ON THE LATTICE MODEL OF THE WEIL REPRESENTATION AND THE
HOWE DUALITY CONJECTURE

SHUICHIRO TAKEDA

Abstract. The lattice model of the Weil representation over non-archimedean local field $F$ of
odd residual characteristic has been known for decades, and is used to prove the Howe duality
conjecture for unramified dual pairs when the residue characteristic of $F$ is odd. In this paper, we
will modify the lattice model of the Weil representation so that it is defined independently of the
residue characteristic. Although to define the lattice model alone is not enough to prove the Howe
duality conjecture for even residual characteristic, we will propose a couple of conjectural lemmas
which imply the Howe duality conjecture for unramified dual pairs for even residual characteristic.
Also we will give a proof of those lemmas for certain cases, which allow us to prove (a version of)
the Howe duality conjecture for even residual characteristic for a certain class of representations for
the dual pair $(O(2n), Sp(2n))$, where $O(2n)$ is unramified. We hope this paper serves as a first step
toward a proof of the Howe duality conjecture for even residual characteristic.

1. Introduction

Let $F$ be a non-archimedean local field of characteristic $0$ and $W$ be a symplectic space over $F$
of dimension $2n$. For an additive character $\psi$ on $F$, we let $\omega_\psi$ be the Weil representation of
the metaplectic cover $\widetilde{Sp}(W)$ of $Sp(W)$. Let $E$ be either $F$ or a quadratic extension of $F$. For $i = 1, 2$,
let $(\mathcal{V}_i, (-,-)_i)$ be an $\epsilon_i$-Hermitian space over $E$ where $\epsilon_i \in \{\pm 1\}$ and $\epsilon_1 \epsilon_2 = -1$, and let $U(\mathcal{V}_i)$
be its isometry group. Assume the pair $(U(\mathcal{V}_1), U(\mathcal{V}_2))$ forms an irreducible dual reductive pair in
$\widetilde{Sp}(W)$, so that $U(\mathcal{V}_1) \cdot U(\mathcal{V}_1)$ is a subgroup of $Sp(W)$. We call the restriction of $\omega_\psi$ to the preimage
of $U(\mathcal{V}_1) \cdot U(\mathcal{V}_2)$ in $\widetilde{Sp}(W)$ also $\omega_\psi$. In this introduction just for notational convenience we assume that
both $U(\mathcal{V}_1)$ and $U(\mathcal{V}_2)$ split in $\widetilde{Sp}(W)$, so we may view $\omega_\psi$ as a representation of $U(\mathcal{V}_1) \cdot U(\mathcal{V}_1)$, or
even as a representation of $U(\mathcal{V}_1) \times U(\mathcal{V}_2)$ via the multiplication map $U(\mathcal{V}_1) \times U(\mathcal{V}_2) \to U(\mathcal{V}_1) \cdot U(\mathcal{V}_2)$.

For an irreducible admissible representation $\pi_1$ of $U(\mathcal{V}_1)$, the maximum $\pi$-isotypic quotient of $\omega_\psi$
as a representation of $U(\mathcal{V}_1) \times U(\mathcal{V}_2)$ has the form

$$\pi \otimes \Theta_\psi(\pi)$$

for some (possibly zero) smooth representation $\Theta_\psi(\pi)$ of $U(\mathcal{V}_2)$. It is known that $\Theta_\psi(\pi)$ is of finite
length and hence is admissible. We let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$. It has
been conjectured by Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is non-zero.
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This conjecture has been known as the Howe duality conjecture, and proven by Howe and Waldspurger
when the residue characteristic of $F$ is odd more than two decades ago ([MVW], [Hi], [Wa]).

The case for even residual characteristic is still widely open in general. To the best of our knowledge,
the only general result for even residual characteristic is the quarter century old result by Kudla ([Ku])
in which he shows if $\pi$ is supercuspidal, then $\Theta_\psi(\pi)$ is always irreducible. Since then, however, it seems
no progress has been made, possibly with the exceptions of the recently result by Li-Sun-Tian ([LST])

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which shows that \( \theta_\psi(\pi) \) is multiplicity free. Also when the ranks for the groups are very small like 2 or 3 at most, one can check the Howe duality by hand.

The proof of the Howe duality conjecture for odd residual characteristic is reproduced in detail by Waldspurger in [Wa]. The proof requires what is known as the generalized lattice model of the Weil representation and is highly complex. However when the dual pair \((U(V_1), U(V_2))\) is “unramified” in the sense of [MVW], which is the same as saying both of the groups \(U(V_1)\) and \(U(V_2)\) are unramified i.e. split over unramified extension of \(F\), one only needs the lattice model of the Weil representation, which is much simpler than the generalized lattice model, and hence the proof becomes significantly simpler. This proof is reproduced in Chapter 5 of [MVW]. Also another version of the proof is given in [MT].

All of those proofs of the Howe duality conjecture for odd residual characteristic require the (generalized) lattice model of the Weil representation. One of the crucial obstructions to apply those proofs to the case of even residual characteristic is the unavailability of such model.

In this paper, we modify the known lattice model so that it can be defined even when the residue characteristic of \(F\) is even. As we will see, however, this is not enough to extend the proof in [MVW] Ch. 5] to the case of even residual characteristic, and various technical difficulties arise if one simply tries to apply the arguments in [MVW] Ch. 5] to the case of even residual characteristic. In particular, one has to prove a couple of lemmas, which we call “the first and second key lemmas on the lattice model”, which are the analogues of Theorem I.4 and Proposition I.5 in Chapter 5 of [MVW, p. 103], respectively.

Let us be more specific. For an unramified dual pair \((U(V_1), U(V_2))\), there exists a self-dual lattice \(L_i\) of \(V_i\) for \(i = 1, 2\), so that \(A := L_1 \otimes_\mathcal{O}_E L_2\) is a self-dual lattice of \(W := V_1 \otimes_\mathcal{O}_E V_2\) with respect to the additive character \(\psi\). First we show

**Theorem 1.1.** There exists the lattice model \((\omega_\psi, S_A)\) of the Weil representation \(\omega_\psi\) even when the residue characteristic of \(F\) is even.

Here the space \(S := S_A\) of the lattice model is a certain set of smooth compactly supported functions \(f : W \times F \to \mathbb{C}\) on \(W \times F\). For each sublattice \(L \subseteq L_1\) of the self-dual lattice \(L_1\), we define \(S_L\) to be the subspace of \(S_A\) consisting of functions whose support is in \((L^\perp \otimes_\mathcal{O}_E L_2) \times F\), where \(L^\perp\) is the dual lattice of \(L\).

The first key lemma roughly says the following: For any sublattice \(L \subset L_1\), we have the equality

\[
\omega_\psi(H_2)S_L = S_{J_1(L)^p}
\]

where \(H_2\) is the spherical Hecke algebra of \(U(V_2)\), and \(S_{J_1(L)^p}\) is the set of functions invariant by the subgroup \(J_1(L)\) of \(U(V_1)\) which is the kernel of the reduction map \(U(L^\perp) \to \text{Aut}(L^\perp/L)\), where \(L^\perp\) is the dual lattice of \(L\). (Strictly speaking we need to modify \(J_1(L)\) by a certain subgroup \(J_1(L)^p\), which will be defined later in the paper.) The same should hold by switching the roles of \(V_1\) and \(V_2\).

The second key lemma is even more of technical nature: First we define \(H_1(L)\) to be the kernel of the reduction map \(U(L_1) \to \text{Aut}(L_1/L)\). Then it turns out that for each \(w \in W\), there is a character \(\psi^w\) which naturally arises in the theory of lattice model. Then the second key lemma says if \(\psi^w = \psi^{w'}\) for \(w, w' \in W\) and both \(w\) and \(w'\) satisfy a certain maximality condition, then \(w\) and \(w'\) have to be in the same orbit under the action of the maximal open compact subgroup of \(U(V_1)\) on \(W/A\). And the same should hold by switching the roles of \(V_1\) and \(V_2\).

With those two conjectural lemmas, we can show the following version of the Howe duality principle:

**Theorem 1.2.** Modulo the above two conjectural lemmas, the Howe duality conjecture holds independently of the residue characteristic, in the sense that if \(\pi\) is an irreducible admissible representation of \(U(V_1)\) and \(\Theta_\psi(\pi) \neq 0\), then \(\Theta_\psi(\pi)\) has a unique irreducible non-zero quotient.
As we already mentioned, we are not able to prove the two key lemmas in full generality. However, if $L$ is of the form $L = \mathcal{O} \mathcal{L}_1$, where $\mathcal{O}$ is a uniformizer of $F$ and $k$ is an integer with $k \geq 1 + e$, where $e$ is the ramification index of 2 in $F$, then the two key lemmas can be proven to the extent necessary to prove

**Theorem 1.3 (Main Theorem).** Let $\pi$ be an irreducible admissible representation of $O(V)$, where $O(V)$ is quasi-split and split over an unramified extension and $\dim V = 2n$. Let $L_1 \subseteq V$ be a self-dual lattice. Assume

- $\pi^{f_1(L)} = 0$ for all $L$ with $L \supseteq \mathcal{O} \mathcal{L}_1$;
- $\pi^{f_1(L)} \neq 0$ for $L = \mathcal{O} \mathcal{L}_1$ for some $k$ with $k > 1 + e$.

Then for the dual pair $(O(2n), Sp(2n))$, if $\Theta_{\psi}(\pi) \neq 0$, it has a unique non-zero irreducible quotient.

In the above theorem we need to assume that the symplectic group has the same rank as the orthogonal group. All the conditions we need to impose on this theorem are all of technical nature. Also we consider the lifting to smaller rank symplectic groups and prove

**Theorem 1.4.** Let $\pi$ be an irreducible admissible representation of $O(V)$, where $O(V)$ is quasi-split and split over an unramified extension, with $\dim V = 2m$ or $\dim V = 2m + 1$. Assume $\pi$ is such that

$$V_{\pi^{f_1(L)}} = 0$$

for all $L \supseteq 2\mathcal{O} \mathcal{L}_1 = \mathcal{O}^{1+e} \mathcal{L}_1$ for a self-dual lattice $L_1 \subseteq V$. Then for the dual pair $(O(V), Sp(2n))$, (or $(O(V), Sp(2n))$) with $m > n$, we always have $\Theta_{\psi}(\pi) = 0$.

The main structure of this paper is the following: In the next section (Section 2), we will go over the formulation of the Heisenberg group and the Weil representation. Our formulation differs from the modern convention, but closely follows the one in the original paper by Weil [W]. In Section 3, we will define the lattice model of the Weil representation, which works independently of the residual characteristic, and make explicit the action of the metaplectic group on this model. In Section 4, we extend some of the lemmas about lattices proven in [MVW, Ch. 5, II] to the case of even residual characteristic, and in Section 5 we will go over the notion of unramified dual pair. In Section 6, we formulate the two (conjectural) key lemmas. In Section 7 and 8 we prove the first and second key lemmas for the special type of lattices mentioned above namely those $L$ with $L \subseteq 2\mathcal{O} \mathcal{L}_1 = \mathcal{O}^{1+e} \mathcal{L}_1$. Then finally in Section 9, we give our proof of the main theorem, and in Section 10 we will prove the last theorem mentioned above.

**Notations**

Throughout the paper, $F$ will be a non-archimedean local field of characteristic 0, and $E$ will be either $F$ or an unramified quadratic extension of $F$. We let $\mathcal{O}$ (resp. $\mathcal{O}_E$) be the ring of integers of $F$ (resp. $E$). We let $\mathcal{O}$ be a chosen uniformizer of $F$, and choose our uniformizer of $E$ to be $\mathcal{O}$ as well. Also we write $\mathcal{P}^n = \mathcal{O}^n$ and $\mathcal{P}_E^n = \mathcal{O}_E^n$ for each integer $n$. We fix an additive character $\psi$ of $F$, and $r$ be the exponential conductor of $\psi$ so that $\psi$ is trivial on $\mathcal{P}^r$. Also we let

$$e = \ord_F(2),$$

so $2 = \mathcal{O}^e \times \text{unit}$, and in particular if the residue characteristic of $F$ is even, $e$ is the ramification index of 2 in $F$.

For each $c \in E$, we denote $\bar{c} = c$ if $E = F$ and $\bar{c} = \tau(c)$ if $E \neq F$ where $\tau$ is the non-trivial element in $\text{Gal}(E/F)$.

For $c \in \{\pm 1\}$, by an $c$-Hermitian space $(V, (-,-))$ over $E$, we mean a finite dimensional vector space $V$ over $E$ equipped with a map $(-,-) : V \times V \to E$ which is linear on the first argument and
antilinear on the second with the property that \( \langle v_1, v_2 \rangle = \epsilon (v_2, v_1) \). We always assume our \( \epsilon \)-Hermitian space is nondegenerate. We let \( U(V) \) be the group of isometries of \( (V, \langle - , - \rangle) \). By a lattice \( L \) of \( V \), we mean a free \( \mathcal{O}_E \)-module \( L \) whose rank is equal to \( \dim_E V \).

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2. The Heisenberg group and the Weil representation

Let \( (W, \langle - , - \rangle) \) be a symplectic space over \( F \) of dimension \( 2n \). In this section, we define the Heisenberg group and the Weil representation for \( W \), but in order to construct the lattice model of the Weil representation that works for the case of even residual characteristic, we need to adapt a different convention. The Heisenberg group \( H(W) \) associated with \( W \) is usually defined to be \( H(W) = W \times F \) as a set with the group structure given by

\[
(w_1, z_1) \cdot (w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \frac{1}{2} \langle w_1, w_2 \rangle)
\]

for \( (w_1, z_1) \in H(W) \), and then one can see that each \( g \in \text{Sp}(W) \) acts on \( H(W) \) by \( g \cdot (w, z) = (gw, z) \).

However the \( \frac{1}{2} \) appearing here makes it impossible to define the lattice model of the Weil representation when the residue characteristic of \( F \) is even. To get around it, we will define the Heisenberg group differently by following the original formulation by Weil (W).

First we need to fix a polarization

\[
W = W^+ \oplus W^-,
\]

and for each element \( w \in W \), we write \( w = w^+ + w^- \) where \( w^+ \in W^+ \) and \( w^- \in W^- \). With respect to this polarization, we define a bilinear form

\[
\beta : W \times W \to F, \quad (w_1, w_2) \mapsto \langle w^+_1, w^-_2 \rangle.
\]

Note that \( \beta \) is indeed bilinear and \( \beta(w_1, w_2) = \langle w^+_1, w^-_2 \rangle = \langle w_1, w_2 \rangle \) but in general \( \beta(w_1, w_2) \neq -\beta(w_2, w_1) \), but instead we have

\[
\beta(w_1, w_2) - \beta(w_2, w_1) = \langle w_1, w_2 \rangle.
\]

For each \( \beta \), we define the Heisenberg group \( H_\beta(W) \) to be the group with underlying set

\[
H_\beta(W) = W \times F
\]

where the group operation is given by

\[
(w_1, z_1) \cdot (w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \beta(w_1, w_2)).
\]

One can check that the center of \( H_\beta(W) \) is \( \{ (0, z) : z \in F \} \).

Proposition 2.1. For any \( \beta \), \( H_\beta(W) \) is isomorphic to the usual Heisenberg group \( H(W) \) defined by \( \frac{1}{2} \langle - , - \rangle \).

Proof. By direct computation, one can check that the map \( H_\beta(W) \to H(W) \) defined by \( (w, z) \mapsto (w, z - \frac{1}{2} \langle w^+, w^- \rangle) \) is an isomorphism. \( \square \)

Proposition 2.2. All the \( H_\beta(W) \) are isomorphic to each other.
Proof. Of course, this immediately follows from the previous proposition because $H(W)$ is independent of $\beta$. But one can also construct an explicit isomorphism as follows: Let $W = X \oplus Y$ be another polarization. Define $\beta' : W \rightarrow F$ analogously with respect to this polarization. By Witt’s extension theorem, there exists $g \in \text{Sp}(W)$ such that $g(W^+) = X$ and $g(W^-) = Y$. Then one can see that the map $H_\beta(W) \rightarrow H_{\beta'}(W)$ defined by $(w, z) \mapsto (g(w), z)$ is an isomorphism. \hfill \Box

If we define our Heisenberg group in this way, however, we no longer have $\beta(gw_1, gw_2) = \beta(w_1, w_2)$ for every $g \in \text{Sp}(W)$, and hence $\text{Sp}(W)$ does not act in any obvious way. Namely the discrepancy $\beta(gw_1, gw_2) - \beta(w_1, w_2)$ has to be taken care of. For this purpose, let us define, for each $g \in \text{Sp}(W)$, $\Sigma_g$ to be the set of all continuous functions $\alpha : W \rightarrow F$ such that

$$\alpha(w_1 + w_2) - \alpha(w_1) - \alpha(w_2) = \beta(gw_1, gw_2) - \beta(w_1, w_2).$$

Such function $\alpha$ is a character of second degree in the sense of [W]. Following Weil ([W]), we define the linear pseudosymplectic group $\text{Ps}(W)$ by

$$\text{Ps}(W) := \{(g, \alpha) : g \in \text{Sp}(W), \alpha \in \Sigma_g\},$$

where the group structure is given by

$$(g_1, \alpha_1) \cdot (g_2, \alpha_2) = (g_1g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2)$$

where $g_2^{-1} \cdot \alpha_1$ is defined by

$$g_2^{-1} \cdot \alpha_1(w) = \alpha_1(g_2w).$$

Then $\text{Ps}(W)$ acts on $H_\beta(W)$ as

$$(g, \alpha) \cdot (w, z) = (gw, z + \alpha(w)).$$

One can verify that this is indeed an action.

We have the obvious map $\text{Ps}(W) \rightarrow \text{Sp}(W)$ given by $(g, \alpha) \mapsto g$. Weil shows that the sequence $0 \rightarrow W^* \rightarrow \text{Ps}(W) \rightarrow \text{Sp}(W) \rightarrow 0$ is exact, where $W^* = \text{Hom}_F(W, F)$ (see [W] p. 150). Moreover, he shows that the exact sequence splits by the following lemma.

**Lemma 2.3.** Let $g \in \text{Sp}(W)$ have the matrix representation

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the polarization $W = W^+ \oplus W^-$. Let

$$(2.4) \quad \alpha_g(w^+ + w^-) = \frac{1}{2}(aw^+ + cw^+) + \frac{1}{2}(bw^- + dw^-) + (bw^-, cw^+).$$

Then $\alpha_g \in \Sigma_g$. Moreover, the map $g \mapsto (g, \alpha_g)$ gives a homomorphism from $\text{Sp}(W)$ to $\text{Ps}(W)$, namely

$$\alpha_{gh} = h^{-1} \cdot \alpha_g + \alpha_h \quad \text{for all } g, h \in \text{Sp}(W).$$

**Proof.** Though this is proven in [W] Sec 4], it is not so easy to read it off from there due to the notational discrepancy. Hence we give a proof here with our notations. First let us show $\alpha_g \in \Sigma_g$. Let $w_1 = w_1^+ + w_1^-$ and $w_2 = w_2^+ + w_2^-$ be in $W = W^+ \oplus W^-$. First we need to show that $\alpha_g \in \Sigma_g$, namely

$$\alpha_g(w_1 + w_2) - \alpha_g(w_1) - \alpha_g(w_2) = \beta(gw_1, gw_2) + \beta(w_1, w_2) = 0.$$

By keeping in mind

$$\beta(w_1, w_2) = \langle w_1^+, w_2^- \rangle = \langle gw_1^+, gw_2^- \rangle = \langle aw_1^+ + cw_1^-, bw_2^- + dw_2^- \rangle = \langle aw_1^+, dw_2^- \rangle + \langle cw_1^+, bw_2^- \rangle,$$
one can show by direct computations that
\[
\alpha_g(w_1 + w_2) - \alpha_g(w_1) - \alpha_g(w_2) - \beta(gw_1, gw_2) + \beta(w_1, w_2) = \frac{1}{2} \langle aw_1^+, cw_2^+ \rangle + \frac{1}{2} \langle aw_2^+, cw_1^+ \rangle - \frac{1}{2} \langle bw_1^-, dw_2^- \rangle + \frac{1}{2} \langle bw_2^-, dw_1^- \rangle.
\]
Here notice that
\[
-\frac{1}{2} \langle aw_1^+, cw_2^+ \rangle + \frac{1}{2} \langle aw_2^+, cw_1^+ \rangle = \frac{1}{2} \langle cw_2^+, aw_1^+ \rangle + \frac{1}{2} \langle aw_2^+, cw_1^+ \rangle = \frac{1}{2} \langle aw_2^+ + cw_2^+, aw_1^+ \rangle + \frac{1}{2} \langle aw_2^+ + cw_2^+, cw_1^+ \rangle = \frac{1}{2} \langle gw_2^+, g w_1^+ \rangle = \frac{1}{2} \langle w_2^+, w_1^+ \rangle = 0.
\]
Similarly one can show that \(-\frac{1}{2} \langle bw_1^-, dw_2^- \rangle + \frac{1}{2} \langle bw_2^-, dw_1^- \rangle = 0\). Hence we have shown \(\alpha_g \in \Sigma_g\).

To show that the map \(g \mapsto (g, \alpha_g)\) is a group homomorphism is even a more straight forward computation, though tedious. \(\square\)

**Remark 2.5.** Let us mention that in the above lemma, if \(b = c = 0\), namely \(g\) is in the Siegel Levi, then \(\alpha_g = 0\).

We would like to describe \((g, \alpha_g)^{-1}\) for each \(g \in \text{Sp}(W)\). For this purpose, define \(\alpha^g = -\alpha_g \circ g^{-1}\). Namely it is the map \(\alpha^g : W \to F\) defined by
\[
\alpha^g(w) = -\alpha_g(g^{-1}w)
\]
for \(w \in W\). Then we have

**Lemma 2.6.** For each \(g \in \text{Sp}(W)\),

1. \(\alpha^g \in \Sigma_{g^{-1}}\);
2. \((g^{-1}, \alpha^g) = (g, \alpha_g)^{-1}\) in \(\text{Ps}(W)\).

**Proof.** (1) For \(w_1, w_2 \in W\), we have
\[
\alpha^g(w_1 + w_2) - \alpha^g(w_1) - \alpha^g(w_2) = -\alpha_g(g^{-1}(w_1 + w_2)) + \alpha_g(g^{-1}w_1) + \alpha_g(g^{-1}w_2)
\]
\[
= -\beta(gg^{-1}w_1, gg^{-1}w_2) + \beta(g^{-1}w_1, g^{-1}w_2)
\]
\[
= \beta(g^{-1}w_1, g^{-1}w_2) - \beta(w_1, w_2).
\]
Hence \(\alpha^g \in \Sigma_{g^{-1}}\).

