ON SOME NEW RESULTS ON ANISOTROPIC SINGULAR PERTURBATIONS OF SECOND-ORDER ELLIPTIC OPERATORS

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Abstract. In this article, we deal with some problems involving a class of singularly perturbed elliptic operators. We prove the asymptotic preserving of a general Galerkin method associated to a semilinear problem. We use a particular Galerkin approximation to estimate the convergence rate on the whole domain, for the linear problem. Finally, we study the asymptotic behavior of the semigroup generated.

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1. INTRODUCTION

Anisotropic singular perturbations problems were introduced by Chipot in [1], these problems can model diffusion phenomena when the diffusion parameters become small in certain directions. We refer the reader to [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] for several works on this topic. In this article, we will study some new theoretical aspects which have not been studied before for these problems.

Let us consider the following perturbed elliptic problem

$$\beta(u_{\varepsilon}) - \text{div}(A_{\varepsilon} \nabla u_{\varepsilon}) = f \text{ in } \Omega,$$

supplemented with the boundary condition

$$u_{\varepsilon} = 0 \text{ on } \partial \Omega.$$

Here, $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1$ and $\Omega_2$ are two bounded open sets of $\mathbb{R}^q$ and $\mathbb{R}^{N-q}$, with $N > q \geq 1$, and $f \in L^2(\Omega)$. We denote by $x = (x_1, \ldots, x_N) = (X_1, X_2) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$ i.e. we split the coordinates into two parts. With this notation we set

$$\nabla = (\partial_{x_1}, \ldots, \partial_{x_N})^T = \left( \begin{array}{c} \nabla X_1 \\ \nabla X_2 \end{array} \right),$$

where

$$\nabla X_1 = (\partial_{x_1}, \ldots, \partial_{x_q})^T \text{ and } \nabla X_2 = (\partial_{x_{q+1}}, \ldots, \partial_{x_N})^T.$$

The function $A = (a_{ij})_{1 \leq i,j \leq N} : \Omega \to \mathcal{M}_N(\mathbb{R})$ satisfies the ellipticity assumptions

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• There exists \( \lambda > 0 \) such that for a.e. \( x \in \Omega \)
\[
A\xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{for any} \quad \xi \in \mathbb{R}^N.
\] (3)

• The coefficients of \( A \) are bounded, that is
\[
a_{ij} \in L^\infty(\Omega) \quad \text{for any} \quad (i,j) \in \{1,2,\ldots,N\}^2.
\] (4)

We have decomposed \( A \) into four blocks
\[
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
\]
where \( A_{11} \), \( A_{22} \) are \( q \times q \) and \( (N-q) \times (N-q) \) matrices respectively. For \( \epsilon \in (0,1] \) we have set
\[
A_\epsilon = \begin{pmatrix}
\epsilon^2 A_{11} & \epsilon A_{12} \\
\epsilon A_{21} & A_{22}
\end{pmatrix}.
\]

The function \( \beta : \mathbb{R} \to \mathbb{R} \) satisfies the following conditions:
\[
\exists M \geq 0 : \forall s \in \mathbb{R}, |\beta(s)| \leq M (1 + |s|).
\] (6)

The weak formulation of problem (1)-(2) is
\[
\begin{cases}
\int_\Omega \beta(u_\epsilon) \varphi dx + \int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \forall \varphi \in H^1_0(\Omega) \\
u_\epsilon \in H^1_0(\Omega),
\end{cases}
\] (7)

where the existence and the uniqueness follow from assumptions (3 - 6). The limit problem is given by
\[
\beta(u) - \text{div}_{\mathcal{X}_2}(A_{22} \nabla u) = f \text{ on } \Omega,
\] (8)
supplemented with the boundary condition
\[
u(X_1, \cdot) = 0 \text{ in } \partial \omega_2, \text{ for } X_1 \in \omega_1.
\] (9)

We introduce the functional space
\[
H^1_0(\Omega; \omega_2) = \{ v \in L^2(\Omega) \text{ such that } \nabla_{\mathcal{X}_2} v \in L^2(\Omega)^{N-q} \text{ and for a.e. } X_1 \in \omega_1, v(X_1, \cdot) \in H^1_0(\omega_2) \},
\]
equipped with the norm \( \| \nabla_{\mathcal{X}_2}(\cdot) \|_{L^2(\Omega)^{N-q}} \). Notice that this norm is equivalent to
\[
\left( \| (\cdot) \|_{L^2(\Omega)}^2 + \| \nabla_{\mathcal{X}_2}(\cdot) \|_{L^2(\Omega)^{N-q}} \right)^{1/2},
\]
thanks to Poincaré’s inequality
\[
\| v \|_{L^2(\Omega)} \leq C_{\omega_2} \| \nabla_{\mathcal{X}_2} v \|_{L^2(\Omega)^{N-q}}, \text{ for any } v \in H^1_0(\Omega; \omega_2).
\] (10)

One can prove that \( H^1_0(\Omega; \omega_2) \) is a Hilbert space. The space \( H^1_0(\Omega) \) will be normed by \( \| \nabla(\cdot) \|_{L^2(\Omega)^N} \). One can check immediately that the embedding \( H^1_0(\Omega) \hookrightarrow H^1_0(\Omega, \omega_2) \) is continuous.
The weak formulation of the limit problem \((8) - (9)\) is given by
\[
\begin{aligned}
\int_{\omega_2} \beta(u)(X_1, \cdot) \psi dX_2 + \int_{\omega_2} A_{22}(X_1, \cdot) \nabla X_2 u(X_1, \cdot) \cdot \nabla X_2 \psi dX_2 \\
= \int_{\omega_2} f(X_1, \cdot) \psi dX_2, \forall \psi \in H_0^1(\omega_2) \\
\end{aligned}
\tag{11}
\]

This problem has been studied in \([9]\), and the author proved the following (see Proposition 4 in the above reference)

**Theorem 1.1.** Under assumptions (3), (4), (5) and (6) we have:
\[
u_\epsilon \to u \text{ in } L^2(\Omega), \quad \epsilon \nabla X_1 u_\epsilon \to 0 \text{ in } L^2(\Omega)^q \text{ and } \nabla X_2 u_\epsilon \to \nabla X_2 u \text{ in } L^2(\Omega)^{N-q},
\]
where \(u_\epsilon\) is the unique solution to \((7)\) in \(H_0^1(\Omega)\) and \(u\) is the unique solution to \((11)\) in \(H_0^1(\Omega; \omega_2)\).

Remark that for \(\varphi \in H_0^1(\Omega; \omega_2)\), and for a.e \(X_1 \in \omega_1\) we have \(\varphi(X_1, \cdot) \in H_0^1(\omega_2)\). By testing with \(\varphi(X_1, \cdot)\) in \((11)\) and by integrating over \(\omega_1\) we get
\[
\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla X_2 u \cdot \nabla X_2 \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in H_0^1(\Omega; \omega_2). \tag{12}
\]

This paper is organized as follows:

- As a first main result, we will prove the asymptotic preserving of the general Galerkin method for the elliptic problem \((1)-(2)\). This concept has been introduced by S. Jin in \([12]\) and it could be illustrated by the following commutative diagram

\[
\begin{array}{ccc}
P_{\epsilon,n} & \xrightarrow{n \to \infty} & P_{\epsilon} \\
\downarrow \epsilon \to 0 & & \downarrow \epsilon \to 0 \\
P_n & \xrightarrow{n \to \infty} & P_0
\end{array}
\]

here, \(P_{\epsilon,n}\) is the Galerkin approximation of the infinite dimensional perturbed problem \(P_{\epsilon}\) and \(P_n\) is the Galerkin approximation of the infinite dimensional limit problem \(P_0\). We will derive an estimation of the error for a general Galerkin method, and by using a Céa type lemmas we prove the asymptotic-preserving of the method.

- As a second main result, we will prove, in the linear case, a new result on the estimation of the global convergence rate, such a result is of the form \(\|\nabla X_2 (u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \leq C \epsilon\). This estimation is an improvement of the local one proved by Chipot and Guesmia in \([3]\). Our arguments are based on the use of a particular Galerkin approximation constructed by a tensor product.

- In section 4, we will prove our third main result on the asymptotic behavior of the semigroup generated by the perturbed elliptic operator \(\text{div}(A, \nabla \cdot)\), and we will give a simple application to linear parabolic problems.

Finally, to make the paper readable, we put some intermediate technical lemmas in the appendix.

2. **Main theorems for the elliptic problem**

**Definition 2.1.** Let \((V_n)\) be a sequence of finite dimensional subspaces of a Hilbert space \(H\). We say that \((V_n)\) approximates \(H\), if for every \(w \in H\),
\[
\inf_{v \in V_n} \|w - v\|_H \to 0 \text{ as } n \to \infty.
\]
For a sequence $(V_n)$ of a finite dimensional spaces of $H^1_0(\Omega)$, and for every $\epsilon \in (0,1]$ and $n \in \mathbb{N}$, we denote $u_{\epsilon,n}$ the unique solution of

\[
\begin{cases}
\int_{\Omega} \partial(u_{\epsilon,n}) \varphi dx + \int_{\Omega} A_{\epsilon,n} \nabla u_{\epsilon,n} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \\ u_{\epsilon,n} \in V_n.
\end{cases}
\]  

(13)

We suppose that

\[\partial_{\epsilon,n}a_{ij} \in L^\infty(\Omega), \partial_{\epsilon,n}a_{ij} \in L^\infty(\Omega) \text{ for } i = 1, ..., q \text{ and } j = q + 1, ..., N.\]  

(14)

We have the following:

**Theorem 2.2.** Let $\Omega = \omega_1 \times \omega_2$, where $\omega_1$ and $\omega_2$ are two bounded open sets of $\mathbb{R}^q$ and $\mathbb{R}^{N-q}$ respectively, with $N > q \geq 1$. Suppose that $f \in L^2(\Omega)$ and assume (3), (4), (5), (6), and (14). Let $(V_n)$ be a sequence of finite dimensional spaces of $H^1_0(\Omega)$ which approximates it in the sense of Definition 2.1. Let $(u_{\epsilon,n})$ be the sequence of solutions of (13) then we have:

\[\lim_{n}(\lim_{\epsilon} u_{\epsilon,n}) = \lim_{\epsilon}(\lim_{n} u_{\epsilon,n}) = u, \text{ in } H^1_0(\Omega; \omega_2),\]

where $u$ is the unique solution of (11) in $H^1_0(\Omega; \omega_2)$.

Our second result concerns the estimation of the rate of convergence for problem (7) in the linear case, this result could be seen as a refinement of the following result proved in [3]:

\[\forall \omega'_1 \subset \subset \omega_1 \text{ open : } \|\nabla X_2(u_{\epsilon} - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(\epsilon), \text{ and } \|\nabla X_1(u_{\epsilon} - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(1).\]  

(15)

In the above reference, the authors have supposed that

\[\nabla X_1 f \in L^2(\Omega)^q,\]  

(16)

assumption (14), and that $\nabla X_1 A_{22} \in L^\infty(\Omega)$. Our contribution consists in extending (15) to the whole domain $\Omega$, to obtain such a result we take some additional hypothesis on $A$ and $f$, namely:

For a.e. $X_2 \in \omega_2 : f(., X_2) \in H^1_0(\omega_1),$

(17)

and

The block $A_{22}$ depends only on $X_2$.

(18)

**Theorem 2.3.** Let $\Omega = \omega_1 \times \omega_2$ where $\omega_1$ and $\omega_2$ are two bounded open sets of $\mathbb{R}^q$ and $\mathbb{R}^{N-q}$ respectively, with $N > q \geq 1$. Suppose that $\beta = 0$, and let us assume that $A$ satisfies (3), (4), (14) and (18). Let $f \in L^2(\Omega)$ such that (16) and (17), then there exists $C > 0$ depending on $f$, $\lambda$, $C_{\omega_2}$ and $A$ such that:

\[\forall \epsilon \in (0,1] : \|\nabla X_2(u_{\epsilon} - u)\|_{L^2(\Omega)^{N-q}} \leq C\epsilon,\]

where $u_{\epsilon}$ is the unique solution of (7) in $H^1_0(\Omega)$ and $u$ is the unique solution to (11) in $H^1_0(\Omega; \omega_2)$. Moreover, we have:

\[u \in H^1_0(\Omega) \text{ and } \nabla X_1(u_{\epsilon} - u) \rightharpoonup 0 \text{ weakly in } L^2(\Omega)^q, \text{ as } \epsilon \to 0.\]

The constant $C$ is of the form $C_1 \|\nabla X_1 f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)}$ where $C_1, C_2$ are dependent on $A, \lambda, C_{\omega_2}$.

The proof of this theorem will be done in two steps. First, we give the proof in the case $f \in H^1_0(\omega_1) \otimes H^1_0(\omega_2)$, and next that we conclude by a density argument. Let us recall this density rule, which will be used throughout this article: If $(E, \tau)$ and $(F, \tau')$ are two topological spaces such that $E \subset F$, and $E$ is dense in $F$ and the canonical injection $E \to F$ is continuous, then every dense subset in $(E, \tau)$ is dense in $(F, \tau')$. 
Remark 2.4. The hypothesis (17) is necessary to obtain the global boundedness of $\nabla X_1(u_\epsilon - u)$. We can observe that through this 2d example, we take

$$A = id, f : (x_1, x_2) \mapsto \cos(x_1) \sin(x_2), \text{ and } \Omega = (0, \pi) \times (0, \pi).$$

In this case, we have $u(x_1, x_2) = \cos(x_1) \sin(x_2)$. The quantity $\|\nabla X_1(u_\epsilon - u)\|_{L^2(\Omega)^N}$ could not be bounded. Indeed, if we suppose the converse then according to Theorem 1.1 there exists a subsequence still labeled $(u_\epsilon)$ such that $\nabla X_1(u_\epsilon - u) \rightarrow 0$ weakly in $L^2(\Omega)^N$, and $\|\nabla X_2(u_\epsilon - u)\|_{L^2(\Omega)^N-q} \rightarrow 0$. Whence $u \in H^1_0(\Omega)$ which is a contradiction.

Let us finish by giving this remark which will be used later in section 4.

Remark 2.5. Suppose that $\beta : s \rightarrow \mu s$, for some $\mu > 0$, and suppose that assumptions of Theorem 2.3 hold, then we have the same results of Theorem 2.3 with the same constants. Assume, in addition, that the block $A_{12}$ satisfies the following:

$$\partial^2_{x_1x_2}a_{ij} \in L^2(\Omega), \text{ for } i = 1, ..., q, j = q + 1, ..., N,$$

then we have:

$$\forall \epsilon \in (0, 1] : \|\nabla X_2(u_\epsilon - u)\|_{L^2(\Omega)^N-q} \leq \frac{\epsilon}{\mu} \left(C'_1 \|\nabla X_1 f\|_{L^2(\Omega)^N} + C'_2 \|f\|_{L^2(\Omega)}\right),$$

where $C'_1, C'_2$ are only dependent on $A, \lambda, C_{\omega_2}$.

