Solutions of Discretized Affine Toda Field Equations for $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_n^{(2)}$ and $D_{n+1}^{(2)}$

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Abstract

It is known that a family of transfer matrix functional equations, the T-system, can be compactly written in terms of the Cartan matrix of a simple Lie algebra. We formally replace this Cartan matrix of a simple Lie algebra with that of an affine Lie algebra, and then we obtain a system of functional equations different from the T-system. It may be viewed as an $X_n^{(a)}$ type affine Toda field equation on discrete space time. We present, for $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_n^{(2)}$ and $D_{n+1}^{(2)}$, its solutions in terms of determinants or Pfaffians.

KEYWORDS: affine Lie algebra, discretized affine Toda field equation, T-system, determinant, pfaffian

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1 Introduction

Kuniba et al. proposed \[1\], a class of functional equations, the T-system. It is a family of functional equations satisfied by a set of commuting transfer matrices \({\{T^a_m(u)\}}\) of solvable lattice models associated with any quantum affine algebras \(U_q(X_r^{(1)})\). Here, the transfer matrix \(T^a_m(u)\) is defined as a trace of monodromy matrix on auxiliary space labeled by \(a\) and \(m\) \((m \in \mathbb{Z}, u \in \mathbb{C}, a \in \{1, 2, \ldots , r\})\).

It was pointed out \[3\] that the T-system is not only a family of transfer matrix functional equations but also an example of Toda field equation \[4, 5\] on discrete space time. Solving the T-system recursively, we obtained solutions \({\{T^a_m(u)\}}\) \[6, 3, 7\] with the following properties:

1. They are polynomials of the initial values \(T^{(1)}_1(u + \text{shift}), \ldots , T^{(r)}_1(u + \text{shift})\).

2. They have determinant or pfaffian expressions.

Using the Cartan matrix of a simple Lie algebra \(X_r\), the T-system is written as a compact form \[2\]. In this sense, the T-system is considered to be a class of functional equations associated with Dynkin diagrams. Then the following natural question arise.

If we formally replace this Cartan matrix of a simple Lie algebra with that of a Kac-Moody Lie algebra, then we obtain a system of functional equations different from the T-system. Can one construct solutions of these equations, which have similar properties as those of the T-system?

The purpose of this paper is to answer this question. So far, we have succeeded to construct solutions for affine Lie algebras \(A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, A^{(2)}_n\) and \(D^{(2)}_{n+1}\), modifying the expressions of solutions \[3, 6, 7\] of the T-system.

The functional equations which we deal with in this paper can be compactly written as follows:

\[
T^a_m(u + \frac{1}{t_a})T^a_m(u - \frac{1}{t_a}) = T^a_{m+1}(u)T^a_{m-1}(u) + \prod_{b=0}^{r} \mathcal{T}(a,b, m, u)^{I_{ab}} \tag{1.1}
\]

Here,

\[
\mathcal{T}(a, b, m, u) = T_{b/m/t_a}(u) \quad \text{for} \quad \frac{t_b}{t_a} = 1, 2, 3, \tag{1.2}
\]
\[ T^{(b)}_m(u) = T^{(b)}_m(u + \frac{1}{2}) T^{(b)}_{m+1}(u) T^{(b)}_{m-1}(u) \text{ for } \frac{t_b}{t_a} = \frac{1}{2}, \quad (1.3) \]
\[ T^{(b)}_m(u + \frac{2}{3}) T^{(b)}_{m+1}(u) T^{(b)}_{m-1}(u) \text{ for } \frac{t_b}{t_a} = \frac{1}{3}, \quad (1.4) \]
\[ \times T^{(b)}_{m-1}(u - \frac{1}{3}) T^{(b)}_{m-1}(u + \frac{1}{3}) T^{(b)}_{m+1}(u - \frac{1}{3}) T^{(b)}_{m+1}(u + \frac{1}{3}) \]
\[ \text{where } a \in \{0, 1, 2, \ldots, r\}; \quad T^{(a)}_0(u) = 1; \quad T^{(a)}_m(u) = 1 \quad (\text{if } m \notin Z); \quad I_{ab} = 2\delta_{ab} - B_{ab}(I_{ab} = 1, \text{if } a \text{ th node of Dynkin diagram is connected with } b \text{ th node of it); } I_{ab} = 0, \text{if } a \text{ th node of Dynkin diagram is disconnected with } b \text{ th node of it); } B_{ab} = C_{ab} t_{ab}; \quad C_{ab} = \frac{2(\alpha_a | \alpha_b)}{\alpha_a | \alpha_a} \quad (C_{ab}: \text{Cartan matrix of affine Lie algebra; } \alpha_a: \text{simple root}); \quad t_a = \frac{2}{\alpha_a | \alpha_a}; \quad t_{ab} = \max(t_a, t_b). \]

This functional equation resembles the T-system except the 0 th component \( T^{(0)}_m(u) \). However, unlike the T-system, the solutions of this functional equation (1.1) have nothing to do with the solvable lattice models to the author’s knowledge. Moreover, as is mentioned in section 2, our functional equation may be viewed as an example of discretized affine Toda field equation. In this sense, the functional equation (1.1) should be viewed as the generalization of a discretized Toda equation rather than that of the T-system. It has beautiful structure as an example of discretized soliton equation [3, 4, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Discrete equations have richer contents than their continuum counterparts. In fact, our new solutions here are not so-called soliton solutions which have continuum counterparts but the ones whose matrix sizes depend on the parameter \( m \) and thus peculiar to difference equations. Solving the functional equation (1.1) recursively, we can express \( T^{(a)}_m(u) \) as a polynomial of \( T^{(1)}_1(u + \text{shift}), \ldots, T^{(r)}_1(u + \text{shift}) \). Combining the similar expressions of the solutions [3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] to the T-system, we present, for \( A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, A^{(2)}_n \) and \( D^{(2)}_{n+1} \), solutions of eq. (1.1) in terms of determinants or pfaffians.

For simple Lie algebras \( B_r \) and \( D_r \), there are at least two types of expressions of the solutions to the T-system, which are equivalent. The one is expressed \textbf{explicitly} by polynomial of the fundamental polynomials \( T^{(1)}_1, \ldots, T^{(r)}_1 \) and is given as a determinant or a pfaffian whose non-zero matrix elements distribute sparsely [3]. The other is expressed by polynomial of auxiliary transfer matrix \( T^a(u) \) (or ‘dress function’ in the analytic Bethe ansatz), and
thus implicitly by polynomial of the fundamental polynomials $T_1^{(1)}, \ldots, T_r^{(r)}$ and is given as a determinant or a pfaffian whose non-zero matrix elements distribute densely \[3, 7\]. On constructing the solution of the functional equation (1.1), we use only the dense type expression of the solution to the T-system. The boundary condition to the function $T^a(u)$ for eq.(1.1) is quite different from that for the T-system. As a result, in spite of the resemblance on appearance between the expression of solution to the functional equation (1.1) and that to the T-system, the solution of the functional equation (1.1) is quite different from that of the T-system. As is often the case in soliton theory, most of the proofs of the determinant or pfaffian formulae reduce to the Jacobi identity.

