Renormalization Group Transformation for the Wave Function

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Abstract

The problem considered here is the determination of the hamiltonian of a first quantized nonrelativistic particle by the help of some measurements of the location with a finite resolution. The resulting hamiltonian depends on the resolution of the measuring device. This dependence is reproduced by the help of a blocking transformation on the wave function. The systems with quadratic hamiltonian are studied in details. The representation of the renormalization group in the space of observables is identified.

1 Introduction

The renormalization group provides us a systematic method to study the dynamics of a subsystem which is embedded in a larger dynamical environment. When the subsystem contains the degrees of freedom characterizing the physics beyond the length scale of a certain ultraviolet cutoff, $\Lambda^{-1}$, then the $\Lambda$ dependence of the dynamics of the subsystem reveals the dependence

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of the fundamental laws and the physical quantities on the scale of the observation, $\Lambda$.

The cutoff can be implemented in a sharp or smooth manner. There are furthermore two slightly different generic realizations of the sharp cutoff depending on whether the degrees of freedoms are left intact by the decomposition of the whole system into the subsystem and its environment. What is usually employed for statistical or second quantized systems \[1 \] \[2 \] is the factorization

$$H = H_s \otimes H_e$$ \hspace{1cm} (1)$$

of the Hilbert space of the whole system $H$. The $\Lambda$ dependent subspace $H_s$ contains the states of the subsystem in question and the states of the environment are in $H_e$. With the proper choice of $H_s$ one can ensure that the subsystem corresponds to usual canonical degrees of freedom. The typical implementation of this factorization in Quantum Field Theory is when $H_s$ is chosen to be the space containing the multiparticle states where the energy or the momentum of each particle is below $\Lambda$.

Another, simpler method \[3 \] \[4 \] \[5 \] better suited for the first quantized systems, starts with the direct sum decomposition

$$H = H_s \oplus H_e.$$ \hspace{1cm} (2)$$

There is now no general argument to guarantee that the subspace $H_s$ contains all state vectors of some canonical degrees of freedom. In some important applications of this procedure, such as the ground state and few low lying resonances in the study of the nuclear collisions \[3 \], or the lowest Landau level in the case of the Quantum Hall Effect \[6 \] $H_s$ consists of the states with (unperturbed) energy less than $\Lambda$.

The goal of the renormalization group is to identify the dynamics of the subsystem by means of the reduced time evolution operator

$$U_s(t) = \text{Tr}_{H_e} e^{-\frac{i}{\hbar} t H},$$ \hspace{1cm} (3)$$

or

$$U_s(t) = \mathcal{P}_{H_s} e^{-\frac{i}{\hbar} t H} \mathcal{P}_{H_s},$$ \hspace{1cm} (4)$$

in the case of the direct product or direct sum decomposition, respectively. Here $\text{Tr}_{H_e}$ stands for the trace in the space $H_e$ and $\mathcal{P}_{H_s}$ denotes the projection operator into the subspace $H_s$. The characteristic feature of such a projected
dynamics is that the time evolution is not unitary because the state vectors in $\mathcal{H}_s$ leak into $\mathcal{H}_e$. This nonunitarity has been confirmed in nuclear physics where the name ”optical potential” was coined to describe its effects.

The physical difference between (1) and (2) is that each state in $\mathcal{H}_s$ has a ”contamination” from $\mathcal{H}_e$ in the case of the direct product. In fact, the low energy particle state in $\mathcal{H}_s$ has the cloud of the virtual high energy particles from $\mathcal{H}_e$ in any interactive Quantum Field Theory. The mathematical origin of this problem is that the elimination of a degree of freedom converts the pure states into mixed ones. This issue can be better understood in the renormalization of Quantum Field Theory where the effective theory in $\mathcal{H}_s$ contains nonlocal vertices up to the distance scale $O(\Lambda^{-1})$ which are described by the higher derivative terms in the effective action. These terms prevent the construction of the transfer matrix and the effective hamiltonian in $\mathcal{H}_s$. As a result the specification of the initial and final pure states of the particles in $\mathcal{H}_s$ is not sufficient to obtain the transition amplitude. So long as the higher order derivative terms are irrelevant $[7][8]$, this problem is naturally negligible.

The sharp cutoff procedure is clearly an idealization, the actual separation is defined by the interactions between the subsystem and its environment. In fact, the higher energy states decouple from the subsystem in a continuous rather than a sharp manner. The smooth cutoff procedure attempts to introduce such a more natural decomposition. This is usually achieved by means of a smearing function which is to make up the assumed gradual decoupling of the short distance modes from the long distance ones. In Quantum Field Theory this happens by the introduction of the blocked field variable

$$\phi(x) \longrightarrow \phi'(x) = \int dy \chi(x-y)\phi(y).$$

The effective action, $S'[\phi']$, is defined by the path integral

$$e^{i\hat{S}[\phi']} = \int D[\phi(x)] \prod_x \delta(\phi'(x) - \int dy \chi(x-y)\phi(y)) e^{i\hat{S}[\phi]}.$$  

The periodic boundary condition for the field variable in its time argument implements the trace in (3).

A natural choice for the smearing function,

$$\chi_a(x) = \Omega_d^{-1}(a)\Theta(a - |x|),$$

3
where \( \Omega_{d-1}(a) \) is the volume of the sphere of radius \( a \) in \( d \)-dimensions, corresponds to a sharp cutoff, \( a \), in real-space. Another simple choice is

\[
\chi_\Lambda(x) = \int_{|q| \leq \Lambda} \frac{d^d q}{(2\pi)^d} e^{iq \cdot x},
\]

which introduces a sharp cutoff in the momentum space. In fact, (3) yields the blocking transformation

\[
\tilde{\phi}(q) \rightarrow \tilde{\phi}'(q) = \tilde{\chi}(q) \tilde{\phi}(q)
\]

for the Fourier transform

\[
\tilde{\phi}(q) = \int \frac{d^d q}{(2\pi)^d} e^{iq \cdot x} \phi(x),
\]

where

\[
\tilde{\chi}_\Lambda(q) = \Theta(\Lambda - |q|).
\]

A cutoff which is smooth both in real and momentum space is given by

\[
\tilde{\chi}_\Lambda(q) = e^{-\frac{q^2}{\Lambda^2}}.
\]

It is worthwhile noting that the procedures are similar in Minkowsky and Euclidean space-time except that the regulator is usually not Lorentz invariant \(^3\). Another characteristic feature of the real-time blocking is that the complex phase factor lends a singular dependence to the running coupling constants on the cutoff \([9]\).

The smooth cutoff can be implemented in (4) by the replacement of the projection operator \( P_{H_e} \) by another operator whose eigenvalue is a smooth function of the momentum or the energy. The price is the loss of the property \( P_{H_e}^2 = P_{H_e} \).

The goal of this paper is to provide a "softening" of the sharp cutoff (2). This allows one to follow the resolution dependence of the dynamics for the first quantized systems in a more realistic and systematic manner. With a slight generalization of (11) we introduce

\[
U_{s,s'}(t) = e^{-\frac{i}{\hbar} H_{s,s'}(t)} = P_{H_e} e^{-\frac{i}{\hbar} H} P_{H_{s'}}
\]

\(^3\)The non-compactness of the Lorentz group makes the cutoff nonrelativistic in a positive definite Hilbert space. The negative norm states allows Lorentz invariant regularization but there is no path integral representation for the transition amplitudes.
which describes the evolution of the system from the subspace \( \mathcal{H}_{s'} \) to \( \mathcal{H}_s \) and change the sharp cutoff of (2) to a smooth one in a manner reminiscent of (3). In realistic situations, the smearing (5) is supposed to account for the effects of the short distance processes such as virtual particle emissions and absorptions. These processes ”dress up” the particles and modify their interactions. This appears as the evolution of the coupling constants as the functions of the cutoff. We shall emulate this regrouping of the dynamics from the degrees of freedom into the choice of the coupling constants in first quantized Quantum Mechanics.

