Universal Sampling Discretization

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Abstract
Let $X_N$ be an $N$-dimensional subspace of $L_2$ functions on a probability space $(\Omega, \mu)$ spanned by a uniformly bounded Riesz basis $\Phi_N$. Given an integer $1 \leq v \leq N$ and an exponent $1 \leq p \leq 2$, we obtain universal discretization for the integral norms $L_p(\Omega, \mu)$ of functions from the collection of all subspaces of $X_N$ spanned by $v$ elements of $\Phi_N$ with the number $m$ of required points satisfying $m \ll v (\log N)^2 (\log v)^2$. This last bound on $m$ is much better than previously known bounds which are quadratic in $v$. Our proof uses a conditional theorem on universal sampling discretization, and an inequality of entropy numbers in terms of greedy approximation with respect to dictionaries.

Keywords Sampling discretization · Universality · Entropy numbers

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1 Introduction

A standard approach to solving a continuous problem numerically – the Galerkin method – suggests looking for an approximate solution from a given finite-dimensional subspace. A typical way to measure an error of approximation is an appropriate $L_p$ norm, $1 \leq p \leq \infty$. Thus, the problem of discretization of the $L_p$ norms of functions from a given finite-dimensional subspace arises in a very natural way. Approximation by elements from a linear subspace falls in the category of linear approximation.

It was understood in numerical analysis and approximation theory that in many problems from signal/image processing it is beneficial to use an $m$-term approximant with respect to a given system of elements (dictionary) $D_N := \{g_i\}_{i=1}^N$. This means that for $f \in X$ we look for an approximant of the form

$$a_m(f) := \sum_{k \in \Lambda(f)} c_k g_k \quad (1.1)$$

where $\Lambda(f) \subset [1, N]$ is a set of $m$ indices which is determined by $f$. The complexity of this approximant is characterized by the cardinality $|\Lambda(f)| = m$ of $\Lambda(f)$. Approximation of this type is referred to as nonlinear approximation because, for a fixed $m$, the approximant $a_m(f)$ comes from different linear subspaces spanned by $g_k, k \in \Lambda(f)$, which depend on $f$. The cardinality $|\Lambda(f)|$ is a fundamental characteristic of $a_m(f)$ called sparsity of $a_m(f)$ with respect to $D_N$. It is now well understood that we need to study nonlinear sparse approximation in order to significantly increase our ability to process (compress, denoise, etc.) large data sets. Sparse approximations of a function are not only a powerful analytic tool but they are utilized in many applications in image/signal processing and numerical computation.

Therefore, here is an important ingredient of the discretization problem, desirable in practical applications. Suppose we have a finite dictionary $D_N := \{g_j\}_{j=1}^N$ of functions from $L_p(\Omega, \mu)$. Applying our strategy of sparse $m$-term approximation with respect to $D_N$ we obtain a collection of all subspaces spanned by at most $m$ elements of $D_N$ as a possible source of approximating (representing) elements. Thus, we would like to build a discretization scheme, which works well for all such subspaces. This kind of discretization falls in the category of universal discretization. The paper is devoted to the problem of universal sampling discretization.

Let $\Omega$ be a nonempty set equipped with a probability measure $\mu$. For $1 \leq p \leq \infty$, let $L_p(\Omega) := L_p(\Omega, \mu)$ denote the real Lebesgue space $L_p$ defined with respect to the measure $\mu$ on $\Omega$, and let $\| \cdot \|_p$ be the norm of $L_p(\Omega)$. By discretization of the $L_p$ norm we understand a replacement of the measure $\mu$ by a discrete measure $\mu_m$ with support on a set $\xi = \{\xi_j\}_{j=1}^m \subset \Omega$. This means that integration with respect to measure $\mu$ is replaced by an appropriate cubature formula. Thus, integration is replaced by evaluation of a function $f$ at a finite set $\xi$ of points. This is why this way of discretization is called sampling discretization. The problem of sampling discretization is a classical problem. The first results in this direction were obtained in the 1930s by Bernstein, by Marcinkiewicz, and by Marcinkiewicz and Zygmund for discretization of the $L_p$ norms of the univariate trigonometric polynomials. Even though this problem is very important in applications, its systematic study has begun only recently (see the
survey paper [5]). We now give explicit formulations of the sampling discretization problem (also known as the Marcinkiewicz discretization problem) and of the problem of universal discretization.

**The sampling discretization problem.** Let \((\Omega, \mu)\) be a probability space and let \(X_N \subset L^p(\Omega, \mu)\) be an \(N\)-dimensional subspace of \(L^p(\Omega, \mu)\) with \(1 \leq p \leq \infty\) (the index \(N\) here, usually, stands for the dimension of \(X_N\)). We shall always assume that every function in \(X_N\) is defined everywhere on \(\Omega\), and

\[
f \in X_N, \quad \|f\|_p = 0 \implies f = 0 \in X_N.
\]

We say that \(X_N\) admits the Marcinkiewicz-type discretization with parameters \(m \in \mathbb{N}\) and \(p\) and positive constants \(C_1 \leq C_2\) if there exists a set \(\xi := \{\xi^j\}_{j=1}^m \subset \Omega\) such that for any \(f \in X_N\) we have in the case of \(1 \leq p < \infty\),

\[
C_1 \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq C_2 \|f\|_p^p,
\]

and in the case of \(p = \infty\),

\[
C_1 \|f\|_\infty \leq \max_{1 \leq j \leq m} |f(\xi^j)| \leq \|f\|_\infty.
\]

**The problem of universal discretization.** Let \(\mathcal{X} := \{X(n)\}_{n=1}^k\) be a collection of finite-dimensional linear subspaces \(X(n)\) of the space \(L^p(\Omega)\) for a given \(1 \leq p \leq \infty\). We say that a set \(\xi := \{\xi^j\}_{j=1}^m \subset \Omega\) provides *universal discretization* for the collection \(\mathcal{X}\) if there are two positive constants \(C_i, i = 1, 2\), such that for each \(n \in \{1, \ldots, k\}\) and any \(f \in X(n)\) we have

\[
C_1 \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq C_2 \|f\|_p^p
\]

in the case of \(1 \leq p < \infty\), and

\[
C_1 \|f\|_\infty \leq \max_{1 \leq j \leq m} |f(\xi^j)| \leq \|f\|_\infty
\]

in the case of \(p = \infty\).

Note that the problem of universal discretization for the collection \(\mathcal{X} := \{X(n)\}_{n=1}^k\) is the sampling discretization problem for the set \(\bigcup_{n=1}^k X(n)\). Also, we point out that the concept of universality is well known in approximation theory. For instance, the reader can find a discussion of universal cubature formulas in [28], Section 6.8.

The problem of universal discretization for some special subspaces of the trigonometric polynomials was studied in [5, 29]. To describe the results in [5, 29] we need
to introduce some necessary notations. First, for a given finite subset \( Q \) of \( \mathbb{Z}^d \) we set
\[
T(Q) := \left\{ f : f = \sum_{k \in Q} c_k e^{i(k,x)}, \ c_k \in \mathbb{C}, \ k \in Q \right\}.
\]
For \( s = (s_1, \ldots, s_d) \in \mathbb{Z}_+^d \) we define
\[
R(s) := \{k \in \mathbb{Z}^d : |k_j| < 2^s_j, \ j = 1, \ldots, d\}.
\]
The following result, proved in \([29]\), solves the universal discretization problem for the collection
\[
\mathcal{C}(n, d) := \{T(R(s)) : s_1 + \cdots + s_d = n\}
\]
of subspaces of trigonometric polynomials.

**Theorem 1.1** \([29]\) For every \( 1 \leq p \leq \infty \) there exists a large enough constant \( C(d, p) \), which depends only on \( d \) and \( p \), such that for any \( n \in \mathbb{N} \) there is a set \( \xi := (\xi_v)_{v=1}^m \subset \mathbb{T}^d \), with \( m \leq C(d, p)2^n \) that provides universal discretization in \( L_p \) for the collection \( \mathcal{C}(n, d) \).

Second, for \( n \in \mathbb{N} \) let
\[
\Pi_n := [-2^{n-1} + 1, 2^{n-1} - 1]^d \cap \mathbb{Z}^d.
\]
For a positive integer \( v \leq |\Pi_n| \) define
\[
S(v, n) := \{Q : Q \subset \Pi_n, \ |Q| = v\}.
\]
Then it is easily seen that
\[
|S(v, n)| = \left(\frac{|\Pi_n|}{v}\right) < 2^{dnv}.
\]
The following theorem provides universal discretization of \( L_1 \) and \( L_2 \) norms for the collection \( \{T(Q) : Q \in S(v, n)\} \).

