Axial Currents of Virtual Charm in Light Quark Processes

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Abstract

The systematic investigations of the role of the virtual charm axial currents in the decay of $B$-mesons to $\eta'$ $K$ and the spin structure of the nucleon are performed. We reduce the divergence of the virtual charm axial current to a specific gluon operator. Since this operator receives the main contribution from topologically nontrivial components of the QCD vacuum, we rederive the Diakonov-Petrov Effective Action, based on the instanton QCD vacuum model. This action is applied to the calculations of the coupling of $\eta'$ to the charm axial current and found $f^{(c)}(\mu \simeq m_c) = -(12.3 \sim 18.4 \text{MeV})$, providing a possibility for the explanations of the recent experimental data on $B \rightarrow \eta' K$-decay. Analogous calculations of the virtual charm content of the nucleon spin leads to $\Delta c(\mu \simeq m_c) = -(0.015 \sim 0.024)$, which is one order of magnitude smaller than the analogous contribution of the strange quark.

13.25.Hw, 14.65.Dw, 12.38.Lg, 12.39.Fe

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I. INTRODUCTION

The processes sensitive to the OZI rule violations and the contributions of the nonvalence components of hadrons become of great interest. In the present paper, we would like to discuss the role of the charm content of the light hadrons in the $B$-meson decay to $\eta'$ and $K$-mesons and in deep–inelastic scattering with polarized charged leptons on polarized nucleons. In both cases there are no charmed hadrons both in the initial and the final states, which means that a charm may give a contribution only through virtual processes.

Decay of $B$-mesons to $\eta' K$

Recently there is a great theoretical interest \[3–13\] on the new experimental data on the branching ratios of the decays of $B \rightarrow \eta' K[1, 2]$:

$$\text{Br}(B^\pm \rightarrow \eta' K^\pm) = (6.5^{+1.5}_{-1.0} \pm 0.9) \times 10^{-5}, \quad (1.1)$$
$$\text{Br}(B^0 \rightarrow \eta' K^0) = (4.7^{+2.7}_{-2.0} \pm 0.9) \times 10^{-5}. \quad (1.2)$$

In the Standard Model, the Cabibbo favored $b \rightarrow \bar{c}c\bar{s}$ elementary process may be followed by conversion of $\bar{c}c$ pair into $\eta'$ through gluons. The amplitude of this process is described by

$$M = \frac{G_F}{\sqrt{2}} V_{cb} V^*_{cs} a_1 \langle \eta' | \bar{c} \gamma_\mu \gamma_5 c | 0 \rangle \langle K | \bar{s} \gamma_\mu b | B (p + q) \rangle. \quad (1.3)$$

Here $G_F$ is the weak coupling constant, $V_{cb}$, $V^*_{cs}$ are Kobayashi-Masukawa matrix elements and $a_1 = 0.25$ is the phenomenological number obtained from a fit to experiments (see \[3\] for the references). The matrix element

$$\langle \eta' | \bar{c} \gamma_\mu \gamma_5 c | 0 \rangle = -i f^{(c)}_{\eta'} p_\mu \quad (1.4)$$

is non-zero due to the virtual $\bar{c}c \rightarrow \text{gluons}$ transitions. Certainly, this matrix element is suppressed by $1/m_c^2$ factor. However, due to the presence of strong nonperturbative gluon fields in the QCD vacuum together with the Cabibbo favored $b \rightarrow c$ transition the suggested $b \rightarrow \bar{c}c\bar{s}$ mechanism (1.3) may be expected to compete appreciably with the other mechanisms of the $B \rightarrow K\eta'$ process \[9, 12\].

If we assume the dominance of the virtual charm mechanism (1.3), the branching ratio is written in terms of $f^{(c)}_{\eta'}$ as \[3\]

$$\text{Br}(B \rightarrow K\eta') \simeq 3.92 \cdot 10^{-3} \cdot \left( \frac{f^{(c)}_{\eta'}}{1 \text{ GeV}} \right)^2. \quad (1.5)$$

Using the experimental data, (1.1), it is found $f^{(c)}_{\eta'} \simeq 140 \text{ MeV}$ ("exp").

This value perfectly coincides with the estimate of Halperin and Zhitnitsky \[3\]:

$$f^{(c)}_{\eta'} = (50 - 180) \text{ MeV}. \quad (1.6)$$

On the other hand, a recent phenomenological study placed another bound on $f^{(c)}_{\eta'}$, namely $-65 \text{ MeV} \leq f^{(c)}_{\eta'} \leq 15 \text{ MeV}$, with $f^{(c)}_{\eta'}$ being consistent with zero by analyzing the $Q^2$
evolution of the $\eta'\gamma$ form factor [7], and more recently it was estimated from observed ratio of $J/\psi$ decay to $\eta'$ and $\eta_c$ the value of $f_{\eta'}^{(c)} = -(6.3 \pm 0.6)\text{MeV}$ [8]. Other similar estimation which leads to $|f_{\eta'}^{(c)}| < 12\text{MeV}$ was made in [13]. Ali et al. considered the complete amplitude for the exclusive $B-$meson decays, including $\eta'K$ channels, where they combined the contribution from the process $b \to s(\bar{c}c) \to s(\text{gluons}) \to s\eta'(\gamma)$ with all the others [5, 6]. Their estimations gave $|f_{\eta'}^{(c)}| \approx 5.8 \text{MeV}[5]$ and $f_{\eta'}^{(c)} = -3.1 (-2.3) \text{MeV}$ (for $m_c$ in the range 1.3 - 1.5 GeV)[6] in agreement with the analysis[7]. They stressed the importance of the sign of $f_{\eta'}^{(c)}$ and found a theoretical branching ratio in the range

$$Br(B \to \eta'K) = (2 - 4) \times 10^{-5},$$

which is somewhat smaller than the experimental one (1.1). The similar analysis made in [10] led the authors to conclude that $f_{\eta'}^{(c)} = -50\text{MeV}$ might provide the explanation of the data.

Having this situation, it is important to recalculate $f_{\eta'}^{(c)}$ to clarify the mechanism of $B \to K\eta'$ decay in the similar framework performed by Halperin and Zhitnitsky [3].

**Polarized DIS**

In the constituent quark model, the spin of the proton is supposed to be carried by $q = u, d$ valence quarks so that $\Delta \Sigma = \Delta u + \Delta d = 1$. The quark spin $\Delta q$ is defined as

$$\Delta q 2m_N s^\mu = \langle p, s | \overline{q} \gamma^\mu \gamma_5 q | p, s \rangle,$$

where $m_N$ and $s^\mu$ are the mass and the spin of the nucleon, respectively. On the other hand,

$$\Delta q = \int_0^1 dx \ (q_R(x) - q_L(x)),$$

where $q_R(L)(x)$ are quark distributions of chirally right-handed (left-handed) quarks in a polarized proton.

Deep inelastic scattering (DIS) with polarized charged leptons on polarized targets provides the investigations of the quark distributions $g_{R(L)}$. These quantities are extracted from the structure function $g_1(x, Q^2)$ measured in polarized DIS using the parton model relation

$$g_1(x, Q^2) = \frac{1}{2} \sum_q q^2_q \ (q_R(x) - q_L(x)).$$

So called "spin crisis" problem is related with a very large disagreement between the experiments and the prediction of the naive constituent quark model for the first moment of the proton(neutron) spin structure function $\Gamma_1^{p(n)}$ defined by

$$\Gamma_1^{p(n)} \equiv \int dx g_1^{p(n)}(x) = +(-) \frac{1}{12} g_A^3 + \frac{1}{36} g_A^8 + \frac{1}{9} g_A^0.$$

Here,

$$g_A^3 = \Delta u - \Delta d = 1.25$$

and

$$g_A^8 = (\Delta u + \Delta d - 2\Delta s) = 0.69$$

are the isovector and the octet axial charges measured from neutron and hyperon decays and

$$g_A^0 = \Delta \Sigma = (\Delta u + \Delta d + \Delta s).$$
The expectation was $g_0^A = 1$ but the measured value of this quantity given by EMC in 1988 implied that $g_0^A \sim 0.12$. It means that only $\sim 12\%$ of the spin of the proton is carried by its quarks! The modern value is $g_0^A \sim 0.3$[15–17].

In all of the consideration of the nucleon spin the contribution of the charm was neglected completely. With the account of this contribution

$$g_0^A = \Delta \Sigma + 2\Delta c$$

and we have now the problem to estimate also $\Delta c$. We may expect sizable value of $\Delta c$, since in this case we are dealing again with axial currents which may be strongly affected by the vacuum nonperturbative gluon fields. The previous calculations of $\Delta c$ by Halperin and Zhitnitsky [18] and Blotz and Shuryak [19] gave quite different results $\Delta c \sim 0.3$ [18] and $\Delta c/\Delta \Sigma = -(0.2 \sim 0.08)[19]$. They contradicted each other in the sign and also absolute value. This is our motivation for the recalculation of the charm contribution to the spin of the nucleon.

Axial currents of virtual charm

The symmetry of the classical lagrangian may be destroyed by quantum fluctuations [20–22]. In gauge theories the axial anomaly arises from noninvariance of the fermionic measure against axial transformations in the path integrals of the theory[23] (see also ref.[24], concerning higher-loop corrections). The present problem is intimately related with this phenomenon.

In the following we will work only with the Euclidean QCD, where its convention is written in the footnote. In the Euclidean QCD the axial anomaly in the light quark axial current in the chiral limit reads

$$\partial_\mu \psi_f^\dagger \gamma_5 \gamma_\mu \psi_f = -i \frac{g^2}{16\pi^2} G \tilde{G}, \quad (1.9)$$

where $\psi_f$ is the light quark field ($f = u, d, s$) and $g$ is the QCD coupling constant. $2G \tilde{G} = \epsilon^{\mu\nu\lambda\sigma} G_{\mu\nu}^a G_{\lambda\sigma}^a$, where $G_{\mu\nu}^a$ is the gluon field strength operator with $a$ being the color index.

