PRODUCTS OF SEQUENTIALLY COMPACT SPACES AND COMPACTNESS WITH RESPECT TO A SET OF FILTERS

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Abstract. We show that, under suitably general formulations, covering properties, accumulation properties and filter convergence are all equivalent notions. This general correspondence is exemplified in the study of products.

Let $X$ be a product of topological spaces. We prove that $X$ is sequentially compact if and only if all subproducts by $\leq s$ factors are sequentially compact. If $s = h$, then $X$ is sequentially compact if and only if all factors are sequentially compact and all but at most $< s$ factors are ultraconnected. We give a topological proof of the inequality $cf s \geq h$. Recall that $s$ denotes the splitting number and $h$ the distributivity number. The product $X$ is Lindelöf if and only if all subproducts by $\leq \omega_1$ factors are Lindelöf. Parallel results are obtained for final $\omega_n$-compactness, $[\lambda, \mu]$-compactness, the Menger and Rothberger properties.

1. Introduction

All sections of the paper are mostly self-contained. The reader interested only in the results about sequential compactness might skip directly to Section 6. The proof of $cf s \geq h$ appears at the beginning of Section 7. The reader interested only in products of Lindelöf spaces or, more generally, finally $\omega_n$-compact or $[\mu, \lambda]$-compact spaces might skip to Section 4 and turn back when needed. Results about the Menger and Rothberger properties are presented in Section 5. In Section 2 we provide a characterization theorem which shows that, under suitably

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general formulations, covering properties, accumulation properties and filter convergence are all sides of the same coin.

The main theme of the present note is the following: given a property $P$ of topological spaces, find some cardinal $\kappa$ such that a product satisfies $P$ if and only if all subproducts by $\leq \kappa$ factors satisfy $P$. We believe that the problem is best seen in terms of the general context of compactness with respect to a set of filters, as introduced in [19, 21], though in many particular cases we get better results by direct means. We are going to briefly review the general notion here.

The notion of filter and ultrafilter convergence plays a key role in the study of products of topological spaces; see the surveys Stephenson [28], Vaughan [32], García-Ferreira and Kočínac [12], and further references there and in [20, 21]. In [17] Kombarov introduced a local notion of ultrafilter convergence, where, by “local”, we mean that the ultrafilter depends on the sequence intended to converge, rather than being fixed in advance. Kombarov put in a general setting the idea by Ginsburg and Saks [14] that countable compactness has an equivalent formulation in this “local” fashion, but cannot be defined by ultrafilter convergence in a “strict” sense (that is, in terms of a single fixed ultrafilter).

In [21] we extended Kombarov notion to filters, and showed that this is a proper generalization, since, for example, sequential compactness can be characterized in terms of such a “local” filter convergence, but all the filters involved, in this case, are necessarily not maximal [21, Section 5]. We called this general notion sequencewise $\mathcal{P}$-compactness (here and in what follows $\mathcal{P}$ is always a family of filters over some set $I$) and in [21] we mentioned that sequencewise $\mathcal{P}$-compactness incorporates many compactness, covering and convergence properties, including sequential compactness, countable compactness, initial $\kappa$-compactness, $[\lambda, \mu]$-compactness and the Menger and Rothberger properties.

In fact, the definition of sequencewise $\mathcal{P}$-compactness already appeared hidden and in different terminology in a remark contained in [19], together with equivalent formulations. In Section 2 after recalling the relevant definitions together with some examples, we state the result in full as a theorem, with some details of the proof. Then in Section 3 we show that a product of topological spaces is sequencewise $\mathcal{P}$-compact if and only if so is any subproduct with $\leq |\mathcal{P}|$ factors (Theorem 3.1). This unifies many former results by W. W. Comfort, J. Ginsburg, V. Saks, C. T. Scarborough, A. H. Stone and possibly others.
In the subsequent sections we apply Theorem 3.1 to many particular cases, usually getting better bounds by additional methods. Actually, in certain situations Theorem 3.1 is not used at all. In details, in Section 4 we find optimal values in the case both of final $\omega_n$-compactness and of $[\omega_n, \lambda]$-compactness, for $\lambda$ singular strong limit of cofinality $\omega_n$. See Theorems 4.1 and 4.3. On the other hand, the values obtained for the general case of $[\mu, \lambda]$-compactness are essentially those given by Theorem 3.1. See Corollary 4.10. There are obstacles to extending, say, Theorem 4.1, which deals with final $\omega_n$-compactness, to cardinals $\geq \omega_\omega$; this is briefly hinted at the end of the section.

Section 5 is concerned with the Menger and Rothberger properties and their versions for countable covers. In Section 6 we deal with sequential compactness and prove the results mentioned in the abstract. We also show that the assumption $s = h$ is necessary in Corollary 6.6 When $s = h$ is not assumed, similar results can only be proved by considering subproducts, rather than factors.

Finally, Section 7 contains a proof that $\text{cf } s \geq h$. The idea is general, and suggests a way of attaching corresponding invariants to every property of topological spaces (or even more general objects). Some of these invariants are cardinals, but it is also natural to consider classes of cardinals. Some very basic properties of such invariants are discussed.

Throughout the paper we shall assume no separation axiom. In order to avoid trivial exceptions, all topological spaces under consideration are assumed to be nonempty. In all the theorems concerning products, if not otherwise mentioned, repetitions are allowed, that is, the same space might occur multiple times as a factor (in other words, we are dealing with products of sequences, not with products of sets).

2. Equivalents of sequencewise $\mathcal{P}$-compactness

We first recall the basic definitions. We refer to [19] [21] for further motivations, examples and references.

If $X$ is a topological space, $I$ is a set, $(x_i)_{i \in I}$ is an $I$-indexed sequence of elements of $X$ and $F$ is a filter over $I$, a point $x \in X$ is an $F$-limit point of the sequence $(x_i)_{i \in I}$ if $\{i \in I \mid x_i \in U\} \in F$, for every open neighborhood $U$ of $x$. If this is the case, we shall also say that $(x_i)_{i \in I}$ $F$-converges to $x$. Notice that, in general, unless the Hausdorff separation axiom is assumed, such an $x$ is not necessarily unique. $X$ is $F$-compact if every $I$-indexed sequence of elements of $X$ $F$-converges to some point of $X$.

**Definition 2.1.** If $\mathcal{P}$ is a family of filters over the same set $I$, a topological space $X$ is sequencewise $\mathcal{P}$-compact if, for every $I$-indexed sequence
of elements of $X$, there is $F \in \mathcal{P}$ such that the sequence has an $F$-limit point.

As observed in [21], sequencewise $\mathcal{P}$-compactness generalizes former notions introduced by Kombarov [17] and García-Ferreira [11] under different names. Considering filters which are not necessarily ultra provides a substantial generalization, as shown in the next remark.

**Remark 2.2.** A sequence $(x_n)_{n \in \omega}$ converges if and only if it has an $F$-limit point, for the Fréchet filter $F$ over $\omega$. This shows that sequential compactness is equivalent to sequencewise $\mathcal{P}$-compactness, for an appropriate $\mathcal{P}$. Just take $\mathcal{P} = \{F_Z | Z \in [\omega]^\omega\}$, where $F_Z = \{W \subseteq \omega | Z \setminus W$ is finite\} and $[\omega]^\omega$ denotes the set of all infinite subsets of $\omega$. See [21] for more details.

We are now going to see that sequencewise $\mathcal{P}$-compactness admits equivalent formulations (a result implicit in [19]), but first we need some definitions.

**Definition 2.3.** Let $A$ be a set, and $B, G \subseteq \mathcal{P}(A)$, where $\mathcal{P}(A)$ denotes the set of all subsets of $A$. A topological space $X$ is $[B, G]$-compact if, whenever $(O_a)_{a \in A}$ is a sequence of open sets of $X$ such that $(O_a)_{a \in K}$ is a cover of $X$, for every $K \in G$, then there is $H \in B$ such that $(O_a)_{a \in H}$ is a cover of $X$.

Covering properties like compactness and countable compactness, which involve just one “starting” cover, can be expressed as particular cases of Definition 2.3 by taking $G = \{A\}$. For example, to get countable compactness take $A$ countable, $G = \{A\}$ and $B = [A]^{<\omega}$, the set of all finite subsets of $A$.

It is useful to consider the general case in which $G$ contains more than one set. The reason is that in this way we can also get covering properties which involve simultaneously many “starting” covers, as is the case for the Menger and Rothberger properties. For example, take $A = \omega$, $G$ a partition of $\omega$ into infinitely many infinite classes, and $B$ the family of those sets that intersect each member of $G$ in a finite (respectively, one-element) set. In this case we get the **Menger** (respectively, **Rothberger** property for countable covers. More generally, if $\lambda, \mu$ are cardinals, take $A = \lambda \cdot \mu$, $G$ a partition of $A$ into $\lambda$-many pieces of cardinality $\mu$, and $B$ the family of those sets that intersect each member of $G$ in a set of cardinality $< \kappa$. Then we get the property that any $\lambda$-sequence of open covers of size $\leq \mu$ admits a $<\kappa$-selection, a property denoted by $R(\lambda, \mu; <\kappa)$ in [20]. Clearly, the Menger and Rothberger properties can be obtained from $R(\omega, \lambda; <\omega)$, respectively $R(\omega, \lambda; <2)$, by letting $\lambda$ be arbitrarily large.
Definition 2.4. Suppose that $I$ is a set, $E \subseteq \mathcal{P}(I)$, $\mathcal{E}$ is a set of subsets of $\mathcal{P}(I)$ and $X$ is a topological space.

If $(x_i)_{i \in I}$ is a sequence of elements of $X$, we say that $x \in X$ is an $E$-accumulation point of $(x_i)_{i \in I}$ if $\{i \in I \mid x_i \in U\} \in E$, for every open neighborhood $U$ of $x$ in $X$.

We say that $x \in X$ is an $E$-accumulation point of $(x_i)_{i \in I}$ if and only if there is $E \in \mathcal{E}$ such that $x$ is an $E$-accumulation point of $(x_i)_{i \in I}$.

We say that $X$ satisfies the $\mathcal{E}$-accumulation property (the $\mathcal{E}$-accumulation property) if every $I$-indexed sequence of elements of $X$ has some $\mathcal{E}$-accumulation point (some $E$-accumulation point).

Remark 2.5. In the particular case when $E$ is a filter, $E$-accumulation points are exactly $E$-limit points; hence in this case $E$-compactness is the same as the $E$-accumulation property. So, if each member of $\mathcal{E}$ is a filter, then the $\mathcal{E}$-accumulation property is the same as sequencewise $\mathcal{E}$-compactness.

Remark 2.6. Countable compactness is another motivating example for our definition of the $\mathcal{E}$-accumulation property. Indeed, a topological space is countably compact if and only if it satisfies the $E$-accumulation property, with $E$ the set of all infinite subsets of $\omega$. But countable compactness is also equivalent to sequencewise $\mathcal{P}$-compactness, for the family $\mathcal{P}$ of all uniform ultrafilters over $\omega$. The equivalent formulations of countable compactness can be seen as a prototypical example of the general equivalence given by Theorem 2.7 below. See [19, Remark 2.5] for a full discussion.

