A dual foliation treatment of General Relativity is presented. The basic idea of the construction is to consider two foliations of a spacetime by spacelike hypersurfaces and relate the two geometries. The treatment is expected to be useful in various situations, and in particular whenever one would like to compare objects represented in different coordinates. Potential examples include the construction of initial data and the study of trapped tubes. It is common for studies in mathematical relativity to employ a double-null gauge. In such studies local well-posedness is treated by referring back, for example, to the generalized harmonic formulation, global properties of solutions being treated in a separate formalism. As a first application of the dual foliation formulation we find that one can in fact obtain local well-posedness in the double-null coordinates directly, which should allow their use in numerical relativity with standard methods. With due care it is expected that practically any coordinates can be used with this approach.

I. INTRODUCTION

For their consideration as an initial value problem the field equations of General Relativity are typically split into a set of evolution and constraint equations. This is done by introducing coordinates $x^\mu = (t, x^i)$. The level sets of the time coordinate $t$ are taken to be spacelike hypersurfaces which foliate the spacetime. The unit normal to the hypersurfaces is used to 3+1 decompose the field equations in the natural way. This results in the vacuum field equations in the textbook form \[1\,2,\]

\[ \begin{align*}
\partial_t \gamma_{ij} & = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}, \\
\partial_t K_{ij} & = -D_i D_j \alpha + \alpha[R_{ij} - 2K_{ij} K_{kl} + K K_{ij}] + \mathcal{L}_\beta K_{ij}, \\
H & = R - K_{ij} K^{ij} + K^2 = 0, \\
M_i & = D^j K_{ij} - D_i K = 0. 
\end{align*} \]

(1)

Given two sets of observers, one associated with $x^\mu$, another with coordinates $X^\mu = (T, X^i)$, what is the relationship between the solutions as expressed in each in the 3+1 picture? Unfortunately a clear presentation of the resulting formalism is not readily available, despite being straightforwardly obtained by space-time decomposition of the four-dimensional Jacobians $J_{\mu \nu} = \partial X^\mu/\partial x^\nu$. The first aim of this work is to give just such a presentation, which can be found in section \[11\]. This dual foliation approach will be useful in numerical relativity, where one expects it will help in the construction of initial data and in the comparison of solutions constructed with different choices of lapse $\alpha$ and shift $\beta^i$.

Amongst the most powerful machinery in mathematical relativity is that of the double-null coordinates. With this choice the field equations exhibit a particular structure that allows the demonstration of the stability of Minkowski spacetime \[4\], and that a certain special class of vacuum initial data will collapse to form blackholes \[5\]. One would thus like to use these coordinates in numerical relativity, preferably with standard methods. A number of applications present themselves; the conjectured instability of Cauchy horizons \[7\], the propagation of weak-null singularities \[7\], and the critical formation of blackholes \[8\]. Unfortunately from this perspective the proofs of these impressive results employ a different formalism for long-term results and local existence. This is a serious bugbear because, as painful experience has taught, a necessary condition for any numerical method to converge is that the PDE problem is locally well-posed. Therefore the second aim of this work is to find such a formulation. In section \[11\] this is attempted in a direct way. The normal approach is to modify the equations by introducing new constraints, coupled in a particular way to the gauge choice, and insodoing uncover, say, a strongly hyperbolic formulation. But we find in a pure gauge analysis that the standard form of the double-null coordinates are only weakly hyperbolic, and so cannot be used in this way \[9\,11\]. With appropriate alterations, there may be such a straightforward formulation, but because a preferred direction is singled out the construction will in any case be complicated. We thus abandon the search.

In section \[11\] we summarize the first order reduction \[12\] of the generalized harmonic gauge (GHG) formulation \[13\,15\] employed in the numerical relativity codes SpEC \[16\] and bamps \[17\,18\]. We use the dual foliation formalism to derive equations of motion for the Jacobian that maps from generalized harmonic to double-null coordinates. These equations are minimally coupled to the field equations, and so their hyperbolicity may be treated easily. Indeed the Jacobians satisfy a set of nonlinear advection equations, and so are hyperbolic. We may consider the full set of fields to be solved for as the GHG system with the Jacobians tacked on. We can subsequently change independent variables from $x^\mu \rightarrow X^\mu$ with $X^\mu$ the double-null coordinates. The punchline is that since the system has at most first derivatives, we can do so without generating any derivatives of the evolved Jacobians. Therefore the PDE properties of the system are unaffected and we end up with a formulation that is symmetric hyperbolic directly in the double-null coordinates. Weaker notions of hyperbolicity are also considered. Concluding remarks are collected in section \[11\].
II. THE DUAL FOLIATION FORMALISM

In this section we work in the intersection of two coordinate patches \( x^\mu = (t, x^i) \) and \( X^\mu = (T, X^i) \), related by the Jacobian \( J^{\mu}_{\nu} = \partial X^\mu / \partial x^\nu \). The two time coordinates define distinct foliations of the spacetime, and with them different notions of spacelike tensors, intrinsic and extrinsic curvatures. These quantities are related in the natural way with a \( 3 + 1 \) split of the Jacobian. Consequently the form of the gravitational field equations in each foliation is related. Throughout latin indices \( a, b, c, d, e \) will be abstract. Greek indices stand for those in coordinates \( x^\mu \), or if underlined in \( X^\mu \). Similarly latin indices \( i, j, k, l, m, p \) stand for spatial components in \( x^\mu \), and when underlined for spatial components in \( X^\mu \). Indices \( n \) and \( v \) denote contraction in that slot with the vectors \( n^a \) and \( v^a \) respectively. The summation convention is always employed.

A. Coordinate freedom

Coordinate change under a \( 3 + 1 \) decomposition: Consider two sets of coordinates \( x^\mu \) and \( X^\mu \) defined in the same region of spacetime. Each of the two time coordinates \( t \) and \( T \) naturally defines a foliation of the spacetime which we will refer to as the lower case and upper case foliations respectively. In the lower case foliation we define the standard lapse, normal vector, time vector, projection operator, and shift vector,

\[
\alpha = (-\nabla_a t \nabla^a t)^{-\frac{1}{2}}, \quad n^a = -\alpha \nabla^a t, \quad t^a \nabla_a t \equiv 1, \quad \perp^a b = \delta^a b + n^a n_b, \quad \beta_a = \perp^b a \beta_b, \quad \beta^a = -\alpha n^a \nabla_a x^t. \tag{2}
\]

With both indices downstairs the projection operator \( \perp^a b \) becomes the natural induced metric \( \gamma_{ab} \) on the lower case foliation. The covariant derivative associated with \( \gamma_{ab} \) is denoted by \( D \) with connection \( \Gamma \). The same definitions are made in the upper case foliation,

\[
A = (-\nabla_a T \nabla^a T)^{-\frac{1}{2}}, \quad N^a = -A \nabla^a T, \quad T^a \nabla_a T \equiv 1, \quad \perp^a b = \delta^a b + N^a N_b, \quad B_a = \perp^b a B_b, \quad B^a = -A N^a \nabla_a X^t. \tag{3}
\]

The covariant derivative associated with \( \perp^a b \) of the \( (N) \) with connection \( (N) \Gamma \).

The Lorentz factor and boost vector: The unit normal vectors of the upper and lower case foliations are related by

\[
N^a = W(n^a + v^a), \tag{4}
\]

where we have defined the Lorentz factor \( W \) and lower case boost vector \( v_a \),

\[
W = -(N^a n_a), \quad v_a = \frac{1}{W} \perp^b a N_b. \tag{5}
\]

TABLE I: A summary of the definitions of the various metrics, time reduction variables the relationship between them. The fourth column gives the object used as a time reduction variable when employing the given metric. The final column gives states the equation numbers relating the curvature of the given metric with that of the others. ‘GCM’ stands for the Gauss-Codazzi-Mainardi equations.

| Metric | Connection | Defn. | Time der. | Curv. | Note |
|--------|------------|-------|-----------|-------|------|
| \( g_{ab} \) | \( \nabla, (N) \Gamma \) | (4) | \( \gamma_{ab} \) | \( D, \Gamma \) | \( \perp_{ab} \) | (6) |
| \( \gamma_{ab} \) | \( D, \Gamma \) | (7) | \( g_{ab} \) | \( D, G \) | \( \perp_{ab} \) | \( \perp_{ab} \) |

Since the normal vectors have unit magnitude the Lorentz factor and boost vector satisfy,

\[
W = \frac{1}{\sqrt{1 - v_i v^i}}, \quad W \geq 1 > \gamma_{ij} v_i v_j \equiv v^2. \tag{6}
\]

This is simply the requirement that the upper and lower case normal observers travel subluminally relative to one another. Observe that we also have,

\[
n^a = W(N^a + V^a), \tag{7}
\]

with the upper case boost vector,

\[
V_a = \frac{1}{W} \perp^b a n_b = (W^{-1} - W)n_a - Wv_a, \tag{8}
\]

so that there is a natural reciprocity between the coordinate systems as expected. We thus also obtain,

\[
W = \frac{1}{\sqrt{1 - v_i v^i}}. \tag{9}
\]

Provided the spatial boost vector, a vector \( S_a \) which is spacelike in the upper case foliation, \( S_a N^a = 0 \), can be reconstructed from its projection into the lower case foliation with,

\[
S_a = (\perp S)_b (\delta^b a + v^b n_a), \tag{10}
\]

and obviously a similar result holds for all tensor valences. Therefore we may restrict our attention of upper case spacelike tensors to their projections into the lower case foliation, and thus look only at the spatial components in the lower case tensor basis.

