Supersymmetry enhancement of heterotic horizons

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Abstract
The supersymmetry of near-horizon geometries in heterotic supergravity is considered. A necessary and sufficient condition for a solution to preserve more than the minimal $N = 2$ supersymmetry is obtained. A supersymmetric near-horizon solution is constructed which is a $U(1)$ fibration of $AdS_3$ over a particular Aloff–Wallach space. It is proven that this solution preserves the conditions required for $N = 2$ supersymmetry, but does not satisfy the necessary condition required for further supersymmetry enhancement. Hence, there exist supersymmetric near-horizon heterotic solutions preserving exactly $N = 2$ supersymmetry.

Keywords: supergravity, supersymmetry, black holes

1. Introduction
The geometric properties of horizons of supersymmetric black holes are very closely linked to the notion of supersymmetry enhancement. In particular, the presence of additional spinors ensures that the black hole near-horizon solutions have certain symmetries. The first well-understood example of this is the case of the BMPV black hole [1]. The bulk geometry of this solution preserves half of the supersymmetry, whereas the near-horizon solution obtained by taking an appropriate decoupling limit is maximally supersymmetric. Indeed, the systematic analysis of the near-horizon geometries of minimal $N = 2, D = 5$ supergravity constructed in [2] found all possible near-horizon solutions of this theory, including flat space and $AdS_3 \times S^2$, which are also maximally supersymmetric. By exploiting certain similarities between heterotic supergravity and $N = 2, D = 5$ supergravity, this near-horizon supersymmetry doubling was

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also proven to hold for heterotic horizons in [3], utilizing the classification of supersymmetric heterotic solutions in [4, 5]. Consequently, the number of supersymmetries of heterotic near-horizon solutions was shown to be even. Further investigation of near-horizon geometries, utilizing generalized Lichnerowicz theorems, extended this result to \( D = 11 \) supergravity [6], and also to type II supergravities in \( D = 10 \) [7–9]. In particular, as a consequence of this, black hole near-horizon geometries generically admit a \( SL(2, \mathbb{R}) \) symmetry algebra.

The conditions for the heterotic horizons are notably similar to those of the \( N = 2, D = 5 \) horizons. They are also rather stronger than those found for near-horizon geometries in type II and \( D = 11 \) supergravity, for both \( N = 2 \) and \( N \geq 4 \) supersymmetry. In particular, the conditions required for near-horizon geometries to preserve \( N = 4 \) supersymmetry in \( D = 11 \) supergravity were found in [10]. As the structure of the heterotic near-horizon solutions are somewhat simpler, it is most straightforward to find explicit examples of such solutions in the heterotic theory. Some progress was made in [11], where a large class of heterotic horizons were found based on del Pezzo surfaces. However, all of these examples preserve at least \( N = 4 \) supersymmetry, and following [3], it is known that the minimal amount of supersymmetry for heterotic near-horizons is \( N = 2 \).

The purpose of this work is to prove that there do exist solutions preserving exactly \( N = 2 \) supersymmetry. As has been proven in [3], the number of supersymmetries of such solutions must be even, so there can be no \( N = 3 \) solutions. However, it has been unclear if there may be some mechanism whereby supersymmetry might be automatically enhanced from \( N = 2 \) to \( N = 4 \), because hitherto there have been no explicit exactly \( N = 2 \) solutions. We shall establish the existence of exactly \( N = 2 \) solutions first by establishing a particularly simple condition which is necessary and sufficient for a \( N = 2 \) solution to preserve additional \( N \geq 4 \) supersymmetry. Then we proceed to construct an explicit near-horizon geometry, which is a \( U(1) \) fibration of \( AdS_5 \) over a certain seven-dimensional Aloff–Wallach space \( M_{k, \ell} = SU(3)/U(1)_{k, \ell} \) equipped with co-closed \( G_2 \) structure. This solution satisfies the conditions required for \( N = 2 \) supersymmetry, but fails to satisfy the condition which is necessary and sufficient for further supersymmetry enhancement. In particular, this solution is an example of a ‘descendant’ solution, for which the gravitino equation holds for all the spinors, but the dilatino equation does not hold for two of the spinors [12, 13]. This establishes the existence of exactly \( N = 2 \) supersymmetric heterotic near-horizon geometries.

The plan of this paper is as follows. In section 2 we summarize some of the key details concerning supersymmetric heterotic near-horizon geometries derived in [3]. In section 3 we demonstrate how the bosonic field equations, Bianchi identities and Killing spinor equations (KSE) reduce to conditions on a seven-dimensional manifold equipped with a conformally balanced \( G_2 \) structure, and we also establish a Lichnerowicz type theorem for the near-horizon geometries. In section 4, utilizing the Lichnerowicz type theorem, we construct a necessary and sufficient condition for a \( N = 2 \) supersymmetric heterotic near-horizon solution to admit \( N \geq 4 \) supersymmetry. In section 5, an explicit construction of an exactly \( N = 2 \) near-horizon geometry is provided, utilizing a \( G_2 \) structure defined on a certain Aloff–Wallach space. Section 6 contains some conclusions. In appendix A, some useful \( G_2 \) identities are listed. In appendix B, some further details relating to the Aloff–Wallach space used to construct the solution in section 5 are given. In appendix C, properties of the Fernández–Gray [14] classification of \( G_2 \) structures are listed.

2. Supersymmetric heterotic horizons

In this section we summarize some particularly important results for supersymmetric heterotic near-horizon geometries, as found in [3]. These results were found by using a bilinear matching
condition to simplify some of the bosonic fields in the solution. However, as has been shown in [7], these conditions can be obtained independently via a compactness argument.

The $D = 10$ heterotic gravitino and dilatino KSE are given by

$$\nabla_\mu \epsilon - \frac{1}{8} H_{\mu \nu_1 \nu_2} \Gamma^{\nu_1 \nu_2} \epsilon = \mathcal{O}(\alpha'^2)$$

(1)

$$\left( \Gamma^\mu \nabla_\mu \Phi - \frac{1}{12} H_{\nu_1 \nu_2 \nu_3} \Gamma^{\nu_1 \nu_2 \nu_3} \right) \epsilon = \mathcal{O}(\alpha'^3),$$

(2)

where $\mu, \nu$ denote $D = 10$ frame indices and in (1), $\nabla$ denotes the $D = 10$ Levi-Civita connection; $H$ is the three-form and $\Phi$ is the dilaton. The gaugino KSE is given by

$$\Gamma^{\mu \nu} F_{\mu \nu \epsilon} = \mathcal{O}(\alpha').$$

(3)

To zeroth order in $\alpha'$ the conditions involving $F$ decouple completely from the remaining equations, and consequently in this work we are counting the number of solutions of (1) and (2), taking $F = 0$.

