Minimum Riesz energy problems with external fields

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Abstract. The paper deals with minimum energy problems in the presence of external fields with respect to the Riesz kernels $|x - y|^{\alpha-n}$, $0 < \alpha < n$, on $\mathbb{R}^n$, $n \geq 2$. For quite a general (not necessarily lower semicontinuous) external field $f$, we obtain necessary and/or sufficient conditions for the existence of $\lambda_{A,f}$ minimizing the Gauss functional

$$\int |x - y|^{\alpha-n} d(\mu \otimes \mu)(x,y) + 2 \int f \, d\mu$$

over all positive Radon measures $\mu$ with $\mu(\mathbb{R}^n) = 1$, concentrated on quite a general (not necessarily closed) $A \subset \mathbb{R}^n$. We also provide various alternative characterizations of the minimizer $\lambda_{A,f}$, analyze the continuity of both $\lambda_{A,f}$ and the modified Robin constant for monotone families of sets, and give a description of the support of $\lambda_{A,f}$. The significant improvement of the theory in question thereby achieved is due to a new approach based on the close interaction between the strong and the vague topologies, as well as on the theory of inner balayage, developed recently by the author.

1. Statement of the problem

1.1. Basic assumptions and general facts. Fix $n \geq 2$ and $0 < \alpha < n$. The present paper deals with minimum energy problems with respect to the $\alpha$-Riesz kernel $\kappa_\alpha(x,y) := |x - y|^{\alpha-n}$ on $\mathbb{R}^n$, evaluated in the presence of external fields $f : \mathbb{R}^n \to [-\infty, \infty]$. (Here $|x - y|$ denotes the Euclidean distance between $x, y \in \mathbb{R}^n$.)

Denote by $\mathcal{M} = \mathcal{M}(\mathbb{R}^n)$ the linear space of all (scalar real-valued Radon) measures $\mu$ on $\mathbb{R}^n$, equipped with the vague ($= \text{weak}^*$) topology of pointwise convergence on the class $C_0(\mathbb{R}^n)$ of all continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ of compact support, and by $\mathcal{M}^+ = \mathcal{M}^+(\mathbb{R}^n)$ the cone of all positive $\mu \in \mathcal{M}$, where $\mu$ is positive if and only if $\mu(\varphi) \geq 0$ for all positive $\varphi \in C_0(\mathbb{R}^n)$. When speaking of a (signed) measure $\mu \in \mathcal{M}$, we always understand that its $\alpha$-Riesz potential

$$U^\mu(x) := \int \kappa_\alpha(x,y) \, d\mu(y), \quad x \in \mathbb{R}^n,$$

is well defined and finite almost everywhere (a.e.) with respect to the Lebesgue measure on $\mathbb{R}^n$; or equivalently, that (cf. [22, Section I.3.7])

$$\int_{|y| > 1} \frac{d|\mu|(y)}{|y|^{n-\alpha}} < \infty, \quad (1.1)$$

where $|\mu| := \mu^+ + \mu^-$, $\mu^+$ and $\mu^-$ being the positive and negative parts of $\mu$ in the Hahn–Jordan decomposition. (This would necessarily hold if $\mu$ were assumed to be bounded, that is, with $|\mu|(\mathbb{R}^n) < \infty$.) Actually, then (and only then) $U^\mu$ is finite quasi-everywhere (q.e.) on $\mathbb{R}^n$, namely, everywhere except for a set of zero outer capacity, cf. [22, Section III.1.1]. (Regarding the concepts of outer and inner $\alpha$-Riesz capacities, denoted by $c^\ast(\cdot)$ and $c_\ast(\cdot)$, respectively, see [22, Section II.2.6].)

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Yet another assumption sufficient for $U^n$ to be finite q.e. on $\mathbb{R}^n$ is that $\mu$ be of finite $\alpha$-Riesz energy (see [15 Corollary to Lemma 3.2.3]), i.e.

$$I(\mu) := \int \kappa_\alpha(x, y) \, d(\mu \otimes \mu)(x, y) < \infty.$$ 

A basic fact to be permanently used in what follows is that the $\alpha$-Riesz kernel is strictly positive definite, which means that the energy of any (signed) $\mu \in \mathcal{M}$ is $\geq 0$ (whenever defined), and it is zero only for $\mu = 0$, see [22 Theorem 1.15]. Then all $\mu \in \mathcal{M}$ with $I(\mu) < \infty$ form a pre-Hilbert space $\mathcal{E} = \mathcal{E}(\mathbb{R}^n)$ with the inner product

$$\langle \mu, \nu \rangle := I(\mu, \nu) := \int \kappa_\alpha(x, y) \, d(\mu \otimes \nu)(x, y)$$

and the energy norm $\|\mu\| := \sqrt{I(\mu)}$, cf. [15 Section 3.1]. The (Hausdorff) topology on $\mathcal{E}$ determined by means of this norm $\| \cdot \|$ is said to be strong.

Another fact crucial to our study is that the cone $\mathcal{E}^+ = \mathcal{E}^+(\mathbb{R}^n) := \mathcal{E} \cap \mathcal{M}^+(\mathbb{R}^n)$ is complete in the induced strong topology, and that the strong topology on $\mathcal{E}^+$ is finer than the vague topology on $\mathcal{E}^+$ (see [7, 10, 11, cf. [22 Section I.4.13]). Thus any strong Cauchy sequence (net) $(\mu_j) \subset \mathcal{E}^+$ converges both strongly and vaguely to the same (unique) $\mu_0 \in \mathcal{E}^+$. The $\alpha$-Riesz kernel is, therefore, perfect [15, Section 3.3].

1.2. Statement of the problem. For any $A \subset \mathbb{R}^n$, let $\mathcal{M}^+(A)$ stand for the class of all $\mu \in \mathcal{M}^+$ concentrated on $A$, which means that $A^c := \mathbb{R}^n \setminus A$ is $\mu$-negligible, or equivalently, that $A$ is $\mu$-measurable and $\mu = \mu|_A$, $\mu|_A := 1_A \cdot \mu$ being the trace of $\mu$ to $A$, cf. [4 Section V.5.7]. (Note that for $\mu \in \mathcal{M}^+(A)$, the indicator function $1_A$ of $A$ is locally $\mu$-integrable.) We also denote

$$\mathcal{M}^+(A) := \{ \mu \in \mathcal{M}^+(A) : \mu(\mathbb{R}^n) = 1 \},$$

$$\mathcal{E}^+(A) := \mathcal{E} \cap \mathcal{M}^+(A), \quad \mathcal{E}^+_V(A) := \mathcal{E} \cap \mathcal{M}^+(A).$$

Given $A \subset \mathbb{R}^n$, fix a universally measurable function $f : \overline{A} \to [-\infty, \infty]$, to be treated as an external field acting on charges (measures) carried by $\overline{A}$ ($:= \text{Cl}_\mathbb{R}^n A$), and let $\mathcal{E}_V^+(A)$ stand for the class of all $\mu \in \mathcal{E}^+(A)$ such that $f$ is $\mu$-integrable [4] (Chapter IV, Sections 3, 4). Then the Gauss functional (= the $f$-weighted energy)

$$I_f(\mu) := I(\mu) + 2 \int f \, d\mu$$

is finite for all $\mu \in \mathcal{E}_V^+(A)$, and hence one can introduce the extremal value

$$w_f(A) := \inf_{\mu \in \mathcal{E}_V^+(A)} I_f(\mu) \in [-\infty, \infty],$$

where

$$\mathcal{E}_V^+(A) := \mathcal{E}_V^+(A) \cap \mathcal{M}^+(A).$$

• Throughout the present paper, we always assume that

$$-\infty < w_f(A) < \infty. \tag{1.2}$$

(See Lemma [130] below for necessary and sufficient conditions for (1.2) to occur.) Then the class $\mathcal{E}_V^+(A)$ is nonempty, and hence the following problem makes sense.

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1It is worth noting here that according to a well-known counterexample by H. Cartan [7], the whole pre-Hilbert space $\mathcal{E}$ is strongly incomplete (cf. [22 Theorem 1.19]).

2As usual, the infimum over the empty set is interpreted as $+\infty$. We also agree that $1/(+\infty) = 0$ and $1/0 = +\infty$. 

Problem 1.1. Does there exist $\lambda_{A,f} \in \mathcal{E}^+_{f}(A)$ with $I_f(\lambda_{A,f}) = w_f(A)$? 

This problem, originated by C.F. Gauss [19] for charges on the boundary surface of a bounded domain in $\mathbb{R}^3$, is sometimes referred to as the inner Gauss variational problem [24]. Recent results on this topic are reviewed in the monographs [3, 23] (see also numerous references therein); for the latest works on Problem 1.1, see [9, 13].

Unlike the previous studies on Problem 1.1 in the present paper we do not require an external field $f$ to be necessarily lower semicontinuous, and a set $A$ to be necessarily closed. The $f$-weighted energy $I_f(\mu)$ is, therefore, no longer vaguely lower semicontinuous, and the class $\mathcal{M}^+(A)$ is no longer vaguely closed. To overcome these additional difficulties, we initiate a new approach based on the close interaction between the strong and the vague topologies on the (strongly complete) cone $\mathcal{E}^+(\mathbb{R}^n)$.

In the particular case where $0 < \alpha \leq 2$, the analysis performed below is also substantially based on the theory of inner $\alpha$-Riesz balayage, established in the author’s recent papers [26, 30]. (Such a theory generalizes the theory of inner Newtonian balayage, originated in the pioneering work by Cartan [8]. For the theory of outer $\alpha$-Riesz balayage, see the monographs [2, 12], the latter dealing with $\alpha = 2$.)

The approach thereby developed enables us to obtain necessary and/or sufficient conditions for the solvability of Problem 1.1 for noncompact (and even nonclosed) sets $A \subset \mathbb{R}^n$, and for quite general (not necessarily lower semicontinuous) external fields $f$ (see Theorems 2.1, 2.7, 2.13, 2.15 and Corollaries 2.2, 2.11).

Furthermore, we provide a number of alternative characterizations of the solution $\lambda_{A,f}$ to Problem 1.1 (see Theorems 1.4 and 2.7), and we analyze the continuity of $\lambda_{A,f}$ as well as that of the so-called inner modified Robin constant for monotone families of sets (see Theorems 2.3, 2.5, 2.17, and 2.18).

Finally, we establish a complete description of the support of $\lambda_{A,f}$ (Theorems 2.19, 2.22), thereby giving an answer to Open question 2.1 by M. Ohtsuka [24, p. 284].

These improve significantly recent results on the topic in question, see Section 2.5 for details. New phenomena thereby discovered are illustrated by means of examples.

1.3. General assumptions on external fields. Preliminary results on Problem 1.1.

For closed $F \subset \mathbb{R}^n$, we denote by $\Phi(F)$ the class of all lower semicontinuous (l.s.c.) functions $\varphi : F \to (-\infty, \infty]$ that are $\geq 0$ unless $F$ is compact.

- Given arbitrary $A \subset \mathbb{R}^n$, in all that follows we assume $f$ to be of the form

$$f = \psi + U^0 + U^\omega;$$

(1.3)

where $\psi$, $\vartheta$, and $\omega$ have the following properties:

(P$_1$) $\psi \in \Phi(\overline{A})$.

(P$_2$) $\vartheta \in \mathcal{E}(\mathbb{R}^n)$.

(P$_3$) $\omega \in \mathcal{M}(\mathbb{R}^n)$ is bounded and such that

$$d(S_\omega, A) := \inf_{x \in S_\omega, y \in A} |x - y| > 0,$$

$S_\omega = S(\omega)$ being the support of $\omega$.

3If the unweighted case $f = 0$ takes place, then we drop the index $f$, and write $w(A)$, $\lambda_A$, etc. in place of $w_f(A)$ and $\lambda_{A,f}$, respectively. (Note that, actually, $c_e(A) = 1/w(A)$.)

4In the previous researches on Problem 1.1 (see e.g. [3, 9, 13, 23] and references therein), an external field was always defined to be a lower semicontinuous function $\varphi$, whereas the presence of an alternative/additional source of energy, generated by signed charges $\vartheta \in \mathcal{E}(\mathbb{R}^n)$ and/or $\omega \in \mathcal{M}(\mathbb{R}^n)$, cf. [4,3], agrees well with the original electrostatic nature of the Gauss variational problem.

5Each of the summands in (1.3) might be 0. (For $\omega = 0$, we have $S_\omega = \varnothing$, hence $d(S_\omega, A) = +\infty$.)
Such an external field \( f \) is well defined as a finite number or \( \pm \infty \) q.e. on \( \overline{A} \), for so are all the summands in (1.3), see Section 1.1. (Here we have used the countable subadditivity of outer capacity on arbitrary sets in \( \mathbb{R}^n \) \cite[Section II.2.6]{22}.)

Similarly, the \( f \)-weighted potential
\[
U_f^\mu := U^\mu + f, \quad \mu \in \mathcal{M},
\]
is well defined as a finite number or \( \pm \infty \) q.e. on \( A \) — provided, of course, that the measure \( \mu \) meets (1.1) (thus in particular if \( \mu \) is bounded or of finite energy).

**Lemma 1.2.** Given an external field \( f \) of form (1.3), \( \int f \, d\mu \) as well as \( I_f(\mu) \) is well defined as a finite number or \( +\infty \) for all bounded \( \mu \in \mathcal{E}^+(A) \).

**Proof.** Observe that \( f \) is \( \mu \)-measurable for all \( \mu \in \mathcal{M}^+(A) \), for so are all the summands \( \psi, U^\varrho, \) and \( U^\omega \) in (1.3). We further note that the inequality \( \int \psi \, d\mu > -\infty \) is obvious whenever \( \psi \geq 0 \), while the remaining case of compact \( A \) is treated by replacing \( \psi \) by \( \psi' := \psi + c \geq 0 \) on \( \overline{A} \), where \( c \in (0, \infty) \), which is always possible since a lower semicontinuous function on a compact set is lower bounded.

