An affine Weyl group characterization of polynomial Heisenberg algebras

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Abstract

We study deformations of the harmonic oscillator algebra known as polynomial Heisenberg algebras (PHAs), and establish a connection between them and extended affine Weyl groups of type $A^{(1)}_m$, where $m$ is the degree of the PHA. To establish this connection, we employ supersymmetric quantum mechanics to first connect a polynomial Heisenberg algebra to symmetric systems of differential equations. This connection has been previously used to relate quantum systems to non-linear differential equations; most notably, the fourth and fifth Painlevé equations. Once this is done, we use previous studies on the Bäcklund transformations of Painlevé equations and generalizations of their symmetric forms characterized by extended affine Weyl groups. This work contributes to better understand quantum systems and the algebraic structures characterizing them.

1 Introduction

Supersymmetric quantum mechanics relates the eigenvalue problem of multiparametric families of quantum systems [1,2]. This tool is particularly powerful in that it allows the use of algebraic methods to study quantum systems [3,4]. In this regard, a supersymmetric transformation produces a realization of the factorization method in quantum mechanics [5,7].

The most common use of supersymmetric quantum mechanics has been the so-called spectral design [7,10]. It consists in obtaining quantum systems with prescribed energy spectrum, departing from a system whose energy
spectrum is already known \[11-32\]. Hamiltonians thus related are known as *supersymmetric partners* and the relations among them are known as *supersymmetric transformations*, *Darboux transformations*, or *intertwining relations* \[22\].

Indeed, the algebraic structures characterizing each supersymmetric partner are also related. One can be regarded as a deformation of the other. The harmonic oscillator and its supersymmetric partners are frequent examples. While the former is characterized by the oscillator algebra, the later are characterized by polynomial Heisenberg algebras (PHAs).

In general, polynomial Heisenberg algebras are deformations of the oscillator algebra, where the commutator between the ladder operators is a polynomial of the hamiltonian. The degree of this polynomial is known as the degree of the PHA. With regards to the eigenvalue problem of the hamiltonians of PHAs, it has been found that their energy spectra consists of a superposition of equidistant *ladders* \[32-35\].

It is also well known that systems ruled by PHA’s of second and third degree are connected to the fourth and fifth Painlevé equations, respectively \[35-43\]. This connection has been used in two ways. First, to produce concrete realizations of PHAs, one can use an specific solution of such Painlevé equations. However, Painlevé IV and V equations are second order nonlinear differential equations. Thus, an alternative and simpler use of the aforementioned connection is in the opposite direction: concrete realizations of second and third degree PHAs can be used to obtain solutions to the Painlevé IV and V equations, respectively \[31,32,44\].

Significant work has already been done in generalizing and exploiting the groups of rational transformations of Painlevé equations and their connection with symmetric systems or dressing chains. For example, in \[48\] the theory of symmetry transformations of Painlevé IV equations is developed. In \[37\] authors introduce the dressing chains as higher order generalizations of the fourth and fifth Painlevé equations. In \[38\] the author defines periodically closed sequences of transformations equivalent to second to sixth Painlevé equations. Symmetries of Painlevé equations and special functions have also been connected in \[50-58\]. In \[45\] the symmetries of Painlevé equations characterized by affine Weyl groups are given and properties studied. More recently, in \[53\] complete classification of rational solutions of the fourth Painlevé equation and its higher order generalizations is provided by means of the Weyl group $A_{2n}$ characterization. Work has also been done in \[31,59\] with respect to to connections between Painlevé equations and quantum
systems.

The main objective of this work is to generalize the connection between polynomial Heisenberg algebras and Painlevé equations. The generalization rests on the structure of the rational transformations of Painlevé equations, and their generalization to the case of the affine Weyl group of type $A^{(1)}_m$. These transformations are instances of the so-called Bäcklund transformations and they allow one to obtain new solutions of Painlevé equations from a given known solution. In short, the generalization of the structure of Bäcklund transformations of Painlevé equations to the case of the affine Weyl group of type $A^{(1)}_m$ can be used to study PHAs of arbitrary degree by means of tools provided by supersymmetric quantum mechanics.

The following diagram illustrates the main idea of this work:

\[
\text{Polynomial Hisenberg algebras} \quad \downarrow \quad \text{connection} \quad \uparrow
\]

\[
\text{Symmetric form of Painlevé equations} \quad \downarrow \quad \text{generalization}
\]

\[
\text{Bäcklund transformations of type } A^{(1)}_m
\]

The application of the first arrow is performed by supersymmetric quantum mechanics while the second arrow is performed by Nuomi’s generalization of the structure of rational transformations of Painlevé equations [45, 60, 61].

This article is organized as follows: In section 2 we briefly review the tool of supersymmetric quantum mechanics and its relation to the factorization method. In section 3 we describe the object of this study, i.e., polynomial Heisenberg algebras. In section 4 we present the manner in which low degree PHAs connect to symmetric forms of differential equations and generalize the results to the case of PHAs of arbitrary degree. Finally, in section 5 we give the final remarks on the results presented here.
2 Supersymmetric transformations

A supersymmetric transformation in quantum mechanics relates two hamiltonians \( H_i = -\frac{1}{2}\frac{d^2}{dx^2} + V_i(x), \) \( i = 0, 1, \) through the following operator equation
\[
H_1 Q^+ = Q^+ H_0 ,
\]
where \( Q^+ \) is a \( k \)-th order differential operator known as the intertwining operator. Units such that \( \hbar = m = 1 \) are assumed.

