.} In such a case, the free-rider has a strict incentive to take over the role of volunteer to kickstart the exploration process at a larger gap. Therefore, in equilibrium, the volunteer must be compensated for acting as a lone explorer at less promising states by bearing relatively
less burden at more promising states.

For arbitrary $N$, we have the following result.

**Proposition 6 (Asymmetric MPE).** The $N$-player exploration game admits Markov perfect equilibria with thresholds $0 \leq a^\ddagger \leq a^\# < a^b < a^*$, such that on $[0, a^\ddagger]$, the players have a common payoff function; on $[0, a^\ddagger]$, all players choose exploration exclusively; on $(a^\ddagger, a^\#)$, the players allocate a common interior fraction of the unit resource to exploration, and this fraction decreases in the gap; on $[a^\#, a^*)$, the intensity of exploration equals 1 with players taking turns exploring on consecutive subintervals; on $[a^b, +\infty)$, all players choose exploitation exclusively. The intensity of exploration is continuous in the gap on $[0, a^\ddagger]$. The average payoff function is strictly decreasing on $[0, a^\ddagger]$, once continuously differentiable on $\mathbb{R}_{++}$, and twice continuously differentiable on $\mathbb{R}_{++}$ except for the cutoff $a^b$. On $[0, a^\ddagger)$, the average payoff is higher than in the symmetric equilibrium, and $a^b$ lies to the right of the threshold $\tilde{a}$ at which all exploration stops in that equilibrium.

### 6.2 Welfare Results

For $N \geq 3$, further improvements can be easily achieved by letting the players take turns exploring, maintaining the intensity of exploration at $K$ whenever $K < u < K + 1 - K/N$ for all $K \in \{1, \ldots, N\}$, rather than for $K = 1$ only as in the Proposition above. However, it is not clear whether such improvements achieve the highest welfare among all MPE of the $N$-player exploration game. For $N = 2$, the asymmetric equilibria of Proposition 6 are the best among all MPE.

**Proposition 7 (Best MPE for $N = 2$).** The average payoff in any Markov perfect equilibrium of the two-player exploration game cannot exceed the average payoff in the equilibria of Proposition 6.

In the construction of the asymmetric MPE depicted in Figure 4, the interval $[a^\#, a^b)$ is not split into subintervals. This assertion can be confirmed by the observation from Figure 5 that the players’ payoff functions match values only at the endpoints of $[a^\#, a^b]$, not in the interior. Our construction allows an arbitrary partition on $[a^\#, a^b)$ during the splitting procedure, thus a trivial partition of $[a^\#, a^b)$, as in Figure 4, suffices. A finer partition, however, produces equilibria in which the players exchange roles more often, which allows them to share the burden of exploration more equally. Sufficiently frequent alternation of roles on $[a^\#, a^b)$ guarantees each player a payoff close enough to the average payoff and thus yields a Pareto improvement over the symmetric equilibrium.\(^{22}\)

\(^{22}\)The payoffs of both players in the asymmetric MPE depicted in Figure 5 are higher than in the
Proposition 8 (Pareto Improvement over the Symmetric MPE). For any \( \epsilon > 0 \), the \( N \)-player exploration game admits Markov perfect equilibria as in Proposition 6 in which each player’s payoff exceeds the symmetric equilibrium payoff on \([0, a^\mu - \epsilon]\).

Recall that the stopping threshold \( \hat{a}_N \) in the symmetric equilibrium remains bounded as \( N \to +\infty \) when the players are too impatient. The reason is that the common payoff function on the interval of interior allocation must satisfy the ODE \( \beta(a, U_i^\mu) = 1 \), which does not depend on \( N \). As a result, the common payoff function in the symmetric equilibrium is constant over the team size when the resource constraints are not binding. By contrast, the average payoff always increases in the number of players in the asymmetric MPE in which the players take turns exploring at states immediately to the left of the stopping cutoff. This is because the burden of keeping the overall intensity at one of these states can be shared among more players in larger teams. As a result, the players would be able to exploit more often on average, which in turn encourages them to explore at less promising states. Therefore, unlike the comparative statics of the symmetric equilibrium, the stopping threshold in the asymmetric equilibria we constructed is not bounded as \( N \) goes up, irrespective of the patience of the players and the underlying landscapes.

Proposition 9. The stopping cutoff \( a_N^\mu \) in the asymmetric MPE of Proposition 6 goes to infinity as \( N \to +\infty \).

Therefore, the amount of exploration, and thus the long-run outcomes, can be improved significantly over the symmetric equilibrium for large \( N \) by letting the players take turns exploring before the exploration fully stops. This positive result, unfortunately, does not extend as far to the welfare, as the rate of exploration might still be too low because of non-binding resource constraints, similar to the situation faced in the symmetric equilibrium. More precisely, for a DRC landscape, the full-intensity threshold \( \hat{a}_N^\mu \) always hits 0 as \( N \to +\infty \); for a CRC or IRC landscape, whether this happens again depends on the patience of the players, just like in the symmetric equilibrium. It can be shown that the results regarding the average payoff and the full-intensity threshold in Corollary 5 extend to the asymmetric MPE of Proposition 6 with a larger \( \hat{a} \).

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symmetric equilibrium on \([0, \hat{a}_E]\); however, this might not be the case in general when the trivial partition of \([a^\mu, a^\nu]\) is used in the construction, as \( \hat{a}_E \) could lie on the left of \( \hat{a} \).

22For an IRC landscape, if the asymmetric MPE of Proposition 6 fail to exist due to unbounded payoffs for \( N \geq 1/(\rho(1 + \theta)) \), then by \( N \to +\infty \) we actually mean \( N \to 1/(\rho(1 + \theta)) \).
7 Discussion

In this section, we discuss our modeling assumptions and assess the extent to which our results rely on them. We argue that our assumptions establish a parsimonious environment, suggesting that our findings regarding the prevalence of the encouragement effect remain robust across various plausible extensions.

Payoffs. We have assumed that players receive payoffs only from exploitation, which serves to highlight the innovation-driven motives for exploration. This assumption deviates from the literature on strategic experimentation such as Bolton and Harris (1999) and Keller, Rady, and Cripps (2005), and the literature on spatial experimentation such as Callander (2011) and Garfagnini and Strulovici (2016), in which the players also receive payoffs directly from experimentation. Notably, allowing the players to benefit from exploration per se barely changes our results. For example, suppose in addition to the flow payoffs from exploitation, the players also receive a flow payoff of \( k_{n,t} \exp(W(X_t)) \, dt \) from exploration. In such a case, the “gap” still serves as a state variable, but some of the closed-form representations in our results may no longer be attainable. The players would then strictly prefer exploration over exploitation when the gap is sufficiently close to zero, as exploration offers a positive option value in addition to a flow payoff that is nearly identical to that of exploitation. In other words, not only does the prospect of technological advancements motivate the players to explore, but also exploration per se. As a result, the players would have a stronger incentive to explore, and thus the encouragement effect would prevail under weaker conditions.

We have also assumed that the qualities of the technologies contribute exponentially to the payoffs from exploitation. This assumption aligns with the exponential growth of total factor productivity, commonly assumed in macroeconomic growth literature dating back to Solow (1956). Some empirical observations, such as Moore’s law (doubling of transistors on integrated circuits every two years), are in line with exponential growth, while others do not (see, e.g., Philippon (2022)). We make the exponential growth assumption mainly for tractability, because it helps reduce the dimension of the state variable. This simplification can also be achieved by choosing a factor depending linearly on the qualities of the technologies (i.e., adopting the best-known technology delivers a flow payoff of \( (1 - k_{n,t}) S_t \, dt \) instead of \( (1 - k_{n,t}) \exp(S_t) \, dt \)), but with the underlying landscape represented by a geometric Brownian motion. All our results continue to hold in such an equivalent formulation, with the ratio \( S_t/W(X_t) \), or its monotone transformation, such as \( \ln(S_t) - \ln(W(X_t)) \), serving as a one-dimensional state variable.
We suspect that it is inevitable to resort to a two-dimensional state variable such as $(w, s) = (W(X_i), S_i)$ for other function forms of the flow payoff. The challenge mainly arises from the lack of homogeneity in payoff functions stated in Lemma 5. Without homogeneity, it would be difficult to pin down the equilibrium payoff functions. The analysis for each given $s$ remains similar to the current setting, but the equilibrium strategies could be hard to analyze and interpret if the strategies are unrestricted along the $s$-coordinate. We also suspect that if the order of growth is lower than the exponential rate, the encouragement effect would be unable to overcome the free-rider effect. Moreover, reward and punishment become possible by conditioning actions on the highest-known quality $s$, which probably leads to more efficient outcomes as in Hörner, Klein, and Rady (2021). In that paper, they demonstrate that inefficiencies disappear entirely in a class of non-Markovian equilibria in a rich environment that encompasses the Brownian and Poisson models. Since our setting lies outside of their environment, it remains an open question whether the insight from their constructive proofs can be applied to our setting to achieve full efficiency, complementary to the positive results that we obtained here by focusing only on MPE.

**Exploration.** The scope of experimentation is certainly limited in our model: Players do not have complete freedom to choose where to explore. Garfagnini and Strulovici (2016) allow the players to experiment with any technologies, but radical experimentation, which involves exploring technologies far away from the feasible ones, is assumed to be more expensive. Exploration in our model can be viewed as an extreme abstraction of their model, where incremental experimentation is costless, while radical experimentation comes at an infinite cost. In practice, such a limited experimentation scope may be more appropriate in the context of technology development. For example, pharmaceutical companies can easily test the efficacy of a medicine once its formula is provided, but creating the formula from scratch is nearly impossible.

It might also be reasonable to assume that multiple research directions emerge during exploration, allowing players to pursue them concurrently or switch direction if one proves fruitless. Such a possibility is beyond the scope of this paper, but it is expected to give rise to a stronger encouragement effect, as restarting opportunities would raise the likelihood of innovation and therefore the value of exploration as well.

8 Concluding Remarks

This paper introduces a novel and tractable framework for examining the incentives of forward-looking agents in knowledge creation. We identify two key effects that shape
the incentives for experimentation: an encouragement effect, unique to strategic and dynamic contexts, and an innovation effect, which is absent from the existing strategic bandit literature where innovation possibilities are often overlooked. We demonstrate that the innovation effect, stemming from the prospect of technological advancements, can amplify the encouragement effect, thereby offsetting the free-rider problem prevalent in large teams. Our analysis further illustrates how these effects impact the trajectory of technological progress and long-run outcomes. The proposed model holds promise for future research, with potential applications in dynamic games of innovation, such as patent races.

Appendix

In this Appendix, we maintain Assumption 1 unless it is explicitly stated otherwise. Proofs related to the complete information setting and the comparative statics are relegated to the Online Appendix.

A Explicit Representation of the Symmetric MPE

**Corollary 6.** The explicit representation for the normalized payoff function \( U^\dagger \) in the unique symmetric equilibrium on \( [a^\dagger, \bar{a}] \) is given by

\[
U^\dagger(a) = \begin{cases} 
1 + \frac{1}{2\rho} (\bar{a} - a)^2, & \text{if } \theta = 0, \\
1 + \frac{1}{\rho\theta} \left( \bar{a} - a + \frac{1}{\theta} \left( e^{-\theta(\bar{a} - a)} - 1 \right) \right), & \text{if } \theta \neq 0.
\end{cases}
\]

If \( a^\dagger > 0 \), \( U^\dagger \) on \( [0, a^\dagger] \) is given by

\[
U^\dagger(a) = \frac{N}{\gamma_2 - \gamma_1} \left( (\gamma_2 + \iota)e^{-\gamma_1(a^\dagger - a)} - (\gamma_1 + \iota)e^{-\gamma_2(a^\dagger - a)} \right),
\]

with \( \gamma_1 < \gamma_2 \) being the roots of the equation \( \gamma(\gamma - \theta) = 1/(N\rho) \) and \( \iota > 0 \) being

\[
\iota := \begin{cases} 
\frac{1}{N\rho\theta} \left( 1 + W_0 \left( -\exp\left( -1 - (N - 1)\rho\theta^2 \right) \right) \right), & \text{if } \theta > 0, \\
\sqrt{\frac{2}{N\rho}} (1 - 1/N), & \text{if } \theta = 0, \\
\frac{1}{N\rho\theta} \left( 1 + W_{-1} \left( -\exp\left( -1 - (N - 1)\rho\theta^2 \right) \right) \right), & \text{if } \theta < 0,
\end{cases}
\]
where \( W_0 \) and \( W_{-1} \) are the real branches of the Lambert W function.\(^{24}\)

If \( t < 1 \), then the equilibrium belongs to the binding case. The full-intensity threshold is given by

\[
a^\dagger = \frac{1}{\gamma_2 - \gamma_1} \left( \ln \left( \frac{1 + \gamma_2}{\nu + \gamma_2} \right) - \ln \left( \frac{1 + \gamma_1}{\nu + \gamma_1} \right) \right),
\]

and the stopping threshold is given by

\[
\tilde{a} = a^\dagger + N\rho (\nu + (1 - 1/N)).
\]

If \( t \geq 1 \), then the equilibrium belongs to the non-binding case with the full-intensity threshold \( a^\dagger = 0 \), whereas the stopping threshold \( \tilde{a} \) is given by

\[
\tilde{a} = \begin{cases} 
\frac{1}{2} \left( 1 + \theta - \theta^2 \rho + W_0 \left( -(1 + \theta) e^{1-\theta^2 \rho} \right) \right), & \text{if } \theta < 0, \\
1 - \sqrt{1 - 2\rho}, & \text{if } \theta = 0, \\
\frac{1}{2} \left( 1 + \theta - \theta^2 \rho + W_{-1} \left( -(1 + \theta) e^{1-\theta^2 \rho} \right) \right), & \text{if } \theta > 0.
\end{cases}
\]

Proof. These expressions follow from Lemma 4 and explicit calculations. \( \square \)

### B Properties of Payoff Functions

**Lemma 5 (Homogeneity).** Player \( n \)'s payoff function for an \( s \)-invariant Markov strategy profile \( k \in \mathcal{K}^N \) can be written as \( v_n(a, s \mid k) = e^s v_n(a, 0 \mid k) \).