As \( \int U^\varrho \, d\mu \) is certainly finite for all \( \mu \in \mathcal{E} \), it remains to verify that
\[
\int U^\omega^- \, d\mu < \infty \quad \text{for all bounded} \quad \mu \in \mathcal{M}^+(\overline{A}),
\]
which however follows immediately from the estimates (cf. (P3))
\[
\int |x - y|^{\alpha-n} d(\omega^- \otimes \mu)(x, y) \leq d(S_\omega, A)^{\alpha-n} \omega^-(\mathbb{R}^n) \mu(\mathbb{R}^n) < \infty \quad (1.4)
\]
by making use of Lebesgue–Fubini’s theorem \cite[Section V.8, Theorem 1(a)]{4}.

Lemma 1.2 enables us to deduce from our earlier paper \cite{25} the following two theorems on Problem 1.1. (In view of their important role in the proofs of the main results of the present study, we provide here their explicit formulations.)

**Theorem 1.3 (cf. \cite[Theorems 1, 2]{25}).** If the solution \( \lambda = \lambda_{A,f} \) to Problem 1.1 exists\textsuperscript{6} then its \( f \)-weighted potential \( U_f^\lambda \) has the properties\textsuperscript{7}
\[
U_f^\lambda \geq c_{A,f} \quad \text{n.e. on} \quad A, \tag{1.5}
\]
\[
U_f^\lambda = c_{A,f} \quad \text{\( \lambda \)-a.e. on} \quad A, \tag{1.6}
\]
where
\[
c_{A,f} := \int U_f^\lambda \, d\lambda = w_f(A) - \int f \, d\lambda \in (-\infty, \infty) \quad (1.7)
\]
is said to be the inner \( f \)-weighted equilibrium constant\textsuperscript{8} If moreover \( f \) is l.s.c. on \( \overline{A} \), then also
\[
U_f^\lambda \leq c_{A,f} \quad \text{on} \quad S(\lambda),
\]
and hence
\[
U_f^\lambda = c_{A,f} \quad \text{n.e. on} \quad A \cap S(\lambda).
\]

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\textsuperscript{6}The solution \( \lambda_{A,f} \) to Problem 1.1 is unique (if it exists), which follows easily from the convexity of the class \( \hat{\mathcal{E}}_0^+(A) \) by use of the parallelogram identity in the pre-Hilbert space \( \mathcal{E} \) (cf. \cite[Lemma 6]{25}). This \( \lambda_{A,f} \) will also be referred to as the inner \( f \)-weighted equilibrium measure.

\textsuperscript{7}The abbreviation n.e. (nearly everywhere) means, as usual, that the set of all \( x \in A \) where the inequality fails is of zero inner capacity. Compare with the concept of quasi-everywhere, where an exceptional set must be of zero outer capacity (see Section 1.1).

\textsuperscript{8}Similarly to \cite[p. 27]{23}, \( c_{A,f} \) is also said to be the inner modified Robin constant.
Relations (1.5) and (1.7), resp. (1.6) and (1.7), characterize the minimizer $\lambda_{A,f}$ uniquely within $\tilde{E}_f^+(A)$. In more detail, the following theorem holds true.

**Theorem 1.4** (cf. [25, Proposition 1]). For $\mu \in \tilde{E}_f^+(A)$ to be the (unique) solution $\lambda_{A,f}$ to Problem (1.1), it is necessary and sufficient that either of the following two characteristic inequalities be fulfilled:

$$U_{f}^\mu \geq \int U_{f}^\mu \ d\mu =: c_1 \text{ n.e. on } A,$$

$$U_{f}^\mu \leq w_f(A) - \int f \ d\mu =: c_2 \text{ } \mu\text{-a.e. on } A. \tag{1.8}$$

If either of (1.8) or (1.9) holds, then equality actually prevails in (1.9), and moreover

$$c_1 = c_2 = c_{A,f},$$

$c_{A,f}$ being the inner $f$-weighted equilibrium constant.

1.4. **When does (1.2) hold?** For (1.2) to be satisfied, it is necessary that $c_*(A) > 0$, since otherwise the class $E_f^+(A)$ would reduce to $\{0\}$ (see e.g. [15, Lemma 2.3.1]). Actually, the following stronger assertion holds true.

**Lemma 1.5.** For (1.2) to hold, it is necessary and sufficient that

$$c_*(\{x \in A : \psi(x) < \infty\}) > 0, \tag{1.10}$$

$\psi$ being the first summand in representation (1.3).

**Proof.** According to [25, Lemma 5], $w_f(A) < \infty$ is fulfilled if and only if

$$c_*(\{x \in A : |f|(x) < \infty\}) > 0,$$

which however is equivalent to (1.10) since the second and the third summands in (1.3) take finite values q.e. (hence n.e.) on $\mathbb{R}^n$ (see Section 1.1).

It thus remains to show that $w_f(A) > -\infty$. For any given $\vartheta \in \mathcal{E}$,

$$I_{U_{\vartheta}}(\mu) = \|\mu\|^2 + 2 \int U_{\vartheta}^\mu \ d\mu = \|\mu + \vartheta\|^2 - \|\vartheta\|^2 \text{ for all } \mu \in \mathcal{E};$$

hence, by the strict positive definiteness of the $\alpha$-Riesz kernel,

$$w_{U_{\vartheta}}(A) \geq -\|\vartheta\|^2 > -\infty.$$

On account of (1.4), the proof will be completed by verifying the inequality

$$\inf_{\mu \in \tilde{E}_f^+(A)} \int \psi \ d\mu > -\infty.$$

This however is obvious if $\psi \geq 0$, while the remaining case of compact $\overline{A}$ is treated as usual, namely, by replacing $\psi$ by $\psi' := \psi + c \geq 0$ on $\overline{A}$, where $c \in (0, \infty)$. □

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9Here we have used the following strengthened version of countable subadditivity for inner capacity: For arbitrary $A \subset \mathbb{R}^n$ and universally measurable $U_j \subset \mathbb{R}^n$, $j \in \mathbb{N}$,

$$c_*(\bigcup_{j \in \mathbb{N}} A \cap U_j) \leq \sum_{j \in \mathbb{N}} c_*(A \cap U_j).$$

(See [15] pp. 157–159 or [8] p. 253, compare with [22] p. 144.)
2. Main results

• Recall that throughout the whole paper, an external field $f$ is of form (1.3), where $\psi$, $\vartheta$, and $\omega$ satisfy (P1)–(P3) as well as (1.10). (These general conventions will not be repeated hereafter.)
• Along with these general assumptions, throughout Sections 2.2–2.4 as well as in Theorem 2.1 we additionally require a set $A \subset \mathbb{R}^n$ to have the following property:
  \((P_4)\) The class $\mathcal{E}^+(A)$ is closed in the strong topology on $\mathcal{E}^+$. (This in particular occurs if $A$ is quasiclosed (quasicompact), that is, if $A$ can be approximated in outer capacity by closed (compact) sets \([16]\); see Theorem 3.9 below.)

2.1. The case of $\alpha \in (0, n)$. Theorems 2.1, 2.2, and Corollary 2.2 do hold for the $\alpha$-Riesz kernels of arbitrary order $\alpha \in (0, n)$.

**Theorem 2.1.** Assume, in addition, that \((P_4)\) is fulfilled. If moreover
\[
c_*(A) < \infty, \tag{2.1}
\]
then the (unique) solution $\lambda_{A,f}$ to Problem 1.1 does exist.

**Corollary 2.2.** Problem 1.1 is solvable for any quasicompact $A \subset \mathbb{R}^n$.

**Proof.** For quasicompact $A$, (2.1) necessarily holds, the $\alpha$-Riesz capacity of a compact set being obviously finite. Since \((P_4)\) is fulfilled as well (Theorem 3.9), the corollary follows directly from Theorem 2.1. \(\square\)

Given $A \subset \mathbb{R}^n$, denote by $\mathcal{C}_A$ the upward directed set of all compact subsets $K$ of $A$, where $K_1 \subseteq K_2$ if and only if $K_1 \subset K_2$. If a net $(x_K)_{K \in \mathcal{C}_A} \subset Y$ converges to $x_0 \in Y$, $Y$ being a topological space, then we shall indicate this fact by writing $x_K \to x_0$ in $Y$ as $K \uparrow A$.

**Theorem 2.3.** Assume that the solution $\lambda_{A,f}$ to Problem 1.1 exists.\(^{10}\) Then
\[
\lambda_{K,f} \to \lambda_{A,f} \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A, \tag{2.2}
\]
$\lambda_{K,f}$ being the solution to Problem 1.1 for $K \in \mathcal{C}_A$ large enough. The inner $f$-weighted equilibrium constant $c_{K,f}$ also varies continuously when $K \uparrow A$, that is,
\[
\lim_{K \uparrow A} c_{K,f} = c_{A,f}. \tag{2.3}
\]

**Theorem 2.4.** Let $(A_j) \subset \mathbb{R}^n$ be an increasing sequence of universally measurable sets with the union $A$. Then
\[
w_f(A_j) \downarrow w_f(A) \text{ as } j \to \infty. \tag{2.4}
\]
If moreover there exist the minimizers $\lambda_{A,f}$ and $\lambda_{A_j,f}$, then also\(^{11}\)
\[
\lambda_{A_j,f} \to \lambda_{A,f} \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } j \to \infty, \tag{2.5}
\]
\[
\lim_{j \to \infty} c_{A_j,f} = c_{A,f}. \tag{2.6}
\]

**Theorem 2.5.** Let $(A_j) \subset \mathbb{R}^n$ be a decreasing sequence of quasiclosed sets with the intersection $A$, and let $c_*(A_{j_0}) < \infty$ for some $j_0 \in \mathbb{N}$. Then
\[
w_f(A_j) \uparrow w_f(A) \text{ as } j \to \infty, \tag{2.7}
\]
and moreover \((2.3), (2.6)\) hold true (the minimizers $\lambda_{A_j,f}$, $j \geq j_0$ and $\lambda_{A,f}$ do exist).

\(^{10}\)See Theorems 2.1, 2.2, 2.3 and Corollaries 2.2, 2.4 for sufficient conditions for this to hold.

\(^{11}\)This holds, for instance, if $A_j$, $j \in \mathbb{N}$, are closed, and $A$, their union, is of finite capacity. See also Theorem 2.11 below.
Remark 2.6. Theorem 2.5 as well as Theorem 2.18 (see below) remains valid for nets in place of sequences, provided that the sets in question are closed.

2.2. The case of $\alpha \in (0, 2]$. Throughout Sections 2.2–2.4, we always assume that:

- $\alpha \in (0, 2]$.
- $(P_0)$ is fulfilled.
- $\psi, \vartheta, \text{ and } \omega$ in (1.3) satisfy the hypotheses $\psi = 0, \vartheta^+ = \omega^+ = 0$.

To simplify notations, write $\tau := \vartheta^-$ and $\sigma := \omega^-; \text{ then } f$ takes the form

$$f = -U^\tau - U^\sigma = -U^\delta, \quad (2.8)$$

where $\delta := \tau + \sigma \in \mathfrak{M}^+(\mathbb{R}^n)$, while $\tau$ and $\sigma$ have the following properties:

$(P'_2)$ $\tau \in \mathcal{E}^+(\mathbb{R}^n)$.

$(P'_3)$ $\sigma \in \mathfrak{M}^+(\mathbb{R}^n)$ is bounded and such that

$$d(S_\sigma, A) > 0. \quad (2.9)$$

Denote by $\mu^{\partial}$ the inner $\alpha$-Riesz balayage of $\mu$ to $Q$, where $\mu \in \mathfrak{M}^+$ and $Q \subset \mathbb{R}^n$ are arbitrary (see [26, 27] for some basic results on this concept, and [28–30] for their further development; cf. also Section 4.6 below).

Theorem 2.7. Under the above assumptions, if moreover

$$\delta(\mathbb{R}^n) \leq 1, \quad (2.10)$$

then Problem 1.1 is solvable if and only if

either $c_*(A) < \infty$, or $\delta(A) = 1. \quad (2.11)$

In the affirmative case, the (unique) solution $\lambda_{A,f}$ to Problem 1.1 has the form

$$\lambda_{A,f} = \begin{cases} \delta^A + \eta_{A,f} \gamma_A & \text{if } c_*(A) < \infty, \\ \delta^A & \text{otherwise,} \end{cases} \quad (2.12)$$

where $\gamma_A$ is the inner capacitary measure on $A$, normalized by $\gamma_A(\mathbb{R}^n) = c_*(A)$, while

$$\eta_{A,f} := \frac{1 - \delta^A(\mathbb{R}^n)}{c_*(A)} \in [0, \infty). \quad (2.13)$$

Furthermore, this $\lambda_{A,f}$ can alternatively be characterized by means of any one of the following three assertions:

(i) $\lambda_{A,f}$ is the unique measure of minimum energy in the class

$$\Lambda_{A,f} := \{ \mu \in \mathfrak{M}^+ : U^\mu_f \geq \eta_{A,f} \text{ n.e. on } A \}, \quad (2.14)$$

$\eta_{A,f}$ being introduced by formula (2.13). That is, $\lambda_{A,f} \in \Lambda_{A,f}$ and

$$I(\lambda_{A,f}) = \min_{\mu \in \Lambda_{A,f}} I(\mu). \quad (2.15)$$

(ii) $\lambda_{A,f}$ is the unique measure of minimum potential in the class $\Lambda_{A,f}$, introduced by means of (2.14). That is, $\lambda_{A,f} \in \Lambda_{A,f}$ and

$$U^{\lambda_{A,f}} = \min_{\mu \in \Lambda_{A,f}} U^\mu \text{ on } \mathbb{R}^n. \quad (2.16)$$

12Due to $\psi = 0$, (1.10) is equivalent to the assumption $c_*(A) > 0$.

