The phase space structure of the oligopoly dynamical system by means of Darboux integrability

Adam Krawiec\textsuperscript{a,e}, Tomasz Stachowiak\textsuperscript{b,c}, Marek Szydlowski\textsuperscript{d,e}

\textsuperscript{a}Institute of Economics, Finance and Management, Jagiellonian University, Lojasiewicza 4, 30-348 Kraków, Poland
\textsuperscript{b}Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, Warszawa, Poland
\textsuperscript{c}Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, 606-8501 Kyoto, Japan
\textsuperscript{d}Astronomical Observatory, Jagiellonian University, Orla 171, 30-244 Kraków, Poland
\textsuperscript{e}Mark Kac Complex Systems Research Centre, Jagiellonian University, Kraków, Poland

Abstract

We investigate the dynamical complexity of Cournot oligopoly dynamics of three firms by using the qualitative methods of dynamical systems to study the phase structure of this model. The phase space is organized with one-dimensional and two-dimensional invariant submanifolds (for the monopoly and duopoly) and unique stable node (global attractor) in the positive quadrant of the phase space (Cournot equilibrium). We also study the integrability of the system. We demonstrate the effectiveness of the method of the Darboux polynomials in searching for first integrals of the oligopoly. The general method as well as examples of adopting this method are presented. We study Darboux non-integrability of the oligopoly for linear demand functions and find first integrals of this system for special classes of the system, in particular, rational integrals can be found for a quite general set of model parameters. We show how first integral can be useful in lowering the dimension of the system using the example of $n$ almost identical firms. This first integral also gives information about the structure of the phase space and the behaviour of trajectories in the neighbourhood of a Nash equilibrium.

1. Introduction

In the economic study of imperfect markets a special place is held by the oligopoly. It is a market structure where a substantial market share is held by a small number of firms. The behaviour of firms in the market, studied first by Cournot [1], consists in the firms producing a homogeneous product and fixing
the price taking into account that any change of price by a rival firm influences its profit. There is an equilibrium in which these firms maximize profits (it is a Nash equilibrium). The special cases of two and three firms are called a duopoly and triopoly, respectively.

From the economic point of view the study of oligopoly dynamics is important as we want to know a mechanism governing the market dynamics. The existence of equilibrium of the market, its genericity, its stability and the dynamical behaviour in the neighbourhood of the equilibrium are crucial for understanding the formation and evolution of markets. In modern economics, the studies of imperfect markets, especially the oligopoly model, is important as it is the prevailing market structure. Accordingly, the oligopoly has been a subject of intensive studies [2]. Many researchers have developed different variants of the classical Cournot model. These investigations pursued both discrete and continuous time scales using the methods of dynamical systems. These studies concentrated on the asymptotic stability. The summary of earlier results can be found in Okuguchi [3] while Bischi et al. presented comprehensive results for dynamics of discrete oligopoly systems [4].

In our analysis of the oligopoly, we consider two main problems. First, we study the structure of the phase space, by modelling the oligopoly as an autonomous $n$ dimensional system of ordinary differential equations. The motivation for such investigation is looking for dynamical complexity of the trajectories’ behaviour in the phase space. Exploration of the phase structure seems to be interesting because our knowledge about trajectories describes the possible (long-term) behaviour of the firms and equilibria which they reach.

The second is the problem of the existence of first integrals of a differential equation systems. It has been studied in many dynamical systems of physical and biological origin. However, in economic theory the problem of finding first integrals of dynamical systems is almost absent. Among the few examples there are the production/inventory model [5] and the multiplier-accelerator model of business cycle [6]. In this paper we consider this problem for the oligopoly model.

To this aim, we propose to apply methods of the Darboux polynomials in finding first and the so called “second” integrals. We then search for rational and algebraic, time-dependent first integrals by means of combinations of the Darboux polynomials. Given enough such quantities, we can also obtain time-independent first integrals of the original oligopoly system. The knowledge of time-independent and functionally independent first integral yields important information about phase space structure of any dynamical model.

If the system is two dimensional (duopoly) then the existence of one first integral means that the system is completely integrable and thus phase space structure is completely characterized. If the first integral does not depend qualitatively on the special value of the model’s parameters, then its phase portrait is completely determined on the phase plane. In this case one can easily study periodic orbits, limit cycles etc.

The advantages of dynamical system methods is that they gives us the possibility of geometrization of the dynamical behaviour and thus visualisation
of dynamics. The evolutional paths of a model are represented by trajectories of a system. The phase space contains all trajectories for all admissible initial conditions. Thus, the global information on dynamics is obtained. In the phase space, trajectories can behave in a regular way or exhibit complex behaviour [8]. The phase space has a structure organised by fixed points, periodic orbits, invariant submanifolds, etc. The main aim of this paper is reveal this structure in the context of oligopoly dynamics. To this aim, we construct 3-dimensional phase portraits representative for the problem and discuss in details integrability of the oligopoly systems. In the discussion of problem of integrability we pay attention to Darboux integrability. The analysis of integrability gives us information about existence or not-existence of first integral. For visualization we construct the level sets of the first integral.

The problem of complex oligopoly dynamics was investigated in [9] for discrete version of oligopoly. In the paper we consider smooth dynamical systems with continuous time. The existence of first integrals gives us information on non-existence of complex chaotic behaviour. Although it is not possible to formulate such conclusions in full generality, it holds for sufficiently different firms. The generic nature of the critical points points towards preservation of the regular phase space structure in the presence of small perturbations. When they exist.

For 3-dimensional dynamical systems the information that there is first integral means that solutions are restricted to the level sets of such a function. If an n-dimensional system admits a time independent first integral (as for example in the case of oligopoly dynamics of n almost identical firms), then the first integral can be used to lower the dimension of the original system by one (of course if we can effectively solve the conservation law for one of the variables, so that it can be excludet from the system). We will illustrate how for 3 almost identical firms the first integral can be used to lower the dimension of the system to dimension two.

2. The oligopoly model

2.1. General model

Let us consider the oligopoly market with three firms. These three players produce a homogeneous product of the total supply \( Q(t) = q_1(t) + q_2(t) + q_3(t) \). The price of the good is \( p \) and the demand is given by a linear inverse demand function

\[
p(Q(t)) = a - bQ(t)
\]

where \( a \) and \( b \) are positive constants. The former is the highest market price of the good. We assume that the firms’ cost function has the quadratic form

\[
C_i(q_i) = c_i + d_i q_i(t) + e_i q_i^2(t), \quad i = 1, 2, 3,
\]

where \( c_i \) is the positive fixed cost of firm \( i \), and \( d_i, e_i \) are constants. As the marginal cost of the firm must be less than the highest market price we have the condition \( d_i + 2e_i q_i < a, \quad i = 1, 2, 3 \).
Additionally, we assume that the speed of adjustment \( \alpha \) where

\[
\text{firms so without loss of generality we take } \alpha = 2.
\]

2.2. Identical firms with linear cost function

Let the cost function be linear and identical for all three firms

\[
C(q_i) = c + dq_i, \quad c > 0, d > 0
\]

Additionally, we assume that the speed of adjustment \( \alpha \) is the same for all three firms so without loss of generality we take \( \alpha = 1 \), then the dynamical system has the form

\[
\dot{q}_1(t) = q_1(t)[a - 2bq_1(t) - bq_2(t) - bq_3(t)], \quad \dot{q}_2(t) = q_2(t)[a - bq_1(t) - 2bq_2(t) - bq_3(t)], \quad \dot{q}_3(t) = q_3(t)[a - bq_1(t) - bq_2(t) - 2bq_3(t)],
\]

