THE COULOMB GAS BEHAVIOUR
OF TWO DIMENSIONAL TURBULENCE

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ABSTRACT

The long-time large-distance behaviour of free decaying two dimensional turbulence is studied. Stochastic solutions of the Navier-Stokes equation are explicitly shown to follow renormalisation group trajectories. It is proven that solutions of the Navier-Stokes equation asymptotically converge to fixed points which are conformal field theories. A particular fixed point is given by the free Gaussian field with a charge at infinity. The stream function is identified with a vertex operator. It happens that this solution also admits constant $n$-enstrophy fluxes in the asymptotic regime, therefore fulfilling all requirements to represent an asymptotic state of two-dimensional turbulence. The renormalisation basin of attraction of this fixed point consists of a charged Coulomb gas. This Coulomb gas gives an effective description of turbulence.

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I Introduction

Two-dimensional turbulence is a simplified version of three-dimensional turbulence. It can nonetheless give qualitative information on phenomena such as atmospheric turbulence. Recently, many field theorists have been attracted by 2d turbulence. This new interest has been aroused by Polyakov's seminal\cite{1} paper. For the first time, it seems that this long-standing problem can be tackled using field theoretic methods\cite{2}. Definite predictions are obtained. One can hope to compare them with numerical simulations\cite{3}.

Most analyses of two-dimensional turbulence aim at computing the two-dimensional analogue of the celebrated Kolmogorov spectrum\cite{4} in the long wavelength limit. An early analysis by Kraichnan\cite{5} showed that the energy spectrum should decrease as $k^{-3}$, due to the transfer of vorticity from large to small scales where dissipation takes place. Nevertheless, this prediction suffers from large infrared divergences and is corrected by logarithms at one loop order. On the other hand, such a correction does not assure convergence at two loop order where double logarithm infrared divergences appear. Alternative predictions have been given by Saffman\cite{6} and Moffat\cite{7} with respective spectra $k^{-4}$ and $k^{-1.5}$. This variety of results shows that the situation is far from being settled. From a field theoretical point of view, these approaches have severe drawbacks, they all assume that dimensional analysis is exact and neglect possible non-trivial anomalous dimensions appearing in short distance expansions of field products. Unfortunately, no precise rationale can be extracted from numerical simulations. It happens that simulations starting from gaussian initial conditions show an increase of the exponent from -4 to -3 for short times. Then coherent vortices emerge. Moreover, a very long time simulation\cite{8} shows that the fluid motion seems to converge to an ensemble of point vortices\cite{9}. The fate of the spectrum exponent as time goes to infinity is certainly not clear.

The Polyakov approach deals with steady states of two-dimensional turbulence. This amounts to imposing boundary conditions and an external pumping force. Turbulence is then only obtained in the inviscid limit where the viscosity goes to zero. Interpreting the Navier-stokes equation as an equation for correlation functions, Polyakov shows that scale-invariant field theories can represent steady states if a certain inequality is satisfied by operator dimensions. This boils down to imposing a vanishing non-linear term in the Navier-Stokes equation. Another constraint is obtained by forcing the enstrophy flux to be a constant. A simplified treatment of the resulting Diophantine equation leads to a solution whose spectrum scales as $k^{-25/7}$. Unfortunately, such a steady state is non-unitary and thus suffers from large infrared problems. The Diophantine equation can be shown to yield thousands of possible non-unitary minimal conformal field theories\cite{10}. As emphasised in ref.\cite{11}, these solutions are hardly physical. In addition to infrared divergences, they fail to provide a non-zero enstrophy flux. Moreover, they only take into account the enstrophy flux and discard fluxes of higher conserved quantities. It has been recently shown\cite{12} that a conformal description of two dimensional steady states is rather unlikely.

In this paper, we strive to tackle a less ambitious problem, i.e free decaying two dimensional turbulence. To do so, we follow closely the Polyakov approach but try to remedy some of its drawbacks by imposing that fluxes of all conserved quantities are constant. As time increases, viscosity effects become less important and solutions of the Navier-Stokes equation reach a quasi-equilibrium containing free coherent vortices. For such a state, it seems reasonable to expect that fluxes as well as the energy are zero. The object of this article is to show that these conclusions can be derived from an effective field theory describing the long-time regime of two-dimensional turbulence.

The paper is arranged as follows. In a first part, some comments on the nature of turbulence are proposed. The second part deals with solutions of Navier-Stokes equations. In the third part, we
impose that fluxes of conserved quantities are constant. Then, the long time behaviour of conserved charges is analysed. In the last section, we propose an effective theory for two dimensional free decaying turbulence.

II A Field Theoretical View of Turbulence

Turbulence is often described as a random behaviour reached by fluids under certain conditions such as high speed and low viscosity. Fluid dynamics describe any fluid motion in terms of the velocity field \( v \). This field is a macroscopic field: it is the average of molecular velocities over a macroscopic size \( a \) ( \( a \) is rather small and can be thought to be of the order of a micron). Below this scale, the Hydrodynamic approximation is not valid. There is another intrinsic scale \( l \); it is set by viscosity effects and represents the scale below which viscosity smoothes variations of the velocity field. Above this scale, the velocity is rapidly varying. A sufficiently low viscosity is a phenomenological condition for the existence of a turbulent regime. In that case, the viscous length \( l \) becomes of the order of the macroscopic length \( a \). We suppose that this criterion is fulfilled, so that there is a single fundamental length scale \( a \).

The equation governing the time evolution of the velocity field is a non-linear equation,

\[
(\partial_t + v \cdot \nabla) v = \nu \Delta v - \nabla p + F
\]

known as the Navier-Stokes equation where \( p \) is the pressure field and \( F \) the external pumping force. Such an equation is simpler in two dimensions as it only involves a pseudo-scalar field instead of a vector field. This allows strong results about classical solutions\(^1\) to be proven. It can be shown that smooth (\( C^\infty \)) initial conditions remain smooth. However, this is not the sort of statement we are interested in. Indeed, we would like to understand the Navier-Stokes equation when the velocity field is far from being smooth.

