Ubiquitous quantum scarring does not prevent ergodicity

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In a classically chaotic system that is ergodic, any trajectory will be arbitrarily close to any point of the available phase space after a long time, filling it uniformly. Using Born’s rules to connect quantum states with probabilities, one might then expect that all quantum states in the chaotic regime should be uniformly distributed in phase space. This simplified picture was shaken by the discovery of quantum scarring, where some eigenstates are concentrated along unstable periodic orbits. Despite that, it is widely accepted that most eigenstates of chaotic models are indeed ergodic. Our results show instead that all eigenstates of the chaotic Dicke model are actually scarred. They also show that even the most random states of this interacting atom-photon system never occupy more than half of the available phase space. Quantum ergodicity is achievable only as an ensemble property, after temporal averages are performed.
A striking feature of the quantum-classical correspondence not recognized in the early days of the quantum theory is the repercussion that measure-zero structures of the classical phase space may have in the quantum domain. A recent example is the effect of unstable fixed points, that cause the exponentially fast scrambling of quantum information in both integrable and chaotic quantum systems. Another better known example is the phenomenon of quantum scarring, which is done using a measure that we introduce to quantify the level of localization of quantum states in the phase space. We say that a quantum state is ergodic if its infinite-time average leads to full delocalization. Under this definition, stationary quantum states are never ergodic, while random states are, and coherent states lie somewhere in between.

Results

Dicke model and chaos. The Hamiltonian of the Dicke model is written as

\[ \hat{H}_D = \omega \hat{a}^\dagger \hat{a} + \omega J_z + \frac{2\gamma}{\sqrt{N}} \hat{J}_x (\hat{a}^\dagger + \hat{a}), \]

where \( h = 1 \). It describes \( N \) two-level atoms with atomic transition frequency \( \omega_0 \) interacting with a single mode of the electromagnetic field with radiation frequency \( \omega \). In the equation above, \( \hat{a} (\hat{a}^\dagger) \) is the bosonic annihilation (creation) operator of the field mode, \( \hat{J}_x, \hat{J}_y, \hat{J}_z \) are the collective pseudo-spin operators, with \( \hat{J}_x, \hat{J}_y, \hat{J}_z \) being the Pauli matrices, and \( \gamma \) is the atom-field coupling strength.

The eigenvalues \( j(j+1) \) of the squared total spin operator \( \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \) specify the different invariant subspaces of the model. We use the symmetric atomic subspace defined by the maximum pseudo-spin \( j = N/2 \), which includes the ground state. When the Dicke model reaches the critical value \( \gamma_c = \sqrt{\omega_0/2} \), it goes from a normal phase \( (\gamma < \gamma_c) \) to a superradiant phase \((\gamma > \gamma_c)\). Our studies are done in the superradiant phase, \( \gamma = 2\gamma_c \), and we choose \( \omega = \omega_0 = 1 \). The rescaled energies are denoted by \( \epsilon = E/t \). For the selected parameters, \( \epsilon_{GS} = -2.125 \) is the ground-state energy.

In systems studied before, scarred eigenstates were thought to be a fraction of the total number of eigenstates. In contrast to that, we show that deep in the chaotic regime of the Dicke model, all eigenstates are scarred. Their phase-space probability distributions always display structures that can be traced back to periodic orbits in the classical limit. Yet, we find eigenstates that are highly localized in phase space and eigenstates that are nearly as much spread out as random states, although none of them, including the random states, can cover more than approximately half of the available phase space.

