Stability and Performance Analysis of Control Based on Incomplete Models

Diogo Rodrigues * Håkan Hjalmarsson *

* Department of Automatic Control, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden (diogor@kth.se, hjalmars@kth.se)

Abstract: The models of chemical and biological processes are typically composed of unknown and known parts. This paper recalls a multivariable control strategy that deals explicitly with the existence of incomplete models. The suggested strategy uses estimation of the unknown rates in the model and feedback linearization to achieve good closed-loop performance with simple tuning of design parameters and without the need of a complete model. The main contribution of this paper is the development of methods for quantitative analysis of the local stability and performance of the control strategy in the continuous-time case, based on a system description using transfer functions. The approach is illustrated via a simulated example of reactor control.

Keywords: control, incomplete model, estimation, feedback linearization, stability, performance

1. INTRODUCTION

In the context of chemical and biological processes, the dynamic models that are used for control are typically composed of two parts: (i) one part that is well known thanks to its macroscopic nature, for instance the flows of material and energy between different units; and (ii) another part that is unknown due to the fact that it results from microscopic processes, such as the reaction kinetics and transport phenomena. Although it is possible to use system identification and first-principles modeling to obtain a relatively accurate description of the macroscopic effects of the second part, this description remains subject to some uncertainty or imprecision and requires experimental work that may be time-consuming and expensive.

Hence, the goals of this paper are: (i) to present a multivariable control structure that deals explicitly and systematically with the fact that process models may be incomplete; and (ii) to develop methods that allow assessing the stability and performance of this structure for a given system. Furthermore, this structure should not only be simple to tune and understand but also ensure good closed-loop performance. In particular, it is suggested here that the concept of rate estimation can be used to estimate the unknown part of the incomplete model without identifying its model. This concept has already been developed and applied to control of continuous reactors, first for temperature control (Rodrigues et al., 2015a), later for control of reaction variants (Zhao et al., 2016), and recently for control of fast transport phenomena. Although it is possible to use this concept to tune and understand but also ensure good closed-loop performance and provide an illustrative example of reactor control.

2. PROBLEM DESCRIPTION

2.1 System dynamics

Consider a nonlinear dynamic system with $n_x$ states $x(t)$, $n_u$ inputs $u(t)$, and $n_y$ outputs $y(t)$. The state dynamics are composed of an unknown part and a known or available part, denoted by the subscripts $u$ and $a$, while the outputs are known linear combinations of the states. This dynamic model is written as:

$$
\dot{x}(t) = f_u(x(t), u(t)) + f_{sa}(x(t), u(t)), \quad x(0) = x_0, \quad (1a)
$$

$$
y(t) = C x(t), \quad (1b)
$$

where $f_u(x(t), u(t))$ is a known function and $f_{sa}(x(t), u(t))$ is an unknown function. In addition, assume that (i) $s_u(y(t), u(t)) := C f_u(x(t), u(t))$ can be computed from the current outputs and inputs, and (ii) it is known that $s_a(x(t)) := C f_{sa}(x(t), u(t))$ is an unknown function of the states only, without any assumptions about $f_u(x(t), u(t))$ and $f_{sa}(x(t), u(t))$. Both assumptions are satisfied for several chemical and biological processes, namely, for reactors.

Without additional assumptions, we write $s_u(x(t))$ as linear combinations of $n_r$ rates $r_u(x(t))$, that is, $s_u(x(t)) = \mathcal{L} r_u(x(t))$, where $r_u(x(t))$ is an unknown function and $\mathcal{L}$ is a known $n_y \times n_r$ matrix, with rank $(\mathcal{L}) = n_r$ (which implies $n_r \leq n_y$). Then, there is an $n_r \times n_y$ matrix $\mathcal{T}$ such that

$$
\mathcal{T} \mathcal{L} = \mathbf{I}_{n_r}. \quad (2)
$$

Hence, we define the $n_r$ states $x_r(t) := \mathcal{T} C x(t)$ with initial conditions $x_{r,0} := \mathcal{T} C x_0$ and corresponding outputs $y_r(t) := \mathcal{T} y(t)$ that are described by the dynamics

$$
\dot{x_r}(t) = r_u(x(t)) + \mathcal{T} s_u(y(t), u(t)), \quad x_r(0) = x_{r,0}, \quad (3a)
$$

$$
y_r(t) = x_r(t), \quad (3b)
$$

* This work was supported by the VINNOVA Competence Centre AdBIOPRO, contract 2016-05181.
where the outputs $\mathbf{y}_r(t)$ are variant with respect to the rates $\mathbf{r}_a(\mathbf{x}(t))$ since they have the interesting property of one-to-one correspondence (Rodrigues et al., 2015b).

