On a sufficient condition for the subharmonicity of functions, satisfying the discrete mean value inequality

D S Telyakovskii
National Research Nuclear University MEPhI (Moscow Engineering Physics Institute),
31 Kashirskoe shosse, 115409 Moscow, Russia
E-mail: dtelyakov@mail.ru

Abstract. We obtain a sufficient condition for subharmonicity of functions of two variables that uses Laplace difference operator of Schwartz type. M.A. Kreines proved that an upper semi-continuous function is subharmonic, if for some \( n \geq 3 \) for every point \( \zeta \) of its domain one can construct of a sufficiently small radius with the center at \( \zeta \) such that the right \( n \)-gon can be inscribed in the circle, while the function value at the point \( \zeta \) does not exceed the arithmetic mean of its values at the vertices of this \( n \)-gon. In this note the semi-continuity condition for the function is replaced by some conditions of function directional continuity and summability (with respect to the planar Lebesgue measure.)

We also provide the position of nodes for the Laplace difference operator such that the proposition similar to the Kreines’s Theorem on a sufficient condition for the subharmonicity is valid for the functions of three variables.

In book of I.I. Privalov [1] is given the following result of M.A. Kreines that Kreines communicated to Privalov.

Theorem (Theorem (Kreines)). Let function \( u(z) = u(x,y) \) be upper semi-continuous in region \( G \subset \mathbb{R}^2 \). If for every point \( \zeta_0 \in G \) for some natural \( n \geq 3 \) in every circle of sufficiently small radius with center at \( \zeta_0 \) we can inscribe a right \( n \)-gon with vertices at points \( \zeta_1, \ldots, \zeta_n \) for which the following inequality holds

\[
u(\zeta_0) \leq \frac{1}{n} \sum_{j=1}^{n} u(\zeta_j),
\]

then the function \( u(z) \) is subharmonic in region \( G \).

Inequality (1) is the discrete analog of the mean value inequality for the subharmonic functions. In the current paper this inequality will be written as a discrete relation condition of Schwartz type for the Laplace equation.

We note that in the formulation of the above theorem given in [1] the function \( u(z) \) is assumed to be continuous, but the same proof would hold if the continuity is replaced by the upper semi-continuity of \( u(z) \).

1 This work is partially supported by the program of increasing of competitiveness of the National Nuclear Research University MEPhI, project No. 02.a03.21.0005 from 27.08.2013
In order to write the inequality (1) in the form of the difference relation condition we introduce the following notation: points \( \zeta_0, \zeta_1, \ldots, \zeta_n \) we call nodes of the difference relation, the point \( \zeta_0 \) is the central node of the set of nodes \( A_{\zeta_0} := \{ \zeta_0, \zeta_1, \ldots, \zeta_n \} \) and to simplify the notation we write \( u_j := u(\zeta_j), j = 0, 1, \ldots, n \).

Define the difference relation \( \Delta^{(s)}u(\zeta_0) \) that we consider in this paper

\[
\Delta^{(s)}u(\zeta_0) = \Delta^{(s)}u(\zeta_0; A_{\zeta_0}) := \frac{1}{nr^2} \left( \sum_{j=1}^{n} u_j - nu_0 \right) = \frac{1}{nr^2} \sum_{j=1}^{n} (u_j - u_0),
\]

where \( r \) is the radius of the circle, the points \( \zeta_1, \ldots, \zeta_n \) are on that circle. Using the expression (2) the inequality (1) can be rewritten as \( \Delta^{(s)}u(\zeta_0; A_{\zeta_0}) \geq 0 \).

In this paper the expression (2) is applied to the set of nodes located not only in the plane, but in space too. In any case those nodes \( \zeta_1, \ldots, \zeta_n \) are located on the same distance \( r \) away from the central node \( \zeta_0 \). In planar case nodes \( \zeta_1, \ldots, \zeta_n \) are always vertices of the right \( n \)-gon with the center at \( \zeta_0 \), this fact will not be mentioned each time separately.

We use the following definitions and terminologies.

As a star \( S_\zeta \) with center at the point \( \zeta \) we will call the collection of several segments with the common end \( \zeta \). We will consider only those stars \( S_\zeta \), that inside of every half plane containing point \( \zeta \) at its boundary, contain at least one ray, and do not contain rays in the angles of a quantity not exceeding \( \pi/2 \), formed by any two other rays. It is clear that under such conditions, a star can not contain less than three or more than seven rays and that the maximum angle between neighboring rays is greater \( \pi/4 \).