Suppose that we measure (up to a global phase) the wave function of a particle that propagates in an unknown potential by an experimental device with space resolution \( \Delta x = \sigma \). How does the potential, \( V_\sigma(x) \), reconstructed by the help of these measurements depend on the space resolution? There is certainly such a dependence because the potential is smeared out and is slowly varying within the distance \( \sigma \). We shall try to take the deformations of the propagation caused by the measuring device, such as the interference or decoherence, into account by a smearing of the wave function rather than by going into the second quantized description. One hopes that if the smearing of the wave function was chosen appropriately then such a simplification still contains the essential elements of the physics.

Our procedure can be summarized in a formal manner as follows. In the Schrödinger functional formalism the Quantum Field Theory for the scalar field \( \phi(x,t) \) is considered as first quantized quantum mechanical system by converting \( \phi(x,t) \) into coordinate, i.e. changing the internal space of the field variable into the external space of the quantum system. The dimension of the latter is the number of degrees of freedom of the field theory model, the number of points in the coordinate or the momentum space. The blocking (9) can be viewed as a rescaling, or a suppression of the coordinates \( \phi(p,\omega) \) which depends on the choice of the type of the coordinate, \( (p,\omega) \). It drives the system into a lower dimensional subspace. Our blocking in the first quantized formalism will be realized by a smearing function \( \chi(x - y) \) for the wave function which depends on the location in space. This would have been a \( \phi(x,t) \)-dependent rescaling in the case of the field theory.

In Section 2. of the paper the blocking, in particular the decimation in time, is reviewed for Quantum Mechanics. A simple motivation of our blocking prescription in space is presented in Section 3. This procedure is applied to different systems with quadratic hamiltonians in Section 4. Section
is devoted to the construction of the representation of the renormalization

group in the space of observables. Finally, a brief summary of our results is

presented in Section 6.

2 Renormalization Group in Quantum Mechanics

We start by reviewing the motivation of using the renormalization group in

Quantum Mechanics. As in Quantum Field Theory, the idea of the reno-

malization group appeared first for systems with ultraviolet divergences. For

hamiltonians with a power-like or Dirac-delta type singular potential, the

regularization and the subsequent renormalization was applied in [10]. A

partial resummation of the perturbation expansion by the introduction of

the running coupling constant was performed in [11]. The singular quantum

propagation inspired the implementation of the renormalization group as a

device to trace down the dependence on the time scale of the observations in

[9]. To understand this latter issue better, consider the wave function

$$
\psi(x, t; y) = \langle x| e^{-\frac{i}{\hbar}tH} | y \rangle = \left( \frac{m}{2\pi i \hbar t} \right)^{1/2} e^{im(x-y)^2/2\hbar t}, \quad (14)
$$

for a free particle in one dimension which satisfies the initial condition

$$
\psi(x, 0; y) = \delta(x - y). \quad (15)
$$

It keeps the relative phase difference between the point $y$ and the sphere with

radius

$$
\ell(t) = \sqrt{\frac{2\hbar \phi}{m}}, \quad (16)
$$

around $y$. The phase velocity,

$$
v_\phi = \sqrt{\frac{\hbar \phi}{2mt}}, \quad (17)
$$

associated to this propagation has an ultraviolet, i.e. $t \to 0$, divergence.

This is in agreement with the observation that the Wiener measure is con-

centrated on nowhere differentiable paths and the velocity obtained form the
trajectories of the path integral in Quantum Mechanics diverges as \(17\) \([12]\). Such a surprising deviation from the analiticity of the classical trajectories suggests that the dependence of the result of a measurement on the time of the observation may be unusual in Quantum Mechanics.

Based on a formal similarity between path integrals for a 0+1 dimensional Quantum Field Theory and Quantum Mechanics, certain elements of the renormalization group method can be replanted from Quantum Field Theory to Quantum Mechanics. The evolution of the observables during the increase of the time scale of the measurement was studied in Ref. \([9]\) by means of the decimation of the integral variables of the path integral formalism. The concept of the running coupling constant was introduced by parametrizing the logarithm of the propagator

\[
\langle x | e^{-\frac{i}{\hbar} \Delta t H} | y \rangle = e^{i \bar{\hbar} S(x,y;\Delta t)},
\]

by the quantum action,

\[
S(x,y;\Delta t) = \sum_{n,m} g_{n,m}(t)(x-y)^n \left(\frac{x+y}{2}\right)^m,
\]

in formal analogy with Quantum Field Theory. Here \(g_{n,m}(t)\) is the coupling constant which is a periodic function of time for a quadratic, hermitean hamiltonian and the decimation in real time creates singularities and zeros as a function of \(t\). At the tree level one finds that the coupling constant \(g_{n,m}(t)\) is relevant, i.e. is an increasing function of \(t\) for

\[
n \leq 2.
\]

It is important to keep in mind that the path integral may have ultraviolet divergences in Quantum Mechanics. We usually call a coupling constant renormalizable if the insertion of the corresponding vertex into a graph reduces the degree of the overall divergence. According to the power counting argument, \(g_{n,m}(t)\) is renormalizable for

\[
\omega_{n,m} = \frac{m-n}{2} + 1 \geq 0.
\]

This comes from the expression

\[
\omega(G) = D + \frac{2-D}{2} E - \sum_v \omega_v
\]
for the overall divergence of a graph in $D$-dimensional Quantum Field Theory with $E$ external legs. The summation is over the vertices $v$ and $\omega_v$ denotes the energy dimension of the vertex

$$\omega_v \equiv \omega_{n,m} = D + m - \frac{D}{2}(n + m).$$

Eq. (22) shows that the degree of divergence of the graph is not increased by the insertion of a renormalizable vertex.

Note that the discrepancy between (20) and (21) indicates that the class of relevant and marginal operators does not agree with the class of the perturbatively renormalizable operators. It is not so surprising that the operators with $n > 2$ and $m > n + 2$ are irrelevant and renormalizable in the same time because the ultraviolet divergences are weaker in low dimensions. The other type of discrepancy, which is in particular represented by the coupling constant $g_{n,m}$ with $(n,m) = (2,0)$, the kinetic energy, is more worrisome. But we believe that the solution of this puzzle will come from a better understanding of the renormalizability below the lower critical dimension, $D < 2$.

Observe that in such dimensions the overall degree of divergence of a graph is increased by adding external legs. Thus there are infinitely many divergent Green functions and the usual BPHZ procedure is not sufficient to remove the cutoff and to renormalize the theory. The more general nonperturbative arguments, which assert the equivalence of the relevant and marginal operators with the renormalizable ones, suggest that similar result will only be established after a more carefully repetition of the renormalization program for $D < 2$.

Furthermore, the irrelevant coupling constants $g_{n,m}$ with $n > 2$ allow the construction of the "effective theories" which are non-renormalizable quantum systems with path integral description but without hamiltonian. This may happen because on the one hand, the path integral is well defined for $\Delta t \neq 0$ after the introduction of the counterterms to remove the divergences, and on the other hand, the Schrödinger equation is obtained in the limit $\Delta t \rightarrow 0$ which does not exist for a non-renormalizable system.

We present in this paper a renormalization group transformation in the coordinate space. The corresponding renormalized trajectory maps out the dependence of the measurements on the resolution of the measuring devices in the coordinate space. The renormalized trajectories for blocking in space or time are actually unrelated for nonrelativistic systems.
3 Measurements and Blocking

The implementation of the smearing of the wave function as a method to separate the subsystem from its environment can be motivated by inspecting a typical measurement process.

