Mader’s conjecture for graphs with small connectivity

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Abstract
Mader conjectured that for any tree $T$ of order $m$, every $k$-connected graph $G$ with minimum degree at least $\left\lceil \frac{3k}{2} \right\rceil + m - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is $k$-connected. In this article, we give a characterization for a subgraph to contain an embedding of a specified tree avoiding some vertex. As a corollary, we confirm Mader’s conjecture for $k \leq 3$.

KEYWORDS
connectivity, embedding, tree

MATHEMATICAL SUBJECT CLASSIFICATION
05C75, 05C05, 05C35

1 | INTRODUCTION

Graphs considered in this article are finite, undirected, and simple (no loops or multiple edges). For a graph $G$, we use $V(G)$, $E(G)$, $\delta(G)$, and $\kappa(G)$ to denote its vertex set, edge set, minimum degree, and connectivity, respectively. The order of a graph $G$ is the cardinality of its vertex set, denoted by $|G|$. By $H \subseteq G$ we mean that $H$ is a subgraph of $G$, and we view any subset of vertices as a subgraph with no edges. For any $U \subseteq G$, the neighborhood of $U$, denoted by $N_G(U)$, is the set of vertices in $V(G) - U$ adjacent to at least one vertex in $U$. If the graph $G$ is clear from the context, the reference to $G$ is usually omitted. For graph-theoretical terminologies and notation not defined here, we follow [1]. The following well-known result due to Chartrand, Kaugars, and Lick [2].

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Theorem 1.1 (Chartrand et al. [2]). Every $k$-connected graph $G$ with $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor$ has a vertex $x$ such that $G - x$ is still $k$-connected.

Fujita and Kawarabayashi [4] studied the question that whether one can choose the minimum degree $\delta(G)$ of a $k$-connected graph $G$ large enough (independent of the number of vertices $|G|$ of $G$) to contain adjacent vertices $u \neq v$ such that $G - \{u, v\}$ is still $k$-connected, and gave a positive answer. Also, in the same paper, they conjectured that one can find a function $f_k(m)$ such that every $k$-connected graph $G$ with $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor + f_k(m)$ contains a connected subgraph $W$ of order $m$ such that $G - V(W)$ is still $k$-connected. Fujita and Kawarabayashi [4] gave examples in [4] to show that $f_k(m) \geq m$ for all positive integers $k, m$. In [8], Mader confirmed the conjecture by showing that $f_k(m) = m$. In fact, Mader showed the connected subgraph $W$ can be chosen to be a path. Based on the result, Mader conjectured that the $W$ can be chosen to be any given tree of order $m$ and made the following conjecture. A graph isomorphism from a graph $G$ to a graph $H$, written $\phi : G \rightarrow H$, is a mapping $\phi : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. When such a mapping $\phi$ exists, we write $G \cong H$ and say that $G$ is isomorphic to $H$.

Conjecture 1.2 (Mader [8]). For any tree $T$ of order $m$, every $k$-connected graph $G$ with $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is $k$-connected.

Concerning to this conjecture, Mader also proved that a lower bound on the minimum degree which is independent of the order of $G$ indeed exists for any $m$ and $k$.

Theorem 1.3 (Mader [9]). For any tree $T$ of order $m$, every $k$-connected graph $G$ with $\delta(G) \geq 2(k - 1 + m)^2 + m - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is $k$-connected.

Until now, Mader’s conjecture has been justified for $k = 1$ by Diwan and Tholiya [3]. For $k = 2$, only some classes of trees have been verified. In [10,11], stars, double-stars, path-stars, and path-double-stars are verified due to Tian, Lai, Xu, and Meng. In [7], trees with diameter at most 4 are verified due to Lu and Zhang. In [6], trees with small internal vertices and quasi-monotone caterpillars are verified due to Hssunuma and Ono. Very recently, caterpillars and spiders are verified due to Hong, Liu, Lu, and Ye [5]. All the methods are concerned with the structure of the trees. In this article, we develop a general way that works for arbitrary trees and confirm Mader’s conjecture for all classes of trees when $k = 2$ or 3. Our method might have some potential generalization into $k$-connected case.

We conclude this section with some notations about trees. A rooted tree is a tree with a specific vertex, which is call the root. An $r$-tree is a rooted tree with root $r$. For an $r$-tree $T$ and a vertex $v \in V(T)$, $rTv$ is defined as the unique path from $r$ to $v$ in $T$. Every vertex in the path $rTv$ is called an ancestor of $v$, and each vertex of which $v$ is an ancestor is a descendant of $v$. An ancestor or descendant of a vertex is proper if it is not the vertex itself. The immediate proper ancestor of a vertex $v$ other than the root is its parent and the vertices whose parent is $v$ are its children. A leaf of a rooted tree is a vertex with no children. The set of leaves of an $r$-tree $T$ is denoted by Leaf $(T)$. For a vertex $v$ in an $r$-tree $T$, the subtree of $T$, denoted by $T_v$, is the tree induced by all its descendant and with root $v$. 

