QUANTUM REIDEMEISTER TORSION OF MONOTONE LAGRANGIAN TORI

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Abstract. We define a notion of Reidemeister torsion for the pearl complex of a monotone Lagrangian submanifold and show that it is always trivial for \( n \)-tori, \( n \geq 2 \). We also compute the torsion of Lagrangian circles and show that it does not always vanish.

1. Introduction

Lagrangian quantum homology \( QH(L) \) is an invariant associated to a closed monotone Lagrangian submanifold in a tame symplectic manifold \((M,\omega)\), see [BC09b]. This invariant can vanish, for example if the Lagrangian can be displaced from itself by a Hamiltonian isotopy, as is the case for any Lagrangians in \( \mathbb{C}^n \), in which case \( L \) is called narrow. Another way to produce narrow Lagrangians is to consider representations \( \varphi \) of \( \pi_2(M,L) \) and their associated quantum homology \( QH^\varphi(L) \). Relevant definitions will be given in the next sections.

Classically, Reidemeister torsion is an invariant extracted from the \( \mathbb{Z}[\pi_1(X)] \)-equivariant cellular complex of the universal cover \( \tilde{X} \) of a CW-complex \( X \). It is a secondary invariant, meaning that it is defined precisely for representations of \( \pi_1(X) \) for which the equivariant cellular homology vanishes. In particular, the Euler characteristic of \( X \) vanishes, \( \chi(L) = 0 \). For an account of that theory and more, see for example Cohen [Coh73], Milnor [Mil66] or Turaev [Tur01].

In this paper, we adapt Reidemeister torsion to the pearl complex of narrow Lagrangians, which we call quantum Reidemeister torsion. This is rather direct over a field, although the homological algebra involved is that of periodic chain complexes (i.e. with a \( \mathbb{Z}/2\mathbb{Z} \) grading), rather than bounded chain complexes, as explained in §2.

Somewhat surprisingly, when trying to adapt the construction over rings, one runs into algebraic difficulties due to the cyclic grading and it is not always possible to define a notion of torsion, at least it is not clear to me how to do so. See §2.3 for more on this.

Other notions of torsion have been used in symplectic topology, although they were concerned with the Floer complex rather than the pearl complex, in the works of Fukaya [Fuk94], Lee [Lee05a, Lee05b], Suarez [Sua14] and Sullivan [Sul02].

Our torsion involves various choices of generic data. We postpone the general study of invariance with respect to these choices to a future paper, although we do prove invariance by brute force.
when the Lagrangian is a torus. Our main result concerns the torsion of Lagrangian $n$-tori, $n \geq 2$. A more precise statement is in §4:

**Theorem 1.0.1.** Let $L$ be a monotone Lagrangian $n$-torus, $n \geq 2$, and $\varphi : \pi_2(M,L) \to \mathbb{F}^\times$ a representation for which $QH^\varphi(L)$ vanishes. Then the quantum Reidemeister torsion of $L$, relative to any generic choice of pearl data, vanishes.

This result says that, as far as representations in fields are concerned, quantum Reidemeister torsion does not provide invariants of narrow Lagrangian tori. It would be interesting to know if higher order invariants can be defined and computed for such tori.

The proof is given in §4. It is algebraic and relies on the fact that the pearl differential is almost entirely determined by its value on codimension one homology classes of the torus, a phenomenon first observed independently by Biran–Cornea in [BC09b] and Buhovski in [Buh10].

In the last section, we compute the quantum torsion for any Lagrangian circle. In contrast to higher dimensional tori, it does not always vanish.

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2. **The torsion of an acyclic chain complex**

2.1. **Bounded complexes.** Following the presentation by Milnor [Mil66], we recall the definition of torsion of an acyclic chain complex. See also [Coh73] and [Tur01]. Let $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ be a bounded chain complex over a field $\mathbb{F}$ such that each $C_i$ has a preferred finite basis $c_i = (c_{i,1}, \ldots, c_{i,r_i})$. Denote by $B_i$ the image of the boundary morphism $d : C_{i+1} \to C_i$ and by $Z_{i+1}$ its kernel.

Choose bases $b_i = (b_{i,1}, \ldots, b_{i,k_i})$ of $B_i$ and assume that $C_\ast$ is acyclic, so that $Z_i = B_i$ and the Euler characteristic is $\chi(C_\ast) = 0$. One then has exact sequences

$$0 \to B_i \to C_i \xrightarrow{d} B_{i-1} \to 0$$

which split since we work over a field. Given splittings $s_{i-1} : B_{i-1} \to C_i$, we get a new basis of $C_i$ obtained by concatenating the bases $s_{i-1}(b_{i-1})$ and $b_i$, which we write $s_{i-1}(b_{i-1})b_i$.