3. The Analysis of a General Galerkin Method

3.1. Preliminaries

Let $V \subset H_0^1(\Omega)$ be a closed subspace of $H_0^1(\Omega, \omega_2)$. Notice that $V$ is closed in $H^1_0(\Omega)$, thanks to the continuous embedding $H^1_0(\Omega) \hookrightarrow H^1_0(\Omega, \omega_2)$. Let $f \in L^2(\Omega)$, we denote by $u_{e,V,f}$ the unique solution of

$$\left\{
\begin{array}{l}
\int_{\Omega} \beta(u_{e,V,f}) \varphi dx + \int_{\Omega} A_x \nabla u_{e,V,f} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\
u_{e,V,f} \in V.
\end{array}
\right.$$

(20)

We denote by $u_{V,f}$ the unique solution of

$$\left\{
\begin{array}{l}
\int_{\Omega} \beta(u_{V,f}) \varphi dx + \int_{\Omega} A_{2x} \nabla X_2 u_{V,f} \cdot \nabla X_2 \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\
u_{V,f} \in V.
\end{array}
\right.$$

(21)

Under assumptions of Theorem 1.1, one can prove by using the Schauder fixed point theorem that $u_{e,V,f}$ exists. For the existence of $u_{V,f}$ see Appendix C. The uniqueness, for the two problems, follows immediately from (3) and (5). Now, let us begin by some preliminary lemmas

**Lemma 3.1.** Under assumptions of Theorem 1.1 and for any $\epsilon \in (0, 1]$, we have the following bounds:

$$\|\nabla u_{e,V,f}\|_{L^2(\Omega)^N} \leq \frac{C_\Omega \|f\|_{L^2(\Omega)}}{\lambda \epsilon^2}, \text{ and } \|\nabla u_{e,f}\|_{L^2(\Omega)^N} \leq \frac{C_\Omega \|f\|_{L^2(\Omega)}}{\lambda \epsilon^2}. \tag{22}$$

$$\|\nabla X_2 u_{e,V,f}\|_{L^2(\Omega)^N-q} \leq \frac{C_\omega \|f\|_{L^2(\Omega)}}{\lambda}, \text{ and } \|\nabla X_2 u_{e,f}\|_{L^2(\Omega)^N-q} \leq \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda}. \tag{23}$$

$$\|\beta(u_{e,V,f})\|_{L^2(\Omega)} \leq \frac{M}{\epsilon^2} \left(\|\Omega\|^2 + \frac{C_1^2 \|f\|_{L^2(\Omega)}}{\lambda}\right), \text{ and } \|\beta(u_{e,f})\|_{L^2(\Omega)} \leq \frac{M}{\epsilon^2} \left(\|\Omega\|^2 + \frac{C_1^2 \|f\|_{L^2(\Omega)}}{\lambda}\right). \tag{24}$$
\[ \| \beta(u_{V,f}) \|_{L^2(\Omega)} \leq M \left( \frac{\Omega}{\lambda} \right) + \frac{C^2_2 \| f \|_{L^2(\Omega)}}{\lambda} \), and \( \| \beta(u_f) \|_{L^2(\Omega)} \leq M \left( \frac{\Omega}{\lambda} \right) + \frac{C^2_2 \| f \|_{L^2(\Omega)}}{\lambda} \). \] (25)

Here, \( C_\Omega \) is the Poincaré constant of \( \Omega \), and \( u_{e,f}, u_f \) are the unique solutions of (7) and (11) respectively.

**Proof.** These bounds follow easily from a suitable choice of the test functions, monotonicity and ellipticity assumptions. Let us prove, for example, the second inequality in (23) and the second inequality in (25). According to Theorem 1.1 one can take \( \varphi = u_f \) in (12), using ellipticity assumption and the fact that \( \int_\Omega \beta(u_f)u_f dx \geq 0 \) (thanks to (5)) we obtain

\[ \lambda \int_\Omega |\nabla u_f|^2 dx \leq \int_\Omega f dx. \]

By the Cauchy-Schwarz inequality and Poincaré's inequality (10), we obtain the second inequality of (23). Now, by using assumption (6), we obtain

\[ |\beta(u_f)|^2 \leq M^2 (1 + |u_f|)^2. \]

Integrating over \( \Omega \) and applying Minkowski inequality, (10), and (23) we obtain the second inequality of (25). \( \square \)

By using the above lemma, one can prove the following Céa type lemma

**Lemma 3.2.** Under assumptions of Theorem 1.1 we have:

\[ \| \nabla u^*_f \|_{L^2(\Omega)^N} \leq C_\text{Céa} \left( \inf_{v \in V} \| \nabla u^*_f (v - u_f) \|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}, \] (26)

and for any \( \epsilon \in (0, 1] \):

\[ \| \nabla (u_{e,f} - u_f) \|_{L^2(\Omega)^N} \leq \frac{C_\text{Céa}'}{\epsilon} \left( \inf_{v \in V} \| \nabla (v - u_{e,f}) \|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}, \] (27)

where

\[ C_\text{Céa}^2 = \frac{1}{\lambda} \left[ 2MC_{C_1} \left( \frac{\Omega}{\lambda} \right)^\frac{1}{2} + \frac{C^2_2 \| f \|_{L^2(\Omega)}}{\lambda} \right] + \| A \|_{L^\infty(\Omega)} \frac{2C_2 \| f \|_{L^2(\Omega)}}{\lambda}, \]

and

\[ C_\text{Céa}' = \frac{1}{\lambda} \left[ 2MC_1 \left( \frac{\Omega}{\lambda} \right)^\frac{1}{2} + \frac{C_{L^2(\Omega)}}{\lambda} \right] + \| A \|_{L^\infty(\Omega)} \frac{2C_1 \| f \|_{L^2(\Omega)}}{\lambda}. \]

**Proof.** The proofs of these two inequalities are similar. So, let us prove the first one. Using the Galerkin orthogonality one has, for \( v \in V \):

\[ \int_\Omega (\beta(u_{V,f}) - \beta(u_f))(uv_{f} - u_f)dx + \lambda \| \nabla u^*_f (uv_{f} - u_f) \|^2_{L^2(\Omega)^{N-1}} \]

\[ \leq \int_\Omega (\beta(u_{V,f}) - \beta(u_f))(v - u_f)dx + \int_\Omega A_{22} \nabla u^*_f (uv_{f} - u_f) \cdot \nabla (v - u_f)dx. \]

Using the fact that \( \int_\Omega (\beta(u_{V,f}) - \beta(u_f))(uv_{f} - u_f)dx \geq 0 \), then by Cauchy-Schwarz and Poincaré’s inequalities we derive

\[ \lambda \| \nabla u^*_f (uv_{f} - u_f) \|^2_{L^2(\Omega)^{N-1}} \leq \left[ C_{C_1} \| \beta(u_{V,f}) - \beta(u_f) \|_{L^2(\Omega)} + \| A_{22} \|_{L^\infty(\Omega)} \| \nabla u^*_f (uv_{f} - u_f) \|_{L^2(\Omega)^{N-1}} \right] \times \| \nabla (v - u_f) \|_{L^2(\Omega)^{N-1}}. \]
Now, by using (23), (25) and the triangle inequality we obtain
\[ \lambda\|\nabla_x(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}} \leq \]
\[ 2 \left[ MC_{\omega_2} \left( \|f\|_{L^2(\Omega)} + \frac{C_{\omega_2}}{\lambda} \right) + \frac{C_{22}}{L^\infty(\Omega)} \frac{C_{\omega_2}}{\lambda} \|f\|_{L^2(\Omega)} \right] \times \|\nabla_x(v - u_f)\|_{L^2(\Omega)^{N-q}}, \]
and (26) follows.

**Remark 3.3.** 1) If \( \beta = 0 \) (the linear case), then we have for any \( \epsilon \in (0,1] \):
\[ \|\nabla_{\epsilon,V,f} - \nabla_{\epsilon,f}\|_{L^2(\Omega)^N} \leq \frac{\|A\|_{L^\infty(\Omega)}}{\lambda\epsilon^2} \inf_{v \in V} \|\nabla v - \nabla_{\epsilon,f}\|_{L^2(\Omega)^N}. \]
\[ \|\nabla X_2 u_{V,f} - \nabla X_2 u_f\|_{L^2(\Omega)^{N-q}} \leq \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V} \|\nabla X_2 v - \nabla X_2 u_f\|_{L^2(\Omega)^{N-q}}. \]
2) If \( \beta \) is Lipschitz, then we can obtain estimations similar to those of the linear case.

### 3.2. Error estimates in the Galerkin method

**Lemma 3.4.** Under assumptions of Theorem 1.1, suppose in addition that (14) holds, then we have for every \( \epsilon \in (0,1] \):
\[ \|\nabla_{\epsilon,V,f} - \nabla_{\epsilon,f}\|_{L^2(\Omega)^N} \leq \epsilon \left( C_1 \|\nabla_{\epsilon,V,f}\|_{L^2(\Omega)^N} + C_2 \|f\|_{L^2(\Omega)} \right), \]
and
\[ \|\nabla X_1 (u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^N} \leq \frac{1}{\sqrt{2}} \left( C_1 \|\nabla_{\epsilon,V,f}\|_{L^2(\Omega)^N} + C_2 \|f\|_{L^2(\Omega)} \right), \]
where
\[ C_1 = \left( \frac{4(C + C')}{\lambda} \right)^{\frac{1}{2}} \text{ and } C_2 = \frac{2\sqrt{C''}C_{\omega_2}}{\lambda^{3/2}}. \]

Here, \( C, C', \) and \( C'' \) are given by (29), (31) and (32). Notice that these constants are independent of \( \epsilon, V \) and \( f \).

**Proof.** By subtracting (21) from (20) we get, for every \( v \in V \):
\[ \int_\Omega (\beta_{\epsilon,V,f} - \beta_{V,f}) v dx + \epsilon^2 \int_\Omega A_{11} \nabla X_1 u_{\epsilon,V,f} \cdot \nabla X_1 v dx \]
\[ + \epsilon \int_\Omega A_{12} \nabla X_2 u_{\epsilon,V,f} \cdot \nabla X_1 v dx + \epsilon \int_\Omega A_{21} \nabla X_1 u_{\epsilon,V,f} \cdot \nabla X_2 v dx \]
\[ + \int_\Omega A_{22} \nabla X_2 (u_{\epsilon,V,f} - u_{V,f}) \cdot \nabla X_2 v dx = 0, \]
Testing with \( v = u_{\epsilon,V,f} - u_{V,f} \), we obtain
\[ \int_\Omega (\beta_{\epsilon,V,f} - \beta_{V,f}) (u_{\epsilon,V,f} - u_{V,f}) dx + \int_\Omega A_{11} \nabla(u_{\epsilon,V,f} - u_{V,f}) \cdot \nabla(u_{\epsilon,V,f} - u_{V,f}) \]
\[ = -\epsilon^2 \int_\Omega A_{11} \nabla X_1 u_{V,f} \cdot \nabla X_1 (u_{\epsilon,V,f} - u_{V,f}) dx - \epsilon \int_\Omega A_{12} \nabla X_2 u_{V,f} \cdot \nabla X_1 (u_{\epsilon,V,f} - u_{V,f}) dx \]
\[ - \epsilon \int_\Omega A_{21} \nabla X_1 u_{V,f} \cdot \nabla X_2 (u_{\epsilon,V,f} - u_{V,f}) dx. \]
whence, by using (5) and the ellipticity assumption we get
\[
\epsilon^2 \lambda \int_\Omega |\nabla X_1(u, v, f - u, v, f)|^2 \, dx + \lambda \int_\Omega |\nabla X_2(u, v, f - u, v, f)|^2 \, dx \leq \\
- \epsilon^2 \int_\Omega A_{11} \nabla X_1 u, v, f \cdot \nabla X_1 (u, v, f - u, v, f) \, dx - \epsilon \int_\Omega A_{12} \nabla X_2 u, v, f \cdot \nabla X_1 (u, v, f - u, v, f) \, dx \\
- \epsilon \int_\Omega A_{21} \nabla X_1 u, v, f \cdot \nabla X_2 (u, v, f - u, v, f) \, dx.
\]

Let us estimate the first and the last term of the second member in the above inequality. By using Young’s inequality we obtain
\[
- \epsilon^2 \int_\Omega A_{11} \nabla X_1 u, v, f \cdot \nabla X_1 (u, v, f - u, v, f) \, dx \\
\leq \epsilon^2 \lambda \frac{2}{\lambda} \int_\Omega |\nabla X_1 (u, v, f - u, v, f)|^2 \, dx + \epsilon^2 \frac{\|A_{11}\|^2_{L^\infty(\Omega)}}{2\lambda} \int_\Omega |\nabla X_1 u, v, f|^2 \, dx,
\]
and
\[
- \epsilon \int_\Omega A_{21} \nabla X_1 u, v, f \cdot \nabla X_2 (u, v, f - u, v, f) \, dx \\
\leq \epsilon \frac{\|A_{21}\|^2_{L^\infty(\Omega)}}{2\lambda} \int_\Omega |\nabla X_2 (u, v, f - u, v, f)|^2 \, dx + \frac{\lambda}{2} \int_\Omega |\nabla X_1 (u, v, f - u, v, f)|^2 \, dx,
\]
thus
\[
\frac{\epsilon^2 \lambda}{2} \|\nabla X_1 (u, v, f - u, v, f)\|^2_{L^2(\Omega)} + \frac{\lambda}{2} \|\nabla X_2 (u, v, f - u, v, f)\|^2_{L^2(\Omega)^{n-q}} \\
\leq C \epsilon^2 \int_\Omega |\nabla X_1 u, v, f|^2 \, dx - \epsilon \int_\Omega A_{12} \nabla X_2 u, v, f \cdot \nabla X_1 (u, v, f - u, v, f) \, dx,
\]
(28)

where
\[
C = \frac{\|A_{21}\|^2_{L^\infty(\Omega)} + \|A_{11}\|^2_{L^\infty(\Omega)}}{2\lambda}.
\]
(29)

