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Abstract
We develop a partial Hamiltonian framework to obtain reductions and closed-form solutions via first integrals of current value Hamiltonian systems of ordinary differential equations (ODEs). The approach is algorithmic and applies to many state and costate variables of the current value Hamiltonian. However, we apply the method to models with one control, one state and one costate variable to illustrate its effectiveness. The current value Hamiltonian systems arise in economic growth theory and other economic models. We explain our approach with the help of a simple illustrative example and then apply it to two widely used economic growth models: the Ramsey model with a constant relative risk aversion (CRRA) utility function and Cobb Douglas technology and a one-sector AK model of endogenous growth are considered. We show that our newly developed systematic approach can be used to deduce results given in the literature and also to find new solutions.

\textbf{keyword:} Current value Hamiltonian, partial Hamiltonian approach, economic growth models

1 Introduction

There has been extensive use of dynamic optimization in economic modeling and many of these models use the current value Hamiltonian whenever the integrand function contains a discount factor. These models range from those used for neoclassical economic growth ([1], [2]) to optimal firm-level investment [3] and human capital and earnings [4]. Pontrygin’s maximum principle provides a set of necessary conditions for the solution of the continuous time optimal control problem involving a current value Hamiltonian and a dynamical system of ODEs is obtained for control, state and costate variables. Beginning with [5] there have been various approaches, both qualitative and quantitative (see [6] for a good account of these), to deal with dynamic economic models arising from current value Hamiltonian system and most of these models were solved using numerical approaches (like [7]) or linear approximations around steady states ([8]). The critical problem is that for the underlying nonlinear dynamical system in economics there is a lack of a general analytical solution procedure not only for higher order systems but even for systems with one state and costate variable.

It is true to say that nonlinear dynamical systems evade closed-form solutions in general. However, the lack of a general procedure inhibits the search for
reductions and solutions of such type of nonlinear equations even when solutions do exist. Having said that, there are some well-known closed-form solutions that appear in the literature (see, e.g. \[9, 10, 11, 12, 13, 14\]). These solutions have been obtained by seemingly disparate approaches. Independent of the knowledge of explicit solutions, dynamic local stability of certain systems (see \[15, 16, 17\]) have been characterized by qualitative or numerical approaches.

Several important contributions have been made in the analysis of nonlinear dynamical systems of economic models. Here we focus on a new approach which yields first reductions and closed-form solutions for such systems of ODEs. We develop a Hamiltonian framework for several control, state and costate variables. Therefore, the method we develop is applicable to an arbitrary system of ODEs. However, we apply it to a system of two ODEs in order to show its effectiveness. In the case of higher order systems of ODEs, the approach may require the use of algebraic computing.

The layout of the paper is as follows. The partial Hamiltonian approach is developed in Section 2. In Section 3 we provide a simple illustrative example to show how our approach works. The Ramsey model with a constant relative risk aversion (CRRA) utility function with Cobb Douglas technology and the one-sector AK model of endogenous growth are studied in Section 4 and known solutions are deduced via our partial Hamiltonian approach. Conclusions are finally presented in Section 5.

2 A Hamiltonian version of the Noether-type theorem

Herein we develop a partial Hamiltonian approach for current value Hamiltonians which do not satisfy the canonical Hamilton equations. This is done for several control, state and costate variables.

Let \( t \) be the independent variable and \((q, p) = (q^1, \ldots, q^n, p_1, \ldots, p_n)\) the phase space coordinates. The derivatives of \( q^i, p_i \) with respect to \( t \) are

\[
\dot{q_i} = D(p_i), \quad \dot{p_i} = D(q_i), \quad i = 1, 2, \ldots, n,
\]

where

\[
D = \frac{\partial}{\partial t} + \dot{q_i} \frac{\partial}{\partial q_i} + \dot{p_i} \frac{\partial}{\partial p_i} + \cdots
\]

is the total derivative operator with respect to \( t \). The summation convention is utilized for repeated indices. The variables \( t, q, p \) are independent and connected only by the differential relations (1).

There are some well-known operators which are defined in the space of the variables \((t, q, p)\) and its prolongations. We introduce them.

In addition to the Euler operator

\[
\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D \frac{\partial}{\partial q^i}, i = 1, 2, \ldots, n,
\]
one also has the variational operator
\[
\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D \frac{\partial}{\partial \dot{p}_i}, \quad i = 1, 2, \ldots, n.
\] (4)

The action of the operators (3) and (4) on
\[ L(t, q, \dot{q}) = p_i \dot{q}^i - H(t, q, p) \] (5)
equated to zero yields the canonical Hamilton equations
\[ \dot{q}^i = \frac{\partial H}{\partial \dot{p}_i}, \]
\[ \dot{\dot{p}}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \] (6)

That is \( \delta L/\delta q^i = 0 \) and \( \delta L/\delta p_i = 0 \) results in (6). Equation (5) is the well-known Legendre transformation which relates the Hamiltonian and Lagrangian, where \( p_i = \partial L/\partial \dot{q}^i \) and \( \dot{q}^i = \partial H/\partial \dot{p}_i \).