(2) By part (1), one knows that indeed \((g^{-1}, \alpha^g) \in \text{Ps}(W)\). Now consider
\[
(g, \alpha_g)(g^{-1}, \alpha^g) = (1, g \cdot \alpha_g + \alpha^g),
\]
and for all \(w \in W\),
\[
(g \cdot \alpha_g + \alpha^g)(w) = \alpha_g(g^{-1}w) - \alpha_g(g^{-1}w) = 0,
\]
and so \(g \cdot \alpha_g + \alpha^g = 0\). Hence \((g^{-1}, \alpha^g) = (g, \alpha_g)^{-1}\). \(\square\)

**Remark 2.7.** Throughout this paper, we view \(\text{Sp}(W)\) as a subgroup of \(\text{Ps}(W)\) via the splitting \(g \mapsto (g, \alpha_g)\), and when we denote an element \(g \in \text{Sp}(W)\) it should be considered as an abbreviation of \((g, \alpha_g)\). In particular \(g^{-1}\) has to be considered as \((g^{-1}, \alpha^g)\) rather than \((g^{-1}, \alpha_{g^{-1}})\). Note that in general \(\alpha_{g^{-1}} \neq \alpha^g\).
Recall

**Theorem 2.8** (Stone-Von-Neumann). For a fixed additive character $\psi$, there is a unique (up to isomorphism) smooth irreducible representation $\rho_W$ of $H_\beta(W)$ such that the element $(0, z)$ in the center acts as multiplication by $\psi(z)$.

Via the splitting $\text{Sp}(W) \to \text{Ps}(W)$, the symplectic group $\text{Sp}(W)$ acts on $H_\beta(W)$, namely
\[
g \cdot (w, z) = (gw, z + \alpha_g(w))
\]
for $g \in \text{Sp}(W)$ and $(w, z) \in H_\beta(W)$. This, combined with the Stone-Von-Neumann theorem, gives rise to the projective representation $\text{Sp}(W) \to \text{PGL}(\rho_W)$. This projective representation defines the metaplectic cover $\tilde{\text{Sp}}(W)$ which is the subgroup of $\text{Sp}(W) \times \text{GL}(\rho_W)$ that consists of pairs $(g, M_g)$ where $g \in \text{Sp}(W)$ and $M_g \in \text{GL}(\rho_W)$ are such that
\[
M_g \circ \rho_W(h) = \rho_W(g \cdot h) \circ M_g
\]
for all $h \in H_\beta(W)$, where $g \cdot h$ is the above mentioned action of $\text{Sp}(W)$ on $H_\beta(W)$. Note that for each fixed $g \in \text{Sp}(W)$, the map $h \mapsto \rho_W(g \cdot h)$ defines another irreducible representation of $H_\beta(W)$, and hence by Stone-Von-Neumann, this is equivalence to $\rho_W$. Let $\omega_\psi$ be the representation of $\tilde{\text{Sp}}(W)$ on the space of $\rho_W$ given by
\[
\omega_\psi(g, M_g) := M_g.
\]
This representation is called the Weil representation of $\tilde{\text{Sp}}(W)$. Also note that we have the short exact sequence
\[
1 \to \mathbb{C}^\times \to \tilde{\text{Sp}}(W) \to \text{Sp}(W) \to 1,
\]
where the map $\mathbb{C}^\times \to \tilde{\text{Sp}}(W)$ is the inclusion $z \mapsto (1, z \text{Id})$ and the map $\tilde{\text{Sp}}(W) \to \text{Sp}(W)$ is the projection $(g, M_g) \mapsto g$.

**Remark 2.9.** The Weil representation $\omega_\psi$ can be shown to be independent of the choice of $\beta$. Also the metaplectic group $\tilde{\text{Sp}}(W)$ can be shown to be independent of $\psi$ and $\beta$.

### 3. The Lattice Model of the Weil representation

Let $A$ be a lattice of $W$, namely a free $\mathcal{O}$-module of $W$ of rank equal to $\dim W$. We define the dual $A^\perp$ of $A$ with respect to an integer $r$ by
\[
A^\perp := \{ w \in W : \langle a, w \rangle \in \mathcal{P}^r \text{ for all } a \in A \}.
\]
We usually assume $r$ to be the exponential conductor of our fixed additive character $\psi$, and then the condition $\langle a, w \rangle \in \mathcal{P}^r$ for all $a \in A$ is equivalent to $\psi(\langle a, w \rangle) = 1$ for all $a \in A$. We say a lattice $A$ is self-dual if $A = A^\perp$. Note that if $A$ is self-dual, then $\langle a_1, a_2 \rangle \in \mathcal{P}^r$ for all $a_1, a_2 \in A$. Given a self-dual lattice $A$ (with respect to $r$) one can always choose the polarization $W = W^+ \oplus W^-$ such that we have the decomposition $A = (A \cap W^+) \oplus (A \cap W^-)$. Conversely for each polarization $W = W^+ \oplus W^-$, one can always find a self-dual lattice $A$ with respect to $r$ so that $A = (A \cap W^+) \oplus (A \cap W^-)$. When a self-dual lattice $A$ is decomposed in this way with respect to the polarization $W = W^+ \oplus W^-$, we say that $A$ is compatible with the polarization. If $A$ is compatible with the polarization defining $\beta$, one can see that $\beta(a_1, a_2) \in \mathcal{P}^r$ for all $a_1, a_2 \in A$.

For a self-dual lattice $A$ compatible with our fixed polarization of $W$, we let
\[
H_\beta(A) := A \times F \subset H_\beta(W).
\]
Then $H_\beta(A)$ is a subgroup of $H_\beta(W)$. Define the character
\[
\psi_A : H_\beta(A) \to \mathbb{C}^\times, \quad (a, z) \mapsto \psi(z).
\]
Remark 3.2. The reader may wonder what would go wrong if one uses the usual Heisenberg group $H_f$ for this matter, this is essentially equivalent to defining the Heisenberg group using $H_{\beta}(A)$. Consider the induced representation

$$S_A := \text{ind}^{H_{\beta}(W)}_{H_{\beta}(A)} \psi_A,$$

where the induction is compact induction. Then we have

**Proposition 3.1.** The above induced representation provides a model of the Heisenberg representation.

**Proof.** It is immediate that the central character is $\psi$. Hence it suffices to show that it is irreducible. The proof is essentially the same as the usual lattice model (see [MVW, p. 29]). Yet, since the proof is not identical, we will give the detail here.

Let $L$ be an open compact subgroup of $W$ and $w \in W$ be fixed. Define a function $f_{w,L} : H_{\beta}(W) \to \mathbb{C}$ by

$$f_{w,L}(w', z) = \begin{cases} \psi(z), & \text{if } w' \in A + w + L, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_{w,L} \in S_A$. As in [MVW, p.29], the space $S_A$ is spanned by functions of this form.

Let $S' \subseteq S_A$ be a non-zero subspace invariant under the action of $H_{\beta}(W)$. Let $w \in W$ be any. Then one can always find $f \in S'$ such that $f(w,0) \neq 0$ by translating. By smoothness, there exists an open compact subgroup $L_w \subseteq W$, viewed as a subgroup of $H_{\beta}(W)$, which fixes $f$. Let $L \subseteq L_w$ be an open subgroup of $L_w$. Now by the theory of Fourier transforms, there exists a locally constant function $\varphi$ on $A$ such that for all $w' \in W$, we have

$$\int_A \psi((w',a))\varphi(a) \, da = \begin{cases} 1, & \text{if } w' \in A + w + L \\ 0, & \text{otherwise,} \end{cases}$$

because one can identify the Pontryagin dual of $W/A$ with $A$. Now define a function $F_{\varphi,f} : W \times F \to \mathbb{C}$ by

$$F_{\varphi,f}(w', z) := \psi(z) \int_A f((w',0))(a,0)\varphi(a) \, da$$

$$= \psi(z) \int_A f((a + w', \beta(w',a)))\varphi(a) \, da$$

$$= \psi(z) \int_A f((a, \beta(w',a) - \beta(a, w'))(w',0))\varphi(a) \, da$$

$$= \psi(z) \left( \int_A \psi(\beta(w',a) - \beta(a, w'))\varphi(a) \, da \right) f(w',0)$$

$$= \psi(z) \left( \int_A \psi(w',a)\varphi(a) \, da \right) f(w',0)$$

$$= \begin{cases} \psi(z)f(w',0), & \text{if } w' \in A + w + L \\ 0, & \text{otherwise.} \end{cases}$$

Hence $F_{\varphi,f}$ is a scalar multiple of $f_{w,L}$. But $w$ is arbitrary and hence $L$ can be arbitrary. As we mentioned, the functions of the form $f_{w,L}$ generate the space $S_A$, i.e. $S' = S$. \qed

**Remark 3.2.** The reader may wonder what would go wrong if one uses the usual Heisenberg group $H(W)$ but with the additive character $\psi_2$, so that the character $(a, z) \mapsto \psi_2(z)$ is defined. (Also for this matter, this is essentially equivalent to defining the Heisenberg group using $\langle -, - \rangle$ instead of $\frac{1}{2}\langle -, - \rangle$.) But if one uses $\psi_2$ (or $\langle -, - \rangle$), the above proof would not work. Indeed, in this case the induced representation would be reducible.
Proposition 3.6. Since the integral defining

\[ s_w = \text{the unique function in } S_A \text{ with } \text{supp}(s_w) = (A + w) \times F \text{ such that } s_w(w,0) = 1. \]

This function plays an important role through the paper.

To make explicit the action of \( M_g \) on the lattice model, we need to use \( \alpha^g \) as in Lemma 2.6. Using \( \alpha^g \), we can describe the action of \( M_g \) on the space \( S_A \) of the lattice model explicitly as follows: For \( f \in S_A \) and \((w, z) \in H_0(W)\), define an endomorphism \( M_g : S_A \to S_A \) by

\[
M_g \circ f(w,z) = \int_{A/A_g} \psi(\beta(a,w)) f(\gamma^{-1}(a+w), z + \alpha^g(a+w)) \, da
\]

where \( w \in W, z \in F, g \in \text{Sp}(W) \) and

\[
A_g = \{ a \in A : ga \in A \text{ and } \alpha_g(a) \in \mathcal{P}^r \}. 
\]

One can check that \( A_g \) is a subgroup of \( A \), because if \( a, b \in A_g \) and so \( ga, gb \in A \), then \( \alpha_g(a + b) = \beta(ga, gb) - \beta(a, b) + \alpha_g(a) + \alpha_g(b) \) and each term here is in \( \mathcal{P}^r \). Note that \( 2(gA \cap A) \subseteq A_g \) and \( A/A_g \) is a finite set. To show the integral can be indeed defined over \( A/A_g \), i.e. the integral is invariant under \( A_g \), is a direct computation.

Remark 3.5. Recall that each \( g \in \text{Sp}(W) \) has to be interpreted as \((g, \alpha_g) \in \text{Ps}(W)\), and so \( g^{-1} \) is actually \((g^{-1}, \alpha^g)\) instead of \((g^{-1}, \alpha_{g^{-1}})\). (See Remark 2.7.) Hence \( M_{g^{-1}} \) is given by

\[
M_{g^{-1}} \circ f(w,z) = \int_{A/A_{g^{-1}}} \psi(\beta(a,w)) f(g(a+w), z + \alpha_g(a+w)) \, da,
\]

where \( A_{g^{-1}} = \{ a \in A : g^{-1}a \in A \text{ and } \alpha^g(a) \in \mathcal{P}^r \} \).

Proposition 3.6. The map \( M_g \) is indeed well-defined, i.e. \( M_g \circ f \in S_A \) for all \( f \in S_A \).

Proof. Since the integral defining \( M_g \circ f \) is a finite sum, it is clear that it is smooth with compact support modulo the center. Hence we have only to show \( M_g \circ f(a'+w,0) = \psi(-\beta(a',w))M_g \circ f(w,0) \) for all \( a' \in A \). But

\[
M_g \circ f(a'+w,0) = \int_{A/A_g} \psi(\beta(a,a'+w)) f(g^{-1}(a+a'+w), \alpha^g(a+a'+w)) \, da
\]

\[
= \int_{A/A_g} \psi(\beta(a-a',a'+w)) f(g^{-1}(a+w), \alpha^g(a+w)) \, da
\]

\[
= \int_{A/A_g} \psi(\beta(a,w) + \beta(a,a') - \beta(a',a') - \beta(a',w)) f(g^{-1}(a+w), \alpha^g(a+w)) \, da
\]

\[
= \psi(-\beta(a',w)) \int_{A/A_g} \psi(\beta(a,w)) f(g^{-1}(a+w), z + \alpha^g(a+w)) \, da
\]

\[
= \psi(-\beta(a',w))M_g \circ f(w,0).
\]
where for the second equality we used the invariance of the measure on \( A \) and for the fourth, we used \( \beta(a, a') \in \mathcal{P}^r \) and \( \beta(a', a') \in \mathcal{P}^r \). \( \square \)

**Proposition 3.7.** For each \( g \in \text{Sp}(W) \), the map \( M_g \) is not identically zero.

**Proof.** Consider the function \( s_0 \in S_A \), i.e. set \( w = 0 \) in (3.3). Then \( s_0(a, 0) = 1 \) for all \( a \in A \). We have

\[
M_g \circ s_0(0, 0) = \int_{\mathbb{A}/\mathbb{A}_g} s_0(g^{-1}a, \alpha^g(a))da
\]

\[
= \int_{g\mathbb{A} \cap \mathbb{A}/\mathbb{A}_g} s_0(g^{-1}a, \alpha^g(a))da
\]

\[
= \int_{g\mathbb{A} \cap \mathbb{A}/\mathbb{A}_g} \psi(\alpha^g(a))da.
\]

Now the map \( \psi \circ \alpha^g \) is a non-degenerate character of second degree on the finite group \( g\mathbb{A} \cap \mathbb{A}/\mathbb{A}_g \) in the sense of [W]. Hence by [R Theorem A.2 (5)], we have

\[
\int_{g\mathbb{A} \cap \mathbb{A}/\mathbb{A}_g} \psi(\alpha^g(a))da = |g\mathbb{A} \cap \mathbb{A}/\mathbb{A}_g|^{-1/2} \gamma(\psi \circ \alpha^g),
\]

where \( \gamma(\psi \circ \alpha^g) \) is the Weil index of \( \psi \circ \alpha^g \). (See [R Appendix] for details.) In particular, it is non-zero. Thus \( M_g \circ s_0(0, 0) \neq 0 \). The proposition follows. \( \square \)

Then we have

**Proposition 3.8.** For the above defined \( M_g \), we have \( M_g \circ \rho_W(h) = \rho_W(g \cdot h) \circ M_g \) for all \( h \in H_\beta(W) \), namely the element \( (g, M_g) \) is indeed in \( \widetilde{\text{Sp}}(W) \) and \( M_g \) defines the action for the Weil representation.

**Proof.** Note that since \( \rho_W \) is irreducible, if we can show \( M_g \circ \rho_W(h) = \rho_W(g \cdot h) \circ M_g \) for all \( h \in H_\beta(W) \), it will imply \( M_g \) is invertible and hence \( (g, M_g) \) is indeed in \( \widetilde{\text{Sp}}(W) \).

Let \( (w, z) \in H_\beta(W) \), and \( f \in S_A \). Also let \( h = (w', z') \in H_\beta(W) \). Recalling \( g \cdot h = g \cdot (w', z') = (gw', z' + \alpha_g(w')) \), we have

\[
(\rho_W(g \cdot h) \circ M_g) \circ f(w, z)
\]

\[
= \rho_W(g \cdot h)(M_g \circ f)(w, z)
\]

\[
= M_g \circ f((w, z)(gw', z' + \alpha_g(w')))
\]

\[
= M_g \circ f(w + gw', z + z' + \alpha_g(w') + \beta(w, gw'))
\]

\[
= \int \psi(\beta(a, w + gw'))f(g^{-1}(a + w + gw'), z + z' + \alpha_g(w') + \beta(w, gw') + \alpha^g(a + w + gw'))da
\]

\[
= \int \psi(\beta(a, w))f(g^{-1}(a + w) + w', z + z' + \alpha_g(w') + \beta(w, gw') + \alpha^g(a + w + gw') + \beta(a, gw'))da
\]

\[
= \int \psi(\beta(a, w))f(g^{-1}(a + w) + w', z + z' + \alpha_g(w') + \alpha^g(a + w + gw') + \beta(w + a, gw'))da,
\]

where all the integrals are over \( A/\mathbb{A}_g \).
On the other hand,
\[ (M_g \circ \rho_W(h))f(w, z) = \int_{A/A_g} \psi(\beta(a, w))\rho(w', z')f(g^{-1}(a + w), z + \alpha^g(a + w)) \, da \]
\[ = \int_{A/A_g} \psi(\beta(a, w))f((g^{-1}(a + w), z + \alpha^g(a + w))(w', z')) \, da \]
\[ = \int_{A/A_g} \psi(\beta(a, w))f(g^{-1}(a + w), w') \, da, \]
where again all the integrals are over \(A/A_g\).

In order for us to show that those two are equal, it suffices to show
\[ \alpha_g(w') + \alpha^g(a + w + gw') + \beta(w + a, gw') = \alpha^g(a + w) + \beta(g^{-1}(a + w), w'), \]
namely
\[ \alpha^g(a + w + gw') - \alpha^g(a + w) + \alpha_g(w') = \beta(g^{-1}(a + w), w') - \beta(w + a, gw'). \]
But this follows because \( \alpha_g(w') = -\alpha^g(g w') \) and \( \alpha^g \in \Sigma_{g^{-1}}. \)

Let us define
\[ \Gamma_A := \{ g \in \Sp(W) : g A \subseteq A \}. \]

One can see that \( \Gamma_A \) is an open compact subgroup of \( \Sp(W) \). Note that the condition \( g A \subseteq A \) implies \( gA = A \) because \( g \) is an isometry and hence preserves volume. (This also applies to any lattice. See for example [O] §82:12.) Also we have \( \Gamma_A = \Sp_{2n}(O) \). If \( g \in \Gamma_A \), then
\[ M_g \circ f(w, z) = \int_{A/A_g} \psi(\beta(a, w) - \beta(g^{-1}a, g^{-1}w) + \alpha^g(a + w))f(g^{-1}w, z) \, da \]
\[ = \int_{A/A_g} \psi(\alpha^g(a) + \alpha^g(w))f(g^{-1}w, z) \, da \]
\[ = \int_{A/A_g} \psi(\alpha^g(a))f(g^{-1}w, z + \alpha^g(w)) \, da \]
\[ = \left( \int_{A/A_g} \psi(\alpha^g(a)) \, da \right) f(g^{-1}w, z + \alpha^g(w)). \]

Proposition 3.9. Each \( g \in \Gamma_A \) acts on \( f \in S_A \) by (non-zero scalar multiple of) translation. To be more precise,
\[ M_g \circ f(w, z) = \left( \int_{A/A_g} \psi(\alpha^g(a)) \, da \right) f(g^{-1}w, z + \alpha^g(w)). \]

From the above integral formula for \( M_g \), it is important to know when we have \( \psi(\alpha^g(a)) = 1 \) for all \( a \in A \) or equivalently \( \alpha^g(a) \in \mathcal{P} \) for all \( a \in A \). For this purpose, let us start with

Lemma 3.10. Let \( B \) be a (not necessarily self-dual) lattice of \( W \). Then the set
\[ G_B := \{ g \in \Sp(W) : \alpha_g(w) \in \mathcal{P} \text{ for all } w \in B \} \]
is open and closed. (We do not know if it is a group.)
Proof. For each \( w \in B \), consider the continuous map \( f_w : \text{Sp}(W) \to F \) defined by \( f_w(g) = \alpha_g(w) \). Then
\[
G_B = \bigcap_{w \in B} f_w^{-1}(P^r).
\]
Note that \( f_w^{-1}(P^r) \) is open and closed, so \( G_B \) is closed. Next by looking at the description of \( \alpha_g(w) \) in \( \ref{2.4} \), one can see that \( f_w^{-1}(P^r) \subseteq f_w^{-1}(P^r) \) for all \( w' \in O_w \). Since \( B \) is compact, we have \( B = \bigcup_i O_{w_i} \) for some finite union. So \( G_B = \bigcap_i f_w^{-1}(P^r) \), which is open. \( \square \)

Now let us define
\[
\Gamma_A^\circ := G_A \cap \Gamma_A = \{ g \in \Gamma_A : \alpha_g(a) \in P^r \text{ for all } a \in A \}.
\]
Here let us emphasize that each \( g \in \Gamma_A \subseteq \text{Sp}(W) \) should be interpreted as \( (g, \alpha_g) \in \text{Ps}(W) \). (See Remark \( \ref{2.7} \))

**Lemma 3.11.** \( \Gamma_A^\circ \) is an open compact subgroup of \( \text{Sp}(W) \).

**Proof.** By the above lemma and the fact that \( \Gamma_A \) is open and compact, one can conclude that \( \Gamma_A^\circ \) is open and compact.

To show it is a subgroup, let \( g, h \in \Gamma_A^\circ \). For \( a \in A \), we have \( \alpha_{gh}(a) = h^{-1} \cdot \alpha_g(a) + \alpha_h(a) = \alpha_g(ha) + \alpha_h(a) \in P^r \) because \( ha \in A \). Hence \( gh \in \Gamma_A^\circ \). Now let \( g \in \Gamma_A^\circ \). To show \( g^{-1} \in \Gamma_A^\circ \), one needs to show \( \alpha^g(a) \in P^r \) for all \( a \in A \). (Here what is needed is not \( \alpha_a(g^{-1}a) \in P^r \)!) See Remark \( \ref{2.7} \). But since \( \alpha^g(a) = -\alpha_g(g^{-1}a) \), and \( g^{-1}a \in A \), we have \( \alpha_g(g^{-1}a) \in P^r \). Hence \( \Gamma_A^\circ \) is a subgroup. \( \square \)

Now for the integral defining \( M_g \), if we choose the measure \( da \) so that the volume of \( A/A_g \) is 1, one has
\[
M_g \circ f(w, z) = f(g^{-1}w, z + \alpha^g(w)) \quad \text{for all } g \in \Gamma_A^\circ.
\]
Note that
\[
M_g^{-1} \circ f(w, z) = f(gw, z + \alpha_g(w)).
\]
by Remark \( \ref{3.3} \). One can check that for \( g, h \in \Gamma_A^\circ \), we have \( M_{gh} \circ f = M_h \circ (M_g \circ f) \) by using \( \alpha_{gh} = h^{-1} \cdot \alpha_g + \alpha_h \). This shows \( \Gamma_A^\circ \) splits in the metaplectic cover \( \widetilde{\text{Sp}}(W) \).

