CANONICAL COHERENT STATES FOR THE RELATIVISTIC HARMONIC OSCILLATOR

V. Aldaya\textsuperscript{2,3} and J. Guerrero\textsuperscript{2,4}

March 14, 1995

Abstract

In this paper we construct manifestly covariant relativistic coherent states on the entire complex plane which reproduce others previously introduced on a given $SL(2,R)$ representation, once a change of variables $z \in C \rightarrow z_D \in$ unit disk is performed. We also introduce higher-order, relativistic creation and annihilation operators, $\hat{a}, \hat{a}^\dagger$, with canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ rather than the covariant one $[\hat{z}, \hat{z}^\dagger] \approx$ Energy and naturally associated with the $SL(2,R)$ group. The canonical (relativistic) coherent states are then defined as eigenstates of $\hat{a}$. Finally, we construct a canonical, minimal representation in configuration space by mean of eigenstates of a canonical position operator.

\textsuperscript{1}Work partially supported by the DGICYT.
\textsuperscript{2}Instituto de Física Teórica y Computacional Carlos I, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, Granada 18002, Spain
\textsuperscript{3}IFIC, Centro Mixto Universidad de Valencia-CSIC, Burjasot 46100-Valencia, Spain.
\textsuperscript{4}Departamento de Física Teórica y del Cosmos, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, Granada 18002, Spain
1 Introduction

Ordinary coherent states were introduced from the beginning of the developments of Quantum Mechanics and Radiation Theory in several different, yet equivalent, ways according to different interesting properties with direct applications to practical, mainly optical, systems (see the pioneer work by Glauber [1]). Essentially, these states can be characterized by a) giving minimal and symmetric $q - p$ uncertainty relations, b) being eigenstates of the annihilation operator $\hat{a}$, or c) as the result of applying the displacement operator $e^{i(\alpha \hat{a}^\dagger - \alpha^* \hat{a})}$ on the vacuum $|0\rangle$, and among the practical properties, we mention the low noise in amplifying applications (as a consequence of a)) and optical coherence (as a consequence of b)) (see for instance [2, 3]).

Relativistic quantum mechanical systems in general are characterized by possessing manifestly covariant commutation relations of the form $[\hat{x}, \hat{p}] \approx \text{Energy}$, so that the uncertainty relations are no longer $\Delta \hat{x} \Delta \hat{p} \geq \frac{\hbar}{2}$, but $\Delta \hat{x}\Delta \hat{p} \geq \frac{\hbar}{2} |< \hat{E}>|$ and, therefore, the absolute minimum can be reached only by the vacuum. In particular, the adopted commutation relations for the basic operators $\hat{x}, \hat{p}$ and $\hat{E}$ corresponding to the quantum relativistic harmonic oscillator are:

$$[\hat{E}, \hat{x}] = -i \frac{\hbar}{m} \hat{p}, \quad [\hat{E}, \hat{p}] = im\omega^2 \hbar \hat{x}, \quad [\hat{x}, \hat{p}] = i\hbar (1 + \frac{1}{mc^2} \hat{E}),$$ (1)

which implement a central (pseudo-)extension of the Lie algebra (see Ref. [4] for a study of the cohomology of Lie algebras) $SL(2, R)$ ($\approx$ Anti-deSitter group in 1+1 dimensions). This algebra, where $\hat{E}$ generates the time translations, $\hat{p}$ the space translations and $\hat{x}$ the boosts, reproduces the pseudo-extended Poincaré algebra in the $\omega \to 0$ limit and the extended Newton (non-relativistic harmonic oscillator) algebra when $c \to \infty$. It should be recalled at this point that it is the pseudo-extended Poincaré group which regains the centrally extended Galilei group in the non-relativistic limit [5] (for a general study of central extensions of groups see Ref. [6]).

The deviation of the relativistic commutation relations between $\hat{x}$ and $\hat{p}$ from the Galilean ones causes the different definitions of coherent states given above to be non-equivalent. The definition c) seems to be more widely adopted at least for those cases with an underlying group structure [7, 8].

