Wigner’s new physics frontier: Physics of two-by-two matrices, including the Lorentz group and optical instruments

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Abstract

According to Eugene Wigner, quantum mechanics is a physics of Fourier transformations, and special relativity is a physics of Lorentz transformations. Since two-by-two matrices with unit determinant form the group $SL(2, c)$ which acts as the universal covering group of the Lorentz group, the two-by-two matrices constitute the natural language for special relativity. The central language for optical instruments is the two-by-two matrix called the beam transfer matrix, or the so-called $ABCD$ matrix. It is shown that the $ABCD$ matrices also form the $SL(2, C)$ group. Thus, it is possible to perform experiments in special relativity using optical instruments. Likewise, the optical instruments can be explained in terms of the symmetry of relativistic particles.

Based on this review article, the last author (YSK) presented papers at a number of conferences, including the 8th International Wigner Symposium (New York, U.S.A., 2003), the 8th International Conference on Squeezed States and Uncertainty Relations (Puebla, Mexico, 2003), the Symmetry Festival (Budapest, Hungary, 2003), and the International Conference on Physics and Control (Saint Petersburg, Russia, 2003).

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1 Introduction

Eugene Paul Wigner received the 1963 Noble prize in physics for his contributions to symmetry problems in physics. Even before the formulation of the present form of quantum mechanics, Wigner perceived the importance of symmetry problems in quantum mechanics. In his book which was published in 1931 [1], Wigner formulated the quantum theory of angular momentum in terms of the three-dimensional rotational symmetries. In so doing, he introduced group theoretical methods to physics.

In 1932 [2], he published a paper concerning thermodynamic corrections to equilibrium systems, and introduced a phase-space picture of quantum mechanics. The phase-space distribution function he introduced in this paper is called the Wigner function. The Wigner function is an indispensable theoretical tool in quantum optics and in the foundations of quantum mechanics [3].

Frederick Seitz was Wigner’s first student at Princeton. In their papers published in 1933 and 1934 [4], they worked out the symmetry of sodium crystals, in terms of group theory. They initiated application of quantum mechanics to matter. This field is now known as condensed matter physics.

By 1936, the spin and isospin symmetries for nucleons were well established in physics. Wigner was interested whether those two separate symmetries come from one big symmetry. In so doing, he formulated the concept of supermultiplets [5]. Wigner’s supermultiplet theory was the grandfather of the quark model which Gell-Mann formulated in 1963 [6].

In 1939, Wigner published his most fundamental paper dealing with space-time symmetries of relativistic particles [7]. In this paper, Wigner introduced the Lorentz group to physics. Furthermore, by introducing his “little groups,” Wigner provided the framework for studying the internal space-time symmetries of relativistic particles. Since this paper was so ahead of his time, it was rejected by three different journals before John von Neumann, then the editor of the Annals of Melodramatics, decided to publish it in his journal. The scientific contents of this paper have not yet been fully recognized by the physics community. We are writing this report as a continuation of the work Wigner initiated in this history-making paper.

While particle physicists are still struggling to understand internal space-time symmetries of elementary particles, Wigner’s Lorentz group is becoming useful to many other branches of physics. Among them is optical sciences, both quantum and classical. In quantum optics, the coherent and squeezed states are representations of the Lorentz group [8]. Recently, the Lorentz group is becoming the fundamental language for classical ray optics. It is gratifying to note that optical components, such as lenses, polarizers, interferometers, lasers, and multi-layers can all be formulated in terms of the Lorentz group which Wigner formulated in his 1939 paper. Classical ray optics is of course a very old subject, but we cannot do new physics without measurements using optical instruments. Indeed, classical ray optics constitutes Wigner’s frontier in physics.

The word “group theory” sounds like an abstract mathematics, but it is gratifying that Wigner’s little groups can be formulated in terms of two-by-two
Table 1: Further contents of Einstein’s $E = mc^2$. Massive and massless particles have different energy-momentum relations. Einstein’s special relativity gives one relation for both. Wigner’s little group unifies the internal space-time symmetries for massive and massless particles. The quark model and the parton model can also be combined into one covariant model.

| Massive, Slow | COVARIANCE | Massless, Fast |
|--------------|-------------|----------------|
| Energy-Momentum | $E = p^2/2m_0$ | $E = \sqrt{p^2c^2 + m_0^2c^4}$ | $E = cp$ |
| Internal space-time symmetry | $S_3$ | Wigner’s Little Group | $S_3$ |
|                | $S_1, S_2$ | Gauge Transformation |               |
| Relativistic Extended Particles | Quark Model | Covariant Model of Hadrons | Partons |

matrices, while classical ray optics is largely a physics of two-by-two matrices. The mathematical correspondence is straight-forward.

In order to see this point clearly, let us start with the following classic example. The second-order differential equation

$$A\frac{d^2q(t)}{dt^2} + B\frac{dq(t)}{dt} + Cq(t) = F\cos(\omega t).$$  \hspace{1cm} (1)

is applicable to a driven harmonic oscillator with dissipation. This can also be used for studying an electronic circuits consisting of inductance, resistance, capacitance, and an alternator. Thus, it is possible to study the oscillator system using the electronic circuit. Likewise, an algebra of two-by-two matrices can serve as the scientific language for two more different branches of physics.

There are many physical systems which can be formulated in terms of two-by-two matrices. If we restrict that their determinant be one, there is a well established mathematical discipline called the group theory of $SL(2, C)$. This aspect was noted in the study of Lorentz transformations. In group theoretical terminology, the group $SL(2, C)$ is the universal covering group of the group of the Lorentz group. In practical terms, to each two-by-two matrix, there corresponds one four-by-four matrix which performs a Lorentz transformation.
in the four-dimensional Minkowskian space. Thus, if a physical system can be
explained in terms of two-by-two matrices, it can be explained with the language
of Lorentz transformations. Conversely, the system can serve as an analogue
computer for Lorentz transformations.

Optical filters, polarizers, and interferometers deal with two independent
optical rays. They superpose the beams, change the relative phase shift, and
change relative amplitudes. The basic language here is called the Jones matrix
formalism, consisting of the two-by-two matrix representation of the $SL(2,\mathbb{C})$
group [8, 9]. The four-by-four Mueller matrices are derivable from the two-by-
two matrices of $SL(2,\mathbb{C})$.

Para-axial lens optics can also be formulated in terms of two-by-two matrices,
applicable to the two-component vector space consisting of the distance from
the optical axis and the slope with respect to the axis. The lens and translation
matrices are triangular, but they are basically representations of the $Sp(2)$ group
which is the real subgroup of the group $SL(2,\mathbb{C})$ [10, 11].

Laser optics is basically multi-lens lens optics. However, the problem here is
how to get simple mathematical expression for the system of a large number of
the same lens separately by equal distance. Here again, group theory simplifies
calculations [12].

In multi-layer optics, we deal with two optical rays moving in opposite di-
rections. The standard language in this case is the S-matrix formalism [13].
This is also a two-by-two matrix formalism. As in the case of laser cavities, the
problem is the multiplication of a large number of matrix chains [14, 15].