The next theorem is implicit in [19]. We give details for the reader’s convenience.

Theorem 2.7. For every class $\mathcal{K}$ of topological spaces, the following conditions are equivalent.

(i) $\mathcal{K}$ is the class of all $[B, G]$-compact spaces, for some set $A$ and sets $B, G \subseteq \mathcal{P}(A)$.

(ii) $\mathcal{K}$ is the class of all the spaces satisfying the $\mathcal{E}$-accumulation property, for some set $I$ and some family $\mathcal{E}$ of subsets of $\mathcal{P}(I)$ such that each member of $\mathcal{E}$ is closed under supersets.

(iii) $\mathcal{K}$ is the class of all sequencewise $\mathcal{P}$-compact spaces, for some $\mathcal{P}$.

Given a class $\mathcal{K}$ and $B, G$ satisfying (i), there is $\mathcal{E}$ such that $|\mathcal{E}| \leq |G|$ and (ii) is satisfied. Conversely, if (ii) is satisfied for some $\mathcal{K}$ and $\mathcal{E}$, there are $B$ and $G$ such that (i) is satisfied and $|G| \leq |\mathcal{E}|$. On the other hand, there are a class $\mathcal{K}$ and some $\mathcal{E}$ such that (ii) holds, but for any $\mathcal{P}$ satisfying (iii) we have $|\mathcal{P}| > |\mathcal{E}|$. 
Before proving Theorem 2.7, we need some lemmas which may be of independent interest.

**Lemma 2.8.** [19] p. 300 A space $X$ is $[B, G]$-compact if and only if, for every sequence $(C_a)_{a \in A}$ of closed sets of $X$, if $\bigcap_{a \in H} C_a \neq \emptyset$, for every $H \in B$, then there is $K \in G$ such that $\bigcap_{a \in K} C_a \neq \emptyset$.

*Proof.* By stating the implication in the definition of $[B, G]$-compactness in contrapositive form and taking complements. □

**Lemma 2.9.** Given $A, B, G$ as in the definition of $[B, G]$-compactness, let $I = B$ and $E = \{E_K \mid K \in G\}$ where, for $K \in G$, we set $E_K = \{Z \subseteq B \mid \text{for every } a \in K, \text{ there is } H \in Z \text{ such that } a \in H\} = \{Z \subseteq B \mid \bigcup_{H \in Z} H \supseteq K\}$.

Under the above definitions, $[B, G]$-compactness is equivalent to the $E$-accumulation property.

*Proof.* Assume that $X$ satisfies the $E$-accumulation property, for $E$ as in the statement of the lemma, and assume that $(C_a)_{a \in A}$ is a sequence of closed sets such that $\bigcap_{a \in H} C_a \neq \emptyset$, for every $H \in B$. For every $H \in B$, pick $x_H \in \bigcap_{a \in H} C_a$. By the $E$-accumulation property, the sequence $(x_H)_{H \in B}$ has an $E_K$-accumulation point $x$, for some $K \in G$ (recall that $I = B$). Thus, for every neighborhood $U$ of $x$, $\{H \in B \mid x_H \in U\} \in E_K$, that is, for every $a \in K$, there is some $H$ such that $x_H \in U$ and $a \in H$. If $a \in H$, then $x_H \in C_a$, by construction; thus every neighborhood of $x$ intersects $C_a$, hence $x \in C_a$, since $C_a$ is closed. This holds for every $a \in K$, hence $x \in \bigcap_{a \in K} C_a$, thus $\bigcap_{a \in K} C_a \neq \emptyset$. This implies $[B, G]$-compactness, by Lemma 2.8.

Conversely, assume that $X$ is $[B, G]$-compact and let $(x_H)_{H \in B}$ be a sequence of elements in $X$. For $a \in A$, let $C_a = \{x_H \mid a \in H\}$, thus $x_H \in \bigcap_{a \in H} C_a$, for every $H \in B$. In particular, $\bigcap_{a \in H} C_a \neq \emptyset$. By Lemma 2.8, $\bigcap_{a \in K} C_a \neq \emptyset$, for some $K \in G$. Let $x \in \bigcap_{a \in K} C_a$. Since $C_a = \{x_H \mid a \in H\}$, then, for every $a \in K$ and every neighborhood $U$ of $x$, there is some $H$ such that $a \in H$ and $x_H \in U$. This means that, for every neighborhood $U$ of $x$, $\bigcup \{H \mid x_H \in U\} \supseteq K$, that is, $x$ is an $E_K$-accumulation point of $(x_H)_{H \in B}$, in particular, an $E$-accumulation point.

Compare the above proof with [19] Theorem 5.8 (1) $\Rightarrow$ (5)]. □

If $E \subseteq \mathcal{P}(I)$, we say that $E$ is closed under supersets (in $I$) if whenever $e \in E$ and $e \subseteq f \subseteq I$, then $f \in E$. We let $E^+_I = \{a \subseteq I \mid a \cap e \neq \emptyset, \text{ for every } e \in E\}$. Usually, the set $I$ will be clear from the context and reference to it shall be dropped. Notice that, in case $E$ is a filter, then $E^+$ is the complement in $\mathcal{P}(I)$ of the dual ideal of $E$.  


This observation justifies the notation. Symmetrically, if $\mathcal{P}(I) \setminus E$ is an ideal, then $E^+$ is the filter dual to this ideal.

**Lemma 2.10.** For every $E \subseteq \mathcal{P}(I)$, we have that $E^+$ is closed under supersets. Moreover, $E^{++} = E$ if and only if $E$ is closed under supersets.

*Proof.* The first statement is immediate from the definition. In particular, $E^{++}$ is closed under supersets, hence if $E^{++} = E$, then $E$ is closed under supersets.

To prove the converse, $E^{++} \supseteq E$ is immediate from the definition. Suppose by contradiction that $E^{++} \subsetneq E$ and $E$ is closed under supersets, thus there is $f \in E^{++}\setminus E$ such that $f \cap a \neq \emptyset$, for every $a \in E^+$. Since $E$ is closed under supersets and $f \notin E$, then, for every $e \in E$, there is some $i_e \in e \setminus f$. Then, by construction, $a = \{i_e \mid e \in E\} \in E^+$, but $a \cap f = \emptyset$, a contradiction. \hfill \Box

**Lemma 2.11.** [19, Proposition 3.11] Suppose that $X$ is a topological space, $x \in X$, $I$ is a set, and $(x_i)_{i \in I}$ is a sequence of elements of $X$. Suppose that $K \subseteq \mathcal{P}(I)$, $E = K^+$, and, for $a \in K$, put $D_a = \{x_i \mid i \in a\}$.

Then the following conditions are equivalent.

(1) $x$ is an $E$-accumulation point of $(x_i)_{i \in I}$.

(2) $x \in \bigcap_{a \in K} D_a$.

*Proof.* If (1) holds and $a \in K$, then, for every neighborhood $U$ of $x$, $e_U = \{i \in I \mid x_i \in U\} \in E$, thus $a \cap e_U \neq \emptyset$, by the definition of $E$. If $i \in a \cap e_U$, then $x_i \in D_a \cap U$, hence $D_a \cap U \neq \emptyset$. Since $D_a$ is closed, and $D_a \cap U \neq \emptyset$, for every neighborhood $U$ of $x$, then $x \in D_a$. Since $a$ was arbitrary in the above argument, we have $x \in \bigcap_{a \in K} D_a$.

If (2) holds and $U$ is a neighborhood of $x$, let $e_U = \{i \in I \mid x_i \in U\}$. For every $a \in K$, by (2), $x \in D_a$ and, by the definition of $D_a$, there is $i \in a$ such that $x_i \in U$. By the definition of $e_U$, $i \in e_U$, thus $i \in e_U \cap a \neq \emptyset$. Thus $e_U \in E = K^+$, for every neighborhood of $x$, and this means that $x$ is an $E$-accumulation point of $(x_i)_{i \in I}$. \hfill \Box

**Lemma 2.12.** Suppose that $I$ is a set, $\mathcal{E}$ is a set of subsets of $\mathcal{P}(I)$ and every $E \in \mathcal{E}$ is closed under supersets. Let $A = \mathcal{P}(I)$, $G = \{E^+ \mid E \in \mathcal{E}\}$ and $B = \{i^c \mid i \in I\}$, where, for $i \in I$, $i^c = \{a \in A \mid i \in a\}$.

Then, for every topological space $X$, the following conditions are equivalent.

(1) $X$ satisfies the $\mathcal{E}$-accumulation property.

(2) $X$ is $[B, G]$-compact.
Proof. (1) ⇒ (2). Using Lemma 2.8 suppose that \((C_a)_{a \in A}\) are closed sets and \(\bigcap_{a \in H} C_a \neq \emptyset\), for every \(H \in B\). Because of the definition of \(B\), this means \(\bigcap \{C_a \mid i \in a\} \neq \emptyset\), for every \(i \in I\). For each \(i \in I\), choose \(x_i \in \bigcap \{C_a \mid i \in a\}\). By the \(\mathcal{E}\)-accumulation property, there are \(E \in \mathcal{E}\) and \(x \in X\) such that \(x\) is an \(E\)-accumulation point of \((x_i)_{i \in I}\). If \(K = E^+\), then \(E = E^{++} = K^+\), by Lemma 2.10 and since \(E\) is closed under supersets, by assumption. If \(D_a = \{x_i \mid i \in a\}\), for \(a \in K\), then, by Lemma 2.11, \(x \in \bigcap_{a \in K} D_a \subseteq \bigcap_{a \in K} C_a\). Since \(K = E^+ \in G\), this shows that \(X\) is \([B, G]\)-compact.

(2) ⇒ (1) Suppose that \((x_i)_{i \in I}\) is a sequence, and set \(C_a = \{x_i \mid i \in a\}\), for \(a \in A\). If \(H \in B\), say, \(H = i^<\), then \(x_i \in \bigcap_{a \in H} C_a\), hence, by \([B, G]\)-compactness and Lemma 2.8 there is \(K \in G\) such that \(\bigcap_{a \in K} C_a \neq \emptyset\). By the definition of \(G\), \(K = E^+\), for some \(E \in \mathcal{E}\), hence \(E = E^{++} = K^+\), by Lemma 2.10 since \(E\) is closed under supersets. Then \((x_i)_{i \in I}\) has an \(E\)-accumulation point, by Lemma 2.11. This proves the \(\mathcal{E}\)-accumulation property.

Proof of Theorem 2.7. (i) ⇒ (ii) follows from Lemma 2.9 noticing that the \(\mathcal{E}_K\)‘s defined there are closed under supersets.

(ii) ⇒ (i) follows from Lemma 2.12.

(ii) ⇒ (iii) If \(E\) is closed under supersets, \(x\) is an \(E\)-accumulation point of the sequence \((x_i)_{i \in I}\) and, for every neighborhood \(U\) of \(x\), we set \(x_U = \{i \in I \mid x_i \in U\}\), then the family \(\{x_U \mid U\}\) a neighborhood of \(x\) generates a filter \(E'\) which is contained in \(E\), and \(x\) is an \(E'\)-accumulation point of the sequence \((x_i)_{i \in I}\).

