A STEINBERG ALGEBRA APPROACH TO ÉTALE GROUPOID C*-ALGEBRAS

LISA ORLOFF CLARK AND JOEL ZIMMERMAN

Abstract. We construct the full and reduced C*-algebras of an ample groupoid from its complex Steinberg algebra. We also show that our construction gives the same C*-algebras as the standard constructions. In the last section, we consider an arbitrary locally compact, second-countable, étale groupoid, possibly non-Hausdorff. Using the techniques developed for Steinberg algebras, we show that every ∗-homomorphism from Connes’ space of functions to $B(ℋ)$ is automatically I-norm bounded. Previously, this was only known for Hausdorff $G$.

1. Introduction

C*-algebras of Hausdorff groupoids were introduced by Renault [18] as a completion of $C_c(G)$, the complex vector space of continuous functions from the groupoid $G$ to $ℂ$ that are zero outside of some compact set. The construction for non-Hausdorff groupoids was developed (simultaneously, it seems!) by Connes in [6, 5]. Instead of taking a completion of $C_c(G)$, Connes takes a completion of $ℂ(G)$, the complex vector space of functions from $G$ to $ℂ$ spanned by elements of $C_c(U)$ for all open Hausdorff subsets $U \subseteq G$. (If $G$ is Hausdorff, $C_c(G)$ can be identified with $ℂ(G)$.) In the full C*-algebra, the completion is taken with respect to the full norm, defined for $f ∈ ℂ(G)$ as is the supremum of the operator norms $\|π(f)\|$ taken over all bounded ∗-homomorphisms $π : ℂ(G) → B(ℋ)$ for some Hilbert space $ℋ$.

Exel gives an alternate approach in [7] for étale groupoids, again by taking completions of $ℂ(G)$. He defines the full norm similarly; however, he takes the supremum over all ∗-homomorphisms, that is, he does not require them to be bounded. He demonstrates that when $G$ is étale, the additional assumption is not needed to define a norm on $ℂ(G)$. This leaves open the question of whether or not Exel’s construction produces the same C*-algebra as Renault and Connes’ constructions.

An ample groupoid is an étale groupoid that has a basis of compact open sets. The class of C*-algebras of ample groupoids contains many important subclasses including graph and higher-rank graph C*-algebras [11, 12], Exel-Pardo algebras [10], inverse semigroup algebras [16] and all Kirchberg algebras [4].

In this paper, we build the full and reduced C*-algebra of an ample groupoid $G$ by taking completions of its Steinberg algebra $A(G)$, the complex vector space of functions from $G$ to $ℂ$ spanned by characteristic functions $1_B$ for all compact open Hausdorff subsets $B \subseteq G$, (see [20, Definition 4.1]). As in [11] and [3], we advance the notion that the Steinberg
algebra is a sort of Rosetta stone that can be used to make connections between a more general étale groupoid its C*-algebra.

Given an ample groupoid \( G \) with Hausdorff unit space, defining its reduced C*-algebra from \( A(G) \) is straightforward; moving from \( \mathcal{C}(G) \) to \( A(G) \) doesn’t cause any problems. To define the full C*-algebra, we use Exel’s approach from [7]. The full norm of a function \( f \in A(G) \) will be the supremum of \( \| \pi(f) \| \) taken over all \(*\)-homomorphisms \( \pi : A(G) \to B(\mathcal{H}) \). However, to show that this supremum is finite (without making a boundedness assumption on \( \pi \)), we require some innovation. In Exel’s original argument, for every compact open set \( U \subseteq G^{(0)} \), the algebra of continuous functions, \( C(U) \) is a C*-algebra sitting inside \( \mathcal{C}(G) \). Thus C*-algebraic tools are available. For us, \( A(U) \) is not a C*-algebra (there is no obvious C*-norm in which \( A(U) \) is complete) so we have to work a little harder. Once we establish that particular kinds of algebra \(*\)-homomorphism are norm decreasing with respect to the uniform norm on \( A(U) \), (see Proposition 4.2), Exel’s strategy for \( \mathcal{C}(G) \) goes through. We hope our construction will be a good starting point to learn about groupoid C*-algebras as there is less technical overhead. Our approach also unifies the Hausdorff and non-Hausdorff construction since the definition of \( A(G) \) is the same regardless of the Hausdorff property.

To reconcile our construction with the standard ones, we start by showing in Proposition 5.2 that any \(*\)-homomorphism \( \pi \) from \( A(G) \) to \( B(\mathcal{H}) \) is automatically bounded with respect to the inductive limit topology (see definition 5.1). Then, for second countable \( G \), Renault’s disintegration theorem [13, Theorem 7.8] implies \( \pi \) is automatically I-norm bounded. This proof depends on the structure of \( A(G) \) and uses the basis of compact open sets in \( G \). But the techniques developed in the proof are our Rosetta stone. By seeing how things played out in \( A(G) \), we can translate to \( \mathcal{C}(G) \) for more general étale groupoids using approximation arguments. In particular, we show in Corollary 6.6 if \( G \) is a second countable étale groupoid and \( \pi \) is a \(*\)-homomorphism from \( \mathcal{C}(G) \) to \( B(\mathcal{H}) \), then \( \pi \) is I-norm bounded. Previously, this was an open question for non-Hausdorff groupoids.

After a brief preliminaries section, we give a self contained construction of the reduced and full groupoid C*-algebra as the completion of \( A(G) \) in Sections 3 and 4. Note that our construction does not require \( G \) to be second countable. In section 5 we show that \(*\)-homomorphisms from \( A(G) \) to \( B(\mathcal{H}) \) are automatically bounded and in Section 6 we consider second-countable étale groupoids that are not necessarily ample. In the last section, we show that the C*-algebras we have constructed are the same as the standard ones (see Theorem 7.1).

## 2. Preliminaries

First some terminology. When we say a subset \( X \) of a topological space is compact, we mean that every open cover has a finite subcover, even for non-Hausdorff \( X \). We say a topological space is locally compact if every point has a compact neighborhood base, again, without making any Hausdorff assumption. The first author has also called this ‘locally locally compact’, see [2] for further discussion.

