HYPERBOLIC COMPONENTS AND CUBIC POLYNOMIALS

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Abstract. In the space of cubic polynomials, Milnor defined a notable curve $S_p$, consisting of cubic polynomials with a periodic critical point, whose period is exactly $p$. In this paper, we show that for any integer $p \geq 1$, any bounded hyperbolic component on $S_p$ is a Jordan disk.

1. Main Theorem

Polynomial maps $f : \mathbb{C} \to \mathbb{C}$, viewed as dynamical systems, yield complicated behaviors under iterations. Their bifurcations, both in the dynamical plane and in the parameter space, are the major attractions in the field of complex dynamics in recent thirty years.

To start, let $P(d)$ be the space of monic and centered polynomials of degree $d \geq 2$. The connectedness locus $C(d)$ consists of $f \in P(d)$ whose Julia set is connected, while the shift locus $S(d)$ consists of maps in $P(d)$ for which all critical orbits escape to infinity under iterations. The topology of $C(d)$ and $S(d)$ attracts lots of people. A well-known theorem states that $C(d)$ is compact and cellular, this is proven by Douady-Hubbard [DH1] for $d = 2$, Branner-Hubbard [BH1] for $d = 3$, and Lavaurs [L], DeMarco-Pilgrim [DP1] for the general cases. On the other hand, the shift locus $S(d)$ is studied from a different viewpoint. It’s fundamental group is studied by Blanchard, Devaney and Keen [BDK], while its simplicial structure and bifurcations are studied comprehensively by DeMarco and Pilgrim [DP1,DP2,DP3].

When $d = 3$, more amazing structure of the parameter space are revealed. Pioneering work of Branner and Hubbard [BH1] exhibited two dynamically meaningful solid tori with linking number 3 in the parameter space. Meanwhile, it is observed by Milnor, and proven by Lavaurs [L] that $C(3)$ is not locally connected. Later, Epstein and Yampolski [EY] proved the existence of products of the Mandelbrot set in $C(3)$. These striking results indicate the complexity of the cubic polynomial space.

In their papers [BH1,BH2], Branner and Hubbard used the following form

$$f_{c,a}(z) = z^3 - 3c^2z + a,$$

Date: September 11, 2018.

2010 Mathematics Subject Classification. Primary 37F45; Secondary 37F10, 37F15.

Key words and phrases. hyperbolic component, periodic curve, cubic polynomials.
where \((c, a) \in \mathbb{C}^2\) is a pair of parameters. This critically marked form are widely adopted by the followers. Note that \(f_{c,a}\) has two critical points \(\pm c\).

To study the parameter space of \(f_{c,a}\), Milnor suggested to study the one dimensional slices, and defined a kind of notable slices called the critically-periodic curves \(S_p, p \geq 1\). The curve \(S_p\) consists of \((c, a) \in \mathbb{C}^2\) for which the critical point \(c\) has period exactly \(p\) under iterations of \(f_{c,a}\):

\[
S_p = \{(c, a) \in \mathbb{C}^2; f^p_{c,a}(c) = c \text{ and } f^k_{c,a}(c) \neq c, \forall 1 \leq k < p\}.
\]

Milnor showed that \(S_p\) is a smooth affine algebraic curve, and asked whether it is irreducible. A proof of irreducibility is recently announced by Arfeux and Kiwi [AK].

The curve \(S_p\) has a remarkable topology. It is known that \(S_1\) is biholomorphic to the \(\mathbb{C}\), \(S_2\) is biholomorphic to the punctured plane \(\mathbb{C}^*\), \(S_3\) has genus one with 8 punctures (see Figure 1), \(S_4\) has genus 15 with 20 punctures. Both the genus \(g_p\) and the number \(N_p\) of punctures of \(S_p\) grow exponentially with \(p\). Bonifant, Kiwi and Milnor [BKM] proved that the Euler characteristic of \(S_p\) (without assuming its irreducibility) is given by

\[
\chi(S_p) = (2 - p)d(p),
\]

where \(d(p)\) is the degree of \(S_p\), satisfying the formula: \(\sum_{n|p} d(n) = 3^{p-1}\). The genus \(g_p\) and the number \(N_p\) of punctures of \(S_p\) have no explicit formulas. An algorithm to compute \(N_p\) (hence also \(g_p\)) is designed by DeMarco and Schiff [DS], building on previous work of DeMarco and Pilgrim [DP3]. By pluripotential methods, Dujardin [Du] showed that

\[
\lim_{p \to +\infty} \frac{\chi(S_p) + N_p}{3^p} \to -\infty.
\]

Assuming the irreducibility [AK], the above behavior implies that the genus \(g_p\) actually grows faster than \(3^p\).

\[
\text{Figure 1. } S_3 \text{ has a non-trivial topology. It is a complex torus with 8 punctures.}
\]
In this article, we study the *bifurcations of dynamical systems* on the algebraic curve $S_p$, see Figure 2. Precisely, the bifurcations on the boundary of stable regions are the main focus. Here, ‘stable’ refers to *hyperbolic*. Recall that, a rational map $f$ is *hyperbolic* if all the critical points are attracted by the attracting cycles. In a holomorphic family of rational maps, hyperbolic maps form an open (and conjecturally dense) subset, each component is called a *hyperbolic component*.

**Figure 2.** Bifurcations on $S_1$ (left) and $S_2$ (right).

In this paper, we establish the following

**Theorem 1.1.** For any integer $p \geq 1$, any bounded hyperbolic component on $S_p$ is a Jordan disk.

Theorem 1.1 is among one of several conjectural pictures of $S_p$, proposed by Milnor [M2, p.13], [M3]. The case $p = 1$ is proven independently by Faught [F] and Roesch [R]. The main analytical tool in their proof is the so-called *para-puzzle* technique, whose philosophy, as interpreted by Douady, is: sowing in the dynamical plane and harvesting in the parameter space. However, the para-puzzle technique loses its power when dealing with the parameter space with a complicated topology. Since the topology of $S_p$ is far beyond understanding when $p$ is large, this makes a tough enemy of the para-puzzles.

Instead of using para-puzzle technique, our approach makes the most of the dynamical puzzles, combinatorial rigidity, and holomorphic motion theory. Our arguments are *local*, this makes our techniques being powerful and serve as a model to study bifurcations on more general algebraic curves: those defined by critical relations, in any critically marked polynomial space.

The strategy and organization of the proof is as follows:

A classification and dynamical parameterization of hyperbolic components, due to Milnor, is given in Section 2. Then some basics of dynamical rays are recalled in Section 3.
Section 4 solves one main technical difficulty in the Branner-Hubbard-Yoccoz puzzle theory: finding a puzzle with a non-degenerate critical annulus. The idea is to discuss the relative position of the critical orbits with respect to the candidate graphs, the proof logics has an independent interest.

In Section 5, we prove the combinatorial rigidity for maps on $S_p$. The ideas and methods for quadratic polynomials [H, L, M5] can not work here. We take advantage of recent development [AKLS, KL1, KL2, KSS, KS] in deriving rigidity phenomenon to treat our situations.

We then prove Theorem 1.1 for two types of hyperbolic components in Section 6 using rigidity and characterization of boundary maps. Finally, we deal with the capture type hyperbolic components in Section 7. Instead of using rigidity there, we make the best use of holomorphic motion theory.

Theorem 1.1 then follows from Theorems 2.1(4), 6.1 and 7.1. Note that its statement is complete because unbounded hyperbolic components on $S_p$ are not Jordan disks, see [BKM]. To the author’s knowledge, Theorem 1.1 is the first complete description of boundary regularity of stable regions whose parameter space has a non-trivial topology.

Acknowledgement The author thanks Yongcheng Yin for helpful discussions on polynomials rays and puzzles. The research is supported by NSFC.

2. Hyperbolic components

In this paper, for any $(c, a) \in S_p$, any point $z$ in the Fatou set of $f_{c,a}$, let $U_{c,a}(z)$ be the Fatou component containing $z$. Let’s use the notations

$$\mathcal{B}_{c,a} = \{U_{c,a}(f_{c,a}^k(c)); 0 \leq k < p\},$$

$$\mathcal{A}_{c,a} = U_{c,a}(c) \cup U_{c,a}(f_{c,a}(c)) \cup \ldots \cup U_{c,a}(f_{c,a}^{p-1}(c)).$$

The boundary of each $V \in \mathcal{B}_{c,a}$, according to the work of Roesch and Yin [RY], is a Jordan curve. The connectedness locus of $S_p$ is denoted by $C(S_p)$. For any $z$, let $\text{orb}(z) = \{f_{c,a}^k(z); k \in \mathbb{N}\}$ be the set of forward orbit of $z$.

According to Milnor [M3], there are four types of bounded hyperbolic components on $S_p$ (see Figures 3 and 4):

**Type A** *(Adjacent critical points)*, with both critical points in the same periodic Fatou component.

**Type B** *(Bitransitive)*, with two critical points in different Fatou components belonging to the same periodic cycle.

**Type C** *(Capture)*, with just one critical point in the cycle of periodic Fatou components. The orbit of the other critical point must eventually land in (or be captured by) this cycle.

**Type D** *(Disjoint attracting orbits)*, with two distinct attracting periodic orbits, each of which necessarily attracts just one critical orbit.

All these four types of hyperbolic components admit the following natural dynamical parameterizations, due to Milnor [M3, Lemma 6.8]. This serves as the first step to study the boundaries of hyperbolic components.
Theorem 2.1. Let $\mathcal{H}$ be a hyperbolic component in $C(S_p)$ of Type-$\omega$.

1. If $\omega = A$, i.e. $-c \in U_{c,a}(c)$, then the map
   \[ \Phi : \mathcal{H} \to \D, \ (c, a) \mapsto B_{c,a}(-c) \]
   is a double cover ramified at a single point, where $B_{c,a}$ is the Böttcher map of $f^p_{c,a}$ defined in a neighborhood of $c$.

2. If $\omega = B$, i.e. $-c \in U_{c,a}(f^l_{c,a}(c))$ for some $1 \leq l < p$, then the map
   \[ \Phi : \mathcal{H} \to \D, \ (c, a) \mapsto B_{c,a}(-c) \]
   is a triple cover ramified at a single point, where $B_{c,a}$ is the Böttcher map of $f^p_{c,a}$ defined in a neighborhood of $f^l_{c,a}(c)$.

3. If $\omega = C$, i.e. $f^l_{c,a}(-c) \in A_{c,a}$ for some smallest integer $l > 0$, then
   \[ \Phi : \mathcal{H} \to \D, \ (c, a) \mapsto B_{c,a}(f^l_{c,a}(-c)) \]
is a conformal isomorphism, where $B_{c,a}$ is the Böttcher map of $f_{c,a}^p$ defined in $U_{c,a}(f_{c,a}^p(-c))$.

4. If $\omega = D$, let $z_{c,a} \in U_{c,a}(-c)$ be the attracting point with period say $q$, then the multiplier map

$$\rho : \mathcal{H} \to \mathbb{D}, \ (c,a) \mapsto (f_{c,a}^q)'(z_{c,a})$$

is a conformal isomorphism. In this case, $\rho$ can be extended to a homeomorphism $\rho : \overline{\mathcal{H}} \to \mathbb{D}$ (implying that $\partial \mathcal{H}$ is a Jordan curve).

We remark that for each $p \geq 1$, there are only finitely many Type-$A$, $B$ components on $\mathcal{S}_p$, but there are infinitely many Type-$C$ or $D$ components.

By Theorem 2.1, for a Type-$\omega \in \{A, B\}$ component $\mathcal{H}$, one has

$$\text{deg}(\Phi) = \text{deg}(f_{c,a}^p|_{U_{c,a}(c)}) - 1,$$

where $f_{c,a}$ is a representative map in $\mathcal{H}$. The above number depends only on the type $\omega \in \{A, B\}$, not the specific component $\mathcal{H}$. For this, we write $d_{\omega} = \text{deg}(\Phi)$. Clearly $d_A = 3, d_B = 2$. The notation $d_{\omega}$ will be used later.

3. Dynamical rays

We introduce the dynamical rays in this section, as a preparation for further discussions. These materials are standard in polynomial dynamics.

3.1. Dynamical internal rays. There are finitely many maps on $\mathcal{S}_p$, for which $-c$ meets the orbit of $c$ [M3, Lemma 5.8]. Let

$$\mathcal{S}'_p = \mathcal{S}_p - \{(c,a) : f_{c,a}^k(c) = -c \text{ for some } 0 < k \leq p\}.$$

Let $(c,a) \in \mathcal{S}'_p$. For any $V \in \mathcal{B}_{c,a}$, its Green function $G_{c,a}^V : V \to [-\infty, 0)$ is defined by

$$G_{c,a}^V(z) = \lim_{n \to +\infty} 2^{-n} \log |f_{c,a}^n(z) - w|,$$

where $w \in \text{orb}(c) \cap V$. One may verify that

$$G_{c,a}(V) \circ f_{c,a} = \begin{cases} 2G_{c,a}^V, & \text{if } V = U_{c,a}(c), \\ G_{c,a}^V, & \text{if } V = U_{c,a}(f_{c,a}^k(c)), 1 \leq k < p. \end{cases}$$

The locus $(G_{c,a}^V)^{-1}(\ell) = \{z \in V ; G_{c,a}^V(z) = \ell\}$ with $\ell < 0$ is called an equipotential curve in $V$. The internal rays are defined as follows.

If $f_{c,a}$ is hyperbolic and $-c \in U := U_{c,a}(f_{c,a}^l(c))$ for some $0 \leq l < p$, then the Böttcher map $B_{c,a}^U$ of $f_{c,a}^p$ is defined in a neighborhood of $f_{c,a}^l(c)$. For any $t \in \mathbb{S}$, the set $R_{c,a}^U(t)$ in $U$ is defined as the orthogonal trajectory (possibly bifurcates) of the equipotential curves, starting from $f_{c,a}^l(c)$ and containing $(B_{c,a}^U)^{-1}((0, e^{2\pi i t}))$ for some $\epsilon \in (0, 1)$. By conformal pushing forward or pulling back via some iterations of $f_{c,a}$, we can define $R_{c,a}^W(t)$ for any $W \in \mathcal{B}_{c,a} - \{U\}$. The set $R_{c,a}^U(t)$ or $R_{c,a}^W(t)$ is called an internal ray if it does not bifurcate, namely $2^n t \neq \arg B_{c,a}^U(-c)$ for any $n \in \mathbb{N}$. The pulling
back procedure allows one to define internal rays in any Fatou component whose orbit meets $U$.

In all other situations, set $V = U_{c,a}(c)$. The Böttcher map $B_{c,a}^V$ of $f_{c,a}^p$ can be defined in $V$, and the internal ray $R_{c,a}^V(t) = (B_{c,a}^V)^{-1}((0,1)e^{2\pi it})$, $\forall t \in \mathbb{S}$.

By conformal pulling back $R_{c,a}^V(t)$ via iterations of $f_{c,a}$, one can define the internal ray $R_{c,a}^{\ell V}(t)$ in any Fatou component $U(\neq V)$ whose orbit meets $V$.

One may verify that

$$f_{c,a}(R_{c,a}^V(t)) = \begin{cases} R_{c,a}^{f_{c,a}(V)}(2t), & \text{if } V = U_{c,a}(c), \\ R_{c,a}^{f_{c,a}(V)}(t), & \text{if } V = U_{c,a}(f_{c,a}(c)), 1 \leq k < p. \end{cases}$$

3.2. Dynamical external rays. For any $(c, a) \in S_p$, let $A^\infty_{c,a}$ be the basin of $\infty$ for $f_{c,a}$. Near $\infty$, the Böttcher map $B^\infty_{c,a}$ is defined as

$$B^\infty_{c,a}(z) = \lim_{n \to +\infty} 3^n \sqrt[n]{f_{c,a}^n(z)}.$$

The Böttcher map $B^\infty_{c,a}$ is unique if we require that it is asymptotic to the identity map at $\infty$. It satisfies $B^\infty_{c,a}(f_{c,a}(z)) = B^\infty_{c,a}(z)^3$ when $|z|$ is large.

The Green function $G^\infty_{c,a}: A^\infty_{c,a} \to (0, +\infty)$ is defined by

$$G^\infty_{c,a}(z) = \lim_{n \to +\infty} 3^{-n} \log |f_{c,a}^n(z)|.$$  

Each locus $(G^\infty_{c,a})^{-1}(\ell) = \{z \in A^\infty_{c,a}; G^\infty_{c,a}(z) = \ell\}$ with $\ell > 0$ is called an equipotential curve. For $t \in \mathbb{R}/\mathbb{Z}$, the set $R^\infty_{c,a}(t)$ is the orthogonal trajectory (possibly bifurcates) of the equipotential curves, starting from $\infty$ and containing $(B^\infty_{c,a})^{-1}((R, \infty)e^{2\pi it})$ for some $R > 0$. It is called an external ray if it does not bifurcate. Clearly $f_{c,a}(R_{c,a}^\ell(t)) = R_{c,a}^\ell(3t)$.