Indeed if the residue characteristic of \( F \) is odd, one can see \( \psi(\alpha^g(a)) = 1 \) for any \( g \in \Gamma_A \) and \( a \in A \). This is because from the explicit description of \( \alpha^g \) as in \( \ref{2.4} \) together with the fact that \( \frac{1}{2} = 1 \) is a unit in \( \mathcal{O} \), one can see that all the three terms in the definition of \( \alpha^g(a) \) are in \( P^r \). Hence \( \Gamma_A = \Gamma_A^\circ \). So this explains the well-known splitting of \( \text{Sp}(\mathcal{O}) \) in \( \widetilde{\text{Sp}}(W) \).

If the residue characteristic of \( F \) is even, we no longer have \( \psi(\alpha^g(a)) = 1 \) for every \( g \in \Gamma_A \) and \( a \in A \). Yet, the above lemma shows that for a sufficiently small open compact subgroup \( \Gamma_A^\circ \) of \( \Gamma_A \), we do have \( \psi(\alpha^g(a)) = 1 \) for any \( g \in \Gamma_A^\circ \) and any \( a \in A \). Hence we have the analogous splitting of this open compact subgroup, which also explains the well-known fact that a certain open compact subgroup of \( \text{Sp}(W) \) splits in \( \widetilde{\text{Sp}}(W) \) for the case of even residual characteristic.

**Remark 3.12.** Not only the group \( \Gamma_A^\circ \) but also various other subgroups of \( \text{Sp}(W) \) are known to be split in the metaplectic cover \( \widetilde{\text{Sp}}(W) \). Whenever \( H \) is a subgroup of \( \text{Sp}(W) \) which splits in \( \widetilde{\text{Sp}}(W) \), for each \( h \in H \) and \( f \in S_A \) we usually denote \( M_h \circ f \) by \( \omega_h(f) \) or simply \( \omega(f) \) because the additive character \( \psi \) is fixed throughout the paper.

4. On lattices

In this section we let \((V, (-, -))\) be an \( \epsilon \)-Hermitian space over \( E \), where \( E \) is either \( F \) or a quadratic extension of \( F \). In particular, we have
\[
\langle cv, c'v' \rangle = c\overline{c'} \langle v, v' \rangle \quad \text{and} \quad \overline{\langle v, v' \rangle} = \epsilon \langle v', v \rangle.
\]
for $v, v' \in \mathcal{V}$ and $c, c' \in E$. (Recall from the notation section that for each $c \in E$, we denote $\bar{c} = c$ if $E = F$, and $\bar{c} = \tau(c)$ where $\tau$ is the nontrivial element in $\text{Gal}(E/F)$ if $E$ is a quadratic extension of $F$.) We always assume that $E$ is unramified over $F$ when $E \neq F$.

By a lattice $L$ of $\mathcal{V}$, we mean a free $\mathcal{O}_E$-module of rank equal to $\dim \mathcal{V}$. For any lattice $L \subseteq \mathcal{V}$, we define the dual lattice $L^\perp$ with respect to an integer $r$ by

$$L^\perp = \{ v \in \mathcal{V} : \langle v, l \rangle \in \mathcal{P}_E^r \text{ for all } l \in L \}.$$ 

(In this paper $r$ is usually reserved for the exponential conductor of $\psi$ but in this section we use $r$ for any fixed integer.) A lattice $L$ is called a self-dual lattice (with respect to $r$) if $L^\perp = L$.

Not every $\epsilon$-Hermitian space has a self-dual lattice, and even when it does, we sometimes need some restriction on $r$. To be specific, we have

**Lemma 4.1.** An $\epsilon$-Hermitian space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ admits a self-dual lattice (with an occasional restriction on $r$) if it is one of the following:

(a) $\mathcal{V}$ is symplectic, namely $E = F$ and $\epsilon = -1$. ($r$ can be any.)

(b) $\mathcal{V}$ is symmetric, namely $E = F$ and $\epsilon = 1$, where the anisotropic part $\mathcal{V}^a$ is one of the following:

- $\mathcal{V}^a = 0$: ($r$ can be any.)
- $\mathcal{V}^a = F$ and $(a, b) = \eta\bar{a}$ for $a, b \in F$ where $\eta \in \mathcal{O}_E^\times$; ($r$ has to be even.)
- $\mathcal{V}^a = F'$ where $F'$ is an unramified quadratic extension of $F$ equipped with the norm form, namely for $x, y \in F'$, we have $\langle x, y \rangle = \frac{1}{\eta}(x\bar{y} + \bar{x}y)$ where the bar is the conjugation for the quadratic extension $F'/F$. ($r$ has to be even.)

(c) $\mathcal{V}$ is Hermitian, namely $E$ is a quadratic unramified extension over $F$ and $\epsilon = \pm 1$, where the anisotropic part $\mathcal{V}^a$ is one of the following:

- $\mathcal{V}^a = 0$: ($r$ can be any.)
- $\mathcal{V}^a = E$ equipped with the norm form if $\epsilon = 1$, namely $\langle x, y \rangle = x\bar{y}$ for $x, y \in E$, and $\eta$ times the norm form if $\epsilon = -1$ where $\eta$ is an element in $\mathcal{O}_E^\times$ such that $\bar{\eta} = -\eta$, namely $\langle x, y \rangle = \eta x\bar{y}$. ($r$ has to be even.)

**Proof.** This list is as in [MVW, p. 100], although there it is always assumed $r = 0$. But for later convenience, let us describe the self-dual lattices for all the cases in detail.

(a) $\mathcal{V}$ is symplectic; Then $\dim \mathcal{V} = \text{even} = 2n$ and $\mathcal{V}$ admits a basis $\{ e_1, \ldots, e_n, f_1, \ldots, f_n \}$ so that $\langle e_i, f_j \rangle = \delta_{ij} \varpi^r$, $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$, and further

$$L = \text{span}_\mathcal{O}\{ e_1, \ldots, e_n, f_1, \ldots, f_n \}.$$ 

(b) $\mathcal{V}$ is symmetric;

- $\mathcal{V}^a = 0$: Then $\dim \mathcal{V} = \text{even} = 2n$ and $\mathcal{V}$ admits a basis $\{ e_1, \ldots, e_n, f_1, \ldots, f_n \}$ so that $\langle e_i, f_j \rangle = \delta_{ij} \varpi^r$ and $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$, and further

$$L = \text{span}_\mathcal{O}\{ e_1, \ldots, e_n, f_1, \ldots, f_n \}.$$ 

- $\mathcal{V}^a = F$: Then $\dim \mathcal{V} = \text{odd} = 2n + 1$ and $\mathcal{V}$ admits a basis $\{ e_1, \ldots, e_n, f_1, \ldots, f_n, v \}$ with $\mathcal{V}^a = \text{span}_F\{ v \}$ where $v = \varpi^{r/2}$ so that $\langle e_i, f_j \rangle = \delta_{ij} \varpi^r$, $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = \langle e_i, v \rangle = \langle f_i, v \rangle = 0$, and $\langle v, v \rangle = \eta \varpi^r$ (assuming $r$ is even), and further

$$L = \text{span}_\mathcal{O}\{ e_1, \ldots, e_n, f_1, \ldots, f_n, v \}.$$ 

- $\mathcal{V}^a = F'$: First note that since $F'$ is unramified over $F$, we can write $F' = F \oplus \eta F$ where $\eta \in F'$ is such that $\bar{\eta} \in \mathcal{O}_E^\times$. Then $\dim \mathcal{V} = \text{even} = 2n + 2$ and $\mathcal{V}$ admits a basis $\{ e_1, \ldots, e_n, f_1, \ldots, f_n, v_1, v_2 \}$ with $\mathcal{V}^a = \text{span}_F\{ v_1, v_2 \}$ where $v_1 = \varpi^{r/2}$ and $v_2 = \eta \varpi^{r/2}$ so that $\langle e_i, f_j \rangle = \delta_{ij} \varpi^r$, $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = \langle e_i, v \rangle = \langle f_i, v \rangle = \langle v_1, v_2 \rangle = 0$, $\langle v_1, v_1 \rangle = \varpi^r$ and $\langle v_2, v_2 \rangle = -\eta \varpi^r$, and further

$$L = \text{span}_\mathcal{O}\{ e_1, \ldots, e_n, f_1, \ldots, f_n, v_1, v_2 \}.$$ 

(c) $\mathcal{V}$ is Hermitian;
Lemma 4.3. Let \( V^a = 0; \) then \( \dim E V = \text{even} = 2n, \) and \( V \) admits an \( E \)-basis \( \{ e_1, \ldots, e_n, f_1, \ldots, f_n \} \) so that \( \langle e_i, f_j \rangle = \delta_{ij} \omega^r \) and \( \langle e_i, e_j \rangle = (f_i, f_j) = 0, \) and further
\[
L = \text{span}_{\mathcal{O}_E} \{ e_1, \ldots, e_n, f_1, \ldots, f_n \}.
\]

\( V^a = E; \) we let \( v = \omega^{r/2}. \) We have \( \langle v, v \rangle = u \omega^r \) for some unit \( u \in \mathcal{O}_E^\times. \) Then \( \dim E V = \text{odd} = 2n + 1, \) and \( V \) admits an \( E \)-basis \( \{ e_1, \ldots, e_n, f_1, \ldots, f_n, v \} \) so that \( \langle e_i, f_j \rangle = \delta_{ij} \omega^r, \)
\[
\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = \langle e_i, v \rangle = (f_i, v) = 0 \quad \text{and} \quad \langle v, v \rangle = u \omega^r, \quad \text{and further}
\]
\[
L = \text{span}_{\mathcal{O}_E} \{ e_1, \ldots, e_n, f_1, \ldots, f_n, v \}.
\]
\( \square \)

From this lemma, one can conclude

Lemma 4.2. Let \( L \subset V \) be a self-dual lattice of \( V. \) Then there is a decomposition \( V = V^+ \oplus V^a \oplus V^-, \) where \( V^a \) is the anisotropic part and both \( V^+ \) and \( V^- \) are totally isotropic of the same dimension, so that \( V^+ \oplus V^- \) is a product of copies of the \( \epsilon \)-Hermitian hyperbolic planes, such that
\[
L = V^+ \cap L \oplus V^a \cap L \oplus V^- \cap L.
\]
We often write
\[
L^+ = V^+ \cap L, \quad L^a = V^a \cap L, \quad L^- = V^- \cap L.
\]

We will recall some of the properties of self-dual lattices, all of which are essentially in \( \mathrm{MVW} \) p.107-112, though in \( \mathrm{MVW} \) it is always assumed \( r = 0. \)

Note that the quotient \( L/\omega L \) is viewed as an \( \epsilon \)-Hermitian space over the residue field \( \mathcal{O}_E/\omega \mathcal{O}_E \) by reducing \( \omega^{-r}(\cdot, \cdot) \) mod \( \omega \mathcal{O}_E. \) For each \( v \in V, \) we denote by \( \bar{v} \) the image of \( v \) in \( L/\omega L. \)

Let us mention a couple of properties on the \( \epsilon \)-Hermitian spaces over the finite field \( \mathcal{O}_E/\omega \mathcal{O}_E. \)

Lemma 4.3. Let \( \overline{L} := L/\omega L \) be the \( \epsilon \)-Hermitian spaces over the finite field \( \mathcal{O}_E/\omega \mathcal{O}_E. \)

(1) Assume \( \overline{X} \) is a totally isotropic subspace of \( \overline{L}. \) Then there exist subspaces \( \overline{Y} \) and \( \overline{L} \) such that \( \overline{Y} \) is totally isotropic with \( \dim \overline{V} = \dim \overline{Y}, \) and we have the decomposition \( \overline{L} = \overline{X} \oplus \overline{L} \oplus \overline{Y} \) where \( \overline{X} \) is orthogonal to \( \overline{L}. \)

(2) Let \( \overline{X} \) and \( \overline{X}' \) be two subspaces of \( \overline{L} \) with \( \dim \overline{X} = \dim \overline{X}'. \) Then any isometry \( \overline{X} \to \overline{X}' \) can be extended to an isometry on \( \overline{L}. \)

Proof. Both of them are well-known when the characteristic of the residue field \( \mathcal{O}_E/\omega \mathcal{O}_E \) is odd. When the characteristic is even, it does not seem to be well-known. First of all, let us denote the corresponding form by \( b(\cdot, \cdot) := \omega^{-r}(\cdot, \cdot). \) If \( E = F, \) then the form \( b \) is always symmetric because \( \epsilon = 1 \) in \( \mathcal{O}_E/\omega \mathcal{O}_E. \) Clearly \( b \) is non-degenerate. Then part (1) is \( \mathrm{K} \) Proposition 1.2.2 and part (2) is \( \mathrm{K} \) Corollary 1.2.1.

If \( E \neq F, \) then Dieudonné in \( \mathrm{D} \) p. 21 has shown that these two properties hold if \( b(x, x) \) is a trace in \( \mathcal{O}_E/\omega \mathcal{O}_E \) for all \( x \in \overline{L}, \) namely for each \( x \in \overline{L} \) there exists \( a \in \mathcal{O}_E/\omega \mathcal{O}_E \) such that \( b(x, x) = a + \bar{a}, \) where \( \bar{a} \) is the Galois conjugate of \( a \) for the quadratic extension \( \mathcal{O}_E/\omega \mathcal{O}_E \) over \( \mathcal{O}/\omega \mathcal{O}. \) But considering \( \epsilon = 1, \) we always have \( b(x, x) = b(x, x) \) and \( b(x, x) \in \mathcal{O}/\omega \mathcal{O}. \) The condition is satisfied because the trace map \( \mathcal{O}_E/\omega \mathcal{O}_E \to \mathcal{O}/\omega \mathcal{O} \) is surjective. \( \square \)

Lemma 4.4. Let \( v_1, \ldots, v_k \in L \) be such that the reductions \( \bar{v}_1, \ldots, \bar{v}_k \) in \( L/\omega L \) are linearly independent over the residue field. Then \( \{ v_1, \ldots, v_k \} \) can be extended to a basis of \( L. \)

Proof. This is an elementary exercise. \( \square \)
The following lemma, which is an extension of [MVW] Proposition II.2, p.107 is crucial to our computations.

**Lemma 4.5.** For a self-dual lattice $L$ (with respect to $r$) of $V$, let $v_1, \ldots, v_s \in L$, $t_1, \ldots, t_s \in \mathbb{Z}$ and $M = (m_{ij})$ an $s \times s$ matrix with coefficients in $\mathcal{O}_E$. Suppose

1. $\bar{v}_i, \ldots, \bar{v}_s$ are linearly independent over the residue field;
2. $1 + e \leq t_1 \leq \cdots \leq t_s$;
3. $m_{ij} = \frac{m_{ji}}{n_{ij}}$ for all $1 \leq i, j \leq s$;
4. $m_{ij} \equiv \langle v_i, v_j \rangle \mod \mathcal{P}_E^{t_i}$ for all $1 \leq i \neq j \leq s$;
5. $m_{ii} \equiv \langle v_i, v_i \rangle \mod \mathcal{P}_E^{t_i+c}$ for all $i \in \{1, \ldots, s\}$.

Then there exist $v'_1, \ldots, v'_s \in L$ such that

1. $v'_i - v_i \in \mathcal{O}_E L$ for all $i$;
2. $m_{ij} = \langle v'_i, v'_j \rangle$ for all $1 \leq i, j \leq s$.

**Proof.** When the residue characteristic of $F$ is odd, this is [MVW] Proposition II.2, p.107 except that in [MVW] it is always assumed $r = 0$. For the case of even residual characteristic, one needs to slightly modify the argument there, and we need the condition (5), which can be absorbed by the condition (4) in the case of odd residual characteristic. In any case, since the proof is only a slight modification of the one given in [MVW], we will repeat only the essentially point. The basic idea is to construct a sequence of vectors $v_i(t)$ for each $i \in \{1, \ldots, s\}$ where $t \in \mathbb{Z}_{\geq 1}$ such that

(a) $v_i(t) - v_i \in \mathcal{O}_E L$;
(b) $v_i(t) - v_i(t - 1) \in \mathcal{O}_E^{t+r-1} L$ for $t \geq 2$;
(c) $\langle v_i(t), v_j(t) \rangle \equiv m_{ij} \mod \mathcal{P}_E^{t+r}$ for all $1 \leq i \neq j \leq s$.
(d) $\langle v_i(t), v_i(t) \rangle \equiv m_{ii} \mod \mathcal{P}_E^{t+r+c}$ for all $i \in \{1, \ldots, s\}$.

The condition (b) guarantees the sequence $v_i(t)$ is Cauchy and hence converges to some $v'_i$, and the conditions (a) (c) and (d) guarantee $v'_i$ has the desired property.

The sequence $v_i(t)$ is constructed by recursion as follows. First by Lemma [4.4] we can extend $\{v_1, \ldots, v_s\}$ to a basis $\{v_1, \ldots, v_n\}$ of $L$. Choose a dual basis $\{v'_1, \ldots, v'_n\}$ of $L$, so that $\langle v_i, v'_j \rangle = \mathcal{P}^{r} \delta_{ij}$. Define $v_i(t)$ by

$$v_i(1) = v_i,$$

$$v_i(t) = \begin{cases} v_i(t - 1) & \text{if } t \leq t_i \\ v_i(t - 1) + \sum_{k=i}^{t} \mathcal{P}^{t+r-1} a_{ki}(t) v'_i & \text{if } t > t_i, \end{cases}$$

where

$$a_{ki}(t) = \mathcal{P}^{1-t-r}(m_{ki} - \langle v_k(t - 1), v_i(t - 1) \rangle) \text{ if } i < k;$$

$$a_{ii}(t) = \frac{1}{2} \mathcal{P}^{1-t-r}(m_{ii} - \langle v_i(t - 1), v_i(t - 1) \rangle).$$

(Let us mention that in [MVW], p.108 there is a typo in the definition of $v_i(t)$. The summation has to start with $i$ instead of $i + 1$.) Note that $a_{ii}(t) \in \mathcal{O}_E$ thanks to the condition (c) and (d). Also note that to obtain (c) and (d) at each step, one needs $t_i \geq 1 + e$. By definition of $v_i(t)$, one has $v_i(t) - v_i \in \mathcal{O}_E L$ if $t \leq t_i$, and $v_i(t) - v_i = \mathcal{O}_E^{t-1} L \subset \mathcal{O}_E^{t-1} L$ if $t > t_i$. So the condition (a) is satisfied.

This lemma is very unfortunate in that the restriction $t_i \geq 1 + e$ will not allow us to apply many of the computations in [MVW] to the case of even residual characteristic. However, if we assume $V$ is symplectic, we have

**Lemma 4.6.** If $V$ is symplectic in the above lemma, one can assume $t_i \geq 1$ instead of $t_i \geq 1 + e$, and can suppress the condition (5).
Proof. If \( V \) is symplectic, one can simply take \( a_{ii}(t) = 0 \) for all \( t \). (Of course, always \( m_{ii} = 0 \).) □

This implies

**Lemma 4.7.** For a self-dual lattice \( L \) (with respect to \( r \)) of \( V \), let \( v_1, \ldots, v_n \in L \, \text{and} \, t \in \mathbb{Z}_{>0} \).

Further assume \( V \) is symplectic if the residue characteristic of \( F \) is even. Suppose

1. \( v_1, \ldots, v_n \) are linearly independent over the residue field;
2. For all \( i, j \in \{1, \ldots, s\} \), \( (v_i, v_j) \equiv 0 \mod P_E^{r+t} \).

Then there exist elements \( v'_1 \ldots v'_n \in L \) and subspaces \( X, V^o, Y \) of \( V \) such that

3. \( \{v'_1 \ldots v'_n\} \) is a basis of \( X \) over \( E \);
4. we have the orthogonal decomposition \( W = X \oplus V^o \oplus Y \) such that \( X \oplus Y \) is totally isotropic;
5. \( L = L \cap X \oplus L \cap V^o \oplus L \cap Y \);
6. for all \( i = 1, \ldots, n \), we have \( v'_i - v \in \omega^i L \).

Proof. This can be proven in the same way as [MVW, Corollary II.3] using the previous lemma and Lemma 4.3. □

One reason we have to assume \( V_2 \) is symplectic in our main theorem is the unavailability of this lemma for the other types of spaces.

**Lemma 4.8.** Let \( L \) be a self-dual lattice of \( V \), where \( V \) is either symplectic or symmetric with \( V^o = 0 \) when the residue characteristic is even. Let \( v_1, \ldots, v_n \) and \( v'_1, \ldots, v'_s \) are vectors in \( V \) such that

1. \( v_1, \ldots, v'_s \) are linearly independent over the residue field;
2. \( v_1, \ldots, v'_s \) are linearly independent over the residue field;
3. \( (v_i, v_j) = (v'_i, v'_j) \) for all \( i, j \in \{1, \ldots, s\} \).

Then there exists \( u \in U(V) \) such that \( u(L) = L \) and \( u(v_i) = v'_i \) for all \( i = 1, \ldots, s \).

Proof. This is [MVW, Corollary II.5, p.111] when the residue characteristic is odd. If the residue characteristic is even and \( V \) is symplectic, one can prove it in the same way as the case of odd residual characteristic by using Lemma 4.6. For the other case, unfortunately, the same proof does not work because of the restriction \( t_i \geq 1 + e \) in Lemma 4.3. However if \( V \) is symmetric and \( V^o = 0 \), then this lemma is simply Witt’s extension theorem for self-dual lattices, which is known to hold. (See, for example, Corollary 5.4.1 of [K].) □

As the last thing in this section, let us introduce the notion of **admissible lattices** and some of their properties.

**Definition 4.9.** Assume \( V \) is such that \( V^o = 0 \). We say a (not necessarily self-dual) lattice \( L \subseteq V \) is “admissible” if \( L = L^+ + L^- \), where recall \( L^+ = L \cap V^+ \) and \( L^- = L \cap V^- \).

