On complex Landsberg and Berwald spaces

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Abstract

In this paper, we study the complex Landsberg spaces and some of their important subclasses. The tools of this study are: the Chern-Finsler, Berwald and Rund complex linear connections. We introduce and characterize the class of generalized Berwald and complex Landsberg spaces. The intersection of these spaces gives the so called $G$-Landsberg class. This last class contains two other kinds of complex Finsler spaces: strong Landsberg and $G$-Kähler spaces. We prove that the class of $G$-Kähler spaces coincides with complex Berwald spaces, in $[2]$’s sense, and it is a subclass of the strong Landsberg spaces. Some special complex Finsler spaces with $(\alpha, \beta)$ - metrics offer examples of generalized Berwald spaces. Complex Randers spaces with generalized Berwald and weakly Kähler properties are complex Berwald spaces.

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1 Introduction

The real Landsberg spaces, in particular the real Berwald spaces, have been a major subject of study for many people over the years. In 1926 L. Berwald introduced a special class of Finsler spaces which took his name in 1964. It is known that a real Finsler space is called a Berwald space if the local coefficients of the Berwald connection depend only on position coordinates. An equivalent condition to this is that the Cartan tensor field is $h$ – parallel to the Berwald connection, i.e. $C_{ijk;r} = 0$, where here $';r$' means the horizontal covariant derivative with respect to Berwald connection. In 1934 Cartan emphasized two weak points of the Berwald connection. One is that it is not metrical. Moreover $g_{ij;k} = -2C_{ijk;0}$ and therefore if $C_{ijk;0} = 0$, then it
becomes metrical. However, such a space was called Landsberg by L. Berwald in 1928.

Many great contributions to the geometry of the real Landsberg and Berwald spaces are due to Z. Szabo [26], M. Matsumoto [20], P. Antonelli [7], A. Bejancu [9], Z. Shen [24]. Every Berwald space is a Landsberg space. The converse, has been a long-standing problem, [17, 27, 13].

Part of the general themes from real Finsler geometry about Landsberg and Berwald spaces can be broached in complex Finsler geometry. However, there are sensitive differences comparing to real reasonings, mainly on account of the fact that in complex Finsler geometry there exist two different covariant derivative for the Cartan tensors, $C_{ijk|h}$ and $C_{ijk|\bar{h}}$. Such reason determined T. Aikou, [2], to request in the definition of a complex Berwald space, beside the natural condition $C_{ijk|h} = 0$, the Kähler condition.

Therefore, the same arguments will be taken into account in the definition of a complex Landsberg space. Using some ideas from the real case, related to the Rund and Berwald connections, our aim in the present paper is to introduce and study the complex Landsberg spaces and some of their subclasses.

We associate to the canonical nonlinear connection, with the local coefficients $N^c_{ij} := \partial_j G^i$, two complex linear connections: one of Berwald type $B\Gamma := (N^c_{ij}, L^c_{jk}, L^c_{j\bar{k}}, 0, 0)$ and another of Rund type $R\Gamma := (N^c_{ij}, L^c_{jk}, L^c_{j\bar{k}}, 0, 0)$. In the real case, a Finsler space is Landsberg if the Berwald and Rund connections coincide. But, in complex Finsler geometry the things are considerably more difficult because in general the $B\Gamma$ and $R\Gamma$ connections are not of $(1, 0)$ - type. Moreover, in the complex case alongside of the horizontal covariant derivative, with respect to $B\Gamma$ connection, we have its conjugate. Here, we speak of complex Landsberg space iff $L^c_{jk} = L^c_{j\bar{k}}$ and various characterizations of Landsberg spaces are proved in Theorem 3.1. Further on we define the class of $G$ - Landsberg spaces. It is in the class of complex Landsberg spaces with $\partial_{\bar{k}} G^i = 0$. Theorem 3.2 reports on the necessary and sufficient conditions for a complex Finsler space to be a $G$ - Landsberg space. A reinforcement of the tensorial characterization for a $G$ - Landsberg space gives rise to a subclass of $G$ - Landsberg, namely strong Landsberg iff $C_{\bar{r}\bar{h}|0}^c = 0$ and $C_{r\bar{h}|0}^c = 0$. Other important properties of the strong Landsberg spaces are contained in Theorem 3.3.

Because any Kähler space is a complex Landsberg space, the substitution of the Landsberg condition with the Kähler condition in the definition of the $G$ - Landsberg spaces leads to another subclass of this, called us $G$ - Kähler.
Inter alia in Theorem 3.4 we prove that it coincides with the category of complex Berwald spaces defined by Aikou in [2]. The strong Landsberg spaces are situated somewhere between complex Berwald spaces and $G$-Landsberg spaces.

The complex Berwald spaces were introduced as a generalization of the real case, but in the particular context of Kähler. Therefore an unquestionable extension of these, directly related on the $B\Gamma$ connection, is called by us generalized Berwald space. It is with the coefficients $L^B_{jk}$ depending only on the position $z$. We give some characterizations for the generalized Berwald space, see Theorem 3.6.

An intuitive scheme with the introduced classes of complex Finsler spaces is in the following figure.

![Figure 1: Inclusions](image)

The general theory on generalized Berwald spaces is completed by some special outcomes for the class of complex Finsler spaces with $(\alpha, \beta)$-metrics. We prove that the complex Berwald spaces under assumptions of generalized Berwald and weakly Kähler are complex Berwald, see Theorem 4.3. A class of complex Kropina spaces which are generalized Berwald is distinguished in Theorem 4.4.

The organization of the paper is as follows. In §2, we recall some preliminary properties of the $n$-dimensional complex Finsler spaces, completed with some others needed for our aforementioned study. In §3, we prove the above mentioned Theorems and we establish interrelations among all classes of complex Finsler spaces. In section §4 we produced some family of complex Finsler spaces with $(\alpha, \beta)$-metrics which are generalized Berwald spaces and in particular, complex Berwald spaces.
2 Preliminaries

In this section we will give some preliminaries about complex Finsler geometry with Chern-Finsler, Berwald and Rund complex linear connections. We will set the basic notions (for more see [11, 22]) and we will prove some important properties of these connections.

2.1 Complex Finsler spaces

Let $M$ be a $n$-dimensional complex manifold, $z = (z_k)_{k=1}^n$ be the complex coordinates in a local chart.

The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z_k, \eta_k)_{k=1}^n$. These are changed into $(z'^k, \eta'^k)_{k=1}^n$ by the rules $z'^k = z^k(z)$ and $\eta'^k = \frac{\partial z^k}{\partial z^l} \eta^l$. A complex Finsler space is a pair $(M, F)$, where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

i) $L := F^2$ is smooth on $\tilde{T}'M := T'M \setminus \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda \eta) = |\lambda| F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $(g^i \bar{\eta}^j(z, \eta))$ is positive definite, where $g^i \bar{\eta}^j := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, it means that the indicatrix is strongly pseudo-convex.

Consequently, from iii) we have $\frac{\partial L}{\partial \eta^k} \eta^k = \frac{\partial L}{\partial \bar{\eta}^k} \bar{\eta}^k = L$, $\frac{\partial g^i \bar{\eta}^j}{\partial \eta^k} \eta^k \bar{\eta}^j = 0$ and $L = g^i \bar{\eta}^i \bar{\eta}^j$.

Roughly speaking, the geometry of a complex Finsler space consists of the study of the geometric objects of the complex manifold $T'M$ endowed with the Hermitian metric structure defined by $g^i \bar{\eta}^j$.

Therefore, the first step is to study sections of the complexified tangent bundle of $T'M$, which is decomposed in the sum $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\frac{\partial}{\partial \eta^k}$, and $VT''M$ be its conjugate.

At this point, the idea of complex nonlinear connection, briefly $(c.n.c.)$, is an instrument in ‘linearization’ of this geometry. A $(c.n.c.)$ is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_uT'M$ is locally spanned by $\{ \frac{\partial}{\partial z^k}, N^i_k \frac{\partial}{\partial \eta^i} \}$, where $N^i_k(z, \eta)$ are the coefficients of the $(c.n.c.)$. The pair $\{ \delta := \frac{\partial}{\partial \eta^k}, \delta^*_k := \frac{\partial}{\partial \eta^i} \}$ will be called the adapted frame of the $(c.n.c.)$ which
obey to the change rules \( \delta_k = \frac{\partial \eta^i}{\partial z^k} \delta^i_j \) and \( \dot{\delta}_k = \frac{\partial \varepsilon^i}{\partial z^k} \dot{\delta}^i_j \). By conjugation every-where we have obtained an adapted frame \( \{ \delta_k, \dot{\delta}_k \} \) on \( T''_u(T'M) \). The dual adapted bases are \( \{ dz^k, \delta \eta^k \} \) and \( \{ d \bar{z}^k, \delta \bar{\eta}^k \} \).

