A NOTE ON APPLICATIONS OF THE $d$-INARIANT AND DONALDSON’S THEOREM

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ABSTRACT. This note contains two remarks about the application of the $d$-invariant in Heegaard Floer homology and Donaldson’s diagonalization theorem to knot theory. The first is the equivalence of two obstructions they give to a 2-bridge knot being smoothly slice. The second carries out a suggestion by Stefan Friedl to replace the use of Heegaard Floer homology by Donaldson’s theorem in the proof of the main result of [Gre13] concerning Conway mutation of alternating links.

Dedicated to the memory of Tim Cochran

1. Introduction.

Donaldson’s diagonalization theorem and Heegaard Floer homology have led to great success in knot theory. In this note, we focus on two specific applications of these tools to knot concordance and mutation that appear in the literature. We show how in both cases they can be used interchangeably towards the same end. The moral is that for the applications considered herein, the $d$-invariant in Heegaard Floer homology simply repackages the information already carried by Donaldson’s theorem. This is not unexpected in light of the close relationship between them (see [OSz03, Section 9]).

We first briefly recall both tools at work. Donaldson’s theorem asserts that if $Z$ is a closed, oriented, smooth 4-manifold whose intersection pairing $Q_Z$ is definite, then $H_2(Z;\mathbb{Z})/\text{Tors}$ admits an orthonormal basis with respect to $Q_Z$ [Don87, Theorem 1]. The $d$-invariant is a highly useful invariant defined by Ozsváth and Szabó in Heegaard Floer homology. It is modeled on the $h$-invariant defined by Frøyshov in Seiberg Witten Floer homology. It assigns a rational number $d(Y,t)$ to a closed, oriented 3-manifold $Y$ equipped with a torsion spin$^c$ structure $t$.

To set the stage for the main results, we recall how both of these tools can be used in order to prove that the pretzel knot $P(-3, 5, 7)$ is not smoothly slice. By contrast, this knot is topologically slice, since it has trivial Alexander polynomial [FQ90]. The existence of topologically slice knots that are not smoothly slice was a sensational early application of Donaldson’s and Freedman’s work. According to the paper of Cochran and Gompf, the line of argument using Donaldson’s theorem is due to Casson [CG88, §1]. The line of argument using Heegaard Floer homology is standard by now. As we shall see, the two proofs are slight variations of one another.
The starting point for both proofs is the following observation. If a knot $K$ bounds a disk $D$ smoothly and properly embedded in $D^4$, then the double cover of $D^4$ branched along $D$ is a smooth, compact 4-manifold $\Sigma(D)$ with the $\mathbb{Z}/2\mathbb{Z}$ (and hence rational) homology groups of a ball, and its boundary $\Sigma(K)$ is the double cover of $S^3$ branched along $K$ \cite[Lemma 2]{CG86}. For the case $K = P(-3, 5, 7)$, the manifold $\Sigma(K)$ is the Brieskorn sphere $\Sigma(3, 5, 7)$.

This space bounds a smooth, compact 4-manifold $X$ obtained by plumbing disk-bundles over spheres and for which $(H^2(X; \mathbb{Z}), Q_X)$ is isometric to the unimodular, definite lattice $D^+_{12}$ (see \cite[Section 3.2]{OS03}).

Now, if $K = P(-3, 5, 7)$ were smoothly slice, then $\Sigma(K)$ would also bound a smooth rational homology ball $W$. The union $Z = X \cup (-W)$ would then be a smooth, closed 4-manifold with $(H^2(Z; \mathbb{Z})/\text{Tors}, Q_Z)$ isometric to $D^+_{12}$, which does not admit an orthonormal basis, in violation of Donaldson’s theorem. Therefore, $K$ is not smoothly slice. Alternatively, following \cite[Section 3.2]{OS03}, a calculation with $D^+_{12}$ shows that $d(Y, t) = -2$, where $t$ denotes the unique spin$^c$ structure on $Y$. Since $t$ is the unique spin$^c$ structure on $\Sigma(K)$, it extends over any smooth, compact 4-manifold filling $\Sigma(K)$. On the other hand, if $t$ extends to a spin$^c$ structure on a rational homology ball $W$ that fills $Y$, then $d(Y, t) = 0$ \cite[Proposition 9.9]{OSz03}. It follows once more that no rational homology ball fills $\Sigma(K)$, so $K$ is not smoothly slice.

