Bondi accretion onto cosmological black holes

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In this paper we investigate a steady accretion within the Einstein-Straus vacuole, in the presence of the cosmological constant. The dark energy damps the mass accretion rate and — above certain limit — completely stops the steady accretion onto black holes, which in particular is prohibited in the inflation era and after (roughly) \(10^{12}\) years from Big Bang (assuming the presently known value of the cosmological constant). Steady accretion would not exist in the late phases of the Penrose’s scenario - known as the Weyl curvature hypothesis - of the evolution of the Universe.

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I. INTRODUCTION

The classical Bondi accretion model \([1]\) describes, in the test gas and stationarity approximations, the spherical accretion/wind of a barotropic fluid onto a newtonian gravity center. That description has been later extended to general-relativistic spacetimes. The steady accretion in the Schwarzschildian geometry has been investigated by Michel \([2]\) and Shapiro and Teukolsky \([3]\), and in a spherical symmetric spacetime with backreaction by Malec \([4]\). A remarkable universal behaviour has been discovered for selfgravitating transonic flows, under suitable boundary conditions: the ratio of respective mass accretion rates \(M_{GR}/M_N\) appears independent of the fractional mass of the gas and depends only on the asymptotic temperature. It is close to 1 in the regime of low asymptotic temperatures and can grow by one order of magnitude at high temperatures \([5]-[6]\). Here \(M_{GR}\) and \(M_N\) are the general relativistic and the Newtonian mass accretion rates, respectively. The stability of steady accretion has been established by Mach and his coworkers \([7]-[9]\).

In this paper we consider the accretion onto a black hole that is immersed in a cosmological universe. We adopt the Einstein-Straus "swiss cheese" model \([10]\). A spherically symmetric black hole surrounded with accreting gas sits inside a vacuole that is matched — through the Darmois-Israel junction conditions \([11], [12]\) — to an external FRW spacetime. We calculate how the presence of a cosmological constant \(\Lambda\) would affect the most important system characteristic — the mass accretion rate \(\dot{M}\) of a fluid onto the center. We found that the mass accretion rate is damped by the presence of the cosmological constant. This effect is stronger for negative values than for positive values of the cosmological constant and it depends on the ratio of the dark energy density to the fluid density.

In the next two sections we remind the Kottler solutions and draw the general picture of the steady accretion, together with suitable equations. Sections 4 and 5 present three analytic sets of results. These show that the matter distribution can be affected by the cosmological system. More importantly, we prove that if the cosmological constant exceeds certain limit, then the steady accretion ceases to exist. Section 6 presents results of numerical integrations. It appears that the presence of dark energy, either negative (attractive) or positive (repulsive), damps the mass accretion rate. It appears - in accordance with analytic predictions - that the steady accretion does not exist above certain values of the cosmological constant. The last section summarizes main results and applies them to two cosmological epochs when the dark energy is dominant. It is pointed out that our results do not agree with the Penrose’s "Weyl curvature hypothesis".

II. ACCRETION IN AN EINSTEIN-STRAUS VACUOLE

We assume relativistic units with \(G = c = 1\). Let a FLRW spacetime be filled with dust and dark energy. Its line element reads

\[
ds^2 = -d\tau^2 + a(\tau) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),
\]

where the variables have obvious meaning \([13]\) and \(k = 0, 1\). We do not need to analyze the FLRW equations, for a reason explained below.

We cut off, following Einstein and Straus \([10]\), a ball from this FLRW spacetime and insert instead a spherical symmetric inhomogeneity — a black hole surrounded by a spherical cloud of gas — satisfying Einsteins equation. It is known that the boundary of the ball must be comoving if the FLRW geometry contains only dust and the dark energy represented by \(\Lambda\), assuming the absence of a boundary layer and continuity of pressure \([11], [15]\). Crenon and Lake show that the no-boundary layer assumption is necessary for having a comoving boundary \([13]\).