*Measurements with finite resolution:* Suppose that we want to determine the location of a particle. The devices, such as the photosensitive film, the steamer chamber or the solid state detectors are based on certain interactions between the particle in question and the material of the detecting device. As far as the latter is concerned, it is useful to distinguish two length scales. One, the total length, $\ell_t$, is the size of the volume element where the detector is sensitive. Another one, $\ell_c << \ell_t$, is the coherence length, the size of the microscopic structure used in the detection process. This is the size of the atom or the cluster of atoms which enters into interaction with our particle. One has to average the quantum amplitude on the scale $x < \ell_c$ and the probability for $\ell_c < x < \ell_t$ [4]. It is assumed here that the motion of the particle is coherent within the whole detector, i.e. the soft particle emission and other sources of decoherence are neglected and the coherence of the measurement is lost for $x > \ell_c$ due to the components of the detector. We introduce the projection operator,

$$P(x) = |\chi_x\rangle \otimes \langle \chi_x|,$$  \hspace{1cm} (24)

$P^2(x) = P(x)$, where $|\chi_x\rangle$ denotes the state of the microscopic part of the measuring device located at $x$. $P(x)$ corresponds to the measurement process performed by the microscopic structure located at the point $x$. The probability of finding the particle with the wave function $\psi(x)$ is approximated by

$$P_m = \int dx \rho(x) \text{tr} P(x) |\psi\rangle \otimes \langle \psi|,$$  \hspace{1cm} (25)

where $\rho(x)$ is to take into account the distribution of the microscopic objects in the detector.

*Blocking:* The formal relation with the Kadanoff-Wilson blocking is established by writing (25) as

$$P_m = \int dx \rho(x) |\psi(x)\rangle |\psi(x)\rangle^2,$$  \hspace{1cm} (26)

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4One encounters similar treatment in the description of the scattering processes where one sums the amplitude for the initial states and the probability for the final states.
where
\[ \psi(x) = B \chi \psi(x) = \int dy \chi^*(x - y) \psi(y). \] (27)

The amplitude \( \chi(x - y) \) defined as
\[ \chi(y) = \langle y | \chi_0 \rangle \] (28)
is the wave function of the microscopic part of the device and is vanishing for \( |y| \geq \ell_c \). The linear operator \( B \chi \) introduced here performs the smearing, i.e., the blocking of the wave function. The sharp cutoff in one-dimensional space is realized by the smearing function
\[ \chi_a(x) = \frac{1}{a} \Theta(a - |x|) \] (29)
where \( a \approx \ell_c \).

Our goal is the study of the dependence on the parameter \( \ell_c \). This will be achieved by finding the quantum mechanics of the particle, i.e., its hamiltonian \( H \chi \), which is reconstructed by the help of the measurements of the coordinates. The dependence on the classical, incoherent part, \( \rho(x) \) is trivial from our point of view so we shall take \( \rho(x) = \delta(x) \) in this work.

Note that the blocking operator, \( B \chi \), introduced in this way is not necessarily unitary or even invertible. These features reflect our intention to lose information during the blocking. In fact, the retaining of the infrared part of a state decreases the norm of its wave function. The deficiency,
\[ 1 - \langle \psi(t) | B_{\chi}^\dagger B_{\chi} | \psi(t) \rangle, \] (30)
will be a complicated, nonmonotonic function of time. It is useful to introduce the properly normalized blocked wave function,
\[ \psi_R(x, t) = \sqrt{Z(t)} \psi_{\chi}(x, t), \] (31)
by the help of the wave function renormalization constant \( Z \).

**Gaussian smearing:** Another blocking procedure is a Gaussian suppression in the momentum space,
\[ B(\sigma^2) = e^{-\frac{\sigma^2}{2\hbar^2} p^2}, \] (32)
where $\sigma^2_\pm = \sigma^2 \pm i\epsilon$ and $0 < \epsilon \to 0$. The normalized wave functions are generated by the nonlinear operator

$$B_R(\sigma^2) = e^{-\frac{\sigma^2}{2\pi^2}p^2-V(\sigma^2)}, \quad (33)$$

where $Z^{1/2} = e^{-V}$ satisfies the inequality

$$\sigma^2 \frac{\partial}{\partial \sigma^2} V = \frac{1}{2} \sigma^2 \frac{\partial}{\partial \sigma^2} \log Z = -\frac{\langle \psi_\sigma | \frac{\sigma^2}{2\pi^2} p^2 | \psi_\sigma \rangle}{\langle \psi_\sigma | \psi_\sigma \rangle} \leq 0. \quad (34)$$

showing that the norm of the blocked wave function is decreased during blocking by the suppression of the higher momentum components of the states. The transformation

$$\tilde{\psi}_\sigma(p) = e^{-\frac{\sigma^2}{2\pi^2} p^2} \tilde{\psi}(p), \quad (35)$$

of the Fourier transform of the wave function $\tilde{\psi}(p)$ results in the blocking

$$\psi_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int dy e^{-\frac{1}{2\sigma^2}(x-y)^2} \psi(y) \quad (36)$$

for the wave function $\psi(x)$.

**Landau pole:** The renormalization in Quantum Field Theory consists of the successive elimination of degrees of freedom from the theory. The effect of a mode under elimination is kept in the system by modifying the dynamics, i.e. in the scale dependence of the blocked hamiltonian for the retained modes. We have a harmonic oscillator associated to each particle mode of the theory. It may happen that the elimination of some modes makes the quadratic part of the hamiltonian negative semidefinite. At this point, at the Landau pole, the perturbation expansion fails and the fluctuations become large. This indicates that the new hamiltonian which includes the effects of the eliminated modes, can not be written in the same functional form as the original one, at least within the framework of perturbation expansion. A Landau pole is called infrared or ultraviolet depending on the way it is encountered during the renormalization procedure.

Quantum mechanical systems display singularities in the function of the cutoff, as well, but they turn out to be the opposite of the Landau poles of the quantum field theory models in that the quantum fluctuations disappear.
there. This is because we make the blocking in coordinate space in quantum mechanics which corresponds to the internal space of the Quantum Field Theory model and such a blocking modifies the noncommuting quadratic pieces of the hamiltonian, kinetic energy and harmonic oscillator potential, differently. We find, in the case of the quadratic systems considered here, that only the kinetic energy vanishes at the Landau pole leaving behind a harmonic oscillator potential.

The explicit blocking construction guarantees the existence of the blocked action so our Gaussian blocking form a semi-group,

\[ B(0) = 1, \quad B(\sigma^2) B(\sigma^2) = B(\sigma^2 + \sigma^2). \tag{37} \]

The ultraviolet Landau pole occurs at \( \sigma_L \) if \( B(\sigma^2_L) \) is not invertible. There is no infrared Landau pole in this formalism.

The inverse of the blocking transformation is

\[ B(\sigma^2)^{-1} = e^{\frac{\sigma^2}{2\pi}p^2}. \tag{38} \]