The two proofs that $P(-3, 5, 7)$ is not smoothly slice are both based on the existence of the 4-manifold $X$, properties of the $D^+_{12}$ lattice, and a suitably sensitive tool in smooth 4-manifold topology. In fact, the result and both proofs generalize to any knot $K$ for which $\Sigma(K)$ bounds a 4-manifold with a positive definite intersection pairing not isometric to the Euclidean lattice $\mathbb{Z}^n$. The proof using Donaldson’s theorem generalizes directly. The proof using Heegaard Floer homology does as well, as the $d$-invariant of such a manifold is negative by \cite[Theorem 9.6]{OSz03} and a theorem of Elkies \cite{Elk95}.

In Section 2 we show that, in much the same way, two obstructions in the literature to a two-bridge knot being smoothly slice, one from Donaldson’s theorem and one from the $d$-invariant, are equivalent. In Section 3 we highlight a novel instance in which Donaldson’s theorem can be used in place of Heegaard Floer homology. This possibility was pointed out by Stefan Friedl. The main result of \cite{Gre13} asserts that if $L$ and $L'$ are alternating links, then $\Sigma(L) \approx \Sigma(L')$ if and only if $L$ and $L'$ are mutants. Moreover, the $d$-invariant of $\Sigma(L)$ is a complete invariant of the mutation type of $L$. The argument hinges on an expression for the $d$-invariant of $\Sigma(L)$ due to Ozsváth and Szabó \cite[Theorem 3.4]{OSz05}. The expression is defined in terms of a lattice $\Lambda(D)$ associated with an alternating link diagram $D$ of $L$. It follows from the invariance of the $d$-invariant that if $D'$ is an alternating link diagram of $L'$ and $\Sigma(L) \approx \Sigma(L')$, then the formulas for the $d$-invariant derived from $\Lambda(D)$ and $\Lambda(D')$ are the same. Friedl’s suggestion was to show that these formulas are the same by an appeal to Donaldson’s theorem instead of Heegaard Floer homology. We carry out the details of this suggestion in Section 3.

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2. Sliceness of two-bridge knots.

Let $Y$ denote an oriented rational homology 3-sphere, and suppose that $Y$ bounds an oriented rational homology 4-ball $W$. As remarked above, Ozsváth and Szabó showed that the invariant $d(Y,t)$ vanishes for any spin$^c$ structure $t$ on $Y$ that extends across $W$. A lot of work has gone into using this fact as an obstruction: given a rational homology sphere $Y$, one attempts to argue that it is not the boundary of any rational homology ball. For instance, if all of the correction terms $d(Y,t)$ are non-zero, then one concludes that $Y$ does not bound a rational ball, as we did above in the case of $\Sigma(3,5,7)$. Variations on this theme are carried out in [GJ11], [GRS08], [JN07], [Lec12], [Lis07a, Lis07b], and [OS12].

Most applications of this idea require somewhat more: one seeks more a priori conditions on which spin$^c$ structures on $Y$ could extend over a putative rational ball, and to combine these conditions with the vanishing of the correction terms. Casson and Gordon observed that the image of the restriction mapping $r_H : H^2(W;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is a subgroup $\text{im}(r_H)$ whose order is the square-root of $|H^2(Y;\mathbb{Z})|$, implying the latter value is a perfect square $m^2$ [CG86 Lemma 3]. This observation holds at the level of Spin$^c$ structures: the image of the restriction mapping $r_S : \text{Spin}^c(W) \to \text{Spin}^c(Y)$ forms a torsor over the subgroup $\text{im}(r_H)$. Furthermore, there is a conjugation action on Spin$^c(W)$ and Spin$^c(Y)$ which commutes with the restriction map. Thus, in order for $Y$ to bound a rational ball, the $d$-invariant vanish on a conjugation-invariant subtorsor of Spin$^c(Y)$ over a subgroup of order $m$.

