On the Application of the Non Linear Sigma Model to Spin Chains and Spin Ladders

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Abstract

We review the non linear sigma model approach (NLSM) to spin chains and spin ladders, presenting new results. The generalization of the Haldane’s map to ladders in the Hamiltonian approach, give rise to different values of the $\theta$ parameter depending on the spin $S$, the number of legs $n_\ell$ and the choice of blocks needed to built up the NLSM fields. For rectangular blocks we obtain $\theta = 0$ or $2\pi S$ depending on wether $n_\ell$, is even or odd, while for diagonal blocks we obtain $\theta = 2\pi S n_\ell$. Both results agree modulo $2\pi$, and yield the same prediction, namely that even (resp. odd) ladders are gapped (resp. gapless).

For even leeged ladders we show that the spin gap collapses exponentially with $n_\ell$ and we propose a finite size correction to the gap formula recently derived by Chakravarty using the 2+1 NSLM, which gives a good fit of numerical results. We show the existence of a Haldane phase in the two legged ladder using diagonal blocks and finally we consider the phase diagram of dimerized ladders.

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Introduction

There have been three major developments in the 80’s and 90’s in Condensed Matter or more specifically in Strongly Correlated Systems. In historical order they are

- Haldane’s conjecture in 1d antiferromagnetic spin chains (1983) [1, 2]
- Discovery of high-$T_c$ superconductivity and antiferromagnetism in doped and undoped cuprate compounds (1986) [3]
- Discovery of Ladder Materials (1987,91) [4, 5]

All these findings have in common various features: low dimensionality, antiferromagnetism, importance of quantum fluctuations and the fact that they constituted theoretical and experimental surprises. Moreover these topics can be studied using sigma model techniques, which establish a methodological link between them. In this talk we shall be mainly concerned with spin chains and ladders.

The first surprise deals with the behaviour of 1d antiferromagnetic Heisenberg spin chains (AFH) as a function of the spin. The Hamiltonian describing the AFH model is given by,

$$H_{\text{chain}} = J \sum_{n=1}^{L} S(n)S(n+1)$$  \hspace{1cm} (1)

where $J > 0$ is the exchange coupling constant and $S(n)$ is a spin-S matrix acting at the $n^{th}$ site of the chain. Haldane’s conjecture concerning the spectrum and spin correlations of the Hamiltonian (1) is given in table 1,

| 2S | Spectrum   | Correlations          |
|----|------------|-----------------------|
| even | gapped    | exponential decay     |
| odd  | gapless    | algebraic decay       |

Table 1: Behaviour of spin chains

Nowadays there is sufficient theoretical, numerical and experimental evidence to support this conjecture, which should rather be called a ”theorem” despite of the lack of a rigorous mathematical proof [6].

The third surprise, in the historical order shown above, deals with the behaviour of spin ladders as function of the number of legs $n_\ell$ [7] ( for a review and references on the subject see [8]). A ladder is an array of $n_\ell$ spin chains coupled as in figure 1. The Hamiltonian of the system is given by,

$$H_{\text{ladder}} = H_{\text{leg}} + H_{\text{rung}}$$

$$H_{\text{leg}} = J \sum_{a=1}^{n_\ell} \sum_{n=1}^{L} S_a(n)S_a(n+1)$$

$$H_{\text{rung}} = J' \sum_{a=1}^{n_\ell-1} S_a(n)S_{a+1}(n)$$  \hspace{1cm} (2)

where $S_a(n)$ are spin-S matrices located in the $a^{th}$ leg at the position $n = 1, \ldots, L$. We consider periodic boundary conditions alongs the legs, i.e. $S_a(n) = S_a(n + L)$, and open BC’s along the rungs. The intraleg coupling constant $J$ is positive but the interleg coupling constant $J'$ can be either positive or negative. The qualitative behaviour of spin 1/2 ladders is given in table 2,
This result holds for both signs of $J'$. The analogy between the integer/half-integer behaviour of spin chains and the even/odd behaviour of spin 1/2 ladders is evident. In the limit where $J'$ goes to minus infinity, the spin ladder Hamiltonian becomes equivalent to the spin chain Hamiltonian for an effective spin equal to $n_\ell/2$, and hence the ladder behaviour given in table 2 follows from the behaviour of the spin chains. What is not so obvious is that this behaviour holds not just for strong ferromagnetic rung couplings but also for antiferromagnetic ones, regardless their magnitude. As for the Haldane conjecture there is by now sufficient evidence to support the ”ladder conjecture” coming from Quantum Monte Carlo, exact diagonalization, mean field theory, experiments on ladder materials like VOPO or cuprates, bosonization, finite size analysis, sigma model, etc [8].

In this talk we shall give a unified description of the behaviour of spin chains and spin ladders utilizing the non linear sigma model (NLSM) [9, 10, 11]. Let us first recall the logic underlying Haldane conjecture. It is based in the following map [1, 2],

\[
\text{Low Energy Modes of the Spin Chain} \longrightarrow \text{Non Linear Sigma Model} \quad (3)
\]

which is obtained in the semiclassical limit where the spin $S$ becomes very large, although (3) can be derived on more general grounds based on symmetry arguments [12]. However these type of arguments miss the theta term in the action, which plays a crucial role in determining the physics of the model.

Hence using the properties of the (NLSM) one derives those of the low energy spectrum of the spin chain.

To arrive at the results of table 2 for the ladders we shall follow the same logic as for spin chains, that is, we shall construct a map

\[
\text{Low Energy Modes of the Spin Ladder} \longrightarrow \text{Non Linear Sigma Model} \quad (4)
\]

and use the properties of the NLSM. In this way the NLSM provides a unified and economic approach to both problems. Actually, the 2d Antiferromagnetic Heisenberg model, can also be mapped into the O(3) NLSM in 2+1 dimensions [13]. The RG analysis at finite temperature of the later model made in [14] has unravelled many of the properties of the undoped cuprate compounds (second surprise). On the other hand the NLSM’s that appear in the r.h.s. of the maps (3) and (4) are defined in 1+1 dimensions. This poses the problem of the crossover between 1d and 2d Heisenberg systems through spin ladders [15].

### The Non Linear Sigma Model: A Primer

Before we construct the maps (3) and (4), it will be convenient to review the basics of the O(3) NLSM. This model was proposed as a toy version of QCD, as can be seen by enumerating its main features: asymptotic freedom, dynamical mass generation, instantons, skyrmions, integrability,
etc (for a review see [16, 17]). For these and other reasons the NLSM still attract the interest of many physicists and mathematicians. In the case of spin chains and ladders the NLSM becomes not just a toy model but a realistic model describing the low energy physics!

The O(3) NLSM is a relativistic quantum field theory in 1+1 dimensions whose field $\Phi$ is a three component vector living on the 2d-sphere $S^2$,

$$\Phi^2 = 1,$$

(5)

The euclidean action of the model is given by,

$$S = \int d^2x \left[ -\frac{1}{2g} (\partial_\mu \Phi)^2 + i \frac{\theta}{8\pi} \epsilon_{\mu\nu} \Phi \cdot (\partial_\mu \Phi \times \partial_\nu \Phi) \right]$$

(6)

where $g > 0$ is the sigma model coupling constant, and $\epsilon_{\mu\nu}$ the 2d Levi-Civita symbol. The quantity

$$W = \frac{1}{8\pi} \int d^2x \epsilon_{\mu\nu} \Phi \cdot (\partial_\mu \Phi \times \partial_\nu \Phi) \in \mathbb{Z}$$

(7)

takes an integer value for field configurations $\Phi(x)$ which go to a fixed value, say $\Phi_0$, when $|x| \to \infty$ (this condition is required to have a finite action). Compactifying the space-time into the sphere $S^2$, the integral (7) gives the winding number of the map $S^2$ (space-time) $\to S^2$ (target space). The parameter $\theta$ enters in the partition function as $e^{i\theta W}$, and therefore the integer character of $W$ implies that $\theta$ is defined modulo $2\pi$. The theta term in the action is a truly topological term, which leads to dramatic non perturbative effects.

