QUANTUM COMPUTATIONAL LOGICS. A SURVEY.

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Abstract. Quantum computation has suggested new forms of quantum logic, called quantum computational logics ([CDCGL02]). The basic semantic idea is the following: the meaning of a sentence is identified with a quregister, a system of qubits, representing a possible pure state of a compound quantum system. The generalization to mixed states, which might be useful to analyse entanglement-phenomena, is due to Gudder ([Gu02]). Quantum computational logics represent non-standard examples of unsharp quantum logic, where the non-contradiction principle is violated, while conjunctions and disjunctions are strongly non-idempotent. In this framework, any sentence \( \alpha \) of the language gives rise to a quantum tree: a kind of quantum circuit that transforms the quregister associated to the atomic subformulas of \( \alpha \) into the quregister associated to \( \alpha \).

1. Introduction

Quantum computation has suggested new forms of quantum logic that have been called quantum computational logics. The main difference between orthodox quantum logic (first proposed by Birkhoff and von Neumann [BVN36]) and quantum computational logics concerns a basic semantic question: how to represent the meanings of the sentences of a given language? The answer given by Birkhoff and von Neumann is the following: the meanings of the elementary experimental sentences of quantum theory (QT) have to be regarded as determined by convenient sets of states of quantum objects. Since these sets should satisfy some special closure conditions, it turns out that, in the framework of orthodox quantum logic, sentences can be adequately interpreted as closed subspaces of the Hilbert space associated to the physical systems under investigation. The answer given in the framework of quantum computational logics is quite different: meanings of sentences are represented by information quantities, a kind of abstract objects that are described in the framework of quantum information theory.

2. From classical to quantum information

As is well known, the unit of measurement in classical information theory is the bit: one bit measures the information quantity that can be either

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transmitted or received whenever one chooses one element from a set consisting of two distinct elements (say, from the set \( \{0, 1\} \)). From an intuitive point of view, both the objects 0 and 1 can be imagined as a well determined state of a classical physical system (for instance, the state of a tape cell in a given machine).

Let us now refer to a quantum computational context, where information is supposed to be elaborated and transmitted by means of a quantum system. According to the standard axiomatization of QT, the pure states of our system are mathematically represented by unit vectors in a convenient Hilbert space \( \mathcal{H} \). Let us refer to the simplest situation, where our Hilbert space \( \mathcal{H} \) has dimension 2; hence \( \mathcal{H} = \mathbb{C}^2 \). In such a case \( \mathcal{H} \) will have a basis consisting of two unit elements, and any vector of the space will be representable as a superposition of the two basis-elements. In quantum computation, it is customary to use Dirac’s notation. Accordingly, the vectors of \( \mathcal{H} \) are indicated by \( |\psi\rangle \), \( |\varphi\rangle \),... ; while the basis-elements are denoted by \( |0\rangle \), \( |1\rangle \). As a consequence, for any unit vector \( |\psi\rangle \) we will have:

\[
|\psi\rangle = a_0|0\rangle + a_1|1\rangle,
\]

where the coefficients \( a_0, a_1 \) are complex numbers (also called amplitudes) such that:

\[
|a_0|^2 + |a_1|^2 = 1.
\]

Let us now try and apply such a formalism to a quantum information theory. The basic idea is to generalize the concept of bit, by introducing the notion of qubit or quantum bit. A qubit is a unit vector in the Hilbert space \( \mathbb{C}^2 \). Thus any qubit will have the form \( |\psi\rangle = a_0|0\rangle + a_1|1\rangle \). The interpretation is determined by an axiom of QT that is usually called the Born rule. Suppose that (like in the classical case) the pure states \( |0\rangle \) and \( |1\rangle \) represent two maximal (and precise) pieces of information. Then the superposition-state \( |\psi\rangle = a_0|0\rangle + a_1|1\rangle \) will represent an information that involves a certain degree of uncertainty. In particular, the number \( |a_0|^2 \) will correspond to the probability-value of the information described by the basic state \( |0\rangle \); while \( |a_1|^2 \) will correspond to the probability-value of the information described by the basic state \( |1\rangle \).

In this context, it makes sense to imagine \( |\psi\rangle \) as an epistemic state that stocks two precise pieces of information in parallel: the information \( |0\rangle \) and the information \( |1\rangle \).

Let us now consider a situation characterized by many bits or qubits. As is well known, in classical information theory, a system consisting of \( n \) bits is naturally represented by a sequence of \( n \) elements belonging to the set \( \{0, 1\} \) (i.e. as an element of the set \( \{0, 1\}^n \)). In the framework of quantum computation, it is convenient to adopt the tensor product formalism, which is used in quantum theory in order to represent compound physical systems. Suppose a two-particle quantum system:

\[
\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2.
\]
For instance, \( S_1 \) and \( S_2 \) might correspond respectively to the two electrons in a given helium atom. In such a case, the Hilbert space \( \mathcal{H}^S \) associated to \( S \) will be the tensor product \( \mathcal{H}^{S_1} \otimes \mathcal{H}^{S_2} \) of the two Hilbert spaces \( \mathcal{H}^{S_1} \) and \( \mathcal{H}^{S_2} \), that are associated to \( S_1 \) and \( S_2 \), respectively. Thus any pure state of \( S \) will be a unit vector in the space \( \mathcal{H}^S \).

A particularly interesting case is represented by those vectors \( |\psi\rangle \) of \( \mathcal{H}^S \) that can be expressed as the tensor product of two vectors \( |\psi_1\rangle \) and \( |\psi_2\rangle \), belonging to \( \mathcal{H}^{S_1} \) and \( \mathcal{H}^{S_2} \), respectively. In other words:

\[
|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle.
\]

In such cases, one usually speak of factorized states. It is worthwhile recalling that not all vectors of \( \mathcal{H}^S \) can be expressed in such a simple form.

How to represent, in this framework, a system consisting of \( n \) qubits? It seems quite natural to describe our system as the pure state of a compound physical system consisting of \( n \) quantum objects. On this basis our \( n \)-qubit system can be identified with a unit vector of the product space

\[
\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2.
\]

Instead of \( \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \) we will also write: \( \otimes^n \mathbb{C}^2 \). Particularly interesting examples will be represented by the vectors of \( \otimes^n \mathbb{C}^2 \) whose form is:

\[
|x_1\rangle \otimes \ldots \otimes |x_n\rangle,
\]

where each \( |x_i\rangle \) is an element of the basis of \( \mathbb{C}^2 \) (i.e., \( |x_i\rangle = |0\rangle \) or \( |x_i\rangle = |1\rangle \)). One can prove that the set of all vectors having this form represents a basis for the product space \( \otimes^n \mathbb{C}^2 \).

How can we deal with the concept of quantum computation, in this framework? The basic idea is to describe a computation by means of that kind of process that corresponds to the dynamic evolution of a quantum system. Suppose a physical system \( S \), whose pure state at time \( t_0 \) is the vector \( |\psi(t_0)\rangle \) (where \( |\psi(t_0)\rangle \) belongs to the Hilbert space \( \mathcal{H}^S \) associated to \( S \)). Owing to Schrödinger’s equation, for any time \( t \) (where either \( t \leq t_0 \) or \( t_0 \leq t \)), there exists an operator \( U_{[t_0,t]} \) that determines the state of the system at time \( t \) as a function of the state of the system at time \( t_0 \). In other words:

\[
|\psi(t)\rangle = U_{[t_0,t]} |\psi(t_0)\rangle.
\]

Any operator \( U_{[t_0,t]} \) of this kind is unitary. Hence, our operator preserves the length of the vectors and the orthogonality relation. Further it is reversible: one can go from \( |\psi(t_0)\rangle \) to \( |\psi(t)\rangle \) and viceversa, without any dissipation of information.

On this ground, it makes sense to represent a quantum computation by means of convenient unitary operators assuming arguments and values in particular sets of qubit systems. Since qubits are generally superposition-states, one obtains some typical parallel configurations.
3. Qubits, Quregisters and Qumixs

As we have seen, qubits and qubit-systems (also called quregisters) correspond to pure states, which are maximal pieces of information of the observer about the quantum system under investigation. In other words, one is dealing with a kind of information that cannot be consistently extended to a richer knowledge (expressed in the same language). In many concrete situations it may be interesting to consider also mixed states (or mixtures), describing pieces of information that are not generally maximal. According to the standard axiomatization of QT such states are mathematically represented by density operators $\rho$ of the Hilbert space $\mathcal{H}$ (associated to the system). Any pure state $|\psi\rangle$ corresponds to a limit-case of a density operator: the projection $P|\psi\rangle$ onto the one-dimensional closed subspace determined by the vector $|\psi\rangle$. Representing quantum information by density operators turns out to be important in order to deal with entanglement-phenomena, which play a fundamental role in teleportation and in quantum cryptography.

We will now sum up some basic formal definitions of quantum computation. Consider the two–dimensional Hilbert space $\mathbb{C}^2$ (where any vector $|\psi\rangle$ is represented by a pair of complex numbers). Let $\mathcal{B}^{(1)} = \{\langle 0|, |1\rangle\}$ be the canonical orthonormal basis for $\mathbb{C}^2$, where $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$.

**Definition 3.1. (Qubit).**
A qubit is a unit vector $|\psi\rangle$ of the Hilbert space $\mathbb{C}^2$.

As we have seen, from an intuitive point of view, any qubit $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ (with $|a_0|^2 + |a_1|^2 = 1$) can be regarded as an uncertain piece of information, where the answer NO has probability $|a_0|^2$, while the answer YES has probability $|a_1|^2$. The two basis-elements $|0\rangle$ and $|1\rangle$ are taken as encoding the classical bit-values 0 and 1, respectively. From a semantic point of view, they can be also regarded as the classical truth-values Falsity and Truth.

An $n$-qubit system (or $n$-quregister) is represented by a unit vector in the $n$-fold tensor product Hilbert space $\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ (where $\otimes^1 \mathbb{C}^2 := \mathbb{C}^2$). We will use $x, y, \ldots$ as variables ranging over the set $\{0,1\}$. At the same time, $|x\rangle, |y\rangle, \ldots$ will range over the basis $\mathcal{B}^{(1)}$. Any factorized unit vector $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ of the space $\otimes^n \mathbb{C}^2$ will be called an $n$–configuration (which can be regarded as a quantum realization of a classical bit sequence of length $n$). Instead of $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ we will simply write $|x_1, \ldots, x_n\rangle$. Recall that the dimension of $\otimes^n \mathbb{C}^2$ is $2^n$, while the set of all $n$–configurations $\mathcal{B}^{(n)} = \{|x_1, \ldots, x_n\rangle : x_i \in \{0,1\}\}$ is an orthonormal basis for the space $\otimes^n \mathbb{C}^2$. We will call this set a computational basis for the $n$–quregisters. Since any string $x_1, \ldots, x_n$ represents a natural number $j \in [0, 2^n - 1]$ (where $j = 2^{n-1}x_1 + 2^{n-2}x_2 + \ldots + x_n$), any unit vector of $\otimes^n \mathbb{C}^2$ can be shortly
expressed in the following form: \( \sum_{j=0}^{n-1} c_j \|j\rangle \), where \( c_j \in \mathbb{C} \), \( \|j\rangle \) is the \( n \)-configuration corresponding to the number \( j \) and \( \sum_{j=0}^{n-1} |c_j|^2 = 1 \).

We will indicate by \( \mathcal{R}(\otimes^n \mathbb{C}^2) \) the set of all quregisters of \( \otimes^n \mathbb{C}^2 \), while \( \mathcal{R} \) will represent the set \( \bigcup_{n=1}^{\infty} \mathcal{R}(\otimes^n \mathbb{C}^2) \). The set \( \mathcal{R}(\otimes^1 \mathbb{C}^2) \) of all qubits will be shortly indicated by \( \mathcal{Q} \).

Consider now the two following sets of natural numbers:

\[ C_1^{(n)} := \{ i : \|i\rangle = |x_1, \ldots, x_n\rangle \text{ and } x_n = 1 \} \]

and

\[ C_0^{(n)} := \{ i : \|i\rangle = |x_1, \ldots, x_n\rangle \text{ and } x_n = 0 \} \].

Let us refer to a generic unit vector of the space \( \otimes^n \mathbb{C}^2 \):

\[ |\psi\rangle = \sum_{i=0}^{2^n-1} a_i \|i\rangle. \]

We obtain:

\[ |\psi\rangle = \sum_{i \in C_0^{(n)}} a_i \|i\rangle + \sum_{j \in C_1^{(n)}} a_j \|j\rangle. \]

Let \( P_1^{(n)} \) and \( P_0^{(n)} \) be the projections onto the span of \( \{ \|i\rangle : i \in C_1^{(n)} \} \) and \( \{ \|i\rangle : i \in C_0^{(n)} \} \), respectively. Clearly, \( P_1^{(n)} + P_0^{(n)} = I^{(n)} \), where \( I^{(n)} \) is the identity operator of \( \otimes^n \mathbb{C}^2 \). Apparently, \( P_1^{(n)} \) and \( P_0^{(n)} \) are density operators iff \( n = 1 \). Let \( k_n = \frac{1}{2^{n-1}} \) be the normalization coefficient such that \( k_n P_1^{(n)} \) and \( k_n P_0^{(n)} \) are density operators. From an intuitive point of view, \( k_n P_1^{(n)} \) can be regarded as a privileged information corresponding to the Truth, while \( k_n P_0^{(n)} \) corresponds to the Falsity. In particular, \( P_1^{(1)} \) represents the bit \( |1\rangle \), while \( P_0^{(1)} \) represents the bit \( |0\rangle \). Let \( \mathcal{D}(\otimes^n \mathbb{C}^2) \) be the set of all density operators of \( \otimes^n \mathbb{C}^2 \) and let \( \mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}(\otimes^n \mathbb{C}^2) \).