Generators of point symmetries in the space \((t, q, p)\) are operators of the form
\[ X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta^i(t, q, p) \frac{\partial}{\partial q^i} + \zeta_i(t, q, p) \frac{\partial}{\partial p_i}. \] (7)
The operator in (7) is a generator of a point symmetry of the canonical Hamiltonian system (6) if
\[
\dot{\eta}^i - \dot{\dot{q}}^i \dot{\xi} - X \left( \frac{\partial H}{\partial \dot{p}_i} \right) = 0, \\
\dot{\zeta}_i - \dot{\dot{p}}_i \dot{\xi} + X \left( \frac{\partial H}{\partial q^i} \right) = 0, \quad i = 1, \ldots, n
\] (8)
on the system (6).

Hamiltonian symmetries in evolutionary or canonical form have been considered ([18]). Furthermore, symmetry properties of the Hamiltonian action have been investigated in the space \((t, q, p)\) by [19] and [20]. In the latter, the authors considered the general form of the symmetries (7) and provided a Hamiltonian version of Noether’s theorem.

The following important results which are analogs of Noether symmetries and the Noether theorem (see [18, 21, 22, 20] for a discussion) were established.

**Theorem 1** (Hamilton action symmetries): A Hamiltonian action
\[ p_i dq^i - H dt \] (9)
is invariant up to gauge \( B(t, q, p) \) with respect to a group generated by (7) if and only if the condition
\[
\zeta_i \frac{\partial H}{\partial \dot{p}_i} + p_i D(\eta^i) - X(H) - HD(\xi) - D(B) = 0,
\] (10)
holds.

**Theorem 2** (Hamiltonian version of Noether’s theorem): The canonical Hamilton system (6) which is invariant has the first integral

\[ I = p_i \eta^i - \xi H - B \]  

for some gauge function \( B = B(t, q, p) \) if and only if the Hamiltonian action is invariant up to divergence with respect to the operator \( X \) given in (7) on the solutions to equations (6).

We now focus our attention on systems of equations which are not in the canonical form (6). Therefore the Theorems 1 and 2 do not apply for these systems. We need an extension of the existing results which we carry out below.

Since the current value Hamiltonian (see, e.g. [23]) satisfies

\[ \dot{q}^i = \frac{\partial H}{\partial p_i}, \]
\[ \dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma_i, \quad i = 1, 2, \ldots, n, \]  

where \( \Gamma_i \) is a nonzero function, we seek an extension of the results relating to the canonical Hamiltonian system to the system (12) so that we can obtain first integrals of system (12) in an algorithmic manner. We also refer to an \( H \) that satisfies (12) as a partial Hamiltonian.

It is opportune to remark that \( X \) as in (7) is a generator of point symmetry of the current value Hamiltonian system (12) if

\[ \dot{\eta}^i - \dot{\xi}^i - X(\frac{\partial H}{\partial p_i}) = 0, \]
\[ \dot{\xi}_i - \dot{p}_i \xi + X(\frac{\partial H}{\partial q^i} - \Gamma_i) = 0, \quad i = 1, \ldots, n \]  

on the system (12). Note that (13) is evidently different from (8) due to the nonzero term \( \Gamma_i \).

We introduce the definition of what we call the partial Hamiltonian operator below. This is motivated by the analogous definition of the partial Noether operator given in [24, 25].

**Definition 1:** An operator \( X \) of the form (7) is a partial Hamiltonian operator corresponding to a current value Hamiltonian as in (12), if there exists a function \( B(t, q, p) \) such that

\[ \zeta_i \frac{\partial H}{\partial p_i} + p_i D(\eta^i) - X(H) - HD(\xi) = D(B) + (\eta^i - \xi \frac{\partial H}{\partial p_i})(-\Gamma_i) \]  

holds.

Note that if \( H \) is a present value Hamiltonian, then equation (14) becomes the usual determining equation for symmetries of the Hamiltonian action since \( \Gamma_i = 0 \) in this case.
Also one can immediately see from (14) that if \( X \) and \( Y \) are partial Hamiltonian operators, then so is a linear combination of these Hamiltonian operators.

We now have the following important theorem on how one constructs first integrals for the system (12). That is we present the partial Hamiltonian approach for current value Hamiltonians. This is achieved for several control, state and costate variables.

**Theorem 3** (partial Hamiltonian version pf the partial Noether theorem): An operator \( X \) of the form (7) is a partial Hamiltonian operator of the current value Hamiltonian \( H \) corresponding to system (12) if and only if (11) is its first integral.

