A REFINEMENT OF BRASCAMP-LIEB-POINCARÉ INEQUALITY IN ONE DIMENSION

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ABSTRACT. In this short note we give a refinement of Brascamp-Lieb [1] in the style of Houdré-Kagan [3] extension for Poincaré inequality in one dimension. This is inspired by works of Helffer [2] and Ledoux [4].

1. THE BRASCAMP-LIEB INEQUALITY

We take a convex potential \( V : \mathbb{R} \to \mathbb{R} \) which is \( C^k \) with \( k \geq 2 \) and the measure \( \mu(dx) = e^{-V(x)}dx \) which we assume it is a probability measure on \( \mathbb{R} \).

Theorem 1 (Brascamp-Lieb [1]). If \( V'' > 0 \) then for any \( C^2 \) compactly supported function \( f \) on the real line

\[
\text{Var}_\mu(\phi) \leq \int \frac{(f')^2}{V''}d\mu.
\]

One of the proofs is due to Helffer [2] and we sketch it here as it is the starting point of our approach. Consider the operator \( L \) acting on \( C^2 \) functions is given by

\[
L = -D^2 + V'D
\]

with \( D\phi = \phi' \). We denote \( \langle \cdot, \cdot \rangle \) the \( L^2(\mu) \) inner product and observe that

\[
\langle L\phi, \phi \rangle = \|\phi'\|^2.
\]

In particular \( L \) can be extended to an unbounded non-negative operator on \( L^2(\mu) \). From this, we get

\[
\|L\phi\|^2 = \langle DL\phi, D\phi \rangle
\]

and then if we take \( f \) a \( C^2 \) compactly supported function such that \( \int f d\mu = 0 \) and replace \( \phi = L^{-1}f \), then we get

\[
\text{Var}_\mu(f) = \langle f', DL^{-1}f \rangle.
\]

Now a simple calculation reveals that

\[
DL = (L + V'')D
\]

and then \( (L + V'')^{-1}D = DL^{-1} \) where the inverses are defined appropriately. Therefore we get

\[
\text{Var}_\mu(f) = \langle (L + V'')^{-1}f', f' \rangle.
\]

Since \( L \) is a non-negative operator \( (L + V'')^{-1} \leq (V'')^{-1} \) and this implies (1.1).

2. REFINEMENTS IN THE CASE OF \( \mathbb{R} \)

We start with (1.3) and iterate it. This is inspired from [4] but without any use of the semigroup theory.

We let \( D \) be the derivation operator and we denote \( D^* = -D + V' \), the adjoint of \( D \) with respect to the inner product in \( L^2(\mu) \). In the sequel, for a given function \( F \), we are going to denote by \( F \) also the multiplication operator by \( F \). The main commutation relations are the content of the following.

Proposition 2. Let \( A \) denote the operator defined on smooth positive functions \( E \) given by

\[
A(E)(x) = \frac{1}{4} \left( 2E''(x) + 2V'(x)E'(x) - \frac{E'(x)^2}{E(x)} + 4E(x)V''(x) \right)
\]

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Proof. (1) We want to find two functions $F$ and $G$ such that
\[ DED^* = F + GD^*G \]
For this, take a function $\phi$ and write
\[ (DE(-D + V'))\phi = (EV')'\phi + (-E' + EV')\phi' - E\phi'' \]
while
\[ F\phi + G(-D + V')DG\phi = (F - GG'' + GG'V')\phi + (G^2V' - 2GG')\phi' - G^2\phi'' \]
therefore it suffices to choose $G$ such that
\[ G^2 = E \quad \text{and} \quad F = GG'' - GG'V' + (EV')' \]
which means $G = E^{1/2}$ and $F = A(E)$.