In fact,

\[
\langle x | B(\sigma^2) | y \rangle = \int \frac{dp}{2\pi\hbar} e^{i\hbar p(x-y) - \frac{\sigma^2}{2\pi}p^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-y)^2},
\]

\[
\langle x | B(\sigma^2)^{-1} | y \rangle = \int \frac{dp}{2\pi\hbar} e^{i\hbar p(x-y) + \frac{\sigma^2}{2\pi}p^2} = \frac{1}{\sqrt{-2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(x-y)^2}, \tag{39}
\]

which yields

\[
\langle x | B(\sigma^2) B(\sigma^2)^{-1} | y \rangle = \int dz \langle x | B(\sigma^2) | z \rangle \langle z | B(\sigma^2)^{-1} | y \rangle = -\frac{1}{2\pi\sigma^2} e^{\frac{\sigma^2}{2\sigma^2} - \frac{\sigma^2}{2\sigma^2}} \times \int dz e^{-\frac{\sigma^2}{2\sigma^2}(\frac{1}{\sigma^2} - \frac{1}{\sigma^2}) + z(\frac{1}{\sigma^2} - \frac{1}{\sigma^2})} = \frac{1}{\sqrt{4\pi i \epsilon}} e^{-\frac{1}{4\epsilon}(x-y)^2}. \tag{40}
\]
\( B(\sigma^2)^{-1} \) is well defined in the Hilbert space span by the wave functions

\[
\psi(x) = o \left( e^{-\frac{1}{2\sigma^2}x^2} \right)
\]

for \( x \to \infty \). If this asymptotic condition (41) is not respected then the multiple integrals appearing in the matrix elements are not uniform convergent indicating an ultraviolet Landau pole. This can clearly be seen in the case of the matrix elements of \( B(\sigma^2)B^{-1}(\sigma^2) \) between the harmonic oscillator eigenstates

\[
\langle n | B(\sigma^2)B^{-1}(\sigma^2) | m \rangle = \frac{1}{\ell \sqrt{\pi}} \int dx dy e^{-\frac{1}{2\sigma^2}(x^2+y^2)}
\]

where \( \ell^2 = \hbar/m\omega \). When the integration over \( x \) and \( z \) in second line of (40) is performed before the \( y \) integration, then (42) is always well defined and finite. On the contrary, when the scalar product between \( B(\sigma^2)^{-1} \) and the state \( |m\rangle \) is made for fixed values of \( x \) and \( z \), then (42) is divergent for \( \sigma^2 < \ell^2 \). The blocking transformation can not be inverted within the physical space of such a harmonic oscillator. We encounter an ultraviolet Landau pole because one can not create a state with the localization \( \Delta x < \ell \) in the Hilbert space span by the harmonic oscillator eigenstates.

The blocking is supposed to loose information and should be noninvertible such as (29). Though the \( i\epsilon \) modification of the Kadanoff-Wilson blocking procedure guarantees the formal inverse in the complete Hilbert space, it may happen that the blocking transformation is not invertible in the subspace determined by the physical problem.

**Blocked dynamics:** We shall introduce the running coupling constants in the hamiltonian and the lagrangian. The time evolution of the original wave function

\[
i\hbar \partial_t \psi = H\psi,
\]

induces

\[
i\hbar \partial_t \psi_x = H_x \psi_x
\]

as the equation of motion for the blocked wave function where

\[
H_x B_x = B_x H.
\]
The Hamiltonian that generates the time evolution for the blocked wave function can be obtained directly,

\[ H_\sigma = B(\sigma^2)HB(\sigma^2)^{-1} = e^{-\frac{\sigma^2}{2\pi}p^2} H e^{\frac{\sigma^2}{2\pi}p^2}, \]  

(46)
as long as the blocking is invertible. Notice that \( H_\sigma \) is not the Hamiltonian sandwiched between wave packet states, it is defined instead as the generator of the time evolution for the wave packets. The way \( H_\sigma \) is obtained corresponds to performing a diffusion process, using the free quantum propagation for

\[ \tau = \frac{\sigma^2 m}{\hbar} \]  

(47)
which is an imaginary time. It is interesting to notice that the choice

\[ B_\tau = e^{-\frac{\tau}{\hbar}H} \]  

(48)
correspond to a special blocking which generates the time evolution

\[ \psi_\chi(x, t) = e^{-\frac{i}{\hbar}(t-i\tau)H} \psi(x, 0). \]  

(49)

Starting with the form \( H = \frac{p^2}{2m} + V(x, p) \) the blocking transformation can be written as

\[ H_\sigma = e^{-\frac{\sigma^2}{2\pi}p^2} \left( \frac{p^2}{2m} + V(x, p) \right) e^{\frac{\sigma^2}{2\pi}p^2} \]
\[ = \frac{p^2}{2m} + V(x, p) + e^{-\frac{\sigma^2}{2\pi}p^2} [V(x, p), e^{\frac{\sigma^2}{2\pi}p^2}]. \]  

(50)
The differential form of the renormalization group equation for the length scale independent operator \( h = \sigma^{-2}H_\sigma \) is

\[ \sigma \frac{\partial}{\partial \sigma} h = -2h + \frac{\sigma^2}{\hbar^2} [h, p^2]. \]  

(51)
The term \( H_{n,m}(x, p) \) which is an \( n \)-th and \( m \)-th order homogeneous function in \( p \) and \( x \), respectively, is represented by the length scale independent form,

\[ h_{n,m}(x, p) = \sigma^{2-n-m}H_{n,m}(x, p) \]  

(52)
in the hamiltonian. The action of an infinitesimal renormalization group transformation is
\[ \mathcal{L} h_{n,m} = \sigma \frac{\partial}{\partial \sigma} h_{n,m} = (2 - n - m) h_{n,m} \frac{\sigma^2}{\hbar^2} [h_{n,m}, p^2]. \] (53)

The scaling operators satisfy the equation
\[ \mathcal{L} h_{n,m} = \nu h_{n,m}. \] (54)

A simple class of the scaling operators is made up by the powers \( p^n \), with critical exponent \( \nu = 2 - n \). The only relevant operators of this class come from a homogeneous vector potential and the kinetic energy. Higher powers of the momentum, like the relativistic corrections are irrelevant.

The quantum action, \( S(x, y; t) \), introduced in (18) is the starting point for the space and time dependent running coupling constants of the lagrangian. The blocking transformation for \( S(x, y; t) \) is given by
\[
e^{\frac{i}{\hbar} S(x, y; t)} = \langle x | e^{-\frac{i}{\hbar} tH} | y \rangle \\
= \langle x | B(\sigma^2) e^{-\frac{i}{\hbar} tH} B(\sigma^2)^{-1} | y \rangle \\
= \langle x | e^{-\frac{i}{\hbar} tH} | y \rangle \\
= e^{\frac{i}{\hbar} S_{\sigma}(x, y; t)}. \] (55)

For the generalized problem in (13) we introduce
\[
e^{\frac{i}{\hbar} S_{\sigma, \sigma'}(x, y; t)} = \langle x | B(\sigma^2) e^{-\frac{i}{\hbar} tH} B(\sigma'^2)^{-1} | y \rangle. \] (56)

The quantum action \( S_{\sigma, \sigma'}(x, y; t) \) describes the propagation of a state with localization \( \sigma' \) at \( y \) into another one with localization \( \sigma \) at \( x \). We shall consider only the case \( \sigma' = 0 \) for simplicity.

**Semiclassical limit:** The commutator in the right hand side of (53) is \( O(\hbar) \) and the nonclassical, second term is thus \( O(\hbar^{-1}) \) indicating that the anomalous dimensions diverge in the semiclassical limit. This can be traced back to the factor \( \hbar^{-2} \) in the exponent of (32).

The dimensional parameter of Quantum Mechanics has two different roles in this formalism. One is related to its usual apparence in the time evolution operator, the other is through its presence in the blocking, (32). The divergence of the anomalous dimension (53) originates from the second role,
the divergence of the diffusion time $t_d$ what realizes the blocking. In other
words, any finite length scale $\sigma^2$ becomes large compared to $\hbar$ when the latter
is assumed to approach zero. In order to decouple the two different roles we
can modify (32) as

$$B(\sigma^2) = e^{-\frac{\sigma^2}{2\hbar^2}},$$

(57)

by introducing a new constant, $\hbar_0$, with the same dimension as $\hbar$ which re-
mains unchanged in the semiclassical limit. This amounts to the replacement

$$\sigma^2 \rightarrow \frac{\hbar^2}{\hbar_0^2} \sigma^2$$

(58)
in the expressions obtained so far and makes the anomalous dimension equiv-
alent with the classical dimension in the semiclassical limit, as expected. We
shall return to the contribution of the last term of (53) in Section 5.

### 4 Quadratic Hamiltonians

We now apply the blocking procedure introduced above for quadratic systems
and obtain their renormalized trajectory.

