Equivalent series theorem and obtaining some new summable numerical series using fast expansion polynomials

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Abstract. A theorem on equivalent uniformly converging series is proved, which allows one to find a set of new spectra of summable series. The classification of number series into three classes is given. By expanding special polynomials in Fourier series in terms of various fundamental system of functions, it is possible to summarize a wide class of new numerical series that have a convenient classical form of the first class. These polynomials are borrowed from the fast expansions method of the authors of this article.

1. Introduction

Some summable numerical series obtained in the early and middle of the 20th century are known. Various authors found particular cases; therefore, the series they obtained were published as accompanying results in separate scientific articles and monographs. These particular cases of numerical series are collected in reference books [1, 2]. Since then no new summable series have been found. These reference books contain a large number of functional series built with help of complex special functions. Recently, some studies of various properties of numerical series [3-6] have been conducted. In [7], it was obtained that the order of decreasing of the norm $L$ in the remainder of the Fourier sine series with monotonic coefficients is expressed through the coefficients of the series in the same way as for the series with convex coefficients.

In this work, to obtain new numerical series, fast polynomials are used, which have a simple form; therefore, they allow you to organize several sets of summed spectra of numerical series. It is convenient to use such series to evaluate some functional series. They can also be used as majorant series, in computational practice and theoretical research.

2. Problem setup

All known summable numerical series are obtained from functional ones, where the variable $x$ is given some characteristic values. Below, the uniformly converging functional series will be written in the form

$$\sum_{n=1}^{\infty} f(n, x), \quad f(n, x) \in C(x \in [a, b], \quad n = 1, 2, \ldots), \quad \lim_{n \to \infty} f(n, x) = 0, \quad \forall x \in [a, b].$$

(1)
The function \( f(n, x) \) depends on two variables: a continuous bounded variable \( x \in [a, b] \) and an integer number \( n = 1, 2, \ldots \). Let function \( f(n, x) \) exists and is continuous for all values of the indicated variables.

In this article, when obtaining new numerical series, we will permute the terms of series (1) in some definite order. The question arises about the legality of such a permutation. In this connection, we prove the theorem.

A theorem on equivalent uniformly converging series. Let the series
\[
\sum_{n=1}^{\infty} f(n, x) = S_n, \quad f(n, x) \in C \quad (\forall n = 1, 2, \ldots; \ x \in [a, b])
\]
(2)
converge uniformly for \( \forall x \in [a, b] \), where \( f(n, x) \) is a continuous and smooth function of two variables: an integer discrete unbounded variable \( n = 1, 2, \ldots \) and a continuous variable \( x \in [a, b] \). Let \( A \) additional rows with sums \( S_1 \div S_A \)
\[
\sum_{n=1}^{\infty} f\left((An-(A-1)), x\right) = S_1(x), \quad \sum_{n=1}^{\infty} f\left((An-(A-2)), x\right) = S_2(x),
\]
(3)
\[
\sum_{n=1}^{\infty} f\left((An-1), x\right) = S_{A-1}(x), \quad \sum_{n=1}^{\infty} f(An, x) = S_A(x), \ x \in [a, b], \ A \in N, \ A \geq 2
\]
converge uniformly for \( \forall x \in [a, b] \), where \( A \in N \) is some arbitrary given natural number, called the numeric period. Then the equality takes place:
\[
\sum_{n=1}^{\infty} f\left((An-A+1), x\right) + \sum_{n=1}^{\infty} f\left((An-A+2), x\right) + \sum_{n=1}^{\infty} f\left((An-A+3), x\right) + \ldots + \sum_{n=1}^{\infty} f\left((An-1), x\right)
\]
\[
+ \sum_{n=1}^{\infty} f(An, x) = \sum_{n=1}^{\infty} f(n, x), \ x \in [a, b], \ A \in N, \ A \geq 2.
\]
(4)
The penultimate row on the left of (4) is equal to \( \sum_{n=1}^{\infty} f\left((An-1), x\right) \). The last row is equal to \( \sum_{n=1}^{\infty} f(An, x) \). Equality (4) means that the sum \( A \) of the series in the left-hand side is equivalent to one row in the right-hand side.

Let us prove the theorem. By the conditions of the theorem, the number of series on the left-hand side of (4) is equal \( A \) and each series in (4) converges uniformly. To prove the theorem, we write down the partial sums of these series in (4) as follows:
\[
[ f(1, x) + f\left((A+1), x\right) + f\left((2A+1), x\right) + \ldots + f\left(K_0A+1, x\right)]
\]
\[
+ [ f(2, x) + f\left((A+2), x\right) + f\left((2A+2), x\right) + \ldots + f\left(K_0A+2, x\right)]
\]
\[
+ [ f(3, x) + f\left((A+3), x\right) + f\left((2A+3), x\right) + f\left(K_0A+3, x\right) + \ldots]
\]
\[
+ \ldots [ f(A, x) + f\left((A+A), x\right) + f\left((2A+A), x\right) + f\left(K_0A+A, x\right) + \ldots]
\]
\[
= f(1, x) + f(2, x) + \ldots + f\left(K_0A+A, x\right).
\]
(5)
On the left-hand side of (5), each term of the first partial sum is not repeated in the remaining partial sums, and each member of the second partial sum is not repeated in the remaining partial sums. The partial sum of each row on the left side of (5) consists of \( K_0 + 1 \) terms. The total number of terms
in the $A$ rows on the left side is equal to $A(K_0+1)$ that is equal to the number of the first terms of the partial sum of the row on the right side. Each term of the partial sum on the right-hand side of (5) is contained once in the left-hand side, i.e. equality (5) is satisfied exactly. Since all the series under consideration converge uniformly, the limit of the sum of partial sums of a finite number $A$ of uniformly converging series on the left is equal to the sum of their limits and is equal to the sum of the series on the right:

$$
\lim_{K_0 \to \infty} \left[ f(1, x) + f((A+1), x) + f((2A+1), x) + \ldots + f(K_0 + A + 1, x) \right] \\
+ \lim_{K_0 \to \infty} \left[ f(2, x) + f((A+2), x) + f((2A+2), x) + \ldots + f(K_0 + A + 2, x) \right] \\
+ \lim_{K_0 \to \infty} \left[ f(3, x) + f((A+3), x) + f((2A+3), x) + f(K_0 + A + 3, x) + \ldots \right] \\
+ \ldots \lim_{K_0 \to \infty} \left[ f(A, x) + f((A+A), x) + f((2A+A), x) + f(K_0 + A + A, x) + \ldots \right] \\
= \lim_{K_0 \to \infty} \left[ f(1, x) + f(2, x) + \ldots + f(K_0 + A + A, x) \right].
$$

(6)

Hence, it follows $S_1 + S_2 + \ldots + S_A = S_0$, i.e. the sum of the series on the left is equal to the sum of the series on the right, and the theorem is proved.

Corollary 1. If series (2) converges uniformly, then it admits an ordered rearrangement of its terms with some numerical period $A$ in accordance with formula (4).

Corollary 2. If in equality (4) the sums of all series except one are known, then the unknown sum of the series can be found from (4) and thus a new summed series can be obtained.

Corollary 3. Setting $A = 2, 3, \ldots$ and using formula (4), we have a system of auxiliary equalities for finding the sums of some series.

Corollary 4. Using this theorem, we can sum up a rather large number of uniformly converging series.

Let us give examples of different types of equation (4) for the most common cases ($A = 2, 3, 4, 5, 6$).

When $A = 2$ we obtain

$$
\sum_{n=1}^{\infty} f(n, x) = \sum_{n=1}^{\infty} f((2n-1), x) + \sum_{n=1}^{\infty} f(2n, x), \quad x \in [a, b].
$$

(7)

When $A = 3$ we get

$$
\sum_{n=1}^{\infty} f(n, x) = \sum_{n=1}^{\infty} f((3n-2), x) + \sum_{n=1}^{\infty} f((3n-1), x) + \sum_{n=1}^{\infty} f(3n, x), \quad x \in [a, b].
$$

(8)

When $A = 4$ we have

$$
\sum_{n=1}^{\infty} f(n, x) = \sum_{n=1}^{\infty} f((4n-3), x) + \sum_{n=1}^{\infty} f((4n-2), x) + \sum_{n=1}^{\infty} f((4n-1), x) + \sum_{n=1}^{\infty} f(4n, x), \quad x \in [a, b].
$$

(9)

When $A = 5$ we get

$$
\sum_{n=1}^{\infty} f(n, x) = \sum_{n=1}^{\infty} f((5n-4), x) + \sum_{n=1}^{\infty} f((5n-3), x) + \sum_{n=1}^{\infty} f((5n-2), x) + \sum_{n=1}^{\infty} f((5n-1), x) \\
+ \sum_{n=1}^{\infty} f(5n, x), \quad x \in [a, b].
$$

(10)

When $A = 6$ we have

$$
\sum_{n=1}^{\infty} f(n, x) = \sum_{n=1}^{\infty} f((6n-5), x) + \sum_{n=1}^{\infty} f((6n-4), x) + \sum_{n=1}^{\infty} f((6n-3), x) + \sum_{n=1}^{\infty} f((6n-2), x) + \sum_{n=1}^{\infty} f((6n-1), x) \\
+ \sum_{n=1}^{\infty} f(6n, x), \quad x \in [a, b].
$$

(11)
\[
\sum_{n=1}^{\infty} f(n, x) = \sum_{n=1}^{\infty} f((6n-5), x) + \sum_{n=1}^{\infty} f((6n-4), x) + \sum_{n=1}^{\infty} f((6n-3), x) + \sum_{n=1}^{\infty} f((6n-2), x) \\
+ \sum_{n=1}^{\infty} f((6n-1), x) + \sum_{n=1}^{\infty} f(6n, x), \quad x \in [a, b].
\] (11)

Below, all new summable numerical series will be obtained using the proved theorem. The validity of this theorem can also be verified by direct summing on a computer of the terms of the series under consideration and calculating the corresponding values of the fast polynomials through which the sums are expressed.

3. Materials and methods

Now we consider fast polynomials and the classification of numerical series. Converging series of numbers can be divided into three classes.

The first class includes numerical series, the sum of which is expressed through elementary functions in a finite form. Such a sum is easily calculated even on a calculator and is convenient for practical use. There are few such numerical series; they are given in reference books [1, 2].