The profit of the \( i \)-th firm is

\[
\Pi_i(q_1(t), q_2(t), q_3(t)) = q_i(t)(a - bQ(t)) - (c + dq_i(t) + e_iq_i^2(t)), \quad i = 1, 2, 3
\]

and its marginal profit is

\[
\frac{\partial \Pi_i(q_1, q_2, q_3)}{\partial q_i} = a - bQ(t) - bq_i(t) - d_i - 2e_iq_i(t), \quad i = 1, 2, 3.
\]

We assume that all the firms have imperfect knowledge of the market. Therefore, they follow a bounded rationality adjustment process based on a local estimate of the marginal profit \( \partial \Pi_i/\partial q_i \) \([7, 10]\). It means that the firm increases its production as long as the marginal profit is positive. When the marginal profit is negative the firm reduces its production. This adjustment mechanism has the form

\[
\frac{dq_i(t)}{dt} = a_iq_i(t)[\frac{\partial \Pi_i(q_1(t), q_2(t), q_3(t))}{\partial q_i}], \quad i = 1, 2, 3,
\]

where \( a_i \) is a positive speed of adjustment. The three-firm oligopoly is given by the system of differential equations

\[
\begin{align*}
\dot{q}_1(t) &= a_1q_1(t)[a - bQ(t) - bq_1(t) - d_1 - 2e_1q_1(t)], \\
\dot{q}_2(t) &= a_2q_2(t)[a - bQ(t) - bq_2(t) - d_2 - 2e_2q_2(t)], \\
\dot{q}_3(t) &= a_3q_3(t)[a - bQ(t) - bq_3(t) - d_3 - 2e_3q_3(t)],
\end{align*}
\]

or

\[
\begin{align*}
\dot{q}_1(t) &= a_1q_1(t)[a_1 - 2bq_1(t) - bq_2(t) - bq_3(t)], \\
\dot{q}_2(t) &= a_2q_2(t)[a_2 - bq_1(t) - 2bq_2(t) - bq_3(t)], \\
\dot{q}_3(t) &= a_3q_3(t)[a_3 - bq_1(t) - bq_2(t) - 2bq_3(t)],
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= a - d_1, & a_2 &= a - d_2, & a_3 &= a - d_3, \\
b_1 &= b + e_1, & b_2 &= b + e_2, & b_3 &= b + e_3.
\end{align*}
\]

The variables \( q_i, i = 1, 2, 3 \) must be non-negative to be economically significant. Below we list more explicitly some simple sub-cases of interest.
Additionally, we assume that the speed of adjustment \( \alpha \) is the same for all three firms so without loss of generality we take \( \alpha = 1 \), then the dynamical system has the form

\[
\begin{align*}
\dot{q}_1(t) &= q_1(t)[\bar{a} - 2bq_1(t) - bq_2(t) - bq_3(t)], \\
\dot{q}_2(t) &= q_2(t)[\bar{a} - bq_1(t) - 2bq_2(t) - bq_3(t)], \\
\dot{q}_3(t) &= q_3(t)[\bar{a} - bq_1(t) - bq_2(t) - 2bq_3(t)],
\end{align*}
\]

where

\[
\bar{a} = a - d > 0, \quad a > 0, \quad d > 0 \quad \text{and} \quad b > 0.
\]

### 2.3. Identical firms with quadratic cost function

Now the cost function is quadratic

\[
C(q_i) = c + dq_i + e_i q_i^2, \quad c > 0, d > 0, e > 0. \tag{11}
\]

Additionally, we assume that the speed of adjustment \( \alpha \) is the same for all three firms and without loss of generality we take \( \alpha = 1 \), then the dynamical system has the form

\[
\begin{align*}
\dot{q}_1(t) &= q_1(t)[\bar{a} - 2bq_1(t) - bq_2(t) - bq_3(t)], \\
\dot{q}_2(t) &= q_2(t)[\bar{a} - bq_1(t) - 2bq_2(t) - bq_3(t)], \\
\dot{q}_3(t) &= q_3(t)[\bar{a} - bq_1(t) - bq_2(t) - 2bq_3(t)],
\end{align*}
\]

where

\[
\bar{a} = a - d > 0 \quad \text{and} \quad \bar{b} = b + e > 0.
\]

### 2.4. Different firms with quadratic cost function

Now the cost function is quadratic and different for each firm

\[
C_i(q_i) = c_i + dq_i + e_i q_i^2, \quad c_i > 0, d_i > 0, e_i > 0 \tag{13}
\]

Additionally, we assume that the speed of adjustment \( \alpha \) is the same for all three firms and without loss of generality we take \( \alpha = 1 \), then the dynamical system has the form

\[
\begin{align*}
\dot{q}_1(t) &= q_1(t)[a_1 - 2b_1 q_1(t) - b_2 q_2(t) - b_3 q_3(t)], \\
\dot{q}_2(t) &= q_2(t)[a_2 - b_1 q_1(t) - 2b_2 q_2(t) - b_3 q_3(t)], \\
\dot{q}_3(t) &= q_3(t)[a_3 - b_1 q_1(t) - b_2 q_2(t) - 2b_3 q_3(t)],
\end{align*}
\]

where

\[
a_i = a - d_i > 0 \quad \text{and} \quad b_i = b + e_i > 0, \\
a > 0, \quad d_i > 0, \quad b > 0, \quad e_i > 0.
\]

### 3. Analysis of the model’s dynamics

#### 3.1. General model

In this section we present the general analysis of dynamical system (7)

\[
\begin{align*}
\dot{q}_1(t) &= \alpha_1 q_1(t)[a_1 - 2b_1 q_1(t) - b_2 q_2(t) - b_3 q_3(t)] = h^1(q_1, q_2, q_3), \\
\dot{q}_2(t) &= \alpha_2 q_2(t)[a_2 - b_1 q_1(t) - 2b_2 q_2(t) - b_3 q_3(t)] = h^2(q_1, q_2, q_3), \\
\dot{q}_3(t) &= \alpha_3 q_3(t)[a_3 - b_1 q_1(t) - b_2 q_2(t) - 2b_3 q_3(t)] = h^3(q_1, q_2, q_3), \tag{15}
\end{align*}
\]
where
\[
\begin{align*}
    a_1 &= a - d_1, & a_2 &= a - d_2, & a_3 &= a - d_3, \\
    b_1 &= b + e_1, & b_2 &= b + e_2, & b_3 &= b + e_3.
\end{align*}
\] (16)

To find a critical points of system (7) we solve the system \( h'(q^*) = 0, \) \( i = 1, 2, 3, \) for \((q_1^*, q_2^*, q_3^*)\). Here, we obtain multiple solutions:

\[
\begin{align*}
    E_1 &= (0, 0, 0) \quad & (17) \\
    E_2 &= \left( \frac{a_1}{2b_1}, 0, 0 \right) \quad & (18) \\
    E_3 &= \left( 0, \frac{a_2}{2b_2}, 0 \right) \quad & (19) \\
    E_4 &= \left( 0, 0, \frac{a_3}{2b_3} \right) \quad & (20) \\
    E_5 &= \left( \frac{2a_1b_2 - ba_2}{4b_1b_2 - b^2}, \frac{2a_2b_1 - ba_1}{4b_1b_2 - b^2}, 0 \right) \quad & (21) \\
    E_6 &= \left( 0, \frac{2a_2b_3 - ba_3}{4b_2b_3 - b^2}, \frac{2a_3b_2 - ba_2}{4b_2b_3 - b^2} \right) \quad & (22) \\
    E_7 &= \left( \frac{2a_1b_3 - ba_3}{4b_1b_3 - b^2}, 0, \frac{2a_3b_1 - ba_1}{4b_1b_3 - b^2} \right) \quad & (23) \\
    E_8 &= \left( \frac{4a_1b_2b_3 + \left(-a_1 + a_2 + a_3\right)b^2 - 2(a_2b_3 + a_3b_2)b}{8b_1b_2b_3 + 2b^3 - 2(b_1 + b_2 + b_3)b^2}, \right. \\
    & \quad \left. \frac{4a_2b_1b_3 + \left(a_1 - a_2 + a_3\right)b^2 - 2(a_1b_3 + a_3b_1)b}{8b_1b_2b_3 + 2b^3 - 2(b_1 + b_2 + b_3)b^2}, \right. \\
    & \quad \left. \frac{4a_3b_1b_2 + \left(a_1 + a_2 - a_3\right)b^2 - 2(a_1b_2 + a_2b_1)b}{8b_1b_2b_3 + 2b^3 - 2(b_1 + b_2 + b_3)b^2} \right) \quad & (24)
\end{align*}
\]