*Proof.* At state \((a, s)\), player \( n \)'s payoff associated to \( k \) is given by

\[
v_n(a, s \mid k) = E \left[ \int_0^\infty e^{-rt} (1 - k_n(A_t)) e^{S_t} \, dt \mid A_0 = a, S_0 = s \right]
= e^s E \left[ \int_0^\infty e^{-rt} (1 - k_n(A_t)) e^{S_t} \, dt \mid A_0 = a, S_0 = s \right]
= e^s E \left[ \int_0^\infty e^{-rt} (1 - k_n(A_t)) e^{S_t} \, dt \mid A_0 = a, S_0 = 0 \right]
= e^s v_n(a, 0 \mid k),
\]

where the second-to-last equality comes from the Markov property of the diffusion process \( \{A_t\}_{t \geq 0} \).

\( \square \)

\(^{24}\)The Lambert W function maps each \( x \geq -1/e \) to the solutions of the equation \( y e^y = x \). It has two real branches. The value of the principal branch \( W_0(x) \) denotes the unique solution \( y \) such that \( y \geq -1 \), whereas the value of the branch \( W_{-1}(x) \) denotes the unique solution \( y \) such that \( y < -1 \).
Lemma 6 (Smoothness). Given an $s$-invariant Markov strategy profile $k \in \mathcal{K}^N$, for each $s$, player $n$'s payoff function $u_n(\cdot \mid k)$ is

1. $C^1$ and piecewise $C^2$ on $(a_L, a_R) \subset \mathbb{R}_+$ if the intensity of exploration $K(a)$ is bounded away from zero on $(a_L, a_R)$;

2. $C^2$ on $(a_L, a_R)$ if both $K(a)$ and $k_n(a)$ are continuous on $(a_L, a_R)$ in addition to the condition above.

Proof. See the Online Appendix.

Lemma 7 (Normal Reflection Condition and Feynman-Kac Equation). Given an $s$-invariant Markov strategy profile, the associated payoff function of each player satisfies Feynman-Kac equation (6) at each state at which it is twice continuously differentiable. Moreover, if the intensity of exploration is bounded away from 0 in some neighborhood of the state $a = 0$, the payoff function also satisfies the normal reflection condition (4).

Proof. A more general version of this lemma is provided in the Online Appendix for Markov strategy profile $k$ with the state variable $(a, s)$. The normal reflection condition $(\partial v_n/\partial a + \partial v_n/\partial s)(0+, s) = 0$ takes form of $u_n(0) + u_n'(0+) = 0$ because of the homogeneity of the payoff functions for $s$-invariant strategies.

C Equilibrium Characterization

C.1 Proof of Lemma 4

Lemma 4 follows from Lemma 8 and 9 below.

Lemma 8 (Sufficiency). Given $k_{-n} \in \mathcal{K}^{N-1}$, if $u_n : \mathbb{R}_+ \to \mathbb{R}$ satisfies Condition 1–4 in Lemma 4, then a piecewise right-continuous function $k^*_n : \mathbb{R}_+ \to [0, 1]$ which maximizes the right-hand side of HJB equation (5) at each continuity point of $u_n''$ is a best response against $k_{-n}$, with $u_n$ being the associated payoff function of player $n$.

Proof. The proof is a standard verification argument. See, e.g., Fleming and Soner (2006), Theorem III.9.1. The role played by the linear growth condition in a standard proof is instead played by Assumption 1.

Given $k_{-n} \in \mathcal{K}^{N-1}$, for any admissible control process $k_n = \{k_{n,t}\}_{t \geq 0} \in \mathcal{A}$, let

$$L := \eta(a, s, k_{n,t}) \frac{\partial}{\partial a} + \frac{1}{2} \alpha^2(a, s, k_{n,t}) \frac{\partial^2}{\partial a^2},$$
with \( \eta(a, s, k) = -\mu(k + K_{\eta}(a)) \) and \( \alpha(a, s, k) = \sigma\sqrt{k + K_{\alpha}(a)} \). Let \( f(a, s, k) = r(1 - k)e^s \), and consider \( v(a, s) = e^s u_n(a) \), where \( u_n : \mathbb{R}_+ \to \mathbb{R} \) satisfies Conditions 1–4 in Lemma 4.

Usually, applying Itô’s formula to \( v \) at \( a > 0 \) requires \( u_n \in C^2(\mathbb{R}_+) \). However, Itô’s formula is still valid for \( C^1 \) functions with absolutely continuous derivatives, which is satisfied by \( u_n \) under Conditions 1 and 2.²⁵

By applying Itô’s formula to \( e^{-rt} v(A_T, S_T) \),²⁶ for fixed \( 0 < T < \infty \) we have

\[
e^{-rt} v(A_T, S_T) - v(a, s) = \int_0^T e^{-rt}(Lv - rv)(A_t, S_t) \, dt - \int_0^T e^{-rt} \alpha(A_t, S_t, k_{n,t}) \frac{\partial v}{\partial a}(A_t, S_t) \, dB_t + \int_0^T e^{-rt} \left( \frac{\partial v}{\partial a} + \frac{\partial v}{\partial s} \right)(A_t, S_t) \, dS_t.
\]

Since \( dS_t = 0 \) whenever \( A_t > 0 \), and by Condition 3 we have \( \left( \frac{\partial v}{\partial a} + \frac{\partial v}{\partial s} \right)(A_t, S_t) = 0 \) when \( A_t = 0 \), the last term is identically zero. Moreover, because \( \alpha(A_t, S_t, k_{n,t}) \) and \( \frac{\partial v}{\partial a} \) are bounded, the second term has mean zero. Taking expectations on both sides, for fixed \( 0 < T < \infty \) we have Dynkin’s formula

\[
v(a, s) = -E_{as} \left[ \int_0^T e^{-rt}(Lv - rv)(A_t, S_t) \, dt \right] + e^{-rT} E_{as} [v(A_T, S_T)],
\]

where the expectations on the right-hand side are finite for each \( T < \infty \).

Note that HJB equation (5) implies

\[
rv(a, s) \geq f(a, s, k_{n,t}) + Lv(a, s),
\]

and therefore we have

\[
v(a, s) \geq E_{as} \left[ \int_0^T e^{-rt} f(A_t, S_t, k_{n,t}) \, dt \right] + e^{-rT} E_{as} [v(A_T, S_T)]. \tag{10}
\]

Let \( T \to \infty \), we have

\[
v(a, s) \geq \liminf_{T \to \infty} E_{as} \left[ \int_0^T e^{-rt} f(A_t, S_t, k_{n,t}) \, dt \right] + \liminf_{T \to \infty} e^{-rT} E_{as} [v(A_T, S_T)].
\]

²⁵See, e.g., Chung and R. J. Williams (2014), Remark 1, p. 187; Rogers and D. Williams (2000), Lemma IV.45.9, p. 105; or Strulovici and Szydlowski (2015), footnotes 73 and 78.

²⁶Note that \( S_t \) is a finite variation process and hence both the quadratic variation \( \langle S, S \rangle_t \) and the covariation \( \langle A, S \rangle_t \) are identically zero. See Shreve (2004), Section 7.4.2 for a non-technical treatment.
Because \( \int_0^T e^{-rt} f(A_t, S_t, k_{n,t}) \, dt \) is nondecreasing in \( T \), as the integrand is nonnegative, we can apply either the monotone convergence theorem or Fatou’s lemma to have

\[
v(a, s) \geq E_{as} \left[ \int_0^\infty e^{-rt} f(A_t, S_t, k_{n,t}) \, dt \right] + \liminf_{T \to \infty} e^{-rT} E_{as} [v(A_T, S_T)]
\]

\[
= v_n(a, s | k_n, k_{-n}) + \liminf_{T \to \infty} e^{-rT} E_{as} [v(A_T, S_T)]
\]

\[
\geq v_n(a, s | k_n, k_{-n}).
\]

Now we repeat the argument by replacing \( k_{n,t} \) with \( k^*_n(A_t) \). Inequality (10) becomes equality, and for \( 0 < T < \infty \) we have

\[
v(a, s) = E_{as} \left[ \int_0^T e^{-rt} f(A_t, S_t, k^*_n) \, dt \right] + e^{-rT} E_{as} [v(A_T, S_T)].
\]

As the first term on the right-hand side is nondecreasing in \( T \), the second term must be nonincreasing, and hence

\[
v(a, s) = E_{as} \left[ \int_0^\infty e^{-rt} f(A_t, S_t, k^*_n) \, dt \right] + \lim_{T \to \infty} e^{-rT} E_{as} [v(A_T, S_T)]
\]

\[
= v(a, s | k^*_n, k_{-n}) + \lim_{T \to \infty} e^{-rT} E_{as} [v(A_T, S_T)].
\]

We finish the proof by showing \( \lim_{T \to \infty} e^{-rT} E_{as} [v(A_T, S_T)] = 0 \) as follows.

Because \( A_t \geq 0 \), we have

\[
E_{as} [v(A_T, S_T)] \leq E_{as} [v(0, S_T)]
\]

\[
= u(0) E_{as} [e^{S_T}]
\]

\[
\leq u(0) E_0 e^{S_T}
\]

\[
= u(0) e^{E_0 [e^{S_T - s}]}
\]

\[
\leq u(0) e^{E M_{NT}},
\]

where \( M_T = \max_{0 \leq t \leq T} \{ \mu t + \sigma B_t \} \). The last inequality comes from the fact that \( S_T - s \leq M_{NT} \) almost surely. Indeed, given that \( A_0 = 0 \), the process \( \{S_T - s\}_{T \geq 0} \) can be viewed as \( \{M_T\}_{T \geq 0} \) with time change, in the sense that \( S_T - s = M_{T'} \) with \( T' = \int_0^T K_t \, dt \). Because \( K_t \in [0, N] \), we have \( T' \leq NT \) and hence \( M_{T'} \leq M_{NT} \).
Then from the Math Appendix in the Online Appendix, we have
\[ E[e^{MNT}] \leq C_1 e^{(\mu+\sigma^2/2)NT} + C_2 \]
for some \( C_1, C_2 \geq 0 \), and therefore
\[ e^{-rT}E_{\alpha_\bar{a}}[v(A_T, S_T)] \leq C_1 e^{(-r+(\mu+\sigma^2/2)N)T} + C_2 e^{-rT}. \]
The right-hand side goes to 0 as \( T \to \infty \) as long as \( (\mu+\sigma^2/2)N < r \), which is precisely Assumption 1.

\[ \square \]

**Lemma 9** (Necessity). In any MPE \( k \in \mathcal{K}^N \), for each player \( n \in \{1, \ldots, N\} \), her payoff function \( u_n(\cdot | k) \) satisfies Conditions 1–4 in Lemma 4, and her equilibrium strategy \( k_n(a) \) maximizes the right-hand side of the HJB equation (5) at each continuity point of \( u_n'' \).

**Proof.** Condition 1: In the interior of the stopping region where \( K(a) = 0 \), it is trivial that the payoff functions are smooth. In the region where \( K(a) \) is bounded away from zero, players’ payoff functions are continuous by standard results,\(^{27}\) and once continuously differentiable by Lemma 6. Because the intensity of exploration in any MPE is bounded away from zero on any compact subset of \([0, \bar{a})\), as we argue in Section 4.2, the only part left to prove is the smooth pasting condition, which states that the payoff functions in any MPE must be once continuously differentiable at the stopping threshold \( \bar{a} \). This is proved in Lemma 10 below.

Condition 2: In the interior of the stopping region where \( K(a) = 0 \), it is again trivial that the payoff functions are smooth. In the region where \( K(a) \) is bounded away from zero, by Lemma 6, players’ payoff functions are twice continuously differentiable at each point at which all players’ strategies are continuous. Condition 2 thus follows from our piecewise Lipschitz-continuity assumption on players’ strategies.

Condition 3 is a property of any payoff functions for any strategy profile with \( K(a) \) bounded away from 0 around \( a = 0 \). See Lemma 7.

Condition 4 is proved in Lemma 11 below.

\[ \square \]

**Lemma 10** (Smooth Pasting Condition). In any MPE with stopping threshold \( \bar{a} > 0 \), each player’s normalized payoff function is continuously differentiable at \( \bar{a} \).

\(^{27}\)See, e.g., Strulovici and Szydlowski (2015), footnote 71.
Proof. Given any MPE \( k = (k_1, \ldots, k_N) \), consider the region \([0, \bar{a})\) in which the intensity of exploration is positive. Clearly \( u_n' (\bar{a}^-) \), the left derivative of the equilibrium payoff function of player \( n \) at \( \bar{a} \), cannot be positive, because otherwise we would have \( u_n(a) < 1 \) for \( a \) immediately to the left of \( \bar{a} \), contradicting the fact that \( k_n \) is a best response.

Suppose by contradiction that \( u_n \) violates the smooth pasting condition at \( \bar{a} \) with \( u_n' (\bar{a}^-) < 0 \). Consider the deviation strategy \( k_\epsilon \) in which \( k_\epsilon (a) = 1 \) on \([\bar{a} - \epsilon, \bar{a} + \epsilon)\) for some small \( \epsilon > 0 \), and \( k_\epsilon = k_n \) otherwise, with \( u_\epsilon \) denoting the associated payoff function of player \( n \). Moreover, write \( w_n := (\ln(u_n))' = u_n'/u_n \) and \( w_\epsilon := (\ln(u_\epsilon))' = u_\epsilon'/u_\epsilon \).