The outline of this paper is given as follows. In section2, we present explicitly our functional equations for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_n^{(2)}$ and $D_n^{(2)}$, whose solutions are given in section4. In section3, we fix the notation. Section5 is dedicated for summary and discussion. AppendixA is a list of functional equations for $G_2^{(1)}$, $F_4^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_6^{(2)}$ and $D_4^{(3)}$, which have been not yet solved.

Finally, it should be noted that, for the twisted quantum affine algebras $U_q(X_n^{(k)})$ ($X_n^{(k)} = A_n^{(2)}, D_n^{(2)}, E_6^{(2)}, D_4^{(3)}$), extensions of the T-system different from our functional equations were proposed by Kuniba and Suzuki [23]. However, they do not reduce to affine Toda field equations in the continuum limit.

### 2 Functional Equations

For the function $T_m^{(a)}(u)$ ($m \in \mathbb{Z}, u \in \mathbb{C}, a \in \{0, 1, \ldots, r\}$), we assume the following boundary condition:

$$T_m^{(a)}(u) = 0 \quad \text{for} \quad m < 0, \quad T_0^{(a)}(u) = 1.$$

Substituting each Cartan matrix of affine Lie algebra into the functional equation (1.1), we obtain explicitly the following functional equations:

For $A_r^{(1)} (r \geq 1)$,

$$T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u) \quad 0 \leq a \leq r$$

(2.1)
with $T_m^{(-1)}(u) = T_m^{(r)}(u)$ and $T_m^{(r+1)}(u) = T_m^{(0)}(u)$.

For $B^{(1)}_r (r \geq 3)$,
\begin{align}
T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(2)}_m(u) \quad a = 0, 1, \quad (2.2) \\
T^{(2)}_m(u-1)T^{(2)}_m(u+1) &= T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u) + T^{(0)}_m(u)T^{(1)}_m(u)T^{(3)}_m(u), \quad (2.3) \\
T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u), \quad 3 \leq a \leq r - 2, \quad (2.4) \\
T^{(r-1)}_m(u-1)T^{(r-1)}_m(u+1) &= T^{(r-1)}_{m+1}(u)T^{(r-1)}_{m-1}(u) + T^{(r-2)}_m(u)T^{(r)}_m(u), \quad (2.5) \\
T^{(r)}_{2m}(u-1/2)T^{(r)}_{2m}(u+1/2) &= T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u) + T^{(r-1)}_m(u-1/2)T^{(r-1)}_m(u+1/2), \quad (2.6) \\
T^{(r)}_{2m+1}(u-1/2)T^{(r)}_{2m+1}(u+1/2) &= T^{(r)}_{2m+2}(u)T^{(r)}_{2m}(u) + T^{(r-1)}_mT^{(r)}_{m+1}(u). \quad (2.7)
\end{align}

If $r = 3$, eqs. (2.3) - (2.5) reduce to the following equation:
\begin{equation}
T^{(2)}_m(u-1)T^{(2)}_m(u+1) = T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u) + T^{(0)}_m(u)T^{(1)}_m(u)T^{(3)}_m(u). \quad (2.8)
\end{equation}

For $C^{(1)}_r (r \geq 2)$,
\begin{align}
T^{(0)}_m(u-1)T^{(0)}_m(u+1) &= T^{(0)}_{m+1}(u)T^{(0)}_{m-1}(u) + T^{(1)}_m(u), \quad (2.9) \\
T^{(1)}_{2m}(u-1/2)T^{(1)}_{2m}(u+1/2) &= T^{(1)}_{2m+1}(u)T^{(1)}_{2m-1}(u) + T^{(0)}_m(u-1/2)T^{(0)}_m(u+1/2)T^{(2)}_m(u), \quad (2.10) \\
T^{(1)}_{2m+1}(u-1/2)T^{(1)}_{2m+1}(u+1/2) &= T^{(1)}_{2m+2}(u)T^{(1)}_{2m}(u) + T^{(0)}_m(u)T^{(0)}_{m+1}(u)T^{(2)}_{m+1}(u), \quad (2.11) \\
T^{(a)}_m(u-1/2)T^{(a)}_m(u+1/2) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad 2 \leq a \leq r - 2, \quad (2.12) \\
T^{(r-1)}_{2m}(u-1/2)T^{(r-1)}_{2m}(u+1/2) &= T^{(r-1)}_{2m+1}(u)T^{(r-1)}_{2m-1}(u) + T^{(r-2)}_m(u-1/2)T^{(r)}_m(u+1/2), \quad (2.13) \\
T^{(r-1)}_{2m+1}(u-1/2)T^{(r-1)}_{2m+1}(u+1/2) &= T^{(r-1)}_{2m+2}(u)T^{(r-1)}_m(u) + T^{(r-2)}_m(u)T^{(r)}_{m+1}(u), \quad (2.14) \\
T^{(r)}_m(u-1)T^{(r)}_m(u+1) &= T^{(r)}_{m+1}(u)T^{(r)}_{m-1}(u) + T^{(r-1)}_m(u). \quad (2.15)
\end{align}
If \( r = 2 \), eqs. (2.10) - (2.14) reduce to the following equations:

\[
T_{2m}^{(1)}(u - 1/2)T_{2m}^{(1)}(u + 1/2) = T_{2m+1}^{(1)}(u)T_{2m-1}^{(1)}(u) + T_{m}^{(0)}(u - 1/2)T_{m}^{(0)}(u + 1/2)T_{m}^{(2)}(u - 1/2)T_{m}^{(2)}(u + 1/2),
\]
\[
T_{2m+1}^{(1)}(u - 1/2)T_{2m+1}^{(1)}(u + 1/2) = T_{2m+2}^{(1)}(u)T_{2m}^{(1)}(u) + T_{m}^{(0)}(u)T_{m+1}^{(0)}(u)T_{m}^{(2)}(u)T_{m+1}^{(2)}(u).
\]

