Fault detection observer design in finite-frequency domain for networked interconnected systems

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Abstract. With the development of network technology, information can flow flexibly, and the research of interconnected systems are booming. The article proposes an $H_\infty / H_\infty$ fault detection observer design means for a networked interconnected system. First, convert a linear networked interconnection systems to linear network systems. Next, the $H_\infty$ index is accustomed to measuring the worst-case sensitivity to failure and $H_\infty$ norm from ungiven inputs to residuals is accustomed to measure disturbance robustness performance, so that the residuals generated by the designed fault detection observer are sensitive to failures and robust to disturbance in finite-frequency domain. Besides, sufficient conditions of the observer design are derived and transformed by a set of LMIs. Lastly, numerical simulations are provided to demonstrate the correctness and effectiveness of the designed means.

1. Introduction
Since the 1970 s, the scale of modern systems has been expanding, the structure is more and more complex, and the requirements for control quality are also increasing. For example, the system models of complex systems such as ships [1], aircraft engine [2], high-speed trains [3], power systems [4], chemical processes [5] and robots [6] all have the form of interconnection. Therefore, interconnected systems attract great interest. Reference [7] studied the distributed tracking control problem of nonlinear interconnected systems by using the distributed dynamic surface control method. Reference [8] used the small gain theorem to analyze the stability of the interconnected discrete-time nonlinear system. Reference [9] proposed an interconnected observer method with coupling term for interconnected linear time-invariant systems with output disturbances. Reference [10] studied the decentralized control strategy for a class of nonlinear interconnected systems based on sliding mode control.

The above research results only studied the stability of interconnected systems, and did not consider the possible faults in the systems. For the interconnected systems, due to its own complexity, its fault may not only occur in the subsystems, but also in the interconnection, which makes the fault detection of the interconnected systems particularly difficult. Once the systems fail, the consequences will be disastrous. In order to ensure that the fault can be quickly detected in any position and avoid catastrophic paralysis of the whole systems, the suitable fault detection scheme is necessary.

Reference [11] analyzed and summarized the existing design methods of fault detection and fault-tolerant control for interconnected systems. Reference [12] constructed a filter to provide residuals
signal for each subsystems of Markov jump interconnected systems by using the information of local measurements and adjacent models. More importantly, the proposed method realized fault detection and isolation at the same time. Reference [13] studied the distributed fault detection mechanism for a class of interconnected nonlinear systems with partial communication links to detect faults online and in real time. Reference [14] proposed a nonlinear observer fault detection method is proposed for a class of nonlinear interconnected systems with modeling error and output measurement noise. Reference [15] proposed a fault separation design method based on interval observer for discrete fuzzy interconnected system with ungiven interconnection to determine the fault location.

On the basis of the above research work, the paper proposes a novel idea for a class of linear interconnected systems, rewriting linear interconnected systems into ordinary linear systems, and use the conclusions applied to linear systems to study the linear interconnected systems. Lastly, a numerical instance is accustomed to verify the feasibility.

Notations: The notations in this paper are standard. \( \mathbb{R}^n \) bespeaks the n-dimensional Euclidean space and \( I \) represents the identity matrix with appropriate dimension. For a matrix \( A \), \( A^T \) and \( A' \) denote the conjugate and complex conjugate transpose of \( A \). \( He(A) \) is accustomed to denote \( He(A) := A + A' \) and \( A > 0 (A < 0) \) means that \( A \) is a positive definite (negative definite). In symmetric matrices, the asterisk \( * \) represents the symmetric block in the matrix. For a signal \( x(t) \), 
\[
\| y(t) \|_2 = \sqrt{\int_0^\infty y^T(t) y(t) dt}
\]
bespeaks its \( F_2 \)-norm.

2. Problem Formulation
Firstly, the following linear networked interconnected systems are taken into account in this paper.
\[
\begin{align*}
\dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + D_i d_i(t) + F_i f_i(t) + \sum_{j=k}^N A_{ij} x_j(t) \\
y_i(t) &= C_i x_i(t) \quad i = 1, 2, \ldots, N
\end{align*}
\]
where \( x_i(t) \in \mathbb{R}^{n_i} \) represents the state vector, \( u_i(t) \in \mathbb{R}^{n_u} \) is the control input vector, \( d_i(t) \in \mathbb{R}^{n_d} \) is the ungiven disturbance, \( f_i(t) \in \mathbb{R}^{n_f} \) is the actuator fault and \( y_i(t) \in \mathbb{R}^{n_y} \) is the measured output vector. \( n_x, n_u, n_d, n_f, n_y \) represent the dimension of each vector. \( A_i, B_i, D_i, F_i, C_i \) are given constant matrices with adequacy dimensions. \( A_{ij} \) is a constant interconnection matrix; for all \( i = 1, 2, \ldots, N \), \((A_i, B_i)\) is observer.

