EXTENDED \((p, q)\)-Mittag-Leffler Function and Its Properties

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Abstract. In this study our aim to define the extended \((p, q)\)-Mittag-Leffler(ML) function by using extension of beta functions and to obtain the integral representation of new function. We also take the Mellin transform of this new function in terms of Wright hypergeometric function. Extended fractional derivative of the classical Mittag-Leffler(ML) function leads the extended \((p, q)\)-Mittag-Leffler(ML) function.

1. Introduction and Preliminaries

The Mittag-Leffler(ML) function occurs naturally in many real world problems, especially in the solution of fractional integro-differential equations having the arbitrary order. The importance of such functions in physics and engineering is steadily increasing. Some applications of the Mittag-Leffler(ML) is carried out in the study of kinetic equation, study of Lorenz system, random walk, Levy flights and complex systems and also in applied problems such as fluid flow and electric network.

We begin with the Prabhakar [9] Mittag-Leffler(ML) function, which is defined by

\[
E_{\gamma}^{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\rho n + \sigma)} z^n, \quad z, \sigma \in \mathbb{C}; \text{ and } \Re(\rho) > 0,
\]

where \((\gamma)_n\) is the Pochhammer Symbol and given by:

\[
(\gamma)_n = \gamma(\gamma + 1) \cdots (\gamma + n - 1),
\]

for \((n \in \mathbb{N}, \gamma \in \mathbb{C})\). In particular if \(n = 0\) then \((\gamma)_0 = 1\).

In theory of special functions, the applications and importance of Mittag-Leffler(ML) functions were studied by many researchers and their extensions are found in [4], [14]-[17]. Shukla and Prajapati [13] (see also [18]) have further introduced the function \(E_{\delta,q}^{\rho,\sigma}(z)\), defined as:

\[
E_{\delta,q}^{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{pq}}{\Gamma(\rho n + \sigma)} z^n,
\]

where \(z, \sigma, \delta \in \mathbb{C}; \text{ and } \Re(\rho) > 0; q > 0\). By considering (1.2) for particular parameters one can deduce several different special functions.

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Özarslan and Yilmaz have defined the following extended Mittag-Leffler (ML) function \( E_{\rho,\sigma}^{\delta,c}(z;p) \) by

\[
E_{\rho,\sigma}^{\delta,c}(z;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\delta + n, c - \delta)}{\beta(\delta, c - \delta)} \frac{(c)_n}{\Gamma(\rho n + \sigma)} \frac{z^n}{n!},
\]

(1.3)

where \( p \geq 0, \Re(c) > \Re(\delta) > 0 \) and \( \beta_p(x,y) \) is extended beta function, see [2, 6] given by

\[
\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-p(t-t^{-1})} dt,
\]

(1.4)

where \( \Re(p) > 0, \Re(x) > 0 \) and \( \Re(y) > 0 \).

Very recently Rahman et al. ([10], [11] and [12]) defined the fractional integrals and differentials formulas and pathway integral formulas of extended Mittag-Leffler (ML) functions.

Choi et al. have defined the following extension of beta function by

\[
\beta(x,y;p,q) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-p(t-t^{-1})} dt
\]

(1.5)

where \( \Re(p) > 0, \Re(q) > 0, \Re(x) > 0, \) and \( \Re(y) > 0 \).

Remark 1.1. In particular

(i) If setting \( p = q \) in (1.6), then deduce to the extended Mittag-Leffler (ML) function as in the equation (1.3).

(ii) If setting \( p = q = 0 \) in (1.6), then reduce to Mittag-Leffler (ML) function as in (1.1).

2. Properties of extended \((p,q)\)-Mittag-Leffler (ML) function

Next, we define the integral representations of (1.6).

**Theorem 2.1.** Let \( c, \alpha, \beta, \gamma \in \mathbb{C}, \Re(c) > \Re(\gamma) > 0, \Re(\alpha) > 0 \) and \( \Re(\beta) > 0 \). Then the following integral representation hold true,

\[
E_{\alpha,\beta}^{\gamma,c}(z;p,q) = \frac{1}{\beta(\gamma,c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\gamma-1} \exp \left( -\frac{p}{t} - \frac{q}{(1-t)} \right) E_{\alpha,\beta}(tz) dt.
\]