For \( D_r^{(1)}(r \geq 4) \),

\[
T_{m}^{(a)}(u - 1)T_{m}^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_{m}^{(2)}(u), \quad a = 0, 1,
\]
\[
T_{m}^{(2)}(u - 1)T_{m}^{(2)}(u + 1) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_{m}^{(0)}(u)T_{m}^{(1)}(u)T_{m}^{(3)}(u),
\]
\[
T_{m}^{(a)}(u - 1)T_{m}^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u)
\]
\[
3 \leq a \leq r - 3,
\]
\[
T_{m}^{(r-2)}(u - 1)T_{m}^{(r-2)}(u + 1) = T_{m+1}^{(r-2)}(u)T_{m-1}^{(r-2)}(u) + T_{m}^{(r-3)}(u)T_{m}^{(r-1)}(u)T_{m}^{(r)}(u),
\]
\[
T_{m}^{(a)}(u - 1)T_{m}^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_{m}^{(r-2)}(u), \quad a = r - 1, r.
\]

If \( r = 4 \), eqs. (2.19) - (2.21) reduce to the following equation:

\[
T_{m}^{(2)}(u - 1)T_{m}^{(2)}(u + 1) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_{m}^{(0)}(u)T_{m}^{(1)}(u)T_{m}^{(r-1)}(u)T_{m}^{(r)}(u).
\]

For \( A_{2r}^{(2)}(r \geq 2) \),

\[
T_{m}^{(0)}(u - 1)T_{m}^{(0)}(u + 1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_{2m}^{(1)}(u),
\]
\[
T_{2m}^{(1)}(u - 1/2)T_{2m}^{(1)}(u + 1/2) = T_{2m+1}^{(1)}(u)T_{2m-1}^{(1)}(u) + T_{m}^{(0)}(u - 1/2)T_{m}^{(0)}(u + 1/2)T_{m}^{(2)}(u),
\]
\[
T_{2m+1}^{(1)}(u - 1/2)T_{2m+1}^{(1)}(u + 1/2) = T_{2m+2}^{(1)}(u)T_{2m}^{(1)}(u) + T_{m}^{(0)}(u)T_{m+1}^{(0)}(u)T_{m}^{(2)}(u),
\]
\[
T_{m}^{(a)}(u - 1/2)T_{m}^{(a)}(u + 1/2) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u)
\]
\[
2 \leq a \leq r - 2,
\]
\[
T_{m}^{(r-1)}(u - 1/2)T_{m}^{(r-1)}(u + 1/2) = T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u).
\]
\[ T_{2m}^{(r)}(u - 1/4)T_{2m}^{(r)}(u + 1/4) = T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u) \]
\[ + T_m^{(r-1)}(u - 1/4)T_m^{(r-1)}(u + 1/4), \quad (2.29) \]
\[ T_{2m+1}^{(r)}(u - 1/4)T_{2m+1}^{(r)}(u + 1/4) = T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u) \]
\[ + T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u). \quad (2.30) \]

If \( r = 2 \), eqs. (2.25) - (2.28) reduce to the following equations:
\[
T_{2m}^{(2)}(u - 1/2)T_{2m}^{(2)}(u + 1/2) = T_{2m+1}^{(2)}(u)T_{2m-1}^{(2)}(u) + T_m^{(1)}(u - 1/2)T_m^{(1)}(u + 1/2)T_{2m}^{(2)}(u), \quad (2.31)
\]
\[
T_{2m+1}^{(2)}(u - 1/2)T_{2m+1}^{(2)}(u + 1/2) = T_{2m+2}^{(2)}(u)T_{2m}^{(2)}(u) + T_m^{(0)}(u)T_{m+1}^{(0)}(u)T_{4m+2}^{(2)}(u). \quad (2.32)
\]

For \( A_{2r-1}^{(2)}(r \geq 3) \),
\[
T_m^{(a)}(u - 1/2)T_m^{(a)}(u + 1/2) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(2)}(u) \quad a = 0, 1, \quad (2.33)
\]
\[
T_m^{(2)}(u - 1/2)T_m^{(2)}(u + 1/2) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_m^{(0)}(u)T_m^{(1)}(u)T_m^{(3)}(u), \quad (2.34)
\]
\[
T_m^{(a)}(u - 1/2)T_m^{(a)}(u + 1/2) = T_{m+1}^{(a)}(u)T_m^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \quad 3 \leq a \leq r - 2 \quad (2.35)
\]
\[
T_{2m}^{(r)}(u - 1/2)T_{2m}^{(r)}(u + 1/2) = T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u) + T_{2m+1}^{(r-2)}(u)T_m^{(r)}(u)T_{m+1}^{(r)}(u), \quad (2.36)
\]
\[
T_{m}^{(r)}(u - 1)T_{m}^{(r)}(u + 1) = T_{m+1}^{(r)}(u)T_{m-1}^{(r)}(u) + T_{2m}^{(r-1)}(u). \quad (2.37)
\]

If \( r = 3 \), eqs. (2.34) - (2.37) reduce to the following equations:
\[
T_{2m}^{(2)}(u - 1/2)T_{2m}^{(2)}(u + 1/2) = T_{2m+1}^{(2)}(u)T_{2m-1}^{(2)}(u) + T_{2m}^{(2)}(u)T_{2m}^{(3)}(u)T_{m}^{(3)}(u), \quad (2.39)
\]
\[
T_{2m+1}^{(2)}(u - 1/2)T_{2m+1}^{(2)}(u + 1/2) = T_{2m+2}^{(2)}(u)T_{2m}^{(2)}(u) + T_{2m+1}^{(2)}(u)T_{m}^{(3)}(u)T_{m+1}^{(3)}(u), \quad (2.40)
\]
For $D_{r+1}^{(2)} (r \geq 2)$,

\begin{align*}
T_{2m}^{(0)}(u - 1/2)T_{2m}^{(0)}(u + 1/2) &= T_{2m+1}^{(0)}(u)T_{2m-1}^{(0)}(u) \\
&\quad + T_m^{(1)}(u - 1/2)T_m^{(1)}(u + 1/2), \quad (2.41) \\
T_{2m+1}^{(0)}(u - 1/2)T_{2m+1}^{(0)}(u + 1/2) &= T_{2m+2}^{(0)}(u)T_{2m}^{(0)}(u) \\
&\quad + T_m^{(1)}(u)T_{m+1}^{(1)}(u), \quad (2.42) \\
T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) &= T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_{2m}^{(0)}(u)T_m^{(2)}(u), \quad (2.43) \\
T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) &= T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u) \quad 2 \leq a \leq r - 2, \quad (2.44) \\
T_m^{(r-1)}(u - 1)T_m^{(r-1)}(u + 1) &= T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u) \\
&\quad + T_m^{(r-2)}(u)T_{m}^{(r)}(u), \quad (2.45) \\
T_{2m}^{(r)}(u - 1/2)T_{2m}^{(r)}(u + 1/2) &= T_{2m+1}^{(r)}(u)T_{2m-1}^{(r)}(u) + T_m^{(r-1)}(u - 1/2)T_m^{(r-1)}(u + 1/2), \quad (2.46) \\
T_{2m+1}^{(r)}(u - 1/2)T_{2m+1}^{(r)}(u + 1/2) &= T_{2m+2}^{(r)}(u)T_{2m}^{(r)}(u) \\
&\quad + T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u). \quad (2.47)
\end{align*}

