Integrated organic inference (IOI): A reconciliation of statistical paradigms

Russell J. Bowater

Independent researcher, Sartre 47, Acatlima, Huajuapan de León, Oaxaca, C.P. 69004, Mexico. Email address: as given on arXiv.org. Twitter profile: @naked_statist
Personal website: [sites.google.com/site/bowaterfospage](sites.google.com/site/bowaterfospage)

Abstract: It is recognised that the Bayesian approach to inference can not adequately cope with all the types of pre-data beliefs about population quantities of interest that are commonly held in practice. In particular, it generally encounters difficulty when there is a lack of such beliefs over some or all the parameters of a model, or within certain partitions of the parameter space concerned. To address this issue, a fairly comprehensive theory of inference is put forward called integrated organic inference that is based on a fusion of Fisherian and Bayesian reasoning. Depending on the pre-data knowledge that is held about any given model parameter, inferences are made about the parameter conditional on all other parameters using one of three methods of inference, namely organic fiducial inference, bispatial inference and Bayesian inference. The full conditional post-data densities that result from doing this are then combined using a framework that allows a joint post-data density for all the parameters to be sensibly formed without requiring these full conditional densities to be compatible. Various examples of the application of this theory are presented. Finally, the theory is defended against possible criticisms partially in terms of what was previously defined as generalised subjective probability.

Keywords: Bayesian; bispatial inference; Fisherian; Gibbs sampler; incompatible conditional densities; Objective and subjective probability; organic fiducial inference; P values.
1. Introduction

The general problem of making inferences about a population on the basis of a small random sample from that population has long been of great interest to scientific researchers. This problem is often addressed by making the assumption that, in the population, the distribution of the measurements being considered is a member of a given parametric family of distributions. Although this assumption can be criticised, we will choose in this paper to examine problems of inference that are constrained by this assumption. Our justification for this is that, first, this class of problems has substantial importance in its own right, and second, resolving such problems can be viewed as a convenient first step towards tackling cases in which making such an assumption is not appropriate. Therefore, let us suppose that the data set to be analysed \( x = \{x_1, x_2, \ldots, x_n\} \) was drawn from a joint density or mass function \( g(x \mid \theta) \) that depends on a set of parameters \( \theta = \{\theta_i : i = 1, 2, \ldots, k\} \), where each \( \theta_i \) is a one-dimensional variable.

A way of classifying the nature of the problem that is encountered in trying to make inferences about the set of parameters \( \theta \) is to do so on the basis of the type of knowledge that was held about these parameters before the data were observed. In this respect, it can be argued that the three most common types of pre-data opinion that, in practice, are naturally held about any given model parameter \( \theta_j \) conditional on all other parameters \( \theta_{-j} = \{\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k\} \) being known are as follows:

1) Nothing or very little is known about the parameter.

2) It is felt that the parameter may well be close to a specific value, which may for example indicate the absence of a treatment effect, or the lack of a correlation between variables, but apart from this nothing or very little is known about the parameter. Some examples of where it would be reasonable to hold this type of pre-data opinion were given in Bowater (2019b).
3) We know enough about the parameter for our opinion about it to be satisfactorily represented by a probability distribution function over the parameter.

Therefore, each of these types of pre-data opinion about the parameter $\theta_j$ will be treated as corresponding to a distinct problem of inference. Nevertheless, since our pre-data opinions about each of the parameters in any given set of parameters $\theta$ may well fall into different categories among the three being considered, it may be necessary to address two or all three of these types of problem in any particular scenario.

These problems are the three problems of inference that will be of principal interest in what follows. More specifically, the aim of the present paper will be to show how these problems can be dealt with in a harmonious manner by using an approach to inference based on a fusion of Fisherian (as attributed to R. A. Fisher) and Bayesian reasoning. Of course, given the obvious incompatibilities that exist between, and to some extent even within, these two schools of reasoning, we will need to be given some liberty in how each of these approaches to inference is interpreted.

In this respect, although the theory that will be outlined is based on a type of probability that is inherently subjective, and therefore not frequentist as in the Fisherian paradigm, it is not the same type of probability that is commonly regarded as underlying subjective Bayesian theory. Instead, it is a generalised form of subjective probability that effectively allows probability distributions to be distinguished according to where they are on a scale that goes from them being virtually objective to them being extremely subjective. This type of probability was referred to as generalised subjective probability in Bowater and Guzmán (2018b). Furthermore, the theory to be presented relies on various concepts that are heavily used by frequentist statisticians, e.g. sufficient and ancillary statistics, point estimators and their distributions, the classical notion of significance, and also one very important idea that during his own lifetime was chiefly advocated by Fisher himself, namely the fiducial argument. We are not suggesting, though, that the
proposed methodology should be judged positively simply because it represents a compromise between competing schools of inference, rather we recommend, quite naturally, that it should be evaluated on the basis of its effectiveness in dealing with the particular inferential challenge that has been set out.

To give a little more detail, each of the three aforementioned problems of inference will be addressed using a method that is specific to the problem concerned, and although this results in the use of three methods that are of a clearly different nature, these methods are nevertheless compatible with the overall framework of inference that will be put forward. In particular, the first type of problem will be tackled using what, in Bowater (2019a), was called organic fiducial inference. On the other hand, the second problem will be addressed using what, in Bowater (2019b), was called bispatial inference. Finally, the third problem will be dealt with using Bayesian inference. The overall framework just referred to provides a way of coordinating these distinct methods of inference so that it is possible to simultaneously make inferences about all of the parameters in the model.

Let us now briefly describe the structure of the paper. In the next five sections, we will present summaries of the fundamental concepts and methods that form the basis of the general theory in question, which will be called integrated organic inference. In particular, in the next two sections we will summarise the theory of generalised subjective probability and the overall framework of integrated organic inference. Furthermore, after clarifying in Section 2.3 the interpretation that will be adopted in this paper of the Bayesian approach to inference, concise accounts of the methods of organic fiducial inference and bispatial inference will be given in Sections 2.4 and 2.5. Various examples of the application of integrated organic inference will then be outlined in detail in Sections 3.1 to 3.5. In the final section of the paper (Section 4), a discussion of this theory of inference will be presented in the form of answers to questions that would be expected to naturally arise about the theory when it is first evaluated.
The theory will be referred to as integrated organic inference (IOI) because it integrates what are often considered to be conflicting approaches to inference into an overall framework that relies, in general, on what can be viewed as being an organic simulation algorithm. Furthermore, the type of inferences that this theory facilitates may, depending on the circumstances, be regarded as being objective or very subjective, but are nevertheless always organic, in the sense that they are intended to be only really understood by living subjects, e.g. humans, rather than primitive robots.

2. Fundamental concepts and methods

2.1. Generalised subjective probability

Overview

Under this definition of probability, a probability distribution is defined by its (cumulative) distribution function and the strength of this function relative to other distribution functions of interest. The distribution function is defined as having the standard mathematical properties of such a function. Let us now briefly outline the notion of the strength of a distribution function and some of the concepts that underlie this notion. Further details and examples of these concepts, and of the notion of strength itself, can be found in Bowater and Guzmán (2018b).

Similarity

As in the aforementioned paper, let \( S(A, B) \) denote the similarity that a given individual feels there is between his confidence (or conviction) that an event \( A \) will occur and his confidence (or conviction) that an event \( B \) will occur. For any three events \( A, B \) and \( C \), it is assumed that an individual is capable of deciding whether or not the orderings \( S(A, B) > S(A, C) \) and \( S(A, B) < S(A, C) \) are applicable. The notation
\( S(A, B) = S(A, C) \) is used to represent the case where neither of these orderings apply.

**Reference set of events**

Let \( O = \{O_1, O_2, \ldots, O_m\} \) be a finite ordered set of \( m \) mutually exclusive and exhaustive events. Now, if for any given three subsets \( O(1), O(2) \) and \( O(3) \) of the set \( O \) that contain the same number of events, the following is true:

\[
S \left( \bigcup_{i \in O(1)} O_i, \bigcup_{i \in O(2)} O_i \right) = S \left( \bigcup_{i \in O(1)} O_i, \bigcup_{i \in O(3)} O_i \right)
\]

then a possible reference set of events \( R \) is defined by

\[
R = \{R(\lambda) : \lambda \in \Lambda\}
\]

where \( R(\lambda) = O_1 \cup O_2 \cup \cdots \cup O_{\lambda m} \) and \( \Lambda = \{1/m, 2/m, \ldots, (m-1)/m\} \). For example, it should be clear that any given individual could easily decide that the set of all the outcomes of randomly drawing a ball out of an urn containing \( m \) distinctly labelled balls could be the set \( O \).

Equation (1) gives the definition of a reference set of events assuming that this set is discrete. For the definition of a continuous reference set of events, see Bowater and Guzmán (2018b).

**External strength of a distribution function**

Let two continuous random variables \( X \) and \( Y \) of possibly different dimensions have elicited or given distribution functions \( F_X(x) \) and \( G_Y(y) \) respectively. Also, we specify the set of events \( \mathcal{F}[a] \) by

\[
\mathcal{F}[a] = \left\{ \{X \in \mathcal{A}\} : \int_{\mathcal{A}} f_X(x) \, dx = a \right\}
\]

where \( \{X \in \mathcal{A}\} \) is the event that \( X \) lies in the set \( \mathcal{A} \) and \( f_X(x) \) is the density function corresponding to \( F_X(x) \), and we specify the set \( \mathcal{G}[a] \) in the same way with respect to the variable \( Y \) instead of \( X \) and the distribution function \( G_Y(y) \) instead of \( F_X(x) \).
Using this notation, we now define that, for a given discrete or continuous reference set of events $R$ that are independent of $X$ and $Y$, the function $F_X(x)$ is externally stronger than the function $G_Y(y)$ at the resolution $\lambda$, where $\lambda \in \Lambda$, if
\[
S_F = \min_{A \in \mathcal{F}[\lambda]} S(A, R(\lambda)) > \max_{A \in \mathcal{G}[\lambda]} S(A, R(\lambda)) = S_G
\]
An interpretation that could be given to this definition is that, if a specific individual judges a function $F_X(x)$ as being externally stronger than a function $G_Y(y)$ then, relative to the reference event $R(\lambda)$, the function $F_X(x)$ could be regarded as representing his uncertainty about the variable $X$ better than $G_Y(y)$ represents his uncertainty about the variable $Y$.

A definition of the internal rather than external strength of a distribution function, and other definitions of the external strength of a distribution function that are applicable to discrete distribution functions and to distribution functions derived by a formal system of reasoning, e.g. derived by applying the standard rules of probability, can be found in Bowater and Guzmán (2018b).

### 2.2. Overall framework of the theory

**Brief outline**

The general aim of the theory to be presented is to construct a joint distribution function of all the model parameters $\theta$ that accurately represents what is known about these parameters after the data have been observed, i.e. it is a post-data distribution function of these parameters. It will be assumed that this is done by first forming the set of full conditional post-data density functions of the parameters $\theta$, i.e. the set of density functions
\[
p(\theta_j | \theta_{-j}, x) \quad \text{for } j = 1, 2, \ldots, k
\]
One of the key features of the approach that will be developed is that it allows any given
one of these density functions to be constructed on the basis of whichever one of the
three distinct methods of inference mentioned in the Introduction is regarded as being
the most appropriate for the task.

More specifically, we will assume that, during the process of determining each of the
full conditional densities in question, the quite natural rule is followed of always treating
the set of conditioning parameters $\theta_{-j}$ as being known constants. However, although by
applying this rule we remove what would otherwise be an important source of conflict
between the three methods of inference being referred to, we do not, in general, eliminate
the possibility that the set of full conditional densities in equation (2) may be determined
using these methods in a way that implies that they are not consistent with any joint
density function of the parameters concerned, i.e. these conditional densities may be
incompatible among themselves. On the other hand, if the conditional densities under
discussion are indeed compatible then, since, under a mild requirement, a joint density
function is uniquely defined by its full conditional densities, these densities will, in general,
define a unique joint post-data density function for the parameters $\theta$.

Therefore, it would be helpful to know the cases in which the conditional densities in
equation (2) are compatible and when they are not, and if they are indeed incompatible,
whether and how the difficulty that this leads to can be addressed. In this regard, we
will propose two different strategies. The first strategy is to establish whether the full
conditional densities in question are compatible using analytical methods. By contrast,
the second strategy is to assume that these conditional densities are incompatible even
when they may not be, and use a computational method to try to find the joint density
function for all the parameters $\theta$ that has full conditional densities that most closely
approximate the densities in equation (2). We now will discuss each of these strategies
in a bit more detail.
Verifying the compatibility of full conditional distributions

Various analytical methods have been proposed for establishing the compatibility of full conditional distribution functions in a general context, see for example Arnold and Press (1989), Arnold, Castillo and Sarabia (2002) and Kuo and Wang (2011). Nevertheless, these methods can largely only be applied to cases where the variables over which these distribution functions are defined can only take a finite number of different values, or where there are only two such variables. There are, though, two such methods that at least potentially are more widely applicable. Therefore, we now will take a look at these two methods.

The first method we will consider is a simple one. In particular, we begin by proposing an analytical expression for the joint density function of the set of parameters $\theta$, then we determine the full conditional density functions for this joint density, and finally we see whether these conditional densities are equivalent to the full conditional densities in equation (2). If this equivalence is achieved, then these latter conditional densities clearly must be compatible. This method has the advantage that, in such circumstances, it directly gives us, under a mild condition, an analytical expression for the unique joint post-data density of the parameters $\theta$, i.e. under this condition, it will be the originally proposed joint density for these parameters.

The second method that we will consider for verifying the compatibility of the set of full conditional densities of interest depends on studying the behaviour of a Gibbs sampling algorithm (Geman and Geman 1984, Gelfand and Smith 1990) that makes transitions on the basis of this set of conditional densities. In particular, let us define a single transition of this type of algorithm as being one that results from randomly drawing a value from each of the full conditional densities in equation (2) according to some given fixed order of these densities, which we will call a fixed scanning order, replacing each time the previous value of the parameter concerned by the value that is generated. To clarify, it
is being assumed that only the set of values for the parameters $\theta$ that are obtained on completing a transition of this kind are recorded as being a newly generated sample, i.e. the intermediate sets of parameter values that are used in the process of making such a transition do not form part of the output of the algorithm.

On the basis of the results in Chen and Ip (2015), it can be deduced that the full conditional densities in equation (2) will be compatible if the Gibbs sampling algorithm just described satisfies the following three conditions:

1) It is positive recurrent for all possible fixed scanning orders. This condition ensures that the sampling algorithm has at least one stationary distribution for any given fixed scanning order.

2) It is irreducible and aperiodic for all possible fixed scanning orders. Together with condition (1), this condition ensures that the sampling algorithm has a limiting distribution for any given fixed scanning order.

3) Given conditions (1) and (2) hold, the limiting density function of the sampling algorithm needs to be the same over all possible fixed scanning orders.

Moreover, when these conditions hold, the joint post-data density function of the parameters $\theta$ implied by the full conditional densities under discussion will be the unique limiting density function of these parameters referred to in condition (3). The sufficiency of the conditions (1) to (3) just listed for establishing the compatibility of any given set of full conditional densities was proved for a special case in Chen and Ip (2015), which is a proof that can be easily extended to the more general case that is currently of interest.

In the context of the full conditional densities in equation (2) being determined using integrated organic inference, let us briefly comment on how easy it is likely to be, in practice, to establish whether or not these conditional densities satisfy each of the three conditions in question. First, it would not be expected to be that difficult, in this context, to determine whether or not condition (1) is satisfied, since a failure of this
condition to hold would only be expected to occur in very pathological cases. Also, the fulfilment of condition (2) will usually be easy to verify through an inspection of the full conditional densities concerned. On the other hand, in the context of interest, it will usually be impossible to determine whether or not condition (3) is satisfied. Despite this substantial drawback, we will nevertheless consider again the strategy that has just been outlined in the next subsection.