We shall be considering supersymmetric solutions which are near-horizon geometries. In what follows, we assume that the eight-dimensional spatial cross section of the event horizon $S$ is smooth and compact without boundary. Following [3], the near-horizon metric can be written in Gaussian null co-ordinates $\{u, r, y^I\}$ where $y^I$ are local co-ordinates on $S$. The metric, in the near-horizon limit, is then

$$ds^2 = 2du(du + rh) + ds^2(S),$$

(4)

where $h$ is a ($u, r$-independent) one-form on $S$, and the metric on $S$ also does not depend on $u, r$.

We remark that the analysis of the supersymmetric near-horizon solutions when $\alpha' \neq 0$ has been done in [17]. It is notable that if $\alpha' \neq 0$, the gaugino equation (3) follows from (1) and (2), together with the bosonic field equations and Bianchi identities. If $\alpha' \neq 0$ then the supersymmetric near-horizon solutions split into two cases. Firstly, if $h$ is covariantly constant, to zero and first order in $\alpha'$, with respect to the metric connection on $S$ whose torsion is equal to the pull-back of $H$ to $S$, then the number of supersymmetries is even (again, to zero and first order in $\alpha'$). However, there is also a case for which this covariant constancy condition on $h$ does not hold to first order in $\alpha'$, and then the number of supersymmetries need not be even. It would be interesting to understand this case better. However, from [3] it is known that the covariant constancy condition on $h$ must hold to zeroth order in $\alpha'$ and consequently at zeroth order in $\alpha'$, the number of supersymmetries is even.

In addition to considering the metric in the near-horizon limit, we assume that the three-form $H$ admits a well-defined near-horizon limit, as considered in [3]. To zeroth order in $\alpha'$, $H$ is closed, with

$$H = du \wedge dr \wedge h + r du \wedge dh \wedge \hat{H},$$

(5)

where $\hat{H}$ is a ($u, r$-independent) closed three-form on $S$. Additional conditions are then obtained from the analysis of the KSE in [3]. These are:

$$\nabla_i H_{ji} = 0$$

(6)

$$\mathcal{L}_H \Phi = 0, \quad \mathcal{L}_H \hat{H} = 0$$

(7)
\[ h^2 = \text{const.} \quad (8) \]

\[ dh = i_h \tilde{H} \quad (9) \]

\[ \nabla^i \tilde{H}_{ij} = 2 \nabla^i \Phi \tilde{H}_{ij} \quad (10) \]

\[ \nabla^2 \Phi = 2 \nabla^i \Phi \nabla_i \Phi + \frac{1}{2} h^2 - \frac{1}{12} \tilde{H}_{ij} \tilde{H}^{ij} \quad (11) \]

\[ R_{ij} = \frac{1}{4} \tilde{H}_{imn} \tilde{H}_{j}^{mn} - 2 \nabla_i \nabla_j \Phi \quad (12) \]

where \( i, j \) are frame indices on \( S \), \( \nabla \) denotes the Levi-Civita connection on \( S \), and \( R_{ij} \) is the Ricci tensor of \( S \). If \( h = 0 \) then it has been shown that \( \tilde{H} = 0 \) and \( \Phi = \text{const.} \), and the near-horizon geometry is \( \mathbb{R}^{1,1} \times S \) where \( S \) is a Spin(7) holonomy manifold. We discard this special case.

In terms of explicitly counting the number of supersymmetries, it is useful to consider some algebraic properties of the Killing spinors, following the analysis of the KSE given in [3]. In particular, the Killing spinor \( \epsilon \) of heterotic near-horizon solutions is determined algebraically in terms of two spinors \( \eta_{\pm} \) on \( S \) via

\[ \epsilon = \frac{u}{2} h \Gamma^{-i} \eta_{-} + \eta_{+} + \eta_{-} \quad (13) \]

where

\[ \Gamma_{\pm} \eta_{\pm} = 0 \quad (14) \]

and \( +, - \) are lightcone directions associated with the Gaussian null co-ordinate system. The \( D = 10 \) KSE decompose into conditions involving only \( \eta_{+} \), and conditions involving only \( \eta_{-} \). Moreover, if \( \eta_{+} \) satisfies the KSE involving \( \eta_{+} \), then \( \eta_{-} \) defined by

\[ \eta_{-} = \Gamma_{-} h \Gamma^{i} \eta_{+} \quad (15) \]

automatically satisfies the KSE involving \( \eta_{-} \). Conversely, if \( \eta_{-} \) satisfies the KSE involving \( \eta_{-} \), then \( \eta_{+} \) defined by

\[ \eta_{+} = \Gamma_{+} h \Gamma^{i} \eta_{-} \quad (16) \]

automatically satisfies the KSE involving \( \eta_{+} \). It follows that the total number of killing spinors \( N \) must be even, \( N = 2N_{+} \), where \( N_{+} \) is equal to the number of positive chirality spinors \( \eta_{+} \).

So, in analyzing the KSE, it suffices to consider the KSE involving only \( \eta_{+} \), which are given by

\[ \nabla_i \eta_{+} = \frac{1}{8} \tilde{H}_{i\bar{k}} \Gamma^{\bar{k}} \eta_{+} \quad (17) \]

\[ dh_{ij} \Gamma^{ij} \eta_{+} = 0 \quad (18) \]

\[ (2 \nabla_i \Phi + h_{i}) \Gamma^{i} \eta_{+} - \frac{1}{6} \tilde{H}_{i\bar{k}} \Gamma^{i\bar{k}} \eta_{+} = 0. \quad (19) \]
We remark that (18) is implied by the other KSE and the bosonic conditions, however we shall retain (18) for convenience.

3. Reduction to seven dimensions

The existence of the isometry generated by $h$ allows one to reduce the KSE (and bosonic field equations) to those on a seven-dimensional manifold $M_7$, where

$$\text{d}s^2(S) = Q^{-2} h \otimes h + \text{d}s^2(M_7)$$

and we have set

$$h^2 = Q^2$$

for non-zero constant $Q$. We shall set

$$h = Q e^8$$

and take frame indices $A, B = 1, \ldots, 7$ to be frame indices on $M_7$. Then from the results of [3],

$$\tilde{H} = Q^{-2} h \wedge \text{d}h + \tilde{H}(\gamma),$$

where $\tilde{H}(\gamma)$ is a three-form on $M_7$, given in terms of the $G_2$ three-form, $\varphi$, by

$$\tilde{H}(\gamma) = Q \varphi + e^{2\Phi} * \gamma \text{d} \left( e^{-2\Phi} \varphi \right).$$