Certainly, a main problem in this geometry is to determine a \((c.n.c.)\) related only to the fundamental function of the complex Finsler space \((M, F)\).

The next step is the action of a derivative law \( D \) on the sections of \( T_C(T'M) \). A Hermitian connection \( D \), of \((1,0)\)– type, which satisfies in addition \( D_{fX}Y = JD_XY \), for all \( X \) horizontal vectors and \( J \) the natural complex structure of the manifold, is so called Chern-Finsler connection (cf. [1]), in brief \( C - F \). The \( C - F \) connection is locally given by the following coefficients (cf. [22]):

\[
N^i_j = g^m_{ji} \frac{\partial g_{mj}}{\partial z^j}; \quad L^i_{jk} = g^i_l \dot{\delta}^j_k = \dot{\delta}_j N^i_k + C^i_jk = g^i_l \dot{\delta}^j_k g^j_l, \quad (2.1)
\]

and \( L^j_{jk} = C^j_{jk} = 0 \), where here and further on \( \delta_k \) is the adapted frame of the \( C - F \) \((c.n.c.)\) and \( D_{\delta_k} \delta_j = L^i_{jk} \delta^i, \quad D_{\delta_k} \dot{\delta}_j = C^i_{jk} \dot{\delta}_i \), etc. The \( C - F \) connection is the main tool in this study.

Denoting by “”, “” | “”, “” and “” | “”, the \( h-, v-, \bar{h}-, \bar{v}- \) covariant derivatives with respect to \( C - F \) connection, respectively, it results

\[
\eta^i_k = \eta^i_k = \eta^i_k = 0; \quad \eta^i_k = \delta^i_k; \quad \eta^i|_k = \delta^i_k; \quad (2.2)
\]

\[g_{ij}|_k = g_{ij} = g_{ij}|_k = g_{ij}|_k = 0.
\]

Now, we consider the complex Cartan tensors: \( C_{i\bar{j}k} = \dot{\delta}_k g_{i\bar{j}} \) and \( C_{i\bar{j}k} = \dot{\delta}_k g_{i\bar{j}} \).

**Lemma 2.1.** For any complex Finsler space \((M,F)\), we have

i) \( C_{i\bar{h}i|k} = (\dot{\delta}_h L^i_{ik})g_{i\bar{f}} \);

ii) \( C_{i\bar{h}h|k} = (\dot{\delta}_h L^i_{ik})g_{i\bar{f}} + (\dot{\delta}_h N^i_{k})C_{i\bar{f}l}. \)

**Proof.** Differentiating \( N^i_k g_{i\bar{f}} = \frac{\partial g_{i\bar{f}}}{\partial z^k} \eta^i \) with respect to \( \eta^i \), gives

\[
L^i_{ik} g_{i\bar{f}} = \frac{\partial g_{i\bar{f}}}{\partial z^k} - N^i_k C_{i\bar{f}l}. \quad (2.3)
\]

Now, differentiating in \((2.3)\) with respect to \( \eta^h \) it results i), and with respect to \( \dot{\eta}^h \) leads to ii). \( \square \)

Let us recall that in [1]’s terminology, the complex Finsler space \((M,F)\) is strongly Kähler iff \( T^i_{jk} = 0 \), Kähler iff \( T^i_{jk} g^j = 0 \) and weakly Kähler iff \( g_{i\bar{f}} T^i_{jk} \bar{\eta}^j = 0 \), where \( T^i_{jk} := L^i_{jk} - L^i_{kj} \). In [11] it is proved that strongly Kähler
and Kähler notions actually coincide. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on $M$, so-called purely Hermitian metrics in [22], (i.e. $g_{i\overline{j}} = g_{i\overline{j}}(z)$), all those nuances of Kähler are same.

On the other hand, as in Aikou’s work [2], a complex Finsler space which is Kähler and $L_{i\overline{jk}} = L_{i\overline{jk}}(z)$ is named a complex Berwald space.

### 2.2 Connections on complex Finsler spaces

It is well known by [1, 22] that the complex geodesics curves are defined by means of Chern-Finsler (c.n.c.). Between complex spray and (c.n.c.) there exists an interdependence, one determining the other. In [22] it is proved that the Chern-Finsler (c.n.c.) does not generally come from a complex spray except when the complex metric is weakly Kähler. On the other hand, its local coefficients $N_{k}^{j} = \frac{\partial g_{ik}}{\partial z^{j}}$ always determine a complex spray with coefficients $G_{i}^{j} = \frac{1}{2} N_{ij}^{k} \eta^{k}$. Further, $G_{i}^{j}$ induce a (c.n.c.) denoted by $c_{N}^{i}_{jk} := \dot{\delta}_{j} G_{i}^{k}$ and called canonical in [22], where it is proved that it coincides with Chern-Finsler (c.n.c.) if and only if the complex Finsler metric is Kähler. Using canonical (c.n.c.) we associate to it the next complex linear connections: one of Berwald type

$$B \Gamma := \left( \begin{array}{cc}
\dot{c}_{N_{i}^{j}}, & L_{i}^{j} := \dot{\delta}_{k} N_{i}^{j}\nB_{i}^{j}, & L_{i}^{j} := \dot{\delta}_{k} N_{ik}^{j}, 0, 0
\end{array} \right)$$

and another of Rund type

$$R \Gamma := \left( \begin{array}{cc}
\dot{c}_{N_{i}^{j}}, & L_{i}^{j} := \frac{1}{2} g^{ji} (\delta_{k} g_{j} + \delta_{j} g_{ki})\nB_{i}^{j}, & L_{i}^{j} := \frac{1}{2} g^{ji} (\delta_{k} g_{j} - \delta_{l} g_{kj}), 0, 0
\end{array} \right),$$

where $\delta_{k} := \frac{\partial}{\partial z^{k}} - N_{k}^{l} \delta_{l}$. $R \Gamma$ is only $h-$metrical and $B \Gamma$ is neither $h-$nor $v-$metrical, (for more details see [22]). Note that the spray coefficients perform $2G_{i}^{j} := N_{i}^{j} \eta^{j} = N_{j}^{i} \eta^{i} = L_{i}^{j} \eta^{i} \eta^{k}$ and $\dot{c}_{j} = \delta_{j} - (N_{j}^{k} - N_{i}^{k}) \dot{\delta}_{k}$.

Moreover, in the Kähler case $\delta_{j} = \delta_{j}$, ([22], p. 68) and hence, $\delta_{j} g_{kh} = \delta_{k} g_{j\overline{h}}$ which contracted with $g^{ij}$ gives $g^{ij} (\delta_{j} g_{kh} - \delta_{k} g_{j\overline{h}}) = 0$, that is $L_{i}^{j} = 0$. By conjugation, it follows $L_{i}^{j} = 0$. Also, in the Kähler case we have $L_{i}^{j} = L_{i}^{j} = L_{j}^{i} \overline{k}$. Further on, everywhere in this paper the Berwald and Rund connections will be specified by a super-index, like above (e.g. $\dot{c}_{k}, L_{i}^{j}, L_{i}^{j} \overline{k}, X_{B}^{i} \overline{k}, etc.$),
while for the Chern-Finsler connection will keep the initial generic notation without super-index (e.g. $\delta_k$, $L^i_{jk}$, $X^i_{[k]}$, etc.).

**Lemma 2.2.** For any complex Finsler space $(M,F)$, the following hold:

i) $L^i_{jh} \tilde{\eta}^k = 0$;

ii) $-C_{\bar{v}h|0}^i n = g_{\bar{v}h|0} + g_{\bar{v}h|l} + L^i_{\bar{v}h} g_{ln} + L^i_{\bar{v}l} g_{hm}$;

iii) $2(\partial_h G^i) g_{\bar{v}r} = C_{\bar{v}h|0}^i = 0$;

iv) $C_{\bar{v}h|k}^i g_{\bar{v}l} = \tilde{\partial}_h (g_{\bar{v}l} g_{\bar{v}k}) + (\tilde{\partial}_h L^i_{\bar{v}k}) g_{\bar{v}l} + (\tilde{\partial}_l L^i_{\bar{v}k}) g_{\bar{v}m} + L^i_{\bar{v}l} C_{\bar{v}m} - L^i_{\bar{v}h} C_{\bar{v}m}$, where $i$ is $h$-covariant derivative with respect to $B\Gamma$.

