RESONANCES FOR DIRAC OPERATORS ON THE HALF-LINE
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Abstract. We consider the 1D Dirac operator on the half-line with compactly supported potentials. We study resonances as the poles of scattering matrix or equivalently as the zeros of modified Fredholm determinant. We obtain the following properties of the resonances: 1) asymptotics of counting function, 2) estimates on the resonances and the forbidden domain.

1. Introduction and main results

1.1. Definition of the Dirac operator. Consider the free Dirac operator $H_0$ acting in the Hilbert space $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ with the Dirichlet boundary condition at $x = 0$ and given by

$$H_0 f = -i\sigma_2 f' + \sigma_3 m f = \begin{pmatrix} mf_1 & -f_2' \\ f_1' & -mf_2 \\ \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ \end{pmatrix}, \quad f_1(0) = 0. \quad (1.1)$$

Here $m > 0$ is the mass and $\sigma_j$, $i = 1, 2, 3$, are the Pauli matrices

$$\sigma_0 = I_2, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Define the perturbed Dirac operator $H$ by

$$H = H_0 + V = \begin{pmatrix} m + p_1 & -\partial_x + q \\ \partial_x + q & -m + p_2 \end{pmatrix}. \quad (1.2)$$

We consider a perturbation of the form

$$V(x) = \begin{pmatrix} p_1 \\ q \\ p_2 \end{pmatrix}(x), \quad x \geq 0, \quad (1.3)$$

where $p_1, p_2$ and $q$ are real-valued functions satisfying

$$p_1, p_2, q \in L^1(\mathbb{R}_+).$$

Later, we shall place further restrictions on these functions.

In [IK13] we considered the similar operator but in the massless case $m = 0$ and on the real line. This paper generalizes some results obtained in [IK13].

We recall some well-known spectral properties of the Dirac operators, see for example [LS88]. The operators $H_0$, $H$ with Dirichlet condition (1.1) are self-adjoint in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$. The spectrum of $H_0$ is absolutely continuous and is given by

$$\sigma(H_0) = \sigma_{ac}(H_0) = \mathbb{R} \setminus (-m, m).$$

The spectrum of $H$ consists of the absolutely continuous part $\sigma_{ac}(H) = \sigma_{ac}(H_0)$ and finite number of simple eigenvalues in the gap $(-m, m)$.
In order to define resonances we will suppose that $V$ has compact support and satisfy the following hypothesis:

**Condition A.** Real-valued functions $p_1, p_2, q \in L^2(\mathbb{R}_+)$ and

$$\text{supp } V := \text{supp } p_1 \cup \text{supp } p_2 \cup \text{supp } q \subset [0, \gamma], \quad \gamma = \text{supp } V > 0.$$  

We denote $\mathbb{C}_\pm = \{ \lambda \in \mathbb{C}; \pm \text{Im } \lambda > 0 \}$. We introduce the quasi-momentum $k(\lambda)$ by

$$k(\lambda) = \sqrt{\lambda^2 - m^2}, \quad \lambda \in \Lambda = \mathbb{C} \setminus [-m, m].$$

The function $k(\lambda)$ is a conformal mapping from $\Lambda$ onto $\mathbb{C} \setminus [\text{im }, -\text{im }]$ and satisfies

$$k(\lambda) = \lambda - \frac{m^2}{2\lambda} + \mathcal{O}(1) \quad \text{as } |\lambda| \to \infty. \quad (1.4)$$

The function $k(\lambda)$ maps the horizontal cut $(-m, m)$ on the vertical cut $[\text{im }, -\text{im }]$. Moreover,

$$k(\mathbb{R}_+ \setminus (-m, m)) = \mathbb{R}_+, \quad k(i \mathbb{R}_+) = i \mathbb{R}_+ \setminus (-\text{im }, \text{im }).$$

The Riemann surface for $k(\lambda)$ is obtained by joining the upper and lower rims of two copies $\mathbb{C} \setminus \sigma_{\text{ac}}(H_0)$ cut along the $\sigma_{\text{ac}}(H_0)$ in the usual (crosswise) way. Instead of this two-sheeted Riemann surface it is more convenient to work on the cut plane $\Lambda$ and half-planes $\Lambda_\pm$ given by

$$\Lambda = \mathbb{C} \setminus [-m, m], \quad \Lambda_\pm = \mathbb{C}_\pm \cup g_\pm.$$

Here we denote $g_+ \subset \Lambda_+$, and $g_- \subset \Lambda_-$, the upper respectively and lower rim of the cut $(-m, m)$ in $\mathbb{C} \setminus [-m, m]$. Here the upper half-plane $\Lambda_+ = \mathbb{C}_+ \cup g_+$ corresponds to the physical sheet and the lower half-plane $\Lambda_- = \mathbb{C}_- \cup g_-$ corresponds to the non-physical sheet.

**Below we consider all functions and the resolvent in $\mathbb{C}_+$ and will obtain their analytic continuation into the cut domain $\Lambda$.**

Note that, equivalently, we could consider the Jost function, the resolvent etc in $\Lambda_-$ (the physical sheet) and obtain their analytic continuation into the whole cut domain $\Lambda$.

We denote $\sqrt{z}$ the principal branch of the square root that is positive for $z > 0$ and with the cut along the negative real axis.

**By abuse of notation, we will think of all functions $f$ as functions of both $\lambda$ and $k$, and will regard notations as $f(x, \lambda)$ and $f(x, k)$ as indistinguishable.**

We consider the Dirac system for a vector-valued function $f(x)$

$$\begin{cases} 
 f_1' + qf_1 - (m - p_2 + \lambda)f_2 = 0 
 \quad \text{and} \quad f_1(0) = 0, \\
 f_2' - qf_2 - (m + p_1 - \lambda)f_1 = 0, 
 \quad f_2(V) = 0, 
\end{cases} \quad \lambda \in \mathbb{C}, \quad f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \quad (1.5)$$

Define the Jost solutions of the free Dirac system by

$$\psi^\pm = e^{\pm i k(\lambda)x} \begin{pmatrix} \pm k_0(\lambda) \\ 1 \end{pmatrix}, \quad k_0(\lambda) = \frac{\lambda + m}{ik(\lambda)}, \quad \lambda \in \mathbb{C}_+$$

as a solution of (1.5) at $V \equiv 0$.

**Remark.** Note that this definition of the Jost solution differs by the factor $k_0(\lambda)$ from the standard one (see [LS88]). The relation with standard Jost solution is given in below (see [LS88]). We adopt this normalization at spatial infinity in order to simplify comparison with the results previously obtained in [HKS88], [HKS89], where the similar problems were considered but on the physical sheet.

The Jost solutions $f^\pm$ for the Dirac system (1.3) are defined using the standard condition

$$f^+(x, \lambda) = \psi^+(x, \lambda) + o(1), \quad x \to \infty, \quad f^-(x, \lambda) = \overline{f^+(x, \lambda)}, \quad \lambda \in \sigma_{\text{ac}}(H_0).$$
The function $f_1^+(0, \lambda)$, which is the first component of the Jost solution $f^+(x, \lambda)$ at point $x = 0$, is called the Jost function. We will most of the time work with the Jost solution $f^+(x, \lambda)$ and often omit the upper index $+$. We define resonances. It is well known that the function $G(\lambda) = ((H - \lambda)^{-1}h, h)$, for each $h \in C_0(\mathbb{R}_+, \mathbb{C}^2)$, has meromorphic continuation from $\mathbb{C}_+$ into $\Lambda$ through the set $\sigma_{ac}(H_0)$ and $G$ does not have poles in $\mathbb{C}_+$.

Definition Let $G(\lambda) = ((H - \lambda)^{-1}h, h)$, $\lambda \in \Lambda$ for some $h \in C_0(\mathbb{R}_+, \mathbb{C}^2)$, $h \neq 0$.
1) If $G(\lambda)$ has pole at some $\lambda_0 \in g_+ \subset \Lambda_+$ we call $\lambda_0$ an eigenvalue.
2) If $G(\lambda)$ has pole at some $\lambda_0 \in \Lambda_-$ we call $\lambda_0$ a resonance. If, in addition, $\lambda_0 \in g_- \subset \Lambda_-$, then we call $\lambda_0$ an anti-bound state.
3) A point $\lambda_0 = m$ or $\lambda_0 = -m$ is called virtual state if the function $z \to G(\lambda_0 + z^2)$ has a pole at 0.
4) A point $\lambda_0 \in \Lambda$ is called a state if it is either an eigenvalue, a resonance or a virtual state. Its multiplicity is the multiplicity of the corresponding pole. If the pole is simple, then the state is called simple. We denote $\sigma_{ac}(H)$ the set of all states.

We will show that the set of resonances coincides with the set of zeros in $\Lambda_-$ of the Jost function $f_1^+(0, \lambda)$, and multiplicity of a resonance is the multiplicity of the corresponding zero.

Proposition 1.1. The free Dirac operator $H_0$ has only one state: the simple virtual state at $\lambda = -m$.

Remark. Note that if $m < 0$ (positron) we would get $\lambda_0 = m$ is a virtual state.

Theorem 1.2. Let $V$ satisfy condition A. Then the states of $H$ satisfy:
1) Let $\lambda^{(1)} \in g_+ \subset \Lambda_+$ be eigenvalue of $H$ and $\lambda^{(2)} \in g_- \subset \Lambda_-$ be the same number but on the ”non-physical sheet”. Then $\lambda^{(2)}$ is not an anti-bound state.
2) Let $\lambda_1 < \lambda_2$ be eigenvalues of $H$ and assume that there are no other eigenvalues on the interval $(\lambda_1, \lambda_2) \subset g_+$. Let $\Omega \subset g_-$ be the same interval but on the ”non-physical sheet” $\Lambda_-$. Then there exists an odd number $\geq 1$ of anti-bound states counted with multiplicities on $\Omega$.

Remark. In the case of Dirac systems we have only one gap. The case of many gaps for compact perturbations of periodic systems in different settings was previously studied in [K11] for Schrödinger operator with periodic plus compactly supported potentials on the real line. Here in general there is any number of gaps. Later on these results were extended to Schrödinger operator with periodic plus compactly supported potentials on the half-line [KS12]. Finally, they were extended to Jacobi operator with periodic plus compactly supported coefficients [K11]. In the present paper we also use technics from [K11], [KS12].

In our case, the free Dirac operator is the simplest example of a Dirac operator with periodic coefficients, which has only one gap, see [K11]. This gap corresponds to the Riemann surface of the function $\sqrt{\lambda^2 - m^2}$. All functions (the Jost function, the resolvent etc ) are analytic on this surface. Nevertheless, we can reduce the analysis of the Jost function on two-sheeted Riemann surface to an entire function $F$, defined by (1.14).

We will show later that
$$f_1(0, i\eta) = e^{i\Omega_0 \frac{\pi}{4}} + o(1), \quad \text{as} \quad \eta \to \infty,$$
where
$$\Omega_0 = \frac{1}{2} \int_0^\infty v(t)dt, \quad v(t) = \text{Tr} V(t).$$

(1.6)
Due to (1.6) we can define the unique branch $\log f_1(0, \lambda)$ in $\mathbb{C}_+$ and define the functions

$$\log f_1(0, \lambda) = \log |f_1(0, \lambda)| + i \arg f_1(0, \lambda), \quad \lambda \in \mathbb{C}_+,$$

where the function $\phi_{sc} = \arg f_1(0, \lambda) + \pi/2$ is called the scattering phase (or the spectral shift function). The scattering matrix $S(\lambda)$, $\lambda \in \sigma_{ac}(H_0)$, for the pair $H, H_0$ is given by

$$S(\lambda) = -\frac{f_1(0, \lambda + i0)}{f_1(0, \lambda + i0)} = e^{-2i \phi_{sc}}, \quad \text{for } \lambda \in \sigma_{ac}(H_0).$$

The minus sign comes from our choice of the normalization the Jost solutions at the spatial infinity (see (4.10) below). Property (1.6) implies

$$\phi_{sc}(\lambda) = \Omega(0) + o(1), \quad \text{as } \text{Im } \lambda \to \infty.$$

We denote $R_0 = (H_0 - \lambda)^{-1}$, $R = (H - \lambda)^{-1}$ the resolvents for $H_0, H$ respectively. Observing that the operator valued function $VR_0(\lambda)$ is in the Hilbert-Schmidt class $\mathcal{B}_2$ but not in the trace class $\mathcal{B}_1$, we introduce the modified Fredholm determinant $D(\lambda)$ (see [GK69]) as follows

$$D(\lambda) = \det \left[ (I + VR_0(\lambda)) e^{-VR_0(\lambda)} \right], \quad \forall \lambda \in \mathbb{C}_+. $$

We will show later that the function

$$\Omega(\lambda) = \frac{1}{2i} \text{Tr} V(R_0(\lambda + i0) - R_0(\lambda - i0)), \quad \text{if } \lambda \in \mathbb{R} \setminus \{ \pm m \}$$

is well defined. Note that $\Omega(\lambda) = 0$ on the interval $(-m, m)$.

We formulate the main results about the modified Fredholm determinant $D$.

**Theorem 1.3.** Let $V \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Then the function $f_1^+(0, \lambda)$ and the determinant $D(\lambda)$ are analytic in $\mathbb{C}_+$, continuous up to $\mathbb{R} \setminus \{ \pm m \}$ and satisfy

$$S(\lambda) = \frac{D(\lambda - i0)}{D(\lambda + i0)} e^{-2i \Omega(\lambda)}, \quad \phi_{sc} = \Omega(\lambda) + \arg D(\lambda), \quad \forall \lambda \in \sigma_{ac}(H_0), \quad \lambda \neq \pm m.$$

Here the function $\Omega$ (defined in (1.7)) is continuous on $\mathbb{R} \setminus \{ \pm m \}$ and satisfies

$$\Omega(\lambda) = \int_0^\infty \left( p_1(t) \frac{1}{k} \sin^2 k t + p_2(t) \frac{k}{\lambda + m} \cos^2 k t + q(t) \sin 2k t \right) dt, \quad (1.8)$$

and $k = k(\lambda)$. If in addition $V' \in L^1(\mathbb{R}_+)$, then the functions $f_1^+(0, \lambda), D(\lambda)$ satisfy for $\lambda \in \mathbb{C}_+$

$$f_1^+(0, \lambda) = k_0(\lambda) D(\lambda) \exp \left( i \Omega(\lambda) + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\Omega(t) - \Omega_0}{t - \lambda} dt \right), \quad \Omega_0 = \frac{1}{2} \int_0^\infty \text{Tr} V(t) dt. \quad (1.9)$$

**Remark.** Note that $\Omega(\lambda) = \Omega_0 + o(1)$, as $\lambda \to \infty$. If in addition $(p_1 - p_2)'$, $q' \in L^1(\mathbb{R}_+)$, then the error term is of order $\mathcal{O}(\lambda^{-1})$ as $\lambda \to \infty$ (see Remark after Proposition 1.3), which implies that $\Omega(\cdot) - \Omega_0 \in L^2(\mathbb{R})$ and that the first integral in (1.9) is well defined.

If $V$ has a compact support, then $D(\cdot)$ has an analytic continuation from $\mathbb{C}_+$ into $\Lambda_-$ through the set $\sigma_{ac}(H_0)$. The zeros of $D(\cdot)$ in $\mathbb{C}_-$ are the complex resonances.

