Kinetic axi-symmetric gravitational equilibria in collisionless accretion disc plasmas

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A theoretical treatment is presented of kinetic equilibria in accretion discs around compact objects, for cases where the plasma can be considered as collisionless. The plasma is assumed to be axi-symmetric and to be acted on by gravitational and electromagnetic fields; in this paper, the particular case is considered where the magnetic field admits a family of toroidal magnetic surfaces, which are locally mutually-nested and closed. It is pointed out that there exist asymptotic kinetic equilibria represented by generalized bi-Maxwellian distribution functions and characterized by primarily toroidal differential rotation and temperature anisotropy. It is conjectured that kinetic equilibria of this type can exist which are able to sustain both toroidal and poloidal electric current densities, the latter being produced via finite Larmor-radius effects associated with the temperature anisotropy. This leads to the possibility of existence of a new kinetic effect - referred to here as a “kinetic dynamo effect” - resulting in the self-generation of toroidal magnetic field even by a stationary plasma, without any net radial accretion flow being required. The conditions for these equilibria to occur, their basic theoretical features and their physical properties are all discussed in detail.

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I. INTRODUCTION

A. Astrophysical background

Accretion discs are observed in a wide range of astrophysical contexts, from the small-scale regions around proto-stars or stars in binary systems to the much larger scales associated with the cores of galaxies and Active Galactic Nuclei (AGN). Observations tell us that these systems contain matter accreting onto a central object, losing angular momentum and releasing gravitational binding energy. This can give rise to an extremely powerful source of energy generation, causing the matter to be in the plasma state and allowing the discs to be detected through their radiation emission. A particularly interesting class of accretion discs consists of those occurring around black holes in binary systems, which give rise to compact X-ray sources. For these, one has both a strong gravitational field and also presence of significant magnetic fields which are mainly self-generated by the plasma current densities. Despite the information available about these systems, mainly provided by observations collected over the past forty years and concerning their macroscopic physical and geometrical properties (structure, emission spectrum, etc.), no complete theoretical description of the physical processes involved in the generation and evolution of the magnetic fields is yet available. While it is widely thought that the magneto-rotational instability (MRI) plays a leading role in generating an effective viscosity in these discs, more remains to be done in order to obtain a full understanding of the dynamics of disc plasmas and the relation of this with the accretion process. This requires identifying the microphysical phenomena involved in the generation
of instabilities and/or turbulence which may represent a plausible source for the effective viscosity which, in turn, is then related to the accretion rates [11, 13]. There is a lot of observational evidence which cannot yet be explained or fully understood within the framework of existing theoretical descriptions and many fundamental questions still remain to be answered [4, 5].

B. Motivations for a kinetic theory

Historically, most theoretical and numerical investigations of accretion discs have been made in the context of hydrodynamics (HD) or magneto-hydrodynamics (MHD) (fluid approaches) [1–3, 6–11]. Treating the medium as a fluid allows one to capture the basic large-scale properties of the disc structure and evolution. An interesting development within this context has been the work of Coppi [12] and Coppi and Rousseau [13] who showed that stationary magnetic configurations in AD plasmas, for both low and high magnetic energy densities, can exhibit complex magnetic structures characterized locally by plasma rings with closed nested magnetic surfaces. However, even the most sophisticated fluid models are still not able to give a good explanation for all of the complexity of the phenomena arising in these systems. While fluid descriptions are often useful, it is well known that, at a fundamental level, a correct description of microscopic and macroscopic plasma dynamics should be formulated on the basis of kinetic theory (kinetic approach) [14, 15], and for that there is still, remarkably, no satisfactory theoretical formulation. Going to a kinetic approach can overcome the problem characteristic of fluid theories of uniquely defining consistent closure conditions [14, 15], and a kinetic formulation is necessarily required, instead of MHD, for correctly describing regimes in which the plasma is either collisionless or weakly collisional [16–18]. In these situations, the distribution function describing the AD plasma will be different from a Maxwellian, which is instead characteristic of highly collisional plasmas for which fluid theories properly apply. An interesting example of collisionless plasmas, arising in the context of astrophysical accretion discs around black holes, is that of radiatively inefficient accretion flows [10, 20]. Theoretical investigations of such systems have suggested that the accreting matter consists of a two-temperature plasma, with the proton temperature being much higher than the electron temperature [1, 3]. This in turn implies that the timescale for energy exchange by Coulomb collisions between electrons and ions must be longer than the other characteristic timescales of the system, in particular the inflow time. In this case, a correct physical description of the phenomenology governing these objects can only be provided by a kinetic formulation. Finally, the kinetic formalism is more convenient for the inclusion of some particular physical effects, including ones due to temperature anisotropies, and is essential for making a complete study of the kinetic instabilities which can play a key role in causing the accretion process [16–18]. We note here that, although there are in principle several possible physical processes which may explain the appearance of temperature anisotropies (see for example [17, 18]), the main reason for their maintenance in a collisionless and non-turbulent plasma may simply be the lack of any efficient mechanism for temperature isotropization.

C. Previous work

Only a few studies have so far addressed the problem of deriving a kinetic formulation of steady-state solutions for AD plasmas.

The paper by Bhaskaran and Krishan [21], based on theoretical results obtained by Mahajan [22, 23] for laboratory plasmas, is a first example going in this direction. These authors assumed an equilibrium distribution function expressed as an infinite power series in the ratio of the drift velocity to the thermal speed (considered as the small expansion parameter) such that the zero-order term coincides with a homogeneous Maxwellian distribution. Assuming a prescribed profile for the external magnetic field and ignoring the self-generated field, they looked for analytic solutions of the Vlasov-Maxwell system for the coefficients of the series (typically truncated after the first few orders). However, the assumptions made strongly limited the applicability of their model.

Another approach proposed recently by Cremaschini et al. [15], gives an exact solution for the equilibrium Kinetic Distribution Function (KDF) of strongly magnetized non-relativistic collisionless plasmas with isotropic temperature and purely toroidal flow velocity. The strategy adopted was similar to that developed by Catto et al. [24] for toroidal plasmas, suitably adapted to the context of accretion discs. The stationary KDF was expressed in terms of the first integrals of motion of the system showing, for example, that the standard Maxwellian KDF is an asymptotic stationary solution only in the limit of a strongly magnetized plasma, and that the spatial profiles of the fluid fields are fixed by specific kinetic constraints [15].

In recent years, the kinetic formalism has also been used for investigating stability of AD plasmas, particularly in the collisionless regime and focused on studying the role and importance of MRI [3, 13, 23, 24]. The main goal of these studies [16–18, 20] was to provide and test suitable kinetic closure conditions for asymptotically-reduced fluid equations (referred to as “kinetic MHD”), so as to allow the fluid stability analysis to include some of the relevant kinetic effects for collisionless plasmas [16–18, 20]. However, none of them systematically treated the issue of kinetic equilibrium, and the underlying unperturbed plasma was usually taken to be described by either a Maxwellian or a bi-Maxwellian KDF.
Finally, increasing attention has been paid to the role of temperature anisotropy and the related kinetic instabilities. Some recent numerical studies have tried to include the effects of temperature anisotropy but, although it is clear that this can give rise to an entirely new class of phenomena, all of these estimates rely on fluid models in which kinetic effects are included in only an approximate way. A kinetic approach is needed rather than a fluid one, in order to give a clear and self-consistent picture, and this needs to be based on equilibrium solutions suitable for accretion discs.

D. Open problems

Many problems remain to be addressed and solved regarding the kinetic formulation of AD plasma dynamics. Among them, we focus on the following:

1. The construction of a kinetic theory for AD plasmas within the framework of the Vlasov-Maxwell description, and the investigation of their kinetic equilibrium properties.

