Inheriting of properties on spectra

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Abstract

We develop the technique of inheriting some properties from spectrum to differential spectrum. Using this machinery, we obtain some new geometric properties of differential algebraic groups.

1 Introduction

The technique presented shows that differential spectrum inherit some properties from spectrum. This idea allows to obtain results automatically. The first step is to find appropriate non differential property, and the second step is to show that differential spectrum inherit this property. The first step presented in statement 2. The reduction of a differential problem to a non differential one follows from statement 9. The main result based on our methods is theorem 13. Using additional geometric technique such as statements 11 and 12 theorem 13 allows to prove geometrical properties of differential algebraic varieties such as statements 23 and 24.

In section 2 we recall basic definition. In section 3 we explain our method and introduce inheriting properties. In section 4 we derive corollaries from results of section 3 such as “going up” and “going down” theorems. The most interesting case for us is the case of differentially finitely generated algebras over a field of characteristic zero. In section 5 we introduce a required technique result – statement 8. Section 6 is devoted to demonstration of the power of the previous statement. In section 7 we translate results obtained to geometrical language of spectra. After that in section 8 we prove the main result. Section 9 is devoted to locally closed points of differential spectrum. This set plays the same role in differential algebraic geometry as a maximal spectrum in algebraic geometry. In section 10 we state geometrical properties of differential algebraic varieties and groups.

The nontrivial part of the proof is concerned with the relation between flatness and properties of differential spectrum. This relation presented in theorem 13.

2 Terms and notation

Throughout the text the word ring means an associative commutative ring with an identity element. All homomorphisms preserve an identity
element. We use the material of the books 11 and 2 as a background, all terms and notation are taken from these sources.

We shall briefly discuss notation. The set of all derivations of a ring will be denoted by \( \Delta \). Let \( A \) and \( B \) be differential rings, \( f: A \to B \) be a differential homomorphism, and \( a \) and \( b \) are differential ideals in \( A \) and \( B \) respectively. Then the ideal \( Bf(a) \) will be denoted by \( a^\circ \) and will be called an extension of \( a \), the ideal \( f^{-1}(b) \) will be denoted by \( b^\circ \) and will be called a contraction of \( b \). Note that contraction and extension of differential ideal is a differential ideal. The set of all prime ideals of \( A \) will be denoted by \( \text{Spec } A \) and called a spectrum of \( A \). The set of all prime differential ideals will be denoted by \( \text{Spec}^\Delta A \) and called a differential spectrum of \( A \).

For any ring homomorphism \( f: A \to B \) corresponding mapping \( q \mapsto f^{-1}(q) \) whenever \( q \in \text{Spec } B \) will be denoted by \( f^* \). The restriction of \( f^* \) onto differential spectrum will be denoted by \( f^*_\Delta \). Additionally, we suppose that both spectra are provided by the Zarissky topology. A differential ring \( A \) will be said to be a Keigher ring if for any differential ideal \( a \) its radical is a differential ideal too. A differential ring will be called a Ritt algebra if it is an algebra over \( \mathbb{Q} \).

### 3 Inheriting of properties

Let \( f: A \to B \) be a differential homomorphism of Keigher rings. Consider the following pair of properties:

- \( A_1 \) is a property of \( f \), where \( f \) is considered as a homomorphism
- \( A_2 \) is a property of \( f \), where \( f \) is considered as a differential homomorphism

such that \( A_1 \) implies \( A_2 \). The idea is the following: we shall find an appropriate pair of the properties and then reduce a differential problem to a non differential one.

Let us demonstrate a first relation between spectrum and differential spectrum.

**Statement 1.** Let \( A \to B \) be a differential homomorphism of Keigher rings and \( \mathfrak{p} \) is a prime differential ideal in \( A \). Then the following conditions are equivalent:

1. \( \mathfrak{p}^c = \mathfrak{p} \)
2. \( (f^*)^{-1}(p) \neq \emptyset \)
3. \( (f^*_\Delta)^{-1}(p) \neq \emptyset \)