We determine the asymptotics of the counting function. We denote the number of zeros of a function $f$ having modulus $\leq r$ by $N(r, f)$, each zero being counted according to its multiplicity.
Theorem 1.4. Assume that potential $V$ satisfies Condition A and $V' \in L^1(\mathbb{R}_+)$. Then $D(\cdot)$ is analytic in $\mathbb{C} \setminus [-m, m]$. The set of zeros of $D$ with negative imaginary part (i.e. complex resonances) satisfy:

$$N(r, D, \Lambda) = \frac{2r\gamma}{\pi}(1 + o(1)) \quad \text{as} \quad r \to \infty. \quad (1.10)$$

For each $\delta > 0$ the number of zeros of $D$ with negative imaginary part with modulus $\leq r$ lying outside both of the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$ is $o(r)$ for large $r$.

Remark. 1) Zworski obtained in [ZS7] similar results for the Schrödinger operator with compactly supported potentials on the real line.

2) Our proof follows from Proposition 6.2 and Levinson Theorem 2.1

An entire function $f(z)$ is said to be of exponential type if there is a constant $A$ such that $|f(z)| \leq \exp(A|z|)$ everywhere. The infimum of the set of $A$ for which inequality holds is called the type of $f(z)$ (see [Koo81]). Section 2 contains more details on the exponential type functions. If $f$ is analytic and satisfies the above inequality only in $\mathbb{C}_+$ or $\mathbb{C}_-$, we will say that $f$ is of exponential type in $\mathbb{C}_\pm$ with the type defined appropriately.

Theorem 1.5. Assume that potential $V$ satisfies Condition A and $V' \in L^1(\mathbb{R}_+)$. Then

i) the Jost function $f_1(0, \cdot)$ satisfies

$$f_1(0, \lambda) = k_0(\lambda)e^{i\Omega_0} \left(1 - \frac{B(\lambda)}{2\lambda} + \mathcal{O}(1)\right), \quad B(\lambda) = B_0 + o(1), \quad (1.11)$$

as $|\lambda| \to \infty$, $\lambda \in \mathbb{C}_+$, where

$$B(\lambda) = B_0 + B_1(\lambda), \quad B_0 = w(0) + \int_0^\gamma (2mp(x) + |w(x)|^2)dx,$$

$$B_1(\lambda) = \int_0^\gamma e^{2i(kx - f_0'(u(s))}(i2vw - w')dx, \quad (1.12)$$

and $p(x) = (p_1 - p_2)/2$, $w(x) = q + ip$.

ii) The scattering phase satisfies

$$\phi_\infty(\lambda) = \Omega_0 + \frac{B(\lambda)}{2\lambda} + \mathcal{O}(\lambda^{-2}), \quad (1.13)$$

as $|\lambda| \to \infty$, $\lambda \in \mathbb{C}_+$.

iii) The Jost function $f_1(0, \cdot)$ has exponential type $0$ in $\mathbb{C}_+$ and $2\gamma$ in $\mathbb{C}_-$.

Remark. By Riemann-Lebesgue Lemma the term $B_1(\lambda)$ in (1.12) is $o(1)$ as $\lambda \to \pm\infty$. The other terms in (1.12) were first obtained in [HKSS9] but with the minus in front of $2pm$, which according to our result is incorrect. The $\lambda^{-1}$ term in [HKSS9] was obtained by integration by parts in the iteration expansion of the solution of an integral equations and no uniform estimate on the rest term was given. In our method (see Lemmas 5.1 and 5.2) we use integration by parts and rearrangement in order to obtain an integral equation which directly leads to the full expansion of the solution in orders of $k^{-1} \sim \lambda^{-1}$, which can be used to obtain coefficient in any order (theoretically: as explicit calculations become technically complicated). Moreover, we get an uniform bound on the rest terms.

Similar to [KS12] we define the function $F$ on the spectrum $\sigma(H_0)$ by

$$F(\lambda) = (\lambda - m)f_1^+(0, \lambda)f_1^-(0, \lambda), \quad \lambda \in \sigma(H_0). \quad (1.14)$$
This function has an analytic extension into the whole complex plane. In Proposition 4.4 we show that $F$ is entire and its zeros coincide, including multiplicity, with the states of $H$.

We describe the position of resonances and the forbidden domain.

**Theorem 1.6.** Assume that potential $V$ satisfies Condition A and $V' \in L^1(\mathbb{R}_+)$.

Let $\lambda_n \in \Lambda_-$, $n \geq 1$, be a resonance. Then

$$\left| \frac{\lambda_n^2}{4} \left( \lambda_n + (m - p(0)) + \frac{1}{4\lambda_n} (|B(\lambda)|^2 - 4mp(0)) \right)^2 - \lambda_n^2 \right| \leq C_1 e^{-2\gamma \Im \lambda_n},$$  \hspace{1cm} (1.15)

where $B$ is given in (1.12) and the constant

$$C_1 = \sup_{\lambda \in \mathbb{R}} \left| \lambda^2 \left( F(\lambda) - \lambda - (m - p(0)) - \frac{1}{4\lambda} (|B(\lambda)|^2 - 4mp(0)) \right) \right| < \infty.$$  

In particular, for any $A > 0$, there are only finitely many resonances in the region

$$\{0 > \Im \lambda \geq -A - \frac{1}{\gamma} \log |\Re \lambda| \}.$$  

**Remark.** The proof follows from Corollary 2.3.

**General comments.** Resonances, from a physicists point of view, were first studied by Regge in 1958 (see [R58]). Since then, the properties of resonances has been the object of intense study and we refer to [SZ91] for the mathematical approach in the multi-dimensional case and references given there. In the multi-dimensional Dirac case resonances were studied locally in [HB92]. We discuss the global properties of resonances in the one-dimensional case.

A lot of papers are devoted to the resonances for the 1D Schrödinger operator, see Froese [F97], Korotyaev [K04], Simon [S00], Zworski [Z87] and references given there. We recall that Zworski [Z87] obtained the first results about the asymptotic distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. Different properties of resonances were determined in [H99], [K11], [S00] and [Z87]. Inverse problems (characterization, recovering, plus uniqueness) in terms of resonances were solved by Korotyaev for the Schrödinger operator with a compactly supported potential on the real line [K05] and the half-line [K04]. The "local resonance" stability problems were considered in [K04s], [MSW10].

Similar questions for Dirac operators are much less studied. In [K12] the estimates of the sum of the negative power of all resonances in terms of the norm of the potential and the diameter of its support are determined. In [IK13] we consider the 1D massless Dirac operator on the real line with compactly supported potentials. It is a special kind of the Zakharov-Shabat operator (see [DEGM], [ZMNP]). Technically, this case is much simpler than the massive Dirac operator studied in the present paper, since in the massless Riemann surface consists of two disjoint sheets $\mathbb{C}$. Moreover, the resolvent has a simple representation. In [IK13] we were able even to prove the trace formulas in terms of resonances. Note that in the massless case the relation between the modified Fredholm determinant $D$ and the Jost function $f_1^+(0, \lambda)$ (corresponding to $a$ for the problem on the line in [IK13], the inverse of the transmission coefficient) is much easier than in the massive case (see Theorem 1.3), namely $D(\lambda) = a(\lambda)$, with no any proportionality factors in between.

The properties of the Jost solutions of (1.2) and Levinson theorem for the number of eigenvalues were studied in [HKSS88] and [HKSS99] under the hypothesis that the potential satisfies $V \in L^1$ or $\int_0^\infty (1 + r)|V(r)|dr$. In particular, if $V' \in L^1$, the large $|\lambda|$ asymptotics for the
Jost function as in (1.11) were obtained (see also Remark after Theorem 1.5). Our choice of normalization for $x \to \infty$ of the Jost solution of (1.5) was motivated by [HKS88] and [HKS89]. The origin of (1.2) is the Dirac equation in $\mathbb{R}^3$ given by

$$-i \sum_{j=1}^{3} \alpha_j \frac{\partial \psi}{\partial x_j} + (V(x) + \beta m)\psi = E\psi, \quad x \in \mathbb{R}^3,$$

(1.16)

which physically describes a relativistic electron of mass $m$ in an electrostatic field $V(x)$. In (1.16) $\psi$ is the four-component wave-function, $\alpha_j, \beta$ are the following matrices

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

in the units $\hbar = c = 1$.

If $V(x) = V(r)$, $r = |x|$, is spherically symmetric (for example the Coulomb potential $\gamma/r$) then (1.16) is spectrally equivalent to the direct sum of the operators of the form (1.2) on $\mathbb{R}_+$ with $p_1 = p_2 = V(r)$, and where $q(r) = \frac{\xi}{r}$, $\kappa \in \mathbb{Z} \setminus \{0\}$ is related to the total angular momentum of the particle.

If $p_1 = -p_2 = p(x)$ the system (1.5) on the line $\mathbb{R}$ has been associated with inverse scattering for nonlinear evolution equations and with certain waveguide problems. This is the choice in [HJKS91].

Note that (see [HJKS91] or [LS88]) if we introduce the orthogonal transformation $y = \mathcal{U}z$, where

$$\mathcal{U}(x) = \begin{pmatrix} \cos W & \sin W \\ -\sin W & \cos W \end{pmatrix}(x), \quad W(x) = \frac{1}{2} \int_{\gamma} (p_1(t) + p_2(t)) dt,$$

then equation $Hy = \lambda y$ where $H$ is given in (1.2) transforms to

$$\begin{pmatrix} m + \tilde{p}(x) & -\partial x + \tilde{q}(x) \\ \partial x + \tilde{q}(x) & -m - \tilde{p}(x) \end{pmatrix} z = \lambda z.$$  (1.17)

Moreover, if $\frac{1}{2} \int_{0}^{\gamma} (p_1 + p_2) dt = 2\pi n$, $n \in \mathbb{Z}$, then equations $Hy = \lambda y$ and (1.17) have the same Jost function and $S$-function, which follows from $y(0) = z(0)$.

If we introduce new functions

$$v := \frac{p_1 + p_2}{2}, \quad p := \frac{p_1 - p_2}{2},$$

then operator $H$ given in (1.3) has the form

$$H = -i\sigma_2 \partial_x + \sigma_1 q + (m + p)\sigma_3 + v\sigma_0.$$  

The plan of our paper is as follows. In Section 2 we recall some results about entire functions and prove Theorem 1.6 referring to the results obtained in Section 5. In Section 3 we study the free (unperturbed) Dirac operator, associated spectral representation and prove useful Hilbert-Schmidt estimates for the “sandwiched” free resolvent, based on an estimate which is proved in Section 8. In Section 4 we describe the properties of fundamental solution of the full Dirac operator $H$, the resolvent and the function $F$.

In Section 5 we obtain uniform estimates on the Jost solution as $\lambda \to \infty$ under the condition that $V' \in L^1(\mathbb{R}_+)$. These results are used in Section 6 in order to prove that function $F$ is in Cartwright class.

In Section 7 we give the properties of the modified Fredholm determinant and prove Theorem 1.3.
Notations For a matrix valued function $E(x)$, we denote the norms $|E(x)| = \sup_{i,j} |E_{ij}(x)|$ and $\|E\|_{L^p} = (\int_0^\infty |E(x)|^p dx)^{1/p}, p \in \mathbb{N}$.

2. Cartwright class of entire functions

In this section we will prove Theorem 1.6. The proof is based on some well-known facts from the theory of entire functions which we recall here. We mostly follow [Koo81]. An entire function $f(z)$ is said to be of exponential type if there is a constant $A$ such that $|f(z)| \leq \text{const} e^{A|z|}$ everywhere. The function $f$ is said to belong to the Cartwright class $\text{Cart}_\rho$ if $f$ is entire, of exponential type, and the following conditions hold true:

$$\int_R \frac{\log(1 + |f(x)|)}{1 + x^2} dx < \infty, \quad \rho_\pm(f) \equiv \limsup_{y \to \infty} \frac{\log |f(\pm iy)|}{y} = \rho > 0,$$

(2.1)

for some $\rho > 0$. Here $\rho_\pm(f)$ is the type of the exponential type function $f$ in $\mathbb{C}_\pm$.

We will be working with the following sub-class of exponential type functions satisfying (2.1). Fix $\rho > 0$.

Definition. Let $\mathcal{E}(\rho, \delta > 0)$ denote the space of exponential type functions $f$, which satisfy the following conditions:

i) $\rho_+(f) = \rho_-(f) = \rho$,
ii) $f \in L^\infty(\mathbb{R})$.

Assume that $f$ belongs to the Cartwright class and denote by $(z_n)_{n=1}^\infty$ the sequence of its zeros $\neq 0$ (counted with multiplicity), so arranged that $0 < |z_1| \leq |z_2| \leq \ldots$. Then we have the Hadamard factorization

$$f(z) = C z^m \lim_{r \to \infty} \prod_{|z_n| \leq r} \left(1 - \frac{z}{z_n}\right), \quad C = \frac{f^{(m)}(0)}{m!},$$

(2.2)

for some integer $m$, where the product converges uniformly in every bounded disc and

$$\sum_{|z_n|} \frac{|\text{Im} z_n|}{|z_n|^2} < \infty.$$

(2.3)

We denote the number of zeros of a function $f$ having modulus $\leq r$ by $\mathcal{N}(r, f)$, each zero being counted according to its multiplicity.

We also denote $\mathcal{N}_+(r, f)$ (or $\mathcal{N}_-(r, f)$) the number of zeros of function $f$ counted in $\mathcal{N}(r, f)$ with non-negative (negative) imaginary part having modulus $\leq r$ by , each zero being counted according to its multiplicity. We need the following well known result (see [Koo81], page 69).

Theorem 2.1 (Levinson). Let the function $f$ belong to the Cartwright class for some $\rho > 0$. Then

$$\mathcal{N}_+(r, f) = \mathcal{N}_-(r, f) = \frac{\rho r}{\pi} (1 + o(1)), \quad r \to \infty.$$  

(2.4)

For each $\delta > 0$ the number of zeros of $f$ with modulus $\leq r$ lying outside both of the two sectors $|\arg z|, |\arg z - \pi| < \delta$ is $o(r)$ for large $r$.

Now, similar to [IK13], we will use some arguments from the paper [K04], where some properties of resonances were proved for the Schrödinger operators. In order to simplify applications of the formulas to our settings we chose $\rho = 2\gamma$, $\gamma > 0$. In order to prove Theorem 1.6 we need
Lemma 2.2. Let \( f \in \mathcal{E}(2\gamma) \) and \( \gamma > 0 \). Assume that for some \( p \geq 0 \) there exists a polynomial \( G_p(z) = z + d_0 + \sum_1^p d_n z^{-n} \) and a constant \( C_p \) such that

\[
C_p = \sup_{x \in \mathbb{R}} |x^{p+1}(f(x) - G_p(x))| < \infty, \tag{2.5}
\]

Then for each zero \( z_n, n \geq 1 \), the following estimate holds true:

\[
|G_p(z_n)| \leq C_p |z_n|^{-p-1} e^{-2\gamma y_n}, \quad y_n = \text{Im } z_n. \tag{2.6}
\]

Proof. We take the function \( f_p(z) = z^{p+1}(f(z) - G_p(z)) e^{-i2\gamma z} \). By condition, the function \( f_p \) satisfies the estimates

1) \( |f_p(x)| \leq C_p \) for \( x \in \mathbb{R} \),
2) \( \log |f_p(z)| \leq O(|z|) \) for large \( z \in \mathbb{C}_- \),
3) \( \lim_{y \to \infty} y^{-1} \log |f_p(-iy)| = 0 \).

