Dynamics of Periodic Monopoles

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Abstract

BPS monopoles which are periodic in one of the spatial directions correspond, via a generalized Nahm transform, to solutions of the Hitchin equations on a cylinder. A one-parameter family of solutions of these equations, representing a geodesic in the 2-monopole moduli space, is constructed numerically. It corresponds to a slow-motion dynamical evolution, in which two parallel monopole chains collide and scatter at right angles.

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1 Introduction

The idea of understanding the slow-motion dynamics of BPS monopoles by studying the geodesics on the moduli space of static monopoles has been extensively exploited for monopoles in \( \mathbb{R}^3 \) — see [1] for a review. For example, much is known about the 4-dimensional moduli space of centred 2-monopoles in \( \mathbb{R}^3 \) (the Atiyah-Hitchin space), about its geodesics, and hence about the mutual scattering of two monopoles. The purpose of this note is to look at an example of the corresponding thing for periodic monopoles, i.e. monopoles on \( \mathbb{R}^2 \times S^1 \). (This could be visualized as the transverse scattering of two parallel chains of BPS monopoles.) In this case [2], the moduli space of centred 2-monopoles is again 4-dimensional, an “ALG gravitational instanton of type \( D_0 \)”, but much less is known about its metric; in particular, the geometry is believed to have no continuous symmetries (unlike for the Atiyah-Hitchin metric). But it does have discrete isometries, and this enables us to identify some geodesic submanifolds and geodesics. We shall see that one such geodesic corresponds to the head-on collision of two monopoles, resulting (as usual) in 90° scattering.

The approach is to use a generalized Nahm transform [3], which says that periodic monopoles correspond to certain solutions of the Hitchin equations. The latter are what one gets by reducing the (4-dimensional) self-dual Yang-Mills equations down to two dimensions. This system has been studied for some time. Long ago, it was noted there there are no finite-energy solutions of the SU(2) Hitchin equations on \( \mathbb{R}^2 \) [4, 5]. Hitchin’s comprehensive paper, investigating the equations on compact Riemann surfaces, appeared in 1987 [6]. For non-compact gauge groups such as SO(2,1), it was pointed out more recently that finite-energy solutions on \( \mathbb{R}^2 \) do exist [7].

In our case [3], we are interested in the SU(2) Hitchin equations on the cylinder \( \mathbb{R} \times S^1 \). These equations are analysed in section 2 below; section 3 reviews how their solutions may be used to construct periodic monopoles, and defines a natural totally-geodesic surface \( S \) in the moduli space; and section 4 describes the monopoles corresponding to points of \( S \), and in particular a geodesic in \( S \) which represents the head-on collision of two periodic monopoles. Finally, in section 5, we show how a particular explicit solution of the Hitchin equations may be obtained as the \( N \to \infty \) limit of the Nahm data for a finite chain of \( N \) monopoles.

Except for this special case, the Hitchin-equation solutions are not explicit, and have to be obtained numerically. The Nahm transform which generates the monopole fields is also implemented numerically, to produce the picture of monopole scattering. This is analogous to what was done to obtain examples of \( N \)-monopole scattering in \( \mathbb{R}^3 \) — see [1] for examples.
The SU(2) Hitchin Equations

The Hitchin equations may be viewed as a 2-dimensional reduction of the 4-dimensional self-dual Yang-Mills equations $F_{12} = F_{34}, F_{13} = F_{42}, F_{14} = F_{23}$. Assume that the gauge potential $A_\mu$ depends only on the two coordinates $(x^1, x^2)$, and write $s = x^1 + i x^2$ and $\Phi = A_3 - i A_4$. Then the SDYM equations reduce to the Hitchin equations

\begin{align*}
D_s \Phi &= 0, \\
F_{12} &= \frac{i}{2} [\Phi, \Phi^*].
\end{align*}

Here $D_s \Phi := \partial_s \Phi + [A_\bar{s}, \Phi]$, and $\Phi^*$ denotes the complex conjugate transpose of $\Phi$. Take the gauge group to be SU(2); the fields $\Phi$ and $A_\bar{s}$ are trace-free $2 \times 2$ complex matrices, and are smooth on some surface $\Sigma$ having local coordinates $(s, \bar{s})$.

