Vertex degrees of Steiner Minimal Trees in $\ell^d_p$ and other smooth Minkowski spaces

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Abstract
We find upper bounds for the degrees of vertices and Steiner points in Steiner Minimal Trees in the $d$-dimensional Banach spaces $\ell^d_p$ independent of $d$. This is in contrast to Minimal Spanning Trees, where the maximum degree of vertices grows exponentially in $d$ (Robins and Salowe, 1995). Our upper bounds follow from characterizations of singularities of SMT’s due to Lawlor and Morgan (1994), which we extend, and certain $\ell_p$-inequalities. We derive a general upper bound of $d + 1$ for the degree of vertices of an SMT in an arbitrary smooth $d$-dimensional Banach space (i.e. Minkowski space); the same upper bound for Steiner points having been found by Lawlor and Morgan. We obtain a second upper bound for the degrees of vertices in terms of 1-summing norms.

1 Introduction
Given a metric space $(X, \rho)$ and a set $S \subseteq X$, a Minimal Spanning Tree (MST) of $S$ is a tree $T$ with vertex set $V(T) = S$ and edge set $E(T)$ such that

$$\sum_{\{x,y\} \in E(T)} \rho(x,y)$$

is minimal among all trees on $S$.

A Steiner Minimal Tree (SMT) of $S$ is a tree $T$ with vertex set $V(T)$ satisfying $S \subseteq V(T) \subseteq X$ such that

$$\sum_{\{x,y\} \in E(T)} \rho(x,y)$$

is minimal among all trees on $S$ with vertex sets satisfying $S \subseteq V(T) \subseteq X$. The elements of $S$ are vertices, and the elements of $V(T) \setminus S$ are Steiner points of the SMT.

Estimates for the largest degrees of MST’s and SMT’s have consequences for the complexities of algorithms that find such trees. For example, it is known that an MST on $n$ points can be calculated in polynomial time [2], while calculating the SMT in the euclidean or rectilinear planes is NP-hard [7, 8]. Upper bounds
for the degrees of vertices and Steiner points are used to reduce the search space
of known exponential time algorithms.

Distance functions other than euclidean or rectilinear are sometimes used. The \( \ell_p \) metrics have been found useful; see \[15\]. We consider general Minkowski spaces, i.e. finite dimensional Banach spaces, and then specialize to \( \ell^d_p \), \( d \)-dimensional real linear space with norm

\[
\| (x_1, \ldots, x_d) \|_p = \left( \sum_{i=1}^{d} |x_i|^p \right)^{1/p}.
\]

It is known that in a Minkowski space, the largest degree of an MST is equal to the so-called Hadwiger number \( H(B) \) of the unit ball \( B \) of the space \[3\]. For each \( 1 \leq p \leq \infty \) there is an exponential lower bound for the Hadwiger number of \( \ell^d_p \), \( H(B^d_p) > (1 + \epsilon_p)^d \) \[19\].

In contrast to this, we show in Section 4 that the degrees of both vertices and Steiner points of an SMT in \( \ell^d_p \) \( (1 < p < \infty) \) are bounded above by functions of \( p \) alone, independent of \( d \). For \( p > 2 \) we derive a general upper bound of 7, with various sharper values for specific \( p \). For \( 1 < p < 2 \) however, we find an upper bound exponential in \( p^* := p/(p-1) \), and a lower bound linear in \( p^* \), as \( p \) tends to 1. Thus with respect to the SMT problem, \( \ell^d_p \) behaves very similarly to euclidean space, where both vertices and Steiner points have degree at most 3.

For general \( d \)-dimensional smooth Minkowski spaces, it is known that the degree of a Steiner point is at most \( d + 1 \) \[14\]. In Section 3 we show that this upper bound also holds for the degree of a vertex in an SMT. The proof has two ingredients. Firstly, in Section 2 we derive a characterization of the local structure of a vertex in an SMT (Theorem 4) similar to the characterization of Steiner points due to Lawlor and Morgan \[14\]. We also rederive their characterization, paying attention to some combinatorial subtleties (Theorem 1). Both derivations are completely elementary. The second ingredient is Theorem 4, which generalizes a result of \[6\] and \[14\], thus answering a question in \[21\].

In Theorem 5 we also obtain an upper bound for the degrees of vertices and Steiner points in terms of the 1-summing norm of the dual of the space.