Decomposition of the Jacobian \( J^{\mu}_{\nu} \): Upon \( 3 + 1 \) decomposition the Jacobian \( J^{\mu}_{\nu} = \partial X^\mu / \partial x^\nu \) can be written,

\[
n^a J^{\mu}_{\nu} N_\nu = -W, \quad n^a J^{\mu}_{\nu} \equiv \pi^a, \quad J^{\mu}_{\nu} N_\nu = Wv_\nu, \quad J^{\mu}_{\nu} \equiv \phi^\nu. \tag{11}
\]

The components \( \pi^a \) are given in terms of the upper case lapse, shift and boost vectors by,

\[
\pi^a = W V^a - W A^{-1} B^a. \tag{12}
\]
although we rarely find that this is the most convenient form for the expression. In matrix form we therefore have the representation,

$$J = \begin{pmatrix}
A^{-1}W(\alpha - \beta v_i) & \alpha \pi_i + \beta \phi_i \\
-\alpha^{-1}Wv_i & \phi_i
\end{pmatrix} . \quad (13)
$$

Note that by introducing the variables $A^{-1}Wv_i$ and $\phi_i$ to replace first order spatial derivatives of the coordinates we have effectively introduced reduction constraints,

$$D_{[i}A^{-1}Wv_{j]} = 0, \quad D_{[i}\phi_{j]} = 0 , \quad (14)$$

which we will refer to as the hypersurface (orthogonality) constraints. Here and in what follows one must be careful to note that the upper case underlined index is to be treated as a simple label rather than an open slot when working in the lower case coordinates. It is straightforwardly checked that the transformation,

$$J^{-1} = \begin{pmatrix}
\alpha^{-1}W(A - B^\alpha_i) & A\Pi^i + B\omega \phi_i \\
-\alpha^{-1}Wv_i & \Phi_i
\end{pmatrix} , \quad (15)
$$

with the various auxiliary quantities defined in the natural way, is indeed the inverse transformation.

**Time development of the Jacobian:** By the equality of mixed partials we have the Hamilton-Jacobi like equations,

$$\partial_t(A^{-1}Wv_i) = -D_i[\alpha(A^{-1}W)] + \mathcal{L}_\beta(A^{-1}Wv_i) ,
\partial_t\phi_i = D_i(\alpha \pi_i) + \mathcal{L}_\beta \phi_i , \quad (16)$$

for the components $J^L_i$ and $J_\beta$. These expressions hold regardless of the upper case coordinate choice, but the remaining four components can be determined only once a particular coordinate choice is known. Perhaps the simplest useful example is the generalized harmonic gauge choice $\Box X_l = H_l$, which results in,

$$\partial_t(A^{-1}W) = \alpha(A^{-1}W)(K + E) - D^i(\alpha A^{-1}Wv_i)
+ \mathcal{L}_\beta(A^{-1}W) ,
\partial_t\pi_i = D^i(\alpha \phi_i) + \alpha E^i + \mathcal{L}_\beta \pi_i , \quad (17)$$

where the gauge source functions are decomposed as $E = (A/W)H^L$ and $E^L = -H_L$. The lower case extrinsic curvature is defined by,

$$K_{ab} = -\mathcal{L}_{\alpha}^a \nabla_c n_b , \quad (18)$$

and likewise in the upper case foliation, except that as elsewhere we append a label $N$. On a given spacetime with coordinates $x^a$ the system \[(10), (17)\] can be viewed as a simple first order reduction of the four wave equations $\Box X_l = H_l$. Other choices will be more conveniently expressed once the relationship between the two induced geometries are known.

**Equations of motion for projected upper case objects:** The equations of motion of the projection of upper case spacelike tensors $S_a$ and $S_{ab}$ projected into the lower case foliation are,

$$\partial_t S_i = \frac{\alpha}{W} \mathcal{L}_N S_i + \mathcal{L}_{(\beta - \alpha \omega)} S_i ,
\partial_t S_{ij} = \frac{\alpha}{W} \mathcal{L}_N S_{ij} + \mathcal{L}_{(\beta - \alpha \omega)} S_{ij} , \quad (19)$$

for vectors and symmetric tensors respectively. Similar expressions hold for arbitrary valences but will not be used in what follows. The projected upper case induced metric defined by $g_{ab} = \gamma^c \gamma^d b^{(n)c}_{\gamma d}$ is,

$$g_{ij} = (\gamma^i)^{(n)}_{\gamma j} = \gamma_{ij} + W^2 v_i v_j . \quad (20)$$

Note that as the sum of a symmetric positive definite matrix and semi-positive definite combination of the boost vector, the projected upper case metric is itself positive definite and thus invertible, and can be considered a metric on leaves of the lower case foliation in its own right, if we so wish. When doing so we will refer to this object as the boost metric, and denote the covariant derivative by $D$ with connection $\mathcal{G}$. The boost metric has inverse,

$$(\mathcal{G}_{ij}^{-1})^{ij} = \gamma^{ij} - v^iv^j , \quad (21)$$

by the Sherman-Morrison formula. We now aim to relate the geometry of the upper and lower case foliations in terms of the lower case normal, Lorentz factor and boost vector. A summary of the relationships between the four different metrics $g_{ab}, \gamma_{ab}, (\gamma_{ab})^{(n)}$ and $g_{ab}$ is given in Table II.

**Connections and curvatures:** The Christoffel symbol associated with the upper case induced metric is given by the standard expression,

$$(\gamma_{ab})^{(n)} = (\gamma_{ab})^{(4)}(\gamma_{ab})^{(4)} , \quad (22)$$

which holds in arbitrary coordinates, and where here and in what follows we use the projection operator without indices to denote the projection on every open slot. The Christoffel symbol associated with the lower case induced metric is defined similarly. By the argument around equation \[(10)\] we need only compute the projection of the upper case Christoffel into the lower case foliation. Using \[(20)\] and \[(22)\] we find that,

$$\mathcal{L}^{(n)}\Gamma^{k}_{ij} = g_{\mathcal{G}^k} g_{\mathcal{G}^i} g_{\mathcal{G}^j} \Gamma^{k}_{i\mathcal{G}^j} m_{\mathcal{G}^{i\mathcal{G}^j}} K_{\mathcal{G}^m}$$
$$+ W^2 v^i g_{\mathcal{G}^i} g_{\mathcal{G}^j} K_{\mathcal{G}^{i\mathcal{G}^j}} + 2W^2 v^i g_{\mathcal{G}^i} g_{\mathcal{G}^{i\mathcal{G}^j}} \alpha_{\mathcal{G}^{i\mathcal{G}^j}}$$
$$+ W^4 (v^i g_{\mathcal{G}^i} g_{\mathcal{G}^{i\mathcal{G}^j}}) \mathcal{G}_{\mathcal{G}^i} \mathcal{G}_{\mathcal{G}^{i\mathcal{G}^j}} K_{\mathcal{G}^m}$$
$$+ W^4 (\alpha_{\mathcal{G}^{i\mathcal{G}^j}} g_{\mathcal{G}^i} g_{\mathcal{G}^{i\mathcal{G}^j}}) \mathcal{G}_{\mathcal{G}^i} \mathcal{G}_{\mathcal{G}^{i\mathcal{G}^j}} \mathcal{G}_{\mathcal{G}^{i\mathcal{G}^j}} K_{\mathcal{G}^m}$$
$$+ W^4 (\alpha_{\mathcal{G}^{i\mathcal{G}^j}} g_{\mathcal{G}^i} g_{\mathcal{G}^{i\mathcal{G}^j}}) \mathcal{G}_{\mathcal{G}^i} \mathcal{G}_{\mathcal{G}^{i\mathcal{G}^j}} \mathcal{G}_{\mathcal{G}^{i\mathcal{G}^j}} K_{\mathcal{G}^m}$$
$$+ W^4 (\alpha_{\mathcal{G}^{i\mathcal{G}^j}} g_{\mathcal{G}^i} g_{\mathcal{G}^{i\mathcal{G}^j}}) \mathcal{G}_{\mathcal{G}^i} \mathcal{G}_{\mathcal{G}^{i\mathcal{G}^j}} \mathcal{G}_{\mathcal{G}^{i\mathcal{G}^j}} K_{\mathcal{G}^m}$$
$$(\gamma_{ab})^{(n)} = (\gamma_{ab})^{(4)}(\gamma_{ab})^{(4)} , \quad (23)$$

where here we use an index $n$ to denote contraction with the lower case unit normal vector $n^a$, and where the acceleration of lower case Eulerian observers is $\alpha_i = D_i n$.
The upper case induced Ricci tensor can be computed from,
\[
\text{(24) } R_{ij}^{(N)} = (N) \| \partial^m (N) \Gamma^k_{im} - (N) \| \partial^m (N) \Gamma^k_{jm} \\
+ (N) \Gamma^m_{ij} (N) \Gamma^k_{mj} - (N) \Gamma^m_{ij} (N) \Gamma^k_{mj},
\]
and likewise for the lower case curvature. The relationships between the upper and lower case connections above \( (22) \) can be used to compute the relationship between the two Ricci curvatures by brute force, but it is more convenient to use the Gauss-Codazzi equations, as described momentarily. The upper case extrinsic curvature projected into the lower case foliation is,
\[
\text{(25) } K_{ij} = W K_{ij} - D_i W v_j - W A_i(v_j) \equiv W(K_{ij} - A_i(v_j)),
\]
where here we also define \( K_{ij} \) which, for lack of a better name, we call the boost extrinsic curvature. Note that we define the trace \( K \equiv (g^{-1})^{ij} K_{ij} \) in a nonstandard way by using the inverse boost metric. The projected upper case acceleration is,
\[
\text{(26) } A_i \equiv A_i = D_i (\ln A) + W^2 v_i \left( \nu^j D_j (\ln A) + L_\alpha (\ln A) \right).
\]

It is most convenient to express the various equations in terms of \( A_i \) rather than using this expression, as we wish to suppress the explicit appearance of a \( L_\alpha (\ln A) \) contribution.