As a matter of fact, the velocity field is a random field. This means that each field configuration is chosen at random according to a probability law. We will restrict ourselves to probability laws which can be expressed as statistical field theories. Finding the probability law amounts to finding the action \( S_t \) giving the Boltzmann weight at every time \( t \). The action \( S_t \) is an effective action valid for distances larger than the macroscopic length \( a \). It is a local action. Indeed, the range of the interaction between two fluid elements (by this we mean a macroscopic neighbourhood of a point) is given by the viscosity length \( l \). Below this scale, the velocity field hardly varies due to viscosity effects; on the contrary, for larger scales, two fluid elements are not directly in contact, and their velocity fields can be almost uncorrelated. But for turbulent fluids, the viscous length is of the order of the macroscopic length. The length \( a \) plays the role of a physical Ultra-Violet cut-off, so that the interaction length is of the order of the UV cut-off. As all fields will be smeared out (regularised) over a distance \( a \), this implies that the Lagrangian composing the action is in fact local, i.e. can be expressed as a function of regularised fields and their derivatives. This does not prevent the existence of any long-range order. As for critical phenomena, long-range order can occur due to collective effects even though interactions are short-ranged.

Two highly different issues could be tackled. On the one hand, the Ultra-Violet problem amounts to solving the Navier-Stokes equation with an external pumping force and given boundary conditions. Most of the time, a large box and a low frequency random pumping force are specified. Turbulence only appears in the inviscid limit where \( \nu \) goes to zero. The velocity field is random because of the random external force. Notice that in field theoretic parlance, this is tantamount
to studying the UV behaviour of the action $S_t$, keeping an Infra-Red cut off fixed. Scale invariant solutions are then time-independent UV fixed points in the renormalisation group sense. They represent stationary turbulence states. On the other hand, the Infra-Red problem deals with infinite space and Dirichlet boundary conditions at infinity. No external pumping is required but random initial conditions are imposed. As shown in section III, the viscosity $\nu$ is finite and does not need to be scaled down to zero. It becomes negligible in the long time regime. This is a free decaying turbulence situation where scale invariance is gradually restored by time evolution. It is quite analogous to a self-criticality regime. This issue has to be faced as an Infra-Red problem, where long-time, large-scale limits are taken, keeping the UV cut-off a fixed. Asymptotic solutions are then IR fixed points. The relation between IR and UV fixed points seems to be a difficult issue.

It is convenient to move from the velocity picture to the vorticity picture. In 2d, the continuity equation reads

$$\nabla v = 0, \quad (2)$$

valid for an incompressible fluid. This entails that $v$ is a curl:

$$v_\alpha = e_{\alpha\beta}\partial\beta\psi \quad (3)$$

where $\psi$ is the stream function. The field $\psi$ is a pseudo-scalar field. In the plane, it is always possible to construct pseudo-scalar fields from scalar fields, they are just two-forms $\psi dz \wedge d\bar{z}$. It is also useful to introduce the vorticity which is the curl of $v$:

$$\omega = e_{\alpha\beta}\partial\alpha v_\beta. \quad (4)$$

Notice that $e_{12} = -e_{21} = 1$ is the antisymmetric tensor, i.e. $\omega = -\Delta \psi$. This is also a pseudo-scalar field. This yields a different form of the Navier-Stokes equation:

$$\partial_t \omega + e_{\alpha\beta}\partial\alpha \psi \partial\beta \Delta \psi = \nu \Delta \omega \quad (5)$$

This equation for the vorticity field will be the main ingredient of our analysis.

Before further delving into the properties of the vorticity field, we will recall a few facts about certain two-dimensional field theories. The velocity field obeys the Navier-Stokes equation. It is not clear what meaning can be given to the product of fields appearing in the non-linear term. The following will provide an answer. Fields will be chosen to be tempered distributions $\psi \in \mathcal{S}'$ (continuous linear forms on the space of rapidly decreasing functions $\mathcal{S}$). A field theory such as the IR fixed point is a measure on the space of tempered distributions$^{[13]}$. To regularise the short distance behaviour of fields, introduce a smoothing function $\rho_\epsilon$:

$$\int d^2x \rho(x) = 1$$

$$\rho_\epsilon(x) = \frac{1}{\epsilon^2} \rho\left(\frac{x}{\epsilon}\right) \quad (7)$$

This function is an approximate delta function. It allows to transform the very rough variations of tempered distributions into perfectly well behaved functions. To do so, introduce the regularised field

$$\psi_\epsilon(x) = \int d^2y \, \psi(y) \rho_\epsilon(y - x) \quad (8)$$
This is now a smooth ($C^\infty$) function. We will interpret macroscopic fields as regularised fields where $\epsilon = a$ the physical UV cut-off. This allows to see the Navier-Stokes equation and its vorticity counterpart as equations for regularised fields. The non-linear term is then a simple product of smooth functions. Obviously care will be needed in removing the cut-off.

The Navier-Stokes equation is now expressed in terms of regularised fields. Physically, this is all the more natural as the velocity field is the average of molecular velocities. This gives:

$$\partial_t \omega_a + e_{\alpha\beta} \partial_\alpha \omega_a \partial_\beta \Delta \psi_a = \nu \Delta \omega_a.$$  \hspace{1cm} (9)

The initial time field theory is specified down to the cut-off $a$. It is the initial state of the Cauchy problem for the evolution equation. It is specified by an initial effective field theory representing the initial probability law of the stream function. At any given time $t$, the effective field theory describing the behaviour of $\psi_a$ is obtained from (9). It represents the probability law describing the fluctuations of $\psi_a$ at time $t$.

