A Stochastic Gronwall Lemma

Michael Scheutzow*

May 11, 2014

Abstract

We prove a stochastic Gronwall lemma of the following type: if $Z$ is an adapted nonnegative continuous process which satisfies a linear integral inequality with an added continuous local martingale $M$ and a process $H$ on the right hand side, then for any $p \in (0, 1)$ the $p$-th moment of the supremum of $Z$ is bounded by a constant $\kappa_p$ (which does not depend on $M$) times the $p$-th moment of the supremum of $H$. Our main tool is a martingale inequality which is due to D. Burkholder. We provide an alternative simple proof of the martingale inequality which provides an explicit numerical value for the constant $c_p$ appearing in the inequality which is at most four times as large as the optimal constant.

2010 Mathematics Subject Classification Primary 60G44  Secondary 60H10, 60E15, 26D15

Keywords. stochastic Gronwall lemma, martingale inequality.

In this note we first state a martingale inequality which is due to D. Burkholder [1] and which estimates the $p$-th moment of the supremum of a continuous local martingale by a constant $c_p$ times the $p$-th moment of its negative infimum for $0 < p < 1$. Then we apply the martingale inequality and prove a stochastic Gronwall lemma for a nonnegative process $Z$. The stochastic Gronwall lemma is useful when proving existence and uniqueness of solutions to stochastic differential equations satisfying only a one-sided Lipschitz condition (where the usual proof using the Burkholder-Davis-Gundy inequality does not apply). The point of the stochastic Gronwall lemma is that it provides an upper bound for the $p$-th moment of $Z$ which does not depend on the local martingale $M$ on the right-hand side of the inequality. The price one has to pay for this uniformity with respect to $M$ is that one has to assume $p < 1$.

We start by formulating the martingale inequality.

Proposition 1. For each $p \in (0, 1)$ and each continuous local martingale $M(t), t \geq 0$ starting at $M(0) = 0$, we have

$$
\mathbb{E}\left(\sup_{t \geq 0} M^p(t)\right) \leq c_p \mathbb{E}\left(\left(-\inf_{t \geq 0} M(t)\right)^p\right),
$$

(1)

where $c_p := \left(4 \land \frac{1}{p}\right) \frac{\pi p}{\sin(\pi p)}$.

*Institut für Mathematik, MA 7-5, Fakultät II, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, FRG; ms@math.tu-berlin.de
Remark 2. The inequality was first proved by D. Burkholder ([1], Theorem 1.4) even a bit more generally (for a larger class of functions than the $p$-th power) but without an explicit estimate of the numerical value of $c_p$. We provide a short and elementary alternative proof below.

Remark 3. It is clear that the previous proposition does not extend to $p \geq 1$: consider the continuous martingale $M(t) := W(\tau_{i-1} \wedge t)$ where $W$ is standard Brownian motion and $\tau_x := \inf\{s \geq 0 : W(s) = x\}$. Then the left hand side of (1) is infinite for each $p \geq 1$ while the right hand side is finite. This example also shows that even though the constant $c_p$ is certainly not optimal, it is at most off from the optimal constant by the factor $4 \wedge (1/p)$ (which converges to one as $p$ approaches one). It is also clear that the proposition does not extend to right-continuous martingales: consider a martingale which is constant except for a single jump at time 1 of height 1 with probability $\delta$ and height $\frac{\pi}{1 - \delta}$ with probability $1 - \delta$ where $\delta \in (0, 1)$. It is straightforward to check that for an inequality of type $(E \sup_{t \geq 0} M^p(t))^{1/p} \leq c_{p,q} (E(-\inf_{t \geq 0} M(t))^q)^{1/q}$ to hold for this class of examples for some finite $c_{p,q}$, we require that $q \geq 1$ irrespective of the value of $p \in (0, 1)$.

Proof of Proposition [1] Since $M$ is a continuous local martingale starting at 0 it can be represented as a time-changed Brownian motion $W$ (on a suitable probability space). We can and will assume that $M$ converges almost surely (otherwise there is nothing to prove), so there exists an almost surely finite stopping time $T$ for $W$ such that $A := \sup_{0 \leq t \leq T} W(t) = \sup_{0 \leq t} M(t)$ and $B := -\inf_{0 \leq t \leq T} W(t) = -\inf_{0 \leq t} M(t)$. Let $0 = a_0 < a_1 < \ldots$ be a sequence which converges to $\infty$ and define

$$
\tau_i := \inf\{t \geq 0 : W(t) = -a_i\}, \quad Y_i := \sup_{\tau_{i-1} \leq t \leq \tau_i} W(t), \quad i \in \mathbb{N}, \quad N := \inf\{i \in \mathbb{N} : \tau_i \geq T\}.
$$

The $Y_i$ are independent by the strong Markov property of $W$ and for $p \in (0, 1)$ and $i \in \mathbb{N}$ we have

$$
\Gamma_i := E(Y_i \vee 0)^p = \frac{a_i - a_{i-1}}{a_i^{1-p}} \int_0^\infty \frac{1}{1 + y^{1/p}} \, dy = \frac{a_i - a_{i-1}}{a_i^{1-p}} \frac{\pi p}{\sin(\pi p)}.
$$