**Constraints under the coordinate change:** Comparing the Hamiltonian and momentum constraints in each foliation, we find that,
\[
\text{(27) } H = W^2 H - 2 W^2 M_v, \\
\text{(27) } M_i = W M_i + 2 W^3 M_v v_i - W^3 H v_i.
\]
An index \( \nu \) denotes contraction with the boost vector \( e^\nu \).
It is thus obvious that the full set of constraints will be satisfied in the lower case foliation if and only if they are satisfied in the upper case foliation. Expanding out the upper and lower case constraints and using projected upper case extrinsic curvature \( (25) \) in combination with \( (20) \), we easily find the relationship between the two spatial Ricci scalars. As it stands, equation \( (27) \) is really of a purely geometric nature and independent of the gravitational field equations, where we think of the symbols as mere shorthands according to \( (11) \), or the upper case foliation equivalent.

**Electric and Magnetic decomposition of the Weyl tensor:** Especially in General Relativity the decomposition of the Weyl tensor \( W_{abcd} \) into two spatial tracefree tensors, Electric and Magnetic parts,
\[
E_{ab} = n^c n^d W_{acbd}, \quad B_{ab} = n^c n^d * W_{acbd},
\]
has special significance in encoding the propagating degrees of freedom of the gravitational field. The dual Weyl tensor here is \( * W_{abcd} = \frac{1}{2} \epsilon^{ef} C_{abef} W_{cdef} \). Evidently this decomposition is foliation dependent, because of the presence of the normal vector \( n^a \). The relationship between the two decompositions is given by,
\[
\text{(28) } E_{ij} = (2 W^2 - 1) E_{ij} - 2 W^2 E_{i(l} v_{j)} + W^2 E_{v l} v^j + 2 W^2 \epsilon_{i(l} k j k), \\
\text{(28) } B_{ij} = W^2 B_{ij} - W^2 \epsilon_{i(l} j k k E_{l},
\]
where \( \epsilon_{abcd} = n_a \epsilon_{abcd} \) stands for the lower case spatial volume form. Furthermore this equation shows trivially that changes of coordinates can not create nor destroy gravitational waves. Since in vacuum the electric and magnetic parts also satisfy a closed evolution system \( (20) \), it is also clear that if we are given initial data with vanishing \( E_{ij} \) and \( B_{ij} \) this will be the case once and for all. The relationship \( (28) \) holds in general, but using the vacuum Einstein equations, we have that,
\[
E_{ab} = R_{ab}^{\text{EF}} - (K^{\text{EF}} a K_{bc})^{\text{TF}} + K_K^{\text{TF}} ab,
\]
and likewise for the upper case Electric part. We can thus relate the tracefree part of the upper case and lower case spatial Ricci tensors as we did for the Ricci scalars, namely by expanding out the projected upper case extrinsic curvature with \( (25) \) and using \( (19) \) to obtain the non-spatial components.