Inside the excision ball itself we can distinguish an inner accretion region such that the areal radius \(R\) does not exceed a limiting radius \(R_{\infty}\) (that in turn should be smaller, or perhaps much smaller, than the areal radius of the boundary of the ball), and the asymptotic vacuum region outside \(R_{\infty}\). In the asymptotic region the metric...
is given by the Kottler (Schwarzschild-de Sitter (SdS, \( \Lambda > 0 \)) and Schwarzschild-anti de Sitter (SAdS, \( \Lambda < 0 \))) spacetime line element \([17]\)

\[
ds^2 = - \left( 1 - \frac{2m}{R} - \frac{\Lambda}{3} R^2 \right) dt^2 + \frac{R^2}{1 - \frac{2m}{R} - \frac{\Lambda}{3} R^2} dr^2 + R^2 d\Omega^2.
\]

The Darmois-Israel gluing conditions \([11], [12]\) demand that along the comoving boundary the first and second fundamental forms are continuous. In the spherically symmetric case the radius \( R \) has to be continuous and in addition: i) the excess mass \( m \) should be exactly equal to the mass of dust in the excised ball; ii) the boundary should be comoving with the Hubble velocity \( H \). This guarantees that the stress-energy tensor \( T_{\mu\nu} \) does not form the surface layer at the junction surface. Balbinot et al. have found an explicit solution that describes the corresponding matching between the Schwarzschild-de Sitter and FLRW (dust+\( \Lambda \)) spacetimes \([18]\). Therefore we stop discussion on FLRW and SdS (SAdS) solutions at this point and focus on the description of the accretion zone.

We assume that in the accretion region the Einstein equations can be approximated by a set of stationary equations in time intervals that are much smaller than a characteristic time \( T \) that is defined in the next section. Let \( R_\infty \) be the size of the cloud of gas and \( a_\infty \) the asymptotic speed of sound. Outside \( R_\infty \) the geometry can be connected smoothly to the SdS or SAdS spacetime geometry by a transient zone of a negligible mass. Let us remark that the assumption of the approximate stationarity can be checked a posteriori, after finding appropriate solutions. Approximately stationary accretion equations have been derived in \([4]\). A quasi-stationary solution, of a similar accretion problem in the asymptotically flat spacetime with a spherical black black hole, appears to be stable under axially symmetric perturbations \([8]\).

### III. EQUATIONS OF STEADY ACCRETION

The metric in the accretion zone is spherically symmetric and has the form

\[
ds^2 = -N^2 dt^2 + \hat{a} dr^2 + R^2 (d\theta^2 + \sin^2(\theta) d\phi^2),
\]

where we use comoving coordinates \( t, r, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi \) — time, coordinate radius and two angle variables, respectively. \( R \) denotes the areal radius and \( N \) is the lapse. \( \hat{a} \) is the radial-radial metric component. The radial velocity of gas is given by \( U = \frac{\hat{a}}{N} \).

The energy-momentum tensor reads \( T^B_{\mu\nu} = (\rho + p) U^\mu U^\nu + pg_{\mu\nu} \) with the time-like normalized four-velocity \( U^\mu, U^\nu U^\mu = -1 \). A comoving observer would measure local mass density \( \rho = T^{\nu\nu} U^\nu U^\nu \). Let \( n_\mu \) be the unit normal to a coordinate sphere lying in the hypersurface \( t = \text{const} \) and let \( k \) be the related mean curvature scalar, \( k = \frac{R}{2} \nabla_n n^\mu = \frac{1}{\sqrt{n}} \partial_t R \). We assume the perfect gas equation of state \( p = (\Gamma - 1) \rho \epsilon \), where \( \epsilon \) is the specific internal energy and \( \Gamma \) is a constant. Assuming that the accretion is isentropic, one can derive the polytropic equation of state \( p = K \rho \epsilon \), with a constant \( K \). The internal energy \( E = \rho_0 \epsilon \) and the rest \( \rho \) and baryonic \( \rho_0 \) mass densities are related by \( \rho = \rho_0 + E = \rho_0 + p/(\Gamma - 1) \).