Note that under this deviation the intensity of exploration \( k_\epsilon + \sum_{\ell \neq n} k_\ell \) is bounded away from zero on \([0, \bar{a} + \epsilon)\) so that \( u_\epsilon \) is once continuously differentiable in this region, which implies \( w_\epsilon \) is continuous on \([0, \bar{a} + \epsilon)\). Also, note that because the strategy profile remains unchanged on \([0, \bar{a} - \epsilon)\), the normal reflection condition \( w_n(0+) = w_\epsilon(0+) = -1 \) and Feynman-Kac equation (6) imply that \( w_n \) and \( w_\epsilon \) coincide on \([0, \bar{a} - \epsilon)\).28 From \( u_n'(\bar{a}^-) < 0 \) we know that \( w_n(\bar{a}^-) < 0 \) as well, and therefore by the continuity of \( w_\epsilon \) we can choose \( \epsilon \) small enough so that \( w_\epsilon < 0 \) on \([\bar{a} - \epsilon, \bar{a} + \epsilon)\). This implies \( u_\epsilon > 1 = u_n \) on \([\bar{a}, \bar{a} + \epsilon)\), because \( u_\epsilon(a) = \exp \left( \int_{\bar{a} + \epsilon}^a w_\epsilon(z) \, dz \right) \) for \( a < \bar{a} + \epsilon \). Therefore, the deviation strategy \( k_\epsilon \) leads to a higher payoff on \([\bar{a}, \bar{a} + \epsilon)\) than the equilibrium strategy \( k_n \), which is a contradiction.

\[\square\]

**Lemma 11 (HJB Equation).** In any MPE \( k = (k_1^*, \ldots, k_N^*) \in \mathcal{K}^N \), player \( n \)'s payoff function \( u_n(\cdot \mid k) \) satisfies, at each continuity point of \( u_n'' \), HJB equation (5) with

\[ k_n^*(a) \in \arg \max_{k_n \in [0, 1]} k_n \{ \beta(a, u_n) - 1 \} \]
We now turn to the case of \( a \in (0, \bar{a}) \). If \( \beta(\cdot, u_n) = 1 \) in some neighborhood of \( a \), then any \( k^n_\ast(a) \in [0, 1] \) is a maximum and there is nothing to prove. Next, we prove for the case of \( \beta(a, u_n) > 1 \). The proof for the case of \( \beta(a, u_n) < 1 \) is analogous and thus omitted.

If \( \beta(a, u_n) > 1 \), suppose by contradiction that \( k^n_\ast(a) < 1 \). Since \( u''_n \) is continuous and \( k^n_\ast \) is right-continuous at \( a \), we assume without loss that on \( (a, a + \epsilon) \) for some \( a + \epsilon < \bar{a}, u''_n \) is continuous, \( k^n_\ast(\cdot) < 1 \), and \( \beta(\cdot, u_n) > 1 \).

Consider the deviation strategy \( k_\varepsilon \) in which \( k_\varepsilon = 1 \) on \([a, a + \epsilon)\) and \( k_\varepsilon = k^n_\ast \) otherwise, with \( u_\varepsilon \) denoting the associated payoff function of player \( n \). Moreover, write \( w_n := (\ln(u_n))' = u'_n/u_n \) and \( w_\varepsilon := (\ln(u_\varepsilon))' = u'_\varepsilon/u_\varepsilon \). Because the intensity of exploration is bounded away from zero on any compact subset of \([0, \bar{a})\), Lemma 6 implies that \( w_n \) and \( w_\varepsilon \) are continuous on \((0, \bar{a})\). Also, note that the strategy remains unchanged on \([0, a)\). As a consequence, the normal reflection condition \( w_n(0+) = w_\varepsilon(0+) = -1 \) and Feynman-Kac equation (6) imply that \( w_n \) and \( w_\varepsilon \) coincide on \((0, a]\).

Rewrite Feynman-Kac equation (6) as \( (u_n(a) - 1 + k_n(a))/(K_{-n}(a) + k_n(a)) = \beta(a, u_n) \). Since \( \beta(a, u_n) > 1 \) by assumption, we have \( u_n(a) - 1 > K_{-n}(a) \). This implies \( (u_n(a) - 1 + k_n)/(K_{-n}(a) + k_n) = \beta(a, u_n) = \rho(u''_n(a) - \theta u'(a)) \) is decreasing in \( k_n \). Because \( u_n, u_\varepsilon \) and their derivatives are continuous at \( a \), and \( k_\varepsilon(a+) = 1 > k^n_\ast(a+) \), we have \( u''_\varepsilon(a+) < u''_n(a+) \). By the definition of \( w_n \) and \( w_\varepsilon \), we can conclude \( w'_\varepsilon(a+) < w'_n(a+) \). As \( w_\varepsilon(a) = w_n(a) \), we can then let \( \epsilon \) be sufficiently small so that \( w_\varepsilon(a + \epsilon) < w_n(a + \epsilon) \). Since both \( w_n \) and \( u_\varepsilon \) solves the same equation (11) in footnote 28 on \((a + \epsilon, \bar{a})\), we must have \( w_\varepsilon(\bar{a} -) < w_n(\bar{a} -) \). This implies \( u'_\varepsilon(\bar{a} -) < u'_n(\bar{a} -) = 0 \), where the equality comes from the smooth pasting condition for equilibrium payoff functions. Therefore, the deviation strategy \( k_\varepsilon \) leads to a higher payoff on \((\bar{a} - \delta, \bar{a})\) for some \( \delta > 0 \) than the equilibrium strategy \( k^n_\ast \), which is a contradiction.

\[ \square \]

## D Cooperative Solution

### D.1 Proof of Proposition 1

**Proof.** The cooperative solution can be viewed as a corollary of Lemma 8 with \( K_{-n} = 0 \). The closed-form expressions for the stopping threshold and the payoff functions follow from the explicit calculation. \[ \square \]

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29See footnote 28. This claim is not affected by the possible discontinuities of the strategies on \([0, \bar{a})\), as the payoff functions are at least once continuously differentiable in this region.
### E Properties of MPE

#### E.1 Proof of Proposition 2

*Proof.* Suppose to the contrary that there is an MPE where player 1 chooses exploitation at all states. Let \( \bar{a} > 0 \) denote the state at which all exploration stops. Then on \((0, \bar{a})\) player 1’s payoff function \( u_1 \) is continuously differentiable and solves the free-rider ODE \( u(a) = 1 + K(a)\beta(a, u) \) with value matching \( u(\bar{a}) = 1 \) and smooth pasting condition \( u'(\bar{a}) = 0 \). It can then be easily verified that \( u_1(a) = 1 \) is the unique solution, which is a contradiction because the normal reflection condition (4) is violated. \( \square \)

#### E.2 Proof of Proposition 3

*Proof.* As the individual payoff functions are bounded from below by the single-agent payoff function \( U_1^* \), it is clear that the stopping threshold \( \bar{a} \) in any MPE is weakly larger than the single-agent cutoff \( a_1^* \). Suppose by contradiction that \( \bar{a} = a_1^* \). Assume without loss of generality that all players’ strategies are continuous on \((a_1^* - \epsilon, a_1^*)\) for some \( \epsilon > 0 \), so that each player’s payoff function is twice continuously differentiable in this region. Note that for each player \( n \), both \( u_n \) and \( U_1^* \) satisfy value matching and smooth pasting at \( a_1^* \). Then from equation (6), we have for each player \( n \),

\[
\rho u_n''(a_1^* -) = \frac{k_n(a_1^* -)}{K(a_1^* -)} \leq 1
\]

and \( \rho U_1^{*''}(a_1^* -) = 1 \). As \( \rho u_n''(a_1^* -) < 1 \) for at least one player, her payoff lower bound \( u_n \geq U_1^* \) is then violated at the states immediately to the left of \( \bar{a} \). \( \square \)

#### E.3 Proof of Proposition 4

*Proof.* Suppose by contradiction that there is an MPE where all players use cutoff strategies. Let player 1 be the one who uses the strategy with the largest cutoff \( \bar{a} \). Then no other player uses the same cutoff \( \bar{a} \) as player 1 for the following reason. Suppose to the contrary that player 2 uses a strategy with the same cutoff \( \bar{a} \), then both player 1 and 2 must have a payoff strictly greater than 1 and lower than 2 at the states immediately to the left of \( \bar{a} \), as their payoff functions solve the explorer ODE \( u(a) = K\beta(a, u) \) for some \( K > 1 \) with the initial condition \( u(\bar{a}) = 1 \) and \( u'(\bar{a}) = 0 \). As a result, exploration with full intensity cannot be optimal for both players at the states immediately to the left of cutoff \( \bar{a} \) according to our characterization of best responses. Therefore, player 1 would
be the lone explorer with \( u_1 > 1 \) on \((\bar{a} - \epsilon, \bar{a})\) for some \( \epsilon > 0 \), whereas all other players free-ride in this region.

Moreover, it is easy to see that player 1’s payoff is weakly lower than the others, because the region in which she collects flow payoffs is the smallest among all players. However, on \((\bar{a} - \epsilon, \bar{a})\) the payoff function \( u_n \) of player \( n \neq 1 \) satisfies the free-rider ODE \( u = 1 + \beta(a, u) \) with the same initial conditions as player 1. This yields the unique solution \( u_n = 1 < u_1 \) on \((\bar{a} - \epsilon, \bar{a})\), a contradiction.

\[ \square \]

**F Symmetric MPE**

**F.1 Proof of Proposition 5**

From Lemma 4, it is not difficult to verify that the strategy profile in Corollary 6 constitutes an equilibrium, and that our proposition properly summarizes the equilibrium payoff function in that corollary.

Uniqueness follows directly from symmetry and Lemma 4. One can check by explicit calculations that, under Assumption 1, the equilibrium in Corollary 6 is the only symmetric \( s \)-invariant strategy profile such that the associated common payoff function satisfies Conditions 1–4 in Lemma 4.

The comparison between the stopping threshold and the cooperative cutoff follows from Lemma 15 in the Online Appendix.

**G Asymmetric MPE**

**G.1 Simple MPE**

Here we present a class of asymmetric MPE similar to the simple MPE in Section 6.1 of Keller, Rady, and Cripps (2005). The construction is almost identical to the asymmetric MPE of Proposition 6, and therefore the proof is omitted.

**Proposition 10** (Asymmetric Simple MPE). The \( N \)-player exploration game admits simple Markov perfect equilibria with \( M \) thresholds \( 0 =: \bar{a}_{M+1} < \bar{a}_M < \cdots < \bar{a}_1 < \bar{a}_0 := +\infty \) for some \( M \in \{1, \ldots, N\} \), such that on \([\bar{a}_{K+1}, \bar{a}_K)\) exact \( K \) players take turns exploring on consecutive subintervals.

The average payoff function \( \bar{u} \) is strictly decreasing on \([0, \bar{a}_1]\), twice continuously differentiable on \( \mathbb{R}_{++} \) except for the states \( \{\bar{a}_K\}_{K=1}^M \). Moreover, we have \( [\bar{u}(0)] = M \), and \( \bar{u} \) solves the ODE \( u = (1 - K/N) + K \rho (u'' - \theta u') \) on \([\bar{a}_{K+1}, \bar{a}_K)\) for each
\( K \in \{0, 1, \ldots, M\} \). The payoff function of each player \( u_n \) is strictly decreasing on \([0, \bar{a}_1]\), once continuously differentiable on \( \mathbb{R}_{++} \), and satisfies \( u_n(\bar{a}_K) = \bar{u}(\bar{a}_K) = K \) for \( K \in \{1, \ldots, M\} \).

G.2 Proof of Proposition 6

Step 1: Construction of the average payoff function.

Let \( \bar{u} : \mathbb{R}_- \to \mathbb{R} \) be once continuously differentiable and solves

\[
\max\{u(a)/N, \min\{1, u(a) - (1 - 1/N)\}\} = \beta(a, u),
\]

with the initial conditions \( u(0) = 1 \) and \( u'(0) = 0 \).

Let \( a^\sharp > 0 \) be such that \( \bar{u}(-a^\sharp) + \bar{u}'(-a^\sharp) = 0 \).

Now, we let the average payoff function \( \bar{u}(a) = \bar{u}(a-a^n) \) for \( a \leq a^\sharp \) and \( \bar{u}(a) = 1 \) for \( a > a^\sharp \). We can check that \( \bar{u} \) satisfies value matching \( \bar{u}(a^\sharp) = 1 \), smooth pasting \( \bar{u}'(a^\sharp) = 0 \), and the normal reflection condition \( \bar{u}(0) + \bar{u}'(0) = 0 \). Let \( a^\sharp = \bar{u}^{-1}(2 - 1/N) \) and \( a^\ddagger = \bar{u}^{-1}(N) \) where \( \bar{u}^{-1}(u) = \inf\{ a \geq 0 \mid \bar{u}(a) \leq u \} \). Under such construction we have \( 0 \leq a^\sharp \leq a^\ddagger < a^\sharp \) and \( \bar{u} : \mathbb{R}_+ \to \mathbb{R} \) is continuously differentiable on \( \mathbb{R}_{++} \) and satisfies, on \((a^\sharp, +\infty)\), \( \bar{u} = 1 \); on \((a^\ddagger, a^\sharp)\), solves \( u = 1 - 1/N + \beta(a, u) \); on \((a^\sharp, a^\ddagger)\), solves \( 1 = \beta(a, u) \); on \((0, a^\sharp)\), solves \( u = N\beta(a, u) \).

Step 2: Construction of the players’ payoff functions and strategies.

For each player \( n \), on \([a^\sharp, +\infty)\), let \( k_n(a) = 0 \) and \( u_n = \bar{u} = 1 \); on \([0, a^\sharp)\), if not empty, let \( k_n(a) = \min\{1, (\bar{u}(a) - 1)/(N - 1)\} \) and \( u_n = \bar{u} \).

