On the Equivalence of Dual Theories

A. Subbotin, I. V. Tyutin

P. N. Lebedev Physics Institute,
Leninsky Prospect 53, 117924, Moscow, Russia.

ABSTRACT

We discuss the equivalence of two dual scalar field theories in 2 dimensions. The models are derived through the elimination of different fields in the same Freedman–Townsend model. It is shown that tree $S$-matrices of these models do not coincide. The 2-loop counterterms are calculated. It turns out that while one of these models is single-charged, the other theory is multi-charged. Thus the dual models considered are non-equivalent on classical and quantum levels. It indicates the possibility of the anomaly leading to non-equivalence of dual models.

$^{1}$E-mail address: tyutin@lpi.ac.ru
1 Introduction

The duality transformations are nowadays widely used in a field theory, providing the description of physical systems on alternative groundings. The first example of this approach is likely to be given by the Kramers-Wannier duality \[1\], relating the low-temperature properties of lattice models with the high-temperature ones. (See \[2\] for a review on the applications of duality transformation in superstring theory, and references therein). The original model and its dual are commonly assumed to be physically equivalent. However, in quantum theory the transformations like changes of variables may induce anomalies, with axial and conformal anomalies being their typical examples. Thus, the question on equivalence of dual theories deserves a more thorough discussion.

The present work is aimed to join this discussion, considering two dimensional Freedman–Townsend model \[3\]. In the general opinion, this model is equivalent to the model of principal chiral field \(\phi^a\) \[3, 4, 5\]. The latter one arises after elimination of vector fields \(A^a_\mu\) using the equations of motion due to variations of the action by a scalar field \(B^a\) (which an antisymmetric tensor in 2 dimensions is reduced to). On the other hand, when the equations of motions due to \(A^a_\mu\) field variations are used, one arrives at the theory written in terms of the field \(B^a\). The two models: the model of the principal chiral field \(\phi^a\), and that of the field \(B^a\), are related by duality transformation \[7, 8, 6, 4\]. Can one treat them as equivalent? We show in this work, that in perturbation theory one can not. To compare the models, the Born scattering amplitudes \(2 \rightarrow 2\), and the 2-loop counterterms are calculated. Though even the Born amplitudes turn out to be different, the arguments based on calculations of \(S\)-matrix elements could be considered dubious, since the massless particles are involved.

The comparison of counterterms provide more powerful ones. The geometry of the principal chiral field model constrains the total renormalization reducing it to a multiplicative renormalization of the coupling and a non-linear renormalization of the field \[9\]. The principal chiral field model is single-charged. The \(B^a\)-field model is also single-charged in one-loop approximation (it has already been known since the work \[8\]), though the charge renormalizations in these models are different. Below we demonstrate, that in two-loop approximation the renormalization in the \(B^a\)-field model is not reduced to the charge and \(B^a\) field renormalizations. In fact, this model is multi-charged. The latter means that these dual models are nonequivalent, calling for explicit checks to be made when the equivalence of any pair of dual models is alleged.

The paper is organized as follows. In section 2A Formal Approachsection.2 the Freedman–Townsend model is described. Restricting ourselves to the \(SU(2)\) group case, we construct the action in terms of the field \(B^a\), and, after a sequence of path integral transformations, demonstrate a formal equivalence of this model to the model of principal chiral fields. Born scattering amplitudes are calculated in section 3Born Amplitudessection.3. In section 4Two-Loop Countertermssection.4 we evaluate two-
loop counterterms. Finally, in section 53-Dimensional Model section 53-Dimensional Model the calculations of Born amplitudes in two different representations of the 3-dimensional Freedman–Townsend model are presented. In this case the amplitudes do coincide! We interpret this fact as an indication that the possible origin of 2-dimensional anomalies lies in ill-defined infrared behavior of the massless theory.

2 A Formal Approach

Consider the Freedman–Townsend model

\[ S = \int d^2x \left( B^i_{\mu\nu} F_i^{\mu\nu} + \frac{1}{2} A_i^\mu A_i^{\mu} \right) \]  

with \( F_i^{\mu\nu} = \partial_\mu A_i^\nu - \partial_\nu A_i^\mu + f^{ijk} A_j^\mu A_k^\nu \), \( i, j, k = 1, 2, 3 \), in two dimensional space-time: \( \mu = 0, 1 \). In this case, any antisymmetric tensor \( B_{\mu\nu} \) is proportional to \( \epsilon^{\mu\nu} \), so that

\[ B_{\mu\nu} = \frac{1}{2} B^i \epsilon^{\mu\nu}, \quad \epsilon^{01} = 1, \]

and the action takes the form

\[ S = \int dx \left( \frac{1}{2} B^i \epsilon^{\mu\nu} F_i^{\mu\nu} + \frac{1}{2} A_i^\mu A_i^{\mu} \right). \]

The path integral for this theory reads

\[ Z = \int DB^i DA_\mu^i \exp\{iS\}. \] 