## 2 Derivation of the singularity characterizations

Theorem 1 below, due to \[14\], provides a characterization of the structure of the neighbourhood of a Steiner point in an SMT in a smooth Minkowski space. We give a similar characterization of the structure of the neighbourhood of a vertex in an SMT in Theorem 4. Both characterizations are in terms of unit vectors in the dual of the Minkowski space.

We now recall some facts about dual spaces. Note that the discussion below pertains to finite dimensional Banach spaces, i.e. Minkowski spaces; see \[23\].

For any \( d \)-dimensional real vector space \( X \), the dual of \( X \), denoted by \( X^* \), is the vector space of linear functionals \( x^* : X \to \mathbb{R} \). This dual is also a \( d \)-dimensional vector space. We denote application of \( x^* \in X^* \) to \( x \in X \) by \( \langle x^*, x \rangle \). If \( X \) is furthermore a Minkowski space with norm \( \| \cdot \| \), then \( \| x^* \|^* = \sup_{\| x \| \leq 1} \langle x^*, x \rangle \) defines a norm on \( X^* \).
We say that a Minkowski space is smooth if
\[
\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t} =: f_x(h)
\]
exists for all \(x, h \in X\) with \(x \neq 0\). It follows easily that \(f_x \in X^*, \|f_x\| = 1\) and \(\langle f_x, x \rangle = \|x\|\). A linear functional \(x^* \in X^*\) is a norming functional of \(x\) if \(x^*\) satisfies \(\langle x^*, x \rangle = \|x\|\) and \(\|x^*\| = 1\). Each non-zero vector in a Minkowski space has a norming functional (the Hahn-Banach theorem). A Minkowski space is smooth iff each non-zero vector has a unique norming functional.

A Minkowski space \(X\) is strictly convex if \(\|x\| = \|y\| = 1\) and \(x \neq y\) imply that \(\|\frac{1}{2}(x + y)\| < 1\), equivalently, that the boundary of the unit ball of \(X\) does not contain any straight line segment. A Minkowski space \(X\) is smooth [strictly convex] iff \(X^*\) is strictly convex [smooth].

The balancing and collapsing conditions in Theorems 1 and 2 thus occur in a strictly convex space. We say that a finite set of unit vectors \(x_1, \ldots, x_m \in X\) satisfies the balancing condition if
\[
\sum_{i=1}^{m} x_i = 0, \quad (1)
\]
and satisfies the collapsing condition if
\[
\left\| \sum_{i \in J} x_i \right\| \leq 1 \text{ for each } J \subseteq \{1, \ldots, m\}. \quad (2)
\]

Note that the above balancing condition is the characterization of the so-called Fermat point of a set of points in a smooth Minkowski space in the non-absorbing case (i.e. where the Fermat point differs from the given points) in terms of norming functionals, derived in [1].

Theorem 1 (Lawlor and Morgan [14]). Let \(a_1, \ldots, a_m\) be distinct non-zero points in a smooth Minkowski space \(X\). For each \(i = 1, \ldots, m\), let \(a_i^*\) be the norming functional of \(a_i\). Then the tree connecting each \(a_i\) to 0 is an SMT of \(S = \{a_1, \ldots, a_m\}\) iff \(\{a_1^*, \ldots, a_m^*\}\) satisfies the balancing and collapsing conditions in \(X^*\).

Proof. \(\Rightarrow\): Since we have an SMT, for any \(x \in X\)
\[
\sum_{i=1}^{m} \|a_i - x\| \geq \sum_{i=1}^{m} \|a_i\|,
\]
i.e. for any unit vector \(e \in X\) the function
\[
\phi_e(t) := \sum_{i=1}^{m} (\|a_i + te\| - \|a_i\|) \geq 0
\]
attains a minimum at \(t = 0\). For sufficiently small \(t, a_i + te \neq 0\), and \(\phi_e(t)\) is differentiable at 0, with \(\phi_e'(0) = 0\). But
\[
\phi_e'(0) = \lim_{t \to 0} \frac{\sum_{i=1}^{m} \|a_i + te\| - \|a_i\|}{t} = \sum_{i=1}^{m} \langle a_i^*, e \rangle.
\]
Therefore, \( \sum_{i=1}^{m} a_i^* = 0 \).

