Non-commutative black holes
in $D$ dimensions

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April, 1994

Abstract

Recently introduced classical theory of gravity in non-commutative
gometry is studied. The most general (four parametric) family of
$D$ dimensional static spherically symmetric spacetimes is identified
and its properties are studied in detail. For wide class of the choices
of parameters, the corresponding spacetimes have the structure of
asymptotically flat black holes with a smooth event horizon hiding the
curvature singularity. A specific attention is devoted to the behavior
of components of the metric in non-commutative direction, which are
interpreted as the black hole hair.

PACS index: 04.20, 04.30

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1. Introduction

General relativity is, in a sense, the most interesting field theory because the propagating field – the metric tensor – encodes the geometrical properties of the space-time in which all other fields propagate. Thus, the gravity field couples to all remaining matter and in this sense it is universal. It was always very tempting to give a geometrical interpretation also to other physically relevant fields, like electromagnetic or Yang-Mills potentials. The oldest such scenario is the Kaluza-Klein one in which the components of the metric tensor corresponding to N additional coordinates of the 4+N-dimensional space-time play the role of the matter fields \[1, 2, 3\]. There are many variants of the Kaluza-Klein approach but all of them have some common features. Namely, the higher dimensional space is “real” and one has to look for realistic dynamical compactification which would lead to vacua with a very tiny size in the N extra dimensions. Thus, the study of the \(D\)-dimensional systems may be something far more than an academic exercise.

Among more modern geometrical theories of matter the important role is played by string theory \[4\]. The dynamics of the massless modes of the string is governed by an effective action which possesses the reparametrization invariance and describes the interaction of the metric tensor with the axion and dilaton fields. The important lesson to be learnt from this consists in emergence of new matter field multiplets (i.e. axion and dilaton) in the theory, reflecting its underlying geometrical structure. In this sense, those new fields can be understood as being of the geometrical origin.

Recently, some activity was devoted to the formulation of general relativity in the framework of non-commutative geometry of A. Connes \[5, 6\]. The fundamental algebra of the non-commutative geometry was chosen to be the algebra of two by two diagonal matrices with \(C^\infty\) functions on some \(D\)-dimensional manifold as the entries. Speaking more intuitively, the space-time consisted of two smooth manifolds—the sheets and it had a component measuring the distance of the sheets (for the details see \[5, 6\]). Thus, apart from the standard general relativity metric, there were components taking into account the relation between two sheets. Indeed those components appeared in the final action as the “matter” fields from the point of view of the standard general relativity. In this way we may try to give geometrical meaning to various matter fields, using an appropriate non-commutative geometry setting.
In the paper by Chamseddine, Felder and Fröhlich [5], the geometrical interpretation was given to the massless scalar field coupled to general relativity. On the other hand, in the work by Klimčík, Pompoš and Souček [6] the evaluation of the non-commutative Einstein-Hilbert action resulted in somewhat exotic coupling of the vector field to standard general relativity Lagrangian:

\[ I = \int_X d^D x \sqrt{g}[2R + Q^{\alpha\beta\gamma\delta}(V) \nabla_\alpha V_\beta \nabla_\gamma V_\delta], \quad (1.1) \]

where

\[ Q^{\alpha\beta\gamma\delta}(V) = 4 \frac{1}{(V^2)^3} \left( V^\alpha V^\beta V^\gamma V^\delta - g^{\alpha\beta} V^\gamma V^\delta V^2 - g^{\gamma\delta} V^\alpha V^\beta V^2 \right), \]

and \( \nabla_\mu \) is covariant derivative. We have written the corresponding action in \( D \) dimensions because the main point of this paper consists in studying the properties of the system in dependence on \( D \). In four dimensions we have found the solutions of the model which the black hole structure with the \( V^\alpha \) field playing the role of hair [7]. It turned out that while the metric was the standard black-hole-like one with the smooth horizon, singularity at the origin and vanishing curvature in the asymptotical region the hair was less “healthy”. Indeed there was a critical radius (which could lie below the horizon) under which the hair became imaginary! This is very peculiar singularity of the hair because otherwise nothing happens at that point - the curvature is smooth and bounded as well as the hair is.

