A WDM model for the evolution of galactic halos

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Abstract

It is a well-known fact that the gravitational effect of dark matter in galaxies is only noticeable when the orbital accelerations drop below $a_0 \simeq 2 \times 10^{-8}$ cm s$^{-1}$ (Milgrom’s Law). This peculiarity of the dynamic behaviour of galaxies was initially ascribed to a modification of Newtonian dynamics (MOND theory) and, consequently, it was used as an argument to criticize the dark matter hypothesis. In our model, warm dark matter is composed by collisionless Vlasov particles with a primordial typical velocity $\simeq 330$ km s$^{-1}$ and, consequently, they evaporated from galactic cores and reorganized in halos with a cusp at a finite distance from the galactic center (in contrast with Cold Dark Matter simulations which predict a cusp at the center of galaxies). This is confirmed by mean-field N-body simulations of the self-gravitating Vlasov dark matter particles in the potential well of the baryonic core. The rest mass of these particles, $\mu$, is determined from a kinetic theory of the early universe with a cosmological constant. We find that $\mu$ is in the range of a few keV. This result makes sterile neutrinos the best suited candidates for the main component of dark matter.

Keywords: dark matter simulations, galaxy evolution, physics of the early universe

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1 Introduction

In the pioneer work of Zwicky [1] he presented the results of the first detailed observations of the dynamics of the Coma galaxy cluster. In this work, it was hypothesized that most of the matter in these clusters must be dark because luminous matter is not sufficient to account for the orbital velocities as deduced from Newtonian dynamics. Zwicky’s dark matter hypothesis was consigned to oblivion by the astrophysical community until the end of the seventies and the beginning of the eighties of the past century. At that time, a series of exhaustive measurements of the Doppler shift for the 21 cm hydrogen line in the galactic gas clouds for the Milky Way and other galaxies proved, beyond any doubt, that the rotation curves of galaxies were anomalous [2, 3, 4]. This anomaly is characterized by a flat asymptotic region in the rotation curve corresponding to a constant orbital velocity of stars and gas clouds, $V_\infty$. This is the case even for distances of thousands of kpcs beyond the luminous core of the galaxy. If galaxies were composed only of the observed luminous matter we will expect a decrease of the orbital velocity $\propto r^{-1/2}$, $r$ being the distance to the galactic centre. Consequently, the most natural explanation of this phenomenon is the presence of Zwicky’s dark matter, inferred from the observation of clusters of galaxies, also inside individual galaxies in the form of disperse halos with a volumetric density decreasing as $r^{-2}$.

Just two years after the acceptance of this evidence by the astrophysical community, Milgrom proposed an alternative explanation for the anomalous rotation curves of galaxies based upon a modification of Newtonian dynamics for small accelerations (MOND theory) [5]. A careful analysis of the rotation curves published by Faber & Gallagher [2] revealed that the effect of the supposed dark matter seems to activate only when the orbital acceleration is smaller than $a_0 \approx 2 \times 10^{-8}$ cm s$^{-2}$ (do not confuse with the curvature radius of the Universe usually denoted by the same symbol). Consequently, Milgrom modified Newton’s second law of dynamics by assuming that the gravitational mass of an object is a function $m(a/a_0)$. From his theory, Milgrom also derived the following equation

$$V_\infty^4 = a_0GM,$$  \hspace{1cm} (1)

where $V_\infty$ is the asymptotic orbital velocity in the rotation curve of a galaxy and $M$ is its observable mass. This equation is consistent with the Tully–Fisher relation between $V_\infty$ and the luminosity of a galaxy, $L \propto V_\infty^\delta$, with
The MOND theory gathered later some attention [7, 8, 9] but it is defended only by a limited number of theoreticians. The preference for the dark matter explanation is not only a consequence of the application of Occam’s razor principle. In recent years the evidence for dark matter has been increasing by means of indirect observations that fit nicely into the dark matter paradigm but which are very difficult to explain by the MOND theory: 

(i) Gravitational lensing by galaxy clusters is far more intense than one could expect from the observable luminous matter [10] 
(ii) The measurements of the cosmic microwave background (CMB) anisotropies by the Boomerang and Wilkinson Microwave Anisotropy probes have placed certain limits on the necessary amount of matter in the Universe to explain the formation of galaxies from the primordial fluctuations at the matter–radiation decoupling era (see [11] for a three-year survey of the recent WMAP project and see [12] for a review of the Boomerang project). Moreover, the peak found at $l_{\text{max}} \simeq 200$ in the CMB spectrum implies a flat Universe [13] and this can only be explained if we have more mass in the Universe than that observed in galaxies and clusters. Consequently, dark matter (DM) has become an essential ingredient of modern cosmology and without it concordance with the observational data from all these independent sources cannot be achieved [14, 15, 16]. The overwhelming evidence for DM has promoted, from the last decade of past century, the proliferation of hypotheses on its nature. The most economic hypothesis attributes this lost mass to bodies which emit little radiation: brown or white dwarfs, neutron stars or black holes. However, the analysis of recent observations of gravitational microlensing events has discarded this source as a significant contribution to DM [17, 18].