All these are points in the phase space which represent stationary states of the system. The critical point \( E_1 \) is trivial without any supply in the market. The critical points \( E_2, E_3 \) and \( E_4 \) correspond to a monopoly market with firm “1”, firm “2” or firm “3”, respectively. The critical points \( E_5, E_6 \) and \( E_7 \) correspond to a duopoly market with firms “1” and “2”, firms “1” and “3”, or firms “2” and “3”, respectively. The last critical point \( E_8 \) corresponds to a three firms oligopoly.

In the critical point \( E_8 \) all three firms choose their supply to be optimal in such a way that their profit is maximized. It corresponds to the Cournot equilibrium for three firms, and is also a Nash equilibrium as no firm wants to change (increase or decrease) its supply as this would lead to a decrease in its profit.

There also exists a generic correspondence between the critical points and sets invariant with respect to the flow. The point \( E_8 \) usually lies outside invariant submanifolds, the monopolies \( E_2, E_3, E_4 \) lie on one-dimensional submanifolds, the duopolies \( E_5, E_6, E_7 \) lie on two dimensional submanifolds, and \( E_1 \) could be
said to lie on a zero-dimensional one, but this is, strictly speaking, true for any critical point.

From (17) we have that in the Cournot equilibrium the total supply is

\[ Q^* = q_1^* + q_2^* + q_3^* \]  

(25)

and the price is

\[ P^* = a - b(q_1^* + q_2^* + q_3^*). \]  

(26)

Let us try local stability analysis of the critical points represented the Cournot equilibrium \((E_3)\). Following the Hartman-Grobman theorem \([11]\) the dynamics in the neighbourhood of this non-degenerate critical point is approximated through linear part of the system. The linearization at \(q^*\) is

\[
\frac{d}{dt}\begin{bmatrix} q_1 - q_1^* \\ q_2 - q_2^* \\ q_3 - q_3^* \end{bmatrix} = M \begin{bmatrix} q_1 - q_1^* \\ q_2 - q_2^* \\ q_3 - q_3^* \end{bmatrix}
\]  

(27)

where \(M\) is the Jacobian calculated at \(q^*\)

\[
M = \begin{bmatrix}
\frac{\partial h_1}{\partial q_1}|_{q=q^*} & \frac{\partial h_1}{\partial q_2}|_{q=q^*} & \frac{\partial h_1}{\partial q_3}|_{q=q^*} \\
\frac{\partial h_2}{\partial q_1}|_{q=q^*} & \frac{\partial h_2}{\partial q_2}|_{q=q^*} & \frac{\partial h_2}{\partial q_3}|_{q=q^*} \\
\frac{\partial h_3}{\partial q_1}|_{q=q^*} & \frac{\partial h_3}{\partial q_2}|_{q=q^*} & \frac{\partial h_3}{\partial q_3}|_{q=q^*}
\end{bmatrix}
\]  

(28)

\[
= \begin{bmatrix}
-2\alpha_1 b_1 q_1^* + \alpha_1 g^1 & -\alpha_1 b q_1^* & -\alpha_1 b q_1^* \\
-\alpha_2 b q_2^* & -2\alpha_2 b_2 q_2^* + \alpha_2 g^2 & -\alpha_2 b q_2^* \\
-\alpha_3 b q_3^* & -\alpha_3 b q_3^* & -2\alpha_3 b_3 q_3^* + \alpha_3 g^3
\end{bmatrix}
\]  

(29)

where

\[ g^1 = g^1(q_1^*, q_2^*, q_3^*) = a_1 - 2b_1 q_1^* - b q_2^* - b q_3^* \]
\[ g^2 = g^2(q_1^*, q_2^*, q_3^*) = a_1 - b q_1^* - 2b_2 q_2^* - b q_3^* \]
\[ g^3 = g^3(q_1^*, q_2^*, q_3^*) = a_1 - b q_1^* - b q_2^* - 2b_3 q_3^* \]

3.2. **Local stability analysis of the general model**

The stability of the critical point as well as its character depends on the eigenvalues of the linearization matrix which are the solutions of the characteristic equation

\[
\det[M - \lambda I] = \lambda^3 - \text{tr}(M)\lambda^2 + \left(\text{tr}(M)^2 - \text{tr}(M^2)\right)\lambda \\
+ \left(\text{tr}(M)^3 + 2\text{tr}(M)\right) - 3\text{tr}(M)\text{tr}(M^2) \\
= m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 = 0.
\]  

(30)

Let us consider critical point \(E_3\). From the economic point of view this point is the equilibrium for oligopoly of three firms. The Jacobi matrix calculated at this critical point is

\[
M = \begin{bmatrix}
-2\alpha_1 b_1 q_1^* & -\alpha_1 b q_1^* & -\alpha_1 b q_1^* \\
-\alpha_2 b q_2^* & -2\alpha_2 b_2 q_2^* & -\alpha_2 b q_2^* \\
-\alpha_3 b q_3^* & -\alpha_3 b q_3^* & -2\alpha_3 b_3 q_3^*
\end{bmatrix}
\]  

(31)
and the characteristic equation is
\[
\det[M - \lambda I] = m_3 \lambda^3 + m_2 \lambda^2 + m_1 \lambda + m_0 = \\
\lambda^3 + 2(\alpha_1 b_1 q_1^* + \alpha_2 b_2 q_2^* + \alpha_3 b_3 q_3^*) \lambda^2 \\
+ [\alpha_1 \alpha_2 q_1^* q_2^* (4b_1 b_2 - b^2) + \alpha_1 \alpha_3 q_1^* q_3^* (4b_1 b_3 - b^2) + \alpha_2 \alpha_3 q_2^* q_3^* (4b_2 b_3 - b^2)] \lambda \\
+ 2\alpha_1 \alpha_2 \alpha_3 q_1^* q_2^* q_3^* (b^3 + 4b_1 b_2 b_3 - b^2 b_1 - b^2 b_2 - b^2 b_3) = 0
\]
\[\text{(32)}\]

The sign of the discriminant \(\Delta\) of the characteristic equation determines whether eigenvalues are real or complex. If the discriminant is negative, then all eigenvalues are real. In turn, from the Routh-Hurwitz stability criterion we have that if all coefficients of characteristic equation are positive \(a_i > 0\) and the condition \(a_2 a_1 > a_3 a_0\) is fulfilled, then the critical point is stable. The first condition of the criterion is always fulfilled because \(b_i > b, i = 1, 2, 3\) and all parameters \(c_i, i = 0, 1, 2, 3\) are positive.

