Quantum Double of Yangian of strange Lie superalgebra $Q_n$ and multiplicative formula for universal $R$-matrix

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Abstract. Main result is the multiplicative formula for universal $R$-matrix for Quantum Double of Yangian of strange Lie superalgebra $Q_n$ type. We introduce the Quantum Double of the Yangian of the strange Lie superalgebra $Q_n$ and define its PBW basis. We compute the Hopf pairing for the generators of the Yangian Double. From the Hopf pairing formulas we derive a factorized multiplicative formula for the universal $R$-matrix of the Yangian Double of the Lie superalgebra $Q_n$. After them we obtain coefficients in this multiplicative formula for universal $R$-matrix.

1. Introduction

The computation of universal $R$-matrix for the infinite dimensional Quantum Algebras be the one of the most important problems of Quantum Algebras theory for applications in mathematical physics and for the theory of exactly solvable models of statistical mechanics and quantum field theory. For example, construction of the transfer-matrix based on the determination of image of universal $R$-matrix of Quantum Double of Yangian by the action of tensor product of irreducible and identical representations. From the point of view of the theory of Yangians, the construction of the transfer matrix uses the image of the universal $R$-matrix of the Quantum Double of the Yangian under the action of the tensor product of an irreducible representation and the identity map. Computation of quantum $R$-matrix, spectrum of Hamiltonian and correlation functions is based also on the using universal $R$-matrix formula and description of irreducible representations of Yangian.

Let’s note that Yangians together with quantized enveloping algebras and elliptic quantum algebras are one of the three most important examples of quantum algebras. The notion Yangian was introduced by V. Drinfel’d in honour to C.N. Yang. But algebras isomorphic to Yangian were used for investigation exactly solvable models by means of Quantum Inverse Scattering Method in the 80-th years by V. Tarasov (see, for example, [31], [32]).
Yangian theory and representations theory of Yangians of simple and reductive Lie algebras is a developed theory, which began to develop until to appearance of term "Yangian"; as above mentioned (see [1], [10], [3], [4], [15], [16], [19]). In contrast with these theory, theory of Yangians of Lie Superalgebras is a young discipline, which appear at the beginning of 90-th years of 20 century ([17], [22]). But in the papers [33], [34], [26], [28], [30] had been done a description of finite dimensional irreducible representations in the most important cases, namely, for Yangian of general Lie superalgebra and special linear superalgebra. In the last years the number of applications of Theory of Yangians of Lie Superalgebras rise, to began research problems of Yang-Mills fields and Quantum Superstring Theory using Yangian Theory technique (see [2], [6], [20], [21]). The notion of universal R-matrix for the most important classes of quasitriangular Hopf algebras: quantized introduced by V. Drinfeld ([7], [8]). V. Drinfel’d also formulated the problem of calculating the universal enveloping algebras, quantum affine algebras and Yangians.

I recall that universal R-matrix of the Yangian Y(g) of the simple Lie algebra g was introduced by V.G. Drinfel’d (see [7], [8]) as a formal power series R(λ) = 1 + ∑∞ k=0 Rk · λ−k−1 with coefficients Rk ∈ Y(g)⊗2, which conjugates the comultiplication ∆ and opposite comultiplication ∆′ = τ ∘ ∆, τ(x ⊗ y) = (−1)p(x)p(y)(y ⊗ x) for Lie superalgebras. Explicit definitions will be given later in the text of this article.) More exactly, R(λ) conjugate the images ∆ and ∆′ under action operator id ⊗ Tλ, where Tλ is a quantum counterpart of shift operator, and id be an identical operator. The R(λ) behave as it is image of some hypothetical R-matrix R under action id⊗Tλ conjugates ∆ and ∆′. Drinfel’d called such formal power series R(λ) the pseudotriangular structure and proved it existence for Y(g), when g be a simple Lie algebra. But explicite formula for R(λ) it hasn’t received up to now. If we shall see on classical counterparts of notions R(λ) and Rk, namely, on classical r-matrices r(λ) and r, then r be an element of a topological tensor square of a classical double and of a classical quantum double. Then we can naturally to expect that and in the quantum case R(λ) will be an image of universal R-matrix R of Quantum Double of Yangian under the action of some shift operator. When V.Drinfel’d defined pseudotriangular structure he didn’t know good description of Yangian Double in terms of generators and defining relations and universal R-matrix of Yangian Double. But in the middle of 90-th S.Khoroshkin and V.Tolstoy received the description of Yangian Double in the terms of generators and defining relations and they computed the multiplicative formula of universal R-matrix of Yangian Double ([13]). The multiplicative formula for universal R-matrix of Quantum Double of Yangian Y(A(m, n)) of special linear superalgebra A(m, n) was obtained in the papers [23], [24], [26]. These works was based on ideas of papers [13], [14].