In the application to knot concordance, we assume that $Y = \Sigma(K)$ for some knot $K \subset S^3$. This has the added feature that $Y$ is a $\mathbb{Z}/2\mathbb{Z}$ homology sphere, and if $K$ is smoothly slice, then $Y$ is filled by the smooth $\mathbb{Z}/2\mathbb{Z}$ homology ball $W = \Sigma(D)$, where $D$ denotes the slice disk. There exists a first Chern class mapping $c_1 : \text{Spin}^c(\cdot) \to H^2(\cdot;\mathbb{Z})$ for both $Y$ and $W$. It is a torsor isomorphism since $H^2(\cdot;\mathbb{Z})$ has no 2-torsion, it commutes with the restriction map, and $c_1(\bar{s}) = -c_1(s)$ for conjugate spin$^c$ structures $s, \bar{s} \in \text{Spin}^c(W)$. If $H^2(Y;\mathbb{Z})$ is a cyclic group of odd order $m^2$, then it follows that $\text{im}(c_1 \circ r_S)$ is the unique cyclic subgroup of order $m$. This identifies $T := c_1^{-1}(m \cdot H^2(Y;\mathbb{Z})) \subset \text{Spin}^c(Y)$ with $\text{im}(r_S)$. Thus, in order to argue that $K$ is not smoothly slice, it suffices to show that the $d$-invariant of $Y$ does not vanish on $T$.

We shall use the following lattice-theoretic description of $T$. Express $Y$ by integral surgery along a framed link $L \subset S^3 = \partial D^4$. Attaching 2-handles to $D^4$ along $L$ produces a 4-manifold $X$ with $\partial X = Y$, $H_1(X;\mathbb{Z}) = 0$, and whose intersection pairing $\Lambda = (H_2(X;\mathbb{Z}), Q_X)$ is presented by the linking matrix of $L$. Now glue $X$ and the putative $\mathbb{Z}/2\mathbb{Z}$ homology ball $-W$ by an orientation-reversing diffeomorphism of their boundaries to produce a closed 4-manifold $Z$. By Poincaré duality, the intersection pairing lattice $\Lambda' = (H_2(Z;\mathbb{Z}), Q_Z)$ is unimodular and integral, and the inclusion $X \hookrightarrow Z$ induces an inclusion $\Lambda \hookrightarrow \Lambda'$. Since $H_1(X;\mathbb{Z}) = 0$, every spin$^c$ structure on $Y$ extends across $X$, so a spin$^c$ structure on $Y$ extends across $W$ if and only if it extends across $Z$. Since $H^1(X;\mathbb{Z}/2\mathbb{Z})$ and $H^1(Z;\mathbb{Z}/2\mathbb{Z})$ vanish, the first Chern class mapping $c_1$ establishes one-to-one correspondences $\text{Spin}^c(X) \leftrightarrow \text{Char}(\Lambda)$ and
Proposition 2.1. Conditions (1) and (2) on a lens space are equivalent.

Proof. (1) $\Rightarrow$ (2): As discussed, $T$ can be identified with $\text{Char}(\Lambda') \mod 2\Lambda$, where $\Lambda'$ is the unique unimodular lattice with $\Lambda_i \subset \Lambda_i' \subset \Lambda_i''$. By (1), $d$ vanishes on this subset, so the $d$-invariant of $\Lambda_i'$ is 0. By Elkies’s theorem \cite{Elk95}, it follows that $\Lambda_i' \approx \mathbb{Z}^{r_i}$, so condition (2) holds.

(1) $\Rightarrow$ (2): Since the $d$-invariant of $\mathbb{Z}^{r_i}$ is zero, it follows from (2) that $d(L(p,q),t)$ and $d(L(p,p-q),t)$ are non-negative for all $t \in T$. On the other hand, $d(L(p,q),t) = -d(L(p,p-q),t)$, so both values vanish, and (1) holds. \hfill \Box

Observe that the statement and proof of Proposition 2.1 extends to any space $Y$ for which $H^2(Y;\mathbb{Z})$ is a cyclic group with order an odd perfect square and both $Y$ and $-Y$ bound positive definite 4-manifolds with vanishing $H_1$. A similar but somewhat more complicated conclusion may be drawn without the assumptions on $H^2(Y;\mathbb{Z})$. 
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3. Mutation of alternating links.

