Generalization Of Separation Of Variables n-Harmonic Equation m Dimension
and Unbounded Boundary Value Problem

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ABSTRACT The method of separation of variables is significant, it has been applied to physics, engineering, chemistry and other fields. It allows to reduce the difficulty of problems by separating the variables from partial differential equation system into ordinary differential equations system. However, this method has complexity in higher order partial differential equations. In this research, we generalize this method by using multinomial theorem of n-harmonic equation to solve n-harmonic equation with m dimension and then solving an important class of partial differential equations with unbounded boundary conditions. Additionaly, application of convolution.

KEYWORDS: Fourier transform, boundary value problem, n-harmonic equation, application of convolution theorem.

1 Introduction

The n-harmonic 2 dimension has been studied by Dοscharin in [1], and neumann Problem 1-harmonic m dimension have been solved by Otuf in [3]. We are interested in n-harmonic equation m dimension which is in the form of

\[ \Delta_m^n u(x) = 0, \]

where \( x = (x_1, x_2, \ldots, x_m) \). This equations has high complexity since it is a partial differential equation of order 2n. However, separation of variables is an interesting approach to solve this kind of problem. At begining, we need to put our attention in driving new concept which decrease the complexity of solving this problem. Multinomial theorem of n-harmonic equation which is coming directly from the original multinomial theorem allows to write the n-harmonic equation as summation of ordinary differential equations. This lead to separate these equations and write them as sequence of equations. Solving these equations will design an important generalization. Moreover, we are interested in solving unbounded boundary value problem that satisfying \( \Delta_m^n u(x) = 0 \) and then application of its convolutions theorem.

2 Multinomial Theorem of n-Harmonic Equation

We formalize multinomial theorem to design short approach to separate the variables of \( \Delta_m^n u(x) \), and write them as summation of ordinary differential equations. Suppose for \( y_i = \frac{X_i(2)}{X_i} \), we have \( y_i^{(n)} = \frac{X_i^{(2n)}}{X_i} \). We let \( u(x) = \prod_{i=1}^{m-1} X_i(x_i)X_m(x_m) \), then apply \( \Delta_m^n \) to \( u(x) \) and then divide by \( \prod_{i=1}^{m-1} X_i(x_i)X_m(x_m) \) leads to the multinomial theorem of n-harmonic equation which is as following.
Where
\[
\left( \sum_{r=1}^{m} y_i \right)^{(n)}_{h_1 + h_2 + \ldots + h_m = n} = \sum_{i=1}^{m} \left( h_1, h_2, \ldots, h_m \right) y_i^{h_i}.
\]

(2)

\[
(n) \text{ is order of derivative and } h_i \neq 0. \text{ Furthermore, multiplication of any two or more ordinary}
\]

differential equations, yields sum of their orders. For example \( X_2^{(2)} X_2^{(4)} = X_2^{(6)} \) We list some

examples For \( \Delta^2_3 u(x) = 0 \), we have
\[
(y_1 + y_2)^{(2)} = y_1^{(2)} + 2y_1 y_2 + y_2^{(2)} = X_1^{(4)} X_1^{(2)} + 2X_1^{(2)} X_2^{(2)} + X_2^{(4)} X_2^{(2)}.
\]

(4)

For \( \Delta^3_3 u(x) = 0 \), we have
\[
(y_1 + y_2)^{(3)} = y_1^{(3)} + 3y_1^{(2)} y_2 + 3y_1 y_2^{(2)} + y_2^{(3)} = X_1^{(6)} + 3X_1^{(4)} X_2^{(2)} + X_1^{(2)} X_2^{(4)} + X_1^{(6)} X_2^{(2)}.
\]

(5)

For \( \Delta^3_3 u(x) = 0 \), we have
\[
(y_1 + y_2 + y_3)^{(3)} = X_1^{(6)} + X_2^{(6)} + X_3^{(6)} + 3X_1^{(4)} X_2^{(2)} + 3X_1^{(2)} X_2^{(4)} + 3X_1^{(2)} X_3^{(2)} + 3X_2^{(4)} X_3^{(2)}.
\]

(6)

\[
+ 3X_1^{(4)} X_3^{(2)} + 3X_1^{(2)} X_3^{(4)} + 3X_2^{(4)} X_3^{(2)} + 3X_2^{(4)} X_3^{(2)} + X_2^{(4)} X_3^{(2)}.
\]

(7)

3 Generalization Of Separation Of Variables

Let \( x = (x_1, x_2, \ldots, x_m) \) be vector in \( \mathbb{R}^m \). The n-harmonic equation is
\[
\Delta^n_m u(x) = 0.
\]