Next, consider any partition \( a^\sharp = a_1 < a_2 < \cdots < a_m < a_{m+1} = a^\sharp \) of interval \([a^\sharp, a^\sharp)\). For each subinterval \([a_j, a_{j+1})\) in the partition \( \{a_j\}_{j=1}^{m+1} \), we use Algorithm 1 to construct payoff function \( u_n \) and strategy \( k_n \) for each player \( n \).

For each subinterval \([a_j, a_{j+1})\), Algorithm 1 calls procedure ActionAssignment, which first calls function Split to split \([a_j, a_{j+1})\) into three subintervals according to Lemma 12 below, and lets the player with the lowest index available free-ride on the subinterval \([a_{M-}, a_{M+})\) in the middle, and explore on the rest two subintervals at both ends. In such a way, the strategy \( k_n \) and payoff function \( u_n \) of this player are defined on \([a_j, a_{j+1})\), and she is then labeled as unavailable. Then ActionAssignment is called recursively on these three subintervals, preserving the total intensity \( K = 1 \) by allocating intensity 0 on the subintervals at both ends, and intensity 1 on \([a_{M-}, a_{M+})\), with \( \bar{u} \) on

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\(^{30}\)It is straightforward to verify that \( \bar{u} \) is strictly decreasing.

\(^{31}\)It can be shown that under Assumption 1 there exists a unique \( 0 < a^\sharp < +\infty \).
these subintervals being replaced by \( \bar{u}_{-n} \), which is the average payoff function among the rest of the available players.

Lemma 12 ensures that function \texttt{SPLIT} partitions any interval \([a_L, a_R]\) into three subintervals in a unique way such that \( u_n \), and therefore also \( \bar{u}_{-n} \), have the same values and derivatives as \( \bar{u} \) at the end points \( a_L \) and \( a_R \).

The termination of Algorithm 1 is trivial because each time \texttt{ACTIONASSIGNMENT} is called, the strategy of one of the players is assigned and she is thereafter removed from the set of the available players. In the case where the allocation of strategies is unambiguous—that is, either the task of exploration has already been assigned or is to be assigned to the last available player—splitting the interval is unnecessary, and the corresponding strategies are set at once for all the available players.

When Algorithm 1 terminates, the strategies and the constructed payoff functions of each player are defined on \([\bar{a}^\sharp, \bar{a}^\flat] \). Also, note that \( \bar{u} \) from Step 1 is indeed the average of the constructed payoffs of all the players, and the input requirement for \( \bar{u} \) in procedure \texttt{SPLIT} is satisfied in each call to the procedure. These conditions are maintained by line 22 during each call of \texttt{ACTIONASSIGNMENT}. Lastly, \( u_n \) is once continuously differentiable on \( \mathbb{R}_{++} \) for each \( n \), since no “kink” is created in Algorithm 1.

**Step 3: Ensuring mutually best responses.**

Because each \( u_n \) is once differentiable on \( \mathbb{R}_{++} \), twice differentiable except for the switch points, and satisfies the normal reflection condition (4), our characterization of MPE in Lemma 4 states that \((k_1, \ldots, k_N)\) constitutes an MPE with \( u_n \) being the associated payoff function of player \( n \) if \( u_n \) solves the HJB equation

\[
u_n(a) = 1 + K_{-n}(a)\beta(a, u_n) + \max_{k_n \in [0, 1]} k_n\{\beta(a, u_n) - 1\}
\]

for each continuity point of \( u_{n}'' \), and the constructed strategy maximizes the right-hand side.

In other words, we need to verify for all \( a > 0 \), \( k_n(a) = 1 \) if \( \beta(a, u_n) > 1 \), and \( k_n(a) = 0 \) if \( \beta(a, u_n) < 1 \). Then the HJB is satisfied by construction.

On \((a^\flat, +\infty)\), \( u_n(a) = 1 \) gives \( \beta(a, u_n) = 0 < 1 \), therefore \( k_n(a) = 0 \) is optimal. On \([0, a^\flat]\), the argument is exactly the same as in the symmetric MPE. Lastly, on \((a^\sharp, a^\flat)\), we have \( K = 1 \) by construction. The monotonicity of \( \bar{u} \) and the construction of \( u_n \) implies \( 1 \leq u_n(a) < \bar{u}(a^\flat) = 2 - 1/N \) for each player \( n \). Our construction of \( u_n \) ensures \( k_n(a) = 1 \) if and only if \( u_n(a) = \beta(a, u) \), and \( k_n(a) = 0 \) if and only if \( u_n(a) = 1 + \beta(a, u) \) for each \( x \in (a^\flat, a^\flat) \). Suppose \( \beta(a, u_n) > 1 \), which implies \( 1 + \beta(a, u_n) > 2 > u_n(a) \). Then by construction it must be \( u_n(a) = \beta(a, u) \), and hence the constructed strategy
Algorithm 1 Payoff construction

Require: $a^d = a_1 < \ldots < a_{m+1} = a^o$, $\bar{u}$ from Step 1, and a finite set of players $\mathcal{N} = \{1, \ldots, N\}$

Ensure: equilibrium strategy profile $\{k_n\}_{n=1}^N$ is defined on $[a_1, a_{m+1}]$, and $\{u_n\}_{n=1}^N$ is the set payoff functions corresponding to strategy profile $\{k_n\}_{n=1}^N$

1: for all $j \in \{1, \ldots, m\}$ do
2: \hspace{1em} ACTIONASSIGNMENT($\mathcal{N}$, 1, $a_j$, $a_{j+1}$, $\bar{u}$) \hspace{1em} \triangleright \text{input and output according to Lemma 12}
3: end for

4: function SPLIT($\mathcal{I}$, $a_L$, $a_R$, $\bar{u}$) \hspace{1em} \triangleright \text{allocate aggregate intensity $\kappa \in \{0, 1\}$ to a set of available players $\mathcal{I}$}
5: \hspace{1em} return $(a_{M-}, a_{M+}, \bar{u})$
6: end function

7: procedure ACTIONASSIGNMENT($\mathcal{I}$, $\kappa$, $a_L$, $a_R$, $\bar{u}$) \hspace{1em} \triangleright \text{let all available players free-ride}
8: \hspace{1em} if $\kappa = 0$ then
9: \hspace{2em} for all $n \in \mathcal{I}$ do
10: \hspace{3em} $k_n|_{(a_L, a_R)} \leftarrow 0$
11: \hspace{3em} $u_n|_{(a_L, a_R)} \leftarrow \bar{u}$
12: \hspace{2em} end for
13: \hspace{1em} else if $\kappa = 1 = |\mathcal{I}| = |\{n\}|$ then \hspace{1em} \triangleright \text{let the only available player explore}
14: \hspace{2em} $k_n|_{(a_L, a_R)} \leftarrow 1$
15: \hspace{2em} $u_n|_{(a_L, a_R)} \leftarrow \bar{u}$
16: \hspace{1em} else \hspace{1em} \triangleright \text{let one of the $|\mathcal{I}| \geq 2$ available player explore}
17: \hspace{2em} $(a_{M-}, a_{M+}, \bar{u}) \leftarrow \text{SPLIT}(|\mathcal{I}|, a_L, a_R, \bar{u})$
18: \hspace{2em} $n \leftarrow \min \mathcal{I}$
19: \hspace{2em} $k_n|_{(a_L, a_{M-}) \cup (a_{M+}, a_R)} \leftarrow 1$
20: \hspace{2em} $k_n|_{(a_{M-}, a_{M+})} \leftarrow 0$
21: \hspace{2em} $u_n|_{(a_L, a_R)} \leftarrow \bar{u}$
22: \hspace{2em} $\bar{u}_{\mathcal{I} \setminus \{n\}}|_{(a_L, a_R)} \leftarrow \frac{|\mathcal{I}| \bar{u} - u_n}{|\mathcal{I}|-1}$ \hspace{1em} \triangleright \text{the average payoff among the rest of the players}
23: \hspace{1em} end if
24: \hspace{1em} end procedure

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$k_n(a) = 1$ is optimal. On the contrary, if $\beta(a, u_n) < 1$, which implies $\beta(a, u_n) < u_n(a)$, then by construction it must be $u_n(a) = 1 + \beta(a, u_n)$, and hence the constructed strategy $k_n(a) = 0$ is optimal.

**Comparison with symmetric MPE**

Note that $\tilde{u}$ solves ODE

$$u'' = \max\{u/N, \min\{1, u - (1 - 1/N)\}\}/\rho + \theta u'$$

on $(0, a^b)$, while the average payoff function $U^\dagger$ of symmetric MPE solves

$$u'' = \max\{u/N, 1\}/\rho + \theta u'$$

on $(0, \bar{a})$, with value matching and smooth pasting at $a^b$ and $\bar{a}$, respectively. Obviously the right-hand side in the first equation is weakly smaller. We can then verify Condition (ii) in Lemma 15 in the Online Appendix with $\tilde{u} = \tilde{u}(a^b) = \min\{\tilde{u}(0), 2 - 1/N\}$ and conclude that $\tilde{a} < a^b$ and $\bar{a} > U^\dagger$ on $[0, a^b)$. Similarly, we can also compare the first equation with the ODE in the cooperative problem $u'' = u/(N\rho) + \theta u'$ and conclude $a^b > a^N$.

□

**Lemma 12 (Split).** Given a strictly decreasing function $\tilde{u} : [a_L, a_R] \rightarrow [u_L, u_R]$, with $\tilde{u}(a_L) = u_L$ and $\tilde{u}(a_R) = u_R \geq 1$, which satisfies the average payoff ODE $u = f + \rho (u'' - \theta u')$ on $(a_L, a_R)$ with $0 < f < 1$ and $\rho > 0$, there exist $a_L < a_{M^-} < a_{M^+} < a_R$, and a function $\hat{u} : [a_L, a_R] \rightarrow [u_L, u_R]$ continuously differentiable on $(a_L, a_R)$ such that $\hat{u}(a_L) = \tilde{u}(a_L)$; $\hat{u}'(a_L) = \tilde{u}'(a_L)$; $\hat{u}(a_R) = \tilde{u}(a_R)$; $\hat{u}'(a_R) = \tilde{u}'(a_R)$; $\hat{u}$ solves the explorer ODE $u = \rho (u'' - \theta u')$ on $(a_L, a_{M^-}) \cup (a_{M^+}, a_R)$ and solves the free-rider ODE $u = 1 + \rho (u'' - \theta u')$ on $(a_{M^-}, a_{M^+})$.

*Proof.* See the Online Appendix. □

**G.3 Proof of Proposition 7**

*Proof.* The average payoff $\hat{u}$ in any MPE must be once continuously differentiable and satisfy the ODE $u'' = g(u, u')$ for some $g$ respecting Feynman-Kac equation (6) whenever $\hat{u} > 1$ and $\hat{u}''$ is continuous. For $N = 2$, the characterization of best responses gives the following minimal requirements on $g$. For each $1 < u < 2$, $g(u, u')$ can only be either $u^{-1/2} + \theta u'$ (one player explores and the other free-rides) or $\frac{1}{\rho} + \theta u'$ (both players choose a common interior allocation); for each $u > 2$, $g$ can only be either $u^{-1/2} + \theta u'$ (one player
explores and the other free-rides) or $\frac{1}{2} + \theta u'$ (both players explore). The asymmetric MPE we construct in Proposition 6 adopts $g(u, u') = \max\{u/2, \min\{1, u-1/2\}\}/\rho + \theta u'$, which is the minimal $g$ that satisfies these constraints, and therefore achieves the highest payoff among all MPE by Lemma 15 in the Online Appendix.

**G.4 Proof of Proposition 8**

Proof. Choose a partition $\{a_j\}_{j=1}^{m+1}$ of $[a^s, a^b]$ in the construction of the equilibria in Proposition 6, so that

$$\max_{1 \leq j \leq m} |\bar{u}(a_{j+1}) - \bar{u}(a_j)| \leq \delta := \frac{1}{2} \min_{a^s \leq a \leq \bar{a}} \{\bar{u}(a) - U^\dagger(a)\},$$

and $|a_m - a_{m+1}| < \epsilon$. Because the average payoff $\bar{u}$ and the players’ payoff functions are monotone and coincide at the endpoints of the subintervals $[a_j, a_{j+1}]$, we have $|u_n(a) - \bar{u}(a)| \leq \delta$ for $a \in [a^s, \bar{a}]$ for all player $n$. Therefore, we have $u_n \geq \bar{u} - \delta > U^\dagger$ on $[a^s, \bar{a}]$. Also, as $u_n(a_m) = \bar{u}(a_m) > 1$ for all player $n$, we have $u_n > 1 = U^\dagger$ on $[\bar{a}, a_m] \supset [\bar{a}, a^b - \epsilon]$. Lastly, $u_n = \bar{u} > U^\dagger$ on $[0, a^s)$, if it is not empty.

□

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Online Appendix

“Strategic Exploration for Innovation”

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Recall that \( \lambda = (r/N - \mu)/(\sigma^2/2) \) in Lemma 1 can also be written as \( \lambda = 1/(N\rho) - \theta \). Moreover, we let \( \delta := r/N \) for the sake of notational convenience.

O.1 Complete Information Setting

O.1.1 Proof of Lemma 1

Proof. Let \( m(W) := \sup_{x > 0}\{W(x) - \delta x\} \) and \( \hat{s}(W) := \max\{s_0, m(W)\} \). Note that for \( x > 0 \), \( W(x) - \delta x - w_0 \) has the same law as the Brownian motion starting from 0 with drift \( \mu - \delta \) and volatility \( \sigma \). Then it is well-known that \( m(W) - w_0 \) has the exponential distribution with mean \( 1/\lambda \) if \( \lambda > 1 \). Otherwise, if \( \lambda \leq 1 \), then \( m(W) = +\infty \) with probability 1, which obviously leads to \( \hat{V}(a, s) = +\infty \).