**Proof.** The implications \( (3) \Rightarrow (1) \Rightarrow (2) \) follow from 1 chapter 3, sec. 2, prop. 3.16]. Let us show that \( (2) \Rightarrow (3) \). From 1 chapter 3, sec. 2, ex. 21 (IV)] the fiber \( (f^*)^{-1}(p) \) is naturally homeomorphic to \( \text{Spec}(B_{\mathfrak{p}}/pB_{\mathfrak{p}}) \). It is easy to see that differential fiber \( (f^*_\Delta)^{-1}(p) \) is naturally homeomorphic to \( \text{Spec}^\Delta(B_{\mathfrak{p}}/pB_{\mathfrak{p}}) \). Since fiber is not empty the ring \( B_{\mathfrak{p}}/pB_{\mathfrak{p}} \) is not zero. Since this ring is a Keigher ring, its differential spectrum is not empty. \( \square \)

We shall introduce a couple of definitions. Let \( f: A \to B \) be a homomorphism of rings.
The homomorphism \( f \) is said to have the going-up property if for every chain of prime ideals \( p_1 \subseteq \ldots \subseteq p_n \) in \( f(A) \) and any nonempty chain of prime ideals \( q_1 \subseteq \ldots \subseteq q_m \) (\( m < n \)) in \( B \) with condition \( q_i \cap f(A) = p_i \) (\( 1 \leq i \leq m \)) the second chain can be extended to a chain \( q_1 \subseteq \ldots \subseteq q_n \) with property \( q_i \cap f(A) = p_i, 1 \leq i \leq n \).

The homomorphism \( f \) is said to have the going-down property if for every chain of prime ideals \( p_1 \supseteq \ldots \supseteq p_n \) in \( f(A) \) and every nonempty chain of prime ideals \( q_1 \supseteq \ldots \supseteq q_m \) (\( m < n \)) in \( B \) with condition \( q_i \cap f(A) = p_i \) (\( 1 \leq i \leq m \)) the second chain can be extended to a chain \( q_1 \supseteq \ldots \supseteq q_n \) with property \( q_i \cap f(A) = p_i, 1 \leq i \leq n \).

If in the previous definitions all ideals are differential then we shall say that \( f \) has the going-up (going-down respectively) property for differential ideals.

**Statement 2.** Let \( f : A \to B \) be a differential homomorphism of Keigher rings and let \( p \) be a differential ideal in \( A \). Then

1. \((f^*)^{-1}(p) \neq \emptyset \Leftrightarrow (f_\Delta^*)^{-1}(p) \neq \emptyset \).
2. \( f^* \) is surjective \( \Rightarrow f_\Delta^* \) is surjective.
3. \( f \) has the going-up property \( \Rightarrow f \) has the going-up property for differential ideals.
4. \( f \) has the going-down property \( \Rightarrow f \) has the going-down property for differential ideals.

**Proof.** Property 1 is a partial case of statement 1. The second condition immediately follows from the first one.

To prove 3 we just need to consider the case of two ideals \( p \subseteq p' \) in \( A \) and ideal \( q \) in \( B \) such that \( q^c = p \). Applying statement 1 to \( A/p \subseteq B/q \) and ideal \( p' \), we get the desired result.

For condition 4 we need to consider the case of two ideals \( p' \subseteq p \) in \( A \) and ideal \( q \) in \( B \) such that \( q^c = p \). Applying statement 1 to \( A/p \subseteq B/q \) and ideal \( p' \), we get the desired result.

### 4 First applications

We shall apply our technique to generalize going-up and going-down theorems to the case of Keigher rings.

**Theorem 3** ("going-up"). Let \( f : A \to B \) be a differential homomorphism of Keigher rings, \( B \) being integral over \( A \). Then \( f \) has the going-up property for differential ideals.

**Proof.** From going-up theorem [1, chapter 5, sec. 2, th. 5.11] it follows that \( f \) has the going-up property. Now we apply statement 4.

**Theorem 4** ("going-down"). Let \( f : A \to B \) be a differential homomorphism of Keigher rings. Suppose that \( B \) is integral domain, \( f(A) \) is integrally closed, and \( B \) is integral over \( A \). Then \( f \) has the going-down property for differential ideals.
Proof. From going-down theorem [1, chapter 5, sec. 3, th. 5.16] it follows that \( f \) has the going-down property. Now we apply statement 2.

These results appeared in [6] but in the case of Ritt algebras. Previous results are not the most interesting case. The most interesting case is the case of differentially finitely generated algebras over a field of characteristic zero. But to advance in our theory we need a special technique for such algebras. This technique will be discussed in the next section.