**Proof.** i) $L^i_{jh} \tilde{\eta}^k = \frac{\partial^2 G^i}{\partial \tilde{\eta}^i \partial \eta^k} \tilde{\eta}^k = \tilde{\partial}_j [\tilde{\partial}_i (\tilde{\partial}_h G^i) \tilde{\eta}^k] = \tilde{\partial}_j [\tilde{\partial}_i (\frac{1}{2} N^i_{jk}) \tilde{\eta}^k]$

$= \frac{1}{2} \tilde{\partial}_j [\tilde{\partial}_i (N^i_{jk}) \eta^k] = \frac{1}{2} \tilde{\partial}_j [\tilde{\partial}_i (g_{\bar{v}l} \frac{\partial g_{mz}}{\partial x^k} \eta^k) \tilde{\eta}^k] = 0$.

ii) $G^i = \frac{1}{2} g^{ij} \eta^j \eta^k$ can be rewrite as follows

$$G^i_{\bar{v}r} = \frac{1}{2} \frac{\partial g_{\bar{v}r}}{\partial \bar{z}^k} \eta^j \eta^k. \quad (2.4)$$

Differentiating $(2.4)$ with respect to $\eta^i$ yields

$$N^i_{\bar{v}r} g_{\bar{v}r} = \frac{1}{2} \left( \frac{\partial g_{\bar{v}r}}{\partial \bar{z}^k} + \frac{\partial g_{\bar{v}r}}{\partial \bar{z}^l} \right) \eta^k - G^i C_{\bar{v}r}. \quad (2.5)$$

Differentiating $(2.5)$ with respect to $\eta^h$ it results

$$B^i_{\bar{v}h} g_{\bar{v}r} = \frac{1}{2} \left( \frac{\partial g_{\bar{v}r}}{\partial \bar{z}^h} + \frac{\partial g_{\bar{v}r}}{\partial \bar{z}^l} \right) + \frac{1}{2} \frac{\partial C_{\bar{v}h}}{\partial \bar{z}^k} \eta^k - G^i (\tilde{\partial}_h C_{\bar{v}r}) - N^i_{\bar{v}r} C_{\bar{v}r} - N^i_{\bar{v}h} C_{\bar{v}r}, \quad (2.6)$$

which leads to

$$- (\tilde{\partial}_h (C_{\bar{v}h}) \eta^k + N^i_{\bar{v}r} C_{\bar{v}r} + N^i_{\bar{v}h} C_{\bar{v}r} = \delta^c_{\bar{v}h} g_{\bar{v}r} + \delta_{\bar{h}} g_{\bar{v}r} - 2 B^i_{\bar{v}h} g_{\bar{v}r}. \quad (2.7)$$

Now, taking into account that $g_{\bar{v}h|0} = \delta^c_{\bar{v}h} g_{\bar{v}r} - L^i_{\bar{v}h} g_{\bar{v}r} - L^m_{\bar{v}h} g_{\bar{m}r}$ and i) it results ii).

iii) Differentiating $(2.4)$ with respect to $\eta^h$ yields

$$2(\partial_h G^i) g_{\bar{v}r} = \frac{\partial C_{\bar{v}h}}{\partial \bar{z}^k} \eta^l \eta^k - N^i_{\bar{v}h} \eta^k C_{\bar{v}r}. \quad (2.8)$$
But, \( \hat{\partial}_h(C_{is\bar{h}}\eta^j) = (\hat{\partial}_h C_{is\bar{h}})\eta^j + C_{is\bar{h}} = \hat{\partial}_h(C_{i\bar{a}}\eta^j) + C_{is\bar{h}} = C_{is\bar{h}} \). Using (2.8) and \( N_j^i\eta^j = N_j^i \eta^j \) we obtain \( 2(\hat{\partial}_h G^i)g_{ij} = \delta_k (C_{i\bar{a}}\eta^j)\eta^k = \delta_k (C_{i\bar{a}}\eta^j)\eta^k \), which together with i) and the \( h \) - covariant derivative rule with respect to \( C - F \) connection gives iii).

iv) Using again \( g_{ij|k}^c = \delta_k g_{ij} - L^B_{ik} g_{ij} - L^B_{jk} g_{im} \), which differentiated with respect to \( \eta^h \), gives

\[
\hat{\partial}_h(g_{ij|k}^c) = \frac{\partial g_{ij|k}^c}{\partial z^B} - L^B_{kk} C_{ijl} - N^l_k (\hat{\partial}_h C_{ijl}) - (\hat{\partial}_h L^B_{ik}) g_{ij} - L^B_{ik} C_{ijl} - (\hat{\partial}_h L^B_{jk}) g_{im} - L^B_{jk} C_{ijm}.
\]

For v) we compute

\[
\hat{\partial}_h(g_{ij|k}^c) = \frac{\partial g_{ij|k}^c}{\partial z^B} - L^B_{kk} C_{ijl} - N^l_k (\hat{\partial}_h C_{ijl}) - (\hat{\partial}_h L^B_{ik}) g_{ij} - L^B_{ik} C_{ijl} - (\hat{\partial}_h L^B_{jk}) g_{im} - L^B_{jk} C_{ijm} - (\hat{\partial}_h L^B_{ik}) g_{ij} - \hat{\partial}_h L^B_{jk} g_{im}.
\]

\[\Box\]

3 The complex Landsberg spaces

In real Finsler geometry the classes of Landsberg and Berwald spaces are related by Rund and Berwald connections, [10]. Namely, a real Finsler space is Landsberg if the Berwald and Rund connections coincide. Nevertheless, in complex Finsler geometry some differences appear. We will speak about of three kinds of complex spaces of Landsberg type.

**Definition 3.1.** Let \((M, F)\) be a \( n \) - dimensional complex Finsler space. \((M, F)\) is called complex Landsberg space if \( L^B_{jk} = L^c_{jk} \).

We remark that any complex Finsler space which is Kähler is a Landsberg space too, because if \((M, F)\) is Kähler, then \( L^B_{jk} = L^c_{jk} \) and so it is
Thus, the Kähler spaces offer a asset family of complex Landsberg spaces.

**Theorem 3.1.** Let $(M, F)$ be a $n$-dimensional complex Finsler space. Then the following assertions are equivalent:

i) $(M, F)$ is a complex Landsberg space;

ii) $C_{i \bar{r} h}^B = 0$;

iii) $2(\partial_i^B \bar{L}^i_{jk}) g_{i \bar{r}} - L^m_{\bar{r}k} C_{j \bar{m} h} - L^m_{\bar{r}j} C_{k \bar{m} h} = C_{j \bar{r} h}^B + C_{k \bar{r} h}^B$;

iv) $g_{i \bar{r}} n_j = (L^m_{jk} - \bar{L}^m_{jk}) g_{i \bar{m}}$.

**Proof.** i) $\Leftrightarrow$ ii). A direct computation gives

$$g_{i \bar{r}} n_j = g_{i \bar{r}} n + L^m_{\bar{r}k} g_{i \bar{m}} + \bar{L}^m_{\bar{r}j} g_{i \bar{m}} = 2(L_{i \bar{r}} g_{i \bar{r}} - L_{i \bar{r}} g_{i \bar{r}})$$

which, together with Lemma 2.2 ii), get the proof.

i) $\Rightarrow$ iii). Because $(M, F)$ is Landsberg we have

$$L^B_{i \bar{r}} g_{i \bar{r}} = \frac{c}{2}(\delta_k g_{i \bar{r}} + \delta_j g_{i \bar{r}}).$$

Differentiating it with respect to $\eta^h$ yields iii).

iii) $\Rightarrow$ ii). Contracting iii) with $\eta^k$ gives $2(\partial_i^B \bar{L}^i_{jk}) g_{i \bar{r}} \eta^k = C_{j \bar{r} h}^B$. On the other hand, $(\partial_i^B \bar{L}^i_{jk}) g_{i \bar{r}} \eta^k = 0$. From here it results ii).

i) $\Leftrightarrow$ iv). $g_{i \bar{r}} n_j = \delta_k g_{i \bar{r}} - \bar{L}^i_{ik} g_{i \bar{r}} - L^m_{\bar{r}j} g_{i \bar{m}}$

$$= g_{i \bar{r}} n_j + (L^i_{ik} - \bar{L}^i_{ik}) g_{i \bar{r}} + (L^m_{\bar{r}j} - \bar{L}^m_{\bar{r}j}) g_{i \bar{m}},$$

where $\Gamma$ is $h-$covariant derivative with respect to $R \Gamma$ connection. But, $R \Gamma$ connection is $h-$metrical, therefore $g_{i \bar{r}} n_j = (L^i_{ik} - \bar{L}^i_{ik}) g_{i \bar{r}} - (L^m_{\bar{r}j} - \bar{L}^m_{\bar{r}j}) g_{i \bar{m}},$ which justifies this equivalence.

