WEIGHTED BERGMAN KERNEL FUNCTIONS
ASSOCIATED TO MEROMORPHIC FUNCTIONS

ROBERT JACOBSON

Abstract. We present a technique for computing explicit, concrete formulas for the weighted Bergman kernel on a planar domain with weight the modulus squared of a meromorphic function in the case that the meromorphic function has a finite number of zeros on the domain and a concrete formula for the unweighted kernel is known. We apply this theory to the study of the Lu Qi-keng Problem.

1. Introduction

The Bergman kernel function has been called a cornerstone of geometric function theory [10] and is an object of considerable study in complex analysis. The problems of computing explicit formulas for this function and determining its zero set are classical problems in complex analysis. Domains for which the associated Bergman kernel is zero-free are called Lu Qi-keng domains, and the problem of determining which domains are Lu Qi-keng is known as the Lu Qi-keng Problem. This problem is of interest in the study of Bergman representative coordinates which require the kernel to be zero-free (see [7, 8]). The Lu Qi-keng Problem for smooth planar domains has been solved [14], but a solution for higher dimensions is not yet known [2]. The property of having a zero-free kernel is also a biholomorphic invariant and hence may be used to distinguish biholomorphic equivalence classes.

The main result of this paper is Theorem 4 which allows one to write certain weighted Bergman kernels on the plane in terms of other weighted Bergman kernels with simpler weights. One consequence of this theorem is that if one has an explicit, concrete formula for an unweighted kernel, then one can compute an explicit, concrete formula for the weighted kernel whenever the weight is the modulus squared of a meromorphic function with finitely many zeros on the associated domain. By the well-known technique of Theorem 1 weighted kernels for domains on the plane are related to unweighted kernels for domains in $\mathbb{C}^2$. Thus, the results presented here that are specific to complex dimension 1 have relevance to the classical problems of the first paragraph, in particular the Lu Qi-keng Problem, in complex dimension 2. With the 2-dimensional Lu Qi-keng Problem in mind, we study the zero sets of weighted kernels in Section 4.

Date: May 22, 2014.

2010 Mathematics Subject Classification. Primary 32A25; Secondary 32A36.

Key words and phrases. Bergman kernel function, Bergman space.

This work was partially supported by a grant from The Foundation To Promote Scholarship & Teaching at Roger Williams University.
The Bergman kernel for a domain \( \Omega \subset \mathbb{C} \) is the unique skew-symmetric sesqui-holomorphic\(^1\) function \( K^\Omega : \Omega \times \Omega \to \mathbb{C} \) with the reproducing property
\[
 f(z) = \langle f, K^\Omega (\cdot, z) \rangle = \int_{\Omega} f(w) K^\Omega (z, w) \, dV_w \quad \text{for all } f \in A^2(\Omega),
\]
where \( dV_w \) is the real \( 2n \)-dimensional Lebesgue volume (or area) measure, and
\( A^2(\Omega) \) is the Hilbert space of square-integrable holomorphic functions on \( \Omega \), called the Bergman space. (When the domain is clear, we will omit it from the superscript of \( K \).) Equivalently, if \( \{ \phi_j \}_{j=0}^\infty \) is an orthonormal Hilbert space basis for \( A^2(\Omega) \), then the Bergman kernel function \( K^\Omega (z, w) \) is given by
\[
 K^\Omega (z, w) := \sum_{j=0}^\infty \overline{\phi_j(z)} \phi_j(w).
\]

Also of present interest is the weighted Bergman kernel with respect to a nonnegative weight function \( \varphi \), which we denote \( K^\Omega_\varphi (z, w) \). Replacing the inner product in (1) with the weighted inner product
\[
 \langle f, g \rangle_\varphi := \int_{\Omega} f(w) g(w) \varphi(w) \, dV_w,
\]
the kernel \( K^\Omega_\varphi (z, w) \) is the unique reproducing kernel for the weighted Bergman space \( A^2_\varphi (\Omega) = \{ f \mid \langle f, f \rangle_\varphi < \infty \text{ and } f \text{ holomorphic} \} \).

The details of this classical theory can be found in many texts on complex analysis, for example in [1, 9].

2. Preliminary theory

To study the Lu Qi-keng Problem in higher dimensions, we would like concrete examples of kernels on domains in \( n \)-dimensional complex space, but obtaining a closed-form formula for the kernel from (2) is possible only for domains with a high degree of symmetry. There are, however, several techniques for relating the kernel of one domain to the kernel of another domain of different complex dimension (see [3]). We shall make crucial use of the following known result.

**Theorem 1.** Let \( D \) be a bounded domain in \( \mathbb{C} \), let \( \varphi(z) : D \to [0, \infty) \) be a weight function on \( D \), and let \( \Omega \) be defined by \( \Omega := \{ (z, w) \in \mathbb{C}^2 \mid z \in D, |w| < \varphi(z) \} \subset \mathbb{C}^2 \). Then \( K_{\pi \varphi^2}(z, w) \equiv K(z, 0, w, 0) \).