3.3. Continuity of dynamical rays. If an internal (or external) ray lands at a repelling point, then they satisfy the following local stability property:

**Lemma 3.1.** Let $(c_0, a_0) \in S_p^*$ so that the dynamical ray $R^\varepsilon_{c_0,a_0}(\theta)$ with $\varepsilon \in B_{c,a} \cup \{\infty\}$ lands at a repelling periodic point $p_{c_0,a_0}$. Then there is a neighborhood $U \subset S_p^*$ of $(c_0, a_0)$ such that for all $(c, a) \in U$,

1. the set $R^\varepsilon_{c,a}(\theta)$ is a ray landing at a repelling periodic point, and

2. the closure $\overline{R^\varepsilon_{c,a}(\theta)}$ moves continuously in Hausdorff topology with respect to $(c, a) \in U$.

**Proof.** We only prove the result for external rays, the argument is similar for internal rays. The idea is to cut the external ray $R^\infty_{c_0,a_0}(\theta)$ into two parts: one near $\infty$ and the other near the repelling point $p_{c_0,a_0}$. Each part moves continuously w.r.t parameters. This implies that, after gluing them together, the external ray itself moves continuously. Here is the detail:

There exist a neighborhood $U$ of $(c_0, a_0)$ and a large number $R > 1$ such that for all $(c, a) \in U$, the Böttcher map $B^R_{c,a}$ is defined in $U^R_{c,a} = \{z \in A^\infty_{c,a}; G^\infty_{c,a}(z) > \log R\}$, and $B^\infty_{c,a}: U^R_{c,a} \to \{w \in \mathbb{C}; |w| > R\}$ is conformal.
For all $t \in \mathbb{R}/\mathbb{Z}$, $(c, a) \in U$ and $k \in \mathbb{N}$, let
\[
L_{c,a}^0(t) = (B_{c,a}^\infty)^{-1}((R, -\infty) e^{2\pi it})
\]
and $L_{c,a}^k(t)$ be the component of $f_{c,a}^{-k}(L_{c,a}^0(3^k t))$ containing $L_{c,a}^0(t)$.

We may shrink $U$ if necessary so that

- after perturbation in $U$, the $f_{c_0,a_0}$-repelling periodic point $p_{c_0,a_0}$ becomes an $f_{c,a}$-repelling periodic point $p_{c,a}$, and
- there is a large integer $s$ (independent of $(c, a) \in U$) so that $E_{c,a} := L_{c,a}^{s+1}(\theta) \setminus L_{c,a}^{a}(\theta)$ is included in a linearized neighborhood $Y_{c,a}$ of $p_{c,a}$.
- $−c \notin \bigcup_{k \geq 0} L_{c,a}^{s+1}(3^k \theta)$ for all $(c, a) \in U$.

Note that $\theta$ is periodic under the angle tripling map $t \mapsto 3t$ (mod $\mathbb{Z}$). Let $l$ be its period. Since the inverse $h = (f_{c,a}^l|_{Y_{c,a}})^{-1}$ is contracting, the arc
\[
T_{c,a} = \bigcup_{k \geq 0} h^k(T_{c,a}^-)
\]
moves continuously with respect to $(c, a) \in U$.

Note that neither $E_{c,a}$ nor $T_{c,a}$ meets the backward orbit of $−c$. Hence the set $R_{c,a}^\infty(\theta)$ defines an external ray.

Finally, the continuity of $(c, a) \mapsto R_{c,a}^\infty(\theta)$ follows from the fact that $R_{c,a}^\infty(\theta) = L_{c,a}^{s+1}(\theta) \cup T_{c,a}$ and the continuity of $L_{c,a}^{s+1}(\theta)$ and $T_{c,a}$.

\section*{3.4. Intersection of attracting components.}

\begin{proposition}
Let $(c, a) \in S_p$, and $V_1, V_2 \in \mathcal{B}_{c,a}$ with $V_1 \neq V_2$. If $\partial V_1 \cap \partial V_2 \neq \emptyset$, then $\partial V_1 \cap \partial V_2$ is a singleton $\{q\}$, satisfying that $f_{c,a}^p(q) = q$.
\end{proposition}

\begin{proof}
Let $U$ be the unbounded component of $\mathbb{C} - \overline{V_1} \cup \overline{V_2}$, and $B = \mathbb{C} - \overline{U}$. Clearly, $V_1 \cup V_2 \subset B$. If $\partial V_1 \cap \partial V_2$ contains at least two points, then $B \cap J(f_{c,a}) \neq \emptyset$. Let’s consider the iterations $\{f_{c,a}^n|_{\partial B}\}_{n \geq 1}$. Note that the iterations $\{f_{c,a}^n|_{\partial B}\}_{n \geq 1}$ is uniformly bounded. By the maximum principle, the iterations $\{f_{c,a}^n|_{\partial B}\}_{n \geq 1}$ is uniformly bounded too. By Montel’s theorem, $\{f_{c,a}^n|_{\partial B}\}_{n \geq 1}$ is a normal family, implying that $B$ is contained in the Fatou set. This contradicts $B \cap J(f_{c,a}) \neq \emptyset$.

If $q \in \partial V_1 \cap \partial V_2$, then obviously $f_{c,a}^p(q) \in \partial V_1 \cap \partial V_2$. Above argument shows that $\partial V_1 \cap \partial V_2$ consists of a singleton, implying that $f_{c,a}^p(q) = q$. $\square$
\end{proof}

\begin{remark}
Assume $-c \notin \mathcal{A}_{c,a}$, and $\partial V_1 \cap \partial V_2 = \{q\}$ for $V_1, V_2 \in \mathcal{B}_{c,a}$. Then $q$ is the common landing point of the internal rays $R_{c,a}^{\frac{s}{3}}(0)$ and $R_{c,a}^{\frac{3s}{3}}(0)$.
\end{remark}

\section*{4. Branner-Hubbard-Yoccoz Puzzle}

In this section, we first introduce some basic definitions for the Branner-Hubbard-Yoccoz Puzzle theory. Among these definitions, the most important one (for this section) is the admissible puzzle. The main result here is to show the existence of admissible puzzles for maps on $S_p$ (Theorem 4.2).
4.1. **Definitions.** Let $X, X'$ be open subsets of $\hat{\mathbb{C}}$, each is bounded by finitely many Jordan curves, such that $X' \subset X \neq \hat{\mathbb{C}}$. A proper holomorphic map $f : X' \to X$ is called a rational-like map. We denote by $\deg(f)$ the topological degree of $f$ and by $K(f) = \bigcap_{n \geq 0} f^{-n}(X)$ the filled Julia set, by $J(f) = \partial K(f)$ the Julia set. The set of critical points on $K(f)$ is denoted by $C(f)$. A rational-like map $f : X' \to X$ is called polynomial-like if $X, X'$ are Jordan disks and $K(f)$ is connected.

A finite graph $\Gamma \subset X$ is called a puzzle of $f$ if it satisfies the conditions: $\partial X \subset \Gamma$, $f(\Gamma \cap X') \subset \Gamma$, and the orbit of each critical point of $f$ avoids $\Gamma$.

The puzzle pieces $P_n$ of depth $n \geq 0$ are the connected components of $f^{-n}(X \setminus \Gamma)$, and the one containing the point $z$ is denoted by $P_n(z)$. Let

$$\Gamma_\infty = \bigcup_{k \geq 0} f^{-k}(\Gamma).$$

For any $z \in J(f) - \Gamma_\infty$, the puzzle piece $P_n(z)$ is well defined for all $n \geq 0$. In this case, let $P_n^*(z) = \overline{P_n(z)}$. For $z \in J(f) \cap \Gamma_\infty$, let $P_n^*(z) = \bigcup P_n$, where the union is taken for those $P_n$ satisfying that $z \in \partial P_n$. The impression $\text{Imp}(z)$ of $z$ is defined by

$$\text{Imp}(z) = \bigcap_{n \geq 0} P_n^*(z).$$

For any $z \in J(f) - \Gamma_\infty$, the tableau $T_f(z)$ is the two-dimensional array $(P_{n,l}(z))_{n,l \geq 0}$ with $P_{n,l}(z) = P_n(f^l(z))$. The tableau $T_f(z)$ is called periodic if there is an integer $k \geq 1$ such that $P_n(z) = f^k(P_{n+k}(z))$ for all $n \geq 0$. Otherwise, $T_f(z)$ is said to be aperiodic. For $n, l \geq 0$, we say the position $(n, l)$ of $T_f(z)$ is critical if $P_{n,l}(z)$ contains some critical point $c \in C(f)$ (in this case, we say $(n, l)$ is $c$-critical). We say the tableau $T_f(z)$ is non-critical if there exists an integer $n_0 \geq 0$ such that $(n_0, j)$ is not critical for all $j > 0$. Otherwise $T_f(z)$ is called critical.

All the tableaus satisfy the following two rules $\text{[BH2][M5]}$:

**R1.** If $P_{n,l}(z) = P_{n}(z')$, then $P_{i,j}(z) = P_{i,j}(z')$ for all $0 \leq i + j \leq n$.

**R2.** Let $c \in C(f)$. Assume $T_f(c)$ and $T_f(z)$ satisfy

(a) $P_{n+1-l,l}(c) = P_{n+1-l}(c')$ for some $c' \in C(f)$ and $n > l > 0$, and $P_{n-l,i}(c)$ contains no critical points for $0 < i < l$.

(b) $P_{n,m}(c) = P_n(c)$ and $P_{n+1,m}(c) \neq P_{n+1}(c)$ for some $m > 0$. Then $P_{n+1-l,m+l}(z) \neq P_{n+1-l}(c')$.

We say the forward orbit of $x$ combinatorially accumulates to $y$, written as $x \overset{f}{\to} y$, if for any $n > 0$, there exists $j > 0$ such that $y \in P_{n,j}(x)$, i.e. $f^j(P_{n+j}(x)) = P_n(y)$. It is clear that if $x \overset{f}{\to} y$ and $y \overset{f}{\to} z$, then $x \overset{f}{\to} z$.

An aperiodic tableau $T_f(c)$ with $c \in C(f)$ is said to be recurrent if $c \overset{f}{\to} c$. Otherwise $T_f(c)$ is called non-recurrent.

For two critical puzzle pieces, we say that $P_{n+k}(c')$ is a child of $P_n(c)$ if $f^{k-1}(P_{n+k-1}(c')) \to P_n(c)$ is a conformal map.
Assume that \( T_f(c) \) is recurrent, let’s define
\[
[c]_f = \{c' \in C(f); c \xrightarrow{\ell} c' \text{ and } c' \xrightarrow{\ell} c\}.
\]
We say that \( T_f(c) \) is persistently recurrent if for any \( c_1 \in [c]_f \) and any \( n \geq 0 \), the piece \( P_n(c_1) \) has only finitely many children. Otherwise, \( T_f(c) \) is said to be reluctantly recurrent.

**Definition 4.1** (Admissible puzzle). Let \( \ell \geq 1 \) be an integer, a puzzle \( \Gamma \) is said \( \ell \)-admissible for \( f \) if it satisfies the conditions:

1. for each \( c \in C(f) \), there is an integer \( d_c \geq 0 \) with \( \overline{P_{d_c+\ell}}(c) \subseteq P_{d_c}(c) \).
2. all periodic points on \( \Gamma \cap J(f) \) are repelling, and
3. each puzzle piece is a Jordan disk.

By definition, an \( \ell \)-admissible puzzle is always \( \ell' \)-admissible, where \( \ell' \geq \ell \).

The existence of an admissible puzzle, when combining with analytic techniques, leads to significant properties of the map \( f \) (e.g. local connectivity of Julia set, rigidity, see Section 5). Our task in next subsection is to show the existence of admissible puzzles for most maps on \( S_p \).

### 4.2. Cubic polynomials.

Define \( \mathcal{C}_0(S_p) \subset \mathcal{C}(S_p) \) by
\[
\mathcal{C}_0(S_p) = \{(c,a) \in \mathcal{C}(S_p); f_{c,a}^{(k)}(-c) \notin \mathcal{A}_{c,a} \text{ for any } k \in \mathbb{N}\}.
\]
Let \( f_{c,a} \in \mathcal{C}_0(S_p) \) and
\[
X_{c,a} = \mathbb{C} \setminus \left( (G_{c,a}^\infty)^{-1}([1,\infty)) \cup \bigcup_{V \in B_{c,a}} (G_{c,a}^V)^{-1}((-\infty,-1]) \right).
\]

Obviously, the set \( X'_{c,a} := f_{c,a}^{-p}(X_{c,a}) \) satisfies \( \overline{X'_{c,a}} \subset X_{c,a} \).

Let \( V = U_{c,a}(q) \) for \( q \in \{f_{c,a}^{(k)}; 0 \leq k < p\} \), and \( \tau \) be the angle doubling map. Given a \( \tau \)-(pre-)periodic angle \( \theta \), let \( \zeta(V,\theta) \) be the landing point of the internal ray \( R_{c,a}^V(\theta) \). The point \( \zeta(V,\theta) \) is either (pre-)repelling or (pre-)parabolic and hence it is also the landing point of finitely many external rays (See [MI], Theorems 18.10 and 18.11), say \( R_{c,a}^\infty(\alpha_1), \ldots, R_{c,a}^\infty(\alpha_m) \). If \( \zeta(V,\theta) \) is periodic and repelling, then these external rays are all periodic with the same period.

We define:
\[
R_{c,a}^\theta = R_{c,a}^V(\theta) \cup R_{c,a}^\infty(\alpha_1) \cup \cdots \cup R_{c,a}^\infty(\alpha_m),
\]
\[
\gamma_{c,a}(\theta) = \bigcup_{k \geq 0} f_{c,a}^{(k)}(R_{c,a}^\theta).
\]

Clearly, when \( \theta \) is \( \tau \)-periodic, the graph \( \gamma_{c,a}(\theta) \) satisfies \( f_{c,a}(\gamma_{c,a}(\theta)) = \gamma_{c,a}(\theta) \). Given two rational angles \( \theta_1 \neq \theta_2 \), let \( S_q(\theta_1, \theta_2) \) be the component of \( \mathbb{C} \setminus (R_{c,a}^\theta(\theta_1) \cup R_{c,a}^\theta(\theta_2)) \) containing the internal rays \( R_{c,a}^V(t) \) with \( \theta_1 \leq t \leq \theta_2 \). Let \( S_q^*(\theta_1, \theta_2) = \mathbb{C} \setminus S_q(\theta_2, \theta_1) \). Clearly, \( S_q^*(\theta_1, \theta_2) \) is a closed set containing \( S_q(\theta_1, \theta_2) \). See Figure 5.
Figure 5. The graph $\gamma_{c,a}(\frac{1}{7})$ in the case $p = 1$. The regions $Q, Q_1, Q_2$ are three components of $C - \gamma_{c,a}(\frac{1}{7})$. Note that $S_q(\frac{1}{7}, \frac{2}{7}) = Q$, $S_q^*(\frac{1}{7}, \frac{2}{7}) = \overline{Q} \cup \overline{Q}_1 \cup \overline{Q}_2$. Here $V = U_{c,a}(q)$.

Figure 6. A possible structure of $\Gamma_{c,a}(\frac{1}{7})$ in the case $p = 2$, here $q = f_{c,a}(c)$.

The graph $\Gamma_{c,a}(\theta)$ induced by $\theta$ is defined as follows (see Figure 6):

$$\Gamma_{c,a}(\theta) = \partial X_{c,a} \cup (X_{c,a} \cap \gamma_{c,a}(\theta)).$$

The following is the main result of this section.
Theorem 4.2. Any map \( f_{c,a} \in C_0(S_p) \) admits a \( p \)-admissible puzzle \( \Gamma \). In fact, at least one of the graphs
\[
\Gamma_{c,a}\left(\frac{1}{7}\right), \Gamma_{c,a}\left(\frac{3}{7}\right), \Gamma_{c,a}\left(\frac{1}{7}\right) \cup \Gamma_{c,a}\left(\frac{3}{7}\right)
\]
is a \( p \)-admissible puzzle.

Before the proof, we explain our strategy.