Let \( h(t), t \geq 0, \) be a modulus of continuity type function and let function \( u(z) \) be defined on some set \( B_\zeta \) for which the point \( \zeta \) is the limiting point. If for some value \( L_\zeta > 0 \) at every point \( z \in B_\zeta \) the inequality \( |u(z) - u(\zeta)| \leq L_\zeta h(|z-\zeta|) \) holds, then we say that function \( u(z) \) is \( h \)-regular in \( \zeta \) with respect to \( B_\zeta \). In the theorem 1 as a condition of the continuity of the function \( u(z) \) in points \( \zeta \in G \) we impose the condition of its \( h \)-regularity with respect to some star \( S_\zeta \).

As usual, the positive and negative parts of the function \( u(z) \) are denoted by \( u^+(z) := \max\{u(z), 0\} \) and \( u^-(z) := -\min\{u(z), 0\} \).

**Theorem 1.** Let function \( u(z) \) be defined in the domain \( G \subset \mathbb{R}^2 \), for each point \( \zeta \in G \) the function \( u(z) \) is \( h \)-regular with respect to some star \( S_\zeta \), for which the maximum angle between neighborhood rays does not exceed \( \alpha \pi \) and the function \( |u(z)|^{\max(2n,1)} \) is locally summable in \( G \). Then, if for each point \( \zeta_0 \in G \) in any neighborhood \( U_{\zeta_0} \) for some \( n \geq 3 \) there exists a set of nodes \( A_{\zeta_0} \) for which the quantity

\[
\left( \Delta^{(s)}u(\zeta_0; A_{\zeta_0}) \right)^- \to 0
\]

when the neighborhood \( U_{\zeta_0} \) is contracted to the point \( \zeta_0 \), then function \( u(z) \) is subharmonic in region \( G \).

Note that in this theorem the number of nodes in the set \( A_{\zeta_0} \) may be different for different points \( \zeta_0 \) and different neighborhoods \( U_{\zeta_0} \). If in the condition (3) we assume the tendency to zero not of the negative part of the difference relation \( \Delta^{(s)}u(\zeta_0; A_{\zeta_0}) \), but of its own, then the theorem 1 gives a sufficient condition for harmonicity.

The proof of the theorem 1 is similar to the proof of theorem from the author’s work [2], so we do not give it. Note that the theorem, which was formulated in [2] as sufficient condition of harmonicity, in fact, gives a sufficient condition subharmonicity.

For upper semicontinuous functions of three variables, we can specify the location five nodes under which the fulfillment of the condition (3) implies subharmonicity. We will consider only
sets of five nodes $A_{x^0}$ in which the point $x^0$ is the center of regular tetrahedron, and the remaining nodes $x^j$ are located at the vertices of this tetrahedron. This location of nodes will not be discussed every time specially.

**Theorem 2.** Let function $u(x) = u(x_1, x_2, x_3)$ be upper semicontinuous in the domain $G \subset \mathbb{R}^3$. If for every point $x^0 \in G$ in any its neighborhood $U_{x^0}$ there exists a collection of nodes $A_{x^0}$ for which the relation (3), then the function $u(x)$ is subharmonic in the domain $G$.

To prove this theorem we need the following lemma. We denote by $\text{tr } d^2u$ the trace of the matrix of the quadratic form of the second differential $d^2u$.

**Lemma 1.** Assume that function $u(x)$ has the second Peano differential at point $x^0 \in \mathbb{R}^3$

$$u(x) = u(x^0) + du(x-x^0) + \frac{1}{2} d^2u(x-x^0) + o(|x-x^0|^2) \quad \text{while } x \to x^0. \quad (4)$$

Then if collection of nodes $A_{x^0}$ is contracted to the point $x^0$

$$\Delta^{(*)} u(x^0; A_{x^0}) \to \frac{1}{3} \text{tr } d^2u. \quad (5)$$

**Proof.** First we show that for an arbitrary first-order polynomial $p(x) = p_0 + \sum p_k x_k$ the quantity $\Delta^{(*)} p(x^0; A_{x^0}) = 0$ at any point $x^0$ and for any collection of nodes $A_{x^0}$.