The idea behind this theorem appears in the literature in various forms. Theorem 1 is essentially Corollary 2.1 of [11] which Ligocka, generalizing an idea found in a proof due to Forelli and Rudin in [5], calls the Forelli–Rudin construction. The term Forelli–Rudin construction appears elsewhere in subsequent literature in reference to similar techniques. Such techniques are surveyed in [3].

Our primary goal is to express a weighted kernel in terms of another weighted kernel that is in some sense simpler than the first. The following theorem is the simplest case of such a theorem and is fundamental to the rest of the theory.

**Theorem 2.** Let \( \Omega \subset \mathbb{C}^n \), let \( K_\varphi(z, w) \) be the weighted Bergman kernel on \( \Omega \) with respect to a weight function \( \varphi \), and let \( g \) be meromorphic on \( \Omega \). Suppose that, after

---

\(^1\)Sesqui-holomorphic means holomorphic in the first variable and conjugate holomorphic in the second variable.
possibly removing singularities, $\frac{K(\cdot,\cdot)}{g(\cdot)}$ is holomorphic in $z$. Then $K_{\varphi} |g|^2(\cdot,\cdot) = \frac{K(\cdot,\cdot)}{g(\cdot)g(\cdot)}$.

Proof. The weight $\varphi$ plays no role in the following argument, so for simplicity of notation, we suppress the subscript $\varphi$ in the calculation. We have that

$$\int_{\Omega} \left| K(z,\cdot) \right|^2 |g(z)|^2 dV_z = \|K(\cdot,\cdot)\|^2 < \infty,$$

so

$$K(z,\cdot) = \frac{K(z,\cdot)}{g(z)} \in A^2|g|^2(\Omega) \text{ (as a function of } z).$$

Also,

$$\int_{\Omega} |K(z,\cdot)|^2 |g(z)|^2 dV_z = \|K_{|g|^2}(\cdot,\cdot)\|^2 < \infty,$$

so

$$K_{|g|^2}(z,\cdot)g(z) \in A^2(\Omega) \text{ (as a function of } z).$$

By (3) and the reproducing property of $K_{|g|^2}(z,\cdot)$, we have

$$\frac{K(z,\cdot)}{g(z)} = \int_{\Omega} \frac{K(\zeta,\cdot)}{g(\zeta)} K_{|g|^2}(z,\cdot) |g(\zeta)|^2 dV_{\zeta}$$

$$= \int_{\Omega} K(\zeta,\cdot) K_{|g|^2}(z,\zeta) \overline{g(\zeta)} dV_{\zeta}$$

$$= \int_{\Omega} K(z,\cdot) K_{|g|^2}(\zeta,z) g(\zeta) dV_{\zeta}.$$

By (3) and the reproducing property of the kernel $K(z,\cdot)$, this last expression is

$$K_{|g|^2}(\cdot,\cdot)g(\cdot) = g(\cdot) K_{|g|^2}(z,\cdot).$$

We have shown that $\frac{K(z,\cdot)}{g(z)} = \frac{g(w)}{g(z)} K_{|g|^2}(z,\cdot)$, from which the theorem follows. \hfill $\square$

This theorem and the ancillary Theorem 14 are the only multidimensional theorems in this paper. The other results are specific to domains of dimension 1. As described above, the 1-dimensional results together with Theorem 1 can be used to study domains in higher dimensions. Indeed, Theorem 2 provides a recipe, illustrated by Example 3, for constructing non Lu Qi-keng domains in $\mathbb{C}^2$. The technique of this example, though elementary, appears to be absent from the literature.

Example 3. Let $c$ be a point in the open unit disk $\mathbb{D}$, and define $\varphi(z) := |z - c|^{-2}$ and $\Omega := \{ (z,w) \in \mathbb{C}^2 \mid z \in \mathbb{D}, |w| < \sqrt{\int \varphi(z)} \} \subset \mathbb{C}^2$. By Theorem 1 the kernel $K_\varphi^D(z,\cdot)$ satisfies $K_\varphi^D(z,w) = K^D(z,0,w,0)$. Theorem 2 gives $K_\varphi^D(z,w) = (z - c)K^D(z,w)(\overline{w} - \overline{c})$, which clearly has zeros whenever $z = c$ or $w = c$. Hence $\Omega$ is not Lu Qi-keng.

The domain in Example 3 is an unbounded domain, but a bounded non Lu Qi-keng domain can be obtained via Ramanadov’s Theorem together with Hurwitz’s Theorem.

Our goal is now to obtain a formula for a weighted kernel explicitly in terms of the unweighted kernel when the weight is the modulus squared of a meromorphic function. Theorem 2 allows us to handle the poles: the poles appear as zeros of the same order in the formula for the weighted kernel given by Theorem 2. On the
other hand, any zeros of the meromorphic function associated to the weight clearly cannot appear as poles in the formula for the weighted kernel since the kernel is holomorphic.