2 | PRELIMINARIES

By greedy strategy, it is easy to see that every graph with minimum degree at least \( m - 1 \) contains every tree of order \( m \). In fact, one may obtain a slightly general form as following two lemmas.

**Lemma 2.1** (Hasunuma and Ono [6]). Let \( T \) be a tree of order \( m \) and \( T_i \) a subtree of \( T \). If a graph \( G \) contains a subtree \( T_i' \cong T_i \), denoting by \( \phi : T_i \rightarrow T_i' \) the isomorphism, such that \( d_G(v) \geq m - 1 \) for any \( v \in V(G) \setminus \{\phi(x) : x \in V(T_i) \} \), then \( G \) contains a subtree \( T' \cong T \) such that \( T_i' \subseteq T' \).

**Lemma 2.2** (Hasunuma and Ono [6]). Let \( T \) be a tree of order \( m \), \( U \subseteq V(T) \) and \( S \) be a subtree obtained from \( T \) by deleting some leaves adjacent to a vertex in \( U \). If a graph \( G \) contains a subtree \( S' \cong S \) such that \( d_G(u) \geq m - 1 \) for any \( u \in U' = \{\phi(x) : x \in U \} \) for an isomorphism \( \phi \) from \( S \) to \( S' \), then \( G \) contains a subtree \( T' \cong T \) such that \( S' \subseteq T' \).

By merging the above two lemmas, we may get the following more general form which will be used to construct trees in our proof.

**Lemma 2.3.** Let \( T_0 \) be a tree of order \( m \) and \( T' \) a subtree of \( T_0 \). Let \( (L_1, ..., L_{k+1}) \) be a partition of \( V(T_0) \setminus V(T') \) such that for \( i = 0, ..., k - 1 \), \( L_{i+1} \subseteq \text{Leaf}(T_i) \) and \( T_{i+1} = T_i - L_{i+1} \) (note that vertices in \( L_{k+1} \) may not be leaves). Assume \( G \) is a graph containing \( T \) as a subgraph \( H' \cong T' \), denoting by \( \phi : T' \rightarrow H' \) the isomorphism, and \( V(G) \setminus V(H') \) has a partition \( (X_1, ..., X_{k+1}) \) such that each of the followings hold, where \( X_{k+2} = \{\phi(v) : v \in V(T'), N_{T_i}(v) \not\subseteq V(T') \} \).

(a) \(|N_G(x) \setminus \bigcup_{j=1}^{k} X_j| \geq m - 1 - \sum_{j=1}^{k} |L_j| \) for each \( x \in X_{k+1} \cup X_{k+2} \);

(b) \(|N_G(x) \setminus \bigcup_{j=i}^{i+2} X_j| \geq m - 1 - \sum_{j=i}^{i+2} |L_j| \) for each \( i = 2, ..., k + 1 \) and for each \( x \in \bigcup_{j=i}^{i+2} X_j \).

Then \( G \) contains a subtree \( H \cong T_0 \) such that \( H' \subseteq H \).

**Proof.** The proof is by induction on \( k \). If \( k = 0 \) then the condition (a) is exactly the degree condition in Lemma 2.1 and the condition (b) is nothing. Thus the result follows Lemma 2.1. So we may assume \( k \geq 1 \) and the result holds for any smaller \( k \). Let \( G' = G - X_i \). Then \( T_i' \) is a subtree of \( T_i \), \( H' \) is a subgraph of \( G' \), \( (L_2, ..., L_{k+1}) \) is a partition of \( V(T_i) \setminus V(T') \) and \( (X_2, ..., X_{k+1}) \) is a partition of \( G' - V(H') \). It is easy to see that the conditions holds for \( T_i \) and \( G' \). Then by induction hypothesis, \( G' \) contains a subgraph \( H_i \cong T_i \) such that \( H' \subseteq H_i \). Denote by \( \phi' : T_i \rightarrow H_i \) the isomorphism.

Let \( L_{k+2} = \{v \in V(T'), N_{T_i}(v) \not\subseteq V(T') \} \). Then \( \phi'(L_{k+2}) = \phi(L_{k+2}) = X_{k+2} \). By (b) with \( i = 2 \), we see that \( d_G(x) \geq m - 1 \) for any \( x \in \bigcup_{j=2}^{k+2} X_i = (V(G') \setminus V(H')) \cup X_{k+2} \). By the assumption of \( L_{k+2} \), \( N_{T_i}(L_i) \subseteq \bigcup_{i=2}^{k+2} L_i \). Then by Lemma 2.2, there is an embedding of \( T \) in \( G \) and the result holds.

In our proof, we always apply Lemma 2.3 with \( k = 2 \). Then we may restate the lemma when \( k = 2 \) as follows.
Corollary 2.4. Let $T$ be a tree of order $m$ and $T'$ a subtree of $T$. Let $L_1 \subseteq \text{Leaf}(T) \setminus V(T')$, $L_2 \subseteq \text{Leaf}(T - L_1) \setminus V(T')$ and $L_3 = V(T) \setminus (V(T') \cup L_1 \cup L_2)$. Assume $G$ is a graph which contains a subgraph $H' \cong T'$, denoting by $\phi : T' \to H'$ the isomorphism, and $V(G) \setminus V(H')$ has a partition $(X_1, X_2, X_3)$ such that each of the followings hold, where $X_4 = \{\phi(v) : v \in V(T'), N_T(v) \subseteq V(T')\}$.