It is shown in this report that the two-by-two representation of the six-
parameter Lorentz group is the underlying common scientific language for all of
the instruments mentioned above. While the abstract group theoretical ideas
make two-by-two matrix calculations more systematic and transparent in optics,
optical instruments can act as analogue computers for Lorentz transformations
in special relativity. It is gratifying to note that special relativity and ray optics
can be formulated as the physics of two-by-two matrices.

In Sec. 2, we discuss the historical significance of Winger’s 1939 paper [7]
on the Lorentz group and its application to the internal space-time symmetries
of relativistic particles. In Sec. 3, we present the basic building blocks for
the two-by-two representation of the Lorentz group in terms of the matrices
commonly seen in ray optics. In Secs. 4, 5, 6, 7, and 8, we discuss polariza-
tion optics, one-lens system, multi-lens system, laser cavities, and multi-layer optics,
respectively.

Since the Lorentz group is relatively new to many who study optics, we
explain how it is possible to represent the group using two-by-two matrices in
Appendix A, we explain how the Lorentz group can be formulated in terms
of four-by-four matrices. It is shown that the group can have six independent
parameters. In Appendix B, we explain how it is possible to formulate the
Lorentz group in terms of two-by-two matrices. It is shown that the four-by-four
transformation matrices can be constructed from those two-by-two matrices. In
Appendix C, it is noted that the four-by-four matrices are real, their two-by-two
counterparts are complex. However, there is a three-parameter subgroup called
It is shown that the complex subgroup $SU(1, 1)$ is equivalent to $Sp(2)$ through conjugate transformation.

The purpose of these appendixes is to give an introduction to group theoretical methods used in this report and in the recent optics literature. In Appendix D it is shown that the Lie group method, in terms of the generators, is not the only method in constructing group representations. For the rotation group and the three-parameter subgroups of the Lorentz group, it is simpler to start with the minimum number of starter matrices. For instance, while there are three generators for the rotation group in the Lie approach, we can construct the most general form of the rotation matrix from rotations around two directions, as Goldstein constructed the Euler angles \cite{29}.

As for the references, we have not made attempts to list all the papers relevant to the present report. However the references are given in the papers in the refereed journals.

2 Wigner’s Little Groups

If the momentum of a particle is much smaller than its rest-mass energy, the energy-momentum relation is $E = p^2/2m_0 + m_0c^2$. If the momentum is much larger than its rest-mass energy, the relation is $E = cp$. These two different relations can be combined into one covariant formula $E^2 = m_0^2c^4 + p^2c^2$. This aspect of Einstein’s $E = mc^2$ is also well known.

In addition, particles have internal space-time variables. Massive particles have spins while massless particles have their helicities and gauge variables. Our first question is whether this aspect of space-time variables can be unified into one covariant concept. The answer to this question is Yes. Wigner’s little group does the job. In addition, particles can have space-time extensions. For instance, in the quark model, hadrons are bound states of quarks. However, the hadron appears as a collection of partons when they move with speed close to the velocity of light. Quarks and partons seem to have quite distinct properties. Are they different manifestations of a single covariant entity? This is one of the main issues in high-energy particles physics.

By “further contents” of Einstein’s $E = mc^2$, we mean that the internal space-time structures of massive and massless particles can be unified into one covariant package, as $E^2 = m_0^2c^4 + p^2c^2$ does for the energy-momentum relation. The mathematical framework of this program was developed by Eugene Wigner in 1939 \cite{7}. He constructed maximal subgroups of the Lorentz group whose transformations will leave the four-momentum of a given particle invariant. These groups are known as Wigner’s little groups. Thus, the transformations of the little groups change the internal space-time variables the particle. The little group is a covariant entity and takes different forms for the particles moving with different speeds.

The space-time symmetry of relativistic particles is dictated by the Poincaré group \cite{4}. The Poincaré group is the group of inhomogeneous Lorentz transformations, namely Lorentz transformations preceded or followed by space-time
translations. Thus, the Poincaré group is a semi-direct product of the Lorentz and translation groups. The two Casimir operators of this group correspond to the \((\text{mass})^2\) and \((\text{spin})^2\) of a given particle. Indeed, the particle mass and its spin magnitude are Lorentz-invariant quantities.

The question then is how to construct the representations of the Lorentz group which are relevant to physics. For this purpose, Wigner in 1939 studied the maximal subgroups of the Lorentz group whose transformations leave the four-momentum of a given free particle \[7\]. These subgroups are called the little groups. Since the little group leaves the four-momentum invariant, it governs the internal space-time symmetries of relativistic particles. Wigner shows in his paper that the internal space-time symmetries of massive and massless particles are dictated by the little groups which are locally isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean groups respectively.

The group of Lorentz transformations consists of three boosts and three rotations. The rotations therefore constitute a subgroup of the Lorentz group. If a massive particle is at rest, its four-momentum is invariant under rotations. Thus the little group for a massive particle at rest is the three-dimensional rotation group. Then what is affected by the rotation? The answer to this question is very simple. The particle in general has its spin. The spin orientation is going to be affected by the rotation! If we use the four-vector coordinate \((t, z, x, y)\), the four-momentum vector for the particle at rest is \((m_0c^2, 0, 0, 0)\), and the three-dimensional rotation group leaves this four-momentum invariant.

This little group is generated by

\[
J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{pmatrix}, \quad
J_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
J_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}.
\]

These are essentially the generators of the three-dimensional rotation group. They satisfy the commutation relations:

\[
[J_i, J_j] = i\epsilon_{ijk}J_k.
\]

If the rest-particle is boosted along the \(z\) direction, it will pick up a non-zero momentum component along the same direction. The above generators will also be boosted. The boost will take the form of conjugation by the boost matrix

\[
B = \begin{pmatrix}
cosh \eta & \sinh \eta & 0 & 0 \\
\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

This boost will not change the commutation relations of Eq.\(\text{[W]}\) for \(O(3)\), and the boosted little group will still leave the boosted four-momentum invariant. Thus, the little group of a moving massive particle is still \(O(3)\)-like.
It is not possible to bring a massless particle to its rest frame. In his 1939 paper [7], Wigner observed that the little group for a massless particle moving along the $z$ axis is generated by the rotation generator around the $z$ axis, namely $J_3$ of Eq.(2), and two other generators which take the form

$$N_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & i & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & -i & 0 & 0 \end{pmatrix}. \quad (5)$$

If we use $K_i$ for the boost generator along the $i$-th axis, these matrices can be written as

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \quad (6)$$

with

$$K_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

The generators $J_3, N_1$ and $N_2$ satisfy the following set of commutation relations.

$$[N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1. \quad (8)$$

In order to understand the mathematical basis of the above commutation relations, let us consider transformations on a two-dimensional plane with the $xy$ coordinate system. We can then make rotations around the origin and translations along the $x$ and $y$ directions. If we write these generators as $L, P_x$ and $P_y$ respectively, they satisfy the commutation relations [16]

$$[P_x, P_y] = 0, \quad [L, P_x] = iP_y, \quad [L, P_y] = -iP_x. \quad (9)$$

This is a closed set of commutation relations for the generators of the $E(2)$ group. If we replace $N_1$ and $N_2$ of Eq.(8) by $P_x$ and $P_y$, and $J_3$ by $L$, the commutations relations for the generators of the $E(2)$-like little group becomes those for the $E(2)$-like little group. This is precisely why we say that the little group for massless particles are like $E(2)$.