**Definition 3.2.** (Qumix).

A qumix is a density operator in \( \mathcal{D} \).

Needless to say, quregisters correspond to particular qumixs that are pure states (i.e. projections onto one-dimensional closed subspaces of a given \( \otimes^n \mathbb{C}^n \)). For any quregister \( |\psi\rangle \), we will indicate by \( P_1 |\psi\rangle \) the pure density operator represented by the projection onto the one-dimensional subspace spanned by the vector \( |\psi\rangle \). The set of all pure density operators will be indicated by \( \mathcal{D}_R \).

Recalling the Born rule, we can now define the probability-value of any qumix.

**Definition 3.3.** (Probability of a qumix).

For any qumix \( \rho \in \mathcal{D}(\otimes^n \mathbb{C}^n) \):

\[ p(\rho) = \text{tr}(P_1^{(n)} \rho). \]
From an intuitive point of view, \( p(\rho) \) represents the probability that the information stocked by the qumix \( \rho \) is true. In the particular case where \( \rho \) is a pure density operator \( P|\psi\rangle \), determined by the qubit \( |\psi\rangle = a_0|0\rangle + a_1|1\rangle \), we obtain that \( p(\rho) = |a_1|^2 \).

For any quregister \( |\psi\rangle \), we will write \( p(|\psi\rangle) \) instead of \( p(P|\psi\rangle) \).

### 4. Quantum Logical Gates

We will now introduce some examples of quantum logical gates. Generally, a quantum logical gate can be described as a unitary operator, assuming arguments and values in a product-Hilbert space \( \otimes^n \mathbb{C}^2 \). First of all we will study the so called Petri-Toffoli gate (\cite{Pe67} and \cite{To80}). It will be expedient to start by analysing the simplest case, where our Hilbert space has the form:

\[
\otimes^3 \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.
\]

In such a case, the Petri-Toffoli gate transforms the vectors of \( \otimes^3 \mathbb{C}^2 \) into vectors of \( \otimes^3 \mathbb{C}^2 \). In order to stress that our operator is defined on the product space \( \otimes^3 \mathbb{C}^2 \), we will indicate it by \( T(1,1,1) \). Since we want to define a unitary operator, it will be sufficient to determine its behaviour for the elements of the basis, having the form \( |x\rangle \otimes |y\rangle \otimes |z\rangle \) (where \( x, y, z \in \{0, 1\} \)).

**Definition 4.1.** *(The Petri-Toffoli gate \( T^{(1,1,1)} \)).*

The Petri-Toffoli gate \( T^{(1,1,1)} \) is the linear operator \( T^{(1,1,1)} : \otimes^3 \mathbb{C}^2 \rightarrow \otimes^3 \mathbb{C}^2 \) that is defined for any element \( |x\rangle \otimes |y\rangle \otimes |z\rangle \) of the basis as follows:

\[
T^{(1,1,1)}(|x\rangle \otimes |y\rangle \otimes |z\rangle) = |x\rangle \otimes |y\rangle \otimes |xy \oplus z\rangle,
\]

where \( \oplus \) represents the sum modulo 2.

From an intuitive point of view, it seems quite natural to “see” the gate \( T^{(1,1,1)} \) as a kind of self-reversible “truth-table” that transforms triples of zeros and ones into triples of zeros and ones. The “table” we obtain is the following:

\[
\begin{align*}
|0, 0, 0\rangle &\mapsto |0, 0, 0\rangle \\
|0, 0, 1\rangle &\mapsto |0, 0, 1\rangle \\
|0, 1, 0\rangle &\mapsto |0, 1, 0\rangle \\
|0, 1, 1\rangle &\mapsto |0, 1, 1\rangle \\
|1, 0, 0\rangle &\mapsto |1, 0, 0\rangle \\
|1, 0, 1\rangle &\mapsto |1, 0, 1\rangle \\
|1, 1, 0\rangle &\mapsto |1, 1, 1\rangle \\
|1, 1, 1\rangle &\mapsto |1, 1, 0\rangle
\end{align*}
\]

In the first six cases, \( T^{(1,1,1)} \) behaves like the identity operator; in the last two cases, instead, our gate transforms the last element of the triple into the opposite element (0 is transformed into 1 and 1 transformed into 0).
One can easily show that $T_{(1,1,1)}$ has been well defined for our aims: one is dealing with an operator that is not only linear but also unitary.

By using $T_{(1,1,1)}$, we can introduce a convenient notion of conjunction. Our conjunction, which will be indicated by $\text{And}$, is characterized as a function whose arguments are pairs of vectors in $\mathbb{C}^2$ and whose values are vectors of the product space $\otimes^3\mathbb{C}^2$.

**Definition 4.2.** $(\text{And})$.
For any $|\varphi\rangle \in \mathbb{C}^2$ and any $|\psi\rangle \in \mathbb{C}^2$:

$$\text{And} (|\varphi\rangle, |\psi\rangle) := T_{(1,1,1)}(|\varphi\rangle \otimes |\psi\rangle \otimes |0\rangle).$$

Clearly, the qubit $|0\rangle$ behaves here as an “ancilla”.

Let us check that $\text{And}$ represents a good generalization of the corresponding classical truth-function. For the arguments $|0\rangle$ and $|1\rangle$ we will obtain the following “truth-table”:

$$\begin{align*}
(0,0) &\mapsto T_{(1,1,1)}((0) \otimes (0) \otimes (0)) = (0) \otimes (0) \otimes (0) \\
(0,1) &\mapsto T_{(1,1,1)}((0) \otimes (1) \otimes (0)) = (0) \otimes (1) \otimes (0) \\
(1,0) &\mapsto T_{(1,1,1)}((1) \otimes (0) \otimes (0)) = (1) \otimes (0) \otimes (0) \\
(1,1) &\mapsto T_{(1,1,1)}((1) \otimes (1) \otimes (0)) = (1) \otimes (1) \otimes (1)
\end{align*}$$

One immediately realizes the difference with respect to the classical case. The classical truth-table represents a typical irreversible transformation:

$$\begin{align*}
(0,0) &\mapsto 0 \\
(0,1) &\mapsto 0 \\
(1,0) &\mapsto 0 \\
(1,1) &\mapsto 1
\end{align*}$$

The arguments of the function determine the value, but not the other way around. As is well known, irreversibility generally brings about dissipation of information. Mathematically, however, any Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}^m$ can be transformed into a reversible function $\hat{f} : \{0,1\}^n \times \{0,1\}^m \rightarrow \{0,1\}^n \times \{0,1\}^m$ in the following way:

$$\forall u \in \{0,1\}^n \forall v \in \{0,1\}^m : \hat{f}(u, v) = (u, v \oplus f(u)),$$

where $\oplus$ is the sum modulo 2 pointwise defined. The function that is obtained by making reversible the irreversible classical “and” corresponds to the Petri-Toffoli gate. The classical “and” is then recovered by fixing the third input bit to 0.

Accordingly, the three arguments $(0,0)$, $(0,1)$, $(1,0)$ turn out to correspond to three distinct values, represented by the triples $(0,0,0)$, $(0,1,0)$, $(1,0,0)$. The price we have paid in order to obtain a reversible situation is the increasing of the complexity of our Hilbert space. The function $\text{And}$
associates to pairs of arguments, belonging to the two-dimensional space $\mathbb{C}^2$, values belonging to the space $\otimes^3 \mathbb{C}^2$ (whose dimension is 2^3).

All this happens in the simplest situation, when one is only dealing with elements of the basis (in other words, with precise pieces of information). Let us examine the case where the function $\text{And}$ is applied to arguments that are superpositions of the basis-elements in the space $\mathbb{C}^2$. Consider the following qubit pair:

\[ |\psi\rangle = a_0 |0\rangle + a_1 |1\rangle,\quad |\varphi\rangle = b_0 |0\rangle + b_1 |1\rangle. \]

By applying the definitions of $\text{And}$ and of $T^{(1,1,1)}$, we obtain:

\[ \text{And}(|\psi\rangle, |\varphi\rangle) = a_1 b_1 |1,1,1\rangle + a_1 b_0 |1,0,0\rangle + a_0 b_1 |0,1,0\rangle + a_0 b_0 |0,0,0\rangle. \]

This result suggests a quite natural logical interpretation. The four basis-elements that occur in our superposition-vector correspond to the four cases of the truth-table for the classical conjunction:

\[ (1,1,1), (1,0,0), (0,1,0), (0,0,0). \]

However here, unlike the classical situation, each case is accompanied by a complex number, which represents a characteristic quantum amplitude. By applying the “Born rule” we will obtain the following interpretation:

\[ |a_1 b_1|^2 \text{ represents the probability-value that both the qubit-arguments are equal to } |1\rangle, \text{ and consequently their conjunction is } |1\rangle. \]

Similarly in the other three cases.

So far we have considered a very special situation, characterized by a Hilbert space having the form $\otimes^3 \mathbb{C}^2$. However, our procedure can be easily generalized. The Petri-Toffoli gate can be defined in any Hilbert space having the form:

\[ (\otimes^n \mathbb{C}^2) \otimes (\otimes^m \mathbb{C}^2) \otimes \mathbb{C}^2 (= \otimes^{n+m+1} \mathbb{C}^2). \]

**Definition 4.3.** *(The Petri-Toffoli gate $T^{(n,m,1)}$).*

The Petri-Toffoli gate $T^{(n,m,1)}$ is the linear operator

\[ T^{(n,m,1)} : (\otimes^n \mathbb{C}^2) \otimes (\otimes^m \mathbb{C}^2) \otimes \mathbb{C}^2 \to (\otimes^n \mathbb{C}^2) \otimes (\otimes^m \mathbb{C}^2) \otimes \mathbb{C}^2, \]

that is defined for any element $|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z\rangle$ of the computational basis of $\otimes^{n+m+1} \mathbb{C}^2$ as follows:

\[ T^{(n,m,1)}(|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z\rangle) = |x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |x_n y_m \oplus z\rangle, \]

where $\oplus$ represents the sum modulo 2.

On this basis one can immediately generalize our definition of $\text{And}$.

**Definition 4.4.** *(And).*

For any $|\varphi\rangle \in \otimes^n \mathbb{C}^2$ and any $|\psi\rangle \in \otimes^m \mathbb{C}^2$:

\[ \text{And}(|\varphi\rangle, |\psi\rangle) := T^{(n,m,1)}(|\varphi\rangle \otimes |\psi\rangle \otimes |0\rangle). \]
How to deal in this context with the concept of negation? A characteristic of quantum computation is the possibility of defining a plurality of negation-operations: some of them represent good generalizations of the classical negation. We will first consider the operator $\text{Not}^{(n)}$ that simply inverts the value of the last element of any configuration of the space $\otimes^n \mathbb{C}^2$. Thus, if $|x_1,\ldots,x_n\rangle$ is a vector of the computational basis $\mathcal{B}^{(n)}$, the result of the application of $\text{Not}^{(n)}$ to $|x_1,\ldots,x_n\rangle$ will be $|x_1,\ldots,x_{n-1},1-x_n\rangle$.

**Definition 4.5.** $\langle \text{Not}^{(n)} \rangle$.
The negation-gate is the linear operator $\text{Not}^{(n)}$ that is defined for any element $|x_1,\ldots,x_n\rangle$ of the computational basis of $\otimes^n \mathbb{C}^2$ as follows:

$$\text{Not}^{(n)}(|x_1,\ldots,x_n\rangle) = |x_1,\ldots,x_{n-1},1-x_n\rangle.$$ 

One can immediately check that $\text{Not}^{(n)}$ represents a good generalization of the classical truth-table. Consider the basis-elements $|0\rangle$ and $|1\rangle$ of the space $\mathbb{C}^2$. In such a case we obtain:

$$\text{Not}^{(1)}(|1\rangle) = |0\rangle;$$
$$\text{Not}^{(1)}(|0\rangle) = |1\rangle.$$ 

The matrix corresponding to $\text{Not}^{(1)}$ is:

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

Both the negation-gate and the Petri-Toffoli gate can be uniformly defined on the set $\mathfrak{R}$ of all quregisters in the expected way:

$$\text{Not}(|\psi\rangle) := \text{Not}^{(n)}(|\psi\rangle), \quad \text{if } |\psi\rangle \in \otimes^n \mathbb{C}^2;$$
$$T(|\psi\rangle,|\varphi\rangle,|\chi\rangle) := T^{(n,m,1)}(|\psi\rangle,|\varphi\rangle,|\chi\rangle), \quad \text{if } |\psi\rangle \in \otimes^n \mathbb{C}^2, |\varphi\rangle \in \otimes^m \mathbb{C}^2 \text{ and } |\chi\rangle \in \mathbb{C}^2.$$ 

Finally, how to introduce a reasonable disjunction? A gate $\text{Or}$ can be naturally defined in terms of $\text{And}$ and $\text{Not}$ via de Morgan.

