Inverse spectral problems for Hill-type operators with frozen argument

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Abstract
The paper deals with nonlocal differential operators possessing a term with frozen (fixed) argument appearing, in particular, in modelling various physical systems with feedback. The presence of a feedback means that the external affect on the system depends on its current state. If this state is taken into account only at some fixed physical point, then mathematically this corresponds to an operator with frozen argument. In the present paper, we consider the operator

\[ L_y \equiv -y''(x) + q(x)y(a), \]

\[ y(0) = y(1), \]

where \( \gamma \in \mathbb{C} \setminus \{0\} \). The operator \( L \) is a nonlocal analog of the classical Hill operator describing various processes in cyclic or periodic media. We study two inverse problems of recovering the complex-valued square-integrable potential \( q(x) \) from some spectral information about \( L \). The first problem involves only single spectrum as the input data. We obtain complete characterization of the spectrum and prove that its specification determines \( q(x) \) uniquely if and only if \( \gamma \neq \pm 1 \). For the rest (periodic and antiperiodic) cases, we describe classes of iso-spectral potentials and provide restrictions under which the uniqueness holds. The second inverse problem deals with recovering \( q(x) \) from the two spectra related to \( \gamma = \pm 1 \). We obtain necessary and sufficient conditions for its solvability and establish that uniqueness holds if and only if \( a = 0, 1 \). For \( a \in (0, 1) \), we describe classes of iso-bispectral potentials and give restrictions under which the uniqueness resumes. Algorithms for solving both inverse problems are provided. In the appendix, we prove Riesz-basisness of an auxiliary two-sided sequence of sines.

Keywords Sturm–Liouville-type operator · Functional-differential operator · Frozen argument · Inverse spectral problem · Processes with feedback · Riesz-basis of sines

Mathematics Subject Classification 34A55 · 34K29

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1 Introduction

In recent years, there appeared a considerable interest in nonlocal differential operators possessing a term with frozen argument (see [1–13] and references therein). Operators of this kind form an important and illustrative class of the so-called loaded differential operators (see, e.g., [14–20]), which often appear in mathematical physics. The corresponding loaded equations can be characterized by the presence of a trace of the unknown function. Below we describe some typical models of physical systems with feedback leading to nonlocal differential operators with frozen argument. The presence of a feedback means that the external affect on the system depends on its current state. If this state is taken into account only at some fixed physical point of the system, then mathematically this corresponds to an operator with frozen argument.

We focus on the boundary value problem \( \mathcal{L}(q(x), a, \gamma) \) of the form

\[
\ell y := -y''(x) + q(x)y(a) = \lambda y(x), \quad 0 < x < 1, \tag{1}
\]

\[
y^{(v)}(0) = \gamma y^{(v)}(1), \quad v = 0, 1, \tag{2}
\]

where \( q(x) \in L_2(0, 1) \) is a complex-valued function, \( \lambda \) is the spectral parameter and \( \gamma \in \mathbb{C}\{0\} \), while \( a \in [0, 1] \). The operator generated by the functional-differential expression \( \ell \) and equipped with boundary conditions (2) is called Hill-type operator with frozen argument. This operator is a nonlocal loaded analog of the classical Hill operator appearing after separating variables in partial differential equations describing various processes in cyclic or periodic media.

In the present paper, we study inverse spectral problems of recovering the potential \( q(x) \). The most complete results in the inverse spectral theory are known for purely differential operators (local), see monographs [21–23]. The inverse problems arise in mathematics, mechanics, physics, geophysics, electronics and other branches of natural sciences and engineering, including nanoscale technology. In particular, for the classical Hill operator inverse problems were studied in [21,22,24–28] and other works. For operators with frozen argument as well as for other types of nonlocal operators, classical methods of the inverse spectral theory do not work.

Equation (1) appears, for example, after applying the Fourier method of separation of variables to the following loaded parabolic partial differential equation:

\[
\frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) + f(x, t), \quad f(x, t) = -q(x)u(a, t), \quad 0 < x < 1, \quad t > 0, \tag{3}
\]

involving the trace \( u(a, t) \) of the unknown function \( u(x, t) \). Consider also the initial condition

\[
u(x, 0) = \varphi(x), \quad 0 < x < 1, \tag{4}
\]

and the homogeneous boundary conditions

\[
\frac{\partial^\alpha}{\partial x^\alpha} u(x, t)\bigg|_{x=0} = \frac{\partial^\beta}{\partial x^\beta} u(x, t)\bigg|_{x=1} = 0, \quad t > 0, \tag{5}
\]

where \( \alpha, \beta \) are nonnegative integers.
where \( \alpha, \beta \in \{0, 1\} \) are fixed. It is well known that the initial-boundary value problem (3)–(5) models heat conduction in a rod of unit length possessing an external distributed heat source \( f(x, t) \). In our case, this heat source is described by the function \(-q(x)u(a, t)\), i.e. its power is proportional to the temperature \( u(a, t) \) at the fixed point \( a \) of the rod. Such a model can be implemented by an electric conductive rod of a constant thermal conductivity but possessing the variable electrical resistance \( R(x) = -q(x) \geq 0 \) independent of the temperature of the rod. In order to create the heat source as in (3), the voltage \( U(t) \) being applied to the ends of the rod should be proportional to \( \sqrt{u(a, t)} \):

\[
U(t) = R \sqrt{u(a, t)}, \quad R = -\int_0^1 q(\xi) \, d\xi,
\]

where \( R \) is the full resistance of the rod. Indeed, the current \( I(t) \) in the rod does not depend on \( x \) and, by Ohm’s law, we have \( I(t) = U(t)/R = \sqrt{u(a, t)} \), while the density of the created heat source in each point \( x \in [0, 1] \) of the rod equals to \( I^2(t)R(x) = -q(x)u(a, t) = f(x, t) \).

Similarly, the loaded hyperbolic equation

\[
\frac{\partial^2}{\partial t^2} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) - q(x)u(a, t), \quad 0 < x < 1, \quad t > 0,
\]

under the initial conditions

\[
u(x, 0) = \varphi(x), \quad \frac{\partial}{\partial t} u(x, t) \Big|_{t=0} = \psi(x), \quad 0 < x < 1,
\]

as well as the boundary conditions (5), with \( q(x) \), \( \varphi(x) \) and \( \psi(x) \) belonging to some appropriate classes, models an oscillatory process under a damping external force that is proportional in each point \( x \) both to the function \( q(x) \) and to the displacement \( u(a, t) \) of the oscillating system in the fixed point \( a \) equipped with a displacement sensor. An example occurs if a vibrating wire is affected by a magnetic field exerting a force per unit mass represented by \(-q(x)u(a, t)\), i.e. depending on the lateral displacement \( u(a, t) \) at the point \( a \) at the time \( t \).

After separating variables, both above-mentioned initial-boundary value problems yield one and the same eigenvalue problem \( B := B(q(x), a, \alpha, \beta) \) consisting of the ordinary functional-differential equation with frozen argument (1) and the separated boundary conditions

\[
y^{(\alpha)}(0) = y^{(\beta)}(1) = 0.
\]

We note that some analogous as well as different models leading to ordinary functional-differential equations with one or several frozen arguments were given, e.g., in [15,18].

Various aspects of inverse problems for operators with frozen argument were studied in [1–7,9–11,13]. In particular, in [10] the problem \( B \) was considered for \( a \in [0, 1] \cap \mathbb{Q} \), i.e. \( a = j/k \) with some mutually prime integers \( j \) and \( k \). It was established that unique recoverability of the potential \( q(x) \) from the spectrum of \( B \) depends on the values \( \alpha, \beta \) as well as on the parity of \( k \). This implies two cases: non-degenerate
and degenerate ones (see also [6, 7]), depending on whether the inverse problem is uniquely solvable or not, respectively. In the degenerate case, asymptotically $k$th part of the spectrum degenerates in the sense that each $k$th eigenvalue carries no information on the potential. In [13], it was shown that each irrational $a \in (0, 1)$ implies the non-degenerate case for any pair of $\alpha, \beta \in \{0, 1\}$.

