Linear vertex-kernels for several dense ranking r-CSPs

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Abstract
A ranking r-constraint satisfaction problem (ranking r-CSP for short) consists of a ground set of vertices $V$, an arity $r \geq 2$, a parameter $k \in \mathbb{N}$ and a constraint system $c$, where $c$ is a function which maps rankings of $r$-sized sets $S \subseteq V$ to $\{0, 1\}$. The objective is to decide if there exists a ranking $\sigma$ of the vertices satisfying all but at most $k$ constraints (i.e. $\sum_{S \subseteq V, |S|=r} c(\sigma(S)) \leq k$). Famous ranking r-CSPs include Feedback Arc Set in Tournaments and Dense Betweenness [4, 13]. We consider such problems from the kernelization viewpoint. We prove that so-called $l_r$-simply characterized ranking r-CSPs admit linear vertex-kernels whenever they admit constant-factor approximation algorithms. This implies that $r$-Dense Betweenness and $r$-Dense Transitive Feedback Arc Set [13], two natural generalizations of the previously mentioned problems, admit linear vertex-kernels. Moreover, we introduce another generalization of Feedback Arc Set in Tournaments, which does not fit the aforementioned framework. Based on techniques from [10] we obtain a 5-approximation, and then provide a linear vertex-kernel.

Introduction
Parameterized complexity is a powerful theoretical framework to cope with NP-Hard problems. The aim is to identify some parameter $k$, independent from the instance size $n$, which captures the exponential growth of the complexity to solve the problem at hand. A parameterized problem is said to be fixed parameter tractable (FPT for short) whenever it can be solved in $f(k) \cdot n^{O(1)}$ time, where $f$ is any computable function [11][16]. In this extended abstract, we focus on kernelization, which is one of the most efficient technique to design parameterized algorithms [6]. A kernelization algorithm (or kernel for short) for a parameterized problem $\Pi$ is a polynomial-time algorithm that given an instance $(I, k)$ of $\Pi$ outputs an equivalent instance $(I', k')$ of $\Pi$ such that $|I'| \leq g(k)$ and $k' \leq k$. The function $g$ is said to be the size of the kernel, and $\Pi$ admits a polynomial kernel whenever $g$ is a polynomial. A well-known result states that a (decidable) parameterized problem is FPT if and only if it admits a kernel [16]. Observe that this theoretical result provides kernels of super-polynomial size. Recently, several results gave evidence that there exist parameterized problems that do not admit polynomial kernels (under some complexity-theoretic assumption, see e.g. [7][8]). Hence, determining the existence of polynomial kernels is of important interest from both the theoretical and practical viewpoints.

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We mainly study ranking $r$-CSPs from the kernelization viewpoint. A ranking $r$-CSP consists of a ground set of vertices $V$, an arity $r \geq 2$ and a constraint system $c$, where $c$ is a function which maps rankings (i.e. orderings) of $r$-sized sets $S \subseteq V$ to $\{0,1\}$. The objective is to find a ranking $\sigma$ of the vertices that minimizes the number of unsatisfied constraints (i.e. $\sum_{S \subseteq V, |S|=r} c(\sigma(S))$). We study the decision version of these problems, where the instance comes together with some parameter $k \in \mathbb{N}$ and the aim is to decide if there exists a ranking of the vertices satisfying all but at most $k$ constraints. We focus on problems where a constraint $S$ can be represented by a set of selected vertices, which determines the conditions that a ranking must verify in order to satisfy $S$. Moreover, we consider such problems on dense instances, where every set of $r$ vertices is a constraint. A lot of well-studied problems can be expressed in such terms [4, 14]. For instance, Feedback Arc Set in Tournaments fits this framework with $r = 2$, any arc $uv$ being satisfied by a ranking $\sigma$ (i.e. $c(\sigma(\sigma)) = 0$) iff $u <_\sigma v$. Such problems can be equivalently stated in terms of modification problems: can we edit at most $k$ constraints to obtain an instance that admits a ranking satisfying all its constraints?

Related results. While a lot of kernelization results are known for graph modification problems [5, 15, 18, 19], fewer results exist regarding directed graph and hypergraph modification problems. An example of polynomial kernel for a directed graph modification problem is the quadratic vertex-kernel for Transitivity Editing [20]. Regarding dense ranking $r$-CSPs, Feedback Arc Set in Tournaments and Dense Betweenness are NP-Complete [2, 3, 9] but fixed parameter tractable [4, 13], and both admit a linear vertex-kernel [17]. Concerning ranking $r$-CSPs, Karpinski and Schudy [14] recently showed PTASs and subexponential parameterized algorithms for so-called (weakly)-fragile ranking $r$-CSPs. A constraint is fragile (resp. weakly-fragile) if whenever it is satisfied by one ranking then making one single move (resp. making one of the following moves: swapping the first two vertices, the last two vertices or making a cyclic move) makes it unsatisfied. These problems contain in particular Feedback Arc Set in Tournaments and Dense Betweenness.

Our results. Following this line of research, we provide several linear vertex-kernels for dense ranking $r$-CSPs. More precisely, we introduce a particular type of ranking $r$-CSPs, called $l_r$-simply characterized, and prove that such problems admit linear vertex-kernels whenever they admit constant-factor approximation algorithms (Section 2). Surprisingly, our kernels mainly use a modification of the classical sunflower reduction rule, which usually provides polynomial kernels [4, 5, 12]. This result improves the size of the kernel for Dense Betweenness [17]. Moreover, it implies linear vertex-kernels for $r$-Dense Betweenness and $r$-Dense Transitive Feedback Arc Set, two natural generalizations of Feedback Arc Set in Tournaments and Dense Betweenness. Both problems were left open by Karpinski and Schudy [14]. Finally, we introduce a different generalization of Feedback Arc Set in Tournaments, which allows more freedom on the satisfiability of a given constraint. Based on ideas used for Feedback Arc Set in Tournaments [10], we prove that this problem admits a 5-approximation algorithm, and then obtain a linear vertex-kernel (Section 3.4).
1 Preliminaries