5 Structure result

The following statement is another point of view on characteristic sets.

**Statement 5.** Let \( A \) be a Ritt algebra, \( B \) being differentially finitely generated over \( A \). Then there exists an element \( s \) in \( B \) such that \( B_s = C[y_\alpha] \), where \( C \) is finitely generated algebra over \( A \) and \( y_\alpha \) is not more than countable set of algebraically independent over \( C \) elements.

**Proof.** The ring \( B \) can be presented as a quotient ring of the ring of differential polynomials by a prime differential ideal 

\[
B = A(y_1, \ldots, y_n)/\mathfrak{p}.
\]

Consider a characteristic set \( F = \{ f_1, \ldots, f_k \} \) of \( \mathfrak{p} \) with respect of some ranking. Let \( s \) be the product of all initials and separants of the set \( F \). Let \( W \) be the set of all derivatives involved in elements of \( F \) and \( Z \) be the set of all derivatives not belonging to \( W \) such that no one element of \( Z \) is a proper derivative of a leader of an element of \( F \). From proposition [2, chapter I, sec. 9, prop. 1] it follows that \( B_s \) can be presented in the required form, where \( C \) is subalgebra in \( B_s \) generated by \( W \) and \( 1/s \) and the family \( y_\alpha \) coincides with \( Z \). Moreover, we have \( C = A[W]/(f_1, \ldots, f_k) \).

6 Demonstration

We shall demonstrate our method on some examples. These results are well-known but we shall show how the method works. The next theorem is a variant of result [3, chapter III, sec. 3, lemma 2].

**Theorem 6.** Let \( A \subseteq B \) be Ritt algebras. Suppose that \( B \) is an integral domain differentially finitely generated over \( A \). Then there exists an element \( s \) in \( A \) such that the corresponding mapping \( \text{Spec}^A B_s \to \text{Spec}^A A_s \) is surjective.

**Proof.** It suffices to show the desired result for the ring \( B_u \) instead of \( B \), where \( u \) is an element of \( B \). Let \( u \) be as in statement 6. We shall show that the corresponding mapping on spectra is surjective. Then statement 2 condition (2) guaranties that the required result holds. Using notation of statement 5 we see that the mapping from spectrum of \( B_u \) to spectrum of \( C_s \) is surjective. From commutative algebra we know that there is an element \( s \) in \( A \) such that the mapping from spectrum of \( C_s \) to spectrum of \( A_s \) is surjective. q. e. d.
Corollary 7. Let $B$ be a simple differentially finitely generated algebra over a field of characteristic zero, $A \subseteq B$ being an arbitrary subalgebra over the same field. Then there exists an element $s$ in $A$ such that $A_s$ is simple.

Remark. Particularly, let $K$ be the mentioned field. Taking an algebra $K\{y\}$, where $y$ is an arbitrary element of $B$, we get that every element of $B$ is differentially dependent over $K$.

The following statement is an analogue of [2, chapter III, sec. 10, prop. 7]

Corollary 8. Let $B$ be a simple differentially finitely generated algebra over a field $K$ of characteristic zero, $F$ being a field of fraction of $B$. Denote by $C_K$, $C_B$, and $C_F$ rings of constants of $K$, $B$, and $F$ respectively. Then $C_B = C_F$ and $C_B$ is algebraic over $C_K$.

Proof. Let us show that $C_B = C_F$. Consider an element $a \in C_F$, then the ideal $(B : a) = \{ r \in B \mid ar \in B \}$ is nontrivial differential ideal in $B$. So, since $B$ is simple, this ideal contain the identity element, hence $a$ in $B$.

Let us show that every element of $C_F$ is algebraic over $C_K$. Applying theorem [to a pair of rings $K\{y\} = K[y]$ and $B$, we get that there is an element $s \in K[y]$ such that $K[y]_s$ is a simple differential ring. Consequently, $y$ is algebraic over $K$. Since $y$ is constant element, $y$ is algebraic over $C_K$.

Remark. Particularly, if subfield of constants of the field $K$ is algebraically closed, then there are no new constant elements in every simple differentially finitely generated algebra over the field $K$.