A popular alternative assigns the main role to particles predicted by extensions of the Standard Model: axions (with a rest mass $\mu \sim 10^{-5}$ eV), magnetic monopoles ($\mu \sim 10^{16}$ GeV), weakly interacting massive particles (WIMPs), the so-called neutralinos, or another supersymmetrical partner of already known particles (with masses in the range $\mu \sim 1–10^{3}$ GeV) or, even, neutrinos heavier than ordinarily assumed ($\mu \sim 10$ eV). This intriguing possibility has boosted an interesting synergy between cosmology, astrophysics and particle physics. In particular, there have been important experimental efforts in the determination of an upper limit for the mass of the electron neutrino. In the Mainz neutrino mass experiment [19] and the Troitsk experiment [20] an absolute limit for the electron neutrino mass is found by investigating in large detail the endpoint of the tritium $\beta$ decay spectrum. A limit $m_{\nu,e}c^2 < 2.8$ eV is found in these experiments. Even more pre-
cise measurements, in the sub-eV range are expected from Katrin project [21]. Fixing confidence intervals for the masses of the muon and tau neutrino is even more difficult as it has to be based on decay processes, such as \( \tau \rightarrow 3\pi^{\pm} + \nu_{\tau} \), of particles produced in accelerators [22]. This way it was estimated that \( m_{\nu,\tau}c^2 < 15 \text{ MeV} \). The phenomenon of neutrino oscillations also put some bounds on the splitting of masses for neutrinos [23]. The maximum mass difference squared, \( \Delta m^2 \), is bounded by \( 2.4 \times 10^{-3} \text{ eV} \). By combining this result with the upper limit on neutrino mass, a scenario in which the three neutrino species have rest masses below the eV range is steadily gaining acceptance among cosmologists and particle physicists. Taking into account that neutrino abundances are fixed in the Big Bang model, the known neutrinos cannot be the main ingredient of DM [14, 15]. Experimental settings devised to detect other candidates for DM particles – axions or neutralinos – have also been unsuccessful to date [17]. The lack of detection of WIMPs has favoured the suggestion of a family of sterile neutrinos – sterile means that they do not interact via neutral or charged currents – which appears in some minimal extensions of the Standard Model [24, 25]. Sterile neutrinos are even more difficult to detect than WIMPs because, essentially, they only interact through gravity. Their masses are predicted to be in the range \( 2 < M_I < 5 \text{ keV} \). These masses, which are relatively large compared with that of ordinary neutrinos, are acquired by means of the so-called seesaw mechanism [26]. It has been proposed that sterile neutrinos could explain pulsar kick velocities [27] or the baryonic asymmetry in the Universe [25].

A pioneer proposal for a sterile neutrino was given by Dodelson and Widrow [28] in 1994. They considered that these neutrinos could be produced by oscillations in the early Universe. However, a mass around 0.1 keV was predicted in this model. Larger masses are desirable in order to confront the main problem associated with light dark matter: the large free-streaming lengths which avoid the accretion of galaxies in the young Universe. Another mechanism was proposed later on by Shi and Fuller [29]. They proposed that sterile neutrinos with masses in a range 0.1-10 keV are produced via a lepton-number-driven resonant conversion of active neutrinos at the big bang nucleosynthesis epoch. With larger masses for the sterile neutrinos the accretion problem could be circumvented. The energy spectrum of these neutrinos resembles a gaussian with a cutoff at \( E/k_B T \approx 0.7 \). Consequently, they are sufficiently cold to condensate on primordial fluctuations and favour galaxy formation. On the other hand, the off-resonance process yields a different
energy spectrum \[28\]:

\[
\frac{f_s}{f_a} = \frac{7.7}{g_s^{1/2}} \left( \frac{m_a}{1 \text{ eV}} \right)^2 \left( \frac{1 \text{ keV}}{m_s} \right) y \int_x^\infty \frac{d\xi}{(1 + y^2 \xi^2)^2},
\]

where \(f_s (f_a)\) is the spectrum of the sterile (active) neutrinos, \(m_s\) and \(m_a\) are their masses, \(g_s\) is a constant, \(y = E/(k_B T)\) and \(x = 78(T/1 \text{ GeV})^3(1 \text{ keV}/m_s)\). For \(T \ll 1 \text{ GeV}\) and a mass for the sterile neutrino of the order of keV the spectrum of sterile neutrinos is proportional to that of the active species. Another alternative has been given by Shaposhnikov and Tkachev \[30\] and Kusenko \[31\]. In these models a heavy scalar decays through some channels into the sterile neutrinos. If this model is correct it would exist some chance for discovering such a scalar in the Large Hadron Collider. More recently, it has been suggested that the decay of sterile neutrinos into ordinary neutrinos and X rays will boost the production of molecular hydrogen in the early Universe \[32\]. Consequently, star formation will also increase despite of the larger free-streaming lengths for Warm dark matter (WDM). In this scenario there is a constraint for X-ray observations which implies that the mass of sterile neutrinos must be smaller than 3 keV \[33\].