The situation with heavy quarks is very different, since we must take into account the contribution of the mass term. The divergence of the axial current of charmed quarks has a form:

$$\partial_\mu c^\dagger \gamma_\mu \gamma_5 c = -i \frac{g^2}{16\pi^2} G \tilde{G} + 2m_c c^\dagger \gamma_5 c, \quad (1.10)$$

The first term in (1.10) again comes from noninvariance of the fermionic measure (or in other words - from Pauli-Villars regularization). The main problem here is to calculate the contribution from the second term in (1.10). It is clear that this one is reduced to the problem of the calculation of the vacuum expectation value of the operator $2m_c c^\dagger \gamma_5 c$ in the presence of a gluon fields.

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$^{1}i\cdot x_{M0} = x_{E4}, \quad x_{M4} = x_{E4}, \quad A_{M0} = iA_{E4}, \quad A_{M4} = -A_{E4}, \quad \psi_M = \psi_E, \quad i\bar{\psi}_M = \psi_E^\dagger, \quad \gamma_{M0} = \gamma_{E4}, \quad \gamma_{M4} = i\gamma_{E1}, \quad \gamma_{M5} = \gamma_{E5}.$ In the following we will omit index $E$. }


In the path integral approach, the calculation of the contribution of this term to any matrix element over light hadrons may be considered in the sequence of the integrations. In the first step, the integration over c-quark is performed, and the next step is the calculation of the integral over gluon fields and finally the integration over light quarks.

We consider here the first step – the integration over c-quarks. The resulting \( \langle 2m_c c\gamma_5 c \rangle \) must be a gauge invariant function of the gauge field \( A \) and therefore must be expressed through the gluon field strength tensor and their covariant derivatives. The matrix elements of such types of the operators are almost completely defined by nonperturbative topologically nontrivial (like instantons) contributions at least for the small virtualities flowing through these operators, which is our case. This one can be clearly justified by the consideration of low-energy theorems for the various matrix elements of the gluonic operator in (1.9) [25]. It was shown that only with account of the instanton-like configurations of vacuum gluonic fields, it is possible to satisfy these theorems and that Diakonov & Petrov (DP) chiral quark Effective Action [28, 29] based on the QCD instanton vacuum model [32, 33] reproduce well these low-energy theorems in the chiral limit but fail beyond this limit [26, 27]. To our present knowledge the instanton structure of QCD vacuum is characterized by the average size \( \rho \) and by the average inter-instanton distance \( R \), which are [33, 32]

\[
\rho = 1/3 \text{ fm}, \quad R = 1 \text{ fm}.
\]

Therefore the packing parameter \( (\rho/R)^4 = 0.012 \) is small, legitimatizing independent averaging over positions and orientations of the instantons.

In the present paper we first calculate gluonic operators in the divergence of the virtual charm axial currents, further we rederive the DP Effective Action starting from the Lee&Bardeen result for the quark propagator in the instanton media and apply this action for the calculations of the correlators with gluonic operators. Finally we apply these results to the calculations of the virtual charm effects in above mentioned problems of the \( B \)-decay and DIS.

II. THE DIVERGENCE OF THE VIRTUAL CHARM AXIAL CURRENTS

We calculate the expectation value of the operator \( 2m_c c\gamma_5 c \) in the presence of gluon field. For this, we take the method, which was developed by Schwinger in electrodynamics many years ago[20], and later was applied to QCD by Vainshtein et al. [34]. The key point of this method is based on an assumption of a possibility of an expansion of the Green function such as \( \langle 2m_c c\gamma_5 c \rangle \) over \( G/m_c^2 \).

We introduce first the coordinate and momentum operators, \( X_\mu \) and \( p_\mu \), respectively, which satisfy \( [p_\mu, X_\nu] = i\delta_{\mu\nu}, [p_\mu, p_\nu] = [X_\mu, X_\nu] = 0 \). We define then the covariant momentum operator \( P_\mu \) satisfying the following commutation relations,

\[
[P_\mu, X_\nu] = i\delta_{\mu\nu}, \quad [P_\mu, P_\nu] = igG^a_{\mu\nu}t^a , \quad (2.1)
\]

where \( t^a \) is a generator of the color group and \( G^a_{\mu\nu} \) is the gluon field strength tensor. Moreover, we introduce a formal complete set of states \( |x\rangle \) as the eigenstates of the coordinate operator \( X_\mu \),

\[
X_\mu |x\rangle = x_\mu |x\rangle , \quad (2.2)
\]
which satisfies
\[ \langle y | x \rangle = \delta^{(4)}(x - y), \quad \int d^4x |x| = 1. \tag{2.3} \]

In this basis, the operator \( P_\mu \) acts as a covariant derivative \( D_\mu \),
\[ \langle y | P_\mu | x \rangle = iD_\mu \langle y | x \rangle \equiv \left( i\partial_\mu + gA_\mu^a(x)t^a \right) \delta^{(4)}(x - y). \tag{2.4} \]

The algebra (2.1) is the basic tool of the Schwinger formalism. We expand the Green functions in the gluon background field, and need to use only this algebra in each order of expansion.

Next, we calculate the expectation value of the operator \( 2m_c c^\dagger(x)\gamma_5 c(x) \), which is the Green function that should be expanded by using the Schwinger method. We consider the integration over \( c \)-quarks. In the path integral approach, we define:
\[ \langle 2m_c c^\dagger(x)\gamma_5 c(x) \rangle = \int DcDc^\dagger 2m_c c^\dagger(x)\gamma_5 c(x) \exp \left\{ \int d^4y c^\dagger(y)(\hat{P} + im_c)c(y) \right\} \tag{2.5} \]

with \( \hat{P} \equiv P_\mu \gamma_\mu \). Since the argument of the exponential is quadratic in the quark field \( c \), the path integral (2.5) can be written in the form,
\[ \langle 2m_c c^\dagger(x)\gamma_5 c(x) \rangle = 2m_c \text{det} \| \hat{P} + im_c \| \langle x | \text{Tr} \gamma_5 \frac{1}{\hat{P} + im_c} | x \rangle, \tag{2.6} \]

where \( \text{Tr} \) denotes the trace over spin and color indices. Since \( x \) is a continuous variable, the operator has an infinite number of matrix elements, and in calculating the determinant of this matrix there arise infinities of various types. Hence, the determinant, \( \text{det} \| \hat{P} + im_c \| \), must be regularized in the standard manner as
\[ \text{det} \| \hat{P} + im_c \| \longrightarrow \text{det} \left\| \frac{(\hat{P} + im_c)(\hat{P} + iM)}{(\hat{P} + im_c)(\hat{P} + iM)} \right\|, \]

where \( M \) is the Pauli-Villars regulator mass. Eq.(2.6) must be gauge invariant and expressed through the gluon field strength tensor and the covariant derivatives.

Eq.(2.6) is expanded in the series of a power of \( G/m_c^2 \) under the assumption that the field strength \( G_{\mu\nu}^a \) is much less than the square of \( c \)-quark mass \( m_c^2 \). Here, we will take into account \( O(G^2) \) and \( O(G^3) \) terms in the expansion of (2.6). We start from the calculation of
\[ H(x) \equiv 2m_c \langle x | \text{Tr} \gamma_5 \frac{1}{\hat{P} + im_c} | x \rangle. \tag{2.7} \]

The calculations of the \( \text{det} \| \hat{P} + im_c \| \) is presented in the Appendix, since it give a contribution to (2.6) starting \( O(G^4) \) terms.

Using the formulas
\[ \hat{P}^2 = \hat{P}^2 + \frac{g}{2} \sigma G \quad \text{and} \quad I = \frac{1}{\hat{P} - im_c} (\hat{P} - im_c), \]

where \( \sigma G \equiv \sigma_{\mu\nu} G_{\mu\nu}, \sigma_{\mu\nu} \equiv \frac{i}{2} [\gamma_\mu, \gamma_\nu] \) and \( I \) is an identity matrix, Eq.(2.7) reduces to
\[ H(x) = 2m_c \langle x | \text{Tr} \gamma_5 \frac{1}{\hat{P} + im_c \hat{P} - im_c} (\hat{P} - im_c) | x \rangle \]
\[ = -2im_c^2 \langle x | \text{Tr} \gamma_5 \frac{1}{\hat{P}^2 + m_c^2 + \frac{2}{2} \sigma G} | x \rangle \]
\[ = -2im_c^2 \langle x | \text{Tr} \gamma_5 \left\{ \frac{1}{\hat{P}^2 + m_c^2} \sigma G \frac{1}{\hat{P}^2 + m_c^2} \sigma G \frac{1}{\hat{P}^2 + m_c^2} \right\} | x \rangle \]
\[ \equiv H_2(x) + H_3(x) + \cdots . \]

It has been used here that the trace of an odd product of \( \gamma \) matrices vanishes.

