ENTROPIC NONEXTENSIVITY: A POSSIBLE MEASURE OF COMPLEXITY

Constantino Tsallis
Centro Brasileiro de Pesquisas Físicas, Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil
Department of Physics, University of North Texas, P.O. Box 311437, Denton, Texas 76203, USA
Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA
tsallis@cbpf.br

"Beauty is truth, truth beauty,"—that is all
Ye know on earth, and all ye need to know.
(John Keats, May 1819)

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I. INTRODUCTION

Would you say that Newtonian mechanics is universal? We almost hear a big and unanimous yes coming from practically all nineteenth-century physicists. And yet, they would all be wrong! Indeed, we know today that if the involved masses are very small, say that of the electron, Newtonian mechanics has to be replaced by quantum mechanics. And if the involved speeds are very high, close to that of light in vacuum, special relativity has to be used instead. And if the masses are very large, like that of the Sun or of a galaxy, general relativity enters into the game. It is our present understanding that only in the $(\hbar, 1/c, G) \to (0, 0, 0)$ limit, Newtonian mechanics appears to be strictly correct. Now, what about Boltzmann-Gibbs statistical mechanics and standard thermodynamics? Are they universal? An attempt to answer such question and to clarify its implications constitutes the main goal of the present effort. Let us see what Ludwig Boltzmann has to tell us about this. In the first page of the second part of his Vorlesungen über Gastheorie, he qualifies the concept of ideal gas by writing: "When the distance at which two gas molecules interact with each other noticeably is vanishingly small relative to the average distance between a molecule and its nearest neighbor – or, as one can also say, when the space occupied by the molecules (or their spheres of action) is negligible compared to the space filled by the gas – ...". Were Boltzmann our contemporary, he would have perhaps told us that he was addressing systems involving what we nowadays call short range interactions! Several decades ago, Laszlo Tisza, in his Generalized Thermodynamics, writes "The situation is different for the additivity postulate [...], the validity of which cannot be inferred from general principles. We have to require that the interaction energy between thermodynamic systems be negligible. This assumption is closely related to the homogeneity postulate [...]. From the molecular point of view, additivity and homogeneity can be expected to be reasonable approximations for systems containing many particles, provided that the intramolecular forces have a short range character", when referring to the usual thermodynamic functions such as internal energy, entropy, free energy, and others. Also, Peter T. Landsberg, in his Statistical Mechanics and Thermodynamics, writes "The presence of long-range forces causes important amendments to thermodynamics, some of which are not fully investigated as yet.". More, during a visit to Rio de Janeiro some years ago, he told me that such restrictions should be in the first pages of all books on thermodynamics... but they are not! The title of the 1993 Nature article by John Maddox focusing on black holes is "When entropy does not seem extensive". What is the viewpoint on such matters of our colleagues mathematicians? Distinguished expert in nonlinear dynamical systems, Floris Takens writes "The values of $p_i$ are determined by the following dogma: if the energy of the system in the $i$th state is $E_i$ and if the temperature of the system is $T$ then: $p_i = \exp(-E_i/kT)/Z(T)$, where $Z(T) = \sum_i \exp(-E_i/kT)$ [...]. This choice of $p_i$ is called Gibbs distribution. We shall give no justification for this dogma; even a physicist like Ruelle disposes of this question as "deep and incompletely clarified"." It is known that mathematicians sometimes use the word dogma when they do not have the theorem! As early as 1964, Michael E. Fisher, and later on with David Ruelle as well as with Joel L. Lebowitz, addressed in detail such questions. They established, for instance, that a $d$-dimensional classical system (say a fluid) including two-body interactions that are nonsingular at the origin and decay, at long distances, like $r^{-\alpha}$, exhibits standard thermostatistical behavior if $\alpha/d > 1$. An interesting question remains open: what happens when $\alpha/d \leq 1$? This is the case of all self-gravitating systems (e.g., galaxies, black holes), a fact which explains well known difficulties in using standard thermodynamics to address them. These difficulties are not without relation with the case of a single hydrogen atom ($d = 3$ and $\alpha = 1$),
for which no tractable Boltzmann-Gibbs thermostatis-
tical calculations are possible (diverging partition func-
tion) unless we confine the hydrogen atom in a box \( \beta \). Of course, no such difficulties are encountered if we fo-
cus on a single harmonic oscillator, or on a single spin in an
external magnetic field, or on a Lennard-Jones fluid
\((d = 3 \text{ and } \alpha = 6)\), or even on a neutral plasma, where
the Coulombian interaction is “dressed” thus becoming a
thermodynamically innocuous, exponentially decaying
interaction. No severe anomalies appear in such cases,
all relevant sums and integrals being finite and perfectly
defined. More generally speaking, a variety of recent
analytical and numerical results (see \([9]\) and references
therein) in classical systems (with no singularities at the
potential origin or elsewhere) suggest that, at the appro-
ch to thermal equilibrium, the internal energy per particle grows, when the number of
particles \( N \) grows, like \( N^{1-\alpha/d}/[1-\alpha/d] \).
We easily verify that \( N \) approaches a finite constant if
\( \alpha/d > 1 \), diverges logarithmically if \( \alpha/d = 1 \), and
diverges like \( N^{1-\alpha/d} \) if \( 0 < \alpha/d < 1 \), thus recovering, for
\( \alpha = 0 \), the well known Molecular Field Approximation
scaling with \( N \) (usually englobed within the coupling
constant whenever dealing with magnetic models such
as the Ising ones, among many others). We trivially see
that the standard thermodynamical extensivity is lost
in such cases. The physical behavior for the marginal case
\( \alpha/d = 1 \) is intimately related to the mathematical fact
that some relevant sums that are absolutely convergent on
the extensive side and divergent on the nonextensive one,
become conditionally convergent in that case. This en-
ables us to understand why the amount of calories to be
provided to a table in order to increase its temperature
in one degree only depends on its weight and material
(iron, wood), whereas the amount of Coulombs we must
provide to a capacitor to generate a one Volt potential
difference at its ends also depends on its shape! Indeed,
the relevant interaction in the latter being the perma-
nent dipole-dipole one (hence \( d = \alpha = 3 \)), the capacity
depends on whether the capacitor is a slab, round, cylin-
drical or otherwise. Such behavior dramatically contrasts
with the extensivity usually focused on in textbooks of
thermodynamics. In fact, it is after all quite natural that
self-gravitating systems provided the first physical ap-
lication of the formalism we are addressing here. Indeed,
physicist Angel Plastino and his son astronomer Angel R.
Plastino showed in 1993 \([10]\) that by sufficiently depart-
ing from BG thermostatistics, it is possible to overcome
an old difficulty, namely to have the physically desirable
feature that total energy, entropy and mass be simulta-
neously finite.

Let us now try to deepen our quest for understanding
the restrictions within which Boltzmann-Gibbs statistical
mechanics is the appropriate formalism for describing a
(fantastic!) variety of systems in nature. Are long-range
interactions the only mechanism which creates anom-
aliies? The answer seems to be no. Indeed, several indica-
tions exist which suggest that mesoscopic dissipative sys-
tems (like granular matter) also are thermostatically
anomalous: the velocity distribution measured in a va-
riety of computational experiments is, as argued by Leo
P. Kadanoff et al \([11]\), Y.-H. Taguchi and H. Takayasu
\([12]\), Hans J. Herrmann \([13]\) and others, far from be-
ning Maxwell’s Gaussian. On general grounds, multifrac-
tally structured systems also tend to exhibit anomalies.
And I believe that it would be surprise for very few that
the same happened with systems based on strongly non-
markovian microscopic memory (e.g., a memory function
decreasing towards the past as a slow enough power-law).

As an attempt to theoretically handle at least some of
these anomalies, I proposed in 1988 \([4]\) the generalization
of the Boltzmann-Gibbs formalism by postulating a
nonextensive entropy \( S_q \) (See Sections I and II) which
reverses the usual logarithmic measure as the \( q = 1 \) par-
cular case. Before going further, let us describe \( S_q \), its
properties and historical context.