A ranking \( r \text{-CSP} \) consists of a ground set of vertices \( V \), an arity \( r \geq 2 \), a parameter \( k \in \mathbb{N} \) and a constraint system \( c \), where \( c \) is a function which maps rankings (i.e. orderings) of \( r \)-sized sets \( S \subseteq V \) to \( \{0,1\} \). In a slight abuse of notation, we refer to a set of vertices \( S \subseteq V, |S| = r \), as a constraint (when we are actually referring to \( c \) applied to rankings of \( S \)). A constraint \( S \) is non-trivial whenever there exists a ranking \( \sigma \) such that \( c(\sigma(S)) = 1 \). In the following, we always mean non-trivial constraints when speaking of constraints. A constraint \( S \) is satisfied by a ranking \( \sigma \) whenever \( c(\sigma(S)) = 0 \), in which case \( S \) is said to be consistent w.r.t. \( \sigma \) (we forget the mention w.r.t. \( \sigma \) whenever the context is clear). Otherwise, we say that \( S \) is inconsistent. Similarly, a ranking \( \sigma \) is consistent with the constraint system \( c \) if it does not contain any inconsistent constraint, and inconsistent otherwise. The objective of a ranking \( r \text{-CSP} \) is to find a ranking of the vertices with at most \( k \) inconsistent constraints. We consider ranking \( r \text{-CSPs} \) where a constraint \( S \) can be represented by a subset \( \text{sel}(S) \subseteq S \) of selected vertices, that determines the conditions that a ranking must verify in order to satisfy \( S \). An equivalent formulation of these problems is the following: is it possible to edit at most \( k \) constraints so that there exists a ranking consistent with the new constraint system? By editing a constraint, we mean that we modify its set of selected vertices.

We consider dense instances, where every subset of \( r \) vertices of \( V \) is a constraint. Let \( R = (V,c) \) be an instance of any ranking \( r \text{-CSP} \). Given a set of vertices \( V' \subseteq V \), we define the instance induced by \( V' \) (and denote it \( R[V'] \)) as the constraint system \( c \) restricted to \( r \)-sized subsets of \( V' \). A set of vertices \( C \subseteq V \) is a conflict if there does not exist any ranking consistent with the instance induced by \( C \). We mainly study the following problems.

| \( r \)-Dense Betweenness: |
| --- |
| **Input:** A set of vertices \( V \), a constraint system \( c \), where a constraint \( S = \{s_1, \ldots, s_r\} \) contains two selected vertices \( s_i \) and \( s_j \), \( 1 \leq i < j \leq r \), and is satisfied by a ranking \( \sigma \) iff \( s_i <_\sigma s_l <_\sigma s_j \) or \( s_j <_\sigma s_l <_\sigma s_i \) holds for \( 1 \leq l \leq r, l \neq \{i, j\} \). |
| **Parameter:** \( k \). |
| **Output:** A ranking \( \sigma \) of \( V \) that satisfies all but at most \( k \) constraints. |

| \( r \)-Dense Feedback Arc Set: |
| --- |
| **Input:** A set of vertices \( V \), a constraint system \( c \), where a constraint \( S \) contains one selected vertex \( s \) and is satisfied by a ranking \( \sigma \) if \( u <_\sigma s \) for any \( u \in S \setminus \{s\} \). |
| **Parameter:** \( k \). |
| **Output:** A ranking \( \sigma \) of \( V \) that satisfies all but at most \( k \) constraints. |

We also consider another generalization of the Feedback Arc Set in Tournaments problem, namely \( r \text{-Dense Transitive Feedback Arc Set} \) (\( r \text{-FAST} \)) [14], where a constraint \( S \) corresponds to an acyclic tournament and is satisfied by a ranking \( \sigma \) if \( \sigma \) is the transitive ranking of the corresponding tournament.
Ordered instances. In the following, we consider instances whose vertices are ordered under some fixed ranking $\sigma$ (i.e. instances of the form $R_\sigma = (V, c, \sigma)$). Given any constraint $S = \{s_1, \ldots, s_r\}$, with $s_i <_{\sigma} s_{i+1}$ for $1 \leq i < r$, $\text{span}(S)$ denotes the set of vertices $\{v \in V : s_1 \leq_{\sigma} v \leq_{\sigma} s_r\}$. A constraint $S$ is unconsecutive if $|\text{span}(S)| > r$, and consecutive otherwise. Given $V' \subseteq V$, $R_\sigma[V']$ denotes the instance $R[V']$ ordered under $\sigma$. Finally, given a ranking $\sigma$ over $V$ and an inconsistent constraint $S$, we say that we edit $S$ w.r.t. $\sigma$ whenever we edit its selected vertices so that it becomes satisfied by $\sigma$.

2 Simple characterization and sunflower

In this section, we describe the general framework of our kernelization algorithms, using a modification of the sunflower rule together with the notion of simple characterization. We first define the notion of sunflower, which has been widely used to obtain polynomial kernels for modification problems \cite{1,4,5,12}. An edition is a set of constraints $F$ such that one can obtain a consistent instance by editing constraints in $F$.

**Definition 2.1.** A sunflower $S$ is a set of conflicts $\{C_1, \ldots, C_m\}$ pairwise intersecting in exactly one constraint $S$, called the center of $S$.

**Lemma 2.2.** Let $R = (V, c)$ be an instance of any ranking $r$-CSP, and $S$ be the center of a sunflower $S = \{C_1, \ldots, C_m\}$, $m > k$. Any edition of size at most $k$ has to edit $S$.

*Proof.* Let $F$ be any edition of size at most $k$, and assume that $F$ does not edit $S$. This means that $F$ must contain one constraint for every conflict $C_i$, $1 \leq i \leq m$. Since $m > k$, we conclude that $F$ contains more than $k$ constraints, a contradiction. \qed

Observe that the sunflower rule cannot be applied directly on ranking $r$-CSPs, $r \geq 3$, since it may be the case that there exist several ways to edit the center of a given sunflower. In order to deal with this issue, we introduce the notion of simple characterization for ranking $r$-CSPs. Roughly speaking, a ranking $r$-CSP is $l_r$-simply characterized if for any ordered instance, any set of $l_r$ vertices which involve exactly one inconsistent constraint is a conflict. We first give a formal definition of this notion, and next describe how the sunflower rule can be modified for such problems.