13This implies immediately that $\lambda_{A,f}$ can also be characterized as the unique measure of minimum $f$-weighted potential in the class $\Lambda_{A,f}$. Now, however, nearly everywhere on $\mathbb{R}^n$:

$$U^\mu_f = \min_{\mu \in \Lambda_{A,f}} U^\mu \text{ n.e. on } \mathbb{R}^n.$$
(iii) \( \lambda_{A,f} \) is the only measure in \( \mathcal{E}^+(A) \) with the property \( U_f^{\lambda_{A,f}} = \eta_{A,f} \) n.e. on \( A \).

In addition, the inner \( f \)-weighted equilibrium constant \( c_{A,f} \), introduced by (1.7), admits an alternative representation

\[
c_{A,f} = \eta_{A,f},
\]
and hence (2.3) can be specified as follows:

\[
c_K,f \downarrow c_{A,f} \text{ in } \mathbb{R} \text{ as } K \uparrow A.
\]

Remark 2.8. By [27] Corollary 5.3, the latter relation in (2.11) is valid e.g. if \( \delta(\mathbb{R}^n) = 1 \) while \( A \) is not inner \( \alpha \)-thin at infinity.

Recall that according to [21][27], \( Q \subset \mathbb{R}^n \) is said to be inner \( \alpha \)-thin at infinity if

\[
\sum_{k \in \mathbb{N}} c_x(Q_k) < \infty,
\]

where \( q \in (1, \infty) \) and \( Q_k := Q \cap \{x \in \mathbb{R}^n : q^k \leq |x| < q^{k+1}\} \); or equivalently, if either \( Q \) is bounded, or \( x = 0 \) is an inner \( \alpha \)-irregular boundary point for the inverse of \( Q \) with respect to \( |x| = 1 \). (For the concept of inner \( \alpha \)-irregular boundary points for arbitrary \( Q \subset \mathbb{R}^n \), see the author’s recent paper [26] Section 6; compare with N.S. Landkof’s book [22] Section V.1, where \( Q \) was required to be Borel.)

Corollary 2.9. Under the assumptions of Theorem 2.7 if moreover (2.11) is fulfilled, then \( \lambda_{A,f} \) is a measure of minimum total mass in the class \( \Lambda_{A,f} \), i.e.

\[
\lambda_{A,f}(\mathbb{R}^n) = \min_{\mu \in \Lambda_{A,f}} \mu(\mathbb{R}^n) \quad (= 1).
\]

Remark 2.10. The extremal property (2.20) cannot, however, serve as an alternative characterization of the minimizer \( \lambda_{A,f} \), for it does not determine \( \lambda_{A,f} \) uniquely within \( \Lambda_{A,f} \). Indeed, take, for instance, \( A := \{ |x| \geq 1 \} \), and let \( f \) be given by (2.8) with \( \delta \in \mathcal{D}(A^c) \). Since \( A \) is not \( \alpha \)-thin at infinity, applying [27] Corollary 5.3] gives

\[
\delta^A(\mathbb{R}^n) = \delta(\mathbb{R}^n) = 1.
\]

Hence \( \eta_{A,f} = 0 \), which in view of the equality \( U^{\delta^A} = U^\delta \) n.e. on \( A \) [27] Eq. (1.11)] yields \( \delta, \delta^A \in \Lambda_{A,f} \). Noting that \( \delta^A \neq \delta \), and taking (2.21) into account, we therefore conclude that there are actually infinitely many measures of minimum total mass in \( \Lambda_{A,f} \), for so are all the measures of the form \( a\delta + b\delta^A \), where \( a, b \in [0, 1] \) and \( a + b = 1 \).

Corollary 2.11. The following two assertions on the existence of \( \lambda_{A,f} \) hold true.

(a) If \( c_\alpha(A) = \infty \) and \( \delta(\mathbb{R}^n) < 1 \), then \( \lambda_{A,f} \) fails to exist.

(b) If \( \delta \in \mathcal{E}^+(A) \), then \( \lambda_{A,f} \) does exist. Moreover, then \( \lambda_{A,f} = \delta^A = \delta \), and hence Theorem 2.7 is fully applicable to both \( \lambda_{A,f} \) and \( c_{A,f} \).

Remark 2.12. Corollary 2.11(a) improves [13] Corollary 2.6(ii)] by showing that the latter remains valid if the closed set \( \Sigma \) involved in it, is just of infinite capacity. (Observe that the set \( \Sigma \) in [13] was required throughout not to be \( \alpha \)-thin at infinity. Regarding the existence of a set with infinite capacity which is, nonetheless, \( \alpha \)-thin at infinity, see [27] Remark 2.2], cf. also Example 2.24 below.)

14For a concept of outer thinness of \( Q \) at infinity when \( \alpha = 2 \), see M. Brelot [8, p. 313] as well as J.L. Doob [12, pp. 175–176]. As shown in [27] Remark 2.3], these two concepts are, actually, different, the latter being less restrictive. But if \( Q \) is Borel, then for \( \alpha = 2 \), the concept of inner thinness, given by (2.19), coincides, in fact, with that of outer thinness introduced by Doob. Therefore, when speaking of inner \( \alpha \)-thinness for Borel sets, the term "inner" might be omitted.
Theorem 2.13. Assume that \( A \) is not inner \( \alpha \)-thin at infinity. Then we have:

(a) If \( \delta(\mathbb{R}^n) = 1 \), \( \lambda_{A,f} \) does exist. Moreover, since then \( \delta^A(\mathbb{R}^n) = 1 \), Theorem 2.14 is fully applicable to both \( \lambda_{A,f} \) and \( c_{A,f} \). In particular,

\[
\lambda_{A,f} = \delta^A \quad \text{and} \quad c_{A,f} = 0. \tag{2.22}
\]

(b) Assume moreover that \( f \) is l.s.c. on \( \overline{A} \)

Then \( \lambda_{A,f} \) exists \( \iff \delta(\mathbb{R}^n) \geq 1 \). \tag{2.23}

In addition, \( c_{A,f} < 0 \) if \( \delta(\mathbb{R}^n) > 1 \). \tag{2.24}

Remark 2.14. The requirement on \( A \) of not being inner \( \alpha \)-thin at infinity is important for the validity of both (a) and (b) in Theorem 2.14 (see Theorem 2.15).

Let \( F \subset \mathbb{R}^n \) be closed and \( \alpha \)-thin at infinity. If moreover \( \alpha = 2 \), then there exists the unique connected component \( \Delta_F \) of the (open) set \( F^c \) such that \( (\Delta_F)^c \not\subset F \) still remains (closed and) \( 2 \)-thin at infinity. \tag{2.25}

For the given \( F \), denote

\[
\Omega_F = \begin{cases} 
F^c & \text{if } \alpha < 2, \\
\Delta_F & \text{if } \alpha = 2.
\end{cases}
\]

Theorem 2.15. Under the hypotheses listed at the beginning of this subsection, assume moreover that \( c_*(A) = \infty \), and that \( \overline{A} \) is \( \alpha \)-thin at infinity. Then Problem 1.11 is unsolvable whenever the following two assumptions are fulfilled:

\[
\delta(\mathbb{R}^n) \leq 1, \quad \delta(\Omega_F) > 0, \quad \Omega_F \text{ being introduced by (2.25) with } F := \overline{A}.
\]

Remark 2.16. As seen from either of Theorems 2.1 and 2.7, the assumption \( c_*(A) = \infty \) is necessary for the validity of both Corollary 2.11 (a) and Theorem 2.15.

2.3. Convergence results. Referring to Theorems 2.3 and 2.5 for convergence results for \( \alpha \in (0, n) \), assume now that \( \alpha \leq 2 \), and that \( f \) is of form (2.8).

Theorem 2.17. Let \( (A_j) \) be an increasing sequence of closed set \( A_j \subset \mathbb{R}^n \) with the union \( A \) which is not \( \alpha \)-thin at infinity, and let each \( A_j \) be either of finite capacity, or not \( \alpha \)-thin at infinity. Assume that either \( \delta(\mathbb{R}^n) = 1 \), or \( U^\delta \) is continuous on \( A \) while \( \delta(\mathbb{R}^n) > 1 \). Then (2.3)–(2.6) do hold (the minimizers \( \lambda_{A_j,f} \) and \( \lambda_{A,f} \) do exist) \( ^{16} \)

Theorem 2.18. Consider a decreasing sequence \( (A_j) \) of quasiclosed \( A_j \) with the intersection \( A \) which is not inner \( \alpha \)-thin at infinity, and let \( f \) be of form (2.8) with \( d(S_\sigma, A_1) > 0 \) and \( \delta(\mathbb{R}^n) = 1 \). Then (2.5)–(2.7) hold true. Furthermore,

\[
U^{\lambda_{A_j,f}} \downarrow U^{\lambda_{A,f}} \quad \text{pointwise on } \mathbb{R}^n \text{ as } j \to \infty. \tag{2.26}
\]

\(^{15}\)This occurs, for instance, if \( S(\tau) \cap \overline{A} = \emptyset \), cf. (2.8) and (P_3).

\(^{16}\)Compare with (2.22).

\(^{17}\)Indeed, for \( F \) compact, \( \Delta_F \) is, in fact, the (unique) unbounded connected component of \( F^c \). For \( F \) noncompact, the origin \( x = 0 \) is 2-irregular for \( F^c \), the inverse of \( F \cup \{\infty_{\mathbb{R}^n}\} \) with respect to \( |x| = 1 \). (Here \( \infty_{\mathbb{R}^n} \) denotes the Alexandroff point of \( \mathbb{R}^n \).) By \( ^9 \) Section VIII.6, Remark 3, there exists, therefore, a unique connected component \( G \) of the (open) set \( (F^c)^c \) such that \( x = 0 \) is 2-irregular for \( G^c \), and the inverse \( \Delta_F \) of this \( G \) with respect to \( |x| = 1 \) is as claimed.

\(^{18}\)In the case where \( c(A) < \infty \), limit relations (2.3)–(2.6) hold true under much more general assumptions — namely, for any \( \alpha \in (0, n) \) and any external field \( f \) of form (1.3) (see footnote 11). Here \( c(A) := c_*(A) = c^*(A) \), Borel sets being capacitable \( ^{22} \) (Theorem 2.8).
2.4. **A description of** \( S(\lambda_{A,f}) \). The **reduced kernel** \( \tilde{A} \) of \( A \) is the set of all \( x \in A \) with \( c_i(B(x,r) \cap A) > 0 \) for any \( r > 0 \), where \( B(x,r) := \{ y \in \mathbb{R}^n : |y - x| < r \} \), see [22, p. 164]. Under the assumptions listed at the beginning of Section 2.2, a description of the support of the minimizer \( \lambda_{A,f} \) is given in Theorems 2.19 and 2.22.

**Theorem 2.19.** Let \( A \) be closed, \( A = \tilde{A} \), \( \delta|_{\tilde{A}} \neq 0 \), and let \((a_2)\) or \((b_2)\) occur:

- \((a_2)\) \( c(A) < \infty \) and \( \delta(\mathbb{R}^n) \leq 1 \). If \( \alpha = 2 \), assume moreover that \( A^c \) is connected.
- \((b_2)\) \( A \) is not \( \alpha \)-thin at infinity, and \( \delta(\mathbb{R}^n) = 1 \).

If \( D \) denotes the union of all connected components \( D_i \) of \( A^c \) with \( \delta|_{D_i} > 0 \), then

\[
S(\lambda_{A,f}) = \begin{cases} A & \text{if } \alpha < 2, \\ S(\delta|_A) \cup \partial_{\mathbb{R}^n} D & \text{if } \alpha = 2. \end{cases}
\]  

(2.27)

**Remark 2.20.** Formula (2.27) gives an answer to [24, p. 284, Open question 2.1] (formulated for compact \( A = K \)) about conditions ensuring the identity \( S(\lambda_{A,f}) = A \).

**Remark 2.21.** Let \( n \geq 3 \), \( A := \mathbb{R}^{n-1} (= : \mathrm{Cl} \mathbb{R}^n \mathbb{R}^{n-1}) \), \( \mathbb{R}^{n-1} \) being immersed in \( \mathbb{R}^n \), and let \( f := -U^{\varepsilon_{x_0}} \), where \( \varepsilon_{x_0} \) is the unit Dirac measure at \( x_0 \in \mathbb{R}^n \setminus \mathbb{R}^{n-1} \). As stated in [13] Section 4.2, case (i)], \( S(\lambda_{A,f}) = \mathbb{R}^{n-1} \). However, for \( \alpha = 2 \), the quoted assertion from [13] is false. (Note that, by (2.27), \( S(\lambda_{A,f}) = \partial \mathbb{R}^n \mathbb{R}^{n-1} \))

To substantiate this, we note that \( \lambda_{A,f} \) is the Newtonian balayage \( \varepsilon_{x_0} \) of \( \varepsilon_{x_0} \) onto \( \mathbb{R}^{n-1} \), see the latter formula in representation (2.12). Denoting by \( K_0 \) the inverse of the one-point compactification of \( \mathbb{R}^{n-1} \) with respect to \( |x-x_0| = r \), \( r > 0 \) being small enough, we further note that \( \varepsilon_{x_0} \) is the Kelvin transform of the Newtonian capacitary measure \( \gamma_{K_0} \) on \( K_0 \) (see [22] Section IV.6.24]). Since \( K_0 \) is compact and coincides with its reduced kernel, while its complement is connected, we have \( S(\gamma_{K_0}) = \partial K_0 \) [22 Section II.3.13], which implies that, indeed, \( S(\lambda_{A,f}) = \partial \mathbb{R}^n \mathbb{R}^{n-1} \) (\( \neq \mathbb{R}^{n-1} \)).

**Theorem 2.22.** Assume that \( A \) is not inner \( \alpha \)-thin at infinity, \( \delta(\mathbb{R}^n) > 1 \), \( U^\delta \) is continuous on \( \tilde{A} \), and there is the limit

\[
\lim_{x \to \infty_{\infty \mathbb{R}^n} x \in A} U^\delta(x).
\]  

(2.28)

Then \( \lambda_{A,f} \) is of compact support.