It is not difficult to associate the rotation generator $J_3$ with the helicity degree of freedom of the massless particle. Then what physical variable is associated with the $N_1$ and $N_2$ generators? Indeed, Wigner was the one who discovered the existence of these generators, but did not give any physical interpretation to these translation-like generators in his original paper [7]. For this reason, for many years, only those representations with the zero-eigenvalues of the $N$ operators were thought to be physically meaningful representations [17]. It was not until 1971 when Janner and Janssen reported that the transformations generated by these operators are gauge transformations [18, 19]. The role of this translation-like transformation has also been studied for spin-1/2 particles, and it was concluded that the polarization of neutrinos is due to gauge invariance [20, 21].
The $O(3)$-like little group remains $O(3)$-like when the particle is Lorentz-boosted. Then, what happens when the particle speed becomes the speed of light? The energy-momentum relation $E^2 = m_0^2c^4 + p^2c^2$ become $E = pc$. Is there then a limiting case of the $O(3)$-like little group? Since those little groups are like the three-dimensional rotation group and the two-dimensional Euclidean group respectively, we are first interested in whether $E(2)$ can be obtained from $O(3)$. This will then give a clue to obtain the $E(2)$-like little group as a limiting case of $O(3)$-like little group. With this point in mind, let us look into this geometrical problem.

In 1953, Inonu and Wigner formulated this problem as the contraction of $O(3)$ to $E(2)$ [22]. Let us see what they did. We always associate the three-dimensional rotation group with a spherical surface. Let us consider a circular area of radius 1 kilometer centered on the north pole of the earth. Since the radius of the earth is more than 6,450 times longer, the circular region appears flat. Thus, within this region, we use the $E(2)$ symmetry group. The validity of this approximation depends on the ratio of the two radii.

How about then the little groups which are isomorphic to $O(3)$ and $E(2)$? It is reasonable to expect that the $E(2)$-like little group can be obtained as a limiting case for of the $O(3)$-like little group for massless particles. In 1981, it was observed by Bacry and Chang [23] and by Ferrara and Savoy [24] that this limiting process is the Lorentz boost to infinite-momentum frame.

In 1983, it was noted by Han et al that the large-radius limit in the the contraction of $O(3)$ to $E(2)$ corresponds to the infinite-momentum limit for the case of $O(3)$-like little group to $E(2)$-like little group. They showed that transverse rotation generators become the generators of gauge transformations in the limit of infinite momentum [25].

Let us see how this happens. The $J_3$ operator of Eq.(2), which generates rotations around the $z$ axis, is not affected by the boost conjugation of the $B$ matrix of Eq.(4). On the other hand, the $J_1$ and $J_2$ matrices become

\[ N_1 = \lim_{\eta \to \infty} e^{-\eta B^{-1}} J_2 B, \quad N_2 = \lim_{\eta \to \infty} -e^{-\eta B^{-1}} J_1 B, \]

and they become $N_1$ and $N_2$ given in Eq.(5). The generators $N_1$ and $N_2$ are the contracted $J_2$ and $J_1$ respectively in the infinite-momentum. In 1987, Kim and Wigner studied this problem in more detail and showed that the little group for massless particles is the cylindrical group which is isomorphic to the $E(2)$ group [26].

This completes the second row in Table I where Wigner’s little group unifies the internal space-time symmetries of massive and massless particles. The transverse components of the rotation generators become generators of gauge transformations in the infinite-momentum limit.

Let us go back to Table I given in Sec. I. As for the third row for relativistic extended particles, the most efficient approach is to construct representations of the little groups using the wave functions which can be Lorentz-boosted. This means that we have to construct wave functions which are consistent with all known rules of quantum mechanics and special relativity. It is possible
to construct harmonic oscillator wave functions which satisfy these conditions. We can then take the low-speed and high-speed limits of the covariant harmonic oscillator wave functions for the quark model and the parton model respectively. This aspect was extensively discussed in the literature [16], and is beyond the scope of the present report.

3 Formulation of the Problem

Let us consider two optical beams propagating along the $z$ axis. We are then led to the column vector:

$$
\begin{pmatrix}
A_1 
\exp \left( -i \left( kz - \omega t + \phi_1 \right) \right) \\
A_2 
\exp \left( -i \left( kz - \omega t + \phi_2 \right) \right)
\end{pmatrix}.
$$

(11)

We can then achieve a phase shift between the beams by applying the two-by-two matrix:

$$
\begin{pmatrix}
e^{i\phi/2} & 0 \\
0 & e^{-i\phi/2}
\end{pmatrix}.
$$

(12)

If we are interested in mixing up the two beams, we can apply

$$
\begin{pmatrix}
\cos(\theta/2) & -\sin(\theta/2) \\
\sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}
$$

(13)

to the column vector.

If the amplitudes become changed by either by attenuation or reflection, we can use the matrix

$$
\begin{pmatrix}
e^{\eta/2} & 0 \\
0 & e^{-\eta/2}
\end{pmatrix}
$$

(14)

for the change. In this paper, we are dealing only with the relative amplitudes, or the ratio of the amplitudes.

Repeated applications of these matrices lead to the form

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix},
$$

(15)

where the elements are in general complex numbers. The determinant of this matrix is one. Thus, the matrix can have six independent parameters.

Indeed, this matrix is the most general form of the matrices in the $SL(2, c)$ group, which is known to be the universal covering group for the six-parameter Lorentz group. This means that, to each two-by-two matrix of $SL(2, c)$, there corresponds one four-by-four matrix of the group of Lorentz transformations applicable to the four-dimensional Minkowski space [16]. It is possible to construct explicitly the four-by-four Lorentz transformation matrix from the parameters $\alpha, \beta, \gamma,$ and $\delta$. This expression is available in the literature [16], and we consider here only special cases.
As is shown in Appendix A, the Lorentz group includes Lorentz boosts along three different directions and rotations around those three different directions. A rotation around the \( z \) axis in the Minowskian space can be written as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \phi & -\sin \phi \\
0 & 0 & \sin \phi & \cos \phi
\end{pmatrix}.
\] (16)

This four-by-four matrix corresponds to the two-by-two matrix of Eq. (13). We use in this paper the four-vector convention \((t, z, x, y)\).

The rotation around the \( y \) axis can be written as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (17)

The two-by-two matrix of Eq. (14) corresponds to the Lorentz boost matrix

\[
\begin{pmatrix}
\cosh \eta & \sinh \eta & 0 & 0 \\
\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (18)

The four-by-four representation of the Lorentz group can be constructed from the two-by-two representation of the \( SL(2, c) \) group, which is known as the universal covering group of the six-parameter Lorentz group. This aspect is discussed in Appendix B.