Notice that $W$ changes its sign under a parity (or time reversal) transformations. Hence the topological term of the action breaks explicitly parity (or time reversal) unless $\theta = 0$ or $\pi$. Indeed if $\theta = 0 \pmod{2\pi}$ the topological term is completely absent while if $\theta = \pi$ the winding number contributes to the action with a sign, i.e. $(-1)^W$. We expect from these properties that the AFH-spin chains and ladders which are parity invariant will be associated with $\theta = 0$ or $\pi$.

The basic properties of the NLSM for these values of $\theta$ are given in table 3 [18, 19, 20, 1, 21].

| $\theta$ (mod $2\pi$) | Spectrum | Correlations |
|------------------------|----------|--------------|
| 0                      | gapped   | exponential decay |
| $\pi$                  | gapless  | algebraic decay |

Table 3: Behaviour of the O(3) NLSM

This behaviour, which is independent of the magnitude of the coupling constant $g$, will allow us to make a quick derivation of table 3 in the strong coupling limit. Before we do that, let’s write the Hamiltonian that follows from the action (6),

$$H_{NLSM} = \frac{c}{2} \int dx \left[ g \left( 1 - \frac{\theta}{4\pi} \Phi' \right)^2 + \frac{1}{g} \Phi'^2 \right]$$

(8)

where $\Phi'(x) = \partial_x \Phi(x)$, $c$ is the “speed of light” and $l(x)$ is the angular momentum density, which is defined as,

$$l = \Phi \times \frac{d\Phi}{dt}$$

(9)
Besides the constraint (5) the fields $\Phi$ and $l$ satisfy,

$$l(x) \cdot \Phi(x) = 0$$

(10)

and the canonical equal-time commutation relations,

$$[l^a(x), l^b(y)] = i\epsilon^{abc}\delta(x - y)l^c(x)$$
$$[l^a(x), \Phi^b(y)] = i\epsilon^{abc}\delta(x - y)\Phi^c(x)$$
$$[\Phi^a(x), \Phi^b(y)] = 0$$

(11)

It is very illustrative to prove the statements given in table 3 in the strong coupling limit ($g >> 1$), for which we shall use a regularized lattice version of the Hamiltonian (8) [20, 21],

$$H_{\text{NLSM}}^{\text{lattice}} = \frac{c}{2} \sum_n \left( g l_n^2 - \frac{2}{g} \Phi_n \Phi_{n+1} \right)$$

(12)

At every site of the lattice there is a "particle" moving on a sphere ($\Phi_n^2 = 1$), with angular momenta $l_n$. If $\theta = 0$ the angular momenta takes all possible integer values, $l_n = 0, 1, \ldots \infty$ [20]. However if $\theta = \pi$ there is a monopole of charge 1 at the center of every sphere, which implies that the possible values of the angular momentum are restricted to half-integers values, $l_n = \frac{1}{2}, \frac{3}{2}, \ldots \infty$ [21].

In the limit $g >> 1$ the kinetic term $g \sum_n l_n^2$ dominates over the potential term $-\frac{2}{g} \sum \Phi_n \Phi_{n+1}$, and the ground state is obtained choosing the smallest possible value of $l_n$ at every site.

If $\theta = 0$ the optimal choice is given by $l_n = 0, \forall n$, which yields a unique ground state. The first excited states are obtained by choosing the irrep $l = 1$ in one site and $l = 0$ in the rest of the chain. Since this can be done at every site, there is a huge degeneracy, which is broken by the potential term, which delocalizes the $l = 1$ excitations. The 3L degenerate first excited states become a band of $l = 1$ magnons, separated from the ground state by a gap, which can be computed in perturbation theory. In [20] this gap was computed up to 6th order in $1/g^2$. The first three terms read,

$$\Delta = cg \left( 1 - \frac{2}{3} \frac{1}{g^2} + 0.074 \frac{1}{g^4} + O(\frac{1}{g^6}) \right)$$

(13)

In the weak coupling limit ($g << 1$) a perturbative RG analysis shows that the gap vanishes exponentially as [18]

$$\Delta \sim \frac{c}{g} e^{-2\pi/g}$$

(14)

The proportionality constant depends on the regularization of the model [22] (see also [23])

Combining the strong and weak coupling analysis and using Pade approximants the authors of [20] concluded that the gap should never vanishes as long as $g$ is non zero.

For $\theta = \pi$ the kinetic energy is minimized by the choice $l_n = 1/2, \forall n$, which still leaves a huge degeneracy for the ground state. This degeneracy is lifted by the potential term, which leads to an effective AFH model when restricted to the subspace $l_n = 1/2$ [21]. Since the $S = 1/2$ AFH model is massless one gets that the NLSM at $\theta = \pi$ is also massless. In reference [21] it was argued that this gapless behaviour persists to all values of $g$. 

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**Haldane map for spin chains**

The map from the Heisenberg model into the NLSM can be done in various ways: using coherent states in the path integral formalism, generalizing the Hubbard-Stratonovich formula in the partition function, or applying gradient expansions in the Hamiltonian formalism. We shall mainly use the latter one, but we shall also briefly explain the path integral approach for ladders.

The starting point of the construction is the spin wave analysis of the AFH model. This consist in the linearization of the evolution equations of the spin operators $S(n)$, around the classical Neel configuration. The equations of motion of the spin operators read,

$$ \frac{dS(n)}{dt} = i[H_{\text{chain}}, S(n)] = -JS(n) \times [S(n + 1) + S(n - 1)] \tag{15} $$

The basic assumption is that $S(n)$ deviates by a small amount from the alternating Neel configuration,

$$ S(n) = (-1)^n S z + s(n) \tag{16} $$

where $S$ is the magnitude of the spin and $z$ is the unit vector in the vertical direction. In the linearized approximation eq.(15) becomes,

$$ \frac{d\zeta(n)}{dt} = i(-1)^{n+1}JS[\zeta(n + 1) + \zeta(n - 1) + 2\zeta(n)] \tag{17} $$

where $\zeta(n) = s^x(n) + is^y(n)$. The spin waves are the plane wave solutions of eq.(17),

$$ \zeta(n) = e^{i(\omega t + kn)}(\psi(k) + (-1)^{n+1}\phi(k)) \tag{18} $$

We are interested in the low energy and long wavelength solutions which are given by,

$$ \omega = vk; \quad \psi(k) \sim Ak \quad \phi(k) \sim B \tag{19} $$

Equation (17) is satisfied by the ansatz (18) if $2A = B$. The spin wave velocity $v$ is given by,

$$ v = 2JS \tag{20} $$

The two transverse spin wave solutions (18) can be identified with the massless goldstone bosons associated with the breaking of the rotational symmetry of the AFH Hamiltonian by the classical Neel state. Eq(18) implies that the spin variables have two slowly varying components around the momenta $k = 0$ (field $\psi$) and $k = \pi$ (field $\phi$), which correspond to the local spin density and the local staggered magnetization respectively.