Now, we estimate \(- \epsilon \int_\Omega A_{12} \nabla X_2 u, v, f \cdot \nabla X_1 (u, v, f - u, v, f) \, dx\). Since \(u, v, f - u, v, f \in L^1_0(\Omega)\) and \(\partial_x a_{ij} \in L^\infty(\Omega)\), \(\partial_x \partial_x a_{ij} \in L^\infty(\Omega)\) for \(i = 1, \ldots, q\) and \(j = q + 1, \ldots, N\), (assumption (14)) then we can show by a simple density argument that for \(i = 1, \ldots, q\) and \(j = q + 1, \ldots, N\), \(\partial_x a_{ij}(u, v, f - u, v, f)) \in L^2(\Omega)\) and:
\[
\partial_x a_{ij}(u, v, f - u, v, f)) = (u, v, f - u, v, f)\partial_x a_{ij} + a_{ij} \partial_x (u, v, f - u, v, f), \text{ for } k = i, j.
\]
Whence

\[- \epsilon \int_{\Omega} A_{12} \nabla X_2 u_{V,f} \cdot \nabla X_2 (u_{e,V,f} - u_{V,f}) dx = - \frac{q}{a} \sum_{i=1}^{N} \sum_{j=q+1}^{N} \int_{\Omega} a_{ij} \partial_{x_j} u_{V,f} \partial_{x_i} (u_{e,V,f} - u_{V,f}) dx \]

\[= - \epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} \partial_{x_i} (a_{ij} (u_{e,V,f} - u_{V,f})) \partial_{x_j} u_{V,f} dx \]

\[+ \epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} (u_{e,V,f} - u_{V,f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V,f} dx \]

\[= - \epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} \partial_{x_i} (a_{ij} (u_{e,V,f} - u_{V,f})) \partial_{x_j} u_{V,f} dx \]

\[+ \epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} (u_{e,V,f} - u_{V,f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V,f} dx, \]

where we have used \( \int_{\Omega} \partial_{x_i} (a_{ij} (u_{e,V,f} - u_{V,f})) \partial_{x_j} u_{V,f} dx = \int_{\Omega} \partial_{x_j} (a_{ij} (u_{e,V,f} - u_{V,f})) \partial_{x_i} u_{V,f} dx \) which follows by a simple density argument (recall that \( u_{V,f} \in H^1_0(\Omega) \)). Therefore

\[- \epsilon \int_{\Omega} A_{12} \nabla X_2 u_{V,f} \cdot \nabla X_2 (u_{e,V,f} - u_{V,f}) dx = - \epsilon \sum_{i=1}^{q} \sum_{j=q+1}^{N} \int_{\Omega} (u_{e,V,f} - u_{V,f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V,f} dx \]

\[= \frac{q}{a} \sum_{i=1}^{N} \sum_{j=q+1}^{N} \int_{\Omega} a_{ij} \partial_{x_j} u_{V,f} \partial_{x_i} (u_{e,V,f} - u_{V,f}) dx \]

By Young’s and Poincaré’s inequalities we obtain

\[- \epsilon \int_{\Omega} A_{12} \nabla X_2 u_{V,f} \cdot \nabla X_2 (u_{e,V,f} - u_{V,f}) dx \leq \frac{\lambda}{4} \int_{\Omega} |\nabla X_2 (u_{e,V,f} - u_{V,f})|^2 dx \]

\[+ C' \epsilon^2 \int_{\Omega} |\nabla X_2 u_{V,f}|^2 dx + C'' \epsilon^2 \int_{\Omega} |\nabla X_2 u_{V,f}|^2 dx, \]

where

\[C' = \frac{3 \left[ C_{\omega_{2,1 \leq i \leq q+1 \leq j \leq N}} \| \partial_{x_j} a_{ij} \|_{L^\infty(\Omega)} (N - q) \right]^2 + 3 \left( \max_{1 \leq i \leq q+1 \leq j \leq N} \| a_{ij} \|_{L^\infty(\Omega)} (N - q) \right)^2}{\lambda}, \quad (31)\]

and

\[C'' = \frac{3 \left[ qC_{\omega_{2,1 \leq i \leq q+1 \leq j \leq N}} \| \partial_{x_i} a_{ij} \|_{L^\infty(\Omega)} \right]^2}{\lambda}. \quad (32)\]
By using (23) we obtain
\[- \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{e,V,f} - u_{V,f}) \, dx \leq \]
\[ \frac{\lambda}{4} \int_{\Omega} |\nabla_{X_2} (u_{e,V,f} - u_{V,f})|^2 \, dx + C'' \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 \, dx + \epsilon^2 C'' \left( \frac{C_{\omega} \| f \|_{L^2(\Omega)}}{\lambda} \right)^2. \]

Combining (28) and (33) we get
\[ \frac{\epsilon^2 \lambda}{2} \| \nabla_{X_1} (u_{e,V,f} - u_{V,f}) \|_{L^2(\Omega)^{N-q}}^2 + \frac{\lambda}{4} \| \nabla_{X_2} (u_{e,V,f} - u_{V,f}) \|_{L^2(\Omega)^{N-q}}^2 \]
\[ \leq \epsilon^2 \left( (C + C'') \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 \, dx + C'' \left( \frac{C_{\omega} \| f \|_{L^2(\Omega)}}{\lambda} \right)^2 \right), \]

and the proof is finished. \[ \square \]

Using the triangle inequality, the above Lemma and (26) we obtain the following estimation of the global error between \( u_{e,V,f} \) and \( u_f \).

**Corollary 3.5.** Under assumptions of Lemma 3.4 we have for any \( \epsilon \in (0,1] \):

\[ \| \nabla_{X_2} (u_{e,V,f} - u_f) \|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 \| \nabla_{X_1} u_{V,f} \|_{L^2(\Omega)^q} + C_2 \| f \|_{L^2(\Omega)} \right) + C_{ce\epsilon} \left( \inf_{v \in V} \| \nabla_{X_2} (v - u_f) \|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}. \]

Now, we give an important remark which will be used to prove the inequality given in Remark 2.5.

**Remark 3.6.** When \( \beta(s) = \mu s \) for some \( \mu > 0 \) and when the block \( A_{12} \) satisfies assumption (19), then by performing some integration by parts in the last term of (30), and by using the fact that

\[ \| u_{V,f} \|_{L^2(\Omega)} \leq \frac{1}{\mu} \| f \|_{L^2(\Omega)}, \]

we can obtain the following estimation:

\[ \forall \epsilon \in (0,1] : \| \nabla_{X_2} (u_{e,V,f} - u_{V,f}) \|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C'_1 \| \nabla_{X_1} u_{V,f} \|_{L^2(\Omega)^q} + C'_2 \frac{\| f \|_{L^2(\Omega)}}{\mu} \right), \]

where \( C'_1, C'_2 > 0 \) are independent of \( f, V, \mu \) and \( \epsilon \).

### 3.3. Proof of Theorem 2.2

Let \( (V_n) \) be a sequence of finite dimensional subspaces which approximates \( H^1_0(\Omega) \) in the sense of Definition 2.1. Using the density of \( H^1_0(\Omega) \) in \( H^1_0(\Omega,\omega_2) \) (Lemma A.1, Appendix A), one can check easily that \( (V_n) \) approximates \( H^1_0(\Omega,\omega_2) \) in the same sense. Therefore, one has:

\[ \text{For every } \epsilon \in (0,1] : \inf_{v \in V_n} \| \nabla (v - u_{e,f}) \|_{L^2(\Omega)^N} \to 0 \text{ as } n \to \infty, \]  

(34)
and
\[ \inf_{v \in V_n} \| \nabla X_2(v - u_f) \|_{L^2(\Omega)^{N - q}} \to 0 \quad \text{as} \quad n \to \infty. \] (35)

According to Lemma 3.4, (26) and (27) we have, for every \( n \in \mathbb{N} \) and \( \epsilon \in (0,1] \):

\[ \| \nabla X_2(u_{\epsilon,V_n,f} - u_{V_n,f}) \|_{L^2(\Omega)^{N - q}} \leq \epsilon \left( C_1 \| \nabla X_1 u_{V_n,f} \|_{L^2(\Omega)^q} + C_2 \| f \|_{L^2(\Omega)} \right), \] (36)

\[ \| \nabla X_2(u_{V_n,f} - u_f) \|_{L^2(\Omega)^{N - q}} \leq C_{\epsilon \delta a} \left( \inf_{v \in V_n} \| \nabla X_2(v - u_f) \|_{L^2(\Omega)^{N - q}} \right)^{1/2}, \] (37)

and

\[ \| \nabla (u_{\epsilon,V_n,f} - u_{\epsilon,f}) \|_{L^2(\Omega)^N} \leq C_{\epsilon \delta a} \left( \inf_{v \in V_n} \| \nabla (v - u_{\epsilon,f}) \|_{L^2(\Omega)^N} \right)^{1/2}. \] (38)

- Fix \( \epsilon \) and pass to the limit in (38) by using (34), we get

\[ u_{\epsilon,V_n,f} \to u_{\epsilon,f} \quad \text{as} \quad n \to \infty \quad \text{in} \quad H^1_0(\Omega), \]

in particular, by using the continuous embedding \( H^1_0(\Omega) \hookrightarrow H^1_0(\Omega, \omega_2) \) we deduce

\[ u_{\epsilon,V_n,f} \to u_{\epsilon,f} \quad \text{as} \quad n \to \infty \quad \text{in} \quad H^1_0(\Omega, \omega_2). \]

Now, passing to the limit as \( \epsilon \to 0 \) by using Theorem 1.1, we get

\[ \lim_{\epsilon} (\lim_n u_{\epsilon,V_n,f}) = u_f \quad \text{in} \quad H^1_0(\Omega, \omega_2). \] (39)

- Fix \( n \) and pass to the limit as \( \epsilon \to 0 \) in (36), we get

\[ u_{\epsilon,V_n,f} \to u_{V_n,f} \quad \text{as} \quad \epsilon \to 0 \quad \text{in} \quad H^1_0(\Omega, \omega_2). \]

Now, passing to the limit as \( n \to \infty \) in (37) by using (35), we get

\[ \lim_{n} (\lim_{\epsilon} u_{\epsilon,V_n,f}) = u_f \quad \text{in} \quad H^1_0(\Omega, \omega_2). \] (40)

Finally, Theorem 2.2 follows from (39) and (40).

### 3.4. Proof of Theorem 2.3

Throughout this subsection, we will suppose that \( \beta = 0 \). The key of the proof of Theorem 2.3 is based on the control of the quantity \( \| \nabla X_1 u_{V,f} \|_{L^2(\Omega)^q} \) independently of \( V \). In fact, we need the following:

**Lemma 3.7.** Let us assume that \( A \) satisfies (3), (4), and that \( A_{22} \) satisfies (18). Let \( V_1 \) and \( V_2 \) be two finite dimensional subspaces of \( H^1_0(\omega_1) \) and \( H^1_0(\omega_2) \) respectively. Let \( f \in V_1 \otimes V_2 \), and let \( u_{V,f} \) be the unique solution in \( V = V_1 \otimes V_2 \) to:

\[ \int_{\Omega} A_{22}(X_2) \nabla X_2 u_{V,f} \cdot \nabla X_2 v dx = \int_{\Omega} f v dx, \quad \forall v \in V_1 \otimes V_2, \] (41)

then we have:

\[ \| \nabla X_1 u_{V,f} \|_{L^2(\Omega)^q} \leq C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q}, \]

where \( C_3 \) is given by \( C_3 = \sqrt{q} \frac{C_{\epsilon \delta a}}{A}. \)
Proof. The proof is based on the difference quotient method (see for instance [13] page 168). Let $v = \varphi \otimes \psi \in V_1 \otimes V_2$. The function \( X_1 \mapsto \int_{\omega_2} A_{22}(X_2) \nabla X_2 u_{V,f}(X_1, X_2) \cdot \nabla X_2 \psi \, dX_2 \) belongs to \( V_1 \). In fact $u_{V,f} = \sum_{f_{\text{finite}}} \varphi_i \otimes \psi_i$, and whence $\int_{\omega_2} A_{22}(X_2) \nabla X_2 u_{V,f} \cdot \nabla X_2 \psi \, dX_2$ is a linear combination of $\varphi_i$, thanks to the linearity of the integral. Similarly, the function \( X_1 \mapsto \int_{\omega_2} f(X_1, X_2) \psi \, dX_2 \) belongs to \( V_1 \). Now, testing with $v$ in (41), we derive:

$$\int_{\omega_1} \left( \int_{\omega_2} \{ A_{22}(X_2) \nabla X_2 u_{V,f} \cdot \nabla X_2 \psi - f \psi \} \, dX_2 \right) \varphi \, dX_1 = 0,$$

thus, when $\varphi$ run through a set of an orthogonal basis of the euclidean space \( V_1 \) equipped with the \( L^2(\omega_1) \)–scalar product, one can deduce that for a.e. \( X_1 \in \omega_1 \):

$$\int_{\omega_2} A_{22}(X_2) \nabla X_2 u_{V,f}(X_1, X_2) \cdot \nabla X_2 \psi \, dX_2 = \int_{\omega_2} f(X_1, X_2) \psi \, dX_2, \quad \forall \psi \in V_2.$$

Now, fix $i \in \{1, \ldots, q\}$. Let $\omega'_i \subset \subset \omega_1$ open, for any $0 < h < d(\omega'_i, \partial \omega_1)$ and for any \( (X_1, X_2) \in \omega'_i \times \omega_2 \) we denote $\tau_h u_{V,f}(x) = u_{V,f}(x_1, \ldots, x_i + h, \ldots, x_q, X_2)$. According to the above equality, we get for a.e. \( X_1 \in \omega'_i \) and for every $\psi \in V_2$:

$$\int_{\omega_2} A_{22}(X_2) \nabla X_2 \{ \tau_h u_{V,f}(X_1, X_2) - u_{V,f}(X_1, X_2) \} \cdot \nabla X_2 \psi \, dX_2 = \int_{\omega_2} \{ \tau_h f(X_1, X_2) - f(X_1, X_2) \} \, \psi \, dX_2.$$

For every $w \in V_1 \otimes V_2$, and for every $X_1$ fixed the function $w(X_1, \cdot)$ belongs to $V_2$, so one can take $\psi = \tau_h u_{V,f}(X_1, \cdot) - u_{V,f}(X_1, \cdot)$ as a test function in the above equality. Therefore, by using the Cauchy-Schwarz inequality, the ellipticity assumption, and Poincaré’s inequality (10), we obtain:

$$\int_{\omega_2} |\tau_h u_{V,f}(X_1, \cdot) - u_{V,f}(X_1, \cdot)|^2 \, dX_2 \leq \frac{C_4}{\lambda^2} \int_{\omega_2} |\tau_h f(X_1, \cdot) - f(X_1, \cdot)|^2 \, dX_2.$$

Now, integrating the above inequality over $\omega'_i$, yields

$$\int_{\omega'_i \times \omega_2} |\tau_h u_{V,f} - u_{V,f}|^2 \, dx \leq \frac{C_4}{\lambda^2} \int_{\omega'_i \times \omega_2} |\tau_h f - f|^2 \, dx.$$

Since $\nabla X_1 f \in L^2(\Omega)^q$, then

$$\int_{\omega'_i \times \omega_2} |\tau_h f - f|^2 \, dx \leq \|\nabla X_1 f\|^2_{L^2(\Omega)} h^2.$$

Finally, we obtain

$$\int_{\omega'_i \times \omega_2} \left| \frac{\tau_h u_{V,f} - u_{V,f}}{h} \right|^2 \, dx \leq \frac{C_4}{\lambda^2} \frac{\|\nabla X_1 f\|^2_{L^2(\Omega)}}{h^2}.$$