\section{Decomposition Theorems}

Theorem 2 needs modification in the case that the meromorphic function in the weight vanishes. The goal of this section is to show how to accomplish this modification in dimension 1. For a general planar domain $\Omega$ and holomorphic function $f$, we are able to express $K^\Omega_{f^2}(z, w)$ in terms of the kernel associated to a “simpler” weight function and the basis functions for the orthogonal complement of $A^2_{|z|^2}(\Omega)$ in a larger space of functions; when $\varphi$ is both bounded and bounded away from zero near $c$, the normalized function $\frac{K^\Omega_{\varphi}(z, c)}{(z-c)^2}$ turns out to span the orthogonal complement of $A^2_{|z-c|^2\varphi(z)}(\Omega)$ in $A^2_{|z-c|^2\varphi(z)}(\Omega \setminus \{c\})$.

\begin{theorem}
Let $\Omega \subset \mathbb{C}$ be a domain, $c \in \Omega$, and $\varphi$ be a weight on $\Omega$ which is bounded in a neighborhood of $c$. Then

\begin{equation}
K^\Omega_{|z-c|^2\varphi}(z, w) = \frac{K^\Omega_{\varphi}(z, w)}{(z-c)(\bar{w}-\bar{c})} - \frac{K^\Omega_{\varphi}(z, c)K^\Omega_{\varphi}(c, w)}{(z-c)(\bar{w}-\bar{c})K^\Omega_{\varphi}(c, c)}.
\end{equation}

\end{theorem}

\textbf{Remark.} The requirement that $\varphi$ be bounded in a neighborhood of $c$ excludes degenerate cases such as $\varphi(z) = \frac{1}{|z|^2}$ with $c = 0$. The right hand side of Equation 3 apparently has singularities at $z = c$ and $w = c$, but these singularities are removable.

\textbf{Proof.} Let $\psi(z) := \frac{K^\Omega_{\varphi}(z, c)}{(z-c)^2}$. Clearly $\psi \in A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$. Our strategy is as follows:

1. $K^\Omega_{|z-c|^2\varphi}$ reproduces elements of $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$.

2. $\psi(z)$ is orthogonal to $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$; as a consequence, $\psi(z)$ is orthogonal to $K^\Omega_{|z-c|^2\varphi}(z, w)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$.

4. From (1) and (2), $Q(z, w) := \frac{K^\Omega_{\varphi}(z, w)}{(z-c)(\bar{w}-\bar{c})} - c_0(w)\psi(z)$ also reproduces elements of $A^2_{|z-c|^2\varphi}(\Omega)$ in $A^2_{|z-c|^2\varphi}(\Omega \setminus \{c\})$, where $c_0(w)$ is arbitrary.

5. Setting $c_0(w) := \frac{\psi(w)}{K^\Omega_{\varphi}(c, c)}$, we have $Q \in A^2_{|z-c|^2\varphi}(\Omega)$; it follows from (1) and the uniqueness of the Bergman kernel that $Q(z, w) \equiv K^\Omega_{|z-c|^2\varphi}(z, w)$.

Once (1) and (2) are proven, (3) and (4) are obvious.
Proof of (1): Let $f \in A^2_{|z-c|^2\varphi}(\Omega)$. We have
\[
\int_{\Omega \setminus \{c\}} f(w)\frac{K^\Omega_\varphi(z,w)}{(z-c)(w-c)}|w-c|^2\varphi(w)\,dV_w
= \frac{1}{z-c} \int_{\Omega} K^\Omega_\varphi(z,w)f(w)(w-c)\varphi(w)\,dV_w
= \frac{1}{z-c}f(z)(z-c)
= f(z).
\]
This proves (1).

Proof of (2): Let $f \in A^2_{|z-c|^2\varphi}(\Omega)$. We have
\[
\int_{\Omega \setminus \{c\}} f(w)\overline{\psi(w)}|w-c|^2\varphi(w)\,dV_w
= \int f(w)\frac{K^\Omega_\varphi(w,c)}{w-c}|w-c|^2\varphi(w)\,dV_w
= \int f(w)(w-c)K^\Omega_\varphi(c,w)\varphi(w)\,dV_w
= 0
\]
(since $f(z)(z-c) \in A^2_\varphi(\Omega)$).
This proves (2).

To finish the proof, observe that for $c_0(w) := \overline{\psi(w)}/K^\Omega_\varphi(c,c)$, we have that
\[
Q(z,w) := \frac{K^\Omega_\varphi(z,w)}{(z-c)(\overline{w-c})} - \frac{K^\Omega_\varphi(z,c)K^\Omega_\varphi(c,w)}{(z-c)(\overline{w-c})K^\Omega_\varphi(c,c)},
\]
which has a removable singularity at $z = c$ and $w = c$. Thus (3) holds, and the theorem is proven.

Theorem 4 combined with Theorem 2 allows one to produce an explicit formula for $K^\Omega_{|z|^2}(z,w)$ in terms of $K^\Omega(z,w)$ in the case that $f$ is meromorphic on $\Omega$ with a finite number of zeros by just iterating the formula of Equation 5. In fact, Theorem 4 is a special case of the following more general theorem.