In addition to the analysis of quantum scarring and phase-space localization, we also provide a definition of quantum ergodicity. This is done using a measure that we introduce to quantify the level of localization of quantum states in the phase space. We say that a quantum state is ergodic if its infinite-time average leads to full delocalization. Under this definition, all quantum states are not ergodic, while random states are, and coherent states lie somewhere in between.
Scarring vs. phase-space localization. To measure the localization of a state in a Hilbert-space basis indexed by some letter \( n \), the most commonly used quantity is the participation ratio \( P_n \). Its inverse is given by \( P_n^{-1} = \sum_{n} P_n \), where \( P_n \) is the probability of finding the state in the \( n \)th basis vector. By analogy, we introduce a measure of localization in phase space that employs as basis the overcomplete set of coherent states within a single energy shell \( \mathcal{M}_\epsilon = \{ x = (q,p,Q,P) \mid h_{\text{cl}}(x) = \epsilon \} \), so that we replace the sum \( \Sigma_k \) with a three-dimensional surface integral \( \int_{\mathcal{M}_\epsilon} \) ds over \( \mathcal{M}_\epsilon \). For a given \( x \in \mathcal{M}_\epsilon \), the probability \( P_x \) of finding the state \( \hat{\rho} \) in the coherent state \( |x\rangle \) is given by the Husimi function \( Q_x(\hat{\rho}) \). With these replacements, we finally obtain a measure of phase-space localization \( \mathcal{U}(\epsilon,\hat{\rho}) \) given by

\[
\mathcal{U}(\epsilon,\hat{\rho})^{-1} = \frac{1}{N} \int_{\mathcal{M}_\epsilon} \text{d}s \ Q_x^2(\hat{\rho}),
\]

where \( N = \int_{\mathcal{M}_\epsilon} \text{d}s \ Q_x^2(\hat{\rho}) \) is a normalization constant and \( \mathcal{V}(\epsilon) = \int_{\mathcal{M}_\epsilon} \text{d}s \ Q_x^2(\hat{\rho}) \) is the volume of \( \mathcal{M}_\epsilon \) (see Methods).

The measure \( \mathcal{U}(\epsilon,\hat{\rho}) \) is an energy-restricted second moment of the Husimi function. It is related to the second-order Rényi-Wehrl entropy, which, in turn, was shown for the Dicke model to be linearly related to the first-order Rényi-Wehrl entropy.

The value of \( \mathcal{U}(\epsilon,\hat{\rho}) \) indicates the fraction of the classical energy shell at \( \epsilon \) that is covered by the state \( \hat{\rho} \). It varies from its minimum value \( \mathcal{U}(\epsilon,\hat{\rho}) \sim (2\pi\hbar\omega)^2 \mathcal{V}(\epsilon) \), which indicates maximum localization, to \( \mathcal{U}(\epsilon,\hat{\rho}) = 1 \), which implies complete delocalization over the energy shell. The former occurs for coherent states, and the latter happens if \( Q_x(\hat{\rho}) \) is a constant for all \( x \in \mathcal{M}_\epsilon \), in which case the projection \( Q_x(\hat{\rho}) \) is also constant for the allowed values of \( Q \) and \( P \).

All eigenstates in Fig. 1 have values of \( \mathcal{U} \) below 1/2. For the eigenstates in Fig. 1a1, a2, a4, b3, these values are very small, since the eigenstates are almost entirely localized around the plotted periodic orbits. As the energy increases and the scar associated to the blue orbit. The localization measure is larger in Fig. 1b1, b2 simply because in these cases the phase space is very small. It is also larger for the states in the high energy region in Fig. 1a5, a6, b4, b5, b6, because they spread beyond the marked periodic orbits. As the energy increases and one approaches the chaotic region, more unstable periodic orbits emerge in the classical limit, enhancing the likelihood that a single quantum eigenstate gets scarred by different periodic orbits. We stress, however, that even for those high-energy states with larger values of \( \mathcal{U} \), the drawn periodic orbits cast a bright green shadow that is clearly visible in the Husimi projections.

