Controllable simulation of topological phases and edge states with quantum walk

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We simulate various topological phenomena in condensed matter, such as formation of different edge states and topological phases, through two types of quantum walk with step-dependent coins in one- and two-dimensional position space. Particularly, we show that quantum walk with step-dependent coin simulates all types of topological phases and edge states. In addition, we show that step-dependent coins provide the number of steps as a controlling factor over the simulations. In fact, with tuning number of steps, we can determine the occurrences of edge states or topological phases, the type of edge state or topological phases and where they should be located. These two features make quantum walks versatile and highly controllable simulators of topological phases, edge states and topological phase transitions. We also report on emergences of cell-like structures for simulated topological phases and edge states. Each cell contains all types of edge states and topological phases.

I. INTRODUCTION

Phenomena, such as the integer Hall effect [11], fractional charges [2] and topological insulators [3–6], emerge in different phases of matter known as topological phases in the literature. In contrast to conventional phases of matter, the topological phases are symmetry-preserving and can be parametrized by global topological orders (topological invariants). These topological invariants characterize the global structure of ground-state wave function of matters and do not change continuously [1]. The topological phases are separated by edge states where topological invariant is usually ill-defined and the gapped energy bands closes up their gap [1]. In general, there are three types of geometry in which the energy bands can close up their gap (three types of edge states) [6–9]: a) linearly closing (linear dispersive behavior) which is known as Dirac cone, b) nonlinearly closing (nonlinear dispersive behavior) which is called Fermi arc and c) flat closing (dispersionless behavior) which is named flat band.

So far, topological phases and edge states have been reported in topological insulators and superconductors [3–6,10], cold atoms in optical lattices [11,12], and phononic states in mechanical oscillators [13]. Recently, it has been reported that quantum walks with photons can be used as simulators of the topological phenomena [14–23].

Quantum walks are universal computational primitives [24] that have been applied to simulate different quantum systems and phenomena [25,26] including topological phases as observed in condensed matter [27]. In particular, quantum walks can simulate all types of topological phases in one and two dimensions [27,31]. In addition, quantum walks enable one to extract topological invariants [32], create anomalous Floquet Chern insulators [33], investigate bulk-boundary correspondence [34] and topological phase transitions [35–37]. The power of quantum walks as simulators arises from their flexibility and controllability. In fact, quantum walks are able to check the robustness of the edge states [14] and helps suppress the limitations on strongly-driven systems.

In this paper, we take another step to show that quantum walks are even better simulators of topological phenomena that was perceived before. In particular, we address two issues; a) simulation of all types of edge states and b) high control over the simulated topological phases, edge states and phase transitions. In previous studies, the emphasis was on the simulation of topological phases but the edge states were neglected [27,31]. Here, we use step-dependent coins in one- and two-dimensional quantum walk with two protocols of simple-step (shift-coin operator) and split-step (shift-coin-shift-coin operator) to simulate topological phases and edge states. We show that such quantum walks simulate all types of edge states in addition to all types of topological phases. Furthermore, the step-dependent coins enable one to readily control the simulations by just choosing a proper number of steps. This controlling factor enables us to determine the presence or absence of edge states or topological phases, their types and where they should be observed. In addition, we report on emergences of cell-like structure in the simulated topological phases and edge states under certain conditions. Each cell contains all types of edge states.

The structure of the paper is as follows. First, we outline the relation between unitary operation of the quantum walk and Hamiltonian governing topological phenomena present in the state of a matter. Next, we use simple-step protocol for quantum walk in one-dimensional position space and show that specific types of topological phases and edge state could be simulated by it. Then, we modify the simple-step protocol to split-step and show that this modification enables us to simulate all types of edge states and topological phases. In

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addition, we generalize our study to two-dimensional position space and investigate the possibility of simulation of different edge states for two protocols simple- and split-step. The paper is concluded by closing remarks.

II. GENERAL SCHEME

Formally, the quantum walk with step-independent coin is generated by a successive application of an unitary protocol upon an initial state of a walker, i. e. a photon or an Ion,

\[ |\phi\rangle_{fin} = |\phi\rangle_T = \prod_{m=1}^{T} \hat{U}(m)|\phi\rangle_{int}, \tag{1} \]

and for quantum walk with step-dependent coins, the final state of the walker is achieved through slightly different successive application of an unitary protocol on similar walkers

\[ |\phi\rangle_{fin} = |\phi\rangle_T = \prod_{m=1}^{T} \hat{U}(m)|\phi\rangle_{int}, \tag{2} \]

where in each step, the coin operator changes.

The (initial) state of the walker is given by walker’s internal degrees of freedom, i. e. polarization \[38\] or spin \[11\] \[39\], and external degrees of freedom, i. e. optical lattice \[31\] \[41\] or angular momentum \[31\] \[43\] \[44\].

The protocol of the quantum walk consists of a number of coin and shift operators and describe in which sequence the coin and shift operators act upon the state of the walker. While the coin operators change the internal degrees of freedom, the shift operators modify the external degrees of freedom based on walker’s internal degrees of freedom (internal states). The number of the coin and shift operators determine the type of protocol that is used in the quantum walk. In literature, two types of protocols have been used for the quantum walk: a) simple-step which has one coin and one shift operator \[37\], b) split-step which consists of at least two different coin and two different shift operators \[27\]. In this paper, we use both of these protocols in sections \[31\] \[41\] and \[37\] to simulate topological phases and edge states.

The protocols of the quantum walk are quite similar to Su-Schrieffer-Heeger model which describes electrons (or fermions) that hop on a one-dimensional lattice with staggered hopping amplitudes \[15\] \[16\]. In addition, since the quantum walk is obtained as a result of repeated application of a unitary operator, we can describe the evolution of the quantum walk in the framework of a time periodic driving system of Floquet theory. Therefore, we can map the evolution of the quantum walk to a stroboscopic evolution under an effective Hamiltonian \[27\] \[37\]

\[ \hat{H}(k) = i \ln \hat{U}(k) = E(k)\mathbf{n}(k) \cdot \mathbf{\sigma}, \tag{3} \]

where \( E(k) \) is the energy dispersion, \( \mathbf{\sigma} \) are Pauli matrices and \( \mathbf{n}(k) \) defines the eigenstates of the energy. With this mapping, we can connect the energy of the Hamiltonian to the eigenvalues of the protocol of quantum walk through

\[ \lambda(k) = e^{-iE(k)}, \tag{4} \]

where \( \lambda(k) \) is the eigenvalue of \( \hat{U} \). The Hamiltonian and its energy are used to investigate topological phases and edge states. Since there is an one to one correspondence between Hamiltonian (energy) and the protocol of the quantum walk (eigenvalues of \( \hat{U} \)), one can use the quantum walk to simulate and investigate the topological phases, edge states and topological phase transitions. The properties of the topological phases and edge states can be characterized by the group velocity of the energy eigenstates \[27\] \[37\]

\[ V(k) = \frac{dE(k)}{dk}. \tag{5} \]