According to the definition of $\Delta^{(*)}$

$$\Delta^{(*)} p(x^0; A_{x^0}) = \frac{1}{4r^2} \sum_{j=1}^4 (p(x^j) - p(x^0)) = \frac{1}{4r^2} \sum_{k=1}^3 \sum_{j=1}^4 (x^j_k - x^0_k). \quad (6)$$

Each of the inner sums on the right-hand side of this formula is equal to the sum of the moments of system of unit masses located at the nodes $x^j$ (that is, at vertices regular tetrahedron with center $x^0$) with respect to the axis that passes through the point $x^0$ and is parallel to the coordinate axis $Ox_k$. Since the central node $x^0$ is the center of gravity of this mass system, then the expression in the formula (6) is zero.

Now let $p(x)$ be a second-order polynomial. According to what has been proved above we can assume that $p(x)$ contains only terms of the second order, and thus $p(x)$ is a quadratic form

$$p(x) = p_{11} x_1^2 + p_{22} x_2^2 + p_{33} x_3^2 + 2p_{12} x_1 x_2 + 2p_{13} x_1 x_3 + 2p_{23} x_2 x_3.$$

Changing the coordinate system is orthogonal coordinate transformation. Since the trace of the matrix of the quadratic form is invariant under the orthogonal transformation, the coordinate system can be chosen arbitrarily.

We choose a coordinate system with the origin at the point $x^0$ in which the nodes $x^1$, $x^2$, $x^3$ and $x^4$ have respectively coordinates $(r, 0, 0)$, $(-r/3, \sqrt{2}r/3, -\sqrt{2}r/3)$ and $(-r/3, -\sqrt{2}r/3, \sqrt{2}r/3)$.

The values of the polynomial $p(x)$ at the nodes of the set $A_{x^0}$ are equal, respectively:

$$p(x^0) = 0, \quad p(x^1) = p_{11} r^2, \quad p(x^2) = \left(\frac{p_{11}}{9} + \frac{8p_{33}}{9} - \frac{4\sqrt{2}p_{13}}{9}\right) r^2;$$

$$p(x^3) = \left(\frac{p_{11}}{9} + \frac{2p_{22}}{3} + \frac{2p_{33}}{9} - \frac{2\sqrt{2}p_{12}}{3\sqrt{3}} + \frac{2\sqrt{2}p_{13}}{9} - \frac{4p_{23}}{3\sqrt{3}}\right) r^2;$$

$$p(x^4) = \left(\frac{p_{11}}{9} + \frac{2p_{22}}{3} + \frac{2p_{33}}{9} + \frac{2\sqrt{2}p_{12}}{3\sqrt{3}} + \frac{2\sqrt{2}p_{13}}{9} + \frac{4p_{23}}{3\sqrt{3}}\right) r^2.$$
Substituting these values into the expression (2) for $\Delta^{(e)} p$ we obtain that for each point $x^0$ and each set of nodes $A_{x^0}$

$$\Delta^{(e)}p(x^0, A_{x^0}) = \frac{1}{3}(p_{11} + p_{22} + p_{33}) = \frac{1}{3}\text{tr} p(x). \quad (7)$$

To prove the lemma it remains to verify that the remainder term in the formula (4) does not affect the value of the limit (5), but this is obvious. The Lemma is proved.

Let real-values functions $u(z)$ and $v(z)$ be defined in the domain $G$ (are the $u(z)$ and $v(z)$ defined on the boundary of the domain isn’t sufficient). Will say that at the boundary point $\zeta \in \partial G$ the function $v(z)$ majorizes function $u(z)$ if $\lim_{z \to \zeta} (v(z) - u(z)) \geq 0$ when $z \to \zeta$, $z \in G$.

We will check the fulfillment of the following definition of subharmonicity.

**Definition.** A function $u(x)$ defined in a domain $G$ is said to be subharmonic, if it is a) upper semicontinuous and not identically $-\infty$ in $G$ and b) for each closed ball $D \subset G$ any harmonic function that majorizes the function $u(x)$ on the frontier $\partial D$ also majorizes $u(x)$ inside $D$.

The proof of the theorem 2 follows the Privalov scheme [3], applied to prove the harmonicity of continuous functions, which satisfy the Laplace equation.

