A Bernstein Type Result for Special
Lagrangian Submanifolds

Mao-Pei Tsui & Mu-Tao Wang

October 25, 2001, revised Jan 21, 2002

email: tsui@math.columbia.edu, mtwang@math.columbia.edu

Abstract

Let Σ be a complete minimal Lagrangian submanifold of \(\mathbb{C}^n\). We identify regions in the Grassmannian of Lagrangian subspaces so that whenever the image of the Gauss map of Σ lies in one of these regions, then Σ is an affine space.

1 Introduction

The well-known Bernstein theorem states any complete minimal surface that can be written as the graph of a function on \(\mathbb{R}^2\) must be a plane. This type of result has been generalized in higher dimension and codimension under various conditions. See [2] and the reference therein for the codimension one case and [1], [3], and [6] for higher codimension case. In this note, we prove a Bernstein type result for complete minimal Lagrangian submanifolds of \(\mathbb{C}^n\). We remark that Jost-Xin [7] obtained similar results from a somewhat different approach.

Recall a submanifold Σ of \(\mathbb{C}^n\) is called Lagrangian if the Kähler form \(\sum_{i=1}^n dx^i \wedge dy^i\) restricts to zero on Σ. If Σ happens to be the graph of a vector-valued function from a Lagrangian subspace \(L\) to its complement \(L^\perp\) in \(\mathbb{C}^n\). Rotating \(\mathbb{C}^n\) by a element in \(U(n)\), we may assume \(L\) is the \(x^i\) subspace and \(L^\perp\) is the \(y^i\) subspace. In this case, there exists a smooth
function $F : \mathbb{R}^n \to \mathbb{R}$ such that $\Sigma$ is defined by the gradient of $F$, $\nabla F$. The minimal Lagrangian equation can be written in terms of $F$.

$$\text{Im}(\det((I + i \text{Hess}(F)))) = \text{constant}$$  \hspace{1cm} (1.1)$$

where $I$ = identity matrix and $\text{Hess} F = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)$.

Such minimal submanifolds were first studied by Harvey and Lawson \cite{5} in the context of calibrated geometry. In fact, they are calibrated by $n$ forms of the type $\text{Re}(e^{i\theta} dz^1 \wedge \cdots \wedge dz^n)$ for some constant $\theta$. They are usually referred as special Lagrangian submanifold (SLg) in literature in a more general sense. Recently, Strominger-Yau-Zaslow \cite{8} established a conjectural relation of fibrations by special Lagrangian tori with mirror symmetry.

In terms of (1.1), a Bernstein type question is to determine under what conditions an entire solution $F$ becomes a quadratic polynomial.

The results in this paper imposes conditions on the image of the Gauss map of $\Sigma$. Recall the set of all Lagrangian subspaces of $\mathbb{C}^n$ is parametrized by the Lagrangian Grassmannian $U(n)/SO(n)$. The Gauss map of a Lagrangian submanifold $\gamma : \Sigma \mapsto U(n)/SO(n)$ assigns to each $x \in \Sigma$ the tangent space at $x$, $T_x \Sigma$.

A particular subset of the Lagrangian Grassmannian consists of the graphs of any symmetric linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. These can be considered as Lagrangians defined by the gradient of quadratic polynomials on $\mathbb{R}^n$.

For any $K > 0$, let $\mathcal{B}_K$ to be the subset of the Lagrangian Grassmannian consisting of graphs of symmetric linear transformations $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ with eigenvalues $|\lambda_i| \leq K$ for each $i$. We remark that if the Gauss map of $\Sigma$ lies in $\mathcal{B}_K$ then $\Sigma$ is the graph of $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with uniformly bounded $|df|$.

**Theorem A** Denote by $\mathfrak{X}$ the subset of the Lagrangian Grassmannian consisting of graphs of symmetric linear transformations $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ with eigenvalues $\lambda_i \lambda_j \geq -1$ for any $i, j$. Let $\Sigma$ be a complete minimal Lagrangian submanifold of $\mathbb{C}^n$. Suppose there exists an element $g \in U(n)$ such that the image of the Gauss map of $g(\Sigma)$ lies in $\mathfrak{X} \cap \mathcal{B}_K$, then $\Sigma$ is an affine space.

