Analysis on the High-order Distributed Consensus under Probabilistic Quantized Communication

Huan-xin PENG

School of Mechanical Engineering, Nanjing Institute of Industry Technology, Nanjing Jiangsu 210046, China

Keywords: Distributed consensus, Quantized communication, Probabilistic quantization.

Abstract. Because the high-order distributed consensus algorithm utilizes the previous state values of non-adjacent nodes to accelerate the convergent rate, under the quantized communication, it means that the high-order distributed consensus algorithm utilizes the quantized information of more non-adjacent nodes, the performance of the high-order algorithm become very important under quantized communication. In the paper, the performance and the mean square errors of the high-order probabilistic quantization distributed consensus algorithm were analyzed. By analysis and simulation, the bigger the length of quantization message is, the better the performance is, and the smaller the square errors of the high-order algorithm is.

Introduction

In the past ten years, the distributed consensus algorithm attracted the much attention[1-6]. In practical applications, the information exchange among the nodes of sensors network is based on digital communication, the impact of the quantization errors on the performance of the distributed consensus algorithms has to be considered. The impact of quantization errors on distributed average consensus is considered firstly in [7]. The nodes in a networks based on the distributed average consensus with uniform quantization [8] usually fail to reach a common value. A dithered quantization scheme is proposed in [9-10]. Distributed average consensus with dithered quantization can reach a consensus, but the consensus is random and not equal to the average of initial states of nodes. A probabilistic quantized scheme is present in [11]. A probabilistic quantized distributed averaging algorithm (PQDA) is proposed in [12]. The uniform probabilistic quantization is equivalent to the dithered quantization [13], every node in a networks based on uniform probabilistic quantization can reach a common value, and the common value is random and not equal to the average of the initial states. In the PQDA algorithm, the state of every node is updated by its quantization value and the quantized information of adjacency nodes.

In the paper, the convergence performance and the mean square errors of the high-order probabilistic quantization distributed consensus were analyzed, by analysis and simulation, the results were show.

High-order Distributed Consensus Algorithm

Assuming ideal links and no quantization, the high-order distributed consensus without distortion in [6] is written by:

\[ x_i(k+1) = x_i(k) - \varepsilon \sum_{j \in N_i} a_{ji} \left\{ (x_i(k) - x_j(k)) + \{ y_j(k) + \sum_{l \in N_j} a_{lj} \left\{ (x_l(k-1) - x_j(k-1)) + \cdots + \sum_{s \in N_j} a_{ls} \left\{ (x_s(k-m+2) - x_j(k-m+2)) \cdots \right\}\right\}\right\} \]

\[ y_j(k+1) = \sum_{l \in N_i} a_{lj} (x_i(k) - x_j(k)) \]
We substitute Eq.2 into Eq.1, and get an equivalent equation, and the collective dynamics of the nodes can be written as:

\[ x(k+1) = (I_n - \varepsilon L_n) x(k) - \cdots - \varepsilon L_m x(k-m+1) \]  

(3)

Where \( L_i \) (\( i = 1, 2, \cdots, m \)) is \( i \)-hop graph Laplacian matrix of graph \( G \). We define:

\[
H = \begin{bmatrix}
I_n - \varepsilon L_1 & -\varepsilon L_2 & -\varepsilon L_3 & \cdots & -\varepsilon L_m \\
I_n & 0 & 0 & 0 & 0 \\
0 & I_n & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & I_n
\end{bmatrix}, \quad \text{and} \quad J = \begin{bmatrix}
K & 0 & \cdots & 0 \\
K & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
K & 0 & \cdots & 0
\end{bmatrix}, \quad \text{where} \quad K = \frac{1}{n} 1^T, 
\]

and \( 1 = [1, 1, \ldots, 1]^T \) is a \( n \) dimensional vect. We can get

\[ X(k+1) = HX(k) \]  

(4)

where \( X(k+1) = [x(k+1)^T, x(k)^T, \cdots, x(k-m+2)^T]^T \).

**High-order Distributed Consensus with Probabilistic Quantization**

Considering the probabilistic quantization, we have the following lemma [11].

**Lemma 1.** For the uniform probabilistic quantization, we can infer that \( E(q_i(k)) = x_i(k) \), \( E(q_i(k) - x_i(k))^2 \leq \varepsilon^2/n^2 \), i.e., the quantization value \( q_i(k) \) is an unbiased representation of \( x_i(k) \).