(a) $d_G(x) \geq m - 1$ for each $x \in X_2 \cup X_3 \cup X_4$.

(b) $|N_G(x) \setminus X_1| \geq m - 1 - |L_1|$ and $|N_G(x) \setminus (X_1 \cup X_2)| \geq m - 1 - |L_1| - |L_2|$ for each $x \in X_3 \cup X_4$.

Then $G$ contains a subtree $H \cong T$ such that $H' \subseteq H$.

3 2-CONNECTED GRAPHS

In this section, we give a characterization for a subgraph to contain an embedding of a specified tree avoiding some vertex. As a corollary, we confirm Mader's conjecture for 2-connected graphs. To apply Corollary 2.4, we need the following counting lemma to satisfy the conditions in Corollary 2.4.

Lemma 3.1. Let $T$ be a tree, $T_0 \subseteq T$ and $T_1, ..., T_k$ be the components of $T - V(T_0)$. If $L_0 = \text{Leaf}(T_0)$, $L_1 = \text{Leaf}(T)$ and $L_2 = \text{Leaf}(T - L_1)$ then $|L_1| \geq |L_0|$ and $|L_1| + |L_2| \geq |L_0| + k$.

Proof. For $i = 1, ..., k$, assume $s_i \in E(T)$ such that $s_i \in V(T_0)$ and $r_i \in V(T_i)$. Note that some $s_i$'s may be the same vertex. Denote $S = \{s_1, ..., s_k\}$ and $R = \{r_1, ..., r_k\}$. Then $|S| \leq k = |R|$.

Since each $T_i$ contains a leave of $T$,

$$|L_1| \geq k + |L_0| - |S| \geq |L_0| + k - |L_0| \cap |S| \geq |L_0|.$$  

Moreover, note that some $T_i$ may contain exactly one vertex, and $V(T_i) \cap L_2 \neq \emptyset$ if $|V(T_i)| \geq 2$. Let $R' = \{r_i \mid V(T_i) \mid \geq 2\}$. Then

$$|L_2| \geq |R'| + |L_2| \cap |S|.$$  

For any $s_i \in S \cap L_0$, if $N_T(s_i) \cap R = \emptyset$ then $N_T(s_i) \setminus V(T_0) \subseteq L_1$ and $s_i \in L_2$. It follows that $N_T(s_i) \cap R' \setminus |S_i| \cap L_2 \geq |S_i| \cap L_0$. Also, the inequality still holds if $s_i \notin L_0$. By the arbitrariness of $s_i$, we see that $|R'| + |S \cap L_2| \geq |S \cap L_0|$. It follows that $|L_1| + |L_2| \geq |L_0| + k - |L_0| \cap |S| + |R'| + |L_2| \cap |S| \geq |L_0| + k$. The proof is completed.

By Corollary 2.4 and Lemma 3.1, we have the following lemma which provides a method to spare a vertex in an embedding of a tree. Applying the lemma several times, we may confirm Mader's conjecture when $k = 2$ or 3.

Lemma 3.2. Let $T_0$ be a tree of order $m$ and $G$ be a graph $\delta(G) \geq \delta \geq m$. Assume $T, B, H$ are disjoint subgraphs of $G$ such that $T \cong T_0$, $V(G) = V(H \cup B \cup T)$. Then one of the follows holds.
(a) \( N_G(H) \cap V(T) = \emptyset \).
(b) \(|N_G(v) \cap V(B)| \leq \delta - m \) for each \( v \in V(H \cup T) \).
(c) There exists \( v \in V(H \cup T) \) such that \(|N_G(v) \cap V(B)| \geq \delta - m + 1 \) and \( H \cup T - v \) contains a tree \( T' \cong T \).

Proof. Assuming none of (a), (b) and (c) holds, we will derive a contradiction. To this end, denote

\[ X = \{ v \in V(H \cup T) \mid |N_G(v) \cap V(B)| \geq \delta - m + 1 \} \quad \text{and} \quad Y = V(T) \cap N_G(H). \]

Then by the assumptions, we see that \( X \neq \emptyset \). We have the following claim.

**Claim 1.** \( X \cup Y = V(T) \).

First, \( X \subseteq V(T) \) is clear since any vertex in \( X \cap V(H) \) satisfies (c). For the second part, suppose that there exists \( z \in V(T) \setminus (X \cup Y) \). Then by the definitions of \( X, Y \), we have both \( |N_G(z) \cap V(B)| \leq \delta - m \) and \( N(z) \cap V(H) = \emptyset \). Thus \( d_G(z) \leq |N_G(z) \cap V(B)| + |N_G(z) \cap V(T)| \leq \delta - m + m - 1 = \delta - 1 \), a contradiction to the assumption on the minimum degree. Claim 1 is proved.