II. THE NONEXTENSIVE ENTROPY \( S_q \)

A. Properties

In the case of a discrete number \( W \) of microscopic con-
fugurations, \( S_q \) is given by \([4]\)

\[
S_q = k \frac{1}{q-1} \sum_{i=1}^{W} p_i^q \left( \sum_{i=1}^{W} p_i = 1; k > 0 \right) \tag{1}
\]

(similar expressions correspond to the cases where we
have a continuum of microscopic configurations, or when
the system is a quantum one).

In the limit \( q \to 1 \) we have \( p_i^{q-1} = e^{(q-1) \ln p_i} \sim 1 + (q - 1) \ln p_i \), hence \( S_1 = -k \sum_{i=1}^{W} p_i \ln p_i \), which is the usual
Boltzmann-Gibbs-Shannon expression. Also, it can be
shown that

(i) \( S_q \geq 0 \), and, for \( q > 0 \), equals zero in the case
of certainty (i.e., when all probabilities vanish but one
which equals unity);

(ii) \( S_q \) attains its extremum (maximum for \( q > 0 \) and
minimum for \( q < 0 \)) for equiprobability (i.e., \( p_i = 1/W \)),
thus becoming \( S_q = k[W^{1-q} - 1]/[1-q] \). This expression
becomes, for \( q = 1 \), \( S_1 = k \ln W \), which is the formula
graved on Boltzmann’s tombstone in the Central
Cemetery in Vienna;

(iii) \( S_q \) is concave (convex) for \( q > 0 \) \((q < 0)\), which is
the basis for thermodynamic stability;

(iv) If two systems \( A \) and \( B \) are independent in the
sense of the theory of probabilities (i.e., \( p_{ij}^{A+B} = p_i^A \times p_j^B \)),
then

\[
\frac{S_q(A + B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q) \frac{S_q(A)}{k} \frac{S_q(B)}{k}, \tag{2}
\]
hence, superextensivity, extensivity or subextensivity occurs when \( q < 1 \), \( q = 1 \) or \( q > 1 \) respectively;

(v) If the \( W \) states of a system are divided into \( W_L \) and \( W_M \) states (\( W_L + W_M = W \)), and we define \( p_I = \sum_{W_L \text{states}} p_I \) and \( p_M = \sum_{W_M \text{states}} p_M \) (\( p_L + p_M = 1 \)), then

\[
S_q(p_I) = S_q(p_L, p_M) + p_L^q S_q(p_I/p_L) + p_M^q S_q(p_I/p_M)
\]

which generalizes Shannon’s celebrated formula;

(vi) It can be easily checked that \(-d(\sum_i p_i^x)/dx|_{x=1} = -\sum_i p_i \ln p_i\). Sumiyoshi Abe, in Funabashi, made in 1997 [15] a simple but very deep remark. He considered the so-called Jackson derivative. The mathematician F. Jackson generalized, one century ago, the usual differential operator. He defined \( D_q f(x) \equiv [f(qx) - f(x)]/[qx - q] \), which reproduces \( d/dx \) in the limit \( q \to 1 \). Abe’s observation consists in the following easily verifiable property, namely \(-D_q(\sum_i p_i^x)/dx|_{x=1} = S_q\) ! Since it is clear that Jackson derivative “tests” how the function \( f(x) \) “reacts” under dilatation of \( x \) in very much the same way the standard derivative “tests” it under translation of \( x \). Abe’s remark opens a wide door to physical insight onto thermodynamics. It was so perceived, I believe, by Murray Gell-Mann, of the Santa Fe Institute. Indeed, one year ago, during a conference in Italy, he asked me whether I would go to conjecture that systems with relevant symmetries other than say translation or dilatation invariances would need entropies other than \( S_1 \) or \( S_q \). My answer was probably yes. Indeed, there are symmetries that appear in general relativity or in string theory that are characterized by a sensible amount of parameters. The concept of thermodynamic information upon such systems, i.e., the appropriate entropy could well be a form involving essentially a similar number of parameters. If so, then symmetry would control entropy, and \( S_q \) would only be the beginning of a presumably long and fascinating story! Since we have learnt from Gell-Mann and others, how deeply symmetry controls energy, we could then say that symmetry controls thermodynamics, that science which nothing is but a delicate balance between energy and entropy. More precisely, symmetry would then determine (see also [14]) the specific microscopic form that Clausius thermodynamic entropy would take for specific classes of systems. Symmetry controlling thermodynamics! If such a conjectural scenario turned out to be correct, this would not be to displease Plato with his unified view of truth and beauty!

(vii) Along the lines of Claude E. Shannon’s theorem, it has been shown by Roberto J. V. dos Santos, in Macceio, Brazil [14], that a set of conditions generalizing (through properties (iv) and (v)) that imposed by Shannon uniquely determines \( S_q \). Analogously, Abe recently showed [13] that consistently generalizing Khinchin’s set of conditions, once again a unique solution emerges, namely \( S_q \).

B. Brief review of the labyrinthine history of the entropies

The entropy and the entropic forms have haunted physicists, chemists, mathematicians, engineers and others since long! At least since Clausius coined the word and wrote “The energy of the universe is constant, its entropy tends to increase” (See Appendix). The whole story is a labyrinth plenty of rediscoveries. After Ludwig Boltzmann and Josiah Willard Gibbs introduced and first analyzed, more than one century ago, their respective expressions [20] of the entropy in terms of microscopic quantities, many generalizations and reformulations have been proposed and used. Let us mention some of the most relevant ones. John von Neumann introduced a quantum expression for the entropy in terms of the density operator; when the operator is diagonalized, the traditional expression (herein referred, because of the functional form, to as the Boltzmann-Gibbs entropy, although these two scientists used to work with different, though consistent, expressions [20]) is recovered. Shannon reinvented, more than half a century ago, a binary version of the B-G functional form and interpreted it in terms of information and communication theory. It is no doubt very interesting to notice that in The mathematical theory of communication he wrote, in reference to the specific form he was using, that “This theorem, and the assumptions required for its proof, are in no way necessary for the present theory. It is given chiefly to lend a certain plausibility to some of our later definitions. The real justification of these definitions, however, will reside in their implications.”. In 1957, Edwin Jaynes attributed to the whole formalism a very generic, and still controversial in spite of an enormous amount of practical applications, anthropomorphic shape. Since those years, people working in cybernetics, information theory, complexity and nonlinear dynamical systems, among others, have introduced close to 20 different entropic forms! This number has almost doubled, this time mainly because of physicists, since the 1988 paper [1] appeared.

In 1960, A. Renyi proposed [21] a form which recovers that of Shannon as a special case. His form is always extensive, but not always concave (or convex) with regard to the probability distributions. It seems that it was a rediscovery; indeed, according to I. Csiszar’s 1974 critical survey [22], that form had essentially already been introduced by Paul-Marcel Schutzenberger in 1954 [23]. In 1967, presumably for cybernetic goals, J. Harvda and F. Charvat introduced [24], although with a different prefactor (fitted for binary variables), the entropic form herein noted \( S_q \), and studied some of its mathematical properties. One year later, I. Vajda further studied [25] this form, quoting Harvda and Charvat. In 1970, in the context of information and control, Z. Daroczy rediscovered [20] this form (he quotes neither Harvda and Charvat, nor Vajda). Later on, in 1975, B.D. Sharma and D.P. Mittal introduced [26] and studied some mathematical proper-
ties of a two-parameter form which recovers both Renyi’s and $S_q$ as particular instances! In his 1978 paper in Reviews of Modern Physics, A. Wehrl mentions the form of $S_q$, quoting Daroczy, but not Harvda, Charvat, Vajda, Sharma and Mittal. In 1988, in the Physics community, a new rediscovery, by myself this time. Indeed, inspired by multifractals, I postulated the form $S_q$ as a possible new path for generalizing Boltzmann-Gibbs statistical mechanics; through optimization of $S_q$ I obtained an equilibrium-like distribution which generically is a power-law, and reproduces Boltzmann’s celebrated factor as the particular limit $q \rightarrow 1$. In my 1988 paper, I quoted only Renyi’s entropy, having never taken notice of all these generalized forms, whose existence (and that of many others!) I have been discovering along the years. In fact, Renyi’s entropy can be shown to be a monotonically increasing function of $S_q$. However, concavity is not preserved through monotonicity. So what?, since the optimizing distribution will be exactly the same. Well, it happens that statistical mechanics is much more than an equilibrium-like distribution optimizing an entropy under some generic constraints! It includes a variety of other relevant points, such as the role played by fluctuations, linear responses, thermodynamical stability, to name but a few. Nobody would, I believe, try to reformulate Boltzmann statistical mechanics; through optimization of $S_q$, $S_1$ is both concave and extensive, the Renyi entropy, $S_q^R$, is extensive but nonconcave (for all $q > 0$; concave only for $0 < q \leq 1$), the one fixed on herein, $S_q$, is nonextensive but concave (for all $q > 0$), and the Sharma-Mittal entropy is nonextensive and nonconcave. The latter is determined by two parameters and contains the other three as particular cases. The Renyi entropy and $S_q$ are determined by one parameter ($q$) and share a common case, namely the BG entropy, which is nonparametric. Since Renyi’s, Sharma-Mittal’s and the present entropy depend from the probability set through one and the same expression, namely $\sum_i p_i^q$, they are all three related through simple functions. In particular, $S_q^R \equiv \frac{k \ln \sum p_i^q}{1 - q} = \frac{k \ln[1 + (1 - q)S_q/k]}{1 - q}$ (\forall q), (4)