So, using our technique we reproved some known result. Moreover, we shall show that, using this machinery, we are able to prove more delicate properties of differential spectra, and therefore we get new geometrical properties of differential algebraic varieties.

7 Geometry

We shall recall some definitions.

A subset $E$ of a topological space $X$ will be said to be constructible if it is of the following form $E = U_1 \cap V_1 \cup \ldots \cup U_n \cap V_n$, where $U_i$ are open and $V_i$ are closed.

The mapping $f: Y \rightarrow X$ of topological spaces is called closed if for every closed subset $E \subseteq Y$ its image $f(E)$ is closed.

The mapping $f: Y \rightarrow X$ of topological spaces is called open if for every open subset $E \subseteq Y$ its image $f(E)$ is open.

Statement 9. Let $A$ be a Ritt algebra, $B$ be differentially finitely generated over $A$, and $\text{Spec}^A A$ is a noetherian space. Denote $X = \text{Spec}^A A$, $Y = \text{Spec}^A B$. Then if $E \subseteq Y$ is a constructible set then $f^A(E)$ is constructible.
Proof. To show the desired result we use the following criterion: the set $E$ of a noetherian topological space is constructible if and only if for every irreducible closed set $X_0$, either $E \cap X_0 \neq X_0$ or else $E \cap X_0$ contains a nonempty open subset of $X_0$. (see [1, chapter 7, ex. 21]). It remains to use theorem [9]. Indeed, without loss of generality we may suppose that $E = U \cap C$, where $U$ is open and $C$ is closed in $Y$. Replacing the ring $B$ by its homomorphic image we may assume that $E$ is open in $Y$. Since $Y$ is noetherian, $E$ is quasi-compact therefore a finite union of open sets of the form $\text{Spec}^\Lambda B_0$. Hence reduce to the case $E = Y$. To show that $f^*_\Lambda(Y)$ is constructible we use the mentioned criterion. Let $X_0$ be an irreducible closed subset of $X$ such that $f^*_\Lambda(Y) \cap X_0$ is dense in $X_0$. Restricting $f^*_\Lambda$ on $(f^*_\Lambda)^{-1}(X_0)$ we may suppose that $X = X_0$. Hence reduce to the case where $A$ is an integral domain and $f$ is injective. If $Y_1, \ldots, Y_n$ are the irreducible components of $Y$, it is enough to show that some $f^*_\Lambda(Y_i)$ contains a nonempty open set in $X$. So finally we are brought down to the situation in which $A, B$ are the integral domains and $f$ is injective. Now we apply theorem [9] to complete the proof.

Lemma 10. Let $f : A \to B$ be a ring homomorphism, $p$ be a minimal prime ideal containing $\ker f$. Then the fibre over $p$ is not empty.

Proof. Replacing $A$ to $A/\ker f$ we reduce to the case $f$ is injective and $p$ is minimal prime ideal. Let $S = A \setminus p$. Consider the exact sequence of rings

$$0 \to A \to B$$

Localizing by $S$ we get the exact sequence

$$0 \to S^{-1}A \to S^{-1}B.$$

Consequently, the ring $S^{-1}B$ is not zero. Since $p$ is minimal prime ideal, $\text{Spec} S^{-1}B$ is naturally homeomorphic to fiber over $p$.

Statement 11. Let $f : A \to B$ be a homomorphism of Keigher rings. Then the following conditions are equivalent:

1. $f$ has the going-up property for differential ideals.
2. $f^*_\Lambda$ is closed.
3. $\forall q \in \text{Spec}^\Lambda B \Rightarrow f^*_\Lambda : \text{Spec}^\Lambda(B/q) \to \text{Spec}^\Lambda(A/q^*)$ is surjective.

Proof. The equivalence of (1) and (3) follows from the definition. Let us show that (2) implies (3). Note that closure of $f^*_\Lambda(\text{Spec}^\Lambda(B/q))$ coincides with $\text{Spec}^\Lambda(A/p)$. Therefore, since $f^*_\Lambda$ is closed, $f^*_\Lambda$ is surjective.