**Boost metric connection and curvature:** We would like to have the equations of motion for upper case objects in the lower case coordinates. But as it is more convenient to work with spatial tensors in the lower case coordinates we instead work with the boost metric and boost extrinsic curvature \( (g_{ij}, K_{ij}) \) to obtain the desired results. The time derivative of the boost vector is conveniently encoded as,
\[
\text{(31) } \partial_t (W v_i) = \alpha W [A_i - A_\alpha v_i - D_i (\ln (\alpha W))] + L_\beta (W v_i).
\]
The relationship between the connection of the boost metric and the lower case spatial curvature is \( C^{k}_{ij} = \mathbf{G}^{k}_{ij} - \Gamma^{k}_{ij} \), with
\[
\text{(32) } C_k^{ij} = W^2 v_i D_j v^k - \frac{1}{2} D^k (W^2 v_i v_j) + \frac{1}{2} v^k L_\nu (g_{ij}) \\
= (g^{-1})^{kl} \left[ v_i D_j (W^2 v_l) - \frac{1}{2} D_l (W^2 v_i v_j) \right] \\
+ \frac{1}{2} v^k L_\nu \gamma_{ij}.
\]
This result can be obtained from the standard expression, see Ch. 7 of \( [21] \), since the lower case spatial metric and boost metric act in the same tangent space; notice that this is not the case when we try to relate the upper and lower case spatial connections. We could now examine how spatial geodesics are deformed in the boost metric, but since these are not often used in practice we elect not to do so here. The standard expression,
\[
R_{ij} = R_{ij} - 2 D_i C^{k}_{ij} + 2 C^{l}_{i[l} C^{k}_{kl]} \\
= R_{ij} - 2 D_i C^{k}_{ij} - 2 C^{l}_{i[l} C^{k}_{kl]},
\]

```
similarly relates the two curvatures. Note that in the three spatial dimensions of the spatial slice the Weyl tensor is identically zero, so we need only consider the Ricci tensors rather than the full spatial Riemann tensors. Projecting upper case covariant derivatives into the lower case foliation immediately reveals the geometric meaning of the boost covariant derivative. Let $S_a$ and $S_b$ denote upper case spatial tensors, related in the standard way to their projections $s_a$ and $s_b$ into the lower case foliation. Then we have,

$$\nabla^{(N)} D_i S = D_i S + W v_i \left( \mathcal{L}_N S \right),$$  

for a scalar $S$, and,

$$\nabla^{(N)} D_i S_j = D_i S_j + W v_i \left( \mathcal{L}_N S_j \right) - X^{k}{}_{ij} s_k,$$

$$\nabla^{(N)} D_i S_{jk} = D_i s_{jk} + W v_i \left( \mathcal{L}_N s_{jk} \right) - X^{l}{}_{ijk} s_l - X^{l}{}_{ikj} s_l,$$

for the tensors, with,

$$X^{k}{}_{ij} = W(g^{-1})^{kl} \left[ \nabla^{(N)} K \right]_{ij} = W^2(g^{-1})^{kl} \left[ K_{ij} v_l - 2 K_{li} v_j + v_l v_j A_i \right].$$  

The general pattern can be read off from these equations. The boost covariant derivative is the part of the projected upper case derivative formed from lower case spatial derivatives and the boost vector. The remainder depends on the Lie derivative along $N^a$ and the boost extrinsic curvature $K_{ij}$. We can relate the projected upper case spatial Ricci tensor and the boost curvature by straightforward, albeit tedious, direct computation,

$$R_{ij} = \nabla^{(N)} R_{ij} + D_k X^{k}{}_{ij} - D_i X^{k}{}_{jk} - X^{k}{}_{kl} X^{l}{}_{ij}$$

$$+ X^{k}{}_{il} X^j{}_{lk} - W v_i \nabla^{(N)} D_j (\nabla^{(N)} K) + A_j (\nabla^{(N)} K)$$

$$- W^{-1} v^k \nabla^{(N)} D_j (\nabla^{(N)} K)_{ij} - 2 \nabla^{(N)} D_i (\nabla^{(N)} K)_{jk}$$

$$+ W^{-1} v^k \nabla^{(N)} A_k (\nabla^{(N)} K)_{ij} - 2 \nabla^{(N)} A_i (\nabla^{(N)} K)_{jk}.$$  

This result holds regardless of the gravitational field equations, but possible further manipulation is possible by the addition of the hypersurface constraints. The projected upper case covariant derivatives here can be replaced using $[35]$, which results in a slightly longer expression in terms of $K_{ij}$.

**Dual foliation formulation of the wave-equation:** As a simple example, we consider a 3 + 1 decomposition of the wave equation $\Box \phi = 0$ using the dual foliation. For the Lie-derivative of the boost metric we define,

$$P_{ij} \equiv \frac{1}{2} W \mathcal{L}_{(W^{-1})} g_{ij},$$

again with the convention that $P \equiv (g^{-1})^{ij} P_{ij}$. Introducing the reduction variable $\mathcal{L}_N \phi = W \pi$. We then obtain,

$$\partial_t \phi = \alpha \pi + \mathcal{L}_{(\beta - \alpha \nu)} \phi,$$

$$\partial_t \pi = \alpha (g^{-1})^{ij} \left[ D_i D_j \phi - X^{k}{}_{ij} D_k \phi + A_i D_j \phi \right]$$

$$+ \alpha \left[ K + K_{uu} + P + L_c \log(W) \right] \pi + \mathcal{L}_{(\beta + \alpha \nu)} \pi.$$  

These equations serve as a prototype when looking at the more complicated systems that follow. In particular the Lie derivative terms for $\pi$ differs from what one might naively expect.

**Gravitational field equations:** We denote $A_{(i} v_{j)} \equiv A \otimes v_{ij}$. Moving now to write the field equations in terms of the boost metric we obtain,

$$\partial_t g_{ij} = -2 \alpha K_{ij} + 2 \alpha A_{(i} v_{j)} + \mathcal{L}_{(\beta - \alpha \nu)} g_{ij},$$

and after delicate use of the first hypersurface constraint [14],

$$\partial_t K_{ij} = \alpha R_{ij} - W^{-1} D_i \left[ W^{-1} \alpha A_{ij} \right] + v^2 \alpha A_{ij},$$

$$- W^{\alpha} (g^{-1})^{kl} (D_k \mathcal{A}_{ij} + (2 - v^2) \mathcal{A}_k A_{ij}) v_{lj}$$

$$- \alpha \mathcal{L}_{(W^{-1})} (W A \otimes v_{ij}) - 2 W^{-2} D_i \left[ W^2 \alpha K_{ij} \right]$$

$$- 2 W^{2 \alpha} (g^{-1})^{kl} (g^{-1})^{mn} (v_{ij} A_{kl} K_{lm} - W^2 \alpha A_{ik} v_{lj})$$

$$- 2 \alpha \mathcal{A}_{ij} K_{jv} + (K + K_{uu} + P + L_c \log(W)) K_{ij}$$

$$+ \alpha K P_{ij} - \alpha (v^2 K + 2 K_{uv} + P + W^{2} A_{ij}) A \otimes v_{ij}$$

$$- 2 \alpha (K - A \otimes v) (K - A \otimes v)_{ij} - 2 K_{(i} \alpha (D_j) \alpha$$

$$- 4 \alpha (g^{-1})^{kl} (K - A \otimes v)_{ijk} (P_{ij}) + \mathcal{L}_{(\beta + \alpha \nu)} K_{ij}.$$  

Notice that we end up here with equations involving principal derivatives of the lower case shift but not the lower case lape. That is, in equation [10] first derivatives of $\beta$ appear; but in equation [11] no second derivatives of $\alpha$ appear, and instead we have derivatives of $A$. This should be compared with the form of the equation [11] in which second derivatives of $\alpha$ are present. Intuitively this happens because we are mixing the use of lower case spatial coordinates with upper case spatial objects. The Hamiltonian constraint can also be expressed in terms of these variables, giving,

$$H = R + (K + K_{uu})^2 - K_{ij} K^{ij} + 2 D_i (v^2 K)$$

$$- 2 (g^{-1})^{ij} D_i (K_{jv} - W^{-1} v^k D_k W_{ij})$$

$$+ W^{-2} v^{ij} (g^{-1})^{kl} D_{ij} W_{kli} D_{ij} W_{vij}.$$  

Likewise the momentum constraint becomes,

$$M_i = (g^{-1})^{jk} D_j (W^{-1} K_{ki}) - D_i (K + K_{uu}) + L_c K_{vi}$$

$$+ W (g^{-1})^{jk} D_j D_k W_{vij} - \frac{2}{2} W_{vij} D_i (W^{-1} v^k D_k W_{vij})$$

$$- 2 (g^{-1})^{jk} v^k D_i [W_{vij} D_{jk} W_{vij}] + R_{uv} + P K_{vi},$$  

where, up to additions of the hypersurface constraints, we define $H = H - 2 M_i$ and $M_i = M_i$. For readability in equations [12] and [15] we write $D_i W_{vij} \equiv D_i [W_{vij}]$. No complications arise in including the stress-energy tensor for nonvacuum spacetimes. We note in passing that by taking $g_{ij}$ and $K_{ij}$ as evolved variables the evolution equation for the metric also looks natural, since the projected upper case acceleration appears as it would in a tetrad formulation when the timelike leg differs from the normal vector $n^a$. With this choice of variables, we also see that the constraints can be written so that there is
no explicit appearance of the projected upper case acceleration $A_i$. Furthermore, there is no explicit appearance of $L_NA_i$ in equation (44). Nevertheless we may be more interested in how projected upper case extrinsic curvature evolves. This slightly more compact expression is trivially obtained by combining (44) and (41). The equations are given in both forms in mathematica notebooks that accompany the paper [23]. One expects to be able to formulate an initial data construction strategy naturally around the boost metric. A natural starting point for this would be to examine conformal flatness of the boost metric in boosted Schwarzschild. See [23] for work along these lines. Observe that the notation in these last equations is made slightly more cumbersome by continuing to insist on raising and lowering indices with the spatial metric $\gamma_{ij}$, but we prefer to do so to avoid confusion with the surrounding calculations.

The Generalized Jang equation: The Jang equation [24] is a quasilinear partial differential equation of minimal surface type, originally introduced as a tool for the proof of the positive energy theorem. Given initial data $(\gamma_{ij}, K_{ij})$ for the initial value problem in General Relativity it reads,

$$\left(\gamma^{ij} - \frac{D^TDT}{1 + D^TDkT}\right) \left(K_{ij} - \frac{D_i D_i T}{(1 + D^TDkT)^{1/2}}\right) = 0,$$

(44)

for the unknown scalar field $T$. This equation was motivated by the characterization of slices of the Minkowski spacetime, in which there exists a scalar function $T$ such that the second bracket of (41) vanishes, and such that the boost metric,

$$\mathbf{g}_{ij} = \gamma_{ij} + D_i T D_j T,$$

is flat. With the present formalism it is clear that the natural curved space generalization to (41) should be,

$$(\mathbf{g}^{-1})^{ij} (K_{ij} - A_i (v_j)) = \mathbf{K} - W^{-2} A_v = 0,$$

(45)

which, remarkably corresponds to the upper case foliation being maximal, because using the inverse boost metric to trace projected upper case quantities reveals the full trace of the original upper case tensor. One furthermore expects an analogous characterization of general asymptotically flat initial data sets to that of flat-space, in roughly the following terms: Consider data $(\gamma_{ij}, K_{ij})$ extracted from a spacetime $(M, g)$, written in coordinates $X^a = (T, X^i)$, on some Cauchy slice, not necessarily a level set of $T$. An initial data set $(\gamma_{ij}, K_{ij})$ corresponds to the same data if and only if there exist vectors $v^i$, $A_i$ satisfying the hypersurface constraint (44), which we may write in the form