Let $\{\lambda_n\}_{n \geq 0}$ be the spectrum of the boundary value problem $\mathcal{L}(q(x), a, \gamma)$. We start with the following inverse problem.

**Inverse Problem 1** Given $\{\lambda_n\}_{n \geq 0}$, $a$ and $\gamma$; find $q(x)$.

For example, the case $\gamma = 1$ corresponds to the initial-boundary value problem for the heat equation (3) under the initial condition (4) as well as the periodic boundary conditions

$$u(0, t) = u(1, t), \quad \frac{\partial}{\partial x} u(x, t) \bigg|_{x=0} = \frac{\partial}{\partial x} u(x, t) \bigg|_{x=1}, \quad t > 0,$$

which models heat conduction in a thin closed ring of unit length parameterized by the variable $x \in [0, 1]$, whose values 0 and 1 correspond to one and the same physical point of the ring. Analogously, by using Eq. (6) along with the initial conditions (7), one can model the corresponding oscillatory process. Each of these two models hints that it is sufficient to study the case $a = 0$. Moreover, this sufficiency remains also for any $\gamma \neq 0$ (see Lemma 4 in Sect. 3). This property allows making no distinguishing between rational and irrational cases, which gives a new quality comparing with the case of separated boundary conditions (8).

For any $a \in [0, 1]$ and $\gamma \neq 0$, we obtain complete characterization of the spectrum and prove that its specification determines $q(x)$ uniquely if and only if $\gamma \neq \pm 1$. For $\gamma = \pm 1$, we establish that precisely half of the spectrum degenerates, and describe classes of iso-spectral potentials. Moreover, we provide restrictions on the potential under which the uniqueness resumes. The proof is constructive and gives algorithms for solving Inverse Problem 1.

Further, we study recovering the potential $q(x)$ from the periodic and the antiperiodic spectra. For $\alpha = 0, 1$, we denote by $\{\lambda_{n, \alpha}\}_{n \geq 0}$ the spectrum of the boundary value problem $\mathcal{L}_\alpha(q(x), a) := \mathcal{L}(q(x), a, (-1)^\alpha)$ and consider the following inverse problem.

**Inverse Problem 2** Given $\{\lambda_{n, \alpha}\}_{n \geq 0}$, $\alpha = 0, 1$, and $a$; find $q(x)$.

We prove a uniqueness theorem and obtain a constructive procedure for solving this inverse problem along with necessary and sufficient conditions for its solvability. In particular, it is established that the solution is unique if and only if $a \in \{0, 1\}$. For $a \in (0, 1)$, we describe classes of iso-bispectral potentials and provide restrictions under which the uniqueness holds. In the latter case, characterization of the spectra, besides asymptotics, includes a restriction on the type of the sum of the characteristic functions being an entire function of order $1/2$.

The paper is organized as follows. In the next section, we study the characteristic function of the problem $\mathcal{L}(q(x), a, \gamma)$ and derive the so-called main equation of the inverse problem. Therein, we also obtain asymptotics of the spectrum. In Sect. 3,
we study Inverse Problem 1, while Inverse Problem 2 is investigated in Sect. 4. In Appendix A, we prove Riesz-basisness of one auxiliary functional system.

## 2 Characteristic function and main equation

Consider the solutions $C(x, \lambda)$ and $S(x, \lambda)$ of equation (1) under the initial conditions

$$C(a, \lambda) = S'(a, \lambda) = 1, \quad S(a, \lambda) = C'(a, \lambda) = 0.$$ 

Having put $\rho^2 = \lambda$, we arrive at the representations

$$C(x, \lambda) = \cos \rho(x - a) + \int_a^x \frac{\sin \rho(x - t)}{\rho} q(t) \, dt, \quad S(x, \lambda) = \frac{\sin \rho(x - a)}{\rho}. \tag{9}$$

Clearly, eigenvalues of the problem (1), (2) coincide with zeros of the entire function

$$\Delta(\lambda) = \left| \begin{array}{cc} C(0, \lambda) - \gamma C(1, \lambda) & S(0, \lambda) - \gamma S(1, \lambda) \\ C'(0, \lambda) - \gamma C'(1, \lambda) & S'(0, \lambda) - \gamma S'(1, \lambda) \end{array} \right|, \tag{10}$$

which is called characteristics function. Consider the Wronski-type determinant

$$W(x, \lambda) := \langle C(x, \lambda), S(x, \lambda) \rangle,$$

where $\langle y(x), z(x) \rangle = y(x)z'(x) - y'(x)z(x)$. Using (9), it is easy to calculate

$$W(x, \lambda) = 1 - \int_0^{a-x} q(a-t) \frac{\sin \rho t}{\rho} \, dt. \tag{11}$$

Unlike the Wronskian for the classical Sturm–Liouville equation, the function $W(x, \lambda)$ depends on $x$ and, moreover, may vanish for some values of $x \in [0, 1]$. However, we need the designation $W(x, \lambda)$ only for brevity. The next lemma gives a fundamental representation for $\Delta(\lambda)$.

**Remark 1** Since the spectra of the problems $\mathcal{L}(q(x), a, \gamma)$ and $\mathcal{L}(q(1-x), 1-a, \gamma^{-1})$, obviously, coincide, without loss of generality, one can assume that $a \in [0, 1/2]$.

**Lemma 1** For any $\gamma$, the characteristic function $\Delta(\lambda)$ of the problem (1), (2) has the form

$$\Delta(\lambda) = 1 + \gamma^2 - 2\gamma \cos \rho - \int_0^1 w(x) \frac{\sin \rho x}{\rho} \, dx, \quad w(x) \in L_2(0, 1). \tag{12}$$

Moreover, the following representation holds:

$$w(x) = \begin{cases} \gamma^2 q(a + x) + q(a - x), & x \in (0, a), \\ \gamma q(a + 1 - x) + \gamma^2 q(a + x), & x \in (a, 1 - a), \\ \gamma(q(a + 1 - x) + q(a - 1 + x)), & x \in (1 - a, 1), \end{cases} \tag{13}$$

where without loss of generality, we assumed $2a \leq 1$. 
According to (10) and the definition of \( W(x, \lambda) \), we calculate
\[
\Delta(\lambda) = W(0, \lambda) + \gamma \Delta_{1,0}(\lambda) - \gamma \Delta_{0,1}(\lambda) + \gamma^2 W(1, \lambda),
\]
where
\[
\Delta_{\alpha,\beta}(\lambda) = \left| \frac{C^{(\alpha)}(0, \lambda)}{C^{(\beta)}(1, \lambda)}-\frac{S^{(\alpha)}(0, \lambda)}{S^{(\beta)}(1, \lambda)} \right|.
\]
By virtue of Lemma 1 in [10], for \( \alpha \neq \beta \) we have
\[
\Delta_{\alpha,\beta}(\lambda) = (-1)^{\alpha} \cos \rho + \int_0^1 w_{\alpha,\beta}(x) \frac{\sin \rho x}{\rho} dx, \quad w_{\alpha,\beta}(x) \in L_2(0, 1),
\]
where (for \( 2a \leq 1 \))
\[
w_{\alpha,\beta}(x) = \frac{1}{2} \begin{cases} 
q(1-a+x) - q(1-a-x), & x \in (0, a), \\
(-1)^{1+\beta}q(1+a-x) - q(1-a-x), & x \in (a, 1-a), \\
(-1)^{1+\beta}q(1+a-x) + q(x-1+a), & x \in (1-a, 1).
\end{cases}
\]
Using (11) and (14)–(16), we arrive at (12) and (13).