A groupoid is a generalisation of a group where the binary operation is only partially defined. See [18] for a precise definition. Let \( G \) be a groupoid and \( G^{(0)} \) be the set of units in \( G \) so that for each \( \gamma \in G \) we have the maps

\[
s(\gamma) = \gamma^{-1} \gamma \quad \text{and} \quad r(\gamma) = \gamma \gamma^{-1}
\]
from $G$ to $G^{(0)} \subseteq G$. An étale groupoid is a topological groupoid such that $s$ (and hence $r$) is a local homeomorphism. In an étale groupoid, $G^{(0)}$ is open in $G$; $G^{(0)}$ is closed in $G$ if and only if $G$ is Hausdorff (see, for example, [9, Proposition 3.10]). We call an open set $B \subseteq G$ an open bisection if $r$ and $s$ restricted to $B$ are homeomorphisms onto open subsets of $G^{(0)}$. It is straightforward to check that a topological groupoid $G$ is étale groupoid if and only if $G$ has a basis of open bisections.

We say $G$ is an ample groupoid if $G$ has a basis of compact open bisections. Equivalently, $G$ is ample if $G$ is locally compact and étale and the unit space $G^{(0)}$ is totally disconnected [8, Proposition 4.1]. The collection of all compact open bisections in an ample groupoid $G$ forms an inverse semigroup with product and inverse

$$BD := \{bd : b \in B, d \in D, s(b) = r(d)\} \quad \text{and} \quad B^{-1} := \{b^{-1} : b \in B\}.$$ 

(See [16, Proposition 2.2.3].) Note that, if $G$ is not Hausdorff, the intersection of compact open bisections need not be compact.

Because ample groupoids are necessarily étale, we have that the restriction of the range map to a compact open bisection $B$ induces a homeomorphism between $B$ and $r(B) \subseteq G^{(0)}$. So if $G^{(0)}$ is Hausdorff, then $B$ is also Hausdorff. In this situation, techniques developed for Hausdorff groupoids can sometimes be employed locally. For example, suppose that $B$ is a compact open bisection and that $\{D_i\}_{i \in I}$ is a finite cover of $B$ such that each $D_i \subseteq B$ is compact open. Then because $B$ is Hausdorff and each $D_i$ is compact, each $D_i$ is closed in the subspace $B$ and hence for any compact open $A \subseteq B$, we have that $D_i \setminus A$ is compact open. So can find a disjoint cover of $B$ by compact open sets $\{D'_i\}_{i \in I}$ where $D'_1 = D_1$ and for $i > 1$

$$D'_i := D_i \setminus \bigcup_{j=1}^{i-1} D_j.$$ 

When we refine a cover in this manner, we say that we disjointify the cover.

Let $G$ be an ample groupoid such that $G^{(0)}$ is Hausdorff. For $B \subseteq G$, we write $1_B$ for the function from $G$ to $\mathbb{C}$ that takes value 1 on $B$ and 0 outside of $B$, that is, the characteristic function of $B$. The complex Steinberg algebra of $G$ is the complex vector space

$$A(G) := \text{span}\{1_B \mid B \text{ is a compact open bisection}\}$$

where addition and scalar multiplication are defined pointwise. To make clear our notation, for each $f \in A(G)$, we can write

$$f = \sum_{B \in F} a_B 1_B$$

such that $F$ is a finite collection of compact open bisections and for each $B \in F$, $a_B \in \mathbb{C}$. Then $A(G)$ is a $\ast$-algebra with convolution and involution of generators is given by

$$1_B 1_D = 1_{BD} \quad \text{and} \quad (1_B)^\ast = 1_{B^{-1}}$$

which distribute to give the usual convolution and involution formulae: for $f, g \in A(G)$, and $\gamma \in G$

$$f \ast g(\gamma) = \sum_{\alpha \beta = \gamma} f(\alpha) g(\beta) \quad \text{and} \quad f(\gamma)^\ast = \overline{f(\gamma)}.$$
We will sometimes write convolution as $fg$ omitting the $\ast$. By [20, Proposition 4.3 and Definition 4.1], $A(G)$ is also equal to the span of $1_B$ ranging over all compact open Hausdorff subsets $B$ of $G$.

Since we are working with potentially non-Hausdorff groupoids, we need to take extra care with compact sets and closures. So for example, if $D$ is a compact open bisection, then $1_D$ is a continuous function on $D$. If $G$ is Hausdorff, the compact subsets of $G$ are closed and therefore $D$ is a clopen subset of $G$, which means that $1_D$ is a continuous on $G$. However, if $G$ is not Hausdorff then $D$ may not be closed in $G$ and then $1_D$ is not continuous on all of $G$. See Example 2.1.

For $f \in A(G)$, we write

$$\text{supp}^o(f) = \{ \gamma \in G : f(\gamma) \not= 0 \}.$$ 

Note that this set might not be open in $G$ and, although it is contained in a compact set (so the function is “compactly supported”), its closure might fail to be compact. See Example 2.1. For more details on Steinberg algebras, see [20].

Example 2.1. A basic non-Hausdorff example to keep in mind is the “two-headed snake” groupoid: the unit space $G^{(0)}$ is the Cantor set (viewed as a subset of $[0,1] \subseteq \mathbb{R}$) and $G = G^{(0)} \cup \{ \gamma_1 \}$ where $s(\gamma_1) = r(\gamma_1) = 0$ and composition $\gamma_1 \gamma_1 = 0$. Equivalently, $G$ is a group bundle over the Cantor set where all the groups are trivial except for the one at 0, which is $\mathbb{Z}_2$. To make $G$ into a topological groupoid, we take the usual topology on $G^{(0)}$, that is, the subspace topology of $\mathbb{R}$. In addition, for every open set $U$ containing 0, we add an open set $U_{1}$ where we remove 0 and add $\gamma_{1}$. Then $G$ is an ample groupoid that has a Hausdorff unit space but is not Hausdorff itself.