(2) We have
\[ (E + D^*D)^{-1} = E^{-1} - E^{-1/2}(I - (I + E^{-1/2}D^*DE^{-1/2})^{-1})E^{-1/2} \]
\[ = E^{-1} - E^{-1}D^*(I + DE^{-1}D^*)^{-1}DE^{-1} \]
where we used the fact that for any operator $T$,
\[ I - (I + T^*T)^{-1} = T^*(I - TT)^{-1}T. \]

(3) From (2.2), we know that $I + DE^{-1}D^* = I + A(E^{-1}) + E^{-1/2}D^*DE^{-1/2} = FE^{-1} + E^{-1/2}D^*DE^{-1/2}$
and from (2.3),
\[ (FE^{-1} + E^{-1/2}D^*DE^{-1/2})^{-1} = E^{1/2}(F + D^*D)^{-1}E^{1/2} = F^{-1}E - E^{-1/2}F^{-1}D^*(I + DF^{-1}D^*)^{-1}DF^{-1}E^{1/2}. \]
\[ \square \]

Now, let us get back to the fact that $L = D^*D$ and that (1.3) gives
\[ \text{Var}_\mu(f) = \langle (V' + D^*D)^{-1}f', f' \rangle. \]

From (2.3) with $E_1 = V''$ we obtain first that
\[ \text{Var}_\mu(f) = \langle (V'')^{-1}f', f' \rangle - \langle (I + DE_1^{-1}D^*)^{-1}D[E_1^{-1}f'], D[E_1^{-1}f'] \rangle. \]

It is interesting to point out that this provides the case of equality in the Brascamp-Lieb if $D[(V'')^{-1}f'] = 0$ which solves for $f = C_1V' + C_2$.

Now we want to continue the inequality in (2.5) by taking $E_1 = E$ and using (2.5) for the case of $E_2 = E_1(I + A(E_1^{-1})) > 0$, thus we continue with
\[ (I + DE_1^{-1}D^*)^{-1} = E_2^{-1}E_1 - E_2^{1/2}E_2^{-1}D*(I + DE_2^{-1}D^*)^{-1}DE_2^{-1}E_2^{1/2}. \]

Hence we can write by setting $f_1 = E_1^{-1}f'$ and $f_2 = E_1^{1/2}D[f_1]$
\[ \text{Var}_\mu(f) = \|E_1^{-1/2}f''\|^2 - \|E_2^{-1/2}f_2\|^2 + \langle (I + DE_2^{-1}D^*)^{-1}D[E_2^{-1}f_2], D[E_2^{-1}f_2] \rangle. \]

Using a similar argument, let $E_3 = E_2(1 + A(E_2^{-1}))$ provided that $E_3$ is positive. Then we can continue with
\[ (I + DE_3^{-1}D^*)^{-1} = I + A(E_3^{-1}) - E_2^{1/2}E_3^{-1}D*(I + DE_3^{-1}D^*)^{-1}DE_3^{-1}E_2^{1/2}. \]
and letting $f_3 = E_2^{1/2} D[f_2]$, we obtain

$$\text{Var}_t(f) = \|E_1^{-1/2} f'|^2 - \|E_2^{-1/2} f_2\|_2^2 + \|E_3^{-1/2} f_3\|_2^2 - \langle (I + DE_3^{-1}D^*)^{-1} D[E_3^{-1} f_3], D[E_3^{-1} f_3] \rangle.$$ 

By induction we can define

$$E_1 = V'' \text{ and } f_1 = E_1^{-1} f'$$

$$E_n = E_{n-1} (1 + A(E_{n-1}^{-1})) \text{ and } f_n = E_n^{1/2} D[f_{n-1}]$$

Notice that here $E_n$ is defined only if $E_{n-1}$ is defined and positive and we will assume the sequence is defined as long as this condition is satisfied. We get the following result.

**Theorem 3.** If $E_1, E_2, \ldots, E_n$ are positive functions, then for any compactly supported function $f$,

$$\text{Var}_t(f) = \|E_1^{-1/2} f'|^2 - \|E_2^{-1/2} f_2\|_2^2 + \cdots + (-1)^{n-1} \|E_n^{-1/2} f_n\|_2^2$$

$$+ (-1)^n \langle (I + DE_n^{-1}D^*)^{-1} D[E_n^{-1} f_n], D[E_n^{-1} f_n] \rangle.$$ 

In particular, for $n$ even,

$$\text{Var}_t(f) \geq \|E_1^{-1/2} f'|^2 - \|E_2^{-1/2} f_2\|_2^2 + \cdots + (-1)^{n-1} \|E_n^{-1/2} f_n\|_2^2$$

and for $n$ odd,

$$\text{Var}_t(f) \leq \|E_1^{-1/2} f'|^2 - \|E_2^{-1/2} f_2\|_2^2 + \cdots + (-1)^{n-1} \|E_n^{-1/2} f_n\|_2^2.$$

For $V(x) = x^2/2 - \log(\sqrt{2\pi})$ this leads to the following version of Houdrè-Kagan [3] due to Ledoux [4].