On the part of experimental particle physics, sterile neutrinos have been used in the sketching of an explanation for the low energy anomaly in the neutrino oscillations studied in the Liquid Scintillator Neutrino Detector or LSND \[34\]. Very recent results from the MiniBooNE collaboration excludes, at a 98 % confidence, neutrino oscillations between two species as an explanation of the LSND anomaly \[35\]. The relevance of the sterile neutrino theory for cosmology is also controversial \[36\].

On the other hand, some recent work by Boyarsky et al. \[41, 42\] determines the lower thresholds for the masses of WDM particles based upon the Lyman-\(\alpha\) forest and the WMAP5 results. This threshold depends on the production mechanism of the sterile neutrino but a mass in the range of a few keV is compatible with the WMAP and Lyman-\(\alpha\) data.

With independence of the nature of the constituent particles of DM, the DM hypothesis is the better explanation of a plethora of astrophysical observations. However, an important point of discrepancy remains, because simulations of cold dark matter (CDM) accretion favour the formation of halos with a cusp at the center of the galaxies (the latest and more precise CDM’s halos simulation is known as Via Lactea II \[37\]). The calculations of galactic density profiles from the rotation curves \[38\] and many observations
of these rotation curves from galaxies $^{39}$ contradict the prediction of an halo with a peak at the center of galaxies. This phenomenon is related with Milgrom’s law because a galactic core almost free from DM will imply that there exists a critical orbit separating the inner regions of the galaxy, where rotation behaviour can be deduced from the observed mass, and the outer regions where DM must be invoked to explain the discrepancy. The increasing and independent evidence on dark matter has made the clarification of this paradox, in the context of dark matter models, urgent. Trying to reconcile DM models with Milgrom’s law could also help us in clarifying the properties of the fundamental particles which form it.

A recent proposal to explain Milgrom’s law in a DM model has been given by Kaplinghat & Turner $^{8}$. These authors suggest that scale-free primordial fluctuations and baryonic dissipation can explain the remarkable numerical coincidence $a_0 \sim cH_0$ between Milgrom’s critical acceleration, $a_0$, and the present value of Hubble constant, $H_0$. Dunkel has also shown that by taking into account the DM gravitation potential a generalized MOND equation can be derived as a special limit $^{40}$. However, the process which formed DM halos is not explained in these models.

We propose a different approach based upon a simple idea: warm dark matter (WDM) particles are trapped in the gravitational field of the galaxies but their velocities are sufficiently large compared with the typical orbital velocities in the galactic cores and also random in their directions. Consequently, they tend to evaporate from the core and distribute in a disperse halo. These WDM particles were trapped back at the time of galaxy formation, when the CMB was roughly at a temperature $k_B T \simeq 3 \text{ eV}$ $^{14}$. A kinetic model for the Hubble expansion cooling of this collisionless gas indicates that the mass of these particles is in the keV range ($\mu \lesssim 3 \text{ keV}$), in good agreement with the sterile neutrino model. We use mean-field N-body simulations for $2 \times 10^4$ DM particles moving in the static spherical potential well of the core in order to study the evolution of the halo and compare with the mass distribution of our galaxy (Section 2). A model with DM and cosmological constant fitting the most recent parameters obtained from WMAP $^{11}$ is used to study the cooling of WDM in the early Universe. The rest mass of the constituents of WDM is obtained by applying the condition that the typical velocity of the matter trapped in primordial inhomogeneities was fixed at the galaxy formation era (Section 3). The reason for the quenching of the velocity distribution is the gravitational trapping of DM in the galaxy. This way the effect of Hubble expansion after the galaxy formation era is
eluded. The papers ends with some conclusions and remarks in Section 4.

2 Self-gravitating model for Vlasov particles and a baryonic core

Models for the matter distribution of galaxies have been developed with a margin of accuracy only for the Milky Way. In this case the mass inside a sphere of radius $r$ from the galactic center has been determined by Kalberla & Kerp [43, 44]. We have plotted these results in Fig. 1 where separated curves for the galactic bulge, the galactic disk and DM are shown. We notice that the DM mass inside an sphere of radius $< 5$ kpc is negligible.

The mass distributions in Fig. 1 have three conspicuous features: most of the visible mass is inside a sphere of radius $R_c \simeq 10$ kpc ($M_c = 75 \times 10^9$ solar masses), there exists a region corresponding to distances $r_0 \sim 5$ kpc $< R_c$ almost free from DM and the halo spans from $r_0$ to very large distances.
\( R_0 \simeq 50 \text{kpc} \). The total mass of the galaxy (bulge, disk and halo) is, approximately, \( M = 300 \times 10^9 \) solar masses.