Now suppose \( \lambda > 1 \). The (ex-ante) value under complete information at state \((a, s) = (s_0 - w_0, s_0)\) can be calculated as

\[
\hat{V}(a, s) = E \left[ \hat{v}(W) \mid W(0+) = w_0 = s - a, W(0) = s \right]
\]

\[
= P(\hat{s}(W) \leq s)e^s + P(\hat{s}(W) > s)E \left[ e^{m(W)} \mid \hat{s}(W) > s \right]
\]

\[
= e^s \left( P(\hat{s}(W) \leq s) + P(\hat{s}(W) > s)E \left[ e^{m(W)} - w_0 \mid \hat{s}(W) > s \right] \right),
\]

where the second equality follows from \( \hat{v}(W) = e^{\hat{s}(W)} \), which is proved in Lemma 13 below.

Note that \( \hat{s}(W) > s \) if and only if \( m(W) - w_0 > a \), and hence we have

\[
P(\hat{s}(W) > s)E \left[ e^{m(W) - w_0} \mid \hat{s}(W) > s \right] = e^{-a} \int_a^\infty e^z \lambda e^{-\lambda z} dz = \frac{\lambda}{1 - \lambda} e^{-\lambda a}.
\]

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Therefore, we have $\hat{V}(a, s) = e^s \left( 1 - e^{-la} - \frac{1}{1-a} e^{-la} \right) =: e^s \hat{U}(a)$ with $\hat{U}(x) = 1 + e^{-la} / (\lambda - 1)$. $

\square$

**Lemma 13.** For a given technological landscape $W$, the average (ex-post) value under complete information is given by $\hat{v}(W) = e^{\hat{s}(W)}$, where $\hat{s}(W) = \sup_{x \geq 0} \{ W(x) - \delta x \}$.

**Proof.** For an arbitrary technology $x \geq 0$, denote by $K_x^\times = \{ K^\times_t \}_{t \geq 0}$ the cutoff strategy in which the players explore with full intensity until $x$ is developed, and exploit $x$ thereafter. In other words, $K^\times_t = N \mathbf{1}_{[0, x/N]}(t)$. By the nature of the problem, it is obvious that we can focus on this class of cutoff strategies without loss.

The average payoff for strategy $K^\times$ can be calculated as

$$\hat{v}(W | K^\times) = \int_{x/N}^{\infty} re^{-rt} e^{W(x)} \, dt = e^{W(x) - \delta x},$$

which gives

$$\hat{v}(W) = \sup_{x \geq 0} \hat{v}(W | K^\times) = e^{\hat{s}(W)},$$

and the proof is complete.

We define the first-best technology $\hat{x}_N(W)$ to be any $\hat{x}_N \in \arg \max_{x \geq 0} \{ W(x) - \delta x \}$ if $\hat{s}(W) < +\infty$. In such a case, the value $\hat{v}(W) < +\infty$ is achieved by strategy $K^\times_{\hat{x}_N}$. The first-best technology is not defined for $W$ with $\hat{s}(W) = +\infty$. $

\square$

**O.2 Long-run Outcomes**

**O.2.1 Proof of Lemma 2**

**Proof.** This lemma can be easily derived from the results in Lehoczky (1977). $

\square$

**O.2.2 Proof of Lemma 3**

**Proof.** By the definition of $\hat{x}_N$ in the proof of Lemma 13, we have $W(\hat{x}_N) - \delta \hat{x}_N \geq W(x) - \delta x$ for all $x \geq 0$. We assume there exists a unique $\hat{x}_N$ without loss. Denote by $\bar{x} := \arg \max_{[0, \bar{x}]} W(x)$ the technology that has the quality $W(\bar{x}) = \bar{s}$. Note that if $\delta x \geq 0$, then $\bar{x} > 0$.

1The value from an optimal strategy described by a function $t \mapsto K(t)$ can always be achieved by a strategy depending only on $x = X_t$. This is because $X_t$ and $t$ is not one-to-one only if there is a time interval in which $K_t = 0$. But the strategy that resumes exploration after stopping temporarily can not strictly dominate all strategies that stop only once, by the dynamic programming principle.

2This assumption holds almost surely under Assumption 1.

3In this proof we write $\bar{x}$ and $\bar{s}$ instead of $\bar{x}(\bar{a})$ and $\bar{s}(\bar{a})$ for the sake of notational convenience.

4By standard results, $\bar{x}(\bar{a})$ is almost surely finite under Assumption 1. See, e.g., equation (1.1) of Taylor (1975).
a < \bar{a} we have \( W(\bar{x}) = \bar{x} - \bar{a} \), otherwise we have \( \bar{x} = 0 \) and thus \( W(\bar{x}) = s - a \).

Suppose that \( \hat{x}_N > \bar{x} \). Then we have \( W(\hat{x}_N) - \delta \hat{x}_N > W(\bar{x}) - \delta \bar{x} = \bar{x} - \delta \bar{x} \geq \bar{x} - \delta \bar{x} \), which implies the event of \( \sup_{x > \bar{x}} \{ W(x) - \delta (x - \bar{x}) \} > \bar{x} \). By the Markov property of the Brownian motion, this event occurs with probability \( P(M_\infty > \max\{a, \bar{a}\}) \), where \( M_\infty \) denotes the global maximum of a Brownian motion starting from zero, with drift \( \mu - \delta \) and volatility \( \sigma \). It is well known that \( M_\infty \) has an exponential distribution with mean \( -\frac{\mu^2}{2(\mu - \delta)} = 1/\lambda \) if \( \lambda > 1 \), which is guaranteed by Assumption 1. Therefore, we have \( q(\bar{a}) = 1 - P_{d_1}(\hat{x}_N > \bar{x}) \geq 1 - P(M_\infty > \max\{a, \bar{a}\}) \to 1 \) as \( \bar{a} \to +\infty \).

\[ \Box \]

O.3 Properties of Payoff Functions

O.3.1 Proof of Lemma 6

Proof. The second part is a standard result and thus the proof is omitted.\(^5\) Here we prove the first part in which \( k \) might be discontinuous.

We first show that given the payoffs \( u_L := u_n(a_L \mid k) \) and \( u_R := u_n(a_R \mid k) \) at the boundaries, if there exists a function \( v \) of class \( C^1 \) and piecewise \( C^2 \) on \( (a_L, a_R) \) which solves the two-point boundary value problem (BVP)

\[
L_v(a) - rv(a) = -f(a),
\]

\[
v(a_L) = u_L,
\]

\[
v(a_R) = u_R,
\]

with \( L_v(a) := -\mu K(a)v'(a) + \frac{1}{2} \sigma^2 K(a)v''(a) \) and \( f(a) := r(1 - k_n(a))e^a \), then we have \( v = u_n \) on \( [a_L, a_R] \). We then argue that such a solution exists and the proof is complete.

Suppose \( v \) is of class \( C^1 \) and piecewise \( C^2 \) and solves the above BVP. Denote the first time that \( A_t \) leaves \( (a_L, a_R) \) by \( \tau := \inf\{ t > 0 \mid A_t \notin (a_L, a_R) \} \), and note that it is a bounded stopping time. We can apply Itô’s formula\(^6\) to \( e^{-rt} v(A_T) \) and have

\[
e^{-r(T^\wedge \tau)} v(A_T) - v(a) = \int_0^{T^\wedge \tau} e^{-rt} (L_v - rv)(A_t) dt - \int_0^{T^\wedge \tau} e^{-rt} \sigma \sqrt{K(A_t)} v'(A_t) dB_t.
\]

We replace \( L_v - rv \) with \( -f \) and rearrange to obtain

\[
e^{-r(T^\wedge \tau)} v(A_T) - v(a) + \int_0^{T^\wedge \tau} e^{-rt} f(A_t) dt = -\int_0^{T^\wedge \tau} e^{-rt} \sigma \sqrt{K(A_t)} v'(A_t) dB_t,
\]

which implies the left-hand side is a continuous martingale starting from 0. We can then

\(^5\)See, e.g., the proof of Theorem 4.5 in Durrett (1996), p. 226.

\(^6\)Itô’s formula applies to any function that is \( C^1 \) and piecewise \( C^2 \).

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apply the optional stopping theorem and have
\[ v(a) = E_\omega \left[ \int_0^\tau e^{-rt} f(A_t) \, dt + e^{-r\tau} v(A_\tau) \right] = E_\omega \left[ \int_0^\tau e^{-rt} f(A_t) \, dt \right] + E_\omega [e^{-r\tau} 1_{\{A_\tau = a_L\}}] u_L + E_\omega [e^{-r\tau} 1_{\{A_\tau = a_R\}}] u_R = u_n(a \mid k), \]
which shows that the solution to the BVP coincides with player \( n \)'s payoff function on \((a_L, a_R)\).

We next prove the existence of a solution of class \( C^1 \) and piecewise \( C^2 \) to the above BVP, using the standard "shooting method" (see, e.g., Strulovici and Szydlowski (2015), p. 1044). Without loss of generality, we assume both \( K(a) \) and \( k_n(a) \) are Lipschitz-continuous on \((a_L, a_M) \cup (a_M, a_R)\).\(^7\) Consider the initial value problem (IVP)

\[ L v(a) - rv(a) = -f(a), \]
\[ v(a_L) = u_L, \]
\[ v'(a_L) = 0, \]
and the IVP with the homogeneous ODE

\[ L v(a) - rv(a) = 0, \]
\[ v(a_L) = 0, \]
\[ v'(a_L) = 1. \]

Since \( K(a) \) is bounded away from zero, standard result (see, e.g., Barbu (2016), Theorem 2.4) shows that both problems admit unique solution defined on \((a_L, a_M)\) as long as the length of the interval \(|a_L - a_M|\) is sufficiently small. By the same argument, we can then uniquely extend these solutions to the whole interval \((a_L, a_R)\), requiring them to be continuously differentiable at \( a_M \). Denote such an extension of the solution to the first IVP by \( v_f \), and to the homogeneous IVP by \( v_0 \). Then \( v_f + \frac{\alpha - v_f(a_R)}{v_0(a_R)} v_0 \) is the solution of class \( C^1 \) and piecewise \( C^2 \) to the BVP as long as \( v_0(a_R) \neq 0 \).

Lastly, we argue that \( v_0(a_R) \neq 0 \). Suppose by contradiction that \( v_0(a_R) = 0 \). Since \( v_0'(a_L) = 1 > 0 \), \( v_0 \) achieves its maximum at some \( \bar{a} \in (a_L, a_R) \) with \( v_0(\bar{a}) > 0 \) and \( v_0'(\bar{a}) = 0 \). As \( v_0 \) solves \( L v - rv = 0 \), we have \( L v(a) - rv(a) \to 0 \) as \( a \to \bar{a}^- \), which implies \( \lim_{a \to \bar{a}^-} v_0''(a) = rv_0(\bar{a})/(\sigma^2 K(\bar{a})/2) > 0 \), contradicting the maximum

\(^7\)Recall that we require \( k_n \), and hence also \( K \), to be right-continuous and piecewise Lipschitz-continuous. Here we assume both functions have only one possible discontinuity point at \( a_M \in (a_L, a_R) \).

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property of \( v_0(\hat{a}) \).

\[ \square \]

### O.3.2 Proof of Lemma 7

Given Markov strategy profile \( k \) with the state variable \((a, s)\) in which the strategies are not necessarily best responses against each other, let

\[
L = \eta(a, s) \frac{\partial}{\partial a} + \frac{1}{2} \alpha^2(a, s) \frac{\partial^2}{\partial a^2}
\]

with \( \eta(a, s) = -\mu K(a, s) \) and \( \alpha(a, s) = \sigma \sqrt{K(a, s)} \). Denote by \( f(a, s) = r(1 - k_n(a, s))e^s \) the flow payoff that player \( n \) receives at state \((a, s)\), and by \( v(a, s) = v_n(a, s \mid k) \) her payoff at state \((a, s)\).