We may partially solve the Hitchin equations as follows. First, note that (1) implies that $\det(\Phi)$ is holomorphic, so we write $\det(\Phi) = -H(s)$, and think of the holomorphic function $H(s)$ as being fixed. Next, we can use the gauge freedom to diagonalize the gauge field $F_{12} = -2i F_{s\bar{s}}$ on $\Sigma$ (as long as there is no global topological obstruction — but for the case we are interested in, namely $\Sigma = \mathbb{R} \times S^1$, there is no such obstruction). So we write $F_{12} = i B \sigma_3$, where $\sigma_3 = \text{diag}(1, -1)$, and where $B$ is a real-valued function on $\Sigma$. Let us assume that $B$ is not identically zero (the case where $B = 0$ is easy to solve locally). Then the residual gauge freedom is a local $O(2)$, consisting of a local $U(1)$

\begin{equation}
\Phi \mapsto U^{-1} \Phi U, \quad U := \exp[i u(s, \bar{s}) \sigma_3],
\end{equation}

plus a reflection. The second Hitchin equation (2) now implies that $\Phi$ has the form

\begin{equation}
\Phi = \begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix},
\end{equation}

where $fg = H$. The residual gauge freedom acts on the functions $f$ and $g$ by

\begin{align*}
&f \mapsto f e^{-2i u}, \quad g \mapsto g e^{2i u} \quad \text{(local $U(1)$)}, \\
&f \mapsto g, \quad g \mapsto f \quad \text{(reflection)}.
\end{align*}

The first Hitchin equation (1) is then equivalent to the gauge potential $A$ having the form $A_\bar{s} = a \sigma_3 + \alpha \Phi$, where $a(s, \bar{s})$ and $\alpha(s, \bar{s})$ are complex-valued functions, and where $a$ given by

\begin{equation}
2a = g^{-1} \partial_s g = -f^{-1} \partial_{\bar{s}} f.
\end{equation}
Finally, what remains of (2) is equivalent to the two equations

$$\Delta [\text{Re} \log(g/f)] = 2(1 + 4|\alpha|^2)(|g|^2 - |f|^2),$$  \hspace{1cm} (7)

where $\Delta = \partial_1^2 + \partial_2^2 = 4 \partial_s \partial_{\bar{s}}$ is the 2-dimensional Laplacian; and

$$f^{-1} \partial_s(|f|^2 \alpha) + g^{-1} \partial_{\bar{s}}(|g|^2 \bar{\alpha}) = 0.$$  \hspace{1cm} (8)

From (6) it follows that, in order for $a$ to be smooth, the functions $f$ and $g$ must have the form

$$f = \mu_+ e^{\psi/2}, \quad g = \mu_- e^{-\psi/2},$$  \hspace{1cm} (9)

where $\mu_\pm$ are holomorphic functions with $\mu_+ \mu_- = H$, and $\psi = \psi(s, \bar{s})$ is a smooth function. Using the residual gauge freedom (4), we can set $\psi$ to be real-valued. Then (7) becomes

$$\Delta \psi = 2(1 + 4|\alpha|^2)(\xi_+ e^{\psi} - \xi_- e^{-\psi}),$$  \hspace{1cm} (10)

where $\xi_\pm = |\mu_\pm|^2$; and (8) can be re-written as

$$e^{\psi/2} \partial_s(e^\psi \mu_+ \alpha) + e^{\psi/2} \partial_{\bar{s}}(e^{-\psi} \bar{\mu_-} \bar{\alpha}) = 0.$$  \hspace{1cm} (11)

To summarize: given $H(s)$, we first choose a holomorphic splitting $H = \mu_+ \mu_-$(essentially, the only choice is which zeros of $H$ to include in $\mu_+$, and which to include in $\mu_-)$). Then the Hitchin equations (10 2) are equivalent to the coupled system (11 for the real function $\psi$ and the complex function $\alpha$. Finally, the functions $f, g, a$ (and hence $\Phi$ and $A_{\bar{s}}$) are determined by (9) and $4a = -\partial_s \psi$. Note that the functions $\psi$ and $\alpha$ are gauge-invariant, except that $\psi$ changes sign under the reflection (5).

