QUASIDETERMINANT SOLUTIONS OF NONCOMMUTATIVE EQUATION OF LANGMUIR OSCILLATIONS

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Abstract. In this article we present noncommutative analogue of equation of Langmuir oscillations and its Darboux solutions in additive form with their N-th generalization in terms of quasideterminants. We also derive the noncommutative ricatti equation from its linear system that yields the Bäcklund transformation for the commutative version of equation of Langmuir oscillations with classical limit. The last section, involves the derivation of reduction of that Bäcklund transformation to the Darboux solutions under commutative limit.

Keywords: Noncommutative equation of Langmuir oscillations, Darboux transformation, Quasideterminants, Riccati equation, Bäcklund transformation

1. Introduction

The differential difference equation

\[ u_{nt} = u_n(u_{n-1} - u_{n+1}) \]  

appears in analysis of spectrum structure associated to the Langmuir oscillations in plasma which completely integrable in classical framework as it possesses Lax representation [1]. The non-abelian analogue of equation (1)

\[ u_{nt} = u_{n-1}u_n - u_nu_{n+1} \]  

has been computed [2] from the compatibility of following linear system

\[ u_n\psi_{n+1} = \lambda\psi_n - \psi_{n-1} \]  

\[ \psi_{(n)t} = -u_nu_{n+1}\psi_{n+2}. \]

with its connection to discrete nonlinear Schrödinger equation [3][4]. Moreover Darboux transformation for that non-abelian version has been presented in [2] in multiplicative form as

\[ u_n[1] = \varphi_{n-2}\varphi_{n}^{-1}u_n(\varphi_{n-1}\varphi_{n+1}^{-1})^{-1}. \]

that generates all solutions with non-zero seed solution, means \( u_n \neq 0 \). Initially, That Darboux method was developed by [5] to find transformations on potential of the Schrödinger equation which satisfies Korteweg-de Vries equation [6] in framework of Lax formalism. Later on few remarkable results on DT were analysed by [7] to reveal its importance in theory of integrable system and the more efficient implementations.
on various nonlinear physical systems were developed by V. Matveev [8] to construct the exact solutions of these systems. The successful implementations of these transformations have been shown in the analysis of various mathematical features of graphene [9] and has also fruitful applications in cavity quantum electrodynamics [10, 11] for the dynamical analysis of the propagation of associated disturbance Moreover these transformations significantly extended to construct the determinantal solutions of noncommutative integrable systems such as in case of noncommutative Painlevé second equation [12] with its associated noncommutative Toda equation [13].

In this article, we construct the Darboux transformation (DT) for the noncommutative analogue of equation (1) in additive form as

\[ u_{n}[1] = u_{n} - \varphi_{n}^{'} \varphi_{n-1}^{-1}. \]  

which generates all possible solutions in zero background as seed solution \( u_{n} = 0 \), that is initial trivial solution of equation (2) and \( \varphi_{n}, \varphi_{n}^{'} \) are the particular solutions at particular values of \( \lambda \). Further the NC DT (6) is generalized to \( N \)-th form in terms of quasideterminant with exact solution in commutative case. The end section encloses the derivation of Ricatti equation associated to noncommutative analogue of equation (1) that yields its Backlund transformation which are reducible to NC DT (6).

2. **Noncommutative Darboux Solutions of Equations of Langmuir Oscillations**

For the noncommutative extension of equation (2), we consider \( u_{t} \) and independent variable \( t \) are purely noncommuting objects such as \([u_{t}, t] \neq 0 \) and the time derivation is defined as \( \partial_{t} f^{-1}(t) = -f^{-1} \partial_{t} ff^{-1} \), further the fields and their derivatives are also noncommuting elements. From the compatibility condition of linear systems (3) and (4) in noncommutative frame work, we obtain

\[ u_{nt} = u_{n-1}u_{n} - u_{n}u_{n+1} \]  

the NC version of equation (2) and the Darboux solution of above equation can be derived through its associated linear systems with Darboux transformation [15] on arbitrary function \( \psi_{n} \) defining in NC framework as

\[ \psi_{n}[1] = \psi_{n} - \varphi_{n} \varphi_{n+1}^{-1} \psi_{n+1} \]  