**Finding compatible approximations to incompatible full conditionals**

In any given situation where it is not easy to establish whether or not the full conditional densities in equation (2) are compatible, let us imagine that we make the pessimistic assumption that they are in fact incompatible. Nevertheless, even though these conditional densities could be incompatible, they may be considered as representing the best information that is available for constructing a joint post-data density function for the parameters $\theta$, i.e. the density $p(\theta | x)$. If this is the case, it would therefore seem appropriate to try to find the form for this joint density that has full conditional densities that most closely approximate those given in equation (2).

Various methods have been proposed for doing this when the random variables concerned can only take a finite number of different values, which means of course that we need to refer to the probability mass rather than density functions of these variables, see for example Arnold, Castillo and Sarabia (2002), Chen, Ip and Wang (2011), Chen and Ip (2015) and Kuo, Song and Jiang (2017). Similar to what was done earlier, here we will again focus attention on a more widely applicable method, in particular the method that simply consists in making the assumption that the joint density of the parameters $\theta$ that most closely corresponds to the set of full conditional densities in equation (2) is equal to the limiting density function of a Gibbs sampling algorithm that is based on these conditional densities with some given fixed or random scanning order of the parameters concerned. This approach relates to more specific methods for addressing the general
problem of interest that were discussed in Chen, Ip and Wang (2011) and Muré (2019).

To clarify, a transition of the type of Gibbs sampler being considered under a random
scanning order will be defined as being one that results from generating a value from one
of the conditional densities in equation (2) that is chosen at random, with the probability
of any given density \( p(\theta_j | \theta_{-j}, x) \) being selected being set equal to some given value \( a_j \),
where of course \( \sum_{i=1}^{k} a_i = 1 \), and then treating the generated value as the updated value
of the parameter concerned.

To measure how close the full conditional densities of the limiting density function
of this Gibbs sampler are to the full conditional densities in equation (2), we can use
a variation on the line of reasoning that underlies the second method for verifying the
compatibility of full conditional densities that was outlined in the last subsection. In
particular, assuming that Condition 1 (positive recurrence condition) and Condition 2
(irreducibility and aperiodicity condition) of this method are satisfied, it would appear
to be useful (with reference to Condition 3 of this method) to analyse how the limiting
density function of the Gibbs sampler under discussion varies over a reasonable number
of very distinct fixed scanning orders of the sampler. If within such an analysis, the
variation of this limiting density with respect to the scanning order of the parameters \( \theta \)
can be classified as small, negligible or undetectable, then this should give us reassurance
that the full conditional densities in equation (2) are, respectively according to such
classifications, close, very close or at least very close, to the full conditional densities of
the limiting density of a Gibbs sampler of the type that is of main interest, i.e. a Gibbs
sampler that is based on any given fixed or random scanning order of the parameters
concerned.

In trying to choose the scanning order of this type of Gibbs sampler such that it has
a limiting density function that corresponds to a set of full conditional densities that
most accurately approximate the densities in equation (2), a good general choice would
arguably be the random scanning order of the parameters \( \theta \) that was defined earlier with the selection probability of any given parameter, i.e. the probability \( a_j \), being set equal to \( 1/k \) for all \( j \), which is what we will call a uniform random scanning order. In a loosely similar context, Muré (2019) recommends, and moreover provides analytical results to support, the use of such an approach to address the issue in question.

However, it can be easily shown that independent of whether or not the set of full conditional densities in equation (2) are compatible, the last full conditional density in this set that is sampled from in completing a given fixed scanning order will be one of the full conditional densities of the limiting density function of the type of Gibbs sampler being considered that uses such a fixed scanning order. This therefore provides a reason for perhaps deciding, in certain applications, that the limiting density of this Gibbs sampler most satisfactorily corresponds to the full conditional densities in equation (2) when a given fixed rather than a uniform random scanning order of the parameters \( \theta \) is used.

As with all Gibbs samplers it is important to verify in implementing any of the aforementioned strategies that the sampler concerned has converged to its limiting density function within the restricted number of transitions of the sampler that can be observed in practice. To do this we can make use of standard methods for analysing the convergence of Monte Carlo Markov chains described in, for example, Gelman and Rubin (1992) and Brooks and Roberts (1998). However, the use of such convergence diagnostics may be considered to be slightly more important in the case of present interest in which the full conditional densities on which the Gibbs sampler is based could be incompatible, since, compared to the case where these densities are known to be compatible, there is likely to be, in practice, a little more concern that the Gibbs sampler may not actually have a limiting density function, even though in reality the genuine risk of this may still be extremely low.
A notable advantage of the general method for finding a suitable joint post-data density for the parameters \( \theta \) that has just been outlined is that it can directly achieve what is often the main goal of a standard application of the Gibbs sampler, namely that of obtaining good approximations to the expected values of functions of the parameters of a model over the post-data or posterior density for these parameters that is of interest, i.e. expected values of the following type

\[
E[h(\theta) \mid x] = \int h(\theta)p(\theta \mid x)d\theta
\]

where \( p(\theta \mid x) \) is a given post-data density function of the parameters \( \theta \), while \( h(\theta) \) is any given function of these parameters. To be more specific, this kind of expected value may, of course, be approximated using the Monte Carlo estimator:

\[
\frac{1}{m-b} \sum_{i=b+1}^{m} h(\theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_k^{(i)})
\]

where \( \theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_k^{(i)} \) is the \( i \)th sample of parameter values among the \( m \) samples generated by the sampler in total, and \( b \) is the number of initial samples that are classified as belonging to the burn-in phase of the sampler.

Finally, it is worth noting that when the sampling model has only two parameters, i.e. \( k = 2 \), it is easy to show that the limiting marginal density functions of a Gibbs sampler that is based on incompatible full conditional post-data densities of the two parameters are not affected by the scanning order of the sampler over both fixed and random scanning orders of the sampler as defined earlier. This property may be of some convenience if the aim is to only determine marginal post-data densities for the two parameters concerned. It is, though, a property that does not generally hold when there are more than two parameters.
2.3. Bayesian inference

As was in effect done so by Bayes in his famous paper Bayes (1763), it will be assumed that Bayesian inference depends on three key concepts. First, Bayes’ theorem as a purely mathematical expression. Second, the justification of the application of this theorem to well-understood physical experiments, e.g. random spins of a wheel or random draws of a ball from an urn of balls. Finally, something which will be referred to as Bayes’ analogy, which is the type of analogy that can be made between the uncertainty that surrounds the outcomes of the kind of physical experiments just mentioned to which Bayes’ theorem can be very naturally applied, and the uncertainty that surrounds what are the true values of any unknown real-world quantities that are of interest.

By using this latter concept, we can justify the use of Bayesian inference in a much wider range of applications than is allowed by only using the first two concepts. However, depending on the type of application, the Bayes’ analogy may be a good analogy or a poor analogy, which is something that needs to be taken into account when assessing the adequacy of any given application of the Bayesian method.

In keeping with the notation defined in the Introduction, the post-data or posterior density function of the parameter $\theta_j$ given all other model parameters $\theta_{-j}$ can be expressed according to Bayes’ theorem as follows:

$$p(\theta_j \mid \theta_{-j}, x) = C_0 g(x \mid \theta)p(\theta_j \mid \theta_{-j})$$

where $p(\theta_j \mid \theta_{-j})$ is the pre-data or prior density function of the parameter $\theta_j$ given the parameters $\theta_{-j}$, while $C_0$ is a normalising constant.

In this paper, we will exclude from consideration two methods of inference that are often referred to as ‘objective’ forms of Bayesian inference. The first of these methods consists in always specifying the prior density $p(\theta_j \mid \theta_{-j})$ as being a uniform or flat density function over all values of $\theta_j$. This implies, though, that the Bayes’ analogy must be broken due to this prior density being improper and/or the method has the serious drawback
that the posterior density $p(\theta_j | \theta_{-j}, x)$ will depend, in general, on the parameterisation of the sampling model. On the other hand, the second type of method entails specifying the prior density in question such that it depends on the sampling model, i.e. allowing what is known about the parameter $\theta_j$ to depend on how we intend to collect more information about this parameter, however doing this clearly again breaks the Bayes’ analogy. A famous example of a type of prior density that is specified in this way is a prior density that is derived by applying Jeffreys’ rule, see Jeffreys (1961), although many other prior densities of this kind have been proposed, see for example, Kass and Wasserman (1996).

In both of the cases just mentioned where the Bayes’ analogy is broken it can be strongly argued, on the basis of the comments made in this section, that the application of the method of inference concerned should not really be regarded as being an application of the Bayesian approach to inference at all.

2.4. Organic fiducial inference

We will now outline some of the key concepts that underlie the theory of organic fiducial inference. Descriptions of other important concepts on which this theory is based, along with further details about the concepts that will be outlined here and about the overall theory itself, can be found in Bowater (2019a). Throughout this section, it will be assumed that the values of the parameters in the set $\theta_{-j}$ are known.

Fiducial statistics

A fiducial statistic $Q(x)$ will be defined as being a univariate statistic of the sample $x$ that can be regarded as efficiently summarising the information that is contained in this sample about the only unknown parameter $\theta_j$, given the values of other statistics that do not provide any information about this parameter, i.e. ancillary statistics. If, in any given case, there exists a univariate sufficient statistic for $\theta_j$, then this would naturally
be chosen to be the fiducial statistic for that case. In other cases, it may well make good sense to choose the statistic \( Q(x) \) to be the maximum likelihood estimator of \( \theta_j \).

For ease of presentation, we will assume, in what follows, that the choice of the fiducial statistic can be justified without reference to any particular ancillary statistics.

**Data generating algorithm**

Independent of the way in which the data set \( x \) was actually generated, it will be assumed that this data set was generated by the following algorithm:

1) Generate a value \( \gamma \) for a continuous one-dimensional random variable \( \Gamma \), which has a density function \( \pi_0(\gamma) \) that does not depend on the parameter \( \theta_j \).

2) Determine a value \( q(x) \) for the fiducial statistic \( Q(x) \) by setting \( \Gamma \) equal to \( \gamma \) and \( Q(x) \) equal to \( q(x) \) in the following expression for the statistic \( Q(x) \), which effectively should fully specify the distribution function of this statistic:

\[
Q(x) = \varphi(\Gamma, \theta_j)
\]  

(3)

where the function \( \varphi(\Gamma, \theta_j) \) is defined so that it satisfies the following conditions:

a) The distribution function of \( Q(x) \) as defined by the expression in equation (3) is equal to what it would have been if \( Q(x) \) had been determined on the basis of the data set \( x \).

b) The only random variable upon which \( \varphi(\Gamma, \theta_j) \) depends is the variable \( \Gamma \).

3) Generate the data set \( x \) by conditioning the sampling density or mass function \( g(x \mid \theta_1, \theta_2, \ldots, \theta_k) \) on the already generated value for \( Q(x) \).

In the context of this algorithm, the variable \( \Gamma \) is referred to as the primary random variable (primary r.v.)

**Strong fiducial argument**

This is the argument that the density function of the primary r.v. \( \Gamma \) after the data have
been observed, i.e. the post-data density function of $\Gamma$, should be equal to the pre-data density function of $\Gamma$, i.e. the density function $\pi_0(\gamma)$ as defined in step 1 of the data generating algorithm just presented.

**Moderate fiducial argument**

It will be assumed that this argument is only applicable if, on observing the data $x$, there exists some positive measure set of values of the primary r.v. $\Gamma$ over which the pre-data density function $\pi_0(\gamma)$ was positive, but over which the post-data density function of $\Gamma$, which will be denoted as the density function $\pi_1(\gamma)$, is necessarily zero. Under this condition, it is the argument that, over the set of values of $\Gamma$ for which the density function $\pi_1(\gamma)$ is necessarily positive, the relative height of this function should be equal to the relative height of the density function $\pi_0(\gamma)$.

**Weak fiducial argument**

This argument will be assumed to be only applicable if neither the strong nor the moderate fiducial argument is considered to be appropriate. It is the argument that, over the set of values of $\Gamma$ for which the post-data density function $\pi_1(\gamma)$ is necessarily positive, the relative height of this function should be equal to the relative height of the pre-data density function $\pi_0(\gamma)$ multiplied by weights on the values of $\Gamma$ determined by a given function of $\theta_j$ that was specified before the data were observed. This latter function is called the global pre-data function of $\theta_j$. Let us now define this function.

**Global pre-data (GPD) function**

The global pre-data (GPD) function $\omega_G(\theta_j)$ is used to express pre-data knowledge, or a lack of such knowledge, about the only unknown parameter $\theta_j$. This function may be any given non-negative and locally integrable function over the space of the parameter $\theta_j$. It is a function that only needs to be specified up to a proportionality constant, in
the sense that, if it is multiplied by a positive constant, then the value of the constant is redundant. Unlike a Bayesian prior density, it is not controversial to use a GPD function that is not globally integrable.

**A principle for defining the fiducial density** \( f(\theta_j | \theta_{-j}, x) \)

Let us now consider a principle for defining the post-data density of \( \theta_j \) conditional on the set \( \theta_{-j} \), which given that it will be derived using a type of fiducial inference, will be called the fiducial density of \( \theta_j \) conditional on \( \theta_{-j} \), and will be denoted as the density \( f(\theta_j | \theta_{-j}, x) \). This principle requires that the following condition holds.

**Condition 1**

Let \( G_x \) and \( H_x \) be respectively the sets of all the values of \( \Gamma \) and \( \theta_j \) for which the density functions of these variables must necessarily be positive in light of having observed only the value of the fiducial statistic \( Q(x) \), i.e. the value \( q(x) \), and not any other information in the data set. Given this notation, the present condition will be satisfied if, on substituting the variable \( Q(x) \) in equation (3) by the value \( q(x) \), this equation would define a bijective mapping between the set \( G_x \) and the set \( H_x \).

Under this condition, the fiducial density \( f(\theta_j | \theta_{-j}, x) \) is defined by setting \( Q(x) \) equal to \( q(x) \) in equation (3), and then treating the value \( \theta_j \) in this equation as being a realisation of the random variable \( \Theta_j \), to give the expression:

\[
q(x) = \varphi(\Gamma, \Theta_j) \quad (4)
\]

except that, instead of the variable \( \Gamma \) necessarily having the density function \( \pi_0(\gamma) \) as defined in step 1 of the data generating algorithm, it will be assumed to have the post-data density function of this variable as defined by:

\[
\pi_1(\gamma) = \begin{cases} 
C_1 \omega_G(\theta_j(\gamma)) \pi_0(\gamma) & \text{if } \gamma \in G_x \\
0 & \text{otherwise}
\end{cases} \quad (5)
\]

where \( \theta_j(\gamma) \) is the value of \( \theta_j \) that maps on to the value \( \gamma \), the function \( \omega_G(\theta_j(\gamma)) \) is the
GPD function, and $C_1$ is a normalising constant.

Notice that if, on substituting the variable $Q(x)$ by the value $q(x)$, equation (3) defines an injective mapping from the set $\{\gamma : \pi_0(\gamma) > 0\}$ to the space of the parameter $\theta_j$, then the GPD function $\omega_G(\theta_j)$ expresses in effect our pre-data beliefs about $\theta_j$ relative to what is implied by the strong fiducial argument. By doing so, it determines whether the strong, moderate or weak fiducial argument is used to make inferences about $\theta_j$, and also the way in which the latter two arguments influence the inferential process.

In the case where nothing or very little was known about the parameter $\theta_j$ before the data were observed, it is natural to choose the GPD function of the parameter $\theta_j$ to be equal to a constant over the entire space of this parameter, which, under the same assumption about equation (3) that was just made, would imply that the density $\pi_1(\gamma)$ will be equal to the density $\pi_0(\gamma)$, i.e. inferences about $\theta_j$ will be made using the strong fiducial argument. The use of the theory being considered in this special case is discussed to some extent in Bowater (2019a), but more extensively in Bowater (2018a), where the more specific theory that is employed was referred to as subjective fiducial inference.