Equivalent statement and strategy. Note that for any \( k \in \mathbb{N} \) and any graph \( \Gamma \in \{\Gamma_{c,a}(\frac{1}{7}), \Gamma_{c,a}(\frac{3}{7}), \Gamma_{c,a}(\frac{1}{7}) \cup \Gamma_{c,a}(\frac{3}{7})\} \), the set \( f_{c,a}^{-k}(\Gamma) \) is connected (because the pre-images of external rays are external rays, connecting the rest parts of \( f_{c,a}^{-k}(\Gamma) \)), so each component of \( f_{c,a}^{-k}(X_{c,a} \setminus \Gamma) \) is a Jordan disk.

Our goal is to show that there is a puzzle \( \Gamma \) among the three candidate graphs, with the property that there is a component \( Q \) of \( f_{c,a}^{-p}(X_{c,a} \setminus \Gamma) \), a component \( P \) of \( X_{c,a} \setminus \Gamma \), satisfying that
\[
\text{orb}(c) \cap Q \neq \emptyset, \quad \overline{Q} \subset P.
\]

Then it’s not hard to see that \( \Gamma \) is a \( p \)-admissible puzzle. In fact, assume \( f_{c,a}^{n}(-c) \in Q \) for some \( n \geq 0 \). The puzzle pieces induced by \( \Gamma \) satisfy that \( P_p(f_{c,a}^{n}(-c)) \subset P_0(f_{c,a}^{n}(-c)) \). By taking \( f_{c,a}^{n} \)-preimages, we see that \( P_{p+n}(-c) \subset P_n(-c) \), implying that \( \Gamma \) is \( p \)-admissible.

The main idea of the proof is to discuss the relative position of the critical orbit with respect to the candidate graphs. The argument has some independent interest. We first treat the case \( p = 1 \) to illustrate the idea, then deal with the more delicate case \( p > 1 \).

Proof of Theorem 4.2 when \( p = 1 \). First note that
\[
\tau^{-1}\left(\left\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right\}\right) = \left\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right\} \cup \left\{\frac{1}{14}, \frac{9}{14}, \frac{11}{14}\right\},
\]
\[
\tau^{-1}\left(\left\{\frac{3}{7}, \frac{5}{7}, \frac{6}{7}\right\}\right) = \left\{\frac{3}{7}, \frac{5}{7}, \frac{6}{7}\right\} \cup \left\{\frac{3}{14}, \frac{5}{14}, \frac{13}{14}\right\}.
\]

We first assume that the graph \( \Gamma = \Gamma_{c,a}(\frac{1}{7}) \cup \Gamma_{c,a}(\frac{3}{7}) \) avoids the orbit of \(-c\), and \( \Gamma \cap J(f_{c,a}) \) contains no parabolic point. Figure 7 will be helpful to understand the proof. We first assert that either we are in Case 1:
\[
\text{orb}(c) \cap S^*_c\left(\frac{9}{14}, \frac{5}{14}\right) \neq \emptyset,
\]
or in Case 2: we can find a component \( Q \) of \( f_{c,a}^{-1}(X_{c,a} \setminus \Gamma) \), a component \( P \) of \( X_{c,a} \setminus \Gamma \), satisfying that
\[
\text{orb}(c) \cap Q \neq \emptyset, \quad \overline{Q} \subset P.
\]

To see this, note that \( C = S_c\left(\frac{5}{14}, \frac{9}{14}\right) \cup S^*_c\left(\frac{5}{14}, \frac{9}{14}\right) \). If \( \text{orb}(c) \cap S^*_c\left(\frac{5}{14}, \frac{9}{14}\right) = \emptyset \), then we have \( \text{orb}(c) \subset S_c\left(\frac{5}{14}, \frac{9}{14}\right) \). Let \( Q \) be a component of \( f_{c,a}^{-1}(X_{c,a} \setminus \Gamma) \) so that \( Q \cap \text{orb}(c) \neq \emptyset \), and let \( P \) be the component of \( X_{c,a} \setminus \Gamma \) that contains \( Q \). Clearly \( Q \subset S_c\left(\frac{5}{14}, \frac{9}{14}\right) \). We claim \( \overline{Q} \subset P \). In fact, if it is not true, then \( \partial Q \cap U_{c,a}(c) \neq \emptyset \). This would imply that \( f_{c,a}(Q) \subset S^*_c\left(\frac{5}{14}, \frac{9}{14}\right) \), and further
orb(−c) ∩ \(S^*_c(\frac{9}{14}, \frac{5}{14})\) ≠ ∅, leading to a contradiction. This shows that we are in Case 2, in which case \(\Gamma\) is 1-admissible and the proof is done.

**Figure 7.** The graphs \(\gamma_{c,a}(\frac{1}{7})\) (black), \(\gamma_{c,a}(\frac{3}{7})\) (red) and their first \(f^k_{c,a}\)-preimages (dashed black, dashed red, respectively) near \(c\). Here \(U_{c,a}(c)\) is bounded by blue curve.

We need further discuss Case 1. In this case, we have

\[ f^k_{c,a}(-c) ∈ S^*_c(\frac{9}{14}, \frac{1}{14}) \cup S^*_c(\frac{1}{14}, \frac{5}{14}) \]

for some integer \(k ≥ 0\). If \(f^k_{c,a}(-c) ∈ S^*_c(\frac{9}{14}, \frac{1}{14})\), then let’s consider the graph \(\Gamma_{c,a}(\frac{1}{7})\). There is a component \(Q\) of \(f_{c,a}^{-1}(X_{c,a} \setminus \Gamma_{c,a}(\frac{1}{7}))\) and a component \(P\) of \(X_{c,a} \setminus \Gamma_{c,a}(\frac{1}{7})\) satisfying that

\[ f^k_{c,a}(-c) ∈ Q \subset P. \]

In this case, the boundaries of \(Q\) and \(P\) will not touch (see Figure), therefore \(Q \subset P\), implying that \(\Gamma_{c,a}(\frac{1}{7})\) is 1-admissible. If \(f^k_{c,a}(-c) ∈ S^*_c(\frac{1}{14}, \frac{5}{14})\), with the similar argument, we see that the graph \(\Gamma_{c,a}(\frac{5}{7})\) is 1-admissible.

Finally, we treat the rest cases. Write \(V = U_{c,a}(c)\).

If \(\Gamma_{c,a}(\frac{1}{7}) \cap \text{orb}(−c) ≠ ∅\) or \(ζ(V, \frac{1}{7})\) (recall that it is the landing point of \(R^V_{c,a}(\frac{1}{7})\)) is parabolic, then \(\Gamma_{c,a}(\frac{5}{7})\) is a puzzle. In the former case, one has
$\zeta(V, \frac{1}{7}) = f_{c,a}^k(-c)$ for some $k \geq 0$. In the latter case, let $W$ be a parabolic basin so that $\zeta(V, \frac{1}{7}) \in \partial W \cap \partial V$, then $f_{c,a}^k(-c) \in W$ for some $k \geq 0$. In either case, there is a component of $f_{c,a}^{-1}(X_{c,a} \setminus \Gamma_{c,a}(\frac{2}{7}))$, say $Q$, containing $f_{c,a}^k(-c)$, and a component $P$ of $X_{c,a} \setminus \Gamma_{c,a}(\frac{3}{7})$ containing $\overline{Q}$. This implies that $\Gamma_{c,a}(\frac{3}{7})$ is $1$-admissible.

If $\Gamma_{c,a}(\frac{2}{7}) \cap \text{orb}(-c) \neq \emptyset$ or $\zeta(V, \frac{2}{7})$ is parabolic, by similar argument as above, we see that $\Gamma_{c,a}(\frac{1}{7})$ is $1$-admissible.

This completes the proof in the case $p = 1$.

To deal with the case $p > 1$, we first prove the following fact:

**Lemma 4.3.** Assume $p > 1$. Let $q = f_{c,a}^k(c)$ for some $0 \leq k < p$ and let $\Gamma$ be one of the graphs

$$
\Gamma_{c,a}(\frac{1}{7}), \quad \Gamma_{c,a}(\frac{3}{7}), \quad \Gamma_{c,a}(\frac{1}{7}) \cup \Gamma_{c,a}(\frac{3}{7}).
$$

Let $Q$ be a component of $f_{c,a}^{-p}(X_{c,a} \setminus \Gamma)$. Suppose that $Q \subset f_{c,a}^p(Q)$ and $\overline{Q} \cap \overline{U_{c,a}(q)} \neq \emptyset$. Then we have

$$
\overline{Q} \subset f_{c,a}^p(Q).
$$

Before the proof, we need a notation. For any integer $k \geq 0$, any component $P$ of $f_{c,a}^{-k}(X_{c,a} \setminus \Gamma)$, let $\nu(P)$ be the number of attracting basins $V \in B_{c,a}$ whose boundary touches $P$:

$$
\nu(P) = \{|V \in B_{c,a} ; \overline{P} \cap V \neq \emptyset|\}.
$$

**Figure 8.** A possible graph structure, $q = f_{c,a}^k(c)$. Here $\nu(Q) = \nu(Q_1) = 1$, $\nu(Q_2) = 0$. Dashed rays are first pre-image of the dynamical rays defining the graph. Equipotential curves are not included.
In either case, one may verify that six choices (say \(\alpha\) that is contained in \((1, 3)\) and \(\zeta\) is of period three under the map \(f_{c,a}\)). This situation can happen only if \(Q = f_{c,a}(Q)\) implies that there is a component \(C\) of \(\tau^{-1}((\alpha_q, \beta_q))\) that is contained in \((\alpha_q, \beta_q)\). This situation can happen only if

\[
(\alpha_q, \beta_q) = \begin{cases} 
(\frac{4}{7}, \frac{1}{7}), & \text{if } \Gamma = \Gamma_{c,a}(\frac{4}{7}), \\
(\frac{6}{7}, \frac{2}{7}), & \text{if } \Gamma = \Gamma_{c,a}(\frac{6}{7}), \\
(\frac{9}{7}, \frac{1}{7}), & \text{if } \Gamma = \Gamma_{c,a}(\frac{9}{7}) \cup \Gamma_{c,a}(\frac{4}{7}).
\end{cases}
\]

In either case, one may verify that \(\mathcal{C} \subset (\alpha_q, \beta_q)\). This implies that if \(\nu(Q) = 1\), then \(Q \subset f_{c,a}(Q)\) and the proof is done (see Figure 8).

If \(\nu(Q) \geq 2\) (see Figure 9), then for any \(q' = f_{c,a}(q)\) with \(q' \neq q\) and \(Q \cap U_{c,a}(q') \neq \emptyset\), by the same argument as above, we see that \(Q \cap U_{c,a}(q') \neq \emptyset\) and \(Q \cap U_{c,a}(q') \subset f_{c,a}(Q)\). Therefore in this case, \(Q \subset f_{c,a}(Q)\).

\(\square\)
Proof of Theorem 4.2 when $p > 1$. Let $\Gamma = \Gamma_{c,a}(\frac{1}{7}) \cup \Gamma_{c,a}(\frac{2}{7})$. We first assume that $\Gamma \cap \text{orb}(-c) = \emptyset$ and $\Gamma \cap J(f_{c,a})$ contains no parabolic point.

Let $Q$ be a component of $f_{c,a}^{-p}(X_{c,a} \setminus \Gamma)$ so that $\text{orb}(-c) \cap Q \neq \emptyset$, and let $P$ be the component of $X_{c,a} \setminus \Gamma$ containing $Q$.

Case 1. $\nu(Q) = 0$. This is equivalent to say that $\partial Q \cap \partial A_{c,a} = \emptyset$. Therefore $\partial Q \subset f_{c,a}^{-p} \cap \Gamma$ and $Q \subset P$.

Case 2. $\nu(Q) \geq 2$. Note that $\nu(f_{c,a}^{-p}(Q')) \geq \nu(Q) \geq 2$. Clearly, both $f_{c,a}^{-p}(Q)$ and $P$ intersect those $\nu$’s with $V \in B_{c,a}$ and $V \cap Q \neq \emptyset$. Since there is only one component of $X_{c,a} \setminus \Gamma$ satisfying this property, we have $f_{c,a}^{-p}(Q) = P$. Then by Lemma 4.3, we get $Q \subset P$.

Case 3. $\nu(Q) = 1$. Let $q \in \{f_{c,a}^k(c); 0 \leq k < p\}$ be the unique point with $\partial Q \cap U_{c,a}(q) \neq \emptyset$. Note that $$S_q\left(\frac{9}{14}, \frac{1}{14}\right) \cup S_q\left(\frac{1}{14}, \frac{5}{14}\right) \cup S_q\left(\frac{5}{14}, \frac{9}{14}\right) = \mathbb{C}.$$

Case 3.1. $Q \subset S_q\left(\frac{9}{14}, \frac{1}{14}\right)$. Let $Q'$ be the component of $f_{c,a}^{-p}(X_{c,a} \setminus \Gamma_{c,a}(\frac{1}{7}))$ containing $Q$, and $P'$ the component of $X_{c,a} \setminus \Gamma_{c,a}(\frac{1}{7})$ containing $Q'$. Clearly, $\nu(P') \geq \nu(Q') \geq \nu(Q) = 1$. If $\nu(Q') \geq 2$, then by the same argument as Case 2, we have $Q' \subset P'$. If $\nu(Q') = 1$, the fact $Q' \cap U_{c,a}(q) \subset P'$ (see Figure 8) implies that $Q' \subset P'$. In either case, the graph $\Gamma_{c,a}(\frac{1}{7})$ is a $p$-admissible puzzle.

Case 3.2. $Q \subset S_q\left(\frac{1}{14}, \frac{5}{14}\right)$. Let $Q'$ be the component of $f_{c,a}^{-p}(X_{c,a} \setminus \Gamma_{c,a}(\frac{2}{7}))$ containing $Q$, and $P'$ the component of $X_{c,a} \setminus \Gamma_{c,a}(\frac{2}{7})$ containing $Q'$. Similarly as Case 3.1, we have $Q' \subset P'$, implying that $\Gamma_{c,a}(\frac{2}{7})$ is a $p$-admissible puzzle.

Case 3.3. $Q \subset S_q\left(\frac{5}{14}, \frac{9}{14}\right)$. In this case, $f_{c,a}^{-p}(Q)$ is a component of $X_{c,a} \setminus \Gamma$ satisfying that $$f_{c,a}^{-p}(Q) \subset S_q\left(\frac{9}{14}, \frac{1}{14}\right) \cup S_q\left(\frac{1}{14}, \frac{5}{14}\right) \cup S_q\left(\frac{5}{14}, \frac{9}{14}\right) \text{ and } \nu(f_{c,a}^{-p}(Q)) \geq 1.$$

Note that $\text{orb}(-c) \cap f_{c,a}^{-p}(Q) \neq \emptyset$. There is a component $Q''$ of $f_{c,a}^{-p}(X_{c,a} \setminus \Gamma)$ satisfying that $Q'' \subset f_{c,a}^{-p}(Q)$ and $\text{orb}(-c) \cap Q'' \neq \emptyset$. Clearly either $Q'' \subset S_q\left(\frac{9}{14}, \frac{1}{14}\right)$ or $Q'' \subset S_q\left(\frac{1}{14}, \frac{5}{14}\right)$, meaning that we are again in Case 3.1 or Case 3.2. With the same argument, we see that $\Gamma_{c,a}(\frac{1}{7})$ or $\Gamma_{c,a}(\frac{2}{7})$ is $p$-admissible.

Rest Cases. Finally, we handle the rest cases: $\Gamma \cap \text{orb}(-c) = \emptyset$ or $\Gamma \cap J(f_{c,a})$ contains a parabolic point.

Suppose that $\Gamma_{c,a}(\frac{1}{7}) \cap \text{orb}(-c) = \emptyset$ or $\Gamma_{c,a}(\frac{1}{7}) \cap J(f_{c,a})$ contains a parabolic point. Let $V = U_{c,a}(c)$, then the landing point $\zeta(V, \frac{1}{7})$ of $R_{V, \frac{1}{7}}$ is either $f_{c,a}^{-p}(-c)$ for some $k \geq 1$, or parabolic. In the latter case, $\zeta(V, \frac{1}{7})$ is on the boundary of some parabolic basin $W$, which contains $f_{c,a}^{-p}(-c)$ for some $k$ (here we use the same $k$ because the two cases can not happen simultaneously). In either case, let $Q$ be the component of $f_{c,a}^{-p}(X_{c,a} \setminus \Gamma_{c,a}(\frac{1}{7}))$
containing $f_{c,a}^k(-c)$, and let $P$ be the component of $X_{c,a} \setminus \Gamma_{c,a}(\frac{3}{4})$ containing $Q$. The fact $Q \cap V \neq \emptyset$ implies that $\nu(Q) \geq 1$. If $\nu(Q) \geq 2$, then by the same argument as in Case 2, we have $Q \subset P$. If $\nu(Q) = 1$, note that $Q \cap V \subset P$, we also have $Q \subset P$. Therefore $\Gamma_{c,a}(\frac{3}{4})$ is $p$-admissible.