The unit space is a compact open subset of $G$ so $1_{G^{(0)}} \in A(G)$ but since $G^{(0)}$ is not closed in $G$, $1_{G^{(0)}}$ is not continuous. Let $B = (G^{(0)} \setminus \{0\}) \cup \{1\}$ which is a compact open bisection in $G$. Then for $f = 1_{G^{(0)}} - 1_B$, we have $\text{supp}^o(f) = \{0, \gamma_1\}$ which is not open.

Since $G$ is compact, for every $f \in A(G)$, the closure $\text{supp}^o(f)$ is compact. However, we can add more heads to the snake to get an example where this fails: let $H$ be the group bundle over the Cantor set where all the groups are trivial except for the one at 0, which is $\mathbb{Z}$. Denote the non-unit elements of $H$ by $\gamma_i$ for $i \in \mathbb{Z} \setminus \{0\}$. For the topology, we have the usual topology on the Cantor set and for each $i \in \mathbb{Z}$ and each open set containing 0, we add an open set where we replace 0 with $\gamma_i$. Then $H$ is an ample groupoid with Hausdorff unit space. Now $1_{G^{(0)}} \in A(G)$ but $\text{supp}^o(1_{G^{(0)}}) = G$ is not compact.

We will need the following lemma in the sequel. It is obvious when $G$ is Hausdorff but more generally it is surprisingly technical.

Lemma 2.2. Let $G$ be an ample groupoid such that $G^{(0)}$ is Hausdorff. Let $f \in A(G)$ with $\text{supp}^o(f) \subseteq C$ where $C \subseteq G$ is a compact open subset. Suppose $f = \sum_{B \in H} a_B 1_B$ where $H$ is a finite collection of compact open bisections. Then we can write $f = \sum_{D \in F} a_D 1_D$ where $F$ is a finite collection of compact open bisections such that for each $D \in F$ we have $D \subseteq C$.

Proof. We do a proof by induction on $|H|$, the size of $H$. The base case $|H| = 1$ is clear. For the inductive step, suppose the lemma holds for functions $f \in A(G)$ with $|H| = n \geq 1$. Fix $f \in A(G)$ with $\text{supp}^o(f) \subseteq C$ such that $f = \sum_{B \in H} a_B 1_B$ where $|H| = n + 1$. Suppose there exists $B_1 \in H$ such that $B_1 \setminus C \not= \emptyset$. (If no such $B_1$ exists, we are done.) Notice
that $B_1 \setminus C$ is closed in $B_1$ and hence is compact. Fix $\gamma \in B_1 \setminus C$. Since $f(\gamma) = 0$, there exists a maximal $S_\gamma \subseteq H$ such that

$$B_1 \in S_\gamma, \quad \gamma \in \bigcap_{B \in S_\gamma} B, \quad \text{and} \quad \sum_{B \in S_\gamma} a_B = 0.$$ 

Because $G$ is ample and elements of $H$ are open, we can find a compact open bisection $D_\gamma$ such that

$$\gamma \in D_\gamma \subseteq \bigcap_{B \in S_\gamma} B.$$ 

The collection $\{D_\gamma\}_{\gamma \in B_1 \setminus C}$ covers $B_1 \setminus C$ and hence has a finite subcover $\{D_\gamma\}_{\gamma \in I}$ for some finite $I \subseteq B_1 \setminus C$. Since we are inside the Hausdorff space $B_1$, we can disjointify and assume this finite cover of $B_1 \setminus C$ is disjoint. Now define compact open bisections for each $B \in H$

$$D_B := B \setminus \bigcup_{\{\gamma \in I : B \in S_\gamma\}} D_\gamma.$$ 

Notice that for our fixed $B_1$ we have $D_{B_1} \subseteq C$. Let $F_1 = \{D_B : B \in H\}$. We claim that

$$f = \sum_{D_B \in F_1} a_B 1_{D_B}.$$ 

Fix $\alpha \in G$. We consider two cases. First suppose $\alpha \notin D_\gamma$ for all $\gamma \in I$. Then

$$\alpha \in B \in F \iff \alpha \in D_B \in F_1$$

so the claim is true. Now suppose there exists $\gamma \in I$ such that $\alpha \in D_\gamma \subseteq \bigcap_{B \in S_\gamma} B$. Since the collection is disjoint there is only one such $\gamma \in I$. By construction $\sum_{B \in S_\gamma} a_B = 0$. So

$$f(\alpha) = \sum_{\{B \in H : \alpha \in B\}} a_B$$

$$= \sum_{\{D_B \in F_1 : \alpha \in D_B\}} a_B + \sum_{B \in S_\gamma} a_B$$

$$= \sum_{\{D_B \in F_1 : \alpha \in D_B\}} a_B$$

proving the claim. Thus

$$f = \sum_{D_B \in F_1} a_B 1_{D_B} = a_{B_1} 1_{D_{B_1}} + \sum_{D_B \in F_1, B \neq B_1} a_B 1_{D_B}$$

with $D_{B_1} \subseteq C$. Now the inductive hypothesis applies to the second term, proving the lemma \qed

3. The reduced C*-algebra

To define the reduced C*-algebra of an ample groupoid $G$ from $A(G)$, the usual approach for $\mathcal{C}(G)$ goes through virtually unchanged so we just outline the process. The idea is to embed $A(G)$ into $B(\mathcal{H})$ for a specific $\mathcal{H}$ using an injective $*$-homomorphism $\pi$ and then define the reduced norm of a function in $A(G)$ to be the operator norm of its image under $\pi$. Then the reduced C*-algebra $C^*_r(G)$ is the completion of $A(G)$ with respect to this norm, and is isomorphic to the closure of $\pi(A(G))$ in $B(\mathcal{H})$. 
The I-norm for $f \in A(G)$ is defined as
\[
\|f\|_I := \sup_{x \in G^{(0)}} \left\{ \sum_{\{\gamma : s(\gamma) = x\}} |f(\gamma)|, \sum_{\{\gamma : r(\gamma) = x\}} |f(\gamma)| \right\}.
\]
We begin by introducing for each $x \in G^{(0)}$, an I-norm bounded \(*\)-homomorphism
\[
\pi_x : A(G) \to B(\ell^2(G_x))
\]
which is called the regular representation of $A(G)$ at $x$. Write the standard orthonormal basis of $\ell^2(G_x)$ as the indicator functions $\{\delta_x\}_{\gamma \in G_x}$. Then if $B$ is a compact open bisection in $G$ and $\gamma \in G_x$, we define $\pi_x$ on generators by the formula
\[
\pi_x(1_B)\delta_\gamma = \delta_{B\gamma}
\]
where $B\gamma$ is shorthand for the product of sets $B\{\gamma\}$. Since $B$ is a bisection, $B\gamma$ is either empty or a singleton. More generally, we have the following. The proof follows similarly to the proof of [19 Proposition 3.3.1].