Let us write down another pair of expressions, formally equivalent to the above. The first one is derived as follows. Perform the formal change of variables \( A_i^\mu \rightarrow (\varphi^i, A_i) \) in \( S \), where

\[ A_1^i = \frac{1}{f} f^{ijk} C_{kj}^{-1}(\varphi) \partial_1 C_{nj}(\varphi) \equiv \Lambda_{jk}(\varphi) \partial_1 \varphi^j , \]

\[ A_0^i = \frac{1}{f} f^{ijk} C_{kj}^{-1}(\varphi) \partial_0 C_{nj}(\varphi) + A_i \equiv \Lambda_{ji}(\varphi) \partial_0 \varphi^j + A_i , \]

\[ C_{ij}(\varphi) = (\exp \varphi)_{ij}, \quad \varphi_{ij} = f^{ijk} \varphi^k, \quad f^{ijk} f^{nji} = f^{inj}; \]

\[ \Lambda_{ij}(\varphi) = \int_0^1 d\tau C_{ij}(\tau \varphi) \left( \frac{\exp \varphi - 1}{\varphi} \right)_{ij}. \]

When \( A^i = 0 \), the fields \( A_\mu^i \) represent a pure gauge. The Jacobian of the change is

\[ \frac{D(A_0^i, A_1^i)}{D(\varphi^i, A^i)} = \prod_x \sqrt{g(\varphi)} \ \mathrm{Det} \ \partial_1 \equiv J(\varphi) \ \mathrm{Det} \ \partial_1 , \]
where \( g(\varphi) = \det g_{ij}(\varphi) \), \( g_{ij}(\varphi) = \Lambda_{ik}(\varphi)\Lambda_{jk}(\varphi) \). The path integral becomes
\[
Z = \int DB^iD\varphi^iDA^iJ(\varphi) \det \partial_1 \exp \left\{ i \int dx \left( B^i \nabla_1^{ij} A^j + \frac{1}{2} \partial_\mu \varphi^i g_{ij} \partial_\mu \varphi^j + O(A) \right) \right\}
\]
\[
= \int D\varphi^iDA^iJ(\varphi) \det \partial_1 \exp \left\{ i \int dx \left( \frac{1}{2} \partial_\mu \varphi^i g_{ij} \partial_\mu \varphi^j + O(A) \right) \right\}
\]
\[
= \int D\varphi^iJ(\varphi) \frac{\det \partial_1}{\det \nabla_1} \exp \left\{ i \int dx \frac{1}{2} \partial_\mu \varphi^i g_{ij} \partial_\mu \varphi^j \right\}.
\]
Here \( \nabla_1^{ij} \) is the covariant derivative
\[
\nabla_1^{ij} = \partial_1 \delta^{ij} + f^{ijk} A_1^k.
\]

One easily checks that \( \det \nabla_1^{ij} = \det \partial_1 \), which finally leads to
\[
Z = \int D\varphi^iJ(\varphi) \exp \left\{ i \int dx \frac{1}{2} \partial_\mu \varphi^i g_{ij} \partial_\mu \varphi^j \right\} \equiv \int D\varphi^iJ(\varphi) \exp \{ i S_{\text{ch}}(\varphi) \}.
\] (3)

The action \( S_{\text{ch}}(\varphi) \) is nothing but an action of the model of principal chiral fields. Hence one concludes that this model is equivalent to the model described by the Freedman–Townsend action \( S_{\text{b}}(B) \).

The other expression for \( Z \) results after a trivial integration over the fields \( A_1^i \):
\[
Z = \int DB^i e^{iS_{\text{b}}(B)} J_1(B)^{-1/2},
\] (4)
\[
S_{\text{b}}(B) = -\frac{1}{2} \int dx \epsilon^{\mu\lambda} \partial_\lambda B^i N_{ij}^{\mu} \epsilon^{\nu\sigma} \partial_\sigma B^j,
\] (5)
\[
N^{\mu\nu} = \delta^{\mu\nu} \eta^{\mu\nu} + \epsilon^{\mu\nu} f^{ijk} B^k,
\]
\[
J_1(B) = \det N, \quad \eta^{\mu\nu} = \text{diag} (1, -1).
\]

The representation \( S_{\text{b}}(B) \) should be interpreted as the partition function for the theory of scalar fields with the action \( S_{\text{b}}(B) \). So, at the formal level, one is tempted to accept the equivalence of the theory of principal scalar fields to the theory \( S_{\text{b}}(B) \). In the following sections we show that such a conclusion is incorrect.