Considering the dimension of the spacetime as a parameter of the physical theories has brought about many important insights about the dynamics of physical systems [8]. The higher-dimensional theories can be relevant in the Kaluza-Klein scenario and lower-dimensional ones describe dynamics of some particular subsets of degrees of freedom of four dimensional models. It is therefore natural to study the dimension dependence of our non-commutative theories.

In the present case we were very much interested what is going to be the destiny of the “imaginary hair” pathology in arbitrary dimension. We have found, by solving quite a complicated system of equations, that due to specific structure of the field equations this pathology is inevitable only in four dimensions! Moreover, the space of the solutions within the static spherically symmetric ansatz is richer comparing with the four dimensional case.
In what follows, we shall find the general $D$-dimensional solution within the ansatz (in sec.2) and will describe both its metric properties (sec.3) and its hair (sec.4). We find many black hole space-times with real hair. The case $D = 3$ we will treat separately. We shall discuss what we have learned from those concrete results in sec.5: Conclusions and Outlook.

2. Non-commutative action in $D$ dimensions

Generalizing the 4-dimensional action obtained from non-commutative geometry to the arbitrary dimensional spacetimes (with metric of signature $- + \cdots +$) we get the new action

$$I = \int_X d^Dx \sqrt{-g} \left[ 2R + 4\kappa \nabla_\beta \left( \frac{V^\mu V^\beta}{\sqrt{V^2}} \right) \nabla_\mu \left( \frac{1}{\sqrt{V^2}} \right) \right],$$

(2.1)

It what follows we will find convenient to define the new fields $f^\alpha$, $\sigma$ as

$$f^\alpha = \frac{V^\alpha}{\sqrt{V^2}}, \quad \sigma = \frac{1}{\sqrt{V^2}}$$

(similarly as in [7]). Then the action becomes

$$I = \int_X d^Dx \sqrt{-g} \left[ 2R - 4\kappa f^\alpha f^\beta \frac{\nabla_\alpha \nabla_\beta \sigma}{\sigma} - 4\kappa \Lambda (f^\alpha f_\alpha - 1) \right].$$

(2.2)

with Lagrange multiplier $\Lambda$. Variations of action (2.2) with respect to $\Lambda$, $f^\alpha$, $\sigma$, $g^{\alpha\beta}$ yield

$$f^\alpha f^\beta g_{\alpha\beta} = 1,$$

$$f^\alpha \left( \frac{\nabla_\mu \nabla_\alpha \sigma}{\sigma} + \Lambda g_{\mu\alpha} \right) = 0,$$

(2.3)

$$\nabla_\beta \left( 2f^\alpha f^\beta \frac{\nabla_\alpha \sigma}{\sigma} - \nabla_\alpha (f^\alpha f^\beta) \right) = 0,$$

(2.4)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = \kappa \left[ g_{\mu\alpha} \left( f^\alpha f^\beta \left( \frac{\nabla_\alpha \nabla_\beta \sigma}{\sigma} + \Lambda g_{\alpha\beta} \right) - \Lambda \right) +
+ 2f^\alpha f_{\ [\mu} \left( \frac{\nabla_\alpha \nabla_{\nu] \sigma}}{\sigma^2} + \frac{\nabla_\alpha \sigma \nabla_{\nu] \sigma}}{\sigma^2} \right) + 2f_{\mu} f_{\nu} \Lambda +
+ \nabla_\alpha \left( f_{\mu} f_{\nu} \frac{\nabla_\alpha \sigma}{\sigma} \right) - 2\nabla_\alpha (f^\alpha f_{\ [\mu} \frac{\nabla_{\nu]} \sigma}{\sigma} \right].$$