The second class includes numerical series, the sum of which is expressed in the final form through special functions, which in turn are expressed by some series. In this case, the calculation of the sum of the series under consideration is replaced by a computer calculation of the series for the special function. The reference books [1, 2] mainly show the ranks of the second class.

The third class should include all other converging numerical series, for which the expression of the sum of the series in the final form through elementary or special functions is unknown. Their sum can be calculated on a computer by direct summation of the members of the series.

Below, a set of spectra of number series of the first class will be obtained, which is a significant addition to the general list of similar series.

For these purposes, polynomials borrowed from the method of fast expansions [7] are used, hereinafter called fast polynomials. First, we indicate the logical path leading to the summation of the numerical series below, so as not to create the impression of the random nature of the results obtained.

When using the method of fast expansions, we represent an arbitrary smooth function \( f(x) \) on an interval \( x \in [0, 1] \) as the sum of the boundary function \( M_q(x) \) and a rapidly converging Fourier series constructed for the difference \( f(x) - M_q(x) \) [7-11], etc. The boundary function \( M_q(x) \) consists of special polynomials \( P_m(x) \), \( Q_m(x) \), \( m = 1 \pm q \), which are called fast polynomials. The number \( q \) can be taken as an arbitrary finite integer. The rate of convergence of Fourier series used in fast expansions depends on it. The expressions of these polynomials and the boundary function are determined by the type of boundary conditions that \( f(x) \) must satisfy at the ends of the segment \( x \in [0, 1] \).

Let some function \( f(x) \) be continuous and smooth and at the ends of the segment \( [0, 1] \) satisfies the Dirichlet boundary conditions:

\[
f(x) \in C^{(2)}(\forall x \in [0, 1]), \quad f(x)\big|_{x=0} = f(0), \quad f(x)\big|_{x=1} = f(1).
\] (12)

Then \( f(x) \) can be represented by fast sine expansions in the form

\[
f(x) = M_{2p}(x) + \sum_{m=1}^{\infty} f_m \sin m\pi x.
\] (13)
Here \( 2p \) is a given order of smoothness, \( f_m \) are the Fourier coefficients for the difference \( (f(x) - M_{2p}(x)) \), and \( M_{2p}(x) \) is the boundary function. In the case of Dirichlet boundary conditions (12), \( M_{2p}(x) \) are defined as follows:

\[
M_{2p}(x) = \sum_{m=0}^{\infty} \left( A_{2m}P_{2m}(x) + B_{2m}Q_{2m}(x) \right), \quad A_{2m} = f^{(2m)}(0), \quad B_{2m} = f^{(2m)}(1), \quad m = 0 \div p . \tag{14}
\]

In (14) \( P_{2m}(x), \ Q_{2m}(x) \) are the fast polynomials calculated by the following recurrent integral formulas

\[
P_0(x) = 1 - x, \quad Q_0(x) = x, \quad P_{2m}(x) = \int_0^1 \int_0^t P_{2m-2}(t) dt_1 dt_2 \left( t_1 - x \right) \int_0^t P_{2m-2}(t) dt_2 \left( t_1 - x \right), \quad Q_{2m}(x) = \int_0^1 \int_0^t Q_{2m-2}(t) dt_1 dt_2 \left( t_1 - x \right) \int_0^t Q_{2m-2}(t) dt_2 \left( t_1 - x \right), \quad m = 1 \div p , \quad 0 \leq t_2 \leq t_1 \leq x , \quad 0 \leq x \leq 1 . \tag{15}
\]

The polynomials \( P_0(x), \ Q_0(x) \) are the generating ones. All the others are found with even indices using double integrals. Fast polynomials \( P_{2m}(x), \ Q_{2m}(x) \) can also be calculated from the solution of boundary value differential problems with zero Dirichlet boundary conditions, for which they were created [7, 8]:

\[
P_{2m}^*(x) = P_{2m-2}(x), \quad P_{2m}(0) = P_{2m}(1) = 0, \quad m \neq 0, \quad m = 1 \div p , \quad Q_{2m}^*(x) = Q_{2m-2}(x), \quad Q_{2m}(0) = Q_{2m}(1) = 0 . \tag{16}
\]

The index \( 2p \) in (13) – (16) is equal to the order of the highest derivative \( f^{(2p)}(x) \) used in the construction \( M_{2p}(x) \). The fast expansions in (13) admit term-by-term differentiation \( 2p \) times, while the Fourier series remain rapidly converging. Due to the special construction of polynomials \( P_{2m}(x), \ Q_{2m}(x) \) in (15), the boundary function \( M_{2p}(x) \) from (14) significantly increases the rate of convergence of the Fourier series in the fast expansions (13). With increasing \( 2p \) order, the rate of convergence increases significantly [9, 10]. Polynomials \( P_{2m}(x), \ Q_{2m}(x) \) with even indices are used in fast sine expansions (13). For fast cosine expansions, polynomials \( P_{2m-1}(x), \ Q_{2m-1}(x) \) with odd indices are used:

\[
f(x) = M_{2p-1}(x) + f_0 + \sum_{n=1}^{\infty} f_n \cos n \pi x , \quad f(x) \in C^{(2p-1)}(\forall x \in [0, 1]). \tag{17}
\]