If $r = 2$, eqs. (2.43) - (2.45) reduce to the following equation :

\begin{equation}
T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_{2m}^{(0)}(u)T_{2m}^{(2)}(u). \quad (2.48)
\end{equation}

These functional equations are neatly connected with Dynkin diagrams, and thus they are reflecting Dynkin diagram symmetries. For example, the functional equations (2.18) and (2.19) are transformed into eqs.(2.22) and (2.21) respectively under the following transformation:

\begin{equation}
T_m^{(b)}(u) \rightarrow T_m^{(r-b)}(u). \quad (2.49)
\end{equation}

It was pointed out \cite{[3]} that the eq.(2.1) with boundary condition

\begin{equation*}
T_m^{(r-1)}(u) = 0, \quad T_m^{(0)}(u) = T_m^{(r+1)}(u) = 1
\end{equation*}

corresponds to Hirota-Miwa equation.

Kuniba et al. \cite{[3]} pointed out that the T-system may be looked upon as a discritized Toda field equation. By the same argument, our functional
equation (1.1) may be viewed as a discretized affine Toda field equation. Set \( u = v/\delta \) and \( m = w/\delta \), for functional equation (1.1). For small \( \delta \), from the coefficient of \( \delta^2 \) on Taylor expansion, we obtain two-dimensional affine Toda field equation of the form

\[
(\partial_v^2 - \partial_w^2) \log \psi_a(v, w) = \text{constant} \prod_{b=0}^r \psi_b(v, w)^{-C_{ab}} \tag{2.50}
\]

where \( \psi_a(v, w) \) denotes a scaled \( T_m^{(a)}(u) \) and \( C_{ab} = 2(\alpha_a|\alpha_b)/(\alpha_a|\alpha_a) \) the Cartan matrix of affine Lie algebra.

3 Notation and Convention

In this section, we present a lot of expressions, which are necessary to construct the solutions. The origin of them go back to earlier work \([3, 6, 7]\) on solutions of the T-system. Set

\[
x_j^{[1]}(k|u) = T_1^{(\delta_j-1)}(u + (2j-2)/k), \quad y_j^{[1]}(k|u) = T_1^{(\delta_j)}(u + (2j-2)/k),
\]

\[
t_{ij}^{[1]}(k|u) = T_1^{(\delta_j)}(u + (i+j-2)/k),
\]

\[
x_j^{[2]}(k|u) = T_1^{(r-\delta_j-1)}(u + (2j-2)/k), \quad y_j^{[2]}(k|u) = T_1^{(r-\delta_j)}(u + (2j-2)/k),
\]

\[
t_{ij}^{[2]}(k|u) = T_1^{(r+i-j-1)}(u + (i+j-2)/k),
\]

\[
a_{ij}^{[p]}(k|u) = x_i^{[p]}(k|u)y_j^{[p]}(k|u) - t_{ij}^{[p]}(k|u),
\]

\[
b_{ij}^{[p]}(k|u) = y_i^{[p]}(k|u)x_j^{[p]}(k|u) - t_{ij}^{[p]}(k|u), \quad p = 1, 2.
\]

where

\[
\delta_i = \begin{cases} 0 & \text{if } i \in 2\mathbb{Z} \\ 1 & \text{if } i \in 2\mathbb{Z} + 1. \end{cases}
\]

By the definition we have

\[
x_i^{[p]}(k|u + 2/k) = y_{i+1}^{[p]}(k|u), \quad y_i^{[p]}(k|u + 2/k) = x_{i+1}^{[p]}(k|u),
\]

\[
t_i^{[p]}(k|u + 2/k) = t_{i+1}^{[p]}(k|u),
\]

\[
a_{ij}^{[p]}(k|u + 2/k) = b_{i+1,j+1}^{[p]}(k|u), \quad b_{ij}^{[p]}(k|u + 2/k) = a_{i+1,j+1}^{[p]}(k|u), \tag{3.2}
\]

for \( p = 1, 2 \). These relations are necessary to prove Lemma 4.9. Now we introduce \( m \times m \) matrices \( T_m^a(k|u) = (T_{ij}^a(k|u))_{1 \leq i, j \leq m} \quad (a \in \mathbb{Z}) \).
(2m) matrices $\mathcal{F}_{ij}^{[p]}(k|u) = (\mathcal{F}_{ij}^{[p]}(k|u))_{1 \leq i,j \leq 2m}$, $(m + 1) \times (m + 1)$ matrices $\mathcal{H}_{ij}^{[p]}(k|u) = (\mathcal{H}_{ij}^{[p]}(k|u))_{1 \leq i,j \leq m+1}$, $\mathcal{S}_{ij}^{[p]}(k|u) = (\mathcal{S}_{ij}^{[p]}(k|u))_{1 \leq i,j \leq m+1}$, $\mathcal{C}_{ij}^{[p]}(k|u) = (\mathcal{C}_{ij}^{[p]}(k|u))_{1 \leq i,j \leq m+1}$ and $\mathcal{R}_{ij}^{[p]}(k|u) = (\mathcal{R}_{ij}^{[p]}(k|u))_{1 \leq i,j \leq m+1}$ ($p = 1,2$) whose $(i,j)$ elements are given by

$$T_{ij}^{a}(k|u) = \mathcal{T}^{a+i-j}(u + (i + j - m - 1)/k),$$  \hfill (3.3)

$$\mathcal{F}_{ij}^{[1]}(k|u) = \begin{cases} 0 : i = j \\ \mathcal{T}^{-i+j-1}(u + (i + j - 2m - 1)/k) : 1 \leq i < j \leq 2m \\ -\mathcal{T}^{-i-1}(u + (i + j - 2m - 1)/k) : 1 \leq j < i \leq 2m, \end{cases}$$  \hfill (3.4)

$$\mathcal{F}_{ij}^{[2]}(k|u) = \begin{cases} 0 : i = j \\ \mathcal{T}^{i-j+r+1}(u + (i + j - 2m - 1)/k) : 1 \leq i < j \leq 2m \\ -\mathcal{T}^{-i+j+r+1}(u + (i + j - 2m - 1)/k) : 1 \leq j < i \leq 2m, \end{cases}$$  \hfill (3.5)

$$\mathcal{H}_{ij}^{[1]}(k|u) = \begin{cases} T_{ij}^{(0)}(u + (2i - 2)/k) : 1 \leq i \leq m + 1 \text{, and } j = 1 \\ \mathcal{T}^{i+j}(u + (i + j - 5/2)/k) : 1 \leq i \leq m + 1, \ 2 \leq j \leq m + 1, \end{cases}$$  \hfill (3.6)

