Edge State Transfer

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Abstract

Let $L$ denote the Laplacian matrix of a graph $G$. We study continuous quantum walks on $G$ defined by the transition matrix $U(t) = \exp(itL)$. The initial state is of the edge state form, $e_a - e_b$ with $(a, b)$ being an edge of $G$. We provide two ways to construct infinite families of graphs that have perfect edge state transfer. We study a “transitivity” phenomenon which cannot occur in vertex state transfer. We characterize perfect edge state transfer on paths and cycles. We also study the case when quantum walks are generated by the unsigned Laplacians of underlying graphs and the initial states are of the plus state form, $e_a + e_b$. When the underlying graphs are bipartite, plus state transfer is equivalent to edge state transfer.

1 Introduction

Let $G$ be a graph. The evolution of a continuous quantum walk on $G$ is given by the matrices

$$U(t) = \exp(itH), \quad (t \in \mathbb{R}).$$

Here $H$ is a matrix, called the Hamiltonian of the walk, and is usually either the adjacency matrix, the Laplacian or the signless Laplacian of $G$. In any case, $H$ is Hermitian and its rows and columns are indexed by the vertices of $G$. If $n = |V(G)|$, then the walk represents a quantum system with an $n$-dimensional state space. We identify the states of the system by density matrices, i.e., positive semidefinite matrices with trace 1. The physically meaningful questions are the form: “Given that the quantum system is initially in state represented
by a density matrix $D_0$, what is the probability that, at time $t$, its state is $D_1$?

Let $e_1, e_2, \cdots, e_n$ denote the standard basis for $\mathbb{C}^n$. Then $D_r = e_r e_r^T$ is a density matrix, and the mathematical questions reduce to question about the absolute value of $U(t)_{r,s}$ (for given vertices $r$ and $s$). Thus if $r \neq s$ and $|U(t)_{r,s}| = 1$ at some time $t$, we say that have perfect state transfer from vertex $r$ to vertex $s$. Perfect state transfer is a potentially useful tool in quantum computation, and so there is a considerable literature on the topic [1]. In [1], it was observed that most of the results on the topic only require the fact that the density matrices $e_r e_r^T$ are real. This suggests strongly that there may be other density matrices of interest. For a graph theorist, this leads to the idea of using density matrices based on edges, not just on vertices; once we are using edges it seems more natural to use the Laplacian or signless Laplacian.

The goal of this paper is to develop the theory of state transfer between state of the form $e_r - e_s$, where $\{r, s\} \in E(G)$, using the Laplacian of $G$ as the Hamiltonian.

Edge state transfer and vertex state transfer share a lot of properties and we will discuss those common properties in Section 3.1 and Section 4. But surprisingly, the following two tables shows that perfect edge state transfer occurs more often than perfect vertex state transfer in the same set of graphs.

| $G_n$ | Total | Adjacency Matrix | Prop. | Laplacian | Prop. |
|-------|-------|-----------------|------|-----------|------|
| $G_5$ | 21    | 7               | 33.3%| 13        | 61.9%|
| $G_6$ | 112   | 10              | 8.9% | 50        | 44.6%|
| $G_7$ | 853   | 23              | 2.7% | 191       | 22.4%|
| $G_8$ | 11117 | 40              | 0.4% | 1265      | 11.4%|

Table 1: Periodic Vertices

Let $G_n$ denote the set of connected graphs on $n$ vertices and the second column shows the cardinality of the corresponding $G_n$. The third column shows the number of graphs with periodic vertices in $G_n$ when the adjacency matrices are the Hamiltonians and the fourth column shows the corresponding proportion. The fifth column shows the number of graphs with periodic vertices in $G_n$ when the Laplacians are the Hamiltonians and the last column shows the corresponding propor-
We will see that periodicity is a necessary condition for a state to have perfect state transfer. Given the rarity of periodic vertices shown in Table 1, one cannot expect that perfect state transfer occurs more often than periodic vertices in vertex state transfer. From Table 2, we can see that there are more periodic edges than periodic vertices.

Perfect state transfer is a significant phenomenon in quantum communication but quite rare in quantum walks. We always want to find more graphs with perfect state transfer. On a fixed number of vertices, there are more graphs with perfect edge state transfer than graphs with perfect vertex state transfer. This is a huge advantage of Laplacian edge state transfer and this is also why Laplacian edge state transfer is interesting.

We prove that perfect edge state transfer is preserved under complementations and taking Cartesian product. This helps us to construct more examples of graphs with perfect edge state transfer.

Transitivity is one phenomenon that only can occurs when the initial state is edge states that will be discussed in Section 6. Although edge state transfer also has monogamy and symmetry properties that limits the evolution behavior, involving characteristic vectors of two vertices as initial states provides more flexibility than the case when the initial state is the characteristic vector of a single vertex.

We prove that perfect edge state transfer occurs on $P_n$ if and only if $n = 3$ or 4. We also prove that $C_4$ is the only cycle that has perfect edge state transfer. So far, those are the only examples we found where perfect state transfer has a strong correspondence with perfect vertex state transfer of their line graphs.

We also study the case when the unsigned Laplacian of the underlying graph is used as Hamiltonian and the initial state of the form
We prove that perfect edge state transfer is equivalent to perfect plus state transfer when the underlying graph is bipartite. This allows to prove the same result for perfect plus state transfer. That is, $P_3, P_4$ are the only paths and $C_4$ are the only cycles that have perfect plus state transfer.

## 2 Preliminaries

### 2.1 Edge State Transfer

Let $G$ be a graph with $n$ vertices. Let $A$ denote the adjacency matrix of $G$ and let $\Delta$ denote the degree matrix of $G$. Then the Laplacian of $G$ is the matrix such that

$$L = \Delta - A.$$ 

When $L$ is the Hamiltonian associated to the quantum walk on $G$, the transition matrix is

$$U(t) = \exp(itL).$$

A edge $(a, b)$ of $G$ represents the quantum edge state

$$e_a - e_b,$$

where $e_a, e_b \in \mathbb{R}^n$ are the characteristic vectors of $a, b$ respectively. We want quantum states to be represented by unit vectors, so when we perform computations about edge state transfer, we use the normalized edge state

$$\frac{1}{\sqrt{2}}(e_a - e_b),$$

but except for that, we always use $e_a - e_b$ to denote our edge states for convenience. Unless explicitly stated otherwise, we use edge states in the continuous quantum walk on $G$ generated by the Laplacian of $G$.

There is perfect edge state transfer from $(a, b)$ to $(c, d)$ if there exists a complex scalar $\gamma$ satisfying that

$$U(t)(e_a - e_b) = \gamma (e_c - e_d)$$

for some non-negative time $t$. In terms of the probability distribution, there is perfect state transfer from $e_a - e_b$ to $e_c - e_d$ at time $t$ if and only if

$$\left| \frac{1}{2}(e_c - e_d)U(t)(e_a - e_b) \right|^2 = 1.$$
We say edge \((a, b)\) is periodic with period \(\tau\) if it has perfect state transfer to itself at time \(\tau\). That is, there exist a complex scalar \(\gamma\) such that
\[ U(\tau)(e_a - e_b) = \gamma(e_c - e_d). \]

Another way to represent a quantum state is density matrices. A density matrix is a semidefinite matrix of trace 1. A density matrix \(D\) represents a pure state if \(\text{rk}(D) = 1\). If \(e_a\) denotes the standard basis vector in \(\mathbb{C}^{|V(G)|}\) indexed by the vertex \(a\) in graph \(G\), then
\[ D = \frac{1}{2}(e_a - e_b)(e_a - e_b)^T \]
is a pure state associated with the edge \((a, b)\) in \(G\), which we call the density matrix of edge \((a, b)\).