**Proposition 3.1.** For each $x \in G^{(0)}$, there exists a \(*\)-homomorphism $\pi_x : A(G) \to B(\ell^2(G_x))$ defined by
\[
\pi_x(f)\delta_\gamma = \sum_{\alpha \in G_{r(\gamma)}} f(\alpha)\delta_{\alpha\gamma}
\]
such that $\|\pi_x(f)\| \leq \|f\|_I$.

It is useful to observe that the formula for $\pi_x$ is the same as the convolution product formula in $A(G)$. Let $H = \bigoplus_{x \in G^{(0)}} \ell^2(G_x)$. Then there is an injective \(*\)-homomorphism $\pi : A(G) \to B(H)$ such that for $(g_x)_{x \in G^{(0)}} \in H$ and $f \in A(G)$ we have
\[
\pi(f)(g_x)_{x \in G^{(0)}} = (\pi_x(f)g_x)_{x \in G^{(0)}} \quad \text{and} \quad \|\pi(f)\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|.
\]
We call $\pi$ the left regular representation of $A(G)$. Now the reduced norm for $f \in A(G)$ is defined such that $\|f\|_r = \|\pi(f)\|$ and $C^*_r(G)$ is identified with $\overline{\pi(A(G))}$ in $B(H)$.

4. **The full C*-algebra**

To construct the full C*-algebra, we show that for $f \in A(G)$, we can define a norm $\|f\|$ with the formula
\[
\sup\{\|\pi(f)\| \mid \pi : A(G) \to B(H) \text{ is a } \ast\text{-homomorphism for some Hilbert space } H\}.
\]
Thus we must show that the set
\[
P_f := \{\|\pi(f)\| \mid \pi : A(G) \to B(H) \text{ is a } \ast\text{-homomorphism for some Hilbert space } H\}
\]
has an upper bound. We do this in stages:
- We first show $P_f$ has an upper bound when $\text{supp}^o(f) \subseteq G^{(0)}$.
- Then we show $P_f$ has an upper bound when $\text{supp}^o(f) \subseteq B$ for some compact open bisection $B$.
- Finally we show $P_f$ has an upper bound for any $f \in A(G)$.

**Functions supported on $G^{(0)}$:** If $U \subseteq G^{(0)}$ is a compact open subset, then $U$ is itself a groupoid that consists entirely of units and $A(U)$ is a complex Steinberg algebra. In the next proposition we show that when $\pi : A(U) \to B(H)$ is a \(*\)-homomorphism, then $\pi$ is automatically norm-decreasing.
Remark 4.1. In Exel’s construction in [7], instead of \( A(U) \), he works with \( C(U) \subseteq C(G) \), which is a C*-algebra. So he can use the fact that any homomorphism between C*-algebras is automatically norm-decreasing, see, for example, [15, Theorem 2.1.7].

**Proposition 4.2.** Let \( G \) be an ample groupoid with Hausdorff unit space and let \( U \subseteq G^{(0)} \) be compact open. Suppose

\[
\pi : A(U) \to B(\mathcal{H})
\]

is a *-homomorphism for some Hilbert space \( \mathcal{H} \). Then \( \pi \) is norm-decreasing with respect to the uniform norm on \( A(U) \) and the operator norm on \( B(\mathcal{H}) \).

**Proof.** Notice that \( A(U) \) is unital with identity \( 1_U \). Let \( B = \pi(A(U)) \). Then \( \pi(1_U) \) is an identity in \( B \). By continuity of multiplication in \( B(\mathcal{H}) \), \( \pi(1_U) \) is an identity in \( B \).

Since \( B \) is a C*-algebra, each element \( b \in B \) has a spectrum \( \sigma_b(b) \) and spectral radius \( r_b(b) \). Fix \( f \in A(U) \). Define the algebraic spectrum of \( f \) to be

\[
\sigma_{A(U)}(f) := \{ \lambda \in \sigma_{C(U)}(f) : f - \lambda 1_U \text{ is not invertible in } A(U) \}
\]

where \( \sigma_{C(U)}(f) \) denotes the usual spectrum of \( f \) in the C*-algebra \( C(U) \) of continuous functions on \( U \). Similarly define the spectral radius

\[
r_{A(U)}(f) := \sup\{|\lambda| : \lambda \in \sigma_{A(U)}(f)\}.
\]

In fact \( \sigma_{A(U)}(f) = \sigma_{C(U)}(f) \) and \( r_{A(U)}(f) = r_{C(U)}(f) \), we are just being careful to observe which algebra we are working in.

We claim that \( r_{\overline{B}}(\pi(f)) \leq r_{A(U)}(f) \). Suppose \( \lambda \notin \sigma_{A(U)}(f) \). Then \( \pi(f - \lambda 1_U)^{-1} \in A(U) \).