Suppose that the effects of viscosity can be neglected in the asymptotic regime. Then the inviscid fluid possesses an infinite number of conserved charges. The first one is obvious, it is the energy per unit mass. Another set of conserved quantities stems from the pseudo-scalar behaviour of the vorticity. Introduce the generalised $n$-enstrophy:

$$H_{n,a} = \int d^2 x \omega_a^n(x).$$  \hspace{1cm} (10)

Only the even enstrophies are non zero. Now compute

$$\frac{dH_{n,a}}{dt} = ne_{\alpha\beta} \int d^2 x \partial_\alpha \psi_a \partial_\beta \Delta \psi_a \omega_a^{n-1}(x).$$  \hspace{1cm} (11)

Integrate by parts using the antisymmetry of $e_{\alpha\beta}$ to obtain:

$$\frac{dH_{n,a}}{dt} = e_{\alpha\beta} \int d^2 x \omega_a^n \partial_\alpha \partial_\beta \Delta \omega_a \equiv 0.$$  \hspace{1cm} (12)

We will need to integrate by parts very frequently. So from now on, we restrict fields to be zero at infinity, ie regularisations $\psi_a$, $\omega_a...$ vanish at infinity. The second moment is simply called the Enstrophy. In section III, we will justify this calculation directly from the viscous Navier-Stokes equation. It will be shown that this result is indeed true in the long-time regime, even in the presence of viscosity. Such conserved charges give rise to conserved currents in Fourier space. These currents represent the flux of n-enstrophies from scale $q$ to $q + dq$. In the usual Kolmogorov approach, the energy flux (called the energy transfer rate $\epsilon$) is supposed to be finite and constant. Physically, these conditions imply that every point in the fluid plays the role of a local source of n-enstrophy and energy. In the free decaying case, we expect these fluxes to be zero if a steady state exists. In the following we will call turbulence any critical solution of Navier-Stokes equations such that the associated fluxes are finite and constant (possibly zero).
III General Asymptotic Properties Of Decaying Turbulence

a) The Neighbourhood of Renormalisation Group Fixed Points

We will suppose that turbulence is specified by a massless field theory. This theory is a fixed point of the evolution equations in the long time-large scale regime. From given initial conditions, it is not clear what the set of fixed points can be. It could be a single theory, and in that case any initial condition would finally be attracted by this fixed point. It could also be a more complicated set such as a manifold.

On physical grounds, 2d turbulence is likely to be represented by a unique fixed point. Indeed, 2d turbulence is isotropic. Then, universality suggests that there must be a unique universality class describing 2d turbulence in the infrared regime. This is certainly true if the notion of universality can be extended to self-organised criticality. In the following, we will try to first analyse the features of a fixed point. Thus, we will show that any conformal theory satisfying a specified inequality can be a fixed point. Fixed points are actually asymptotic solutions of Navier-Stokes if they are stable fixed points, i.e. if nearby deformations of a fixed point converge in the renormalisation group sense.

First observe that the viscosity defines a length scale and a time scale, i.e. by engineering dimensional analysis:

\[ \nu = \frac{a^2}{\tau}. \] (13)

The scale \( a \) is the UV cut-off whereas typical times are measured in units of \( \tau \). We want to investigate the large distance \( r \gg a \) long time \( t \gg \tau \) regime. Let us first define a fixed point as a continuum conformal field theory satisfying the Euler equation:

\[ \lim_{\lambda \to \infty} \left[ \frac{d\omega}{dt} + e_{\alpha\beta} \partial_\alpha \psi \Delta \psi \right] = 0. \] (14)

Notice that such an equation involves the product at shorter and shorter distances of two regularised fields. In order to further analyse the Euler equation, we need to evaluate short distance expansions. In a nutshell, the expansion

\[ \psi(y)\phi(z) = \sum_{\alpha} C_\alpha(y,z)\mathcal{O}_\alpha(z) \] (15)

is valid in any correlation function. The coefficient functions are in fact distributions which scale as \( v = |y-z| \to 0 \) as:

\[ C_\alpha(y,z) \sim v^{d\phi - d\psi}. \] (16)

Moreover, the most singular term on the right-hand side can be retained as the most significant piece. The Euler equation involves a non-linear term whose continuum limit is far from being obvious. Nevertheless, it can be computed using the operator product of two \( \psi \). Indeed, the non-linear term reads (we put \( \epsilon = \frac{\lambda}{\lambda} \))

\[ e_{\alpha\beta} \partial_\alpha \psi(x) \partial_\beta \Delta \psi(x) = \int d^2 u d^2 v e_{\alpha\beta} \partial_\alpha \rho_\epsilon(u-x) \partial_\beta \Delta \rho_\epsilon(v-x) \psi(u)\psi(v). \] (17)

In the continuum limit \( \epsilon \to 0 \), it is safe to retain the dominant term in the operator product expansion:

\[ \int d^2 u d^2 v e_{\alpha\beta} \rho_\epsilon(u-x) \rho_\epsilon(v-x) \partial_\alpha \rho_\epsilon \partial_\beta \Delta \rho_\epsilon \sum_\gamma C_\gamma(u,v)\psi_\gamma^2(v), \] (18)
where $\psi_2^\gamma$ are the fields appearing in the short distance expansion. The only way of getting a non-zero term is to act on the possibly non-symmetric term under $\alpha \rightarrow \beta$. One of the coefficient functions must satisfy $C(u_1, u_2, v_1, v_2) = -C(v_2, v_1, u_2, u_1)$. This can only be obtained from descendant terms, where the leading operator is constructed from a pseudo-scalar combination of descendant fields $L_{-n_1} \ldots L_{-n_m} \psi$. The Virasoro generators $L_n = \frac{1}{2\pi} \int dz z^{n+1} T(z)$ are the generators of the 2-d conformal momentum tensor. Truncating the series yields:

$$\int d^2 v d^2 u \rho_\gamma(v) \rho_2(u - x) e_{\alpha\beta} \partial_\alpha \partial_\beta \Delta^u C_\gamma(u + v, u) \psi_2(u)$$  \hspace{1cm} (19)

After changing variable $v \rightarrow \epsilon v$, and using the fact that the cut-off is small:

$$\int d^2 v e_{\alpha\beta} \partial_\alpha \partial_\beta \Delta^u C_\gamma(u + \epsilon v, u) \sim \epsilon^d \psi_2^\gamma - 2d \psi_2 - 4 \int d^2 v e_{\alpha\beta} \partial_\alpha \partial_\beta \Delta^u C_\gamma(u + v, u).$$  \hspace{1cm} (20)

We will denote the constant by $C$, neglecting its dependence on $u$:

$$e_{\alpha\beta} \partial_\alpha \psi_2 \partial_\beta \Delta \psi_2 \sim \epsilon^d \psi_2^\gamma - 2d \psi_2 - 4 C \psi_2^\gamma(x).$$  \hspace{1cm} (21)