The objective of this section is to propose a dynamical model for the interaction of DM Vlasov particles and a spherical baryonic core to explain qualitatively the DM distribution in Fig. 1. Self-gravitating systems have been an area of intense study for more than forty years [45, 46, 47, 48]. Interest into these models has been mainly spurred by their applications to globular clusters of stars [49]. Recently, the Smoluchowski-Poisson equation for self-gravitating random walkers has been proposed as an adequate model for describing accretion of planetesimals in the solar nebulae where dissipation and turbulence plays an important role [50, 51]. In these works the qualitative behaviour of self-gravitating systems is derived from a mean-field theory approach in which the gravitational force acting upon a particle is calculated by means of Gauss’s theorem (for a spherically symmetric system). This way we avoid a rigorous N body simulation which implies an unaffordable computational cost.

In order to develop our mean-field model for the evolution of the galactic halo we will take two assumptions for granted: (i) The galactic core is reasonably well represented by an sphere with radius \( R_c \) which contains most of the baryonic mass. This core was formed very early after the condensation around primordial inhomogeneities (ii) In the early Universe DM followed the distribution of baryonic matter more closely than today. In particular, we will assume that all the DM was uniformly distributed inside the core. The velocity modulus, \( \bar{v} \) is taken the same for all DM particles and the angular distribution is homogeneous in the unit sphere. Alternatively, we can consider this initial condition as an extreme perturbation in the configuration space without historical significance. In any case, we will find an evolution towards a fixed point in the self-gravitation dynamics whose robustness is tested by exploring two initial velocity distributions a Dirac delta and a truncated parabola.

Starting from these initial conditions and, taking into account that DM particles are Vlasov particles that only interacts gravitationally among themselves and with the core, the time evolution is deduced from the numerical integration of Newton equations by Euler method:

\[
\begin{align*}
X_i(t+h) & = X_i(t) + hV_i(t) + O(h^2) \\
V_i(t+h) & = V_i(t) + hA_i(X_1, \ldots, X_n) + O(h^2), \quad i = 1, \ldots, N
\end{align*}
\]

where \( X_i(t) \) and \( V_i(t) \) are, respectively, the position and velocity of the \( i \)-th
particle at time $t$, $N$ is the total number of particles, $\mathbf{A}_i(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ is the acceleration of the $i$-th particle due to the joint gravitational attraction of the other Vlasov particles and the baryonic core and $h$ is the time step of the numerical method. It is convenient to scale these parameters in terms of characteristic parameters referred to the core. So, we will measure distances in units of $R_c$, masses in units of the mass of the core, $M_c$ and velocities in terms of the escape velocity of a particle from the edge of the core in the absence of any more mass, $V_c = \sqrt{2GM_c/R_c}$. The rest of units are derived: $R_c/V_c$ is the unit of time and $V_c^2/R_c$ is our unit of acceleration. Iteration of Eq. (3) is straightforward if we assume that spherical symmetry is preserved by the evolution (the acceleration for each particle always points towards the center of the core). This way, we can easily calculate a mean-field estimation of the modulus of $\mathbf{A}_i$ as follows:

$$ |\mathbf{A}_i| = \begin{cases} \frac{r_i}{2} + \frac{n(r_i)m}{2r_i^2} & \text{if } r_i < 1 \\ \frac{1 + n(r_i)m}{2r_i^2} & \text{if } r_i \geq 1 \end{cases} \tag{4} $$

where $n(r)$ is the number of particles at a distance from the center of the core smaller than $r$, $r_i$ is the distance of the $i$-th particle from the center of the core at time $t$. $m = M_H/N$ is the scaled mass of a single particle in the simulation, $M_H$ being the mass of the DM halo. We must take into account that these particles would represent many real DM particles because $N$ is small. Notice than in Eq. (4) we have already used the scaled parameters defined above. We will use this units in the following.

In order to grasp the behaviour of the halo for large times we have performed simulations for several values of the typical velocity of DM particles and $N = 20000$. Timestep is taken as $h = 10^{-4}$. The results for the evolution radial density as a function of time for $\bar{V} = 1.5$ are shown in Fig. 2. The radial density, $R(r) = 4\pi r^2 \rho(r)$ ($\rho(r)$ being the volumetric density), was calculated using the following approximation $R(r) \approx (n(r + \Delta r) - n(r))m/\Delta r$ and taking $\Delta r = 0.1$. We find that for $\bar{V} = 1.5$ and larger typical velocities the DM evaporates from the baryonic core. In this process a diffusive wave with increasing width develops. This spherical wave propagates at almost constant velocity expanding forever into intergalactic space. Globular clusters and dwarf galaxies should have lost most of their DM content through this evaporation process. However, dwarf galaxies are also DM dominated. The mechanism in this case could be peculiar and related with the evolution
of this type of galaxies as we discuss in Sec. If we apply the result of this simulation to the case of ω Centauri using the data given by Meylan and Mayor [52], $R_c = 30$ pc and $M_c = 18$ millions solar masses, we find that the process in Fig. 2 happened in only 8 million years. In the course of this time the halo has expanded to a scale of $\simeq 300$ pc. Later on, the halo continued its expansion being finally captured by the potential well of the Milky Way or dispersed into the intergalactic space of the Local Group. This is a similar process to the formation of nebulae, such as the Crab or the Ring nebulae, by the explosion of supernovaes [53]. In this processes the debris of the supernova explosion, moving at a speed larger than the escape velocity from the original star, disperse in a diffuse halo over the centuries to form the nebulae we see today.