This elementary spin wave analysis is very useful in order to construct the map from the Heisenberg model into the NLSM. This map can be mathematically formulated as a change of variables. First of all, one has to divide the chain into blocks of two sites (see figure 2). The
block, say \((2n, 2n+1)\), is given the coordinate \(x= 2n+1/2\) and for every block one performs the following change of variables,

\[
S(2n) = l(x) - S\Phi(x) \\
S(2n+1) = l(x) + S\Phi(x)
\]

The inverse of eqs.\((21)\) are

\[
l(x) = \frac{1}{2}[S(2n + 1) + S(2n)] \\
\Phi(x) = \frac{1}{2S}[S(2n + 1) - S(2n)]
\]

These relations imply that \(l\) is a local spin density and \(\Phi\) is a local staggered magnetization. Using \((21)\) and the eqs \(S^2(n) = S(S + 1)\) one gets,

\[
l^2(x) + S^2\Phi^2(x) = S(S + 1) \\
l \cdot \Phi = 0
\]

In the semiclassical limit \(S >> 1\) eqs.\((23)\) become the sigma model conditions \((5)\) and \((10)\). The commutators \((11)\) can also be obtained in the limit \(S >> 1\) from the commutation relations of the spin matrices, with the following identification of the Dirac’s delta function and the Kronecker’s delta symbol in the limit where the lattice spacing \(\delta\) goes to zero,

\[
\delta(x - y) = \lim_{\delta \to 0} \frac{1}{2\delta} \delta_{x,y}
\]

The term \(2\delta\) in the denominator of eq.\((24)\) is the lattice spacing in the \(x\)-variables (from now on we shall set \(\delta = 1\)).

The AFH Hamiltonian \((1)\) reads in terms of the new variables \(l\) and \(\Phi\),

\[
H_{\text{chain}} = \sum_x [-S(S + 1) + 2l^2(x) + l(x) l(x + 2) + S(\Phi(x) l(x + 2) - l(x) \Phi(x + 2)) - S^2\Phi(x) \Phi(x + 2)]
\]

Making the assumption that the fields \(l(x)\) and \(\Phi(x)\) vary slowly in the \(x\) coordinate, we can perform a gradient expansion and truncation of the higher derivative terms. Keeping terms with at most two spatial or time derivatives in the field \(\Phi\), we obtain the following Hamiltonian,

\[
H^{(\text{Continuum})}_{\text{chain}} = \int dx \left[2l^2 - S(4l' + \Phi' + \Phi' l) + S^2(\Phi')^2\right]
\]

which coincides with the NLSM Hamiltonian given in \((8)\) upon the identifications,

\[
\theta = 2\pi S, \quad g = \frac{2}{S}, \quad c = 2JS
\]

This is the desired result which permits to derive table 1 from table 3. From eqs \((14)\) and \((27)\) we can estimate the value of the spin gap and the correlation length as functions of the spin,
\[ \Delta_S \sim J S^2 e^{-\pi S}, \xi \sim \frac{2}{S} e^{\pi S} \] (28)

In table 4 we show the numerical values of the gap, correlation length and spin velocity of the \( S=1 \) and 2 Heisenberg chains, obtained using Quantum Monte Carlo \[24, 25\] and DMRG methods \[26, 27\].

| Spin | \( \Delta/J \) | \( \xi \) | \( c/J \) |
|------|---------------|----------|-------|
| \( S = 1 \) | 0.4107 | \( \sim 6 \) | 2.47 |
| \( S = 2 \) | 0.049-0.085 | 49 | 4.16 |

Table 4: Spin gap of the \( S = 1 \) and 2 chains.

This table shows the semi-quantitative agreement between the numerical (exact) results and the NLSM estimates \[28\]. In the semiclassical limit where \( S \to \infty \) the NLSM predicts that the gap should go to zero exponentially fast, recovering in that way the classical gapless behaviour. This shows that the existence of the gap is a truly quantum mechanical effect.

To end up our review of the spin chains, we would like to make some comments on the non-uniqueness of the map \[3\]. Indeed eqs\[21\] and \[22\], which give the map between the spin and the sigma model variables, depend on the choice of the 2 site blocks. The other possible choice, given by the blocks \((2n+1, 2n+2)\), leads two another couple of variables \( \tilde{l} \) and \( \tilde{\Phi} \), which are linearly related to \( l \) and \( \Phi \), by the eqs,

\[ \tilde{l} = 1 - S \, \Phi' \]
\[ \tilde{\Phi} = \Phi \] (29)

These eqs. leave invariant the constraints \[(5), (10)\], and in fact they can be obtained by the following transformation \[2\],

\[ \tilde{l} = e^{-iS \int dx \, \beta' \cos \alpha} \, l \, e^{iS \int dx \, \beta' \cos \alpha} \]
\[ \tilde{\Phi} = e^{-iS \int dx \, \beta' \cos \alpha} \, \Phi \, e^{iS \int dx \, \beta' \cos \alpha} \] (30)

where \( \alpha \) and \( \beta \) are the spherical coordinates that parametrize the staggered field

\[ \Phi = (\sin \alpha \, \cos \beta, \sin \alpha \, \sin \beta, \cos \alpha) \] (31)

However the Hamiltonian \[26\] is not left invariant, but the change affects only the theta parameter which for the new variables becomes,

\[ \tilde{\theta} = -2\pi S \] (32)

This change in the value of \( \theta \), depending on the blocking, can also be seen as due to a parity transformation. Recall that parity changes the sign of the topological term \( W \). The two blockings \((2n, 2n+1)\) and \((2n+1, 2n+2)\) are indeed related by parity. In the next section we shall consider more general blockings which will lead to changes in \( \theta \) but that are not related by parity transformations. If \( \theta = \pi \) the change \[32\] leaves invariant the sign factor \((-1)^W\) appearing in the partition function.
Haldane map for spin ladders

There are two ways to study the problem of spin ladders using NLSM methods. The first one consists in the application of the Haldane’s map to every chain forming the ladder, obtaining a system of $n_\ell$ sigma model fields coupled by rung interactions. This approach is similar in spirit to the bosonization studies of spin ladders made in references [28], and its applicability is reasonable in the weak coupling regime $J'/J << 1$. The other possible approach is to built up a unique sigma model field describing the low energy modes of the spin ladder as a whole [9, 10, 11]. This approach should be appropriate to study the intermediate coupling regime ($J'/J \sim O(1)$). The strong coupling regime ($J'/J >> 1$) has been mainly studied using perturbative [29, 30, 31] and mean field methods [32]. We shall see that the later approaches are related to the discrete version of the NLSM based on the Hamiltonian [12].

The map from the ladder into the NLSM follows the same steps as that of the spin chain. We shall review below the approach of reference [10].