**Lemma 14** (Normal Reflection Condition and Feynman-Kac Equation). If \( a \mapsto v(a, s) \) is twice continuously differentiable at \( a \), then it satisfies the Feynman-Kac formula

\[
rv(a, s) = Lv(a, s) + f(a, s) \tag{13}
\]

Moreover, if \( v \) is twice continuously differentiable on \( \{a = 0\} \), and \( K(a, s) \) is bounded away from 0 on some open set containing \( \{a = 0\} \), then \( v \) satisfies the normal reflection condition

\[
\frac{\partial v(0+, s)}{\partial a} + \frac{\partial v(0+, s)}{\partial s} = 0 \tag{14}
\]

**Proof.** Given strategy profile \( k \), consider player \( n \)'s continuation value at time \( T \), that is,

\[
v(A_T, S_T) = \mathbb{E} \left[ \int_T^\infty e^{-rt} f(A_t, S_t) \, dt \bigg| \mathcal{F}_T \right],
\]

and her total discounted payoff at time 0, evaluated conditionally on the information available at time \( T \), that is,

\[
Z_T := \mathbb{E} \left[ \int_0^\infty e^{-rt} f(A_t, S_t) \, dt \bigg| \mathcal{F}_T \right] = \mathbb{E} \left[ \int_T^\infty e^{-rt} f(A_t, S_t) \, dt \bigg| \mathcal{F}_T \right] + \int_0^T e^{-rt} f(A_t, S_t) \, dt = e^{-rT} v(A_T, S_T) + \int_0^T e^{-rt} f(A_t, S_t) \, dt.
\]

\[\text{Here we mean } v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \text{ can be extended to a function that is twice continuously differentiable on some open set in } \mathbb{R}^2 \text{ containing } \{a = 0\}.\]
Note that \( \{Z_t\}_{t \geq 0} \) is a martingale.\(^9\) Indeed, for \( 0 \leq \tau \leq T \), we have

\[
E[Z_T|\mathcal{F}_\tau] = E \left[ E \left[ \int_0^\infty e^{-rt} f(A_t, S_t) \, dt \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_\tau \right] \\
= E \left[ \int_0^\infty e^{-rt} f(A_t, S_t) \, dt \bigg| \mathcal{F}_\tau \right] \\
= Z_\tau.
\]

If Itô’s formula can be applied to \( e^{-rT}v(A_T, S_T) \), we would have

\[
e^{-rT}v(A_T, S_T) - v(a, s) = \int_0^T e^{-rt} (Lv - rv)(A_t, S_t) \, dt \\
- \int_0^T e^{-rt} \alpha(A_t, S_t) \frac{\partial v}{\partial a}(A_t, S_t) \, dB_t \\
+ \int_0^T e^{-rt} \left( \frac{\partial v}{\partial a} + \frac{\partial v}{\partial s} \right)(A_t, S_t) \, dS_t.
\]

Because \( Z_T \) is a martingale starting from \( v(a, s) \), we can then conclude the process

\[
Z_T - v(a, s) + \int_0^T e^{-rt} \alpha(A_t, S_t) \frac{\partial v}{\partial a}(A_t, S_t) \, dB_t \\
= \int_0^T e^{-rt} ((Lv - rv + f)(A_t, S_t)) \, dt + \int_0^T e^{-rt} \left( \frac{\partial v}{\partial a} + \frac{\partial v}{\partial s} \right)(A_t, S_t) \, dS_t
\]

is a continuous martingale starting from 0. Observe that the right-hand side is a finite variation process. As a consequence, it must be 0 for all \( T \geq 0 \) almost surely. This claim continues to hold if we replace \( T \) with a bounded stopping time. Itô’s formula can then be applied to \( e^{-r(T \wedge \tau_1)}v(A_{T \wedge \tau_1}, S_{T \wedge \tau_1}) \), if we let \( \tau_1 := \inf \{ t > 0 \mid (A_t, S_t) \notin O \times \{s\} \} \) for some neighborhood \( O \) of \( a \) in which \( a \mapsto v(a, s) \) is twice continuously differentiable. Feynman-Kac formula (13) then follows from the fact that the second integral on the right-hand side is 0 for all \( T \geq 0 \) and thus the first integral is almost surely 0 for all \( T \geq 0 \) as well.

Now, consider \( \tau_2 := \inf \{ t > 0 \mid (A_t, S_t) \notin [0, \epsilon_1) \times [s, s + \epsilon_2) \} \), for some \( \epsilon_1, \epsilon_2 > 0 \) such that \( a \in [0, \epsilon_1) \), \( K(\cdot) \) is bounded away from zero on \( [0, \epsilon_1) \), and \( v \) has a \( C^2 \) extension on \( (-\epsilon_1, \epsilon_1) \times (s - \epsilon_2, s + \epsilon_2) \) by our assumption. We can then apply Itô’s formula to \( e^{-r(t \wedge \tau_2)}v(A_{T \wedge \tau_2}, S_{T \wedge \tau_2}) \), and conclude that for all \( T \geq 0 \), almost surely the process on the right-hand side of the equation above with \( T \) replaced with \( T \wedge \tau_2 \) is 0. Moreover, note that the Lebesgue measure \( dt \) and the measure \( dS_t \) on \( (0, T \wedge \tau_2], \mathcal{B}([0, T \wedge \tau_2]) \) are mutually singular almost surely. Indeed, the set \( D := \{ t \in [0, T \wedge \tau_2] \mid A_t = 0 \} \) is

\(^9\)Assumption 1 guarantees all expectations in this proof are finite.
a null set under Lebesgue measure $dt$, and $[0, T \land \tau_2] \setminus D$ is a null set under measure $dS_t$ almost surely. Because for all $T \geq 0$, almost surely the sum of the integrals on the right-hand side is zero and $dt \perp dS_t$, we know that for all $T \geq 0$ each integrand is a.s. a.e. 0 on $[0, T \land \tau_2]$. This yields the normal reflection condition (14). \hfill \Box

### 0.4 Comparative Statics

**Lemma 15** (Welfare Comparison). Consider normalized payoff functions $u_j : \mathbb{R}_+ \to [1, +\infty)$ for $j \in \{l, h\}$, with stopping thresholds $\bar{a}_j := \sup\{a \geq 0 \mid u_j(a) > 1\} < +\infty$. Suppose for each $j \in \{l, h\}$, $u_j$ satisfies the following conditions:

1. $C^1$ on $(0, \bar{a}_j)$ with $u'_l(\bar{a}_l-) = u'_h(\bar{a}_h-) \leq 0$;
2. piecewise $C^2$ on $\mathbb{R}_{++}$;
3. there exists some function $g_j : (1, +\infty) \times \mathbb{R}_- \to \mathbb{R}_{++}$ with $g_j(z_1, z_2)/z_2$ nondecreasing in $z_1$ and nonincreasing in $z_2$ such that
   
   (a) $u''_j(a) = g_j(u_j(a), u'_j(a))$ for all $a \in (0, \bar{a}_j)$ at which $u''_j(a)$ is continuous;
   
   (b) $u'_j(a) + g_j(u_j(a), u'_j(a)) > 0$ for all $a \in (0, \bar{a}_j)$.

If $g_l \geq g_h$, then $\bar{a}_l \leq \bar{a}_h$ and $u_l \leq u_h$. Additionally, if for some $\bar{u} \in (1, u_h(0)]$ we have $g_l = g_h$ on $(\bar{u}, u_h(0)] \times \mathbb{R}_-$ and for all $z_2 \leq 0$ at least one of the following two conditions holds:

(i) $g_l(\bar{u}-, z_2) > g_h(\bar{u}-, z_2)$;

(ii) \(\frac{\partial g_l(\bar{u}-, z_2)}{\partial z_1} < \frac{\partial g_h(\bar{u}-, z_2)}{\partial z_1}\) and \(\frac{\partial g_l(\bar{u}-, z_2)}{\partial z_2} \geq \frac{\partial g_h(\bar{u}-, z_2)}{\partial z_2}\);

then $\bar{a}_l \leq \bar{a}_h$, and $u_l \leq u_h$ on $[0, \bar{a}_h)$.

**Remark.** Almost all payoff functions in this paper fulfill Conditions 1–3 above under Assumption 1 with value matching and smooth pasting at $\bar{a}$. These include

- the cooperative solution: $g(z_1, z_2) = \frac{z_1}{N\rho} + \theta z_2$;
- the symmetric MPE: $g(z_1, z_2) = \max\{z_1/N, 1\}/\rho + \theta z_2$.

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99By a.e. 0 on $[0, T \land \tau_2]$ we mean the integrands are 0 on all non-null sets w.r.t. the corresponding measure.

10We require $K(a, s) > 0$ in some neighborhood containing $(a = 0)$, as otherwise all measurable subsets of $[0, T \land \tau_2]$ would be null sets w.r.t. $dS_t(\omega)$, in which case we cannot conclude that $v$ satisfies the normal reflection condition (14).
• the average payoff in the asymmetric MPE:

\[ g(z_1, z_2) = \max\{z_1/N, \min\{1, z_1 - (1 - 1/N)\}\}/\rho + \theta z_2; \]

• the average payoff in the simple asymmetric MPE of Proposition 10:

\[ g(z_1, z_2) = \begin{cases} 
\frac{z_1 - (1-K/N)}{K\rho} + \theta z_2, & \text{if } K < z_1 \leq K + 1 \text{ for } K = 1, \ldots, N - 1, \\
\frac{z_1}{N\rho} + \theta z_2, & \text{if } z_1 \geq N. 
\end{cases} \]

Proof of Lemma 15. From \( u_j''(a) = g_j(u_j(a), u_j'(a)) > 0 \) and \( u_j'(\tilde{a}_j) \leq 0 \), we know \( u_j' \) is negative on \((0, \tilde{a}_j)\). Therefore, \( u_j^{-1}(u) \), the inverse of \( u_j \) at \( u \), is uniquely defined for \( u \in [1, u_j(0)] \). To simplify notations, we denote the derivatives (one-side if necessary) of \( u_j \) at level \( u \in [1, u_j(0)] \) by \( \tilde{u}_j'(u) \) and \( \tilde{u}_j''(u) \), where \( \tilde{u}_j' = u_j' \circ u_j^{-1} \) and \( \tilde{u}_j'' = u_j'' \circ u_j^{-1} \).

We first show that for all \( u \in [1, \min\{u_l(0), u_h(0)\}] \), we have \( \tilde{u}_j'(u) \leq \tilde{u}_h'(u) \). Suppose by contradiction that \( \tilde{u}_j'(\hat{u}) > \tilde{u}_h'(\hat{u}) \) for some \( 1 \leq \hat{u} \leq \min\{u_l(0), u_h(0)\} \). Let \( \tilde{a} := \sup\{ u \leq \tilde{a} | \tilde{u}_j'(u) \leq \tilde{u}_h'(u) \} \). Obviously we have \( 1 \leq \tilde{a} < \hat{u} \) since \( \tilde{u}_j'(1) = \tilde{u}_h'(1) \), and by the continuity of \( \tilde{u}_j' \) we have \( \tilde{u}_j'(\tilde{a}) = \tilde{u}_h'(\tilde{a}) < 0 \). We then compare their derivatives on \((\tilde{a}, \tilde{a} + \epsilon)\) for some small enough \( \epsilon > 0 \) and have

\[ \frac{d\tilde{u}_j'(u)}{du} = \frac{\tilde{u}_j''(u)}{\tilde{u}_j'(u)} \leq \frac{g_l(u, \tilde{u}_j'(u))}{\tilde{u}_j'(u)} \leq \frac{g_h(u, \tilde{u}_h'(u))}{\tilde{u}_h'(u)} = \frac{\tilde{u}_h'(u)}{\tilde{u}_h''(u)} = \frac{d\tilde{u}_h'(u)}{du}, \]

where the first inequality comes from the monotonicity of \( g_l(z_1, z_2)/z_2 \) in \( z_2 \), and the second inequality comes from \( \tilde{u}_h'(u) < 0 \) and our assumption that \( g_l \geq g_h > 0 \). This, together with \( \tilde{u}_j'(\tilde{a}) = \tilde{u}_h'(\tilde{a}) \), implies \( \tilde{u}_j'(u) \leq \tilde{u}_h'(u) \) on \((\tilde{a}, \tilde{a} + \epsilon)\), contradicting the definition of \( \tilde{a} \).

Next we show that \( u_l(0) \leq u_h(0) \). Suppose by contradiction that \( u_l(0) > u_h(0) \). Then the normal reflection conditions \( u_j(0) = -u_j'(0) \) and Condition 3b in the lemma

\[ 12 \text{We define } u_j^{-1}(1) := \tilde{a}_j. \]
imply the following strict inequality

\[
\begin{align*}
    u_I(0) = -u_I'(0) = -\tilde{u}_I'(u_I(0)) &= -\tilde{u}_I'(u_h(0)) - \int_{u_h(0)}^{u_I(0)} \tilde{u}_I''(u)/\tilde{u}_I'(u) \, du \\
    &= -\tilde{u}_I'(u_h(0)) - \int_{u_h(0)}^{u_I(0)} g_I(u, \tilde{u}_I'(u))/\tilde{u}_I'(u) \, du \\
    &\geq -\tilde{u}_h'(u_h(0)) - \int_{u_h(0)}^{u_I(0)} g_I(u, \tilde{u}_I'(u))/\tilde{u}_I'(u) \, du \\
    &> -\tilde{u}_h'(u_h(0)) + \int_{u_h(0)}^{u_I(0)} 1 \, du \\
    &= -u_h'(0) + u_I(0) - u_h(0) = u_I(0),
\end{align*}
\]

which is a contradiction.

Since we have \( \tilde{u}_I'(u) \leq \tilde{u}_h'(u) \leq 0 \) for all \( u \in [1, u_I(0)] \) with \( u_I(0) \leq u_h(0) \), it is obvious that \( \tilde{a}_I \leq \tilde{a}_h \) and \( a_I \leq u_h \).

To prove the strict inequalities, suppose in addition to \( g_I \geq g_h \) we also have \( g_I = g_h \) on \( (\bar{u}, u_h(0)) \times \mathbb{R}_- \) for some \( \bar{u} \in (1, u_h(0)] \). Moreover, assume that for all \( z_2 \leq 0 \), we have either (i) \( g_I(\bar{u}^-, z_2) > g_h(\bar{u}^-, z_2) \), or we have (ii) \( \frac{\partial g_I(\bar{u}^-, z_2)}{\partial z_1} < \frac{\partial g_h(\bar{u}^-, z_2)}{\partial z_1} \) and \( \frac{\partial g_I(\bar{u}^-, z_2)}{\partial z_2} \geq \frac{\partial g_h(\bar{u}^-, z_2)}{\partial z_2} \). We next show that \( u_I(0) < u_h(0) \), which together with the weak inequalities proved above yields the strict inequalities in the Lemma.

Suppose by contradiction that \( u_I(0) = u_h(0) \). Note that either of (i) or (ii) implies \( g_I(z_1, z_2) > g_h(z_1, z_2) \) for \( z_1 \) immediately below \( \bar{u} \), which together with \( \tilde{u}_I'(\bar{u}) \leq \tilde{u}_h'(\bar{u}) \) proved above implies that \( \tilde{u}_I' < \tilde{u}_h' < 0 \) on \( (\bar{u} - \eta, \bar{u}) \) for some \( \eta > 0 \). Moreover, by the normal reflection conditions, our assumption \( u_I(0) = u_h(0) \) implies \( u_I'(0) = u_h'(0) \).