**Remark 2.23.** Theorem 2.22 is **sharp** in the sense that \( \delta(\mathbb{R}^n) > 1 \) cannot in general be replaced by \( \delta(\mathbb{R}^n) = 1 \). Indeed, if \( A \) is closed and not \( \alpha \)-thin at infinity, \( \delta(\mathbb{R}^n) = 1 \), and \( \delta(A^c) > 0 \), then for \( \alpha < 2 \), \( S(\lambda_{A,f}) \) is always noncompact, cf. (2.27).

**Example 2.24.** On \( \mathbb{R}^3 \), consider the kernel \( 1/|x-y| \) and the rotation bodies

\[
F_i := \{ x \in \mathbb{R}^3 : 0 \leq x_1 < \infty, x_2^2 + x_3^2 \leq \varrho_i^2(x_1) \}, i = 1, 2, 3,
\]

where

\[
\varrho_1(x_1) := x_1^{-s} \text{ with } s \in [0, \infty),
\]

\[
\varrho_2(x_1) := \exp(-x_1^3) \text{ with } s \in (0, 1],
\]

\[
\varrho_3(x_1) := \exp(-x_1^3) \text{ with } s \in (1, \infty).
\]

As seen from estimates in [22] Section V.1, Example], \( F_1 \) is not 2-thin at infinity, \( F_2 \) is 2-thin at infinity, though has infinite Newtonian capacity, whereas \( F_3 \) is of finite Newtonian capacity. Therefore, \( \lambda_{F_i,f} \) exists for any \( f \) of form (1.3) (Theorem 2.21).

Let now \( f \) be of form (2.23), where \( S(\delta) \subset F_i^c \) is compact. Then \( \lambda_{F_i,f} \) exists if and only if \( \delta(\mathbb{R}^n) \geq 1 \) (Theorem 2.13(b.1)). If moreover \( \delta(\mathbb{R}^n) = 1 \), then \( S(\lambda_{F_i,f}) = \partial F_i \) (Theorem 2.19(b.2)) while \( c_{F_1,f} = 0 \) (Theorem 2.13(a.1)); whereas for \( \delta(\mathbb{R}^n) > 1 \),
Figure 1. The set $F_1$ in Example 2.24 with $g_1(x_1) = 1/x_1$.

$S(\lambda_{F_1,f})$ is compact (Theorem 2.22) while $c_{F_1,f} < 0$ (Theorem 2.13(b)). If now $\delta(\mathbb{R}^n) \leq 1$, then $S(\lambda_{F_3,f}) = \partial F_3$ (Theorem 2.19(a)), whereas $\lambda_{F_2,f}$ fails to exist (Theorem 2.15).

Figure 2. The set $F_2$ in Example 2.24 with $g_2(x_1) = \exp(-x_1)$.

2.5. Remark. In the stated generality, all the results thus obtained seem to be new; however, a few of them were known before for some very particular $f$ and/or $A$. For instance, if $\alpha \in (0, 2]$, $A$ is closed while $\delta$, the measure in representation (2.8), is just $q \varepsilon_{x_0}$, where $q \in (0, \infty)$ and $x_0 \notin A$, then (2.23) and Theorem 2.22 were given in [13, Corollary 2.6] (cf. also [11, Theorem 2.1]). See also Remarks 2.12 and 2.21 above.

Such a significant improvement of the theory in question is due to a new approach, based on the close interaction between the strong and the vague topologies on the strongly complete cone $\mathcal{E}^+$, as well as on the author’s recent theory of inner balayage.
3. Preliminary results

3.1. On convergent nets. We shall first establish several auxiliary results related to convergent nets \((\mu_s)_{s \in S}\) of positive measures.\(^{19}\)

**Lemma 3.1.** Assume that a set \(F \subset \mathbb{R}^n\) is closed, a function \(g : F \to (-\infty, \infty]\) is l.s.c. and lower bounded, and a net \((\mu_s)_s \subset \mathcal{M}^+(F)\) converges vaguely to \(\mu_0.\)\(^{20}\)

If moreover either \(g \geq 0\) on \(F,\) or

\[
\lim_{s \in S} \mu_s(\mathbb{R}^n) = \mu_0(\mathbb{R}^n) \in (0, \infty), \tag{3.1}
\]

then

\[
\int g \, d\mu_0 \leq \liminf_{s \in S} \int g \, d\mu_s. \tag{3.2}
\]

**Proof.** For \(g \geq 0,\) this is well known (see [1] Section IV.1, Proposition 4), applied to \(F\) treated as a locally compact subspace of \(\mathbb{R}^n\). Otherwise, there exists \(c \in (0, \infty)\) such that \(g' := g + c\) is l.s.c. and positive on \(F.\) Applying (3.2) to \(g',\) and then subtracting (3.1) multiplied by \(c\) from the inequality thereby obtained, we get the lemma. \(\square\)

**Remark 3.2.** Assumption (3.1) is certainly fulfilled if \(F = K\) is compact, for the mapping \(\mu \mapsto \mu(K)\) is vaguely continuous on \(\mathcal{M}^+(K)\). Yet another possibility for (3.1) to hold is described in Lemma 3.3 below.

**Lemma 3.3.** Given an arbitrary set \(A \subset \mathbb{R}^n\) with \(c_s(A) < \infty,\) consider a net \((\mu_s)_s \subset \hat{\mathcal{E}}^+(A)\) converging strongly (hence vaguely) to \(\mu_0 \in \mathcal{E}^+.\) Then

\[
\mu_0(\mathbb{R}^n) = 1. \tag{3.3}
\]

**Proof.** Taking a subnet if necessary and changing notation, we can certainly assume \((\mu_s)_s \in S\) to be strongly bounded:

\[
\sup_{s \in S} \|\mu_s\| < \infty. \tag{3.4}
\]

Furthermore, since \(\mu_s \to \mu_0\) vaguely, Lemma 3.1 with \(F := \mathbb{R}^n\) and \(g := 1\) gives

\[
\mu_0(\mathbb{R}^n) \leq \liminf_{s \in S} \mu_s(\mathbb{R}^n) = 1, \tag{3.5}
\]

whereas [1] Section IV.4.4, Corollary 3\] yields

\[
\int 1_K \, d\mu_0 \geq \limsup_{s \in S} \int 1_K \, d\mu_s \text{ for every compact } K \subset \mathbb{R}^n, \tag{3.6}
\]

the indicator function \(1_K\) being bounded, of compact support, and upper semicontinuous on \(\mathbb{R}^n.\) Combining (3.5) and (3.6) with

\[
\mu_0(\mathbb{R}^n) = \lim_{K \uparrow \mathbb{R}^n} \int 1_K \, d\mu_0,
\]

we get

\[
1 \geq \mu_0(\mathbb{R}^n) \geq \limsup_{(s,K) \in S \times \mathcal{C}} \int 1_K \, d\mu_s = 1 - \liminf_{(s,K) \in S \times \mathcal{C}} \int 1_{A \setminus K} \, d\mu_s, \tag{3.7}
\]

\(\mathcal{C} := \mathcal{C}_{\mathbb{R}^n}\) being the upward directed set of all compact subsets \(K\) of \(\mathbb{R}^n,\) and \(S \times \mathcal{C}\) the directed product of the directed sets \(S\) and \(\mathcal{C},\) cf. [20] p. 68]. Note that the

\(^{19}\)Since \(\mathcal{M}(\mathbb{R}^n)\) equipped with the vague topology satisfies the first axiom of countability (see e.g. [20] Lemma 4.4), it is usually enough to consider convergent sequences of measures. Nonetheless, in the study of inner potential theoretical concepts, it is often convenient to operate with nets, in particular when dealing with the upward directed set \(\mathcal{C}_A\) of all compact subsets \(K\) of \(A.\)

\(^{20}\)Then necessarily \(\mu_0 \in \mathcal{M}^+(F), \mathcal{M}^+(F)\) being vaguely closed [4] Section III.2, Proposition 6].
equality in (3.7) is implied by the fact that each $\mu_s$ is a positive measure of unit total mass concentrated on $A$.) The proof of (3.3) is thus reduced to that of

$$\liminf_{(s,K) \in S \times \mathcal{E}} \int 1_{A \setminus K} \, d\mu_s = 0. \quad (3.8)$$

By [22, Theorem 2.6] applied to $A \setminus K$, $K \in \mathcal{C}$ being arbitrarily chosen, there exists the (unique) inner capacitary measure $\gamma_{A\setminus K}$, minimizing the energy $\| \cdot \|^2$ over the (convex) class $\Gamma_{A\setminus K}$ of all $\mu \in \mathcal{E}^+$ with the property

$$U^\mu \geq 1 \text{ n.e. on } A \setminus K.$$ 
For any $K' \in \mathcal{C}$ such that $K \subset K'$, we have $\Gamma_{A\setminus K} \subset \Gamma_{A\setminus K'}$, and [22, Lemma 2.2] therefore gives

$$\| \gamma_{A\setminus K} - \gamma_{A\setminus K'} \|^2 \leq \| \gamma_{A\setminus K} \|^2 - \| \gamma_{A\setminus K'} \|^2. \quad (3.9)$$

Since $\| \gamma_{A\setminus K} \|^2 = c_s(A \setminus K)$ [22, Theorem 2.6], $\| \gamma_{A\setminus K} \|^2$ decreases as $K$ ranges through $\mathcal{C}$, which together with (3.9) implies that the net $(\gamma_{A\setminus K})_{K \in \mathcal{E}} \subset \mathcal{E}^+$ is Cauchy in the strong topology on $\mathcal{E}^+$. Noting that $(\gamma_{A\setminus K})_{K \in \mathcal{E}}$ converges vaguely to zero, we get

$$\gamma_{A\setminus K} \to 0 \text{ strongly in } \mathcal{E}^+ \text{ as } K \uparrow \mathbb{R}^n, \quad (3.10)$$

the $\alpha$-Riesz kernel being perfect (Section 1.1).

It follows from the above that

$$U^{\gamma_{A\setminus K}} \geq 1_{A\setminus K} \text{ n.e. on } A \setminus K, \quad (3.11)$$

hence $\mu_s$-a.e. for all $s \in S$, the latter being derived from Lemma 3.5 (see below) due to the fact that $A \setminus K$ along with $A$ is $\mu_s$-measurable, and therefore so is the set $E$ of all $x \in A \setminus K$ having the property $U^{\gamma_{A\setminus K}}(x) < 1$. Integrating (3.11) with respect to $\mu_s$ we thus obtain, by the Cauchy–Schwarz (Bunyakovski) inequality,

$$\int 1_{A \setminus K} \, d\mu_s \leq \int U^{\gamma_{A\setminus K}} \, d\mu_s \leq \| \gamma_{A\setminus K} \| \cdot \| \mu_s \| \text{ for all } K \in \mathcal{C} \text{ and } s \in S,$$

which combined with (3.4) and (3.10) establishes (3.8), whence (3.3). 

\[\square\]

**Definition 3.4** (Landkof [22, Section II.1.2]). A measure $\mu \in \mathfrak{M}$ is said to be $C$-absolutely continuous if $\mu(K) = 0$ for any compact set $K \subset \mathbb{R}^n$ of zero capacity. Every $\mu \in \mathcal{E}$ is $C$-absolutely continuous, but not conversely.

**Lemma 3.5.** Given a $C$-absolutely continuous measure $\mu \in \mathfrak{M}^+$, let $E \subset \mathbb{R}^n$ be a $\mu$-measurable set with $c_s(E) = 0$. Then $E$ is $\mu$-negligible, that is, $\mu^*(E) = 0$.

**Proof.** Since $\mathbb{R}^n$ is representable as a countable union of compact sets, it is enough to show that $E$ is locally $\mu$-negligible, or equivalently, that $\mu_s(E) = 0$. In view of the $C$-absolute continuity of $\mu$, this follows immediately from $c_s(E) = 0$. \[\square\]

**Theorem 3.6.** Given $A \subset \mathbb{R}^n$ with $c_s(A) < \infty$, assume $(P_4)$ is fulfilled. Then $\mathcal{E}^+(A)$ along with $\mathcal{E}^+(A)$ is strongly closed, and hence strongly complete.

**Proof.** Since according to $(P_4)$, $\mathcal{E}^+(A)$ is strongly closed, applying Lemma 3.3 shows that so is $\mathcal{E}^+(A)$. It remains to observe that any strongly closed subset of the strongly complete cone $\mathcal{E}^+$ must be strongly complete as well. \[\square\]

\[\text{21}\text{Indeed, for any given } \varphi \in C_0(\mathbb{R}^n), \text{ there exists a relatively compact, open set } G \subset \mathbb{R}^n \text{ such that } \varphi(x) = 0 \text{ for all } x \notin \overline{G}. \text{ Hence, } \gamma_{A\setminus K}(\varphi) = 0 \text{ for all } K \in \mathcal{C} \text{ with } K \supset \overline{G}, \text{ and the claim follows.}\]
3.2. Continuity properties of the \( f \)-weighted energy. Recall that an external field \( f \) is assumed to be of form \( (I.3) \), where \( \psi, \vartheta, \) and \( \omega \) satisfy \( (P_1)-(P_3) \). Then the \( f \)-weighted energy \( I_f(\mu) \) along with \( \int f \, d\mu \) is well defined as a finite number or \( +\infty \) for all bounded \( \mu \in \mathcal{E}^+(\overline{A}) \), see Lemma 3.2.