Let $[s_{i-1}(b_{i-1})b_i/c_i]$ denote the matrix expressing $s_{i-1}(b_{i-1})b_i$ with respect to the basis $c_i$. Notice that given splittings $s_{i-1}$ and $s'_{i-1}$, we have $\det[s_{i-1}(b_{i-1})b_i/c_i] = \det[s'_{i-1}(b_{i-1})b_i/c_i]$. Since we are concerned mostly with determinants, we will often omit the section and write simply $b_{i-1}b_i$ for the new basis.
Definition 2.1.1. The torsion of the chain complex \( C_\ast \) with respect to the bases \( \{ c_i \} \) is
\[
\tau(C_\ast, c_\ast) = \prod_{i=0}^{n} \det [b_{i-1}b_i/c_i]^{(-1)i} \in \mathbb{F}^\times = \mathbb{F} \setminus \{0\}.
\]

Reordering the basis vectors in \( c_\ast \) multiplies \( \tau \) by \( \pm 1 \), thus we consider the value of \( \tau \in \mathbb{F}^\times / \pm 1 \).

As in [Mil66], if we choose different bases \( b'_i \) of \( B_i \), we get
\[
\prod_{i=0}^{n} \det [b'_{i-1}b'_i/c_i]^{(-1)i} = \prod_{i=0}^{n} (\det [b_{i-1}b_i/c_i] \ast \det [b'_i/b_i] \ast \det [b'_{i-1}/b_{i-1}])^{(-1)i}
\]
and the last two terms multiply up to 1, so \( \tau \) is independent of \( b_i \).

Finally, the torsion depends on the choice of basis \( c_\ast \). Indeed, choosing another basis \( c'_\ast \), we get
\[
\tau(C_\ast, c'_\ast) = \tau(C_\ast, c_\ast) \prod_{i=0}^{n} \det [c_i/c'_i]^{(-1)i}
\]

2.1.1. Non-acyclic complexes: Milnor’s definition. If \( C_\ast \) is not acyclic, then one has the following exact sequences:
\[
\begin{align*}
0 & \longrightarrow Z_i \longrightarrow C_i \overset{d}{\longrightarrow} B_{i-1} \longrightarrow 0 \\
0 & \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0
\end{align*}
\]
which combine to yield a new basis \( b_{i-1}h_{i}b_i \) of \( C_i \).

Definition 2.1.2. The torsion is defined as
\[
\tau(C_\ast, c_\ast) = \prod_{i=0}^{n} \det [h_i b_i b_{i-1}/c_i]^{(-1)i} \in \mathbb{F}^\times / \pm 1.
\]

We will not adapt this definition to the pearl complex in this paper, but we will use it for the Morse complex in §4.

2.2. Periodic complexes. Consider now a 2-periodic chain complex \( C[i] \) over a field \( \mathbb{F} \):
\[
\begin{array}{c}
C[i] \\
\circlearrowleft
\end{array}
\begin{array}{c}
\overset{d}{\longrightarrow} \\
\overset{d}{\longrightarrow}
\end{array}
\begin{array}{c}
C[0] \\
\circlearrowleft
\end{array}
\]

The primary example we have in mind is the pearl complex with a cyclic grading, see §3.3. A similar version using maximal abelian torsion for the Floer complex has been considered by Lee [Lee05a, §2.2.3].

Assuming as before that \( C[i] \) is acyclic and choosing bases \( c[i] \) and \( b[i] \), we have:

Definition 2.2.1. The torsion of a 2-periodic acyclic chain complex is
\[
\tau_2(C[i], c[i]) = \frac{\det [b[1]b[0]/c[0]]}{\det [b[0]b[1]/c[1]]} \in \mathbb{F}^\times / \pm 1.
\]
Equation (1) still applies to prove that \( \tau_2 \) is independent of the choice of bases \( b_{[i]} \). Choosing different sections again leaves \( \tau_2 \) invariant.

This torsion generalizes the one defined in the previous section. Given a bounded chain complex \( C_* \), define a 2-periodic complex by setting
\[
(C_{[0]},c_{[0]}) = \bigoplus_{k \text{ even}} (C_k,c_k), \quad C_{[1]} = \bigoplus_{k \text{ odd}} (C_k,c_k)
\]
with the differential being simply the direct sum of the differentials of \( C_* \). Note that \( \chi(C_{[i]}) = \chi(C_*) \).