**Other ways of defining the fiducial density $f(\theta_j | \theta_{-j}, x)$**

In cases where the aforementioned principle can not be applied (i.e. when Condition 1 does not hold), we may well be able to define the fiducial density $f(\theta_j | \theta_{-j}, x)$ using the alternative principle for this purpose that was presented in Section 3.4 of Bowater (2019a) as Principle 2, or it may well be considered acceptable to define this density using the kind of variations on this latter principle that were discussed in Sections 7.2 and 8 of this earlier paper. The alternative principle in question, which is particularly useful in cases where the data are discrete or categorical, relies on the concept of a local pre-data (LPD) function for expressing additional information concerning the pre-data beliefs that were held about the parameter $\theta_j$ to that which is expressed by the GPD function for $\theta_j$. This concept is also detailed in Bowater (2019a).
2.5. Bispatial inference

The type of bispatial inference that will be integrated into the theory being developed in the present paper will be the special form of bispatial inference that was outlined in Section 3 of Bowater (2019b). Let us now outline the key concepts that underlie this type of bispatial inference. Further details about these concepts and about the overall method of inference itself can be found in Bowater (2019b). As in the previous section, the values of the parameters in the set \( \theta_{-j} \) will be assumed to be known.

Scenario of interest

This scenario is characterised by there having been a substantial degree of belief, before the data were observed, that the only unknown parameter \( \theta_j \) lay in a narrow interval \([\theta_{j0}, \theta_{j1}]\), but conditional on \( \theta_j \) not lying in this interval, there having been no or very little pre-data knowledge about the parameter \( \theta_j \). Among the three common types of pre-data opinion about \( \theta_j \) that were highlighted in the Introduction, this scenario is clearly consistent with holding the second type of opinion.

Test statistics

In the context of bispatial inference, a test statistic \( T(x) \) (which will be also denoted simply as \( t \)) is specified such that it satisfies two criteria. First, this statistic must fit within the broad definition of a fiducial statistic that was given in the previous section. Therefore, this could mean that the choice of the statistic \( T(x) \) can only be justified with reference to given ancillary statistics, however, similar to the previous section, we will assume here, for ease of presentation, that this is not the case.

The second criterion is that if \( F(t \mid \theta_j) \) is the cumulative distribution function of the unobserved test statistic \( T(X) \) evaluated at its observed value \( t \) conditional on a value for the parameter \( \theta_j \), i.e. \( F(t \mid \theta_j) = P(T(X) \leq t \mid \theta_j) \), and if \( F'(t \mid \theta_j) \) is equal to the probability \( P(T(X) \geq t \mid \theta_j) \), then it is necessary that, over the set of allowable values
for $\theta_j$, the probabilities $F(t \mid \theta_j)$ and $1 - F'(t \mid \theta_j)$ strictly decrease as $\theta_j$ increases.

**Parameter and sampling space hypotheses**

Under this definition of a test statistic $T(x)$, if the condition

$$F(t \mid \theta_j = \theta_{j0}) \leq F'(t \mid \theta_j = \theta_{j1})$$

(6)

holds, where the values $\theta_{j0}$ and $\theta_{j1}$ are as defined at the start of this section, then the parameter space hypothesis $H_P$ and the sampling space hypothesis $H_S$ will be defined as:

$$H_P : \theta_j \geq \theta_{j0}$$

(7)

$$H_S : \rho(T(X^*) \leq t) \leq F(t \mid \theta_j = \theta_{j0})$$

(8)

where $X^*$ is an as-yet-unobserved second sample of values drawn from the density function $g(x \mid \theta)$ that is the same size as the observed (first) sample $x$, i.e. it consists of $n$ observations, and where $\rho(A)$ is the unknown population proportion of times that condition $A$ is satisfied, assuming that $\theta_j$ is fixed but unknown.

On the other hand, if the condition in equation (6) does not hold, then the hypotheses in question will be defined as:

$$H_P : \theta_j \leq \theta_{j1}$$

(9)

$$H_S : \rho(T(X^*) \geq t) \leq F'(t \mid \theta_j = \theta_{j1})$$

(10)

It can easily be seen that, given the way the test statistic $T(x)$ was defined, the hypothesis $H_S$ is equivalent to the hypothesis $H_P$ both here, and as these hypotheses were defined in equations (7) and (8). Also, observe that the probabilities $F(t \mid \theta_j = \theta_{j0})$ and $F'(t \mid \theta_j = \theta_{j1})$ in equations (8) and (10) respectively would be standard one-sided P values if the hypotheses $H_P$ that have been defined alongside each of the hypotheses $H_S$ given by these two equations were regarded as being the null hypotheses.
Inferential process

It will be assumed that inferences are made about the parameter $\theta_j$ by means of the following three-step process:

Step 1: An assessment of the likeliness of the hypothesis $H_P$ being true using only pre-data knowledge about $\theta_j$, with special attention being given to evaluating the likeliness of the hypothesis that $\theta_j$ lies in the interval $[\theta_{j0}, \theta_{j1}]$, which is an hypothesis that is always included in the hypothesis $H_P$. It is not necessary that this assessment is expressed in terms of a formal measure of uncertainty, e.g. a probability does not need to be assigned to the hypothesis $H_P$.

Step 2: An assessment of the likeliness of the hypothesis $H_S$ being true, leading to the assignment of a probability to this hypothesis, which will be denoted as the probability $\kappa$. This evaluation of course can only be done after the data $x$ have been observed. In carrying out such an assessment, all relevant factors ought to be taken into account including, in particular: (a) the size of the one-sided P value that appears in the definition of the hypothesis $H_S$, i.e. the value $F(t \mid \theta_j = \theta_{j0})$ or $F'(t \mid \theta_j = \theta_{j1})$, (b) the assessment made in Step 1, and (c) the known equivalency between the hypotheses $H_P$ and $H_S$ after the data have been observed.

Step 3: Conclusion about the probability of $H_P$ being true on the basis of the data $x$. This is directly implied by the assessment made in Step 2.

In combination with organic fiducial inference

It was described in Bowater (2019b) how bispatial inference can be extended from allowing us to simply determine a post-data probability for the hypothesis $H_P$ being true to allowing us to determine an entire post-data density function for the parameter $\theta_j$. As was the case in this earlier paper, we will again favour doing this in an indirect way by combining bispatial inference as has just been detailed with organic fiducial inference as
was summarised in Section 2.4. In particular, the method that we will choose to adopt to achieve the goal in question will be essentially the method that was put forward in Section 4.2 of Bowater (2019b). Let us now give briefly outline this method.

To begin with, in applying the method concerned, we assume that both the post-data density function of $\theta_j$ conditional on $\theta_j$ lying in the interval $[\theta_{j0}, \theta_{j1}]$, and the post-data density function of $\theta_j$ conditional on $\theta_j$ not lying in this interval are derived under the paradigm of organic fiducial inference, i.e. they are fiducial density functions. Since it has been assumed that, under the condition that $\theta_j$ does not lie in the interval $[\theta_{j0}, \theta_{j1}]$, nothing or very little would have been known about $\theta_j$ before the data were observed, it would seem quite natural, in deriving the latter of these two fiducial densities, i.e. the density $f(\theta_j | \theta_j \notin [\theta_{j0}, \theta_{j1}], x)$, to use a GPD function for $\theta_j$ that has the following form:

$$\omega_G(\theta_j) = \begin{cases} 0 & \text{if } \theta_j \in [\theta_{j0}, \theta_{j1}] \\ a & \text{otherwise} \end{cases}$$

(11)

where $a > 0$, which would be classed as a neutral GPD function using the terminology of Bowater (2019a).

On the basis of this GPD function, the fiducial density $f(\theta_j | \theta_j \notin [\theta_{j0}, \theta_{j1}], x)$ can often be derived by applying the moderate fiducial argument under the principle that was briefly outlined in Section 2.4, i.e. Principle 1 of Bowater (2019a), or as also advocated in this earlier paper, can be more generally defined by the following expression:

$$f(\theta_j | \theta_j \notin [\theta_{j0}, \theta_{j1}], x) = C_2 f_S(\theta_j | x)$$

(12)

where $C_2$ is a normalising constant, and $f_S(\theta_j | x)$ is a fiducial density for $\theta_j$ derived using either Principle 1 or Principle 2 of Bowater (2019a) that would be regarded as being suitable in a general scenario where there is no or very little pre-data knowledge about $\theta_j$ over all possible values of $\theta_j$.

To construct the fiducial density conditional on $\theta_j$ lying in the interval $[\theta_{j0}, \theta_{j1}]$, i.e. the density $f(\theta_j | \theta_j \in [\theta_{j0}, \theta_{j1}], x)$, the method being considered relies on quite a general
type of GPD function for $\theta_j$. In particular, it is assumed that this GPD function has the following form:

$$\omega_G(\theta_j) = \begin{cases} 
1 + \tau h(\theta_j) & \text{if } \theta_j \in [\theta_{j0}, \theta_{j1}] \\
0 & \text{otherwise}
\end{cases}$$

(13)

where $\tau \geq 0$, and $h(\theta_j)$ is a continuous unimodal density function on the interval $[\theta_{j0}, \theta_{j1}]$ that is equal to zero at the limits of this interval. On the basis of this GPD function, the fiducial density $f(\theta_j | \theta_j \in [\theta_{j0}, \theta_{j1}], x)$ can often be derived by applying the weak fiducial argument under the principle of Section 2.4 (i.e. Principle 1 of Bowater 2019a), or as also advocated in Bowater (2019a), can be more generally defined by the expression:

$$f(\theta_j | \theta_j \in [\theta_{j0}, \theta_{j1}], x) = C_3 \omega_G(\theta_j) f_S(\theta_j | x)$$

(14)

where $f_S(\theta_j | x)$ is the same fiducial density function that appeared in equation (12), and $C_3$ is a normalising constant.

Now, if using the method of bispatial inference outlined in the preceding subsections, the hypothesis $H_P$, i.e. the hypothesis in equation (7) or equation (9), is assigned a sensible post-data probability, i.e. a probability above a very low limit that is defined in Bowater (2019b), then given the two conditional post-data densities for $\theta_j$ that have just been specified, i.e. the fiducial densities $f(\theta_j | \theta_j \in [\theta_{j0}, \theta_{j1}], x)$ and $f(\theta_j | \theta_j \notin [\theta_{j0}, \theta_{j1}], x)$, we have sufficient information to determine a valid post-data density function of $\theta_j$ over all values of $\theta_j$. Hopefully, it is fairly clear why this is the case, nevertheless a more detailed account of the derivation of this latter post-data density function is given in Bowater (2019b). In the rest of this paper, we will denote such a post-data density function as the density $b(\theta_j | \theta_{-j}, x)$ to indicate that it was derived using bispatial inference.

However, there is an important final issue that needs to be resolved, which is how the value of the constant $\tau$ in equation (13) is chosen. Using the method being discussed, this constant must be chosen such that the overall post-data density $b(\theta_j | \theta_{-j}, x)$ is made equivalent to a fiducial density function for $\theta_j$ that is based on a continuous GPD func-
tion for $\theta_j$ over all values of $\theta_j$, but except for the way in which this GPD function is specified, is based on the same assumptions as were used to derive the fiducial density $f_S(\theta_j \mid x)$. In practice, this condition on the choice of $\tau$ can be viewed as not placing a substantial restriction on the way we are allowed to express our pre-data knowledge about the parameter $\theta_j$, while it ensures that the density function $b(\theta_j \mid \theta_j, x)$ possesses, in general, the usually desirable property of being continuous over all values of $\theta_j$.

**Post-data opinion curve**

Observe that the probability that is assigned to the hypothesis $H_S$ in either equation (8) or equation (10) should arguably depend, in general, on the values of the parameters in the set $\theta_{-j}$. As a result, to implement the method that has just been outlined within the overall framework for determining a joint post-data density of all the model parameters $\theta$ that was put forward in Section 2.2, we will generally wish to assign not just one, but various probabilities to the hypothesis $H_S$ conditional on the parameters $\theta_{-j}$.

It is possible though to simplify matters greatly by assuming that, the probability that is assigned to any given hypothesis $H_S$, i.e. the probability $\kappa$, will be the same for any fixed value of the one-sided P value that appears in the definition of this hypothesis, i.e. $F(t \mid \theta_j = \theta_{j0})$ or $F'(t \mid \theta_j = \theta_{j1})$ no matter what values are actually taken by the parameters in the set $\theta_{-j}$. By making this assumption, which is arguably a reasonable assumption in many practical situations, the probability $\kappa$ becomes a mathematical function of the one-sided P value that appears in the definition of the hypothesis $H_S$ concerned, which is a P value that, in the following sections, will be denoted generically as $\eta$. As was the case in Bowater (2019b), this function will be called the post-data opinion (PDO) curve for the parameter $\theta_j$ conditional on the parameters $\theta_{-j}$.
3. Examples

We will now present various examples of the application of the overall theory that was outlined in previous sections, i.e. the theory of integrated organic inference.

3.1. Inference about a univariate normal distribution

Let us begin by considering what can be referred to as Student’s problem, that is, the standard problem of making inferences about the mean $\mu$ of a normal density function, when its variance $\sigma^2$ is unknown, on the basis of a sample $x$ of size $n$, i.e. $x = \{x_1, x_2, \ldots, x_n\}$, drawn from the density function concerned.

If $\sigma^2$ was known, a sufficient statistic for $\mu$ would be the sample mean $\bar{x}$, which therefore, in applying the theory outlined in Section 2.4, can naturally be assumed to be the fiducial statistic $Q(x)$ for this case. Based on this assumption and given a value for $\sigma^2$, equation (3) can be expressed as

$$\bar{x} = \phi(\Gamma, \mu) = \mu + (\sigma/\sqrt{n})\Gamma$$

(15)

where the primary r.v. $\Gamma \sim N(0, 1)$. If nothing or very little was known about $\mu$ before the data $x$ were observed, then in keeping with what was mentioned in Section 2.4, it would be quite natural to specify the GPD function for $\mu$ as follows: $\omega_G(\mu) = a$, $\mu \in (-\infty, \infty)$, where $a > 0$. Using the principle outlined in this earlier section, and in particular using equation (4), this would imply that the fiducial density of $\mu$ given $\sigma^2$ can be expressed as:

$$f(\mu | \sigma^2, x) = \phi((\mu - \bar{x})\sqrt{n}/\sigma)$$

(16)

where $\phi(y)$ is the standard normal density function evaluated at the value $y$.

On the other hand, if $\mu$ was known, a sufficient statistic for $\sigma^2$ would be $\hat{\sigma}^2 = (1/n) \sum_{i=1}^{n}(x_i - \mu)^2$, which therefore will be assumed to be the statistic $Q(x)$ for this case. Based on this assumption and given a value for $\mu$, equation (3) can be expressed
as
\[ \hat{\sigma}^2 = \varphi(\Gamma, \sigma^2) = (\sigma^2/n)\Gamma \] (17)
where the primary r.v. \( \Gamma \sim \chi^2_n \). If there was no or very little pre-data knowledge about \( \sigma^2 \), it would be quite natural to specify the GPD function for \( \sigma \) as follows:

\[ \omega_G(\sigma^2) = b \quad \text{if } \sigma^2 \geq 0 \text{ and } 0 \text{ otherwise} \] (18)

where \( b > 0 \). Again using the principle outlined in Section 2.4, this would imply that the fiducial density \( f(\sigma^2 | \mu, x) \) is defined by

\[ \sigma^2 | \mu, x \sim \text{Inv-Gamma} \left( n/2, n\hat{\sigma}^2/2 \right) \] (19)
i.e. it is an inverse gamma density function with shape parameter equal to \( n/2 \) and scale parameter equal to \( n\hat{\sigma}^2/2 \).