with $S_1^R = S_1 = -k \sum_i p_i \ln p_i$.

For completeness, let us close this section by mentioning that expressions of the type $\sum_i p_i^q$ have also been, long ago, discussed on mathematical grounds by Hardy, Littlewood and Polya (and, for $q = 2$, by the Pythagoreans!)

III. EQUILIBRIUM-LIKE STATISTICS

The next step is to follow along Gibbs path. For instance, to formally obtain the ”equilibrium” distribution associated with the canonical ensemble (i.e., a conservative system at ”equilibrium” with a thermostat, where the physical meaning of this state will turn out to be, as we shall see later on, close to a stationary one), we need to impose a constraint on total energy. We shall adopt that $\langle \mathcal{H} \rangle_q \equiv \sum_i \epsilon_i p_i = U_q$, where $U_q$ is fixed and finite. $\{\epsilon_i\}$ is the set of energy levels of the system Hamiltonian $\mathcal{H}$, and the escort distribution $\{P_i\}$ is defined as $P_i \equiv p_i^q/\sum p_i^q$; from now on we refer to $\langle \ldots \rangle_q$ as the normalized $q$-expectation value. At the present stage, a plethora of mathematical reasons exist for adopting this particular generalization of the familiar concept of internal energy. Not the least of these reasons is the fact that such definition enables relevant sums and integrals to be finite, which would otherwise diverge (e.g., the second moment of Lévy distributions diverges, whereas the second $q$-moment with $q$ appropriately chosen is finite). We can say that in this sense the theory becomes regularized. The optimization of $S_q$ with this constraint (to which we associate a Lagrange parameter $\beta$) leads to

$$p_i = [1 - (1 - q)\beta_q (\epsilon_i - U_q)]^{1/(1 - q)}/Z_q \quad (\beta_q \equiv \beta/Z_q^{1 - q})$$ (5)

with $Z_q \equiv \sum_j [1 - (1 - q)\beta_q (\epsilon_j - U_q)]^{1/(1 - q)}$ (See Fig. 1). We easily verify that $q \rightarrow 1$ recovers the celebrated, exponential Boltzmann factor

$$p_i = e^{-\beta \epsilon_i}/Z_1 \quad (Z_1 \equiv \sum_j e^{-\beta \epsilon_j}).$$ (6)

For $q > 1$ a power-law tail emerges; for $q < 1$ the formalism imposes a high-energy cutoff, i.e., $p_i = 0$ whenever the argument of the power function becomes negative. One comment is worthy: this distribution is generically not an exponential law, i.e., it is generically not factorizable (under sum in the argument), and nevertheless is invariant under the choice of zero energy for the energy spectrum! (this is in fact one of the aside benefits of defining the constraints in terms of normalized distributions like the escort ones).

Eq. (5) can be rewritten in the following convenient form:

$$p_i = [1 - (1 - q)\beta' \epsilon_i]^{1/(1 - q)}/Z_q'$$ (7)

with $Z_q' \equiv \sum_j [1 - (1 - q)\beta' \epsilon_j]^{1/(1 - q)}$, where $\beta'$ is a simple function of $\beta_q$ and $U_q$.

It can be shown, along the lines of the 1991 paper with Evaldo M.F. Curado, that this formalism satisfies, for arbitrary $q$, $1/T \equiv \partial \ln S_q/\partial U_q (T \equiv 1/(k\beta))$, and the entire Legendre transform structure of standard thermodynamics. Moreover, the relations which provide
\[ U_q \quad \text{and} \quad F_q = U_q - T S_q \quad \text{in terms of} \quad Z_q \quad (\text{i.e., the connection between the microscopic and macroscopic descriptions of the system}), \quad \text{remain basically the same as} \quad \text{the usual ones, the role played by the logarithmic function} \quad \ln x \quad \text{being now played by} \quad (x^{1-q} - 1)/(1-q). \quad \text{Finally, various important thermostatistical theorems can be shown to be} \quad q\text{-invariant.} \]

Among them, we must mention the Boltzmann H-theorem (macroscopic time irreversibility, \text{i.e., essentially the second principle of thermodynamics, first tackled within this context by Ananias M. Mariz in Natal, Brazil} [33]), Ehrenfest theorem (correspondence principle between classical and quantum mechanics), Einstein 1910 factorizability of the likelihood function in terms of the entropy (this \text{q-generalization was first realized in 1993 during a private discussion with Manuel O. Caceres, from Bariloche, Argentina}), Onsager reciprocity theorem (macroscopic time reversibility), Kramers and Wannier relations (causality), and, apparently, even Pesin relation for nonlinear dynamical systems, although the last one remains at the level of plausibility (no proof available). The \text{q-generalization of the fluctuation-dissipation theorem was first addressed by Anna Chame and Evandro V.L. de Mello, in Niteroi, Brazil} [33], and then, along with the generalization of Kubo’s linear response theory, by A.K. Rajagopal, in the Naval Research Laboratory in Washington [32]. Since then, a variety of important statistical mechanical approaches such as the Bogolyubov inequality, the many-body Green function and path integral formalisms have been generalized by Renio S. Mendes in Maringa, Brazil, Ervin K. Lenzi in Rio de Janeiro, and collaborators [32].

Nonextensive versions of Fermi-Dirac and Bose-Einstein statistics have been discussed [38] by Fevzi Buyukkilic, Dogan Demirhan and Ugur Tarmaki in Izmir, Turkey, and their collaborators. Finally, it is certainly important to mention that, very recently, Rajagopal and Abe have done an exhaustive review [34] of apparently all the traditional methods for obtaining the BG equilibrium distribution, namely the Balian-Balazs counting method within a microcanonical basis, the Darwin-Fowler steepest descent method, and the Khinchin’s large-numbers method. They q-generalized all three, and systematically obtained the same power-law distribution obtained above through the Gibbs’ method, \text{i.e., the optimization of an appropriately constrained entropy.}

In order to see the nonextensive formalism at work, let us address an important question which long remained without satisfactory solution, namely, why Lévy distributions are ubiquitous in nature, in a manner which by all means is similar to the ubiquity of Gaussians? To introduce the point, let us first address the Gaussian case. If we optimize the BG entropy \( S_1 \) by imposing a fixed and finite average \( \langle x^2 \rangle \), we obtain a Gaussian one-jump distribution. By convoluting \( N \) such distributions we obtain a \( N \)-jump distribution which also is a Gaussian. From this immediately follows that \( \langle x^2 \rangle (N) \propto N \).

