Fractal properties of relaxation clusters and phase transition in a stochastic sandpile automaton

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Abstract

We study numerically the spatial properties of relaxation clusters in a two dimensional sandpile automaton with dynamic rules depending stochastically on a parameter $p$, which models the effects of static friction. In the limiting cases $p = 1$ and $p = 0$ the model reduces to the critical height model and critical slope model, respectively. At $p = p_c$, a continuous phase transition occurs to the state characterized by a nonzero average slope. Our analysis reveals that the loss of finite average slope at the transition is accompanied by the loss of fractal properties of the relaxation clusters.

Keywords

Self-organized criticality, sandpile automata, relaxation cluster, fractal dimension
1 Introduction

The term self-organized criticality (SOC) (Bak 1987, Bak 1988) applies to certain nonlinear extended dynamical systems which tend to self-organize into a steady state with long range spatial correlations, analogous to the critical state close to equilibrium second-order phase transitions. The dynamics of these systems consists of series of events (avalanches) in which the system is repeatedly perturbed and let to relax according to certain microscopic dynamic rules. After a certain large number of time steps, the system "learns" its response to the external perturbation. Clusters of relaxed sites have been shown to have fractal properties (Jánosi 1994). Therefore, the study of spatial properties of relaxation clusters offers an additional possibility to analyze the character of the dynamical process itself.

Sandpile automata models (Bak 1987, Dhar 1989, Manna 1991) are well known prototype models exhibiting SOC. By adding a particle from the outside, a sandpile is perturbed and the perturbation may lead to instabilities at neighboring sites if the local height of sand exceeds a critical value $h_c$ — critical height model (CHM) (Bak 1987, Dhar 1989), or if the local slopes exceed a critical value $\sigma_c$ — critical slope model (CSM) (Lübeck 1993, Manna 1991). The self-tuning of the system in the case of the CHM is accompanied with a state of a zero average slope. A finite average slope characterizes a CSM.

In the present work we study a two dimensional sandpile-type automaton with preferred direction in which the updating rules are tuned continuously by changing a parameter $p$ between CHM (for $p = 1$) and CSM (for $p = 0$). For all values $0 < p < 1$ the dynamics is stochastic. According to these dynamic rules, the relaxation clusters may have holes and dendritic forms. We present the results of numerical investigations of the spatial properties, i.e. distributions of size and length, and fractal dimension of the relaxation clusters. Our conclusion is that there is a threshold value of the probability parameter, $p = p_c$, above which the system behaves like a CHM, characterized by zero net slope. However, for $p < p_c$ the dynamics leads to the state with a nonzero net slope. The phase transition between these two states of the automaton is characterized by a continuous appearance of the average slope at $p \leq p_c$. At the same time, on approaching $p = p_c$ from above, the fractality of the relaxation clusters disappears.
2 Model and Numerical Simulations

We consider a two-dimensional sandpile model on a square lattice of size \( L \times L \) and integer variables \( h(i, j) \), representing the local height. We assume a directed dynamics, i.e., particles are restricted to flow in the downward direction (increasing \( i \)). According to the widespread 'sandpile language' the first row \((i = 1)\) and the last row \((i = L)\) represent the top and the bottom of the pile, respectively. To minimize the influence of the horizontal boundaries we limit our investigations to periodic boundary conditions in this horizontal direction \((j\text{-direction})\). Any site of the lattice has two downward and two upward next neighbours, namely

\[
h(i + 1, j_{\pm}) \quad \text{and} \quad h(i - 1, j_{\pm}) \quad \text{with} \quad j_{\pm} = j \pm \frac{(1 \pm (-1)^{i})}{2}. \tag{1}
\]

We perturb the system by adding particles at a random place on the top of the pile according to

\[
h(1, j) \mapsto h(1, j) + 1, \quad \text{with random} \; j. \tag{2}
\]

A site is called unstable if the height \( h(i, j) \) or at least one of the two slopes

\[
\sigma(i, j_{\pm}) = h(i, j) - h(i + 1, j_{\pm}) \tag{3}
\]

exceeds a critical value, i.e., if \( h(i, j) \geq h_c \) or \( \sigma(i, j_{\pm}) \geq \sigma_c \), respectively.

In the case of the critical slope condition toppling takes place until both slopes become subcritical. In this toppling process particles drop alternative to the downward next neighbours if both slopes are unstable. If only one slope exceeds \( \sigma_c \) particles drop to the corresponding downward neighbour.

In contrast to the critical slope the critical height conditions has a stochastic character. If the local height exceeds the critical value \( h_c \), toppling occurs only with the probability \( p \), and then two particles drop to the two downward neighbours.