Since \( \pi \) is a unital homomorphism from \( A(U) \) into \( \overline{B} \), \( \pi((f - \lambda 1_U)^{-1}) \) is an inverse for \( \pi(f - \lambda 1_U) \) and so by linearity, \( \lambda \notin \sigma_{\overline{B}}(\pi(f)) \). Thus \( \sigma_{\overline{B}}(\pi(f)) \subseteq \sigma_{A(U)}(f) \) and the claim follows. Now because \( \overline{B} \) is a C*-algebra we use [15, Theorem 2.1.1] to compute

\[
\|\pi(f)\|^2 = \|\pi(f)\|_{\overline{B}}^2 = \|\pi(f^*f)\|_{\overline{B}} = r_{\overline{B}}(\pi(f^*f)),
\]

further, we have

\[
r_{A(U)}(f^*f) = r_{C(U)}(f^*f) = \|f\|_\infty^2;
\]

and by our claim and the above calculations, we observe that \( \|\pi(f)\|^2 \leq \|f\|_\infty^2 \) as required. \( \square \)

For compact open \( U \subseteq G^{(0)} \), we identify \( A(U) \) with its isomorphic image in \( A(G) \) under the inclusion map \( i : A(U) \to A(G) \) where

\[
i(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in U \\ 0 & \text{otherwise.} \end{cases}
\]

The next lemma shows that our intuition about “functions supported on the unit space” is accurate.

**Lemma 4.3.** Let \( G \) be an ample groupoid such that \( G^{(0)} \) is Hausdorff. If \( f \in A(G) \) with \( \text{supp}^o(f) \subseteq G^{(0)} \), then there is a compact open subset \( U \subseteq G^{(0)} \) such that \( \text{supp}^o(f) \subseteq U \).

**Proof.** Write \( f = \sum_{B \in F} a_B 1_B \) where \( F \) is a finite collection of compact open bisections. Then

\[
\text{supp}^o(f) \subseteq K := \bigcup_{B \in F} B.
\]
Notice that $K$ is compact open. We have $\text{supp}^o(f) \subseteq G^{(0)}$ by assumption, so $s(\text{supp}^o(f)) = \text{supp}^o(f)$. Now

$$\text{supp}^o(f) \subseteq s(K)$$

which is compact open because $s$ is a local homeomorphism. So $U = s(K)$ suffices.

We can now verify that $P_f$ is bounded by $\|f\|_\infty$ for and function $f$ supported on $G^{(0)}$.

**Lemma 4.4.** Let $\pi : A(G) \rightarrow B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If $f \in A(G)$ with $\text{supp}^o(f) \subseteq G^{(0)}$, then $\|\pi(f)\| \leq \|f\|_\infty$.

**Proof.** We have $\text{supp}^o(f) \subseteq U$ for some compact open $U \subseteq G^{(0)}$ by Lemma 4.3. Observe that $\pi|_{A(U)}$ is a $*$-homomorphism from $A(U)$ to $B(\mathcal{H})$. Then $\|\pi(f)\| = \|\pi|_{A(U)}(f)\| \leq \|f\|_\infty$ by Proposition 4.2.

**Functions supported on a bisection:** If $f \in A(G)$ with $\text{supp}^o(f) \subseteq B$ for some compact open bisection $B$, then once again we get that $P_f$ has upper bound $\|f\|_\infty$.

**Lemma 4.5.** Let $\pi : A(G) \rightarrow B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If $f \in A(G)$ with $\text{supp}^o(f) \subseteq B$ for some compact open bisection $B$, then $\|\pi(f)\| \leq \|f\|_\infty$.

**Proof.** Fix $f \in A(G)$ such that $\text{supp}^o(f)$ is contained in a compact open bisection. Then $\text{supp}^o(f^*f) \subseteq G^{(0)}$ by Prop 3.12. Now Lemma 4.4 gives the result:

$$\|\pi(f)\|^2 = \|\pi(f^*f)\| \leq \|f^*f\|_\infty = \|f\|^2_\infty.$$ 

An arbitrary element of $A(G)$: For an arbitrary $f \in A(G)$, $\|f\|_\infty$ is no longer an upper bound for $P_f$; however, since $f$ is a sum of things that are each bounded, we can still find a bound for $P_f$.

**Lemma 4.6.** Let $\pi : A(G) \rightarrow B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If $f \in A(G)$ then there exists $M_f \geq 0$ such that $\|\pi(f)\| \leq M_f$.

**Proof.** Write $f = \sum_{D \in F} a_D 1_D$ where each $D \in F$ is a compact open bisection and each $a_D \in \mathbb{C}$. Now we use that $\pi$ is a homomorphism and apply Lemma 4.5 to get

$$\|\pi(f)\| = \|\pi(\sum_{D \in F} a_D 1_D)\| \leq \sum_{D \in F} \|\pi(a_D 1_D)\| \leq \sum_{D \in F} |a_D|.$$

So $M_f = \sum_{D \in F} |a_D|$ suffices.

**Theorem 4.7.** Let $G$ be an ample groupoid with Hausdorff unit space. For $f \in A(G)$, the formula

$$\|f\| = \sup\{\|\pi(f)\| \mid \pi : A(G) \rightarrow B(\mathcal{H}) \text{ is a } *\text{-homomorphism for some Hilbert space } \mathcal{H}\}$$

is a $C^*$-norm on $A(G)$.

**Proof.** That the supremum exists follows from Lemma 4.6. It is straightforward to check that this is a $C^*$-seminorm by noting that the maps $f \mapsto \|\pi(f)\|$ are all $C^*$-norms in their own right, and by noting that properties such as the triangle inequality, submultiplicativity, and the $C^*$-identity are all preserved after taking the supremum. If $f$ is nonzero, then $\|f\|$ is nonzero since the left regular representation is injective.

**Definition 4.8.** Let $G$ be an ample groupoid with Hausdorff unit space. Define $C^*(G)$ to be the completion of $A(G)$ in the norm $\| \cdot \|$ from Theorem 4.7.
5. \textit{\textbf{\,-}}\textit{-\textbf{HOMOMORPHISMS ARE AUTOMATICALLY BOUNDED}}

\textbf{The full C*-algebra:} In order to show our full C*-algebra is the same as the standard one, we establish Proposition \ref{proposition:5.2}. We could have used this proposition to avoid all of the work above to get a bound on \(\|\pi(f)\|\) for any \textit{-}homomorphism \(\pi : A(G) \rightarrow B(\mathcal{H})\). However, the above construction is much less technical and gives a simplified, self-contained construction of \(C^*(G)\) which was one of our goals.