Then the Phragmen-Lindelöf Theorem (see [Koo81], page 23) implies \( |f_p(z)| \leq C_p \) for \( z \in \mathbb{C}_- \). Hence at \( z = z_n \) we obtain

\[
|z^{p+1}G_p(z)e^{-i2\gamma z}| = |f_p(z)| = |z^{p+1}(f(z) - G_p(z))e^{-i2\gamma z}| \leq C_p,
\]

which yields (2.6). \( \blacksquare \)

Corollary 2.3. Let \( f \in \mathcal{E}(2\gamma) \) and \( \gamma > 0 \). Let \( z_n, n \geq 1 \), be zeros of \( f \).

i) Assume that \( C_0 = \sup_{x \in \mathbb{R}} |x^2(f(x) - x - d_0)| < \infty \). Then each zero \( z_n, n \geq 1 \), satisfies

\[
|z_n(z_n + d_0)| \leq C_0 e^{-2\gamma y_n}. \tag{2.8}
\]

ii) Assume that \( C_1 = \sup_{x \in \mathbb{R}} |x^2(f(x) - x - d_0 x^{-1})| < \infty \) for some \( A \). Then each zero \( z_n, n \geq 1 \), satisfies

\[
|z_n^2(z_n + d_0 + d_1 z_n^{-1})| \leq C_1 e^{-2\gamma y_n}. \tag{2.9}
\]

Proof of Theorem 1.6. Note that in Proposition 4.4 it is proved that function \( F(\lambda) \) belongs to \( \mathcal{E}(2\gamma) \). Moreover, if \( V \) satisfies Condition A and \( V' \in L^1(\mathbb{R}) \), then \( F(\lambda) \) satisfy uniform bound (6.2) which follows from Corollary 6.1 and therefore the conditions of Corollary 2.3 are satisfied. \( \blacksquare \)

3. Free Dirac system.

3.1. Properties of quasi-momentum. Here we recall the properties of \( k(\lambda) \). Recall that the brunch of \( k(\lambda) \) is chosen so that \( k(\lambda) > 0 \) for real \( \lambda > m \) and \( k(\lambda) < 0 \) for \( \lambda < -m \). In particular, we have the following properties:

\[
\text{Im } k(\lambda) > 0 \iff \lambda \in \Lambda^+,
\]

for \( \lambda \in \mathbb{C} \setminus [-m, m] \):

- \( k(\lambda) = -k(-\lambda) = \overline{k(\lambda)} \),
- \( k(\lambda \pm i0) = \pm i|m^2 - \lambda^2|^{\frac{1}{2}} \),
- \( k(\lambda) = \pm |\lambda^2 - m^2|^{\frac{1}{2}}, \quad \pm \lambda \geq m \)

and

\[
k_0(\lambda) = \frac{\lambda + m}{ik(\lambda)} = -i \left( 1 + \frac{m}{\lambda} + \frac{O(1)}{\lambda^2} \right), \quad |\lambda| \to \infty.
\]
3.2. Preliminaries. We consider the free Dirac system $H_0f = \lambda f$ for a vector valued function $f(x)$

\[
\begin{align*}
  f'_1 - (m + \lambda)f_2 &= 0, \\
  f'_2 - (m - \lambda)f_1 &= 0,
\end{align*}
\]

where $f_1, f_2$ are the functions of $x \in \mathbb{R}_+$. We introduce a basis of solutions $\psi^\pm$ for the unperturbed problem \([3.2]\)

\[
\psi^\pm(x, \lambda) = e^{\pm ik(x) x} \left( \pm \frac{k_0(\lambda)}{1} \right), \quad k_0(\lambda) = \frac{\lambda + m}{ik(\lambda)}, \quad \lambda \in \mathbb{C}_+,
\]

where the function $k = k(\lambda) = \sqrt{\lambda^2 - m^2}$ is quasi-momentum.

We consider the fundamental matrix of solutions \([3.2]\)

\[
\mathcal{M}_0(x, \lambda) := \begin{pmatrix} \vartheta_1 & \varphi_1 \\ \vartheta_2 & \varphi_2 \end{pmatrix}(x, \lambda) = \begin{pmatrix} \cos k(\lambda) x & i k_0(\lambda) \sin k(\lambda) x \\ \frac{1}{k_0(\lambda)} \sin k(\lambda) x & \cos k(\lambda) x \end{pmatrix}, \tag{3.3}
\]

where $\vartheta(x, \lambda), \varphi(x, \lambda)$ are fundamental solutions of \([3.2]\) satisfying

\[
\vartheta(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

From definition \([3.3]\) it follows that $\mathcal{M}_0(x, \cdot)$ is entire function.

For two vector-functions $f, g$ we define the Wronskian as $\det(f, g) = f_1g_2 - f_2g_1$.

Now, the integral kernel of the free resolvent $R_0(\lambda) := (H_0 - \lambda)^{-1}$ is given by

\[
R_0(x, y, \lambda) = \begin{cases} \frac{1}{k_0(\lambda)} \psi^+(x, \lambda) (\varphi(y, \lambda))^T & \text{if } y < x, \\ \frac{1}{k_0(\lambda)} \varphi(x, \lambda) (\psi^+(y, \lambda))^T & \text{if } x < y, \end{cases}
\]

where $k_0(\lambda) = \det (\varphi^+, \varphi)$ is the Wronskian and

\[
R_0(x, y, \lambda) = e^{ik(x) y} \begin{pmatrix} i k_0(\lambda) \sin k(\lambda) x & \cos k(\lambda) y \\ i \sin k(\lambda) x & \cos k(\lambda) y \end{pmatrix} \quad \text{if } y < x, \tag{3.4}
\]

and

\[
R_0(x, y, \lambda) = e^{ik(y) x} \begin{pmatrix} i k_0(\lambda) \sin k(\lambda) x & i \sin k(\lambda) x \\ \cos k(\lambda) y & \cos k(\lambda) x \end{pmatrix} \quad \text{if } x < y. \tag{3.5}
\]

We have

\[
\psi^+(x, \lambda) = k_0(\lambda) \vartheta + \varphi = k_0(\lambda) (\vartheta + m_0(\lambda) \varphi), \quad \psi^-(x, \lambda) = \overline{\psi^+(x, \lambda)}, \quad \lambda \in \sigma_{ac}(H_0), \tag{3.6}
\]

where $m_0(\lambda) = \frac{1}{k_0(\lambda)} = \frac{ik(\lambda)}{x + m}$ is the Titchmarsh-Weyl function for $H^0$, and it satisfies $m_0(\lambda) = i + O(1/\lambda)$ as $\lambda \to \pm \infty$.

**Proof of Proposition 1.1.** Note that for $\lambda \in \Lambda_+^+$

\[
k(\lambda) = i \sqrt{2m} \sqrt{\epsilon} (1 - O(\epsilon)), \quad k_0(\lambda) = -\frac{\sqrt{2m}}{\sqrt{\epsilon}} (1 + O(\epsilon)), \quad \lambda = m - \epsilon, \quad \epsilon \to 0+, \tag{3.7}
\]

\[
k(\lambda) = i \sqrt{2m} \sqrt{\epsilon} (1 - O(\epsilon)), \quad k_0(\lambda) = -\frac{\sqrt{\epsilon}}{\sqrt{2m}} (1 + O(\epsilon)), \quad \lambda = -m + \epsilon, \quad \epsilon \to 0+.
\]

Using \([3.4], [3.5]\) and \([3.7]\) we get that all but $\frac{1}{k_0(\lambda)} \cos k(\lambda) r_\lambda$, $r_\lambda = \min\{x, y\}$, entries of the resolvent matrix are analytic in $\Lambda$. Taking $\lambda = m - \epsilon$, as $\epsilon \to 0+$, we get $\frac{1}{k_0(\lambda)} \cos k(\lambda) r_\lambda = -\frac{\sqrt{\epsilon}}{\sqrt{2m}} (1 + O(\epsilon))$. If $\lambda = -m + \epsilon$, $\epsilon \to 0+$, then we get $\frac{1}{k_0(\lambda)} \cos k(\lambda) r_\lambda = -\frac{\sqrt{\epsilon}}{\sqrt{2m}} (1 + O(\epsilon))$. 

Therefore, \( \lambda_0 = -m \) is a virtual state, but not \( \lambda_0 = m \).

3.3. Spectral representation. In this section we follow the classical ideas of spectral representation for Dirac operators as presented for example in \([LS88]\). Let as before

\[
\varphi(x, \lambda) = \left( \frac{ik_0(\lambda) \sin k(\lambda)x}{\cos k(\lambda)x} \right)
\]

be the fundamental solution satisfying the Dirichlet condition: \( \varphi(0, \lambda) = 0 \). Then there exists a non-decreasing function \( \rho(s) \), \( s \in \mathbb{R} \), such that for any vector-function \( f \in L^2(\mathbb{R}_+) \) there exists scalar-valued function \( F \in L^2(\mathbb{R}, d\rho) \) such that

\[
\mathcal{F}(s) = \int_0^\infty f^T(x)\varphi(x, s)dx, \quad f(x) = \int_{-\infty}^\infty \mathcal{F}(s)\varphi(x, s)d\rho(s).
\] (3.8)

and

\[
\int_0^\infty (f_1^2(x) + f_2^2(x))dx = \int_{-\infty}^\infty \mathcal{F}^2(s)d\rho(s).
\] (3.9)

Here, \( \mathcal{F} \) is the generalized Fourier transform of the vector-function \( f \) with respect to the eigenfunctions of the Dirac equation (3.2) with the Dirichlet boundary condition. We denote the generalized Fourier transform by \( \Phi \) and write \( F = \Phi f \). Formula (3.9) is the Parseval’s identity and it shows that \( \Phi \) is isometry of \( H \).

Function \( \rho \) is the spectral function. It satisfies the finiteness condition \( \int_{-\infty}^\infty (1 + s^2)^{-1}d\rho(s) < \infty \). As the discreet spectrum of \( H_0 \) is empty, then \( \rho(s) = 0 \) for \( s \in (-m, m) \). For \( \lambda \in \sigma_{ac}(H_0) \) the function \( \rho \) can be easily derived from the Weyl function \( m_0(\lambda) = \frac{ik(\lambda)}{\lambda + m} \). For \( s \in \sigma_{ac}(H_0) = (-\infty, -m) \cup [m, +\infty) \) we have

\[
d\rho(s) = \rho'(s)ds = \frac{1}{\pi} \text{Im} m_0(s + i0)ds = \frac{1}{\pi} \frac{k(\sigma)}{s + m} ds.
\]

It is convenient to introduce the modified transform \( \hat{\Phi} \) as follows.

Using that \( \rho'(s) = \frac{1}{\pi} \frac{k(\sigma)}{s + m} \) is positive for \( s \in (-\infty, -m) \cup [m, \infty) \) and by introducing the functions \( \hat{\varphi}(x, s) = \varphi(x, s)\sqrt{\rho'(s)} \), \( \hat{\mathcal{F}}(s) = \mathcal{F}(s)\sqrt{\rho'(s)} \), we get

\[
\hat{\mathcal{F}}(s) = \int_0^\infty f^T(x)\hat{\varphi}(x, s)dx, \quad f(x) = \int_{-\infty}^m \hat{\mathcal{F}}(s)\varphi(x, s)ds + \int_m^\infty \hat{\mathcal{F}}(s)\hat{\varphi}(x, s)ds.
\] (3.10)

and

\[
\int_0^\infty (f_1^2(x) + f_2^2(x))dx = \int_{-\infty}^m \hat{\mathcal{F}}^2(s)ds + \int_m^\infty \hat{\mathcal{F}}^2(s)ds.
\] (3.11)

The modified generalized Fourier transform \( \hat{\Phi} : \hat{\mathcal{F}}(s) = (\hat{\Phi} f)(s) \) is isometry of \( \mathcal{H} = (L^2(\mathbb{R}_+))^2 \) onto

\[
\hat{\mathcal{H}} = \hat{\Phi}(\mathcal{H}) = L^2((-\infty, -m], \rho'(s)ds) \oplus L^2([m, +\infty), \rho'(s)ds).
\]

Moreover, as for any \( f \in \mathcal{H} \), \( g \in \hat{\mathcal{H}} \), we have

\[
\langle \hat{\Phi} f, g \rangle_{\hat{\mathcal{H}}} = \langle f, \hat{\Phi}^{-1}g \rangle_{\mathcal{H}}
\]

and \( \hat{\Phi}^{-1} \) is the formal adjoint \( (\hat{\Phi})^* \) of \( \hat{\Phi} \). Here \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) denotes the scalar product in Hilbert space \( \mathcal{H} \).
Let
\[ \mathcal{E}(x, s) = \begin{pmatrix} \hat{\varphi}_1(x, s) & 0 \\ 0 & \hat{\varphi}_2(x, s) \end{pmatrix} \]
and \( \sigma = \sigma_{\text{ac}}(H_0) = (-\infty, -m] \cup [m, \infty) \). Then it follows from (3.11)
\[ \int_0^\infty \left| \int_{\sigma} \hat{\cal{F}}(s) \mathcal{E}(y, s) ds \right|^2 dy = \int_{\sigma} |\hat{\cal{F}}(s)|^2 ds \quad (3.12) \]
As
\[ \hat{\varphi}(x, s) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} \sqrt{\frac{s+m}{k(s)}} \sin k(s)x \\ \sqrt{\frac{k(s)}{s+m}} \cos k(s)x \end{pmatrix}, \]
where we chose the principal branch of square root: \( \sqrt{z} > 0, z > 0 \), then
\[ \pi |\mathcal{E}(x, s)|^2 = |k_0(s)| \sin^2 k(s)x + \left| \frac{1}{k_0(s)} \right| \cos^2 k(s)x \leq \max \left( |k_0(s)|, \left| \frac{1}{k_0(s)} \right| \right) =: \mathcal{K}_1(s). \quad (3.13) \]

3.4. Hilbert-Schmidt norms. We define the sets
\[ Z_c^\pm = \{ \lambda \in \mathbb{C}_\pm \; | \; -m, m]; \; |\lambda \pm m| \geq \epsilon \}, \quad Z_c = Z_c^+ \cup Z_c^-, \quad \epsilon > 0. \quad (3.14) \]
We denote by \( \| \cdot \|_{\mathcal{B}_2} \), the trace \( (k = 1) \) and the Hilbert-Schmidt \( (k = 2) \) operator norms.
For a Banach space \( \mathcal{X} \) let \( AC(\mathcal{X}) \) denote the set of all \( \mathcal{X} \)-valued analytic functions on \( \mathbb{C}_+ \), continuous in \( \mathbb{C}_+ \setminus \{ \pm m \} \).