**Lemma 3.7.** For any \( A \subset \mathbb{R}^n \) with \( c_*(A) < \infty \), assume a net \( (\mu_s)_{s \in S} \subset \hat{\mathcal{E}}^+(A) \) converges strongly to \( \mu_0 \in \mathcal{E}^+ \). Then

\[
I_f(\mu_0) \leq \liminf_{s \to S} I_f(\mu_s).
\]

*Proof.* Since the energy norm is strongly continuous on \( \mathcal{E} \), we only need to verify that

\[
\int f \, d\mu_0 \leq \liminf_{s \to S} \int f \, d\mu_s.
\] (3.12)

Applying the Cauchy–Schwarz inequality to the (signed) measures \( \vartheta \) and \( \mu_s - \mu_0 \), elements of the pre-Hilbert space \( \mathcal{E} \), we get

\[
|\langle \vartheta, \mu_s - \mu_0 \rangle| \leq \| \vartheta \| \cdot \| \mu_s - \mu_0 \|,
\]

which yields, by the strong convergence of \( (\mu_s)_{s \in S} \) to \( \mu_0 \),

\[
\langle \vartheta, \mu_0 \rangle = \lim_{s \to S} \langle \vartheta, \mu_s \rangle,
\] (3.13)

whence (3.12) for \( U^\vartheta \) in place of \( f \).

Since \( c_*(A) < \infty \) while \( (\mu_s)_{s \in S} \subset \hat{\mathcal{E}}^+(A) \) converges strongly (hence vaguely) to \( \mu_0 \), Lemma 3.3 gives \( \mu_0(\mathbb{R}^n) = 1 \), whence (3.11). Therefore, applying Lemma 3.1 with \( F := \overline{A} \) to each of the functions \( \psi, U^\omega^+, \) and \( -U^\omega^- \), and then combining the inequalities obtained with (3.13), we get (3.12). (For \( \psi \), we use the fact that a l.s.c. function on a compact set is lower bounded. For \( -U^\omega^- \), we observe from \( (P_3) \) that its restriction to \( \overline{A} \) is continuous (hence l.s.c.) and lower bounded.) \( \square \)

3.3. Quasiclosed sets. For any \( A \subset \mathbb{R}^n \), let \( \mathcal{E}'(A) \) stand for the closure of \( \mathcal{E}^+(A) \) in the strong topology on \( \mathcal{E}^+ \). Being a strongly closed subcone of the strongly complete cone \( \mathcal{E}^+, \mathcal{E}'(A) \) is likewise complete in the (induced) strong topology.

We shall be mainly interested in the case when \( (P_4) \) is fulfilled (see Section 2), or equivalently, when

\[
\mathcal{E}'(A) = \mathcal{E}^+(A).
\]

A sufficient condition for this to occur is given in Theorem 3.9 below. (In general,
\[
\mathcal{E}'(A) \subset \mathcal{E}^+(\overline{A}),
\] (3.14)

\( \mathcal{E}^+(\overline{A}) \) being strongly closed according to Theorem 3.9)

**Definition 3.8** (B. Fuglede [14]). A set \( A \subset \mathbb{R}^n \) is said to be quasiclosed if

\[
\inf \left\{ c^*(A \triangle F) : F \text{ closed, } F \subset \mathbb{R}^n \right\} = 0,
\]

\( \triangle \) being the symmetric difference. Replacing here "closed" by "compact", we arrive at the concept of quasicompact sets.

**Theorem 3.9.** If \( A \subset \mathbb{R}^n \) is quasiclosed (or in particular quasicompact), then the cone \( \mathcal{E}^+(A) \) is strongly closed.

*Proof.* Given a sequence \( (\mu_j) \subset \mathcal{E}^+(A) \) converging strongly (hence vaguely) to \( \mu_0 \in \mathcal{E}^+ \), we only need to show that \( \mu_0 \) is concentrated on \( A \). For closed \( A \), the cone \( \mathfrak{M}^+(A) \) is vaguely closed according to [4] Section III.2, Proposition 6], whence \( \mu_0 \in \mathfrak{M}^+(A) \).
For quasiclosed $A$, note that for every $q \in (0, \infty)$, $\mathcal{E}_q^+ := \{ \mu \in \mathcal{E}^+ : \|\mu\| \leq q \}$ is hereditary [17, Definition 5.2] and vaguely compact [15, Lemma 2.5.1], the Riesz kernels being strictly pseudo-definite, cf. [15, p. 150]. Since a strongly convergent sequence is certainly strongly bounded, $(\mu_j) \subset \mathcal{E}_{q'}^+$ for some $q' \in (0, \infty)$. Applying [17, Corollary 6.2] with $J := \mathcal{E}_{q'}^+ \cap \mathcal{M}^+(A)$ is vaguely compact, and, therefore, $(\mu_j) \subset \tilde{J}$ has a vague limit point $\nu_0 \in \tilde{J}$. The vague topology being Hausdorff, $\nu_0 = \mu_0$, whence $\mu_0 \in \tilde{J} \subset \mathcal{M}^+(A)$ as desired. □

4. PROOFS OF THE MAIN RESULTS

- Throughout Sections 1.3, 1.5, $\alpha \in (0, n)$ is arbitrary. Furthermore, an external field $f$ is of form [15.3], where $\psi$, $\vartheta$, and $\omega$ satisfy $(P_1)$–$(P_3)$ as well as [14.10].

4.1. Extremal measures. A net $(\mu_s)_{s \in S} \subset \hat{\mathcal{E}}_f^+(A)$ is said to be minimizing (in Problem 1.1) if

$$\lim_{s \in S} I_f(\mu_s) = w_f(A).$$

(4.1)

We denote by $\mathbb{M}_f(A)$ the (nonempty) set of all those nets $(\mu_s)_{s \in S}$.

**Lemma 4.1.** There is the unique $\xi_{A,f} \in \mathcal{E}^+$ such that, for each $(\mu_s)_{s \in S} \in \mathbb{M}_f(A)$,

$$\mu_s \to \xi_{A,f} \text{ strongly and vaguely in } \mathcal{E}^+. \quad (4.2)$$

This $\xi_{A,f}$ is said to be the extremal measure (in Problem 1.1).

**Proof.** We shall first show that for any $(\mu_s)_{s \in S}$ and $(\nu_t)_{t \in T}$ from $\mathbb{M}_f(A)$,

$$\lim_{(s,t) \in S \times T} \|\mu_s - \nu_t\| = 0, \quad (4.3)$$

$S \times T$ being the directed product of the directed sets $S$ and $T$. In fact, due to the convexity of the class $\hat{\mathcal{E}}_f^+(A)$, for any $(s,t) \in S \times T$ we have

$$4w_f(A) \leq 4I_f(\frac{\mu_s + \nu_t}{2}) = \|\mu_s + \nu_t\|^2 + 4 \int f d(\mu_s + \nu_t).$$

Applying the parallelogram identity in the pre-Hilbert space $\mathcal{E}$ therefore gives

$$0 \leq \|\mu_s - \nu_t\|^2 \leq -4w_f(A) + 2I_f(\mu_s) + 2I_f(\nu_t),$$

which together with (1.2) and (4.1) yields (4.3) by letting $(s,t)$ range through $S \times T$.

Taking the two nets in (1.3) to be equal, we infer that every $(\nu_t)_{t \in T} \in \mathbb{M}_f(A)$ is strong Cauchy. The cone $\mathcal{E}^+$ being strongly complete, this $(\nu_t)_{t \in T}$ converges strongly to some $\xi_{A,f} \in \mathcal{E}^+$. The same (unique) $\xi_{A,f}$ also serves as the strong limit of any other $(\mu_s)_{s \in S} \in \mathbb{M}_f(A)$, which is obvious from (1.3). The strong topology on $\mathcal{E}^+$ being finer than the vague topology on $\mathcal{E}^+$, $(\mu_s)_{s \in S}$ must converge to $\xi_{A,f}$ also vaguely. □

**Remark 4.2.** In general, the extremal measure $\xi_{A,f}$ is *not* concentrated on $A$. What is clear so far is that

$$\xi_{A,f} \in \mathcal{E}'(A) \subset \mathcal{E}^+(A),$$

the former relation being obvious from (1.2), and the latter from (2.4).

**Remark 4.3.** Another consequence of (4.2) is that

$$\xi_{A,f}(\mathbb{R}^n) \leq 1, \quad (4.4)$$

$\mu \mapsto \mu(\mathbb{R}^n)$ being vaguely l.s.c. on $\mathcal{M}^+$ (Lemma 3.1 with $F := \mathbb{R}^n$ and $g := 1$). Equality necessarily prevails in (4.4) if $A$ is compact [14, Section III.1.9, Corollary 3].
Corollary 4.4. If the solution \( \lambda_{A,f} \) to Problem 1.1 exists, then necessarily
\[
\lambda_{A,f} = \xi_{A,f}.
\] (4.5)

Proof. The trivial sequence \( (\lambda_{A,f}) \) being obviously minimizing:
\[
(\lambda_{A,f}) \in \mathcal{M}_f(A),
\]
it must converge strongly in \( \mathcal{E}^+ \) to the extremal measure \( \xi_{A,f} \) (Lemma 1.1), as well as to \( \lambda_{A,f} \). Since the strong topology on \( \mathcal{E} \) is Hausdorff, (4.5) follows. \( \square \)

4.2. Proof of Theorem 2.1. Since under the assumptions of the theorem, \( \mathcal{E}^+(A) \) is strongly closed, see (P4), so must be \( \hat{\mathcal{E}}^+(A) \) (Theorem 3.6). Fix a minimizing net \( (\mu_s) \in \mathcal{M}_f(A) \); according to Lemma 4.1, it converges strongly and vaguely to the extremal measure \( \xi_{A,f} \). As \( (\mu_s) \subset \hat{\mathcal{E}}^+(A) \), the strong closedness of \( \hat{\mathcal{E}}^+(A) \) yields
\[
\xi_{A,f} \in \hat{\mathcal{E}}^+(A).
\] (4.6)

Applying Lemma 3.7 we further obtain
\[
I_f(\xi_{A,f}) \leq \lim_{s \in S} I_f(\mu_s) = w_f(A),
\] (4.7)
the equality being valid by virtue of (4.1). As \( I_f(\xi_{A,f}) > -\infty \) (Lemma 1.2), we infer from (1.2), (2.1), and (4.7) that, actually,
\[
\xi_{A,f} \in \hat{\mathcal{E}}_f^+(A),
\] (4.8)
whence \( I_f(\xi_{A,f}) \geq w_f(A) \), which combined with (4.7) gives
\[
I_f(\xi_{A,f}) = w_f(A).
\]

This together with (4.8) shows that the extremal measure \( \xi_{A,f} \) serves as the solution \( \lambda_{A,f} \) to Problem 1.1; thereby completing the proof of the theorem.

4.3. Proof of Theorem 2.3. We first show that for any \( g \in \Phi(\overline{A}) \) and \( \mu \in \mathcal{M}^+(A) \),
\[
\int g \, d\mu = \lim_{K \uparrow A} \int g \, d\mu|_K,
\] (4.9)
\( \mu|_K \) being the trace of \( \mu \) to \( K \). Indeed, if \( g \geq 0 \), this is implied by [15, Lemma 1.2.2], noting that for \( \mu \in \mathcal{M}^+(A) \), the set \( A \) is \( \mu \)-measurable and, moreover, \( \mu = \mu|_A \). Otherwise, \( \overline{A} \) must be compact, and the claim follows by applying (4.9) to a function \( g' := g + c \geq 0 \) on \( A \), where \( c \in (0, \infty) \), and then by making use of the fact that
\[
\lim_{K \uparrow A} \mu(K) = \mu(A) < \infty,
\]
the measures on (compact) \( \overline{A} \) being bounded.

For any \( \mu \in \hat{\mathcal{E}}_f^+(A) \), \( \mu(K) \uparrow 1 \) as \( K \uparrow A \). Applying (4.9) to each of the \( \mu \)-integrable functions \( \kappa_\alpha, \psi, U^{\theta^+}, U^{\theta^-}, U^{\omega^+}, \) and \( U^{\omega^-} \) therefore gives
\[
I_f(\mu) = \lim_{K \uparrow A} I_f(\mu|_K) = \lim_{K \uparrow A} I_f(\nu_K) \geq \lim_{K \uparrow A} w_f(K),
\]
where \( \nu_K := \mu|_K/\mu(K) \in \hat{\mathcal{E}}_f^+(K), K \geq K_0 \). Letting \( \mu \) range over \( \hat{\mathcal{E}}_f^+(A) \) we thus get
\[
w_f(A) \geq \lim_{K \uparrow A} w_f(K),
\]
whence
\[
w_f(K) \downarrow w_f(A) \quad \text{as} \quad K \uparrow A,
\] (4.10)
the net \( (w_f(K))_{K \in \mathcal{E}_A} \) being decreasing and bounded from below by \( w_f(A) \).
In view of (1.2), there exists, therefore, $K_0 \in \mathcal{C}_A$ such that, for each $K \geq K_0$, $w_f(K)$ is finite, and hence, according to Corollary 2.2 Problem [1.1] with $A := K$ is solvable. Noting from (1.10) that those solutions $\lambda_{K,f}$ form a minimizing net:

$$(\lambda_{K,f})_{K \geq K_0} \in M_f(A),$$

we obtain, by virtue of Lemma 4.1

$$\lambda_{K,f} \to \xi_{A,f} \text{ strongly and vaguely as } K \uparrow A. \tag{4.11}$$

Since $\lambda_{A,f}$ exists by assumption, $\xi_{A,f} = \lambda_{A,f}$ according to Corollary 4.4 which substituted into (4.11) proves (2.2).

Thus, by (2.2),

$$\lim_{K \uparrow A} \|\lambda_{K,f}\|^2 = \|\lambda_{A,f}\|^2.$$

On the other hand, (4.10) can be rewritten in the form

$$\lim_{K \uparrow A} \left(\|\lambda_{K,f}\|^2 + 2 \int f d\lambda_{K,f}\right) = \|\lambda_{A,f}\|^2 + 2 \int f d\lambda_{A,f}.$$

The last two relations combined imply (2.3), cf. (1.7), thereby completing the proof.