If we consider that \( N \) is proportional to time, we have the central result of Einstein’s 1905 celebrated paper on Brownian motion, which provided at the time strong evidence in favor of Boltzmann’s ideas. We may reword this result by saying that the foundation of Gaussians in nature lies upon two pillars, namely the BG entropy and the central limit theorem. What about Lévy distributions? The difficulty comes from the fact that all Lévy distributions have infinite second moment, due to their long tail (\( \propto 1/|x|^{1+\gamma} \) with \( 0 < \gamma < 2 \)). Consequently, what simple auxiliary constraint to use along with the optimization of the entropy? In 1994, Pablo Alemany and Damian H. Zanette in Bariloche, Argentina, settled the basis for the solution, namely optimization of \( S_q \) while imposing a finite \( q \)-expectation value of \( x^2 \). The one-jump distribution thus obtained is proportional to \( [1 - (1-q)\beta x^2]^{1/(1-q)} \), which respectively reproduces the Gaussian, Lorentzian (Cauchy for the mathematicians) and completely flat distributions for \( q \) = 1, 2 and 3. For \( q < 1 \) the support is compact, and for \( q > 1 \) a power-law tail is obtained. The proof was completed one year later [35], when it was argued that the Lévy-Gnedenko central limit theorem was applicable for \( q > 5/3 \), thus approaching, for \( N \gg 1 \) precisely Lévy distributions for the \( N \)-jump distribution. We may then summarize this result by saying that the foundation of Lévy distributions in nature also has two pillars, namely \( S_q \) and the Lévy-Gnedenko theorem. It follows that \( \gamma = 2 \) for \( q \leq 5/3 \) and \( \gamma = (3-q)/(q-1) \) for \( 5/3 < q < 3 \). Since \( \gamma \) can also be interpreted as the fractal dimension to be associated with Lévy walks, this turned out to be the first exact connection between nonextensive statistical mechanics and scaling. Similar ideas have been successfully put forward by Plastino and Plastino [35] and by Lisa Borland [38] and others, concerning nonlinear as well as fractional-derivative Fokker-Planck-like equations. In all these various types of anomalous diffusion, the Gaussian solution associated with Jean Baptiste Joseph Fourier’s bicentennial heat equation is recovered as the \( q \to 1 \) limit. Let me also mention that Hermann Haken, in Stuttgart, and collaborators have recently applied related ideas within a theoretical model of the human brain [38].

In the same vein which led to the above discussion of Lévy distributions, if we optimize \( S_q \) imposing, besides normalization of \( p(x) \), finite values for the normalized \( q \)-expectation values of both \( x \) and \( x^2 \), we obtain

\[
p_q(x) \propto [1 - (1-q)(\beta^1 x^1 + \beta^2 x^2)]^{1/(1-q)},
\]

where \( \beta^1 \) and \( \beta^2 \) are determined by the constraints. This distribution contains, as particular cases, a considerable amount of well (and not so well) known distributions, such as the exponential, the Gaussian, the Cauchy-Lorentz, the Edgeworth [40] (for \( \beta^1 = 1, \beta^2 = 0 \), and \( x \geq 0 \)), the \( \nu \)- and the Student’s \( \nu \)-ones [41], among others. Let us end with a short and amusing historical remark concerning that which can be considered as the most famous of all nontrivial distributions, namely the normal one. In contrast with what almost everybody would naively think, it was [12] first introduced by Abraham De Moivre in 1733, then by Pierre Simon de Laplace...
IV. MICROSCOPIC DYNAMICAL FOUNDATION

To have a deeper understanding of what this generalized formalism means, it is highly convenient to focus on the mixing properties.

A. Mixing in one-dimensional dissipative maps

Let us now address a very simple type of system, which will nevertheless provide important hints about the physical and mathematical significance of the present formalism. Let us consider the well known logistic map, namely

\[ x_{t+1} = 1 - ax_t^2 \quad (t = 0, 1, 2, \ldots; 0 \leq a \leq 2) \]  

For \( a < a_c = 1.40115519 \ldots \), the attractors are finite cycles (fixed points, cycle-2, cycle-4, etc) and the Lyapunov exponent (hereafter noted \( \lambda_1 \)) is almost everywhere negative (i.e., strongly insensitive to the initial conditions and any intermediate rounding). For \( a_c \leq a \leq 2 \), the attractor is chaotic for most values (i.e., it has an infinite number of elements, analogously to an irrational number) and the \( \lambda_1 \) exponent is consistently positive (i.e., strongly sensitive to the initial conditions and any intermediate rounding). There is however, in the interval \([0, 2]\), an infinite number of values of \( a \) for which \( \lambda_1 = 0 \). What happens with the sensitivity to the initial conditions in those marginal cases? Let us focus on this problem by first recalling the definition of \( \lambda_1 \). If two initial conditions \( x_0 \) and \( x'_0 \) differ by the small amount \( \Delta x_0 \), we can follow the time evolution of \( \Delta x_t \) through the quantity \( \xi \equiv \lim_{\Delta x_0 \to 0} (\Delta x_t/\Delta x_0) \). The most frequent situation is that \( \xi = e^{\lambda_1 t} \), which defines the Lyapunov exponent \( \lambda_1 \). We can trivially check that \( \xi(t) \) is the solution of \( dx_t/dt = \lambda_1 \xi \). We have conjectured [43] that, whenever \( \lambda_1 \) vanishes for this map (and similar ones), the controlling equation becomes \( dx_t/dt = \lambda_q \xi^q \), hence \( \xi = [1 + (1-q)\lambda_q t^{1/(1-q)}]^{1/(1-q)} \). This result recovers the usual case at the \( q = 1 \) limit; also, it defines a generalized Lyapunov coefficient \( \lambda_q \), which, like \( \lambda_1 \), inversely scales with time, but now within a law which generically is a power, instead of the standard exponential. The two paradigmatic cases are \( q < 1 \) (with \( \lambda_q > 0 \)), hereafter referred to as weakly sensitive to the initial conditions, and \( q > 1 \) (with \( \lambda_q < 0 \)), hereafter referred to as weakly insensitive to the initial conditions. Within this unified scenario, we would have strong chaos for \( q = 1 \) and \( \lambda_1 > 0 \) (i.e., exponential mixing) and weak chaos for \( q < 1 \), \( \lambda_1 = 0 \) and \( \lambda_q > 0 \) (i.e., power-law mixing). All these various possibilities have indeed been observed for the logistic map. If we start with \( x_0 = 0 \), we have (i) \( \lambda_1 < 0 \) for \( a = a_c - 10^{-3} \), (ii) \( \lambda_1 > 0 \) for \( a = a_c + 10^{-3} \), (iii) \( \lambda_1 = 0 \), \( q > 1 \) and \( \lambda_q < 0 \) for a doubling-period bifurcation (e.g., \( a = 3/4 \)) as well as for a tangent bifurcation (e.g., \( a = 7/4 \)), and finally (iv) \( \lambda_1 = 0 \), \( q < 1 \) and \( \lambda_q > 0 \) at the edge of chaos (i.e., \( a = a_c \)). The last case is, for the present purposes, the most interesting one. This power-law mixing was first observed and analyzed by P. Grassberger, A. Politi, H. Mori and collaborators [44]. Its importance seems to come mainly from the fact that, in the view of many authors, “life appeared at the edge of chaos”! Moreover, this type of situation seems to be a paradigm for vast classes of the so-called complex systems, much in vogue nowadays. Let us say more about this interesting point. The precise power-law function \( \xi \) indicated above appears to be the upper bound of a complex time-dependence of the sensitivity to the initial conditions, which includes considerable and ever lasting fluctuations. From a log-log representation we can numerically get the slope and, since this is to be identified with \( 1/(1-q) \), we obtain the corresponding value \( q^* \). For the logistic map we have obtained [43] \( q^* = 0.24 \ldots \) If we apply these concepts to a more general map, like \( x_{t+1} = 1 - ax_t^q \) \((0 \leq a \leq 2; z > 1)\), we can check, and this is long known, that the edge of chaos depends on \( z \). More precisely, when \( z \) increases from 1 to infinity, \( a_c \) increases from 1 to 2. If for \( a_c \) we check the sensitivity to the initial conditions, we obtain that \( q^* \) increases from \(-\infty\) close to 1 (though smaller than 1), when \( z \) increases from 1 to infinity. Simultaneously, the fractal dimension \( d_f \) of the chaotic attractor increases from 0 to close to 1 (though smaller than 1). If instead of the logistic family of maps, we use some specific family of circle maps also characterized by an inflexion exponent \( z \), it can be shown that, once again, \( q^* \) increases with \( z \) for all studied values of \( z \) (\( z \geq 3 \)). The interesting feature in this case is that \( d_f = 1 \) for all these values of \( z \). What we learn from this is that the index \( q \) characterizes essentially the speed of the mixing, and only indirectly how “filled” is the phase space within which this mixing is occuring.