After assuming \( w(x) \) to be known, relation (13) can be considered as a linear functional equation with respect to \( q(x) \), which we refer to as the main equation of the inverse problem.

For \( \gamma \neq 0 \), denote
\[
\alpha := \frac{1}{\pi} \arccos \frac{1 + \gamma^2}{2\gamma} \in \{ z : \text{Re} \ z \in [0, 1], \ \text{Im} \ z \geq 0 \} \cup \{ z : \text{Re} \ z \in (0, 1), \ \text{Im} \ z < 0 \}.
\]
The following theorem describes behavior of the spectrum.

**Theorem 1** The spectrum \( \{ \lambda_n \}_{n \geq 0} \) of the problem \( L(q(x), a, \gamma) \) has the form
\[
\lambda_{2k} = (2k + \alpha)^2 \pi^2 + \varpi_{2k}, \quad k \geq 0,
\]
\[
\lambda_{2k-1} = (2k - \alpha)^2 \pi^2 + \varpi_{2k-1}, \quad k \geq 1,
\]
\[
\{ \varpi_n \}_{n \geq 0} \in l_2.
\]
Moreover, if \( \gamma = \pm 1 \) (i.e. if \( \alpha = 0, 1 \)), then
\[
\varpi_{2k-1} = 0, \quad k \geq 1,
\]
i.e. in both the periodic and the antiperiodic cases, the spectrum degenerates in the sense of (19).
**Proof** Rewrite (12) in the form

\[ \Delta(\lambda) = 4\gamma \sin \frac{\rho + \pi\alpha}{2} \sin \frac{\rho - \pi\alpha}{2} - \int_0^1 w(x) \frac{\sin \rho x}{\rho} dx, \quad w(x) \in L_2(0, 1). \quad (20) \]

Thus, by the known method involving Rouché’s theorem (see, e.g., [23]) one can prove that any function of the form (12) has infinitely many zeros \( \lambda_n, \, n \geq 0 \), which with account of multiplicities have the form

\[ \lambda_n = \rho_n^2, \quad \rho_{2k} = (2k + \alpha)\pi + \varepsilon_{2k}, \quad k \geq 0, \]
\[ \rho_{2k-1} = (2k - \alpha)\pi + \varepsilon_{2k-1}, \quad k \geq 1, \quad \varepsilon_n = o(1). \quad (21) \]

Further we consider two cases.

(i) Let \( \gamma \neq \pm 1 \), i.e. \( \sin \pi\alpha \neq 0 \). Substituting (21) into (20), we arrive at the relation

\[ 4\gamma \left( (-1)^n \sin \pi\alpha \cos \frac{\varepsilon_n}{2} + \frac{1 + \gamma^2}{2\gamma} \sin \frac{\varepsilon_n}{2} \right) \sin \frac{\varepsilon_n}{2} = \int_0^1 w(x) \frac{\sin \rho_n x}{\rho_n} dx. \]

Since \( \sin \rho = \rho + O(\rho^3) \) and \( \cos \rho = 1 + O(\rho^2) \) as \( \rho \to 0 \), we refine \( \{n\varepsilon_n\}_{n \geq 0} \in l_2 \), which along with (21) gives (18) for \( \gamma \neq \pm 1 \).

(ii) Let \( \gamma \in \{-1, 1\} \). Then, by virtue of (13), we have

\[ w(x) = (-1)^{1-\gamma} w(1-x). \quad (22) \]

For \( \gamma = 1 \), this implies

\[ \int_0^1 w(x) \sin \rho x dx = \int_0^{1/2} w(x) \left( \sin \rho x + \sin \rho (1 - x) \right) dx \]
\[ = 2 \sin \frac{\rho}{2} \int_0^{1/2} w \left( \frac{1}{2} - x \right) \cos \rho x dx. \]

Since \( \alpha = 0 \), we rewrite (20) in the form

\[ \Delta(\lambda) = \frac{\rho}{2} \sin \frac{\rho}{2} \left( 2\rho \sin \frac{\rho}{2} - \int_0^{1/2} w \left( \frac{1}{2} - x \right) \cos \rho x dx \right), \quad (23) \]

which yields (18) and (19) for \( \gamma = 1 \). Similarly, for \( \gamma = -1 \), formula (22) gives

\[ \int_0^1 w(x) \sin \rho x dx = \int_0^{1/2} w(x) \left( \sin \rho x - \sin \rho (1 - x) \right) dx \]
\[ = -2 \cos \frac{\rho}{2} \int_0^{1/2} w \left( \frac{1}{2} - x \right) \sin \rho x dx, \]
which along with (20) and α = 1 implies
\[
\Delta(\lambda) = 2 \cos \frac{\rho}{2} \left( 2 \cos \frac{\rho}{2} + \int_0^{\frac{1}{2}} w \left( \frac{1}{2} - x \right) \sin \frac{\rho x}{\rho} \, dx \right).
\]
The latter formula gives (18) and (19) for γ = −1.

The characteristic function is uniquely determined by the spectrum. In more details, the following lemma holds.

**Lemma 2** Any function \(\Delta(\lambda)\) of the form (12) is uniquely determined by its zeros \(\{\lambda_n\}_{n \geq 0}\). Moreover, the following representation holds:

\[
\Delta(\lambda) = \begin{cases} 
(1 - \gamma)^2 \prod_{n=0}^{\infty} \frac{\lambda_n - \lambda}{\lambda_{n,0}^0}, & \gamma \neq 1, \\
(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{\lambda_{n,0}^0}, & \gamma = 1,
\end{cases}
\]

where

\[
\rho_{n,0}^{0} = (\rho_{n,0}^{0})^2, \quad n \geq 0, \quad \rho_{2k,\alpha}^{0} = (2k + \alpha)\pi, \quad k \geq 0, \quad \rho_{2k-1,\alpha}^{0} = (2k - \alpha)\pi, \quad k \geq 1.
\]

**Proof** By virtue of Hadamard’s factorization theorem (see, e.g., [29]), formula (12) implies

\[
\Delta(\lambda) = C \lambda^s \prod_{\lambda_n \neq 0} \left( 1 - \frac{\lambda}{\lambda_n} \right),
\]

where \(C\) is some constant, while \(s\) is the algebraic multiplicity of the null zero \(\lambda_n = 0\). In particular, we have

\[
\Delta_0(\lambda) := 1 + \gamma^2 - 2\gamma \cos \rho = \begin{cases} 
(1 - \gamma)^2 \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{\lambda_{n,0}^0} \right), & \gamma \neq 1, \\
\lambda \prod_{n=1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_{n,0}^0} \right), & \gamma = 1.
\end{cases}
\]

Let \(\gamma \neq 1\). Then, dividing (27) by (26), we obtain

\[
\frac{\Delta_0(\lambda)}{\Delta(\lambda)} = \frac{(1 - \gamma)^2}{C} \prod_{\lambda_n = 0} \left( \frac{1}{\lambda} - \frac{1}{\lambda_{n,0}^0} \right) \prod_{\lambda_n \neq 0} \frac{\lambda_n}{\lambda_{n,0}^0} \prod_{\lambda_n \neq 0} \frac{\lambda_{n,0}^0 - \lambda}{\lambda_n - \lambda},
\]

while (12) and (27) imply \(\Delta_0(\lambda)/\Delta(\lambda) \to 1\) as \(\lambda \to -\infty\), which along with (28) gives

\[
C = (1 - \gamma)^2 (-1)^s \prod_{\lambda_n = 0} \frac{1}{\lambda_{n,0}^0} \prod_{\lambda_n \neq 0} \frac{\lambda_n}{\lambda_{n,0}^0}.
\]
Substituting this into (26), we arrive at (24) for $\gamma \neq 1$. The case $\gamma = 1$ is treated similarly. 