By Claim 1, pick \( u \in Y \) such that \( X \setminus \{ u \} \) is as large as possible. Then \( X \setminus \{ u \} = \emptyset \), for otherwise, \( X = \{ u \} \) and \( Y \setminus \{ u \} = \emptyset \), implying \( m = 1 \). Then each vertex in \( H \) forms a tree isomorphic to \( T_0 \) and \( u \) is the desired vertex in (c). Consider \( T \) as a tree rooted at \( u \).

By Claim 1, there exists an edge \( w_1w_2 \in E(T) \) such that \( w_1 \in Y, w_2 \in X \) and \( w_1 \) is the parent of \( w_2 \). Subject to this, we may further assume that \( w_1w_2 \) is such an edge that the number of vertices of the rooted subtree \( T_{w_2} \) is as small as possible. Under the conditions on \( w_1w_2 \), we see that no edge of \( T_{w_2} \) satisfies the conditions of \( w_1w_2 \), that is, for any edge \( w_1'w_2' \) of \( T_{w_2} \) such that \( w_1' \) is the parent of \( w_2' \), either \( w_1' \notin Y \) or \( w_2' \notin X \).

As a corollary, for any \( w \in X \cap V(T_{w_2}) \), all the ancestors of \( w \) in \( T_{w_2} \) are in \( X \). Thus there is a subtree \( T_2 \subseteq T_{w_2} \) such that \( V(T_{w_2}) \cap X = V(T_2) \). Let \( L_0 = \text{Leaf}(T_2) \) and \( T_3 = T_2 - L_0 \). By the choice of \( w_1w_2 \),

\[ N_G(H) \cap V(T_3) = \emptyset. \tag{1} \]

Let \( R_1, \ldots, R_p \) be the rooted trees obtained from \( T_{w_2} \) by deleting the vertices in \( T_2 \). For \( i = 1, \ldots, p \), assume \( r_i \) is the root of \( R_i \) and \( s_i \) is the parent of \( r_i \) in \( T_{w_2} \).

**Claim 2.** \( N_G(T_3) \cap \bigcup_{i=1}^p V(R_i - r_i) \neq \emptyset \).

Suppose, to the contrary, that the claim is not true. Let \( G' = G - (V(B) \cup \{ w_2 \}) \) and pick \( h \in N_G(w_1) \cap V(H) \). Let \( T'_4 = T - V(T_{w_2} - w_2) \) and \( T'_4 \) is obtained from \( T_4 \) by replacing \( w_2 \) with \( h \). Then \( T'_4 \cong T_4 \subseteq T \).

Define \( L_1 = \text{Leaf}(T_{w_2}), L_2 = \text{Leaf}(T_{w_2} - L_1) \) and \( L_3 = V(T_{w_2} - w_2) \setminus (L_1 \cup L_2) \). Then by Lemma 3.1, \((L_1, L_2, L_3)\) is a partition of \( V(T) \setminus V(T_4) \) such that

\[ |L_1| \geq |L_0| \quad \text{and} \quad |L_1| + |L_2| \geq |L_0| + p. \]
Let \( X_1 = V(T_2 - w_2), \) \( X_2 = \{r_1, ..., r_p\} \), and \( X_3 = V(G') \setminus (V(T'_4) \cup X_1 \cup X_2) \). Then \((X_1, X_2, X_3)\) is a partition of \( V(G') \setminus V(T'_4) \) such that \(|N_G(x) \cap X_1| \leq |L_0| \leq |L_1|\) for any \( x \in X_3 \cup \{h\} \) by (1) and by the contradiction assumption. Now we verify the degree conditions in Corollary 2.4. In fact, for \( x \in X_2 \cup X_3 \cup \{h\}, \) 
\[ d_G(x) \geq d_G(x) - |N_G(x) \cap (V(B) \cup \{w_2\})| \geq \delta - (\delta - m + 1) = m - 1 \] since \( x \notin X \), and for any \( x \in X_3 \cup \{h\}, \) 
\[ |N_G(x) \cap X_1| = d_G(x) - |N_G(x) \cap X_1| \geq m - 1 - |L_0| \geq m - 1 - |L_1|, \] and \( |N_G(x) \cap (X_1 \cup X_3)| \geq m - 1 - |L_0| - |X_1| = m - 1 - |L_0| - |L_1| \geq m - 1 - |L_1| - |L_2| \). Then by Corollary 2.4, \( G' \) contains a tree \( T' \cong T \). Then \( w_2 \) is a desired vertex in (c), a contradiction. The claim is proved.