2. The inclusion of finite Larmor-radius (FLR) effects in the MHD equations. For magnetized plasmas, this can be achieved by making a kinetic treatment and representing the KDF in terms of gyrokinetic variables (Bernstein and Catto). The gyrokinetic formalism provides a simplified description of the dynamics of charged particles in the presence of magnetic fields, thanks to the symmetry of the Larmor gyratory motion of the particles around the magnetic field lines. Therefore kinetic and gyrokinetic theory are both fundamental tools for treating FLR effects in a consistent way.

3. The determination of suitable kinetic closure conditions to be used in the fluid description of the discs. These should include the kinetic effects of the plasma dynamics in a consistent way.

4. Extension of the known solutions to more general contexts, with the inclusion of important effects such as temperature anisotropy.

5. Development of a kinetic theory for stability analysis of AD plasmas. As already mentioned, this could throw further light on the physical mechanism giving rise to the effective viscosity and the related accretion processes. This is particularly interesting for collisionless plasmas with temperature anisotropy, since only kinetic theory could be able to explain how instabilities can originate and grow to restore the isotropic properties of the plasma.

E. Goals of the paper

The aim of this paper is to extend the investigation of kinetic equilibria developed in an earlier paper to a more general class of solutions. In particular, we pose here the problem of constructing analytic solutions for exact kinetic and gyrokinetic axi-symmetric gravitational equilibria (see definition below) in accretion discs around compact objects. The solution presented is applicable to collisionless magnetized plasmas with temperature anisotropy and mainly toroidal flow velocity. The kinetic treatment of the gravitational equilibria necessarily requires that the KDF is itself a stationary solution of the relevant kinetic equations. Ignoring possible weakly-dissipative effects, we shall assume - in particular - that the KDF and the electromagnetic (EM) fields associated with the plasma obey the system of Vlasov-Maxwell equations. The only restriction on the form of the KDF, besides assuming its strict positivity and it being suitably smooth in the relevant phase-space, is due to the requirement that it must be a function only of the independent first integrals of the motion or the adiabatic invariants for the system.

The paper is organized as follows. In Section 2 we discuss the conditions for the existence of a kinetic equilibrium and we introduce the basic assumptions for the formulation of the kinetic theory. Section 3 deals with the first integrals of motion and the gyrokinetic adiabatic invariants of the system. In Section 4 we construct the equilibrium KDF for AD plasmas with non-isotropic temperature, giving also a useful asymptotic expansion for this in the limit of strong magnetic fields. Section 5 deals with calculation of the fluid moments of the stationary KDF. The limit of isotropic temperature is then investigated in Section 6, while Section 7 is devoted to analyzing the Maxwell equations. Finally, Section 8 contains conclusions and a summary of the main results.

II. KINETIC THEORY FOR ACCRETION DISC PLASMAS: BASIC ASSUMPTIONS

We first discuss what is meant by asymptotic kinetic equilibria in the present study and what are the physical conditions under which they can be realized.

An asymptotic kinetic equilibrium must be one obtained within the context of kinetic theory and must be described by the stationary Vlasov-Maxwell equations. This means that the generic plasma KDF, $f$, must be a solution of the stationary Vlasov equation, as will be the case if $f_{s}$ is expressed in terms of exact first integrals of the motion or adiabatic invariants of the system, which in turn implies that $f_{s}$ for each species must be an exact first integral of the motion or an adiabatic invariant. The stationarity condition means that the equilibrium KDF cannot depend explicitly on time, although in principle it could contain an implicit time dependence via its fluid moments (in which case the kinetic equilibrium
does not correspond to a fluid equilibrium and there are non-stationary fluid fields).

In the following we shall take the AD plasma to be: a) non-relativistic, in the sense that it has non-relativistic species flow velocities, that the gravitational field can be treated within the classical Newtonian theory; b) collisionless, so that the mean free path of the plasma particles is much longer than the largest characteristic scale length of the plasma; c) axi-symmetric, so that the relevant dynamical variables characterizing the plasma (e.g., the fluid fields) are independent of the toroidal angle $\varphi$, when referred to a set of cylindrical coordinates $(R, \varphi, z)$; d) acted on by both gravitational and EM fields.

Also, we will focus on the situation where the equilibrium magnetic field $\mathbf{B}$ admits, at least locally, a family of nested axi-symmetric closed toroidal magnetic surfaces $\{\psi(r)\} \equiv \{\psi(r) = \text{const.}\}$, where $\psi$ denotes the poloidal magnetic flux of $\mathbf{B}$ (see [12, 13] for a proof of the possible existence of such configurations in the context of astrophysical accretion discs; see also [14, 15] for further discussions in this regard and Fig.1 for a schematic view of such a configuration). In this situation, a set of magnetic coordinates $(\psi, \varphi, \vartheta)$ can be defined locally, where $\vartheta$ is a curvilinear angle-like coordinate on the magnetic surfaces $\psi(r) = \text{const.}$ Each relevant physical quantity $A(r)$ can then be expressed as a function of these magnetic coordinates, i.e. $A(r) = A(\psi, \vartheta)$, where the $\varphi$ dependence has been suppressed due to the axi-symmetry. It follows that it is always possible to write the following decomposition: $A = A^\sim + \langle A \rangle$, where the oscillatory part $A^\sim \equiv A - \langle A \rangle$ contains the $\vartheta$-dependencies and $\langle A \rangle$ is the $\psi-$surface average of the function $A(r)$ defined on a flux surface $\psi(r) = \text{const.}$ as $\langle A \rangle = \xi^{-1} \int d\vartheta A(r)/|\mathbf{B} \cdot \nabla \vartheta|$, with $\xi$ denoting $\xi = \int d\vartheta/|\mathbf{B} \cdot \nabla \vartheta|$.

For definiteness, we shall consider here a plasma consisting of at least two species of charged particles: one species of ions and one of electrons.

We also introduce some convenient dimensionless parameters which will be used in constructing asymptotic orderings for the relevant quantities of the theory. The first one, which enters into the construction of the gyrokinetic theory, is defined as $\varepsilon_M \equiv \max \{1, s = i, e\}$, where $r_{i,s} = v_{i,s}/\Omega_{i,s}$ is the species average Larmor radius, with $v_{i,s} = T_{i,s}/M_i s^{1/2}$ denoting the species thermal velocity perpendicular to the magnetic field direction and $\Omega_{i,s} = Z_s e B/M_i c$ denoting the species Larmor frequency. Here $L$ is the characteristic length-scale of the inhomogeneities of the EM field, defined as $L \sim L_B \sim L_E$, where $L_B$ and $L_E$ are the characteristic lengths of the gradients of the absolute values of the magnetic field $\mathbf{B}(r,t)$ and the electric field $\mathbf{E}(r,t)$, defined as

$$
\frac{1}{L_B} \equiv \max \left\{ \frac{\partial}{\partial r_i} \ln B^i, i = 1, 3 \right\}
$$

and

$$
\frac{1}{L_E} \equiv \max \left\{ \frac{\partial}{\partial r_i} \ln E^i, i = 1, 3 \right\}.
$$

For typical temperatures and magnetic fields in AD plasmas, $0 < \varepsilon_M \ll 1$.

The second parameter is the inverse aspect ratio defined as $\delta \equiv \frac{r_{\text{max}}}{R_0}$, where $R_0$ is the radial distance from the vertical axis to the center of the nested magnetic surfaces and $r_{\text{max}}$ is the average cross-sectional poloidal radius of the largest closed toroidal magnetic surface; see Fig.1 for a schematic view of the configuration geometry and the meaning of the notation introduced here. Then, we impose the requirement $0 < \delta \ll 1$, which is referred to as "small inverse aspect ratio ordering". The main motivation for introducing this ordering is that we are discussing only local solutions where this asymptotic condition holds; this property also follows from the results presented in [12, 13], and has already been used in other previous work on the subject [14, 15]. The requirement $\delta \ll 1$ is also needed in order to satisfy the constraint condition imposed by Ampere's law, as discussed in Sec. VII. We stress that the $\delta-$ordering here introduced is consistent with the assumption of nested and closed magnetic surfaces that are assumed to be localized in space.