Given a density matrix \(D\) as the initial state of a continuous quantum walk, then the state that \(D\) is transferred to at time \(t\) is given by
\[ D(t) = U(t)DU(-t), \]
where \(U(t) = \exp(itL)\) is the usual transition matrix associated with \(G\) whose Laplacian matrix is \(L\). There is perfect state transfer between density matrices \(P\) and \(Q\) means that there is a time \(t\) such that
\[ Q = U(t)PU(-t). \]
We say a state \(P\) is periodic if there is a time \(t\) such that
\[ P = U(t)PU(-t). \]

### 2.2 Transition Matrices

Let \(G\) be a graph with the Laplacian matrix \(L\). We assume that the eigenvalues of \(L\) are \(\theta_1, \theta_2, \cdots, \theta_n\). We know that \(L\) is a real symmetric matrix. Then the spectral decomposition of \(L\) is
\[ L = \sum_{i=1}^{n} \theta_i E_i, \tag{1} \]
where the matrices \(E_1, E_2, \cdots, E_n\) satisfy:
(i) \(\sum_{i=1}^{n} E_i = I,\)
(ii) \(E_r E_s = \begin{cases} 0, & \text{if } r \neq s; \\ E_r, & \text{if } r = s. \end{cases}\)
A matrix $E$ is an idempotent if $E^2 = E$. The matrices $E_1, E_2, \cdots, E_n$ in Equation 1 are called spectral idempotents and $E_r$ represents the orthogonal projection onto the $\theta_r$-eigenspace of $L$.

2.1 Theorem. Let $M$ be a real symmetric matrix and let $\sum_{i=1}^{n} \theta_i E_i$ denote the spectral decomposition of $M$. If $f(x)$ is an analytic function defined on the eigenvalues of $M$, then

$$f(M) = \sum_{i=1}^{n} f(\theta_i) E_i.$$ 

This standard algebraic graph theory result helps us to obtain the spectral decomposition of the transition matrix $U(t)$, which brings the spectrum of the underlying graph into the picture of continuous quantum walk.

Let $\sum_{i=1}^{n} \theta_i E_i$ denote the spectral decomposition of the Laplacian matrix of a graph $G$. Then the transition matrix of edge state transfer on $G$ is

$$U(t) = \sum_{i=1}^{n} e^{it \theta_i} E_i. \tag{2}$$

For example, the spectral decomposition of the Laplacian $L$ of $P_3$ is

$$L = 0 \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} + 1 \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + 3 \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$ 

Then the transition matrix associated is

$$U(t) = e^{0it} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} + e^{it} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + e^{3it} \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} + \frac{1}{2} e^{it} + \frac{1}{6} e^{3it} \\ \frac{1}{3} - \frac{1}{2} e^{it} + \frac{1}{6} e^{3it} \\ \frac{1}{3} - \frac{1}{2} e^{3it} + \frac{1}{6} e^{3it} \end{pmatrix}.$$
When $t = \frac{\pi}{2}$, the transition matrix of $P_3$ is

$$U \left( \frac{\pi}{2} \right) = \begin{pmatrix}
\frac{1}{3} + \frac{1}{3}i & \frac{1}{3} - \frac{2}{3}i \\
\frac{1}{3} + \frac{1}{3}i & \frac{1}{3} + \frac{1}{3}i \\
\frac{1}{3} - \frac{2}{3}i & \frac{1}{3} + \frac{1}{3}i \\
\frac{1}{3} + \frac{1}{3}i & \frac{1}{3} + \frac{1}{3}i
\end{pmatrix}$$

Then we can see that

$$| \frac{1}{2} (e_1 - e_2) U \left( \frac{\pi}{2} \right) (e_0 - e_1) |^2$$

$$= \left| \frac{1}{2} \left( \frac{1}{3} + \frac{1}{3}i - \left( \frac{1}{3} - \frac{2}{3}i \right) - \left( \frac{1}{3} - \frac{2}{3}i \right) + \frac{1}{3} + \frac{1}{3}i \right) \right|^2$$

$$= \left| \frac{1}{2} (2i) \right|^2$$

$$= 1,$$

which implies that there is perfect edge state transfer from $e_0 - e_1$ to $e_1 - e_2$ at time $\frac{\pi}{2}$ in $P_3$. Later we will prove actually $P_3$ and $P_4$ are the only paths that have perfect edge state transfer.

### 2.3 Eigenvalue Supports

From Equation 2, we also can see that the Laplacian eigenvalues of $G$ play a large role in the edge state transfer. Let $E_r$ be a spectral idempotent such that

$$E_r(e_a - e_b) = 0.$$

Then we can see that when we talk about the state transfer started in the state $e_a - e_b$, the eigenvalue $\theta_r$ and its idempotent $E_r$ contribute nothing to the evolution.

The eigenvalue support of the state $e_a - e_b$ is the set of Laplacian eigenvalues $\theta_r$ such that the corresponding idempotent $E_r$ satisfies

$$E_r(e_a - e_b) \neq 0.$$
Thus, when we talk about quantum state transfer initialized in the state $e_a - e_b$, we only care about the eigenvalues in the eigenvalue support of $e_a - e_b$. Recall the example of $P_3$ in previous section. From the spectral decomposition of the Laplacian of $P_3$, we can see that the eigenvalue supports of edge states $e_0 - e_1$ and $e_1 - e_2$ are the same, that is, $\{1, 3\}$.

We say two states $e_a - e_b$ and $e_c - e_d$ are strongly cospectral in $G$ if and only if for each spectral idempotent $E_r$ of the Laplacian of $G$, we have

$$E_r(e_a - e_b) = \pm E_r(e_c - e_d).$$

Thus, we can see that if two states are strongly cospectral in graph $G$, then their eigenvalue supports are the same. From the definition of strong cospectrality, the theorem and the corollary below follows immediately.

2.2 Theorem. If there is a perfect state transfer between $e_a - e_b$ and $e_c - e_d$ in graph $G$, then $e_a - e_b$ and $e_c - e_d$ are strongly cospectral. \hfill \qed

2.3 Corollary. If there is perfect state transfer between $e_a - e_b$ and $e_c - e_d$ in $G$, then $e_a - e_b$ and $e_c - e_d$ have the same eigenvalue support. \hfill \qed

Now we let $\Lambda_{ab,cd}^+$ denote the set of eigenvalues such that

$$E_r(e_a - e_b) = E_r(e_c - e_d)$$

and let $\Lambda_{ab,cd}^-$ denote the set of eigenvalues such that

$$E_r(e_a - e_b) = -E_r(e_c - e_d).$$

It is easy to see that

$$\Lambda_{ab} = \Lambda_{cd} = \Lambda_{ab,cd}^+ \cup \Lambda_{ab,cd}^-,$$

$$\Lambda_{ab,cd}^+ \cap \Lambda_{ab,cd}^- = \emptyset.$$

Using strongly cospectrality, we can derive a characterization of perfect edge state transfer. Since the proof is similar to the proof of perfect vertex state transfer, we omit the proof here. One can refer to [2, Theorem 2.4.2] for details.

2.4 Lemma. Let $X$ be a graph and $(a,b), (c,d) \in E(X)$. Perfect edge state transfer between $e_a - e_b$ and $e_c - e_d$ occurs at time $\tau$ if and only if all of the following conditions hold.
(a) Edge states $e_a - e_b$ and $e_c - e_d$ are strongly cospectral. Let $	heta_0 \in \Lambda_{ab,cd}^+$. 

(b) For all $\theta_r \in \Lambda_{ab,cd}^+$, there is a $k$ such that $\tau(\theta_0 - \theta_r) = 2k\pi$.

(c) For all $\theta_r \in \Lambda_{ab,cd}^-$, there is a $k$ such that $\tau(\theta_0 - \theta_r) = (2k+1)\pi$. $\square$

3 Perfect State Transfer & Periodicity

In this section, we introduce symmetry and monogamy properties of perfect edge state transfer and give a characterization of periodicity in terms of eigenvalues. We also give a characterization of a fixed state, which is a special case of periodicity.

3.1 Basic Properties

The original results in this section can be found in [6] stated and proved in terms of vertex states. Since proofs of the results using edge states are very similar to the proofs using vertex states, here we just state the results without proof. One can see [1] for detailed proofs using edge states.

Just like perfect vertex state transfer, symmetry and monogamy are two basic properties of perfect edge state transfer.