Therefore

$$\|D_x u_{V,f}\|_{L^2(\Omega)} \leq \frac{C_2}{\lambda} \|\nabla X_1 f\|_{L^2(\Omega)},$$

and hence

$$\|\nabla X_1 u_{V,f}\|_{L^2(\Omega)} \leq C_3 \|\nabla X_1 f\|_{L^2(\Omega)},$$

with $C_3 = \frac{\sqrt{n^2}}{2 \lambda}$. \(\square\)
Remark 3.8. We have a similar result when (41) is replaced by

\[ \mu \int_{\Omega} u_{V,f} v dx + \int_{\Omega} A_{22}(X_2)\nabla X_2 u_{V,f} \cdot \nabla X_2 v dx = \int_{\Omega} f v dx, \quad \forall v \in V_1 \otimes V_2, \]

where \( \mu > 0 \). In this case, we obtain the following:

\[ ||\nabla X_1 u_{V,f}||_{L^2(\Omega)^q} \leq \frac{\sqrt{q}}{\mu} ||\nabla X_1 f||_{L^2(\Omega)^q}. \]

Now, we can refine the estimations of Lemma 3.4 as follows

Lemma 3.9. Under assumptions of Lemmas 3.4 and 3.7 we have:

\[ ||\nabla X_2 u_{e,V,f} - \nabla X_2 u_f||_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 ||\nabla X_1 f||_{L^2(\Omega)^q} + C_2 ||f||_{L^2(\Omega)} \right) \]

\[ + \frac{||A_{22}||_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} ||\nabla X_2 v - \nabla X_2 u_f||_{L^2(\Omega)^{N-q}}, \]

and

\[ ||\nabla X_1 u_{e,V,f}||_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left( C_1 C_3 ||\nabla X_1 f||_{L^2(\Omega)^q} + C_2 ||f||_{L^2(\Omega)} \right) + C_3 ||\nabla X_1 f||_{L^2(\Omega)^q}. \]

Proof. We have

\[ ||\nabla X_2 u_{e,V,f} - \nabla X_2 u_f||_{L^2(\Omega)^{N-q}} \leq ||\nabla X_2 u_{e,V,f} - \nabla X_2 u_{V,f}||_{L^2(\Omega)^{N-q}} + ||\nabla X_2 u_{V,f} - \nabla X_2 u_f||_{L^2(\Omega)^{N-q}}. \]

By using Lemma 3.4 and Lemma 3.7 we obtain that

\[ ||\nabla X_2 u_{e,V,f} - \nabla X_2 u_{V,f}||_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 ||\nabla X_1 f||_{L^2(\Omega)^q} + C_2 ||f||_{L^2(\Omega)} \right), \]

and by using Remark 3.3, we deduce

\[ ||\nabla X_2 u_{V,f} - \nabla X_2 u_f||_{L^2(\Omega)^{N-q}} \leq \frac{||A_{22}||_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} ||\nabla X_2 v - \nabla X_2 u_f||_{L^2(\Omega)^{N-q}}. \]

By using the above inequalities, we get the expected result. The second inequality follows from the triangle inequality and Lemmas 3.4 and 3.7.

Remark 3.10. Let \( \beta(s) = \mu s \), for a certain \( \mu > 0 \). Under assumptions of the above Lemma and (19) we obtain, by combining Remarks 3.6 and 3.8, the estimation:

\[ \forall \epsilon \in (0, 1]: ||\nabla X_2 (u_{e,V,f} - u_{V,f})||_{L^2(\Omega)} \leq \frac{\epsilon}{\mu} \left( \sqrt{q} C_1' ||\nabla X_1 f||_{L^2(\Omega)^q} + C_2' ||f||_{L^2(\Omega)} \right). \]

Now, we are able to give the first convergence result

Lemma 3.11. Suppose that assumptions of Lemmas 3.4 and 3.7 hold. Let \( f \in H_0^1(\omega_1) \otimes H_0^1(\omega_2) \), then we have for any \( \epsilon \in (0, 1]: \)

\[ ||\nabla X_2 u_{e,f} - \nabla X_2 u_f||_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 ||\nabla X_1 f||_{L^2(\Omega)^q} + C_2 ||f||_{L^2(\Omega)} \right), \]

and

\[ ||\nabla X_1 u_{e,f}||_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left( C_1 C_3 ||\nabla X_1 f||_{L^2(\Omega)^q} + C_2 ||f||_{L^2(\Omega)} \right) + C_3 ||\nabla X_1 f||_{L^2(\Omega)^q}. \]
Proof. Let \((V_n^{(1)})_{n \geq 0}\) and \((V_n^{(2)})_{n \geq 0}\) be two nondecreasing sequences of finite dimensional subspaces of \(H^1_0(\omega_1)\) and \(H^1_0(\omega_2)\) respectively, such that \(\bigcup V_n^{(1)}\) and \(\bigcup V_n^{(2)}\) are dense in \(H^1_0(\omega_1)\) and \(H^1_0(\omega_2)\) respectively, and such that \(f \in V_n^{(1)} \otimes V_n^{(2)}\), such sequences always exit. Indeed, let \(\{e_i^{(1)}\}_{i \in \mathbb{N}}\) and \(\{e_i^{(2)}\}_{i \in \mathbb{N}}\) be Hilbert bases of \(H^1_0(\omega_1)\) and \(H^1_0(\omega_2)\) respectively, then \(\bigcup_{n \geq 0} \text{span}(e_0^{(1)}, \ldots, e_n^{(1)})\) and \(\bigcup_{n \geq 0} \text{span}(e_0^{(2)}, \ldots, e_n^{(2)})\) are dense in \(H^1_0(\omega_1)\) and \(H^1_0(\omega_2)\) respectively, in the other hand we have \(f = \sum_{i=0}^{m} f_i^{(1)} \times f_i^{(2)}\) for some \(m \in \mathbb{N}\) and \(f_i^{(1)} \in H^1_0(\omega_1)\), \(f_i^{(2)} \in H^1_0(\omega_2)\) for \(i = 0, \ldots, m\), then we set, for every \(n \in \mathbb{N}\):

\[
V_n^{(1)} := \text{span}(e_0^{(1)}, \ldots, e_n^{(1)}, f_0^{(1)}, \ldots, f_m^{(1)}),
\]

\[
V_n^{(2)} := \text{span}(e_0^{(2)}, \ldots, e_n^{(2)}, f_0^{(2)}, \ldots, f_m^{(2)}).
\]

Now, since \(f\) belongs to each \(V_n^{(1)} \otimes V_n^{(2)}\) then according to Lemma 3.9 one has, for every \(\epsilon \in (0, 1]\), \(n \in \mathbb{N}\):

\[
\|\nabla x_2 u_{\epsilon, V_n, f} - \nabla x_2 u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 \|\nabla x_1 f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right)
+ \|A_2\|_{L^\infty(\Omega)} \inf_{v \in V_n} \|\nabla x_2 v - \nabla x_2 u_f\|_{L^2(\Omega)^{N-q}},
\]

where \(V_n := V_n^{(1)} \otimes V_n^{(2)}\). According to Corollary A.5 in Appendix A, \(\bigcup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)})\) is dense in \(H^1_0(\Omega)\). Using the fact that the sequence \((V_n)_{n \geq 0}\) is nondecreasing, then we obtain that

\[
\forall \epsilon \in (0, 1] : \lim_{n \to \infty} \inf_{v \in V_n} \|\nabla v - \nabla u_{\epsilon, f}\|_{L^2(\Omega)^N} = 0,
\]

and therefore, by using (27) we get

\[
\forall \epsilon \in (0, 1] : \lim_{n \to \infty} \|\nabla u_{\epsilon, V_n, f} - \nabla u_{\epsilon, f}\|_{L^2(\Omega)^N} = 0,
\]

and thus

\[
\forall \epsilon \in (0, 1] : \lim_{n \to \infty} \|\nabla x_2 u_{\epsilon, V_n, f} - \nabla x_2 u_{\epsilon, f}\|_{L^2(\Omega)^{N-q}} = 0, \quad \text{and} \quad \lim_{n \to \infty} \|\nabla x_1 u_{\epsilon, V_n, f} - \nabla x_1 u_{\epsilon, f}\|_{L^2(\Omega)^q} = 0.
\]

Using the fact that \(H^1_0(\Omega)\) is dense in \(H^1_0(\Omega, \omega_2)\) (Lemma A.1, Appendix A) and that the embedding \(H^1_0(\Omega) \hookrightarrow H^1_0(\Omega, \omega_2)\) is continuous then \(\bigcup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)})\) is dense in \(H^1_0(\Omega, \omega_2)\). Using the fact that the sequence \((V_n)_{n \geq 0}\) is nondecreasing, then we obtain that

\[
\lim_{n \to \infty} \inf_{v \in V_n} \|\nabla x_2 v - \nabla x_2 u_f\|_{L^2(\Omega)^{N-q}} = 0.
\]

Now, passing to the limit, as \(n \to \infty\), in the above inequality we deduce

\[
\forall \epsilon \in (0, 1] : \|\nabla x_2 u_{\epsilon, f} - \nabla x_2 u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left( C_1 C_3 \|\nabla x_1 f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right).
\]

Finally, by using the second inequality of Lemma 3.9 we get

\[
\forall \epsilon \in (0, 1] : \|\nabla x_1 u_{\epsilon, V_n, f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left( C_1 C_3 \|\nabla x_1 f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla x_1 f\|_{L^2(\Omega)^q},
\]

and the passage to limit as \(n \to \infty\) shows the second estimation of the lemma. \(\square\)
Now, we are able to give the proof of Theorem 2.3. Let us introduce the space

$$H^1_0(\Omega; \omega_1) = \{ v \in L^2(\Omega) \text{ such that } \nabla X_1 v \in L^2(\Omega) \}$$


	normed by the Hilbertian norm $\| \nabla X_1 (\cdot) \|_{L^2(\Omega)}$. We have the Poincaré’s inequality

$$\| v \|_{L^2(\Omega)} \leq C_{\omega_1} \| \nabla X_1 v \|_{L^2(\Omega)} \text{ for any } v \in H^1_0(\Omega; \omega_1)$$

(42)

Let $f \in L^2(\Omega)$ such that (16) and (17), thus $f \in H^1_0(\Omega; \omega_1)$. According to Lemma A.3 of Appendix A $H^1_0(\omega_1) \otimes H^1_0(\omega_2)$ is dense in $H^1_0(\Omega)$, and according to Remark A.2 of Appendix A $H^1_0(\Omega)$ is dense in $H^1_0(\Omega; \omega_1)$, then it follows that $H^1_0(\omega_1) \otimes H^1_0(\omega_2)$ is dense in $H^1_0(\Omega; \omega_1)$, thanks to the continuous embedding $H^1_0(\Omega) \hookrightarrow H^1_0(\Omega; \omega_1)$. Therefore, for $\delta > 0$ there exists $g_\delta \in H^1_0(\omega_1) \otimes H^1_0(\omega_2)$ such that

$$\| \nabla X_1 (f - g_\delta) \|_{L^2(\Omega)} \leq \delta.$$  

(43)

Let $u_{e,g_\delta}$ be the unique solution of (7) with $f$ replaced by $g_\delta$. Testing with $u_{e,f} - u_{e,g_\delta}$ in the difference of the weak formulations (recall that $\beta = 0$)

$$\int_{\Omega} A(e) \nabla(u_{e,f} - u_{e,g_\delta}) \cdot \nabla \varphi dx = \int_{\Omega} (f - g_\delta) \varphi dx, \forall \varphi \in H^1_0(\Omega),$$

we obtain

$$\| \nabla X_2 u_{e,f} - \nabla X_2 u_{e,g_\delta} \|_{L^2(\Omega)} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta,$$

and

$$\| \nabla X_1 u_{e,f} - \nabla X_1 u_{e,g_\delta} \|_{L^2(\Omega)} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta,$$

where we have used the ellipticity assumption, Poincaré’s inequalities (10), (42), and (43). By passing to the limit as $\epsilon \to 0$ in the first inequality above, using Theorem 1.1, we get

$$\| \nabla X_2 u_{f} - \nabla X_2 u_{g_\delta} \|_{L^2(\Omega)} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta.$$  

Applying Lemma 3.11 on $u_{e,g_\delta}$ and $u_{g_\delta}$ we obtain

$$\| \nabla X_2 u_{e,g_\delta} - \nabla X_2 u_{g_\delta} \|_{L^2(\Omega)} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 g_\delta \|_{L^2(\Omega)} + C_2 \| g_\delta \|_{L^2(\Omega)} \right),$$

and from (43) we derive

$$\| \nabla X_2 u_{e,g_\delta} - \nabla X_2 u_{g_\delta} \|_{L^2(\Omega)} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)} + \delta \right) \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)} + \delta \right) + C_2 \| g_\delta \|_{L^2(\Omega)}.$$

Notice that $\| g_\delta \|_{L^2(\Omega)} \to \| f \|_{L^2(\Omega)}$ as $\delta \to 0$, thanks to (43) and Poincaré’s inequality (42). Finally, the triangle inequality gives

$$\| \nabla X_2 u_{e,f} - \nabla X_2 u_{f} \|_{L^2(\Omega)} \leq \| \nabla X_2 u_{e,f} - \nabla X_2 u_{e,g_\delta} \|_{L^2(\Omega)} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)} + \delta \right) + C_2 \| g_\delta \|_{L^2(\Omega)}.$$  

Finally, the triangle inequality gives

$$\| \nabla X_2 u_{e,f} - \nabla X_2 u_{f} \|_{L^2(\Omega)} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)} + \delta \right) + C_2 \| g_\delta \|_{L^2(\Omega)} + \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta.$$  

Passing to the limit as $\delta \to 0$ we obtain

$$\| \nabla X_2 u_{e,f} - \nabla X_2 u_{f} \|_{L^2(\Omega)} \leq \epsilon \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)} + C_2 \| f \|_{L^2(\Omega)} \right),$$

now. We are able to give the proof of Theorem 2.3. Let us introduce the space
which is the estimation given in Theorem 2.3.

For the estimation in the first direction, we have
\[
\forall \epsilon \in (0, 1] : \| \nabla X_1 u_{e, f} \|_{L^2(\Omega)^q} \leq \| \nabla X_1 u_{e, f} - \nabla X_1 u_{e, g} \|_{L^2(\Omega)^q} + \| \nabla X_1 u_{e, g} \|_{L^2(\Omega)^q}
\]
\[
\leq \frac{C_{e_1} C_{e_2}}{\lambda^2} \delta + \frac{1}{\sqrt{2}} \left( C_1 C_3 \| \nabla X_1 g \|_{L^2(\Omega)^q} + C_2 \| g \|_{L^2(\Omega)^q} \right) + C_3 \| \nabla X_1 g \|_{L^2(\Omega)^q},
\]
where we have applied the triangle inequality and Lemma 3.11. Passing to the limit as \( \delta \to 0 \) by using (43), we obtain
\[
\forall \epsilon \in (0, 1] : \| \nabla X_1 u_{e, f} \|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left( C_1 C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q} + C_2 \| f \|_{L^2(\Omega)^q} \right) + C_3 \| \nabla X_1 f \|_{L^2(\Omega)^q}.
\]

Hence, passing to the limit in \( L^2(\Omega) - \text{weak} \) as \( \epsilon \to 0 \), up to a subsequence, we show that \( u_f \) belongs to \( H_0^1(\Omega) \), and by a contradiction argument, using the metrizability (for the weak topology) of weakly compact subsets in separable Hilbert spaces, one can show that the global sequence \( (\nabla X_1 u_{e, f})_\epsilon \) converges weakly to \( \nabla X_1 u_f \) in \( L^2(\Omega)^q \), and this completes the proof of Theorem 2.3.