Polynomials \( P_{2m-1}(x), \ Q_{2m-1}(x) \) with odd indices are expressed in terms of derivatives of polynomials \( P_{2m}(x), \ Q_{2m}(x) \) with even indices

\[
P_{2m-1}(x) = P_{2m}^*(x), \quad Q_{2m-1}(x) = Q_{2m}^*(x), \quad x \in [0, 1], \quad m = 1, 2, ... \tag{18}
\]

Below in (19) there are ten examples of polynomials with even indices and in (20) there are eight more examples of polynomials with odd indices:
\[ P_0(x) = 1 - x, \quad Q_0(x) = x, \quad P_2(x) = \frac{x^2}{2} - \frac{x^3}{6} - \frac{x}{3}, \quad Q_2(x) = \frac{x^3}{6} - \frac{x}{6}, \]
\[ P_4(x) = \frac{x^4}{24} - \frac{x^5}{120} + \frac{x}{18} + \frac{x^3}{45}, \quad Q_4(x) = \frac{x^5}{120} - \frac{x^3}{36} + \frac{7x}{360}, \]
\[ P_6(x) = \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^5}{3!} + \frac{x^3}{45 \cdot 3!} - \frac{2x}{945}, \quad Q_6(x) = \frac{x^7}{7!} - \frac{x^5}{6!} + \frac{7x^3}{3 \cdot 6!} - \frac{31x}{3 \cdot 7!}, \]
\[ P_8(x) = \frac{x^8}{8!} - \frac{x^9}{9!} + \frac{x^7}{3 \cdot 7!} + \frac{5400}{4725}, \]
\[ Q_8(x) = \frac{x^9}{9!} - \frac{x^7}{6 \cdot 7!} + \frac{7x^5}{3 \cdot 6!} - \frac{31x^3}{18 \cdot 7!} + \frac{127x}{604800}, \]
\[ (19) \]

and
\[ P_1(x) = x - \frac{x^2}{2} - \frac{1}{3}, \quad Q_1(x) = \frac{x^2}{2} - \frac{1}{6}, \]
\[ P_3(x) = \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^2}{6} + \frac{1}{45}, \quad Q_3(x) = \frac{x^4}{24} - \frac{x^2}{12} + \frac{7}{360}, \]
\[ P_5(x) = \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^4}{24} - \frac{x^2}{90} - \frac{2}{945}, \quad Q_5(x) = \frac{x^6}{6!} - \frac{5x^4}{6!} + \frac{7x^2}{6} - \frac{31}{3 \cdot 7!}, \]
\[ P_7(x) = \frac{x^7}{7!} - \frac{x^8}{8!} + \frac{x^6}{3 \cdot 6!} + \frac{x^4}{1080} - \frac{x^2}{945} - \frac{1}{4725}, \]
\[ Q_7(x) = \frac{x^8}{8!} - \frac{x^6}{6 \cdot 6!} + \frac{7x^4}{12 \cdot 6!} - \frac{31x^2}{6 \cdot 7!} + \frac{127}{604800}. \]
\[ (20) \]

Fast polynomials expressions (19) and (20) do not depend on the order of the boundary function. They have special properties; one of them is used to obtain new summable wonderful numerical series by expanding into corresponding Fourier series.

Let us consider construction of numerical series using fast polynomials. Numerical series will be constructed by expanding polynomials (19) in \( \sin n\pi x \) and polynomials (20) in \( \cos n\pi x \) at \( x \in [0, 1] \).

Let us write the Fourier series of polynomials \( P_{2m}(x), Q_{2m}(x) \) in \( \sin n\pi x \):
\[ P_{2m}(x) = \sum_{n=1}^{\infty} P_n^{(2m)} \sin n\pi x, \quad P_n^{(2m)} = 2 \int_0^1 P_{2m}(x) \sin n\pi x dx \]
\[ Q_{2m}(x) = \sum_{n=1}^{\infty} Q_n^{(2m)} \sin n\pi x, \quad Q_n^{(2m)} = 2 \int_0^1 Q_{2m}(x) \sin n\pi x dx. \]
\[ (21) \]

Polynomials \( P_{2m}(x), Q_{2m}(x) \) have a special construction. Therefore, the Fourier coefficients \( P_n^{(2m)}, Q_n^{(2m)} \) have a special convenient form. To calculate the coefficients \( P_n^{(2m)}, Q_n^{(2m)} \) for any given \( m \) we use the integrals of the generating polynomials:
\[ \int_0^1 P_0(x) \sin n\pi x dx = \frac{1}{n\pi}, \quad \int_0^1 Q_0(x) \sin n\pi x dx = \frac{(-1)^{n+1}}{n\pi}. \]
\[ (22) \]

Using (21) and (22), we get the formulas:
\[ \int_0^1 P_{2m}(x) \sin n\pi x \, dx = \frac{(-1)^m}{n^{2m+1} \pi^{2m+1}}, \quad \int_0^1 Q_{2m}(x) \sin n\pi x \, dx = \frac{(-1)^{m+1}}{n^{2m+1} \pi^{2m+1}}. \]  

Using integrals (23), the Fourier series for the fast polynomials \( P_{2m}(x), \) \( Q_{2m}(x) \) with even indices take the form:

\[
P_{2m}(x) = \sum_{n=1}^{\infty} \frac{P(n)}{n^{2m+1}} \sin n\pi x, \quad x \in [0, 1],
\]

\[
Q_{2m}(x) = \frac{2(-1)^m}{\pi^{2m+1}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2m+1}} \sin n\pi x, \quad m = 0, 1, 2, \ldots
\]