Spin Wave Analysis of Ladders

This analysis shows that at the linearized level there are two massless modes corresponding to two Goldstone bosons. These modes describe the gapless deviations from the classical Neel solution, which in the case of AF-coupling along the rungs reads,

$$S_a^{\text{class}}(n) = (-1)^{a+n} S$$

In the rest of the talk we shall confine ourselves to this case. The transverse components of the spin waves, i.e. $\zeta_a(n) = s^x_a(n) + i s^y_a(n)$, are given by

$$\zeta_a(n) = e^{i(\omega t + kn)} (A_a k + (-1)^{a+n+1} B)$$

where $\omega = v k$. The quantities $A_a$ represent the fraction of total spin carried out by the $a^{th}$-leg, and we shall normalize them as,

$$\sum_a A_a = 1$$

The spin wave velocity $v$ and the $A'_a$'s are given by,

$$\left(\frac{v}{S}\right)^2 = J n_\ell / \sum_{a,b} L^{-1}_{a,b}$$

$$A_a = \sum_b L^{-1}_{a,b} / \sum_{c,d} L^{-1}_{c,d}$$

where $L^{-1}$ is the inverse of a matrix $L$ defined as follows,

$$L_{a,b} = \begin{cases} 4J + J' & a = b = 1 \text{ or } n_\ell \\ 4J + 2J' & 1 < a = b < n_\ell \\ J' & |a - b| = 1 \end{cases}$$

Besides the 2 Goldstone bosons there are, at the linearized level, $2(n_\ell - 1)$ massive modes. As an example we give below the values of their masses for the $n_\ell = 2$ and 3 ladders,
\[ m_{n\ell=2} = S\sqrt{8JJ'} \]
\[ m_{n\ell=3} = S\sqrt{J'(J+4J')} = S\sqrt{J'(J+12J')} \] (38)

Of course in the limit when \( J' \to 0 \) the masses of the massive modes (38) disappear and all the modes become massless, as should be the case for a set of \( n\ell \) uncoupled chains. The basic assumption in the construction of [10] is the truncation of the massive modes, keeping only the gapless ones, which is justified if the energy scales, generated non perturbatively, are lower than the gap of the massive modes. One may expect that this condition is satisfied in the intermediate and strong coupling regimes.

**Map: Spin Ladders \( \to NLSM \)**

The spin wave analysis serves as a preparation for the more complicated job of finding the map from the spin ladder into the sigma model. In the Hamiltonian formulation the first step is to split the ladder into blocks of \( n_B \) sites, defining for each block an average angular momentum \( \mathbf{l} \) and staggered magnetization \( \Phi \) as follows,

\[
\mathbf{l}(x) = \frac{n\ell}{n_B} \sum_{(a,n) \in B(x)} S_a(n)
\]
\[\Phi(x) = \frac{1}{S n_B} \sum_{(a,n) \in B(x)} (-1)^{a+n} S_a(n)\] (39)

where \( x \) denotes the center of mass position of the block \( B(x) \) along the leg axis, and the prefactor \( n\ell/n_B \) is required in order that \( \mathbf{l} \) satisfy the canonical commutation relations (11) in the continuum limit. The relation (24) reads in the more general case,

\[
\delta(x-y) = \lim_{\delta \to 0} \frac{n\ell}{n_B \delta} \delta_{x,y} \] (40)

For single chains there are only two types of blockings related by parity. However for ladders there are many different types of blockings, which may in principle lead to different results. In order to choose those blockings that are physically acceptable we shall impose the following conditions

- The blocks must have an even number of sites.
- Every block must contain a site (or more) of every leg the same number of times, which implies that \( n_B \) should be a multiple of \( n\ell \).
- Every site in a block must have a nearest neighbour belonging to the same block.

Motivated by the spin wave solution (34), we shall propose the following ansatz for the spin operators in terms of the sigma model fields,

\[ S_a(n) = A_a \mathbf{l}(x) + (-1)^{a+n} S \Phi(x) \text{ for } (a,n) \in B(x) \] (41)

The consistency between eqs. (39) and (41) is guaranteed by the following identities,
\[
\sum_{(a,n) \in B(x)} (-1)^{a+n} = 0 \\
\sum_{(a,n) \in B(x)} A_a = \frac{n_B}{n_\ell} \\
\sum_{(a,n) \in B(x)} (-1)^{a+n} A_a = 0
\]  

which can be proved from the conditions on the allowed blocks given above and the symmetry properties of \( A_a \).

Strictly speaking we should add to the r.h.s. of (41) a set of fields describing the massive modes whose existence we discussed in the spin wave analysis. Their inclusion makes the map (41) more rigorous from a mathematical point of view, allowing a careful derivation of the effective Hamiltonian governing the dynamics of the fields \( I \) and \( \Phi \). Having found the effective Hamiltonian the massive fields are discarded, so that we shall not consider them from now on.

Let us now consider a few examples ladder’s blockings.

**Columnar Blocks** ( \( n_B = n_\ell \)) For ladders with an even numbers of legs the smallest possible blocks, satisfying the conditions given above, coincide with the rungs (see Fig 3), i.e. \( n_B = n_\ell \). The partition of the ladders into rungs is what one effectively does in the study of the strong coupling limit of ladders [29, 30, 31]. In that sense one may expect to find relations between the NSLM approach and the strong coupling analysis.

Replacing the ansatz (41) into the ladder Hamiltonian (2), and taking the continuum limit we get the following expression,

\[
H = \int dx \left[ \frac{1}{2} L_{a,b} A_a A_b \Phi^2 + \frac{1}{2} JS^2 n_\ell \Phi^2 \right]
\]  

Comparing (43) with the NLSM Hamiltonian (8) we find,

\[
\theta = 0 \\
\left( \frac{\Phi}{\Phi_0} \right)^2 = J n_\ell / \sum_{a,b} L_{a,b}^{-1} \\
g = 1 / \left( S \sqrt{JS^2 n_\ell \sum_{a,b} L_{a,b}^{-1}} \right)
\]  

Since \( \theta = 0 \) we see that the NLSM is gapped, which implies that even ladders should always be gapped for any value of the spin. The velocity \( c \) coincides with the spin wave velocity \( v \) [36]. Finally, the coupling constant \( g \) has a non trivial dependence of the number of legs and the ratio \( J'/J \). This is interesting because we can derive from the eq.(44) the dependence of the spin gap on the ladder’s parameters.

For odd ladders the minimal blocks satisfying the conditions above must have at least \( n_B = 2n_\ell \) sites. Unlike the blocks with \( n_B = n_\ell \) there are now more geometries. We shall in what follows consider the cases of even and odd values of \( n_\ell \). In figures 4 and 5 we show two of them. The choice of figure 4 was the one studied in [10]. We shall give for completeness the results obtained there. Later on we shall study the blocks of figure 5.

**Rectangular Blocks** ( \( n_B = 2n_\ell \)) The Hamiltonian that one obtains after taking the continuum limit is (fig 4),
\[ H = \int \frac{dx}{2} \left[ \sum_{a,b} L_{a,b} A_a A_b \right] 1^2 + 2JS^2 n_\ell \Phi'^2 + 2SJ \sum_{a=1}^{n_\ell} (-1)^a A_a \left( \Phi' 1 + 1\Phi' \right) \] (45)

In [10] it was shown that the value of \( \theta \) corresponding to (45) is given by,
\[
\theta = \begin{cases} 
0 & n_\ell : \text{even} \\
2\pi S n_\ell & n_\ell : \text{odd} 
\end{cases} \tag{46}
\]

The vanishing of \( \theta \) for even ladders follows simply from the symmetry property \( A_a = A_{n_\ell+1-a} \), while the value \( \theta = 2\pi S \) for odd ladders requires some more work [10]. The other NLSM parameters of (45) are given by,
\[
\left( \frac{c}{S} \right)^2 = \frac{2JS}{\sum_{a,b} L_{a,b} - 1} \tag{47}
\]

where \( \delta_{n_\ell,\text{odd}} = 1 \) (resp. 0) if \( n_\ell \) is odd (resp. even). For \( n_\ell \) even the delta terms appearing in these eqs. are absent and we recover [14], except for a renormalization of \( c \) and \( g \) (\( c_{\text{rect}} = \sqrt{2} c_{\text{col}}, g_{\text{rect}} = g_{\text{col}}/\sqrt{2} \)).