Equation (1) relates the energy spectra of the system \( H_1 \) with the one of \( H_0 \) [7–10]. This spectral relations are obtained by noticing that if \( \psi_n(x) \) solves the stationary Schrödinger equation for \( H_0 \psi_n(x) = E_n \psi_n(x) \), then \( \phi_n \propto Q^+ \psi_n(x) \) solves \( H_1 \phi_n(x) = E_n \phi_n(x) \). Thus, there might exist values in the energy spectrum of \( H_0 \) that are also in the energy spectrum of \( H_1 \). One must notice that, in order to obtain an appropriate supersymmetric transformation, both eigenvalue problems, for \( H_0 \) and \( H_1 \), must possess the same domain of definition and boundary conditions.

As a simple example consider the harmonic oscillator \( V_0 = \frac{x^2}{2} \). The energy spectrum of the hamiltonian with potential \( V_0 \) is given by an infinite equidistant ladder \( E_n = n + \frac{1}{2} \), where \( n = 0, 1, ..., \) , the corresponding eigenfunctions are
\[
\psi_n(x) = \frac{1}{\pi^{1/4} \sqrt{n!}} e^{-x^2/2} h_n(x) , \tag{2}
\]
where \( h_n(x) \) is the \( n \)-th Hermite polynomial.

On the other hand, the energy spectra of the supersymmetric partners of the harmonic oscillator \( H_1 \) consist in general of an isospectral subset \( E_n = n + \frac{1}{2}, \ n = 0, 1, ..., \) , whose corresponding eigenfunctions are
\[
\phi_n(x) = \frac{Q^+ \psi_n(x)}{\sqrt{(E_n - \epsilon_1)\ldots(E_n - \epsilon_k)}}, \quad n = 0, 1, .... \tag{3}
\]
However, in general, for a \( k \)-th order transformation, i.e. one where operator \( Q^+ \) is of \( k \)-th order, \( k \) new values \( \epsilon_j, j = 1, ..., k, \) may be added to the energy spectra of \( H_1 \), associated to the eigenfunctions
\[
\phi_{\epsilon_j} \propto \frac{W(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)}{W(u_1, \ldots, u_k)}, \quad j = 1, ..., k, \tag{4}
\]
respectively. \( W(u_1, \ldots, u_k) \) is the wronskian of \( k \) functions \( u_j, \ j = 1, ..., k, \) that solve the stationary Schrödinger equation \( H_0 u_j = \epsilon_j u_j \). Although these \( u_j \)'s
do not necessarily satisfy the boundary conditions of the eigenvalue problem for $H_1$. Thus, the general form they take is given by

$$u_j(x) = e^{-x^2/2} \left[ \, _1F_1 \left( \frac{1 - 2\epsilon_j}{4}, \frac{1}{2}, x^2 \right) + 2\nu x \frac{\Gamma \left( \frac{3-2\epsilon_j}{4} \right)}{\Gamma \left( \frac{1-2\epsilon_j}{4} \right)} \, _1F_1 \left( \frac{3 - 2\epsilon_j}{4}, \frac{3}{2}, x^2 \right) \right],$$

and not (2). $\Gamma(x)$ is the gamma function, $\, _1F_1(a,b,x)$ is the hypergeometric confluent function of $x$ with parameters $a$ and $b$, and $\nu$ is a constant.

The potential in $H_1$ is

$$V_1(x) = \frac{x^2}{2} - \frac{d}{d\epsilon} \frac{d}{d\epsilon} \ln W(u_1, ..., u_k),$$

where $' = \frac{d}{dx}$. In order to have a non-singular transformation and, in turn, to not break the domain of definition of the eigenvalue problem, the wronskian $W(u_1, ..., u_k)$ must not possess zeroes in the domain of definition of the problem, which in this case is the whole real line $x \in \mathbb{R}$. Let us note that the expression for $V_1$, eq. (6), defines a multiparametric family of potentials. The parameters are precisely the possibly added energy values $\epsilon_1, ..., \epsilon_k$.

Supersymmetric transformations can be also used to realize the so-called factorization method, that consists in identifying hamiltonians whose energy spectra can be obtained algebraically [46, 47]. First, consider the hermitian conjugate of equation (1),

$$QH_1 = H_0Q,$$  

where $Q = (Q^+)\dagger$ is the hermitian conjugate of $Q^+$. Then one can express the products of the intertwining operators $Q$ and $Q^+$ as the polynomials

$$QQ^+ = \prod_{i=1}^{k} (H_0 - \epsilon_i), \quad Q^+Q = \prod_{i=1}^{k} (H_1 - \epsilon_i),$$

where $k$ is the differential order of $Q$ and $Q^+$, and $\epsilon_i, i = 1, \ldots, k$ are the roots of the operator polynomials in $H_0$ and $H_1$. These are often called factorization energies.

