A CONSTRUCTION OF SOME IDEALS IN AFFINE VERTEX ALGEBRAS

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Abstract. Let $N_k(g)$ be a vertex operator algebra (VOA) associated to the generalized Verma module for an affine Lie algebra of type $A_{\ell-1}^{(1)}$ or $C_{\ell}^{(1)}$. We construct a family of ideals $J_{m,n}(g)$ in $N_k(g)$, and a family $V_{m,n}(g)$ of corresponding quotient VOAs. These families include VOAs associated to the integrable representations, and VOAs associated to the admissible representations at half-integer levels investigated in [A]. We also explicitly identify the Zhu’s algebras $A(V_{m,n}(g))$ and find a connection between these Zhu’s algebras and Weyl algebras.

1. Introduction

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated to the finite-dimensional simple Lie algebra $\mathfrak{g}$. Then on the generalized Verma module $N_k(g)$, $k \in \mathbb{C}$, exists a natural structure of a vertex operator algebra (VOA) (cf. [FZ], [FFr], [MP], [Li], [K2]). Moreover, every $\hat{\mathfrak{g}}$-submodule $I$ of $N_k(g)$ becomes an ideal in the VOA $N_k(g)$, and on the quotient $N_k(g)/I$ exists a structure of a VOA. Let $N_k^1(g)$ be the maximal ideal in $N_k(g)$. Then the quotient $L_k(g) = N_k(g)/N_k^1(g)$ is a simple VOA. If $k$ is an integer, then $N_k^1(g)$ is generated by one singular vector (cf. [K1], [FZ]). The similar situation is in the case when $k$ is an admissible rational number (cf. [A], [AN], [DLM], [FM], [KW]).

In order to study annihilating ideals of highest weight representations, it is very important to understand the ideal lattice of the VOA $N_k(g)$. This problem was initiated [FM]. It is known fact that in the case $\mathfrak{g} = sl_2$ and $k \neq -2$, $N_k^1(sl_2(\mathbb{C}))$ is the unique ideal in the VOA $N_k(sl_2(\mathbb{C}))$. Different situation is in the case of critical level ($k = -h^\vee$, here $h^\vee$ denotes the dual Coxeter number). In this case there exists a very rich structure of ideals of $N_{-h^\vee}(\mathfrak{g})$, which implies the existence of infinitely many non-isomorphic VOAs (cf. [FFr]).

In the present paper we will introduce a family of VOAs which are quotients of the VOA $N_k(g)$ in the cases of affine Lie algebras $A_{\ell-1}^{(1)}$ and $C_{\ell}^{(1)}$ at integer and half-integer levels. This family includes VOAs associated to the admissible representations studied in [A]. The basic
step in our construction is the construction of one infinite family of singular vectors in $N_k(\mathfrak{g})$. As a consequence we get some new non-trivial ideals in VOAs $N_k(\mathfrak{g})$. We also begin the study of the representation theory of these VOAs by identifying the corresponding Zhu’s algebras explicitly. We demonstrate that the VOA $N_k(\mathfrak{g})$ for $k \in \mathbb{N}$ can have a nontrivial quotient which has infinitely many irreducible modules from the category $\mathcal{O}$. These representations are parameterized with certain algebraic curves.

2. Vertex operator algebra $N_k(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot, \cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ be a triangular decomposition for $\mathfrak{g}$. Let $\theta$ be the highest root for $\mathfrak{g}$, and $e_\theta$ the corresponding root vector. Define $\rho$ as usual. The affine Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $c$ is the canonical central element $\mathbb{K}$ and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}c,$$

$$[d, x \otimes t^n] = nx \otimes t^n$$

for $x, y \in \mathfrak{g}$. We will write $x(n)$ for $x \otimes t^n$.

The Cartan subalgebra $\mathfrak{h}$ and subalgebras $\hat{\mathfrak{g}}_+, \hat{\mathfrak{g}}_-$ of $\hat{\mathfrak{g}}$ are defined by

$$\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \hat{\mathfrak{g}}_\pm = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}].$$

Let $P = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be upper parabolic subalgebra. For every $k \in \mathbb{C}$, $k \neq -h^\vee$, let $\mathcal{C}v_k$ be 1–dimensional $P$–module such that the subalgebra $\mathfrak{g} \otimes \mathbb{C}[t] + \mathbb{C}d$ acts trivially, and the central element $c$ acts as multiplication with $k \in \mathbb{C}$. Define the generalized Verma module $N_k(\mathfrak{g})$ as

$$N_k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(P)} \mathcal{C}v_k.$$ Then $N_k(\mathfrak{g})$ has a natural structure of a vertex operator algebra (VOA). Vacuum vector is $\mathbf{1} = 1 \otimes v_k$.