**Definition 3.2.** Let $(M, F)$ be a $n$-dimensional complex Finsler space. $(M, F)$ is called $G$-Landsberg space if it is Landsberg and the spray coefficients $G^i$ are holomorphic in $\eta$, i.e. $\partial_k G^i = 0$.

Some immediately consequences follow below.
Proposition 3.1. If \((M,F)\) is a \(G\) - Landsberg space then the connection 
\(B\Gamma\) is of \((1,0)\) - type.

Corollary 3.1. \(G^i\) are holomorphic in \(\eta\) if and only if the connection \(B\Gamma\) is of \((1,0)\) - type.

Proposition 3.2. \(G^i\) are holomorphic in \(\eta\) if and only if \(L^i_{jk}\) depend only on \(z\).

Proof. If \(G^i\) are holomorphic functions in \(\eta\), then 
\[\dot{\partial}_k G^i = 0\] 
which leads to 
\[\dot{\partial}_k N^i_h = 0\] 
and more 
\[\dot{\partial}_k L^i_{jh} = 0\]. Hence, the functions \(L^i_{jh}\) are holomorphic in \(\eta\), too.

Now, we make a similar reasoning like that in [12], Proposition 1.1, but for the 0−homogeneous functions \(L^i_{jh}\) in \(\eta\). We consider 
\[D_\varepsilon := \{\eta \in T^*_z M \mid G(\eta, \bar{\eta}) < \varepsilon^2, \varepsilon > 0\}\] 
and we study the functions \(L^i_{jh}\) on the domain 
\[D_\varepsilon \setminus D_{\frac{1}{\varepsilon}}\]. Because these functions are 0−homogeneous, their modulus achieve maximum at an interior point of 
\[D_\varepsilon \setminus D_{\frac{1}{\varepsilon}}\]. Thus, we can apply the strong maximum principle which gives that the functions \(L^i_{jh}\) are constant with respect to \(\eta\) on 
\[D_\varepsilon \setminus D_{\frac{1}{\varepsilon}}\]. Now, making \(\varepsilon \to \infty\) it results the functions \(L^i_{jk}\) depend only on \(z\), too. Thus, globally we have \(L^i_{jk}(z)\).

Conversely, if \(L^i_{jk}(z)\) then 
\[\dot{\partial}_k L^i_{jh} = 0\], which contracted by \(\eta^i \eta^h\) complete the proof.

\hfill \Box

Corollary 3.2. \(L^i_{jk}\) depend only on \(z\) if and only if 
\[\dot{\partial}_k L^i_{jh} = 0\].

Proof. It is obvious that \(L^i_{jk}\) depend only on \(z\) implies 
\[\dot{\partial}_k L^i_{jh} = 0\]. Conversely, if \(\dot{\partial}_k L^i_{jh} = 0\), by conjugation we have 
\[\dot{\partial}_k L^i_{jh} = 0\], i.e. \(L^i_{jh}\) are holomorphic in \(\eta\). But, the functions \(L^i_{jh}\) are also 0−homogeneous and so, by same arguments as in the above Proposition it results that 
\[\dot{\partial}_k L^i_{jh} = 0\], and by conjugation 
\[\dot{\partial}_k L^i_{jh} = 0\]. Applying again above Proposition we obtain \(L^i_{jk}(z)\).

\hfill \Box
Theorem 3.2. Let \((M, F)\) be a \(n\)-dimensional complex Finsler space. Then the following assertions are equivalent:

i) \((M, F)\) is a \(G\)-Landsberg space;

ii) \(L^c_j^k_{j,j} = L^c_j^k_z(z)\);

iii) \(C_{\bar{k}h|0} = 0\) and \(C_{\bar{j}h|0} = 0\);

iv) \(g_{0\bar{j}h} = L^c_j^m g_{jm} \) and \(\hat{\eta}_h G^i = 0\).

v) \(C_{\bar{j}k|0} + C_{\bar{k}h|0} = 0\) and \(C_{\bar{j}h|0} + C_{\bar{k}h|0} = 0\).

Proof. i) \(\Leftrightarrow\) ii) is obtained by Proposition 3.1. i) \(\Leftrightarrow\) iii) results by Lemma 2.2 iii) and Theorem 3.1 ii). Under assumptions \(\hat{\eta}_h G^i = 0\), the equivalence i) \(\Leftrightarrow\) iv) from Theorem 3.1 gets the proof for i) \(\Leftrightarrow\) iv).

i) \(\Rightarrow\) v) If \((M, F)\) is a \(G\)-Landsberg space then by Lemma 2.2 ii) and v) we have \(g_{0\bar{j}h} = L^c_j^m g_{jm} \) and \(\hat{\eta}_h G^i = 0\), so that

\[
C_{\bar{j}k|0} + C_{\bar{k}h|0} = \hat{\eta}_h (g_{0\bar{j}h} + g_{0\bar{k}h}) = 0
\]
and by conjugation it leads to

\[
C_{\bar{j}h|0} + C_{\bar{k}h|0} = 0.
\]

Now, using Lemma 2.2 iv) and Proposition 3.2 it results \(C_{\bar{j}k|0} + C_{\bar{k}h|0} = 0\).

v) \(\Rightarrow\) i) First, contracting with \(\eta^k\) in the identity \(C_{\bar{j}h|0} + C_{\bar{k}h|0} = 0\) it results \(C_{\bar{j}h|0} = 0\), i.e. the space is Landsberg. On the other hand the contraction by \(\eta^j\) of the identity \(C_{\bar{j}h|0} + C_{\bar{k}h|0} = 0\) gives \(2C_{\bar{j}h|0} = 0\) and by conjugation it is \(2C_{\bar{k}h|0} = 0\). But, by Lemma 2.2 iii) we have \(2(\hat{\eta}_h G^i)g_{0\bar{j}h} = C_{\bar{j}h|0}\) from here we obtain \(\hat{\eta}_h G^i = 0\), which completes the proof.

Now, having in mind the tensorial characterization iii) from Theorem 3.2 for a \(G\)-Landsberg space, it give rise to another class of complex Landsberg spaces.

Definition 3.3. Let \((M, F)\) be a \(n\)-dimensional complex Finsler space. \((M, F)\) is called a strong Landsberg space if \(C_{\bar{j}h|0} = 0\) and \(C_{\bar{k}h|0} = 0\).

Theorem 3.3. Let \((M, F)\) be a \(n\)-dimensional complex Finsler space. Then the following assertions are equivalent:

i) \((M, F)\) is a strong Landsberg space;

ii) \(g_{0\bar{j}h}(z)\) and \(\hat{\eta}_h G^i = 0\);

iii) \(C_{\bar{j}h|0} = 0\) and \(\hat{\eta}_h G^i = 0\);

iv) \(C_{\bar{j}h|0} = 0\).
Proof. i) \(\Rightarrow\) ii). If \((M,F)\) is a strong Landsberg space, then by Theorem 3.2 iii) it is \(G\)-Landsberg. Therefore, Lemma 2.2 iv) and v) become

\[
C_{ijh|k}^B = \hat{\partial}_h(g_{ij|k}^B) \quad \text{and} \quad C_{\ell r k|j}^B = \hat{\partial}_h(g_{\ell r|k}^B),
\]

which contracted by \(\eta^k\) implies

\[
0 = \hat{\partial}_h(g_{ij|k}^B)\eta^k; \quad 0 = \hat{\partial}_h(g_{\ell r|k}^B)\eta^k.
\]

(3.1)

Differentiating the second equation of (3.1) with respect to \(\eta^s\), it yields

\[
0 = \hat{\partial}_\ell (g_{\ell r|k}^B)\eta^k + \hat{\partial}_h(g_{\ell r|s}^B).
\]

Now, using the first relation from (3.1) it results

\[
\hat{\partial}_h(g_{\ell r|k}^B) = 0.
\]

Because \(g_{\ell r|s}^B\) are holomorphic with respect to \(\eta\) and homogeneous of zero degree, these give that \(g_{\ell r|s}^B\) depends on \(z\) only, i.e. \(g_{\ell r|s}^B(z)\).