Let us now address, for maps like the logistic one, a second manner of obtaining \( q^* \), this time based on the multifractal geometry of the chaotic attractor. This geometry can be, and usually is, characterized by the so called \( f(\alpha) \) function, a down-wourd parabola-like curve which generically is below (and tangential to) the bisector in the \((\alpha, f)\) space, is concave, its maximal value is \( d_f \), and, in most of the cases, vanishes at two points, namely \( \alpha_{min} \) and \( \alpha_{max} \) (typically, \( 0 < \alpha_{min} < d_f < \alpha_{max} \)). For a fractal, one expects, \( \alpha_{min} = d_f = \alpha_{max} = f(\alpha_{min}) = f(d_f) = f(\alpha_{max}) \). For a so-called fat (multi)fractal, \( d_f \) attains the value of the Euclidean space within which the (multi)fractal is embedded (this is the case of the family of circle maps mentioned above). Marcelo L. Lyra, in Maceio, and myself used (imitating the successful Ben Widom style!) in 1998 [44] some simple scaling arguments and obtained the following relation:
of probabilities $p_i$ and windows (time $\{t \}$). We will observe a set $\{N_i(t)\}$ of points inside the $W$ windows ($\sum_{i=1}^{W} N_i(t) = N, \forall t$). We next define a set of probabilities $p_i = N_i/N (i = 1, 2, ..., W)$, which enable in turn the calculation of $S_1(t) = -\sum_{i=1}^{W} p_i \ln p_i$. This entropy starts, by construction, from zero at $t = 0$, and tends to linearly increase as time goes on, finally saturating at a value which, of course, cannot exceed $\ln W$. The Kolmogorov-Sinai-like entropy rate is then defined through $K_1 = \lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} S_1(t)/t$. We expect, according to the Pesin equality, $K_1 = \lambda_1$. Indeed, we numerically obtain $K_1 = \ln 2$ (see Fig. 2(a)).

We can analogously define a generalized quantity, namely $K_q = \lim_{t \to \infty} \lim_{W \to \infty} \lim_{N \to \infty} S_q(t)/t$. And we can numerically verify that, for $a = 2$, $K_q$ vanishes (diverges) for any value of $q > 1$ ($q < 1$), being finite only for $q = 1$. Let us now turn onto the edge of chaos, i.e., $a = a_c$. Vito Latora and Andrea Rapisarda, in Catania, Italy, Michel Baranger, at MIT, and myself [43], have numerically verified the same structure, excepting for the fact that now $K_q$ vanishes (diverges) for any value $q > q^*$ ($q < q^*$), being finite only for $q = q^* = 0.24... !$ (see Fig. 2(b)). Tímalki et al [44] have also verified the validity of this structure for various typical values of $z$ within the logistic-like family of maps.

Let us summarize this section by emphasizing that we have seen that it is possible to arrive to one and the same nontrivial value of $q^*$ through three different and suggestive roads, namely sensitivity to the initial conditions, multifractal structure, and rate of loss of information. By so doing, we obtain insight onto the mathematical and physical meaning of $q$ and its associated nonextensive formalism. Moreover, a pleasant unification is obtained with the concepts already known for standard, exponential mixing. A very interesting question remains open though, namely the generalization of Pesin equality. It might well be that, as conjectured in [15], $K_q$ equals $\lambda_q$ (or, more generically, an appropriate average of $\lambda_q(x)$, whenever nonuniformity is present). However, a mathematical basis for properties such as this one is heavily needed. Some recent numerical results along these lines by Paolo Grigolini, in Denton, Texas, and collaborators [45] seem promising. In the same vein, it is known that whenever escape exists, as time evolves, out of a specific region of the phase space of a chaotic system, $K_1$ becomes larger than $\lambda_1$, the difference being precisely the escape rate [47]. All these quantities are defined through exponential time-dependences. What happens if all three are power laws instead? A natural speculation emerges, namely that perhaps a similar relation exists for $q \neq 1$.

**B. Mixing in many-body dissipative maps**

In the previous section we described a variety of interesting nonextensive connections occurring in one-dimensional dissipative systems. A natural subsequent question then is what happens when the dissipative system has many, say $N >> 1$, degrees of freedom? Such is the case of essentially all the models exhibiting Per Bak’s self-organized criticality (SOC). Francisco Tamarit...
and Sergio Cannas, in Cordoba, Argentina, and myself have discussed this issue for the Bak-Sneppen model for biological evolution. By numerically studying the time evolution of the Hamming distance for increasingly large systems, a power-law was exhibited. By identifying the exponent with $1/(1 - q^*)$, the value $q^* \simeq -2.1$ was obtained. Totally analogous results are obtained for the Suzuki-Kaneko model for imitation games (e.g., bird singings). It would be of appreciable importance if the same numbers were achieved through the other two procedures, namely the determination of the $f(\alpha)$ function and of the loss of information. Such consistency would be of great help for further understanding. These tasks remain, however, to be done.

C. Mixing in many-body Hamiltonian systems

Let us know address what might be considered as the Sancta Sanctorum of statistical mechanics, namely the systems that Boltzmann himself was mainly focusing on, i.e., classical many-body Hamiltonian systems. We know that no severe thermostatistical anomalies arrive for $N$-body $d$-dimensional systems whose (say two-body) interactions are neither singular at the origin nor long-ranged. A most important example which violates both restrictions is of course Newtonian gravitation. We shall here restrict to a simpler case, namely interactions that are well behaved at the origin, but which can be long-ranged. For specificity, let us assume a two-body potential which (attractively) decays at long distances like $1/r^\alpha$ with $0 \leq \alpha$ (for historical reasons we use $\alpha$ for this exponent; however, it has clearly nothing to do with the $\alpha$ appearing in the characterization of multifractals previously discussed). The limit $\alpha \to \infty$ corresponds to very short interactions; the other extreme ($\alpha = 0$) corresponds to a typical Mean Field Approximation. No thermostatistical basic difficulties are expected for $\alpha > \alpha_c$ but the situation is quite different for $0 \leq \alpha \leq \alpha_c$. As already mentioned, Fisher and collaborators established that $\alpha_c = d$ for classical systems ($\alpha_c \geq d$ is expected for quantum systems). Therefore, the nontrivial case is $0 \leq \alpha/d < 1$. Two physically different situations are to be understood, namely what happens at the $\lim_{N \to \infty} \lim_{t \to \infty}$ situation, and what happens at the $\lim_{t \to \infty} \lim_{N \to \infty}$ one. The two orderings are expected to be essentially equivalent for $\alpha/d > 1$, and dramatically different otherwise. Let us first focus on the $\lim_{N \to \infty} \lim_{t \to \infty}$ ordering. This should correspond to the traditional, BG-like, concept of thermal equilibrium, though with anomalous scalings. Let us be more specific and consider a fluid thermodynamically described by

$$\frac{G(T,p,N)}{N} = \frac{U(T,p,N)}{N} - \frac{T}{N} S(T,p,N) + \frac{p}{N} V(T,p,N),$$

hence, using the quantity $\tilde{N}$ introduced in Section I,

$$G(T,p,N) = \frac{U(T,p,N)}{\tilde{N}} - \frac{T}{\tilde{N}} S(T,p,N) + \frac{p}{\tilde{N}} V(T,p,N).$$