To deal with the second condition, we consider three possible positions of the point \(E_8\) which give additional inequalities for the parameters constraints

- there is a critical point inside the positive quadrant of the phase space (point \(E_8\))
  
  \[\text{case I} \quad q_1^* > 0 \quad i = 1, 2, 3.\]

- there is no critical point inside the positive quadrant of the phase space; the critical point \(E_8\) is located on a 2-dimensional invariant manifold
  
  \[\text{case IIa} \quad q_i^* > 0, \quad q_j^* > 0, \quad q_k^* = 0, \quad i \neq j \neq k \neq i\]

- there is no critical point inside the positive quadrant of the phase; the critical point \(E_8\) is located on a 1-dimensional invariant manifold
  
  \[\text{case IIb} \quad q_i^* > 0, \quad q_j^* = q_k^* = 0, \quad i \neq j \neq k \neq i.\]

For case I, the phase portrait of system (7) is presented in fig. 1 with conditions
\[
\begin{align*}
q_1^* > 0 & \quad \text{i.e.} \quad 4a_1 b_2 b_3 + (-a_1 + a_2 + a_3)b^2 - 2(a_2 b_3 + a_3 b_2)b > 0, \\
q_2^* > 0 & \quad \text{i.e.} \quad 4a_2 b_1 b_3 + (a_1 - a_2 + a_3)b^2 - 2(a_1 b_3 + a_3 b_1)b > 0, \\
q_3^* > 0 & \quad \text{i.e.} \quad 4a_3 b_1 b_2 + (a_1 + a_2 - a_3)b^2 - 2(a_1 b_2 + a_2 b_1)b > 0.
\end{align*}
\]
\[\text{(33)}\]

The stable node \(E_8\) is inside the positive quadrant of the phase space. Under the above conditions, the set of initial conditions which lead to this point is the whole positive quadrant. This point represents the Cournot equilibrium where three firms, maximizing profits, coexist in the market. This is the unique point in the positive quadrant which is a global attractor.

For case IIa, the phase portrait of system (7) with conditions
\[
\begin{align*}
q_1^* > 0 & \quad \text{i.e.} \quad 4a_1 b_2 b_3 + (-a_1 + a_2 + a_3)b^2 - 2(a_2 b_3 + a_3 b_2)b > 0, \\
q_2^* > 0 & \quad \text{i.e.} \quad 4a_2 b_1 b_3 + (a_1 - a_2 + a_3)b^2 - 2(a_1 b_3 + a_3 b_1)b > 0, \\
q_3^* = 0 & \quad \text{i.e.} \quad 4a_3 b_1 b_2 + (a_1 + a_2 - a_3)b^2 - 2(a_1 b_2 + a_2 b_1)b = 0.
\end{align*}
\]
\[\text{(34)}\]
is presented in fig. 2. In this case there is no critical point in the positive quadrant of the phase space. The set of initial conditions constituting the positive quadrant leads to the critical point located on the two-dimensional invariant submanifold \( q_3 = 0 \). It means that the marginal cost of one of the firms (parameter \( d \)) is sufficiently greater than marginal costs of other firms, such that as time goes to infinity the firm reduces its production to zero.

This case describes evolutionary scenario that one of the firms gradually withdraws from the market due to costs higher than for the competition.

Case IIb is a variant of case IIa, where two firms have higher marginal costs than the third firm. Both these firms gradually withdraw from market and only one firm, the monopolist, is left.

For deeper understanding of the phase portrait, the additional map of eigenvalues calculated at the critical points is presented in both previous figures. These vectors span the plane tangent to the stable and unstable manifolds.

To show the explicit implication of Routh-Hurwitz criterion, let us consider a simpler case of identical firms with a linear cost function and speed adjustment to market is equal 1, i.e.

\[
\alpha_1 = \alpha_2 = \alpha_3 = 1, \quad b_1 = b_2 = b_3 = b
\]

and the critical point \( E_8 \) has the coordinates

\[
q_1^* = q_2^* = q_3^* = q^* = \frac{a}{4}
\]

where \( a = a - d \) and \( d = d_1 = d_2 = d_3 \). Now the characteristic equation has the form

\[
\lambda^3 + \frac{3}{2} \bar{a} \lambda^2 + \frac{9}{16} \bar{a}^2 \lambda + \frac{1}{16} \bar{a}^3 = c_3 \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0.
\]
Let us state the theorem describing two possibilities for eigenvalues in the system (10).

**Theorem 1.** For system (10) equilibrium point $E_8$ is a stable node.

The discriminant of the third order polynomial (37) is

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = 0 \quad (38)$$

where

$$p = -\frac{3}{16} \bar{a}^2, \quad q = \frac{1}{32} \bar{a}^3.$$

Both $p$ and $q$ are different from zero so then there is

a) one real eigenvalue of multiplicity one

b) and one real eigenvalue of multiplicity two.

Let us check the Routh-Hurwitz criterion of stability of the critical point. All the coefficients of the characteristic equation (37) are positive. The condition $(m_2m_1 > m_3m_0)$ is also fulfilled for any parameter $\bar{a}$.

Hence, the critical point (36) is a stable node.

### 3.3. Invariant submanifolds

So far, the dynamics of the system was considered inside the phase space $\{(q_1, q_2, q_3): q_i > 0, i = 1, 2, 3\}$. Now, let us consider the two-dimensional planes in this 3-dimensional phase space defined as $\{(q_1, q_2, q_3) = 0\}$, $\{(q_1, q_3), q_2 = 0\}$, and $\{(q_2, q_3), q_1 = 0\}$. The system (27) possesses at least three invariant submanifolds on which it assumes the form of two-dimensional autonomous dynamical system.
For the first invariant submanifold the system has the form
\[
\begin{align*}
\dot{q}_1(t) &= 0, \\
\dot{q}_2(t) &= \alpha_2 q_2(t)[a_2 - bq_1(t) - 2b_2 q_2(t) - b q_3(t)], \\
\dot{q}_3(t) &= \alpha_3 q_3(t)[a_3 - bq_1(t) - b q_2(t) - 2b_3 q_3(t)],
\end{align*}
\] (39)

For the second invariant submanifold the system has the form
\[
\begin{align*}
\dot{q}_1(t) &= q_1(t)[a_1 - 2b_1 q_1(t) - b q_2(t) - b q_3(t)], \\
\dot{q}_2(t) &= 0, \\
\dot{q}_3(t) &= \alpha_3 q_3(t)[a_3 - bq_1(t) - b q_2(t) - 2b_3 q_3(t)],
\end{align*}
\] (40)

For the third invariant submanifold the system has the form
\[
\begin{align*}
\dot{q}_1(t) &= q_1(t)[a_1 - 2b_1 q_1(t) - b q_2(t) - b q_3(t)], \\
\dot{q}_2(t) &= \alpha_2 q_2(t)[a_2 - bq_1(t) - 2b_2 q_2(t) - b q_3(t)], \\
\dot{q}_3(t) &= 0,
\end{align*}
\] (41)

where \(a_i = a - d_i\) and \(b_i = b + e_i\).

These exist regardless of the values of the parameters, but we will be able to find other linear submanifolds under some general assumptions – we defer this discussion to the section on Darboux polynomials.

Let us choose the following values of the parameters
\[
\alpha_1 = 1, \ a_1 = 10, \ a_2 = 20, \ a_3 = 30, \ b = 0.5, \ e_i = 0.
\] (42)
to present the dynamics on the invariant submanifolds.

Let us consider the first system on invariant submanifold (39) and perform the dynamical analysis for it. This system has four critical points
\[
\begin{align*}
E_1 &= (0, 0) \\
E_2 &= \left( \frac{a_1}{2b_1}, 0 \right) \\
E_3 &= \left( 0, \frac{a_2}{2b_2} \right) \\
E_4 &= \left( \frac{2a_1 b_2 - ba_2}{4b_1 b_2 - b^2}, \frac{2a_2 b_1 - ba_1}{4b_1 b_2 - b^2} \right)
\end{align*}
\] (43-46)
The phase portrait for this system is presented on the plane \(\{ (q_2, q_3), q_1 = 0 \}\) in Fig. 3.