3.1 Theorem. There is perfect state transfer from $e_a - e_b$ to $e_c - e_d$ in graph $G$ at time $\tau$ if and only if there is perfect state transfer from $e_c - e_d$ to $e_a - e_b$ at time $\tau$. $\square$

3.2 Theorem. Suppose that $e_a - e_b$ has perfect state transfer at time $\tau$ in graph $G$. Then $e_a - e_b$ is periodic at time $2\tau$. $\square$

Thus, being periodic is a necessary condition for a state to have perfect state transfer.

3.3 Theorem. If there is perfect state transfer between edge $(a, b)$ and $(c, d)$ in graph $G$, then both edges $(a, b)$ and $(c, d)$ are periodic with the same minimum period. If the minimum period is $\sigma$, then perfect state transfer between the two edges occurs at time $\frac{1}{2}\sigma$. $\square$
Since perfect state transfer occurs exactly at half of the period, the monogamy property of perfect state transfer follows immediately.

### 3.4 Corollary
For any edge \((a, b)\), there is at most one edge \((c, d)\) such that there is perfect state transfer from \((a, b)\) to \((c, d)\).  

Being periodic is the necessary condition for a state to have perfect state transfer and the period of a state involved in perfect state transfer can tell us the exact time when the perfect state transfer occurs. So periodicity of states provides a useful tool for analysis of perfect state transfer.

### 3.5 Theorem (the Ratio Condition)
Let \(U(t)\) be the transition matrix corresponding to a graph \(G\). Let \(\sum_r \theta_r E_r\) be the spectral decomposition of the Laplacian of \(G\). Then \(e_a - e_b\) is periodic in \(G\) if and only if

\[
\frac{\theta_r - \theta_s}{\theta_k - \theta_l} \in \mathbb{Q}
\]

for any \(\theta_r, \theta_s, \theta_k, \theta_l\) in the eigenvalue support of \((e_a - e_b)\) with \(\theta_l \neq \theta_k\).

The following theorem can be viewed as a corollary of the ratio condition. The original proof can be found in Coutinho and Godsil \[3\] stated in terms of vertex states. Detailed proof using edge state transfer can be found in \[1\].

### 3.6 Theorem
Let \(G\) be a graph with the Laplacian matrix \(L\) and let \((a, b)\) be an edge of \(G\) with eigenvalue support \(S\). Then \(e_a - e_b\) is periodic in \(G\) if and only if either:

1. All the eigenvalues in \(S\) are integers;
2. There is a square-free integer \(\Delta\) such that all eigenvalues in \(S\) are quadratic integers in \(\mathbb{Q}(\sqrt{\Delta})\), and the difference of any two eigenvalues in \(S\) is an integer multiple of \(\sqrt{\Delta}\).

### 3.7 Corollary
If \(e_a - e_b\) is periodic in graph \(G\), then any two distinct eigenvalues in the eigenvalue supports of \(e_a - e_b\) differ by at least one.

### 3.8 Corollary
If an edge is periodic in graph \(G\) with period \(\tau\), then \(\tau \leq 2\pi\).
Using the ration condition, we give the following necessary and sufficient conditions for perfect edge state transfer. Due to the similarity of the proof, we omit the proof here. One can refer to [2, Theorem 2.4.4] for details.

3.9 Theorem. Let $X$ be a graph. Then $X$ admits perfect edge state transfer between $e_a - e_b$ and $e_c - e_d$ if and only if all of the following conditions hold.

(i) Edge states $e_a - e_b$ and $e_c - e_d$ are strongly cospectral. Let $\theta_0$ be an eigenvalue in $\Lambda_{ab,cd}^+$.  

(ii) The eigenvalues in the eigenvalue support of $e_a - e_b$ are either all integers or all quadratic integers. Moreover, there is a square-free integer $\Delta$ such that all eigenvalues in the eigenvalue support are quadratic integers in $\mathbb{Q}(\sqrt{\Delta})$, and the difference of any two eigenvalues in the eigenvalue support is an integer multiple of $\sqrt{\Delta}$.

(iii) Let $g = \gcd \left( \left\{ \frac{\theta_0 - \theta_r \sqrt{\Delta}}{\sqrt{\Delta}} \right\} \right)$. Then

(a) $\theta_r \in \Lambda_{ab,cd}^+$ if and only if $\frac{\theta_0 - \theta_r}{g \sqrt{\Delta}}$ is even, and

(b) $\theta_r \in \Lambda_{ab,cd}^-$ if and only if $\frac{\theta_0 - \theta_r}{g \sqrt{\Delta}}$ is odd.

3.2 Fixed Edge States

Let $v$ be a vertex of graph $G$. We use $N(v)$ to denote the neighbours of $v$ in $G$. We prove that an edge state $e_a - e_b$ is fixed, i.e., for all non-negative $t \in \mathbb{R}$,

$$U(t)(e_a - e_b) = \gamma(e_a - e_b)$$

with $\gamma$ being a norm-one complex scalar if and only if vertices $a, b$ are twins in $G$, which means it can never have perfect edge state transfer. Notice that a fixed state can be viewed as a state that is periodic at $t$ for any non-negative $t \in \mathbb{R}$

3.10 Lemma. The edge state $e_a - e_b$ is fixed in $G$ if and only if the density matrix of $e_a - e_b$ and the Laplacian matrix $L$ of $G$ commute.

Proof. Let $D$ denote the density matrix of $e_a - e_b$. For any non-negative $t$, we have

$$U(t)DU(-t) = D$$

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if and only if
\[ U(t)D = DU(t), \]
which means that \( D \) commutes with \( U(t) \) for all \( t \). This is equivalent to that \( D \) commutes with \( L \) as \( U(t) = \sum_{m} \frac{(it)^m}{m!} L \).

**3.11 Lemma.** Let \( D \) denote the density matrix of \( e_a - e_b \) and let \( L \) denote the Laplacian matrix of graph \( G \). Then \( LD = DL \) if and only if vertices \( a, b \) are twins in \( G \).

**Proof.** We know that \( DL = LD \) if and only if \( LD \) is symmetric, which is equivalent to \( N(a) = N(b) \). \( \square \)

**3.12 Theorem.** The edge state \( e_a - e_b \) is fixed if and only if vertices \( a, b \) are twins in \( G \). \( \square \)

**3.13 Corollary.** Let \( a, b \) be two vertices in \( G \). If \( N(a) = N(b) \), then \( e_a - e_b \) has no perfect edge state transfer.

**Proof.** From previous theorem, we know that \( e_a - e_b \) is fixed. \( \square \)

**3.14 Theorem.** Edge states \( e_a - e_b \) are fixed if and only if there is only one eigenvalue in the eigenvalue support of \( e_a - e_b \) and the eigenvalue must be an integer.

**Proof.** Let \( \sum_r \theta_r E_r \) denote the spectral decomposition of the Laplacian of the graph \( G \). For any time \( t \), we have
\[
U(t)(e_a - e_b) = \sum_r \theta_r E_r(e_a - e_b) = \gamma(e_a - e_b)
\]
for some complex scalar \( \gamma \) with \( |\gamma| = 1 \). This is equivalent to that for all eigenvalues \( \theta_r \), we have
\[
e^{it\theta_r} E_r(e_a - e_b) = \gamma E_r(e_a - e_b),
\]
which gives us that
\[
\gamma = e^{it\theta_r}
\]
for all \( \theta_r \) at any time \( t \).

It follows that \( e_a - e_b \) is fixed if and only if all the eigenvalues in the eigenvalue support coincide. If \( \theta_r \) is an eigenvalue in the eigenvalue support, then all the algebraic conjugates of \( \theta_r \) are also in the eigenvalue suppport. So we can conclude that \( e_a - e_b \) is fixed if and only if the eigenvalue support of \( e_a - e_b \) is \( \{\theta\} \) for some integer \( \theta \). \( \square \)
3.15 Corollary. In graph $G$, vertices $a$ and $b$ are twins if and only if the eigenvalue support of $e_a - e_b$ consists of one integer eigenvalue. □

This is a feature that distinguishes vertex state transfer and edge state transfer. In a connected graph with at least two vertices, the eigenvalue support of a vertex state must have size at least two while the eigenvalue support of an edge state can have size one.

4 Algebraic Properties

In this section, we show how algebraic properties of the underlying graphs can help us to get more information about state transfer.