2.2 Control objective
Suppose that the objective is to control $\mathbf{y}_c(t)$, the $n_c$ linear combinations of outputs given by the $n_c \times n_y$ matrix $\mathbf{S}$, with rank $(\mathbf{S}) = n_c$ (which implies $n_c \leq n_y$), to the setpoints $\mathbf{y}_r(t)$. Hence, we define the $n_c$ states $\mathbf{x}_c(t) := \mathbf{S}\mathbf{x}(t)$ with initial conditions $\mathbf{x}_{c,0} := \mathbf{S}\mathbf{x}_0$ and corresponding outputs $\mathbf{y}_c(t) := \mathbf{S}\mathbf{y}(t)$ that are described by the dynamics

$$\dot{\mathbf{x}}_c(t) = \mathbf{H}\mathbf{r}_a(\mathbf{x}(t)) + \mathbf{b}_a(\mathbf{y}(t), \mathbf{u}(t)), \quad \mathbf{x}_{c,0} = \mathbf{x}_{c,0}, \quad \text{(4a)}$$

$$\mathbf{y}_c(t) = \mathbf{c}_a(\mathbf{y}(t)), \quad \text{(4b)}$$

where $\mathbf{H} := \mathbf{S}\mathbf{C}$ and $\mathbf{h}_a(\mathbf{y}(t), \mathbf{u}(t)) := \mathbf{S}\mathbf{s}_a(\mathbf{y}(t), \mathbf{u}(t))$. If one assumes that the known part of the dynamics is affine in the inputs, (4a) can be reformulated as:

$$\dot{\mathbf{x}}_c(t) = \mathbf{H}\mathbf{r}_a(\mathbf{x}(t)) + \mathbf{b}_a(\mathbf{y}(t))\mathbf{u}(t).$$

Note that $\mathbf{y}_c(t)$ and $\mathbf{y}_c(t)$ are not necessarily related, although both $\mathbf{y}_c(t)$ and $\mathbf{y}_c(t)$ can be computed from $\mathbf{y}(t)$ and their dimension is smaller than the dimension of $\mathbf{y}(t)$.

3. CONTROL STRATEGY
3.1 Control via feedback linearization
If the dynamic model (5) were perfectly known and the matrix $\mathbf{B}_a(\mathbf{y}(t))$ were invertible (which implies $n_c = n_u$ and is typically true for certain choices of controlled outputs $\mathbf{y}_c(t)$), one could use the feedback linearization law

$$\mathbf{u}(t) = \mathbf{B}_a(\mathbf{y}(t))^{-1}(\mathbf{v}(t) - \mathbf{H}\mathbf{r}_a(\mathbf{x}(t)) - \mathbf{b}_a(\mathbf{y}(t)))$$

(6) to set the rates of variation $\mathbf{v}(t)$ for these states as follows:

$$\dot{\mathbf{x}}_c(t) = \mathbf{v}(t).$$

(7)

For the rates of variation $\mathbf{v}(t)$, one can choose among other alternatives the proportional control law with gain $\tau_c^{-1}$

$$\mathbf{v}(t) = \tau_c^{-1}(\mathbf{y}_r(t) - \mathbf{y}_c(t)),$$

(8) which would ensure exponential convergence of $\mathbf{y}_c(t)$ to $\mathbf{y}_r(t)$ with the time constant $\tau_c$ in the ideal case of known rate model $\mathbf{r}_a(\mathbf{x}(t))$ and no input and output disturbances.

Unfortunately, as mentioned, the rate model $\mathbf{r}_a(\mathbf{x}(t))$ is unknown. Furthermore, there is a difference between the system inputs $\mathbf{u}(t)$ and the actuator inputs $\tilde{\mathbf{u}}(t)$ due to the input disturbances $\mathbf{d}(t) := \mathbf{u}(t) - \tilde{\mathbf{u}}(t)$ and a difference between the sensor outputs $\tilde{\mathbf{y}}(t)$ and the system outputs $\mathbf{y}(t)$ due to the output disturbances $\mathbf{w}(t) := \tilde{\mathbf{y}}(t) - \mathbf{y}(t)$.