Thus if each \(E \in \mathcal{E}\) is closed under supersets, then the \(\mathcal{E}\)-accumulation property is equivalent to the \(\mathcal{E}'\)-accumulation property, where \(\mathcal{E}'\) is the set of all filters which are contained in some element of \(\mathcal{E}\). Then \(\mathcal{E}'\) contains only filters, and the \(\mathcal{E}'\)-accumulation property is the same as sequencewise \(\mathcal{E}'\)-compactness, by Remark 2.5. The implication (iii) ⇒ (ii) is trivial from the same remark.

Again by Lemmas 2.9 and 2.12 the sets \(\mathcal{E}\) and \(G\) can be chosen to satisfy the cardinality requirements. On the other hand, as mentioned, countable compactness can be characterized as the \(\mathcal{E}\)-accumulation property, for some one-element \(\mathcal{E}\), but countable compactness is not equivalent to sequencewise \(\mathcal{P}\)-compactness, for any one-element \(\mathcal{P}\). Indeed, this would mean \(F\)-compactness, for the single filter \(F\) belonging to \(\mathcal{P}\), but it is well known that \(F\)-compactness is preserved under products, while countable compactness is not.

Corollary 2.13. [19 Corollaries 3.5 and 3.10] For every class \(\mathcal{K}\) of topological spaces, the following conditions are equivalent.
(i) $\mathcal{K}$ is the class of all $[B, \{A\}]$-compact spaces, for some $A$ and $B \subseteq \mathcal{P}(A)$.

(ii) $\mathcal{K}$ is the class of all the spaces satisfying the $E$-accumulation property, for some set $I$ and some $E \subseteq \mathcal{P}(I)$ closed under supersets.

At first sight one could be tempted to believe that condition (iii) in Theorem 2.7 is always preferable to condition (ii), since $\mathcal{P}$ in (iii) contains only filters, which are surely more manageable subsets of $\mathcal{P}(I)$ than the members of $E$ in (ii), which are only supposed to be closed under supersets. However the last statement in Theorem 2.7 shows that there are cases in which the $E$ in (ii) has the advantage of having much smaller cardinality than $\mathcal{P}$.

The proof of Theorem 2.7 gives explicit constructions, which are of some use even in particular cases. For example, the proof of 2.7 (i) $\Rightarrow$ (ii) can be used to express the Menger properties as some kind of accumulation properties, as we explicitly worked out in $[20]$ Lemma 2.2(3)]. See also $[19]$ Corollary 5.13], which, however, is stated in non-standard terminology: there we used the expressions “Menger property”, respectively “Rothberger property”, in place of their versions for countable covers, that is, $R(\omega, \omega; <\omega)$, respectively $R(\omega, \omega; <2)$. Then in $[20]$ Theorem 2.3] the Menger properties are explicitly described as sequencewise $\mathcal{P}$-compactness, for some appropriate $\mathcal{P}$ consisting only of ultrafilters. That the Menger properties can be described as sequencewise $\mathcal{P}$-compactness, for some $\mathcal{P}$, follows directly from Theorem 2.7; the main point in $[20]$ is that the members of $\mathcal{P}$ can be chosen to be ultrafilters; this follows also abstractly from $[21]$ Corollary 5.3.

In the other direction, the proof of 2.7 (ii) $\Rightarrow$ (i) can be used to provide alternative formulations in terms of open covers both of $D$-compactness $[19]$ Proposition 1.3], stated here as Proposition 4.5, and of sequential compactness $[19]$ Corollaries 5.12 and 5.15]. See $[19]$ Corollaries 2.6, 3.14 and 5.15] for further results of this kind and $[19]$ Section 4 and Theorems 5.9 and 5.11] for further theorems dealing with pseudocompact-like generalizations.

We do not know whether the technical assumption that the members of $E$ are closed under supersets is necessary in condition (ii) in Theorem 2.7, namely, whether, for every $E$, there is some $E'$ such that the $E'$-accumulation property is equivalent to the $E'$-accumulation property and all members of $E'$ are closed under supersets.

3. CHECKING COMPACTNESS BY MEANS OF SUBPRODUCTS

Recall the definition of sequencewise $\mathcal{P}$-compactness from Definition 2.1. If $\prod_{j \in J} X_j$ is a product of topological spaces, a subproduct is a
space of the form $\prod_{j \in K} X_j$, for some $K \subseteq J$. Formally, if $K = \emptyset$, the corresponding subproduct is a one-element space (hence it satisfies all reasonable compactness properties). Otherwise, the reader might always exclude the case of subproducts with respect to an empty index set.

**Theorem 3.1.** Let $\mathcal{P}$ be a nonempty family of filters over some set $I$. A product of topological spaces is sequencewise $\mathcal{P}$-compact if and only if so is any subproduct with respect to a nonempty index set.

**Proof.** The only if part is immediate from the observation that sequencewise $\mathcal{P}$-compactness is preserved under continuous surjective images.

For the other direction, by contraposition, suppose that $X = \prod_{j \in I} X_j$ is not sequencewise $\mathcal{P}$-compact, thus there is a sequence $(x_i)_{i \in I}$ of elements of $X$ such that, for no $F \in \mathcal{P}$, $(x_i)_{i \in I}$ $F$-converges in $X$. Notice that a sequence in a product $\prod_{j \in J} X_j$ of topological spaces $F$-converges if and only if, for every $j \in J$, the projection of the sequence into $X_j$ $F$-converges in $X_j$. Hence, for every $F \in \mathcal{P}$, there is some $j_F \in J$ such that the projection of $(x_i)_{i \in I}$ into $X_{j_F}$ does not $F$-converge in $X_{j_F}$. Choose one such $j_F$ for each $F \in \mathcal{P}$, and let $K = \{j_F \mid F \in \mathcal{P}\}$, thus $|K| \leq |\mathcal{P}|$.

Let $X' = \prod_{j \in K} X_j$, and let $(x'_i)_{i \in I}$ be the natural projection of $(x_i)_{i \in I}$ into $X'$. We claim that the sequence $(x'_i)_{i \in I}$ witnesses that $X'$ is not sequencewise $\mathcal{P}$-compact. Indeed, for every $F \in \mathcal{P}$, we have that $(x'_i)_{i \in I}$ does not $F$-converge in $X'$, since the projection of $(x'_i)_{i \in I}$ into $X_{j_F}$ (which is the same as the projection of $(x_i)_{i \in I}$ into $X_{j_F}$) does not $F$-converge in $X_{j_F}$. Thus we have found a subproduct with $\leq |\mathcal{P}|$ factors which is not sequencewise $\mathcal{P}$-compact. \qed

**Remark 3.2.** Notice that the particular case $\mathcal{P} = \{F\}$ of Theorem 3.1 states that a product is $F$-compact if and only if each factor is $F$-compact (however, this does not follow from Theorem 3.1 since it is used in the proof). Thus Theorem 3.1 incorporates Tychonoff theorem, since a topological space is compact if and only if it is $D$-compact, for every ultrafilter $D$.

Apparently, besides Tychonoff theorem, the first result of the form of Theorem 3.1 has been proved by Scarborough and Stone [26, Theorem 5.6], asserting that a product is countably compact, provided that all subproducts by at most $2^{2^\omega}$ factors are countably compact. Scarborough and Stone [26, Corollary 5.7] also obtained the improved value $2^{2^\omega}$ for the particular case of first countable factors. Ginsburg and Saks [14, Theorem 2.6] then obtained the improved bound $2^{2^\omega}$ for powers of
a single space, and Comfort [7] and Saks [25] observed that the methods from [14] give the result for arbitrary factors, a result which is a particular case of Theorem 3.1 by Remark 2.6 and since there are $2^{\omega}$ nonprincipal ultrafilters over $\omega$.

Saks [25, Theorem 2.3] also proved that a product satisfies $\text{CAP}_\lambda$ if and only if each subproduct by $\leq 2^{\lambda}$ factors satisfies it; actually, he stated the result in terms of an interval of cardinals and in different terminology. Recall that a topological space is said to satisfy $\text{CAP}_\lambda$ if every subset $Y$ of cardinality $\lambda$ has a complete accumulation point, that is, a point each neighborhood of which intersects $Y$ in a set of cardinality $\lambda$. Saks’ result, too, can be obtained as a consequence of Theorem 3.1 but some care should be taken of the case when $\lambda$ is singular.

Concerning a related property, Caicedo [6, Section 3] essentially gave, in the present terminology, a characterization of $[\mu, \lambda]$-compactness as sequencewise $\mathcal{P}$-compactness, for an appropriate $\mathcal{P}$. This will be recalled in Theorem 4.4 below. Theorem 3.1 can then be applied in order to provide a characterization of those products which are $[\mu, \lambda]$-compact. We shall work this out in Corollary 4.10. In the particular cases of initial $\omega_n$-compactness and of $[\omega_n, \lambda]$-compactness, for $\lambda$ singular strong limit, better results can be obtained using further arguments, as we will show in Theorems 4.1 and 4.3.

Other possible examples of applications of Theorem 3.1 deal with the Menger, the Rothberger and the related properties mentioned after Definition 2.3. However, in this case, too, best results about these properties are obtained by direct means: see [20] and also Section 5 here. A similar situation occurs with regard to sequential compactness, as we shall show in Section 6.

Notice that the equivalence of conditions (i) and (ii) in [21, Theorem 2.1] can be obtained as an immediate consequence of Theorem 3.1.

Let us remark that Theorem 3.1 stresses the importance of studying the problem when sequencewise $\mathcal{P}$-compactness is equivalent to sequencewise $\mathcal{P}'$-compactness, for various sets $\mathcal{P}$ and $\mathcal{P}'$, as already mentioned in [21]. In particular, given $\mathcal{P}$, Theorem 3.1 implies that it is useful to characterize the minimal cardinality of some $\mathcal{P}'$ such that the above equivalence holds. The cardinality of such a “minimal” $\mathcal{P}'$ is also connected with some other invariants, see Definition 7.2 and Proposition 7.3 below.

Let $F$ be the trivial filter over $\kappa$, that is, $F = \{\kappa\}$. Then a topological space $X$ is $F$-compact if and only if, for every subset $Y$ of $X$ of cardinality $\leq \kappa$, there is $x \in X$ such that every neighborhood of $x$
contains the whole of $Y$. Such spaces are called $\kappa^+$-filtered. See Brandhorst and Erné [5] for further details and characterizations. Trivially, if $\mathcal{P}$ is a nonempty family of filters over $\kappa$, then any $\kappa^+$-filtered space is sequencewise $\mathcal{P}$-compact. The next lemma is trivial, but it has some use (see the proof of Proposition 5.1).

**Lemma 3.3.** If $\mathcal{P}$ is a family of filters over $\kappa$ and $X_1$ is a $\kappa^+$-filtered topological space, then a product $X_1 \times X_2$ is sequencewise $\mathcal{P}$-compact if and only if $X_2$ is sequencewise $\mathcal{P}$-compact.