4.1 Theorem. Let $G$ be a graph and $(a, b)$ is an edge of $G$. If there is a permutation $\sigma \in \text{Aut}(G)$ such that $\sigma(e_a - e_b) = e_c - e_d$, then $e_a - e_b$ and $e_c - e_d$ have the same eigenvalue support.

Proof. Let $P$ denote the permutation matrix associated with $\sigma \in \text{Aut}(G)$. Since $\text{Col}(A)$ is invariant under $P$, we have that

$$PA = AP.$$ 

Let $\Delta$ denote the degree matrix of $G$. We know that $\Delta$ is a diagonal matrix, so that $P$ commutes with $\Delta$. Let $L$ denote the Laplacian matrix of $G$. Thus, we have that

$$LP = (\Delta - A)P = P(\Delta - A) = PL.$$ 

Let $\sum_r \theta_r E_r$ denote the spectral decomposition of $L$. Since $E_r$ is a polynomial in $L$ and hence, we know that $L$ commutes with $E_r$. Then we have that

$$PE_r(e_a - e_b) = E_rP(e_a - e_b) = E_r(e_c - e_d).$$ 

We know that $\theta_r$ is not in the eigenvalue support of $e_a - e_b$ if and only if

$$E_r(e_a - e_b) = 0.$$ 

Since $P$ acts on $E_r(e_a - e_b)$ by permuting its entries, we can see that $E_r(e_a - e_b) = 0$ if and only if

$$PE_r(e_a - e_b) = E_r(e_c - e_d) = 0.$$ 

Thus, we can conclude that $\theta_r$ is not in the eigenvalue support of $e_a - e_b$ if and only if $\theta_r$ is not in the eigenvalue support of $e_c - e_d$. □
One immediate consequence is that all the edge states in an edge-transitive graph have the same eigenvalue support. Actually the eigenvalue support of an edge state of a edge-transitive graph is consist of all the non-zero eigenvalues.

4.2 Theorem. If $G$ is a connected edge-transitive graph, then the eigenvalue support of edge states of $G$ consists of all the non-zero eigenvalues.

Proof. Let $\sum_r \theta_r E_r$ denote the spectral decomposition of the Laplacian matrix of $G$. Assume towards contradictions that $E_s$ is the spectral idempotent corresponding to a non-zero eigenvalue $\theta_s$ such that

$$E_s(e_a - e_b) = 0$$

for all $(a, b) \in E(G)$. Since $G$ is connected, for any $u, v \in V(G)$, there exist a path $P$ from $u$ to $v$. Then for any two vertices $i, j$ on $P$, we must have

$$E_se_i = E_se_j.$$

We can conclude that all the columns of $E_s$ are equal, which contradict to that $\theta_s$ is a non-zero eigenvalue. Thus, we know that a non-zero eigenvalue $\theta_r$ must in the eigenvalue support of some edge state of $G$. Since $G$ is edge-transitive, we know that the eigenvalue supports of $e_a - e_b$ for all $(a, b) \in E(G)$ are equal. Therefore, all the non-zero eigenvalues are in eigenvalue support of edge states of $G$. \hfill \square

In the proof of Theorem 4.1, we prove that

$$LP = PL.$$

The transition matrix associated with graph $G$ is a polynomial in $L$. So for any permutation matrix $P$ from $\text{Aut}(G)$, we have

$$U(t)P = PU(t).$$

Using this, the following Lemma is proved in [5, Corollary 9.2].

4.3 Lemma. If a graph $G$ admits perfect state transfer between $e_a - e_b$ to $e_c - e_d$, then the stabilizer of $e_a - e_b$ is the same as the stabilizer of $e_c - e_d$ in $\text{Aut}(G)$. \hfill \square
4.4 Lemma. If graph $G$ admits perfect state transfer between $e_a - e_b$ to $e_c - e_d$, then all the edge states in the orbit of $e_a - e_b$ under $\text{Aut}(G)$ have perfect state transfer.

Proof. Assume there exist time $\tau$ such that $U(\tau)(e_a - e_b) = \gamma(e_c - e_d)$ for some $|\gamma| = 1$. Let $P$ denote a permutation matrix associated with a $\sigma \in \text{Aut}(G)$ and $P(e_a - e_b) = e_{a'} - e_{b'}$. By Lemma 4.3, we know $P$ does not fix $(e_c - e_d)$ and assume $P(e_c - e_d) = e_{c'} - e_{d'}$. Then we have

$$U(\tau)P(e_a - e_b) = \gamma P(e_c - e_d)$$

$$U(\tau)(e_{a'} - e_{b'}) = \gamma(e_{c'} - e_{d'}).$$

Thus, there is also perfect state transfer between $e_{a'} - e_{b'}$ and $e_{c'} - e_{d'}$ at time $\tau$.

By the monogamy of perfect state transfer, the following results follow immediately.

4.5 Corollary. If there is perfect state transfer between $e_a - e_b$ and $e_c - e_d$ in graph $G$, then the orbit of $e_a - e_b$ and the orbit of $e_c - e_d$ under $\text{Aut}(G)$ must have the same size.

4.6 Corollary. Given an edge-transitive graph $G$, if perfect edge state transfer occurs in $G$, then all the edges have perfect state transfer.

By monogamy property of perfect state transfer, we know that perfect state edge transfer in an edge-transitive graph partition edges into pairs.

4.7 Corollary. Let $G$ be an edge-transitive graph with $n$ edges. If $n$ is odd, there is no edge perfect state transfer in $G$.

5 Constructions

In this section, we show how to use complements and Cartesian products to build infinite families of graphs with perfect edge state transfer.
5.1 Complements

We use standard algebraic graph theory result to show that complementation preserves perfect Laplacian edge state transfer. Let $\overline{G}$ denote the complement of a graph $G$.

5.1 Lemma. Let $G$ be a graph with $n$ vertices and $L$ denote the Laplacian matrix of $G$. Then every Laplacian eigenvector of $G$ with non-zero eigenvalue $\theta$ is a Laplacian eigenvector of $\overline{G}$ with eigenvalue $n - \theta$. $\square$

5.2 Theorem. There is perfect state transfer between $(e_a - e_b)$ and $(e_c - e_d)$ in graph $G$ if and only if there is perfect state transfer between $(e_a - e_b)$ and $(e_c - e_d)$ in $\overline{G}$.

Proof. Let $S = \{\theta_1, \theta_2, \ldots, \theta_r\}$ denote the eigenvalue support of $(e_a - e_b)$ and $(e_c - e_d)$ in $G$ and $\sum_r \theta_r E_r$ denote the spectral decomposition of the Laplacian of $G$. Let

$$a_j = (E_j)_{ac} + (E_j)_{ad} - (E_j)_{bc} + (E_j)_{bd}$$

for all eigenvalue $\theta_j \in S$. Then we have that

$$\left| \frac{1}{2}(e_c - e_d)^T U(t)(e_a - e_b) \right|^2 = \left| \frac{1}{2} \sum_{j=1}^r e^{it\theta_j} ((E_j)_{ac} + (E_j)_{ad} - (E_j)_{bc} + (E_j)_{bd}) \right|^2 = \left| \frac{1}{2} \left( a_1 e^{it\theta_1} + a_2 e^{it\theta_2} + \cdots + a_r e^{it\theta_r} \right) \right|^2 = \frac{1}{4} \left( (a_1 \cos(\theta_1 t) + a_2 \cos(\theta_2 t) + \cdots + a_r \cos(\theta_r t))^2 + (a_1 \sin(\theta_1 t) + a_2 \sin(\theta_2 t) + \cdots + a_r \sin(\theta_r t))^2 \right)

= \frac{1}{4} \left( a_1^2 + a_2^2 + \cdots + a_r^2 + \sum_{r \neq s} 2a_r a_s \cos((\theta_r - \theta_s) t) \right)$$