Finally we introduce a parameter $\delta_{T,s}$ which measures the magnitude of the species temperature anisotropy and is defined as $\delta_{T,s} \equiv T_{i,s}/T_s$, where $T_{//,s}$ and $T_{\perp,s}$ denote the parallel and perpendicular temperatures, as measured with respect to the magnetic field direction.

Note that, in the following, we will use a prime " $'$ " to denote a dynamical variable defined at the guiding-center position.

### A. The treatment of EM and gravitational fields

In the following, we shall assume that the magnetic field is of the form

$$
\mathbf{B} \equiv \nabla \times \mathbf{A} = \mathbf{B}^{\text{self}}(r,t) + \mathbf{B}^{\text{ext}}(r,t),
$$

where $\mathbf{B}^{\text{self}}$ and $\mathbf{B}^{\text{ext}}$ denote the self-generated magnetic field produced by the AD plasma and a non-vanishing external magnetic field produced by the central object. We also impose the following relative ordering between the

![FIG. 1: Schematic view of the configuration geometry.](image-url)
two components of the total magnetic field: \[ \frac{|\mathbf{B}^\text{ext}|}{|\mathbf{B}\text{eff}|} \sim O(\varepsilon^k_M), \]
with \( k \geq 1 \). This means that the self-field is the dominant component: the magnetic field is primarily self-generated. Also, the overall magnetic field is assumed to be slowly varying in time, i.e., to be of the form \( \mathbf{B}(r, \varepsilon_M t) \), while \( \mathbf{B}^\text{eff} \) and \( \mathbf{B}^\text{ext} \) are defined as

\[ \mathbf{B}^\text{ext} = \nabla \psi_D(r, \varepsilon_M t) \times \nabla \varphi, \]

\[ \mathbf{B}^\text{eff} = \nabla \psi_p(r, \varepsilon_M t) \times \nabla \varphi, \]

where \( \mathbf{B}_T = \nabla \psi(r, \varepsilon_M t) \nabla \varphi \) and \( \mathbf{B}_P = \nabla \psi_p(r, \varepsilon_M t) \times \nabla \varphi \) are the toroidal and poloidal components of the self-field, with \( (\psi_p, \varphi, \theta) \) defining locally a set of magnetic coordinates. Moreover, the external magnetic field \( \mathbf{B}^\text{ext} \) is assumed to be purely poloidal and defined in terms of the vacuum potential \( \psi_D(r, \varepsilon_M t) \). In particular, we notice here that for typical astrophysical applications of interest, the function \( \psi_D(r, \varepsilon_M t) \) can be conveniently identified with the flux function of a dipolar magnetic field. It follows that the magnetic field can also be written in the form

\[ \mathbf{B} = \nabla \psi + \nabla \psi_p \times \nabla \varphi. \]

where the function \( \psi(r, \varepsilon_M t) \) is defined as \( \psi(r, \varepsilon_M t) \equiv \psi_p(r, \varepsilon_M t) + \psi_D(r, \varepsilon_M t) \), and \( (\psi, \varphi, \theta) \) define a set of local magnetic coordinates, as implied by the equation \( \mathbf{B} \cdot \nabla \psi = 0 \) which is identically satisfied. In addition, it is assumed that the charged particles of the plasma are subject to the action of effective EM potentials \( \{ \Phi^\text{eff}(r, \varepsilon_M t), \mathbf{A}(r, \varepsilon_M t) \} \), where \( \mathbf{A}(r, \varepsilon_M t) \) is the vector potential corresponding to the magnetic field of Eq. (4), while \( \Phi^\text{eff}(r, \varepsilon_M t) \) is given by

\[ \Phi^\text{eff}(r, \varepsilon_M t) = \Phi(r, \varepsilon_M t) + \frac{M_s}{Z e c} \Phi_G(r, \varepsilon_M t), \]

with \( \Phi^\text{eff}(r, \varepsilon_M t), \Phi(r, \varepsilon_M t) \) and \( \Phi_G(r, \varepsilon_M t) \) denoting the effective electrostatic potential and the electrostatic and generalized gravitational potentials (the latter, in principle, being produced both by the central object and the accretion disc). Finally, both the equilibrium effective electric field \( \mathbf{E}^\text{eff} \), generated by the combined action of the effective EM potentials and defined as

\[ \mathbf{E}^\text{eff} = -\nabla \Phi^\text{eff}, \]

and the magnetic field \( \mathbf{B} \) are also assumed to be axi-symmetric.

**III. FIRST INTEGRALS OF MOTION AND GUIDING-CENTER ADIABATIC INVARIANTS**

In the present formulation, assuming axi-symmetry and stationary EM and gravitational fields, the exact first integrals of motion can be immediately recovered from the symmetry properties of the single charged particle Lagrangian function \( \mathcal{L} \). In particular, these are the total particle energy

\[ E_s = \frac{M_s}{2} \dot{v}^2 + Z e \Phi^\text{eff}_s(r), \]

and the canonical momentum \( p_{cs} \) (conjugate to the ignorable toroidal angle \( \varphi \))

\[ p_{cs} = M_s \mathbf{v} \cdot \mathbf{e}_\varphi + \frac{Z e c}{e} \psi = \frac{Z e c}{e} \psi_s. \]

Gyrokinetic theory allows one to derive the adiabatic invariants of the system \( [28, 29] \), by construction, these are quantities conserved only in an asymptotic sense, i.e., only to a prescribed order of accuracy. As is well known, gyrokinetic theory is a basic prerequisite for the investigation both of kinetic instabilities (see for example \( [32, 34] \)) and of equilibrium flows occurring in magnetized plasmas \( [24, 35–38] \). For astrophysical plasmas close to compact objects, this generally involves the treatment of strong gravitational fields which needs to be based on a covariant formulation (see \( [39–42] \)). However, for non-relativistic plasmas (in the sense already discussed), the appropriate formulation can also be directly recovered via a suitable reformulation of the standard (non-relativistic) theory for magnetically confined laboratory plasmas \( [24, 31, 43–49] \). In connection with this, consider again the Lagrangian function \( \mathcal{L} \) of charged particle dynamics. By performing a gyrokinetic transformation of \( \mathcal{L} \), accurate to the prescribed order in \( \varepsilon_M \), it follows that - by construction - the transformed Lagrangian \( \mathcal{L}' \) becomes independent of the guiding-center gyrophase angle \( \phi' \). Therefore, by construction, the canonical momentum \( p'_{\theta} = \partial \mathcal{L}' / \partial \phi' \), as well as the related magnetic moment \( m'_{\theta} \), are adiabatic invariants. As shown by Kruskal (1962 \[ 50 \]) it is always possible to determine \( \mathcal{L}' \) so that \( m'_{\theta} \) is an adiabatic invariant of arbitrary order in \( \varepsilon_M \), in the sense that \( \frac{\Delta m'_{\theta}}{m'_{\theta}} = 0 + O(\varepsilon_M^{n+1}) \), where \( \Omega_{cs} = Z e c B / M_s c \) denotes the Larmor frequency evaluated at the guiding-center and the integer \( n \) depends on the approximation used in the perturbation theory to evaluate \( m'_{\theta} \). In addition, the guiding-center invariants corresponding to \( E_s \) and \( \psi_s \) (denoted as \( E'_s \) and \( \psi'_s \) respectively) can also be given in terms of \( \mathcal{L}' \). These are also, by definition, manifestly independent of \( \phi' \).

This basic property of the magnetic moment \( m'_{\theta} \) is essential in the subsequent developments. Indeed, we shall prove that it allows the effects of temperature anisotropy to be included in the asymptotic stationary solution.