To show that (1) implies (2) we use lemma [10]. Let $b$ be a differential ideal in $B$ and $a = b^c$ is its contraction to $A$. Let us show that for any prime differential ideal $p$ containing $a$ there exists a prime ideal $q$ containing $b$ and contracting to $p$. It is well-known that any minimal prime ideal containing differential ideal is differential (all rings are Keigher rings). Let $p$ be minimal prime ideal (therefore differential) containing $a$. Applying lemma [10] to $A/a \to B/b$, we get the required result for $p$. If $p$ is an arbitrary prime differential ideal containing $a$ then it contains a
minimal prime differential ideal $p'$. From the previous arguments there is an ideal $q'$ in $B$ contracting to $p'$. Now we apply the going-up property for differential ideals.

**Statement 12.** Let $A$ be a Ritt algebra, $B$ be differentially finitely generated algebra over $A$, and $\text{Spec}^\Delta A$ is a noetherian topological space. Then the following conditions are equivalent:

1. $f$ has the going-down property for differential ideals.
2. $f_\Delta$ is open.
3. $\forall q \in \text{Spec}^\Delta B \Rightarrow f_\Delta^*: \text{Spec}^\Delta(B_q) \to \text{Spec}^\Delta(A_{q'})$ is surjective.

**Proof.** The equivalence of (1) and (3) follows from the definition. Let us show that (2) implies (3). Observe that $B_q$ is a direct limit of the rings $B_t$, where $t \in (B \setminus q)$. Then, applying statement 2 to result of [1, chapter 3, ex. 26] we see, that for Keigher rings the following holds

$$f^*(\text{Spec}^\Delta(B_q)) = \bigcap_t f_\Delta^*(\text{Spec}^\Delta(B_t))$$

Since $\text{Spec}^\Delta(B_q)$ is an open neighborhood of the point $q$ and $f_\Delta^*$ is open, $f^*(\text{Spec}^\Delta(B_q))$ is an open neighborhood of $p = q^+$ and therefore contains $\text{Spec}^\Delta(A_p)$.

To show that (1) implies (2) we use the criterion: the subset $E$ of a noetherian topological space is open if and only if for every irreducible closed subset $X_0$, either $E \cap X_0 = \emptyset$ or else $E \cap X_0$ contains a nonempty open subset of $X_0$ (see [1, chapter 7, ex. 22]). As in theorem 9 we reduce the problem to the case $E = f_\Delta^*(Y)$. The going-down property shows that if $p \in E$ and $p' \subseteq p$, then $p' \in E$, that is if $X_0$ is irreducible closed subset of $X$ intersecting $E$, then $E \cap X_0$ is dense in $X_0$. Since every constructible set contains an open in its closure, the set $E \cap X_0$ contains open in $X_0$. From the mentioned criterion it follows that $E$ is open.

8 Inheriting of geometrical properties

Using statements 2 and 12 we are able to amplify theorem 6.

**Theorem 13.** Let $A \subseteq B$ be differential Ritt algebras, $B$ being an integral domain differentially finitely generated over $A$. Then there exists an element $s$ in $B$ such that the embedding $A \subseteq B$ has a going-down property for differential ideals.

**Proof.** Let $u$ be as in statement 5. Then the algebra $B_u$ is a free $C$ module, and therefore is faithfully flat over $C$. Since $C$ is finitely generated over $A$ there exists an element $t$ in $A$ that $C_t$ is free $A_t$-module (see [5, chapter 8, sec. 22, th. 52]). So $B_{tu}$ is faithfully flat over $A_t$. Moreover $A_t$ is flat over $A$. From statement 11 chapter 5, ex. 11 it follows that the embedding $A \subseteq B_{tu}$ has going-down property. Statement 2 completes the proof.
Corollary 14. Let $A \subseteq B$ be differential Ritt algebras, $B$ being an integral domain differentially finitely generated over $A$. Suppose that differential spectrum of $A$ is noetherian. Then there exists an element $s$ in $B$ such that the mapping $\text{Spec}^\Delta B_s \to \text{Spec}^\Delta A$ is open.

Proof. The statement immediately follows from the previous theorem and statement 12.

9 Locally closed points of differential spectrum

The set of all maximal differential ideals of $A$ will be denoted by $\text{Max}^\Delta A$. This set is not necessarily very dense subset of differential spectrum (the definition see in [1, chapter 5, ex. 26]). Define the following subset in differential spectrum:

$$\text{SMax}^\Delta A = \{ p \in \text{Spec}^\Delta A \mid \exists s \in A : (A/p)_s \text{ is simple} \}$$

This is the set of all locally closed points of differential spectrum. It is a topological subspace in differential spectrum. The main aim is to show that this set plays the same role in the case of differentially finitely generated algebras over a differential field as the maximal spectrum plays in the case of finitely generated algebras over a field.