**Definition 2.3** (Simple characterization). Let $\Pi$ be a ranking $r$-CSP, $R_\sigma = (V, c, \sigma)$ be any ordered instance of $\Pi$, and $l_r \in \mathbb{N}$. The ranking $r$-CSP $\Pi$ is $l_r$-simply characterized iff any set $C \subseteq V$ of $l_r$ vertices such that $R_\sigma[C]$ contains exactly one inconsistent constraint is a conflict.
**Definition 2.4** (Simple sunflower). Let $R_{\sigma} = (V, c, \sigma)$ be an ordered instance of a $l_r$-simply characterized ranking $r$-CSP. A sunflower $S = \{C_1, \ldots, C_m\}$ of $R_{\sigma}$ is simple if its center is the only inconsistent constraint in $R_{\sigma}[C_i]$, $1 \leq i \leq m$.

**Rule 2.1.** Let $\Pi$ be a $l_r$-simply characterized ranking $r$-CSP. Let $R_{\sigma} = (V, c, \sigma)$ be an ordered instance of $\Pi$ and $S = \{C_1, \ldots, C_m\}$, $m > k$, be a simple sunflower of center $S$. Edit $S$ w.r.t. $\sigma$ and decrease $k$ by 1.

**Lemma 2.5.** Rule 2.1 is sound.

**Proof.** Let $F$ be any edition of size at most $k$: by Lemma 2.2, $F$ must contain $S$. Since $|F| \leq k$ and $m > k$, there exists $1 \leq i \leq m$ such that $S$ is the only constraint edited by $F$ in $R[C_i]$. Assume that $S$ was not edited w.r.t. $\sigma$: since no other constraint has been edited in $R[C_i]$, $R_{\sigma}[C_i]$ still contains exactly one inconsistent constraint (namely $S$). Since $\Pi$ is $l_r$-simply characterized, it follows that $C_i$ defines a conflict, contradicting the fact that $F$ is an edition.

The main problem that remains is to compute such a sunflower in polynomial time. The following result will allow us to do so, providing that $V$ contains sufficiently many vertices (w.r.t. parameter $k$).

**Lemma 2.6.** Let $\Pi$ be a $l_r$-simply characterized ranking $r$-CSP, and $R_{\sigma} = (V, c, \sigma)$ be an ordered instance of $\Pi$ with at most $p \geq 1$ inconsistent constraints. If $|V| > p(l_r - r) + (l_r - r) \cdot (k + 1) + r$, there exists a simple sunflower $\{C_1, \ldots, C_m\}$, $m > k$, that can be found in polynomial time.

**Proof.** Let $S$ be any inconsistent constraint of $R_{\sigma}$. Since $R_{\sigma}$ contains at most $p$ inconsistent constraints, there are at most $p$ disjoint sets $P_i$, $1 \leq i \leq p$, such that $|P_i| = l_r - r$ and $R_{\sigma}[S \cup P_i]$ contains more than one inconsistent constraint. It follows that there exist at least $m > k$ disjoint sets $\{S_1, \ldots, S_m\}$ of size $l_r - r$ such that: (i) $C_i = S \cup S_i$ contains $l_r$ vertices and (ii) $R_{\sigma}[C_i]$ contains exactly one inconsistent constraint, $1 \leq i \leq m$. Since $\Pi$ is $l_r$-simply characterized, $C_i$ defines a conflict for every $1 \leq i \leq m$. It follows that $\{C_1, \ldots, C_m\}$ is a simple sunflower of center $S$.

In order to obtain the ranking necessary to apply Lemma 2.6, we rely on the existence of a constant-factor approximation algorithms for the problem at hand.

**Theorem 2.7.** Let $\Pi$ be a $l_r$-simply characterized ranking $r$-CSP that admits a $q$-factor approximation algorithm for some constant $q > 0$. Then $\Pi$ admits a kernel with at most $k[(q+1) \cdot (l_r - r)] + l_r$ vertices.

**Proof.** Let $R = (V, c)$ be an instance of $\Pi$. We start by computing a ranking $\sigma$ containing $p$ inconsistent constraints using the $q$-factor approximation algorithm. Observe that we can assume that $p > k$, since otherwise we simply return a small trivial Yes-instance. Similarly, we can assume that $p \leq qk$, since otherwise we return a small trivial No-instance. We now consider $R_{\sigma} = (V, c, \sigma)$ and assume that $|V| > p(l_r - r) + (l_r - r) \cdot (k + 1) + r$: by Lemma 2.6, it follows that there exists a simple sunflower that can be found in polynomial time, and hence Rule 2.1 can be applied. Since conditions of Lemma 2.6 still hold after an application of Rule 2.1 repeating this process on $R_{\sigma}$.
implies that every inconsistent constraint must be edited. Since \( p > k \), we return a small trivial No-instance in such a case. This means that \(|V| \leq qk(l_r - r) + (l_r - r) \cdot (k + 1) + r\), implying the result. 

3 Simple characterization of several ranking \( r \)-CSPs

3.1 Dense Betweenness

As a first consequence of Theorem 2.7, we improve the size of the linear vertex-kernel for BIT from \( 5k \) \[17\] to \((2 + \epsilon)k + 4\) for any \( \epsilon > 0 \). The result directly follows from the fact that BIT admits a PTAS \[14\] and is 4-simply characterized \[17\].

**Corollary 3.1.** Dense Betweenness admits a kernel with at most \((2 + \epsilon)k + 4\) vertices.

3.2 \( r \)-Dense Betweenness (\( r \geq 4 \))

We now consider the \( r \)-BIT problem with constraints of arity \( r \geq 4 \). The main difference with the case \( r = 3 \) lies in the fact that there is no longer a unique way to rank the vertices of a consistent instance in order to satisfy all constraints. In particular, this means that the problem is not \((r+1)\)-simply characterized. However, as we shall see in Lemma 3.4, \( r \)-BIT is \( 2^r \)-simply characterized. To see this, we first need the following result.