It is straightforward to calculate the second term of the right hand side of (2.8), if neglect the noncommutativity of the operators \( \mathcal{P}_\mu \) and \( G_{\mu\nu} \) in (2.8). Thus,
\[ H_3(x) = \frac{ig^3 m_c^2}{2^2} \langle x | \text{Tr} \gamma_5 \frac{1}{(\hat{P}^2 + m_c^2)^4} (\sigma G)^3 | x \rangle . \]

In that case, since the operator \( \mathcal{P}_\mu \) can be replaced by the ordinary momentum \( p_\mu \), we can use the evident formulas,
\[ \langle x | \frac{1}{(\mathcal{P}^2 + m_c^2)^n} | x \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(\mathcal{P}^2 + m_c^2)^n} |_{A=0} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m_c^2)^n} = \frac{1}{2^n n! (n-1)! (n-2)!} m_c^{2(n-2)} \]
and
\[ \text{Tr} \gamma_5 (\sigma G)^3 = 2^5 i \text{tr}_c G\bar{G}G - 2^3 f_{abc} G^a \bar{G}b G^c , \]
where \( G\bar{G}G = G_{\mu\nu} \bar{G}_{\nu\alpha} G_{\alpha\mu} \) and \( f_{abc} \) is the structure constant of the color group. As a result,
\[ H_3(x) = -\frac{ig^3}{2^4 \cdot 3\pi^2 m_c^2} f_{abc} G^a \bar{G}b G^c . \]

We still have to calculate \( H_2(x) \) which contains the gluon field strength \( G_{\mu\nu} \) to the second power. However, the calculation of \( H_2(x) \) needs much more efforts owing to the noncommutativity of the operators, which is not negligible. For the sake of the systematic momentum integration of \( H_2(x) \), one must transfer all operators containing \( \mathcal{P}_\mu \) to the left hand side in the trace. Using the following expansion :
\[ \sigma G \frac{1}{\mathcal{P}^2 + m_c^2} \sigma G = \frac{1}{\mathcal{P}^2 + m_c^2} \sigma G + \frac{1}{(\mathcal{P}^2 + m_c^2)^2} [\mathcal{P}^2, \sigma G] \frac{1}{\mathcal{P}^2 + m_c^2} \sigma G + \frac{1}{(\mathcal{P}^2 + m_c^2)^3} [\mathcal{P}^2, [\mathcal{P}^2, \sigma G]] \frac{1}{\mathcal{P}^2 + m_c^2} \sigma G + \cdots , \]
we can rewrite $H_2(x)$ as

$$
H_2(x) = -\frac{ig^2m_c^2}{2}\langle x|\text{Tr}\gamma_5\left\{\frac{1}{(\mathcal{P}^2 + m_c^2)^3}(\sigma G)^2 + \frac{1}{(\mathcal{P}^2 + m_c^2)^4}(2[\mathcal{P}^2, \sigma G]\sigma G + \sigma G[\mathcal{P}^2, \sigma G])
\right. \\
\left. + \frac{1}{(\mathcal{P}^2 + m_c^2)^3}(3[\mathcal{P}^2, [\mathcal{P}^2, \sigma G]]\sigma G + 3[\mathcal{P}^2, \sigma G]^2 + \sigma G[\mathcal{P}^2, [\mathcal{P}^2, \sigma G]]) + \cdots \right\}|x\rangle \\
\equiv h_1(x) + h_2(x) + h_3(x) + \cdots .
$$

(2.12)

The calculation of the commutators is performed systematically. For an arbitrary operator $\mathcal{Q}$, which satisfies $[\mathcal{P}_\mu, \mathcal{Q}] = i\mathcal{D}_\mu \mathcal{Q}$,

$$
[\mathcal{P}^2, \mathcal{Q}] = \mathcal{D}^2 \mathcal{Q} + 2i\mathcal{P}_\mu \mathcal{D}_\mu \mathcal{Q}.
$$

(2.13)

Repetitive use of this identity leads to

$$
[\mathcal{P}^2, [\mathcal{P}^2, \sigma G]] = \mathcal{D}^4 \sigma G + 2i\mathcal{D}_\rho \mathcal{G}_{\rho\mu} \cdot \mathcal{D}_\mu \sigma G + 2i\mathcal{P}_\nu (\mathcal{D}_\mu \mathcal{D}^2 + \mathcal{D}^2 \mathcal{D}_\mu)\sigma G \\
-4\mathcal{P}_\nu \mathcal{G}_{\nu\mu} \mathcal{D}_\mu \sigma G - 4\mathcal{P}_\nu \mathcal{P}_\mu \mathcal{D}_\nu \mathcal{D}_\mu \sigma G .
$$

(2.14)

In the following, we assume the quasi-classical vacuum and the source $\rho$, which is defined by $[\mathcal{P}_\mu, \mathcal{G}_{\mu\nu}] = i\mathcal{D}_\mu \mathcal{G}_{\mu\nu} = \rho$, is sufficiently small. So, the second term in (2.14) may be negligible. Since other higher commutators contribute in $H_2(x)$ as at least third power of $\mathcal{G}_{\mu\nu}$, we can neglect the corresponding terms which are denoted by dots in (2.12).

The momentum integration of each term can be performed easily since we need only both $\mathcal{O}(G^2)$ and $\mathcal{O}(G^3)$ terms. The calculation of the first term in $H_2(x)$, $h_1(x)$, is somewhat technical. By using the translational invariance of

$$
\langle x|\text{Tr}\gamma_5\frac{1}{\mathcal{P}^2 + m_c^2}(\sigma G)^2|x\rangle
$$

with an arbitrary momentum $q_\mu$ and extracting all $q^2$-terms as in Appendix, $h_1(x)$ is given as

$$
h_1(x) = -\frac{ig^2m_c^2}{2}\langle x|\text{Tr}\gamma_5\frac{1}{(\mathcal{P}^2 + m_c^2)^3}(\sigma G)^2|x\rangle = \frac{ig^2}{2\pi^2}\mathcal{G}^a\tilde{\mathcal{G}}^a + \mathcal{O}(G^4) .
$$

(2.15)

However, $\mathcal{O}(G^4)$ term may be neglected for our purpose. So we may regard the calculation of $h_1(x)$ as an integral over an ordinary momentum $p_\mu$, as we have calculated in (2.9). The term $h_1(x)$ contributes to cancel the axial anomaly from the noninvariance of the fermionic measure.

The second term in $H_2(x)$, $h_2(x)$, is rewritten as

$$
\begin{align*}
    h_2(x) &= -\frac{ig^2m_c^2}{2}\langle x|\text{Tr}\gamma_5\frac{1}{(\mathcal{P}^2 + m_c^2)^4}(2[\mathcal{P}^2, \sigma G]\sigma G + \sigma G[\mathcal{P}^2, \sigma G]) |x\rangle \\
    &= -\frac{ig^2m_c^2}{2}\left\{\langle x|\frac{1}{(\mathcal{P}^2 + m_c^2)^4}\text{Tr}\gamma_5\sigma G \cdot \mathcal{D}^2 \sigma G \\
    &+ 6i \langle x|\frac{1}{(\mathcal{P}^2 + m_c^2)^4}\mathcal{P}_\mu |x\rangle\text{Tr}\gamma_5\sigma G \cdot \mathcal{D}_\mu \sigma G \right\} + \text{(Total derivative)} .
\end{align*}
$$

(2.16)
We can omit here also the terms which contain a single operator $P_\mu$. The reason is that the matrix elements, $\langle x| (P^2 + m^2)^{-n} P_\mu | x \rangle$, must be denoted in terms of $G_{\mu\nu}$ and $D_\nu$. The first nonvanishing term which can give a contribution to this matrix element is $D_\mu G^2$. It is clear that it leads to $\mathcal{O}(G^3)$, which we do not calculate here.

By using the Bianchi identity, it is easy to show that

$$D^2 G_{\mu\nu} = -ig [G_{\alpha\mu}, G_{\alpha\nu}] + D_\mu D_\alpha G_{\alpha\nu} - D_\nu D_\alpha G_{\alpha\mu}. \quad (2.18)$$

where the second and third terms in the right-hand side may be neglected. The reason is

$$\text{Tr}\gamma_5 \sigma G \sigma_{\mu\nu} D_\mu D_\alpha G_{\alpha\nu} = \text{tr}_L \gamma_5 \sigma_{\lambda\rho} \sigma_{\mu\nu} \cdot \text{tr}_C G_{\lambda\rho} D_\mu D_\alpha G_{\alpha\nu}$$

$$= -4\varepsilon_{\lambda\mu\nu} \text{tr}_C G_{\lambda\rho} D_\mu D_\alpha G_{\alpha\nu}$$

$$= 4\varepsilon_{\lambda\mu\nu} \text{tr}_C D_\mu G_{\lambda\rho} \cdot D_\alpha G_{\alpha\nu} + (\text{Total derivative})$$

$$= 8\text{tr}_C D_\mu \tilde{G}_{\mu\nu} \cdot D_\alpha G_{\alpha\nu} + (\text{Total derivative})$$

$$= 0 + (\text{Total derivative}),$$

because of the Bianchi identity, $D_\mu \tilde{G}_{\mu\nu} = 0$. Then, $D^2 G_{\mu\nu}$ is the second power of $G$, and

$$D^2 \sigma G = -2ig \sigma_{\mu\nu} G_{\alpha\mu} G_{\alpha\nu}. \quad (2.19)$$

The solution of $h_2(x)$ is obtained as

$$h_2(x) = \frac{ig^3}{24 \cdot 3\pi^2 m_c^2} f_{abc} G^a \tilde{G}^b G^c = -H_3(x). \quad (2.20)$$

Hence, $h_2(x)$ cancels with $H_3(x)$ in (2.8). Thus, it will be only the term $h_3(x)$ in $H_2(x)$, which contribute to the divergence of the axial current of charmed quarks.