$$(D \times Wv)_i = (D \ln A \times A^{-1} Wv)_i,$$

(46)

and a projected Jacobian transformation $\phi_\perp$ with inverse $(\phi^\perp)^i$ such that,

$$\mathbf{g}_{ij} = \gamma_{ij} + W^2 v_i v_j,$$

$$K_{ij} = K_{ij} - W^{-1} D_i (Wv)_j,$$

(47)

satisfy,

$$(\gamma^{N}_{ij}) = (\phi^{-1})^i J_{\perp} (\phi^{-1})^j \mathbf{g}_{ij},$$

$$(K_{ij}) = W (\phi^{-1})^i J_{\perp} (\phi^{-1})^j (K_{ij} - A_i (v_j)),$$

(48)

everywhere on the constant-$t$ slice. The relationship between the projected and true Jacobian transformation is that,

$$\phi_\perp = (\phi^{-1})^i J_{\perp},$$

$$\phi^\perp = J_{\perp} (J^{-1})^i \mu + W v^\mu V^m,$$

(49)

The transformation $\phi_\perp$, has the property that it maps $N$-space-like contravariant tensor indices in $X^a$ coordinates to $x^i$ coordinates and simultaneously projects the object into the lower case slice, and vice-versa for covariant n-space-like indices in coordinates $x^i$. The inverse property is easily verified by direct computation. The equivalent transformation $\phi^\perp$ in the opposite direction is defined in the obvious way. In the special case that the upper case coordinates are global inertial on Minkowski spacetime this characterization reduces to that stated above motivating the Jang equation. We propose that this, rather than the conformal transformation, as is sometimes claimed, constitutes the relation that should be used to build the natural equivalence class over physically equivalent solutions to metric-based formulations of GR.

Discussion of the dual foliation initial value problem: Given a boost metric $\mathbf{g}_{ij}$, projected extrinsic curvature $K_{ij}$, boost vector $v_i$, and acceleration $A_i$ satisfying the hypersurface, Hamiltonian and momentum constraints we have a suitable set of initial data for vacuum GR. None of these quantities are invariant under changes of the upper case time coordinate $T$, but the form of the field equations is nevertheless invariant under this change. One way to view the resulting additional freedom is that by breaking the correspondence between the spatial metric and the projection operator onto slices of the foliation, we gain the freedom to take the spatial metric of other foliations as the evolved variable. The subsequent projection into the foliation to obtain the boost metric is the most convenient way to deal with the variable in the $3 + 1$ language, and fits nicely with earlier work such as the Jang equation. It is natural to compare this reformulation with the freedom in the Maxwell equations, whose gauge can be altered without changing the coordinates on spacetime. The electric and magnetic fields are invariant under such changes, as they depend only on the Faraday tensor and the choice of coordinates. We have exactly the same status with the boost freedom; the electric and magnetic parts of the Weyl tensor are determined purely by the choice of coordinates, according to [23], and thus independent of the choice in the boost freedom. But the form of the field equations is invariant under changes to the boost.

Working with the upper case spatial tensor basis: Allowing the boost freedom decouples, in the highest
derivatives, the lower case lapse from the evolved variables. Therefore it is natural to ask whether such a decoupling can also be obtained in the lower case shift by keeping all tensors in the $\mathbf{X}^2$ coordinate basis. We therefore now wish to drop the Jang-equation style use of two time coordinates with everything expressed in the coordinate basis $x^i$ vectors, and instead use the $\mathbf{X}^2$-basis tensor components, whilst computing derivatives in the $x^k$ coordinates. The time derivative of the projected Jacobian is,

$$
\partial_t \varphi^i_{\alpha} = L_{(\beta - \alpha)} \varphi^i_{\alpha} - \alpha (A W)^{-1} \varphi^i_{\alpha} \partial_\mu B^\mu, \\
\partial_t (\varphi^{-1})^i_{\alpha} = L_{(\beta - \alpha)} (\varphi^{-1})^i_{\alpha} + \alpha (A W)^{-1} (\varphi^{-1})^i_{\alpha} \partial_\mu B^\mu. 
$$

(50)

Given the time derivative of an upper case spatial tensor in lower case coordinates we can now use (50) to compute the lower case time derivative in the upper case basis. Take for example $S_{ij}$ symmetric upper case spatial, again with projection $s_{ij}$ into the lower case foliation. Suppose we have,

$$
\partial_t s_{ij} = \alpha X_{ij} + \mathcal{L}_\beta s_{ij}, 
$$

(51)

then it follows that,

$$
\partial_t S_{ij} = \alpha (\varphi^{-1})^i_{\alpha} (\varphi^{-1})^j_{\alpha} X_{ij} + 2 \alpha (A W)^{-1} S_{L(\alpha)} B^\alpha + \mathcal{L}_{(\beta - \alpha)} S_{ij}. 
$$

(52)

Taking $S_{ab}$ as the upper case spatial metric, and looking at (10) we immediately see that indeed the lower case shift does become decoupled in the highest derivatives. It is sufficient to consider only this evolution equation because this is the only place where the shift is coupled in the principal part. The relationship between the upper case Christoffel symbol and that of the boost metric is,

$$
\varphi_{ij} (\varphi^{-1})^k_{\alpha} \Gamma^\alpha_{ij} = X_{ij} + \mathcal{D}_{ij}, \\
+ (\varphi^{-1})^k_{\alpha} \partial_\alpha X_{ij} - A^{-1} W v_{(i} \varphi_{j)} \partial_\mu B^\mu, 
$$

(53)

with $X_{ij}$ defined as above. Note that (53) can be rewritten as,

$$
\mathcal{D}_{ij} + \mathcal{D}_{ij} = \mathcal{D}_{ij} = X_{ij} + \mathcal{D}_{ij} - A^{-1} W v_{(i} \varphi_{j)} \partial_\mu B^\mu, 
$$

(54)

which in the setting with vanishing boost vector can be interpreted as the statement that the Levi-Civita connection of the spatial metric is the gauge covariant derivative associated with spatial diffeomorphisms. The full field equations are given in this mixed basis form in [25], together with a canonical Hamiltonian treatment.

III. DOUBLE-NULL FORMULATION

In this section we work in coordinates $x^\nu = (t, x^i)$. Throughout latin indices $a, b, c, d, e$ will be abstract as in the previous section, and likewise latin indices $i, j, k, l, m, p$ stand for spatial components in $x^i$ as before. We perform a $2 + 1$ decomposition against $r$ on the spatial slice, and take $x^\nu = (t, r, b^k)$ to be adapted coordinates. Thus upper case latin indices $A, B, C, D$ stand for those in the level-sets of $r$.

A. 2 + 1 + 1 Decomposition

**Motivation:** We now turn our attention towards finding coordinates suitable for studying the collapse of gravitational waves. It is known that sufficiently small perturbations of the Minkowski spacetime are long-lived in the pure harmonic gauge [24, 27], but this class of initial data presumably does not include every possible data set that eventually asymptote to the Minkowski spacetime; for sufficiently strong data, or indeed sufficiently strong pure gauge perturbations, coordinate singularities are expected to form. Indeed there are examples of this phenomenon [28], and it may be that some of the difficulties in evolving strong Brill waves in [29] were caused by the use of the pure harmonic slicing. Therefore we look elsewhere. Empirically the generalized harmonic gauges [30] have been found very robust in both binary blackhole and collapse scenarios. However, from the mathematical point of view, the strongest results concerning collapse to a blackhole employ a double-null foliation [2]. It is known that a particular type of initial data will form an apparent horizon before any coordinate singularity forms. It is not clear that close to the critical threshold of blackhole formation these coordinates are well-behaved. It is also rather doubtful that the double-null foliation will be useful in the strong-field region in binary-blackhole spacetimes. But since these coordinates naturally conform to the causal structure of the spacetime, and there is likewise no guarantee of nice behavior for any other coordinate system, they seem to be in the best shape for consideration. In particular, in both $3 + 1$-spherical symmetry [31] and a $2 + 1$ dimensional setting [32] such coordinates have been effectively used in studies of critical collapse, neatly sidestepping the need for mesh-refinement. We thus look at related gauges suitable for the initial value problem.

The $2 + 1$ decomposition: In a double-null foliation there are two crucial coordinates, optical functions, whose level sets are incoming and outgoing null surfaces. In the $3 + 1$ setting we have however only singled out the time-coordinate for special treatment. A given pair of these null hypersurfaces intersect in a spacelike two-sphere. Given a spatial slice of constant $t$, complete with spatial metric $\gamma_{ij}$ and extrinsic curvature $K_{ij}$ let us define a new coordinate $r$, which we will use to perform a $2 + 1$ split. The idea is that the level sets of $r$ should become the spheres on which the null surfaces intersect. Obviously the calculations that follow are essentially the same as those of the standard $3 + 1$ split, as described in detail in [3]. The coordinate $r$ defines a unit normal $s^i$
to a surface of constant \( r \) according to,
\[
L^{-2} = \gamma^{ij}(D_1 r)(D_2 r), \quad s^i = \gamma^{ij} L D_j r.
\]  
(55)

We will call \( L \) the length scalar. The normal vector \( s^i \) naturally defines the induced metric in the two-dimensional level set,
\[
q_{ij} = \gamma_{ij} - s_i s_j.
\]  
(56)

Likewise we have the extrinsic curvature,
\[
\chi_{ij} = \frac{q^k}{2} D_k s_j,
\]  
(57)

so the first derivatives of the spatial normal vector are in total,
\[
D_i s_j = \chi_{ij} - s_i \partial_j \ln L.
\]  
(58)

We denote the covariant derivative compatible with the induced metric \( q_{ij} \) by \( \partial_j \), and likewise use \( \partial_i \) for the partial derivative projected into the surface. Note the relative change in sign in the definition of this extrinsic curvature and that of the slice \( K_{ij} \). Let the vector \( r^i \) be tangent to lines of constant \( \theta^A \), the two spatial coordinates in the level set. We have