(2.5)
where \([\alpha \beta]\) means the symmetrization in the indices, i.e. \(V_{[\alpha}V_{\beta]} = \frac{1}{2}(V_{\alpha}V_{\beta} + V_{\beta}V_{\alpha})\). As we are interested in static, spherically symmetric solutions of this system, let our metric have the form
\[
 ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2d\Omega_{D-2}^2,
\]
where \(d\Omega_{D-2}^2\) is the standard round metric on the sphere \(S^{D-2}\). In agreement with (2.3) we assume that only \(f^0\) and \(f^1\) components of vector \(f^\alpha\) are non-zero. In this ansatz the equations of motion are
\[
- e^\nu f^0 f^0 + e^\lambda f^1 f^1 = 1, \tag{2.7}
\]
\[
\frac{\sigma' \nu'}{\sigma} e^{-\lambda} + \Lambda = 0, \tag{2.8}
\]
\[
\frac{1}{\sigma} \left( \sigma'' - \frac{\lambda'}{2} \sigma' \right) + e^\lambda \Lambda = 0, \tag{2.9}
\]
\[
r^{D-2}e^{\frac{\nu + \lambda}{2}} \left[ \left( f^1 f^1 \right)' + f^1 f^1 \left( \frac{\nu'}{2} + \lambda' - \frac{2\sigma'}{\sigma} + \frac{D-2}{r} \right) + f^0 f^0 \frac{\nu'}{2} e^{\nu - \lambda} \right] = A, \tag{2.10}
\]
\[
\frac{(D-2)(D-3)}{2r^2} - (D-2)e^{-\lambda} \left( \frac{D-3}{2r^2} - \frac{\lambda'}{2} \right) = \kappa \left( f^0 f^0 \right)' \frac{\sigma'}{\sigma} e^{\nu - \lambda} +
\]
\[
+ \kappa \left[ f^0 f^0 \left( -\frac{\nu'}{2} e^{\nu - \lambda} \frac{\sigma'}{\sigma} - \Lambda e^{\nu} \right) + f^1 f^1 \left( \frac{\sigma''}{\sigma} - \frac{\lambda'}{2} \frac{\sigma'}{\sigma} + \Lambda e^{\lambda} \right) - \Lambda \right] +
\]
\[
+ \kappa f^0 f^0 e^{\nu - \lambda} \left[ \frac{\sigma'}{\sigma} \left( \frac{5}{2} \nu' - \frac{\sigma'}{\sigma} - \frac{\lambda'}{2} + \frac{D-2}{r} \right) + \frac{\sigma''}{\sigma} + 2\Lambda e^{\lambda} \right], \tag{2.11}
\]
\[
\frac{(D-2)(D-3)}{2r^2} - (D-2)e^{-\lambda} \left( \frac{(D-3)}{2r^2} + \frac{\nu'}{2r} \right) = \kappa \left( f^1 f^1 \right)' \frac{\sigma'}{\sigma} +
\]
\[
+ \kappa \left[ f^0 f^0 \left( -\frac{\nu'}{2} e^{\nu - \lambda} \frac{\sigma'}{\sigma} - \Lambda e^{\nu} \right) + f^1 f^1 \left( \frac{\sigma''}{\sigma} - \frac{\lambda'}{2} \frac{\sigma'}{\sigma} + \Lambda e^{\lambda} \right) - \Lambda \right] +
\]
\[
+ \kappa f^1 f^1 \left[ \frac{\sigma'}{\sigma} \left( -\frac{\sigma'}{\sigma} + \frac{5}{2} \lambda' + \frac{\nu'}{2} + \frac{(D-2)}{r} \right) - \frac{3}{2} \frac{\sigma''}{\sigma} - 2\Lambda e^{\lambda} \right] +
\]
\[
+ \kappa f^0 f^0 \nu' e^{\nu - \lambda} \frac{\sigma'}{\sigma}, \tag{2.12}
\]
\[ \frac{D-4}{4r} e^{-\lambda} \left( \nu' - \lambda' + \frac{2}{r} (D-3)(1-e^\lambda) \right) + \frac{e^{-\lambda}}{4} \left( 2 \nu'' - \nu' \lambda' - \nu' \right) + \]
\[ + \frac{D-2}{r} (\nu' - \lambda') = \kappa \Lambda, \quad (2.13) \]

(where (2.10) is actually the first integral of (2.4) with an integration constant \( A \)).