4.4. Proof of Theorem 2.4. Let $A$ be the union of an increasing sequence of universally measurable sets $A_j$. For any l.s.c. $g : \overline{A} \to [0, \infty]$ and any $\mu \in \mathfrak{M}^+(A)$,

$$\int g \, d\mu = \lim_{j \to \infty} \int 1_{A_j} g \, d\mu = \lim_{j \to \infty} \int g \, d\mu_{|A_j}, \tag{4.12}$$

where the former equality holds true by the monotone convergence theorem [4, Section IV.1, Theorem 3] applied to the increasing sequence $(1_{A_j}, g)$ of positive functions with the upper envelope $g$, and the latter by [4, Propositions 4.14.1(b) and 4.14.6(3)]

If now $g \in \Phi(\overline{A})$ and $g \not\equiv 0$, then $\overline{A}$ must be compact, and the proof of (4.12) runs as usual — namely, by replacing $g$ by $g' := g + c$, where $c \in (0, \infty)$, and then by making use of the fact that, due to (4.12) with $g := 1$,

$$\lim_{j \to \infty} \mu(A_j) = \mu(A) < \infty.$$

Having thus established (1.12) for any $g \in \Phi(\overline{A})$, we further arrive at (2.4) by means of the same arguments as in the proof of (1.10).

The rest of the proof runs similarly to that in Section 1.3. To be precise, since the minimizers $\lambda_{A_j,f}$ and $\lambda_{A,f}$ exist by assumption, it follows from (2.4) that $\lambda_{A_j,f}$ form a minimizing sequence, which, according to Lemma 4.1 and Corollary 4.4 must converge to $\lambda_{A,f}$ strongly and vaguely. This establishes (2.5). Finally, (2.4) and (2.5) result in (2.6) in exactly the same way as in the last paragraph of Section 1.3.

4.5. Proof of Theorem 2.5. We begin by noting that $A$, being a countable intersection of quasiclosed sets $A_j$, is likewise quasiclosed [10, Lemma 2.3]. Since $(P_4)$ is fulfilled for quasiclosed sets (Theorem 3.9), applying Theorem 2.1 shows that the minimizers $\lambda_{A_j,f}$, $j \geq j_0$, as well as $\lambda_{A,f}$ do exist.

By the convexity of $\hat{\mathcal{E}}^+_f(A_j)$, $(\lambda_{A_j,f} + \lambda_{A_k,f})/2 \in \hat{\mathcal{E}}^+_f(A_j)$ for all $k \geq j$, hence

$$4w_f(A_j) \leq 4I_f(\frac{\lambda_{A_j,f} + \lambda_{A_k,f}}{2}) = \|\lambda_{A_j,f} + \lambda_{A_k,f}\|^2 + 4 \int f d(\lambda_{A_j,f} + \lambda_{A_k,f}).$$

Using the parallelogram identity in the pre-Hilbert space $\mathcal{E}$ therefore gives

$$\|\lambda_{A_j,f} - \lambda_{A_k,f}\|^2 \leq -4w_f(A_j) + 2I_f(\lambda_{A_j,f}) + 2I_f(\lambda_{A_k,f}) = 2w_f(A_k) - 2w_f(A_j). \tag{4.13}$$
The sequence \((w_f(A_k))\) being increasing and bounded from above by \(w_f(A) \in \mathbb{R}\), we infer from (4.13) that \((\lambda_{A_k,f})_{k \geq j} \subset \mathcal{E}^+(A_j)\) is strong Cauchy, and hence it converges strongly and vaguely to some \(\lambda_0 \in \mathcal{E}^+\). We claim that \(\lambda_0 = \lambda_{A,f}\).

Since \(A_j\) is quasiclosed and of finite inner capacity, \(\mathcal{E}^+(A_j)\) is strongly closed (Theorems 3.6 and 3.9), and therefore \(\lambda_0 \in \mathcal{E}^+(A_j)\) for all \(j \geq j_0\). Being thus a countable union of \(\lambda_0\)-negligible sets \(A_j\), the set \(A^c\) is likewise \(\lambda_0\)-negligible, whence

\[
\lambda_0 \in \mathcal{E}^+(A) .
\]

Furthermore,

\[-\infty < I_f(\lambda_0) \leq \lim \inf_{j \to \infty} I_f(\lambda_{A_j,f}) = \lim_{j \to \infty} w_f(A_j) \leq w_f(A) < \infty ,
\]

where the first and the second inequalities hold true by virtue of Lemmas 1.2 and 3.7 respectively. In view of (4.14) and (4.15),

\[
\lambda_0 \in \mathcal{E}^+_f(A),
\]

whence \(I_f(\lambda_0) \geq w_f(A)\), which substituted into (4.15) shows that, actually,

\[
\lim_{j \to \infty} w_f(A_j) = w_f(A) = I_f(\lambda_0) .
\]

As seen from (4.16) and (4.17), the measure \(\lambda_0\), the strong and the vague limit of the sequence \((\lambda_{A_j,f})_{j \geq j_0}\), serves, indeed, as \(\lambda_{A,f}\). To complete the proof, it remains to verify that \(c_{A_j,f} \to c_{A,f}\) as \(j \to \infty\), but this follows from the above in exactly the same manner as in the last paragraph of Section 4.3.

4.6. The case of \(\alpha \in (0, 2]\). Auxiliary results. In the remainder of the paper, \(\alpha \in (0, 2]\).

Among the variety of equivalent definitions of inner \(\alpha\)-Riesz balayage (see [26, 27], cf. also [28–30]), here we have chosen the following one to start with.

**Definition 4.5.** For any \(\mu \in \mathcal{M}^+\) and any \(A \subset \mathbb{R}^n\), the inner \(\alpha\)-Riesz balayage \(\mu^A \in \mathcal{M}^+\) is the vague limit of the sequence \((\mu_j^A) \subset \mathcal{E}^+\), where \((\mu_j) \subset \mathcal{E}^+\) is such that

\[
U^{\mu_j} \uparrow U^{\mu} \text{ pointwise on } \mathbb{R}^n \text{ as } j \to \infty ,
\]

while \(\mu_j^A\) denotes the only measure in \(\mathcal{E}'(A)\), the strong closure of \(\mathcal{E}^+(A)\) (Section 3.3), having the property \(U^{\mu_j^A} = U^{\mu_j} \) n.e. on \(A\).

The inner balayage \(\mu^A\) thus defined does exist, is unique, and it does not depend on the choice of the above sequence \((\mu_j)\), cf. [26, Section 3.3]. Furthermore,

\[
U^{\mu^A} = U^{\mu} \text{ n.e. on } A ,
\]

\[
U^{\mu^A} \leq U^{\mu} \text{ on } \mathbb{R}^n .
\]

The same \(\mu^A\) is uniquely characterized within \(\mathcal{M}^+\) by the symmetry relation

\[
I(\mu^A, \chi) = I(\mu, \chi^A) \text{ for all } \chi \in \mathcal{E}^+ ,
\]

where \(\chi^A\) denotes the only measure in \(\mathcal{E}'(A)\) with \(U\chi^A = U\chi\) n.e. on \(A\) (footnote 22).

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22 A sequence \((\mu_j) \subset \mathcal{E}^+\) satisfying (4.13) does exist (see [22, p. 272], cf. [8, p. 257]). Furthermore, for each \(\chi \in \mathcal{E}^+\), there exists the unique measure \(\chi^A \in \mathcal{E}'(A)\) such that \(U\chi^A = U\chi\) n.e. on \(A\) (29, Theorem 3.1(c)); it can alternatively be characterized as the orthogonal projection of \(\chi\) in the pre-Hilbert space \(\mathcal{E}\) onto the convex, strongly complete cone \(\mathcal{E}'(A) \subset \mathcal{E}^+\) (29, Theorem 3.1(b)). Regarding the concept of orthogonal projection in a pre-Hilbert space, see e.g. [14, Theorem 1.12.3].
Remark 4.6. In general, $\mu^A$ is not concentrated on $A$ (unless, of course, $A$ is closed). It is also worth noting that, even for closed $A$, there may exist infinitely many $\nu \in \mathfrak{M}^+(A)$ with $U^\nu = U^\mu$ n.e. on $A$; hence, (4.19) cannot, in general, serve as a characteristic property of balayage. Compare with Theorem 4.7(iii).

- In all that follows, we assume that $\mathcal{E}^+(A)$ is strongly closed (or in particular that $A$ is quasiclosed), so that $\mathcal{E}^+(A) = \mathcal{E}'(A)$, and that $f$ is of form (2.8), i.e.

$$f = -U^\tau - U^\sigma = -U^\delta,$$

where $\tau \in \mathcal{E}^+$, $\sigma \in \mathfrak{M}^+$ is a bounded measure with $d(S_\sigma, A) > 0$, and $\delta := \tau + \sigma$. That is, $(P_4)$, $(P'_4)$, and $(P'_3)$ are required to hold (see the beginning of Section 2.2).

It is crucial to the analysis below that for these $A$ and $\delta$, the inner balayage $\delta^A$ is a measure of finite energy, concentrated on $A$ (see Theorem 4.7 for details).

Theorem 4.7. For the above $A$ and $\delta$, the inner balayage $\delta^A$ can equivalently be determined by means of any one of the following assertions.

(i) $\delta^A$ is the only measure of minimum energy in the class

$$\Gamma_{A, \delta} := \{ \nu \in \mathfrak{M}^+ : U^\nu \geq U^\delta \text{ n.e. on } A \}. \quad (4.20)$$

That is, $\delta^A \in \Gamma_{A, \delta}$ and

$$I(\delta^A) = \min_{\nu \in \Gamma_{A, \delta}} I(\nu).$$

(ii) $\delta^A$ is the only measure of minimum potential in the class $\Gamma_{A, \delta}$, introduced by means of (4.20). That is, $\delta^A \in \Gamma_{A, \delta}$ and

$$U^{\delta^A} = \min_{\nu \in \Gamma_{A, \delta}} U^\nu \text{ on } \mathbb{R}^n.$$  

(iii) $\delta^A$ is the only measure in the class $\mathcal{E}^+(A)$ having the property

$$U^{\delta^A} = U^\delta \text{ n.e. on } A.$$

(iv) $\delta^A$ can alternatively be characterized by any one of the three limit relations

$$\delta^K \to \delta^A \text{ strongly in } \mathcal{E}^+ \text{ as } K \uparrow A, \quad (4.21)$$

$$\delta^K \to \delta^A \text{ vaguely in } \mathfrak{M}^+ \text{ as } K \uparrow A, \quad (4.22)$$

$$U^{\delta^K} \uparrow U^{\delta^A} \text{ pointwise on } \mathbb{R}^n \text{ as } K \uparrow A,$$

where $\delta^K$ denotes the only measure in $\mathcal{E}^+(K)$ with $U^{\delta^K} = U^\delta$ n.e. on $K$.

Proof. To simplify notations, for $\mu \in \mathfrak{M}^+$ write $\mu' := \mu^\top$. We begin by showing that

$$\delta' \in \mathcal{E}^+(\overline{A}). \quad (4.23)$$

Since obviously $\tau' \in \mathcal{E}^+(\overline{A})$, this reduces to $\sigma' \in \mathcal{E}^+(\overline{A})$. As $S_\sigma \cap \overline{A} = \emptyset$, $\sigma'$ is $C$-absolutely continuous (see [13, Corollary 3.19] or, more generally, [27, Corollary 5.2]), and therefore $U^{\sigma'} = U^{\sigma} \sigma'$-a.e. on $\mathbb{R}^n$, cf. (4.19). This implies by integration

$$\int U^{\sigma'} \, d\sigma' = \int U^{\sigma} \, d\sigma' \leq d(S_\sigma, A)^{n-n} \sigma(\mathbb{R}^n)^2 < \infty,$$

cf. (2.9), whence the claim.

\[23\] Here we use the fact that for any $\mu \in \mathfrak{M}^+$ and any $Q \subset \mathbb{R}^n$, $\mu^Q(\mathbb{R}^n) \leq \mu(\mathbb{R}^n)$ [20, Corollary 4.9].
But, by Corollary 4.2 (["balayage with a rest"])\footnote{See also Landkof’s book \cite{23} p. 264], where the sets in question were assumed to be Borel, while the measures bounded.}

\[
\delta^A = (\delta')^A.
\]

Therefore, applying Theorem 3.1(c) to \(\delta' \in \mathcal{E}^+\), cf. (4.23), we obtain (iii) by making use of the equality \(\mathcal{E}^+(A) = \mathcal{E}'(A)\) as well as of the fact that for any \(\nu \in \mathcal{M}^+_1\),

\[
U'' = U^{\delta'} \text{ n.e. on } A \implies U'' = U^\delta \text{ n.e. on } A,
\]

which in turn is implied by (4.19) (applied to \(A\) and \(\delta\)) and the strengthened version of countable subadditivity for inner capacity (footnote 9).

The remaining assertions (i), (ii), and (iv) follow from Definition 3.1 and Theorem 3.1, (a) and (d) in the same manner as above. \(\square\)

**Lemma 4.8.** For the above \(A\) and \(\delta\), we have

\[
\xi_{A,f} \in \mathcal{E}^+(A) \quad \text{and} \quad I_f(\xi_{A,f}) = w_f(A),
\]

(4.24)

\(\xi_{A,f}\) being the extremal measure. Problem 1.1 is therefore solvable if and only if

\[
\xi_{A,f}(\mathbb{R}^n) = 1,
\]

(4.25)

and in the affirmative case \(\xi_{A,f} = \lambda_{A,f}\).

**Proof.** Fix \((\mu_s)_{s \in S} \in \mathcal{M}_f(A)\). Since \(\mu_s \rightarrow \xi_{A,f}\) strongly (Lemma 4.1), whereas \(\mathcal{E}^+(A)\) is strongly closed, see (P4), the former relation in (4.24) is obvious.