**Theorem 1.2** \([5, 27, \text{Theorem 7.4}]\) For positive integers \( n \) and \( 1 \leq v \leq |\Pi_n| \) let
\[
M_p(n, v) := \begin{cases} v^2 n^{9/2}, & \text{if } p = 1, \\ v^2 n, & \text{if } p = 2. \end{cases}
\]
Then there exist three positive constants \( C_i(d) \), \( i = 1, 2, 3 \), such that for any \( n, v \in \mathbb{N} \) with \( v \leq |\Pi_n| \), and for \( p = 1 \) and \( p = 2 \) there is a set \( \xi := (\xi_v)_{v=1}^m \subset \mathbb{T}^d \), with
m \leq C_1 d M_\rho(n, v) \text{ such that for any } f \in \bigcup_Q \sigma_{\mathcal{S}(v, n)} \mathcal{T}(Q)

C_2(d) \| f \|_p^p \leq \frac{1}{m} \sum_{\nu=1}^{m} | f(\xi^\nu) |^p \leq C_3(d) \| f \|_p^p.

Let us denote by \( \mathcal{D}_N = \{ g_i \}_{i=1}^{N} \) a system of functions from \( L_\rho \). Denote the set of all \( v \)-term approximants with respect to \( \mathcal{D}_N \) as

\[ \Sigma_v(\mathcal{D}_N) := \left\{ f : f = \sum_{i \in G} c_i g_i, \text{ with } G \subset [1, N] \text{ such that } |G| = v \right\} . \]

Theorem 1.2 provides universal discretization for the collection \( \{ \mathcal{T}(Q) : Q \in \mathcal{S}(v, n) \} \), which is equivalent to the sampling discretization of the \( L_\rho \) norm of elements from the set \( \Sigma_v(\mathcal{D}_N) \) with \( N = |\Pi_n|, \mathcal{D}_N = \{ e^{i(k,x)} \}_{k \in \Pi_n} \). The proof of Theorem 1.2 in the case \( p = 2 \) is based on deep results on random matrices and in the case \( p = 1 \) is based on the chaining technique. We point out that in both cases \( p = 2 \) and \( p = 1 \) Theorem 1.2 provides universal discretization with the number of points growing as \( v^2 \).

On the other hand, while Theorem 1.1 provides universal discretization for the subcollection \( \mathcal{C}(n, d) \) of \( \{ \mathcal{T}(Q) : Q \in \mathcal{S}(v, n+1) \} \) (rather than the whole collection \( \{ \mathcal{T}(Q) : Q \in \mathcal{S}(v, n+1) \} \) with \( v = 2^n \) and

\[ \mathcal{D} := \{ e^{i(k,x)} : k \in \mathbb{Z}^d, |k_j| < 2^n, 1 \leq j \leq d \}, \]

it gives a better estimate \( m \leq C v \) on the number of points, which is linear in \( v \), and applies to the full range of \( 1 \leq p \leq \infty \).

In this paper we prove the following estimate (see below for definitions and notations).

**Theorem 1.3** Let \( 1 \leq p \leq 2 \). Assume that \( \Phi_N \) is a uniformly bounded Riesz basis of \( X_N := \text{span} (\Phi_N) \) satisfying (2.8) for some constants \( 0 < R_1 \leq R_2 \). Then for a large enough constant \( C = C(p, R_1, R_2) \) and any integer \( 1 \leq v \leq N \) there exist \( m \) points \( \xi^1, \cdots, \xi^m \in \Omega \) with

\[ m \leq C v (\log N)^2 (\log(2v))^2 \]

such that for any \( f \in \Sigma_v(\Phi_N) \) we have

\[ \frac{1}{2} \| f \|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} | f(\xi^j) |^p \leq \frac{3}{2} \| f \|_p^p. \tag{1.4} \]

In particular, Theorem 1.3 gives the order of bound

\[ m \ll v (\log N)^2 (\log(2v))^2, \]
which is linear in $v$ with extra logarithmic terms in $N$ and $v$. This bound is much better than previously known bounds (see Theorem 1.2), which provided quadratic in $v$ bounds. Note that even for each individual subspace from $\{T(Q) : Q \in S(v, n)\}$ we have the lower bound $m \geq v$ for the sampling discretization.

Finally, we point out that very recent progress related to universal discretization has been made in our follow-up papers [8, 9]. More precisely, in [8] we prove that in the setting of Theorem 1.3 independent random points $\xi_1, \cdots, \xi_m \in \Omega$ that are identically distributed according to a given probabilistic measure $\mu$ provide the universal discretization (1.4) with high probability under a slightly weaker condition than the condition in Theorem 1.3 on the number of points

$$m \ll v(\log N)(\log(2v))^2(\log(2v) + \log \log N).$$

Also, in [8] we relaxed the condition on the Riesz basis $\Phi_N$. In [9] we show how universal discretization can be applied to deduce interesting results on sparse sampling recovery. In particular, we demonstrate that a simple greedy type algorithm based on good points for universal discretization provides good recovery in the square norm.

The rest of this paper is organized as follows. Sections 2 and 3 are devoted to estimating the entropy numbers $\varepsilon_k(\Sigma_p^\mu(\Phi_N), L_\infty)$ of the sets

$$\Sigma_p^\mu(\Phi_N) := \{f \in \Sigma_v(\Phi_N) : \|f\|_p \leq 1\}, \quad 1 \leq p \leq 2,$$

in the $L_\infty$-norm, where $\Phi_N$ is a uniformly bounded Riesz basis of $X_N := [\Phi_N] \subset L_2$ and $1 \leq v \leq N$ is an integer. Such estimates play an important role in the proof of Theorem 1.3. To be more precise, in Sect. 2 we prove under the additional condition (2.9) on the space $X_N = \text{span}(\Phi_N)$ that for $p = 2$,

$$\varepsilon_k(\Sigma_2^\mu(\Phi_N), L_\infty) \leq C(\log N)\left(\frac{v}{k}\right)^{1/2}, \quad k = 1, 2, \ldots \quad (1.5)$$

The proof of (1.5) uses a known result from Greedy approximation in smooth Banach spaces and its connection with entropy numbers. In Sect. 3, we show how the estimate (1.5) can be extended to the case $1 \leq p < 2$ under condition (2.9). This extension step is based on a general inequality for the entropy, which is given in Lemma 3.1 and appears to be of independent interest. In Sect. 4 we prove Theorem 1.3, using the estimates on entropy numbers established in the previous two sections and a conditional theorem on sampling discretization. A main step in the proof is to show that the condition (2.9) that is assumed in our estimates of entropy numbers can be dropped in sampling discretization. The conditional Theorem 2.2 used in the proof of Theorem 1.3 is given in Sect. 2 without proof. In Sect. 5, we prove a refined conditional theorem for sampling discretization of all integral norms $L_p$ of functions from a subset $W \subset L_\infty$ satisfying certain conditions, which allows us to estimate the number of points required for the sampling discretization in terms of an integral of the $\varepsilon$-entropy $H_\varepsilon(W, L_\infty)$, $\varepsilon > 0$. This is an extension of the conditional result proved in [7, 27] for the unit ball of the space $X_N \subset L_p$. In particular, it also allows us to prove a refined version of Theorem 1.3, where the constants $\frac{1}{2}$ and $\frac{3}{2}$ in (1.4) are replaced by $1 - \varepsilon$ and $1 + \varepsilon$. 

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respectively for an arbitrarily given \( \varepsilon \in (0, 1) \). Finally, in Sect. 6 we give a few remarks on universal sampling discretization of \( L_p \) norms for \( p > 2 \).

Throughout this paper the letter \( C \) denotes a general positive constant depending only on the parameters indicated as arguments or subscripts. We use the notation \( |A| \) to denote the cardinality of a finite set \( A \).

### 2 Some General Entropy Bounds and the Case \( p = 2 \)

It is well known that bounds of the entropy numbers of the unit ball of an \( N \)-dimensional subspace \( X_N \subset L_p \)

\[
X_N^p := \{ f \in X_N : \| f \|_p \leq 1 \}
\]

play an important role in sampling discretization of the \( L_p \) norm of elements of \( X_N \) (see [6, 26, 27], and [7]).

Recall the definition of entropy numbers in Banach spaces. Let \( X \) be a Banach space and \( B_X(g, r) \) denote the closed ball \( \{ f \in X : \| f - g \| \leq r \} \) with center \( g \in X \) and radius \( r > 0 \). Given a positive number \( \varepsilon \), the covering number \( N_\varepsilon(A, X) \) of a compact set \( A \subset X \) is defined as

\[
N_\varepsilon(A, X) := \min \left\{ n \in \mathbb{N} : \exists g^1, \ldots, g^n \in A, A \subset \bigcup_{j=1}^{n} B_X(g^j, \varepsilon) \right\}.
\]

We denote by \( \mathcal{N}_\varepsilon(A, X) \) the corresponding minimal \( \varepsilon \)-net of the set \( A \) in \( X \); namely, \( \mathcal{N}_\varepsilon(A, X) \) is a finite subset of \( A \) such that \( A \subset \bigcup_{y \in \mathcal{N}_\varepsilon(A, X)} B_X(y, \varepsilon) \) and \( \mathcal{N}_\varepsilon(A, X) = |\mathcal{N}_\varepsilon(A, X)| \). The \( \varepsilon \)-entropy \( \mathcal{H}_\varepsilon(A, X) \) of the compact set \( A \) in \( X \) is defined as \( \log_2 N_\varepsilon(A, X) \), and the entropy numbers \( \varepsilon_k(A, X) \) of the set \( A \) in \( X \) are defined as

\[
\varepsilon_k(A, X) := \inf \{ \varepsilon > 0 : \mathcal{H}_\varepsilon(A, X) \leq k \}, \quad k = 1, 2, \ldots
\]

The following conditional result was proved in [27] for \( p = 1 \) and in [6] for the full range of \( 1 \leq p < \infty \).