The polynomials \( P_{2m}(x), Q_{2m}(x) \) in the left-hand sides of (24) are obtained from the recurrent expressions (15) and therefore their exact value can be calculated at \( \forall x \in [0, 1] \). In expansions (24), we will sequentially take the values of the variable \( x = 1/6, 1/4, 1/3, 1/2, 2/3, 3/4, 5/6 \). Here the values \( x = 0, x = 1 \) are not taken, since then the equalities \( P_{2m}(x), Q_{2m}(x) \) for in (24) become identities. Assuming \( x = 1/6 \) in (24), we obtain the first spectrum of the summed numerical series for different values \( m = 0, 1, 2, \ldots \), which we write as follows:

\[
P_{2m}\left(\frac{1}{6}\right) = \frac{2(-1)^m}{\pi^{2m+1}} \sum_{n=1}^{\infty} \frac{1}{n^{2m+1}} \sin n\pi \frac{1}{6} = \frac{2(-1)^m}{\pi^{2m+1}} \left( \frac{1}{1^{2m+1}} + \frac{1}{2^{2m+1}} + \frac{\sqrt{3}}{3^{2m+1}} + \frac{1}{4^{2m+1}} \right)
\]

\[
+ \frac{1}{5^{2m+1}} - \frac{1}{7^{2m+1}} - \frac{1}{8^{2m+1}} - \frac{1}{9^{2m+1}} + \frac{\sqrt{3}}{10^{2m+1}} - \frac{1}{11^{2m+1}} + \frac{1}{12^{2m+1}} + \ldots
\]

Using Corollary 1, we group the terms into three groups with factors \( 1/2, \sqrt{3}/2 \) and 1:

\[
P_{2m}\left(\frac{1}{6}\right) = \frac{2(-1)^m}{\pi^{2m+1}} \sum_{n=1}^{\infty} \frac{1}{n^{2m+1}} \sin n\pi \frac{1}{6} = \frac{2(-1)^m}{\pi^{2m+1}} \left( \frac{1}{2^{2m+1}} + \frac{1}{4^{2m+1}} - \frac{1}{8^{2m+1}} + \frac{1}{10^{2m+1}} + \ldots \right)
\]

\[
+ \frac{\sqrt{3}}{2} \left( \frac{1}{2^{2m+1}} + \frac{1}{4^{2m+1}} - \frac{1}{8^{2m+1}} + \frac{1}{10^{2m+1}} + \ldots \right) + \left( \frac{1}{3^{2m+1}} + \frac{1}{9^{2m+1}} + \frac{1}{15^{2m+1}} + \ldots \right)
\]

Here the right side can be represented by the following rows:

\[
\sum_{n=1}^{\infty} \left( \frac{(-1)^n}{(6n-5)^{2m+1}} + \frac{(-1)^{n+1}}{(6n-1)^{2m+1}} \right) + \sqrt{3} \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{(6n-4)^{2m+1}} + \frac{(-1)^{n+1}}{(6n-2)^{2m+1}} \right)
\]

\[
+ 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(6n-3)^{2m+1}} = (-1)^m \pi^{2m+1} P_{2m}\left(\frac{1}{6}\right), \quad m = 0, 1, 2, \ldots
\]

This equality is a consequence of formula (11) at \( A = 6 \), which after simplification takes the form:

\[
\sum_{n=1}^{\infty} \left( \frac{(-1)^n}{(6n-5)^{2m+1}} + \frac{(-1)^{n+1}}{(6n-1)^{2m+1}} \right) + \frac{\sqrt{3}}{2} \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{(3n-2)^{2m+1}} + \frac{(-1)^{n+1}}{(3n-1)^{2m+1}} \right)
\]

\[
+ 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^{2m+1}} = (-1)^m \pi^{2m+1} P_{2m}\left(\frac{1}{6}\right), \quad m = 0, 1, 2, \ldots
\]
In series (25) and in subsequent series, the index $m$ is a spectral parameter that takes any natural value.

A rearrangement of the terms was used, admissible by Corollary 1 at $A = 6$. The validity of equality (25) is easily verified by calculating the partial sum of the series in its final expression and calculating the value of the polynomials $P_{2m}(x)$ at the point $x = 1/6$. A similar check can be done in all the numerical series obtained below. We do not write the series expression for $Q_{2m}(x)$ at $x = 1/6$, since it does not lead to the original series.

We find another spectrum of numerical series corresponding to the value of the numerical period $A = 3$ for $x = 1/4$ from (24):

$$
\frac{\sqrt{2}}{2} \sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{(3n-2)^{2m+1}} + \frac{(-1)^{n+1}}{(3n-1)^{2m+1}} \right) + \frac{1}{2^{m}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{m+1}} = \left( -1 \right)^{m} \frac{\pi^{2m+1}}{2^{m}} P_{2m} \left( \frac{1}{4} \right), \quad m = 0, 1, 2, \ldots \ (26)
$$

We have the spectrum of numerical series by setting $x = 1/3$ in (24):

$$
\sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{(3n-2)^{2m+1}} + \frac{(-1)^{n+1}}{(3n-1)^{2m+1}} \right) = \left( -1 \right)^{m} \frac{\pi^{2m+1}}{\sqrt{3}} P_{2m} \left( \frac{1}{3} \right), \quad m = 0, 1, 2, \ldots, \quad (27)
$$