It is important to make it clear that scarring and localization, despite related, are not synonyms. Of course, there is no eigenstate with a large value of \( \mathcal{U} \) that would at the same time have a high value of our scarring measure, but there is more to the relationship between these two concepts. A highly localized eigenstate is scarred by few periodic orbits of a particular family and therefore has a high value of the scarring measure for that family, although it has low values for the other families of periodic orbits. Furthermore, even the most delocalized
17,000 eigenstates with energy are provided in the Supplementary Note 2). The difference the low-energy regular regime, chaotic region at higher energies, the distribution of equivalent to a Peres lattice for expectation values of different families.

eigenstates are still scarred, but now by periodic orbits from classical periodic orbits. Conversely, as the system enters the chaotic region with the periodic orbits get better deformed. Their existence is supported by the shape of the Husimi projections, which display all the features of periodic orbits: they always cross the line perpendicularly, they are symmetric along both the horizontal and vertical axes, and they visibly form closed loops. There is no quantum effect other than scarring that would produce such patterns. One therefore deduces that deep in the chaotic region of the Dicke model, all quantum eigenstates are scarred, although they have different degrees of scarring, as seen by the patterns, and they have different degrees of localization, ranging from strong localization (\(\mathcal{L} \sim 0.1\)) to states that are nearly as much delocalized as random states.

Our results are sharpened as one approaches the semiclassical limit. The patterns indicating scarring do not fade away as the system size increases. Quite the opposite, as \(j = 1/\hbar_{\text{eff}}\) increases, the periodic orbits get better defined in the Husimi projections (cf. the figures for \(j = 30\) and \(j = 100\) in the Supplementary Note 3). To study the dependence of the level of phase-space localization on system size, we show the distributions of \(\mathcal{L}\) for random states (Fig. 2b) and for eigenstates in the chaotic region (Fig. 2c) for different values of \(j\). For the random states, \(\mathcal{L}\) is concentrated around 1/2 and the width of the distribution between random states and eigenstates is clear. The Husimi projections of the random states do not show structures that resemble closed periodic orbits, so they are not scarred, while the Husimi projections of all eigenstates in the chaotic region do show those structures. In contrast to Fig. 1 (see also ref. 45), we have not identified the periodic orbits associated with Fig. 2s1-s22, but the visible circular patterns are clear evidence of periodic orbits. Their existence is supported by the shape of the Husimi projections and by knowing the generic direction of the classical Hamiltonian flow. The patterns display all the features of periodic orbits: they always cross the line \(P = 0\) perpendicularly, they are symmetric along both the horizontal and vertical axes, and they visibly form closed loops. There is no quantum effect other than scarring that would produce such patterns. One therefore deduces that deep in the chaotic region of the Dicke model, all quantum eigenstates are scarred, although they have different degrees of scarring, as seen by the patterns, and they have different degrees of localization, ranging from strong localization (\(\mathcal{L} \sim 0.1\)) to states that are nearly as much delocalized as random states.

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Quantum ergodicity. We have so far discussed two concepts—quantum scarring and phase-space localization—that are related, but are not equal. How about their relationship with quantum ergodicity? In the classical limit, a system is ergodic if the trajectories cover the energy shell homogeneously. We then adopt the same definition for quantum ergodicity. To quantify how much of the energy shell is visited on average by the evolved state \( \hat{\rho}(t) = e^{-i\mathcal{H}t}\hat{\rho} e^{i\mathcal{H}t} \), we consider the infinite-time average \( \overline{\rho} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \hat{\rho}(t) \), and compute \( \overline{\mathcal{V}}(\epsilon, \hat{\rho}) \equiv \overline{\mathcal{V}}(\epsilon, \overline{\rho}) \) with Eq. (3). If the whole energy shell at \( \epsilon \) is homogeneously visited by \( \hat{\rho} \), then \( \overline{\mathcal{V}}(\epsilon, \hat{\rho}) = 1 \). We thus say that a quantum state \( \hat{\rho} \) is ergodic over the energy shell \( \epsilon \) if \( \overline{\mathcal{V}}(\epsilon, \hat{\rho}) = 1 \). According to this definition, all stationary states in the chaotic region of the Dicke model are non-ergodic, since \( \overline{\mathcal{V}}(\epsilon_t, \hat{\rho}_k) = \overline{\mathcal{V}}(\epsilon_t, \hat{\rho}_k) \leq 1/2 \), as shown above.