For proving solvability of the inverse problems we will need also the following lemma.

**Lemma 3** Let $\gamma \neq \pm 1$. Then for any complex sequence $\{\lambda_n\}_{n \geq 0}$ of the form (18), the function $\Delta(\lambda)$ constructed by formula (24) has the form (12) with some function $w(x) \in L_2(0, 1)$.

Let $\gamma \in \{-1, 1\}$ (i.e. $\alpha \in \{0, 1\}$). Then for any complex sequence $\{\lambda_n\}_{n \geq 0}$ of the form (18) and satisfying the degeneration condition (19), the function $\Delta(\lambda)$ constructed by formula (24) has the form (12) with some function $w(x) \in L_2(0, 1)$ obeying (22).

**Proof** Let us first consider the case $\gamma = 1$, in which one should actually prove representation (23), where we have

$$\frac{2}{\rho} \sin \frac{\rho}{2} = \frac{\lambda}{(2\pi k)^2}. \quad (17)$$

Thus, according to (18) with $\alpha = 0$ and (19), it is sufficient to prove the relation

$$(\lambda - \lambda_0) \prod_{k=1}^{\infty} \frac{\lambda_{2k} - \lambda}{(2\pi k)^2} = 2\rho \sin \frac{\rho}{2} - \int_0^{\frac{1}{2}} w\left(\frac{1}{2} - x\right) \cos \rho x \, dx,$$

which, in turn, can be established similarly to Lemma 3.3 in [30] or obtained as its corollary.

The case $\gamma = -1$ can be treated analogously. Assume now that $\gamma \neq \pm 1$ and let us show that $\{\rho_{n, \alpha}^0, \Delta(\lambda_{n, \alpha}^0)\}_{n \geq 0} \in l_2$. Indeed, according to (24), (25) and (27), we have

$$\Delta(\lambda) = \Delta_0(\lambda) \prod_{k=0}^{\infty} \frac{\lambda_{k, \alpha} - \lambda}{\lambda_0^{k, \alpha} - \lambda} = \frac{\lambda_n - \lambda}{\rho_{n, \alpha}^0 + \rho} \cdot \frac{\Delta_0(\lambda)}{\prod_{k \neq n}^{\infty} \lambda_{k, \alpha} - \lambda_0^{k, \alpha} - \lambda}.$$

Substituting $\lambda = \lambda_{n, \alpha}^0$ therein and using (18), we get $\rho_{n, \alpha}^0 \Delta(\lambda_{n, \alpha}^0) = a_n b_n \kappa_n$, where

$$a_n = \frac{1}{2} \lim_{\rho \to \rho_{n, \alpha}^0} \frac{\Delta_0(\lambda)}{\rho_{n, \alpha}^0 - \rho} = \gamma (-1)^{n+1} \sin \alpha \pi, \quad b_n = \prod_{k \neq n}^{\infty} \frac{\lambda_{k, \alpha} - \lambda_{n, \alpha}^0}{\lambda_{k, \alpha}^0 - \lambda_{n, \alpha}^0}.$$

Using (17), it is easy to show that $|\lambda_{k, \alpha}^0 - \lambda_{n, \alpha}^0| \geq C_\gamma (k + 1)$ for $k \neq n$, where $C_\gamma > 0$ depends only on $\gamma$. Hence, $|b_n| \leq C$. Further, note that the functional system...
\[
\{\sin \rho_0^n x \}_{n \geq 0} \text{ coincides up to signs with the system } \{\sin (2n + \alpha)\pi x \}_{n \in \mathbb{Z}}, \text{ which, in turn, forms a Riesz basis in } L_2(0, 1) \text{ as soon as } \alpha \notin \mathbb{Z}. \text{ Indeed, for real non-integer } \alpha \text{'s this fact was proved in [32], while for the general case the corresponding proof is provided in Appendix A (see Lemma A1). Thus, there exists a unique function } \psi(x) \in L_2(0, 1) \text{ such that }
\]
\[
\Delta(\lambda_{n, \alpha}) = -\int_0^1 \psi(x) \frac{\sin \rho_0^n x}{\rho_0^n} \, dx, \quad n \geq 0.
\]

Consider the function
\[
\theta(\lambda) = -\int_0^1 \psi(x) \frac{\sin \rho x}{\rho} \, dx.
\]

It remains to show that \( \Delta(\lambda) = \Delta_0(\lambda) + \theta(\lambda). \) For this purpose, we introduce the function
\[
\sigma(\lambda) = \frac{\Delta(\lambda) - \Delta_0(\lambda) - \theta(\lambda)}{\Delta_0(\lambda)} = F(\lambda) - 1 - \frac{\theta(\lambda)}{\Delta_0(\lambda)}, \quad F(\lambda) = \prod_{k=0}^{\infty} \frac{\lambda_k - \lambda}{\lambda_{k, \alpha} - \lambda}.
\]

After removing singularities, the function \( \sigma(\lambda) \) is entire in \( \lambda. \) As in the proof of Lemma 3.3 in [30], one can show that \( |F(\lambda)| < C_\delta \) for \( \lambda \in G_\delta := \{ \lambda = \rho^2 : |\rho - (2n + \alpha)\pi| \geq \delta, n \in \mathbb{Z} \}, \delta > 0, \) and \( F(\lambda) \to 1 \) as \( \lambda \to -\infty. \) Moreover, we have \( \rho \theta(\lambda) = o(\Delta_0(\lambda)) \) in \( G_\delta \) as soon as \( |\lambda| \to \infty. \) Thus, by virtue of the maximum modulus principle, the function \( \sigma(\lambda) \) is bounded and consequently, according to Liouville’s theorem, it is constant. Since \( \sigma(\lambda) \to 0 \) as \( \lambda \to -\infty, \) we get \( \sigma(\lambda) \equiv 0, \) which finishes the proof.

3 Recovering from one spectrum

As mentioned in Introduction, when \( \gamma = 1, \) studying Inverse Problem 1 for any \( a \) can, obviously, be reduced to the case \( a = 0. \) The following lemma reveals this possibility also for any other \( \gamma \neq 0. \)

**Lemma 4** For any \( a \in (0, 1], \) the spectrum of the problem \( L(q(x), a, \gamma) \) coincides with the spectrum of \( L(q_a, 0, \gamma), \) where the function \( q_a(x) \) is determined by the formula
\[
q_a(x) = \begin{cases} 
q(x + a), & x \in (0, 1 - a), \\
\frac{1}{\gamma} q(x + a - 1), & x \in (1 - a, 1). 
\end{cases}
\]
\textbf{Proof} Denoting
\[
y_a(x) = \begin{cases} 
y(x + a), & x \in [0, 1 - a], \\
\frac{1}{\gamma}y(x + a - 1), & x \in (1 - a, 1],
\end{cases}
\] (30)
we note that, by virtue of the boundary conditions (2), \(y_a(x) \in W^2_2[0, 1]\). Moreover, since \(y(x) \in W^2_2[0, 1]\), we have
\[
y_a^{(v)}(0) = \gamma y_a^{(v)}(1), \quad v = 0, 1.
\] (31)
Further, using (29) and (30), we rewrite Eq. (1) in the form
\[
- y_a''(x) + q_a(x) y_a(0) = \lambda y_a(x), \quad 0 < x < 1.
\] (32)
It remains to note that one and the same \(\lambda\) can be eigenvalue of the boundary value problem (1), (2) and the problem (31), (32) only simultaneously and only of the same multiplicity. \(\square\)

Now we proceed directly to studying Inverse Problem 1. Here and in the subsequent section, along with the boundary value problem \(\mathcal{L} := \mathcal{L}(q(x), a, \gamma)\) we consider a problem \(\tilde{\mathcal{L}} := \mathcal{L}(\tilde{q}(x), \tilde{a}, \tilde{\gamma})\) of the same form but with a different potential \(\tilde{q}(x)\). We agree that if a certain symbol \(\beta\) denotes an object related to the problem \(\mathcal{L}\), then this symbol with tilde \(\tilde{\beta}\) will denote the corresponding object related to \(\tilde{\mathcal{L}}\). The following uniqueness theorem holds.