Since \( g_I = g_h \) on \( (\bar{u}, u_h(0)) \times \mathbb{R}_- \), and the initial conditions for \( u''_I = g_I(u_I, u'_I) \) are the same for both \( j = l, h \), we must have \( \tilde{a}_I'(\bar{u}) = \tilde{a}_h'(\bar{u}) =: \bar{z}_2 < 0 \). We can then compare the left derivatives of \( \tilde{a}_I' \) at \( \bar{u} \) to have

\[
    \frac{d\tilde{a}_I'(\bar{u}-)}{du} = \frac{\tilde{u}_I''(\bar{u}-)}{\tilde{u}_I'(\bar{u})} \leq \frac{g_I(\bar{u}-, \bar{z}_2)}{\bar{z}_2} = \frac{\tilde{u}_h''(\bar{u}-)}{\tilde{u}_h'(\bar{u})} = \frac{d\tilde{a}_h'(\bar{u}-)}{du}.
\]

If \( g_I(\bar{u}-, \bar{z}_2) > g_h(\bar{u}-, \bar{z}_2) \), then this inequality is strict, which together with \( \tilde{a}_I'(\bar{u}) = \tilde{a}_h'(\bar{u}) \), contradicts the fact that \( \tilde{u}_I' < \tilde{u}_h' \) on \( (\bar{u} - \eta, \bar{u}) \). Otherwise, it must be \( g_I(\bar{u}-, \bar{z}_2) =
we know $g_h(\bar{u}, \bar{z}_2)$ and hence $\frac{\partial g_j(\bar{u} - \bar{z}_2)}{\partial z_1} < 0$. Then from their second-order derivatives we can conclude from $\frac{d^2 \tilde{u}_j'(\bar{u} - \bar{z}_2)}{du^2}$. This, together with $\tilde{u}_j'(\bar{u}) = \tilde{u}_h'(\bar{u})$ and $\frac{\partial g_j(\bar{u} - \bar{z}_2)}{\partial z_1} \geq \frac{\partial g_j(\bar{u} - \bar{z}_2)}{\partial z_2} = \frac{g_j(\bar{u} - \bar{z}_2)}{d(\tilde{u}_j'(\bar{u} - \bar{z}_2))}$, contradicts the fact that $\tilde{u}_j' < \tilde{u}_h'(\bar{u})$. □

### O.4.1 Proof of Corollary 1

**Proof.** Here we prove the limit results only. Other results can be easily derived from explicit calculations. Write $\hat{w} = (\ln \tilde{U})'$ and $w^* = (\ln U^*)'$. It is not difficult to verify that $\hat{w}' = -\left(\frac{1}{N^*_{\rho}} - \theta\right) \hat{w} - \hat{w}^2$ on $\mathbb{R}_{++}$; $w'' = \frac{1}{N^*_{\rho}} + \theta w^* - w^* - 2$ on $(0, a_N^*)$ and $w^* = 0$ on $[a_N^*, +\infty)$.

Suppose that $\theta \leq -1$. Then on $(0, a_N^*)$, both ODEs above converge to $w' - \theta w + w^2 = 0$ as $N \to +\infty$. Because the normal reflection condition (4) implies $\hat{w}(0) = w^*(0) = -1$, we must have $\lim w = \lim \hat{w}$ on $(0, a_N^*)$. However, note that $\lim \hat{w}(a) < 0$ for all $a \geq 0$. Therefore, we must have $a_N^* \to +\infty$ by the continuity of $w^*$ on $\mathbb{R}_{+}$. This, together with the convergence of $w^*$ and $\hat{w}$, implies $\lim U^*(a) = \lim \tilde{U}(a)$ for each $a \geq 0$. In particular for $\theta = -1$, from Lemma 1 we know $\tilde{U}(a) \to +\infty$ as $N \to +\infty$, and hence $U^*(a) \to +\infty$ for each $a \geq 0$. Then the results for the case of $\theta > -1$ follow from the monotonicity of $U^*(a)$ in $\theta$, which can be easily verified by Lemma 15. □

### O.4.2 Proof of Corollary 3

**Proof.** Since $r = \sigma^2/(2\rho)$, the comparative statics with respect to $r$ directly follow from the following proof of the comparative statics with respect to $\rho$. Recall that the common payoff function $U_{\rho}^\dagger$ in the symmetric MPE satisfies ODE $u'' = g(u, u')$ on $(0, \bar{a})$ where $g(z_1, z_2) = \max\{z_1/N, 1\}/\rho + \theta z_2$. Then the comparative statics of $U_{\rho}^\dagger$ and the stopping threshold $\bar{a}_\rho$ follow directly from Lemma 15.

Because $U_{\rho}^\dagger(a)$ is decreasing on $[0, \bar{a})$ and increasing in $\rho$ for each $a \in [0, \bar{a})$, the full-intensity threshold $a_{\rho}^\dagger = U_{\rho}^{\dagger-1}(N)$ is weakly increasing in $\rho$. 

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Lastly, because \( k^\dagger_\rho(a) = (U^\dagger_\rho(a) - 1)/(N - 1) \) on \((a^\dagger_\rho, \tilde{a}_\rho)\), the comparative statics results above imply that \( k^\dagger_\rho(a) \) is weakly increasing in \( \rho \) for all \( a \geq 0 \).

\( \square \)

O.4.3 Proof of Corollary 4

Proof. For the binding case, similar to the proof of Corollary 3, the comparative statics of \( U^\dagger_N \) and \( \tilde{a}_N \) with respect to \( N \) are immediate from Lemma 15.

For the non-binding case, Lemma 15 implies that both \( U^\dagger_N \) and \( \tilde{a}_N \) are constant over \( N \), as \( g(z_1, z_2) = 1/\rho + \theta z_2 \) does not depend on \( N \). Then it is obvious that \( k_N(x) = (U^\dagger_N(a) - 1)/(N - 1) \) is weakly decreasing in \( N \) for each \( a \geq 0 \).

\( \square \)

O.4.4 Proof of Corollary 5

For the sake of notational convenience, we are going to prove the results in terms of \( \rho = \sigma^2/(2r) > \hat{\rho} \) for some \( \hat{\rho} \), which is equivalent to \( r < \hat{r} = \sigma^2/(2\hat{\rho}) \). We let \( \hat{\rho} = (\theta - \ln(1 + \theta))/\theta^2 \) if \( \theta > -1 \).

To show the statement for the case that \( \rho > \hat{\rho} \) and \( \theta > -1 \), we first show in Lemma 16 below that \( |a^*_{N} - a^\dagger_{N}| \to \Delta \) for some \( \Delta < +\infty \) as \( N \to \tilde{N} := 1/(\rho(1 + \theta)) \). Because we know from Corollary 1 that \( a^*_{N} \to +\infty \), we then conclude that \( a^\dagger_{N} \to +\infty \), and hence \( \tilde{a}_N \to +\infty \) as well. The convergence of \( q_N(\tilde{a}_N) \) then follows from Lemma 3.

After showing that \( \lim_{N \to \tilde{N}} \left\| U^*_N / U^\dagger_N \right\|_\infty < +\infty \) in Lemma 17, we can then conclude \( U^*_N \to +\infty \) from the fact that \( U^\dagger_N \to +\infty \) from Corollary 1.

The opposite case that \( \rho \leq \hat{\rho} \) or \( \theta \leq -1 \) follows from Lemma 18 below.

Lemma 16. If \( \rho > \hat{\rho} \), we have \( |a^*_{N} - a^\dagger_{N}| \to \Delta \in (0, +\infty) \) as \( N \to \tilde{N} \).

Proof. Note that when \( \rho > \hat{\rho} \), the expression of \( \tilde{a} \) for the non-binding case in Corollary 6 cannot be applied. Therefore, the symmetric MPE must belong to the binding case for all \( N \in (1, \tilde{N}) \). Moreover, for \( \rho > \hat{\rho} \), we can verify that \( t_N \) in Corollary 6 converges to some \( t_{\tilde{N}} \in (0, 1) \). Then from explicit calculations, we know that \( \gamma_2 \to 1 + \theta > 0 \) and

\[^{13}\text{If } \theta = 0 \text{ we set } \hat{\rho} = 1/2, \text{ the limit of the right-hand side as } \theta \to 0.\]
Lemma 17. If \(\rho > \hat{\rho}\), we have \(\lim_{N \to \hat{N}} \|U_N^*/U_N^\dagger\|_\infty < +\infty\) as \(N \to \hat{N}\).

Proof. We first show that \(\|U^*/U^\dagger\|_\infty = U^*(0)/U^\dagger(0)\) by showing \(U^*(a)/U^\dagger(a)\) is nonincreasing in \(a\).

Write \(w^* = (\ln U^*)'\) and \(w^\dagger = (\ln U^\dagger)'\). It is not difficult to verify that

\[
\frac{1}{N_\rho} + \theta w^* - w^{*2}, \quad \text{on } (0,a^*),
\]

and

\[
\frac{1}{U_\rho} + \theta w^\dagger - w^{\dagger2}, \quad \text{on } (0,a^\dagger),
\]

on \((a^\dagger,\hat{a})\), we have \(w^{\dagger\prime} \geq w'^\prime\). This, together with the normal reflection condition \(w^*(0) = w^\dagger(0) = -1\), implies \(w^* \leq w^\dagger\). Therefore, \(w^*(a) - w^\dagger(a) = (\ln(U^*(a)/U^\dagger(a)))' \leq 0\), which implies \(\ln(U^*(a)/U^\dagger(a))\) is nonincreasing in \(a\) and hence so is \(U^*(a)/U^\dagger(a)\). Therefore, we have \(\|U^*/U^\dagger\|_\infty = U^*(0)/U^\dagger(0)\).

Moreover, because \(w^*(0) = w^\dagger(0) = -1\) and \(w'^* = w^{\dagger\prime}\) on \((0,a^*_N)\), we have \(w^* = w^\dagger\) on \((0,a^*_N)\) and hence \(U^*/U^\dagger\) is constant on \((0,a^*_N)\). Therefore, we can write

\[
\frac{U_N^*(0)}{U_N^\dagger(0)} = \frac{U_N^*(a^*_N)}{U_N^\dagger(a^*_N)} = \frac{1}{N\gamma_2 - \gamma_1} \left( \gamma_2 e^{-\gamma_1(a^* - a)} - \gamma_1 e^{-\gamma_2(a^* - a)} \right).
\]

Since we know that \(\gamma_2 \to 1 + \theta > 0\) and \(\gamma_1 \to -1\), we have

\[
\|U_N^*/U_N^\dagger\|_\infty = \frac{U_N^*(0)}{U_N^\dagger(0)} \to \frac{1}{N(2 + \theta)} \left( (1 + \theta)e^\Delta + e^{-(1+\theta)\Delta} \right) < +\infty.
\]

Lemma 18. If either \(\rho \leq \hat{\rho}\) or \(\theta \leq -1\), then there exists \(\hat{N} > 1\) such that \(a_N^\dagger = 0\) and \(U_N^\dagger = U_N^\dagger < +\infty\) for all \(N \geq \hat{N}\).
Here we do not impose Assumption 1 in Lemma 18. Therefore, this lemma states that when $N$ reaches $\bar{N}$ as increased from 1, the symmetric equilibrium falls into the non-binding case and continues to be an equilibrium for all $N > \bar{N}$, even when Assumption 1 is violated for large $N$ in the case of $\theta > -1$. As a consequence, we have $\bar{a}_N = \bar{a}_N$ for all $N \geq \bar{N}$, and thus the stopping threshold $\bar{a}_N$ is bounded from above as $N \to +\infty$. This implies $|U_N^\ast(a) - U_N^\dag(a)|$ is bounded away from zero in the limit, because for large enough $N$, we have that $U_N^\dag(a)$ is constant over $N$, and that $U_N^\ast(a)$ is increasing in $N$ for each $a \geq 0$.

Moreover, since $\bar{a}_N$ is constant over $N$ for large enough $N$, the amount of exploration $\bar{x}(\bar{a}_N)$ is also constant for large $N$. The limit result for $q_N(\bar{a}_N)$ then follows from the fact that the first-best technology $\bar{x}_N$ is nondecreasing in $N$ for each given Brownian path $W$ and each initial state $(a, s)$.

Next, we prove the above lemma.

**Proof of Lemma 18.** Suppose either $\rho \leq \hat{\rho}$ or $\theta \leq -1$. We construct $U_N^\dag$ according to the closed-form expression of the equilibrium payoff function for the non-binding case in Corollary 6. This is possible only when either $\rho \leq \hat{\rho}$ or $\theta \leq -1$, as otherwise the expression for $\bar{a}$ in that corollary is not well-defined. Let $\bar{N} = U_N^\dag(0)$, and for all $N \geq \bar{N}$, let $k^\dag_N = (U_N^\dag - 1)/(N - 1)$.

We now verify that the strategy profile $k^\dag_N$ with each player playing $k^\dag_N$ constitutes a symmetric MPE with non-binding resource constraints in the $N$-player exploration game for all $N \geq \bar{N}$, with $U_N^\dag$ being the associated payoff function. Note that we cannot apply Lemma 4 directly because it relies on Assumption 1. Nevertheless, from the proof of Lemma 8, we know that $U_N^\dag$ is an upper bound on player $n$’s achievable payoffs against $k^\dag_{-n}$ (this fact does not rely on Assumption 1). Moreover, $U_N^\dag$ is indeed the payoff function associated with $k^\dag_N$, since $U_N^\dag$ is the only function that satisfies both of the properties in Lemma 7. As a result, the upper bound on the payoff functions when the player plays against $k^\dag_{-n}$ is achieved by $k^\dag_N$, and therefore $k^\dag_N$ is a symmetric MPE.

\[\square\]

**O.4.5 Proof of Proposition 9**

**Proof.** Suppose $\theta < -1$. Then Assumption 1 is always satisfied and the asymmetric MPE in Proposition 6 exist for all $N$.