Therefore,

$$
EA^p \leq \sum_{n=1}^\infty E(\sup\{Y_1, \ldots, Y_n\}^p 1_{N=n}) \leq \sum_{n=1}^\infty \sum_{i=1}^n E((Y_i \vee 0)^p 1_{N=n}) \leq \sum_{i=1}^\infty E((Y_i \vee 0)^p 1_{N \geq i}) = \sum_{i=1}^\infty \Gamma_i P\{N \geq i\},
$$

where the last equality again follows from the strong Markov property. Inserting the formula for $\Gamma_i$, choosing the particular values $a_i = c_i\gamma^i$ for some $c > 0$ and $\gamma > 1$, and observing that
\[ P\{N \geq i\} \leq P\{B \geq a_{i-1}\}, \] we get
\[
E A^p \leq \frac{\pi p}{\sin(\pi p)} e^p \left( \gamma^p + \left( 1 - \frac{1}{\gamma} \right) \sum_{i=2}^{\infty} \gamma^i p P\{B \geq c\gamma^{i-1}\} \right)
\]
\[
= \frac{\pi p}{\sin(\pi p)} e^p \left( \gamma^p + \left( 1 - \frac{1}{\gamma} \right) \sum_{j=2}^{\infty} P\{B \in [c\gamma^{j-1}, c\gamma^j]\} \left( \frac{\gamma^j - 1}{\gamma^p - 1} - 1 - \gamma^p \right) \right)
\]
\[
\leq \frac{\pi p}{\sin(\pi p)} \left( e^p \gamma^p + \left( 1 - \frac{1}{\gamma} \right) \gamma^p \sum_{i=2}^{\infty} E B^p - e^p \left( 1 - \frac{1}{\gamma} \right) \left( \frac{1}{\gamma^p - 1} + 1 + \gamma^p \right) P\{B \geq c\} \right).
\]

Dropping the last (negative) term, letting \( c \to 0 \) and observing that the function of \( \gamma \) in front of \( EB^p \) converges to \( 1/p \) as \( \gamma \to 1 \) and that \( \inf_{\gamma > 1} \gamma^{2p}/(\gamma^p - 1) = 4 \) we obtain the assertion. \( \square \)

Next, we apply the martingale inequality to prove a stochastic Gronwall lemma. A similar stochastic Gronwall lemma was proved and used in \([2]\) in order to prove existence and uniqueness of a solution to a stochastic functional differential equation satisfying a one-sided Lipschitz condition only. That result was slightly more general in the sense that on the right hand side of equation \((2)\) \( Z \) was replaced by its running supremum, but it was less general concerning the function \( \psi \) and it required higher moments of \( H^* \). The proof did not explicitly use a martingale inequality.

For a real-valued process denote \( Y^*(t) := \sup_{0 \leq s \leq t} Y(s) \).

**Theorem 4.** Let \( c_p \) be as in Theorem \([7]\). Let \( Z \) and \( H \) be nonnegative, adapted processes with continuous paths and assume that \( \psi \) is nonnegative and progressively measurable. Let \( M \) be a continuous local martingale starting at 0. If
\[
Z(t) \leq \int_0^t \psi(s) Z(s) \, ds + M(t) + H(t) \quad (2)
\]
holds for all \( t \geq 0 \), then for \( p \in (0, 1) \), and \( \mu, \nu > 1 \) such that \( \frac{1}{\mu} + \frac{1}{\nu} = 1 \) and \( p \nu < 1 \), we have
\[
E \sup_{0 \leq s \leq t} Z^p(s) \leq (c_p + 1)^{1/\nu} \left( E \exp\{p\mu \int_0^t \psi(s) \, ds\} \right)^{1/\mu} \left( E (H^*(t))^{p \nu} \right)^{1/\nu}. \quad (3)
\]
If \( \psi \) is deterministic, then
\[
E \sup_{0 \leq s \leq t} Z^p(s) \leq (c_p + 1) \exp\{p \int_0^t \psi(s) \, ds\} \left( E (H^*(t))^p \right), \quad (4)
\]
and
\[
EZ(t) \leq \exp\{\int_0^t \psi(s) \, ds\} E H^*(t). \quad (5)
\]

**Proof.** Let \( L(t) := \int_0^t \exp\{- \int_0^s \psi(u) \, du\} \, dM(s) \). Applying the usual Gronwall Lemma (for each fixed \( \omega \in \Omega \)) to \( Z \) and integrating by parts, we obtain
\[
Z(t) \leq \exp\{\int_0^t \psi(s) \, ds\} (L(t) + H^*(t)). \quad (6)
\]
Since $Z$ is nonnegative, we have $-L(t) \leq H^*(t)$ for all $t \geq 0$. Therefore, using Proposition 1 and Hölder’s inequality, we get

$$E(Z^*)^p(t) \leq \left( E\exp\{p\mu \int_0^t \psi(s) \, ds\}\right)^{1/\mu} \left( E(L^*(t))^{p\nu} + E(H^*(t))^{p\nu}\right)^{1/\nu},$$

which is (3). Inequality (4) follows similarly. The final statement follows by applying (6) to $\tau_n \wedge t$ for a sequence of localizing stopping times $\tau_n$ for $L$ and applying Fatou’s Lemma.

**Acknowledgement.** It is a pleasure to thank R. Bañuelos for drawing my attention to the cited paper of Donald Burkholder.

**References**

[1] Burkholder, D. L. (1975). One-sided maximal functions and $H^p$, *J. Func. Anal.* **18**, 429–454.

[2] v. Renesse, M., Scheutzow, M. (2010). Existence and uniqueness of solutions of stochastic functional differential equations, *Random Oper. Stoch. Equ.* **18**, 267–284.