Statement 15. Let $A$ be a Keigher ring. Then the set $\text{SMax}^\Delta A$ is very dense in $\text{Spec}^\Delta A$.

Proof. It is enough to show that every prime differential ideal can be presented as an intersection of ideals of $\text{SMax}^\Delta A$. Indeed, if ideal $p$ is in $\text{SMax}^\Delta A$ we have nothing to prove. Suppose that $p$ is not in $\text{SMax}^\Delta A$, then for any element $t \in A \setminus p$ there is a maximal prime ideal $q_t$ such that $p \subset q_t$ and $t \notin q_t$. From the definition we have $q_t \in \text{Max}^\Delta A$ and $p = \bigcap_{t \in A \setminus p} q_t$.

From now we shall fix a differential field $K$ of characteristic zero.

Statement 16. Let $A$ and $B$ be differential algebras over $K$. Then for any differential homomorphism $f : A \to B$ we have $f^*_\Delta(\text{Max}^\Delta B) \subseteq \text{Max}^\Delta A$.

Proof. This is an immediate consequence of theorem 6.

Consequently, in the case of differentially finitely generated algebras over a field the contraction mapping can be restricted to the set of all locally closed points of differential spectrum. Moreover, the following universal property holds.

Statement 17. Let $A$ be a differentially finitely generated algebra over a field $K$. Then among subsets of $\text{Spec}^\Delta A$ with the property: for every differentially finitely generated algebra $B$ over $K$ and every differential homomorphism $f : A \to B$ we have $f^*(\text{Max}^\Delta B) \subseteq \text{Spec}^\Delta A$, there exists the smallest subset coinciding with $\text{SMax}^\Delta A$. 
Proof. The proof is a partial case of theorem 6.

In the case of differentially finitely generated algebras statements 11 and 12 can be amplified.

**Statement 18.** Let $f : A \rightarrow B$ be a differential homomorphism. Then

1. $f$ has the going-up property for differential ideals if and only if the mapping $f^*_\Delta : \text{SMax}^\Delta B \rightarrow \text{SMax}^\Delta A$ is closed.

2. $f$ has the going-down property for differential ideals if and only if the mapping $f^*_\Delta : \text{SMax}^\Delta B \rightarrow \text{SMax}^\Delta A$ is open.

3. $E$ is a constructible subset of $\text{SMax}^\Delta B$ then $f^*_\Delta(E)$ is a constructible subset of $\text{SMax}^\Delta A$.

Proof. (1)$\Rightarrow$. If $f$ has the going-up property for differential ideals then from statement 11 it follows that $f^*_\Delta : \text{Spec}^\Delta B \rightarrow \text{Spec}^\Delta A$ is a closed map. Let show that restriction of $f^*_\Delta$ to $\text{SMax}^\Delta B$ is closed. Let $b$ be a differential ideal in $B$ and $a$ its contraction to $A$. Let $V(a) \cap \text{SMax}^\Delta B$ is a closed set in $\text{SMax}^\Delta B$, then its image belongs to $V(a) \cap \text{SMax}^\Delta A$. Let show that the image coincides with the last set. Let $m$ be an ideal of $V(a) \cap \text{SMax}^\Delta A$, then there is a prime differential ideal $q \in V(b)$ contracting to $m$. From the definition of $m$ it follows that there exists an element $s$ such that $m$ is maximal differential ideal not meting $\{s^n\}$. Consider the set of all differential ideals containing $q$ and not meting $\{s^n\}$. Zorn’s lemma guaranties that there is a maximal ideal $n$ with that property. It is clear that $n$ is the desired ideal.

(1)$\Leftarrow$. Consider a closed set $V(b)$ in $\text{Spec}^\Delta B$ and let $a$ be a contraction of $b$. Let us show that the image of $V(b)$ coincides with $V(a)$, hence is closed and statement 11 implies the required result.