**Definition 4.6.** $\langle \text{Or} \rangle$.
For any quregisters $|\varphi\rangle$ and $|\psi\rangle$:

$$\text{Or}(|\varphi\rangle,|\psi\rangle) = \text{Not} (\text{And} (\text{Not}(|\varphi\rangle), \text{Not}(|\psi\rangle))).$$

At first sight, $\text{And}$ and $\text{Or}$ may look as irreversible transformations. However, it is important to recall that, in this framework, $\text{And}(|\psi\rangle,|\varphi\rangle)$ should be regarded as a mere metalinguistic abbreviation for $T(|\psi\rangle,|\varphi\rangle,|0\rangle)$ (where $T$ is reversible). Similarly $\text{Or}$.

The quantum logical gates we have considered so far are, in a sense, “semiclassical”. A quantum logical behaviour only emerges in the case where our gates are applied to superpositions. When restricted to classical registers,
our gates turn out to behave as classical truth-functions. We will now investigate genuine quantum gates that may transform classical registers into quregisters that are superpositions.

One of the most significant genuine quantum gates is the square root of the negation, which will be indicated by $\sqrt{\text{Not}}$. As suggested by the name, the characteristic property of the gate $\sqrt{\text{Not}}$ is the following: for any quregister $|\psi\rangle$,

$$\sqrt{\text{Not}}(\sqrt{\text{Not}}(|\psi\rangle)) = \text{Not}(|\psi\rangle).$$

In other words: applying twice the square root of the negation “means” negating.

Interestingly enough, the gate $\sqrt{\text{Not}}$ has some natural physical models (and implementations). As an example, consider an idealized atom with a single electron and two energy levels: a ground state (identified with $|0\rangle$) and an excited state (identified with $|1\rangle$). By shining a pulse of light of appropriate intensity, duration and wavelength, it is possible to force the electron to change energy level. As a consequence, the state (bit) $|0\rangle$ is transformed into the state (bit) $|1\rangle$, and viceversa:

$$|0\rangle \mapsto |1\rangle; \quad |1\rangle \mapsto |0\rangle.$$

We have obtained a typical physical model for the gate $\text{Not}$. Now, by using a light pulse of half the duration as the one needed to perform the $\text{Not}$ operation, we effect a half-flip between the two logical states. The state of the atom after the half pulse is neither $|0\rangle$ nor $|1\rangle$, but rather a superposition of both states. As observed by Deutsch, Ekert, Lupacchini ([DEL00]):

Logicians are now entitled to propose a new logical operation $\sqrt{\text{Not}}$. Why? Because a faithful physical model for it exists in nature.

The physical models of the gate $\sqrt{\text{Not}}$ naturally suggest the following logical interpretation: $\sqrt{\text{Not}}$ represents a kind of “tentative negation”. By applying twice our “attempt” to negate, we obtain a full negation.

Interestingly enough, the gate $\sqrt{\text{Not}}$ seems to have also some linguistic “models”. For instance, consider the French language. Put:

$\sqrt{\text{Not}} = "\text{ne}" = "\text{pas}".$

We obtain:

$$\sqrt{\text{Not}}\sqrt{\text{Not}} = "\text{ne...pas}" = \text{Not}.$$ 

Needless to observe, our linguistic example is only a partial model of the gate $\sqrt{\text{Not}}$. In French, neither the expression “il ne pleut” nor the expression “il pleut pas” are grammatically correct sentences. And in the spoken language “il pleut pas” is simply used as an abbreviation for the correct “il ne pleut pas”. In quantum computation, instead, for any quregister $|\psi\rangle$, the vector $\sqrt{\text{Not}}(|\psi\rangle)$ is a quregister that is essentially different from the quregister $\text{Not}(|\psi\rangle)$.

Let us now give the mathematical definition of $\sqrt{\text{Not}}$. 
Definition 4.7. (The square root of the negation).
The square root of the negation on $\otimes^n \mathbb{C}^2$ is the linear operator $\sqrt{\text{Not}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $B^{(n)}$:

$$\sqrt{\text{Not}}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \left(\frac{1+i}{2}|x_n\rangle + \frac{1-i}{2}|1-x_n\rangle\right)$$

(where $i$ is the imaginary unit).

In other words, $\sqrt{\text{Not}}^{(n)}$ transforms the last element $x_n$ of any configuration $|x_1, \ldots, x_n\rangle$ into the element $\frac{1+i}{2}|x_n\rangle + \frac{1-i}{2}|1-x_n\rangle$. As a consequence, for the two bits $|0\rangle$ and $|1\rangle$ (“living in the space $\mathbb{C}^2$) we obtain:

$$\sqrt{\text{Not}}^{(1)}(|0\rangle) = \frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle;$$

$$\sqrt{\text{Not}}^{(1)}(|1\rangle) = \frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle.$$

One can easily show that $\sqrt{\text{Not}}^{(n)}$ is a unitary operator, which satisfies the following condition:

for any $|\psi\rangle \in \otimes^n \mathbb{C}^2$, $\sqrt{\text{Not}}^{(n)}(\sqrt{\text{Not}}^{(n)}(|\psi\rangle)) = \text{Not}^{(n)}(|\psi\rangle)$.

In other words, applying twice the square root of the negation means negating.

It turns out that the matrix associated to $\sqrt{\text{Not}}^{(1)}$ is

$$\begin{pmatrix}
\frac{1+i}{2} & \frac{1-i}{2} \\
\frac{1-i}{2} & \frac{1+i}{2}
\end{pmatrix}$$

Like the negation, also the square root of the negation can be uniformly defined on the set $\mathcal{R}$ of all quregisters:

$$\sqrt{\text{Not}}(|\psi\rangle) := \sqrt{\text{Not}}^{(n)}(|\psi\rangle),$$

if $|\psi\rangle \in \otimes^n \mathbb{C}^2$.

As expected, the square root of the negation has no Boolean counterpart.

Lemma 4.1. There is no function $f : \{0, 1\} \to \{0, 1\}$ such that for any $x \in \{0, 1\} : f(f(x)) = 1 - x$.

Proof. Suppose, by contradiction, that such a function $f$ exists. Two cases are possible: (i) $f(0) = 0$; (ii) $f(0) = 1$.

(i) By hypothesis, $f(0) = 0$. Thus, $1 = f(f(0)) = f(0) = 0$, contradiction.

(ii) By hypothesis, $f(0) = 1$. Thus, $1 = f(f(0)) = f(1)$. Hence, $f(0) = f(1)$. Therefore, $1 = f(f(0)) = f(f(1)) = 0$, contradiction.

Interestingly enough, $\sqrt{\text{Not}}$ does not even have any fuzzy counterpart, represented by a continuous function ([DCGLL02]).
Lemma 4.2. There is no continuous function $f : [0, 1] \to [0, 1]$ such that for any $x \in [0, 1] : f(f(x)) = 1 - x$.

Proof. Suppose, by contradiction, that such a function $f$ exists. First, we prove that $f(\frac{1}{2}) = \frac{1}{2}$. By hypothesis, $f(f(\frac{1}{2})) = 1 - \frac{1}{2} = \frac{1}{2}$. Hence, $f(f(f(\frac{1}{2}))) = f(\frac{1}{2})$. Thus, $1 - f(\frac{1}{2}) = f(\frac{1}{2})$. Therefore, $f(\frac{1}{2}) = \frac{1}{2}$. Consider now $f(0)$. One can easily show: $f(0) \neq 0$ and $f(0) \neq 1$. Clearly, $f(0) \neq \frac{1}{2}$ since otherwise we would obtain $1 = f(f(0)) = f(\frac{1}{2}) = \frac{1}{2}$. Thus, only two cases are possible: (i) $0 < f(0) < \frac{1}{2}$; (ii) $\frac{1}{2} < f(0) < 1$.

(i) By hypothesis, $0 < f(0) < \frac{1}{2} < 1 = f(f(0))$. Consequently, by continuity, $\exists x \in (0, f(0))$ such that $\frac{1}{2} = f(x)$. Accordingly, $\frac{1}{2} = f(\frac{1}{2}) = f(f(x)) = 1 - x$. Hence, $x = \frac{1}{2}$, which contradicts $x < f(0) < \frac{1}{2}$.

(ii) By hypothesis, $\frac{1}{2} < f(0) < 1 = f(f(0))$. By continuity, $\exists x \in (\frac{1}{2}, f(0))$ such that $f(x) = f(0)$. Thus, $1 - x = f(f(x)) = f(f(0)) = 1$. Hence, $x = 0$, which contradicts $x > \frac{1}{2}$. $\square$

The gates considered so far can be naturally generalized to qumixs. When our gates will be applied to density operators, we will write: NOT, $\sqrt{\operatorname{NOT}}$, AND, OR (instead of NOT, $\sqrt{\operatorname{NOT}}$, AND, OR).

Definition 4.8. (The negation).
For any qumix $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$,

$$\operatorname{NOT}^{(n)} \rho = \operatorname{Not}^{(n)} \rho \operatorname{Not}^{(n)}.$$

Definition 4.9. (The square root of the negation).
For any qumix $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$,

$$\sqrt{\operatorname{NOT}}^{(n)} \rho = \sqrt{\operatorname{Not}^{(n)}} \rho \sqrt{\operatorname{Not}^{(n)}}^*$$

(where $\sqrt{\operatorname{Not}^{(n)}}^*$ is the adjoint of $\sqrt{\operatorname{Not}^{(n)}}$).

It is easy to see that for any $n \in \mathbb{N}^+$, both $\operatorname{NOT}^{(n)}(\rho)$ and $\sqrt{\operatorname{NOT}}^{(n)}(\rho)$ are qumixs of $\mathcal{D}(\otimes^n \mathbb{C}^2)$. Further: $\operatorname{NOT}^{(n)} \operatorname{NOT}^{(n)} = I^{(n)}$.

Definition 4.10. (The conjunction).
Let $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$ and $\sigma \in \mathcal{D}(\otimes^m \mathbb{C}^2)$.

$$\operatorname{AND}(\rho, \sigma) = T^{(n, m, 1)}(\rho \otimes \sigma \otimes P_0^{(1)}) T^{(n, m, 1)}.$$

Like in the quregister-case, the gates NOT, $\sqrt{\operatorname{NOT}}$, AND, OR can be uniformly defined on the set $\mathcal{D}$ of all qumixs.

The following theorem sums up some basic properties of our gates:

Theorem 4.1.

(i) $\sqrt{\operatorname{NOT}} \sqrt{\operatorname{NOT}} \rho = \operatorname{NOT}\rho$;
(ii) $p(\operatorname{NOT}\rho) = 1 - p(\rho)$;
(iii) $p(\sqrt{\operatorname{NOT}} \operatorname{NOT}\rho) = p(\operatorname{NOT}\sqrt{\operatorname{NOT}}\rho)$;
(iv) $p(\operatorname{AND}(\rho, \sigma)) = p(\rho)p(\sigma)$;
\( p(\sqrt{\text{NOT}} \land (\rho, \sigma)) = \frac{1}{2}. \)

**Proof.** [Gu02] and [CDCGL03] \( \square \)

5. **Reversible and irreversible quantum computational structures**

An interesting feature of the qumix system is the following: any real number \( \lambda \in [0, 1] \subset \mathbb{R} \) uniquely determines a qumix \( \rho^{(n)}_\lambda \) (for any \( n \in \mathbb{N}^+ \)):

\[
\rho^{(n)}_\lambda := (1 - \lambda)k_nP_0^{(n)} + \lambda k_nP_1^{(n)}. \quad (5.1)
\]

Clearly, \( \rho^{(n)}_\lambda \in \mathcal{D}(\otimes^n\mathbb{C}^2) \). From an intuitive point of view, \( \rho^{(n)}_\lambda \) represents a *mixture of pieces of information* that might correspond to the *Truth* with probability \( \lambda \).

From a physical point of view, \( \rho^{(n)}_\lambda \) corresponds to a particular preparation of the system such that the quantum system might be in the state \( k_nP_0^{(n)} \) with probability \( 1 - \lambda \) and in the state \( k_nP_1^{(n)} \) with probability \( \lambda \). It is worthwhile recalling that the random polarized states of the photon are represented by the density operator \( \rho^{(1)}_{1/2} = \frac{1}{2}I^{(1)} \).

Two important properties of the qumix \( \rho^{(n)}_\lambda \) are described by the following lemma:

**Lemma 5.1.**

(i) \( \forall n \in \mathbb{N}^+ \forall \lambda \in [0, 1] : p(\rho^{(n)}_\lambda) = \lambda; \)

(ii) \( p(\sqrt{\text{NOT}}\rho^{(n)}_\lambda) = \frac{1}{2} \).

**Proof.** [CDCGL03] \( \square \)

We will now introduce two interesting relations that can be defined on the set of all qumixs. Both of them turn out to be a preorder-relation. We will speak of *weak* and of *strong preorder*, respectively.