In this paper we consider a set of states of the relativistic harmonic oscillator -our relativistic coherent states- obtained, in a natural way, following a group approach to quantization [4, 10], which reproduce those previously introduced by Perelomov [7] after the change of variables

$$z_D = \sqrt{\frac{2}{N}} \frac{z}{1 + \kappa}, \quad N \equiv \frac{mc^2}{\hbar \omega}, \quad \kappa \equiv \sqrt{1 + \frac{2zz^*}{N}},$$ (2)
from $C$ to the unit disk $D$, has been performed. Then, using a generalization of the concept of Polarization in Geometric Quantization \cite{[11],[12]}, we are able to find higher-order creation and annihilation operators $\hat{a}, \hat{a}^\dagger$ in terms of the basic generators $\hat{z}, \hat{z}^\dagger$ of $SL(2, R)$, as a non-polynomic function, satisfying canonical (yet relativistic) commutation relations, and allowing for a conventional way of defining canonical, relativistic coherent states as eigenstates of the new higher-order annihilation operator $\hat{a}$. The new relativistic coherent states thus satisfy properties fully analogous to those of ordinary (non-relativistic) coherent states, although defined in terms of canonical or Darboux \cite{[13]} co-ordinates.

Our construction of the operators $\hat{a}$ and $\hat{a}^\dagger$ in terms of the $SL(2, R)$ generators $\hat{z}$ and $\hat{z}^\dagger$ is invertible, but the inverse (non-polynomical) relation, i.e. $\hat{z}$ and $\hat{z}^\dagger$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ must not be confused, however, with the “standard” quadratic realization of the $SL(2, R)$ generators as $\hat{K}_- = \frac{1}{2} (\hat{a})^2$, $\hat{K}_+ = \frac{1}{2} (\hat{a}^\dagger)^2$ and $\hat{K}_3 = \frac{1}{2} \hat{a}^\dagger \hat{a}$, appearing, for instance, in Quantum Optics. This quadratic construction close algebra with the operators $\hat{a}$ and $\hat{a}^\dagger$ themselves but possesses the drawback that only two particular irreducible representations of $SL(2, R)$ can be obtained, namely those with Bargmann index $k = \frac{1}{4}; \frac{3}{4}$.

2 Relativistic coherent states (RCS).

In the group-quantization scheme, the coherent states (generalizing the standard non-relativistic coherent states \cite{[1]}), as well as the corresponding wave functions, are defined by mean of infinitesimal relations (differential polarization equations) \cite{[14]}, rather than a finite group action on the vacuum, associated with a previously given representation of the group \cite{[7],[8]} (see \cite{[15],[16],[17]} for a more general study of over complete families of states non-necessarily associated with groups). These are defined simply as \cite{[14],[18]}:

$$|z> \equiv \sum_{n=0}^{\infty} \bar{\Phi}_n^N(z, z^*)^*|N, n> \leftrightarrow \bar{\Phi}_n^N(z, z^*) =<z|N, n>$$

$$\bar{\Phi}_n^N(z, z^*) \equiv \frac{1}{\pi \sqrt{n!}} \sqrt{\frac{(2N)_n}{(2N)^n}} \sqrt{\frac{2N-1}{2N}} \frac{(1+\kappa)}{2}^{-N-n} \sqrt{n!} (2N)^n (2N)^n \bar{z}^* n$$

where the states $|N, n>$ constitute the Fock space for the relativistic harmonic oscillator, i.e.

$$<0, N|N, 0> = 1, \quad |N, n>= \frac{(\hat{z}^\dagger)^n|N, 0>}{\sqrt{n!(2N)_n}}$$

$$\hat{z}|N, n> = \sqrt{n(1+\frac{n-1}{2N})}|N, n-1>$$

2
\[ \hat{\zeta}^\dagger |N, n > = \sqrt{(n + 1)(1 + \frac{n}{2N})} |N, n + 1 > \]
\[ \hat{H} |N, n > = n |N, n > \]