**Theorem 3.1.** Let \( \chi, \tilde{\chi} \in L^2(\mathbb{R}_+; \mathbb{C}^2) \) and \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Then it follows:

i) Functions \( \chi R_0(\lambda), R_0(\lambda) \chi, \chi R_0(\lambda) \tilde{\chi} \) are analytic \( \mathcal{B}_2 \)-valued operator-functions in \( \mathbb{C}_\pm \) satisfying the following properties:
\[
\| \chi R_0(\lambda) \|_{\mathcal{B}_2}^2 = \| R_0(\lambda) \chi \|_{\mathcal{B}_2}^2 \leq \left[ \frac{4\pi}{|\text{Im} \lambda|} \left| \text{Re} \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \right| + O \left( \max \left\{ \frac{1}{|\lambda|^2}, \frac{1}{|\lambda \pm m|^2} \right\} \right) \right] \| \chi \|_{\mathcal{B}_2}^2, \quad (3.15) \]
\[
\| \chi R_0(\lambda) \tilde{\chi} \|_{\mathcal{B}_2} \leq \frac{c}{\epsilon} \| \chi \|_2 \| \tilde{\chi} \|_2 \quad \text{for } \lambda \in Z_c, \quad (3.16) \]
\[
\| \chi R_0(\lambda) \tilde{\chi} \|_{\mathcal{B}_2} \rightarrow 0 \quad \text{as } |\text{Im} \lambda| \rightarrow \infty. \]
Moreover, for each \( \lambda \in \mathbb{C}_+ \), the operator-function \( \chi R_0(\lambda) \tilde{\chi} \in AC(\mathcal{B}_2) \).

ii) For each \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), operator \( \chi R_0^r(\lambda) \tilde{\chi} = \chi R_0^r(\lambda) \tilde{\chi} \in AC(\mathcal{B}_2) \) is the \( \mathcal{B}_2 \)-valued operator-functions satisfying
\[
\| \chi R_0^r(\lambda) \tilde{\chi} \|_{\mathcal{B}_2} \leq \frac{c}{|\epsilon^2|} \| \chi \|_2 \| \tilde{\chi} \|_2 \quad \text{for } \lambda \in Z_c, \quad (3.17) \]
\[
\| \chi R_0^r(\lambda) \tilde{\chi} \|_{\mathcal{B}_2} \rightarrow 0 \quad \text{as } |\text{Im} \lambda| \rightarrow \infty. \]

**Remark.** Note that in the massless case [IK13] in the similar to (3.17) estimate we got a stronger bound with the gain of \( |\text{Im} \lambda|^{-1} \).

**Proof.** i) In order to prove the estimates for \( \text{Im} \lambda \neq 0 \) we use the generalized Fourier transform \( \Phi \). Let \( \sigma = (-\infty, -m] \cup [m, \infty) \). Denote \( \hat{R}_0 \) the Fourier transformed free resolvent
acting in the $s$--space $L^2(\mathbb{R}, d\rho(s))$. Then $\hat{R}_0$ is the operator of multiplication by $\frac{1}{s - \lambda}$ and we have

$$R_0(\lambda)f(x) = \int_{-\infty}^{\infty} \frac{1}{s - \lambda} \left( \int_{0}^{\infty} f^T(t) \varphi(t, s) dt \right) \varphi(x, s) d\rho(s)$$

$$= \int_{\sigma} \frac{1}{s - \lambda} \left( \int_{0}^{\infty} f^T(t) \hat{\varphi}(t, s) dt \right) \hat{\varphi}(x, s) ds = \int_{\sigma} R_0(x, s, \lambda) \int_{0}^{\infty} \mathcal{E}(y, s) f(y) dy ds,$$

where

$$R_0(x, s, \lambda) = \frac{\mathcal{E}(x, s)}{s - \lambda}, \quad \mathcal{E}(x, s) = \begin{pmatrix} \hat{\varphi}_1(x, s) & 0 \\ 0 & \hat{\varphi}_2(x, s) \end{pmatrix}. \tag{3.18}$$

Moreover, we have (3.12)

$$\int_{\sigma} \int_{0}^{\infty} |\hat{\mathcal{F}}(s) \mathcal{E}(y, s) ds|^2 dy = \int_{\sigma} |\hat{\mathcal{F}}(s)|^2 ds,$$

where $\hat{\mathcal{F}}$ is as in (3.10).

Let $\chi \in L^2(\mathbb{R}_+; \mathbb{C}^2)$ and note that it is enough to take $\chi$ diagonal matrix. We get

$$\|\chi R_0(\lambda)\|_{B_2}^2 = \int_{\sigma} \int_{0}^{\infty} \left| \int_{\sigma} \chi(x) R_0(x, s, \lambda) \mathcal{E}(y, s) ds \right|^2 dy dx = \int_{\sigma} \int_{0}^{\infty} \left| \chi(x) R_0(x, s, \lambda) \right|^2 ds dx$$

$$= \int_{\sigma} \int_{\sigma} \chi(x) \mathcal{E}(x, s) \left\| \frac{1}{s - \lambda} \right\|^2 ds dx \leq \frac{1}{\pi} \int_{\sigma} \chi(x)^2 dx \int_{\sigma} \mathcal{K}_1(s) \left( \frac{1}{s - \lambda} \right)^2 ds,$$

where $\mathcal{K}_1(s) = \max \left\{ |k_0(s)|, \left| \frac{1}{k_0(s)} \right| \right\}$ is as in (3.13).

We have

$$\int_{\sigma} \frac{\mathcal{K}_1(s)}{|s - \lambda|^2} ds = \int_{-\infty}^{m} \frac{1}{|s - \lambda|^2} \left| k(s) \right| ds + \int_{m}^{\infty} \frac{1}{|s - \lambda|^2} \left| \frac{s + m}{k(s)} \right| ds = I_1 + I_2. \tag{3.20}$$

We estimate $I_2$ first. By the change of variable

$$\kappa(s) = \xi, \quad s = \lambda(\xi) := \sqrt{\xi^2 + m^2}, \quad ds = \frac{\xi}{\sqrt{\xi^2 + m^2}} d\xi,$$

we get

$$I_2 = \int_{m}^{\infty} \frac{1}{|s - \lambda|^2} \left| \frac{s + m}{k(s)} \right| ds = \int_{0}^{\infty} \frac{1}{\sqrt{\xi^2 + m^2}^2 - \lambda^2} \frac{\sqrt{\xi^2 + m^2} + m}{\sqrt{\xi^2 + m^2}^2} d\xi$$

$$< 2 \int_{0}^{\infty} \frac{1}{|\lambda(\xi) - \lambda|^2} d\xi.$$

For $I_1$ we get $\kappa(s) = \xi, \quad s = -|\sqrt{\xi^2 + m^2}|, \quad ds = -\frac{\xi}{|\sqrt{\xi^2 + m^2}|} d\xi,$

$$I_1 = \int_{-\infty}^{m} \frac{1}{|s - \lambda|^2} \left| k(s) \right| ds = \int_{-\infty}^{0} \frac{1}{|\lambda(\xi) - \lambda|^2} \left( \frac{\xi^2}{|\sqrt{\xi^2 + m^2}|^2 - m} \right) \frac{s}{|\sqrt{\xi^2 + m^2}|} d\xi$$

$$< 2 \int_{-\infty}^{0} \frac{1}{|\lambda(\xi) - \lambda|^2} d\xi.$$
These bounds and the dominated convergence Lebesgue theorem yields (3.16).

and together with (3.19), (3.20) this yields (3.15).

Moreover they satisfy asymptotics (3.7) as $\lambda \to \pm m$.

Now, we will prove (3.16). Recall that

$$R_0(x, y, \lambda) = \frac{1}{2} e^{ik(x|y|)} \left( \frac{k_0(\lambda)}{e^{2ik(\lambda)r_c} + \text{sgn}(x - y)} \right),$$

where $r_c = \min\{x, y\}$.

Let $\text{Im} \ \lambda > 0$ (for $\text{Im} \ \lambda < 0$ the proof is similar). The functions $k_0, 1/k_0$ are continuous and bounded on $\mathbb{Z}_+^*$. Moreover they satisfy asymptotics (3.7) as $\lambda \to \pm m$.

It is enough to prove (3.17) for one entry of the matrix (3.23),

$$E(x, y, \lambda) = \frac{1}{2} e^{ik(x|y|)} \frac{1}{k_0(\lambda)} \left( e^{2ik(\lambda)r_c} + 1 \right),$$

for the other entries the estimates are similar. Let $\chi, \widetilde{\chi} \in L^2(\mathbb{R}_+; \mathbb{C})$. Then for $\lambda \in \mathbb{Z}_+$,

$$\| \chi E \widetilde{\chi} \|_{B_2}^2 \leq \frac{c}{\epsilon} \int_{\mathbb{R}_+} |\chi(x)|^2 \int_{\mathbb{R}_+} e^{-2|x-y|\text{Im} \lambda} |\overline{\chi}(y)|^2 dydx \leq \frac{c}{\epsilon} \| \chi \|^2_{L^2} \| \widetilde{\chi} \|^2_{L^2}.$$

These bounds and the dominated convergence Lebesgue theorem yields (3.16).

Moreover, these arguments show that for each $\lambda \in \mathbb{R}$, $\lambda \neq \pm m$, $\chi R_0(\lambda \pm i0)\widetilde{\chi} \in B_2$.

That $\chi R_0 \widetilde{\chi} \in AC(B_2)$, follows from (3.22).

ii) Estimate (3.17) is easily verified as in the proof of i).

4. Perturbed Dirac systems.

4.1. Properties of the Jost solutions. We consider the Dirac system (1.5) for a vector valued function $f(x)$:

$$\begin{align*}
& f_1' + qf_1 - (m - p_2 + \lambda)f_2 = 0, \\
& f_2' - qf_2 - (m + p_1 - \lambda)f_1 = 0,
\end{align*}$$

The Jost solutions $f^\pm$ of (1.5) are defined using the following condition

$$f^+(x, \lambda) = \psi^+(x, \lambda) + o(1), \quad x \to \infty, \quad f^-(x, \lambda) = \overline{f^+(x, \lambda)}, \quad \lambda \in \sigma_{ac}(H_0).$$
Note that for any two solutions $f(x, \lambda), g(x, \lambda)$ of (1.3) the Wronskian $\det(f, g) = f_1g_2 - f_2g_1$ is independent of $x$. Therefore, the Wronskian of the pair $f^+, f^-$ is given by

$$\det(f^+, f^-) = \det(\psi^+, \psi^-) = \begin{vmatrix} k_0 & -k_0 \\ 1 & 1 \end{vmatrix} = 2k_0. \quad (4.2)$$

For a function $f(\lambda)$ on $\Lambda$ denote $f^*(x, \lambda) = f^+(x, \lambda)$. Note that $(f^+)^* = f^-$. The Jost function $f^+$ is the unique solution of the integral equation

$$f^+(x, \lambda) = \psi^+(x, \lambda) + i \int_x^\infty \mathcal{M}_0(x-t, \lambda)\sigma_2 V(t)f^+(t, \lambda)dt, \quad (4.3)$$

where $\mathcal{M}_0$ is the fundamental matrix of monodromy and was defined in (3.3). Recall that for each $x \geq 0$, $\mathcal{M}_0(x, \cdot)$ is entire.

The function $\chi(x, \lambda) = e^{-ik(\lambda)x}f^+(x, \lambda)$ satisfies

$$\chi(x, \lambda) = \chi^0 + \int_x^\infty G(x, t, \lambda)i\sigma_2 V(t)\chi(t, \lambda)dt, \quad \chi^0 = \left(\begin{array}{c} k_0(\lambda) \\ 1 \end{array}\right),$$

where

$$G(x, t, \lambda) = \frac{1}{2} \left( \begin{array}{cc} (1 + e^{-2ik(\lambda)(x-t)}) & k_0(\lambda) \left(1 - e^{-2ik(\lambda)(x-t)}\right) \\ (1 - e^{-2ik(\lambda)(x-t)}) & (1 + e^{-2ik(\lambda)(x-t)}) \end{array} \right).$$

Thus we have the power series

$$\chi(x, \lambda) = \sum_{n \geq 0} \chi^n(x, \lambda), \quad \chi^{n+1}(x, \lambda) = \int_x^\gamma G(x, t, \lambda)i\sigma_2 V(t)\chi^n(t, \lambda)dt, \quad \chi^0 = \left(\begin{array}{c} k_0(\lambda) \\ 1 \end{array}\right). \quad (4.4)$$

We have the following standard result generalizing [IK13].

**Lemma 4.1.** Let $\eta := \Im k(\lambda)$ and $K_1(\lambda)$ be as in (3.13). Denote

$$K_2(\lambda) := \max \left( |k_0(\lambda)|, 1 \right).$$

i) Suppose $V \in L^1(\mathbb{R}_+, \mathbb{C}^2)$. Then for each $x \in \mathbb{R}_+$, the function $\chi(x, \cdot)$ is analytic in $\mathbb{C}_+$ and continuous up to $\mathbb{C}_+ \setminus \{\pm m\}$. For each $x \in \mathbb{R}_+$, $\Im \lambda \geq 0$, $\lambda \neq \pm m$, the functions $\chi_n$, $\chi$ satisfy the following estimates:

$$|\chi^n(x, \lambda)| \leq K_2(\lambda) \frac{K_1(\lambda)}{n!} \left( \int_x^\infty |V(t)|dt \right)^n, \quad \forall \ n \geq 1, \quad (4.5)$$

$$|\chi(x, \lambda)| \leq K_2(\lambda) e^{K_1(\lambda) \int_x^\infty |V(t)|dt}. \quad (4.6)$$

ii) If $V$ satisfies Condition A, then for each $x \in \mathbb{R}_+$ the function $\chi(x, \cdot)$ is analytic in $\Lambda$. For each $x \in [0, \gamma]$ and $\lambda \in \Lambda = \mathbb{C} \setminus [-m, m]$, the functions $\chi_n$, $\chi$ satisfy the following estimates:

$$|\chi^n(x, \lambda)| \leq e^{(\gamma-x)(|\eta|-\eta)} K_2(\lambda) \frac{K_1(\lambda)}{n!} \left( \int_x^\gamma |V(t)|dt \right)^n, \quad \forall \ n \geq 1, \quad (4.7)$$

$$|\chi(x, \lambda)| \leq K_2(\lambda) e^{(\gamma-x)(|\eta|-\eta)} e^{K_1(\lambda) \int_x^\gamma |V(t)|dt}, \quad (4.8)$$
Proof. We will prove only part 2) as the estimates in part 1) are similar and can be easily obtained by adapting the proof of part 2).

Let $\eta_- = \frac{\langle \eta, \eta \rangle}{2}$. Then using (4.4) we obtain

$$\chi^n(x, \lambda) = \int_{x=0 < t_1 < t_2 < \ldots < t_n < \gamma} \left( \prod_{1 \leq j \leq n} G(t_{j-1}, t_j, \lambda)i\sigma_2 V(t) \right) \chi^0 dt_1 dt_2 \ldots dt_n,$$

which yields

$$|\chi^n(x, \lambda)| \leq \int_{x=0 < t_1 < t_2 < \ldots < t_n < \gamma} \left( \prod_{1 \leq j \leq n} \left( e^{2\eta_- (t_j - t_{j-1})} - 1 \right) K_1(\lambda)|V(t_j)| \right)^n |\chi|^0 dt_1 \ldots dt_n$$

$$\leq K_2(\lambda) \int_{x=0 < t_1 < t_2 < \ldots < t_n < \gamma} \left( \prod_{1 \leq j \leq n} \left( e^{2\eta_- (t_j - t_{j-1})} K_1(\lambda)|V(t_j)| \right)^n dt_1 \ldots dt_n$$

$$= K_2(\lambda) \int_{x < t_1 < t_2 < \ldots < t_n < \gamma} \left( \prod_{1 \leq j \leq n} |V(t_j)| \right) \frac{1}{n!} \left( K_1(\lambda) \int_x^\gamma |V(t)| dt \right)^n dt_1 \ldots dt_n$$

(4.9)

which yields (4.7).

This shows that the series (4.3) converges uniformly on bounded subsets of $\mathbb{C} \setminus \{ \pm m \}$. Each term of this series is analytic function in $\Lambda$. Hence the sum is an analytic function in $\Lambda$ and continuous up to the boundary. Summing the majorants we obtain estimate (4.8).