For \( n_\ell \) odd, the formula (47) does not coincide with the spin wave velocity (36), except for the case of the spin chains ( \( n_\ell = 1 \)!). For large ladders the velocity approaches the value \( 4JS \) ( \( J' = J \) ) which differs by a factor of \( \sqrt{2} \) with respect to the 2d value which is \( 2\sqrt{2}JS \).

**Diagonal Blocks** \((n_B = 2n_\ell)\) Within the gradient approximation which we are using, it is consistent to assign to all the points within a diagonal block (fig. 5) the same coordinate \( x \). Under this assumption we get the following Hamiltonian,
\[
H = \int \frac{dx}{2} \left[ \sum_{a,b} L_{a,b} A_a A_b \right] 1^2 + 2S^2 \sum_{a=1}^{n_\ell} p_a \Phi'^2 + 2S \sum_{a=1}^{n_\ell} A_a p_a \left( \Phi' 1 + 1\Phi' \right) \] (48)

where \( p_a \) is defined as
\[
p_a = \begin{cases} 
J + J'/2 & a = 1 \text{ or } n_\ell \\
J + J' & \text{else} 
\end{cases} \tag{49}
\]

The value of \( \theta \) that comes out from (48) is given by,
\[
\theta = 8\pi S \frac{\sum_a A_a p_a}{\sum_{b,c} L_{b,c} A_b A_c} = 8\pi S \sum_a p_a L_{a,b}^{-1} \tag{50}
\]

To evaluate \( \theta \) it is convenient to write (49) in matrix form as follows,
\[
\theta = 8\pi S n_\ell < F | PL^{-1} | F > \tag{51}
\]

where \( | F > \) denotes a normalized \( n_\ell \)-component vector with all its entries equal to \( 1/\sqrt{n_\ell} \) and \( P \) is a diagonal matrix with its entries given by \( p_a \). The trick is now to write the matrix \( L \) in the form \( L = 4P - K^- \), where \( K^- \) is defined as,
Making a "Dyson" decomposition of $L^{-1}$

$$L = \frac{1}{4 P} - \frac{1}{4 P} K^- \frac{1}{4 P - K^-}$$

and using the fact that $K^-$ annihilates the vector $|F> \text{ we finally get,}$

$$\theta = 2 \pi S n_\ell$$

It is quite remarkable that all the coupling constants in the expression (50) just combined in order to produce a result which only depends on "global" data $S$ and $n_\ell$. Eq.(54) agrees with (46) mod $2 \pi$, implying that the choice of block does not affect this parameter. Let us see what are the values of $g$ and $c$ for the Hamiltonian (48),

$$g = \frac{1}{S} \sqrt{2 \sum_a b, c p_a L_{b,c}^{-1} - (2 \sum_{a,b} p_a L_{a,b}^{-1})^2}$$

Path Integral Derivation of Haldane Map for Ladders

In this section we shall apply coherent state techniques to derive the map from the spin ladders into the NLSM (for a review see reference [33]). The map for the 2 legged ladder was first worked out in [9]. Our results agree with the result of this reference and extend them to generic values of $n_\ell$ (see also [11] for an alternative derivation of the results given below).

The action of the spin ladder system in the path integral formulation can be written in the following form,

$$S = \int d\tau \left\{ i \sum_a \sum_{n=1}^{n_\ell} A(S_a(n)) \cdot \frac{dS_a(n)}{d\tau} - J \sum_n \sum_{a=1}^{n_\ell} S_a(n) S_a(n+1) - J' \sum_n \sum_{a=1}^{n_\ell-1} S_a(n) S_{a+1}(n) \right\}$$

where $S_a(n)$ is a classical spin variable satisfying $S_a(n)^2 = S^2$ and $A(S_a(n))$ is the vector potential that fulfills the constraint $\text{rot} A(n) = n \quad (n = S/S)$. The long wavelength limit of the ladder’s spin variables $S_a(n)$ is given, according to (41), by

$$S_a(n) = \delta A_a l(n) + (-1)^{a+n} S \Phi(n) \left( 1 - \frac{\delta^2}{S^2} A_a^2 \right)^{1/2}$$

where the term with the square root is needed for the correct normalization of the variable $S_a(n)$. For spin chains this later term can be replaced by 1, but for ladders it gives a non trivial contribution when considering the coupling between rungs.

Introducing (57) into (56) and performing the standard gradient expansions one finds the following NLSM parameters
\[ \theta = 2\pi S \delta_{n,\text{odd}} \]
\[ \left( \frac{2^2}{\pi} \right)^2 = \frac{Jn_\ell}{\sum_{a,b} L_{a,b}^{-1}} \]
\[ g = \frac{1}{S\sqrt{Jn_\ell \sum_{a,b} L_{a,b}^{-1}}} \]

(58)

which coincide with those obtained using columnar blocks (44) for even legged ladders. For odd ladders the results of the path integral and those obtained with rectangular blocks differ, except in the case of spin chains (\(n_\ell = 1\)). This poses the problem between the relation between the Hamiltonian formalism and the path integral formalism for this type of ladders. In any case, notice that the path integral approach gives for even and odd legged ladders a value of \(c\) identical to the spin wave velocity (36).

Before we extract more consequences from the mapping of the ladders into the NLSM, it is convenient to express \(c\) and \(g\) in terms of

\[ f_n(\ell) \]

Let us define \(f_n(\ell)\) as \(10\),

\[ f_n(\ell)(J'/J) = \frac{J'}{n_\ell} \sum_{a,b} L_{a,b}^{-1} \]
\[ = \frac{1}{n_\ell} \left[ \delta_{n,\text{odd}} + 2 \sum_{m=1,3,...,n_\ell-1} \sin^{-2} \left( \frac{\pi m}{2n_\ell} \right) \left( 1 + \frac{J'}{J} \cos^2 \frac{\pi m}{2n_\ell} \right)^{-1} \right] \]

(59)

Its explicit expression and numerical values for \(z = 1\) in the cases \(n_\ell = 1, 2, 3, 4\) and \(\infty\) is given in table 5.

| \(n_\ell\) | 1     | 2     | 3     | 4     | \(\infty\) |
|-----------|-------|-------|-------|-------|-----------|
| \(f_n(z)\) | 1     | \(1/(1+z)\) | \(1+\frac{3z}{4}\) | \(1+\frac{3z}{4}\) | 0.5882    | 0.5     |
| \(f_n(1)\) | 1     | 0.666 | 0.6190| 0.6178| 0.5884    | 0.5     |
| \(\frac{1}{2} \left( 1 + \frac{1}{n_\ell \sqrt{2}} \right)\) | 0.853 | 0.677 | 0.6178| 0.5884 | 0.5     |

Table 5

\(f_n(z)\) is a monotonically decreasing function in \(z\), which varies from 1 to \(1/n_\ell^2\). \(\delta_{n,\text{odd}}\) as \(z\) varies from 0 to \(\infty\). Interesting enough, the sum appearing in eq. (59) can be performed yielding,

\[ f_n(z) = \frac{1}{1+z} \left[ 1 + \frac{z}{n_\ell (1+z)^{1/2}} \left( 1 + \frac{2z}{1+z} \right)^{n_\ell} - (-1)^{n_\ell} \right] \]

(60)

In the isotropic case (\(z = 1\)) we get,

\[ f_n(1) = \frac{1}{2} \left( 1 + \frac{1}{n_\ell \sqrt{2}} \left( 3 + 2\sqrt{2} \right)^{n_\ell} - (-1)^{n_\ell} \right) \]

(61)

For \(n_\ell > 2\), a good approximation of (61) is given by (see table 5),
\[ f_{n\ell}(1) \sim \frac{1}{2} \left( 1 + \frac{1}{n\ell \sqrt{2}} \right) \]  

(62)

In eqs (60) and (61) the difference between even and odd ladders dissapears exponentially as \( n\ell \) increases.