They carry an irreducible representation (with Bargmann index \( k = N \)) of the \( SL(2, R) \) algebra

\[ [\hat{H}, \hat{\zeta}] = -\hat{\zeta}, \quad [\hat{H}, \hat{\zeta}^\dagger] = \hat{\zeta}^\dagger, \quad \hat{\zeta} \hat{\zeta}^\dagger = \hat{1} + \frac{1}{N} \hat{H} \]

where these operators are essentially the right generators of the \( SL(2, R) \) group \([14, 19]\). The relation with the standard notation for (abstract) \( SL(2, R) \) generators is \( \hat{\zeta} = \frac{1}{\sqrt{2N}} \hat{K}_- = \frac{1}{\sqrt{2N}} (\hat{K}_1 - i \hat{K}_2) \), \( \hat{\zeta}^\dagger = \frac{1}{\sqrt{2N}} \hat{K}_+ = \frac{1}{\sqrt{2N}} (\hat{K}_1 + i \hat{K}_2) \) and \( \hat{H} = \hat{K}_3 - N \). However, our generators have the advantage of a proper non-relativistic limit.

The associated wave functions \( < z | z' > \) are:

\[ < z' | z > = \sum_{n=0}^{\infty} \tilde{\Phi}^N_n (z', z^*) \tilde{\Phi}^N_n (z, z^*)^* = \frac{1}{\pi} \frac{2N - 1}{2N} \frac{(1 + \kappa)}{2}^{-N} \frac{(1 + \kappa')}{2}^{-N} \sum_{n=0}^{\infty} \frac{(2N)_n}{n!(2N)^n} \frac{(2z'^* - 2z)}{1 + \kappa' + 1 + \kappa} \]

As in the non-relativistic case, the RCS constitute an overcomplete set and satisfy the reproducing kernel property with respect to the group measure:

\[ I = \int \frac{dzdz^*}{\kappa} |z > < z| \]

\[ |z' > = \int \frac{dzdz^*}{\kappa} |z > < z| z' > \]

The expectation values of \( \hat{\zeta} \) and \( \hat{\zeta}^\dagger \) on the coherent states are \( < \hat{\zeta} > \equiv \frac{< z | \hat{\zeta} | z >}{< z | z >} = z \)
and \( < \hat{\zeta}^\dagger > = z^* \), making the variables \( z, z^* \in C \) especially suitable to describe the Bargmann-Fock-like representation. Defining the operators \( \hat{x} \) and \( \hat{p} \) in the usual way, i.e.

\[ \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{\zeta} + \hat{\zeta}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} (\hat{\zeta}^\dagger - \hat{\zeta}), \]

we get \( < \hat{x} > = x \), \( < \hat{p} > = p \), where \( x \) and \( p \) are defined in the same way, constituting the phase-space coordinates for Anti-deSitter space-time (see \([19]\) where an adequate choice of time is discussed). We observe that these expectation values follow the classical trajectories (geodesics) of the motion.
Repeating the group quantization in the new variables we obtain the manifestly-covariant $x$-representation. The states $|x,t>$ are defined as:

$$|x,t> \equiv \sum_{n=0}^{\infty} \Psi_n^N(x,t)^*|N,n>,$$

where

$$\Psi_n^N(x,t) \equiv e^{-in\omega t}\Phi_n^N(x)$$

$$\Phi_n^N(x) = \sqrt{\frac{\omega}{2\pi}} \left( \frac{m\omega}{\hbar \pi} \right)^{\frac{1}{4}} \frac{1}{2^n \sqrt{n!}} \left( \frac{(2N)^n}{(2N)_n} \right) \frac{\Gamma(N)}{\sqrt{N\Gamma(N-\frac{1}{2})}} \alpha^{-(N+n)} H_n^N(\chi),$$

$\alpha \equiv \sqrt{1 + \frac{\omega^2}{c^2} x^2}$, $\chi \equiv \sqrt{\frac{m\omega}{\hbar}} x$ and $H_n^N(\chi)$ are the Relativistic Hermite polynomials [20, 19]. These states are not eigenstates of the boost operator $\hat{x}$ in the same manner that the states $|z,t> \equiv |e^{-i\omega t} z>$ are not eigenstates of the annihilation operator $\hat{\rho}$. The integration measure is $dxdt$, coming from the group measure once the $p$-integration has been regularized [19].