The last cases are $\Gamma_{c,a}(\frac{3}{4}) \cap \text{orb}(-c) \neq \emptyset$ or $\Gamma_{c,a}(\frac{3}{4}) \cap J(f_{c,a})$ contains a parabolic point. Similarly as above, we have that $\Gamma_{c,a}(\frac{1}{4})$ is $p$-admissible.

The proof of the theorem is completed.

5. Rigidity via puzzles

This section is devoted to proving the combinatorial rigidity for maps on $S_p$. Rigidity is one of the most remarkable phenomena in holomorphic dynamics. One of its applications is to study the boundaries of hyperbolic components in the next section. To simplify notations, write $f = f_{c,a}, \tilde{f} = f_{\tilde{c},\tilde{a}}, X = X_{c,a}$, $c^* = -c$.

Here, $X_{c,a}$ is defined in Section 4.2. The corresponding objects (graphs, puzzles, tableau, etc) for $\tilde{f}$ are marked with tilde.

In this section, we assume $f, \tilde{f} \in C_0(S_p)$. Let’s take a $p$-admissible puzzle $\Gamma$ for $f$, given by Theorem 4.2. The puzzle pieces and tableau (in particular $T_f(c^*)$) are induced by $\Gamma$. Write $\Gamma_k = f^{-k}(\Gamma)$ for $k \geq 0$, the collection $P_k$ of puzzle pieces of depth $k$ consists of the connected components of $P_k := f^{-k}(X - \Gamma)$. Let $\tilde{\Gamma}$ be the graph of $\tilde{f}$ with the same structure as $\Gamma$.

We first define the combinatorial equivalence between $f$ and $\tilde{f}$. Roughly speaking, it means that the two maps have the same puzzle structures at any depth. Rigorous definition goes as follows. Let $\phi : \Gamma \rightarrow \tilde{\Gamma}$ be a homeomorphism, written as the identity map in the Böttcher coordinates. For an integer $k \geq 1$, we say that $f$ and $\tilde{f}$ have the same combinatorics up to depth $k$, if there is a homeomorphism $\phi_k : \Gamma_k \rightarrow \tilde{\Gamma}_k$ so that $\tilde{f} \circ \phi_k = \phi \circ f$ on $\Gamma_k$ and $\phi_k|_{\Gamma_k \cap \Gamma} = \phi|_{\Gamma_k \cap \Gamma}$. We say that $f$ and $\tilde{f}$ are combinatorially equivalent if they have the same combinatorics up to any depth. If furthermore $\phi$ can be extended to a quasi-conformal map $\Phi : \mathbb{C} \rightarrow \mathbb{C}$, we say that $f$ and $\tilde{f}$ are qc-combinatorially equivalent. Combinatorial equivalence allows one to extend $\phi$ as a homeomorphism

$$\phi : \bigcup_{k \geq 0} \Gamma_k \rightarrow \bigcup_{k \geq 0} \tilde{\Gamma}_k$$

by setting $\phi|_{\Gamma_k} = \phi_k|_{\Gamma_k}$ for any $k$. Further, $\phi$ induces a bijection $\phi_*$ between puzzle pieces:

$$\phi_* : \bigcup_{k \geq 0} P_k \rightarrow \bigcup_{k \geq 0} \tilde{P}_k,$$

here $\phi_*(P_k)$ is defined to be the puzzle piece of $\tilde{f}$ bounded by $\phi(\partial P_k)$.

**Theorem 5.1.** If $f, \tilde{f} \in C_0(S_p)$ are qc-combinatorially equivalent and $T_f(c^*)$ is aperiodic, then $f = \tilde{f}$.
The assumption that $T_f(c^*)$ is aperiodic implies that $f$ is not renormalizable\(^1\) and $J(f) = K(f)$. The proof of Theorem 5.1 actually gives more:

**Theorem 5.2.** Let $f \in \mathcal{C}_0(S_\rho)$, suppose that $T_f(c^*)$ is aperiodic. Then

(1). The Julia set $J(f)$ is locally connected.

(2). $f$ carries no invariant line fields on $J(f)$.

Here, a line field $\mu$ supported on $E$ is a Beltrami differential $\mu = \mu(z)\frac{dz}{dz}$ supported on $E$ with $|\mu| = 1$. A line field $\mu$ is called measurable if $\mu(z)$ is a measurable function. We say that $f$ carries an invariant line field if there is a measurable line field $\mu = \mu(z)\frac{dz}{dz}$ supported on a positive measurable subset of $J(f)$ such that $f^*\mu = \mu$ almost everywhere.

Theorem 5.1 and Theorem 5.2 (1) generalize Yoccoz’s famous theorem to cubic maps $f \in \mathcal{C}_0(S_\rho)$. In fact, in the case $p = 1$, Yoccoz’s proof of local connectivity [H,M5], Lyubich’s proof of zero measure [L] for quadratic Julia sets both work here. However, their arguments will break down for cubic $f \in \mathcal{C}_0(S_\rho)$ in the persistently recurrent case when $p \geq 2$. This is because the existence of a $p$-admissible puzzle with $p \geq 2$ makes the situation essentially as complicated as the multicritical case. The principle nest of critical puzzle pieces (see Theorem 5.8) will be involved to deal with this case.

This section is organized as follows. We first recall some analytic lemmas to be used in our approach (Section 5.1). For further discussions, we distinguish $T_f(c^*)$ into the persistently recurrent case and the other (non-recurrent, reluctantly recurrent) cases. We will recall the principal nest in Section 5.2 and use it to deal with the persistently recurrent case in Section 5.3. Finally, we treat the rest cases in Section 5.4. The methods for these cases are slightly different.

### 5.1. Analytic tools

To prove Theorems 5.1 and 5.2, we need some analytic tools, including a qc-extension lemma (Lemma 5.3); a criterion of no invariant field (Lemma 5.4); a qc-criterion (Lemma 5.5); an analytic fact on Lebesgue density and geometry (Lemma 5.6). The first two will be used in the persistently recurrent case, while the last two take effect in other cases.

**Lemma 5.3** (see [AKLS] Lemma 3.2). For every number $\rho \in (0,1)$ and integer $d \geq 2$, there exist numbers $r = r(\rho,d) \in (\rho,1)$ and $K_0 = K_0(\rho,d)$ with the following property. Let $G, \tilde{G} : \mathbb{D} \to \mathbb{D}$ be proper holomorphic maps of degree $d$. Let $h_1, h_2 : \partial \mathbb{D} \to \partial \mathbb{D}$ be such that $\tilde{G} \circ h_2 = h_1 \circ G$. Assume that

(1). $|G(0)|, |\tilde{G}(0)| \leq \rho$;

(2). The critical values of $G, \tilde{G}$ are contained in $\mathbb{D}_\rho$;

(3). $h_1$ has a $K_1$-qc extension $H_1 : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_\rho$.

---

\(^1\)We say that $f$ is renormalizable, there exist an integer $k \geq 0$, two open disks $U, V$ with $U \subset V$, such that $f^k : U \to V$ is a polynomial-like map of degree $\geq 2$, with connected Julia set, which is not equal to $J(f)$.

\(^2\)To see this, note that $K(f) \neq J(f) \implies$ there is a periodic Fatou component $U \subset K(f) \implies$ there exists puzzles pieces $P_n \supset P_{n+k} \supset U \implies$ deg$(f^k : P_{n+k} \to P_n) \geq 2$ (by Schwarz Lemma) $\implies c^*$ is in the cycle of $U \implies T_f(c^*)$ is periodic.
Then $h_2$ admits a $K_2$-qc extension $H_2 : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where $K_2 = \max\{K_1, K_0\}$.

Lemma 5.3 is a variant of [AKLS] Lemma 3.2 which require that $G(0) = G(0) = 0$. The rewritten condition (1) here allows more flexible applications. Their proofs are essentially same.

For a topological disk $U \subset \mathbb{C}$ and a point $z \in U$, the shape of $U$ with respect to $z$ is a quantity to measure the geometry of $U$, defined by

$$S(U, z) = \sup_{w \in \partial U} |w - z|/\min_{w \in \partial U} |w - z|. $$

Lemma 5.4 (see [Sh] Prop. 3.2). Let $R$ be a rational map of degree $\geq 2$ with $\infty \notin J(R)$. Let $z \in J(R)$. If there exist a constant $C \geq 1$, positive integers $N \geq 2$, $n_k$’s, and proper maps $h_k = R^{n_k}|_{U_k} : U_k \to V_k$, $k \geq 1$ with the following properties:

(1). $U_k, V_k$ are topological disks in $\mathbb{C}$ and as $k \to \infty$

$$\text{diam}(U_k) \to 0, \ \text{diam}(V_k) \to 0. $$

(2). $2 \leq \deg(h_k) \leq N$, for all $k \geq 1$.

(3). For some $u \in U_k$ with $h_k'(u) = 0$ and for $v = h_k(u)$, we have

$$S(U_k, u), \ S(V_k, v) \leq C.$$ 

(4). $d(U_k, z) \leq C\text{diam}(U_k), d(V_k, z) \leq C\text{diam}(V_k)$. Here $\text{diam}$ and $d$ denote the Euclidean diameter and distance.

Then for any line field $\mu$ with $R^*\mu = \mu$, either $z \notin \text{supp}(\mu)$ or $\mu$ is not almost continuous at $z$.

The following qc-criterion is a simplified version of [KSS] Lemma 12.1, with a slightly difference in the second assumption (that is, we replace a sequence of round disks in [KSS] by a sequence of disks with uniformly bounded shape), and the original proof goes through without any problem.

Lemma 5.5 (see [KSS] Lemma 12.1]). Let $\phi : \Omega \to \tilde{\Omega}$ be a homeomorphism between two Jordan domains, $k \in (0, 1)$ be a constant. Let $X$ be a subset of $\Omega$ such that both $X$ and $\phi(X)$ have zero Lebesgue measures. Assume:

1. $|\partial \phi| \leq k|\partial \phi|$ a.e. on $\Omega \setminus X$.

2. There is a constant $M > 0$ such that for all $x \in X$, there is a sequence of open topological disks $D_1 \supseteq D_2 \supseteq \cdots$ containing $x$, satisfying that

(a). $\bigcap_j D_j = \{x\}$, and

(b). $\sup_j S(D_j, x) \leq M$, $\sup_j S(\phi(D_j), \phi(x)) < \infty$.

Then $\phi$ is a $K$-quasi-conformal map, where $K$ depends on $k$ and $M$.

Lastly, the following fact is useful when dealing with the non persistently recurrent cases, see [QWY] Prop. 6.1] and [QRWY] Lemma 9.4.

Lemma 5.6. Let $R$ be a rational (or rational-like) map with $\infty \notin J(R)$. Let $z \in J(R)$. Suppose there exist integers $D_z > 0$ and $0 \leq n_1 < n_2 < \cdots$, a sequence of disk neighborhoods $U'_j \subset U_j$ of $z$, two disks $V'_z \subset V_z$ so that $R^{n_j} : U'_j \to V'_z$ and $R^{n_j} : U'_j \to V'_z$ are proper maps of degree $\leq D_z$. Then
where $C(D_z, m_z)$ depends on $D_z$ and $m_z = \{\mod(V_z - V''_z), \mod(V'_z - V''_z)\}$.

5.2. Principal nest. We assume that $T_f(c^*)$ is persistently recurrent. Since $\Gamma$ is a $p$-admissible puzzle for $f$, we see that $P_{d_0+p}(c^*) \subseteq P_{d_0}(c^*)$ for some $d_0 \geq 0$. The recurrence of $T_f(c^*)$ allows us to find infinitely many integers $L \geq d_0$ so that $P_{L+p}(c^*) \subset P_L(c^*)$.

Sometimes, we work with $g = f^p$. It’s critical set $C(g) = \bigcup_{0 \leq k < p} f^{-k}(c^*)$. View $\Gamma$ as a graph of $g$, one can define the puzzle pieces of $g$ induced by $\Gamma$. The tableau $T_g(z)$ consists of the $pn \times pn$-positions of the tableau $T_f(z)$.

We may decompose $C(g) = C_0(g) \sqcup C_1(g)$, where

\[ C_0(g) = \{\zeta \in C(g); \zeta \notin \zeta\}, \quad C_1(g) = C(g) - C_0(g). \]

**Lemma 5.7.** Assume that $T_f(c^*)$ is persistently recurrent, then $c^* \in C_0(g)$.

**Proof.** The recurrence of $T_f(c^*)$ implies that there is an integer $0 \leq l \leq p$, so that the $(l+p\mathbb{N})$-columns of the tableau $T_f(c^*)$ contain $c^*$-positions of arbitrarily large depth. It follows that one can find $\zeta \in f^{-l}(c^*)$, so that the $p\mathbb{N}$-columns of the tableau $T_f(c^*)$ contain $\zeta$-positions of arbitrarily large depth. This means that $\zeta \nrightarrow \zeta$. By tableau rules, $f^l(\zeta) \nrightarrow f^l(\zeta)$. □

For any $\zeta \in C_0(g)$, clearly $\zeta \nrightarrow c^*$. On the other hand, by Lemma 5.7 and tableau rules, we see that $c^* \nrightarrow \zeta$. So we have $[c^*]_g = \{\zeta \in C_0(g); c^* \nrightarrow \zeta\}$. Let $\operatorname{orb}_g([c^*]_g) = \bigcup_{k \in \mathbb{N}} g^k([c^*]_g)$. Clearly $\operatorname{orb}_g([c^*]_g) \subset \operatorname{orb}(c^*) \cup C(g)$.

We may assume the graph $\Gamma$ (by choosing $L_0 \geq d_0$ suitably) satisfy that

A1. $P_{L_0+p}(c^*) \subset P_{L_0}(c^*)$.

A2. For any $\zeta_1, \zeta_2 \in C(g)$ with $\zeta_1 \neq \zeta_2$, one has $P_{L_0}(\zeta_1) \neq P_{L_0}(\zeta_2)$.

A3. For any $\zeta_1, \zeta_2 \in C(g)$ (not necessarily distinct), if they do not satisfy $\zeta_2 \nrightarrow \zeta_1$, then the $\{L_0\} \times p\mathbb{N}^+$ positions of $T_f(\zeta_2)$ are not $\zeta_1$-positions.

The assumption A1 implies that the puzzle piece $Y_0(c^*) = P_{L_0+p}(c^*)$ satisfies $g(\partial Y_0(c^*)) \cap \partial Y_0(c^*) = \emptyset$. In literature, a puzzle piece $Y$ satisfying $g(\partial Y) \cap \partial Y = \emptyset$ is called nice. Nice puzzle piece allows one to construct the principle nest, whose significant properties are summarized as follows

**Theorem 5.8.** Assume $T_f(c^*)$ is persistently recurrent and the puzzle $\Gamma$ satisfies A1, A2, A3. Then there exist a nest of $c^*$-puzzle pieces

\[ Y_0(c^*) \ni Y_1(c^*) \ni Y'_1(c^*) \ni Y_2(c^*) \ni Y'_2(c^*) \ni \cdots, \]

each is a suitable pull back of $Y_0(c^*)$, satisfying the following properties:

1. There exist integers $D_0 > 0$, $n_j > m_j \geq 1$ for all $j \geq 1$, so that

\[ g^{m_j}: Y'_j(c^*) \rightarrow Y_j(c^*), \quad g^{n_j}: Y_{j+1}(c^*) \rightarrow Y_j(c^*) \]
Remark 5.9. are proper maps of degree $\leq D_0$, and $g^{n_j}(Y^j_{j+1}(c^*)) \subset Y^j_j(c^*)$.

(2) For all $j \geq 1$,

$$ (Y^j_j(c^*) - \overline{Y}^j_j(c^*)) \cap \text{orb}_g([c^*]_g) = \emptyset. $$

(3) There is a constant $\nu > 0$ so that for all $j \geq 1$,

$$ \text{mod}(Y^j_j(c^*) - \overline{Y}^j_j(c^*)) \geq \nu. $$

(4) There is a constant $C_0 > 0$ so that for all $j \geq 1$,

$$ S(Y^j_j(c^*), c^*) \leq C_0. $$

Moreover, for $j \geq 1$, there is another $c^*$-piece $Y''_j(c^*) \subset Y^j_j(c^*)$ with

$$ (Y^j_j(c^*) - \overline{Y}''_j(c^*)) \cap \text{orb}_g([c^*]_g) = \emptyset \text{ and mod}(Y^j_j(c^*) - \overline{Y}''_j(c^*)) \geq \nu. $$

The construction of the principal nest is attributed to Kahn-Lyubich [KL1] in the unicritical case, Kozlovski-Shen-van Strien [KSS] in the multicritical case. The complex bounds are proven by Kahn-Lyubich [KL1,KL2] (unicritical case), Kozlovski-van Strien [KS] and Qiu-Yin [QY] independently (the multicritical case). The bounded geometry property (4) is derived by Yin-Zhai [YZ] Lemma 6 and Prop.1[3]. See these references for a detail construction of the nest and the proof of its properties.