Because the vector of the systems are written by
\[
\begin{align*}
\begin{bmatrix}
x_1^T(t) \\ x_2^T(t) \\ \vdots \\ x_N^T(t)
\end{bmatrix}, \quad 
\begin{bmatrix}
y_1^T(t) \\ y_2^T(t) \\ \vdots \\ y_N^T(t)
\end{bmatrix}, \quad 
\begin{bmatrix}
u_1^T(t) \\ u_2^T(t) \\ \vdots \\ u_N^T(t)
\end{bmatrix}, \quad 
\begin{bmatrix}
d_1^T(t) \\ d_2^T(t) \\ \vdots \\ d_N^T(t)
\end{bmatrix}, \quad 
\begin{bmatrix}
f_1^T(t) \\ f_2^T(t) \\ \vdots \\ f_N^T(t)
\end{bmatrix}
\end{align*}
\]
so the networked interconnected systems can be rewritten as
\[
\begin{align*}
\dot{x}(t) &= \tilde{A} x(t) + \tilde{B} u(t) + \tilde{D} d(t) + \tilde{F} f(t) \\
y(t) &= \tilde{C} x(t)
\end{align*}
\]
where \( x(t) \in \mathbb{R}^{n_x \times N} \) represent the state vectors, \( u(t) \in \mathbb{R}^{n_u \times N} \) are the control input vectors, \( d(t) \in \mathbb{R}^{n_d \times N} \) are the ungiven noise, \( f(t) \in \mathbb{R}^{n_f \times N} \) are the failure of actuators and \( y(t) \in \mathbb{R}^{n_y \times N} \) are is
the output vectors of measurement. $n_x$, $n_u$, $n_d$, $n_f$, $n_y$ represent the dimension of each vector. $\bar{A}$, $\bar{B}$, $\bar{D}$, $\bar{F}$, $\bar{C}$ are given constant matrices with adequacy dimensions and

$$
\bar{A} = \begin{bmatrix}
A_1 & A_{21} & \cdots & A_{2N} \\
A_{12} & A_2 & \cdots & A_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
A_{N1} & A_{N2} & \cdots & A_N
\end{bmatrix}, \quad 
\bar{B} = \begin{bmatrix}
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_N
\end{bmatrix}, \quad 
\bar{C} = \begin{bmatrix}
C_1 & 0 & \cdots & 0 \\
0 & C_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_N
\end{bmatrix}, \quad 
\bar{D} = \begin{bmatrix}
D_1 & 0 & \cdots & 0 \\
0 & D_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_N
\end{bmatrix}, \quad 
\bar{F} = \begin{bmatrix}
F_1 & 0 & \cdots & 0 \\
0 & F_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & F_N
\end{bmatrix}
$$

The failure of actuators and ungiven disturbance are supposed to pertain a range with finite-frequency in this essay, and the finite-frequency range which might be uniformly described as

$$
\Pi := \{ \omega \in \mathbb{R} | (\omega - \omega_t)(\omega - \omega_h) \leq 0 \} \quad (3)
$$

when $\tau = 1$, $-\omega_t = \omega_t = \omega_h$, the $\Pi$ in (3) is amount to the low-frequency range, $|\omega| \leq \omega_t$, $\omega_t$ is bounded of the low-frequency range;
when $\tau = 1$, $\omega_t < \omega_t$, the $\Pi$ in (3) is amount to the middle-frequency range, $\omega_t < \omega < \omega_h$;
when $\tau = -1$, $-\omega_t = \omega_t = \omega_h$, the $\Pi$ in (3) is amount to the high-frequency range, $|\omega| \geq \omega_h$, $\omega_h$ is bounded of the high-frequency range.

Detecting the failure of actuator for system (2), a fault detection observer is supposed as

$$
\begin{align*}
\dot{x}(t) &= \bar{A}x(t) + \bar{B}u(t) + L(y(t) - \hat{y}(t)) \\
\hat{y}(t) &= \bar{C}x(t) \\
r(t) &= y(t) - \hat{y}(t)
\end{align*} \quad (4)
$$

so that the output residuals of the fault detection observer are sensitive to the fault and robust to the ungiven disturbance. Using $H_\infty$ norm performance of the system (2). Where $x(t) \in \mathbb{R}^{n_x}$ are vectors of the state estimation, $r(t) \in \mathbb{R}^{n_y}$ are the vectors of residuals used to detect fault, and the gain matrix $L \in \mathbb{R}^{(n_y \times n_x)}$ is needed to be designed.