As discussed in Bowater (2018a, 2019a), the full conditional fiducial densities \( f(\mu | \sigma^2, x) \) and \( f(\sigma^2 | \mu, x) \) as they have just been specified represent a definition of a joint density function of \( \mu \) and \( \sigma^2 \), i.e. the fiducial density \( f(\mu, \sigma^2 | x) \). In particular, the marginal density of \( \mu \) over the joint density \( f(\mu, \sigma^2 | x) \) is given by:

\[ \mu | x \sim \text{Non-standardised} \ t_{n-1}(\bar{x}, s/\sqrt{n}) \] (20)
where \( s \) is the sample standard deviation, i.e. it is a non-standardised Student t density function with \( n - 1 \) degrees of freedom, location parameter equal to \( \bar{x} \) and scaling parameter equal to \( s/\sqrt{n} \) (which are settings that of course make it a very familiar member of this particular family of density functions), while the marginal density of \( \sigma^2 \) over this joint density is given by:

\[ \sigma^2 | x \sim \text{Inv-Gamma} \left( (n - 1)/2, (n - 1)s^2/2 \right) \] (21)

All the main results that have just been outlined were previously given in Bowater (2019a). In what follows, the results that will be presented are generally original.
results, i.e. results not discussed in earlier papers, although various references will be made to examples that have been detailed previously.

In the scenario currently being considered, let us now turn our attention to the case where we have important pre-data knowledge about either of the parameters $\mu$ or $\sigma^2$ that can be adequately represented by a probability density function over the parameter concerned conditional on the other parameter being known. To give an example, let us assume that our pre-data opinion about $\sigma^2$ conditional on $\mu$ being known can be adequately represented by the density function of $\sigma^2$ conditional on $\mu$ that is defined by

$$
\sigma^2 | \mu \sim \text{Inv-Gamma} \left( \alpha_0, \beta_0 \right)
$$

where $\alpha_0 > 0$ and $\beta_0 > 0$ are given constants. Treating this density function as a prior density function, and combining it with the likelihood function in this case, under the Bayesian paradigm, leads to a posterior density of $\sigma^2$ conditional on $\mu$ that is defined by

$$
\sigma^2 | \mu, x \sim \text{Inv-Gamma} \left( \alpha_0 + \frac{n}{2}, \beta_0 + \frac{n\bar{s}^2}{2} \right)
$$

If there was no or very little pre-data knowledge about $\mu$, then it would be quite natural to let the full conditional fiducial density $f(\mu | \sigma^2, x)$ defined by equation (16), and the full conditional posterior density $p(\sigma^2 | \mu, x)$ defined in the equation just given, form the basis for using the framework described in Section 2.2 to determine the joint post-data density of $\mu$ and $\sigma^2$, i.e. the density $p(\mu, \sigma^2 | x)$. In fact, it can be analytically shown that these full conditional densities are compatible, and therefore directly define a unique joint density for $\mu$ and $\sigma^2$. This joint density function is therefore the post-data density $p(\mu, \sigma^2 | x)$. Furthermore, the marginal density of $\mu$ over this joint post-data density is given by

$$
\mu | x \sim \text{Non-standardised } t_{2\alpha_0+n-1} \left( \bar{x}, \frac{2\beta_0 + (n-1)s^2}{(2\alpha_0 + n-1)n} \right)^{0.5},
$$

while the marginal density of $\sigma^2$ over the joint density in question is given by

$$
\sigma^2 | x \sim \text{Inv-Gamma} \left( \alpha_0 + \frac{(n-1)}{2}, \beta_0 + \frac{(n-1)\bar{s}^2}{2} \right)
$$
To illustrate this example, Figure 1 shows a number of plots that are relevant to the application of the calculations just described to the analysis of a data set \( x \) that is summarised by the values \( n = 9, \bar{x} = 2.7 \) and \( s^2 = 9 \), given specific prior information. In particular, it shows a plot of the prior density \( p(\sigma \mid \mu) \) as defined by equation (22), which is represented by the short-dashed curve in Figure 1(b), a plot of the marginal post-data density \( p(\mu \mid x) \) as defined by equation (24), which is represented by the long-dashed (rather than the dot-dashed) curve in Figure 1(a), and a plot of the marginal post-data density \( p(\sigma \mid x) \) as given by equation (25), which is represented by the long-dashed curve in Figure 1(b). To complete the specification of the prior density \( p(\sigma \mid \mu) \), the constants \( \alpha_0 \) and \( \beta_0 \) in equation (22) were set to be 4 and 64 respectively. These settings imply that this prior density would be equal to the marginal fiducial density of \( \sigma \) defined by equation (21) if this latter density was based on having observed a variance of 16 in a preliminary sample of 9 observations drawn from a population having the same unknown variance \( \sigma^2 \) that is currently being considered. Notice that this interpretation makes good sense if the mean \( \mu \) of this population is not only assumed to be unknown, but is assumed not to be the same as the mean \( \mu \) of present interest. On the basis of only the data set of main concern, i.e. the data set \( x \), and for comparison with the plots being considered, the solid curves in Figures 1(a) and 1(b) represent, respectively, the marginal fiducial density \( f(\mu \mid x) \) as defined by equation (20) and the marginal fiducial density \( f(\sigma \mid x) \) as given by equation (21).

Let us now change the state of knowledge about both the parameters \( \mu \) and \( \sigma^2 \) before the data were observed. In particular, let us begin by imagining that we have important pre-data knowledge about the mean \( \mu \) that can be adequately represented by a probability density function over \( \mu \) conditional on \( \sigma^2 \) being known, i.e. the density \( p(\mu \mid \sigma^2) \). To give an example, let this density function be defined by:

\[
\mu \mid \sigma^2 \sim \text{Non-standardised } t_{\nu_0}(\mu_0, \sigma_0) \tag{26}
\]
where \( \nu_0 > 0, \sigma_0 > 0 \) and \( \mu_0 \) are given constants. Treating this choice of the density \( p(\mu | \sigma^2) \) as a prior density under the Bayesian paradigm leads to a posterior density of \( \mu \) conditional on \( \sigma^2 \) that is defined by

\[
p(\mu | \sigma^2, x) \propto (1 + (1/\sigma_0^2 \nu_0)(\mu - \mu_0)^2)^{-(\nu_0 + 1)/2} \exp(-n(\bar{x} - \mu)^2/2\sigma^2)
\]

If now we assume that there was no or very little pre-data knowledge about \( \sigma^2 \), then it would be quite natural to use the full conditional fiducial density \( f(\sigma^2 | \mu, x) \) given by equation (25), and the full conditional posterior density \( p(\mu | \sigma^2, x) \) that has just been defined, as the basis for determining the joint post-data density of \( \mu \) and \( \sigma^2 \), i.e. the density \( p(\mu, \sigma^2 | x) \). Similar to the previous example, it can be analytically shown that these full conditional densities are compatible, and therefore directly and uniquely define the density \( p(\mu, \sigma^2 | x) \). This density may be more explicitly expressed as follows:

\[
p(\mu, \sigma^2 | x) = (1/\sigma^2)^{(n/2)+1}(1 + (1/\sigma_0^2 \nu_0)(\mu - \mu_0)^2)^{-(\nu_0 + 1)/2} \exp(-(1/2\sigma^2)n\tilde{\sigma}^2)
\]

(27)

To illustrate this example, Figure 1 shows, along with the plots that were mentioned earlier, a plot of the prior density \( p(\mu | \sigma^2) \) as defined by equation (26), which is represented by the short-dashed curve in Figure 1(a), and plots of the marginal densities of \( \mu \) and \( \sigma \) over the joint post-data density \( p(\mu, \sigma^2 | x) \) given in equation (27), which are represented by the dot-dash curves in Figure 1(a) and Figure 1(b) respectively. These marginal densities were obtained by numerical integration over the joint density \( p(\mu, \sigma^2 | x) \). The plots in question are based on the same data \( x \) as in the previous example. To complete the specification of the prior density \( p(\mu | \sigma^2) \), the constants in equation (26) were given the settings \( \nu_0 = 17, \mu_0 = -0.3 \) and \( \sigma_0 = 4/3 \). These settings imply that this prior density would be equal to the marginal fiducial density of \( \mu \) given by equation (20) if this latter density was based on having observed a mean of \(-0.3\) and a variance of 32 in a preliminary sample of 18 observations drawn from a population having the same unknown mean \( \mu \) that is currently being considered. Similar to earlier, such an inter-
pretation holds up well if the variance $\sigma^2$ of this population is not only assumed to be unknown, but is assumed not to be the same as the variance $\sigma^2$ of present interest.

Finally, in the case where we have important pre-data knowledge about both $\mu$ and $\sigma^2$ that can be adequately represented by full conditional probability densities over each of these parameters, i.e. the densities $p(\mu \mid \sigma^2)$ and $p(\sigma^2 \mid \mu)$, it would seem reasonable, assuming that these conditional densities are compatible, to treat these densities as being conditional prior densities, and to use exclusively the standard Bayesian approach to make inferences about $\mu$ and $\sigma^2$. Since Bayesian inference is a well-known form of inference, no further discussion of this particular case will be given here.

3.2. Alternative solution to Student’s problem

In the previous section, Student’s problem was tackled by incorporating organic fiducial inference and Bayesian inference into the framework outlined in Section 2.2, now let us consider a case in which it would seem appropriate to address the same problem by also
incorporating bispatial inference into this framework.

In particular, let us assume that conditioned on $\sigma^2$ being known, the scenario of interest of Section 2.5 would apply if the general parameter $\theta_j$ was taken as being the mean $\mu$, with the interval $[\theta_{j0}, \theta_{j1}]$ in this scenario being denoted now as the interval $[\mu_0 - \varepsilon, \mu_0 + \varepsilon]$, where $\varepsilon \geq 0$ and $\mu_0$ are given constants. We will therefore construct the post-data density of $\mu$ conditional on $\sigma^2$ using the type of bispatial inference described in Section 2.5.

To do this, the test statistic $T(x)$ as defined in Section 2.5 will be assumed to be the sample mean $\bar{x}$. Therefore, in the case where the mean $\bar{x}$ is greater than zero, which will be assumed to be the case of particular interest, the hypotheses $H_P$ and $H_S$ will be as defined in equations (9) and (10), which implies that, for the present example, they can be more specifically expressed as:

$$H_P : \mu \leq \mu_0 + \varepsilon$$
$$H_S : \rho(\bar{X}^* > \bar{x}) \leq 1 - \Phi((\bar{x} - \mu_0 - \varepsilon)\sqrt{n}/\sigma) \quad (= \eta)$$

where $\bar{X}^*$ is the mean of an as-yet-unobserved sample of $n$ additional values drawn from the normal density function in question, and $\Phi(y)$ is the cumulative density of a standard normal distribution at the value $y$. Also, it will be assumed, quite reasonably, that the fiducial density $f_S(\theta_j | x)$, which is required by equations (12) and (14), i.e. the density $f_S(\mu | \sigma^2, x)$ in the present case, is the fiducial density for $\mu$ given $\sigma^2$ that was defined in equation (16).

To complete the specification of the post-data density of $\mu$ given $\sigma^2$, i.e. in keeping with earlier notation, the density $b(\mu | \sigma^2, x)$, let us now make some more specific assumptions. In particular, let us assume that $\mu_0 = 0$ and $\varepsilon = 0.2$, and that the density function $h(\theta_j)$ that appears in equation (13), i.e. the density $h(\mu)$ in the present case, is defined by

$$\mu \sim \text{Beta}(4, 4, -0.2, 0.2)$$

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Figure 2: Histograms representing marginal post-data densities of the mean $\mu$ and standard deviation $\sigma$ of a normal distribution i.e. it is a beta density function for $\mu$ on the interval $[-0.2, 0.2]$ with both its shape parameters equal to 4. Furthermore, we will assume that the data is summarised as it was in the previous section, i.e. by $n = 9$, $\bar{x} = 2.7$ and $s^2 = 9$. Finally, it will be assumed that the PDO curve for $\mu$ conditional on $\sigma^2$, i.e. the curve that specifies the probabilities $\kappa$ that we would assign to the hypothesis $H_S$ in equation (28) for different values of the one-sided P value $\eta$ in this equation, or equivalently in this case, for different values of $\sigma$, is defined by: $\kappa = \eta^{0.6}$. These assumptions fully specify the post-data density $b(\mu \mid \sigma^2, x)$ according to the methodology outlined in Section 2.5.

In Bowater (2019b), precisely this full conditional density function and the full conditional fiducial density $f(\sigma^2 \mid \mu, x)$ given by equation (19), with the data set $x$ assumed to be as it has just been specified, were used as the basis for determining the joint post-data density of $\mu$ and $\sigma^2$ within the same type of framework as outlined in Section 2.2. As mentioned earlier, the use of the latter full conditional density would be quite natural if it was assumed there was no or very little pre-data knowledge about $\sigma^2$. However this as-
assumption will not be made here. Instead, let us assume that we have important pre-data knowledge about $\sigma^2$ that in fact is adequately represented by the density function for $\sigma^2$ conditional on $\mu$ being known that is defined by equation (22), with the same variable settings as were used earlier to express pre-data knowledge about $\sigma^2$ conditional on $\mu$, i.e. with $\alpha_0 = 4$ and $\beta_0 = 64$. Treating this density function as a prior density function under the Bayesian paradigm, leads therefore to the posterior density of $\sigma^2$ given $\mu$, i.e. $p(\sigma^2 | \mu, x)$, that is defined by equation (23).

Figure 2 shows some results from running a Gibbs sampler on the basis of the full conditional post-data densities that have just been defined, i.e. the post-data density $b(\mu | \sigma^2, x)$ and the posterior density $p(\sigma^2 | \mu, x)$, with a uniform random scanning order of the parameters, as this term was defined in Section 2.2. In particular, the histograms in Figures 2(a) and 2(b) represent the marginal density functions of $\mu$ and $\sigma$, respectively, over a single run of six million samples of $\mu$ and $\sigma$ generated by the Gibbs sampler after an initial two thousand samples of its output were excluded due to the values concerned being classified as belonging to its burn-in phase. The sampling of the density $b(\mu | \sigma^2, x)$ was based on the Metropolis algorithm (Metropolis et al. 1953), while each value drawn from the density $p(\sigma^2 | \mu, x)$ was independent from the preceding iterations.

In addition to this analysis, the Gibbs sampler was run various times from different starting points, and a careful study of the output of these runs using appropriate diagnostics provided no evidence to suggest that the sampler does not have a limiting density function, and showed, at the same time, that it would appear to generally converge quickly to this density function. Furthermore, the Gibbs sampling algorithm was run separately with each of the two possible fixed scanning orders of the parameters, i.e. the one in which $\mu$ is updated first and then $\sigma$ is updated, and the one that has the reverse order, in accordance with how a single transition of such an algorithm was defined in Section 2.2 i.e. single transitions of the algorithm incorporated updates of both param-
eters. In doing this, no statistically significant difference was found between the samples of parameter values aggregated over the runs of the sampler using each of these scanning orders after excluding the burn-in phase of the sampler, e.g. between the correlations of $\mu$ and $\sigma$, even when the runs concerned were long. This implies that the full conditional densities of the limiting density function of the original Gibbs sampler, i.e. the one with a uniform random scanning order, should at, the very least, be close approximations to the full conditional densities on which the sampler is based, i.e. the densities $b(\mu \mid \sigma^2, x)$ and $p(\sigma^2 \mid \mu, x)$.

Each of the curves overlaid on the histograms in Figures 2(a) and 2(b), which are distinguished by being plotted with short-dashed, long-dashed and solid lines, is identical to the curve plotted using the same line type in Figures 1(a) and 1(b) respectively. By comparing these histograms with the curves in question, it can be seen that the forms of the marginal post-data densities of $\mu$ and $\sigma$ that are represented by the histograms are consistent with what we would have intuitively expected given the pre-data beliefs about $\mu$ and $\sigma$ that have been taken into account. It may also be to some extent informative to compare Figures 2(a) and 2(b) with Figures 4(a) and 4(b) of Bowater (2019b), since these latter figures relate to the example from this earlier paper that was mentioned midway through the present section.

3.3. Inference about a trinomial distribution

We will now consider the problem of making inferences about the parameters $\pi = (\pi_1, \pi_2, \pi_3)'$ of a trinomial distribution, where $\pi_i$ is the proportion of times that outcome $i$ is generated in the long run, based on observing a sample of counts $x = (x_1, x_2, x_3)'$ from the distribution concerned, where $x_i$ is the number of times outcome $i$ is observed. Since of course $\pi_1 + \pi_2 + \pi_3 = 1$, this model has effectively only two parameters, which we will assume to be the proportions $\pi_1$ and $\pi_2$. 