**Theorem 2** Let \(\gamma \neq \pm 1\). Then coincidence of the spectra: \(\{\lambda_n\}_{n \geq 0} = \{\tilde{\lambda}_n\}_{n \geq 0}\) implies coincidence of the potentials: \(q(x) = \tilde{q}(x)\) a.e. on \((0, 1)\).

Let \(\gamma \in \{-1, 1\}\). Assume that there exists an operator \(K : L^2(0, 1/2) \to L^2(0, 1/2)\) with injective \(I + \gamma K\), such that
\[
q_a\left(\frac{1}{2} - x\right) = K\left(q_a\left(\frac{1}{2} + x\right)\right), \quad \tilde{q}_a\left(\frac{1}{2} - x\right) = K\left(\tilde{q}_a\left(\frac{1}{2} + x\right)\right) \text{ a.e. on } (0, \frac{1}{2}),
\] (33)
where \(I\) is the identity operator and the function \(q_a(x)\) is determined by formula (29). Then \(\{\lambda_n\}_{n \geq 0} = \{\tilde{\lambda}_n\}_{n \geq 0}\) implies \(q(x) = \tilde{q}(x)\) a.e. on \((0, 1)\).

**Proof** Let \(\{\lambda_n\}_{n \geq 0} = \{\tilde{\lambda}_n\}_{n \geq 0}\). Then, by virtue of (12) and (24), we always have \(w(x) = \tilde{w}(x)\) a.e. on \((0, 1)\). According to Lemmas 1 and 4, the main equation (13) takes the form
\[
w(x) = \gamma q_a(1 - x) + \gamma^2 q_a(x), \quad x \in (0, 1),
\] (34)
where \(q_a(x)\) is determined by (29). This can also be checked directly using (13) and (29). Clearly, Eq. (34) is equivalent to the linear system
\[
\begin{align*}
 w(x) &= \gamma q_a(1 - x) + \gamma^2 q_a(x), \\
w(1 - x) &= \gamma^2 q_a(1 - x) + \gamma q_a(x),
\end{align*}
\] (35)

\[x \in \left(0, \frac{1}{2}\right),\]
whose determinant equals to $\gamma^2(1 - \gamma^2)$. Thus, the system (35) is non-degenerate if and only if $\gamma \neq \pm 1$. Hence, assuming $\gamma \neq \pm 1$ and inverting (29), we get $q(x) = \tilde{q}(x)$ a.e. on $(0, 1)$.

Let $\gamma \in \{-1, 1\}$. Then, with accordance to (22), the equations in (35) are equivalent. Making in the first equation of (35) the change of variable and only if $\gamma \neq \pm 1$, we get

$$w\left(\frac{1}{2} - x\right) = \gamma q_a\left(\frac{1}{2} + x\right) + K\left(q_a\left(\frac{1}{2} + x\right)\right),$$

$$w\left(\frac{1}{2} - x\right) = \gamma \tilde{q}_a\left(\frac{1}{2} + x\right) + K\left(\tilde{q}_a\left(\frac{1}{2} + x\right)\right)$$

(36)
a.e. on $(0, 1/2)$. Since $I + \gamma K$ is injective, we get $q_a(x) = \tilde{q}_a(x)$ a.e. on $(1/2, 1)$. Finally, using (33) again, we arrive at $q_a(x) = \tilde{q}_a(x)$ a.e. on $(0, 1/2)$, which finishes the proof. □

**Remark 2** Using (17) and (18), it is easy to show that specification of the spectrum $\{\lambda_n\}_{n \geq 0}$ determines also the constant $\gamma$ uniquely if $\gamma \in \{-1, 1\}$ and up to inversion if $\gamma \neq \pm 1$.

**Remark 3** Clearly, the second part of Theorem 2 remains true also if one applies the operator $K$ to the left-hand sides of the equalities in (33) instead of the right-hand ones.

For $\gamma \in \{1, -1\}$, condition (33) may mean, in particular, the evenness (for $\gamma = 1$) or the oddness (for $\gamma = -1$) of the function $q_a(x)$ with respect to the midpoint of the interval $(0, 1)$, i.e. $q_a(1/2 - x) = \gamma q_a(1/2 + x)$, $0 < x < 1/2$. However, the case $K = -\gamma I$ is not covered by condition (33) and not eligible. Indeed, according to (36), in this case the spectrum of $\mathcal{L}(q(x), a, \gamma)$ coincides with the one of $\mathcal{L}(0, a, \gamma)$ and, hence, carries no information on the potential $q(x)$. The case of constant $K$ (i.e. when $K(f)$ is independent of $f$) corresponds to a priori specification of $q_a(x)$ on the subinterval $(0, 1/2)$.

The next theorem means that Theorem 1 gives complete characterization of the spectrum.

**Theorem 3** Let $\gamma \neq \pm 1$. Then for any fixed $a \in [0, 1]$ and any sequence of complex numbers $\{\lambda_n\}_{n \geq 0}$ of the form (18), there exists a unique function $q(x) \in L_2(0, 1)$ such that $\{\lambda_n\}_{n \geq 0}$ is the spectrum of the corresponding boundary value problem $\mathcal{L}(q(x), a, \gamma)$.

Let $\gamma \in \{-1, 1\}$. Then for any fixed $a \in [0, 1]$ and any complex sequence $\{\lambda_n\}_{n \geq 0}$ of the form (18), obeying the degeneration condition (19), there exists $q(x) \in L_2(0, 1)$ (not unique) such that $\{\lambda_n\}_{n \geq 0}$ is the spectrum of the corresponding problem $\mathcal{L}(q(x), a, \gamma)$.

**Proof** Using $\{\lambda_n\}_{n \geq 0}$, we construct the function $\Delta(\lambda)$ by formula (24). By virtue of Lemma 3, it has the representation (12) with a certain function $w(x) \in L_2(0, 1)$. If $\gamma \in \{-1, 1\}$, then this $w(x)$ additionally obeys condition (22). Thus, in any case, the linear system (35) is consistent. Consider some its solution $q_a(x) \in L_2(0, 1)$, which
is not unique if and only if $\gamma \in \{-1, 1\}$. Find the function $q(x) \in L_2(0, 1)$ determined by formula (29) and consider the corresponding problem $L := \mathcal{L}(q(x), a, \gamma)$. Let us show that $\{\lambda_n\}_{n\geq0}$ is its spectrum.

Indeed, by virtue of Lemma 4, the spectrum of $L$ coincides with the one of the problem $L_a := \mathcal{L}(q_a, 0, \gamma)$, where the function $q_a(x)$ is determined by formula (29). Let $\Delta_a(\lambda)$ be the characteristic function of $L_a$. According to Lemma 1, it has the representation

$$
\Delta_a(\lambda) = 1 + \gamma^2 - 2\gamma \cos \rho - \int_0^1 w_a(x) \frac{\sin \rho x}{\rho} \, dx,
$$

where $w_a(x) = \gamma q_a(x) + \gamma^2 q_a(x)$, $x \in (0, 1)$. Comparing this with (34), we get $w_a(x) = w(x)$ a.e. on $(0, 1)$. Thus, we have $\Delta_a(\lambda) \equiv \Delta(\lambda)$, i.e. the spectrum of $L_a$ as well as the one of $L$ coincide with $\{\lambda_n\}_{n\geq0}$. $\square$

The proof of Theorem 3 is constructive and gives algorithms for solving the inverse problem. First, we provide an algorithm for the case $\gamma \neq \pm 1$.

