ON THEOREMS OF BRAUER-NESBITT AND BRANDT FOR CHARACTERIZATIONS OF SMALL BLOCK ALGEBRAS

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Abstract. In 1941, Brauer-Nesbitt established a characterization of a block with trivial defect group as a block $B$ with $k(B) = 1$ where $k(B)$ is the number of irreducible ordinary characters of $B$. In 1982, Brandt established a characterization of a block with defect group of order two as a block $B$ with $k(B) = 2$. These correspond to the cases when the block is Morita equivalent to the one-dimensional algebra and to the non-semisimple two-dimensional algebra, respectively.

In this paper, we redefine $k(A)$ to be the codimension of the commutator subspace $K(A)$ of a finite-dimensional algebra $A$ and prove analogous statements for arbitrary (not necessarily symmetric) finite-dimensional algebras. This is achieved by extending the Okuyama refinement of the Brandt result to this setting. To this end, we study the codimension of the sum of the commutator subspace $K(A)$ and $n$th Jacobson radical $\operatorname{Rad}^n(A)$. We prove that this is Morita invariant and give an upper bound for the codimension as well.

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1. Introduction

Modular representation theory of finite groups aims to understand the structure of a block of a finite group algebra and its invariants. The complexity of the representations of a block $B$ is measured by invariants including the defect group $P$, the number of irreducible ordinary characters $k(B)$, the number of irreducible modular characters $\ell(B)$, and the Cartan matrix $C_B$ of the block $B$. A semisimple block, a block of the simplest kind, is characterized as a block of defect zero; The
characterization has the following equivalent condition, which was essentially due to Brauer-Nesbitt [1, Section 11].

\[ P \cong 1 \iff k(B) = 1. \]  
(1.1)

A few decades later Brandt [2, Theorem A] proved similar.

\[ P \cong \mathbb{Z}/2\mathbb{Z} \iff k(B) = 2 \text{ (and } \ell(B) = 1) \]  
(1.2)

This condition characterizes a block that is Morita equivalent to the group algebra of \( \mathbb{Z}/2\mathbb{Z} \). For textbook accounts for these, see the Landrock textbook [11, Section I.16]. We call a block \( B \) small if \( k(B) \) is small.

The aim of this paper is to gain a better understanding of the above results by generalizing to arbitrary finite-dimensional algebras. Throughout this paper, \( A \) denotes a finite-dimensional algebra over a field \( F \) and \( \{ S_i \mid 1 \leq i \leq \ell(A) \} \) a complete set of pairwise non-isomorphic simple right \( A \)-modules. An obstacle for generalization is that there is no established notion of \( k(A) \). To overcome this difficulty we introduce the following definition.

**Definition 1.1.** Let \( K(A) \) be the **commutator subspace** of \( A \) (i.e., the \( F \)-subspace of \( A \) spanned by \( xy - yx \) for all \( x, y \in A \)). Then we define

\[ k(A) := \text{codim} K(A). \]  
(1.3)

The new definition of \( k(A) \) for an algebra \( A \) coincides with the old \( k(B) \) for a block \( B \) by Külshammer [8, Lemma A(ii)]. Hence \( k(A) \) is supposed to measure the complexity of the representations of the algebra \( A \). Indeed, we are able to prove that this is the case if \( k(A) \) is small.

**Theorem 1.2** (See also Theorem A.1). Suppose that \( F \) is a splitting field for \( A \). Then the following statements are equivalent.

(i) \( A \) is Morita equivalent to \( F \).

(ii) \( k(A) = 1 \).

**Theorem 1.3** (See also Theorem A.1). Suppose that \( F \) is a splitting field for \( A \). Then the following statements are equivalent.

(i) \( A \) is Morita equivalent to \( F[X]/(X^2) \).

(ii) \( k(A) = 2 \) and \( \ell(A) = 1 \).

Note that we cannot omit the condition \( \ell(A) = 1 \) in the above (Note 4.4). If, among others, \( A \) is non-semisimple symmetric then we can omit the condition (Proposition 4.5). We also remark that a natural candidate \( k^*(A) := \dim Z(A) \), the dimension of the center, does not work here. Consider the \( F \)-algebra \( T_n \) of lower triangular matrices of degree \( n \in \mathbb{N} \). Then \( k^*(T_n) = 1 \) and the analog is no longer true.

