Additional bending moment of thick-walled elliptical tube under internal pressure

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Abstract. With the development of industry, the elliptic-sectional tube is widely used in some special situation. In this paper, a theoretical model is developed to calculate the additional bending moment of thick-walled elliptical tube under the internal pressure, based on the structural static force method. The effects of internal pressure and ellipticity on the additional moment were discussed. At last, finite element analyses were conducted and the results were compared with the theoretical results to verify the correctness of this theoretical model. This model can provide a guideline for determining the maximum additional bending moment of elliptic, and help in estimating the failure of the elliptical tube under the internal pressure.

1. Introduction

At present, the pressure pipeline and vessel typically adopt the circular cross-section, but the elliptical section is also widely used in some special industrial fields, such as electricity, petroleum and chemical industry [1]. Compared with the circular section, the elliptical section tube has shorter service life. The plastic failure controlled by limit load is the general failure mode. Because of the large additional bending moments generated by the internal pressure applied on the elliptical section, the circumferential stress caused by the bending moments is the main factor causing the failure of the elliptical section tube [2-3]. Therefore, it is great significant to accurately obtain the stress distribution on the tube section for the judgment of tube failure.

Much of the researchers have put forward the mathematical method to calculate the ultimate bearing capacity of the pressured tube. For example, Chen [4] developed a simplified numerical method for both lower and upper bound limit loads of the 3-D pipeline with one or two part-through slots of various geometrical configurations are calculated by the proposed method, using the finite element method and mathematical programming techniques. Lei [5] reviewed the limit load solutions for axially cracked cylinders, and developed new limit solutions for thick-walled cylinders with axial cracks under internal pressure to overcome problems in the existing solutions. In addition, Kanninen et.al [6] put forward the net section collapse criterion, considering the strain hardening phenomenon of tube material. Then, Kurihara et.al [7] proposed several simple limit-load criteria to predict in a light water reactor pressure boundary piping the failure bending moment of a tough pipe. Kim et.al [8-9] proposed plastic limit load solutions for thin-walled branch junctions under internal pressure and in-plane bending, based on finite element limit loads resulting from three-dimensional FE limit analyses using elastic-perfectly plastic materials.

The finite element method is an effective method to solve the tube stress, but it is not easy to implement under any conditions due to the restrictions of element division, data preparation and computer capacity. In this paper, a theoretical model was developed to estimate the additional bending moment of thick-walled elliptical tube under internal pressure, and the effects of internal pressure and
ellipticity on the additional moment were discussed. At last, the result was compared with finite element analysis. This model can be used to calculate the additional moment at any point on the elliptical tube is obtained by a short program, and help in estimating the failure life of the elliptical tube.

2. Theoretical analysis

Figure 1(a) shows the schematic diagram of an elliptical tube under internal pressure. Due to the symmetry of its shape and stress state when the internal pressure $p$ was applied, the model can be simplified to a quarter of the section for stress analysis, as shown in figure 1(b). Section A is the intersection of the major axis with the coordinate x-axis. In order to facilitate the subsequent calculation process, in this paper, the major and minor half axis of the geometric neutral layer of the ellipse is defined as $a_i$ and $b_i$; the major and minor half axis of the outer layer is $a_o$ and $b_o$, the inner layer is $a_i$ and $b_i$; the wall thickness is $t$. Obviously, there is a geometric relationship as:

$$a_i = a - t/2, \quad b_i = b - t/2; \quad a_o = a + t/2, \quad b_o = b + t/2$$

(1)

Figure 1. Analyzed model of ellipse (a) the whole model and (b) simplified model.