Secondly, given \( J \subseteq \{1, \ldots, m\} \), define a tree \( T_J \) as follows: Connect \( \{a_i : i \in J\} \) to an arbitrary point \( x \), connect \( \{a_i : i \notin J\} \) to 0, and connect \( x \) to 0. Then the total length of \( T_J \) is not smaller than \( \sum_{i=1}^{m} \|a_i\| : \sum_{i \in J} \|a_i - x\| + \sum_{i \notin J} \|a_i\| + \|x\| \geq m \sum_{i=1}^{m} \|a_i\| \),

i.e. for any unit vector \( e \) the function

\[
\psi_e(t) := \sum_{i \in J} (\|a_i - te\| - \|a_i\|) + |t| \geq 0
\]

attains a minimum at \( t = 0 \). However, \( \psi_e \) is not differentiable at 0. Circumventing this difficulty, we calculate

\[
0 \leq \lim_{t \to 0^+} \frac{\psi_e(t)}{t} = \lim_{t \to 0^+} \sum_{i \in J} \frac{\|a_i - te\| - \|a_i\|}{t} + 1 \]

\[
= \sum_{i \in J} \langle a_i^*, -e \rangle + 1
\]

and \( \langle \sum_{i \in J} a_i^*, e \rangle \leq 1 \) for all unit \( e \). Thus \( \|\sum_{i \in J} a_i^*\|^* \leq 1 \).

\[
\Leftarrow: \quad \text{Let } a_1^*, \ldots, a_m^* \in X^* \text{ satisfy } (1) \text{ and } (2), \text{ and let } T \text{ be any SMT of } \{a_1, \ldots, a_m\}. \text{ We have to show that}
\]

\[
\sum_{\{x, y\} \in E(T)} \|x - y\| \geq \sum_{i=1}^{m} \|a_i\| .
\]

For \( i \geq 2 \), let \( P_i \) be any non-overlapping path in \( T \) from \( a_1 \) to \( a_i \), i.e. \( P_i = x_1^{(i)} x_2^{(i)} \ldots x_k^{(i)} \) with \( x_1^{(i)} = a_1, x_k^{(i)} = a_i, \) and \( \{x_j^{(i)}, x_{j+1}^{(i)}\} \) distinct edges in \( E(T) \) for \( j = 1, \ldots, k_i - 1 \). Note that each edge of \( T \) is used in some \( P_i \), since the union of the paths is a connected subgraph of \( T \). For each edge \( e \in E(T) \) we assign a direction depending on the way \( e \) is traversed in some \( P_i \) containing \( e \). This direction is unambiguous, since if two paths would give conflicting directions, their union would contain a cycle. We denote a directed edge from \( x \) to \( y \) by \( (x, y) = \vec{e} \) and the set of directed edges by \( \vec{E}(T) \). For each \( \vec{e} \in \vec{E}(T) \), let
\[ S_e := \{ i \geq 2 : \vec{e} \in P_i \}. \] Then
\[
\sum_{i=1}^{m} \|a_i\| = \sum_{i=1}^{m} \langle a_i^*, a_i \rangle
\]
\[ = \sum_{i=2}^{m} \langle a_i^*, a_i - a_1 \rangle \quad \text{(by the balancing condition)} \]
\[ = \sum_{i=2}^{m-1} \sum_{j=2}^{k_i-1} \langle a_i^*, x_j^{(i)} - x_j^{(i)} \rangle \]
\[ = \sum_{\vec{e}=(x,y) \in \vec{E}(T)} \sum_{i \in S_e} \langle a_i^*, y - x \rangle \]
\[ \leq \sum_{\vec{e}=(x,y) \in \vec{E}(T)} \left\| \sum_{i \in S_e} a_i^* \right\| \|x - y\| \]
\[ \leq \sum_{(x,y) \in \vec{E}(T)} \|x - y\| \quad \text{(by the collapsing condition)}.
\]

As mentioned in [14], the balancing and collapsing conditions are still sufficient for the tree in the above theorem to be an SMT in non-smooth spaces, if (1) and (2) holds for \( a_i^* \) for each \( a_i \). A similar remark holds for the next theorem.

**Theorem 2.** Given points \( a_1, \ldots, a_m \neq 0 \) in a smooth Minkowski space \( X \), let \( a_i^* \) be the norming functional of \( a_i \). Then the tree connecting each \( a_i \) to 0 is an SMT of \( S = \{ 0, a_1, \ldots, a_m \} \) iff \( \{ a_1^*, \ldots, a_m^* \} \) satisfies the collapsing condition in \( X^* \).

**Proof.** Similar to the proof of the previous theorem. Note that there is no balancing condition, since we cannot perturb 0, as 0 is in this case a vertex of the SMT.