**Proof**: The result follows by straightforward differentiation of the first integral formula (11) on the solutions of system (12). However, one has to remember that the terms involving \( \dot{p}_i \) need to be replaced by the right hand side of the second equation of system (12) which has a non-zero function \( \Gamma_i \). Another way to show this to be the case is to utilize

\[
D(H)|_{\dot{q}^i = \frac{\partial}{\partial p_i}, \dot{p}^i = -\frac{\partial}{\partial q^i} + \Gamma_i} = H_t + \Gamma_i \frac{\partial H}{\partial p_i}
\]

as well as the identity

\[
\zeta_i \dot{q}^i + p_i D(\eta^i) - X(H) - HD(\xi) - D(B) - (\eta^i - \xi \frac{\partial H}{\partial p_i})(-\Gamma_i) \\
= \xi(D(H) - H_t - \Gamma_i \frac{\partial H}{\partial p_i}) - \eta^i(\dot{p}^i + \frac{\partial H}{\partial q^i} - \Gamma_i) + \zeta_i(\dot{q}^i - \frac{\partial H}{\partial p_i}) \\
+ D(p_i \eta^i - \xi H - B)
\]

which holds for any smooth functions \( H(t, q, p) \) and suitable functions \( B(t, q, p) \) and \( \Gamma_i \). This identity follows from direct computations.

**Remark**: An approach in proving Theorem 3 is by invoking the Legendre transformation (5) on the partial Noether operators and partial Noether theorem given in [24].

### 3 A Simple illustrative example

Consider the following mathematical example:

Maximize

\[
\int_0^\infty [\alpha q - \beta q^2 - \alpha u^2 - \gamma u] e^{-rt} dt
\]

subject to

\[
\dot{q} = u,
\]

where \( \alpha, \beta, \gamma \) are all positive, \( r \) is a discount factor, \( q(t) \) is the state variable and \( u(t) \) is the control variable.
Hamiltonian function and maximum principle:
The current value Hamiltonian function is defined as
\[ H(t, q, p, u) = \alpha q - \beta q^2 - \alpha u^2 - \gamma u + pu \] (19)
where \( p(t) \) is called the costate variable. The necessary first order conditions for optimal control are [23]:
\[
\begin{align*}
\frac{\partial H}{\partial u} &= 0 \quad (20) \\
\dot{q} &= \frac{\partial H}{\partial p} \quad (21) \\
\dot{p} &= -\frac{\partial H}{\partial q} + rp \quad (22)
\end{align*}
\]
Equation (20)-(22) with \( H \) given by (19) yields
\[
\begin{align*}
p &= 2\alpha u + \gamma \quad (23) \\
\dot{q} &= u \quad (24) \\
\dot{p} &= 2\beta q - \alpha + pr \quad (25)
\end{align*}
\]
Equations (23)-(25) need to be solved for \( p(t), q(t), u(t) \). Of course, the direct way to solve this problem is to eliminate \( p, u \) by utilizing (23)-(25) in order to obtain a scalar linear second order ordinary differential equation in \( q \), which is amenable to straightforward integration. We explain here how we can find the solution by using the partial Hamiltonian approach introduced above.

Determination of Partial Hamiltonian operators:
The partial Hamiltonian operator determining equation is given in (14) Expansion of equation (14) yields
\[
p(\xi_t + \dot{q}\xi_q) - \eta(\alpha - 2\beta q) - (\alpha q - \beta q^2 - \alpha u^2 - \gamma u + pu)(\xi_t + \dot{q}\xi_q) \\
= B_t + \dot{q}B_q + (\eta - \xi u)(-rp), \quad (26)
\]
in which we assume that \( \xi = \xi(t, q), \eta = \eta(t, q), B = B(t, q) \).

Note that one can also assume these functions to be dependent on \( p \). We have chosen \( (t, q) \) dependence to simplify the calculations here and in the models considered in Section 4 with the purpose of deriving solutions. This assumption leads to at least one partial Hamiltonian operator. More general assumptions are used in the event that one does not obtain an operator.

With the help of (23)-(24), Equation (26) can be written as
\[
(2\alpha u + \gamma)(\xi_t + \dot{u}\xi_q) - (\xi_t + u\xi_q)(\alpha q - \beta q^2 - \alpha u^2 - \gamma u + 2\alpha u^2 + \gamma u) \\
- \eta(\alpha - 2\beta q) = B_t + uB_q + (\eta - \xi u)(-2\alpha u - \gamma r), \quad (27)
\]
Separating equation (27) with respect to powers of \( u \) as \( \xi, \eta, B \) do not contain \( u \), we have

\[
\begin{align*}
  u^3 : -\alpha \xi q &= 0, \\
  u^2 : 2\alpha \eta q - \alpha \xi t &= 2\alpha r \xi, \\
  u : 2\alpha \eta q + \gamma \eta q &= B_q - 2\alpha r \eta + \gamma r \xi, \\
  u^0 : \gamma \eta q - \eta(\alpha - 2\beta q) - \xi(\alpha q - \beta q^2) &= B_t - r \gamma \eta.
\end{align*}
\]