### 3 Born Amplitudes

In this section we calculate \( 2 \rightarrow 2 \) scattering amplitudes in the theory of principal chiral fields, and in the theory described by the action \( S_{\text{b}}(B) \). So, at the formal level, one is tempted to accept the equivalence of the theory of principal scalar fields to the theory \( S_{\text{b}}(B) \). In the following sections we show that such a conclusion is incorrect.
The Born $2 \to 2$ scattering amplitude $A_{\text{ch}}$ is given by a single diagram

\[ p_1, i \quad p_3, k \]

\[ p_2, j \quad p_4, l \]

and equals to

\[ A_{\text{ch}} = \frac{i}{6} \left[ f^{ijn} f^{klm} (p_1 p_3 - p_1 p_4) + f^{ijn} f^{klm} (p_1 p_2 - p_1 p_4) + f^{ilm} f^{jkn} (p_1 p_2 - p_1 p_3) \right]. \tag{7} \]

Of course, this expression is valid for any space-time dimension.

The action for $B^i$ field up to an order required is

\[
S_{b}(B) = \int d^{2}x \left( \frac{1}{2} \partial_\mu B^i \partial_\mu B^i - \frac{1}{2} \epsilon_{\mu\nu} f^{ijk} B^i \partial_\mu B^j \partial_\nu B^k - \frac{1}{2} f^{ijk} B^i \partial_\mu B^j f^{ilm} B^l \partial_\mu B^m + O(B^5) \right).
\]

In this theory, the total scattering amplitude $A_B$ equals to a sum of the amplitudes

\[ A_B = A_4 + A_{33}, \]

where $A_4$ is represented by the diagram $\text{[3]}$, while $A_{33}$ is given by the sum of diagrams

\[ p_1, a \quad p_3, c \quad p_1, a \quad p_2, b \quad p_1, a \quad p_2, b \]

\[ p_2, b \quad p_4, d \quad p_3, c \quad p_4, d \quad p_4, d \quad p_3, c \]

(8)

The calculations result in

\[ A_4 = 12 A_{\text{ch}} \]

\[ A_{33} = -27 A_{\text{ch}}, \]

so that

\[ A_B = -15 A_{\text{ch}}, \]

and thus the Born scattering amplitudes differ in these models. Of course, any arguments based on calculations of scattering amplitudes may seem unreliable since the scattering particles are massless and, strictly speaking, do not exist in two dimensions. More serious arguments will be given in the section below.
4 Two-Loop Counterterms

In this section we study the structure of counterterms in the model \[ \mathcal{B} \]. Both the model of principal chiral fields and the \( B^i \)-field model are essentially non-linear. Being renormalizable by their divergence indices (the counterterm dimensions do not exceed 2), they generally admit an infinite number of counterterm structures. The existence of a global symmetry group in the first model allows to prove \[ \mathcal{I} \] that the renormalization is reduced to a renormalization of the coupling constant (an overall factor before the total action), and a non-linear renormalization (reparametrization) of the fields\[ \mathcal{I} \]. The \( B^i \)-field model do not possess any geometric background, thus, no symmetry restrictions on the choice of imaginable counterterms do exist. However, having taken on truth the equivalence of the two models, one should expect a renormalization to be reduced (modulo renormalization of the field \( B^i \)) solely to the renormalization of the charge, i.e. the factor before the total action. In other words, the renormalized action should have the form

\[
S_b(B) + \frac{\eta}{\epsilon} S_1(B) + \eta^2 \left( \frac{1}{\epsilon} S_{21}(B) + \frac{1}{\epsilon^2} S_{22}(B) \right) = \lambda(\eta, \epsilon) S_b(B) (\mathcal{B}(B, \eta, \epsilon)) + O(\eta^3),
\]

where \( \eta \) is the loop expansion parameter (two loops will suit for what follows),

\[
\lambda(\eta, \epsilon) = 1 + \frac{\eta}{\epsilon} \lambda_1 + \eta^2 \left( \frac{1}{\epsilon} \lambda_{21} + \frac{1}{\epsilon^2} \lambda_{22} \right),
\]

\[
B^a(B, \eta, \epsilon) = B^a + \frac{\eta}{\epsilon} F^a_1(B) + \eta^2 \left( \frac{1}{\epsilon} F^a_{21}(B) + \frac{1}{\epsilon^2} F^a_{22}(B) \right),
\]

and \( \epsilon \) is the parameter of dimensional regularization we use below. Comparing the coefficients before \( 1/\epsilon \) in left and right-hand sides of the relation \[ \mathcal{I} \], we see that the following equalities must hold

\[
S_1(B) = \lambda_1 s_b(B) + \frac{\delta S_b(B)}{\delta B^i} F^i_1(B),
\]

\[
S_{21}(B) = \lambda_{21} s_b(B) + \frac{\delta S_b(B)}{\delta B^i} F^i_{21}(B).
\]

