Computer Simulation of Station Keeping Costs of Halo Orbits in Sun-Earth system

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Abstract. Spacecraft missions to the collinear libration points of the Sun-Earth system have long proved scientific merit by providing valuable observational data. The unstable nature of the orbits around the mentioned points in space requires the development of control strategies for such missions in order to hold the spacecraft in vicinity of the libration points. We investigate a possible control strategy for the spacecraft on the halo orbits around the L1 and L2 collinear libration points aiming to evaluate the station keeping performance. The station-keeping strategy is based on the elimination of unstable component of motion by periodical velocity corrections performed in unstable directions. The simulations were performed in circular restricted three body problem. The impact of spacecraft position and velocity measurement accuracy, as well as the precision of the control maneuvers, on the station keeping costs, was calculated and studied. It was found that the measurement accuracy of the velocity components of motion and the precision of control burns implementation have the highest impact on the performance of the control strategy.

1. Introduction

Spacecraft mission development relies on the selection of the target orbit, selection of the transfer strategy to the target orbit and the station-keeping strategy for the spacecraft on the target orbit. The latter problem plays important role for the libration point missions because of unstable nature \cite{1} of orbits around the collinear libration points $L_1$, $L_2$, and $L_3$.

Known impulsive station-keeping techniques include: targeting strategies \cite{2, 3}, Floquet mode approach \cite{4, 5, 6} and others \cite{7, 8}. The station-keeping technique employed in current study utilizes bisection method \cite{9} to eliminate instability of nonlinear motion by performing impulsive maneuvers in unstable direction.

Monte-Carlo simulation was used to introduce position and velocity measurement errors, correction impulse execution error and to estimate overall station-keeping costs. The Circular restricted three-body problem (CRTBP) model \cite{10} was utilized to study overall patterns of the spacecraft trajectories near the $L_1$ and $L_2$ libration points. The realistic simulation of the station-keeping strategy was performed by applying the randomized errors to the initial spacecraft state-vector and computing the correction impulse needed to reduce the divergence of spacecraft trajectory. The divergence of the positional and velocity components of the spacecraft state vector correspond to the measurement errors. The correction maneuver execution errors were also applied to simulate the finite precision of the spacecraft propulsion systems.

All simulations were performed using SciPy \cite{11} library for Python.
2. Circular Restricted Three Body Problem

The Circular Restricted Three Body Problem (CRTBP) [10] defines the mathematical model for the state vector of the spacecraft at the given moment of time in the gravitational field of two massive bodies moving in the circular orbits around their common barycenter.

The origin of the rotating coordinate frame is at the barycenter of the primary and secondary bodies, the $OX$ axis is directed towards the center of the secondary body, the $OY$ axis lies on an ecliptic plane and is collinear to the velocity vector of the secondary body. The $OZ$ axis is perpendicular to the ecliptic plane and is directed to form a right handed coordinate system.

The equations of motion of the spacecraft on the given coordinate system can be written as (1).

\[
\begin{align*}
\dot{x} - 2n\dot{y} - n^2 x &= - \left[ \mu_1 \frac{x + \mu_2}{r_1^3} + \mu_2 \frac{x - \mu_1}{r_2^3} \right] \\
\dot{y} + 2n\dot{x} - n^2 y &= - \left[ \mu_1 \frac{1}{r_1^3} + \mu_2 \frac{1}{r_2^3} \right] y \\
\dot{z} &= - \left[ \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} \right] z
\end{align*}
\]

where $r_1 = \sqrt{(x + \mu_2)^2 + y^2 + z^2}$ and $r_2 = \sqrt{(x - \mu_1)^2 + y^2 + z^2}$.

Since this work studies the spacecraft motion around the $L_1$ and $L_2$ libration points of the Sun-Earth system, the constants in the (1) are as follows: $\mu_2 = \frac{m_2}{m_1 + m_2} = 3.00341 \times 10^{-6}$, $\mu_1 = 1 - \mu_2$. Additionally, the term $n$ represents the mean motion. For current model $n = 1$.

3. Unstable directions of the spacecraft motion

The State Transition Matrix (STM) can be obtained through the linearization of the CRTBP equations. Given the initial spacecraft state vector $\vec{X} = (x, y, z, \dot{x}, \dot{y}, \dot{z})^T$ the initial system (1) can be rewritten as (2).

\[
\vec{X} = \vec{f}(\vec{X})
\]

Vector-valued function $\vec{f}$ is given by the following equation (3).

\[
\vec{f}(\vec{X}) = \left( x, \dot{y}, \dot{z}, 2x\dot{y} + \partial U / \partial x, -2nx + \partial U / \partial y, \partial U / \partial z \right)^T
\]

where the $U = \frac{\mu_1^2}{2}(x^2 + y^2) + \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}$, a pseudo-potential.