These theorems are obtained from a study of the codimension of the \( F \)-subspace of \( A \) defined by

\[ KR^n(A) := K(A) + \text{Rad}^n(A) \quad (n \in \mathbb{N}) \]

where \( \text{Rad}^n(A) \) denotes the \( n \)th Jacobson radical of \( A \). The following has been known for finite-dimensional symmetric algebras, while we extend it for arbitrary finite-dimensional algebras.

**Theorem 1.4** (See also Note 3.5). Suppose that \( F \) is a splitting field for \( A \). Then the following holds.
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(i) \( \text{codim} KR^1(A) = \ell(A). \)
(ii) \( \text{codim} KR^2(A) = \ell(A) + \sum_{1 \leq i \leq \ell(A)} \dim \text{Ext}^1(S_i, S_i). \)
(iii) \( \ell(A) + \sum_{1 \leq i \leq \ell(A)} \dim \text{Ext}^1(S_i, S_i) \leq k(A) \leq \text{tr} C_A. \)

This paper is organized as follows. First, Section 2 is devoted to prove Morita invariance (Theorem 2.1). In Section 3 we establish an upper bound for the codimension of \( KR^n(A) \) in Theorem 3.4 and prove Theorem 1.4. Finally, Section 4 provides a characterization of truncated polynomial algebras (Theorem 4.2) using Theorems 1.4 and 2.1. Then characterizations of small algebras (Theorems 1.2 and 1.3) follow immediately.

2. Morita invariance

This section is devoted to prove Morita invariance of the codimension of \( KR^n(A) \). See also Sakurai [14, Theorem 2.3] for its “dual”.

**Theorem 2.1.** Let \( A \) and \( B \) be Morita equivalent finite-dimensional algebras over a field \( F \). Then there is an \( F \)-linear isomorphism \( A/K(A) \to B/K(B) \) inducing the \( F \)-linear isomorphism \( A/KR^n(A) \to B/KR^n(B) \) for every \( n \in \mathbb{N} \).

**Proof.** It suffices to prove the case for \( A \) and its basic algebra \( B := eAe \) because Morita equivalent basic algebras are isomorphic. Since the basic idempotent \( e \in A \) is full, there exist elements \( u_k, v_k \in A \) such that
\[
1 = \sum_k u_k e v_k.
\]
Then define \( F \)-linear maps
\[
A/K(A) \xrightarrow{\tau} B/K(B)
\]
by
\[
\tau(a + K(A)) = \sum_k e v_k a u_k e + K(B) \quad (a \in A)
\]
\[
\sigma(b + K(B)) = b + K(A) \quad (b \in B).
\]
We claim that \( \tau \) and \( \sigma \) are well-defined and mutually inverse. Indeed, for \( x, y \in A \) we have
\[
\sum_k e v_k (xy - yx) u_k e = \sum_k e v_k x u_k e e v_k y u_k e - \sum_l e v_l y u_l e e v_k x u_l e = \sum_{k,l} e v_k x u_l e e v_l y u_k e - \sum_{l,k} e v_l y u_l e e v_k x u_k e = \sum_{k,l} (e v_k x u_l e)(e v_l y u_k e) - (e v_l y u_l e)(e v_k x u_k e) \in K(B).
\]
Hence \( \tau \) is well-defined. It is routine to check the other parts.
These linear isomorphisms induce linear maps $\tau_n$ and $\sigma_n$ commuting the following diagram where $\pi_{n,-}$ denotes the canonical epimorphism.

$$
\begin{array}{c}
A/K(A) \xrightarrow{\tau_n} B/K(B) \\
\downarrow \pi_{n,A} \quad \downarrow \pi_{n,B} \\
A/KR^n(A) \xrightarrow{\tau_n} B/KR^n(B)
\end{array}
$$

These maps are well-defined and diagram chase reveals that these maps are mutually inverse since $\pi_{n,-}$ is an epimorphism. \hfill \square

3. Upper bound

In this section we establish an upper bound for the codimension of $KR^n(A)$, which extends the result of Otokita \cite{otokita} (Theorem 3.4). As a corollary we prove Theorem 1.4.