**Remark 5.9.** Theorem 5.8 with the assumptions A2, A3, implies that

$$(Y^j_j(c^*) - \overline{Y}''_j(c^*)) \cap \text{orb}(c^*) = \emptyset.$$  

5.3. **Proof of Theorems 5.1 and 5.2:** persistently recurrent case. Assume $T_f(c^*)$ is persistently recurrent. Recall that $P_k = f^{-k}(X - \Gamma)$ and $\phi : \bigcup_{k \geq 0} P_k \to \bigcup_{k \geq 0} \overline{P}_k$ is a homeomorphism induced by the combinatorial equivalence. Our proof follows the strategy of [AKLS] and has six steps.

**Step 1:** Construction of qc maps at any depth. For any $n \geq 0$, there is a qc-map $\phi_n : \mathbb{C} \to \mathbb{C}$, so that $f \circ \phi_n = \phi_n \circ f$ on $\mathbb{C} - P_n$.

Note that $\phi|_{\mathbb{C} - P_0}$ is the identity map in Böttcher coordinates. The restriction $\phi|_{\mathbb{C} - P_0}$ can be extended to a qc map $\phi_0 : (\mathbb{C}, f(c^*)) \to (\mathbb{C}, \tilde{f}(\tilde{c}^*))$. Then there is a qc map $\phi_1 : \mathbb{C} \to \mathbb{C}$ so that $f \circ \phi_1 = \phi_0 \circ f$ and $\phi_1|_{\mathbb{C} - P_0} = \phi_0|_{\mathbb{C} - P_0}$. We may adjust $\phi_1$ so that $\phi_1(f(c^*)) = \tilde{f}(\tilde{c}^*)$. This allows us to get a lift $\phi_2$ of $\phi_1$, in the way that $f \circ \phi_2 = \phi_1 \circ f$ and $\phi_2|_{\mathbb{C} - P_1} = \phi_1|_{\mathbb{C} - P_1}$. By induction, for any $n$, there is a qc-map $\phi_{n+1}$, such that $f \circ \phi_{n+1} = \phi_n \circ f$ and $\phi_{n+1}|_{\mathbb{C} - P_n} = \phi_n|_{\mathbb{C} - P_n}$. We remark that the dilatations of $\phi_n$'s might not be uniformly bounded, to overcome this, we prove:

**Step 2:** Bounding dilatation by critical piece. For any $j \geq 1$, if $\phi|_{\partial Y_j(c^*)}$ has a $K$-qc-extension $\Phi_j : \overline{Y}_j_j(c^*) \to \overline{Y}_j_j(c^*)$, then it has a further $K$-qc-extension $H_j : \mathbb{C} \to \mathbb{C}$ such that $\tilde{f} \circ H_j = H_j \circ f$ on $\mathbb{C} - Y_j_j(c^*)$.

---

3The notations $Y_j_j(c^*), Y'_j_j(c^*), Y''_j_j(c^*)$ here correspond to $K'_n, K_n, \tilde{K}_n$ in [YZ].
To prove the implication, let’s define
\[ Z = \bigcup_{k \geq 1} f^{-k}(Y_j(c^*)) - Y_j(c^*). \]

For each component \( U \) of \( Z \), there is an integer \( l = l(U) \geq 1 \) (called return time) so that \( f^l : U \to Y_j(c^*) \) is conformal (clearly its counterpart \( \tilde{f}^l : \tilde{U} \to \tilde{Y}_j(c^*) \) is also conformal). We define \( H_j|_U : U \to \tilde{U} \) by
\[ H_j|_U = (\tilde{f}^l|_U)^{-1} \circ \Phi_j \circ \tilde{f}^l|_U. \]

On \( F(f) - Z \) (where \( F(f) \) is the Fatou set of \( f \)), we may define \( H_j \) to be identity map in the Böttcher coordinates, hence conformal. These maps match on the common boundary of the pieces \( U \). Since the residual set
\[ J_{res} = \bigcap_{k \geq 0} f^{-k}(J(f) - Y_j(c^*)) \]
is hyperbolic hence has zero Lebesgue measure, we see that \( H_j \) admits a qc-extension to the plane, with the same dilatation as that of \( \Phi_j \).

**Step 3. An induction procedure.** For any \( j \geq 1 \), we have that
\[
\begin{array}{c}
\phi \big|_{\partial Y_j(c^*)} \text{ has a qc-extension} \\
\Phi_j : Y_j(c^*) \to \tilde{Y}_j(c^*)
\end{array} \implies
\begin{array}{c}
\phi \big|_{\partial Y_{j+1}(c^*)} \text{ has a qc-extension} \\
\Phi_{j+1} : Y_{j+1}(c^*) \to \tilde{Y}_{j+1}(c^*)
\end{array}
\]
and the dilatations \( K_j, K_{j+1} \) of \( \Phi_j, \Phi_{j+1} \) satisfy
\[ K_{j+1} \leq \max\{K_j, K(\nu, \tilde{\nu}, D_0)\} \leq \max\{K_1, K(\nu, \tilde{\nu}, D_0)\}, \]
where \( \nu, \tilde{\nu}, D_0 \) are given by Theorem 5.8, and \( K(\nu, \tilde{\nu}, D_0) \) depends on them.

For each \( j \geq 1 \), let \( \psi_j : (Y_j(c^*), c^*) \to (\mathbb{D}, 0) \) be a conformal map, then \( G_j = \psi_j \circ g^{\nu_j} \circ \psi_j^{-1} : \mathbb{D} \to \mathbb{D} \) is proper holomorphic, and \( 2 \leq \deg(G_j) \leq D_0 \).

Let \( \Omega_j = \psi(Y_j(c^*)) \) for \( j \geq 1 \). By Theorem 5.8, we have \( G_j(\Omega_{j+1}) \subset \Omega_j \), the critical values of \( G_j \) are in \( \Omega_j \) (in particular \( G_j(0) \in \Omega_j \)), and
\[ \text{mod}(\mathbb{D} - \Omega_j) = \text{mod}(Y_j(c^*) - \overline{Y_j(c^*)}) \geq \nu. \]

So there is a constant \( \rho(\nu) \in (0, 1) \) with \( \Omega_j \subset \mathbb{D}_{\rho(\nu)}. \)

Let’s define \( h_j, h_{j+1} : \partial \mathbb{D} \to \partial \mathbb{D} \) by
\[ h_j = \tilde{\psi}_j \circ \phi \big|_{\partial Y_j(c^*)} \circ \psi_j^{-1}, \quad h_{j+1} = \tilde{\psi}_{j+1} \circ \phi \big|_{\partial Y_{j+1}(c^*)} \circ \psi_{j+1}^{-1}. \]

Clearly \( h_j \circ G_j = \tilde{G}_j \circ h_{j+1} \). By induction hypothesis, \( h_j \) has an extension \( L_j = \tilde{\psi}_j \circ \Phi_j \circ \psi_j^{-1} \). Note that \( G_j, \tilde{G}_j, h_j, h_{j+1} \) satisfy the assumptions in Lemma 5.3

Let
\[ \rho = \max\{\rho(\nu), \rho(\tilde{\nu})\}, \quad r_0 = \min\{r(\rho, d); 2 \leq d \leq D_0\} \in (\rho, 1), \]
where \( r(\rho, d) \)’s are given by Lemma 5.3. Assume that \( L_j \) is identity\(^4\) on \( \mathbb{D}_{r_0} \), then by Lemma 5.3 \( h_{j+1} \) has a qc extension \( L_{j+1} \), which is identity

\(^4\)This assumption is satisfied by making \( L_1 \) satisfy \( L_1|_{\mathbb{D}_{r_0}} = \text{id} \) and by induction.
on $\mathbb{D}_{r_0}$, with dilatation $K_{j+1} = \max\{K_j, K(\nu, \nu, D_0)\}$. Finally, we take $\Phi_{j+1} = \Psi_{j+1}^{-1} \circ L_{j+1} \circ \Psi_{j+1}$ and get an extension of $\phi|_{\partial Y_{j+1}(e^*)}$.

**Step 4: Conjugacy via taking a limit.**

By Step 3, the map $\phi|_{\partial Y_{j}(c^*)}$ has aqc extension $\Phi_j : Y_j(c^*) \to \tilde{Y}_j(c^*)$, with dilatation $K_* = \max\{K_1, K(\nu, \nu', D_0)\}$. By Step 2, there is an extension $H_j$ of $\Phi_j$, conjugate $f$ to $\tilde{f}$ on $\mathbb{C} - Y_j(c^*)$, without increasing the dilatation of $\Phi_j$. Then $\{H_j; j \geq 1\}$ is a normal family, whose limit is a $K_*$-qc map $H$, satisfying $\tilde{f} \circ H = H \circ f$ on the Fatou set of $f$. Since $J(f)$ has no interior, $H$ is a conjugacy on $\mathbb{C}$ by continuity.

**Step 5: $f$ carries no invariant line fields on $J(f)$.**

First note that the set

$$X_\infty := \bigcup_{j \geq 1} \bigcap_{k \geq 1} f^{-k}(J(f) - Y_j(c^*))$$

has Lebesgue measure zero and $\Gamma_\infty \cap J(f) \subset X_\infty$. Suppose that $f$ carries an invariant line field $\mu$. Let $z \in \text{supp}(\mu) \cap (J(f) - X_\infty)$. Clearly, $T_f(z)$ is critical. Let $Y_j(c^*), Y_j'(c^*), Y_j''(c^*)$ be given by Theorem 5.8 and write

$$Y_j(c^*) = P_{d_j}(c^*), Y_j'(c^*) = P_{d_j'}(c^*), Y_j''(c^*) = P_{d''}(c^*).$$

Let $s_j \geq 0$ be the first integer with $f^{s_j}(P_{d''+s_j}(z)) = P_{d''}(c^*)$. By Theorem 5.8 (2)(4), one has $f^{s_j}(P_{d''+s_j}(z)) = P_{d_j}(c^*), f^{s_j}(P_{d''+s_j}(z)) = P_{d_j'}(c^*)$, and $f^{s_j} P_{d''+s_j}(z)$ is conformal (in particular, if $z \in P_{d''}(c^*)$, then $s_j = 0$ and $f^{s_j} = id$), we have

$$\text{mod}(P_{d_j+s_j}(z) - \overline{P_{d''+s_j}(z)}) \geq \nu, \text{mod}(P_{d_j'+s_j}(z) - \overline{P_{d''+s_j}(z)}) \geq \nu.$$ 

It follows that

$$S(P_{d_j+s_j}(z), z) \leq C_1(\nu) S(Y_{d_j'}(c^*), f^{s_j}(z)) \leq C_1(\nu)C_2(\nu) S(Y_{d_j'}(c^*), c^*) \leq C_1(\nu)C_2(\nu)C_0,$$

where $C_1(\nu), C_2(\nu)$ are constants depending on $\nu$, and $C_0$ is given by Theorem 5.8. To apply Lemma 5.4, we take $V_j = P_{d_j+s_j}(z)$. It remains to find $U_j$. Let $t_j > 0$ be the first integer such that $f^{t_j}(P_{d''+t_j}(c^*)) = P_{d''}(c^*)$ and $r_j \geq 0$ be the first integer such that $f^{r_j}(P_{d_j'+t_j+r_j}(z)) = P_{d_j'+t_j}(c^*)$. Again Theorem 5.8 (2)(4) assert that $f^{r_j}(P_{d_j'+t_j+r_j}(z)) = P_{d_j'+t_j}(c^*)$ and $f^{r_j}(P_{d_j'+t_j+r_j}(z)) = P_{d_j'+t_j}(c^*)$. Similarly as above, one has

$$S(P_{d_j'+t_j+r_j}(z), z) \leq C_1(\nu)C_2(\nu)C_0.$$
We take \( U_j = P_{d_j+r_j}^j(z) \), and \( h_j = f_j^{r_j-s_j}|_{U_j} \) (one may verify that \( s_j \leq r_j \)). Then \( h_j : U_j \to V_j \) is of degree two, and satisfies the assumptions of Lemma 5.4. Hence \( \mu \) is not almost continuous at \( z \), which gives a contradiction. The proof of Step 5 is completed.

It follows that the qc conjugacy \( H \) obtained in Step 4 is conformal, and \( H(z) = z + O(1) \) near \( \infty \), therefore \( H(z) = z \) and \( f = \tilde{f} \). The proof of Theorem 5.1 in the persistently recurrent case is finished.

**Step 6: Local connectivity of \( J(f) \).** Note that for any \( z \in J(f) \) and any \( n \geq 0 \), the intersection \( P_n^\ast(z) \cap J(f) \) is connected (because each connected component of \( P_n^\ast(z) \setminus J(f) \) is simply connected). It suffices to show that \( \text{Imp}(z) = \{ z \} \). Let \( d_j, d_j', d_j'' \) be given in Step 5.

For \( z \in J(f) - \Gamma_\infty \) with \( T_f(z) \) critical, let \( l_j \geq 0 \) be an integer so that \( f^j : P_{d_j+l_j}^j(\bar{z}) \to P_{d_j}^j(e^*) \) is conformal. Clearly \( f^j : P_{d_j+l_j}^j(z) \to P_{d_j}^j(e^*) \) is also conformal, by Theorem 5.8 (2). Therefore \( \text{mod}(P_{d_j+l_j}^j(z) - P_{d_j}^j(z)) \geq \nu \), and hence \( \text{Imp}(z) = \{ z \} \). In particular, \( \text{Imp}(e^*) = \{ e^* \} \).

For \( z \in J(f) - \Gamma_\infty \) with \( T_f(z) \) non critical, or \( z \in \Gamma_\infty \cap J(f) \), the proof of the fact \( \text{Imp}(z) = \{ z \} \) is the same as the quadratic case \(^{[M4]}\). This case involves the so called *thickened puzzle piece* technique, see \(^{[M4]}\) for its construction and \(^{[M4]}\) Lemmas 1.6 and 1.8, Theorem 1.9 for its applications. For this, we skip the details.

The proof of Theorem 5.2 in the persistently recurrent case is finished. \( \square \)

### 5.4. Proof of Theorems 5.1 and 5.2: other cases.

In this part, we assume \( T_f(e^*) \) is either reluctantly recurrent or non recurrent. For Theorem 5.2 (2), a stronger fact that \( J(f) \) has zero Lebesgue measure is proven. Lemmas 5.5 and 5.6 will take effect in the proof. To verify the assumptions of these lemmas, we first show:

**Lemma 5.10.** For any \( z \in J(f) \), there exist integers \( D, m > 0 \) (both independent of \( z \)) and \( n_j \)'s, Jordan disks \( U_j(z) \supseteq U_j^1(z) \supseteq U_j^m(z) \)'s and \( V_z \supseteq V_z' \supseteq V_z'' \) such that

1. \( \{ f^{n_j}(z); j \geq 1 \} \subseteq V_z'' \).
2. \( \deg(f^{n_j}) : U_j(z) \to V_z \leq D \) for all \( j \geq 1 \).
3. \( \text{mod}(V_z - \overline{V_z'}) \geq m \), \( \text{mod}(V_z' - \overline{V_z''}) \geq m \).

**Proof.** Recall that \( P_{L_0+q}^\ast(e^*) \in P_{L_0}(e^*) \). Let \( q > 0 \) be an integer (to be determined later). We first treat the points in \( J(f) - \Gamma_\infty \). For \( z \in J(f) - \Gamma_\infty \), we deal with points in \( J(f) \cap \Gamma_\infty \) or in \( J(f) - \Gamma_\infty \) whose tableau is non critical. The choices of \( D, n_j \)'s, and the Jordan disks can be seen in the proof. The numbers \( m \) and \( q \) will be determined in the final step.