The Yangian of the strange Lie superalgebra was introduced by M. Nazarov (see [18]) in the middle of 90-th years of last century. In the paper [25] the Yangian of the strange Lie superalgebra was defined in the terms of the Drinfel’d generators and in [27] Yangian of the strange Lie superalgebra was defined within a more general approach. The Quantum Double of Drinfel’d Yangian of the strange Lie superalgebra was introduced in [29]. The following natural problem is to compute the universal R-matrix of Quantum Double DY(Qn) is solving in this article. Let’s note that in the papers [28], [30] was obtained the classification of the irreducible finite dimensional representations of Yangian Y(A(m, n)) of special linear superalgebra A(m, n). (Let’s note that A(m, n) is an one more analog of general linear Lie algebra gln , along with the strange Lie superalgebra Qn). But similar classification for finite dimensional representations of Yangian Y(Qn) of strange Lie superalgebra yet hasn’t been obtain in present time. When such classification will be obtained we can be systematically describe all possible quantum R-matrices and transfer matrices connecting with Yangian symmetries of Y(Qn) type.

Some words about structure of this article. In the second section we recall definition of the Lie Superalgebra Qn in terms of root generators and defining relations. After them we define of the Yangian of Lie Superalgebra Qn in terms of generators and defining relations. We define
it in terms of current system of generators as in the author’s paper [28]. We also introduce Quantum Double $DY(Q_n)$ of Yangian of the strange Lie superalgebra. In the third section we recall construction of PBW bases and give a sketch of the proof of this theorem and describe the normal (convex) orders on the root generators of Yangian and its Quantum Double. In the forth section we formulate pairing formula for root generators of Quantum Double $DY(Q_n)$. In last section we formulate the main theorem – the multiplicative formula for the universal $R$-matrix of Quantum Double $DY(Q_n)$.

We’ll use the following notations. We denote by $\mathbb{C}$ the field of complex numbers, by $K[u]$ the ring of polynomials with coefficients in ring $K$, by $\mathbb{Z}_+ = \{0, 1, 2, \ldots \}$ the set of non negative integers, which is union of zero and set of natural numbers $\mathbb{N}$, and by $\mathbb{Z}$ the set of integer numbers.

2. Definition of Yangian of Lie Superalgebra $Q_n$

2.1. Strange Lie Superalgebra $Q_n$

Let me recall basic definitions from Lie Superalgebra theory, related to the Lie Superalgebra $Q_n$ (see also [11], [12]).

Let $C(n|n) = \mathbb{C}^n \oplus \mathbb{C}^n$ be a $\mathbb{Z}_2$-graded $(n|n)$ dimensional vector space (superspace) over complex numbers field $\mathbb{C}$. Let’s choose a standard basis in this superspace. We’ll numerate the elements of this basis by integer numbers from segment $[−n, n]$ (without zero).

So, let $(e_{−n}, e_{−n+1}, \ldots, e_{−1}, e_{1}, \ldots, e_n)$ be a standard basis in $C(n|n)$, and $End(C(n|n))$ be a superalgebra of linear operators acting in $C(n|n)$. Basis in $End(C(n|n))$ is formed by matrices $E_{a,b}, −n \leq a, b \leq n, ab \neq 0$ and parity $p$ for $E_{a,b}$ is defined by formula:

$$p(E_{a,b}) = |a| + |b|,$$

where $|a| = p(a) = 0$, if $a < 0$ and $|a| = p(a) = 1$, for $a > 0$, $|a|, |b| \in \mathbb{Z}_2$.

Recall, that general linear Lie superalgebra $gl(n|n)$ is defined as vector space $End(C(n|n))$ with (super)bracket $[,]$ defined by formula:

$$[x, y] = x \cdot y - (-1)^{|x||y|} y \cdot x.$$  \hfill (2)

Super trace on $gl(n|n)$ defined by formula $str(E_{ab}) = \delta_{ab}(-1)^{1+|a|}$ on base elements and extends linearly to all other elements of the vector $gl(n|n)$.

Recall also a definition of special linear Lie superalgebra $sl(n|n)$:

$$sl(n|n) = \{ A \in gl(n|n)| str(A) = 0 \}.$$  \hfill (3)

Let

$$\sigma : gl(n|n) \to gl(n|n),$$

be an involutive automorphism given by the formula:

$$\sigma : E_{a,b} \mapsto E_{−a,−b}.$$ 

Let’s note that $\sigma$ maps $sl(n|n)$ into $sl(n|n)$. Let’s note also that on $sl(n, n)$ is defined nondegenerate bilinear invariant form $(\cdot, \cdot)$, given by formula:

$$(A, B) = str(AB).$$  \hfill (4)

The form (4) is different from Killing form.

Let also

$$J = \sum_{i=1}^{n} (E_{i,i} - E_{−i,i}).$$
Definition 1 Lie superalgebra $q_n$ can be defined by the two following equivalent ways: as centralizer of $J$ in $gl(n|n)$, or as the fixed points set of involutive automorphism $\sigma : gl(n|n) \rightarrow gl(n|n)$. Let also $sq_n = [q_n, q_n]$.

Let’s note, that $sq_n$ be a Lie subsuperalgebra of Lie superalgebra $q_n$, consisting from $2n \times 2n$-matrices which left lower and upper right $n \times n$-blocks have a zero trace. Easy to see, that $sq_n$ contains matrix $E_{2n}$.