Now we are ready to prove the result. In fact, by Claim 2, there exists an edge \( w_3r \in E(G) \) such that \( w_3 \in V(T_3) \) and \( r \in V(R_i - \eta_i) \) for some \( i \). Let \( w_4 \) be a child of \( w_3 \) in \( T_2 \). Furthermore, we may assume such \( w_3r \) and \( w_4 \) satisfying that the subtree \( T_{w_4} \) of \( T \) is as small as possible. Without loss of generality, we may assume that \( i = 1 \). By the choice of \( w_3 \),

\[ N_{G(r_1)} \cap V(T_{w_4}) \cap V(T_3) = \emptyset \] for any \( r' \in \bigcup_{i=1}^{p} V(R_i - \eta_i). \] Let \( G' = G - (V(B) \cup \{w_4\}) \). We consider the following two cases.

**Case 1.** \( V(T_{w_4}) \cap V(R_1) \neq \emptyset \).

In this case, \( V(R_1) \subseteq V(T_{w_4}) \) since both them are rooted subtrees of \( T \). Let \( T_5 = T - V(T_{w_4} - w_4) \) and \( T_5 = T - V(T_{w_4} - r) + w_3r. \) Then \( T_5 \cong T_5 \subseteq T \).

Define \( L_1 = \text{Leaf}(T_{w_4}), \) \( L_2 = \text{Leaf}(T_{w_4} - L_1), \) and \( L_3 = V(T_{w_4} - w_4) \setminus (L_1 \cup L_2) \). Then by Lemma 3.1, \((L_1, L_2, L_3)\) is a partition of \( V(T) \setminus V(T_5) \) such that \( |L_1| \geq |L_0 \cap V(T_{w_4})| \) and \( |L_1 + |L_2| \geq |L_0 \cap V(T_{w_4}) + \{r_1, ..., r_p\} \cap V(T_{w_4})| \). Let \( X_1 = V(T_{w_4} - w_4) \cap V(T_5), \) \( X_2 = \{r_1, ..., r_p\} \cap V(T_{w_4}) \) and \( X_3 = V(G') \setminus (V(T'_2) \cup X_1 \cup X_2) \). Then \((X_1, X_2, X_3)\) is a partition of \( V(G') \setminus V(T'_2) \) such that \( |N_G(x) \cap X_1| \leq |L_0 \cap V(T_{w_4})| \leq |L_1| \) for any \( x \in X_3 \cup \{r\} \) by (1) and (2). Now we verify the degree conditions in Corollary 2.4. In fact, noting that \( \{v \in N_T(v) \subseteq V(T_5)\} = \{w_4\} \), for any \( x \in X_2 \cup X_3 \cup \{r\}, \) 
\[ d_G(x) \geq d_G(x) - |N_G(x) \cap (V(B) \cup \{w_4\})| \geq m - 1, \] and for any \( x \in X_2 \cup \{r\}, \) 
\[ |N_G(x) \setminus X_1| = d_G(x) - |N_G(x) \cap X_1| \geq m - 1 - |L_0| \cap V(T_{w_4})| \geq m - 1 - |L_1|, \] \( |N_G(x) \cap (X_1 \cup X_3)| \geq m - 1 - |L_0 \cap V(T_{w_4})| - |X_1| = m - 1 - |L_0 \cap V(T_{w_4}) - |L_1|). \) Then by Corollary 2.4, \( G' \) contains a tree \( T' \cong T \). Thus, \( w_4 \) is a desired vertex in (c), a contradiction.

**Case 2.** \( V(T_{w_4}) \cap V(R_1) = \emptyset \).

In this case, let \( T_6 = T \setminus (V(T) \setminus (V(T_{w_4} - w_4) \cup V(R_1 - \eta_1) \cup N_{R_1}(\eta_1)))) \) and \( T'_6 \) be obtained from \( T_6 \) by replacing \( r \) with a neighbor of \( \eta_1 \) in \( H \) if \( r \in N_{R_1}(\eta_1) \), and then replacing \( w_4 \) with \( r \). Then \( T'_6 \cong T_6 \subseteq T \).

Define \( L_1 = \text{Leaf}(T_{w_4}), \) \( L_2 = \text{Leaf}(T_{w_4} - L_1) \) and \( L_3 = V(T) \setminus (V(T_6) \cup L_1 \cup L_2). \) Then by Lemma 3.1, \((L_1, L_2, L_3)\) is a partition of \( V(T) \setminus V(T_6) \) such that