How about non-stationary states, such as coherent states or random states, are they ergodic? We study the evolution of initial coherent states \( |\Psi(0)\rangle = |\xi_0\rangle = \sum_k c_k |E_k\rangle \) with mean energies \( \epsilon = -0.5 \), that are in the chaotic region (see Methods). We select both coherent states that are highly localized and delocalized in the energy eigenbasis, with the degree of delocalization measured by the participation ratio \( P_R = \langle \sum_k |c_k|^4 \rangle^{-1} \). Their energy distributions are shown in Fig. 3a1–g1. The components of the states with low \( P_R \) are bunched around specific energy levels (Fig. 3a1, b1), exhibiting the comb-like structure typical of scarred states. As \( P_R \) increases, the coherent states become more spread in the energy eigenbasis, looking more similar to the random state \( |\Psi(0)\rangle = |\mathcal{R}_\xi\rangle \) shown in Fig. 3h1, whose mean energy is also \( \epsilon = -0.5 \).

The evolution of the survival probability, \( S_P(t) = |\langle \Psi(0)|e^{-i\mathcal{H}t}|\Psi(0)\rangle|^2 \), for the coherent states with low \( P_R \) leads to large revivals before the saturation of the dynamics, as seen in Fig. 3a2, b2, c2. This contrasts with the evolution of the chaotic states with large \( P_R \), such as those in Fig. 3d2–g2, and the evolution of the random state in Fig. 3h2. In these cases, the approach to the asymptotic value of \( S_P(t) \) is much smoother and exhibits the so-called correlation hole, which corresponds to the ramp towards saturation. The correlation hole reflects the presence of correlated eigenvalues and is a quantum signature of classical chaos.

The values of \( \overline{\mathcal{V}}(\epsilon, \hat{\rho}) \) for the states in Fig. 3a1–h1 are indicated in Fig. 3a3–h3. The random state is indeed ergodic, reaching \( \overline{\mathcal{V}} \sim 1 \). For the coherent states, \( \overline{\mathcal{V}} \) increases as \( P_R \) does, but even for the states with the largest values of the participation ratio, \( \overline{\mathcal{V}} \) is still slightly under 1. The analysis of the dependence of \( \overline{\mathcal{V}} \) on system size is done in Fig. 3i, j. The distribution of \( \overline{\mathcal{V}} \) for random states (Fig. 3j) gets narrower and better centered at \( \overline{\mathcal{V}} = 1 \) as \( j \) increases, confirming that these states indeed behave ergodically. The distribution for the coherent states (Fig. 3h) is much broader. The center moves towards larger values as \( j \) increases, but it is not clear whether there will ever be a significant portion of the initial coherent states with \( \overline{\mathcal{V}} \sim 1 \).

In Fig. 3a3–h3, we plot the projected time-averaged Husimi distributions \( \overline{\mathcal{Q}}_{\epsilon, \phi}(Q, P) \) for the states in Fig. 3a1–h1. Remarkably, even for the coherent states with high \( P_R \), which do not exhibit any comb-like structure in their energy distributions and do not show revivals in the evolution of their survival probability, we still see an enhancement around unstable periodic orbits, as revealed by a careful inspection of Fig. 3d3–g3 and in contrast with the absence of any pattern for the random state in Fig. 3h3. This unexpected manifestation of dynamical scarring can only be observed in phase space, having no identifiable signature in the energy distribution of the initial states or in the evolution of the survival probability. Thus, revivals in the long-time quantum dynamics are signs of a scarred initial state, but lack of revivals do not exclude the presence of scarring, it just indicates that if it exists it is at a low degree.