In the positive quadrant there is only one critical point: the stable node, which corresponds to the duopoly equilibrium. Only two firms exist on the market and they reach the stable equilibrium with fixed production level. The condition for the existence of this point is
\[
2a_1 b_2 - ba_2 > 0 \quad \text{and} \quad 2a_2 b_1 - ba_1 > 0.
\] (47)
For the chosen parameter values (42) we have $2a_2b_1 - ba_1 = 0$. In this case there is no critical point inside the positive quadrant as well as on the respective submanifolds, which are the planes $\{(q_1, q_3), q_2 = 0\}$ and $\{(q_1, q_2), q_3 = 0\}$. Markets with initial conditions in those planes tend towards monopolies on axes $q_2$ or $q_3$ as seen in fig. 3, the exception being the $q_1$ axis which is an invariant submanifold itself.

3.4. Lapunov function

For the system (41) we will look for the Lapunov function. We assume that both conditions $2a_1b_2 - ba_2 > 0$ and $2a_2b_1 - ba_1 = 0$ are fulfilled and $q^*_1 > 0$ and $q^*_2 > 0$, so we consider the critical point (46).

Let us take the Taylor expansion of the right-hand sides of (41) in the neighbourhood of the critical point $(q^*_1, q^*_2)$ given by (46) and write the system in the form \[ \dot{x} = A_{2 \times 2} x + g(x) \] (48)

where $x$ is a vector of components $x_1 = q_1 - q^*_1$ and $x_2 = q_2 - q^*_2$. The matrix $A$ and the vector $g(x)$ has the form

$A = \begin{bmatrix} -2a_1b_1q^*_1 & -a_1bq^*_1 \\ -a_2bq^*_2 & -2a_2b^2q^*_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (49)

where

$p = \text{tr} A = -2(a_1b_1q^*_1 + a_2b_2q^*_2) < 0$ (50)

$q = \det A = a_1a_2q^*_1q^*_2(4b_1b_2 - b^2) > 0$ (51)

as $b_1 > b$ and $b_2 > b$, and

$g(x) = \begin{bmatrix} -2a_1(2b_1x_1^2 + bx_1x_2) \\ -2a_2(2b_2x_2^2 + bx_1x_2) \end{bmatrix}$. (52)

We define the Lapunov function \[ V(x) = x^T K x \] (53)

where $K$ is a $2 \times 2$ constant symmetric matrix which is a solution to $A^T K + KA = -I_{2 \times 2}$ in the form $K = m(A^T)^{-1}A^{-1} + nI_{2 \times 2}$. It takes place if $m = -q/2p$ and $n = -1/2p$. Hence, we find that

$K = -\frac{1}{2pq} \begin{bmatrix} a_{21}^2 + a_{22}^2 + q & -a_{11}a_{12} - a_{21}a_{22} \\ -a_{11}a_{12} - a_{21}a_{22} & a_{11}^2 + a_{12}^2 + q \end{bmatrix}$ (54)

And the Lapunov function

$V = -\frac{(a_{22}x_1 - a_{12}x_2)^2 + (a_{21}x_1 - a_{11}x_2)^2 + q(x_1^2 + x_2^2)}{2pq} > 0 \quad \text{for} \quad x \neq 0$ (55)
Figure 3: The co-existence of three two-dimensional submanifold of system (1) representing the planes of \( \{(q_1, q_2), q_3 = 0\} \), \( \{(q_1, q_3), q_2 = 0\} \) and \( \{(q_2, q_3), q_1 = 0\} \).
is positive definite.

Given that the Lapunov function is of the form (53) and matrix $K$ (54) we get
\[
\dot{V} = -x_1^2 - x_2^2 + 2g^T K x
\]
where
\[
2g^T K x = \frac{1}{|p|q} \{-2\alpha_1 (2b_1 x_1^2 + bx_1 x_2) [(a_{21}^2 + a_{22}^2 + q)x_1 - (a_{11} a_{12} + a_{21} a_{22}) x_2] \\
- 2\alpha_2 (2b_2 x_2^2 + bx_1 x_2) \{(a_{11} a_{12} - a_{21} a_{22}) x_1 + (a_{11}^2 + a_{12}^2 + q) x_2 \}.
\]

For any $p$ and $q$, the first two terms $-x_1^2 - x_2^2$ predominate over the third term $2g^T K x$ in the neighbourhood of the origin, and $\dot{V}$ is negative definite. Therefore, system (41) has a strict Lapunov function.

4. The first integral analysis

For the purpose of the subsequent sections it will be convenient to adopt a redefined set of parameters. As the system has the general form
\[
\frac{dq_i}{dt} = \alpha_i q_i (a - d_i - (b + 2e_i) q_i - bQ), \quad i = 1, 2, 3,
\]
where $Q := q_1 + q_2 + q_3$, one can see that $b = 0$ makes the equation decouple and the system completely solvable. It thus makes sense to rescale the time by $t \to t/b$ and introduce new parameters:
\[
f_i := \frac{\alpha_i}{b} (a - d_i), \\
\epsilon_i := \frac{2\alpha_i}{b} (b + e_i).
\]

We can then make use of the following matrix notation
\[
\frac{dq_i}{dt} = q_i \left( f_i + \sum_{j=1}^{3} \beta_{ij} q_j \right),
\]
where
\[
\beta := \begin{bmatrix} \epsilon_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \epsilon_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \epsilon_3 \end{bmatrix}.
\]

4.1. A general result regarding analytic first integrals

In a polynomial system it is natural to look for polynomial first integrals, which are a special case of analytic ones, but even the latter class is very restrictive. In particular, if one demands that the conserved function $I$ be
analytic in all variables, then each hyperbolic critical point provides additional necessary conditions. For around it, the system can be linearized to be

$$\dot{q}_i = \lambda_i q_i + O(q^2), \quad i = 1, \ldots, 3,$$

(62)

assuming the point is at $q = 0$. The derivative of the lowest monomial in the expansion of $I$ around that point is then

$$\frac{d}{dt}(q_1^{n_1} q_2^{n_2} q_3^{n_3}) = (\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3)(q_1^{n_1} q_2^{n_2} q_3^{n_3}) + O(q^{n_1+n_2+n_3+1}).$$

(63)

It follows that for $\dot{I}$ to vanish, $\lambda_i$ must be linearly dependend over $\mathbb{N}$. The general statement, due to Poincaré, is that the same holds when $\lambda_i$ are the eigenvalues of the linearization matrix around any critical point, even if there are Jordan blocks.

In our case this immediately leads to the stringent result:

**Theorem 2.** If the system has a first integral analytic at the origin, the quantities $f_1$, $f_2$ and $f_3$ must be linearly dependent over the non-negative integers.

In particular this means that $f_i$ cannot all have the same sign because $n_1 f_1 + n_2 f_2 + n_3 f_3$ cannot vanish for any positive integers $n_i$. The cases 4.1 and 4.2 are thus precluded from having an analytic integral.

The same can be done around all the other points, the most straightforward ones being $E_2$, $E_3$ and $E_4$, for which we have the following sets of eigenvalues

$$\left\{ -f_i, f_j - \frac{\alpha_j f_i}{\epsilon_i}, f_k - \frac{\alpha_k f_i}{\epsilon_i} \right\},$$

(64)

where $(i, j, k)$ are cyclic permutation of $(1, 2, 3)$. These triples must be independent over $\mathbb{N}$ for there to exist a first integral analytic at $E_2$, $E_3$ and $E_4$, respectively.

The drawback of this approach is that the analysis has to be done separately at each point, and the sets of integer coefficients $n_i$ is different every time. Also, the requirement of analyticity is rather restrictive and we shall see in the next section that under fairly general assumptions there can still exist rational first integrals.