The $G_2$ three-form, $\varphi$, has components given by

$$\| \eta_+ \|^2 \varphi_{ABC} = \langle \eta_+, \Gamma_8^{ABC} \eta_+ \rangle$$

and

$$\| \eta_+ \|^2 \gamma \varphi_{ABCD} = \langle \eta_+, \Gamma_{ABCD} \eta_+ \rangle.$$  

The Bianchi identity $d\tilde{H} = 0$ then implies

$$d\tilde{H}(\gamma) = -Q^{-2} \text{d}h \wedge \text{d}h$$

and $\text{d}h$ is a closed two-form on $M_7$, with

$$\text{d}h \in \mathfrak{g}_2$$

as a consequence of the supersymmetry. In addition, the $G_2$ structure must be conformally co-calibrated

$$d \left( e^{-2\Phi} * \gamma \varphi \right) = 0$$

which implies that

$$\hat{\nabla}_A \Phi = \frac{1}{12} \varphi_A R_{B_1 B_2} \hat{\nabla}^{B_1} \varphi_{DB_1 B_2},$$

1 We remark that in [3], the constant $Q$ was referred to as $k$; however in this work we shall instead reserve $k$ to denote a parameter in the Aloff–Wallach space considered later in section 5.
where $\hat{\nabla}$ denotes the Levi-Civita connection on $M_7$. The gauge field equation (10) is
\[
d (e^{-2\Phi} \ast_7 \tilde{H}(7)) = 0,
\] (31)
which is satisfied as a consequence of (24) and (29). The dilaton equation (11) is equivalent to
\[
\hat{\nabla}^A \hat{\nabla}_A \Phi - 2 \hat{\nabla}^A \Phi \hat{\nabla}_A \Phi + \frac{1}{4} Q^{-2} \, dh_{AB} \, dh^{AB} \\
+ \frac{1}{12} (\tilde{H}(7))_{ABC} \, (\tilde{H}(7))^{ABC} - \frac{1}{2} Q^2 = 0
\] (32)
and (12) implies that
\[
\hat{R}_{AB} = \frac{1}{4} (\tilde{H}(7))_{AMN} (\tilde{H}(7))^{MN}_B + Q^{-2} \, dh^M \, dh_{MB} - 2 \hat{\nabla}_A \hat{\nabla}_B \Phi,
\] (33)
where $\hat{R}_{AB}$ denotes the Ricci tensor of $M_7$. These conditions imply that
\[
\hat{R} = \frac{5}{12} (\tilde{H}(7))_{ABC} (\tilde{H}(7))^{ABC} + \frac{3}{2} Q^{-2} \, dh_{AB} \, dh^{AB} - 4 \hat{\nabla}^A \Phi \hat{\nabla}_A \Phi - Q^2.
\] (34)
The constant $Q$ is not free; it is determined by the $G_2$ structure via
\[
Q = -\frac{1}{144} \ast_7 \phi^{ABCD} d\phi_{ABCD}
\] (35)
as a consequence of the dilatino KSE. In particular, for nontrivial solutions with $Q \neq 0$, one cannot have $d\phi = 0$.
We remark that these conditions can be used to simplify certain components of the Bianchi identity (27). In particular, $dh \in g_2$ implies that $dh^7 = 0$, and hence $(dh \wedge dh)^7 = 0$. This, when combined with (27), implies that the 7 part of $dH_7$ must also vanish, or equivalently
\[
(d\tilde{H}(7))_{AB_1B_2B_3} \, \phi^{B_1B_2B_3} = 0.
\] (36)
This condition can be rewritten as
\[
36Q \hat{\nabla}^N \phi + 6 \hat{\nabla}^N \phi \phi^{B_2B_1} (\ast_7 d\phi)_{AB_1B_2} \\
+ \ast_7 \phi^{B_1B_2B_3} \left( -2 \hat{\nabla}^L \phi \hat{\nabla}^{L} \phi \hat{\nabla}^{B_2B_1} + \hat{\nabla}^L \hat{\nabla}^{B_1B_2} \phi \right) = 0.
\] (37)
Furthermore, it is possible to see that (37) also follows as a consequence of integrability conditions associated with the decomposition of $d\phi$ in terms of torsion classes. In particular, on taking the exterior derivative of (68), dualizing the resulting five-form, and then taking the 7 projection of this expression, one obtains after some computation (37).
Next, we consider the reduction of the KSE. First, note that the $i = 8$ component of (17) can be rewritten, using (19) as
\[
\nabla_8 \eta_+ = 0.
\] (38)
Furthermore, with (18), this is equivalent to
\[
\mathcal{L}_8 \eta_+ = 0,
\] (39)
where \( \mathcal{L} \) here denotes the spinorial Lie derivative. The remaining content of (17)–(19) is equivalent to

\[
\hat{\nabla}_A \eta_+ + \left( -\frac{1}{16} \Gamma_A (2\hat{\nabla}_B \Phi \Gamma^B + Q \Gamma_8) + \frac{1}{96} \Gamma_A B_1 B_2 B_3 (\tilde{H}(7)) B_1 B_2 B_3 \right) \eta_+ = 0
\]

and

\[
\frac{3}{32} (\tilde{H}(7)) A B_1 B_2 B_3 \eta_+ = 0
\]

(40)

and

\[
\frac{3}{32} (\tilde{H}(7)) A B_1 B_2 B_3 \eta_+ = 0
\]

(41)

and

\[
d h_{AB} \Gamma^{AB} \eta_+ = 0.
\]

(42)

The form of (40) is somewhat arbitrary, in the sense that one can without loss of generality add any multiple of \( \Gamma_A \times \) (41) to (40) and obtain an equivalent set of KSEs. However, the particular form of (40) is inspired by the standard embedding of heterotic solutions into IIB theory, as it is known how to formulate a Lichnerowicz type theorem in IIB [7, 18]. We next consider the heterotic Lichnerowicz type theorem, which is derived using the same techniques utilized for the IIB horizons.

