THE NUMBER OF EQUATIONS NEEDED TO DEFINE AN ALGEBRAIC SET

GENNADY LYUBEZNIK

Let \( B \) be a commutative Noetherian ring, \( X = \text{Spec} \, B \) the associated affine scheme, \( I \subset B \) an ideal and \( V = V(I) \subset X \) the closed subset defined by \( I \).

**Definition.** Elements \( f_1, \ldots, f_s \in I \) define \( V \) set-theoretically (equivalently, \( V \) is defined set-theoretically by \( s \) equations \( f_1 = 0, f_2 = 0, \ldots, f_s = 0 \)) if \( \sqrt{(f_1, \ldots, f_s)} = \sqrt{I} \).

Hilbert's Nullstellensatz implies that in the case when \( B \) is a finitely generated algebra over an algebraically closed field \( k \), this definition agrees with the usual one, i.e., all \( f_1, \ldots, f_s \) vanish at a \( k \)-rational point if and only if it belongs to \( V \). In the sequel "defined" always means "defined set-theoretically".

The question we are dealing with here concerns the minimum number of equations needed to define a given \( V \subset X \). A classical result that goes back to L. Kronecker [Kr] says that if \( B \) is \( n \)-dimensional, then \( n+1 \) equations would suffice for every \( V \subset X \). Our first theorem describes those \( V \subset X \) which can be defined by \( n \) equations.

**Theorem A.** Let \( k \) be an algebraically closed field, \( X \) a smooth affine \( n \)-dimensional variety over \( k \) with coordinate ring \( B \), and \( V = V' \cup P_1 \cup P_2 \cup \cdots \cup P_r \) an algebraic subset of \( X = \text{Spec} \, B \), where \( V' \) is the union of irreducible components of positive dimensions and \( P_1, P_2, \ldots, P_r \) some isolated closed points (which do not belong to \( V' \)). Then \( V \) can be defined by \( n \) equations if and only if one of the following conditions holds.

(i) \( r = 0 \), i.e., \( V \) consists only of irreducible components of positive dimension.

(ii) \( V' \) is empty, i.e., \( V \) consists only of closed points and there exist positive integers \( n_1, n_2, \ldots, n_r \) such that \( n_1 P_1 + n_2 P_2 + \cdots + n_r P_r = 0 \) in \( A_0(X) \).

(iii) \( V' \) is nonempty, \( r \geq 1 \) and there exist positive integers \( n_1, n_2, \ldots, n_r \) such that \( n_1 P_1 + n_2 P_2 + \cdots + n_r P_r \) belongs to the image of the natural map \( A_0(V') \to A_0(X) \) induced by the inclusion \( V' \to X \).

Here \( A_0(\cdot) \) stands for the group of zero-cycles modulo rational equivalence [Fu].

**Sketch of Proof.** Our proof consists of three steps. In Step 1 we construct an ideal \( I \subset B \) such that \( \sqrt{I} \) is the defining ideal of \( V \), and in addition \( I \) has some other special properties. In Step 2 we, in a special way, pick some ideals \( Q_1, \ldots, Q_n \) such that \( \sqrt{Q_i} \) is a maximal ideal containing \( I \) for each \( i \) and \( J/J^2 \) is \( n \)-generated, where \( J = I \cap Q_1 \cap \cdots \cap Q_n \). In Step 3 we...
prove that the 0-dimensional Segre class of $J$ equals 0, so by [Mu2, Theorem 2] and the Suslin cancellation theorem [Su], $J$ is $n$-generated.

If $B$ is not assumed to be regular, the result of Theorem A need not hold. In fact, for every $n \geq 0$, we have constructed an example of a finitely generated $n$-dimensional algebra $B$ over a suitable algebraically closed field, such that its singular locus consists of just one closed point and for every $d$ between 1 and $n - 1$ it contains a $d$-dimensional subvariety which cannot be defined by $n$ equations.

Storch [St] and Eisenbud-Evans [EE] proved that every algebraic set in $\mathbb{A}_k^n$ can be defined by $n$ equations, while Cowsik-Nori [CN] proved that every curve in $\mathbb{A}_k^n$, where $\text{char} k = p > 0$, can be defined by $n - 1$ equations. Our next theorem sharpens these results.

**Theorem B.** If $\text{char} k = p > 0$, then every algebraic set $V \subset \mathbb{A}_k^n$ consisting only of irreducible components of positive dimensions can be defined by $n - 1$ equations.

**Sketch of Proof.** The main idea of the proof is contained in the following lemma.

**Lemma.** Let $f_1, \ldots, f_r, g_1, \ldots, g_r \in I$ and $a \in B$ such that
(i) $f_1, \ldots, f_r$ generates $I_a$ up to radical.
(ii) $g_1, \ldots, g_r$ generate $(I + (a))/(a) \subset B/(a)$ up to radical. Then there exist $r + 1$ elements $h_1, \ldots, h_{r+1}$ which generate $I$ up to radical.