Let us go back to Eq. (15), the \( SL(2, c) \) group represented by this matrix has many interesting subgroups. If the matrices are to be Hermitian, then the subgroup is \( SU(2) \) corresponding three-dimensional rotation group. If all the elements are real numbers, the group becomes the three-parameter \( Sp(2) \) group. This subgroup is equivalent to \( SU(1, 1) \) which is the primary scientific language for squeezed states of light [27, 3].

We can also consider the matrix of Eq. (15) when one of its off-diagonal elements vanishes. Then, it takes the form

\[
\begin{pmatrix}
\exp (i\phi/2) & 0 \\
\gamma & \exp (-i\phi/2)
\end{pmatrix},
\] (19)

where \( \gamma \) is a complex number with two real parameters. In 1939 [7], Wigner observed this form as one of the subgroups of the Lorentz group. He observed further that this group is isomorphic to the two-dimensional Euclidean group, and that its four-by-four equivalent can explain the internal space-time symmetries of massless particles including the photons.

In ray optics, we often have to deal with this type of triangular matrices, particularly in lens optics and stability problems in laser and multilayer optics. In the language of mathematics, dealing with this form is called the Iwasawa decomposition [28]. This aspect of the Lorentz group is discussed in Appendix C.
4 Polarization Optics and Interferometers

In polarization physics, we are studying optical rays with two independent components. Thus the two-by-two matrix given in Sec 1 is directly applicable. We are quite familiar with Pauli spinors and Pauli matrices with three independent parameters. They deal with rotations. If we add Lorentz boosts, the group becomes the six-parameter $SL(2,c)$ group. The representation is still two-by-two.

From the group theoretical point of view, the Jones matrices constitute the $SL(2,c)$ group or the universal covering group for the Lorentz group. The Jones vectors and Jones matrices are nothing more and less than the $SL(2,C)$ representation and the $SL(2)$ spinors.

There is a specific procedure to construct the Minkowskian four-vectors and the four-by-four Lorentz transformation matrices. Then they become Stokes parameters and the Mueller matrices.

If the Jones matrix contains all the parameters for the polarized light beam, why do we need the mathematics in the four-dimensional space? The answer to this question is well known. In addition to the basic parameter given by the Jones vector, the Stokes parameters give the degree of coherence between the two rays.

Let us write Eq. (11) as a Jones spinor of the form

$$
\left( \psi_1(z,t) \right)_1 \psi_2(z,t),
$$

and construct the quantities:

$$S_{11} = <\psi_1^* \psi_1>, \quad S_{12} = <\psi_1^* \psi_2>,$n
$$S_{21} = <\psi_2^* \psi_1>, \quad S_{22} = <\psi_2^* \psi_2>. \quad (21)$$

Then the Stokes vector consists of

$$S_0 = S_{11} + S_{22}, \quad S_1 = S_{11} - S_{22},$$
$$S_2 = S_{12} + S_{21}, \quad S_3 = -i(S_{11} + S_{22}). \quad (22)$$

The four-component vector $(S_0, S_1, S_2, S_3)$ transforms like the space-time four-vector $(t, z, x, y)$ under Lorentz transformations. The Mueller matrix is therefore like the Lorentz-transformation matrix.

As in the case of special relativity, let us consider the quantity

$$M^2 = S_0^2 - S_1^2 - S_2^2 - S_3^2. \quad (23)$$

Then $M$ is like the mass of the particle while the Stokes four-vector is like the four-momentum.

If $M = 0$, the two-beams are in a purely state. As $M$ increases, the system becomes mixed, and the entropy increases. If it reaches the value of $S_0$, the system becomes completely random. It is gratifying to note that this mechanism can be formulated in terms of the four-momentum in particle physics [9].
5 One-lens System

In analyzing optical rays in para-axial lens optics, we start with the lens matrix:

\[
L = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix},
\]

and the translation matrix

\[
T = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},
\]

assuming that the beam is propagating along the Z direction.

Then the one-lens system consists of

\[
\begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}.
\]

If we perform the matrix multiplication,

\[
\begin{pmatrix} 1 - z_2/f & z_1 + z_2 - z_1 z_2/f \\ -1/f & 1 - z_1/f \end{pmatrix}.
\]

If we assert that the upper-right element be zero, then

\[
\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f},
\]

and the image is focussed, where \(z_1\) and \(z_2\) are the distance between the lens and object and between the lens and image respectively. They are in general different, but we shall assume for simplicity that they are the same: \(z_1 = z_2 = z\). We are doing this because this simplicity does not destroy the main point of our discussion, and because the case with two different values has been dealt with in the literature [15]. Under this assumption, we are left with

\[
\begin{pmatrix} 1 - z/f & 2z - z^2/f \\ -1/f & 1 - z/f \end{pmatrix}.
\]

The diagonal elements of this matrix are dimensionless. In order to make the off-diagonal elements dimensionless, we write this matrix as

\[
-\begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1/\sqrt{z} \end{pmatrix}\begin{pmatrix} 1 - z/f & z/f - 2 \\ z/f & 1 - z/f \end{pmatrix}\begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1/\sqrt{z} \end{pmatrix}.
\]

Indeed, the matrix in the middle contains dimensionless elements. The negative sign in front is purely for convenience. We are then led to study the core matrix

\[
C = \begin{pmatrix} x - 1 & x - 2 \\ x & x - 1 \end{pmatrix}.
\]
Here, the important point is that the above matrices can be written in terms of transformations in the Lorentz group. In the two-by-two matrix representation, the Lorentz boost along the $z$ direction takes the form

$$B(\eta) = \begin{pmatrix} \exp (\eta/2) & 0 \\ 0 & \exp (-\eta/2) \end{pmatrix},$$

(32)

and the rotation around the $y$ axis can be written as

$$R(\theta) = \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix},$$

(33)

and the boost along the $x$ axis takes the form

$$X(\chi) = \begin{pmatrix} \cosh(\chi/2) & \sinh(\chi/2) \\ \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}.$$  

(34)

Then the core matrix of Eq.(31) can be written as

$$B(\eta)R(\theta)B(-\eta),$$

(35)

or

$$\begin{pmatrix} \cos(\phi/2) & -e^{-\eta}\sin(\phi/2) \\ e^{\eta}\sin(\phi/2) & \cos(\phi/2) \end{pmatrix},$$

(36)

if $1 < x < 2$. If $x$ is greater than 2, the upper-right element of the core is positive and it can take the form

$$B(\eta)X(\chi)B(-\eta),$$

(37)

or

$$\begin{pmatrix} \cosh(\chi/2) & e^{-\eta}\sinh(\chi/2) \\ e^{\eta}\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}.$$  

(38)

The expressions of Eq.(35) and Eq.(37) are a Lorentz boosted rotation and a Lorentz-boosted boost matrix along the $x$ direction respectively. These expressions play the key role in understanding Wigner’s little groups for relativistic particles.

Let us look at their explicit matrix representations given in Eq.(36) and Eq.(38). The transition from Eq.(36) to Eq.(38) requires the upper right element going through zero. This can only be achieved through $\eta$ going to infinity. If we like to keep the lower-left element finite during this process, the angle $\phi$ and the boost parameter $\chi$ have to approach zero. The process of approaching the vanishing upper-right element is necessarily a singular transformation. This aspect plays the key role in unifying the internal space-time symmetries of massive and massless particles. This is like Einstein’s $E = \sqrt{(pc)^2 + m_0^2c^4}$ becoming $E = pc$ in the limit of large momentum.