**Proof of Theorem 2.** Suppose that the theorem is not true and the function $u(x)$ is not subharmonic everywhere in $G$. Then there is a ball $D \subset G$ and harmonic inside of the ball $D$ function $v(x)$, which majorizes the function $u(x)$ on $\partial D$, but does not majorizes $u(x)$ inside $D$. We set $w(x) := u(x) - v(x)$.

The function $w(x)$ is upper semicontinuous in $D$, so it reaches a maximum on $D$. Since at some interior points of $D$ the function $w(x)$ is positive, this maximum is positive, and since on $\partial D$ the function $w(x)$ does not exceed zero, the point of maximum $w(x)$ belongs to interior of $D$. Let $\max_D w(x) = w(x^*)$.

Inside the ball $D$ we denote as

$$q(x) := w(x) + \frac{w(x^*)}{2\text{diam}^2 D}((x_1-x_1^*)^2 + (x_2-x_2^*)^2 + (x_3-x_3^*)^2).$$

By definition, the function $q(x)$ is upper semicontinuous on the ball $D$, therefore, it reaches a maximum on $D$, and since $q(x) \geq w(x)$, then $\max_D q(x) \geq \max_D w(x) = w(x^*)$. Since the function $q(x)$ does not exceed $\frac{w(x^*)}{2}$ on the frontier $\partial D$, then $q(x)$ takes the maximum value strictly inside the ball $D$. Let $\max_D q(x) = q(x^0)$.

By the hypothesis of the Theorem, there exists a sequence of sets of nodes $\{A^n_{x^0}\}$ lying inside the ball $D$ that contracts to the point $x^0$ for which $(\Delta^{(e)}u(x^0, A^n_{x^0}))^{-} \to 0$ as $n \to \infty$. Since the function $v(x)$ is harmonic in some neighborhood of the point $x^0$, then, by virtue of the Lemma, for any sequence of sets of knots contracting to $x^0$, including the sequence $\{A^n_{x^0}\}$, is satisfied $\Delta^{(e)}v(x^0, A^n_{x^0}) \to 0$. By the definition $w(x) = u(x) - v(x)$, then $\Delta^{(e)}w(x^0, A^n_{x^0}) = \Delta^{(e)}u(x^0, A^n_{x^0}) - \Delta^{(e)}v(x^0, A^n_{x^0})$ and, therefore,

$$\lim_{n \to \infty} (\Delta^{(e)}w(x^0, A^n_{x^0}))^{-} = \lim_{n \to \infty} (\Delta^{(e)}u(x^0, A^n_{x^0}))^{-} \to 0. \quad (8)$$

It follows from the equality (7) that for a polynomial of the second order $\frac{w(x^*)}{2\text{diam}^2 D}((x_1-x_1^*)^2 + (x_2-x_2^*)^2 + (x_3-x_3^*)^2)$ for each of the sets of nodes $A^n_{x^0}$ the value of $\Delta^{(e)}$ is equal to the same
positive number $\frac{w(x^*)}{6\text{diam}^2 D}$. From this and from (8) it follows that, starting with a sufficiently large number, the quantity $\Delta^{(*)}q(x^0, A^n_{x^0})$ is positive:

$$\Delta^{(*)}q(x^0, A^n_{x^0}) = \Delta^{(*)}w(x^0, A^n_{x^0}) + \frac{w(x^*)}{6\text{diam}^2 D} > 0.$$ 

Therefore, with such numbers, at least one term in the expression (2) for $\Delta^{(*)}q(x^0, A^n_{x^0})$ is positive. This contradicts the assumption that $x^0$ is the maximum point of the function $q(x)$ in the ball $D$.

Therefore our assumption, that the function $u(x)$ is not subharmonic everywhere inside of the domain $G$ is not true. The Theorem is proved.

References

[1] I I Privalov Subgarmonicheskie funkii (Subharmonic functions). Moscow, Leningrad: ONTI Publ., 1937, 199 p. (in Russian).

[2] D S Telyakovskii A sufficient condition for the harmonicity of a function of two variables satisfying the Laplace difference equation. Proc. of IMM UB RAS, vol. 22, no 4, 2016, 269–283 (in Russian).

[3] I Priwaloff Sur les fonctions harmoniques. Mat. Sb., 1925, vol. 23, no. 3, pp. 464–471.