\[ |L_1| \geq |L_0 \cap V(T_{w_4})| \text{ and } |L_1 + |L_2| \geq |L_0 \cap V(T_{w_4}) + \{r_2, ..., r_p\} \cap V(T_{w_4})|. \]
Let $X_1 = (V(T_w - w_4)) \cap V(T_2), \ X_2 = \{r_2, ..., r_p\} \cap V(T_{wa})$ and $X_3 = V(G') \setminus (V(T_3) \cup X_1 \cup X_2)$. Then $(X_1, X_2, X_3)$ is a partition of $V(G') \setminus V(T_3)$ such that $|N_G(x) \cap X_i| \leq |L_0 \cap V(T_{wa})| \leq |L_1|$ for any $x \in V(H) \cup \bigcup_{i=1}^p V(R_i - \eta)$ by (1) and (2). Now we verify the degree conditions in Corollary 2.4. In fact, we have, letting $X_4 = \{r\} \cup (N_{T_w}(\eta) \cap (V(H) \cup V(R_i))),$ for $x \in X_2 \cup X_3 \cup X_4,$ $d_{G'}(x) \geq d_G(x) - |N_G(x) \cap (V(B) \cup \{w_4\})| \geq m - 1,$ and for any $x \in X_1 \cup X_4,$ $N_{G'}(x) \setminus X_i \geq d_{G'}(x) - |L_0 \cap V(T_{wa})| \geq m - 1 - |L_1|$ and $|N_{G'}(x) \setminus (X_1 \cup X_2)| \geq m - 1 - |L_0 \cap V(T_{wa})| - |X_2| \geq m - 1 - |L_1| - |L_2|.$ Then by Corollary 2.4, $G'$ contains a tree $T' \cong T.$ However, $G[V(B) \cup \{w_4\}]$ is contained in a block of $G - V(T'),$ a contradiction, which completes the proof.

By Lemma 3.2, we may confirm Conjecture 1.2 for $k = 2.$

**Theorem 3.3.** For every tree $T_0$ of order $m$, each 2-connected graph $G$ with $\delta(G) \geq m + 2$ contains a subtree $T \cong T_0$ such that $G - V(T)$ is 2-connected.

**Proof.** Since $\delta(G) \geq m + 2$, there exists a subtree $T \cong T_0$. Then we may assume $T$ is such a tree that the maximum block $B$ of $G - V(T)$ has order as large as possible. Let $H = G - V(B \cup T).$ We are to prove $V(H) = \emptyset.$ In fact, by contradiction, assume that $V(H) \neq \emptyset$.

Then $|N_G(v) \cap V(B)| \leq 1$ for each $v \in V(H)$ by the assumption of $B$. Also, since $G$ is 2-connected and $B \cup H$ is not 2-connected, $N_{G_0}(H) \cap V(T) \neq \emptyset.$ If there is a vertex $v \in V(H \cup T)$ such that $|N_G(v) \cap V(B)| \geq 2$ and $H \cup T - v$ contains a tree $T' \cong T_0$, then $T'$ is a subtree of $G$ such that $G[V(B) \cup \{v\}]$ is contained in a block of $G - V(T'),$ a contradiction to the choice of $T$. So applying Lemma 3.2 with $\delta = m + 1$, we see that for any $v \in V(T \cup H),$ $|N_{G}(v) \cap V(B)| \leq m + 1 - m = 1.$

Since $G$ is 2-connected, there is a shortest path $Q$ with both ends in $B$ and internal vertices not in $V(B)$ such that $|V(Q)| \geq 3.$ By the assumption of $Q,$ $|N_G(v) \cap V(Q)| \leq 3$ for each $v \notin V(B \cup Q).$ Thus $|N_G(v) \cap V(B \cup Q)| \leq 4$. In fact, if the equality holds then a shorter path $Q'$ with ends in $B$ by using $v$ can be found easily, a contradiction. So $|N_G(v) \cap V(B \cup Q)| \leq 3$ and thus $\delta(G - V(B \cup Q)) \geq m + 2 - 3 = m - 1.$ This implies there is a $T' \cong T_0$ in $G - V(B \cup Q)$ such that $B \cup Q$ is contained in a block of $G - V(T'),$ a contradiction to our assumption. This completes the proof.

4 | 3-CONNECTED GRAPHS

In this case, we deal with 3-connected graphs. The main method is similar to the one for 2-connected graphs. However, in the proof of 2-connected graphs the subgraph $B$ is always 2-connected, since every 2-connected graph has an ear-decomposition. This no longer works for 3-connected graphs. Instead, we maintain a subdivision of some simple 3-connected graph in proof. In the end, by the assumption of the minimum degree, the subdivision is in fact a 3-connected graphs. Let $G$ be a subdivision of some simple 3-connected graph. An *ear* of $G$ is a maximal path $P$ each of whose internal vertex has degree 2 in $G$. Then any two ears of $G$ has different pair of ends. Write $t(G) = |\{vd_G(v) \geq 3\}|.$ Then $G$ is 3-connected if and only if
\( t(G) = |G|. \) Let \( X, Y \subseteq V(G). \) An \((X, Y)\)-path is path with one end in \( X \), the other end in \( Y \) and internal vertices not in \( X \cup Y \). If \(|X| = |x|\) then we use \((x, Y)\)-path instead of \(|x|, Y\)-path.

**Theorem 4.1.** For every tree \( T_0 \) of order \( m \), each 3-connected graph \( G \) with \( \delta(G) \geq m + 3 \) contains a subtree \( T \cong T_0 \) such that \( G - V(T) \) is 3-connected.