**Notation 3.1.** For a basic set of primitive idempotents $\{e_i \mid 1 \leq i \leq \ell(A)\}$ of $A$, set

$$
aCyc(A) := \sum_{1 \leq i \leq \ell(A)} e_i Ae_j \quad \text{and} \quad \text{Cyc}^{\geq n}(A) := \sum_{1 \leq i \leq \ell(A)} e_i \text{Rad}^n(A)e_i.
$$

**Lemma 3.2.** If $A$ is basic then

$$
\sum_{1 \leq i \leq \ell(A)} \dim e_i Ae_i/e_i \text{Rad}^n(A)e_i = \text{codim} \left( aCyc(A) + \text{Cyc}^{\geq n}(A) \right).
$$

**Proof.**

\[
\begin{align*}
\sum_i \dim e_i Ae_i/e_i \text{Rad}^n(A)e_i &= \dim \sum_i e_i Ae_i/\sum_i e_i \text{Rad}^n(A)e_i \\
&= \dim \sum_{i,j} e_i Ae_j/\left( \sum_i e_i Ae_j + \sum_i e_i \text{Rad}^n(A)e_i \right) \\
&= \text{codim} \left( aCyc(A) + \text{Cyc}^{\geq n}(A) \right).
\end{align*}
\]

\hfill \square

**Proposition 3.3.** If $A$ is basic then the following statements are equivalent.

(i) $\text{codim} KR^n(A) = \sum_{1 \leq i \leq \ell(A)} \dim e_i Ae_i/e_i \text{Rad}^n(A)e_i.$

(ii) $K(A) \subseteq aCyc(A) + \text{Cyc}^{\geq n}(A).$

**Proof.** First, note that

\begin{equation}
KR^n(A) \supseteq aCyc(A) + \text{Cyc}^{\geq n}(A). \tag{3.1}
\end{equation}

\[\textbf{[1]} \implies \textbf{[2]} \] By (3.1) and Lemma 3.2 we have $K(A) \subseteq KR^n(A) = aCyc(A) + \text{Cyc}^{\geq n}(A).$

\[^1\text{See \cite{ref} p. 305}.\]
(ii) \implies (i): By the hypothesis, we have the following.

\[ KR^n(A) \subseteq a\text{Cyc}(A) + \text{Cyc}^\geq_n(A) + \text{Rad}^n(A) \]

\[ = a\text{Cyc}(A) + \sum_{i \neq j} e_i \text{Rad}^n(A)e_i + \text{Cyc}^\geq_n(A) + \sum_i e_i \text{Rad}^n(A)e_i \]

\[ = a\text{Cyc}(A) + \text{Cyc}^\geq_n(A). \]

Then \( KR^n(A) = a\text{Cyc}(A) + \text{Cyc}^\geq_n(A) \) by \((3.1)\). Thus, by Lemma 3.2, we have

\[ \text{codim } KR^n(A) = \text{codim } (a\text{Cyc}(A) + \text{Cyc}^\geq_n(A)) \]

\[ = \sum_i \dim e_i A e_i / e_i \text{Rad}^n(A)e_i. \] □

**Theorem 3.4** (See Otokita [13] for blocks). Let \( \{ e_i \mid 1 \leq i \leq \ell(A) \} \) be a basic set of primitive idempotents of \( A \). Then the following holds for every \( n \in \mathbb{N} \).

\[ \text{codim } KR^n(A) \leq \sum_{1 \leq i \leq \ell(A)} \dim e_i A e_i / e_i \text{Rad}^n(A)e_i \]

Furthermore, if the equality holds for some \( n \) then so does for every \( 1 \leq m \leq n \); if the equality does not hold for some \( n \) then so does not for every \( m \geq n \).

**Proof.** By Theorem 2.1, we can assume \( A \) is basic. We claim the map

\[ \pi: \bigoplus_{1 \leq i \leq \ell(A)} e_i A e_i / e_i \text{Rad}^n(A)e_i \to A / KR^n(A), \]

\[ \sum_i (a_i + e_i \text{Rad}^n(A)e_i) \mapsto \sum_i a_i + KR^n(A) \] (3.2)

is surjective. Let \( a + KR^n(A) \in A / KR^n(A). \) Then we have the following.

\[ a + KR^n(A) = \sum_i e_i a e_i + KR^n(A) + \sum_{i \neq j} e_i a e_j + KR^n(A) \]

\[ = \sum_i e_i a e_i + KR^n(A) + \sum_{i \neq j} (e_i a e_j - e_j e_i a) + KR^n(A) \]

\[ = \sum_i e_i a e_i + KR^n(A) \]

\[ = \pi \left( \sum_i e_i a e_i + e_j \text{Rad}^n(A)e_i \right). \]

Hence the proof of the inequality completes.