Discussion

The three main concepts investigated and compared in this work were quantum scarring, phase-space localization, and quantum ergodicity. We showed that for the Dicke model, all eigenstates in the chaotic region are scarred, although with different degrees of scarring and different levels of phase-space localization. Evidently, an eigenstate that is strongly scarred is also highly localized in phase space, but a single eigenstate may be scarred by different periodic orbits and reach levels of delocalization almost as high as a random state. We also showed that any pure state—even without any trace of scarring—is localized in phase space, and that ergodicity is an ensemble property, achievable only through temporal averages. Thus, scarring, localization and lack of ergodicity are not synonyms, although connections exist.

The ubiquitous scarring of the eigenstates does not immediately translate into the breaking of quantum ergodicity. All eigenstates are certainly non-ergodic, since they never reach complete delocalization in phase space, but if a non-stationary state visits on average the available phase-space homogeneously, then it is ergodic. Random states, for example, are ergodic. The analysis of initial coherent states showed that some are heavily scarred, resulting in the strong breaking of ergodicity that translates into the usual revivalss of the survival probability. More interesting is the subtle behavior of the majority of the initial coherent states, which do not display revivals in the quantum dynamics or the comb-like structure in their energy distributions, but yet show some degree of scarring.

Analyses that focus on the Hilbert space, such as the energy distribution of the initial states or the fluctuations of eigenstate expectation values in Peres lattices and comparisons with thermodynamic averages, that are often done in studies of the eigenstate thermalization hypothesis (ETH), may miss the ubiquitous scarring observed in this work. For this feature to be revealed, one needs to look at the structures of the states in phase space.

Our results for the Dicke model are, of course, important, due to the widespread theoretical interest in this model and the fact that it is employed to describe experiments with trapped ions and...
superconducting circuits. But the repercussion of the findings
discussed here goes beyond the limit of spin-boson systems by
raising the question of whether scattering in other quantum
systems is also the rule and not the exception. Our work provides the
appropriate tools to address this question. The phase-space
method that we developed is applicable to any quantum system
that has a tractable phase-space. Whether these studies could
eventually be extended also to interacting many-body quantum
systems, such as interacting spin-1/2 models, will depend on the
viability of their semiclassical analysis, and some recent works
give reasons for optimism.

Methods

Classical Hamiltonian. To construct the classical Hamiltonian in a four-
dimensional phase space $\mathcal{M}$ with coordinates $x = (q, p; Q, P)$, we use the Glauber-
Bloch coherent states $|x\rangle = |q, p\rangle \otimes |Q, P\rangle$ [30,43-46]. They are tensor products of
the bosonic Glauber coherent states $|q, p\rangle = e^{-(1/2)(q^2 + p^2)} e^{(1/(\sqrt{2} \pi))}\sqrt{2} q |0\rangle$ and the
pseudo-spin Bloch coherent states $|Q, P\rangle = \left(1 - \frac{2}{Z^2}ight)^{1/2} e^{(1/4)Z^2 \frac{Q}{\sqrt{J_x}} + \frac{P}{\sqrt{J_y}}} |J_x, J_y\rangle$, where
$Z^2 = Q^2 + P^2$, $|0\rangle$ denotes the photon vacuum, $|J_x, J_y\rangle$ designates the state with all
atoms in their ground state, and $J_z$ is the raising atomic operator. The classical
Hamiltonian is given by

$$h_0(x) = \frac{\omega}{2} (q^2 + p^2) + \omega_0 \omega^2 / 2 + 2 y Q q_i \sqrt{1 - \frac{2 p^2}{4} - \omega_0}. \tag{5}$$

Efficient basis and system sizes. The efficient basis is the Dicke Hamiltonian (1)
eigenbasis in the limit $\omega_0 \rightarrow 0$, which can be analytically obtained by a displacement
of the bosonic operator $A = a + (2 y / (\sqrt{\omega N})) J_z$ and a rotation of $-\pi / 2$ around the