### 3 Polynomial Heisenberg algebras

Polynomial Heisenberg algebras (PHA) are deformations of the algebra of the harmonic oscillator [32]. These are defined by three operators, $H, L^+$


and \( L^- \), together with the commutation relations

\[
[H, L^\pm] = \pm L^\pm, \quad [L^-, L^+] = N(H + 1) - N(H) = P_m(H),
\]

(9)

where \( N(H) = L^+L^- \) is the analogue of the number operator of the harmonic oscillator and \( P_m(H) \) is a polynomial of degree \( m \) of the hamiltonian \( H \).

Notice that, indeed, the first commutation relations are the defining ones for ladder operators \( L^\pm \), while the commutator between \( L^+ \) and \( L^- \) characterize the deformation of the harmonic oscillator algebra. The degree \( m \) of the polynomial \( P_m(H) \) defines the so-called degree of the PHA. If the differential order of the ladder operators is \( m + 1 \) and the hamiltonian is of second differential order, as is usual in quantum mechanics, then the corresponding polynomial Heisenberg algebra is in general of degree \( m \), also denoted as \( m \)-PHA. Indeed,

\[
N(H) = \prod_{i=1}^{m+1} (H - \mathcal{E}_i),
\]

(10)

where \( \mathcal{E}_i \) are the roots of the polynomial \( N(H) \).

This algebras describe rather interesting physical systems, since their energy spectra generally consists of \( m + 1 \) equidistant sets of values (ladders). To see how this happens consider the functions \( \psi(x) \) in the kernel of the operator \( L^- \), i.e., those that satisfy \( L^-\psi = 0 \). Then, \( L^+L^-\psi = 0 \) and

\[
\prod_{i=1}^{m+1} (H - \mathcal{E}_i)\psi = 0.
\]

(11)

Now, from the relations (9) we obtain

\[
L^-H\psi = (H + 1)L^-\psi = 0,
\]

(12)

which in turn implies that the kernel of \( L^- \) is invariant under the action of \( H \). Thus, by choosing the linearly independent functions \( \psi_i \) generating the \( \ker(L^-) \) as the eigenfunctions of \( H \) corresponding to the eigenvalues \( \mathcal{E}_i \). This are usually called extremal states of \( H \). Finally, departing from each extremal state, the system described by the \( m \)-PHA possesses \( m + 1 \) eigenenergy ladders, obtained from the repeated action of \( L^+ \) on such states.

Examples of PHAs can be readily obtained by performing a supersymmetric transformation on the harmonic oscillator. First, we note that the
hamiltonian $H_1$, with potential given by (6), has ladder operators defined in a natural way by

$$\ell^+ = Q^+a^+Q, \quad \ell^- = Q^+a^-Q,$$

(13)

where $a^\pm = (x \mp \frac{d}{dx})/\sqrt{2}$ are the usual ladder operators of the harmonic oscillator. Then, while $Q^+$ and $Q$ are differential operators of $k$-th order, $\ell^+$ and $\ell^-$ are differential operators of $2k + 1$-th order.

The definition of $\ell^\pm$, together with the commutation relations between $H_0$ and $a^\pm$ for the harmonic oscillator, i.e. $[H_0, a^\pm] = \pm a^\pm$, imply

$$H_1\ell^\pm = \ell^\pm(H_1 \pm 1).$$

(14)

Hence, $[\tilde{H}, \ell^\pm] = \pm \ell^\pm$, and $\ell^\pm$ are indeed ladder operators for $H_1$. However, the commutation relation between $\ell^+$ and $\ell^-$ satisfy

$$\ell^-\ell^+ = \left(H - \frac{1}{2}\right)\prod_{i=1}^{k}(H - \epsilon_i)(H - \epsilon_i - 1).$$

(15)

In contrast to the case of the harmonic oscillator, where ladder operators satisfy $[a^-, a^+] = 1$, for its supersymmetric partners $[\ell^-, \ell^+]$ is a polynomial $P_m(\tilde{H})$.

Based on the examples just mentioned, we can see that supersymmetric transformations can be used to better understand PHAs. The key idea that we will follow now is to employ the supersymmetric transformations to decompose the ladder operators into a product of first-order intertwining operators. This will allow us to connect the PHAs to symmetric systems of equations characterized by extended affine Weyl groups.

4 $A_n^{(1)}$ characterization of polynomial Heisenberg algebras

Before tackling the general case, let us build some intuition about how the connection between PHAs and affine Weyl groups occur for low degrees. This will also help us to obtain general quantum systems realizing the corresponding $m$-PHA. As described in the previous section, PHAs are defined by the commutation relations (9), where $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$ and $L^\pm$ are $m + 1$-th order differential operators.
The PHA with the lowest degree, i.e. zero, is obtained by considering first-order ladder operators

\[ L^+ = \frac{1}{2^{1/2}} \left( \frac{d}{dx} - f_1 \right), \quad (16) \]

\[ L^- = \frac{1}{2^{1/2}} \left( - \frac{d}{dx} - f_1 \right), \quad (17) \]

where \( f_1 = f_1(x) \) is a function of the coordinate \( x \). Thus, \( f_1 \) fixes \( L^\pm \).