5. Darboux Polynomials

The analysis of rational integrals for homogeneous systems relies on the so called Darboux polynomials $F$, which are defined by the following property

$$\dot{F} = \sum_{i=0}^{3} V_i \partial_x, F =: PF,$$

(65)

where $P$ is called the cofactor, and is also necessarily a polynomial. The polynomials $F$ are sometimes called partial or second integrals, since they are
not conserved in general but each of them is conserved on the set defined by \( F = 0 \) as seen above. If the initial conditions are such that the system starts in such a set it will remain there for all later times. If the cofactor is zero, then the polynomial is just a first integral.

The first fundamental fact that we will use here is that if the system has a rational integral \( R = R_1/R_2 \) with \( R_1 \) and \( R_2 \) relatively prime, they must both be Darboux polynomials with the same cofactor. This is immediately seen from direct differentiation.

Secondly, any product of Darboux polynomials is itself a Darboux polynomial

\[
\frac{d}{dt} \left( \prod_{k=1}^{n} F_k^\gamma_k \right) = \left( \sum_{k=1}^{n} \gamma_k P_k \right) \prod_{k=1}^{n} F_k^\gamma_k, \quad \gamma_k \in \mathbb{N}. \tag{66}
\]

Note that if the domain of motion is such that the above functions are well defined, any real \( \gamma_k \) are admissible in a construction of first integrals, although it might not be rational or even algebraic then. In other words, if there are enough Darboux polynomials, so that one can find numbers \( \gamma_k \) such that the cofactor in (66) vanishes, a first integral can be found.

Let us now apply the above to our system. It is immediately visible that each of the dependent variables \( q_i \) is itself a Darboux polynomial with cofactors given by \( f_i + \sum_j \beta_{ij} q_j =: f_i + L_i \). Their linear dependence is a major factor in determining integrability.

Since these are three polynomials in three variables, they will generically be linearly independent. In fact, dependence does not occur for economically viable values of the parameters, but we include this case nevertheless for the sake of completeness – the result applies to any system of the prescribed form.

5.1. The case \( \det(\beta) = 0 \)

If it so happens that

\[
-\frac{\det(\beta)}{\alpha_1 \alpha_2 \alpha_3} = 2 - \frac{\epsilon_1}{\alpha_1} - \frac{\epsilon_2}{\alpha_2} - \frac{\epsilon_3}{\alpha_3} + \frac{\epsilon_1 \epsilon_2 \epsilon_3}{\alpha_1 \alpha_2 \alpha_3} = 0, \tag{67}
\]

a linear combination of \( L_i \) will vanish and this will lead to at least a time-dependent first integral. To see this, let us notice that when the determinant of \( \beta \) is zero, it has a null left eigenvector \( \gamma_k \) which is not the zero vector

\[
\sum_{k=1}^{3} \gamma_k \beta_{kj} = 0, \quad j = 1, \ldots, 3. \tag{68}
\]

The coefficients \( \gamma_k \) might not be integer, but in any case the following function will be defined for positive \( q_j \):

\[
I_0 := \prod_{j=1}^{3} q_j^{\gamma_j}, \tag{69}
\]
whose cofactor, by (66) will be

$$ P_0 = \sum_{j=1}^{3} \gamma_j \left( f_j + \sum_{k=1}^{3} \beta_{jk} q_k \right) = \sum_{j=1}^{3} \gamma_j f_j. \quad (70) $$

Because this is constant, the equation \( \dot{I}_0 = P_0 I_0 \) can be immediately integrated to yield the time-dependent integral

$$ e^{-P_0 t} I_0 = \text{const.} \quad (71) $$

It might additionally happen that \( P_0 = 0 \), so there is no time dependence, or that there are two zero eigenvalues of \( \beta \) such that two independent null vectors \( \gamma_k, \theta_k \) can be found. In the second case there will be two first integrals and time can be eliminated from at least one of them, because both depend on it exponentially.

To illustrate the above, let us look at the case when all \( \alpha_i \) are equal. This is not a crucial assumption in this case, since they must all be non-zero, so their values do not change the condition that \( \det(\beta) = 0 \). The other parameters must satisfy some constraint for the condition to be true, and there are many possibilities due to the large number of parameters, but equation (68) can always be solved. For example, in the generic case of \( \gamma_k \neq 0 \), the following parameters give vanishing determinant:

$$ e_i = -b \frac{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_i}{2 \gamma_i}, \quad (72) $$

The corresponding the first integral is

$$ I = e^{-(f_1 \gamma_1 + f_2 \gamma_2 + f_3 \gamma_3) t} q_1^\gamma_1 q_2^\gamma_2 q_3^\gamma_3. \quad (73) $$

After taking logarithm both sides and differentiating with respect to time we obtain that the rates of growth are linearly dependent and satisfy the following condition

$$ f_1 \gamma_1 + f_2 \gamma_2 + f_3 \gamma_3 = \gamma_1 \frac{\dot{q}_1}{q_1} + \gamma_2 \frac{\dot{q}_2}{q_2} + \gamma_3 \frac{\dot{q}_3}{q_3}. \quad (74) $$

The rates of growth of firms’ production are strictly related as a given firm need to adjust its production rate of growth with respect to production rates of growth of two other firms. This relation is a dynamical constraint imposed on the production rates of growth at any moment. It is analogous to the reaction curves analysis [2].

In a special case of constant rate of growth there is a solution

$$ q_1(t) \propto e^{f_1 t}, \quad q_2(t) \propto e^{f_2 t}, \quad q_3(t) \propto e^{f_3 t}. \quad (75) $$

6. The case \( \det(\beta) \neq 0 \).

We can next proceed to the generic, and more difficult, case when \( \det(\beta) \neq 0 \), noting first that any cofactor of our system must be of the form

$$ P_4 = p_0 + \sum_{i=1}^{3} p_i q_i, \quad (76) $$

17
because the right-hand sides of the system are quadratic.

We will first try to find all linear Darboux polynomials

\[ F_4 = w_0 + \sum_{i=1}^{3} w_i q_i, \quad (77) \]

which do not reduce to just \( q_i \) or a constant. Calculating \( \dot{F}_4 - P_4 F_4 \) and equating coefficients of different monomials to zero, gives system of polynomial equations for \( w_i \) and \( p_i \):

\[ q^0 : \quad w_0 p_0 = 0, \quad (78) \]

\[ q^1 : \begin{align*}
 w_0 p_1 + w_1 (p_0 - f_1) &= 0, \\
 w_0 p_2 + w_3 (p_0 - f_2) &= 0, \\
 w_0 p_3 + w_3 (p_0 - f_3) &= 0,
\end{align*} \quad (79) \]

\[ q^2 : \begin{align*}
 w_1 (\alpha_1 + p_2) + w_2 (\alpha_2 + p_1) &= 0, \\
 w_2 (\alpha_2 + p_3) + w_3 (\alpha_3 + p_2) &= 0, \\
 w_3 (\alpha_3 + p_1) + w_1 (\alpha_1 + p_3) &= 0,
\end{align*} \quad (80) \]

In the generic situation all \( w_i \) are non-zero, which then leads to a homogeneous linear system:

\[
\begin{bmatrix}
\alpha_1 - \epsilon_2 & \alpha_2 - \epsilon_1 & 0 \\
\alpha_1 - \epsilon_3 & 0 & \alpha_3 - \epsilon_1 \\
0 & \alpha_2 - \epsilon_3 & \alpha_4 - \epsilon_2
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix} = 0, \quad (81)
\]

whose non-zero solution requires the determinant to vanish

\[ D_1 := (\alpha_1 - \epsilon_3)(\alpha_2 - \epsilon_1)(\alpha_3 - \epsilon_2) + (\alpha_1 - \epsilon_2)(\alpha_2 - \epsilon_2)(\alpha_3 - \epsilon_1) = 0. \quad (82) \]

If a solution of the system exists, the set of equations (79) gives three constraints

\[ f_i = p_0 - \frac{w_0}{w_i} \epsilon_i, \quad (83) \]

which have to take into account that \( w_0 p_0 = 0 \). So either \( w_0 = 0 \) and \( p_0 = f_i \), which must all be equal; or \( w_0 = 1 \) (\( F \) is always defined up to a constant factor), \( p_0 = 0 \) and \( f_i w_i = \epsilon_i \).