**Proof.** The “only if” part is trivial.

For the other direction, suppose that $X_1$ is $\kappa^+$-filtered and $X_2$ is sequencewise $\mathcal{P}$-compact. Let $(x_\alpha)_{\alpha \in \kappa}$ be a sequence of elements of $X_1 \times X_2$. Since $X_2$ is sequencewise $\mathcal{P}$-compact, there is $F \in \mathcal{P}$ such that the second projection of $(x_\alpha)_{\alpha \in \kappa}$ $F$-converges in $X_2$. Since $X_1$ is $\kappa^+$-filtered, the first projection of $(x_\alpha)_{\alpha \in \kappa}$ $F$-converges in $X_1$, hence $(x_\alpha)_{\alpha \in \kappa}$ $F$-converges in $X_1 \times X_2$. □

A more significant result shall be proved in Corollary 4.7, where the assumption of being $\kappa^+$-filtered shall be replaced by initial $2^\kappa$-compactness, provided that all members of $\mathcal{P}$ are ultrafilters.

4. **Final $\mu$-compactness and $[\mu, \lambda]$-compactness**

**Final $\omega_n$-compactness.** In this section we present some generalizations of the following theorem, which can be obtained as consequence of results from [18] (it just needs a small elaboration besides [18, Corollary 33]). Recall that a topological space is finally $\mu$-compact if every open cover has a subcover of cardinality $< \mu$.

In what follows we shall freely use the categorical properties of products and, in case there is no risk of confusion, we shall identify, say, $\prod_{j \in J} Y_j$ with $\prod_{j \in H} Y_j \times \prod_{j \in J \setminus H} Y_j$, for $H \subseteq J$.

**Theorem 4.1.** If $X$ is a product of topological spaces, then the following conditions are equivalent.

(i) $X$ is finally $\omega_n$-compact.

(ii) All subproducts of $X$ by $\leq \omega_n$ factors are finally $\omega_n$-compact.

(iii) All but $< \omega_n$ factors of $X$ are compact, and the product of the non compact factors, if any, is finally $\omega_n$-compact.

(iv) The product of the non compact factors (if any) is finally $\omega_n$-compact.

**Proof.** (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial.

(ii) $\Rightarrow$ (iii) If $n = 0$, this is immediate from Tychonoff theorem. If $n > 0$, suppose by contradiction that there are (at least) $\omega_n$ factors
which are not compact. Theorem 2 in [18] (in contrapositive form) asserts that their product is not finally \( \omega_n \)-compact, contradicting (ii). Hence all but \( < \omega_n \) factors of \( X \) are compact, and the product of the remaining factors is finally \( \omega_n \)-compact, by (ii).

(iv) \( \Rightarrow \) (i) Letting apart the trivial improper cases, group together the compact factors, on one hand, and the non compact factors, on the other hand. Then we get by Tychonoff theorem that \( X \) is (homeomorphic to) a product of a compact space with a finally \( \omega_n \)-compact space, and a standard argument shows that any such product is finally \( \omega_n \)-compact (anyway, a more general result shall be proved in Corollary 4.8 below).

Since Lindelöfness is the same as final \( \omega_1 \)-compactness, we get the following corollary which might be known, though we know no reference for it.

**Corollary 4.2.** A product is (linearly) Lindelöf if and only if all subproducts by \( \leq \omega_1 \) factors are (linearly) Lindelöf, if and only if all but countably many factors are compact and the product of the non compact factors, if any, is (linearly) Lindelöf.

Recall that a topological space is linearly Lindelöf if every open cover which is linearly ordered by inclusion has a countable subcover (some authors use the term chain-Lindelöf). The linear Lindelöf case of Corollary 4.2 follows from [18, Theorem 3], arguing as in the proof of Theorem 4.1.

**\([\omega_n, \lambda]\)-compactness.** We now combine the arguments in Theorem 4.1 with some classical methods from Stephenson and Vaughan [29] in order to get a similar characterization of \([\omega_n, \lambda]\)-compact products, for \( \lambda \) a singular strong limit cardinal having cofinality \( \geq \omega_n \). Recall that a topological space \( X \) is \([\mu, \lambda]\)-compact if every open cover by at most \( \lambda \) sets has a subcover of cardinality \( < \mu \). Initial \( \lambda \)-compactness is \([\omega, \lambda]\)-compactness.

**Theorem 4.3.** Suppose that \( n \in \omega, \lambda \) is a singular strong limit cardinal, and \( \text{cf} \lambda \geq \omega_n \). If \( X \) is a product of topological spaces, then the following conditions are equivalent.

(i) \( X \) is \([\omega_n, \lambda]\)-compact.

(ii) Every subproduct of \( X \) by \( \leq \omega_n \) factors is \([\omega_n, \lambda]\)-compact.

(iii) All but \( < \omega_n \) factors of \( X \) are initially \( \lambda \)-compact, and the product of the non initially \( \lambda \)-compact factors (if any) is \([\omega_n, \lambda]\)-compact.

(iv) The product of the non initially \( \lambda \)-compact factors (if any) is \([\omega_n, \lambda]\)-compact.
Some auxiliary results are needed before we can give the proof of Theorem 4.3. By $[\lambda]^\mu$ we denote the set of all subsets of $\lambda$ of cardinality $< \mu$. This is now a quite standard notation, but notice that some authors (including the present one) sometimes used alternative notations for this, such as $S_\mu(\lambda)$, $P_{<\mu}(\lambda)$ and other. We say that an ultrafilter $D$ over $[\lambda]^{<\mu}$ covers $\lambda$ in case $\{Z \in [\lambda]^{<\mu} | \alpha \in Z\} \in D$, for every $\alpha \in \lambda$.

**Theorem 4.4.** (Caicedo [6]) A topological space is $[\mu, \lambda]$-compact if and only if it is sequencewise $P$-compact, for the family $P$ of the ultrafilters over $[\lambda]^{<\mu}$ which cover $\lambda$.

If $\lambda$ is regular, then a topological space is $[\lambda, \lambda]$-compact if and only if it is sequencewise $P$-compact, for the family $P$ of the uniform ultrafilters over $\lambda$.

Theorem 4.4 is essentially proved in Caicedo [6, Section 3]. Full details for the first statement can be found in [20, Theorem 2.3], considering the particular case $\lambda = 1$ therein: see the remark at the bottom of [20, p. 2509].

The second statement is much simpler; actually, it is a reformulation of some remarks from Saks [25], in particular, (i) on pp. 80–81 therein. Notice that, for $\lambda$ regular, $[\lambda, \lambda]$-compactness is equivalent to $\text{CAP}_\lambda, C[\lambda, \lambda]$ in Saks’ notation.

**Proposition 4.5.** If $D$ is an ultrafilter over $I$, then a topological space $X$ is $D$-compact if and only if, for every open cover $(O_Z)_{Z \in D}$ of $X$, there is some $i \in I$ such that $(O_Z)_{i \in Z \in D}$ is a cover of $X$.

See [19, Proposition 1.3 and Remark 3.12] for a proof of Proposition 4.5.

**Corollary 4.6.** If $X$ is an initially $\lambda$-compact topological space and $2^\kappa \leq \lambda$, then $X$ is $D$-compact, for every ultrafilter $D$ over some set of cardinality $\leq \kappa$.

**Proof.** We use Proposition 4.5. Let $(O_Z)_{Z \in D}$ be an open cover of $X$. Since $|D| \leq 2^\kappa \leq \lambda$, then by initial $\lambda$-compactness $(O_Z)_{Z \in D}$ has a finite subcover, say $O_{Z_1}, \ldots, O_{Z_m}$. Since $D$ is (in particular) a filter, $Z_1 \cap \cdots \cap Z_m \neq \emptyset$. If $i \in Z_1 \cap \cdots \cap Z_m$, then $(O_Z)_{i \in Z \in D}$ is a cover of $X$. By Proposition 4.5, $X$ is $D$-compact. □

Corollary 4.6 can also be obtained as a consequence of implications (8) and (5) in [28, Diagram 3.6].

**Corollary 4.7.** Suppose that $2^\kappa \leq \lambda$ and $P$ is a family of ultrafilters over some set $I$ of cardinality $\leq \kappa$. Then the product of an initially
λ-compact and of a sequencewise $\mathcal{P}$-compact topological space is sequencewise $\mathcal{P}$-compact.

Proof. Let $X_1$ be initially $\lambda$-compact, $X_2$ be sequencewise $\mathcal{P}$-compact, and let $(x_i)_{i \in I}$ be a sequence in $X_1 \times X_2$. By the sequencewise $\mathcal{P}$-compactness of $X_2$, there is some $D \in \mathcal{P}$ such that the second projection of $(x_i)_{i \in I}$ $D$-converges in $X_2$. Since $2^\kappa \leq \lambda$, the first projection of $(x_i)_{i \in I}$ $D$-converges in $X_1$, by Corollary 4.6. Hence $(x_i)_{i \in I}$ $D$-converges in $X_1 \times X_2$, thus $X_1 \times X_2$ is sequencewise $\mathcal{P}$-compact. □

By $\nu^{< \mu}$ we denote $\sup_{\mu' < \mu} \nu^{\mu'}$. Notice that $[\nu]^{< \mu}$ has cardinality $\nu^{< \mu}$.

Corollary 4.8. If $2^{\nu^{< \mu}} \leq \lambda$, then the product $X_1 \times X_2$ of a $[\mu, \nu]$-compact space $X_1$ and an initially $\lambda$-compact space $X_2$ is $[\mu, \nu]$-compact.

If the interval $[\mu, \nu]$ consists only of regular cardinals, the assumption $2^{\nu^{< \mu}} \leq \lambda$ above can be relaxed to $2^\nu \leq \lambda$.

Proof. By Theorem 4.4, $[\mu, \nu]$-compactness is equivalent to sequencewise $\mathcal{P}$-compactness, for a family $\mathcal{P}$ of ultrafilters over $[\nu]^{< \mu}$, a set of cardinality $\nu^{< \mu}$. Hence the first statement is immediate from Corollary 4.7 with $\kappa = \nu^{< \mu}$.

To prove the last statement, recall that $[\mu, \nu]$-compactness is equivalent to $[\mu', \mu']$-compactness, for every $\mu'$ such that $\mu \leq \mu' \leq \nu$. From the second statement in Theorem 4.4 and applying again Corollary 4.7, we get that $X_1 \times X_2$ is $[\mu', \mu']$-compact, for every $\mu'$ as above, since $2^{\mu'} \leq 2^\nu \leq \lambda$. Hence $X_1 \times X_2$ is $[\mu, \nu]$-compact. □

Proof of Theorem 4.3. (i) ⇒ (ii) and (iii) ⇒ (iv) are trivial.

(ii) ⇒ (iii) The case $n = 0$ follows from Stephenson and Vaughan’s Theorem [29, Theorem 1.1], asserting that if $\lambda$ is a singular strong limit cardinal, then any product of initially $\lambda$-compact topological spaces is still initially $\lambda$-compact. If $n > 0$, suppose by contradiction that there are $\geq \omega_n$ factors which are not initially $\lambda$-compact. By (ii), each such factor is $[\omega_n, \lambda]$-compact, hence not initially $\omega_n-1$-compact, otherwise it would be initially $\lambda$-compact. Hence we have at least $\omega_n$ factors which are not initially $\omega_n-1$-compact, and, by [18, Theorem 6], their product is not $[\omega_n, \omega_n]$-compact, hence not $[\omega_n, \lambda]$-compact, contradicting (ii).