Obviously, \( B_{[0]} = \oplus_{k \text{ even}} B_k \) and \( B_{[1]} = \oplus_{k \text{ odd}} B_k \). The split sequences
\[
0 \rightarrow B_i \rightarrow C_i \rightarrow B_{i-1} \rightarrow 0
\]
can be added up by defining \( s_{[0]} = \oplus s_{\text{even}} \), \( s_{[1]} = \oplus s_{\text{odd}} \), to yield two splittings
(4) \[
0 \rightarrow B_{[i]} \rightarrow C_{[i]} \rightarrow B_{[i-1]} \rightarrow 0
\]
After a reordering of the bases, we get block diagonal matrices \( [b_{[i]}b_{[0]}/c_{[0]}] = \oplus_{k \text{ even}} [b_{k+1}b_k/c_k] \) and \( [b_{[0]}b_{[1]}/c_{[1]}] = \oplus_{k \text{ odd}} [b_{k+1}b_k/c_k] \), therefore
\[
\tau(C_*,c_*) = \tau_2(C_{[i]},c_{[i]}).
\]

**Remark.** If an acyclic complex has a \( \mathbb{Z}/2\mathbb{Z} \)-grading, then one can cook up a \( \mathbb{Z}/2\mathbb{Z} \)-graded acyclic complex out of it, by reducing the grading modulo 2, just as above, hence there is no need to define the notion for such complexes, 2-periodic complexes are enough for this purpose. It is not clear how to adapt these definitions to \( \mathbb{Z}/(2k+1)\mathbb{Z} \)-graded complexes.

2.3. **Stably-free modules and Whitehead torsion.** It is possible to define the torsion of bounded acyclic complexes over rings \( R \) satisfying the invariant basis property (IBP) - e.g. commutative rings and group rings - as an element of the Whitehead group of that ring, denoted by \( K_1(R) \). This is ultimately possible because the boundary modules \( B_i \) are stably-free (meaning that there exists positive integers \( s_i \) and \( k_i \) such that \( B_i \oplus R^{s_i} \cong R^{k_i} \)), by a simple induction argument. See e.g. Cohen [Coh73, §13] or [Mil66, §4] for more on this.

Unfortunately, the same procedure does not work for periodic complexes over such rings, since the boundary modules \( B_{[i]} \) are not automatically stably-free, in fact not even projective, as the following examples show. Moreover, it is easy to see that \( B_{[0]} \) is stably-free if and only if \( B_{[1]} \) is.

**Examples.** (1) Let \( B_{[0]} \) be a projective module that is not stably-free over some ring \( R \) satisfying the (IBP) property (these exist), so that there exists a complement \( B_{[1]} \) to \( B_{[0]} \) in a free module, i.e. \( B_{[0]} \oplus B_{[1]} \cong R^k \), for some \( k \geq 1 \). Set \( C_{[0]} = C_{[1]} = R^k \) and define differentials by projecting to each factor
\[
d : C_{[1]} \rightarrow C_{[0]} \\
(b_0,b_1) \mapsto (b_0,0)
\]
\[
\delta : C_{[0]} \rightarrow C_{[1]} \\
(b_0,b_1) \mapsto (0,b_1)
\]
Then $C_{[4]}$ is acyclic but the boundary modules are not stably-free, since $\text{im } d \cong B_{[0]}$ and $\text{im } \delta \cong B_{[1]}$. 

(2) Take $R = \mathbb{Z}[\mathbb{Z}/p\mathbb{Z}] \cong \mathbb{Z}[t]/(t^p - 1)$, the group ring of $\mathbb{Z}/p\mathbb{Z}$, for $p$ an odd prime number. The polynomials $t - 1$ and $1 + t + \cdots + t^{p-1}$ are zero divisors in this ring. Set $C_{[0]} = C_{[1]} = R$ with differentials

$$d: C_{[1]} \to C_{[0]} \quad \delta: C_{[0]} \to C_{[1]}$$

$$r \mapsto r(t - 1) \quad r \mapsto r(1 + t + \cdots + t^{p-1})$$

This complex is acyclic. The boundaries are $R$-submodules of $R$ given by $B_{[0]} = (t - 1)R$, $B_{[1]} = (1 + t + \cdots + t^{p-1})R$. These modules are not projective, hence not stably-free, since projective submodules correspond to idempotents, and the only idempotents of $\mathbb{Z}[G]$, with $|G|$ finite, are 0 and 1, see e.g. Weibel [Wei13, Corollary 2.5.3].

3. The pearl complex and its torsion

We refer to Biran–Cornea’s papers [BC09a, BC09b, BC12] for foundations and applications of Lagrangian quantum homology. The version we use here (with oriented moduli spaces of pearls) is adapted from [BC12].

3.1. Setting. Throughout the text, $(M, \omega)$ is a $2n$-dimensional symplectic manifold that is connected and tame. The space of $\omega$-compatible almost complex structures on $M$ is denoted by $\mathcal{J}_\omega$.

All Lagrangian submanifolds $L \subset (M, \omega)$ are closed and connected. Moreover, they are endowed with a fixed choice of orientation and spin structure which we do not write. From §3.3 onwards, all Lagrangians have vanishing Euler characteristic, in order for torsion to be defined.