$$\mathcal{H}_{ij}^{[2]}(k|u) = \begin{cases} T_{ij}^{(r)}(u + (2i - 2)/k) : 1 \leq i \leq m + 1 \text{ and } j = 1 \\ \mathcal{T}^{r+i-j}(u + (i + j - 5/2)/k) : 1 \leq i \leq m + 1, \ 2 \leq j \leq m + 1, \end{cases}$$  \hfill (3.7)

$$\mathcal{S}_{ij}^{[1]}(k|u) = \begin{cases} 0 : i = j = 1 \\ T_{ij}^{(\delta)}(u + (2j - 4)/k) : i = 1 \text{ and } 2 \leq j \leq m + 1 \\ -T_{ij}^{(\delta)}(u + (2i - 4)/k) : 2 \leq i \leq m + 1 \text{ and } j = 1 \\ -\mathcal{T}^{-i+j}(u + (i + j - 4)/k) : 2 \leq i, j \leq m + 1, \end{cases}$$  \hfill (3.8)

$$\mathcal{S}_{ij}^{[2]}(k|u) = \begin{cases} 0 : i = j = 1 \\ T_{ij}^{(r-\delta)}(u + (2j - 4)/k) : i = 1 \text{ and } 2 \leq j \leq m + 1 \\ -T_{ij}^{(r-\delta)}(u + (2i - 4)/k) : 2 \leq i \leq m + 1 \text{ and } j = 1 \\ -\mathcal{T}^{r+i-j}(u + (i + j - 4)/k) : 2 \leq i, j \leq m + 1, \end{cases}$$  \hfill (3.9)
Note that the matrices $F_{ij}^{[1]}(k|u)$, $H_{ij}^{[1]}(k|u)$, $S_{ij}^{[1]}(k|u)$, $C_{ij}^{[1]}(k|u)$ and $R_{ij}^{[1]}(k|u)$ are transformed into $F_{ij}^{[2]}(k|u)$, $H_{ij}^{[2]}(k|u)$, $S_{ij}^{[2]}(k|u)$, $C_{ij}^{[2]}(k|u)$ and $R_{ij}^{[2]}(k|u)$ respectively under the following transformation:

$$
T^{(b)}_1(u) \rightarrow T^{(r-b)}_1(u), \quad T^b(u) \rightarrow T^{r-b}(u). \quad (3.12)
$$

For any matrix $M(u)$, we shall let $M \begin{bmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{bmatrix}(u)$ denote the minor matrix getting rid of $i_i$'s rows and $j_j$'s columns from $M(u)$. We introduce the following pfaffian and determinant expressions, which will be used as the components of the solutions.

$$
T^a_m(k|u) = \det[\mathcal{T}^a_m(k|u)] \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \quad (3.13)
$$

$$
F^0_m(k|u) = \text{pf}[\mathcal{F}^{[1]}_{2m}(k|u)] \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \quad (3.14)
$$

$$
F^{(r)}_m(k|u) = \text{pf}[\mathcal{F}^{[2]}_{2m}(k|u)] \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \quad (3.15)
$$

$$
G^{(0)}_m(k|u) = \begin{cases} 
\text{pf}[\mathcal{C}^{[1]}_{m+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (k|u + (-m + 1)/k)] & \text{for } m \in 2\mathbb{Z}_{\geq 0} \\
\text{pf}[\mathcal{C}^{[1]}_{m+1}(k|u + (-m + 1)/k)] & \text{for } m \in 2\mathbb{Z}_{\geq 0} + 1,
\end{cases} \quad (3.16)
$$

$$
G^{(1)}_m(k|u) = \begin{cases} 
\text{pf}[\mathcal{C}^{[1]}_{m+2} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} (k|u + (-m - 1)/k)] & \text{for } m \in 2\mathbb{Z}_{\geq 0} \\
\text{pf}[\mathcal{C}^{[1]}_{m+2}(k|u + (-m - 1)/k)] & \text{for } m \in 2\mathbb{Z}_{\geq 0} + 1,
\end{cases} \quad (3.17)
$$
For a ∈ ℤ, k ∈ ℤ − {0} and u ∈ ℂ, we introduce the following functional relations, combinations of which will be used in solving the eq.(1.1):

\[ T^a(u) + T^{1-a}(u) = T_1^{(0)}(u + (a - 1/2)/k)T_1^{(0)}(u + (-a + 1/2)/k), \] (3.22)

\[ T^a(u) + T^{2r-1-a}(u) = T_1^{(r)}(u + (-r + a + 1/2)/k)T_1^{(r)}(u + (r - a - 1/2)/k), \] (3.23)

\[ T^a(u) + T^{-2-a}(u) = 0, \] (3.24)

\[ T^a(u) + T^{2r-a+2}(u) = 0, \] (3.25)

\[ T^a(u) + T^{2-a}(u) = T_1^{(0)}(u + (a - 1)/k)T_1^{(0)}(u + (-a + 1)/k) + T_1^{(1)}(u + (a - 1)/k)T_1^{(0)}(u + (-a + 1)/k), \] (3.26)

\[ T^a(u) + T^{2r-a-2}(u) = T_1^{(r)}(u + (r - a - 1)/k)T_1^{(r)}(u + (-r + a + 1)/k) + T_1^{(r-1)}(u + (r - a - 1)/k)T_1^{(r-1)}(u + (-r + a + 1)/k). \] (3.27)

Note that the functional equations (3.22), (3.24) and (3.26) are transformed into eqs. (3.23), (3.25) and (3.27) respectively under the following transformation (3.12).

The origin of the function \( T^a(u) \) in eq.(3.23) goes back to auxiliary transfer matrix (or ‘dress function’ in the analytic Bethe ansatz) for \( U_q(B^{(1)}_r) \) vertex model. Actually, the functional relation (3.23) for \( k = 1 \) with the boundary condition

\[ T^a(u) = \begin{cases} 
0 & a < 0 \\
1 & a = 0 \\
T_1^{(a)}(u) & 1 \leq a \leq r - 1
\end{cases} \] (3.28)
was firstly introduced in ref. [3]. The origin of the function $T^a(u)$ in eq. (3.27) goes back to auxiliary transfer matrix (or ‘dress function’ in the analytic Bethe ansatz) for $U_q(D^{(1)}_r)$ vertex model. Actually, the functional relation (3.27) for $k = 1$ with the boundary condition

$$T^a(u) = \begin{cases} 
0 & a < 0 \\
1 & a = 0 \\
T^a_1(u) & 1 \leq a \leq r - 2
\end{cases}$$

(3.29)

was firstly introduced in ref. [7].

