Fermion masses and mixing in models with $SO(10) \times A_4$ symmetry

Federica Bazzocchi$^1$, Michele Frigerio$^2$ and Stefano Morisi$^1$

$^1$ AHEP Group, Instituto de Física Corpuscular – C.S.I.C./Universitat de València
Edificio Institutos de Paterna, Apt 22085, E-46071 Valencia, Spain

$^2$ Institut de Physique Théorique, CEA-Saclay, 91191 Gif-sur-Yvette Cedex, France

We study the flavour sector in models where the three families of matter are unified in a $(16, 3)$ representation of the $SO(10) \times A_4$ group. The necessary ingredients to realize tri-bi-maximal mixing in the lepton sector are identified systematically. The non-renormalizable operators contributing to the fermion mass matrices play an important role. We also present a mechanism to explain the inter-family mass hierarchy of quarks and charged leptons, which relies on a ‘universal seesaw’ mechanism and is compatible with tri-bi-maximal mixing.

I. INTRODUCTION

Before the experimental determination of neutrino oscillation parameters, a common theoretical prejudice was the expectation of small lepton mixing angles, by analogy with the CKM angles in the quark sector. This was thought to be a generic prediction of Grand Unification Theories, as they incorporate some form of quark-lepton symmetry. On the contrary, experiments have shown that the $2 \rightarrow 3$ lepton mixing is close to maximal, the $1 \rightarrow 2$ mixing is large, while the $1 \rightarrow 3$ angle can be as large as the Cabibbo angle [1, 2]. With the benefit of hindsight, this disparity between quarks and leptons was interpreted as a manifestation of the Majorana nature of neutrinos. From then on, many grand unified models have been built, in which large lepton mixing angles appear naturally (see e.g. [3, 4, 5, 6, 7]).

Several conjectures on special ‘symmetric’ values of the lepton mixing angles have been also proposed. For example, the bi-maximal mixing pattern ($\theta_{23} = \theta_{12} = \pi/4$ and $\theta_{13} = 0$) was extensively studied but it is by now excluded, since solar neutrino experiments require $\theta_{12} = (35 \pm 4)^\circ$ at 3-$\sigma$. Current data are in very good agreement, instead, with the so-called tri-bi-maximal (TBM) mixing scheme [8], where the second and third neutrino mass eigenstates have the following flavour content: $\nu_2 = (\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$ and $\nu_3 = (\nu_\mu - \nu_\tau)/\sqrt{2}$. Such special values of the angles may point to a non-abelian family symmetry. In particular, the TBM scheme seems to require that the three families of lepton doublets belong to the same dimension three representation, the simplest suitable symmetry being the discrete group $A_4$ [9, 10] (early applications of $A_4$ as a family symmetry are discussed in [11, 12]).

While generically large lepton mixing angles found their place in the framework of Grand Unification, it is a much tougher task to justify special values of such angles. The first few attempts to build unified models with TBM mixing have encountered several difficulties and concrete realizations so far require many degrees of complication. In most $A_4$ models for TBM mixing, the lepton doublets and singlets transform differently under the family symmetry, so that the allowed embedding is $SU(5)$ unification [13, 14, 15], or the Pati-Salam group [16]. The alternative option, to take all leptons transforming as triplets under $A_4$, was later considered [17] and it was shown [18] to be suitable to achieve TBM mixing. This assignment is compatible with $SO(10)$ unification of the gauge interactions. Few models based on the $SO(10) \times A_4$ group
have been studied \cite{19,20}. A different scheme to achieve TBM mixing in \( SO(10) \) unification makes use of an \( SU(3) \) family symmetry \cite{21} or its discrete subgroup \( \Delta(27) \) \cite{22}.

In this paper, we investigate systematically the structure of fermion masses and mixing in models with \( SO(10) \) unification and \( A_4 \) family symmetry. This is the minimal framework for a complete unification of the three families of matter, since the three \( SO(10) \) spinors of dimension 16 can transform as a triplet under \( A_4 \). The scheme for realizing TBM lepton mixing consists of breaking \( A_4 \) into its \( Z_3 \) and \( Z_2 \) subgroups in the charged lepton and neutrino sectors, respectively. This misalignment results in TBM values for the lepton mixing angles. On the other hand, the flavour alignment of the up and down quark sectors should be approximately maintained, in order to explain the smallness of the CKM angles. Since in our framework all matter fields belong to the same representation of \( SO(10) \times A_4 \), the Yukawa couplings of the different sectors are strictly related and it is particularly challenging to satisfy all constraints.

More specifically, one faces the need to generate two independent structures for the up-quark mass matrix \( M_u \) and the neutrino Dirac mass matrix \( M_\nu \), which in \( SO(10) \) are strictly related. It has been shown \cite{19} that a sufficiently complicated arrangement of higher dimensional operators can disentangle the two structures. Another way out \cite{20} is to assume that the light neutrino mass matrix, \( m_\nu \), is independent from the form of \( M_\nu \) and it is instead generated by the coupling of two lepton doublets to a Higgs triplet. However, in the context of \( SO(10) \) theories this assumption cannot be realized exactly, as discussed later. Another difficulty of \( A_4 \) models, which becomes even more severe in \( SO(10) \), is the need to introduce non-vanishing CKM parameters without generating too large deviations from the TBM lepton mixing. We will reconsider these issues, generalizing previous results and identifying new solutions.

Perhaps the most serious shortcoming of \( A_4 \) flavour models is that the hierarchy between the masses of the three families can be accommodated, but is not explained. This problem can be addressed \cite{23} in \( A_4 \) models where lepton doublets transform as \( L_i \sim 3 \) while charged lepton singlets transform as \( e^c_i \sim 1, 1', 1'' \). In this case one can introduce an extra family symmetry that distinguishes the three families. For example, a \( U(1) \) family symmetry of the Froggatt-Nielsen type \cite{24}, with different charges for the three \( e^c_i \), can explain the hierarchy of the charged lepton masses. This mechanism can be promptly extended in an \( SU(5) \)-invariant fashion to the down and up quarks. The origin of the inter-family hierarchy is more problematic in \( SO(10) \), since both chiralities of matter transform as \( A_4 \) triplets. As a consequence, a universal mass term of the type \( m\psi_i \psi^c_i \) is allowed \cite{25} and no extra symmetry can distinguish the families. We will argue that a natural solution for this problem emerges from the mixing with heavy vector-like families. The hierarchy arises in the quark and charged lepton sectors precisely because there \( A_4 \) is broken to \( Z_3 \) and each family transforms differently under this residual symmetry.

The paper is organized as follows. In section II (and in appendix A) we recall the structure of the fermion mass matrices which lead to TBM mixing in \( A_4 \) models. In section III (and in appendix B1) we study \( SO(10) \times A_4 \) models with renormalizable Yukawa couplings and investigate the possibility to realize the TBM mixing scheme. In section IV (and in appendix B2) we extend our analysis to dimension five operators, which overcomes several difficulties of the renormalizable models. In section V we present the mechanism to explain the mass hierarchy between the three families, in a way that is compatible with TBM mixing. In section VI we summarize our main results.

II. THE ROUTE TO TRI-BI-MAXIMAL MIXING

In this section we review a scheme to achieve TBM lepton mixing, that was employed in the construction of several \( A_4 \) models \cite{23,26,27,28,29,30,31,32,33,34,35,36,37,38,39} and turns out to be the one
suitable in the context of $SO(10)$ unification \[18, 19, 20, 40\].

Let us introduce a set of simple mass matrix structures which can accommodate all fermion masses, leading at the same time to TBM lepton mixing and vanishing quark mixing. We adopt a supersymmetric notation, with $-\mathcal{L}_m = u M_u u^c + d M_d d^c + e M_e e^c + \frac{1}{2} \nu m_{\nu} \nu$, where

$$M_f = \begin{pmatrix} A_f & B_f & C_f \\ C_f & A_f & B_f \\ B_f & C_f & A_f \end{pmatrix}, \quad f = u, d, e,$$

$$m_\nu = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}. \quad (1)$$

These charged fermion and neutrino Majorana mass matrices can be diagonalized as follows:

$$D_f = U_f^T M_f U_f = \text{diag}(A_f + B_f + C_f, A_f + \omega^2 B_f + \omega C_f, A_f + \omega B_f + \omega^2 C_f) \equiv \text{diag}(m_{f1}, m_{f2}, m_{f3}),$$

$$d_\nu = U_\nu^T m_\nu U_\nu = \text{diag}(a + b, c, a - b) \equiv \text{diag}(m_1, m_2, m_3), \quad (2)$$

where $\omega \equiv e^{2\pi i/3} = (-1 + \sqrt{3}i)/2$ and the unitary mixing matrices are given by

$$U_f = U_\omega \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}. \quad (3)$$

The matrix $U_\omega$ is known as ‘magic’ mixing matrix. We will refer to the matrix structures in Eq. (1) as ‘magic’ and ‘cross’ mass matrix, respectively.

There is no observable quark mixing, since $U_{CKM} \equiv U_u^T U_d = U_\nu^T U_\omega = I_3$. On the contrary, the leptons mix tri-bi-maximally:

$$U_{\text{lepton}} \equiv U_e^T U_\nu = U_\nu^T U_\omega = \text{diag}(1, \omega^2, \omega) \ U_{TBM} \ \text{diag}(1, 1, -i), \quad U_{TBM} \equiv \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix}. \quad (4)$$

The diagonal phase matrices can be absorbed by rephasing the charged lepton fields and $m_3$. Notice that several different choices of the pair of unitary matrices $U_e$ and $U_\nu$ may reproduce TBM mixing. However, as shown in Appendix A, the choice made in Eq. (3) is the only one justified by the $A_4$ family symmetry.

In ordinary $SO(10)$ models, all light matter fields reside in three dim-16 representations, which also contain right-handed neutrinos. Then, the light Majorana neutrino mass matrix is given in general by

$$m_\nu = M_L - M_\nu M_R^{-1} M_\nu^T, \quad (5)$$

where $M_L$ is a direct Majorana mass term for the left-handed neutrinos, $M_\nu$ is the Dirac-type neutrino mass matrix and $M_R$ is the Majorana mass matrix of right-handed neutrinos. The second term in Eq. (5) is the outcome of the type I seesaw mechanism \[41, 42, 43, 44, 45\], while the first term may arise from a type II seesaw \[46, 47, 48, 49\]. The latter can be sizable in models with at least one $126$ Higgs multiplet and a scalar potential such that the $SU(2)_L$ triplet component of $126$ develops a vacuum expectation value (VEV). In all other cases the type I seesaw dominates and then the cross structure of $m_\nu$ in Eq. (5) should emerge from the interplay of the $M_L$ and $M_R$ structures. The simplest possibility is, of course,

$$M_\nu = \begin{pmatrix} A_\nu & 0 & B_\nu \\ 0 & C_\nu & 0 \\ B_\nu & 0 & A_\nu \end{pmatrix}, \quad M_R = \begin{pmatrix} A_R & 0 & B_R \\ 0 & C_R & 0 \\ B_R & 0 & A_R \end{pmatrix}, \quad (6)$$
with at least one among $B_{\nu}$ and $B_{R}$ being non-zero. Another possibility which may be justified by the $A_{4}$ family symmetry is given by

$$M_{\nu} = \begin{pmatrix} 0 & A_{\nu} & 0 \\ -A_{\nu} & 0 & A_{\nu} \\ 0 & -A_{\nu} & 0 \end{pmatrix},$$

with $M_{R}$ as in Eq. (4). In this case one recovers the form of $m_{\nu}$ in Eq. (1) with $b = -a$, which implies a normal neutrino mass hierarchy.

### III. MODELS WITH ONLY RENORMALIZABLE YUKAWA COUPLINGS

In this section we consider the Yukawa couplings $Y\Psi\Psi\Phi$ invariant under $SO(10) \times A_{4}$ and investigate whether they can generate the mass matrix structures in Eq. (1) and thus realize TBM lepton mixing.