This ends the review of the properties of \( f_{n\ell} \) and we return now to our general discussion.

Summary of Results and Conclusions

In table 6 we summarize the results obtained so far for the three types of blocks we have considered so far,

| Block      | \( n\ell \)   | \( \theta \) | \( c \)                          | \( g \)                          |
|------------|---------------|--------------|----------------------------------|----------------------------------|
| Columnar Even 0 | \( 2 JS \sqrt{f_{n\ell}} \) | \( 2/Sn\ell \sqrt{f_{n\ell}} \) |                                  |                                  |
| Rectangular Even 0 | \( 2\sqrt{2} JS \sqrt{f_{n\ell}} \) | \( \sqrt{2}/Sn\ell \sqrt{f_{n\ell}} \) |                                  |                                  |
| Rectangular Odd 2\pi S | \( \frac{2\sqrt{7} JS}{\sqrt{f_{n\ell}}} \left( 1 - \frac{1}{2n^2 f_{n\ell}} \right)^{1/2} \) | \( \frac{\sqrt{2}}{Sn\ell \sqrt{f_{n\ell}}} \left( 1 - \frac{1}{2n^2 f_{n\ell}} \right)^{-1/2} \) |                                  |                                  |
| Diagonal Both \( 2\pi Sn\ell \) | \( \frac{2JS}{f_{n\ell}} \left[ 2f_{n\ell}(1 + (1 - \frac{1}{n\ell})J') - 1 \right]^{1/2} \) | \( \frac{2}{Sn\ell} \left[ 2f_{n\ell}(1 + (1 - \frac{1}{n\ell})J') - 1 \right]^{-1/2} \) |                                  |                                  |
| Path Int. Both \( 2\pi S\delta_{n\ell,\text{odd}} \) | \( 2JS \sqrt{f_{n\ell}} \) | \( 2/Sn\ell \sqrt{f_{n\ell}} \) |                                  |                                  |

Table 6

From this table we can extract the following conclusions,

- The values of \( \theta, c \) and \( g \) are block dependent. For example, for \( n\ell \) even we get \( g_{\text{rect}} = \frac{1}{\sqrt{2}} g_{\text{col}} \). Since the size of both blocks is different we could interpreted the previous relation as the RG relation bewtween \( g' \)s at two different length scales (\( n_B(\text{rect})/n_B(\text{col}) = 2 \)). Curiously enough the value of \( g_{\text{rec}} \) is smaller than the value of \( g_{\text{col}} \), suggesting that this ”one step RG flow” is similar to the RG flow of the 2+1 NLSM, where the coupling constant \( g \) decreases at longer distances in the ”renormalized classical region” [14]. After this first RG step, which truncates the ladder’s degrees of freedom down to those of the 1+1 NLSM, the value of \( g \) will start growing as a consequence of the 1+1 NLSM RG equations, so that the physics of the system will be dominated by the strong coupling regime.

From table 6 we find two ”RG-invariant” quantities: the \( \theta \) parameter and the perpendicular spin susceptibility.

- The invariance of the \( \theta \) parameter relies on its periodicity, which implies that for all purposes we may take \( \theta = 2\pi Sn\ell \). This is really a topological result for it only depends on the ”global data” \( S \) and \( n\ell \), and not on the values of the ladder coupling constants \( J \) and \( J' \). For odd ladders this result must be a consequence of the generalization, due to Affleck [15], of the well known theorem by Lieb-Schultz-Mattis [30], that asserts that in the infinite length limit the odd ladders must either have a degenerate ground state or else there are gapless excitations.
• The value of the bare perpendicular susceptibility \( \chi^0 \) is given for all the block choices by,

\[
\chi^0 = \frac{1}{c g} = \frac{n_\ell f_{n\ell}}{4J}
\]  

(63)

Notice that for large \( n_\ell \) the susceptibility per site \( \chi^0/n_\ell \) goes to a finite value.

It is quite interesting to compare (63) and the spin wave velocity (64),

\[
c = \frac{2JS}{\sqrt{f_{n\ell}}}
\]

(64)

with the corresponding expressions of the bare perpendicular spin susceptibility and spin wave velocities of a \( d \) dimensional sigma model [37],

\[
\chi^0 = \frac{1}{4dJa^d}, \quad c = 2\sqrt{dJa}
\]

(65)

where \( a \) denotes the lattice spacing.

The comparison of (63), (64) and (65) suggest somehow that ladders behave as \( d_{ladder} \) dimensional spin systems with,

\[
d_{ladder} = \frac{1}{f_{n\ell}}
\]

(66)

This naive definition of ”fractal” dimension of ladders helps to explain some numerical facts. First of all, if we choose \( J' = 0 \), then \( f_{n\ell}(0) = 1 \) and we get \( d_{ladder} = 1 \), which indeed corresponds to 1d chains. For the isotropic models and \( n_\ell \) large we get,

\[
d_{ladder} \sim \frac{2}{1 + \frac{1}{n_\ell \sqrt{2}}}
\]

(67)

which converges towards \( d = 2 \) (plane) from below.

A less heuristic proposal is to associate \( f_{n\ell} \) with a ”finite size” effect in the renormalization constants \( Z_c \) and \( Z_\chi \) of the ladder. Combining the recent work of Chakravarty [15], together with the results presented above, we shall propose a finite size correction of the spin susceptibility, spin velocity and spin-stiffness renormalization constants of ladders as follows,

\[
\begin{align*}
Z_\chi(S, n_\ell) &= 2f_{n\ell}Z_\chi(S) \\
Z_c(S, n_\ell) &= Z_c(S)/\sqrt{2f_{n\ell}} \\
Z_{\rho_s}(S, n_\ell) &= Z_{\rho_s}(S)
\end{align*}
\]

(68)

where \( Z_\chi(S), Z_c(S) \) and \( Z_{\rho_s}(S) \) are those of the 2d spin system (i.e. \( n_\ell = \infty \)). The last eq. in (68) follows from the first two thanks to the relation \( Z_{\rho_s} = Z_\chi Z_c^2 \). We shall give some numerical support to (68) in the next section.

The general formulation we have introduced above, will allow us to discuss certain important issues concerning the ladders.
Spin Gap of Even Ladders

Choosing the columnar-block description of the even ladders we deduce from table 6 and eq. (14) the following expression for the spin gap,

$$\Delta_{(1+1)}^{(1+1)} \sim J S^2 n_\ell \exp \left( -\pi S n_\ell \sqrt{f_{n_\ell}} \right)$$  \hspace{1cm} (69)

which predicts an exponential decay of the gap as a function of the number of legs [10]. This implies in particular that the spin gaps of the 2, 4 and 6 legged ladders should be related. Indeed one finds from numerical results of the isotropic ladders

$$\frac{\Delta_2 \Delta_6}{\Delta_4} \sim 1$$  \hspace{1cm} (70)

where we have used the data of table 7 together with the value $\Delta_2/J = 0.504$ [29, 38].