**Theorem 5.** Let $\Omega$ be a planar domain, $\{c_j\}^m_{j=1}$ a sequence of $m$ distinct points in $\Omega$, $\{\alpha_j\}^m_{j=1}$ a sequence of positive integers, and $\varphi$ a weight such that for all $j$, $\varphi$ is both bounded and bounded away from zero in a neighborhood of $c_j$. Define the following polynomials:
\[
p(z) := (z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\cdots(z-c_m)^{\alpha_m};
p_{j,k}(z) := (z-c_1)^{\alpha_1}(z-c_2)^{\alpha_2}\cdots(z-c_{j-1})^{\alpha_{j-1}}(z-c_j)^{\alpha_j}(z-c_{j+1})^{\alpha_{j+1}}(z-c_{j+2})^{\alpha_{j+2}}\cdots(z-c_m)^{\alpha_m},
\]
Then
\[
K^\Omega_{|p(z)|^2\varphi}(z,w) = \frac{K^\Omega_\varphi(z,w)}{p(z)p(w)} - \sum_{j=1}^m \sum_{k=1}^{\alpha_j} \frac{K^\Omega_{|p_{j,k}(z)|^2\varphi}(c_j,c_j)K^\Omega_{|p_{j,k}(z)|^2\varphi}(c_j,w)}{p_{j,k}(z)p_{j,k}(w)K^\Omega_{|p_{j,k}(z)|^2\varphi}(c_j,c_j)}.
\]
Remark 6. By the $L^2$-version of the Riemann Removable Singularity Theorem [13], when a weight $\psi$ is both bounded and bounded away from zero in a neighborhood of $c$, then $K_\psi^{\Omega\setminus\{c\}}(z, w) \equiv K_\psi^\Omega(z, w)$.

Proof. We wish to show that the functions $\psi_{j,k}(z) := \frac{K_{|z-c_j|^2\varphi}^{\Omega}(z, c_j)}{p_{j,k}(z)}$ form a basis for the orthogonal complement of $A^2_{|p|^2\varphi}(\Omega)$ in $A^2_{|p|^2\varphi}(\Omega \setminus \{c_j\}_{j=1}^m)$. We prove only that the $\psi_{j,k}$ are mutually orthogonal, the rest of the proof being an easy exercise.

For $\psi_{j_0,k_0}$ and $\psi_{j_1,k_1}$ distinct, we may assume $j_0 > j_1$ or else $j_0 = j_1$ and $k_0 > k_1$. Then

$$p_{j_0,j_1}(z) = p_{j_1,k_1}(z)(z - c_{j_1})^{\alpha_{j_1} - k_1}(z - c_{j_1+1})^{\alpha_{j_1+1}} \cdots (z - c_{j_0})^{k_0},$$

and

$$\langle \psi_{j_0,k_0}(z), \psi_{j_1,k_1}(z) \rangle_{|p|^2\varphi} \equiv \int_{\Omega \setminus \{c_j\}_{j=1}^m} \frac{K_{|q_{j_0,k_0}|^2\varphi}(z, c_{j_0})}{p_{j_0,k_0}(z)} \frac{K_{|q_{j_1,k_1}|^2\varphi}(c_{j_1}, z)}{p_{j_1,k_1}(z)} |p(z)|^2 \varphi(z) \, dV_z \equiv \int_{\Omega} K_{|q_{j_0,k_0}|^2\varphi}(z, c_{j_0}) \times K_{|q_{j_1,k_1}|^2\varphi}(c_{j_1}, z) |(z - c_{j_1})^{\alpha_{j_1} - k_1}(z - c_{j_1+1})^{\alpha_{j_1+1}} \cdots (z - c_{j_0})^{k_0}| \times |q_{j_0,k_0}(z)|^2 \varphi(z) \, dV_z \equiv 0.$$

The proof does not depend on $m$ being finite; we can still construct an orthonormal basis for the orthogonal complement of $A^2_{|p|^2\varphi}(\Omega)$ in $A^2_{|p|^2\varphi}(\Omega \setminus \{c_j\}_{j=1}^m)$. However, this is of limited practical value since in that case Theorem 5 fails to give a closed form expression for the original weighted kernel. Moreover, in practice the simpler Theorem 4 is sufficient.

4. Zeros of weighted kernels

We now study the relationship of the zeros of these weighted kernels have to the zeros of the simpler kernels.

Theorem 7. Let $\Omega$ be a domain in $\mathbb{C}$, let $c, z_0, w_0 \in \Omega$, and let $\varphi$ be a weight on $\Omega$ that is bounded and bounded away from zero in some neighborhood of $c$. Suppose $K_{|z-c|^2\varphi}(z_0, w_0) = 0$. Then $K_\varphi(z_0, w_0) = 0$ if and only if either $K_\varphi(z_0, c) = 0$ or $K_\varphi(c, w_0) = 0$.