Now, the conditions \(g_{\ell r|s}^B(z)\) and \(\hat{\partial}_h G^i = 0\) substituted into Lemma 2.2 iv), gives \(C_{ijh|k}^B = 0\). So that, we have proved ii) \(\Rightarrow\) iii).

To prove iii) \(\Rightarrow\) iv) we use again Lemma 2.2 iv). Under assumptions iii), it is \(\hat{\partial}_h(g_{ij|k}^B) = 0\) and by conjugation \(\hat{\partial}_h(g_{ji|k}^B) = 0\). This means that \(g_{ij|k}^B\) is holomorphic in \(\eta\) which together its 0-homogeneity implies \(g_{ij|k}^B(z)\) and so its conjugate \(g_{ji|k}^B\) depends on \(z\) only. Therefore, v) from Lemma 2.1 leads to \(C_{ijh|k}^B = 0\), i.e. iv).

The proof is complete if we show that iv) \(\Rightarrow\) i). Indeed, \(C_{ijh|k}^B = 0\) implies \(C_{ijh|k}^B = 0\) and \(\hat{\partial}_h G^i = 0\), by Lemma 2.1 iii).

Lemma 2.2 v) gives \(\hat{\partial}_h(g_{ij|k}^B) = 0\) and so \(g_{ij|k}^B(z)\). Thus, by Lemma 2.2 iv) we obtain \(C_{ijh|k}^B = 0\), which contracted by \(\eta^k\) yields \(C_{ijh|k}^B = 0\). So that the space is strong Landsberg.

\(\square\)

Remark 3.1. By Theorem 3.2 iv) and Theorem 3.3 ii) it results that a \(n\)-dimensional complex Finsler space is strong Landsberg if and only if \((L_{\ell j}^m g_{\ell mn})(z)\) and \(\hat{\partial}_h G^i = 0\).

Now recall that according to Aikou, [2], a complex Berwald space is a Finsler space which is Kähler and \(L_{\ell j}^m = L_{ij}(z)\).

Having in the mind that any Kähler complex Finsler space is Landsberg, we can introduce another generalization for the \(G\)-Landsberg spaces. So, by replacing the Landsberg condition from definition of the \(G\)-Landsberg space with the Kähler condition we obtain:
Definition 3.4. Let \((M, F)\) be a \(n\) - dimensional complex Finsler space. \((M, F)\) is called \(G\) - Kähler space if it is Kähler and the spray coefficients \(G^i\) are holomorphic in \(\eta\).

Some necessary and sufficient conditions for \(G\) - Kähler spaces are contained in the next theorem.

Theorem 3.4. Let \((M, F)\) be a \(n\) - dimensional complex Finsler space. Then the following assertions are equivalent:

\(\text{i)}\) \((M, F)\) is \(G\) - Kähler;
\(\text{ii)}\) \(L^c_{jk} = L^B_{jk} = 0\);
\(\text{iii)}\) \(L^c_{jk} = L^B_{jk}(z)\);
\(\text{iv)}\) \((M, F)\) is a complex Berwald space;
\(\text{v)}\) \(g \eta_{ij\mid k} = 0\) and \(\partial_k G^i = 0\).

Proof. i) \(\iff\) ii). If \((M, F)\) is \(G\) - Kähler then \(L^c_{jk} = 0\) and \(L^B_{jk} = 0\) which imply \(L^c_{jk} = L^B_{jk} = 0\). Conversely, if \(L^c_{jk} = L^B_{jk}\) then \(\partial_k N^j = 0\) and \(\delta_k g^i_{j\mid k} = 0\) which imply that \((M, F)\) is Kähler, as well as \(\partial_k N^j = 0\). The contraction of \(\partial_k N^i = 0\) by \(\eta^j\) gives \(\partial_k G^i = 0\), i.e. \(G^i\) does not depend on \(\eta^j\).

Taking into account \((M, F)\) is Kähler if and only if \(L^c_{jk} = L^B_{jk} = 0\), and using Propositions 3.2, it follows the proof for i) \(\iff\) iii) and i) \(\iff\) iv).

i) \(\iff\) v). It is obvious that if \((M, F)\) is \(G\) - Kähler then \(g \eta_{ij\mid k} = g_{ij\mid k} = 0\). Conversely, if \(g \eta_{ij\mid k} = 0\) and \(\partial_k G^i = 0\), then \(L_{jk}^c = L_{jk}^B = 0\). But, \(L_{jk}^c = L_{jk}^B\) which implies \(\partial_k g^i_{j\mid k} = \partial_k g_{j\mid k}\) as well as \(\partial_k g^i_{j\mid k} = 0\), i.e. Kähler. \(\square\)

An immediately consequence of above Theorem follows.

Proposition 3.3. \((M, F)\) is a complex Berwald space if and only if the connections \(BT\) and \(RT\) are of \((1, 0)\) - type.

Lemma 3.1. For any complex Finsler space \((M, F)\), \(C_{\alpha\beta k} = 0\) if and only if \(C_{\alpha\beta k} = 0\).
Proof. If $C_{lrhl} = 0$, then Lemma 2.1 i) induces $\hat{\partial}_h L^i_{jk} = 0$ and its conjugated $\hat{\partial}_l L^i_{lk} = 0$. This means that $L^i_{jk}$ are holomorphic in $\eta$, which together with their 0 - homogeneity gives $L^i_{jk}(z)$. Thus, by Lemma 2.1 ii) it results $\hat{\partial}_h L^i_{lk} = 0$. Conversely, if $C_{lrhl} = 0$ then ii) from Lemma 2.1 becomes $(\hat{\partial}_h L^i_{lk})_{\eta^i} + (\hat{\partial}_l N^i_k)_{C_{lrhl}} = 0$. It contracted by $\eta^j$ gives $\hat{\partial}_h N^i_k = 0$ and so, $\hat{\partial}_h L^i_{lk} = 0$ which implies $L^i_{jk}(z)$. Now, using i) from Lemma 2.1, we obtain $C_{lrhl} = 0$. 

We note that $C_{lrhl} = 0$ or $C_{lrhl} = 0$ implies $\hat{\partial}_h G^i = 0$, but the converse is not true. $\hat{\partial}_h G^i$ is 0 together with the Kähler condition gives $C_{lrhl} = 0$ or $C_{lrhl} = 0$. Therefore, some tensorial characterizations for complex Berwald spaces are contained in the next theorem.

Theorem 3.5. $(M, F)$ is a complex Berwald space if and only if it is Kähler and either $C_{lrhl} = 0$ or $C_{lrhl} = 0$.

In the remainder of this section we return to the notion of the real Berwald space, [9]. It is a real Finsler space for which the coefficients of the (real) Berwald connection depend only on the position. Our problem is to see whether there exist a corespondent of this real assertion in complex Finsler spaces. Taking into account Theorem 3.4 we have $L^i_{jk}(z)$, for any complex Berwald space. Nevertheless the converse is not true, will see below there are complex Finsler spaces with $L^i_{jk}$ depending only on $z$ which are not Berwald.

Therefore, it comes into view another class of complex Finsler spaces.

Definition 3.5. Let $(M, F)$ be a $n$ - dimensional complex Finsler space. $(M, F)$ is called generalized Berwald if the Berwald connection coefficients $L^i_{jk}$ depend only on the position $z$.

Using Corollary 3.1 and Proposition 3.2, we have proved the following.

Theorem 3.6. Let $(M, F)$ be a $n$ - dimensional complex Finsler space. Then the following assertions are equivalent:

i) $(M, F)$ is generalized Berwald;

ii) $G^i$ are holomorphic in $\eta$;

iii) $B \Gamma$ is of $(1, 0)$ - type.

Corollary 3.3. If $(M, F)$ is a complex Finsler space with the $C - F$ coefficients $L^i_{jk}$ depending only on $z$, then the space is generalized Berwald.