We obtain a numerical series at $A = 2$ from (24) at $x = 1/2$:

$$
\sum_{n=1}^{\infty} \left( \frac{1}{(3n-2)^{2m+1}} - \frac{1}{(3n-1)^{2m+1}} \right) = \left( -1 \right)^{m} \frac{\pi^{2m+1}}{2} Q_{2m} \left( \frac{1}{2} \right), \quad m = 0, 1, 2, \ldots \ (28)
$$

When writing the Fourier cosine series, one should take into account the equalities for polynomials $P_{2m-1}(x), Q_{2m-1}(x)$ with odd indices:

$$
\int_{0}^{1} P_{2m-1} (x) \, dx = 0, \quad \int_{0}^{1} Q_{2m-1} (x) \, dx = 0 \ , \quad (29)
$$

which follow from properties (18) and (16). Taking into account (29), the Fourier series for $P_{2m-1}(x), Q_{2m-1}(x)$ are written by the expressions:

$$
P_{2m-1} (x) = \frac{2(-1)^{m}}{\pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}} \cos n\pi x, \quad Q_{2m-1} (x) = \frac{2(-1)^{m}}{\pi^{2m}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2m}} \cos n\pi x, \quad (30)
$$

which can be obtained by calculating the derivative of (24).

From the expansion (30) in the Fourier series of polynomials $P_{2m-1}(x), Q_{2m-1}(x)$ with odd indices, we find the following numerical series.

Setting $x = 0$ in (30) and then $x = 1$, we have

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \left( -1 \right)^{m} \frac{\pi^{2m}}{2} P_{2m-1} (0), \quad \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^{2m}} = \left( -1 \right)^{m} \frac{\pi^{2m}}{2} P_{2m-1} (1). \quad (31)
$$

If we take $x = 1/3$ in (30), we get a series for $A = 3$:
We supplement the set of numerical series (25) – (28), (31), (32) using the fast polynomials $R_{2m}(x)$ and $R_{2m-1}(x)$ applied for fast expansions in problems with mixed boundary conditions. These polynomials are determined by the integral expressions:

\[
\int_0^1 R_{m-2}(t_2) dt_2,
\]

\[
0 \leq t_2 \leq t_1, \quad 0 \leq t_1 \leq x, \quad 0 \leq x \leq 1, \quad m = 3, 4, ..., (2p + 2).
\]

Here $R_1(x)$, $R_2(x)$ are the generating polynomials, the rest $R_{2m}(x)$ and $R_{2m-1}(x)$ are obtained by the recurrent formula (33). Here are eight examples of polynomials $R_{2m}(x)$, $R_{2m-1}(x)$:

\[
R_1(x) = 1, \quad R_2(x) = x, \quad R_3(x) = \frac{1}{2} x^2 - x, \quad R_4(x) = \frac{1}{6} x^3 - \frac{1}{2} x,
\]

\[
R_5(x) = \frac{1}{24} x^4 - \frac{1}{6} x^3 + \frac{1}{3} x, \quad R_6(x) = \frac{1}{120} x^5 - \frac{1}{12} x^3 + \frac{5}{24} x,
\]

\[
R_7(x) = \frac{1}{720} x^6 - \frac{1}{120} x^5 + \frac{1}{18} x^3 - \frac{2}{15} x, \quad R_8(x) = \frac{1}{5040} x^7 - \frac{1}{240} x^6 + \frac{5}{144} x^3 - \frac{61}{720} x.
\]

In this case, to obtain new numerical series, we also expand $R_{2m}(x)$ and $R_{2m-1}(x)$ into a Fourier series according to functions $\sin \pi (n + 1/2) x$:

\[
R_{2m-1}(x) = \sum_{n=0}^{\infty} R_{n}^{(2m-1)} \sin \left(n \pi + \frac{\pi}{2}\right) x, \quad R_{2m}(x) = \sum_{n=0}^{\infty} R_{n}^{(2m)} \sin \left(n \pi + \frac{\pi}{2}\right) x.
\]

Firstly, we determine the Fourier coefficients $R_{n}^{(1)}$, $R_{n}^{(2)}$ for the generating polynomials $R_1(x)$, $R_2(x)$ presented in (34):

\[
R_{n}^{(1)} = \frac{2}{\pi} \left[ R_1(x) \sin \left(n \pi + \frac{\pi}{2}\right) x \right] dx = \frac{2}{\pi (n + 1/2)},
\]

\[
R_{n}^{(2)} = \frac{2}{\pi^2} \left[ R_2(x) \sin \left(n \pi + \frac{\pi}{2}\right) x \right] dx = \frac{2 (-1)^n}{\pi^2 (n + 1/2)^2}.
\]
\[ R_n^{(2m)} = 2 \int_0^1 R_{2m}(x) \sin \left( n\pi + \frac{\pi}{2} \right) dx = (-1)^n \frac{2(-1)^{n+1} 2^{2m}}{\pi^{2m} (2n+1)^{2m}}, \]  
\[ R_n^{(2m-1)} = 2 \int_0^1 R_{2m-1}(x) \sin \left( n\pi + \frac{\pi}{2} \right) dx = \frac{(-1)^{n+1} 2^{2m}}{\pi^{2m-1} (2n+1)^{2m-1}}. \]  
(37)