The matter-related and geometric quantities satisfy Einstein equations and the baryonic mass conservation. (But let us point out that for isentropic flows the conservation of baryonic mass also follows from Einstein equations \([4]\).) One can find the mean curvature \( k \) from the Einstein constraint equations \( G_{\mu\nu} = 8\pi \rho_0 \)[4]

\[
k = \sqrt{1 - \frac{2m(R)}{R} - \frac{\Lambda}{3} R^2 + U^2},
\]

where \( m(R) \) is the quasilocal mass,

\[
m(R) = m - 4\pi \int_R^{R_\infty} d\tau \hat{a} \rho.
\]

In the line element \([3]\) we have the comoving time. In the polar gauge foliation one has a new time \( t_s(t, r) \) with \( \partial_\delta = \partial_t - N U \partial_R \). The field \( \partial_\delta \) is tangent to the cylinder of constant areal radius, \( \partial_\delta R = 0 \).

One can show that

\[
\dot{M}_\rho \equiv \partial_\delta m(R) = 4\pi N U R^2 (\rho + p).
\]

The mass contained in an annulus \( (R, R_\infty) \) changes if the fluxes on the right hand side, one directed outward and the other inward, do not cancel. The baryonic current density reads \( j^\mu = \rho_0 U^\mu \). Its continuity equation reads

\[
\nabla_\mu j^\mu = 0.
\]

For stationary flows the local baryonic flux

\[
\dot{M} = -4\pi U R^2 \rho_0.
\]

is time-independent at a fixed \( R \).

We say that the accretion process is stationary (or quasi-stationary) if all relevant physically observables, that are measured at a fixed areal radius \( R \), remain approximately constant during time intervals much smaller than the runaway instability time scale \( T = M/M \). That means that \( \partial_\delta X \equiv (\partial_\delta - N U \partial_R) X = 0 \) for \( X = \rho_0, \rho, j, U, \ldots, M \).

For quasi-stationary flows we have \( \partial_R \dot{M}_\rho = 0 \) and \( \partial_R \dot{M} = 0 \)[4]. Therefore both \( \dot{M}_\rho \) and \( \dot{M} \) are equal, modulo a constant factor; they can be identified. The quantity \( \dot{M} \) will be called the mass accretion rate. One obtains from \([8]\) an expression

\[
\partial_R U^2 = -4U^2 - 2U^2 \partial_R \ln (\rho_0)
\]
The speed of sound is defined as \( a = \sqrt{\partial_{\rho} p} \). It is a useful and straightforward exercise to express the hydrodynamic quantities in terms of \( a \):

\[
\begin{align*}
\rho &= \rho_0 \frac{\Gamma - 1}{\Gamma - 1 - a^2}, \\
p &= \rho_0 \frac{\Gamma}{\Gamma - 1 - a^2} - \rho,
\end{align*}
\]

\( \rho_0 = \rho_0\left( \frac{a}{a_\infty} \right)^{\frac{2}{\Gamma-1}} \left( \frac{1 - \frac{a^2}{a_\infty^2}}{1 - \frac{a^2}{a_\infty^2}} \right)^{\frac{1}{\Gamma-1}}. \) \hspace{1cm} (10)

Here (and below) quantities with the suffix \( \infty \) do refer to their asymptotic values (i.e., at \( R_\infty \)). Notice that \( \rho, \rho_0 \) and \( a^2 \) show the same monotonicity behaviour: \( \partial_R \rho_0 = C_1 \partial_R p = C_2 \partial_R a^2 \), where \( C_1, C_2 \) are strictly positive functions.

There are two conservation equations that originate from the contracted Bianchi identities \( \nabla_\mu T^\mu_\nu = 0 \). One of them can be eliminated, if we choose instead the baryonic mass conservation. The case when \( \nu = 0 \) is the relativistic version of the Euler equation,

\[
N \frac{d}{dR} p + (\rho + \rho_0) \frac{d}{dR} N = 0. \hspace{1cm} (11)
\]

One can solve Eq. (11), using Eq. (10):

\[
N = C \left( \Gamma - 1 - a^2 \right). \hspace{1cm} (12)
\]

The whole system of algebraic equations (4) — (12) closes dynamic quantities in terms of useful and straightforward exercise to express the hydrodynamic function of the areal radius is called the sonic radius, denoted as \( a \).