(2.1)
Proof. Using (1.5) in equation (1.6), we have

\[ E_{\gamma,c}^{\alpha,\beta}(z; p, q) = \sum_{n=0}^{\infty} \left\{ \int_{0}^{1} t^{\gamma+n-1}(1-t)^{c-\gamma-1} \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) dt \right\} \]

which can be written as

\[ E_{\gamma,c}^{\alpha,\beta}(z; p, q) = \int_{0}^{1} t^{\gamma-1}(1-t)^{c-\gamma-1} \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) \times \sum_{n=0}^{\infty} \frac{(c)_n}{\beta(\gamma, c - \gamma) \Gamma(\alpha n + \beta) n!} (tz)^n dt. \]

Using (1.1) in above equation, we get the desired integral representation. \(\square\)

**Corollary 2.1.** Substituting \(t = \frac{u}{1+u}\) in Theorem 2.1 we get

\[ E_{\gamma,c}^{\alpha,\beta}(z; p, q) = \frac{1}{\beta(\gamma, c - \gamma)} \int_{0}^{\infty} \frac{u^{\gamma-1}}{(u+1)^c} \exp \left( -\frac{p(1+u)}{u} - q(1+u) \right) \times E_{\alpha,\beta}^{c}(\frac{uz}{1+u}) du. \]

**Corollary 2.2.** Setting \(t = \sin^2 \theta\) in Theorem 2.1 we get following integral representation

\[ E_{\gamma,c}^{\alpha,\beta}(z; p, q) = \frac{1}{\beta(\gamma, c - \gamma)} \left[ 2 \int_{0}^{\infty} \sin^{2\gamma-1} \theta \cos^{2c-1} \theta \exp \left( -\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta} \right) \right] \times E_{\alpha,\beta}^{c}(z \sin^2 \theta) d\theta. \]

Kurulay and Bayram [5] defined the following,

\[ E_{\alpha,\beta}^{c}(tz) = \beta E_{\alpha,\beta+1}^{c}(1z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{c}(1z). \]

By using (2.4) and (2.1), we obtained the following relation:

**Corollary 2.3.** Let \(p, q \geq 0, \mathbb{R}(c) > \mathbb{R}(\gamma) > 0, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0\), then following relation holds:

\[ E_{\gamma,c}^{\alpha,\beta}(z; p, q) = \beta E_{\alpha,\beta+1}^{\gamma,c}(z; p, q) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,c}(z; p, q) \]

In the next theorem, we apply the Mellin transforms on (1.6) and obtained the result in form of Wright hypergeometric function which is defined in (see [19]-[21]) by

\[ p \Psi_q(z) = \left[ \frac{(\alpha_i, \mu_i)_{1,p}}{(\beta_j, \lambda_j)_{1,q}} ; z \right] \]

\[ = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \mu_1 n) \cdots \Gamma(\alpha_p + \mu_p n)}{\Gamma(\beta_1 + \lambda_1 n) \cdots \Gamma(\beta_q + \lambda_q n)} \frac{z^n}{n!} \]
where \( \beta_r \) and \( \alpha_s \) belongs to \( \mathbb{R}^+ \) such that

\[
1 + \sum_{s=1}^{q} \lambda_s - \sum_{r=1}^{p} \mu_r \geq 0.
\]

**Theorem 2.2.** The Mellin transform of (1.6) given by

\[
\mathcal{M}\left\{ E_{\alpha,\beta}^{\gamma,c}(z; p, q); p \to s, q \to r \right\} = \frac{\Gamma(s)\Gamma(r)\Gamma(c + r - \gamma)}{\Gamma\gamma,c - \gamma} \times 2 \Psi_2 \left[ \begin{array}{c} (c, 1), (\gamma + s, 1), \\
(\beta, \gamma), (c + s + r, 1), \\
\end{array} \right],
\]

(2.7)

where \( p, q \geq 0 \), \( \Re(c) > \Re(\gamma) > 0 \), and \( \Re(\alpha) > 0 \), \( \Re(\beta) > 0 \).

**Proof.** Applying the Mellin transform to (1.6),

\[
\mathcal{M}\left\{ E_{\alpha,\beta}^{\gamma,c}(z; p, q); p \to s, q \to r \right\} = \int_0^\infty \int_0^\infty p^{s-1}q^{r-1}E_{\alpha,\beta}^{\gamma,c,\lambda,\rho}(z; p, q)dpdq.
\]

(2.8)

and using (2.1) in (2.8), we get

\[
\mathcal{M}\left\{ E_{\alpha,\beta}^{\gamma,c}(z; p, q); p \to s, q \to r \right\} = \frac{1}{\beta(\gamma, c - \gamma)} \int_0^\infty \int_0^\infty p^{s-1}q^{r-1}\left\{ \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\exp\left( - \frac{p}{t} - \frac{q}{1-t} \right) \right\} dt.
\]