We restrict ourselves to the case of \( SU(2) \) group, when the action \( S_b(B) \) can be written out explicitly. After the transition to Euclidean space-time, it reads

\[
S_b(B) = \frac{1}{2} \int d^2 x \, \bar{\psi} \gamma^i \partial_\mu B^i \left[ \frac{\delta_{ij} + B^i B^j}{1 + B^2} \delta_{\mu\nu} + i \epsilon_{\mu\nu} \epsilon^{ijk} B^k \right] \partial_\nu B^j.
\]

\[ \text{Under a suitable choice of parametrization of the group manifold the renormalization of the fields becomes multiplicative} \[ \mathcal{I}, \mathcal{I} \]. \]
Due to conservation of global $SU(2)$ group, the renormalizations have the following general structure

\[
S_1(B) = \frac{1}{2} \int d^2x \partial_\mu B^i [\delta_{\mu\nu} \delta_{ij} A_1(z) + \delta_{\mu\nu} B^i B^j C_1(z) \\
+ i\epsilon^{\mu\nu} \epsilon^{ijk} B^k D_1(z)] \partial_\nu B^j,
\]

\[
S_{21}(B) = \frac{1}{2} \int d^2x \partial_\mu B^i [\delta_{\mu\nu} \delta_{ij} A_{21}(z) + \delta_{\mu\nu} B^i B^j C_{21}(z) \\
+ i\epsilon^{\mu\nu} \epsilon^{ijk} B^k D_{21}(z)] \partial_\nu B^j,
\]

\[
F_i(B) = B^i f_1(z), \quad F_{21}^i(B) = B^i f_{21}(z), \quad z = B^i B^i \equiv B^2.
\]

Eq. [10,11] imply the relations between the functions introduced above:

\[
A_1(z) = \frac{2 f_1(z)}{(1 + z)^2} + \frac{\lambda_1}{1 + z}, \quad D_1(z) = \frac{(3 + z) f_1(z)}{(1 + z)^2} + \frac{\lambda_1}{1 + z},
\]

\[
C_1(z) = \frac{4 + 2z}{(1 + z)^2} f_1(z) + 4 f'_1(z) + \frac{\lambda_1}{1 + z}, \quad (13)
\]

\[
A_{21}(z) = \frac{2 f_{21}(z)}{(1 + z)^2} + \frac{\lambda_{21}}{1 + z}, \quad D_{21}(z) = \frac{(3 + z) f_{21}(z)}{(1 + z)^2} + \frac{\lambda_{21}}{1 + z},
\]

\[
C_{21}(z) = \frac{4 + 2z}{(1 + z)^2} f_{21}(z) + 4 f'_{21}(z) + \frac{\lambda_{21}}{1 + z}. \quad (14)
\]

Below we present the results of calculations of the functions $A_1, C_1, D_1, A_{21}, C_{21}, D_{21}$, and the solutions to the equations [13,14].

It is known [12, 13] that up to a non-linear change of variables, the counterterms are expressed in terms of the geometric objects, namely, through the metric, curvature, torsion and their covariant derivatives. In our case the action may be rewritten in the form (in Euclidean space-time)

\[
S_b(B) = \frac{1}{2} \int d^2x \partial_\mu B^i [g_{ij}(B) \delta_{\mu\nu} + i\epsilon^{\mu\nu} h_{ij}(B)] \partial_\nu B^j,
\]

where

\[
g_{ij}(B) = \frac{\delta_{ij} + B_i B_j}{1 + B^2}, \quad h_{ij} = \frac{\epsilon^{ijk} B^k}{1 + B^2}.
\]

We will need the following expressions:

\[
g^{ij} = (1 + B^2) \delta_{ij} - B^i B^j,
\]

\[
\Gamma^{i}_{jk} = \frac{1}{2} g^{in} (\partial_j g_{nk} + \partial_k g_{jn} - \partial_n g_{jk})
\]

\[
= \frac{(2 + B^2) B^i \delta_{jk} - (1 + B^2) B^j \delta_{ik} - (1 + B^2) B^k \delta_{ij} + B^i B^j B^k}{(1 + B^2)^2},
\]
\[ H_{ijk} = \partial_k h_{ij} + \partial_j h_{ki} + \partial_i h_{jk} = 3 + B^2 \frac{(1 + B^2)^2}{2} \delta_{ijk} \]
\[ R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{nk} \Gamma^m_{jl} - \Gamma^i_{nl} \Gamma^m_{jk} \]
\[ = (3 \delta_{il} \delta_{jk} - 3 B^2 \delta_{il} \delta_{jk} - B^4 \delta_{il} \delta_{jk} + 3 \delta_{lk} \delta_{jl} + 3 B^2 \delta_{lk} \delta_{jl} + B^4 \delta_{lk} \delta_{jl} - 3 \delta_{jl} B^2 B^k - B^2 \delta_{jl} B^i B^k + B^2 \delta_{il} B^j B^k + 3 \delta_{jk} B^i B^l + B^2 \delta_{jk} B^i B^l - B^2 \delta_{ik} B^j B^l)/(1 + B^2)^3. \]

4.1 One-Loop Approximation

The one-loop counterterm (modulo the change of fields) equals \[ S_1(B) = -\frac{1}{2} \int d^2 x \partial_\mu B^i \left( \hat{R}_{(ij)} + i \epsilon^{\mu\nu} \hat{R}_{[ij]} \right) \partial_\nu B^j \]
\[ \hat{R}_{(ij)} = \frac{1}{2\pi} \left( R_{ij} - \frac{1}{4} H_{imn} H_{mj}^{mn} \right), \]
\[ \hat{R}_{[ij]} = -\frac{1}{4\pi} D^k H_{ijk}. \]