4.2. Characterization of states. Let $\tilde{\vartheta}, \tilde{\varphi}$ be solutions of (1.5) satisfying

$$(\tilde{\vartheta}, \tilde{\varphi}) = (\vartheta, \varphi) + o(1) \quad \text{as } x \to +\infty.$$

Using that the fundamental matrix of monodromy $\mathcal{M}_0(x, \cdot) = (\vartheta(x, \cdot), \varphi(x, \cdot))$ (see (3.3)) is entire we get

**Lemma 4.2.** Suppose $V \in L^1(\mathbb{R}_+, \mathbb{C}^2)$. Then for each $x \geq 0$, $\tilde{\vartheta}(x, \cdot), \tilde{\varphi}(x, \cdot)$ are entire functions.

Now, using (3.6) we get

$$f^+(x, \lambda) = k_0(\lambda) \tilde{\vartheta} + \tilde{\varphi} = k_0(\lambda) \left( \tilde{\vartheta} + m_0(\lambda) \tilde{\varphi} \right), \quad f^-(x, \lambda) = \tilde{f}^+(x, \lambda), \quad \lambda \in \mathbb{R}.$$ (4.10)

We see that all singularities of $f^+$ coincide with the singularities of $k_0(\lambda)$ and do not depend on $x > 0$. As $k_0 = \frac{\lambda + m}{\lambda - m}$, the only such singularity is at $\lambda = m$.

Now, the integral kernel of the resolvent $R(\lambda) := (H - \lambda)^{-1}$ is given by

$$R(x, y, \lambda) = \begin{cases} \frac{1}{\det(f^+(\cdot, \lambda))} f^+(x, \lambda)(\phi(y, \lambda))^T & \text{if } y < x, \\ \frac{1}{\det(f^+(\cdot, \lambda))} \phi(x, \lambda)(f^+(y, \lambda))^T & \text{if } x < y, \end{cases}$$
where $\phi(x, \lambda)$ is solution of (1.3) satisfying the Dirichlet boundary condition $\phi_1(0, \lambda) = 0$. We have $\det(f^+, \phi) = f_1^+(0, \lambda)$. As $\Phi$ is entire, the essential part of the resolvent is

$$R(x, \lambda) = \frac{1}{k_0(\lambda)\vartheta_1(0, \lambda) + \phi_1(0, \lambda)} \left( k_0(\lambda)\bar{\vartheta}(x, \lambda) + \bar{\phi}(x, \lambda) \right)$$

The singularities of $R(x, \lambda)$ are independent of $x$ and are either zeros of the Jost function $f_1^0(0, \lambda)$ or poles of $k_0$, i.e. $\lambda = -m$. Therefore, we have the following equivalent characterization of $\sigma(H)$ (we omit the complete proof as it mimics similar proofs presented in [KSI2] and later on in [IK11]).

**Lemma 4.3.** 1) A point $\lambda_0 \in g^+$ is an eigenvalue iff $f_1^+(0, \lambda) = 0$. 2) A point $\lambda_0 \in \Lambda_1^-$ is a resonance iff $f_1^+(0, \lambda) = 0$. 3) The multiplicity of an eigenvalue or a resonance is the multiplicity of the corresponding zero. 4) The point $\lambda = -m$ is a virtual state iff $\phi_1(0, -m) = 0$. The point $\lambda = m$ is a virtual state iff $\vartheta_1(0, m) = 0$.

### 4.3. Function $F$

Following the ideas as in [IK11] and [KSI2] we introduce an entire function whose zeros contain the states of $H$. We define

$$F(\lambda) = (\lambda - m)f_1^+(0, \lambda)f_1^-(0, \lambda). \tag{4.11}$$

For a function $g = g(\lambda, x) \in C$, $x \geq 0$, we denote $\dot{g} = \partial_\lambda g$ and $g' = \partial_x g$. We have

$$F = (\lambda - m) \left( k_0(\lambda)\bar{\vartheta} + \bar{\varphi} \right) \left( k_0^*(\lambda)\vartheta + \varphi \right) = (\lambda + m) \left( \vartheta_1 + m_0(\lambda)\varphi_1 \right) \left( \bar{\vartheta}_1 + m_0^*(\lambda)\bar{\varphi}_1 \right),$$

where $g^*(\lambda) := \overline{g(\lambda)}$. Using that for $\lambda \in \sigma_{ac}(H_0)$ we have $k^*(\lambda) = k(\lambda)$ and

$$k_0(\lambda) = \frac{\lambda + m}{ik(\lambda)}, \quad k_0^*(\lambda) = -k_0(\lambda)$$

$$m_0(\lambda)m_0^*(\lambda) = \frac{\lambda - m}{\lambda + m}, \quad m_0(\lambda) + m_0^*(\lambda) = 0,$$

we get

$$F(x, \lambda) = (\lambda + m)\bar{\vartheta}_1^2(x, \lambda) + (\lambda - m)\bar{\varphi}_1^2(x, \lambda) \tag{4.12}$$

and in unperturbed case $H = H_0$ we have $F = F^0 = (\lambda - m)k_0k_0^* = \lambda + m$.

**Proposition 4.4.** Assume that potential $V$ satisfies Condition A. Then function $F$ has the following properties:

i) $F(\cdot)$ is entire.

ii) $F(\cdot)$ is real on $\mathbb{R}$. The set of zeros of $F$ is symmetric with respect to the real line. Moreover, $F(\lambda) > 0$ for $\lambda \in ]-\infty, -m[ \cup ]m, +\infty[$, and $F$ can have only even number of zeros in $[-m, m]$.

iii) If $\lambda_1$ is an eigenvalue of $H$ then

$$\dot{F}(\lambda_1) = -2|k(\lambda_1)| \frac{\|f_1^+(\cdot, \lambda_1)\|_2^2}{\|f_1^+(0, \lambda_1)\|_2^2} < 0. \tag{4.13}$$
Theorem 1.2.}

Proof. Properies i), ii) follows from definition of $F$ and formula (1.12).

The proof of iii) is based on the following result which can be checked by direct calculation: If $f = (f_1, f_2)^T = f(x, \lambda)$ is solution of the Dirac equation $Hf = \lambda f$ (1.5), then

$$\left(\det(\dot{f}, f)\right)^T = f_1^2 + f_2^2 \quad \text{for any } x \in \mathbb{R}_+ \text{ and } \lambda \in \mathbb{C} \setminus \{\pm m\}. \quad (4.14)$$

Let $\lambda_1 \in \sigma_{ba}(H)$. Then $\dot{F}(\lambda_1) = (\lambda_1 - m)\dot{f}_1^+(\lambda_1)f_1^-(\lambda_1)$. Using the wronskian identity (4.2) we get $-f_2^+(\lambda_1)f_1^-(\lambda_1) = 2k_0(\lambda_1)$ and therefore

$$F(\lambda_1) = (\lambda_1 - m)\dot{f}_1^+(\lambda_1)\frac{-2k_0(\lambda_1)}{(f_2^+(\lambda_1))}f_2^+(\lambda_1), \quad k_0(\lambda_1) = -\frac{\lambda_1 + m}{\sqrt{m^2 - \lambda_1^2}} \quad (4.15)$$

Now, (4.14) is equivalent to

$$\begin{vmatrix} \dot{f}_1(x, \lambda) & f_1(x, \lambda) \\ \dot{f}_2(x, \lambda) & f_2(x, \lambda) \end{vmatrix} = C(\lambda) + \int_0^x (f_1^2(t, \lambda) + f_2^2(t, \lambda))dt$$

with an arbitrary function $C = C(\lambda)$.

Now, put $f = f^+(x, \lambda), \lambda \in g^+$ (the upper rim of the gap $(-m, m)$ in $\mathbb{C} \setminus [-m, m]$). Then $\det(\dot{f}^+(x, \lambda), f^+(x, \lambda)) \to 0$ as $x \to \infty$ and we get

$$C(\lambda) = -\int_0^{\infty} (f_1^2(t, \lambda) + f_2^2(t, \lambda))dt$$

and

$$\begin{vmatrix} \dot{f}_1(x, \lambda) & f_1(x, \lambda) \\ \dot{f}_2(x, \lambda) & f_2(x, \lambda) \end{vmatrix} = -\int_x^{\infty} (f_1^2(t, \lambda) + f_2^2(t, \lambda))dt.$$ 

Putting $\lambda = \lambda_1 \in \sigma_{ba}(H), x = 0$, we get

$$\dot{f}_1^+(\lambda_1)f_2^+(\lambda_1) = -\int_0^{\infty} (f_1^+(t, \lambda_1))^2 + (f_2^+(t, \lambda_1))^2)dt = -\|f^+(\cdot, \lambda_1)\|_{L^2}^2,$$

and (4.15) implies (4.13). \hfill \blacksquare

Proof of Theorem 1.2. 2) follows from the wronskian identity (4.2) which implies that if $f_1^+(\lambda_1) = 0, \lambda \neq -m$, then $f_1^-(\lambda_1) \neq 0$.

3) follows from identity (4.13), Proposition 4.4. \hfill \blacksquare

5. Uniform estimates on the Jost solutions

In order to get uniform estimates on the Jost function as $|\lambda| \to \infty$ we need to transform the Dirac system (1.5) to the more convenient form. Put

$$\mathcal{M} = \left(\begin{array}{cc} 0 & M \\ M & 0 \end{array}\right), \quad M(t, \lambda) = e^{-it\int_0^\infty (q + i\rho)ds} \left[ (q + ip) + \frac{i}{2} (p_2a(\lambda) - p_1b(\lambda)) \right], \quad (5.1)$$

$$\mathcal{N} = \left(\begin{array}{cc} N & 0 \\ 0 & -N \end{array}\right), \quad N(t, \lambda) = \frac{i}{2} (p_2(t)a(\lambda) + p_1(t)b(\lambda)), \quad (5.2)$$

where

$$a(\lambda) = \begin{cases} \frac{k(\lambda)}{\lambda + m} - 1, & \lambda \in \sigma_{ac}(H_0), \\
(\lambda + m) k(\lambda) - 1, & \end{cases} \quad p(x) = \frac{p_1(x) - p_2(x)}{2}, \quad \lambda \in \sigma_{ac}(H_0).$$

We denote by the same letters the natural analytic continuation of $\mathcal{M}(t, \cdot), \mathcal{N}(t, \cdot)$ into the $\lambda$–plane $\Lambda = \mathbb{C} \setminus [-m, m]$ or the corresponding $k$–plane. Recall that, by abuse of notation, we
Lemma 5.1. Suppose $V$ satisfy Condition A.

Let $f^+(x, \lambda)$ be the Jost solution and let the vector-function $Y = Y(x, \lambda)$ be defined via
\[
    f^+(x, \lambda) = e^{\Omega_0} \begin{pmatrix} k_0(\lambda) & k_0(\lambda) \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\int_0^x v(t)dt} & 0 \\ 0 & e^{i\int_0^x v(t)dt} \end{pmatrix} Y, \quad \lambda \in \sigma_{ac}(H_0).
\]

Then $Y = Y(x)$ satisfies the differential equation
\[
    Y' = (ik\sigma_3 - (\mathcal{N} + \mathcal{M}))Y, \quad Y(x, \lambda)|_{x \geq \gamma} = e^{ikx\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
which is equivalent to the integral equation
\[
    Y(x) = e^{ikx\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^\gamma e^{ik\sigma_3(x-t)}(\mathcal{N}(t) + \mathcal{M}(t))Y(t)dt, \quad \lambda \in \sigma_{ac}(H_0). \quad (5.3)
\]

Lemma 5.2. Suppose $V$ satisfy Condition A and in addition $V' \in L^1(\mathbb{R}_+; \mathbb{C})$. We denote
\[
    \mathcal{W}(t) = 2ik \mathcal{N}(t) + \mathcal{M}'(t) - \mathcal{M} \mathcal{N} - |M|^2, \quad \mathcal{A}(x) = I - \frac{1}{2ik}\sigma_3 \mathcal{M}(x).
\]

Denote $w(x) = q + ip$. The function $\mathcal{W}(t)$ has the following asymptotics as $|k| \to \infty$:
\[
    \mathcal{W} = \left( \begin{array}{cc} -2pm - |w|^2 & e^{2\int_0^t v(s)ds} [i2vw + w'] \\ e^{-2\int_0^t v(s)ds} [-i2vw' + w] & 2pm - |w|^2 \end{array} \right) + \mathcal{O}(k^{-1}).
\]

Then for $|k| \geq \sup_{x \in \mathbb{R}_+} |M(x)|$ the matrix $\mathcal{A}(x) = I - \frac{1}{2ik}\sigma_3 \mathcal{M}(x)$ has bounded inverse $\mathcal{B}(x)$ and $Y = Y(x)$ satisfies
\[
    Y = Y^0 + (2ik)^{-1} \mathcal{B}KY, \quad Y^0(x) = e^{ikx} \mathcal{B}(x, k) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad KY = \int_x^\gamma e^{i\sigma_3(kx-t)}\mathcal{W}(t)Y(t)dt,
\]
and is given by the expansion in powers of $(2ik)^{-1}$
\[
    Y = Y^0 + \sum_{n \geq 1} Y^n, \quad Y^n = \frac{1}{(2ik)^n} (\mathcal{B}K)^n Y^0,
\]
where
\[
    |Y^n(x, k)| \leq \frac{2}{n!|k|^n} e^{\frac{\Im k}{2}(2\gamma - x)} \left( \int_x^\gamma |\mathcal{W}(s)|ds \right)^n.
\]

Proof of Lemma 5.1. Firstly, similar to [HKS88], [HKS89], by a chain of transformations of the Dirac equation (we omit the details here), we introduce a new vector-function $X$ related to the Jost solution $f^+$ via
\[
    f^+ = ik \begin{pmatrix} k_0 & k_0 \\ 1 & -1 \end{pmatrix} e^{i\sigma_3(\int_0^x v(t)dt)} X.
\]

Then $X$ satisfies the differential equation
\[
    X' = -\mathcal{W}X, \quad X|_{x \geq \gamma} = \frac{1}{ik(\lambda)} e^{i\int_0^\gamma v(s)ds} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathcal{W}(t) = e^{-ik\sigma_3 (\mathcal{N} + \mathcal{M})} e^{ik\sigma_3}.
\]

Now, the function
\[
    Y(x) = ike^{ikx\sigma_3} e^{-i\int_0^x v(s)ds} X(x)
\]
satisfies (5.3) and (5.4). 

Note that problem (5.6) is equivalent to the integral equation

\[ X(x) = X^0 + \int_x^\gamma \tilde{W}(t)X(t)dt. \]  

(5.7)

Equation (5.7) implies the following relations between components of vector-function \( X \).

\[
\begin{align*}
X_1(x) &= \frac{1}{ik(\lambda)} e^{i\int_0^x v(t)dt} + \int_x^\gamma (NX_1(t) + e^{-2iktM}X_2) dt, \\
X_2(x) &= \int_x^\gamma \left( e^{2ikt}X_1(t) + \overline{N}X_2(t) \right) dt.
\end{align*}
\]

Now \( f_1^+(0, \lambda) = (\lambda + m) (X_1(0, \lambda) + X_2(0, \lambda)) \), \( f_2^+(0, \lambda) = ik(\lambda) (X_1(0, \lambda) - X_2(0, \lambda)) \), and we get

\[
f_1^+(0) = (\lambda + m) \left\{ \frac{1}{ik(\lambda)} e^{i\int_0^x v(t)dt} + \int_0^\gamma \left[ (N + e^{2iktM})X_1(t) + (\overline{N} + e^{-2iktM})X_2(t) \right] dt \right\}.
\]

(5.8)

Formula (5.8) will be applied in the Froese Lemma 5.3.