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In particular, let us begin by applying organic fiducial inference as outlined in Section 2.4 to make inferences about \( \pi_2 \) conditional on \( \pi_1 \) being known. In this regard, observe that if \( \pi_1 \) was known, sufficient statistics for \( \pi_2 \) would be \( x_2 \) and \( x_2 + x_3 \). However, \( x_2 + x_3 \) is an ancillary complement of \( x_2 \), and therefore, according to the more general definition of the fiducial statistic \( Q(x) \) given in Bowater (2019a), the count \( x_2 \) can justifiably be assumed to be the statistic \( Q(x) \). Based on this assumption and given a value for \( \pi_1 \), equation (3) can naturally be redefined as

\[
x_2 = \varphi(\Gamma, \pi_2) = \min\{y : \Gamma < \sum_{j=0}^{y} g_0(j | \pi_2)\}
\]

(30)

where the primary r.v. \( \Gamma \sim U(0, 1) \), and the function \( g_0(j | \pi_2) \) is given by

\[
g_0(j | \pi_2) = \binom{x_2 + x_3}{j} \left( \frac{\pi_2}{1 - \pi_1} \right)^j \left( \frac{1 - \pi_1 - \pi_2}{1 - \pi_1} \right)^{x_2 + x_3 - j}
\]

in which the statistic \( x_2 + x_3 \) is treated as having already been generated.

Given that it will be assumed that there was no or very little pre-data knowledge about \( \pi_2 \), the GPD function for \( \pi_2 \) will be quite reasonably specified as follows: \( \omega_{G}(\pi_2) = a \) if \( 0 \leq \pi_2 \leq 1 - \pi_1 \) and 0 otherwise, where \( a > 0 \). However, since for whatever choice is made for this GPD function, equation (30) will not, for any sample \( x \), satisfy Condition 1, the principle outlined in Section 2.4 can not be employed to determine the fiducial density of \( \pi_2 \) given \( \pi_1 \), i.e. the density \( f(\pi_2 | \pi_1, x) \). This density can instead, though, be determined by applying Principle 2 of Bowater (2019a), which as mentioned in Section 2.4, is a principle that relies on the concept of a local pre-data (LPD) function. In particular, to make use of this principle in the present case, we need to specify a LPD function for \( \pi_2 \). Further details about how the principle in question is applied are given in Bowater (2019a).

As also discussed in this earlier paper, the type of method being considered could be used to obtain a complete set of full conditional fiducial densities for all the \( k - 1 \) population proportions of a multinomial distribution with \( k \) categories, which could then
be used to determine a joint fiducial density of these proportions using the type of framework outlined in Section 2.2 of the present paper. In relation to this issue, a detailed example was presented in Bowater (2019a) of how a joint fiducial density of all four of the population proportions of a multinomial distribution with five categories could be obtained using such an approach.

However, in the present case, it will not be assumed that the full conditional post-data densities of both the proportions $\pi_1$ and $\pi_2$ belong to the class of fiducial densities under discussion. This is because, in contrast to the kind of scenario where the type of approach just mentioned is most applicable, it will be assumed that we have important pre-data knowledge about the proportion $\pi_1$, and that this pre-data knowledge can, in particular, be adequately represented by a probability density function over $\pi_1$ conditional on $\pi_2$ being known, i.e. the density $p(\pi_1 \mid \pi_2)$. To give an example, let this density function be defined by:

$$p(\pi_1 \mid \pi_2) = \begin{cases} C_4(\pi_1)^{\alpha-1}(1-\pi_1)^{\beta-1} & \text{if } 0 \leq \pi_1 \leq 1-\pi_2 \\ 0 & \text{otherwise} \end{cases}$$

(31)

where $\alpha > 0$ and $\beta > 0$ are given constants, and $C_4$ is a normalising constant. Treating this choice of the density $p(\pi_1 \mid \pi_2)$ as a prior density and combining it with the likelihood function in this case, under the Bayesian paradigm, leads to a posterior density of $\pi_1$ given $\pi_2$ that is defined by

$$p(\pi_1 \mid \pi_2, x) = \begin{cases} C_5(\pi_1)^{\alpha+x_1-1}(1-\pi_1-\pi_2)^{n-x_1-x_2}(1-\pi_1)^{\beta-1} & \text{if } 0 \leq \pi_1 \leq 1-\pi_2 \\ 0 & \text{otherwise} \end{cases}$$

(32)

where $C_5$ is a normalising constant.

To illustrate this example, Figure 3 shows some results from running a Gibbs sampler on the basis of the full conditional post-data densities that have just been referred to, i.e. the fiducial density $f(\pi_2 \mid \pi_1, x)$ and the posterior density $p(\pi_1 \mid \pi_2, x)$, with a uniform random scanning order of the parameters. In particular, the histograms in Figures 3(a)
and 3(b) represent the marginal density functions of $\pi_1$ and $\pi_2$, respectively, over a single run of six million samples of $\pi_1$ and $\pi_2$ generated by the Gibbs sampler after allowing for its burn-in phase by excluding an initial one thousand samples of its output. The sampling of the density $p(\pi_1 | \pi_2, x)$ was based on the Metropolis algorithm, while the sampling of the density $f(\pi_2 | \pi_1, x)$ was independent from the preceding iterations.

Moreover, the observed counts on which the inferential procedure concerned is based were set as $x_1 = 4$, $x_2 = 2$ and $x_3 = 6$. Also, the LPD function for $\pi_2$ was set as being

$$\omega_L(\pi_2) = \begin{cases} b & \text{if } 0 \leq \pi_2 \leq 1 - \pi_1 \\ 0 & \text{otherwise} \end{cases}$$

(33)

which is in keeping with the choices that were made for this function in the aforementioned example in Bowater (2019a) of the use of fiducial inference in this type of situation. Finally, the specification of the prior density $p(\pi_1 | \pi_2)$ was completed by making the assignments $\alpha = 1.5$ and $\beta = 11.5$ in equation (31).

Observe that these choices for the variables $\alpha$ and $\beta$ imply that the prior density
\( p(\pi_1 \mid \pi_2) \) is equal to the density function of \( \pi_1 \) that is defined by

\[
p(\pi_1) \propto (\pi_1)^{0.5}(1 - \pi_1)^{10.5} \text{ if } 0 \leq \pi_1 \leq 1 \text{ and equal to 0 otherwise}
\]

conditioned on the inequality \( \pi_1 \leq 1 - \pi_2 \), which clearly must always hold, but is of course a condition that can only be applied if \( \pi_2 \) is known. Furthermore, this latter density \( p(\pi_1) \) is equivalent to the posterior density of \( \pi_1 \) that would be formed after observing the counts \( x_1 = 1 \) and \( x_2 + x_3 = 11 \) (for which, we can see, membership of categories 2 and 3 is not distinguished) if the prior density of \( \pi_1 \) was the Jeffreys prior that corresponds to conducting the binomial experiment that produced these counts. However, since as mentioned in Section 2.3, posterior densities formed on the basis of prior densities that are dependent on the sampling model, such as the Jeffreys prior, are controversial, it is arguably of more interest to note that this posterior density of \( \pi_1 \) is a close approximation to forms of the fiducial density of \( \pi_1 \) that would be naturally constructed on the basis of the two counts in question by applying the methodology in Bowater (2019a) if nothing or very little was known about \( \pi_1 \) before these counts were observed. This type of approximation was discussed both in this previous paper and in Bowater (2019b).

In addition to this analysis, the Gibbs sampler was also run various times from different starting points, and there was no suggestion from using appropriate diagnostics that the sampler does not have a limiting density function. Furthermore, after excluding the burn-in phase of the sampler, no statistically significant difference was found between the samples of parameter values aggregated over the runs of the sampler in using each of the two fixed scanning orders of the parameters \( \pi_1 \) and \( \pi_2 \) that are possible, with a single transition of the sampler defined in the same way as in the example discussed in the previous section, even when the runs concerned were long. Therefore, the full conditional densities of the limiting density function of the original random-scan Gibbs sampler should, at the very least, be close approximations to the full conditional densities.
on which the sampler is based, i.e. the densities \( p(\pi_1 | \pi_2, x) \) and \( f(\pi_2 | \pi_1, x) \).

The solid curves overlaid on the histogram in Figures 3(a) and 3(b) are plots of the marginal densities of \( \pi_1 \) and \( \pi_2 \) respectively over the joint posterior density of \( \pi_1 \) and \( \pi_2 \) that would be formed after having observed the same counts \( x \) if the joint prior density of these parameters was the Jeffreys prior for this case. It can be shown that this joint posterior density, which is in fact defined by the expression

\[
p(\pi_1, \pi_2 | x) = \begin{cases} 
    C_6(\pi_1)^{x_1 - 0.5}(\pi_2)^{x_2 - 0.5}(1 - \pi_1 - \pi_2)^{x_3 - 0.5} & \text{if } \pi_1, \pi_2 \in [0,1] \text{ and } \pi_1 + \pi_2 \leq 1 \\
    0 & \text{otherwise}
\end{cases}
\]

where \( C_6 \) is a normalising constant, is a close approximation to forms of the joint fiducial density of \( \pi_1 \) and \( \pi_2 \) that would be naturally constructed on the basis of this data by applying the methodology in Bowater (2019a) if there was no or very little pre-data knowledge about \( \pi_1 \) and \( \pi_2 \). The dashed curve overlaid on the histogram in Figure 3(a) is a plot of the unconditioned prior density of \( \pi_1 \) given in equation (34).

By comparing the locations and degrees of dispersion of the histograms in Figures 3(a) and 3(b), it can be seen that it is beyond dispute that generally more precise conclusions can be drawn about the proportion \( \pi_1 \) than the proportion \( \pi_2 \) after the counts \( x \) have been observed, which, on the basis of comparing these histograms with the curves overlaid on them, can be clearly attributed to the incorporation, under the Bayesian paradigm, of substantial prior information about \( \pi_1 \) into the construction of the joint post-data density of \( \pi_1 \) and \( \pi_2 \).

### 3.4. Inference about a linear regression model

Let us now turn our attention to the problem of making inferences about all the parameters \( \beta_0, \beta_1, \beta_2, \beta_3 \) and \( \sigma^2 \) of the normal linear regression model defined by

\[
Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2)
\] (35)
where $Y$ is the response variable and $x_1$, $x_2$ and $x_3$ are three covariates, on the basis of a data set $y_+ = \{(y_i, x_{1i}, x_{2i}, x_{3i}) : i = 1, 2, \ldots, n\}$, where $y_i$ is the value of $Y$ generated by this model for the $i$th case in this data set given values $x_{1i}$, $x_{2i}$ and $x_{3i}$ of $x_1$, $x_2$ and $x_3$ respectively.

Observe that sufficient statistics for each of the parameters $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$ and $\sigma^2$ conditional on all parameters except the parameter itself being known are respectively:

$$
\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} x_{1i}y_i, \sum_{i=1}^{n} x_{2i}y_i, \sum_{i=1}^{n} x_{3i}y_i \text{ and } \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i})^2
$$

(36)

In Bowater (2018a), all except the fourth statistic here were used as fiducial statistics $Q(y_+)$ to derive, under the strong fiducial argument, a complete set of full conditional fiducial densities of the model parameters in the special case where the model in equation (35) is a quadratic regression model, i.e. where $x_2 = (x_1)^2$ and the coefficient $\beta_3$ is set to zero (hence the lack of a need for the fourth statistic). Also, it was shown in this earlier paper that, since these full conditional densities are compatible, they directly define a unique joint density for $\beta_0$, $\beta_1$, $\beta_2$ and $\sigma^2$, which is therefore a joint fiducial density for these parameters. Furthermore, it is fairly clear from this previous analysis how the inferential procedure concerned can be extended to address the problem of making inferences about the parameters of the normal linear regression model as it is more generally defined by equation (35).

However, this specific type of method is not going to be directly applicable to the case that will be presently considered. This is because, although it will be assumed that nothing or very little was known about the parameters $\beta_0$, $\beta_2$ and $\sigma^2$ before the data were observed, by contrast it is going to be assumed that there was a substantial amount of pre-data knowledge about the parameters $\beta_1$ and $\beta_3$. Let us begin though by clarifying how the full conditional post-data densities of $\beta_0$, $\beta_2$ and $\sigma^2$ will be constructed.

With this aim, notice that if the sufficient statistics for $\beta_0$ and $\beta_2$ presented in equation (36) are treated as the fiducial statistics $Q(y_+)$ in making inferences about these two
parameters respectively, then given that the sampling distributions of these statistics are normal, the functions \( \varphi(\Gamma, \beta_0) \) and \( \varphi(\Gamma, \beta_2) \), as generally defined by equation (3), can be expressed in a similar way to how the function \( \varphi(\Gamma, \mu) \) was expressed in equation (15). Also, if the sufficient statistic for \( \sigma^2 \) presented in equation (36) is treated as the statistic \( Q(y_+) \) under the condition that \( \sigma^2 \) is the only unknown parameter, then given that this statistic divided by \( \sigma^2 \) has a chi-squared sampling distribution with \( n \) degrees of freedom, the function \( \varphi(\Gamma, \sigma^2) \) can be expressed in a similar way to how the function \( \varphi(\Gamma, \sigma^2) \) was expressed in equation (17). Furthermore, given what has been assumed, it would be quite natural to specify the GPD function for \( \sigma^2 \) in the same way as the GPD function for the variance (also denoted as \( \sigma^2 \)) was defined in equation (18), and to specify the GPD functions for \( \beta_0 \) and \( \beta_2 \) as follows:

\[
\omega_{G}(\beta_i) = a, \quad \beta_i \in (-\infty, \infty), \quad a > 0.
\]

This leads to the full conditional fiducial densities for \( \beta_0, \beta_2 \) and \( \sigma^2 \) being defined as follows:

\[
\beta_0 \mid \beta_{-0}, \sigma^2, y_+ \sim N \left( \frac{\sum_{i=1}^{n} y_i/n - \beta_0 \sum_{i=1}^{n} x_{1i}/n - \beta_2 \sum_{i=1}^{n} x_{2i}/n - \beta_3 \sum_{i=1}^{n} x_{3i}/n}{\sigma^2/n} \right)
\]

\[
\beta_2 \mid \beta_{-2}, \sigma^2, y_+ \sim N \left( \frac{\sum_{i=1}^{n} x_{2i}y_i - \beta_0 \sum_{i=1}^{n} x_{2i} - \beta_1 \sum_{i=1}^{n} x_{1i}x_{2i} - \beta_3 \sum_{i=1}^{n} x_{2i}x_{3i} - \beta_2 \sum_{i=1}^{n} x_{2i}^2}{\sum_{i=1}^{n} x_{2i}^2} \right)
\]

\[
\sigma^2 \mid \beta_0, ..., \beta_3, y_+ \sim \text{Inv-Gamma} \left( \frac{n}{2}, \sum_{i=1}^{n} \left( y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i} - \beta_3 x_{3i} \right)^2 / 2 \right)
\]

Now let us provide more details about what was known about the coefficient \( \beta_3 \) before the data were observed. In particular, let us assume that conditioned on all other parameters in the model being known, the scenario of interest of Section 2.5 would apply if the general parameter \( \theta_j \) was taken as being \( \beta_3 \), with the interval \([\theta_{j0}, \theta_{j1}]\) in this scenario now being specified as simply the interval \([-\delta, \delta]\), where \( \delta \geq 0 \). We will therefore construct the full conditional post-data density of \( \beta_3 \) using the type of bispatial inference outlined in Section 2.5, which implies that, from now, this density will be denoted as \( b(\beta_3 \mid \beta_{-3}, \sigma^2, x) \).