#### 4.1 Free Particle

The blocked wave function is

$$\psi_\sigma(x, t; y) = e^{\frac{i}{\hbar} S_{\sigma, 0}(x; y, t)}$$

$$= \sqrt{\frac{m}{2\pi i \hbar t}} \int \sqrt{\frac{2\pi}{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} (x-z)^2 + \frac{im(\sigma, 0)}{2\hbar t} (z-y)^2}$$

$$= \sqrt{\frac{m(\sigma, 0)}{2\pi i \hbar t}} e^{\frac{im(\sigma, 0)}{2\hbar t} (x-y)^2},$$

(59)

where

$$\ell^2 = \frac{\hbar m}{\ell^2}$$

(60)

and the mass parameter is defined by

$$m(\sigma, 0) = m \frac{1}{1 - i \frac{\sigma^2}{\ell^2}}.$$
The probability density of the blocked wave function is
\[ |\psi_\sigma(x,t; y)|^2 = N^{-1} e^{-\frac{1}{\Delta x^2} (x-y)^2}, \] (62)
with
\[ \Delta x^2 = \ell^2 \left( \frac{\ell^2}{\sigma^2} + \frac{\sigma^2}{\ell^2} \right), \]
\[ N = 2\pi \ell^2 \sqrt{1 + \frac{\sigma^4}{\ell^4}}. \] (63)

The norm of the blocked wave function gives
\[ Z = 2\sqrt{\pi} \sigma. \] (64)

It is easy to understand that \( \Delta x^2 \) reaches its minimum at \( \sigma = \sigma_{cr} = \ell \).

For the under-smeared case with \( \sigma \ll \sigma_{cr} \), the destructive interference is building up in (53) as we increase \( x - y \) when the phase of the original wave function changes by \( \pi \) within the interval \( [x + \sigma, x] \),
\[ \frac{(x + \sigma - y)^2 - (x - y)^2}{\ell^2} \approx \pi, \] (65)
which yields
\[ \Delta x = x - y \approx \frac{\pi}{2} \frac{\ell^2}{\sigma} - \frac{\sigma}{2} \approx \frac{\ell^2}{\sigma}. \] (66)
For \( \sigma >> \sigma_{cr} \), we have over-smearing because the interference is now as destructive as it can be even within the longest period length at \( x - y \approx 0 \). But since the integration is done in the distance while the periodicity is in the distance square the cancellation within a period length is more complete for short period length, i.e. \( |x - y| >> \ell \). The dominant contribution coming from \( x - y \approx 0 \) is thus suppressed without any further interference simply by the smearing,
\[ \Delta x^2 \approx \sigma^2. \] (67)

The two opposite effects are balanced at the crossover \( \sigma = \sigma_{cr} \).

The effective mass read off from the lagrangian or hamiltonian is renormalization group invariant since \([H, B(\sigma^2)] = 0\). The translation invariant propagation between states with the same smearing, \( S_{\sigma, \sigma}(x, y; t) \), is scale invariant, \( m(\sigma, \sigma) = m \).
4.2 Harmonic Oscillator

Wave function: The quantum action,

$$i\hbar S(x, y; t) = \ln\langle x|e^{-\frac{i\hbar}{\hbar}H}|y\rangle = \frac{i}{2} \cot t\omega \frac{x^2 + y^2}{\ell^2} - \frac{i}{\sin \omega t} \frac{xy}{\ell^2} - \frac{1}{2} \ln 2\pi i \ell^2 \sin \omega t,$$

(68)

where

$$\ell^2 = \frac{\hbar}{m\omega},$$

(69)
corresponds to the hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 = \frac{p^2}{2m} + \frac{\hbar \omega}{2} \frac{x^2}{\ell^2}.$$  

(70)

The blocked wave function is then given by

$$\psi_\sigma(x, t; y) = e^{iS_{\sigma, 0}(x, y; t)}$$

$$= \frac{1}{\sqrt{4\pi^2 i \ell^2 \sigma^2 \sin \omega t}} \int dz e^{-\frac{1}{2\sigma^2} (x-z)^2 - \frac{1}{2} \frac{x^2+y^2}{\ell^2} - bxy}$$

$$= \frac{1}{\sqrt{2\pi i \rho^{-1} \ell^2 \sin \omega t}} e^{-\rho \left[\frac{1}{2}(x^2+y^2)+bxy+\frac{1}{4}\sigma^2(x^2+y^2)\right]},$$  

(71)

where

$$a = -\frac{i}{\ell^2} \cot \omega t, \quad b = \frac{i}{\ell^2 \sin \omega t}, \quad \rho = \frac{1 + \frac{\sigma^2}{\ell^4} \cot^2 \omega t}{1 + \frac{\sigma^4}{\ell^4} \cot^2 \omega t}.$$  

(72)

The probability distribution is

$$|\psi_\sigma(x, t; y)|^2 = \mathcal{N}^{-1} e^{-\frac{1}{2\sigma^2}(x-y/\cos \omega t)^2},$$  

(73)

with

$$\Delta^2 x = \ell^2 \left( \frac{\ell^2}{\sigma^2} \tan^2 \omega t + \frac{\sigma^2}{\ell^2} \right),$$

$$\mathcal{N} = 2\pi \ell^2 \sin \omega t \sqrt{1 + \frac{\sigma^4}{\ell^4} \cot^2 \omega t}.$$  

(74)
Note that the time reversed process starts with a wave packet and the $y$ dependence describes the oscillation of its center. The trigonometric functions in our expressions are due to the multiple reflection from the potential.

The width of the distribution in $x$ is minimal, $\Delta x = \sigma$, for focusing or anti-focusing,

$$\begin{align*}
t \approx \begin{cases} 
  nT & \text{focusing,} \\
  (n + \frac{1}{2})T & \text{anti-focusing,} 
\end{cases}
\end{align*}$$

(75)

where $T = \frac{2\pi}{\omega}$. The localization is very strong after blocking for small $\sigma$ at focusing or anti-focusing because the original wave function is perfectly localized. The crossover at

$$\sigma_{cr} = \ell \sqrt{|\tan \omega t|}$$

(76)

indicates that the system is over-smeared close to focusing or anti-focusing and under-smeared otherwise. We found

$$Z = 2\sqrt{\pi} \sigma |\cos \omega t|$$

(77)

for the wave function renormalization constant. The norm of the blocked wave function, $Z^{-1/2}$, shows minima for focusing and anti-focusing when the wave function is well localized, i.e. is concentrated in the ultraviolet regime.

**Hamiltonian:** One can easily find the blocked version of the hamiltonian, satisfying the relation

$$H_\sigma e^{-\frac{p^2}{2m(\sigma)^2} \frac{\omega^2}{\ell^2}}} = e^{-\frac{p^2}{2m(\sigma)^2} \frac{\omega^2}{\ell^2}}} \left( \frac{p^2}{2m} + \frac{m\omega^2}{2\ell^4} x^2 \right).$$

(78)

The direct computation gives

$$H_\sigma = \frac{p^2}{2m(\sigma)} + \frac{m(\sigma)\omega^2}{2} x^2 - \frac{m(\sigma)\omega^2\sigma^4}{2\ell^4} x^2 + i \frac{\omega \sigma^2}{2\ell^2} (xp + px),$$

(79)

with

$$m(\sigma) = m \frac{1}{1 - \frac{\sigma^2}{\ell^2}}.$$ 

(80)

The coefficient of $x^2$ turns out to be renormalization group invariant. Nevertheless it is reasonable to keep the frequency independent of $\sigma$ because the blocking in space can not influence the period length in time, the quantities
without length dimension, in general. The effective mass has a singularity at \( \sigma = \ell \) and is negative for \( \sigma > \ell \). The inverse of the naive blocking transformation (36) is well defined for \( \sigma > \ell \) and the system has an ultraviolet Landau pole at \( \sigma = \ell \) where the inverse of the coefficient of the kinetic energy, \( m(\sigma) \), diverges. The movement of the particle is more and more difficult to observe as space resolution, \( \sigma \), approaches the localization length, \( \ell \), from below. The motion disappears completely at \( \sigma = \ell \) because the localization of the particle happens to be compensated for exactly by the smearing.