In the thermodynamic limit $N \to \infty$ we expect to obtain finite quantities as follows:

$$g(\tilde{T},\tilde{p}) = u(\tilde{T},\tilde{p}) - \tilde{T} s(\tilde{T},\tilde{p}) + \tilde{p} v(\tilde{T},\tilde{p}),$$

with $\tilde{T} \equiv T/\tilde{N}$ and $\tilde{p} \equiv p/\tilde{N}$. So, when $\alpha/d > 1$, $\tilde{N}$ is a constant, and we recover the usual concepts that $G$, $U$, $S$, and $V$ are extensive variables since they scale like $N$, whereas $T$ and $p$ are intensive variables since they are invariant with respect to $N$. If, however, $0 \leq \alpha/d \leq 1$, the extensivity-intensivity concepts loose their usual simple meanings. Indeed, there are now three, and not two, thermodynamical categories, namely the energetic ones ($G$, $F$, $U$, ...) which scale like $N\tilde{N}$, the pseudo-intensive ones ($T$, $p$, $H$, ...) which scale like $\tilde{N}$, and their conjugate pseudo-extensive variables ($S$, $V$, $M$, ...) which scale like $N$ (if and only if expressed in terms of the pseudo-intensive variables). I can unfortunately not prove the general validity of the above scheme, but it has been shown to be true in all the models that have been numerically or analytically studied (one- and two-dimensional Ising and Potts magnets, one-dimensional bond percolation and XY coupled rotators, one-, two- and three-dimensional Lennard-Jones-like fluids, among others). Also, for the case $\alpha = 0$, we recover the traditional scalings of Mean Field Approximations, where the Hamiltonian is artificially made extensive by dividing the coupling constant by $N$. By so doing, we are obliged to pay a conceptually high price, namely to have microscopic couplings which depend on $N$! This long standing tradition in magnetism has, for very good reasons, never been adopted in astronomy: we are not aware of any astronomer dividing the universal gravitational constant $G$ by any power of $N$ in order to artificially make extensive a Hamiltonian which clearly is not! The thermodynamical scheme that has been described above escapes from such unpleasant criticism. I sometimes refer to the above discussion of long-ranged systems as weak violation of BG statistical mechanics. Indeed, it becomes necessary to conveniently rescale the thermodynamic variables (using $\tilde{N}$) but the $q = 1$ approach remains the adequate one, i.e., energy distributions still are of the exponential type.

Let us now address the much more complex $\lim_{t \to \infty} \lim_{N \to \infty}$ situation. At the present moment, the reliable informations that we have are mainly of numerical nature, more precisely computational molecular dynamics. A paradigmatic system has been intensively addressed during last years, namely a chain of coupled inertial planar rotators (ferromagnetic XY-like interaction), the interaction decaying as $1/r^\alpha$. The system being classical, it is thermodynamically extensive for $\alpha > 1$ and nonextensive for $0 \leq \alpha \leq 1$. Important informations were established by Ruffo, in Florence, and collaborators, who focused on the $\alpha = 0$ case. If we note $E_N$ the
total energy associated with \( N \) rotators, it can be shown that a second order phase transition occurs at the critical point \( e \equiv E_N/(N\tilde{N}) = 3/4 \equiv e_c \). For \( e > e_c \), the rotators are nearly decoupled and for \( e < e_c \) they tend to clusterize. Not surprising, \( e_c \) depends on \( \alpha \), and vanishes above a critical value of \( \alpha \) which is larger than unity. For \( \alpha > 1 \), nothing very anomalous occurs. But, in the interval \( 0 \leq \alpha \leq 1 \), curious phenomena happen on both sides of \( e_c(\alpha) \).

For \( e > e_c \) (say \( e = 5 \)), the (conventionally scaled) maximum Lyapunov exponent \( \lambda_{\text{max}}(N) \) goes to zero as \( N^{-\kappa} \), where \( \kappa(\alpha) \) was numerically established, by Celia Anteneodo and myself \(^{33} \) (see Fig. 3). Nothing like this is observed for \( \alpha > 1 \), where the numerical evidence strongly points that \( \lim_{N \to \infty} \lambda_{\text{max}}(N) > 0 \). This of course suggests that, for a thermodynamically large system, mixing is not exponential, but possibly a power-law, i.e., weak mixing, as discussed above for the one-dimensional maps. The \( \kappa(\alpha) \) dependence has been recently studied by Andrea Giansanti, in Rome, and collaborators for \( d = 2 \) and \( 3 \) as well. If plotted against \( \alpha/d \) (see Fig. 3), an universal curve seems to emerge. This evidence is reinforced by the fact that it is possible to analytically prove, for this and other models, that \( \kappa(0) = 1/3 \). It is consistently obtained \( \kappa = 0 \) for \( \alpha/d \geq 1 \).

Let us now focus on the anomalies below \( e_c \). The case \( e = 0.69 \) for \( \alpha = 0 \) has been and is being further studied in detail. The maximal Lyapunov exponent is here positive in the limit \( N \to \infty \) (for all \( 0 < e < e_c \), in fact) but, suggestively enough, the usual (BG) canonical ensemble equilibrium is preceded by a long, metastable-like, quasi stationary state (QSS), whose duration \( \tau \) diverges with \( N \!). This means that, if \( N \) is of the order of the Avogadro number, during a time that might be longer than the age of the universe, only this curious QSS will be observable. So, we better understand it! As can be seen in Fig. 4, the average kinetic energy per particle \( e_k \) depends on both \( N \) and time \( t \). Only in the \( \lim_{N \to \infty} \lim_{t \to \infty} \) the BG canonical temperature \( T \) is attained. If we take the limits the other way around a completely different result is obtained, namely \( e_k \) approaches a finite value below \( T \), and \( \tau \) diverges like \( N^\kappa \). It becomes possible to speculate that \( \rho(\alpha) \) decreases from a finite positive value to zero, when \( \alpha/d \) increases from zero to unity; thereafter, of course expect \( \rho = 0 \), the system being then a well behaved BG one. To make this scenario more robust, consistent evidence is available concerning diffusion of the \( \alpha = 0 \) rotators. The numerical study of this phenomenon has shown \(^{22,56} \) an average second moment of the angle increasing like \( t^\beta \), where \( \beta \) depends on both \( N \) and \( t \). For fixed \( N \), \( \beta > 1 \) (i.e., superdiffusion) for a long time until, at \( t = \tau_D \) (\( D \) stands for diffusion), makes a crossover to unity (normal diffusion). Analogously to what was described above, \( \tau_D \) increases like \( N^\kappa \). Once again, it is possible to speculate that \( \sigma(\alpha) \) decreases from some finite positive value to zero when \( \alpha/d \) increases from zero to unity. The door remains open to whether there is a simple connection between \( \sigma \) and \( \rho \). All these results enable to conjecture that a scenario close to what is shown in Fig. 5 could indeed occur for some physical systems. Before closing this section, let me emphasize what Fig. 4 strongly suggests is the possible validity of the zeroth principle of thermodynamics even out of a canonical BG scenario! Indeed, for large enough \( N \), systems can share the same "temperature", thus being at thermal equilibrium, though this equilibrium is not the familiar one. This is quite remarkable if we take into account that there is nowadays quite strong numerical evidence that the distribution of velocities in such systems is far from Gaussian, i.e., the familiar Maxwellian distribution of velocities is by all means violated.

**V. COMPARISON WITH EXPERIMENTAL RESULTS**

During this last decade many types of comparisons have been done between the present theory and experimental data. They have different epistemological status, and range from simple fittings (with sometimes clear, sometimes rather unclear physical insight), up to completely closed theories, with no fitting parameters at all. Let me review a few among them.