Given a topological space \(U\), recall that \(C_c(U)\) is the set of continuous functions from \(U\) to \(\mathbb{C}\) that are zero outside of some compact set in \(U\). So if \(U\) is a Hausdorff subspace of some bigger space \(X\) and \(f \in C_c(U)\), then \(\text{supp}\,f(U)\), where closure is with respect to the subspace topology on \(U\), is compact. Recall that

\[
\mathcal{C}(G) = \text{span}\{f \in C_c(U) \mid U \subseteq G \text{ is open and Hausdorff}\}.
\]

We view each \(C_c(U)\) as sitting inside the vector space of all functions from \(G\) to \(\mathbb{C}\) by defining them to be zero outside of \(U\) and hence \(A(G) \subseteq \mathcal{C}(G)\).

\textbf{Definition 5.1.} Following \cite{13}:

- We say a net of functions \(\{f_n\}\) converges to \(f\) in the inductive limit topology if \(f_n \to f\) uniformly and \(\text{supp}\,f_n \subseteq K\) eventually for some compact \(K\). Like Muhly and Williams, we are not claiming there is a topology where these are the only convergent sequences.
- We will also say a map with domain \(A(G)\) (or \(\mathcal{C}(G)\)) is continuous with respect to the inductive limit topology if it takes convergent nets in the inductive limit topology to convergent nets in the codomain.

\textbf{Proposition 5.2.} Let \(G\) be an ample groupoid with Hausdorff unit space. Suppose \(\pi : A(G) \rightarrow B(\mathcal{H})\) is a \textit{-}homomorphism for some Hilbert space \(\mathcal{H}\). Then \(\pi\) is continuous with respect to the inductive limit topology on \(A(G)\).

Before presenting the proof, we need to make sure functions in \(A(G)\) can be written in a somewhat efficient way. When \(G\) is Hausdorff, this is easy because functions in \(A(G)\) are locally constant and have finite range.

\textbf{Lemma 5.3.} Let \(B, D\) be compact open bisections with \(B \subseteq D\). If \(f \in A(G)\) with \(\text{supp}\,f \subseteq D\), then \(f|_B \in A(G)\).

\textit{Proof.} Using Lemma \ref{lemma:2.2} write \(f = \sum_{C \in F} a_C 1_C\) where each \(C \in F\) is a compact open bisection in \(D\). Since \(D\) is an open bisection, it is Hausdorff thus for each \(C \in F\), \(B \cap C\) is also a compact open bisection. Now

\[
f|_B = (\sum_{C \in F} a_C 1_C)|_B = \sum_{C \in F} a_C 1_{C \cap B} \in A(G).
\]

\[\Box\]

\textbf{Lemma 5.4.} Let \(K \subseteq G\) be compact and \(f \in A(G)\) with \(\text{supp}\,f \subseteq K\). Suppose \(B_1, \ldots, B_k\) are compact open bisections that cover \(K\). Suppose \(\epsilon > 0\). Then for each \(i, 1 \leq i \leq k\), there exists \(f_i \in A(G)\) with \(\text{supp}\,f_i \subseteq B_i\) such that

1. \(\|f_i\|_{\infty} \leq \|f\|_{\infty} + \epsilon\) and
2. \(f = \sum_{i=1}^{k} f_i\).
Proof. Fix $\epsilon > 0$. Write $f = \sum_{D \in F} f_D$ where $F$ is a finite collection of compact open bisections and for each $D \in F$, $f_D = a_D 1_D$ with nonzero $a_D \in \mathbb{C}$. We can assume each $D \subseteq \bigcup_{i=1}^k B_i$ by Lemma (2.2). Fix $D \in F$. Since $D \subseteq K$, the collection $\{D \cap B_i : 1 \leq i \leq k\}$ is a cover of $D$ by open sets. Since $G$ is ample and hence as a basis of compact open bisections, for each $\gamma \in D$, there exist $1 \leq i_+ \leq k$ and a compact open bisection $D_{\gamma,i_+}$ such that $\gamma \in D_{\gamma,i_+} \subseteq D \cap B_{i_+}$. The collection $\{D_{\gamma,i_+} \}_{\gamma \in D}$ covers $D$ and hence has a finite subcover. Since each $D_{\gamma,i_+}$ is compact open in the the Hausdorff subspace $D$, it is clopen in $D$. So we can disjointify (and relabel) to get a finite disjoint cover of $D$ by compact open bisections $\{D_{\gamma,i_+} \}_{\gamma \in I_D}$ for a finite set $I_D \subseteq D$ where each $D_{\gamma,i_+} \subseteq D \cap B_{i_+}$.

So

$$f_D = \sum_{\gamma \in I_D} (f_D)|_{D_{\gamma,i_+}}$$

and each summand is in $A(G)$ by Lemma 5.3.

For pairs $(\gamma, i)$ with $\gamma \in I_D$ and $i \neq i_+$, define $D_{\gamma,i}$ to be the empty set and write

$$(5.1) \quad f = \sum_{D \in F} \left( \sum_{\gamma \in I_D} (f_D)|_{D_{\gamma,i_+}} \right) = \sum_{D \in F} \left( \sum_{i_+} \sum_{\gamma \in I_D} (f_D)|_{D_{\gamma,i_+}} \right) = \sum_{i_+} \left( \sum_{D \in F} \sum_{\gamma \in I_D} (f_D)|_{D_{\gamma,i_+}} \right).$$

We start out by defining $f_i^0 \in C_c(B_i) \cap A(G)$ to be the function in the $i$th summand in the last expression for $f$ in (5.1). Then (2) holds for the $f_i^0$'s. Since each $f_i^0$ is defined on an open Hausdorff subset, it is locally constant.

Here things get more technical. In what follows, we adjust each of these functions $f_i^0$ so that (2) still holds in order to ensure (1) holds as well. The argument that follows does not depend on the exact definition of $f_i^0$, it only uses that each $f_i^0$ is a locally constant element of $A(G)$ that is zero outside $B_i$ and that (2) holds.