\[
r^i = L s^i + b^i,
\]  
(59)

with \( b^i s_i = 0 \). We call the two-dimensional vector \( b^i \) the slip vector. Note the relation \( r^i D_i r = 1 \). With this notation we can express the spatial metric as,
\[
dl^2 = L^2 dr^2 + q_{AB}(d\theta^A + b^A dr)(d\theta^B + b^B dr).
\]  
(60)

When performing the 3 + 1 decomposition one finds that the lapse scalar and shift vector are freely specifiable, which is of course not the case with the analogous length scalar and slip vector. The four-dimensional metric is,
\[
ds^2 = -\alpha^2 dt^2 + L^2 (dr + L^{-1} \beta^i dt)^2
\]  
\[
+ q_{AB}(d\theta^A + b^A dr + \beta^A dt)(d\theta^B + b^B dr + \beta^B dt).
\]  
(61)

The coordinate light speeds in the increasing and decreasing \( r \) directions are,
\[
c^4_+ = (-\beta^i + \alpha) L^{-1},
\]  
(62)

whilst in the transverse directions we have,
\[
c^4_- = -\beta^A \mp b^A \alpha L^{-1}.
\]  
(63)

Although they may not necessarily be associated with spherical-polar coordinates we will call these transverse directions ‘angular’. An obvious choice is \( \alpha = L \), and \( \beta^i = 0 \), under which the coordinate light-speeds are \( c^4_\pm = \pm 1 \). With this choice the combinations \( u = t - r \) and \( v = t + r \) are the optical outgoing and incoming null coordinates alluded to earlier, and the coordinates are naturally adapted to the causal structure of the spacetime in the null \( n^a \pm s^a \) directions.

The extrinsic curvatures: We immediately split the extrinsic curvature \( \chi_{AB} \) into a trace and tracefree part,
\[
\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{2} q_{AB} \chi.
\]  
(64)

We will similarly use the notation \( q \) for the determinant of the two-metric. Finally we decompose the extrinsic curvature of the spacelike surface as embedded in the spacetime by
\[
K_{ij} = s_i s_j K_{ss} + \frac{1}{2} q_{ij} K_{qq} + 2 s_i q^i j K_A + \hat{K}_{AB},
\]  
(65)

where in accordance with the previous notation \( \hat{K}_{AB} \) stands for the projected, tracefree part. We use indices \( s \) to denote contraction with \( s^a \), and \( qq \) for a trace taken with the two-dimensional metric \( q_{AB} \). We write \( K_{AB} = q^i A q^j B K_{ij} \) for the projected extrinsic curvature, and occasionally still use \( K = K_{ss} + K_{qq} \) as a shorthand for the trace of the extrinsic curvature.

The Christoffel symbol: The spatial Christoffel symbol is readily decomposed as,
\[
\Gamma^s_{ij} = L_s (\ln L), \quad \Gamma^s_{ij} q^j A = \hat{\phi} A (\ln L),
\]
\[
\Gamma^s_{ij} q^j A q^j B = \hat{\chi}_{AB}, \quad \Gamma^s_{ij} q^j A q^j B q^k C = \hat{F}^C_{AB},
\]
\[
\Gamma^s_{ss} q^i A = -\hat{\chi} (\ln L) + \frac{1}{2} (\mathcal{L}_L b^A - b^B \partial_B b^A),
\]
\[
\Gamma^s_{ss} q^i A q^j B q^k A = \hat{\chi}_{AB} + \frac{1}{2} q_{AC} \partial_B b^C.
\]  
(66)

with \( \Gamma \) denoting the Christoffel symbols of the two-metric \( q_{AB} \). The spatial contracted Christoffel symbols \( \Gamma^i = \gamma^{jk} \Gamma_{jk} \) are therefore,
\[
\Gamma^s = L_s (\ln L) - \chi,
\]  
\[
\Gamma^s q^j A = \Gamma^A + \frac{1}{2} (L_s b^A - b^B \partial_B b^A) - \mathcal{D} A (\ln L).
\]  
(67)

Curvature: As in \([53]\) the spatial Ricci curvature \( R_{ij} \) can be decomposed according to,
\[
R_{ss} = -\frac{1}{2} \mathcal{D} A \mathcal{D} A L - L_s \chi - \chi A B C A,
\]
\[
R_{qq} = -\frac{1}{2} \mathcal{D} A \mathcal{D} A L - L_s \chi - \chi^2 - 2 \chi A B C A,
\]
\[
R_{ss} = \mathcal{D} b^A b^A - \mathcal{D} A \chi, \quad \hat{R}_{AB} = -L_s \chi A B - \frac{1}{2} \mathcal{D} A \mathcal{D} A L + 2 \chi C A \chi B C A,
\]  
(68)

by the classical Gauss-Codazzi-Mainardi equations.

Vacuum field equations: The \( 2 + 1 + 1 \) decomposed Einstein equations are given first by the constraints,
\[
H = R_{ss} - \frac{1}{2} \mathcal{D} A \mathcal{D} A L - 2 L_s \chi - \chi A B C A - \frac{2}{3} \chi^2 - \frac{1}{2} K_{qq}^2 - 2 K_{qq} K_{ss} + 2 K_{qq} K_A + \hat{K}_{AB} K^{AB},
\]
\[
M_s = -L_s K_{qq} + \mathcal{D} A K_A + 2 K_A \mathcal{D} A (\ln L) + \chi K_{ss} - \frac{1}{2} \chi K_{qq} + \chi A B C A \chi B C A,
\]
\[
M_A = L_s K_A + \mathcal{D} A \hat{K}_{AB} - \frac{1}{2} \mathcal{D} A K_{qq} - \mathcal{D} A K_{ss} + \chi K_A - K_{ss} \mathcal{D} A (\ln L) + \frac{1}{2} K_{qq} \mathcal{D} A (\ln L) + \hat{K}_{AB} \mathcal{D} A (\ln L).
\]  
(69)

Since null geodesic expansions in the \( n^a \pm s^a \) directions are given by \( \chi \pm K_{qq} \), we can view the \( H \) and \( M_s \) constraints
as dictating how the expansions vary over the slice. Next we have evolution equations for the metric,

\[
\partial_t (\ln L) = -\alpha K_{ss} + \beta^A \Psi_A (\ln L) + D_s \beta^s,
\]

\[
\partial_t b^A = -2\alpha L^A + L^2 \Psi(L^{-1} \beta^s) + L_r \beta^A + \mathcal{L}_B b^A,
\]

\[
\partial_t q_{AB} = -2\alpha K_{AB} + 2\beta^s \chi_{AB} + \mathcal{L}_B q_{AB}.
\] (70)

and for the decomposed extrinsic curvature,

\[
\partial_t K_{ss} = -\mathcal{L}_s \mathcal{L}_s \alpha + \alpha [R_{ss} + 2K_A K^A + K_{qq} K_{ss} + K_{ss}^2] - \Psi^A (\ln L) \mathcal{D}_{\alpha} \Psi_A - 2L_K D_s (L^{-1} \beta^s)
\]

\[
- 2L^A \Psi_A (L^{-1} \beta^s) + \beta^s D_s K_{ss} + \beta^A \mathcal{D}_A K_{ss},
\]

\[
\partial_t K_{qq} = -\Psi^A \mathcal{D}_{\alpha} \Psi_A + \alpha [R_{qq} + K^2_{qq} + 2K_A K^A + K_{qq} K_{ss}]
\]

\[
- \chi \mathcal{L}_s \alpha + 2L^2 \Psi (L^{-1} \beta^s) + \beta^A \mathcal{D}_B K_{qq} + \beta^A \Psi_A K_{qq},
\]

\[
\partial_t K_{A} = -\Psi^A \mathcal{D}_{\alpha} \Psi_A + \alpha [R_{A} + K_{qq} K_{A}] + \chi B^A \Psi_B \alpha
\]

\[
- L^A K_{A} D_s (L^{-1} \beta^s) - L^B K^A \mathcal{D}_B (L^{-1} \beta^s)
\]

\[
+ 2L^A K_{A} \mathcal{D}_A K_{ss},
\]

\[
\partial_t K_{AB} = -\Psi_A \mathcal{D}_{\alpha} \Psi_B - \chi_{AB} \mathcal{L}_s \alpha + 2L^A K_B (\mathcal{D}_B (L^{-1} \beta^s)
\]

\[
+ \alpha [R_{AB} - 2K^C K_{BC} + (K_A K_B) \mathcal{D}_F (L^{-1} \beta^s)]
\]

\[
+ \beta^A \mathcal{L}_s K_{AB} + \mathcal{L}_B K_{AB}.
\] (71)

Here we have defined the Lie-derivative in the level-set of \( r \), in the obvious way, and where it is understood that the vector argument must be projected with \( q_{AB} \), which allows us to write, for example,

\[
\mathcal{L}_B K_{AB} = q^k A^i D (q^j A^s \mathcal{D}_k (q^i \beta^s)).
\] (72)

Finally the equation of motion for the extrinsic curvature \( \chi_{ij} \) can be computed from the relation \( 2\chi_{ij} = \mathcal{L}_s q_{ij} \). The result is,

\[
\chi_{ij} = -\mathcal{L}_s (\alpha K_{AB}) + \alpha \mathcal{L}_s K_{AB} + 2K (\mathcal{D}_B (L^{-1} \beta^s)
\]

\[
+ \alpha \chi_{ij} \mathcal{L}_s \alpha + 2\alpha K_{ij} (\mathcal{D}_B (L^{-1} \beta^s) - \mathcal{D}_B \mathcal{L}_B (L^{-1} \beta^s))
\]

\[
- \partial_A \mathcal{L}_B \beta^s + \beta^A \mathcal{L}_s \chi_{AB} + \mathcal{L}_B \chi_{AB}.
\] (73)

If we want to treat \( r \) as a radial coordinate, and the remaining \( \beta^A \) as angular coordinates we obtain regularity conditions at \( r = 0 \). These conditions will be discussed elsewhere.

**B. Double-null formulation**

We now look at the field equations imposing the double-null gauge explicitly. The aim here is, first, to present the simplified form of the field equations in this gauge, and second, to examine whether or not hyperbolicity of the full system can be obtained.

**Pure gauge analysis:** Let us examine the behavior of infinitesimal perturbations to coordinates satisfying the optical conditions \( \alpha = L \) and \( \beta^s = 0 \) above, additionally taking \( \beta^A = b^A \). This sign is chosen assuming that the gravitational wave is traveling mostly in the minus \( r \) direction consistent with earlier work. This is Christodoulou’s gauge choice in [8], rewritten in a time-space rather than a double-null form. The expressions [10] for the time development of the perturbations to the time and space coordinates are,

\[
\partial_t \theta = U - \psi^j D_j \alpha + \beta^s \partial_t \theta,
\]

\[
\partial_t \psi^i = V^i + \alpha D^j \theta - \theta D^j \alpha + \mathcal{L}_B \psi^i.
\] (74)

Then we have,

\[
U \equiv \Delta [\alpha] = \Delta [L] = \Delta [(\gamma ij) D_i r D_j r]^{-1/2}
\]

\[
= L \mathcal{L}_s \psi - \theta L K_s + \psi^B \mathcal{D}_B L,
\]

\[
V^s \equiv s_i \Delta [\beta^s],
\]

\[
V^A \equiv \Delta [\beta^A] = \Delta [b^A]
\]

and obtain the pure gauge subsystem,

\[
\partial_t [L^{-1} \theta] = L D_s [L^{-1} \psi] + b^A \mathcal{D}_A [L^{-1} \theta],
\]

\[
\partial_t [L^{-1} \psi] = L D_s [L^{-1} \theta] + b^A \mathcal{D}_A [L^{-1} \psi],
\]

\[
\partial_t (q \cdot \psi)^A = (q \cdot \psi)^A + L^2 \Psi [L^{-1} (\theta + \psi)]
\]

\[
- 2L^A \theta (\theta + \psi).
\] (76)

This first order PDE system is only weakly hyperbolic. The arguments presented in [10], building on those of [4, 5], can be used to show that no strongly hyperbolic formulation can be built with this gauge condition, at least if the formulation is constructed under the standard free-evolution approach. Therefore the double-null gauge can not be directly used in numerical relativity in the standard way, but requires a more subtle approach, or some modification. Note that the problem here comes from the choice \( \beta^A = b^A \), and there are simple modifications under which strong hyperbolicity of the pure gauge subsystem can be obtained. Nevertheless, building a formulation of GR which is at least strongly hyperbolic with one of these good, modified, conditions will be more involved because here the \( s^0 \) direction is singled out for special treatment. Therefore in the following sections we instead look for a simpler approach employing the dual foliation formalism.

**Fixing the gauge:** Despite the shortcomings unearthed by the pure gauge analysis, for completeness we present the full field equations with in the double-null form. As above we choose \( \alpha = L, \beta^s = 0 \) and \( \beta^A = b^A \). This choice has no effect on the constraints [90], but the evolution equations become,

\[