From now on, all fields will be normalised in such a way that finite constants like $C$ are equal to one. It is now necessary to identify the leading operator. Notice that due to the term $e_{\alpha\beta}$ the result is a pseudo-scalar. Furthermore it has to be antisymmetric under $u_1 \rightarrow u_2$. Using complex coordinates:

$$\psi_2^\gamma(i\bar{z}, iz) = -\psi_2^\gamma(z, \bar{z})$$

$$\psi_2^\gamma(\bar{z}, z) = -\psi_2^\gamma(z, \bar{z})$$  \hspace{1cm} (22)

Under the first transformation the energy momentum tensor transforms as:

$$T \rightarrow -\bar{T}$$

$$\bar{T} \rightarrow -T$$  \hspace{1cm} (23)

This yields the transformations of Virasoro generators $L_n \rightarrow i^n \bar{L}_n$. The lowest order anti-invariant combination is:

$$\psi_2^\gamma = (L_{-2} \bar{L}_{-1}^2 - \bar{L}_{-2} \bar{L}_{-1}^2) \psi_2.$$  \hspace{1cm} (24)

This immediately gives the leading behaviour in (20) as

$$d_{\psi_2^\gamma} = d_{\psi_2} + 4.$$  \hspace{1cm} (25)

This calculation can be extended by induction to higher order short distance expansions. After this lengthy computation, we obtain the conditions for the existence of a fixed point. Suppose that the fixed point theory is time invariant. Then the Euler equation is satisfied if the non-linear term vanishes altogether in the continuum limit. This implies that\[1]:

$$d_{\psi_2} > 2d_{\psi}$$  \hspace{1cm} (26)

This is the Polyakov inequality. Hence, all conformal field theories satisfying this equation are solutions of the Euler equation. However, they may not be valid representation of 2d turbulence. To select the expected unique fixed point modelling 2d turbulence, we will have to impose that fluxes of $n$-enstrophies are constant in the infra-red regime.
We are now in position to prove that fixed points are asymptotic solutions of the Navier-Stokes equation. Suppose that, after a transient period of order $O(\tau)$, the field theory obtained solving the Cauchy problem from a given initial condition falls within the renormalisation group neighbourhood of a fixed point (we will call this neighbourhood the basin of attraction of the fixed point). This means that this field theory converges in the large distance limit towards the fixed point. Let us prove that such renormalisation group trajectories coincide with the Navier-Stokes evolution. Suppose that large distance long time trajectories are renormalisation group trajectories.

To implement this idea, rescale $x \rightarrow \lambda x$ and $t \rightarrow \lambda T t$ where $T$ is a yet unknown constant. Then define a new field $\tilde{\psi}$ as the regularised field following the renormalisation trajectories of the solution $\psi$ at time $t$. This means that the regularised field $\tilde{\psi}_a(\lambda x, \lambda^T t)$ obeys the following:

$$\tilde{\psi}_a(\lambda x, \lambda^T t) = \lambda^{-d_\psi} \psi_{\lambda^{-1} a}(x, t).$$

(27)

The field $\tilde{\psi}$ is the effective field obtained after integrating the action $S_t$ over distances in the range $[\lambda^{-1} a, a]$, whereas the field $\psi_{\lambda^{-1} a}$ is the regularised field defined with the action $S_t$. Notice that this ansatz $\tilde{\psi}$ is a solution of Navier-Stokes equation provided that, when plugging $\tilde{\psi}$ in (9), one ends up with the original the Navier-Stokes equation for $\psi$ at time $t$ and cut-off $\lambda^{-1} a$. Obviously, such an equation should have a modified viscosity $\nu \rightarrow \lambda^{2+d_\psi} \nu$ as the typical time $\tau$ is also rescaled to $\lambda^T \tau$.

Putting this ansatz back in Navier-Stokes equation after adjusting the time rescaling $t \rightarrow \lambda^{2+d_\psi} t$, one obtains

$$\partial_t \omega_{\lambda^{-1} a} + e_{\alpha\beta} \partial_\alpha \psi_{\lambda^{-1} a} \partial_\beta \Delta \psi_{\lambda^{-1} a} = \lambda^{-2+T} \nu \Delta \omega_{\lambda^{-1} a}.$$  

(28)

This is indeed the expected result. This proves that $\tilde{\psi}$ is a solution. Notice that scale invariance is asymptotically achieved if and only if the dimension $d_\psi$ is negative. Moreover $T$ has to be positive in order to ensure that time flows towards infinity. This implies that $-2 < d_\psi < 0$. Hence, we retrieve Polyakov’s idea that turbulence requires non-unitary theories. This entails an extreme boundary condition sensitivity. We will end up considering charges at infinity which in a sense epitomise the non-trivial infrared properties of turbulence. Notice that viscosity explicitely breaks scale invariance. The effective action in the basin of attraction of the fixed point will reflect this fact, i.e. it will be the sum of the fixed point action and irrelevant operators flowing to zero in the infrared regime. To conclude, the field $\psi_{a\lambda^{-1}}$ tends to the continuum limit field $\psi$ whereas the solution $\tilde{\psi}$ converges to the fixed point along renormalisation group trajectories. This proves that fixed points are asymptotic solutions of Navier-Stokes equation $^1$. Amongst all these fixed points, we will explicitly find an example admitting finite fluxes in the infra-red regime.

c) The Effects of Viscosity on N-Enstrophies

As viscosity is introduced, it is no longer true that $n$-enstrophies are conserved. Nevertheless, they are slowly varying. In order to assess the effect of viscosity, let us compute the $n$-enstrophies and their time derivatives in the asymptotic regime. By (10),

$$H_{n,a}(\lambda^T t) = \int d^2x \, \omega^n_a(x, \lambda^T t),$$

(29)