On the other hand, galaxies have conserved a great amount of DM. This behaviour is obtained for smaller typical velocities for the DM particles. In Fig. 3 we have plotted the results for the radial density of the DM distribution in four time snapshots from $t = 5$ to $t = 50$ with $\bar{V} = 1.1$. The development
Figure 3: The radial density of DM particles for $t = 5$ (solid line), 20 (dashed line), 35 (dashed-dotted line) and 50 (dotted line). Initial velocity modulus was $\bar{V} = 1.1$.

of a characteristic cusp structure, with a cusp at a finite distance from the galactic center, is clearly observed. Moreover, the Vlasov particles do not disperse but, equally important, they do not collapse at the center of the core. An stationary state is achieved with a maximum radial density at the edge of the baryonic core. The resulting DM halo keeps bound by the joint attraction of the core and its own self-gravitation. In the case of the Milky Way we may assume $R_c = 10$ kpc and $M_c = 75 \times 10^9$ solar masses (using this radius most of the baryonic mass is contained inside the core). The time evolution of the simulations in Fig. (3) should correspond to, approximately, 2000 millions years.

An slice of the halo at $t = 10$ obtained by plotting only the particles with coordinate $z$ in the range $[-0.2, 0.2]$ is shown in Fig. 4. We observe the development of an inner spherical region with it is already almost devoid of DM particles.

In order to compare with the Kalberla & Kerp density model for out
Figure 4: Particles in the DM halo at $t = 10$ with a coordinate $-0.2 < z < 0.2$ (initial velocity $\bar{V} = 1$). The circle delimits the extension of the baryonic core.
Galaxy \cite{43, 44} we have fitted their data by a Padé approximant as follows:

\[
\mathcal{M}_H(r) = \frac{M_H r^3}{\xi_0 + \xi_1 r + \xi_2 r^2 + r^3},
\]

(5)

where \(\mathcal{M}_H(r)\) is the DM mass inside a sphere of radius \(r\), \(\mathcal{M}_H\) is the total mass of the halo and \(\xi_0, \xi_1\) and \(\xi_2\) are constants. If we measure the mass in units of \(10^{12}\) solar masses and the distance in kpc we get \(\mathcal{M}_H = 0.225\), \(\xi_0 = 5874.07\), \(\xi_1 = -40.17\) and \(\xi_2 = -0.2539\). These values were obtained by imposing the conditions that the result of Eq. (5) coincides with the data in Fig. 1 for \(r = 5, 10\) and \(20\) kpc. The mass of the halo is \(M_H = M - M_c = 225 \times 10^9\) solar masses.

In Fig. 5 we compare the results of the Padé approximant for the radial density, \(R(r) = d\mathcal{M}_H(r)/dr\) with the simulation results for three different initial velocities of the DM particles (distances and masses are conveniently scaled with \(R_c = 10\) kpc and \(M_c = 75 \times 10^9\) solar masses) after 500000 time steps (\(t = 50\)). The initial mass of the DM was \(M_H = 3\) (in units of the mass of the baryonic core). The position and radial density at the cusp are qualitatively well described by the distribution obtained in the simulations for \(\bar{V} = 1.15\). However, the tail of the halo is probably composed by particles with an initially larger velocity and cannot be fitted by considering a Dirac delta velocity distribution. However, we obtain an estimation of \(\bar{V} = 1.3V_c = 330\) km s\(^{-1}\) for the primordial typical velocities of DM particles.

We also notice that the area under the predicted profiles in Fig. 5 is smaller than the initial mass of the DM. The reason for that is the evaporation to intergalactic space of a 20 \% of the DM particles after the virialization of the halo.