The curve of the neutral layer can be expressed by the parametric equation of the ellipse:

$$x = a \cos(\alpha)$$

$$y = b \sin(\alpha)$$

(2)

The point $(x, y)$ was defined by parameter $\alpha$. The angle between the normal line of the point $(x, y)$ and the x-axis is $\beta$, and the point $(x_i, y_i)$ is the intersection of the normal line and the inner surface, as shown in Fig. 1b.

$$x_i = x - t/2 \cos(\beta)$$

$$y_i = y - t/2 \sin(\beta)$$

(3)

The arc length of the neutral layer is:

$$ds = \sqrt{x'^2 + y'^2} d\alpha = \sqrt{(a \sin(\alpha))^2 + (b \cos(\alpha))^2} d\alpha$$

(4)

There are circumferential stress $\sigma_\theta$ and the additional bending moment $M_A$ in section A. For the circumferential $\sigma_\theta$, the elliptical tube can be regarded as a thick-walled cylinder with internal radius $a_i$ and external radius $a_o$. So the circumferential stress $\sigma_\theta$ can be expressed as:

$$\sigma_\theta = \frac{pa_i^2}{a_o^2 - a_i^2} \left(1 + \frac{a_i^2}{\rho^2}\right) = \frac{pa_i^2}{2at} \left(1 + \frac{a_i^2}{\rho^2}\right)$$

(5)

Where $\rho$ is the distance from any point at the section A to the coordinate origin.

The additional bending moment $M_A$ is the maximum because the section A is on the major axis. Consequently, the moment $M_A$ is needed to be solved, and it can be obtained from the canonical equations of force method. Since the structure is statically indeterminate at this time, the canonical equation can be obtained by using the deformation compatibility condition with zero rotation angle of section A.
\[
\int (M_\sigma + M_\rho) ds + M_A \int ds = 0
\]  
(6)

In the Eq. (6), \(M_\rho\) is the bending moment acting on the point \((x, y)\) by internal pressure, and \(M_\sigma\) is the bending moment caused by circumferential stress. Then \(M_A\) can be expressed as:

\[
M_A = -\int (M_\sigma + M_\rho) ds \times \left(\int ds\right)^{-1}
\]  
(7)

The moment \(M_\sigma\) relative to the point \((x, y)\) can be expressed as Eq. (8), where the clockwise as positive.

\[
M_\sigma = \int_0^{\sigma} \sigma_o (\rho - x) d\rho = \frac{pa_o}{2a} \left( at - \frac{2ax}{a} + a_i^2 \ln \left( \frac{a_o}{a_i} \right) \right)
\]  
(8)

The bending moment \(M_\rho\) can be decomposed into the moment \(M_{\rho x}\) caused by internal pressure in horizontal direction and the moment \(M_{\rho y}\) caused by internal pressure in vertical direction, that is:

\[
M_{\rho x} = -\frac{1}{2} pt \left( a_i - x \right)^2 + \frac{1}{2} p \left( x - x_i \right)^2 = \frac{pt^2}{8} \cos^2 (\beta) - \frac{1}{2} p \left( a_i - x \right)^2
\]

\[
M_{\rho y} = -\frac{1}{2} p y^2 + \frac{1}{2} p \left( y - y_i \right)^2 = \frac{pt^2}{8} \sin^2 (\beta) - \frac{1}{2} p y^2
\]  
(9)

Thus,

\[
M_\rho = M_{\rho x} + M_{\rho y} = \frac{pt^2}{8} \left( (a_i - x)^2 + y^2 \right)
\]  
(11)

Finally the \(M_A\) can be solved by Matlab program based on the Eq. (12), and the direction is counter clockwise.

\[
M_A = \left( \int_0^{\sigma/2} M_\sigma ds + \int_0^{\sigma/2} M_\rho ds \right) \left( \int_0^{\sigma/2} ds \right)^{-1}
\]  
(12)

For other section, the additional moment \(M(\alpha)\) is:

\[
M(\alpha) = M_A + M_\rho + M_\sigma
\]  
(13)

3. Validation of the analytical model

To validate the analytical model, the finite element analysis (FEA) was used and the results were compared with those of the theoretical solution. To simulate the process, the software ABAQUS 6.13-1 and static, general step was used. Considering the symmetry of the forming parts, the right part along the axisymmetric line is taken as the research object, and the plane strain model is established. The tube pipe is defined as isotropic elastic-plastic model with CPE4R 4-node bilinear plane strain quadrilateral elements. The maximum element size was chosen to be 0.25mm to accurately represent the bending effect. The idealized material was used for the simulation and the Young modulus is assumed as E=1000GPa.