Proof. By the hypothesis and Theorem 4

$$0 = \frac{K_\varphi(z_0, w_0)}{(z_0 - c)(w_0 - c)} - \frac{K_\varphi(z_0, c)K_\varphi(c, w_0)}{(z_0 - c)(w_0 - c)K_\varphi(c, c)},$$

from which the theorem is evident. □

Requiring that $\varphi$ be bounded and bounded away from zero in a neighborhood of $c$ determines the order of the zero of the weight $|z - c|^2\varphi(c)$ to be two, a fact to which there are two significant consequences. First, as a consequence of the $L^2$-version of the Riemann Removable Singularity Theorem, $K_\varphi^{\Omega\setminus\{c\}}(z, w) \equiv K_\varphi^\Omega(z, w)$.
on \((\Omega \setminus \{c\}) \times (\Omega \setminus \{c\})\). We employ this fact in the next several theorems without comment. Second, for zeros of higher orders in the weight, we would need to use Theorem 5 rather than Theorem 4 which would not give the conclusion of Theorem 7.

Theorem 7 says the value of \(K^\Omega_\varphi(z, w)\) at \(c\) affects the zero set of \(K^\Omega_{|z-c|^2\varphi}(z, w)\). Compare this to the case that \(c \not\in \Omega\), in which case Theorem 2 says that the zero sets of both kernels coincide.

Theorem 7 assumes \(K^\Omega_{|z-c|^2\varphi}(z, w)\) has a zero and then says when \(K^\Omega_\varphi(z, w)\) has a zero. The next theorem assumes \(K_\varphi(z, w)\) has a zero and then says when \(K^\Omega_{|z-c|^2\varphi}(z, w)\) has a zero.

**Theorem 8.** Let \(\Omega\) be a domain in \(\mathbb{C}\), let \(z_0, c \in \Omega\) with \(z_0 \neq c\), and let \(\varphi\) be a weight on \(\Omega\) that is bounded and bounded away from zero in some neighborhood of \(c\). Suppose \(K_\varphi(z_0, c) = 0\). Then \(K^\Omega_{|z-c|^2\varphi}(z_0, w)\) has a zero of order \(m - 1\) at \(w = c\) if and only if \(K_\varphi(z_0, w)\) has a zero of order \(m\) at \(w = c\).

**Proof.** By Theorem 4

\[
K^\Omega_{|z-c|^2\varphi}(z_0, w) = \frac{K_\varphi(z_0, w)}{(z_0 - c)(\overline{w} - \overline{c})} - \frac{K_\varphi(z_0, c)K_\varphi(c, w)}{(z_0 - c)(\overline{w} - \overline{c})}K_\varphi(c, c)
\]

\[
= \frac{1}{z_0 - c} \frac{K_\varphi(z_0, w)}{\overline{w} - \overline{c}}
\]

If \(m\) is the order of the zero of \(K_\varphi(z_0, w)\) at \(w = c\), then this last expression has a zero of order \(m - 1\) at \(w = c\). \(\Box\)

**Theorem 9.** Let \(\Omega\) be a domain in \(\mathbb{C}\), let \(c_0, c_1, c_2 \in \Omega\) be distinct, and let \(\varphi\) be a weight on \(\Omega\) that in some neighborhood of \(c_0\) is bounded and bounded away from zero. Suppose either \(K_\varphi(c_0, c_1) = 0\) or \(K_\varphi(c_0, c_2) = 0\). Then \(K^\Omega_{|z-c_0|^2\varphi}(c_1, c_2) = 0\) if and only if \(K_\varphi(c_1, c_2) = 0\).

**Proof.** By Theorem 4

\[
K^\Omega_{|z-c_0|^2\varphi}(c_1, c_2) = \frac{K_\varphi(c_1, c_2)}{(c_1 - c_0)(c_2 - c_0)} - \frac{K_\varphi(c_1, c_0)K_\varphi(c_0, c_2)}{(c_1 - c_0)(c_2 - c_0)K_\varphi(c_0, c_0)}
\]

\[
= \frac{1}{(c_1 - c_0)(c_2 - c_0)} \cdot K_\varphi(c_1, c_2),
\]

from which the theorem is evident. \(\Box\)

**Theorem 10.** Let \(\Omega\) be a domain in \(\mathbb{C}\), and let \(\varphi\) be a weight on \(\Omega\). Suppose that for some \(c_0 \in \partial \Omega\) and some sequence \(\{c_j\}_{j=1}^\infty\) in \(\Omega\) converging to \(c_0\), we have \(\frac{K_\varphi(z_j, c_0)}{K_\varphi(c_j, c_0)} \to 0\) as \(j \to \infty\) for all fixed \(z \in \Omega\). Suppose also that there exist \(z_0, w_0 \in \Omega\) such that \(K_\varphi(z_0, w_0) = 0\) and that \(K_\varphi(z, c_j)\) is bounded away from 0 when \(j\) is large enough and \(z\) is in a compact subset of \(\Omega\). Then for sufficiently large \(j\) (i.e., for \(c_j\) sufficiently close to \(c_0 \in \partial \Omega\)), there exists a \(z_1 = z_1(c_j) \in \Omega\) near \(z_0\) such that \(K^\Omega_{|z-c_j|^2\varphi}(z_1, w_0) = 0\).
Proof. Define the following for all $\zeta, \omega, z \in \Omega$ and $\varepsilon > 0$:

$$g_{\zeta, \omega}(z) := \frac{K_\varphi(z, \omega)}{K_\varphi(z, \zeta)};$$

$$\alpha(\zeta) := |g_{\zeta, \omega_0}(\zeta)| = \frac{|K_\varphi(\zeta, \omega_0)|}{|K_\varphi(\zeta, \zeta)|};$$

$$B(z, \varepsilon) := \{w \in \Omega \mid |z - w| < \varepsilon\} \text{ (the usual open } \varepsilon\text{-ball about } z).$$