Let $\omega: \pi_2(M, L) \to \mathbb{R}$ be given by the symplectic area of discs and $\mu: \pi_2(M, L) \to \mathbb{Z}$ denote the Maslov index. The positive generator of the image of $\mu$ is called the minimal Maslov index and is denoted by $N_L$. Lagrangians are assumed monotone, that is, there exists a constant $\tau > 0$ such that

- $\omega = \tau \mu$
- $N_L \geq 2$

Since $L$ is orientable, $N_L$ is even.

3.2. The 2-periodic pearl complex. Fix a triple $\mathcal{D} = (f, \rho, J)$, where $\rho$ is a Riemannian metric, $f: L \to \mathbb{R}$ a Morse-Smale function and $J \in \mathcal{J}_\omega$.

Set

$$C_k = C_k(\mathcal{D}) = \mathbb{Z}[\pi_2(M, L)](\text{Crit}_k f), \quad k = 0, \ldots, n = \dim L,$$

where $\text{Crit}_k f$ is the set of critical points with Morse index $k$ and $\mathbb{Z}[G]$ is the group ring of a group $G$. We write the Morse index of a critical point $x$ as $|x|$.
For a generic triple $\mathcal{D}$, the pearl differential is defined by
\[
d : \bigoplus C_k(\mathcal{D}) \rightarrow \bigoplus C_k(\mathcal{D})
\]
\[
\text{Crit } f \ni x \mapsto \sum_y \left( \sum_{A \in \pi_2(M,L)} \#(\mathcal{P}(x,y,A)) A \right) y
\]
where $\#(\mathcal{P}(x,y,A))$ is the (signed) number of pearls in the homotopy class $A$ going from $x$ to $y$. When $\mu(A) = 0$, a pearl is simply a negative gradient flow line of $f$. This morphism decomposes as a finite sum
\[
d = d_M + d_1 + \ldots
\]
where $d_M : C_k \rightarrow C_{k-1}$ is the Morse differential and $d_i : C_k \rightarrow C_{k-1+iN_L}$ counts pearls of Maslov index $iN_L$. Note that $d_i$’s are not differentials, they do not square to zero.

Since $N_L$ is even, $k$ and $k-1+iN_L$ have different parity, hence there is a 2-periodic pearl complex over $\mathbb{Z}[\pi_2(M,L)]$, defined by
\[
C_{[\ast]}(\mathcal{D}) = \bigoplus_{k \equiv [\ast] \mod 2} C_k(\mathcal{D}), \quad [\ast] = 0, 1
\]
with an induced differential $d : C_{[\ast]} \rightarrow C_{[\ast-1]}$.

The homology of this complex is called the quantum homology of $L$, denoted by $QH_{[\ast]}(L)$, or simply $QH(L)$. It is independent of generic choices of $\mathcal{D}$. If $QH(L) = 0$, we say that $L$ is narrow.

### 3.3. Narrow representations and torsion

Fix a field $\mathbb{F}$ and a ring morphism (by convention, ring morphisms map 1 to 1)
\[
\varphi : \mathbb{Z}[\pi_2(M,L)] \rightarrow \mathbb{F},
\]
so that $\mathbb{F}$ becomes a $\mathbb{Z}[\pi_2(M,L)]$-module. This defines a 2-periodic chain complex over $\mathbb{F}$ by setting
\[
C_{[\ast]}^\varphi(\mathcal{D}) = C_{[\ast]}(\mathcal{D}) \otimes_{\mathbb{Z}[\pi_2(M,L)]} \mathbb{F}, \quad d^\varphi = d \otimes 1.
\]
As above, the homology of this new complex, denoted by $QH^\varphi(L)$, does not depend on $\mathcal{D}$. If it vanishes, we say that $L$ is $\varphi$-narrow, which implies $\chi(L) = 0$. Notice that $QH(L) = 0$ implies $QH^\varphi(L) = 0$ for every $\varphi$.

The set of narrow representations of $L$ over $\mathbb{F}$ is defined by
\[
\mathcal{N}(L,\mathbb{F}) = \{ \varphi : \mathbb{Z}[\pi_2(M,L)] \rightarrow \mathbb{F} \mid L \text{ is } \varphi\text{-narrow} \}.
\]
Given $\varphi \in \mathcal{N}(L,\mathbb{F})$ and $\mathcal{D}$ a generic set of data, there is a preferred basis for $C_{[\ast]}^\varphi(\mathcal{D})$ given by $\text{Crit}_{[\ast]} f$.