### 3 Upper bounds for smooth Minkowski spaces

For a Minkowski space \( X \), let \( v(X) \) be the largest degree of a vertex of an SMT in \( X \), and \( s(X) \) the largest degree of a Steiner point in an SMT.

In [14] it is shown that \( s(X) \leq d + 1 \) if \( X \) is smooth and \( d \)-dimensional. This inequality is sharp in the sense that there are spaces and SMT’s where the degree of \( d + 1 \) is attained. We give a similar bound for \( v(X) \):

**Theorem 3.** For a smooth Minkowski space \( X \) of dimension \( d \geq 2 \),
\[ 3 \leq s(X) \leq v(X) \leq d + 1. \]

The outer inequalities are sharp in general.

**Proof.** Theorems [1] and [2] immediately imply \( s(X) \leq v(X) \).

In any 2-dimensional subspace of the dual \( X^* \) we can find two unit vectors \( x^*, y^* \) such that \( \|x^* - y^*\|^* = 1 \). Then the set \( \{ x^*, -y^*, y^* - x^* \} \) satisfies (1) and (2).
The euclidean spaces $X = \ell^2_d$ are examples where $s(X) = v(X) = 3$.

The rest of the theorem now follows from Theorem 2 and Theorem 4 below. An example where $v(X) = d + 1$ may be constructed in the same way as for $s(X)$, as is done in [14, Lemma 4.3].

The following theorem, suggested in [21], sharpens results from [6] and [14] by eliminating the balancing condition from the hypotheses.

**Theorem 4.** Let $X$ be a strictly convex $d$-dimensional Minkowski space. If $x_1, \ldots, x_m \in X$ are unit vectors satisfying the collapsing condition, then $m \leq d + 1$. Furthermore, if the balancing condition is not satisfied, i.e. $\sum_{i=1}^m x_i \neq 0$, then $m \leq d$.

**Proof.** Let $x_i^* \in X^*$ be norming functionals of $x_i$. Firstly, for $i \neq j$ we have

$$1 + \langle x_i^*, x_j \rangle = \langle x_i^*, x_i + x_j \rangle \leq \|x_i + x_j\| \leq 1$$

by the collapsing condition, and thus

$$\langle x_i^*, x_j \rangle \leq 0 \text{ for } i \neq j.$$

Secondly,

$$0 \leq \langle x_i^*, -\sum_{j \neq i} x_j \rangle \leq \left\| \sum_{j \neq i} x_j \right\| \leq 1.$$

If $\langle x_i^*, -\sum_{j \neq i} x_j \rangle = 1$, then $x_i^*$ is also a norming functional of $-\sum_{j \neq i} x_j$, which is now a unit vector. Then, since $X$ is strictly convex, it easily follows that $x_i = -\sum_{j \neq i} x_j$.

Thus, if $\sum_{i=1}^m x_i \neq 0$, then

$$0 \leq \langle x_i^*, -\sum_{j \neq i} x_j \rangle < 1,$$

and the diagonal of the matrix $A = [\langle x_i^*, x_j \rangle]_{i,j=1}^m$ majorizes the rows. Thus $A$ is invertible. Since $A$ has rank at most $d$, we obtain $m \leq d$.

If however $\sum_{i=1}^m x_i = 0$, the above argument applied to $x_1, \ldots, x_{m-1}$ gives $m - 1 \leq d$.

Note that in the above proof, we do not nearly use the full force of the collapsing condition.

For the next bound, we recall a notion from the local theory of Banach spaces. The absolutely summing constant or the 1-summing norm (of the identity operator on) a Minkowski space $X$ is defined to be

$$\pi_1(X) := \inf \left\{ c > 0 : \forall x_1, \ldots, x_m \in X : \sum_{i=1}^m \|x_i\| \leq c \max_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^m \epsilon_i x_i \right\| \right\}.$$ 

This notion has been studied extensively; see e.g. [16, 5, 20, 12, 9, 13]. Note that the quantity $(2\pi_1(X))^{-1}$ has also been called the Macphail constant in the literature.

**Theorem 5.** For a smooth Minkowski space $X$,

$$s(X) \leq v(X) \leq 2\pi_1(X^*).$$
Proof. Let \( x_1^*, \ldots, x_m^* \in X^* \) be unit vectors satisfying the collapsing condition, with \( m = v(X) \). Then, for any sequence of signs \( \epsilon_i = \pm 1, i = 1, \ldots, m \) we have \( \| \sum_i \epsilon_i x_i^* \| \leq 2 \), hence

\[
m = \sum_{i=1}^m \| x_i^* \|^* \geq \frac{m}{2} \max_{\epsilon_i = \pm 1} \left\| \sum_i \epsilon_i x_i^* \right\|^*,
\]

implying that \( \frac{m}{2} \leq \pi_1(X^*) \).