We will indicate with $(R, \tilde{R})$ a multiplet transforming in the representation $R$ of $SO(10)$ and $\tilde{R}$ of $A_{4}$. Matter fields are unified into a single multiplet $\Psi$ under $SU(5) \times \text{SO}(10)$. The structure of the mass matrices depends on the assignment of the Higgs multiplets $\Phi$ under $SO(10) \times A_{4}$ and on the components of $\Phi$ which acquire a non-zero VEV. We will indicate with $\langle r^\Phi \rangle \equiv v_{r}^\Phi$ the VEV of the $\Phi$ component in the representation $r$ of $SU(5)$. All the possible matrix structures are listed in Appendix B.

Let us discuss first the mass matrices of charged fermions. As follows from Eq. (2), the masses of the three families can be accommodated only if the matrix entries $A_{f}$, $B_{f}$ and $C_{f}$ are all non-zero, of the same order of magnitude, and different from each other. One can generate the up-quark mass matrix $M_{u}$ introducing the Higgs multiplets $\phi \sim (10, 1)$, $\eta \sim (10, 3)$ and $\rho \sim (120, 3)$. The VEVs

$$\langle 5^\phi \rangle = v_{5}^\phi, \quad \langle 5^\eta \rangle = (v_{5}^\eta, v_{\xi 5}^\eta, v_{\zeta 5}^\eta), \quad \langle 45^\rho \rangle = (v_{45}^\rho, v_{45}^\rho, v_{45}^\rho)$$

(8)

generate $A_{u}$, $B_{u} + C_{u}$ and $B_{u} - C_{u}$, respectively. Alternatively, $\phi$ and/or $\eta$ can be replaced by $\Delta \sim (126, 1)$ and/or $\Omega \sim (126, 3)$ with the same VEV alignment.

In the down quark and charged lepton sector, there are two classes of contributions to the mass matrices: those of the type $\delta M_{u}^T = \delta M_{d}$ can be generated by the VEVs

$$\langle 15^\Delta \rangle = v_{15}^\Delta, \quad \langle 15^{\Omega}_{1} \rangle = (v_{15}^{\Omega}_{1}, v_{15}^{\Omega}_{2}, v_{15}^{\Omega}_{3}), \quad \langle 15^{\Omega}_{2} \rangle = (v_{45}^{\Omega}_{1}, v_{45}^{\Omega}_{2}, v_{45}^{\Omega}_{3})$$

(9)

and $\delta M_{u}^T = -3 \delta M_{d}$ can be generated by the VEVs

$$\langle 15^{\Delta} \rangle = v_{15}^{\Delta}, \quad \langle 15^{\Omega}_{1} \rangle = (v_{45}^{\Omega}_{1}, v_{45}^{\Omega}_{2}, v_{45}^{\Omega}_{3}), \quad \langle 15^{\Omega}_{2} \rangle = (v_{45}^{\Omega}_{1}, v_{45}^{\Omega}_{2}, v_{45}^{\Omega}_{3})$$

(10)

The first, second and third term in Eqs. (9) and (10) contribute to $A_{d,e}$, $B_{d,e} + C_{d,e}$ and $B_{d,e} - C_{d,e}$, respectively. When all six contributions are present, the three masses of down quarks and charged leptons can be fitted. If some of the VEVs are zero, non-trivial relations between the masses are predicted. For example, an economical scenario with no VEVs in the Higgs doublets of the $126$ multiplets would imply $A_{e} = A_{d}$ and $B_{e} + C_{e} = B_{d} + C_{d}$. In this limit one predicts (i) $|e^{i \varphi_{\mu}} m_{\mu} + m_{\gamma}| = |e^{i \varphi_{s}} m_{s} + m_{b}|$ with $\varphi_{\mu,s}$ arbitrary, which may be compatible with the GUT scale values (see e.g. [50]), and (ii) $m_{e} = m_{d}$, which requires corrections of the order of one MeV.

Let us now move to the neutrino sector. Consider first the term $M_{L}$ in Eq. (5). It is generated when the $SU(2)_{L}$ Higgs triplets in $126$ multiplets receive a tiny VEV through a type II seesaw mechanism [46, 47, 48, 49]. Taking

$$\langle 15^{\Delta} \rangle = v_{15}^{\Delta}, \quad \langle 15^{\Omega}_{1} \rangle = (0, v_{15}^{\Omega}_{1}, 0),$$

(11)
$M_L$ acquires the structure of $m_\nu$ in Eq. (1) with $a = c$. Therefore, if one assumes $m_\nu = M_L$, the TBM lepton mixing is realized with one constraint on the light neutrino mass spectrum [23].

One should stress, however, that the type I seesaw contribution (second term in Eq. (5)) is necessarily present in $SO(10)$ models and it turns out to be incompatible with the TBM structure. In fact, the VEVs $v_5^\nu$ and $v_3^\nu$, which are necessary to generate $M_u$, also contribute to the neutrino Dirac mass matrix $M_\nu$, which takes the form

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu & B_\nu \\ B_\nu & A_\nu & B_\nu \\ B_\nu & B_\nu & A_\nu \end{pmatrix}, \tag{12}$$

with $A_\nu = A_u$ and $B_\nu = (B_u + C_u)/2$. The right-handed neutrino mass matrix $M_R$ is generated by the VEVs of the $SU(5)$ singlet components of the $126$ Higgs multiplets. By constructing the combination $M_\nu M_R M_\nu^T$ for all the possible $A_4$ structures of $M_R$ (see Table VI) and barring fine-tuning of independent couplings, we find that the term $B_\nu \propto v_3^\nu$ is never compatible with the structure of $m_\nu$ given in Eq. (14). This shows that the exact realization of the TBM mixing is not possible.

In order to estimate the deviation from TBM mixing, a reasonable hypothesis is that $M_R$ acquires a structure analog to $M_L$ from the VEVs

$$\langle 1^\Delta \rangle = v_1^\Delta, \quad \langle 1^\Omega \rangle = (0, v_1^\Omega, 0). \tag{13}$$

In this case $M_R$ has the form in Eq. (6) with $A_R = C_R$ and one finds

$$m_\nu = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} + \begin{pmatrix} 0 & \epsilon & \epsilon \\ \epsilon & 0 & \epsilon \\ \epsilon & \epsilon & 0 \end{pmatrix}. \tag{14}$$

The effect of type I seesaw is to make $a \neq c$ and to add the $\epsilon$ term. It is easy to check that, in the basis in which the charged leptons are diagonal, $m_\nu$ is still $\mu - \tau$ symmetric, therefore the values of the mixing angles $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ are preserved. On the contrary, the TBM value of $\theta_{12}$ and the light neutrino mass spectrum are corrected in a correlated way. Taking for simplicity all parameters real, one finds

$$\tan 2\theta_{12} = \frac{2\sqrt{2}}{1 + \frac{g_\epsilon}{a + b - c}}. \tag{15}$$

The TBM value of $\theta_{12}$ is recovered for $\epsilon = 0$. In order to accommodate the experimental value of $\theta_{12}$ at the 2-$\sigma$ level [1], one needs $-0.03 < \epsilon/(a + b - c) < 0.04$, so that a few percent type I seesaw contribution can be tolerated. In the case $v_1^\Omega = 0$, the same discussion holds with $a = c$.

We have shown that, if $M_u$ is given by Eq. (1), then the structure of $M_\nu$ is not compatible with exact TBM mixing. Vice versa, it is instructive to consider the structures of $M_\nu$ leading to TBM mixing and see what are the implications for the up quark sector. A first possibility is to take the $M_R$ structure of Eq. (6), generated by the VEVs in Eq. (13), and the $M_\nu$ structure of Eq. (14) by the VEV

$$\langle 5^\nu \rangle = (v_5^\nu, 0, v_5^\nu). \tag{16}$$

Even though this VEV alignment does not preserve any subgroup of $A_4$, we expect that it may be justified dynamically; the analog alignment has been obtained e.g. in a model with $SU(3)$ family symmetry [51]. This would be sufficient to achieve TBM mixing in the lepton sector. In the context of $SO(10)$, however, the only VEV that may contribute to $M_u$ without modifying the structure of $M_\nu$ in Eq. (7) is $v_{45}^\nu$ (see table VI).
Therefore $M_u$ would be purely antisymmetric ($A_u = 0$ and $B_u = -C_u$) and thus would have one zero and two equal eigenvalues, which clearly is not acceptable. A second possibility to realize exact TBM mixing is to generate the structure of $M_\nu$ in Eq. (6) by modifying the VEV alignment of $5^\eta_i$ in Eq. (8) as follows:

$$\langle 5^\eta_i \rangle = (0, v_\eta^5, 0).$$

(17)

In this case the up quark mass matrix takes the form

$$M_u = \begin{pmatrix} A_u & (B_u - C_u)/2 & C_u \\ (C_u - B_u)/2 & A_u & (B_u - C_u)/2 \\ B_u & (C_u - B_u)/2 & A_u \end{pmatrix}. \quad (18)$$

Since $M_d$ has the structure in Eq. (1), the requirement to have $V_{CKM} \approx \mathbb{I}_3$ would imply $C_u \approx -B_u$ and thus lead to the wrong relation $m_c \approx m_t$ (see Eq. (2)). Vice versa, the requirement to fit the three up quark masses would force the $V_{CKM}$ angles to be large. We conclude that the strong departures from the $M_u$ structure in Eq. (1), required to achieve exact TBM lepton mixing, are not viable phenomenologically.

A comment is in order on the pattern of $SO(10) \times A_4$ spontaneous breaking. The triplet VEVs of the type $(1,1,1)$, introduced in Eqs. (8)-(10), break $A_4$ to a $Z_3$ subgroup at the electroweak scale. The VEV of the type $(0,1,0)$, introduced in Eq. (11), breaks $A_4$ to a $Z_2$ subgroup at the scale of light neutrino masses (notice that the large $A_4$ breaking VEV $v_\Omega^1$ in Eq. (13) is not needed in a minimal scenario). This misalignment between the charged fermion sector and the neutrino sector is a crucial ingredient to explain TBM mixing.

On the $SO(10)$ side, one faces the problem to generate VEVs only in some specific components of the Higgs multiplets, specifically the Higgs doublets in 10 and 120 multiplets and the Higgs triplets and singlets in 126 multiplets. It is a difficult task to arrange for an appropriate scalar potential, also in view of the different energy scales of different sets of VEVs. The analysis of such potential, which is beyond the scope of this paper, may in principle reveal a connection between the VEV alignment dynamics in the $A_4$ and $SO(10)$ sectors.

In the rest of this section, we will show that a relatively economic model can be built by using only $SO(10)$ Higgs multiplets which are singlet under $A_4$, plus a set of gauge singlet flavon fields responsible for the breaking of the family symmetry. In this approach the problem of VEV alignments can be treated separately in the $SO(10)$ and $A_4$ sectors. Another advantage is that the number of $SO(10)$ multiplets can be considerably reduced, which is desirable to maintain the theory perturbative well above the GUT scale.