Let us now define the concept of gyrokinetic and equilibrium KDFs.

**Def. - Gyrokinetic KDF (GK KDF)**

A generic KDF \( f_s(r, v, t) \) will be referred to as gyrokinetic if its Lagrangian time-derivative \( \frac{\partial}{\partial t} f_s(r, v, t) \) is independent of the gyrophase angle \( \phi' \) evaluated the guiding-center position when its state \( x = (r, v) \) is expressed as a function of an arbitrary gyrokinetic state \( z = (\varphi', \phi') \). More generally, in the following \( f_s(r, v, t) \)
will be referred to as an asymptotic-GK KDF if, neglecting corrections of order \(O(\varepsilon_M^{n+1})\), \(\frac{d}{dt} f_s(r,v,t)\) is independent of \(\psi'\).

**Def. - Equilibrium KDF**

A generic KDF \(f_s(r,v,t)\) will be referred to as an equilibrium KDF if it identically satisfies the Vlasov equation \(\frac{d}{dt} f_s(r,v,t) = 0\) and if \(f_s\) is also independent of time, namely \(f_s = f_s(r,v)\). More generally, \(f_s(r,v,t)\) will be referred to as an asymptotic-equilibrium KDF if, neglecting corrections of order \(O(\varepsilon_M^{n+1})\), \(\frac{d}{dt} f_s(r,v,t) = 0\) and to the same order \(f_s\) is independent of \(t\).

Let us first provide an example of a GK equilibrium KDF. This can be obtained by assuming that \(f_s\) depends only on the exact invariants, namely that it is of the form 
\[
f_s = f_s(E_s,\psi_{ss}),
\]
The presence of a temperature anisotropy means that any GK equilibrium KDF depending on the guiding-center magnetic moment \(m'\). In particular, the form of the stationary KDF which we are going to introduce is characterized by the following properties: 1) it is analytically tractable; 2) it affords an explicit determination of the relevant kinetic constraints to be imposed on the fluid fields (see the discussion after Eq. (12)); 3) it represents a possible kinetic model which is consistent with fluid descriptions of collisionless plasmas characterized by temperature anisotropy; 4) it is suitable for comparisons with previous literature, in which astrophysical plasmas have been treated by means of a Maxwellian or a bi-Maxwellian KDF (see for example [15, 18, 26]). Then, following [15, 24], a convenient solution is given by
\[
\tilde{f}_{ss} = \frac{\tilde{\beta}_{ss}}{(2\pi/M_s)^{3/2} (T_{||ss})^{1/2}} \exp \left\{ \frac{H_{ss}}{T_{||ss}} - m'_s \tilde{\alpha}_{ss} \right\}
\]

(*Generalized bi-Maxwellian KDF*), where
\[
\begin{align*}
\tilde{\beta}_{ss} &\equiv \frac{\eta_s}{T_{\perp s}}, \\
\tilde{\alpha}_{ss} &\equiv \frac{B'}{\Delta T_s}, \\
H_{ss} &\equiv E_s - \frac{Z_s e c}{c} \psi_{ss} \Omega_{ss},
\end{align*}
\]
while \(E_s\) is given by Eq. (7), \(\psi_{ss}\) is given by Eq. (8) and \(\Delta T_s \equiv \frac{1}{T_{\perp s}} - \frac{1}{T_{||s}}\). In order for the solution (10) to be a function of the integrals of motion and the adiabatic invariants, the functions \(\tilde{\beta}_{ss}, \tilde{\alpha}_{ss}, T_{||ss}\) and \(\Omega_{ss}\) must depend on the constants of motion by themselves. In general this would require a functional dependence on both the total particle energy and the canonical momentum. However, in the following, for simplicity, we shall consider the case in which only a dependence on \(\psi_{ss}\) is retained [15, 24]. Namely, \(\tilde{f}_{ss}\) depends, by assumption, on the flux functions \(\{\tilde{\beta}_{ss}, T_{||ss}, \tilde{\alpha}_{ss}, \Omega_{ss}\}\):
\[
\begin{align*}
\tilde{\beta}_{ss} &= \tilde{\beta}_{ss}(\psi_{ss}), \\
T_{||ss} &= T_{||ss}(\psi_{ss}), \\
\tilde{\alpha}_{ss} &= \tilde{\alpha}_{ss}(\psi_{ss}), \\
\Omega_{ss} &= \Omega_{ss}(\psi_{ss}),
\end{align*}
\]
which in the following will be referred to as kinetic constraints. From these considerations it is clear that the KDF $f_{ss}$ is itself an adiabatic invariant, and is therefore an asymptotic solution of the stationary Vlasov equation, whose order of accuracy is uniquely determined by the magnetic moment, as already anticipated.

From definition (13), it follows immediately that an equivalent representation for $f_{ss}$ is given by:

$$
\hat{f}_{ss} = \frac{\beta^2 \exp \left[ \frac{X_{ss}}{T_{ss}} \right]}{(2\pi/M_s)^{3/2} (T_{ss})^{1/2}} \times \exp \left\{ -\frac{M_s \left( \mathbf{v} - \mathbf{V}_{ss} \right)^2}{2T_{ss}} - m_s^2 \alpha_{ss} \right\},
$$

(18)

where $\mathbf{V}_{ss} = e_s R \Omega_{ss} (\psi_{ss})$ and

$$
X_{ss} \equiv \left( \frac{M_s}{2} \frac{\left| \mathbf{V}_{ss} \right|^2}{c^2} + \frac{Z_s e}{c} \psi_{ss} - Z_s e \Phi^s_{eff} \right).
$$

(19)

The same kinetic constraints (14)-(17) also apply to the solution (18). Note that the functions $\beta_{ss} \exp \left[ \frac{X_{ss}}{T_{ss}} \right]$, $\mathbf{V}_{ss}$ and $T_{ss}$ cannot be regarded as fluid fields, since they have a dependence on the particle velocity via the canonical momentum $\psi_{ss}$. On the other hand, fluid fields must be computed as integral moments of the distribution function over the particle velocity $\mathbf{v}$.

Next we show that a convenient asymptotic expansion for the adiabatic invariant $\hat{f}_{ss}$ can be properly obtained in the following suitable limit. Consider, in fact, the quantity $\varepsilon$ defined as $\varepsilon \equiv \max \{ \varepsilon_{s,s}, s = i, e \}$, with $\varepsilon_s \equiv \left| \frac{L_{ss}}{r_{Ls}} \right| \equiv \left| \frac{M_s \Omega_{ss}}{2 \Delta \nu_{ve}} \right|$, where we have used the definition (13) with $v_{ss} \equiv \mathbf{v} \cdot \mathbf{e}_s$, and where $L_{ss}$ denotes the species particle angular momentum. We can give an average upper limit estimate for the magnitude of $\varepsilon_s$ in terms of the species thermal velocity and the inverse aspect ratio previously defined. To do this, we first set $\psi \sim B_{ss} r^2$, which is appropriate for the domain of closed nested magnetic surfaces. Recall that here $r$ is the average poloidal radius of a generic nested magnetic surface. In this evaluation, the species thermal velocities $v_{ths}$ and the toroidal flow velocities $R \Omega_{ss}$ are considered to be of the same order with respect to the $\varepsilon$-expansion, i.e. $v_{ths}/R \Omega_{ss} \sim O (\varepsilon^0)$ (referred to as sonic flow). Therefore, assuming $v_{ss} \sim v_{ths}$ it follows immediately that $\varepsilon_s \sim \frac{r_{Ls}}{r_\delta}$, where $r_{Ls}$ is the species Larmor radius and $L_{ss} \equiv r_\delta$, with $\delta$ the inverse aspect ratio. We shall say that the AD plasma is strongly magnetized whenever $0 < \varepsilon \ll 1$. This condition is realized if $r \geq r_{\min}$, where $r_{\min} = \min \left\{ \varepsilon_s, s = i, e \right\}$ is the minimum average poloidal radius of the toroidal nested magnetic surfaces for which $\varepsilon \ll 1$ is satisfied. In this case $\varepsilon$ can be taken as a small parameter for making a Taylor expansion of the KDF and its related quantities, by setting $\psi_{ss} \equiv \psi + O (\varepsilon^k)$, $k \geq 1$. From the above discussion, it is clear that this asymptotic expansion is valid for $r$ within an interval $r_{\min} \leq r \leq r_{\max}$, where the lower bound is fixed by the condition of having a strongly magnetized plasma, while the upper bound is given by the geometric properties of the system and the small inverse aspect ratio ordering. For the purpose of this paper, in performing the asymptotic expansion we retain the leading-order expression for the guiding-center magnetic moment $m_s^2 \equiv \mu_s^2 = \frac{M_s w^2}{2 \mu^2 B_s}$. Then, it is straightforward to prove that for strongly magnetized plasmas, the following relation holds to first order in $\varepsilon$ (i.e., retaining only linear terms in the expansion): $f_{ss} = f_s [1 + h_{Ds}] + O (\varepsilon^n)$, $n \geq 2$. Here, the zero order distribution $f_s$ is expressed as