In Fig. 3 Dynamical behavior of coherent and random states. a1-g1 Energy distribution for coherent states with different values of $P_R$ and centered at $\epsilon = -0.5, j = 100$. h1 Energy distribution for a random state $|R\rangle$ with energy width $\sigma = 0.3$ centered at $\epsilon = -0.5, j = 100$. a2-h2 Quantum survival probability (gray solid line), its running average (blue solid line) and its asymptotic value (black dashed line) for the corresponding states of panels a1-h1. a3-h3 Projected Husimi distributions of the time-averaged ensemble, $Q_{\text{max}} = 0.5$, for the corresponding states of panels a1-h1 with the values of $\overline{E}(\epsilon = -0.5, \rho)$ indicated. i Distribution of $\overline{E}(\epsilon, \rho)$ for a set of 1551 coherent states that are evenly distributed along the atomic variables ($Q, P$) at $\epsilon = -0.5$ and $\rho = 0$. This is done for system sizes $j = 20, 30, 40, 50, 100$, as indicated. j Distribution of $\overline{E}(\epsilon, \rho)$ for 500 random states with $\epsilon = -0.5$ and $\sigma = 0.3$ for $j = 20, 30, 40, 50, 100$. The y axis of the collective pseudo-spin operators $(J_x, J_y, J_z) \rightarrow (\hat{y}_x, \hat{y}_y, \hat{y}_z)$.

$$|N;j, m\rangle = \frac{1}{\sqrt{N!}} |N = 0; j, m\rangle,$$  

where $N$ is the eigenvalue of the modified bosonic number operator $\hat{A} \hat{A}$ and $m = m_i$ is the eigenvalue of the original collective pseudo-spin operator $J_z$. The modified bosonic vacuum states in $|N = 0; j, m\rangle = |N = 0 \oplus |j, m\rangle$ are Glauber coherent states $|N = 0\rangle = -2 y m / (\sqrt{\omega N})$. The Hilbert space dimension of this
basis is given by $D = (2j + 1)(N_{\text{max}} + 1)$, where $N_{\text{max}}$ designates a cutoff of the
modified bosonic subspace.

This basis allows to work with larger values of $j$ by reducing the value of $N_{\text{max}}$
required for convergence of the high-energy eigenstates. With $j = 100$ and $N_{\text{max}} = 300$ ($D = 60,501$), we are able to get 30,825 converged eigenstates covering
the whole energy spectrum up to $\epsilon = 0.853$. Having converged eigenstates in
such a high-energy regime would be infeasible with the usual Fock basis for
$j = 100$ [40-49].

Husimi projection and localization measure. To compute the Husimi projection
given in Eq. (2) and the localization measure given by Eq. (3), one has to compute integrals of the form

$$\overline{f}(Q) = \int dQ dP \delta (\epsilon - h_0(x)) f(x). \tag{7}$$

where $x = (q, p; Q, P)$ and $f(x)$ is a non-negative function in the phase space. For
the localization measure, note that $\delta (\epsilon - h_0(x)) = \frac{1}{(2\pi)^{3/2}} \int dQ dP f_1(Q, P) f_2(Q, P)$ with $f_1 = Q_0$ and $f_2 = Q_1$,

$N = \int dQ dP f_1(Q, P)^2 \sqrt{\nu}(\epsilon) f_1(Q, P)$ with $f_1 = Q_0$ and $f_2 = f_1$.
By using properties of the $\delta$ function, those integrals are reduced to
\[
\tilde{f}(Q, P) = \int_{0}^{r_{\text{p}}} \frac{d\phi}{2\pi} \sum_{n=1}^{\infty} f(q_{n}, p; Q, P) \sqrt{\Delta(e, p, Q, P)},
\]
where $q_{n}$ are the two solutions in $q$ of the second-degree equation $h_{2}(q, p; Q, P) = e$.
\[
\Delta(e, p, Q, P) = \left| \frac{\partial q_{n}}{\partial q} (q_{n}, p; Q, P) \right|^2
\]
\[= 2\omega_{0} \left( \frac{1}{e_{0}^2} + \frac{1}{2Q^2 + P^2} + 4y^2 Q^2 \left( 1 - \frac{Q^2 + P^2}{4} \right) - \omega^2 P^2 \right),
\]
and $p_{n}$ are the two solutions in $p$ of the second-degree equation $\Delta(e, p, Q, P) = 0$.