Defined $e(t) = x(t) - \hat{x}(t)$, the estimation error dynamics system is obtained as

$$
\begin{align*}
\dot{e}(t) &= (\bar{A} - L\bar{C})e(t) + \bar{F}f(t) + \bar{D}d(t) \\
r(t) &= \bar{C}e(t)
\end{align*} \quad (5)
$$

The observer designed in this paper should satisfy the following requirements

- the system (5) is stable in state space;
- In the domain (3), the residuals $r(t)$ are robust to ungiven disturbance $d(t)$, the following $H_\infty$ norm performance holds

$$
\int_0^\infty r^T(t)r(t)dt \leq \gamma^2 \int_0^\infty d^T(t)d(t)dt \quad (6)
$$

- In the domain (3), the residuals $r(t)$ are sensing to failures of actuator $f(t)$, under zero-initial values, the $H_\infty$ fault sensitivity condition holds

$$
\int_0^\infty r^T(t)r(t)dt \geq \beta^2 \int_0^\infty f^T(t)f(t)dt \quad (7)
$$
and satisfying
\[ \int_0^\infty \tau \left( \alpha_j e(t) + j \dot{e}(t) \right) \left( \alpha_j e(t) + j \dot{e}(t) \right) dt \leq 0 \]  
(8)

where \( \alpha_j \), \( \alpha_k \) reflecting the frequency domain of disturbance and failure of actuator \( f(t) \), \( \beta \) is a known positive scalar to test the residuals sensitive, and \( \tau \) is +1 or −1, relying on the considered frequency range.

We will use the Lemma 1 in the following paper.

**Lemma 1.** [16] The finite-frequency domain (3) which we consider, if a symmetric positive definite matrices \( Q = Q^T \) is existed, so the following inequalities will be hold
\[ \int_0^\infty tr\left[ He\left(W\right)Q_2 \right] dt = \int_0^\infty tr\left[ He\left( \left( \alpha_j e(t) + j \dot{e}(t) \right) \left( \alpha_j e(t) + j \dot{e}(t) \right) \right)^T Q \right] dt \leq 0 \]
(9)

### 3. Main Results

#### 3.1. Design of Fault detection observer

In this part, we design an \( H_+ / H_- \) fault detection observer for system (1) in the finite-frequency domain to make the generated residuals \( r(t) \) sensitive to failure of actuator, meanwhile robust against disturbances.

The supposition that the disturbance \( d(t) \) and fault \( f(t) \) are supposed to pertain a finite-frequency range (3), the sufficient conditions of \( H_+ \) performance index (6) and \( H_- \) fault sensitivity condition (7) are given in the following Theorem.

**Theorem 1:** For the system (2), given a scalar \( \gamma > 0 \), if there exist symmetric positive definite matrices \( P_1 = P_1^T > 0 \in \mathbb{R}^{(n \times n)^\otimes(n \times n)} \) , \( P_2 = P_2^T > 0 \in \mathbb{R}^{(n \times n)^\otimes(n \times n)} \) , and symmetric matrices \( Q_d = Q_d^T > 0 \in \mathbb{R}^{(n \times n)^\otimes(n \times n)} \) , \( Q_f = Q_f^T > 0 \in \mathbb{R}^{(n \times n)^\otimes(n \times n)} \) satisfying \( \tau Q_d > 0 \), \( \tau Q_f > 0 \) such that the following inequalities hold
\[
\begin{bmatrix}
\lambda_{11} & \lambda_{12} \\
* & -\gamma^2 I - DQ_d \bar{D}
\end{bmatrix} < 0
\]
(10)
\[
\begin{bmatrix}
\pi_{11} & \pi_{12} \\
* & -\bar{F}Q_f \bar{F} + \beta^2 I
\end{bmatrix} < 0
\]
(11)
then the observer (4) is the \( H_+ / H_- \) observer of system (5), and the residuals of the observer output are sensitive to failure of actuator and robust against disturbances.

where
\[
\lambda_{11} = He\left( P_1 \left( \bar{A} - L\bar{C} \right) \right) - \alpha_1 \alpha_2 Q_d - j \alpha_2 Q_d\left( \bar{A} - L\bar{C} \right)^T Q_d - \left( \bar{A} - L\bar{C} \right)^T Q_d \left( \bar{A} - L\bar{C} \right) + \bar{C}^T C
\]
\[
\lambda_{12} = P_1 \bar{D} - \left( \bar{A} - L\bar{C} \right)^T Q_d \bar{D} - j \alpha_2 Q_d \bar{D}
\]
\[
\pi_{11} = He\left( P_2 \left( \bar{A} - L\bar{C} \right) \right) - \alpha_1 \alpha_2 Q_f - j \alpha_2 Q_f\left( \bar{A} - L\bar{C} \right)^T Q_f - \left( \bar{A} - L\bar{C} \right)^T Q_f \left( \bar{A} - L\bar{C} \right) - \bar{C}^T C
\]
\[
\pi_{12} = P_2 \bar{F} - \left( \bar{A} - L\bar{C} \right)^T Q_f \bar{F} - j \alpha_2 Q_f \bar{F}
\]
\[
\alpha_b = \frac{1}{2} (\alpha_1 + \alpha_2)
\]