Because each relaxing cell changes the heights and slopes of its four next neighbouring sites the stability conditions are applied at these four 'activated' sites in the next updating step. Toppling may take place after adding a particle on the first row. We first apply the slope and then the height stability condition in parallel for each activated site. An avalanche stops if all activated sites are stable. Then we start again according to Eq. (2). One can interpret \( 1 - p \) as being due to static friction between the sand grains, which prevents toppling even if the height exceeds the critical value. In the case \( p = 1 \), our model is identical to the model of Dhar and Ramaswamy (Dhar 1989), which exhibits a robust SOC behavior. In the limit \( p = 0 \), our directed rules lead to a steady state where all slopes are equal, i.e., \( \sigma(i, j_{\pm}) = \sigma_c - 1 \). Thus, if a particle is added at the first row it
performs a directed random walk downward the pile until it reaches the boundary and
drops out of the system — no SOC can occur in this limit.

We fix the critical height \( h_c \) and the critical slope \( \sigma_c \) and restrict ourselves to \( h_c = 2 \)
and \( \sigma_c = 8 \), although it should be emphasized that the results depend on the choice of
these parameters.

Figure 1  Snapshots of sandpiles for \( p = 1 \) (upper left), 0.7 (upper right), 0.3 (lower
left), and 0.1. In all cases the upper left boundary corresponds to the first row of the pile
\( (i = 1) \).

Starting from an empty lattice of linear size \( L \) we add approximately \( L^3 \) particles to
equilibrated the system before taking measurements. In each case the stationarity of the
system was checked by determining that the average height \( \langle h(i,j) \rangle \) has reached a con-
stant value.
3 Results

In Figure 1 we show four snapshots of the sandpile for different values of \( p \). The upper left picture represents the pile in a steady state of the pure CHM \((p = 1)\). Only the heights 0 and 1 are present in the critical state. With decreasing \( p < 1 \) a certain number of heights with values \( h(i, j) > h_c \) remain in the interior of an avalanche leading to a rough surface of the pile (see e.g. the case \( p = 0.7 \) of Figure 1).

For still lower values of \( p \) the roughness becomes more dramatic and eventually for \( p = 0.3 \) (lower left) the surface shows large fluctuations, indicating a change of the system behavior. In the case \( p = 0.1 \), displayed in the lower right picture, the heights grow up to the full size required by the CSM rules with fluctuations around the well defined average slope. We now analyze this behavior quantitatively by considering the average height \( \langle h(i) \rangle \) as a function of distance from the top row \( i \), defined as

\[
\langle h(i) \rangle = L^{-1} \sum_{j=1}^{L} \langle h(i, j) \rangle,
\]

where \( \langle h(i, j) \rangle \) is determined as the average over total number of time steps. In the interior of the pile the average height is independent of \( i \) for all values of \( p \geq 0.3 \), i.e. \( \langle h(i) \rangle = \langle h \rangle \), where \( \langle h \rangle \) depends on the probability parameter \( p \). The normalized average heights are plotted in Figure 2a vs. the distance \( i \). Except of the close vicinity of the boundaries the curves corresponding to different values of \( p > 0.3 \) collapse into a single curve \( \langle h(i) \rangle / \langle h \rangle = 1 \).

However, this is no longer the case for \( p < 0.3 \), where the deviations from \( \langle h \rangle \) characteristic for the behavior at the boundaries proliferates into the interior of the pile. Instead of constant values of the average heights we find for \( p < 0.3 \) constant values of the average slopes, which are defined as \( \sigma(i) = \langle h(i) \rangle - \langle h(i+1) \rangle \). In Figure 2b the normalized average slopes are shown. Except of the deviations due to the boundaries all curves more or less collapse to \( \sigma(i)/\sigma(i = 50) = 1 \). Simulations for some values of \( p \) and different automata sizes \((L = 50, 100, 200)\) show that these deviations are independent of \( L \), i.e. they are true boundary effects. In the interior part of the automaton the average slopes are independent of \( i \).

For \( p < 0.3 \), the slopes obey a translation invariance similar to that of the heights in the opposite limit. The average slopes become a function of \( i \) if the probability parameter \( p \) is close to \( p = 0.3 \).
Figure 2  (a) Normalized average height $\langle h(i) \rangle$ at distance $i$ from the top of the pile plotted vs. the distance $i$ for various values of $p$ between 0.9 (flat curve) and 0.295 (curve with profile). (b) Normalized average slope $\langle \sigma(i) \rangle$ for various values of $p$ between 0.1 (flat curve) and 0.29 (curve with profile).