In this paper, we focus on topological phases with chiral symmetry. Therefore, there is an unitary and Hermitian operator, \( \hat{\Gamma} \), that satisfies two conditions of \( \hat{\Gamma}^2 = I \) and \( \hat{\Gamma}\hat{H}\hat{\Gamma} = -\hat{H} \). Presence of chiral symmetry limits the \( \mathbf{n}(k) \) to lie on a great circle of the Bloch sphere and winding number is interpreted as a topological invariant. The winding number is defined as number of times \( \mathbf{n}(k) \) winds around the origin as \( k \) transverses through the first Brillouin zone, \([-\pi, \pi]\). One can obtain the chiral symmetry operator, \( \hat{\Gamma} \), as

\[ \hat{\Gamma} = \mathbf{A} \cdot \mathbf{\sigma}, \tag{6} \]

where \( \mathbf{A} \) is a vector labeling a point on the Bloch sphere and perpendicular to \( \mathbf{n}(k) \) for all \( k \).

In this paper, we are interested in the quantum walks that have one dimensional external degree of freedom and two internal degrees of freedom. Therefore, the Hilbert space of the walk is given by tensor product of two subspaces, \( \mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C \) where the position Hilbert space (\( \mathcal{H}_P \)) and the coin Hilbert space (\( \mathcal{H}_C \)) are spanned by \( \{|i\}_P : i \in \mathbb{Z}\} \) and \( \{|0\} , |1\} \), respectively.

The protocols that we will consider for quantum walks are Hermitian and their determinants are 1. Consequently, the Hamiltonians associating to the protocols are traceless which leads the bands of the energy to have symmetry of \( E(k) = E(-k) \) and the energy’s value traverses in \([-\pi, \pi]\). Since there are two internal states, there will be two bands of energy. The topological phases are where these two bands of energy are gapped. The edge states are determined by gapless bands of energy.

In general, there are three types of geometry in which the energy bands can close up their gap \[8\]. This essentially means three types of edge states. The first type is known as Dirac cone where the energy bands are linear functions of the momentum, \( k \) and linearly close up.
the gap. The second type is Fermi arc where the gapless energy bands close up the gap nonlinearly. Finally, we have flat bands. In the flat bands, energy is constant and independent of momentum. It should be noted that all of the closing gaps happen at \( E = 0 \) and \( \pm \pi \).

## III. SIMPLE-STEP QUANTUM WALK

In this section, we employ the simple-step protocol for the quantum walk (simple-step quantum walk) to simulate topological phases, edge states and topological phase transitions.

### A. General details

The simple-step protocol consists of one coin and one shift operator

\[
\hat{U} = \hat{S}_{\uparrow \downarrow} \hat{C}_\theta,
\]

which shows that one step of the quantum walk comprises rotation of internal states with \( \hat{C}_\theta \) and displacement of its position with \( \hat{S}_{\uparrow \downarrow} \). The coin operator is step-dependent and considered as \([37, 48]\)

\[
\hat{C}_\theta = e^{-i\frac{T\theta}{2} \sigma_z},
\]

where \( \theta \) is the rotation angle spanning \([0, 2\pi]\), \( T \) is the number of steps characterizing step-dependency of the coin operator. The shift operator is

\[
\hat{S}_{\uparrow \downarrow} = |\uparrow\rangle \langle \uparrow| \otimes \sum_x |x+1\rangle \langle x| + |\downarrow\rangle \langle \downarrow| \otimes \sum_x |x-1\rangle \langle x|.
\]

We can use Discrete Fourier Transformation, \( |k\rangle = \sum_x e^{-i\frac{2\pi}{T} k x} |x\rangle \), to rewrite the shift operator in diagonalized form

\[
\hat{S}_{\uparrow \downarrow} = e^{i k \sigma_z}.
\]

It is a matter of calculation to find the eigenvalues of the \( \hat{U} \) in Eq. (7) as

\[
\lambda(k) = \cos(k) \cos\left(\frac{T \theta}{2}\right) \pm \sqrt{\cos(k) \cos\left(\frac{T \theta}{2}\right)^2 - 1}, \tag{11}
\]

which by using Eq. (4), we can obtain the energy as \([37]\)

\[
E(k) = \pm \cos^{-1}\left[ \cos\left(\frac{T \theta}{2}\right) \cos(k) \right]. \tag{12}
\]

The positive and negative branches of energy correspond to the two bands of energy. These two bands of energy are gapped and their gap could close up only at \( E = 0 \) and \( \pm \pi \). The bands of energy close up their gaps at \( E = \pi \) if \( \theta \) admits \([37]\)

\[
\theta_{E=\pi} = \frac{\pm 2 \cos^{-1}[\pm \sec(k)] + 4 \pi c}{T} = \begin{cases} 
\frac{4 \pi c}{T} & k = -\pi \\
\pm 2 \pi + 4 \pi c & k = 0 \\
\frac{4 \pi c}{T} & k = \pi
\end{cases}, \tag{13}
\]

where \( c \) is an integer. Similarly, for closing the gap with \( E = 0 \), we have \([37]\)

\[
\theta_{E=0} = \frac{\pm 2 \cos^{-1}[\sec(k)] + 4 \pi c}{T} = \begin{cases} 
\frac{4 \pi c}{T} & k = -\pi \\
\pm 2 \pi + 4 \pi c & k = 0 \\
\frac{4 \pi c}{T} & k = \pi
\end{cases}. \tag{14}
\]

On the other hand, we can find cases where energy is independent of momentum and constant, \( E = \frac{\pi}{2} \). This takes place step dependently for

\[
\theta_{E=\text{const}} = \frac{4 \pi c \pm \pi}{T}. \tag{15}
\]

It should be noted that \( c \) must be chosen in a way that obtained solutions are within \([0, 2\pi]\). The flat bands represent existences of degenerate localized states. In the next step, we find \( n(k) \) as

\[
n(k) = \frac{(\cos\left(\frac{T \theta}{2}\right) \sin(k), \sin\left(\frac{T \theta}{2}\right) \cos(k), -\cos\left(\frac{T \theta}{2}\right) \sin(k))}{\sin E(k)},
\]

which confirms that when the bands of energy close up their gap, \( n(k) \) becomes ill-defined. Therefore, the edge states are characterized by ill-defined \( n(k) \). Since the winding number is the topological invariant, we can conclude that the winding number is also ill-defined at edge states which is in agreement with our expectation.