By Lemma 5.1, we know that the eigenvalue support $\overline{S}$ of $(e_a - e_b)$ and $(e_c - e_d)$ in $\overline{G}$ is $\{n - \theta_1, n - \theta_2, \ldots, n - \theta_r\}$. Since zero is never in the eigenvalue support, the spectral idempotent $E_r$ of $L$ with eigenvalue $n - \theta_r$ is the same as $E_r$ with eigenvalue $\theta_r$ of $L$ for all eigenvalues in
the eigenvalue support of $e_a - e_b$ in $G$. Let $\overline{U}(t) = \exp(it\overline{L})$ be the
transition matrix associated with $\overline{G}$. We have that

$$\left| \frac{1}{2} (e_c - e_d)^T \overline{U}(t)(e_a - e_b) \right|^2$$

$$= \left| \frac{1}{2} \sum_{j=1}^{r} e^{it(n-\theta_j)} ((E_j)_{ac} + (E_j)_{ad} - (E_j)_{bc} + (E_j)_{bd}) \right|^2$$

$$= \frac{1}{4} \left( a_1^2 + a_2^2 + \cdots + a_r^2 + \sum_{r \neq s} 2a_r a_s \cos ((n - \theta_r)t - (n - \theta_s)t) \right)$$

Since cosine is an even function, we get that

$$\left| \frac{1}{2} (e_c - e_d)^T U(t)(e_a - e_b) \right|^2 = \left| \frac{1}{2} (e_c - e_d)^T U(t)(e_a - e_b) \right|^2.$$

Therefore, there is perfect state transfer between $(e_a - e_b)$ and $(e_c - e_d)$
in graph $G$ if and only if there is perfect state transfer between them
in the complement of $G$.

Let $G_1, G_2$ be two graphs. Let $E'$ denote the set of all the edges
with one end in $V(G_1)$ and the other end in $V(G_2)$. The join graph
of $G_1$ and $G_2$ is a graph $G$ such that

$$V(G) = V(G_1) \cup V(G_2), E(G) = E(G_1) \cup E(G_2) \cup E'.$$

and the with vertex set $V(G)$.

5.3 Corollary. Let $G$ be a graph and $a, b, c, d$ are vertices in $G$. There
is perfect state transfer between $e_a - e_b$ and $e_c - e_d$ in $G$ if and only if
there is perfect state transfer between $e_a - e_b$ and $e_c - e_d$ in the join
graph of $G$ and $H$ for a graph $H$.

Notice that when $H$ is a simple graph with one vertex, the join
graph of a graph $G$ and $H$ is a cone graph $G$. So we can see that if
there is perfect edge state transfer in a graph $G$, using Theorem 5.2
we can easily construct a cone graph of $G$ to obtain a new graph that
admits perfect edge state transfer.

Theorem 5.2 also allows us to characterize perfect state transfer in
some graphs with special structure.
5.4 Corollary. Let $K_n$ be a complete graph on $n$ vertices and $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Let $G$ denote the graph obtained from $K_n$ by deleting edge $(v_1, v_2)$. Then there is perfect state transfer between $e_1-e_i$ and $e_2-e_i$ for all $i \in \{3, 4, \cdots, n\}$.

5.2 Cartesian Products

Cartesian product is also an operator that we can use to construct infinite families of graphs with perfect edge state transfer.

If $G, H$ are two graphs, their Cartesian product has vertex set $V(G) \times V(H)$, where $(g_1, h_1)$ is adjacent to $(g_2, h_2)$ if and only if either

(i) $g_1 = g_2$ in $G$ and $h_1$ is adjacent to $h_2$ in $H$, or  
(ii) $g_1$ is adjacent to $g_2$ in $G$ and $h_1 = h_2$ in $H$.

5.5 Lemma. Let $G, H$ be graphs with Laplacian matrices $L_G$ of order $n \times n$, $L_H$ of order $m \times m$ respectively. Let $G \Box H$ denote the Cartesian product of $G$ and $H$ with the Laplacian matrix $L_{G \Box H}$. Then $L_{G \Box H} = L_G \otimes I + I \otimes L_H$. □

5.6 Lemma. Let $G, H$ be two graphs with transition matrices $U_G(t) = \exp(itL_G)$ and $U_H(t) = \exp(itL_H)$ respectively. Let $U_{G \Box H}(t) = \exp(itL_{G \Box H})$ denote the transition matrix of $G \Box H$. Then $U_{G \Box H}(t) = U_G(t) \otimes U_H(t)$. 

Proof. Let $L_G$ be a matrix of order $n \times n$ and let $L_H$ be a matrix of order $m \times m$. If $M$ is a matrix of order $m$ and $N$ is a matrix of order $n \times n$, the Kronecker sum of $M$ and $N$ is

$$M \oplus N = M \otimes I_n + I_m \otimes N.$$ 

Using the Kronecker sum and previous lemma, we have

$$U_{G \Box H}(t) = \exp(itL_{G \Box H})$$
$$= \exp \left( it(L_G \otimes I_m + I_n \otimes L_H) \right)$$
$$= \exp \left( it(L_G \oplus L_H) \right)$$
$$= \exp(itL_G) \otimes \exp(itL_H)$$
$$= U_G(t) \otimes U_H(t).$$ □
5.7 Theorem. Let $G, H$ be two graphs, let $a, b$ be two vertices in $G$ and let $\alpha, \beta, \gamma, \kappa$ be vertices in $H$. There is perfect state transfer between the edge $\{(a, \alpha), (a, \beta)\}$ and the edge $\{(b, \gamma), (b, \kappa)\}$ in $G \Box H$ at time $t$ if and only if both of the following conditions hold:

(i) There is perfect Laplacian vertex state transfer between vertices $a$ and $b$ in $G$ at time $t$.

(ii) There is perfect edge state transfer between edges $(\alpha, \beta)$ and $(\gamma, \kappa)$ in $H$ at time $t$.

Proof. The state that denotes the edge $\{(a, \alpha), (a, \beta)\}$ is

$$\frac{1}{\sqrt{2}} (e_a \otimes (e_\alpha - e_\beta))$$

and then we can see that the density matrix of this edge is $D_a \otimes D_{\alpha\beta}$.

Similarly, the density matrix of the edge $\{(b, \gamma), (b, \kappa)\}$ is $D_b \otimes D_{\gamma\kappa}$. There is perfect state transfer between edge $\{(a, \alpha), (a, \beta)\}$ and edge $\{(b, \gamma), (b, \kappa)\}$ at time $t$ if and only if

$$U_{G \Box H}(t) \cdot D_a \otimes D_{\alpha\beta} \cdot U_{G \Box H}(-t) = D_b \otimes D_{\gamma\kappa}.$$

By previous corollary, we have that

$$U_{G \Box H}(t) \cdot D_a \otimes D_{\alpha\beta} \cdot U_{G \Box H}(-t)$$

$$= U_{G \Box H}(t) \cdot D_a \otimes D_{\alpha\beta} \cdot (U_G(-t) \otimes U_H(-t))$$

$$= (U_G(t) \otimes U_H(t)) \cdot (D_a U_G(-t) \otimes D_{\alpha\beta} U_H(-t))$$

$$= (U_G(t) D_a U_G(-t)) \otimes (U_H(t) D_{\alpha\beta} U_H(-t))$$

$$= D_b \otimes D_{\gamma\kappa},$$

which is equivalent to that there is perfect Laplacian state transfer between vertex $a$ and $b$ in $G$ at time $t$ and at the same time there is perfect state transfer between edge $(\alpha, \beta)$ and $(\gamma, \kappa)$ in $H$.

The example given by Coutinho in [2] Section 2.4 shows that $P_2$ admits perfect state transfer with respect to its Laplacian matrix between its vertices at time $\frac{\pi}{2}$. Later on we will prove Theorem 7.7 which tells us that $P_3$ admits perfect edge state transfer between its edges at time $\frac{\pi}{2}$. By Theorem 5.7, there is perfect state transfer from $e_3 - e_5$ to $e_1 - e_0$ and from $e_2 - e_3$ to $e_1 - e_4$ in Figure 2.
6 Transitivity

The graph in Figure 3 is the complement of the graph in Figure 2. We know that there is perfect edge state transfer from $e_3 - e_5$ to $e_1 - e_0$ and also from $e_2 - e_3$ to $e_1 - e_4$ in Figure 2. By Theorem 5.2, we know that there are also perfect edge states transfer from $e_3 - e_5$ to $e_1 - e_0$ and from $e_2 - e_3$ to $e_1 - e_4$ in Figure 3.