We remark that the gradient of $g(\Sigma)$ is not necessarily bounded. Indeed, the most general theorem of this type is the following.
Let $\mathcal{M}$ be the set of graphs of all symmetric linear transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ whose eigenvalues ($\lambda_i$) satisfy the following two conditions:

1. 

$$F(h_{ijk}) = \sum_{i,j,k} h_{ijk}^2 + \sum_{k,i} \lambda_i^2 h_{iik}^2 + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ijk}^2 \geq 0$$

for any trace-free symmetric three tensor $h_{ijk}$.

2. 

$$F(h_{ijk}) = 0$$

if and only if $h_{ijk} = 0$ for all $i, j, k$.

Here $h_{ijk}$ is any element in $\otimes^3 \mathbb{R}^n$ that is symmetric in $i, j$ and $k$. $h_{ijk}$ being trace-free means $\sum_{i=1}^n h_{iik} = 0$ for any $k$. In fact, $h_{ijk}$ corresponds to the second fundamental form of a Lagrangian submanifold. The trace-free condition corresponds to vanishing mean curvature vector. It is clear that $\Xi$ is a subset of $\mathcal{M}$.

**Theorem B** The conclusion for Theorem A holds for $\mathcal{M}_K$, the subset of the Lagrangian Grassmannian consisting of graphs of symmetric linear transformations in $\mathcal{M} \cap \mathcal{B}_K$.

These theorems are proved by maximum principle. When $\Sigma$ is the graph over a Lagrangian subspace $L$, we calculate the Laplacian of $\ln *\Omega$ where $*\Omega$ is the Jacobian of the projection from $\Sigma$ to $L$. This is a positive function and when the Gauss map of $\Sigma$ satisfies the above conditions it is indeed superharmonic. The parabolic version of this equation was first derived in [9] in the study of higher co-dimension mean curvature flow.

We wish to thank Professor D. H. Phong and Professor S.-T. Yau for their encouragement and support.

## 2 Proof of Theorem

Let $\Sigma$ be a complete submanifold of $\mathbb{R}^{2n}$. Around any point $p \in \Sigma$, we choose orthonormal frames $\{e_i\}_{i=1,\ldots,n}$ for $T\Sigma$ and $\{e_\alpha\}_{\alpha=n+1,\ldots,2n}$ for $N\Sigma$, the
normal bundle of $\Sigma$. The convention that $i, j, k, \ldots$ denote tangent indexes and $\alpha, \beta, \gamma \ldots$ denote normal indexes is followed.

The second fundamental form of $\Sigma$ is denoted by $h_{\alpha ij} = \langle \nabla e_i e_j, e_\alpha \rangle$.

The following formula was essentially derived in [9]. To apply to the current situation, we note that minimal submanifold corresponds to stationary phase of mean curvature flow.

**Proposition 2.1** Let $\Sigma$ be a the graph of $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $(\lambda_i)$ be the eigenvalues of $\sqrt{(df)^T df}$. If $\Sigma$ is a minimal submanifold, then $\ast \Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}}$ satisfies the following equation.

\[
\Delta \ast \Omega = -\ast \Omega \left\{ \sum_{i,j,k} h_{\alpha ijk}^2 - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{n+i,ik} h_{n+j,jk} + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{n+j,ik} h_{n+i,jk} \right\}
\]

(2.1)

where $\Delta$ is the Laplace operator of the induced metric on $\Sigma$.

Geometrically, $\ast \Omega$ is the Jacobian of the projection from $\Sigma$ to the domain $\mathbb{R}^n$.

**Proof of Theorem A.** First we show if the Gauss map of $\Sigma$ lies in $\Xi \cap \mathfrak{B}_K$, then $\Sigma$ is an affine space. The general case follows from the following observation: if $g \in U(n)$ then $g(\Sigma)$ is again a minimal Lagrangian submanifold.

We rewrite equation (2.1) in the Lagrangian case. Hence the tangent bundle is canonically isomorphic to the normal bundle by the complex structure $J$. We define

\[
h_{ijk} = \langle \nabla e_i e_j, J(e_k) \rangle
\]

then $h_{ijk}$ is symmetric in $i, j$ and $k$.