The field content of the model is given in Table I. In order to couple the appropriate flavon fields only to certain $SO(10)$ operators, we introduced a $Z_N$ symmetry, the minimal choice being $N = 3$ with charges $m = 1$ and $r = 2$. The Yukawa superpotential is then given by

$$W_Y = \frac{y_1 \sigma + y_3\chi}{\Lambda} \bar{\Psi} \Psi \phi + \frac{g_3 \chi}{\Lambda} \bar{\Psi} \Psi \rho + \frac{f_1 \tau + f_3 \varphi}{\Lambda} \bar{\Psi} \Psi \Delta,$$

(19)
where $\Lambda$ is a large energy scale where the $A_4 \times Z_N$ symmetry is realized. The flavon fields are assumed to acquire the VEVs

$$
\langle \sigma \rangle = \sigma, \quad \langle \chi_i \rangle = (\chi, \chi, \chi), \quad \langle \tau \rangle = \tau, \quad \langle \varphi_i \rangle = (0, \varphi, 0),
$$

which break $A_4$ as well as $Z_N$ at some scale $\sim \lambda \Lambda$, with $\lambda < 1$. The problem to achieve dynamically such VEV alignment has been addressed and solved in several papers [23, 25, 27, 30]. Notice that in Eq. (19) we included only the leading order operators, linear in the flavon fields. Higher dimensional operators with $1 + n$ flavons generate corrections to the mass matrices of relative order $\lambda^n$, which may be significant for small $n$ and $\Lambda$ close to one. However, all the operators up to some given $n$ can be forbidden by using a $Z_N$ with a sufficiently large $N$ and choosing carefully the charges $m$ and $r$ in Table I.

We suppose that all the Higgs doublets in $\phi$ and $\rho$ acquire a VEV at the electroweak scale, except the one in $5^\prime$. Also, $1^\Delta$ acquires a VEV at the GUT scale and the Higgs triplet in $15^\Delta$ takes a VEV of the order of the light neutrino mass scale. Then, the mass matrices of charged fermions are explicitly given by

$$
M_u = \frac{1}{\Lambda} \begin{pmatrix}
y_1 \sigma \phi^0 & y_3 \chi \phi^0 + g_3 \chi \nu_{15}^0 & y_3 \chi \phi^0 - g_3 \chi \nu_{15}^0 \\
y_3 \chi \nu_5^0 - g_3 \chi \nu_{15}^0 & y_1 \sigma \phi^0 & y_3 \chi \nu_{15}^0 + g_3 \chi \nu_{15}^0 \\
y_3 \chi \nu_5^0 + g_3 \chi \nu_{15}^0 & y_3 \chi \nu_{15}^0 - g_3 \chi \nu_{15}^0 & y_1 \sigma \phi^0
\end{pmatrix},
$$

$$
M_d = \frac{1}{\Lambda} \begin{pmatrix}
y_1 \sigma \phi^0 & y_3 \chi \nu_5^0 + g_3 \chi (\nu_5^0 + \nu_{15}^0) & y_3 \chi \nu_5^0 - g_3 \chi (\nu_5^0 + \nu_{15}^0) \\
y_3 \chi \nu_5^0 - g_3 \chi (\nu_5^0 + \nu_{15}^0) & y_1 \sigma \phi^0 & y_3 \chi \nu_5^0 + g_3 \chi (\nu_5^0 + \nu_{15}^0) \\
y_3 \chi \nu_5^0 + g_3 \chi (\nu_5^0 + \nu_{15}^0) & y_3 \chi \nu_5^0 - g_3 \chi (\nu_5^0 + \nu_{15}^0) & y_1 \sigma \phi^0
\end{pmatrix},
$$

$$
M_e = \frac{1}{\Lambda} \begin{pmatrix}
y_1 \sigma \phi^0 & y_3 \chi \nu_5^0 - g_3 \chi (\nu_5^0 + 3 \nu_{15}^0) & y_3 \chi \nu_5^0 + g_3 \chi (\nu_5^0 + 3 \nu_{15}^0) \\
y_3 \chi \nu_5^0 - g_3 \chi (\nu_5^0 + 3 \nu_{15}^0) & y_1 \sigma \phi^0 & y_3 \chi \nu_5^0 + g_3 \chi (\nu_5^0 + 3 \nu_{15}^0) \\
y_3 \chi \nu_5^0 + g_3 \chi (\nu_5^0 + 3 \nu_{15}^0) & y_3 \chi \nu_5^0 - g_3 \chi (\nu_5^0 + 3 \nu_{15}^0) & y_1 \sigma \phi^0
\end{pmatrix}. \tag{21}
$$

The matrix $M_u$ accommodates the three up quark masses and, combined with $M_d$, leads to vanishing CKM mixing. The matrices $M_d$ and $M_e$ depend only on four complex parameters, so that non-trivial relations are predicted among their eigenvalues, as discussed after Eq. (19). If one insists to relax this constraint in order to fit independently down quark and charged lepton masses, it suffices to introduce an extra Higgs multiplet $\Delta' \sim (\overline{126}, 1, \alpha^m)$ with a VEV in the $45^\Delta$ component. The neutrino mass matrices are given by

$$
M_\nu = \frac{1}{\Lambda} \begin{pmatrix}
y_1 \sigma & y_3 \chi & y_3 \chi \\
y_3 \chi & y_1 \sigma & y_3 \chi \\
y_3 \chi & y_3 \chi & y_1 \sigma
\end{pmatrix} \nu^\phi_5, \quad M_L = \frac{1}{\Lambda} \begin{pmatrix}
f_1 \tau & 0 & f_3 \varphi \\
0 & f_1 \tau & 0 \\
f_3 \varphi & 0 & f_1 \tau
\end{pmatrix} \nu_1^\Lambda, \quad M_R = \frac{1}{\Lambda} \begin{pmatrix}
f_1 \tau & 0 & 0 \\
f_3 \varphi & 0 & 0 \\
0 & f_1 \tau & 0
\end{pmatrix} \nu_1^\Lambda. \tag{22}
$$

The light neutrino mass matrix has the form in Eq. (13). The departure from the TBM value of the 1-2 lepton mixing angle is controlled by the ratio $(v_5^\phi)^2/(v_1^\Lambda v_1^\Lambda)$, where we assume that the couplings in Eq. (19) and also the flavon VEVs in Eq. (20) are all of the same order. As discussed after Eq. (13), this ratio must be smaller than few percents, which is realized e.g. for $v_5^\phi \approx 100$ GeV, $v_1^\Lambda \approx 10^{16}$ GeV and $v_1^\Lambda \approx 0.1$ eV.

In the above model there is no mixing in the quark sector. Let us briefly review possible mechanisms to generate non-vanishing CKM parameters in $A_4$ models where $V_{CKM} = \mathbb{1}_3$ at leading order. In order to introduce the quark mixing, people considered (i) small explicit $A_4$ breaking Yukawas [52]; (ii) $A_4$ breaking in the soft supersymmetry breaking terms [10]; (iii) one-loop corrections that communicate to the quark sector the effect of the $A_4$ spontaneous breaking in the $(0,1,0)$ direction [31, 40]; (iv) spontaneous $A_4$ breaking by a triplet with VEV $(v, v, v_3)$ with $v \neq v_3$ [33]; (v) extra flavons transforming as $1'$ or $1''$ coupled to the quarks [13, 34]. In most of these cases it is problematic to generate a sufficiently large quark mixing, without introducing too large deviations from TBM lepton mixing [40]. One possible exception is provided by $SU(5)$
GUT models, where the CKM parameters can be introduced as a left-handed rotation in the down quark sector, which corresponds to a right-handed one in the charged lepton sector and therefore does not affect lepton mixing \[15\]. However, in our analysis of $SO(10)$ GUT models we did not find a contribution to $M_d$ and $M_T$ that introduces only a mixing on the left. At the end of section IV we will present a different mechanism to accommodate non-zero CKM parameters, which does not perturb the TBM lepton mixing.

IV. MODELS WITH NON-RENORMALIZABLE OPERATORS

In the previous section we have shown that the TBM lepton mixing cannot be exactly realized by Yukawa couplings symmetric under $SO(10) \times A_4$. In this section we solve this difficulty by considering the effect of non-renormalizable operators that contribute to the fermion mass matrices. In addition, the analysis of such operators will provide a new tool to accommodate the CKM parameters.

Higher dimensional operators are proportional to powers of $M_{GUT}/\Lambda$, where the cutoff $\Lambda$ can be identified with the Planck or the string scale, or with the mass of vector-like matter multiplets. For definiteness, in the following we study this last possibility, by considering the superpotential

$$W = y_A \Phi \Sigma + y_B \Phi \Sigma - M \Phi \Phi,$$

where $\Phi_A$ and $\Phi_B$ are Higgs multiplets whose components may acquire VEVs $a$ and $b$, respectively, $(\Sigma, \Sigma^c)$ is a vector-like pair of matter multiplets and we assume $a, b \ll M$. In this case $\Sigma$ and $\Sigma^c$ can be integrated out and one is left with an effective operator

$$W_{\text{eff}} = \frac{y_A y_B}{M} (\Phi_A \Sigma \Phi_B \Sigma^c),$$

where the subscripts specify how $\Phi$ and $\Phi_{A,B}$ are contracted. The VEVs $a$ and $b$ can be either of the order of $M_{GUT}$, if they participate to the $SO(10)$ symmetry breaking to the SM, or of the order of the electroweak scale, if they break the SM gauge group.

The flavour structure generated by the operator in Eq. (24) is determined by the $A_4$ assignments of $\Phi_A, \Phi_B, \Sigma$ and $\Sigma^c$. All the possibilities are analyzed in Appendix B2 and displayed in Table VII. Beside the structures already possible with renormalizable Yukawa couplings, several new flavour structures can be realized (compare with Table V). This provides new options for model-building with $A_4$ symmetry. Here we will focus on the realization of TBM lepton mixing, but alternative scenarios can be studied on the same footing.

By surveying all the possible operators listed in Table VII one can identify the flavour structures compatible with the TBM mixing scheme. The cross structure of the neutrino Dirac and Majorana mass matrices, shown in Eqs. (11) and (12), can be built with the following components:

- $\text{diag}(1, 1, 1)$ from the operator A;
- $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ from the operator B with VEV alignment $(0, a, 0)$ and no antisymmetric term ($g_A^a = 0$);
- $\text{diag}(0, 1, 0)$ from the operator C (or $C'$) with VEV alignment $(0, a, 0)$ and $(0, b, 0)$;
- $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ from the operator C with VEV alignment $(a, 0, a)$ and $(b, 0, b)$;
• $\text{diag}(1,0,1)$ from the operator $D$ with VEV alignment $(0,a,0)$ and $(0,b,0)$ and either $y^a_{A,B} = 0$ or $y^b_{A,B} = 0$;

\[
\begin{pmatrix}
1 & 0 & \pm 1 \\
0 & 2 & 0 \\
\pm 1 & 0 & 1
\end{pmatrix}
\] from the operator $D$ with VEV alignment $(a,0,a)$ and $(b,0,b)$ and either $y^a_{A,B} = 0$ or $y^b_{A,B} = 0$.

The magic structure of the charged fermion mass matrices, shown in Eq. (11), can be built with the following components:

• $\text{diag}(1,1,1)$ from the operator $A$;

• $y^a_A \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + y^b_A \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ from the operator $B$ with VEV alignment $(a,a,a)$;

• $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ from the operator $C$ with VEV alignment $(a,a,a)$ and $(b,b,b)$;

• $\begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}$ from the operator $C'$ with VEV alignment $(a,a,a)$ and $(b,b,b)$;

• $y^a_B y^b_B \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} + y^a_B y^b_B \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + (y^a_A y^b_B - y^a_B y^a_B) \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ from the operator $D$ with VEV alignment $(a,a,a)$ and $(b,b,b)$.

The contribution of the operator in Eq. (24) to the mass matrices of the different sectors is determined by the $SO(10)$ assignments of $\Phi_A$, $\Phi_B$, $\Sigma$ and $\bar{\Sigma}$. Higher dimensional operators have been often used to build realistic $SO(10)$ models without large representations, which may be disfavoured theoretically. We thus perform a systematic analysis of dim-5 operators involving only $SO(10)$ multiplets of size less than or equal to 120. There are only six such operators, that are analyzed in Appendix B2 and displayed in Table VIII. Several qualitative new relations between the various sectors are possible, with respect to the case of renormalizable Yukawa couplings (compare with Table VI).