The calculation gives
\[ \hat{R}_{(ij)} = -\frac{3 + B^4}{4\pi(1 + B^2)} \delta_{ij} + \frac{3 + 8 B^2 + B^4}{4\pi(1 + B^2)^3} B^i B^j, \]
\[ \hat{R}_{[ij]} = -\frac{1}{\pi} \frac{1}{(1 + B^2)} \epsilon^{ijk} B^k. \]

The equations \[ 13 \] in this case have the solution
\[ f_1(z) = -\frac{1 - B^2}{4\pi(1 + B^2)}, \quad \lambda_1 = -\frac{1}{4\pi}. \]

So, the renormalization of the model with the action \[ 12 \] is reduced to the renormalization of a single parameter (the factor before the action) exactly as it has been in a chiral theory; the renormalizations in these models are, however, different (chiral theory had \( \lambda_1 = -1/\pi \) \[ 17, 18 \]).

4.2 Two-Loop Approximation

The metric renormalization will turn out to be sufficient for our purposes, i.e. only the function \( \beta^{(2)}_{ij}(B) \) in the expression for the two loop counterterm
\[ S_{21}(B) = \frac{1}{2} \int d^2 x \partial_\mu B^i \left( -\frac{1}{2} \beta^{(2)}_{ij}(B) \delta_{\mu\nu} - \frac{i}{2} \beta^{(2)}_{h_{ij}}(B) \epsilon_{\mu\nu} \right) \partial_\nu B^j \]
matters.

The expression for $\beta^{(2)}_{g}ij$ we used was taken from the work \[15\] (see also \[16\])

$$
\beta^{(2)}_{g}ij = \frac{1}{4\pi^2} \left\{ \frac{1}{2} R_{abc} R_{ij}^{abc} - \frac{3}{2} R_{(i}^{abc} H_{j)ab} l - \frac{1}{2} R_{abrs} H_{iab} H_{jrs} + \frac{1}{8}(H^4)_{ij} \right. \\
+ \frac{1}{4} D_l H_{iab} D^l H_{ab} ^{ij} + \frac{1}{12} D_l H_{abc} D_j H_{abc} + \frac{1}{8} H_{iab} H_{j} (H^2)_{ij} \\
+ p_1 \left[ R_{iab}(H^2)_{ij} + 2 R_{(i}^{abc} H_{j)ab} l + R_{abrs} H_{abi} H_{rsj} \\
- D_l H_{iab} D^l H_{ab} ^{ij} \right] + \frac{1}{2} D_l (D^l H_{j} H_{ij}^{2}) + \frac{1}{2} D_l (D_i H_{j} H_{ij}^{2}) \\
+ p_2 H_{ab(i} \left( D_{j})D_l H_{iab} + 2 D^a D_i H_{iab}^{jl} \right) \right\}.
$$

(16)

Here

$$(H^2)_{ij} \equiv H_{iab} H_{ab}^{ij},$$

$$(H^4)_{ij} \equiv \frac{1}{2} H_{iab} H_{j}^{arb} H_{iab} H_{j} + (i \leftrightarrow j).$$

The coefficients $p_1$ and $p_2$ are arbitrary; they reflect the freedom in the definition of $\epsilon^{\mu
\nu}$ in dimensional regularization, and the possibility of finite metric renormalizations \[15\].

The calculation of separate terms of the expression \[16\] gives

$$t_1 \equiv R_{abc} R_{ij}^{abc} = 2(18\delta_{ij} + 18B^2\delta_{ij} + 15B^4\delta_{ij} + 6B^6\delta_{ij} + B^8\delta_{ij} \\
- 15B^2B^2B^i - 6B^4B^iB^j - B^6B^iB^j)/(1 + B^2)^5$$

$$t_2 \equiv -(3/2)R_{abc} H_{j} H_{ij}^{abc} = 3(-54\delta_{ij} - 63B^2\delta_{ij} - 33B^4\delta_{ij} - 9B^6\delta_{ij} - B^8\delta_{ij} \\
- 27B^2B^2 - 9B^2B^2B^j + 3B^4B^iB^j + B^6B^iB^j)/(1 + B^2)^5$$

$$t_3 \equiv -(1/2)R_{abc} H_{iab} H_{jrs} = -2(27\delta_{ij} + 18B^2\delta_{ij} + 3B^4\delta_{ij} \\
+ 54B^2B^j + 72B^2B^iB^j + 3B^4B^iB^j + 10B^6B^iB^j + B^8B^iB^j)/(1 + B^2)^5$$