### 3 Nahm Transform

In this section, we review how certain solutions of the SU(2) Hitchin equations correspond, via a generalized Nahm transform, to centred periodic monopoles of charge 2. Such a periodic monopole solution consists of an SU(2) gauge field $\hat{A}_j = \hat{A}_j(x, y, z)$, and a Higgs field $\hat{\Phi} = \hat{\Phi}(x, y, z)$ in the adjoint representation, satisfying

- $\hat{D}_j \hat{\Phi} = -\frac{i}{2} \varepsilon_{jkl} \hat{F}_{kl}$, where $\hat{F}_{jk}$ is the gauge field obtained from $\hat{A}_j$;
- $\hat{\Phi}$ and $\hat{A}_j$ are smooth, and periodic in $z$ with period $2\pi$;
- locally in some gauge, $\hat{\Phi}$ and $\hat{A}_j$ satisfy the boundary conditions $\hat{\Phi} - (i/\pi)(\log \rho)\sigma_3 \rightarrow 0, \hat{A}_x \rightarrow 0, \hat{A}_y \rightarrow 0, \hat{\Phi} - (i\theta/\pi)\sigma_3 \rightarrow 0$ as $\rho \rightarrow \infty$. (Here $\rho$ and $\theta$ are polar coordinates: $x + iy = \rho e^{i\theta}$.)
A generalized Nahm transform [3] relates such monopoles to certain solutions of the SU(2) Hitchin equations on the cylinder $\mathbb{R} \times S^1$. Let $r \in \mathbb{R}$, and $t$ with period 1, denote the coordinates on this cylinder. The solutions are required to satisfy

- $H(s) = 2 \cosh(2\pi s) - K$, for some constant $K \in \mathbb{C}$; and
- $F_{12} \to 0$ as $r \to \pm \infty$.

In the first of these conditions, we could introduce a couple of parameters, by writing $H(s) = 2C \cosh(2\pi s) - K$, where $C$ is a complex constant. Then $|C|$ would determine the ratio between the monopole size and the spatial period, while $\text{arg}(C)$ would determine a spatial orientation. In the monopole picture, these two real parameters would show up by a slight alteration in the form of the boundary conditions on $\hat{\Phi}$ and $\hat{A}_j$. For simplicity, both of the parameters are omitted here, but it would be straightforward to re-introduce them. It is also worth pointing out that restricting to SU(2) rather than U(2) Hitchin fields has the effect of centring the corresponding monopoles, i.e. removing the translation freedom in $x, y, z$.

The details of the transform by which one obtains $\hat{\Phi}$ and $\hat{A}_j$ from $\Phi$ and $A_\bar{s}$ are as follows. Write $\zeta = x + iy$. Let $\Psi_+$ and $\Psi_-$ be $2 \times 2$ matrices satisfying the linear system

\begin{align}
2\partial_s \Psi_+ + 2A_\bar{s}\Psi_+ - z\Psi_+ + \zeta \Psi_- - \Phi \Psi_- &= 0,
\tag{12}
\end{align}

\begin{align}
2\partial_s \Psi_- + 2A_\bar{s}\Psi_- + z\Psi_- + \overline{\zeta} \Psi_+ - \Phi^\ast \Psi_+ &= 0,
\tag{13}
\end{align}

as well as the normalization condition $\langle \Psi_+, \Psi_+ \rangle + \langle \Psi_-, \Psi_- \rangle = I$. Here $I$ is the $2 \times 2$ identity matrix, and $\langle , \rangle$ is the $L^2$ inner product $\langle \Theta, \Gamma \rangle = \int_{-\infty}^{\infty} \int_0^1 \Theta^\ast \Gamma \, dt \, dr$. Let $\Psi$ be the $4 \times 2$ matrix obtained by adjoining $\Psi_+$ and $\Psi_-$, in other words

$\Psi = \begin{bmatrix} \Psi_+ \\ \Psi_- \end{bmatrix}$.