Proceeding as in §2.2, we have:

**Definition 3.3.1.** The quantum Reidemeister torsion of $L$ is
\[
\tau_\varphi(L,\mathcal{D}) = \tau_2(C_{[\ast]}^\varphi(\mathcal{D}), \text{Crit}_{[\ast]} f) \in \mathbb{F}^\times / \pm 1
\]
Remarks. a. Narrow representations do not always exist (for example, take any \( L \) with non-vanishing Euler characteristic). However, for tori, ”most” representations are narrow, and they can be detected by computing partial derivatives of the Landau-Ginzburg superpotential, see Biran–Cornea [BC12, §3.3] for more on this.
b. It is possible to give a \( \mathbb{Z} \)-grading to the pearl complex by introducing a Novikov variable that keeps track of Maslov indices. However, doing this makes the pearl complex unbounded, and the differential becomes periodic. Thus, to define torsion in this context, one needs to take an infinite product of determinants which repeat themselves every multiple of \( N_L \). To avoid this type of issue, we chose to get rid of the Novikov variable altogether and use a \( \mathbb{Z}/2\mathbb{Z} \)-grading.

3.3.1. Invariance. In this paper, we prove by direct computations that \( \tau_\varphi(L, D) \) is independent of \( D \), whenever \( L \) is an \( n \)-torus, \( n \geq 1 \).

However, we do not study the behavior of \( \tau_\varphi(L, D) \) with respect to different choices of \( D \) for other Lagrangians, which would require a bifurcation analysis of the pearl complex and will be investigated in another paper.

Remark. There are other contexts in which the behaviour of torsion under changes of data has been studied. See for example Hutchings [Hut02] for the Morse-Novikov complex of circle-valued Morse functions, Lee [Lee05a, Lee05b] for the Hamiltonian Floer complex, Suarez [Sua14] for exact Lagrangian cobordisms and Sullivan [Sul02] for a version of the Lagrangian Floer complex. In the first two cases, torsion is \textit{not} invariant, while in the last two it is.

4. Quantum torsion of the \( n \)-torus, \( n \geq 2 \)

Throughout this section, \( L = T^n = S^1 \times \cdots \times S^1 \) (\( n \) times) is a monotone torus, where \( n \geq 2 \). Moreover, we fix \( \rho \) a Riemannian metric, \( f : T^n \to \mathbb{R} \) a Morse-Smale function and \( J \in \mathcal{J}_\omega \) such that the associated triple \( D = (f, \rho, J) \) is generic.

Theorem A. Fix a field \( \mathbb{F} \). Given any narrow representation \( \varphi \in \mathcal{N}(T^n, \mathbb{F}) \), we have

\[ \tau_\varphi(T^n, D) = 1 \in \mathbb{F}^\times / \pm 1 \]

As we show in §5, \( \tau_\varphi(S^1, D) \) is not always trivial and does not depend on \( D \).

4.1. Proof of Theorem A. The proof will be divided in many steps, which allow us to compute the pearl differential as explicitly as possible.

Step 1:
Results of Buhovski [Buh10] and Biran–Cornea [BC09b, Theorem 1.2.2], show that \( L = T^n \) is \( \varphi \)-narrow only if \( N_L = 2 \). Recall from (5) that the differential decomposes as a sum \( d^\varphi =: d = \)
\[d_M + d_1 + \ldots, \text{ where } d_i: C_{k} \to C_{k-1+2i}, \text{ since } N_L = 2. \] In matrix form, \(d: C_{[i]} \to C_{[i-1]}\) is then:

\[
\begin{pmatrix}
   \ldots & C_{i-2} & C_i & C_{i+2} & \ldots \\
   \vdots & \ldots & \ldots & \ldots & \ldots \\
   C_{i-1} & \ldots & d_1 & d_M & 0 & \ldots \\
   C_{i+1} & \ldots & d_2 & d_1 & d_M & \ldots \\
   C_{i+3} & \ldots & d_3 & d_2 & d_1 & \ldots \\
\end{pmatrix}
\]

(6)

**Step 2:** Change the basis of \(C_{i}\) to a more manageable one.

Applying Milnor’s definition from §2.1.1 to the Morse differential \(d_M\), we obtain a new basis of \(C_{i}\) given by \(s(h_i)b_i^M s_M(b_{i-1}^M)\), where \(s\) and \(s_M\) are sections which we will sometimes not write. The letter \(M\) stands for Morse. The specific basis \(h_i\) we choose for the Morse homology \(H_i(L)\) is as follows. \([L]\) is the fundamental class, i.e. the generator of \(H_n(L)\). The rest of the homology is generated, as a unitary ring endowed with the Morse intersection product, by codimension 1 classes \(x_i \in H_{n-1}(L), \ i = 1 \ldots n\). The unit is \([L]\). Each element is thus expressed as a linear combination of monomials \(x_{i_1\ldots i_k} := x_{i_1} \cdot x_{i_2} \cdots x_{i_k}\), with the convention that \(1 \leq i_1 < i_2 < \cdots < i_k \leq n\).