In this realistic case, it is not possible to use directly (6) and (8). However, one can approximate them by replacing $\mathbf{u}(t)$ and $\mathbf{y}(t)$ by $\tilde{\mathbf{u}}(t)$ and $\tilde{\mathbf{y}}(t)$ as well as the unknown rates $\mathbf{r}_a(\mathbf{x}(t))$ by their estimates $\hat{\mathbf{r}}_a(t)$, which results in:

$$\dot{\mathbf{u}}(t) = \mathbf{B}_a(\tilde{\mathbf{y}}(t))^{-1}(\mathbf{v}(t) - \mathbf{H}\hat{\mathbf{r}}_a(t) - \mathbf{b}_a(\tilde{\mathbf{y}}(t))),$$

$$\mathbf{v}(t) = \tau_c^{-1}(\mathbf{y}_r(t) - \mathbf{y}_c(t)).$$

(9)

(10)

The estimation of the rates $\mathbf{r}_a(\mathbf{x}(t))$ is investigated next.

3.2 Estimation of unknown rates
One can use (3) to estimate the values $\hat{\mathbf{r}}_a(t)$ of the unknown rate signals by applying an FIR (finite impulse response) filter to the measured variants $\tilde{\mathbf{y}}(t) := \mathbf{T}\tilde{\mathbf{y}}(t)$ and the available rates $\hat{\mathbf{s}}_a(t)$ computed from $\tilde{\mathbf{y}}(t)$ and $\tilde{\mathbf{u}}(t)$:

$$\mathbf{r}_a(t) = \int_{0}^{\Delta t} \frac{c(\tau)}{\Delta t^2} \mathbf{y}_r(t - \Delta t + \tau) d\tau$$

$$- \int_{0}^{\Delta t} \frac{b(\tau)}{\Delta t} \mathbf{r}_a(t - \Delta t + \tau) d\tau,$$

(11)

where $c(\tau)$ and $b(\tau)$ are convolution functions and $\Delta t$ is the size of the filter window. For the sake of simplicity, all the signals are represented here as functions of time $t$ only.

Obviously one can also apply an IIR filter to $\tilde{\mathbf{y}}(t)$ and $\hat{\mathbf{s}}_a(t)$, which results in a linear observer (Perrier et al., 2000). However, the following results about rate estimation refer to the FIR filter. These results were discussed previously for the discrete-time case (Rodrigues et al., 2018) and are simply adapted here to the continuous-time case.

Assuming that $\mathbf{y}(t)$ are Lipschitz continuous signals with respect to $t$ (since they depend on the states $\mathbf{x}(t)$ that are typically Lipschitz continuous) and the disturbances that affect $\hat{\mathbf{s}}_a(t)$ are negligible with respect to those that affect $\tilde{\mathbf{y}}(t)$, one can show that the rate estimates are given by

$$\mathbf{r}_a(t) = \int_{0}^{\Delta t} \frac{c(\tau)}{\Delta t^2} \mathbf{w}(t - \Delta t + \tau) d\tau$$

$$+ \int_{0}^{\Delta t} \frac{b(\tau)}{\Delta t} \mathbf{r}_a(t - \Delta t + \tau) d\tau,$$

(12)

if $c(\tau)$ and $b(\tau)$ are such that $c(\tau) = -\Delta t \frac{d c(\tau)}{d \tau}$ and $b(0) = b(\Delta t) = 0$. If in addition $\int_{0}^{\Delta t} \frac{b(\tau)}{\Delta t} d\tau = 1$, one can observe from (12) that the rate estimates $\mathbf{r}_a(t)$ consist of the effect of the output disturbances $\mathbf{w}(t)$ and a weighted average of the unknown rates $\mathbf{r}_a$ in the interval $[t - \Delta t, t]$. Hence, the estimates $\mathbf{r}_a(t)$ provide an accurate result if the output disturbances $\mathbf{w}(t)$ correspond to zero-mean and uncorrelated noise and $\Delta t$ is chosen such that the rates $\mathbf{r}_a$ do not vary much in $[t - \Delta t, t]$, which is possible since they also depend on the Lipschitz continuous states $\mathbf{x}(t)$.