**Definition 5.1.** (Weak preorder).

\( \rho \leq \sigma \iff p(\rho) \leq p(\sigma). \)

**Definition 5.2.** (Strong preorder).

\( \rho \preceq \sigma \iff \) the following conditions hold:

(i) \( p(\rho) \leq p(\sigma); \)

(ii) \( p(\sqrt{\text{NOT}}\sigma) \leq p(\sqrt{\text{NOT}}\rho). \)

Clearly, \( \rho \preceq \sigma \) implies \( \rho \leq \sigma \), but not the other way around. One immediately shows that both \( \leq \) and \( \preceq \) are reflexive and transitive, but not antisymmetric. Counterexamples can be easily found in \( \mathcal{D}(\mathbb{C}^2) \).

Consider now the following structure:

\[
\left( \mathcal{D}, \preceq, \land, \text{NOT}, \sqrt{\text{NOT}}, P_0^{(1)}, P_1^{(1)}, \rho^{(1)}_{1/2} \right). \quad (5.2)
\]
We will call such a structure the standard reversible quantum computational structure (shortly the RQC-structure).

In the following we will generally write $I, P_0, P_1$ and $\rho_{1/2}$ instead of $I^{(1)}, P_0^{(1)}, P_1^{(1)}, \rho_{1/2}^{(1)}$. From an intuitive point of view, $P_0, P_1$ and $\rho_{1/2}$ represent privileged pieces of information that are true, false, indeterminate, respectively. Generally, our qumixs fail to satisfy Duns Scotus law: $P_0$ and $P_1$ are not the minimum and the maximum element of the RQC-structure. Hence, in this situation, it is interesting to isolate the elements that have a Scotian behaviour.

**Definition 5.3. (Down and up scotian qumixs).**

Let $\rho$ be a qumix of $\mathcal{D}$.

(i) $\rho$ is down Scotian iff $P_0 \preceq \rho$;
(ii) $\rho$ is up Scotian iff $\rho \preceq P_1$;
(iii) $\rho$ is Scotian iff $\rho$ is both down and up Scotian.

**Lemma 5.2.**

(i) $\rho \preceq \sqrt{\text{NOT}} P_1$ iff $p(\rho) \leq \frac{1}{2}$;
(ii) $\sqrt{\text{NOT}} P_0 \preceq \rho$ iff $p(\rho) \geq \frac{1}{2}$.

**Proof.** [CDCGL03]

**Theorem 5.1.**

(i) $\rho$ is down Scotian iff $p(\sqrt{\text{NOT}} \rho) \leq \frac{1}{2}$ iff $\sqrt{\text{NOT}} P_1 \preceq \sqrt{\text{NOT}} \rho$;
(ii) $\rho$ is up Scotian iff $\frac{1}{2} \leq p(\sqrt{\text{NOT}} \rho)$ iff $\sqrt{\text{NOT}} P_0 \preceq \sqrt{\text{NOT}} \rho$;
(iii) $\rho$ is Scotian iff $p(\sqrt{\text{NOT}} \rho) = \frac{1}{2}$;
(iv) $\forall n \in \mathbb{N}^+: k_n P_0^{(n)}, k_n P_1^{(n)}, \rho_{1/2}^{(n)}$ are Scotian;
(v) For any $n \in \mathbb{N}^+$, the set $\mathcal{D}(\otimes^n \mathbb{C}^2)$ contains uncountably many Scotian density operators.

**Proof.** [CDCGL03]

The gates we have considered so far represent typical reversible logical operations. From a logical point of view, it might be interesting to consider also some irreversible operations. An important example is represented by a Lukasiewicz-like disjunction.

**Definition 5.4. (The Lukasiewicz disjunction).**

Let $\tau \in \mathcal{D}(\otimes^n \mathbb{C}^2)$ and $\sigma \in \mathcal{D}(\otimes^m \mathbb{C}^2)$.

$$\tau \oplus \sigma := \rho^{(1)}_{p(\tau) \oplus p(\sigma)},$$

where $\oplus$ in $p(\tau) \oplus p(\sigma)$ is the Lukasiewicz “truncated sum” defined on the real interval $[0, 1]$ (i.e. $p(\tau) \oplus p(\sigma) = \min\{1, p(\tau) + p(\sigma)\}$) ([Za34]).
The following lemmas sum up some basic properties of the Łukasiewicz disjunction:

**Lemma 5.3.**

(i) \( \tau \oplus \sigma = \begin{cases} \rho_p^{(1)}(\tau) + \rho_p^{(1)}(\sigma) ; & \text{if } p(\tau) + p(\sigma) \leq 1 ; \\ P_1^{(1)} ; & \text{otherwise} ; \end{cases} \)

(ii) \( p(\tau \oplus \sigma) = p(\tau) \oplus p(\sigma) ; \)

(iii) \( p(\sqrt{\text{NOT}}(\tau \oplus \sigma)) = \frac{1}{2} . \)

**Proof.** [CDCGL03] \( \Box \)

**Lemma 5.4.** Let \( \rho \in \mathcal{D}(\otimes^n \mathbb{C}^2) . \)

(i) \( \forall n \in \mathbb{N}^+ : \rho \oplus k_n P_1^{(n)} = P_1^{(1)} ; \)

(ii) \( \forall n \in \mathbb{N}^+ : \rho \oplus k_n P_0^{(n)} = \rho_p^{(1)} ; \)

(iii) \( \rho \oplus \text{NOT} \rho = P_1^{(1)} . \)

**Proof.** Straightforward. \( \Box \)

From Lemma 5.4 it follows that \( p(\rho \oplus k_n P_1^{(n)}) = 1 , p(\rho \oplus k_n P_0^{(n)}) = p(\rho) \) and \( p(\rho \oplus \text{NOT} \rho) = 1 . \)

The preorder \( \preceq \) permits us to define on the set of all qumixs an equivalence relation in the expected way.

**Definition 5.5.** (The strong equivalence relation).

\( \rho \cong \sigma \iff \rho \preceq \sigma \text{ and } \sigma \preceq \rho . \)

Clearly, \( \cong \) is an equivalence relation. Let

\[ [\mathcal{D}]_{\cong} := \{ [\rho]_{\cong} : \rho \in \mathcal{D} \} . \]

We will omit \( \cong \) in \( [\rho]_{\cong} \) if no confusion is possible.

Unlike the qumixs (which are only preordered by \( \preceq \)), the equivalence-classes of \( [\mathcal{D}]_{\cong} \) can be partially ordered in a natural way.

**Definition 5.6.**

\[ [\rho] \preceq [\sigma] \iff \rho \preceq \sigma . \]

The relation \( \preceq \) (which is well defined) is a partial order.

**Lemma 5.5.**

(i) \( \forall n \in \mathbb{N}^+ : [P_1] = \left[ k_n P_1^{(n)} \right] ; \)

(ii) \( \forall n \in \mathbb{N}^+ : [P_0] = \left[ k_n P_0^{(n)} \right] ; \)

(iii) \( \forall n \in \mathbb{N}^+ \forall \lambda \in [0, 1] : \left[ \rho_p^{(1)} \right] = \left[ \rho^{(n)} \right] . \)

**Proof.** [CDCGL03] \( \Box \)
On this basis, one can naturally define on the set $[\mathcal{D}]_{\equiv}$ a conjunction, a negation, the square root of the negation, a Łukasiewicz disjunction:

**Definition 5.7.** Let $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$ and $\sigma \in \mathcal{D}(\otimes^m \mathbb{C}^2)$.

(i) $[\rho]_{\equiv} \wedge [\sigma]_{\equiv} = [\wedge(\rho, \sigma)]_{\equiv}$;
(ii) $[\neg \rho]_{\equiv} = [\neg \rho]_{\equiv}$;
(iii) $\sqrt{\neg \rho} = \sqrt{\neg \rho}$;
(iv) $[\rho]_{\equiv} \oplus [\sigma]_{\equiv} = [\rho \oplus \sigma]_{\equiv}$.

**Lemma 5.6.** The operations of Definition 5.7 are well defined.

*Proof.* [CDCGL03] □

**Lemma 5.7.**

(i) The operation $\wedge$ is associative and commutative;
(ii) The operation $\oplus$ is associative and commutative;
(iii) $\neg \neg [\rho]_{\equiv} = [\rho]_{\equiv}$;
(iv) $\sqrt{\neg} \sqrt{\neg} [\rho]_{\equiv} = \neg [\rho]_{\equiv}$;
(v) $\sqrt{\neg \neg} [\rho]_{\equiv} = \neg \sqrt{\neg} [\rho]_{\equiv}$.

*Proof.* Straightforward. □

Consider now the structure

$$
\left( [\mathcal{D}]_{\equiv}, \wedge, \oplus, \neg, \sqrt{\neg}, [P_0]_{\equiv}, [P_1]_{\equiv}, [\rho/2] \right). \quad (5.3)
$$

We will call such a structure the standard irreversible quantum computational algebra (shortly the IQC-algebra).

As happens in the case of $\preceq$, also the weak preorder $\leq$ permits us to define an equivalence relation, which will be called weak equivalence relation.

**Definition 5.8.** (Weak equivalence relation).

$\rho \equiv \sigma$ iff $\rho \leq \sigma$ and $\sigma \leq \rho$.

Clearly, $\equiv$ is an equivalence relation. Let

$$
[\mathcal{D}]_{\equiv} := \{ [\rho]_{\equiv} : \rho \in \mathcal{D} \}.
$$

Also $[\mathcal{D}]_{\equiv}$ can be partially ordered in a natural way.

**Definition 5.9.**

$[\rho]_{\equiv} \leq [\sigma]_{\equiv}$ iff $\rho \leq \sigma$.

One can easily show that the relation $\leq$ (which is well defined) is a partial order.

A conjunction, a Łukasiewicz disjunction, a negation (but not the square root of the negation!) can be naturally defined on $[\mathcal{D}]_{\equiv}$.

**Definition 5.10.** Let $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$ and $\sigma \in \mathcal{D}(\otimes^m \mathbb{C}^2)$.

(i) $[\rho]_{\equiv} \wedge [\sigma]_{\equiv} = [\wedge(\rho, \sigma)]_{\equiv}$;
(ii) $[\neg \rho]_{\equiv} = [\neg \rho]_{\equiv}$;
(iii) $[\rho]_{\equiv} \oplus [\sigma]_{\equiv} = [\rho \oplus \sigma]_{\equiv}$.
Lemma 5.8. The operations of Definition 5.10 are well defined.

Proof. [CDCGL03] □

Unlike $\cong$, the relation $\equiv$ is not a congruence with respect to $\sqrt{\text{NOT}}$. In fact, the following situation is possible: $[\rho]_\equiv = [\sigma]_\equiv$ and $[\sqrt{\text{NOT}}\rho]_\equiv \neq [\sqrt{\text{NOT}}\sigma]_\equiv$.

Consider for example the following unit vectors of $\mathbb{C}^2$: $|\psi\rangle := \frac{\sqrt{3}}{2} |0\rangle + \frac{\sqrt{2}}{2} |1\rangle$ and $|\varphi\rangle := \frac{\sqrt{3}}{2} |0\rangle + \frac{1+i}{2} |1\rangle$.

Let $P_{|\psi\rangle}$ and $P_{|\varphi\rangle}$ be the projections onto the unidimensional spaces spanned by $|\psi\rangle$ and $|\varphi\rangle$, respectively. It turns out that $p(P_{|\psi\rangle}) = p(P_{|\varphi\rangle}) = \frac{1}{2}$. Accordingly, $[P_{|\psi\rangle}]_\equiv = [P_{|\varphi\rangle}]_\equiv$. However, $p(\sqrt{\text{NOT}} P_{|\psi\rangle}) = \frac{1}{2}$ and $p(\sqrt{\text{NOT}} P_{|\varphi\rangle}) = \frac{1}{2} - \frac{\sqrt{3}}{4} \approx 0.146447$. Consequently, $[P_{|\psi\rangle}]_\equiv \neq [P_{|\varphi\rangle}]_\equiv$.

An interesting relation between the weak and the strong preorder is described by the following theorem.

Theorem 5.2. For any $\rho, \sigma \in \mathcal{D}$:

$[\rho]_\equiv \leq [\sigma]_\equiv$ iff $[\rho]_\cong \text{AND } [P_1]_\cong \leq [\sigma]_\cong \text{AND } [P_1]_\cong$.

Proof. [CDCGL03] □

6. The Poincaré Quantum Computational Structures

We will now restrict our analysis to the qumixs living in the two-dimensional space $\mathbb{C}^2$. As is well known, every density operator of $\mathcal{D}(\mathbb{C}^2)$ has the following matrix representation:

$$\frac{1}{2} (I + r_1 X + r_2 Y + r_3 Z),$$

where $r_1, r_2, r_3$ are real numbers such that $r_1^2 + r_2^2 + r_3^2 \leq 1$ and $X, Y, Z$ are the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It turns out that a density operator $\frac{1}{2} (I + r_1 X + r_2 Y + r_3 Z)$ is pure iff $r_1^2 + r_2^2 + r_3^2 = 1$. Consequently,

- Pure density operators are in 1 : 1 correspondence with the points of the surface of the Poincaré sphere;
- Proper mixtures are in 1 : 1 correspondence with the inner points of the Poincaré sphere.

Let $\rho$ be a density operator of $\mathcal{D}(\mathbb{C}^2)$. We will denote by $\bar{\rho}$ the point of the Poincaré sphere that is univocally associated to $\rho$.