Note that for any lattice \( L \) with \( L^o = 0 \), we always have \( L \subseteq L^+ + L^- \), but the inclusion might be strict. A self-dual lattice is always admissible.

We need to quote a few lemmas:

**Lemma 4.10.** Assume \( V \) is such that \( V^o = 0 \). A lattice \( L \) of \( V \) is admissible if and only if \( L^\perp \) is.

Proof. Assume \( L \) is admissible, so \( L = L^+ + L^- \). Let \( l \in L^\perp \). We need to show \( l^+ \in L^\perp \). Let \( m = m^+ + m^- \in L \). Note that since \( L \) is admissible, \( m^- \in L \), and so \( \langle l, m^- \rangle \in P^r \). So we have

\[
\langle l^+, m \rangle = \langle l^+, m^+ + m^- \rangle = \langle l^+, m^- \rangle = \langle l^+ + l^-, m^- \rangle = \langle l, m^- \rangle \in P^r.
\]

Thus \( l^+ \in L^\perp \). So \( (L^\perp)^+ \subseteq L^\perp \). Similarly \( (L^\perp)^- \subseteq L^\perp \). Hence \( (L^\perp)^+ + (L^\perp)^- \subseteq L^\perp \), which shows \( L^\perp \) is admissible.
Conversely assume $L^\perp$ is admissible. From the above argument, $L^\perp \perp$ is admissible. But $L^\perp \perp = L$, so $L$ is admissible.

**Lemma 4.11.** Let $L_1$ be a self-dual lattice of $V$ and $L \subseteq L_1$ a sublattice of $L_1$. Assume $\dim V = n$. Then there exist a basis $\{e_1, \ldots, e_n\}$ of $L_1$ and an integer $s$ with $0 \leq s \leq n$ such that

$$\{e_1, \ldots, e_s, \varpi^{t_i+1} e_{s+1}, \ldots, \varpi^{t_n} e_n\}$$

is a basis of $L$, where $t_i \geq 1$ for $s + 1 \leq i \leq n$.

**Proof.** This is a part of Lemma in [MVW, p.112]. □

This lemma implies

**Lemma 4.12.** Assume $V$ is such that $\dim V = 0$. Let $L_1$ be a self-dual lattice of $V$ and $L \subseteq L_1$ an admissible sublattice of $L_1$. Assume $\dim V = 2n$. Then there exist a basis $\{e_1, \ldots, e_n\}$ of $L_1^+$, a basis $\{f_1, \ldots, f_n\}$ of $L_1^-$ and integers $s^+, s^-$ with $0 \leq s^+ \leq n$ and $0 \leq s^- \leq n$ such that

$$\{e_1, \ldots, e_s, \varpi^{t_i+1} e_{s+1}, \ldots, \varpi^{t_n} e_n\}$$

is a basis of $L^+$, where $t_i \geq 1$ for $s^+ + 1 \leq i \leq n$, and

$$\{f_1, \ldots, f_{s^-}, \varpi^{u_i-1} f_{s^-+1}, \ldots, \varpi^{u_n} f_n\}$$

is a basis of $L^-$, where $u_i \geq 1$ for $s^- + 1 \leq i \leq n$.

**Proof.** By the above lemma, there exist a basis $\{v_1, \ldots, v_{2n}\}$ of $L_1$ and an integer $s$ such that $\{v_1, \ldots, v_s, \varpi^{t_1+1} v_{s+1}, \ldots, \varpi^{t_n} v_{2n}\}$ is a basis of $L$. Since $L$ is admissible, each $\varpi^{t_i} v_i^+$ (where $t_i = 0$ if $i \leq s$) is in $L^+$, and hence $\{v_1^+, \ldots, v_s^+, \varpi^{t_1+1} v_{s+1}^+, \ldots, \varpi^{t_n} v_{2n}^+\}$ is a generator of the free $\mathcal{O}_E$-module $L^+$. Then we can shrink it to a basis of $L^+$. Similarly for $L^-$. □

5. Unramified dual pairs

For $i = 1, 2$, let $(V_i, \langle \ , \ \rangle_i)$ be an $\epsilon_i$-Hermitian space over $E$ where $\epsilon_i \in \{\pm 1\}$. If $\epsilon_1 \epsilon_2 = -1$, the space

$$W = V_1 \otimes_F V_2$$

becomes a symplectic space of dimension $\dim_F V_1 \cdot \dim_F V_2$ with the symplectic form defined by

$$\langle v_1 \otimes v_2, v_1' \otimes v_2' \rangle = \text{tr}_{E/F}(\langle v_1, v_1' \rangle_1 \langle v_2, v_2' \rangle_2).$$

We have the natural map $U(V_1) \times U(V_2) \to \text{Sp}(W)$, and say the pair $(U(V_1), U(V_2))$ is a dual pair.

Assume both $(V_1, \langle \ , \ \rangle_1)$ and $(V_2, \langle \ , \ \rangle_2)$ admit self-dual lattices $L_1 \subseteq V_1$ and $L_2 \subseteq V_2$ with respect to the integers $r_1$ and $r_2$, respectively. We fix a decomposition

$$V_i = V_i^+ \oplus V_i^0 \oplus V_i^-$$

as in Lemma 4.2. Notice that the lattice

$$A := L_1 \otimes \mathcal{O}_E L_2,$$

viewed as an $\mathcal{O}$-module, is a self-dual lattice of $W = V_1 \otimes_F V_2$ with respect to the integer $r_1 + r_2$. In what follows, let us assume $V_i^0 = 0$ for either $i = 1$ or $2$.

We would like to choose our polarization $W = W^+ \oplus W^-$ for $W$ in such a way that $A = A \cap W^+ \oplus A \cap W^-$. For this, we consider the following two cases:
Case 1: $V_2^g = 0$; In this case we choose
\[
W^+ = V_1 \otimes V_2^+
\]
\[
W^- = V_1 \otimes V_2^-
\]
so that
\[
A \cap W^+ = L_1 \otimes L_2^+
\]
\[
A \cap W^- = L_1 \otimes L_2^-.
\]
Certainly we have $A = A \cap W^+ \oplus A \cap W^-$. 

Case 2: $V_1^g = 0$; In this case we choose
\[
W^+ = V_1^+ \otimes V_2
\]
\[
W^- = V_1^- \otimes V_2
\]
so that
\[
A \cap W^+ = L_1^+ \otimes L_2
\]
\[
A \cap W^- = L_1^- \otimes L_2.
\]
Certainly we have $A = A \cap W^+ \oplus A \cap W^-$. 

If $V_1^g = V_2^g = 0$, we can choose the polarization of $W$ in either way. For this reason, let us make the following definition.

**Definition 5.1.** For $W = V_1 \otimes V_2$, if the polarization is chosen as in Case 1 above, we call it “Type 1 polarization”. If it is chosen as in Case 2, we call it “Type 2 polarization”.

**Remark 5.2.** One important thing to be noted is that if $W$ is given Type 1 polarization, the group $U(V_1)$ is in the Siegel Levi (with respect to this polarization), and hence $\alpha_g = 0$ for all $g \in U(V_1)$. If $W$ is given Type 2 polarization, $U(V_2)$ is in the Siegel Levi and $\alpha_g = 0$ for all $g \in U(V_2)$.

**Remark 5.3.** For a self-dual lattice $A$ in $W$, if a polarization $W = W^+ \oplus W^-$ is so chosen that $A = A \cap W^+ \oplus A \cap W^-$, we say that the polarization of $W$ is compatible with the self-dual lattice $A$. The above discussion shows that for our $V_1$ and $V_2$ with fixed self-dual lattices $L_1$ and $L_2$, both types of polarization are compatible with the self-dual lattice $L_1 \otimes L_2$.

Each element $w \in V_1 \otimes_F V_2$ can be viewed as an element in $\text{Hom}_E(V_1, V_2)$ in the standard way as follows: For $w = \sum v_1 \otimes v_2$, define $w : V_1 \to V_2$ by
\[
w(v) = \sum \langle v, v_1 \rangle_1 v_2
\]
for $v \in V_1$. Also $w$ can be viewed as an element in $\text{Hom}_E(V_2, V_1)$ by
\[
w(v) = \sum \langle v, v_2 \rangle_2 v_1
\]
for $v \in V_2$. Which one is meant is always clear from the context.

We define $P^+ : W \to W^+ \subseteq W$ to be the projection on $W^+$ and $P^- : W \to W^- \subseteq W$ the projection on $W^-$, so we have
\[
\beta(w, w') = \langle P^+(w), P^-(w') \rangle,
\]
provided $\beta$ is defined with respect to the polarization $W = W^+ \oplus W^-$. Now for each $i$, define $P^+_i, P^-_i : V_i \to V_i$ to be the projections on $V_i^+, V_i^-$ and $V_i^-$, respectively. If we view each element $w \in V_1 \otimes_F V_2$ as $w \in \text{Hom}_E(V_1, V_2)$ then $P^+(w)$ and $P^-(w)$ as elements in $\text{Hom}_E(V_1, V_2)$ are to be interpreted as follows.
Recall from Section 3 that \( \Gamma \) polarization. As before, we denote the space of the lattice model by \( \mathcal{V}_1 \). Let \( \psi \) be the symplectic space as before. We let \( K = U(\mathcal{V}_1) \cap \Gamma \). Note that if the residue characteristic is odd, we always have \( K \) even when the residue characteristic is even.

We often write \( w^+ := P^+(w) \) and \( w^- := P^-(w) \). Also for \( v \in \mathcal{V}_1 \) and we write \( v^+ := P_i^+(v) \), \( v^- := P_i^-(v) \) and \( v^a := P_i^a(v) \). So for example, if \( w = v_1 \otimes v_2 \in \mathcal{V}_1 \otimes \mathcal{V}_2 \) and \( W \) is given Type 1 polarization, then \( w^+ = v_1 \otimes v_2^+ \) and \( w^- = v_1 \otimes v_2^- \), and if it is given Type 2 polarization, then \( w^+ = v_1^+ \otimes v_2 \) and \( w^- = v_1^- \otimes v_2 \).

6. Two key lemmas

In this section, we will formulate the two key lemmas which would imply the Howe duality conjecture for unramified dual pairs, if proven. They are the analogues of Theorem I.4 and Proposition I.5 in [MVW]. We closely follow the notations in [MVW].

As before, we fix \( (\mathcal{V}_1, \langle \ , \rangle_1) \) and \( (\mathcal{V}_2, \langle \ , \rangle_2) \) together with self-dual lattices \( L_1 \) and \( L_2 \) with respect to \( r_1 \) and \( r_2 \), respectively, where \( r_1 + r_2 = r \) is the exponential conductor of \( \psi \), and let

\[
W = \mathcal{V}_1 \otimes \mathcal{V}_2
\]

be the symplectic space as before. We let

\[
A := L_1 \otimes L_2 \subseteq W,
\]

which is a self-dual lattice of \( W \) with respect to \( r \).

In this section, we do not assume anything specific about the polarization of \( W \), and hence in particular, it might be neither Type 1 nor Type 2. For example if \( \mathcal{V}_1 \neq 0 \) and \( \mathcal{V}_2 \neq 0 \), the polarization is neither of the two. But in this section, we include such cases in our consideration. We always realize the Weil representation \( \omega_\psi \) of \( \hat{Sp}(W) \) in the lattice model with respect to \( A \) and the chosen polarization. As before, we denote the space of the lattice model by \( S_A \) or sometimes simply \( S \).

We let

\[
K_i = U(\mathcal{V}_i) \cap \Gamma_A \\
K_i^o = U(\mathcal{V}_i) \cap \Gamma^o_A.
\]

Recall from Section 3 that \( \Gamma_A = \{ g \in \text{Sp}(W) : gA = A \} \) and \( \Gamma^o_A = \{ g \in \Gamma_A : \alpha_g(a) \in \mathcal{P}^r \text{ for all } a \in A \} \). Note that if the residue characteristic is odd, we always have \( K_i = K_i^o \) and it is the usual maximal open compact subgroup of \( U(\mathcal{V}_i) \). Also if the polarization of \( W \) is chosen to be Type 1 (resp. Type 2), then \( K_1 = K_1^o \) (resp. \( K_2 = K_2^o \)) even when the residue characteristic is even.

For any lattice \( L \) of \( \mathcal{V}_1 \), we let

\[
B(L) = L^\perp \otimes_{\mathcal{O}_E} L_2 \subseteq W.
\]

One can check

\[
B(L)^\perp = B(L^\perp),
\]

where the \( \perp \) for \( B(L) \) is with respect to \( r = r_1 + r_2 \) and the one for \( L \) is with respect to \( r_1 \). Also note that \( L \subseteq L_1 = L_1^\perp \subseteq L^\perp \), and for each \( u \in U(L_1) \), we have \( (uL)^\perp = u(L^\perp) \).

For each sublattice \( L \subseteq L_1 \), we define

\[
K_i(L)^o := \{ u \in K_i^o : uL^\perp \subseteq L^\perp \},
\]


type 1 polarization: \( \mathcal{V}_1 \neq 0 \);

\[
\begin{align*}
P^+(w) &= P_2^+ \circ w \\
P^- (w) &= P_2^- \circ w.
\end{align*}
\]

Type 2 polarization: \( \mathcal{V}_2 \neq 0 \);

\[
\begin{align*}
P^+(w) &= w \circ P_1^- \\
P^-(w) &= w \circ P_1^+.
\end{align*}
\]
where we view $K_1$ as a subgroup of $\mathrm{Sp}(W)$. Note that $uL^\perp \subseteq L^\perp$ implies $uL^\perp = L^\perp$ and the condition $uL = L$ is equivalent to $uL^\perp = L^\perp$. Hence $K_1(L^\perp)^o = K_1(L)^\circ$.

One can check that the group $K_1(L)^\circ$ is an open compact subgroup of $U(V_1)$. Of course $K_1(L_1)^\circ = K_1^\circ$.

Also define

$$J_1(L) = \{ u \in K_1(L)^\circ : (u-1)L^\perp \subseteq L \}$$

$$H_1(L) = \{ u \in K_1(L)^\circ : (u-1)L^\perp \subseteq L \}.$$ 

Both of them are open compact subgroups of $U(V_1)$ with $J_1(L) \subseteq H_1(L)$. In particular, we have

$$0 \to J_1(L) \to K_1(L)^\circ \to \mathrm{Aut}(L^\perp/L);$$

$$0 \to H_1(L) \to K_1(L)^\circ \to \mathrm{Aut}(L_1/L).$$

Let us note that our $J_1(L)$ and $H_1(L)$ differ from the ones in [MVW] in that we always require each element $u \in J_1(L)$ or $H_1(L)$ be in $K_1(L)^\circ$, so that $\alpha_a(a) \in \mathcal{P}_r$ for all $a \in A$, though for example if the polarization of $W$ is Type 1, we always have $\alpha_a(a) \in \mathcal{P}_r$, and hence our $J_1(L)$ and $H_1(L)$ coincide with those of [MVW].

For a sublattice $M \subseteq L_2$, one defines open compact subgroups $K_2(M)^\circ, H_2(M)$ and $J_2(M)$ of $U(V_2)$ analogously.

We define

$$S_L := \{ f \in S_A : \text{supp } f \subseteq B(L) \}$$

Also for any subgroup $G \subseteq U(V_1)$ which splits in the cover $\overline{\mathrm{Sp}}(W)$, we define

$$S_G := \{ f \in S_A : \omega \psi(g) \cdot f = f \text{ for all } g \in G \}.$$ 

Recall from [M] that for each $w \in W$, we have defined $s_w$ to be the unique function in $S_A$ with $\text{supp}(s_w) = (A + w) \times F$ such that $s_w(w,0) = 1$. Then the following is an easy exercise.

**Lemma 6.1.** The space $S_L$ is generated by $s_w$ where $w \in B(L)$.

We have the following lemma, which is the analogue of part of Lemma in [MVW] p. 102.

**Lemma 6.2.** Assume $L$ is any sublattice if $L_1$. For $w \in B(L)$ and $h \in H_1(L)$, we have the equality

$$\omega(h)s_w = \psi(-\beta(h^{-1}w - w, w) + \alpha^h(w))s_w.$$ 

In particular, the map

$$\psi^w : h \mapsto \psi(-\beta(h^{-1}w - w, w) + \alpha^h(w))$$

is a character on $H_1(L)$.

**Proof.** By the above lemma, the space $S_L$ is generated by the functions $s_w$ for $w \in B(L)$. Hence the first part implies the second part. To show the first part, let $h \in H_1(L)$. For $w' \in W$, we have $\omega(h)s_w(w', z) = s_w(h^{-1}w', z + \alpha^h(w'))$, which is non-zero if (and only if) $h^{-1}w' \in A + w$, i.e. $w' \in A + hw$. Since $h \in H_1(L)$, we have $hw - w \in A$, i.e. $hw \in A + w$. Thus the support of $\omega(h)s_w$ is contained in that of $s_w$. Hence $\omega(h)s_w$ is proportional to $s_w$. To determine the constant of proportionality, choose $w' = w$. Keeping in mind $h^{-1}w - w \in A$, we have

$$\omega(h)s_w(w, z) = s_w(h^{-1}w, z + \alpha^h(w))$$

$$= s_w(h^{-1}w - w, z + \alpha^h(w))$$

$$= \psi(-\beta(h^{-1}w - w, w) + \alpha^h(w))s_w(w, z).$$

Thus $\omega(h)s_w = \psi(-\beta(h^{-1}w - w, w) + \alpha^h(w))s_w$. 

\qed
We define
\[ J_1(L)^\circ := \bigcap_{w \in B(L)} \ker \psi^w_1. \]

Of course by definition of \( J_1(L)^\circ \), we have

**Proposition 6.3.** For any sublattice \( L \) of \( L_1 \),
\[ S_L \subseteq S_{J_1(L)^\circ}. \]

**Remark 6.4.** the group \( J_1(L)^\circ \) is an open and compact subgroup of \( U(V_1) \). To see this, the reader can verify that if the polarization of \( W \) is Type 1, \( J_1(L) = J_1(L)^\circ \), and if the polarization of \( W \) is Type 2 and the lattice \( L \) is admissible, then \( J_1(L)^\circ = \{ u \in J_1(L) : \alpha^u(w) \in \mathcal{P} \text{ for all } w \in B(L) \} \).

Hence in either case, \( J_1(L)^\circ \) is an open compact subgroup of \( U(V_1) \). (To show the latter case, use Lemma 3.1.) If \( W \) is of Type 2 but \( L \) is not admissible, one can always find an admissible lattice \( L' \subseteq L \), and \( J_1(L')^\circ \subseteq J_1(L)^\circ \), which shows \( J_1(L)^\circ \) is open and compact.

For a sublattice \( M \subseteq L_2 \) and \( w \in B(M) \), we can analogously define \( \psi^w_2 \) and \( J_2(M)^\circ \), and have
\[ S_M \subseteq S_{J_2(M)^\circ}. \]

Now let \( H_1 \) (resp. \( H_2 \)) be the Hecke algebra for \( \tilde{U}(V_1) \) (resp. \( \tilde{U}(V_2) \)) as in [MVW] and \( \omega(H_2)S_L \) the subspace of \( S_L \) generated by the elements of the form \( \omega(\varphi)f \) for \( \varphi \in H_2 \) and \( f \in S_L \). The first of the two key lemmas for the proof of the Howe duality is

**Conjecture 6.5** (First Key Lemma). For any sublattice \( L \subseteq L_1 \) we have
\[ \omega(H_2)S_L = S_{J_1(L)^\circ}. \]

Also for each \( w \in B(L) \), let
\[ M = M_w = (w(\varpi^{-r_2}L_1) + L_2)^\perp. \]

Then
\[ \omega(H_1)S_M = S_{J_2(M)^\circ}. \]

Apparently this is the analogue of Theorem in [MVW] p. 103. But the inclusions \( \omega(H_2)S_L \subseteq S_{J_1(L)^\circ} \) and \( \omega(H_1)S_M \subseteq S_{J_2(M)^\circ} \) immediately follow from the previous lemma because the actions of \( H_1 \) and \( H_2 \) commute. The hard part is to show the other inclusion, which we do not know how to prove in full generality.

If \( M \) is as in the above conjectural lemma, one can verify that \( w \in B(M) \). With this said, let us state the second key lemma, which is the analogue of Proposition [MVW] p.103].

**Conjecture 6.6** (Second Key Lemma). Let \( w, w' \in B(L) \) be such that
(1) \( w(\varpi^{-r_2}L_2) + L_1 = w'(\varpi^{-r_2}L_2) + L_1 = L^\perp; \)
(2) \( \psi^w_1 = \psi^{w'}_1 \) as characters on \( H_1(L) \).
Then there exists \( k \in K_1 \) such that \( A + w = k(A + w') \).

Let \( M = M_w \) be as in the first key lemma, and \( w' \in B(M) \) be such that
(1) \( w'(\varpi^{-r_1}L_1) + L_2 = M^\perp; \)
(2) \( \psi^{w'}_2 = \psi^{w'}_2 \) as characters on \( H_2(M) \).
Then there exists \( k \in K_2 \) such that \( A + w = k(A + w') \).

Finally

**Theorem 6.7.** The above two conjectural lemmas imply the Howe duality conjecture in the sense that for any irreducible admissible representation \( \pi \) of \( \tilde{U}(V_1) \), if \( \Theta_\psi(\pi) \neq 0 \), then \( \Theta_\psi(\pi) \) has a unique non-zero irreducible quotient.
Proof. If we believe the above two lemmas, one can simply trace the proof for the Howe duality conjecture for odd residual characteristic as in [MVW] p.103-106. We give the details for this derivation in our special case where $L$ is of the form $L = \varpi^k L_1$ for $k \geq 1 + e$ in a later section.

\[\square\]

7. A proof of a special case of the first key lemma

In this section we prove the inclusion $S^J(L)\subseteq \omega(H_2)S_L$ and hence the first key lemma when $V_2$ is symplectic (and hence $V_1$ is symmetric) and $L \subseteq 2\varpi L_1$. However we need this assumption only near the end of the section, and we do not even need to assume that one of $V_1^\varpi$ and $V_2^\varpi$ is zero. Hence at the beginning, we do not make any assumption on $V_1$ and $V_2$ except, of course, that the pair is unramified.

Our proof is a modification of the one given in [MVW, Chapter 5.III]. Though we try to make our proof as self-contained as possible, the reader is always advised to compare ours with the one in [MVW].