On the other hand, the core matrix of Eq.(31) is an analytic function of the variable $x$. Thus, the lens matrix allows a parametrization which allows the transition from massive particle to massless particle analytically. The lens
optics indeed serves as the analogue computer for this important transition in particle physics.

From the mathematical point of view, Eq. (36) and Eq. (38) represent circular and hyperbolic geometries, respectively. The transition from one to the other is not a trivial mathematical procedure. It requires a further investigation.

Let us go back to the core matrix of Eq. (31). The \( x \) parameter does not appear to be a parameter of Lorentz transformations. However, the matrix can be written in terms of another set of Lorentz transformations. This aspect has been discussed in the literature [11].

6 Multi-lens Problem

Let us consider a co-axial system of an arbitrary number of lens. Their focal lengths are not necessarily the same, nor are their separations. We are then led to consider an arbitrary number of the lens matrix given in Eq. (24) and an arbitrary number of translation matrix of Eq. (25). They are multiplied like

\[
T_1 L_1 T_2 L_2 T_3 L_3 \ldots \ldots T_N L_N, \tag{39}
\]

where \( N \) is the number of lenses.

The easiest way to tackle this problem in to use the Lie-algebra approach. Let us start with the generators of the Sp(2) group:

\[
B_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad J = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{40}
\]

Since the generators are pure imaginary, the transformation matrices are real.

On the other hand, the \( L \) and \( T \) matrices of Eq. (24) and Eq. (25) are generated by

\[
X_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}. \tag{41}
\]

If we introduce the third matrix

\[
X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{42}
\]

all three matrices form a closed set of commutation relations:

\[
[X_1, X_2] = iX_3, \quad [X_1, X_3] = -iX_1, \quad [X_2, X_3] = iX_2. \tag{43}
\]

Thus, these generators also form a closed set of Lie algebra generating real two-by-two matrices. What group would this generate? The answer has to be \( Sp(2) \). The truth is that the three generators given in Eq. (43) can be written as linear combinations of the generators of the \( Sp(2) \) group given in Eq. (40) [11]. Thus, the \( X_i \) matrices given above can also act as the generators of the \( Sp(2) \) group, and the lens-system matrix given in Eq. (39) is a three-parameter matrix of the form of Eq. (18) with real elements.
The resulting real matrix is written as
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]
and is called the \textit{ABCD} matrix. According to Bargamann \cite{30}, this three-parameter matrix can be decomposed into
\[
\begin{pmatrix}
\cos(\alpha/2) & -\sin(\alpha/2) \\
\sin(\alpha/2) & \cos(\alpha/2)
\end{pmatrix}
\begin{pmatrix}
e^{\gamma/2} & 0 \\
0 & e^{-\gamma/2}
\end{pmatrix}
\begin{pmatrix}
\cos(\beta/2) & -\sin(\beta/2) \\
\sin(\beta/2) & \cos(\beta/2)
\end{pmatrix},
\]
which can be written as the product of one symmetric matrix resulting from
\[
\begin{pmatrix}
\cos(\alpha/2) & -\sin(\alpha/2) \\
\sin(\alpha/2) & \cos(\alpha/2)
\end{pmatrix}
\begin{pmatrix}
e^{\gamma/2} & 0 \\
0 & e^{-\gamma/2}
\end{pmatrix}
\begin{pmatrix}
\cos(\alpha/2) & -\sin(\alpha/2) \\
\sin(\alpha/2) & \cos(\alpha/2)
\end{pmatrix},
\]
and one rotation matrix:
\[
\begin{pmatrix}
\cos[(\beta - \alpha)/2] & -\sin[(\beta - \alpha)/2] \\
\sin[(\beta - \alpha)/2] & \cos[(\beta - \alpha)/2]
\end{pmatrix}.
\]

We can then decompose each of these two matrices into the lens and translation matrices. The net result is that we do not need more than three lenses to describe the lens system consisting of an arbitrary number of lenses. The detailed calculations are given in Ref. \cite{10}.

7 Laser Cavities

In a laser cavity, the optical ray makes round trips between two mirrors. One cycle is therefore equivalent to a two-lens system with two identical lenses and the same distance between the lenses. Let us rewrite the matrix corresponding to the one-lens system given in Eq.(31).
\[
C = \begin{pmatrix} x - 1 & x - 2 \\ x & x - 1 \end{pmatrix}.
\]

Then one complete cycle consists of \(C^2\). For \(N\) cycles, the expression should be \(C^{2N}\). However this calculation, using the above expression, will not lead to a manageable form. However, we can resort to the expressions of Eq.(35) and Eq.(37). Then one cycle consists of
\[
C^2 = [B(\eta)R(\phi/2)B(-\eta)][B(\eta)R(\phi/2)B(-\eta)] = B(\eta)R(\phi)B(-\eta),
\]
if the upper-right element is negative. If it is positive, the expression should be
\[
C^2 = [B(\eta)X(\chi/2)B(-\eta)][B(\eta)X(\chi/2)B(-\eta)] = B(\eta)X(\chi)B(-\eta).
\]
If these expressions are repeated $N$ times,
\[ C^{2N} = B(\eta)R(N\phi)B(-\eta), \] (51)
if the upper-right element is negative. It it is positive, the expression should be
\[ C^{2N} = B(\eta)X(N\chi)B(-\eta). \] (52)
As $N$ becomes large, $\cosh(N\chi)$ and $\sinh(N\chi)$ become very large, the beam deviates from the laser cavity. Thus, we have to restrict ourselves to the case given in Eq. (51).

The core of the expression of Eq.(51) is the rotation matrix
\[ R(N\phi) = [R(\phi)]^N. \] (53)
This means that one complete cycle in the cavity corresponds to the rotation matrix $R(\phi)$. The rotation continues as the beam continues to repeat the cycle.

Let us go back to Eq.(49). The expression corresponds to a Lorentz boosted rotation, or bringing a moving particle to its rest frame, rotate, and boost back to the original momentum. The rotation associated with the momentum-preserving transformation is called the Wigner’s little-group rotation, which is related to the Wigner rotation commonly mentioned in the literature [10].