Proof. We have $\hat{\partial}_h L^i_{jk} = 0$, which contracted by $\eta^j \eta^k$ gives $\hat{\partial}_h G^i = 0$. 

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An example of generalized Berwald space is given by the complex version of Antonelli - Shimada metric

$$F_\text{AS}^2 = L_{\text{AS}}(z, w; \eta, \theta) := e^{2\sigma (|\eta|^4 + |\theta|^4)^{\frac{1}{2}}},$$

with $\eta, \theta \neq 0$, \( (3.2) \)
on a domain $D$ from $\tilde{T}^{\prime}M$, $\dim_CM = 2$, such that its metric tensor is non-degenerated. We relabeled the local coordinates $z^1, z^2, \eta^1, \eta^2$ as $z, w, \eta, \theta$, respectively. $\sigma(z, w)$ is a real valued function and $|\eta^i|^2 := \eta^i\overline{\eta}^i$, $\eta^i \in \{\eta, \theta\}$, \( (22) \). A direct computation leads to

$$L_{zz}^z = L_{ww}^w = 2\frac{\partial \sigma}{\partial z}; \quad L_{zw}^z = L_{ww}^w = 2\frac{\partial \sigma}{\partial w},$$

which depend only on $z$ and $w$.

Summing up all the results proved above we have the inclusions from Fig. 1.

The intersection between the set of Landsberg spaces and those of the generalized Berwald spaces gives the class of $G$ - Landsberg spaces.

Trivial examples of such spaces are given by the purely Hermitian and local Minkowski manifolds. In the next section we came with some nice families of generalized Berwald spaces.

4 Generalized Berwald spaces with $(\alpha, \beta)$ - metrics

We consider $z \in M$, $\eta \in T^\prime_zM$, $\eta = \eta^i \frac{\partial}{\partial z^i}$, $\tilde{a} := a_{ij}(z)dz^i \otimes d\bar{z}^j$ a purely Hermitian metric and $b = b_i(z)dz^i$ a differential $(1, 0)-$ form. By these objects we have defined (for more details see [3, 4]) the complex $(\alpha, \beta)-$ metric $F$ on $T^\prime M$

$$F(z, \eta) := F(\alpha(z, \eta), |\beta(z, \eta)|),$$

where

$$\alpha(z, \eta) := \sqrt{a_{ij}(z)\eta^i \overline{\eta}^j} \quad \text{and} \quad \beta(z, \eta) = b_i(z)\eta^i.$$  \( (4.4) \)

Let us recall that the coefficients of the $C - F$ connection corresponding to the purely Hermitian metric $\alpha$ are

$$N_{jk}^i := \alpha^m_k \frac{\partial a_{lm}}{\partial z^j} \eta^l; \quad L_{jk}^i := \alpha^i_{\bar{j}a} (\delta_k^a a_{j\bar{a}}); \quad C_{jk}^i = 0$$

and we consider the settings

$$b^i := \tilde{b}^j b_j^i; \quad ||b||^2 := a^{\bar{i}}b_{\bar{i}}b_j^i; \quad \tilde{b}^i := \tilde{b}^i.$$  \( (4.5) \)
Lemma 4.1. \[3\] Let \((M, F)\) be a complex Finsler space with \((\alpha, \beta)\)–metrics which satisfy 
\[
\frac{\partial |\beta|^2}{\partial \bar{z}^i} = ||b||^2 \frac{\partial \alpha^2}{\partial \bar{z}^i}.
\] The following statements are equivalent:

1. \( ||b||^2 b^r \frac{\partial a_{\bar{r}m}}{\partial \bar{z}^i} \eta^m = \bar{\beta} b^m b^r \frac{\partial a_{\bar{r}m}}{\partial \bar{z}^i} \);

2. \( ||b||^2 \frac{\partial b_{\bar{m}}}{\partial \bar{z}^i} \eta^m = \bar{\beta} \frac{\partial b_{\bar{m}}}{\partial \bar{z}^i} \);

3. \( b^r \frac{\partial b_{\bar{m}}}{\partial \bar{z}^i} \eta^m = \bar{\beta} b^r \frac{\partial b_{\bar{m}}}{\partial \bar{z}^i} \);

4. \( \bar{\beta} \left( \frac{\partial b_{\bar{m}}}{\partial \bar{z}^i} \eta^l - 2b_l \ G^i \right) + \beta \frac{\partial b_{\bar{m}}}{\partial \bar{z}^i} \eta^m \eta^l = 0 \), where \( G^i := \frac{1}{2} N^i_j \eta^j \).

Proposition 4.1. \[3\] Let \((M, F)\) be a complex Finsler space with \((\alpha, \beta)\)–metrics. If \( \frac{\partial |\beta|^2}{\partial \bar{z}^i} = ||b||^2 \frac{\partial \alpha^2}{\partial \bar{z}^i} \) and one of the equivalent conditions from Lemma 4.1 holds, then \( N^i_j = N^i_j \). Moreover, if \( \alpha \) is Kähler, then \( F \) is Kähler.

Theorem 4.1. Let \((M, F)\) be a complex Finsler space with \((\alpha, \beta)\)–metrics. If \( \frac{\partial |\beta|^2}{\partial \bar{z}^i} = ||b||^2 \frac{\partial \alpha^2}{\partial \bar{z}^i} \) and one of the equivalent conditions from Lemma 4.1 holds, then the space is generalized Berwald. Moreover if \( \alpha \) is Kähler then the space is Berwald.

Proof. By Proposition 4.1 we have \( G^i = a^m \bar{\beta} a_{\bar{m}a} \eta^l \eta^j \) which is holomorphic in \( \eta \), i.e. the space is generalized Berwald. Adding the Kähler property for \( a_{ij} \) then the space becomes one complex Berwald.

Further on, we asunder focus on two classes of complex \((\alpha, \beta)\)–metrics, namely the complex Randers metrics \( F := \alpha + |\beta| \) and the complex Kropina metrics \( F := \frac{\alpha}{|\beta|} \), \( |\beta| \neq 0 \).

### 4.1 Complex Randers metric \( F := \alpha + |\beta| \)

For the complex Randers metric \( F := \alpha + |\beta| \) we have, (4.4)

\[
\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} l^i ; \quad \frac{\partial |\beta|}{\partial \eta^i} = \bar{\beta} \frac{1}{2|\beta|} b^i ; \quad \eta_i := \frac{\partial L}{\partial \eta^i} = \frac{F}{\alpha} l_i + \frac{F \bar{\beta}}{|\beta|} b_i, \quad (4.6)
\]

\( N^i_j = N^i_j + \frac{1}{\gamma} \left( l^i \frac{\partial b_r}{\partial \bar{z}^j} - \frac{\beta^2}{|\beta|^2} \frac{\partial b_r}{\partial \bar{z}^j} \eta^l \right) \xi^j + \frac{\beta}{2|\beta|} k^r_i \frac{\partial b_r}{\partial \bar{z}^j} \),

where \( k^r_i := 2a \bar{\alpha} \bar{j}_i + \frac{2a}{\gamma} b^i \bar{b}^r \eta^l \eta^j - \frac{2a}{\gamma} b^i \bar{b}^r \eta^j (\bar{\beta} \eta^l \bar{b}^r + \beta \eta^l \bar{b}^r) \), \( \gamma := L + \alpha^2 (||b||^2 - 1) \), \( \xi^j := \bar{\beta} \eta^j + \alpha^2 b^j \) and so the spray coefficients are

\[
G^i = \frac{a}{2\gamma} \left( l^i \frac{\partial b_r}{\partial \bar{z}^j} - \frac{\beta^2}{|\beta|^2} \frac{\partial b_r}{\partial \bar{z}^j} \eta^l \right) \xi^j \eta^l + \frac{\beta}{4|\beta|} k^r_i \frac{\partial b_r}{\partial \bar{z}^j} \eta^j. \quad (4.7)
\]

Moreover, for the weakly Kähler complex Randers spaces we have proven.
Proposition 4.2. ([4]) A complex Randers space \((M, F)\) is weakly Kähler if and only if
\[
\frac{\alpha^2 |\beta|}{\gamma \delta} \left[ \beta \frac{\alpha |b|^2 + |\beta|}{|\beta|} \frac{\partial b_m}{\partial z^r} \bar{\eta}^m + \bar{\beta} \left( \frac{\partial b_r}{\partial z^i} - b^m r \frac{\partial a_{im}}{\partial z^r} \right) \eta^i - \alpha |\beta| \frac{\partial b_m}{\partial z^r} \frac{\partial b_m}{\partial z^r} \right] \eta^r C_k
\]
\[
- \left( \alpha \bar{\beta} F_{kl} + \alpha b_{kl} \frac{\partial b_r}{\partial z^k} \bar{\eta}^r + 2 |\beta| a_{lr} \Gamma_{jk} \bar{\eta}^j \right) \eta^l + \alpha b_k \frac{\partial b_m}{\partial z^r} \eta^m \eta^r = 0, \quad (4.8)
\]
where \(C_j := C_{j \bar{k}k} g_{\bar{h}k} = \delta \left( \frac{1}{\alpha^2} l_j - \frac{\bar{\beta}}{|\beta|} b_j \right) \) with \(\delta := \frac{\alpha^2 |b|^2 - |\beta|^2}{2 \gamma} - \frac{n |\beta|}{2 \gamma^2} \), \(\Gamma_{ji}^l := \frac{1}{2} a_{rk} \left( \frac{\partial a_{kl}}{\partial z^r} - \frac{\partial a_{kr}}{\partial z^l} \right) \) and \(F_{kl} := \frac{\partial b_k}{\partial z^l} - \frac{\partial b_l}{\partial z^k} \).