Now consider an auxiliary Hamiltonian \( H_2 = H + \lambda \), where \( \lambda \in \mathbb{R} \), intertwined with \( H_1 = H \) as

\[ H_2 L^+ = L^+ H_1. \quad (18) \]

Alternatively one could use the Hermitian conjugate \( H_1 L^- = L^- H_2 \). This, by means of equation (8), leads to the following factorizations:

\[ H_1 = L^- L^+ + \epsilon_1, \quad H_2 = L^+ L^- + \epsilon_1, \quad (19) \]

recall that \( \epsilon_1 \) is a factorization energy. Using these factorizations in the intertwining (4.1) yields the following expression for potential of \( H \) in terms of \( f_1 \):

\[ V = f_1' + f_1^2 + \epsilon_1, \quad (20) \]

where we have used the notation \( ' = \frac{d}{dx} \).

We can see that, in general, \( f_1 \) specifies a realization of a 0-PHA through equations (16), (17) and (20). However, equation (20) is not the only result from and (8). The following equation for \( f_1 \) is also obtained:

\[ f_1' = \lambda. \quad (21) \]

This indeed shows that the general quantum system described by a 0-PHA is given by a potential \( V(x) \) quadratic in \( x \). The harmonic oscillator is the main representative of this family of physical systems. A 0-PHA is indeed the oscillator algebra where the commutation relation between \( L^- \) and \( L^+ \) can be rescaled.
4.2 1-PHA

A first-degree PHA is obtained by considering second-order ladder operators

\[ L^+ = Q_2^+ Q_1^+ = \frac{1}{2} \left( \frac{d}{dx} - f_2 \right) \left( \frac{d}{dx} - f_1 \right), \]
(22)

\[ L^- = Q_1^- Q_2^- = \frac{1}{2} \left( -\frac{d}{dx} - f_1 \right) \left( -\frac{d}{dx} - f_2 \right), \]
(23)

where we have factorized \( L^\pm \) in terms of first-order operators \( Q_i^\pm, i = 1, 2 \).

This decomposition is used to produce a series of intertwinings

\[ H_3 Q_2^+ = Q_2^+ H_2, \quad H_2 Q_1^+ = Q_1^+ H_1, \]
(24)

with auxiliar Hamiltonians \( H_2 \) and \( H_3 \), and \( H_1 = H \).

We also consider the closure relation \( H_3 + \lambda = H_1 \), where \( \lambda \neq 0 \) is taken as a real number again. Thus, we can write the following factorizations of the hamiltonians:

\[ H_1 = Q_1^- Q_1^+ + \epsilon_1, \]
(25)

\[ H_2 = Q_1^+ Q_1^- + \epsilon_1 = Q_2^- Q_2^+ + \epsilon_2, \]
(26)

\[ H_3 = Q_2^+ Q_2^- + \epsilon_2. \]
(27)

By using (25)-(27) in (24), we obtain, once more, that the potential in the hamiltonian \( H \) is given by (20). However, instead of equation (21) we obtain that functions \( f_1 \) and \( f_2 \) satisfy

\[ f_1' + f_2' = f_1^2 - f_2^2 + 2(\epsilon_1 - \epsilon_2), \]

\[ f_1' + f_2' = f_2^2 - f_1^2 + 2(\epsilon_2 - \epsilon_1 + \lambda). \]
(28)

Furthermore, these lead to

\[ (f_1 + f_2)' = \lambda, \]
(29)

\[ f_1^2 - f_2^2 + 2(\epsilon_1 - \epsilon_2) = \lambda, \]
(30)

that, in turn, yield an expression for \( f_2 \) in terms of \( f_1 \):

\[ f_2 = \lambda x + c_0 - f_1, \]
(31)

where \( c_0 \) is an integration constant; and a general expression for \( f_1 \):

\[ f_1 = \lambda x + c_0 + \frac{2(\epsilon_2 - \epsilon_1) + \lambda}{\lambda x + c_0}. \]
(32)
In a similar fashion as for a 0-PHA, \( f_1 \) specifies a realization of a 1-PHA. Then, by explicitly writing the resulting expression for the potential of \( H \), we can see that the general quantum system described by a 1-PHA possesses potentials of the form

\[
V(x) = \left( \frac{2\epsilon_2 - 2\epsilon_1 + \lambda}{c_0 + \lambda x} + c_0 + \lambda x \right)^2 - \frac{\lambda(2\epsilon_2 - 2\epsilon_1 + \lambda)}{(c_0 + \lambda x)^2} + \epsilon_1 + \lambda,
\]

(33)

with the radial oscillator as the main representative of such systems [32].

Evenmore, system (28) admits rational transformations characterized by the extented affine Weyl group of type \( A_1^{(1)} \); namely, \( \tilde{W}(A_1^{(1)}) = \langle s_0, s_1, \pi \rangle \).

These transformations are given by

\[
s_j(f_j) = f_j + \frac{\alpha_j}{f_i + f_{j+1}}, \quad \alpha_j = -\alpha_j,
\]

(34)

\[
s_j(f_{j+1}) = f_{j+1} - \frac{\alpha_j}{f_i + f_{j+1}}, \quad \alpha_j = \alpha_{j\pm 1} + 2\alpha_j,
\]

(35)

\[
\pi(f_j) = f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1},
\]

(36)

where \( i, j = 0, 1 \) and \( \alpha_i = 2(\epsilon_{i+1} - \epsilon_{i+2}) \). Notice that the closure relation previously imposed implies that \( \epsilon_3 + \lambda = \epsilon_1 \).