• the operator $16_M 16_M 16_H 16_H$ contributes only to the down (up) sector; here the subscripts distinguish matter ($M$) from Higgs ($H$) multiplets;

• the operator $16_M 16_M 16_H 16_H$ can generate Majorana masses for left-handed and right-handed neutrinos, with no need of any $120_H$ multiplet;

• the operator $(16_M 16_H)_{1}(16_M 16_H)_{1}$ contributes to $M_\nu$ without affecting $M_u$;

• the operator $(16_M 120_H)_{16}(16_M 45_H)_{16}$ provides two independent contributions to $M_\nu$ and $M_u$, proportional to $\langle 5^{20}_{H} \rangle$ and $\langle 45^{120}_{H} \rangle$, respectively.

We are now in the position to build a minimal model which realizes TBM lepton mixing using the dim-5 operators. Notice first that, in the absence of $120_H$ multiplets, the type I seesaw is the dominant source of
light neutrino masses. In fact, the type II contribution to \( m_\nu \) generated by the operator \( 16_M 16_M 16_H \) is of the order \( \langle 5_H \rangle^2 / M \lesssim (100 \text{ GeV})^2 / M_{\text{GUT}} \sim 10^{-3} \text{ eV} \), that is negligible.

In section [III] we showed that, in the case of type I seesaw, one cannot reproduce TBM lepton mixing by using renormalizable Yukawa couplings. The technical reason is the absence of an operator that provides an off-diagonal symmetric term, needed for the magic structure of \( M_\nu \), without modifying the cross structure of \( M_\nu \). This difficulty can be overcome by using dim-5 operators. The mechanism to generate the required contribution to \( M_\nu \) is most easily described in \( SU(5) \) language: the VEV of the up-type doublet in a 45 Higgs multiplet couples (antisymmetrically) to two 10 matter multiplets generating \( M_\nu \), while it does not contribute to \( M_\nu \).

In order to make this contribution to \( M_\nu \) not antisymmetric, one needs the insertion of the VEV of a 24 Higgs multiplet, which couples to 10 and \( \overline{10} \) matter multiplets. Since the SM singlet in 24 is in the hypercharge direction, the Clebsch-Gordan coefficients for the \( Q \) and \( u^c \) components are different (by a relative factor \(-4\)), thus making \( M_\nu \) not antisymmetric. This mechanism is embedded in \( SO(10) \) by the operator \( (16_M 120_H)_16 (16_M 45_H)_{\overline{16}} \), with \( 45_H^{120} \) and \( 24_{120}^{45} \) acquiring a VEV. By inspecting Table [VIII] this is in fact the only possible contribution to \( M_\nu \) that does not affect \( M_\nu \), at least with multiplets of dim \( \leq 120 \). Notice that such contribution to \( M_\nu \) is symmetric when the VEV of 45\(_H\) is in the \( B - L \) direction (to see this, replace Eq. (22) in the row VIII of Table VIII).

For completeness, let us mention another \( SU(5) \) mechanism that allows to generate only \( M_\nu \) and not \( M_\nu \): one may employ the VEV of a 75 Higgs multiplet, which couples to 10 and \( \overline{10} \) (but not to 5 and \( \overline{5} \)) matter multiplets. The minimal \( SO(10) \) embedding is provided by the operator \( (16_M 10_H)_16 (16_M 210_H)_{\overline{16}} \), with \( 5_H^{10} \) and \( 75_{210}^{10} \) acquiring a VEV. However, the 75 component that acquires the VEV couples with opposite Clebsch-Gordan coefficients to the \( Q \) and \( u^c \) components. As a consequence the contribution to \( M_\nu \) of this operator is antisymmetric. A symmetric contribution requires an extra antisymmetric coupling, which can be introduced by replacing \( 10_H \) with \( 120_H \).

Our strategy to build an explicit model is to start from the renormalizable model discussed at the end of section [III] which is defined by the superpotential in Eq. (19), and replace some of the couplings with the appropriate higher dimensional operators. The field content of the model is given in Table [III] where we introduced a \( Z_N \) symmetry. The Yukawa superpotential invariant under \( SO(10) \times A_4 \times Z_N \) is given by

\[
W_Y = \frac{y_1 \sigma}{\Lambda} \Psi \phi + \frac{g_3 \chi}{\Lambda} \Psi \rho + \frac{h_3 \chi}{\Lambda} \Sigma_2 A - M_2 \Sigma_2 \Sigma_2 + h \Sigma_2 \Psi \rho + \frac{f_1 \tau + f_3 \varphi}{\Lambda} \Psi_1 \xi - M_1 \Sigma_1 \Sigma_1 + f \Sigma_1 \Psi_1 \xi .
\] (25)

In order to forbid all other couplings, up to terms quadratic in the flavons, the \( Z_N \) charges in Table [III] must be carefully chosen. One viable choice is given by \( N = 8 \), with \( m = 1, r = 2 \) and \( n = 5 \). By taking \( N \) large enough, it is possible to forbid unwanted couplings up to higher order in the flavons. Integrating out the heavy messenger fields \( \Sigma_i, \Sigma_i \), for \( i = 1, 2 \), the effective superpotential takes the form

\[
W_Y^{\text{eff}} = \frac{y_1 \sigma}{\Lambda} \Psi \phi + \frac{g_3 \chi}{\Lambda} \Psi \rho + \frac{h_3 \chi}{\Lambda} \frac{(\Psi \rho)_{\Sigma_2} (\Psi A)_{\Sigma_2}}{M_2} + \frac{(f_1 \tau + f_3 \varphi)f (\Psi \xi)_{\Sigma_1} (\Psi \xi)_{\Sigma_1}}{M_1}.
\] (26)

For brevity, in Eqs. (25) and (26) we indicated with \( h_3 \) and \( f_3 \) two independent couplings \( h_{6a} \) and \( h_{6a} \), which correspond to the two possible contractions of \( A_4 \) indexes in the product \( 3 \times 3 \times 3 \) (see case B in Table [VIII]). The first operator in \( W_Y^{\text{eff}} \) contributes to the Dirac mass matrices of the four sectors, the second and the third only to the charged fermion mass matrices (we assume \( \langle 5^0 \rangle = \langle 5^r \rangle = 0 \)) and the fourth only to the Dirac and Majorana neutrino mass matrices. The flavons take the same VEVs as in Eq. (20).

The charged fermion mass matrices acquire the magic structure

\[
M_f = A_f \mathbb{1}_3 + \frac{B_f + C_f}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \frac{B_f - C_f}{2} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},
\] (27)
where the coefficients have the following expression:

\[
A_u = \frac{y_1 \sigma}{\Lambda} v_5^\phi, \\
B_u + C_u = \frac{1}{2} - \frac{h_3 a_5}{\Lambda} M_2, \\
B_u - C_u = \frac{1}{2} v_5^\phi + \frac{h_3 a_5}{\Lambda} 2 v_1^A - 3 v_2^\phi v_4^\phi, \\
A_d = \frac{y_1 \sigma}{\Lambda} v_5^\phi, \\
B_d + C_d = \frac{1}{2} - \frac{h_3 a_5}{\Lambda} M_2, \\
B_d - C_d = \frac{1}{2} (v_5^\phi + v_4^\phi) - \frac{h_3 a_5}{\Lambda} 2 v_1^A - 3 v_2^\phi v_4^\phi, \\
A_e = \frac{y_1 \sigma}{\Lambda} v_5^\phi, \\
B_e + C_e = \frac{1}{2} - \frac{h_3 a_5}{\Lambda} M_2, \\
B_e - C_e = \frac{1}{2} (v_5^\phi + v_4^\phi) - \frac{h_3 a_5}{\Lambda} 2 v_1^A - 3 v_2^\phi v_4^\phi.
\]

These 9 quantities are independent, except for \(A_e = A_d\), therefore the 9 masses of quarks and charged leptons can be accommodated with this one constraint, that was already discussed in section III. The hierarchy between the three families requires \(A_f, B_f + C_f\) and \(B_f - C_f\) to be of the same order. In addition, since the Yukawa of the top is close to one, in the up quark sector the three parameters should be close to the electroweak scale. This implies that the flavon VEVs \(\sigma\) and \(\chi\) are not much smaller than \(\Lambda\) and the VEV of the adjoint Higgs multiplet, \(\langle A \rangle \sim M_{GUT}\), is not much smaller than \(M_2\). We will discuss in more detail the effect of vector-like matter fields with mass close to \(M_{GUT}\) in section V.

The neutrino mass matrices take the form

\[
M_\nu = \frac{1}{\Lambda} \begin{pmatrix}
 y_1 \sigma v_5^\phi + 2 f_1 \tau v_5^\phi & \xi v_1^\phi & 2 f_3 \phi v_5^\phi \\
0 & y_1 \sigma v_5^\phi + 2 f_1 \tau v_5^\phi & 0 \\
2 f_3 \phi v_5^\phi & 0 & y_1 \sigma v_5^\phi + 2 f_1 \tau v_5^\phi
\end{pmatrix},
\]

\[
M_R = \frac{1}{\Lambda} \begin{pmatrix}
f_1 \tau & 0 & f_3 \phi \\
0 & f_1 \tau & 0 \\
f_3 \phi & 0 & f_1 \tau
\end{pmatrix} \begin{pmatrix}
(v_5^\phi)^2 \\
(v_1^\phi)^2 \\
(v_3^\phi)^2
\end{pmatrix} M_1,
\]

\[
M_L = \frac{1}{\Lambda} \begin{pmatrix}
f_1 \tau & 0 & f_3 \phi \\
0 & f_1 \tau & 0 \\
f_3 \phi & 0 & f_1 \tau
\end{pmatrix} \begin{pmatrix}
(v_5^\phi)^2 \\
(v_1^\phi)^2 \\
(v_3^\phi)^2
\end{pmatrix} M_1.
\]

TABLE II: Chiral superfields of a minimal non-renormalizable model for TBM mixing. Here \(\alpha^N = 1\) and \(1 \leq m \neq n \neq r < N\).
Light neutrino masses of the correct order of magnitude require $M_1 \gg v_1^f \sim M_{GUT}$. In this case the second term in Eq. (5) can be sufficiently large to accommodate oscillation data. The TBM lepton mixing is exactly realized, as desired.

We conclude this section by showing that the dim-5 operators offer also the opportunity to generate non-zero quark mixing while preserving TBM lepton mixing. The idea is to introduce small deviations from the magic matrix structure in the up quark sector. If these deviations were introduced in the down quark sector, that would affect also the charged lepton mass matrix and, therefore, the TBM values of the lepton mixing angles. We do not explore this possibility in this paper.

Interestingly, the Cabibbo angle is naturally induced if the $A_4$ triplet VEV aligned in the direction $(0, 1, 0)$, which is needed in the neutrino sector, also contributes to $M_u$. This can be easily achieved by considering the same model as in Table II but with $\Sigma_1$ and $\Sigma_1$ transforming as 45 under $SO(10)$. In this case the fourth operator in Eq. (26) generates the neutrino sector as before (up to irrelevant Clebsch-Gordan factors, see Table VIII), but also contributes to the up quark sector with

$$\delta M_u = \frac{1}{\Lambda} \begin{pmatrix} f_1 \tau & 0 & f_3 \varphi \\ 0 & f_1 \tau & 0 \\ f_3 \varphi & 0 & f_1 \tau \end{pmatrix} \frac{16 v_1^f v_5^f}{v_1^T}.$$  \hspace{1cm} (30)

We absorb the diagonal entries of $\delta M_u$ in the definition of $A_u$ in Eq. (28). The off-diagonal entries, instead, do not respect the magic structure. In the basis where $M_4$ is diagonal, the up quark mass matrix is given by

$$U_\omega^T (M_u^{\text{magic}} + \delta M_u) U_\omega^* = \begin{pmatrix} m_{u1} + 2 \epsilon & -\omega^2 \epsilon & -\epsilon \\ -\omega \epsilon & m_{u2} - \epsilon & 2 \omega^2 \epsilon \\ -\omega^2 \epsilon & 2 \omega \epsilon & m_{u3} - \epsilon \end{pmatrix},$$  \hspace{1cm} (31)

where $m_{ui}$ are defined in Eq. (2) and $\epsilon \equiv (16/3)(f_3 \varphi/\Lambda)(v_1^f/M_1)v_5^f$. Notice that all entries of $M_u$ receive corrections of the same order. Taking $\epsilon \ll m_{u2,3}$, which is natural since $v_1^f/M_1 \ll 1$, one can accommodate the $1 - 2$ quark mixing angle, $\theta_{12}^a \approx |\epsilon/m_{u2}| \approx |\epsilon|/m_c$. A similar idea to accommodate the Cabibbo angle was used in [40]. The values of the other two quark mixing angles are given by $\theta_{23}^a \sim \theta_{13}^a \sim (m_c/m_t)\theta_{12}^a$, which unfortunately are too small to explain the experimental values.