The latter statements follow from Proposition 3.3. □

**Proof of Theorem 1.4.** Since the last statement \((iii)\) is clear from \((i)\) and Theorem 3.4, we prove \((i)\) and \((ii)\) in the following.
We first claim that equality holds in Theorem 3.4 for \( n \leq 2 \). By Theorem 2.1, we may assume that \( A \) is basic. Then we have
\[
K(A) = \sum_{i,j,s,t} [e_i A e_j, e_s A e_t]
\]
\[
\subseteq a\text{Cyc}(A) + \sum_{i,j} [e_i A e_j, e_j A e_i]
\]
\[
= a\text{Cyc}(A) + \sum_{i \neq j} \sum_{i} K(e_i A e_i)
\]
\[
\subseteq a\text{Cyc}(A) + \text{Cyc}^{\geq 2}(A) + \sum_{i} K(e_i A e_i)
\]
\[
\subseteq a\text{Cyc}(A) + \text{Cyc}^{\geq 2}(A) + \sum_{i} \text{Rad}^2(e_i A e_i)
\]
Now the claim follows from Proposition 3.3. Then we have the following.
\[
\text{codim} KR^1(A) = \sum_i \dim e_i A e_i / e_i \text{Rad}(A) e_i
\]
\[
= \ell(A);
\]
\[
\text{codim} KR^2(A) = \sum_i \dim e_i A e_i / e_i \text{Rad}^2(A) e_i
\]
\[
= \sum_i \dim e_i A e_i / e_i \text{Rad}(A) e_i
\]
\[
+ \sum_i \dim e_i \text{Rad}(A) e_i / e_i \text{Rad}^2(A) e_i
\]
\[
= \ell(A) + \sum_i \dim \text{Ext}^1(S_i, S_i).
\]

**Notes 3.5.**

(i) If \( F \) is an algebraically closed field of positive characteristic \( p \), a well-known theorem of Brauer [3, (3A)] states \( \ell(A) = \text{codim} T(A) \) where
\[
T(A) := \{ a \in A \mid a p^n \in K(A) \text{ for some } n \in \mathbb{N} \}.
\]
Since \( T(A) = KR^1(A) \) by Külshammer [8, Lemma B], Theorem 1.4 follows in this case. In fact, the part (i) is certainly not our contribution although it is sometimes stated only for positive characteristic case; It can be found, among others, in [11, Proposition I.13.3(i)].

(ii) Okuyama [12] proves essentially the same statement of Theorem 1.4 for a block of a finite group algebra over an algebraically closed field of positive characteristic. (See also Koshitani [7], which is written in English.)

(iii) The inequality \( \ell(A) + \sum_{1 \leq i \leq \ell(A)} \dim \text{Ext}^1(S_i, S_i) \leq k(A) \) is an extension of Brandt [2] Theorem B. (For a symmetric algebra, it can be proved that the equality holds if and only if its Loewy length is at most two. See also [14, Lemma 2.1].) For a block \( B \) of a finite group algebra, the inequality \( k(B) \leq \text{tr} C_B \) is a direct consequence of \( C_B = D_B \cdot D_B \) where \( D_B \) is the decomposition matrix of \( B \). See also Külshammer-Wada [10] for refinements in this case.
Corollary 3.6. For a radical square zero algebra $A$ over a splitting field, we have $k(A) = \text{tr } C_A$.

Proof. It suffices to prove $k(A) \geq \text{tr } C_A$ by Theorem 1.4(iii)

$$k(A) \geq \ell(A) + \sum_i \dim \text{Ext}^1(S_i, S_i)$$ (By Theorem 1.4(iii))

$$= \ell(A) + \sum_i \dim e_i \text{Rad}(A)e_i$$ (By $\text{Rad}^2(A) = 0$)

$$= \sum_i (1 + \dim e_i \text{Rad}(A)e_i)$$

$$= \sum_i \dim e_i Ae_i = \text{tr } C_A. \square$$

4. SMALL ALGEBRAS

This section provides a characterization of truncated polynomial algebras $F[X]/(X^n)$ using Theorems 1.4 and 2.1. Then characterizations of small algebras (Theorems 1.2 and 1.3) follow immediately.