**Proof of Lemma 5.2.** Consider equation

\[
Y(x) = e^{ikx\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^\gamma e^{ikx\sigma_3(x-t)} N(t)Y(t)dt + \int_x^\gamma e^{ikx\sigma_3(x-t)} M(t)Y(t)dt.
\]

(5.9)

Note that \( \mathcal{N} = \mathcal{O}(\lambda^{-1}) \),

\[
N(t, \lambda) = \frac{ip(t)m}{\lambda} + \mathcal{O}(1)/\lambda^2, \quad M(t, \lambda) = e^{-i2\int_0^t v(s)ds} \left[ q + ip(t) - \frac{i\lambda(t)m}{\lambda} + \mathcal{O}(1)/\lambda^2 \right],
\]

(5.10)

and

\[
e^{-ikx\sigma_3} \mathcal{N} = \mathcal{N} e^{-ikx\sigma_3}, \quad e^{ikx\sigma_3} \mathcal{M} = \mathcal{M} e^{-ikx\sigma_3}.
\]

(5.11)

In the last term in (5.9) we use the second commutation relation in (5.11) and integrate by parts:

\[
I = \int_x^\gamma e^{ikx\sigma_3(x-t)} \mathcal{M}(t)Y(t)dt = \int_x^\gamma e^{ikx\sigma_3(x-2t)} \mathcal{M}(t) \left( e^{-ikx\sigma_3(Y(t))} \right) dt = \left[ -\frac{1}{2ik} \sigma_3 e^{ikx\sigma_3(x-2t)} \mathcal{M}(t) \left( e^{-ikx\sigma_3(Y(t))} \right) \right]_{t=x}^{\gamma} + \frac{1}{2ik} \int_x^\gamma e^{ikx\sigma_3(x-2t)} \{ \mathcal{M}'(t)e^{-ikx\sigma_3} - \mathcal{M} e^{-ikx\sigma_3} (\mathcal{N} + \mathcal{M}) \} Y(t)dt,
\]

where we used that

\[
\tilde{X}' = -\tilde{W} \tilde{X}, \quad \tilde{X} = e^{-ikx\sigma_3} Y, \quad \tilde{W} = e^{-ikx\sigma_3} (\mathcal{N} + \mathcal{M}) e^{ikx\sigma_3}.
\]

By using (5.11) and \( \mathcal{M}^2 = |M|^2 J_2 \) we get

\[
I = \frac{1}{2ik} \sigma_3 \mathcal{M}(x) Y(x) + \frac{1}{2ik} \int_x^\gamma e^{ikx\sigma_3(X(t))} \left( \mathcal{M}'(t) - \mathcal{M} \mathcal{N} - |M|^2 \right) Y(t)dt.
\]
Inserting it in (5.9) we get
\[ Y(x) = e^{ikx \sigma_3} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \frac{1}{2ik} \sigma_3 \mathcal{M}(x) Y(x) + \]
\[ + \frac{1}{2ik} \int_x^\gamma e^{isk(x-t)} \left( 2ik \mathcal{N}(t) + \mathcal{M}'(t) - \mathcal{M} \mathcal{N} - |M|^2 \right) Y(t) dt. \]

We denote
\[ \mathcal{W}(t) = 2ik \mathcal{N}(t) + \mathcal{M}'(t) - \mathcal{M} \mathcal{N} - |M|^2, \quad \mathcal{A}(t) = I - \frac{1}{2ik} \sigma_3 \mathcal{M}(x). \]

Then \( Y \) satisfies
\[ \mathcal{A}(x,k) Y(x) = e^{ikx \sigma_3} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \frac{1}{2ik} \int_x^\gamma e^{isk(x-t)} \mathcal{W}(t) Y(t) dt. \]

Using that
\[ \sup_{t \in \mathbb{R}} |\mathcal{A}^{-1}| \leq 2 \quad \text{for} \quad |k| \geq \sup |\sigma_3 \mathcal{M}|, \quad (5.12) \]

we get the integral equation
\[ Y(x) = Y^0 + \frac{1}{2ik} (a(x)^{-1} KY, \quad Y^0(x) = \mathcal{A}^{-1}(x,k) e^{ikx \sigma_3} \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]
\[ KY = \int_x^\gamma e^{isk(x-t)} \mathcal{W}(t) Y(t) dt. \]

By iterating we get
\[ Y = Y^0 + \sum_{n \geq 1} Y^n, \quad Y^n = \frac{1}{(2ik)^n} (\mathcal{A}^{-1} K)^n Y^0. \]

Let \( t = (t_j)_1^n \in \mathbb{R}^n \) and \( \mathcal{D}_t(n) = \{ x = t_0 < t_1 < t_2 < ... < t_n < \gamma \}. \)

\[ Y^n = \frac{1}{(2ik)^n} \int_{\mathcal{D}_t(n)} \prod_{j=1}^n (\mathcal{A}(t_{j-1}))^{-1} e^{iks(t_j-t_{j-1})} \mathcal{W}(t_j)(\mathcal{A}(t_n))^{-1} e^{ikt_{n+1} \sigma_3} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) dt. \]

Now, using (5.12) we get
\[ |Y^n(x,k)| \leq \frac{2}{|k|^n} e^{\text{Im} k(2\gamma-x)} \prod_{j=1}^n |\mathcal{W}(t_j)| dt = \frac{2}{n! |k|^n} e^{\text{Im} k(2\gamma-x)} \left( \int_x^\gamma |\mathcal{W}(s)| ds \right)^n. \]

**Proof of Theorem 1.5.** (i) We will calculate the first two terms in the expansion in orders of \( k^{-1} \) of
\[ f_1^+(0, \lambda) = k_0 e^{i \int_0^\gamma \nu(t) dt} (Y_1(0) + Y_2(0)), \quad (5.13) \]

where
\[ Y_1(0) + Y_2(0) = Y_1^0(0) + Y_2^0(0) + Y_1^1(0) + Y_2^1(0) + \mathcal{O}(k^{-2}). \]

We will need the following asymptotics:
\[ N(t, \lambda) = ip(t)m\lambda^{-1} + \mathcal{O}(\lambda^{-2}), \quad M(t, \lambda) = e^{-i2 \int_0^\gamma \nu(s) ds} \left[ w(t) - iv(t)m\lambda^{-1} + \mathcal{O}(\lambda^{-2}) \right], \]
\[ |M|^2 = |w|^2 + O(k^{-1}), \quad \mathcal{W} = \mathcal{W}_0 + O(k^{-1}), \] where
\[
\mathcal{W}_0(t) = \begin{pmatrix}
-2pm - |w|^2 & e^{i2\int_0^t v(s) ds} [i2v\overline{w} + \overline{w}'] \\
negative term & 2pm - |w|^2
\end{pmatrix}.
\]
The inverse of \( \mathcal{A} \) has the following formula
\[
\mathcal{A}^{-1} = \mathcal{B} = -\frac{2ki}{4k^2 - |M|^2} \begin{pmatrix}
2ki & \overline{M} \\
-M & 2ki
\end{pmatrix} = \frac{1}{1 - |M|^2/(4k^2)} \begin{pmatrix}
1 & \frac{M}{2ki} \\
-M/2ki & 1
\end{pmatrix},
\]
and using the above asymptotics for \( M \) we get
\[
\mathcal{B}(t) = \begin{pmatrix}
-e^{ikx} & 1 \\
0 & \frac{e^{i2\int_0^t v(s) ds} \overline{w(t)}}{2ki}
\end{pmatrix} + O(k^{-2})
\]
We have
\[
Y^0(x) = \mathcal{B} \begin{pmatrix}
e^{ikx} \\
0
\end{pmatrix} = -e^{ikx} \frac{2ki}{4k^2 - |M|^2} \begin{pmatrix}
2ki \\
-M
\end{pmatrix} = e^{ikx} \frac{1}{2} + O(k^{-1}) = e^{ikx} \left(-e^{i2\int_0^t v(s) ds} \frac{w(t)}{2ki}\right) + O(k^{-2}),
\]
\[
Y^0_1(0) + Y^0_2(0) = 1 - \frac{w(0)}{2ki} + O(k^{-2}). \tag{5.14}
\]
Now, \( Y^1 = (2ik)^{-1} \mathcal{B} \mathcal{K} Y^0, \mathcal{B} = I + O(k^{-1}), \)
\[
2ikY^1(0) = \int_0^\gamma e^{-ikt\sigma_3} \mathcal{W}_0(t)e^{ikt} \begin{pmatrix}
1 & 0 \\
0 & e^{-2ikt}
\end{pmatrix} \begin{pmatrix}
-2pm - |w|^2 \\
e^{-i2\int_0^t v(s) ds} [-i2vw + w']
\end{pmatrix} dt + O(k^{-2})
\]
\[
= \int_0^\gamma \begin{pmatrix}
1 & 0 \\
0 & e^{-2ikt}
\end{pmatrix} \begin{pmatrix}
2pm + |w|^2 \\
e^{i2\int_0^t v(s) ds} [i2vw - w']
\end{pmatrix} dt.
\]
Now,
\[
Y^1_1(0) + Y^1_2(0) = \frac{1}{2ik} \int_0^\gamma 2pm + |w|^2 + e^{2i(kt-\int_0^t v(s) ds)} [i2vw - w'] dt + O(k^{-2}). \tag{5.15}
\]
Combining (5.13), (5.14) and (5.15) we get
\[
f^+_1(0, \lambda) = k_0 e^{i\int_0^\gamma v(t) dt} \left\{ 1 - \frac{1}{2ik} \left( (w(0)) + \int_0^\gamma 2pm + |w|^2 + e^{2i(kt-\int_0^t v(s) ds)} [i2vw - w'] dt \right) + O(k^{-2}) \right\},
\]
which implies expansion (1.11).

Now, note that by Riemann-Lebesgue lemma
\[
\int_0^\gamma e^{2ikt} e^{-i2\int_0^t v(s) ds} [i2vw - w'] dt \to 0 \text{ as } k \in \mathbb{R}, |k| \to \infty.
\]
Thus we get for real \( k, |k| \to \infty \)
\[
f^+_1(0, \lambda) = k_0 e^{i\int_0^\gamma v(t) dt} \left( 1 - \frac{1}{2ik} \mathcal{B}_0 + o(k^{-1}) \right), \quad \mathcal{B}_0 = w(0) + \int_0^\gamma (2pm + |w|^2) dt, \tag{5.16}
\]
which is less precise version of \((1.11)\).

Note that in [HKSS9] the sign in front of 2\(pm\) was minus, which according to 
(5.16) is not correct.

We write \((5.16)\) in a short form \(f_1(0, k) = a_0 + a_1\lambda^{-1} + o(\lambda^{-1})\) (with \(a_i, i = 0, 1,\) identified from \((5.16)\)). Now, by Lemma 4.1 it can be seen that \(f_1(0, k)\) is bounded on \(\mathbb{C}_+\). Thus the function \(\lambda(f_1(0, k) - a_0) - a_1\) is exponentially bounded in \(\mathbb{C}_+\) and it is bounded for \(|\lambda| \gg m\), \(\lambda \in \mathbb{R}\). By the Phragmen-Lindelöf principle \(\lambda(f_1(0, k) - a_0) - a_1\) is bounded in \(\mathbb{C}_+\). However, this function is actually \(o(1)\) as \(\lambda \to \infty\) through real values. Another version of the Phragmen-Lindelöf principle guarantees that \(\lambda(f_1(0, k) - a_0) - a_1\to 0\) along any curve approaching \(\infty\) in the upper half \(\lambda\) plane, and this completes the proof of \((1.11)\) in Theorem 1.5

ii) Expansion for the scattering phase \((1.13)\) follows immediately from \((1.11)\) by applying \(\phi_{sc} = \arg f_1(0, \lambda) + \pi/2\).

iii) We will need the following Lemma by Froese (see [97], Lemma 4.1). Even though the original lemma was stated for \(V \in L^\infty\) the argument also works for \(V \in L^2\) and we omit the proof.

Lemma 5.3 (Froese).
Suppose \(V \in L^2(\mathbb{R})\) has compact support contained in \([0, 1]\), but in no smaller interval. Suppose \(g(x, \lambda)\) is analytic for \(\lambda\) in the lower half plane, and for real \(\lambda\) we have \(g(x, \lambda) \in L^2([0, 1]) dx, \mathbb{R}d\lambda\). Then \(\int e^{ik\lambda}V(1 + g(x, \lambda)) dx\) has exponential type at least 1 for \(\lambda\) in the lower half plane.

Put \(\tilde{X} = ik(\lambda)e^{-i\int_0^t v(t)dt}X(t) = e^{-ikx_{sc}}Y\). From equations \((5.3), (5.7)\) it follows the representation \((5.8)\):

\[
f_1(0, k) = \frac{(\lambda + m)}{ik(\lambda)}e^{ik\int_0^t v(t)dt} \left( 1 + \int_0^\gamma \left[ (N + e^{2iktM}) \tilde{X}_1(t) + (\overline{N} + e^{-2ikt\overline{M}}) \tilde{X}_2(t) \right] dt \right).
\]

Now, from the properties of \(Y\) as in the proof of Lemma \(5.2\) it follows that \(\tilde{X}_1 = 1 + g(t, k), g(t, k) = O(\lambda^{-1}),\) and \(\tilde{X}_2 = O(\lambda^{-1})\). We write the integral in the right hand side above in the more convenient form

\[
\int_0^\gamma \left[ (N + e^{2iktM}) \tilde{X}_1(t) + (\overline{N} + e^{-2ikt\overline{M}}) \tilde{X}_2(t) \right] dt =
\]

\[
= \int_0^\gamma e^{2iktM_0(t)}(1 + g(t, k))dt + \int_0^\gamma e^{2ikt(M(t, k) - M_0(t))}\tilde{X}_1(t, k)dt +
\]

\[
+ \int_0^\gamma \left[ N\tilde{X}_1(t) + (\overline{N} + e^{-2ikt\overline{M}}) \tilde{X}_2(t, k) \right] dt
\]

where \(M_0(t) = e^{-i\int_0^t v(s)ds}w(t)\). Note that \(M(t, k) - M_0(t) = O(\lambda^{-1})\) by \((5.10)\). Let \(K(\lambda) = \int_0^\gamma e^{2iktM_0(t)}(1 + g(t, k))dt\). Now, it is enough to apply a version \(5.3\) of Lemma of Froese to \(K(\lambda + i), \lambda \in \mathbb{C}_-\), where we shift the argument of function \(K\) in order to avoid the singularities at \(\lambda = \pm m\), and using that \(\sup_{t\in[0,\gamma]} |g(t, k(\tau - i))| = O(\tau^{-1})\) as \(\tau \to \pm \infty\). Thus the function \(f_1(0, k)\) has exponential type \(2\gamma\) in the half plane \(\mathbb{C}_-\).

The second statement of the Theorem \(1.5\) is proved. \(\blacksquare\)
6. Function $F$ is in Cartwright class

Recall that $F(\lambda) = (\lambda - m)f^+_1(0, \lambda) = f_1(0, \lambda)$, $f^-_2(0, \lambda) = f_1(0, \lambda)$, $\lambda \in \sigma_{ac}(H_0)$, are given in Theorem 1.5. Formula (6.1) and imply

\[
F(\lambda) = (\lambda - m)f^+_1(0, \lambda)f^-_2(0, \lambda) = (\lambda + m) \left(1 - \frac{p(0)}{\lambda} + \frac{1}{4\lambda^2} |B|^2 + O(\lambda^{-4})\right),
\]

as $\lambda \to \infty$.

In Proposition 4.4 we showed that $F$ is entire in $\mathbb{C}$. Now, Theorem 1.5 ii), implies that $F$ is of exponential type $2\gamma$.