Note, that Lie superalgebra $sq_n$ is denoted also by $\tilde{Q}_{n-1}$.

Simple Lie superalgebra $Q_{n-1}$ is defined (see [12]) as factor superalgebra of $sq_n$ by one-dimensional center:

$$Q_{n-1} = sq_n/C_{2n}.$$ 

Let’s introduce the following notations

$$\tilde{h}_{i,0} := E_{i,i} + E_{-i,-i}, \quad h_{i,0} := \tilde{h}_{i,0} - h_{i+1,1}, \quad (5)$$
$$\tilde{h}_{i,1} := E_{i,-i} + E_{-i,i}, \quad h_{i,1} := \tilde{h}_{i,1} - h_{i+1,1}, \quad (6)$$
$$x^+_{i,0} := E_{i+1,i} + E_{-i,-i}, \quad x^-_{i,0} := E_{i+1,i} + E_{-i,-i-1}, \quad (7)$$
$$x^+_{i,1} := E_{i+1,-i} + E_{-i,i}, \quad x^-_{i,1} := E_{i,-i-1} + E_{-i,i+1}. \quad (8)$$

Proposition 1 Lie superalgebra $sq_n$ is isomorphic to Lie superalgebra $g$, generated by generators $x^\pm_{i,j}, h_{i,1}, 1 \leq i \leq n-1, j = 0, 1$ and with second index $j$ are odd and with second index $j = 0$ are even. These generators are satisfy the following system of defining relations:

$$[h_{i,1}, h_{i,2}] = 2\delta_{i,i_2} (h_{i,0} + h_{i+1,0}) - 2\delta_{i_1,i_2} h_{i,2}, \quad i_1 \leq i_2;$$
$$[h_{i,j}, h_{i,j_2}] = 0, \quad (j_1, j_2) \neq (1, 1),$$
$$[h_{i,0}, x^\pm_{i,j,2}] = \pm (\delta_{i,j_2} - \delta_{i,j_2+1}) x^\pm_{i,j_2},$$
$$[h_{i,1}, x^\pm_{i,j_2}] = (\pm 1)^j \delta_{j,j_2} (2\delta_{i,j_2} - (j_1 - j_2) \delta_{i,j_2+1} - \delta_{i,j_2+1} - \delta_{i,j_2+1} - \delta_{i,j_2+1}) x^\pm_{i,j_2},$$
$$[x^+_{i,j_1}, x^-_{i,j_2} = \delta_{j,j_2} (h_{i,0} - (j_1 + j_2 + 1) x^\pm_{i,j_2},$$
$$[x^\pm_{i,j_1}, x^\pm_{i,j_2}] = \delta_{j,j_2} (h_{i,0} - (j_1 + j_2 + 1) x^\pm_{i,j_2},$$
$$[x^\pm_{i,j_1}, x^\pm_{i,j_2}] = (\pm 1)^{j_1 + j_2} [x^\pm_{i,j_1}, x^\pm_{i,j_2}], \quad i_1 \leq i_2;$$
$$ad(x^\pm_{i,j_1})^2(x^\pm_{i,j_2}) = 0, \quad 1 \leq i_1, i_2 \leq n - 1, \quad j_1, j_2 = 0, 1, \quad |i - i_2| = 1,$$
$$[x^\pm_{i,j_1}, x^\pm_{i,j_2}] = 0, \quad 1 \leq i_1, i_2 \leq n - 1, \quad j_1, j_2 = 0, 1, \quad |i - i_2| \neq 1.$$

The definitions of loop algebra $A(n,n)[u, u^{-1}]$ and twisted loop algebra $A^{(2)}(n,n)[u, u^{-1}]$ are similar (see [25], [29]).

Let be below $g = Q_n$.

2.2. Definition of Yangian

First, we recall the definition of Yangian $Y(g) = Y(Q_n)$ (see also [25], [29]).

Yangian, as other quantum algebras, is defined as a result of deformation (in the case of Yangian) of universal enveloping (super)algebra of current Lie superalgebra (with values in Lie superalgebra) in the class of Hopf superalgebras. In the paper [25] was shown that this definition is equivalent other definition using so called current system of generators. This system of generators is convenient for representation theory of Yangians. Therefore we’ll consider it further.
**Definition 2** Let $Y(g)$ be an associative superalgebra (over complex number field $\mathbb{C}$), generated by generators $\tilde{h}_{i,j,m}$, $\tilde{h}_{i,j,m} = \tilde{h}_{i,j,m} - \tilde{h}_{i+1,j,m}$, $x^\pm_{i,j,m}$, $i \in I = \{1, \ldots, n-1\}$, $m \in \mathbb{Z}_+$, $j \in \mathbb{Z}_2 = \{0,1\}$, $p(x^\pm_{i,1,m}) = 1$, $p(x^\pm_{i,0,m}) = \tilde{h}_{i,j,m} = \tilde{h}_{i,j,m} = 0$, which satisfied the following defining relations:

$$[\tilde{h}_{i,0,m_1}, \tilde{h}_{i,0,m_2}] = 0,$$

$$[\tilde{h}_{i,1,2m_1}, \tilde{h}_{i,2,2m_2}] = 0,$$

$$[\tilde{h}_{i,1,2m_1+1}, \tilde{h}_{i,2,2m_2+1}] = (-1)^i2\delta_{i_1,i_2}(\tilde{h}_{i,0,2(m_1+m_2+t)} + \tilde{h}_{i,1,2(m_1+m_2+t)}) - 2(-1)^i(\delta_{i_1,1,i_2} + \delta_{i_1-1,i_2})\tilde{h}_{i,2,0,2(m_1+m_2+t)},$$

$$[\tilde{h}_{i,1,2m_1+1}, \tilde{h}_{i,2,2m_2}] = [\tilde{h}_{i,1,2m_1}, \tilde{h}_{i,2,2m_2+1}] = 2\delta_{i_1,i_2}(\tilde{h}_{i,1,2(m_1+m_2+1)} + \tilde{h}_{i,1,2(m_1+m_2+1)} - 2(\delta_{i_1+1,i_2} + \delta_{i_1-1,i_2})\tilde{h}_{i,2,0,2(m_1+m_2+1)},$$

$$[\tilde{h}_{i,1,2m_1+1}, \tilde{h}_{i,2,2m_2}] = [\tilde{h}_{i,1,2m_1}, \tilde{h}_{i,2,2m_2+1}] = 0,$$

$$[x^\pm_{i_1,j_1,m_1}, x^\pm_{i_2,j_2,m_2}] = [x^\pm_{i_1,j_1,m_1}, x^\pm_{i_2,j_2,m_2+1}] = \frac{\alpha_{i_1,i_2}}{2}(\tilde{h}_{i,0,m_1}x^\pm_{i_2,j_2,m_2+1} + x^\pm_{i_2,j_2,m_2}h_{i,0,m_1}),$$

$$[x^\pm_{i_1,j_1,m_1}, x^\pm_{i_2,j_2,m_2+1}] = \frac{\alpha_{i_1,i_2}}{2}(x^\pm_{i_1,j_1,m_1}x^\pm_{i_2,j_2,m_2} + x^\pm_{i_2,j_2,m_2}x^\pm_{i_1,j_1,m_1}).$$

$$[x^\pm_{i_1,j_1,m_1}, x^\pm_{j_1,r+1,r}] = \pm \frac{\alpha_{i_1,j_1}}{2}(x^\pm_{i_1,j_1,m_1}x^\pm_{j_1,r}, x^\pm_{i_1,j_1,m_1}),$$

$$[x^\pm_{i_1,j_1,m_1}, x^\pm_{i_2,j_2,m_2}] = [x^\pm_{i_1,j_1,m_1}, x^\pm_{i_2,j_2,m_2+1}] = \frac{\alpha_{i_1,i_2}}{2}(\tilde{h}_{i,1,1,m_1}x^\pm_{i_2,j_2,m_2+1} + x^\pm_{i_2,j_2,m_2}h_{i,1,1,m_1}),$$

$$[[x^\pm_{i_1,j_1,0}, x^\pm_{i_2,j_2,m_2}], x^\pm_{i_3,j_3,m_3}] + [x^\pm_{i_1,j_1,0}, [x^\pm_{i_2,j_2,m_2}, x^\pm_{i_3,j_3,m_3}]] = 0,$$

$$[x^\pm_{i_1,1,0}, x^\pm_{i_1,1,0}] = 0, \quad t_1 \in \mathbb{Z}_+.$$

2.3. **Definition of Yangian Double**

I recall the definition of a Quantum Double of Hopf superalgebra. Let $A$ be an arbitrary Hopf superalgebra. Then the quantum double $DA$ of the Hopf superalgebra $A$ is a quasitriangular Hopf superalgebra $(DA, R)$ such that $DA$ contains $A$, $A^0$ as Hopf subsuperalgebras; $R$ is the image of the canonical element in $A \otimes A^0$, corresponding to the unit operator, under the embedding of $A \otimes A^0$ in $DA \otimes DA$. Linear mapping $A \otimes A^0 \to DA$, $a \otimes b \to ab$

be an isomorphism.

We describe Quantum Double $DY(Q_n)$ of Yangian $Y(Q_n)$. First step, we describe the dual Hopf superalgebra $Y^*(Q_n)$ to Yangian $Y(Q_n)$. We’ll denote the Hopf superalgebra $Y^*(Q_n)$ with the opposite comultiplication by $Y^0(Q_n)$. We actually describe the structure of the associative superalgebra on $Y^0(Q_n)$. This description will be based on the pairing formulas between the root generators of the superalgebra $Y(Q_n)$ that form the Poincare-Birkhoff-Witt basis and the dual generators of the superalgebra $Y^0(Q_n)$. But first we carried out the computation between the generators of the Hopf superalgebra $Y(Q_n)$ and the dual superalgebra $Y^0(Q_n)$.
Theorem 1 The Quantum Double $DY(Q_n)$ of Yangian $Y(Q_n)$ be a superalgebra (over complex number field $\mathbb{C}$), generated by generators $\tilde{h}_{1,j,m}, \tilde{h}_{i,j,m} = \tilde{h}_{i,j,m} - \tilde{h}_{i+1,j,m}, \quad x_{i,j,m}^+, \quad i \in I = \{1, \ldots, n-1\}, \quad m \in \mathbb{Z}, \quad j \in \mathbb{Z}_2 = \{0, 1\}$, which satisfied the following above defining relations (9) - (21) (where $m \in \mathbb{Z}$).