Hence the set of factors which are not initially $\lambda$-compact has cardinality $< \omega_n$, and their product is $[\omega_n, \lambda]$-compact by (ii).

(iv) ⇒ (i) By the mentioned Stephenson and Vaughan’s theorem [29, Theorem 1.1], the product of the initially $\lambda$-compact factors, if any, is still initially $\lambda$-compact. By (iv), the product of the non initially $\lambda$-compact factors, if any, is $[\omega_n, \lambda]$-compact. Hence, excluding the
improper cases, \(X\) is (homeomorphic to) the product of an initially \(\lambda\)-compact space with an \([\omega_n, \lambda]\)-compact one. By Corollary 4.8 and since \(\lambda\) is strong limit, then, for every \(\nu < \lambda\), \(X\) is \([\omega_n, \nu]\)-compact. Since \(\lambda > \text{cf} \lambda \geq \omega_n\), then \(X\) is \([\omega_n, \lambda]\)-compact, by the well-known fact that \([\omega_n, \nu]\)-compactness, for every \(\nu < \lambda\), together with \([\text{cf} \lambda, \text{cf} \lambda]\)-compactness imply \([\omega_n, \lambda]\)-compactness.

\([\mu, \lambda]\)-compactness.

Remark 4.9. Certain values obtained in Theorems 4.1 and 4.3 are much better than the values which could be obtained by a simple direct application of Theorems 3.1 and 4.4. For example, if \(\lambda\) is a singular strong limit cardinal, and \(\text{cf} \lambda \geq \omega_n\), then there are \(2^{2^\lambda}\) ultrafilters over \(\lambda = \lambda^{\omega_n}\). Then Theorems 3.1 and 4.4 imply that some product \(X\) is \([\omega_n, \lambda]\)-compact if and only if all subproducts of \(X\) by \(\leq \kappa\) factors are finally \([\omega_n, \lambda]\)-compact. However, Theorem 4.3 shows that the value of \(\kappa\) can be improved to \(\omega_n\). See the next subsection for related comments.

In the more general case of arbitrary \(\mu\) and \(\lambda\), we have the following corollary of Theorem 4.3, a corollary in which we essentially get the values given by Theorem 3.1, sometimes with minor improvements.

Corollary 4.10. A product of topological spaces is \([\mu, \lambda]\)-compact if and only if so is any subproduct by \(\leq 2^{2^\nu}\) factors, where \(\kappa = \lambda^{<\mu}\). The value of \(\kappa\) can be improved to \(\kappa = \lambda\) in case the interval \([\mu, \lambda]\) contains only regular cardinals.

More generally, a product is \([\mu, \lambda]\)-compact if and only if so is any subproduct by \(< \theta\) factors, where \(\theta\) is the smallest cardinal such that both

(a) \(\theta > 2^{2^\nu}\), for every regular \(\nu\) such that \(\mu \leq \nu \leq \lambda\), and

(b) \(\theta > 2^{2^\nu^{<\mu}}\), for every singular \(\nu\) of cofinality \(< \mu\) such that \(\mu \leq \nu \leq \lambda\).

Proof. The first two statements are immediate from Theorems 3.1 and 4.4, since there are \(2^{2^\nu}\) ultrafilters over \([\lambda]^{<\mu}\), respectively, \(2^{2^\nu}\) ultrafilters over \(\nu\). Here \(\nu\) varies among the cardinals such that \(\mu \leq \nu \leq \lambda\), and we are using again the mentioned fact that \([\mu, \lambda]\)-compactness is equivalent to \([\nu, \nu]\)-compactness, for every \(\nu\) such that \(\mu \leq \nu \leq \lambda\).

In order to prove the last statement, recall that, for every \(\nu\), \([\text{cf} \nu, \text{cf} \nu]\)-compactness implies \([\nu, \nu]\)-compactness. Using this property, together with the fact mentioned at the end of the previous paragraph, it is easy to see that \([\mu, \lambda]\)-compactness is equivalent to the conjunction of

(i) \([\nu, \nu]\)-compactness, for every regular \(\nu\) with \(\mu \leq \nu \leq \lambda\), and
(ii) $[\mu, \nu]$-compactness, for every singular $\nu$ of cofinality $< \mu$ and such that $\mu \leq \nu \leq \lambda$.

Now we get the result by applying, for each $\nu$, the corresponding (and already proved) statements in the first paragraph of the corollary (with $\nu$ in place of $\lambda$).

□

Remark 4.11. Corollary 4.10 complements [25, Theorem 2.3], which asserts that a product satisfies $\text{CAP}_\nu$, for every $\nu \in [\mu, \lambda]$, if and only if so does every subproduct by $\leq 2^{2^\lambda}$ factors. Notice that if the interval $[\mu, \lambda]$ contains only regular cardinals, then [25, Theorem 2.3] and Corollary 4.10 overlap, since it is well-known that, if $\nu$ is a regular infinite cardinal, then $\text{CAP}_\nu$ and $[\nu, \nu]$-compactness are equivalent notions.

Short remarks about final $\mu$-compactness for arbitrary $\mu$. Under special set-theoretical assumptions, we know improvements of all the results proved in the present section. However, we cannot go exceedingly far. Of course, the equivalence of (i) and (iv) both in Theorem 4.1 and in Theorem 4.3 holds for every infinite cardinal in place of $\omega_\eta$. However, the other equivalences do not necessarily remain true, when $\omega_\eta$ is replaced by some larger cardinal.

For example, if $\kappa$ is a strongly compact cardinal, then every power of $\omega$ with the discrete topology is finally $\kappa$-compact. This is a consequence of a classical result by Mycielski [22], asserting that if $\kappa$ is strongly compact, then every product of finally $\kappa$-compact spaces is still finally $\kappa$-compact. This can be obtained also from Theorem 4.4 together with the ultrafilter characterization of strong compactness. Thus if $\kappa$ is strongly compact, then the analogue of Theorem 4.1 (i) $\Leftrightarrow$ (iii) with $\kappa$ in place of $\omega_\eta$ badly fails, since every power of $\omega$ is finally $\kappa$-compact, but $\omega$ is not compact.

Concerning condition 4.1(ii), first define, for every infinite cardinal $\mu$, the cardinal $s(P_\mu)$ as the smallest cardinal, if it exists, such that some product is finally $\mu$-compact if and only if so is every subproduct by $< s(P_\mu)$ factors. Here $P_\mu$ is intended to be the property of being finally $\mu$-compact, as we want the notation to be consistent with the general one we shall introduce in Definition 7.2. With this terminology, clearly $s(P_\omega) = 2$, as a reformulation of Tychonoff theorem. Moreover, Theorem 4.1 (i) $\Leftrightarrow$ (ii) implies that if $n > 0$, then $s(P_{\omega_n}) = \omega_{n+1}$ Indeed, $\omega_{n-1}$ is finally $\omega_n$-compact, but not every power of $\omega_{n-1}$ is, hence the value given by Theorem 4.1 cannot be improved. Contrary to the case of $\omega_\eta$, we know examples in which, under certain set theoretical constraints, $s(P_\mu)$ is far larger than $\mu$. Full details shall be presented elsewhere, since they involve deep set theoretical problems. On the
other hand, Mycielski’s Theorem mentioned above implies that if \( \kappa \) is a strongly compact cardinal, then \( s(P_\kappa) = 2 \).

We also remark that a characterization of Lindelöf products in terms of factors, rather than subproducts must necessarily involve deep structural properties of the factors. Even the product of two Lindelöf spaces may turn out to be very incompact. Moreover, there are three regular Lindelöf spaces whose product has very large Lindelöf number, while every pairwise product of two of them is still Lindelöf. See Usuba [30]. On the other hand, under a weak set-theoretical assumption, sequentially compact products can be characterized in terms of factors. See Corollary 6.6(ii) below.

Notice that if we apply Theorem 4.4 to final \( \mu \)-compactness, we get a proper class \( \mathcal{P} \), since final \( \mu \)-compactness is equivalent to \([\mu, \lambda] \)-compactness, for every \( \lambda \geq \mu \), alternatively, equivalent to \([\lambda, \lambda] \)-compactness, for every \( \lambda \geq \mu \). However, we can take good advantage of the theorem by Mycielski mentioned above, in order to find bounds for \( s(P_\mu) \), when \( \mu \) is smaller than some strongly compact cardinal.

**Proposition 4.12.** Suppose that \( \mu \) is an infinite cardinal, \( \mu \leq \theta \) and \( \theta \) is strongly compact. If \( X \) is a product of topological spaces, then the following conditions are equivalent.

(i) \( X \) is finally \( \mu \)-compact.

(ii) All subproducts of \( X \) by \( \leq \theta \) factors are finally \( \mu \)-compact.

**Proof.** Suppose that (ii) holds; in particular, all factors are finally \( \theta \)-compact, since \( \mu \leq \theta \). By Mycielski Theorem, \( X \) is finally \( \theta \)-compact. By Corollary 4.10, for every \( \lambda \) with \( \mu \leq \lambda < \theta \), \( X \) is \([\lambda, \lambda] \)-compact, since strongly compact cardinals are inaccessible. Hence \( X \) is finally \( \mu \)-compact. \( \square \)

5. **Menger and Rothberger**

Recall that a topological space \( X \) satisfies the Rothberger property (respectively, the Rothberger property for countable covers) if, given a countable family of open covers (resp., of countable open covers) of \( X \), one can obtain another cover of \( X \) by selecting an open set from each one of the given covers. We get the Menger property when we allow to select a finite number of open sets from each cover. Recall that we are not assuming any separation axiom.

Recall that a topological space is \( \kappa \)-filtered if, for every subset \( Y \) of \( X \) of cardinality \( < \kappa \), there is \( x \in X \) such that every neighborhood of \( x \) contains \( Y \). A space is supercompact if it is \( \kappa \)-filtered for all \( \kappa \). Equivalently, a space \( X \) is supercompact if and only if it has a dense
point (a point whose closure is the whole of $X$), if and only if $X$ is $[2, \infty]$-compact. Here $[2, \infty]$-compact is a shorthand for $[2, \lambda]$-compact, for every cardinal $\lambda$.

Notice that if a product of $T_1$ spaces is Rothberger, then all but finitely many spaces are one-element. Indeed, a $T_1$ space with more than one element contains a closed copy of the two-element discrete topological space $2$, and $2^\omega$ is not Rothberger. Hence most results in the present section are significant only in the (quite exotic) context of spaces satisfying little or no separation axiom. We present the results since the proofs need very little special efforts and, on the other hand, they might be of some interest due to renewed interest in spaces satisfying few separation axioms, for example, in connection with the specialization (pre)order, which becomes trivial for $T_1$ spaces, and because of significant applications to theoretical computer science. See, e. g., Gierz, Hofmann, Keimel, Lawson, Mislove, and Scott [13] and Goubault-Larrecq [Go]. Compare also Vickers [34]. See Nyikos [23] for an interesting recent manifesto in support of the study of spaces satisfying lower separation axioms from a purely topological point of view.