We find the group velocity using Eqs. (5) and (12)

\[
V(k) = \pm \frac{\cos\left(\frac{T \theta}{2}\right) \sin(k)}{\sqrt{1 - \cos\left(\frac{T \theta}{2}\right) \cos(k)}}, \tag{17}
\]

which has symmetry of \( V(k) = -V(-k) \). Obtained group velocity becomes ill-defined at roots \( \theta' \) of denominator of the group velocity \([37]\)

\[
\theta' = \begin{cases} 
\frac{\pm 2 \cos^{-1}[\pm \sec(k)] + 4 \pi c}{T} & \text{for } k = -\pi \\
\pm 2 \pi + 4 \pi c & \text{for } k = 0 \\
\frac{4 \pi c}{T} & \text{for } k = \pi
\end{cases}, \tag{18}
\]

which are coincidence with the calculated points where the energy bands are gapless.
Finally, in order to obtain the chiral symmetry operator, we find \( A = (\cos(T \theta_2), 0, \sin(T \theta_2)) \) which is perpendicular to \( n \) for arbitrary \( k \) and gives us the chiral symmetry operator in form of

\[
\hat{\Gamma} = \begin{pmatrix}
  \sin(T \theta_2) & \cos(T \theta_2) \\
  \cos(T \theta_2) & -\sin(T \theta_2)
\end{pmatrix},
\]

which admits both conditions of \( \hat{\Gamma}^2 = I \) and \( \hat{\Gamma} H \hat{\Gamma} = -\hat{H} \).

### B. Results

The major consequence of the step-dependent coin is dynamical nature (step-dependency) for Hamiltonian, bands of energy, group velocity, \( n \) and even chiral symmetry operator. This is in complete contrast to simple-step quantum walk with step-independent coin where the evolution of the quantum walk has no effect on these properties.

The step-dependent bands of energy shows that through the evolution of the quantum walk, the bands of energy associating to each rotation angle will change. Therefore, we can use the number of steps as a controlling factor and by tuning it, we can decide if the bands of energy for specific rotation angle close up their gaps, or in which topological phases they should be located (Fig. 1).

Consequently, with step-dependent coin, we can engineer the number of topological phases, their sizes, edge states and their places (phase-transition points). This provides us with outstanding level of control over simulations.

In our study, we established that edge states (gapless points) are where \( n \) and their places (phase-transition points). This provides the number of topological phases, their sizes, edge states. The type of edge states observed for the quantum walk into a split-step one and investigate the types of topological phases and edge states that the split-step quantum walk can simulate.

### IV. Split-Step Quantum Walk

In this section, we modifies the simple-step protocol of the quantum walk into a split-step and investigate the types of topological phases and edge states. The split-step quantum walk can simulate all different types of topological phases and edge states, we should use the split-step protocol for the quantum walk (split-step quantum walk).

#### A. General details

The split-step protocol has certain flexibility since we can have several different coin and shift operators. In this paper, we consider the following split-step protocol

\[
\hat{U} = \hat{S}_\uparrow \hat{C}_\alpha \hat{S}_\downarrow \hat{C}_\beta,
\]

in which, one step of the quantum walk includes rotation of internal states with \( \hat{C}_\beta \), displacement of its position with \( \hat{S}_\downarrow \), a second rotation of internal states with \( \hat{C}_\alpha \) and finally, its displacement with \( \hat{S}_\uparrow \). The coin operators are given by

\[
\hat{C}_\alpha = e^{-\frac{i\pi}{2} \sigma_y},
\]

\[
\hat{C}_\beta = e^{-\frac{i\pi}{2} \sigma_y},
\]

where \( \alpha \) and \( \beta \) are rotation angles spanning \([-\pi, \pi]\) and \( T \) characterizes step-dependency of the coins. The \( \hat{S}_\uparrow \) and \( \hat{S}_\downarrow \) are shift operators

\[
\hat{S}_\uparrow = \sum_x [\langle \uparrow \rangle \otimes |x + 1\rangle \langle x | + |\downarrow \rangle \langle \downarrow \rangle \otimes |x\rangle \langle x |],
\]

\[
\hat{S}_\downarrow = \sum_x [\langle \uparrow \rangle \otimes |x\rangle \langle x | + |\downarrow \rangle \langle \downarrow \rangle \otimes |x - 1\rangle \langle x |],
\]

which by using Discrete Fourier Transformation, we can rewrite them as \( \hat{S}_\uparrow = e^{\frac{i\pi}{T} \sigma_x} \) and \( \hat{S}_\downarrow = e^{\frac{i\pi}{T} \sigma_x} \). Finally, we obtain the eigenvalues of the \( \hat{U} \) in Eq. (20) as

\[
\lambda(k) = \gamma \pm \sqrt{\gamma^2 - 1},
\]

where \( \gamma = \cos(k) \cos(T \theta_2) \cos(T \theta_2) - \sin(T \theta_2) \sin(T \theta_2) \). It is a matter of calculation to find the energy as

\[
E(k) = \pm \cos^{-1}(\gamma),
\]
FIG. 1: Simple-step quantum walk (one-dimensional): Modification of energy (a) and group velocity (b) (both positive branches) as functions of momentum and rotation angle $\theta$ for subsequent steps of $T = 2, \ldots, 8$. In (a), we observe only Dirac cone gapless energy bands. As number of steps increases, the distance between two closing gaps decreases. It is evident that by tuning the number of steps, we can engineer the position of edge states and the type of topological phases for each rotation angle. In (b), we see that the group velocity at gapless points is ill-defined. In contrast, the group velocity is constant around gapless point and its signature swaps from positive to negative and vice versa depending on the energy of gapless point.

in which $\pm$ corresponds to two bands of energy separated by a gap. The energy spans $[-\pi, \pi]$ and the gap between two bands of energy can only close up at $E = 0$ and $E = \pm \pi$. Based on obtained energy \(26\), there are three possible geometries for gapless points which are determined by the following conditions:

I) If $\sin(\frac{T\alpha}{2}) \sin(\frac{T\beta}{2}) = 0$, the energy bands close up their gaps linearly. Therefore, the edge states are Dirac cones.