Lemma 4.1. Let $n \in \mathbb{N}$. Then the following statements are equivalent.

(i) $A$ is isomorphic to $F[X]/(X^n)$.
(ii) $A$ is $n$-dimensional basic local Nakayama algebra.

Theorem 4.2. Let $n \in \mathbb{N}$ and suppose that $F$ is a splitting field for $A$. Then the following statements are equivalent.

(i) $A$ is Morita equivalent to $F[X]/(X^n)$.
(ii) $k(A) = n$, codim $KR^2(A) \leq 2$, and $\ell(A) = 1$.

Proof. (i) $\implies$ (ii) Clear by Theorem 2.1. 

(ii) $\implies$ (i) We may assume $A$ is basic. Since $\ell(A) = 1$, we have the unique simple right $A$-module $S := A/\text{Rad}(A)$. By Theorem 1.4,

$$\dim \text{Rad}(A)/\text{Rad}^2(A) = \dim \text{Ext}^1(S, S)$$

$$= \ell(A) + \dim \text{Ext}^1(S, S) - \ell(A)$$

$$= \text{codim } KR^2(A) - \text{codim } KR^1(A)$$

$$\leq 2 - 1 = 1.$$

Thus $A$ is a basic local Nakayama algebra. By Lemma 4.1 we have $A \cong F[X]/(X^m)$ for some $m \in \mathbb{N}$. Since $A$ is commutative,

$$n = \text{codim } K(A) = \dim A = \dim F[X]/(X^m) = m.$$ 

Therefore $A \cong F[X]/(X^n)$. \square

Proof of Theorem 1.2. (i) $\implies$ (ii) Clear from Theorem 2.1. 

(ii) $\implies$ (i) Clear from Theorem 1.2. \square

Proof of Theorem 1.3. (i) $\implies$ (ii) Clear from Theorem 2.1. 

(ii) $\implies$ (i) Clear from Theorem 1.2. \square
Notes 4.3. Consider the four-dimensional basic local Frobenius algebra defined by

\[ A_q := F \langle X, Y \rangle / (X^2, Y^2, XY - qYX) \quad (q \in F \setminus \{0, 1\}). \]

Since \( A_q \) is non-commutative, we have \( k(A_q) = 3 \) by Theorems 1.2 and 1.3. Hence, we may have infinitely many non-equivalent finite-dimensional algebras \( A \) with \( k(A) = 3 \) and \( \ell(A) = 1 \) unlike the case \( k(A) = 1, 2 \). (This is because \( A_q \cong A_r \) if and only if \( \{q^\pm1\} = \{r^\pm1\} \) for \( q, r \in F \setminus \{0\} \). See [15, p. 865].)

Notes 4.4. Consider the \( n \)-Kronecker algebra — the path algebra \( FQ_n \) of the \( n \)-Kronecker quiver \( Q_n \) defined by

\[ \circ - \circ - \circ - \circ \cdots \circ \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n. \]

Then \( k(FQ_n) = \ell(FQ_n) = 2 \). Hence, we have infinitely many non-equivalent finite-dimensional algebras \( A \) with \( k(A) = \ell(A) = 2 \).

We have determined small algebras in Theorems 1.2 and 1.3. The other extreme case is that the algebra \( A \) with \( k(A) = \ell(A) \). These algebras are close to be semisimple to some extent.

Proposition 4.5. Suppose that \( F \) is a splitting field for \( A \). Then the following statements are equivalent.

(i) \( k(A) = \ell(A) \)

(ii) \( \text{Rad}(A) \subseteq K(A) \)

In particular, \( k(A) = \ell(A) \) holds if \( A \) is semisimple. The converse also holds if \( A \) is symmetric, commutative, or local.

Proof. \([\text{(i) } \iff \text{(ii)}]\) Evidently \( k(A) = \ell(A) \) if and only if \( K(A) = KR^1(A) \), i.e., \( \text{Rad}(A) \subseteq K(A) \).

Symmetric case: Assume \( A \) is symmetric with a symmetrizing form \( \tau : A \to F \) and \( k(A) = \ell(A) \). Then \( \text{Rad}(A) \subseteq K(A) \subseteq \ker \tau \). Hence we have \( \text{Rad}(A) = 0 \).