Both representations are related through the Relativistic Bargmann transform [18], the kernel of which is nothing but the configuration-space wave function of the coherent states $|z,t>$ defined above, and have the proper non-relativistic limit.

The time variable can be factorized out (non-trivially) from the manifestly-covariant $x$-representation, giving rise to a minimal $x$-representation $|x>$. The new integration measure turns out to be $dx/\alpha^2$ and it is this measure that makes the Relativistic Hermite polynomials (multiplied by the partial weights $\alpha^{-N-n}$) a set of orthogonal functions [19, 21, 22].

The uncertainty relations for the operators $\hat{x}$ and $\hat{\rho}$ on the $|z>$ states are:

$$\Delta \hat{x} \Delta \hat{\rho} = \frac{\hbar}{2} \sqrt{\frac{\kappa^2}{4} + \frac{1}{4N^2} [4|z|^4 - (z^2 + z^*2)^2]} \geq \frac{1}{2} \hbar \kappa = \frac{1}{2} <[\hat{x},\hat{\rho}]>.$$

The equality holds for $z = |z|e^{in\pi/2}$, i.e. $z$ real or pure imaginary, defining the so-called "intelligent states" [23], but only for $z = 0$ (the vacuum) we reach the absolute minimum.

Our RCS correspond to definition c) in Sec. 1, and can be identified with the generalized coherent states on the unit complex disk [7] once the change of variables $z_D = \sqrt{\frac{2}{N}} \frac{z}{1+\kappa} \in D$ ($z \in C$), and the identification $k \equiv N$ have been made, where $D$ is the unit complex disk and $k$ is the Bargmann index characterizing the irreducible representations of $SL(2, R)$. For a calculation of the uncertainty relations in the unit Disk see [24].

It would be natural to ask whether a definition analogous to b) could also be given. In fact, there exists a solution to the relativistic eigenvalue problem, $\hat{\rho} |\rho> = \rho |\rho>$.  

4
Using the commutation relations (3) and the Fock-space representation (6), we obtain:

\[ |\rho > = c_0 \sum_{n=0}^{\infty} \frac{(2\sqrt{N}\rho)^n}{\sqrt{n!(2N)^n}} |N, n >, \]

where \( c_0 \) is an arbitrary normalization factor (a function of \( \rho \) actually). In the \( c \to \infty \) limit these states also reproduce the standard non-relativistic coherent states. They are related to others previously defined by Barut and Girardello [25] through the change \( \rho \to \rho/\sqrt{N} \), and choosing \( c_0 = 1 \). The connection between the corresponding generators is \( \hat{\bar{z}} = \frac{1}{2\sqrt{N}} \hat{L}_- \) and \( \hat{\bar{z}}^\dagger = \frac{1}{2\sqrt{N}} \hat{L}_+ \).

Very recently, [26], it has been shown that both sets of coherent states (Perelomov’s and Barut and Girardello’s) are particular cases of a more general definition of coherent states, the generalized intelligent states, which minimize the Robertson-Schrödinger uncertainty relation [27]. In the particular case of operators satisfying canonical commutation relations the states that minimize the Robertson-Shrödinger uncertainty relation had been called correlated states in [28], although it has been proved more recently that they coincide with ordinary squeezed states (see [26] and references therein).

3 Canonical (higher-order) creation and annihilation operators: canonical, relativistic coherent states.

The definition of polarization in group quantization (given in terms of left-invariant vector fields \( \tilde{X}_L \)) can be generalized so as to admit operators in the left enveloping algebra. This generalization has already been exploited in finding a position operator for the free relativistic particle [29], and for obtaining a new momentum operator canonically associated with the boosts operator [30] (as well as in solving anomalous problems [10]). In the present case it also makes sense to look for basic operators satisfying canonical (versus manifestly covariant) commutation relations. Let us then seek power series in \( \hat{X}^L_z \) and \( \hat{X}^L_{z^*} (\eta = e^{i\omega t/2}) \) [14, 19],