**1.** \( z \in J(f) - \Gamma_\infty \) and \( T_f(z) \) is critical.

**Case 1.** \( T_f(e^*) \) is not recurrent. In this case,

\[
D_{e^*} := \sup_k \deg(f^k|_{P_k(e^*)}) < +\infty.
\]
Let \((L_0 + p + q, n_j), j \geq 1\) be all the \(c^*\)-positions in the tableau \(T_f(z)\). By the tableau rules, we have that for all \(j \geq 1\),
\[
\deg(f^{n_j}) : P_{n_j+L_0}(z) \to P_{L_0}(c) \leq D_{c^*}.
\]
It suffices to take
\[
(U_j(z), U_j'(z), U_j''(z)) = (P_{n_j+L_0}(z), P_{n_j+L_0+p}(z), P_{n_j+L_0+p+q}(z)),
\]
\[(V_z, V'_z, V''_z) = (P_{L_0}(c^*), P_{L_0+p}(c^*), P_{L_0+p+q}(c^*)).
\]

**Case 2.** \(T_f(c^*)\) is reluctantly recurrent. The recurrence of \(T_f(c^*)\) implies that there is an integer \(L \geq L_0\) so that \(P_{L+p}(c^*) \in P_L(c^*)\) and \(P_L(c^*)\) has infinitely many children, say \(P_{L+n_j}(c^*), j \geq 1\). Let \(J\) be the collection of indices \(j \in \mathbb{N}\) so that \(P_{L+p}(c^*)\) has a child \(P_{L+p+q}(c^*)\) with \(l \in [n_j, n_{j+1}) \cap \mathbb{N}\). For each \(j \in J\), let \(m_j \in [n_j, n_{j+1}) \cap \mathbb{N}\) be the first integer so that \(P_{L+m_j+p}(c^*)\) is a child of \(P_{L+p}(c^*)\). Define \(J' \subset J\) by
\[
J' = \{ j \in J; P_{L+p+q}(c^*) \text{ has a child } P_{L+l+p+q}(c^*) \text{ with } l \in [m_j, n_j) \cap \mathbb{N}\}.
\]
The recurrence of \(T_f(c^*)\) implies that \(J'\) is an infinite set.

For each \(j \in J'\), let \(l_j \in [m_j, n_{j+1}) \cap \mathbb{N}\) be the first integer so that \(P_{L+l_j+p+q}(c^*)\) is a child of \(P_{L+p+q}(c^*)\). The choices of \(m_j, l_j\) imply that
\[
\deg(f^{l_j} : P_{L+l_j}(c^*) \to P_L(c^*)) \leq 2 \cdot 2^{p+q}, \forall j \in J'.
\]
For \(c^*\), we take \((V_z, V'_z, V''_z) = (P_L(c^*), P_{L+p}(c^*), P_{L+p+q}(c^*))\) and
\[
(U_j(c^*), U_j'(c^*), U_j''(c^*)) = (P_{L+l_j}(c^*), P_{L+p+l_j}(c^*), P_{L+p+q+l_j}(c^*)).
\]
Let \(z \in J(f) - \Gamma_\infty\) with \(z \neq c^*\) and \(T_f(z)\) critical. For each \(j \in J'\), let \(k_j \geq 0\) be the first integer so that \(f^{k_j} : P_{L+l_j+k_j}(z) \to P_{L+l_j}(c^*)\) is conformal. Fix \(k_j\), let \(s_j \geq k_j\) be the first integer so that \(f^{s_j} : P_{L+p+l_j+s_j}(z) \to P_{L+p+l_j}(c^*)\) is conformal. Fix \(s_j\), let \(t_j \geq s_j\) be the first integer so that \(f^{t_j} : P_{L+p+q+l_j+t_j}(z) \to P_{L+p+q+l_j}(c^*)\) is conformal. Then the degree of \(f^{t_j} : P_{L+l_j+t_j}(z) \to P_{L+l_j}(c^*)\) is bounded by \(2^p+q\). It follows that the degree of \(f^{l_j+t_j} : P_{L+l_j+t_j}(z) \to P_{L}(c^*)\) is bounded by \(2 \cdot 2^p+q \cdot 2^p+q = 2 \cdot 4^p+q\).

We may take
\[
(U_j(z), U_j'(z), U_j''(z)) = (P_{L+l_j+t_j}(z), P_{L+p+l_j+t_j}(z), P_{L+p+q+l_j+t_j}(z)),
\]
\[(V_z, V'_z, V''_z) = (P_L(c^*), P_{L+p}(c^*), P_{L+p+q}(c^*)).
\]

(2). \(z \in J(f) - \Gamma_\infty\) and \(T_f(z)\) is non critical, or \(z \in J(f) \cap \Gamma_\infty\).

Suppose \(z \in J(f) - \Gamma_\infty\) and \(T_f(z)\) is non critical. By the same argument as step 6 in Section 5.3, one can show that \(\text{Imp}(z) = \{z\}\). Based on this fact, we can find integers \(d', d'' > d, n_j\)'s, three puzzle pieces \(P_d, P_{d'}, P_{d''}\) of depths \(d, d', d''\) respectively, with the following properties:
- \(\text{mod}(P_d - P_{d'}) \geq 1\) and \(\text{mod}(P_{d'} - P_{d''}) \geq 1\).
- For \(j \geq 1\) and \(l \in \{d, d', d''\}\), the map \(f^{n_j} : P_{l+n_j}(z) \to P_l\) is proper.
- \(\{f^{n_j}(z); j \geq 1\} \subset P_{d''}\) and \(\deg(f^{n_j}) : P_{d+n_j}(z) \to P_d \leq 2\).
Set $V_z = P_d, V_z' = P_{d'}, V_z'' = P_{d''}$ and
\[
U_j(z) = P_{d+n_j}(z), U_j'(z) = P_{d'+n_j}(z), U_j''(z) = P_{d''+n_j}(z).
\]

For $z \in J(f) \cap \Gamma_\infty$, we replace $P_l$ by $P_l^\infty$ and the argument is similar.

(3). The choice of $q$ and $m$. In fact, for all $z \in J(f)$, we have already chosen puzzle pieces $U_j(z) \supseteq U_j'(z)$ so that $\text{mod}(U_j(z) - \overline{U}_j(z))$ has a lower bound independent of $j$. This implies that $\text{Imp}(z) = \bigcap U_j(z) = \{z\}$. In particular, we have $\text{Imp}(c^*) = \{c^*\}$ implying that $\text{diam}(P_j(c^*)) \to 0$ as $j \to \infty$. It suffices to take $q$ with $P_{l_0+p+q}(c^*) \in P_{l_0+p}(c^*)$. It follows that
\[
m := \inf_{z \in J(f)} \min\{\text{mod}(V_z - \overline{V}_z), \text{mod}(V_z' - \overline{V}_z')\} > 0.
\]
The proof is completed.

Proof of Theorems 5.2 and 5.1 (other cases). For any $z \in J(f)$, the sets $U_j(z), U_j'(z)$’s given by Lemma 5.10 are puzzle pieces satisfying that $\bigcap U_j(z) = \{z\}$ and $\overline{U}_j \cap J(f)$’s are connected. The local connectivity of $J(f)$ at $z$, hence at all points, follows immediately.

By the proof Theorem 5.2 (1), one has $\text{Imp}(z) = \{z\}$ for any $z \in J(f)$. Recall that $\phi_c$ is a bijection between puzzle pieces. Combining these facts, we get a natural extension $\Phi : \mathbb{C} \to \mathbb{C}$ of $\phi : \bigcup \Gamma_k \to \bigcup \overline{\Gamma}_k$ as follows: on the Julia set, we define $\Phi(z)$ as the intersection point of $\bigcap \phi_k(P_k(z))$; in the Fatou components, we define $\Phi(z)$ inductively by $f \circ \Phi = \Phi \circ f$. One may verify that $\Phi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, satisfying $\hat{f} \circ \Phi = \Phi \circ f$ in $\mathbb{C}$.

Lemmas 5.10 and 5.5 imply that $J(f)$ has zero measure. To show that $\Phi$ is quasi-conformal, by Lemma 5.5, it suffices to verify the assumption 2(b). By Lemma 5.6, it reduces to show that $D_z, S(V_z', f^{n_k}(z))$ are bounded by constants independent of $z, k$. This follows from Lemma 5.10.

6. Boundary regularity

In this section, we show

**Theorem 6.1.** Every Type-A or B hyperbolic component is a Jordan disk.

Let $H$ be a hyperbolic component of Type-$\omega \in \{A, B\}$. Recall that $\ell(H)$ is the integer $k \in [0, p) \cap \mathbb{N}$, so that $-c \in U_{c,a}(f_{c,a}^k(c))$.

6.1. Maps on the boundary of $H$. We first show

**Proposition 6.2.** The boundary $\partial H$ consists of parameters $(c, a)$ for which the map $f_{c,a}$ has either a parabolic point or the critical point $-c$ on $\partial A_{c,a}$.

*Proof.* Let $(c, a) \in \partial H$, assume the restriction $f_{c,a}^p : \partial U_{c,a}(c) \to \partial U_{c,a}(c)$ has neither critical point nor parabolic point. Write $g = f_{c,a}^p$ and $V = U_{c,a}(c)$.

By assumption, one has $\overline{V} \cap (g^{-1}(V) - V) = \emptyset$. Let $\phi : \mathbb{C} \setminus \overline{V} \to \mathbb{C} \setminus \overline{D}$ be the Riemann mapping fixing $\infty$, then $G = \phi \circ g \circ \phi^{-1}$ is defined in $\phi(\mathbb{C} \setminus g^{-1}(V))$ and $G(\partial D) = \partial \overline{D}$. By Schwartz reflection, this $G$ can be defined in an annular neighborhood $U$ of $\partial |D|$. 

By assumption, $G$ has no critical point on $\partial \mathbb{D}$ and $\deg(G|_{\partial \mathbb{D}}) = 2$. If $G$ has a non-repelling periodic point, say $q$ with period $k$, on $\partial \mathbb{D}$. The multiplier $\lambda = (G^k)'(q)$ is real because $G(\partial \mathbb{D}) = \partial \mathbb{D}$. It turns out that $q$ is either attracting or parabolic. Let $B$ be its immediate basin, and $A = \phi^{-1}(B) = \phi^{-1}(B \cap (\mathbb{C} \setminus \mathbb{D}))$. Then $A$ is bounded in $\mathbb{C}$ and is stable by $g^k$.

Note that every point of $B$ is attracted to $q$ under iterations of $G^k$, meaning that every point of $A$ is attracted to a boundary point of $\partial V$ by $g^k$, this means that $q$ has a parabolic point on $\partial V$. Contradiction.

Hence the analytic map $G$ has neither critical point nor non-repelling point on $\partial \mathbb{D}$. By Mañé’s theorem [Ma], $\partial \mathbb{D}$ is a hyperbolic set of $G$: there are constants $C > 0$ and $\nu > 1$ such that for all $n \geq 1$

$$||(G^n)'(z)|| \geq C \nu^n.$$

Then one can find an integer $\ell \geq 1$ and two annular neighborhoods $X, Y$ of $\partial \mathbb{D}$ with $X \subset Y \subset U$, such that $G^\ell : X \to Y$ is a proper map of degree $2^\ell$. By pulling back $X \setminus \mathbb{D}$, $Y \setminus \mathbb{D}$ via $\phi$, we get a polynomial-like map $f_{c,a}^\ell : Z_{c,a} \to Y_{c,a}$, where

$$Z_{c,a} = \phi^{-1}(X \setminus \mathbb{D}) \cup V, \quad Y_{c,a} = \phi^{-1}(Y \setminus \mathbb{D}) \cup V.$$

It’s clear that there is a neighborhood $U$ of $(c, a)$, such that for all $(\tilde{c}, \tilde{a}) \in U$, the map $f_{\tilde{c}, \tilde{a}}^\ell$ has exactly one critical value in $\overline{Y}_{c,a}$. This critical value is nothing but $\tilde{c}$. Thus the component $Z_{\tilde{c}, \tilde{a}}$ of $f_{\tilde{c}, \tilde{a}}^-\ell(Y_{c,a})$ that contains $\tilde{c}$ is a disk. Since $Z_{\tilde{c}, \tilde{a}}$ moves holomorphically with respect to $(\tilde{c}, \tilde{a}) \in U$, we may shrink $U$ if necessary so that $Z_{\tilde{c}, \tilde{a}} \in Y_{c,a}$ for all $(\tilde{c}, \tilde{a}) \in U$. In this way, we get a polynomial-like map $f_{c,a}^\ell : Z_{c,a} \to Y_{c,a}$ of degree $2^\ell$ for all $(c, a) \in U$.

However when $(\tilde{c}, \tilde{a}) \in U \cap \mathcal{H}$, its clear that $\pm \tilde{c} \in A_{\tilde{c}, \tilde{a}}$, and the degree of $f_{c,a}^\ell : Z_{\tilde{c}, \tilde{a}} \to Y_{c,a}$ is $(d_\omega + 1)^\ell > 2^\ell$. It’s a contradiction. \hfill $\square$

**Proposition 6.3.** Let $(c, a) \in \partial \mathcal{H}$, $f = f_{c,a}$ and let $\Gamma$ be the $p$-admissible puzzle for $f$ (by Theorem 4.2). Then $-c \in \partial A_{c,a}$ iff $T_f(-c)$ is aperiodic.

**Proof.** If $-c \in \partial W$ for some $W \in \mathcal{B}_{c,a}$, then this $W$ is unique (by Proposition 3.2). For all $d \geq 1$, let $P_d(-c)$ be the puzzle piece of depth $d$ containing $-c$. By the puzzle structure, $\partial P_d(-c) \cap W$ contains two sections of internal rays $R_{c,a}^W(\alpha_d)$ and $R_{c,a}^W(\beta_d)$ with $0 \leq \alpha_d < \beta_d \leq 1$. Clearly when $d$ is large,

$$\alpha_d \leq \alpha_{d+1} \leq \cdots \leq \beta_{d+1} \leq \beta_d.$$

If $T_f(-c)$ is periodic, then its period $l = sp$ for some $s \geq 1$. We have

$$\beta_{d+s} - \alpha_{d+s} = 2^{-s}(\beta_d - \alpha_d)$$

for large $d$. So $\alpha_k$’s and $\beta_k$’s have a common limit $t$, which satisfies $2^st = t \mod Z$. The fact $\bigcap (P_k(-c) \cap \partial W) = \{-c\}$ implies that $R_{c,a}^W(t)$ lands at $-c$, meaning that $-c$ is a periodic point on $J(f)$. This is a contradiction.
If \(-c \notin \partial \mathcal{A}_{c,a}\), then by Proposition 6.2, \(f\) has a parabolic point \(\zeta \in \partial \mathcal{A}_{c,a}\). Clearly \(T_f(\zeta)\) is periodic, with period say \(n \geq 1\). We claim that
\[-c \in P_{d+n}(\zeta) \cup P_{d+n-1}(f(\zeta)) \cup \cdots \cup P_{d+1}(f^{n-1}(\zeta)).\]
In fact, if it is not true, then \(f^n : P_{d+n}(\zeta) \to P_d(\zeta)\) is conformal, implying that \(\zeta\) is repelling (by Schwarz Lemma). This contradicts that \(\zeta\) is parabolic. Hence we get the claim, which implies that \(T_f(-c)\) is periodic. 

6.2. Parameter rays. By Theorem 2.1, the map
\[
\Phi : \mathcal{H} \to \mathbb{D}, \ (c,a) \mapsto B_{c,a}(-c)
\]
is a \(d_\omega\)-fold cover ramified at a single point, where \(B_{c,a}\) is the Böttcher map of \(f_{c,a}^\ell\) defined near \(f_{c,a}^\ell(\mathcal{H})(c)\). Taking \(\Psi = \frac{d\sqrt{T}}{d\zeta}\) yields a conformal map \(\Psi : \mathcal{H} \to \mathbb{D}\). There are \(d_\omega\) choices of \(\Psi\), we may fix one of them.

The parameter ray \(\mathcal{R}(t)\) of angle \(t \in [0,1)\) in \(\mathcal{H}\) is defined by
\[
\mathcal{R}(t) := \Psi^{-1}((0,1)e^{2\pi it}).
\]
The impression of \(\mathcal{R}(t)\) is \(\mathcal{I}(t) := \bigcap_{k \geq 1} \mathcal{S}_k(t)\), where
\[
\mathcal{S}_k(t) = \Psi^{-1}(\{re^{2\pi it}; r \in (1 - 1/k, 1), \theta \in (t - 1/k, t + 1/k)\}).
\]

The following proposition is a sharper version of Proposition 6.2.

**Proposition 6.4.** Let \(t \in [0,1)\) and \((c,a) \in \mathcal{I}(t)\). Write \(V = U_{c,a}(f_{c,a}^\ell(\mathcal{H})(c))\).