(8)

Let \( \lambda_{ij} \) to be an constants in \( \mathbb{R} \), where \( i = 1, 2, 3, \ldots, m \), and \( j = 1, 2, \ldots, n \). We seek solution of the form
\[
u(x) = \prod_{i=1}^{m-1} X_i (x_i) X_m (x_m).
\]

(9)

Apply method of separation of variables to get multinomial theorem of n-harmonic equation that is
\[
\left( \sum_{r=1}^{m} y_i \right)^{(n)}_{h_1 + h_2 + \ldots + h_m = n} = \sum_{i=1}^{m} \left( h_1, h_2, \ldots, h_m \right) y_i^{h_i}.
\]

(10)

Where
\[
y_i = X_i^{(2)} X_i^{(2)}.
\]

(11)

and
\[
y_i^{(n)} = X_i^{(2n)} X_i^{(2n)}.
\]

(12)

Let
\[
\lambda_{ij} = X_i^{(2j)} X_i^{(2j)}.
\]

(13)
we get ordinary differential equations those are

\[ X_i^{(2j)} - \lambda_i X_i = 0, \quad (14) \]

and

\[ \sum_{k=0}^{(n)} \binom{n}{k} X_m^{(2n-2k)} K_m^{(2k)} = 0. \quad (15) \]

Where \( X_m^{(0)} = X \), \((2j)\) is the order of derivative, and \( k_m = \sum_{i=1}^{m-1} \lambda_{i_1} \). Let solve (14) if \( j = 1 \), we get

\[ X_i^{(2)} - \lambda_{i_1} X_i = 0. \quad (16) \]

If \( \lambda_{i_1} < 0 \), we have

\[ X_i(x_i) = a_i \cos \sqrt{-\lambda_{i_1}} x_i + b_i \sin \sqrt{-\lambda_{i_1}} x_i. \quad (17) \]

for \( j > 1 \), we will let \( \lambda_{i_j} = (\lambda_{i_1})^j \) in (14), then the solutions of this forms have two terms. First one is (17), and second terms in which their constant forced to be zeros. The final result is (17). Similarly if \( \lambda_{i_1} > 0 \), we have

\[ X_i(x_i) = a_i \cosh \sqrt{\lambda_{i_1}} x_i + b_i \sinh \sqrt{\lambda_{i_1}} x_i. \quad (18) \]

If \( \lambda_{i_1} = 0 \), we have

\[ X_i(x_i) = a_i + b_i x. \quad (19) \]

The solution of (15), elementary differential equation yields

\[ \sum_{k=0}^{n} (X_m^2 + k_m)^{(n)} = 0. \quad (20) \]

Where \( X^{(0)} = X \). This gives

\[ \sum_{k=0}^{n} (s^2 + k_m)^n = 0. \quad (21) \]

If \( k_m < 0 \), we have

\[ X_m(x_m) = \sum_{r=1}^{2n} x_m^{r-1} [c_r \cosh \sqrt{-k_m} x_m + d_r \sinh \sqrt{-k_m} x_m]. \quad (22) \]

If \( k_m > 0 \), we have

\[ X_m(x_m) = \sum_{r=1}^{2n} x_m^{r-1} [c_r \cos \sqrt{k_m} x_m + d_r \sin \sqrt{k_m} x_m]. \quad (23) \]

Finally if \( k_m = 0 \), we have

\[ X_m(x_m) = \sum_{r=1}^{2m} c_r x_r^{r-1}. \quad (24) \]

The final solutions depends on choice of \( \lambda_{i_1} \) for each \( i = 1, 2, \ldots, m - 1 \). We have

\[ u(x) = \prod_{i=1}^{m-1} X_i(x_i) X_m(x_m). \quad (25) \]

Remark: We may have different signs for \( \lambda_{i_1} \).
4 Boundary Value Problem Unbounded Domain

We consider the Boundary value problem (BVP) replace \( x = (x_1, x_2, \ldots, x_{m-1}, x_m) \) by \( (x, x_m) = (x_1, x_2, \ldots, x_{m-1}, x_m) \).

\[
\Delta_m^n u(x, x_m) = 0, \quad -\infty < x_i < \infty; \quad L < x_m < \infty,
\]

\[
u(x, 0) = f(x), \quad |u(x, x_m)| < M.
\]

Write \( u(x, x_m) = \prod_{i=1}^{m-1} X_i(x_i) X_m(x_m) \). Apply generalization of separation of variables to get

\[
X_i^{(2)} - \lambda_i X_i = 0,
\]

and

\[
\sum_{k=0}^{n} (X_m^2 + k_m)^{(n)} = 0.
\]