$$
f_s = \frac{n_s}{(2\pi/M_s)^{3/2} (T_{ss})^{1/2} T_{\perp s}} \times \exp \left\{ -\frac{M_s \left( \mathbf{v} - \mathbf{V}_{ss} \right)^2}{2T_{ss}} - \frac{M_s w^2}{2 \Delta T_s} \right\},
$$

(20)

which we will here call the bi-Maxwellian KDF, where $\frac{1}{\Delta T_s} \equiv \frac{1}{T_{ss}} - \frac{1}{T_{\perp s}}$, the number density $n_s = \eta_s \exp \left[ \frac{X_{ss}}{T_{ss}} \right]$ and

$$
X_s \equiv \left( \frac{M_s R^2 \Omega_s^2}{2} + \frac{Z_s e}{c} \psi_s - Z_s e \Phi^s_{eff} \right)
$$

(21)

with $\eta_s$ denoting the pseudo-density. Then, $\mathbf{V}_s = e_s R \Omega_s (\psi_s)$ and the following kinetic constraints are implied from (14)-(17): $\beta_s = \beta_s (\psi) = \frac{\psi_s}{\psi_A}$, $T_{||s} = T_{\perp s} (\psi)$, $\alpha_s = \alpha_s (\psi) = \frac{\psi_s}{\psi_A}$, $\Omega_s = \Omega_s (\psi)$. As can be seen, the functional form of the leading order number density, the flow velocity and the temperatures carried by the bi-Maxwellian KDF is naturally determined. In particular, note that the flow velocity is species-dependent, while the related angular frequency $\Omega_s$ must necessarily be constant on each nested toroidal magnetic surface $\{ \psi (r) = const. \}$. Finally, the quantity $h_{Ds}$ represents the diamagnetic part of the KDF $f_{ss}$, given by

$$
h_{Ds} = \left\{ cM_s R \frac{Y_1 + M_s R}{T_{\perp s} Y_2} \right\} (\mathbf{v} \cdot \mathbf{e}_s),
$$

(22)

with $Y_1 \equiv \left[ A_{1s} + A_{2s} \left( \frac{\mu_s}{\mu_A} - \frac{1}{2} \right) - \mu_s^2 A_{4s} \right]$, $H_s \equiv E_s - \frac{2 \Delta \nu}{\Delta \nu_{ve}} \psi_s \Omega_s (\psi_s) + Y_s = \Omega_s (\psi) [1 + \psi A_{3s}]$, where we have introduced the following definitions: $A_{1s} \equiv \frac{\partial \ln T_{\perp s}}{\partial \ln \nu}$, $A_{2s} \equiv \frac{\partial \ln T_{\perp s}}{\partial \ln \nu}$, $A_{3s} \equiv \frac{\partial \ln \Omega_s (\psi)}{\partial \ln \nu}$, $A_{4s} \equiv \frac{\partial \ln \psi_s}{\partial \ln \nu}$. We remark here that: 1) in the $\varepsilon$-expansion of (10), performed around (20), no magnetic or electric field scale lengths enter, as can be seen from Eq. (22), 2) we also implicitly assume the validity of the ordering $\frac{\Delta \nu}{\Delta \nu_{ve}} \ll 1$, which will be discussed below (see next section). For this reason, corrections of $O (\varepsilon^k)$, with $k \geq 1$, to (22) have been neglected; 3) in this $\varepsilon$–expansion we have also assumed that the scale-length $L$ is of the same order (with respect
to $\varepsilon$) as the characteristic scale-lengths associated with the species pseudo-densities $\eta_s$, the temperatures $T_{\perp,s}$ and $T_{\parallel,s}$, and the toroidal rotational frequencies $\Omega_s$.

To conclude this section we point out that the very existence of the present asymptotic kinetic equilibrium solution and the realizability of the kinetic constraints implied by it, must be checked for consistency also with the constraints imposed by the Maxwell equations, as discussed in Sec. VII.

V. MOMENTS OF THE KDF

It is well known that, given a distribution function, it is always possible to compute the fluid moments associated with it, which are defined through integrals of the distribution over the velocity space. Although an exact calculation of the fluid moments could be carried out (e.g., numerically) for prescribed kinetic closures, in this section we want to take advantage of the asymptotic expansion of the KDF in the limit of strongly magnetized plasmas to evaluate them analytically, thanks to the properties of the bi-Maxwellian KDF. In the following, we provide approximate expressions for the number density and the flow velocity, which allow one to write the Poisson and Ampere equations for the EM fields in a closed form, and for the non-isotropic species pressure tensor. Since these fluid fields are then known (in terms of suitable kinetic flux functions and with a prescribed accuracy), the closure problem characteristic of the fluid theories is then naturally solved as well.

The main feature of this calculation is that the number density and flow velocity are computed by performing a transformation of all of the guiding-center quantities appearing in the asymptotic equilibrium KDF to the actual particle position, to leading order in $\varepsilon_M$ (according to the order of accuracy of the adiabatic invariant used), and they are then determined up to first order in $\varepsilon$, in agreement with the order of expansion previously set for the KDF. Terms of higher order, i.e. $O(\varepsilon^M_M)$, with $n \geq 1$, and $O(\varepsilon^k)$, with $k \geq 2$, as well as mixed terms of order $O(\varepsilon M_M^n)$ with $n \geq 1$, are therefore neglected in the present calculation. This approximation clearly holds if $\frac{\varepsilon_M}{\varepsilon} \ll 1$, which is consistent with the present assumptions. In fact, from the definitions given for these two small dimensionless parameters it follows that

$$\frac{\varepsilon_M}{\varepsilon} \sim O(\delta) \ll 1.$$  \hspace{1cm} (23)