Theorem 4.2. Let \((M, F)\) be a connected complex Randers space. Then, \((M, F)\) is a generalized Berwald space if and only if \((\bar{\beta} l_r \frac{\partial b_r}{\partial z^j} + \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^r) \eta^j = 0 \)

Proof. If \((M, F)\) is generalized Berwald then \(2 G^i = L_{jk}^i (z) \eta^j \eta^k \), which means that \(G^i \) is quadratic in \(\eta \). Thus, using (4.7) we have
\[
\beta \left[ (\alpha^2 |b|^2 + |\beta|^2) a_{ij} + |b|^2 \bar{\eta}^j \eta^i - \alpha^2 \bar{b}^i \bar{b}^j - \bar{\beta} \bar{\eta}^i \bar{b}^j - \beta \bar{b}^i \eta^j \right] \frac{\partial b_r}{\partial z^j} \eta^j
\]
\[
= 4 |\beta|^2 (G^i - \bar{G}^i) \quad \text{and} \quad (\bar{\beta} l_r \frac{\partial b_r}{\partial z^j} + \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^r) \eta^j \eta^i + \alpha^2 \beta \left( \beta l_r \frac{\partial b_r}{\partial z^j} - \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^r \right) \eta^j b^i + 2 \alpha^2 \beta a_{ij} \frac{\partial b_r}{\partial z^j} \eta^j \]
\[
= 2 (\alpha^2 |b|^2 + |\beta|^2) (G^i - \bar{G}^i).
\]

Their contractions by \(b_i\) and \(l_i\) yield
\[
(G^i - \bar{G}^i) b_i = 0; \quad (4.9)
\]
\[
4 |\beta|^2 (G^i - \bar{G}^i) l_i + 2 \beta a^2 (|b|^2 \bar{\eta}^r - \bar{\beta} \bar{b}^r) \frac{\partial b_r}{\partial z^j} \eta^j = 0;
\]
\[
\bar{\beta} (\alpha^2 |b|^2 + |\beta|^2) l_i \bar{b}^r \frac{\partial b_r}{\partial z^j} \eta^j - \beta (\alpha^2 |b|^2 - |\beta|^2) \frac{\partial b_r}{\partial z^j} \bar{\eta}^r \eta^i + 2 \alpha^2 |\beta|^2 \bar{b}^r \frac{\partial b_r}{\partial z^j} \eta^j = 0;
\]
\[
(\alpha^2 |b|^2 + |\beta|^2) (G^i - \bar{G}^i) l_i + \alpha^2 (l_r \frac{\partial b_r}{\partial z^j} + \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^r) \eta^j = 0.
\]
Adding the second and the third relations from (4.9), we obtain

\[ 4|\beta|^2(G^i - G^i)l_i + (\alpha^2||b||^2 + |\beta|^2)(\bar{\beta}l_e \frac{\partial \bar{m}}{\partial z^j} + \beta \frac{\partial b_m}{\partial z^j} \bar{\eta}^i)\eta^j = 0. \]

This together with the fourth equation from (4.9) implies \( G^i \neq G^i \), \( \bar{a} \). Moreover, \( \beta \frac{\partial b_m}{\partial z^j} \eta^j = 0. \)

Conversely, if \( (\bar{\beta}l_e \frac{\partial \bar{m}}{\partial z^j} + \beta \frac{\partial b_m}{\partial z^j} \bar{\eta}^i)\eta^j = 0 \), by derivation with respect to \( \bar{\eta} \)

we deduce \( (l_e \frac{\partial \bar{m}}{\partial z^j} b_m + \beta \frac{\partial b_m}{\partial z^j} \bar{\eta}^i)\eta^j = 0. \) The last two relations give

\[ a^m \frac{\partial b_m}{\partial z^j} \eta^j = \frac{\beta}{|\beta|^2} \frac{\partial b_e}{\partial z^j} \bar{\eta}^e \bar{\eta}^i \eta^j \] and \( \bar{b}^m \frac{\partial b_m}{\partial z^j} \eta^j = ||b||^2 \frac{\beta}{|\beta|^2} \frac{\partial b_e}{\partial z^j} \bar{\eta}^e \bar{\eta}^i \eta^j \),

which substituted into (4.7) imply \( G^i = G^i \) and so, \( G^i \) is holomorphic in \( \eta \), i.e. the space is generalized Berwald.

**Theorem 4.3.** Let \( (M, F) \) be a connected complex Randers space. Then, \( (M, F) \) is a complex Berwald space if and only if it is in the same time generalized Berwald and weakly Kähler.

**Proof.** If \( (M, F) \) is Berwald then it is obvious that the space is generalized Berwald and weakly Kähler.

Now, we prove the converse. On the one hand, if the space is generalized Berwald, by Theorem 4.2, it results \( (\bar{\beta}l_e \frac{\partial \bar{m}}{\partial z^j} + \beta \frac{\partial b_m}{\partial z^j} \bar{\eta}^i)\eta^j = 0 \), which can be rewritten as

\[ \bar{\beta} \left( \frac{\partial b_e}{\partial z^j} \eta^j \right) - 2b_l \frac{\partial a}{\partial z^j} G^i = 0. \] (4.10)

Moreover, (4.10) implies

\[ ||b||^2 \bar{\beta} \left( \frac{\partial b_i}{\partial z^j} \eta^j \right) + \frac{\partial b_m}{\partial z^j} \bar{\eta}^m \bar{\eta}^i \eta^j = 0. \] (4.11)

On the second hand, the space is supposed weakly Kähler. Therefore, (4.10) and (4.11) substituted into (4.8) lead to

\[ \alpha^2 \left( \bar{\beta} F_{kli} \eta^l + \beta \frac{\partial b_e}{\partial z^k} \bar{\eta}^e - b_k \frac{\partial b_m}{\partial z^k} \bar{\eta}^m \eta^i \right) + 2\alpha |\beta| a_l \Gamma_{jk}^e \eta^j = 0, \] (4.12)

which contains two parts: the first is rational and the second is irrational. It results

\[ \bar{\beta} F_{kli} \eta^l + \beta \frac{\partial b_e}{\partial z^k} \bar{\eta}^e - b_k \frac{\partial b_m}{\partial z^k} \bar{\eta}^m \eta^i = 0 \] and \( a_l \Gamma_{jk}^e \eta^j = 0. \) (4.13)