The Lagrangian condition also implies $\langle df(X), J(Y) \rangle$ is symmetric in $X, Y$. We can find an orthonormal basis $\{e_i\}_{i=1 \ldots n}$ for $T_p \Sigma$ so that $df(e_i) = \lambda_i J(e_i)$ and $\{J(e_i)\}_{i=1 \ldots n}$ becomes an orthonormal basis for the normal bundle. Equation (2.1) becomes

\[
\Delta \ast \Omega = -\ast \Omega \left\{ \sum_{i,j,k} h_{ijk}^2 - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{iik} h_{jjk} + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{jik} h_{ijk} \right\}
\]

(2.2)
We shall calculate

$$\Delta (\ln *\Omega) = \frac{\partial \Omega \Delta (\ast \Omega) - |\nabla \ast \Omega|^2}{|\ast \Omega|^2} \quad (2.3)$$

The covariant derivative of $\ast \Omega$ can be calculated as in equation (3.1) in \cite{9}.

$$(\ast \Omega)_k = - \ast \Omega (\sum \lambda_i h_{iik})$$

Plug this and equation (2.2) into equation (2.3) and we obtain

$$\Delta (\ln \ast \Omega) = -\{ \sum_{i,j,k} h_{ijk}^2 + \sum_{k,i} \lambda_i^2 h_{iik}^2 + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ijk}^2 \} \quad (2.4)$$

If the Gauss map of $\Sigma$ lies in $\Xi$, then it is obvious that $\Delta (\ln \ast \Omega) \leq 0$. The condition $|\lambda_i| \leq K$ means $\Sigma$ is the graph of a vector-valued function with bounded gradient. $\sum_{i,j,k} h_{ijk}^2 + \sum_{k,i} \lambda_i^2 h_{iik}^2 + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ijk}^2 = 0$ forces $h_{ijk} = 0$ for any $i, j, k$ by symmetry consideration. This immediately implies any minimal Lagrangian cone satisfies the assumption of the theorem is flat by maximum principle. For the general case, we can apply the standard blow-down and Allard regularity theorem to conclude $\Sigma$ is totally geodesic and thus an affine space.

If $\Sigma$ is minimal Lagrangian, so is any $g(\Sigma)$ for $g \in U(n)$. This is because $U(n)$ is contained in the isometry group of $\mathbb{C}^n$ and it preserves the standard Kähler form. This completes the proof of Theorem A.

$\square$

Proof of Theorem B.

This follows immediately from the definition of the set $\Xi'_K$ and equation (2.4). Because $\Sigma$ is minimal, we only need to consider trace-free $h_{ijk}$.

$\square$

In the rest of the paper, we identify another region of the Lagrangian Grassmanian where the Bernstein-type theorem also applies. This is not as general as the region in Theorem B. However, we expect it will provide a better estimate in future application.
**Theorem C** The conclusion for Theorem A holds for $\Xi' \cap \mathcal{B}_K$ where $\Xi'$ is the subset of Lagrangian Grassmannian consisting of graphs of symmetric linear transformations $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ with eigenvalues $\lambda_i \lambda_j + \lambda_i \lambda_k + \lambda_j \lambda_k \geq 0$ for any pairwise distinct $i, j, k$.

**Proof of Theorem C.**

We rewrite the right hand side of equation (2.4).

$$\left\{ \sum_{i,j,k} h^2_{ijk} + \sum_{k,i} \lambda_i^2 h^2_{iik} + 2 \sum_{k,j \leq i} \lambda_i \lambda_j h^2_{ijk} \right\}$$

$$= \left\{ \sum_{i,j,k} h^2_{ijk} + \sum_{i} \lambda_i^2 h^2_{iii} + \sum_{i < k} (\lambda_i^2 + 2 \lambda_i \lambda_k) h^2_{iik} + \sum_{i > k} (\lambda_i^2 + 2 \lambda_i \lambda_k) h^2_{iik} + 2 \sum_{i < j < k} (\lambda_i \lambda_j + \lambda_j \lambda_k + \lambda_k \lambda_i) h^2_{ijk} \right\}$$

(2.5)

Since $\Sigma$ is minimal, the mean curvature vector $\sum_{i=1}^n h_{iik} = 0$ for each $k$, we have

$$\sum_{i} \lambda_i^2 h^2_{iii}$$

$$= \sum_{i < j} \lambda_i^2 h^2_{ijj} + \sum_{i > j} \lambda_i^2 h^2_{ijj} + 2 \sum_{i \neq j, i \neq l, j < l} \lambda_i^2 h_{ijj} h_{iil}$$

