EXTENSIONS OF QUASI-MORPHISMS TO THE SYMPLECTOMORPHISM GROUP OF THE DISK

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Abstract. On the group $\text{Symp}(D, \partial D)$ of symplectomorphisms of the disk which are the identity near the boundary, there are homogeneous quasi-morphisms called the Ruelle invariant and Gambaudo-Ghys quasi-morphisms. In this paper, we show that the above homogeneous quasi-morphisms extend to homogeneous quasi-morphisms on the whole group $\text{Symp}(D)$ of symplectomorphisms of the disk. As a corollary, we show that the second bounded cohomology $H^2_b(\text{Symp}(D))$ is infinite-dimensional.

1. Introduction

Let $G$ be a group. A function $\phi : G \to \mathbb{R}$ is called a quasi-morphism if there exists a constant $C$ such that the condition $|\phi(gh) - \phi(g) - \phi(h)| \leq C$ holds for any $g, h \in G$. A quasi-morphism $\phi$ is homogeneous if, for any integer $n \in \mathbb{Z}$ and any $g \in G$, the condition $\phi(g^n) = n\phi(g)$ holds. Let $Q(G)$ denote the vector space of all homogeneous quasi-morphisms on $G$. The homogenization $\overline{\phi}$ of a quasi-morphism $\phi$ is defined by $\overline{\phi}(g) = \lim_{n \to \infty} \phi(g^n)/n$. This homogenization $\overline{\phi}$ is a homogeneous quasi-morphism.

Let $K$ be a subgroup of $G$. It is natural to ask whether given homogeneous quasi-morphism $\psi : K \to \mathbb{R}$ can be extended to a homogeneous quasi-morphism on $G$. This extension problem of quasi-morphisms is studied in some papers ([11], [7], [8]). In this paper, we consider the extension problem of the Ruelle invariant and Gambaudo-Ghys quasi-morphisms on $\text{Symp}(D, \partial D)$ to the group $\text{Symp}(D)$.

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be a unit disk and $\omega = dx \wedge dy$ a standard symplectic form. Let $\text{Symp}(D)$ denote the symplectomorphism group of the disk $D$ and $\text{Symp}(D, \partial D)$ the subgroup of $\text{Symp}(D)$ consisting of symplectomorphisms which are the identity near the boundary $\partial D$. There are many quasi-morphisms on $\text{Symp}(D, \partial D)$. For example, in [10], Ruelle constructed a homogeneous quasi-morphism on $\text{Symp}(D, \partial D)$, which is called the Ruelle invariant. In [11] and [5], it was shown that the vector space $Q(\text{Symp}(D, \partial D))$ of homogeneous quasi-morphisms on $\text{Symp}(D, \partial D)$ is infinite-dimensional. In [5], Gambaudo and Ghys constructed countably many quasi-morphisms on $\text{Symp}(D, \partial D)$ by integrating on the disk the signature quasi-morphism on pure braid group $P_n$ on $n$-strands. Brandenbursky [3] generalized this idea to any quasi-morphisms on $P_n$ and defined the linear map $\Gamma_n : Q(P_n) \to Q(\text{Symp}(D, \partial D))$. We call a homogeneous quasi-morphism in $\Gamma_n(Q(P_n))$ a Gambaudo-Ghys quasi-morphism.
The well-definedness of the Ruelle invariant and Gambaudo-Ghys quasi-morphisms comes from the fact that the group $\text{Symp}(D, \partial D)$ is contractible. Since the group $\text{Symp}(D)$ is not contractible, both constructions cannot be applied to the group $\text{Symp}(D)$. However, we show that the following theorem holds.

**Theorem 1.1.** The Ruelle invariant and Gambaudo-Ghys quasi-morphisms on the group $\text{Symp}(D, \partial D)$ extend to homogeneous quasi-morphisms on the group $\text{Symp}(D)$. In particular, the vector space $Q(\text{Symp}(D))$ is infinite-dimensional.

Let $B_n$ denote the braid group on $n$-strands. Ishida\[6\] showed that the restriction $\Gamma_n|_{Q(B_n)} : Q(B_n) \to Q(\text{Symp}(D, \partial D))$ and the induced map $EH^2_b(B_n) \to EH^2_b(\text{Symp}(D, \partial D))$ are injective. Here the symbol $EH^2_b(\cdot)$ denotes the second exact bounded cohomology defined in section 2. Together with this Ishida’s theorem and the fact that $EH^2_b(B_n)$ is infinite-dimensional\[1\], we obtain the following corollary.