So we can maximise the \( H_+ \) index \( \beta \) to find the observer, and then the optimisation problem is shown as follows
\[
\max \beta \\
\text{s.t. (10) and (11)}
\] (12)

**Proof:**

1) \(H_\infty\) norm performance condition

Because of the Lyapunov stability theory, putting \(f(t) = 0\) in (5), the system (5) becomes
\[
\begin{align*}
\dot{e}(t) &= (\bar{A} - L\bar{C})e(t) + \bar{D}d(t) \\
\dot{r}(t) &= \bar{C}e(t)
\end{align*}
\] (13)

The following Lyapunov function is used to prove the \(H_\infty\) norm performance condition
\[
V_i(t) = e^T(t)P_1e(t), \quad P_1 = P_1^T > 0
\] (14)

Next, \(V_i(t)\) along the system (13) is gotten as
\[
V_i(t) = e^T(t)H(e^T(t)P_1(\bar{A} - L\bar{C}))e(t) + 2e^T(t)P_1\bar{D}d(t)
\] (15)

In order to satisfy the \(H_\infty\) norm performance in the finite-frequency domain, namely
\[
\int_0^\infty [r^T(t)r(t) - \gamma^2d^T(t)d(t)]dt \leq 0
\] (16)

one rule is defined as
\[
J_\infty = \int_0^\infty r^T(t)r(t) - \gamma^2d^T(t)d(t) = -\text{tr}\left\{\frac{1}{2}He(W)Q_d\right\} + V_i(t)dt
\] (17)

where \(W = (\omega_2 e(t) + j\dot{e}(t))(\omega_2 e(t) + j\dot{e}(t))^*\), \(Q_d = Q_d^T\) is a symmetric matrix.

Under zero initial condition, because of \(V_i(t) > 0\), if \(J_\infty < 0\) is established, it can be deduced that
\[
\int_0^\infty r^T(t)r(t) - \gamma^2d^T(t)d(t)dt = -\text{tr}\left\{\frac{1}{2}He(W)Q_d\right\}dt < 0
\] (18)

It can be seen from Lemma 1
\[
\int_0^\infty \text{tr}\left\{\frac{1}{2}He(W)Q_d\right\}dt \leq 0
\] (19)

So, when (18) is established, it can be derived
\[
\int_0^\infty r^T(t)r(t) - \gamma^2d^T(t)d(t)dt < \int_0^\infty \text{tr}\left\{\frac{1}{2}He(W)Q_d\right\}dt \leq 0
\] (20)

which implies when \(J_\infty < 0\), the system (13) satisfies \(H_\infty\) norm performance condition (6).

According to the operational properties of trace operator, it can be obtained that
\[
\text{tr}\left\{\frac{1}{2}He\left[(\omega_2 e(t) + j\dot{e}(t))(\omega_2 e(t) + j\dot{e}(t))^*\right]Q_d\right\} = \text{tr}\left[\omega_2\omega_2 e(t)e(t)Q_d + \dot{e}(t)e(t)Q_d + j\omega_2\dot{e}(t)e(t)Q_d - j\omega_2 e(t)\dot{e}(t)Q_d\right]
\] (21)

where \(\omega_0 = \frac{1}{2}(\omega_1 + \omega_2)\).

By inserting (13) into (21), we can get
\[
tr \left\{ \frac{1}{2} He \left[ (\omega_1 e(t) + j\dot{e}(t))(\omega_2 e(t) + j\dot{e}(t))^\top \right] Q_p \right\} \\
= e^T(t) \omega_1 Q_p e(t) + e^T(t) (A - L\bar{C})^T Q_p (A - L\bar{C}) e(t) + 2e^T(t) (A - L\bar{C})^T Q_p \bar{D}(t) + d^T(t) \bar{D}^T Q_p \bar{D}(t) + \omega_2^T Q_p e(t) - d^T(t) \omega_2 \bar{D}^T Q_p e(t)
\]

(22)

Then, substituting (13), (15) and (22) into (18), \( J_\infty < 0 \) is equal to

\[
\left[ \begin{array}{c} e(t) \\ d(t) \end{array} \right]^\top \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
& \ddots \\
& & \lambda_\infty
\end{bmatrix} \left[ \begin{array}{c} e(t) \\ d(t) \end{array} \right] < 0
\]

(23)

where

\[
\lambda_1 = He\left\{ P_1 (A - L\bar{C}) - \omega_1 Q_p - j\omega_2 Q_p (A - L\bar{C}) + j\omega_1 (A - L\bar{C})^T Q_p (A - L\bar{C}) + \bar{C}^T \bar{C} \right\}
\]

\[
\lambda_2 = P_2 \bar{D} - (A - L\bar{C})^T Q_p \bar{D} - j\omega_2 Q_p \bar{D}
\]