Now under above transformation (8) the linear system (3) can be written in following form

\[ u_{n}[1] \psi_{n+1}[1] = \lambda \psi_{n}[1] - \psi_{n-1}[1]. \]  

Now after substituting the values for transformed eigenfuctions from (8) into above transformed expression and then with the help of system (4) the resulting expression yields Darboux transformation on \( u_{n} \) as below

\[ u_{n}[1] = u_{n} - \lambda \varphi_{n} \varphi_{n-1}^{-1}. \]
we can also express above result as follow, taking \( \lambda \varphi_n = \varphi'_n \)

\[
u_n[1] = u_n - \varphi'_n \varphi^{-1}_{n-1}.
\]
(11)

The above transformation involves new solution \( u_n[1] \), old solution \( u_n \) of equation (2) also called the seed solution and the particular solutions of linear systems (3) and (4).

Here the comparison of Darboux solution (11) and with result on Darboux solution obtained in [2] shows a difference, the transformations (11) are additive holds for all seed solution even for the trivial solution \( u_n = 0 \) of equation (7) in NC frame as well as in non-abelian case and also in classical framework under commutative limit.

The one fold Darboux transformation (3) with its second iteration can be expressed in form of quasideterminant as below with setting \( \psi_n = \psi_0, \psi_{n+1} = \psi'_0 \) and \( \varphi_n = \varphi_1, \varphi_{n+1} = \varphi'_1 \) and defining \( \lambda \psi = \psi' \), then we can present one fold Darboux transformation in terms of quasideterminant as

\[
\psi_n[1] = \begin{vmatrix}
\psi_0 & \psi_1 \\
\lambda_0 \psi_0 & \lambda_1 \psi_1
\end{vmatrix}
\]
(12)

and the two fold NC Darboux transformation can be evaluated as

\[
\psi_n[2] = \begin{vmatrix}
\psi_0 & \psi_1 & \psi_2 \\
\lambda_0 \psi_0 & \lambda_1 \psi_1 & \lambda_2 \psi_2 \\
\lambda^2_0 \psi_0 & \lambda^2_1 \psi_1 & \lambda^2_2 \psi_2
\end{vmatrix}
\]
(13)

further can be generalized to \( N \)-th form as below

\[
\psi_n[N] = \begin{vmatrix}
\psi_0 & \psi_1 & \cdots & \psi_{N-1} & \psi_N \\
\lambda_0 \psi_0 & \lambda_1 \psi_1 & \cdots & \lambda_{N-1} \psi_{N-1} & \lambda_N \psi_N \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\lambda^{N-1}_0 \psi_0 & \lambda^{N-1}_1 \psi_1 & \cdots & \lambda^{N-1}_{N-1} \psi_{N-1} & \lambda^N_N \psi_N
\end{vmatrix}
\]
(14)

Now in the similar way the one fold Darboux solution (3) can be generalized to \( N \)-th form in terms of quasideterminants as

\[
u_n[N + 1] = u_n[N] - \psi'_n[N] \psi^{-1}_{n-1}[N].
\]
(15)

Here for \( N = 0 \), we have \( u_n[0] = u_n \), initial solution and \( \psi_n[0] = \varphi_n \), \( \psi_{n-1}[0] = \varphi_{n-1} \) are the particular solutions. Further, we may construct the \( N \) fold expression for \( \psi_{n-1}[N] \) with the help of (3) with the replacement of \( n \) by \( n-1 \) and setting \( \psi_{n-1} = \lambda_0 \psi_0 \)

\[
\psi_{n-1}[N] = \begin{vmatrix}
\psi_N & \psi_{N-1} & \cdots & \psi_1 & \psi_0 \\
\lambda_N \psi_N & \lambda_{N-1} \psi_{N-1} & \cdots & \lambda_1 \psi_1 & \lambda_0 \psi_0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\lambda^{N-1}_N \psi_N & \lambda^{N-1}_{N-1} \psi_{N-1} & \cdots & \lambda^{N-1}_1 \psi_1 & \lambda^{N-1}_0 \psi_0 \\
\lambda^N_N \psi_N & \lambda^N_{N-1} \psi_{N-1} & \cdots & \lambda^N_1 \psi_1 & \lambda^N_0 \psi_0
\end{vmatrix}
\]
(16)
here we can assume $\psi_{n-1}[0] = \varphi_{n-1}$ as the initial untransformed solution, that is particular solution of the linear system.