Now assume all irreducible components of $V$ have dimension $d \geq 1$ and use induction on $d$, the case $d = 1$ being settled by [CN]. Let $I = I(V) \subset B = k[x_1, \ldots, x_n]$ be the defining ideal of $V$. By a change of variables we can assume that $k[x_n]$ has zero intersection with every minimal prime overideal of $I$. By induction the extension of $I$ in $k(x_n)[x_1, \ldots, x_{n-1}]$ can be generated up to radical by $n - 2$ elements. Thus there exists a square-free polynomial $a(x_n)$ such that $I_a(x_n)$ can be generated up to radical by $n - 2$ elements. Since $k[x_n]/a(x_n)$ is a product of fields, by induction $(I + (a))/(a) \subset (k[x_n]/a(x_n))[x_1, \ldots, x_{n-1}]$ also can be generated up to radical by $n - 2$ elements. Now by the lemma, $I$ can be generated up to radical by $n - 1$ elements.

If $V$ is not equidimensional, the proof is considerably harder, but the main idea remains the same.

The rest of this paper is devoted to a question of M. P. Murthy [Mu1], who asked whether every locally complete intersection (l.c.i.) subscheme $V \subset \mathbb{A}_k^n$ is a set-theoretic complete intersection. The answer is known to be positive if $\dim V = 1$ [Fe, Sz, Bo, MK].

We generalize this result as follows:

**Theorem C.** Every l.c.i. subscheme $V \subset \mathbb{A}_k^n$ of constant positive dimension can be defined by $n - 1$ equations.

A proof of this theorem is similar to that of the equidimensional case of Theorem B. □
Moreover, we have obtained the following:

**Proposition.** Let $k$ be an algebraically closed field of characteristic $p > 0$, $I$ a l.c.i. of equidimension $d$ in $B = k[x_1, \ldots, x_n]$ and $n \geq 3d$. Set $A = B/\sqrt{I}$. Then there exists a l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and $J/J \cdot \sqrt{I}$, the reduced conormal module of $J$, is isomorphic to $P \oplus A^{n-2d+1}$, where $P$ is a projective $A$-module of rank $d - 1$ with trivial determinant.

**Sketch of Proof.** By the Ferrand construction [Fe, p. 24; Va, p. 89] we are reduced to the case when the determinant of the conormal module of $I$ is trivial. Let $c_d \in F^d(Spec \ A)$ be the top Chern class of the reduced conormal module of $I$, where $F^d(Spec \ A)$ is the $d$th component of the Grothendieck $\gamma$-filtration [FL, p. 48]. Since $F^d(Spec \ A)$ is divisible, we can write $c_d = (1-p^d)c$. Let $Q$ be a projective $A$-module of rank $d$ such that $Q - A^d \in F^d(Spec \ A)$ and the top Chern class of $Q$ equals $c$. Set $I/I \cdot \sqrt{I} \approx M \oplus Q$, where $M$ is projective of rank $n - 2d \geq d$. We show that there exists a l.c.i. $J_1 \subset I$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of $J_1$ is $M \oplus F(Q)$, where $F(Q)$ is the Frobenius of $Q$. It is straightforward to compute that the top Chern class of $M \oplus F(Q)$ is zero. Taking the iterated Frobenius of $J_1$, if necessary, we obtain, by [Mu2, Theorem 5] an l.c.i. $J$ with required properties. □

**Theorem D.** Let $k$ be any algebraically closed field of characteristic $p > 0$ and $V \subset A^n_k$ a locally complete intersection subscheme of constant dimension $d$, such that $2 \leq d \leq n - 4$. Then $V$ can be defined by $n - 2$ equations. In particular, every locally complete intersection surface in $A^n_k$, where $n \geq 6$, is a set-theoretic complete intersection.

**Proof.** By induction, using the lemma we reduce to the case $d = \dim V = 2$. By the proposition there exists l.c.i. $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of $J$ is free. Now by [MK, Theorem 5] $J$ is $(n-2)$-generated. □

**Theorem E.** Let $k$ be any algebraically closed field of characteristic $2$ and $V \subset A^n_k$ a locally complete intersection subscheme of constant dimension $d$, such that $3 \leq d \leq n - 6$. Then $V$ can be defined by $n - 3$ equations. In particular, every locally complete intersection threefold in $A^n_k$, where $n \geq 9$, is a set-theoretic complete intersection.

**Sketch of Proof.** By an inductive argument with the help of the lemma we are reduced to the case $d = 3$. By the proposition, we get $J \subset I$ such that $\sqrt{J} = \sqrt{I}$ and the reduced conormal module of $J$ is $P \oplus A^{n-5}$, where $P$ is projective of rank $2$ with trivial determinant. By a special argument in characteristic $2$, we then obtain a new l.c.i. ideal $J_1 \subset J$ such that $\sqrt{J_1} = \sqrt{I}$ and the reduced conormal module of $J_1$ is free. Now we are done by [MK, Theorem 5]. □

Complete proofs will appear elsewhere.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637