\( \omega_\infty = \frac{1}{2}(\omega_1 + \omega_2) \)

2) \( H_\infty \) fault sensitivity condition

Because of the Lyapunov stability theory, putting \( d(t) = 0 \) in (5), the system (5) becomes

\[
\begin{align*}
\dot{e}(t) &= (A - L\bar{C}) e(t) + \bar{F} f(t) \\
r(t) &= \bar{C} e(t)
\end{align*}
\]

(24)

The following Lyapunov function is used to prove the \( H_\infty \) fault condition

\[
V_2(t) = e^T(t) P_2 e(t), \quad P_2 = P_2^T > 0
\]

(25)

Then, the \( V_2(t) \) along the system (24) is gotten as

\[
\dot{V}_2(t) = e^T(t) He\left\{ P_2 (A - L\bar{C}) \right\} e(t) + 2e^T(t) P_2 \bar{F} f(t)
\]

(26)

To meet the above conditions, namely

\[
\int_0^T \left[ \beta^2 f^T(t) f(t) - r^T(t) r(t) \right] dt \leq 0
\]

(27)

one rule is defined as

\[
J_\infty = \int_0^T \beta^2 f^T(t) f(t) - r^T(t) r(t) \left( -tr\left[ \frac{1}{2} He[W] Q_f \right] + \dot{V}_2(t) \right) dt
\]

(28)

where \( W = (\omega_1 e(t) + j\dot{e}(t))(\omega_1 e(t) + j\dot{e}(t))^\top, \quad Q_f = Q_f^T \) is a symmetric matrix.

Under zero initial condition, because of \( V_2(t) > 0 \), if \( J_\infty < 0 \) is established, it can be deduced that

\[
\int_0^T \beta^2 f^T(t) f(t) - r^T(t) r(t) -tr\left[ \frac{1}{2} He[W] Q_f \right] dt < 0
\]

(29)

It can also be seen from Lemma 1 and when (18) is established, it can be derived

\[
\int_0^T \beta^2 f^T(t) f(t) - r^T(t) r(t) \leq \int_0^T tr\left[ \frac{1}{2} He[W] Q_f \right] dt \leq 0
\]

(30)

which implies when \( J_\infty < 0 \), the system (24) satisfies \( H_\infty \) norm performance condition (7).

By inserting (24) into (21), we have
\[
\text{tr}\left\{ \frac{1}{2} He\left[ (\omega_1 e(t) + j \dot{e}(t))(\omega_2 e(t) + j \dot{e}(t))^T\right] Q_f \right\} \\
= e^T(t) \omega_1 \omega_2 Q_f e(t) + e^T(t) (\bar{A} - \bar{L}C)^T Q_f (\bar{A} - \bar{L}C) e(t) + 2e^T(t) (\bar{A} - \bar{L}C)^T Q_f \bar{F}_f(t) + f^T(t) \bar{F}_f(t) + f^T(t) \bar{F}_f(t) e(t)
\]
\[
e^T(t) j \omega_2 Q_f (\bar{A} - \bar{L}C) e(t) + e^T(t) j \omega_2 Q_f \bar{F}_f(t) - e^T(t) j \omega_2 (\bar{A} - \bar{L}C)^T Q_f e(t) - f^T(t) j \omega_2 \bar{F}_f(t) e(t)
\]
\[
\text{(31)}
\]

Then, substituting (24), (26) and (31) into (29), \( J < 0 \) is equal to
\[
\begin{bmatrix}
e(t) \\
f(t)
\end{bmatrix}^T
\begin{bmatrix}
\pi_{11} & \pi_{12} \\
\pi_{12}^* & -\bar{F}_f + \beta^2 I
\end{bmatrix}
\begin{bmatrix}
e(t) \\
f(t)
\end{bmatrix} < 0
\]
\[
\text{(32)}
\]

where
\[
\pi_{11} = He \left\{ P_2 (\bar{A} - \bar{L}C)^T Q_f (\bar{A} - \bar{L}C) + j \omega_2 (\bar{A} - \bar{L}C)^T Q_f (\bar{A} - \bar{L}C) - \bar{C}^T \bar{C} \right\}
\]
\[
\pi_{12} = P_2 \bar{F} - (\bar{A} - \bar{L}C)^T Q_f \bar{F} - j \omega_2 Q_f \bar{F}
\]
\[
\alpha_0 = \frac{1}{2} (\alpha_1 + \alpha_2)
\]

The proof is over.

Remark 1: Theorem 1 does not consider the stability condition of the state estimation error dynamics system, and the stability condition will be given in Theorem 2.

Remark 2: The inequality (10) (11) in Theorem 1 is not a LMI which is difficult to solve. So, we will facilitate \( H_\infty \) observer design in the following theorem.