Because of the form of the weight $1/\sqrt{\Delta}$, the integral given by Eq. (8) may be computed efficiently using a Choyebshv-Gauss quadrature method.

It is worth noting that a quantum state with a Wigner distribution that is constant in an energy shell will lead to a Husimi function that needs not to be constant within the same energy shell. This is because the Husimi distribution is the convolution of the Wigner distribution with the Gaussian Wigner distributions of the coherent states, which have different energy widths due to the geometry of the energy shells in the phase space. This rather marginal effect may be seen in Fig. 3h3, where there is a barely visible weak concentration towards the center of the plot causing $Z$ to be slightly under 1. We stress that this effect is not related to quantum scarring. It is just a manifestation of the phase-space geometry in the Husimi distributions.

Coherent states.

The coherent states $|\alpha\rangle$ are described by four coordinates $x = (q, p; Q, P)$. To select coherent states at a given energy $e$, we solve the second-degree equation $h_{2}(q, p; Q, P) = e$ for $q$, which yields two solutions $q_{\pm}(e, p; Q, P) = q_{\pm}(e, p; Q, P)[59]$. All of the initial coherent states shown in Fig. 3 have bosonic variables given by $q_{\pm}(e, p; Q, P)$ and $p = 0$. The coherent states shown in Fig. 3a-g1 have atomic coordinates given by $Q_{\pm}(e, p; Q, P) = (1.75, 0) a_{1}, (0.5, 0.75) b_{1}, (0.75, 0.5) c_{1}, (1.25, 0.25) d_{1}, (-1.25, 1) e_{1}, (-1.25, 0.75) f_{1}$, and $(0.75, 0.5) g_{1}$. The atomic coordinates of the 1551 coherent states whose distributions of $\mathcal{U}$ are shown in Fig. 3i were selected by constructing a rectangular grid with step $\Delta Q = \Delta P = 0.05$ from $(Q_{\pm}, P_{\pm}) = (-2.0, 2.0)$ to $(Q_{\pm}, P_{\pm}) = (2.0, 2.0)$. Of the 3321 points inside of this grid, 1551 have allowed values of $Q, P$ (i.e. they satisfy $Q^2 + P^2 \leq 4$) and fall inside of the energy shell at $e = 0.5$ (i.e. there exists $q_{\pm}(e, p; Q, P)$). We use these 1551 coherent states to compute the distributions in Fig. 3i.

Random states.

The state $|R_{i}\rangle = \sum_{n} e^{i\theta_{n}}|E_{n}\rangle$ is built by sampling random numbers $\theta_{n}$ from an exponential distribution $\lambda_{0}^{-1}$ and random phases $\theta_{n}$ from a uniform distribution in $[0, 2\pi]$. We use
\[
\theta_{n} = \frac{\int_{0}^{2\pi} e^{i\theta} |E_{n}\rangle \langle E_{n}| \lambda_{0}}{\int_{0}^{2\pi} |E_{n}\rangle \langle E_{n}| \lambda_{0}},
\]
where $|E_{n}\rangle$ is the density of states, $\rho(E)$ is a Gaussian profile of width $\delta_{0}$ centered at energy $E_{c}$, and $M$ ensures normalization. This way, $|R_{i}\rangle$ has a defined energy center $e$, where the Husimi projection and the localization measure are calculated[59,70].

Data availability

All the data that support the plots within this paper and other findings of this study are available from the corresponding authors upon request.

Code availability

All the computational codes that were used to generate the data presented in this paper are available from the corresponding authors upon request.

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**Competing interests**
The authors declare no competing interests.

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