II) If $\cos(k) \cos(\frac{T\alpha}{2}) \cos(\frac{T\beta}{2}) = 0$, energy becomes independent of momentum $k$ and the gapless bands of energy are flat bands.

III) If the two previous conditions are not met, the bands of energy close up their gaps nonlinearly and the edge states are Fermi arcs.

Therefore, depending on these conditions, the closing gaps observed for the bands of energy could be in different geometries which means the possibility of simulation of all types of the edge states. If the rotation angles, $\alpha$ and $\beta$ are independent of each other, only one type of edge state is observed in each step (See Fig. 2). In contrast, if these rotation angles are linearly related to each other, it is possible to simulate all of the geometries for gapless bands of energy altogether at specific steps (See Fig. 3). Therefore, we can simulate all types of the edge states at specific steps if $\beta = s_1\alpha + s_2$ where $s_i$ are real numbers. In the next section, we give detailed explanations regarding the physical interpretations of these results.

Next, we find $\mathbf{n}(k)$ in the following form

$$n_x(k) = \frac{\cos(\frac{T\alpha}{2}) \sin(\frac{T\beta}{2}) \sin(k)}{\sin E(k)},$$

$$n_y(k) = \frac{\sin(\frac{T\alpha}{2}) \cos(\frac{T\beta}{2}) + \sin(k) \cos(\frac{T\alpha}{2}) \sin(\frac{T\beta}{2})}{\sin E(k)},$$

$$n_z(k) = -\frac{\cos(\frac{T\alpha}{2}) \cos(\frac{T\beta}{2}) \sin(k)}{\sin E(k)} \quad (27)$$

which confirms that the gapless points for the energy bands are where $\mathbf{n}$ becomes ill-defined. Therefore, similar to the simple-step case, the edge states are characterized by ill-defined $\mathbf{n}$ and topological invariants. In addition, we obtain the group velocity as

$$V(k) = \pm \frac{\cos(\frac{T\alpha}{2}) \cos(\frac{T\beta}{2}) \sin(k)}{\sqrt{1 - \gamma^2}} \quad (28)$$

Here, the characteristic behavior of the group velocity at the edge states depends on the type of the edge states.
For gapless energy bands with Dirac cone and Fermi arc geometries, the group velocity is ill-defined at edge states (see Figs. 2, 3). Whereas for gapless bands of energy with flat band geometry, the group velocity is zero. This is due to fact that energy is independent of momentum, \( k \), for flat bands of energy.

Since we are focused on chiral symmetry for our quantum walks, we find \( \mathbf{A} = \left( \cos(\frac{T_\beta}{2}), 0, \sin(\frac{T_\beta}{2}) \right) \) and by using it with Eq. (6), we have the chiral symmetry operator as

\[
\hat{\Gamma} = \begin{pmatrix}
\sin(\frac{T_\beta}{2}) & \cos(\frac{T_\beta}{2}) \\
\cos(\frac{T_\beta}{2}) & -\sin(\frac{T_\beta}{2})
\end{pmatrix},
\]

(29)

B. Results

Similar to simple-step quantum walk, Hamiltonian, energy bands, group velocity, \( \mathbf{n}(k) \) and chiral symmetry operator are dynamical since they are step-dependent. The chiral symmetry operator and \( \mathbf{A} \) are functions of rotation angle \( \beta \) and independent of \( \alpha \). Chronologically, the coin operator with rotation angle \( \beta \) is applied first in the split-step protocol (see (22)). Therefore, the symmetrical properties governing the topological phases are dominated by the first coin operator that is applied on internal states of the walker.

One of the major differences between simple-step and split-step quantum walks is the presence of \( \sin(\frac{T_\alpha}{2}) \sin(\frac{T_\beta}{2}) \) term in the energy bands of the split-step quantum walk (compare (12) and (23)). This indeed enriches the topological phases and edge states that the quantum walk can simulate. In fact, this term is one of the main reasons that the quantum walk with split-step protocol can simulate all types of topological phases and edge states. We further enriched the capability of the quantum walk to simulate topological phases and edge states by considering step-dependent coin.

As we pointed it out before, the geometry of gapless bands of energy depends on certain conditions governed by the rotation angles, \( \alpha \) and \( \beta \). In fact, based on these conditions, we can program our simulations to have specific geometry for gapless bands of energy through the whole simulation (Figs. 2 and 3). On the other hand, the dynamicality of the energy bands enables us to determine how many edge states or topological phases and where these edge states or topological phases should be as rotation angles traverses \( [-\pi, \pi] \). The controlling factor is the number of steps. These two properties give us high level of controllability over simulation of topological phenomena with quantum walk.

The most interesting simulations of topological phases are those when the two rotation angles are linearly related to each other (\( \beta = s_1 \alpha + s_2 \)). First of all, it is possible to simulate all three geometries of Dirac cone, Fermi arc and flat bands for gapless energy bands in a single step (Fig. 3). This is in contrast to previous case where each geometry for gapless energy bands could be simulated in different steps and not all together (compare Figs. 2 and 3). The second issue is the coexistence of these three gapless energy bands together in a single step. Finally, the emergences of cell-like topological structures in specific steps. These cell-like structures are characterized by two flat gapless energy bands which are the cell’s walls. The interior part of the cell contains two Fermi arcs and a Dirac cone between them. Therefore, a single cell contains all three types of edge states and corresponding topological phase transitions.

The protocol considered in this paper also simulates all types of topological phases. To prove this, we use the following argument: At the first step, \( T = 1 \), the obtained Hamiltonian, energy bands and other properties yield split-step quantum walk with step-independent coin which was investigated by Kitagawa et al in Ref. [27]. Kitagawa et al showed that the split-step quantum walk considered by them simulates all types of topological phases in one dimensions. Therefore, split-step quantum walk with step-dependent coin in this paper also simulates all types of topological phases. In addition, since the topological phase transitions are between different phases (over edge states) and cell-like structures contain all types of edge states, we can conclude that all types of topological phases are simulated in each cell.