In particular to do this, the test statistic \( T(x) \) as defined in Section 2.5, which now
needs to be denoted as \( T(y_+) \), will be assumed to be the least squares estimator of \( \beta_3 \) under the condition that all other parameters are known, i.e. the estimator

\[
\hat{\beta}_3 = \frac{\sum_{i=1}^{n} x_{3i}y_i - \beta_0 \sum_{i=1}^{n} x_{3i} - \beta_1 \sum_{i=1}^{n} x_{1i}x_{3i} - \beta_2 \sum_{i=1}^{n} x_{2i}x_{3i}}{\sum_{i=1}^{n} x_{3i}^2}
\]

(40)

which is a reasonable assumption to make since, under this condition, it is a sufficient statistic for \( \beta_3 \) that satisfies the second criterion given in Section 2.5 for being the statistic \( T(y_+) \). Observe that this estimator has a sampling distribution that is defined by

\[
\hat{\beta}_3 \sim N \left( \beta_3, \frac{\sigma^2}{\sum_{i=1}^{n} x_{3i}^2} \right)
\]

Therefore, the hypotheses \( H_P \) and \( H_S \) defined in Section 2.5 that are applicable in the case where \( \hat{\beta}_3 \leq 0 \), i.e. the hypotheses in equations (7) and (8), can now be expressed as:

\[
H_P: \beta_3 \geq -\delta
\]

\[
H_S: \rho(\hat{B}_3^* < \hat{\beta}_3) \leq \Phi \left( (\hat{\beta}_3 + \delta)(1/\sigma)\sqrt{\sum_{i=1}^{n} x_{3i}^2} \right) \quad (= \eta)
\]

(41)

where \( \hat{B}_3^* \) is the estimator \( \hat{\beta}_3 \) calculated exclusively on the basis of an as-yet-unobserved sample of \( n \) additional data points \( Y_+^* = \{(Y_{i}^*, x_{1i}, x_{2i}, x_{3i}) : i = 1, 2, \ldots, n\} \) from the regression model in equation (35), where the values of the covariates \( x_1, x_2 \) and \( x_3 \) are assumed to the same as in the original sample. On the other hand, the hypotheses \( H_P \) and \( H_S \) that apply if \( \hat{\beta}_3 > 0 \), i.e. the hypotheses in equations (9) and (10), can be re-expressed as:

\[
H_P: \beta_3 \leq \delta
\]

\[
H_S: \rho(\hat{B}_3^* > \hat{\beta}_3) \leq 1 - \Phi \left( (\hat{\beta}_3 - \delta)(1/\sigma)\sqrt{\sum_{i=1}^{n} x_{3i}^2} \right) \quad (= \eta)
\]

(42)

Also, let us assume, quite reasonably, that the fiducial density \( f_S(\theta_j \mid x) \) that is required by equations (12) and (14), i.e. the density \( f_S(\beta_3 \mid \beta_{-3}, \sigma^2, x) \) in the present case, is derived on the basis of the strong fiducial argument with the fiducial statistic \( Q(y_+) \) specified as being a sufficient statistic for \( \beta_3 \), such as the ones given in equation (36) and equation (40).
Under these assumptions, this fiducial density is determined in a similar way to how the fiducial densities in equations (37), (38) and (39) were determined, and in particular is given by the expression:

$$\beta_3 | \beta_{-3}, \sigma^2, y_+ \sim N \left( \hat{\beta}_3, \sigma^2 / \sum_{i=1}^{n} x_{3i}^2 \right)$$  \hspace{0.5cm} (43)

On the other hand, it will be assumed that we knew enough about the coefficient $\beta_1$ before the data were observed such that it is possible to adequately represent our pre-data knowledge about this coefficient by placing a probability density function over this coefficient conditional on all other parameters being known, i.e. the density $p(\beta_1 | \beta_{-1}, \sigma^2)$.

To give an example, let this density function be defined by:

$$\beta_1 | \beta_{-1}, \sigma^2 \sim N(\mu_0, \sigma_0^2)$$  \hspace{0.5cm} (44)

where $\mu_0$ and $\sigma_0 > 0$ are given constants. Treating this choice of the density $p(\beta_1 | \beta_{-1}, \sigma^2)$ as a prior density and combining it with the likelihood function in this case, under the Bayesian paradigm, leads to a full conditional posterior density of $\beta_1$, i.e. the density $p(\beta_1 | \beta_{-1}, \sigma^2, y_+)$, that can be expressed as:

$$\beta_1 | \beta_{-1}, \sigma^2, y_+ \sim N \left( \sigma_1^2 \left[ \frac{\hat{\beta}_1 \sum_{i=1}^{n} x_{1i}^2}{\sigma^2} + \frac{\mu_0}{\sigma_0^2} \right], \sigma_1^2 \right)$$

where

$$\sigma_1^2 = (\sum_{i=1}^{n} x_{1i}^2/\sigma^2) + (1/\sigma_0^2))^{-1}$$

and

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_{1i}y_i - \beta_0 \sum_{i=1}^{n} x_{1i} - \beta_2 \sum_{i=1}^{n} x_{1i}x_{2i} - \beta_3 \sum_{i=1}^{n} x_{1i}x_{3i}}{\sum_{i=1}^{n} x_{1i}^2}$$

To illustrate this example, Figure 4 shows some results from running a Gibbs sampler on the basis of the full conditional post-data densities that have just been described, i.e. the fiducial densities $f(\beta_0 | \beta_{-0}, \sigma^2, x)$, $f(\beta_2 | \beta_{-2}, \sigma^2, x)$ and $f(\sigma^2 | \beta_0, ..., \beta_3, x)$, the post-data density $b(\beta_3 | \beta_{-3}, \sigma^2, x)$ and the posterior density $p(\beta_1 | \beta_{-1}, \sigma^2, x)$, with a uniform random scanning order of the parameters concerned. In particular, the histograms in Figures 4(a) to 4(d) represent the marginal density functions of $\beta_1$, $\beta_2$, $\beta_3$ and $\sigma$, respectively, over a single run of ten million samples of all five model parameters generated by
Figure 4: Conditional prior density of one parameter $\beta_1$ and marginal post-data densities of four parameters $\beta_1$, $\beta_2$, $\beta_3$ and $\sigma$ of a normal linear regression model

the Gibbs sampler after allowing for its burn-in phase by excluding an initial five thousand samples of its output. (To save space, a histogram of the generated values of $\beta_0$ is not given.) The sampling of the density $b(\beta_3 | \beta_{-3}, \sigma^2, x)$ was based on the Metropolis algorithm, while the sampling of each of the other four full conditional post-data densities was independent from the preceding iterations.
Moreover, the values for the response variable \( Y \) in the observed data set \( y_+ \) were a typical sample of \( n = 18 \) such values from the regression model in equation (35) with \( \beta_0 = 0, \beta_1 = 5, \beta_2 = -2, \beta_3 = 1 \) and \( \sigma = 1.5 \), and with the values of the covariates \( x_1, x_2 \) and \( x_3 \) in this data set chosen without replacement from the 27 combinations of values for these covariates that are possible if each covariate can only take the value \(-1, 0 \) or \( 1 \). In particular, for the selected covariates values, \( \sum x_{1i} = -1, \sum x_{2i} = 2, \sum x_{3i} = 1, \sum x_{1i}x_{2i} = 3, \sum x_{1i}x_{3i} = 4 \) and \( \sum x_{2i}x_{3i} = -3 \). In addition, the specification of the posterior density \( p(\beta_1 | \beta_{-1}, \sigma^2, x) \) was completed by setting the constants \( \mu_0 \) and \( \sigma_0 \), i.e. the constants that control the choice of the prior density of \( \beta_1 \) in equation (44), to be 4.4 and 0.6 respectively. On the other hand, with regard to how the post-data density \( b(\beta_3 | \beta_{-3}, \sigma^2, x) \) was fully determined, it was assumed that \( \delta = 0.1 \) and that the probabilities \( \kappa \) that would be assigned to the hypothesis \( H_S \) in both equations (41) and (42) for different values of the conditioning parameters concerned, i.e. all parameters except \( \beta_3 \), would be given by the PDO curve with the formula: \( \kappa = \eta^{0.6} \), where \( \eta \) is the one-sided P value in these equations. Also, in determining this latter density, the density function \( h(\theta_j) \) that appears in equation (13), i.e. the density \( h(\beta_3) \) in the present case, was defined, similar to an earlier example, by the expression \( \beta_3 \sim \text{Beta}(4, 4, -0.1, 0.1) \), where the notation here is the same as used in equation (29).

Supplementary to this analysis, there was no suggestion from applying appropriate diagnostics to multiple runs of the Gibbs sampler from different starting points that it did not have a limiting density function. Furthermore, the Gibbs sampling algorithm was run separately with various very distinct fixed scanning orders of the five model parameters in accordance with how a single transition of such an algorithm was defined in Section 2.2. In doing this, no statistically significant difference was found between the samples of parameter values aggregated over the runs of the sampler, after excluding the burn-in phase of the sampler, in using each of the scanning orders concerned, e.g.
between the various correlation matrices of the parameters and between the various marginal densities of each parameter, even when the runs in question were long. It would therefore be reasonable to conclude that the full conditional densities of the limiting density function of the original random-scan Gibbs sampler should, at the very least, be close approximations to the full conditional densities on which the sampler is based, i.e. the densities $f(\beta_0 | \beta_{-0}, \sigma^2, x)$, $p(\beta_1 | \beta_{-1}, \sigma^2, x)$, $f(\beta_2 | \beta_{-2}, \sigma^2, x)$, $b(\beta_3 | \beta_{-3}, \sigma^2, x)$ and $f(\sigma^2 | \beta_0, \ldots, \beta_3, x)$.

The solid curves overlaid on the histograms in Figures 4(a) to 4(d) are plots of the marginal densities of $\beta_1$, $\beta_2$, $\beta_3$ and $\sigma$ respectively over the joint fiducial density of all the parameters in the model that is directly defined by the set of compatible full conditional densities that consists of the fiducial densities given by equations (37), (38) and (39), the fiducial density $f_S(\beta_3 | \beta_{-3}, \sigma^2, x)$ given by equation (43), and the full conditional fiducial density for $\beta_1$ that results from making assumptions that are analogous to those on which the aforementioned full conditional fiducial densities for the other regression coefficients are based. On the other hand, the dashed curve overlaid on the histogram in Figure 4(a) is a plot of the prior density of $\beta_1$ given in equation (44).

By comparing the histograms in Figures 4(a) to 4(d) with the curves overlaid on them, it can be seen that the forms of the marginal post-data densities of $\beta_1$, $\beta_2$, $\beta_3$ and $\sigma$ that are represented by the histograms are consistent with what could have been intuitively expected given the pre-data beliefs about all of the model parameters that have been taken into account.

### 3.5. Inference about a bivariate normal distribution

To give a final detailed example of the application of integrated organic inference, let us consider the problem of making inferences about all five parameters of a bivariate normal density function, i.e. the means $\mu_x$ and $\mu_y$ and the variances $\sigma^2_x$ and $\sigma^2_y$, respectively, of
the two random variables concerned $X$ and $Y$, and the correlation $\tau$ of $X$ and $Y$, on the basis of a sample of values of $X$ and $Y$, i.e. the sample $z = \{(x_i, y_i) : i = 1, 2, \ldots, n\}$, where $x_i$ and $y_i$ are the $i$th realisations of $X$ and $Y$ respectively.

In Bowater (2018a), as a way of addressing this problem, full conditional fiducial densities were derived either exactly or approximately for each of the parameters $\mu_x$, $\mu_y$, $\sigma^2_x$, $\sigma^2_y$ and $\tau$ by using appropriately chosen fiducial statistics under the strong fiducial argument, and then it was illustrated how, on the basis of these conditional densities, a joint fiducial density of these parameters can be obtained by using the Gibbs sampler within the type of framework outlined in Section [2.2] of the present paper. However, similar to what was discussed in the previous section, it will not be possible, in the case that will be presently considered, to directly apply this specific method of inference. In particular, this is due to the fact that, although we will assume that nothing or very little was known about the means $\mu_x$ and $\mu_y$ before the data were observed, by contrast we are going to assume that there was a substantial amount of pre-data knowledge about the variances $\sigma^2_x$ and $\sigma^2_y$ and the correlation coefficient $\tau$. To begin with though, let us clarify how the full conditional post-data densities of $\mu_x$ and $\mu_y$ will be constructed.

Observe that sufficient statistics for the parameters $\mu_x$ and $\mu_y$ conditional on all parameters except the parameter itself being known are:

$$q_x = \bar{x} - \tau(\sigma_x/\sigma_y)\bar{y} \quad \text{and} \quad q_y = \bar{y} - \tau(\sigma_y/\sigma_x)\bar{x},$$

respectively, where $\bar{x} = \sum_{i=1}^n x_i$ and $\bar{y} = \sum_{i=1}^n y_i$. Therefore, these two statistics $q_x$ and $q_y$ will be assumed to be the fiducial statistics $Q(z)$ that will be used in making inferences about $\mu_x$ and $\mu_y$ respectively. Under this assumption, if $\mu_x$ is the only unknown parameter in the model, then equation (3) will now have the form $q_x = \varphi(\Gamma, \mu_x)$, and more specifically can be expressed as:

$$\bar{x} - \tau\left(\frac{\sigma_x}{\sigma_y}\right)\bar{y} = \mu_x - \tau\left(\frac{\sigma_x}{\sigma_y}\right)\mu_y + \Gamma\left(\frac{\sigma^2_x(1-\tau^2)}{n}\right)^{0.5}$$
where the primary r.v. $\Gamma \sim N(0, 1)$. Also, given what has been assumed about the case of interest, it would be quite natural to specify the GPD function for $\mu_x$ as follows: $\omega_G(\mu_x) = a$, for $\mu_x \in (-\infty, \infty)$, where $a > 0$. This implies that the full conditional fiducial density of $\mu_x$ is defined by:

$$
\mu_x | \mu_y, \sigma^2_x, \sigma^2_y, \tau, z \sim N \left( \bar{x} + \tau \left( \frac{\sigma_x}{\sigma_y} \right) (\mu_y - \bar{y}), \frac{\sigma_x^2 (1 - \tau^2)}{n} \right)
$$

(45)

Furthermore, due to the symmetrical nature of the bivariate normal distribution, it should be clear that, using a similar type of GPD function for $\mu_y$, the full conditional fiducial density of $\mu_y$ would be defined by:

$$
\mu_y | \mu_x, \sigma^2_x, \sigma^2_y, \tau, z \sim N \left( \bar{y} + \tau \left( \frac{\sigma_y}{\sigma_x} \right) (\mu_x - \bar{x}), \frac{\sigma_y^2 (1 - \tau^2)}{n} \right)
$$

(46)

With regard to what was known about the variances $\sigma^2_x$ and $\sigma^2_y$ before the data were observed, we will assume that it is possible to adequately represent such knowledge by placing a probability density function over each of these parameters conditional on all parameters except the parameter itself being known, i.e. the densities $p(\sigma^2_x | \mu_x, \mu_y, \sigma^2_y, \tau)$ and $p(\sigma^2_y | \mu_x, \mu_y, \sigma^2_x, \tau)$ respectively. To give an example, let these density functions for $\sigma^2_x$ and $\sigma^2_y$ be defined respectively by:

$$
\sigma^2_x \sim \text{Inv-Gamma} (\alpha_x, \beta_x) \quad \text{and} \quad \sigma^2_y \sim \text{Inv-Gamma} (\alpha_y, \beta_y)
$$

(47)

where $\alpha_x > 0$, $\beta_x > 0$, $\alpha_y > 0$ and $\beta_y > 0$ are given constants.