Our remaining work is to calculate only the third term in $H_2(x)$, namely,

$$h_3(x) = -\frac{ig^2 m_c^2}{2} \langle x | \text{Tr}\gamma_5 \frac{1}{(P^2 + m_c^2)^5} \left( 3[P^2, [P^2, \sigma G]] \sigma G \right. \left. + 3[P^2, \sigma G]^2 + \sigma G [P^2, [P^2, \sigma G]] \right) | x \rangle. \quad (2.21)$$

By using the expansion of the double commutator (2.14), products of commutators and $\sigma G$ satisfy the relation,

$$[P^2, [P^2, \sigma G]] \sigma G = -[P^2, \sigma G]^2 = \sigma G [P^2, [P^2, \sigma G]] = -4P_\nu P_\mu \sigma G D_\nu D_\mu \sigma G \quad (2.22)$$

except both higher order terms and nonessential total derivatives which are not needed in our purpose. So the first term in (2.21) cancels with the second term. Hence, $h_3(x)$ is reduced to

$$h_3(x) = 2ig^2 m_c^2 \text{Tr} \langle x | \frac{1}{(P^2 + m_c^2)^5} P_\nu P_\mu | x \rangle \gamma_5 \sigma G D_\nu D_\mu \sigma G. \quad (2.23)$$

Here, owing to the algebraic relation (2.5), an extra $G_{\mu\nu}$ could appear as $P_\nu P_\mu = \frac{1}{2} \{P_\nu, P_\mu\} + igG_{\nu\mu}$. However, since there is the $\mathcal{O}(G^3)$ content in the trace, we can neglect the noncommutativity in the matrix element of (2.23). Then the operator $P_\mu$ can be replaced by $p_\mu$ as we performed in the calculation of $H_3(x)$ before.
\[ \langle x \rangle \frac{1}{(P^2 + m_c^2)^n} P_\nu P_\mu \bigg|_{A=0} \langle x \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{p_\nu p_\mu}{(p^2 + m_c^2)^n} \delta_{\mu\nu} \bigg|_{A=0} = \frac{2^n \pi^2 (n-1)(n-2)(n-3)m_c^{2n-3}}{2^n \pi^2 (n-1)(n-2)(n-3)m_c^{2n-3}}, \]  \tag{2.24}

Substituting Eqs. (2.19) and (2.24) for (2.23), we obtain

\[ h_3(x) = -\frac{ig^3}{2^5 \cdot 3^n \pi^2 m_c^2} f_{abc} G^a \tilde{G}^b G^c. \]  \tag{2.25}

Hence,

\[ H_2(x) = h_1(x) + h_2(x) + h_3(x) \]
\[ = \frac{ig^2}{2^4 \pi^2} G^a \tilde{G}^a + \frac{ig^3}{2^5 \cdot 3^n \pi^2 m_c^2} f_{abc} G^a \tilde{G}^b G^c. \]  \tag{2.26}

Finally, since \( h_2(x) \) cancels \( H_3(x) \), we get

\[ H(x) = h_1(x) + h_3(x) \]
\[ = \frac{ig^2}{2^4 \pi^2} G^a \tilde{G}^a - \frac{ig^3}{2^n \pi^2 m_c^2} f_{abc} G^a \tilde{G}^b G^c. \]  \tag{2.27}

As expected, the first term in \( H(x) \) cancels with the first term in (1.10), which is the contribution from noninvariance of the measure and the rest part leads to the divergence of the \( c \)-quark axial current in the form

\[ \langle \partial_\mu c^\dagger(x) \gamma_\mu \gamma_5 c(x) \rangle = -\frac{ig^3}{2^n \pi^2 m_c^2} f_{abc} G^a \tilde{G}^b G^c. \]  \tag{2.28}

We would like to stress that our answer for \( \langle \partial_\mu c^\dagger(x) \gamma_\mu \gamma_5 c(x) \rangle \) is 6 times less than that of Halperin and Zhitnitsky [3].

### III. MATRIX ELEMENTS OF GLUONIC OPERATORS IN EFFECTIVE ACTION APPROACH

We face now the problem of the calculations of various matrix elements of the gluonic operators like those in (1.9) and (2.28).

First we rederive the Diakonov-Petrov (DP) Effective Action [28] as it was suggested in [27]. It is natural to choose the singular gauge for the instantons in describing many instanton effects in the propagation of the quarks. In the case of a small packing parameter it is possible to do the following sum ansatz for the background instanton field:

\[ A_{\mu}(x) = \sum_{+}^{N_+} A_{+,\mu}(x; \xi_+) + \sum_{-}^{N_-} A_{-,\mu}(x; \xi_-), \quad (\xi_\pm = (z_\pm, U_\pm, \rho_\pm)), \]  \tag{3.1}

where \( z_i, U_i \) and \( \rho_i \) are the position, orientation and size of the \( i \)-th instanton. The canonical partition function of the \( N_+ \) instantons and \( N_- \) anti-instantons can be schematically written as
\[ Z_{N_+,N_-} = \int \det_N \exp(-V_g) \prod_i d^4z_i dU_i d\rho_i, \]  

(3.2)

where \( V_g \) is the instanton-(anti)instanton interaction potential generated by the gluon field action and \( \det_N \) is a quark determinant in the instanton field. The main assumption of the instanton model is that \( V_g \) is repulsive at small distances between instanton and anti-instanton. This should provide the stabilization of the instanton sizes and of the inter-instanton distances. We mainly deal with \( \det_N \), which describes the influence of light quarks.

Lee&Bardeen [30] (LB) calculated the quark propagator in a more sophisticated approximation than DP, finding

\[ \det_N = \det B, \quad B_{ij} = im\delta_{ij} + a_{ji}, \]  

(3.3)

where \( a_{ij} \) is the overlapping matrix element of the quark zero-modes \( \Phi_{\pm,0} \) generated by instantons. This matrix element is nonzero only between instantons and anti-instantons (and vice versa) due to the chiral factor in \( \Phi_{\pm,0} \), i.e.,

\[ a_{++} = -\langle \Phi_{-,0} | i\hat{\theta}| \Phi_{+,0} \rangle. \]  

(3.4)

The overlap of the quark zero-modes causes quarks jumping from one instanton to another during their propagation.

Eq. (3.3) implies that for \( N_+ \neq N_- \) \( \det_N \sim m^{|N_+ - N_-|} \), so the fluctuations of \( |N_+ - N_-| \) are strongly suppressed due to the presence of light quarks. Therefore we assume \( N_+ = N_- = N/2 \).

Let us rewrite the \( \det_N \) following the idea suggested in [31]. First, by introducing the Grassmanian \((N_+,N_-)\) vector

\[ \Omega = (u_1...u_{N_+},v_1...v_{N_-}) \]

and

\[ \bar{\Omega} = (\bar{u}_1...\bar{u}_{N_+},\bar{v}_1...\bar{v}_{N_-}) \]

we can rewrite

\[ \det_N = \int d\Omega d\bar{\Omega} \exp(\bar{\Omega}B\Omega), \]  

(3.5)

where

\[ \bar{\Omega}B\Omega = \bar{\Omega}(im + a^T)\Omega = i \sum_+ m\bar{u}_+u_+ + i \sum_- m\bar{v}_-v_- + \sum_+(-\bar{u}_+v_-a_{--} + \bar{v}_-u_+a_{++}). \]  

(3.6)

The product \( \bar{u}_+v_-a_{--} \) can be rewritten in the form

\[ \bar{u}_+v_-a_{--} = -\langle i\hat{\theta}\Phi_{-,0}v_- | (i\hat{\theta})^{-1} | i\hat{\theta}\Phi_{+,0}u_+ \rangle. \]  

(3.7)

\( \Phi_{\pm,0} \) is the solution of the Dirac equation \( \hat{P}\Phi_{\pm,0} = 0 \) in the instanton(anti-instanton) field \( A_{\pm,\mu}(x;\xi_{\pm}) \).
The next step is to introduce $N_+, N_-$ sources $\eta = (\eta_+, \eta_-)$ and $N_-, N_+$ sources $\bar{\eta} = (\bar{\eta}_-, \bar{\eta}_+)$ defined as:

$$\bar{\eta}_- = \langle i \hat{\partial} \Phi_{-0} v_- \rangle, \bar{\eta}_+ = \langle i \hat{\partial} \Phi_{+0} u_+ \rangle$$

$$\eta_+ = |i \hat{\partial} \Phi_{+0} u_+ \rangle, \eta_- = |i \hat{\partial} \Phi_{-0} v_- \rangle.$$ 

Then $\exp(\Omega a^T \Omega)$ can be rewritten as

$$\exp(\Omega a^T \Omega) = \exp \int (-\bar{\eta}(i \hat{\partial})^{-1} \eta) = \left(\det(i \hat{\partial})\right)^{-1} \int \psi^\dagger(x) i \hat{\partial} \psi(x) - \bar{\eta}(x) \psi(x) + \psi^\dagger(x) \eta(x) \right) \quad (3.8)$$

Integrating over Grassmanian variables $\Omega$ and $\bar{\Omega}$ and taking into account the $N_f$ flavors $\det_N = \prod_f \det B_f$ this provides the fermionized representation of the Lee & Bardeen’s result for $\det_N$ in the form:

$$\det_N = \int \psi^\dagger \psi \exp \left( \int d^4 x \sum_f \psi_f^\dagger i \hat{\partial} \psi_f \right)$$

$$\times \prod_f \left\{ \left( \text{im}_f + V_+[\psi_f^\dagger, \psi_f] \right) \left( \text{im}_f + V_-[\psi_f^\dagger, \psi_f] \right) \right\}, \quad (3.9)$$

where

$$V_+[\psi_f^\dagger, \psi_f] = \int d^4 x \left( \psi_f^\dagger(x) i \hat{\partial} \Phi_{+0}(x; \xi_\pm) \right) \int d^4 y \left( \Phi_{+0,0}^\dagger(y; \xi_\pm) i \hat{\partial} \psi_f(y) \right). \quad (3.10)$$

Eq. (3.9) coincides with the ansatz for the fixed $N$ partition function postulated by DP, except for the sign in front of $V_\pm$. Keeping in mind the low density of the instanton media, which allows independent averaging over positions and orientations of the instantons, Eq. (3.9) leads to the partition function

$$Z_N = \int \psi^\dagger \psi \exp \left( \int d^4 x \psi^\dagger i \hat{\partial} \psi \right) W_+^{N_+} W_-^{N_-}, \quad (3.11)$$

where

$$W_\pm = \int d^4 \xi_\pm \prod_f \left( V_\pm[\psi_f^\dagger, \psi_f] + \text{im}_f \right)$$

$$(-i)^{N_f} \left( \frac{4 \pi^2 \rho^2}{N_c} \right)^{N_f} \int \frac{d^4 z}{V} \det_f \left( i J_{\pm}(z) - m N_c \right) \quad (3.12)$$

and

$$J_{\pm}(z)_{fg} = \int \frac{d^4 k d^4 l}{(2 \pi)^8} e^{-i(k-l)z} F(k^2) F(l^2) \psi_f^\dagger(k) \frac{1}{2}(1 \pm \gamma_5) \psi_g(l). \quad (3.13)$$