**Theorem 2.1** [27], [6, Theorem 1.3] Let \( 1 \leq p < \infty \). Suppose that a subspace \( X_N \subset L_p(\Omega, \mu) \) satisfies the condition

\[
\varepsilon_k(X_N^p, L_\infty) \leq B(N/k)^{1/p}, \quad 1 \leq k \leq N, \tag{2.1}
\]

where \( B \geq 1 \). Then for a large enough constant \( C(p) \) there exist \( m \) points \( \xi^1, \ldots, \xi^m \in \Omega \) with

\[
m \leq C(p)NB^p(\log_2(2BN))^2
\]
such that for any \( f \in X_N \) we have

\[
\frac{1}{2} \| f \|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^p \leq \frac{3}{2} \| f \|_p^p.
\]

As we explained above the problem of universal discretization of the collection \( \{X(n)\}_{n=1}^{k} \) is equivalent to the sampling discretization of the union \( \bigcup_{n=1}^{k} X(n) \) of the corresponding subsets. Therefore, instead of bounds of the entropy numbers of the unit ball \( X_N^p \) we are interested in the entropy bounds of the "unit ball"

\[
\Sigma^p_v(D_N) := \{ f \in \Sigma_v(D_N) : \| f \|_p \leq 1 \},
\]

which is the union of the corresponding unit balls.

The following version of Theorem 2.1 follows directly from its proof.

**Theorem 2.2** Let \( 1 \leq p < \infty \) and \( 1 \leq v \leq N \). Suppose that a dictionary \( D_N \) is such that the set \( \Sigma^p_v(D_N) \) satisfies the condition

\[
\varepsilon_k(\Sigma^p_v(D_N), L_\infty) \leq B_1(v/k)^{1/p} \| f \|_p, \quad 1 \leq k < \infty, \tag{2.2}
\]

where \( B_1 \geq 1 \). Assume in addition that there exists a constant \( B_2 \geq 1 \) such that

\[
\| f \|_\infty \leq B_2 v^{1/p} \| f \|_p, \quad \forall f \in \Sigma^p_v(D_N). \tag{2.3}
\]

Then for a large enough constant \( C(p) \) there exist \( m \) points \( \xi^1, \ldots, \xi^m \in \Omega \) with

\[
m \leq C(p) B_1^p v(\log(2B_2v))^2
\]

such that for any \( f \in \Sigma_v(D_N) \) we have

\[
\frac{3}{4} \| f \|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^p \leq \frac{5}{4} \| f \|_p^p.
\]

Theorem 2.2 also follows from a more general conditional theorem that will be proved in Sect. 5 (see Corollary 5.1).

**Remark 2.1** We point out that (2.2) implies

\[
\| f \|_\infty \leq 3B_1 v^{1/p} \| f \|_p, \quad \forall f \in \Sigma^p_v(D_N). \tag{2.4}
\]

Therefore, assumption (2.3) can be dropped with \( B_2 \) replaced by \( 3B_1 \) in the bound on \( m \). However, in applications the constant \( B_2 \) in (2.3) may be significantly smaller than \( 3B_1 \). For example, if \( D_N \) is a uniformly bounded orthonormal system with \( \max_{f \in D_N} \| f \|_\infty = 1 \), then we can take \( B_2 = 1 \).
Proof of (2.4). For $\Lambda \subset [1, N] \cap \mathbb{N}$ denote $X(\Lambda) := \text{span}(g_i)_{i \in \Lambda}$ and $X(\Lambda)^{p} := \{ f \in X(\Lambda) : \| f \|_p \leq 1 \}$. Clearly, (2.2) implies the same bound for each $X(\Lambda)^{p}$ with $|\Lambda| = v$. Thus, it is sufficient to prove (2.4) for a $v$-dimensional subspace $X_v$. With a slightly worse constant $4B_1$ instead of $3B_1$ it was proved in [7, Remark 1.1]. We now show how to get a better constant. Setting $\varepsilon_1 := \varepsilon_1(X_0^p, L_\infty)$, we can find two functions $f_1, f_2 \in X_0^p$ such that $X_0^p \subset B_{L_\infty}(f_1, \varepsilon_1) \cup B_{L_\infty}(f_2, \varepsilon_1)$. Since $0 \in X_0^p$, it was proved in [7, Remark 1.1]. We may assume $0$ is contained in one of the two balls. Without loss of generality we may assume that $0 \in B_{L_\infty}(f_1, \varepsilon_1)$ so that $\| f_1 \|_\infty \leq \varepsilon_1$. Since $-f_2 \in X_0^p$, we have either $-f_2 \in B_{L_\infty}(f_1, \varepsilon_1)$ or $-f_2 \in B_{L_\infty}(f_2, \varepsilon_1)$, which implies $\| f_2 \|_\infty \leq 2\varepsilon_1$. It then follows that $\| f \|_\infty \leq 3\varepsilon_1$ for all $f \in X_0^p$. This together with (2.2) proves (2.4). □

Theorem 2.2 motivates us to estimate the characteristics $\varepsilon_\ell(\Sigma_0^p(D_N), L_\infty)$. We now recall some known general results, which turn out to be useful for that purpose. Let $D_N = \{ g_j \}_{j=1}^N$ be a system of elements of cardinality $|D_N| = N$ in a Banach space $X$. Consider the best $m$-term approximations of $f$ with respect to $D_N$

$$\sigma_m(f, D_N)_X := \inf_{\{c_j : |\Lambda| = m\}} \| f - \sum_{j \in \Lambda} c_j g_j \|_X.$$ 

For a set $W \subset X$ we define

$$\sigma_m(W, D_N)_X := \sup_{f \in W} \sigma_m(f, D_N)_X, \ m = 1, 2, \ldots ,$$

and $\sigma_0(W, D_N)_X = \sup_{f \in W} \| f \|_X$. The following Theorem 2.3 was proved in [24] (see also [28], p.331, Theorem 7.4.3).

Theorem 2.3 Let a compact $W \subset X$ be such that there exist a system $D_N \subset X$ with $|D_N| = N$, and a number $r > 0$ such that

$$\sigma_m(W, D_N)_X \leq (m + 1)^{-r}, \ m = 0, 1, \ldots , N.$$ 

Then for $k \leq N$

$$\varepsilon_k(W, X) \leq C(r) \left( \frac{\log(2N/k)}{k} \right)^r . \quad (2.5)$$

For a given set $D_N = \{ g_j \}_{j=1}^N$ of elements we introduce the octahedron (generalized octahedron)

$$A_1(D_N) := \left\{ f : f = \sum_{j=1}^N c_j g_j, \sum_{j=1}^N |c_j| \leq 1 \right\} \quad (2.6)$$

and the norm $\| \cdot \|_A$ on $X_N$

$$\| f \|_A := \inf \left\{ \sum_{j=1}^N |c_j| : f = \sum_{j=1}^N c_j g_j \right\}. \quad \square$$
We now use a known general result for a smooth Banach space. For a Banach space \( X \) we define the modulus of smoothness
\[
\rho(u) := \rho(X, u) := \sup_{\|x\| = \|y\| = 1} \left( \frac{1}{2} (\|x + uy\| + \|x - uy\|) - 1 \right).
\]

The uniformly smooth Banach space is the one with the property
\[
\lim_{u \to 0} \frac{\rho(u)}{u} = 0.
\]

In this paper we only consider uniformly smooth Banach spaces with power type moduli of smoothness \( \rho(u) \leq \gamma u^s, 1 < s \leq 2 \). The following bound is a corollary of greedy approximation results (see, for instance [28], p.455).