Finally the line element, in \((t, R)\) coordinates, is given by

\[
ds^2 = -\left( N^2 - U^2 \right) dt^2 - 2N \frac{dtdR}{k} + \frac{dR^2}{k^2} + R^2 d\Omega^2. \hspace{1cm} (14)
\]

We shall study transonic accretion flows. For them the principal object of interest is a sonic point, where both the denominator \( a^2 - U^2 \) and the numerator \( \frac{m(R)}{R} - 2U^2 + 4\pi R^2 p - \frac{\Lambda R^2}{3} \) of Eq. (13) vanish. The corresponding value of the areal radius is called the sonic radius, denoted as \( R_s \). A closer inspection of the sonic point shows that it is a critical point, with branching pairs of solutions describing accretion or wind. In the accretion branch, below the sonic point the infall velocity \(|U|/k\) is bigger than \( a \), while outside the sonic sphere the converse is true. This analysis is very much standard, replicating the work done in the Newtonian case by Bondi [2] and in the general-relativistic case by Malec [4].

IV. QUALITATIVE ANALYSIS OF SOLUTIONS: SDS BLACK HOLES

In this section we shall investigate some properties of transsonic solutions for positive values of \( \Lambda \). The speed of sound \( a \) (and thus also \( \rho_0 \) and \( p \)) decreases as a function of \( R \) — for accreting solutions — when the cosmological constant is constant. We shall show, in Lemma 2, that for large values of \( \Lambda R^2 \) the converse is possible.

**Lemma 1.**

Assume \( \Lambda < \frac{3m}{R_5^2} \). Then the speed of sound, the pressure and the mass density \( \rho_0 \) are decreasing outside the sonic sphere, \( \frac{dR}{dX} < 0 \) for \( X = a^2, p \) and \( \rho_0 \).

**Proof of Lemma 1.**

Let us define \( C(R) = \frac{m(R)}{R} - 2U^2(R) + 4\pi R^2 p - \frac{\Lambda R^2}{3} \) and let \( C_\infty \equiv C(R_\infty) \). One easily obtains from this definition, using equation (9), in order to eliminate \( \partial_R U^2 \), that

\[
\frac{d}{dR} C(R) = \frac{-4C}{R} + 4\pi R \left( \rho + 6p - \frac{\Lambda}{2\pi} \right) + 3m \frac{R}{R^2} + 4U^2 \frac{\partial p}{\partial \rho_0} + 4\pi R^2 \frac{d}{dR} b. \hspace{1cm} (15)
\]

The condition \( \Lambda < \frac{3m}{R_5^2} \) guarantees that \( C_\infty > 0 \) and — using the argument of continuity — the positivity of the function \( C(R) \) in an open interval \((R_s, R_\infty)\). We show that \( R_s = R_s \).

Notice that while at the sonic point \( C(R_s) = 0 \), the function \( \partial_R C \) must be strictly positive. Indeed, assume the opposite; from continuity \( C \) would have to be negative just above \( R_s \). Then from Eq. (15) it would follow that both terms with derivatives must be negative, that is — see the remark below (10) — \( \partial_R a^2 < 0 \). But Eq. (13) gives \( \partial_R a^2 > 0 \), if \( C < 0 \) and \( R > R_s \). This contradiction proves our statement and implies the positivity of \( C \) in an interval \((R_s, R_\infty)\).

Now, assume that \( C \) changes sign at \( R_b \), that is \( C(R_b) = 0 \) and \( \frac{d}{dR} C(R)_{R=R_b} < 0 \). But if \( C(R_b) = 0 \) then — from Eq. (13) — the derivative \( \partial_R a^2 \) would have to vanish. From (10) would then follow \( \partial_R \rho_0 = \partial_R p = 0 \). Thus Eq. (15) yields at \( R_b \)

\[
\frac{d}{dR} C = 4\pi R b \left( \rho(R_b) + 6p(R_b) - \frac{\Lambda}{2\pi} \right) + \frac{3m}{R_5^2}. \hspace{1cm} (16)
\]

but this implies \( \frac{d}{dR} C > 0 \) at \( R_b \), due to the condition \( \Lambda < \frac{3m}{R_5^2} \). This contradiction proves that \( C(R) \) is strictly positive outside the sonic sphere.