Then $\hat{\Phi}$ and $\hat{A}_j$ are given by

$\hat{\Phi} = i\langle \Psi, r\Psi \rangle, \quad \hat{A}_j = \langle \Psi, \partial_j \Psi \rangle$.

The moduli space $\hat{M}$ of centred periodic 2-monopoles is 4-dimensional, and it has a natural hyperkähler metric [2]. This space is the same (via the Nahm transform) as the moduli space $M$ of solutions of the Hitchin equations satisfying the various conditions listed above. (Strictly speaking, it has not been proved that the natural hyperkähler metrics on $\hat{M}$ and $M$ are isometric, but this is expected, on general grounds, to be true.) Two of the moduli consist of the real and imaginary parts of $K$, and the remaining two determine the function $\alpha$. 

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In what follows, we shall concentrate on the surface $S$ in $\hat{\mathcal{M}} \cong \mathcal{M}$ corresponding to $\alpha = 0$. This surface $S$ is a totally-geodesic submanifold of $\mathcal{M}$: to see this, we may reason as follows. Note, first, that rotation of the monopole field by $\pi$ about the $z$-axis (i.e. $\zeta \mapsto -\zeta$ or $\theta \mapsto \theta + \pi$) preserves the boundary condition on $(\hat{\Phi}, \hat{A}_j)$; the change in asymptotic behaviour can be compensated by a gauge transformation. So this map is an isometry of $\hat{\mathcal{M}}$. From (12, 13) we see that it corresponds to $\Psi_+ \mapsto \Psi_+$ and $\Psi_- \mapsto -\Psi_-$, together with the map $\Phi \mapsto -\Phi$, $\bar{A}_\phi \mapsto \bar{A}_\phi$ $\Leftrightarrow \psi \mapsto \psi$, $\alpha \mapsto -\alpha$. (15)

So (15), which clearly preserves the Hitchin equations and their boundary conditions, is also an isometry of the moduli space $\mathcal{M}$. It follows that the surface $S$ corresponding to $\alpha = 0$ is a totally-geodesic submanifold of $\mathcal{M}$; or equivalently that the surface corresponding to monopoles invariant under rotations by $\pi$ about the periodic axis, is a totally-geodesic submanifold of $\hat{\mathcal{M}}$.

4 Monopoles and Scattering

For our subfamily $S$ of solutions with $\alpha = 0$, the Hitchin equations reduce, in effect, to

$$\Delta \psi = 2(\xi_+ e^{\psi} - \xi_- e^{-\psi}).$$

In this section, we examine solutions of (16), and describe the corresponding periodic monopoles; in particular, a geodesic in $S$ representing a head-on scattering process.

First, there is the matter of defining $\mu_+ (s)$ and $\mu_- (s)$, which effectively means choosing their zeros. The function $H(s)$ has two zeros, and we shall adopt the most obvious choice of allocating one of them to $\mu_+(s)$ and the other to $\mu_-(s)$: set $\mu_\pm = e^{\pi s} \pm \lambda_\pm e^{-\pi s}$, where $\lambda_\pm = (K \pm \sqrt{K^2 - 4})/2$. In the square root, take $\sqrt{K^2 - 4}$ to be continuous on the complement of the cut $K \in (-2, 2)$, with Im$(\sqrt{K^2 - 4}) \geq 0$ for Im$(K) \geq 0$.

The boundary condition $F_{12} \to 0$ as $r \to \pm \infty$ is equivalent to a boundary condition on $\psi$, namely $\xi_+ e^{\psi} - \xi_- e^{-\psi} \to 0$ as $r \to \pm \infty$; with our choice of $\mu_\pm (s)$ this gives $\psi \to 0$ as $r \to +\infty$ and $\psi \to \log |\lambda_-/\lambda_+|$ as $r \to -\infty$. It is easy to see that, with this boundary condition (and fixing $K$), any small perturbation $\psi \mapsto \psi + \delta \psi$ of a solution $\psi$ of (16) has to be trivial. For if $\delta \psi$ is such a perturbation, with $\delta \psi \to 0$ as $r \to \pm \infty$, then $L(\delta \psi) = 0$, where $L$ is the strictly-negative operator $L = \Delta - 2(\xi_+ e^{\psi} + \xi_- e^{-\psi})$; so the only solution is $\delta \psi = 0$.