We recall that a function $f$ is said to belong to the Cartwright class $\text{Cart}_\omega$ if $f$ is entire, of exponential type, and satisfies

\[
\rho_\omega(f) \equiv \limsup_{y \to \infty} \frac{\log |f(\pm iy)|}{y} = \omega > 0, \quad \int_{\mathbb{R}} \frac{\log(1 + |f(x)|)}{1 + x^2} dx < \infty.
\]

We summarize the obtained results following from Theorem 1.5 in the following Corollary:

**Corollary 6.1.** Let $V$ satisfy Condition A and $V' \in L^1(\mathbb{R}_+)$. Then function $F \in \text{Cart}_2\gamma$ and for $\lambda \in \mathbb{R}$ it satisfies

\[
F(\lambda) = \lambda + (m - p(0)) + \frac{1}{4\lambda} (|B(\lambda)|^2 - 4mp(0)) + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \pm \infty,
\]

where $B$ was defined in (1.12).

We determine the asymptotics of the counting function. We denote $N(r, f)$ the total number of zeros of $f$ of modulus $\leq r$ (each zero being counted according to its multiplicity).

We also denote $N_+(r, f)$ (or $N_-(r, f)$) the number of zeros of function $f$ counted in $N(r, f)$ with non-negative (negative) imaginary part having modulus $\leq r$, each zero being counted according to its multiplicity.

**Proposition 6.2.** Assume that potential $V$ satisfies Condition A and $V' \in L^1(\mathbb{R}_+)$. Then $F(\cdot)$ is entire. The set of zeros of $F$ is symmetric with respect to the real line. The set of zeros of $F$ with negative imaginary part (i.e. resonances) satisfy:

\[
N(r, F) = 2N_-(r, f^+_1(0)) = \frac{4r\gamma}{\pi}(1 + o(1)) \quad \text{as} \quad r \to \infty.
\]

For each $\delta > 0$ the number of zeros of $F$ with negative imaginary part with modulus $\leq r$ lying outside both of the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$ is $o(r)$ for large $r$.

7. Modified Fredholm determinant

Let $R_0(\lambda) = (H_0 - \lambda)^{-1}$, $R(\lambda) = (H - \lambda)^{-1}$ denote the resolvent for the operator $H_0$, $H$ respectively. We factorize the potential $V = V_1V_2$, where the choice of $V_2$ we leave open for the moment. Later we will show that we can choose $V_2 = V$.

Let $Y_0(\lambda) = V_2R_0(\lambda)V_1$, $Y(\lambda) = V_2R(\lambda)V_1$. Then we have

\[
Y(\lambda) = Y_0(\lambda) - Y_0(\lambda)[I + Y_0(\lambda)]^{-1}Y_0(\lambda), \quad Y = I - (1 + Y_0)^{-1}
\]

and

\[
(I + Y_0(\lambda))(I - Y(\lambda)) = I.
\]
As \( Y_0(\lambda) := V_2 R_0(\lambda) V_1 \in B_2 \) is Hilbert-Schmidt but is not trace class, we define the modified Fredholm determinant

\[
D(\lambda) = \det \left[ (I + Y_0(\lambda)) e^{-Y_0(\lambda)} \right], \quad \lambda \in \mathbb{C}_+.
\]

**Corollary 7.1.** Let \( V \in L^2(\mathbb{R}_+) \) and let \( \text{Im} \, \lambda \neq 0 \). Then

i) \[
\| V R_0(\lambda) \|_{B_2}^2 \leq \left[ 4\pi \frac{1}{|\text{Im} \, \lambda|} \left| \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \right| + O \left( \max \left\{ \frac{1}{|\lambda|^2}, \frac{1}{|\lambda \pm m|^2} \right\} \right) \right] \| V \|_2^2, \quad (7.3)
\]

ii) The operator \( R(\lambda) - R_0(\lambda) \) is of trace class and satisfies

\[
\| R(\lambda) - R_0(\lambda) \|_{B_1} \leq \frac{C_1}{|\text{Im} \, \lambda|} \left| \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \right| + C_2 \left( \max \left\{ \frac{1}{|\lambda|^2}, \frac{1}{|\lambda \pm m|^2} \right\} \right), \quad (7.4)
\]

for some constants \( C_{1,2} \).

iii) Let, in addition, \( V = V_1 V_2 \in L^2(\mathbb{R}_+) \) with \( V_1, V_2 \in L^2(\mathbb{R}_+) \). Then for each \( \epsilon > 0 \), we have \( Y_0, Y, Y_0', Y' \in AC(B_2) \), and the following estimates are satisfied:

\[
\| Y_0(\lambda) \|_{B_2} \leq \frac{C}{\epsilon} \| V_1 \|_2 \| V_2 \|_2, \quad \forall \lambda \in \mathbb{Z}_+; \quad \| Y_0(\lambda) \|_{B_2} \to 0 \quad \text{as} \quad |\text{Im} \, \lambda| \to \infty. \quad (7.5)
\]

\[
\| Y_0'(\lambda) \|_{B_2} \to 0 \quad \text{as} \quad |\text{Im} \, \lambda| \to \infty. \quad (7.6)
\]

**Proof.** i) Identity \((7.3)\) follows from \((3.15)\). ii) Denote \( J_0(\lambda) = I + Y_0(\lambda) \). For \( \text{Im} \, \lambda \neq 0 \), operator \( J_0(\lambda) \) has bounded inverse and the operator

\[
R(\lambda) - R_0(\lambda) = -R_0(\lambda) V_1 [J_0(\lambda)]^{-1} V_2 R_0(\lambda), \quad \text{Im} \, \lambda \neq 0,
\]

is trace class and the estimate follows from \((7.3)\).

iii) That \( Y_0, Y \in AC(B_2) \) follows as in the proof of Lemma 3.1 resolvent identity \((7.1)\) and ii), bound \((7.4)\), we get

\[
Y'(\lambda) = (I - Y(\lambda)) Y_0'(\lambda)(I - Y(\lambda)) \in AC(B_2).
\]

Formula \((7.5)\) follows from Lemma 3.1.

**Lemma 7.2.** Let \( V \in L^2(\mathbb{R}) \). Then the following facts hold true.

i) For each \( \epsilon > 0 \), the function \( D \) belongs to \( AC(\mathbb{C}) \) and satisfies:

\[
D'(\lambda) = -D(\lambda) \text{Tr} [Y(\lambda) Y_0'(\lambda)] \quad \forall \lambda \in \mathbb{C}_+; \quad (7.7)
\]

\[
|D(\lambda)| \leq e^{\frac{1}{\epsilon} \| Y_0 \|_{B_2}^2}, \quad \forall \lambda \in \mathbb{C}_+; \quad (7.8)
\]

\[
D(\lambda) \to 1 \quad \text{as} \quad |\text{Im} \, \lambda| \to \infty. \quad (7.9)
\]

ii) For each \( \epsilon > 0 \), the functions \( \log D(\lambda) \) and \( \frac{d}{d\lambda} \log D(\lambda) \) belong to \( AC(\mathbb{C}) \), and the following identities hold true:

\[
- \log D(\lambda) = \sum_{n \geq 2} \frac{\text{Tr} (-Y_0(\lambda))^n}{n}, \quad (7.10)
\]

where the series converges absolutely and uniformly for \( \lambda \) in the domain

\[
\mathcal{L} = \{ \lambda \in \mathbb{C}; \ \text{Im} \, \lambda > \epsilon \| V \|_{B_2} \}
\]

for some constant \( \epsilon > 0 \) large enough, and

\[
\left| \log D(\lambda) + \sum_{n \geq 2} \frac{\text{Tr} (-Y_0(\lambda))^n}{n} \right| \leq \frac{\varepsilon_\lambda^{N+1}}{(N+1)(1 - \varepsilon_\lambda)}, \quad \lambda \in \mathcal{L}, \quad \varepsilon_\lambda = C_2(\lambda) \| V \|_{B_2}, \quad (7.11)
\]
for any $N \geq 1$. Here $C_\lambda$ is given in (7.14). Moreover, $\frac{d^k}{d\lambda^k} \log D(\lambda) \in AC(\mathbb{C})$ for any $k \in \mathbb{N}$ and the function $D$ is independent of factorization of $V = V_1V_2$ in $Y_0 = V_2R_0V_1$, so we can choose $Y_0 = VR_0$.

**Proof.** Formula (7.7) is well-known (see for example Krein) and together with the above results it implies that the functions $\log D(\lambda)$ and $\frac{d}{d\lambda} \log D(\lambda)$ belong to $AC(\mathbb{C})$. Estimate (7.8) follows from the inequality (Gohberg-Krein [GK69], page 212, (2.2) in russian edition)

$$|D(\lambda)| \leq e^{\frac{1}{2}\text{Tr}[Y_0^{\ast}(\lambda)Y_0(\lambda)]}.$$  \hfill (7.12)

Property (7.9) will follow from estimate (7.11) in the part ii) of the Lemma. We will prove it now.

We suppose first that $Y_0 = VR_0$ which corresponds to the choice $V_1 = I$ in the factorization $V = V_1V_2$. Denote by $F(\lambda)$ the series in (7.10). We show that this series converges absolutely and uniformly.

Indeed, using that for any Hilbert-Schmidt operator $A$ we have

$$|\text{Tr} A^n| \leq \|A^{n-1}A\|_{\mathcal{B}_1} \leq \|A^{n-1}\|_{\mathcal{B}_1}\|A\| \leq \|A\|\|A\|^{-1} \leq \|A\|^{\frac{n}{2}}.$$  \hfill (7.13)

From (7.3) it follows

$$|\text{Tr} (VR_0(\lambda))^n| \leq \|VR_0(\lambda)\|_{\mathcal{B}_2} \leq \varepsilon_\lambda^n,$$  \hfill (7.14)

where

$$\varepsilon_\lambda = C_\chi^{\frac{1}{2}}(\lambda)\|V\|_{\mathcal{B}_2}, \quad C_\lambda = \left[4\pi \frac{1}{\text{Im} \lambda} \left|\text{Re} \frac{\lambda}{\sqrt{\lambda^2 - m^2}}\right| + O\left(\max\left\{\frac{1}{|\lambda|^2}, \frac{1}{|\lambda \pm m|^2}\right\}\right)\right].$$  \hfill (7.14)

We have

$$\varepsilon_\lambda < \frac{1}{2} \iff C_\lambda\|V\|_{\mathcal{B}_2}^2 < \frac{1}{4} \iff \text{Im} \lambda > c\|V\|_{\mathcal{B}_2}^2,$$

where constant $c$ can be chosen by fixing any $\epsilon > 0$ and defining

$$c = \sup_{\text{Im} \lambda > 0, |\lambda \pm m| > \epsilon > 0} \left[16\pi \left|\text{Re} \frac{\lambda}{\sqrt{\lambda^2 - m^2}}\right| + O\left(\max\left\{\frac{\text{Im} \lambda}{|\lambda|^2}, \frac{\text{Im} \lambda}{|\lambda \pm m|^2}\right\}\right)\right].$$

Then $F(\lambda)$ is an analytic function in the domain $\Omega$. Moreover, by differentiating $F$ and using (7.2) we get

$$F'(\lambda) = -\lim_{m \to \infty} \sum_{n \geq 2} \text{Tr}(-Y_0'(\lambda))^{n-1}Y_0'(\lambda) = \text{Tr} Y(\lambda)Y_0'(\lambda), \quad \lambda \in \mathcal{L},$$

and then the function $F = \log D(\lambda)$, since $F(i\tau) = o(1)$ as $\tau \to \infty$. Using (7.10) and (7.13) we obtain (7.11).

Now, we have that for $\lambda \in \mathcal{L}$, $\log D(\lambda)$ and thus $D(\lambda)$ is independent of the choice of factorization $V = V_1V_2$. Using the fact that $D(\lambda) \in AC(\mathbb{C})$ we get that $D(\lambda)$ is independent of factorization $V = V_1V_2$.

**Proposition 7.3.** Suppose $V \in L^1(\mathbb{R}_+), \lambda \in \sigma_{ac}(H_0), \lambda \neq \pm m$. Then the function

$$\Omega(\lambda) = \frac{1}{2i} \text{Tr} V (R_0(\lambda + i0) - R_0(\lambda - i0))$$
satisfies, for \( \lambda \in (-\infty, -m) \cup (m, +\infty) \),
\[
\Omega(\lambda) = \int_0^\gamma p_1(y) \frac{\lambda + m}{k(\lambda)} \sin^2 ky dy + \int_0^\gamma p_2(y) \frac{k(\lambda)}{\lambda + m} \cos^2 ky dy + \int_0^\gamma q(y) \sin 2ky dy,
\]
and for \( \lambda \in (-m, m) \), \( \Omega(\lambda) = 0 \).

**Remark.** As \( \lambda \to \pm \infty \) we have
\[
\Omega(\lambda) = \int_0^\gamma v(y) dy - \int_0^\gamma p(y) \cos 2\lambda y dy + \int_0^\gamma q(y) \sin 2\lambda y dy + O(\lambda^{-1}),
\]
where \( v = \frac{p_1 + p_2}{2} \), \( p = \frac{p_1 - p_2}{2} \). Now, if in addition \( p', q' \in L^1(\mathbb{R}_+) \), then by integration by parts \( \Omega(\lambda) = \int_0^\gamma v(y) dy + O(\lambda^{-1}) \) as \( \lambda \to \pm \infty \).