Where \( X^{(0)} = X \). From the generalization we need only \( j = 1 \) i.e.

\[
X_i^{(2)} - \lambda_i X_i = 0,
\]

and by letting \( \lambda_i = -w_i^2 \). We get

\[
X_i^{(2)} + w_i^2 X_i = 0.
\]

The solutions is

\[
X_i(x_i) = a_i \cos w_i x_i + b_i \sin w_i x_i.
\]

The solutions for the \( m \)th variable is in the form

\[
X_m(x_m) = \sum_{r=1}^{2n} x_m^{r-1} [q_r e^{-\sqrt{-k_m} x_m} + f_r e^{\sqrt{-k_m} x_m}].
\]

The final solutions

\[
u(x, x_m) = \prod_{i=1}^{m-1} [a_i \cos w_i x_i + b_i \sin w_i x_i] \sum_{r=1}^{2n} x_m^{r-1} [q_r e^{-\sqrt{-k_m} x_m} + f_r e^{\sqrt{-k_m} x_m}].
\]

Where \( k_m = -\sum_{i=1}^{m-1} w_i^2 \). Since \( w_i > 0 \), and \( x_m \to \infty \), we must have \( f_r = 0 \). It follows that

\[
u(x, x_m) = \int_{(0, \infty)^{m-1}} \prod_{r=1}^{m-1} [A_i x_m^{r-1} \cos w_i x_i + x_m^{r-1} B_i \sin w_i x_i] \left[e^{-\sqrt{-k_m} x_m}\right] d w_1 d w_2 \cdots d w_{m-1}.
\]

Since \( u(x, L) = f(x) \), then

\[
f(x) = \int_{(0, \infty)^{m-1}} \prod_{r=1}^{m-1} L^{r-1} e^{\sqrt{-k_m} L} [A_i \cos w_i x_i + A_i \sin w_i x_i] d w_1 d w_2 \cdots d w_{m-1}.
\]

Where \( w = (w_1, w_2, \ldots, w_{m-1}) \). Thus, we have

\[
A_i(w) = \frac{\sum_{r=1}^{2n} L^{r-1} e^{\sqrt{-k_m} L} \int_{(-\infty, \infty)^{m-1}} f(x) \prod_{i=1}^{m-1} \cos w_i x_i d x_1 d x_2 \cdots d x_{m-1}}{\pi^{m-1}}.
\]
\[ B_n(w) = \sum_{r=1}^{2n} \frac{L^{1-r}e^{\pm iw_1L}}{\pi^{m-1}} \int_{(-\infty,\infty)^m} f(x) \prod_{i=1}^{m-1} \sin w_i x_i \, dx_1 dx_2 \cdots dx_m. \] 

(38)

Where \( r = 1, 2, \ldots, 2n. \)

5 Application of Convolution Theorem

The case 1-harmonic 2 dimension has been done in [21], we extended to n-harmonic 2 dimension. Let us the result of previous problem when \( m=2, \)

\[ u(x_1, x_2) = \int_0^\infty \sum_{r=1}^{2n} [A_{1r} x_2^{r-1} \cos w_1 x_1 + x_2^{r-1} B_{1r} \sin w_1 x_1]e^{-w_1 x_2} \, dw_1. \]

(39)

where

\[ A_{1r}(w) = \sum_{r=1}^{2n} \frac{L^{1-r}e^{w_1L}}{\pi} \int_{-\infty}^{\infty} f(x) \cos w_1 x_1 \, dx_1. \]

(40)

\[ B_{1r}(w) = \sum_{r=1}^{2n} \frac{L^{1-r}e^{w_1L}}{\pi} \int_{-\infty}^{\infty} f(x) \sin w_1 x_1 \, dx_1. \]

(41)

After we plug the coefficients into the general formula, we get

\[ u(x_1, x_2) = \sum_{r=1}^{2n} \frac{L^{1-r}e^{w_1L}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{r=1}^{2n} [x_2^{r-1} \cos w_1 x_1 \cos w_1 z_1 f(z)] \, dw_1 \, dz_1 \]

(42)

\[ + x_2^{r-1} \sin w_1 x_1 \sin w_1 z_1 f(z_1) \, [e^{-w_1 x_2}] \, dw_1 \, dz_1. \]

(43)

\[ = \sum_{r=1}^{2n} \frac{L^{1-r}e^{w_1L}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{r=1}^{2n} [x_2^{r-1}e^{-w_1 x_2} \cos w_1 (z_1 - x_1) f(z_1)] \, dw_1 \, dz_1. \]