As in [MVW], we always identify $W$ with $\text{Hom}(V_1, V_2)$ or $\text{Hom}(V_2, V_1)$, and which one is meant is always clear from the context. Also we abbreviate

\[J = J_1(L), \quad J^o = J_1(L)^o, \quad H = H_1(L), \quad B = B(L),\]

and the sublattice $L \subseteq L_1$ will be fixed throughout. For $w \in W$, we define a function $s[w] : H_{\beta}(W) \to \mathbb{C}$ by

\[s[w](w', z) = \int_{J^o} \omega(u)s_w(w', z) \, du = \int_{J^o} s_w(u^{-1}w', z + \alpha^u(w')) \, du.\]

where $(w', z) \in H_{\beta}(W)$. (Recall from (3.3) that $s_w$ is the unique function in $S$ whose support is $(A + w) \times F$ and $s_w(w, 0) = 1$.)

Lemma 7.1. The function $s[w]$ is in the space $S^J$.

Proof. To check $s[w] \in S$ it suffices to show that

\[s[w](a + w', z) = \psi(-\beta(a, w'))s[w](w', z)\]

for all $a \in A$ and $(w', z) \in H_{\beta}(W)$. But

\[s[w](a + w', z) = \int_{J^o} \omega(u)s_w(a + w', z) \, du\]

\[= \int_{J^o} s_w(u^{-1}a + u^{-1}w', z + \alpha^u(a + w')) \, du\]

\[= \int_{J^o} \psi(-\beta(u^{-1}a, u^{-1}w') + \alpha^u(a + w'))s_w(u^{-1}w', z) \, du\]

\[= \int_{J^o} \psi(-\beta(u^{-1}a, u^{-1}w') + \beta(u^{-1}a, u^{-1}w') - \beta(a, w') + \alpha^u(a) + \alpha^u(w'))s_w(u^{-1}w', z) \, du\]

\[= \int_{J^o} \psi(-\beta(a, w'))s_w(u^{-1}w', z + \alpha^u(w')) \, du\]

\[= \psi(-\beta(a, w')) \int_{J^o} \omega(u)s_w(w', z) \, du\]

\[= \psi(-\beta(a, w'))s[w](w', z),\]

where for the fourth equality we used

\[
\alpha^u(a + w') - \alpha^u(a) - \alpha^u(w') = \beta(u^{-1}a, u^{-1}w') - \beta(a, w'),
\]

and for the fifth we used $\psi(\alpha^u(a)) = 1$. 

Next we will show that $s[w] \in S^{J^0}$, namely $\omega(v)s[w] = s[w]$ for $v \in J^0$. But
\[
\omega(v)s[w](w', z) = s[w](v^{-1}w', z + \alpha^v(w'))
\]
\[
= \int_{J^0} \omega(u)s_w(v^{-1}w', z + \alpha^v(w'))
\]
\[
= \int_{J^0} s_w(u^{-1}v^{-1}w', z + \alpha^v(w') + \alpha^u(v^{-1}w')
\]

Now one has
\[
\alpha^v(w') + \alpha^u(v^{-1}w') = \alpha^{vu}(w'),
\]
because
\[
\alpha^{vu}(w') = -\alpha_{vu}(u^{-1}v^{-1}w')
\]
\[
= -u^{-1}\cdot \alpha_v(u^{-1}v^{-1}w') - \alpha_u(u^{-1}v^{-1}w')
\]
\[
= -\alpha_v(v^{-1}w') - \alpha_u(u^{-1}v^{-1}w')
\]
\[
= \alpha^v(w') + \alpha^u(v^{-1}w').
\]

Hence the integral above is written as
\[
\int_{J^0} s_w(u^{-1}v^{-1}w', z + \alpha^{vu}(w'))
\]

which, by the change of variable $vu \mapsto u$, becomes
\[
\int_{J^0} s_w(u^{-1}w', z + \alpha^u(w'))
\]

Hence $\omega(v)s[w] = s[w]$. \hfill $\square$

Let us note

**Lemma 7.2.** $s[w] \neq 0$ if and only if $s[w](w, 0) \neq 0$.

**Proof.** Assume $s[w] \neq 0$, i.e. $s[w](w', z') \neq 0$ for some $w' \in W$ and $z' \in F$. But if $s[w](w', z') \neq 0$ for some $z' \in F$, then $s[w](w', z) \neq 0$ for all $z \in F$. Then there exists $u \in J^0$ such that $(u^{-1}w', z)$ is in the support of $s_w$, i.e. $u^{-1}w' \in A + w$. Considering $uA \subseteq A$, we have $w' \in A + uw$. So $w' = a + uw$ for some $a \in A$. So $s[w](a + uw, z) = 0$. But we know $s[w](a + uw, z) = \psi(-\beta(a, uw))s[w](uw, z)$ and $\psi(-\beta(a, uw)) \neq 0$, and hence $s[w](uw, z) \neq 0$. Let $z = \alpha_u(w)$. Then $s[w](uw, \alpha_u(w)) = \omega(u^{-1})s[w](w, 0) \neq 0$. (See Remark 2.27) But $\omega(u^{-1})s[w] = s[w]$, and so $s[w](w, 0) \neq 0$. The converse is obvious. \hfill $\square$

For each $w \in W$, let
\[
C(w) = \bigcup_{u \in J^0} u(A + w).
\]

The union here is actually finite because of the compactness of $J^0$.

**Lemma 7.3.** If $s[w] \neq 0$, the support of $s[w]$ is $C(w) \times F$.

**Proof.** If $(w', z)$ is in the support of $s[w]$, then the integrand in $s[w](w', z)$ is non-zero for some $u \in J^0$, and hence $u^{-1}w' \in A + w$, which implies $(w', z) \in C(w) \times F$. \hfill $\square$
Now we will show that for any element \((w', z) \in C(w), s[w](w', z) \neq 0\). We may assume \(z = 0\), and assume \(w' = a + uw\) for some \(a \in A\), considering \(uA \subseteq A\). But then

\[
s[w](w', z) = \psi(-\beta(a, uw))s[w](uw, 0)
\]

\[
= \psi(-\beta(a, uw) - \alpha_u(w))s[w](uw, \alpha_u(w))
\]

\[
= \psi(-\beta(a, uw) - \alpha_u(w))\omega(u^{-1})s[w](w, 0)
\]

\[
= \psi(-\beta(a, uw) - \alpha_u(w))s[w](w, 0),
\]

and by the above lemma \(s[w](w, 0) \neq 0\).

If \(s[w] \neq 0\), then it is a unique (up to constant) function in \(S^J\) with support equal to \(C(w) \times F\), because if \(s \in S^J\) has support equal to \(C(w) \times F\), its values are determined by the value at \((w, 0)\).

Also we have

**Lemma 7.4.** The space \(S^J\) is generated by the functions of the form \(s[w]\).

**Proof.** Let \(s \in S^J\), and \(C \times F \subseteq W \times F\) its support. Then \(C\) is a finite disjoint union of the sets of the form \(A + w\) for some \(w \in W\). But since \(s \in S^J\), we have \(uC = C\) for all \(u \in J^o\). Hence if \(A + w \subseteq C\), then \(u(A + w) \subseteq C\) for all \(u \in J^o\). Hence \(C\) is written as a finite disjoint union of the sets of the form \(C(w)\). Hence \(s\) must be a linear combination of functions whose supports are of the form \(C(w) \times F\). The lemma follows.

We need to be more specific about when the function \(s[w]\) is nonzero. First since the support of \(s_w\) is \(A + w\), the integrand of the integral defining \(s[w]\) is zero unless \(u^{-1}w' \in A + w\), and hence we have

\[
s[w](w', z) = \int_{u^{-1}w' \in A + w} \omega(u)s_w(w', z) \, du.
\]

But as we mentioned above, the support of \(s[w]\) is \(C(w) \times F\), and hence we may assume \(w' \in (A + w)\) for some \(w' \in J^o\). Hence the above integral is over the set

\[
\{u \in J^o : u^{-1}w'(A + w) = A + w\}.
\]

But one can see

\[
\{u \in J^o : u^{-1}w'(A + w) = A + w\} = \{u \in J^o : u(A + w) = A + w\}
\]

via the map \(u \mapsto u^{-1}w'\). Now define a sublattice \(L_w \subseteq L_1\) by

\[
L_w^1 = L_1 + w(\omega^{-1}L_2).
\]

Then \(w \in B(L_w)\) and the stabilizer in \(K_1\) of \(A + w\) is \(H_1(L_w)\). Hence the integral for \(s[w](w', z)\) is over \(J^o \cap H_1(L_w)\) and by Lemma [6.2] we have

\[
s[w](w', z) = \int_{u \in J^o \cap H_1(L_w)} \psi(-\beta(u^{-1}w - w, w) + \alpha^u(w))s_w(w', z) \, du.
\]

Note that the map \(u \mapsto \psi(-\beta(u^{-1}w - w, w) + \alpha^u(w))\) is a character on \(H_1(L_w)\). Hence by the orthogonality of characters, we have

**Proposition 7.5.** For each \(w \in W\),

\[
s[w] \neq 0 \quad \text{if and only if} \quad \psi(-\beta(u^{-1}w - w, w) + \alpha^u(w)) = 1
\]

for all \(u \in J^o\) such that \(u(A + w) = A + w\). (Here the condition \(u(A + w) = A + w\) is equivalent to \(uw \in A + w\).)
An in [MVW], we would like to write the condition \( \psi(-\beta(u^{-1}w - w, w) + \alpha(w)) = 1 \) in a more explicit form by using Cayley transforms, whose notion we will recall now. For an \( \epsilon \)-Hermitian space \((\mathcal{V}, \langle \cdot, \cdot \rangle)\) over \(E\), we let \(\mathfrak{u}(\mathcal{V})\) be the Lie algebra of \(U(\mathcal{V})\), namely

\[
\mathfrak{u}(\mathcal{V}) = \{ c \in \text{End}(\mathcal{V}) : \langle cv, v' \rangle + \langle v, cv' \rangle = 0 \}.
\]

There is a bijection

\[
\{ c \in \mathfrak{u}(\mathcal{V}) : 1 + c \text{ is invertible} \} \cong \{ g \in U(\mathcal{V}) : 1 + g \text{ is invertible} \}
\]

where the bijection is given by

\[
g = (1 - c)(1 + c)^{-1}, \quad c = (1 - g)(1 + g)^{-1}.
\]

We call \((1 - c)(1 + c)^{-1}\) the Cayley transform of \(c\). For \(x, y \in \mathcal{V}\), define \(c_{x,y} \in \text{End}_E(\mathcal{V})\) by

\[
c_{x,y}(v) = \langle v, y \rangle x - \epsilon(v, x)y.
\]

One can check that \(c_{x,y} \in \mathfrak{u}(\mathcal{V})\). Assume \(1 + c_{x,y}\) is invertible. We let \(u_{x,y}\) be the Cayley transform of \(c_{x,y}\), namely

\[
u_{x,y} = (1 - c_{x,y})(1 + c_{x,y})^{-1}.
\]

Now let \(x, y \in \mathcal{V}_1\) be given. We would like to know when \((1 + c_{x,y})^{-1}\) and hence the Cayley transform exist, and if it does exist when it is in the group \(H\) or \(J\). For this purpose we need to introduce the notion of order with respect to a lattice. Namely for an \(\epsilon\)-Hermitian space \((\mathcal{V}, \langle \cdot, \cdot \rangle)\) over \(E\) and a lattice \(M \subseteq \mathcal{V}\), we define “\(M\)-order” \(\text{ord}_M : \mathcal{V} \to \mathbb{Z}\) by

\[
\text{ord}_M(v) = \max \{ m \in \mathbb{Z} : v \in \mathcal{W}^m M \},
\]

i.e. \(v \in \mathcal{W}^m M \setminus \mathcal{W}^{m+1} M\). Namely if we choose an \(O_E\) basis \(\{v_1, \ldots, v_n\}\) of \(M\) and write \(v = a_1 v_1 + \cdots + a_n v_n\), then \(\text{ord}_M(v) = \min \{|a_1|\}\). (We assume \(\text{ord}_M(0) = \infty\).) Apparently \(\text{ord}_M(x + y) \geq \min \{\text{ord}_M(x), \text{ord}_M(y)\}\).

Next we need to go back to the dual pair situation where \(L\) is a sublattice of our self-dual lattice \(L_1 \subseteq \mathcal{V}_1\) and \(W = \mathcal{V}_1 \otimes \mathcal{V}_2\).

**Proposition 7.6.** For \(x, y \in \mathcal{V}_1\) and \(w \in W\), consider the following conditions:

(i) \(\text{ord}_{L_1}(x) + \text{ord}_{L_1}(y) \geq 1 - r_1\);
(ii) \(\text{ord}_{L_1}(x) + \text{ord}_{L_1}(y) \geq -r_1 - c\);
(iii) \(\text{ord}_{L_1}(x) + \text{ord}_{L_1}(y) \geq -r_1 - c\) and \(\text{ord}_{L_1}(x) + \text{ord}_{L_1}(y) \geq -r_1 - c\);
(iv) \(\text{ord}_{L_1}(x) + \text{ord}_{L_2}(w) + \text{ord}_{L_1}(y) \geq -c\) and \(\text{ord}_{L_2}(wx) + \text{ord}_{L_1}(y) \geq -c\).

Depending on which of those conditions the pair \((x, y)\) satisfies, the Cayley transform \(u_{x,y}\) satisfies the following:

(1) The condition (i) implies \(u_{x,y}\) exists and \(u_{x,y}L_1 \subseteq L_1\);
(2) The conditions (i) and (ii) imply \((u_{x,y}^{-1})L_1 \subseteq L\), and hence if the condition \(\psi(\alpha_{u_{x,y}}(a)) = 1\) for all \(a \in A\) is satisfied, we have \(u_{x,y} \in J\);
(3) The conditions (i) and (iii) imply \((u_{x,y}^{-1})L_1 \subseteq L_1\), and hence if the condition \(\psi(\alpha_{u_{x,y}}(a)) = 1\) for all \(a \in A\) is satisfied, we have \(u_{x,y} \in H\);
(4) The conditions (i) and (iv) imply \(u_{x,y}w \in A + w\).
Proof. (1) Assume the condition (i) is satisfied. Let \( r_x = \operatorname{ord}_{L_1}(x) \) and \( r_y = \operatorname{ord}_{L_2}(y) \), and so \( r_x + r_y \geq 1 - r_1 \). Then for \( l \in L_1 \), we have

\[
c_{x,y}(l) = (l, y)_1 x - \epsilon_1 (l, x)_1 y
= \omega^{r_x + r_y} (l, \omega^{-r_y})_1 \omega^{-r_x} x - \epsilon_1 \omega^{r_x + r_y} (l, \omega^{-r_x})_1 \omega^{-r_y} y,
\]

where both \( (l, \omega^{-r_y})_1 \omega^{-r_x} x \) and \( (l, \omega^{-r_x})_1 \omega^{-r_y} y \) are in \( \omega^{n_1} L_1 \). Hence \( c_{x,y}(l) \in \omega^{r_x + r_y + r_1} L_1 \subseteq \omega L_1 \). Hence \( c_{x,y}(L_1) \in \omega L_1 \) and \( 1 + c_{x,y} \) is invertible. (To see \( 1 + c_{x,y} \) is invertible, notice that for any nonzero \( v \in V_1 \), we have \( \operatorname{ord}_{L_1}(c_{x,y}(v)) > \operatorname{ord}_{L_1}(v) \) and so \( (1 + c_{x,y})(v) \neq 0 \).)

Let

\[
u_{x,y} = (1 - c_{x,y})(1 + c_{x,y})^{-1}.
\]

Notice that since \( c_{x,y}(L_1) \in \omega L_1 \), if we set the topology of \( V \) to be induced from that of \( E \), the sequence \( c_{x,y}(v) \) converges to 0 as \( n \to \infty \), and thus we obtain the geometric series

\[
(1 + c_{x,y})^{-1} = \sum_{n=0}^{\infty} (-1)^n c_{x,y}^n,
\]

which gives

\[
u_{x,y} = (1 - c_{x,y})(1 + c_{x,y})^{-1} = 1 - 2c_{x,y} + 2c_{x,y}^2 - 2c_{x,y}^3 + \cdots.
\]

Hence one can see \( u_{x,y} L_1 \subseteq L_1 \).

(2) Assume the conditions (i) and (ii). From the above series expansion of \( u_{x,y} \), one can tell that to show \( (u_{x,y} - 1)L_1 \subseteq L \) it suffices to show \( 2c_{x,y} L_1 \subseteq L \). Let \( r_x = \operatorname{ord}_{L_1}(x) \) and \( r_y = \operatorname{ord}_{L_2}(y) \), and so \( r_x + r_y \geq -r_1 \) and both \( \omega^{-r_x} x \) and \( \omega^{-r_y} y \) are in \( L \). Then for \( l \in L_1 \), we have

\[
2c_{x,y}(l) = 2((l, y)_1 x - \epsilon_1 (l, x)_1 y)
= 2(\omega^{r_x + r_y} (l, \omega^{-r_y})_1 \omega^{-r_x} x - \epsilon_1 \omega^{r_x + r_y} (l, \omega^{-r_x})_1 \omega^{-r_y} y),
\]

where both \( (l, \omega^{-r_y})_1 \omega^{-r_x} x \) and \( (l, \omega^{-r_x})_1 \omega^{-r_y} y \) are in \( \omega^{r_1} L \). Hence \( 2c_{x,y}(l) \in \omega^{r_1 + r_1 + r_1} L \).

If \( r_x + r_y \geq -r_1 - \epsilon \), we have \( 2c_{x,y}(l) \in L \).

(3) This case is completely analogous to (2).

(4) Finally assume \( u_{x,y} \) exists and the condition (iv) is satisfied. We need to show \( (u_{x,y} - 1)w \in A \). Again by the series expansion of \( u_{x,y} \), it suffices to show \( 2c_{x,y}(w) \in A \). Note that here we are viewing \( c_{x,y} \) as an operator on \( W \) rather than just on \( V_1 \) in the obvious way. Also we may assume \( w = v_1 \otimes v_2 \) where \( v_1 \in V_1 \) and \( v_2 \in V_2 \). Then

\[
2c_{x,y}(w) = 2(\langle v_1, y \rangle_1 x \otimes v_2 - \epsilon_1 \langle v_1, x \rangle_1 y \otimes v_2)
= 2x \otimes \langle v_1, y \rangle_1 v_2 - 2 \epsilon_1 y \otimes \langle v_1, x \rangle_1 v_2.
\]

Note that \( \operatorname{ord}_{L_2}(\langle v_1, y \rangle_1 v_2) = \operatorname{ord}_{L_2}(wy) \) and \( \operatorname{ord}_{L_2}(\langle v_1, x \rangle_1 v_2) = \operatorname{ord}_{L_2}(wx) \). Then arguing as above, one can see that the condition \( \operatorname{ord}_{L_1}(x) + \operatorname{ord}_{L_2}(wy) \geq -\epsilon \) implies \( 2x \otimes wy \in A \) and the condition \( \operatorname{ord}_{L_2}(wx) + \operatorname{ord}_{L_1}(y) \geq -\epsilon \) implies \( 2y \otimes wy \in A \).

\( \square \)

The following special case will be also needed.

**Proposition 7.8.** Assume \( \epsilon_1 = -1 \). Let \( x, y \in V_1 \) be such that \( x = ay \) for \( a \in E \) with \( \bar{a} = a \). Also let \( w \in W \). Consider the following conditions:

(i) \( \operatorname{ord}_{L_1}(x) + \operatorname{ord}_{L_2}(y) \geq 1 - r_1 - \epsilon; \)
(ii) \( \operatorname{ord}_{L_1}(x) + \operatorname{ord}_{L_2}(y) \geq -r_1 - 2\epsilon; \)
(iii) \( \operatorname{ord}_{L_1}(x) + \operatorname{ord}_{L_2}(y) \geq -r_1 - 2\epsilon \) and \( \operatorname{ord}_{L_1}(x) + \operatorname{ord}_{L_2}(y) \geq -r_1 - 2\epsilon; \)
(iv) \( \operatorname{ord}_{L_1}(x) + \operatorname{ord}_{L_2}(wy) \geq -2\epsilon \) and \( \operatorname{ord}_{L_1}(wx) + \operatorname{ord}_{L_2}(y) \geq -2\epsilon. \)

Depending on which of those conditions the pair \( (x, y) \) satisfies, the Cayley transform \( u_{x,y} \) satisfies the following:
(1) The condition (i) implies $u_{x,y}L_1 \subseteq L_1$;
(2) The conditions (i) and (ii) imply $(u_{x,y} - 1)L_1 \subseteq L$, and hence if the condition $\psi(\alpha_{u_{x,y}}(a)) = 1$ for all $a \in A$ is satisfied, we have $u_{x,y} \in J$;
(3) The conditions (i) and (iii) imply $(u_{x,y} - 1)L_1 \subseteq L_1$, and hence if the condition $\psi(\alpha_{u_{x,y}}(a)) = 1$ for all $a \in A$ is satisfied, we have $u_{x,y} \in H$;
(4) The conditions (i) and (iv) imply $u_{x,y}w \in A + w$.

Proof. The proof is the same as the one for the previous proposition, except that this time we have $c_{x,y}(v) = 2(v, x)_{1}y$ for $v \in V_1$.

Lemma 7.9. Let $x, y \in V_1$ satisfy the conditions (i) and (iv) of Proposition 7.6 or Proposition 7.8 if $x = ay$ with $a = a$. Then

$\beta(u_{x,y}^{-1}w - w, w) \equiv 2\beta(c_{x,y}(w), w) \mod 4P^r$.

Proof. For notational convenience, let $u = u_{x,y}$ and $w' = (1 - c_{x,y})^{-1}w$. Then we have

$\beta(u^{-1}w - w, w) = \beta(u^{-1}w, w) - \beta(w, w)$

$= \beta((1 + c_{x,y})w', (1 - c_{x,y})w') - \beta(w, w)$

$= \beta((1 - c_{x,y})w', (1 - c_{x,y})w') - \beta(w, w) + 2\beta(c_{x,y}w', (1 - c_{x,y})w') - \beta(w, w)$

$= \beta(w, w) + 2\beta(c_{x,y}w', w') - 2\beta(c_{x,y}w', c_{x,y}w') - \beta(w, w)$

$= 2\beta(c_{x,y}w', w') - 2\beta(c_{x,y}w', c_{x,y}w')$.