Recall that there is one-to-one correspondence between irreducible modules of the VOA $V$ and the irreducible modules for the corresponding Zhu’s algebra $A(V)$ (cf. [2], [4Z]). The Zhu’s algebra of the VOA $N_k(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})$ (cf. [FZ]). Let $F : U(\hat{\mathfrak{g}}_-) \to U(\mathfrak{g})$ be the projection map defined with

$$F(a_1(-i_1 - 1) \cdots a_n(-i_n - 1)) = (-1)^{i_1+\cdots+i_n}a_na_{n-1} \cdots a_1,$$

for every $a_1, \ldots, a_n \in \mathfrak{g}$, $i_1, \ldots, i_n \in \mathbb{Z}_+$, $n \in \mathbb{N}$. Assume that $J$ is an ideal in the VOA $N_k(\mathfrak{g})$. Let $\langle F(J) \rangle$ be a two-sided ideal of $U(\mathfrak{g})$.
generated by the set \{F(w) | w \in U(\hat{g}), \ w1 \in J\}. Then the Zhu’s algebra of the quotient VOA \(N_k(g) / F(J)\) is isomorphic to the quotient algebra \(U(g) / \langle F(J) \rangle\) (for more details see [FZ]).

3. Lie algebras \(sp_{2\ell}(\mathbb{C})\) and \(sl_{\ell}(\mathbb{C})\)

Consider now two \(\ell\)-dimensional vector spaces \(A_1 = \sum_{i=1}^{\ell} C a_i, A_2 = \sum_{i=1}^{\ell} C a_i^*\). Let \(A = A_1 + A_2\). The Weyl algebra \(W(A)\) is the associative algebra over \(C\) generated by \(A\) and relations

\[ [a_i, a_j] = [a_i^*, a_j^*] = 0, \quad [a_i, a_i^*] = \delta_{i,j}, \quad i, j \in \{1, 2, \ldots, \ell\}. \]

Define the normal ordering on \(A\) by

\[ :xy: = \frac{1}{2}(xy + yx) \quad x, y \in A. \]

Then (cf. [B] and [FF]) all such elements \(xy\) : span a Lie algebra isomorphic to \(g = sp_{2\ell}(\mathbb{C})\) with a Cartan subalgebra \(\mathfrak{h}\) spanned by

\[ h_i = - : a_i a_i^*: \quad i = 1, 2, ..., \ell. \]

Let \(\{\epsilon_i | 1 \leq i \leq \ell\} \subset \mathfrak{h}^*\) be the dual basis such that \(\epsilon_i(h_j) = \delta_{i,j}\). The root system of \(g\) is given by

\[ \Delta = \{ \pm(\epsilon_i \pm \epsilon_j), \pm2\epsilon_i | 1 \leq i, j \leq \ell, i < j \} \]

with \(\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{\ell-1} = \epsilon_{\ell-1} - \epsilon_\ell, \alpha_\ell = 2\epsilon_\ell\) being a set of simple roots. The highest root is \(\theta = 2\epsilon_1\). We fix the root vectors :

\[ X_{\epsilon_i - \epsilon_j} = : a_i a_j^*:; \quad X_{\epsilon_i + \epsilon_j} = : a_i a_j^*:; \quad X_{-(\epsilon_i + \epsilon_j)} = : a_i^* a_j^*:; \]

Assume that \(\ell \geq 2\). Then the simple Lie algebra \(sl_{\ell}(\mathbb{C})\) is a Lie subalgebra \(g_1\) of \(g\) generated by the set

\[ \{X_{\epsilon_i - \epsilon_j} | i, j = 1, \ldots, \ell; i \neq j\}. \]

The Cartan subalgebra \(\mathfrak{h}_1\) is spanned by

\[ \{h_i - h_j | i, j = 1, \ldots, \ell; i \neq j\}. \]

From the above construction we conclude that there are non-zero homomorphisms

\[ \Phi : U(g) \rightarrow W(A), \quad \Phi_1 = \Phi|_{U(g_1)} : U(g_1) \rightarrow W(A). \]
4. Ideals in the VOA $\mathcal{N}_k(sp_{2\ell}(\mathbb{C}))$