The positivity of \( C \) allows one to conclude — from Eq. (13) — that \( \partial_R a^2 \) is strictly decreasing outside the sonic sphere. This in turn implies the decrease of the pressure \( p \) and the baryonic mass density \( \rho_0 \) (see the remark under Eq. (10)).

**Lemma 2.**

Assume \( \Lambda R^2 > \frac{18\pi a^2}{\Gamma-1-a^2} \) and define \( R_0 = (\frac{6m}{\Lambda})^{\frac{1}{3}} \). The necessary condition for the existence of solutions is \( \alpha \leq 1 \).
If \( \alpha > 1/3 \) then \( R_0 \ll R_\infty \) and the speed of sound, the pressure and the mass densities are increasing in the interval \( (R_0, R_\infty) \):

\[
\frac{d}{dR} X > 0 \quad \text{for} \quad X = a^2, p \quad \text{and} \quad \rho_0.
\]

**Proof of the Lemma 2.**

The estimate \( R_0 \ll R_\infty \) follows immediately from the boundary condition \( a^2_\infty \gg m/R_\infty \) and the assumed value of \( \Lambda \). In the asymptotic end \( (R_0, R_\infty) \) the function \( -C(R) \) can be bounded from below as follows,

\[
-C(R) = -\frac{m(R)}{R} + 2U^2(R) - 4\pi R^2 p + \frac{\Lambda R^2}{3} \geq \frac{-2m}{R} + \frac{\Lambda R^2}{3}.
\]

(17)

The second line follows from \( U^2 \geq 0 \) and from the observation that \( p \leq \rho - \Gamma \) thus \( 4\pi R^2 p \leq 4\pi \int_0^{R_\infty} drr\rho \) and \( 4\pi \int_0^{R_\infty} drr\rho \leq \frac{m}{\pi} \).

Taking into account the last estimate, we can estimate from below the right hand side of Eq. (13) by

\[
D(R) = \frac{\Gamma - 1 - a_{\infty}^2}{a^2_{\infty}} \left( -\frac{2m}{R^2} + \frac{\Lambda R^2}{3} \right).
\]

(18)

We obtain from Eqs (13) and (18) the inequality

\[
\frac{d}{dR} a^2 \geq \left( \Gamma - 1 - a_{\infty}^2 \right) \left( -\frac{2m}{R^2} + \frac{\Lambda R^2}{3} \right).
\]

(19)

It is clear that i) \( R_0 \) is the null point of \( D(R) \); ii) the function \( D(R) \) is strictly positive for \( R \in (R_0, R_\infty) \). Therefore \( a^2 \) and consequently the remaining gas characteristics \( p \) and \( \rho_0 \) do increase in this interval, provided that a solution does exist.

The necessary condition for the existence of a solution is that \( a(R_0) \) should be strictly positive. Integration of Eq. (19) between \( R_0 \) and \( R_\infty \), yields

\[
a_{\infty}^2 - a^2(R_0) \geq (\Gamma - 1 - a_{\infty}^2) \times \left( \frac{2m}{R_0} - \frac{\Lambda R_0^2}{6} + \frac{2m}{R_\infty} + \frac{\Lambda R_\infty^2}{6} \right).
\]

(20)

We can replace \( \frac{2m}{R_0} \) by \( \frac{\Lambda R_0^2}{3} \), using the definition of \( R_0 \), and get the following

\[
a^2(R_0) \leq (\Gamma - 1 - a_{\infty}^2) \frac{\Lambda R_0^2}{18} \left( \frac{6R_0^2}{R_\infty^2} - 1 \right) + a_{\infty}^2 - (\Gamma - 1 - a_{\infty}^2) \frac{2m}{R_\infty} \leq a_{\infty}^2 + (\Gamma - 1 - a_{\infty}^2) \frac{\Lambda R_\infty^2}{18} \left( 6 \times 36\frac{1}{4} \frac{(m/R_\infty)^{\frac{1}{2}}}{(\Lambda R_\infty^2)^{\frac{1}{2}}} - 1 \right).
\]

(21)