Now by writing $w' = (1 - c_{x,y})^{-1}w$ in terms of geometric series as in (7.7), one can see that

$2\beta(c_{x,y}w', w') - 2\beta(c_{x,y}w', c_{x,y}w') = 2\beta(c_{x,y}w, w) + \text{(higher terms)}$,

where (higher terms) is the sum of the terms of the form $4\beta(c_{x,y}^{k}w, c_{x,y}^{l}w)$ for $k \geq 1$ and $l \geq 1$. Since $(x, y)$ satisfies the condition (iv), $c_{x,y}w \in A$. Hence those higher terms are in $P^r$, which proves the lemma.

Now we can write $\beta(c_{x,y}(w), w)$ more concretely in terms of $\langle -, \cdots \rangle_2$ as long as the polarization of $W$ is chosen to be either Type 1 or Type 2. Accordingly, from now on we will assume either $V_1 = 0$ or $V_2 = 0$, and if the former is the case, we assume the polarization is Type 2 and the latter Type 1. If both $V_1$ and $V_2$ are zero, the polarization can be taken to be either Type 1 or Type 2.

Lemma 7.10. $\beta(c_{x,y}(w), w)$ can be computed as follows:

Type 1 ($V_2 = 0$):

$\beta(c_{x,y}(w), w) = - \text{tr}_{E/F} \left( \langle w(x), w(y) \rangle_2 \right)$.

Type 2 ($V_1 = 0$):

$\beta(c_{x,y}(w), w) = - \text{tr}_{E/F} \left( \langle w(x^+), w(y) \rangle_2 + \langle w(x), w(y^+) \rangle_2 \right)$.

Proof. Let $w = \sum_v v_{1,i} \otimes v_{2,i}$, where $v_{1,i} \in V_1$ and $v_{2,i} \in V_2$. Then

$\beta(c_{x,y}(w), w) = \sum_i c_{x,y}(v_{1,i}) \otimes v_{2,i}$

$= \sum_i \langle v_{1,i}, y \rangle_1 x - \epsilon_1 \langle v_{1,i}, x \rangle_1 y \otimes v_{2,i}$
and hence

\[
\beta(c_{x,y}(w), w) = \beta(\sum_i (\langle v_{1,i}, y \rangle x - \epsilon_1(v_{1,i}, x) y) \otimes v_{2,i}, \sum_j v_{1,j} \otimes v_{2,j}) = \sum_i \sum_j \beta((\langle v_{1,i}, y \rangle x - \epsilon_1(v_{1,i}, x) y) \otimes v_{2,i}, v_{1,j} \otimes v_{2,j}).
\]

Here

\[
\beta((\langle v_{1,i}, y \rangle x - \epsilon_1(v_{1,i}, x) y) \otimes v_{2,i}, v_{1,j} \otimes v_{2,j}) = \beta((\langle v_{1,i}, y \rangle x - \epsilon_1(v_{1,i}, x) y) \otimes v_{2,i}, v_{1,j} \otimes v_{2,j}).
\]

Now we have to argue case-by-case.

**Type 1 ($V^+_2 = 0$):** Recall $W^+ = V_1 \otimes V^+_2$ and $W^- = V_1 \otimes V^-_2$. Then

\[
\beta((\langle v_{1,i}, y \rangle x \otimes v_{2,i}, v_{1,j} \otimes v_{2,j}) - \epsilon_1((\langle v_{1,i}, x \rangle y \otimes v_{2,i}, v_{1,j} \otimes v_{2,j}).
\]

\[
= -\text{tr}_{E/F}\left((-((\langle v_{1,i}, y \rangle x, v_{1,j})_1 v_{2,i}^+, v_{2,j}^+)_2 + \epsilon_2((\langle v_{1,i}, x \rangle y, v_{1,j})_1 v_{2,i}^+, v_{2,j}^+)_2\right)
\]

\[
= -\text{tr}_{E/F}\left((y, v_{1,i})_1 (x, v_{1,j})_1 v_{2,i}^+, v_{2,j}^+) + (x, v_{1,i})_1 (y, v_{1,j})_1 v_{2,i}^+, v_{2,j}^+)\right)
\]

\[
= -\text{tr}_{E/F}\left((x, v_{1,i})_1 v_{2,i}^+, (y, v_{1,i})_1 v_{2,i}^+, (y, v_{1,j})_1 v_{2,j}^+)\right)
\]

Hence

\[
\sum_i \sum_j \beta((\langle v_{1,i}, y \rangle x - \epsilon_1(v_{1,i}, x) y) \otimes v_{2,i}, v_{1,j} \otimes v_{2,j})
\]

\[
= \sum_i \sum_j -\text{tr}_{E/F}\left((x, v_{1,i})_1 v_{2,i}^+, (y, v_{1,i})_1 v_{2,i}^+, (y, v_{1,j})_1 v_{2,j}^+)\right)
\]

\[
= -\text{tr}_{E/F}\left((w(x)_1 w(y)_1)_2 + (w(x)_1 w(y)_1)_2\right)
\]

**Type 2 ($V^-_2 = 0$):** Recall $W^+ = V_1^+ \otimes V_2$ and $W^- = V_1^- \otimes V_2$. Then

\[
\beta((\langle v_{1,i}, y \rangle x \otimes v_{2,i}, v_{1,j} \otimes v_{2,j}) - \epsilon_1((\langle v_{1,i}, x \rangle y \otimes v_{2,i}, v_{1,j} \otimes v_{2,j})
\]

\[
= -\text{tr}_{E/F}\left((\langle v_{1,i}, y \rangle x^+, v_{1,j}^-)_1 (v_{2,i}^+, v_{2,j}^-)_2 + \epsilon_1((\langle v_{1,i}, x \rangle y^+, v_{1,j}^-)_1 (v_{2,i}^+, v_{2,j}^-)_2\right)
\]

\[
= -\text{tr}_{E/F}\left((y, v_{1,i})_1 (x^+, v_{1,j}^-)_1 (v_{2,i}^+, v_{2,j}^-)_2 + (x, v_{1,i})_1 (y^+, v_{1,j}^-)_1 (v_{2,i}^+, v_{2,j}^-)_2\right)
\]

\[
= -\text{tr}_{E/F}\left((x^+, v_{1,i}^-)_1 v_{2,i}^+, (y, v_{1,i}^-)_1 v_{2,i}^+, (y^+, v_{1,j}^-)_1 v_{2,j}^-\right)
\]

Hence by taking $\sum_i \sum_j$ as above, one obtains the desired formula.
Next we would like to know when we have $\alpha^u(w) \in \mathcal{P}^r$ for all $w \in B$ when $u$ is a Cayley transform. Let us start with

**Lemma 7.11.** Let $w \in W$ be any.

1. Suppose $W$ is equipped with Type 1 polarization. Assume $u_{x,y}$ exists for $x, y \in \mathcal{V}_1$. Then

$$\alpha^{u_{x,y}}(w) = 0.$$  

2. Suppose $W$ is equipped with Type 2 polarization. Assume $u_{x,y}$ exists where $x, y \in \mathcal{V}_1^\perp$. Then

$$\alpha^{u_{x,y}}(w) = \begin{cases} (c_{x,y}(w^-), w^-) = -2 \text{tr}_{E/F}(\langle w(x), w(y) \rangle) & \text{if } x, y \in \mathcal{V}_1^+; \\ \langle w^+, c_{x,y}(w^+) \rangle = 2 \text{tr}_{E/F}(\langle w(x), w(y) \rangle) & \text{if } x, y \in \mathcal{V}_1^-; \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** (1) This case is obvious because, if $W$ is given Type 1 polarization, then $\alpha_g = 0$ for any $g \in U(\mathcal{V}_1)$.

(2) First assume $x \in \mathcal{V}_1^+$ and $y \in \mathcal{V}_1^-$ (or $x \in \mathcal{V}_1^-$ and $y \in \mathcal{V}_1^+$). Then one can see that $u_{x,y}$ is in the Siegel Levi of $\text{Sp}(W)$ and hence $\alpha^{u_{x,y}} = 0$.

Next assume $x, y \in \mathcal{V}_1^+$, and for notational simplicity, let us write $u = u_{x,y}$. Viewed as an operator on $W$, $c_{x,y}|_{W^+} = 0$ and the image of $c_{x,y}$ is in $W^+$. Hence $c_{x,y}^2 = 0$ and $c_{x,y}$ may be viewed as a map from $W^-$ to $W^-$. Then one can see

$$u = \begin{pmatrix} 1 & -2c_{x,y} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad u^{-1} = \begin{pmatrix} 1 & 2c_{x,y} \\ 0 & 1 \end{pmatrix},$$

where the matrix is chosen with respect to the polarization $W = W^+ + W^-$. Now for $w = w^+ + w^- \in W$,

$$\alpha^u(w) = -\alpha_u(u^{-1}(w^+ + w^-))$$
$$= -\alpha_u(w^+ + 2c_{x,y}(w^-) + w^-)$$
$$= -\frac{1}{2} \langle -2c_{x,y}(w^-), w^- \rangle \quad \text{by (2.4)}$$
$$= \langle c_{x,y}(w^-), w^- \rangle.$$

Let us write $w = \sum_i v_{1,i} \otimes v_{2,i}$. Since the polarization is Type 2, we have $w^- = \sum_i v_{1,i}^- \otimes v_{2,i}$. Thus

$$\langle c_{x,y}(w^-), w^- \rangle = \sum_i \sum_j \langle c_{x,y}(v_{1,i}^- \otimes v_{2,i}), v_{1,j}^- \otimes v_{2,j} \rangle$$

Here

$$\langle c_{x,y}(v_{1,i}^- \otimes v_{2,j}), v_{1,i}^- \otimes v_{2,j} \rangle$$
$$= \text{tr}_{E/F} \left( c_{x,y}(v_{1,i}^-), v_{1,j}^- \right) \left( v_{2,i} \otimes v_{2,j} \right)$$
$$= \text{tr}_{E/F} \left( (v_{1,i}^- \otimes y) x - \epsilon_1 (v_{1,i}^- \otimes x) y, v_{1,j}^- \right) \left( v_{2,i} \otimes v_{2,j} \right)$$
$$= \text{tr}_{E/F} \left( v_{1,i}^- \otimes y \right) \left( x, v_{1,j}^- \right) \left( v_{2,i} \otimes v_{2,j} \right) - \epsilon_1 (v_{1,i}^- \otimes x) \left( y, v_{1,j}^- \right) \left( v_{2,i} \otimes v_{2,j} \right)$$
$$= \text{tr}_{E/F} \left( (v_{1,i}^- \otimes y) v_{2,i}, x, v_{1,j}^- \right) \left( v_{2,j} \right) - \left( (x, v_{1,i}^- \otimes y) v_{2,i}, y, v_{1,j}^- \right) \left( v_{2,j} \right).$$
Now by taking $\sum_i \sum_j$, one can see that
\[
\langle c_{x,y}(w^-), w^- \rangle = \text{tr}_{E/F} \left( \epsilon_1(w^-_{(y)}, w^-_{(x)})_2 - \langle w^-_{(x)}, w^-_{(y)} \rangle_2 \right)
= -\text{tr}_{E/F} \left( \langle w^-_{(x)}, w^-_{(y)} \rangle_2 + \langle w^-_{(x)}, w^-_{(y)} \rangle_2 \right)
= -2\text{tr}_{E/F} \left( \langle w^-_{(x)}, w^-_{(y)} \rangle_2 \right).
\]

Now since $x, y \in \mathcal{V}_1^\pm$, we have $w^-_{(x)} = w(x)$ and $w^-_{(y)} = w(y)$. Thus the lemma follows.

The case $x, y \in \mathcal{V}_1^-$ is almost identical. \hfill \Box

This lemma immediately implies

Lemma 7.12. Let $w \in B$ and $x, y \in \mathcal{V}_1$. Assume $u_{x,y}$ exits.

1. Assume that $W$ is given Type 1 polarization. Then $\alpha^{u_{x,y}}(w) \in \mathcal{P}^r$;
2. Assume that $W$ is given Type 2 polarization and $x, y \in \mathcal{V}_1^\pm$ and $L$ is admissible. Assume further that, if $x, y \in \mathcal{V}_1^+$ or $x, y \in \mathcal{V}_1^-$, then the condition (ii) of Proposition 7.6 is satisfied. Then $\alpha^{u_{x,y}}(w) \in \mathcal{P}^r$.

Proof. (1) is obvious. For (2), the only case that is not too obvious is when $x, y \in \mathcal{V}_1^+$ or $x, y \in \mathcal{V}_1^-$. Assume $x, y \in \mathcal{V}_1^+$. If the condition (ii) is satisfied, we have $c_{x,y}w^- \in L \otimes L_2$. (Note that since $L$ is admissible, if $w \in B$ then $w^- \in B$.) Hence $\alpha^{u_{x,y}}(w) = \langle c_{x,y}(w^-), w^- \rangle \in \mathcal{P}^r$. The proof for case $x, y \in \mathcal{V}_1^-$ is essentially the same. \hfill \Box

One consequence of the lemma, especially for Type 2 polarization, is the following.

Proposition 7.13. Let $x, y \in \mathcal{V}_1$ be such that the pair $(x, y)$ satisfies (i) and (ii) of Proposition 7.6.

1. If $W$ is given Type 1 polarization, then $u_{x,y} \in J_0$.
2. If $W$ is given Type 2 polarization and $L$ is admissible, then $u_{x,y} \in J_0$ as long as $x, y \in \mathcal{V}_1^\pm$.

Proof. We prove only (2). Let $u = u_{x,y}$. Assume $W$ is given Type 2 polarization. By the above lemma, $u \in J_0$ if and only if $\beta(u, w, w) \in \mathcal{P}^r$ for all $w \in B$. But this happens if and only if $w\cdot w \in L \otimes L_2$ for all $w \in B$, because $L$ is admissible. Hence the proposition follows by Proposition 7.6 (2). (The reader should notice that this does not necessarily follow if $L$ is not admissible.) \hfill \Box

We should mention that if $W$ is given Type 1 polarization, we actually have $J_0 = J$.

For $w \in W$ and $t \in \mathbb{Z}$ with $t \geq 0$, let us define the condition (\textit{wt}) as follows:

\textit{(wt)} $\varpi^{-r_2}w(2\varpi L_1 \cap L) \subseteq \varpi^{-t}L_2$.

We define
\[
S_t := \{ s \in S : \text{(wt) holds for all } (w, 0) \in \text{supp}(s) \}.
\]
Let us note that $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$, and $S = \bigcup_{t \geq 0} S_t$. The space $S_t$ is stable under the action of $J_0$ because $J_0$ preserves the space $2\varpi L_1 \cap L$. Hence we have
\[
S_{J_0} = \bigcup_{t \geq 0} S_t^{J_0}.
\]
One can also check that if $w \in W$ satisfies the condition (\textit{wt}), then $w \in S_t^{J_0}$.

In what follows, we will show

Proposition 7.14. Assume $\mathcal{V}_2$ is symplectic (if the residue characteristic of $F$ is even), and $L$ is admissible if the polarization of $W$ is Type 2. (If the polarization of $W$ is Type 1, $L$ does not have to be admissible.) We have the inclusion $S_{J_0} \subseteq \omega(\mathcal{H}_2)S_{J_0}$.
Before going into the proof, let us mention that this immediately implies

**Proposition 7.15.** Under the assumption of the previous proposition, if \( L \subseteq 2\pi L_1 \), we have the equality

\[
S^J = \omega(\mathcal{H}_2)S_L,
\]

which is nothing but the first key lemma.

*Proof.* If \( L \subseteq 2\pi L_1 \), then \( S_0 = S_L \). \( \square \)

To prove Proposition 7.14 recall from Proposition 6.3 that for each \( w \in W \) we have shown \( s[w] \neq 0 \) if and only if \( \psi(-\beta(u^{-1}w-w, w) + \alpha^w(w)) = 1 \) for all \( u \in J^o \) such that \( u^{-1}w \in A + w \).

**Lemma 7.16.** Let \( w \in W \) be such that \( s[w] \neq 0 \). Also let \( x, y \in \mathcal{V}_1 \). Further assume \( x, y \in \mathcal{V}_1^\pm \) and \( L \) is admissible in case \( W \) is given Type 2 polarization. Then if the pair \((x,y)\) satisfies \((i),(ii)\) and \((iv)\) of Proposition 7.6 then

\[
2\langle w(x), w(y) \rangle_2 \in \mathcal{P}_E^r.
\]

(Here \( \mathcal{V}_2 \) does not have to be symplectic.)

*Proof.* Recall that if the pair \((x,y)\) satisfies \((i),(ii)\) and \((iv)\) of Proposition 7.6 then \( u_{x,y} \in J^o \).

Assume \( W \) is given Type 1 polarization. By Lemmas 7.9 and 7.10 one has

\[
\psi(2\text{tr}_{E/F} \langle w(x), w(y) \rangle_2) = 1.
\]

By replacing \((x,y)\) by \((ax,x)\) for any \( a \in O_E^x \), we still have the same, namely

\[
\psi(2\text{tr}_{E/F} \langle w(ax), w(y) \rangle_2) = \psi(2\text{tr}_{E/F} a \langle w(x), w(y) \rangle_2) = 1 \quad \text{for all } a \in O_E^x,
\]

which implies

\[
2\langle w(x), w(y) \rangle_2 \in \mathcal{P}_E^r.
\]

Assume \( W \) is given Type 2 polarization. Note that

\[
\langle w(x), w(y) \rangle_2 = \langle w(x^+), w(y^+) \rangle_2 + \langle w(x^-), w(y^-) \rangle_2 + \langle w(x^+), w(y^-) \rangle_2 + \langle w(x^-), w(y^+) \rangle_2,
\]

and so it suffices to show that 2 times each of those terms is in \( \mathcal{P}_E^r \).

First consider the pair \((x^+, y^-)\). Since \( L \) is admissible, the pair \((x^+, y^-)\) also satisfies the conditions \((i),(ii)\) and \((iv)\) of Proposition 7.6. By Lemma 7.12 we know \( a_{u_{x^+, y^-}} = 0 \). Thus by Lemmas 7.9 and 7.10 one has

\[
\psi(2\text{tr}_{E/F} \langle w(x^+), w(y^-) \rangle_2) = 1.
\]

Arguing as in the previous case, this implies

\[
2\langle w(x^+), w(y^-) \rangle_2 \in \mathcal{P}_E^r.
\]

The case for \((x^-, y^+)\) is the same.

Next consider the pair \((x^+, y^+)\). By Lemma 7.12 we know \( u_{x^+, y^+} \in J^o \). By Lemma 7.11 together with Lemmas 7.9 and 7.10 we conclude that

\[
\psi(-\beta(u^{-1}w-w, w) + \alpha^w(w)) = \psi(4\text{tr}_{E/F} \langle w(x^+), w(y^+) \rangle_2 - 2\text{tr}_{E/F} \langle w(x^+), w(y^+) \rangle_2)
\]

\[
= \psi(2\text{tr}_{E/F} \langle w(x^+), w(y^+) \rangle_2)
\]

\[
= 1.
\]

Arguing as before, this implies

\[
2\langle w(x^+), w(y^+) \rangle_2 \in \mathcal{P}_E^r.
\]

The case for \((x^-, y^-)\) is essentially the same. \( \square \)

Using this lemma, we have
Lemma 7.17. Let $t > 0$ be fixed, and let $w \in W$ be such that $s[w] \neq 0$ and $s[w] \in S^t_1$, so $w$ satisfies the condition $(wt)$. Assume $x, y \in 2\mathcal{W}L_1 \cap L$ and assume further that $x, y \in V_1^\pm$ and $L$ is admissible if $W$ is given Type 2 polarization. Then we have

$$\langle \varpi^{-t} w(x), \varpi^{-t} w(y) \rangle_2 \in \mathcal{P}_{E_t}^{r+2} \subseteq \mathcal{P}_{E_t}^{r+1}.$$  

(Again $V_2$ does not have to be symplectic.)

Proof. First notice that the pair $(\frac{1}{2} \varpi^{-t} x, y)$ satisfies the conditions (i), (ii) and (iv). So by the above lemma, we have $2\{w(\frac{1}{2} \varpi^{-t} x), w(y)\}_2 \in \mathcal{P}_{E_t}^r$, which gives

$$\langle \varpi^{-t} w(x), \varpi^{-t} w(y) \rangle_2 \in \mathcal{P}_{E_t}^{r+1},$$  

where recall $r = r_1 + r_2$. The case for Type 2 polarization can be proven in the same way.  

Now we are ready to prove Proposition 7.14. Though our proof is almost identical to the one given in [MVW] p.118-119, we give details here for the sake of completeness.

Proof of Proposition 7.14 We only give a proof for the case of Type 2 polarization, leaving the Type 1 case to the reader. Hence for what follows, $L$ is admissible.

Since we have $S^t_t = \bigcup_{t \geq 0} S^t_{t+1}$, it suffices to show $S^t_{t+1} \subseteq \omega(H_2)S^t_i$ for $t \geq 1$. From Lemma 7.4 we know that the functions of the form $s[w]$ where $w$ satisfies $(wt)$, i.e. $\varpi^{-t} w(2\varpi L_1 \cap L) \subseteq \varpi^{-t} L_2$, generate the space $S^t_t$. Hence we have to show that for such $s[w]$, there exists $u \in U(V_2)$ such that $\omega(\tilde{u})s[w] \in S^t_{t+1}$, where $\tilde{u}$ is some metaplectic preimage of $u$. (Note that if $\omega(\tilde{u})s[w] \in S^t_{t+1}$, then $\omega(\tilde{u})s[w] \in S^t_{t+2}$ due to the commutativity of the actions of $U(V_1)$ and $U(V_2)$.)

Let $\tilde{X}$ be the image of $\varpi^{-t} w(2\varpi L_1 \cap L)$ in $L_2/\varpi L_2$. Let $x_1, \ldots, x_s \in 2\varpi L_1 \cap L$ be such that the reductions of $\varpi^{-t} w(x_1), \ldots, \varpi^{-t} w(x_s)$ in $L_2/\varpi L_2$ form a basis of $\tilde{X}$. (Here we may assume each $x_i$ is in $(2\varpi L_1 \cap L)^\pm$ for the following reason: If the reductions of $\varpi^{-t} w(x_1), \ldots, \varpi^{-t} w(x_s)$ form a basis of $\tilde{X}$, then the reductions of $\varpi^{-t} w(x_1^0), \ldots, \varpi^{-t} w(x_s^0)$ span $\tilde{X}$, and hence can be reduced to a basis. Moreover since $L$ and $L_1$ are admissible, $x_i^0 \in 2\varpi L_1 \cap L$.)