**Theorem 2.4** Let \( X \) be \( s \)-smooth: \( \rho(X, u) \leq \gamma u^s, 1 < s \leq 2 \). Then for any normalized system \( \mathcal{D}_N \) of cardinality \( |\mathcal{D}_N| = N \) we have
\[
\sigma_m(A_1(\mathcal{D}_N), X) \leq C(s)\gamma^{1/s}m^{1/s-1}.
\]

Note that it is known that in the case \( X = L^p \) we have
\[
\rho(L^p, u) \leq (p-1)u^2/2, \quad 2 \leq p < \infty.
\]

We now proceed to a special case when \( X = L^p \) and \( \mathcal{D}_N = \Phi_N := \{\varphi_j\}_{j=1}^N \) is a uniformly bounded Riesz basis of \( X_N := [\Phi_N] := \text{span}(\varphi_1, \ldots, \varphi_N) \). Namely, we assume that \( \|\varphi_j\|_\infty \leq 1, 1 \leq j \leq N \) and for any \( (a_1, \ldots, a_N) \in \mathbb{R}^N \)
\[
R_1 \left( \sum_{j=1}^N |a_j|^2 \right)^{1/2} \leq \left\| \sum_{j=1}^N a_j \varphi_j \right\|_2 \leq R_2 \left( \sum_{j=1}^N |a_j|^2 \right)^{1/2},
\]
where \( 0 < R_1 \leq R_2 < \infty \). Assume in addition that for any \( f \in X_N \) we have
\[
\|f\|_\infty \leq C_0\|f\|_{\log N}.
\]

**Theorem 2.5** Assume that \( \Phi_N \) is a uniformly bounded Riesz basis of \( X_N := [\Phi_N] \) satisfying (2.9). Then we have
\[
\varepsilon_k(\Sigma_v^2(\Phi_N), L_\infty) \leq C(R_1, C_0)(\log N)\left( \frac{v}{k} \right)^{1/2}, \quad k = 1, 2, \ldots.
\]

**Proof** First of all, for any \( f = \sum_{j \in G} a_j \varphi_j, |G| = v \) we get
\[
\|f\|_A \leq \sum_{j \in G} |a_j| \leq v^{1/2} \left( \sum_{j \in G} |a_j|^2 \right)^{1/2} \leq R_1^{-1}v^{1/2}\|f\|_2.
\]
Therefore,

$$\Sigma_v^2(\Phi_N) \subset R_1^{-1} v^{1/2} \Sigma_v^A(\Phi_N),$$

where

$$R \Sigma_v^A(\Phi_N) := \{ f \in \Sigma_v(\Phi_N) : \| f \|_A \leq R \}.$$  

By Theorem 2.4 with $s = 2$ and by (2.7) we have that for $p \in [2, \infty)$

$$\sigma_m(\Sigma_v^2(\Phi_N), L_p) \leq CR_1^{-1} v^{1/2} \sqrt{pm}^{-1/2}, \quad m = 1, 2, \ldots, N. \tag{2.12}$$

Thus, Theorem 2.3 implies that for $p \in [2, \infty)$

$$\varepsilon_k(\Sigma_v^2(\Phi_N), L_p) \leq C(R_1)(p \log(2N/k))^{1/2}(v/k)^{1/2}, \quad k = 1, 2, \ldots, N. \tag{2.13}$$

Second, by (2.9) we obtain

$$\varepsilon_k(\Sigma_v^2(\Phi_N), L_\infty) \leq C_0\varepsilon_k(\Sigma_v^2(\Phi_N), L_{\log N}). \tag{2.14}$$

Combining (2.13) and (2.14) we get

$$\varepsilon_k(\Sigma_v^2(\Phi_N), L_\infty) \leq C(R_1, C_0)(\log N)(v/k)^{1/2}, \quad k = 1, 2, \ldots, N. \tag{2.15}$$

Finally, for $k > N$ we use the inequalities

$$\varepsilon_k(W, L_\infty) \leq \varepsilon_N(W, L_\infty)\varepsilon_{k-N}(X_N^{\infty}, L_\infty)$$

and

$$\varepsilon_n(X_N^{\infty}, L_\infty) \leq 3(2^{-n/N}), \quad 2^{-x} \leq 1/x, \quad x \geq 1,$$

to obtain (2.15) for all $k$. This completes the proof. \hfill \Box

### 3 A Step From $p = 2$ to $1 \leq p < 2$

In this section we show how Theorem 2.5 proved in Sect. 2 for $p = 2$ can be extended to the case $1 \leq p < 2$. This extension step is based on a general inequality for the entropy. For convenience, we set $\Sigma_v(\mathcal{D}_N) = X_N := [\mathcal{D}_N]$ for $v > N$.

**Lemma 3.1** For $v = 1, 2, \ldots, N$, $1 \leq p < 2 < q \leq \infty$, and

$$\theta := (\frac{1}{2} - \frac{1}{q})/(\frac{1}{p} - \frac{1}{q})$$

we have for $\varepsilon > 0$
\[ \mathcal{H}_e(\Sigma^p_v(\mathcal{D}_N); L_q) \leq \sum_{s=0}^\infty \mathcal{H}_{2^{-3}2^{-s-1}e^{\theta}}(\Sigma^2_v(\mathcal{D}_N); L_q) + \mathcal{H}_{e^{\theta}}(\Sigma^2_v(\mathcal{D}_N); L_q), \quad (3.1) \]

where \( a = a(\theta) = 2^{\frac{\theta}{\alpha}} \).

**Proof** In the case when \( v = N \) Lemma 3.1 was proved in [7, Lemma 3.3]. A slight modification of the proof there works equally well for a general case of \( 1 \leq v \leq N \). For completeness, we include the proof of this lemma here. First, we note that for any \( \varepsilon_1, \varepsilon_2 > 0 \)

\[ \mathcal{H}_{\varepsilon_1\varepsilon_2}(\Sigma^p_v(\mathcal{D}_N); L_q) \leq \mathcal{H}_{\varepsilon_1}(\Sigma^p_v(\mathcal{D}_N); L_2) + \mathcal{H}_{\varepsilon_2}(\Sigma^2_v(\mathcal{D}_N); L_q). \quad (3.2) \]

To see this, let \( x_1, \ldots, x_{N_1} \in \Sigma^p_v(\mathcal{D}_N) \) and \( y_1, \ldots, y_{N_2} \in \Sigma^2_v(\mathcal{D}_N) \) be such that

\[ \Sigma^p_v(\mathcal{D}_N) \subset \bigcup_{i=1}^{N_1} (x_i + \varepsilon_1 B_{L_2}) \quad \text{and} \quad \Sigma^2_v(\mathcal{D}_N) \subset \bigcup_{j=1}^{N_2} (y_j + \varepsilon_2 B_{L_q}), \]

where \( N_1 = N_{\varepsilon_1}(\Sigma^p_v(\mathcal{D}_N), L_2) \) and \( N_2 = N_{\varepsilon_2}(\Sigma^2_v(\mathcal{D}_N), L_q) \). Since \( \Sigma^p_v(\mathcal{D}_N) \subset \Sigma^2_v(\mathcal{D}_N) \), we have

\[ \Sigma^p_v(\mathcal{D}_N) \subset \bigcup_{i=1}^{N_1} (x_i + \varepsilon_1 B_{L_2}) \cap \Sigma^p_v(\mathcal{D}_N) \subset \bigcup_{i=1}^{N_1} (x_i + \varepsilon_1 \Sigma^2_v(\mathcal{D}_N)) \]

\[ \subset \bigcup_{i=1}^{N_1} \bigcup_{j=1}^{N_2} (x_i + \varepsilon_1 y_j + \varepsilon_1 \varepsilon_2 B_{L_q}). \]

Inequality (3.2) then follows.

Next, setting \( \varepsilon_1 := \varepsilon^{1-\theta} \) and \( \varepsilon_2 = \varepsilon^\theta \) in (3.2), we reduce the problem to showing that

\[ \mathcal{H}_{\varepsilon_1}(\Sigma^p_v(\mathcal{D}_N); L_2) \leq \sum_{s=0}^\infty \mathcal{H}_{2^{-3}2^{-s-1}e^{\theta}}(\Sigma^2_v(\mathcal{D}_N); L_q). \quad (3.3) \]

It will be shown that for \( s = 0, 1, \ldots \),

\[ \mathcal{H}_{2^{s+1}\varepsilon_1}(\Sigma^p_v(\mathcal{D}_N); L_2) - \mathcal{H}_{2^{s}\varepsilon_1}(\Sigma^p_v(\mathcal{D}_N); L_2) \leq \mathcal{H}_{2^{-3}2^{-s-1}e^{\theta}}(\Sigma^2_v(\mathcal{D}_N); L_q), \quad (3.4) \]

from which (3.3) will follow by taking the sum over \( s = 0, 1, \ldots \).