Theorem 2: For the system (2), given a scalar \( \gamma > 0 \), \( \alpha_0, \alpha_1, \alpha_2, \xi_0 \) and matrix \( V_f \in \mathbb{R}^{(n_N)(n_N)} \), \( V_f \in \mathbb{R}^{(n_N)(n_N)} \), matrix \( P_0 \in \mathbb{R}^{(n_N)(n_N)} \), \( P_0 = P_0^T > 0 \in \mathbb{R}^{(n_N)(n_N)} \), \( P_1 = P_1^T > 0 \in \mathbb{R}^{(n_N)(n_N)} \), \( P_2 = P_2^T > 0 \in \mathbb{R}^{(n_N)(n_N)} \) are exist, and symmetric matrices \( Q_d = Q_d^T > 0 \in \mathbb{R}^{(n_N)(n_N)} \), \( Q_f = Q_f^T > 0 \in \mathbb{R}^{(n_N)(n_N)} \) satisfying \( \tau Q_d > 0 \), \( \tau Q_f > 0 \), \( M \in \mathbb{R}^{(n_N)(n_N)} \), \( W \in \mathbb{R}^{(n_N)(n_N)} \) and such that the following inequalities hold
\[
\begin{bmatrix}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
* & \lambda_{22} & V_d^T M + \alpha_1 \bar{D}^T M^T \\
* & * & Q_d - \alpha_1 M - \alpha_2 M^T
\end{bmatrix} < 0
\]
\[
\text{(33)}
\]
\[
\begin{bmatrix}
\pi_{11} & \pi_{12} & \pi_{13} \\
* & \pi_{22} & V_f^T M + \alpha_2 \bar{F}^T M^T \\
* & * & Q_f - \alpha_2 M - \alpha_3 M^T
\end{bmatrix} < 0
\]
\[
\text{(34)}
\]
\[
\begin{bmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} \\
* & \Delta_{22} & \alpha_1 \bar{D} \bar{M} \bar{M}^T \\
* & * & \alpha_2 \bar{D} \bar{M} \bar{M}^T
\end{bmatrix} < 0
\]
\[
\text{(35)}
\]

then the observer (4) is the \( H_\infty \) observer of system (5), and the observer residuals are sensitive to failure meanwhile robust against disturbances and in a given range, where
\[ \lambda_1 = \text{He}(M\bar{A} - W\bar{C}) - \omega_1 \omega_2 Q_x + \bar{C}^T \bar{C} \quad \lambda_2 = M\bar{D} + \left( \bar{A}^T M^T - \bar{C}^T W^T \right) V_d \]
\[ \lambda_{13} = P_1 - M - j\omega_0 Q_x + \alpha_1 \left( \bar{A}^T M^T - \bar{C}^T W^T \right) \quad \lambda_{22} = -\gamma' I + V_j^T M\bar{D} + \bar{D}^T M^T V_d \]
\[ \pi_{11} = \text{He}(M\bar{A} - W\bar{C}) - \omega_1 \omega_2 Q_x + \bar{C}^T \bar{C} \quad \pi_{12} = M\bar{F} + \left( \bar{A}^T M^T - \bar{C}^T W^T \right) V_f \]
\[ \pi_{13} = P_1 - M - j\omega_0 Q_x + \alpha_1 \left( \bar{A}^T M^T - \bar{C}^T W^T \right) \quad \pi_{22} = \beta^2 I + V_j^T M\bar{F} + \bar{F}^T M^T V_f \]
\[ \Delta_{11} = \text{He} \left( \xi_1 (M\bar{A} - W\bar{C}) \right) \quad \Delta_{12} = P_0 - \xi_0 M - j\omega_0 Q_x + \alpha_0 \left( \bar{A}^T M^T - \bar{C}^T W^T \right) \]
\[ \omega_0 = \frac{1}{2}(\omega_1 + \omega_2) \]

So we can maximise the \( H_\infty \) index \( \beta \) to find the observer, and then the optimisation problem is shown as follows

\[
\max \beta \\
\text{s.t. (33), (34) and (35)}
\]

Then, we can solve the matrix \( L \) of observer gain in system (4) by \( L = M^{-1} W \).

**Proof:**

Firstly, formula (10) can be rewritten by

\[
M_1 = \begin{bmatrix}
-\omega_1 \omega_2 Q_x + \bar{C}^T \bar{C} & 0 \\
0 & -\gamma^2 I
\end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix}
\bar{A} - L\bar{C} & \bar{D}
\end{bmatrix}, \quad P_1 = \begin{bmatrix}
P_1 - j\omega_0 Q_x
\end{bmatrix}
\]

Inequality (37) is equivalent to

\[
\begin{bmatrix}
M_1 + \delta_1 \bar{A}_1 + \bar{A}^T \delta_1^T & -\delta_1 + \bar{P}_1 + \bar{A}^T \bar{G}^T
\end{bmatrix} < 0
\]

We can obtain the above inequality by pre- and post-multiplying (46) with \( \begin{bmatrix} I & \bar{A}_1^T \end{bmatrix} \) and its transpose, respectively. We choose

\[
\delta_1 = \begin{bmatrix}
M \\
V_j^T M
\end{bmatrix}, \quad G_d = \alpha_a M, \quad W = ML
\]

By inserting (40) to (39), the inequality can be obtained in (33).