$$t_4 \equiv (1/8)(H^4)_{ij} = (81\delta_{ij} + 108B^2\delta_{ij} + 54B^4\delta_{ij} + 12B^6\delta_{ij} + B^8\delta_{ij} \\
+ 81B^2B^j + 108B^2B^iB^j + 54B^4B^iB^j + 12B^6B^iB^j + B^8B^iB^j)/(1 + B^2)^5$$

$$t_5 \equiv (1/4) D_l H_{iab} D^l H_{iab} = 8B^2(\delta_{ij} + B^iB^j)/(1 + B^2)^5$$

$$t_6 \equiv (1/12) D_l H_{abc} D_j H_{abc} = 8B^2B^j/(1 + B^2)^4$$

$$t_7 \equiv (1/8) H_{iab} H_{j} (H^2)_{ij} = (81\delta_{ij} + 108B^2\delta_{ij} + 54B^4\delta_{ij} + 12B^6\delta_{ij} + B^8\delta_{ij} \\
+ 81B^2B^j + 108B^2B^iB^j + 54B^4B^iB^j + 12B^6B^iB^j + B^8B^iB^j)/(1 + B^2)^5$$

$$t_8 \equiv R_{iabj}(H^2)_{ij} = 2(-54\delta_{ij} - 63B^2\delta_{ij} - 33B^4\delta_{ij} - 9B^6\delta_{ij} - B^8\delta_{ij} \\
- 27B^2B^j - 9B^2B^iB^j + 3B^4B^iB^j + B^6B^iB^j)/(1 + B^2)^5$$

$$t_9 \equiv H_{kl}(D_j (H_{iab}) = 2D^k(D_m(H_{iab}) = 24(-3\delta_{ij} + 2B^2\delta_{ij} + B^4\delta_{ij} \\
- 3B^2B^j + 2B^2B^iB^j + B^4B^iB^j)/(1 + B^2)^5$$
Thus, the final expression for $\beta^{(2)}_{ij}$ is

$$
\beta^{(2)}_{ij} = \frac{1}{4\pi^2} \left( \frac{1}{2} (t_1 + t_2 + t_3 + t_4 + t_5 + t_7) + p_1 \left( t_8 - \frac{4}{3} t_2 - 2 t_3 - 4 t_5 \right) + p_2 t_9 \right)
$$

$$
= \left( \frac{-477 + 1728p_1 - 576p_2 - 400B^2 + 1328p_1 B^2 + 384p_2 B^2 - 138B^4}{27 + 90B} \right) 
+ \frac{1}{2} \frac{1}{2} \left( -5 - 112B^2 + 18B^4 + 8B^6 + B^8 \right) / 8(1 + B^2)^5,
$$

$$
\hat{\beta}^{(2)}_{ij} = \hat{\Gamma}^{l}_{ij} = \Gamma^{l}_{ij} - \frac{1}{2} H^{l}_{ij}.
$$

As noted in [13], for $p_1 = 1/4$, $p_2 = 0$, the expression [16] coincides with the symmetric part of

$$
\hat{\beta}^{(2)}_{ij} = \frac{1}{2\pi^2} \left( \hat{\Gamma}^{abc}_{(j} \hat{\Gamma}_{i)abc} - \frac{1}{2} \hat{\Gamma}^{bca}_{(j} \hat{\Gamma}_{i)abc} + \frac{1}{2} \hat{R}_{a(ij)b}(H^2)^{ab} \right) 
\equiv \frac{1}{4\pi^2} (\hat{t}_1 + \hat{t}_2 + \hat{t}_3),
$$

where $\hat{R}_{bca}$ is given by its standard formula twisted with the change

To cross-check the validity of calculations, we have also evaluated the expression [18]:

$$
\hat{t}_1 = ((9 - 32B^2 + 42B^4 + 24B^6 + 5B^8) \delta_{ij} + (24 - 48B^2 - 8B^4) \epsilon_{ijk} B^k 
+ (69 + 16B^2 + 18B^4 + 8B^6 + B^8) B^i B^j) / 8(1 + B^2)^5,
$$

$$
\hat{t}_2 = ((9 - 32B^2 + 42B^4 + 24B^6 + 5B^8) \delta_{ij} + (24 - 48B^2 - 8B^4) \epsilon_{ijk} B^k 
+ (-59 - 112B^2 + 18B^4 + 8B^6 + B^8) B^i B^j) / 8(1 + B^2)^5,
$$

$$
\hat{t}_3 = (-27 + 18B^2 + 12B^4 + 6B^6 + B^8) \delta_{ij} - (36 + 24B^2 + 4B^4) \epsilon_{ijk} B^k 
+ (27 + 90B^2 + 60B^4 + 14B^6 + B^8) B^i B^j) / 2(1 + B^2)^5,
$$

$$
\hat{\beta}^{(2)}_{ij} = \left( \frac{-45 - 68B^2 + 18B^4 + 12B^6 + 3B^8}{2} \delta_{ij} 
- (48 + 96B^2 + 16B^4) \epsilon_{ijk} B^k 
+ (59 + 132B^2 + 138B^4 + 36B^6 + 3B^8) B^i B^j) / 32\pi^2(1 + B^2)^5.
$$