System (28)-(30) yields

\[
\xi = a(t), \quad \eta = \left(\frac{1}{2} \ddot{a} + ra\right)q + b(t), \quad B = \alpha \left(\frac{1}{2} \dddot{a} + r\dot{a}\right)q^2 + \alpha r \left(\frac{1}{2} \ddot{a} + ra\right)q + 2a\dot{b}q + 2\alpha rbq + \frac{1}{2} \gamma \dot{a}q + d(t).
\]

Substituting \( \xi, \eta, B \) from (32) in (31) and then separating w.r.t powers of \( q \) we have

\[
\begin{align*}
  q^2 : \frac{1}{2} \alpha \dddot{a} + 3\alpha \alpha \dot{a} + (\alpha r - 2\beta) \dot{a} - 2\alpha r a &= 0, \\
  q : \frac{3}{2}(r\gamma - \alpha) \dot{a} + r(r\gamma - \alpha) a + 2b\beta &= 2a\dddot{b} + 2\alpha \dot{b}, \\
  q^0 : \dot{b} - ab + \gamma rb &= \dot{d}.
\end{align*}
\]

The solution of equations (33)-(35) for \( a, b, d \) with general \( \alpha, \beta, \gamma, r \) is purely formal and depend on the roots of the characteristic equation. Clearly there are three lengthy solutions for \( a \) and two for \( b \). To be transparent, we have selected values. Therefore we seek a solution of equations (33)-(35) for \( a, b, d \) with specific values \( \alpha = \gamma = r = 1 \) and \( \beta = 2 \) and we arrive at

\[
\begin{align*}
  a(t) &= c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-4t}, \\
  b(t) &= c_4 e^t + c_5 e^{-2t}, \\
  d(t) &= c_4 e^t + c_5 e^{-2t} + c_6,
\end{align*}
\]

where \( c_1, \ldots, c_6 \) are arbitrary constants. Finally, we obtain the following \( \xi, \eta, B \) after substituting \( a, b, d \) from (36) into (32)

\[
\begin{align*}
  \xi &= c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-4t}, \\
  \eta &= \left(\frac{1}{2} c_1 e^{-t} + 2c_2 e^{2t} - c_3 e^{-4t}\right)q + c_4 e^t + c_5 e^{-2t}, \\
  B(t) &= (6c_2 e^{2t} + 3c_3 e^{-4t})q^2 + \left(-\frac{1}{2} c_1 e^{-t} + c_2 e^{2t} - 2c_3 e^{-4t} + 4c_4 e^t - 2c_5 e^{-2t}\right)q + c_4 e^t + c_5 e^{-2t} + c_6.
\end{align*}
\]

The generators \( X_i \) form a vector space. By choosing one of the constants as one and the rest as zero in turn we have the following five operators and gauge
terms:

\[ X_1 = e^{-t} \frac{\partial}{\partial t} + \frac{1}{2} q e^{-t} \frac{\partial}{\partial q}; \quad B_1 = -\frac{1}{2} e^{-t} q \]
\[ X_2 = e^{2t} \frac{\partial}{\partial t} + 2q e^{2t} \frac{\partial}{\partial q}; \quad B_2 = 6q^2 e^{2t} + q e^{2t} \]
\[ X_3 = e^{-4t} \frac{\partial}{\partial t} - q e^{-4t} \frac{\partial}{\partial q}; \quad B_3 = 3q^2 e^{-4t} - 2q e^{-4t} \]
\[ X_4 = e^{t} \frac{\partial}{\partial q}; \quad B_4 = 4q e^t + e^t \]
\[ X_5 = e^{-2t} \frac{\partial}{\partial q}; \quad B_5 = -2q e^{-2t} + e^{-2t}. \]

In general the \( X_i \)’s are not symmetries of the system

\[ \dot{q} = \frac{1}{2} p - \frac{1}{2} t, \]
\[ \dot{p} = 4q + p - 1, \]  

which considered as a non-canonical Hamiltonian system (23)-(25) admits the operators (38). For example in the case of \( X_4 \) we have that the first of equations (13) gives \( \zeta = 2e^t \). However, the second equation of (13) is not satisfied as easily can be verified.

**Construction of first integrals from partial Hamiltonian operators and gauge terms:**

Now, first integrals satisfying \( DI = 0 \), on the solutions, corresponding to operators and gauge terms given in (38) can be computed from (11) and the following integrals result.

\[ I_1 = \left[ \frac{1}{2} pq - (q - 2q^2 - u^2 - u + pu) \right] + \frac{1}{2} e^{-t}, \]
\[ I_2 = \left[ 2pq - (q - 2q^2 - u^2 - u + pu) - 6q^2 - q \right] e^{2t}, \]
\[ I_3 = \left[ -pq - (q - 2q^2 - u^2 - u + pu) - 3q^2 + 2q \right] e^{-4t}, \]
\[ I_4 = \left[ p - 4q - 1 \right] e^t, \]
\[ I_5 = \left[ p + 2q - 1 \right] e^{-2t}. \]

There are five first integrals, two of which are functionally independent.