**Lemma 3.2.** Let \( R_\sigma = (V, c, \sigma) \) be an ordered instance of \( r \)-BIT, and \( C = \{s_1, \ldots, s_{r+1}\} \) be a set of \( r+1 \) vertices such that \( s_i <_\sigma s_{i+1}, 1 \leq i \leq r \). Assume that \( R_\sigma[C] \) contains exactly one inconsistent constraint \( S \). Then \( C \) is a conflict if and only if:

(i) \( S \) is unconsecutive or,

(ii) \( S = \{s_1, \ldots, s_r\} \) and \( sel(S) \neq \{s_1, s_l\} \) with \( 2 < l < r \) (resp. \( S = \{s_2, \ldots, s_{r+1}\} \) and \( sel(S) \neq \{s_1, s_{r+1}\} \) with \( 2 < l < r \)).

**Proof.** Assume first that the vertices of \( S \) are unconsecutive (i.e. there exists \( 1 < i < r + 1 \) such that \( S \) is equal to \( C \setminus \{s_i\} \)).

- **Case 1.** Assume that a single move on a selected vertex is sufficient to make \( S \) consistent. To be more precise, we assume that we have \( sel(S) = \{s_l, s_{r+1}\} \) for some \( 2 \leq l \leq r, l \neq i \) (the case \( sel(S) = \{s_1, \ldots, s_l\} \), \( 2 \leq l \leq r \) and \( l \neq i \) being similar). Since all constraints induced by \( C \) but \( S \) are consistent w.r.t. \( \sigma \), we know that \( sel(C \setminus \{s_j\}) = \{s_1, s_{r+1}\} \), with \( 2 \leq j \leq r, j \neq \{i, l\} \) (which is well-defined since \( r \geq 4 \)).

In order to satisfy \( S \), any consistent ranking \( \rho \) must rank \( s_l \) between \( s_l \) and \( s_{r+1} \), while \( C \setminus \{s_j\} \) implies that \( s_l \) must be between \( s_1 \) and \( s_{r+1} \). Since no ranking can satisfy both these properties, it follows that there does not exist any ranking consistent with \( C \).

- **Case 2.** Next, assume that the two selected vertices must be moved to obtain a consistent ranking for \( S \). In other words, we have \( sel(S) = \{s_l, s_{l'}\} \) for some \( 2 \leq l < l' \leq r, \{l, l'\} \neq i \).
Since all constraints but $S$ are consistent, we know that $\text{sel}(C \setminus \{s_{r+1}\}) = \{s_1, s_r\}$. In order to satisfy $S$, any consistent ranking must rank $s_1$ between $s_l$ and $s_r$, while $C \setminus \{s_{r+1}\}$ implies that $s_1$ must be before (or after) both $s_l$ and $s_{r'}$, which cannot be.

Assume now that the vertices of the inconsistent constraint $S$ are $\{s_1, \ldots, s_r\}$. Moreover, assume that a single move on a selected vertex is sufficient to make $S$ consistent (see Figure 2 for an illustration).

- **Case 1.** If we have $\text{sel}(S) = \{s_1, s_l\}$ with $2 < l < r$, then swapping $s_l$ and $s_r$ will yield a consistent ranking. Indeed, observe that since all constraints but $S$ are consistent, it follows that the only constraint where $s_l$ is a selected vertex is $S$, and that $(s_1, s_{r+1})$ and $(s_2, s_{r+1})$ are the selected vertices of the other constraints. Hence swapping $s_l$ and $s_r$ will not modify the consistency of any constraint but $S$. This is the only case where we can get a consistent ranking.

- **Case 2.** Assume now that $\text{sel}(S) = \{s_1, s_l\}$ with $l = 2$. Then we have $\text{sel}(C \setminus \{s_1\}) = \{s_l, s_{r+1}\}$ and $\text{sel}(C \setminus \{s_l\}) = \{s_1, s_{r+1}\}$. Observe here that in order to satisfy $S$ and $C \setminus \{s_1\}$, any ranking $\rho$ must rank $s_r$ between $s_1$ and $s_l$ and between $s_l$ and $s_{r+1}$. W.l.o.g., this means that we must have $s_1 < s_{r+1} < s_r < s_l$, which is inconsistent with $C \setminus \{s_l\}$. It follows that $C$ is a conflict.

- **Case 3.** Assume next that we have $\text{sel}(S) = \{s_l, s_r\}$, $1 < l < r$, and $\text{sel}(C \setminus \{s_l\}) = \{s_1, s_{r+1}\}$. We know that $\text{sel}(C \setminus \{s_j\}) = \{s_1, s_{r+1}\}$, with $2 \leq j < r$, $j \neq l$. Once again, this means that any consistent ranking $\rho$ must rank $s_1$ between $s_r$ and $s_l$ in order to satisfy $S$, and $s_l$ between $s_1$ and $s_{r+1}$ in order to satisfy $C \setminus \{s_j\}$ (recall that $l < r$). W.l.o.g., this means that we must have $s_r < s_1 < s_l < s_{r+1}$, which is inconsistent with $C \setminus \{s_l\}$. Hence $C$ is a conflict.

- **Case 4.** Finally, assume that the two selected vertices must be moved to obtain a consistent ranking for $S$. In other words, we have $\text{sel}(S) = \{s_l, s_{l'}\}$ with $1 < l < l' < r$. We have $\text{sel}(C \setminus \{s_r\}) = \{s_1, s_{r+1}\}$. This implies that any consistent ranking must rank $s_1$ between $s_l$ and $s_{l'}$ (to satisfy $S$) and $s_l$ and $s_{l'}$ between $s_1$ and $s_{r+1}$ (to satisfy $C \setminus \{s_r\}$). Since no ranking can satisfy both these properties, it follows that $C$ is a conflict.