$^1$ This result is also valid in three dimensions. In that case, solutions of the Navier-Stokes equation still follow renormalisation group trajectories. In the infrared limit, solutions converge to non-unitary conformal fixed points defined by a straightforward generalisation of (14). On the contrary, the Polyakov inequality seems to require a non-trivial extension.
Change \( x \rightarrow \lambda x \) and use the scaling properties of \( \omega \) to obtain

\[
H_{n,a}(\lambda^T t) = \lambda^{2-2n-nd_\nu} \int d^2 x \, \omega_{\lambda^{-1}a}^n(x,t).
\] (30)

The integrand tends to be the product of \( n \) fields at the same point, so it is necessary to evaluate higher order short distance expansions. These products are defined by induction. They can be written in a compact way using conformal families:

\[
[\psi][\psi_{n-1}] = [\psi_n] + \ldots
\] (31)

where \( \psi_n \equiv \text{Dom}(\psi \psi_{n-1}) \) is the leading part of the short distance expansion \( \psi \psi_{n-1} \). This entails that inductively:

\[
\omega_{\lambda^{-1}a}^n(x,t,t) \sim \lambda^{a^{-1}} (\lambda a^{-1})^{-d_\psi + nd_\psi + 2n} \psi_{n,\lambda^{-1}a}(x,t).
\] (32)

Hence the \( n \)-enstrophy:

\[
H_{n,a}(\lambda^T t) = \lambda^{2-d_\psi} a^{d_\psi - nd_\psi - 2n} \int d^2 x \, \psi_{n,\lambda^{-1}a}(x,t).
\] (33)

\( N \)-Enstrophies are simply integrals of higher order fields.

Similarly, the time derivative reads

\[
\frac{dH_{n,a}(\lambda^T t)}{dt} = n \lambda^{2-2n-nd_\psi} \int d^2 x \, \nu \lambda^{d_\psi} \Delta \omega_{\lambda^{-1}a}(x,t) \omega_{\lambda^{-1}a}^{n-1}(x,t)
\] (34)

where the rescaled version (28) of Navier-Stokes equation is used. The right-hand side is always negative as integration by parts yields \( -(n-1) \nu \int d^2 x \, (\partial \omega_{\lambda^{-1}a})^2 \omega_{\lambda^{-1}a}^{n-2} \). Finally,

\[
\frac{dH_{n,a}(\lambda^T t)}{dt} = -n \nu \lambda^{4+d_\psi} a^{-2(n+1)-nd_\psi + d_\psi} \int d^2 x \, \psi_{n,\lambda^{-1}a}(x,t).
\] (35)

Using the scaling properties of the fields, this yields in the infrared regime,

\[
\frac{dH_{n,a}(t)}{dt} = -n \nu \frac{1}{a^2} H_{n,a}(t)
\] (36)

where \( t \) is now the \textit{asymptotic physical time}. Notice that the transient time \( \frac{1}{\tau} \) appears. The \( n \)-enstrophies go to zero in the asymptotic regime. Similarly, one gets

\[
E_a(t) = E_a(\tau) \exp(-2\frac{t-\tau}{\tau})
\] (37)

in the asymptotic regime. Notice that the energy is exponentially damped to zero unless there is no viscosity.

So we have found that in the infrared regime solutions of Navier-Stokes equation have vanishing conserved quantities. This does not imply however that solutions are trivial.

c) The Energy Spectrum
In the same vein, one can obtain an expression for the scaling exponent of the energy spectrum. The spectrum is nothing but the Fourier transform of the velocity connected 2-point function:

\[ E(k) = 2\pi k \int d^2x \exp(-i2\pi k.x) \langle v_{a,a}(0)v_{a,a}(x) \rangle \]  

such that

\[ E_a = \int dk E(k) \]  

The scaling behaviour of \( E(k) \) in the inertial range can be deduced as follows:

\[ \langle \partial_\alpha \psi_a(\lambda x, \lambda^T t) \partial_\beta \psi_a(\lambda y, \lambda^T t) \rangle \sim \lambda^{-2-2d} \langle \partial_\alpha \psi_{\lambda^{-1}a}(x, t) \partial_\alpha \psi_{\lambda^{-1}a}(y, t) \rangle . \]  

The two point function can be evaluated in the continuum limit as \( \lambda \) goes to infinity:

\[ \langle \partial_\alpha \psi_a(\lambda x, \lambda^T t) \partial_\alpha \psi_a(\lambda y, \lambda^T t) \rangle \sim (\lambda x - \lambda y)^{-2-2d}. \]  

Putting \( y = 0 \) yields the large scale behaviour of the velocity two point function. The Fourier transform scales as

\[ E(k) \sim k^{2d_\psi+1}. \]  

Notice that this is the infra-red spectrum. Once a fixed point is specified, the infrared spectrum follows. As expected, no prediction on the UV spectrum is derived.

d) Some Examples

Let us now give the simplest possible solution of the Navier-Stokes equation: the Gaussian free field with charges at infinity\(^{16}\). Choose the Lagrangian to be:

\[ S = \frac{1}{4\pi} \int d^2x (\partial \phi)^2 \]  

and identify \( \psi =: \exp i\phi : \). We explicitly suppose that there is a charge \( Q \) at infinity. This conformal field theory can be abstractly defined by its energy-momentum tensor and the operator product algebra. One can also use also a formal device to write down the action of the deformed Gaussian theory with a charge at infinity. Indeed, it is sufficient to add to the free Gaussian action a source term \( \int d^2x J_\infty(x)\phi \) where the source is \( J_\infty = iQ\delta^2(x - \infty) \). This formal source at infinity implies that correlation functions of vertex operators are non-zero provided the sum of the exponents of each operator is equal to \( Q \). Then the field \( \psi_2 =: \exp 2i\phi : \) has dimension \( 2 + Q \) whereas \( d_\psi = 0.5(1 + Q) \). So vertex operators are good candidates to represent the asymptotic regime of the stream function. The charge at infinity is restricted to \(-5 \leq Q < -1\). This theory is quite peculiar. Indeed, all UV properties, such as short distance expansions, are highly dependent on the value of the charge at infinity. In a sense, we can say that this reflects the non-unitarity of the theory and the increase of correlation functions with distance, i.e. large distance properties influence small scale characteristics.