To first order in $\varepsilon$, the total number density $n^{\text{tot}}_s$ is given by $n^{\text{tot}}_s \equiv \int d\mathbf{v} \tilde{f}_{ss} \simeq n_s [1 + \Delta_n]$. Note here that the resulting number density has two distinct contributions: $n_s$ is the zero order term given in the previous section, while $\Delta_n$ represents the term of $O(\varepsilon)$ which carries all of the corrections due to the asymptotic expansion of the KDF for strongly magnetized plasmas. The full expression for $\Delta_n$ is given in Appendix A. Finally, a similar integral can be performed to compute the total flow velocity $\mathbf{V}^{\text{tot}}_s$. This has the form

$$n^{\text{tot}}_s \mathbf{V}^{\text{tot}}_s \equiv \int d\mathbf{v} \mathbf{v} \tilde{f}_{ss} \simeq n_s [\mathbf{V} + \Delta \mathbf{U}_s],$$

where by definition $\mathbf{V} = \Omega_s(\psi) \mathbf{R}_s$ and $\Delta \mathbf{U}_s$ represents the self-consistent FLR velocity corrections given by:

$$\Delta \mathbf{U}_s \equiv \Delta \mathbf{\varphi}_s \mathbf{e}_\varphi + \frac{\Delta \mathbf{\varphi}_s}{B} \mathbf{\nabla} \psi \times \mathbf{\nabla} \varphi,$$  \hspace{1cm} (24)

where $\Delta \mathbf{\varphi}_s \equiv \Delta_n \mathbf{\Omega}_s \mathbf{R} + \Delta_{2s} + \Delta_{3s} \frac{\mathbf{M}}{\mathbf{R} \mathbf{B}}$. Note that in Eq. (24) the terms proportional to $\Delta_{n_s}$, $\Delta_{2s}$ and $\Delta_{3s}$ come from the asymptotic expansion of the KDF for strongly magnetized plasmas and are of $O(\varepsilon)$ with respect to the toroidal velocity $\mathbf{\Omega}_s \mathbf{R}$. The full expressions for $\Delta_{n_s}$, $\Delta_{2s}$ and $\Delta_{3s}$ are given in Appendix A. The first-order term $\Delta \mathbf{U}_s$ provides corrections to the zero-order toroidal flow velocity with components in all of the three space directions and so we can conclude that, although the dominant fluid velocity is mainly toroidal, there is also a poloidal component of order $\varepsilon$, associated with the term $\frac{\Delta \mathbf{\varphi}_s}{B} \mathbf{\nabla} \psi \times \mathbf{\nabla} \varphi$. However, this is not necessarily an accretion velocity, especially under the hypothesis of closed nested magnetic surfaces which define a local domain in which the disc plasma is confined. Moreover, note that the ratio between the toroidal and poloidal velocities depends also on $\partial \tau_s$, in the sense that $\frac{|\Delta \mathbf{\varphi}_s \mathbf{e}_\varphi|}{|\mathbf{V}|} \sim O(\varepsilon) O(\partial \tau_s)$. The magnitude of the temperature anisotropy can therefore be relevant in further decreasing the poloidal velocity in comparison with the toroidal one, which on the contrary is not affected by $\partial \tau_s$. However, the real importance of this result in connection with the astrophysics of collisionless AD plasmas is, instead, the fact that this poloidal velocity is a primary source for a poloidal current density which in turn can generate a finite toroidal magnetic field (see the section on the Maxwell equations). This means that, even without any net accretion of disc material (which would require at least a redistribution of the angular momentum), the kinetic equilibrium solution provides a mechanism for the generation of a toroidal magnetic field, with serious implications for the stability analysis of these equilibria. The physical mechanism responsible for this poloidal drift is purely kinetic and is essentially due to the conservation of the canonical toroidal momentum and the FLR effects associated with the temperature anisotropy. As a last point, consider the species non-isotropic pressure tensor, which is defined by the following moments of the KDF: $\Pi = \int d\mathbf{v} M_s (\mathbf{v} - \mathbf{V}^{\text{tot}}_s)(\mathbf{v} - \mathbf{V}^{\text{tot}}_s)^{\perp} \tilde{f}_{ss}$. Then, the overall pressure tensor of the system is obtained by summing the single species pressure tensors: $\Pi = \sum_{s=1}^n \Pi_s$. A direct analytical calculation retaining only the zero-order terms (with respect to all of the small dimensionless parameters) in the Taylor expansion of the KDF $\tilde{f}_{ss}$ (whose leading-order expression coincides with the bi-Maxwellian KDF), shows that the corresponding species tensor pressure is non-isotropic to leading order and in this approximation is given by:

$$\Pi_s = p_{\perp,s} \mathbf{I} + (p_{\parallel,s} - p_{\perp,s}) \mathbf{b}_s.$$  \hspace{1cm} (25)
where \( p_{\perp s} = n_s T_{\perp s} \) and \( p_{|| s} = n_s T_{|| s} \) represent the leading-order perpendicular and parallel pressures. The divergence of the species pressure tensor is of particular interest; this is given by:

\[
\nabla \cdot \Pi_s = \nabla p_{\perp s} + b B \cdot \nabla \left( \frac{p_{|| s} - p_{\perp s}}{B} \right) - \Delta p_s Q,
\]

where \( Q \equiv [b b \cdot \nabla \ln B + \frac{4 \pi}{3} b \times J - \nabla \ln B] \) and \( \Delta p_s \equiv (p_{|| s} - p_{\perp s}) \).

VI. THE CASE OF ISOTROPIC TEMPERATURE

In this section, we consider the case of isotropic temperature for the equilibrium distribution \( \bar{f}_s \). When the condition \( T_{|| s} = T_{\perp s} = T_s \) is satisfied, the stationary KDF reduces to \( f_s \), where

\[
f_s = \frac{n_s}{\pi^{3/2} (2 T_s / M_s)^{3/2}} \exp \left\{ -\frac{H_s}{T_s} \right\}
\]

is referred to as the Generalized Maxwellian Distribution with isotropic temperature \([13]\). Here, \( H_s \) retains its definition \([13]\), while the kinetic constraints are expressed for the quantities \( n_s \) and \( T_s \), whose functional dependence is \( \eta_s = \eta_s(\psi_s) \) and \( T_s = T_s(\psi_s) \). By construction, this distribution function is expressed only in terms of the first integrals of motion of the system and is therefore an exact kinetic equilibrium solution. Performing an asymptotic expansion in the limit of strong magnetic field, as done before for \( \bar{f}_s \), gives the following result:

\[
f_s = f_{Ms} [1 + h_{Ds}] + O(\varepsilon^n), \quad n \geq 2,
\]

where

\[
f_{Ms} = \frac{n_s}{\pi^{3/2} (2 T_s / M_s)^{3/2}} \exp \left\{ \frac{M_s (v - V_s)^2}{2 T_s} \right\}
\]

is the zero-order term of the series, which coincides with a drifted Maxwellian KDF with \( T_s = T_s(\psi) \), \( V_s = \Omega_s(\psi) R \Omega x \) and \( n_s = n_s(\psi) \exp \left[ \frac{X_s}{T_s} \right] \). In this case, the function \( h_{Ds} \) is given by

\[
h_{Ds} = \left\{ c M_s R_s Z_s e Y_1 + \frac{M_s R_s}{T_s} Y_2 \right\} (\mathbf{v} \cdot \mathbf{e}_x),
\]

with

\[
Y_1 = \left( A_{1s} + A_{2s} \left( \frac{H_s}{T_s} - \frac{2}{9} \right) \right) \quad \text{and} \quad Y_2 = \Omega_s(\psi) [1 + \psi A_{3s}],
\]

where \( A_{1s} = \frac{\partial \ln n_s}{\partial \psi}, \quad A_{2s} = \frac{\partial \ln T_s}{\partial \psi}, \quad A_{3s} = \frac{\partial \ln \Omega_s(\psi)}{\partial \psi} \). Finally, as shown in \([13]\), the angular frequency is given to leading order by \( \Omega_s(\psi) = \frac{\partial \psi}{\partial \psi} \), where \( \chi \equiv c d \psi / f + \frac{\partial \psi}{\partial \psi} \ln n_s \).

Before concluding this section, we stress again that the case with isotropic temperature represents a result whose accuracy is not limited by dependence on any gyrokinetic invariant and that it does not require any guiding-center variable transformation. For this reason, there are no restrictions of applicability of the solution \([27]\) which, in principle, holds also in the limit of vanishing magnetic field.