Now by the above lemma applied to the pair $(x_i, x_j)$, we have

$$\langle \varpi^{-t} w(x_i), \varpi^{-t} w(x_j) \rangle_2 \in \mathcal{P}_{E_t}^{r+2} \subseteq \mathcal{P}_{E_t}^{r+1}$$

for all $i, j \in \{1, \ldots, s\}$. Hence by Lemma 7.7 there exist elements $e_1, \ldots, e_s \in L_2$ and subspaces $X, Y$ of $V_2$ and $Y$ of $V_2$ such that

- $Y = X \oplus V_2$;
- $\{e_1, \ldots, e_s\}$ is a basis of $X$;
- $X$ and $Y$ are totally isotropic;
- $X + Y$ is orthogonal to $V_2$;
- $L_2 = L_X \oplus L_2^Y \oplus L_Y$, where $L_X = X \cap L_2, L_2^Y = V_2 \cap L_2$ and $L_Y = Y \cap L_2$;
- $e_i \equiv \varpi^{-t} w(x_i) \mod \varpi^{t+1} L_2$ for $i \in \{1, \ldots, r\}$.

Let $x \in (2\varpi L_2 \cap L)^\pm$. Since $L$ and $L_1$ are admissible, $x \in 2\varpi L_1 \cap L$. Let us write

$$\varpi^{-t} w(x) = v_X + v^0 + v_Y \in L_X \oplus L_2^Y \oplus L_Y,$$

where $v_X \in L_X, v^0 \in L_2^Y$ and $v_Y \in L_Y$. Since the reduction of $\varpi^{-t} w(x_i)$ is in $\tilde{X}$, the reductions of $v^0$ and $v_Y$ are zero, and in particular $v_Y \in \varpi L_2$. For each $i \in \{1, \ldots, r\}$ we have

$$\langle e_i, v_Y \rangle_2 = \langle e_i, \varpi^{-t} w(x_i) \rangle_2 \quad \text{by the decomposition} \quad L_2 = L_X \oplus L_2^Y \oplus L_Y$$

$$= \langle \varpi^{-t} w(x_i) + \varpi^{t+1} l, \varpi^{-t} w(x) \rangle_2 \quad \text{for some} \ l \in L_2$$

$$\equiv \langle \varpi^{-t} w(x_i), \varpi^{-t} w(x) \rangle_2 \mod \mathcal{P}_{E_t}^{r+1}$$

$$\equiv 0 \mod \mathcal{P}_{E_t}^{r+1} \quad \text{by the above lemma applied to the pair} \ (x_i, x).$$
Since \( \{e_1, \ldots, e_s\} \) is a basis of \( X \) and \( Y \cong \text{Hom}_{O_k}(L_X, \mathbb{F}_k^2) \) via the map \( y \mapsto (-, y)_2 \), we obtain \( v_Y \in \varpi^{t+1}L_Y \). Therefore

\[
\varpi^{t-1} w(2\varpi L_1 \cap L) \subseteq L_X \oplus \varpi L_2^0 \oplus \varpi^{t+1} L_Y.
\]

Now let \( w' \in C(w) \), so we may assume \( w' = uw + a \) for some \( a \in A \) and \( u \in J^0 \). Then

\[
\varpi^{t-1} w'(2\varpi L_1 \cap L) = \varpi^{t-1} (uw + a)(2\varpi L_1 \cap L) = \varpi^{t-1} \left((uw(2\varpi L_1 \cap L) + a(2\varpi L_1 \cap L))\right) = \varpi^{t-1} \left(w(u^{-1}(2\varpi L_1 \cap L)) + a(2\varpi L_1 \cap L)\right) \subseteq L_X + \varpi L_2^0 \oplus \varpi^{t+1} L_Y + 2\pi^{t+1} L_2 \subseteq L_X \oplus \varpi L_2^0 \oplus \varpi^{t+1} L_Y.
\]

Let us define the operator \( u \in U(V_2) \) by

\[
u = \varpi \text{Id}_X \oplus \text{Id}_{V_2} \oplus \varpi^{-1} \text{Id}_Y,
\]

where \( \text{Id}_X, \text{Id}_{V_2} \) and \( \text{Id}_Y \) are the identity operators on the corresponding spaces. (It is clear that \( u \) is indeed in \( U(V_2) \).) Fix a preimage \( \hat{u} \) of \( u \) in \( \hat{U}(V_2) \). Define \( s := \omega(\hat{u})s[w] \in \omega(\hat{U}(V_2))S_1 \). We will show \( s \in S_{l-1} \), which will imply \( s[w] = \omega(s^{-1})s \in \omega(S_2)S_{l-1} \), and will complete the proof. For this, it suffices to show that for all \( w' \in W \) such that \( s(w', 0) \neq 0 \) we have \( \varpi^{-1} w'(2\varpi L_1 \cap L) \subseteq \varpi^{-1} L_2 \). Now from \([13, 1]\) one can see that \( s(w', 0) \) is (a scalar multiple of)

\[
M_u \circ s[w](w', 0) = \int_{\mathbb{A}_A / \mathbb{A}_u} \psi(\beta(a, w'))s[w](u^{-1}(a + w'), \alpha^u(a + w')) \, da.
\]

Hence in order for this integral to be non-zero, we must have

\[
u^{-1}(a + w') \in C(w) \text{ for some } a \in A.
\]

But if \( u^{-1}(a + w') \in C(w) \) as we have shown above, we have

\[
\varpi^{t-1} u^{-1}(a + w')(2\varpi L_1 \cap L) \subseteq L_X + \varpi L_2^0 \oplus \varpi^{t+1} L_Y.
\]

By multiplying \( u \) to both sides, one obtains

\[
\varpi^{t-1}(a + w')(2\varpi L_1 \cap L) \subseteq \varpi L_X \oplus \varpi L_2^0 \oplus \varpi L_Y = \varpi L_2,
\]

which implies

\[
\varpi^{t-1} w'(2\varpi L_1 \cap L) \subseteq \varpi L_2
\]

because \( \varpi^{t-1} a(2\varpi L_1 \cap L) \subseteq \varpi L_2 \). This gives \( \varpi^{-1} w'(2\varpi L_1 \cap L) \subseteq \varpi^{-1} L_2 \). The proposition is proven.

As in Proposition \([13, 1]\) if \( L \subseteq 2\varpi L_1 \), Proposition \([13, 1]\) implies the desired equality \( S^{J^0} = \omega(\mathcal{H}_2)S_L \).

Now, unfortunately it seems that (if the residue characteristic is even) we cannot proceed further than this simply by following the arguments in \([13, 1]\) to show \( S^{J^0} = \omega(\mathcal{H}_2)S_L \) if \( L \nsubseteq 2\varpi L_1 \) for several reasons. For example, the analogue of Lemma II. 7 in \([13, 1]\) cannot be shown for the case of even residual characteristic.
8. A proof of a special case of the second key lemma

In this section, we will give a proof of the second key lemma to the extend necessary to prove our main theorem. Let us start with a couple of lemmas:

**Lemma 8.1.** Let $L \subseteq L_1$ be a lattice. For $w \in B(L)$, $w(\varpi^{-r_2}L_2) + L_1 \neq L^\perp$ if and only if there exists a lattice $L'$ such that $L \subseteq L' \subseteq L_1$ and $w \in B(L')$.

**Proof.** Assume $w(\varpi^{-r_2}L_2) + L_1 \neq L^\perp$. Note that it is always true that $w(\varpi^{-r_2}L_2) + L_1 \subseteq L^\perp$. Hence we assume $w(\varpi^{-r_2}L_2) + L_1 \subseteq L^\perp$. Then we can take $L'$ to be such that

$$L'^\perp = w(\varpi^{-r_2}L_2) + L_1,$$

so $L'^\perp \subseteq L^\perp$, which implies $L \subseteq L'$. But one can see $w \in B(L')$.

Conversely, assume there exists a lattice $L'$ such that $L \subseteq L' \subseteq L_1$ and $w \in B(L')$. Then $L'^\perp \subseteq L^\perp$, which implies $w(\varpi^{-r_2}L_2) + L_1 \subseteq L'^\perp \subseteq L^\perp$, which implies $L'^\perp = w(\varpi^{-r_2}L_2) + L_1$. □

**Lemma 8.2.** For a lattice $L \subseteq L_1$ and $w \in B(L)$, $w(\varpi^{-r_2}L_2) + L_1 = L^\perp$ if and only if $w^{-1}(\varpi^{r_1}L_2) \cap L_1 = L$, where $w^{-1}(\varpi^{r_1}L_2)$ is the inverse image of $\varpi^{r_1}L_2$ when $w$ is viewed as a map $w : \mathcal{V}_1 \to \mathcal{V}_2$.

**Proof.** Assume $w(\varpi^{-r_2}L_2) + L_1 = L^\perp$. If $\{e_i\}_i$ is a basis of $L_2$, then we can write $w = \sum_i m_i \otimes e_i$, where $m_i \in L^\perp$. We then see that

$$w(\varpi^{-r_2}L_2) + L_1 = \text{span} \{m_i\}_i + L_1 = L^\perp.$$  

Now let $l \in w^{-1}(\varpi^{r_1}L_2) \cap L_1$. So $w(l) = \sum_i \langle l, m_i \rangle e_i \in \varpi^{r_1}L_2$, which means $\langle l, m_i \rangle \in \mathcal{P}_{E_1}^r$ for all $i$. Thus for any $x \in \text{span} \{m_i\}_i + L_1 = L^\perp$, we have $\langle x, 1 \rangle \in \mathcal{P}_{E_1}^r$, which implies $l \in L$. The other inclusion $L \subseteq w^{-1}(\varpi^{r_1}L_2) \cap L_1$ is clear because $w \in B(L)$.

Conversely, assume $w^{-1}(\varpi^{r_1}L_2) \cap L_1 = L$. But assume $w(\varpi^{-r_2}L_2) + L_1 \neq L^\perp$. By the above lemma, there exists a lattice $L'$ such that $L \subseteq L'$ and $w \in B(L')$, which implies $L' \subseteq w^{-1}(\varpi^{r_1}L_2) \cap L_1$, and hence $L \subseteq w^{-1}(\varpi^{r_1}L_2) \cap L_1$, which is a contradiction. □

Now we specialize to the orthogonal-symplectic dual pair.

**Proposition 8.3.** Assume the pair $(\mathcal{V}_1, (-,-)_1)$ and $(\mathcal{V}_2, (-,-)_2)$ is such that $\mathcal{V}_2$ is symplectic. We let $\dim \mathcal{V}_1 = m$ and $\dim \mathcal{V}_2 = 2n$. We give $W = \mathcal{V}_1 \otimes \mathcal{V}_2$ Type 1 polarization, i.e. $W^+ = \mathcal{V}_1 \otimes \mathcal{V}_2^+$ and $W^- = \mathcal{V}_1 \otimes \mathcal{V}_2^-$. Assume $L$ is a sublattice of $L_1$ which admits a basis of the form

$$\{e_1, \ldots, e_s, \varpi^{r+s+1}e_{s+1}, \ldots, \varpi^m e_m\}$$

for some $s \in \{0, \ldots, m-1\}$, where $\{e_1, \ldots, e_m\}$ is a basis of $L_1$ and $t_i \geq 1 + \epsilon$ for all $i \in \{r+1, \ldots, m\}$. Let $w \in B(L)$ be such that $w(\varpi^{-r_2}L_2) + L_1 = L^\perp$. Let $w' \in B(L)$ be such that

- $w'(\varpi^{-r_2}L_2) + L_1 = w(\varpi^{-r_2}L_2) + L_1 = L^\perp$;
- $\psi_1^w = \psi_1^{w'}$ as characters of $H_3(L)$.

Then there exists $k \in K_2$ such that $A + w = k(A + w')$.

**Proof.** Though this can be proven simply by modifying the proof of Proposition I.5 of [MWV] p.124-125, we will give the details.

First note that for each $i \in \{1, \ldots, s\}$, we have $w(e_i), w'(e_i) \in L_2$. Hence by adding some elements in $A$ to $w$ and $w'$, respectively, one can assume $w(e_i) = w'(e_i) = 0$ for all $i \in \{1, \ldots, r\}$. Also of course we may assume $t_i \leq t_j$ for $i \leq j$. Let

$$z_i = w(\varpi^{-r_1}e_i) \quad \text{and} \quad z'_i = w'(\varpi^{t_i-r_1}e_i)$$

for all $i \in \{1, \ldots, s\}$.
for each \( i \in \{s+1, \ldots, m\} \). Since \( w, w' \in B(L) \), \( z_i, z_i' \in L_2 \). Moreover, the reductions \( \tilde{z}_i \) in \( L_2/\mathcal{W}L_2 \) are linearly independent over the residue field. To show this, consider

\[
\sum_{i=s+1}^{m} d_i \tilde{z}_i \in \mathcal{W}L_2
\]

for \( d_i \in \mathcal{O} \). Then

\[
\sum_{i=s+1}^{m} d_i \varepsilon^{i-r_i} e_i \in w^{-1}(\mathcal{W}L_2) \cap \varepsilon^{1-r_1} L_1 = \varepsilon^{1-r_1} L,
\]

where we used the above lemma to obtain the equality. Considering the vectors \( \varepsilon^{i-r_i} e_i \) are basis vectors of \( L \), we have \( d_i \in \mathcal{W}\mathcal{O} \). Hence the reductions \( \tilde{z}_i \) are linearly independent over the residue field. Similarly the reductions \( z'_i \) are linearly independent.

Now for \( i, j \in \{s+1, \ldots, m\} \) with \( i \leq j \) (so \( t_i \leq t_j \)), let \( x = \frac{1}{2} e_i \) and \( y = \varepsilon^{t_j-r_j} e_j \). Then the pair \((x, y)\) satisfies the conditions (i) and (iii) of Proposition 7.6, and so \( u_{x,y} \in H_1(L) \). So we have

\[
\psi_1^w(u_{x,y}) = \psi_1^{w'}(u_{x,y}),
\]

i.e.

\[
\psi(-\beta(u_{x,y}^{-1} w - w, w) + \alpha^{x,y}(w)) = \psi(-\beta(u_{x,y}^{-1} w' - w', w') + \alpha^{x,y}(w')).
\]

Also the pair \((x, y)\) satisfies the condition (iv) of Proposition 7.6 so by Lemmas 7.9, 7.10 and 7.11, this gives

\[
\psi(2\langle w(x), w(y) \rangle_2) = \psi(2\langle w'(x), w'(y) \rangle_2).
\]

By replacing \((x, y)\) by \((ax, y)\) for any \( a \in \mathcal{O}^\times \), we still have the same equality, and hence have

\[
2\langle w(x), w(y) \rangle_2 \equiv 2\langle w'(x), w'(y) \rangle_2 \mod \mathcal{P}^r.
\]

Thus we have

\[
\langle w(\varepsilon^{i-r_i} e_i), w(\varepsilon^{t_j-r_j} e_j) \rangle_2 \equiv \langle w' \varepsilon^{i-r_i} e_i, w' \varepsilon^{t_j-r_j} e_j \rangle_2 \mod \mathcal{P}^{r_2 + t_1},
\]

i.e.

\[
\langle z_i, z_j \rangle_2 \equiv \langle z'_i, z'_j \rangle_2 \mod \mathcal{P}^{r_2 + t_1}.
\]

By taking \( m_{ij} = \langle z_i, z_j \rangle_2 \) in Lemma 4.16 there exist elements \( z''_i, \ldots, z''_{i} \in L_2 \) such that

\[
\langle z''_i, z''_i \rangle_2 = \langle z_i, z_j \rangle_2 \quad \text{for all } i \in \{s+1, \ldots, m\};
\]

\[
\langle z''_i, z''_i \rangle_2 = \langle z_i, z_j \rangle_2 \quad \text{for all } i, j \in \{r + 1, \ldots, m\}.
\]

By Lemma 4.8 there exist \( k \in K_2 \) such that \( k z_i = k z''_i \) for all \( i \in \{s+1, \ldots, m\} \). Define \( a \in A \) by

\[
a(e_i) = \begin{cases} 
0 & \text{for all } i \in \{1, \ldots, s\}; \\
\varepsilon^{-r_i} (z''_i - z''_i) & \text{for all } i \in \{s+1, \ldots, m\}.
\end{cases}
\]

Indeed \( a \in A \) thanks to \( z''_i - z''_i \in \mathcal{P}^{r_i} \). One can verify that

\[
(w' + a)(e_i) = kw(e_i) \quad \text{for all } i \in \{1, \ldots, m\},
\]

so \( w' + a = kw \). This completes the proof. \( \square \)

**Remark 8.4.** In the above proof, the fact that the reductions \( \tilde{z}_i \) are linearly independent actually implies that \( \dim \mathcal{V}_2 \geq m - r \). Namely if \( \dim \mathcal{V}_2 < m - r \), simply there is no \( w \) with \( w(L_2) + L_1 = L_2 \). So in particular, if \( L \subseteq 2\mathcal{W}L_1 \), then we must that \( \dim \mathcal{V}_2 \geq \dim \mathcal{V}_1 \).

**Proposition 8.5.** Assume the pair \((\mathcal{V}_1, (-,-)_1)\) and \((\mathcal{V}_2, (-,-)_2)\) is such that \( \mathcal{V}_2 \) is symplectic and \( \dim \mathcal{V}_1 = \dim \mathcal{V}_2 = 2n \). We give \( W = \mathcal{V}_1 \otimes \mathcal{V}_2 \) Type 1 polarization, i.e. \( W^+ = \mathcal{V}_1 \otimes \mathcal{V}_2^+ \) and \( W^- = \mathcal{V}_1 \otimes \mathcal{V}_2^- \). Assume \( L = \mathcal{W}^k L_1 \) for some \( k \geq 1 + e \). Let \( w \in B(L) \) be such that \( w(\mathcal{W}^{-r_2} L_2) + L_1 = L_2 \). Let \( M = M_w = (w(\mathcal{W}^{-r_2} L_1) + L_2)\), so \( w \in B(M) \). Assume \( w' \in W \) is such that
• \( w'(\varpi^{-r_1}L_1) + L_2 = w(\varpi^{-r_1}L_1) + L_2 = M^\perp \);
• \( \psi_2^{w'} = \psi_2^w \) as characters of \( H_2(M) \).

Then there exists \( k \in K_1 \) such that \( A + w = k(A + w') \).

Proof. Assume \( \dim V_1 = \dim V_2 = 2n \). Let \( \{e_1, \ldots, e_{2n}\} \) be a basis of \( L_1 \) dual to \( \{e_1, \ldots, e_{2n}\} \), i.e. \( \langle e_i, e_j \rangle = \varpi^r \delta_{ij} \) for all \( i, j \in \{1, \ldots, 2n\} \), so \( L_1^\perp = \text{span}\{\varpi^{-k}e^*_r, \ldots, \varpi^{-k}e^*_{2n}\} \). Then \( w \) can be written as

\[
 w = \sum_{i=1}^{2n} \varpi^{-k}e^*_i \otimes v_i,
\]

where \( v_i \in L_2 \). Hence

\[
 w(\varpi^{-r_1}L_1) + L_2 = \text{span}_O\{\varpi^{-k}v_1, \ldots, \varpi^{-k}v_{2n}\} + L_2.
\]

We define \( z_i = w(\varpi^k e_i) \) as in the previous lemma. Since the reductions \( \bar{z}_i \) are linearly independent, one can see that \( \{v_1, \ldots, v_{2n}\} \) can be extended to an \( O \)-basis of \( L_2 \) by Lemma 4.4. But because \( \dim O L_1 = 2n \), all \( \{v_1, \ldots, v_{2n}\} \) is an \( O \)-basis of \( L_2 \). Thus we have

\[
 w(\varpi^{-r_1}L_1) + L_2 = \text{span}_O\{\varpi^{-k}v_1, \ldots, \varpi^{-k}v_{2n}\}.
\]

So if we let \( \{v_1, \ldots, v_{2n}\} \) be a basis dual to \( \{v_1, \ldots, v_{2n}\} \), i.e. \( \langle v_i, v_j \rangle = \varpi^r \delta_{ij} \), then

\[
 M = (w(\varpi^{-r_1}L_1) + L_2)^\perp = \text{span}_O\{\varpi^k v_1, \ldots, \varpi^k v_{2n}\} = \varpi^k L_2.
\]

Clearly \( M \) is admissible. Let \( \{f_1, \ldots, f_{2n}\} \) be a symplectic basis of \( V_2 \), so for each \( i \), either \( f_i \in L_2^\perp \) or \( f_i \in L_2^\perp \). Note that

\[
 M = \text{span}_O\{\varpi^k f_1, \ldots, \varpi^k f_{2n}\}.
\]

Now let \( w' \in B(M) \) be as in the proposition and let

\[
 u_i = w(\varpi^k f_i) \quad \text{and} \quad u'_i = w'(\varpi^k f_i).
\]

Let \( i, j \in \{1, \ldots, 2n\} \) be such that \( i < j \). Also it satisfies the condition (iv). By arguing as in the previous proposition, we obtain

\[
 \langle u_i, u_j \rangle_1 = \langle u'_i, u'_j \rangle_1 \mod \mathcal{P}^{r_1+k}.
\]

(Let us mention that in the previous proposition, to use Lemmas 7.10 and 7.12 we did not need to assume \( L \) is admissible because the polarization of \( W \) is of Type 1. But this time, we do need \( M \) to be admissible, because here we switch the roles of \( V_1 \) and \( V_2 \), i.e. “from the point of view of \( V_2^\ast \), the polarization of \( W \) is of Type 2. This is why we have to assume \( L \) is of the form \( \varpi^k L_1 \).”)