It is known that \( \sqrt{d} \leq \pi_1(X) \leq d \) for any \( d \)-dimensional \( X \) \([12]\). We thus obtain an upper bound worse than that of Theorem 3, although it is of the same order. It is however possible in principle to obtain bounds better than that of Theorem 3 for specific spaces. However, we cannot do better than \( 2\sqrt{d} \).

4 Upper bounds for \( \ell^d_p \)

Restricting ourselves to the smooth case \( 1 < p < \infty \), we recall that the dual of \( \ell^d_p \) is \( (\ell^d_p)^* = \ell^d_{p'} \), where \( 1/p + 1/p' = 1 \). We use the Khinchin inequalities with the best constants, due to \([22]\) and \([10, 11]\).

Khinchin’s inequalities. For any \( 1 \leq q < \infty \) there exist constants \( A_q, B_q > 0 \) such that for any \( a_1, \ldots, a_n \in \mathbb{R} \) we have

\[
A_q \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \leq \left( 2^{-n} \sum_{\epsilon_i = \pm 1} \left| \sum_{i=1}^n \epsilon_i a_i \right|^q \right)^{1/q} \leq B_q \left( \sum_{i=1}^n a_i^2 \right)^{1/2}.
\]

For \( q \geq 2 \) we have \( A_q = 1 \), \( B_q = \sqrt{2} (\Gamma(\frac{q+1}{2})/\sqrt{\pi})^{1/q} \), and for \( 1 \leq q \leq 2 \), \( B_q = 1 \),

\[
A_q = \begin{cases} 2^{1/2-1/q} & \text{if } q < q_0, \\ \sqrt{2} (\Gamma(\frac{q+1}{2})/\sqrt{\pi})^{1/q} & \text{if } q \geq q_0, \end{cases}
\]

where \( q_0 \approx 1.8474 \) is defined by \( \Gamma\left(\frac{2q_0+1}{2}\right) = \frac{2^{q_0}}{\sqrt{\pi}} \), \( 1 < q_0 < 2 \).

The following lemma is analogous to \([11]\) Hilfsatz 4. We omit the proof, which easily follows from calculus.

Lemma 6. Let \( x, y \in \mathbb{R} \) and \( 1 \leq q \leq 2 \). Then

\[
|x + y|^q \geq 2^{q-2} (|x|^{q/2} \text{sgn } x + |y|^{q/2} \text{sgn } y)^2.
\]

The earliest reference we could find to the following lemma is Rankin \([17]\).

Lemma 7. Let \( x_1, \ldots, x_m \in \ell^2_{\mathbb{Z}} \) satisfy \( \|x_i\|_2 = 1 \) and \( \langle x_i, x_j \rangle < -1/n \) for \( i \neq j \), where \( n \) is a positive integer. Then \( m \leq n \).

Proof. 

\[
0 \leq \left\| \sum_{i=1}^m x_i \right\|_2^2 = \sum_{i=1}^m \| x_i \|_2^2 + 2 \sum_{i<j} \langle x_i, x_j \rangle \\
< m - m(m-1)/n.
\]
The next two theorems show that the largest degree of a vertex \(v(\ell^d_p)\) and the largest degree of a Steiner point \(s(\ell^d_p)\) in an SMT in \(\ell^d_p\) are both relatively small and independent of \(d\). In particular, for \(p \geq 2\) we have a general upper bound of 7. For \(2 \leq p \leq 3.40942\) we furthermore obtain the exact values of \(v(\ell^d_p)\) and \(s(\ell^d_p)\). For \(p < 2\) we only obtain a lower bound linear in \(p^*\) and an upper bound exponential in \(p^*\). It is not clear what the correct order of growth should be in this case.