The form factor $F$ is related to the zero–mode wave function in momentum space $\Phi_{\pm}(k; \xi_\pm)$ and is equal to
\[
F(k^2) = -i \frac{d}{dt} [I_0(t)K_0(t) - I_1(t)K_1(t)], \quad t = \frac{1}{2} \sqrt{k^2 \rho}. \tag{3.14}
\]

**Correlators in the DP effective action**

In quasiclassical (saddle point) approximation any gluon operator receives its main contribution from instanton background. In the following the operator \(g^2G\tilde{G}(x)\) will be considered for the illustration of the method. Owing to the low density of the instanton medium, it is possible to neglect the overlap of the fields of different instantons. In that case, the matrix element of \(g^2G\tilde{G}(x)\) with any other quark operator \(Q\) is

\[
\langle g^2G\tilde{G}(x)Q \rangle_N = Z_N^{-1} \int D\psi D\psi^\dagger \exp \left( \int d^4x \psi^\dagger i\hat{\partial} \psi \right)
\times \left\{ N_+ \left( W_{G\tilde{G}+}(x)Q \right) W_{N+}^{N+} W_{N-}^{N-} + N_- \left( W_{G\tilde{G}-}(x)Q \right) W_{N+}^{N+} W_{N-}^{N-} \right\}, \tag{3.15}
\]

where

\[
W_{G\tilde{G} \pm} = \pm \left( \frac{4\pi^2 \rho^2}{N_c} \right)^{N_f} \int \frac{d^4z}{V} f_2(x-z) \det f \left( J_{\pm}(z) + \frac{m N_c}{4\pi^2 \rho^2} \right). \tag{3.16}
\]

and \(f_2(x-z)\) is defined as

\[
(g^2G\tilde{G}(x))_{\pm} = \pm f_2(x-z) = \pm \frac{192\rho^4}{[\rho^2 + (x-z)^2]^4}. \tag{3.17}
\]

It is useful to introduce the external field \(\kappa_{2(3)}(x)\), coupled respectively to \(g^2G\tilde{G}\) and \(g^3f_{abc}G^a\tilde{G}^bG^c\). Starting from (3.15) and (3.16), we find the partition function \(\hat{Z}[\kappa_{2(3)}]\) describing mesons[26] in presence of such external field:

\[
\hat{Z}[\kappa_{2(3)}] = \int D\Phi_+ D\Phi_- \exp (-W[\Phi_+, \Phi_-]), \tag{3.18}
\]

where

\[
W[\Phi_+, \Phi_-] = \int d^4x (w_a + w_b - w_c),\]

\[
w_a = (N_f - 1) \frac{N}{2V} \left( \prod_f M_f^{-1} \det \Phi_+ \right)^{N_f-1} + (\Phi_+ \rightarrow \Phi_-),
\]

\[
w_b = \frac{N_c}{4\pi^2 \rho^2} \text{Tr} \{m(\Phi_+ + \Phi_-)\},
\]

\[
w_c = \sum_f \text{Tr} \ln \frac{i\hat{\partial} + iF^2 (\Phi_+ \beta_+ + \Phi_- \beta_-)}{i\hat{\partial} + im_f},
\]

\[
\beta_\pm = \left[ \left( 1 \pm (\kappa_{2(3)} f_{2(3)}) \right)^{N_f-1} \right] \frac{1}{2}(1 \pm \gamma_5).
\]
The two remarkable formulas

\[(ab)^N = \int d\lambda \exp(N\ln\frac{aN}{\lambda} - N + \lambda b) \quad (N \gg 1). \quad (3.20)\]

and

\[\exp(\lambda \det[iA]) = \int d\Phi \exp \left[-(N_f - 1)\lambda^{-\frac{1}{N_f - 1}}(\det \Phi)^{\frac{1}{N_f - 1}} + itr(\Phi A)\right] \quad (3.21)\]

have been used here. It is possible to check these formulas by the saddle point approximation of the integrals. They were proposed in [29] and we followed this approach.

The saddle point of the integral (3.18) is located at \((\Phi_\pm)_f g = M_f \delta_{fg}\), a self-consistency condition for the effective quark mass, i.e.,

\[4N_c V \int \frac{d^4k}{(2\pi)^4} \frac{M_f^4(k^2)}{M_f^4(k^2) + k^2} = N + \frac{m_f M_f V N_c}{2\pi^2 \rho^2}, \quad (3.22)\]

being imposed, which describes also the shift of the effective mass of the quark \(M_f\) due to current mass \(m_f\). In the following we will neglect \(m_f\), since this model fails to reproduce properly the low-energy theorems beyond chiral limit [27].

The solution of a self-consistency equation (3.22) in chiral limit correspond to \(M_0 = 340\text{ MeV}\) assuming the parameters \(\rho\) and \(R\), which are the values given in (1.11).

The action \(W[\Phi_+, \Phi_-]\) in (3.19) has imaginary part, in general, which is reduced to Wess-Zumino term in long-wave limit \((k \ll M\), where \(k\) is a mesons momentum).

**IV. VIRTUAL CHARM IN B \to K\eta' DECAY**

We apply the Effective Action (3.19) and the formula for the divergence of the c-quark axial current (2.28) to the calculation of \(f_{\eta'}^{(c)}\) and compare with the analogous quantity \(f_{\eta'}^{(u)}\), which is defined in the similar way as \(f_{\eta'}^{(c)}\) in (1.4). These quantities certainly are defined in Minkowski space. The eqs. (1.9) and (2.28) may be easily translated from Euclidean to Minkowski space accordingly the above-given prescription. For instance \(((G^a \tilde{G}^a)^E \to (G^a \tilde{G}^a)_M)\) and \(((f_{abc}G^a \tilde{G}^b G^c)_E \to -(f_{abc}G^a \tilde{G}^b G^c)_M)\), which lead to

\[m_{\eta'}^2 f_{\eta'}^{(u)} = <0|\frac{g^2}{16\pi^2}(G^a \tilde{G}^a)_M|\eta'>. \quad (4.1)\]

and

\[m_{\eta'}^2 f_{\eta'}^{(c)} = -<0|\frac{g^3}{2^53\pi^2 m_c^2}(f_{abc}G^a \tilde{G}^b G^c)_M|\eta'>. \quad (4.2)\]

The phenomenological way of the estimation of the \(f_{\eta'}^{(u)}\) is the application of the QCD+QED axial anomaly equation together with data on \(\eta' \to 2\gamma\) decay leads to

\[f_{\eta'}^{(u)} = 63.6 \text{ MeV}, \quad (4.3)\]
which was used in [6]. On the other hand the calculation of the matrix elements (4.1), (4.2) may be reduced to the calculation of the correlators

\[ \langle G^a \tilde{G}^a(x) G^a \tilde{G}^a(y) \rangle, \quad \langle f_{abc} G^a \tilde{G}^b G^c(x) G^a \tilde{G}^a(y) \rangle \]

respectively. The calculation of the correlators are naturally performed in Euclidean space. These correlators have almost the same dependence on the large relative distances \(|x - y|\) and the ratio of these correlators become almost constant, at least in DP chiral quark model [28].

In the Effective Action approach, the abovementioned gluonic operators vertices in the correlators are generated by differentiation of the Effective Action (3.19) over \(\kappa_2(3)\), which leads to the vertices

\[ i f_{2(3)} F^2 N_f^{-1} \frac{1}{2}(\Phi_+ + \Phi_-) + (\Phi_+ - \Phi_-) \gamma_5 \]

(4.4)

So, the calculations of \(f_{2(c)}^c\) and also \(f_{2(u)}^u\) may be reduced to the calculations of the similar two-point \(\gamma_5\)-singlet correlators with additional form-factor \(f_3(q)\) or \(f_2(q)\) (3.17), which are simply the momentum representation of the instanton contribution to the operators \(g^3 f_{abc} G^a \tilde{G}^b G^c\) and \(g^2 G^a \tilde{G}^a\) (see Eq. (3.17)), respectively. They are defined as

\[ f_3(q) = -1536 \rho^6 \int d^4x \exp(iqx) (\rho^2 + x^2)^{-6}, \]

\[ f_2(q) = 192 \rho^4 \int d^4x \exp(iqx) (\rho^2 + x^2)^{-4}, \]

(4.5)

and

\[ f_2(0) = 32\pi^2, \quad f_3(0) = -\frac{12}{5\rho^2} f_2(0) \]

(4.6)

It is easy to reduce the integrals in (4.5) to the Bessel functions as

\[ \int d^4x \exp(iqx) (\rho^2 + x^2)^{-n} = \frac{\pi^2}{(n-1)!} \left( \frac{q^2}{\rho^2} \right)^{(n-3)/2} K_{n-3} \left( (q^2 \rho^2)^{1/2} \right). \]

(4.7)

With the Effective Action (3.19) we calculate everything in Euclidean space with further analytical continuation to the Minkowski region.

On the other hand the ratio of abovementioned correlators in the Effective Action approach equal the ratio of form-factors \(f_3(q)\) and \(f_2(q)\). This ratio has a weak dependence on the argument (like 10% on the scale \(q \sim \rho^{-1}\)). As a result, this provides the possibility to calculate the ratio \(f_{2(c)}^{(c)} / f_{2(u)}^{(u)}\) at small Euclidean \(q^2\) within this accuracy.