Since a zero net slope characterizes a CHM and a non zero net slope a CSM, respectively, we interpret the average slopes

$$\langle \sigma \rangle = \frac{1}{L - 40} \sum_{i=20}^{L-20} \langle \sigma(i) \rangle$$

as an order parameter, which describes the transition from the critical height to the critical slope regime. In this definition we cut off the first and last 20 values in order to minimize the influence of the boundaries. The results for $\langle \sigma \rangle$ which are obtained in this way are independent of $L$ as additional simulations for $L = 200$ have shown. Of course it is also irrelevant if the cut off length is larger than 20.
The true nature of the transition as well as the transition point \( p_c \) should be determined from the \( p \)-dependence of the average slope itself in the vicinity of \( p_c \). We determined the average net slope according to Eq. (5) for different values of \( p \). The results are shown in Figure 3. As the average net slope plays the role of an order parameter in our model, we see from Figure 3 that a continuous transition occurs between the two regimes of the system. Assuming the following form

\[
\langle \sigma \rangle \sim (p_c - p)^\alpha,
\]

we determine the transition point \( p_c = 0.293 \pm 0.002 \) and the exponent \( \alpha = 0.8 \pm 0.05 \) from the data in Figure 3.

![Figure 3](image-url)

**Figure 3** Average slope of the pile \( \langle \sigma \rangle \) plotted vs. probability \( p \). The solid line of the inset corresponds to a fit according to equation (6) with \( p_c = 0.293 \) and \( \alpha = 0.8 \).

Next we consider the spatial structure of the avalanches. One relaxation cluster consists of all sites that topple during one event following one particle being added at the top row. In Figure 4 we show six representative snapshots of the relaxation clusters taken from simulations for different values of \( p \). For \( p = 1 \) the avalanches are compact. With decreasing \( p \) some supercritical sites may remain in the interior of the avalanche due to the stochastic character of the dynamic rules, leading to holes and branching of the clusters. For values of \( p \) which are lower than \( p_c \approx 0.3 \) this structure of the avalanches is lost. In this region the net nonzero slope is developed and the probability of triggering an after-avalanche increases dramatically. The system relaxes occasionally through huge avalanches involving almost all sites. In the limiting case of \( p = 0 \) the system shows rather simple behavior, since the rules of the CSM are deterministic and fluctuations...
around the average slope $\sigma_c - 1$ are absent. Each added particle performs a random walk down the whole automaton.

![Snapshots of relaxation clusters for a few values of $p$ for a linear system size of $L = 80$. Sites which toppled during an avalanche are marked as black. The base of every picture represents the bottom of the pile ($i = L$).](image)

**Figure 4** Snapshots of relaxation clusters for a few values of $p$ for a linear system size of $L = 80$. Sites which toppled during an avalanche are marked as black. The base of every picture represents the bottom of the pile ($i = L$).

We next analyze the scaling properties of the clusters, which we expect to reflect these observations. We calculate the average size of all clusters of length $l$, $\langle s \rangle (l)$, which scales as a power of the length $l$, i.e.,

$$\langle s \rangle (l) \sim l^{d_\parallel} .$$  

(7)

We emphasize that this definition of $d_\parallel$ describes only the scaling along the preferred direction. Actually the avalanches in our model are self-affine fractals, i.e. the fractal
dimension along the perpendicular direction \( d_{\perp} \) is not equal to \( d_{\parallel} \). A detailed discussion of this anisotropic behavior will be published elsewhere.

![Figure 5](image_url)  

**Figure 5**  (a) Double logarithmic plot of the average size of clusters \( \langle s \rangle (l) \) vs. length \( l \) for various values of \( p \) as indicated. (b) Parallel fractal dimension \( d_{\parallel} \) of the relaxation clusters plotted vs. probability \( p \). For \( p \geq 0.33 \) the error-bars are smaller than the symbols.

The exponent in Eq. (7), which determines the scaling properties of relaxation clusters, depends on the parameter \( p \) (see Figure 5). We examined this \( p \)-dependence of the fractal dimension intensively, the obtained results are shown in Figure 5. In agreement with the pictures of the avalanche shapes in Figure 4 \( d_{\parallel} \) increases with decreasing \( p \). From \( d_{\parallel} = 1.469 \) for \( p = 1 \), which is in a good agreement with the known value \( d_{\parallel} = \frac{3}{2} \) for the pure directed CHM (Tadić 1992), the fractal dimension grows with decreasing \( p \) and reaches its maximum value \( d_{\parallel} = 2 \) for \( p \leq 0.3 \). In this way the critical height regime
is accompanied by a self-affine fractal structure of the relaxation cluster. This fractal behavior vanishes below the transition point $p_c$ where the critical slope condition rules the dynamics.

4 Conclusions

In conclusion, we have shown that our stochastic sandpile automaton exhibits a continuous phase transition in its steady state, as the probability parameter $p$ is varied through the transition point $p_c = 0.293$. The steady states for $p < p_c$ are characterized by a finite net average slope. For $p > p_c$ the average slope remains zero. Fractality of the relaxation clusters, measured by the deviation of their fractal dimension $d_{\parallel}$ from the Euclidean dimension $d = 2$, is shown to vanish at the transition point, strongly suggesting that the fractal character of the relaxation clusters is closely related to the CHM regime.

The analysis of the avalanche distributions and especially the question whether the model displays SOC behavior for $p < 1$ remains beyond the scope of the present paper and will be published elsewhere.

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