**Corollary 1.2.** The exact bounded cohomology $EH^2_b(\text{Symp}(D))$ and therefore the second bounded cohomology $H^2_b(\text{Symp}(D))$ are infinite-dimensional.

This paper is organized as follows. In section 2, we recall the bounded cohomology. In section 3, we consider the Ruelle’s construction and Gambaudo-Ghys’s construction on the universal covering of the group $\text{Symp}(D)$. In section 4, we show the main theorem. In section 5, we deal with the extension problem of homomorphisms, which relate to the Calabi invariant.

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2. Bounded cohomology

Let $G$ be a group. Let $C^p(G)$ denote the set of maps from $p$-fold product $G^p$ to $\mathbb{R}$ for $p > 0$ and let $C^0(G) = \mathbb{R}$. The coboundary operator $\delta : C^p(G) \to C^{p+1}(G)$ is defined by

$$\delta c(g_1, \ldots, g_{p+1}) = c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1})$$

and the cohomology of $(C^*(G), \delta)$ is called the **group cohomology of $G$** and denoted by $H^*(G)$. Note that, by definition, the first cohomology $H^1(G)$ is equal to the vector space $\text{Hom}(G, \mathbb{R})$ of homomorphisms from $G$ to $\mathbb{R}$.

Let $(C^*_b(G), \delta)$ denote the subcomplex of $(C^*(G), \delta)$ consisting of all bounded functions. Its cohomology is called the **bounded cohomology of $G$** and denoted by $H^*_b(G)$. The inclusion $C^*_b(G) \hookrightarrow C^*(G)$ induces the map $H^*_b(G) \to H^*(G)$ called the **comparison map**. The kernel of the comparison map is called the **exact bounded cohomology of $G$** and denoted by $EH^*_b(G)$. Then there is an exact sequence

$$0 \to H^1(G) \to Q(G) \to EH^2_b(G) \to 0,$$

where the map $Q(G) \to EH^2_b(G)$ is given by $\phi \mapsto [\delta \phi]$. In other words, the second exact bounded cohomology is isomorphic to the quotient $Q(G)/H^1(G)$. 


3. Homogeneous quasi-morphisms on $\widehat{\text{Symp}}(D)$

On the group $\text{Symp}(D, \partial D)$, many homogeneous quasi-morphisms are constructed [4, 3]. In this section, we apply to the universal covering $\widehat{\text{Symp}}(D)$ the methods explained in [5] to obtain homogeneous quasi-morphisms.

3.1. Ruelle invariant. For $x \in D$ and a path $\{g_t\}_{t \in [0,1]}$ in $\text{Symp}(D)$ with $g_0 = \text{id}$, let $u_t(x) \in \mathbb{R}^2 \setminus (0,0)$ denote the first column of $dg_t(x) \in SL(2, \mathbb{R})$. Then the variation of the angle of $u_t(x)$ depends on $x$ and the homotopy class of the path $\{g_t\}_{t \in [0,1]}$ relatively to fixed ends. Thus, for $\alpha \in \widehat{\text{Symp}}(D)$ represented by the path $\{g_t\}_{t \in [0,1]}$, we denote the variation of the angle of $u_t(x)$ by $\text{Ang}_\alpha(x)$. For $\alpha, \beta \in \widehat{\text{Symp}}(D)$ and a path $\{h_t\}_{t \in [0,1]}$ which represents $\beta$, the inequality

\[ |\text{Ang}_{\alpha \beta}(x) - \text{Ang}_\beta(x) - \text{Ang}_\alpha(h_1(x))| < 1/2 \]

holds for any $x \in D$, where we consider $S^1$ as $\mathbb{R}/\mathbb{Z}$. By the above inequality (3.1), the function $r : \text{Symp}(D) \to \mathbb{R}$ defined by

\[ r(\alpha) = \int_D \text{Ang}_\alpha \cdot \omega \]

is a quasi-morphism on $\widehat{\text{Symp}}(D)$. Let $\tau$ denote the homogenization of $r$. By construction, the restriction of $\tau$ to $\text{Symp}(D, \partial D) = \widehat{\text{Symp}}(D, \partial D)$ coincides with the classical Ruelle invariant on the disk. Since the Ruelle invariant is non-trivial homogeneous quasi-morphism on $\text{Symp}(D, \partial D)$, so is $\tau : \text{Symp}(D) \to \mathbb{R}$.