Given that $\psi(r, t)$ is to be periodic in $t$, we need to change gauge in order for $f$
and \( g \) to be periodic: instead of the expressions (9), we set

\[
f = \mu_+ e^{\psi/2} e^{-i\pi t} e^{i\omega}, \quad g = \mu_- e^{-\psi/2} e^{i\pi t} e^{-i\omega},
\]

where \( \omega \) is some real constant. Notice that if \( \lambda_+ \) and \( \lambda_- \) are interchanged, then \( \psi \mapsto -\psi \), and \( f \) and \( g \) are interchanged. This is a gauge transformation (5), so the corresponding solutions of the Hitchin equations are gauge-equivalent. In particular, there is no ambiguity (up to gauge-equivalence) on the cut \( K \in (-2, 2) \).

If \( K = \pm 2 \), then (16) admits the solution \( \psi = 0 \), and we get explicit solutions of the Hitchin equations for which \( F_{12} \) is identically zero. Conversely, it is straightforward to show that these are the only solutions for which \( F_{12} \) is identically zero. These solutions correspond to taking the explicit Hitchin-equation solution for the periodic 1-monopole [3, 8], and recycling it by simply doubling the period. They can also be obtained as limits of finite monopole chains, as we show in section 5.

For \( K \neq \pm 2 \), it is not as easy to find explicit solutions of (16). For the discussion that follows, the equation was solved numerically, for various values of \( K \), by minimizing the functional

\[
E[\psi] = \int_{\Sigma} \left[ \frac{1}{4} (\partial_j \psi)^2 + \xi_+ e^\psi + \xi_- e^{-\psi} - P \right] \, dr \, dt.
\]

(18)

Here \( P = P(r,t) \) is a fixed function, depending on \( K \), introduced simply in order to ensure that the integral converges. Note that (16) is the Euler-Lagrange equation for \( E[\psi] \), so any local minimum will be a solution of (16). The functional (18) was modelled using spectral methods: Chebyshev with \( n_r \) grid points for \( r \in [-L, L] \), and Fourier with \( n_t \) grid points for \( t \). The field \( \psi \) approaches its (finite) boundary values very rapidly — for example, \( \psi = o(e^{-2\pi r}) \) as \( r \to \infty \) — and there is little loss of accuracy in taking \( r \) to lie in a finite range \([-L, L]\), with \( L \) of order unity, and with \( \psi \) attaining its boundary values at \( r = \pm L \). The numerical minimization was effected using a conjugate-gradient method, for various values of the quantities \( L, n_r \) and \( n_t \), and the resulting fields \( \psi \) compared. For any two such fields \( \psi \) and \( \tilde{\psi} \), we typically obtain \( \| \psi - \tilde{\psi} \| < 0.01 \| \psi \| \), where \( \| \cdot \| \) denotes the supremum norm \( \| \psi \| = \max_{(r,t)} |\psi| \). On the basis of this, we believe that for each \( K \), our numerical solution \( \psi \) is within 1% of the actual solution.