However, many choices of $c(\tau)$ and $b(\tau)$ are still possible. One can show that, under all the assumptions stated previously, the functions $c(\tau)$ and $b(\tau)$ that minimize the effect of the measurement noise in $\mathbf{w}(t)$ are the following:

$$c(\tau) = 12 \frac{\Delta t}{2} - 6,$$

$$b(\tau) = -6 \left(\frac{\Delta t}{2}\right)^2 + 6 \frac{\Delta t}{2}.$$  

(13a)  

(13b)

Note that the function $c(\tau)$ corresponds to the continuous-time version of the differentiatization Savitzky-Golay filter of order 1, which was developed originally for the discrete-time case (Savitzky and Golay, 1964) and has been called algebraic time-derivative estimation in the continuous-time case (Reger and Jouffroy, 2009). Note also that the use of a smaller window size $\Delta t$ of the differentiatization filter amplifies the measurement noise in $\mathbf{w}(t)$.

3.3 Design parameters
The presented control strategy requires only two design parameters with a well-defined meaning: (i) $\Delta t$ is the window size of the differentiatization filter, which should be small enough so that the unknown rates $\mathbf{r}_a(\mathbf{x}(t))$ are approximately constant in this window, but not too small to prevent amplification of measurement noise; and (ii) $\tau_c$ is the inverse of the controller gain and is expected to
be approximately equal to the dominant closed-loop time constant if the rate estimation is accurate enough.

4. STABILITY AND PERFORMANCE

We can now address the main goal of this paper, which is the assessment of the stability and performance of the control scheme. Although several other criteria exist, the performance is assessed here only in terms of the speed of the closed-loop response. The concepts of transfer functions and eigenvalues are used next for this assessment. Although it was assumed for controller design that part of the model is completely unknown, it is assumed for stability and performance assessment that there is some knowledge of $f_n(x(t), u(t))$, but it is uncertain or imprecise and described by the unknown plant parameters $\theta$, for which it is only known that they can take any value in a compact set $\Theta$.

For each signal $f(t)$, it is assumed that the initial conditions $f(0)$ correspond to the steady state $\bar{f}$, deviation variables are defined as $\delta f(t) := f(t) - \bar{f}$, which implies that the initial conditions $\delta f(0)$ are zero and can be omitted, and the corresponding Laplace transforms are denoted as $F(s)$. Then, the plant and the controller are linearized to allow expressing the open-loop and closed-loop systems as transfer functions, which are accurate approximations of the true nonlinear systems if the deviations are not too large.

4.1 Open-loop transfer functions

We start by describing the plant in (1) in terms of deviation variables, which yields

$$\delta \dot{x}(t) = A_p \delta x(t) + B_p \delta u(t),$$

$$\delta y(t) = C_p \delta x(t),$$

with

$$A_p = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u}),$$

$$B_p = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u}),$$

$$C_p = C.$$

The transfer function of the plant is computed as $G_p(s)$.

The estimation of $\hat{r}_n(t)$ in (11) uses the inputs $\tilde{y}(t)$ and $\delta z_n(t)$. The computation of $\delta z_n(t)$ from $\tilde{y}(t)$ and $\hat{u}(t)$ implies

$$\delta z_n(t) = D_{\delta y} \tilde{y}(t) + D_{\delta u} \delta \hat{u}(t),$$

with

$$D_{\delta y} = \left. \frac{\partial f}{\partial y}(\bar{y}, \bar{u}) \right|_{\bar{u}}.$$ 

By applying the Laplace transform to (11), one shows that

$$\hat{R}_n(s) = \left( \frac{6(s \Delta t + 2) \exp(-s \Delta t) + (s \Delta t - 2)}{s^2 \Delta t} \right) \tilde{y}(s) - \left( \frac{6(s \Delta t + 2) \exp(-s \Delta t) + (s \Delta t - 2)}{s^2 \Delta t} \right) \delta \hat{u}(s).$$

Then, one can use the Padé approximation

$$\exp(-s \Delta t) \approx \frac{1 - \frac{s \Delta t}{2} + \frac{s^2 \Delta t^2}{8}}{1 + \frac{s \Delta t}{2} + \frac{s^2 \Delta t^2}{8}},$$

to express (18) as a rational function of $s$, which results in

$$\hat{R}_n(s) = \frac{\frac{1}{\Delta t} T \tilde{y}(s) - \frac{1}{\Delta t} T \delta \hat{u}(s)}{s^2 + \frac{\Delta t}{\Delta t} s + \frac{\Delta t^2}{\Delta t^2}}.$$  