Recall that \( \tilde{W}(A_{n-1}^{(1)}) \) is the group defined by the generators \( s_0, s_1, \ldots, s_{n-1} \) and \( \pi \), together with the fundamental relations

\[
s_i^2 = 1,
\]

(37)

\[(s_is_j)^2 = 1 \quad (j \neq i, i \pm 1),
\]

(38)

\[(s_is_j)^3 = 1 \quad (j = i \pm 1),
\]

(39)

\[
\pi s_i = s_{i+1}\pi \quad (i = 0, 1, ..., n-1),
\]

(40)

\[
\pi^n = 1.
\]

(41)

4.3 2-PHA

Now let us consider the case of third order ladder operators

\[
L^+ = A_3^+ A_2^+ A_1^+ = \frac{1}{2^{3/2}} \left( \frac{d}{dx} - f_3 \right) \left( \frac{d}{dx} - f_2 \right) \left( \frac{d}{dx} - f_1 \right),
\]

(42)

\[
L^- = A_1^- A_2^- A_3^- = \frac{1}{2^{3/2}} \left( -\frac{d}{dx} - f_1 \right) \left( -\frac{d}{dx} - f_2 \right) \left( -\frac{d}{dx} - f_3 \right),
\]

(43)
in a second degree PHA. Again, we have factorized both ladder operators in terms of first order operators \( Q_i^\pm \), \( i = 1, 2, 3 \), and use them to produce the intertwining relations

\[
H_4 Q_3^+ = Q_3^+ H_3, \quad H_3 Q_2^+ = Q_2^+ H_2, \quad H_2 Q_1^+ = Q_1^+ H_1, \tag{44}
\]

where \( H_2, H_3 \) and \( H_4 \) are auxilar Hamiltonians, and \( H_1 = H \).

This time the closure relation is set to be \( H_4 + \lambda = H_1 \), \( \lambda \in \mathbb{R} \). The resulting factorizations of the hamiltonians are given as

\[
H_1 = Q_1^- Q_1^+ + \epsilon_1, \quad H_2 = Q_1^+ Q_1^- + \epsilon_1 = Q_2^- Q_2^+ + \epsilon_2, \quad H_4 = Q_3^+ \beta_3 + \epsilon_3, \quad H_3 = Q_2^- Q_2^+ + \epsilon_2 = Q_3^- Q_3^+ + \epsilon_3. \tag{45}
\]

Equations (45) and (56) can be combined to obtain equation (20), as well as the following system of equations:

\[
\begin{align*}
    f_1' + f_2' & = f_1^2 - f_2^2 + 2(\epsilon_2 - \epsilon_1), \\
    f_2' + f_3' & = f_2^2 - f_3^2 + 2(\epsilon_3 - \epsilon_1), \\
    f_3' + f_1' & = f_3^2 - f_1^2 + 2(\epsilon_3 - \epsilon_1 + \lambda). \\
\end{align*} \tag{46}
\]

Adding these, we can obtain the equation

\[
(f_1 + f_2 + f_3)' = \lambda. \tag{47}
\]

However, this time, obtaining the equations that functions \( f_i \), \( i = 1, 2, 3 \), satisfy requires some lenghty calculations that can be found in [32, 44]. The result yields

\[
\begin{align*}
    2f_2 & = (\lambda x + c_0 - f_1) - \frac{f_1' + 2(\epsilon_2 - \epsilon_3) - \lambda}{\lambda x + c_0 - f_1}, \tag{48} \\
    2f_3 & = (\lambda x + c_0 - f_1) + \frac{f_1' + 2(\epsilon_2 - \epsilon_3) - \lambda}{\lambda x + c_0 - f_1}, \tag{49}
\end{align*}
\]

that show how to obtain \( f_2 \) and \( f_3 \) from \( f_1 \), while \( f_1 \) satisfies is a second order non-linear differential equation that, upon the sustitution \( f_1 = g(x) - \lambda x + c_0 \), can be shown to be equivalent to the Painlevé IV equation

\[
g'' = \frac{(g')^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2(x^2 - b_0)g + \frac{b_1}{g}, \tag{50}
\]

where \( b_0 \) and \( b_1 \) are constants.
One can see that, similarly to the previous cases, $V, f_2$ and $f_3$ can be obtained from $f_1$, that in turn can be obtained by solving the Painlevé IV equation. On the other hand, this time the general quantum system described by a 2-PHA possesses a potential of the form

$$V(x) = \frac{f_1^2}{2} - \frac{f_1'}{2} + x f_1 + \frac{x^2}{2} + c,$$

where $c$ is a constant.