Now it is clear that with the Effective Action (3.19) the matrix element in (4.2) may be reduced to the calculation of the matrix element in (4.1) with an additional factor \(-\frac{12}{5\rho^2} f_2(0)\). So, the ratio of Eqs. (4.2) and (4.1) is equal in this model to:

\[ \frac{f_{2(c)}^{(c)}}{f_{2(u)}^{(u)}} = -\frac{12}{5\rho^2} \frac{1}{6m_c^4} \sim -0.1, \]

(4.8)
where the value for $\rho$ is given in (1.11). By taking into account the estimation (4.3) (we use $m_c(\mu \simeq m_c) \simeq 1.25$ GeV on the scale $\mu \simeq m_c$ for the numerical estimates), we find

$$f_{\eta'}^{(c)} = -6 \text{ MeV}.$$  \hspace{1cm} (4.9)

This number is close to the one of [5], $|f_{\eta'}^{(c)}| = 5.8$ MeV and the sign and the order of the value coincide with the estimations of [6, 8, 13].

Recently, Shuryak and Zhitnitsky [4] performed direct numerical evaluations of the various correlators of the operators $g^2 G^a \tilde{G}^a$, $g^3 f_{abc} G^a \tilde{G}^b G^c$ in the Interacting Instanton Liquid Model (IILM). Their calculations lead to:

$$\langle 0 | g^2 G^a \tilde{G}^a | \eta' \rangle = 7 \text{ GeV},$$  \hspace{1cm} (4.10)

(which leads to $f_{\eta'}^{(u)} = 48.3$ MeV) and

$$\frac{1}{|\langle 0 | g^2 G^a \tilde{G}^a | \eta' \rangle|} \langle 0 | g^3 f_{abc} G^a \tilde{G}^b G^c | \eta' \rangle \approx (1.5 \sim 2.2) \text{ GeV}^2.$$  \hspace{1cm} (4.11)

The later is somewhat large than their simple estimate for this ratio of matrix elements

$$\frac{12}{5} \left( \frac{1}{\rho^2} \right) \approx (1 \sim 1.5) \text{ GeV}^2.$$  \hspace{1cm} (4.12)

With the use of (4.11) and (2.28) we arrive at

$$\frac{f_{\eta'}^{(c)}}{f_{\eta'}^{(u)}} = -0.17 \sim -0.25.$$  \hspace{1cm} (4.13)

This ratio gives $f_{\eta'}^{(c)} = -8.2 \sim -12.3$ MeV at the scale of the size of the instanton $\mu \simeq \rho^{-1}$. The abovementioned experimental numbers (1.1) are given at the scale $\mu \simeq m_c$, which is different from the scale of this instanton calculation. The account of the anomalous dimension of the $g^3 G\tilde{G}G$ operator [35] leads to correction [4]

$$f_{\eta'}^{(c)}(\mu \simeq m_c) \simeq 1.5 f_{\eta'}^{(c)}(\mu \simeq \rho^{-1}).$$  \hspace{1cm} (4.14)

The account of this scale factor leads to

$$f_{\eta'}^{(c)}(\mu \simeq m_c) = -12.3 \sim -18.4 \text{ MeV}.$$  \hspace{1cm} (4.15)

Hence, using (4.11), the result of more sophisticated calculations of Shuryak and Zhitnitsky, we get the number (4.15) which is 2-3 times larger than our simple estimation (4.9).

These numbers (4.9), (4.15) are in agreement with the phenomenological bounds [7, 13] and almost in agreement in the sign and the value with [6, 8] but six-ten times less than the estimations given by [3] (see (1.6)).

By using the numerical analysis of the branching ratio for $B^\pm \rightarrow \eta'K^\pm$ given at [5] (Fig.17 of [5]) we expect that the value of $f_{\eta'}^{(c)}$ given in (4.15) may provide a more satisfactory explanation of the experimental data.
V. VIRTUAL CHARM IN POLARIZED DIS

From the definition Eq. (1.7) and from Eqs. (1.9), (2.28) it is easy to find

$$\Delta \Sigma 2m_N \bar{p}i\gamma_5 p = \langle p, s | \frac{N_f g^2}{16\pi^2} G\tilde{G} | p, s \rangle,$$ 

(5.1)

$$\Delta c 2m_N \bar{p}i\gamma_5 p = -\langle p, s | i\frac{g^3}{253^2\pi m_c^2} f_{abc} G^a \tilde{G}^b G^c | p, s \rangle$$

(5.2)

(Theese formulas are defined in Minkowski space). As it was mentioned before $\Delta \Sigma = 1$ in the constituent quark model. On the other hand it was shown that Skyrme soliton model gives zero for this quantity. Chiral quark-soliton model (see recent review of this model [38]) interpolate between these constituent quark and Skyrme models and give the value of $\Delta \Sigma$ in the range [36, 37]

$$\Delta \Sigma = 0.3 \sim 0.5.$$  

(5.3)

The calculation of the same quantity by QCD sum rules approach give the same answer (see the review [17]).

Chiral quark-soliton model is essentially based on the Effective Action like (3.19), where all the degrees of freedom are frozen except constituent quarks and pions. In that case, by restoring of the quark degrees of freedom in (3.19), we find

$$\hat{Z}_N[\kappa_{2(3)}] = \int D\psi D\bar{\psi} DU \exp \int \psi^\dagger \left\{ i\hat{\partial} + iF^2 MU^{\gamma_5} \left( 1 + \gamma_5 (\kappa_{2(3)} f) \right) N_f^{-1} \right\} \psi,$$  

(5.4)

where $U^{\gamma_5} = \frac{1}{2M}((\Phi_+ + \Phi_-) + (\Phi_+ - \Phi_-)\gamma_5)$, $\Phi_{\pm} = M \exp(\pm i\phi)$ and the usual decomposition for the pions $\phi = \sum_1^3 \tau_i \phi_i$ may be used.

The mass of nucleon is calculated from Euclidean large-distance asymptotes of two-point correlator of the composite quark operators $\Gamma_N(x)$ with nucleon quantum numbers and certainly by using $\hat{Z}_N[0]$.

The nucleon mass receive the contributions from the $N_c$ products of the constituent quark propagators in the external pion field and the polarization of the constituent quarks vacuum by this field(effective action for the pions) integrated over this field. The saddle-point condition in this path integral means topologically nontrivial pion field $\phi_s$ like famous skyrmion (see for example [38]).

The calculation of the nucleon matrix elements of the any combination of the gluonic operators $g^2 G^a \tilde{G}^a$ and $g^3 f_{abc} G^a \tilde{G}^b G^c$ may be reduced to the differentiation over $\kappa_{2(3)}$ of the two-point correlator of the composite quark operators $\Gamma_N(x)$ calculated now by using $\hat{Z}_N[\kappa_{2(3)}]$.  

This is the way to calculate $\Delta \Sigma$ and $\Delta c$, which are essentially reduced to the nucleon matrix elements of the gluonic operators $g^2 G^a \tilde{G}^a$ and $g^3 f_{abc} G^a \tilde{G}^b G^c$ respectively. It is clear, that these operators lead to the analogous (4.4) vertices

$$i f_{2(3)} M F^2 U^{\gamma_5} N_f^{-1} \gamma_5.$$  

(5.5)

The form-factors $f_2$ and $f_3$ are defined in (4.5). In the present case we need these form-factors at $q^2 = 0$. 

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On the very general ground, it is possible to prove that the calculations of $\Delta \Sigma$ with using $\hat{Z}_N[\kappa_2]$ leads exactly the result of Wakamatsu and Yoshiki [36]. In that paper authors calculated the nucleon matrix element of the operator $u^\dagger \gamma_\mu \gamma_5 u + d^\dagger \gamma_\mu \gamma_5 d$ by using the Effective Action like $\hat{Z}_N[0]$.

Let us change variables in $\hat{Z}_N[\kappa_2]$, Eq. (5.4), accordingly:

$$\psi' = \exp(\alpha(z)\gamma_5)\psi, \quad \psi'^\dagger = \psi^\dagger \exp(\alpha(z)\gamma_5). \quad (5.6)$$

If we choose

$$\alpha = \frac{1}{2}(\kappa_2 f_2)^{-1}$$

then most important $O(\kappa_2)$ term in $\hat{Z}_N[\kappa_2]$ will appear now in another form as:

$$\hat{Z}_N[\kappa_2] = \int D\psi' D\psi' J DU \times \exp \int dz \psi'^\dagger(z) \left\{ i\hat{\partial} + \frac{1}{2} \gamma_\mu \gamma_5 \left( \int dx \kappa_2(x) \partial_{x,\mu} f_2(x - z) \right)^{-1} + iF^2MU\gamma_5 \right\} \psi'(z) \quad (5.7)$$

where $J$ is a Jacobian of the transformations of the measure. Such type of Jacobians is responsible for the chiral anomaly, in general. In the present case we are dealing with the action in (3.19), (5.4), which has an imaginary part, since Dirac operator

$$D = i\hat{\partial} + iF^2MU\gamma_5$$

is not hermitian. To calculate the Jacobian $J$ we must properly define the measure.

First of all we must choose the total systems of the eigenfunctions of some hermitian operators. It is natural to take the operators:

$$D^+ D = -\partial^2 + F^4 M^2 - F^2 M(\hat{\partial}U\gamma_5), \quad DD^+ = -\partial^2 + F^4 M^2 + F^2 M(\hat{\partial}U\gamma_5^+), \quad (5.8)$$

and define a set of the eigenfunctions:

$$D^+ D \phi_n = \lambda_n^2 \phi_n, \quad DD^+ \Phi_n = \lambda_n^2 \Phi_n. \quad (5.9)$$

It is easy to show that:

$$D \phi_n = \eta_n \Phi_n,$$

where $|\eta_n| = |\lambda_n|$.