It is possible to introduce other corrections to $M_u$ in such a way that all CKM parameters can be accommodated, without affecting the structure of the other mass matrices. For this purpose one needs a 120 Higgs multiplet $\rho_{up}$ with VEV only in the $SU(5)$ component 45. Notice that in the model of Table II one can make the identification $\rho \equiv \rho_{up}$, since the VEVs $v_5^f$ and $v_5^f$ are not necessary for the model to work. Then, the operators $\Phi \Phi \rho_{up}$ or $\Phi \Phi A \rho_{up}$ should couple to the flavon $\varphi$ in order to modify the magic structure of $M_u$. The flavour structure of such corrections can be sufficiently rich to accommodate all the CKM parameters, e.g. employing the operator D in Table VII. However, we did not find any simple way to explain the hierarchy among the values of the quark mixing angles, therefore we refrain from presenting further details.

V. A MECHANISM TO EXPLAIN THE INTER-FAMILY MASS HIERARCHY

In the previous sections, we analyzed the structure of fermion mixing. We did not address yet the origin of the strong hierarchy between the masses of the three families of quarks and charged leptons. In general, such hierarchy is not explained by the $SO(10) \times A_4$ symmetry by itself. In fact, the mass matrix structure $M_f$ in Eq. (11) accommodates three arbitrary mass eigenvalues, which are therefore free parameters like in the Standard Model. Moreover, as we discussed in the Introduction, since in $SO(10)$ both fermion chiralities
transform as 3 under $A_4$, one cannot use an extra family symmetry besides $A_4$ in order to distinguish the three families.

The purpose of this section is to present an elegant mechanism to generate the hierarchy between the three families in $SO(10) \times A_4$ models. Such mechanism emerges from the analysis of dim-5 operators performed in section IV and it turns out to be closely related to ‘universal seesaw’ models [53, 54, 55, 56]. We will show that it is compatible with the generation of TBM lepton mixing discussed in the previous sections.

The masses of the three families of charged fermions are given in Eq. (2). They are linear combinations, with coefficients of unit modulus, of the mass matrix elements $A_f, B_f, C_f$. When the equality $A_f = \omega B_f = \omega^2 C_f \equiv m_{f3}/3$ holds, the mass eigenvalues are $(0, 0, m_{f3})$. When $A_f = (m_{f3} + m_{f2})/3$, $B_f = (\omega m_{f3} + \omega^2 m_{f2})/3$ and $C_f = (\omega m_{f3} + \omega^2 m_{f2})/3$, the mass eigenvalues are $(0, m_{f2}, m_{f3})$. If the above relations were approximatively realized, one might explain the inter-family hierarchy. At first sight such relations seem completely ad hoc because, in the $A_4$ models built so far, the parameters $A_f, B_f, C_f$ are generated by independent $A_4$ invariant operators. To remove this problem, let us begin by writing the mass matrix structure in Eq. (1) as a sum over three rank-1 components:

$$M_f = \frac{m_{f1}}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_{f2}}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix} + \frac{m_{f3}}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix}.$$  \hspace{1cm} (32)

The three flavour structures in Eq. (32) cannot be generated in $A_4$ models with renormalizable Yukawa couplings. However, we have shown in section IV that several new flavour structures are possible when the light matter fields $\Psi$ mix with heavy matter multiplets. This mixing may actually lead to the three requisite structures in Eq. (32). Specifically, the “democratic” structure is obtained from the dim-5 operator by taking $(\Sigma, \Sigma')$ singlets under $A_4$ and $\Phi_A$ and $\Phi_B$ transforming as $A_4$ triplets with VEVs $(a, a, a)$ and $(b, b, b)$, corresponding to the case C in Table VII. The second (third) flavour structure in Eq. (32) is obtained similarly, but with the pair of matter multiplets $\Sigma'' \sim 1''$ and $\Sigma' \sim 1'$ ($\Sigma' \sim 1'$ and $\Sigma'' \sim 1''$), corresponding to the case $C'$ in Table VII.

There is a simple group theoretical interpretation of Eq. (32). The $A_4$ representations decompose under the residual $Z_3$ symmetry (preserved by the vacuum alignment $(1, 1, 1)$) as $1_{A_4} = 1_{Z_3}, 1'_{A_4} = 1'_{Z_3}, 1''_{A_4} = 1''_{Z_3}$ and $3_{A_4} = 1_{Z_3} + 1'_{Z_3} + 1''_{Z_3}$. The mass eigenstates are precisely those three orthogonal combinations of $\Psi_i \sim 3_{A_4}$ which transform in a given representation of $Z_3$ and, therefore, they separately mix with $\Sigma, \Sigma'$ and $\Sigma''$, respectively. The three terms in Eq. (32) correspond to the $Z_3$ invariants $1_{Z_3} \times 1_{Z_3}, 1'_{Z_3} \times 1'_{Z_3}$ and $1''_{Z_3} \times 1''_{Z_3}$.

For definiteness, we take the two Higgs VEVs $a \sim M_{EW}$ and $b \sim M_{GUT}$. Since the first and second generation Yukawa couplings are much smaller than one, they are well described by the dim-5 operators obtained decoupling $(\Sigma, \Sigma')$ and $(\Sigma'', \Sigma')$ at the mass scales $M_1 \gg M_2 \gg M_{GUT}$. On the contrary, the third generation Yukawa couplings (in particular the top) are large, therefore we are led to consider a vector-like pair of matter multiplets $(\Sigma', \Sigma'')$ with mass $M_3$ of the same order as $M_{GUT}$. In this case it is not appropriate to integrate out these states and treat their effect in terms of higher dimensional operators. We shall instead consider explicitly the mixing of $\Sigma'$ with $\Psi$ and show how this generates the third term in Eq. (32). The ‘inverse’ hierarchy $M_1 \gg M_2 \gg M_3$ may be justified e.g. by a Froggatt-Nielsen $U(1)$ symmetry with different charges for $\Sigma, \Sigma''$ and $\Sigma'$.

The above discussion applies to any model with the specified $A_4$ assignment of fields. In the case of $SO(10)$ models, one should carefully choose the $SO(10)$ operators and symmetry breaking pattern, in order to generate the structure of $M_{u,d,e}$ as in Eq. (32) and preserve, at the same time, the cross structure of $m_{\nu}$. Let us consider, to begin with, the matter multiplets $\Sigma' \sim (16, 1')$ and $\Sigma'' \sim (\overline{16}, 1'')$, together with the
Higgs multiplets $\phi_i \sim (10, 3)$ and $A_i \sim (45, 3)$ (the discussion below can be easily generalized to different $SO(10) \times A_4$ assignments of the fields). The superpotential reads

$$ W = -M_3 \Sigma' \Sigma'' + y(\Psi_1 \phi_1 + \omega \Psi_2 \phi_2 + \omega^2 \Psi_3 \phi_3) \Sigma' + g(\Psi_1 A_1 + \omega^2 \Psi_2 A_2 + \omega \Psi_3 A_3) \Sigma'' .$$

(33)

The direct contribution to the light fermion mass matrices from the couplings $\Psi_i \phi_j \phi_k$ is assumed to be sub-dominant. It can be forbidden e.g. by a parity symmetry. Right-handed isospin (the $B$ right-handed isospin symmetry). When

$$ \text{SO}(10)-\text{breaking VEV of } A$$

we assume that $A_i$ acquire VEVs, such that $\langle A_i \rangle \Psi_{ia} = V k a \Psi_{ia}$, where $\alpha$ runs over the 16 components of $\Psi_i$ and $k a$ are Clebsch-Gordan coefficients which depend on the $SO(10)$ direction associated with $V$. Then, the following linear combination of the 16 multiplet components becomes heavy:

$$ \Psi^h_\alpha = c_\alpha \Sigma'_\alpha - \frac{s_\alpha}{\sqrt{3}} (\Psi_{1\alpha} + \omega^2 \Psi_{2\alpha} + \omega \Psi_{3\alpha} ) ,$$

(34)

where

$$ c_\alpha \equiv \frac{M_3}{\sqrt{|M_3|^2 + 3g V k a|^2}} , \quad s_\alpha \equiv \frac{\sqrt{3} g V k a}{\sqrt{|M_3|^2 + 3g V k a|^2}} .$$

(35)

Notice that $|c_\alpha|^2 + |s_\alpha|^2 = 1$. The light fermions $\psi^l_{ia}, i = 1, 2, 3$, are given by the three linear combinations orthogonal to $\Psi^h_\alpha$. The unitary $4 \times 4$ mixing matrix is defined by

$$ \begin{pmatrix} \Psi_{1\alpha} \\ \Psi_{2\alpha} \\ \Psi_{3\alpha} \\ \Sigma'_\alpha \end{pmatrix} = \begin{pmatrix} P_\alpha & -s_\alpha \sqrt{3} \\ -\omega s_\alpha \sqrt{3} & -\omega^2 s_\alpha \sqrt{3} \\ -\omega^2 s_\alpha \sqrt{3} & -\omega s_\alpha \sqrt{3} \end{pmatrix} \begin{pmatrix} \psi^l_{1\alpha} \\ \psi^l_{2\alpha} \\ \psi^l_{3\alpha} \end{pmatrix} .$$

(36)

Since the choice of basis for $\psi^l_{ia}$ is arbitrary, the matrices $P_\alpha$ and $Q_\alpha$ are determined up to a $3 \times 3$ unitary rotation from the right. We choose the basis where $\psi^l_{ia} \rightarrow \Psi_{ia}$ in the limit $s_\alpha \rightarrow 0$. In this case one finds

$$ P_\alpha = I_3 + \frac{|c_\alpha| - 1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix} , \quad Q_\alpha = \frac{s_\alpha c_\alpha^*}{\sqrt{3}|c_\alpha|} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix} .$$

(37)

In the following we take $c_\alpha$ real and positive, without loss of generality.