\partial_t (\ln L) = -L K_{ss} + b^A \mathcal{D}_A (\ln L),
\]

\[
\partial_t b^A = -2L^2 K^A + \mathcal{L}_B b^A,
\]

\[
\partial_t q_{AB} = -2\alpha K_{AB} + \mathcal{L}_B q_{AB}.
\] (77)
for the metric components, and
\[ \partial_t K_{ss} = -L_s L_t + L[R_{ss} + 2K_A K^A + K qq K_{ss} + K^2_{ss}] - \varphi^A (\ln L) \varphi_A L_t + b^A \varphi_A K_{ss}, \]
\[ \partial_t K_{qq} = -\varphi^A \varphi_A L_t + L[R_{qq} + 2K_A K^A + K qq K_{qq}] - \chi L_s L_t + b^A \varphi_A K_{qq}, \]
\[ \partial_t K_A = -\varphi_A \varphi^A L_t + L[R_s A + K qq K_A] + \chi B A \varphi_B L_t + \varphi A \varphi A K_A. \]
for the extrinsic curvature. Finally we have,
\[ \partial_t \chi_{AB} = -L_s (L K_{AB}) + \varphi^B \varphi^B L_t + 2K_A \varphi_B L_t + 2L_{ss} \chi_{AB} + \varphi A \varphi A \chi_{AB}. \]

Taking linear combinations of these variables one can rewrite so that all of the equations take the form of ‘transport equations’ in the \( n^A \pm s^A \) directions, but with transverse derivatives appearing as sources. Particularly relevant are the combinations \( \chi \pm K_{qq} \), as they reveal the Raychaudhuri equations. Up to this trivial change of variables and the split into time-space derivatives, rather than the double-null choice, this is the same system presented in Ch. 3 of \([\text{1]}\), where it was also noted that this system is not hyperbolic.

**IV. COORDINATE SWITCHED FIRST ORDER GENERALIZED HARMONIC GAUGE**

**A. Double-null Jacobians**

**Time and Radial coordinates:** Let us now abandon the idea of evolving in double-null coordinates directly, and instead examine how the spacetime could be constructed in these coordinates a posteriori, having constructed the spacetime locally in the harmonic gauge, for example. This is similar to the strategy employed in \([\text{1}]\). It has also been used in numerically, in for example \([\text{34}]\). Let us work in lower case coordinates \( x^a \), We would like the upper case coordinates to satisfy the double-null conditions. As elsewhere,
\[ N^a = -A \nabla^a T, \quad S^a = L \nabla^a R. \]  
(80)
with \( \mathcal{L} = (T, R, \Theta_B) \). We choose \( A = L \) to impose the double-null gauge, regardless of the angular coordinates. Under this condition we define ingoing and outgoing null vectors \( L^a, K^a \),
\[ -N^a + S^a = L^a = L \l^a, \quad -N^a - S^a = K^a = L \k^a. \]  
(81)
The renormalized vectors \( \hat{L}^a \) and \( \hat{K}^a \) generate null geodesics in the \( L^a \) and \( K^a \) directions. It is natural to define two lower case spatial vectors \( v^a_\pm \) and \( s^a_\pm \) from the Jacobian according to,
\[ v^a_i = E^a_\pm s^a_i = \phi^a_i \mp L^{-1} W v_i. \]  
(82)
The vectors \( s^a_\pm \) have unit magnitude. Indices \( s^a_\pm \) stand for contraction with these vectors. The scalars \( E^a_\pm \) are the energy of the ingoing and outgoing congruences as measured by the Eulerian observers \( n^a \). In terms of the Jacobian they are,
\[ E^a_\pm = L^{-1} W \mp \pi^a_\pm. \]  
(83)
The null geodesic vectors are then,
\[ \hat{K}^a = -E_-(n^a + s^a_\pm), \quad \hat{L}^a = -E_+(n^a - s^a_\pm). \]  
(84)
Straightforward computation then reveals evolution equations,
\[ \partial_t E^a_\pm = \mathcal{L}(\beta \pm \alpha s^a_\pm) \ln E^a_\pm + \alpha (K_{ss} s^a_\pm \pm s^a_\pm \ln \alpha), \]
\[ \partial_t v^a_i = \pm \alpha D(s^a_\pm v^a_i) \mp E^a_\pm D(\alpha) \mp \beta v^a_i. \]  
(85)
The hypersurface constraints were used freely to arrive at this result. Interestingly the term appearing as a source in the first equation is essentially a characteristic variable of the (first order in time, second order in space) GHG formulation. Notice furthermore that, after adjusting the normalization of \( v^a_i \), these can be compared with the results of \([\text{35}]\), and are of course compatible. Note also that the equations for \( E^a_\pm \) follow from \( \gamma^i v^i_\pm v^i_\pm = E^2_\pm \). The equations \([\text{35}]\) form a symmetric hyperbolic system in \((E^a_\pm, v^a_\pm)\) which is equivalent to a system in the scalar components \( L^{-1} W \) and \( \pi^a_\pm \) and the vector parts \( \phi^a_\pm \) and \( L^{-1} W v_i \) of the Jacobian.

**Angular coordinates:** To complete the equations of motion for the Jacobian we require a choice for \( \phi^a_\pm \) and \( \pi^a_\pm \). We have already seen that from the pure gauge point of view the choice \( 'b^A = b^A' \) is problematic. To understand how this issue appears in the Jacobian formulation, let us assume that the component \( \pi^a_\pm \) takes the form,
\[ \pi^a_\pm = -m^a \phi^a_\pm, \]  
(86)
for some known lower case spatial vector \( m^a \). This corresponds to determining the angular coordinates by Lie-dragging so that \( (n^a + m^a) V_\varphi A \Theta A = 0 \). This family includes \( 'b^A = \pm b^A' \) by choosing \( m^a = \pm s^a_\pm \). Plugging the relation \([\text{30}]\) into the Jacobian equation of motion \([\text{10}]\) and using the hypersurface constraints gives the compact expression,
\[ \partial_t \phi^a_\pm = \mathcal{L}(\beta - \alpha m^a) \phi^a_\pm. \]  
(87)
If the vector \( m^a \) is given a priori this equation is hyperbolic. On the other hand if we wish to make \( m^a \) dependent on the other components of the Jacobian, the derivative of \( m^a \) in the Lie-derivative is problematic, as it is a one-way coupling, in the sense that the
equations of motion for the angular coordinates explicitly depend upon the \((T,R)\) coordinates in the principal part, but not vice-versa. This type of coupling is dangerous because it can leave non-trivial Jordan blocks in the principal symbol if the speeds associated with the \(T,R\) and \(\Theta\) blocks clash. Fortunately this discussion suggests two alternatives. The first is to construct the angular coordinates using just vectors associated with lower case coordinates. For example we could choose \(n^a \nabla_a \Theta = 0\), or even \(v^a \nabla_a \Theta = 0\), in which case the angular coordinates could correspond to those built directly from the lower case coordinates. One possibility would be to use a reference metric to build a first order GHG formulation directly in spherical-polar coordinates. Then the upper case angular coordinates could be exactly those of the lower case system. The second option is to make sure that the speeds of the two subsystems do not coincide, or that if they do the principal symbol is nevertheless diagonalizable. For this we might try \(N^a \nabla_a \Theta = W(n^a + v^a) \nabla_a \Theta = 0\), or in other words \(m^a = v^a\), which is also equivalent to \(D \Theta = 0\) when working in upper case coordinates. Since,

\[
v^i = (E_+ + E_-)^{-1} (v_-^i - v_+^i),
\]

we arrive at,

\[
\partial_t \phi_{\pm a} = -\alpha v^i D_i \phi_{\pm a} + \frac{1}{2} \alpha \nu W^{-1} \phi_{\pm a}, D_i (v_+^i - v_-^i) - \frac{1}{2} \alpha \nu W^{-1} \pi D_i (E_+ + E_-) + \nu D_i \alpha + \nu L \phi_{\pm a},
\]

after expanding the Lie-derivative.

**Hyperbolicity of the Double-Null Jacobians:** The double-null Jacobian system with \(m^a\) given a priori is trivially symmetric hyperbolic. Choosing instead \(m^a = v^a\) as in \([30]\), the principal symbol for the subsystem with the components \((E_\pm, v_+^i, v_-^i, \phi_{\pm a})\) is,

\[
P^a = \alpha A^a + \beta^a \mathbf{1},
\]

with,

\[
A^a = \begin{pmatrix}
\pm s_+ & 0 & 0 & 0 \\
0 & \pm s_+ & 0 & 0 \\
0 & 0 & \pm s_- & 0 \\
\frac{1}{2} \nu W & \pm \frac{1}{2} \nu \phi_{\pm a} & \pm \frac{1}{2} \nu \phi_{\pm a} & -s^a
\end{pmatrix},
\]

with an index ‘\(s\)’ denoting contraction with an arbitrary lower case unit spatial vector \(s^a\) and \(q_{ab} = \gamma_{ab} = s_a s_b\) as elsewhere. The remaining block of the full principal symbol, associated with \(\phi_{\pm a}\), is decoupled, and has no affect on the following discussion. The eigenvalues of the principal symbol are \(\beta^a \pm \alpha s_\pm^a\) and \(\beta^a - \alpha v_\pm^a\). For strong hyperbolicity we need that the symbol is diagonalizable for every \(s\). Choose for example \(s^a\) such that,

\[
(v^i - s_i^i) s_i = 0,
\]

then the ‘\(s^a\)’ and ‘\(v^a\)’ eigenvalues coincide, and the principal symbol is missing eigenvectors. Therefore the system is again only weakly hyperbolic. This type of degeneracy occurs if \(m^a\) is taken to be any vector constructed from \(v_{\pm}^i\), thus we must abandon the second alternative suggested by the above discussion. This leaves the option to fix \(m^a\) a priori, or to choose a gauge of a different form for the angular coordinates, such as the generalized harmonic option \([17]\). At least for the choice \(\beta^a = b^a\) this result was expected, because the pure gauge subsystem we examined before is closely related to the system for the Jacobians.

**B. First order generalized harmonic gauge**

There is much work about formulations of general relativity in first order form. For a review, see \([36]\). Of particular interest in recent years has been the first order reduction of the GHG formulation \([12, 37]\). To a large extent this interest was driven by use inside the SpEC numerical code. Here we summarize just the relevant features. The first order GHG system is a quasilinear, first order symmetric hyperbolic system of the form,

\[
\partial_t u = A^a \partial_a u + S.
\]

The formulation has a set of constraints compatible with the evolution equations. The key feature of the system is that the principal part takes the form of a first order reduction of the wave equation. This is attained by carefully coupling the coordinate choice \(\Box x^a = H^a\) to the field equations \([13]\) and then reducing to first order. Gravitational radiation controlling, constraint preserving boundary conditions for the system have been studied and implemented \([29, 38, 39]\). The evolution system is now used frequently in the evolution of compact binary spacetimes. Of crucial importance in the present work is that in the first order reduction, all first derivatives of the metric can be expressed in terms of evolved variables without taking any derivatives. So, for example if we were to evolve the double-null Jacobians \([30]\) alongside the GHG variables, we can formulate the system so that no coupling occurs through derivatives. In other words the Jacobians are minimally coupled to the GHG formulation.