**A. Turbulence**

X.-P. Huang and C.F. Driscoll exhibited in 1994 \(^{57} \) some quite interesting nonneutral electronic plasma experiments done in a metallic cylinder in the presence of an axial magnetic field. They observed a turbulent axisymmetric metaequilibrium state, the electronic density radial distribution of which was measured. Its average (over typically 100 shots) monotonically decreased with the radial distance, disappearing at some radius sensibly smaller than the radius of the container (i.e., a cut-off was observed). They also proposed four phenomenological theories trying to reproduce the experimentally observed profile. The Restricted Minimum Enstrophy one provided a quite satisfactory first-approximation fitting. Bruce M. Boghosian, in Boston, showed in 1996 \(^{58} \) that the Huang and Driscoll successful attempt precisely corresponds to the optimization of \( S_q \) with \( q = 1/2 \), the necessary cut-off naturally emerging from the formalism. Since then, improved versions of his calculation \(^{14,59} \), as well as controversial arguments \(^{58} \), have been published. The merit of first connecting the present formalism to turbulence remains however doubtless.

In recent months, Fernando M. Ramos and collaborators, in São José dos Campos, Brazil \(^{61} \), Toshihiko Arimitsu and N. Arimitsu, in Tokyo \(^{62} \), as well as Christian Beck, in London \(^{4} \), have exhibited very interesting connections with fully developed turbulence. Beck’s theory is a closed one, with no fitting parameters. He
obtained a quite impressive agreement with the experimental distribution of the differences (at distance \( r \)) of radial velocities, including its well known slight skewness (see Fig. 6). The entropic parameter is given by \( 1/(q-1) = 1 + \log_2(r/\eta) \) where \( \eta \) is the viscosity. The fact that \( q \) is not universal certainly reminds us the criticality of the two-dimensional XY ferromagnet, with which nonextensivity presents in fact various analogies.

\[ \text{B. High-energy collisions} \]

Ignacio Bediaga and collaborators [64], in Rio de Janeiro, have worked out, along Rolf Hagedorn’s 1965 lines [3], a phenomenological theory for the distribution of transverse momenta in the hadronic jets emerging from electron-positron annihilation after central collisions at energies ranging from 14 to 161 GeV. Indeed, since early ideas of Fermi, and later of Field and Feynman, a thermodynamical equilibrium scenario has been advanced for this distribution of transverse momenta. Hagedorn developed a full theory based on the BG distribution. The success was only partial. Indeed, a central ingredient of the physical picture is that higher collision energies do not increase the transverse momenta temperature \( T \) but increase instead the number of involved bosons that are produced (like water boiling at higher flux of energy, where only the rate of vapor production is increased, but not the temperature). It is this central physical expectation that was not fulfilled by Hagedorn’s calculation; indeed, to fit the curves associated with increasingly high energies, increasingly high values for \( T \) become necessary. Furthermore, an inflexion point clearly emerges in the distribution, which by no means can be reproduced by the BG exponential distribution. The adequation of only two parameters \((T\text{ and }q)\) have enabled Bediaga et al [64] to fit amazingly large sets of experimental data (see Fig. 7). And, in order to make the expected picture complete, the central demand of \( T \) being independent from the collision energy is fulfilled!

Grzegorz Wilk, in Warsaw, and collaborators [56], as well as D.B. Walton and J. Rafelski in Tucson [67], have provided further evidences of the applicability of the present ideas to high energy physics, where nonmarkovian processes and long-range interactions are commonly accepted hypothesis. Consistent evidence has also been provided by D.B. Ion and M.L.D. Ion, in Bucharest, very especially in hadronic scattering [58].

\[ \text{C. Solar neutrino problem} \]

Piero Quarati and collaborators, in Torino and Cagliari, have been advancing since 1996 [69] an interesting possibility concerning the famous solar neutrino problem. Indeed, it is known that calculations within the so-called Standard Solar Model (SSM) provide a neutrino flux to be detected on the surface of the Earth which is roughly the double of what is actually measured in several laboratories around the world. This enigmatic discrepancy intrigues the specialists since over 20-30 years ago. At least two, non exclusive, possible causes are under current analysis. One of them concerns the nature of neutrinos: they could oscillate in such a way that only part of them would be detectable by the present experimental devices. The second one addresses the possibility that the SSM could need to be sensibly improved in what concerns the production of neutrinos at the solar core plasma. It is along this line that Quarati’s suggestion progresses. Indeed, the neutrino flux is related to the total area of the so-called Gamow peak, which is in turn due to the product of the thermal equilibrium BG distribution function (which decreases with energy) and the penetration factor (which increases with energy). The position of the peak is at energies 10 times larger than \( kT \), therefore only the far tail of the thermal distribution is concerned. Quarati et al argue that very slight departures from \( q = 1 \) (of the order of 0.01) are enough to substantially modify (close to the desired factor two) the area of the Gamow peak. They also show that this degree of departure remains within the experimental accuracy of other independent measurements, such as the helioseismographic ones. This slight nonextensivity would be related to quite plausible nonmarkovian processes and other anomalies.

\[ \text{D. Others} \]

Many other phenomena have been, during recent years, temptatively connected to the present nonextensivity. Let me briefly mention some of them. The experimental, clearly non Gaussian, distribution of velocities in various systems has been shown to be very satisfactorily fitted by \( q \neq 1 \) curves. Such is the case of the distribution (observed from the COBE satellite) of peculiar velocities of spiral galaxies, shown [70], in 1998 by Quarati, myself and collaborators, to be very well fitted by \( q \approx 0.24 \). Such is also the case of the distribution of James Glazier, in Notre Dame University, and collaborators [71], measured for the horizontal velocities of Hydra Vulgaris living in physiological solution. They were remarkably well fitted using \( q \approx 1.5 \). Such is finally the case of the computationally simulated granular matter involving mesoscopic inelastic collisions. Indeed, the distribution function that Y.-H. Taguchi and H. Takayasu used to fit their 1995 data [12] precisely is a typical \( q > 1 \) one.

In 1995 [72], F.C. Sa Barreto, in Belo Horizonte, Brazil, E.D. Loh, in Michigan State University, and myself advanced the possibility of explaining very tiny departures from the black-body radiation Planck’s law for fitting the COBE satellite data for the cosmic background microwave radiation. Indeed, values of \( q \) departing from unity by the order of \( 10^{-5} \) were advanced. This possibility has since then been further analyzed by a variety
of scientists, the conclusion still remaining basically the same. Only more precise (perhaps 10 times more precise would be enough) experimental data would confirm or reject such a possibility, physically motivated by present or ancient subtle effects of gravitation, which could even lead ultimately to a modification of our understanding of the nature of spacetime (at very small scales, it could very well be noncontinuous and even nondifferentiable, in contrast with our usual perceptions!).

Molecules like CO and O2 in some hemoproteins can be dissociated from their natural positions by light flashes, as many classical experiments have shown. Under specific circumstances, these molecules then tend to re-associate with a time-dependent rate. The experimental rate has been shown, in 1999 by G. Bemski, R.S. Mendes and myself [3], to be very well fitted by the solutions of a nonextensive-inspired differential equation. The intrinsic fractality of such proteins is believed to be the physical motivation for such approach. Numbers of citations of scientific publications can also be well fitted with \( q > 1 \) curves, consistently with S. Denisov’s 1997 recovering, within the present formalism, of the so-called Zipf-Mandelbrot law for linguistics [4]. Oscar Sotolongo-Costa, in La Habana, and collaborators [75], as well as K.K. Gudima, in Caen, France, and collaborators [76], have recently invoked the present formalism for describing multifragmentation.

Last but not least, it is worthy mentioning that the present framework has induced [4] quite quicker versions of the Simulated Annealing methods for global optimization. This methodology has found successful applications in theoretical chemistry, in particular for studying proteins, their folding and related phenomena. Such procedures are currently being used by John E. Straub, at the Boston University, and collaborators, by Kleber C. Mundim, in Salvador, Brazil, Donald E. Ellis, in Chicago, and collaborators, by Yuko Okamoto, in Olazaki, Ulrich H.E. Hansmann, in Houghton, and collaborators, among others [77].