Suppose that $D$ is a connected, reduced, alternating diagram of an alternating link $L \subset S^3$. Color the regions of $D$ in chessboard fashion according to the convention shown in Figure 1. Let $G(D)$ denote the Tait graph whose vertices correspond to black regions of $D$ and whose edges correspond to crossings where a pair of regions touch. Denote by $\Lambda(D)$ the lattice of flows on $G(D)$. This is a positive definite, integral lattice.

The $d$-invariant of a positive definite, integral lattice was defined in [Gre13, Section 2.4] (see also [Gre12, Section 2]). The definition is intended to mimic the formula of Ozsváth and Szabó in [OSz05, Theorem 3.4]. According to it, the $d$-invariant of $-\Sigma(L)$ is isomorphic to the $d$-invariant of the lattice $\Lambda(D)$ (the notion of isomorphism of $d$-invariants is codified in [Gre13, Definition 2.2]). As a consequence, if $D$ and $D'$ are connected, reduced alternating diagrams of a pair of links $L, L'$ for which $\Sigma(L) \approx \Sigma(L')$, then the $d$-invariants of $\Lambda(D)$ and $\Lambda(D')$ are isomorphic. In other words, [OSz05, Theorem 3.4] implies the following result.

**Proposition 3.1.** The isomorphism type of the $d$-invariant of $\Lambda(D)$ is an invariant of the oriented homeomorphism type of $\Sigma(L)$.

This is the sole input from Heegaard Floer homology used in the proof of the main result of [Gre13] (specifically, see the use of [Gre13, Theorem 4.7] in the proof of [Gre13, Theorem 1.1]). Following Friedl’s suggestion, we will derive Proposition 3.1 as an application of Donaldson’s theorem, without reference to Heegaard Floer homology.

Before proving Proposition 3.1 we require some more preparation. Let $F \subset S^3$ denote the spanning surface for $L$ corresponding to the black regions. Note that $F$ deformation retracts onto $G(D)$. Let $D^4$ denote a 4-ball that fills $S^3$ and push int($F$) into int($D^4$) to obtain a properly embedded surface $F'$ that fills $L$. Let $X(D)$ denote the double-cover of $D^4$ branched along $F'$. Its boundary is $\Sigma(L)$, and it follows from the work of Gordon and Litherland that $(H_2(X(D)); \mathbb{Z})$, equipped with its intersection pairing, is isometric to $\Lambda(D)$ [GL78, Theorems 1 & 3]. The additional notation in the following proof comes from [Gre13] (cf. [Gre12]).

**Proof.** We must show that if $D$ and $D'$ are connected, reduced, alternating diagrams of a pair of links $L$ and $L'$, and $\Sigma(L) \approx \Sigma(L')$ as oriented manifolds, then $(X(\Lambda(D)), d) \approx (X(\Lambda(D')), d)$.

Let $D'$ denote the mirror of the diagram $D$ and $L$ the mirror of the link $L$. We have $\partial X(D) \approx \Sigma(L) \approx -\Sigma(L') \approx -\partial X(D')$. Fix an orientation-reversing homeomorphism $\phi : \partial X(D) \to \partial X(D')$. Following the discussion in Section 2 we can identify $X(\Lambda(D))$ and...
\[ \mathcal{X}(\Lambda(D')) \text{ with } \text{Spin}^c(\Sigma(L)) \text{ (note that } H^2(X(D); \mathbb{Z}) \text{ does not contain 2-torsion). The map } \phi \text{ then descends to a torsor isomorphism} \]
\[ \varphi : \mathcal{X}(\Lambda(D)) \to \mathcal{X}(\Lambda(D')). \]

We seek to show that \( \varphi \) establishes an isomorphism between the \(-d\)-invariant of \( \Lambda(\overline{D}) \) and the \(d\)-invariant of \( \Lambda(D') \): that is, \( d(\varphi(x)) = -d(x) \) for all \( x \in \mathcal{X}(\Lambda(\overline{D})) \).