The cases where the vertices of the inconsistent constraint are $\{s_2, \ldots, s_{r+1}\}$ is similar to the previous one, the only case yielding a consistent ranking being when $\text{sel}(S) = \{s_l, s_{r+1}\}$ and $2 \leq l < r$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{An illustration of some cases for $r = 4$. The gray boxes represent the inconsistent constraint $S$. (a) $\text{sel}(S) = \{s_1, s_l\}$ with $2 < l < r$; (b) $\text{sel}(S) = \{s_1, s_l\}$ with $l = 2$ and (c) $S$ is unconsecutive. In the former case $\sigma$ is not a conflict, while $\sigma$ is a conflict in the latter ones.}
\end{figure}
Compatible constraints. Given an ordered instance \( R_{\sigma} = (V, c, \sigma) \) of r-BIT, we say that an inconsistent constraint \( S = \{s_1, \ldots, s_r\}, s_1 <_{\sigma} \ldots <_{\sigma} s_r, \) is right- (resp. left-) compatible whenever \( \text{sel}(S) = \{s_1, s_l\} \) with \( 2 < l < r \) (resp. \( \text{sel}(S) = \{s_l, s_r\} \) with \( 1 < l < r - 1 \)), and right- (resp. left-) incompatible otherwise. The intuition behind this notion is the following: for any vertex \( u \) lying before (resp. after) \( S \) in \( \sigma \) such that \( S \) is the only inconsistent constraint in \( R_{\sigma}[S \cup \{u\}] \), the set \( S \cup \{u\} \) does not define a conflict (see Figure 3). The following result directly follows by definition of a compatible constraint.

**Observation 3.3.** Any right- (resp. left-) compatible constraint is left- (resp. right-) incompatible.

**Lemma 3.4.** The r-BIT problem is \( 2r \)-simply characterized.

**Proof.** Observe that compatible constraints correspond to the cases of Lemma 3.2 that fail to define a conflict. Let \( R_{\sigma} = (V, c, \sigma) \) be an ordered instance of r-BIT and \( C = \{s_1, \ldots, s_{2r}\} \) be a set of \( 2r \) vertices such that \( s_i <_{\sigma} s_{i+1} \) for \( 1 \leq i < 2r \). Assume that \( R_{\sigma}[C] \) contains exactly one inconsistent constraint \( S \). We need to prove that \( C \) is a conflict. By Lemma 3.2 the result holds if \( S \) is neither right nor left-compatible. So we assume w.l.o.g. that \( S \) is right-compatible. By Lemma 3.2 we can also assume that the vertices of \( S \) are consecutive and are the first of the ranking, since otherwise \( C \) is a conflict by Lemma 3.2 and we are done (recall that \( S \) is left-incompatible by Observation 3.3). Hence we may assume that \( S = \{s_1, \ldots, s_r\} \) and \( \text{sel}(S) = \{s_1, s_l\} \) for \( 2 < l < r \). Moreover, the constraints \( S_2 = \{s_l, \ldots, s_r, s_{l+r}\} \) and \( S_3 = \{s_1, \ldots, s_l, s_{r-l}, \ldots, s_{l+r}\} \) (with \( |S_3| = r \)) have as selected vertices \( \text{sel}(S_2) = \{s_l, s_{l+r}\} \) and \( \text{sel}(S_3) = \{s_1, s_{l+r}\} \). In order to be consistent with \( S \) and \( S_2 \), any ranking \( \rho \) must rank \( s_r \) between \( \{s_1, s_{l+r}\} \) and \( s_l \), which is inconsistent with the last constraint (which forces \( s_r \) to be between \( s_1 \) and \( s_{l+r} \)). \( \square \)

**Corollary 3.5.** r-BIT admits a kernel with at most \( (2 + \varepsilon)rk + 2r \) vertices.

### 3.3 r-Dense Transitive Feedback Arc Set

Karpinski and Schudy [13] considered a particular generalization of the Feedback Arc Set in Tournaments problem, where every constraint \( S \) is satisfied by a one particular ranking \( \sigma \) and no other (i.e. \( S \) corresponds to an acyclic tournament). We show that the r-TFAST problem admits a linear vertex-kernel as a particular case of fragile ranking r-CSP [14]. We say that a ranking r-CSP is strongly fragile whenever a constraint is satisfied by one particular ranking and no other.
Lemma 3.6. Let \( \Pi \) be any strongly fragile ranking \( r \)-CSP, \( r \geq 3 \). Then \( \Pi \) is \((r + 1)\)-simply characterized.

Proof. Let \( R_\sigma = (V, c, \sigma) \) be an ordered instance of \( \Pi \), and \( C \) be a set of \( r + 1 \) vertices such that \( R_\sigma[C] \) contains exactly one inconsistent constraint \( S \). We need to prove that \( C \) is a conflict. Assume for a contradiction that this is not the case, i.e. that there exists a ranking \( \rho \) consistent with \( C \). In particular, there exist two vertices \( u, v \in S \) such that \( u <_\sigma v \) and \( v <_\rho u \). Let \( S' \neq S \) be any constraint of \( R_\sigma[C] \) such that \( \{u, v\} \subset S' \) (observe that \( S' \) is well-defined since \( r \geq 3 \)). Since \( S' \) was consistent in \( \sigma \) and since \( \Pi \) is strongly fragile, \( S' \) is inconsistent in \( \rho \); a contradiction.

Corollary 3.7. Any strongly fragile ranking \( r \)-CSP admits a kernel with at most \((2 + \epsilon)k + (r + 1)\) vertices.

3.4 \( r \)-Dense Feedback Arc Set (\( r \geq 3 \))

As mentioned previously, the \( r \)-TFAST problem deals with constraints that are given by a transitive tournament and are thus satisfied by one particular ranking and no other. To allow more freedom on the satisfiability of a constraint, we consider a different generalization of this problem, namely \( r \)-FAST. Recall that, in this problem, any constraint \( S \) contains a selected vertex \( s \) and is satisfied by a ranking \( \sigma \) if \( u <_\sigma s \) for any \( u \in S \setminus \{s\} \). Observe that \( r \)-FAST is not (weakly-)fragile, since swapping the first two vertices of any consistent ranking yields a consistent ranking. Hence, we cannot directly apply the PTASs from [14]. However, the results needed to obtain a 5-approximation for Feedback Arc Set in Tournaments [10] can be generalized to the \( r \)-Dense Feedback Arc Set problem.

3.4.1 Approximation algorithm

Definition 3.8. Let \( R = (V, c) \) be any instance of \( r \)-FAST, and \( v \in V \). The in-degree of \( v \) is the number of constraints where \( v \) is selected.

Algorithm [INC-DEGREE] Order the vertices of \( R \) according to their increasing in-degrees.