**Optimal solution via first integrals:**

Equations (23)-(25) need to be solved for \( p(t), q(t), u(t) \) with \( \alpha = \gamma = r = 1, \beta = 2 \). We demonstrate here how one can find a solution by using first integrals.

We derive the solution associated with the first integral \( I_4 \). As \( DI = 0 \), on the solutions, and thus \( I = \text{constant} \), we have

\[ [p - 4q - 1] e^t = A_1, \]  

8
where $A_1$ is an arbitrary constant and this gives

$$p(t) = 4q + 1 + A_1 e^{-t}. \quad (42)$$

From (23), $u = \frac{e^{-1}}{2}$ and after using $p$ from (42), we have

$$u(t) = \frac{4q + A_1 e^{-t}}{2}. \quad (43)$$

Thus if $q(t)$ is known we can get the optimal path $p(t)$ and $u(t)$ from (42) and (43). Equation (24) with $u$ from (43) yields

$$\dot{q} = \frac{4q + A_1 e^{-t}}{2}, \quad (44)$$

and this is a first order linear equation in $q(t)$. The solution of equation (44) is

$$q(t) = \frac{A_1}{2} e^{-t} + A_2 e^{2t}, \quad (45)$$

where $A_1$ and $A_2$ are arbitrary constants which we can specify if we have given initial and terminal conditions. One can use any one of the first integrals (40) to obtain the general solution to this linear problem. Generally a first integral provides a reduction of order of the system by one. One can also achieve reduction to quadratures of the system by invoking first integrals as we see in the examples that follow.

## 4 Optimal path of some economic models

### 4.1 Ramsey neoclassical model with CRRA utility function

We consider the following Ramsey neoclassical growth model [8], [26], where the representative consumer’s utility maximization problem is defined as

$$Max \int_0^\infty e^{-rt} c^{1-\sigma} dt, \sigma \neq 0, 1 \quad (46)$$

subject to the capital accumulation equation and parameter restriction

$$\dot{k}(t) = k^\beta - \delta k - c, \quad k(0) = k_0, \quad 0 < \beta < 1, \quad (47)$$

where $c(t)$ is the consumption per person, $k(t)$ is the capital labor ratio, $\beta, \delta, r$ are the capital share, depreciation rate, rate of time preferences respectively. The intertemporal elasticity of substitution is given by $1/\sigma$ and $k_0$ is the initial capital stock.

The current value Hamiltonian function for this model is defined as

$$H(t, c, k, \lambda) = c^{1-\sigma} + \lambda(k^\beta - \delta k - c), \quad (48)$$
where \( \lambda(t) \) is the costate variable. The necessary first order conditions for optimal control are

\[
\lambda = (1 - \sigma)c^{-\sigma} \quad (49)
\]

\[
\dot{k} = k^\beta - \delta k - c \quad (50)
\]

\[
\dot{\lambda} = -\lambda(\beta k^{\beta - 1} - \delta) + \lambda r \quad (51)
\]

and the transversality condition is

\[
\lim_{t \to \infty} e^{-rt}\lambda(t)k(t) = 0. \quad (52)
\]

From (49) and (51), the growth rate of consumption is given by

\[
\ddot{c} = \frac{\beta}{\sigma}k^{\beta-1} - \frac{1}{\sigma}(\delta + r). \quad (53)
\]

We seek a solution \( \lambda(t), k(t), c(t) \) of equations (49)-(51) by utilizing the Hamiltonian approach. The partial Hamiltonian determining equation (14) for the Hamiltonian (48) yields

\[
\lambda(\eta_t + \dot{k}\eta_k) - \eta\lambda(\beta k^{\beta-1} - \delta) - [c^{1-\sigma} + \lambda(k^\beta - \delta k - c)](\xi_t + \dot{k}\xi_k)
\]

\[
= B_t + \dot{k}B_k + (\eta - \xi \frac{\partial H}{\partial \lambda})(-r\lambda), \quad (54)
\]

in which we assume that \( \xi = \xi(t, k), \eta = \eta(t, k), B = B(t, k) \). The same reason applies here as for the illustrative example. Equation (54) with the help of (49)-(51) can be written as

\[
(1 - \sigma)c^{-\sigma}[\eta_t + (k^\beta - \delta k - c)\eta_k] - \eta(1 - \sigma)c^{-\sigma}(\beta k^{\beta - 1} - \delta)
\]

\[
- [c^{1-\sigma} + (1 - \sigma)c^{-\sigma}(k^\beta - \delta k - c)](\xi_t + (k^\beta - \delta k - c)\xi_k)
\]

\[
= B_t + (k^\beta - \delta k - c)B_k - r(1 - \sigma)c^{-\sigma}[\eta - \xi(k^\beta - \delta k - c)]. \quad (55)
\]