After these calculations, the Fourier series for \( R_{2m}(x) \), \( R_{2m-1}(x) \) take the form:

\[ R_{2m}(x) = 2(-1)^m \frac{2^{2m}}{\pi^{2m}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^{2m}} \sin \pi (2n+1) \frac{1}{2} x, \quad m = 1, 2, ..., \]  
\[ R_{2m-1}(x) = (-1)^{m+1} \frac{2^{2m}}{\pi^{2m-1}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2m-1}} \sin \pi (2n+1) \frac{1}{2} x. \]  
(38)

In the series (38), to obtain new numerical series, the variable \( x \) is assumed to be equal \( 1; 2/3; 1/2; 1/3 \), for which the number series are obtained in a convenient form. If we take \( x = 0 \), then expressions (38) turn into identities. For \( x = 1 \), we find the following two spectra of number rows:

\[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2m}} = (-1)^m \frac{\pi^{2m}}{2^{2m+1}} R_{2m}(1), \]  
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2m-1}} = (-1)^{m+1} \frac{\pi^{2m-1}}{2^{2m-1}} R_{2m-1}(1), \quad m = 1, 2, ... \]  
(39)

When \( x = 2/3 \) we obtain:

\[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(6n+1)^{2m}} - \frac{(-1)^{n+1}}{(6n+5)^{2m}} = (-1)^m \frac{\pi^{2m}}{2^{2m+1}} R_{2m} \left( \frac{2}{3} \right), \]  
\[ \sum_{n=0}^{\infty} \frac{1}{(6n+1)^{2m+1}} - \frac{1}{(6n+5)^{2m+1}} = (-1)^{m+1} \frac{\pi^{2m-1}}{2^{2m-1}} R_{2m-1} \left( \frac{2}{3} \right), \quad m = 1, 2, ... \]  
(40)

When \( x = 1/2 \) we get

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)^{2m}} - \frac{(-1)^n}{(4n+1)^{2m}} = (-1)^m \frac{\pi^{2m}}{2^{2m+1}} R_{2m} \left( \frac{1}{2} \right), \quad m = 1, 2, ... \]  
(41)

When \( x = 1/3 \) we have:

\[ \sum_{n=0}^{\infty} \frac{1}{(6n+1)^{2m}} + \frac{1}{(6n+5)^{2m}} - \frac{2}{3^{2m+1}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2m}} = (-1)^m \frac{\pi^{2m}}{2^{2m}} R_{2m} \left( \frac{1}{3} \right), \quad m = 1, 2, ... \]  
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)^{2m+1}} + \frac{(-1)^n}{(6n+5)^{2m+1}} = \frac{2}{3^{2m-1}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2m-1}} = (-1)^{m+1} \frac{\pi^{2m-1}}{2^{2m-1}} R_{2m-1} \left( \frac{1}{3} \right). \]  
(42)

Among the obtained series, the expressions for (27), (28), (31), (39) – (41) have rather simple form. Their left-hand sides contain sums that cannot be simplified further. The remaining expressions of the
obtained series (25), (26), (32), (42) can be simplified as follows. To simplify them in expression (25), we exclude the second and third sums using the first series from (27) and the second series from (39), respectively ($m = 0, 1, 2, \ldots$):

$$\sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{(6n-5)^{2n+1}} + \frac{(-1)^{n+1}}{(6n-1)^{2n+1}} \right) = \left(-1\right)^m \pi^{2m+1} \left[ P_{2m} \left( \frac{1}{6} \right) - \frac{1}{2} P_{2m} \left( \frac{1}{3} \right) - \frac{1}{6^{2m+1}} R_{2m+1} \left( 1 \right) \right]. \quad (43)$$

Expression (26) can also be simplified. For this, we exclude the second sum from (26) using the second series from (39):

$$\sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{(4n-3)^{2n+1}} + \frac{(-1)^{n+1}}{(4n-1)^{2n+1}} \right) = \left(-1\right)^m \pi^{2m+1} \left( \frac{1}{4} P_{2m} \left( \frac{1}{3} \right) - \frac{1}{4^{2m+1}} R_{2m+1} \left( 1 \right) \right), \quad m = 0, 1 \ldots \quad (44)$$

For further simplifications, we exclude the second sum from the first equality (32) using the first equality (31), and from the second equality (32) we exclude the second sum using the second equality (31):

$$\sum_{n=1}^{\infty} \left( \frac{1}{(3n-2)^{2n}} + \frac{1}{(3n-1)^{2n}} \right) = \left(-1\right)^m \pi^{2m} \left( Q_{2m-1} \left( \frac{1}{3} \right) + \frac{1}{3^{2m}} P_{2m-1} \left( 0 \right) \right), \quad (45)$$

$$\sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{(3n-2)^{2n}} - \frac{(-1)^{n+1}}{(3n-1)^{2n}} \right) = \left(-1\right)^m \pi^{2m} \left( P_{2m-1} \left( 0 \right) - \frac{1}{3^{2m}} P_{2m-1} \left( 1 \right) \right), \quad m = 1, 2 \ldots \quad (46)$$