3.2. Gambaudo–Ghys construction. Let $X_n$ denote the $n$-fold configuration space of $D$. For $\alpha \in \text{Symp}(D)$ and for almost all $x = (x_1, \ldots, x_n) \in X_n$, a pure braid $\gamma(\alpha; x) \in P_n$ is defined as follows. Let us fix a base point $z = (z_1, \ldots, z_n) \in X_n$. Take a path $\{g_t\}_{t \in [0,1]}$ which represents $\alpha \in \widehat{\text{Symp}}(D)$. Then we obtain a loop $l(\alpha; x)$ in $X_n$ by

\[ l(\alpha; x) = \begin{cases} (1 - 3t)z + 3tx & (0 \leq t \leq 1/3) \\ g_t(x) & (1/3 \leq t \leq 2/3) \\ (3 - 3t)g_1(x) + (3t - 2)z & (2/3 \leq t \leq 1), \end{cases} \]

where $g_t(x) = (g_t(x_1), \ldots, g_t(x_n)) \in X_n$. This loop is well-defined for almost all $x \in X_n$ and its homotopy class is independent of the choice of representatives of $\alpha$. Thus we define $\gamma(\alpha; x) \in P_n$ as the braid represented by the loop $l(\alpha; x)$.

For a (homogeneous) quasi-morphism $\phi$ on the pure braid group $P_n$, we define a function $\hat{\Gamma}_n(\phi) : \text{Symp}(D) \to \mathbb{R}$ by

\[ \hat{\Gamma}_n(\phi)(\alpha) = \int_{X_n} \phi(\gamma(\alpha; x)) dx, \]
where $dx$ is the volume form on $X_n$ induced from the volume form $\omega^n$ on $D^n$. The proof of the integrability of the function $\phi(\gamma(\alpha; x))$ and the fact that $\hat{\Gamma}_n(\phi)$ is a quasi-morphism is the same as in [3 Lemma 4.1]. Let $\hat{\Gamma}_n(\phi)$ denote the homogenization of $\Gamma_n(\phi)$, then we have a linear map $\hat{\Gamma}_n : Q(B_n) \to Q(\text{Symp}(D))$.

Let $\iota : \text{Symp}(D, \partial D) = \text{Symp}(D, \partial D) \hookrightarrow \text{Symp}(D)$ be the inclusion. Then, by construction, the composition $\iota^* \circ \hat{\Gamma}_n : Q(B_n) \to Q(\text{Symp}(D, \partial D))$ coincides with the original Gambaudo-Ghys construction on $\text{Symp}(D, \partial D)$. Put $\Gamma_n = \iota^* \circ \hat{\Gamma}_n$. Ishida [6] showed that the restriction

$$\Gamma_n|_{Q(B_n)} : Q(B_n) \to Q(\text{Symp}(D, \partial D))$$

is injective. Thus the linear map $\tilde{\Gamma}_n|_{Q(B_n)} : Q(B_n) \to Q(\text{Symp}(D))$ is also injective. Since the vector space $Q(B_n)$ is infinite-dimensional [4], the following proposition holds.

**Proposition 3.1.** The vector space $Q(\text{Symp}(D))$ is infinite-dimensional.

Ishida showed in [6] that the linear map $\Gamma_n|_{Q(B_n)} : Q(B_n) \to Q(\text{Symp}(D, \partial D))$ induces the injective map $\hat{\Gamma}_n^* : EH^2_b(B_n) \to EH^2_b(\text{Symp}(D, \partial D))$. Thus we have the injection $\tilde{\Gamma}_n^* : EH^2_b(B_n) \to EH^2_b(\text{Symp}(D))$. Since the $EH^2_b(B_n)$ is infinite-dimensional for $n > 2$ (see [4]), the following theorem holds.

**Theorem 3.2.** The second exact bounded cohomology $EH^2_b(\text{Symp}(D))$ and therefore the bounded cohomology $H^2_b(\text{Symp}(D))$ are infinite-dimensional.

### 4. Homogeneous Quasi-Morphisms on $\text{Symp}(D)$

In this section, we show that the Ruelle invariant and the Gambaudo-Ghys quasi-morphisms on $\text{Symp}(D, \partial D)$ extend to homogeneous quasi-morphisms on $\text{Symp}(D)$.