3. The solution of the field equations in \( D \) dimensions

Extracting \( \Lambda \) from Eq. (2.7) and Eq. (2.9) and integrating the obtained relation we (as in [7]) arrive at

\[ e^\nu = \sigma^2 e^{-\lambda} e^{-2K}, \quad (3.1) \]

where \( K \) is a constant. Substituting Eq. (3.1) and Eq. (2.7) into Eq. (2.10) we obtain

\[ \left( f^1 f^1 \right)' + 2 f^1 f^1 \left( \frac{D-2}{2r} + \frac{\sigma''}{\sigma'} - \frac{\sigma'}{\sigma} \right) = \frac{A e^K}{r^{D-2} \sigma^2} + e^{-\lambda} \left( \frac{\sigma''}{\sigma'} - \frac{\lambda'}{2} \right). \quad (3.2) \]

The difference of Eq. (2.11) and Eq. (2.12) results in

\[ \frac{\sigma''}{\sigma} + \kappa \left( \frac{\sigma'^2}{\sigma^2} - \frac{r}{D-2} \frac{\sigma'^3}{\sigma^3} \right) = 0, \quad (3.3) \]

and Eq. (2.12), combined with Eq. (3.2), yields

\[ f^1 f^1 = \left( \frac{(D-2)(D-3)\sigma^2}{2 \kappa r^2 \sigma'^2} - \frac{(D-2)\sigma'^2}{2 \kappa} \right) \left( \frac{(D-3)\sigma^2}{r^2 \sigma'^2} + \frac{2 \sigma'^2 \sigma''}{r \sigma'^3} \right) - \]
\[ - \frac{\lambda' \sigma^2}{r^2 \sigma'^2} \right) - \frac{A \sigma e^K}{r \sigma'^2 D-2}. \quad (3.4) \]

We assume \( \sigma' \neq 0 \), because otherwise we would end up with standard Schwarzschild case (see [8]).
Using Eq. (2.7) and inserting \( f_1 f_1 \) from Eq. (3.4) into Eq. (2.4), we obtain

\[
\frac{\sigma^2}{\sigma'^2} Q'' + \left( \frac{\kappa}{\sigma'} - \frac{\sigma''}{\sigma'^3} \right) Q' - \frac{\kappa (D-3)Q}{D-2} - (D-3)(D-4)e^{-2K} \sigma^2 r^{D-5} = 0,
\]

where we have introduced \( Q(r) \), defined by

\[
Q(r) \equiv e^{\nu(r)} r^{D-3}.
\]

The equation (3.3) can be easily solved after changing variable \( r \rightarrow \sigma \), as then it becomes

\[
\frac{d^2 r}{d\sigma^2} - \frac{\kappa}{\sigma} \frac{dr}{d\sigma} + \kappa r = 0.
\]

The general solution is

\[
r = c_1 \sigma^{\alpha_1} + c_2 \sigma^{\alpha_2},
\]

where

\[
\alpha_{1,2} = \frac{1}{2} \left( 1 + \kappa \pm \sqrt{1 + 2\kappa \frac{D-4}{D-2} + \kappa^2} \right).
\]

It is straightforward to solve the Eq. (3.5) in \( D = 3 \). The general solution in this case is

\[
Q(\sigma) = \begin{cases} 
k_1 \ln \sigma + k_2 & \kappa = 1, \\
\frac{\sigma^{1-\kappa}}{1-\kappa} + k_2 & \kappa \neq 1.
\end{cases}
\]

In higher dimensions we change the variable \( r \rightarrow \sigma \), and obtain

\[
\frac{d^2 Q}{d\sigma^2} + \frac{\kappa}{\sigma} \frac{dQ}{d\sigma} - \frac{\kappa (D-3)Q}{(D-2)\sigma^2} = (D-3)(D-4)e^{-2K} r^{D-5}.
\]

This equation is the Euler’s differential equation with non-zero right-hand-side. The general solution of Eq. (3.10) is given by

\[
Q(\sigma) = k_1 \sigma^{\beta_1} + k_2 \sigma^{\beta_2} + \sum_{j=0}^{D-5} A_j \sigma^{\omega_j},
\]
where

\[ \beta_{1,2} = \frac{1}{2} \left( 1 - \kappa \pm \sqrt{1 + \frac{2 \kappa (D - 4)}{D - 2} + \kappa^2} \right), \]

\[ \omega_j = j(\alpha_2 - \alpha_1) + (D - 5)\alpha_1 + 2, \]

\[ A_j = \frac{(D - 5)(D - 3)(D - 4)c_1^{D-5-j} c_2^j e^{-2\kappa}}{\omega_j(\omega_j - 1) + \kappa(\omega_j - \frac{D-3}{D-2})}, \]

c_{1,2} and k_{1,2} are (real) constants, and \( j = 0, 1, 2, \ldots, D - 5 \).