**Remark 3.12.** In the case \( \beta(s) = \mu s \) with \( \mu > 0 \), we repeat the same arguments of this subsection by using Remark 3.10, and then we obtain the estimation of Remark 2.5.

4. **Anisotropic Perturbations of semigroups**

4.1. **Preliminaries**

For the standard basic theory of semigroups of bounded linear operators, we refer the reader to [14]. Let us begin by some reminders. Let \( E \) be a real Banach space. An unbounded linear operator \( A : D(A) \subset E \to E \) is said to be closed if for every sequence \( (x_n) \) of \( D(A) \) such that \( (x_n) \) and \( (A(x_n)) \) converge in \( E \), we have \( \lim x_n \in D(A) \) and \( \lim A(x_n) = A(\lim x_n) \). An operator is said to be densely defined on \( E \) if its domain \( D(A) \) is dense in \( E \). Let \( \mu \in \mathbb{R} \), we said that \( \mu \) belongs to the resolvent set of \( A \) if \( (\mu I - A)^{-1} : E \to D(A) \subset E \) is a bounded operator on \( E \). Notice that \( R_\mu \) and \( A \) commute on \( D(A) \), that is \( \forall x \in D(A) : R_\mu A x = A R_\mu x \). Let \( A \) be a densely defined closed operator. The bounded operator
\[
A_\mu = \mu A (\mu I - A)^{-1} = \mu A R_\mu = \mu^2 R_\mu - \mu I,
\]
is called the Yosida approximation of \( A \). We check immediately that \( A_\mu \) and \( A \) commute on \( D(A) \) that is for every \( x \in D(A) \) we have \( A_\mu x \in D(A) \) and \( A A_\mu x = A_\mu A x \). Furthermore, since \( A \) is closed then \( e^{t A_\mu} \) and \( A \) commute on \( D(A) \), that is
\[
\forall t \in \mathbb{R}, \forall x \in D(A), e^{t A_\mu} x \in D(A),
\]
and
\[
A e^{t A_\mu} x = e^{t A_\mu} A x = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A_\mu)^k A x,
\]
indeed, we can check by induction that if \( x \in D(A) \) then \( (A_\mu)^k x \in D(A) \), and that \( (A_\mu)^k \) and \( A \) commute on \( D(A) \). Recall also that if \( (\mu I - A)^{-1} \) exists for \( \mu > 0 \) and such that \( \| (\mu I - A)^{-1} \| \leq \frac{1}{\mu} \) then
\[
\forall t \geq 0 : \| e^{t A_\mu} \| = \left\| e^{t \mu^2 R_\mu} \right\| \times \left\| e^{-\mu t} \right\| \leq e^{\mu^2 \| R_\mu \|} \times e^{-\mu t} \leq 1,
\]
where \( \| \cdot \| \) is the operator norm of \( \mathcal{L}(E) \). A \( C_0 \) semigroup of bounded linear operators on \( E \) is a family of bounded operators \( (S(t))_{t \geq 0} \) of \( \mathcal{L}(E) \) such that: \( S(0) = I \), for every \( t, s \geq 0 : S(t + s) = S(t) S(s) \), and for every \( x \in E : \| S(t)x - x \|_E \to 0 \) as \( t \to 0 \). \( (S(t))_{t \geq 0} \) is called a semigroup of contractions if for every
\( t \geq 0 : \|S(t)\|_E \leq 1 \). Now, let us recall the well-known Hill-Yosida theorem in its Hilbertian (real) version: An unbounded operator \( \mathcal{A} \) is the infinitesimal generator of a \( C_0 \) semigroup of contractions \( (S(t))_{t \geq 0} \) if and only if \( \mathcal{A} \) is maximal dissipative, that is when \( \mu I - \mathcal{A} \) is surjective for every \( \mu > 0 \) and for every \( x \in D(\mathcal{A}) : (\mathcal{A}x, x) \leq 0 \). Recall that, in this case \( D(\mathcal{A}) \) is dense and \( \mathcal{A} \) is closed and its resolvent set contains \([0, +\infty]\). Furthermore, for every \( t \geq 0 \), \( e^{t\mathcal{A}} \) converges, in the strong operator topology, to \( S(t) \), as \( \mu \to +\infty \) i.e. \( \forall x \in E : e^{t\mathcal{A}}x \to S(t)x \) in \( E \) as \( \mu \to +\infty \).

Let \( \Omega \) as in the introduction. The basic Hilbert space in the sequel is \( E = L^2(\Omega) \). For \( \epsilon \in (0, 1] \), we introduce the operator \( \mathcal{A}_\epsilon \) acting on \( L^2(\Omega) \) and given by the formula

\[
\mathcal{A}_\epsilon u = \text{div}(A, \nabla u),
\]

where \( A \) is given as in the introduction of this paper. The domain of \( \mathcal{A}_\epsilon \) is given by

\[
D(\mathcal{A}_\epsilon) = \{ u \in H^1_0(\Omega) \mid \text{div}(A, \nabla u) \in L^2(\Omega) \},
\]

where \( \text{div}(A, \nabla u) \in L^2(\Omega) \) is taken in the distributional sense. Now, we introduce the operator \( \mathcal{A}_0 \) defined on \( D(\mathcal{A}_0) = \{ u \in H^1_0(\Omega; \omega_2) \mid \text{div}_{X_2}(A_{22} \nabla X_2 u) \in L^2(\Omega) \} \), by the formula

\[
\mathcal{A}_0 u = \text{div}_{X_2}(A_{22} \nabla X_2 u).
\]

We check immediately, by using assumptions (3 – 4), that \( \mathcal{A}_\epsilon \) and \( \mathcal{A}_0 \) are maximal dissipative and therefore, they are the infinitesimal generators of \( C_0 \) semigroups of contractions on \( L^2(\Omega) \), denoted \( (S_\epsilon(t))_{t \geq 0} \) and \( (S_0(t))_{t \geq 0} \) respectively. For \( \mu > 0 \) we denote by \( R_{\epsilon, \mu} \) the resolvent of \( \mathcal{A}_\epsilon \). Similarly, we denote by \( R_{0, \mu} \) the resolvent of \( \mathcal{A}_0 \).

For \( f \in L^2(\Omega) \), we denote \( u_{\epsilon, \mu} \) the unique solution in \( H^1_0(\Omega) \) to

\[
\mu \int_\Omega u_{\epsilon, \mu} \varphi dx + \int_\Omega A_{\epsilon} \nabla u_{\epsilon, \mu} \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in H^1_0(\Omega),
\]

we have \( R_{\epsilon, \mu} f = u_{\epsilon, \mu} \) and \( \|R_{\epsilon, \mu}\| \leq \frac{1}{\mu} \), where \( \| \cdot \| \) is the operator norm of \( \mathcal{L}(L^2(\Omega)) \). Similarly, let \( u_{0, \mu} \) be the unique solution in \( H^1_0(\Omega; \omega_2) \) to

\[
\mu \int_\Omega u_{0, \mu} \varphi dx + \int_\Omega A_{22} \nabla X_2 u_{0, \mu} \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in H^1_0(\Omega; \omega_2),
\]

we have \( R_{0, \mu} f = u_{0, \mu} \) and \( \|R_{0, \mu}\| \leq \frac{1}{\mu} \). According to Remark 2.5, we have the following

**Lemma 4.1.** Assume (3), (4), (14), (18) and (19). Let \( f \in H^1_0(\Omega; \omega_1) \), then there exists \( C_{A, \Omega} > 0 \) depending only on \( A \) and \( \Omega \) such that:

\[
\forall \epsilon \in (0, 1], \forall \mu > 0 : \quad \|R_{\epsilon, \mu} f - R_{0, \mu} f\|_{L^2(\Omega)} \leq C_{A, \Omega} \times \frac{\epsilon}{\mu} \times \left( \|\nabla X_1 f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).
\]

4.2. The asymptotic behavior of the perturbed semigroup

In this subsection, we study the relationship between the semigroups \( (S_\epsilon(t))_{t \geq 0} \) and \( (S_0(t))_{t \geq 0} \). We will assume that

\[
A \in W^{1,\infty}(\Omega)^{N^2}.
\]

Notice that (47) shows that, for any \( \epsilon \in (0, 1] \):

\[
H^1_0(\Omega) \cap H^2(\Omega) \subset D(\mathcal{A}_0) \cap D(\mathcal{A}_\epsilon).
\]

Remark also that (47) implies (14). Now, we can give the main theorem of this section.
Theorem 4.2. Let $\Omega = \Omega_1 \times \Omega_2$ be a bounded domain of $\mathbb{R}^q \times \mathbb{R}^{N-q}$. Assume (3), (4), (18), (19) and (47). Let $g \in L^2(\Omega)$ and $T \geq 0$, we have:

$$\sup_{t \in [0,T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0.$$ 

In particular, for $g \in (H^1_0 \cap H^2(\Omega_1)) \otimes (H^1_0 \cap H^2(\omega_2))$ there exists $C_{g,A,\Omega} > 0$ such that :

$$\forall \epsilon \in (0,1] : \sup_{t \in [0,T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times T \times \epsilon.$$ 

Let us begin by this important lemma

Lemma 4.3. Suppose that assumptions of Theorem 4.2 hold. Let $f \in H^1_0(\Omega) \cap D(A_0)$ such that

$$\text{div}_1(A_{11} \nabla X_1 f), \text{div}_1(A_{12} \nabla X_2 f), \text{div}_2(A_{21} \nabla X_1 f) \in L^2(\Omega),$$

then there exists a constant $C_{f,A,\Omega} > 0$ such that for every $\mu > 0$, $\epsilon \in (0,1]$ we have:

$$\| A_{\epsilon,\mu} f - A_{0,\mu} f \|_{L^2(\Omega)} \leq C_{f,A,\Omega} \times \epsilon,$$

where $A_{\epsilon,\mu}$ and $A_{0,\mu}$ are the Yosida approximations of $A_\epsilon$ and $A_0$ respectively. The constant $C_{f,A,\Omega}$ is given by:

$$C_{f,A,\Omega} = \| \text{div}_1(A_{11} \nabla X_1 f) \|_{L^2(\Omega)} + \| \text{div}_1(A_{12} \nabla X_2 f) \|_{L^2(\Omega)} + \| \text{div}_2(A_{21} \nabla X_1 f) \|_{L^2(\Omega)} + C_{A,\Omega} \left( \| \nabla X_1 A_0 f \|_{L^2(\Omega)} + \| A_0 f \|_{L^2(\Omega)} \right).$$

Proof. Let $\epsilon \in (0,1]$ and $\mu > 0$. The bounded operators $A_{\epsilon,\mu}$, $A_{0,\mu}$ of $L(L^2(\Omega))$ are given by:

$$A_{\epsilon,\mu} = \mu A_\epsilon R_{\epsilon,\mu} \text{ and } A_{0,\mu} = \mu A_0 R_{0,\mu}.$$ 

Now, under the above hypothesis we obtain that $f \in D(A_\epsilon) \cap D(A_0)$. We have:

$$\| A_{\epsilon,\mu} f - A_{0,\mu} f \|_{L^2(\Omega)} = \mu \| A_\epsilon R_{\epsilon,\mu} f - A_0 R_{0,\mu} f \|_{L^2(\Omega)} = \mu \| R_{\epsilon,\mu} A_\epsilon f - R_{0,\mu} A_0 f \|_{L^2(\Omega)} \leq \mu \| R_{\epsilon,\mu} f - R_{\epsilon,\mu} A_0 f \|_{L^2(\Omega)} + \mu \| R_{0,\mu} A_0 f - R_{0,\mu} A_0 f \|_{L^2(\Omega)} \leq \mu \| R_{\epsilon,\mu} f - A_0 f \|_{L^2(\Omega)} + \mu \| R_{0,\mu} A_0 f - R_{0,\mu} A_0 f \|_{L^2(\Omega)}.$$ 

Since $A_0 f \in H^1_0(\Omega_1; \omega_1)$ by hypothesis, then by using (46) (where we replace $f$ by $A_0 f$) and the fact that $\| R_{\epsilon,\mu} \| \leq \frac{1}{\mu}$, we obtain

$$\| A_{\epsilon,\mu} f - A_{0,\mu} f \|_{L^2(\Omega)} \leq \| A_\epsilon f - A_0 f \|_{L^2(\Omega)} + \epsilon C_{A,\Omega} \left( \| \nabla X_1 A_0 f \|_{L^2(\Omega)} + \| A_0 f \|_{L^2(\Omega)} \right)$$

$$= \epsilon \left( \| \text{div}_1(A_{11} \nabla X_1 f) \|_{L^2(\Omega)} + \| \text{div}_1(A_{12} \nabla X_2 f) \|_{L^2(\Omega)} + \| \text{div}_2(A_{21} \nabla X_1 f) \|_{L^2(\Omega)} + C_{A,\Omega} \left( \| \nabla X_1 A_0 f \|_{L^2(\Omega)} + \| A_0 f \|_{L^2(\Omega)} \right) \right) \leq C_{f,A,\Omega} \times \epsilon,$$

where we have used the identity:

$$A_\epsilon f - A_0 f = c^2 \text{div}_1(A_{11} \nabla X_1 f) + \epsilon \text{div}_1(A_{12} \nabla X_2 f) + \epsilon \text{div}_2(A_{21} \nabla X_1 f),$$

and the proof of the lemma is finished. \qed
Lemma 4.4. Under assumptions of Theorem 4.2, we have for any \( g \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2)) \):

\[
\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \|e^{tA_{\epsilon, \mu}}g - e^{tA_{0, \mu}}g\|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times t \times \epsilon,
\]

where \( C_{g, A, \Omega} \) is independent of \( \mu \) and \( \epsilon \).

Proof. Let \( \mu > 0 \) and \( t \geq 0 \) and \( \epsilon \in (0, 1] \), we have

\[
e^{tA_{0, \mu}} - e^{tA_{\epsilon, \mu}} = \int_0^t \frac{d}{ds} (e^{(t-s)A_{\epsilon, \mu}}e^{sA_{0, \mu}}) ds
\]

\[
= \int_0^t e^{(t-s)A_{\epsilon, \mu}}(A_{0, \mu} - A_{\epsilon, \mu})e^{sA_{0, \mu}} ds.
\]

Hence, for \( g \in L^2(\Omega) \) we have

\[
\|e^{tA_{\epsilon, \mu}}g - e^{tA_{0, \mu}}g\|_{L^2(\Omega)} \leq \int_0^t \|A_{0, \mu}e^{sA_{0, \mu}}g - A_{\epsilon, \mu}e^{sA_{0, \mu}}g\|_{L^2(\Omega)} ds,
\]

where have used \( \|e^{(t-s)A_{\epsilon, \mu}}\| \leq 1 \), since \( t - s \geq 0 \).