3. Root vectors, Poincare-Birkhoff-Witt theorem and normal orders
First of all, we describe in more detail the Poincare-Birkhoff-Witt basis (PBW-basis) for $DY(Q_n)$ (see [29]). Let, as above $\Delta, \Delta_+$ denotes the set of roots, respectively, the set of positive roots of the Lie superalgebra $A(n-1, n-1)$. We also consider the set of real roots of the corresponding affine twisted Lie superalgebra $A(n-1, n-1)(\mathbb{R})$ (see [11]) $\hat{\Delta}^{re}$. For the generators $x_{i,j,k}^{\pm}$ of the quantum double $DY(Q_n)$ we use the following notation:

$$x_{\alpha_i+k\delta,0} = x_{\alpha_i+k\delta} := x_{i,0,k}^+, \quad x_{-\alpha_i+k\delta,0} = x_{-\alpha_i+k\delta} := x_{i,0,k}^-;$$

$$x_{\alpha_i+k\delta,1} = x_{\alpha_i+k\delta} := x_{i,1,k}^+, \quad x_{-\alpha_i+k\delta,1} = x_{-\alpha_i+k\delta} := x_{i,1,k}^-; i \in I, k \in \mathbb{Z}, \alpha_i \in \Delta_+.$$

Here (as above), we denote by $\hat{\alpha}$ an odd root in the root system $Q_n$ that is an odd double of the even root $\alpha$.

Note that in this case $\pm \alpha_i + k\delta \in \hat{\Delta}^{re}$. Let $\Xi \subset \hat{\Delta}^{re}$. The linear order $\preceq$ on $X \iota$ is said to be convex (normal) if for any roots $\alpha, \beta, \gamma \in \Xi$ and such that $\gamma = \alpha + \beta$ has place one of the following two relations of order:

$\alpha \preceq \gamma \preceq \beta \Leftrightarrow \beta \preceq \gamma \preceq \alpha$.

Let's introduce the subsets $\Xi_+, \Xi_-$ of set $\hat{\Delta}^{re}$:

$$\Xi_\pm := \{ \gamma + k\delta : \gamma \in \hat{\Delta}^{re}_+ \}.$$

We introduce on $\Xi_+, \Xi_-$ convex orders $\preceq_+, \preceq_-$, satisfying the following conditions:

$$\gamma + k\delta \preceq_+ \gamma + l\delta \quad \text{and} \quad -\gamma - l\delta \preceq_- -\gamma - k\delta,$$

if $k \leq l$, for $\forall \gamma \in \Delta_+$.

We now define the root vectors $x_{\pm \beta}, \beta \in \Xi_+ \cup \Xi_-$ by induction as follows. Let the vectors $x_{\beta_1}, x_{\beta_2}$ have already been constructed. If the root $x_{\beta_3}$ is such that:

$$x_{\beta_1} \preceq_+ x_{\beta_3} \preceq_+ x_{\beta_2}$$

and there are no roots in the interval $(x_{\beta_1}, x_{\beta_2})$ for which the root vectors are already constructed, then we define the root vector $x_{\pm \beta_3}$ by the formulas:

$$x_{\beta_3} = [x_{\beta_1}, x_{\beta_2}], \quad x_{-\beta_3} = [x_{-\beta_2}, x_{-\beta_1}].$$

We note that a convex (normal) order is associated with the natural ordering of elements of the affne Weyl group. We need the following description of $DY(Q_n)$, which is an analog of the description of a twisted quantized affine superalgebra. First, we fix the following normal order on roots of $A(n, n)$:

$$(\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \ldots, \epsilon_1 - \epsilon_{2n+2}), (\epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \ldots, \epsilon_2 - \epsilon_{2n+2}), \ldots (\epsilon_{2n+1} - \epsilon_{2n+2}).$$
Here $\epsilon_i - \epsilon_j = \alpha_i + \cdots + \alpha_j$.

We add the affine root $\alpha_0 = \delta - \theta$ to the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, \alpha_{2n-1}$. Here $\theta := \alpha_1 + \cdots + \alpha_{2n-1} = \epsilon_1 - \epsilon_{2n-1}$ is the highest root, $\delta$ is the minimal imaginary root. Consider the following normal order on the set $\hat{A}^{re}$ of affine real roots:

$$(\alpha_1, \alpha_1 + \delta, \alpha_1 + 2\delta, \ldots, \alpha_1 + m\delta, \ldots), (\alpha_1 + \alpha_2 + m\delta, \ldots, \alpha_1 + \alpha_2 + \delta, \alpha_1 + \alpha_2),$$

$$(\alpha_1 + \alpha_2 + \alpha_3 + m\delta, \ldots, \alpha_1 + \alpha_2 + \alpha_3 + \delta, \alpha_1 + \alpha_2 + \alpha_3), \ldots, (\alpha_{2n-1} + n\delta, \ldots, \alpha_{2n-1} + \delta, \alpha_{2n-1}).$$

4. Triangular decomposition and pairing formulas

We need explicit pairing formulas in the Quantum Double $DY(Q_n)$ of Yangian of the strange Lie superalgebra.