In this section let $\mathfrak{g} = sp_{2\ell}(\mathbb{C})$. We will present one construction of singular vectors in $\mathcal{N}_k(\mathfrak{g})$ for integer and half-integer values of $k$. We will use the notation as in the Section 3. This construction generalizes the construction of singular vectors at half-integer levels from $[A]$. For $m \in \mathbb{N}$, $m \leq \ell$ we define matrices $C_m$ and $C_m(-1)$ by

$$C_m = \begin{bmatrix}
X_{2\ell_1} & X_{\ell_1+\epsilon_2} & \cdots & X_{\ell_1+\epsilon_m} \\
X_{\ell_1+\epsilon_2} & X_{2\ell_2} & \cdots & X_{\ell_2+\epsilon_m} \\
\vdots & \vdots & \ddots & \vdots \\
X_{\ell_1+\epsilon_m} & \cdots & \cdots & X_{2\ell_m}
\end{bmatrix},$$

$$C_m(-1) = \begin{bmatrix}
X_{2\ell_1}(-1) & X_{\ell_1+\epsilon_2}(-1) & \cdots & X_{\ell_1+\epsilon_m}(-1) \\
X_{\ell_1+\epsilon_2}(-1) & X_{2\ell_2}(-1) & \cdots & X_{\ell_2+\epsilon_m}(-1) \\
\vdots & \vdots & \ddots & \vdots \\
X_{\ell_1+\epsilon_m}(-1) & \cdots & \cdots & X_{2\ell_m}(-1)
\end{bmatrix}.$$ 

As usual let $C^i_m$ (resp. $C^i_m(-1)$) be $(m - 1) \times (m - 1)$ matrix obtained by deleting $i^{th}$ row and $j^{th}$ column of matrix $C_m$ (resp. $C_m(-1)$).

Define next

$$\Delta_m(-1) = \det(C_m(-1)) = \sum_{\sigma \in \text{Sym}_m} (-1)^{\text{sign}(\sigma)} \prod_{i=1}^m X_{\ell_i+\epsilon_{\sigma(i)}}(-1),$$

$$\Delta_m = \det(C_m) = \sum_{\sigma \in \text{Sym}_m} (-1)^{\text{sign}(\sigma)} \prod_{i=1}^m X_{\ell_i+\epsilon_{\sigma(i)}}.$$ 

Set $\Delta^i_m(-1) = \det(C^i_m(-1)).$

By using the definition and the properties of determinants, one sees the following lemma.

**Lemma 4.1.**

1. $[X_{\alpha_i}, \Delta_m(-1)] = 0$ for $i = 1, \ldots, \ell$.
2. $[X_{\epsilon_i-\epsilon_j}(0), \Delta_m(-1)] = 0$ for $i, j = 1, \ldots, m$, $i \neq j$.
3. $[X_{-2\ell_1}(1), \Delta_m(-1)] = -(2(m - 1) + 4(c - h_1)) (\Delta^{1,1}_m(-1)) + f$ where

$$f \in \sum_{i \neq j} U(\hat{\mathfrak{g}}-) X_{\ell_i-\epsilon_j}(0).$$

**Theorem 4.1.** For every $m, n \in \mathbb{N}$, $m \leq \ell$, set $k_{m,n} = n - \frac{m+1}{2}$. Then $(\Delta_m(-1))^n \mathbf{1}$ is a singular vector in $\mathcal{N}_{k_{m,n}}(\mathfrak{g})$. 

Proof. From Lemma 4.1 directly follows that
\[ X_{\alpha_i}(0)(\Delta_{m}(-1))^n 1 = 0, \text{ for } i = 1, \ldots, \ell. \]
Again using Lemma 4.1 we get
\[ X_{-\theta}(1)(\Delta_{m}(-1))^n 1 = X_{-2\epsilon_1}(1)(\Delta_{m}(-1))^n 1 = -4n(c - k_{m,n})(\Delta^1_m(-1))(-1)^{n-1} 1 = 0, \]
which proves the theorem.

Define the ideal \( J_{m,n}(g) \) in the VOA \( N_{k_{m,n}}(g) \) with
\[ J_{m,n}(g) = U(\hat{g})((\Delta_{m}(-1))^n 1). \]
Let
\[ V_{m,n}(g) = \frac{N_{k_{m,n}}(g)}{J_{m,n}(g)} \]
be the quotient VOA.