The result of the direct computation of the blocked hamiltonian

\[
H_\sigma = \frac{p^2}{2m} + V(x) + e^{-\frac{\sigma^2}{2\sigma^2}p^2}[V(x), e^{\frac{\sigma^2}{2\sigma^2}p^2}],
\]

performed at the infrared side of the Landau pole reproduces (79). This demonstrates that the \( i\epsilon \) prescription makes the blocking transformation (38) invertible and the Landau pole of the naive blocking procedure disappears.

**Action:** The transformation of the quantum action under blocking is

\[
e^{-\frac{\sigma^2}{4\sigma^2}(x^2+y^2)-bzy} \rightarrow \frac{1}{\sqrt{-4\pi^2\sigma^4}} \int dz_1 dz_2
\]

\[
\times e^{-\frac{1}{2\sigma^2}(x-z_1)^2-\frac{\sigma^2}{4}(z_1^2+z_2^2)-bz_1z_2+\frac{1}{2\sigma^2}(z_2-y)^2}.
\]

The gaussian integral results

\[
\frac{i}{\hbar} S_\sigma(x, y; t) = -\frac{1}{1 - \frac{\sigma^4}{\ell^4}} \left[ \frac{a}{2}(x^2 + y^2) + bxy - \frac{\sigma^2}{2\ell^4}(x^2 - y^2) \right]
\]

\[
-\frac{1}{2} \ln 2\pi\ell^2(1 - \frac{\sigma^4}{\ell^4}) \sin \omega t
\]

which is the logarithm of the transition amplitude for the wave packets. The unitary part of the propagation can be obtained from (88) by replacing the mass by its running value (80). The probability density,

\[
\left| e^{i\hbar S_\sigma(x, y; t)} \right|^2 = \mathcal{N}^{-1} e^{-\frac{1}{2\sigma^2}(x^2 - y^2)},
\]

with

\[
\Delta x^2 = \ell^2 \left( \frac{\sigma^2}{\ell^2} - \frac{\ell^2}{\sigma^2} \right),
\]

\[
\mathcal{N} = \left| 2\pi\ell^2 \left( 1 - \frac{\sigma^4}{\ell^4} \right) \sin \omega t \right|,
\]

20
shows the presence of the Landau pole at $\sigma = \ell$ where the sign of the kinetic term changes. The time dependence drops from the blocking transformation because both the initial and the final states are smeared in the same manner.

The blocked action yields normalizable wave functions only for the over-smeared case where the localization of the blocked states is weaker than the localization supported by the dynamics. In the operator formalism the usual positive-mass Hamiltonian was obtained on the ultraviolet side of the Landau pole because in that case the energy, rather than the states with a certain localization, was followed.

Note that the piece $1 - \sigma^4/\ell^4$ in the quantum action can be interpreted as a multiplicative factor to $\hbar$. As we approach the Landau pole the path integral becomes dominated by the vicinity of the classical trajectory and we recover the semiclassical limit, the center of the blocked wave packets follow the classical equation of motion. The absence of the quantum fluctuations at the Landau pole can be seen in the operator formalism, too. There the vanishing of the kinetic energy reduces the quantum problem into a classical one at the Landau pole.

The lesson of this simple system is that the system possesses a Landau pole where the localization length agrees with the block size. The effective, running parameters of the Hamiltonian change slowly and stay reasonable so long as $\sigma < < \sigma_L = \ell$. The fundamental change of the dynamics around the Landau pole, $\sigma \approx \ell$, is due to the fact that the particle is better localized than the space resolution of the observation for $\sigma >> \sigma_L$. One loses sight of the motion and the particle propagation appears anomalous at the infrared side of this crossover. Thus the sudden changes or singularities in the mass indicates that the localization length is passed by the observational scale.

### 4.3 Electron in Homogeneous Magnetic Field

The free particle gave renormalization group invariant dynamics and the harmonic oscillator yielded a nontrivial renormalized trajectory with Landau pole. These different behaviors suggest the investigation of the propagation of an electron in homogeneous magnetic field. We start with the two dimen-
sional hamiltonian in the presence of the background field \((A_x, A_y) = (0, Bx)\),

\[
H = \frac{1}{2m} p_x^2 + \frac{1}{2m} \left( p_y - \frac{eB}{c} x \right)^2
\]

whose eigenfunctions are

\[
\psi_{p,n}(x, y) = \frac{1}{\sqrt{\pi} \cdot 2^n n!} e^{ip y} H_n \left( \frac{x - x_p}{\ell} \right) e^{-\frac{(x-x_p)^2}{2\ell^2}}
\]

with the eigenvalues

\[
E = \hbar \omega_L \left( n + \frac{1}{2} \right)
\]

where

\[
x_p = \frac{cp_y}{eB}, \quad \omega_L = \frac{|e|B}{mc}, \quad \ell^2 = \frac{\hbar}{m\omega_L}.
\]

The corresponding quantum action can be written as

\[
\frac{1}{\hbar} S(u, v; t) = \frac{1}{2\ell^2} (u - v)^2 + \frac{1}{2\ell^2} (u_x + v_x)(u_y - v_y)
+ i \log 4\pi i \ell^2 \left| \sin \frac{\omega_L t}{2} \right|
\]

by means of the notation

\[
\tilde{\ell}^2 = 2\ell^2 \tan \frac{\omega_L t}{2}.
\]

**Wave function:** A possible gauge covariant generalization of the blocked wave function is

\[
\psi_{\chi}(u) = B_{\chi}[A] \psi(u) = \int d\nu \chi^*(\nu) e^{\frac{i\nu}{\omega_L}} \int_{u+v}^{u} dw A(w) \psi(u+v),
\]

where the integration in the exponent is performed along the straight line connecting \(u + v\) and \(u\). This definition yields

\[
\psi_{\sigma}(u) = \int \frac{d\nu}{2\pi \sigma^2} e^{-\frac{1}{2\sigma^2}(u-v)^2 + \frac{i\nu}{\omega_L}} \int_{v}^{u} dw A(w) \psi(v)
\]

for the Gaussian blocking. Note that this blocking does not correspond any more to a diffusion process, i.e. the propagation in imaginary time because
the gauge phase factor is evaluated along a given path instead of summed over all trajectories. The blocked action reads as

\[ e^{i \frac{\hbar}{\sigma^2} S_\sigma(u,v,t)} = -\int \frac{dw dz}{(2\pi \sigma^2)^2} \exp \left\{ -\frac{1}{2\sigma^2} (u - w)^2 + \frac{ie}{\hbar c} \int_w^u dA(w') \right. \]

\[ + \frac{i}{\hbar} S(w,z,t) + \frac{1}{2\sigma^2} (z - v)^2 - \frac{ie}{\hbar c} \int_z^v dz' A(z') \} \right\}. \] (94)

We take the wave function

\[ \psi(u, t; v) = e^{i \frac{\hbar}{\sigma^2} S(u,v,t)} \] (95)

and perform the blocking,

\[ \psi_\sigma(u, t; v) = e^{i \frac{\hbar}{\sigma^2} S_\sigma(u,v,t)} \]

\[ = \int \frac{dw}{2\pi \sigma^2} e^{-\frac{1}{2\sigma^2} (u - w)^2 + \frac{i}{\sigma^2} (u_x + w_x)(u_y - w_y)} \psi(w, t; v) \] (96)

with the result

\[ \psi_\sigma(u, t; v) = \left( 4\pi i \ell^2 \left( 1 - i \frac{\sigma^2}{\ell^2} \right) \left| \sin \frac{\omega_L t}{2} \right| \right)^{-1} \]

\[ \times \exp \left[ \frac{i}{2\ell^2} \frac{1 + i \frac{\sigma^2}{4\ell^2}}{1 - i \frac{\sigma^2}{\ell^2}} (u - v)^2 \right] \]

\[ + \frac{i}{2\ell^2} (u_x + v_x)(u_y - v_y). \] (97)