3.1. Reduced Lichnerowicz theorems

We shall show that there exist near-horizon Dirac operators \( \mathcal{D}^{(\pm)} \) with the property that \( \mathcal{D}^{(\pm)} \eta_\pm = 0 \) if and only if \( \eta_\pm \) satisfies (40)–(42). To proceed we define

\[
\hat{\nabla}_A \eta_\pm \equiv \hat{\nabla}_A \eta_\pm + \left( -\frac{1}{16} \Gamma_A (2\hat{\nabla}_B \Phi \Gamma^B \pm Q \Gamma_8) + \frac{1}{96} \Gamma_A B_1 B_2 B_3 (\tilde{H}(7)) B_1 B_2 B_3 \right) \eta_\pm
\]

and

\[
\hat{\nabla}_A \eta_\pm \equiv \hat{\nabla}_A \eta_\pm + \left( -\frac{1}{16} \Gamma_A (2\hat{\nabla}_B \Phi \Gamma^B \mp Q \Gamma_8) + \frac{1}{96} \Gamma_A B_1 B_2 B_3 (\tilde{H}(7)) B_1 B_2 B_3 \right) \eta_\pm.
\]

(43)

The reduced horizon Dirac operators associated with (43) are

\[
\mathcal{D}^{(\pm)} \eta_\pm \equiv \Gamma^A \hat{\nabla}_A \eta_\pm + \left( -\frac{1}{24} (\tilde{H}(7)) A B_1 B_2 B_3 \Gamma^{AB} \Phi \mp Q \Gamma_8 \right) \eta_\pm.
\]

(45)

Note that \( \hat{\nabla}_A \Phi \) is the supercovariant derivative appearing in (40), and \( \mathcal{D}^{(\pm)} \) is its associated reduced horizon Dirac equation. Suppose that \( \eta_\pm \) satisfies

\[
\mathcal{D}^{(\pm)} \eta_\pm = 0.
\]

(46)

Then using, (27), (32), (33) and (34), it is straightforward to show, after some computation, that
\[ \nabla^A \nabla_A \| \eta_\pm \|^2 - 2 \nabla^A \Phi \nabla_A \| \eta_\pm \|^2 = 2 \langle \nabla^{(\pm)} \eta_\pm, \nabla^{(\pm)} \eta_\pm \rangle + \frac{1}{8} Q^{-2} \| dh_{\lambda \beta} \Gamma^A \eta_\lambda \|^2 + \frac{9}{128} \| B^{(\pm)} \eta_\pm \|^2. \] (47)

On applying the maximum principle, one obtains
\[ \| \eta_\pm \|^2 = \text{const}. \] (48)

and
\[ \nabla^{(\pm)} \eta_\pm = 0, \quad dh_{\lambda \beta} \Gamma^A \eta_\lambda = 0, \quad B^{(\pm)} \eta_\pm = 0. \] (49)

It follows that if \( \eta_\pm \) satisfies the reduced horizon Dirac equation \( D^{(\pm)} \eta_\pm = 0 \) given by (45), then \( \eta_\pm \) satisfies (49). Conversely, it is straightforward to show that if \( \eta_\pm \) satisfies (49) then \( \eta_\pm \) satisfies the reduced horizon Dirac equation \( D^{(\pm)} \eta_\pm = 0 \) given by (45).

4. A condition for supersymmetry enhancement

In this section we consider the necessary and sufficient conditions for supersymmetry enhancement from \( N = 2 \) to \( N \geq 4 \). To establish the simplest form for such a condition, we shall utilize the Lichnerowicz type theorem which holds for the Killing spinor equation when reduced to \( M_7 \). From [3], it is known that if a \( N = 2 \) solution described by spinors \( \{ \eta_+, \eta_- \} \) admits supersymmetry enhancement to \( N \geq 4 \), then extra spinors are given by \( \{ \Gamma_8 V A \Gamma^A \eta_+, \Gamma_8 V A \Gamma^A \eta_- \} \), where \( V \) is a certain vector field on \( M_7 \).

We aim to obtain the minimal set of conditions on such a \( V \) which are necessary and sufficient to impose supersymmetry enhancement from \( N = 2 \) to \( N \geq 4 \). In particular, suppose that \( \eta_+ \) is a solution of the ‘+’ chirality KSE (17)–(19). Define
\[ \eta'_+ = \Gamma_8 V A \Gamma^A \eta_+. \] (50)

By construction, \( \eta_+, \eta'_+ \) are linearly independent as they are orthogonal with respect to \( \langle , \rangle \). Now consider the KSE. The Lichnerowicz type theorems established previously imply that it suffices to consider \( D^{(\ast)} \eta'_+ \), which is given by
\[ D^{(\ast)} \eta'_+ = \Gamma_8 \left( -\frac{1}{2} dV_{A B} \Gamma^A \Gamma^B + Q V_8 \Gamma^A \right) \eta_+ + \Gamma_8 \left( -\nabla^A \Phi + 2V^A \nabla_A \Phi \right) \eta_+. \] (51)

Hence, there is supersymmetry enhancement if and only if
\[ \left( \frac{1}{2} dV_{A B} \Gamma^A \Gamma^B + Q V_8 \Gamma^A \right) \eta_+ = 0 \] (52)

and
\[ d_{*\gamma} (e^{-2\Phi} V) = 0. \] (53)

The condition (52) is equivalent to
\[ Q V_A - \frac{1}{2} \varphi_{[A} B_{1} B_{2}} \, dV_{B_{1} B_{2}} = 0. \] (54)
or equivalently
\[ dV - \frac{1}{3} QiV \varphi \in g_2. \]  
(55)

Moreover, the condition (29), together with (54), implies (53). Hence, the necessary and sufficient condition for there to be supersymmetry enhancement is that there exists a vector field \( V \) on \( M_7 \) which satisfies (55).

### 5. An exactly \( N = 2 \) heterotic near-horizon solution

In this section, we shall construct explicitly an example of a heterotic near-horizon solution which satisfies the conditions for \( N = 2 \) supersymmetry, but for which the supersymmetry enhancement condition (55) does not hold. This is therefore the first known example of a heterotic near-horizon geometry preserving exactly \( N = 2 \) supersymmetry.

Before presenting the solution, we remark that a number of possible Riemannian manifolds equipped with \( G_2 \) structures are incompatible with the conditions required for \( N = 2 \) supersymmetry. For example, the \( G_2 \) structures constructed in [19, 20] satisfy \( \varphi \wedge d\varphi = 0 \), and hence \( Q = 0 \) as a consequence of (35). We therefore discard this case. Alternatively, we may consider co-calibrated solutions, for which \( d\varphi = \lambda \star \gamma \varphi \). Then (35) implies that \( \lambda = -\frac{1}{7}Q \).