Similarly, formula (11) can also be rewritten by

\[
M_2 = \begin{bmatrix}
-\omega_1 \omega_2 Q_x - \bar{C}^T \bar{C} & 0 \\
0 & \beta^2 I
\end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix}
\bar{A} - L\bar{C} & F
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
P_2 - j\omega_0 Q_x
\end{bmatrix}
\]

Inequality (41) is equivalent to

\[
\begin{bmatrix}
M_2 + \delta_2 \bar{A}_2 + \bar{A}_2^T \delta_2^T & -\delta_2 + \bar{P}_2 + \bar{A}^T \bar{G}^T
\end{bmatrix} < 0
\]

We choose

\[
\delta_2 = \begin{bmatrix}
M \\
V_j^T M
\end{bmatrix}, \quad G_j = \alpha_a M, \quad W = ML
\]

By inserting (40) to (39), the inequality can be obtained in (34).

The above design of \( H_\infty \) fault detection observer does not consider the stability conditions, the stability analysis will be proved below.
For the system (5), the stability condition is if there exist symmetric positive definite matrices $P_0 = P_0^T > 0 \in \mathbb{R}^{(n+N)^2 \times (n+N)}$ satisfies

$$P_0 (\bar{A} - LC)^T + (\bar{A} - LC)^T P_0 < 0$$

(45)

It is equivalent to

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ * & -G_0^T - G_0 \end{bmatrix} < 0$$

(46)

where

$$\Delta_{11} = (\bar{A} - LC)^T \delta_0^T + \delta_0 (\bar{A} - LC) \Delta_{12} = P_0 - \delta_0 + (\bar{A} - LC)^T G_0^T$$

(47)

we choose

$$\delta_0 = \xi_\theta M \cdot G_0 = \alpha_\theta M$$

(48)

By inserting (48) to (47), the inequality (35) is obtained.

The proof is over.

3.2. Threshold Selection

Due to the influence of external disturbance and model uncertainty, the residuals generated by fault detection observer are not zero even under fault-free conditions. Therefore, it is necessary to use appropriate threshold for residuals evaluation to avoid false alarms. In this paper, the root mean square (RMS) value of residuals signal is selected for threshold calculation to carry out residuals evaluation.

We define the residuals signal’s RMS as

$$J(t) = \left\| r(t) \right\|_{RMS} = \sqrt{\frac{1}{\Delta T} \int_{-\Delta T}^{T} r^T(\tau) r(\tau) d\tau}$$

(49)

In order to obtain the appropriate threshold, the $H_\infty$ performance index of formula (6) is approximately

$$\left\| r(t) \right\|_{RMS} \leq \gamma \left\| d(t) \right\|_{RMS}$$

(50)

where $\left\| d(t) \right\|_{RMS}$ represents the disturbance’s average energy of the chosen time window. Assumed we choose the upper bound of $\left\| d(t) \right\|_{RMS}$ as

$$\bar{d}_{RMS} = \sup_{r \in \Delta T} \left\| d(t) \right\|_{RMS}$$

(51)

So the threshold of the fault detection can be chosen as

$$J_{th} = \gamma \bar{d}_{RMS}$$

(52)

Then the following fault decision logic can be proposed

$$\begin{cases} J(t) > J_{th} \Rightarrow \text{Faulty} \\ J(t) \leq J_{th} \Rightarrow \text{Free} - \text{Fault} \end{cases}$$

(53)

4. Simulation Results

This section uses the example model in [17] to prove the correctness of the above mean. The example model is

$$\begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + D_i d_i(t) + F_i f_i(t) + \sum_{j \neq i}^{N} A_{ij} x_j(t) \\ y_i(t) = C_i x_i(t) \quad i = 1, 2, \cdots, N \end{cases}$$

(54)

where
\begin{equation}
A_1 = \begin{bmatrix}
-1 & 3 & 0 \\
-1 & -2 & 1 \\
0 & 1 & -7
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
0.01
\end{bmatrix}, \quad A_{12} = \begin{bmatrix}
0.1 & 0 & 0 \\
0.1 & 0.2 & 0.1 \\
0 & 0.1 & 0
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad F_1 = B_1;
\end{equation}

\begin{equation}
A_2 = \begin{bmatrix}
-3 & 2 & 0 \\
1 & -2 & 1 \\
-1 & -1 & -5
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad D_2 = \begin{bmatrix}
0.01
\end{bmatrix}, \quad A_{21} = \begin{bmatrix}
0 & 0.3 & 0 \\
0 & 0.1 & 0 \\
0.2 & 0 & 0.1
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad F_2 = B_2.
\end{equation}