The three light states acquire a mass when the Higgs doublets $H^u,d$ in $\phi_i$ acquire VEVs at the electroweak scale. Let us begin from the down quark sector. The second term in Eq. (33) contains the couplings $y(Q_i H^d + \omega Q_2 H^d + \omega^2 Q_3 H^d) d^c_3$, as well as $g Q_3 \Sigma_1' (d_1 H^d + \omega d_2 H^d + \omega^2 d_3 H^d)$. Assuming the VEV alignment $(v_d, v_d, v_d)$ and using Eq. (36) for $\Psi_{ia} = Q_i (d_1')$ and $\Sigma'_a = Q_3 \Sigma_1' (d_3')$, we obtain the following mass term for the three light generations:

$$ W \supset (d_1' d_2' d_3') \frac{y v_d}{\sqrt{3}} \begin{pmatrix} s_Q & 1 & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix} s_d^c \begin{pmatrix} d_1'^c \\ d_2'^c \\ d_3'^c \end{pmatrix} .$$

(38)

The charged lepton mass term has this same form with the obvious replacements $Q \rightarrow L$ and $d^c \rightarrow e^c$. The two flavour structures in Eq. (38) are exactly those required to reproduce the second and third generation masses as in Eq. (32). The relative hierarchy between the two terms is determined by the direction of the $SO(10)$-breaking VEV of $A$. In general $\langle A \rangle = v_3^A T_{3R} + v_3^A T_{B-L}$, where $T_{3R}$ ($T_{B-L}$) is the generator of the right-handed isospin (the $B-L$ symmetry). When $v_3^A = 0$, the VEV of $A$ points in the right-handed isospin direction and one has $s_Q = s_L = 0$ as well as $s_{e^c} = s_{e^c}$. In this limit only the third family is massive and $b - \tau$ unification is realized.
Let us now consider the up-quark and Dirac neutrino sectors. If also $H_u$ acquire non-zero VEVs $(v_u, v_u, v_u)$, then $M_u$ ($M_d$) takes the same form as in Eq. (33), with the replacements $v_d \rightarrow v_u$ and $d^{c} \rightarrow u^{c}$ ($Q \rightarrow L$, $d^{c} \rightarrow u^{c}$). However, such contribution to $M_d$ would spoil the TBM lepton mixing. The way to generate only $M_u$ and not $M_d$ is the same described in section IV the up-type Higgs doublets which are kept light and acquire a VEV should not be those in $\phi \sim (10,3)$, but rather those in the $SU(5)$ 45-components of $\rho \sim (120, 3)$. In this case the above derivation can be still applied replacing the coupling $y \Sigma \phi$ by $h \Sigma \phi$. When $v_{15}^A = 0$, only the top quark acquires a mass.

Next, let us discuss the masses of the second family of charged fermions. One contribution may come from a non-zero $v_{15}^A \ll v_3^A$. In this case $s_Q = -s_L / 3 \sim (v_{15}^A / v_3^A) s_d$, so that the hierarchy between the two terms in Eq. (38) follows from a hierarchy between the $SU(4)_C$ and the $SU(2)_R$ symmetry breaking scales. (A recent Pati-Salam model to obtain a flavour hierarchy from a gauge hierarchy without any family symmetry was recently proposed in Ref. [570].) Notice also that the $s$ and $\mu$ masses are generated with the appropriate Georgi-Jarlskog factor, due to their relative $B - L$ number. However, if $v_{15}^A$ contributes also to the up quark sector, one finds $m_c / m_t \sim m_s / m_b$, in strong disagreement with data. The steepest hierarchy in the up quarks cannot be accommodated in this minimal setup.

In fact, there is yet another reason to take the VEV of $\Sigma^\prime \rightarrow \Sigma$ (16, 1) basis. This is guaranteed by taking $v_{15}^A = 0$, because in this case $s_L = 0$ and therefore the lepton doublets $L_i$ do not mix with $L_{5Y}$. We set $v_{15}^A = 0$ in the following. In this case, the second family masses may be introduced, as already mentioned, by the mixing with a pair $\Sigma'' \sim (16, 1')$ and $\Sigma' \sim (\overline{16}, 1)$ with mass $M_2 \gg M_3$. One may repeat the analysis done in Eqs. (38)-(38) with some obvious replacements, in particular swapping $\omega$ with $\omega^2$ everywhere. Assuming that $H_d$ resides in $\phi$ and $H_u$ in $\rho$, consistently with the generation of the third family masses, one can accommodate $m_c / m_t \ll m_s / m_b$ by tuning the independent Yukawa couplings. Some more effort will be needed to make $m_s \neq m_\mu$, as discussed later.

Analogously, one can generate the democratic term in Eq. (32) for the first family masses, by the mixing with a pair $\Sigma \sim (16, 1)$ and $\Sigma' \sim (\overline{16}, 1)$ with mass $M_1 \gg M_2$. The derivation is again very similar to Eqs. (38)-(38), replacing everywhere $\omega$ and $\omega^2$ with 1. Let us remind, however, that since the first family masses are tiny, they may be generated even by non-democratic contributions, from operators close to the Planck scale, which would not affect significantly the values of the mixing angles.

Notice that no CKM mixing between the light quark families is generated by the mechanism described above, as long as the $A_4$ VEV alignment $(1, 1, 1, 1)$ is preserved in all the operators contributing to $M_u$ and $M_d$. In this case a $Z_3$ subgroup is unbroken and, as mentioned above, each of the heavy matter multiplets $\Sigma, \Sigma', \Sigma''$ mixes only with the orthogonal combination of the $\Psi_i$ fields that transforms in the same way under $Z_3$. As a consequence, each light family of quarks does not mix with the others. In order to generate the CKM parameters, one should resort to extra contributions to the quark mass matrices which break $Z_3$.

Let us implement in an explicit model the mechanism to generate the inter-family mass hierarchy together with TBM lepton mixing and non-zero Cabibbo mixing. In order to correctly describe the masses of third and second family of quarks and charged leptons, we found that the minimal set of multiplets is the one given in Table II. The Yukawa superpotential has the form

$$
W_Y = \Psi \left( \frac{y_1 \chi}{\Lambda} \rho_1 + \frac{y_3 \chi}{\Lambda} \rho_2 \right) \Sigma' - M_3 \Sigma \Sigma'' + \Sigma' \frac{y_3 \chi}{\Lambda} A \Psi \\
+ \Psi \left( \frac{y_2 \chi}{\Lambda} \rho_1 + \frac{y_2 \chi}{\Lambda} \rho_2 \right) \Sigma'' - M_2 \Sigma \Sigma' + \Sigma' \frac{y_2 \chi}{\Lambda} A \Psi.
$$

(39)
A $Z_N$ symmetry was introduced to forbid all other couplings, up to terms quadratic in the flavons; the minimal choice is $N = 4$ with charge $m = 1$. The adjoint Higgs multiplet $A$ has the VEV $v_3^A$ in the right-handed isospin direction and the triplet flavon has the VEV $(\chi_i) = (\chi, \chi, \chi)$. In order to reproduce correctly all the mass ratios, we introduced two 120 multiplets $\rho_1$ and $\rho_2$ with the VEVs

$$\langle 45^{\rho_1} \rangle = v_{45}^\rho, \quad \langle 3^{\rho_1} \rangle = v_3^\rho, \quad \langle 3^{\rho_2} \rangle = v_3^{\rho_2}. \quad (40)$$

Following the derivation of Eqs. (33)-(38), it is straightforward to compute the mass eigenvalues defined by Eq. (32):

\[
\begin{align*}
m_t &= \frac{\sqrt{3} g_2 \chi v_{45}^\rho s_3}{\Lambda}, \\
m_b &= -\left( \frac{\sqrt{3} g_2 \chi v_{45}^\rho}{\Lambda} + \frac{\sqrt{3} h_3 \chi v_{3}^{\rho_2}}{\Lambda} \right) s_3, \\
m_c &= \frac{\sqrt{3} g_2 \chi v_{45}^\rho s_2}{\Lambda}, \\
m_s &= -\left( \frac{\sqrt{3} g_2 \chi v_{45}^\rho}{\Lambda} - \frac{\sqrt{3} h_3 \chi v_{3}^{\rho_2}}{\Lambda} \right) s_2, \\
m_\mu &= \left( \frac{\sqrt{3} g_2 \chi v_{45}^\rho}{\Lambda} - \frac{3 \sqrt{3} h_3 \chi v_{3}^{\rho_2}}{\Lambda} \right) s_2.
\end{align*}
\]

where the parameters $s_3, s_2$ control the mixing between $\Psi$ and $\Sigma', \Sigma''$ and are given by

\[
s_3 \equiv \frac{\sqrt{3} g_2 \chi v_{45}^A}{\sqrt{|M_3 A|^2 + \sqrt{3} g_2 \chi v_{3}^{A}|^2}}, \quad s_2 \equiv \frac{\sqrt{3} g_2 \chi v_{45}^A}{\sqrt{|M_2 A|^2 + \sqrt{3} g_2 \chi v_{3}^{A}|^2}}. \quad (42)
\]

The heaviens of the top requires to take $M_3 \sim v_3^A$ so that $s_3 \sim 1$. The hierarchy between second and third generation masses is then explained by taking $M_2 \gg v_3^A$, so that $s_2 \ll 1$. Approximate $b - \tau$ unification is realized when the first term in $m_b$ and $m_\tau$ dominates over the second. The ratio $m_s/m_\mu \sim 1/3$ is realized when the second term in $m_s$ and $m_\mu$ dominates over the first. More precisely, the six masses in Eq. (41) are constrained by one non-trivial relation, $(3m_s - m_\mu) = (m_c/m_\tau)(3m_b - m_\tau)$, which connects the two phenomenological facts $m_s/m_b \gg m_c/m_\tau$ and $m_\mu \approx 3m_s$ at the GUT scale. At this level the first family masses are vanishing.

As for the neutrino sector, we need to generate the cross structure of $m_\nu$, as required by TBM mixing. One may think that is sufficient to introduce the last operator in Eq. (29), however the size of neutrino masses is too small in this case, as can be seen by inspecting Eq. (29) (in the present scenario the term $y_1 \sigma v_5^{\phi}$ in $M_\nu$ is absent). To solve this problem, the neutrino Dirac mass matrix $M_D$ and the right-handed neutrino Majorana mass matrix $M_R$ should be generated by two independent operators. We found that the minimal set of multiplets to achieve this purpose is the one given in Table [IV] with two $16$ Higgs multiplets taking VEVs only in the directions $\langle 1^{EM} \rangle = v_1^{EM}$ and $\langle 5^{ED} \rangle = v_5^{ED}$. We introduced an auxiliary symmetry $Z_N'$ such that all fields charged under $Z_N$ are neutral under $Z_{N'}$ and vice versa. Then, the superpotential in Eq. (38) is extended to include the extra terms

\[
W_Y \supset \Psi \left( \frac{f_{1M} \tau + f_{3M} \phi}{\Lambda} \xi_M + \frac{f_{\mu} \sigma}{\Lambda} \xi_D \right) \Sigma_M - M_{M} \Sigma_{M} \Sigma_{M} + f_{M} \Sigma_{M} \xi_{M} \Psi
\]
TABLE IV: Chiral superfields needed to generate the cross structure of the neutrino mass matrix and the Cabibbo mixing angle. Here $\alpha^{N'} = 1$ and $1 \leq n \neq r < N'$.

\[
\begin{array}{cccccc}
SO(10) & \text{matter fields} & \text{Higgs fields} & \text{flavons} \\
\Psi & \Sigma_D & \Sigma_D & \Sigma_M & \Sigma_M & \xi_D & \xi_M \\
16 & 1 & 1 & 45 & 45 & 1 & 1 \\
A_4 & 3 & 3 & 3 & 3 & 1 & 1 \\
Z_{N'} & 1 & \alpha^{-n} & \alpha^n & \alpha^{-r} & \alpha^r & 1 \\
& 1 & \alpha^n & \alpha^{-r} & 1 & \alpha^r & \alpha^n & \alpha^{-n} & \alpha^f & \alpha^{2f} \\
\end{array}
\]

The $Z_{N'}$ assignments of Table IV forbid all other $SO(10) \times A_4$ invariant couplings (a minimal viable choice is $N' = 5$, with charges $n = 2$ and $r = 4$). Once the messenger fields ($\Sigma_D, \Sigma_M$) and ($\Sigma_M, \Sigma_M$) are integrated out, the effective superpotential contains two relevant operators:

\[
W_{\text{eff}} \supset \frac{\Psi f_\sigma}{\Lambda} \xi_M \Sigma_D - M_D \Sigma_D \Sigma_D + f_D \Sigma_D \xi_D \Psi.
\]

The first operator generates the cross structure of $M_R$, while the second generates $M_\nu \propto M_D^3$ (remember that only the singlet in $\xi_M$ and the doublet in $\xi_D$ acquire a non-zero VEV). By taking $M_M \gg M_D \geq M_{\text{GUT}}$, one obtains right-handed neutrino masses significantly below $M_{\text{GUT}}$ and keeps the Dirac neutrino masses close to electroweak scale. In this way sufficiently large neutrino masses may be generated by type I seesaw.