Now, we suppose that \( g \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2)) \) (remark that \( g \in D(A_0) \)). For \( s \geq 0 \) and \( \mu > 0 \) we set:

\[
f_{g, s, \mu} := e^{sA_{0, \mu}}g
\]

We can prove that \( f_{g, s, \mu} \) fulfills the same hypothesis satisfied by the function \( f \) of Lemma 4.3. Moreover, for every \( i, j = 1, ..., q \) we have \( D_{x, x, j}^2 f_{g, s, \mu} \in L^2(\Omega) \) with:

\[
\left\|D_{x, x, j}^2 f_{g, s, \mu}\right\|_{L^2(\Omega)} \leq \left\|D_{x, x, j}^2 g\right\|_{L^2(\Omega)} \leq \left\|D_{x, g, s, \mu}\right\|_{L^2(\Omega)} \leq \|D_{x, g}\|_{L^2(\Omega)},
\]

and:

\[
\left\|(A_0 f_{g, s, \mu})\right\|_{L^2(\Omega)} \leq \|A_0 g\|_{L^2(\Omega)}, \quad \|D_{x, i}(A_0 f_{g, s, \mu})\|_{L^2(\Omega)} \leq \|D_{x, i}(A_0 g)\|_{L^2(\Omega)}.
\]

also for every \( i = 1, ..., q, j = q + 1, ..., N \) we have \( D_{x, x, j} f_{g, s, \mu} \in L^2(\Omega) \) with:

\[
\|D_{x, f_{g, s, \mu}}\|_{L^2(\Omega)} \leq \left\|D_{x, g}\right\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} \quad \text{and} \quad \|D_{x, x, j} f_{g, s, \mu}\|_{L^2(\Omega)} \leq \frac{1}{\lambda} \|D_{x, g}\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.
\]

The proof of these assertions follows from the identity \( e^{sA_{0, \mu}}(g_1 \otimes g_2) = g_1 \otimes e^{sA_{0, \mu}}g_2 \) (see Appendix B). Notice that the above bounds are independent of \( s, \epsilon, \) and \( \mu \).

Now, apply Lemma 4.3, we get

\[
\|A_{0, \mu}e^{sA_{0, \mu}}g - A_{\epsilon, \mu}e^{sA_{0, \mu}}g\|_{L^2(\Omega)} \leq \epsilon \left( \|\text{div}_{X_1}(A_{11} \nabla X_1 f_{g, s, \mu})\|_{L^2(\Omega)} + \|\text{div}_{X_1}(A_{12} \nabla X_1 f_{g, s, \mu})\|_{L^2(\Omega)} + \|\text{div}_{X_1}(A_{21} \nabla X_1 f_{g, s, \mu})\|_{L^2(\Omega)} + \|\text{div}_{X_1}(A_{22} \nabla X_1 f_{g, s, \mu})\|_{L^2(\Omega)} + C_{A, \Omega} \left( \|\nabla X_1 A_0 f_{g, s, \mu}\|_{L^2(\Omega)} + \|A_0 f_{g, s, \mu}\|_{L^2(\Omega)} \right) \right).
\]

By using (49 – 51) with (47), one can show that the quantity in parentheses in the above inequality is bounded by some \( C_{g, A, \Omega} > 0 \) independent of \( s, \epsilon, \) and \( \mu \), thus

\[
\|A_{0, \mu}e^{sA_{0, \mu}}g - A_{\epsilon, \mu}e^{sA_{0, \mu}}g\|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times \epsilon.
\]

Finally, integrate the above inequality in \( s \) over \([0, t]\), and use (48), we get the desired result. \( \square \)
Now, we are able to prove Theorem 4.2. First we prove the case when \( g \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2)) \) and we conclude by a density argument. So, let \( g \) as mentioned above, by Lemma 4.4 we have
\[
\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \| e^{t A_{\mu}^0} g - e^{t A_0^0} g \|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times t \times \epsilon. \tag{52}
\]
Passing to the limit in (52) as \( \mu \to +\infty \) we get (see the preliminaries, the abstract part)
\[
\forall t \geq 0, \forall \epsilon \in (0, 1] : \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times t \times \epsilon,
\]
whence for \( T \geq 0 \) fixed we obtain
\[
\forall \epsilon \in (0, 1] : \sup_{t \in [0, T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times T \times \epsilon. \tag{53}
\]
Whence
\[
\sup_{t \in [0, T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0. \tag{54}
\]
Now, let \( g \in L^2(\Omega) \) and let \( \delta > 0 \), by density there exists \( g_\delta \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2)) \) such that
\[
\| g - g_\delta \|_{L^2(\Omega)} \leq \frac{\delta}{4}.
\]
According to (54) there exists \( \epsilon_\delta > 0 \) such that
\[
\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \| S_\epsilon(t)g_\delta - S_0(t)g_\delta \|_{L^2(\Omega)} \leq \frac{\delta}{2}.
\]
Whence, by the triangle inequality we get
\[
\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq \frac{\delta}{2} + \sup_{t \in [0, T]} (\| S_\epsilon(t)\| + \| S_0(t)\|) \times \| g_\delta - g \|_{L^2(\Omega)}.
\]
Using the fact that the semigroups \( (S_\epsilon(t))_{t \geq 0} \) and \( (S_0(t))_{t \geq 0} \) are of contractions, we get
\[
\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \leq \delta.
\]
So, \( \sup_{t \in [0, T]} \| S_\epsilon(t)g - S_0(t)g \|_{L^2(\Omega)} \to 0 \) as \( \epsilon \to 0 \). The second assertion of the theorem is given by (53) and the proof of Theorem 4.2 is completed.

4.3. An application to linear parabolic equations

Theorem 4.2 gives an opening for the study of anisotropic singular perturbations of evolution partial differential equations from the semigroup point of view. In this subsection, we give a simple and short application to the linear parabolic equation
\[
\frac{\partial u_\epsilon}{\partial t} - \text{div}(A_\epsilon \nabla u_\epsilon) = 0, \tag{55}
\]
supplemented with the boundary and the initial conditions
\[
\begin{align*}
u_\epsilon(t, \cdot) &= 0 \text{ in } \partial \Omega \text{ for } t \in (0, +\infty) \tag{56} \\
u_\epsilon(0, \cdot) &= u_{\epsilon, 0}. \tag{57}
\end{align*}
\]
The limit problem is
\[ \frac{\partial u}{\partial t} - \text{div} X_2(A_{22} \nabla X_2 u) = 0, \] (58)
supplemented with the boundary and the initial conditions
\[ u(t, \cdot) = 0 \text{ in } \omega_1 \times \partial \omega_2 \text{ for } t \in (0, +\infty) \] (59)
\[ u(0, \cdot) = u_0. \] (60)

The operator forms of (55 – 57) and (58 – 60) read
\[ \frac{du}{dt} - A_\epsilon u = 0, \text{ with } u_\epsilon(0) = u_{\epsilon,0}, \] (61)
and
\[ \frac{du}{dt} - A_0 u = 0, \text{ with } u(0) = u_0. \] (62)

Suppose that \( u_0 \in D(A_0) \) and \( u_{\epsilon,0} \in D(A_\epsilon) \). Assume that (3), (4) hold, then it follows that (61), (62) have unique classical solutions
\[ u_\epsilon \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(A_\epsilon)), \text{ and } u \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(A_0)) \].

We have the following convergence result.

**Proposition 4.5.** Suppose that \( u_0 \in D(A_0) \) and \( u_{\epsilon,0} \in D(A_\epsilon) \) such that \( u_{\epsilon,0} \to u_0 \) in \( L^2(\Omega) \), then under assumptions of Theorem 4.2, we have for any \( T \geq 0 \):
\[ \sup_{t \in [0,T]} \| u_\epsilon(t) - u(t) \|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0. \] (63)

Moreover, if \( u_{\epsilon,0} \) and \( u_0 \) are in \( H^2(\Omega) \) such that \( (u_\epsilon,0) \) is bounded in \( H^2(\Omega) \) and \( \| \nabla X_2(u_{\epsilon,0} - u_0) \|_{L^2(\Omega)} \to 0 \), \( \| \nabla^2 X_2(u_{\epsilon,0} - u_0) \|_{L^2(\Omega)} \to 0 \) as \( \epsilon \to 0 \), then:
\[ \sup_{t \in [0,T]} \left\| \frac{d}{dt} (u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} \to 0. \]

**Proof.** It is well known that the solutions \( u_\epsilon, u \) are given by
\[ u_\epsilon(t) = S_\epsilon(t)u_{\epsilon,0} \text{ and } u_0(t) = S_0(t)u_0, \] for every \( t \geq 0 \).

Let \( T \geq 0 \), we have
\[ \sup_{t \in [0,T]} \| u_\epsilon(t) - u(t) \|_{L^2(\Omega)} \leq \sup_{t \in [0,T]} \| S_\epsilon(t)u_{\epsilon,0} - S_\epsilon(t)u_0 \|_{L^2(\Omega)} + \sup_{t \in [0,T]} \| S_\epsilon(t)u_0 - S_0(t)u_0 \|_{L^2(\Omega)} \]
\[ \leq \| u_{\epsilon,0} - u_0 \|_{L^2(\Omega)} + \sup_{t \in [0,T]} \| S_\epsilon(t)u_0 - S_0(t)u_0 \|_{L^2(\Omega)}. \]

Passing to the limit as \( \epsilon \to 0 \) by using Theorem 4.2, we get \( \sup_{t \in [0,T]} \| u_\epsilon(t) - u(t) \|_{L^2(\Omega)} \to 0. \)

For the second affirmation, we have:
\[ \left\| \frac{d}{dt} (u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} = \| S_\epsilon(t)A_\epsilon u_{\epsilon,0} - S_0(t)A_0u_0 \|_{L^2(\Omega)} \]
\[ \leq \| A_\epsilon u_{\epsilon,0} - A_0u_0 \|_{L^2(\Omega)} + \sup_{t \in [0,T]} \| S_\epsilon(t)A_0u_0 - S_0(t)A_0u_0 \|_{L^2(\Omega)}. \]
As \((u_\varepsilon,0)\) is bounded in \(H^2(\Omega)\), \(u_0 \in H^2(\Omega)\) and \(\|\nabla X_2 (u_{\varepsilon,0} - u_0)\|_{L^2(\Omega)} \to 0\), \(\|\nabla^2 X_2 (u_{\varepsilon,0} - u_0)\|_{L^2(\Omega)} \to 0\) as \(\varepsilon \to 0\), then by using (47) we get immediately \(\|\mathbf{A}_n u_{\varepsilon,0} - \mathbf{A}_0 u_0\|_{L^2(\Omega)} \to 0\) as \(\varepsilon \to 0\), and we conclude by applying Theorem 4.2.

\[\square\]

Remark 4.6. Consider the nonhomogeneous parabolic equations associated to (55) and (58) with second member \(f(t,x)\). Suppose that \(f\) is regular enough, for example \(f \in \text{Lip}(\Omega)\), then the associated classical solutions \(u_\varepsilon\) and \(u\) exist and they are unique. In this case, we have the same convergence result (63). The proof follows immediately from the use of the following integral representation formulas

\[u_\varepsilon(t) = S_\varepsilon(t)u_{\varepsilon,0} + \int_0^t S_\varepsilon(t-r)f(r)dr, \quad u(t) = S_0(t)u_0 + \int_0^t S_0(t-r)f(r)dr, \quad t \in [0,T],\]

Theorem 4.2, and Lebesgue’s theorem.

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Appendix A. Density lemmas

Let \(\omega_1\) and \(\omega_2\) be two open bounded subsets of \(\mathbb{R}^q\) and \(\mathbb{R}^{N-q}\) respectively. Recall that

\[H^1_0(\Omega; \omega_2) = \{ u \in L^2(\Omega) \mid \nabla X_2 u \in L^2(\Omega)^{N-q}, \text{ and for a.e.} X_1 \in \omega_1, u(X_1, \cdot) \in H^1_0(\omega_2) \},\]

normed by \(\|\nabla X_2 (\cdot)\|_{L^2(\Omega)}\). We have the following

Lemma A.1. The space \(H^1_0(\Omega)\) is dense in \(H^1_0(\Omega; \omega_2)\).

Proof. Let \(u \in H^1_0(\Omega; \omega_2)\) fixed. Let \(l\) be the linear form defined on \(H^1_0(\Omega)\) by

\[\forall \phi \in H^1_0(\Omega) : l(\phi) = \int_\Omega \nabla X_2 u \cdot \nabla X_2 \phi dx.\]

\(l\) is continuous on \(H^1_0(\Omega)\), indeed we have

\[\forall \phi \in H^1_0(\Omega) : |l(\phi)| \leq \|\nabla X_2 u\|_{L^2(\Omega)} \|\nabla X_2 \phi\|_{L^2(\Omega)},\]

and then,

\[\forall \phi \in H^1_0(\Omega) : |l(\phi)| \leq \|\nabla X_2 u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}.\]

For every \(n \in \mathbb{N}^*\), we denote \(u_n\) the unique solution to

\[
\begin{cases}
\n\frac{1}{n^2} \int_\Omega \nabla X_1 u_n \cdot \nabla X_1 \phi dx + \int_\Omega \nabla X_2 u_n \cdot \nabla X_2 \phi dx = l(\phi), \quad \forall \phi \in H^1_0(\Omega),
\end{cases}
\]

where the existence and the uniqueness follow from the Lax-Milgram theorem. Testing with \(u_n\) in (64) we get,

\[
\frac{1}{n^2} \int_\Omega |\nabla X_1 u_n|^2 dx + \int_\Omega |\nabla X_2 u_n|^2 dx \leq \|\nabla X_2 u\|_{L^2(\Omega)} \|\nabla X_2 u_n\|_{L^2(\Omega)},
\]

then, we deduce that

\[\forall n \in \mathbb{N}^* : \|\nabla X_2 u_n\|_{L^2(\Omega)} \leq \|\nabla X_2 u\|_{L^2(\Omega)},\]

and

\[\forall n \in \mathbb{N}^* : \frac{1}{n} \|\nabla X_1 u_n\|_{L^2(\Omega)} \leq \|\nabla X_2 u\|_{L^2(\Omega)}.\]
Using (65) and Poincaré’s inequality we obtain:

\[ \forall n \in \mathbb{N}^* : \|u_n\|_{L^2(\Omega)} \leq C_{\omega_2} \|\nabla X_2 u\|_{L^2(\Omega)} \ldots (67) \]

Reflexivity of \( L^2(\Omega) \) shows that there exists, \( u_\infty, u_\infty', u_\infty'' \in L^2(\Omega) \) and a subsequence still labeled \( (u_n) \) such that

\[ u_n \rightharpoonup u_\infty, \nabla X_2 u_n \rightharpoonup u_\infty' \text{ and } \frac{1}{n} \nabla X_1 u_n \rightharpoonup u_\infty'' \text{ in } L^2(\Omega), \text{ weakly.} \]

Using the continuity of derivation on \( D'(\Omega) \) we get

\[ u_n \rightharpoonup u_\infty, \nabla X_2 u_n \rightharpoonup \nabla X_2 u_\infty \text{ and } \frac{1}{n} \nabla X_1 u_n \rightharpoonup 0 \text{ in } L^2(\Omega), \text{ weakly.} \ldots (68) \]

1) we have \( u_\infty \in H^1_0(\Omega; \omega_2) \) : By the Mazur Lemma, there exists a sequence \( (U_n) \) of convex combinations of \( \{u_n\} \) such that

\[ \nabla X_2 U_n \rightharpoonup \nabla X_2 u_\infty \text{ in } L^2(\Omega) \text{ strongly,} \ldots (69) \]

then by the Lebesgue theorem there exists a subsequence \( (U_{n_k}) \) such that:

For a.e. \( X_1 \in \omega_1 : \nabla X_2 U_{n_k}(X_1, \cdot) \rightharpoonup \nabla X_2 u_\infty(X_1, \cdot) \) in \( L^2(\omega_2) \) strongly.