The linearization is then performed by expanding the variation of the initial state vector $\vec{X} + \delta \vec{X}$ to the first-order Taylor polynomial (4) and rewriting it by discarding the higher-order terms (5).

\[
\vec{X} + \delta \vec{X} = \vec{f}(\vec{X}) + \frac{\partial \vec{f}(\vec{X})}{\partial \vec{X}} \delta \vec{X} + O((\delta \vec{X})^2)
\]

\[
\delta \vec{X} = \frac{\partial \vec{f}(\vec{X})}{\partial \vec{X}} \delta \vec{X}
\]

Defining the Jacobi matrix $A(t) = \frac{\partial \vec{f}(\vec{X})}{\partial \vec{X}}$ and the spacecraft state vector at the initial moment of time $\vec{X}(t_0) = \vec{X}_0$ we arrive at (6). The solution for this equation can be given in the form of (7).

\[
\delta \vec{X} = A(t)\delta \vec{X}
\]

\[
\delta \vec{X}(t) = \Phi(t, t_0)\delta \vec{X}_0, \quad t \geq t_0
\]

The term $\Phi(t, t_0)$ is a $6 \times 6$ STM matrix of the linearized system (6) and can be split into multiple $3 \times 3$ matrices (8). Taking into account the variation of the spacecraft state-vector
\( \delta \vec{X} \) and the initial state-vector \( \delta \vec{X}_0 \) the equation (7) can be rewritten as (9). The positional components of the disturbed spacecraft trajectory at the given moment of time \( t \) can be obtained by the equation (10).

\[
\begin{align*}
\Phi(t,t_0) &= \begin{pmatrix} \Phi_{1,1}(t,t_0) & \Phi_{1,2}(t,t_0) \\ \Phi_{2,1}(t,t_0) & \Phi_{2,2}(t,t_0) \end{pmatrix} \\
\delta \vec{x}(t) &= \begin{pmatrix} \begin{pmatrix} \Phi_{1,1}(t,t_0) & \Phi_{1,2}(t,t_0) \end{pmatrix} \delta \vec{x}_0 + \Phi_{1,2}(t,t_0) \delta \vec{v}_0 \\ \begin{pmatrix} \Phi_{2,1}(t,t_0) & \Phi_{2,2}(t,t_0) \end{pmatrix} \delta \vec{x}_0 + \Phi_{2,2}(t,t_0) \delta \vec{v}_0 \end{pmatrix} \\
\delta \vec{v}(t) &= \Phi_{1,1}(t,t_0) \delta \vec{x}_0 + \Phi_{1,2}(t,t_0) \delta \vec{v}_0 \\
\end{align*}
\]

Taking into account that we vary only the velocity components \( \delta \vec{v}_0 \) of the initial spacecraft state vector it can be seen that the variation of positional components of the initial spacecraft state vector \( \delta \vec{x}_0 = 0 \). Then the (10) can be rewritten as (11).

\[
\delta \vec{v}(t) = \Phi_{1,2}(t,t_0) \delta \vec{v}_0
\]

The resulting divergence can then be obtained by computing the following cross product:

\[
|\delta \vec{v}(t)|^2 = (\delta \vec{v}(t), \delta \vec{v}(t)) = (\delta \vec{v}_0 + \delta \vec{v}_0)^T \Phi_{1,2}(t,t_0)^T \Phi_{1,2}(t,t_0) \delta \vec{v}_0
\]

The maximum and minimum values of the quadratic form given by the term (12) correspond to the maximum and minimum eigenvalues \( \lambda_{\text{max}}, \lambda_{\text{min}} \). The unstable direction of the spacecraft motion is then computed by projecting the \( v_{\text{max}}^T \) eigenvector corresponding to the \( \lambda_{\text{max}} \) eigenvalue to the XY plane and computing the angle between the projection and the OX axis.

4. Station-keeping costs for halo orbits around the \( L_1 \) point

Station-keeping of the spacecraft on the halo orbits around the \( L_1 \) libration point is performed by the periodical correction impulses. Two strategies were investigated in this study. According to the first strategy correction impulses was performed along the fixed direction of 28 degrees in the XY plane, while the second strategy calculates unstable direction for each correction impulse. As stated in the [12], the fixed 28 degrees direction in the XY plane corresponds to the unstable direction at the \( L_1 \) libration point.

To take into account the finite accuracy of the real spacecraft measuring instruments the uniform random noise was applied to the spacecraft state vector before the calculation of each correction impulse. Given the spacecraft state-vector \( \vec{X} = (x, y, z, v_x, v_y, v_z) \) and the random vector \( \delta \vec{X} = (\delta x, \delta y, \delta z, \delta v_x, \delta v_y, \delta v_z)^T \), where \( |(\delta x, \delta y, \delta z)^T| \leq \delta \vec{x} \) and \( |(\delta v_x, \delta v_y, \delta v_z)^T| \leq \delta \vec{v} \), the randomized values are applied to the spacecraft state-vector \( \vec{X}_{\text{estimated}} = \vec{X} + \delta \vec{X} \). Magnitude of the correction impulse \( \Delta \vec{v} \) is then computed for the vector \( \vec{X}_{\text{estimated}} \) while direction of \( \Delta \vec{v} \) was selected according to strategy. Then the randomized correction maneuver errors are applied to the computed correction vector \( \Delta \vec{v} + \vec{u} \), where \( \vec{u} \) is randomly pointed vector and \( |\vec{u}| \leq \delta v_{\text{corr}} |\Delta \vec{v}| \). The resulting correction impulse is then applied to the initial spacecraft state-vector \( \vec{X} \).