Remark 1: Under the functional relation (3.26) (resp., (3.27)) the following relations for $p = 1$ (resp., $p = 2$) hold.

$$C^{[p]}_{m+1}(k|u) = S^{[p]}_{m+1}(k|u) \prod_{j=2}^{m+1} P(1, j; -y^{[p]}_{j-1}(k|u)),$$

(3.30)

$$R^{[p]}_{m+1}(k|u) = \left( \prod_{i=2}^{m+1} P(i, 1; y^{[p]}_{i-1}(k|u)) \right) S^{[p]}_{m+1}(k|u)$$

(3.31)

where

$$P(i, j; c) = I + cI_{ij}$$

(3.32)

is the $m+1$ by $m+1$ matrix with $I$ the identity and $I_{ij}$ the matrix unit. Namely, the matrices $C^{[p]}_{m+1}(k|u)$ and $R^{[p]}_{m+1}(k|u)$ can be derived from elementary transformations to the matrix $S^{[p]}_{m+1}(k|u)$.

We further have

$$F^{[1]}_{2m}(k|u) = ^t T^{-1}_{2m}(k|u) = -T^{-1}_{2m}(k|u)$$

(3.33)

under the functional relations (3.24),

$$F^{[2]}_{2m}(k|u) = T^{r+1}_{2m}(k|u)$$

(3.34)

under the functional relations (3.25), where the index $^t$ denotes transposition of a matrix.

Remark 2: In deriving the solutions in section 4, the parameter $k$ in this section is determined by the value of $t_a = 2/(\alpha_a|\alpha_a)$, where $t_a = 1$ if $\alpha_a$ is the longest simple root of affine Lie algebra.
4 Solutions

The solutions of our functional equations (2.1)-(2.48) are given as follows.

Theorem 4.1 (The $A_r^{(1)}$ case, $r \geq 1$) For $m \in \mathbb{Z}_{\geq 0}$,

$$T_m^{(a)}(u) = T_m^a(1|u)$$

(4.1)

solves the functional equations (2.1) under the condition

$$T^{b+g}(u) = T^b(u), \quad g = r + 1, \quad b \in \mathbb{Z}.$$  

(4.2)

Theorem 4.2 (The $B_r^{(1)}$ case, $r \geq 3$) For $m \in \mathbb{Z}_{\geq 0}$,

$$T_m^{(0)}(u) = G_m^{(0)}(1|u), \quad T_m^{(1)}(u) = G_m^{(1)}(1|u),$$

$$T_m^{(a)}(u) = T_m^a(1|u), \quad 2 \leq a \leq r - 1,$$

$$T_m^{(r)}(u) = T_m^r(1|u), \quad T_{2m}^{(r)}(u) = H_{m+1}^{(r)}(1|u)$$

(4.3)

solves the functional equations (2.4-2.8) under the relations (3.23) and (3.24) for $k = 1$.

Theorem 4.3 (The $C_r^{(1)}$ case, $r \geq 2$) For $m \in \mathbb{Z}_{\geq 0}$,

$$T_m^{(0)}(u) = F_m^{(0)}(1|u),$$

$$T_m^{(a)}(u) = T_m^a(2|u), \quad 1 \leq a \leq r - 1$$

$$T_m^{(r)}(u) = F_m^{(r)}(2|u)$$

(4.4)

solves the functional equations (2.9-2.23) under the relations (3.24) and (3.27).

Theorem 4.4 (The $D_r^{(1)}$ case, $r \geq 4$) For $m \in \mathbb{Z}_{\geq 0}$,

$$T_m^{(0)}(u) = G_m^{(0)}(1|u), \quad T_m^{(1)}(u) = G_m^{(1)}(1|u),$$

$$T_m^{(a)}(u) = T_m^a(1|u), \quad 2 \leq a \leq r - 2,$$

$$T_m^{(r-1)}(u) = G_m^{(r-1)}(1|u), \quad T_m^{(r)}(u) = G_m^{(r)}(1|u)$$

(4.5)

solves the functional equations (2.18-2.23) under the relations (3.26) and (3.27) for $k = 1$. 

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Theorem 4.5 (The $A^{(2)}_{2r}$ case, $r ≥ 2$) For $m ∈ Z_{≥0}$,
\[ T^{(0)}_{m}(u) = F^{(0)}_{m}(2u), \quad T^{(a)}_{m}(u) = T^{a}_{m}(2u), \quad 1 ≤ a ≤ r - 1, \quad (4.6) \]
\[ T^{(r)}_{2m}(u) = T^{r}_{m}(2u), \quad T^{(r)}_{2m+1}(u) = H^{(r)}_{m+1}(2u) \]
solves the functional equations (2.24-2.32) under the relations (3.23) and (3.24) for $k = 2$.

Theorem 4.6 (The $A^{(2)}_{2r-1}$ case, $r ≥ 3$) For $m ∈ Z_{≥0}$,
\[ T^{(0)}_{m}(u) = G^{(0)}_{m}(2u), \quad T^{(1)}_{m}(u) = G^{(1)}_{m}(2u), \quad T^{(a)}_{m}(u) = T^{a}_{m}(2u), \quad 2 ≤ a ≤ r - 1, \quad (4.7) \]
\[ T^{(r)}_{m}(u) = F^{(r)}_{m}(2u) \]
solves the functional equations (2.33-2.40) under the relations (3.25) and (3.26) for $k = 2$.

Theorem 4.7 (The $D^{(2)}_{r+1}$ case, $r ≥ 2$) For $m ∈ Z_{≥0}$,
\[ T^{(0)}_{2m}(u) = T^{0}_{m}(1u), \quad T^{(0)}_{2m+1}(u) = H^{(0)}_{m+1}(1u), \quad (4.8) \]
\[ T^{(a)}_{2m}(u) = T^{a}_{m}(1u), \quad 1 ≤ a ≤ r - 1, \quad T^{(r)}_{2m}(u) = T^{r}_{m}(1u), \quad T^{(r)}_{2m+1}(u) = H^{(r)}_{m+1}(1u) \]
solves the functional equations (2.41-2.48) under the relations (3.22) and (3.23) for $k = 1$.

Remark 1: In eqs. (4.1)-(4.8), the following type of boundary conditions are valid.
\[ \text{For } \alpha ≤ a ≤ \beta, \quad T^{a}(u) = T^{a}_{1}(u) \]
where $(\alpha, \beta) = (0, r)$ for (4.1); $(\alpha, \beta) = (2, r - 1)$ for (4.3); $(\alpha, \beta) = (0, r)$ for (4.4); $(\alpha, \beta) = (2, r - 2)$ for (4.5); $(\alpha, \beta) = (0, r - 1)$ for (4.6); $(\alpha, \beta) = (2, r)$ for (4.7); $(\alpha, \beta) = (1, r - 1)$ for (4.8).