The remaining exceptional cases occur when one of \( w_i \) is zero, and because the system has complete symmetry with respect to the indices, it is enough to consider \( w_1 = 0 \). Direct computation then shows that the only non-trivial solution is

\[ F_4 = (\alpha_2 - \epsilon_2) q_2 + (\epsilon_3 - \alpha_2) q_3, \quad P_4 = f_2 - \alpha_2 q_1 - \epsilon_2 q_2 - \epsilon_3 q_3. \quad (84) \]
with the additional constraints of

\[ f_2 = f_3, \quad \alpha_2 = \alpha_3. \tag{85} \]

All the other solutions (and constraints) can be obtained by cyclic permutations of the indices: \(1 \rightarrow 2 \rightarrow 3 \rightarrow 1\).

One should next proceed with all higher-degree Darboux polynomials, or, ideally solve the problem for general degree \(n\). Direct computation shows that there are no new second or third order polynomials, other than products of all the linear ones. The authors have been unable to proceed with the proof for the general case, i.e., when all the parameters are independent. At the same time, imposing some restrictions immediately produces linear Darboux polynomials and first integrals, so it seems reasonable to formulate the following

**Conjecture 1.** All Darboux polynomials of the system in question for general values of parameters are generated by the linear polynomials.

So far we have not made use of the \(\det(\beta) \neq 0\) condition, and if Darboux polynomials can be found via the procedure above, each of them defines an invariant set \(F = 0\) by itself. But now that the matrix \(\beta\) is not singular, the linear forms \(L_i\) form a basis, so the linear part of the cofactor \(P_4\) can be decomposed in full analogy with the previous section:

\[ L_4 = \sum_i p_i q_i = \sum_i \gamma_i L_i, \quad \gamma_i = \sum_j p_j [\beta^{-1}]_{ji}. \tag{86} \]

Then the function

\[ I_0 := F_4 \prod_i x_i^{-\gamma_i}, \tag{87} \]

has a constant cofactor

\[ \dot{I}_0 = \left( p_0 - \sum_i f_i \gamma_i \right) I_0 =: P_0 I_0, \tag{88} \]

and a time-dependent first integral is, as before, \(I = e^{-P_0 t} I_0\).

Coming back to the simpler cases of the system (7), we can consider all firms identical i.e. their cost function are the same and/or the speed of adjustment to market \(\alpha_i\) are the same.

### 6.1. Identical firms with linear or quadratic cost

The first two subcases, 2.2 and 2.4, of identical firms can in fact be treated together. That is, regardless of whether the cost function is linear or quadratic we can find first integrals.

Taking all \(\alpha_i\) to be the same, and the cost function

\[ C(q_i) = c + dq_i + eq_i^2, \quad c > 0, d > 0, e > 0, \tag{89} \]
the parameters (17) have the form

\[
\begin{align*}
f_i &= \frac{a_i}{b_i}, \\
\epsilon_i &= \frac{2}{b_i}(b_i + e_i).
\end{align*}
\] (90)

The matrix \( \beta \) in this case is

\[
\det \beta = -\begin{vmatrix} \epsilon & 1 & 1 \\ 1 & \epsilon & 1 \\ 1 & 1 & \epsilon \end{vmatrix} = -8 \left( \frac{b + e}{b} \right)^3 + 6 \frac{b + e}{b} - 2 \neq 0
\] (91)

if \( b > 0, \epsilon > 0 \) so that \( \frac{b + e}{b} > 1 \).

This is a degenerate case, because both \( D_1 \) vanishes and all \( f_i \) are equal. There are thus three additional linear Darboux polynomials, together with their cofactors they are:

\[
\begin{align*}
F_4 &= q_1 - q_2, \quad P_4 = f - 2q_1 - 2q_2 - 2q_3, \\
F_5 &= q_2 - q_3, \quad P_5 = f - q_1 - 2q_2 - 2q_3, \\
F_6 &= q_3 - q_1, \quad P_6 = f - 2q_3 - q_2 - 2q_1.
\end{align*}
\] (92)

According to the general treatment of section 8, we thus have three time dependent integrals

\[
\begin{align*}
I_4 &= e^{ft/4}q_3(q_1 - q_2)(q_1q_2q_3)^{-3/4}, \\
I_5 &= e^{ft/4}q_1(q_2 - q_3)(q_1q_2q_3)^{-3/4}, \\
I_6 &= e^{ft/4}q_2(q_3 - q_1)(q_1q_2q_3)^{-3/4},
\end{align*}
\] (93)

and time can be eliminated from two of them, to yield a first integral

\[
I_1 = \frac{q_1(q_2 - q_3)}{q_3(q_1 - q_2)},
\] (94)

and we note that cyclic permutations of indices produce also first integrals, but they are all functionally dependent.

6.2. Firms with different quadratic terms of the cost functions

This is a subcase of the situation of 2.4, and could be called “almost identical” firms, as the parameters coincide in the linear parts (\( d_i \) and \( \alpha_i \) are equal), but the quadratic terms of the cost functions are different for each company. The previous two subcases are just restrictions of this one, and we state here the general results regarding invariant submanifolds and stability.

The parameters \( f \) and \( \epsilon_i \) (17) now have the form

\[
\begin{align*}
f_i &= \frac{a - d}{b}, \\
\epsilon_i &= \frac{2}{b}(b + e_i).
\end{align*}
\] (95)

and we note that cyclic permutations of indices produce also first integrals, but they are all functionally dependent.
The matrix $\beta$ in this case is

$$
\det \beta = - \begin{vmatrix} \epsilon_1 & 1 & 1 \\ 1 & \epsilon_2 & 1 \\ 1 & 1 & \epsilon_3 \end{vmatrix} = -\epsilon_1 \epsilon_2 \epsilon_3 + (\epsilon_1 + \epsilon_2 + \epsilon_3) - 2. \tag{96}
$$

Because $b > 0$ and $\epsilon_i > 0$ we have that $\epsilon_i > 2$ and hence $\det \beta < 0$.