In the second inequality we dropped the positive term with \( m/R_\infty \) and used the definition of \( R_0 \). We already assumed that \( m/R_\infty \ll a_{\infty}^2 \), which ensures that the first term within second bracket is small and the right hand side of the last inequality in (21) is approximated by

\[
E(R) \equiv a_{\infty}^2 - \frac{\Lambda R_\infty^2}{18} \left( \Gamma - 1 - a_{\infty}^2 \right).
\]

Therefore if the cosmological constant is given by \( \Lambda R_\infty^2 = 18\alpha \frac{a_{\infty}^2}{1-\alpha a_{\infty}^2} \), with \( \alpha > 1 \) then \( E(R) \) must be negative and the square of the speed of sound at \( R_0 \) must be nonpositive, which means that solutions are absent. On the other hand solutions can exist for \( 0.01 < \alpha < 1 \); in this case \( C(R) < 0 \) between \( (R_0, R_\infty) \), which implies that \( \frac{d}{dR} X < 0 \) for \( X = a^2, p, \rho_0 \) and \( \rho \).

The two lemmas proven in this section have a transparent physical interpretation. Lemma 1 assumes that the dark energy density \( \Lambda/4\pi \) is smaller than one half of the averaged matter density \( 3m/4\pi R_\infty^2 \). The presence of \( \Lambda \) is expected to have some quantitative impact on accretion mass rate \( M \), but qualitative features of the flow are not influenced — in particular the gas characteristics \( a, \frac{|U|}{R}, \rho_0 \) and \( \rho \) are all decreasing functions.

Lemma 2 assumes the opposite — that the dark energy density \( \Lambda/4\pi \) is much larger than the average matter density \( 3m/4\pi R_\infty^2 \). Above a particular value — that depends on the boundary characteristics of the flow, its volume and the black hole mass — steady accreting solutions do not exist. This is intuitively understandable — large \( \Lambda \) implies a large Hubble expansion velocity which can obstruct or even prohibit accretion, through the junction condition on the boundary of the vacuole. Therefore one can expect a significantly smaller mass accretion rate \( M \) and even its absence.

V. QUALITATIVE ANALYSIS OF SOLUTIONS: SADS BLACK HOLES

**Lemma 3.** Assume \( \Lambda < 0 \). Then:

i) for \( R > R_\ast \) the speed of sound, the pressure and the mass densities \( \rho_0 \) and \( \rho \) are decreasing, \( \frac{d}{dR} X < 0 \) for \( X = a^2, p \) and \( \rho_0 \).

ii) Solutions are absent if \( \Lambda R_\infty^2 \leq 6\Gamma - 1 - a_{\infty}^2 \).

**Proof.**

Part i) The key observation is that the function \( C \) is strictly positive, if \( \Lambda < 0 \). Indeed, let \( R_b \geq R_\ast \) be a smallest radius at which \( C(R_b) \) vanishes, and assume \( C < 0 \) in the open interval \( (R_\ast, R_b) \) (notice that \( C(R_\ast) \) vanishes). But Eq. (15) would imply in such a case that both terms with derivatives become negative in an subinterval \((R_\ast, R_1)\) of \( (R_\ast, R_b) \), that is — see the remark below (10) — \( \frac{d}{dR} a^2 < 0 \). But this in turn would contradict Eq. (13) — if \( C < 0 \) then \( \frac{dx}{dt} a^2 > 0 \) outside the sonic sphere. Thus we arrive to the contradiction in this sub-interval \((R_\ast, R_1)\) and the argument of continuity and the definition of \( R_b \) imply the positivity of \( C \) in the open interval \((R_\ast, R_b)\).

In the next step we show that \( C \) cannot vanish outside the sonic sphere. Indeed, (13) and (15) imply that \( \frac{dC}{dR} \) is strictly positive whenever \( C = 0 \) for \( R > R_\ast \); that in turn yields (by the argument of continuity) the strict positivity of \( C \) outside the sonic sphere. Eq. (13) allows
one to conclude that the speed of sound is monotonically decreasing, and that implies also decrease of the mass density $\rho_0$ and the pressure.