**Corollary 4.** For $V(x) = x^2/2 - \log(\sqrt{2\pi})$ and $f$ which is $C^n$ with compact support, the following holds true

$$\text{Var}_t(f) = \|f'|^2 - \frac{1}{2^1} \|f''\|^2 + \cdots + \frac{(-1)^{n-1}}{(n-1)!} \|f^{(n-1)}\|^2 + \frac{(-1)^{n}}{(n-1)!} \langle (n + L)^{-1} f^n , f^n \rangle.$$ 

Another particular case is the following which is a reverse type Brascamp-Lieb.

**Corollary 5.**

$$\text{Var}_t(f) \geq \langle (V^n)^{-1} f' , f' \rangle - \langle (1 + A((V^n)^{-1}))^{-1} D((V^n)^{-1} f'), D((V^n)^{-1} f') \rangle$$

provided $1 + A((V^n)^{-1}) > 0$ which is equivalent to which amounts to

$$3V^{(3)}(x)^2 + 8V''(x)^3 - 2V^{(4)}(x)V''(x) - 2V^{(3)}(x)V''(x) V'(x) > 0.$$ 

For instance in the case $a, b > 0$ and

$$V(x) = ax^2/2 + bx^4/4 + C$$

(where $C$ is the normalizing constant which makes $\mu$ a probability) the condition (2.9) reads as

(*)

$$2a^3 - 3ab + (15a^2b + 18b^2) x^2 + 42ab^2 x^4 + 45b^3 x^6 > 0$$

for any $x$. In particular, for $x = 0$, this gives $3b < 2a^2$ which turns out to be enough to guarantee (*) for any other $x$. For the next corrections the condition that $1 + A(E_2^{-1}) > 0$ becomes equivalent to

$$4a^9 - 18a^7b + 27a^5b^2 + (90a^6b - 225a^5b^2 + 504a^4b^3 + 540a^2b^4) x^2 + (916a^7b^2 - 756a^6b^3 + 4203a^3b^4 - 162a^2b^5) x^4$$

$$+ (5563a^6b^3 + 2172a^5b^4 + 11124a^4b^5 + 1944b^6) x^6 + (22326a^5b^4 + 23868a^3b^6 + 720a^2b^7) x^8$$

$$+ (61689a^4b^5 + 74817a^2b^6 - 5832b^7) x^{10} + (11784a^3b^6 + 109026ab^7) x^{12} + (150741a^2b^7 + 63180b^8) x^{14}$$

$$+ 117450ab^8 x^{16} + 42525b^9 x^{18} > 0$$

for all $x$. This turns out be equivalent to $b < \frac{1}{3} (-1 + \sqrt{3}) a^2$. In general, for higher corrections the condition $E_n > 0$ appears to be equivalent to a condition of the form $b < a^2 t_n$ for some $t_n > 0$ which is decreasing in $n$ to $0$. We do not have a solid proof of this, but some numerical simulations suggests this conclusion.
Another example is the potential \( V(x) = \frac{x^2}{2} - a \log(x^2) + C \) with \( a > 0 \), for which condition (2.9) becomes equivalent to
\[
4a^3 - 3ax^2 + 12a^2x^2 + 7ax^4 + x^6 > 0
\]
for all \( x \). This turns out to be equivalent to \( a > a_0 \), where \( a_0 \) is the solution in \((0, 1)\) of the equation
\[
108 - 855a + 144a^2 + 272a^3 = 0
\]
and numerically is \( a_0 \approx 0.129852 \). For the second order correction a numerical simulation indicates that we need to take \( a > a_1 \) with \( a_1 \approx 0.314584 \). Some numerical approximations suggest that \( E_n > 0 \) is equivalent to \( a > a_n \) with \( a_n \) being an increasing sequence to infinity.

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