(44)

Let

\[ g(x_1, x_2) = \int_0^\infty \sum_{r=1}^{2n} \cos w_1 (x_1) x_2^{r-1} e^{-w_1 x_2} \, dw_1. \]

(45)

\[ = \sum_{r=1}^{2n} \int_0^\infty e^{iw_1 x_1} + e^{-iw_1 x_1} 2 e^{-w_1 x_2} x_2^{r-1} \, dw_1 \]

(46)

\[ = \sum_{r=1}^{2n} x_2^{r-1} e^{-w_1 x_2} \frac{1}{2} \left( \frac{1}{x_2 - ix_1} + \frac{1}{x_2 + ix_1} \right). \]

(47)

\[ = \sum_{r=1}^{2n} x_2^{r-1} e^{-w_1 x_2} \frac{x_2}{x_2^2 + x_1^2}. \]

(48)

Define the convolution theorem in the form of

\[ u(x_1, x_2) = \sum_{r=1}^{2n} \frac{L^{1-r}e^{w_1L}}{\pi} \int_{-\infty}^{\infty} f(z_1)g(x_1 - z_1, x_2) \, dz_1. \]

(49)

Let

\[ f(z_1) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{if } z < 0. \end{cases} \]

(50)

Now apply the theorem to get
\[ u(x_1, x_2) = \sum_{r=1}^{2n} \frac{L^{1-r} e^{w_1 L}}{\pi} \int_{-\infty}^{\infty} \sum_{r=1}^{2n} x_2^{r-1} e^{-w_1 x_2} \frac{x_2}{x_2^2 + (x_1 - z_1)^2} dz_1. \]  
\[ = \sum_{r=1}^{2n} \frac{L^{1-r} e^{w_1 L}}{\pi} \sum_{r=1}^{2n} x_2^{r-1} e^{-w_1 x_2} \int_{-\infty}^{\infty} e^{-\sqrt{k_m x_m}} \frac{x_m}{x_m^2 + (x_i - z_i)^2} dz_i. \]
\[ = \sum_{r=1}^{2n} \frac{L^{1-r} e^{w_1 L}}{\pi} \sum_{r=1}^{2n} x_2^{r-1} e^{-w_1 x_2} \left[ \frac{\pi}{2} + \tan^{-1} \left( \frac{x_1}{x_2} \right) \right]. \]

6 Parabolic Version Of n-Harmonic Equation

The parabolic version of n-harmonic equation is

\[ \alpha \Delta_m^n u(x, t) = u_t(x, t). \]  

Where \((x, t) = (x_1, x_2, \ldots, x_m, t)\), \(t > 0\), \(\alpha \in \mathbb{R}\), and \((x, t) \in \mathbb{R}^m \times (0, \infty)\). The between difference between the elliptic version of n harmonic equation and prarabolic one is similar to difference between heat equation and 1-harmonic equation. The solution can presented as \(u(x) = \prod_{i=1}^{m} X_i(x_i)T(t)\). \(X_i(x_i)\) has same solutions as the one in n harmonic equation. For \(T(t)\) we have

\[ T'(t) - \alpha k_{m+1} T(t) = 0. \]

The solutions is

\[ T(t) = Ae^{\alpha k_{m+1} t}. \]

7 Hyperbolic Version Of n-Harmonic Equation

The hyperbolic version of n-harmonic equation is

\[ \beta^2 \Delta_m^n u(x, t) = u_t(x, t). \]  

Where \((x, t) = (x_1, x_2, \ldots, x_m, t)\), \(t > 0\), \(\alpha \in \mathbb{R}\), and \((x, t) \in \mathbb{R}^m \times (0, \infty)\). The difference between the elliptic version of n harmonic equation and hyperbolic one is similar to difference between wave equation and 1-harmonic equation. The solution can written as \(u(x) = \prod_{i=1}^{m} X_i(x_i)T(t)\). \(X_i(x_i)\) has same solutions as the one in n harmonic equation. For \(T(t)\) we have

\[ T''(t) - \beta^2 k_{m+1} T(t) = 0. \]

The solutions is

\[ T(t) = Ce^{\beta \sqrt{k_{m+1} t}} + De^{-\beta \sqrt{k_{m+1} t}}. \]

8 Conclusion

Sepations of variables is powerful method that assist to reduce the difficulty of partial differetion equation problems and solve them. The n-harmonic in dimension has been generalized. Moreover, unbounded boundary value problem has been solved and applications of its convolution. Hyperbolic version and parabolic version are dicussed.

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