Part ii) In order to prove the necessary existence condition, notice that Eq. (13) gives, after integration,

$$|a_0^2 - a^2(R)| \geq \int_R^{R_\infty} dr \left| (\Gamma - 1 - a^2) \frac{1}{r} \left( -2U^2 - \frac{\Lambda R^2}{3} \right) \right|.$$  (22)

Notice that $a^2 < \Gamma - 1$; thus $|a_0^2 - a^2(R)| \leq \Gamma - 1 - a_0^2$. The conservation of the mass accretion rate $M$ and $\frac{d}{dr} \rho_0 \leq 0$ lead to the estimate $\frac{d}{dr} \left( U^2 R^4 \right) \geq 0$, and this in turn implies inequality

$$U^2 \leq U_\infty^2 \frac{R_\infty^4}{R^4}.$$  (23)

Collecting this information, one finally obtains

$$\Gamma - 1 - a^2 \geq \left( \Gamma - 1 - a^2(R) \right) \left( \frac{U^2}{2} - \frac{\Lambda R^2}{6} \left( 1 - \frac{R^2}{R_\infty^2} \right) \right).$$  (24)

Let $R = R_*$ and $\frac{\Delta R^2}{6} < \frac{-1 - a^2}{1 - a^2}$; then — taking into account that $\frac{d^2}{dr} \rho_0$ is negligibly small, and the same is true for the velocity term $U^2$ — we conclude that the right hand side of Eq. (22) exceeds its left hand side. This contradiction proves the necessary condition of Lemma 3.

VI. DESCRIPTION OF NUMERICS

Our aim is to find transsonic accretion flows. We assume the total mass $m = 1$, $\Gamma = 4/3$ and $R_\infty = 10^6$. The boundary values at $R_\infty$ are $k = N = \sqrt{1 - \frac{2m}{R_\infty} - U_\infty^2 - \frac{\Lambda}{3} R_\infty^2}$ and $a_0^2 = 2 \times 10^{-4}$. There are three remaining parameters that are free: the cosmological constant $\Lambda$, the mass accretion rate $M$ and the asymptotic density $\rho_0$ — after specifying them, equations are integrated inward, starting from $R_\infty$. These quantities should be such as to ensure the asymptotic conditions: $|U_\infty| \ll \sqrt{2m/R_\infty} \ll a_0$.

A numerical run, with the above boundary data, has been regarded as successful, if:

i) the integration started from $R_\infty$ and reached the outermost apparent horizon, that is, the point $R_i$, such that $\frac{2m(R_i)}{R_i} + \frac{\Lambda R_i^2}{3} = 1$, along the accretion branch (with the increasing — in the direction to the centre — infall velocity $|U|$);

ii) there appears a sonic point at a radius $R_s \geq R_i$.

More specifically, in order to find a single transsonic flow, we fix $\Lambda$ and the mass accretion rate $M$, and treat the asymptotic value of the baryonic mass density as a free parameter. For a random choice of $\rho_0$, there are three possibilities, one exceptional and two generic:

a) an almost unlikely event, that the numerical run is successful in the sense defined above and we find the sought accretion flow;

b) the solution numerically fails — a singularity appears during integration;

c) there exists a numerical solution without a sonic point — a piece of the solution describes accelerating accreting flow, while the other corresponds to a decelerating accreting gas.

In the case a) the job is done. If one of the remaining generic alternatives occurs — say b), with the asymptotic density $\rho_0$ — we change $\rho_0$ until we get the situation described in c) with the asymptotic density $\rho_0$. Now we use the bisection method in order to find an intermediate value $\rho_{0i}$ between the two values $\rho_0$ and $\rho_{0i}$, for which the numerical run is successful and the transsonic flow exists.

The numerics itself is standard — we use commonly known algorithms of the 8th order.

The calculations had been performed for a few dozens of values — negative and positive — of the cosmological constant $\Lambda$. In the sector of negative cosmological constant we take $\Lambda \in (-5 \times 10^{-13}, -10^{-13})$: the absolute value over bound is larger by one order of magnitude than the value found in Lemma 3 as a necessary existence condition. In the positive part of the spectrum we choose $\Lambda \in (10^{-19}, 7.2 \times 10^{-15})$: this is quite close to the value established as the necessary condition in Lemma 2. For each fixed value of $\Lambda$ we find about 100 solutions corresponding to different values of $M$.