1. If \(f_{c,a}\) has a parabolic point, then \(R_{c,a}^{V_{\omega}}(d_\omega t)\) lands at a parabolic point.

2. If \(f_{c,a}\) has no parabolic point, then \(R_{c,a}^{V_{\omega}}(d_\omega t)\) lands at \(-c\).

**Proof.** Let \(\gamma_{c,a} = \gamma_{c,a}(\theta)\) be a graph so that \(\gamma_{c,a} \cap \text{orb}(-c) = \emptyset\) and \(\gamma_{c,a} \cap \mathcal{J}(f_{c,a})\) consists of repelling points. If \(R_{c,a}^{V_{\omega}}(d_\omega t)\) lands at neither \(-c\) (non parabolic case) nor a parabolic point \(p_{c,a}\) (parabolic case), then by Lemma 3.1, there is a neighborhood \(\mathcal{V}\) of \((c,a)\) and an integer \(l > 0\), satisfying that

(a). the graph \(f_{c,a}^{-l}(\gamma_{c,a})\) separates \(R_{c,a}^{V_{\omega}}(d_\omega t)\) from the critical point \(-c\);

(b). \(\gamma_{c',a'} = \gamma_{c',a'}(\theta)\) is a graph moving continuously for \((c',a') \in \mathcal{V}\);

(c). \(-c' \notin f_{c',a'}^{-l}(\gamma_{c',a'})\) for all \((c',a') \in \mathcal{V}\).

By (b) and (c), for all \((c',a') \in \mathcal{R}(t) \cap \mathcal{V}\), the graph \(f_{c',a'}^{-l}(\gamma_{c',a'})\) separates \(R_{c',a'}^{V'}(d_\omega t)\) from \(-c'\), where \(V' = U_{c',a'}(f_{c',a'}^\ell(\mathcal{H}')(c'))\). But this would contradict the fact that when \((c',a') \in \mathcal{R}(t) \cap \mathcal{V}\), one has \(-c' \in R_{c',a'}^{V'}(d_\omega t). \square

**Remark 6.5.** Let \((c,a) \in \mathcal{I}(t)\) and \(f = f_{c,a}\). Proposition 6.4 asserts that if \(f\) has a parabolic cycle, then the landing point \(z\) of \(R_{c,a}^{V_{\omega}}(d_\omega t)\) is parabolic. By Proposition 6.3, \(T_f(-c)\) is periodic, with period say \(m\). Then \(f^m : P_{d+m}(-c) \to P_d(-c)\) defines a renormalization of \(f\). Its filled Julia set \(K = \bigcap P_k(-c) = \text{Imp}(-c)\) satisfies that \(\partial K \cap \partial \mathcal{V} = \{z\}.\)
6.3. Proof of Theorem 6.1. There are three main ingredients in the proof:

- Characterization of the maps on \( \partial \mathcal{H} \) (Proposition 6.4).
- Combinatorial rigidity (Theorem 5.1).
- Holomorphic motion theory [Sl].

Proof. Define \( \mathcal{I}_0(t) \subset \mathcal{I}(t) \) by

\[
\mathcal{I}_0(t) = \{ (c,a) \in \mathcal{I}(t); f_{c,a} \text{ has no parabolic cycle} \}.
\]

Fix \( (c_0, a_0) \in \mathcal{I}_0(t) \), let \( \Gamma_{c_0,a_0} \) be the \( p \)-admissible puzzle of \( f_{c_0,a_0} \) given by Theorem 4.2. By Lemma 3.1, there is a neighborhood \( \mathcal{U} \) of \( (c_0, a_0) \), so that \( \Gamma_{c_0,a_0} \) admits a holomorphic motion \( \Gamma_{c,a} \) for \( (c,a) \in \mathcal{U} \).

Precisely, there is a continuous map \( h : \mathcal{U} \times ((\mathbb{C} - X_{c_0,a_0}) \cup \Gamma_{c_0,a_0}) \to \mathbb{C} \) defined in the way that for any \( ((c,a), z) \in \mathcal{U} \times ((\mathbb{C} - X_{c_0,a_0}) \cup \Gamma_{c_0,a_0}) \), the point \( h((c,a), z) \) is in the dynamical plane of \( f_{c,a} \), with the same equipotential and internal (or external) angle as that of \( z \) in the dynamical plane of \( f_{c_0,a_0} \). In other words, \( z \) and \( h((c,a), z) \) have the same ‘dynamical position’ in their corresponding attracting basins. One may verify that \( h \) is a holomorphic motion parameterized by \( \mathcal{U} \), with base point \( (c_0, a_0) \) (namely \( h((c_0, a_0), \cdot) = id \)). By [Sl], \( h \) can be extended to a holomorphic motion \( \Phi : \mathcal{U} \times \mathbb{C} \to \mathbb{C} \). In particular, for any \( (c,a) \in \mathcal{U} \), the map \( \Phi((c,a), \cdot) : \mathbb{C} \to \mathbb{C} \) is quasiconformal.

For any \( (c,a) \in \mathcal{I}_0(t) \cap \mathcal{U} \), let \( \phi = \Phi((c,a), \cdot)|_{\mathbb{C}} \). The above \( \Gamma_{c,a} \) is nothing but \( \phi(\Gamma_{c_0,a_0}) \). By Proposition 6.4, the critical point \( -c \) for \( f_{c,a} \) has the same ‘dynamical position’ as that of \( -c_0 \) for \( f_{c_0,a_0} \). This implies that for any \( k \geq 1 \), there is a homeomorphism \( \phi_k : \Gamma_{c_0,a_0}^{k} \to \Gamma_{c_0,a_0}^{k} \), which matches \( \phi \) on \( \Gamma_{c_0,a_0}^{k} \cap \Gamma_{c_0,a_0} \). Equivalently, \( f_{c_0,a_0} \) and \( f_{c,a} \) have the same combinatorics up to depth \( k \). Since \( k \) is arbitrary, the maps \( f_{c_0,a_0}, f_{c,a} \) are qc combinatorially equivalent. By Theorem 5.1, we have \( (c,a) = (c_0,a_0) \). This means that \( \mathcal{I}_0(t) \) is at most a singleton. The discreteness of \( \mathcal{I}(t) - \mathcal{I}_0(t) \) and the connectivity of \( \mathcal{I}(t) \) imply that \( \mathcal{I}(t) \) is a singleton. Since \( t \) is arbitrary, we have that \( \partial \mathcal{H} \) is locally connected.

To finish, we show that \( \partial \mathcal{H} \) is a Jordan curve. If this is not true, then \( \mathcal{I}(t_1) = \mathcal{I}(t_2) = \{(c_0,a_0)\} \) for some \( 0 \leq t_1 < t_2 < 1 \) (here \( (c_0,a_0) \) is the same symbol as we used above, without assuming that \( f_{c_0,a_0} \) has no parabolic cycle). Then by Lemma 6.4 and Remark 6.5, the internal rays \( R_{c_0,a_0}(d_\omega t_1) \) and \( R_{c_0,a_0}(d_\omega t_2) \) land at the same point, which is exactly the unique intersection point of \( \text{Imp}(-c) \cap \partial V \), here \( V = U_{c_0,a_0}(f_{c_0,a_0}^{(c_0)}) \). Since \( \partial V \) is a Jordan curve [RY], this implies that

\[
d_\omega t_1 = d_\omega t_2 \mod \mathbb{Z}.
\]

Let \( \mathcal{U}, H \) be defined as above. We may shrink \( \mathcal{U} \) if necessary so that for all \( (c,a) \in \mathcal{U} \), one has \( f_{c,a}(-c) \notin \Gamma_{c,a} \). It follows that \( f_{c,a}^{-1}(\Gamma_{c,a}) \) moves continuously with respect to \( (c,a) \in \mathcal{U} \), and avoids \( -c \) along the motion. Choose \( (c_1,a_1) \in \mathcal{R}(t_1) \cap \mathcal{U} \) and \( (c_2,a_2) \in \mathcal{R}(t_2) \cap \mathcal{U} \) with

\[
\Phi(c_1,a_1) = \Phi(c_2,a_2).
\]
Note that $f_{c_1,a_1}$ and $f_{c_2,a_2}$ are hyperbolic. Let
\[
\psi = H((c_2, a_2), \cdot) \circ H((c_1, a_1), \cdot)^{-1}.
\]

Clearly $\psi$ is a quasi-conformal map from $\mathbb{C}$ to $\mathbb{C}$, holomorphic in $\mathbb{C} - X_{c_1,a_1}$, and $\psi(\Gamma_{c_1,a_1}) = \Gamma_{c_2,a_2}$. We may get a modification $\psi_0$ of $\psi$ so that $\psi_0$ matches $\psi$ in $(\mathbb{C} - X_{c_1,a_1}) \cup \Gamma_{c_1,a_1}$, and $\psi_0$ is the identity map under the Böttcher coordinates defined in $Y = f_{c_1,a_1}^{-M}(\mathbb{C} - X_{c_1,a_1})$, here the integer $M \geq 0$ is chosen so that $f_{c_1,a_1}(-c_1) \in Y$. In this way $\psi_0$ gives a conjugacy between $f_{c_1,a_1}$ and $f_{c_2,a_2}$ on the postcritical set of $f_{c_1,a_1}$.

The relation $\Phi(c_1, a_1) = \Phi(c_2, a_2)$ implies that $f_{c_1,a_1}, f_{c_2,a_2}$ have the same critical dynamical positions. This allows one to get a sequence of qc-maps $\psi_k$’s by the lifting process
\[
f_{c_2,a_2} \circ \psi_{k+1} = \psi_k \circ f_{c_1,a_1}.
\]
so that $\psi_{k+1}$ and $\psi_k$ are isotopic rel the postcritical set of $f_{c_1,a_1}$, holomorphic and identical on $f_{c_1,a_1}^{-k}(Y)$. The dilatations of $\psi_k$’s are uniformly bounded, so they have a limit $\psi$, which is a quasi-conformal map on $\mathbb{C}$, holomorphic in the Fatou set of $f_{c_1,a_1}$. Since $f_{c_1,a_1}$ is hyperbolic, its Julia set has zero measure, we conclude that $\psi$ is a conformal map. One has $\psi = id$ and $(c_1, a_1) = (c_2, a_2)$. This contradicts the fact that $(c_1, a_1) \neq (c_2, a_2)$. \hfill $\square$

**Remark 6.6.** By Theorem 6.1, one can state Proposition 6.4 as follows: Suppose the parameter ray $R(t)$ lands at $(c, a)$. Write $V = U_{c,a}(f_{c,a}^{\ell}(c))$.

1. If $d_{\mathcal{S}}t$ is 2-periodic, then $f_{c,a}$ has a parabolic point on $\partial V$.
2. If $d_{\mathcal{S}}t$ is not 2-periodic, then $-c \in \partial V$.

Lastly, the following fact has some independent interest.

**Proposition 6.7.** Let $\text{Bif}(\mathcal{S}_p)$ be the bifurcation locus of $\mathcal{S}_p$ and $\mathcal{E}$ be a compact subset of $\mathcal{C}(\mathcal{S}_p)$. For any component $\mathcal{U}$ of $\mathcal{S}_p - \mathcal{E}$, either

1. $\mathcal{U}$ is unbounded in $\mathcal{S}_p$, or
2. $\mathcal{U} \subset \mathcal{C}(\mathcal{S}_p)$ and $\text{Bif}(\mathcal{S}_p) \cap \mathcal{U} = \emptyset$.

**Proof.** Assume that $\mathcal{U}$ is bounded in $\mathcal{S}_p$, we will show that $\text{Bif}(\mathcal{S}_p) \cap \mathcal{U} = \emptyset$. To this end, consider the holomorphic maps $F_k : \mathcal{U} \to \mathbb{C}$ defined by
\[
F_k(c, a) = f_{c,a}^k(-c), \quad k \geq 0.
\]

Clearly $\partial \mathcal{U} \subset \mathcal{C}(\mathcal{S}_p)$. So for any $(c, a) \in \partial \mathcal{U}$, one has $F_k(c, a) \in K(f_{c,a})$ (the filled Julia set). By univalent function theory, one has $|\zeta - c| \leq 4$, $\forall \zeta \in K(f_{c,a})$. Therefore
\[
|F_k(c, a)| \leq |c| + 4 \leq \sup_{(c', a') \in \partial \mathcal{U}} |c'| + 4, \quad \forall (c, a) \in \partial \mathcal{U}.
\]

By the maximum modulus principle, the above inequality holds for all $(c, a) \in \mathcal{U}$. Then Montel’s theorem asserts that $\{F_k\}$ is a normal family in $\mathcal{U}$. Equivalently, $\text{Bif}(\mathcal{S}_p) \cap \mathcal{U} = \emptyset$. \hfill $\square$
7. Capture component

The main purpose of this section is to show

**Theorem 7.1.** Every Type-C hyperbolic component is a Jordan disk.

Let $\mathcal{H}$ be a Type-C component and $f_{c,a} \in \mathcal{H}$. Let $l > 0$ be the smallest integer so that $f_{c,a}^l(-c) \in \mathcal{A}_{c,a}$. By Theorem 2.1, the map

$$\Phi : \mathcal{H} \to \mathbb{D}, \ (c, a) \mapsto B_{c,a}(f_{c,a}^l(-c))$$

is a conformal isomorphism, where $B_{c,a}$ is the Böttcher map of $f_{c,a}$ defined in $U_{c,a}(f_{c,a}^l(-c))$. Let $\kappa \in (0, p)$ be the unique integer so that

$$U_{c,a}(f_{c,a}^l(-c)) = U_{c,a}(f_{c,a}^\kappa(c)).$$

The parameter ray $\mathcal{R}(t)$ of angle $t \in \mathbb{R}/\mathbb{Z}$ in $\mathcal{H}$ is $\Phi^{-1}((0, 1)e^{2\pi it})$. The impression $\mathcal{I}(t)$ of $\mathcal{R}(t)$ is defined as $\mathcal{I}(t) = \bigcap_{k \geq 1} \mathcal{S}_k(t)$, where

$$\mathcal{S}_k(t) = \Phi^{-1}(\{re^{2\pi it}, r \in (1 - 1/k, 1), \ t \in (t - 1/k, t + 1/k)\}).$$

Let $v_{c,a} = f_{c,a}(-c)$ be the free critical value. For $(c, a) \in \mathcal{H}$, let $V_{c,a}$ be the Fatou component of $f_{c,a}$ containing $v_{c,a}$. Clearly, the center $\sigma = \sigma(c, a)$ of $V_{c,a}$, defined as the unique point $\sigma \in V_{c,a}$ satisfying $f_{c,a}^{-1}(\sigma) = f_{c,a}^\kappa(c)$, moves continuously with respect to $(c, a) \in \mathcal{H}$. The center map $(c, a) \mapsto \sigma$ has a continuous extension to $\partial \mathcal{H}$. Therefore, when $(c, a) \in \partial \mathcal{H}$, the point $\sigma(c, a)$ and Fatou component $V_{c,a}$ containing $\sigma(c, a)$ are well-defined.

Let $\mathcal{H}_{AB}$ be the union of all Type-A and Type-B hyperbolic components. Clearly, $\mathcal{H}_{AB}$ is compact because it is a closed subset of $\mathcal{C}(\mathcal{S}_p)$.

**Lemma 7.2.** For any $t \in [0, 1)$ and any $(c, a) \in \mathcal{I}(t) \setminus \mathcal{H}_{AB}$, the dynamical internal ray $R_{V_{c,a}}^c(t)$ lands at $v_{c,a} \in \partial V_{c,a}$.

**Proof.** Note that for any $(c_0, a_0) \in \mathcal{I}(t) \setminus \mathcal{H}_{AB}$, there is a disk neighborhood $\mathcal{U}$ of $(c_0, a_0)$ contained in $\mathcal{S}_p \setminus \mathcal{H}_{AB}$.

Let $W_{c,a} := U_{c,a}(f_{c,a}^\kappa(c))$ for $(c, a) \in \mathcal{U}$. We first claim that $\partial W_{c,a}$ moves holomorphically with respect to $(c, a) \in \mathcal{U}$. To see this, we define a map $h : \mathcal{U} \times W_{c_0,a_0} \to \mathbb{C}$ by $h((c, a), z) = (B_{c,a}^{W_{c,a}})^{-1} \circ B_{c_0,a_0}^{W_{c,a}}(z)$, where $B_{c,a}^{W_{c,a}}$ is the Böttcher map of $f_{c,a}$ defined in $W_{c,a}$. It satisfies:

1. Fix any $z \in W_{c_0,a_0}$, the map $(c, a) \mapsto h((c, a), z)$ is holomorphic;
2. Fix any $(c, a) \in \mathcal{U}$, the map $z \mapsto h((c, a), z)$ is injective;
3. $h((c_0, a_0), z) = z$ for all $z \in W_{c_0,a_0}$.