Observe that by hypothesis, $\alpha(c_j) \to 0$ as $j \to \infty$. Let $d := \frac{1}{2} \text{dist}(z_0, \partial \Omega)$. Choose a $j_0 \in \mathbb{N}$ so that the following hold:

1. $\frac{1}{j_0} < d$, and
2. $|c_j - c_0| < \frac{1}{j_0}$ for all $j > j_0$.

By (1) and the definition of $d$,

3. the closed ball $B\left(z_0, \frac{1}{j_0}\right)$ is contained in $\Omega$.

By hypothesis, for $j$ large enough, $K_\varphi(z, c_j)$ is bounded away from zero for $z \in B\left(z_0, \frac{1}{j_0}\right)$. Thus for $j$ large enough, the zeros of $g_{c_j, \omega_0}(z) := \frac{K_\varphi(z, \omega_0)}{K_\varphi(z, c_j)}$ correspond to the zeros of $K_\varphi(z, \omega_0)$ on $B\left(z_0, \frac{1}{j_0}\right)$. So by possibly increasing $j_0$, we can choose $j_0$ large enough so that we also have

4. $B\left(z_0, \frac{1}{j_0}\right)$ contains a single zero of $g_{c_j, \omega_0}(z)$ when $j > j_0$, namely $z_0$.

Now choose $j_1 \geq j_0$ such that

5. $\alpha(c_j) < \frac{1}{j_0}$ for all $j \geq j_1$, and

6. $\alpha(c_j) < \inf\left\{|g_{c_{j_1}, \omega_0}(z)| \mid z \in \partial B\left(z_0, \frac{1}{j_0}\right)\right\}$ for all $j \geq j_1$.

Now we argue that $C_0 := g_{c_{j_1}, \omega_0}(\partial B\left(z_0, \frac{1}{j_0}\right))$ is a closed curve about the origin and the point $g_{c_{j_1}, \omega_0}(c_{j_1})$. Since $z_0$ is a zero of the holomorphic function $g_{c_{j_1}, \omega_0}(z)$ and $\partial B\left(z_0, \frac{1}{j_0}\right)$ is a closed curve about $z_0$, it follows from the argument principle of the elementary theory of holomorphic functions that $C_0$ is a closed curve about the origin. Moreover, $\alpha(c_{j_1}) < \inf\left\{|g_{c_{j_1}, \omega_0}(z)| \mid z \in \partial B\left(z_0, \frac{1}{j_0}\right)\right\}$ by (6), and so $C_0$ also encloses a region containing $g_{c_{j_1}, \omega_0}(c_{j_1})$, that is, $|g_{c_{j_1}, \omega_0}(c_{j_1})| < |g_{c_{j_1}, \omega_0}(z)|$ on $\partial B\left(z_0, \frac{1}{j_0}\right)$. By Rouché’s Theorem [4, p. 110], it follows that the function $g_{c_{j_1}, \omega_0}(z) - g_{c_{j_1}, \omega_0}(c_{j_1})$ has a zero in $B\left(z_0, \frac{1}{j_0}\right)$. Hence for some $z_1 \in B\left(z_0, \frac{1}{j_0}\right)$, we have $g_{c_{j_1}, \omega_0}(z_1) = g_{c_{j_1}, \omega_0}(c_{j_1})$, which is equivalent to $\frac{K_\varphi(z_1, \omega_0)}{K_\varphi(z_1, c_{j_1})} = K_\varphi(c_{j_1}, \omega_0)$. Since both $|z_0 - z_1| < d$ and $|c_0 - c_{j_1}| < d$, it must be that $z_1 \neq c_{j_1}$. Therefore $K_{|z-c_{j_1}|^2 \varphi(z_1, \omega_0)} = 0$.

When $c \notin \Omega$, then $K_{|z-c|^2 \varphi}(z, w) = \frac{K_\varphi(z, w)}{(z-c)^2}$, by Theorem [2] so the zero set of $K_{|z-c|^2 \varphi}(z, w)$ corresponds to the zero set of $K_\varphi(z, w)$ in that case. An interpretation of Theorem [10] is that for $c \in \Omega$ as $c$ approaches the boundary of $\Omega$, the zero set of $K_{|z-c|^2 \varphi}(z, w)$ approaches the zero set of $K_\varphi(z, w)$. The following corollary to Theorem [8] does not assume that $c$ is near the boundary of $\Omega$, though unlike in Theorem [10] we assume $c$ is adapted to a zero of the kernel.
Corollary 11 (Corollary to Theorem 8). Let $\Omega$ be a domain in $\mathbb{C}$, and let $\varphi$ be a weight on $\Omega$. Suppose $c, w_0 \in \Omega$ such that $K_\varphi(z, w_0)$ has a zero of order $m > 1$ at $z = c$. Then there exist $z_1, z_2, \ldots, z_{m-1}, w_1 \in \Omega$ with the $z_j$ near $z_0$ and $w_1$ near $w_0$ such that $K_{|z-c|^2\varphi}(z, w_1) = 0$ for $j = 1, \ldots, m - 1$.