The ranking of the orbits around the libration points is performed by computing the amplitude values. For the halo orbits the current study utilizes the \( A_{x+} \) orbit characteristic which can be defined as the maximum value of the \( z \) coordinate for the given full revolution: \( A_{x+} = \max(|z| \leq 0, y = 0) \).

Figure 1 represent the results of the evaluation of the two investigated control strategies for three halo orbits with amplitudes in positive direction of Z-axis \( A_{z+} \): 296003.07 km, 603820.43 km, 750544.09 km. Magnitude of errors for this evaluation was: \( \delta \vec{x} = 20 \text{ km}, \delta \vec{v} = 15 \text{ cm/s}, \)
\[ \delta_{\text{corr}} = 10\%. \] Average values were calculated from 30 Monte-Carlo experiments per each orbit-strategy pair. As can be seen the second strategy requires less station-keeping impulse compared to the fixed-direction strategy and this advantage grows with increasing amplitude \( A_{z+} \): 3.66%, 11.05%, 18.66%. The \( \Delta V \) values represent the sum of all the absolute values of spacecraft correction impulses for the given period of time on orbit.

**Figure 1.** Overall station-keeping \( \Delta V \) for the different correction strategies. Dashed line represents the fixed direction strategy, solid line represents the unstable direction strategy.

Figure 2 shows the change of unstable direction depending on the orbit amplitude and the time on orbit before the correction was performed. As can be seen the higher amplitude orbits generally have higher variance of the unstable directions. The data on Figure 1 and Figure 2 indicate that the station-keeping strategy based on the unstable directions of spacecraft motion require less overall station-keeping costs than the strategy utilizing fixed impulse directions. The difference in overall spacecraft impulse increases for the halo orbits of larger amplitudes.

**Figure 2.** Unstable directions for the halo orbits of different amplitudes.

Figure 3 and Figure 4 show the impact of correction period on station-keeping costs for halo-orbit around \( L_1 \) point.

**Figure 3.** The impact of correction period on station-keeping costs for halo-orbit around \( L_1 \) point.

**Figure 4.** The impact of velocity errors on station-keeping costs for halo-orbit around \( L_1 \) point.
components measurement errors and the spacecraft correction maneuver execution errors on the resulting station-keeping costs.

Overall the provided data indicates that the spacecraft state vector velocity components measurement errors and the errors of control impulse execution have the highest impact on the effectiveness of the resulting correction strategy.

5. Station-keeping costs for halo orbits around the $L_2$ point

The station-keeping costs evaluation for the halo orbits around the $L_2$ libration point is performed similarly to the calculations for the $L_1$ point. The simulations were performed for the halo orbit with the amplitude in positive direction of $Z$-axis $A_z^+ = 200000$ km.

Figure 7 and Figure 8 show the impact of correction period and the spacecraft velocity measurement errors on the overall station-keeping costs for the halo orbit around the $L_2$ point. Figure 9 and Figure 10 show the impact of positional errors and correction maneuver execution errors on the effectiveness of the station-keeping strategy for the halo-orbits around the $L_2$ libration point.

Figure 7. The impact of correction period on station-keeping costs for halo-orbit around $L_2$ point.

Figure 8. The impact of velocity errors on station-keeping costs for halo-orbit around $L_2$ point.
6. Conclusion
The research studies the possible control strategies for the spacecraft around the $L_1$ and $L_2$ libration points of the Sun-Earth system aiming to evaluate the overall correction impulse costs. The station-keeping strategy based on the unstable components of motion was utilized and the impacts of the spacecraft state-vector measurement errors and the maneuver execution errors were studied. The results indicate that correction strategies utilizing the unstable directions of the spacecraft trajectory on the halo orbits prove to be more effective than the control strategies utilizing the constant direction correction impulses. Advancement of unstable direction strategy grows with increasing amplitude of halo orbit from $\approx 3.7\%$ for amplitude $\approx 300000$ km to $\approx 18.7\%$ for amplitude $\approx 750000$ km.

Additionally, it was found out that the measurement accuracy of the velocity components of motion and the correction maneuver execution precision have the highest impact on the overall spacecraft station-keeping costs.

7. Acknowledgements
The study was funded by Basic Research Program at the National Research University Higher School of Economics (HSE) in 2020 (grant number TZ-103).

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