Theorem 3.9. INC-DEGREE is a 5-approximation for \( r \)-FAST.

We prove Theorem 3.9 by proving a series of Lemmata. For the sake of simplicity, we let \( V = \{1, \ldots, n\} \) in the remaining of this Section. For any vertex \( v \) of \( V \), we call left constraint (resp. above constraint) any constraint containing \( v \) and vertices before (resp. after or both before and after) \( v \), and let \( L_\sigma(v) \) (resp. \( A_\sigma(v) \)) be the set of left constraints (resp. above constraints) of \( v \). Moreover, we set \( l_\sigma(v) = |L_\sigma(v)| \). Finally, given an ordered instance \( R_\sigma = (V, c, \sigma) \), we define \( B_\sigma \) as the set of inconsistent constraints of \( R_\sigma \), and let \( b_\sigma = |B_\sigma| \). We need to define a distance function between two rankings \( \rho \) and \( \gamma \):

\[
K(\rho, \gamma) = \sum_{S \subseteq V, |S| = r} 1_{(c(\rho(S)) = 1 \land c(\gamma(S)) = 0) \lor (c(\rho(S)) = 0 \land c(\gamma(S)) = 1)}
\]
where 1 denotes the indicative function. This distance gives the number of constraints which are consistent in exactly one out of the two rankings, and thus generalizes the Kendall-Tau distance between two rankings \([10]\).

**Lemma 3.10.** Let \(\rho : V \to V\) be any ranking. The following holds:

\[
2 \cdot b_{\rho} \geq \sum_{v \in V} |l_{\rho}(v) - In(v)|
\]

*Proof.* Let \(v \in V\) be any vertex. We set:

\[
\begin{align*}
W_{L}^{-}(v) &= |\{S \in L_{\rho}(v) : v = \text{sel}(S)\}| \\
W_{L}^{+}(v) &= |\{S \in L_{\rho}(v) : v \neq \text{sel}(S)\}| \\
W_{R}^{-}(v) &= |\{S \in A_{\rho}(v) : v = \text{sel}(S)\}|
\end{align*}
\]

These numbers respectively represent the left constraints where \(v\) is the selected vertex, those where \(v\) is not the selected vertex and finally the above constraints where \(v\) is the selected vertex. Observe that in the last two cases, the considered constraints are inconsistent. By definition, we have that

\[
W_{L}^{-}(v) + W_{R}^{-}(v) = In(v).
\]

Moreover, we have \(l_{\rho}(v) = W_{L}^{+}(v) + W_{L}^{-}(v)\). Now, observe that:

\[
2 \cdot b_{\rho} = \sum_{v \in V} (W_{L}^{+}(v) + W_{R}^{-}(v))
\]

To see this, observe that any inconsistent constraint \(S = \{s_{1}, \ldots, s_{r}\}\) with \(s_{i} <_{\rho} s_{i+1}\), \(1 \leq i < r\), will be counted exactly twice in the sum: (i) \(S\) is counted in \(W_{R}^{-}(\text{sel}(S))\) and (ii) in \(W_{L}^{+}(s_{r})\), and these are the only vertices for which \(S\) will be taken into account.

We conclude the proof by showing that \(W_{L}^{+}(v) + W_{R}^{-}(v) \geq |l_{\rho}(v) - In(v)|\). By the previous observations, we have:

\[
W_{L}^{+}(v) + W_{R}^{-}(v) = l_{\rho}(v) + In(v) - 2W_{L}^{-}(v) = |l_{\rho}(v) - In(v)| + 2(\min\{l_{\rho}(v), In(v)\} - W_{L}^{-}(v)) \geq |l_{\rho}(v) - In(v)|
\]

\(\square\)

In the following, we denote by \(\sigma_{A}\) the ranking returned by \textsc{Inc-Degree}, and by \(\sigma_{C}\) the ranking returned by any optimal solution.

**Lemma 3.11.** Let \(\rho : V \to V\) be any ranking. The following holds:

\[
\sum_{v \in V} |l_{\rho}(v) - In(v)| \geq \sum_{v \in V} |l_{\sigma_{A}}(v) - In(v)|
\]

*Proof.* Observe that this result is trivial if \(\rho\) orders the vertices by increasing in-degrees. So assume this is not the case. This means that there exists \(i \in V\) such that \(In(u) > In(v)\) and \(u = \rho^{-1}(i)\),
Let \( \rho, \gamma : V \to V \) be two rankings. The following holds:

\[
\sum_{v \in V} |l(\rho, v) - l(\gamma, v)| \geq |b(\rho) - b(\gamma)|
\]
Proof. We consider the set of constraints which are inconsistent in $\rho$ but not in $\gamma$ and denote it by $B_{\rho \setminus \gamma}$. Similarly, we consider $B_{\gamma \setminus \rho}$. First observe that:

$$\sum_{S \in B_{\rho \setminus \gamma}} 1 + \sum_{S \in B_{\gamma \setminus \rho}} 1 \geq |b_\gamma - b_\rho|$$

Moreover, we know by definition that $K(\rho, \gamma) \geq \sum_{S \in B_{\rho \setminus \gamma}} 1 + \sum_{S \in B_{\gamma \setminus \rho}} 1$. We need the following result.

Claim 3.13. The following holds: $\sum_{v \in V} |l_\rho(v) - l_\gamma(v)| \geq K(\rho, \gamma)$.