(2.9)

Changing the order of integrations in equation (2.9), we have

\[
\mathcal{M}\left\{ E_{\alpha,\beta}^{\gamma,c}(z; p, q); p \to s, q \to r \right\} = \frac{1}{\beta(\gamma, c - \gamma)} \int_0^\infty \int_0^\infty p^{s-1}\exp\left( - \frac{p}{t} \right)dp \int_0^\infty q^{r-1}\exp\left( - \frac{q}{1-t} \right)dq dt.
\]

(2.10)

Now, taking \( u = \frac{p}{t} \) in the second integral of equation (2.10), we get

\[
\int_0^\infty p^{s-1}\exp\left( - \frac{p}{t} \right)dp = \int_0^\infty u^{s-1}t^{s}(1-t)^{s}e^{-u}du = t^{s}(1-t)^{s} \int_0^\infty u^{s-1}e^{-u}du = t^{s}(1-t)^{s}\Gamma(s),
\]

(2.11)

Similarly, taking \( v = \frac{q}{1-t} \) in the third integral of equation (2.10), we have

\[
\int_0^\infty q^{r-1}\exp\left( - \frac{p}{1-t} \right)dq = (1-t)^{r}\Gamma(r).
\]

(2.12)

Using equation (2.11) and (1.1) in equation (2.10), we get

\[
\mathcal{M}\left\{ E_{\alpha,\beta}^{\gamma,c}(z; p, q); p \to s, q \to r \right\}
\]


The well-known R-L fractional derivative of order

Definition 3.1. The well-known R-L fractional derivative of order \( \lambda \) is defined by

\[
\mathcal{D}_x^\lambda \{ f(z) \} = \frac{1}{\Gamma(-\lambda)} \int_0^z f(\tau)(z-\tau)^{-\lambda-1}d\tau, \Re(\lambda) > 0.
\] (3.1)
If \( m - 1 < \Re(\lambda) < m \) where \( m = 1, 2, \ldots \), then it becomes
\[
\mathcal{D}_x^\lambda \left\{ f(z) \right\} = \frac{d^m}{dx^m} \mathcal{D}_x^{\lambda - m} \left\{ f(z) \right\}
\]
\[
= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\lambda + m)} \int_0^x f(\tau)(x - \tau)^{-\lambda + m - 1} d\tau \right\}, \Re(\lambda) > 0. \tag{3.2}
\]

**Definition 3.2.** (see [8]) The extended R-L fractional derivative of order \( \lambda \) is defined by
\[
\mathcal{D}_x^\lambda \left\{ f(z); p \right\} = \frac{1}{\Gamma(-\lambda)} \int_0^x f(\tau)(x - \tau)^{-\lambda - 1} \exp \left( - \frac{px^2}{\tau(x - \tau)} \right) d\tau, \Re(\lambda) > 0. \tag{3.3}
\]
if \( m - 1 < \Re(\lambda) < m \) where \( m = 1, 2, \ldots \), then it follows
\[
\mathcal{D}_x^\lambda \left\{ f(z); p, q \right\} = \frac{d^m}{dx^m} \mathcal{D}_x^{\lambda - m} \left\{ f(z); p, q \right\}
\]
\[
= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\lambda + m)} \int_0^x f(\tau)(x - \tau)^{-\lambda + m - 1} \exp \left( - \frac{px^2}{\tau(x - \tau)} \right) d\tau \right\}, \Re(\lambda) > 0. \tag{3.4}
\]
Recently, Baleanu et al. [1] give the extension of R-L as

**Definition 3.3.**
\[
\mathcal{D}_x^\lambda \left\{ f(z); p, q \right\} = \frac{1}{\Gamma(-\lambda)} \int_0^x f(\tau)(x - \tau)^{-\lambda - 1} \exp \left( - \frac{px}{\tau} - \frac{qz}{\tau} \right) d\tau, \tag{3.5}
\]
where \( \Re(\lambda) > 0, \Re(p) > 0 \) and \( \Re(q) > 0 \). If \( m - 1 < \Re(\lambda) < m \) where \( m = 1, 2, \ldots \), it follows
\[
\mathcal{D}_x^\lambda \left\{ f(z); p, q \right\} = \frac{d^m}{dx^m} \mathcal{D}_x^{\lambda - m} \left\{ f(z); p, q \right\}
\]
\[
= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\lambda + m)} \int_0^x f(\tau)(x - \tau)^{-\lambda + m - 1} \exp \left( - \frac{px}{\tau} - \frac{qz}{\tau} \right) d\tau \right\}, \Re(\lambda) > 0. \tag{3.6}
\]

Obviously, if \( p = q \), then definition 3.3 reduces to extended fractional derivative 3.2. Similarly, if \( p = 0 = q \), then definition 3.3 reduces to R-L fractional derivative 3.1.