By formula (2), we have

\[
\tau(C_{[s]}, h_{[s]}b_s^M b_{s-1}^M) = \tau_{\varphi}(T^n, D) \prod_{i=0}^n \det[c_i/h_i b_i^M b_{i-1}^M]^{(-1)^i}
\]

Notice that each matrix \([c_i/h_i b_i^M b_{i-1}^M]\) has entries in \(\varphi(\mathbb{Z})\), the reason is that the exact sequences (3) are actually split exact over the integers, since the Morse boundaries are free (they are subgroups of a free abelian group) and the singular homology of the torus is free over \(\mathbb{Z}\). Moreover, \(\varphi\) induces a map from \(GL(\mathbb{Z})\) to \(GL(\varphi(\mathbb{Z}))\). Therefore, the determinant of these matrices is \(1 \in \mathbb{F}^{\times} / \pm 1\) and

\[
\tau(C_{[s]}, h_{[s]}b_s^M b_{s-1}^M) = \tau_{\varphi}(T^n, D). \quad \text{Our task is now to compute the torsion on the left-hand-side of this equality.}
\]

**Step 3:**

With respect to the bases \(h_i b_i^M b_{i-1}^M\), the Morse differential \(d_M: C_{i} \to C_{i-1}\) is given by:

\[
\begin{pmatrix}
   s(H_i) & B_i^M & s_M(B_{i-1}^M) \\
   s(H_{i-1}) & 0 & 0 & 0 \\
   B_{i-1} & 0 & 0 & I \\
   s_M(B_{i-2}^M) & 0 & 0 & 0 \\
\end{pmatrix}
\]

(7)

where \(I\) is the identity matrix, as a quick check shows.

**Step 4:** Write the matrix for \(d_1: C_{i} \to C_{i+1}\) in the new basis.

Since \(d^2 = 0\), we get that \(d_M \circ d_1 + d_1 \circ d_M = 0\), hence \(d_1\) induces a map on Morse homology, which commutes with the section \(s: H_i \to \mathbb{Z}^M_i\), denoted by \(d_1: H_i \to H_{i+1}\). The map \(d_1: C_{i} \to C_{i+1}\) is
then, in matrix form:

\[
\begin{pmatrix}
    s(H_i) & B_i^M & s_M(B_i^M) \\
    s(H_{i+1}) & d_{1*} & 0 \\
    B_{i+1}^M & ? & ? \\
    s_M(B_i^M) & 0 & 0
\end{pmatrix}
\]

where ? denote some matrices which do not matter for our purpose. The map \( d_{1*} \) coincides with the differential on the first page of Oh’s spectral sequence in Floer homology [Oh96]. We refer for details to Buhovski’s paper [Buh10].

**Remark.** In [Buh10], the coefficient ring used is a Novikov ring with a variable to keep track of Maslov indices; setting \( t = 1 \), we recover the coefficients that we use in the present paper.

We will use the following facts from [Buh10], adapting the notation to our setting:

- \( d_{1*} : H_{n-1} \to H_n \), \( d_{1*}(x_i) \mapsto r_i[L] \) for some \( r_i \in \mathbb{F} \), is a surjective map, hence one of the \( r_i \)’s is different from zero. Since permuting the \( x_i \)’s will only change the torsion by \( \pm 1 \), we assume without loss of generality that \( r_1 \neq 0 \).

- \( d_{1*} \) satisfies the Leibniz rule with respect to the Morse intersection product, i.e. \( d_{1*}(x \cdot y) = d_{1*}(x) \cdot y + (-1)^{n-|x|} x \cdot d_{1*}(y) \). In [Buh10], this is expressed by the fact that the product on the first page of the spectral sequence coincides with the Morse intersection product.

Using these facts, and since \([L]\) is a unit, we get

\[
(9) \quad d_{1*}(x_{i_1 \cdots i_k}) = \sum_{m=1}^{k} (-1)^{m+1} r_{i_m} x_{i_1} \cdots \hat{x}_{i_m} \cdots x_{i_k},
\]

where a ”hat” above a variable means it is omitted in the product.

**Lemma 4.1.1.** A basis for the image of \( d_{1*} : H_{n-k} \to H_{n-k+1} \) is given by the set of vectors \( \{d_{1*}(x_{i_2 \cdots i_k})\} \), for all \( 2 \leq i_2 < \cdots < i_k \leq n \).

**Proof.** Using formula (9), a direct computation shows that, for \( j_1 \geq 2 \), we have

\[
d_{1*}(x_{j_1 \cdots j_k}) = \frac{1}{r_1} \sum_{m=1}^{k} (-1)^{m+1} r_{j_m} d_{1*}(x_{1} x_{j_1} \cdots \hat{x}_{j_m} \cdots x_{j_k})
\]

(recall that \( r_1 \neq 0 \)). Therefore, applying \( d_{1*} \) to monomials starting with a product with \( x_1 \) spans the image of \( d_{1*} \). To see that the vectors are linearly independent, one stares at the following matrix, obtained by using formula (9):

\[
\begin{pmatrix}
    \{d_{1*}(x_{i_2 \cdots i_k})\} \\
    \{x_{i_2 \cdots i_k}\} \\
    \{x_{1j_2 \cdots j_k-1}\}
\end{pmatrix}
\begin{pmatrix}
    \text{diag}(r_1) \\
    ?
\end{pmatrix}
\]

where \text{diag}(r_1) means a diagonal matrix with diagonal elements all equal to \( r_1 \). \( \square \)
Step 5: Find a basis for $d(C_{[i]})$.