Notice that, for the case being considered, the likelihood functions of each of the parameters $\sigma^2_x$ and $\sigma^2_y$ assuming that all parameters except the parameter itself are known are given by the expressions:

$$
L(\sigma^2_x | \mu_x, \mu_y, \sigma^2_y, \tau, z) = \frac{1}{\sigma_x^n} \exp \left( \frac{-1}{2(1 - \tau^2)} \left( \frac{\sum (x_i^2)}{\sigma_x^2} \right) + \frac{\tau \left( \frac{\sum x_i^2}{\sigma_x \sigma_y} \right)}{1 - \tau^2} \right)
$$

(48)

and

$$
L(\sigma^2_y | \mu_x, \mu_y, \sigma^2_x, \tau, z) = \frac{1}{\sigma_y^n} \exp \left( \frac{-1}{2(1 - \tau^2)} \left( \frac{\sum (y_i^2)}{\sigma_y^2} \right) + \frac{\tau \left( \frac{\sum x_i y_i}{\sigma_x \sigma_y} \right)}{1 - \tau^2} \right)
$$

(49)
respectively, where \( x'_i = x_i - \mu_x \) and \( y'_i = y_i - \mu_y \). Therefore, if the choices of the densities 
\[ p(\sigma_x^2 \mid \mu_x, \mu_y, \sigma_y^2, \tau) \] and 
\[ p(\sigma_y^2 \mid \mu_x, \mu_y, \sigma_x^2, \tau) \]
in equation (47) are treated as prior densities under the Bayesian paradigm, it can easily be seen how, by combining these prior densities with the likelihood functions in equations (48) and (49), the full conditional posterior densities of \( \sigma_x^2 \) and \( \sigma_y^2 \) can be numerically computed, i.e. the densities 
\[ p(\sigma_x^2 \mid \mu_x, \mu_y, \sigma_y^2, \tau, z) \]
and 
\[ p(\sigma_y^2 \mid \mu_x, \mu_y, \sigma_x^2, \tau, z) \].

On the other hand, with regard to the beliefs that were held about the correlation coefficient \( \tau \) before the data were observed, let us assume that conditioned on all other parameters being known, the scenario of interest of Section 2.5 would apply if the general parameter \( \theta_j \) was taken as being \( \tau \), with the interval \([\theta_{j0}, \theta_{j1}]\) in this scenario now being specified as the interval \([-\varepsilon, \varepsilon]\), where \( \varepsilon \geq 0 \). As a result we will now discuss how the full conditional post-data density of \( \tau \) will be constructed by using the type of bispatial inference outlined in Section 2.5, which implies that it will be denoted as the density 
\[ b(\tau \mid \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, z) \].

Since if all parameters except \( \tau \) are known, there exists no sufficient set of univariate statistics for \( \tau \) that contains only one statistic that is not an ancillary statistic, it would seem reasonable to assume that the test statistic \( T(z) \), as generally defined in Section 2.5, is the maximum likelihood estimator of \( \tau \) given that all other parameters are known. It can be shown that this maximum likelihood estimator is the value \( \hat{\tau} \) that solves the following cubic equation:

\[-n\hat{\tau}^3 + \left( \frac{\sum_{i=1}^{n} x'_iy'_i}{\sigma_x \sigma_y} \right) \hat{\tau}^2 + \left( n - \frac{\sum_{i=1}^{n} (x'_i)^2}{\sigma_x^2} - \frac{\sum_{i=1}^{n} (y'_i)^2}{\sigma_y^2} \right) \hat{\tau} + \frac{\sum_{i=1}^{n} x'_iy'_i}{\sigma_x \sigma_y} = 0 \]

Now, it is well known that a maximum likelihood estimator of a parameter is usually asymptotically normally distributed with mean equal to the true value of the parameter, and variance equal to the inverse of the Fisher information with respect to that parameter. For this reason, if \( n \) is large, the sampling density function of the estimator \( \hat{\tau} \) can be
approximately expressed as follows:

\[ \hat{\tau} \sim N(\tau, 1/I(\tau)) \]

where \( I(\tau) \) is the Fisher information of the likelihood function in this example with respect to \( \tau \), which is in fact given by:

\[ I(\tau) = \frac{n(1 + \tau^2)}{(1 - \tau^2)^2} \]

Using this approximation, the hypotheses \( H_P \) and \( H_S \) defined in Section 2.5 that are applicable in the case where \( \hat{\tau} \leq 0 \), i.e. the hypotheses in equations (7) and (8), can now be expressed as:

\[ H_P : \tau \geq -\varepsilon \quad (50) \]
\[ H_S : \rho(\hat{T}^* < \hat{\tau}) \leq \Phi\left((\hat{\tau} + \varepsilon)\sqrt{I(\varepsilon)}\right) \quad (= \eta) \quad (51) \]

where \( \hat{T}^* \) is the estimator \( \hat{\tau} \) calculated exclusively on the basis of an as-yet-unobserved sample of \( n \) additional data points \( \{(X_i^*, Y_i^*) : i = 1, 2, \ldots, n\} \) drawn from the bivariate normal density function in question. On the other hand, the hypotheses \( H_P \) and \( H_S \) that apply if \( \hat{\tau} > 0 \), i.e. the hypotheses in equations (9) and (10), can be re-expressed as:

\[ H_P : \tau \leq \varepsilon \quad (52) \]
\[ H_S : \rho(\hat{T}^* > \hat{\tau}) \leq 1 - \Phi\left((\hat{\tau} - \varepsilon)\sqrt{I(\varepsilon)}\right) \quad (= \eta) \quad (53) \]

Observe that, if the definitions of the hypotheses \( H_P \) and \( H_S \) that have just been given are interpreted in terms of the approximation under discussion, they are precisely valid under the more general definitions of these two types of hypothesis given in Bowater (2019b), since it can be shown that, even having only constructed the hypotheses \( H_S \) in equations (51) and (53) in an approximate manner, they are nevertheless equivalent, under this interpretation, to the hypotheses \( H_P \) in equations (50) and (52) respectively.

To determine the fiducial density \( f_S(\theta_j | x) \) that is required by equations (12) and (14), i.e. the density \( f_S(\tau | \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, z) \) in the present case, let us begin by assuming that
the estimator \( \hat{\tau} \) is the fiducial statistic \( Q(z) \), which is actually the choice that was made for this statistic in the aforementioned example in Bowater (2018a) of the use of fiducial inference in this type of situation. However, instead of assuming that the sampling density function of \( \hat{\tau} \) is a normal density as has just been done, and as was done in the context just highlighted in Bowater (2018a), let us assume that it is a transformation of \( \hat{\tau} \) that is normally distributed, namely the function \( \tanh^{-1}(\hat{\tau}) \). The reason for doing this is that it can be shown that, under this latter assumption, a generally better approximation to the sampling density of \( \hat{\tau} \) can be obtained than under the former assumption, except, that is, when \( \tau \) is close to zero. Notice that this exception is the reason why this alternative assumption was not used to form an approximation to the hypotheses \( H_S \). More specifically, it will be assumed that the density function of \( \tanh^{-1}(\hat{\tau}) \) is directly specified (and the density function of \( \hat{\tau} \) is therefore indirectly specified) by the expression:

\[
\tanh^{-1}(\hat{\tau}) \sim N(\tanh^{-1}(\tau), 1/I(\tanh^{-1} \tau))
\]

where \( I(\tanh^{-1} \tau) \) is the Fisher information with respect to the quantity \( \tanh^{-1}(\tau) \), which is in fact given by:

\[
I(\tanh^{-1} \tau) = n(1 + \tau^2)
\]

Allowing \( \tanh^{-1}(\hat{\tau}) \) to take the role of the statistic \( Q(z) \), and using the approximation to the density function of this statistic just given, we can therefore approximate equation (3) in the case where \( \tau \) is the only unknown parameter as follows:

\[
tanh^{-1}(\hat{\tau}) = \varphi(\Gamma, \tau) = \tanh^{-1}(\tau) + \frac{\Gamma}{\sqrt{n(1 + \tau^2)}} \tag{54}
\]

where the primary r.v. \( \Gamma \sim N(0, 1) \). Although it can be shown that this equation does not generally satisfy Condition 1 of Section 2.4, it is the case, on the other hand, that if \( \Gamma \) is generated from a standard normal density function truncated to lie in a given interval \((-v, v)\) where \( v > 0 \), then this condition will be satisfied for very large values of \( v \) under the restriction that \( n \) is not too small and \( \hat{\tau} \) is not very close to \(-1\) or \(1\). For example, if

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\( n = 100 \) and \( |\hat{\tau}| < 0.999 \), then Condition 1 will be satisfied not only for small values of \( v \), but even if \( v \) is chosen to be as high as 36, and will be satisfied by substantially larger values of \( v \) as \( |\hat{\tau}| \) becomes smaller.

We will therefore make use of equation (54) under the assumption that the primary r.v. \( \Gamma \) follows the truncated normal density function just mentioned with \( v \) chosen to be equal to or not far below the largest possible value of \( v \) that is consistent with equation (54) satisfying Condition 1. Also, since the fiducial density \( f_S(\tau | \mu_x, \mu_y, \sigma^2_x, \sigma^2_y, z) \) needs to be derived under the assumption that, given the values of the conditioning parameters, there would have been no or very little pre-data knowledge about \( \tau \), it will be assumed that the GPD function of \( \tau \) is specified as follows: \( \omega_G(\tau) = b \), for \( \tau \in [-1, 1] \), where \( b > 0 \). Applying the principle outlined in Section 2.4 under the assumptions that have just been made leads to an approximation to the full conditional fiducial density of \( \tau \) that is given by

\[
f_S(\tau | \mu_x, \mu_y, \sigma^2_x, \sigma^2_y, z) = \psi_t(\gamma) \left| \frac{d\gamma}{d\tau} \right| \text{ if } \tau \in (\tau_0, \tau_1) \text{ and is zero otherwise} \quad (55)
\]

where \( \gamma \) is the value of \( \Gamma \) that solves equation (54) for the given value of \( \tau \), i.e.

\[
\gamma = (\tanh^{-1}(\hat{\tau}) - \tanh^{-1}(\tau))n^{0.5}(1 + \tau^2)^{0.5}
\]

while \( \psi_t(\gamma) \) is the standard normal density function truncated to lie in the interval \((-v, v)\) evaluated at \( \gamma \), and finally \((\tau_0, \tau_1)\) is the interval of values of \( \tau \) that, according to equation (54), correspond to \( \gamma \) lying in the interval \((-v, v)\). Under the assumption that the fiducial density \( f_S(\tau | \mu_x, \mu_y, \sigma^2_x, \sigma^2_y, z) \) is approximately determined in this manner, it can be easily seen how the specification of the post-data density \( b(\tau | \mu_x, \mu_y, \sigma^2_x, \sigma^2_y, z) \) can be completed by using the criteria of Section 2.5.

To illustrate this example, Figure 5 shows some results from running a Gibbs sampler with a uniform random scanning order of all the parameters being considered on the basis of the complete set of full conditional post-data densities that consists of
the fiducial densities \( f(\mu_x | \mu_y, \sigma_x^2, \sigma_y^2, \tau, z) \) and \( f(\mu_y | \mu_x, \sigma_x^2, \sigma_y^2, \tau, z) \) that were defined in equations (45) and (46), along with the posterior densities \( p(\sigma_x^2 | \mu_x, \mu_y, \sigma_y^2, \tau, z) \) and \( p(\sigma_y^2 | \mu_x, \mu_y, \sigma_x^2, \tau, z) \), and the post-data density \( b(\tau | \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, z) \) that were defined in the latter part of this section. In particular, the histograms in Figures 5(a) to 5(e) represent the marginal density functions of \( \mu_x, \mu_y, \sigma_x^2, \sigma_y^2 \) and \( \tau \), respectively, over a single run of ten million samples of the parameters concerned generated by the Gibbs sampler after allowing for its burn-in phase by excluding an initial five thousand samples of its output. The sampling of the densities \( p(\sigma_x^2 | \mu_x, \mu_y, \sigma_y^2, \tau, z) \), \( p(\sigma_y^2 | \mu_x, \mu_y, \sigma_x^2, \tau, z) \) and \( b(\tau | \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, z) \) was based on the Metropolis algorithm, while the sampling of each of the densities \( f(\mu_x | \mu_y, \sigma_x^2, \sigma_y^2, \tau, z) \) and \( f(\mu_y | \mu_x, \sigma_x^2, \sigma_y^2, \tau, z) \) was independent from the preceding iterations.

Moreover, the observed data set \( z \) was a typical sample of \( n = 100 \) data points from the bivariate normal distribution with \( \mu_x = 0, \mu_y = 0, \sigma_x = 1, \sigma_y = 1 \) and \( \tau = 0.3 \). Also, the specification of the posterior densities \( p(\sigma_x^2 | \mu_x, \mu_y, \sigma_y^2, \tau, z) \) and \( p(\sigma_y^2 | \mu_x, \mu_y, \sigma_x^2, \tau, z) \) were completed by giving the constants \( \alpha_x, \beta_x, \alpha_y \) and \( \beta_y \), i.e. the constants that control the choice of the prior densities of \( \sigma_x^2 \) and \( \sigma_y^2 \) in equation (47), the settings \( \alpha_x = 49.5, \beta_x = 48, \alpha_y = 49.5 \) and \( \beta_y = 34 \). On the other hand, with regard to how the post-data density \( b(\tau | \mu_x, \mu_y, \sigma_x^2, \sigma_y^2, z) \) was fully determined, it was assumed that \( \varepsilon = 0.02 \) and that the probabilities \( \kappa \) that would be assigned to the hypothesis \( H_S \) in both equations (51) and (53) for different values of the parameters \( \mu_x, \mu_y, \sigma_x^2 \) and \( \sigma_y^2 \) would again be given by the PDO curve with the formula: \( \kappa = \eta^{0.6} \), where \( \eta \) is the one-sided P value in these equations. Also, in determining this latter density, the density function \( h(\theta_j) \) that appears in equation (13), i.e. the density \( h(\tau) \) in the present case, was defined, similar to earlier examples, by the expression \( \tau \sim \text{Beta}(4, 4, -0.02, 0.02) \), where the notation here is again as used in equation (29).

Supplementary to this analysis, there was no suggestion from applying appropriate
Figure 5: Conditional prior densities of two parameters ($\sigma_x$ and $\sigma_y$) and marginal post-data densities of all five parameters of a bivariate normal distribution
diagnostics to multiple runs of the Gibbs sampler from different starting points that it
did not have a limiting density function. Furthermore, after excluding the burn-in phase
of the sampler, no statistically significant difference was found between the samples of
parameter values aggregated over the runs of the sampler in using various very distinct
fixed scanning orders of the five parameters concerned, with a single transition of the
sampler defined in the same way as in previous examples, even when the runs in question
were long. We can reasonably conclude, therefore, that the full conditional densities of the
limiting density function of the original random-scan Gibbs sampler should, at the very
least, be close approximations to the full conditional densities on which the sampler is
based, i.e. the densities $f(\mu_x | \mu_y, \sigma^2_x, \sigma^2_y, \tau, z)$, $f(\mu_y | \mu_x, \sigma^2_x, \sigma^2_y, \tau, z)$, $p(\sigma^2_x | \mu_x, \mu_y, \sigma^2_y, \tau, z)$, $p(\sigma^2_y | \mu_x, \mu_y, \sigma^2_x, \tau, z)$ and $b(\tau | \mu_x, \mu_y, \sigma^2_x, \sigma^2_y, z)$.

The solid curves overlaid on the histograms in Figures 5(a) and 5(c) are plots of
the marginal fiducial densities of the parameters $\mu$ and $\sigma$, respectively, as defined by
equations (20) and (21) that would apply if the data set of interest only consisted of
the observed values of the variable $X$, i.e. $\{x_i : i = 1, 2, \ldots, 100\}$, while in Figures 5(b)
and 5(d), the solid curves represent, respectively, the marginal fiducial densities of $\mu$ and
$\sigma$ defined in the same way except that these densities correspond to observing only the
realisations of the variable $Y$, i.e. the set of values $\{y_i : i = 1, 2, \ldots, 100\}$ (and hence this
set is now the set $x$ in the equations being discussed). On the other hand, the dashed
curves overlaid on the histograms in Figures 5(c) and 5(d) are plots of the prior densities
for $\sigma_x$ and $\sigma_y$, respectively, as defined in equation (47).