Proof. Apply Hurwitz’s Theorem to the conclusion of Theorem 8. □

Theorem 12.

A. Suppose $\Omega \subset \mathbb{C}$ is a domain, and $\varphi$ a weight, and $\{c_j\}_{j=1}^\infty$ is a sequence in $\Omega$ converging to a point $c_0 \in \partial\Omega$ such that for fixed $z$, $\frac{K_\varphi(z, c_j)}{\sqrt{K_\varphi(c_j, c_j)}} \to 0$ as $j \to \infty$. Suppose also that $K_{|z-c|^2\varphi}(z_0, w_0) = 0$ for all $c \in \Omega$. Then either (a) both $K_\varphi(z_0, w) \equiv 0$ and $K_{|z-c|^2\varphi}(z_0, w) \equiv 0$ as functions of $w$ for all $c$; or (b) both $K_\varphi(z, w_0) \equiv 0$ and $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as functions of $z$ for all $c$.

B. For any domain $\Omega$ and weight $\varphi$, if $K_\varphi(z, w_0) \equiv 0$ as a function of $z$, then for all $c \in \mathbb{C}$, $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as well.

Remark. Part (B) is similar to Theorem 7 and follows from Theorem 7, the hypothesis that $K_\varphi(z, w_0) \equiv 0$, and continuity.

Proof. We prove part (A) first. The proof of part (B) will be obvious from the proof of part (A) and is omitted.

Let $c \in \Omega$. Assume first that $z_0 \neq c$ and $w_0 \neq c$. Then by Theorem 4 we must have

$\frac{K_\varphi(z_0, w_0)}{K_\varphi(c, c)} = \frac{K_\varphi(z_0, c)K_\varphi(c, w_0)}{K_\varphi(c, c)}$.

The right hand side of Equation 6 vanishes when we replace $c$ with $c_j$ and let $j \to \infty$. Hence $K_\varphi(z_0, w_0) = 0$, and therefore either $K_\varphi(z_0, c) = 0$ or $K_\varphi(c, w_0) = 0$. One of these two conditions must hold for a set of values of $c$ having an accumulation point, hence for all $c$. Assume without loss of generality that $K_\varphi(c, w_0) = 0$ for all $c$. Thus $K_\varphi(z, w_0) \equiv 0$ as a function of $z$. But then

$K_\varphi(z, w_0) = \frac{K_\varphi(z, c)K_\varphi(c, w_0)}{K_\varphi(c, c)} = 0$ for all $z$,

and hence (by Theorem 3) $K_{|z-c|^2\varphi}(z, w_0) \equiv 0$ as a function of $z$. □

Since Theorems 11 and 12 have a hypothesis requiring or implied by the condition

$\frac{K_\varphi(z, c)}{\sqrt{K_\varphi(c, c)}} \to 0$ as $c \to c_0 \in \partial\Omega$,

we state sufficient conditions on a domain for this limit condition to be satisfied. Below is [6, Lemma 4.1 part 2] which is “implicit in work of Pflug (see [8, Section 7.6]) and Ohsawa [12] on the completeness of the Bergman metric” according to Fu and Straube [9].

Theorem 13. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Suppose $p_0$ is a point in the boundary of $\Omega$ satisfying the following outer cone condition:

there exist $r \in (0, 1]$, $a \geq 1$, and a sequence $\{w_\ell\}_{\ell=1}^\infty$ of points $w_\ell \notin \Omega$ with $\lim_{\ell \to \infty} w_\ell = p_0$ and $\Omega \cap B(w_\ell, r \|w_\ell - p_0\|^a) = \emptyset$.
Then for any sequence \( \{p_j\}_{j=1}^{\infty} \subset \Omega \) converging to \( p_0 \),

\[
\lim_{j \to \infty} \frac{K_\Omega(z, p_j)}{\sqrt{K_\Omega(p_j, p_j)}} = 0.
\]

The outer cone condition of Theorem 13 is satisfied when \( \Omega \) has \( C^1 \) boundary, for example. Pseudoconvexity is a central notion in several complex variables which reduces to a triviality for domains of a single complex dimension: every domain in the plane is pseudoconvex \([9]\). Because we wish to also have the conclusion of the above theorem for certain weighted kernels, we show that the property addressed by the theorem is preserved when the weight of a kernel is multiplied by the modulus squared of a linear factor.

**Theorem 14.** Suppose \( \Omega \subset \mathbb{C} \) is a domain, \( p_0 \in \partial \Omega \), and \( \{p_j\}_{j=1}^{\infty} \subset \Omega \) is a sequence with \( p_j \to p_0 \) as \( j \to \infty \) such that \( \frac{K_\varphi(z, p_j)}{\sqrt{K_\varphi(p_j, p_j)}} \to 0 \) as \( j \to \infty \) locally uniformly. Then for any \( c \in \Omega \) with \( K_\varphi(c, c) \neq 0 \), \( \frac{K_\varphi(z, c)}{\sqrt{K_\varphi(p_j, p_j)}} \to 0 \) as \( j \to \infty \) locally uniformly.