These properties imply that $h$ is a holomorphic motion parameterized by $\mathcal{U}$, with base point $(c_0, a_0)$. By the Holomorphic Motion Theorem [3], there is a holomorphic motion $H : \mathcal{U} \times \mathbb{C} \to \mathbb{C}$ extending $h$ and for any $(c, a) \in \mathcal{U}$, we have $H((c, a), \partial W_{c,a}) = \partial W_{c,a}$. Therefore $\partial W_{c,a}$ moves holomorphically with respect to $(c, a) \in \mathcal{U}$. The claim is proved.

It follows that for any $k \geq 0$, the set $f_{c,a}^{-k}(\partial W_{c,a})$ moves continuously in Hausdorff topology with respect to $(c, a) \in \mathcal{U}$. By the assumption $(c_0, a_0) \in$
By the continuity of \((c, a) \mapsto \partial W_{c,a}\) (implying the continuity of the ray \(R^W_{c,a}(t)\) with respect to \((c, a), t) \in U \times S\), we have \(f^l_{c_0,a_0}(-c_0) \in \partial W_{c_0,a_0}\) and the internal ray \(R^W_{c_0,a_0}(t)\) lands at \(f^l_{c_0,a_0}(-c_0)\). By the continuity of \((c, a) \mapsto f^l_{c,a}(\partial W_{c,a})\) and the fact \(f^{l-1}_{c,a}(R^W_{c,a}(t)) = R^W_{c,a}(t)\) for \((c, a) \in \overline{H} \cap U\), we have that \(v_{c_0,a_0} \in \partial V_{c_0,a_0}\) and \(R^{V_{c_0,a_0}}_{c_0,a_0}(t)\) lands at \(v_{c_0,a_0}\). \(\square\)

To show that \(\partial H\) is a Jordan curve, we need some lemmas, whose proofs are very technical. For each \(t\), let’s define

\[ I^*(t) = I(t) \setminus \overline{H_{AB}}. \]

**Lemma 7.3.** For all \((c, a) \in I^*(t)\), we have either \(f^{l+p}_{c,a}(-c) = f^l_{c,a}(-c)\), or \(f^{l-1}_{c,a}(-c) \notin \partial A_{c,a}\) and \(f^l_{c,a}(-c) \in \partial A_{c,a}\).

**Proof.** By Lemma 7.2, we know \(f^{l-1}_{c,a}(-c) \in \partial f^{l-2}_{c,a}(V_{c,a})\). If \(f^{l-1}_{c,a}(-c) \in \partial A_{c,a}\), then \(f^{l-1}_{c,a}(-c) \in \partial f^{l-2}_{c,a}(V_{c,a}) \cap \partial U_{c,a}(f^{m}_{c,a}(c))\) for some \(m\). Let \(W_1 = f^{l-1}_{c,a}(V_{c,a})\) and \(W_2 = U_{c,a}(f^{m}_{c,a}(c))\), clearly \(W_1, W_2 \in B_{c,a}\). If \(W_1 = W_2\), then necessarily \(f^{l-1}_{c,a}(-c) = -c\), implying that \(-c\) is a periodic point on Julia set, impossible! So we have \(W_1 \neq W_2\) and \(f^l_{c,a}(-c) \in \partial W_1 \cap \partial W_2\). By Proposition 3.2, we get \(f^{l+p}_{c,a}(-c) = f^l_{c,a}(-c)\). \(\square\)

**Lemma 7.4.** For each \(t\), the set \(I^*(t)\) is either empty or a singleton.

**Proof.** If it is not true, then there exist a connected and compact subset \(E\) of \(I^*(t)\) containing at least two points.

By Lemma 7.2, the internal ray \(R^W_{c,a}(t)\) lands at \(v_{c,a}\) for all \((c, a) \in E\). By Thurston’s theorem [DH2], there are only finitely many pairs \((c, a) \in S_p\) for which \(f^{l+p}_{c,a}(-c) = f^l_{c,a}(-c)\). By Lemma 7.3 (and shrink \(E\) if necessary), we may assume all \((c, a) \in E\) satisfy \(f^{l-1}_{c,a}(-c) \notin \partial A_{c,a}\) and \(f^l_{c,a}(-c) \in \partial A_{c,a}\).

By continuity, there is Jordan disk \(D\) with \(E \subset D \subset S_p - \overline{H_{AB}}\), so that for all \((c, a) \in D\), we have \(f^{l-1}_{c,a}(-c) \notin \overline{A_{c,a}}\).

Now take two different pairs \((c_1, a_1), (c_2, a_2) \in E\). Let

\[ J = \{ f^j_{c_1,a_1}(v_{c_1,a_1}); 0 \leq j \leq l - 2 \} \cup A^\infty_{c_1,a_1} \cup A^\infty_{c_1,a_1}. \]

It’s clear that \(J\) contains the post-critical set of \(f_{c_1,a_1}\). We define a continuous map \(h: D \times J \to \overline{C}\) satisfying that

\[ h((c, a), z) = (B^\infty_{c,a})^{-1} \circ B^\infty_{c_1,a_1}(z) \]
for all \((c,a), (c_1,a_1) \in \mathcal{D} \times A_{c_1,a_1}^\infty\); and
\[
h((c,a), z) = (f_{c,a}^{-i}|_{U(c,a)}(f_{c,a}^i(z)))^{-1} \circ B_{c,a}^{-1} \circ B_{c_1,a_1} \circ f_{c_1,a_1}^{-i}(z)
\]
for all \(((c,a), z) \in \mathcal{D} \times U_{c_1,a_1}(f_{c_1,a_1}(c_1))\), \(1 \leq i \leq \kappa\); and
\[
h((c,a), z) = f_{c,a}^{-i} \circ B_{c,a}^{-1} \circ B_{c_1,a_1} \circ (f_{c_1,a_1}^{-i}|_{U(c,a)}(f_{c_1,a_1}(c_1)))^{-1}(z)
\]
for all \(((c,a), z) \in \mathcal{D} \times U_{c_1,a_1}(f_{c_1,a_1}(c_1))\), \(\kappa < i \leq \rho\); and
\[
h((c,a), f_{c_1,a_1}^j(v_{c_1,a_1})) = f_{c,a}^j(v_{c,a})
\]
for all \((c,a) \in \mathcal{D}\) and \(0 \leq j \leq \ell - 2\).

One may verify that \(h\) is a holomorphic motion, parameterized by \(\mathcal{D}\) and with base point \((c_1,a_1)\) (i.e. \(h((c_1,a_1), \cdot) = id\)). By the Holomorphic Motion Theorem [SI], there is a holomorphic motion \(H : \mathcal{D} \times \mathbb{C} \to \mathbb{C}\) extending \(h\).

We consider the restriction \(H_0 = H|_{\mathcal{E} \times \hat{C}}\) of \(H\). Note that fix any \((c,a) \in \mathcal{E}\), the map \(z \mapsto H((c,a), z)\) sends the post-critical set of \(f_{c_1,a_1}\) to that of \(f_{c,a}\), preserving the dynamics on this set. By the lifting property, there is a unique continuous map \(H_1 : \mathcal{E} \times \hat{C} \to \hat{C}\) such that \(f_{c,a}(H_1((c,a), z)) = H_0((c,a), f_{c_1,a_1}(z))\) for all \(((c,a), z) \in \mathcal{E} \times \hat{C}\) and \(H_1((c_1,a_1), \cdot) = id\). It’s not hard to see that \(H_1\) is also a holomorphic motion.

Set \(\psi_0 = H_0((c_2,a_2), \cdot)\) and \(\psi_1 = H_1((c_2,a_2), \cdot)\). Both \(\psi_0\) and \(\psi_1\) are quasi-conformal maps, satisfying \(f_{c_2,a_2} \circ \psi_1 = \psi_0 \circ f_{c_1,a_1}\). One may verify that \(\psi_0\) and \(\psi_1\) are isotopic rel \(J\). Again by the lifting property, there is a sequence of quasi-conformal maps \(\psi_j\) satisfying that

(a). \(f_{c_2,a_2} \circ \psi_{j+1} = \psi_j \circ f_{c_1,a_1}\) for all \(j \geq 0\),

(b). \(\psi_{j+1}\) and \(\psi_j\) are isotopic rel \(f_{c_1,a_1}(J)\).

The maps \(\psi_j\)’s form a normal family because their dilatations are uniformly bounded above. The limit map \(\psi_\infty\) of \(\psi_j\)’s is quasi-conformal in \(\mathbb{C}\), holomorphic in the Fatou set \(\mathcal{F}(f_{c_1,a_1})\) of \(f_{c_1,a_1}\) and satisfies \(f_{c_2,a_2} \circ \psi_\infty = \psi_\infty \circ f_{c_1,a_1}\) in \(\mathcal{F}(f_{c_1,a_1})\). By continuity, \(\psi_\infty\) is a conjugacy on \(\mathbb{C}\). By Theorem 5.2, the Julia set of \(f_{c_1,a_1}\) carries no invariant line fields. Therefore \(\psi_\infty\) is an affine map: \(\psi_\infty(\zeta) = a\zeta + b\). Since \(\psi_\infty\) is tangent to the identity map near \(\infty\), we have \(\psi_\infty = id\) and \((c_1,a_1) = (c_2,a_2)\). This contradicts our choice of \((c_i,a_i)\).

It follows from Lemma 7.4 that the impression \(\mathcal{I}(t)\) is either a singleton or contained in \(\partial \mathcal{H}_{AB}\). To analyze the latter case, we shall prove

Lemma 7.5. If \(\mathcal{I}(t) \subset \partial \mathcal{H}_{AB}\), then \(\mathcal{I}(t)\) is a singleton.

Proof. Choose \((c,a) \in \mathcal{I}(t) \subset \partial \mathcal{H}_{AB}\) so that \(f_{c,a}\) has no parabolic cycle. By assumption, there is a hyperbolic component \(\mathcal{H}_\omega\) of Type-\(\omega \in \{A,B\}\), so that \((c,a) \in \partial \mathcal{H}_\omega \cap \partial \mathcal{H}\). Since \(\mathcal{H}_\omega\) is a Jordan disk (by Theorem 6.1), there is a parameter ray, say \(R_\omega(a)\) in \(\mathcal{H}_\omega\), landing at \((c,a)\).

By Proposition 6.2, the critical point \(-c \in \partial Y\) for some attracting basin \(Y \in \mathcal{B}_{c,a}\). Such \(Y \in \mathcal{B}_{c,a}\) is unique (if not, then by Remark 5.2, we have \(f_{c,a}^p(-c) = -c\), which is impossible). By Proposition 6.4, the internal ray
\(R_{c,a}^y(d_o \alpha)\) lands at \(-c\). It follows that the internal ray \(R_{c,a}^y(\varepsilon(Y)d_o \alpha)\) land at the critical value \(v_{c,a} = f_{c,a}(-c) \in \partial W\), where

\[W = f_{c,a}(Y) \text{ and } \varepsilon(Y) = 1 \text{ if } Y \neq U_{c,a}(c); \varepsilon(Y) = 2 \text{ if } Y = U_{c,a}(c).\]

Let \(L \geq 0\) be the first integer with \(f_{c,a}^L(V_{c,a}) = W\), and \(W' = f_{c,a}^L(W)\). Then

\[f_{c,a}^L(R_{c,a}^y(\varepsilon(Y)d_o \alpha)) = R_{c,a}^y(\varepsilon(Y)2^M d_o \alpha), \quad f_{c,a}^L(R_{c,a}^{V_{c,a}}(t)) = R_{c,a}^y(2^N t),\]

where

\[N = \#\{0 \leq j < L; f_{c,a}^j(V_{c,a}) = U_{c,a}(c)\},\]

\[M = \#\{0 \leq j < L; f_{c,a}^j(W) = U_{c,a}(c)\}.\]

It’s clear that \(N \in \{0, 1\}\) and \(M < L/p + 1\).

In the following, we will show that

\[\varepsilon(Y)2^M d_o \alpha = 2^N t \mod \mathbb{Z}.\]

There are two possibilities:

If \(W' \neq W\) and the rays \(R_{c,a}^y(\varepsilon(Y)2^M d_o \alpha), R_{c,a}^y(2^N t)\) land at the same point, then by Remark 3.3 we have

\[\varepsilon(Y)2^M d_o \alpha = 2^N t = 0 \mod \mathbb{Z}.\]

If \(R_{c,a}^y(\varepsilon(Y)2^M d_o \alpha)\) and \(R_{c,a}^y(2^N t)\) land at different points (no matter \(W = W'\) or not). Let \(\gamma_{c,a}(\theta)\) (see Section 4.2 for definition) be a graph avoiding the orbit of \(-c\) and so that \(\gamma_{c,a}(\theta) \cap J(f_{c,a})\) is a repelling points. By Lemma 3.1 there exist an integer \(K > 0\) and a neighborhood \(V\) of \((c, a)\), with the following three properties:

(a). \(\gamma_{c',a'}(\theta)\) is well-defined and moves continuously when \((c', a') \in V\);

(b). \(v_{c',a'} = f_{c',a'}(-c') \notin f_{c',a'}^{-K-L}(\gamma_{c',a'}(\theta))\) for all \((c', a') \in V\);

(c). when \((c', a') = (c, a)\), the graph \(f_{c,a}^{-K}(\gamma_{c,a}(\theta))\) separates the internal rays \(R_{c,a}^y(\varepsilon(Y)2^M d_o \alpha)\) and \(R_{c,a}^y(2^N t)\).

Note that \(R_{c,a}^y(\varepsilon(Y)d_o \alpha)\) lands at \(v_{c,a}\). By (c), we see that \(f_{c,a}^{-K-L}(\gamma_{c,a}(\theta))\) separates \(R_{c,a}^y(\varepsilon(Y)d_o \alpha) \cup \{v_{c,a}\}\) from \(R_{c,a}^{V_{c,a}}(t)\). Then by properties (a) and (b), we see that \(f_{c,a}^{-K-L}(\gamma_{c,a}(\theta))\) separates \(v_{c,a'}\) from the set \(R_{c,a}^{V_{c,a}}(t)\) for all \((c', a') \in V \cap \mathcal{H}\). This contradicts the fact that when \((c', a') \in V \cap \mathcal{R}(t),\) the critical value \(v_{c,a'} \in R_{c,a}^{V_{c,a}}(t)\).

Therefore each \((c, a) \in I(t)\) either corresponds to a map \(f_{c,a}\) with a parabolic cycle or a landing point of the parameter ray \(\mathcal{R}_\omega(\alpha)\) of a Type-\(\omega \in \{A, B\}\) component \(\mathcal{H}_\omega\), with \(\varepsilon(Y)2^M d_o \alpha = 2^N t \mod \mathbb{Z}\). So \(I(t)\) is a discrete set. The connectedness of \(I(t)\) implies that it is a singleton. \(\square\)

**Proof of Theorem 7.4** By Lemmas 7.4 and 7.5 \(\partial \mathcal{H}\) is locally connected. Assume that two parameter rays \(\mathcal{R}(t_1), \mathcal{R}(t_2)\) with \(t_1 \neq t_2 \mod \mathbb{Z}\), land at the same point \((c, a) \in \partial \mathcal{H}\). Let’s look at the dynamical plane of \(f_{c,a}\). Note that \(\partial V_{c,a}\) is a Jordan curve \(\partial \mathcal{Y}\), the internal rays \(R_{c,a}^{V_{c,a}}(t_1)\) and \(R_{c,a}^{V_{c,a}}(t_2)\) would land at different points. Similar as the proof as Lemma 7.5, let
Hyperbolic components and cubic polynomials

Let \( \gamma_{c,a}(\theta) \) be a graph avoiding the orbit of \(-c\) and so that \( \gamma_{c,a}(\theta) \cap J(f_{c,a}) \) consists of repelling points. By Lemma 3.1, there exist an integer \( q \geq 0 \) and a neighborhood \( V \) of \((c,a)\), satisfying that

(a). the graph \( f_{c,a}^{-q}(\gamma_{c,a}(\theta)) \) separates \( R_{c,a}^{V}(t_1) \) and \( R_{c,a}^{V}(t_2) \).
(b). \( \gamma_{c',a'}(\theta) \) is well-defined and moves continuously when \((c',a') \in V\);
(c). \(-c' \notin f_{c',a'}^{-q}(\gamma_{c',a'}(\theta)) \) for all \((c',a') \in V\);

The continuity of \( f_{c',a'}^{-q}(\gamma_{c',a'}(\theta)) \) implies that it would separated the sets \( R_{c',a'}^{V}(t_1) \) and \( R_{c',a'}^{V}(t_2) \) (with one possibly bifurcating). This implies that \((c,a)\) can not be the landing point of the parameter rays \( \mathcal{R}(t_1) \) and \( \mathcal{R}(t_2) \) simultaneously. This contradicts our assumption.

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