Five simulation models were performed with the different parameter and the detailed dimensions are shown in Table 1. Take the Model 1 as the example, the major half axis \(a=10\)mm and the minor \(b=10\)mm, thickness \(t=2\)mm and the internal pressure \(p=1\)MPa. The finite element model is shown in figure 2. In the post processing of Abaqus, the moment and force in the section A can be shown by using view cut, the simulation result as shown in figure 3. It is shown that the moment of FEA result \(M_A=9.53\)N/mm, while the theoretical result is \(M_A=9.32\)N/mm, and the error is about 2.2%.

| Model | \(a/\text{mm}\) | \(b/\text{mm}\) | \(t/\text{mm}\) | \(p/\text{MPa}\) | \(M_A(\text{Theory})\) | \(M_A(\text{FEA})\) | error |
|-------|---------------|--------------|-------------|-------------|----------------|----------------|-------|
| 1     | 10            | 8            | 2           | 2           | 9.32           | 9.53           | 2.2%  |
| 2     | 10            | 6            | 2           | 5           | 88.11          | 88.02          | -0.1% |
| 3     | 10            | 6            | 3           | 5           | 86.08          | 89.91          | 3.5%  |
| 4     | 30            | 20           | 2           | 2           | 273.96         | 265.09         | -3.3% |
| 5     | 30            | 25           | 2           | 5           | 357.66         | 335.87         | 6.5%  |
The results of models 2, 3, 4 and 5 were shown in Table 1. It is obvious that the error is about 3%, although the error of model 5 is 6.5%, this is because as the increase of internal pressure, the geometric axis of the ellipse is changed and FEA result is deviated. Figure 4 shows the FEA result of other models.

Figure 2. Finite element model.

Figure 3. The FEA result of model 1.

Figure 4. FEA results of models 2, 3, 4 and 5.
Figure 5 shows the variation curves of moment $M_A$ changed with the internal pressure $p$. It can be seen that the moment $M_A$ is proportional to the pressure $p$, and with the increasing of internal pressure, the simulation error is increasing. Moreover, the FEA result of Young modulus $E=5000\text{GPa}$ is more close to the theory result.

Figure 6 shows the additional bending moment $M_A$ by theory changed with the ellipticity $\omega$, where the ellipticity is defined as $\omega=(a_o-b_o)/r$ and the nominal radius $r=25\text{m}$; the thickness $t=2\text{mm}$ and the internal pressure $p=1\text{MPa}$. It can be seen that additional bending moment has a significant linear correlation with the ellipticity $\omega$.

![Figure 5](image1.png) ![Figure 6](image2.png)

**Figure 5.** The additional moment $M_A$ vs. pressure $p$. **Figure 6.** The moment $M_A$ vs. ellipticity.

4. Conclusions

A mathematical model was developed successfully to calculate the additional bending moment of elliptic-sectional tube under the internal pressure, based on the structural static force method. This analytical model provides an accurate method to determine the additional moment and the stress distribution. The effects of internal pressure and ellipticity on the additional moment were discussed. At last, finite element analyses were conducted and the results were compared with the theoretical results to verify the correctness of this theoretical model. This model can provide a guideline for determining the maximum additional bending moment of elliptic, and help to design the boiler pipeline, and predict the failure and service life of the pipeline with the creep theory.

5. References

[1] Naphon P and Wongwises S 2006 Renew Sust Energ Rev 10(5) pp 463-90
[2] Kanninen MF, Zahoor A, Wilkoski G, Abousayed I and Marschall 1982 EPRI NP-2347 Electric Power Research Institute, Palo Alto
[3] Xuan FZ, Li PN and Tu ST 2006 Int J Mech Sci, 48 pp 460-467
[4] Chen H F and Shu D W 2000 Int J Pres Ves Pip 77(1) pp 17-25
[5] Lei Y 2008 Int J Pres Ves Pip 85(12) pp 825-850
[6] Kanninen MF. 1976 EPRI NP-192
[7] Kurihara R, Ueda S and Sturm D 1988 Nucl Eng Des 106(2) pp 265-273
[8] Kim Y J, Lee K H and Park C Y 2008 Int J Pres Ves Pip 85(6) pp 360-367
[9] Kim Y J, Shim D J, Nikbin K, Kin Y J, Hwang S S and Kin J S 2003 Int J Pres Ves Pip 80 pp 527-40