The connection between system (46) and Painlevé IV equation resides in the fact that the former is precisely the symmetric form of the latter. System (46) possesses a set of Bäcklund transformations given by

$$s_j(f_j) = f_j + \frac{\alpha_j}{f_i + f_{j+1}}, \quad s_j(\alpha_j) = -\alpha_j,$$

$$s_j(f_{j+1}) = f_{j+1} - \frac{\alpha_j}{f_i + f_{j+1}}, \quad s_j(\alpha_{j\pm1}) = \alpha_{j\pm1} + \alpha_j,$$

$$\pi(f_j) = f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1},$$

where $i,j = 0, 1, 2$ and $\alpha_i = 2(\epsilon_{i+1} - \epsilon_{i+2})$, with $\epsilon_4 + \lambda = \epsilon_1$. Notice the change of a coefficient in the transformation $s_j(\alpha_{j\pm1})$ with respect to the previous case. These transformations are characterized by the extended affine Weyl group $\tilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$.

### 4.4 3-PHA

Now that the pattern has been emerging as we reviewed the cases of PHAs of degree 0, 1 and 2, let us just state the results for a third degree PHA. First we consider fourth order ladder operators

$$L^+ = Q_4^+ Q_3^+ Q_2^+ Q_1^+, \quad L^- = Q_1^- Q_2^- Q_3^- Q_4^-,$$

where $Q_i^\pm = \frac{1}{\sqrt{2}} (\pm \frac{d}{dx} - f_i), i = 1, 2, 3, 4$ are used to produce intertwinings

$$H_5 Q_4^+ = Q_4^+ H_4, \quad H_4 Q_3^+ = Q_3^+ H_3, \quad H_3 Q_2^+ = Q_2^+ H_2, \quad H_2 Q_1^+ = Q_1^+ H_1,$$

among the auxiliar hamiltonians $H_i, i = 1, 2, 3, 4$, such that $H_1 = H$ and $H_5 + \lambda = H_1, \lambda \neq 0$. By means of the factorization method equation (20)
holds and we obtain the set of equations

\[
\begin{align*}
\dot{f}_1 + f'_2 & = f_1^2 - f_2^2 + 2(\epsilon_1 - \epsilon_2), \\
\dot{f}_2 + f'_3 & = f_2^2 - f_3^2 + 2(\epsilon_2 - \epsilon_3), \\
\dot{f}_3 + f'_4 & = f_3^2 - f_4^2 + 2(\epsilon_3 - \epsilon_4), \\
\dot{f}_4 + f'_1 & = f_4^2 - f_1^2 + 2(\epsilon_4 - \epsilon_1 + \lambda).
\end{align*}
\]  

(57)

This set of equations can be reduced to the Painlevé V equation [32, 44]

\[
\frac{d^2 w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2}{z^2} \left( c_1 w + \frac{c_2}{w} + c_3 \frac{w}{z} + c_4 \frac{w(w+1)}{w-1} \right),
\]  

(58)

where \( c_i, i = 1, \ldots, 4 \) are constants and a change of variables, \( f_1 = f_1(w) \) and \( x = x(z) \), is used.

Quantum systems described by a 3-PHA can be obtained by means of equation (20), together with particular solutions of the fifth Painlevé equation and \( f_1 = f_1(w), x = x(z) \). Indeed, system (57) is the symmetric form of Painlevé V equation, and its rational Bäcklund transformations are characterized by the extended affine Weyl group \( \tilde{W}(A_3^{(1)}) = \langle s_0, s_1, s_2, s_3, \pi \rangle \). On the other hand, their rational Bäcklund transformations, eqs. (66)-(68), where, in the general case, \( i, j = 0, \ldots, 3 \) and \( \alpha_i = 2(\epsilon_{i+1} - \epsilon_{i+2}) \). Of course, \( \epsilon_5 + \lambda = \epsilon_1 \).

### 4.5 Higher order PHA

In this subsection we present the general case of an \( m \)-PHA, \( m > 1 \). We will build on the cases of low degree PHAs presented in the previous sections. Again, consider the defining relations of an \( m \)-PHA (9), such that \( H = -\frac{1}{2} \frac{d^2}{dx^2} + V \) and

\[
\begin{align*}
L_{m+1}^+ & = \prod_{i=1}^{m+1} Q_{m+2-i}^+, \\
L_{m+1}^- & = \prod_{i=1}^{m+1} Q_i^-. 
\end{align*}
\]  

(59)

(60)

where \( Q_i^\pm = \frac{1}{2x^2} \left( \pm \frac{d}{dx} - f_i \right) \), \( f_i \in \mathbb{R}, i = 1, \ldots, m + 1 \). Operators \( Q_i^\pm \) fulfill the intertwining relations

\[
H_j Q_j^- = Q_j^- H_{j+1}, \quad H_{j+1} Q_j^+ = Q_j^+ H_j,
\]  

(61)
where $H_j$ are auxiliar Hamiltonians and $H_1 = H$. This, in turn, leads to the factorizations

$$H_j = Q_j^- Q_j^+ + \epsilon_j, \quad H_{j+1} = Q_j^+ Q_j^- + \epsilon_j,$$  \hfill (62)

These, together with the closure relation $H_{m+2} + \lambda = H_1$, lead to

$$V(x) = f'_1 + f'_2 + \epsilon_1.$$  \hfill (63)

and the system of equations

$$f'_1 + f'_2 = f'_1 - f'_2 + 2(\epsilon_1 - \epsilon_2),$$

$$\vdots$$

$$f'_m + f'_{m+1} = f'_m - f'_{m+1} + 2(\epsilon_m - \epsilon_{m+1}),$$

$$f'_{m+1} + f'_1 = f'_{m+1} - f'_1 + 2(\epsilon_{m+1} - \epsilon_1 + \lambda).$$  \hfill (64)