**Proof.** Since \( \delta(G) \geq m + 3 \), there exists a subtree \( T \cong T_0 \). Let \( B \) be a induced subgraph of \( G - V(T) \) isomorphic to a subdivision of some simple 3-connected graph. Such \( B \) does exist. In fact, since \( \delta(G - V(T)) \geq m + 3 - |T| = 3 \), thus \( G - V(T) \) contains a subdivision of \( K_4 \). Then the minimum induced subgraph containing a subdivision of \( K_4 \) is a candidate of \( B \). Furthermore, we may assume \( T, B \) are such subgraphs such that \( t(B) \) is as large as possible and then \(|B|\) is as small as possible. \( t(B) \) is as large as possible and then \(|B|\) is as small as possible. \( (3) \)

Let \( H = G - V(B \cup T) \). If \( V(H) = \emptyset \) then \( B = G - V(T) \) is a subdivision of some simple 3-connected graph. In fact, if there exists a vertex \( v \in V(B) \) such that \( \delta_B(v) = 2 \) then \( \delta(v) = \delta_B(v) + |N_G(v) \cap V(T)| \leq 2 + m < \delta(G) \), a contradiction. This implies \( B \) is 3-connected and the result holds. So, by contradiction, we may assume that \( V(H) \neq \emptyset \). Let \( X = \{v \in V(B)|\delta_B(v) \geq 3\} \) and \( Y = V(B) \setminus X \). Then \( t(B) = |X| \).

**Claim 1.** For any \( u \in V(H \cup T) \), if \(|N_G(u) \cap V(B)| \geq 3\) and \( H \cup T - u \) contains a subtree \( T' \cong T_0 \) then there exists an ear \( Q \) of \( B \) such that \( N_G(u) \cap V(B) \subseteq V(Q) \).

In fact, if there exists such a vertex \( u \) such that \( N_G(u) \cap V(B)| \geq 3 \) and no ear of \( Q \) contains \( N_G(u) \cap V(B) \) then, it is easy to find three \((u, X)\)-paths intersecting with each other only at \( u \). Thus \( B_1 = G[V(B) \cup \{u\}] \) is the subdivision of some simple 3-connected graph with \( t(B_1) > t(B) \), a contradiction to \( (3) \). Claim 1 is proved.

**Claim 2.** For any \( u \in V(H \cup T) \), if \( H \cup T - u \) contains a subtree \( T' \cong T_0 \) then \(|N_G(u) \cap V(B)| \leq 3\).

Suppose, to the contrary, that \( u \) is such a vertex such that \(|N_G(u) \cap V(B)| \geq 4\). Then by Claim 1, there exists an ear \( Q \) such that \( N_G(u) \cap V(B) \subseteq V(Q) \). Denote \( Q = u_1u_2...u_q \). Let \( a = \min\{|i|u_i \in N_G(u)|\} \) and \( b = \max\{|i|u_i \in N_G(u)|\}. \) If \( b - a \geq 3 \) then let \( B' = G[V(B - \{v_{b+1}, ..., v_{b-1}\}) \cup \{u\}] \) and we see that \( B' \) is a subdivision of some simple 3-connected graph such that \( t(B') = t(B) \) and \(|B'| - |B| = 1 - (b - a - 1) < 0\), a contradiction to \( (3) \). So \( b - a \leq 2 \) and Claim 2 is proved.

**Claim 3.** For any \( v \in V(H \cup T) \), \(|N_G(v) \cap V(B)| \leq 3\).

Suppose, to the contrary, that there exists \( v \in V(H \cup T) \) such that \(|N_G(v) \cap V(B)| \geq 4\). By Claim 2, \( v \in V(T) \). Apply Lemma 3.2 and by Claim 2, we see that \( N_G(H) \cap V(T) = \emptyset \). Then again by Claim 2, \( \delta(H) \geq m + 3 - 3 = m \). It follows that \( H \) contains \( T' \cong T_0 \). Then by Claim 2, \( N_G(v) \cap V(B)| \leq 3 \), a contradiction. The claim is proved.

**Claim 4.** \( Y \neq \emptyset \).
Suppose, to the contrary, that \( Y = \emptyset \). Then \( B \) is 3-connected. Since \( G \) is 3-connected, for any \( u \notin V(B) \), there exist three \((u, B)\)-paths \( P_1, P_2, P_3 \) intersecting with each other only at \( u \). Furthermore, we may assume that \( P_1, P_2, P_3 \) are such paths such that \( \sum_{i=1}^{3}|P_i| \) is as small as possible. Let \( B' = G[V(B) \cup V(P_1 \cup P_2 \cup P_3)] \). Then \( t(B') > t(B) \) and \( B' \) is still the subdivision of some simple 3-connected graph by the minimality of \( P_1 \)'s. Let \( h \) be a vertex of \( G - V(B') \) with minimum degree. If \( \delta(G - V(B')) < m - 1 \) then \( |N_G(h) \cap V(B')| \geq d_G(h) - \delta(G - V(B')) \geq 5 \). For \( i = 1, 2, 3 \), let \( v_i \in N_G(h) \cap V(B') \) such that \( v_i \notin \{v_1, ..., v_{i-1}\} \) and the distance from \( v_i \) to \( B \) in \( B' - \{v_1, ..., v_{i-1}\} \) is as small as possible. Then there exist three \((h, V(B'))\)-paths in \( G[V(B') \cup \{h\}] \) using edges \( hv_1, hv_2, hv_3 \) and avoiding two neighbors of \( h \) in \( \bigcup_{i=1}^{3} P_i \), a contradiction to the choice of \( P_1 \)'s. Thus \( \delta(G - V(B')) \geq m - 1 \) and then there exists a subgraph \( T' \cong T_0 \) in \( G - V(B') \) such that \( t(B') > t(B) \), a contradiction to (3). Claim 4 is proved.