Fix $\epsilon > 0$. If

$$(5.2) \quad \|f_i^0\|_\infty \leq \|f\|_\infty + \epsilon$$

is true for all $i$, then for each $i$, set $f_i = f_i^0$ and we are done. Otherwise, fix $i$ such that (5.2) fails for $i$. Thus $\|f_i^0\|_\infty > \|f\|_\infty + \epsilon$. Consider the set

$$O = \{ \gamma \in B_i : |f_i^0(\gamma)| < \|f\|_\infty + \epsilon \},$$

Then $O$ is open as it is the inverse image of an open set under the composition of two continuous maps. Since $\|f_i^0\|_\infty > \|f\|_\infty + \epsilon$, the set $B_i \setminus O$ is nonempty. Further $B_i \setminus O$ is closed in $B_i$ and hence compact.

Fix $\gamma \in B_i \setminus O$. Then $|f_i^0(\gamma)| > \|f\|_\infty$ so in particular $f(\gamma) \neq f_i^0(\gamma)$ and hence there exists a maximal nonempty

$$S_{\gamma} \subseteq \{1, ..., i - 1, i + 1, ..., k\}$$

such that

$$\gamma \in B_i \cap \left( \bigcap_{j \in S_{\gamma}} B_j \right)$$

and

$$f(\gamma) = f_i^0(\gamma) + \sum_{j \in S_{\gamma}} f_j^0(\gamma).$$
Choose a compact open bisection $C_{\gamma,S} \subseteq B_i \cap \left( \bigcap_{j \in S_{\gamma}} B_j \right)$ such that $\gamma \in C_{\gamma,S}$ and for each $j \in S_{\gamma}$, $f_j^0$ is constant on $C_{\gamma,S}$. Then the collection $\{C_{\gamma,S}\}_{\gamma \in B_i \setminus O}$ covers $B_i \setminus O$ so there exists a finite subcover. Since this is all taking place inside of the Hausdorff subspace $B_i$, we can disjointify to get a finite disjoint subcover $\{C_{\gamma,S}^p\}_{p=1}^* \subseteq B_i \setminus O$ such that for any $\alpha \in C_{\gamma,S}^p$ we have

\begin{equation}
\tag{5.3}
f_i^0(\alpha) + \sum_{j \in S_p} f_j^0(\alpha) = f(\gamma_p).
\end{equation}

Now, we adjust the functions. First, we define $f_i$ as follows:

$$f_i = f_i^0 + \sum_{j=1}^k \sum_{\{p : j \in S_p\}} (f_j^0)|_{C_{\gamma_p,S_p}}.$$

To ensure (2) still holds, for $j \neq i$ define

$$f_j = f_j^0 - \sum_{\{p : j \in S_p\}} (f_j^0)|_{C_{\gamma_p,S_p}}.$$

In each case, we stay inside of $A(G)$ by Lemma 5.3.

Notice for $j \neq i$ we have $\|f_j\|_\infty \leq \|f_j^0\|_\infty$ as we are just forcing it to be 0 on a larger set. We claim that $\|f_i\|_\infty \leq \|f\|_\infty + \epsilon$. To prove the claim, fix $\alpha \in G$. It suffices to show $|f_i(\alpha)| \leq \|f\|_\infty + \epsilon$. If for every $p$, $\alpha \notin C_{\gamma_p,S_p}$, then $\alpha \in O$ and

$$|f_i(\alpha)| = |f_i^0(\alpha)| < \|f\|_\infty + \epsilon.$$  

On the other hand, if there exists $q$ such that $\alpha \in C_{\gamma_q,S_q}$, then $q$ is unique since the collection is disjoint. Now

$$f_i(\alpha) = f_i^0(\alpha) + \sum_{j=1}^k \sum_{\{p : j \in S_p\}} (f_j^0)|_{C_{\gamma_p,S_p}}(\alpha) = f_i^0(\alpha) + \sum_{j \in S_q} (f_j^0)(\alpha) = f(\gamma_p)$$

by (5.3). Thus $|f_i(\alpha)| \leq \|f\|_\infty$ proving the claim. Thus

$$\|f_i\|_\infty < \|f\|_\infty + \epsilon.$$  

Now check if there are any $j$ such that (5.2) fails. If so, pick one such $j$ and repeat the above process to get a new collection of functions continuing until no such $j$ exists and hence (1) holds. \hfill \Box

We present one last lemma before presenting the proof of Proposition 5.2. It is well-known but we provide the details as they don’t seem to appear explicitly in the literature. It clarifies that we are in a situation where we can apply the disintegration theorem [13, Theorem 7.8] in Corollary 5.6.

**Lemma 5.5.** Let $G$ be an ample groupoid with $G^{(0)}$ Hausdorff. Then $A(G)$ is dense in $\mathcal{C}(G)$ with respect to the inductive limit topology in that for every $f$ in $\mathcal{C}(G)$, there exists a net $(f_n) \subseteq A(G)$ such that $f_n \to f$ in the inductive limit topology.
Proof. Fix $f \in \mathcal{C}(G)$. Then $f = \sum_{i=1}^{n} f_i$ where each $f_i \in C_c(B_i)$ for compact open bisections $B_i$ by [7, Proposition 3.10]. For each $i$, $B_i$ is a compact Hausdorff space and hence $f_i$ is the uniform limit of a net of functions $(f_{i,n}) \subseteq A(G) \cap C_c(B_i)$ by the Stone Weierstraß theorem. Thus $\sum_i f_{i,n}$ converges uniformly to $f$ inside the compact set $\bigcup_{i=1}^{n} B_i$ and the result follows. \hfill \Box