In agreement with [39], Chakravarty has recently derived the exponential fall off of the gap with $n_\ell$ using the 2+1 NLSM [15]. In his approach a spin ladder of width $n_\ell$ at zero temperature and periodic boundary conditions along the rungs, is equivalent to a Heisenberg plane of infinite extent at a finite temperature inversely proportional to $n_\ell$. This allows the use of the 2+1 NLSM results to study ladder systems. In particular the expressions for the spin gap and correlation length are given by (isotropic ladders $J' = J$), [15]

$$\Delta_{(2+1)}^{(2+1)} = \frac{16\pi^2 Z_{\rho_S}}{e} J S Z_{\rho_S} \exp \left( -\frac{\pi^2 Z_{\rho_S}}{\sqrt{2} Z_{\rho_S} L} \right) \left( 1 - \frac{1}{\pi \sqrt{2 S Z_{\rho_S} L}} \right)^{-1}$$  \hspace{1cm} (71)

$$\xi_{(2+1)}^{(2+1)} = \frac{4\pi^2 Z_{\rho_S} L}{\sqrt{2} Z_{\rho_S}} \exp \left( \frac{\pi^2 Z_{\rho_S}}{\sqrt{2} Z_{\rho_S} L} \right) \left( 1 - \frac{1}{\pi \sqrt{2 S Z_{\rho_S} L}} \right)$$

where $L$ and $a$ are the width and lattice spacing of the ladder ($L/a = n_\ell$). The exponential behaviour of eqs (71) and (71) agree in the classical limit $S \rightarrow \infty$ provided we choose $Z_c$ and $Z_{\rho_S}$ as the renormalization constants $Z_c(S, n_\ell)$ and $Z_{\rho_S}(S, n_\ell)$ defined in (68). This gives further support to the finite size correction propose in [15].

In tables 7 and 8 we give the values of the gap and correlation length of the 4 and 6 legged ladders obtained using: i) numerical methods (Quantum Monte Carlo [39, 10] and DMRG [38]), ii) the NLSM in 2+1 ( [15]) and iii) the finite size correction of the NLSM results of [15]. For the two later set of data the values of the renormalization constant for $S=1/2$ are chosen as $Z_c = 1.18$, $Z_{\rho_S} = 0.724$ [11].

| $n_\ell$ | DMRG | QMC | NLSM (2 + 1) | NLSM (2 + 1)+finite size |
|---|---|---|---|---|
| 4 | 0.190 | 0.16 - 0.17 | 0.268 | 0.209 |
| 6 | ? | 0.055 - 0.05 | 0.064 | 0.050 |

Table 7: Ladder’s spin gap.

| $n_\ell$ | DMRG | QMC | NLSM (2 + 1) | NLSM (2 + 1)+finite size |
|---|---|---|---|---|
| 4 | 5-6 | 10.3 | 6.23 | 7.37 |
| 6 | ? | ~ 30 | 26.2 | 31.6 |

Table 8: Ladder’s correlation length.

We observe from tables 7 and 8 that the finite size modification of the Chakravarty formulas (71) seems to give a rather good agreement with the numerical results. Further work needs to be done to settle this matter.
Limits of Applicability of the Ladder’s Map

We mentioned at the beginning of the construction of the map from the ladder into a unique NSLM field, that it would be valid for the intermediate coupling region \((J'/J \sim O(1))\). We shall next explain this point in more detail.

First of all let us consider the weak coupling region \(J'/J << 1\). As shown in eqs.(38) the masses of the higher modes of the ladder, at the linearized level, are of order \(\sqrt{JJ'}\). On the other hand the mass generated non perturbatively is given by (69). A consistent truncation of the massive modes then requires that the mass of these modes should be larger than the mass generated non perturbatively, which leads to,

\[
Ae^{-Bn} << \frac{J'}{J}
\]  

(72)

This equation implies that there is a lower critical value, \((J'/J)_c\) below which the truncation of the high energy modes, at least in the way it is done here, is not valid. In particular, in the weak coupling region the spin gap is approximately proportional to \(J'\) (for \(n = 2, \Delta \sim 0.41J'\)) [29, 42, 40]. This behaviour is not consistent with (69). On the other hand eq.(72) suggests that the range of applicability of our model is bigger as \(n\) increases.

Let us now consider the strong coupling regime \((J'/J >> 1)\), and choose as an example the two legged ladder. For the columnar block, the map (41) reads,

\[
S_1(n) = \frac{1}{2} I(n) + S(-1)^n \Phi(n) \\
S_2(n) = \frac{1}{2} I(n) - S(-1)^n \Phi(n)
\]

(73)

which plugged into the ladder Hamiltonian leads, without making any approximation, to

\[
H_{ladder} = \sum_n \frac{J'}{2} I(n)^2 + J \left( \frac{1}{2} I(n) I(n+1) - 2S^2 \Phi(n) \Phi(n+1) \right)
\]

(74)

If we now apply a gradient expansion in the fields \(I\) and \(\Phi\), we obtain the results given in (44) (see also table 9), for the case \(n = 2\). It is interesting to compare (74) with the lattice NSLM Hamiltonian (12). We see that the gradient expansion and truncation of the \(I\)’s field is crucial in order that (74) becomes a NSLM Hamiltonian. In the strong coupling regime, as can be seen from the mean field analysis of \(g\) and \(c\),

\[
c_{\text{latt}} = S\sqrt{2JJ'} \\
g_{\text{latt}} = \frac{1}{S}\sqrt{JJ'}
\]

(75)

If we replace these parameters into the gap formula obtained in the strong coupling regime (13) we get

\[
\Delta/J' = 1 - \frac{4S^2}{3} \frac{J}{J'} + 0.296S^4 \left( \frac{J}{J'} \right)^2
\]

(76)

If we replace in (76) \(S^2\) by \(S(S+1)\) and we particularize to the case \(S = 1/2\), then we get the correct behaviour of the gap in perturbation theory to order \(J'/J\) [30].

\[
\Delta/J' = 1 - \frac{J}{J'} + \frac{1}{2} \left( \frac{J}{J'} \right)^2 + \frac{1}{4} \left( \frac{J}{J'} \right)^3 + O\left( \left( \frac{J}{J'} \right)^4 \right)
\]

(77)
The previous discussion illustrates that in the strong coupling limit one has to be careful in making gradient expansions of the NLSM fields. A more detailed analysis is needed to clarify this matter.

**Haldane Phase in the 2-Legged Ladder**

An important question concerning even spin ladders is whether these systems are in a fundamentally new state or they are in a more familiar state, as for example the integer spin chains. This problem has been addressed by various authors arriving at different conclusions [45, 44]. In this section we shall apply the NLSM techniques to clarify this issue, finding support for the view of [44] that the 2 legged ladder is in the same phase than the $S = 1$ chain [43]. A more detailed discussion is delayed to the future.