Here, the gapless energy bands are also characterized by ill-defined \( \mathbf{n}(k) \). This is independent of the geometry of gapless energy bands. Therefore, it is a general method to detect edge states but not distinguish them from one another. On the other hand, the group velocity is related to \( \mathbf{n}(k) \) through \( n_z = -|V(k)| \) which shows that the group velocity spans \([-1, 1]\). The behavior of group velocity around gapless points depends on the energy of gapless points and the geometry of gapless energy bands.

For Dirac cone gapless energy bands (see Fig. 4), the group velocity around the edge state is constant and at gapless point it swaps from positive to negative (for \( E = \pi \)) or vice versa (for \( E = 0 \)). In case of Fermi arcs, around the gapless point, the group velocity depends on momentum, \( k \), and its signature swaps from positive to negative (for \( E = \pi \)) or vice versa (for \( E = 0 \)) at gapless point. Finally, for gapless energy bands with flat band geometry, group velocity is zero and independent of momentum. These three distinctive behaviors enable us to recognize the type of edge states.

The group velocity characterizes the motion of a wave packet associating to the walker. Accordingly, the edge states that are in form of Dirac cones are linearly dispersing. The Fermi arc edge states show nonlinear dispersive behaviors. In contrast, the flat band edge states are dispersionless. In fact, the group velocities that are independent of \( k \) are known as dispersionless transport and dependency on \( k \) indicates that we have dispersive transportation. Finally, we should point it out that
FIG. 2: Split-step quantum walk (one-dimensional): Modification of energy (a) and group velocity (b) (both positive branches) as functions of momentum and rotation angle $\alpha$ ($\beta = \pi/3$) for subsequent steps of $T = 2, \ldots, 8$. In (a), we observe that only one type of the geometry for edge states is present at each step. A network of flat bands occurs in step 3 of the walk where energy bands are independent of $k$. Correspondingly, in (b) we see that the group velocity for network of flat bands is zero. The places of topological phases and edge states are step-dependent. Therefore, we can choose the type and place of topological phases and edge states by number of steps.

the presence of flat bands signals to existence of a macroscopic number of degenerate localized states [51].

C. Further possibilities

As a final remark in this section, we return to the flexibility in the split-step protocol. The split-step protocol considered in this paper is a straightforward generalization from simple-step quantum walk to split-step one. We can consider the situations where the first or the second coin operator is step-independent in split-step protocol. This significantly will change the Hamiltonian, energy and other properties of the system, hence simulated topological phenomena by the quantum walk.

A simple example is if we use a step-independent coin for the first coin operator in Eq. [20]. Previously, we showed that the chiral symmetry operator and $A$ are determined only by the first coin operator in the protocol. Accordingly, these two quantities would be step-independent and lose their dynamical natures while the other properties (Hamiltonian, energy and etc.) would be dynamical.

In addition, we have the possibility to include position-dependency in the protocol of the walk [32] or the set of protocols that were introduced and used in Ref. [15]. Each of these protocols are employed to simulate specific topological phenomena. We can use the step-dependent coins in these protocols and have additional phenomena simulated by the quantum walks.

V. GENERALIZATION TO TWO-DIMENSIONAL POSITION SPACE

In this section, we generalize our quantum walks to two-dimensional position space, keep its internal states as two and investigate topological phenomena that can be simulated by such generalization. First, we study a two-dimensional quantum walk with simple-step protocol and then move on to split-step two-dimensional quantum walks.

A. simple-step Quantum walk

The protocol of the simple-step quantum walk with two-dimensional position space is given by

$$\hat{U} = \hat{S}_{t\uparrow} \hat{C}_{\theta},$$

(30)

where the shift operator is comprised of two other shift operators [28].
FIG. 3: Split-step quantum walk (one-dimensional): Modification of energy (a) and group velocity (b) (both positive branches) as functions of momentum and rotation angle $\alpha$ ($\beta = (\alpha + \pi)/3$) for subsequent steps of $T = 2, ..., 8$. In (a), we notice the emergences of all three types of edge states for specific steps and formation of cell-like structures. These cells are marked by two flat bands acting as the cells’ walls and a Dirac cone located between two Fermi arcs. Therefore, all types of edge states are presented in the same step. The place of these edge states is step-dependent, so we can use number of steps as a tuning factor to engineer desired edge state or topological phase for specific rotation angle. In (b), we see that group velocity is ill-defined at Dirac cone and Fermi arc gapless energy bands while it is zero for flat bands.

FIG. 4: Energy (left panels) and group velocity (right panels) (both positive branches) for 6th (a) and 8th (b) steps with $\beta = (\alpha + \pi)/3$. The Dirac cones show linear dispersion. The Fermi arcs have nonlinear dispersive behaviors whereas the flat bands are dispersionless.

\[ \hat{S}_{\uparrow\downarrow} = \hat{S}_{\uparrow\downarrow}(y) \hat{S}_{\uparrow\downarrow}(x), \quad (31) \]

in which

\[ \hat{S}_{\uparrow\downarrow}(x) = |\uparrow\rangle \langle \uparrow| \sum_{x,y} |x + 1, y\rangle \langle x, y| |\downarrow\rangle \langle \downarrow| \sum_{x,y} |x - 1, y\rangle \langle x, y|, \]
\( \hat{S}_{\uparrow \downarrow}(y) = |\uparrow\rangle \otimes \sum_{x,y} |x,y+1\rangle \langle x,y| \downarrow \rangle \otimes \sum_{x,y} |x,y-1\rangle \langle x,y| \)

A single step of the walk includes rotation of internal states of the walker with \( \hat{C}_y \), displacement of its position first in \( x \) position space followed by another displacement in \( y \) position space. Using Discrete Fourier Transformation, we can find these shift operators as \( \hat{S}_{\uparrow \downarrow}(x) = e^{ik_x \sigma_z} \) and \( \hat{S}_{\uparrow \downarrow}(y) = e^{ik_y \sigma_z} \). It is a matter of calculation to find eigenvalues of Eq. (30) as

\[
\lambda(k_x, k_y) = \cos(k_x + k_y) \cos\left(\frac{T\theta}{2}\right) \pm \frac{1}{\sqrt{(\cos(k_x + k_y) \cos\left(\frac{T\theta}{2}\right))^2 - 1}},
\]

and consequently, we obtain the energy as

\[
E(k_x, k_y) = \pm \cos^{-1}\left[\cos\left(\frac{T\theta}{2}\right) \cos(k_x + k_y)\right].
\]