Form the closed, oriented, smooth, definite 4-manifold \( Z = X(\overline{D}) \cup_\phi X(D') \). By Donaldson’s theorem, \( (H_2(Z; \mathbb{Z}), Q_Z) \) is isometric to the Euclidean lattice \( \mathbb{Z}^n \), where \( n = \text{rk}(\Lambda(\overline{D})) + \text{rk}(\Lambda(D')) \). Given a class \( x \in \mathcal{X}(\Lambda(\overline{D})) \), select \( \chi \in \text{Short}(\Lambda(\overline{D})) \) with \( [\chi] = x \) and \( \chi' \in \text{Short}(\Lambda(D')) \) with \( [\chi'] = \varphi(x) \). Then \( \chi + \chi' \in \text{Char}(H_2(Z)) = \text{Char}(\mathbb{Z}^n) \), so \( |\chi + \chi'| \geq n \). It follows that
\[
 d(x) + d(\varphi(x)) = \frac{1}{4}(|\chi| - \text{rk}(\Lambda(\overline{D}))) + \frac{1}{4}(|\chi'| - \text{rk}(\Lambda(D'))) \geq 0, \quad \forall x \in \mathcal{X}(\Lambda(\overline{D})).
\]

Similarly, consideration of the pair \( (D, \overline{D}) \) yields a torsor isomorphism
\[ \psi : \mathcal{X}(\Lambda(D)) \to \mathcal{X}(\Lambda(\overline{D})) \]
with the property that
\[ d(y) + d(\psi(y)) \geq 0, \quad \forall y \in \mathcal{X}(\Lambda(D)). \]

Adding the two collections of inequalities above over all \( x \) and \( y \) shows that the sum of the total \(d\)-invariants of \( \Lambda(D), \Lambda(D'), \Lambda(\overline{D}), \) and \( \Lambda(\overline{D'}) \) is non-negative.

On the other hand, since \( G(D) \) and \( G(\overline{D}) \) are planar dual to one another, there exists an isomorphism
\[ (\mathcal{X}(\Lambda(D)), d) \approx (\mathcal{X}(\Lambda(\overline{D})), -d) \]
by \cite{Gre13} Corollary 3.4. Similarly, there exists an isomorphism between \( (\mathcal{X}(\Lambda(D')), d) \) and \( (\mathcal{X}(\Lambda(\overline{D}')), -d) \). Thus, the sum of the total \(d\)-invariants of the four lattices vanishes, and it follows that \( d(x) + d(\varphi(x)) = 0, \forall x \in \mathcal{X}(\Lambda(D)). \) Therefore, \( \varphi \) establishes an isomorphism
\[ (\mathcal{X}(\Lambda(\overline{D})), -d) \approx (\mathcal{X}(\Lambda(D')), d). \]

Combining the last two indented expressions, it follows that the \(d\)-invariants of \( \Lambda(D) \) and \( \Lambda(D') \) are isomorphic, as desired. \( \square \)

It is intriguing to consider the role of Donaldson’s theorem in the proof of Proposition 3.1. Is it possible to supply a more direct topological argument that \( (H_2(Z, \mathbb{Z}), Q_Z) \approx \mathbb{Z}^n \) for the specific type of 4-manifold \( Z \) appearing in it?

Lastly, we point out another line of argument for establishing \cite[Theorem 3.4]{OSz03}. It is sufficient to show that the 4-manifold \( X(D) \) is sharp. By the combinatorial argument \cite[Corollary 3.4]{Gre13}, the \(d\)-invariants of \( \Lambda(D) \) and \( \Lambda(\overline{D}) \) are negative one another. The first gives a lower bound on the \(d\)-invariant of \( \Sigma(L) \), while the second gives a lower bound on the \(d\)-invariant of \( \Sigma(\overline{L}) \), both by \cite[Theorem 9.6]{OSz03}. The \(d\)-invariants of these manifolds are negative one another by orientation-reversal. It follows that both lower bounds are sharp, so \( X(D) \) is sharp.
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