Furthermore, the conformal co-calibration condition (29) implies that \( \Phi = \text{const.} \) It follows that
\[ \tilde{H}(7) = \frac{Q}{7} \varphi \]  
(56)
and hence the Bianchi identity (27) implies
\[ \frac{6}{49} Q \star \gamma \varphi = dh \wedge dh. \]  
(57)

However, this condition can never hold, because in seven dimensions, there must be a non-vanishing vector field \( Z \) such that \( i_Z dh = 0 \), which would imply that \( i_Z \star \gamma \varphi = 0 \), and hence \( Z = 0 \). Hence, there are no co-calibrated solutions which are consistent with the conditions required for \( N = 2 \) supersymmetry. Having eliminated these as possible candidates for constructing a solution, we shall present a \( G_2 \) structure which is compatible with the conditions of \( N = 2 \) supersymmetry. The solution is obtained from the \( G_2 \) structure considered in [21], in which \( M_7 \) is taken to be a compact Aloff–Wallach space \( M_7 = M_{k,\ell} = SU(3)/U(1)_{k,\ell} \), where \( U(1)_{k,\ell} \) is the circle subgroup of \( SU(3) \) given by
\[ U(1)_{k,\ell} := \left\{ \begin{bmatrix} e^{i k \theta} & 0 & 0 \\ 0 & e^{i \ell \theta} & 0 \\ 0 & 0 & e^{-i (k+\ell) \theta} \end{bmatrix} \mid k, \ell \in \mathbb{Z}, |k| + |\ell| \neq 0, \theta \in \mathbb{R} \right\}. \]  
(58)

An example of such a structure was constructed in [22], however for the particular choice of parameters used in that solution, the Bianchi identity (27) fails to hold, for essentially the same reason as in the consideration of co-calibrated structures above.
5.1. $G_2$ structure and the $N=2$ heterotic solution

Let us consider $M_7 = M_{1,0} = SU(3)/U(1)_{1,0}$, where $M_{1,0}$ is an Aloff–Wallach [23] space with $k = 1$ and $\ell = 0$. Then, as is shown in appendix B, the equations of the $G_2$ structure on $M_7$ are given by

$$
\begin{align*}
\text{de}^1 & = -e^{23} - \sqrt{3}e^{45} + e^{57} + e^5 \wedge \chi \\
\text{de}^2 & = e^{13} + \sqrt{3}e^{60} - e^{37} + e^6 \wedge \chi \\
\text{de}^3 & = -e^{12} + e^{56} - 2e^7 \wedge \chi \\
\text{de}^4 & = \sqrt{3}e^{15} - \sqrt{3}e^{26} \\
\text{de}^5 & = -\sqrt{3}e^{14} + e^{27} - e^{36} - e^1 \wedge \chi \\
\text{de}^6 & = -e^{17} + \sqrt{3}e^{24} + e^{35} - e^2 \wedge \chi \\
\text{de}^7 & = e^{16} - e^{25} + 2e^3 \wedge \chi \\
\text{d}\chi & = e^{15} + e^{26} - 2e^{37},
\end{align*}
$$

(59)

where $\{e^1, e^2, \ldots, e^7\}$ is an orthonormal co-frame, namely

$$
\text{d}s^2(M_7) = (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2
$$

(60)

and $\chi$ is a one-form on $M_7$. The fundamental three-form $\varphi$ of the $G_2$ structure is given by

$$
\varphi = e^{123} - e^{167} + e^{257} - e^{356} + e^{145} + e^{246} + e^{347}
$$

(61)

and the dual of $\varphi$ reads

$$
\star_7 \varphi = e^{4567} - e^{2345} - e^{1346} + e^{1247} + e^{2367} + e^{1357} + e^{1256},
$$

(62)

where $\epsilon_{1234567} = +1$. Using (59) and (62), it is straightforward to show that

$$
\text{d}\star_7 \varphi = 0.
$$

(63)

Comparing (63) with (29), it follows that the dilaton is constant, that is $\text{d}\Phi = 0$. Moreover, using (35), we compute

$$
Q = -4.
$$

(64)

The constancy of the dilaton implies that (24) simplifies to

$$
\tilde{H}_(7) = Q\varphi + \star_7 \text{d}\varphi.
$$

(65)

Using (59) and (61), we compute $\tilde{H}_(7)$ via (65), obtaining

$$
\tilde{H}_(7) = -e^{123} + e^{167} + e^{257} + e^{356} - \sqrt{3}e^{145} + \sqrt{3}e^{246}.
$$

(66)

Furthermore, let us define $\text{d}h = Q\text{d}\chi$. Using (59) and (64), it follows that

$$
\text{d}h = -4(e^{15} + e^{26} - 2e^{37}).
$$

(67)
It is straightforward to show that $dh$, given by (67), satisfies (28). Moreover, we have found that the Bianchi identities (27) and the bosonic field equations (31)–(33) are fulfilled by the above configuration.

It is interesting to inquire in which class of [14] our solution lies within. In particular, we investigate which torsion classes vanish. It was shown in [14] that there are 16 distinct classes of $G_2$ manifolds, which can be described in terms of the irreducible representations of the covariant derivative of the $G_2$ fundamental three-form $\varphi$. It is also possible to characterize each class in terms of the irreducible representations of $d\varphi$ and $d\star_7\varphi$ [28]. To be precise, for any $G_2$ structure on a seven-dimensional orientable manifold $M_7$, there exist unique differential forms $\tau_0 \in \Lambda^0(M_7)$, $\tau_1 \in \Lambda^1(M_7)$, $\tau_2 \in \Lambda^2(M_7)$, $\tau_3 \in \Lambda^3(M_7)$ such that

$$d\varphi = \tau_0 \star_7 \varphi + 3\tau_1 \wedge \varphi + \star_7 \tau_3$$

(68)

and

$$d\star_7 \varphi = 4\tau_1 \wedge \star_7 \varphi + \star_7 \tau_2.$$  

(69)

Notice that

$$\tau_0 = \frac{1}{7} \star_7 (\varphi \wedge d\varphi)$$

(70)

and

$$\tau_1 = \frac{1}{12} \theta_\varphi^{(7)}$$

where $\theta_\varphi^{(7)}$ is the Lee form on $M_7$

$$\theta_\varphi^{(7)} = \star_7 (\varphi \wedge \star_7 d\varphi).$$

(71)

The differential forms $\tau_0$, $\tau_1$, $\tau_2$, $\tau_3$ are called intrinsic torsion forms of the $G_2$ structure. Since each of them can be zero or non-zero, there are $2^4 = 16$ distinct classes of $G_2$ structures, as set out in table C1 in appendix C.

To begin the analysis of the torsion classes for our solution, comparing (63) with (69), it follows immediately that $\tau_1 = \tau_2 = 0$. Moreover, comparing (35) with (68), we get

$$\tau_0 = -\frac{6}{7}Q = \frac{24}{7}$$

(72)

thus $\tau_0 \neq 0$. Furthermore, an explicit computation shows that $\tau_3 = \star_7 d\varphi - \tau_0 \varphi$ is given by

$$\tau_3 = \frac{3}{7} (-e^{123} + e^{167} - e^{257} + e^{356}) + \left(\frac{4}{7} - \sqrt{3}\right) e^{145} + \left(\frac{4}{7} + \sqrt{3}\right) e^{246} + \frac{4}{7} e^{347}$$

(73)

thus $\tau_3 \neq 0$. Consulting table C1 of appendix C, it follows that our solution belongs to the $W_1 \oplus W_3$ class (co-closed $G_2$ structure).