Given such a solution \( \psi \), we then implemented the Nahm transform numerically, to obtain the monopole fields \((\hat{\Phi}, \hat{A}_j)\). For each point \((x, y, z)\) of a finite 3-dimensional grid, the equations (12, 13) were solved using a relaxation method, and the monopole fields were then computed from (14). Since \( \Psi \to 0 \) rapidly as \( r \to \pm \infty \), it is once again reasonable to restrict \( r \) to a finite range \([-L, L]\). The partial derivative in (14) is approximated by a simple finite difference on the grid. As a cross-check, we then
sampled the Bogomolny equations $\hat{D}_j \hat{\Phi} + \frac{1}{2} \epsilon_{jkl} \hat{F}_{kl} = 0$ at various points on the grid, and specifically near the location of the monopoles, again using finite differences for the derivatives. For a sufficiently fine grid (both in $r, t$ and in $x, y, z$) the accuracy is better than 1%, in the sense that $|\hat{D}_1 \hat{\Phi} + \hat{F}_{23}| < 0.01 |\hat{D}_1 \hat{\Phi}|$ (and analogously for the $\hat{D}_2 \hat{\Phi}$ and $\hat{D}_3 \hat{\Phi}$ equations). As a further check, this time of the boundary condition, we computed the quantity $\pi |\hat{\Phi}| / \log x$ on a segment of the $x$-axis; and verified that, as required, it approaches unity as $x$ becomes large. (In the $K = -2.3$ case, for example, the numerical result is that $\pi |\hat{\Phi}| / \log x = 0.986$ at $x = 6.5$.) These checks allow us to be reasonably confident that our numerical procedures give an accurate reflection of the true solution.

The results of these numerical investigations gives the following picture. For each $K \in \mathbb{C}$, there is exactly one solution $\psi$ of (16); and solutions for distinct $K$ are not gauge-equivalent. So the surface $\mathcal{S}$ is diffeomorphic to the plane $\mathbb{C}$, on which $K$ is a global coordinate. A solution with $|K| \gg 1$ corresponds to well-separated monopoles, located at the points $x + iy = \pm \sqrt{-K}$, $z = \pi$ (meaning that these are the points in $\mathbb{R}^2 \times S^1$ where the Higgs field $\hat{\Phi}$ is zero). Each monopole is roughly spherical in shape (see, for example, the $K = \pm 4$ pictures in Figure 1). If, at the other extreme, $K$ lies on the segment $[-2, 2]$ of the real line, then the monopoles are located on the periodic axis $x = y = 0$, at $z = \pi/4$ and $z = 3\pi/4$. They are elongated in the $x$-direction if $-2 \leq K < 0$ (see the $K = -2$ picture in Figure 1), and elongated in the $y$-direction if $0 < K \leq 2$.

To see the transition between the large-$|K|$ and small-$|K|$ regimes, one may examine the monopoles corresponding to the one-parameter family $K \in \mathbb{R}$. In fact, this represents a geodesic in $\mathcal{S}$ (as we shall see below), and hence also a geodesic in $\mathcal{M}$. So this family describes a slow-motion dynamical evolution of the system [9, 11], and provides yet another example of $90^\circ$ scattering following a head-on collision — this time between two parallel monopole chains, scattering transversely. See Figure 1, which plots the surface $|\hat{\Phi}|^2 := \text{tr}(\hat{\Phi} \hat{\Phi}^\ast) = 0.004$, for the monopole solutions corresponding to various real values of $K$. The interpretation in terms of parallel monopole chains, is that the chains approach each other in the $x$-direction; the individual monopoles coalesce and then separate to form a single chain of 1-monopoles; and finally the monopoles in this chain re-coalesce and then separate to form two parallel chains which move apart in the $y$-direction.

To see that $K \in \mathbb{R}$ is a geodesic, we show that the map $K \mapsto \overline{K}$ is an isometry. First, note that the isometry of $\mathbb{R}^2 \times S^1$ given by $x \mapsto x$, $y \mapsto -y$, $z \mapsto 2\pi - z$ preserves the field equation and boundary condition for periodic monopoles. So this map induces an isometry on $\widehat{\mathcal{M}}$. The corresponding map in the Hitchin-equation space is $\Phi(r,t) \mapsto \Phi(r,1-t)^\ast$, $A_r(r,t) \mapsto A_r(r,1-t)$, $A_t(r,t) \mapsto -A_t(r,1-t)$,
$K \mapsto \overline{K}$, which preserves the Hitchin equations and its associated constraints. The corresponding map on $\Psi_\pm$ is $\Psi_\pm(r, t; x, y, z) = \Psi_\mp(r, 1 - t; x, -y, 2\pi - z)$. So $K \mapsto \overline{K}$ is an isometry of $\mathcal{S} \subset \mathcal{M}$, and its fixed set $\text{Im}(K) = 0$ is a geodesic. These monopole solutions are invariant under the Klein group $D_2$, consisting of rotations by $\pi$ about each of the three coordinate axes. The solution for $K = 0$ has additional symmetry, namely rotation by $\pi/2$ about the $z$-axis.