This means that the rate estimation can be described by the minimal realization of the transfer function (20), which yields the following linear system with $n_z := 2n_r$ states:

$$\delta \dot{z}(t) = A_z \delta z(t) + B_z \delta \tilde{y}(t) + B_z \delta \hat{u}(t),$$

$$\delta \hat{u}(t) = C_z \delta z(t),$$

with

$$A_z = \left[ \begin{array}{c} 0_{n_r \times n_r} - \frac{\Delta t}{\Delta t} I_{n_r} \\ I_{n_r} - \frac{\Delta t}{\Delta t} I_{n_r} \end{array} \right],$$

$$B_z^e = \left[ \begin{array}{c} 0_{n_r \times n_r} \\ \frac{\Delta t}{n_r} T \end{array} \right],$$

$$B_z^c = \left[ \begin{array}{c} 0_{n_r \times n_r} \\ 0_{n_r \times n_r} \end{array} \right],$$

$$C_z = [0_{n_r \times n_r}, I_{n_r}].$$

Furthermore, the feedback-linearizing controller in (9) and (10) can be written as

$$\delta \hat{u}(t) = D_{\delta y} \delta \tilde{y}(t) + D_y \delta \hat{y}(t) + D_{ye} \delta \hat{e}(t),$$

with

$$D_{\delta y} = -B_u (\bar{y})^{-1} H,$$

$$D_y = -B_u (\bar{y})^{-1} \left( S_{\bar{y}e}^{-1} + \frac{\partial h}{\partial y}(\bar{y}, \bar{u}) \right),$$

$$D_{ye} = B_u (\bar{y})^{-1} \bar{e}_c - 1.$$  

The combination of (16)–(17), (21)–(22), and (23)–(24) leads to the state-space representation of the controller as

$$\delta \hat{u}(t) = A_k \delta z(t) + B_k \delta \tilde{y}(t) + B_k \delta \hat{e}(t),$$

with

$$A_k = A_c + B_k D_{\delta y} D_n D_c C_c,$$

$$B_k^e = B_u + B_k^e D_n + B_k^c D_n D_c,$$

$$B_k^c = B_u D_n C_c,$$

$$C_k = D_k^c C_c,$$

$$D_k^e = D_y,$$

$$D_k^c = D_c.$$  

The transfer functions of the actuator inputs $\hat{u}(t)$ with respect to the sensor outputs $\delta \hat{y}(t)$ (superscript $y$) and the setpoints $\dot{y}(t)$ (s) are computed as $G_k^e(s)$ and $G_k^c(s)$.

Also, one should consider the following relationships between inputs and outputs of the plant and the controller:

$$\delta u(t) = \delta \hat{u}(t) + \delta d(t),$$

$$\delta \tilde{y}(t) = \delta \hat{y}(t) + \delta w(t).$$  

Fig. 1 shows a schematic of the closed-loop system.

4.2 Closed-loop transfer functions

For the stability and performance analysis, we would like to obtain an explicit state-space representation of the closed-loop system with states $\delta x_c(t) := \left[ \begin{array}{c} \delta x(t) \\ \delta z(t) \end{array} \right]$, outputs

$$\delta y_c(t) := \left[ \begin{array}{c} \delta \hat{y}(t) \\ \delta \hat{u}(t) \end{array} \right],$$

and inputs $\delta u_c(t) := \left[ \begin{array}{c} \delta \hat{e}(t) \\ \delta \hat{d}(t) \end{array} \right]$. One could compute first the closed-loop transfer functions from the transfer functions of the plant and the controller, but this computation is not the ideal method since it requires inverting $I_{n_x} - G_{cd}(s)$, where $G_{cd}(s) := G_n^c(s) G_p(s)$ is the open-loop transfer function. Then, it is better to combine...
Fig. 1. Schematic of the closed-loop system that results from control based on incomplete models of the plant, by using estimation of unknown rates and feedback linearization.