Actually the graph in Figure 3 also admits perfect edge state transfer between $e_2 - e_5$ and $e_0 - e_4$. This is what we call “the transitivity phenomenon”. This phenomenon can never happen in perfect vertex state transfer due to the monogamy and symmetry properties of perfect state transfer.
6.1 Theorem. Suppose there is perfect state transfer between \(e_a - e_b\) and \(e_\alpha - e_\beta\) at time \(\tau\) in \(G\) and there is also perfect state transfer between \(e_b - e_c\) and \(e_\beta - e_\gamma\) at the same time \(\tau\) in \(G\). Then there is perfect state transfer between \(e_a - e_c\) and \(e_\alpha - e_\gamma\) at time \(\tau\) in \(G\).

Proof. Let \(D_{ab}\) denote the density matrix of \(e_a - e_b\) and \(D_{bc}\) denote the density matrix of \(e_b - e_c\). We have that

\[
D_{ab} = \frac{1}{2}(e_a - e_b)(e_a - e_b)^T, \quad D_{bc} = \frac{1}{2}(e_b - e_c)(e_b - e_c)^T.
\]

Using that

\[
(e_a - e_b)^T(e_b - e_c) = (e_b - e_c)^T(e_a - e_b) = -1,
\]

we can write the density matrix of \(e_a - e_c\) in terms of \(D_{ab}\) and \(D_{bc}\) in the following way.

\[
D_{ac} = \frac{1}{2}(e_a - e_c)(e_a - e_c)^T
= \frac{1}{2}((e_a - e_b) + (e_b - e_c))((e_a - e_b) + (e_b - e_c))^T
= \frac{1}{2}((e_a - e_b)(e_a - e_b)^T + (e_b - e_c)(e_b - e_c)^T
+ (e_a - e_b)(e_b - e_c)^T + (e_b - e_c)(e_a - e_b)^T)
= \frac{1}{2}((e_a - e_b)(e_a - e_b)^T + (e_b - e_c)(e_b - e_c)^T
- (e_a - e_b)(e_a - e_b)^T(e_b - e_c)(e_b - e_c)^T
- (e_b - e_c)(e_b - e_c)^T(e_a - e_b)(e_a - e_b)^T)
= D_{ab} + D_{bc} - 2D_{ab}D_{bc} - 2D_{bc}D_{ab}
\]

As the above shows, we have

\[
D_{ac} = D_{ab} + D_{bc} - 2D_{ab}D_{bc} - 2D_{bc}D_{ab}.
\]

Similarly, we have

\[
D_{\alpha\gamma} = D_{\alpha\beta} + D_{\beta\gamma} - 2D_{\alpha\beta}D_{\beta\gamma} - 2D_{\beta\gamma}D_{\alpha\beta}.
\]

Now consider \(U(\tau)D_{ac}U(-\tau)\). Since we know that

\[
U(\tau)D_{ab}U(-\tau) = D_{\alpha\beta} \quad \text{and} \quad U(\tau)D_{bc}U(-\tau) = D_{\beta\gamma},
\]

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we have

\[ U(\tau)D_{ac}U(-\tau) = U(\tau)(D_{ab} + D_{bc} - 2D_{ab}D_{bc} - 2D_{bc}D_{ab})U(-\tau) \]
\[ = D_{\alpha\beta} + D_{\beta\gamma} - 2U(\tau)D_{ab}D_{bc}U(-\tau) - 2U(\tau)D_{bc}D_{ab}U(-\tau). \]

Using \(U(-\tau) \cdot U(\tau) = 1\), we get

\[ U(\tau)D_{ab}D_{bc}U(-\tau) = U(\tau)D_{ab}U(-\tau) \cdot U(\tau)D_{bc}U(-\tau) = D_{\alpha\beta}D_{\beta\gamma} \]

and similarly,

\[ U(\tau)D_{bc}D_{ab}U(-\tau) = U(\tau)D_{bc}U(-\tau) \cdot U(\tau)D_{ab}U(-\tau) = D_{\beta\gamma}D_{\alpha\beta}. \]

Thus, we get that

\[ U(\tau)D_{ac}U(-\tau) = D_{\alpha\beta} + D_{\beta\gamma} - 2D_{\alpha\beta}D_{\beta\gamma} - 2D_{\beta\gamma}D_{\alpha\beta} = D_{\alpha\gamma}. \]

Therefore, there is perfect state transfer between \(e_a - e_c\) and \(e_\alpha - e_\gamma\) at time \(\tau\).

\[ \square \]

7 Special Classes

We show that \(C_4\) is the only cycle and \(P_3, P_4\) are the only paths that have perfect edge state transfer. We observe an interesting correspondence of perfect state transfer between graphs and their line graphs when graphs are paths and cycles.

7.1 Cycles

We use \(C_n\) to denote the cycle on \(n\) vertices and \(A(C_n), L(C_n)\) to denote the adjacency and Laplacian matrix of \(C_n\) respectively.

We use a bound on \(n\) such that \(C_n\) can have a periodic edge state to eliminate the cases when \(C_n\) can have perfect state transfer. We show that \(C_4\) is the only cycle that has perfect state transfer.

7.1 Lemma. Laplacian eigenvectors of \(C_n\) are

\[
v_k = \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(n-1)k} \end{pmatrix}
\]
for $k = 0, 1, \cdots, n - 1$ where $\omega = e^{\frac{2\pi i}{n}}$ with eigenvalues 

$$2 - 2 \cos \frac{2\pi k}{n}$$

for $k = 0, 1, \cdots, n - 1$. 

Since we have 

$$\cos \frac{2\pi (n - r)}{n} = \cos \left(2\pi - \frac{2\pi r}{n}\right) = \cos \frac{2\pi r}{n},$$

we know that $k = r$ and $k = n - r$ produce the same eigenvalue for $r \in \{1, 2, \cdots, n - 1\}$. Thus, we can conclude that $L(C_n)$ has \( \lfloor \frac{n}{2} \rfloor \) distinct non-zero eigenvalues.

Using Theorem 4.2, the Lemma below follows immediately.

7.2 Lemma. every edge state of $C_n$ has eigenvalue support of size $\lfloor \frac{n}{2} \rfloor$. 

7.3 Theorem. There is perfect edge state transfer in $C_n$ if and only if $n = 4$.

Proof. By Lemma 7.1 we know that the Laplacian eigenvalues of $C_n$ are 

$$0 \leq 2 - 2 \cos \left(\frac{2\pi k}{n}\right) \leq 4$$

for $k = 0, 1, \cdots, n - 1$. By Corollary 3.7 we know that for an edge state to be periodic, the size of eigenvalue support must be at most 4. Then by Lemma 7.2 we know that for $C_n$ to have a periodic edge state, we must have $3 \leq n \leq 9$.

Using Theorem 3.6 we can find that there are no periodic edge states in $C_n$ when $n = 7, 8, 9$ which implies that there is no perfect edge state transfer in $C_n$. Since cycles are edge-transitive, by Corollary 4.7 we know there is no perfect state transfer in $C_3$ and $C_5$. Computing 

$$\left| \frac{1}{2}(e_a - e_b)^T U(t)(e_c - e_d) \right|^2$$

for all distinct edges $(a, b), (c, d)$ in $E(C_n)$ when $n = 4, 6$, we can conclude that the only cycle that has perfect edge state transfer is $C_4$.  

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7.2 Paths

We let $P_n$ denote the path on $n$ vertices such that $V(P_n) = \{1, 2, \cdots, n\}$. We use $A(P_n)$ to denote the adjacency matrix of $P_n$ and $L(P_n)$ to denote the Laplacian matrix of $P_n$. We show that $P_3, P_4$ are the only two paths where perfect edge state transfer can occur.