We have two bands of energy corresponding to negative and positive branches of energy. \( k_x \) and \( k_y \) traverse the first Brillouin zone. Energy’s value is limited within \([−\pi, \pi] \) and the gap between the two bands of energy closes up at \( E = 0 \) and \( \pm \pi \) for

\[
\theta_{E=\pi} = \frac{\pm 2 \cos^{-1}[−\sec(k_x + k_y)] + 4\pi c}{T},
\]

\[
\theta_{E=0} = \frac{\pm 2 \cos^{-1}[\sec(k_x + k_y)] + 4\pi c}{T}
\]

where \( c \) is an integer. In addition, the energy could be independent of momentum and constant, \( E = \frac{\pi}{2} \) which happens for

\[
\theta_{E=cte=\pi/2} = \frac{4\pi c \pm \pi}{T}.
\]

Next, we find \( \mathbf{n}(k_x, k_y) \) as

\[
n_x(k_x, k_y) = \frac{\cos\left(\frac{T\theta}{2}\right) \sin(k_x + k_y)}{\sin E(k_x, k_y)},
\]

\[
n_y(k_x, k_y) = \frac{\sin\left(\frac{T\theta}{2}\right) \cos(k_x + k_y)}{\sin E(k_x, k_y)},
\]

\[
n_z(k_x, k_y) = -\frac{-\cos\left(\frac{T\theta}{2}\right) \sin(k_x + k_y)}{\sin E(k_x, k_y)}.
\]

It is evident that \( \mathbf{n}(k_x, k_y) \) becomes ill-defined when the bands of energy close up their gap. It should be noted that \( \mathbf{n}(k_x, k_y) \) maps a small area on a two dimensional torus to a small area on a Bloch sphere [28]. The winding number in one-dimensional quantum walk is no longer the topological invariant for two-dimensional quantum walk. In contrast, an integer-valued number known as the Chern number is the topological invariant. The Chern number measures number of times that \( \mathbf{n}(k_x, k_y) \) covers the Bloch sphere as \( k_x \) and \( k_y \) traverse the first Brillouin zone.

As for the group velocity, since the walker’s wave function propagates in two dimensions, we can find two group velocities associating to each positions space by

\[
V_{k_x} = V_{k_y} = \pm \frac{\cos\left(\frac{T\theta}{2}\right) \sin(k_x + k_y)}{\sqrt{1 - \left[\cos\left(\frac{T\theta}{2}\right) \cos(k_x + k_y)\right]^2}},
\]

in which we have used \( V(k_x) = \partial E(k_x, k_y)/\partial k_x \) and likewise for \( V(k_y) \). Evidently, the group velocity becomes ill-defined at

\[
\theta' = \begin{cases} 
\pm 2 \cos^{-1}[−\sec(k_x + k_y)] + 4\pi c/T, \\
\pm 2 \cos^{-1}[\sec(k_x + k_y)] + 4\pi c/T,
\end{cases}
\]

which are the points where the energy bands are gapless.

B. split-step Quantum walk

In this section, we modifies the simple-step protocol of the two-dimensional quantum walk into a split-step one and investigate the types of topological phases and edge states that the split-step quantum walk can simulate.

There are several types of split-step protocol that we can employ. In this paper, we use the following split-step protocol [22, 27, 28]

\[
\hat{U} = \hat{S}_{\uparrow \downarrow}(y)\hat{C}_x\hat{S}_{\uparrow \downarrow}(x)\hat{C}_\beta,
\]

where in a single step of the quantum walk, the internal states are rotated by \( \hat{C}_\beta \), displacement of its position in \( x \) space with \( \hat{S}_{\uparrow \downarrow}(x) \), additional rotation of internal states with \( \hat{C}_x \) and finally, displacement of its position in \( y \) space with \( \hat{S}_{\uparrow \downarrow}(y) \).

The coin operators are given in Eqs. [21] and [22], while the shift operators are \( \hat{S}_{\uparrow \downarrow}(x) = e^{ik_x \sigma_z} \) and \( \hat{S}_{\uparrow \downarrow}(y) = e^{ik_y \sigma_z} \). Therefore, we obtain the eigenvalues of the \( \hat{U} \) in Eq. (40) as

\[
\lambda(k) = \gamma \pm \sqrt{\gamma^2 - 1},
\]

where \( \gamma = \cos(k_x + k_y) \cos\left(\frac{T\alpha}{2}\right) \cos\left(\frac{T\beta}{2}\right) - \cos(k_x - k_y) \sin\left(\frac{T\alpha}{2}\right) \sin\left(\frac{T\beta}{2}\right) \) and consequently, we find the energy as
\[ E(k) = \pm \cos^{-1}(\gamma), \]  
(42)
in which \( \pm \) represents existence of two bands of energy separated by a gap. The energy traverses \([-\pi, \pi]\) and the two bands of energy become gapless only at \(E = 0\) and \(E = \pm \pi\). In the next section, we show that such quantum walk can simulate three different types of edge states. For now, we highlight three interesting cases (\(c\) is integer):

I) If \(\alpha = \beta = \pm 2c\pi/T\), then energy bands become linearly dependent on \(k_x\) and \(k_y\) given by \(E = \pm (k_x + k_y)\).

II) If \(\alpha = \beta = \pm (2c + 1)\pi/T\), then energy bands again are linear functions of \(k_x\) and \(k_y\) given by \(E = \pm \cos^{-1}(-\cos(k_x - k_y))\).

III) If \(\alpha = \pm 2c\pi/T\) and \(\beta = \pm 2c\pi/T\) or vice versa, the energy bands become flat with \(E = \pm \pi/2\).