To simplify equalities (32), we exclude the first sums in these expressions using two equalities (31):

$$\sum_{n=1}^{\infty} \left( \frac{1}{(3n-2)^{2n}} + \frac{1}{(3n-1)^{2n}} \right) = \left(-1\right)^m \pi^{2m} \left( \frac{1}{3} P_{2m-1} \left( 0 \right) - P_{2m-1} \left( \frac{2}{3} \right) \right), \quad (46)$$

$$\sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{(3n-2)^{2n}} - \frac{(-1)^{n+1}}{(3n-1)^{2n}} \right) = \left(-1\right)^m \pi^{2m} \left( Q_{2m-1} \left( \frac{2}{3} \right) + \frac{1}{3^{2m}} P_{2m-1} \left( 1 \right) \right), \quad m = 1, 2 \ldots \quad (46)$$

Equalities (42) after simplification by means of two series from (39) are also simplified:

$$\sum_{n=0}^{\infty} \left( \frac{1}{(6n+1)^{2n}} + \frac{1}{(6n+5)^{2n}} \right) = \left(-1\right)^{w+1} \pi^{2w} \left( R_{2w} \left( \frac{1}{3} \right) + \frac{1}{3^{2w}} R_{2w} \left( 1 \right) \right), \quad m = 1, 2, \ldots, \quad (47)$$

$$\sum_{n=0}^{\infty} \left( \frac{(-1)^{n}}{(6n+1)^{2n+1}} + \frac{(-1)^{n}}{(6n+5)^{2n+1}} \right) = \left(-1\right)^{m+1} \pi^{2m+1} \left( R_{2m+1} \left( \frac{1}{3} \right) - \frac{1}{3^{2m+1}} R_{2m+1} \left( 1 \right) \right). \quad (47)$$

### 4. Results and discussion

Thus, new spectra of the first class series are obtained: (27), (28), (31), (39) – (41), (43) – (47). Summation of series (31) and (39) is known in reference books, however, their sum in [1, 2] is expressed through the values of the Bernoulli numbers, which are calculated through definite integrals of complex form. In this article, the sums of these series are expressed in terms of fast polynomials, the calculation of which is elementary. The rest of the indicated number series are original, they are obtained in this article for the first time.

In addition to the series presented here, using the theorem on equivalent uniformly converging
series, one can obtain a set of new numerical and uniformly convergent functional series. Let us consider the following examples.

To the first number series from (39), we apply the theorem on equivalent uniformly converging series in the form of formula (8) for \( A = 3 \):

\[
(-1)^n \frac{\pi^{2m}}{2^{2m+1}} R_{2m} (1) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2m}} = \sum_{n=1}^{\infty} \frac{1}{(2(3n-2)-1)^{2m}} + \sum_{n=1}^{\infty} \frac{1}{(2(3n-1)-1)^{2m}} + \sum_{n=1}^{\infty} \frac{1}{(2 \cdot 3n-1)^{2m}}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(6n-5)^{2m}} + \sum_{n=1}^{\infty} \frac{1}{(6n-3)^{2m}} + \sum_{n=1}^{\infty} \frac{1}{(6n-1)^{2m}} = \sum_{n=1}^{\infty} \frac{1}{(6n-5)^{2m}} + \frac{1}{3 \cdot 2^{2m}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2m}},
\]

Hence, after reducing similar terms, we obtain a new series

\[
\sum_{n=1}^{\infty} \frac{1}{(6n-5)^{2m}} + \sum_{n=1}^{\infty} \frac{1}{(6n-3)^{2m}} = \left(1 - \frac{1}{3^{2m}}\right) (-1)^{m} \frac{\pi^{2m}}{2^{2m+1}} R_{2m} (1), \quad m = 1, 2, ...
\]

As an example of a uniformly converging functional series, let us consider the series [1, 2]

\[
\arctg x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}, \quad |x| \leq 1.
\]

We apply formula (8) to this series with \( A = 3 \):

\[
\arctg x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{(3n-2)-1}}{2(3n-2)-1} x^{2(3n-2)-1} + \sum_{n=1}^{\infty} \frac{(-1)^{(3n-1)-1}}{2(3n-1)-1} x^{2(3n-1)-1} + \sum_{n=1}^{\infty} \frac{(-1)^{3n-1}}{6n-1} x^{6n-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n}}{6n-5} x^{6n-5} + \sum_{n=1}^{\infty} \frac{(-1)^{n}}{6n-3} x^{6n-3} - \sum_{n=1}^{\infty} \frac{(-1)^{n}}{6n-1} x^{6n-1} = -\sum_{n=1}^{\infty} \frac{(-1)^{n}}{6n-5} x^{6n-5} - \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^{n}}{6n-1} x^{6n-1}, \quad |x| \leq 1.
\]

After reducing similar terms from the formula (49), we get new series:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{6n-5} x^{6n-5} + \frac{\infty}{n=1} \frac{(-1)^{n-1}}{6n-1} x^{6n-1} = \arctg x + \frac{1}{3} \arctg x^3, \quad |x| \leq 1.
\]

5. Conclusions

Thus, the consideration of fast polynomials borrowed from the method of fast expansions, together with the theorem of equivalent series, made it possible to obtain a large number of new summable series of the first class. The reliability of each obtained series can be checked by direct calculation of the left and right sides of the series on a computer.

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