One immediate result from this system is the equation

$$(f_1 + f_2 + \ldots + f_m + f_{m+1})' = \lambda.$$  \hfill (65)

Rational Bäcklund transformations of system (64) are characterized by the extended affine Weyl group of type $A_m^{(1)}$, i.e., $\tilde{W}(A_m^{(1)}) = \langle s_0, s_1, \ldots, s_m, \pi \rangle$. These Bäcklund transformations take the explicit form of (66)-(68), that we reproduce here for completeness:

$$s_j(f_j) = f_j + \frac{\alpha_j}{f_i + f_{j+1}}, \quad s_j(\alpha_j) = -\alpha_j,$$  \hfill (66)

$$s_j(f_{j+1}) = f_{j+1} - \frac{\alpha_j}{f_i + f_{j+1}}, \quad s_j(\alpha_{j+1}) = \alpha_{j+1} + \alpha_j,$$  \hfill (67)

$$\pi(f_j) = f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1},$$  \hfill (68)

where $i, j = 0, 1, \ldots, m - 1$ and $\alpha_i = 2(\epsilon_{i+1} - \epsilon_{i+2})$. The closure relation implies that $\epsilon_{m+2} + \lambda = \epsilon_1$.

Results thus far show the explicit way in which extended affine Weyl groups characterize polynomial Heisenberg algebras. Thus, one could talk about the Bäcklund transformations of an $m$-PHA characterized by the extended affine Weyl group of type $A_m^{(1)}$. Concrete realizations of a PHA of a fixed degree can be obtained by performing said transformations.

The fact that for $m = 0, 1$ PHAs do not follow the characterization by extended affine Weyl groups seems to derive from the facts that 1) the cyclic
dressing chains (64) possess different behaviors depending on the parity of \( m \), and 2) for each of these classes \( m = 0,1 \) represent the trivial cases, respectively. Thus, for \( m > 3 \), the system (64) is regarded as higher order generalizations of either the fourth or fifth Painlevé equations, according to \( m \)’s parity.

As mentioned in the introduction, the connection between PHAs and nonlinear differential equations can be used to study solutions of the latter, here presented in symmetric form. Now, by means of the rational transformations, work has been done trying to connect rational solutions of such equations; in particular, for Painlevé IV and V. On the other hand, nonrational hierarchies of solutions have also been obtained, e.g., confluent hypergeometric function hierarchy and error function hierarchy [44]. Results presented here can be used to study connections among solutions in these hierarchies, by using nonrational instead of rational seed solutions, for example.

5 Final remarks

We have seen that polynomial Heisenberg algebras of \( m \)-th order (\( m \)-PHA) are deformations of the oscillator algebra, where commutation relations between the Hamiltonian and the ladder operators are the usual ones as in the harmonic oscillator. However, the commutation relation between the ladder operators is an \( m \)-th degree polynomial of the Hamiltonian. By means of appropriate supersymmetric transformations, one can obtain a symmetric system of equations (64) equivalent to the \( m \)-PHA known as cyclic dressing chains. Upon solving it, we can obtain the most general quantum system described by the polynomial Heisenberg algebra.

Even more, the rational Bäcklund transformations of system (64) are characterized by the extended affine Weyl group of type \( A^{(1)}_m \), i.e., \( \tilde{W}(A^{(1)}_m) = \langle s_0, s_1, \cdots s_m, \pi \rangle \). For \( m = 1 \), system (64) admits rational Bäcklund transformations characterized by the extended affine Weyl group of type \( A^{(1)}_1 \). For \( m = 2 \), (64) is known as the symmetric form of Painlevé IV equation; whose rational Bäcklund transformations and characterized by the extended affine Weyl group of type \( A^{(1)}_2 \). For \( m = 3 \), system (64) is known as the symmetric form of Painlevé V equation; and its rational Bäcklund transformations are characterized by the extended affine Weyl group of type \( A^{(1)}_3 \).

We can see that as the degree of the polynomial Heisenberg algebra increases, solving the system (64) becomes more difficult, e.g., for \( m > 3 \), the
non-linear differential equation equivalent to \[ \text{(64)} \] is of order greater than two. Connections described here may help to better study quantum systems described by PHAs, on one hand, and (non-linear) differential equations and affine Weyl groups, on the other.