4. The analysis of the solution in \( D \) dim.

At this moment we have only “raw” solutions of equations of motion. In principle, we know everything about the system. From (3.7) we can obtain \( \sigma \). Formulae (3.11) and (3.1) give us the metric of the spacetime. \( V^0 \) and \( V^1 \) can be obtained using (3.4) and the definitions of \( f^0 \), \( f^1 \). Now we shall study deeper the properties of the solutions. Let us start with 3-dimensional case.

4.1 Black-hole metric and the scalar curvature in 3-dim.

The properties of the obtained spacetime vary with the choice of the integration constants. In what follows, we shall consider only the cases which are the most obvious candidates for the black hole metric.

Case I. Let \( c_2 = 0 \) and \( \kappa > 1 \). Then \( \sigma = \frac{\nu}{c_1} \) and

\[ e^\nu(r) = k_1 c_1^{\kappa-1} \frac{r^{1-\kappa}}{1-\kappa} + k_2, \quad e^\lambda(r) = \frac{e^{-2\kappa} c_1^{-2}}{k_1 c_1^{\kappa-1} \frac{r^{1-\kappa}}{1-\kappa} + k_2}. \] (4.1)

It is easy to see that \( e^\nu \to k_2 \), \( e^\lambda \to \frac{e^{-2\kappa}}{c_2^2 k_2} \) for \( r \to \infty \). We know that \( \sigma \) is positive, thus \( c_1 > 0 \). In the asymptotic region \( (r \to \infty) \) we need \( g_{00} < 0 \) and, therefore \( k_2 > 0 \). Then \( g_{11} > 0 \) for \( r \to \infty \), as it should. Moreover, there is an event horizon if \( k_1 > 0 \).
Case II. Let $c_1 = 0$ and $0 < \kappa < 1$. Then $\sigma(r) = c_2^{-1} r^\frac{1}{\kappa}$,

\[ e^\nu(r) = k_1 c_2^{-\frac{\kappa-1}{\kappa}} r^\frac{1}{1-\kappa} + k_2, \quad e^\lambda(r) = \frac{e^{-2K} \kappa^{-2} c_2 \frac{\kappa-1}{\kappa} r^\frac{2-2\kappa}{\kappa} + k_2}{k_1 c_2^{-\frac{\kappa-1}{\kappa}} r^\frac{1}{1-\kappa}}. \quad (4.2) \]

From $\sigma > 0$ it follows $c_2 > 0$ and from the asymptotic behavior of $g_{00}$ we have $k_1 > 0$. Then the horizon does exist if $k_2 < 0$ and it is given by

\[ r_H = c_2 \left( -\frac{k_2(1-\kappa)}{k_1} \right)^{\frac{1}{1-\kappa}}. \quad (4.3) \]

Is there any curvature singularity hidden behind the horizon? The formula for scalar curvature in $D$-dimensional spacetime with metric of form (2.6) is

\[ R = (D-2)(D-3) - e^{-\lambda} \left( \frac{(D-2)(D-3)}{r^2} \right) + \frac{\nu'^2 + \nu''}{2} + (D-2) \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2}. \quad (4.4) \]

Using this formula for $D = 3$ we can calculate that $R \to \infty$ for $r \to 0$, $R \to 0$ for $r \to \infty$ and $R$ is finite for $r \neq 0$. Thus in both “right” cases (I, II) the spacetime is asymptotically flat, smooth at the horizon and singular at $r = 0$.

Now we are going to analyze the behavior of hair.