In the asymptotic region $|K| \gg 1$, the metric on the moduli space is explicit, and simple to write down [2]; but in the interior region, it is likely to be rather complicated. As a starting-point, we have seen that there is a totally-geodesic surface $\mathcal{S}$ in the moduli space, representing to periodic 2-monopoles which are invariant under rotation by $\pi$ about the periodic axis; and that these correspond, via the Nahm transform, to real-valued functions $\psi(r, t)$ satisfying [10], together with appropriate boundary conditions. One geodesic in $\mathcal{S}$ represents a head-on collision of two
monopole chains. It should be feasible to extend the procedure used in this note, to learn more about the metric and its geodesics, and hence about more general scattering processes.

5 Limits of Finite Chains

In this section, we will examine how the infinite chain of 1-monopoles arises as the \( N \to \infty \) limit of a finite chain of \( N \) single monopoles. One knows that this limit is delicate, and that care is needed to avoid divergences (cf. [10, 3, 11]). The aim here is to describe the limit in the Nahm-transformed picture. The Nahm data for the \( N \)-chain are known [12], and we will see how to take the \( N \to \infty \) limit, thereby obtaining the Hitchin data corresponding to an infinite chain. In fact, one may regard this monopole field as being a chain of \( m \)-monopoles, for any integer \( m \geq 1 \), with \( U(m) \)-valued Hitchin data; we shall do the two cases \( m = 1 \) and \( m = 2 \), and the latter therefore links up with our earlier material on 2-monopole chains.

The Nahm data for a monopole of charge \( N \) consists of three anti-Hermitian matrix-valued functions \( T_a(u) \), for \( a = 1, 2, 3 \) and \( u \in (-1, 1) \), which satisfy the Nahm equation \( dT_a/du = \frac{1}{2} \epsilon_{abc} [T_b, T_c] \), plus appropriate boundary conditions. The Nahm data for an \( N \)-chain have the form \( T_a(u) = -if_a(u)J_a \), where \( J_a \) are the generators of the \( N \)-dimensional irreducible representation of \( SU(2) \), and satisfy \([ J_a, J_b ] = i \epsilon_{abc} J_c \). Explicitly, we may set \( J_1 = \frac{1}{2}(J_+ + J_-) \), \( J_2 = \frac{1}{2}i(J_+ - J_-) \) and \( (J_3)_{ij} = -j \delta_{i,j} \), where

\[
(J_+)_{ij} = \frac{1}{2} \delta_{i+1,j} \sqrt{(N + 1 + 2i)(N + 1 - 2j)},
\]

\[
(J_-)_{ij} = \frac{1}{2} \delta_{i,j+1} \sqrt{(N + 1 + 2j)(N + 1 - 2i)}.
\]

The indices \( i, j \) run from \( -(N - 1)/2 \) to \( (N - 1)/2 \), and take values in \( \mathbb{Z} \) or \( \mathbb{Z} + 1/2 \) according to whether \( N \) is odd or even. The functions \( f_a \) are specified in terms of Jacobi elliptic functions by

\[
f_1(u) = K k' \text{nc}(Ku; k), \quad f_2(u) = K k' \text{sc}(Ku; k), \quad f_3(u) = K \text{dc}(Ku; k),
\]

where \( k \) is the elliptic modulus, \( k' = \sqrt{1 - k^2} \) the complementary modulus, and \( K(k) \) the complete elliptic integral of the first kind. These Nahm data correspond to a finite-length collinear chain of \( N \) single monopoles, and the parameter \( k \) determines the ratio between the monopole size and the distance between adjacent monopoles in the chain.