Separating equation (55) with respect to powers of the control variable \( c \), we have

\[
c^{2-\sigma} : -\sigma\xi_k = 0, \quad (56)
\]

\[
c^{1-\sigma} : -\eta(1 - \sigma) - \sigma\xi_t + r(1 - \sigma)\xi = 0, \quad (57)
\]

\[
c^{-\sigma} : \eta_t + (k^\beta - \delta k)\eta_k - \eta(\beta k^{\beta - 1} - \delta)
\]

\[
- (k^\beta - \delta k)\xi_t + r\eta - r\xi(k^\beta - \delta k) = 0, \quad (58)
\]

\[
c, c^0 : B_k = 0, B_t = 0. \quad (59)
\]

Equations (56), (57) and (59) result in

\[
\xi = a_1(t), \quad \eta = (\frac{\sigma}{1 - \sigma}\dot{a}_1 + ra_1)k + a_2(t), \quad B = 0. \quad (60)
\]
Equation (58) with $\xi, \eta, B$ from (60) gives $a_2 = 0$ and then reduces to
\[
k^\beta : \dot{a}_1 - \frac{\beta r(1 - \sigma)}{\beta \sigma - 1} a_1 = 0, \quad \beta \sigma \neq 1, \quad (61)
k : -\sigma \dot{a}_1 + [r(1 - 2\sigma) + \delta(1 - \sigma)]a_1 + r(1 - \sigma)(r + \delta) a_1 = 0. \quad (62)
\]
Equation (61) is valid if $\sigma \beta \neq 1$, for the case where the capital’s share is not equal to the intertemporal elasticity of substitution. Equations (61) and (62) yield
\[
a_1(t) = c_1 e^{\delta \beta (1 - \sigma) t} \quad (63)
\]
with
\[
\sigma = \frac{r + \delta}{\beta \delta} \quad (64)
\]
The restriction on the parameters (64) is the same as the one given in [8, 26] and our approach yields this during the solution process. Now $\xi, \eta$ and $B$ are given by
\[
\xi = c_1 e^{\delta \beta (1 - \sigma) t}, \quad \eta = -c_1 \delta e^{\delta \beta (1 - \sigma) t} k, \quad B = 0, \quad (65)
\]
and the only partial Hamiltonian operator is
\[
X = e^{\delta \beta (1 - \sigma) t} \frac{\partial}{\partial t} - \delta e^{\delta \beta (1 - \sigma) t} k \frac{\partial}{\partial k}, \quad B = 0. \quad (66)
\]
The following first integral corresponding to the partial Hamiltonian operator and gauge terms given in (66) can be computed from (11):
\[
I = e^{\delta \beta (1 - \sigma) t} \left[-\sigma c^{1 - \sigma} + (\sigma - 1) e^{-\sigma k^\beta}\right]. \quad (67)
\]
We write (67) as a constant, i.e.
\[
-\sigma c^{1 - \sigma} + (\sigma - 1) e^{-\sigma k^\beta} = A_1 e^{\delta \beta (\sigma - 1) t}. \quad (68)
\]
From equation (68), we have
\[
k = \left[\frac{A_1}{\sigma - 1} e^{\sigma e^{\delta \beta (\sigma - 1) t}} + \frac{\sigma}{\sigma - 1} c\right]^{\frac{1}{\beta}}. \quad (69)
\]
Our next goal is to get either $c$ or $k$. If $A_1 = 0$ we arrive at the well-known solution given in [8, 26]. Equation (69) for $A_1 = 0$ yields
\[
c(t) = (1 - \frac{\beta \delta}{r + \delta}) k^\beta \quad (70)
\]
where $\frac{\sigma - 1}{\sigma} = 1 - \frac{\beta \delta}{r + \delta}$ by (64). Substituting $c$ from equation (70) in Equation (50) results in
\[
\dot{k} + \delta k = \left(\frac{\beta \delta}{r + \delta}\right) k^\beta. \quad (71)
\]
The solution of equation (71) subject to the initial condition $k(0) = k_0$ is given by
\[
k(t) = \left[\frac{\beta}{r + \delta} + (k_0^{1 - \beta} - \frac{\beta}{r + \delta}) e^{-(1 - \beta) \delta t}\right]^{\frac{1}{1 - \beta}}. \quad (72)
\]
The solutions (70) and (72) are the same as the ones derived in [8, 26] and satisfy the transversality condition given by (52). This guarantees that our approach works. For $A_1 \neq 0$, we can get more solutions. Now we substitute (69) into equation (53) determining $c$, viz.