Its symmetric part is seen to coincide with [17] for $p_1 = 1/4$, $p_2 = 0$. The functions $A_{21}(z)$ and $C_{21}(z)$ turn out to be

$$
A_{21}(z) = (477 - 1728p_1 + 576p_2 + 400z - 1328p_1 z - 384p_2 z 
+ 138z^2 - 624p_1 z^2 - 192p_2 z^2 + 24z^3 - 144p_1 z^3
$$
\[ C_{21}(z) = \frac{(481 - 2160p_1 + 576p_2 + 416z - 2192p_1z - 384p_2z}{(1 + z)^5} + 162z^2 - 1200p_1z^2 - 192p_2z^2 + 40z^3 - 304p_1z^3 + 5z^4 - 32p_1z^4)!64\pi^2(1 + z)^5. \]

From the Eq. 14 which includes the function \( A_{21}(z) \), one finds

\[ f_{21}(z) = \frac{(477 + 8\lambda_{21} - 1728p_1 + 576p_2 + 400z + 32\lambda_{21}z - 1328p_1z - 384p_2z}{(1 + z)^5} + 138z^2 + 48\lambda_{21}z^2 - 624p_1z^2 - 192p_2z^2 + 24z^3 + 32\lambda_{21}z^3 - 144p_1z^3 + z^4 + 8\lambda_{21}z^4 - 16p_1z^4)!128\pi^2(1 + z)^3. \]

Substituting this into the Eq. 14, where \( C_{21}(z) \) enters, and taking the derivative \( f_{21}'(z) \) under assumption \( \lambda_{21} = \text{const} \), we get

\[ \lambda_{21}(z) = \frac{(-1589 + 6416p_1 - 3648p_2 - 660z + 1920p_1z}{(1 + z)^5} + 384p_2z - 6z^2 - 96p_1z^2 + 192p_2z^2 + 28z^3 - 256p_1z^3 + 3z^4 - 48p_1z^4)!/(192\pi^2(1 + z)^4) \]

\[ \lambda_{21}(z) = \frac{-1589 + 6416p_1 - 3648p_2}{192\pi^2} + \frac{89 - 371p_1 + 234p_2}{3\pi^2} \]

\[ + \frac{-1657 + 7048p_1 - 4728p_2}{24\pi^2} \cdot z^2 + \frac{1577 - 6812p_1 + 4752p_2}{12\pi^2} \cdot z^3 + O(z^4). \]

The condition of vanishing of the coefficients before \( z \) and \( z^2 \) gives:

\[ p_1 = \frac{5509}{17476}, \quad p_2 = \frac{4175}{34952}. \]

Under that, the coefficient before \( z^3 \) is non-zero and equals to \( 997/8738\pi^2 \).

Thus, no choice of \( p_1 \) and \( p_2 \) might make \( \lambda_{21} \) constant. It means that the \( B \)-field action corresponds to a theory with multiple (finite or infinite) number of coupling constants, and hence can not be equivalent to the model of principal chiral fields where this number is one.

## 5 3-Dimensional Model

In this section we calculate \( 2 \rightarrow 2 \) Born scattering amplitude for the three dimensional Freedman–Townsend model. It is described by the action

\[ S = \frac{1}{2} \int \left( B^{i\mu}_{\mu} \epsilon^{\mu\nu\lambda} F_{\nu\lambda}^i + A^{i\mu}_{\mu} A_{i\mu} \right) d^3x, \]

where \( \epsilon^{\mu\nu\lambda} \) is the totally antisymmetric tensor, \( \epsilon^{012} = 1 \).
This is a gauge theory. The gauge transformations read
\[ \delta A^i_\mu = 0, \quad \delta B^i_\mu = (\partial_\mu \delta^i_j + f^{ikj} A^k_\mu) \xi^j. \]

Formally, the model is again equivalent to the model of principal chiral fields. The simplest way to convince oneself in it—is to evaluate the path integral in a definite gauge. Choosing for example \( B^i_2 = 0 \), which requires the Faddeev-Popov determinant \( \Delta = \text{Det} | \partial_2 \delta^i_j + f^{ijk} A^k | \), the integral over \( B^i_\mu \) gives
\[ \delta (G_{12}) \delta (G_{02}). \]

After the change of integration variables \( A^i_\mu \rightarrow \phi^i_0, a^i_0, a^i_1 \)
\[ A^i_2 = \Lambda^j_i \partial_2 \phi^j, \quad A^i_0 = \Lambda^j_i \partial_0 \phi^j + a^i_0, \quad A^i_1 = \Lambda^j_i \partial_1 \phi^j + a^i_1, \]
we get, in analogy with section 2A Formal Approach section 2, the same expression [3] for \( Z \). Correspondingly, the Born scattering amplitude of two particles \( \phi \) is given by Eq. 7.