Furthermore,

\[
\lim_{s \in S} \int f \, d\mu_s = \int f \, d\xi_{A,f},
\]

(4.26)

because

\[
\lim_{s \in S} \int U^\delta \, d\mu_s = \lim_{s \in S} \langle \delta^A, \mu_s \rangle = \langle \delta^A, \xi_{A,f} \rangle = \int U^\delta \, d\xi_{A,f}.
\]

Indeed, the last (similarly, the first) equality holds true since \(U^\delta = U^{\delta^A}\) n.e. on \(A\), hence \(\xi_{A,f}\)-a.e. (see the former relation in (4.24) as well as Lemma 3.5); whereas the second equality follows from (1.2) by making use of the Cauchy–Schwarz inequality, applied to \(\delta^A\) and \(\mu_s - \xi_{A,f}\), elements of the pre-Hilbert space \(\mathcal{E}\) (cf. Theorem 4.7(iii)).

Adding (4.26) multiplied by 2 to

\[
\lim_{s \in S} \|\mu_s\|^2 = \|\xi_{A,f}\|^2,
\]

(4.27)

cf. (1.2), on account of (1.1) we get

\[
I_f(\xi_{A,f}) = \lim_{s \in S} I_f(\mu_s) = w_f(A),
\]

whence the latter relation in (1.24). Thus, actually, \(\xi_{A,f} \in \mathcal{E}^+_f(A)\).

Now, assuming that (4.25) is fulfilled, we derive from (1.24) that \(\xi_{A,f}\) must indeed serve as the minimizer \(\lambda_{A,f}\). For the "only if" part, see Corollary 4.4. \(\square\)

**Lemma 4.9.** For the extremal measure \(\xi = \xi_{A,f}\), we have

\[
U^\xi_f \geq C_\xi \text{ n.e. on } A,
\]

(4.28)

where

\[
C_\xi := \int U^\xi \, d\xi = w_f(A) - \int f \, d\xi \in (-\infty, \infty).
\]

(4.29)
If moreover $f$ is l.s.c. on $\overline{A}$, then also
\[ U_j^\xi \leq C_\xi \text{ on } S(\xi), \tag{4.30} \]
and hence
\[ U_j^\xi = C_\xi \text{ } \xi\text{-a.e.} \tag{4.31} \]

Proof. By Corollary 2.2, there exists the (unique) solution $\lambda_{K,f}$ to Problem 1.1 with $A := K$, where $K \in \mathcal{C}_A$ is large enough ($K \geq K_0$), while according to Theorem 1.3
\[ U_f^{\lambda_{K,f}} \geq \int U_f^{\lambda_{K,f}} d\lambda_{K,f} \text{ n.e. on } K. \tag{4.32} \]
Since $w_f(K) \downarrow w_f(A)$ as $K \uparrow A$, see (4.10), these $\lambda_{K,f}$ form a minimizing net, i.e.
\[ (\lambda_{K,f})_{K \geq K_0} \in \mathcal{M}_f(A), \]
and hence $(\lambda_{K,f})$ converges strongly in $\mathcal{E}^+$ to the extremal measure $\xi$ (Lemma 1.1). Relations (4.26) and (4.27), both with $(\lambda_{K,f})_{K \in \mathcal{C}_A}$ in place of $(\mu_s)_{s \in S}$, give
\[ \lim_{K \uparrow A} \int U_f^{\lambda_{K,f}} d\lambda_{K,f} = \int U_j^\xi d\xi =: C_\xi \in (-\infty, \infty), \tag{4.33} \]
the finiteness of $C_\xi$ being clear from the latter relation in (4.24).

Fix $K_* \in \mathcal{C}_A$. The strong topology on $\mathcal{E}^+$ being first-countable, one can choose a subsequence $(\lambda_{K_j,f})_{j \in \mathbb{N}}$ of the net $(\lambda_{K,f})_{K \in \mathcal{C}_A}$ such that
\[ \lambda_{K_j,f} \to \xi \text{ strongly (hence vaguely) in } \mathcal{E}^+ \text{ as } j \to \infty. \tag{4.34} \]
There is certainly no loss of generality in assuming $K_* \subset K_j$ for all $j$, for if not, we replace $K_j$ by $K'_j := K_j \cup K_*$; then, by the monotonicity of $(w_f(K))_{K \in \mathcal{C}_A}$, the sequence $(\lambda_{K'_j,f})_{j \in \mathbb{N}}$ remains minimizing, and hence also converges strongly to $\xi$.

Due to the arbitrary choice of $K_* \in \mathcal{C}_A$, (4.28) will follow once we show that
\[ U_f^\xi \geq C_\xi \text{ n.e. on } K_* \tag{4.35} \]
Passing if necessary to a subsequence and changing the notation, we conclude from (4.33), by making use of [15] p. 166, Remark], that
\[ U_j^\xi = \lim_{j \to \infty} U_j^{\lambda_{K_j,f}} \text{ n.e. on } \mathbb{R}^n. \tag{4.36} \]
Therefore, applying (4.32) to each $K_j$, and then letting $j \to \infty$, on account of (4.33) and (4.35) we arrive at (4.36). (Here the countable subadditivity of inner capacity on universally measurable sets has been utilized, see e.g. [22] p. 144.)

Assume now that $f$ is l.s.c. on $\overline{A}$. By Theorem 1.3 then
\[ U_j^{\lambda_{K_j,f}} \leq \int U_j^{\lambda_{K_j,f}} d\lambda_{K_j,f} \text{ on } S(\lambda_{K_j,f}), \]
where $(K_j) \subset \mathcal{C}_A$ is the sequence chosen above. Since $(\lambda_{K_j,f})$ converges to $\xi$ vaguely, see (4.34), for every $x \in S(\xi)$ there exist a subsequence $(K_{j_k})$ of $(K_j)$ and points $x_{j_k} \in S(\lambda_{K_{j_k},f})$, $k \in \mathbb{N}$, such that $x_{j_k}$ approach $x$ as $k \to \infty$. Thus
\[ U_f^{\lambda_{K_{j_k},f}}(x_{j_k}) \leq \int U_f^{\lambda_{K_{j_k},f}} d\lambda_{K_{j_k},f} \text{ for all } k \in \mathbb{N}. \]
Letting here $k \to \infty$, in view of (4.33) and the lower semicontinuity of the mapping $(x, \mu) \mapsto U(\mu)(x)$ on $\mathbb{R}^n \times \mathfrak{M}^+$, where $\mathfrak{M}^+$ is equipped with the vague topology [15 Lemma 2.2.1(b)], we obtain (4.30). Finally, combining (4.30) with (4.28) yields $U_j^\xi = C_\xi$ n.e. on $S(\xi) \cap A$, whence (4.31), for $\xi \in \mathcal{E}^+(A)$ (Lemma 4.8). \qed
4.7. Proof of Theorem 2.7. Under the hypotheses \((P'_2), (P'_3),\) and \((P_4),\) assume moreover that \((2.10)\) is fulfilled, where \(\delta\) is the measure appearing in \((2.8)\).

The proof is mainly based on Theorems 1.4 and 4.7 and it is given in four steps.

Step 1. Assume first that \(c_*(A) < \infty.\) Due to \((P'_2), (P'_3),\) and \((P_4),\) \(\delta^A\) and \(\gamma_A,\) the inner balayage of \(\delta\) to \(A\) and the inner capacitary measure of \(A,\) respectively, are both of finite energy, and they are concentrated on \(A,\) i.e.

\[
\delta^A, \gamma_A \in \mathcal{E}^+(A)
\]  

(see Theorem 4.7(iii) and [29, Theorem 7.2], respectively) \(^{25}\) We aim to show that

\[
\beta := \delta^A + \eta_{A,f} \gamma_A,
\]

\(\eta_{A,f}\) being introduced in \((2.13),\) serves as the (unique) solution to Problem 1.1. (Note that this would provide an alternative proof of the solvability of Problem 1.1; compare with Theorem 2.1 and its proof, given in Section 4.2.)

Indeed, [26, Corollary 4.9], quoted in footnote 23 above, yields

\[
0 \leq \delta^A(\mathbb{R}^n) \leq \delta(\mathbb{R}^n) \leq 1,
\]

(4.39)

the last inequality being valid by virtue of \((2.10).\) Thus \(\eta_{A,f} \in [0, \infty),\) and combining \(c_*(A) = \gamma_A(\mathbb{R}^n)\) (see [22, Theorem 2.6]) with \((2.13), (1.37),\) and \((4.38)\) shows that, actually, \(\beta \in \mathcal{E}^+(A).\) Moreover, \(\beta = \mathcal{E}^+_f(A),\) for

\[
\int U^\beta d\beta = \langle \delta^A, \beta \rangle < \infty.
\]

According to Theorem 1.4 \(\beta = \lambda_{A,f}\) will therefore follow once we verify the inequality

\[
U^\beta_f \geq \int U^\beta d\beta \text{ n.e. on } A.
\]  

(4.40)

By the strengthened version of countable subadditivity for inner capacity, we infer from \((1.38), U^{\gamma_A} = 1 \text{ n.e. on } A \text{ [22 p. 145]},\) and \((1.19)\) (applied to \(\delta\)) that

\[
U^\beta_f = U^\beta - U^\delta = U^{\delta^A - \delta} + \eta_{A,f} U^{\gamma_A} = \eta_{A,f} \text{ n.e. on } A,
\]

(4.41)

hence \(\beta\)-a.e. Therefore,

\[
\int U^\beta d\beta = \eta_{A,f} \beta(\mathbb{R}^n) = \eta_{A,f},
\]

which substituted into \((4.41)\) gives \((4.40).\) Thus the solution \(\lambda_{A,f}\) to Problem 1.1 does indeed exist, and moreover \(\lambda_{A,f} = \beta\) and \(c_{A,f} = \eta_{A,f} c_{A,f}\) being the inner \(f\)-weighted equilibrium constant. This establishes \((2.17)\) as well as the representation

\[
\lambda_{A,f} = \delta^A + \eta_{A,f} \gamma_A.
\]  

(4.42)

As \((\gamma_A)^A = \gamma_A\) [29, Lemma 9.1], identity \((4.42)\) can be rewritten in the form

\[
\lambda_{A,f} = (\delta + \eta_{A,f} \gamma_A)^A.
\]

Applying Theorem 4.7 (i) and (ii), to \(\delta + \eta_{A,f} \gamma_A\) in place of \(\delta,\) which is possible because \(\eta_{A,f} \gamma_A \in \mathcal{E}^+,\) we therefore conclude that \(\lambda_{A,f}\) can be characterized as the unique measure of minimum energy, resp. of minimum potential, within the class of all \(\nu \in \mathfrak{M}^+\) having the property \(U^\nu \geq U^\delta + \eta_{A,f} U^{\gamma_A}\) n.e. on \(A,\) or equivalently

\[
U^\nu_f \geq \eta_{A,f} \text{ n.e. on } A.
\]

This establishes assertion (i), resp. (ii), of Theorem 2.7.

\(^{25}\) Also note that \(\gamma_A\) is nonzero, which is obvious from \(c_*(A) > 0,\) cf. \((1.10).\)
Similarly, Theorem 4.7 (iii) applied to $\delta + \eta_{A,f}\gamma_A$ results in Theorem 2.7 (iii).

**Step 2.** Assume now that $c_*(A) = \infty$, and that condition (2.11) is fulfilled; then necessarily $\delta^A(\mathbb{R}^n) = 1$. Actually, $\delta^A \in \tilde{\mathcal{E}}^+(A)$, see Theorem 4.7 (iii), whence
\begin{equation}
\delta^A \in \tilde{\mathcal{E}}^+_f(A),
\end{equation}
for
\[ \int U_\delta \ d\delta^A = \langle \delta^A, \delta^A \rangle < \infty. \]

We aim to show that $\delta^A$ serves as the (unique) solution to Problem 1.1, i.e.
\begin{equation}
\delta^A = \lambda_{A,f}.
\end{equation}
Noting that
\[ U_\delta^A = U_\delta^A - U_\delta = 0 \ \text{n.e. on } A,
\]
hence $\delta^A$ a.e., we get
\[ \int U_\delta^A \ d\delta^A = 0,
\]
which substituted into (4.45) gives
\[ U_\delta^A = \int U_\delta^A \ d\delta^A \ \text{n.e. on } A.
\]
By Theorem 1.4, this together with (4.43) yields (4.44) as well as $c_{A,f} = 0$. Furthermore, observing from (2.13) that $\eta_{A,f}$ along with $c_{A,f}$ equals 0, we arrive at (2.17).

We finally note that, due to the equalities $\lambda_{A,f} = \delta^A$ and $\eta_{A,f} = 0$ thus obtained, assertions (i)–(iii) of Theorem 2.7 follow directly from Theorem 4.7 (i)–(iii).

**Step 3.** The aim of this step is to show that assumption (2.11) is not only sufficient, but also necessary for the existence of the solution $\lambda_{A,f}$. Assume to the contrary that $\lambda_{A,f}$ exists, but (2.11) fails to hold; in view of (4.39), then necessarily
\begin{equation}
c_*(A) = \infty \quad \text{and} \quad \delta^A(\mathbb{R}^n) < 1.
\end{equation}
According to Corollary 2.2, for each $K \in \mathcal{C}_A$ large enough ($K \geq K_0$), there is the solution $\lambda_{K,f}$ to Problem 1.1 with $A := K$; and moreover the net $(\lambda_{K,f})_{K \geq K_0}$ converges strongly and vaguely to the extremal measure $\xi_{A,f}$, determined by Lemma 4.1.
We claim that, due to the former relation in (4.46),
\begin{equation}
\xi_{A,f} = \delta^A.
\end{equation}
By (2.13) and (1.22), both applied to $K \geq K_0$,
\[ \lambda_{K,f} = \delta^K + \tilde{\eta}_{K,f} \lambda_K,
\]
where $\lambda_K := \gamma_K/c(K)$ is the solution to Problem 1.1 with $A := K$ and $f := 0$, and
\[ \tilde{\eta}_{K,f} := 1 - \delta^K(\mathbb{R}^n). \]
But the net $(\tilde{\eta}_{K,f})_{K \geq K_0} \subset \mathbb{R}$ is bounded since, in view of (1.39) with $A := K$,
\[ 0 \leq \delta^K(\mathbb{R}^n) \leq \delta(\mathbb{R}^n) \leq 1 \quad \text{for all } K \in \mathcal{C}_A.
\]
Furthermore, by virtue of (1.21) and (1.22),
\[ \delta^K \to \delta^A \quad \text{strongly (and vaguely) in } \mathcal{E}^+ \text{ as } K \uparrow A.
\]
Thus, if we show that
\[ \lambda_K \to 0 \quad \text{strongly in } \mathcal{E}^+ \text{ as } K \uparrow A,
\]
identity (4.47) will follow from (4.48) by passing to the limit as $K \uparrow A$, and making use of the triangle inequality in the pre-Hilbert space $\mathcal{E}$.