We would like to interpret this theory in a probabilistic way where operators are replaced by fields. This require that we identify \( \exp i\phi : \) with its formal adjoint \( \exp(-(1+Q)i\phi) : \). Therefore, we deduce that\(^2\):

\[ Q = -2. \]  

\(^2\) The central charge is \( c = -11 \)
Notice that in this case the infrared spectrum behaves as
\[ E \sim k^0, \]  
and in fact it happens this apparently innocuous solution is also turbulent. This demonstrates the idea that two dimensional turbulence is a problem where large scales are relevant.

IV- Turbulent Fixed Points

a) Currents in the Infra-Red Limit

The conventional picture of turbulence requires that some physical entity (eg the energy) is transferred from large to small scales. This means that currents are non zero constants. We have proven that in the continuum limit \( n \)-enstrophies and the energy are conserved quantities. The conserved quantities can be written down as reciprocal-space integrals:

\[ H_{n,a} = \int dq \ h_{n,a}(q) \]  

The right-hand side is only modulus dependent. The integrand represents the conserved quantity per unit mode. As \( H_{n,a} \) is asymptotically conserved, there is a conserved current as well. More precisely,

\[ \frac{dh_{n,a}}{dt} + \partial_q R_{n,a}(q) = 0 \]  

where \( R_{n,a} \) is the current. The current involves the Fourier transform of \( \omega_n^a(x) \) and reads

\[ h_{n,a}(q) = 2\pi q L(-q)(\omega_n^a(q)) \]

\[ R_{n,a}(q) = \frac{d}{dt} \int_{k>q} h_{n,a}(k) \]  

where \( L(q) \) is the Fourier transform of the characteristic function of a bounded domain of integration \( L \). The next step is to come back to real space. Introduce the structure factor

\[ \theta_q(x) = \int_{k>q} d^2k \ e^{i2\pi k \cdot x}. \]  

Then

\[ R_{n,a} = \frac{d}{dt} \int_L d^2x \ \theta_q * \omega_n^a. \]  

The infinite volume limit can now be taken. Notice that the currents can be written as integrals of densities:

\[ R_{n,a}(q) = \int d^2x \ r_{n,a}(x) \]

\[ r_{n,a}(x) = \theta_q * \frac{d}{dt} \omega_n^a(x) \]
The time-derivative is explicitly taken as in the Euler equation. Currents appear as integrals of current densities which depend on $q$. In the Kolmogorov theory, current densities are constant and finite. We would like to evaluate current densities in the asymptotic regime. So rescale space and time such that
\[ r_{n,a}(\lambda x, \lambda^T t, q) = \lambda^{-d} r_{n,\lambda^{-1}a}(x, t) \]  
where $d$ is the dimension of $r$. The current density is given by a convolution; it can be simplified by noticing the scaling property:
\[ \theta_q(x - y) = \lambda^2 \theta_{\lambda q}(\lambda x - \lambda y). \]  
Then
\[ r_{n,\lambda^{-1}a}(x, t, q) = -n \int d^2 y \lambda^2 \theta_{\lambda q}(\lambda x - \lambda y) e_{\alpha\beta} \partial_\alpha \psi_{\lambda^{-1}a}(y) \partial_\beta \Delta \psi_{\lambda^{-1}a}(y) \omega_{\lambda^{-1}a}^n(y). \]  
We can now use the fact that, for large $\lambda$, we have convergence of the approximate $\delta$-function $\theta_{\lambda q} \sim \delta$ to obtain
\[ r_{n,\lambda^{-1}a}(x, t) = -n e_{\alpha\beta} \partial_\alpha \psi_{\lambda^{-1}a}(x) \partial_\beta \Delta \psi_{\lambda^{-1}a}(x) \omega_{\lambda^{-1}a}^n(x). \]  
This is $q$-independent. So we have proven that in the asymptotic regime currents are $q$ independent and represent local sources of turbulence.

b) The Polyakov Equation

Current densities are composite fields. Their field theoretic dimensions are the sum of two terms: the canonical dimension which is 2 as they are densities, and the anomalous dimension which accounts for the singularities appearing in short distance expansions. More precisely, the dimension of densities is given by
\[ d = 2 - d_\psi - d_{\psi_{n-1}} + (n + 1)d_\psi. \]  
The anomalous dimension cancels all divergences appearing in the short distance expansions of $\frac{d\omega_{\lambda q}}{dt}$ and $\omega_{\lambda}^{n-1}$. Writing $R_{n,a}(\lambda^T t) = \int d^2 x \lambda^2 - d r_{n,\lambda^{-1}a}(x, t)$, we are interested in the asymptotic behaviour of the rescaled current densities $\tilde{r}_{n,\lambda^{-1}a}(x, t) = \lambda^2 - d r_{n,\lambda^{-1}a}(x, t)$. The rescaled current density is a product of two fields:
\[ \tilde{r}_{n,\lambda^{-1}}(x, t) = a^d \psi_{n-1} + d_\psi - 2(n-1) - (n+1)d_\psi \lambda^{2(n-1)} L \psi_{2,\lambda^{-1}a} \psi_{n-1,\lambda^{-1}a} \]  
where $L = (L_1 L_2 - L_2 L_1)$.

The first power of $\lambda$ is the remnant of the derivatives which do not contribute to the anomalous dimension. This product is still divergent in the continuum limit. Only a fine tuning of the field dimensions can preserve a finite limit. Now, suppose that the leading part of short distance expansions at the fixed point defines an associative algebra, i.e. the leading part $\text{Dom}(\psi \phi)$ of field products satisfies $\text{Dom}(\psi \text{Dom}(\psi_2 \phi_3)) = \text{Dom}(\text{Dom}(\psi_1 \psi_2) \phi_3)$

This allows $[\psi_2][\psi_{n-1}]$ and $[\psi] [\psi_n]$ to be related, leading to
\[ \tilde{r}_{n,\lambda^{-1}a}(x, t) = \lambda^{d_\psi - n_1} + d_\psi - 2(n-1) - (n+1)d_\psi \lambda^{2(n-1)} L \psi_{n+1,\lambda^{-1}a}. \]  

3 The operator algebra is always associative. If the stream function is a simple current \cite{17}, then the leading part of short distance expansions defines an associative algebra.
The field on the right hand side is present as it is the lowest dimensional descendent field of \( \psi_{n+1} \) which turns out to be a scalar (recall that \( \psi_{n+1} \) is pseudo-scalar). The right hand side converges provided

\[
d_{\psi_{n+1}} = d_{\psi_{n-1}} + d_{\psi_2} + 2(n - 1).
\]

A solution can be easily found:

\[
d_{\psi_n} = \frac{1}{2} n(n + 1)
\]

(59)

Notice that \( n = 2 \) gives the Polyakov equation. This generalised Polyakov equation was already mentioned in ref.[11]. We have proven that the current densities converge in the infrared limit. The scaling properties of the fields \( L_{\psi_{n+1}} \) imply that currents vanish:

\[
R_n = 0
\]

(60)

This implies that the infrared behaviour of two-dimensional turbulence is not represented by a cascade.