(14)–(15), (25)–(26), and (27), which allows obtaining directly the state-space representation

\[
\dot{x}_{cl}(t) = A_{cl} \delta x_{cl}(t) + B_{cl} \delta u_{cl}(t), \quad \delta y_{cl}(t) = C_{cl} \delta x_{cl}(t) + D_{cl} \delta u_{cl}(t),
\]

with

\[
A_{cl} = \begin{bmatrix} A_p + B_c D_k^c C_p & B_c C_k \\ B_k^c C_p & A_k \end{bmatrix},
B_{cl} = \begin{bmatrix} B_c D_k^c & B_c & B_k^c \\ B_k^c C_p & 0_{n_u \times n_u} & B_k^c \end{bmatrix},
C_{cl} = \begin{bmatrix} C_p \\ D_k^c C_p & C_k \end{bmatrix},
\]

\[
D_{cl} = \begin{bmatrix} 0_{n_u \times n_u} & 0_{n_u \times n_u} \\ D_k^c & I_{n_u} \end{bmatrix}.
\]

One can compute the transfer functions of the plant outputs \( y(t) \) (superscript \( y \)) and inputs \( u(t) \) (superscript \( u \)) of the closed-loop system with respect to the setpoints \( y^*_s(t) \), the input disturbances \( d(t) \), and the output disturbances \( w(t) \) (superscript \( w \)) as \( G_{cl, y}^{\theta}(s) \), \( G_{cl, u}^{\phi}(s) \), \( G_{cl, y}^{\theta}(s) \), \( G_{cl, u}^{\phi}(s) \), \( G_{cl, y}^{\theta}(s) \), \( G_{cl, u}^{\phi}(s) \). These transfer functions are the so-called "gang of six" that completely specifies the closed-loop behavior.

4.3 Closed-loop eigenvalues and poles

One could use the eigenvalues of \( A_{cl} \) to assess the local stability and performance of the closed-loop system. If the real parts of all the eigenvalues of \( A_{cl} \) are less than a value \(-\gamma\), with \( \gamma \geq 0 \), the closed-loop system is locally stable and all the closed-loop time constants are less than \( \gamma^{-1} \).

However, this might be more conservative than needed, because some eigenvalues of \( A_{cl} \) might correspond to unobservable closed-loop poles due to zero-pole cancellation. In that case, one constructs the decomposition \( A_{cl} = V_{cl} \Lambda_{cl} V_{cl}^{-1} \), where \( V_{cl} \) is a matrix whose columns are the generalized eigenvectors of \( A_{cl} \) and \( \Lambda_{cl} \) is the corresponding matrix in Jordan normal form. Then, the closed-loop poles of \( G_{cl}(s) \) are the diagonal elements of \( \Lambda_{cl} \) that correspond to the nonzero columns of \( C_{cl} V_{cl} \).

On the other hand, it is necessary to keep in mind that the model of the plant is incomplete, which means that \( A_{cl} \) is a function of not only the design parameters \( \Delta t \) and \( \tau_c \) but also the unknown plant parameters \( \theta \) in the set \( \Theta \). Consequently, the robust stability assessment depends on \( \lambda_{\max}(A_{cl}(\Delta t, \tau_c, \theta)) \), the maximum real part of the eigenvalues of \( A_{cl}(\Delta t, \tau_c, \theta) \), while the robust performance assessment is given by \( s_{\max}(G_{cl}(s, \Delta t, \tau_c, \theta)) \), the maximum real part of the closed-loop poles of \( G_{cl}(s, \Delta t, \tau_c, \theta) \).

This can be expressed by the conditions

\[
\lambda_{\max}(A_{cl}(\Delta t, \tau_c, \theta)) < 0, \quad \forall \theta \in \Theta,
\]

\[
s_{\max}(G_{cl}(s, \Delta t, \tau_c, \theta)) < -\gamma, \quad \forall \theta \in \Theta,
\]

which are nonconvex in general but can be verified easily if the number of parameters \( \theta \) is not too large. The corresponding maximum closed-loop time constant is

\[
\tau_{\max}(G_{cl}(s, \Delta t, \tau_c, \theta)) = s_{\max}(G_{cl}(s, \Delta t, \tau_c, \theta))^{-1}.
\]