In this section we consider an analog for Yangian and Yangian Double of a Gaussian decomposition for a Lie superalgebra (and also a Lie superalgebra), and also for its universal enveloping algebra (superalgebra).

Let $Y'_+,$ $Y'_0,$ $Y'_-$ be subsuperalgebras (without unit) of $Y(Q_n')$ are generated, correspondingly, by elements $x_{i,0,k}^+ = x_{i,k}^+,$ $x_{i,1,k}^+$ $h_{i,0,k} = h_{i,k},$ $h_{i,1,k} = k_{i,k}$ and $x_{i,0,k}^- = x_{i,k}^-,$ $x_{i,1,k}^- = \bar{x}_{i,k}^-$ ($(i \in I, k \in \mathbb{Z}_+)$. Let $Y'_+,$ $Y'_0,$ $Y'_-$ be unital subsuperalgebras resulting from subsuperalgebras $Y'_+,$ $Y'_0,$ $Y'_-$ adding to units.

**Proposition 2** Multiplication in $Y(Q_n)$ induces an isomorphism of vector superspaces:

$$Y'_+ \otimes Y'_0 \otimes Y'_- \to Y(Q_n).$$

We are going to describe the the Hopf superalgebra $Y^0(Q_n)$ which is dual Hopf superalgebra $Y^*(Q_n)$ (with opposite comultiplication) to the Yangian $Y(Q_n)$. We first describe the structure of the associative superalgebra on $Y^0(Q_n)$. This description will be based on the pairing formulas between the root generators of the superalgebra $Y(Q_n)$ that form the Poincare-Birkhoff-Witt basis and the dual generators of the superalgebra $Y^0(Q_n)$. But first we calculate pairing between the generators of the Hopf superalgebra $Y(Q_n)$ and the dual generators of superalgebra $Y^0(Q_n)$. We note that the pairing between these subsuperalgebras is a Hopf pairing and the properties of the Hopf pairing are at the heart of the computations of defining relations of the dual Yangian $Y^0(Q_n)$. We calculate the pairing formulas for the root vectors. Let

$$h_{i,0,k}^*, e_{i,0,k}^*, f_{i,0,k}^*, h_{i,1,k}^*, e_{i,1,k}^*, f_{i,1,k}^*$$

are generators of subsuperalgebra $Y^* = Y_-$. Let

$$e_{i,0,k} := x_{i,0,k}^+, f_{i,0,k} := x_{i,0,k}^-,$$ $e_{i,1,k} := x_{i,1,k}^+, f_{i,1,k} := x_{i,1,k}^-,$

are generators of Yangian $Y = Y(Q_n)$. Then we have the following proposition

**Proposition 3** The following two conditions are equivalent.

1) $$(e_{i,0,k}, e_{j,0,-l-1}) = -\delta_{i,j}\delta_{k,l};$$ $(f_{i,0,k}, f_{j,0,-l-1}^*) = -\delta_{i,j}\delta_{k,l};$

$$(h_{i,0,k}, h_{j,0,-l-1}^*) = -\frac{a_{ij}k!}{l(k-l)!};$$ $(e_{i,1,k}, e_{j,1,-l-1}^*) = -\delta_{i,j}\delta_{k,l};$

$$\langle f_{i,1,k}, f_{j,1,-l-1}^* \rangle = -\delta_{i,j}\delta_{k,l};$$ $\langle h_{i,1,k}, h_{j,1,-l-1}^* \rangle = -\frac{a_{ij}s!}{l(s-l)!}$ for $s \geq l \geq 0.$
Here we consider two Hopf superalgebra structures on $A$.

First, we recall that a universal $R$-matrix of Quantum Double $D\bar{Y}(Q_n)$

5. Computation of Universal $R$-matrix of Quantum Double $D\bar{Y}(Q_n)$

5.1. Computation of Universal $R$-matrix of Quantum Double $D\bar{Y}(Q_n)$.

General plan

Here we consider two Hopf superalgebra structures on $Y(Q_n)$ and study relation between these structures.