As a matter of fact, the Gaussian field theory with a charge \( Q = -2 \) at infinity satisfies (60). Moreover, vertex operators are simple currents; therefore the leading part of the operator algebra is associative. We have found a fixed point such that fluxes of \( n \)-enstrophies are finite in the infrared limit. If the leading part of the operator algebra of the fixed point defines an associative algebra, the deformed Gaussian theory is the unique asymptotic description of 2d turbulence.

V- The Coulomb Gas Picture

a) The effective action

The Gaussian fixed point fulfils all requirements to represent an asymptotic state of two-dimensional turbulence. Nevertheless, we would like to describe not only the fixed point but the effective theory in its basin of attraction. Recall that this theory is a perturbed conformal theory whose mass is getting smaller and smaller as one gets closer to the fixed point.

Let us briefly sum up the time evolution of solutions of the Navier-stokes equation. First of all, there is a transient period when \( t \leq \tau \) where solutions evolve from the initial condition to a field theory characterised by an action \( S_\tau \). Two possibilities are to be envisaged. Suppose that a solution never falls within the basin of attraction of a conformal fixed point, then it can keep wandering in function space never reaching any steady state. This is an unattractive case. On the other hand, suppose that a solution falls within the basin of attraction of a fixed point after a transient period. We will show that the most general effective field theory in the basin of attraction of the Gaussian fixed point with a charge at infinity admits vortices.

Let us now describe the effective field theory in the vicinity of the fixed point. This effective theory should respect all the symmetries of the fixed point. We assume that \( S_\tau \) is in the basin of attraction of the deformed Gaussian fixed point. Let us notice that the deformed Gaussian fixed point is invariant under two discrete symmetries which leave the stream function invariant:

\[
\phi \rightarrow -\phi, \quad Q \rightarrow -Q
\]

\[
\phi \rightarrow \phi + 2\pi
\]

(62)

The first invariance ensures that \( \psi \) is a real field whereas the second leaves the vertex operator invariant. The interaction term of the effective theory being an even periodic function can be
expanded in a Fourier series:

\[ S_{\text{eff}} = \frac{1}{2\pi} \int d^2x \left( \frac{1}{2} \partial \tilde{\phi}_a^2 - \sum_{n \geq 1} z_n \cos n \tilde{\phi}_a \right) \tag{63} \]

where \( \tilde{\psi}_a \) is now an effective field defined with a cut-off \( a \). The interaction term can be separated in two terms, the first one containing relevant and marginal fields, the second containing irrelevant fields. In the infra-red regime, the deformed Gaussian theory is a renormalisation group fixed point. It can only be perturbed by irrelevant fields. The coupling constant of each irrelevant fields behaves as \( z_n = z_{n_0}(\lambda^{-1}a)^{\frac{n(n+Q)}{2}} - 2 \). Thus we end up with an effective potential:

\[ V = - \sum_{n \geq n_0} z_{n_0}(a\lambda^{-1})^{\frac{n(n+Q)}{2}} - 2 \cos n \tilde{\phi}_a \tag{64} \]

where \( n_0 \) is the minimal integer such that \( n_0(n_0 + Q) > 4 \); i.e. \( n_0 = 4 \). In the infrared regime the effective potential goes to zero. The constants \( z_{n_0} \) explicitly depend on the initial probability law.

To understand the physical meaning of this effective theory, it is convenient to draw an analogy with a Coulomb gas.

b) Turbulence as a Coulomb Gas

We will show that the effective theory is equivalent to a neutral Coulomb Gas\[^{[16]}\]. This is a gas of charged particles allowed to appear and disappear in a grand canonical fashion. This result stems from the expansion of the partition function:

\[ Z = \int D\phi \exp(-S_{\text{eff}}), \tag{65} \]

which can be written:

\[ Z = \sum_{i \geq n_0} \prod_{m_i,n_i=0}^{\infty} \frac{z_i^{m_i+n_i}}{n_i!m_i!} \int d^2x_i \prod_{i<j} \frac{|x_i - x_j|}{a} \tag{66} \]

where the product is not zero provided that the sum of all \( q_i \) is \(-Q\). This is nothing but the grand canonical partition function of a Coulomb gas where charges \( q_i = -\infty \ldots -n_0, n_0 \ldots \infty \) interact according to a Coulomb potential:

\[ Z = \sum_{i \geq 4} \prod_{m_i,n_i=0}^{\infty} \frac{z_i^{m_i+n_i}}{n_i!m_i!} \int d^2x_i \exp(-2\pi |\sum_{i<j} q_i q_j V(x_i - x_j)|) \tag{67} \]

where \( V(x) = -\frac{1}{2\pi} \ln |x| \) is the Coulomb potential. The gas is charged as the sum of the charges adds up to \(-Q\). Each choice of \( m_i \) particles of charge \(-q_i\), and \( n_i \) of charge \( q_i \) gives a factorial factor to take into account the indistinguishability of charges. The temperature of this gas is \( \frac{1}{2\pi} \) and particles of different charges have different fugacities \( z_i \). Notice that the fugacities are very small.