**Algorithm 1** Let the spectrum $\{\lambda_n\}_{n\geq0}$ of some problem $\mathcal{L}(q(x), a, \gamma)$, $\gamma \neq \pm 1$, be given.

1. Construct the function $\Delta(\lambda)$ by formula (24).
2. Calculate the function $w(x) \in L_2(0, 1)$, inverting the Fourier transform in (12):

$$
w(x) = 2 \sum_{k=1}^{\infty} f(\pi k) \sin \pi k x,
$$

where $f(\rho) = \rho(1 + \gamma^2 - 2\gamma \cos \rho - \Delta(\rho^2))$.
3. Find $q_a(x) \in L_2(0, 1)$ by solving the non-degenerate linear system (35):

$$
q_a(x) = \frac{\gamma w(x) - w(1-x)}{\gamma^3 - \gamma}.
$$

4. Construct the potential $q(x)$ by inverting (29):

$$
q(x) = \begin{cases} 
\gamma q_a(x-a+1), & x \in (0, a), \\
q_a(x-a), & x \in (a, 1).
\end{cases}
$$

The following algorithm deals with the case $\gamma \in \{-1, 1\}$. Unlike the previous one, for definiteness it requires specifying also an operator $K$ appeared in Theorem 2.

**Algorithm 2** Assume that $\gamma \in \{-1, 1\}$ and let the spectrum $\{\lambda_n\}_{n\geq0}$ of a boundary value problem $\mathcal{L}(q(x), a, \gamma)$ along with the operator $K$ in (33) with bijective $I + \gamma K$ be given.

1. Construct the function $w(x)$ by implementing the first two steps of Algorithm 1.
2. Find the function \( q_a(x) \) on the interval \((0, 1/2)\) by the formula

\[
q_a\left(\frac{1}{2} - x\right) = K \left( (I + \gamma K)^{-1} \left( \gamma w\left(\frac{1}{2} - x\right) \right) \right), \quad 0 < x < \frac{1}{2},
\]

3. Find the function \( q_a(x) \) on the interval \((1/2, 1)\) by the formula

\[
q_a\left(\frac{1}{2} + x\right) = \gamma w\left(\frac{1}{2} - x\right) - \gamma q\left(\frac{1}{2} - x\right), \quad 0 < x < \frac{1}{2},
\]

and construct the potential \( q(x) \) by formula (37).

The latter algorithm also allows one to describe the set of all iso-spectral potentials \( q(x) \), i.e. of those for which the corresponding problems \( L(q(x), a, \gamma) \) (with fixed \( a \in [0, 1] \) and \( \gamma \in \{-1, 1\} \)) have one and the same spectrum \( \{\lambda_n\}_{n \geq 0} \). For this purpose, on the second step of Algorithm 2 one should use a constant operator \( K \), i.e. for which there exists a function \( p(x) \in L^2(0, 1/2) \) such that

\[
K(f(x)) = p(x) \tag{38}
\]

for all \( f(x) \in L^2(0, 1/2) \). Indeed, the following theorem holds.

**Theorem 4** If the function \( p(x) \) in (38) ranges over \( L^2(0, 1/2) \), then the corresponding functions \( q(x) \) constructed by Algorithm 2 form the set of all iso-spectral potentials for the given spectrum \( \{\lambda_n\}_{n \geq 0} \).

**Proof** It is clear that, for the operator \( K \) of the form (38) and for any \( p(x) \in L^2(0, 1/2) \), Algorithm 2 gives iso-spectral potentials \( q(x) \). On the other hand, by virtue of Theorem 2, no other iso-spectral potentials exist. \( \square \)

**4 Recovering from two spectra**

As can be seen in the preceding section, the spectrum of the problem \( L(q(x), a, \gamma) \) possesses sufficient (and necessary) information for unique reconstruction of the potential \( q(x) \) if and only if \( \gamma \neq \pm 1 \). Here, we study Inverse Problem 2, dealing with recovering \( q(x) \) from both spectra \( \{\lambda_{n,\alpha}\}_{n \geq 0}, \alpha = 0, 1, \) of the periodic and the antiperiodic boundary value problems \( L_{\alpha}(q(x), a) = L(q(x), a, (-1)^{\alpha}), \alpha = 0, 1. \)

Denote by \( \Delta_{\alpha}(\lambda) \) the characteristic function of \( L_{\alpha}(q(x), a) \). By virtue of Lemma 1, we have

\[
\Delta_{\alpha}(\lambda) = 2 - 2(-1)^{\alpha} \cos \rho - \int_0^1 w_{\alpha}(x) \frac{\sin \rho x}{\rho} \, dx, \quad w_{\alpha}(x) \in L^2(0, 1), \tag{39}
\]
where the functions $w_\alpha(x)$ are determined by the formulae

\[
w_\alpha(x) = \begin{cases} 
q(a + x) + q(a - x), & x \in (0, a), \\
(-1)^\alpha q(1 + a - x) + q(a + x), & x \in (a, 1 - a), \\
(-1)^\alpha q(1 + a - x) + q(x - 1 + a), & x \in (1 - a, 1), 
\end{cases} \tag{40}
\]

which were referred as the main equations for Inverse Problem 1. Moreover, Lemma 2 gives

\[
\Delta_0(\lambda) = (\lambda - \lambda_{0,0}) \prod_{n=1}^{\infty} \frac{\lambda_{n,0} - \lambda}{\lambda_{n,0}^0}, \quad \Delta_1(\lambda) = 4 \prod_{n=0}^{\infty} \frac{\lambda_{n,1} - \lambda}{\lambda_{n,1}^0}, \tag{41}
\]

where $\lambda_{n,0}$, $n \geq 0$, $\alpha = 0, 1$, are determined by (25).

In particular, we will show that Inverse Problem 2 is uniquely solvable without any additional assumptions on $q(x)$ if and only if $a \in \{0, 1\}$. Moreover, the following theorem along with Theorem 1 gives necessary and sufficient conditions for its solvability in this special case.

**Theorem 5** Let $a \in \{0, 1\}$. Then for any complex sequences $\{\lambda_{n,\alpha}\}_{n \geq 0}$, $\alpha = 0, 1$, of the form

\[
\lambda_{2k,\alpha} = (2k + \alpha)^2 \pi^2 + \varepsilon_{k,\alpha}, \quad \{\varepsilon_{k,\alpha}\} \in l_2, \quad k \geq 0,
\]

\[
\lambda_{2k-1,\alpha} = (2k - \alpha)^2 \pi^2, \quad k \geq 1, \quad \alpha = 0, 1, \tag{42}
\]

there exists a unique complex-valued function $q(x) \in L_2(0, 1)$ such that the given sequences are the spectra of the corresponding boundary value problems $L_{\alpha}(q(x), a)$, $\alpha = 0, 1$.