In the following, we shall prove that the solution we have found above preserves exactly $N = 2$ supersymmetries. By contradiction, let us assume that it preserves $N \geq 4$ supersymmetries. Then, as we have shown in section 4, this implies that there exists a non-zero vector $V$ on $M_7$ such that (55) is satisfied. Moreover, since $\eta^+$ and $\eta^+_{\gamma} = \Gamma_{\gamma} \mathcal{N} \eta^+$ are Killing spinors, then

$$\mathcal{N} \phi h \eta^+ = 0$$

(74)

and

$$\phi h \mathcal{N} \eta^+ = 0.$$  

(75)
Taking the difference of (74) and (75), we get
\[ i_V dh = 0. \]  
(76)

Substituting (67) into (76), it follows that
\[ V = V_4 e^4, \]  
(77)

where in (77) we use the same symbol \( V \) to denote the one-form which is dual to the vector \( V \). \( V_4 \) is a non-zero constant, since \( \eta'_+ \) has constant norm. Substituting (77) into (55) and using (64), we obtain
\[ -4V_A - \frac{1}{2} V_4 (d e^4)_{B_1 B_2} \epsilon_{A B_1 B_2} = 0. \]  
(78)

Using (59) and (61), it is easy to check that \( d e^4 \in \mathfrak{g}_2 \), which in turn implies that \( V_A = 0 \) by means of (78). Thus, by assuming that our solution preserves \( N = 4 \) supersymmetries, we have reached a contradiction; this means that our solution preserves exactly \( N = 2 \) supersymmetries.

To conclude this section, let us make an additional remark. Using (59), (66) and (77), it can be shown that \( V = e^4 \) is covariantly constant with respect to the connection with torsion, that is
\[ \hat{\nabla}_A V_B = \frac{1}{2} V^C (\hat{H}_7)_{C A B} \]  
(79)

which in turn implies that \( \eta'_+ \) satisfies the minimal \( N = 4 \) gravitino KSE
\[ \hat{\nabla}_A \eta'_+ = \frac{1}{8} (\hat{H}_7)_{A B C} \Gamma^{B C} \eta'_+. \]  
(80)

The failure of (55) to be satisfied corresponds to the fact that \( \eta'_+ \) does not satisfy the \( N = 4 \) dilatino KSE
\[ \left( 2 \hat{\nabla}^B \Phi^B + Q \Gamma_8 \right) \eta'_+ - \frac{1}{6} (\hat{H}_7)_{A B C} \Gamma^{A B C} \eta'_+ = 0. \]  
(81)

Such ‘descendant’ solutions, for which the gravitino equation holds for all the spinors, but the dilatino equation does not hold for all of the spinors satisfying the gravitino equation, have also been considered in [12, 13].

6. Conclusion

We have found that there exists a near-horizon solution of heterotic supergravity preserving exactly \( N = 2 \) supersymmetry, utilizing the family of \( G_2 \) structures constructed in [21]. Although many supersymmetric heterotic near-horizon geometries had previously been found [11], these solutions all preserved at least \( N = 4 \) supersymmetry. The solution found in this paper is the first near-horizon geometry to preserve the minimal \( N = 2 \) supersymmetry. This demonstrates that there is not some additional mechanism for supersymmetry enhancement of near-horizon solutions in the heterotic theory, which would have meant that the minimal amount of supersymmetry preserved would be \( N = 4 \) and not \( N = 2 \).

We remark that this \( N = 2 \) solution has constant dilaton. Indeed, all known supersymmetric heterotic near-horizon solutions have constant dilaton. It would be interesting to determine whether or not this is a generic property, analogous to an attractor mechanism argument.
constancy of scalars for supersymmetric near-horizon geometries holds for some theories, such as ungauged $N = 2, D = 4$ supergravity coupled to $U(1)$ vector multiplets [24], but not for others, e.g. gauged $N = 2, D = 4$ supergravity coupled to $U(1)$ vector multiplets [24–27]. It is therefore not a priori apparent whether or not one might expect the heterotic dilaton $\Phi$ to be generically constant in the near-horizon limit. However, it would be interesting to determine if the 7 component of the Bianchi identity, given in (37), can be used to obtain additional conditions on the dilaton.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Useful $G_2$ identities

A seven dimensional orientable Riemannian manifold $M_7$ with a $G_2$ structure admits a three-form $\varphi$, with Hodge dual $*7\varphi$. These forms satisfy several algebraic identities;

$$\varphi_{ABJ}\varphi^{CDJ} = 2\delta_{CD}^{AB} - *7\varphi_{AB}^{CD}$$

and hence

$$\varphi_{ACD}\varphi^{BCD} = 6\delta_{B}^{A}$$

and

$$\varphi_{ABJ}*7\varphi^{CDLJ} = 6\delta_{DL}^{[A}\varphi^{B]}_{\mathcal{A}]}$$

where $A, B = 1, \ldots, 7$. In addition, we have

$$\epsilon_{A_1A_2A_3}B_1B_2B_3B_4 = -\varphi_{[A_1A_2}[B_1*7\varphi_{A_3]}B_2B_3B_4] + 3\varphi_{[B_1B_2}[A_1*7\varphi_{A_3A_2]}B_3B_4]$$

and

$$*7\varphi_{A_1A_2A_3C}*7\varphi^{R_1B_2B_3C} = 6\delta_{A_1A_2A_3}^{B_1B_2B_3} - 9\delta_{[A_1}[B_1}*7\varphi_{A_2A_3]}B_2B_3] - \varphi_{A_1A_2A_3}\varphi^{B_1B_2B_3}$$
and
\[
\varphi_{A_1A_2A_3}{}^* \varphi_{B_1B_2B_3} = 5\varphi_{A_1A_2}{}^* \varphi_{B_1B_2B_3} + 3\varphi_{A_1B_2}{}^* \varphi_{A_2B_3}^* B_3B_4 - 3\varphi_{B_1B_2B_3}{}^* \varphi_{A_1A_2A_3},
\]
(A.6)

and
\[
\varphi_{[A_1A_2]}^* \varphi_{B_1B_2B_3} = \varphi_{B_1B_2B_3}{}^* \varphi_{[A_1A_2]}^* B_4 - 6\delta_{A_1A_2}^* \varphi_{B_1B_2B_3}^* A_3^*.
\]
(A.7)