The state of node is updated by its quantization value and quantization values of adjacency nodes [12,13]. The high-order distributed consensus algorithm based on uniform probabilistic quantization can be written by:

\[ x(k+1) = (I_n - \varepsilon L_n) q(k) - \cdots - \varepsilon L_m q(k-m+1) \]  

(5)

where \( q_i(k) \) is the quantization value of node \( i \) in step \( k \), and \( q(k) = [q_1(k), \cdots, q_n(k)]^T \). We define the quantized error \( v_i(k) = x_i(k) - q_i(k) \), then \( v(k) = x(k) - q(k) \), where \( v(k) = [v_1(k), \cdots, v_n(k)]^T \). Then, Eq.5 can be written as:

\[ x(k+1) = (I_n - \varepsilon L_n) x(k) - \cdots - \varepsilon L_m x(k-m+1) - (I_n - \varepsilon L_n) v(k) + \cdots + \varepsilon L_m v(k-m+1) \]  

(6)

based on Eq.4, then Eq.6 can be written as:

\[
X(k+1) = HQ(k) = H \{ X(k) - V(k) \} H^{k+1} X(0) - \sum_{j=0}^{k} H^{k+1-j} V(j) \} 
\]

(7)

where \( Q(k) = [q(k) \cdots]^T \), \( V(k) = [v(k) \cdots v(k-m+1)]^T \).

We suppose the initial condition is \( X(0) = [x^T(0), \cdots]^T \). If the spectral radius \( \rho(H - J) < 1 \), then \( H \) has only one eigenvalue 1, the other eigenvalues \( \lambda(H) < 1 \). For the eigenvalue 1, there is a corresponding left eigenvector \( w_j = [1^T, 0^T, \cdots]^T \) ( \( 0^T \) is a vector), and right eigenvector \( w_j = [1^T, \cdots, 1^T]^T \), \( H w_j = w_j H = w_j^T \), and \( \lim_{k \to \infty} H^k = w_j w_j^T = J \). From (10), we then have the following important result.

**Theory1.** For the high-order distributed consensus with probabilistic quantization, we have.

\[ E \{ \lim_{k \to \infty} x(k+1) \} = \frac{1}{n} 11^T x(0) \]
Proof: \[ E\{\lim_{k \to \infty} X(k + 1)\} = \lim_{k \to \infty} E(X(k + 1)) = \lim_{k \to \infty} \{HX(k) + E(HV(k))\} = JX(0) + \lim_{k \to \infty} E(HV(k)) \]

noting that \(E(V(k) = 0\), so we can obtain \(E\{\lim_{k \to \infty} x(k + 1)\} = \frac{1}{n} F^T x(0)\).

The theory\(^1\) shows that the asymptotic expectation of the states of nodes converge to the average of initial states as system evolves over time.

Because of the quantization effects, the values of nodes are not expected to converge to the average of initial states. What we can hope is that agents reach states which are close to each other and close to the average of initial states. In order to measure the asymptotic disagreement, we model the quantization errors as spatially and temporally uncorrelated random variables \(^{\text{12}}\). From Eq.7, we define the performance index as Euclidean norm. We rewrite Eq.7 as:

\[
\parallel X(k + 1) - JX(0) \parallel^2_F = (H^{k+1}X(0) - JX(0)) (H^{k+1}X(0) - JX(0)) + E(\sum_{j=0}^{k} H^{k+1-j}V(j)) (H^{k+1-j}V(j))
\]

Noting that \((H - J^T)(H - J) = H^2 - J^2\), so we can get \((H - J)^{k+1} = H^{k+1} - J\). Noting that \(JV(j) = 0\). So we can get \(H^{k+1-j}V(j) = (H^{k+1-j} - J)V(j)\). the quantization errors as spatially and temporally uncorrelated random variables, so we can get

\[
\sum_{j=0}^{k} \parallel H^{k+1-j}V(j) \parallel^2_F = \sum_{j=0}^{k} (\parallel H - J \parallel^{k+1-j}V(j) \parallel^2_F) \text{ the mean square errors is}
\]

\[
\parallel X(k + 1) - JX(0) \parallel^2_F = (H^{k+1}X(0) - JX(0)) (H^{k+1}X(0) - JX(0)) + \sum_{j=0}^{k} (\parallel H - J \parallel^{k+1-j}V(j) \parallel^2_F) \leq \rho^{2k+2} (H - J) \parallel X(0) \parallel^2_F
\]

+ \sum_{j=0}^{k} (\rho^{2(k+1-j)} (H - J) \parallel V(j) \parallel^2_F)

Where \(\parallel \cdot \parallel_F\) is denoted as the Euclidean norm or its inductive norm, based on lemma 1, we can get

\[
\parallel V(j) \parallel^2_F \leq \frac{nm}{4} \Delta^2.
\]

The disagreement of the distributed consensus can be written by:

\[
\parallel X(k + 1) - JX(0) \parallel^2_F \leq \rho^{2k+2} (H - J) \parallel X(0) \parallel^2_F + \frac{nm \Delta^2}{4} \sum_{j=0}^{k} \rho^{2(k+1-j)} (H - J)
\]

(8)

Theorem 2. if graph \(G\) is an undirected graph or a directed strongly connected and balance graph, and the high-order algorithm is convergent, then, the worst case is:

\[
\lim_{k \to \infty} \parallel x(k + 1) - Kx(0) \parallel^2_F \leq \frac{n \Delta^2}{4(1 - \rho^2(H - J))}
\]