It is also interesting to consider a more realistic initial velocity distribution for the DM particles. In this spirit, we propose a distribution proportional to the square of the velocity modulus with a cut-off as follows:

\[
f(v) = \begin{cases} 
81 \frac{v^2}{\bar{v}^3} & \text{if } v < 4\bar{v}/3 \\
0 & \text{if } v > 4\bar{v}/3
\end{cases}
\]

(6)

The cut-off is chosen to verify the normalization condition and \(\bar{v}\) is the average velocity. Sterile neutrinos produced by the resonant conversion of active neutrinos are supposed to end up with an energy spectrum of this kind in the protogalaxies \cite{29}.
Figure 5: The asymptotic form of the galactic radial density of the DM halo for $\bar{V} = 1$ (dotted line), $\bar{V} = 1.15$ (solid line) and $\bar{V} = 1.3$ (dashed-dotted line). Circles correspond to the Padé approximant for the density of the Milky Way DM halo as derived from Eq. (5). Initial DM mass is three times larger than the baryonic mass and we use $10^4$ particles. The radial distance $r$ is measured in units of $R_c = 10$ Kpc.
Figure 6: The same as Fig. 5 but for an initial distribution of velocities as given in Eq. 6. The dotted line corresponds to $\bar{v} = 1.2$ and $M_H = 3$ and the solid line corresponds to $\bar{v} = 1.3$ and $M_H = 4$. We used 500000 time steps to achieve the stationary state and $10^6$ DM particles. The unit of distance is $R_c = 10$ Kpc.
In Fig. 6 we have plotted the simulation results for an initial homogeneous distribution of $10^5$ DM particles moving with initial velocities drawn according to Eq. (6). The agreement with the radial density of our galaxy is specially good if we consider a primordial typical velocity $\bar{V} = 1.3V_c$, a mass of the DM halo four times larger than the mass of the baryonic matter.

In the next section we deduce the mass of DM particles from this typical velocity by considering that they cooled by the expansion of the Universe before being captured by the protogalaxies.

3 Hubble cooling of dark matter in the early Universe

In this section we discuss a kinetic model for weakly interacting massive particles or free Vlasov particles only interacting gravitationally as they move in a $\Lambda$CDM expanding Universe. We assume a flat Universe filled with the critical mass according to accepted concordance with the CMB observations [11, 13]. The space-time metric is given by:

$$ds^2 = d\tau^2 - a^2(\tau) \left[ dx^2 + dy^2 + dz^2 \right].$$  \hspace{1cm} (7)

A standard result of cosmological models [54, 55] is the following set of equations for the cosmological radius, $a(\tau)$, in terms of the energy density, $\epsilon$, and the pressure, $p$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^4} (\epsilon + 3p)$$ \hspace{1cm} (8)

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^4} \epsilon,$$  \hspace{1cm} (9)

where the dots denote successive derivatives with respect to $\tau$. The parameter $H = \dot{a}/a$ is the so-called Hubble constant whose value has been largely constrained by the WMAP project [11] and is currently accepted to be given by $H = 73$ km s$^{-1}$ Mpc$^{-1}$. From Eq. (8) we find that the energy density of the Universe is related with $H$ as $\epsilon = 3c^2H^2/8\pi G$. In the $\Lambda$CDM model, the energy content of the Universe comes essentially, from dark energy (with an equation of state $p = -\epsilon$) and dark matter or baryonic matter (whose pressure is negligible compared with the energy density and is, usually, ignored, $p = 0$). In percentage terms, WMAP results are compatible with the values:
\( \Omega_\Lambda = 0.72, \Omega_{\text{DM}} = 0.23 \) and \( \Omega_B = 0.05 \) for the fractions of dark energy, dark matter and baryonic matter, respectively.

Integration of the system in Eq. (8) yields:

\[
a^*(\tau) = \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda} \right)^{1/3} \sinh^{2/3} \left[ \frac{3\tau}{2\tau_\Lambda} \right], \tag{10}
\]

where \( a^*(\tau) \) is the cosmological radius at cosmological time \( \tau \) measured in units of the present radius and \( \tau_\Lambda = 1/(H\sqrt{\Omega_\Lambda}) \approx 15.8 \times 10^9 \) years. From the condition \( a^*(\tau_0) = 1 \) we get the age of the Universe, \( \tau_0 \approx 13.2 \times 10^9 \) years. The absolute curvature radius of the Universe at present, \( a(\tau_0) \), can be inferred from the deceleration parameter \([56, 57]\) but the value of \( a(\tau_0) \) is not necessary in our kinetic model.

Standard Big Bang Cosmology placed the era of formation of galaxies at the time corresponding to a temperature of the CMB \( k_B T \approx 3 \) eV. At that time, matter energy density exceeded radiation energy density and collapse around primordial fluctuations started \([14]\). The CMB temperature is governed by a red-shift law: \( T(\tau) = 2.7277/a^* \) K which in connection with Eq. (10) implies that the protogalaxies formed 12000 years after the Big Bang. If dark matter were composed by weakly interacting massive particles it is also generally accepted by cosmologists that decoupling took place at a CMB temperature \( k_B T \approx 1 \) MeV, i.e. 30 minutes after the Big Bang.