**Proof** According to Remark 1, it is sufficient to consider the case $a = 0$. For $\alpha = 0, 1$, construct the entire functions $\Delta_{\alpha}(\lambda)$ by formulae (41). According to Lemma 3, these functions have the form (39) with some functions $w_\alpha(x) \in L_2(0, 1)$ obeying condition (22), which takes the form

\[
w_\alpha(x) = (-1)^\alpha w_\alpha(1 - x), \quad 0 < x < 1, \quad \alpha = 0, 1. \tag{43}
\]

Solving the system of main equations (40), which, in the present case $a = 0$, takes the form

\[
w_0(x) = q(1 - x) + q(x), \quad w_1(x) = -q(1 - x) + q(x), \quad 0 < x < 1, \tag{44}
\]

we get

\[
q(x) = \frac{w_0(x) + w_1(x)}{2}. \tag{45}
\]
Denote by $\tilde{\Delta}_\alpha(\lambda)$ the characteristic functions of the constructed problems $\mathcal{L}_\alpha(q(x), 0)$, $\alpha = 0, 1$. According to Lemma 1, they have the form

$$\tilde{\Delta}_\alpha(\lambda) = 2 - 2(-1)^\alpha \cos \rho - \int_0^1 \tilde{w}_\alpha(x) \frac{\sin \rho x}{\rho} dx, \quad \alpha = 0, 1,$$  \quad (46)

where the functions $\tilde{w}_\alpha(x)$ are determined by the formulae

$$\tilde{w}_0(x) = q(1-x) + q(x), \quad \tilde{w}_1(x) = -q(1-x) + q(x), \quad 0 < x < 1.$$  

Comparing them with (39) and (44), we arrive at $\tilde{\Delta}_\alpha(\lambda) = \Delta_\alpha(\lambda), \alpha = 0, 1$, i.e. the spectra of the constructed problems coincide with the given sequences, respectively. The uniqueness follows from uniqueness of solution of the system (44). \quad \square

The following algorithm gives solution of Inverse Problem 2 for $a = 0$. According to Remark 1, in the case $a = 1$ this algorithm gives $q(1-x)$ instead of $q(x)$.

**Algorithm 3** For $\alpha = 0, 1$, let the spectra $\{\lambda_{n,\alpha}\}_{n \geq 0}$ of the problems $\mathcal{L}_\alpha(q(x), 0)$ with some common potential $q(x) \in L_2(0, 1)$ be given. Then it can be found by the following steps.

1. Construct the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ by formulae (41).
2. Calculate the functions $w_0(x)$ and $w_1(x)$, inverting the Fourier transform in (39):

$$w_\alpha(x) = 2 \sum_{k=1}^\infty f_\alpha(\pi k) \sin \pi k x, \quad \alpha = 0, 1,$$

where $f_\alpha(\rho) = \rho(2 - 2(-1)^\alpha \cos \rho - \Delta_\alpha(\rho^2))$.
3. Find the function $q(x)$ by formula (45).

As mentioned above, we claim that 0 and 1 are unique values of $a$ in the segment $[0, 1]$ for which Inverse Problem 2 is uniquely solvable. This will actually be established by Theorem 7 below. In order to achieve the uniqueness for $a \in (0, 1)$, one should put some additional restrictions on the potential $q(x)$. Specifically, the following uniqueness theorem holds. According to Remark 1, for definiteness, we consider the case $a \in (0, 1/2]$.

**Theorem 6** Let $a \in (0, 1/2]$ and let there exist an operator $P : L_2(0, a) \to L_2(0, a)$ with injective $I + P$, such that

$$q(a-x) = P(q(a+x)), \quad \tilde{q}(a-x) = P(\tilde{q}(a+x)) \quad \text{a.e. on } (0, a). \quad (47)$$

Then coincidence of the spectra: $\{\lambda_{n,\alpha}\}_{n \geq 0} = \{\tilde{\lambda}_{n,\alpha}\}_{n \geq 0}, \alpha = 0, 1$, implies coincidence of the potentials: $q(x) = \tilde{q}(x)$ a.e. on $(0, 1)$.

**Proof** The coincidence of the spectra implies $w_\alpha(x) = \tilde{w}_\alpha(x)$ a.e. on $(0, 1)$ for $\alpha = 0, 1$. Thus, by virtue of the first line in (40) and (47), we get

$$P(q(a+x)) + q(a+x) = P(\tilde{q}(a+x)) + \tilde{q}(a+x) \quad \text{a.e. on } (0, a),$$
which along with the injectivity of $I + P$ and (47) gives $q(x) = \tilde{q}(x)$ a.e. on $(0, 2a)$.

Further, summing up (40) for $x \in (a, 1 - a)$ and for different $a$’s, we get

$$q(a + x) = \frac{w_0(x) + w_1(x)}{2} = \frac{\tilde{w}_0(x) + \tilde{w}_1(x)}{2} = \tilde{q}(a + x), \text{ a.e. on } (a, 1 - a),$$
i.e. $q(x) = \tilde{q}(x)$ a.e. on $(2a, 1)$, which finishes the proof.  

\textbf{Remark 4} According to Remark 1, for $a > 1/2$, condition (47) takes the form

$$q(a + x) = P(q(a - x)), \quad \tilde{q}(a + x) = P(\tilde{q}(a - x)) \text{ a.e. on } (0, 1 - a). \quad (48)$$

Moreover, as in Remark 3, the operator $P$ can be applied alternatively to the left-hand sides of the equalities in (47) and (48).

The following theorem gives necessary and sufficient conditions for solvability of Inverse Problem 2 in the case $a \in (0, 1)$.

\textbf{Theorem 7} Fix $a \in (0, 1)$. Then for arbitrary sequences of complex numbers $\{\lambda_{n,0}\}_{n \geq 0}$ and $\{\lambda_{n,1}\}_{n \geq 0}$ to be the spectra of the problems $\mathcal{L}_0(q(x), a)$ and $\mathcal{L}_1(q(x), a)$, respectively, with a common potential $q(x) \in L_2(0, 1)$ (not unique) it is necessary and sufficient to have the form (42) and to satisfy the condition

$$\Delta_0(\lambda) + \Delta_1(\lambda) = O(\exp(\max\{1 - a, a\}|\rho|)), \quad \lambda \to -\infty, \quad (49)$$

where the functions $\Delta_0(\lambda)$ and $\Delta_1(\lambda)$ are determined by formulae (41).

\textbf{Proof} According to Remark 1, it is sufficient to consider the case $a \in (0, 1/2]$. By necessity, (42) is already established in Theorem 1. Further, by virtue of (39), we get

$$\Delta_0(\lambda) + \Delta_1(\lambda) = 4 - \int_0^1 (w_0(x) + w_1(x)) \frac{\sin \rho x}{\rho} \, dx, \quad (50)$$

which along with (40) gives

$$\Delta_0(\lambda) + \Delta_1(\lambda) = 4 - \int_0^{1-a} (w_0(x) + w_1(x)) \frac{\sin \rho x}{\rho} \, dx,$$

The latter implies (49) and finishes the proof of the necessity.

Let some sequences $\{\lambda_{n,0}\}_{n \geq 0}$ and $\{\lambda_{n,1}\}_{n \geq 0}$, obeying the conditions of the theorem be given. For $a = 0, 1$, construct the entire functions $\Delta_\alpha(\lambda)$ by formulae (41). According to Lemma 3, they have the form (39) with some functions $w_\alpha(x) \in L_2(0, 1)$ satisfying condition (43). Since the indicator diagrams of the functions $\Delta_0(\rho^2)$ and $\Delta_1(\rho^2)$ lie on the imaginary axis of the $\rho$-plane, condition (49) implies that the type of their sum does not exceed $\max\{1 - a, a\} = 1 - a$. According to the Paley–Wiener theorem, representation (50) implies

$$w_0(x) + w_1(x) = 0 \text{ a.e. on } (1 - a, 1). \quad (51)$$
Choose any function \( q(x) \in L_2(0, 2a) \) that obeys the functional equation
\[
w_0(x) = q(a + 1 - x) + q(a - 1 + x), \quad x \in (1 - a, 1).
\]
(52)

If \( a \neq 1/2 \), then we consider also the linear system
\[
w_0(x) = q(1+a-x)+q(a+x), \quad w_1(x) = -q(1+a-x)+q(a+x), \quad x \in (a, 1-a),
\]
(53)
which has the unique solution
\[
q(x) = \frac{w_0(x-a) + w_1(x-a)}{2}, \quad x \in (2a, 1).
\]
(54)

Thus, we constructed some function \( q(x) \). Note that (43) and (51)–(53) imply (40).

For \( \alpha = 0, 1 \), consider the boundary value problems \( \mathcal{L}_\alpha(q(x), a) \) with this \( q(x) \).