Let $p \in V(a)$. Since $\text{SMax}^\Delta A$ is very dense in $\text{Spec}^\Delta A$, then

$$q = \bigcap_{q \subseteq m} m$$

where $m$ are locally closed. Then for every $m$ there exists a $n$ in $V(b)$ contracting to $m$. Thus intersection of all such $n$ contracts to $q$. Therefore $q^{SC} = q$. Statement 11 completes the proof.

(2)$\Rightarrow$. Since $f$ has the going-down property, statement 12 implies that $f^*_\Delta : \text{Spec}^\Delta B \rightarrow \text{Spec}^\Delta A$ is open. Let $s \in A$ be an arbitrary element, then

$$f^*_\Delta((\text{Spec}^\Delta B)_s) = \bigcup_i (\text{Spec}^\Delta A)_{u_i}$$

Let us show that

$$f^*_\Delta((\text{SMax}^\Delta B)_s) = \bigcup_i (\text{SMax}^\Delta A)_{u_i}$$

The inclusion $\subseteq$ is obvious. Let us show the other one. Let $m$ belongs to $(\text{SMax}^\Delta A)_{u_i}$. Then from the definition there is an element $t$ such that $m$ is maximal differential ideal not meting $\{t^n\}$. Moreover, there exists
$\mathfrak{q} \in (\text{Spec}^\Delta B)_s$ such that $f^\Delta_\mathfrak{q}(\mathfrak{q}) = \mathfrak{m}$. Consider the set of all differential ideals containing $\mathfrak{q}$ and not meting $\{(st)^n\}_{n \in \mathbb{N}}$. A maximal element of this set is a desired one. So, the image of any principal open set is open. Therefore the image of any open set is open.

(2)$\implies$. Conversely,

$$f^\Delta_\mathfrak{q}((\text{Max}^\Delta B)_s) = \bigcup_i (\text{Max}^\Delta A)_{u_i}$$

Let us show that

$$f^\Delta_\mathfrak{q}((\text{Spec}^\Delta B)_s) = \bigcup_i (\text{Spec}^\Delta A)_{u_i}$$

The inclusion $\subseteq$. Let $\mathfrak{q}$ in $(\text{Spec}^\Delta B)_s$, then it lays in some ideal $\mathfrak{m}$ of $(\text{Max}^\Delta B)_s$, q. e. d.

Conversely, the inclusion $\supseteq$. Statement 16 implies that any ideal $\mathfrak{p} \in (\text{Max}^\Delta A)_{u_i}$ can be presented as an intersection

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \subseteq \mathfrak{n}} \mathfrak{m}$$

where $\mathfrak{m}$ are in $(\text{Max}^\Delta A)_{u_i}$. For every $\mathfrak{m}$ there is a pre-image $\mathfrak{n}$ in $(\text{Max}^\Delta B)_s$. The intersection of mentioned pre-images contracts to $\mathfrak{q}$. Therefore there exists a prime differential ideal contracting to $\mathfrak{q}$.

(3). Let $X = \text{Spec}^\Delta A$ and $Y = \text{Spec}^\Delta B$. Since $E$ is constructible, it has the following form

$$E = (V(b_1) \cap Y_{i_1} \cup \ldots \cup V(b_n) \cap Y_{i_n}) \cap \text{Max}^\Delta B.$$ 

Then the image of the set $V(b_1) \cap Y_{i_1} \cup \ldots \cup V(b_n) \cap Y_{i_n}$ is constructible (statement 11), i.e., is of the following form

$$V(a_1) \cap X_{s_1} \cup \ldots \cup V(a_n) \cap X_{s_m}.$$ 

Let us show that

$$f^\Delta_\mathfrak{q}(E) = (V(a_1) \cap X_{s_1} \cup \ldots \cup V(a_n) \cap X_{s_m}) \cap \text{Max}^\Delta A.$$ 

Indeed, the inclusion $\subseteq$ is obvious. We show the other one. Let $\mathfrak{m}$ be a locally closed point of differential spectrum of $A$ belonging to the right part. Then there is a prime differential ideal $\mathfrak{p}$ contracting to $\mathfrak{m}$ and belonging to $V(b_1) \cap Y_{i_1} \cup \ldots \cup V(b_n) \cap Y_{i_n}$. Let $\mathfrak{p}$ is in $V(b_i) \cap Y_{i_i}$ and let $s$ is an elements such that $\mathfrak{m}$ is maximal differential ideal not meting $\{s^n\}$. Then the maximal differential ideal in $B$ containing $\mathfrak{p}$ and not meting $\{(st)^n\}$ is the desired one. This set is not empty because $\mathfrak{p}$ is a prime differential ideal in $B$ not containing $s$ and $t_i$.  