**Proof.** From Theorem 14 we get

\[
\frac{K(z, c)}{\sqrt{K(p_j, p_j)}} = \frac{K(z, p_j)K(z, c) - K(z, c)K(p_j, p_j)}{(z, c)|K(p_j, p_j)|^{1/2}}.
\]

The first factor approaches a constant as \( j \to \infty \). In the second factor, every fraction in the numerator and the denominator approaches zero as \( j \to \infty \) locally uniformly by hypothesis, so the second factor approaches zero as \( j \to \infty \) locally uniformly. This proves the theorem.

5. **Further questions**

Consider the (unweighted) kernel \( K(z, w) \) for the unit disk \( \mathbb{D} \). By summing an appropriate orthonormal basis in Equation (2), it can be shown that for any real \( \alpha \) greater than \(-2\),

\[
K_{|z|^\alpha}(z, w) = K(z, w) + \frac{\alpha}{2\pi(1 - z\overline{w})} = \left(1 + \frac{\alpha}{2} - \frac{\alpha}{2}z\overline{w}\right)K(z, w).
\]

(The reader might verify that this formula agrees with Theorem 14 when \( \alpha = 2p \), \( p \in \mathbb{N} \).) Now let \( c \in D \) and \( p \in \mathbb{N} \). Using this formula, the classical change of
variables theorem for Bergman kernels, and Theorem 2, one obtains

$$K_{|z-c|^p}(z, w) = \frac{K_{\mu_c}(z, w)}{(1 - cz)^p(1 - cw)^p}$$

$$= ((p + 1) - p\mu_c(z)\mu_c(w)) \frac{K(z, w)}{(1 - cz)^p(1 - cw)^p}.$$ 

What is the formula if $p$ is allowed to be real? That is, what is the formula for $K_{|z-c|^\alpha}(z, w)$, $\alpha \in \mathbb{R}$? In particular, what is the formula when $\alpha = 1$?

Generalizing the previous question, is there a technique for computing $K_{\Omega}(z, w)$ explicitly in terms of $K_{\Omega}(z, w)$ in the case that $\varphi$ is the modulus of a meromorphic function rather than the square of the modulus a meromorphic function? Is there such a technique when $\varphi$ is harmonic?

References

1. Steven R. Bell, *The Cauchy transform, potential theory, and conformal mapping*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
2. Harold P. Boas, *Lu Qi-Keng’s problem*, Journal of the Korean Mathematical Society 37 (2000), no. 2, 253–267, Several complex variables (Seoul, 1998).
3. Harold P. Boas, Siqi Fu, and Emil J. Straube, *The Bergman kernel function: explicit formulas and zeroes*, Proceedings of the American Mathematical Society 127 (1999), no. 3, 805–811.
4. Ralph P. Boas, *Invitation to complex analysis*, second edition. Revised by Harold P. Boas ed., MAA Textbooks, Mathematical Association of America, Washington, DC, 2010.
5. Frank Forrester and Walter Rudin, *Projections on spaces of holomorphic functions in balls*, Indiana University Mathematics Journal 24 (1974), 593–602.
6. Siqi Fu and Emil J. Straube, *Compactness of the $\bar{\partial}$-Neumann problem on convex domains*, Journal of Functional Analysis 159 (1998), no. 2, 629–641.
7. Marek Jarnicki and Peter Pflug, *Invariant distances and metrics in complex analysis*, de Gruyter Expositions in Mathematics, vol. 9, Walter de Gruyter & Co., Berlin, 1993.
8. ______, *Invariant distances and metrics in complex analysis—revisited*, Dissertations Mathematicae (Rozprawy Matematyczne) 430 (2005), 192 pp.
9. Steven G. Krantz, *Function theory of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1992 edition.
10. ______, *A new proof and a generalization of Ramadanov's theorem*, Complex Variables and Elliptic Equations. An International Journal 51 (2006), no. 12, 1125–1128.
11. Ewa Ligocka, *On the Forellu-Rudin construction and weighted Bergman projections*, Polka Akademia Nauk. Instytut Matematyczny. Studia Mathematica 94 (1989), no. 3, 257–272.
12. Takeo Ohsawa, *A remark on the completeness of the Bergman metric*, Japan Academy. Proceedings. Series A. Mathematical Sciences 57 (1981), no. 4, 238–240.
13. R. Michael Range, *Holomorphic functions and integral representations in several complex variables*, 1st ed. 1986. corr. 2nd printing ed., Springer, December 2010.
14. Nobuyuki Suita and Akira Yamada, *On the Lu Qi-keng conjecture*, Proceedings of the American Mathematical Society 59 (1976), no. 2, 222–224.

Roger Williams University, One Old Ferry Road, Bristol, RI 02809

E-mail address: rljacobson@member.ams.org