Denote by \( \tilde{\Delta}_\alpha(\lambda) \) their characteristic functions. According to Lemma 1, they have the form (46) with the functions \( \tilde{w}_\alpha(x) \) determined by the formulae
\[
\tilde{w}_\alpha(x) = \begin{cases} 
q(a + x) + q(a - x), & x \in (0, a), \\
(1)^\alpha q(1 + a - x) + q(a + x), & x \in (a, 1 - a), \\
(1)^\alpha q(1 + a - x) + q(x - 1 + a), & x \in (1 - a, 1), 
\end{cases}
\]
\( \alpha = 0, 1. \)

Comparing this with (40) and using (39) and (46), we arrive at \( \tilde{\Delta}_\alpha(\lambda) = \Delta_\alpha(\lambda) \)
for \( \alpha = 0, 1 \), i.e. the spectra of the constructed problems coincide with the given sequences, respectively.

\textbf{Remark 5} According to (43) and (51), Inverse Problem 1 for \( \gamma \in \{-1, 1\} \) is equivalent to
Inverse Problem 2 as soon as \( a = 1/2 \). Indeed, in this case, specification of the spectrum \( \{\lambda_{n,0}\}_{n \geq 0} \)
is equivalent to the specification of \( \{\lambda_{n,1}\}_{n \geq 0} \).

By virtue of Remarks 1 and 5, it is sufficient to provide an algorithm for solving
Inverse Problem 2 only for \( a \in (0, 1/2) \).

\textbf{Algorithm 4} For \( \alpha = 0, 1 \), let the spectra \( \{\lambda_{n,\alpha}\}_{n \geq 0} \) of the problems
\( \mathcal{L}_\alpha(q(x), a) \) with \( a \in (0, 1/2) \) and \( q(x) \in L_2(0, 1) \) be given along with the operator \( P \) in (47) with
bijective \( I + P \). Then the potential \( q(x) \) can be found by the following steps.

1. Calculate the functions \( w_0(x) \) and \( w_1(x) \), implementing the first two steps of
   Algorithm 3.
2. Find \( q(x) \) on the interval \((0, a)\) by the formula
   \[
   q(a - x) = P((I + P)^{-1}(w_0(x))), \quad 0 < x < a.
   \]
3. Find \( q(x) \) on the interval \((a, 2a)\) by the formula
   \[
   q(x + a) = w_0(x) - q(a - x), \quad 0 < x < a.
   \]
4. Finally, construct \( q(x) \) on \((2a, 1)\) by formula (54).

This algorithm allows one to describe the set of iso-bispectral potentials \( q(x) \), i.e. of those for which the corresponding problems \( L_\alpha(q(x), a) \), \( \alpha = 0, 1 \), have one and the same pair of the spectra \( \{\lambda_{n,\alpha}\}_{n \geq 0}, \alpha = 0, 1 \). For definiteness, let \( a \in (0, 1/2] \). Then on the second step of Algorithm 4 one should use the constant operator

\[
P(f(x)) \equiv p(x),
\]

where \( p(x) \in L_2(0, a) \) is fixed.

**Theorem 8** Let \( a \in (0, 1/2] \). If the function \( p(x) \) ranges over \( L_2(0, a) \), then the corresponding functions \( q(x) \) constructed by Algorithm 4 form the set of all iso-bispectral potentials for the given pair of spectra \( \{\lambda_{n,0}\}_{n \geq 0} \) and \( \{\lambda_{n,1}\}_{n \geq 0} \).

**Proof** It is clear that for the operator \( P \) of the form (55) and for any \( p(x) \in L_2(0, a) \), Algorithm 4 gives iso-bispectral potentials \( q(x) \). On the other hand, by virtue of Theorem 6, no other iso-bispectral potentials exist. \( \square \)

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**Appendix A**

Here we prove the following auxiliary assertion.

**Lemma A1.** For \( \alpha \notin \mathbb{Z} \), the system \( \Lambda_\alpha := \{\sin(2n + \alpha)x\}_{n \in \mathbb{Z}} \) is a Riesz basis in \( L_2(0, \pi) \).

For this purpose, we need the following theorem of Levin and Ljubarskii (see [31]).

**Theorem A1.** The system of functions \( \{\exp(i\pi z_k x)\}_{k \in \mathbb{N}} \), where \( \{z_k\}_{k \in \mathbb{N}} \) is the set of zeros of a sine-type function and \( \inf_{k \neq n} |z_k - z_n| > 0 \), is a Riesz basis in \( L_2(-\pi, \pi) \).

We remind that an entire function of exponential type \( S(\rho) \) is called a sine-type function if, for some positive constants \( c, C \) and \( K \), the inequalities \( c < |S(\rho)| \exp(-|\text{Im}\rho| \pi) < C \) hold as soon as \( |\text{Im}\rho| > K \).
**Proof of Lemma A1.** For real $\alpha \notin \mathbb{Z}$, this assertion was proved in [32]. Here we adapt its proof to cover also the case $\text{Im} \alpha \neq 0$. Firstly note, that $\Lambda_{\alpha}$ is complete in $L^2(0, \pi)$. Indeed, since for any $f(x) \in L^2(0, \pi)$ we have

$$
\frac{1}{\sigma(\rho)} \int_0^\pi f(x) \sin \rho x \, dx = o(1), \quad \rho \to \infty, \quad \rho \in G_{\delta,1},
$$

where $G_{\delta,1} = \{ \rho : |\rho - 2n - \alpha| \geq \delta, n \in \mathbb{Z} \}$, $\delta > 0$ and

$$
\sigma(\rho) = \sin \frac{\rho + \alpha}{2} \pi \cdot \sin \frac{\rho - \alpha}{2} \pi,
$$

then $f(x) = 0$ a.e. on $(0, \pi)$ as soon as $\overline{f}(x)$ is orthogonal to all elements of $\Lambda_{\alpha}$. Thus, it remains to show the existence of positive constants $A_1$ and $A_2$ such that for any sequence $\{c_n\} \in l_2$, the following two-sided estimate holds:

$$
A_1 \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \left\| \sum_{n=-\infty}^{\infty} c_n \sin(2n + \alpha)x \right\|^2_{L^2(0,\pi)} \leq A_2 \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{56}
$$

Since $2n \pm \alpha, n \in \mathbb{Z}$, are all zeros of the sine-type function $\sigma(\rho)$, Theorem A1 implies that the system $\{\exp(i(2n + (-1)^v\alpha)x)\}_{n \in \mathbb{Z}, v=0,1}$ is a Riesz basis in $L^2(-\pi, \pi)$. Hence, there exist positive constants $B_1$ and $B_2$ such that for any $\{b_n\} \in l_2$, the two-sided estimate

$$
B_1 \sum_{n=-\infty}^{\infty} |b_n|^2 \leq \left\| \sum_{v=0}^{1} \sum_{n=-\infty}^{\infty} b_{2n+v} \exp(i(2n + (-1)^v\alpha)x) \right\|^2_{L^2(-\pi,\pi)} \leq B_2 \sum_{n=-\infty}^{\infty} |b_n|^2 \tag{57}
$$

holds. In particular, with $b_{2n} = -ic_n/2$ and $b_{2n+1} = ic_{-n}/2$ for $n \in \mathbb{Z}$, estimates (57) imply

$$
\frac{B_1}{2} \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \left\| \sum_{n=-\infty}^{\infty} c_n \sin(2n + \alpha)x \right\|^2_{L^2(-\pi,\pi)} \leq \frac{B_2}{2} \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{58}
$$

Since $\|f\|_{L^2(-\pi,\pi)} = \sqrt{2}\|f\|_{L^2(0,\pi)}$ for any odd (as well as for any even) function $f(x) \in L^2(-\pi,\pi)$, estimates (58) give (56) with $A_j = B_j/4$, $j = 1, 2$, which finishes the proof. \(\Box\)

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