7.4 Lemma. The Laplacian eigenvector with eigenvalue $2 - 2\cos \frac{\pi r}{n}$ of $P_n$ is

$$
2\sin \frac{r\pi}{2n} \begin{pmatrix}
\cos \left(\frac{1\pi}{2n}\right) \\
\cos \left(\frac{3\pi}{2n}\right) \\
\cos \left(\frac{5\pi}{2n}\right) \\
\vdots \\
\cos \left(\frac{(2n - 3)\pi}{2n}\right) \\
\cos \left(\frac{(2n - 1)\pi}{2n}\right)
\end{pmatrix}
$$

for $r = 0, 1, \cdots, n - 1$. \hfill \Box

Using automorphisms of path graphs and Theorem 4.1, we can prove the symmetry of the eigenvalue supports of the edge states of $P_n$.

7.5 Lemma. Let $(k, k+1)$ be an edge of $P_n$ with $1 \leq k \leq n - 1$. Then the eigenvalue supports of the edge states associated with $(k, k+1)$ and $(n-k, n-k+1)$ are the same. \hfill \Box

7.6 Lemma. Let $S$ denote the eigenvalue support of an edge state in $P_n$. Then

$$|S| \geq \frac{n}{2}.$$

Proof. We want to prove that there are at most $n/2$ eigenvalues that are not in the eigenvalue support of an edge state in $P_n$.

Let $E_r$ denote the spectral idempotent of $L(P_n)$ with eigenvalue $2 - 2\cos (\pi r/n)$. Since 0 is never in the eigenvalue support of any edge state, we may assume that $2 - 2\cos (\pi r/n)$ is a non-zero eigenvalue that is not in the eigenvalue support of $e_k - e_{k+1}$ for some integer $1 \leq r \leq n - 1$. Let $v_r$ denote the eigenvector of $L(P_n)$ such that

$$v_r v_r^T = E_r.$$

Assume that $2 - 2\cos (\pi r/n)$ is not in the eigenvalue support of $(k, k+1)$, which means that

$$E_r (e_k - e_{k+1}) = 0.$$
Then we know that 

\[ v_r e_k^T = v_r e_{k+1}^T. \]

By Lemma 7.4, we must have that 

\[ \cos((2k - 1) \frac{r\pi}{2n}) = \cos \left( (2k + 1) \frac{r\pi}{2n} \right). \]

Using the trigonometric identity 

\[ \cos(x) - \cos(y) = -2 \sin \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right), \]

we know that \( r \) must satisfy 

\[ \cos((2k - 1) \frac{r\pi}{2n}) - \cos \left( (2k + 1) \frac{r\pi}{2n} \right) = -2 \sin \left( \frac{4kr\pi}{4n} \right) \sin \left( \frac{r\pi}{2n} \right) = 0. \]

Thus, we know that either \( kr/n \) or \( r/2n \) is an integer. But \( 1 \leq r \leq n - 1 \) and so 

\[ \frac{kr}{n} = z \]

for some positive integer \( z \).

Since \( 1 \leq r \leq n - 1 \), we know that \( z \) must satisfy that 

\[ 1 \leq \frac{n}{k} z \leq n - 1. \]

The number of values of \( z \) satisfying the inequality above is the number of non-zero eigenvalues not in the eigenvalue support of \( e_k - e_{k+1} \). By Lemma 7.5, we only need to consider the cases when \( k = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor \).

Since the number of valid \( z \) increases as the value of \( k \) increases and when \( k = \lfloor \frac{n}{2} \rfloor \), the values that \( z \) can take is at most 

\[ \lfloor \frac{n}{2} \rfloor - 1 \leq \frac{n}{2} - 1. \]

As stated before, zero is never in the eigenvalue support of an edge state and so, we can conclude that there are at most \( n/2 \) eigenvalues that are not in the eigenvalue support of an edge state in \( P_n \). Therefore, the size of the eigenvalue support of an edge state is at least \( n/2 \). \( \square \)

**7.7 Theorem.** A path graph on \( n \) vertices has perfect edge state transfer if and only if \( n = 3, 4 \).
Proof. By Lemma 7.4, we know that $P_n$ has Laplacian eigenvalue

$$0 \leq 2 - 2 \cos \frac{\pi r}{n} \leq 4$$

for $r = 0, 1, \ldots, n - 1$. By Corollary 3.7, we know that if an edge state of $P_n$ is periodic, then its eigenvalue support has size at most four. Lemma 7.6 tells us that the eigenvalue support of an edge state of $P_n$ is at least $n/2$. Thus, we know that for $n \geq 9$, there is no periodic edge states in $P_n$, which implies that there is no perfect state transfer in $P_n$ when $n \geq 9$. Thus, we only need to consider the cases when $n = 3, 4, 5, 6, 7, 8$.

Using Theorem 3.6 we find that when $n = 5, 7, 8, 9$, there is no periodic edge states in $P_n$ and also that, there is only one periodic edge state in $P_6$. If the vertices of $P_6$ from one end to the other are 0, 1, $\ldots$, 5, then $e_2 - e_3$ is the only periodic state with period $2\sqrt{3}\pi/3$. Since if an edge state has perfect state transfer, then it must be periodic, which tells us that when $n = 5, 6, 7, 8, 9$, there is no perfect edge state transfer in $P_n$.

By computing

$$\left| \frac{1}{2}(e_a - e_b)^T U(t)(e_c - e_d) \right|^2$$

for all distinct edges $(a, b), (c, d)$ in $E(P_3)$ and $E(P_4)$, we find that there is perfect state transfer in $P_3$ and $P_4$. Therefore, there is perfect state transfer in $P_n$ if and only if $n = 3, 4$. $\square$

When $n = 3$, there is perfect state transfer between its edges in $P_3$ at time $\pi/2$. When $n = 4$, perfect state transfer occurs between two edges on its ends in $P_4$ at time $\sqrt{2}\pi/2$.

7.3 Comments

Stevanović [7] and Godsil [5] prove that $P_n$ admits perfect vertex state transfer relative to adjacency matrices if and only if $n = 2$ or 3. Perfect vertex state transfer in $P_2$ happens between its two vertices at time $\pi/2$ and perfect vertex state transfer in $P_3$ happens between its end-vertices at time $\sqrt{2}\pi/2$.

We proved that $P_n$ admits perfect edge state transfer only when $n = 3$ or 4 and

(i) there is perfect state transfer between its edges in $P_3$ at time $\pi/2$. 

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(ii) when $n = 4$, perfect state transfer occurs between two edges on its ends in $P_4$ at time $\sqrt{2}\pi/2$.

Later in Section 8, we will prove an analogous result for quantum walks relative to the unsigned Laplacians in paths with initial states of the form $e_a + e_b$. That is, $P_3, P_4$ are the only paths where perfect state transfer relative to the unsigned Laplacians occurs and it occurs between the end-edges of $P_3, P_4$ at time $\pi/2, \sqrt{2}\pi/2$ respectively.

Notice also that $P_2, P_3$ are the line graphs of $P_3, P_4$ respectively. In $P_3$ and its line graph $P_2$, perfect state transfer always occurs at the same time $\pi/2$ between the same pair of edges and their corresponding pair of vertices in the line graph. This happens regardless of our choice of Hamiltonian or form of the initial state. We can make the same observations about perfect state transfer in $P_4$ and its line graph $P_3$.

We know that $C_4$ is the only cycle that admits perfect state transfer relative to adjacency matrices, Laplacians. We will show in next section that $C_4$ is the only cycle that admits perfect state transfer relative to the unsigned Laplacians. No matter our choice of Hamiltonians and form of initial state, perfect state transfer in $C_4$ happens at the same time $\pi/2$ between pairs of opposite edges or vertices.

Let $G$ be a regular graph with valency $k$, then the Laplacian matrix of $G$ is

$$L = kI - A.$$ 

Then the transition matrix for edge state transfer is

$$U(t) = \exp (it(kI - A)) = e^{itk}e^{-itA}.$$ 

A similar argument works for the unsigned Laplacians. Thus we can conclude that the continuous quantum walks generated by the adjacency matrices, the Laplacians and the unsigned Laplacians are equivalent up to a phase factor.

Although the transition matrices with respect to the adjacency matrices, the Laplacians and the unsigned Laplacians of cycles are equivalent, it is still surprising that different forms of initial states actually do not affect perfect state transfer in $C_4$. Also, notice that $C_4$ is the line graph of itself and there is a correspondence between pairs of PST-edges and pairs of PST-vertices.