The obtained information is summarized in figures 1 and 2. The absissa shows the ratio $x = M_\gamma/m$ of the mass $M_\gamma = 4\pi \int_{R_\infty}^{R_\infty} dr \rho$ of gas to the total mass $m$. The total mass has been normalized to 1. We obtain, although in a less transparent form, the same result as in (5) — the maximum of the mass accretion rate corresponds to $x \approx 1/3$. For a fixed value of $M_\gamma$ there exist two flows, one with heavier center and lighter gas, and the other with opposite characteristics. Just around the maximum of $M_\gamma$ there are blank points — we find that in this region it is difficult to find a numerical solution. We think that this a purely numerical artefact. There is a slight shift of this maximum towards smaller values of $x$ with the increase of the absolute value of the cosmological constant.

The only new relevant physically information is that concerning the mass accretion rate $M$. It appears that the presence of the cosmological constant manifests only when the dark energy density $\Lambda/(4\pi)$ exceeds the averaged matter density $3m/(4\pi R_\infty^3) \approx 4 \times 10^{-19}$ by at least three orders of magnitude. The mass accretion rate $M$ is decreasing with the increase of the absolute value of the cosmological constant. It is clear that the rate of the falloff depends on the sign of $\Lambda$ — it is faster for positive values — but the falloff itself occurs both for positive and negative cosmological constant.

VII. COSMOLOGICAL IMPLICATIONS

Results that are reported in the preceding section demonstrate that while the mass accretion rate depends
on the cosmological constant, this effect becomes significant only when the dark energy fraction $\Omega_\Lambda$ is much larger than the material fraction $\Omega_m$. On the other hand, this impact can be dramatic, a sevenfold increase of $\Lambda$ — starting from its value corresponding to $\Omega_\Lambda/\Omega_m \approx 10^3$ — leads to the diminishing of $\dot{M}$ by seven orders in magnitude, as seen in Fig. 1. This is so for the accretion onto SdS black hole, but the phenomenon is strong also in the case of SAdS (Fig. 2). When applying these results to our Universe, we have to take into account, that $\Omega_\Lambda \gg \Omega_m$ in two epochs — during inflation and after $10^{11}$ years from Big Bang.

Inflation strictly excludes steady accretion onto primordial black holes, due to Lemmas 2 and 3, because during inflation era $\Omega_\Lambda$ exceeds $\Omega_m$ by something like 50 orders of magnitude. This fact is concordant with the well known freezing of structures that are bigger than a particle horizon $19$, but the difference is that now it applies to small accretion systems that are located well within the horizon.

In the present Universe the dark energy and material densities are roughly the same, which suggests that the effect of dark energy is negligible. The mass density of largest bound structures - galactic superclusters - exceeds the cosmological mass density by one order of magnitude. Notice, however, that the tenfold ageing of the Universe would increase the ratio $\Omega_\Lambda/\Omega_m$ by a factor of 100. That means that in the Universe older than $10^{11}$ years the steady accretion would become less efficient and at the time $t \rightarrow 10^{12}$ years its efficiency goes to zero.

Penrose conjectured the so-called Weyl curvature hypothesis $20$. It asserts, in its informal version, that a Friedman-Lemaitre-Robertson-Walker spacetime, where the Ricci curvature is nonzero but the Weyl curvature vanishes, evolves towards a vacuum spacetime filled with black holes and gravitational radiation — with nonvanishing Weyl curvature and negligible Ricci tensor. (A more formal scenario has been worked out by Tod $21$). This picture assumes that initially small inhomogeneities of FLRW universe accrete matter and transform themselves into black holes, that gradually merge, leaving at the end a net of huge black holes and a lot of gravitational radiation. A cautious interpretation of our findings would be to say that the role of steady accretion in the realization of this scenario, in the presence of dark energy, is insignificant. It might well happen, however, that this property of dark energy of diminishing the steady spherical accretion signals a more general feature — that dark energy damps any accreting process. In such a case there arises another fundamental question: how dark energy impacts this scenario outlined in the Weyl curvature hypothesis?

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