VII. THE MAXWELL EQUATIONS AND THE “KINETIC DYNAMO”

In this section we write the Poisson and Ampere equations for the EM fields explicitly, pointing out the consequences of the kinetic treatment developed in Sections IV-VI. In particular we prove that, besides a self-generated poloidal magnetic field, the kinetic equilibrium can sustain also a toroidal field (which may be thought of as being a kinetic dynamo), thanks to the combined effects of FLR corrections and temperature anisotropies. For definiteness, let us consider the Poisson equation for the electrostatic potential \( \Phi \), expressed as

\[
\nabla^2 \Phi = -4 \pi \sum_{s=i,e} q_s n_s [1 + \Delta_{n_s}],
\]

where \( \Delta_{n_s} \) is written out explicitly in Appendix A. In the limit of a strongly magnetized plasma and considering the accuracy of the previous asymptotic analytical expansions, we shall say that the plasma is quasi-neutral if the ordering \( \frac{\Delta_{n_s}}{\varepsilon} = 0 + O(\varepsilon^k) \), with \( k \geq 2 \) holds, whereas we call it weakly non-neutral if \( \frac{\Delta_{n_s}}{\varepsilon} = 0 + O(\varepsilon) \). The kinetic equilibrium for a weakly non-neutral plasma is referred to as a Hall kinetic equilibrium \([15]\), and the corresponding fluid configuration is referred to as a Hall Gravitational MHD (Hall-GMHD) fluid equilibrium \([14]\).

Next, we show that quasi-neutrality (in the sense just defined) can be locally satisfied by imposing a suitable constraint on the electrostatic (ES) potential \( \Phi \). It can be shown that this constraint can always be satisfied since the leading-order contribution to the ES potential remains unaffected. The result follows by neglecting higher-order corrections to the number density \( \Delta_{n_s} \) and setting \( q_i = Z e \) and \( q_e = -e \). Thanks to the arbitrariness in the choice of the flux functions introduced by the kinetic constraints (see Sec. IV), it is possible to show that quasi-neutrality implies the following constraint for the oscillatory part \( \Phi^\omega \) of the ES potential, i.e., correct to both \( O(\varepsilon^n) \) and \( O(\varepsilon^m) \):

\[
\Phi^\omega(\psi, \vartheta) = \Phi - \langle \Phi \rangle = \frac{S^\omega}{c (\frac{Z}{T^\vartheta} + \frac{T^\vartheta}{Z})},
\]

where \( S^\omega \equiv \ln \left( \frac{Z}{T^\vartheta} \right) + \left[ \frac{T^\vartheta}{Z} - \frac{Z}{T^\vartheta} \right] \), and \( \chi_s \equiv \left( M_s \frac{\partial \eta_s}{\partial \psi} + \frac{Z e \phi \Omega_s(\psi)}{M_s \Phi_G} \right) \). In particular, the arbitrariness in the coefficient \( \frac{\partial \eta_s}{\partial \psi} \) can be used to satisfy the constraint \( \langle S^\omega \rangle = 0 \). In fact, in view of Eq. \([14]\), it follows that

\[
\eta_s = \frac{\beta_s(\psi) T_{|| s}(\psi)}{1 + \alpha_s(\psi) T_{|| s}(\psi)},
\]

where \( \alpha_s(\psi) \) is related to \( \hat{\alpha}_s(\psi) \) as outlined in Appendix A and the flux functions still remain arbitrary. In conclusion, Eq. \([31]\) determines only \( \Phi^\omega \) and not the total ES...
potential. Note that this solution for the electrostatic potential \( \Phi^* \) can be shown to be consistent with earlier treatments appropriate for Tokamak plasma equilibria\textsuperscript{[24,31]} This can be exactly recovered thanks to the arbitrariness in defining the pseudo-densities and by taking the limit of isotropic temperatures and zero gravitational potential, as in the case of laboratory plasmas. In this limit the species pseudo-densities become flux functions\textsuperscript{[24]}. Then, because of this arbitrariness, by taking

\[
\frac{\eta e}{Z \eta_i} = 1, \tag{33}
\]

it follows that Eq.(31) reduces to the form presented in\textsuperscript{[24]}, which can only be used to determine the poloidal variation of the potential.

Let us now consider the Ampere equation. Adopting the Taylor analytical expansion of the asymptotic equilibrium KDF and neglecting corrections of \( O(\varepsilon^k) \), with \( k \geq 2 \), and \( O(\varepsilon^0 M^1) \), with \( n \geq 0 \), this can be approximately written as follows:

\[
\nabla \times B^{s elf} = \frac{4\pi}{c} \sum_{s=i,e} q_s n_s [V_s + \Delta U_s], \tag{34}
\]

where \( B^{s elf} \) is as defined in Eq.(4) and the expression for \( \Delta U_s \) is given by Eq.(23). The toroidal component of this equation gives the generalized Grad-Shafranov equation for the poloidal flux function \( \psi_p \):

\[
\Delta^* \psi_p = -\frac{4\pi}{c} R \sum_{s=i,e} q_s n_s [\Omega_s (\psi) R + \Delta \varphi s], \tag{35}
\]

where the elliptic operator \( \Delta^* \) is defined as \( \Delta^* \equiv R^2 \nabla \cdot (R^{-2} \nabla) \) \textsuperscript{[51]}. The remaining terms in Eq.(34) give the equation for the toroidal component of the magnetic field \( l(\psi,\varnothing) \). In the same approximation, this is:

\[
\nabla I(\psi,\varnothing) \times \nabla \varphi = \frac{4\pi}{c} \sum_{s=i,e} q_s n_s \frac{\Delta \varphi s}{B} \nabla \psi \times \nabla \varphi, \tag{36}
\]

where \( \Delta \varphi s \), given in Appendix A, contains the contributions of the species temperature anisotropies. For consistency with the approximation introduced, in the small inverse aspect ratio ordering, it follows that \( \frac{\partial I(\psi,\varnothing)}{\partial \varnothing} = 0 + O(\delta^k) \), i.e., to leading order in \( \delta \) : \( I = I(\psi) + O(\delta^k) \), with \( k \geq 1 \). This in turn also requires that the corresponding current density in Eq.(36) is necessarily a flux function. Then, correct to \( O(\varepsilon), O(\varepsilon^0 M) \) and \( O(\delta^0) \), the differential equation for \( I(\psi) \) becomes:

\[
\frac{\partial I(\psi)}{\partial \psi} = \frac{4\pi}{c} \sum_{s=i,e} q_s n_s \frac{\Delta \varphi s}{B}, \tag{37}
\]

which uniquely determines an approximate solution for the toroidal magnetic field. This result is remarkable because it shows that there is a stationary “kinetic dynamo effect” which generates an equilibrium toroidal magnetic field without requiring any net accretion and in the absence of any possible instability/turbulence phenomena. This new mechanism results from poloidal currents arising due to the FLR effects and temperature anisotropies which are characteristic of the equilibrium KDF. We remark that the self-generation of the stationary magnetic field is purely diamagnetic. In particular, the toroidal component is associated with the drifts of the plasma away from the flux surfaces. In the present formulation, possible dissipative phenomena leading to a non-stationary self field have been ignored. Such dissipative phenomena probably do arise in practice and could occur both in the local domain where the equilibrium magnetic surfaces are closed and nested and elsewhere. Temperature anisotropies are therefore an important physical property of collisionless AD plasmas, giving a possible mechanism for producing a stationary toroidal magnetic field. We stress that this effect disappears altogether in the case of isotropic temperatures, as demonstrated in Appendix A. Finally, we consider the ratio between the toroidal and poloidal current densities (\( J_T \) and \( J_P \)). In the small inverse aspect ratio ordering, neglecting corrections of order \( O(\delta^k) \) with \( k \geq 1 \), this provides an estimate of the magnitude of the corresponding components of the magnetic field. In fact, in this limit we can write \( \frac{\nabla \times B^T}{| \nabla |} \sim \frac{\| B_T \|}{\| B_P \|} \sim \frac{J_T}{J_P} \) and so conclude that, although for the single species velocity, the ordering \( \frac{\Delta \psi}{\nabla \times \nabla \psi} \sim O(\varepsilon) O(\delta_T \varepsilon) \) holds (see Sec. VI), this might no longer be the case for the magnetic field, which instead depends on the ratio between the total toroidal and poloidal current densities. In particular, the possibility of having finite stationary toroidal magnetic fields is, in principle, allowed by the present analysis, depending on the properties of the overall solution describing the system.