Remark 2: In eq. (4.1), if $T^{(a)}_{1}(u)$ does not depend on $u ∈ C$ for all $a$, then the following truncation holds:
\[ T^{(b)}_{g}(u) = 0 \quad \text{for any } \quad b. \]

Now we enumerate lemmas that are necessary for the proofs of the theorems.
Lemma 4.8 For \( m \in \mathbb{Z}_{\geq 0} \) and \( u \in \mathbb{C} \), \( F_m^{(0)}(k|u) \) (resp., \( F_m^{(r)}(k|u) \)) satisfy the following relations for \( \alpha = 0 \) (resp., \( \alpha = r \)) under the relation (3.24) (resp., (3.23)).

\[
F_m^{(\alpha)}(k|u - 1/k) F_m^{(\alpha)}(k|u + 1/k) = T_{2m}^{\alpha}(k|u), \quad (4.9)
\]
\[
F_m^{(\alpha)}(k|u) F_{m+1}^{(\alpha)}(k|u) = T_{2m+1}^{\alpha}(k|u). \quad (4.10)
\]

Lemma 4.9 For \( m \in \mathbb{Z}_{\geq 1} \), \( G_m^{(1)}(k|u) \) (3.17) and \( G_m^{(0)}(k|u) \) (3.14) (resp., \( G_m^{(r-1)}(k|u) \) (3.18) and \( G_m^{(r)}(k|u) \) (3.18)) satisfy the following relations for \( (p, \alpha, \beta) = (1, 1, 0) \) (resp., \( (p, \alpha, \beta) = (2, r - 1, r) \)) under the relation (3.24) (resp., (3.23)).

\[
G_m^{(\alpha)}(k|u + 1/k) G_{m-1}^{(\beta)}(k|u) = \begin{cases} 
\det[S_{m+1}^{[\beta]} \begin{bmatrix} m+1 \\ 1 \end{bmatrix} (k|u + (-m + 2)/k)] & \text{for } m \in 2\mathbb{Z}_{\geq 1} \\
\det[S_{m+2}^{[\beta]} \begin{bmatrix} m+2 \\ 2 \end{bmatrix} (k|u - m/k)] & \text{for } m \in 2\mathbb{Z}_{\geq 0} + 1,
\end{cases} \quad (4.11)
\]

\[
G_m^{(\alpha)}(k|u) G_{m+1}^{(\beta)}(k|u) = (-1)^m \det[S_{m+1}^{[\beta]} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (k|u + (-m + 1)/k)], \quad (4.12)
\]

\[
G_m^{(\alpha)}(k|u + 1/k) G_{m+1}^{(\alpha)}(k|u) = (-1)^{m+1} \det[S_{m+2}^{[\alpha]} \begin{bmatrix} 1 \\ 2 \end{bmatrix} (k|u - m/k)], \quad (4.13)
\]

\[
G_{m-1}^{(\alpha)}(k|u) G_m^{(\alpha)}(k|u + 1/k) = (-1)^m \det[S_{m+1}^{[\beta]} \begin{bmatrix} 1 \\ 2 \end{bmatrix} (k|u - m/k)], \quad (4.14)
\]

\[
G_{m-1}^{(\beta)}(k|u - 1/k) G_m^{(\beta)}(k|u) = (-1)^m \det[S_{m+1}^{[\beta]} \begin{bmatrix} 1 \\ m+1 \end{bmatrix} (k|u + (-m + 1)/k)], \quad (4.15)
\]

\[
G_m^{(\alpha)}(k|u) G_{m-1}^{(\beta)}(k|u + 1/k) = (-1)^m \det[S_{m+2}^{[\beta]} \begin{bmatrix} 1 \\ 2 \end{bmatrix} (k|u + (-m - 1)/k)]. \quad (4.16)
\]
Remark 3: Lemma 4.8 for $\alpha = r, k = 1$ with different boundary condition for $T^a(u)$ was firstly proved in ref. [6].

Remark 4: Lemma 4.9 for $(p, \alpha, \beta, k) = (2, r - 1, r, 1)$ with different boundary condition (3.29) for $T^a(u)$ was firstly proved in ref. [7].

The proofs of the theorems and lemmas in this section are quite similar to those in earlier work on solutions of the T-system [1, 3, 7]. That is, most of them reduce to the Jacobi identity:

$$\det M \begin{bmatrix} b \\ c \end{bmatrix} \det M \begin{bmatrix} c \\ b \end{bmatrix} - \det M \begin{bmatrix} b \\ c \end{bmatrix} \det M \begin{bmatrix} c \\ b \end{bmatrix} = \det M \begin{bmatrix} b & c \\ b & c \end{bmatrix} \det M,$$

$(b \neq c)$.

So we shall not go into the detailed proofs here.

5 Summary and Discussion

In this paper, we have given the generalization of a discretized Toda equation. Although the origin of it goes back to the T-system, a family of transfer matrix functional equations, it has nothing to do with the solvable lattice models to the author’s knowledge. It can be looked upon as a discretized affine Toda field equation. Solving it recursively, we have given, for $A^{(1)}_n$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_n$ and $D^{(2)}_n$, its solutions in terms of determinants or pfaffians.

From the point of view of representation theory, the solutions of the T-system [3, 6, 7] are considered to be Yangian analogue of the Jacobi-Trudi formulae, that is, Yangian character formulae in terms of fundamental representations. Then, what kind of underlying structures our new solutions here indicating? Whether we can understand them within the framework of Yangian (or quantum affine algebras) or not is an open problem. There are related papers refs. [3, 6, 20, 27, 28, 29, 30].

There remain problems to solve the functional equations associated with other Kac-Moody Lie algebras.

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A List of functional equations for other affine Lie algebras

For $G_2^{(1)}$,

\begin{align}
T_m^{(0)}(u-1)T_m^{(0)}(u+1) &= T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \\
T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(0)}(u)T_{3m}^{(2)}(u) \\
T_{3m}^{(2)}(u-1/3)T_{3m}^{(2)}(u+1/3) &= T_{3m+1}^{(2)}(u)T_{3m-1}^{(2)}(u) + T_m^{(1)}(u-2/3)T_m^{(1)}(u+2/3) \\
T_{3m+1}^{(2)}(u-1/3)T_{3m+1}^{(2)}(u+1/3) &= T_{3m+2}^{(2)}(u)T_{3m}^{(2)}(u) + T_m^{(1)}(u-1/3)T_m^{(1)}(u+1/3) \\
T_{3m+2}^{(2)}(u-1/3)T_{3m+2}^{(2)}(u+1/3) &= T_{3m+3}^{(2)}(u)T_{3m+1}^{(2)}(u) + T_m^{(1)}(u)T_{m+1}^{(1)}(u-1/3)T_{m+1}^{(1)}(u+1/3)
\end{align}