## 8 Multilayer Optics

The most efficient way to study multilayer optics is to use the S-matrix formalism [13]. We consider a system of two different optical layers. For convenience, we start from the boundary from medium 2 to medium 1. We can write the boundary matrix as
\[ B(\eta) = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix}, \] (54)
taking into account both the transmission and reflection of the beam. As the beam goes through the medium 1, the beam undergoes the phase shift represented by the matrix
\[ P(\phi_1) = \begin{pmatrix} e^{-i\phi_1/2} & 0 \\ 0 & e^{i\phi_1/2} \end{pmatrix}. \] (55)
When the wave hits the surface of the second medium, the boundary matrix is
\[ B(-\eta) = \begin{pmatrix} \cosh(\eta/2) & -\sinh(\eta/2) \\ -\sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix}, \] (56)
which is the inverse of the matrix given in Eq.(54). Within the second medium, we write the phase-shift matrix as
\[ P(\phi_2) = \begin{pmatrix} e^{-i\phi_2/2} & 0 \\ 0 & e^{i\phi_2/2} \end{pmatrix}. \] (57)
Then, when the wave hits the first medium from the second, we have to go back to Eq. (54). Thus, the one cycle consists of

\[
\begin{pmatrix}
\cosh(\eta/2) & \sinh(\eta/2) \\
\sinh(\eta/2) & \cosh(\eta/2)
\end{pmatrix}
\begin{pmatrix}
ev^{-i\phi_1/2} & 0 \\
0 & ev^{i\phi_1/2}
\end{pmatrix}
\times
\begin{pmatrix}
\cosh(\eta/2) & -\sinh(\eta/2) \\
-\sinh(\eta/2) & \cosh(\eta/2)
\end{pmatrix}
\begin{pmatrix}
ev^{-i\phi_2/2} & 0 \\
0 & ev^{i\phi_2/2}
\end{pmatrix}.
\] (58)

The above matrices contain complex numbers. However, it is possible to transform simultaneously

\[
\begin{pmatrix}
\cosh(\eta/2) & \sinh(\eta/2) \\
\sinh(\eta/2) & \cosh(\eta/2)
\end{pmatrix}
\] (59)
to

\[
\begin{pmatrix}
\exp(\eta/2) & 0 \\
0 & \exp(-\eta/2)
\end{pmatrix},
\] (60)
and transform

\[
\begin{pmatrix}
ev^{-i\phi_1/2} & 0 \\
0 & ev^{i\phi_1/2}
\end{pmatrix}
\] (61)
to

\[
\begin{pmatrix}
\cos(\phi_1/2) & -\sin(\phi_1/2) \\
\sin(\phi_1/2) & \cos(\phi_1/2)
\end{pmatrix},
\] (62)
using a conjugate transformation. It is also possible to transform these expressions back to their original forms. This transformation property has been discussed in detail in Ref. [14].

As a consequence, the matrix of Eq. (58) becomes

\[
\begin{pmatrix}
ev^{\eta/2} & 0 \\
0 & ev^{-\eta/2}
\end{pmatrix}
\begin{pmatrix}
\cos(\phi_1/2) & -\sin(\phi_1/2) \\
\sin(\phi_1/2) & \cos(\phi_1/2)
\end{pmatrix}
\times
\begin{pmatrix}
ev^{-\eta/2} & 0 \\
0 & ev^{\eta/2}
\end{pmatrix}
\begin{pmatrix}
\cos(\phi_2/2) & -\sin(\phi_2/2) \\
\sin(\phi_2/2) & \cos(\phi_2/2)
\end{pmatrix}.
\] (63)

In the above expression, the first three matrices are of the same mathematical form as that of core matrix for the one-lens system given in Eq. (56). The fourth matrix is an additional rotation matrix. This makes the mathematics of repetition more complicated, but this has been done [15].

As a consequence the net result becomes

\[B(\mu)R(N\alpha)B(-\mu),\] (64)
or

\[B(\mu)X(N\xi)B(-\mu),\] (65)
where the parameters \(\mu, \alpha\) and \(\xi\) are to be determined from the input parameters \(\eta, \phi_1\) and \(\phi_2\). Detailed calculations are given in Ref. [15].

It is interesting to note that the Lorentz group can serve as a computational device also in multilayer optics.
9 Concluding Remarks

We have seen in this report that the Lorentz group provides convenient calculational tools in many branches of ray optics. The reason is that ray optics is largely based on two-by-two matrices. These matrices also constitute the group $SL(2, c)$ which serves as the universal covering group of the Lorentz group.

The optical instruments discussed in this report are the fundamental components in optical circuits. In the world of electronics, electric circuits form the fabric of the system. In the future high-technology world, optical components will hold the key to technological advances. Indeed, the Lorentz group is the fundamental language for the new world.

It is by now well known that the Lorentz group is the basic language for quantum optics. Coherent and squeezed states are representations of the Lorentz group. It is challenging to see how the Lorentz nature of the above-mentioned optical components will manifest itself in quantum world.

The Lorentz group was introduced to physics by Einstein and Wigner to understand the space-time symmetries of relativistic particles and the covariant world of electromagnetic fields. It is gratifying to note that the Lorentz group can serve as the language common both to particle physics and optical sciences.

Appendix

A Lorentz Transformations

Let us consider the space-time coordinates $(t, z, x, y)$. Then the rotation around the $z$ axis is performed by the four-by-four matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \cos \theta & -\sin \theta \\
0 & 0 & \sin \theta & \cos \theta \\
\end{pmatrix}.
$$

This transformation is generated by

$$
J_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
\end{pmatrix}.
$$

Likewise, we can write down the generators of rotations $J_1$ and $J_2$ around the $x$ and $y$ axes respectively.

$$
J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
$$
These three generators satisfy the closed set of commutation relations

\[ [J_i, J_j] = i\epsilon_{ijk}J_k. \] (69)

This set of commutation relations is for the three-dimensional rotation group.

The Lorentz boost along the \( z \) axis takes the form

\[
\begin{pmatrix}
\cosh \eta & \sinh \eta & 0 & 0 \\
\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\] (70)

which is generated by

\[ K_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (71)

Likewise, we can write generators of boosts \( K_1 \) and \( K_2 \) along the \( x \) and \( y \) axes respectively, and they take the form

\[ K_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \] (72)

These boost generators satisfy the commutation relations

\[ [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k. \] (73)

Indeed, the three rotation generators and the three boost generators satisfy the closed set of commutation relations given in Eq.(69) and Eq.(73). These three commutation relations form the starting point of the Lorentz group. The generators given in this Appendix are four-by-four matrices, but they are not the only set satisfying the commutation relations. We can construct also six two-by-two matrices satisfying the same set of commutation relations. The group of transformations constructed from these two-by-matrices is often called \( SL(2,c) \) or the two-dimensional representation of the Lorentz group. Throughout the present paper, we used the two-by-two transformation matrices constructed from the generators of the \( SL(2,c) \) group.

### B Spinors and Four-vectors in the Lorentz Group

In Appendix A, we have noted that there are four-by-four and two-by-two representations of the Lorentz group. The four-by-four representation is applicable to covariant four-vectors, while the two-by-two transformation matrices are applicable to two-component spinors which in the present case are Jones vectors. The question then is whether we can construct the four-vector from the spinors.
In the language of polarization optics, the question is whether it is possible to construct the coherency matrix \[38, 39\] from the Jones vector.

With this point in mind, let us start from the following form of the Pauli spin matrices.