The probability density is then

\[ |\psi_\sigma(u, t, v)|^2 = N^{-1} e^{-\frac{1}{\Delta x^2} (u - v)^2}, \] (98)

with

\[ \Delta x^2 = \ell^2 \cos^2 \frac{\omega_L t}{2} \left( 4 \frac{\ell^2}{\sigma^2} \tan^2 \frac{\omega_L t}{2} + \frac{\sigma^2}{\ell^2} \right), \]

\[ N = 16\pi^2 \ell^4 \left( 1 + \frac{\sigma^4}{4\ell^4} \cot^2 \frac{\omega_L t}{2} \right) \sin^2 \frac{\omega_L t}{2}. \] (99)
One finds the usual focusing and antifocusing phenomena at \( t = nT \) and \( t = (n + \frac{1}{2})T \), respectively, where \( T = \frac{2\pi}{\omega_L} \). The crossover is at

\[
\sigma_{cr} = \ell \sqrt{2 \left| \tan \frac{\omega_L t}{2} \right|}. \tag{100}
\]

The wave function renormalization constant is found to be

\[
Z = 4\pi \sigma^2. \tag{101}
\]

One can see that the gauge invariant absolute magnitude of the wave function preserves translation invariance and the norm of the blocked wave function follows the \( \sigma \) dependence of the free particle. Nevertheless the harmonic oscillator implicit in the system makes its appearance in \( \sigma \) and the time dependence of the Gaussian peak of the absolute magnitude.

**Hamiltonian:** The gauge invariant generalization of blocking (46), by using

\[
B(\sigma^2) = e^{-\frac{\sigma^2}{2\ell^2}(p - eA)^2}, \tag{102}
\]

yields scale invariant hamiltonian as in the case of the free particle.

**Action:** The straightforward integration yields

\[
\frac{1}{\hbar} S_\sigma(u, v; t) = \frac{1}{2\ell^2}(u - v)^2 + \frac{1}{2\ell^2}(u_x + v_x)(u_y - v_y) + i \log \left[ 4\pi i \ell^2 \left( 1 - \frac{\sigma^4}{4\ell^4} \right) \left| \sin \frac{\omega_L t}{2} \right| \right]. \tag{103}
\]

The homogeneity of the magnetic field protects the action against picking up scale dependence during the blocking as in the case of the free particle and the only \( \sigma \) dependence is in the additive constant. Notice that this result is not as trivial as the scale invariance of the hamiltonian because the gauge invariant smearing (103) does not corresponds to (10) and (102). The difference allows us to detect the harmonic oscillator of the system by the singular scale dependence of the additive constant in the blocked action.

The distinguished feature of systems with quadratic hamiltonians is that the imaginary part of the quantum action, \( S(x, y; t) \) of (18), is independent of the coordinates and the probability density

\[
\rho(x) = \left| e^{iS(x, y; \Delta t)} \right|^2 \tag{104}
\]
is $x$ and $y$ independent. This is a rather surprising result, it means for example that a particle which is perfectly localized at the bottom of a harmonic potential will be found any time later anywhere with homogeneous probability distribution. Such an unphysical spread of the wave function is due to the perfect localization \(^{15}\). If one spreads the states by means of blocking \(^{27}\), then the interference will introduce a realistic propagation pattern without invoking relativistic or multiparticle effects. The essential modification of the dynamics by this spreading is indicated by the drastic change of the propagation for weak smearing, $\sigma \to 0$, c.f. \(^{66}\). But the nonharmonic terms of the hamiltonian strongly influence the probability distribution and may alone remove these singularities. When the propagation between states with the same smearing is considered then the singularity at $\sigma \to 0$ disappears.

5 Representations of the Renormalization Group

We study in this Section the renormalized trajectory

$$H(x,p) \longrightarrow H_\sigma(x,p) = e^{-\frac{\sigma^2}{8\pi^2}p^2} H(x,p)e^{\frac{\sigma^2}{8\pi^2}p^2}$$

\(^{105}\)

for the hamiltonians which are the sum of the products of the canonical operators $x$ and $p$. We are interested in the quantum corrections to the evolution equations. Thus we do not perform the trivial rescaling of the operators by the observational length scale and we have the second term only in the right hand side of \(^{53}\). The simplest nontrivial case is

$$x \longrightarrow x_\sigma = e^{-\frac{\sigma^2}{8\pi^2}p^2}xe^{\frac{\sigma^2}{8\pi^2}p^2} = x + i\kappa p,$$

\(^{106}\)

where

$$\kappa = \frac{\sigma^2}{\hbar}.$$  

\(^{107}\)

This implies the transformation rule

$$H(x, p) \longrightarrow H_\sigma (x, p) = H(x + i\kappa p, p).$$

\(^{108}\)

For each n long string made up by the operators $x$ and $p$, we associate an n-index tensor. The tensor indices take the values 1 and 2 and the only
nonvanishing matrix element is where the value of the j-th index is 1 (2) when the j-th operator in the product is x (p). The value of this matrix element is the c-number coefficient of the product of x and p. In this manner the nonvanishing matrix elements of the tensor corresponding for example to the operator $xxpx - 2pxpx$ are $T^{1121} = 1$ and $T^{2121} = -2$. The blocking is then represented by the linear transformation

$$T^{a_1, \ldots, a_n} \rightarrow g^{a_1, b_1} \ldots g^{a_n, b_n} T^{b_1, \ldots, b_n},$$

where

$$g(\kappa) = \begin{pmatrix} 1 & i\kappa \\ 0 & 1 \end{pmatrix}.$$

It is useful to write

$$g(\kappa) = e^{i\kappa \sigma_+} = 1 + i\kappa \sigma_+,$$

where

$$\sigma_+ = \sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The $g(\kappa)$ matrix gives the "fundamental representation" of the renormalization group,

$$g(\kappa_1)g(\kappa_2) = g(\kappa_1 + \kappa_2)$$

what is a representation of a one dimensional subgroup of $sl(2, c)$.

The operator algebra generated by the products of x and p or the related tensor algebra provides some linear representations of the renormalization group. But one should not forget that the coefficients of the operators, the matrix elements of the tensors are nonlinear functions of the blocking parameter, $\sigma^2$. It is obvious that the subspace $O_n$ which is span by the products of n operator of x or p, i.e. the subspace of the n-index tensors, remains invariant under the blocking. Another invariant subspace of the renormalization group $O_n(Y)$, where Y denotes an irreducible representation of the symmetric group $S_n$, is span by such combination of products of n x or p operators which have the given symmetry with respect to the permutations of the operators in the products. The antisymmetrisation with respect to the exchange of two x or p operators yields 0 or $i\hbar$. Thus the antisymmetrisation of an operator from $O_n$ yields either another operator from $O_{n-2}$ or zero. The formal similarity of this observation with the contraction of the indices by the help of the metric tensor in the tensor representation of the group
sl(2, c) is evident. Thus it is enough to study the subspaces $O_n(S)$ given in terms of the symmetrised Weyl-products. Since \([\Pi]\) is nonunitary, $O_n(S)$ is not necessarily the direct sum of irreducible representations.