Commutative case: Assume \( A \) is commutative and \( k(A) = \ell(A) \). Then \( \text{Rad}(A) \subseteq K(A) = 0 \).

Local case: Assume \( A \) is local and \( k(A) = \ell(A) \). Then \( k(A) = \ell(A) = 1 \) and \( A \) is Morita equivalent to \( F \) by Theorem 1.2. Hence we have \( \text{Rad}(A) = 0 \). \( \square \)

Appendix A. The Chlebowitz theorem

Chlebowitz already studied similar problems and she proved the following stronger facts, which are essentially extensions of the previous works [9, 6] to non-symmetric cases.

Theorem A.1 (Chlebowitz [9]). Let \( A \) be a finite-dimensional local algebra over an algebraically closed field. Then the following holds.

(i) If \( k(A) \leq 3 \) then \( \dim A \leq 4 \).

(ii) If \( k(A) = 4 \) then \( \dim A \leq 10 \).

(iii) If \( k(A) = 5 \) and \( \dim \text{Rad}(A)/\text{Rad}^2(A) \leq 2 \) then \( \dim A \leq 12 \).
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References

[1] F. W. Anderson and K. W. Fuller, *Rings and Categories of Modules, 2nd edn* (Springer-Verlag, New York, 1992). doi:10.1007/978-1-4612-4418-5 MR 1245487 Zbl 0765.16001

[2] J. Brandt, A lower bound for the number of irreducible characters in a block, *J. Algebra* 74 (1982) 509–515. doi:10.1016/0021-8693(82)90036-9 MR 647221 Zbl 0478.20009

[3] R. Brauer, Zur Darstellungstheorie der Gruppen endlicher Ordnung (German), *Math Z.* 63 (1956) 406–444. doi:10.1007/BF01187950 MR 75953 Zbl 0073.03101

[4] R. Brauer and C. Nesbitt, On the modular characters of groups, *Ann. of Math.* 42 (1941) 556–590. doi:10.2307/1968918 MR 4042 Zbl 0027.15202

[5] M. Chlebowitz, Über Abschätzungen von Algebreninvarianten (German), PhD Thesis, Universität Augsburg, 1991.

[6] M. Chlebowitz and B. Külshammer, Symmetric local algebras with 5-dimensional center, *Trans. Amer. Math. Soc.* 329 (1992) 715–731. doi:10.2307/2153960 MR 1025752 Zbl 0758.16004

[7] S. Koshitani, Endo-trivial modules for finite groups with dihedral Sylow 2-groups, *RIMS Kôkyûroku* 2003 (2016) 128–132.

[8] B. Külshammer, Bemerkungen über die Gruppenalgebra als symmetrische Algebra (German), *J. Algebra* 72 (1981) 1–7. doi:10.1016/0021-8693(81)90308-2 MR 634613 Zbl 0472.16007

[9] B. Külshammer, Symmetric local algebras and small blocks of finite groups, *J. Algebra* 88 (1984) 190–195. doi:10.1016/0021-8693(84)90097-8 MR 741039 Zbl 0567.20007

[10] B. Külshammer and T. Wada, Some inequalities between invariants of blocks, *Arch. Math.* 79 (2002) 81–86. doi:10.1007/s000130200308-4 MR 1925739 Zbl 1011.20009

[11] P. Landrock, *Finite Group Algebras and Their Modules,* (Cambridge University Press, Cambridge, 1983). doi:10.1017/CBO9781107325524 MR 737910 Zbl 0523.20001

[12] T. Okuyama, Ext1(S,S) for a simple kG-module S (Japanese), in *Proceedings of the Symposium “Representations of Groups and Rings and Its Applications,”* ed. S. Endo (1981), pp. 238–249.

[13] Y. Otokita, On diagonal entries of Cartan matrices of p-blocks, preprint (2016) 4pp. arXiv 1605.07937v2

[14] T. Sakurai, Central elements of the Jennings basis and certain Morita invariants, preprint (2017) 11pp. arXiv 1701.03799v3

[15] K. Yamagata, Frobenius algebras, in *Handbook of Algebra, vol. 1,* ed. M. Hazewinkel (Elsevier, Amsterdam, 1996), pp. 841–887, doi:10.1016/S1570-7954(96)80082-3 MR 1421820 Zbl 0879.16008