### 4.2 Hair in 3-dim.

Let us first study the case I when $c_2 = 0$ and $\kappa > 1$. Inserting the metric (4.1) into Eq. (3.4) for $D = 3$ we obtain a strange kind of singularity. Our $f^1$, $f^0$ are real and it implies that $f^1 f^1$ and $f^0 f^0$ have always to be positive. But for our choice of integration constants, $f^1 f^1$ cannot be positive in all spacetime. Nevertheless, demanding $A > \frac{1-\kappa}{2\kappa} c_1 k_2 e^K$ we can arrange that the region where $f^1 f^1$ is negative is hidden behind the horizon. Then if we require the asymptotical positivity of $f^0 f^0$, we will obtain another condition, viz. $A <$
\( c_1 k_2 e^K \), which has to be also satisfied. It is clear that both inequalities cannot be satisfied at the same time for whatever \( \kappa > 0 \). Therefore this case is out of our interest.

The next (and in \( D = 3 \) the most interesting) case is when \( c_1 = 0 \) and \( 0 < \kappa < 1 \). Inserting the metric (4.2) into the (3.4) we obtain

\[
 f^1 f^1 = -e^K \kappa^2 c_2^2 (e^{K \kappa^2} c_2^2 + A) r^{1-\frac{1}{\kappa}}.
\]

Using the definition \( f^1 = \sigma V^1 \) we have

\[
 V^1 V^1 = -e^K \kappa^2 c_2^2 (e^{K \kappa^2} c_2^2 + A) r^{1-\frac{1}{\kappa}}. \tag{4.5}
\]

Let

\[
 A \leq -\frac{1}{2} \kappa e^K c_2^2. \tag{4.6}
\]

From equation (4.3) it is then clear that \( V^1 V^1 \geq 0 \) everywhere, in all space-time and that \( V^1 V^1 \to \infty \) when \( r \to 0 \). Using Eq. (2.7), we can see that

\[
 V^0 V^0 = -g_{11} g^{00} V^1 V^1 + g^{00} \sigma^2, \tag{4.7}
\]

so that \( V^0 V^0 \) is positive below the horizon. Because \( V^0 V^0 \sim -\frac{c_2^2}{k_2} r^{-\frac{2}{\kappa}} \) for \( r \to 0 \), it is clear that \( V^0 V^0 \to \infty \) for \( r \to 0 \). Can \( V^0 V^0 \) be negative somewhere above the horizon? Suppose that \( V^0 V^0 = 0 \) in some \( r = r_{\text{crit}} \), i.e.

\[
 Y r_{\text{crit}}^{-\frac{2}{\kappa}-2} Z r_{\text{crit}}^{1-\frac{1}{\kappa}} e^{-2\nu(r_{\text{crit}})} - e^{-\nu(r_{\text{crit}})} = 0, \tag{4.8}
\]

where we have introduced the constants \( Y, Z \) as follows

\[
e^{\nu(r)} = Y r^{1-\frac{1}{\kappa}} + k_2, \quad \text{and} \quad f^1 f^1 = Z r^{1-\frac{1}{\kappa}}
\]

\[
 Y = k_1 c_2^{\frac{\kappa-1}{\kappa}} \frac{1}{1-\kappa}, \quad Z = -e^K \kappa^2 c_2^2 (e^{K \kappa^2} c_2^2 + A).
\]

Inserting the metric (4.2) into the Eq. (4.8), we obtain

\[
 -k_2 = (X - Y Z) r_{\text{crit}}^{1-\frac{1}{\kappa}},
\]
where 
\[ e^{\lambda(r)} = X r^{\frac{D-5}{2}} e^{-\nu}, \quad \text{i.e.} \quad X = e^{-2K} \kappa^{-2} c_2^{-\frac{2}{\kappa}}. \]

Let
\[ A < -\frac{1}{2} \kappa e^{\frac{1}{2} \nu} - \frac{e^{-K} c_2 k_1}{1 - \kappa}. \tag{4.9} \]

Calculating then the coefficient \((X - Y Z)\), we see that it is negative and this implies that \( r_{\text{crit}} \) does not exist, because \( k_2 \) is also negative. This means that the \( V^0 V^0 \) does not change its sign above the horizon. The sign is plus as it is seen from the asymptotical behavior of \( V^0 V^0 \). Note also that only one condition for \( A \) is to be fulfilled because (4.9) implies (4.6).

We conclude that in the case II the hair is real everywhere. Recall that in four dimensions there was no such solution [8].