\[
\begin{align*}
\hat{X}^{L, HO}_{z^*} &= \hat{X}^{L*} + \frac{\alpha}{N} \hat{X}^L \hat{X}^L \hat{X}^L + \ldots \\
\hat{X}^{L, HO}_\eta &= \hat{X}^L - \mu \hat{X}^L \hat{X}^L - \frac{\nu}{N} \hat{X}^L \hat{X}^L \hat{X}^L \hat{X}^L + \ldots,
\end{align*}
\]

such that \( \mathcal{P}^{HO} =< \hat{X}^{L, HO}_\eta, \hat{X}^{L, HO}_{z^*} > \) contains \( \hat{X}^{L}_\eta \) and excludes the central generator. The coefficients of the power series are determined by the requirement that \( \mathcal{P}^{HO} \) is a polarization and the corresponding right operators define a unitary action on the wave functions \( \Psi \) which fortunately are the same as before.

More specifically,

\[
\begin{align*}
[\hat{X}^{L, HO}_\eta, \hat{X}^{L, HO}_{z^*}] &= -2\hat{X}^{L, HO}_{z^*} \\
[\hat{X}^{R, HO}_{z^*}, \hat{X}^{R, HO}_{z^*}] &= \hat{1}
\end{align*}
\]
The resulting higher-order (canonical) creation and annihilation operators are:

\[ \begin{align*}
\hat{a} & = \hat{z} - \left( \frac{1}{4N} - \frac{3}{32N^2} \right) \hat{z}\hat{\hat{z}} + \frac{7}{32N^2} \hat{z}^\dagger \hat{\hat{z}} \hat{\hat{z}} + ... \equiv \sqrt{\frac{2}{1 + \kappa}} \hat{\hat{z}} \\
\hat{a}^\dagger & = \sqrt{\frac{2}{1 + \kappa}} \hat{\hat{z}}^\dagger
\end{align*} \] (17)

and the energy operator is:

\[ \hat{H}^{HO} = N (\kappa - 1) = \hat{a}^\dagger \hat{a} \] (18)

where \( \kappa \equiv \sqrt{1 + \frac{2}{N} (\hat{z}^\dagger \hat{z})} \) and the operator \( \sqrt{\frac{2}{1 + \kappa}} \) must be considered to be functions of the single operator \( (\hat{z}^\dagger \hat{z}) \). We keep the notation \( \hat{H}^{HO} \), even though this operator is only quadratic in \( \hat{a}, \hat{a}^\dagger \), to remind the reader its higher-order origin (in terms of the covariant operators \( \hat{z}, \hat{z}^\dagger \)).

The commutation relations between \( \hat{H}^{HO}, \hat{\hat{a}}, \) and \( \hat{a}^\dagger \) are the non-relativistic (canonical) ones, and the action of these new operators on the relativistic Fock states reproduces the non-relativistic harmonic oscillator representation, even though the states \( |N, n> \) are the same relativistic energy eigenstates as before.

**Canonical coherent states:**

It seems quite natural to define canonical coherent states \( |a> \) as the eigenstates of the canonical annihilation operator, \( \hat{a}|a> = a|a> \), with solutions:

\[ |a> = e^{-|a|^2/2} \sum_n \frac{a^n}{\sqrt{n!}} |N, n> \],

and to introduce a “non-relativistic” Bargmann-Fock space in the usual way:

\[ <a|N, n> = <n, N|a>^* = e^{-|a|^2/2} \frac{a^* n}{\sqrt{n!}} \equiv \tilde{\Phi}^{N,R}_n (a)^* \],

with measure just \( dada^* \).

The connection to the relativistic Bargmann-Fock space is given by

\[ \begin{align*}
\tilde{\Phi}_a(z) & = <z|a> = \sum_{n=0}^{\infty} <z|N, n><n, N|a> = \sum_{n=0}^{\infty} \Phi_n(z) \tilde{\Phi}_n^{N,R} (a)^* \\
& = \frac{1}{\pi} \sqrt{\frac{2N - 1}{2N}} e^{-|a|^2/2} \left( \frac{1 + \kappa}{2} \right)^{-N} \sum_{n=0}^{\infty} \frac{1}{n!} \sqrt{\frac{2N}{(2N)^n}} (\frac{2az^*}{(2N)(1 + \kappa)})^n
\end{align*} \] (21)
This series is convergent in the entire complex plane, thus defining an integral function, as can be checked by standard criteria. A power expansion in terms of $\frac{1}{N}$ can be computed:

$$<z|a> \approx \frac{1}{\pi} e^{-|a|^2/2} e^{-|z|^2/2} e^{az} \left\{ 1 - \frac{1}{4N} \left[ 1 - \frac{1}{2} \left( |z|^2 - az^* \right) \right] \right\} + \ldots$$ (22)