$$\frac{d}{dt}(ce^{\beta t}) = \frac{\beta}{\sigma}ce^{\beta t} - \frac{A_1}{\sigma - 1}c^\sigma e^{\beta(\sigma - 1)t} + \frac{\sigma}{\sigma - 1}c^{1-\frac{1}{\sigma}}. \tag{73}$$

Introducing

$$S = ce^{\beta t}, \tag{74}$$

equation (73) directly results in

$$\frac{\beta}{\sigma}e^{\delta(1-\beta)t}dt = \frac{dS}{S\left(\frac{A_1}{\sigma - 1}S^\sigma + \frac{\sigma}{\sigma - 1}S^{1-\frac{1}{\sigma}}\right)}, \tag{75}$$

which provides the general solution. Here, one operator and thus one first integral was sufficient to work out the solution. In general, one requires two, as we have a system of two first order equations, which we wish to solve.

### 4.2 One-Sector Model of Endogenous Growth: The AK Model

We consider the following one-sector model of endogenous growth presented in [8] where the representative consumer’s utility maximization problem is

$$\text{Max} \int_0^\infty e^{-(\rho-n)t}c^{1-\theta} - \frac{1}{1-\theta}dt, \theta > 0, \theta \neq 1 \tag{76}$$

subject to

$$\dot{a}(t) = (r - n)a + w - c, \quad c(0) = c_0, \tag{77}$$

where $c(t)$ is the consumption per person, $a(t)$ is the assets per person, $r(t)$ is the interest rate, $w(t)$ is the wage rate, and $n$ is the growth rate of population.

Suppose firms have the linear production function

$$y = f(k) = Ak \tag{78}$$

where $A > 0$. The marginal product of capital is not diminishing, i.e. $f'' = 0$ and this property makes it different from neoclassical production function. The marginal product of capital is the constant $A$ and the marginal product of labor is zero. Thus

$$r = A - \delta, \quad w = 0 \tag{79}$$

where $\delta \geq 0$ is the depreciation rate. It is assumed that the economy is closed and $a(t) = k(t)$ holds. If we take $a = k$, $r = A - \delta$ and $w = 0$ then our optimal control problem is to maximize (76) subject to

$$\dot{k} = (A - \delta - n)k - c, \quad c(0) = c_0. \tag{80}$$
The current value Hamiltonian function is defined as

\[ H(t, c, A, \lambda) = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda[(A - \delta - n)k - c], \quad (81) \]

where \( c(t) \) is control variable, \( k(t) \) is the state variable and \( \lambda(t) \) is the costate variable. The necessary first order conditions for optimal control are

\[ \lambda = c^{-\theta} \quad (82) \]
\[ \dot{k} = (A - \delta - n)k - c \quad (83) \]
\[ \dot{\lambda} + (A - \delta - \rho)\lambda = 0. \quad (84) \]

The transversality condition is

\[ \lim_{t \to \infty} e^{-(\rho-n)t}\lambda(t)k(t) = 0 \quad (85) \]

and from (82) and (84), the growth rate of consumption is given by

\[ \frac{\dot{c}}{c} = \frac{1}{\theta}(A - \delta - \rho). \quad (86) \]

Now we solve this model by utilizing our partial Hamiltonian approach. The partial Hamiltonian operator determining equation (14) for Hamiltonian (81) with \( \xi = \xi(t, k), \eta = \eta(t, k), B = B(t, k) \) can be written as

\[ c^{-\theta}[\eta_t + ((A - \delta - n)k - c)\eta_k] - \eta c^{-\theta}(A - \delta - n) \]
\[ -\left[\frac{c^{1-\theta} - 1}{1-\theta}\right] + c^{-\theta}((A - \delta - n)k - c)[\xi_t + ((A - \delta - n)k - c)\xi_k] \]
\[ = B_t + B_k((A - \delta - n)k - c) - c^{-\theta}(\rho - n)[\eta - \xi((A - \delta - n)k - c)], \quad (87) \]

where we have used equations (82)-(84). By following the same procedure for equation (87) as described in the previous two examples, we finally have

\[ \xi = a_1(t), \eta = \left[\frac{\theta}{1-\theta}\right]a_1 + (\rho - n)a_1 + a_2(t), B = \frac{1}{1-\theta}a_1(t), \quad (88) \]
\[ -\left[\frac{\theta}{1-\theta}\right]\ddot{a}_1 + (\rho - A + \delta - \frac{(\rho - n)\theta}{1-\theta})\dot{a}_1 + (\rho - n)(\rho - A + \delta)a_1 = 0, \quad (89) \]
\[ \dot{a}_2 + (\rho - A + \delta)a_2 = 0. \quad (90) \]