It is seen from (4.10) that the net $(\lambda_K)_{K \geq K_0}$ is minimizing in Problem 1.1 with $f = 0$, i.e. $(\lambda_K)_{K \geq K_0} \in \mathcal{M}(A)$. Applying Lemma 2.11 we therefore conclude that there exists the unique extremal measure $\xi_A$ in Problem 1.1 with $f = 0$, and moreover $\lambda_K \to \xi_A$ strongly in $\mathcal{E}^+$. This yields

$$\|\xi_A\|^2 = \lim_{K \uparrow A} \|\lambda_K\|^2 = \lim_{K \uparrow A} w(K) = 0,$$

the last equality being caused by $c_*(A) = \infty$. Since the $\alpha$-Riesz kernel is strictly positive definite, we thus have $\xi_A = 0$, which proves (4.49), whence (4.47).

Since $\lambda_{A,f}$ is assumed to exist, Corollary 4.4 together with (4.47) gives

$$\lambda_{A,f} = \xi_{A,f} = \delta^A,$$

which however is impossible, for $\delta^A(\mathbb{R}^n) < 1$ by (4.46).

Step 4. To complete the proof of the theorem, it remains to establish (2.18). Applying (2.13) and (2.17) to each $K \in \mathcal{C}_A$ large enough $(K \geq K_0)$, we get

$$\lambda_{K,f} = \frac{1 - \delta^K(\mathbb{R}^n)}{c(K)}.
$$

In view of (2.3), (2.18) will therefore follow once we show that the net $(\delta^K(\mathbb{R}^n))_{K \in \mathcal{C}_A}$ increases. But this is obvious from [20] (Corollaries 4.2 and 4.9), because for any $K, K' \in \mathcal{C}_A$ such that $K' \geq K$, we have $\delta^K = (\delta^{K'})^K$, whence $\delta^K(\mathbb{R}^n) \leq \delta^{K'}(\mathbb{R}^n)$.

4.8. Proof of Corollary 2.9 As noted in Theorem 2.7, $\lambda_{A,f} \in \Lambda_{A,f}$, the class $\Lambda_{A,f}$ being introduced in (2.14). We thus only need to show that for any given $\mu \in \Lambda_{A,f}$,

$$\lambda_{A,f}(\mathbb{R}^n) \leq \mu(\mathbb{R}^n).$$

(4.50)

But according to Theorem 2.7(ii), then necessarily

$$U^{\lambda_{A,f}} \leq U^\mu \text{ everywhere on } \mathbb{R}^n,$$

and (4.50) follows by use of the principle of positivity of mass [18, Theorem 3.11].

4.9. Proof of Corollary 2.11 Let the assumptions of (a) be fulfilled. Due to the inequality $\delta^A(\mathbb{R}^n) \leq \delta(\mathbb{R}^n)$ [20, Corollary 4.9], then necessarily $\delta^A(\mathbb{R}^n) < 1$, which implies, by use of Theorem 2.7 that Problem 1.1 indeed has no solution.

Let now $\delta \in \hat{\mathcal{E}}^+(A)$. In view of Theorem 2.7 it is enough to consider the case where $c_*(A) = \infty$. The orthogonal projection of $\delta$ onto the (strongly closed by $\mathcal{P}_d$, hence strongly complete) cone $\mathcal{E}^+(A) = \mathcal{E}'(A)$ is certainly the same $\delta$, which means, by virtue of [29, Theorem 3.1(b)], that $\delta^A = \delta$, whence $\delta^A(\mathbb{R}^n) = 1$. Applying Theorem 2.7 once again, we conclude that the solution $\lambda_{A,f}$ does exist, and moreover $\lambda_{A,f} = \delta^A = \delta$, cf. the latter equality in (2.12). This completes the whole proof.

4.10. Proof of Theorem 2.13 Let $A$ be not inner $\alpha$-thin at infinity; then necessarily $c_*(A) = \infty$. For (a1), assume $\delta(\mathbb{R}^n) = 1$. Since, by virtue of [27, Corollary 5.3],

$$\delta^A(\mathbb{R}^n) = \delta(\mathbb{R}^n) = 1,$$

Theorem 2.7 shows that $\lambda_{A,f}$ does indeed exist, and moreover $\lambda_{A,f} = \delta^A$ (cf. the latter equality in (2.12)). Also, $c_{A,f} = 0$, which is obvious from (2.13) and (2.17).

For (b1), assume that $f$ is l.s.c. on $\overline{A}$. As seen from (a1) and Corollary 2.11(a), formula (2.23) will be proved once we verify the solvability of Problem 1.1 in the case $\delta(\mathbb{R}^n) > 1$. (4.51)
For the extremal measure $\xi = \xi_{A,f}$, suppose first that $C_\xi \geq 0$, where $C_\xi$ is the (finite) constant appearing in Lemma 4.9. Then, by (1.28),

$$U_\xi ^* \geq U^\delta + C_\xi \geq U^\delta \text{ n.e. on } A.$$ 

The set $A$ not being inner $\alpha$-thin at infinity, an application of the strengthened version of the principle of positivity of mass [31, Theorem 1.2] gives

$$1 < \delta (\mathbb{R}^n) \leq \xi (\mathbb{R}^n),$$

cf. (4.51), which however contradicts the inequality $\xi (\mathbb{R}^n) \leq 1$ (Remark 4.3). Thus

$$C_\xi < 0. \quad (4.52)$$

But, by (4.31), $U_\xi ^* = C_\xi$ holds true $\xi$-a.e., which implies by integration

$$\int U_\xi ^* d\xi = C_\xi \cdot \xi (\mathbb{R}^n).$$

Since $C_\xi \neq 0$,

$$\xi (\mathbb{R}^n) = \frac{\int U_\xi ^* d\xi}{C_\xi},$$

whence $\xi (\mathbb{R}^n) = 1$, by (1.29). By virtue of Lemma 4.8, the extremal measure $\xi$ serves, therefore, as the solution $\lambda_{A,f}$. Substituting $\lambda_{A,f} = \xi$ into (1.7) we finally obtain

$$c_{A,f} = \int U_\xi ^* d\xi,$$

which combined with (4.29) and (4.52) proves (2.24).

4.11. **Proof of Theorem 2.15.** Under the assumptions of the theorem, $c_*(A) = \infty$ and $\delta (\mathbb{R}^n) \leq 1$. Therefore, by Theorem 2.7, $\lambda_{A,f}$ exists if and only if $\delta^A (\mathbb{R}^n) = 1$. We aim to show that, due to the remaining requirements of the theorem, we actually have $\delta^A (\mathbb{R}^n) \neq 1$, and so $\lambda_{A,f}$ fails to exist. Since $\overline{A}$ is $\alpha$-thin at infinity, there exists the $\alpha$-Riesz equilibrium measure $\gamma$ of $\overline{A}$, treated in an extended sense where $I(\gamma)$ as well as $\gamma (\mathbb{R}^n)$ may be infinite (for more details, see [22, Section V.1.1], cf. [26, Section 5] and [27, Sections 1.3, 2.1]). Applying [26, Theorem 8.7], we therefore get

$$(\delta |_{\Omega_\gamma})^{\overline{A}} (\mathbb{R}^n) < \delta |_{\Omega_\gamma} (\mathbb{R}^n),$$

whence

$$\delta^A (\mathbb{R}^n) < 1,$$

for, in consequence of [26, Corollaries 4.2, 4.9],

$$(\delta |_{\Omega_\gamma})^A (\mathbb{R}^n) = \left( (\delta |_{\Omega_\gamma})^{\overline{A}} (\mathbb{R}^n) \right)^A (\mathbb{R}^n) \leq \delta |_{\Omega_\gamma} (\mathbb{R}^n).$$

4.12. **Proof of Theorem 2.17.** By Theorems 2.1 and 2.13, the minimizers $\lambda_{A_j,f}$, $j \in \mathbb{N}$, and $\lambda_{A,f}$ do exist. (Here we use the fact that for closed sets, $(\mathcal{P}_A)$ necessarily holds, cf. Theorem 3.9.) Therefore, applying Theorem 2.4 we obtain (2.4)–(2.6).
4.13. **Proof of Theorem 2.18.** Under the assumptions of the theorem, \( \lambda_{A_j,f}, j \in \mathbb{N} \), and \( \lambda_{A,f} \) do exist, see Theorem 2.13(a). (Here it should be taken into account that for quasiclosed sets, \((\mathcal{P}_4)\) necessarily holds, see Theorem 3.9, and that a countable intersection of quasiclosed sets is likewise quasiclosed, see [16, Lemma 2.3].) Fix \( k \in \mathbb{N} \). Applying the former relation in (2.22) to each of \( A_j \) as well as to \( A \) gives

\[
\lambda_{A_j,f} = \delta^{A_j} = \left( \delta^{A_k} \right)^{A_j} \quad \text{for all } j \geq k,
\]

(4.53)

\[
\lambda_{A,f} = \delta^A = \left( \delta^{A_k} \right)^A,
\]

(4.54)

the latter equality in (4.53), resp. (4.54), being valid by virtue of [26, Corollary 4.2]. But according to [28, Theorem 4.10] applied to \( \delta^{A_k} \in \mathcal{E}^+ \), cf. (4.23),

\[
\left( \delta^{A_k} \right)^{A_j} \to \left( \delta^{A_k} \right)^A \quad \text{strongly and vaguely in } \mathcal{E}^+ \text{ as } j \to \infty,
\]

\[
U\left( \delta^{A_k} \right)^{A_j} \downarrow U\left( \delta^{A_k} \right)^A \quad \text{pointwise on } \mathbb{R}^n \text{ as } j \to \infty,
\]

which combined with (4.53) and (4.54) establishes (2.5) and (2.26).

By making use of (2.5), in the same manner as in the proof of Lemma 4.8 we get

\[
\lim_{j \to \infty} \int U^\delta \, d\lambda_{A_j,f} = \lim_{j \to \infty} \left( \delta^A, \lambda_{A_j,f} \right) = \left( \delta^A, \lambda_{A,f} \right) = \int U^\delta \, d\lambda_{A,f},
\]

which together with

\[
\lim_{j \to \infty} \|\lambda_{A_j,f}\|^2 = \|\lambda_{A,f}\|^2
\]

results in (2.7).

To complete the proof, observe that the remaining relation (2.6) is obvious, because under the stated assumptions, \( c_{A_j,f} = c_{A,f} = 0 \), see the latter relation in (2.22).

4.14. **Proof of Theorem 2.19.** As seen from [27, Corollary 5.3], in either of the cases (a) or (b), (2.11) is fulfilled. Therefore, by Theorem 2.17, \( \lambda_{A,f} \) does exist; it is representable by the former equality in (2.12) if (a) occurs, or by the latter equality otherwise. Applying [26, Theorems 7.2, 8.5], providing descriptions of the supports of capacitary and swept measures, we obtain (2.27). (Regarding the first term on the right-hand side in the latter equality, take into account that, due to \( (\mathcal{P}_2') \) and \( (\mathcal{P}_3') \), we have \( \delta|A \in \mathcal{E}^+(A) \), whence \( (\delta|A)^A = \delta|A, (\delta|A)^A \) being the orthogonal projection of \( \delta|A \) onto the strongly closed by \( (\mathcal{P}_4) \), hence strongly complete, convex cone \( \mathcal{E}^+(A) \).)

4.15. **Proof of Theorem 2.22.** Under the hypotheses of the theorem, \( \lambda_{A,f} \) does exist, see Theorem 2.13(b). Assuming to the contrary that \( S(\lambda_{A,f}) \) is noncompact, we conclude by making use of (1.6) that there must exist a sequence \( (x_j) \subset A \) approaching the Alexandroff point \( \infty_{\mathbb{R}^n} \) as \( j \to \infty \), and such that

\[
f(x_j) \leq U^{\lambda_{A,f}}_f(x_j) = c_{A,f} < 0 \quad \text{for all } j \in \mathbb{N},
\]

the latter inequality being valid by (2.24). On account of (2.28), we thus have

\[
\lim_{x \to \infty_{\mathbb{R}^n}, x \in A} U^\delta(x) > 0.
\]

On the other hand, it is clear from [27, Theorem 2.1(ii)] that this limit must be equal to 0, the set \( A \) being not inner \( \alpha \)-thin at infinity. Contradiction.
5. ON THE POSSIBLE EXTENSIONS OF THE ESTABLISHED THEORY

If \( \omega = 0 \), where \( \omega \in \mathfrak{M}(\mathbb{R}^n) \) is the measure appearing in representation (1.3), then some of the results of this study have already been extended to an arbitrary perfect kernel \( \kappa \) on a locally compact Hausdorff space \( X \), satisfying the first and the second maximum principles, see [32, Theorems 1.2, 1.5]. However, a further progress in this direction would require the development of a theory of balayage for Radon measures on \( X \) of infinite energy, which is so far an open question, cf. [30, Problem 7.1].

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