The \(q\)-forms on \(M_7\) decompose w.r.t. various irreps of \(G_2\)
\[
\begin{align*}
\Lambda^1 &= \Lambda^1_7 \\
\Lambda^2 &= \Lambda^2_3 \oplus \Lambda^2_{14} \\
\Lambda^3 &= \Lambda^3_1 \oplus \Lambda^3_3 \oplus \Lambda^3_{27} \\
\Lambda^4 &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \\
\Lambda^5 &= \Lambda^5_3 \oplus \Lambda^5_{14} \\
\Lambda^6 &= \Lambda^6_7.
\end{align*}
\]
(A.8)

For our purposes, the projections associated with the two-forms, three-forms and four-forms are of most interest, and we find that
\[
\begin{align*}
(P^7\alpha)_{A_1A_2} &= \frac{1}{3} \alpha_{A_1A_2} - \frac{1}{6} (\varphi_{A_1A_2}^* \varphi_{B_1B_2B_3}^* \alpha_{B_1B_2}), \\
(P^{14}\alpha)_{A_1A_2} &= \frac{2}{3} \alpha_{A_1A_2} + \frac{1}{6} (\varphi_{A_1A_2}^* \varphi_{B_1B_2B_3}^* \alpha_{B_1B_2}). 
\end{align*}
\]
(A.9)

where \(\alpha\) is a two-form. In particular, \(\alpha \in \mathfrak{g}_2\) iff \(\alpha^7 = 0\), which is equivalent to the condition
\[
\varphi_{A}^{BC} \alpha_{BC} = 0.
\]
(A.10)

For the three forms,
\[
\begin{align*}
(P^3\alpha)_{A_1A_2A_3} &= \frac{1}{42} \varphi_{A_1B_2}^* \varphi_{A_2B_3}^* \varphi_{A_3B_1}^* \alpha_{A_1A_2A_3}, \\
(P^7\alpha)_{A_1A_2A_3} &= \frac{1}{3} \alpha_{A_1A_2A_3} - \frac{1}{24} \varphi_{A_1B_2B_3}^* \varphi_{B_1B_2B_3}^* \varphi_{A_1A_2A_3} - \frac{3}{8} \alpha_{A_1B_2A_3} \varphi_{A_1A_2A_3}^* B_1B_2, \\
(P^{27}\alpha)_{A_1A_2A_3} &= \frac{3}{4} \alpha_{A_1A_2A_3} + \frac{1}{56} \varphi_{A_1B_2B_3}^* \varphi_{B_1B_2B_3}^* \varphi_{A_1A_2A_3} + \frac{3}{8} \alpha_{A_1B_2A_3} \varphi_{A_1A_2A_3}^* B_1B_2, 
\end{align*}
\]
(A.11)

where \(\alpha\) is a three-form; and for the four forms
\[
\begin{align*}
(P^3\alpha)_{A_1A_2A_3A_4} &= \frac{1}{168} \alpha_{A_1A_2A_3A_4} (\varphi_{B_1B_2B_3}^* \varphi_{B_1B_2B_3}^* \varphi_{A_1A_2A_3A_4}), \\
(P^7\alpha)_{A_1A_2A_3A_4} &= \frac{1}{4} \alpha_{A_1A_2A_3A_4} - \frac{3}{4} (\varphi_{B_1B_2}^* \alpha_{B_1B_2}^* \varphi_{A_1A_2A_3A_4}^* B_1B_2).
\end{align*}
\]
\[
\begin{align*}
(p^{27} \alpha)_{A_1 A_2 A_3 A_4} &= \frac{1}{96} \alpha_{B_1 B_2 B_3} (\star \gamma \varphi)_{B_1 B_2 B_3 B_4} (\star \gamma \varphi)_{A_1 A_2 A_3 A_4} \\
&+ \frac{3}{4} \alpha_{A_1 A_2 A_3 A_4} + \frac{3}{4} (\star \gamma \varphi)_{B_1} \left[ (\star \gamma \varphi)_{B_1 B_2 B_3 B_4} (\star \gamma \varphi)_{A_1 A_2 A_3 A_4} \right] \\
&+ \frac{1}{224} \alpha_{B_1 B_2 B_3 B_4} (\star \gamma \varphi)_{B_1 B_2 B_3 B_4} (\star \gamma \varphi)_{A_1 A_2 A_3 A_4},
\end{align*}
\]

where \( \alpha \) is a four-form. In particular, for a four-form \( \alpha \), \( \alpha^7 = 0 \) if and only if

\[
\alpha_{ABCL} \varphi^{BCL} = 0.
\]

**Appendix B. Derivation of the \( G_2 \) structure**

In this appendix, we present further details of how the \( G_2 \) structure presented in section 5.1 is derived, following [21]. The structure equations on the eight-dimensional horizon \( S \) are given by

\[
\begin{align*}
\text{df}^1 &= -f^{23} - \sqrt{3} f^{45} + f^{67} + f^{58} \\
\text{df}^2 &= f^{13} + \sqrt{3} f^{16} - f^{57} + f^{68} \\
\text{df}^3 &= -f^{12} + f^{56} - 2 f^{78} \\
\text{df}^4 &= \sqrt{3} f^{15} - \sqrt{3} f^{26} \\
\text{df}^5 &= -\sqrt{3} f^{14} + f^{27} - f^{56} - f^{18} \\
\text{df}^6 &= -f^{17} + \sqrt{3} f^{24} + f^{35} - f^{28} \\
\text{df}^7 &= f^{16} - f^{25} + 2 f^{38} \\
\text{df}^8 &= f^{15} + f^{26} - 2 f^{37}.
\end{align*}
\]

In (B.1), to simplify the solution we construct, we have set the parameters of the \( G_2 \) structure in [21] to the following values

\[
a = b = c = d = \frac{1}{\sqrt{2}}, \quad k = 1, \quad \ell = 0.
\]

The metric on \( S \) is given by

\[
\text{d}s^2(S) = (f^8)^2 + \text{d}s^2(M_7),
\]

where

\[
\text{d}s^2(M_7) = (f^1)^2 + (f^2)^2 + (f^3)^2 + (f^4)^2 + (f^5)^2 + (f^6)^2 + (f^7)^2
\]

and the fundamental three-form \( \varphi \) of the \( G_2 \) structure is given by

\[
\varphi = f^{123} - f^{167} + f^{257} - f^{356} + f^{145} + f^{246} + f^{357}.
\]