Using the matrices (7) and (8), we see that a part of matrix (6) looks as follows:

\[
\begin{pmatrix}
C_{i-2} & C_i & C_{i+2} \\
\begin{pmatrix}
d_{1*} & 0 & ? \\
? & ? & ? \\
0 & 0 & ?
\end{pmatrix} & \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & I \\
0 & 0 & 0
\end{pmatrix} & \begin{pmatrix}
? & 0 & 0 \\
? & ? & ? \\
0 & 0 & 0
\end{pmatrix}
\end{pmatrix}
\]

(10)

Therefore, all vectors in the set $V = \{d(x_{1j_2 \ldots j_k}), d(s_M(b^M_{i-1}))\}$, where $i$ varies over all integers of the same parity, and $2 \leq j_2 < \cdots < j_k \leq n$ cover all possibilities (also with $k$ having the right parity), are linearly independent (the case where some $B^M_i$ are empty is not excluded).

Claim: These vectors span all of $d(C_{[i]})$, hence give a basis $b_{[i-1]}$ for $B_{[i-1]}$, the image of $d: C_{[i]} \rightarrow C_{[i-1]}$.

Proof. The set of linearly independent vectors $C_0 = \{x_{1j_2 \cdots j_n-(i-1)}, s_M(b^M_{i-2})\}$, where, again, the indices cover all possibilities of the appropriate parity, is in the cokernel of $d: C_{[i]} \rightarrow C_{[i-1]}$, because $d: C_{[i-1]} \rightarrow C_{[i]}$ is injective on these vectors and the complex is acyclic. Counting dimensions, we get dim $V + \text{dim } C_0 = \sum_{[m] \equiv [i-1] \mod 2} \dim C_m = \dim C_{[i-1]}$. Therefore, $V$ has to span the image.

To understand the first equality, notice that in the matrix below, the third and fifth row are in $C_0$, whereas the second and fourth one are parts of $V$, hence $V$ cannot contain more vectors.

\[
\begin{pmatrix}
\bigoplus_{l \leq i-3, [l] \equiv [i-1]} C_l & d(x_{1j_1 \cdots j_n-(i-1)}) & d(s_M(b^M_{i-2})) \\
H_{i-1} \ni x_{j_2 \cdots j_n-(i-1)} & \text{diag}(r_1) & 0 \\
H_{i-1} \ni x_{1j_2 \cdots j_n-(i-1)} & ? & 0 \\
B^M_{i-1} & ? & I \\
C_l & \bigoplus_{l \geq i+1, [l] \equiv [i+1]} C_l & ? & ?
\end{pmatrix}
\]

\[
\square
\]

Step 6:

After all these preliminary steps, we are now ready to compute the torsion. The map

$s: B_{[i-1]} \rightarrow C_{[i]}$

$d(x_{1j_1 \cdots j_n-(i-1)}) \mapsto x_{1j_1 \cdots j_n-(i-1)}$

d($s_M(b^M_{i-1})$) $\mapsto s_M(b^M_{i-1})$
defines a splitting of the exact sequence

\[ 0 \rightarrow B_{[i]} \rightarrow C_{[i]} \rightarrow B_{[i-1]} \rightarrow 0 \]

Therefore, the matrix \([b_{[i]}s(b_{[i-1]})/h_{[i]}b^M_{[i]}s_M(b^M_{[i-1]})]\) is lower triangular and has the following shape:

\[
\begin{pmatrix}
\oplus_{l \leq i-2, [l] \equiv [i]} C_l & \cdots & d(x_{1j_1 \ldots j_{n-i}}) & s(d(x_{1k_1 \ldots k_{n-(i-1)}})) & d(s_M(b^M_{i})) & s(d(s_M(b^M_{i-1}))) & \cdots \\
\oplus_{l \geq i+2, [l] \equiv [i]} C_l & \cdots & \text{diag}(r_1) & 0 & 0 & 0 & 0 \\
H_i \ni x_{j_{n-i}} & \cdots & ? & I & 0 & 0 & 0 \\
B^M_{i} & \cdots & ? & 0 & I & 0 & 0 \\
s_M(B^M_{i-1}) & \cdots & ? & 0 & ? & I & 0 \\
& \cdots & \text{ ? & ? & ? & ? & \cdots} \\
\end{pmatrix}
\]