The second condition from (4.13) gives the Kähler property for \( \alpha \). Thus, deriving (4.10) with respect to \( \eta^k \) it follows

\[ b_k \frac{\partial b_m}{\partial z^j} \eta^j = -\beta \frac{\partial b_e}{\partial z^k} \bar{\eta}^e - \bar{\beta} \left( \frac{\partial b_i}{\partial z^k} + \frac{\partial b_k}{\partial z^j} \right) \eta^l + 2\bar{\beta} b_k \frac{\partial a}{\partial z^k} \eta^l. \] (4.14)
Now, (4.14) together with the first condition from (4.13) implies
\[
\bar{\beta}l_r \frac{\partial \bar{y}}{\partial z^k} + \beta \frac{\partial b_r}{\partial z^k} \bar{\eta} = 0
\]
and from here results its derivative with respect to $\bar{\eta}^m$
\[
l_r \frac{\partial \bar{y}}{\partial z^k} b_m + \beta \frac{\partial b_m}{\partial z^k} = 0.
\]
Moreover, (4.15) and (4.16) imply
\[
a^{ni} \frac{\partial b_m}{\partial z^k} = \frac{\beta}{|\beta|^2} \frac{\partial b_r}{\partial z^k} \bar{\eta}^n b^i
\]
and so, the spray coefficients
\[
\bar{b}^m \frac{\partial b_m}{\partial z^k} = \frac{||b||^2}{|\beta|^2} \frac{\beta}{\partial z^k} \bar{\eta}^r.
\]
Plugging (4.15) and (4.17) into (4.16), we obtain $N^i_j = \frac{\alpha^2}{|\beta|}$ and so, $L^i_{kj} = L^a_{ij} = L^i_{jk}$, i.e. the Randers space is Kähler which proves our claim.

\[
4.2 \text{ Complex Kropina metric } F := \frac{\alpha^2}{|\beta|}, |\beta| \neq 0
\]
For the complex Kropina metric $F := \frac{\alpha^2}{|\beta|}, |\beta| \neq 0$, we have, (4.18)
\[
\frac{\partial \alpha}{\partial \eta^i} = \frac{1}{2\alpha} l_i; \quad \frac{\partial |\beta|}{\partial \eta^i} = \frac{\beta}{|\beta|^2} b_i; \quad \eta_i := \frac{\partial L}{\partial \eta^i} = 2q^2 l_i - q^4 \bar{\beta} b_i; \quad q := \frac{\alpha}{|\beta|};
\]
\[
N^i_j = \frac{\alpha^2}{|\beta|^2} l_i \frac{\partial \bar{y}}{\partial z^j} \eta^j - \frac{q^2 \beta}{2} \bar{T}^i \frac{\partial b_r}{\partial z^j},
\]
where $\bar{T}^i := a^i + \frac{2-q^2|\beta|^2}{|\beta|^2} \bar{\eta} \bar{\eta} \eta^i + \frac{1}{|\beta|^2} (\bar{\beta} \bar{\eta} \bar{\eta} - \beta b^j \bar{\eta}^j)$ and so, the spray coefficients are
\[
G^i_j = G^i_j = \frac{\bar{\beta}}{2|\beta|^2} l_i \frac{\partial \bar{y}}{\partial z^j} \eta^j \eta^i - \frac{q^2 \beta}{4} \bar{T}^i \frac{\partial b_r}{\partial z^j}. \quad (4.19)
\]

**Proposition 4.3.** Let $(M, F)$ be a connected complex Kropina space. Then, $G^i_j = G^i_j$ if and only if $(\bar{\beta} l_r \frac{\partial \bar{y}}{\partial z^j} + \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^j) \eta^i = 0$.

**Proof.** Using (4.19), we have
\[
4|\beta|^2 (\frac{\alpha^2}{|\beta|^2} - G^i_j) l_i = 2 \alpha^2 (\bar{\beta} l_r \frac{\partial \bar{y}}{\partial z^j} + \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^j) \eta^i + \alpha^4 \frac{\partial b_r}{\partial z^j} \eta^i (b^r - \frac{\beta}{|\beta|^2} ||b||^2 \bar{\eta})
\]
and
\[
4|\beta|^2 (\frac{\alpha^2}{|\beta|^2} - G^i_j) b_i = 2 \alpha (\bar{\beta} l_r \frac{\partial \bar{y}}{\partial z^j} + \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^j) \eta^i + 2 \alpha^2 \beta \frac{\partial b_r}{\partial z^j} \eta^i (b^r - \frac{\beta}{|\beta|^2} ||b||^2 \bar{\eta}).
\]

Thus, if $G^i_j = \frac{\alpha^2}{|\beta|^2} - G^i_j$ then $(\bar{\beta} l_r \frac{\partial \bar{y}}{\partial z^j} + \beta \frac{\partial b_r}{\partial z^j} \bar{\eta}^j) \eta^i = 0$ implies $a^{ni} \frac{\partial b_m}{\partial z^k} \eta^j = \frac{\beta}{|\beta|^2} \frac{\partial b_r}{\partial z^j} \bar{\eta}^j b^i$ and $\bar{b}^m \frac{\partial b_m}{\partial z^k} \eta^j = ||b||^2 \frac{\beta}{|\beta|^2} \frac{\partial b_r}{\partial z^j} \bar{\eta}^j \eta^i$, which substituted into (4.19) it results $G^i_j = G^i_j$. \qed
Theorem 4.4. If \((M, F)\) is a connected complex Kropina space with property 
\[
(\beta_l \frac{\partial \bar{v}^l}{\partial z^j} + \beta \frac{\partial \bar{v}}{\partial z^l} \bar{\eta}^r) \bar{\eta}^j = 0,
\]
then it is generalized Berwald.

Proof. Indeed, by Proposition 4.3 we have \(G^i = G^i\) which gives \(\hat{\partial}_k G^i = 0\). □

Proposition 4.4. Let \((M, F)\) be a connected complex Kropina space. Then, \(N_j^a = N_j^a\) if and only if \(\bar{\beta}_l \frac{\partial \bar{v}^l}{\partial z^j} + \beta \frac{\partial \bar{v}}{\partial z^l} \bar{\eta}^r = 0\).

Proof. Taking into account (4.18), we obtain
\[
2|\beta|^2(N_j^a - N_j^b)I_i = 2\alpha^2(\beta_l \frac{\partial \bar{v}^l}{\partial z^j} + \beta \frac{\partial \bar{v}}{\partial z^l} \bar{\eta}^r) + \alpha^4 \frac{\partial \bar{v}}{\partial z^l} (b^r - \frac{\beta}{|\beta|^2} ||b||^2 \bar{\eta}^r) and
\]
\[
2|\beta|^2(N_j^a - N_j^b)I_i = 2\beta(\beta_l \frac{\partial \bar{v}^l}{\partial z^j} + \beta \frac{\partial \bar{v}}{\partial z^l} \bar{\eta}^r) + 2\alpha^2 \beta \frac{\partial \bar{v}}{\partial z^l} (b^r - \frac{\beta}{|\beta|^2} ||b||^2 \bar{\eta}^r),
\]
which give \(\beta_l \frac{\partial \bar{v}^l}{\partial z^j} + \beta \frac{\partial \bar{v}}{\partial z^l} \bar{\eta}^r = 0\), under assumption \(N_j^a = N_j^b\).

Conversely, if \(\beta_l \frac{\partial \bar{v}^l}{\partial z^j} + \beta \frac{\partial \bar{v}}{\partial z^l} \bar{\eta}^r = 0\) then \(a^a \frac{\partial \bar{v}^a}{\partial z^b} = \frac{\beta}{|\beta|^2} \bar{\eta} \bar{\eta} b^r \) and \(\bar{b}^m \frac{\partial \bar{v}^m}{\partial z^a} = ||b||^2 \beta \frac{\partial \bar{v}^a}{\partial z^b} \bar{\eta}^r\), which together (4.18) lead to \(N_j^a = N_j^b\). □

Theorem 4.5. Let \((M, F)\) be a connected complex Kropina space. If \(\alpha\) is Kähler and \(\beta_l \frac{\partial \bar{v}^l}{\partial z^j} + \beta \frac{\partial \bar{v}}{\partial z^l} \bar{\eta}^r = 0\), then the space is Berwald.

Proof. By Proposition 4.4 we have \(N_j^a = N_j^a\) and so, \(\hat{\partial}_k G^i = 0\), i.e. the space is generalized Berwald. But \(\alpha\) is supposed Kähler, therefore it results that \(F\) is also Kähler. The proof is complete. □

Certainly, as it is expected, our study is far from being complete. Here we tried to point out some classes of complex Finsler spaces with special properties for Cartan tensors, having in mind an analogy with the real case. There is not enough space here to prove with other examples that our classification of these complex Finsler spaces is proper. Otherwise, we don’t have at hand too many examples of complex Finsler spaces, in fact the study of these spaces could be considered rather at a first stage. It is our goal to look for other significant examples from the new class of complex Randers spaces, [4 12], in particular for two dimensional case, recently studied by us. Keeping in mind that in the real case the relations between Landsberg and Berwald spaces give rise to some questions which had been open for a long time ([13 27 18]), it is possible that the same takes place for our setting. However, from our point of view this classification seems natural.

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