To show (3.4), for each nonnegative integer \( s \) let \( \mathcal{F}_s \subset \Sigma^p_v(\mathcal{D}_N) \) be a maximal \( 2^s \varepsilon_1 \)-separated subset of \( \Sigma^p_v(\mathcal{D}_N) \) in the metric \( L_2 \); that is \( \| f - g \|_2 \geq 2^s \varepsilon_1 \) for any two distinct functions \( f, g \in \mathcal{F}_s \) and \( \Sigma^p_v(\mathcal{D}_N) \subset \bigcup_{f \in \mathcal{F}_s} B_{L_2}(f, 2^s \varepsilon_1) \). Then

\[ \mathcal{H}_{2^{s+1}\varepsilon_1}(\Sigma^p_v(\mathcal{D}_N); L_2) \leq \log_2 |\mathcal{F}_s| \leq \mathcal{H}_{2^{-3}2^{-s-1}e^{\theta}}(\Sigma^p_v(\mathcal{D}_N); L_2). \quad (3.5) \]
Let \( f_s \in \mathcal{F}_{s+2} \) be such that
\[
\left| B_{L_2}(f_s, 2^{s+2} \varepsilon_1) \cap \mathcal{F}_s \right| = \max_{f \in \mathcal{F}_{s+2}} \left| B_{L_2}(f, 2^{s+2} \varepsilon_1) \cap \mathcal{F}_s \right|.
\]
Since
\[
\mathcal{F}_s = \bigcup_{f \in \mathcal{F}_{s+2}} \left( B_{L_2}(f, 2^{s+2} \varepsilon_1) \cap \mathcal{F}_s \right) \subset \Sigma^p_v(D_N),
\]
it follows that
\[
|\mathcal{F}_s| \leq |\mathcal{F}_{s+2}| \left| B_{L_2}(f_s, 2^{s+2} \varepsilon_1) \cap \mathcal{F}_s \right|.
\]
(3.6)

Set
\[
\mathcal{A}_s := \left\{ \frac{f - f_s}{2^{s+2} \varepsilon_1} : f \in B_{L_2}(f_s, 2^{s+2} \varepsilon_1) \cap \mathcal{F}_s \right\} \subset \Sigma^p_v(D_N).
\]

Clearly, for any \( g \in \mathcal{A}_s \)
\[
\|g\|_2 \leq 1, \quad \|g\|_p \leq (2^{s+1} \varepsilon_1)^{-1}.
\]
(3.7)

On the one hand, using (3.5) and (3.6), we obtain that
\[
\log_2 |\mathcal{A}_s| \geq \log_2 |\mathcal{F}_s| - \log_2 |\mathcal{F}_{s+2}|
\]
\[
\geq \mathcal{H}_{2^{s+1} \varepsilon_1}(\Sigma^p_v(D_N); L_2) - \mathcal{H}_{2^{s+1} \varepsilon_1}(\Sigma^p_v(D_N); L_2).
\]
(3.8)

On the other hand, since \( \frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{q} \), using (3.7) and the fact that \( \mathcal{F}_s \) is \( 2^{s} \varepsilon_1 \)-separated in the \( L_2 \)-metric, we have that for any two distinct \( g', g \in \mathcal{A}_s \)
\[
2^{-2} \leq \|g' - g\|_2 \leq \|g' - g\|_p \|g - g\|_q^{1-\theta} \leq \left( 2^{s+1} \varepsilon_1 \right)^{-\theta} \|g - g\|_q^{1-\theta},
\]
which implies that
\[
\|g' - g\|_q \geq 2^{-2}(2^{s-1} \varepsilon_1)^{\frac{\theta}{1-\theta}} = 2^{-2(2^{s-1} \varepsilon_1)^{\frac{\theta}{1-\theta}}}.
\]

This together with (3.7) means that \( \mathcal{A}_s \) is a \( 2^{-2(2^{s-1} \varepsilon_1)^{\frac{\theta}{1-\theta}}} \)-separated subset of \( \Sigma^2_v(D_N) \) in the metric \( L_q \). We obtain
\[
\log_2 |\mathcal{A}_s| \leq \mathcal{H}_{2^{-3(2^{s-1} \varepsilon_1)^{\frac{\theta}{1-\theta}}}}(\Sigma^2_v(D_N); L_q).
\]
(3.9)

Thus, combining (3.9) with (3.8), we prove inequality (3.4). \( \square \)

Lemma 3.1 with \( 1 \leq p < 2, q = \infty, \theta = p/2 \) and Theorem 2.5 imply the following bound for the entropy numbers.
Theorem 3.1 Assume that $\Phi_N$ is a uniformly bounded Riesz basis of $X_N := [\Phi_N]$ satisfying (2.9). Then for $1 \leq p \leq 2$ we have

$$
\varepsilon_k(\Sigma^p_v(\Phi_N), L_\infty) \leq C(p, R_1, C_0)(\log N)^{2/p} (v/k)^{1/p}, \quad k = 1, 2, \ldots
$$

(3.10)

4 Proof of Theorem 1.3

Theorem 3.1 provides bounds on the entropy numbers $\varepsilon_k(\Sigma^p_v(\Phi_N), L_\infty)$ under additional assumption (2.9). Thus, a combination of Theorem 3.1 with Theorem 2.2 implies the statement of Theorem 1.3 under extra assumption (2.9). However, (2.9) is not assumed in Theorem 1.3. Below we give a proof of Theorem 1.3.

We need the following lemma proved in [7].

Lemma 4.1 [7, Lemma 4.3] Let $1 \leq p < \infty$ be a fixed number. Assume that $X_N$ is an $N$-dimensional subspace of $L_\infty(\Omega)$ satisfying the following condition: For some parameter $\beta > 0$ and constant $K \geq 2$

$$
\|f\|_\infty \leq (KN)^{\beta/p} \|f\|_p, \quad \forall f \in X_N.
$$

(4.1)

Let $\{\xi_j\}_{j=1}^\infty$ be a sequence of independent random points distributed in accordance with $\mu$. Then there exists a positive constant $C_\beta$ depending only on $\beta$ such that for any $0 < \varepsilon \leq 1/2$ and any integer

$$
m \geq C_\beta K^{\beta}\varepsilon^{-2} (\log 2/\varepsilon)N^{\beta+1} \log N
$$

(4.2)

the inequality

$$
(1 - \varepsilon)\|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi_j)|^p \leq (1 + \varepsilon)\|f\|_p^p
$$

(4.3)

holds with probability $\geq 1 - m^{-N/\log K}$.

For a set $\Omega_m := \{x_1, \ldots, x_m\} \subset \Omega$ and a function $f : \Omega_m \to \mathbb{R}$ we define $
\|f\|_{L_\infty(\Omega_m)} := \max_{1 \leq j \leq m} |f(x_j)|$ and

$$
\|f\|_{L_p(\Omega_m)} := \left( \frac{1}{m} \sum_{j=1}^m |f(x_j)|^p \right)^{1/p}
$$

for $p < \infty$.

Now we turn to the proof of Theorem 1.3. Recall that we do not assume (2.9). First, since the Riesz basis $\Phi_N := \{\varphi_j\}_{j=1}^N$ is uniformly bounded by 1 on $\Omega$, we have by (2.8)

$$
\|f\|_\infty \leq R_1^{-1} N^{1/2} \|f\|_2, \quad \forall f \in X_N,
$$
which in turn implies that
\[ \|f\|_\infty \leq C(R_1)N^{\frac{1}{p}}\|f\|_p, \quad \forall f \in X_N, \quad 1 \leq p \leq 2. \]

Thus, by Lemma 4.1 with \( \beta = 1 \) there exists a discrete set \( \Omega_{m_1} := \{\xi^1, \ldots, \xi^{m_1}\} \subset \Omega \) with
\[ C^{-1}N^2 \log N \leq m_1 \leq CN^2 \log N \]
such that for all \( f \in X_N \)
\[ \frac{4}{5} \|f\|_p^p \leq \|f\|_{L_p(\Omega_{m_1})}^p \leq \frac{6}{5} \|f\|_p^p \quad \text{and} \quad \frac{4}{5} \|f\|_2^2 \leq \|f\|_{L_2(\Omega_{m_1})}^2 \leq \frac{6}{5} \|f\|_2^2, \quad (4.4) \]
where \( C > 1 \) is an absolute constant.

Second, we consider the discrete norm \( \|\cdot\|_{L_p(\Omega_{m_1})} \) instead of the norm \( \|\cdot\|_{L_p(\Omega)} \). By (4.4) \( \Phi_N \) is a uniformly bounded Riesz basis of the space \( (X_N, \|\cdot\|_{L_2(\Omega_{m_1})}) \) and moreover
\[ \|f\|_{L_\infty(\Omega_{m_1})} \leq C(p, R_1, R_2)\nu^{1/p}\|f\|_{L_p(\Omega_{m_1})}, \quad \forall f \in \Sigma_v(\Phi_N). \]

Since \( \log m_1 \sim \log N \), by the regular Nikol’skii inequality for the norms \( \ell_p^m \), \( 1 \leq p \leq \infty \), we also have
\[ \|f\|_{L_\infty(\Omega_{m_1})} \leq C\|f\|_{L_{\log N}(\Omega_{m_1})}, \quad \forall f \in X_N, \]
where \( C > 1 \) is an absolute constant. Thus, by Theorem 2.2 and Theorem 3.1 applied to the discrete norm \( \|\cdot\|_{L_p(\Omega_{m_1})} \) we can find a subset \( \Omega_m \subset \Omega_{m_1} \) with
\[ m = |\Omega_m| \leq C(p, R_1, R_2)\nu(\log N)^2(\log(2\nu))^2 \]
such that for any \( f \in \Sigma_v(\Phi_N) \)
\[ \frac{3}{4} \|f\|_{L_p(\Omega_{m_1})}^p \leq \|f\|_{L_p(\Omega_m)}^p \leq \frac{5}{4} \|f\|_{L_p(\Omega_{m_1})}^p, \quad (4.5) \]
Combining (4.4) with (4.5), we obtain the stated result of Theorem 1.3.