4.3 Black-hole metric and the scalar curvature in dimensions \( D \geq 5 \)

Consider \( c_1 = 0 \). Then \( \sigma = \left( \frac{r}{c_2} \right)^{\frac{1}{\alpha_2}} \) and the metric
\[ e^\nu = \left[ A_{D-5} \left( \frac{r}{c_2} \right)^{D-5 + \frac{\alpha_1}{\alpha_2}} + k_1 \left( \frac{r}{c_2} \right)^{\frac{\alpha_1}{\alpha_2}} + k_2 \left( \frac{r}{c_2} \right)^{\frac{\alpha_2}{\alpha_2}} \right] r^{3-D}, \tag{4.10} \]
and, consequently
\[ e^\lambda = \left[ A_{D-5} \left( \frac{r}{c_2} \right)^{D-5 + \frac{\alpha_1}{\alpha_2}} + k_1 \left( \frac{r}{c_2} \right)^{\frac{\alpha_1}{\alpha_2}} + k_2 \left( \frac{r}{c_2} \right)^{\frac{\alpha_2}{\alpha_2}} \right]^{-1} \frac{r^{D-5}}{e^{2K} \alpha_2} \left( \frac{r}{c_2} \right)^{\frac{\alpha_2}{\alpha_2}}. \tag{4.11} \]

Note that \( A_{D-5} > 0 \). We would like to find the black hole solution. Let us set one of the integration constants \( k_1, \ k_2 \) equal to zero. Start with the case when \( k_1 = 0 \). Then the horizon does exist if \( k_2 < 0 \):
\[ r_H = c_2 \left( -\frac{A_{D-5}}{k_2} \right)^{\frac{\alpha_2}{2-(D-5)\alpha_2-2}}. \]

The another case is when \( k_2 = 0 \). Then the horizon is located at
\[ r_H = c_2 \left( -\frac{A_{D-5}}{k_1} \right)^{\frac{\alpha_1}{2-(D-5)\alpha_2-2}}. \]
The direct computation using (4.4) shows in both cases that $R$ diverges for $r \to 0$, tends to zero for $r \to \infty$ and is finite for $r \neq 0$. The conclusion is that our horizons hide the curvature singularities, i.e. we have the standard black-holes.

4.4 The hair in $D \geq 5$

First of all, let us study the behavior of $V^2 = V^\alpha V_\alpha$, the only scalar which can be constructed from our vector field $V^\alpha$. Because we are interested in the case when $c_1 = 0$ we have

$$V^2 = \left(\frac{r}{c_2}\right)^{-\frac{2}{\alpha^2}}.$$ 

From this expression it is clear that $V^2$ is positive everywhere, tends to zero for $r \to \infty$ and is smooth and bounded at the horizon.

Let us go to study the components of vector $V^\alpha$. Let $k_1 = 0$. Inserting the metric (4.10), (4.11) into (3.4) we can see the behavior of $f_1 f^1$. $f^1 f^1 \to$ finite positive constant and $f^1 f^1 \to -\infty$ when $r \to \infty$ and $r \to 0$ respectively. There is one null point of $f^1 f^1$ in $r = r_{\text{crit}}$, which depends on integration constants. Choosing the appropriate integration constant $A$, we can move this null point below the horizon. Then $f^1 f^1 > 0$ for $r > r_{\text{crit}}$. The analysis of $f^0 f^0$ shows that $f^0 f^0 \to \infty$ and $f^0 f^0 \to 0$ for $r \to 0$ and $r \to \infty$ respectively, but it is not necessarily positive everywhere. To ensure the positivity of $f^0 f^0$ at least for $r > r_{\text{crit}}$ (while $r_{\text{crit}} < r_H$) we have to satisfy the following condition (cf. the similar case in $D = 4$ [7])

$$\kappa \leq 1 + \frac{2}{D-4}.$$ 

The behavior of $V^0 V^0$ is qualitatively the same as the behavior of $f^0 f^0$. Note that in this case of choice of integration constant the behavior of hair is very similar to 4-dimensional case (see [7]).