Remark 4.1. For \( m = 1 \) Theorem 4.1 gives the known fact that
\( X_{2\epsilon_1}(-1)^s 1 \) is a singular vector in \( N_{s-1}(g) \). Moreover, this vector generates the submodule \( J_{1,s}(g) \) which coincides with the maximal submodule of \( N_{s-1}(g) \).

For \( m = 2 \) Theorem 4.1 reconstructs the result from [A], Theorem 3.1, that
\[ (X_{2\epsilon_1}(-1)X_{2\epsilon_2}(-1) - X_{\epsilon_1+\epsilon_2}(-1)^2)^s 1 \]
is a singular vector in \( N_{s-\frac{3}{2}}(g) \). Again, the corresponding submodule \( J_{2,s}(g) \) is the maximal submodule of \( N_{s-\frac{3}{2}}(g) \).

Assume that \( m_1 \neq m, n_1 \neq n \) and \( k_{m_1,n_1} = k_{m,n} \). This implies that \((\Delta_{m_1}(-1))^{n_1} 1\) and \((\Delta_{m}(-1))^{n} 1\) are different singular vectors in \( N_{k_{m,n}}(g) \). Then using Theorem 4.1 and Remark 4.1 we get the following corollary on the structure of the maximal submodule of \( N_{k}(g) \).

Corollary 4.1. Assume that \( \ell, s \in \mathbb{N} \) and \( \ell \geq 3 \). The maximal submodule \( J_{1,s}(g) \) of \( N_{s-1}(g) \) is reducible. If \( \ell \geq 4 \) then the maximal submodule \( J_{2,s}(g) \) of \( N_{s-\frac{3}{2}}(g) \) is reducible.

The following result will identify the Zhu’s algebra of the VOA \( V_{m,n}(g) \).

Theorem 4.2.
(1) The Zhu’s algebra \( A(V_{m,n}(g)) \) is isomorphic to \( \frac{U(g)}{(\Delta_{m})^n} \), where \((\Delta_{m})^n\) is a two-sided ideal in \( U(g) \) generated by the vector \((\Delta_{m})^n\).
(2) For $m \geq 2$, there is a nontrivial homomorphism
\[ \Phi : A(V_{m,n}(\mathfrak{g})) \to W(A). \]
In particular, if $\pi : W(A) \to \text{End}(M)$ is any nontrivial $W(A)$–module then $\pi \circ \Phi$ is a module for the Zhu’s algebra $A(V_{m,n}(\mathfrak{g}))$.

**Proof.** The proof of the statement (1) follows from the fact that the projection map $F$ maps $(\Delta_m(-1))^n$ to $(\Delta_m)^n$.

In order to prove (2), we consider the (non-trivial) homomorphism $\Phi : U(\mathfrak{g}) \to W(A)$ defined in Section 3. For $m \geq 2$ we have
\[ \Phi(\Delta_m) = \det \begin{bmatrix} a_1^2 & a_1 a_2 & \cdots & a_1 a_m \\ a_1 a_2 & a_2^2 & \cdots & a_2 a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_1 a_m & \cdots & \cdots & a_m^2 \end{bmatrix} = 0, \]
and $\Phi((\Delta_m)^n) = 0$, which implies that there is a nontrivial homomorphism $\Phi : A(V_{m,n}(\mathfrak{g})) \to W(A)$. \qed

**Remark 4.2.** Since the Weyl algebra $W(A)$ has a rich structure of irreducible representations, Theorem 4.2 implies that for $m \geq 2$, the Zhu’s algebra $A(V_{m,n}(\mathfrak{g}))$ has infinitely many irreducible representations, and therefore the VOA $V_{m,n}(\mathfrak{g})$ has infinitely many irreducible representations.

One very interesting question is the classification of irreducible modules in the category $\mathcal{O}$. In the case $m = 2$ irreducible representations in the category $\mathcal{O}$ of the VOA $V_{2,n}(\mathfrak{g})$ were classified in [A]. It was proved that any $V_{2,n}$–module from the category $\mathcal{O}$ is completely reducible.

**Remark 4.3.** A beautiful application of the theory of vertex operator algebras to integrable highest weight modules was made in [MP]. It was proved that any singular vector in the VOA $N_k(\mathfrak{g})$ corresponds to a loop $\hat{\mathfrak{g}}$–module acting on highest weight representations of level $k$.