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References

[1] Witten E 1981 *Nucl. Phys. B* 185 513
[2] Witten E 1982 *Nucl. Phys. B* 202 253
[3] Mielnik B 1984 *J. Math. Phys.* 25 3387
[4] M.M. Nieto, *Phys. Lett. B* 145 (1984) 208
[5] P.A.M. Dirac, The Principles of Quantum Mechanics, 4th edn, *Clarendon, Oxford* (1957)
[6] V.A. Fock, Fundamentals of Quantum Mechanics, *URSS Publishers* (1931)
[7] Fernández D J 2010 *AIP Conf. Proc.* 1287 3
[8] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* 251 267
[9] Andrianov A A and Ioffe M V 2012 *J. Phys. A: Math. Theor.* 45 503001
[10] Contreras-Astorga A and Fernández D J 2008 *J. Phys. A: Math. Theor.* 41 475303
[11] Quesne C 2011 *Mod. Phys. Lett. A* 26 1843
[12] Gómez-Ullate D, Grandati Y and Milson R 2014 *J. Phys. A: Math. Theor.* 47 015203
[13] Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1995 *Int. J. Mod. Phys. A* **10** 2683

[14] D.J. Fernández and E. Salinas- Hernández, *Phys. Lett. A* **338** (2005) 13

[15] Andrianov A A, Ioffe M V and Spiridonov V P 1993 *Phys. Lett. A* **174** 273

[16] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374

[17] Fernández D J, Glasser M L and Nieto L M 1998 *Phys. Lett. A* **240** 15

[18] Fernández D J, Hussin V and Mielnik B 1998 *Phys. Lett. A* **244** 309

[19] Junker G and Roy P 1998 *Ann. Phys.* **270** 155

[20] Márquez I F, Negro J and Nieto L M 1998 *J. Phys. A: Math. Gen.* **31** 4115

[21] Quesne C and Vansteenkiste N 1999 *Helv. Phys. Acta* **72** 71

[22] Samsonov B F 1999 *Phys.Lett. A* **263** 274

[23] Mielnik B, Nieto L M and Rosas-Ortiz O 2000 *Phys. Lett. A* **269** 70

[24] Aoyama H, Sato M and Tanaka T 2001 *Nucl. Phys. B* **619** 105

[25] Marquette I 2009 *J. Math. Phys.* **50** 095202

[26] Cariñena J F, Ramos A and Fernández D J 2001 *Ann. Phys.* **292** 42

[27] D.J. Fernández and E. Salinas- Hernández, *J. Phys. A: Math. Gen.* **36** (2003) 2537

[28] Mielnik B and Rosas-Ortiz O 2004 *J. Phys. A: Math. Gen.* **37** 10007

[29] Fernández D J and Fernández-García N 2005 *AIP Conference Proceedings* **744** 236

[30] Marquette I 2012 *J. Math. Phys.* **53** 012901

[31] Bermúdez D and Fernández D J 2011 *SIGMA* **7** 025
[32] Carballo J M, Fernández D J, Negro J and Nieto L M 2004 J. Phys. A: Math. Gen. 37 10349
[33] Fernández A J, Negro J and Nieto L M 2004 Phys. Lett. A 324 139
[34] D.J. Fernández, and V. Hussin, J. Phys. A, Math. Gen. 32 (1999) 3603
[35] Mateo J and Negro J 2008 J. Phys. A: Math. Theor. 41 045204
[36] Shabat A 1992 Inverse Problems 8 303
[37] Veselov A P and Shabat A B 1993 Funct. Anal. Appl. 27 81
[38] Adler V E 1994 Physica D 73 335
[39] Dubov S Y, Eleonskii V M and Kulagin N E 1994 Chaos 4 47
[40] Eleonskii V M, Korolev V G and Kulagin N E 1994 Chaos 4 583
[41] Andrianov A A, Ioffe M V and Nishnianidze N D 1995 Phys. Lett. A 201 103
[42] Sukhatme U P, Rasinariu C and Khare A 1997 Phys. Lett. A 234 401
[43] Andrianov A A, Cannata F, Ioffe M and Nishnianidze D 2000 Phys. Lett. A 266 341
[44] Bermúdez D and Fernández 2014 AIP Conference Proceedings 1575 50
[45] Noumi M 2004 Painlevé Equations through Symmetry Translations of Mathematical Monographs AMS 223 Rhode Island, USA
[46] Infeld L 1941, Phys. Rev. 59 737
[47] Infeld L and Hull T E 1951, Rev. Mod. Phys. 23 21
[48] Okamoto K 1986, Math. Ann. 275, 221
[49] Gómez-Ullate D, Grandati Y and Milson R 2021, Adv. Math. 385, 107770
[50] Willox R and Hietarinta J 2003, J. Phys. A 36, 10615
[51] Van Assche W 2018, *Orthogonal Polynomials and Painlevé Equations*, Australian Mathematical Society Lecture Series 27

[52] Fukutani S, Okamoto K and Umemura H 2000, *Nagoya Math. J.* 159 179

[53] García-Ferrero MÁ, Gómez-Ullate D and Milson R 2021, *SIGMA* 17 16

[54] Clarkson P A and Jordaan K 2014, *Constr. Approx.* 39 223

[55] Clarkson P A 2006, *Eur. J. Appl.* 17 293

[56] Clarkson P A 2003, *J. Comput. Appl. Math.* 153 127

[57] Clarkson P A 2003, *J. Math. Phys.* 44 5350

[58] Filipuk G and Clarkson P A 2008, *Stud. Appl. Math.* 121 157

[59] Marquette I and Quesne C 2016, *J. Math. Phys.* 57 052101

[60] Noumi M and Yamada Y 1998, *Funck. Ekvacioj* 41 483

[61] Noumi M and Yamada Y 1999, *Nagoya Math. J.* 153 53