Proof: if graph \(G\) is an undirected graph or a directed strongly connected and balance graph, and the high-order algorithm is convergent, then \(\lim_{k \to \infty} \rho^k (H - J) = 0\), we can get:

\[
\lim_{k \to \infty} \parallel X(k + 1) - JX(0) \parallel^2_F \leq \frac{nm \Delta^2}{4(1 - \rho^2(H - J))}
\]

Noting that \(E \parallel X(k + 1) - JX(0) \parallel^2_F = \sum_{j=k-m+1}^{k} E \parallel x(j) - Kx(0) \parallel^2_F\). Considering that \(E \parallel X(k + 1) - JX(0) \parallel^2_F = E \parallel X(k) - JX(0) \parallel^2_F\), then, \(\lim_{k \to \infty} \parallel x(k + 1) - Kx(0) \parallel^2_F \leq \frac{n \Delta^2}{4(1 - \rho^2(H - J))} \).
Theory 2 shows that there is an upper bound for the worst case of the high-order algorithm, and the upper bound depends on the topology of graph $G$ and the quantized interval.

**Simulation Results**

In order to verify the efficiency of the high-hop distributed consensus algorithm under quantized communication, we test it on two different networks listed in Figure 1, Figure 2, denoted as $G_1$ and $G_2$. Topology $G_1$ is an undirected regular graph, and $G_2$ is a directed strongly and balanced network with 25 nodes. For the sake of simplicity, we assume the weights $a_{ij} = 1$ for any links, $\varepsilon = 0.02$ s, and the initial state of every node in graph $G_1$ is a random value from 1 to 25.

Figure 3 and Figure 4 show the simulation results of the four-order algorithm with probabilistic quantization under graph $G_1$. Figure 5 and Figure 6 the four-order algorithm with probabilistic quantization under graph $G_2$. Figure 3 and Figure 4 show the simulation results of the four-order algorithm with probabilistic quantization under graph $G_1$. Figure 5 and Figure 6 the four-order algorithm with probabilistic
quantization under graph $G_2$. obviously, the MSE of the high-order algorithm become smaller while $q$ become bigger.

Summary
Under the quantization communication, we analyzed the convergence and the MSE of the high-order algorithm. By analyses and simulations, although the high-order algorithm fails to convergence to the average of initial values, the mean square error of the high-order algorithm is small and the curve chart of the proposed algorithm is similar to the curve chart of the distributed average consensus without distortion.

References
[1] R. Olfati-Saber, R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” IEEE Trans. Automat. Contr. vol. 49, no. 9, pp. 1520–1533, 2004.
[2] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and Cooperation in Networked Multi-Agent Systems,” Proc. IEEE, vol. 95, no. 1, pp. 215–233, Jan. 2007.
[3] L. Xiao, S. Boyd, “Fast linear iterations for distributed averaging,” Systems and Control Letters, vol. 53, pp. 65–78, 2004.
[4] T. C. Aysal, B. N. Oreshkin, and M. J. Coates, “Accelerated Distributed Average Consensus via Localized Node State Prediction,” IEEE Transactions on signal processing, vol. 57, no. 4, pp. 1563–1576, 2009.
[5] Gang Xiong, S. Kishore, “Linear high-order distributed average consensus algorithm in wireless sensor networks,” IEEE/SP 15th Workshop on Statistical Signal Processing, 2009, pp. 529-532.
[6] Peng Huanxin, Qi Guoqing, Sheng Andong. “Pseudo multi-hop distributed consensus algorithm,” Control Theory and Applied, vol. 29, no. 5, pp. 623-628, 2012.
[7] A. Kashyap, T. Basar, R. Srikant, “Quantized consensus,” Automatica, vol. 43, pp. 1192-1203, 2007.
[8] M. E. Yildiz, A. Scaglione, “Differential nested lattice encoding for consensus problems,”Proceedings of the information processing in sensor networks, Cambridge, MA: IEEE, 2007, pp. 89-98.
[9] T. C. Aysal, M. J. Coates, M. G. Rabbat, “Distributed average consensus with dithered quantization,” IEEE transaction on signal processing, vol., 56, no. 10, pp. 4905-4918, 2008.
[10] S. Kar, J. M. F. Moura, “Distributed consensus algorithms in sensor networks: quantized data and random link failure,” IEEE transaction on signal processing, vol. 58, no.3, pp. 1383-1400, 2010.
[11] J. J. Xiao, Z. Q. Luo, “Decentralized estimation in an inhomogeneous sensing environment,” IEEE transaction on information theory, vol. 51, no. 10, pp. 3564-3575, 2005.
[12] T. C. Aysal, M. J. Coates, M. G.Rabbat, “Distributed average consensus using probabilistic quantization,” IEEE/SP 14th workshop on statistical signal processing. Madison, Wisconsin: IEEE, 2007, pp. 640-644.
[13] T. C. Aysal, M. J. Coates, M. G.Rabbat, “ Distributed average consensus with dithered quantization,” IEEE transaction on signal processing, vol. 56, no. 10, pp. 4905-4918, 2008.