Denote by $\bar{a}_N$ the average payoff function in the asymmetric MPE of Proposition 6 for an $N$-player exploration game. Because $\bar{a}_N$ satisfies the ODE $u = 1 - 1/N + \rho(u'' - \theta u')$.

\[14\text{More precisely, when Assumption 1 is violated and the uniqueness of the symmetric MPE is not guaranteed, here we mean there exists a sequence of symmetric MPE indexed by } N, \text{ with } \bar{a}_N \text{ bounded from above as } N \to +\infty.\]
whenever $1 < u < 2 - 1/N$, we can verify that $	ilde{w}_N := (\ln \tilde{a}_N)'$ satisfies the second-order ODE

$$w'' - \theta w' + 3w w' + w(w^2 - \theta w - 1/\rho) = 0$$ (15)

on $(a^\#, a^\delta)$, with the initial conditions $w(a^\delta) = 0$ and $w'(a^\delta) = 1/(N\rho)$, and $	ilde{w}_N = 0$ on $[a^\#, +\infty)$. By the continuity of the solution to ODE (15) in the initial data (see, e.g., Theorem 2.14 of Barbu (2016)), as $N \to +\infty$, $w_N$ converges to 0 on $(a^\#, a^\delta)$, because $w = 0$ is the unique solution to ODE (15) with the initial conditions $w(a^\delta) = 0$ and $w'(a^\delta) = \lim 1/(N\rho) = 0$. Suppose by contradiction that $a^\#$ is bounded from above as $N \to +\infty$. This implies for sufficiently large $N$ we have $\bar{u}_N(a) = \exp \left( \int_{a^\delta}^a w_N \right) < 2 - 1/N$ for all $a \in [a^\#, a^\delta]$, and thus by the construction of $\bar{u}_N$ we have $a^\# = 0$. Then the fact that $w_N \to 0$ on $(0, a^\delta)$ contradicts $w_N(0) = -1$ imposed by the normal reflection condition (4).

Now suppose $\theta \geq -1$. If the constructed asymmetric MPE exist for all $N > 1$, then $a^\delta \to +\infty$ as $N \to +\infty$. This is because for fixed $N > 1$ and $a \geq 0$, $\bar{u}_N(a)$ is nondecreasing in $\theta$, and thus the claim follows from the case of $\theta < -1$ as we have shown above. Otherwise, it can be shown that as $N \to 1/(\rho(1+\theta))$ we have $\bar{u}_N \to +\infty$ and $a^\# \to +\infty$ by an argument similar to the proof of Corollary 5 for the symmetric equilibrium.

\[ \square \]

### O.5 Lemmas in the Construction of Asymmetric MPE

**Proof of Lemma 12.** For any $a_L \leq a_1 \leq a_2 \leq a_R$, consider $u(\cdot \mid a_1, a_2)$ of class $C^1((a_L, a_R))$ that solves the explorer ODE $u = \rho(u'' - \theta u')$ on $(a_L, a_1) \cup (a_2, a_R)$, and the free-rider ODE $u = 1 + \rho(u'' - \theta u')$ on $(a_1, a_2)$, with the initial conditions $u(a_R) = \bar{u}(a_R)$ and $u'(a_R) = \bar{u}'(a_R)$. The existence and uniqueness of $u(\cdot \mid a_1, a_2)$ is guaranteed by standard results. We write $u^0_L(a_1, a_2) := u(a_L \mid a_1, a_2)$, and $u^1_L(a_1, a_2) := u'(a_L \mid a_1, a_2)$ for the value and the derivative of $u(\cdot \mid a_1, a_2)$ evaluated at $a_L$.

Note that the functions $u^0_L, u^1_L : \{(a_1, a_2) \mid a_L \leq a_1 \leq a_2 \leq a_R\} \to \mathbb{R}_+$ are continuous (see, e.g., Theorem 2.14 in Barbu (2016)). Moreover, from Lemma 19 below we know that they have the following properties: $u^0_L(a, a) > u_L, u^1_L(a, a) < \bar{u}'(a_L) < 0$ for all $a \in [a_L, a_R]$; $u^0_L(a_L, a_R) < u_L, 0 \geq u^1_L(a_L, a_R) > \bar{u}'(a_L)$; $u^0_L(a_1, a_2)$ is strictly increasing in $a_1$ and strictly decreasing in $a_2$.

By construction, $u(\cdot \mid a_1, a_2)$ and $\bar{u}$ match value and derivative at $a_R$. Next we choose $a_1$ and $a_2$ so that they match value and derivatives at $a_L$ as well.

Because $u^0_L(a_L, a_R) < u_L$ and $u^0_L(a_R, a_R) > u_L$, there exists a unique $\hat{a}_1 \in (a_L, a_R)$ such that $u^0_L(\hat{a}_1, a_R) = u_L$, and $u^0_L(a_1, a_R) < u_L$ for all $a_1 \in [a_L, \hat{a}_1)$. Therefore, for
each \( a_1 \in [a_L, \hat{a}_1] \), because \( u^0_L(a_1, a_1) > u_L \), there exists a unique \( \hat{a}_2(a_1) \in (a_1, a_R) \), such that \( u^0_L(a_1, \hat{a}_2(a_1)) = u_L \), with \( \hat{a}_2(\hat{a}_1) = a_R \) by the definition of \( \hat{a}_1 \). It is obvious that the function \( \hat{a}_2 \) is continuous on its domain \([a_L, \hat{a}_1]\).

In other words, \( u(\cdot | \hat{a}_1, a_R) \) solves the free-rider ODE on \((\hat{a}_1, a_R)\) and the explorer ODE on \((a_L, \hat{a}_1)\), while \( u(\cdot | a_L, \hat{a}_2(a_1)) \) solves the explorer ODE on \((\hat{a}_2(a_1), a_R)\) and the free-rider ODE on \((a_L, \hat{a}_2(a_1))\), with both \( u(\cdot | \hat{a}_1, a_R) \) and \( u(\cdot | a_L, \hat{a}_2(a_1)) \) having the same value as \( \tilde{u} \) at \( a_L \) and \( a_R \), and the same derivatives as \( \tilde{u} \) at \( a_R \).

From Lemma 19 below we can conclude that \( u^1_L(\hat{a}_1, a_R) < \tilde{u}'(a_R) < u^1_L(a_L, \hat{a}_2(a_1)) \).\(^{15}\)

Therefore, there exists a \( \tilde{a}_1 \in (a_L, \hat{a}_1) \) such that \( u^1_L(\tilde{a}_1, \hat{a}_2(\tilde{a}_1)) = \tilde{u}'(a_R) \) from the continuity of \( u^1_L \) and \( \hat{a}_2 \). Because \( \tilde{a}_1 < \tilde{a}_1 \), we have \( u^0_L(\tilde{a}_1, a_R) < u_L \), which implies \( \hat{a}_2(\tilde{a}_1) \neq a_R \) and thus it must be \( \hat{a}_2(\tilde{a}_1) < a_R \).

Finally, let \( a_{M-} := \tilde{a}_1, a_{M+} := \hat{a}_2(\tilde{a}_1) \) and \( \tilde{u}(\cdot) := u(\cdot | a_{M-}, a_{M+}) \). We have shown that \( \tilde{u}(a_L) = u_L, \tilde{u}'(a_L) = \tilde{u}'(a_R) \), and by construction \( a_L < a_{M-} < a_{M+} < a_R \). The range of \( \tilde{u} \) follows from the monotonicity of \( \tilde{u} \), which is implied by Lemma 19 with \( \tilde{u}'(a_R) = \tilde{u}'(a_R) \leq 0 \).

\(^{15}\)Lemma 19. Let \( u(\cdot | f, u^0, u^1) \) be the solution of the ODE \( u = f + \rho(u' - \theta u') \) with the initial conditions \( u(a_R) = u^0 \) and \( u'(a_R) = u^1 \), for some \( \rho > 0 \) and \( f, \theta \in \mathbb{R} \). For any \( a_L < a_R \), \( u(a_L | f, u^0, u^1) \) is strictly increasing in \( u^0 \) and strictly decreasing in \( f \) and \( u^1 \), whereas \( u'(a_L | f, u^0, u^1) \) is strictly decreasing in \( u^0 \) and strictly increasing in \( f \) and \( u^1 \). Moreover, if \( u^0 \geq f \) and \( u^1 \leq 0 \), then \( u'(a_L | f, u^0, u^1) \leq 0 \), with \( u'(a_L | f, u^0, u^1) = 0 \) if and only if \( u^0 = f \) and \( u^1 = 0 \).

Proof. Let \( \eta \) denote either \( f, -u^0 \), or \( u^1 \). For \( a_L < a_R \), to show \( u'(a_L | \eta) \) is strictly increasing in \( \eta \), let \( \eta_1 < \eta_2 \). For the purpose of contradiction, suppose \( u'(a_L | \eta_1) \geq u'(a_L | \eta_2) \). Let \( \tilde{a} = \sup\{a < a_R | u'(a | \eta_1) = u'(a | \eta_2)\} \).

For \( \eta \) being either \( f \) or \( -u^0 \), because \( \eta_1 < \eta_2 \) and \( u''(a_R | \eta) = \frac{\theta^2 - f}{\rho} + \theta u^1 \), we have \( u''(a_L | \eta_1) > u''(a_L | \eta_2) \). Therefore, we have \( u'(a | \eta_1) < u'(a | \eta_2) \) for all \( a \in (a_R - \epsilon, a_R) \) for some \( \epsilon > 0 \), which obviously also holds when \( \eta \) denotes \( u^1 \) by the
continuity of \( u'(· \mid \eta) \). Therefore, we have \( a_L \leq \hat{a} < a_R \), and from the continuity of 
\( u'(· \mid \eta) \), we have \( u'(\hat{a} \mid \eta_1) = u'(\hat{a} \mid \eta_2) \).

Because \( u'(a \mid \eta_1) < u'(a \mid \eta_2) \) for all \( a \in (\hat{a}, a_R) \), we have

\[
\begin{align*}
&u(\hat{a} \mid \eta_1) = u(a_R \mid \eta_1) - \int_{\hat{a}}^{a_R} u'(a \mid \eta_1) \, da \\
&> u(a_R \mid \eta_2) - \int_{\hat{a}}^{a_R} u'(a \mid \eta_2) \, da = u(\hat{a} \mid \eta_2),
\end{align*}
\]

which implies

\[
\begin{align*}
u''(\hat{a} \mid \eta_1) &= \frac{u(\hat{a} \mid \eta_1) - f|\eta_1|}{\rho} + \theta u'(\hat{a} \mid \eta_1) \\
&> \frac{u(\hat{a} \mid \eta_2) - f|\eta_2|}{\rho} + \theta u'(\hat{a} \mid \eta_2) = u''(\hat{a} \mid \eta_2).
\end{align*}
\]

But this is a contradiction, as \( u'(\hat{a} \mid \eta_1) = u'(\hat{a} \mid \eta_2) \) together with \( u'(a \mid \eta_1) < u'(a \mid \eta_2) \) for all \( a \in (\hat{a}, a_R) \) implies \( u''(\hat{a} \mid \eta_1) \leq u''(\hat{a} \mid \eta_2) \).

As \( a_L \) is chosen arbitrarily, we have shown that for any \( a < a_R \), \( u'(a \mid \eta) \) is strictly increasing in \( \eta \). Therefore, as \( u(a_L \mid \eta) = u(a_R \mid \eta) - \int_{a_L}^{a_R} u'(a \mid \eta) \, da \), we have \( u(a_L \mid \eta) \) is strictly decreasing in \( \eta \).

Finally, if \( u^0 = f \) and \( u^1 = 0 \), then \( u(a \mid f, u^0, u^1) = f \) is the solution of the ODE and hence \( u'(a_L \mid f, u^0, u^1) = 0 \) for any \( a_L \leq a_R \). Otherwise, if either \( u^0 > f \) or \( u^1 < 0 \), from the strict monotonicity of \( u'(a_L \mid f, u^0, u^1) \) in \( u^0 \) and \( u^1 \), we have
\[
u'(a_L \mid f, u^0, u^1) < 0.
\]

\[\square\]

**O.6 Math Appendix**

**Lemma 20.** Let \( M_T = \max_{0 \leq t \leq T} \{ \mu + \sigma B_t \} \) be the running maximum of the standard Brownian motion \( \{ B_t \}_{t \geq 0} \) starting from 0.

Then there exist \( C_1, C_2 \geq 0 \) which do not depend on \( T \), such that

\[
E[e^{M_T}] \leq C_1 e^{(\mu+\sigma^2/2)T} + C_2.
\]

Moreover, as \( T \to \infty \),

\[
E[e^{M_T}] \to \frac{\mu}{\mu + \sigma^2/2}
\]

if \( \mu + \sigma^2/2 < 0 \).

**Proof.** From Corollary 9.1 in Choe (2016), or equation (7.27) in Shreve (2004), we can
calculate $E[e^{M_T}]$ explicitly to have
\[
E[e^{M_T}] = \left(\mu + \frac{\sigma^2}{2}\right)^{-1} \left((\mu + \sigma^2)e^{(\mu + \sigma^2/2)T} \left(1 - N\left(-\frac{\mu + \sigma^2}{\sigma^2} \sqrt{T}\right)\right) + \mu N\left(-\frac{\mu}{\sigma^2} \sqrt{T}\right)\right)
\leq C_1 e^{(\mu + \sigma^2/2)T} + C_2
\leq |C_1| e^{(\mu + \sigma^2/2)T} + |C_2|,
\]
where $N$ is the CDF of standard normal distribution. The convergence result follows from straightforward calculations.

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