Let us now interpret the role of these charges for turbulence. The correlation functions of vertex operators are easily computed in the Coulomb Gas theory. Indeed, \( \exp i\phi_a(x) \) is represented by an insertion of a positive charge at point \( x \). Correlation functions of vertex operators are given by
correlation functions of charges. Therefore, two-dimensional turbulence is equivalent to a Coulomb
gas where the probabilistic properties of the stream function are given by the statistical behaviour
of a positive charge. In view of numerical simulations, our aim is to determine the classical
configurations of the stream function and the vorticity. To do so, we need a further modification
of the Coulomb gas picture where the role of charges as vortices will be clear.

c) Two Dimensional Turbulence as a Gas of Vortices

We have just have shown that a charge \( q = 1 \) represents the stream function in a probabilistic
sense, i.e. correlation functions of the stream function are equal to those of this charge in a Coulomb
gas. Nevertheless, this does not give the final description of two-dimensional turbulence as a gas
of charges. Indeed, we are about to see that these charges generate classical field configurations
whose statistical behaviour is exactly the one expected for a Coulomb gas.

Consider the \( \mathcal{O}(2) \) sigma model for stream function configurations, i.e. an action given by

\[
S_{\text{eff}} = \frac{1}{4\pi} \int d^2 x (\partial \psi)^2
\]

(68)

where the stream function \( \psi \) is now taken to be periodic. Notice that this is nothing but the kinetic
energy of a configuration. As expected, the effective action can be expressed as a local action in
the velocity field. The stream field is probabilistically almost a free field. Each stream function
configuration is weighted with a Gaussian action. We allow singular stream functions: different sin-
gularities correspond to sectors in the theory, and the total partition function is obtained summing
over all sectors. The field \( \psi \) is a sum of a random part and a classical part:

\[
\psi = \phi + \psi_c.
\]

(69)

More precisely, the classical field is solution of the equation

\[
\Delta \psi_c = - \sum_i q_i \delta(x_i)
\]

(70)

where defects of charge \( q_i \) have been considered and \( \delta \)-functions are smeared over a distance \( a \). A
charge \( Q \) is put at infinity, the gas is neutral if all the charges add up to \( -Q \). The classical solution
reads

\[
\psi_c(x) = \sum_i q_i \arctan \left( \frac{x-x_i}{a} \right).
\]

(71)

The integer \( q_i \) is the charge of the vortex. The random part \( \phi \) will not contribute to defect
correlation functions. This decomposition tells us that a stream function configuration is entirely
specified by a countable number of vortices.

The energy of each configuration is given by:

\[
S = \frac{1}{4\pi} \int d^2 x (\partial \phi + \partial \psi_c)^2.
\]

(72)

This can be easily computed and gives:

\[
S = \frac{1}{4\pi} \left( \int d^2 x (\partial \phi)^2 - 4\pi \sum_{i<j} q_i q_j \ln \left| \frac{x_i-x_j}{a} \right| \right).
\]

(73)
The energy is finite if and only if:
\[ \sum_i q_i + Q = 0 \] (74)

This is nothing but the energy of a charged Coulomb gas. To complete this correspondence, let us choose to restrict charges to \(-\infty...-n_0,n_0...\infty\). As already shown, the stream function is represented in the Coulomb gas picture by the insertion of a charge in the medium. It is even simpler in the \(O(2)\) sigma model. In that case, the charge \(q_x\) at \(x\) interacts with other free charges. The averaged response of the medium consists of a classical field \(\psi_c\) which depends on the fluctuating number of free charges. The correlation function of two charges representing the correlation function of the stream function is simply given by the average over the fluctuating number of particles of the weight \(\exp\{-S(\psi_c(x,y))\}\) depending on the position of the two fixed charges. This is nothing but the weight of the classical field \(\psi_c(x,y)\). This result can be extended to an arbitrary correlation function. We conclude that these classical configurations represent the stream function of a two-dimensional turbulent flow.

Finally, we can interpret the charge \(q_i\). Recall that the vorticity is minus the Laplacian of the stream field so that for the classical part we have:
\[ \omega_c(x) = \sum_i q_i \delta(x-x_i), \] (75)
i.e. each configuration of two-dimensional turbulence is specified by a finite number of vortices, as shown by numerical simulations. The typical size of each vortex is \(a\). Moreover, these vortices are quantised. Notice that the kinetic energy of a configuration is \(-2\pi \sum_{i<j} q_i q_j \ln |x_i - x_j|\) and the n-enstrophies are proportional to \(\sum_i q_i^n\). In the infrared regime, the probability of charge creation is zero so the energy as well as the n-enstrophies are zero. The convergence of energy and n-enstrophies to zero is with probability one.

We have now proven that in the vicinity of the deformed Gaussian fixed point, the solutions of the Navier-Stokes equations are given by a gas of singularities where the stream function is a sum of vortices. The non-trivial infrared limit is due to the presence of an integer vortex at infinity. This provides a solvable example of two-dimensional turbulence. Moreover, the role of boundary conditions is made explicit.

CONCLUSION

We have shown that the asymptotic regime of two dimensional turbulence in the basin of attraction of the deformed Gaussian fixed point is described by a Coulomb gas. This is an interacting neutral gas filled with charges \(|q| \geq 4\). We have explicitly shown that if the leading part of the operator algebra of the fixed point is associative, the deformed Gaussian fixed point is the unique asymptotic description of 2d turbulence. If universality arguments can be applied to self-organised criticality, it is clear that this must be the unique solution.

Let us comment on the comparison with numerical simulations. First of all, it is conspicuous that the Coulomb gas has a natural description in terms of vortices. This is indeed what simulations obtain, i.e. coherent and stable vortices surrounded by a motionless flow. As initial conditions are always chosen to be Gaussian, we expect that as soon as defects appear, solutions fall within the basin of attraction of the Gaussian fixed point with charges at infinity. These defects are likely to be remnants from large initial fluctuations of the vorticity field. It is then natural to see solutions
flow rapidly along renormalisation trajectories towards a Coulomb gas behaviour. To support this explanation, it would be relevant to know if the infrared energy spectrum is universal. We predict that the infrared energy spectrum is flat, i.e. behaves as $k^0$. It would also be relevant to describe the early development of solutions and to prove that defects appear in a finite time. This would support our claim that the subsequent evolution of solutions is given by a Coulomb gas.

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