**Proposition 5.1.** If $X$ is a product of topological spaces, then the following conditions are equivalent.

(i) $X$ satisfies the Rothberger property.

(ii) Every subproduct of $X$ by countably many factors satisfies the Rothberger property.

(iii) All but a finite number of factors of $X$ are supercompact, and the product of the non supercompact factors (if any) satisfies the Rothberger property.

(iv) The product of the non supercompact factors of $X$ (if any) satisfies the Rothberger property.

**Proof.** (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial (as will be the case for all the corresponding implications throughout the present section).

(ii) $\Rightarrow$ (iii) If by contradiction there is an infinite number of factors which are not supercompact, i. e., not $[2, \infty]$-compact, then their product is not Rothberger, by [20 Proposition 3.1]. Hence the number of factors which are not supercompact is finite, and their product is Rothberger by (ii).

(iv) $\Rightarrow$ (i) As a particular case of Theorem 2.7 (i) $\Leftrightarrow$ (iii) we have that, for every $\lambda$, the Rothberger property for covers of cardinality $\leq \lambda$ is equivalent to sequencewise $P$-compactness, for some $P$ (an explicit description of such a $P$ can be found in [20 Proposition 4.1]). Thus,
by Lemma 3.3, and since any product of supercompact spaces is supercompact, we get that, for every \( \lambda \), \( X \) satisfies the Rothberger property for covers of cardinality \( \leq \lambda \). This means exactly that \( X \) satisfies the Rothberger property.

**Proposition 5.2.** If \( X \) is a product of topological spaces, then the following conditions are equivalent.

(i) \( X \) satisfies the Rothberger property for countable covers.
(ii) Every subproduct of \( X \) by countably many factors satisfies the Rothberger property for countable covers.
(iii) In every factor of \( X \) every sequence converges, except possibly for a finite number of factors, and the product of such factors (if any) satisfies the Rothberger property for countable covers.
(iv) The product of the factors of \( X \) (if any) in which there exists a nonconverging sequence satisfies the Rothberger property for countable covers.

**Proof.**

(ii) \( \Rightarrow \) (iii) It is enough to show that if we are given an infinite number of topological spaces, each with a nonconverging sequence, then their product does not satisfy the Rothberger property for countable covers. By [21, Lemma 4.1 (iv) \( \Rightarrow \) (i)], if some topological space \( Y \) has a nonconvergent sequence, then \( Y \) is not \([2, \omega]\)-compact, hence an infinite product of such spaces does not satisfy the Rothberger property for countable covers, by [20, Proposition 3.1].

(iv) \( \Rightarrow \) (i) The property that every sequence converges is preserved under products. Hence \( X \) is the product of a space in which every sequence converges and of a space satisfying the Rothberger property for countable covers. By Theorem 2.7 (i) \( \Leftrightarrow \) (iii), the Rothberger property for countable covers can be characterized as sequencewise \( \mathcal{P} \)-compactness, for some \( \mathcal{P} \), and \( \mathcal{P} \) can be chosen to consist of filters over \( \omega \), by [20 Proposition 4.1], taking \( \kappa = 2 \) and \( \lambda = \mu = \omega \) there. On the other hand, a space is \( \omega_1 \)-filtered if and only if in it every sequence converges. This is proved in Brandhorst [4] or Brandhorst and Erné [5, Lemma 5.1], and can be also proved as in [21 Lemma 4.1]. Since \( X \) is the product of a space in which every sequence converges and of a space satisfying the Rothberger property for countable covers, then Lemma 3.3 shows that \( X \) satisfies the Rothberger property for countable covers.

Given any infinite cardinal \( \lambda \), Proposition 5.2 can be generalized to deal with the *Rothberger property for covers of cardinality \( \leq \lambda \). We leave the generalization to the reader.
Corollary 5.3. If $X$ is a product of topological spaces, then the following conditions are equivalent.

(i) $X$ satisfies the Menger property.
(ii) Every subproduct of $X$ by countably many factors satisfies the Menger property.
(iii) All but a finite number of factors of $X$ are compact, and the product of the non compact factors (if any) satisfies the Menger property.
(iv) The product of the non compact factors of $X$ (if any) satisfies the Menger property.

Proof. (ii) $\Rightarrow$ (iii) By [20, Proposition 3.1], a product of infinitely many non compact spaces is not Menger, thus if (ii) holds, then there is only a finite number of non compact spaces, and their product is Menger.

(iv) $\Rightarrow$ (i) The product of the compact factors is compact, hence $X$ is the product of a compact space with a Menger space, and any such product is Menger. □

Results related to the present section appear in [20], e. g., Proposition 3.3 and Corollaries 2.5, 3.4 and 4.2 there. We do not know whether results similar to the ones presented in this section can be proved for the Menger property for countable covers. However, it follows from [20, Corollary 2.5] that a product satisfies the Menger property for countable covers if and only if so does any subproduct by $\leq 2^{2^\omega}$ factors. Again, we do not know whether this is the best possible value. Notice that a better value does work in the case of powers of a single space, or, more generally, in case we consider all possible products of spaces in a given family: see [20, Corollary 3.2].

6. Sequential compactness

We now exemplify Theorem 3.1 in the case of sequential compactness, actually, getting better bounds by direct computations. Compare the analogue situation in [21]. In particular, the previous parts of the paper are not necessary for understanding the present section, apart from a few comments.

Recall that a space $X$ is called ultraconnected if no pair of nonempty closed sets of $X$ is disjoint.

Definition 6.1. The splitting number $s$ is the least cardinal such that $2^s$ is not sequentially compact, where $2$ is the two-element discrete topological space. Usually the definition of $s$ is given in equivalent forms, but the present one is the most suitable for our purposes. See Booth [3, Theorem 2] or van Douwen [8, Theorem 6.1] for a proof of the
equivalences, and [8], Vaughan [33] and Blass [2] for further information about $\mathfrak{s}$.

A proof of the next lemma can be found in [21, Lemmata 4.1 and 4.2].

**Lemma 6.2.** (i) A topological space $X$ is both ultraconnected and sequentially compact if and only if every sequence in $X$ converges.

(ii) A product of $\geq \mathfrak{s}$ spaces which are not ultraconnected is not sequentially compact.

**Proposition 6.3.** If a product is sequentially compact, then the set of factors with a nonconverging sequence has cardinality $< \mathfrak{s}$.

*Proof.* Suppose by contradiction that there are $\geq \mathfrak{s}$ factors with a nonconverging sequence. Since each factor is sequentially compact, then, by Lemma 6.2(i), there are $\geq \mathfrak{s}$ factors which are not ultraconnected, and Lemma 6.2(ii) gives a contradiction. $\square$

**Corollary 6.4.** A product of topological spaces is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact.

*Proof.* Necessity is trivial, since we assume that all the spaces are nonempty and sequential compactness is preserved by taking images of surjective continuous functions. For the other direction, suppose that each subproduct of $X = \prod_{j \in J} X_j$ by $\leq \mathfrak{s}$ factors is sequentially compact, and let $J' = \{ j \in J \mid X_j$ has a nonconverging sequence$\}$. If $|J'| \geq \mathfrak{s}$, choose $J'' \subseteq J'$ with $|J''| = \mathfrak{s}$. By assumption, $\prod_{j \in J''} X_j$ is sequentially compact, and we get a contradiction from Proposition 6.3. Thus $|J'| < \mathfrak{s}$. Now $X$ is homeomorphic to $\prod_{j \in J'} X_j \times \prod_{j \in J \setminus J'} X_j$. The first factor is sequentially compact by assumption, since we proved that $|J'| < \mathfrak{s}$. For each $j \in J \setminus J'$, we have that every sequence on $X_j$ converges, thus in $\prod_{j \in J \setminus J'} X_j$, too, every sequence converges; a fortiori, $\prod_{j \in J \setminus J'} X_j$ is sequentially compact. Then $X$ is sequentially compact, being the product of two sequentially compact spaces. $\square$

In the context of $T_1$ spaces Corollary 6.4 is an immediate consequence of Definition 6.1 since any nontrivial $T_1$ space contains a closed subspace isomorphic to 2. Thus if a product of $T_1$ spaces is sequentially compact, then all but $< \mathfrak{s}$ factors are one-element spaces. Then Corollary 6.4 follows, since if all subproducts of $\leq \mathfrak{s}$ factors are sequentially compact, then all but $< \mathfrak{s}$ factors are one-element spaces and the product of the nontrivial factors is sequentially compact by hypothesis. Thus the main point of Corollary 6.4 is the case of spaces satisfying few separation axioms.
The value \( s \) in Corollary 6.4 is the best possible value: by Definition 6.1 all subproducts of \( 2^s \) by \( < s \) factors are sequentially compact, but \( 2^s \) is not.

Notice that if in Corollary 6.4 we replace \( s \) by the rougher estimate \( c \), then the corollary is a consequence of Theorem 3.1 since, by Remark 2.2 there is some \( \mathcal{P} \) of cardinality \( c \) such that sequential compactness is equivalent to sequencewise \( \mathcal{P} \)-compactness. As we mentioned in [21, Problem 4.4], we do not know the value of the smallest cardinal \( ms \) such that sequential compactness is equivalent to sequencewise \( \mathcal{P} \)-compactness, for some \( \mathcal{P} \) with \( |\mathcal{P}| = ms \). Of course, if \( ms \) were equal to \( s \), then Corollary 6.4 would be a direct consequence of Theorem 3.1.

It follows from Remark 2.2 that \( ms \leq c \). Moreover, \( ms \geq s \). If, to the contrary, \( ms < s \), then, by Theorem 3.1 we could prove Corollary 6.4 for the improved value \( ms \) in place of \( s \). However, as we mentioned above, \( s \) is the best possible value. Also the comment after [21, Problem 4.4] shows, in different terminology, that \( ms \geq s \).

We now show that, under a relatively weak cardinality assumption, we can replace “subproducts” with “factors” in Corollary 6.4.

**Definition 6.5.** The distributivity number \( h \) is the smallest cardinal such that there are \( h \) sequentially compact spaces whose product is not sequentially compact. Usually, the definition of \( h \) is given in some equivalent form: see Simon [27] for the proof of the equivalence, and Vaughan [33], Blass [2] for further information. Obviously, \( h \leq s \). It is known that \( h < s \) is relatively consistent.

**Corollary 6.6.** Assume that \( h = s \). If \( X \) is a product of topological spaces, then the following conditions are equivalent.

(i) \( X \) is sequentially compact.

(ii) All factors of \( X \) are sequentially compact, and the set of factors with a nonconverging sequence has cardinality \( < s \).

(iii) All factors of \( X \) are sequentially compact, and all but at most \( < s \) factors are ultraconnected.

**Proof.** Conditions (ii) and (iii) are equivalent by Lemma 6.2(i).

Condition (i) implies Condition (ii) by Proposition 6.3.