When also $\langle 5 \xi_M \rangle = \langle \xi_M \rangle$ is different from zero, the first operator in Eq. (44) gives a negligible contribution to $M_\nu$, because $M_M \gg M_D$. However, it plays an important role in the up quark sector, since it provides a small correction to $M_u$ in a form completely analog to Eq. (30). This correction generates the Cabibbo mixing angle, in the same way as discussed at the end of section IV.

In summary, the model defined by the superpotential in Eqs. (39) and (43) accounts for TBM mixing in the lepton sector and the $1 - 2$ mixing in the quark sector, as well as for the hierarchical values of the quark and charged lepton masses.

Before concluding, let us remark that, in most models of ‘universal seesaw’ with a realistic phenomenology, the mass terms $M \Sigma \Sigma$ of the messenger fields are sensitive to the $SO(10)$ breaking VEVs, thus giving different masses to the different components of $\Sigma$. One may also introduce a non-trivial mixing between the heavy matter families. In our construction we barred these extra possibilities for simplicity. A recent model of ‘universal seesaw’ making use of rank-1 flavour structures, together with more references, can be found in [58]. Heavy messenger fields are also employed in a similar fashion in an $SO(10) \times SU(3)$ model for TBM mixing [21].

VI. CONCLUSIONS

In this paper we performed a systematic analysis of fermion mass matrices in models of Grand Unification based on the gauge symmetry $SO(10)$ with the discrete family symmetry $A_4$. In these models all light fermions and right-handed neutrinos may be unified in a single multiplet $(16, 3)$ of the group $SO(10) \times A_4$. We demonstrated that, even though this scenario is very constrained, it is possible to understand the disparity between the quark and lepton mixing angles as well as the strong hierarchy between the three families of charge fermion masses.

In models with renormalizable Yukawa couplings, we found that the exact TBM lepton mixing can be obtained only if the type I seesaw contribution to $m_\nu$ is neglected. The specific effect of such contribution is
to shift the 1 − 2 lepton mixing angle from the tri-maximal value. The non-zero values of the CKM mixing angles cannot be accommodated without introducing further departures from TBM lepton mixing.

A much richer flavour structure appears once the effect of higher dimensional operators is taken into account. We considered dim-5 operators generated by the mixing of light families with heavy vector-like matter multiplets: the corresponding structures of the mass matrices are listed in Appendix B2 for all $A_4$ representations and all the $SO(10)$ representations of size $\leq 120$. This classification proves to be a useful tool for model-building.

We found that the dim-5 operators help to evade several difficulties in the construction of a satisfactory $SO(10) \times A_4$ model of flavour. One crucial ingredient is provided by those operators which contribute differently to the Dirac mass matrices $M_u$ and $M_\nu$. First, it is possible to obtain exact TBM lepton mixing from a type I seesaw. Second, there are contributions which generate non-zero CKM mixing angles without disturbing the TBM pattern in the lepton sector.

Moreover, we have shown that the mixing of the three light families with vector-like matter multiplets provides a natural explanation of the inter-family mass hierarchy of quarks and charged leptons. In fact, rank-1 contributions to the mass matrices are generated by the mixing with heavy dim-16 multiplets, that transform in a dim-1 representation of $A_4$. The flavour structures of such contributions are exactly those necessary to achieve TBM mixing. A hierarchy in the masses of the heavy families and/or in the VEVs breaking $SO(10)$ reflects into a hierarchy of the masses of the light families.

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APPENDIX A: THE GROUP $A_4$ AND ITS BREAKING PATTERN

The pair of lepton mass matrix structures defined in in Eq. (11) leads to exact TBM mixing in the lepton sector. In this Appendix we prove that such realization of TBM mixing is the only one that can be obtained by the spontaneous breaking of an $A_4$ flavour symmetry. We follow an approach already applied to the group $A_4$ in Ref. [31] and to the group $T'$ in Ref. [59] (see also [60]).

The discrete group $A_4$ is formed by the even permutations of four objects. It can be defined by two generators $S$ and $T$ such that

$$S^2 = T^3 = (ST)^3 = 1.$$  \hfill (A1)

The 12 elements of $A_4$ belong to four conjugacy classes

$$C_1 : I ;$$
$$C_2 : T, ST, TS, STS ;$$
$$C_3 : T^2, ST^2, T^2S, TST ;$$
$$C_4 : S, T^2ST, TST^2 .$$  \hfill (A2)
Each element $g$ of $C_{2,3}$ ($C_4$) satisfies $g^3 = 1$ ($g^2 = 1$) and, therefore, generates a subgroup $Z_3$ ($Z_2$) of $A_4$. There are three dim-1 and one dim-3 irreducible representations. We adopt the basis where the generators of the dim-3 representation are given by

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$  

(A3)

We assume that both lepton doublets and singlets transform as $A_4$ triplets, as dictated by $SO(10)$. The lepton mixing matrix $U_{\text{lepton}} = U_{e}^{T}U_{\nu}$ depends on how $A_4$ is spontaneously broken in the charged lepton and in the neutrino sector. If $A_4$ were unbroken in one sector, the mass matrix would be proportional to the identity. If $A_4$ were broken completely in one sector, the mass matrix would be generic and the values of the mixing angles arbitrary. Therefore, the only non-trivial case occurs when $A_4$ is broken to a subgroup in the charged lepton sector and to another subgroup in the neutrino sector. There are three possibilities:

- the VEV of an $A_4$ triplet with alignment $(1,1,1)$ breaks $A_4$ to $Z_3$;
- the VEV of an $A_4$ triplet with alignment $(0,1,0)$ breaks $A_4$ to $Z_2$;
- the VEV of a singlet in the $A_4$ representation breaks $A_4$ to $Z_2 \times Z_2$.

The relevant choice to obtain TBM mixing is to preserve $Z_3$ in one sector and $Z_2$ in the other, as we now show (one may check in a similar way that all other choices do not lead to TBM mixing).

If $g$ is the generator of a subgroup of $A_4$, the mass term $\psi M \psi^c$ is invariant with respect to $g$ if and only if $g M g^T = M$. For definiteness, and without loss of generality, let us consider the $Z_3$ and $Z_2$ subgroups generated by $T$ and $S$, respectively. There are two possibilities

(i) $T M_\nu T^T = M_\nu \Rightarrow M_\nu = \begin{pmatrix} A_\nu & B_\nu & C_\nu \\ C_\nu & A_\nu & B_\nu \\ B_\nu & C_\nu & A_\nu \end{pmatrix}; \quad S m_\nu S^T = m_\nu \Rightarrow m_\nu = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & d & 0 \end{pmatrix};$  

(ii) $S M_\nu S^T = M_\nu \Rightarrow M_\nu = \begin{pmatrix} A_\nu & 0 & C_\nu \\ 0 & D_\nu & 0 \\ B_\nu & 0 & E_\nu \end{pmatrix}; \quad T m_\nu T^T = m_\nu \Rightarrow m_\nu = \begin{pmatrix} a & b & b \\ a & b & b \\ b & b & a \end{pmatrix}.$  

(A4)

In the case (i) we recognize the charged lepton mass matrix $M_\nu$ of Eq. (11). Notice that $m_\nu$ is more general than in Eq. (11), because the $Z_2$ invariance allows different entries on the diagonal, in particular $a \neq d$. This is because one may couple to the neutrinos also a Higgs transforming as $1'$ or $1''$, and still maintain $m_\nu$ invariant under $Z_2$ [31]. Therefore this $A_4$ breaking pattern is not sufficient to predict TBM mixing. If one does not introduce such $1'$ and $1''$ fields in the model, one finds $a = c = d$ and the neutrino mass matrix of Eq. (11) is recovered in the special limit $a = c$. Then exact TBM lepton mixing is obtained.

For what concerns the case (ii), one can obtain TBM mixing in the neutrino sector alone, by taking $M_\nu$ diagonal. This scenario is realized by coupling to the charged leptons three Higgs singlets $1,1',1''$ and no triplet (the $A_4$ subgroup $Z_2 \times Z_2$ is preserved in this case). However, even though $m_\nu$ is diagonalized by $U_{TBM}$, one has $m_1 = m_3$, in strong disagreement with oscillation experiments. Therefore the case (ii) is not viable.
Consider the Yukawa couplings $Y_{ij} \Psi_i \Psi_j \Phi$, where the matter multiplets transform as $\Psi \sim (16,3)$ under $SO(10) \times A_4$ and $\Phi$ is a Higgs multiplet. For all possible $SO(10) \times A_4$ assignments of $\Phi$, let us analyze the structure of the quark and lepton mass matrices that are generated when a given component of $\Phi$ acquires a VEV $v$.

The $A_4$ assignment of $\Phi$ determines the flavour structure of the mass matrices, as shown in Table V. When $\Phi \sim 3$ (case B in the Table), there are two $A_4$-invariants, one symmetric and the other antisymmetric in the flavour indexes. The $SO(10)$ assignment of $\Phi$ determines the relative contribution to the mass matrices of the different sectors, as shown in Table VI. The Yukawa coupling matrices $Y_{10}$ and $Y_{120}$ ($Y_{126}$) are (anti)symmetric in the flavour indexes.

For a given $SO(10) \times A_4$ assignment of $\Phi$, the contribution to the mass matrices is obtained combining the corresponding rows in Tables V and VI. Such contribution is non-zero only if both rows contain terms (anti)symmetric in the flavour indexes.

### APPENDIX B: STRUCTURES OF MASS MATRICES WITH $SO(10) \times A_4$ SYMMETRY

#### 1. Renormalizable Yukawa couplings

Consider the Yukawa couplings $Y_{ij} \Psi_i \Psi_j \Phi$, where the matter multiplets transform as $\Psi \sim (16,3)$ under $SO(10) \times A_4$ and $\Phi$ is a Higgs multiplet. For all possible $SO(10) \times A_4$ assignments of $\Phi$, let us analyze the structure of the quark and lepton mass matrices that are generated when a given component of $\Phi$ acquires a VEV $v$.

The $A_4$ assignment of $\Phi$ determines the flavour structure of the mass matrices, as shown in Table V. When $\Phi \sim 3$ (case B in the Table), there are two $A_4$-invariants, one symmetric and the other antisymmetric in the flavour indexes. The $SO(10)$ assignment of $\Phi$ determines the relative contribution to the mass matrices of the different sectors, as shown in Table VI. The Yukawa coupling matrices $Y_{10}$ and $Y_{120}$ ($Y_{126}$) are (anti)symmetric in the flavour indexes.

For a given $SO(10) \times A_4$ assignment of $\Phi$, the contribution to the mass matrices is obtained combining the corresponding rows in Tables V and VI. Such contribution is non-zero only if both rows contain terms (anti)symmetric in the flavour indexes.
2. Non-renormalizable operators

Consider the generic dimension-5 operator \( c_{ij} \Psi_i \Psi_j \Phi_A \Phi_B / M \), where \( \Phi_A, \Phi_B \) are two (possibly equal) Higgs multiplets and \( M \) is a cutoff scale. When the appropriate components of \( \Phi_A \) and \( \Phi_B \) acquire VEVs \( a \) and \( b \), respectively, one generates contributions to the quark and lepton mass matrices. Such operator may arise from the exchange of a pair of vector-like matter multiplets \( (\Sigma, \Sigma) \) of mass \( M \), with the superpotential given in Eq. (23). If one is interested in physics at scales much smaller than \( M \) (that is, for \( a, b \ll M \)), the pair \( (\Sigma, \Sigma) \) can be integrated out and one is left with the effective superpotential

\[
W_{\text{eff}} = \frac{y_A y_B}{M} (\Psi \Phi_A \Sigma)(\Psi \Phi_B \Sigma).
\]

(B1)

The dim-5 operators may also arise from the exchange of heavy Higgs multiplets \( \Phi' \). We do not consider this possibility, since in this case the operators involve the Yukawa coupling \( Y_{ij} \Psi_i \Psi_j \Phi' \), therefore they do not generate new mass matrix structures, besides those already discussed in Appendix B1.