(2.6)

Plug equations (2.6) into (2.5),

$$\Delta (\ln * \Omega)$$

$$= - \left\{ \sum_{i,j,k} h^2_{ijk} + 2 \sum_{i \neq j, i \neq l, j < l} \lambda_i^2 h_{ijj} h_{iil} + \sum_{i < k} (\lambda_i^2 + 2 \lambda_i \lambda_k + \lambda_k^2) h^2_{iik} + \sum_{i > k} (\lambda_i^2 + 2 \lambda_i \lambda_k + \lambda_k^2) h^2_{iik} + 2 \sum_{i < j < k} (\lambda_i \lambda_j + \lambda_j \lambda_k + \lambda_k \lambda_i) h^2_{ijk} \right\}$$

(2.7)
\[ 2 \sum_{i \neq j, i \neq l, j < l} \lambda_i^2 h_{ijj} h_{iil} + \sum_{p \neq q} (\lambda_p + \lambda_q)^2 h_{ppq} \]
\[ \geq \sum_{i \neq j, i \neq l, j < l} 2 \lambda_i^2 h_{ijj} h_{iil} + (\lambda_i + \lambda_l)^2 h_{iil}^2 + (\lambda_i + \lambda_j)^2 h_{ijj} \]  \quad (2.8)

If \( j \neq i \neq l \) and \( \lambda_i \lambda_j + \lambda_j \lambda_l + \lambda_l \lambda_i \geq 0 \) then

\[ 2 \lambda_i^2 h_{ijj} h_{iil} + (\lambda_i + \lambda_l)^2 h_{iil}^2 + (\lambda_i + \lambda_j)^2 h_{ijj} \geq 0. \]

Thus \( \Delta (\ln \Omega) \leq -|A|^2 \). The rest is identical to that of Theorem A and B.

We remark the condition \( \lambda_i \lambda_j + \lambda_j \lambda_l + \lambda_l \lambda_i \geq 0 \) for any pairwise distinct \( i, j, k \) is void in two-dimension. Indeed, this is true even without the Lagrangian assumption and hence rediscover the results of [1] (see also [3]) that an non-parametric minimal cone of dimension three must be flat.

\[ \square \]

References

[1] Barbosa, Jo ao Lucas Marquis, An extrinsic rigidity theorem for minimal immersions from \( S^2 \) into \( S^n \), J. Differential Geom. 14 (1979), no. 3, 355–368 (1980).

[2] K. Ecker and G. Huisken, A Bernstein result for minimal graphs of controlled growth., J. Differential Geom. 31 (1990), no. 2, 397–400.

[3] D. Fischer-Colbrie, Some rigidity theorems for minimal submanifolds of the sphere., Acta Math. 145 (1980), no. 1-2, 29–46.

\[ ^1 \text{After this paper was finished, we were informed by Yu Yuan that he has also derived formula (2.4) from a different point of view. In his paper "A Bernstein problem for special Lagrangian equations", Yu Yuan had the following interesting observation: the linear transformation } (x^i, y^i) \mapsto \left( \frac{x^i + y^i}{\sqrt{2}}, \frac{-x^i + y^i}{\sqrt{2}} \right) \text{ takes a convex function } F \text{ to a function } F' \text{ with } -I \leq D^2 F' \leq I. \text{ Since this transformation (so called Lewy transformation) is an element of } U(n), \text{ our Theorem A implies a convex entire solution to equation (1.1) is a quadratic polynomial. This result was also proved in Yu Yuan’s paper. We would like to thank Yu Yuan for sending us his preprint before publication.} \]
[4] S. Hildebrandt, J. Jost and K.-O Widman, *Harmonic mappings and minimal submanifolds*, Invent. Math. 62 (1980/81), no. 2, 269–298.

[5] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. 148 (1982), 47–157.

[6] J. Jost and Y. L. Xin, *Bernstein type theorems for higher codimension*, Calc. Var. Partial Differential Equations 9 (1999), no. 4, 277–296.

[7] J. Jost and Y. L. Xin, *A Bernstein theorem for special Lagrangian graphs*, preprint, 2001.

[8] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243–259.

[9] M-T. Wang, *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension*, to appear in Invent. Math.