**C. Coordinate switch and hyperbolicity**

**Coordinate switch:** Suppose, without loss of generality, that we are given a first order quasilinear system of the form,

\[
\partial_t u = (A A^2 + B E^2) \partial_a u + A S.
\]

Now consider the effect of a change of independent coordinates on the whole system, so that rather than using the upper case \(X^a\) coordinates we employ the lower
case $x^\mu$ ones. In the new coordinates the system of course retains the same functional form, but now with,
\begin{equation}
(1 + A^\mu_\nu) \partial_\nu u = \alpha W^{-1} \left( A^\mu_\nu (\varphi^{-1})^\nu_\rho - (1 + A^\mu_\nu) \Pi^\nu_\rho \right) \partial_\rho u + \alpha W^{-1} S .
\end{equation}
(95)
Recall here the various components of the inverse Jacobian defined by \([10]\) and \([9]\), and the shorthand $A^\mu_\nu = A^\mu_\nu V_\nu$. It is easily confirmed that any closed constraint subsystem remains closed.

**Symmetric hyperbolicity:** The original system is assumed to be symmetric hyperbolic, that is, there exists a symmetric positive definite symmetric tensor $H$ such that
\begin{equation}
H \left( A A^\varsigma_\varsigma + B^\varsigma_\varsigma 1 \right) ,
\end{equation}
is symmetric for each $p$. In the present context, this system can be taken as a first order generalized harmonic formulation coupled, minimally, to the double-null coordinates \([59]\). Under a smallness assumption on $V_\mu$, symmetric hyperbolicity is unaffected by the change of coordinates. Taking the system in the form \([55]\), the symmetrizer is unaffected by the change of coordinates. If we insist on multiplying on the left by the inverse of $(1 + A^\mu_\nu)$, then we pick up exactly a factor of this matrix on the right of the modified symmetrizer. But to obtain energy estimates for metric components using norms formed from the lower case coordinate basis components of tensors more effort may be necessary.

**Strong hyperbolicity:** The principal symbol of the system in $X^\mu$ is,
\begin{equation}
P^S_X = A A^\varsigma_\varsigma + B^\varsigma_\varsigma 1 ,
\end{equation}
where on the right hand side superscript $S$ denotes contraction with an arbitrary unit upper case spatial covector $S_\mu$. Multiplying \([60]\) by the inverse of $(1 + A^\mu_\nu)$, the principal symbol in $x^\mu$ can be read off,
\begin{equation}
P^S_x = \alpha W^{-1} \left( (1 + A^\mu_\nu)^{-1} A^\mu_\nu - \Pi^\mu 1 \right) .
\end{equation}
(97)
On the right hand side superscript $s$ denotes contraction with an arbitrary unit lower case spatial covector $s_\mu$, and underlined superscript $\underline{\varsigma}$ denotes the same contraction, but pushed through the transformation $(\varphi^{-1})^\mu_\nu$. Strong hyperbolicity is the requirement that for each $s_\mu$ there exists a symmetrizer $H^\mu$, uniformly symmetric positive definite in $s_\mu$, such that the product of the symmetrizer with the principal symbol is symmetric. For quasilinear problems strong hyperbolicity, together with some smoothness conditions on the symmetrizer are sufficient to guarantee local well-posedness of the initial value problem. These smoothness conditions are sometimes included in the definition of strong hyperbolicity. The existence of a symmetrizer for fixed $s_\mu$ is equivalent to the requirement that the principal symbol has a complete set of eigenvectors with real eigenvalues. Assuming that the system is strongly hyperbolic in the upper case coordinates $X^\mu$ this necessary condition can be seen to hold in lower case coordinates $x^\mu$ as follows. This discussion is adapted from \([40]\). Fix $s_i$. The characteristic polynomial of $P^S_X$ has real eigenvalues, and thus we have a hyperbolic polynomial with respect to $N^\mu_s$. It follows that if the boost velocity is sufficiently small then $P^S_x$ also has real eigenvalues \([11]\). Estimates of the range over which this condition is satisfied can be given. In the presence of a simple block structure of $A^\mu_\nu$, the eigenvalues will simply be multiplied by those of $(1 + A^\mu_\nu)$. Take one of these eigenvalues $\lambda$. Strong hyperbolicity in $X^\mu$ implies the existence of symmetric positive definite $H$ with,
\begin{equation}
H L \equiv H \left( A^\varsigma_\varsigma - (1 + A^\mu_\nu) (\lambda + \Pi^\mu) \right) ,
\end{equation}
(98)
symmetric. Suppose that $u$ is a non-vanishing vector in the nullspace of $L$, and thus either an eigenvector or generalized eigenvector of $P^S_x$. Suppose that it is a generalized eigenvector, so that $u = \lambda v$ for some $v$. Then,
\begin{equation}
u^T H u = u^T H L v = v^T H L u = 0 .
\end{equation}
(99)
The second equality holds by symmetry of $H L$ and the third because $L u = 0$. Since $H$ is symmetric positive definite we then have that $u = 0$. Therefore if $u$ is in the nullspace of $L$ it is a true eigenvector of $P^S_x$. Thus $P^S_x$ has a complete set of eigenvectors. Details concerning the remaining uniformity condition in $s_i$ can be found in \([40]\).

**Discussion:** The advantage of the dual foliation is now clear. In section \([III]\) we were unable to find a hyperbolic formulation using the double-null coordinates directly. Using the dual foliation however the construction of such a formulation is essentially trivial. Since the whole system is symmetric hyperbolic we do not lose regularity when mapping between the two coordinate systems. Regularity will however need to be looked at carefully if we are to use the double-null radial coordinate all the way to the origin, but this issue is only that of using spherical polar coordinates, and not related to the dual foliation strategy. With a little more bookkeeping we can instead work from a standard first order in time, second order in space \([42, 43]\) formulation of GR and avoid the first order reduction, but postpone any presentation thereof. One expects that this formulation could be treated according to \([44]\) to give an economical demonstration of a ‘neighborhood theorem’ for the GR characteristic initial value problem. Physically what has happened is that the coordinate and gauge degrees of freedom GR were decoupled as much as possible. Note that in earlier work a symmetric hyperbolic formulation using a double-null gauge was constructed using frame variables coupled to the Bianchi equations \([45]\). In contrast here the aim was to arrive at such a formulation with as few changes as possible to standard formulations used in numerical work. From this practical point of view there is always the danger that the matrix $(1 + A^\mu_\nu)$ becomes singular. We can mitigate against this by simply evolving for a short but finite coordinate time, and then resetting the Jacobian to the identity by transforming
all the fields into the upper case tensor basis. For sufficiently regular data hyperbolicity guarantees short time existence so that this procedure can be performed iteratively. Coupled with strong theorems on continuation of solutions, this strategy could perhaps even be used to guarantee that numerical calculations progress successfully into extreme regions of spacetime, although the double-null gauge might have to be replaced with some other suitable choice. For the numerical relativist, probably the simplest summary of the double-null Jacobian result is that it is the generalization of the dual coordinate frames approach \cite{10} employed in the SpEC code \cite{16} to the situation in which two differing time coordinates are considered, and where the Jacobians satisfy dynamical equations rather than arising as derivatives of algebraic relationships between the coordinates.

V. CONCLUSION

Making a dual foliation approach, we have shown that it is possible to effectively decouple the choice of coordinates from local well-posedness of the field equations of general relativity. This was done by evolving a first order reduction of the generalized harmonic formulation alongside Jacobians mapping from the desired coordinate system to the generalized harmonic one. The important example of the double-null gauge was considered. But in fact the set of coordinate choices resulting in equations of motion for Jacobians that are minimally coupled to the remaining field equations is extremely large, and local well-posedness inside this class follows from well-posedness of the coordinate choice alone. It is thus expected that with due care, this observation will allow us to evolve the generalized harmonic formulation using say, the Maximal slicing, and quasi-isotropic spatial coordinates, for example. We hope that this would allow for direct comparison of the results of \cite{17} in a modern numerical relativity code with minimal changes. See \cite{18–52} for recent work on the problem. It is important to realize that the role of the generalized harmonic formulation in the construction is completely auxiliary. In fact any well-behaved formulation of GR is amenable to the same trick, perhaps after a suitable reduction to first order, if we wish to avoid complicated book-keeping. The generalized harmonic formulation is simply the most convenient example.

For numerical applications it will be necessary to translate the constraint preserving boundary conditions \cite{53} of the harmonic system into the new set of variables and coordinates, but no major difficulty is expected in doing so. The obvious next step is to systematically implement and test the approach, preferably in a simple context. Afterwards we expect to use the double-null coordinates to study the critical collapse of gravitational waves using the \texttt{bamps} code \cite{17,10}.

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