(70)

Now, since \( (U_{n_k}) \in H^1_0(\Omega)^N \) then

\[ \text{For a.e. } X_1 \in \omega_1 : (U_{n_k}(X_1, \cdot)) \in H^1_0(\omega_2)^N. \ldots (71) \]

Combining (70) and (71) we deduce:

For a.e. \( X_1 \in \omega_1, u_\infty(X_1, \cdot) \in H^1_0(\omega_2), \)

and the proof of \( u_\infty \in H^1_0(\Omega; \omega_2) \) is finished.

2) we have \( u_\infty = u \) : Passing to the limit in (64) by using (68) we obtain

\[ \int_{\Omega} \nabla X_2 u_\infty \cdot \nabla X_2 \varphi dx = \int_{\Omega} \nabla X_2 u \cdot \nabla X_2 \varphi dx, \forall \varphi \in H^1_0(\Omega). \ldots (72) \]

For every \( \varphi_1 \in H^1_0(\omega_1) \) and \( \varphi_2 \in H^1_0(\omega_2) \) take \( \varphi = \varphi_1 \otimes \varphi_2 \) in (72) we obtain, for a.e. \( X_1 \in \omega_1 \)

\[ \int_{\omega_2} \nabla X_2 u_\infty(X_1, \cdot) \cdot \nabla X_2 \varphi_2 dX_2 = \int_{\omega_2} \nabla X_2 u(X_1, \cdot) \cdot \nabla X_2 \varphi_2 dX_2, \forall \varphi_2 \in H^1_0(\omega_2). \]

For a.e. \( X_1 \in \omega_1 \), take \( \varphi_2 = u_\infty(X_1, \cdot) - u(X_1, \cdot) \) (which belongs to \( H^1_0(\omega_2) \)) in the above equality, we get:

\[ \int_{\omega_2} |\nabla X_2 (u_\infty(X_1, \cdot) - u(X_1, \cdot))|^2 dX_2 = 0. \]

Integrating over \( \omega_1 \) we deduce

\[ \int_{\Omega} |\nabla X_2 (u_\infty - u)|^2 dx = 0. \]

Finally, since \( \|\nabla X_2 (\cdot)\|_{L^2(\Omega)} \) is a norm on \( H^1_0(\Omega; \omega_2) \) we get,

\[ u_\infty = u. \ldots (73) \]

Combining (69) and (73) we get the desired result. \[ \square \]
Remark A.2. By symmetry, \( H_0^1(\Omega) \) is dense in the space
\[
H_0^1(\Omega; \omega_1) = \{ u \in L^2(\Omega) \mid \nabla X_1 u \in L^2(\Omega), \text{ and for a.e. } X_2 \in \omega_2, u(\cdot, X_2) \in H_0^1(\omega_1) \},
\]
normed by \( \| \nabla X_1 (\cdot) \|_{L^2(\Omega)} \).

Lemma A.3. The space \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) is dense in \( H_0^1(\Omega) \).

Proof. It is well known that \( D(\omega_1) \otimes D(\omega_2) \) is dense in \( D(\omega_1 \times \omega_2) \). Here, \( D(\omega_1 \times \omega_2) \) is equipped with its natural topology (the inductive limit topology). It is clear that the injection of \( D(\omega_1 \times \omega_2) \) in \( H_0^1(\omega_1 \times \omega_2) \) is continuous, thanks to the inequality
\[
\forall u \in D(\Omega) : \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq \sqrt{N \times \text{mes}(\Omega)} \times \left( \max_{1 \leq i \leq N} \sup \{ |\partial_x u| \} \right).
\]

Hence, by the density rule we obtain the density of \( D(\omega_1) \otimes D(\omega_2) \) in \( H_0^1(\Omega) \), and the lemma follows. \( \square \)

Lemma A.4. Let \( (V_n^{(1)}) \) and \( (V_n^{(2)}) \) be two sequences of subspaces (not necessarily of finite dimension) of \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively. If \( \cup V_n^{(1)} \) and \( \cup V_n^{(2)} \) are dense in \( H_0^1(\omega_1) \) and \( H_0^1(\omega_2) \) respectively, then \( \text{vect} \left( \bigcup (V_n^{(1)} \otimes V_m^{(2)}) \right) \) is dense in \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) for the induced topology of \( H_0^1(\Omega) \). In particular, if \( (V_n^{(1)}) \) and \( (V_n^{(2)}) \) are nondecreasing then \( \bigcup_n (V_n^{(1)} \otimes V_n^{(2)}) \) is dense in \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \).

Proof. Let us start by a useful inequality. For \( u \otimes v \) in \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) we have:
\[
\| u \otimes v \|^2_{H_0^1(\Omega)} = \int_{\Omega} |\nabla X_1 (u \otimes v)|^2 \, dx + \int_{\Omega} |\nabla X_2 (u \otimes v)|^2 \, dx
\]
\[
= \left( \int_{\omega_2} v^2 \, dX_2 \right) \times \left( \int_{\omega_1} |\nabla X_1 u|^2 \, dX_1 \right) + \left( \int_{\omega_1} u^2 \, dX_1 \right) \times \left( \int_{\omega_2} |\nabla X_2 v|^2 \, dX_2 \right)
\]
\[
\leq C \| u \|^2_{H_0^1(\omega_1)} \times \| v \|^2_{H_0^1(\omega_2)}, \tag{74}
\]
where we have used Fubini’s theorem and Poincaré’s inequality. Here, \( C = C_{\omega_1}^2 + C_{\omega_2}^2 > 0 \). Now, fix \( \eta > 0 \) and let \( \varphi \otimes \psi \in H_0^1(\omega_1) \otimes H_0^1(\omega_2) \), by density of \( \cup V_n^{(1)} \) in \( H_0^1(\omega_1) \) there exists \( n \in \mathbb{N} \) and \( \varphi_n \in V_n^{(1)} \) such that:
\[
\| \psi \|_{H_0^1(\omega_2)} \times \| \varphi_n - \varphi \|_{H_0^1(\omega_1)} \leq \frac{\eta}{2\sqrt{C}}.
\]
Similarly by density of \( \cup V_n^{(2)} \) in \( H_0^1(\omega_2) \), there exits \( m \in \mathbb{N} \) (which depends on \( n \) and \( \psi \)) and \( \psi_m \in V_m^{(2)} \) such that:
\[
\| \varphi_n \|_{H_0^1(\omega_1)} \times \| \psi - \psi_m \|_{H_0^1(\omega_2)} \leq \frac{\eta}{2\sqrt{C}}.
\]
Whence, by using the triangle inequality and (74) we obtain
\[
\| \varphi \otimes \psi - \varphi_m \otimes \psi_m \|_{H_0^1(\Omega)} \leq \eta. \tag{75}
\]
Now, since every element of \( H_0^1(\omega_1) \otimes H_0^1(\omega_2) \) could be written as \( \sum_{i=1}^l \varphi_i \otimes \psi_i \), then by using the triangle inequality and using (75) with \( \eta \) replaced by \( \frac{\eta}{l} \), one gets the desired result. \( \square \)
Corollary A.5. \( \text{vec} \left( \bigcup_{n,m} (V_n^{(1)} \otimes V_m^{(2)}) \right) \) is dense in \( H_0^1(\Omega) \). in particular, if \( (V_n^{(1)}) \) and \( (V_n^{(2)}) \) are nondecreasing, then \( \bigcup_{n} (V_n^{(1)} \otimes V_n^{(2)}) \) is dense in \( H_0^1(\Omega) \).

**APPENDIX B. SEMIGROUPS**

**Lemma B.1.** Assume (3), (4), (18) and let \( f_1 \in L^2(\omega_1) \), \( f_2 \in L^2(\omega_2) \), then for every \( \mu > 0 \) we have:

\[
R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes (R_{0,\mu}f_2).
\]

Notice that \( R_{0,\mu}f_2 \in H_0^1(\omega_2) \). Moreover, we have

\[
A_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes (A_{0,\mu}f_2).
\]

Notice also that \( A_{0,\mu}f_2 \in L^2(\omega_2) \). Here, \( A_{0,\mu} \) is the Yosida approximation of \( A_0 \), recall that \( A_{0,\mu} = \mu A_0 R_{0,\mu} \).

**Proof.** Let \( v_2 \in H_0^1(\omega_2) \) be the unique solution in \( H_0^1(\omega_2) \) to

\[
\mu \int_{\omega_2} v_2 \varphi_2 dX_2 + \int_{\omega_2} A_{22}(X_2) \nabla_X v_2 \cdot \nabla_X \varphi_2 dX_2 = \int_{\omega_2} f_2 \varphi_2 dX_2, \quad \forall \varphi_2 \in H_0^1(\omega_2), \tag{76}
\]

Let \( \varphi \in H_0^1(\Omega; \omega_2) \), then \( \varphi(X_1, \cdot) \in H_0^1(\omega_2) \) for a.e. \( X_1 \in \omega_1 \). Let \( f_1 \in L^2(\omega_1) \), multiplying (76) by \( f_1 \), testing in (76) with \( \varphi(X_1, \cdot) \) and integrating over \( \omega_1 \) yields

\[
\mu \int_{\Omega} f_1 v_2 \varphi dx + \int_{\Omega} A_{22}(X_2) \nabla_X (f_1 v_2) \cdot \nabla_X \varphi dx = \int_{\Omega} f_1 f_2 \varphi dx.
\]

It is clear that \( f_1 v_2 \in H_0^1(\Omega; \omega_2) \) whence, \( R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes v_2 \), in particular when \( f_1 = 1 \) we have \( R_{0,\mu}(f_2) = v_2 \), and therefore \( R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes R_{0,\mu}(f_2) \). The second assertion follows immediately from the first one, in fact

\[
A_{0,\mu}(f_1 \otimes f_2) = \mu A_0 R_{0,\mu}(f_1 \otimes f_2) = \mu A_0(f_1 \otimes R_{0,\mu}f_2).
\]

We have \( R_{0,\mu}f_2 \in D(A_0) \cap H_0^1(\omega_2) \) then by using (18) we get

\[
A_0(f_1 \otimes R_{0,\mu}f_2) = f_1 \otimes A_0(R_{0,\mu}f_2),
\]

Notice that the operator \( A_0 \) is independent of the \( X_1 \) direction and that \( A_0(R_{0,\mu}f_2) \in L^2(\omega_2) \). Finally we get

\[
A_{0,\mu}(f_1 \otimes f_2) = \mu f_1 \otimes A_0(R_{0,\mu}f_2) = f_1 \otimes A_{0,\mu}(f_2).
\]

Now, let \( s \geq 0, \mu > 0 \) and \( g \in L^2(\Omega) \). To simplify the notations, we denote \( f_g := e^{sA_{0,\mu}}g \) instead of \( f_{g,s,\mu} \).

**Lemma B.2.** Assume (3), (4), (18). Let \( g = g_1 \otimes g_2 \in L^2(\omega_1) \otimes L^2(\omega_2) \), then for \( s \geq 0, \mu > 0 \) we have:

\[
f_g = g_1 \otimes e^{sA_{0,\mu}}g_2.
\]

Notice that \( e^{sA_{0,\mu}}g_2 \in L^2(\omega_2) \).
Proof. we have

\[ f_g = e^{sA_{0,\mu}}g = \sum_{k=0}^{\infty} \frac{s^k}{k!} A_{0,\mu}^k g, \]

where the series converges in \( L^2(\Omega) \). By an immediate induction we get by using Lemma B.1

\[ \forall k \in \mathbb{N} : A_{0,\mu}^k g = g_1 \otimes A_{0,\mu}^k g_2, \]

with \( A_{0,\mu}^k g_2 \in L^2(\omega_2) \) for every \( k \in \mathbb{N} \), and the Lemma follows. \( \square \)

**Lemma B.3.** Assume (3), (4), (18). Let \( g \in H^2(\omega_1) \otimes L^2(\omega_2) \), then for \( s \geq 0, \mu > 0, i, j = 1, \ldots, q \) we have \( D^2_{x,x_i} f_g, D_{x,x_i} f_g \in L^2(\Omega) \), with:

\[ D^2_{x,x_i} f_g = e^{sA_{0,\mu}}(D^2_{x,x_i} g), \quad D_{x,x_i} f_g = e^{sA_{0,\mu}}(D_{x} g). \]  \hspace{1cm} (77)

\[ \| D^2_{x,x_i} f_g \|_{L^2(\Omega)} \leq \| D^2_{x,x_i} g \|_{L^2(\Omega)}, \quad \| D_{x,x_i} f_g \|_{L^2(\Omega)} \leq \| D_{x,x_i} g \|_{L^2(\Omega)}. \]  \hspace{1cm} (78)

**Proof.** 1) Suppose the simple case when \( g = g_1 \otimes g_2 \). So, let \( g = g_1 \otimes g_2 \in H^2(\omega_1) \otimes L^2(\omega_2) \) and let us prove assertions (77). By Lemma B.2 we get

\[ f_g = g_1 \otimes e^{sA_{0,\mu}}(g_2), \]

with \( e^{sA_{0,\mu}} g_2 \in L^2(\omega_2) \). Hence, for \( i, j = 1, \ldots, q \) we get \( D^2_{x,x_i} f_g \in L^2(\Omega) \) and \( D^2_{x,x_i} f_g = (D^2_{x,x_i} g_1) \otimes e^{sA_{0,\mu}} g_2 \), and applying Lemma B.2 we get

\[ D^2_{x,x_i} f_g = e^{sA_{0,\mu}}(D^2_{x,x_i} g_1). \]

Similarly we get \( D_{x,x_i} f_g = e^{sA_{0,\mu}}(D_{x,x_i} g_1) \), and assertion (77) follows when \( g = g_1 \otimes g_2 \).