Let $(r_1, r_2, r_3)$ be a point of the Poincaré sphere. We will denote by $(r_1, r_2, r_3)$ the density operator univocally associated to $(r_1, r_2, r_3)$.
Lemma 6.1. Let \( \rho \in \mathcal{D}(\mathbb{C}^2) \) such that \( \bar{\rho} = (r_1, r_2, r_3) \). The following conditions hold:

(i) \( p(\rho) = \frac{1 - r_3}{2} \) and \( p(\sqrt{\text{NOT}}\rho) = \frac{1 - r_2}{2} \);

(ii) \( 0 < p(\rho) < 1 \) and \( 0 < p(\sqrt{\text{NOT}}\rho) < 1 \), whenever \( \rho \) is a proper mixture.

Proof.

(i) Easy computation;

(ii) Since proper mixtures are in 1:1 correspondence with inner points of the Poincaré sphere, we have: \( r_1^2 + r_2^2 + r_3^2 < 1 \). Hence: \( r_2^2, r_3^2 < 1 \) and \( -1 < r_2, r_3 < 1 \). Consequently: \( 0 < p(\rho) = \frac{1 - r_3}{2} < 1 \) and \( 0 < p(\sqrt{\text{NOT}}\rho) = \frac{1 - r_2}{2} < 1 \).

An irreversible conjunction can be now naturally defined on the set of all qumixes of \( \mathcal{D}(\mathbb{C}^2) \).

Definition 6.1. (The irreversible conjunction).

Let \( \tau, \sigma \in \mathcal{D}(\mathbb{C}^2) \).

\[
\text{IAND}(\tau, \sigma) := \rho_{p(\tau)p(\sigma)}^{(1)}
\]

Interestingly enough, the density operator \( \text{IAND}(\tau, \sigma) \) can be described in terms of the partial trace. Suppose we have a compound physical system consisting of three subsystems, and let

\[ \mathcal{H} = (\otimes^n \mathbb{C}^2) \otimes (\otimes^m \mathbb{C}^2) \otimes (\otimes^r \mathbb{C}^2) \]

be the Hilbert space associated to our system. Then, for any density operator \( \rho \) of \( \mathcal{H} \), there is a unique density operator \( \text{tr}_{1,2}(\rho) \) that represents the partial trace of \( \rho \) on the space \( \otimes^r \mathbb{C}^2 \) (associated to the third subsystem). The two operators \( \rho \) and \( \text{tr}_{1,2}(\rho) \) are statistically equivalent with respect to the third subsystem. In other words, for any self-adjoint operator \( A^{(r)} \) of \( \otimes^r \mathbb{C}^2 \):

\[
\text{tr}(\text{tr}_{1,2}(\rho) A^{(r)}) = \text{tr}(\rho (I^{(n)} \otimes I^{(m)} \otimes A^{(r)})).
\]

The density operator \( \text{tr}_{1,2}(\rho) \), obtained by “tracing out” the first and the second subsystem, is also called the reduced state of \( \rho \) on the third subsystem.

One can prove that:

\[
\text{IAND}(\tau, \sigma) = \text{tr}_{1,2}(\text{AND}(\tau, \sigma)).
\]

In other words, \( \text{IAND}(\tau, \sigma) \) represents the reduced state of \( \text{AND}(\tau, \sigma) \) on the third subsystem.

An interesting situation arises when both \( \tau \) and \( \sigma \) are pure states. For instance, suppose that:

\( \tau = P_{|\psi\rangle} \) and \( \sigma = P_{|\varphi\rangle} \),
where $|\psi\rangle$ and $|\phi\rangle$ are proper qubits. Then,
\[
\text{AND}(\tau, \sigma) = P_{T^{(1,1,1)}(|\psi\rangle\otimes|\phi\rangle\otimes|0\rangle)},
\]
which is a pure state. At the same time, we have:
\[
I_{\text{AND}}(\tau, \sigma) = \text{tr}_{1,2}(P_{T^{(1,1,1)}(|\psi\rangle\otimes|\phi\rangle\otimes|0\rangle)}),
\]
which is a proper mixture. Apparently, when considering only the properties of the third subsystem, we lose some information. As a consequence, we obtain a final state that does not represent a maximal knowledge. As is well known, situations where the state of a compound system represents a maximal knowledge, while the states of the subsystems are proper mixtures, play an important role in the framework of entanglement-phenomena.

Lemma 6.2.

(i) $I_{\text{AND}}$ is associative and commutative;
(ii) $I_{\text{AND}}(\rho, P_0) = P_0$;
(iii) $I_{\text{AND}}(\rho, P_1) = \rho P(\rho)$;
(iv) $p(I_{\text{AND}}(\rho, \sigma)) = p(\rho) p(\sigma)$;
(v) $p(\sqrt{\text{NOT}} I_{\text{AND}}(\rho, \sigma)) = \frac{1}{2}$.

Proof. Easy.\qed

Consider now the structure
\[
\left( \mathcal{D}(C^2), I_{\text{AND}}, \oplus, \text{NOT}, \sqrt{\text{NOT}}, P_0, P_1, \rho_{1/2} \right).
\]
We will call such a structure the Poincaré irreversible quantum computational algebra (shortly the Poincaré IQC-algebra).

We can refer to the relation $|\equiv$, representing the restriction of $\approx$ to $\mathcal{D}(C^2)$. For any $\rho \in \mathcal{D}(C^2)$, let
\[
[\rho]_\equiv := \{ \sigma \in \mathcal{D}(C^2) : \rho \approx \sigma \}.
\]
Further define
\[
[\mathcal{D}(C^2)]_\equiv := \{ [\rho]_\equiv : \rho \in \mathcal{D}(C^2) \}.
\]
The operations $I_{\text{AND}}, \oplus, \text{NOT}, \sqrt{\text{NOT}}$ and the relation $\preceq$ can be defined on $[\mathcal{D}(C^2)]_\equiv$ in the expected way.

Consider now the quotient-structure
\[
\left( [\mathcal{D}(C^2)]_\equiv, I_{\text{AND}}, \oplus, \text{NOT}, \sqrt{\text{NOT}}, [P_0]_\equiv, [P_1]_\equiv, [\rho_{1/2}]_\equiv \right).
\]
We will call such a structure the contracted Poincaré irreversible quantum computational algebra (shortly the contracted Poincaré IQC-algebra).

Theorem 6.1. The contracted Poincaré IQC-algebra is isomorphic to the IQC-algebra, via the map $g : [\mathcal{D}(C^2)]_\equiv \to [\mathcal{D}]_\equiv$ such that $\forall \rho \in \mathcal{D}(C^2)$:
\[
g([\rho]_\equiv) = [\rho]_\equiv.
\]
Further, for any $\rho, \sigma \in \mathcal{D}(C^2)$: $[\rho]_\equiv \preceq [\sigma]_\equiv$ iff $g([\rho]_\equiv) \preceq g([\sigma]_\equiv)$.
Proof. [CDCGL03]

One can prove that any density operator $\rho$ in $\mathcal{D}(\mathbb{C}^2)$ is associated to a qubit $|\psi_\rho\rangle$ that is “statistically equivalent” to $\rho$. In a sense, $|\psi_\rho\rangle$ represents a “purification” of $\rho$.

**Lemma 6.3.** For any $\rho \in \mathcal{D}(\mathbb{C}^2)$ such that $\bar{\rho} = (r_1, r_2, r_3)$, there exists a qubit $|\psi_\rho\rangle$ that satisfies the following conditions:

1. $p(\rho) = p(|\psi_\rho\rangle)$;
2. $p(\sqrt{\text{Not}}\rho) = p(\sqrt{\text{Not}}(|\psi_\rho\rangle))$.

**Proof.** Let $\rho \in \mathcal{D}(\mathbb{C}^2)$ such that $\bar{\rho} = (r_1, r_2, r_3)$. Consider the vector

$$|\psi_\rho\rangle = \frac{\sqrt{1-r_2^2-r_3^2-ir_2}}{\sqrt{2(1-r_3)}}|0\rangle + \frac{\sqrt{1-r_3}}{2}|1\rangle,$$

which turns out to be a qubit. An easy computation shows that

$$p(|\psi_\rho\rangle) = \frac{1-r_3}{2} \quad \text{and} \quad p(\sqrt{\text{Not}}|\psi_\rho\rangle) = \frac{1-r_2}{2}.$$

Thus by Lemma 6.1 (i), we can conclude that

$$p(|\psi_\rho\rangle) = p(\rho) \quad \text{and} \quad p(\sqrt{\text{Not}}|\psi_\rho\rangle) = p(\sqrt{\text{Not}}\rho).$$

As an interesting application of Lemma 6.3 consider a density operator whose form is: $\rho_\lambda = (1-\lambda)P_0 + \lambda P_1$. Then, by Lemma 6.3, there exists a qubit $|\psi_{\rho_\lambda}\rangle$ such that $p(|\psi_{\rho_\lambda}\rangle) = \lambda$. It turns out that

$$|\psi_{\rho_\lambda}\rangle = \sqrt{1-\lambda}|0\rangle + \sqrt{\lambda}|1\rangle.$$

**Theorem 6.2.** Let $f : \mathcal{D}^n \to \mathcal{D}(\mathbb{C}^2)$. Consider the set $\Omega$ of all qubits. Then, there exists a map

$$f_\Omega : \Omega^n \to \Omega$$

such that for any qubits $|\psi_1\rangle, \ldots, |\psi_n\rangle$ the following conditions hold:

1. $p(f_\Omega(|\psi_1\rangle, \ldots, |\psi_n\rangle)) = p(f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle}))$;
2. $p(\sqrt{\text{Not}}(f_\Omega(|\psi_1\rangle, \ldots, |\psi_n\rangle))) = p(\sqrt{\text{Not}}f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle}))$.

**Proof.** Let $|\psi_1\rangle, \ldots, |\psi_n\rangle \in \Omega$. Then $P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle} \in \mathcal{D}$ and $f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle}) \in \mathcal{D}(\mathbb{C}^2)$. By lemma 6.3, there exists a qubit $|\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle$ such that

$$p(f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})) = p(|\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle)$$

and

$$p(\sqrt{\text{Not}}f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})) = p(\sqrt{\text{Not}}(|\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle)).$$

Thus, we can put $f_\Omega(|\psi_1\rangle, \ldots, |\psi_n\rangle) := |\psi_{f(P_{|\psi_1\rangle}, \ldots, P_{|\psi_n\rangle})}\rangle$.

As a significant application of Theorem 6.2, we obtain that a Łukasiewicz disjunction $\oplus_\Omega$ and an irreversible conjunction $\text{IAnd}_\Omega$ can be naturally
defined for any qubits $|\varphi\rangle = a_0|0\rangle + a_1|1\rangle$ and $|\chi\rangle = b_0|0\rangle + b_1|1\rangle$:

$$|\varphi\rangle \oplus_\Delta |\chi\rangle := \begin{cases} \sqrt{1 - |a_1|^2 - |b_1|^2}|0\rangle + \sqrt{|a_1|^2 + |b_1|^2}|1\rangle, & \text{if } |a_1|^2 + |b_1|^2 \leq 1; \\ |1\rangle, & \text{otherwise}; \end{cases}$$

$$\text{IAnd}_\Delta(|\varphi\rangle, |\chi\rangle) := \sqrt{1 - |a_1b_1|^2}|0\rangle + |a_1b_1||1\rangle.$$  

From an intuitive point of view, it is interesting to compare $\text{IAnd}_\Delta(|\varphi\rangle, |\chi\rangle)$ with $\text{IAnd}(P_{|\varphi\rangle}, P_{|\chi\rangle})$ and with $\text{And}(|\varphi\rangle, |\chi\rangle)$. As we already know, $\text{And}(|\varphi\rangle, |\chi\rangle)$ represents a pure state of a compound physical system (living in the space $\otimes^3 \mathbb{C}^2$). Hence, one is dealing with a maximal knowledge, that also includes a maximal knowledge about the component systems (described by the pure states $|\varphi\rangle$ and $|\chi\rangle$, respectively). Further, the transformation $(|\varphi\rangle, |\chi\rangle) \mapsto \text{And}(|\varphi\rangle, |\chi\rangle)$ is irreversible. The state $\text{IAND}(P_{|\varphi\rangle}, P_{|\chi\rangle})$, instead, is generally a proper mixture: a non-maximal knowledge about a (non-decomposed) system, representing the output of a computation, where the original information about the component systems (the inputs) has been lost. The transformation $(P_{|\varphi\rangle}, P_{|\chi\rangle}) \mapsto \text{IAnd}(P_{|\varphi\rangle}, P_{|\chi\rangle})$ is typically irreversible. The state $\text{IAnd}_\Delta(|\varphi\rangle, |\chi\rangle)$ represents a "purification" of $\text{IAND}(P_{|\varphi\rangle}, P_{|\chi\rangle})$: one is dealing with a maximal knowledge about the output, that does not preserve the original information about the inputs.