Next for \( i = j \in \{1, \ldots, 2n\} \), let \( x = \frac{1}{k} f^*_i \) and \( y = \varpi^{-k} f^*_i \). The pair \((x, y)\) satisfies the conditions (i) and (iii) of Proposition 7.8 and hence \( u_{x,y} \in H_2(M) \). By arguing as before, we obtain

\[
 \langle u_i, u_j \rangle_1 = \langle u'_i, u'_j \rangle_1 \mod \mathcal{P}^{r_1+k+r}.
\]

Hence by Lemma 4.3 there exist \( u''_1, \ldots, u''_{2n} \in L_1 \) such that

\[
 u''_i - u'_i \in \mathcal{P}^{t_i} \quad \text{for all} \quad i \in \{1, \ldots, 2n\}:
\]

\[
 \langle u''_i, u''_j \rangle_1 = \langle u_i, u_j \rangle_1 \quad \text{for all} \quad i, j \in \{1, \ldots, 2n\}.
\]

Let \( k \in K_2 \) be defined by \( kw_i = ku''_i \) for all \( i \in \{1, \ldots, 2n\} \). (Note that here unlike the previous proposition, we do not need Witt’s extension theorem for lattices). Define \( a \in A \) by

\[
 a(e_i) = \varpi^{-t_i}(u''_i - u'_i) \quad \text{for all} \quad i \in \{1, \ldots, 2n\}.
\]

Indeed \( a \in A \) thanks to \( u''_i - u'_i \in \mathcal{P}^{t_i} \). One can verify that

\[
 (w' + a)(e_i) = kw(e_i) \quad \text{for all} \quad i \in \{1, \ldots, 2n\},
\]

so \( w' + a = kw \). This completes the proof. \( \square \)
9. A proof of the main theorem

We are ready to give a proof of our main theorem on the Howe duality conjecture. Throughout this section, we assume that the unramified pair \((V_1, (-, -)_1)\) and \((V_2, (-, -)_2)\) is such that

- \(V_2\) is symplectic;
- \(\dim V_1 = \dim V_2 = 2n\);
- \(W = V_1 \otimes V_2\) is given Type 1 polarization.

Accordingly we write \(U(V_1) = \text{O}(2n)\) and \(U(V_2) = \text{Sp}(2n)\). It is well-known that the group \(\text{O}(2n) \cdot \text{Sp}(2n)\) splits in the metaplectic cover \(\widetilde{\text{Sp}}(W)\), and hence we view it as a subgroup of \(\widetilde{\text{Sp}}(W)\). Also we may consider the Hecke algebra \(H_1\) (resp. \(H_2\)) as the one for \(U(V_1)\) (resp. \(U(V_2)\)) rather than for \(\widetilde{U}(V_1)\) (resp. \(\widetilde{U}(V_2)\)).

Let us make the following definition.

**Definition 9.1.** Fix a self-dual lattice \(L_1 \subseteq V_1\) with respect to a fixed integer \(r_1\). We define the conductor of an irreducible admissible representation \((\pi, V_\pi)\) of \(U(V_1)\) to be the sublattice \(L\) of \(L_1\) such that

- \(V_\pi^{J_1(L)} \neq 0\);
- \(V_\pi^{J_1(L')} = 0\) for all \(L' \subseteq L_1\) with \(L' \supseteq L\).

Since the groups \(J_1(L)\) form a fundamental system of neighborhood of the identity of \(U(V_1)\), every irreducible admissible representation has a conductor. Also note that if \(L = \varpi^k L_1\) for some integer \(k\), then \(J_1(L)\) fits in the exact sequence

\[0 \to J_1(L) \to \text{O}(2n)(\mathcal{O}) \to \text{O}(2n)(\mathcal{O}/\varpi^{2k}\mathcal{O}).\]

Then the main theorem is

**Theorem 9.2.** Assume \((\pi, V_\pi)\) is an irreducible admissible representation of \(U(V_1)\) whose conductor is of the form \(\varpi^k L_1\) with \(k \geq 1 + e\). If \(\Theta_\psi(\pi) \neq 0\), then it has a unique non-zero irreducible quotient.

The rest of the section is devoted to a proof of this theorem. Our proof follows the arguments in [MVW] p.103-106.

Let us start with the following definition: For any lattice \(L \subseteq L_1\), define

\[\Psi(L) := \{\psi^w_1 : w \in B(L)\text{ and } w(\varpi^{-r_2}L_2) + L_1 = L^\perp\}\]

Recall that for each \(w \in B(L)\), \(\psi^w_1\) is a character on \(H_1(L)\). Also we define

\[V_\pi[H_1(L), \psi_1] := \{v \in V_\pi : \pi(h)v = \psi_1(h)v\},\]

i.e. the \(H_1(L)\)-isotypic component of \(\pi\) of type \(\psi_1\). Then we have

**Lemma 9.3.** Let \(L \subseteq 2\varpi L_1\) be a conductor of an irreducible admissible representation \((\pi, V_\pi)\) of \(\text{O}(2n)\). Assume \(\Theta_\psi(\pi) \neq 0\). Also let \(\Psi'(L) \subseteq \Psi(L)\) be the non-empty subset defined by

\[\Psi'(L) = \{\psi_1 \in \Psi(L) : V_\pi[H_1(L), \psi_1] \neq 0\}.

Then

\[V_\pi^{J_1(L)} = \bigoplus_{\psi_1 \in \Psi'(L)} V_\pi[H_1(L), \psi_1].\]

**Proof.** This is Lemma in [MVW] p.104. But we will give a proof for the sake of completeness. First let

\[V' = \bigoplus_{\psi_1 \in \Psi(L)} V_\pi[H_1(L), \psi_1]\]
We must show $V_{\pi}^{J_1(L)} = V'$. But the inclusion $V' \subseteq V_{\pi}^{J_1(L)}$ is clear because the character $\psi^w_{\pi}$ is trivial on $J_1(L)$ for all $w \in B(L)$.

To show the inclusion $V_{\pi}^{J_1(L)} \subseteq V'$, it suffices to show $V_{\pi}^{J_1(L)} \otimes \Theta \psi(\pi) \subseteq V' \otimes \Theta \psi(\pi)$. To do this, we use the key first lemma (Proposition 7.15) and the maximality of the lattice $L$ with respect to the property $V_{\pi}^{J_1(L)} \neq 0$ in the following way: Let $p : S \to \pi \otimes \Theta \psi(\pi)$ be the surjection. Since $V_{\pi}^{J_1(L)} \neq 0$, the map $p$ restricts to a nonzero surjective map

$$p : S^{J_1(L)} \to V_{\pi}^{J_1(L)} \otimes \Theta \psi(\pi).$$

By Proposition 7.15, we have $S^{J_1(L)} = \omega(H_2)S_L$, so we have the surjective map

$$p : \omega(H_2)S_L \to V_{\pi}^{J_1(L)} \otimes \Theta \psi(\pi).$$

Hence the space $V_{\pi}^{J_1(L)} \otimes \Theta \psi(\pi)$ is generated by the vectors of the form $p(s_w)$ for $w \in B(L)$ under the action of the Hecke algebra $H_2$. For each $w \in B(L)$ such that $p(s_w) \neq 0$, if we show $p(s_w) \in V_{\pi}[H_1(L), \psi^w_{\pi}] \otimes \Theta \psi(\pi)$, we will be done because this will imply $V_{\pi}^{J_1(L)} \otimes \Theta \psi(\pi) \subseteq V' \otimes \Theta \psi(\pi)$. Assume to the contrary that $p(s_w) \in V_{\pi}[H_1(L), \psi^w_{\pi}] \otimes \Theta \psi(\pi)$. Then we have $w(\pi^{-r_2}L_2) + L_1 \subseteq L_L$. By Lemma 8.1 there exists a lattice $L'$ such that $L \subseteq L' \subseteq L_1$ and $w \in B(L')$. Then $s_w \in S_{L'} \subseteq S^{J_1(L')}$, which implies $p(s_w) \in V_{\pi}^{J_1(L')} \otimes \Theta \psi(\pi)$. But by the maximality property of $L$, we have $V_{\pi}^{J_1(L')} = 0$. \)

For what follows, we fix $w \in B(L)$ to be such that

- $w(\pi^{-r_2}L_2) + L_1 = L_L$;
- $V_{\pi}[H_1(L), \psi^w_{\pi}] \neq 0$, or equivalently $p(s_w) \neq 0$ where $p : S \to \pi \otimes \Theta \psi(\pi)$ is the surjective map.

Also as we did in the previous section

- $M = M_w := (w(\pi^{-r_2}L_1) + L_2)^{-1}$, so $w \in B(M)$.

Let us note that $\psi^w_{\pi}$ is a character on $H_2(M)$.

For each smooth but not necessarily irreducible representation $(\sigma_1, V_{\sigma_1})$ of $U(V_1)$, we define

$$\nabla_{\sigma_1} := \{ v \in V_{\sigma_1} : \sigma_1(h)v = \psi_{\pi}^w(h)v \ \text{for all} \ h \in H_1(L) \}. $$

Similarly for each smooth representation $(\sigma_2, V_{\sigma_2})$ of $U(V_2)$, we define

$$\nabla_{\sigma_2} := \{ v \in V_{\sigma_2} : \sigma_2(h)v = \psi_{\pi}^w(h)v \ \text{for all} \ h \in H_2(M) \}. $$

For each $i = 1, 2$, let $e_i \in H_i$ be the idempotent defined by

$$e_i(u) = \begin{cases} [K_i : H_i]\psi^w_{\pi}(u)^{-1} & \text{for} \ u \in H_i \\ 0 & \text{for} \ u \in U(V_i) \text{ but } u \notin H_i, \end{cases}$$

where $H_1 = H_1(L)$ and $H_2 = H_2(M)$. Let us put

$$\nabla_i = e_iH_1e_i.$$

**Proposition 9.4.** Let $(\sigma_2, V_{\sigma_2})$ be a (not necessarily irreducible) smooth non-zero representation of $U(V_2)$, and $p : S \to \pi \otimes \Theta \psi(\pi)$ a surjective $U(V_1) \times U(V_2)$-intertwining map. Then $\nabla_2 \neq 0$ and we have the equality

$$\nabla_1 \otimes \nabla_2 = \pi(\nabla_1)p(s_w) = \sigma_2(\nabla_2)p(s_w),$$

where $w$ is as fixed previously.

This lemma corresponds to Lemma [MVWW, p.104]. But since we do not have the $V_2$-analogue of Proposition 7.15, our proof slightly differs from the one in [MVWW]. We need a couple of lemmas to prove the proposition.

**Lemma 9.5.** Let $M = \varpi^kL_2$ with $k \geq 1 + e$ and $N$ be a lattice with $N \subseteq M$. Then $H_2(N) \subseteq H_2(M)$. 


Proof. In what follows, we will construct an element in $H_2(M)$ not in $H_2(N)$. Since $N \subset M$, there exists $v \in L^1_2$ with $\text{ord}_{L_2}(v) = 0$ and $w^k v \in M \setminus N$. There exists an integer $s > 0$ such that $w^{k+s} v \in N$ but $w^{k+s-1} v \notin N$. Then $w^{-k-s} v^* \in N^1 \setminus M^1$, where $v^*$ is the vector dual to $v$, i.e., $\langle v, v^* \rangle_2 = w^{r_2}$.

So $v^* \in L^2_2$ and $\text{ord}_{L_2}(v^*) = 0$. Let $x = \frac{w^{k-r_2}}{2} v$ and $y = v^*$. The pair $(x, y)$ satisfies the conditions (i) and (iii) of Proposition 7.6 (with, of course, $L$ replaced by $M$, $L_2$ and $r_2$, respectively), and hence $u_{x,y}$ exists and $(u_{x,y}-1)M^1 \subseteq L_2$. Also since $x \in L^1_2$ and $y \in L^2_2$, one can see $u_{x,y}$ is in the Siegel Levi of $\text{Sp}(W)$ for our choice of polarization, so $\alpha_{x,y} = 0$. Hence $u_{x,y} \in H_2(M)$. Now

$$u_{x,y} - 1 = 2 c_{x,y} + \text{higher terms},$$

and

$$2 c_{x,y}(w^{-k-s} v^*) = 2(\langle w^{-k-s} v^*, \frac{w^{k-r_2}}{2} v \rangle_2 y + 2(\langle w^{-k-s} v^*, w^{k-r_2} v \rangle_2 \frac{w^{k-r_2}}{2} v$$

$$= \langle w^{-k-s} v^*, \frac{w^{k-r_2}}{2} v \rangle_2 y$$

$$= w^{-s} y \notin L_2.$$

Hence $u_{x,y} \notin H_2(N)$. \hfill $\square$

Lemma 9.6. Assume $M = w^k L_2$. Let $w \in W$ be arbitrary. If $H_2(M)$ acts on $s_w$ via some character, then $w \in B(M)$ i.e. $s_w \in S_M$.

Proof. If $H_2(M)$ acts on $s_w$ via a character, we have $w \in \text{supp}(\omega(h)s_w)$ for all $h \in H_2(M)$, i.e. $\omega(h)s_w(w,0) \neq 0$. But $\omega(h)s_w(w,0) = s_w(h^{-1}w,\alpha^w(w)) \neq 0$, which implies $h^{-1}w \in A + w$. So $(h-1)w \in A$. Now assume $w \notin B(M)$. Then $w w^{-ri} L_1 \notin M^1$. Let $N = (M^1 + w w^{-ri} L_1)_{\perp}^1$, so $N \subset M$ and $w \in B(N)$. Then $(h-1)N^1 \subset L_2$, and so $h \in H_2(N) \forall h \in H_2(M)$, which would imply $H_2(N) = H_2(M)$. By the above lemma, we must have $N = M$. But $N \subset M$, which is a contradiction. So $w \in B(M)$. \hfill $\square$

Now we are ready to prove Proposition 9.4.

**Proof of Proposition 9.4.** First we will show $\nabla_1 \otimes \nabla_{\sigma_2} = \pi(\mathcal{H}_1)p(s_w)$. The argument is the same as in [MVW], but we will repeat it here.

Since $\nabla_{\sigma_1} \subset V_{\sigma_1}^{L_2}$, and we know $S_{\sigma_1}(L) = \omega(\mathcal{H}_2)S_L$ by Proposition 7.15, by taking Lemma 9.3 into account, we can conclude that $\nabla_1 \otimes \nabla_{\sigma_2}$ is generated under the action of $H_2$ by the elements of the form $\omega(s_w)$ with $w' \in B(M)$, $w' w^{-ri} \perp L_1 \perp L^1$ and $\psi_{w'} = \psi_{w}$. By Proposition 8.3 there exists $k \in K_2$ such that $A + w = k(A + w')$, which implies $s_w$ is proportional to $\omega(k)s_{w'}$. Thus $\nabla_1 \otimes V_{\sigma_2}$ is generated by $\omega(s_{w})$ under $H_2$. The group $H_2(M)$ acts on $s_w$ via the character $\psi_{w}^{2}$, and so $\nabla_{\sigma_2} \neq 0$. So we have

$$\nabla_1 \otimes V_{\sigma_2} = \sigma_2(\mathcal{H}_2)p(s_w) = \sigma_2(\mathcal{H}_2 e_2)p(s_w),$$

because $e_2 = 1$ and $\sigma_2(e_2)p(s_w) = p(s_w)$. Considering $\sigma_2(e_2)V_2 = \nabla_{\sigma_2}$, we have

$$\nabla_1 \otimes V_{\sigma_2} = \sigma_2(\mathcal{H}_2 e_2)V_2 = \mathcal{H}_2 p(s_w).$$

Next we will show the equality

$$\nabla_1 \otimes V_{\sigma_2} = \pi(\mathcal{H}_1)p(s_w).$$

Let $w' \in W$ be such that $p(s_{w'}) \neq 0$. (Note that viewed as a representation of the compact group $H_2(M)$, the space $S$ of the Weil representation can be decomposed as $S = S' \oplus \ker p$ for some subspace $S'$, where both $S'$ and $\ker p$ are spaces of representations of $H_2(W)$. Hence we may assume $s_{w'} \in S'$.) If $p(s_{w'}) \in V_{\sigma_1} \otimes V_{\sigma_2}$, then the group $H_2(M)$ has to act on $p(s_{w'})$ via the character $\psi_{w'}$, which implies $H_2(M)$ acts via the character $\psi_{w'}^2$ on $s_{w'}$ because we assume $s_{w'} \in S'$. By Lemma 9.6, $s_{w'} \in S_M$, which implies the space $S_M$ surjects on $V_1 \otimes V_{\sigma_2}$. Hence the space $V_1 \otimes V_{\sigma_2}$ is generated under $H_1$.\hfill $\square$
by \( p(s_w) \) with \( w' \in B(M) \) and \( \psi_w = \psi_w' \). We will show if \( p(s_{w'}) \neq 0 \), then \( w'(w^{-1}L_1) + L_2 = M^\perp \).

Assume \( w'(w^{-1}L_1) + L_2 \not\subseteq M^\perp \), i.e. \( w'(w^{-1}L_1) + L_2 \subseteq M^\perp \). Let \( L' \subseteq (w'(w^{-1}L_2) + L_1^\perp) \). Note that \( w'(w^{-1}L_2) + L_1 \subseteq L^\perp \), i.e. \( L \subseteq L' \). Also for any \( w_0 \in W \) we have the bijection

\[
L_1/(w_0(w^{-1}L_2) + L_1^\perp) \cong (w_0(w^{-1}L_1) + L_2)/L_2.
\]

(See [MVW, Sous-lemme, p.105].) Hence by choosing \( w_0 = w' \) we have

\[
[L_1 : L'] = [w'(w^{-1}L_1) + L_2 : L_2],
\]

and by choosing \( w_0 = w \), we have

\[
[L_1 : L] = [M^\perp : L_2].
\]

But \( w'(w^{-1}L_1) + L_2 \subseteq M^\perp \) by our assumption on \( w' \), so \( [L_1 : L'] < [L_1 : L] \). Hence \( L \subseteq L' \).

Now \( w' \in B(L') \), so \( J_1(L') \) acts trivially on \( s_{w'} \) and hence on \( p(s_{w'}) \). Hence \( V_{\pi\omega_{J_1}(L')} = 0 \). But by our assumption \( V_{\pi\omega_{J_1}(L')} = 0 \) for any \( L' \) with \( L \subseteq L' \), which is a contradiction.

Therefore we have \( w'(w^{-1}L_1) + L_2 = M^\perp \). Using Lemma 9.3 as above, one can see that \( s_{w'} \) is a scalar multiple of \( \omega(k)s_w \) for some \( k \in K_1 \). Arguing as above, one have the desired equality. \( \square \)

Once Proposition 9.4 is proven, the rest follows from the following general fact.

**Lemma 9.7.** Let \( E \) be a complex vector space, and \( A, B \) be subalgebras of the endomorphism algebra \( \text{End}_C(E) \), viewed as a \( C \)-algebra. Assume \( A \) and \( B \) commute pointwise and there exists \( e \in E \) such that \( Ae = Be \). Then \( A \) is the centralizer of \( B \) in \( \text{End}_C(E) \) and vice versa.

**Proof.** This is nothing but Lemma in p.106 of [MVW]. \( \square \)

We apply this lemma as follows. First assume \( \Theta_\psi(\pi) \) has more than two non-zero irreducible quotients, say \( (\pi_2, V_{\pi_2}) \) and \( (\pi_2', V_{\pi_2'}) \). Let \( (\sigma, V_{\sigma}) \) be the representation of \( U(V_2) \) given by \( \sigma = \pi_2 + \pi_2' \) and \( V_{\sigma} = V_{\pi_2} + V_{\pi_2'} \), so we have the surjection

\[
p : S \to V_{\pi} \otimes V_{\sigma}.
\]

By Proposition 9.4, we have

\[
\nabla_{\pi} \otimes V_{\sigma} = \nabla_{\pi} \otimes V_{\pi_2} + \nabla_{\pi_2} \otimes V_{\pi_2'} = \pi(H_1)p(s_w) = \sigma(H_2)p(s_w).
\]

Let

\[
q : \nabla_{\pi} \otimes V_{\sigma} \to \nabla_{\pi} \otimes V_{\pi_2}
\]

be the projection on the first component. Then \( q \) commutes with the action of \( H_1 \). By applying Lemma 9.7 with \( A = \pi(H_1) \), \( B = \sigma(H_2) \), \( E = \nabla_{\pi} \otimes V_{\sigma} \) and \( e = p(s_w) \), one can conclude that \( q \in \pi(H_1) \), i.e. \( q = \pi(\varphi) \) for some \( \varphi \in \tilde{H}_1 \). But then

\[
q(\nabla_{\pi} \otimes V_{\sigma}) = \pi(\varphi)\nabla_{\pi} \otimes V_{\sigma} = \nabla_{\pi} \otimes \nabla_{\pi_2},
\]

which is a contradiction. This completes the proof of the main theorem.

10. Lifting to smaller rank groups

We will close up this paper with the following theorem.

**Theorem 10.1.** Let \( (U(V_1), U(V_2)) \) be an unramified dual pair with \( V_2 \) symplectic. Further assume \( \dim V_1 < \dim V_2 \). (\( \dim V_1 \) can be even or odd.) If \( (\pi, V_{\pi}) \) is an irreducible admissible representation of \( U(V_1) \) with the property that

\[
V_{\pi\omega_{J_1}(L)} = 0
\]

for all \( L \supseteq 2\pi L_1 \), i.e. the conductor is smaller than \( 2\pi L_1 \). Then \( \Theta_\psi(\pi) = 0 \).
Proof. Let $L$ be the conductor of $\pi$, so $L \subseteq 2\varpi L_1$. Assume $\Theta_\psi(\pi) \neq 0$, and let $p : S \to V_\pi \otimes \Theta_\psi(\pi)$ be the surjection. By Proposition 7.15, we know $S_{J_1(L)} = \omega(H_2)^{SL}$, and hence under $p$, the space $\omega(H_2)^{SL}$ surjects on $V_{\pi}\otimes \Theta_\psi(\pi)$. So for some $w \in B(L)$, we have $p(s_w) \neq 0$. The group $H_1(L)$ acts on $p(s_w)$ via the character $\psi^w$. Hence by Lemma 9.3 we must have $w(\varpi^{-v_2}L_2 + L_1 = L^\perp$. (Note that the proof of Lemma 9.3 goes through without requiring $V_1$ be even.) Then the conditions for Proposition 8.3 are satisfied, and hence by reasoning as in the first part of the proof of the proposition, one can see that the images of $w(\varpi^{-v_2}L)$ in $L_2/\varpi L_2$ have to span the $\text{dim} V_1$-dimensional space over the residue field. But this is impossible because $\text{dim} L_2/\varpi L_2 = \text{dim} V_2 < \text{dim} V_1$. (Also see the first remark after Proposition 8.3.) The theorem follows. 

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