The \( A_4 \) assignments of \( \Phi_A, \Phi_B \) and \( (\Sigma, \Sigma) \) determine the flavour structure of the mass matrices. There are 9 possible cases listed in Table VII. The 5 cases \( B', B'', C, C', D \) provide new flavour structures with respect to the Yukawa couplings in Table V. Let us illustrate some new possibilities:

- In the case \( C \) (and \( C' \)) each entry of the mass matrix is proportional to a different pair of VEVs, therefore matrices with only one non-zero entry (row, column) can be generated choosing the appropriate VEV alignment.

- In the case \( D \), by taking the alignment \((0,a,0)\) and \((0,b,0)\) only the entries \( (11) \) and \( (33) \) are generated, with relative coefficients \( (y_A^a \pm y_A^0)(y_B^a \pm y_B^0) \).

- In the case \( D \), by taking the alignment \((a,a,a)\) and \((b,0,0)\), only the entries 12, 13, 22, 33 are generated; if \( y_A^a = 0 \), the 4 entries are equal.

The \( SO(10) \) assignments of \( \Phi_A, \Phi_B \) and \( (\Sigma, \Sigma) \) determine the relative contribution to the mass matrices of the different sectors. In Table VIII we classify all dim-5 operators involving representations with size \( \leq 120 \). It is useful to recall that an \( SO(10) \) adjoint Higgs multiplet \( 45_H \) can acquire VEVs in two independent directions. In the Table we indicate with \( b_1 \) (\( b_24 \)) the VEV in the \( SU(5) \) singlet (adjoint) direction. Equivalently, one can write the VEV of \( 45_H \) as a combination of a VEV \( b_3 \) in the right-handed isospin direction and a VEV \( b_{15} \) in the \( B - L \) direction, by using

\[
b_1 = \frac{1}{5}(b_3 + 3b_{15}) , \quad b_{24} = \frac{1}{5}(-b_3 + 2b_{15}) .
\]

(B2)

The Clebsch-Gordan coefficients in Table VIII can be derived e.g. with the help of Refs. 61, 62, 63, 64, 65.

For a given \( SO(10) \times A_4 \) assignment of \( \Phi_A, \Phi_B, \Sigma \) and \( \Sigma \), the contribution to the mass matrices is obtained by combining the corresponding rows of Tables VII and VIII.

Operators with dimension larger than 5 can also correct significantly fermion masses, in particular when the cutoff \( M \) is not much larger than \( M_{GUT} \). An \( SO(10) \times A_4 \) model which crucially relies on a dim-6 operator was built in [19]. In this model the operator VIII of Table VIII is used, with \( 120_H \) replaced by a product \( 45_H 10_H \).
| case | $A_4$ operator | mass matrix structure |
|------|----------------|----------------------|
| A | $(3M_{1H})_3(3M_{1H'})_3$ | $y_{AB}y_{BC} \ diag(1,1,1) \frac{ab}{M}$ |
| $A'$ | $(3M_{1H})_3(3M_{1H'})_3$ | $y'_{AB}y'_{BC} \ diag(1,\omega,\omega') \frac{ab}{M}$ |
| $A''$ | $(3M_{1H})_3(3M_{1H'})_3$ | $y''_{AB}y''_{BC} \ diag(1,\omega',\omega) \frac{ab}{M}$ |
| B | $(3M_{3H})_3(3M_{3H'})_3$ | $\begin{bmatrix} y_{AB} \begin{pmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{pmatrix} + y'_{AB} \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \end{bmatrix} \frac{b}{M}$ |
| $B'$ | $(3M_{3H})_3(3M_{3H'})_3$ | $\begin{bmatrix} y_{AB} \begin{pmatrix} 0 & a_3\omega & a_2\omega'^2 \\ a_3 & 0 & a_1\omega \\ a_2 & a_1 & 0 \end{pmatrix} + y'_{AB} \begin{pmatrix} 0 & a_3\omega & -a_2\omega'^2 \\ -a_3 & 0 & a_1\omega \\ a_2 & -a_1 & 0 \end{pmatrix} \end{bmatrix} \frac{b}{M}$ |
| $B''$ | $(3M_{3H})_3(3M_{3H'})_3$ | $\begin{bmatrix} y_{AB} \begin{pmatrix} 0 & a_3\omega^2 & a_2\omega \\ a_3 & 0 & a_1\omega \\ a_2 & a_1 & 0 \end{pmatrix} + y'_{AB} \begin{pmatrix} 0 & a_3\omega^2 & -a_2\omega \\ -a_3 & 0 & a_1\omega \\ a_2 & -a_1 & 0 \end{pmatrix} \end{bmatrix} \frac{b}{M}$ |
| C | $(3M_{3H})_1(3M_{3H'})_1$ | $y_{AB} \begin{pmatrix} a_{1b} & a_{1b2} & a_{1b3} \\ a_{2b} & a_{2b2} & a_{2b3} \\ a_{3b} & a_{3b2} & a_{3b3} \end{pmatrix} \frac{1}{M}$ |
| $C'$ | $(3M_{3H})_1(3M_{3H'})_1'$ | $y_{AB} \begin{pmatrix} a_{1b} & a_{1b2}\omega & a_{1b3}\omega \\ a_{2b} & a_{2b2}\omega & a_{2b3}\omega \\ a_{3b} & a_{3b2}\omega & a_{3b3}\omega \end{pmatrix} \frac{1}{M}$ |
| D | $(3M_{3H})_3(3M_{3H'})_3$ | $\begin{bmatrix} y_{AB} \begin{pmatrix} a_{2b} + a_{3b3} & a_{2b1} & a_{2b2} \\ a_{1b} & a_{1b2} & a_{1b3} + a_{1b1} \\ a_{3b} & a_{3b2} & a_{3b3} + a_{3b1} + a_{2b2} \end{pmatrix} \\ +y_{AB} \begin{pmatrix} a_{2b} + a_{3b3} & a_{2b1} & a_{2b2} \\ a_{1b} & a_{1b2} & a_{1b3} + a_{1b1} \\ a_{3b} & a_{3b2} & a_{3b3} + a_{3b1} + a_{2b2} \end{pmatrix} \\ +y_{AB} \begin{pmatrix} a_{2b} + a_{3b3} & a_{2b1} & a_{2b2} \\ a_{1b} & a_{1b2} & a_{1b3} + a_{1b1} \\ a_{3b} & a_{3b2} & a_{3b3} + a_{3b1} + a_{2b2} \end{pmatrix} \end{bmatrix} \frac{1}{M}$ |

TABLE VII: The $A_4$ flavour structures arising from dim-5 operators, defined as in Eq. [31]. In the cases A, $A'$, $A''$, C and D one may identify both the two Higgs multiplets, $\Phi_A \equiv \Phi_B$, and the pair of messengers, $\Sigma \equiv \Sigma$. If these identifications are made, one has to take $y_A \equiv y_B$ (with possible superscripts understood).

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TABLE VIII: The contributions to the mass matrices from \(SO(10)\)-invariant dim-5 operators, defined as in Eq. (11). Here we introduced \(K \equiv y_{A}y_{B}/M\) and \(K_{s} \equiv (K + K^{T})/2\). This list exhausts all possibilities with \(\Phi_{A}, \Phi_{B}, \Sigma\) and \(\Sigma\) in representations with size \(\leq 120\). Different VEVs of the same \(SO(10)\) Higgs multiplet carry a subscript indicating the \(SU(5)\) component they belong to.

| Case | \(SO(10)\) operator | Mass matrices |
|------|------------------------|--------------|
| IV   | \((16M16_{H})_{10}(16M16_{H})_{10}\) | \(M_{d} = K\alpha_{1}b_{1} + K^{T}\alpha_{1}b_{1}\) |
|      |                        | \(M_{e}^{T} = K\alpha_{1}b_{1} + K^{T}\alpha_{1}b_{1}\) |
| V    | \((16M\overline{16}_{H})_{1}(16M\overline{16}_{H})_{1}\) | \(M_{\nu} = K\alpha_{b_{1}} + K^{T}\alpha_{a_{b_{1}}}\) |
|      |                        | \(M_{L} = K_{4}b_{5}\) |
|      |                        | \(M_{R} = K_{4}b_{1}\) |
| VI   | \((16M\overline{16}_{H})_{45}(16M\overline{16}_{H})_{45}\) | \(M_{u} = 8K_{2}(a_{b_{1}} + a_{b_{1}})\) |
|      |                        | \(M_{\nu} = 3(K\alpha_{b_{1}} + K^{T}\alpha_{a_{b_{1}}})\) |
|      |                        | \(M_{L} = -5K_{a_{b_{5}}}b_{5}\) |
|      |                        | \(M_{R} = -5K_{a_{b_{1}}}b_{1}\) |
| VII  | \((16M10_{H})_{16}(16M45_{H})_{10}\) | \(M_{u} = K\alpha_{b_{1}}(b_{1} - 4b_{24}) + K^{T}\alpha_{b_{1}}(b_{1} + b_{24})\) |
|      |                        | \(M_{\nu} = 5K\alpha_{b_{1}} + K^{T}\alpha_{b_{1}}(-3b_{1} - 3b_{24})\) |
|      |                        | \(M_{d} = K\alpha_{b_{1}}(-3b_{1} + 2b_{24}) + K^{T}\alpha_{b_{1}}(b_{1} - b_{24})\) |
|      |                        | \(M_{e}^{T} = K\alpha_{b_{1}}(-3b_{1} - 3b_{24}) + K^{T}\alpha_{b_{1}}(b_{1} - 6b_{24})\) |
| VIII | \((16M120_{H})_{16}(16M45_{H})_{10}\) | \(M_{u} = K\alpha_{b_{1}}(b_{1} - 4b_{24}) - K^{T}\alpha_{b_{1}}(b_{1} + b_{24})\) |
|      |                        | \(M_{\nu} = 5K\alpha_{b_{1}} + K^{T}\alpha_{b_{1}}(-3b_{1} - 3b_{24})\) |
|      |                        | \(M_{d} = K(\alpha_{b_{1}} + \alpha_{b_{1}})(-3b_{1} + 2b_{24}) - K^{T}(\alpha_{b_{1}} + \alpha_{b_{1}})(b_{1} + b_{24})\) |
|      |                        | \(M_{e}^{T} = K(\alpha_{b_{1}} - 3\alpha_{b_{1}})(-3b_{1} - 3b_{24}) - K^{T}(\alpha_{b_{1}} - 3\alpha_{b_{1}})(b_{1} + 6b_{24})\) |
| IX   | \((16M16_{H})_{120}(16M16_{H})_{120}\) | \(M_{d} = K(\alpha_{b_{1}}b_{1} + 2\alpha_{b_{1}}b_{1}) + K^{T}(\alpha_{b_{1}}b_{1} + 2\alpha_{b_{1}}b_{1})\) |
|      |                        | \(M_{e}^{T} = K(\alpha_{b_{1}}b_{1} + 2\alpha_{b_{1}}b_{1}) + K^{T}(\alpha_{b_{1}}b_{1} + 2\alpha_{b_{1}}b_{1})\) |

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