### 7. Quantum Computational Logics

The quantum computational structures we have investigated suggest a natural semantics, based on the following intuitive idea: any sentence $\alpha$ of the language is interpreted as a convenient qumix, that generally depends on the logical form of $\alpha$; at the same time, the logical connectives are interpreted as operations that either are gates or can be conveniently simulated by gates. We will consider a minimal (sentential) quantum computational language $\mathcal{L}$ that contains a privileged atomic sentence $\mathbf{f}$ (whose intended interpretation is the truth-value Falsity) and the following primitive connectives: the negation ($\neg$), the square root of the negation ($\sqrt{\neg}$), the conjunction ($\wedge$). Let $\text{Form}^\mathcal{L}$ be the set of all sentences of $\mathcal{L}$. We will use the following metavariables: $q, r, \ldots$ for atomic sentences and $\alpha, \beta, \ldots$ for sentences. The connective disjunction ($\lor$) is supposed to be defined via de Morgan ($\alpha \lor \beta := \neg(\neg\alpha \land \neg\beta)$), while the privileged sentence $t$ representing the Truth is defined as the negation of $\mathbf{f}$ ($t := \neg\mathbf{f}$). This minimal quantum computational language can be extended to richer languages containing other primitive connectives (for instance, a connective corresponding to the Lukasiewicz irreversible disjunction $\oplus$) that we will not consider here.

We will first introduce the notion of reversible quantum computational model (shortly, $RQC$-model).

**Definition 7.1.** ($RQC$-model).
A RQC-model of $L$ is a function $\text{Qum} : \text{Form}^L \rightarrow \mathcal{D}$ (which associates to any sentence $\alpha$ of the language a qumix):

$$\text{Qum}(\alpha) := \begin{cases} 
\text{a density operator of } \mathcal{D}(\mathbb{C}^2) & \text{if } \alpha \text{ is an atomic sentence;} \\
\mathbb{1} & \text{if } \alpha = \text{f}; \\
\text{NOT } \text{Qum}(\beta) & \text{if } \alpha = \neg \beta; \\
\sqrt{\text{NOT}} \text{Qum}(\beta) & \text{if } \alpha = \sqrt{\neg \beta}; \\
\text{AND}(\text{Qum}(\beta), \text{Qum}(\gamma)) & \text{if } \alpha = \beta \land \gamma. 
\end{cases}$$

The concept of RQC-model seems to have a “quasi intensional” feature: the meaning $\text{Qum}(\alpha)$ of the sentence $\alpha$ partially reflects the logical form of $\alpha$. In fact, the dimension of the Hilbert space where $\text{Qum}(\alpha)$ “lives” depends on the number of occurrences of atomic sentences in $\alpha$.

**Definition 7.2.** (The atomic complexity of $\alpha$).

$$\text{At}(\alpha) = \begin{cases} 
1 & \text{if } \alpha \text{ is an atomic sentence;} \\
\text{At}(\beta) & \text{if } \alpha = \neg \beta \text{ or } \alpha = \sqrt{\neg \beta}; \\
\text{At}(\beta) + \text{At}(\gamma) + 1 & \text{if } \alpha = \beta \land \gamma. 
\end{cases}$$

(Recall that: $\text{Qum}(\beta \land \gamma) = T^{(n,m,1)}(\text{Qum}(\beta) \otimes \text{Qum}(\gamma) \otimes \text{Qum}(\text{f}))T^{(n,m,1)}$, if $\text{Qum}(\beta) \in \otimes^n \mathbb{C}^2$ and $\text{Qum}(\gamma) \in \otimes^m \mathbb{C}^2$).

**Lemma 7.1.** If $\text{At}(\alpha) = n$, then $\text{Qum}(\alpha) \in \mathcal{D}(\otimes^n \mathbb{C}^2)$.

**Proof.** Straightforward. □

Given a reversible quantum computational model $\text{Qum}$, any sentence $\alpha$ has a natural probability-value, which can be also regarded as its extensional meaning with respect to $\text{Qum}$.

**Definition 7.3.** (The probability-value of $\alpha$ in a model $\text{Qum}$).

$$p_{\text{Qum}}(\alpha) := p(\text{Qum}(\alpha)).$$

As we already know, qumixs are naturally preordered by two basic relations: the strong preorder $\preceq$ and the weak preorder $\preceq$. This suggests to introduce two different consequence relations: the strong and the weak consequence.

**Definition 7.4.** (Strong and weak consequence in a model $\text{Qum}$).

1. A sentence $\beta$ is a strong consequence in a model $\text{Qum}$ of a sentence $\alpha$ ($\alpha \models_{\text{Qum}} \beta$) iff $\text{Qum}(\alpha) \preceq \text{Qum}(\beta)$;
2. A sentence $\beta$ is a weak consequence in a model $\text{Qum}$ of a sentence $\alpha$ ($\alpha \models_{\text{Qum}} \beta$) iff $\text{Qum}(\alpha) \preceq \text{Qum}(\beta)$.

The notions of strong and weak truth, strong and weak logical consequence, strong and weak logical truth can be now defined in the expected way.

**Definition 7.5.** (Strong and weak truth in a model $\text{Qum}$).

1. A sentence $\alpha$ is strongly true in a model $\text{Qum}$ iff $t \models_{\text{Qum}} \alpha$;
2. A sentence $\alpha$ is weakly true in a model $\text{Qum}$ iff $t \models_{\text{Qum}} \alpha$. 
Definition 7.6. (Strong and weak logical consequence).

1. A sentence $\beta$ is a strong logical consequence of a sentence $\alpha$ ($\alpha \vdash \beta$) iff for any model $\text{Qum}$, $\alpha \vdash_{\text{Qum}} \beta$;
2. A sentence $\beta$ is a weak logical consequence of a sentence $\alpha$ ($\alpha \vDash \beta$) iff for any model $\text{Qum}$, $\alpha \vDash_{\text{Qum}} \beta$.

Definition 7.7. (Strong and weak logical truth).

1. A sentence $\alpha$ is a strong logical truth iff for any model $\text{Qum}$, $\alpha$ is strongly true in $\text{Qum}$;
2. A sentence $\alpha$ is a weak logical truth iff for any model $\text{Qum}$, $\alpha$ is weakly true in $\text{Qum}$.

The strong and the weak logical consequence relations ($\vdash$ and $\vDash$) permit us to characterize semantically two different forms of quantum computational logic. We will indicate by $\sqrt{\neg} \text{QCL}$ the logic that is semantically characterized by the strong logical consequence relation $\vdash$. At the same time, the logic that is characterized by the weak consequence relation will be indicated by $\text{QCL}$. In other words, we have:

- $\beta$ is a logical consequence of $\alpha$ in the logic $\sqrt{\neg} \text{QCL}$ ($\alpha \vdash_{\text{QCL}} \beta$) iff $\beta$ is a strong logical consequence of $\alpha$;
- $\beta$ is a logical consequence of $\alpha$ in the logic $\text{QCL}$ ($\alpha \vdash_{\text{QCL}} \beta$) iff $\beta$ is a weak logical consequence of $\alpha$.

Clearly, $\sqrt{\neg} \text{QCL}$ is a sublogic of $\text{QCL}$. For:

\[ \alpha \vDash \beta \text{ implies } \alpha \vDash_{\text{Qum}} \beta. \]

But not the other way around!

An interesting relation between the two logics $\sqrt{\neg} \text{QCL}$ and $\text{QCL}$ is described by the following theorem:

Theorem 7.1. $\alpha \vdash_{\text{QCL}} \beta$ iff $\alpha \land t \vdash_{\text{QCL}} \beta \land t$.

Proof. The theorem is a direct consequence of the definition of $\sqrt{\neg} \text{QCL}$ and $\text{QCL}$ and of Theorem 5.2. \qed

Let us now turn to the concept of irreversible quantum computational model (shortly, $\text{IQC-model}$), where the “quasi-intensional” character of reversible models is lost. In fact, the interpretation of a sentence in an irreversible model does not generally reflect the logical form of our sentence: the meaning of the whole does not include the meanings of the parts. In spite of this, we will prove that reversible and irreversible models turn out to characterize the same logic.

Definition 7.8. ($\text{IQC-model}$).
An IQC-model of $\mathcal{L}$ is a function $\text{Qum}^{C^2} : \text{Form}^\mathcal{L} \to \mathcal{D}(C^2)$ (which associates to any sentence $\alpha$ of the language a qumix of $C^2$):

$$\text{Qum}^{C^2}(\alpha) :=
\begin{cases}
  P_0 & \text{if } \alpha = f; \\
  \text{NOT} \text{Qum}^{C^2}(\beta) & \text{if } \alpha = \neg \beta; \\
  \sqrt{\text{NOT} \text{Qum}^{C^2}(\beta)} & \text{if } \alpha = \sqrt{\neg} \beta; \\
  \text{IAND}(\text{Qum}^{C^2}(\beta), \text{Qum}^{C^2}(\gamma)) & \text{if } \alpha = \beta \land \gamma.
\end{cases}$$

The (strong and weak) notions of consequence, truth, logical consequence, logical truth are defined like in the reversible case, mutatis mutandis. We will shortly speak of strong irreversible logical consequence and of weak irreversible logical consequence. The logic that is determined by the strong irreversible logical consequence will be indicated by $\sqrt{\neg} \text{IQCL}$, while $\text{IQCL}$ will represent the logic determined by the weak irreversible logical consequence.

We will now prove that $\sqrt{\neg} \text{QCL}$ and $\sqrt{\neg} \text{IQCL}$ are the same logic.

**Lemma 7.2.** Let $\text{Qum}$ be a RQC-model and let $\text{Qum}^{C^2}$ be an IQC-model such that for any atomic sentence $q$: $\text{Qum}(q) = \text{Qum}^{C^2}(q)$. Then, for any sentence $\alpha \in \text{Form}^\mathcal{L}$:

$$p(\text{Qum}(\alpha)) = p(\text{Qum}^{C^2}(\alpha)).$$

**Proof.** The proof is by induction on the length (i.e. the number of connectives) of $\alpha$.

(i) $\alpha = q$. Trivial.

(ii) $\alpha = \neg \beta$.

$$p(\text{Qum}(\alpha)) = p(\text{Qum}(\neg \beta))$$
$$= p(\text{NOT} \text{Qum}(\beta))$$
$$= 1 - p(\text{Qum}(\beta)) \quad \text{(Theorem 4.1(ii))}$$
$$= 1 - p(\text{Qum}^{C^2}(\beta)) \quad \text{(Induction hypothesis)}$$
$$= p(\text{NOT} \text{Qum}^{C^2}(\beta))$$
$$= p(\text{Qum}^{C^2}(\neg \beta)).$$

(iii) $\alpha = \sqrt{\neg} \beta$. The following subcases are possible: (iiiia) $\beta = q$; (iiiib) $\beta = \gamma \land \delta$; (iiiic) $\beta = \sqrt{\neg} \gamma$; (iiid) $\beta = \neg \gamma$.

(iiiia) $\beta = q$. The proof follows from the assumption $\text{Qum}(q) = \text{Qum}^{C^2}(q)$.

(iiiib)

$$p(\text{Qum}(\alpha)) = p(\text{Qum}(\sqrt{\neg} \beta))$$
$$= p(\sqrt{\text{NOT} \text{Qum}(\gamma \land \delta)})$$
$$= p(\sqrt{\text{NOT} \text{AND}(\text{Qum}(\gamma), \text{Qum}(\delta)))}$$
$$= \frac{1}{2} \quad \text{(Theorem 4.1(v))}$$
By induction hypothesis and by Lemma 6.2(v), we have:

\[ p(QumC^2(\alpha)) = p(QumC^2(\sqrt{\neg(\gamma \land \delta)})) = p(\sqrt{\neg IAND(QumC^2(\gamma), QumC^2(\delta))}) = \frac{1}{2} = p(Qum(\alpha)). \]

(iiic) The proof follows from induction hypothesis and Theorem 4.1(ii).

(iv) \( \alpha = \beta \land \gamma \).

\[ p(Qum(\alpha)) = p(Qum(\beta \land \gamma)) = p(Qum(\beta))p(Qum(\gamma)) \] (Theorem 4.1 (iv))

\[ = p(QumC^2(\beta))p(QumC^2(\gamma)) \] (Induction hypothesis)

\[ = p(IAND(QumC^2(\beta), QumC^2(\gamma))) \] (Lemma 6.2 (iv))

\[ = p(QumC^2(\beta \land \gamma)). \]

Corollary 7.1.

(i) For any RQC-model Qum, there exists an IQC-model QumC^2 such that for any \( \alpha \in \text{Form}^\mathcal{C} : \)

\[ p(Qum(\alpha)) = p(QumC^2(\alpha)); \]

(ii) For any IQC-model QumC^2 there exists a RQC-model Qum such that for any \( \alpha \in \text{Form}^\mathcal{C} : \)

\[ p(QumC^2(\alpha)) = p(Qum(\alpha)). \]

Theorem 7.2. \( \alpha \models_{\sqrt{\neg}QCL} \beta \) iff \( \alpha \models_{\sqrt{\neg}IQCCL} \beta \).

Proof. The theorem is a direct consequence of Corollary 7.1. \( \square \)

Hence, \( \sqrt{\neg}QCL \) and \( \sqrt{\neg}IQCCL \) are the same logic. Similarly one can prove that \( QCL \) and \( IQCL \) are the same logic.

So far we have considered (reversible and irreversible) models, where the meaning of any sentence is represented by a qumix. A natural question arises: do density operators have an essential role in characterizing the logics \( \sqrt{\neg}QCL \) and \( QCL \)? This question has a negative answer. In fact, one can
prove that quregisters are sufficient for our logical aims in the case of the
minimal quantum computational language $\mathcal{L}$.