Next, it is a matter of calculation to find \(n(k_x + k_y)\) and different component of the group velocity as

\[
n_x(k) = \frac{\sin(k_x + k_y) \cos(Ta/2) \sin(Tb/2) - \sin(k_x - k_y) \sin(Ta/2) \cos(Tb/2)}{\sin E(k_x + k_y)}, \]

\[
n_y(k) = \frac{\cos(k_x + k_y) \cos(Ta/2) \sin(Tb/2) + \cos(k_x - k_y) \sin(Ta/2) \cos(Tb/2)}{\sin E(k_x + k_y)}, \]

\[
n_z(k) = -\frac{\sin(k_x + k_y) \cos(Ta/2) \cos(Tb/2) - \sin(k_x - k_y) \sin(Ta/2) \sin(Tb/2)}{\sin E(k_x + k_y)}, \]

\[
V(k_x) = \pm \frac{\sin(k_x - k_y) \sin(Ta/2) \sin(Tb/2) - \sin(k_x + k_y) \cos(Ta/2) \cos(Tb/2)}{\sqrt{1 - \gamma^2}}, \]

\[
V(k_y) = \pm \frac{-\sin(k_x - k_y) \sin(Ta/2) \sin(Tb/2) - \sin(k_x + k_y) \cos(Ta/2) \cos(Tb/2)}{\sqrt{1 - \gamma^2}}. \]
(43)

Evidently, the \(n(k_x, k_y)\) and group velocity become ill-defined at the gapless points of the energy bands. Therefore, the edge states could be recognized by ill-defined \(n(k_x, k_y)\), topological invariants and group velocity.

**C. Results**

Similar to its on-dimensional counterpart, irrespective of simple- or split-step protocol, the Hamiltonian, energy bands, group velocity and \(n(k)\) are step-dependent. Therefore, the dynamicality in simulated topological phenomena that was observed for one-dimensional quantum walk is also present for two-dimensional quantum walk. The presence of the additional momentum provides us with two options regarding geometry of the gapless energy bands: a) we can fix one of the momenta and let the other one varies through the first Brillouin zone. b) both of the momenta traverse through the first Brillouin zone. We will look at the gapless energy bands for these two specific scenarios.

For simple-step protocol (see Fig. 6), if we fix one of the momenta, the energy bands close up their gap linearly, similar to Dirac cones in one-dimensional quantum walk. In contrast, if both of the momenta varies, we observe characteristic flat bands for gapless energy bands. The major differences between these flat bands with their one-dimensional counterpart is that they are momenta dependent and obtained only when \(k_x + k_y = 0\) and \(\pm \pi\). Therefore, we observe two types of geometries for gapless energy bands depending on which approach we take. In addition, due to dependency of energy bands on steps, the places of gapless energy bands and their number, as \(\theta\) traverses \([0, 2\pi]\), change step dependently. This shows that we can use the step number as a mean to engineer the place of topological phases and edge states.

Modification to split-step protocol results into an additional term in obtained energy bands (\(\cos(k_x - k_y) \sin(Ta/2) \sin(Tb/2)\)). This additional term plays major role in simulation of edge states in which the energy bands close their gap nonlinearly, hence simulation of Fermi arc edge states. Therefore, the emergence of this additional term enables our quantum walk in two-dimensional position space to simulate different edge states.

In split-step two dimensional quantum walk, if we fix one of the rotation angles and let the other one varies (see Fig. 7), we observe that in each step, the simulated
FIG. 5: Simple-step quantum walk (two-dimensional): Energy as a function of rotation angle and momenta for $T = 8$. In (a), we observe the existence of several gapless energy bands, hence edge states with similar geometries.

In (b) and (c), we set rotation angles $\theta = \pi/2$ and $\theta = \pi/4$, respectively. If we consider $k_y = -\pi$ and varies $k_x$ through the first Brillouin zone, energy bands close their gap linearly, hence Dirac cone edge state. For variations of both momenta through the first Brillouin zone, momenta-dependent flat bands edge states are formed. In (d), the rotation angle is $\theta = \pi/5$ which results into gaped energy bands, hence topological phases. For $\theta = \pi/8$, we observe flat bands in (e).

topological phenomena has specific characteristic behavior that differs from other steps. This is the reminiscence of similar one-dimensional quantum walk where in each step, only one type of edge state was simulated (see Fig. 2). The major differences is that in the two-dimensional case for each step, we simulate two types of edge states together. The Dirac cones are observed in different steps and it happens if we fix one of the momenta and modifies the other one. On the other hand, when both of the momenta traverse the first Brillouin zone, the energy bands could close their gap nonlinearly, hence formation of Fermi arc edge states or flat bands edge states are observed (see Fig. 5).

Inspired by similar one-dimensional split-step quantum walk, we can consider one of the rotation angles to be linearly related to the other one ($\beta = s_1 \alpha + s_2$). The first important consequence of such consideration is simulation of edge states with three different geometries of Dirac cone, flat bands and Fermi arcs in a single step (see Fig. 7). The Dirac cones happens if we fix one of the momenta whereas we observe the other two geometries when both of the momenta scan the first Brillouin zone. The second important consequence of such consideration is observation of another type of cell-like structure for simulated topological phenomena (see Fig. 7 for $T = 6$ and $\alpha \in [-\pi/2, \pi/2]$). Each cell is characterized by two flat band edge states playing the role of the cell’s walls, two Fermi arc edge states and one additional flat band edge state located between the Fermi arc edge states. It should be noted that the flat band between the Fermi arcs is different from those of cell’s wall. Each cell also contains Dirac cones provided that one of the momenta is fixed and the other one varies. Therefore, in a single cell, we can simulate edge states with three different geometries. Finally, using the dynamicality of the energy bands through tuning step number, we can determine the number of edge states or topological phases, their type and where they should be. Therefore, we have high level of controllability over simulation of topological phenomena.

It is worthwhile to mention that the both of protocols that we used for two-dimensional quantum walk yield topological phases with Chern number 0. At the edge states, the Chern number is ill-defined since $n(k_x, k_y)$ is ill-defined. At the first step, $T = 1$, the obtained Hamiltonian, energy bands and other properties results into split-step two-dimensional quantum walk with step-independent coin which was investigated by Kitagawa et al in Ref. 28. In order to simulate topological phases with different Chern number, one should employ another type of the protocol for quantum walk which was introduced by Kitagawa et al in Refs. 27 28.
FIG. 6: Split-step quantum walk (two-dimensional): Energy as a function of rotation angle and momenta for \( \beta = \pi/3 \), \( T = 3 \) (a), \( T = 6 \) (b) and \( T = 8 \) (c). We observe that the simulated topological phenomena for each of these steps has specific characteristic behavior that differ from other steps. In (d), \( T = 3 \) and \( \alpha = 5\pi/3 \) which result into simulation of Dirac cone and flat bands edge states. For (e), we set \( T = 6 \) and \( \alpha = \pi/3 \) which also leads to formation of Dirac cone and flat band edge states. But the flat bands in (d) differ from those in (e) due to different conditions on momenta for having flat bands edge states. In (f) and (g), we consider \( T = 8 \) with rotation angles \( \alpha = 2\pi/3 \) (f) and \( \alpha = 5\pi/12 \) (f), respectively and observe simulation of Dirac cone and Fermi arc edge states. In (h), we observe topological phases (gaped energy bands) for \( T = 8 \), \( \alpha = \pi/5 \).