The expectation value $<a|\hat{z}|a>$ defines a classical function $z = z(a)$ relating the variables $a, a^*$ and $z, z^*$ as follows:

$$<a|\hat{z}|a> = a \sum_{n=0}^{\infty} c_n <a| (\hat{a}^{\dagger} \hat{a})^n |a>$$ (23)

where $c_n$ are the coefficients of the power series of $f(u) = \sqrt{1 + \frac{u}{2N}}$. Then we define:

$$z(a) = \sqrt{1 + \frac{|a|^2}{2N} a}$$ (24)

Note that although $<a| (\hat{a}^{\dagger} \hat{a})^n |a> \neq <a|\hat{a}^{\dagger n} \hat{a}^n|a> = |a|^{2n}$, any operator of the form $\hat{F} = \hat{O} \hat{a}^m$ (or $\hat{G} = \hat{a}^{\dagger p} \hat{O}$), where $[\hat{H}, \hat{O}] = 0$, defines a classical function $F(a)$ (or $G(a)$) by the formula:

$$F(a) = a^m \sum_n o_n |a|^{2n}, \quad G(a) = a^{*n} \sum_n o_n |a|^{2n},$$ (25)

where $<a|\hat{O}|a> = \sum_n o_n <a| (\hat{H} \hat{O})^n |a>$. The functions

$$a(z) = \sqrt{\frac{2}{1 + \kappa}} z, \quad a^*(z) = \sqrt{\frac{2}{1 + \kappa}} z^*,$$ (26)

the inverse relation of (24), turn out to be the Darboux coordinates taking the symplectic form $\Omega \equiv \frac{1}{\kappa} dz \wedge dz^*$ to the canonical form $\Omega = da \wedge da^*$.

Finally, we define

$$\hat{q} \equiv \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^{\dagger})$$

$$\hat{\pi} \equiv i \sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^{\dagger} - \hat{a})$$ (27)

satisfying

$$[\hat{q}, \hat{\pi}] = i\hbar \hat{1},$$ (28)

as well as their corresponding classical functions $q$ and $\pi$. For these operators we obviously obtain

$$\Delta q \Delta \pi = \frac{\hbar}{2}$$ (29)
on the $|a>$ states.

A new minimal representation in configuration space can be introduced which will be
called the canonical, minimal representation or the $q$-representation. The corresponding
states, $|q>$, are the eigenstates of the position operator $\hat{q}$. They prove to be

$$|q> = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{1}{2^{n}\sqrt{n!}} e^{-\xi^2/2} H_n(\xi) |N, n>$$

where $\xi \equiv \sqrt{\frac{m\omega}{\hbar}} q$ and $H_n$ are the ordinary, non-relativistic Hermite polynomials. The
integration measure is just $dq$.

The analogue to the transformation kernel $<z|a>$ (21) in configuration space, i.e.
$<x|q>$, also makes sense and relates the Hermite polynomials and the Relativistic
Hermite polynomials, and can be worked out in a similar way:

$$<x|q> = \sum_{n=0}^{\infty} <x|N, n><n, N|q> = \sum_{n=0}^{\infty} \Phi_n^*(x) \Phi_n^N.R.(q)*$$

$$= \sqrt{\frac{m\omega}{\hbar \pi}} e^{-\xi^2/2} \alpha^{-N} \sqrt{\frac{\Gamma(N)}{\sqrt{N\Gamma(N - \frac{1}{2})}}} \sum_{n=0}^{\infty} \frac{\alpha^{-n}}{2^n n!} \frac{(2N)^n}{(2N)_n} H_n^N(\chi) H_n(\xi)$$