Now define a new variable \( r = Ku/(2\pi) \). Then \( T'_a = 2\pi T_a/K \) solve the rescaled Nahm equation

\[
\frac{d}{dr} T'_a = \frac{1}{2} \epsilon_{abc} [T'_b, T'_c].
\]
We restrict attention to the cases where \( N \) is odd, and consider the limit \( N \to \infty, k' \to 0 \), with \( \pi Nk' \to L \) for some constant \( L \in \mathbb{R} \). In this limit the matrices \( T_a' \) become infinite-dimensional, and naturally operate on functions \( u(r,t) \), periodic in \( t \), with Fourier expansion \( u = \sum_{j \in \mathbb{Z}} u_j(r) \exp(2\pi ijt) \). Note that the range of \( r \) becomes \((-\infty, \infty)\) in the limit. Using known limits of elliptic functions, we obtain

\[
(\lim T_1') \cdot u = -A_4(r,t)u(r,t),
(\lim T_2') \cdot u = A_3(r,t)u(r,t),
(\lim T_3') \cdot u = \partial_t u(r,t),
\]

where \( A_3(r,t) = iL \sinh(2\pi r) \sin(2\pi t) \) and \( A_4(r,t) = iL \cosh(2\pi r) \cos(2\pi t) \). The limit of the Nahm equation (19) can be rewritten as the operator equation

\[
[\partial_r, \lim T_a'] = \frac{1}{2} \varepsilon_{abc} [\lim T_b', \lim T_c'],
\]

which is equivalent to the Hitchin equations (1, 2) under the identifications \( A_1 = 0 \), \( \partial_t + A_2 = \lim T_3' \), and \( \Phi = A_3 - iA_4 \). Explicitly, we obtain the known solution of the Hitchin equations corresponding to a charge 1 periodic monopole, namely \( A_1 = 0 \), \( A_2 = 0 \), \( \Phi = L \cosh(2\pi s) \).

As mentioned above, one may vary this procedure to obtain Hitchin data for an infinite chain of \( m \)-monopoles, starting from the same Nahm data; here is the \( m = 2 \) version. This time, define \( r = Ku/\pi \), so that \( T_a' = \pi T_a/K \) solve (19). Restrict attention to the cases where \( N \) is even, and consider again the limit where \( N \to \infty, k' \to 0 \), with \( \pi Nk' \to L \) constant. The resulting infinite matrices act naturally on vectors \( u_j \) with \( j \in \mathbb{Z} + \frac{1}{2} \), and hence on functions

\[
u(r,t) = \sum_{k \in \mathbb{Z}} \begin{pmatrix} u_{2k-\frac{1}{2}}(r) \\ u_{2k+\frac{1}{2}}(r) \end{pmatrix} \exp(2\pi ikt).
\]

A direct calculation yields

\[
(\lim T_1') \cdot u = -A_4(r,t)u(r,t),
(\lim T_2') \cdot u = A_3(r,t)u(r,t),
(\lim T_3') \cdot u = \left( \frac{\partial}{\partial t} - \frac{i\pi}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) u(r,t),
\]

\[
A_4(r,t) := \frac{iL}{2} \cosh(\pi r) \cos(\pi t) \begin{pmatrix} 0 & e^{\pi it} \\ e^{-\pi it} & 0 \end{pmatrix},
A_3(r,t) := \frac{iL}{2} \sinh(\pi r) \sin(\pi t) \begin{pmatrix} 0 & e^{\pi it} \\ e^{-\pi it} & 0 \end{pmatrix}.
\]
Hence we obtain Hitchin data

\[ A_1 = 0, \quad A_2 = -i\frac{\pi}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi = \frac{L}{2} \cosh(\pi s) \begin{pmatrix} 0 & e^{\pi i t} \\ e^{-\pi i t} & 0 \end{pmatrix}. \]

With \( L = 4 \), these are gauge-equivalent to the explicit solution \( \alpha = 0, \ K = -2, \ \psi = 0 \) mentioned in section 4. The above discussion justifies the claim that, in this case, the 2-monopole chain is just a 1-monopole chain in disguise.

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