First, we recall that a universal $R$-matrix for a quasitriangular Hopf superalgebra $A$ is an invertible element $R$ lying in some completion of the tensor product $A \hat{\otimes} A$ and satisfying the following conditions:

\[
\Delta^\text{op}(x) = R \Delta(x) R^{-1}, \quad \forall x \in A;
\]

\[
(\Delta \otimes \text{id}) R = R^{13} R^{23}, \quad (\text{id} \otimes \Delta) R = R^{13} R^{12},
\]

where $\Delta^\text{op} = \sigma \circ \Delta, \sigma(x \otimes y) = (-1)^{p(x)p(y)} y \otimes x, R^{12} = \text{id} \otimes \text{id}, R^{23} = \text{id} \otimes R, R^{13} = \sum a_i \otimes \text{id} \otimes b_i$, if $R = \sum a_i \otimes b_i$. 

\[
\delta_{ij}(-1)^k (\tilde{h}^*_{i,0,k+l+1} + \tilde{h}^*_{i+1,0,k+l}) + \delta_i + 1, j \tilde{h}^*_{j,1,k+l}, \quad k - l \in 2\mathbb{Z} + 1,
\]

\[
[\tilde{h}^*_{i,1,k}, \tilde{h}^*_{j,1,l}] = \delta_{ij}(-1)^k (\tilde{h}^*_{i,0,k+l+1} + \tilde{h}^*_{i+1,0,k+l}) + \delta_i + 1, j \tilde{h}^*_{j,1,k+l}, \quad k - l \in 2\mathbb{Z} + 1,
\]

\[
[\tilde{h}^*_{0,0,k}, \tilde{h}^*_{i,0,k+l}] = \delta_{ij}(-1)^k (\tilde{h}^*_{0,0,k+l+1} + \tilde{h}^*_{0,0,k+l+1}) + \delta_i + 1, j \tilde{h}^*_{j,1,k+l}.
\]
We recall, also, the formula of universal $R$-matrix of Quantum Double $DA$ of topological Hopf superalgebra $A$:

$$R = \sum_i e_i \otimes e^i,$$

(23)

here $\{e_i\}$ is a basis in topological Hopf superalgebra $A$, a $\{e^i\}$ is a dual basis in the Hopf superalgebra $A^0$. Convergence here is understood in a $\hat{h}$-adic topology. We will also use the triangle decomposition of the Yangian described above, and the induced decomposition of the quantum double of Yangian $DY(Q_n)$:

$$DY(Q_n) \cong Y_+ \otimes \hat{Y} \otimes Y^0 \otimes \hat{Y}^0_+.$$

(24)

Here the isomorphism is induced simply by multiplication in the quantum double. A tensor product is a topological tensor product.

5.2. Computation of Universal $R$-matrix

First, we reformulate the above definition of the quantum double $DY(Q_n)$ in a more convenient form for us. To describe $DY(Q_n)$, it is convenient to use the following generating functions:

$$x^\pm_{i,j}(u) = \sum_{k \geq 0} x^\pm_{i,j,k} u^{-k-1}, \quad x^\pm_{i,j}(u) = -\sum_{k < 0} x^\pm_{i,j,k} u^{-k-1},$$

$$h^+_{i,j}(u) = 1 + \sum_{k \geq 0} h_{i,j,k} u^{-k-1}, \quad h^-_{i,j}(u) = 1 - \sum_{k < 0} h_{i,j,k} u^{-k-1}, \quad j \in \mathbb{Z}_2.$$

First we formulate a theorem that gives an idea of the general structure of the multiplicative formula for the universal $R$-matrix of the quantum double of the Yangian of the strange Lie superalgebra. We note that here we understand the tensor product of superalgebras as a topological tensor product. We will also work in the category of topological superalgebras. We understand the formula itself as a formal series.

**Theorem 2** Let $g = Q_n$. The universal $R$-matrix $R$ of the Quantum Double $DY(Q_n)$ can be can be represented in the following factorized form

$$R = R_+ R_0 R_-,$$

(25)

where

$$R_+ = \prod_{\beta \in \Sigma_+} \exp\left(-(-1)^{\theta(\beta)} a(\beta) e_\beta \otimes e_{-\beta}\right),$$

(26)

$$R_- = \prod_{\beta \in \Sigma_-} \exp\left(-(-1)^{\theta(\beta)} a(\beta) e_\beta \otimes e_{-\beta}\right),$$

(27)

where the product is taken in accordance with the above defined normal orders $\Sigma_+, \Sigma_-$, which is constructed from the root system of the affine Lie superalgebra $A(n,n)^{(2)}$. In addition, this ordering is convex.

The normalizing constants $a(\beta)$ can be found from the following conditions:

$$[e_\beta, e_{-\beta}] = (a(\beta))^{-1} h_\gamma \quad e\mathbb{C} \mathbb{A} \quad \beta = \gamma + n\delta \in \Sigma_+, \gamma \in \Delta_+(A(n,n)^{(2)}),$$

$$[e_{-\beta}, e_\beta] = (a(\beta))^{-1} h_\gamma \quad e\mathbb{C} \mathbb{A} \quad \beta = -\gamma + n\delta \in \Sigma_-, \gamma \in \Delta_+(A(n,n)^{(2)}),$$

and $\theta(\beta) = \text{deg}(e_\beta) = \text{deg}(e_{-\beta})$ denotes parity of element $e_{\pm \beta}$.