Let $\phi \in Q(\text{Symp}(D))$ be either the Ruelle invariant or a Gambaudo-Ghys quasi-morphism. Let us consider the short exact sequence

$$0 \to \mathbb{Z} = \pi_1(\text{Symp}(D)) \to \tilde{\text{Symp}}(D) \xrightarrow{\rho} \text{Symp}(D) \to 1.$$ 

Since all homogeneous quasi-morphisms on abelian groups are homomorphism, the restriction $\phi|_{\pi_1(\text{Symp}(D))}$ is a homomorphism. Put $a_\phi(1) \in \mathbb{R}$ where $1 \in \pi_1(\text{Symp}(D))$ is the full rotation of the disk $D$. Let $\hat{\text{Diff}}_+(S^1)$ be the universal covering of $\text{Diff}_+(S^1)$ the orientation preserving diffeomorphisms of the circle and $\rho : \text{Symp}(D) \to \hat{\text{Diff}}_+(S^1)$ the restriction to the boundary. On the group $\hat{\text{Diff}}_+(S^1)$, there is a quasi-morphism $\text{rot} : \hat{\text{Diff}}_+(S^1) \to \mathbb{R}$ called the rotation number. Note that the $\rho(1) \in \pi_1(\hat{\text{Diff}}_+(S^1)) = \mathbb{Z}$ is the full rotation of the circle $S^1$ and thus $\rho(1) = \text{rot}(\rho(1)) = 1.$

Remark 4.1. In general, $a_\phi$ is non-zero value. For example, let $\phi$ be the Ruelle invariant $\tau$ defined in subsection 3.1 then $a_\phi$ is equal to the symplectic area of the disk $D$. 


Lemma 4.2. The homogeneous quasi-morphism $\phi - a_\phi \cdot \rho^* \text{rot}$ on $\widetilde{\text{Symp}}(D)$ descends to a homogeneous quasi-morphism on $\text{Symp}(D)$, that is, there exists a homogeneous quasi-morphism $\psi$ on $\text{Symp}(D)$ satisfying $p^*\psi = \phi - a_\phi \cdot \rho^* \text{rot}$.

Proof. By definition of $a_\phi$, the homogeneous quasi-morphism $\phi - a_\phi \cdot \rho^* \text{rot}$ is equal to 0 on $\pi_1(\text{Symp}(D))$. Thus, by the following Shtern’s theorem, the lemma follows. □

Theorem 4.3. Let $1 \to K \to G \xrightarrow{p} H \to 1$ be a short exact sequence and $\phi$ a homogeneous quasi-morphism on $G$. If $\phi|_K = 0$, then there is a homogeneous quasi-morphism $\psi$ on $H$ such that $\phi = p^*\psi$.

Theorem 4.4. The Ruelle invariant and the Gambaudo-Ghys quasi-morphisms on $\text{Symp}(D,\partial D)$ extend to homogeneous quasi-morphisms on $\text{Symp}(D)$. In particular, the vector space $Q(\text{Symp}(D))$ of homogeneous quasi-morphisms is infinite-dimensional.

Proof. By definition of $\phi$, we have to prove that the restriction $\phi|_{\text{Symp}(D,\partial D)}$ extends to a homogeneous quasi-morphism on $\text{Symp}(D)$. By Lemma 4.2 take the homogeneous quasi-morphism $\psi$ on $\text{Symp}(D)$ satisfying $p^*\psi = \phi - a_\phi \cdot \rho^* \text{rot}$. The composition of $p : \widetilde{\text{Symp}}(D) \to \text{Symp}(D)$ and the inclusion $\iota : \text{Symp}(D,\partial D) \to \widetilde{\text{Symp}}(D)$ is equal to the inclusion $\iota : \text{Symp}(D,\partial D) \to \text{Symp}(D)$ and the composition $\rho : \text{Symp}(D) \to \text{Diff}_+(S^1)$ and $\iota$ is equal to 0. Then we have $\phi|_{\text{Symp}(D,\partial D)} = \iota^*(p^*\psi - a_\phi \cdot \rho^* \text{rot}) = \psi|_{\text{Symp}(D,\partial D)}$ and this implies that the homogeneous quasi-morphism $\psi$ is an extension of $\phi|_{\text{Symp}(D,\partial D)}$. □

The above theorem implies that the image $i^*(Q(\text{Symp}(D)))$ contains the image $\Gamma_n(Q(B_n))$ of the Gambaudo-Ghys construction. Thus the image $i^*(EH^2_b(\text{Symp}(D)))$ also contains $\Gamma_n(EH^2_b(B_n))$. Since $\Gamma_n(EH^2_b(B_n))$ is infinite-dimensional, the following holds.