5. ILLUSTRATIVE EXAMPLE

5.1 System description

The theory in the previous sections is illustrated here by the example of a continuous stirred-tank reactor, that is, a perfectly mixed reactor with constant volume. This particular example concerns a nonisothermal reactor with 4 species (A, B, C, D) and 2 reactions (A + B → C + 2B → D). The \( n_x := 5 \) states \( x(t) \) are the heat \( Q(t) := V \rho c_p (T(t) - T_{ref}) \) and the numbers of moles \( n_s(t) := V c_s(t) \) of the species \( s = A, \ldots, D \). Here, \( V \) is the constant volume, \( c_s \) is the concentration of the species \( s \), \( T(t) \) is the reactor temperature, \( T_{ref} \) is the reference temperature, \( \rho \) is the constant density, and \( c_p \) is the constant specific heat capacity. The \( n_u := 3 \) inputs \( u(t) \) are the exchanged heat power \( q_{ex}(t) \) and the volumetric flowrates of two inlets at the temperature \( T_{ref} \). One inlet is fed with the concentration of \( A \) \( c_{in,A} \) and the flowrate \( F_A(t) \) and the other inlet is fed with the concentration of \( B \) \( c_{in,B} \) and the flowrate \( F_B(t) \). The outlet flowrate is the sum of the inlet flowrates. Since the \( n_y := 3 \) outputs and the controlled outputs are \( y(t) = y_s(t) := [Q(t) \; n_A(t) \; n_B(t)]^T \), one can also define the matrices \( \dot{C} := \begin{bmatrix} 1_t & 0_{3 \times 2} \end{bmatrix} \) and \( S := I_4 \).

Hence, the dynamic model of the plant can be written as

\[
\dot{Q}(t) = -\Delta H_{r_1} r_{v_1}(t) - \Delta H_{v_2} r_{v_2}(t) + q_{ex}(t) - \omega(t) Q(t),
\]

\[
\dot{n}_A(t) = -r_{v_1}(t) + c_{in,A} F_A(t) - \omega(t) n_A(t),
\]

\[
\dot{n}_B(t) = -r_{v_1}(t) - 2 r_{v_2}(t) + c_{in,B} F_B(t) - \omega(t) n_B(t),
\]

\[
\dot{n}_C(t) = r_{v_1}(t) - \omega(t) n_A(t),
\]

\[
\dot{n}_D(t) = r_{v_2}(t) - \omega(t) n_B(t),
\]
where \( r_{c,i}(t) := V r_i(x(t)) \), with \( r_i(x(t)) \) the rate of the \( i \)th reaction, \( \Delta H_{r,i} \) is the enthalpy of the \( i \)th reaction at \( T_{ref} \), and \( \omega(t) := \frac{E_2(t) + E_0(t)}{V} \) is the inverse of the residence time.

The part of the dynamic model shown in (33)–(37) is constant in a window of \( \Delta t \) chosen such that their sum is 5 times less than the desired time constant of the open-loop system \( \tau_{max}(G_p(s, \theta)) \). This paper has presented a control scheme based on incomplete models, which takes advantage of the knowledge about part of the dynamic model of the plant. Although only incomplete knowledge of the plant model is used and the controller possesses only two design parameters that are rather simple to tune thanks to their clear meaning, this control scheme converges quickly to its setpoints and description using transfer functions, which shows that the description using transfer functions approximates well the dynamics of the nonlinear system.

Future work could provide new criteria to assess the stability and performance of the closed-loop system in terms of its maximum time constant \( \tau_{max}(G_p(s, \theta)) \) for values of the unknown plant parameters around their true values, that is, for \( \tau_c \in [1, 10] \) min and \( \Delta t \in [1, 10] \) min. Although the performance of the closed-loop system is not compromised for any values of these parameters. Moreover, Fig. 4 shows \( \tau_{max}(G_{cl}(s, \Delta t, \tau_c, \theta)) \) for values of the unknown plant parameters around their true values, that is, for \( k_{1,ref} \in [0.2, 2] \) L mol\(^{-1}\) min\(^{-1}\) and \( k_{2,ref} \in [0.4, 4] \) L mol\(^{-1}\) min\(^{-1}\). The effect of the unknown plant parameters \( E_{a,1} \) and \( E_{a,2} \) on \( \tau_{max}(G_{cl}(s, \Delta t, \tau_c, \theta)) \) is not shown here since it is much weaker than the effect of \( k_{1,ref} \) and \( k_{2,ref} \). Again, the performance of the closed-loop system becomes worse with increasing values of \( k_{1,ref} \) and \( k_{2,ref} \), but its stability is not compromised.