From the formula (2), we can obtain
\begin{equation}
\bar{A} = \begin{bmatrix}
-1 & 3 & 0 & 0.1 & 0 & 0 \\
-1 & -2 & 1 & 0.1 & 0.2 & 0.1 \\
0 & 1 & -7 & 0 & 0.1 & 0 \\
0 & 0.3 & 0 & -3 & 2 & 0 \\
0 & 0.1 & 0 & 1 & -2 & 1 \\
0 & 0.1 & 0 & -1 & -1 & -5
\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \bar{C} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \bar{D} = \begin{bmatrix}
0.01 & 0 & 0.01 & 0 \\
0.01 & 0 & 0.01 & 0 \\
0.01 & 0 & 0.01 & 0 \\
0.01 & 0 & 0.01 & 0 \\
0.01 & 0 & 0.01 & 0 \\
0.01 & 0 & 0.01 & 0
\end{bmatrix},
\end{equation}

\begin{equation}
F = \bar{B}.
\end{equation}

In simulation, the ungiven disturbance and failure of actuator are assumed to in the finite-frequency range [-0.1, 0.1]. We choose \(\alpha_0 = 1\), \(\alpha_1 = 4\), \(\alpha_2 = 0.5\), \(\alpha_3 = 1\) and matrix \(V_f = -7F\), \(V_\gamma = -7D\), \(\gamma = 0.3017\). By solving the optimisation problem (36), the maximum \(H_\infty\) index is determined as \(\beta = 5.52\), and the observer gain matrix \(L\) is calculated as
\begin{equation}
L = \begin{bmatrix}
-0.7428 & 2.9986 & 0.1000 & -6.7146 \\
-0.9915 & -1.7317 & 0.1010 & 0.2010 \\
-0.0646 & 0.9245 & -6.5609 & -0.0998 \\
2.4594 & 0.2999 & -2.7428 & 1.9984 \\
1.7854 & 0.1000 & 1.0088 & -1.7329 \\
0.2009 & 0.0013 & -1.0470 & -1.0454
\end{bmatrix}
\end{equation}

Assuming that the external disturbance signal is a random vector bound by [-0.1, 0.1], we can obtain \(\Delta_{ext} = 0.4080\). and based on (52), we can get the threshold of fault detection \(J_\alpha = 0.1231\). Suppose the initial state of subsystem 1 is \(x_1(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T\), and the initial state of subsystem 2 is \(x_2(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T\). To prove the effectiveness of the mean, the simulations are performed on the abrupt fault and the slow time-varying fault, respectively, and the comparison experiment is carried out with the \(H_\infty\) fault detection observer. In the first scene, we assume the abrupt fault is
\begin{equation}
f(t) = \begin{cases}
f_{11} = 0, & 0 < t \leq 20s \\
f_{12} = 0.5, & 20s < t \leq 160s \\
0, & 160s < t \leq 200s
\end{cases}
\end{equation}

In the first scene, we assume the time-varying fault is
The Figure 1-8 show results of simulation. In all figures, black dotted line is the fault detection threshold, and the red solid line denotes the RMS value of the residuals. Figures 1-4 shows the dynamic change of the RMS value of the residuals when the abrupt fault occurs in the networked interconnected system. The subsystem 1 is shown in Figure 1 and Figure 2 and Figure 3 and Figure 4 show the subsystem 2. We can get the information from Figure 2 that the faults are detected successfully by proposed means while the $H_{\infty}$ observer is failed. And according to the simulation image, it can also be seen that the fault type is an abrupt fault.

$$f(t)=\begin{cases} 0 & 0<t \leq 30s \\ 0.4 + 0.1\sin (0.1\pi (t-10)) & 30s<t \leq 150s \\ 0 & 150s<t \leq 200s \end{cases}$$

(56)

Figures 5-8 shows the dynamic change of the RMS value of the residuals when the time-varying fault happens in the networked interconnected system. The subsystem 1 is shown in Figure 5 and Figure 6 and Figure 7 and Figure 8 show the subsystem 2. We can get the information from Figure 7 that the faults are detected successfully by proposed means while the $H_{\infty}$ observer is failed. And according to the simulation image, it can also be seen that the fault type is a time-varying fault.
5. Conclusion
In the essay, we propose an $H_\infty / H_\infty$ fault detection observer mean for the networked interconnected system. We propose $H_\infty$ norm performance and $H_\infty$ fault sensitivity condition in the finite-frequency to make the residuals which generated by observer sensitive to fault and robust against disturbance., respectively. The given conditions were transformed into linear matrix inequality. Finally, simulation examples prove the correctness of the mean.

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