VIII. CONCLUSIONS

Getting a complete understanding of the dynamical properties of astrophysical accretion discs still represents a challenging task and there are many open problems remaining to be solved before one can get a full and consistent theoretical formulation for the physical processes involved.

The present investigation provides some important new results for understanding the equilibrium properties of accretion discs, obtained within the framework of a kinetic approach based on the Vlasov-Maxwell description. The derivation presented applies for collisionless non-relativistic and axisymmetric AD plasmas under the influence of both gravitational and EM fields. A wide range of astrophysical scenarios can be investigated with the present theory, thanks to the possibility of properly setting the different parameters which characterize the physical and geometrical properties of the
model. A possible astrophysical context is provided, for example, by radiatively inefficient accretion flows onto black holes, where the accreting material is thought to consist of a plasma of collisionless ions and electrons with different temperatures, in which the dominant magnetic field is generated by the plasma current density. We have considered here the specific case in which the structure of the magnetic field is locally characterized by a family of closed nested magnetic surfaces within which the plasma has mainly toroidal flow velocity. For this, we have proved that a kinetic equilibrium exists and can be described by a stationary KDF expressed in terms of the exact integrals of motion and the magnetic moment prescribed by the gyrokinetic theory, which is an adiabatic invariant. Many interesting new results have been pointed out; the most relevant ones for astrophysical applications are the following: 1) the possibility of including the effects of a non-isotropic temperature in the stationary KDF; 2) the proof that the Maxwellian and bi-Maxwellian KDFs are asymptotic stationary solutions, i.e., they can be regarded as approximate equilibrium solutions in the limit of strongly magnetized plasmas; 3) the possibility of computing the stationary fluid moments to the desired order of accuracy in terms of suitably prescribed flux functions; 4) the proof that a toroidal magnetic field can be generated in a stationary configuration even in the absence of any net accretion flow if and only if the plasma has a temperature anisotropy.

This last point, in particular, is of great interest because it gives a mechanism for generating a stationary toroidal field in the disc, independent of instabilities related to the accretion flow. The consistent kinetic formulation developed here permits the self-generation of such a field by the plasma itself, associated with localized poloidal drift-currents on the nested magnetic surfaces as a consequence of temperature anisotropies. This stationary poloidal motion is made possible in the framework of the FLR effects.

Appendix A: Calculation of the fluid moments

In this appendix we give the detailed expressions for the coefficients which determine the fluid moments of the KDF, obtained by integrating the KDF over the velocity space. For the number density, we have found that $n_s^{\text{tot}} = n_s [1 + \Delta_{n_s}]$. The term $\Delta_{n_s}$ is given by

$$\Delta_{n_s} = V_s \left[ \gamma_1 + \gamma_3 \left( \frac{T_{||s}}{M_s^2} - \frac{4T_{\perp s}}{M_s} + V_s^2 \right) \right] + \left( A1 \right)$$

$$+ \frac{2\gamma_3^2}{B^2} \left( T_{||s} - T_{\perp s} \right) V_s - \frac{\gamma_2}{B} V_s T_{\perp s},$$

where $V_s = R\Omega_s(\psi)$ and

$$\gamma_1 \equiv \left\{ \frac{cM_s R}{Z_s e} K + \frac{M_s V_s}{T_{||s}} \right\} [1 + \psi A_3],$$

$$\gamma_2 \equiv \left\{ \frac{cM_s R}{Z_s e} A_{1s} \right\},$$

$$\gamma_3 \equiv \left\{ \frac{cM_s^2 R}{Z_s e} A_{2s} \right\} 2T_{||s},$$

in which $K = \left[ A_{1s} + A_{2s} \left( \frac{Z_s e \omega_{ce} - \frac{2\pi e^2 \psi}{T_{||s}}}{} \right) 1/2 \right]$ and $A_{4s} \equiv \frac{\partial \psi}{\partial \phi}$, with $\alpha_s(\psi) \equiv \frac{B}{A_{4s}}$. Note that here $\alpha_s(\psi)$ differs from $\hat{\alpha}_s(\psi)$ because of the guiding-center transformation of the magnetic field $B$.

Proceeding in the same way for evaluating the second moment of the KDF, it can be shown that the first order correction $D_U_s$ to the toroidal flow velocity can be written as $D_U_s \equiv \Delta_{\phi s} \Phi \varphi + \Delta_{\psi s} \Phi \psi \times \nabla \varphi$, where we recall that

$$\Delta_{\phi s} \equiv \Delta_{n_s} \Omega_s R + \Delta_{2s} + \Delta_{3s} \frac{I}{RB},$$

Here $\Delta_{n_s}$ is as given in Eq. [A1], while $\Delta_{2s}$ and $\Delta_{3s}$ are given by

$$\Delta_{2s} \equiv \frac{T_{||s}}{M_s} \left( \gamma_1 + 3\gamma_3 V_s^2 \right) - \gamma_2 \frac{2T_{\perp s}}{BM_s} + \left( A6 \right)$$

$$+ \frac{\gamma_3}{M_s^2} \left( T_{||s} + 4T_{\perp s} \right),$$

$$\Delta_{3s} \equiv \frac{I}{RB^2 M_s} \left( 2T_{\perp s} - T_{||s} \right) + \left( A7 \right)$$

$$+ \frac{I}{RB M_s} \left( \gamma_1 + 3\gamma_3 V_s^2 \right) +$$

$$+ \frac{I}{RB^2 M_s} \left( 3T_{||s} - 4T_{\perp s}^2 + T_{||s} T_{\perp s} \right).$$

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Finally, in the limit of isotropic temperatures, the solution for the number density is

\[ \Delta_{n_s} \equiv V_s \left[ \gamma_1 + \gamma_3 \left( \frac{5T_s}{M_s} + V_s^2 \right) \right], \quad (A8) \]

where now \( \gamma_2 = 0 \),

\[ \gamma_1 \equiv \left\{ \frac{cMR}{Z_s e} K + \frac{M_s V_s}{T_s} \left( 1 + \psi \Omega_3 s \right) \right\}, \quad (A9) \]

\[ \gamma_3 \equiv \left\{ \frac{cM_s^2 R A_{2s}}{Z_s e} \right\}, \quad (A10) \]

and \( K \equiv \left[ A_{1s} + A_{2s} \left( \frac{Z_s e \Phi_{11} s}{T_s} - \frac{2s \psi \Omega_3 (\psi)}{T_s} - \frac{1}{2} \right) \right] \). For calculating the flow velocity from Eq.(A5), in the same limit, \( \Delta_{n_s} \) is as given by Eq.(A8), \( \Delta_{2s} \) reduces to

\[ \Delta_{2s} \equiv \frac{T_s}{M_s} \left( \gamma_1 + 3 \gamma_3 V_s^2 \right) + \frac{5 \gamma_3 T_s^2}{M_s^2} \quad (A11) \]

and \( \Delta_{3s} \equiv 0 \) since, in Eq.(A7), the second and third terms on the right hand side necessarily vanish, while the first one is proportional to \( A_{1s} \equiv \partial n_s / \partial \psi \), where \( \alpha_s (\psi) \equiv \frac{\partial}{\partial \psi} \Omega_s \equiv 0 \), and hence vanishes too.

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