By Claim 4 and the fact that \( G \) is 3-connected, there exists a \((Y, B)\)-path \( P \) edge-disjoint with \( B \) such that the two ends of \( P \) lie in no ear of \( B \). Furthermore, we may assume \( B, P \) are such subgraphs and paths such that (3) holds and then \( |P| \) is as small as possible. Let \( B' = G[V(B) \cup V(P)] \) and we will show that \( \delta(G - V(B')) \geq m - 1 \).

Suppose, to the contrary that \( \delta(G - V(B')) \leq m - 2 \). Let \( h \) be a vertex of \( G - V(B') \) with minimum degree and then \( |N_G(h) \cap V(B')| \geq 5 \). Assume \( P \) is a \((u, v)\)-path such that \( u \in Y \). Let \( Q_1 \) be an ear of \( B \) such that \( u \in V(Q_1) \). By Claim 3, \( |N_G(h) \cap V(P)| \geq |N_G(h) \cap V(B')| - 3 \geq 2 \). Let \( v_1, v_2 \in N_G(h) \cap V(P) \) closest to \( u, v \) in \( P \), respectively. By the minimality of \( P \), \( N_G(h) \cap V(P) \leq 3 \) and thus \( |N_G(h) \cap V(B)| \geq 2 \). If \( N_G(h) \cap V(B) = V(Q_1) \setminus Y \) then let \( B_1 = G[V(B) \cup \{h\} \setminus (V(Q_1) \cap Y)] \) and \( P_1 = hv_1Pv \). Then \( t(B_1) = t(B), |B_1| \leq |B| \) and \( |P_1| < |P| \), a contradiction to the choice of \( B \) and \( P \). So we may assume there exists \( w \in N_G(h) \cap (V(B) \setminus (V(Q_1) \setminus Y)) \). If \( w \notin V(Q_1) \) then let \( P_1 = uPv_1hw \) and if \( w \in V(Q_1) \) then let \( P_1 = whv_1Pv \). Then \( P_1 \) is a \((Y, B)\)-path such that no ear contains the two end of \( P \). By the choice of \( P \), \( |P_1| \geq |P| \). Thus \( N_G(h) \cap (V(P) \setminus V(B)) = \{v_1, v_2\} \) and then \( |P_1| \geq 4 \) and \( |N_G(h) \cap V(B)| \geq 3 \). Let \( w_1, w_2, w_3 \in N_G(h) \cap V(B) \). If \( w_1, w_2, w_3 \in X \) then let \( B_2 = G[V(B) \cup \{h\}] \). We have \( t(B_2) > t(B_1) \) and \( G - V(B_2) \) contains a subgraph isomorphic to \( T \) (since \( \delta(G - V(B_2)) \geq m + 3 - (3 + 1) = m - 1 \)), a contradiction. So we may assume that \( w_1 \in Y \) and \( w_1 \) lie in an ear \( Q_2 \) of \( B \). If \( w_2 \notin V(Q_2) \) or \( w_3 \notin V(Q_2) \) then \( w_1hw_2 \) or \( w_1hw_3 \) is a candidate of \( P \) with length 2, a contradiction. So we have \( w_1, w_2, w_3 \in V(Q_2) \) and we may assume \( w_2 \) is the middle neighbor of \( h \) on \( Q_2 \). Then \( w_1, w_2, w_3 \) are consecutive on \( Q_2 \). Let \( B_3 = G[V(B - w_2) \cup \{h\}] \). Then \( B_3 \) is also a subdivision of some simple 3-connected graph and \( t(B_3) = t(B) \) and \( |B_3| = |B| \). If \( Q_2 \neq Q_1 \) then let \( P_2 = hv_1Pu \) and if \( Q_2 = Q_1 \) then let \( P_2 = hv_2Pv \). In either case, \( P_2 \) is a \((Y', B_2)\)-path shorter than \( P \) (where \( Y' = Y \cup \{h\} \setminus \{w_2\} \)), a contradiction. Hence, \( \delta(G - V(B')) \geq m - 1 \).

It follows that there exists a subgraph \( T' \cong T_0 \) in \( G - V(B') \) such that \( t(B') > t(B) \), a contradiction to (3). The proof is completed. \[ \square \]

In view of Lemma 3.2, we believe one can prove there exists a function \( f(k) \) such that every \( k \)-connected graph with minimum degree at least \( f(k) + m \) contains a subtree \( T \) isomorphic to any given tree of order \( m \) such that \( G - V(T) \) is \( k \)-connected.
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