**Theorem 8.** Let \(2 \leq p < \infty\) and \(d \geq 3\).

\[
\begin{align*}
\text{Let } p &< \log 8 \Rightarrow \text{for } \langle \cdot, \cdot \rangle \text{ we get for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle. \quad (3) \\
\text{Let } p &< \log 8 \Rightarrow \text{for } \langle \cdot, \cdot \rangle \text{ we get for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle \text{ we obtain for } \langle \cdot, \cdot \rangle. \quad (4) \\
4 \leq s(\ell^d_p) &\leq s(\ell^d_p) \leq 5 \text{ for } \frac{\log 3}{\log 3 - \log 2} \leq p < \frac{\log 4}{\log 4 - \log 3} \approx 4.81884, \quad (5) \\
4 \leq s(\ell^d_p) &\leq s(\ell^d_p) \leq 6 \text{ for } \frac{\log 3}{\log 4 - \log 3} \leq p < \frac{\log 4}{\log 4 - \log 3} \approx 10.3818, \quad (6) \\
4 \leq s(\ell^d_p) &\leq s(\ell^d_p) \leq 7 \text{ for all } p \geq \frac{\log 3}{\log 3 - \log 2}. \quad (7)
\end{align*}
\]

**Proof.** Let \(q := p^*/(p - 1)\). The lower bound of 3 for \(s(X)\) and \(v(X)\) comes from Theorem 3. For \(p \geq (\log 3)/(\log 3 - \log 2)\), i.e. for \(q \leq (\log 3)/(\log 2)\), we obtain 4 unit vectors in \(\ell_q^d\) satisfying the balancing and collapsing conditions as follows:

\[
\begin{align*}
x_1 &:= 3^{-1/q}(1, 1, 1), & x_2 &:= 3^{-1/q}(1, -1, -1), \\
x_3 &:= 3^{-1/q}(-1, 1, -1), & x_4 &:= 3^{-1/q}(-1, -1, 1).
\end{align*}
\]

For the upper bounds, let \(x_1, \ldots, x_m \in \ell_q^d\) be unit vectors satisfying the collapsing condition.

We first use a “twisting” technique used in the Geometry of Numbers; see 18. Denote the coordinates of \(x_i\) as \(x_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,d})\). Define \(\tilde{x}_i = (\tilde{x}_{i,1}, \tilde{x}_{i,2}, \ldots, \tilde{x}_{i,d})\) by \(\tilde{x}_{i,n} := |x_{i,n}|^{1/2} \text{sgn } x_{i,n}\). Note that \(\|\tilde{x}_i\|_2 = 1\), i.e. we have twisted \(x_i\) to become a euclidean unit vector. By Lemma 4 we obtain for \(i \neq j\) that

\[
1 \geq \|x_i + x_j\|_q^q \geq 2^{q - 2} \|\tilde{x}_i + \tilde{x}_j\|_2^2 = 2^{q - 2} (2 + (\tilde{x}_i, \tilde{x}_j)),
\]

where \(\langle \cdot, \cdot \rangle\) denotes the standard euclidean inner product. Thus \(\langle \tilde{x}_i, \tilde{x}_j \rangle \leq 2^{1-q} - 1 < 0\). If \(p < (\log 3)/(\log 3 - \log 2)\), i.e. \(q > (\log 3)/(\log 2)\), then \(2^{1-q} - 1 < -\frac{1}{3}\).

By Lemma 7 we obtain \(m \leq 3\), and (3) follows. Similarly, if \(p < (\log 8 - \log 3)/(\log 4 - \log 3)\), then \(2^{1-q} - 1 < -\frac{1}{3}\), hence \(m \leq 4\), and (4) follows.

For the remaining estimates we apply Khinchin’s inequalities. We may assume in the light of (3) and (4) that \(p \geq (\log 8 - \log 3)/(\log 4 - \log 3)\), i.e. \(q \leq (\log 8 - \log 3)/(\log 2) < \eta_0\). Thus \(A_q = 2^{1/2 - 1/q}\). By (2) we have for any
sequence of signs $\epsilon_i = \pm 1, i = 1, \ldots, m$ that $\|\sum_{i=1}^m \epsilon_i x_i\|_q \leq 2$. Therefore,

$$2^q \geq \sum_{n=1}^d 2^{-m} \left| \sum_{i=1}^m \epsilon_i x_{i,n} \right|^q$$

$$\geq \sum_{n=1}^d A_q^q \left( \sum_{i=1}^m x_{i,n}^2 \right)^{q/2} \quad \text{(Khinchin's inequality)}$$

$$= A_q^q \left( \sum_{n=1}^d \left( \sum_{i=1}^m |x_{i,n}|^q \right)^q \right)^{2/q} \quad \text{(where $\left( |x_{i,n}|^q \right)^m_{i=1} \in \ell_{2/q}^m$)}$$

$$\geq A_q^q \left( \sum_{n=1}^d \|x_{i,n}\|^q \right)^{2/q} \quad \text{(triangle inequality in $\ell_{2/q}^m$)}$$