For $F_4^{(1)}$,

\begin{align}
T_m^{(0)}(u-1)T_m^{(0)}(u+1) &= T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \\
T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(0)}(u)T_{2m}^{(2)}(u) \\
T_m^{(2)}(u-1)T_m^{(2)}(u+1) &= T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_m^{(1)}(u)T_{2m}^{(3)}(u) \\
T_{2m}^{(3)}(u-1/2)T_{2m}^{(3)}(u+1/2) &= T_{2m+1}^{(3)}(u)T_{2m-1}^{(3)}(u) + T_m^{(2)}(u-1/2)T_m^{(2)}(u+1/2)T_{2m}^{(4)}(u) \\
T_{2m+1}^{(3)}(u-1/2)T_{2m+1}^{(3)}(u+1/2) &= T_{2m+2}^{(3)}(u)T_{2m}^{(3)}(u) + T_{m+1}^{(2)}(u)T_{2m+1}^{(4)}(u) \\
T_m^{(4)}(u-1/2)T_m^{(4)}(u+1/2) &= T_{m+1}^{(4)}(u)T_{m-1}^{(4)}(u) + T_m^{(3)}(u)
\end{align}

For $E_6^{(1)}$,

\begin{align}
T_m^{(0)}(u-1)T_m^{(0)}(u+1) &= T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(6)}(u) \\
T_m^{(a)}(u-1)T_m^{(a)}(u+1) &= T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u),
\end{align}

We had removed this appendix from the published version.
\[
T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(2)}(u) \quad (A.14)
\]
\[
T_m^{(3)}(u - 1)T_m^{(3)}(u + 1) = T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_m^{(2)}(u)T_m^{(4)}(u)T_m^{(6)}(u) \quad (A.15)
\]
\[
T_m^{(5)}(u - 1)T_m^{(5)}(u + 1) = T_{m+1}^{(5)}(u)T_{m-1}^{(5)}(u) + T_m^{(4)}(u) \quad (A.16)
\]
\[
T_m^{(6)}(u - 1)T_m^{(6)}(u + 1) = T_{m+1}^{(6)}(u)T_{m-1}^{(6)}(u) + T_m^{(0)}(u)T_m^{(3)}(u) \quad (A.17)
\]

For \(E_7^{(1)}\),
\[
T_m^{(0)}(u - 1)T_m^{(0)}(u + 1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \quad (A.18)
\]
\[
T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u), \quad a = 1, 2, 4, 5 \quad (A.19)
\]
\[
T_m^{(3)}(u - 1)T_m^{(3)}(u + 1) = T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_m^{(2)}(u)T_m^{(4)}(u)T_m^{(7)}(u) \quad (A.20)
\]
\[
T_m^{(6)}(u - 1)T_m^{(6)}(u + 1) = T_{m+1}^{(6)}(u)T_{m-1}^{(6)}(u) + T_m^{(5)}(u) \quad (A.21)
\]
\[
T_m^{(7)}(u - 1)T_m^{(7)}(u + 1) = T_{m+1}^{(7)}(u)T_{m-1}^{(7)}(u) + T_m^{(3)}(u) \quad (A.22)
\]

For \(E_8^{(1)}\),
\[
T_m^{(0)}(u - 1)T_m^{(0)}(u + 1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(7)}(u) \quad (A.23)
\]
\[
T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u), \quad a = 2, 4, 5, 6 \quad (A.24)
\]
\[
T_m^{(1)}(u - 1)T_m^{(1)}(u + 1) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(2)}(u) \quad (A.25)
\]
\[
T_m^{(3)}(u - 1)T_m^{(3)}(u + 1) = T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_m^{(2)}(u)T_m^{(4)}(u)T_m^{(8)}(u) \quad (A.26)
\]
\[
T_m^{(7)}(u - 1)T_m^{(7)}(u + 1) = T_{m+1}^{(7)}(u)T_{m-1}^{(7)}(u) + T_m^{(0)}(u)T_m^{(6)}(u) \quad (A.27)
\]
\[
T_m^{(8)}(u - 1)T_m^{(8)}(u + 1) = T_{m+1}^{(8)}(u)T_{m-1}^{(8)}(u) + T_m^{(3)}(u) \quad (A.28)
\]

For \(E_6^{(2)}\),
\[
T_m^{(0)}(u - 1/2)T_m^{(0)}(u + 1/2) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \quad (A.29)
\]
\[
T_m^{(1)}(u - 1/2)T_m^{(1)}(u + 1/2) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(0)}(u)T_m^{(2)}(u) \quad (A.30)
\]
\[
T_m^{(2)}(u - 1/2)T_m^{(2)}(u + 1/2)
\]

19
\begin{align}
T_m^{(2)}(u - 1/2)T_m^{(2)}(u + 1/2) &= T_{2m+2}^{(2)}(u)T_{2m}^{(2)}(u) + T_{2m+1}^{(1)}(u)T_m^{(3)}(u)T_{m+1}^{(3)}(u) \\
T_m^{(3)}(u - 1)T_m^{(3)}(u + 1) &= T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_{2m}^{(2)}(u)T_m^{(4)}(u) \\
T_m^{(4)}(u - 1)T_m^{(4)}(u + 1) &= T_{m+1}^{(4)}(u)T_{m-1}^{(4)}(u) + T_m^{(3)}(u)
\end{align}

For $D_4^{(3)}$, 
\begin{align}
T_m^{(0)}(u - 1/3)T_m^{(0)}(u + 1/3) &= T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \\
T_{3m}^{(1)}(u - 1/3)T_{3m}^{(1)}(u + 1/3) &= T_{3m+1}^{(1)}(u)T_{3m-1}^{(1)}(u) + T_{3m}^{(0)}(u)T_m^{(2)}(u - 2/3)T_m^{(2)}(u)T_m^{(2)}(u + 2/3) \\
T_{3m+1}^{(1)}(u - 1/3)T_{3m+1}^{(1)}(u + 1/3) &= T_{3m+2}^{(1)}(u)T_{3m}^{(1)}(u) + T_{3m+1}^{(0)}(u)T_m^{(2)}(u - 1/3)T_m^{(2)}(u + 1/3)T_{m+1}^{(2)}(u) \\
T_{3m+2}^{(1)}(u - 1/3)T_{3m+2}^{(1)}(u + 1/3) &= T_{3m+3}^{(1)}(u)T_{3m+1}^{(1)}(u) \\
+ T_{3m+2}^{(0)}(u)T_m^{(2)}(u)T_{m+1}^{(2)}(u - 1/3)T_{m+1}^{(2)}(u + 1/3) \\
T_m^{(2)}(u - 1)T_m^{(2)}(u + 1) &= T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_{3m}^{(1)}(u)
\end{align}
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