With this basis we may expand:

$$\psi(x) = \sum_n a_n \phi_n, \quad \psi^\dagger(x) = \sum_n b^\dagger_n \Phi_n^+, \quad (5.10)$$

where $a_n$ and $b^\dagger_n$ are Grassmanian numbers. So, the measure may be defined as

$$D\psi = \prod_n da_n, \quad D\psi^\dagger = \prod_n db^\dagger_n.$$ 

The Jacobian $J$ then is given by

$$J = \exp(\int dx \alpha(x) A(x)). \quad (5.11)$$
According to Fujikawa [23], the regularized expression for the anomaly $A(x)$ is given by

$$A(x) = \sum_n \phi_n(x) \gamma_5 \exp\left(-\frac{D^+D}{\mu^2}\right) \phi_n(x) + \sum_n \Phi_n(x) \gamma_5 \exp\left(-\frac{D^+D}{\mu^2}\right) \Phi_n(x)$$

$$= \lim_{\mu \to \infty} tr \int \frac{d^4k}{(2\pi)^4} \exp(-ikx) \gamma_5 \left\{ \exp\left(-\frac{DD^+}{\mu^2}\right) + \exp\left(-\frac{DD^+}{\mu^2}\right) \right\} \exp(ikx).$$  \hspace{1cm} (5.12)

Redefining the variable of the integration in (5.12) $k \to k/\mu$ we arrive at:

$$A(x) = tr \int \frac{d^4k}{(2\pi)^4} \exp(-k^2) \gamma_5 \left\{ (F^2M(\hat{U}\gamma_5))^2 + (F^2M(\hat{U}\gamma_5^+))^2 \right\} = 0,$$  \hspace{1cm} (5.13)

due to the trace over Dirac matrices. So, Jacobian $J$ is equal to one and now it is absolutely clear that the calculations with Eqs. (5.7), (5.4) leads to the same result in [36, 37].

The $SU(3)$ extensions [37] of the model (5.4) will leads to the same result for $\Delta \Sigma$, since the valence $u$, $d$ quarks give the most essential contribution to this quantity, while the vacuum quarks are negligible.

So, the ratio $\Delta c/\Delta \Sigma$ can be easily calculated in the same line as for $f^{(c)}_{\eta'}/f^{(u)}_{\eta'}$ (4.8) which leads to

$$\Delta c/\Delta \Sigma = -\frac{12}{5N_f/6m_c^2} \sim -0.033.$$  \hspace{1cm} (5.14)

With the use of (5.3) it means

$$\Delta c = -(0.01 \sim 0.016).$$  \hspace{1cm} (5.15)

Again the quantity $\Delta c$ in (5.15) is given at the scale of the size of the instanton. On the scale $\approx m_c$ the account of the anomalous dimension of the $g^3G\bar{G}G$ operator [35] leads to the same correction as in (4.14) and we have

$$\Delta c(\mu \simeq m_c) \simeq 1.5\Delta c(\mu \simeq \rho^{-1}) = -(0.015 \sim 0.024).$$  \hspace{1cm} (5.16)

VI. CONCLUSION

The problem of the virtual charm axial current contribution has been reduced to the calculations of the specific gluon operator matrix elements by the application of the operator Schwinger method, developed for QCD by the ITEP group. Due to the specific structure of this operator these matrix elements receive the main contribution from the instanton background. The DP effective action approach, based on instanton model of QCD vacuum, has been rederived and applied to the calculations of such types of matrix elements in chiral limit, since the reliable answer may be obtained only in this limit.

We have calculated the coupling of $\eta'$ with virtual charm axial current $f^{(c)}_{\eta'}$ (4.15). The obtained value may provide a satisfactory explanation of the experimental data on the branching ratio for $B^{\pm} \to \eta'K^{\pm}$-decay as was shown recently by Cheng and Tseng [11]. They demonstrated an impressive good explanation of these data using our value (4.15) for $f^{(c)}_{\eta'}$. 

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This approach has been applied also to the problem of the charm content of the nucleon spin $\Delta c$. We conclude that $\Delta c (5.16)$ is one order of magnitude smaller than the analogous contribution of the strangeness $\Delta s = -0.11 \pm 0.03$ [15].

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APPENDIX

We calculate here the determinant of the Dirac operator which appears from the Gaussian integral over c-quarks. This calculation was performed by using of the expansion of the effective action in the series of \( G/m_c^2 \) by Vainshtein et al. in Ref. [34]. However, we show here that careful calculations give a different result from that of Vainshtein et al [34]. Throughout this appendix, we use the Euclidean convention and consider up to the third order of \( G \) in the expansion. The contribution of the source \( j_\mu \) which satisfies the field equation \( D_\nu G_{\nu\mu} = j_\mu \) may be negligible, as being consistent to the main part of this paper.

As explained in the calculation of \( \langle 2m_c\gamma_5^\dagger c\gamma_5 c \rangle \), the determinant of the Dirac operator, \( \det \| \hat{P} + im_c \| \), must be regularized by the regulator mass \( M \),

\[
D \equiv \det \left\| \left( \hat{P} + im_c \right) \left( \hat{P} + iM \right) \right\| .
\]  

(A.1)

Instead of the direct calculation of \( D \), it is more convenient to calculate the logarithm of \( D \). The relation, \( \ln \det \| A \| = \text{Tr} \ln \| A \| \), is satisfied formally with the infinite-dimensional matrix \( A \),

\[
\ln D = \text{Tr} \ln \left\| \left( \hat{P} + im_c \right) \left( \hat{P} + iM \right) \right\| .
\]  

(A.2)

Note that the symbol \( \text{Tr} \) in this appendix contains the trace over Lorentz and color indices and the integral over the coordinate, i.e.

\[
\text{Tr}(\cdots) \equiv \text{tr}_{L+C} \int d^4 x \langle x| \cdots |x \rangle ,
\]  

(A.3)

while in the main part of this paper it has been written as \( \text{Tr} \equiv \text{tr}_{L+C} \) only.

Let us consider the derivative of \( \ln D \) with respect to the c-quark mass \( m_c \),

\[
\frac{1}{m_c} \frac{d}{dm_c} \ln D = \frac{1}{m_c} \text{Tr} \left( \frac{i}{\hat{P} + im_c} - \frac{i}{\hat{p} + im_c} \right) = \text{Tr} \left( \frac{1}{\hat{p}^2 + m_c^2} - \frac{1}{\hat{p}^2 + m_c^2} \right)
\]

\[
= \text{Tr} \left( \frac{1}{\hat{p}^2 + m_c^2} - \frac{1}{\hat{p}^2 + m_c^2} \right)
\]

\[
+ \text{Tr} \left( \frac{1}{\hat{p}^2 + m_c^2} \frac{g}{2} \sigma G \frac{1}{\hat{p}^2 + m_c^2} \frac{g}{2} \sigma G \frac{1}{\hat{p}^2 + m_c^2} \right)
\]

\[
- \text{Tr} \left( \frac{1}{\hat{p}^2 + m_c^2} \frac{g}{2} \sigma G \frac{1}{\hat{p}^2 + m_c^2} \frac{g}{2} \sigma G \frac{1}{\hat{p}^2 + m_c^2} \right) + \cdots
\]  

(A.4)

\[
\equiv I_0 + I_2 + I_3 + \cdots,
\]

where the single \( G_{\mu\nu} \) term vanishes because of the trace of the single \( \sigma_{\mu\nu} \).

Calculations of both \( I_2 \) and \( I_3 \) are performed as we have done in the calculation of \( \langle 2m_c\gamma_5^\dagger c\gamma_5 c \rangle \), except for \( \gamma_5 \). Since we need up to \( \mathcal{O}(G^3) \) in our purpose, each of them is calculated as follows:
Here we have used the following relations,

\[ I_3(m_c^2) = -\text{Tr} \left( \frac{1}{\mathcal{P}^2 + m_c^2} \frac{g_2 \sigma G}{\mathcal{P}^2 + m_c^2} \frac{g_2 \sigma G}{\mathcal{P}^2 + m_c^2} \frac{g_2 \sigma G}{\mathcal{P}^2 + m_c^2} \right) \]

\[ = -\text{Tr} \left( \frac{1}{(\mathcal{P}^2 + m_c^2)^3} \right) \]

\[ = -\int d^4x \langle x | \frac{1}{(\mathcal{P}^2 + m_c^2)^3} | x \rangle \text{tr}_{L+C} \left\{ \frac{g_2 \sigma G(x)}{2} \right\}^3 \]

\[ = -\frac{g^3}{2^5 \cdot 3\pi^2 m_c^2} \int d^4x f_{abc} G^a G^b G^c(x) , \quad (A.5) \]

and

\[ I_2(m_c^2) = \text{Tr} \left( \frac{1}{\mathcal{P}^2 + m_c^2} \frac{g_2 \sigma G}{\mathcal{P}^2 + m_c^2} \frac{g_2 \sigma G}{\mathcal{P}^2 + m_c^2} \frac{g_2 \sigma G}{\mathcal{P}^2 + m_c^2} \right) \]

\[ = \frac{g^2}{4} \int d^4x \left\{ \langle x | \frac{1}{(\mathcal{P}^2 + m_c^2)^3} | x \rangle \text{tr}_{L+C} (\sigma G)^2 + \langle x | \frac{1}{(\mathcal{P}^2 + m_c^2)^4} | x \rangle \text{tr}_{L+C} \sigma G \cdot D^2 \sigma G \right\} \]

\[ = \frac{g^2}{4} \int d^4x \left\{ \frac{1}{2^5 \pi^2 m_c^2} \text{tr}_{L+C} (\sigma G)^2 + \frac{1}{2^6 \cdot 3\pi^2 m_c^4} \text{tr}_{L+C} \sigma G \cdot D^2 \sigma G \right\} \]

\[ = \frac{g^2}{2^5 \pi^2 m_c^2} \int d^4x G^a G^a(x) + \frac{g^3}{2^6 \cdot 3\pi^2 m_c^4} \int d^4x f_{abc} G^a G^b G^c(x) . \quad (A.6) \]

Here we have used the following relations,

\[ \text{tr}_{L+C} (\sigma G)^3 = -2^5 i \cdot \text{tr}_{C} GGG = 2^3 f_{abc} G^a G^b G^c \]

\[ \text{tr}_{L+C} (\sigma G)^2 = 2^2 \text{tr}_{C} GGG = 2^2 G^a G^a \]

\[ D^2 \sigma G = 2ig\sigma_{\mu\nu} G_{\mu\alpha} G_{\alpha\nu} \quad (A.7) \]

under neglecting the source \( j_\mu \) which satisfies \( D_\nu G_{\nu\mu} = j_\mu \).