Corollary 4.5. The exact bounded cohomology $EH^2_b(\text{Symp}(D))$ and therefore $H^2_b(\text{Symp}(D))$ are infinite-dimensional.

Remark 4.6. In $Q(B_n)$, there is the abelianization homomorphism $a : B_n \to \mathbb{Z}$. It is known that $\Gamma_n(a)$ is equal to the Calabi invariant up to constant multiple. Thus, by the above argument, we can show that the Calabi invariant extend to a homogeneous quasi-morphism on $\text{Symp}(D)$. This extendability of the Calabi invariant is shown in [9].

5. HOMOMORPHISMS ON SYMP(D)

In this section, we deal with homomorphisms from $\text{Symp}(D)$ to $\mathbb{R}$, which relate to the Calabi invariant. It is known that the restriction map $\text{Symp}(D) \to \text{Diff}_+(S^1)$ is surjective (see [13]). Put $\text{Symp}(D)_{\text{rel}} = \text{Ker}(\text{Symp}(D) \to \text{Diff}_+(S^1))$. Let us consider the short exact sequence

$$1 \to \text{Symp}(D)_{\text{rel}} \to \text{Symp}(D) \to \text{Diff}_+(S^1) \to 1.$$
On the group $\text{Symp}(D)_{\text{rel}}$, there is a surjective homomorphism $\text{Cal} : \text{Symp}(D)_{\text{rel}} \to \mathbb{R}$ called the Calabi invariant. Let us consider a part of five-term exact sequence

\begin{equation}
H^1(\text{Symp}(D); \mathbb{R}) \to H^1(\text{Symp}(D)_{\text{rel}}; \mathbb{R}) \xrightarrow{\delta} H^2(\text{Diff}_+ (S^1); \mathbb{R}),
\end{equation}

where $H^1(\text{Symp}(D)_{\text{rel}}; \mathbb{R})$ is $\text{Symp}(D)$-invariant homomorphisms on $\text{Symp}(D)_{\text{rel}}$. Then it is shown in [2] that the element $\delta(\text{Cal})$ is equal to the real Euler class $e_\mathbb{R}$ of $\text{Diff}_+ (S^1)$ up to non-zero constant multiple. Thus, by the exactness of (5.1), the Calabi invariant cannot extend to a homomorphism on $\text{Symp}(D)$. Let us normalize the Calabi invariant as $\delta(\text{Cal}) = e_\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous homomorphism satisfying $f(q) = 0$ for any $q \in \mathbb{Q}$. Since the Calabi invariant is surjective to $\mathbb{R}$, the composition $f \circ \text{Cal}$ is non-trivial if $f \neq 0$.

**Theorem 5.1.** The composition $f \circ \text{Cal}$ extends to a homomorphism on $\text{Symp}(D)$. In particular, the cohomology $H^1(\text{Symp}(D))$ is infinite-dimensional.

**Proof.** By the exactness of (5.1), we have to show that the image $\delta(f \circ \text{Cal})$ is equal to 0 in $H^2(\text{Diff}_+ (S^1); \mathbb{R})$. For any group $\Gamma$, let $f_* : H^*(\Gamma; \mathbb{R}) \to H^*(\Gamma; \mathbb{R})$ denote the coefficients change by $f$. Since the five-term exact sequence is natural with respect to coefficients changes, we obtain

$$
\delta(f \circ \text{Cal}) = \delta(f_*(\text{Cal})) = f_* e_\mathbb{R}.
$$

Let $\iota : \mathbb{Z} \to \mathbb{R}$ be the inclusion and $e_\mathbb{Z} \in H^2(\text{Diff}_+ (S^1); \mathbb{Z})$ the integral Euler class of $\text{Diff}_+ (S^1)$. Since the real Euler class $e_\mathbb{R}$ is equal to $\iota_*(e_\mathbb{Z})$, we have

$$
f_* e_\mathbb{R} = (f\iota)_* e_\mathbb{Z} = (0)_* e_\mathbb{Z} = 0
$$

and the theorem follows. \[\square\]

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