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{74}
\]

These matrices are written in a different convention. Here \(\sigma_3\) is imaginary, while \(\sigma_2\) is imaginary in the traditional notation. Also in this convention, we can construct three rotation generators

\[
J_i = \frac{1}{2} \sigma_i, \tag{75}
\]

which satisfy the closed set of commutation relations

\[
[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{76}
\]

We can also construct three boost generators

\[
K_i = \frac{i}{2} \sigma_i, \tag{77}
\]

which satisfy the commutation relations

\[
[K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{78}
\]

The \(K_i\) matrices alone do not form a closed set of commutation relations, and the rotation generators \(J_i\) are needed to form a closed set:

\[
[J_i, K_j] = i\epsilon_{ijk} K_k. \tag{79}
\]

The six matrices \(J_i\) and \(K_i\) form a closed set of commutation relations, and they are like the generators of the Lorentz group applicable to the (3 + 1)-dimensional Minkowski space. The group generated by the above six matrices is called \(SL(2, c)\) consisting of all two-by-two complex matrices with unit determinant.

In order to construct four-vectors, we need two different spinor representations of the Lorentz group. Let us go to the commutation relations for the generators given in Eqs.\((6), \(85\) and \(79\). These commutators are not invariant under the sign change of the rotation generators \(J_i\), but are invariant under the sign change of the squeeze operators \(K_i\). Thus, to each spinor representation, there is another representation with the squeeze generators with opposite sign. This allows us to construct another representation with the generators:

\[
\dot{J}_i = \frac{1}{2} \sigma_i, \quad \dot{K}_i = -\frac{i}{2} \sigma_i. \tag{80}
\]
We call this representation the “dotted” representation. If we write the transformation matrix \( L \) of Eq. (15) in terms of the generators as

\[
L = \exp \left\{ -\frac{i}{2} \sum_{i=1}^{3} (\theta_i \sigma_i + i\eta_i \sigma_i) \right\},
\]

then the transformation matrix in the dotted representation becomes

\[
\dot{L} = \exp \left\{ -\frac{i}{2} \sum_{i=1}^{3} (\theta_i \sigma_i - i\eta_i \sigma_i) \right\}.
\]

In both of the above matrices, Hermitian conjugation changes the direction of rotation. However, it does not change the direction of boosts. We can achieve this only by interchanging \( L \) to \( \dot{L} \), and we shall call this the “dot” conjugation.

Likewise, there are two different set of spinors. Let us use \( u \) and \( v \) for the up and down spinors for “undotted” representation. Then \( \dot{u} \) and \( \dot{v} \) are for the dotted representation. The four-vectors are then constructed as

\[
\begin{align*}
    u\dot{u} &= -(x - iy), \quad v\dot{v} = (x + iy), \\
    u\dot{v} &= (t + z), \quad v\dot{u} = -(t - z)
\end{align*}
\]

leading to the matrix

\[
C = \begin{pmatrix}
    u\dot{v} & -u\dot{u} \\
    v\dot{v} & -v\dot{u}
\end{pmatrix} = \begin{pmatrix}
    u \\
    v
\end{pmatrix} \begin{pmatrix}
    \dot{v} & -\dot{u}
\end{pmatrix},
\]

where \( u \) and \( \dot{u} \) are one if the spin is up, and are zero if the spin is down, while \( v \) and \( \dot{v} \) are zero and one for the spin-up and spin-down cases. The transformation matrix applicable to the column vector in the above expression is the two-by-two matrix given in Eq. (15). What is then the transformation matrix applicable to the row vector \((\dot{v}, -\dot{u})\) from the right-hand side? It is the transpose of the matrix applicable to the column vector \((\dot{v}, -\dot{u})\). We can obtain this column vector from

\[
\begin{pmatrix}
    \dot{v} \\
    -\dot{u}
\end{pmatrix},
\]

by applying to it the matrix

\[
g = -i\sigma_3 = \begin{pmatrix}
    0 & -1 \\
    1 & 0
\end{pmatrix}.
\]

This matrix also has the property

\[
g\sigma_i g^{-1} = -(\sigma_i)^T,
\]

where the superscript \( T \) means the transpose of the matrix. The transformation matrix applicable to the column vector of Eq. (85) is \( \dot{L} \) of Eq. (82). Thus the matrix applicable to the row vector \((\dot{v}, -\dot{u})\) in Eq. (84) is

\[
\left\{ g^{-1} \dot{L} g \right\}^T = g^{-1} \dot{L}^T g.
\]
This is precisely the Hermitian conjugate of \( L \).

In optics, this two-by-two matrix form appears as the coherency matrix, and it takes the form

\[ C = \begin{pmatrix} <E_x^* E_x> & <E_x^* E_y> \\ <E_y^* E_x> & <E_y^* E_y> \end{pmatrix}, \quad (89) \]

where \(<E_i^* E_j>\) is the time average of \( E_i^* E_j \). This matrix is convenient when we deal with light waves whose two transverse components are only partially coherent. In terms of the complex parameter \( w \), the coherency matrix is proportional to

\[ C = \begin{pmatrix} 1 & re^{-i\delta} \\ re^{i\delta} & r^2 \end{pmatrix}, \quad (90) \]

if the \( x \) and \( y \) components are perfectly coherent with the phase difference of \( \delta \). If they are totally incoherent, the off-diagonal elements vanish in the above matrix.

Let us now consider its transformation properties. As was noted by Opatrny and Perina [37], the matrix of Eq. (89) is like

\[ C = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}, \quad (91) \]

where the set of variables \((x, y, z, t)\) is transformed like a four-vector under Lorentz transformations. Furthermore, it is known that the Lorentz transformation of this four-vector is achieved through the formula

\[ C' = LCL^\dagger, \quad (92) \]

where the transformation matrix \( L \) is that of Eq. (15). The construction of four-vectors from the two-component spinors is not a trivial task [38, 39]. The two-by-two representation of Eq. (91) requires one more step of complication.

### C Conjugate Transformations

The core matrix of Eq. (51) contains the chain of the matrices

\[ W = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} e^{-i\xi} & 0 \\ 0 & e^{i\xi} \end{pmatrix}. \quad (93) \]

The Lorentz group allows us to simplify this expression under certain conditions.

For this purpose, we transform the above expression into a more convenient form, by taking the conjugate of each of the matrices with

\[ C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (94) \]

Then \( C_1 WC_1^{-1} \) leads to

\[ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}. \quad (95) \]
In this way, we have converted $W$ of Eq. (93) into a real matrix, but it is not simple enough.

Let us take another conjugate with
\[
C_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]
(96)

Then the conjugate $C_2 C_1 W C_1^{-1} C_2^{-1}$ becomes
\[
\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^n & 0 \\ 0 & e^{-n} \end{pmatrix} \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}.
\]
(97)
The combined effect of $C_2 C_1$ is
\[
C = C_2 C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ -e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix},
\]
(98)
with
\[
C^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/4} & -e^{i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix}.
\]
(99)

After multiplication, the matrix of Eq. (97) will take the form
\[
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
(100)
where $A, B, C,$ and $D$ are real numbers. If $B$ and $C$ vanish, this matrix will become diagonal, and the problem will become too simple. If, on the other hand, only one of these two elements become zero, we will achieve a substantial mathematical simplification and will be encouraged to look for physical circumstances which will lead to this simplification.