The calculation of \( I_0 \) is rather technical since it needs careful treatments of traces of infinite-dimensional matrices. First, we consider the translational invariance of \( I_0 \). It means that \( I_0 \) does not change under a shift of the operator \( \mathcal{P}_\mu \) by an arbitrary vector \( q_\mu \),

\[ I_0(m_c^2) = \text{Tr} \left( \frac{1}{\mathcal{P}^2 + m_c^2} - \frac{1}{p^2 + m_c^2} \right) \]

\[ = \text{Tr} e^{i\mathcal{P}X} \left( \frac{1}{\mathcal{P}^2 + m_c^2} - \frac{1}{p^2 + m_c^2} \right) e^{-i\mathcal{P}X} \quad (A.8) \]

\[ = \text{Tr} \left\{ \frac{1}{(\mathcal{P} - \mathcal{Q})^2 + m_c^2} - \frac{1}{(p - \mathcal{Q})^2 + m_c^2} \right\} . \]

Here \( e^{i\mathcal{P}X} \) is the shift operator in momentum space, and \( X_\mu \) is the coordinate operator which satisfies the algebra (2.1). Now, we expand the right hand side of (A.8) into the series in \( q_\mu \). The momentum shift of \( (\mathcal{P}^2 + m_c^2)^{-1} \) is expanded as
$$e^{iqX} \frac{1}{P^2 + m_c^2} e^{-iqX}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} [iqX, [iqX, \cdots [iqX, \frac{1}{P^2 + m_c^2} \cdots ]]$$

\[ \text{iqX appears n times} \]

$$= \frac{1}{P^2 + m_c^2} + [iqX, \frac{1}{P^2 + m_c^2}] + \frac{1}{2} [iqX, [iqX, \frac{1}{P^2 + m_c^2}]] + \cdots \quad \text{(A.9)}$$

When one apply the trace $\text{Tr}$ with both sides of this equation, the first term of the right hand side is equal to the left hand side and both terms do not depend at all on the vector $q_\mu$. This means that other terms which depend on any power of $q$ are identically equal to zero, i. e.

$$\text{Tr} [iqX, \frac{1}{P^2 + m_c^2}] = \text{Tr} [iqX, [iqX, \frac{1}{P^2 + m_c^2}]] = \cdots = 0 \quad \text{for any } q_\mu \quad \text{(A.10)}$$

Consider the coefficient of the term which is the second order of $q_\mu$:

$$\text{Tr} [iqX, [iqX, \frac{1}{P^2 + m_c^2}]] - \frac{1}{p^2 + m_c^2}] ]

= -2q_\nu q_\nu \left[ \delta_{\mu \nu} \text{Tr} \left( \frac{1}{(P^2 + m_c^2)^2} - \frac{1}{(p^2 + m_c^2)^2} \right) - 4\text{Tr} \left( \frac{1}{(P^2 + m_c^2)^2} P_\mu P_\nu (P^2 + m_c^2)^{-2} P_\nu - \frac{p_\mu p_\nu}{(p^2 + m_c^2)^2} \right) \right]. \quad \text{(A.11)}$$

Since $q_\mu$ is an arbitrary vector, we may choose the average over the direction of $q_\mu$, i. e. $q_\mu q_\nu = \frac{1}{4} q^2 \delta_{\mu \nu}$. As explained already, the coefficient of $q^2$ is equal to zero. The corresponding equation looks as

$$\text{Tr} \left\{ \frac{1}{(P^2 + m_c^2)^2} - \frac{1}{(p^2 + m_c^2)^2} \right\} = \text{Tr} \left\{ \frac{1}{(P^2 + m_c^2)^3} p_\mu (P^2 + m_c^2) - \frac{p^2}{(p^2 + m_c^2)^3} \right\}. \quad \text{(A.12)}$$

On the other hand, using the relation $(P^2 + m_c^2)(P^2 + m_c^2)^{-1} = 1$,

$$\text{Tr} \left\{ \frac{1}{(P^2 + m_c^2)^2} - \frac{1}{(p^2 + m_c^2)^2} \right\} = \text{Tr} \left\{ \frac{1}{(P^2 + m_c^2)^3} P^2 - \frac{p^2}{(p^2 + m_c^2)^3} \right\} + m_c^2 \text{Tr} \left\{ \frac{1}{(P^2 + m_c^2)^3} - \frac{1}{(p^2 + m_c^2)^3} \right\}. \quad \text{(A.13)}$$

Hence, substituting it to Eq. (A.12), we obtain the relation

$$\text{Tr} \left\{ \frac{1}{(P^2 + m_c^2)^3} - \frac{1}{(p^2 + m_c^2)^3} \right\} = \frac{1}{m_c^2} \text{Tr} \left\{ \frac{1}{(P^2 + m_c^2)^2} P_\mu (P^2 + m_c^2) P_\mu \right\}$$

$$= \frac{1}{m_c^2} \text{Tr} \sum_{n=1}^{\infty} \frac{(-1)^n}{(P^2 + m_c^2)^{n+3}} P_\mu [P^2, [P^2, \cdots [P^2, P_\mu \cdots ]] \right\]. \quad \text{(A.14)}$$
\[ = \frac{1}{m_c^2} \text{Tr} \left\{ \frac{1}{(p^2 + m_c^2)^4} \right\} \left\{ -\mathcal{P}_\mu [\mathcal{P}^2, \mathcal{P}_\mu] - \frac{1}{p^2 + m_c^2} [\mathcal{P}^2, \mathcal{P}_\mu] [\mathcal{P}^2, \mathcal{P}_\mu] \right. \\
+ \frac{1}{(p^2 + m_c^2)^2} [\mathcal{P}^2, \mathcal{P}_\mu] [\mathcal{P}^2, [\mathcal{P}^2, \mathcal{P}_\mu]] \right. \\
+ \left. \frac{1}{(p^2 + m_c^2)^3} [\mathcal{P}^2, [\mathcal{P}^2, \mathcal{P}_\mu]] [\mathcal{P}^2, [\mathcal{P}^2, \mathcal{P}_\mu]] \right\}. \] (A.14)

To integrate the right hand side over momentum space up to the desired order of \( G \), we need only two commutation relations under \( j_\mu \) being neglected,

\[ [\mathcal{P}^2, \mathcal{P}_\mu] = 2ig\mathcal{P}_\mu G_{\nu\mu}, \]
\[ [\mathcal{P}^2, [\mathcal{P}^2, \mathcal{P}_\mu]] = -4g\mathcal{P}_\alpha \mathcal{P}_\nu D_\alpha G_{\nu\mu} - 4g^2 \mathcal{P}_\alpha G_{\alpha\nu} G_{\nu\mu} + 2ig\mathcal{P}_\alpha D^2 G_{\alpha\mu}, \] (A.15)

since the product of \( \mathcal{P}_\mu \) and the higher commutators which could give the \( \mathcal{O}(G^3) \) terms can be replaced by the product of these commutators (A.15) with a suitable sign in the trace \( \text{Tr} \). Thus,

\[ \text{Tr} \left\{ \frac{1}{(p^2 + m_c^2)^3} - \frac{1}{(p^2 + m_c^2)^3} \right\} \]
\[ = -\frac{g^2}{2^5 \cdot 3\pi^2 m_c^2} \int d^4x G^a G^a(x) - \frac{g^3}{2^3 \cdot 3^2 \cdot 5\pi^2 m_c^4} \int d^4 x f_{abc} G^a G^b G^c(x). \] (A.16)

The integration of this result over squared masses twice gives

\[ I_0(m_c^2) = 2 \int_{m_t^2}^{m_c^2} dm_2^2 \int_{m_1^2}^{m_2^2} dm_1^2 \text{Tr} \left\{ \frac{1}{(p^2 + m_1^2)^3} - \frac{1}{(p^2 + m_2^2)^3} \right\} \]
\[ = -\frac{g^2}{2^5 \cdot 3\pi^2 m_c^2} \int d^4x G^a G^a(x) - \frac{g^3}{2^3 \cdot 3^2 \cdot 5\pi^2 m_c^4} \int d^4 x f_{abc} G^a G^b G^c(x). \] (A.17)

Finally, we obtain the determinant

\[ \mathcal{D} = \exp \frac{1}{2} \int_{m_c^2}^{M^2} dm^2 \left\{ I_0(m_c^2) + I_2(m_c^2) + I_3(m_c^2) \right\} \]
\[ = \exp \int d^4x \left\{ \frac{g^2}{2^5 \cdot 3\pi^2} \ln \frac{M^2}{m_c^2} G^a G^a(x) - \frac{23g^3}{2^7 \cdot 3^2 \cdot 5\pi^2 m_c^4} f_{abc} G^a G^b G^c(x) \right\}. \] (A.18)

To compare with the result of Vainshtein et al. [34], we needs to obtain the effective action written in the Minkowski metric. In our case, the Euclidean effective action is defined as

\[ S_{\text{eff}}^E = \ln \mathcal{D}. \] (A.19)

According to the previous convention for the transition between the Minkowskian and the Euclidean space,

\[ (GG)_E = (GG)_M, \quad (GGG)_E = -(GGG)_M. \] (A.20)

Then the Minkowskian effective action is

\[ S_{\text{eff}}^M = \int d^4x \left\{ -\frac{g^2}{2^2 \cdot 3\pi^2} \ln \frac{M^2}{m_c^2} \text{tr}_C GG(x) + \frac{23i g^3}{2^5 \cdot 3^2 \cdot 5\pi^2 m_c^4} \text{tr}_C GGG(x) \right\}. \] (A.21)

Here, \( \text{tr}_C \) is explicitly written in order to compare with the convention of Vainshtein et al. [34]. The coefficient of the second term is different from their result [34] with factor 23/2.
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