On the contrary, integrating over \( A^i_\mu \) first will produce the theory of \( B^i_\mu \) fields with the action
\[ S_0(B) = -\frac{1}{2} \int d^3 x \epsilon^{\mu\lambda\sigma} \partial_\lambda B^i_\mu N^{-1}_{i\mu|j\nu} \epsilon^{\nu\gamma\delta} \partial_\gamma B^j_\delta \]
\[ = \frac{1}{2} \int d^3 x \{ B^i_\mu (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) B^j_\nu - \frac{1}{2} \epsilon^{\mu\nu\lambda} f^{abc} B^i_\nu B^j_\lambda B^k_\sigma \}
\[ - f^{ijk} B^j_\mu B^k_\nu f^{i\mu\nu} B^l_\sigma B^n_\lambda \} + O(B^5), \]
\[ N^i_{\mu|j\nu} = \eta_{\mu\nu} \delta^i_j + \epsilon^{\mu\nu\lambda} f^{ijk} B^l_\lambda, \quad B^i_\mu \equiv \partial_\mu B^i_\nu - \partial_\nu B^i_\mu. \]

Of course, this theory is still a gauge theory; to gauge-fix it we add
\[ -\frac{1}{2} \int d^3 x \partial_\mu B^i_\mu \partial_\nu B^i_\nu \]
to the action. The ghosts action doesn’t matter for what follows. The scattering amplitude is expressed through the vertex function \( \Gamma^{ijkl}_{\mu\nu\lambda\sigma} \) as
\[ A = \xi_\mu(p_1) \xi_\nu(p_2) \xi_\lambda(p_3) \xi_\sigma(p_4) \Gamma^{ijkl}_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4), \]
where all \( p_n \) are taken on mass shell, \( p_n^2 = 0, \sum p_n = 0 \), and \( \xi_\mu(p) \) is the polarization vector for a physical state
\[ \xi_\mu = \frac{1}{p_0} \epsilon^{\mu\nu} p_\nu. \]

The diagram of the type 3 gives the following contribution to the scattering amplitude
\[ A_4 = \frac{i}{P_1 P_2 P_3 P_4} \{ f^{ijmn} f^{klm} [(p_1 p_2)^2 (p_1 p_4 + p_2 p_3 - p_1 p_3 - p_2 p_4) \} \]

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\begin{align*}
+ (p_1 p_3) & \left( p_{20}^2 p_{20} p_{40} + p_{10} p_{30} p_{40}^2 + p_{20} p_{20}^2 p_{40} + p_{10} p_{20}^2 p_{30} - p_{20}^2 p_{20}^2 - p_{20}^2 p_{40} \\
& - p_{20}^2 p_{20} p_{30} - p_{20} p_{20} p_{30}^2 - p_{10} p_{20}^2 p_{40} - p_{10} p_{20}^2 p_{40} \right) \\
- (p_1 p_4) & \left( p_{10} p_{20} p_{40}^2 + p_{10} p_{20} p_{30} + p_{20} p_{10} p_{40}^2 + p_{10} p_{30} p_{40}^2 - p_{20}^2 p_{20} - p_{20}^2 p_{40} \\
& - p_{10}^2 p_{20} p_{30} - p_{10}^2 p_{20} p_{40} - p_{10} p_{20}^2 p_{40} - p_{10} p_{20}^2 p_{40} \right) \\
+ (j, 2 \leftrightarrow k, 3) + (j, 2 \leftrightarrow l, 4) \right\}.
\end{align*}

The diagrams of the type 8 contribute as

\begin{align*}
A_{33} &= \frac{i}{p_{10} p_{20} p_{30} p_{40}} \left\{ f^{ij} f^{klm} \left[ (p_1 p_2)^2 (p_{10} p_{40} + p_{20} p_{30} - p_{10} p_{30} - p_{20} p_{40}) \\
& + (p_1 p_4) (p_{30} p_{40} - p_{10} p_{20} - p_{10}^2 - p_{20}^2) (p_{10} p_{40} + p_{20} p_{30}) - (p_1 p_3) (p_{30} p_{40} \\
& - p_{10} p_{20} - p_{20}^2) p_{10} p_{30} + p_{20} p_{40} - \frac{(p_1 p_3) - (p_2 p_4)}{2} p_{10} p_{20} p_{30} p_{40} \right] \\
& + (j, 2 \leftrightarrow k, 3) + (j, 2 \leftrightarrow l, 4) \right\}.
\end{align*}

Summing up these expressions, we finally get

\begin{align*}
A &= A_4 + A_{33} = \frac{i}{6} \left[ f^{12i} f^{34i} (p_1 p_3 - p_1 p_4) + f^{13i} f^{42i} (p_1 p_4 - p_1 p_2) \\
& + f^{14i} f^{23i} (p_1 p_2 - p_1 p_3) \right],
\end{align*}

which is nothing but a scattering amplitude in the theory of the principal chiral fields (see [5]).

This example confirms the hypothesis that the two dimensional anomaly discussed above is caused by severe infrared singularities of massless fields.

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