Let us summarize. We started in this section with the matrix representation $W$ given in Eq. (93). This form can be transformed into the $V$ matrix of Eq. (97) through the conjugate transformation
\[
V = C W C^{-1},
\]
(101)
where $C$ is given in Eq. (98). Conversely, we can recover the $W$ representation by
\[
W = C^{-1} V C.
\]
(102)
For calculational purposes, the $V$ representation is much easier because we are dealing with real numbers. On the other hand, the $W$ representation is of the form for the S-matrix we intend to compute. It is gratifying to see that they are equivalent.

Let us go back to Eq. (97) and consider the case where the angles $\phi$ and $\xi$ satisfy the following constraints.
\[
\phi + \xi = 2\theta, \quad \phi - \xi = \pi/2,
\]
(103)
thus
\[ \phi = \theta + \pi/4, \quad \xi = \theta - \pi/4. \] (104)

Then in terms of \( \theta \), we can reduce the matrix of Eq. (97) to the form
\[
\begin{pmatrix}
(cosh \eta) \cos(2\theta) & \sinh \eta - (cosh \eta) \sin(2\theta) \\
\sinh \eta + (cosh \eta) \sin(2\theta) & (cosh \eta) \cos(2\theta)
\end{pmatrix}.
\] (105)

Thus the matrix takes a surprisingly simple form if the parameters \( \theta \) and \( \eta \) satisfy the constraint
\[ \sinh \eta = (cosh \eta) \sin(2\theta). \] (106)

Then the matrix becomes
\[
\begin{pmatrix}
1 & 0 \\
2 \sinh \eta & 1
\end{pmatrix}.
\] (107)

This aspect of the Lorentz group is known as the Iwasawa decomposition [28], and has been discussed in the optics literature [34, 9].

The matrices of the form is not so strange in optics. In para-axial lens optics, the translation and lens matrices are written as
\[
\begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
u & 1
\end{pmatrix},
\] (108)

respectively. These matrices have the following interesting mathematical property [3],
\[
\begin{pmatrix}
1 & u_1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & u_2 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & u_1 + u_2 \\
0 & 1
\end{pmatrix},
\] (109)

and
\[
\begin{pmatrix}
1 & 0 \\
u_1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
u & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
u_1 + u_2 & 1
\end{pmatrix}.
\] (110)

We note that the multiplication is commutative, and the parameter becomes additive. These matrices convert multiplication into addition, as logarithmic functions do.

## D Euler versus Lie Representations

In this paper, we restricted ourselves to the algebra of two-by-two matrices and avoided as much as possible group theoretical languages. In order to explain where those algebraic tricks came from, we give in this Appendix A group theoretical interpretation of what we did in this paper.

The group \( SL(2, c) \) consists of two-by-two unimodular matrices whose elements are complex. There are therefore six independent parameters, and thus six generators of the Lie algebra. This group is locally isomorphic to the six-parameter Lorentz group or \( O(3, 1) \) applicable to the Minkowskian space of three space-like directions and one time-like direction.

Like the Lorentz group, the \( SL(2, c) \) has a number of interesting subgroups. The subgroup most familiar to us is \( SU(2) \) which is locally isomorphic to
the three-dimensional rotation group. In addition, this group contains three subgroups which are locally isomorphic to the group $O(2,1)$ applicable to the Minkowskian space of two space-like and one time-like dimensions.

One of the subgroups of $SL(2, c)$ is $SL(2, r)$ consisting of matrices with real elements. This subgroup is also called the $Sp(2)$ group which we used in this paper in order to carry out the Iwasawa decomposition. Another interesting subgroup is the one we used for computing the $S$ matrix, which starts with the boundary matrix of Eq. (54) and the phase-shift matrix of Eq. (55). This group is called $SU(1,1)$. The present paper exploits the isomorphism between $Sp(2)$ and $SU(1,1)$. While the physical world is describable in terms of $SU(1,1)$, we carry out the Iwasawa decomposition in the $Sp(2)$ regime.

Indeed, the conjugate transformation of Eq. (93) to Eq. (95) is from $SU(1,1)$ to $Sp(2)$, while the transition from Eq. (96) to Eq. (97) is within the $Sp(2)$ group. Thus, the transition from Eq. (93) to Eq. (97) is a conjugate transformation from the $SU(1,1)$ subgroup to the subgroup $Sp(2)$ of $SL(2, r)$.

In this paper, we are concerned with the decomposition of the $Sp(2)$ and $SU(1,1)$ matrices. Unlike the traditional approach to group theory which starts from the generators of the Lie algebra, we used in this paper an approach similar to what Goldstein did for the three-dimensional rotation group in terms of the Euler angles [29]. There are three-generators for the rotation group, but Goldstein starts with rotations around the $z$ and $x$ directions. Rotations around the $y$ axis and the most general form for the rotation matrix can be constructed from repeated applications of those two starting matrices. Let us call this type of approach the “Euler construction.”

There are three basic advantages of this approach. First, the number of “starter” matrices is less than the number of generators. For example, we need only two starters for the three-parameter rotation group. In our case, we started with two matrices for the three-parameter group $Sp(2)$ and also for $SU(1,1)$. Second, each starter matrix takes a simple form and has its own physical interpretation.

The third advantage can be stated in the following way. Repeated applications of the starter matrices will lead to a very complicated expression. However, the complicated expression can decomposed into the minimum number of starter matrices. For example, this number is three for the three-dimensional rotation group. This number is also three for $SU(2)$ and $Sp(2)$. We call this the Euler decomposition. The present paper is based on both the Euler construction and the Euler decomposition.

Among the several useful Euler decompositions, the Iwasawa decomposition plays an important role in the Lorentz group. We have seen in this paper what the decomposition does to the two-by-two matrices of $Sp(2)$, but it has been an interesting subject since Iwasawa’s first publication on this subject [28]. It is beyond the scope of this paper to present a historical review of the subject. However, we would like to point out that there are areas of physics where this important mathematical theorem was totally overlooked.

For instance, in particle theory, Wigner’s little groups dictate the internal space-time symmetries of massive and massless particles which are locally iso-
morphic to $O(3)$ and $E(2)$ respectively \cite{wigner39}. The little group is the maximal subgroup of the Lorentz group whose transformations do not change the four-momentum of a given particle \cite{kim91}. The $E(2)$-like subgroup for massless particles is locally isomorphic to the subgroup of $SL(2, c)$ which can be started from one of the matrices in Eq. (108) and the diagonal matrix of Eq. (55). Thus there was an underlying Iwasawa decomposition while the the $E(2)$-like subgroup was decomposed into rotation and boost matrices \cite{han99}, but the authors did not know this. One of those authors is one of the authors of the present paper.

In optics, there are two-by-two matrices with one vanishing off-diagonal element. It was generally known that this has something to do with the Iwasawa effect, but Simon and Mukunda \cite{simon85} and Han et al. \cite{han99} started treating the Iwasawa decomposition as the main issue in their papers on polarized light.

In para-axial lens optics, the matrices of the form given in Eq. (108) are the starters \cite{han99}, and repeated applications of those two starters will lead to the most general form of $Sp(2)$ matrices. It had been a challenging problem since 1985 \cite{han99} to write the most general two-by-two matrix in lens optics in terms the minimum number of those starter matrices. This problem has been solved recently \cite{baskal01}, and the central issue in the problem was the Iwasawa decomposition.

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