2) Now, let \( g \in H^2(\omega_1) \otimes L^2(\omega_2) \), since \( g \) is a finite sum of elements of the form \( g_1 \otimes g_2 \), then by linearity we get \( D^2_{x,x_i} f_g, D_{x,x_i} f_g \in L^2(\Omega) \) and

\[ D^2_{x,x_i} f_g = e^{sA_{0,\mu}}(D^2_{x,x_i} g_1), \quad D_{x,x_i} f_g = e^{sA_{0,\mu}}(D_{x,x_i} g_1), \]  \hspace{1cm} for \( i, j = 1, \ldots, q \),

therefore

\[ \| D^2_{x,x_i} f_g \|_{L^2(\Omega)} \leq \| e^{sA_{0,\mu}} \| \| D^2_{x,x_i} g_1 \|_{L^2(\Omega)} \leq \| D^2_{x,x_i} g_1 \|_{L^2(\Omega)}, \]  \hspace{1cm} for \( i, j = 1, \ldots, q \),

and similarly we obtain the second inequality of (78). \( \square \)

**Lemma B.4.** Assume (3), (4), (18) and (47). Let \( g \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2)), \) then for \( s \geq 0, \mu > 0 \) we have:

\[ f_g \in D(A_0), \quad A_0(f_g) \in H^1_0(\Omega; \omega_1), \quad \text{and} \quad D_{x_i}(A_0 f_g) = e^{sA_{0,\mu}}(D_{x_i} A_0 g), \]  \hspace{1cm} (79)

\[ \| A_0 f_g \|_{L^2(\Omega)} \leq \| A_0 g \|_{L^2(\Omega)} \quad \text{and} \quad \| D_{x_i}(A_0 f_g) \|_{L^2(\Omega)} \leq \| D_{x_i} A_0 g \|_{L^2(\Omega)}, \]  \hspace{1cm} (80)

**Proof.** 1) Suppose \( g = g_1 \otimes g_2 \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2)) \) and let us prove (79). Since \( g \in D(A_0) \), thanks to (47), then \( f_g = e^{sA_{0,\mu}} g \in D(A_0) \) and \( A_0 f_g = e^{sA_{0,\mu}} A_0 g \) ( thanks to (44) ). Now, we have

\[ A_0 f_g = A_0(e^{sA_{0,\mu}} g) = A_0 (g_1 \otimes e^{sA_{0,\mu}} g_2). \]

Notice that, \( g_2 \in D(A_0) \), thanks to (47), then \( e^{sA_{0,\mu}} g_2 \in D(A_0) \) ( thanks to (44) ), hence

\[ A_0 f_g = g_1 A_0 e^{sA_{0,\mu}} g_2, \]

where we have used the fact that \( A_0 \) is independent of the \( X_1 \) – direction. Since \( e^{sA_{0,\mu}} \) and \( A_0 \) commute on \( D(A_0) \), then

\[ A_0 f_g = g_1 e^{sA_{0,\mu}} A_0 g_2. \]
Now, we have $A_0g_2 \in L^2(\omega_2)$ then $e^{sA_0,\mu}A_0g_2 \in L^2(\omega_2)$ (thanks to Lemma B.2), however $g_1 \in H^1_0(\omega_1)$, then $A_0f_g \in H^1_0(\Omega;\omega_1)$. Whence, for $i = 1, \ldots, q$ we have

$$D_{x_i}(A_0f_g) = D_{x_i}g_1 \otimes e^{sA_0,\mu}A_0g_2,$$

and hence by, Lemma B.2 we get

$$D_{x_i}(A_0f_g) = e^{sA_0,\mu}(D_{x_i}g_1 \otimes A_0g_2) = e^{sA_0,\mu}(D_{x_i}A_0g).$$

(Remark that $D_{x_i}A_0g \in L^2(\Omega)$ since $g_1 \in H^1_0(\omega_1)$ and $A_0g_2 \in L^2(\omega_2)$).

2) Now, for a general $g \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2))$, assertion (79) follows by linearity. Finally, we show (80). We have

$$\left\| (A_0f_g) \right\|_{L^2(\Omega)} = \left\| e^{sA_0,\mu}(A_0g) \right\|_{L^2(\Omega)} \leq \left\| e^{sA_0,\mu} \right\| \left\| A_0g \right\|_{L^2(\Omega)} \leq \left\| A_0g \right\|_{L^2(\Omega)}.$$ 

For $i = 1, \ldots, q$ we get

$$\left\| D_{x_i} (A_0f_g) \right\|_{L^2(\Omega)} = \left\| e^{sA_0,\mu}(D_{x_i}A_0g) \right\|_{L^2(\Omega)} \leq \left\| e^{sA_0,\mu} \right\| \left\| D_{x_i}A_0g \right\|_{L^2(\Omega)} \leq \left\| D_{x_i}A_0g \right\|_{L^2(\Omega)}.$$

\[ \Box \]

**Lemma B.5.** Assume (3), (4), (18) and (47). Let $g \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2))$, then for $s \geq 0$, $\mu > 0$, $i = 1, \ldots, q$, $j = q + 1, \ldots, N$ we have $D_{x_j}f_g, D_{x_j}^2f_g \in L^2(\Omega)$ with:

$$\left\| D_{x_j}f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\| A_0g \right\|_{L^2(\Omega)} \left\| g \right\|_{L^2(\Omega)}, \quad \left\| D_{x_j}^2f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\| D_{x_j}A_0g \right\|_{L^2(\Omega)} \left\| D_{x_j}g \right\|_{L^2(\Omega)}. \tag{81}$$

**Proof.** 1) Let us show the first inequality of (81). Suppose $g \in (H^1_0 \cap H^2(\omega_1)) \otimes (H^1_0 \cap H^2(\omega_2))$. Notice that $g \in D(A_0)$, thanks to (47), then according to (44) we have $f_g \in D(A_0) \subset H^1_0(\Omega;\omega_2)$, hence for $j \in \{q+1, \ldots, N\}$ the ellipticity assumption gives

$$\left\| D_{x_i}f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\langle -A_0f_g, f_g \right\rangle_{L^2(\Omega)} \leq \frac{1}{\lambda} \left\| A_0f_g \right\|_{L^2(\Omega)} \left\| f_g \right\|_{L^2(\Omega)}.$$

We have, $\left\| A_0f_g \right\|_{L^2(\Omega)} = \left\| A_0e^{sA_0,\mu}g \right\|_{L^2(\Omega)} = \left\| e^{sA_0,\mu}A_0g \right\|_{L^2(\Omega)} \leq \left\| A_0g \right\|_{L^2(\Omega)}$, and $\left\| f_g \right\|_{L^2(\Omega)} \leq \left\| g \right\|_{L^2(\Omega)}$, therefore

$$\left\| D_{x_j}f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\| A_0g \right\|_{L^2(\Omega)} \left\| g \right\|_{L^2(\Omega)}.$$

2) Now, let $1 \leq i \leq q$ fixed, then according to Lemma B.3 we have $D_{x_i}f_g = e^{sA_0,\mu}D_{x_i}g$, notice that $D_{x_i}g \in D(A_0)$ and hence, $D_{x_i}f_g \in D(A_0)$, in particular $D_{x_i}f_g \in H^1_0(\Omega;\omega_2)$, and for $q + 1 \leq j \leq N$ we get

$$\left\| D_{x_j}^2f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \left\langle -A_0D_{x_i}f_g, D_{x_j}f_g \right\rangle_{L^2(\Omega)} \leq \frac{1}{\lambda} \left\| A_0D_{x_i}f_g \right\|_{L^2(\Omega)} \left\| D_{x_j}f_g \right\|_{L^2(\Omega)}.$$
We have,
\[\|A_0 D_x f_g\|_{L^2(\Omega)} = \|A_0 e^{s A_0 x} (D_x g)\|_{L^2(\Omega)} = \|e^{s A_0 x} (A_0 D_x g)\|_{L^2(\Omega)} \leq \|(D_x, A_0 g)\|_{L^2(\Omega)}.\]

Finally, by using (78) and the above inequality we obtain
\[\left\|D_{x_j}^2 f_g\right\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda} \|(D_x, A_0 g)\|_{L^2(\Omega)} \|D_x g\|_{L^2(\Omega)}.\]

**Lemma B.6.** Under assumptions of Lemma B.5, we have for \(g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2)):\)
\[f_g \in H_0^1(\Omega) \cap D(A_0),\]
and
\[\text{div}_{X_1}(A_{11} \nabla X_1 f), \text{ div}_{X_1}(A_{12} \nabla X_2 f), \text{ div}_{X_2}(A_{21} \nabla X_1, f) \in L^2(\Omega).\]

**Proof.** Let us prove (82). In Lemma B.4 we proved that \(f_g \in D(A_0).\) Let us show that \(f_g \in H_0^1(\Omega).\) Suppose the simple case \(g = g_1 \otimes g_2,\) we have \(f_g = g_1 \otimes e^{s A_0 x} g_2.\) Since \(g_2 \in D(A_0),\) then \(e^{s A_0 x} g_2 \in D(A_0),\) in particular we have \(e^{s A_0 x} g_2 \in H_0^1(\Omega; \omega_2)\) however, according to Lemma B.2 \(e^{s A_0 x} g_2 \in L^2(\omega_2),\) hence \(e^{s A_0 x} g_2 \in H_0^1(\omega_2).\) Finally as \(g_1 \in H_0^1(\omega_1)\) we get \(f_g \in H_0^1(\Omega).\) For a general \(g\) in the tensor product space, the proof follows by linearity.

Now, let us show (83). According to Lemmas B.3, B.5 all these derivatives \(D_x f_g, D_{x_j}^2 f_g\) for \(1 \leq i, j \leq q,\) and \(D_x f_g, D_{x_j}^2 f_g\) for \(1 \leq i \leq q, q + 1 \leq j \leq N\) belong to \(L^2(\Omega).\) Whence, combining this with (47) we get (83).

**APPENDIX C. Existence theorem**

Let \(V \subset H_0^1(\Omega)\) be a subspace. We consider the problem
\[
\begin{align*}
\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla X_2 u \cdot \nabla X_2 \varphi dx &= \int_{\Omega} f \varphi dx, \quad \forall \varphi \in V, \\
u \in V,
\end{align*}
\]
with \(A_{22}\) and \(\beta\) as in the introduction.

**Proposition C.1.** If \(V\) is closed in \(H_0^1(\Omega; \omega_2),\) then there exists a solution to (84).

**Proof.** We consider the perturbed problem
\[
\begin{align*}
\int_{\Omega} \beta(u_\varepsilon) \varphi dx + \int_{\Omega} \tilde{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla \varphi dx &= \int_{\Omega} f \varphi dx, \quad \forall \varphi \in V, \\
u_\varepsilon \in V,
\end{align*}
\]
with
\[
\tilde{A}_\varepsilon = \begin{pmatrix} \varepsilon^2 I_q & 0 \\ 0 & A_{22} \end{pmatrix}.
\]
The space \(V\) is closed in \(H_0^1(\Omega),\) thanks to the continuous embedding \(H_0^1(\Omega) \hookrightarrow H_0^1(\Omega; \omega_2).\) The function \(\tilde{A}_\varepsilon\) is bounded and coercive, then by using the Schauder fixed point theorem, one can show the existence of a solution \(u_\varepsilon\) to (85). This solution is unique in \(V\) thanks to (5) and coercivity of \(\tilde{A}_\varepsilon.\) Testing with \(u_\varepsilon\) in (85) we obtain
\[
\varepsilon \|\nabla X_1 u_\varepsilon\|_{L^2(\Omega)}, \quad \|\nabla X_2 u_\varepsilon\|_{L^2(\Omega)}, \quad \|u_\varepsilon\|_{L^2(\Omega)} \leq C,
\]
where $C$ is independent of $\epsilon$, we have used that $\int_{\Omega} \beta(u) u dx \geq 0$ (thanks to (5)). By using (6), we get
\[ \|\beta(u)\|_{L^2(\Omega)} \leq M(|\Omega|^2 + C). \]

So, there exist $v \in L^2(\Omega)$, $u \in L^2(\Omega)$ with $\nabla X_1 u \in L^2(\Omega)$, and a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that
\[ \beta(u_{\epsilon_k}) \rightharpoonup v, \quad \epsilon_k \nabla X_1 u_{\epsilon_k} \rightharpoonup 0, \quad \nabla X_2 u_{\epsilon_k} \rightharpoonup \nabla X_2 u, \quad u_{\epsilon_k} \rightharpoonup u \text{ in } L^2(\Omega) \text{-weak}. \]

Passing to the limit in (85) we get
\[ \int_{\Omega} v \varphi dx + \int_{\Omega} A_{22} \nabla X_2 u \cdot \nabla X_2 \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in V. \]

Take $\varphi = u_{\epsilon_k}$ in (87) and passing to the limit we get
\[ \int_{\Omega} vu dx + \int_{\Omega} A_{22} \nabla X_2 u \cdot \nabla X_2 u dx = \int_{\Omega} f u dx \tag{88} \]

Let us consider the quantity
\[ 0 \leq I_k = \int_{\Omega} \epsilon^2 |\nabla X_1 u_{\epsilon_k}|^2 dx + \int_{\Omega} A_{22} \nabla X_2 (u_{\epsilon_k} - u) \cdot \nabla X_2 (u_{\epsilon_k} - u) dx \]
\[ + \int_{\Omega} (\beta(u_{\epsilon_k}) - \beta(u))(u_{\epsilon_k} - u) dx \]
\[ = \int_{\Omega} f u_{\epsilon_k} dx - \int_{\Omega} A_{22} \nabla X_2 u_{\epsilon_k} \cdot \nabla X_2 u_{\epsilon_k} dx - \int_{\Omega} A_{22} \nabla X_2 u \cdot \nabla X_2 u_{\epsilon_k} dx \]
\[ + \int_{\Omega} f u dx - \int_{\Omega} vu dx - \int_{\Omega} \beta(u) u_{\epsilon_k} dx \]
\[ - \int_{\Omega} \beta(u_{\epsilon_k}) u_{\epsilon_k} dx + \int_{\Omega} \beta(u) u dx \]

Remark that this quantity is nonnegative, thanks to the ellipticity and monotonicity assumptions. Passing to the limit as $k \to \infty$ using (86), (88) we get
\[ \lim_{k \to \infty} I_k = 0. \]

Therefore, the ellipticity assumption shows that
\[ ||\epsilon_k \nabla X_1 u_{\epsilon_k}||_{L^2(\Omega)}, ||u_{\epsilon_k} - u||_{L^2(\Omega)}, ||\nabla X_2 (u_{\epsilon_k} - u)||_{L^2(\Omega)} \to 0, \tag{89} \]

and hence, by a contradiction argument one has
\[ \beta(u_{\epsilon_k}) \to \beta(u) \text{ strongly in } L^2(\Omega). \]

Whence (87) becomes
\[ \int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla X_2 u \cdot \nabla X_2 \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in V. \]

Finally, $||\nabla X_2 (u_{\epsilon_k} - u)||_{L^2(\Omega)} \to 0$ shows that $u \in H_0^1(\Omega; \omega_2)$, and therefore as $V$ is closed in $H_0^1(\Omega; \omega_2)$ we obtain that $u \in V$. \qed
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