In our kinetic model, we consider that a DM particle colliding with a baryon (interacting via weak fields) at the time before galaxy formation thermalizes and, consequently, its kinetic energy raises to \( k_B T_{a^*} \) eV, because baryons and radiation are in thermal equilibrium. The total energy of a DM particle after an interaction is taken to be

\[
E = \mu c^2 + \frac{k_B T}{a^*}, \tag{11}
\]

where \( \mu \) is the rest mass of the DM particle. Linear momentum at that time is given by

\[
p^2 = \frac{k_B T}{a^*} \left[ 2\mu + \frac{k_B T}{a^* c^2} \right]. \tag{12}
\]

Linear momentum of a free particle in an expanding Universe decreases as \( a^{-1}(\tau) \) \([58]\). Consequently, the kinetic energy of a DM particle at time \( t \), that thermalized at time \( t' \) and never collided again, can be expressed as follows

\[
Q(t, t') = \sqrt{\mu^2 c^4 + \left( \frac{a^*(t')}{a^*(t)} \right)^2 \epsilon_{\text{rad}}(t') \left[ \epsilon_{\text{rad}}(t') + 2\mu c^2 \right] - \mu c^2}, \tag{13}
\]
where $\epsilon_{\text{rad}}(t) = \epsilon_0/a^*(t)$ and $\epsilon_0 = 2.35046 \times 10^{-4}$ eV is the average energy of a CMB photon in present day Planck spectrum. The average kinetic energy of a DM particle at time $t$, assuming that it was thermalized at an earlier time $t'$, is approximately given by

$$\langle K(t) \rangle = e^{-\langle \lambda \rangle(t,t')}(t-t')Q(t,t') + \int_\nu^t d\eta \lambda(\eta)e^{-\langle \lambda \rangle(\eta,t')(t-\eta)}Q(t,\eta),$$

(14)

where $\langle \lambda \rangle(t,t')$ is the average collision frequency in the time interval $(t',t)$ for a DM particle in the baryonic background. Notice that the first term gives us the contribution for those DM particles that do not collide in that interval of time and the second corresponds to DM particles colliding one of several times with baryons (with the most recent collision at time $\eta$).

Collision frequency for weakly interacting dark matter particles in a baryonic background is estimated as follows:

$$\lambda(t) = \sigma(E)n_B(t)v_B(t),$$

(15)

where $\sigma(E)$ is the cross section for weak interactions [15]:

$$\sigma(E) \approx (\hbar c)^{-4}G_F^2E^2.$$

(16)

For sterile neutrinos weak interactions are suppressed by a factor $\sin^2 \theta$, $\theta$ being the mixing angle with ordinary neutrinos [32]. In the oscillation production theories an upper limit has been derived in order to avoid the crowding of the early Universe with sterile neutrinos [59]: $\theta < 1.3 \times 10^{-4}(1 \text{ keV}/M_s)^{0.8}$, where $M_s$ is the mass of the sterile neutrino. In the early Universe the energy, $E$, is commonly identified with the CMB background energy scale, $E = \epsilon_0/a^*(t)$. $G_F = 1.166 \times 10^{-5}(\hbar c)^3 \text{ GeV}^{-2}$ is Fermi’s constant. The baryonic density decreases with the cube of the Universe radius:

$$n_B(t) = \frac{n_0}{a^*(t)},$$

(17)

using a present day baryon density, $n_0 \simeq 3$ baryons $\text{ m}^{-3}$ as readily deduced from the abundance $\Omega_B \simeq 0.05$ for baryons at the $\Lambda$CDM model. Finally, $v_B(t)$ is the relative velocity between DM particles and baryons. If we take it as the baryon velocity (a rough approximation) we get

$$v_B(t) = c\sqrt{\frac{\epsilon_0(t)(\epsilon_0(t) + 2m_Bc^2)}{\epsilon_0(t) + m_Bc^2}} \approx \sqrt{\frac{2\epsilon_0}{m_Bc^2}}a^{-1/2},$$

(18)
where \( m_B c^2 \simeq 939 \text{ MeV} \) is the baryon mass. Inserting Eqs. (18), (17) and (16) into the expression for the collision frequency in Eq. (15) yields

\[
\lambda(t) = (G_F \epsilon_0)^2 \langle n_0 \rangle \sqrt{\frac{2 \epsilon_0}{m_B}} a^{-11/2}.
\]  

(19)

From Eqs. (19) and (10) we can obtain the time average collision frequency as follows

\[
\langle \lambda \rangle(t, t') = \frac{1}{t - t'} \int_{t'}^t \lambda(t) dt.
\]  

(20)

This integral can be evaluated exactly in terms of hypergeometric functions yielding

\[
\langle \lambda \rangle(t, t') = \frac{2}{3} \frac{\tau_A}{t - t'} (G_F \epsilon_0)^2 \langle n_0 \rangle \frac{2 \epsilon_0}{m_B} \left( \frac{\Omega_A}{1 - \Omega_A} \right)^{11/6} \left[ \psi \left( \frac{3t}{2\tau_A} \right) - \psi \left( \frac{3t'}{2\tau_A} \right) \right],
\]  

(21)

where

\[
\psi(x) = \frac{1}{16} \cosh x \left[ 5e^{-2n/3} \, _2F_1 \left( 1/2, 1/3, 3/2, \cosh^2 x \right) \right. \\
+ \left. 3 \frac{5 \sinh^2 x - 2}{\sinh^{8/3} x} \right]
\]  

(22)

\(_2F_1\) being an hypergeometric function of order \((2, 1)\) [61].