As a consequence of our construction in present paper, we get a new family of such loop-modules. Let $R_{m,n} = U(\mathfrak{g})(\Delta_m(-1))^n1$ be the top level of the ideal $J_{m,n}(\mathfrak{g})$. It is clear that $R_{m,n}$ is an irreducible finite-dimensional $\mathfrak{g}$–module with the highest weight $2n\omega_m$ (here $\omega_1, \ldots, \omega_\ell$ denote the fundamental weights for $\mathfrak{g}$). Then $\overline{R}_{m,n} = R_{m,n} \otimes \mathbb{C}[t, t^{-1}]$ is a loop module which acts on highest weight representations of level $k = k_{m,n}$ (for definitions see [MP]). Loop modules $\overline{R}_{m,n}$ for $m = 1$ were constructed in [MP], and for $m = 2$ in [A]. The results of the next section will also provide new examples of loop modules acting on integer level highest weight representations in the case of affine Lie algebra $A_{\ell-1}^{(1)}$. We should also mention that these new loop modules are
"bigger" than the loop modules which characterize integrable highest weight modules.

**Example 4.1.** Let \( \mathfrak{g} = sp_6(\mathbb{C}) \). Set \( R = R_{3,1} = U(\mathfrak{g})\Delta_{3,1} \). Then \( R \) is a finite-dimensional irreducible \( \mathfrak{g} \)-module with the highest weight \( 2\omega_3 \). Using the similar arguments to those in [A], [AM] and [M], one gets that an irreducible highest weight \( \mathfrak{g} \)-module \( V(\lambda) \) with the highest weight \( \lambda \) is a module for the Zhu’s algebra \( A(V_{3,1}(\mathfrak{g})) \) if and only if

\[
R_0 v_\lambda = 0,
\]

where \( v_\lambda \) is the highest weight vector, and \( R_0 \) is the zero-weight subspace of \( R \). Moreover, for every \( u \in R_0 \), there is a unique polynomial \( p_u \in U(\mathfrak{h}) \) such that \( uv_\lambda = p_u(h)v_\lambda \). Since \( \dim R_0 = 4 \), we get that the irreducible highest weight \( \mathfrak{g} \)-modules are parameterized with the zeros of four polynomials \( p_1(h), p_2(h), p_3(h), p_4(h) \). Using the similar considerations to those in [A], Section 5, we get that these polynomials are

\[
\begin{align*}
p_1(h_1, h_2, h_3) &= (h_1 + 1)(h_2 + \frac{1}{2})h_3, \\
p_2(h_1, h_2, h_3) &= (h_1 + 1)(4h_3 + (h_2 + h_3)(h_2 + h_3 - 1)), \\
p_3(h_1, h_2, h_3) &= h_3(4(h_2 + 1) + (h_1 + h_2 + 2)(h_1 + h_2 - 1)), \\
p_4(h_1, h_2, h_3) &= 4h_3(h_2 + 1) + (h_1 + h_3 - 1)(h_2 + h_3 + h_2(h_1 + h_3)).
\end{align*}
\]

Now, it is easy to find the zeros of these polynomials. The classification of irreducible modules follows from the Zhu’s algebra theory. Finally, we obtain the following complete list of irreducible modules from the category \( \mathcal{O} \):

\[
\{L((-x - 1)\Lambda_0 + x\Lambda_1)|x \in \mathbb{C}\} \cup \{L((-x - 1)\Lambda_1 + x\Lambda_2)|x \in \mathbb{C}\} \cup \\
\{L((-x - 1)\Lambda_2 + x\Lambda_3)|x \in \mathbb{C}\} \cup \\
\{L(-2\Lambda_0 + \Lambda_2), L(\Lambda_1 - 2\Lambda_3), L(-\frac{1}{2}\Lambda_0 - \frac{1}{2}\Lambda_3), L(-\frac{1}{2}\Lambda_0 + \Lambda_2 - \frac{3}{2}\Lambda_3), \\
L(-\frac{3}{2}\Lambda_0 + \Lambda_1 - \frac{1}{2}\Lambda_3), L(-\frac{3}{2}\Lambda_0 + \Lambda_1 + \Lambda_2 - \frac{3}{2}\Lambda_3)\}.
\]

It is also important to see that the irreducible modules are parameterized with a union of one finite set and a union of three lines in \( \mathbb{C}^4 \).