**5 A Refined Version of the Conditional Theorem**

Let us first recall some notations. Let \((\Omega, \mu)\) be a probability space. For \( 1 \leq p \leq \infty \) denote by \( L_p(\Omega) \) the usual Lebesgue space \( L_p \) defined with respect to the measure \( \mu \) on \( \Omega \) and by \( \|\cdot\|_p \) the norm of \( L_p(\Omega) \). We also set
\[ B_{L_p} := \{f \in L_p(\Omega) : \|f\|_p \leq 1\}, \quad 1 \leq p \leq \infty. \]
In this section we prove a refined version of the conditional Theorem 2.2 for sampling discretization of all integral norms $L_p$ of functions from a more general subset $\mathcal{W} \subseteq L_\infty$, which allows us to estimate the number of points needed for the sampling discretization in terms of an integral of the $\varepsilon$-entropy $H_\varepsilon(\mathcal{W}, L_\infty)$, $\varepsilon > 0$.

**Theorem 5.1** Let $1 \leq p < \infty$ and let $\mathcal{W}$ be a set of uniformly bounded functions on $\Omega$ with

$$1 \leq R := \sup_{f \in \mathcal{W}} \sup_{x \in \Omega} |f(x)| < \infty.$$  

Assume that $H_t(\mathcal{W}, L_\infty) < \infty$ for every $t > 0$, and

$$(\lambda \cdot \mathcal{W}) \cap B_{L_p} \subseteq \mathcal{W} \subseteq B_{L_p}, \quad \forall \lambda > 0. \quad (5.1)$$

Then there exist positive constants $C_p, c_p$ depending only on $p$ such that for any $\varepsilon \in (0, 1)$ and any integer

$$m \geq C_p \varepsilon^{-5} \left( \int_{10^{-1} \varepsilon^{1/p}}^R u^{\frac{p}{2} - 1} \left( \int_{u}^R \frac{H_{c_p t} \mathcal{W}, L_\infty}{t} \, dt \right)^{\frac{1}{2}} \, du \right)^2, \quad (5.2)$$

there exist $m$ points $x_1, \ldots, x_m \in \Omega$ such that for all $f \in \mathcal{W}$,

$$(1 - \varepsilon) \|f\|_p \leq \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^p \leq (1 + \varepsilon) \|f\|_p. \quad (5.3)$$

In particular, Theorem 5.1 allows us to prove refined versions of Theorem 2.2 and Theorem 1.3, where the constants in the Marcinkiewicz type discretization are replaced by $1 - \varepsilon$ and $1 + \varepsilon$ for an arbitrarily given $\varepsilon \in (0, 1)$.

First, we have the following refined version of Theorem 2.2.

**Corollary 5.1** Let $1 \leq p < \infty$ and $1 \leq v \leq N$. Suppose that a dictionary $\mathcal{D}_N$ is such that

$$\varepsilon_k(\Sigma^p_v(\mathcal{D}_N), L_\infty) \leq B_1 (v/k)^{1/p}, \quad k = 1, 2, \ldots, \quad (5.4)$$

where $B_1 \geq 1$. Assume in addition that there exists a constant $B_2 \geq 1$ such that

$$\|f\|_\infty \leq B_2 v^{1/p} \|f\|_p, \quad \forall f \in \Sigma^p_v(\mathcal{D}_N). \quad (5.5)$$

Then for a large enough constant $C(p)$ and any $\varepsilon \in (0, 1)$ there exist $m$ points $\xi^1, \ldots, \xi^m \in \Omega$ with

$$m \leq C(p) \varepsilon^{-5 - p} v B_1^p \left( \log(B_2 v/\varepsilon) \right)^2.$$
such that for any \( f \in \Sigma_v(D_N) \)

\[
(1 - \varepsilon) \| f \|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq (1 + \varepsilon) \| f \|_p^p.
\]

**Proof of Corollary 5.1** We apply Theorem 5.1 to \( \mathcal{W} := \Sigma^p_v(D_N) \) and \( R = B_2 v^{1/p} \). It is clear that \( \mathcal{W} \) satisfies (5.1). Furthermore, (5.4) implies that (see by [7, Lemma 2.1])

\[
\mathcal{H}_t(\mathcal{W}, L_\infty) \leq C(p) v \cdot (B_1/t)^p, \quad t > 0. \tag{5.6}
\]

Finally, a straightforward calculation using (5.6) then shows that

\[
\varepsilon^{-5} \left( \int_{B_2 v^{1/p}} u^{p-1} \left( \int_u^{B_2 v^{1/p}} \mathcal{H}_{e^t} \left( \mathcal{W}, L_\infty \right) \frac{dt}{t} \right)^{\frac{1}{2}} du \right)^2 
\leq C(p) \varepsilon^{-5-p} B_1^p v (\log(B_2 v/e))^2.
\]

Corollary 5.1 then follows from Theorem 5.1. \( \square \)

Using Corollary 5.1 and following the proof in Sect. 4, we can also obtain the \( \varepsilon \)-version of Theorem 1.3.

**Corollary 5.2** Let \( \Phi_N \) be a uniformly bounded Riesz basis of \( X_N := \text{span}(\Phi_N) \subset L_2(\Omega) \) satisfying (2.8) for some constants \( 0 < R_1 \leq R_2 \). Let \( 1 \leq p \leq 2 \) and let \( 1 \leq v \leq N \) be an integer. Then for a large enough constant \( C = C(p, R_1, R_2) \) and any \( \varepsilon \in (0, 1) \) there exist \( m \) points \( \xi^1, \ldots, \xi^m \in \Omega \) with

\[
m \leq C \varepsilon^{-p-5} v (\log N)^2 (\log(2ve^{-1}))^2.
\]

such that for any \( f \in \Sigma_v(\Phi_N) \) we have

\[
(1 - \varepsilon) \| f \|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq (1 + \varepsilon) \| f \|_p^p.
\]

The rest of this section is devoted to the proof of Theorem 5.1, which is close to the proof of Theorem 1.3 of [6]. We need the following lemma:

**Lemma 5.1** [6, Lemma 2.4] Let \( \{ F_j \}_{j \in G} \) be a collection of finite sets of bounded functions from \( L_1(\Omega, \mu) \). Assume that for each \( j \in G \) and all \( f \in F_j \) we have

\[
\| f \|_1 \leq 1, \quad \| f \|_\infty := \sup_{x \in \Omega} |f(x)| \leq M_j.
\]
Suppose that positive numbers $\eta_j \in (0, 1)$ and a natural number $m$ satisfy the condition
\[
2 \sum_{j \in G} |F_j| \exp \left( -\frac{m \eta_j^2}{8 M_j} \right) < 1.
\]
Then there exists a set $\xi = \{\xi^v\}_{v=1}^m \subset \Omega$ such that for each $j \in G$ and for all $f \in F_j$ we have
\[
\left| \|f\|_1 - \frac{1}{m} \sum_{v=1}^m |f(\xi^v)| \right| \leq \eta_j.
\]

**Proof of Theorem 5.1** Let
\[
\mathcal{W}_1 := \{f / \|f\|_p : f \in \mathcal{W} \setminus \{0\}\}.
\]
Clearly, $\mathcal{W}_1 \subset \mathcal{W}$, and it suffices to prove (5.3) for all $f \in \mathcal{W}_1$. Let $c^* = c^*_p \in (0, \frac{1}{2})$ be a sufficiently small constant depending only on $p$. Let $a := c^* \varepsilon$. Let $J, j_0$ be two integers such that $j_0 < 0 \leq J$,
\[
(1 + a)^{J-1} \leq R < (1 + a)^J \quad \text{and} \quad (1 + a)^{j_0 p} \leq \frac{1}{5} \varepsilon \leq (1 + a)^{(j_0 + 1)p}.
\]
(5.7)

For $j \in \mathbb{Z}$, let
\[
\mathcal{A}_j := \mathcal{N}_{2a(1+a)^j} (\mathcal{W}_1, L_{\infty}) \subset \mathcal{W}_1
\]
denote the minimal $2a(1+a)^j$-net of $\mathcal{W}_1$ in the norm of $L_{\infty}$. For $j \in \mathbb{Z}$ and $f \in \mathcal{W}_1$ we define $\mathcal{A}_j(f)$ to be the function in $\mathcal{A}_j$ that is closest to $f$ in the $L_{\infty}$ norm. Thus, $\|\mathcal{A}_j(f) - f\|_{\infty} \leq 2a(1+a)^j$ for all $f \in \mathcal{W}_1$ and $j \in \mathbb{Z}$.

Next, for $f \in \mathcal{W}_1$ and $j > j_0$ define
\[
U_j(f) := \{x \in \Omega : |\mathcal{A}_j(f)(x)| \geq (1 + a)^{j-1}\},
\]
and
\[
D_j(f) := U_j(f) \setminus \bigcup_{k \geq j+1} U_k(f).
\]

We also set
\[
D_{j_0}(f) := \Omega \setminus \bigcup_{k > j_0} U_k(f).
\]
Note that by (5.7) \( U_j(f) = \emptyset \) for \( j > J \). Thus, \( \{ D_j(f) : j = j_0, \ldots, J \} \) forms a partition of the domain \( \Omega \). Define

\[
h(f, x) := \sum_{j=j_0+1}^{J} (1 + a)^j \chi_{D_j(f)}(x),
\]

where \( \chi_E(x) \) is a characteristic function of a set \( E \).