The proof that (ii) implies (i) is similar to the proof of Corollary 6.4. Suppose that (ii) holds, and that \( X = \prod_{j \in J} X_j \). Split \( X \) as \( \prod_{j \in J'} X_j \times \prod_{j \in J \setminus J'} X_j \), where \( J' = \{ j \in J \mid X_j \text{ has a nonconverging sequence} \} \). By (ii) and the assumption, \( |J'| < s = h \), hence, by the very definition of \( h \) (the one we have presented), \( \prod_{j \in J'} X_j \) is sequentially compact. Moreover \( \prod_{j \in J \setminus J'} X_j \) is sequentially compact, since in it every sequence converges, hence also \( X \) is sequentially compact. \( \square \)
Under the stronger assumption of the Continuum Hypothesis, we have learned of the equivalence of (i) and (ii) in Corollary 6.6 from Brandhorst [4]. See also Brandhorst and Erné [5]. Notice that, when restricted to $T_1$ spaces, Corollary 6.4 follows immediately from Definition 6.1 since any nontrivial $T_1$ space contains a closed subspace isomorphic to 2. Similarly, Definition 6.5 implies that if $h = s$, then a product $X$ of $T_1$ spaces is sequentially compact if and only if all factors are sequentially compact and the set of nontrivial factors has cardinality $< s$. On the other hand, we are not aware of any former result of this kind when no separation axiom is assumed, apart from the mentioned partial result in [4].

Notice that the assumption $h = s$ is necessary in Corollary 6.6. Indeed, it is now almost immediate to show that Conditions (i) and (ii) in Corollary 6.6 are equivalent if and only if $h = s$.

**Corollary 6.7.** The following conditions are equivalent.

(i) $h = s$

(ii) For every product $X$ of topological spaces, condition (i) in Corollary 6.6 holds if and only if condition (ii) there holds.

(iii) For every product $X$ with $h$ factors, condition (ii) in Corollary 6.6 implies condition (i) there.

**Proof.** (i) ⇒ (ii) is given by Corollary 6.6 itself, and (ii) ⇒ (iii) is trivial.

To prove (iii) ⇒ (i) we shall prove the contrapositive. Suppose that (i) fails. By the definition of $h$ there is a not sequentially compact product $X$ by $h$ sequentially compact factors. If $h < s$, then condition (ii) in Corollary 6.6 trivially holds for such an $X$, while condition (i) there fails. Thus condition (iii) in the present corollary fails. □

7. A TOPOLOGICAL PROOF THAT $\text{cf } s \geq h$ AND A GENERALIZATION

We begin this section by giving a curious and purely topological proof of the inequality $\text{cf } s \geq h$. The proof does not use any of the results proved before, but relies heavily on the characterizations of the cardinals $s$ and $h$ that we have presented as Definitions 6.1 and 6.5. See Blass [11, Corollary 2.2] for another proof of $\text{cf } s \geq h$. Andreas R. Blass (personal communication, June 2014) has kindly communicated us a direct simple proof which uses the standard definitions of $s$ and $h$.

By the way, Dow and Shelah [9] have recently showed that it is consistent that $s$ is singular, solving a longstanding problem.

**Corollary 7.1.** $\text{cf } s \geq h$. 
Proof. Suppose by contradiction that $\text{cf } s = \lambda < h$, hence we can express $s$ as $\bigcup_{\alpha \in \lambda} s_\alpha$, with $|s_\alpha| < s$, for $\alpha \in \lambda$; moreover, without loss of generality, we can take the $s_\alpha$’s to be pairwise disjoint. Thus $2^s$ is (homeomorphic to) $\prod_{\alpha \in \lambda} 2^{s_\alpha}$. By the definition of $s$ (the one we have given) and since $|s_\alpha| < s$, for $\alpha \in \lambda$, then each $2^{s_\alpha}$ is sequentially compact. By the definition of $h$, and since $\lambda < h$, we have that $\prod_{\alpha \in \lambda} 2^{s_\alpha}$ is sequentially compact. But then $2^s \cong \prod_{\alpha \in \lambda} 2^{s_\alpha}$ would be sequentially compact, contradicting the definition of $s$. □

The arguments in the proofs of Corollary 7.1 have a general form and suggest the idea of attaching some invariants analogue to $s$ and $h$ to every property $P$ of topological spaces. The arguments are relatively simple, but there is the possibility that the arguments and the general framework might turn out to be a useful paradigm for many disparate situations.

In fact, the arguments we are going to hint have really little to do with topology. Everything works as well for some property of objects in a category in which some infinite products or coproducts are defined. However, the right ambient in which the results can be stated in their full generality appears to be the context of partial infinitary semigroups. We shall sketch the details in Remark 7.6.

Definition 7.2. Let $P$ be any property of (nonempty) topological spaces or, more generally, a property defined on a class of objects in which some notion of an infinite product is defined. By definiteness, we shall assume that subproduct means product of a nonempty set of factors. Alternatively, we can assume that any one-element space (in general, the neutral element) satisfies $P$.

We denote by $S(P)$ the class of all cardinals $\kappa \geq 2$ such that there is some product with $\kappa$ factors with the property that the product does not satisfy $P$, but all subproducts by $< \kappa$ factors satisfy $P$.

Notice that (if $P$ is preserved under homeomorphisms) $S(P) = \emptyset$ means exactly that a nontrivial product satisfies $P$, whenever all factors satisfy $P$.

We denote by $S^*(P)$ the class of all cardinals $\kappa \geq 2$ such that there is some product with $\kappa$ factors with the property that all factors satisfy $P$, but the product does not satisfy $P$ (of course, in many cases, $S^*(P)$ is either empty or an unbounded interval of cardinals). Notice also that if $S^*(P)$ is nonempty and $h(P) = \inf S^*(P)$, then every product of $< h(P)$ spaces satisfying $P$ still satisfies $P$.

We denote by $S_1(P)$ the class of all cardinals $\kappa \geq 2$ such that there is some space $Y$ with the property that $Y^\lambda$ satisfies $P$, for every $\lambda < \kappa$. 
but $Y^\kappa$ does not satisfy $P$. Notice that if $\mathcal{H}_1(P)$ is nonempty, $0 \neq \kappa < \mathfrak{h}_1(P) = \inf \mathcal{H}_1(P)$, and $Y$ satisfies $P$, then $Y^\kappa$ satisfies $P$.

We denote by $\mathcal{H}_f(P)$ the class of all cardinals $\kappa \geq 2$ for which there is some nonempty class $\mathcal{K}$ of topological spaces such that every product of $< \kappa$ members from $\mathcal{K}$ satisfies $P$ (in particular, every member of $\mathcal{K}$ satisfies $P$) but some product of $\kappa$ members from $\mathcal{K}$ does not satisfy $P$ (in all the above products we allow repetitions, that is, each member of $\mathcal{K}$ might appear multiple times).

Notice that the above classes might be empty, for example, this happens when $P$ is compactness. On the other hand, finite cardinals might belong to these classes, for example 2, belongs to each class, when $P$ is countable compactness.

We denote by $s(P)$ the smallest cardinal $\kappa \geq 2$ such that the following holds: for every product $X$, if all subproducts of $X$ by $< \kappa$ factors satisfy $P$, then $X$ satisfies $P$ ($s$ stands for subproducts). If no such cardinal exists, we conventionally put $s(P) = \infty$ and, by convention, we assume that $\lambda < \infty$, for every cardinal $\lambda$.

We denote by $s_1(P)$ the smallest cardinal $\kappa \geq 2$ such that, for every space $Y$, the following holds: if all powers $Y^\lambda$, for each $0 \neq \lambda < \kappa$, satisfy $P$, then all powers of $Y$ satisfy $P$. We apply the same conventions as above, if no such cardinal exists.

We denote by $s_f(P)$ the smallest cardinal $\kappa \geq 2$ such that, for every class $\mathcal{K}$ of topological spaces, the following holds: if all products of $< \kappa$ members from $\mathcal{K}$ (allowing repetitions) satisfy $P$, then all products of members from $\mathcal{K}$ satisfy $P$.

To state parts of the next proposition more concisely, we shall also introduce the following convention. If $\mathcal{R}$ is a nonempty class of cardinals, we let

1. $\sup^+ \mathcal{R} = \infty$ if $\mathcal{R}$ has no supremum;
2. $\sup^+ \mathcal{R} = \sup \mathcal{R}$ if the supremum of $\mathcal{R}$ exists but it is not reached, that is, it is not a maximum. Of course, this can happen only when $\sup \mathcal{R}$ is a limit cardinal.
3. $\sup^+ \mathcal{R} = (\sup \mathcal{R})^+$ if the supremum of $\mathcal{R}$ exists and it is the maximum of $\mathcal{R}$.

We also set $\sup^+ \emptyset = 2$ in case $\mathcal{R} = \emptyset$. This might look unnatural, but shall simplify some statements.

We are now going to prove some simple facts about the above classes and cardinals, including a generalization of Corollary 7.1.

**Proposition 7.3.** Let $P$ be a property of topological spaces, and suppose that $P$ is invariant under homeomorphisms.

(i) $\mathcal{H}_1(P) \subseteq \mathcal{H}_f(P) \subseteq \mathcal{H}(P) \subseteq \mathcal{H}^*(P)$. 


(ii) If \( \kappa \in \mathcal{S}(P) \), then \( 1 + \text{cf} \kappa \in \mathcal{S}^*(P) \).

(iii) If \( \mathcal{S}^*(P) \neq \emptyset \), then

(a) \( \inf \mathcal{S}^*(P) \in \mathcal{S}(P) \), thus

(b) \( \mathcal{S}(P) \neq \emptyset \),

(c) \( \inf \mathcal{S}^*(P) = \inf \mathcal{S}(P) \), and

(d) \( \inf \mathcal{S}^*(P) \) is a regular cardinal.

(iv) \( s(P) = \sup^+ \mathcal{S}(P) \).

(v) \( s_1(P) = \sup^+ \mathcal{S}_1(P) \).

(vi) \( s_f(P) = \sup^+ \mathcal{S}_f(P) \).

(vii) \( s_1(P) \leq s_f(P) \leq s(P) \).

(viii) If \( P \) is sequencewise \( \mathcal{P} \)-compactness, for some \( \mathcal{P} \), then \( s(P) \leq |\mathcal{P}|^+ \).

Proof. (i) is clear.

(ii) is similar to Corollary 7.1. If \( \kappa \) is infinite regular, then (ii) follows from (i).

If \( \kappa = n \geq 2 \) and \( \prod_{i<n} X_i \) witnesses \( \kappa \in \mathcal{S}^*(P) \), then \( X_{n-1} \times \prod_{i<n-1} X_i \) witnesses \( 1 + \text{cf} n = 2 \in \mathcal{S}(P) \).

Suppose that \( \kappa \) is singular, thus \( \kappa = \bigcup_{\alpha \in \text{cf} \kappa} a_\alpha \), for some \( a_\alpha \)'s such that \( |a_\alpha| < \kappa \), for \( \alpha \in \text{cf} \kappa \); moreover, without loss of generality, the \( a_\alpha \)'s can be taken to be disjoint. Let \( X = \prod_{\gamma \in \kappa} X_\gamma \) witness \( \kappa \in \mathcal{S}(P) \) and, for \( \alpha \in \text{cf} \kappa \), let \( Y_\alpha = \prod_{\gamma \in a_\alpha} X_\gamma \). Since \( |a_\alpha| < \kappa \), for \( \alpha \in \text{cf} \kappa \), then, by the definition of \( \mathcal{S}(P) \), each \( Y_\alpha \) satisfies \( P \). Now notice that \( X \) is homeomorphic to \( \prod_{\alpha \in \text{cf} \kappa} Y_\alpha \), and (this realization of) \( X \) witnesses \( \text{cf} \kappa \in \mathcal{S}^*(P) \).