There are again three additional Darboux polynomials

$$
F_4 = (\epsilon_1 - 1)q_1 - (\epsilon_2 - 1)q_2, \quad P_4 = f - \epsilon_1 q_1 - \epsilon_2 q_2 - q_3,
$$

$$
F_5 = (\epsilon_2 - 1)q_2 - (\epsilon_3 - 1)q_3, \quad P_5 = f - q_1 - \epsilon_2 q_2 - \epsilon_3 q_3,
$$

$$
F_6 = (\epsilon_3 - 1)q_3 - (\epsilon_1 - 1)q_1, \quad P_6 = f - \epsilon_3 q_3 - q_2 - \epsilon_1 q_1. \tag{97}
$$

All the time dependent first integrals can then be obtained by cyclic index permutations from the following one

$$
I_4 = e^{-P_0 t} q_3 ( (\epsilon_1 - 1)q_1 - (\epsilon_2 - 1)q_2 ) q_1^{-\gamma_1} q_2^{-\gamma_2} q_3^{-\gamma_3}, \tag{98}
$$

where

$$
\gamma_i = \frac{\epsilon_i - 1}{\det \beta} \left( 1 - \frac{\epsilon_1 \epsilon_2 \epsilon_3}{\epsilon_i} \right), \tag{99}
$$

and

$$
P_0 = \frac{(\epsilon_1 - 1)(\epsilon_2 - 1)(\epsilon_3 - 1)}{\det \beta} f. \tag{100}
$$

The elimination of time yields an ordinary first integral of

$$
I_4 = \frac{q_3 ((\epsilon_1 - 1)q_1 - (\epsilon_2 - 1)q_2) q_1^{-\gamma_1} q_2^{-\gamma_2} q_3^{-\gamma_3}}{q_1 ((\epsilon_2 - 1)q_2 - (\epsilon_3 - 1)q_3)}. \tag{101}
$$

As mentioned before, each Darboux polynomial defines an additional invariant submanifold $F_i = 0$, which in this case are planes through the origin. In fact all three intersect along a line because $F_6 = F_4 + F_5$ and their normal vectors are not independent. Its equation is

$$
q = \left( \frac{q_4}{\epsilon_1 - 1}, \frac{q_4}{\epsilon_2 - 1}, \frac{q_4}{\epsilon_3 - 1} \right), \quad q_4 \in (0, \infty). \tag{102}
$$

The Nash equilibrium $E_8$ lies exactly on this line and is an attracting node.

We can say even more thanks to the explicit formula for $I_4$ and the remaining two integrals. In the positive quadrant $q_1 q_2 q_3$ is nonzero, while $P_0 < 0$ by the restrictions on the parameters. Thus solving the conservation law $I_4 = \text{const.}$ we immediately obtain from (98) that $F_4 \to 0$ as $t \to \infty$, and similarly for the other two.

This means that trajectories are attracted by all the invariant submanifolds (planes) and hence also by the line. On the line itself it is straightforward to check that

$$
q_4 = q_4 \left( f - \left( \frac{1}{\epsilon_1 - 1} + \frac{1}{\epsilon_2 - 1} + \frac{\epsilon_3}{\epsilon_3 - 1} \right) q_4 \right) = f q_4 (1 - q_4/q_4^*), \tag{103}
$$

21
where $q_4^*$ is the position of $E_8$ in this coordinate. The Nash equilibrium is thus seen to be the attractor of two trajectories on the line, which in turn are attracting trajectories for the whole positive quadrant.

The level sets of $I_1$, as seen in Figure 4 are surfaces also intersecting at the central line. Taking the first integral as one coordinate, the system becomes just two dimensional on any such leaf, by solving $I_1 = C$, eliminating one of the original variables, e.g.

$$q_3 = \frac{C(\epsilon_2 - 1)q_1q_2}{(1 - \epsilon_2)q_2 + (\epsilon_1 - 1 + C(\epsilon_3 - 1))q_1} =: Q_3(q_1, q_2),$$

(104)

and substituting into the original system:

$$\dot{q}_1 = h^1(q_1, q_2, Q_3(q_1, q_2))$$

$$\dot{q}_2 = h^2(q_1, q_2, Q_3(q_1, q_2)).$$

(105)
6.3. Extension to n-dimensions

The special shape of the Darboux polynomials and first integrals suggests a straightforward generalisation to n dimensions, and indeed it is easy to check, that for n almost identical (in the above sense) firms, one has a family of Darboux \( n(n-1)/2 \) polynomials

\[
F_{ij} = (\epsilon_i - 1)q_i - (\epsilon_j - 1)q_j, \quad P_{ij} = f - \epsilon_i q_i - \epsilon_j q_j - \sum_{k \neq i,j} q_k. \tag{106}
\]

Like before, there exist time-dependent first integrals

\[
I_{ij} = e^{-P_0 t} F_{ij} \prod_k q_k^{-\gamma_{ij}^k}, \tag{107}
\]

where \( \gamma_{ij}^k \) is the k-th component of the vector

\[
\gamma_{ij} = -[1, \ldots, 1, \epsilon_i, 1, \ldots, 1, \epsilon_j, 1, \ldots, 1] \beta^{-1}, \tag{108}
\]

and it so happens that for all pairs (i, j), the sum of the components is the same giving

\[
P_0 = f - f \sum_k \gamma_{ij}^k = \frac{f}{\det \beta} \prod_k (\epsilon_k - 1). \tag{109}
\]

The ratios of these integrals are thus time-independent, but again not all are functionally independent. E.g., for \( I_{123} := I_{12}/I_{23} \) and \( I_{312} = I_{31}/I_{12} \) we have

\[
I_{312} = \frac{1 - \epsilon_1 + I_{123}(1 - \epsilon_3)}{I_{123}(\epsilon_2 - 1)}. \tag{110}
\]

Finally, because \( \det \beta < 0 \) all \( F_{ij} \) tend to zero exponentially with time, so all trajectories approach the line through the Nash equilibrium, as before.

7. Conclusions

We used the qualitative methods of dynamical systems to study the phase structure of the oligopoly model. Exploring the example of three firms oligopoly we analyse the complexity of the model dynamics. The phase space is organized with one-dimensional and two-dimensional invariant submanifolds (on which the system reduces to monopoly and duopoly) and a unique stable node (global attractor, Cournot and Nash equilibrium) in the positive quadrant of the phase space.

Its inset is the positive quadrant of the phase space \( \{(q_1, q_2, q_3): q_1 > 0, q_2 > 0, q_3 > 0\} \). The boundaries of this quadrant are three two-dimensional submanifolds where the dynamics of duopolies is restricted to. Trajectories from the bulk space depart asymptotically from unstable invariant 2-dimensional submanifolds. In a generic case they reach the Nash equilibrium point. The dynamics of monopolies is restricted to the axes of coordinates systems. On this
line there exits stable critical points (A, C, F in Fig. 1) they are a equilibrium points of monopolies.

From our dynamical analysis of the system one can derive general conclusion that the system under consideration is structurally stable, because its phase portrait in the generic case contains saddle and nodes. It seems to be important in the economic context because its structural stability means that dynamics cannot be destroyed by small perturbations [11].

The problem of integrability of the model was also addressed using Darboux polynomials, since this is the natural setting for polynomial systems. Because of the large number of parameters, the full analysis was not possible, but some general cases of interest were shown to posses additional linear Darboux polynomials and also time-dependent conserved quantities. This allowed to give qualitative analysis of the asymptotic behaviour and dimensional reduction in the general case of firms different at the quadratic level.

The following points of interest were also addressed:

- We formulate a criterion of asymptotic stability of a Cournot equilibrium which indicate that phase space interior of positive octant in $\mathbb{R}_+^3$. The phase space structure is organised as follows: the global attractor and three repelling invariant submanifolds. They form the boundary of this octant. From the economic point of view they represent the dynamics of duopolies.

- The Cournot equilibrium is a global attractor in the phase space for generic class of initial conditions and model parameters.

- We found the new algebraic interpretation of reaction curves for some forms of oligopoly system admitting time dependent first integral. This constraint assumes the form of linear combination of production rates of growth.

- We demonstrate how the dimension of the system can be lowered by one due to existence of first integral in the case of firms with different quadratic term in cost function.

- When a first integral exists it extends our knowledge of invariant submanifolds beyond the linear case (planes) as depicted in Fig. 4. Additionally it points to the lack of chaotic behaviour in the system.

- The relations between growth rates and the variables themselves obtained thanks to the first integrals provide immediate observable constraints that can be tested against data.

Acknowledgments

This work has been supported by the grant No. DEC-2013/09/B/ST1/04130 of the National Science Centre of Poland. The authors thanks Franciszek Humieja for comments and remarks.
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