In Fig. 6 we have plotted the typical velocity of DM particles as a function of its rest mass deduced from its kinetic energy in Eq. (14): \( v/c = \sqrt{\langle K(t) \rangle + 2 \langle K(t) \rangle \mu c^2 / \langle K(t) \rangle + \mu c^2} \). The initial time is approximately \( t_i = 1 \text{ second} \) after the Big Bang (corresponding to \( k_B T \simeq 150 \text{ MeV} \), the estimation for the decoupling era of sterile neutrinos produced via oscillations [28, 29]). Notice that the average kinetic energy will be given by Eq. (13) because sterile neutrinos are collisionless Vlasov particles after their decoupling. Collision frequency is very small indeed: if we consider that the mixing angle for sterile neutrinos is \( \theta \simeq 10^{-4} \), according to a recent estimation for active-sterile neutrino oscillations [59, 60], the average collision frequency in the interval \( t_i < t < 1.1t_i \), calculated from Eq. (21), is of the order of a collision every \( 10^7 \) years. In our scenario, the primordial typical velocities must be \( \simeq 300 \text{ km s}^{-1} \) which implies, according to Fig. (7), that their rest
Figure 7: Typical velocity of DM particles at the galaxies formation era as a function of its rest mass. Solid line is the result for sterile neutrinos (which we assume to decouple at a temperature corresponding to 150 MeV).
mass is roughly 3 keV, in good agreement with the estimation for the mass of sterile neutrinos. On the other hand, the results in Fig. 7 are very robust concerning the era of decoupling of sterile neutrinos or their mixing angle (we conclude that sterile neutrinos were free streaming particles almost from their appearance). Consequently, the decision about the scenario most suitable for the formation of sterile neutrino: off-resonance [28], on-resonance [29] or by the decay of heavy scalars [30, 31] is to be given by particle physics. On the other hand, a recent lower bound $m_{RP} = 2$ keV for the resonance production mechanism and $m_{NPR} = 1.77$ keV for the non-resonant mixing with active neutrinos obtained from a phase-space analysis of DM distribution in dwarf spheroidal galaxies [41] agrees with our estimate from the proposed dynamical self-gravitating model.

4 Conclusions and Remarks

We propose that dark matter trapped in galaxies is warm and evaporates from the galactic cores. The plausibility of this scenario is investigated in the context of a self-gravitation model for the halo of DM particles moving in the potential well of a spherical baryonic core.

The distribution of DM in galaxies is inferred from their rotation curves [2, 3, 4]. There is some agreement about that DM does not concentrate in the core of the galaxies, as one could naively expect, but their mass density peaks at a finite distance from the galaxies center – varying from a galaxy to another – and few DM particles are found inside the galactic core. This is confirmed by recent observations of 26 low surface brightness galaxies carried out by de Blok & Bosma [39] and by calculations of galactic density profiles inferred from rotation curves [38]. This is the so-called cuspy halo problem. Nevertheless, there is also a controversy about the interpretation of galactic rotation curves and some authors claim that central rotation velocities are underestimated by conventional techniques [62, 63].

In our theory, Milgroms’ law and the cuspy halo problem are related. We show that it can be explained within the dark matter paradigm by simply assigning a primordial typical velocity to dark matter particles within the order of 300 km s$^{-1}$. The self-gravitation model predicts the evolution towards a cuspy halo (with a cusp at a finite distance from the galactic center) starting from an initial homogeneous distribution. In this process an inner spherical region, with a radius similar to the radius of the core, appears. This region
is almost empty of DM particles and, consequently, anomalies in the orbits of visible matter are only detected at a critical radius corresponding to Milgrom’s orbital acceleration. Assuming that the velocity distribution of the dark matter gas was quenched at the time of the formation of galaxies, we have derived the typical mass of dark matter particles to fulfill the condition above. Their mass ($\sim 3$ keV) coincides with recent estimations for sterile neutrinos which arise in extensions of the standard model of elementary particles. In order to develop a better insight into the evolution of galaxies some simplifying assumptions used in our model could be removed in more realistic implementations, i.e., we can consider a rotating disk of baryonic matter in addition to the core, the parallel evolution of the halo of Vlasov particles and the dissipative clouds of baryons. Another problem is the dominance of DM over baryonic matter in dwarf galaxies with typical velocities in the flat region of rotation curves $\simeq 60$ km s$^{-1}$.[39]. An special mechanism could be considered in this case such as a WDM+CDM model or the accretion of low velocity DM particles in the outskirts of large galaxies nearby. With these features a better understanding on the formation of galaxies and, in turn, on the nature of DM could be obtained. Work along these lines is in progress and will be published elsewhere.

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