(iii)(a) Suppose that \( \mathcal{S}^*(P) \neq \emptyset \). Let \( \kappa = \inf \mathcal{S}^*(P) \) and let \( \prod_{\gamma \in \kappa} X_\gamma \) witness \( \kappa \in \mathcal{S}^*(P) \). By assumption, \( \kappa \geq 2 \) and each \( X_\gamma \) satisfies \( P \). If there is \( J \subseteq \kappa \) such that \( 2 \leq |J| < \kappa \) and \( \prod_{j \in J} X_j \) does not satisfy \( P \), then \( \prod_{j \in J} X_j \) witnesses \( |J| \in \mathcal{S}^*(P) \), contradicting the minimality of \( \kappa \). Thus, for every \( J \subseteq \kappa \) with \( 1 \leq |J| < \kappa \), we have that \( \prod_{j \in J} X_j \) satisfies \( P \). This means that \( \prod_{\gamma \in \kappa} X_\gamma \) witnesses \( \kappa \in \mathcal{S}(P) \).

(b) follows trivially from (a). (c) follows from (a) and (i). Finally, (d) follows from (c) and (ii).

(iv) It is trivial from the definitions that if \( \kappa \in \mathcal{S}(P) \) then \( \kappa < s(P) \), using our conventions in the case when \( \mathcal{S}(P) \) is either unbound or empty. Thus \( s(P) \geq \sup^+ \mathcal{S}(P) \). On the other hand, if \( 2 \leq \kappa < s(P) \), then there is a product \( X \) which does not satisfy \( P \), but all subproducts by \( < \kappa \) factors satisfy \( P \). Choose some subproduct \( X' \) of \( X \) with a minimal number of factors, say \( \kappa' \) factors, and in such a way that \( X' \) still witnesses \( \kappa < s(P) \). As in the proof of (iii)(a), by the minimality of \( \kappa' \), we have \( \kappa' \in \mathcal{S}(P) \). Since, by construction, \( \kappa \leq \kappa' < s(P) \), we get \( s(P) \leq \sup^+ \mathcal{S}(P) \).
(v) and (vi) are similar.
(vii) follows from (i) and (iv)-(vi).
(viii) is from Theorem 3.1.

\[ \square \]

Remark 7.4. Throughout this remark, let \( P \) be sequential compactness. By Definition 6.3 we have \( h = \inf H^*(P) \). Thus Proposition 7.3(ii)(d) generalizes the well-known result that \( h \) is a regular cardinal. By Definition 6.1 we have \( s \in H_1(P) \), thus \( s \in H(P) \), by 7.3(i). By 7.3(ii) we get \( \text{cf} s \in H^*(P) \), hence \( \text{cf} s \geq h \), since \( h = \inf H^*(P) \). This shows that Proposition 7.3 generalizes Corollary 7.1. By Corollary 6.4 and 7.3(iv) we get \( s^+ = \text{sup}^+ H(P) \), that is, \( s = \text{sup} H(P) \), thus \( H(P) \) is contained in the interval \([h, s]\), since \( h = \inf H^*(P) = \inf H(P) \), by 7.3(iii)(c). We do not know the possible general structure of \( H(P) \) (of course, it is trivial in case \( h = s \)). It is not difficult, using Frolík sums [10], Juhász and Vaughan [16], to show that \( H_1(P) = H_f(P) \) and that \( h = \inf H_1(P) \). This comes close to showing that \( H(P) = H_1(P) \), but this is a conjecture, so far.

Remark 7.5. As an application of Proposition 7.3, one can consider chain compactness. If \( \lambda \leq \mu \) are infinite cardinals, a topological space \( X \) is \([\lambda, \mu]\)-chain compact [31] if, for every cardinal \( \nu \) such that \( \lambda \leq \nu \leq \mu \), every \( \nu \)-indexed sequence of elements of \( X \) has a converging cofinal subsequence. Thus \([\omega, \omega]\)-chain compactness is the same as sequential compactness.

A product of countably many \([\lambda, \mu]\)-chain compact spaces is still \([\lambda, \mu]\)-chain compact [31]. Thus if \( P_{[\lambda, \mu]\text{-c}} \) is the property of being \([\lambda, \mu]\)-chain compact, then \( h(P_{[\lambda, \mu]\text{-c}}) = \inf H^*(P_{[\lambda, \mu]\text{-c}}) > \omega \). By Proposition 7.3, \( h(P_{[\lambda, \mu]\text{-c}}) \) is a regular cardinal, and if \( \kappa \in H(P_{[\lambda, \mu]\text{-c}}) \), then \( \text{cf} \kappa \geq h(P_{[\lambda, \mu]\text{-c}}) \). To the best of our knowledge, it is an open problem to explicitly characterize the cardinal \( h(P_{[\lambda, \mu]\text{-c}}) \) and the class \( H(P_{[\lambda, \mu]\text{-c}}) \).

Some results about products of \([\omega, \mu]\)-chain compact spaces can be found in [24]. If follows from Theorem 3.1 that \( \sup H(P_{[\lambda, \mu]\text{-c}}) \leq 2^\mu \).

In general, under fairly weak hypotheses on \( P \), we know that \( \inf H_1(P) = \inf H(P) \) and \( H_1(P) = H_f(P) \). We shall present details elsewhere.

For every \( P \), let \( \text{ms}(P) = \inf \{|P'| \mid \text{sequencewise } P\text{-compactness is equivalent to sequencewise } P'\text{-compactness} \} \). By Proposition 7.3(viii), if \( P \) is the property of being sequencewise \( P \)-compact, then \( s(P) \leq (\text{ms}(P))^+ \).

The problem of evaluating exactly the above cardinals and describing the classes defined in 7.2 might be very difficult even in special cases, and in general will involve set theory.
Remark 7.6. As hinted before, all the above notions and results can be applied in the context of partial infinitary semigroups, formally, Σ-algebras satisfying properties (U) and (P) in the terminology from [15].

For short, in a partial infinitary semigroup we have a partially defined infinitary operation \( \sum_{i \in I} a_i \), for every index set \( I \). Property (U) asserts that if \( |I| = 1 \), then \( \sum_{i \in I} a_i \) is defined and its outcome is the only element \( a_i \) of the sequence.

Property (P) asserts that if \( \sum_{i \in I} a_i \) is defined, then, for every partition \((J_k)_{k \in K}\) of \( I \), all the sums in the following equality are defined, and equality actually holds: \( \sum_{i \in I} a_i = \sum_{k \in K} \sum_{i \in J_k} a_i \).

With the customary foundational caution, homeomorphism classes of topological spaces with the Tychonoff product form a partial infinitary semigroups.

If \( S \) is such a Σ-algebra and \( P \subseteq S \), let \( \mathcal{S}(P) \) be the class of all cardinals \( \kappa \geq 2 \) such that there are some \( I \) of cardinality \( \kappa \) and some \( \sum_{i \in I} a_i \), which is defined, but its outcome is not in \( P \), while \( \sum_{i \in J} a_i \in P \), for every \( J \subseteq I \) with \( |J| < \kappa \). Notice that property (P) implies that if \( \sum_{i \in I} a_i \) is defined, then \( \sum_{i \in J} a_i \) is defined, for every nonempty \( J \subseteq I \).

Let \( \mathcal{S}^*(P) \) be the class of all cardinals \( \kappa \geq 2 \) such that there are some \( I \) of cardinality \( \kappa \) and some \( \sum_{i \in I} a_i \), which is defined, but its outcome is not in \( P \), while \( a_i \in P \), for every \( i \in I \).

Let all the other invariants be defined in a similar way.

Then Proposition 7.3 holds in this context, as well. A few details are made explicit in the next proposition.

Proposition 7.7. Suppose that \( S \) is a partial infinitary semigroup and \( P \subseteq S \). Then

(i) \( \mathcal{S}(P) \subseteq \mathcal{S}^*(P) \).
(ii) If \( \kappa \in \mathcal{S}(P) \), then \( 1 + \text{cf} \ \kappa \in \mathcal{S}^*(P) \).
(iii) If \( \mathcal{S}^*(P) \) is not empty, then \( \inf \mathcal{S}^*(P) \in \mathcal{S}(P) \), thus \( \mathcal{S}(P) \neq \emptyset \), \( \inf \mathcal{S}^*(P) = \inf \mathcal{S}(P) \), and \( \inf \mathcal{S}^*(P) \) is a regular cardinal.

Proof. (i) follows from the definitions and Property (U).

(ii) If \( \kappa \) is an infinite regular cardinal, then \( \kappa = \text{cf} \ \kappa = 1 + \text{cf} \ \kappa \), hence (ii) follows from (i).

If \( \kappa \) is finite, say, \( \kappa = n \geq 2 \) and \( \sum_{i < n} a_i \) witnesses \( \kappa \in \mathcal{S}(P) \), then \( a_{n-1} + \sum_{i < n-1} a_i \) witnesses \( 1 + \text{cf} \ n = 1 + 1 = 2 \in \mathcal{S}^*(P) \).

The remaining case is similar to Proposition 7.1. Suppose that \( \kappa \) is singular, thus \( \kappa = \bigcup_{k \in K} J_k \) for some sets \( K \) and \( J_k \) such that \( |K|, |J_k| < \kappa \), for \( k \in K \). Let \( c = \sum_{\gamma \in \kappa} a_\gamma \) witness \( \kappa \in \mathcal{S}(P) \). For \( k \in K \), let \( b_k = \sum_{\gamma \in J_k} a_\gamma \). Since \( |J_k| < \kappa \), for \( k \in K \), then, by the definition of \( \mathcal{S}(P) \), each \( b_k \) is in \( P \). By Property (P), \( c = \sum_{k \in K} b_k \) and this sum witnesses \( \text{cf} \ \kappa \in \mathcal{S}^*(P) \).
(iii) Let $\kappa = \inf \mathfrak{H}^*(P)$ and let $\sum_{\gamma \in \kappa} a_\gamma$ witness $\kappa \in \mathfrak{H}^*(P)$. By assumption, $\kappa \geq 2$ and each $a_\gamma$ is in $P$. If there is $J \subseteq \kappa$ such that $2 \leq |J| < \kappa$ and $\sum_{j \in J} a_j \notin P$, then $\sum_{j \in J} a_j$ witnesses $|J| \in \mathfrak{H}^*(P)$, contradicting the minimality of $\kappa$. Thus, by (U), for every $J \subseteq \kappa$ with $1 \leq |J| < \kappa$, we have $\sum_{j \in J} a_j \in P$. This means that $\sum_{\gamma \in \kappa} a_\gamma$ witnesses $\kappa \in \mathfrak{H}(P)$. The rest follows from (i) and (ii). \hfill $\Box$

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