$$= A_q^q \left( \sum_{i=1}^m \|x_i\|^q \right)^{q/2} = A_q^q m^{n/2},$$

and $m \leq 4/A_q^2 = 2^{3-2/p} < 8$. Estimates (5), (6) and (7) now follow. □

**Theorem 9.** Let $1 < p < 2$ and $d \geq 3$. Then

$$\min(d, f(p^*)) \leq s(\ell_p^d), v(\ell_p^d) \leq \min(d + 1, 2^p^*),$$

where for $q > 2$,

$$f(q) := \max\{d : 2(d - 2)^q + (d - 2)2^q \leq (d - 1)^q + d - 1\}.$$  \hspace{1cm} (8)

In particular,

$$f(q) \geq 3 \text{ for } q > 2,$$

$$f(q) \geq 4 \text{ for } q \geq 3.21067,$$

$$f(q) \geq 5 \text{ for } q \geq 3.40093,$$

$$f(q) \geq \left\lceil q/\log 2 \right\rceil \text{ for } q \geq 3.69247.$$  

**Proof.** Let $q := p^* = p/(p - 1)$.

The upper bound follows from Theorem 3 and an application of Khinchin’s inequalities:

$$2^q \geq \sum_{n=1}^d 2^{-m} \left| \sum_{i=1}^m \epsilon_i x_{i,n} \right|^q$$

$$\geq \sum_{n=1}^d \left( \sum_{i=1}^m x_{i,n}^2 \right)^{q/2} \quad \text{(Khinchin's inequality)}$$

$$= \sum_{n=1}^d \|x_{i,n}\|^q_2$$

$$\geq \sum_{n=1}^d \|x_i\|^q_q = m \quad \text{(monotonicity of } q\text{-norms)}.$$

For the lower bound we may assume that $d \geq 4$. Let $x_i$ be the vector in $\ell_q^d$ with $d - 1$ in its $i$'th coordinate, and $-1$ in the remaining coordinates,
for \( i = 1, \ldots, d \). Let \( \hat{x}_i := \|x_i\|_q^{-1} x_i \). Then \( \{\hat{x}_i : i = 1, \ldots, d\} \) satisfies the balancing condition (1). This set will also satisfy the collapsing condition iff for all \( 2 \leq k \leq d/2 \),

\[
g(k, d, q) := k(d-k)^q + (d-k)k^q \leq (d-1)^q + d - 1 = g(1, d, q).
\]

By differentiating with respect to \( q \) and using \( 2 \leq k \leq d/2 \), it is easily seen that if \( g(k, d, q) \leq g(1, d, q) \) holds for some \( q = q' \), then it will hold for all \( q \geq q' \). The following numerical facts are easily verified:

\[
\begin{align*}
g(2, d, q) &\leq g(2, d, q) \text{ for } 4 \leq d \leq 7, 2 \leq k \leq d/2 \text{ and } p \geq 3.2, \\
g(2, 4, q) &\leq g(1, 4, q) \text{ for } q \geq 3.21066 \ldots, \\
g(2, 5, q) &\leq g(1, 5, q) \text{ for } q \geq 3.40092 \ldots, \\
g(2, 6, q) &\leq g(1, 6, q) \text{ for } q \geq 3.69246 \ldots, \text{ and} \\
g(2, 7, q) &\leq g(1, 7, q) \text{ for } q \geq 4.09345 \ldots.
\end{align*}
\]

It is now sufficient to show for \( d \geq 8 \) and \( q = (d-1) \log 2 \) that \( g(k, d, q) \leq g(2, d, q) \leq g(1, d, q) \) for all \( 2 \leq k \leq d/2 \). Firstly, note that in this case \( g(2, d, q) \leq g(1, d, q) \) is equivalent to

\[
2^{1+(d-1)\log(d-2)} + (d-2)2^{(d-1)\log 2} \leq 2^{(d-1)\log(d-1)} + d - 1,
\]

which is easily verified for \( d \geq 8 \).