It may seem that there is a correspondence between perfect edge state transfer in a graph and perfect vertex state transfer in its line graph. However, that is not true for most graphs. So far, paths and cycles are the only examples we have found where the correspondence can be observed.
8 Unsigned Laplacian

Let $G$ be a graph. The unsigned Laplacian of $G$ is matrix $L_+(G)$ such that

$$L_+(G) = \Delta(G) + A(G).$$

When we use $L_+(G)$ as Hamiltonian in a quantum walk, the edge $(a, b)$ of $G$ is associated with the state

$$e_a + e_b,$$

which we call “plus state”.

Every time we refer to plus states, we use the unsigned Laplacian of a graph as Hamiltonian unless stated explicitly otherwise. We define analogously that there is perfect plus state transfer between $e_a + e_b$ and $e_c + e_d$ if and only if

$$U(t)(e_a + e_b) = \exp(itL_+) (e_a + e_b) = \gamma(e_c + e_d),$$

for some complex constant $\gamma$. Also, a plus state $e_a + e_b$ is periodic if and only if it has perfect plus state transfer to itself at some time $t$.

Since the main case of interest in this paper is the case when the Laplacian of a graph is used as Hamiltonian, it is natural to question if there will be perfect state transfer between a edge state and a plus state when we use the Laplacian as Hamiltonian. The answer is no.

8.1 Theorem. Let $G$ be a graph with $a, b, c, d \in V(G)$. There is no perfect state transfer between a state of the form $e_a + e_b$ and a state of the form $e_c - e_d$ in $G$ when the Laplacian of $G$ is used as Hamiltonian of the quantum walk.

Proof. We know that 0 will always be a eigenvalue of the Laplacian of $G$ with the all-ones vector being its eigenvector. Thus, we know 0 will never be in the eigenvalue support of $e_a - e_b$ while 0 is always in the eigenvalue support of $e_c + e_d$. It follows that $e_a - e_b$ and $e_c + e_d$ do not have the same eigenvalue support, which implies that they are not strongly cospectral. By Theorem 2.2 we can conclude that there is no perfect state transfer between a state of the form $(e_a - e_b)$ and a state of the form $(e_c + e_d)$ using Laplacian as Hamiltonian. \qed
Table 3: the Number of Graphs with PST and Periodic States

Despite the huge gap between the number of PST pairs in terms of edge state transfer and plus state transfer, when the underlying graph is a bipartite graph, perfect state transfer in terms of edge states and plus states are equivalent.

8.2 Lemma. Let $G$ be a bipartite graph with two parts $B_1, B_2$ and $A, \Delta$ denote the adjacency matrix and the degree matrix of $G$ respectively. Let $D$ be block matrix such that

$$D = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

indexed by the vertices of $B_1, B_2$ in the order. Then we have

$$D(\Delta - A)D = \Delta + A.$$ 

8.3 Theorem. Let $G$ be a bipartite graph with parts $B_1, B_2$ and vertices $a, c \in B_1$ and $b, d \in B_2$. There is perfect edge state transfer between $(e_a - e_b)$ and $(e_c - e_d)$ if and only if there is perfect plus state transfer between $(e_a + e_b)$ and $(e_c + e_d)$.

Proof. Let $\Delta$ denote the degree matrix of $G$ and $A$ denote the adjacency matrix of $G$. From the Lemma 8.2 we know that

$$D(\Delta - A)D = \Delta + A$$

and inserting $DD = I$ between $m$ copies of $\Delta - A$, we have

$$D(\Delta - A)^mD = D(\Delta - A)DD(\Delta - A)DD \cdots (\Delta - A)D$$

$$= (\Delta + A)^m$$
for any non-negative integer $m$. Then we see that

$$DU(t)D = D \exp(itL)D = D \sum_{m=0}^{\infty} \left( \frac{(it)^m}{m!} (\Delta - A)^m \right) D$$

$$= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} D (\Delta - A)^m D$$

$$= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} (\Delta + A)^m$$

$$= \exp(itL_+)$$

Note that since $a, c \in B_1$ and $b, d \in B_2$, we have that

$$D(e_a - e_b) = -(e_a + e_b), \quad D(e_c - e_d) = -(e_c + e_d).$$

There is perfect state transfer between $(e_a - e_b)$ and $(e_c - e_d)$ using Laplacian if and only if there exist $\tau$ such that

$$U(\tau)(e_a - e_b) = \gamma(e_c - e_d)$$

for some $|\gamma| = 1$. Applying $D$ on both sides of the equation above, we have that

$$DU(\tau)(e_a - e_b) = D(\gamma(e_c - e_d)).$$

Again using $DD = I$, we can rewrite the equation above as

$$DU(\tau)DD(e_a - e_b) = \gamma D(e_c - e_d).$$

This gives us that

$$- \exp(itL_+)(e_a + e_b) = -\gamma(e_c + e_d),$$

$$\exp(itL_+)(e_a + e_b) = \gamma(e_c + e_d),$$

which is equivalent to perfect plus state transfer between $(e_a + e_b)$ and $(e_c + e_d)$ using unsigned Laplacian. This completes our proof.

Since we proved that $C_4$ is the only cycle where perfect edge state transfer occurs, one immediate result from the theorem above is that there is no perfect plus state transfer in $C_n$ when $n$ is even and the
only exception is when \( n = 4 \). In \( C_4 \), there is perfect plus state transfer between opposite edges at time \( \frac{\pi}{2} \).

A plus-state analogue of Corollary 4.7 implies that there is no perfect plus state transfer in \( C_n \) when \( n \) is odd. Therefore, we get the following theorem.

**8.4 Theorem.** There is perfect plus state transfer in \( C_n \) if and only if \( n = 4 \).

Since \( P_3, P_4 \) are the only paths where perfect edge state transfer occurs, Theorem 8.3 gives the result below.

**8.5 Theorem.** A path graph on \( n \) vertices has perfect plus state transfer if and only if \( n = 3, 4 \).

Perfect edge state transfer and perfect plus state transfer happen between the same pairs of edges at the same time in \( P_3 \) and \( P_4 \). When \( n = 3 \), there is perfect plus state transfer between its edges in \( P_3 \) at time \( \pi/2 \). When \( n = 4 \), perfect plus state transfer occurs between two edges on its ends in \( P_4 \) at time \( \sqrt{2}\pi/2 \).

**9 Open Questions**

Checking all the trees on up to 16 vertices, there is no perfect edge state transfer and there is only four types of graphs that contain periodic edges:

1. Star graphs \( K_{1,n} \);
2. Double stars;
3. Paths;
4. The figure below.
Coutinho and Liu in [4] proved that there is no perfect vertex state transfer in trees with more than two vertices when the Laplacian is the Hamiltonian. We suspect a similar result for edge state transfer. Our personal feeling is that there is no perfect edge state transfer on trees, but we have not found a way to prove this.

Another question we would like to answer is that how different Hamiltonians and different initial states affect state transfer.

Table 4: Periodic States in Different Settings

Table 4 shows that the number of graphs with adjacency periodic vertex states, Laplacian periodic edge states and unsigned Laplacian periodic plus states followed with the corresponding proportions from left to right in order.

From Table 4 as well as Table 11 we can see that different choices of Hamiltonian and different forms of initial state largely affect the number of periodic states in a graph and hence, affect the number of states that have perfect state transfer.
However, as shown in Section 7 and Section 8, when underlying graphs are bipartite graphs, cycles and paths, perfect edge state transfer and perfect plus state transfer are equivalent. We can see that perfect state transfer in some certain classes of graphs is invariant under different Hamiltonians and initial states.

Given a specific graph, we are curious about how to choose Hamiltonian and the initial state so that the quantum walk over the graph will have desired phenomena, such as perfect state transfer between some certain pairs. On the other hand, given Hamiltonian and specific initial state, we would like to know how one can construct a graph with perfect state transfer. Also, we would like to know what conditions a graph has to satisfy to admit “stable” perfect state transfer under different Hamiltonians and different forms of initial states.

References

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