Qualitatively new (and in $D \geq 5$ the most interesting) case occurs for $k_2 = 0$. Then the metric has the form

$$e^\nu = A_{D-5} \left(\frac{1}{c_2}\right)^{D-5+\frac{\beta_1}{\alpha^2}} r^{-2+\frac{\beta_1}{\alpha^2}} + k_1 \left(\frac{1}{c_2}\right)^{\frac{\beta_1}{\alpha^2}-\frac{\beta_2}{\alpha^2}-(D-3)},$$

Footnote 3 from page 9 applies here too.
\[
e^\lambda = \left[ A_{D-5} \left( \frac{r}{c_2} \right)^{D-5+\frac{2}{\alpha_2}} + k_1 \left( \frac{r}{c_2} \right)^{\frac{2}{\alpha_2}} \right]^{-1} \frac{r^{D-5}}{e^{2K} \alpha_2^2} \left( \frac{r}{c_2} \right)^{\frac{2}{\alpha_2}}.
\]

Inserting this metric into the expression

\[
V^1 V^1 = \frac{(D-2)(D-3)}{2\kappa r^2 \sigma'^2} - \frac{(D-2)e^{-\lambda}}{2\kappa} \left( \frac{(D-3)}{r^2 \sigma'^2} + \frac{2\sigma''}{r \sigma'^3} - \frac{\lambda'}{r \sigma'^2} \right) - \frac{A e^K}{\alpha_2^2 r^{D-2}},
\]

we obtain that

\[
V^1 V^1 = \frac{D-2}{2\kappa \sigma'^2} \left[ C r^{-2} + \left( -A c_2^{\frac{1}{\alpha_2}} \frac{2Ke^K}{D-2} + E \right) r^{-(D-2)-\frac{1}{\alpha_2}} \right],
\]

where

\[
C = D - 1 - \frac{1}{\alpha_2} - A_{D-5}(D-3)e^{2K} \alpha_2^2 c_2^{5-D}
\]

and

\[
E = k_1 \alpha_2^2 \left( -\frac{1}{\alpha_2} + 1 \right) e^{2K} \frac{2-\beta_2}{c_2^{\alpha_2^2}}.
\]

It can be shown that \( C \) is a positive constant. From (4.13) it is clear that using appropriate constant \( A \), namely \( A < c_2^{-\frac{D-2}{2\kappa c_2}} E \), our hair \( V^1 V^1 \) is positive in all spacetime. What about the behavior of \( V^0 V^0 \)? From the formula (4.7) it follows that \( V^0 V^0 \) is positive below the horizon, and using (4.13), it can be shown that \( V^0 V^0 \) is positive also everywhere above the horizon. Thus this case the hair \( V^0 V^0 \) and \( V^1 V^1 \) are positive everywhere.

Note that \( V^0 V^0 \) diverges at the horizon, though the invariant \( V^\alpha V^\alpha \) is smooth everywhere, including the horizon. It is not clear, however, how pathological is the divergent behavior of the non-invariant components of the field. It may be, that other propagating field coupled to \( V^\alpha \) do not feel any singularity at the horizon (as it is the case of another well-known case with the singular hair, i.e. the Bekenstein black hole\(^5\)[9]). The problem certainly requires a deeper analysis.

\(^5\)In that case, however, even a truly invariant quantity was singular at the horizon.
5. Conclusions and outlook

We have obtained the general $D$-dimensional static spherically symmetric solutions of the specific vector $\sigma$-model coupled to the Einstein gravity. The model arises in the studies of pure gravity in the non-commutative geometry setting. We have found a large subclass of the solutions having the structure of standard asymptotically flat black holes with a smooth horizon covering the curvature singularity. We have interpreted the “non-commutative” components of the metric as the hair and we found the dependence of its properties on the choice of integration constants. The latter turned out to be nontrivial and, in fact, quite restrictive. Unlike the case of $D = 4$ [7] (where the hair was necessarily imaginary near the singularity), there are the black hole solutions with the real hair everywhere! Thus, the peculiar singularity of the four dimensional black hole space-times can be removed by increasing the dimension.

We believe that the model (2.1) deserves further study, because maintaining the reparametrization invariance it decreases the propagating degrees of freedom of the standard gravity. Indeed, it can be seen from Eq. (2.3), that the “direction” fields $\check{f}^{\alpha}$ play essentially the role of the Lagrange multiplier. This property may be very important for further quantization. It can be also very interesting to find the consequences of the fact that the theory has the geometrical origin in the sense of non-commutative geometry.

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