Secondly, to show that \( g(k, d, q) \leq g(2, d, q) \) it is sufficient to show that

\[
f(x) := x(1-x)^q + (1-x)x^q, \quad \frac{2}{d} \leq x \leq \frac{1}{2}
\]

attains its maximum at \( x = \frac{2}{d} \). To see this, it is in turn sufficient to show that \( f'(x) \leq 0 \) for \( 2/d \leq x \leq 1/2 \). By setting \( y = (1-x)/x \) we find that it is sufficient to show that for \( 1 \leq y \leq d/2 - 1 \),

\[
x^{-q}f'(x) = y^q - qy^{q-1} - 1 + qy =: h(y) \leq 0.
\]

By calculating the first and second derivatives of \( h(y) \) and recalling that \( q > 3 \), it is seen that \( h(y) \) does not attain its maximum if \( 1 < y < d/2 - 1 \). Since \( h(1) = 0 \), we only have to show that \( h(d/2 - 1) \leq 0 \), which easily follows from \( q \geq 4 \) and \( d \geq 8 \).

\[\square\]

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**References**

[1] G. D. Chakerian and M. A. Ghandehari, The Fermat Problem in Minkowski Spaces, *Geom. Dedicata* 17 (1985), 227–238.

[2] D. Cheriton and R. E. Tarjan, Finding minimum spanning trees, *SIAM J. Comput.* 5 (1976), 724–742.
[3] D. Cieslik, Knotengrade kürzester Bäume in endlichdimensionalen Banachräumen, *Rostock Math. Kolloq.* **39** (1990), 89–93.

[4] J. G. van der Corput and G. Schuake, Anwendung einer Blichfeldtschen Beweismethode in der Geometrie der Zahlen, *Acta Arithmetica* **2** (1937), 152–160.

[5] A. Dvoretzky and C. A. Rogers, Absolute and unconditional convergence in normed linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **36** (1950), 192–197.

[6] Z. Füredi, J. C. Lagarias and F. Morgan, Singularities of minimal surfaces and networks and related extremal problems in Minkowski space, in *Discrete and Computational Geometry*, (J. E. Goodman, R. Pollack, and W. Steiger, eds.), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 6, Amer. Math. Soc., Providence, RI, 1991, pp. 95–109.

[7] M. R. Garey, R. L. Graham and D. S. Johnson, The complexity of computing Steiner minimal trees, *SIAM J. Appl. Math.* **32** (1977), 835–839.

[8] M. R. Garey and D. S. Johnson, The rectilinear Steiner tree problem is NP-complete, *SIAM J. Appl. Math.* **32** (1977), 826–834.

[9] D. J. H. Garling and Y. Gordon, Relations between some constants associated with finite dimensional Banach spaces, *Israel J. Math.* **9** (1971), 346–361.

[10] U. Haagerup, Les meilleurs constantes de l’inégalité de Khintchine, *C. R. Acad Sci. Paris* **A286** (1978), 259–262.

[11] U. Haagerup, The best constants in the Khintchine inequality, *Studia Math.* **70** (1982), 231–283.

[12] M. I. Kadets and M. G. Snobar, Certain functionals on the Minkowski compactum, *Mat. Zametki* **10** (1971), 453–457 (Russian). English transl. *Math. Notes* **10** (1971), 694–696.

[13] H. König and N. Tomczak-Jaegermann, Bounds for projection constants and 1-summing norms, *Trans. Amer. Math. Soc.* **320** (1990), 799–823.

[14] G. R. Lawlor and F. Morgan, Paired calibrations applied to soap films, immiscible fluids, and surfaces and networks minimizing other norms, *Pacific J. Math.* **166** (1994), 55–82.

[15] R. F. Love and J. G. Morris, Modelling inter-city road distances by mathematical functions, *J. Oper. Res. Soc.* **23** (1972), 61–71.

[16] M. S. Macphail, Absolute and unconditional convergence, *Bull. Amer. Math. Soc.* **53** (1947), 121–123.

[17] R. A. Rankin, On the closest packing of spheres in $n$ dimensions, *Ann. of Math.* **48** (1947), 1062–1081.

[18] R. A. Rankin, On sums of powers of linear forms I, II, *Ann. of Math.* **50** (1949), 691–704.
[19] G. Robins and J. S. Salowe, Low-degree minimum spanning trees, *Discrete Comput. Geom.* **14** (1995), 151–165.

[20] D. Rutovitz, Some parameters associated with finite dimensional Banach spaces, *J. London Math. Soc.* **40** (1965), 241–255.

[21] K. J. Swanepoel, Extremal problems in Minkowski space related to minimal networks, *Proc. Amer. Math. Soc.* **124** (1996), 2513–2518.

[22] S. J. Szarek, On the best constants in the Khintchine inequality, *Studia Math.* **58** (1976), 197–208.

[23] A. C. Thompson, Minkowski Geometry, Encyclopedia of Mathematics and its Applications 63, Cambridge University Press, 1996.