For \( x \in D_{j_0}(f) \) we have

\[
|f(x)| \leq |A_{j_0+1} f(x)| + 2a(1 + a)^{j_0+1} \leq (1 + a)^{j_0} + 2a(1 + a)^{j_0+1} \leq (1 + a)^{j_0}(1 + 4a),
\]

which in turn implies that

\[
|f(x)|^p \leq (1 + a)^{j_0p}(1 + 4a)^p \leq (1 + a)^{j_0p}(1 + C_p a) \leq \frac{\varepsilon}{10} (1 + C_p a).
\]

On the other hand, for \( x \in D_j(f) \) and \( j_0 < j \leq J \) we have

\[
|f(x)| \geq |A_j f(x)| - 2a(1 + a)^j \geq (1 + a)^j (1 - 3a) \quad \text{and} \quad |f(x)| \leq |A_{j+1} f(x)| + 2a(1 + a)^{j+1} \leq (1 + a)^{(j+1)}(1 + 3a),
\]

which implies

\[
(1 + 3a)^{-p} |f(x)|^p \leq |h(f, x)|^p \leq (1 - 3a)^{-p} |f(x)|^p.
\]

Therefore, choosing \( c^* = c_p^* \) small enough, we have

\[
\left(1 - \frac{\varepsilon}{8}\right) |f(x)|^p \leq |h(f, x)|^p \leq \left(1 + \frac{\varepsilon}{8}\right) |f(x)|^p, \quad \forall x \in \bigcup_{j_0 < j \leq J} D_j(f), \quad (5.9)
\]

and

\[
|f(x)|^p \leq \frac{\varepsilon}{8}, \quad \forall x \in D_{j_0}(f).
\]

In particular, this implies that for any probability measure \( \nu \) on \( \Omega \) and any \( f \in \mathcal{W}_p' \)

\[
\left| \|h(f)\|^p_{L_p(\nu)} - \|f\|^p_{L_p(\nu)} \right| \leq \frac{\varepsilon}{8} \|f\|^p_{L_p(\nu)} + \frac{\varepsilon}{8}. \quad (5.10)
\]

For \( j_0 + 1 \leq j \leq J \) let

\[
\mathcal{F}_j^p := \left\{ (1 + a)^p j \chi_{D_j(f)} : f \in \mathcal{W}_1 \right\}.
\]

Our aim is to find \( m \) points \( \xi_1, \ldots, \xi_m \in \Omega \) for each \( m \) satisfying (5.2)
so that the following inequality holds for all \( f \in \mathcal{F}_j^p \) and \( j_0 < j \leq J \):

\[
\left| \frac{1}{m} \sum_{k=1}^{m} f(\xi^k) - \int_{\Omega} f(x) d\mu(x) \right| \leq \varepsilon_j, \tag{5.11}
\]

where \( \{\varepsilon_j\}_{j=j_0+1}^J \subset (0, 1) \) satisfies \( \sum_{j=j_0+1}^J \varepsilon_j \leq \varepsilon/4 \). Once (5.11) is proved, we obtain by (5.8) that

\[
\left| \frac{1}{m} \sum_{j=1}^{m} |h(f, \xi^j)|^p - \|h(f)\|_p^p \right| \leq \frac{\varepsilon}{4}, \tag{5.12}
\]

which, applying (5.10), will prove the desired inequality (5.3).

To see this, we apply Lemma 5.1 for the collection of the above sets \( \mathcal{F}_j^p \) and notice that for \( j_0 < j \leq J \)

\[
\| (1 + a)^{p_j} \chi_{D_j(f)} \|_1 \leq \| h(f) \|_p^p \leq \| f \|_p^p + \frac{\varepsilon}{4} \leq 2
\]

and

\[
\| (1 + a)^{p_j} \chi_{D_j(f)} \|_{\infty} \leq (1 + a)^{p_j} =: M_j.
\]

Thus, by Lemma 5.1 it suffices to show that for each integer \( m \) satisfying (5.2) one can find a sequence \( \{\varepsilon_j\}_{j_0 < j \leq J} \subset (0, 1) \) such that

\[
\sum_{j_0 < j \leq J} \varepsilon_j \leq \frac{\varepsilon}{4} \tag{5.13}
\]

and

\[
\sum_{j=j_0+1}^J |\mathcal{F}_j^p| \exp\left( -\frac{m \varepsilon_j^2}{8M_j} \right) < \frac{1}{2} \tag{5.14}
\]

To this end we need to estimate the cardinalities of the sets \( \mathcal{F}_j^p \). By definition, for each \( j_0 < j \leq J \), the set \( D_j(f) \) is uniquely determined by the functions \( A_k(f) \in \mathcal{A}_k \), \( j \leq k \leq J \). As a result, we have

\[
|\mathcal{F}_j^p| \leq |\mathcal{A}_j| \times \cdots \times |\mathcal{A}_j| =: L_j,
\]

and

\[
\log L_j \leq \sum_{k=j}^J \log |A_k| \leq \sum_{k=j}^J \mathcal{H}_{a(1+a)^k}(W, L_\infty)
\]

\[
\leq \frac{1}{\log(1+a)} \sum_{k=j}^J \frac{(1+a)^k}{k} \mathcal{H}_t(W, L_\infty) \frac{dt}{t}.
\]
\[
\leq C \varepsilon^{-1} \int_{(1+a)^j}^R \mathcal{H}_{at} (W, L_\infty) \frac{dt}{t}.
\]  
(5.15)

For each \( j_0 < j \leq J \) we choose \( \varepsilon_j > 0 \) so that
\[
\log(\lambda L_j) = \frac{m \varepsilon_j^2}{16M_j}, \quad \text{that is} \quad \varepsilon_j := 4 \sqrt{M_j \sqrt{\log(\lambda L_j) m^{-1}}}.
\]  
(5.16)

where \( \lambda > 1 \) is a large absolute constant to be specified later. Then
\[
\sum_{j=j_0+1}^J \varepsilon_j = 4m^{-1/2} \sum_{j=j_0+1}^J (M_j \log(\lambda L_j))^{1/2}
\]
\[
\leq 4m^{-1/2} \sqrt{\log \lambda} \sum_{j=j_0+1}^J (M_j \log(L_j))^{1/2}
\]

and hence (5.13) is ensured once
\[
m \geq \varepsilon^{-2} \left( \sqrt{\log \lambda} \sum_{j=j_0+1}^J (M_j \log(L_j))^{1/2} \right)^2.
\]  
(5.17)

However, using (5.15), we have
\[
\sum_{j=j_0+1}^J (M_j \log L_j)^{1/2} \leq C \varepsilon^{-1/2} \sum_{j=j_0+1}^J (1 + a)^{p/2} (\int_{(1+a)^j}^R \mathcal{H}_{at} (W, L_\infty) \frac{dt}{t})^{1/2}
\]
\[
\leq C_p \varepsilon^{-1/2} \sum_{j=j_0+1}^J \int_{(1+a)^j}^{(1+a)^{j-1}} u^{p/2-1} \left( \int_u^R \mathcal{H}_{at} (W, L_\infty) \frac{dt}{t} \right)^{1/2} du
\]
\[
\leq C_p \varepsilon^{-1/2} \int_{10^{-1} \varepsilon^{1/p}}^{1} u^{p/2-1} \left( \int_u^R \mathcal{H}_{cpest} (W, L_\infty) \frac{dt}{t} \right)^{1/2} du.
\]

This combined with (5.17) implies that (5.13) is ensured by (5.2).

Finally, we prove (5.14). Indeed, using (5.16), we have
\[
\sum_{j=j_0+1}^J |\mathcal{F}_j| \exp \left( -\frac{m \varepsilon_j^2}{8M_j} \right) \leq \lambda \sum_{j=j_0+1}^J L_j \exp \left( -\frac{m \varepsilon_j^2}{8M_j} \right)
\]
\[
= \sum_{j=j_0+1}^J \exp \left( \log(\lambda L_j) - \frac{m \varepsilon_j^2}{8M_j} \right) = \sum_{j=j_0+1}^J \exp \left( -\log(\lambda L_j) \right) = \frac{1}{\lambda} \sum_{j=j_0+1}^J \frac{1}{L_j}
\]
\[
\leq \frac{1}{\lambda} \sum_{j=j_0+1}^J \frac{1}{N_{\alpha(1+a)j} (W, L_\infty)},
\]
where the last step uses the fact that
\[ L_j \geq |A_j| = N_{a(1+a)/j}(\mathcal{W}, L_\infty). \]

We claim that
\[ \mathcal{H}(\mathcal{W}, L_\infty) \geq \log \frac{R}{4t}, \quad \forall 0 < t < R. \] (5.18)

To see this, let \( f^* \in \mathcal{W} \) be such that \( \|f^*\|_\infty = R \). Let \( k = \lfloor \frac{R}{t} \rfloor \). Define \( f_j = \frac{2jt}{R} f^* \) for \( 0 \leq j \leq k/2 \). Then \( \{f_j\}_{0 \leq j \leq k/2} \subset \mathcal{W} \) is \( 2t \)-separated in \( L_\infty \)-norm. It follows that
\[ N_t(\mathcal{W}, L_\infty) \geq \frac{k}{2} \geq \frac{1}{4} \frac{R}{t}, \]
which shows (5.18).