**Proof.** The integral kernel of the free resolvent \( R_0 \) is given in (3.4) and (3.5):
\[
R_0(x, y, \lambda) = e^{ik(\lambda)x} \left( \begin{array}{c} ik_0(\lambda) \sin k(\lambda)y \\ i \sin k(\lambda)y \frac{\cos k(\lambda)y}{k_0(\lambda)} \cos k(\lambda)y \end{array} \right) \quad \text{if } y < x,
\]
and
\[
R_0(x, y, \lambda) = e^{ik(\lambda)y} \left( \begin{array}{c} ik_0(\lambda) \sin k(\lambda)x \\ i \sin k(\lambda)x \frac{\cos k(\lambda)x}{k_0(\lambda)} \cos k(\lambda)x \end{array} \right) \quad \text{if } x < y.
\]
Let \( y < x \). Denote \( \varphi_0(y, \lambda) = ik_0(\lambda) \sin k(\lambda)y = \frac{\lambda + m}{k(\lambda)} \sin k(\lambda)y \). In order to obtain \( R_0(\lambda + i0) - R_0(\lambda - i0) \) we calculate
\[
e^{ikx} \frac{\lambda + m}{k} \sin ky - e^{-ikx} \frac{\lambda + m}{k} \sin ky = 2i \frac{\lambda + m}{k} \sin ky \sin kx,
\]
\[
e^{ikx} \cos ky - e^{-ikx} \cos ky = 2i \sin kx \cos ky,
\]
\[
i \left( e^{ikx} \sin ky + e^{-ikx} \sin ky \right) = 2i \cos kx \sin ky,
\]
\[
e^{ikx} \frac{ik}{\lambda + m} \cos ky + e^{-ikx} \frac{ik}{\lambda + m} \cos ky = \frac{2ik}{\lambda + m} \cos kx \cos ky
\]
and
\[
V(R_0(\lambda + i0) - R_0(\lambda - i0)) = 2iV \left( \begin{array}{cc} \frac{\lambda + m}{k} \sin ky \sin kx & \sin kx \cos ky \\ \cos kx \sin ky & \frac{k}{\lambda + m} \cos kx \cos ky \end{array} \right) \theta(x - y).
\]
Now, we represent \( V = vI + V_0 \) as follows
\[
V = \left( \begin{array}{c} p_1 \\ q \\ p_2 \end{array} \right) = \left( \begin{array}{c} v \\ 0 \\ 0 \end{array} \right) + \left( \begin{array}{c} p \\ q \\ -p \end{array} \right), \quad v = \frac{p_1 + p_2}{2}, \quad p = \frac{p_1 - p_2}{2}
\]
and get
\[
\text{Tr}_{x\to y} v(R_0(\lambda + i0) - R_0(\lambda - i0)) = 2i \int_0^x v(y) \left( \frac{\lambda + m}{k} \sin^2 ky + \frac{k}{\lambda + m} \cos^2 ky \right) dy,
\]
\[
\text{Tr}_{x\to y} V_0(R_0(\lambda + i0) - R_0(\lambda - i0)) = 2i \int_0^x p(y) \left( \frac{\lambda + m}{k} \sin^2 ky - \frac{k}{\lambda + m} \cos^2 ky \right) dy + 2i \int_0^x q(y) \sin 2ky dy
\]
Let $y > x$. Then
\[
V(\lambda + i0) - R_0(\lambda - i0) = 2iV\left(\frac{\lambda m \sin ky \sin kx}{\cos kx \sin ky} \sin kx \cos ky \quad \frac{k}{\lambda + m} \cos kx \cos ky\right) \theta(y - x).
\]
and we get
\[
\frac{1}{2i} \text{Tr} V(\lambda + i0) - R_0(\lambda - i0) = \\
\int_0^\gamma v(y) \left(\frac{\lambda + m}{k} \sin^2 ky + \frac{k}{\lambda + m} \cos^2 ky\right) dy + \\
\int_0^\gamma p(y) \left(\frac{\lambda + m}{k} \sin^2 ky - \frac{k}{\lambda + m} \cos^2 ky\right) dy + \int_0^\gamma q(y) \sin 2ky dy = \\
= \int_0^\gamma p_1(y) \frac{\lambda + m}{k} \sin^2 ky dy + \int_0^\gamma p_2(y) \frac{k}{\lambda + m} \cos^2 ky dy + \int_0^\gamma q(y) \sin 2ky dy.
\]

Recall the definition of the scattering matrix
\[
S(\lambda) = \frac{\mathcal{F}_1((0, \lambda + i0))}{\mathcal{F}_1((0, \lambda - i0))} = e^{-2i\varphi_{ac}}, \quad \text{for } \lambda \in \sigma_{ac}(H_0).
\]

**Proof of Theorem 1.3.** Let $V \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. 

i) We will prove that $D \in \text{AC}(\mathbb{C})$, $S(\lambda) = \frac{D(\lambda - i0)}{D(\lambda + i0)} e^{-2i\Omega(\lambda)}$, $\forall \lambda \in \sigma_{ac}(H_0)$, $\lambda \neq \pm m$.

We adapt arguments from [IsK11, IK13]. Let $\lambda \in \mathbb{C}_+$. Denote $\mathcal{J}_0(\lambda) = I + Y_0(\lambda)$, $\mathcal{J}(\lambda) = I - Y(\lambda)$. Then $\mathcal{J}_0(\lambda)\mathcal{J}(\lambda) = I$ due to (7.2). Now, put $S_0(\lambda) = \mathcal{J}_0(\lambda)\mathcal{J}(\lambda)$. Then we have
\[
S_0(\lambda) = I - (Y_0(\lambda) - Y_0(\lambda)) (I - Y(\lambda)).
\]
Now, by the Hilbert identity,
\[
Y_0(\lambda) - Y_0(\lambda) = (\lambda - \lambda)V_2 R_0(\lambda) R_0(\lambda) V_1
\]
is trace class and
\[
\det S_0(\lambda) = S(\lambda), \quad \lambda \in \sigma_{ac}(H_0).
\]

Let $z = i\tau$, $\tau \in \mathbb{R}_+$ and $D(\lambda) = \det (J_0(\lambda) J(z))$, $\lambda \in \mathbb{C}_+$. 

It is well defined as $J_0(\lambda) J(z) - I \in \text{AC}(\mathcal{B}_1)$. The function $D(\lambda)$ is entire in $\mathbb{C}_+$ and $D(z) = I$.

We put
\[
f(\lambda) = \frac{D(\lambda)}{D(z)} e^{\text{Tr}(Y_0(\lambda) - Y_0(z))}, \quad \lambda \in \mathbb{C}_+,
\]
where
\[
D(\lambda) = \det [(I + Y_0(\lambda)) e^{-Y_0(\lambda)}].
\]
We have $D(\lambda) = f(\lambda)$, $\lambda \in \mathbb{C}_+$. Now, using that $J_0(\lambda) J(\lambda) = I$, we get
\[
\det S_0(\lambda) = \det J_0(\lambda) J(z) \cdot \det (J(z)^{-1} J(\lambda)) = \frac{D(\lambda)}{D(z)} e^{\text{Tr}(Y_0(\lambda) - Y_0(\lambda))}.
\]
As by Proposition 7.3 we have $\text{Tr}(Y_0(\lambda + i0) - Y_0(\lambda - i0)) = 2i\Omega(\lambda)$ for $\lambda \in \sigma_{ac}(H_0)$, then we get
\[
S(\lambda) = \lim_{\epsilon \downarrow 0} \frac{D(\lambda - i\epsilon)}{D(\lambda + i\epsilon)} e^{-2i\Omega(\lambda)} , \quad \lambda \in \sigma_{ac}(H_0).
\]
Now, using that
\[ f_1(0, k) = \frac{\lambda + m}{ik(\lambda)} e^{i f_0^k v(t) dt} + O(\lambda^{-1}) \]
and that for \( \lambda \in \sigma_{ac}(H_0) \), \( k(\lambda - i0) = -k(\lambda + i0) \in \mathbb{R} \), we get
\[ S(\lambda) = -\frac{f_1(0, \lambda + i0)}{f_1(0, \lambda + i0)} = e^{-2k f_0^k v(t) dt} + O(\lambda^{-1}). \]

ii) We have
\[ \frac{f_1(0, \lambda + i0)}{f_1(0, \lambda + i0)} = \frac{D(\lambda - i0)}{D(\lambda + i0)} e^{-2i\Omega(\lambda)}, \quad \lambda \in \sigma_{ac}(H_0). \]

Here,
\[ \Omega(\lambda) = \frac{1}{2i} \text{Tr} V(\sigma(t) - R_0(\lambda + i0) - R_0(\lambda - i0)) \in \mathbb{R}, \]
\[ \Omega(\lambda) = \int_0^\gamma v(t) dt + O(\lambda^{-1}), \quad \lambda \to \pm \infty, \quad \lambda \in \sigma_{ac}(H_0). \]

Let \( \lambda \in \sigma_{ac}(H_0) \setminus \{ \pm m \} \) and write
\[ \frac{f_1(0, \lambda + i0)}{f_1(0, \lambda + i0)} = \frac{D(\lambda + i0)e^{i\Omega(\lambda)}}{D(\lambda + i0)e^{i\Omega(\lambda)}} \iff \frac{f_1(0, \lambda + i0)}{D(\lambda + i0)e^{i\Omega(\lambda)}} = -\frac{f_1(0, \lambda + i0)}{D(\lambda + i0)e^{i\Omega(\lambda)}}. \]

Therefore,
\[ e^{i2\arg f_1(0, \lambda) + i\pi} = e^{i2\arg D(\lambda) e^{i2\Omega(\lambda)}}, \quad \lambda \in \sigma_{ac}(H_0) \setminus \{ \pm m \}. \]

Moreover, using that
\[ f_1(0, k) = k_0(\lambda) e^{i f_0^k v(t) dt} + O(\lambda^{-1}) \]
and that for \( \lambda \in \sigma_{ac}(H_0) \), \( k(\lambda - i0) = -k(\lambda + i0) \in \mathbb{R} \), we get
\[ S(\lambda) e^{2i\Omega_0} = \frac{g(\lambda + i0)}{g(\lambda + i0)} = \frac{D(\lambda + i0)e^{i(\Omega(\lambda) - \Omega_0)}}{D(\lambda + i0)e^{i(\Omega(\lambda) - \Omega_0)}}, \quad \lambda \in \sigma_{ac}(H_0), \]
where
\[ \Omega_0 = \int_0^\gamma v(t) dt; \quad g(z) = \frac{f_1(0, z)}{k_0(z)e^{i\Omega_0}}. \]

Therefore,
\[ e^{i2\arg g(\lambda)} = e^{i2\arg D(\lambda) e^{i2(\Omega(\lambda) - \Omega_0)}}, \quad \lambda \in \sigma_{ac}(H_0) \setminus \{ \pm m \}. \]

We know the following facts:
1) \( g(\cdot), D(\cdot) \in AC(\mathbb{C}) \), i.e. they are analytic functions on \( \mathbb{C}_+ \), continuous in \( \mathbb{C}_+ \setminus \{ \pm m \} \).
2) \( g(z) \to 1, \quad D(z) \to 1, \quad \text{Im} z \to \infty, \quad \Omega_0 = \int_0^\gamma v(x) dx. \)

Then the functions \( \log g(z), \log D(z) \) are uniquely defined on \( C_+ \) and \( (-\infty, -m), (m, +\infty) \); and continued from above to the gap \( (-m, m) \). Thus \( \log g(z), \log D(z) \in AC(\mathbb{C}) \) and we have
\[ 2 \arg g(\lambda) = 2 \arg D(\lambda) + 2(\Omega(\lambda) - \Omega_0), \quad \lambda \in \mathbb{R} \setminus \{ \pm m \}, \quad \text{and} \quad \Omega(\lambda) = 0 \text{ for } \lambda \in (-m, m). \]

By Cauchy formula, for \( z \in \mathbb{C}_+ \setminus \{ \pm m \} \),
\[ \log g(z) = \frac{1}{\pi} \int \frac{\arg g(t)}{t - z} dt = \frac{1}{\pi} \int \frac{\arg D(t) + \Omega(t) - \Omega_0}{t - z} dt = \log D(z) + \frac{1}{\pi} \int_\mathbb{R} \frac{(\Omega(t) - \Omega_0)}{t - z} dt, \]
where the the first two integrals are understood in the principal value sense and the last integral is well defined due to \( \Omega(\cdot) - \Omega_0 \in L^2(\mathbb{R}) \). Thus we get (1.9).
8. Appendix. Relativistic integral.

In order to prove Theorem 3.3.1 we need the following result

**Lemma 8.1.** Let $\text{Im } \lambda \neq 0$. Then

\[
I := \int_R \frac{dk}{|\lambda(k) - \lambda|^2} = \frac{2\pi}{|\text{Im } \lambda|} \left| \text{Re } \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \right| + \mathcal{O} \left( \max \left\{ \frac{1}{|\lambda|^2}, \frac{1}{|\lambda \pm m|^2} \right\} \right).
\]  

(8.1)

**Remark.** Here $\lambda(k) = \sqrt{k^2 + m^2}$, $k \in \mathbb{R}$, is the relativistic hamiltonian, and $\lambda(k)$ has analytic continuation to the Riemann surface $\mathbb{K} := \mathbb{C} \setminus [i m, -i m]$.

**Proof.** Suppose $\text{Im } \lambda > 0$ (for $\text{Im } \lambda < 0$ similar). Let $\gamma$ be the contour following the loop clockwise alone the left and right rims of the cut $[0, im]$. As $\sqrt{R^2 + m^2}$ is defined on $\mathbb{K} := \mathbb{C} \setminus [i m, -i m]$ and has branching points at $k = \pm im$ we get

\[
\int_{\gamma} \frac{dk}{|\lambda(k) - \lambda|^2} = \text{Im } \int_0^1 \left( \frac{1}{|1 - m \sqrt{1 - t^2} - \lambda|^2} - \frac{1}{|m \sqrt{1 - (1 - t)^2} - \lambda|^2} \right) dt
\]

(8.2)

\[
= \mathcal{O} \left( \max \left\{ \frac{1}{|\lambda|^2}, \frac{1}{|\lambda \pm m|^2} \right\} \right).
\]

Let $R > 0$ be large enough. Let $\Gamma_R$ be a closed contour followed anti-clockwise consisting of $[-R, 0] \cup \gamma \cup [0, R]$ and of a half-circle contour $C_R$ in $\mathbb{C}_+$ from the point $R$ to $-R$.

Note that for $k \in (-\infty, 0) \cup \gamma \cup (0, +\infty)$ it follows $\lambda(k) \in \mathbb{R}$, and then

\[
\frac{1}{|\lambda(k) - \lambda|^2} = \frac{1}{2i \text{Im } \lambda} \left( \frac{1}{\lambda(k) - \lambda} - \frac{1}{\lambda(k) + \lambda} \right).
\]  

(8.3)

The function $g(k) = \left( \frac{1}{\lambda(k) - \lambda} - \frac{1}{\lambda(k) + \lambda} \right)$ has analytic continuation to $\mathbb{K} := \mathbb{C} \setminus [i m, -i m]$, and we have

\[
\lim_{R \to \infty} \int_{C_R} g(k) dk = 0.
\]  

(8.4)

Now, by deforming the contour and by using (8.3), (8.2) and (8.4), we get

\[
I = \int_R \frac{dk}{|\lambda(k) - \lambda|^2} = \frac{1}{2i \text{Im } \lambda} \int_R \left( \frac{1}{\lambda(k) - \lambda} - \frac{1}{\lambda(k) + \lambda} \right) dk
\]

\[
= \lim_{R \to \infty} \frac{1}{2i \text{Im } \lambda} \int_{\Gamma_R} g(k) dk - \int_{\gamma} \frac{dk}{|\lambda(k) - \lambda|^2}
\]

\[
= \lim_{R \to \infty} \frac{1}{2i \text{Im } \lambda} \int_{\Gamma_R} g(k) dk + \mathcal{O} \left( \max \left\{ \frac{1}{|\lambda|^2}, \frac{1}{|\lambda \pm m|^2} \right\} \right).
\]

The integrand $g(k)$ is analytic inside $\Gamma_R$ and by applying the residue Theorem we get

\[
\int_{\Gamma_R} g(k) dk = 2\pi i \sum \text{Res}(g(k)),
\]

where the sum is over all poles of $g(k)$ inside $\Gamma_R$. For $\text{Im } \lambda > 0$ the function

\[
\frac{g(k)}{2i \text{Im } \lambda} = \frac{1}{2i \text{Im } \lambda} \left( \frac{1}{\lambda(k) - \lambda} - \frac{1}{\lambda(k) + \lambda} \right)
\]

\[
g(k) = \frac{1}{\lambda(k) - \lambda} - \frac{1}{\lambda(k) + \lambda}
\]
has two simple poles in $\mathbb{C}_+ : k_1 = \sqrt{\lambda^2 - m^2}$, $k_2 = -\sqrt{\lambda^2 - m^2}$, with residues

\[
\frac{\lambda}{\sqrt{\lambda^2 - m^2}}, \quad \frac{\lambda}{\sqrt{(\lambda)^2 - m^2}}
\]

respectively. Therefore, by the residue Theorem (for $R$ large enough) we get

\[
\int_{\Gamma_R} \frac{dk}{|\lambda(k) - \lambda|^2} = \frac{1}{2i \text{Im} \lambda} \int_{\Gamma_R} g(k)dk = \frac{2\pi}{\text{Im} \lambda} \text{Re} \frac{\lambda}{\sqrt{\lambda^2 - m^2}}.
\]

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