In order to reduce the structure equations (B.1) down to \( M_7 \), consider the frame transformation

\[
f^A \to e^A = X^A_B f^B,
\]
where $A, B = 1, 2, \ldots, 7$ and $X \in SO(7)$. Enforcing the requirement
\[ \mathcal{L}_X e^A = 0 \] (B.7)
amounts to imposing the differential equation
\[ \frac{\partial}{\partial \tau} X^A_B - X^A_C C^C_{B8} = 0, \] (B.8)
where the constants $C^C_{B8}$ are defined by $(i, j = A, 8)$
\[ \text{df}^i = \frac{1}{2} C^i_{\mu f j} f^j \wedge f^k \] (B.9)
and we have set $\partial = \frac{\partial}{\partial \tau}$, for some local coordinate $\tau$. Moreover, we take the one-form dual to $\partial$ to be
\[ f^8 = d\tau + \chi, \] (B.10)
where $\chi$ is a one-form on $M_7$. Notice that $\text{df}^8 = d\chi$. The solution of (B.8) is given by
\[
X^A_B = \begin{bmatrix}
\cos \tau & 0 & 0 & 0 & \sin \tau & 0 & 0 \\
0 & \cos \tau & 0 & 0 & 0 & \sin \tau & 0 \\
0 & 0 & \cos(2\tau) & 0 & 0 & 0 & -\sin(2\tau) \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\sin \tau & 0 & 0 & 0 & \cos \tau & 0 & 0 \\
0 & -\sin \tau & 0 & 0 & 0 & \cos \tau & 0 \\
0 & 0 & \sin(2\tau) & 0 & 0 & 0 & \cos(2\tau)
\end{bmatrix}.
\] (B.11)
Notice that $X = (X^A_B) \in G_2$. In order to show that, rewrite (B.8) as follows
\[ C^M_{B8} = (X^{-1})^M_A \frac{\partial}{\partial \tau} X^A_B. \] (B.12)
Using (B.1), it can be easily checked that $\omega_{AB} := C_{AB8} = \delta_{AC} C^C_{B8}$ is a two-form on $M_7$ and $\omega_{AB} C^C_{AB} = 0$, thus $\omega \in g_2$. Equation (B.12) then implies that $X \in G_2$. In turn, this means that the three-form $\varphi$, defined by (B.5), is left unchanged by (B.6), that is
\[ \varphi = e^{123} - e^{167} - e^{357} - e^{356} + e^{145} + e^{246} + e^{357} \] (B.13)
which coincides with (61). Eventually, using (B.1), (B.6) and (B.11), a tedious but straightforward computation yields (59).

**Appendix C. The 16 classes of $G_2$ manifolds**

The 16 distinct classes of $G_2$ structures, determined in terms of their torsion classes, are summarized in the following table [14]:

where $A, B = 1, 2, \ldots, 7$ and $X \in SO(7)$. Enforcing the requirement
\[ \mathcal{L}_X e^A = 0 \] (B.7)
amounts to imposing the differential equation
\[ \frac{\partial}{\partial \tau} X^A_B - X^A_C C^C_{B8} = 0, \] (B.8)
where the constants $C^C_{B8}$ are defined by $(i, j = A, 8)$
\[ \text{df}^i = \frac{1}{2} C^i_{\mu f j} f^j \wedge f^k \] (B.9)
and we have set $\partial = \frac{\partial}{\partial \tau}$, for some local coordinate $\tau$. Moreover, we take the one-form dual to $\partial$ to be
\[ f^8 = d\tau + \chi, \] (B.10)
where $\chi$ is a one-form on $M_7$. Notice that $\text{df}^8 = d\chi$. The solution of (B.8) is given by
\[
X^A_B = \begin{bmatrix}
\cos \tau & 0 & 0 & 0 & \sin \tau & 0 & 0 \\
0 & \cos \tau & 0 & 0 & 0 & \sin \tau & 0 \\
0 & 0 & \cos(2\tau) & 0 & 0 & 0 & -\sin(2\tau) \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\sin \tau & 0 & 0 & 0 & \cos \tau & 0 & 0 \\
0 & -\sin \tau & 0 & 0 & 0 & \cos \tau & 0 \\
0 & 0 & \sin(2\tau) & 0 & 0 & 0 & \cos(2\tau)
\end{bmatrix}.
\] (B.11)
Notice that $X = (X^A_B) \in G_2$. In order to show that, rewrite (B.8) as follows
\[ C^M_{B8} = (X^{-1})^M_A \frac{\partial}{\partial \tau} X^A_B. \] (B.12)
Using (B.1), it can be easily checked that $\omega_{AB} := C_{AB8} = \delta_{AC} C^C_{B8}$ is a two-form on $M_7$ and $\omega_{AB} C^C_{AB} = 0$, thus $\omega \in g_2$. Equation (B.12) then implies that $X \in G_2$. In turn, this means that the three-form $\varphi$, defined by (B.5), is left unchanged by (B.6), that is
\[ \varphi = e^{123} - e^{167} - e^{357} - e^{356} + e^{145} + e^{246} + e^{357} \] (B.13)
which coincides with (61). Eventually, using (B.1), (B.6) and (B.11), a tedious but straightforward computation yields (59).

**Appendix C. The 16 classes of $G_2$ manifolds**

The 16 distinct classes of $G_2$ structures, determined in terms of their torsion classes, are summarized in the following table [14]:
### Table C1. The 16 classes of \( G_2 \) manifolds in terms of the intrinsic torsion forms \( \tau_i \).

| Class       | Defining relation |
|-------------|-------------------|
| \( \mathcal{P} \)       | \( \tau_0 = \tau_1 = \tau_2 = \tau_3 = 0 \) |
| \( \mathcal{W}_1 \)    | \( \tau_1 = \tau_2 = \tau_3 = 0 \) |
| \( \mathcal{W}_2 \)    | \( \tau_0 = \tau_1 = \tau_3 = 0 \) |
| \( \mathcal{W}_3 \)    | \( \tau_0 = \tau_1 = \tau_2 = 0 \) |
| \( \mathcal{W}_4 \)    | \( \tau_0 = \tau_2 = \tau_3 = 0 \) |
| \( \mathcal{W}_1 \oplus \mathcal{W}_2 \) | \( \tau_1 = \tau_3 = 0 \) |
| \( \mathcal{W}_1 \oplus \mathcal{W}_3 \) | \( \tau_1 = \tau_2 = 0 \) |
| \( \mathcal{W}_1 \oplus \mathcal{W}_4 \) | \( \tau_0 = \tau_1 = 0 \) |
| \( \mathcal{W}_2 \oplus \mathcal{W}_3 \) | \( \tau_2 = \tau_3 = 0 \) |
| \( \mathcal{W}_2 \oplus \mathcal{W}_4 \) | \( \tau_0 = \tau_3 = 0 \) |
| \( \mathcal{W}_3 \oplus \mathcal{W}_4 \) | \( \tau_0 = \tau_2 = 0 \) |
| \( \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \) | \( \tau_1 = 0 \) |
| \( \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4 \) | \( \tau_3 = 0 \) |
| \( \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \) | \( \tau_2 = 0 \) |
| \( \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \) | \( \tau_0 = 0 \) |
| \( \mathcal{W} \)         | No relation |

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