We also notice that every module for the Zhu’s algebra \( A(V_{3,1}(\mathfrak{g})) \) is also a module for the Zhu’s algebra \( A(V_{3,n}(\mathfrak{g})) \) for every \( n \in \mathbb{N} \). In this way the previous arguments give that for every \( n \in \mathbb{N} \) the VOA \( A(V_{3,n}) \) has uncountably many irreducible modules from the category \( \mathcal{O} \).
5. Ideals in the VOA \(N_k(sl_\ell(\mathbb{C}))\)

In this section let \(\mathfrak{g} = sl_\ell(\mathbb{C})\). We will present one construction of singular vectors in \(N_k(\mathfrak{g})\) for integer values of \(k\). The proofs will be omitted since they are completely analogous to the proofs from Section 4.

As before, we will use the notation from Section 3. For \(m \in \mathbb{N}\), \(2m \leq \ell\) we define matrices \(A_m\) and \(A_m(-1)\) by

\[
A_m = \begin{bmatrix}
X_{\epsilon_1-\epsilon_\ell} & X_{\epsilon_1-\epsilon_{\ell-1}} & \cdots & X_{\epsilon_1-\epsilon_{\ell-m+1}} \\
X_{\epsilon_2-\epsilon_\ell} & X_{\epsilon_2-\epsilon_{\ell-1}} & \cdots & X_{\epsilon_2-\epsilon_{\ell-m+1}} \\
\vdots & \vdots & \ddots & \vdots \\
X_{\epsilon_m-\epsilon_\ell} & \cdots & \cdots & X_{\epsilon_m-\epsilon_{\ell-m+1}}
\end{bmatrix},
\]

and

\[
A_m(-1) = \begin{bmatrix}
X_{\epsilon_1-\epsilon_\ell}(-1) & X_{\epsilon_1-\epsilon_{\ell-1}}(-1) & \cdots & X_{\epsilon_1-\epsilon_{\ell-m+1}}(-1) \\
X_{\epsilon_2-\epsilon_\ell}(-1) & X_{\epsilon_2-\epsilon_{\ell-1}}(-1) & \cdots & X_{\epsilon_2-\epsilon_{\ell-m+1}}(-1) \\
\vdots & \vdots & \ddots & \vdots \\
X_{\epsilon_m-\epsilon_\ell}(-1) & \cdots & \cdots & X_{\epsilon_m-\epsilon_{\ell-m+1}}(-1)
\end{bmatrix}.
\]

Let \(\Delta_m = \det(A_m)\) and \(\Delta_m(-1) = \det(A_m(-1))\).

**Theorem 5.1.** For every \(m, n \in \mathbb{N}\), \(2m \leq \ell\), set \(k_{m,n} = n - m\). Then \((\Delta_m(-1))^n1\) is a singular vector in \(N_{k_{m,n}}(\mathfrak{g})\).

Define the ideal \(J_{m,n}(\mathfrak{g})\) in the VOA \(N_{k_{m,n}}(\mathfrak{g})\) with

\[
J_{m,n}(\mathfrak{g}) = U(\mathfrak{g})(\Delta_m(-1))^n1.
\]

Let \(V_{m,n}(\mathfrak{g}) = \frac{N_{k_{m,n}}(\mathfrak{g})}{J_{m,n}(\mathfrak{g})}\) be the quotient VOA.

**Remark 5.1.** For \(m = 1\) Theorem 5.1 gives the known fact that \(X_{\epsilon_1-\epsilon_\ell}(-1)^s1\) is a singular vector in \(N_{s-1}(\mathfrak{g})\). Moreover, this vector generates the submodule \(J_{1,s}(\mathfrak{g})\) which is the maximal submodule of \(N_{s-1}(\mathfrak{g})\).

We have:

**Corollary 5.1.** Assume that \(\ell, s \in \mathbb{N}\), \(\ell \geq 4\). Then the maximal submodule \(J_{1,s}(\mathfrak{g})\) of \(N_{s-1}(\mathfrak{g})\) is reducible.

Similarly as in Section 4 we can explicitly identify Zhu’s algebras \(A(V_{m,n}(\mathfrak{g}))\) and find a connection with the Weyl algebra \(W(A)\).

**Theorem 5.2.**
The Zhu’s algebra $A(V_{m,n}(g))$ is isomorphic to the quotient algebra
\[
\frac{U(g)}{(\langle (\Delta_m)^n \rangle)}
\]
where $\langle (\Delta_m)^n \rangle$ is a two-sided ideal in $U(g)$ generated by $(\Delta_m)^n$.

For $m \geq 2$, there is a nontrivial homomorphism
\[
\Phi : A(V_{m,n}(g)) \rightarrow W(A).
\]
In particular, every module for Weyl algebra $W(A)$ can be lifted to a module for the Zhu’s algebra $A(V_{m,n}(g))$.

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