VI. SYMMETRIES

In this section, we briefly discuss the symmetries that different effective Hamiltonian, hence protocols of quantum walk possess. We will confirm that these effective Hamiltonians have three symmetries of particle-hole, chiral and time-reversal.

The matrix elements of the protocols that we used for our quantum walks (Eqs. (7), (20), (30) and (40)) are all real. Therefore, the complex conjugate of these protocols are equal to them (\( \hat{U}^* = \hat{U} \)). Based on the relationship between the protocol of the quantum walk and effective Hamiltonian \( \hat{H} \), we find

\[
\hat{H}^*(k) = -\hat{H}(k). \tag{46}
\]

This indicates that our effective Hamiltonians (irrespective of being simple- or split-step and being one- or two-dimensional) possess particle-hole symmetry, since we can find an antiunitary operator, \( \hat{P} \equiv \hat{K} \) in which \( \hat{K} \) is the complex conjugation operator satisfying \( \hat{P} \hat{H}(k) \hat{P}^{-1} = -\hat{H}(k) \). \tag{47}

The effective Hamiltonians that we simulated by different protocols of the quantum walk have time-reversal symmetry as well. The existence of this symmetry
FIG. 7: Split-step quantum walk (two-dimensional): Energy as a function of rotation angle and momenta for $\beta = (\alpha + \pi)/3$, $T = 6$ (a) and $T = 8$ (b). In (c), we set $T = 6$ with $\alpha = \pi/2$ while in (d) and (e), we consider $T = 8$ with $\alpha = -\pi/4$ and $\alpha = \pi/8$, respectively. We observe simulation of edge states in forms of Dirac cones (provided $k_y = -\pi$) and flat bands (provided $k_x$ and $k_y$ traverse first Brillouin zone). In (f), by setting $T = 8$ with $\alpha = 5\pi/16$, quantum walks simulate Dirac cone and Fermi arc edge states. Focusing on (a) for $\alpha \in [-\pi/2, \pi/2]$, we notice a cell-like structure for simulated topological phenomena. The cell could be recognized by two flat bands edge states playing the role of the cell’s walls, two Fermi arc edge states and one additional flat bands edge state located between the Fermi arcs. Each cell also contains Dirac cones if one of the momenta is fixed ($k_y = -\pi$) and the other one varies. Therefore, a single cell contains three different types of edge states including Dirac cone, flat bands and Fermi arcs.

could be confirmed through two methods. First of all, we showed the obtained energies for different effective Hamiltonians (Eqs. (12), (26), (33) and (42)) have a property in which $E(k) = E(-k)$. This is one of the requirements for the presence of time-reversal symmetry (27). In addition, we confirmed that the protocols of the one-dimensional quantum walks have chiral symmetry. It is straightforward to prove the presence of such symmetry for two-dimensional cases as well. The presence of particle-hole and chiral symmetries guarantees that there is an antiunitary operator, $\hat{T}$, which satisfies (27)

$$\hat{T} \hat{H}(k) \hat{T}^{-1} = \hat{H}(k),$$

where $\hat{T} \equiv \hat{\Gamma} \hat{\mathcal{P}}$.

Finally, it is worthwhile to take a look at the chiral symmetry operator of one-dimensional split-step quantum walk (29) at different types of edge state when $\beta = (\alpha + \pi)/3$. In case of Dirac cone edge states, for the rotation angles where energy bands close their gap, the chiral symmetry operator reduces to $\hat{\Gamma} = (-1)^{T/2} \sigma_x$. As for the flat band edge states, we obtain the chiral symmetry as $\hat{\Gamma} = -\sigma_z$. Additionally, if the edge states are Fermi arcs, the chiral symmetry operator yield $\hat{\Gamma} = \frac{1}{\sqrt{2}} [-(-1)^{T/2} \sigma_x - \sigma_z]$. We observe that for each type of edge state, we have specific characteristics that enable us to distinguish them from one another and put each edge state in its proper category.

VII. CONCLUSION

In this paper, we used the step-dependent coin in the protocols of the one- and two-dimensional quantum walk to simulate topological phases and edge states. We considered two types of protocol for the quantum walks known as simple-step and split-step protocols.

The one-dimensional quantum walks with simple-step protocol simulated only edge states with Dirac cone geometry and two non-trivial topological phases. The step-dependent coin provided the number of steps as a dynamical factor which changes the properties of the simulated topological phases (Hamiltonian, energy and etc.). In fact, by tuning the number of steps, we can engineer the
type of topological phases, the size of each topological phase and the place of edge states, hence phase transitions. Therefore, with simple-step quantum walk, we have highly controllable simulation of non-trivial topological phases and edge states with Dirac cone geometry.

The modification of the simple-step protocol to split-step one significantly enriched the capability of the quantum walk to simulate topological phases and edge states. First of all, the one-dimensional quantum walk with split-step protocol simulated all types of the edge states including Dirac cone, Fermi arc and flat bands. In addition, we showed that this quantum walk also simulates all types of topological phases. Therefore, the quantum walk with split-step protocol and step-dependent coin can simulate all types of the topological phases and edge states (and corresponding topological phase transitions).

In generalization to two-dimensional quantum walk with simple-step protocol, we simulated two types of edge state at each step: Dirac cones if we fix one of the momenta and let the other one varies through the first Brillouin zone, and momenta-dependent flat bands edge states if both of the momenta traverse the first Brillouin zone. In contrast, we showed that if simple-step protocol is modified to split-step in two-dimensional quantum walk, we can simulate all three types of Dirac cone, flat bands and Fermi arc edge states.

Finally, we showed that the rotation angles of coin operators and number of steps are controlling factors. They can be used to enforce simulation of only one (two) type(s) of the edge state at each step or all three types of the edge states together at specific steps. Moreover, we were able to simulate exotic cell-like structures for the topological phases and edge states. Each cell contained all types of edge states (and corresponding topological phase transitions). The dynamicality introduce by step-dependent coins enabled us to determine the place of edge states or topological phases, their types and their numbers. This provides us with highly controllable simulation of topological phenomena and allows one to achieve the universal simulator of topological phases and edge states with quantum walk.

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