In the $c \rightarrow \infty$ limit ($N \rightarrow \infty$) $<x|q> = \delta(x-q)$, as corresponds to the nonrelativistic harmonic oscillator. For large $N$ we can compute a power series expansion in $\frac{1}{N}$:

$$<x|q> \approx \delta(x-q) + \sqrt{\frac{m\omega}{\hbar}} \frac{1}{64N} \left\{ 12(1 + \xi^2) \delta(\xi - \chi) + 4 \left[ 9\xi + 3\chi + \xi^3 + \chi^3 \right] \times \right.$$  

$$\delta'(\xi - \chi) + 6 \left[ (\xi^2 - \chi^2) + 2 \right] \delta''(\xi - \chi) + 4(\xi + \chi) \delta'''(\xi - \chi) \right\} + ... (32)$$

For finite $N$, we can study the convergence of the series having into account that this
depends on the large $n$ behaviour. For $n >> N$ we can use the asymptotic expression for
the relativistic Hermite polynomials given in [22] (as well as the usual one for the Hermite polynomials) and we get (except for factors not depending on $n$)

$$\Phi_n^N(x) \Phi_n^N.R.(q)* \sim n^{-\frac{3}{4}} \cos \left[ \sqrt{2n + 1} \xi - n \frac{\pi}{2} \right] \cos \left[ (n + N) \arcsin \frac{1}{\alpha} - N \frac{\pi}{2} \right]$$

In the particular case of $\xi = \chi = 0$ the resulting series is $\sum (2n)^{-\frac{3}{4}}$, which is of course divergent. For the general case the convergence of the series is assured by the convergence of the integral

$$\int_{\mu}^{\infty} x^{-\frac{3}{4}} e^{i[a x + b \sqrt{2x + 1}]} dx$$

with ($\mu \geq 1$).
The existence of Galilean-like creation and annihilation operators along with the \( SL(2, R) \) operators \( \hat{\xi}, \hat{\xi}^\dagger \), looks rather tricky at first sight and thus deserves some comment. First of all, the co-existence of both type of operators is possible only because the spectra of the non-relativistic and relativistic harmonic oscillator are the same and the Hamiltonian \( \hat{H} \) is shared by the two systems, although written in two different manners: 
\[
\hat{H} = \hat{a}^\dagger \hat{a} = \sqrt{1 + \frac{2}{N} \hat{z}^\dagger \hat{z}}.
\]
The common (phase space) Poisson algebra contains two sub-algebras \((\hat{H}, a^\dagger, a)\) and \((\hat{H}, z^\dagger, z)\) intersecting at \( \hat{H} \) even though \((\hat{H}, a^\dagger, a, z^\dagger, z)\) does not close. The situation is in certain aspects similar to the case of the Schrödinger group \([3],[10]\) which (in 1+1 dimensions) is generated by an analogous set of operators with the only difference that the commutators between \( \hat{a}, \hat{a}^\dagger \) and \( \hat{z}, \hat{z}^\dagger \) close and, therefore, it is possible to find a quantum representation in which \( \hat{z} \) and \( \hat{z}^\dagger \) are written only as quadratic functions of \( \hat{a} \) and \( \hat{a}^\dagger \). This quantum representation is realized only for the special values of the \( SL(2, R) \) Bargmann index \( k = \frac{1}{4}, \frac{3}{4} \), as a consequence of the \textit{anomalous} character of the Schrödinger group \([10]\), with direct physical application in two-photon quantum optics \([3]\).

Needless to say that the non-relativistic harmonic oscillator also support the construction of higher-order operators \( \hat{\xi}, \hat{\xi}^\dagger \) as functions of the operators \( \hat{a}, \hat{a}^\dagger \) (the inverse of \((17)\)):
\[
\hat{\xi} = \sqrt{1 + \frac{1}{2N} \hat{H}} \hat{a}, \quad \hat{\xi}^\dagger = \hat{a}^\dagger \sqrt{1 + \frac{1}{2N} \hat{H}},
\]
thus realizing the \( SL(2, R) \) group on states \(| n \rangle\) and for any value of \( N \) (or Bargmann index \( k \)) and not just for \( N = \frac{1}{4}, \frac{3}{4} \) as in the case of the Schrödinger group.