3. **NC Riccati equation**

To construct the NC riccati equation, let us start with setting

$$R_n = \psi_{n}\psi_{n-1}^{-1}$$

(17)

and taking time derivation of above expression (17), we obtain

$$R'_n = u_{n-1}R_n - \lambda u_n + \lambda^2 R_n - \lambda R_n^2 - R_n u_n$$

(18)

by using the values from linear systems (2) and (3), where $R' = R_t$. Now substituting the value for $(u_n - \lambda R_n)$ from system (3), finally we get NC riccati equation associated to equation (7) as below

$$R'_n = u_{n-1}R_n - R_n u_n + \lambda R_{n-1} - \lambda R_n^2$$

(19)

In next section, we show the reduction of above NC riccati equation (19) into Bäcklund transformation under the commutative limit, further can be simplified to the Darboux transformation for the classical analogue of equation (7).

4. **NC Ricatti equation with commutative limit**

This can be shown that for the classical version (11) the Darboux solution (6) will take the following form

$$u_n[1] = u_n - \frac{\varphi'_n}{\varphi_{n-1}}.$$  

(20)

where $u_n$ and $\varphi_n$ are scalars with same linear systems (3) and (4). Now the NC ricatti equation under the commutative limit becomes

$$R'_n = (u_{n-1} - u_n)R_n + \lambda R_{n-1} - \lambda R_n^2.$$  

(21)

The above ricatti equation (21) under charge parity time reversal (CPT) symmetry transformation [16] becomes

$$- R'_n = (u_{n-1} [1] - u_n [1])R_n + \lambda R_{n-1} - \lambda R_n^2.$$

(22)

where $u_n[1]$ is new solution and $R'_n$ is replaced by $-R'_n$. Now subtracting (21) from (22), we get

$$u_{n-1}[1] - u_{n-1} = (u_n[1] - u_n) + (\lambda [1] - \lambda) \frac{R_{n-1}}{R_n} - (\lambda [1] - \lambda) R_n.$$  

(23)

that equation may be regarded as Bäcklund transformation for equation (11) with Darboux solution (20). Now we assume that $\lambda [1] - \lambda = \epsilon$ a very small difference, then above expression can be written as
\[
\frac{u_{n-1}[1] - u_n[1]}{\epsilon} = -\left(\frac{u_{n-1} - u_n}{\epsilon}\right) + \frac{R_{n-1}}{R_n} - R_n \tag{24}
\]
and under the limiting case \(\epsilon \to 0\), then finally we get
\[
\frac{du_n[1]}{dt} = \frac{du_n}{dt} + \frac{R_{n-1}}{R_n} - R_n. \tag{25}
\]
The above expression becomes equivalent to the Darboux solution \([20]\), \(u_n[1] = u_n - \frac{\phi_n'}{\phi_{n-1}}\), with condition \(\frac{R_{n-1}}{R_n} - R_n = \frac{d}{dt} \frac{\phi_n'}{\phi_{n-1}}\) and taking constant of integration as zero.

Here it has been shown that the commutative version of riccati equation yields the Darboux solution through the Bäcklund transformation in classical framework.

5. Conclusion:

In this paper, the noncommutative analogue equations of Langmuir oscillations has been presented with its Darboux transformation in additive structure as most of the integrable posses in noncommutative as well as in classical frameworks. Further \(N\)-fold darboux solutions have presented in term of quasideterminants applicable for zero background seed solution. Moreover, the associated NC ricatti equation is presented which reduced to the Darboux expression through the Bäcklund transformation in classical framework. Further motivation involves its connection to Discrete noncommutative NLS equation as possesses in classical case and also to investigate its solutions in terms of quantum determinants for its matrix version.

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