Solving equation (89) for \( a_1(t) \), we have

\[ a_1(t) = c_1e^{-(\rho - n)t} + c_2e^{\frac{(\rho - A + \delta)(1-\theta)}{\theta}t}, \quad (91) \]

and equation (90) yields

\[ a_2(t) = c_3e^{(A - \delta - \rho)t}. \quad (92) \]
Thus $\xi, \eta, B$ are given by

$$\begin{align*}
\xi &= c_1 e^{-(\rho-n)t} + c_2 e^{\left(\frac{A-\delta}{\theta}(1-\rho)\right)t}, \\
\eta &= \frac{\rho - n}{1-\theta} c_1 k_1 e^{-(\rho-n)t} - (n - A + \delta)c_2 e^{\left(\frac{A-\delta}{\theta}(1-\rho)\right)t} + c_3 e^{(A-\delta-\rho)t}, \\
B &= \frac{1}{1-\theta}[c_1 e^{-(\rho-n)t} + c_2 e^{\left(\frac{A-\delta}{\theta}(1-\rho)\right)t}],
\end{align*}$$

(93)

The following first integrals corresponding to operators and gauge terms given in (93) are computed from (11):

$$\begin{align*}
I_1 &= e^{-(\rho-n)t} c^{-\theta} k_1 \left[\frac{\rho - n \theta + (\theta - 1)(A - \delta)}{1-\theta}\right] - \frac{\theta}{1-\theta} e^{\theta} e^{-(\rho-n)t}, \\
I_2 &= \frac{\theta}{\theta - 1} c^{-\theta} e^{\left(\frac{A-\delta}{\theta}(1-\rho)\right)t}, \\
I_3 &= c^{-\theta} e^{-(\rho-A+\delta)t}.
\end{align*}$$

(94)

Now we explain how to solve the AK model using the first integrals $I_1, I_3$. Setting $I_1 = a_1$ and $I_3 = a_2$ after some simplifications and using the initial condition $c(0) = c_0$, we have

$$k(t) = \frac{1 - \theta}{\phi \theta} a_1 e^{(\rho-n)t} c^{-\theta}(t) + \frac{1}{\phi} c(t),$$

(95)

where

$$\phi = \frac{1}{\theta} \left[\rho - n \theta + (\theta - 1)(A - \delta)\right]$$

and

$$c(t) = c_0 e^{\left(\frac{A-\delta}{\theta}(1-\rho)\right)t}, c_0 = a_2^{-\frac{1}{\phi}}.$$  

(96)

(97)

Note that $(A - \delta - \rho) > 0$ the consumption $c(t)$ given in (97) increases with time. Substituting the value of consumption $c(t)$ in (97) into (95), capital is given by

$$k(t) = \frac{1 - \theta}{\phi \theta} a_1 c_0 e^{(A-\delta-n)t} + \frac{1}{\phi} c_0 e^{\left(\frac{A-\delta-n}{\theta}\right)t},$$

(98)

where the constant $a_1$ can be determined from the transversality condition. The solutions (97) and (98) are the same as given in [8] and here we deduced these by utilizing our partial Hamiltonian approach. The transversality condition can be rewritten as

$$\lim_{t \to \infty} e^{-(\rho-n)t} c^{-\theta} k = 0$$

(99)

and we need to show $\lim_{t \to \infty} e^{-(\rho-n)t} c^{-\theta} k$ is zero. Using (97) and (98) we have

$$\lim_{t \to \infty} \frac{1 - \theta}{\phi \theta} a_1 + \frac{1}{\phi} c_0 e^{-\phi t}$$

(100)

and it tends to zero only if we choose constant $a_1 = 0$ and assume that $\phi = \frac{1}{\theta} \left[\rho - n \theta + (\theta - 1)(A - \delta)\right] > 0$. This further results in the following restriction on the parameters

$$\rho + \delta > (1-\theta)(A-\delta) + n\theta + \delta$$

(101)
and (95) reduces to

\[ k(t) = \frac{1}{\phi}c(t). \]  

(102)

The detailed interpretations of the solutions obtained here are given in [8].

5 Concluding remarks

A systematic way to obtain reductions and closed-form solutions via first integrals of Hamiltonian systems commonly arising in economic growth theory and other economic models is developed. This is an algorithmic approach and can be applied to many state and costate variables of the current value Hamiltonian. However, we applied our method to systems with one control, one state and one costate variable. The approach was explained with the help of one simple illustrative example. We first studied two economic growth models, the Ramsey model with a constant relative risk aversion (CRRA) utility function and Cobb Douglas technology and the one-sector AK model of endogenous growth. For the Ramsey model, the solutions derived from our methodology were the same as those derived by [8, 26]. The restriction on the parameters was obtained in a systematic way during the solution process unlike in other models where it was assumed. The solutions were valid for \( \sigma \beta \neq 1 \), the case where the capital’s share is not equal to the intertemporal elasticity of substitution. The first integrals and closed-form solutions for the one-sector AK model of endogenous growth were also re-derived by our partial Hamiltonian approach.

We have shown that our systematic approach can be used to deduce results given in the literature and we also found new solutions for a variety of models.

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