10 Differential algebraic varieties

The information about differential algebraic varieties and groups can be found in [4]. We use this work as a background. All terms are taken from [4]. Let fix differentially closed field $K$ of characteristic zero. From
the definition of differentially closed field it follows that any simple differentially finitely generated algebra over $K$ coincides with $K$. Consequently, for any algebra $A$ differentially finitely generated over $K$ we have $\text{Max}^A A = \text{SMax}^A A$. From the other hand, algebra $A$ can be presented as a quotient ring $A = K\{y_1, \ldots, y_n\}/a$. Then the ideal $a$ defines a differential algebraic variety $V(a)$ in $K^n$. The points of $V(a)$ corresponds to differential homomorphisms of $A$ to $K$. The kernel of every such homomorphism is maximal differential ideal. Since algebra $A$ contains a field $K$, we have a bijection between $V(a)$ and $\text{Max}^A A$.

The example [4, chapter I, sec. 5, pp. 901] shows that regular functions do not necessarily belong to coordinate ring. Thus we need an auxiliary statement.

**Statement 19.** Let $X$ be a differential algebraic variety, $A = K\{X\}$ be its coordinate ring, and $f$ is a regular function on $X$. Let $Y$ be a graph of $f$ then $X$ is isomorphic to $Y$.

**Proof.** Consider a variety $X \times K$, its coordinate ring coincides with a polynomial ring $A\{y\}$. The graph of $f$ is defined by the equation $y - f = 0$. It is clear that mappings $X \to Y$ by the rule $(x, f(x))$ and $Y \to X$ defined by $(x, y) \mapsto x$ are inverse to each other isomorphisms.

**Corollary 20.** Let $\varphi: X \to Y$ be a morphism of differential algebraic varieties. Then there exist a coordinate rings $K\{X\}$ and $K\{Y\}$ and a homomorphism $\phi: K\{Y\} \to K\{X\}$ such that $\varphi = \phi^*$.

**Proof.** Consider some coordinate rings $K\{X\}$ and $K\{Y\}$. The ring $K\{Y\} = K\{y_1, \ldots, y_n\}$ is differentially finitely generated over $K$. Consider regular functions $x_i = \varphi^*(y_i) = y_i \circ \varphi$. From the previous statement it follows that $K\{X\}\{x_1, \ldots, x_n\}$ is a coordinate ring of $X$ with required properties.

We shall restate proved statements on geometrical language. The following result is a geometrical version of theorem 6.

**Statement 21.** Let $X \to Y$ be a morphism of differential algebraic varieties, $X$ being irreducible. Then the image of $X$ contains an open in its closure.

From statement 18 condition (3) we have the following.

**Statement 22.** Every morphism of differential algebraic varieties maps constructible sets to constructible sets.

Two previous statements are well-known. Our aim was to show how new technique works in this situation. The following two results show new geometrical properties of differential algebraic varieties. Corollary 14 with statement 18 imply the following.

**Statement 23.** Let $\varphi: X \to Y$ be a dominant morphism of irreducible differential algebraic varieties. Then there exists an open subset $U \in X$ such that the restriction $\varphi|_U$ is open.
Previous statement gives a good result in the case of differential algebraic groups.

**Statement 24.** Let $\varphi: G \to H$ be a dominant morphism of differential algebraic groups. Then $\varphi$ is a surjective open mapping.

**Proof.** The fact that $\varphi$ is surjective is well-known. We need to show that $\varphi$ is open. It is enough to consider the identity components $\varphi: G_0 \to H_0$. From the previous statement it follows that there exist an open subset $U \subseteq G_0$ such that restriction of $\varphi$ on $U$ is open. But the sets of the form $gU$, where $g \in G_0$, cover $G_0$. Since multiplying on $g$ is a homeomorphism, $\varphi$ is open everywhere.

**Remark 25.** The last statement allows to calculate inverse images of sheaves on differential algebraic groups.

**References**

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