The middle term has more complicated structure.
Theorem 3 Let $\varphi_i^\pm(u) = \ln(h_i^\pm(u))$, $\tilde{\varphi}_i^\pm(u) = \ln(h_i^\pm(u))$, $A = (a_{ij})_{i,j=1}^n$ be a symmetric Cartan matrix for Lie algebra $A_n$, $A = a_{ij}(q) = [a_{ij}]_q = \frac{q^{a_{ij}} - q^{-a_{ij}}}{q - q^{-1}}$, $A(q) = (a_{ij}(q))_{i,j=1}^n$. Let also $C(q) = (c_{ij}(q))_{i,j=1}^n$ be a matrix with elements proportional to corresponding elements of matrix $A(q)^{-1}$ and $T$ be a shift operator: $Tf(v) = f(v+1)$. Similarly, $\tilde{A} = (\tilde{a}_{ij})_{i,j=1}^n$, $\tilde{a}_{ij} = (\alpha_i, \alpha_j) = \delta_{i,j+1} - \delta_{i+1,j}$, $\tilde{a}_{ij}(q) = [\tilde{a}_{ij}]_q = \frac{q^{\tilde{a}_{ij}} - q^{-\tilde{a}_{ij}}}{q - q^{-1}}$, $\tilde{A}(q) = (\tilde{a}_{ij}(q))_{i,j=1}^n$. $\tilde{C}(q) = (\tilde{c}_{ij}(q))_{i,j=1}^n$ is a matrix proportional to the matrix $\tilde{A}(q)^{-1}$. Then

$$R_0 = R_0^e R_0^o = \prod_{m \geq 0} \left( \exp \left( \sum_{1 \leq i,j \leq n} \text{Res}_{u,v}(\varphi_{ij}^+(u)) \otimes c_{ij}(T^{-1/2}) \varphi_{ij}^-(v + (n + 1/2))) \right) \right),$$

(28)

where $c_{ij}(T^{-1/2})$, $\tilde{c}_{ij}(T^{-1/2})$ substitutes $T^{-1/2}$ into $c_{ij}(q)$, $\tilde{c}_{ij}(q)$, respectively, instead of $q$.

We briefly describe the plan for calculating the term $R_0$. We show that it is also factorized and can be represented in the form of the product $R_0 = R_0^e \cdot R_0^o$. The proof of this fact is based on the fact that the pairing between the even generators of the Cartan subsuperalgebra $h_{i,0,k}$ and the odd generators $h_{i,k}, i \in I, k \in \mathbb{Z}_+$ in the Cartan subsuperalgebra dual in the quantum double is zero. Therefore, we can carry out separate calculations for the terms $R_0^e$ and $R_0^o$. The scheme of reasoning in both cases is very similar and essentially coincides with the calculation scheme realized in the papers [23], [24], although the formulas themselves differ somewhat.

To describe the terms $R_0^e$ and $R_0^o$, we introduce the following notions.

Along with the Yangian Double $DY(g)$, we consider the Hopf superalgebra $\hat{DY}(g)$, which is isomorphic as an associative superalgebra to the Quantum Double $DY(g)$, but with a different structure of the cosuperalgebra determined by the following simple formulas:

$$\Delta(h_{ij}^\pm(u)) = h_{ij}^\pm(u) \otimes h_{ij}^\pm(u),$$

(29)

$$\Delta(e_{ij}(u)) = e_{ij}(u) \otimes 1 + h_{ij}^-(u) \otimes e_{ij}(u),$$

(30)

$$\Delta(f_{ij}(u)) = 1 \otimes f_{ij}(u) + f_{ij}(u) \otimes h_{ij}^+(u).$$

(31)

We can to check that comultiplications $\Delta$ and $\hat{\Delta}$ are conjugated by the following operator

$$\hat{t}^\infty := \lim_{n \to \infty} \hat{t}^n, \quad \hat{t}(e_{ij,k}) = e_{ij,k+1}, \quad \hat{t}(f_{ij,k}) = f_{ij,k-1},$$

$$\hat{t}(h_{ij,k}) = h_{ij,k}, \quad j \in \{0, 1\}, \quad k \in \mathbb{Z},$$

$$\hat{\Delta}(x) = \lim_{n \to \infty} (\hat{t}^n \otimes \hat{t}^{-n}) \Delta(\hat{t}^{-n}(x)),$$

(32)

for all $x \in DY(g)$.

Let also $\{\phi_{i,m}\}, \{\tilde{\phi}_{i,m}\}$ are dual bases in the spaces of primitive Cartan vectors in $\hat{DY}_0(g)$ (These subspaces we denote by $\hat{\Phi}^+, \hat{\Phi}^-$, correspondingly).

Proposition 4 The element $R_0$ can be can be represented in the following form

$$R_0 = R_0^e R_0^o = \exp \left( \sum_{i,m} \tilde{\phi}_{i,m} \otimes \tilde{\phi}_{i,m}^* \right) \exp \left( - \sum_{i,m} \phi_{i,m} \otimes \phi_{i,m}^* \right).$$

(33)

Further we deduce from the proposition 4, using the formula (32) Theorem 3. The proof of Theorem 2 is somewhat more complicated and more technical.
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