Finally, the solid curve overlaid on the histogram in Figure 5(e) is a plot of a confi-
dence density function for the parameter $\tau$. In general, a density function of this type
corresponds to a set of confidence intervals with a varying coverage probability for the
parameter concerned, see for example Efron (1993) for further clarification. More specif-
ically, for the plot being considered, these confidence intervals for $\tau$ were constructed
on the basis of assuming that the Fisher transformation of the sample correlation coefficient \( r \), i.e. the transformation \( \tanh^{-1}(r) \), has a normal sampling distribution with mean \( \tanh^{-1}(\tau) \) and variance \( 1/(n-3) \), which is a standard approximation that is used in practice to form such intervals for \( \tau \).

Similar to earlier examples, it can be seen from comparing the histograms in Figures 5(a) to 5(d) with the curves overlaid on them that the forms of the marginal post-data densities of \( \mu_x \), \( \mu_y \), \( \sigma_x \) and \( \sigma_y \) that are represented by the histograms are consistent with what we would have intuitively expected given the pre-data beliefs about these parameters and the correlation \( \tau \) that have been taken into account. Furthermore, we can observe that the marginal post-data density for \( \tau \) represented by the histogram in Figure 5(e) differs substantially from the curve overlaid on this histogram, i.e. the aforementioned type of confidence density function for \( \tau \), particularly with regard to the amount of probability mass that these two density functions assign to values of \( \tau \) close to zero. This arguably gives an indication of how inadequate it would be, in this example, to attempt to make inferences about the correlation \( \tau \) using the standard type of confidence intervals for \( \tau \) on which the overlaid curve in question is based.

3.6. Summary of other examples

As part of the discussion of the examples presented in the preceding sections, reference was made to additional examples from Bowater (2018a), Bowater (2019a) and Bowater (2019b) that fit within the inferential framework that has been put forward in the present paper. Here the opportunity will be taken to highlight examples of a similar kind from these earlier papers that have not been mentioned up to this point.

To begin with, let us remark that in Bowater (2019a), organic fiducial inference was applied to the problem of making post-data inferences about discrete probability distributions that naturally only have one unknown parameter, in particular the binomial and
Poisson distributions, and as a result, a fiducial density for the parameter concerned was determined. With regard to the binomial distribution, this application of the method of inference in question represents, of course, a special case of the type of scenario discussed in Section 3.3, i.e. the case where one of the population proportions in this latter example is set to zero. Furthermore, the problem of making post-data inferences about a binomial proportion was addressed in Bowater (2019b) by using organic fiducial inference as a supplement to bispatial inference in the way that was described in Section 2.5.

On the other hand, in Bowater (2018a), it was demonstrated how joint post-data densities for the two parameters of the Pareto, gamma and beta distributions can be determined by using the type of framework that was outlined in Section 2.2 on the basis of full conditional post-data densities of the parameters concerned that are formed by applying, in effect, organic fiducial inference, i.e. all these full conditional and joint post-data densities of the parameters were, in fact, fiducial densities. In addition, it was shown in Bowater (2019b) how the post-data density for a relative risk $\pi_t/\pi_c$ can be determined by using the kind of framework of Section 2.2 on the basis of full conditional post-data densities for the binomial proportions $\pi_t$ and $\pi_c$ that are formed by applying the type of bispatial inference outlined in Section 2.5 in a way that allows dependence to exist between $\pi_t$ and $\pi_c$ in the joint post-data density of these parameters. Finally, in Bowater (2018a), a method that was, in effect, organic fiducial inference was applied to the problem of making post-data inferences about the difference between the means of two normal density functions that have unknown variances on the basis of independent samples from the two density functions concerned, i.e. the Behrens-Fisher problem.

4. Defence and discussion of the theory

There now follows a discussion of the theory put forward in the present paper, i.e. integrated organic inference, arranged in a series of questions that one might expect would be
naturally raised as a reaction to first reading about this theory, and immediate responses
to each of these questions.

**Question 1.** *Why not always use the Bayesian approach to inference?*

As comments were already made in Section 2.3 regarding the flawed nature of two
common ‘objective’ forms of Bayesian inference, let us consider the proposal of always
making post-data inferences about model parameters using the standard or subjective
Bayesian paradigm.

The difficulty with the Bayesian paradigm has always been in choosing a prior density
function for the model parameters that adequately represents what was known about
these parameters before the data were observed. According to the definition of proba-
bility being adopted in this paper, i.e. the definition outlined in detail in Bowater and
Guzmán (2018b) that was summarised in Section 2.1, carrying out this task in an un-
satisfactory manner (which can reasonably be regarded as often being unavoidable) is
formally indicated by a low ranking being attributed to the external strength of the prior
distribution function under the assumption, which also will be made in what follows,
that the event $R(\lambda)$ is a given outcome of a well-understood physical experiment (such
as drawing a ball out of an urn of balls) and the resolution level $\lambda$ is some value in
the interval $[0.05, 0.95]$. Furthermore, it can be argued that, if we only apply Bayesian
reasoning, then this assessment of external strength should, in turn, generally result in a
similar low ranking being attributed to the external strength of the posterior distribution
function of the parameters that is based on the prior distribution function concerned.

We can observe that it is often claimed that the choice of a prior distribution function
is not such an important issue, if over a set of ‘reasonable choices’ for this distribution
function, the posterior distribution function to which it corresponds is not ‘greatly af-
fected’ by this choice. However, it is difficult for such an argument to escape the issue
that has just been raised, which, in the present context, is the question of how externally
strong should we regard any particular posterior distribution function that corresponds
to a prior distribution function that belongs to the aforementioned set assuming that we
can apply only Bayesian reasoning. Moreover, in response to the claim being considered,
it can be argued that if, for example, we had no or very little pre-data knowledge about
the parameters concerned, then the set of ‘reasonable choices’ for the prior density func-
tion would need to be so diverse that the corresponding posterior density function would
indeed be very greatly affected by which density function is chosen from this set.

Of course, if a prior density function can be found that is genuinely considered to be a
good representation of our pre-data knowledge about any given set of parameters, then
we would naturally feel much less uneasy about the appropriateness of using the Bayesian
method to make inferences about these parameters. This is the reason why this method
of inference is a critical component of the integrated framework for data analysis that
has been described in the present paper.

A more detailed discussion of the lines of reasoning that have just been presented can
be found in Bowater (2017, 2018a) and Bowater and Guzmán (2018b). Moreover, it was
also argued in detail in Bowater and Guzmán (2018b) and Bowater (2019a) that a very
high ranking of external strength may be justifiably attributed to fiducial distribution
functions that are derived using the strong or moderate fiducial argument as outlined
in Section 2.4, assuming that there was no or very little pre-data knowledge about the
parameters concerned over their permitted range of values. Partially on the basis of
this kind of reasoning, it could be argued in addition that often, in practice, similar
high rankings should be attributed to the external strengths of post-data distribution
functions derived using the type of bispatial inference described in Section 2.5, assuming
that the scenario of interest specified in this earlier section is strictly applicable.
Question 2. What about Lindley’s criticism regarding the incoherence of fiducial inference?

With reference to Fisher’s fiducial argument, it was shown in Lindley (1958) that, if the fiducial density of a parameter $\theta$ that is formed on the basis of a data set $x$ is treated as a prior density of $\theta$ in forming, in the usual Bayesian way, a posterior density of $\theta$ on the basis of a second data set $y$, then, in general, this posterior density will not be the same as the one that would be formed by repeating the same operation but with $y$ as the first data set, and $x$ as the second data set, i.e. fiducial inference generally fails to satisfy a seemingly reasonable coherency condition.

As a reaction to this, it can be remarked that fiducial inference, whether it is Fisher’s version of this type of inference, or the version outlined in the present paper, relies on pre-data knowledge, or an expression of the lack of such knowledge, being incorporated into the inferential process within the context of the observed data. Therefore, while it may be loosely acceptable, in general, to apply a blanket rule such as the strong fiducial argument without concern for the data actually observed, it is perhaps unsurprising that doing this could lead to the type of phenomenon that has just been highlighted. Also, the act of expressing pre-data knowledge is rarely going to be a completely 100% precise act no matter what paradigm of inference is adopted, therefore the door is always open for inconsistencies in the inferential process such as the one identified in Lindley (1958). Furthermore, if indeed we are in a scenario where the coherency condition being considered is not satisfied, then at least with respect to the type of fiducial inference outlined in the present paper, i.e. organic fiducial inference, it would be expected that good approximate adherence to this condition would usually be achieved providing that the data sets $x$ and $y$ referred to above are at least moderately sized. In other words, it can be argued that the practical consequences of the anomaly in question should generally be regarded as being quite small.
Observe that the same kind of anomaly is clearly also going to apply when post-data densities of given model parameters are constructed by relying on bispatial inference in the way that was described in Section 2.5. Similar arguments can be made, though, in response to the criticism in question with regard to this case as have just been presented.

Finally, we ought to mention an important issue that is related to this criticism. In particular, given that it is regarded as being appropriate in a particular context to form a post-data density function for the parameters of a model by incorporating organic fiducial inference, and possibly also bispatial inference, into the framework that has been detailed in the present paper, would it not be best to use one or both of these methods of inference to construct such a density function on the basis of a minimal part of the data set that has actually been observed, and as a next step, use this density function as a prior density in analysing the rest of the data under only the Bayesian paradigm? Although, at first sight, this strategy may appear to be a reasonable one, it has the drawback that post-data density functions constructed using organic fiducial inference on its own, or combined with bispatial inference, may well be regarded as being less adequate representations of the post-data uncertainty that is felt about the parameters concerned if they are based on a small rather than a large amount of data. For example, even if there was very little pre-data knowledge about a parameter of interest and the fiducial statistic $Q(x)$ is a sufficient statistic, it may be less appropriate to apply the strong fiducial argument to make inferences about the parameter if the data set is small rather than large. Also, with regard to bispatial inference, there is generally less chance, of course, that the one-sided P value in the hypothesis $H_S$ defined by equation (8) or (10) will be small if it is calculated on the basis of a small rather than a large data set, and as a result more chance perhaps that the interpretation of this value will be a little complicated. We are therefore led again to an issue that was discussed in the answer to Question 1 of this section, in particular the question of whether we can justifiably
attribute a very high ranking to the external strength of the prior density that is used in the second part of the type of strategy being considered and, if we can only apply Bayesian reasoning in this second stage, whether we can justifiably attribute a very high ranking to the external strength of the posterior density that results from the whole analysis?

**Question 3.** *If the choice for the fiducial statistic is not obvious, how should this statistic be chosen?*

The definition of a fiducial statistic was given in Section 2.4. As alluded to in this section, if there is not a sufficient statistic for the parameter concerned that is a natural choice for the fiducial statistic, then a fairly general choice for this statistic, which has a good deal of intuitive appeal, is the maximum likelihood estimator of the parameter. Nevertheless, it would appear that more sophisticated criteria for choosing the fiducial statistic could be easily developed so that, in general, the effect of any arbitrariness in the choice of this statistic could be assured as being negligible. Such a development though will be left for future work.

**Question 4.** *Can the results obtained from applying integrated organic inference depend on the parameterisation of the sampling model?*

There are two reasons why the parameterisation of the sampling model may possibly affect the inferences made about population quantities of interest. First, related to a point made in the answer to Question 2 in this section, it may be possible to achieve a more representative expression of pre-data knowledge about the parameters of a model using one parameterisation of the model rather than another. In this case, it is fairly obvious that ideally, out of all possible parameterisations of the model, the one should be chosen with regard to which the most representative expression of pre-data knowledge about the parameters can be achieved.
The second reason why inferences may be possibly affected by model parameterisation is related to the answer given to Question 3 in this section. In particular, it is that parameterisations may exist with regard to which fiducial statistics can be found that make more efficient use of the information contained in the data than those that can be found with regard to other parameterisations. However, it would be expected that, in general, this would not have more than a negligible effect on post-data inferences made about quantities of interest, and where its effect is more than negligible then, in the context of the first point made about the choice of model parameterisation, there clearly should be a preference for those parameterisations that allow fiducial statistics to be chosen that make the best use of the information that is in the data.

**Question 5.** In cases where the framework of Section 2.2 produces a joint post-data density for the parameters of the model that has full conditional densities that only approximate the full conditional post-data densities that were originally constructed by directly using the methods in Sections 2.3, 2.4 and 2.5, how good in general are these approximations?

In the examples in the present paper where this type of question is relevant, i.e. in the examples discussed in Sections 3.2 to 3.5, justifications were given, on the basis of a general line of reasoning outlined in Section 2.2, as to why, in each of these examples, it can be concluded that the approximations of the type referred to in this question should be very good approximations, assuming of course that are indeed only approximations, i.e. assuming that the originally constructed full conditional post-data densities are incompatible.

This concludes the discussion of the theory put forward in the present paper, i.e. integrated organic inference. It is hoped that it will be appreciated that this theory modifies, generalises and extends Fisherian inference, and naturally combines it with Bayesian inference in a way that constitutes a major advance on the level of sophistication.
of either of these two older schools of inference.

References

Arnold, B. C., Castillo, E., Sarabia, J. M. (2002). Exact and near compatibility of discrete conditional distributions. *Computational Statistics and Data Analysis*, 40, 231–252.

Arnold, B. C. and Press, S. J. (1989). Compatible conditional distributions. *Journal of the American Statistical Association*, 84, 152–156.

Bayes, T. (1763). An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society*, 53, 370–418.

Bowater, R. J. (2017). A defence of subjective fiducial inference. *AStA Advances in Statistical Analysis*, 101, 177–197.

Bowater, R. J. (2018a). Multivariate subjective fiducial inference. *arXiv.org (Cornell University), Statistics*, arXiv:1804.09804.

Bowater, R. J. and Guzmán, L. E. (2018b). On a generalized form of subjective probability. *arXiv.org (Cornell University), Statistics*, arXiv:1810.10972.

Bowater, R. J. (2019a). Organic fiducial inference. *arXiv.org (Cornell University), Statistics*, arXiv:1901.08589.

Bowater, R. J. (2019b). Sharp hypotheses and bispatial inference. *arXiv.org (Cornell University), Statistics*, arXiv:1911.09049.

Brooks, S. P. and Roberts, G. O. (1998). Convergence assessment techniques for Markov chain Monte Carlo. *Statistics and Computing*, 8, 319–335.
Chen, S-H. and Ip, E. H. (2015). Behaviour of the Gibbs sampler when conditional distributions are potentially incompatible. *Journal of Statistical Computation and Simulation*, **85**, 3266–3275.

Chen, S-H., Ip, E. H. and Wang, Y. J. (2011). Gibbs ensembles for nearly compatible and incompatible conditional models. *Computational Statistics and Data Analysis*, **55**, 1760–1769.

Efron, B. (1993). Bayes and likelihood calculations from confidence intervals. *Biometrika*, **80**, 3–26.

Gelfand, A. E. and Smith, A. F. M. (1990). Sampling-based approaches to calculating marginal densities. *Journal of the American Statistical Association*, **85**, 398–409.

Gelman, A. and Rubin, D. B. (1992). Inference from iterative simulation using multiple sequences. *Statistical Science*, **7**, 457–472.

Geman, S. and Geman, D. (1984). Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **6**, 721–741.

Jeffreys, H. (1961). *Theory of Probability*, 3rd edition, Oxford University Press, Oxford.

Kass, R. E. and Wasserman, L. (1996). The selection of prior distributions by formal rules. *Journal of the American Statistical Association*, **91**, 1343–1370.

Kuo, K-L., Song, C-C. and Jiang, T. J. (2017). Exactly and almost compatible joint distributions for high-dimensional discrete conditional distributions. *Journal of Multivariate Analysis*, **157**, 115–123.

Kuo, K-L. and Wang, Y. J. (2011). A simple algorithm for checking compatibility among discrete conditional distributions. *Computational Statistics and Data Analysis*, **55**, 67
Lindley, D. V. (1958). Fiducial distributions and Bayes’ theorem. *Journal of the Royal Statistical Society, Series B, 20*, 102–107.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H. and Teller, E. (1953). Equation of state calculations by fast computing machines. *Journal of Chemical Physics, 21*, 1087–1092.

Muré, J. (2019). Optimal compromise between incompatible conditional probability distributions, with application to objective Bayesian kriging. *ESAIM: Probability and Statistics, 23*, 271–309.