VI. FINAL COMMENTS

Many issues remain, at the present moment, partially or fully open to better understanding concerning the present attempt of adequately generalizing Boltzmann-Gibbs statistical mechanics and thermodynamics. These include (i) the direct checking of the energy distribution which generalizes the BG factor (and consistently the connection between \( q \) and \((\alpha, d)\) for long-range interacting Hamiltonian systems (both finite and large), and similar connections for nonconservative ones); (ii) appropriate understanding within the renormalization group framework (Alberto Robledo, in Mexico, has recently initiated this line [4]; see also [3]); (iii) possible firm connections of this formalism with the quantum group formalism for generalizing quantum mechanics (the great analogies that these two nonextensive formalisms exhibit are since long being explored, but not yet deeply understood); (iv) possible connections with the Lévy-like problematic reviewed recently [81] by George M. Zaslavsky; (v) the strict conditions (in terms of the \( N \to \infty \) and \( t \to \infty \) limits) under which the zero\(^{th}\)-principle of thermodynamics (i.e., the criterion for thermal equilibrium) holds; (vi) the microscopic and mathematical interpretation of the direct and escort distributions (which possibly reflect the choice of a multi-fractal description, or of a description in terms of Lebesgue-integrable quantities); (vii) the general connection between symmetry and entropic form adapted to measure information about systems having that particular symmetry; (viii) the rigorous basis for the generalization herein described of concepts such as Kolmogorov-Sinai entropy, Lyapunov exponents, Pesin equality, escape rates, etc; (ix) the possibility of having, for long-range interacting systems, a special value for \( q \) which, through the use of appropriately rescaled variables, would enable a new kind of extensivity in the sense that \( S_q(A + B) = S_q(A) + S_q(B) \), where the large subsystems \( A \) and \( B \) would of course be not independent (in the sense of theory of probabilities) but, on the contrary, strongly coupled; (x) the connection between the index \( q \) appearing in properties such as the sensitivity to the initial conditions, and that appearing in properties such as the microscopic distribution of energies (the most typical values for the former appear to be smaller than unity, whereas they are larger than unity for the latter); (xi) the direct checking of the type of mixing which occurs in the phase-space of many-body long-range interacting large systems; (xii) the deep connection with quantum entanglement through its intrinsic nonlocality. The last point certainly appears as very promising and relevant. An important step forward was very recently done by Abe and Rajagopal [82]; indeed, they succeeded in recovering, through use of the present formalism, the \( x = 1/3 \) Peres (necessary and sufficient) condition, which is known to be the strongest one (stronger than Bell inequality) for having local realism (some degree of separability of the full density matrix into those associated with subsystems \( A \) and \( B \)) for the bipartite spin-\(\frac{1}{2} \) system (Werner-Popescu state). The probability of this approach being fundamentally correct is enhanced nowadays by contributions produced from quite different viewpoints [83].

As we can see, the number of important issues still needing enlightening is quite large. This makes the enterprise but more stimulating! However, caution is recommended in spite of the sensible amount of objective successes of the proposal. Indeed, given the enormous impact of the thermodynamical concepts in Physics, one should address all these points with circumspection. Nevertheless, an important new perspective seems to be already acquired. Nikolai S. Krylov showed, half a century ago, that the key concept in the foundation of standard statistical mechanics is not ergodicity but mixing. Indeed, he showed that the Lyapunov exponents essen-
tially control the relaxation times towards BG equilibrium. The entropy associated with such systems is vastly known to be the BG logarithmic one. What emerges now is an even more fascinating possibility, namely the type of mixing seems to determine the microscopic form of the entropy to be used. If the mixing is of the exponential type (strong chaos), then the entropy of course is the BG one (i.e., $q = 1$). If the mixing is of the power type (weak chaos), then the mixing exponent $(1/(1 - q)$ apparently) would determine the anomalous value of $q$ to be used for the entropy, which would in turn determine the thermal equilibrium distribution, as well as all of its thermodynamical consequences! The kingdom of the exponentials would then be occasionally replaced by the kingdom of the power laws, with their relevance in biology, economics and other complex systems!

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APPENDIX

Scene at the restaurant

AU LABYRINTHE DES ENTROPIES

- Would you have some fresh entropies today, for me and my friends?
- Absolutely Sir! We have them extensive or not, with definite concavity or not, nonnegative defined or otherwise, quantum, classical, relative, cross or mutual, included in several others with a small supplement, composable or not, expansible or not, totally optimized or a little rare, even completely out of equilibrium... single-trajectory-based or ensemble-based...

You can have them at the good old Boltzmann magnificent style, dorée à la Gibbs, very subtle, or von Neumann,..., with pepper à la Jaynes, or the popular Shannon, oh, my God, I was forgetting the esoteric, superb, macroscopic Clausius, the surprising Fisher, the refined Kolmogorov-Sinai, à la Kullback-Leibler for comparison, Renyi with multifractal dressing, with cybernetic sauce à la Harada and Charvat, Vajda and Daroczy, or even the all-taste Sharma and Mittal...

In 1988 we started serving them with Brazilian touch, if you wish to try, it leaves a tropical arrière-gout in your mouth! And since then, our chefs have introduced not less than ten new recipes... Curado, with exponentials, Antecodo with tango flavor, Plastino’s, excellent as family dish, Landsberg, Papa, Johal with curry, Borges and

Roditi, Rajagopal and Abe...

They are all delicious! How many do I serve you today?

FIGURES

Fig. 1 - Equilibrium-like probability versus energy for typical values of $q$, namely, from top to bottom at low energies, $q = 0$, $1/4$, $1/2$, $2/3$, $1$, $3$, $\infty$ (the latter collapses onto the ordinate at the origin, and vanishes for all positive energies). For $q = 1$ we have Boltzmann exponential factor; for $q > 1$ we have a power-law tail; for $q < 1$ there is a cutoff above which the probability vanishes.

Fig. 2 - Time evolution of $S_q$: (a) $a = 2$ ($q^* = 1$); (b) $a = a_c$ ($q^* = 0.2445$; $R$ measures the degree of nonlinearity of the curves in the relevant intermediate region).

Fig. 3 - Coupled planar rotators: $\kappa$ versus $\alpha/d$ ($d = 1$: $\Box$; $d = 2, 3$: $\circ$). The solid line is a guide to the eye.

Fig. 4 - Coupled planar rotators ($\alpha = 0$, $d = 1$): Time evolution of $T$. From Latora and Rapisarda [55].

Fig. 5 - Conjectural time dependence of the probability $p$ of having an energy $E$ for classical systems ($N \equiv N^* + 1$).

Fig. 6 - Distribution of radial velocity differences in fully developed turbulence (from top to bottom: $r/\eta = 3.3, 23.6, 100$). From Beck [63].

Fig. 7 - Distribution of hadronic transverse momenta: (a) cross-section (dotted line: Hagedorn’s theory; solid lines: $q \neq 1$ theory); (b) Fitting parameters $q$ and $T_0$. From Bediaga et al [54].
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$a = 1.40115519$

$q = 0.05$

$q = 0.2445$

$q = 0.5$

$R$

$q$

TIME

ENTROPY
$u = 0.69$

$T_{\text{can}} = 0.4757$

$N = 1000$

$N = 5000$

$N = 10000$

$N = 25000$

Diagram showing the temperature evolution over time for different system sizes. The inset shows the temperature versus the canonical ensemble variable $u$. The dashed line represents the QSS at $N=20000$. The canonical ensemble curve is shown with a solid line.
EXTENSIVE SYSTEMS \( (\alpha > d) \)

\[
\lim_{N \to \infty} \lim_{t \to \infty} = \lim_{t \to \infty} \lim_{N \to \infty} p(t,N,E) = (q = 1)
\]

\[
\propto g(E) \exp(-\beta E)
\]

\( (N^* \text{ not needed}) \)

NONEXTENSIVE SYSTEMS \( (0 \leq \alpha < d) \)

\[
\lim_{t \to \infty} \lim_{N \to \infty} p(t,N,E) = (q \neq 1)
\]

\[
\propto g(E) [1 - (1-q) \beta^* E]^{q/(1-q)}
\]

\( (N^* \text{ needed}) \)

with

\[
\lim_{N \to \infty} \tau(N) = \infty \quad \text{and} \quad N^* \equiv \frac{N^{1-\alpha/d} - 1}{1 - \alpha/d}
\]
Fig. 2