\textbf{Acknowledgement.} The authors wish to thank the referee for valuable suggestions.

\textbf{References}

[1] R.J. Glauber, Phys. Rev. \textbf{130}, 2529 (1963); \textbf{131} 2766 (1963)

[2] H.P. Yuen, Phys. Rev. A \textbf{13}, 2226 (1976)

[3] V.V. Dodonov, O.V. Man’ko, V.I. Man’ko and L. Rosa, \textit{Thermal noise and oscillations of photon distribution for squeezed and correlated light}, INFN-NA-93/31

[4] N. Jacobson, \textit{Lie algebras}, Interscience, New York (1962)

[5] E.J. Saletan, J. Math. Phys. \textbf{2}, 1 (1961)

[6] V. Bargmann, Ann. Math. \textbf{59}, 1 (1954)

[7] A.M. Perelomov, Commun. Math. Phys. \textbf{26}, 22 (1972)
[8] A.M. Perelomov, *Generalized Coherent States and their Applications*, Springer, Berlin (1986)

[9] V. Aldaya and J.A. de Azcárraga, J. Math. Phys. **23**, 1297 (1982)

[10] V. Aldaya, J. Bisquert, R. Loll and J. Navarro-Salas, J. Math. Phys. **33**, 3087 (1992)

[11] J.M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris (1970)

[12] B. Kostant, *Quantization and Unitary Representations*, Lecture Notes in Math. 170, Springer-Verlag, Berlin (1970)

[13] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, W.A. Benjamin, INC. (1967)

[14] V. Aldaya, J.A. de Azcárraga, J. Bisquert and J.M. Cerveró, J. Phys. **A23**, 707 (1990)

[15] J.R. Klauder, J. Math. Phys. **4**, 1055 (1963); **5**, 177 (1964)

[16] J.R. Klauder and J. Mac Kenna, J. Math. Phys. **5**, 878 (1964)

[17] J.R. Klauder and E.C.G. Sudarshan, *Fundamentals of Quantum Optics*, Benjamin, New York (1968)

[18] V. Aldaya and J. Guerrero, J. Phys. **A26**, L1175 (1993)

[19] V. Aldaya, J. Bisquert, J. Guerrero and J. Navarro-Salas, *Group-theoretical construction of the quantum relativistic harmonic oscillator*, UG-FT-34/93

[20] V. Aldaya, J. Bisquert and J. Navarro-Salas, Phys. Lett. **A156**, 351 (1991)

[21] A. Zarzo, J.S. Dehesa and J. Torres, *On a new set of polynomials representing the wave functions of the relativistic harmonic oscillator*. Preprint Granada 1993.

[22] B. Nagel, J. Math. Phys. **35**, 1549 (1994)

[23] C. Aragone, G. Guerri, S. Salamo, and J.L. Tani, J. Phys. **A15**, L149 (1974)

[24] K. Wódkiewicz and J.H. Eberly, J. Opt. Soc. Am. **B2**, 458 (1985)

[25] A.O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971)

[26] D.A. Trifonov, J. Math. Phys. **35**, 2297 (1994)

[27] H.P. Robertson, Phys. Rev. **35**, 667 (1930); E. Schrödinger, Sitzungsber. Preuss. Akad. Wiss. p. 296, Berlin 1930
[28] V.V. Dodonov, E.V. Kurmyshev and V.I. Man’ko, Phys. Lett. A 79, 150 (1980)
[29] V. Aldaya, J. Bisquert, J. Guerrero and J. Navarro-Salas, J. Phys. A26, 5375 (1993)
[30] V. Aldaya and J. Guerrero, J. Phys. A28, L137 (1995)
[31] U. Niederer, Helv. Phys. Acta 45, 802 (1972); 46, 191 (1973); 47, 167 (1974)
[32] V. Aldaya, J. Guerrero (in preparation)
[33] D.J. Navarro and J. Navarro-Salas, A Relativistic Harmonic Oscillator Simulated by an Anti-de Sitter Background, FTUV/94-27, IFIC/94-24