Let us first introduce the notion of (reversible) qubit-model (which is the
basic concept of the qubit-semantics described in [CDCGL02] and [DCGLL02]).

**Definition 7.9. (Reversible qubit-model).**
A reversible qubit-model of $\mathcal{L}$ is a function $\text{Qub} : \text{Form}_{\mathcal{L}} \to \mathfrak{R}$ (which
associates to any sentence $\alpha$ of the language a quregister):

$$
\text{Qub}(\alpha) := \begin{cases} 
\text{a qubit in } \mathbb{C}^2 & \text{if } \alpha \text{ is an atomic sentence;} \\
|0\rangle & \text{if } \alpha = f; \\
\text{Not}(\text{Qub}(\beta)) & \text{if } \alpha = \neg \beta; \\
\sqrt{\text{Not}}(\text{Qub}(\beta)) & \text{if } \alpha = \sqrt{\neg} \beta; \\
\text{And}(\text{Qub}(\beta), \text{Qub}(\gamma)) & \text{if } \alpha = \beta \land \gamma.
\end{cases}
$$

The notions of (weak and strong) consequence, truth, logical consequence,
logical truth are defined like in the case of reversible qumix models, mutatis
mutandis. We will write $\alpha \models_{\text{Qub}}^{\text{QCL}} \beta$, when $\beta$ is a strong logical consequence
of $\alpha$ in the qubit-semantics. Similarly, we will write $\alpha \models_{\text{Qub}}^{\text{QCL}} \beta$
when $\beta$ is a weak logical consequence in the same semantics.

Instead of the class $\mathfrak{R}$ of all quregisters, we could equivalently refer to
the class $\mathfrak{D}_{\mathfrak{R}}$ of all pure density operators having the form $P_{|\psi\rangle}$, where $|\psi\rangle$
is a quregister. One can easily show that $\mathfrak{D}_{\mathfrak{R}}$ is closed under the gates
$\text{NOT}, \sqrt{\text{NOT}}, \text{AND}$. At the same time, $\mathfrak{D}_{\mathfrak{R}}$ is not closed under $\text{IAND}$, because (as
we have seen) $\text{IAND}(P_{|\psi\rangle}, P_{|\varphi\rangle})$ is, generally, a proper mixture.

**Lemma 7.3.** Consider a reversible qubit-model $\text{Qub}$ and let $\text{Qum}$ be a RQC-
model such that for any atomic sentence $q$, $\text{Qum}(q) = P_{\text{Qub}(q)}$. Then, for any
sentences $\alpha$:

$$
\text{Qum}(\alpha) \cong P_{\text{Qub}(\alpha)}.
$$

*Proof.* Easy. □

On this basis we can prove that the qubit-semantics and the qumix-
semantics characterize the same logics.

**Theorem 7.3.**

1. $\alpha \models_{\text{QCL}}^{\text{QCL}} \beta$ iff $\alpha \models_{\text{Qub}}^{\text{QCL}} \beta$;
2. $\alpha \models_{\text{Qub}}^{\text{QCL}} \beta$ iff $\alpha \models_{\text{Qub}}^{\text{QCL}} \beta$.

*Proof.*

1. Suppose that $\alpha \models_{\text{QCL}}^{\text{QCL}} \beta$. Then for any RQC-model $\text{Qum}$:

$$
\text{Qum}(\alpha) \leq \text{Qum}(\beta).
$$

Consequently, by Lemma 7.3, for any qubit-model $\text{Qub}$:

$$
\text{Qub}(\alpha) \leq \text{Qub}(\beta).
$$
(1.2) Suppose, by contradiction, that $\alpha \models_{\text{Qub}} QCL$ and $\alpha \not\models_{\text{QCL}} QCL$ $\beta$. Then, by Theorem 7.2 there exists an irreversible model $Qum_c^2$ such that $Qum_c^2(\alpha) \neq Qum_c^2(\beta)$. By Lemma 6.3, there exists a qubit-model $Qub$ such that for any sentential letter $q$: $p(Qub(q)) = p(Qum_c^2(q))$ and $p(\neg\text{Not}(Qub(q))) = p(\neg\text{Not}Qum_c^2(q))$. One can easily prove that for any $\alpha$, $p(Qub(\alpha)) = p(Qum_c^2(\alpha))$ and $p(\neg\text{Not}(Qub(\alpha))) = p(\neg\text{Not}Qum_c^2(\alpha))$ (by induction on the length of $\alpha$).

Consequently, $\alpha \not\models_{\text{Qub}} QCL$, contradiction.

(2) Similarly.

$\square$

Needless to observe, Theorem 7.3 does not imply that the qumix-semantics is useless. First of all, qubit-models and qumix-models might characterize different logics for languages that are richer than $L$. At the same time, even in the case of our minimal language $L$, qumixs represent an important tool in order to describe entanglement-phenomena.

A remarkable property of the logics $\sqrt{QCL}$ and $QCL$ is the following: our logics do not admit any “genuine” logical truth. In other words, any sentence $\alpha$, that does not contain the atomic sentence $f$, cannot be a logical truth. By Theorem 7.3, it is sufficient to prove that no “genuine” logical truths exist in the framework of the qubit-semantics.

Let us first prove the following theorem ([DCGLL02]):

**Theorem 7.4.** Let $Qub$ be a reversible qubit-model and let $\alpha$ be any sentence. If $p(Qub(\alpha)) \in \{0, 1\}$, then there is an atomic subformula $q$ of $\alpha$ such that $p(Qub(q)) \in \{0, \frac{1}{2}, 1\}$.

**Proof.** Suppose that $p(Qub(\alpha)) \in \{0, 1\}$. The proof is by induction on the length of $\alpha$.

(i) $\alpha$ is an atomic sentence. The proof is trivial.

(ii) $\alpha = \neg\beta$. By Theorem 4.1(ii), $p(Qub(\alpha)) = 1 - p(Qub(\beta)) \in \{0, 1\}$. The conclusion follows by induction hypothesis.

(iii) $\alpha = \sqrt{\neg}\beta$. By hypothesis and by Theorem 4.1(v), $\beta$ cannot be a conjunction. Consequently, only the following cases are possible: (iiia) $\beta = q$; (iiib) $\beta = \neg\gamma$; (iiic) $\beta = \sqrt{\neg}\gamma$.

(iiiia) $\beta = q$. By hypothesis, $p(\sqrt{\neg}\beta) \in \{0, 1\}$. Hence, $\sqrt{\neg}\text{Not}(Qub(q)) = c|x$, where $|x\rangle \in \{|0\rangle, |1\rangle\}$ and $|c\rangle = 1$. We have:

Not$(Qub(q)) = \sqrt{\neg}\text{Not}(\sqrt{\neg}\text{Not}(Qub(q))) = \sqrt{\neg}\text{Not}(c|x\rangle)$. One can easily show that $p(\sqrt{\neg}\text{Not}(c|x\rangle) = \frac{1}{2}$. As a consequence, $p(Qub(\neg q)) = \frac{1}{2} = p(Qub(q))$.

(iiib) $\beta = \neg\gamma$. By Theorem 4.1(iii), $p(Qub(\neg\gamma)) = p(Qub(\neg\sqrt{\neg}\gamma)) = 1 - p(Qub(\sqrt{\neg}\gamma))$. The conclusion follows by induction hypothesis.

(iiic) $\beta = \sqrt{\neg}\gamma$. Then $p(Qub(\sqrt{\neg}\gamma)) = p(Qub(\neg\gamma)) = 1 - p(Qub(\gamma))$. The conclusion follows by induction hypothesis.
(iv) \( \alpha = \beta \land \gamma \). By Theorem 4.1(iv),
\[
p(Qub(\beta \land \gamma)) = p(Qub(\beta))p(Qub(\gamma)) \in \{0, 1\}.
\]
The conclusion follows by induction hypothesis. \( \square \)

As a consequence, we immediately obtain the following Corollary.

**Corollary 7.2.** If \( \alpha \) does not contain \( f \), then \( \alpha \) is not a logical truth either of \( \sqrt{\neg} \text{QCL} \) or of \( \text{QCL} \).

**Proof.** Suppose, by contradiction, that \( \alpha \) is a logical truth either of \( \sqrt{\neg} \text{QCL} \) or of \( \text{QCL} \). Then, in both cases, we obtain that: \( p(\alpha) = 1 \). Let \( q_1, \ldots, q_n \) be the atomic sentences occurring in \( \alpha \). Since \( \alpha \) does not contain \( f \), there exists a qubit-model \( Qub \) such that for any \( i \) (\( 1 \leq i \leq n \)), \( p(Qub(q_i)) \notin \{0, \frac{1}{2}, 1\} \). Then, by Theorem 7.4, \( p(Qub(\alpha)) \notin \{0, 1\} \), contradiction. \( \square \)

We will now list some interesting logical consequences and rules that hold for the logics \( \sqrt{\neg} \text{QCL} \) and \( \text{QCL} \). We will indicate by \( \alpha \models \beta \) the logical consequence relation that refers either to \( \sqrt{\neg} \text{QCL} \) or to \( \text{QCL} \). According to the usual notation we will write:
\[
\frac{\alpha_1 \models \beta_1, \ldots, \alpha_n \models \beta_n}{\gamma \models \delta},
\]
to be read as: if \( \alpha_1 \models \beta_1, \ldots, \alpha_n \models \beta_n \), then \( \gamma \models \delta \). We will also write \( \alpha \equiv \beta \) as an abbreviation for: \( \alpha \models \beta \) and \( \beta \models \alpha \).

Since \( \sqrt{\neg} \text{QCL} \) is a sublogic of \( \text{QCL} \), any logical consequence that holds in \( \sqrt{\neg} \text{QCL} \) will also hold in \( \text{QCL} \). At the same time, some rules that hold in \( \sqrt{\neg} \text{QCL} \) may be violated in \( \text{QCL} \) (and, of course, vice versa).

**Theorem 7.5** (Logical consequences and rules of both \( \sqrt{\neg} \text{QCL} \) and \( \text{QCL} \)).

1. \( \alpha \models \alpha \);
   (identity)

2. \( \frac{\alpha \models \beta, \beta \models \gamma}{\alpha \models \gamma} \);
   (transitivity)

3. \( \alpha \equiv \neg \neg \alpha \);
   (double negation)

4. \( \frac{\alpha \models \beta}{\neg \beta \models \neg \alpha} \);
   (contraposition for the negation)

5. \( \sqrt{\neg} \sqrt{\neg} \alpha \equiv \neg \alpha \);
   (the double square root of the negation principle)
(6) \( \neg \sqrt{-\alpha} \equiv \sqrt{-\neg \alpha} \);
(permutation of the negations)

(7) \( \sqrt{-f} \models \sqrt{-t} \);
(a “tentative negation” of the falsity implies a “tentative negation” of the truth)

(8) \( \alpha \land \beta \equiv \beta \land \alpha, \quad \alpha \lor \beta \equiv \beta \lor \alpha \);
(commutativity)

(9) \( \alpha \land (\beta \land \gamma) \equiv (\alpha \land \beta) \land \gamma, \quad \alpha \lor (\beta \lor \gamma) \equiv (\alpha \lor \beta) \lor \gamma \);
(associativity)

(10) \( \neg (\alpha \land \beta) \equiv \neg \alpha \lor \neg \beta, \quad \neg (\alpha \lor \beta) \equiv \neg \alpha \land \neg \beta \);
(de Morgan)

(11) \( \alpha \land (\beta \lor \gamma) \models (\alpha \land \beta) \lor (\alpha \land \gamma), \quad (\alpha \lor \beta) \land (\alpha \lor \gamma) \models \alpha \lor (\beta \land \gamma) \);
(distributivity 1)

(12) \( f \land f \equiv f, \quad t \land t \equiv t \);
(idempotence for the truth and the falsity)

(13) \( f \land t \equiv f, \quad f \lor t \equiv t \);

(14) \( \frac{\alpha \equiv \beta}{\alpha \lor \beta \equiv \beta \land \beta} \);
(logical equivalence is a congruence for the negation)

(15) \( \frac{\alpha \equiv \gamma_1, \quad \beta \equiv \gamma_2, \quad \delta \equiv \gamma_3}{\gamma_1 \land \gamma_2 \equiv \delta} \);
(logical equivalence is a congruence for the conjunction)

(16) \( \sqrt{-}(\alpha \land \beta) \models \sqrt{-}t \);

(17) \( \frac{\sqrt{-}\alpha \models \sqrt{-}\beta}{\alpha \land \beta \models \beta} \);

(18) \( \frac{\alpha \models \sqrt{-}\beta}{\alpha \land \beta \models \beta} \).
(Weak Duns Scotus)
Proof. Easy.

Let us now consider examples of logical consequences and rules that hold in \( QCL \) and are violated in \( \sqrt{\neg QCL} \).