**Theorem 3.1.** Let \( p, q \geq 0, \Re(\lambda) > \Re(\delta) > 0, \Re(\alpha) > 0, \) and \( \Re(\beta) > 0 \). Then
\[
\mathcal{D}_z^{\delta - \lambda} \left\{ z^{\delta - 1} E_{\alpha, \beta}^c(z); p, q \right\} = \frac{z^{\lambda - 1 - \beta} \beta(\delta, c - \delta)}{\Gamma(\mu - \delta)} E_{\alpha, \beta}^{\lambda, \delta}(z; p, q) \tag{3.7}
\]

**Proof.** Setting \( \lambda \) by \( \delta - \lambda \) in definition (3.3), we have
\[
\mathcal{D}_z^{\delta - \lambda} \left\{ z^{\delta - 1} E_{\alpha, \beta}^c(z); p, q \right\}
\]
\[
= \frac{1}{\Gamma(\lambda - \delta)} \int_0^z \tau^{\delta - 1} E_{\alpha, \beta}^c(\tau)(z - \tau)^{-\delta + \mu - 1} \exp \left( - \frac{pz}{\tau} - \frac{qz}{\tau(z - \tau)} \right) d\tau
\]
\[
= \frac{z^{-\delta + \mu - 1}}{\Gamma(\mu - \delta)} \int_0^z \tau^{\delta - 1} E_{\alpha, \beta}^c(\tau)(1 - \frac{\tau}{z})^{\delta + \lambda - 1} \exp \left( - \frac{pz}{\tau} - \frac{qz}{\tau(z - \tau)} \right) d\tau.
\]
Taking \( u = \frac{\tau}{z} \) in above equation, we have
\[
\mathcal{D}_z^{\delta - \mu} \left\{ z^{\delta - 1} E_{\alpha, \beta}^c(z); p, q \right\}
\]
Comparing equation (3.8) with equation (2.1), then completes the proof. □

Now, we define the following derivative properties of extended \((p, q)\)-Mittag-Leffler function.

**Theorem 3.2.** The following derivative formula holds for the extended Mittag-Leffler (ML) function,

\[
\frac{d^n}{dz^n} \left\{ E^{\gamma,c}_{\alpha,\beta}(z; p, q) \right\} = (c)_n E^{\gamma+n,c+n}_{\alpha,\beta+n}(z; p, q).
\]

**Proof.** Taking derivative of equation (1.6) respect to \(z\), then we have

\[
\frac{d}{dz} \left\{ E^{\gamma,c}_{\alpha,\beta}(z; p, q) \right\} = c E^{\gamma+1,c+1}_{\alpha,\beta+1}(z; p, q).
\]

Again taking derivative of equation (3.10), respect to \(z\), we have

\[
\frac{d^2}{dz^2} \left\{ E^{\gamma,c}_{\alpha,\beta}(z; p, q) \right\} = c(c+1) E^{\gamma+2,c+2}_{\alpha,\beta+2}(z; p, q).
\]

Continuing in this way up to \(n\), we obtain the result. □

**Theorem 3.3.** Following differentiation formula hold true:

\[
\frac{d^n}{dz^n} \left\{ z^{\beta-1} E^{\gamma,c}_{\alpha,\beta}(\mu z^\alpha; p, q) \right\} = z^{\beta-n-1} E^{\gamma+n,c+n}_{\alpha,\beta-n}(\mu z^\alpha; p, q).
\]

**Proof.** Replacing \(z\) by \(\mu z^\alpha\) in equation (3.9) and multiply by \(z^{\beta-1}\) and differentiate \(n\) times respect to \(z\), we obtain the required result. □

### 4. CONCLUSION

In this study, we established a new extension of Mittag-Leffler (ML) and some of its results. We conclude that if \(p = q\), then we get the results of extended Mittag-Leffler (ML) function defined in [7]. Furthermore, if \(p = q = 0\), then we have the classical Mittag-Leffler (ML) function.

**Conflict of Interests:**

There is no conflict of interests.

**Authors contributions:**

The authors contributed equally. All the authors read and approved the final manuscript.
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