### Table 1. Plant parameters and operating conditions.

| Variable      | Value     | Units     |
|---------------|-----------|-----------|
| \( k_{1,ref} \) | 0.53      | L mol\(^{-1}\) min\(^{-1}\) |
| \( k_{2,ref} \) | 1.28      | L mol\(^{-1}\) min\(^{-1}\) |
| \( E_{a,1} \)  | 20000     | J mol\(^{-1}\) |
| \( E_{a,2} \)  | 10000     | J mol\(^{-1}\) |
| \( R \)        | 8.314     | J mol\(^{-1}\) K\(^{-1}\) |
| \( T_{ref} \)  | 298.15    | K         |
| \( -\Delta H_{r,1} \) | 70        | kj mol\(^{-1}\) |
| \( -\Delta H_{r,2} \) | 100       | kj mol\(^{-1}\) |
| \( c_{in,A} \) | 2.0       | mol L\(^{-1}\) |
| \( c_{in,B} \) | 1.5       | mol L\(^{-1}\) |
| \( V \)        | 500       | L         |
| \( V_{ref} \)  | 736.3     | kj K\(^{-1}\) |

### 6. CONCLUSION

This paper has presented a control scheme based on incomplete plant models, which takes advantage of the knowledge about part of the dynamic model of the plant. Although only incomplete knowledge of the plant model is used and the controller possesses only two design parameters that are rather simple to tune thanks to their clear meaning, this control scheme converges quickly to its setpoints and can eliminate steady-state error without any integral term. These features have been demonstrated by an illustrative example. Furthermore, some robust and quantitative criteria to analyze the stability and performance of this control strategy have been provided for the continuous-time case. Future work could provide new criteria to assess the stability and performance in a robust sense and extend these results to the discrete-time case. It would also be useful to formulate conditions that express the stability and performance criteria in a way that is as explicit and computationally cheap as possible. Finally, it would be rather interesting to investigate how much one can improve the closed-loop performance by using complete models that result from system identification and how this control scheme compares with the use of techniques for robust control design in terms of stability and performance.
Fig. 2. Closed-loop response to a step increase in the setpoint $n_s^*(t)$. The blue line represents the response obtained via linearization of the system and description using transfer functions, while the dashed green line represents the response obtained via numerical simulation of the nonlinear system.

Fig. 3. Contour plot of the maximum closed-loop time constant as a function of the design parameters $\tau_c$ and $\Delta t$.

Fig. 4. Contour plot of the maximum closed-loop time constant as a function of the unknown plant parameters $k_{1,ref}$ and $k_{2,ref}$.

REFERENCES
Farschman, C.A., Viswanath, K.P., and Ydstie, B.E. (1998). Process systems and inventory control. AIChE J., 44(8), 1841–1857.
Perrier, M., Feyo de Azevedo, S., Ferreira, E.C., and Dochain, D. (2000). Tuning of observer-based estimators: theory and application to the on-line estimation of kinetic parameters. Control Eng. Practice, 8(4), 377–388.
Renger, J. and Jouffroy, J. (2009). On algebraic time-derivative estimation and deadbeat state reconstruction. In 48th IEEE CDC. Shanghai, P.R. China.
Rodrigues, D., Amrhein, M., Billeter, J., and Bonvin, D. (2018). Fast estimation of plant steady state for imperfectly known dynamic systems, with application to real-time optimization. Ind. Eng. Chem. Res., 57(10), 3699–3716.
Rodrigues, D., Billeter, J., and Bonvin, D. (2015a). Control of reaction systems via rate estimation and feedback linearization. Comput. Aided Chem. Eng., 37, 137–142.
Rodrigues, D., Srinivasan, S., Billeter, J., and Bonvin, D. (2015b). Variant and invariant states for chemical reaction systems. Comput. Chem. Eng., 73, 23–33.
Savitzky, A. and Golay, M.J.E. (1964). Smoothing and differentiation of data by simplified least squares procedures. Anal. Chem., 36(8), 1627–1639.
Zhao, Z., Wassick, J.M., Ferrio, J., and Ydstie, B.E. (2016). Reaction variants and invariants based observer and controller design for CSTRs. IFAC-PapersOnLine, 49(7), 1091–1096.