Now using (5.18), we obtain
\[ \frac{1}{\lambda} \sum_{j=j_0+1}^{J} \frac{1}{N_{a(1+a)/j}(\mathcal{W}, L_\infty)} \leq C R^{-1} \lambda^{-1} \sum_{j=j_0+1}^{J} (1+a)^j \leq C \lambda^{-1} < 1, \]
provided that \( \lambda > 1 \) is large enough. This proves (5.14). \( \square \)

6 Concluding Remarks on Sampling Discretization of \( L_p \) norms for \( 2 < p < \infty \)

In this section, we give a few remarks on sampling discretization of \( L_p \) norms for \( 2 < p < \infty \).

1. The following Nikolskii type inequality plays an important role in the proof of Theorem 1.3:
\[ \|f\|_\infty \leq C v^{1/p} \|f\|_p, \quad \forall f \in \Sigma_v(\Phi_N), \] (6.1)
where the constant \( C \) is independent of \( f \), \( v \) and \( N \). This inequality holds for \( 1 \leq p \leq 2 \) whenever \( \Phi_N \) is a uniformly bounded Riesz basis of \( X_N \). However, this is no longer true for \( p > 2 \). For example, take \( N = 2^v \) and consider the system
\[ \Phi_N = \{e^{2\pi i j x}\}_{j=1}^N \]
on the interval \([0, 1]\) equipped with the usual Lebesgue measure. By the Littlewood-Paley inequality we have that for \( f(x) = \sum_{j=1}^{v} e^{2\pi i j x} \in \Sigma_v(\Phi_N) \) and \( 2 < p < \infty \),
\[ \|f\|_\infty = v > C v^{1/p} \|f\|_p \approx v^{1/p} \frac{1}{p}. \]
2. Let \( \Phi_N \) be a uniformly bounded Riesz basis of \( X_N \subset L_2 \) satisfying (2.9). By monotonicity of the \( L_p \) norms, we have that for any integer \( 1 \leq v \leq N \),

\[
\Sigma^p_v(\Phi_N) \subset \Sigma^2_v(\Phi_N), \quad p > 2,
\]

which in particular implies that

\[
\sup_{f \in \Sigma^p_v(\Phi_N)} \| f \|_\infty \leq \sup_{f \in \Sigma^2_v(\Phi_N)} \| f \|_\infty \leq C v^{1/2}.
\]

Moreover, using Theorem 2.5, we have that for \( p > 2 \) and all integer \( k \geq 1 \),

\[
\epsilon_k(\Sigma^p_v(\Phi_N), L_\infty) \leq \epsilon_k(\Sigma^2_v(\Phi_N), L_\infty) \leq C \cdot (\log N) \left( \frac{v}{k} \right)^{1/2},
\]

which also yields

\[
\mathcal{H}_t(\Sigma^p_v(\Phi_N), L_\infty) \leq C(p) \cdot \left( \frac{\log N}{t} \right)^2, \quad \forall t > 0.
\]

On the other hand, a straightforward calculation shows that for any \( \epsilon \in (0, 1) \) and \( p > 2 \),

\[
\frac{1}{\epsilon^5} \left( \int_{\mathbb{R}^+} v^{1/2} \left( \int_{\mathbb{R}^+} h_{p, \epsilon t}(\Sigma^p_v(\Phi_N), L_\infty) \frac{1}{t} dt \right)^2 \right)^{1/2}
\]

\[
\leq C(p) \epsilon^{-7} v^{p/2} (\log N)^2.
\]

Thus, an application of Theorem 5.1 leads to

**Theorem 6.1** Assume that \( \Phi_N \) is a uniformly bounded Riesz basis of \( X_N := \text{span}(\Phi_N) \) satisfying (2.8) for some constants \( 0 < R_1 \leq R_2 \). Let \( 2 < p < \infty \) and let \( 1 \leq v \leq N \) be an integer. Then for a large enough constant \( C = C(p, R_1, R_2) \) and any \( \epsilon \in (0, 1) \) there exist \( m \) points \( \xi^1, \cdots, \xi^m \in \Omega \) with

\[
m \leq C \epsilon^{-7} v^{p/2} (\log N)^2
\]

such that for any \( f \in \Sigma_v(\Phi_N) \)

\[
(1 - \epsilon) \| f \|_p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^p \leq (1 + \epsilon) \| f \|_p.
\]

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References

1. Batson, J., Spielman, D.A., Srivastava, N.: Twice-Ramanujan Sparsifiers. SIAM J. Comput. 41, 1704–1721 (2012)
2. Belinskii, E.S.: Decomposition theorems and approximation by a “floating” system of exponentials. Trans. Amer. Math. Soc. 350(1), 43–53 (1998)
3. Bourgain, J., Lindenstrauss, J., Milman, V.: Approximation of zonoids by zonotopes. Acta Math. 162, 73–141 (1989)
4. Carl, B.: Entropy numbers, s-numbers, and eigenvalue problems. J. Funct. Anal. 41, 290–306 (1981)
5. Dai, F., Prymak, A., Temlyakov, V.N., Tikhonov, S.: Integral norm discretization and related problems. Russ. Math. Surv. 74, 579–630 (2019)
6. Dai, F., Prymak, A., Shadrin, A., Temlyakov, V.N., Tikhonov, S.: Sampling discretization of integral norms. Constr. Approx. 54(3), 455–471 (2021)
7. Dai, F., Prymak, A., Shadrin, A., Temlyakov, V.N., Tikhonov, S.: Entropy numbers and Marcinkiewicz-type discretization. J. Funct. Anal. 281(6), 109090 (2021)
8. Dai, F., Temlyakov, V.: Random points are good for universal discretization; arXiv:2301.12536
9. Dai, F., Temlyakov, V.: Universal discretization and sparse sampling recovery; arXiv:2301.05962
10. DeVore, R.A., Temlyakov, V.N.: Hyperbolic Cross Approximation, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, (2018); arXiv:1601.03978v2
11. Hinrichs, A., Prochno, J., Vybiral, J.: Entropy numbers of embeddings of Schatten classes. J. Functional Analysis 273, 3241–3261 (2017); arXiv:1612.08105v1
12. Johnson, W.B., Schechtman, G.: Finite dimensional subspaces of $L_p$. Handbook of the geometry of Banach spaces, Vol. 1, 837–870, North-Holland, Amsterdam (2001)
13. Kosov, E.: Marcinkiewicz-type discretization of $L_p$-norms under the Nikolskii-type inequality assumption. J. Math. Anal. Appl. 504(1), 125358 (2021)
14. Marcus, A., Spielman, D.A., Srivastava, N.: Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem. Annal. Math. 182, 327–350 (2015)
15. Nitzan, S., Olevskii, A., Ulanovskii, A.: Exponential frames on unbounded sets. Proc. Am. Math. Soc. 144, 109–118 (2016)
16. Pajor, A., Tomczak-Jaegermann, N.: Subspaces of small codimension of finite-dimensional Banach spaces. Proc. Am. Math. Soc. 97, 637–642 (1986)
17. Rudelson, M.: Almost orthogonal submatrices of an orthogonal matrix. Israel J. Math. 111, 143–155 (1999)
18. Schechtman, G.: Tight embedding of subspaces of $L_p$ in $\ell_p^n$ for even $p$. Proc. Am. Math. Soc. 139(12), 4419–4421 (2011)
19. Schechtman, G., Shnirelman, M.: More on embedding subspaces of $L_p$ in $\ell_p^n$. Compos. Math. 61(2), 159–169 (1987)
20. Schütt, C.: Entropy numbers of diagonal operators between symmetric Banach spaces. J. Approx. Theor. 40, 121–128 (1984)
21. Sudakov, V.N.: Gaussian random processes and measures of solid angles in Hilbert spaces. Sov. Math. Dokl. 12, 412–415 (1971)
22. Temlyakov, V.N.: Greedy Approximation. Cambridge University Press, Cambridge (2011)
23. Temlyakov, V.N.: On the entropy numbers of the mixed smoothness function classes. J. Approx. Theor. 207, 26–56 (2017)
24. Temlyakov, V.N.: The Marcinkiewicz-type discretization theorems. Constr. Approx. 38, 331–360 (2018)
25. Temlyakov, V.N.: A remark on entropy numbers. Studia Math. 263(2), 199–208 (2022); arXiv:2008.13030
26. Zygmund, A.: Trigonometric Series. Cambridge University Press, Cambridge (1959)
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