**Theorem 7.6** (Logical consequences and rules of \( QCL \) that fail in \( \sqrt{\neg QCL} \)).

1. \( \alpha \land \beta \models_{QCL} \alpha, \quad \alpha \land \beta \models_{QCL} \beta; \)

2. \( \alpha \models_{QCL} \alpha \lor \beta, \quad \beta \models_{QCL} \alpha \lor \beta; \)

3. \( \alpha \land \alpha \models_{QCL} \alpha, \quad \alpha \models_{QCL} \alpha \lor \alpha; \) (semi-idempotence 1)

4. \( f \models_{QCL} \alpha. \) (Duns Scotus)

**Proof.** Easy.

**Theorem 7.7** (A rule that holds in \( \sqrt{\neg QCL} \) and fails in \( QCL \)).

\[
\alpha \equiv \beta \quad \vdash_{\sqrt{\neg}} \neg \alpha \equiv \sqrt{\neg} \beta
\]

**Proof.** Easy.

In other words, logical equivalence is a congruence for the square root of the negation.

**Theorem 7.8** (Logical consequences that fail both in \( QCL \) and \( \sqrt{\neg QCL} \)).

1. \( \alpha \not\models \alpha \land \alpha; \) (semi-idempotence 2)

2. \( t \not\models \alpha \lor \neg \alpha; \) (excluded middle)

3. \( t \not\models \neg (\alpha \land \neg \alpha); \) (non contradiction)

4. \( (\alpha \lor \beta) \lor (\alpha \lor \gamma) \not\models \alpha \lor (\beta \lor \gamma), \quad \alpha \lor (\beta \lor \gamma) \not\models (\alpha \lor \beta) \lor (\alpha \lor \gamma); \) (distributivity 2)

**Proof.** Easy.

Apparently, the logics \( QCL \) and \( \sqrt{\neg QCL} \) turn out to be non standard forms of quantum logic. Conjunction and disjunction do not correspond to lattice operations, because they are not generally idempotent. Unlike Birkhoff and von Neumann’s quantum logic, the weak distributivity principle \( ((\alpha \land \beta) \lor (\alpha \land \gamma) \models \alpha \land (\beta \lor \gamma)) \) breaks down. At the same time, the strong distributivity \( (\alpha \land (\beta \lor \gamma) \models (\alpha \land \beta) \lor (\alpha \lor \gamma)) \), that is violated in orthodox quantum logic, is here valid. Both the excluded middle and the non contradiction principles are violated. As a consequence, one can say...
that the logics arising from quantum computation represent, in a sense, new examples of fuzzy logics.

The axiomatizability of $\mathbf{QCL}$ and $\sqrt{\neg} \mathbf{QCL}$ is an open problem.

8. Quantum trees

An interesting feature of the quantum computational semantics is the following: the meaning and the probability-value of any molecular sentence $\alpha$ can be naturally described (and calculated) by means of a convenient quantum tree, that illustrates a kind of reversible transformation of the atomic subformulas of $\alpha$. By theorem 7.3, we know that we can refer to the qubit-semantics (instead of the qumix-semantics), without any loss of generality. For the sake of technical simplicity, we will first slightly modify our language. The new language $\mathcal{L}^\wedge$ contains, besides the atomic sentence $f$ and the two negations ($\neg$ and $\sqrt{\neg}$), a ternary conjunction $\wedge$ (whose semantic behaviour is “close” to the Petri-Toffoli gate). For any sentences $\alpha$ and $\beta$, the expression $\wedge(\alpha, \beta, f)$ is a sentence of $\mathcal{L}^\wedge$. In this framework, the usual conjunction $\alpha \land \beta$ is dealt with as metalinguistic abbreviation for the ternary conjunction $\wedge(\alpha, \beta, f)$. The occurrence of $f$ as the third element in the formula $\wedge(\alpha, \beta, f)$ is called a non-genuine occurrence of $f$. The semantic definition of qubit-model of the language $\mathcal{L}^\wedge$ is then modified in the expected way. Besides the old conditions concerning the interpretation of $f$ and of the two negations ($\neg$, $\sqrt{\neg}$), we require that for any Qub:

$$\text{Qub}(\wedge(\alpha, \beta, f)) = T(\text{Qub}(\alpha), \text{Qub}(\beta), \text{Qub}(f)).$$

Needless to stress, the logics $\mathbf{QCL}$ and $\sqrt{\neg} \mathbf{QCL}$ can be equivalently formalized either in the language $\mathcal{L}$ or in $\mathcal{L}^\wedge$. In case where the language is $\mathcal{L}^\wedge$, Corollary 7.2 shall be formulated as follows: if $\alpha$ does not contain any genuine occurrence of $f$, then $\alpha$ is not a logical truth either of $\sqrt{\neg} \mathbf{QCL}$ or of $\mathbf{QCL}$.

Before dealing with quantum trees, we will first introduce the notion of syntactical tree of a sentence $\alpha$ (abbreviated as $\mathbf{STree}^\alpha$). Consider all subformulas of $\alpha$.

Any subformula may be:

- an atomic sentence $q$ (possibly $f$);
- a negated sentence $\neg \beta$;
- a square-root negated sentence $\sqrt{\neg} \beta$;
- a conjunction $\wedge(\beta, \gamma, f)$.

The intuitive idea of syntactical tree can be illustrated as follows. Every occurrence of a subformula of $\alpha$ gives rise to a node of $\mathbf{STree}^\alpha$. The tree consists of a finite number of levels and each level is represented by a sequence of subformulas of $\alpha$: $\text{Level}_k(\alpha)$.

$$\vdots$$

$\text{Level}_1(\alpha)$. 
Figure 1.
Branching rules for the construction of syntactical trees.

The root-level (denoted by \( \text{Level}_1(\alpha) \)) consists of \( \alpha \). From each node of the tree at most 3 edges may branch according to the branching-rule (Figure 1).

The second level (\( \text{Level}_2(\alpha) \)) is the sequence of subformulas of \( \alpha \) that is obtained by applying the branching-rule to \( \alpha \). The third level (\( \text{Level}_3(\alpha) \)) is obtained by applying the branching-rule to each element (node) of \( \text{Level}_2(\alpha) \), and so on. Finally, one obtains a level represented by the sequence of all atomic occurrences of \( \alpha \). This represents the last level of \( \text{STree}^\alpha \). The height of \( \text{STree}^\alpha \) (denoted by \( \text{Height}(\alpha) \)) is then defined as the number of levels of \( \text{STree}^\alpha \).

A more formal definition of syntactical tree can be given by using some standard graph-theoretical notions.

**Example 8.1.** The syntactical tree of \( \alpha = \neg q \land (r \land \sqrt{\neg} q) \) is the following: Clearly the height of \( \text{STree}^\alpha \) is 4.

For any choice of a qubit-model \( \text{Qub} \), the syntactical tree of \( \alpha \) determines a corresponding sequence of quregisters. Consider a sentence \( \alpha \) with \( n \) atomic occurrences (\( q_1, \ldots, q_n \)). Then \( \text{Qub}(\alpha) \in \otimes^n \mathbb{C}^2 \). We can associate a quregister \( |\psi_i\rangle \) to each \( \text{Level}_i(\alpha) \) of \( \text{STree}^\alpha \) in the following way. Suppose that:

\[
\text{Level}_i(\alpha) = (\beta_1, \ldots, \beta_r).
\]

Then:

\[
|\psi_i\rangle = \text{Qub}(\beta_1) \otimes \ldots \otimes \text{Qub}(\beta_r).
\]

Hence:

\[
\begin{cases}
|\psi_1\rangle = \text{Qub}(\alpha) \\
|\psi_i\rangle \rightarrow \text{Qub}(q_1) \otimes \ldots \otimes \text{Qub}(q_n)
\end{cases}
\]
where all $|\psi_i\rangle$ belong to the same space $\otimes^n \mathbb{C}^2$.

From an intuitive point of view, $|\psi_{\text{Height}(\alpha)}\rangle$ can be regarded as a kind of epistemic state, corresponding to the input of a computation, while $|\psi_1\rangle$ represents the output.

We obtain the following correspondence:

\[
\begin{align*}
\text{Level}_{\text{Height}(\alpha)}(\alpha) & \quad \Longleftrightarrow \quad |\psi_{\text{Height}(\alpha)}\rangle: \text{the input} \\
\ldots & \quad \ldots \\
\text{Level}_1(\alpha) & \quad \Longleftrightarrow \quad |\psi_1\rangle: \text{the output}
\end{align*}
\]

The notion of quantum tree of a sentence $\alpha$ ($QTree^\alpha$) can be now defined as a particular sequence of unitary operators that is uniquely determined by the syntactical tree of $\alpha$. As we already know, each $\text{Level}_i(\alpha)$ of $STree^\alpha$ is a sequence of subformulas of $\alpha$. Let $\text{Level}_i^j(\alpha)$ represent the $j$-th element of $\text{Level}_i(\alpha)$. Each node $\text{Level}_i^j(\alpha)$ (where $1 \leq i < \text{Height}(\alpha)$) can be naturally associated to a unitary operator $\text{Op}_i^j$, according to the following operator-rule:

\[
\text{Op}_i^j := \begin{cases} 
I^{(1)} & \text{if } \text{Level}_i^j(\alpha) \text{ is an atomic sentence,} \\
\text{Not}^{(r)} & \text{if } \text{Level}_i^j(\alpha) = \neg \beta \text{ and } \text{Qub}(\beta) \in \otimes^r \mathbb{C}^2; \\
\sqrt{\text{Not}}^{(r)} & \text{if } \text{Level}_i^j(\alpha) = \sqrt{\neg} \beta \text{ and } \text{Qub}(\beta) \in \otimes^r \mathbb{C}^2; \\
T^{(r,s,1)} & \text{if } \text{Level}_i^j(\alpha) = \bigwedge(\beta, \gamma, \phi), \text{Qub}(\beta) \in \otimes^r \mathbb{C}^2 \text{ and } \text{Qub}(\gamma) \in \otimes^s \mathbb{C}^2.
\end{cases}
\]

On this basis, one can associate an operator $U_i$ to each $\text{Level}_i(\alpha)$ (such that $1 \leq i < \text{Height}(\alpha)$):

\[
U_i := \bigotimes_{j=1}^{\text{|Level}_i(\alpha)|} \text{Op}_i^j,
\]

where $|\text{Level}_i(\alpha)|$ is the length of the sequence $\text{Level}_i(\alpha)$.

Being the tensor product of unitary operators, every $U_i$ turns out to be a unitary operator. One can easily show that all $U_i$ are defined in the same space $\otimes^n \mathbb{C}^2$, where $n$ is the atomic complexity of $\alpha$.

The notion of quantum tree of a sentence can be now defined as follows.

**Definition 8.1.** (The quantum tree of $\alpha$).

The quantum tree of $\alpha$ (denoted by $QTree^\alpha$) is the operator-sequence

\[
(U_1, \ldots, U_{\text{Height}(\alpha) - 1})
\]

that is uniquely determined by the syntactical tree of $\alpha$. 
As an example, consider the following sentence: \( \alpha = q \land \neg q = \land (q, \neg q, f) \).
The syntactical tree of \( \alpha \) is the following:

\[
\begin{align*}
\text{Level}_1(\alpha) &= \land (q, \neg q, f); \\
\text{Level}_2(\alpha) &= (q, \neg q, f); \\
\text{Level}_3(\alpha) &= (q, q, f).
\end{align*}
\]

In order to construct the quantum tree of \( \alpha \), let us first determine the operators \( Op_i \) corresponding to each node of \( Stree^\alpha \). We will obtain:

- \( Op_1^1 = T^{(1,1,1)} \), because \( \land (q, \neg q, f) \) is connected with \( (q, \neg q, f) \) (at \( Level_2(\alpha) \));
- \( Op_1^1 = I^{(1)} \), because \( q \) is connected with \( q \) (at \( Level_3(\alpha) \));
- \( Op_2^2 = \text{Not}^{(1)} \), because \( \neg q \) is connected with \( q \) (at \( Level_3(\alpha) \));
- \( Op_3^3 = I^{(1)} \), because \( f \) is connected with \( f \) (at \( Level_3(\alpha) \)).

The quantum tree of \( \alpha \) is represented by the operator-sequence \((U_1, U_2)\), where:

\[
\begin{align*}
U_1 &= Op_1^1 = T^{(1,1,1)}; \\
U_2 &= Op_2^2 \otimes Op_2^2 \otimes Op_2^2 = I^{(1)} \otimes \text{Not}^{(1)} \otimes I^{(1)}.
\end{align*}
\]

Apparently, \( QTree^\alpha \) is independent of the choice of \( Qub \).

**Theorem 8.1.** Let \( \alpha \) be a sentence whose quantum tree is the operator-sequence \((U_1, \ldots, U_{\text{Height}(\alpha)-1})\). Given a quantum computational model \( Qub \), consider the quregister-sequence \((|\psi_1\rangle, \ldots, |\psi_{\text{Height}(\alpha)}\rangle)\) that is determined by \( Qub \) and by the syntactical tree of \( \alpha \). Then, \( U_i(|\psi_{i+1}\rangle) = |\psi_i\rangle \) (for any \( i \) such that \( 1 \leq i < \text{Height}(\alpha) \)).

**Proof.** Straightforward. \( \square \)

The quantum tree of \( \alpha \) can be naturally regarded as a quantum circuit that computes the output \( Qub(\alpha) \), given the input \( Qub(q_1), \ldots, Qub(q_n) \) (where \( q_1, \ldots, q_n \) are the atomic occurrences of \( \alpha \)). In this framework, each \( U_i \) is the unitary operator that describes the computation performed by the \( i \)-th layer of the circuit.

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