The $n_l = 2$ ladder can be studied using columnar, rectangular and diagonal blocks. The values of the corresponding NLSM parameters are given in table 9.

| Block          | $\theta$ | $c$                           | $g$                        |
|----------------|----------|-------------------------------|---------------------------|
| Columnar       | 0        | $2JS \left(1 + \frac{J'}{2J}\right)^{1/2}$ | $\frac{1}{S} \left(1 + \frac{J'}{2J}\right)^{1/2}$ |
| Rectangular    | 0        | $2\sqrt{2}JS \left(1 + \frac{J'}{2J}\right)^{1/2}$ | $\frac{1}{S\sqrt{2}} \left(1 + \frac{J'}{2J}\right)^{1/2}$ |
| Diagonal       | $4\pi S$ | $2JS \left(1 + \frac{J'}{2J}\right)$ | $\frac{1}{S}$ |

Table 9: NLSM parameters of the 2 legged ladder

From table 9 we see that the parameters obtained using the diagonal blocks coincide with those of an effective spin chain with spin and exchange coupling given by,

$$S_{\text{eff}} = 2S, \quad J_{\text{eff}} = \frac{1}{4}(2J + J')$$

This is not an accidental fact. An alternative way to arrive to (78) is to construct an effective model in terms of the spin $2S$ states formed out by symmetrization of two spin $S$ states located in diagonal positions of the ladder (see fig. 5) [44], i.e.

$$S_{\text{eff}}(n) = S_1(n) + S_2(n - 1)$$

The Hamiltonian governing this effective spins is given, to lowest order in perturbation theory by the chain Hamiltonian,

$$H_{\text{eff}} = \frac{1}{4}(2J + J') \sum_n S_{\text{eff}}(n) S_{\text{eff}}(n + 1)$$

The NLSM parameters corresponding to this model (27) are precisely the ones given in table 9 for the diagonal blocks. Hence the 2 legged ladder is mapped into the same NLSM model as the $S = 1$ chain, which suggests that both systems are in the same phase. The relationship between the "standard" phase of the ladder, given by the RVB picture of [38], and that of the spin 1 chain appears, from the point of view of the NLSM, as the relation between two different types of blockings, namely the rectangular and the diagonal ones. This relation is given by,

$$\tilde{I} = 1 + S\Phi', \quad \Phi = \Phi$$

(81)
where $l, \Phi$ (resp. $\tilde{l}, \tilde{\Phi}$) are the NLSM fields associated to rectangular (resp. diagonal) blocks. Eq. (81) is a canonical transformation, identical to (29), which changes $\theta$ from 0 into $4\pi S$, leaving invariant the values of $c$ and $g$. This poses a puzzle since the later parameters are different for rectangular and diagonal blocks, except in the case $J' = 2J$. This discrepancy must be understood along the lines of our discussion about the different values of $g$ for columnar and rectangular blocks, and very likely do not affect the conclusion about the equivalence between the 2 legged ladder and the spin 1 chain.

**Spin Ladders with Dimerization**

The main property of uniform spin ladders is that there are no phase transitions as one varies the ratio $J'/J$. It is thus interesting to investigate the existence of new phases by enlarging the parameter space of the model. There are plenty of possibilities at hand: dimerization, frustration, spin defects, etc. We shall review here the first one. There are also many possible types of dimerization in a ladder. We shall consider below dimerizations only along the legs. In this case we may distinguish between columnar and staggered dimerizations, for which the intraleg coupling constant is given by,

$$J_a(n) = \begin{cases} 
J(1 + (-1)^n\gamma) & \text{(columnar)} \\
J(1 + (-1)^{n+a}\gamma) & \text{(staggered)} 
\end{cases}$$  \hspace{1cm} (82)

The dimerization parameter $\gamma$ will be chosen to vary in the interval $(1,-1)$ in order not to change the AF character of the legs. The rung coupling constant $J'$ may be positive or negative, so all together there are 4 types of models. In references [46] we studied the models with columnar dimerization and ferromagnetic rung coupling. If $J' \rightarrow -\infty$ the dimerized ladder becomes effectively a dimerized chain and one can apply the results known for chains.

A chain with spin $S$ and alternation parameter $\gamma$ can be mapped into a NLSM with $\theta$ given by [2, 47],

$$\theta = 2\pi S(1 + \gamma)$$  \hspace{1cm} (83)

The criteria that the NLSM models with $\theta = \pi \mod 2\pi$ are massless [4, 21] implies the existence of $2S$ critical points [17]. Indeed, as $\gamma$ varies from -1 to 1, the parameter $\theta$, given in (83), passes $2S$ times through $\pi$. If $S=1/2$ there is a single critical point corresponding to the non dimerized chain (i.e. $\gamma = 0$). If $S=1$ there should exists two critical points for $\gamma_c = \pm1/2$. Numerical computations show that there are indeed two critical points located at $\gamma_c \sim \pm0.25$ [48]. Hence the NLSM predicts the existence of these critical points but it is not precise about their localization. The NLSM prediction has also been confirmed for $S= 3/2$ [19] and $S=2$ [50].

Returning to ladders with columnar dimerization we expect, from the above discussion, that there should exist two critical lines in the plane $(\gamma, J'/J)$. In the example of the spin 1/2 two legged ladder there should exist two of these lines emanating at $J'/J = -\infty, \gamma_c = \pm0.25$ and ending at the origin (i.e. $J'/J = 0, \gamma = 0$) [49]. This critical lines separate the Haldane phase associated to the strong ferromagnetic ladder and a dimerized phase of weakly coupled chains.

The staggered dimerization with AF rung couplings has been studied in [71]. The behaviour of these ladders is very interesting and it is based on the map into the NLSM. The value of the $\theta$ parameter of a staggered ladder with AF rung coupling is given by [51],

\[ \theta = 2\pi S(1 + \gamma) \]
\[ \theta = 2\pi S n_\ell (1 + \gamma f_{n_\ell}) \]  

This formula contains (83) as a particular case (for \( n_\ell = 1 \)). Using the properties of \( f_{n_\ell} \) one can conjecture from (84) the existence of \( 2S n_\ell \) critical lines in the plane \((\gamma, J'/J)\) [51]. In the particular case of the spin 1/2 two legged ladder there should exist two critical lines separating also the RVB phase of the uniform ladders and a dimerized phase of weakly couple chains (see figure 6). The phase diagram of the 3 legged ladder (\( S=1/2 \)) contains besides the uniform critical line (\( \gamma = 0 \)) two more critical lines ending at the walls \(|\gamma| = 1\) (see figure 7).

The two cases analysed above (i.e. columnar/F-rung and staggered/AF-rung) have a very similar phase diagram which suggests some kind of relationship between them [52]. Other types of dimerizations will be considered in [52].

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Figure Captions

Fig.1.- A generic spin ladder with $n_{\ell}$ legs.
Fig.2.- Blocking of the spin chain needed to define the NLSM variables out of the spin ones.
Fig.3.- Columnar Blocking ($n_{\ell} = 4$).
Fig.4.- Rectangular Blocking ($n_{\ell} = 3$).
Fig.5.- Diagonal Blocking ($n_{\ell} = 3$).
Fig.6.- Phase diagrams of the 2 legged ladder with staggered dimerization for $S=1/2$ and $S=1$ (inset). If $|\gamma| = 1$ the ladders degenerate into the “snake” chains which are depicted in the margins of the figure.
Fig.7.- Phase diagram of the 3 legged ladder with columnar (lower half plane) and staggered dimerizations (upper half plane). Observe the existence of 3 critical lines that emerge from the origin.
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Figure 6
Figure 7