Proof. Let $v \in V$ be any vertex. First, observe that since we are considering a dense instance, we have (assuming w.l.o.g. that $\rho(v) > \gamma(v)$):

$$|l_\rho(v) - l_\gamma(v)| = \left( \frac{\rho(v) - 1}{r - 1} \right) - \left( \frac{\gamma(v) - 1}{r - 1} \right)$$

$$= \sum_{i=0}^{r-2} \binom{\gamma(v) - 1}{i} \cdot \binom{\rho(v) - \gamma(v)}{(r - 1) - i}$$

We now count the number of constraints whose consistency may have been modified by the change of position of $v$ in $\rho$ and $\gamma$. Let $S'$ be the set $\{v' \in V : \rho(v') < \rho(v)\}$. Moreover, we define the following sets:

$$S_1 = \{s' \in S' : \gamma(s') < \gamma(v)\}$$
$$S_2 = \{s' \in S' : \gamma(v) \leq \gamma(s') \leq \rho(v)\}$$
$$S_3 = S' \setminus (S_1 \cup S_2)$$

![Figure 4: $S_1, S_2$ and $S_3$ defined according to $v \in V$.](image)

Let $S_v$ be the set of constraints of $\subseteq S_1 \cup S_2 \cup \{v\}$ whose consistency have been modified between the two rankings due to the change of position of $v$. Observe that any constraint $S$ in $S_v$ must contain $v$. Moreover, $S$ cannot contain $r - 1$ vertices in $S_1$, since otherwise its consistency is not modified by moving $v$ (which would be the leftmost vertex of $S$ in both $\rho$ and $\gamma$). Since we are on a dense instance, we know that:

$$|S_v| \leq \sum_{i=0}^{r-2} \binom{\gamma(v) - 1}{i} \cdot \binom{\rho(v) - \gamma(v)}{(r - 1) - i}$$

$$\leq |l_\rho(v) - l_\gamma(v)|$$
A similar counting argument can be applied when \( \rho(v) < \gamma(v) \) (observe that the consistency of any constraint is not modified by \( v \) when \( \rho(v) = \gamma(v) \)). Altogether, we thus have:

\[
\sum_{v \in V} |S_v| \leq \sum_{v \in V} |l_\rho(v) - l_\gamma(v)|
\]

To conclude the proof, we show that \( \sum_{v \in V} |S_v| \geq K(\rho, \gamma) \). To see this, notice that any constraint that belongs to \( K(\rho, \gamma) \) must contain two vertices \( u \) and \( v \) such that (w.l.o.g.) \( u <_\rho v \) and \( v <_\gamma u \), with the additional property that \( S \in L_\rho(v) \) and \( S \in L_\gamma(u) \). Indeed, by definition of \( r\text{-Dense Transitive Feedback Arc Set} \), the leftmost vertex of \( S \) in \( \rho \) cannot be the leftmost vertex of \( S \) in \( \gamma \), since otherwise we would have \( c(\rho(S)) = c(\gamma(S)) \). In particular, this means that \( S \) will be counted in (at least) \( S_{max\{\rho(v),\gamma(u)\}} \).

This concludes the proof of Lemma 3.12.

We are now ready to prove the main result of this section:

\[\textbf{Proof of Theorem 3.9} \] By the previous Lemmata, we have the following:

\[
4b_{\sigma_o} \geq \sum_{v \in V} |l_{\sigma_o}(v) - In(v)| + \sum_{v \in V} |l_{\sigma_o}(v) - In(v)|
\]
\[
\geq \sum_{v \in V} |l_{\sigma_o}(v) - In(v)| + \sum_{v \in V} |l_{\sigma_A}(v) - In(v)|
\]
\[
= \sum_{v \in V} (|l_{\sigma_o}(v) - In(v)| + |l_{\sigma_A}(v) - In(v)|)
\]
\[
\geq \sum_{v \in V} |l_{\sigma_o}(v) - l_{\sigma_A}(v)|
\]
\[
\geq b_{\sigma_A} - b_{\sigma_o}
\]

Hence we have \( b_{\sigma_A} \leq 5b_{\sigma_o} \), which implies the result.

\[\text{\textbf{3.4.2 Kernelization algorithm}}\]

In order to design our kernelization algorithm for this problem, we need to study the topology of conflicts that contain exactly one inconsistent constraint. As we shall see, the configuration for \( r\text{-Dense Feedback Arc Set} \) is slightly different to the ones previously observed. In particular, as we shall see, the problem is not \( l_r \)-simply characterized. However, the addition of a new reduction rule will allow us to conclude as in the other cases.

\[\textbf{Lemma 3.14}.\ Let \( R_\sigma = (V, c, \sigma) \) be an ordered instance of \( r\text{-FAST} \), and \( C = \{s_1, \ldots, s_{r+1}\} \) be a set of \( r+1 \) vertices such that \( s_i <_\sigma s_{i+1} \) for every \( 1 \leq i \leq r \). Assume \( R_\sigma[C] \) contains exactly one inconsistent constraint \( S \). Then \( C \) is a conflict iff \( S \) is unconsecutive or \( S = \{s_2, \ldots, s_{r+1}\} \).

\[\textbf{Proof}.\ Assume first that \( S \) is unconsecutive (i.e. there exists \( 2 \leq i \leq r \) such that \( S = C \setminus \{s_i\} \)). Since any constraint different from \( S \) is consistent w.r.t. \( \sigma \), it follows that \( sel(C \setminus \{s_j\}) = s_{r+1} \) for any \( 1 \leq j \leq r, j \neq \{l, i\} \). Together with the fact that \( sel(S) = s_l \) for \( l < r+1 \), this implies that any consistent ranking \( \rho \) must verify \( s_l <_\rho s_{r+1} \) and \( s_{r+1} <_\rho s_l \) which cannot be. We now deal with the case where \( S \) is induced by \( \{s_2, \ldots, s_{r+1}\} \). Once again, this means that \( sel(S) = s_l, 2 \leq l < r+1, \) and \( sel(C \setminus \{s_j\}) = s_{r+1} \) for any \( 2 \leq j \leq r, j \neq l \), which is inconsistent with any
Finally, assume that $S$ is induced by $\{s_1, \ldots, s_r\}$. Now, by swapping the vertices $\text{sel}(S)$ and $s_r$ (notice that there is no constraint where $s_r$ is the selected vertex), we obtain a consistent ranking, implying that $C$ is not a conflict in this case.

**Corollary 3.15.** There does not exist $l_r \in \mathbb{N}$ such that $r$-FAST is $l_r$-simply characterized.

**Proof.** Let $R = (V, c)$ be an instance of $r$-FAST and $q \in \mathbb{N}$, $q > r$. Let $C = \{s_1, \ldots, s_q\}$ be any set of vertices ordered under some ranking $\sigma$ such that $s_i <_\sigma s_{i+1}$ for $1 \leq i < q$. Assume that $S = \{s_1, \ldots, s_r\}$ is the only inconsistent constraint of $R_\sigma[C]$. By Lemma 3.14, we know that $C$ is not a conflict, implying that $r$-FAST is not $q$-simply characterized.