Let us consider the cases $n = 2$ and 4 as examples where the most general operator in $O_2(S)$ and $O_4(S)$ are

\[
H_2 = g_{2,0}h_{2,0} + ig_{1,1}h_{1,1} + g_{0,2}h_{0,2},
\]
\[
H_4 = \frac{1}{\sigma^2} \left( g_{4,0}h_{4,0} + ig_{3,1}h_{3,1} + g_{2,2}h_{2,2} + ig_{1,3}h_{1,3} + g_{0,4}h_{0,4} \right),
\]

where

\[
\begin{align*}
  h_{2,0} &= p^2, \\
  h_{1,1} &= xp + px, \\
  h_{0,2} &= x^2, \\
  h_{4,0} &= p^4, \\
  h_{3,1} &= p^3x + p^2xp + ppx^2 + xp^3, \\
  h_{2,2} &= p^2x^2 + x^2p^2 + pxp + pxpx + xp^2x + px^2p, \\
  h_{1,3} &= px^3 + px^2x + x^2px + x^3p, \\
  h_{0,4} &= x^4.
\end{align*}
\]

By the help of the relations

\[
\begin{align*}
[e^{-\frac{\sigma^2}{2\pi}p^2}, x^n] &= (\lambda h_{1,n-1} + \lambda^2 h_{2,n-2} + \cdots \lambda^n h_{n,0})e^{-\frac{\sigma^2}{2\pi}p^2}, \\
[e^{-\frac{\sigma^2}{2\pi}p^2}, h_{1,1}] &= 2\lambda h_{2,0}e^{-\frac{\sigma^2}{2\pi}p^2}, \\
[e^{-\frac{\sigma^2}{2\pi}p^2}, h_{3,1}] &= 4\lambda h_{4,0}e^{-\frac{\sigma^2}{2\pi}p^2}, \\
[e^{-\frac{\sigma^2}{2\pi}p^2}, h_{2,2}] &= (3\lambda h_{3,1} + 6\lambda^2 h_{4,0})e^{-\frac{\sigma^2}{2\pi}p^2}, \\
[e^{-\frac{\sigma^2}{2\pi}p^2}, h_{1,3}] &= (2\lambda h_{2,2} + 3\lambda^2 h_{3,1} + 4\lambda^3 h_{4,0})e^{-\frac{\sigma^2}{2\pi}p^2},
\end{align*}
\]

where $\lambda = i\kappa$ we obtain the renormalization group transformation

\[
\begin{align*}
g_{2,0}(\sigma^2) &= g_{2,0} - 2\kappa g_{1,1} - \kappa^2 g_{0,2}, \\
g_{1,1}(\sigma^2) &= g_{1,1} + \kappa g_{0,2}.
\end{align*}
\]
\begin{align*}
g_{0,2}(\sigma^2) &= g_{0,2}, \\
g_{4,0}(\sigma^2) &= g_{4,0} - 4\kappa g_{3,1} - 6\kappa^2 g_{2,2} + 4\kappa^3 g_{1,3} + \kappa^4 g_{0,4}, \\
g_{3,1}(\sigma^2) &= g_{3,1} + 3\kappa g_{2,2} - 3\kappa^2 g_{1,3} - \kappa^3 g_{0,4}, \\
g_{2,2}(\sigma^2) &= g_{2,2} - 2\kappa g_{1,3} - \kappa^2 g_{0,4}, \\
g_{1,3}(\sigma^2) &= g_{1,3} + \kappa g_{0,4}, \\
g_{0,4}(\sigma^2) &= g_{0,4}.
\end{align*}

The nonvanishing of $g_{2,2}(\sigma^2)$ when $g_{0,4}(0) \neq 0$ shows that the the anharmonic terms make the particle to appear as propagating on a curved manifold at the Landau pole, defined by $g_{2,0}(\sigma^2_L) = 0$.

The renormalized trajectory for the harmonic oscillator is realized by

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}x^2 + ig(xp + px).$$

The only eigenvector is the hamiltonian of the free particle. The renormalization group flow is

$$\begin{pmatrix} m^{-1}(\sigma) \\ \omega^2(\sigma) \\ g(\sigma) \end{pmatrix} = \begin{pmatrix} m^{-1} \left(1 - \frac{\sigma^2}{\ell^2}\right) - 4g \frac{\sigma^2}{\hbar} \\ \omega^2 m m^{-1}(\sigma) \\ g + \frac{m\omega^2}{2} \frac{\sigma^2}{\hbar} \end{pmatrix}.$$

Bear in mind that $\omega(\sigma)$ is a parameter only, the true frequency of the motion is left unchanged by blocking due to the nonhermitean piece of the hamiltonian.

Our hamiltonian is ultraviolet finite so any point in the coupling constant space $(m^{-1}, \omega^2, g)$ is an ultraviolet fixed point. The plane $\omega^2 = 0, \ell = \infty$ is invariant under the action of blocking. The flow with $\omega^2 > 0$ has more structure. In fact, the mass is not a monotonic function of $\sigma$ when $g < 0$ and one can identify an ultraviolet and an infrared scaling region separated by a crossover at $\sigma^2_{\ell} = -2g\hbar/m\omega^2$ where the non-hermitean part of the hamiltonian changes sign. The usual harmonic oscillator is at the crossover and its evolution follows the $g > 0$ infrared scaling regime.

$U_\ell(g(\kappa))$, the $\ell$ angular momentum representation of (110) is an upper triangular matrix in the usual $|\ell, m\rangle$ basis so

$$\det[U_\ell(g(\kappa)) - \lambda] = (1 - \lambda)^{2\ell+1}$$

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and all eigenvalues of the representation of the renormalization group is 1. One can verify easily that there is one eigenvector only in each space $O_n(S)$ for even $n$, namely $h_{n,0} = p^n$. There is no eigenvector for odd $n$ as expected since the gaussian smearing cancels the operators with odd space inversion parity. The Taylor expansion of

$$U(\kappa) = e^{iU(\sigma_+)} = 1 + i\kappa U(\sigma_+) + \cdots + \frac{(i\kappa)^{2\ell}}{2\ell!} U^{2\ell}(\sigma_+),$$  \hspace{1cm} (125)

shows that an operator from $O_{2\ell+1}(S)$ is made of $m + \ell \times$ operators where $-\ell \leq m \leq \ell$ contains the $m + \ell$-th power of $\kappa$ in its transformation law under the renormalization group transformation. Thus the operator with the largest weight in the infrared regime will be $h_{n,0} = p^n, n = 2\ell + 1$.

6 Summary

A systematic method to incorporate in Quantum Mechanics, the dependence of the dynamics on the space resolution was presented in the spirit of the renormalization group. The blocking step appears as the formal analogy of the Kadanoff-Wilson blocking procedure of the second quantized quantum field variable repeated on the level of the wave function. This was motivated by a simplified description of the measurement process where the extended structure of the coherent part of the measuring device was taken into account by the blocking of the wave function. The scale dependence of the wave function and the dynamics extracted from it can be mapped out easily by borrowing ideas from the renormalization group.

The transformation of the operator algebra under blocking gives rise to a representation of the renormalization group what happened to be a nonunitary representation of a one dimensional subgroup of $sl(2,c)$. Several irreducible representations were identified. It would be interesting to find all of them in order to classify the possible "elementary dynamics". In the representations provided by the homogeneous functions of the canonical variables, the momentum-dependent operators turned out to be the most relevant in the infrared limit. This is reasonable since on the one hand, the momentum-dependent terms modify the propagation at arbitrary large distances and on the other hand, the polynomial potentials generate bound states at finite energies whose localization suppresses the infrared effects of the coordinate dependent potential.
The smearing of the wave function is necessary in the case of the quadratic Hamiltonians to arrive at acceptable transition amplitudes when a maximally localized Dirac-delta initial condition is used. Three quadratic systems, the free particle, harmonic oscillator and a two dimensional charged particle in homogeneous magnetic field are considered in a detailed manner. The transition amplitudes from a maximally localized state into a wave packet possess a crossover where the resolution reaches the characteristic length scale of the problem. The blocked dynamics of the free particle and the electric charge in a homogeneous magnetic field is found to be scale invariant.

We finally note that the blocked propagator (56) provides a convenient way to study the simultaneous dependence of the propagation both on the space resolution and on the observation time what might be useful in mesoscopic systems.

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