Finally, a simple counting argument yields

\[
\tau_\varphi(T^n, D_f) = \tau(C_{[s]}, h_{[s]}b^M_{[s]}b^M_{[s-1]}) = \frac{\det\[b_{[0]}s(b_{[1]})/h_{[0]}b^M_{[0]}s_M(b^M_{[1]})\]}{\det\[b_{[1]}s(b_{[0]})/h_{[1]}b^M_{[1]}s_M(b^M_{[0]})\]}
\]

\[
= \frac{\sum_{k=0}^n \dim H_k(T^n)/4}{\sum_{k=0}^n \dim H_k(T^n)/4} = 1.
\]

This concludes the proof of Theorem A. \qed

5. The circle

Consider first a monotone circle \(S^1 \subset S^2\), i.e. a circle dividing \(S^2\) in two parts (or hemispheres) of equal area. Then \(\pi_2(S^2, S^1) \cong \mathbb{Z}^2\), with generators given by the classes of the hemispheres, denoted by \(A\) and \(B\). Fix a Morse-Smale function \(f_n: S^1 \rightarrow \mathbb{R}\) with \(\text{Crit}_0 f_n = \{y_i \mid i = 1, \ldots, n\}\) and \(\text{Crit}_1 f_n = \{x_i \mid i = 1, \ldots, n\}\), as on Figure 1.

![Figure 1. \(f_n: S^1 \rightarrow \mathbb{R}\), the ”rock-and-roll” function](image)
By varying \( n \), this gives all possible Morse-Smale functions on \( S^1 \). Fix also a generic compatible almost complex structure \( J \in J_\omega \).

The pearl differential on \( \mathbb{Z}[\pi_2(S^2, S^1)](\text{Crit} \, f_n) \) is then given by

\[
d(x_i) = y_{i-1} - y_i, \quad d(y_i) = (A - B) \sum_{j=1}^{n} x_j.
\]

To understand the shape of \( d(y_i) \), note that, given any two generic points on \( S^1 \) (e.g. \( y_i \) and \( x_j \)) and any generic \( J \), there is one pseudoholomorphic disc in the class \( A \) and minus one in the class \( B \), going through \( x_i \) and \( y_j \), counted with appropriate signs.

Given \( \varphi: \pi_2(S^2, S^1) \to \mathbb{F}^\times \), \( A \mapsto z_1 \), \( B \mapsto z_2 \), we get an induced representation \( \varphi: \mathbb{Z}[\pi_2(S^2, S^1)] \to \mathbb{F} \) and \( S^1 \) is \( \varphi \)-narrow if and only if \( r = z_1 - z_2 \neq 0 \), since \( d(y_i) = r \sum x_j \).

A quick check shows that bases \( b_0 \) and \( b_1 \) for \( d(\text{Crit} \, f_1) \) and \( d(\text{Crit} \, f_0) \) are given by

\[
b_0 = \{ y_i - y_{i+1} \mid i = 1, \ldots, n - 1 \}, \quad b_1 = \sum_{i=1}^{n} x_i.
\]

Moreover, splittings of (4) are given by

\[
s: b_0 \to C_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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If $\varphi$ is defined by a representation $\psi: \pi_1(S^1) \to \mathbb{F}^\times, \ 1 \mapsto z \neq 1$, via $\varphi = \partial \circ \psi$, where
\[ \partial: \pi_2(S^2, S^1) \to \pi_1(S^1) \]
is the connecting morphism, then
\[ \tau_\varphi(S^1) = \frac{1}{z - 1} = \frac{z}{z^2 - 1} = \frac{z}{(z-1)(z+1)} \]
In analogy with the classical notion of Reidemeister torsion (or R-torsion), we get
\[ \tau_\varphi(S^1) = \frac{1}{z^2 - 1} \in \mathbb{F}^\times / \pm \varphi(\pi_2(S^2, S^1)) = \mathbb{F}^\times / \pm \psi(\pi_1(S^1)). \]
This does not coincide with the usual value of R-torsion for the circle: $\Delta_\psi(S^1) = \frac{1}{z - 1} \in \mathbb{F}^\times / \pm \psi(\pi_1(S^1))$ (see e.g. [Mil66, §8]). Note however that $\Delta_\psi(S^1)$ divides $\tau_\varphi(S^1)$ (compare e.g. with [HL99, Theorem 1.12] or [Lee05a, Corollary 2.3.4]).

**Remark.** By setting $B = 0$, we get the torsion of a contractible circle - these are always monotone - in any Riemann surface $\Sigma$ different from $S^2$, closed or not: $\tau_\varphi(S^1) = 1 \in \mathbb{F}^\times / \varphi(\pi_2(\Sigma, S^1))$ and $\Delta_\psi(S^1)$ does not divide 1!

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