We need the following rule, which implies that the last vertex of any ordered instance of $r$-FAST belongs to an inconsistent constraint.

**Rule 3.1.** Let $v$ be any vertex which is selected in every constraint containing it. Remove $v$ and any constraint containing it from $R = (V, c)$.

**Lemma 3.16.** Rule 3.1 is sound and can be applied in polynomial time.

**Proof.** First, observe that any edition of size at most $k$ for the original instance will yield an edition for the reduced one. In the other direction, assume that $R_v = (V \setminus \{v\}, c)$ admits an edition $F$ of size at most $k$, and let $\sigma$ be the consistent ranking obtained after editing the constraints of $F$. Since adding $v$ to the end of $\sigma$ does not introduce any inconsistent constraint, $F$ is also an edition for the original instance.

Observe that a given instance can contain at most one such vertex. We thus iteratively apply this rule until no vertex selected in every constraint containing it remains. Given any constraint $S$ of an ordered instance $R_\sigma = (V, c, \sigma)$, $\text{span}^-(S)$ denotes the set containing $\text{span}(S)$ and all vertices lying before $S$ in $\sigma$. Observe that by Lemma 3.14 any set $C \subseteq V$ of $r + 1$ vertices such that $R_\sigma[C]$ contains exactly one inconsistent constraint is a conflict iff $C \subseteq \text{span}^-(S)$.

**Rule 3.2.** Let $R_\sigma = (V, c, \sigma)$ be an ordered instance of $r$-FAST and $S = \{C_1, \ldots, C_m\}$, $m > k$, be a simple sunflower of center $S$. Edit $S$ w.r.t. $\sigma$ and decrease $k$ by 1.

**Lemma 3.17.** Rule 3.2 is sound.

**Proof.** Let $F$ be any edition of size at most $k$: by Lemma 2.2, $F$ must contain $S$. Since $|F| \leq k$, there exists $1 \leq i \leq m$ such that $S$ is the only constraint edited by $F$ in $R[C_i]$. Assume that $S$ was not edited w.r.t. $\sigma$: since no other constraint has been edited in $R[C_i]$, $R_\sigma[C_i]$ still contains exactly one inconsistent constraint (namely $S$). Observe now that, by definition of a simple conflict, $\bigcup_{i=1}^{m} C_i \subseteq \text{span}^-(S)$ holds for any simple sunflower $S$. Hence Lemma 3.14 implies that $C_i$ is a conflict, contradicting the fact that $F$ is an edition.

Here again, we can prove the existence of a simple sunflower under a certain cardinality assumption.
Lemma 3.18. Let \( R_{\sigma} = (V, c, \sigma) \) be an ordered instance of \( r \)-FAST with at most \( p \geq 1 \) inconsistent constraints, and \( S \) be an inconsistent constraint with \(|\text{span}^{-}(S)| > p + k + r\). Then \( S \) is the center of a simple sunflower \( \{C_1, \ldots, C_m\}, m > k \).

Proof. Since there at most \( p \) vertices \( v \) such that \( R_{\sigma}[S \cup \{v\}] \) contains more than one inconsistent constraint, there exist \( k + 1 \) vertices \( \{s_1, \ldots, s_{k+1}\} \subseteq \text{span}^{-}(S) \) such that \( C_i = S \cup \{s_i\} \) contains exactly one inconsistent constraint, \( 1 \leq i \leq k + 1 \). Since \( s_i \in \text{span}^{-}(S) \) for \( 1 \leq i \leq k + 1 \), Lemma 3.14 implies that \( C_i \) is a conflict for \( 1 \leq i \leq k + 1 \). It follows that \( S \) is the center of the simple sunflower \( \{C_1, \ldots, C_{k+1}\} \).

Theorem 3.19. \( r \)-FAST admits a kernel with at most \( 6k + r \) vertices.

Proof. Let \( R = (V, c) \) be an instance of \( r \)-FAST. We start by running the constant-factor approximation (Theorem 3.9) on \( R \), obtaining a ranking \( \sigma \) of \( V \) with at most \( p \) inconsistent constraints. Notice that we may assume \( p > k \) and \( p \leq 5k \), since otherwise we return a small trivial Yes- (resp. No-)instance. Assume that \(|V| > p + k + r \) and let \( S \) be any inconsistent constraint containing \( v_n \) (thus \( \text{span}^{-}(S) = V \)). Observe that \( S \) is well-defined since \( R \) is reduced by Rule 3.1 (and hence \( v_n \) must belong to an inconsistent constraint). By Lemma 3.18 it follows that we can find a simple sunflower \( \{C_1, \ldots, C_m\}, m > k \) in polynomial time. We thus apply Rule 3.2 and edit \( S \) w.r.t. \( \sigma \). We now apply Rule 3.1 and repeat this process until we either do not find a large enough simple sunflower or \( k < 0 \). In the former case, Lemma 3.18 implies that \(|V| \leq p + k + r \leq 6k + r \), while in the latter case we return a small trivial No-instance.

4 Conclusion

In this paper, we considered the kernelization of several dense ranking \( r \)-CSPs. In particular, we proved that any ranking \( r \)-CSP that can be simply characterized and that admits a constant factor-approximation algorithm also admits a linear vertex-kernel. Interestingly, our kernelization algorithm makes use of a modification of the classical sunflower rule, which usually provides polynomial kernels. As a main consequence of this result, we improved the size of the kernel for Dense Betweenness, and described the first polynomial kernels for \( r \)-BIT and \( r \)-TFAST. A natural question is thus whether these techniques be applied to dense ranking \( r \)-CSP for (weakly-)fragile constraints [14]? Such problems are known to be fixed-parameter tractable [13], but the existence of a polynomial kernel remains an open problem. We also considered another generalization of Feedback Arc Set in Tournaments, namely \( r \)-Dense Feedback Arc Set. We obtained a 5-approximation algorithm as well as a linear vertex-kernel for this problem. Investigating analogy with Feedback Arc Set in Tournaments, it would be interesting to determine whether this problem admits a PTAS.

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