Proof of Proposition 5.2. Fix a *-homomorphism $\pi : A(G) \to B(\mathcal{H})$. Suppose we have a net $(f_n)_{n \in (J, \leq)}$ in $A(G)$ such that $f_n \to f \in A(G)$ uniformly and $\text{supp}^0(f_n) \subseteq K$ eventually for some compact $K$. Since $K$ is compact, there exists a finite collection of compact open bisections $B_1, \ldots, B_k$ that cover $K$. Fix $\epsilon > 0$ and apply Lemma 5.4 to write each $f_n - f = \sum_{i=1}^{k} f_{n,i}$ such that each $f_{n,i} \in A(G)$ with $\text{supp}^0(f_{n,i}) \subseteq B_i$ and

$$\|f_{n,i}\|_{\infty} \leq \|f_n - f\|_{\infty} + \frac{\epsilon}{2k}.$$ 

Choose $\alpha$ so that

$$\|f_n - f\|_{\infty} \leq \frac{\epsilon}{2k}$$

for $\alpha \leq n$. Now we compute, applying Lemma 4.5 at the second inequality

$$\|\pi(f_n - f)\| \leq \sum_{i=1}^{k} \|\pi(f_{n,i})\| \leq \sum_{i=1}^{k} \|f_{n,i}\|_{\infty} \leq k(\|f_n - f\|_{\infty} + \frac{\epsilon}{2k}) \leq \epsilon$$

when $\alpha \leq n$. \hfill \Box

Corollary 5.6. Let $G$ be a second countable ample groupoid with Hausdorff unit space. Suppose $\pi : A(G) \to B(\mathcal{H})$ is a *-homomorphism for some Hilbert space $\mathcal{H}$. Then $\pi$ is bounded with respect to the $I$-norm on $A(G)$.

Proof. From Proposition 5.2 we have that $\pi$ is continuous with respect to the inductive limit topology. Since $A(G)$ is dense in $\mathcal{C}(G)$ with respect to the inductive limit topology by Lemma 5.5 we can extend $\pi$ to a *-homomorphism $\tilde{\pi} : \mathcal{C}(G) \to B(\mathcal{H})$ that is also continuous with respect to the inductive limit topology. Since $G$ is second countable, Renaut’s disintegration theorem [13] Theorem 7.8] gives that $\tilde{\pi}$ is bounded with respect to $\|\cdot\|_I$ and hence so is $\pi$. \hfill \Box

6. Étale groupoids

In this section, we will move away from ample groupoids and consider second countable, locally compact étale groupoids with Hausdorff unit spaces. By [7] Proposition 3.10, $\mathcal{C}(G)$ is the linear span of functions that are each in some $C_c(U)$ for an open bisection $U$ that is contained in a compact set. We show that every *-homomorphism from $\mathcal{C}(G)$ to $B(\mathcal{H})$ is bounded with respect to the $I$-norm. Previously this was only known for Hausdorff étale groupoids, see for example [19] Lemma 2.3.3]. Note that we add the assumption that $G$ is second countable so that we have that $G$ is locally normal. This also means we can work with sequences in $\mathcal{C}(G)$ instead of nets. The main result of this section is the following.

Theorem 6.1. Let $G$ be a second countable, locally compact étale groupoid with $G^{(0)}$ Hausdorff. Suppose $\pi : \mathcal{C}(G) \to B(\mathcal{H})$ is a *-homomorphism for some Hilbert space $\mathcal{H}$. Then $\pi$ is continuous with respect to the inductive limit topology on $\mathcal{C}(G)$. 


The proof of Theorem 6.1 will follow after we establish some lemmas that parallel the ideas of Lemma 2.2 and Lemma 6.3.

**Lemma 6.2.** Let $G$ be a second-countable, locally compact étale groupoid with $G^{(0)}$ Hausdorff. Let $f \in \mathcal{C}(G)$ such that $f = \sum_{V \in H} f_{V}$ where $H$ is a finite collection of open bisections and each $f_{V} \in C_c(V)$. Suppose $\text{supp}^o(f) \subseteq O$ for some open set $O \subseteq G$. Then there exists a finite collection of open bisections $F$ and functions $f_{V} \in C_c(U)$ for each $U \in F$ such that $\text{supp}^o(f_{V}) \subseteq O$ and $f = \sum_{U \in F} f_{U}$.

**Proof.** We proceed by induction on the size of $H$. For the base case, suppose $H$ has a single element $V$. Then $f \in C_c(U)$ for some open bisection $U$ and by assumption $\text{supp}^o(f) \subseteq O$ so $F = H$ suffices.

For the inductive step, suppose the lemma is true for functions that can be written as a sum of $n$ functions, each in some $C_c(V)$ for some $V$. Fix $f \in \mathcal{C}(G)$ such that $\text{supp}^o(f) \subseteq O$ and $f = \sum_{V \in H} f_{H}$ with $|H| = n + 1$.

Fix $V \in H$. Let

$$U_V = \text{supp}^o(f_V) \cap O$$

and for each $W \in H$, $W \neq V$, let

$$U_W = \text{supp}^o(f_V) \cap \text{supp}^o(f_W).$$

Since $\text{supp}^o(f) \subseteq O$, if $\gamma \in \text{supp}^o(f_V) \setminus O$ then $\gamma \in U_W$ for some $W \in H$, $W \neq V$. So the collection $\{U_W\}_{W \in H}$ is an open cover of $\text{supp}^o(f_V)$.

Since the open subset $\text{supp}^o(f_V)$ is a normal topological space with respect to the subspace topology, we can find a partition of unity $\{\rho_W\}_{W \in H}$ in $C(\text{supp}^o(f_V))$ (the continuous functions on $\text{supp}^o(f_V)$) subordinate to $\{U_W\}_{W \in H}$ such that for each $\gamma \in \text{supp}^o(f_V)$ we have $\sum_{W \in H} \rho_W(\gamma) = 1$ (see, for example, [14, Theorem 5.1]). So $f_V = \sum_{W \in H} \rho_W f_V$ (with pointwise multiplication). We claim that

1. For all $W \in H$, $\rho_W f_V \in C_c(V)$,
2. for $W \neq V$, $\rho_W f_V \in C_c(W)$.

For item (1), fix $W \in H$. Then $\rho_W f_V$ is continuous on $\text{supp}^o(f_V)$. We show it is continuous on all of $V$. Let $(\alpha_n)$ be a sequence in $V$ such that $\alpha_n \to \alpha \in V$. If $\alpha \in \text{supp}^o(f_V)$, then $\alpha_n \in \text{supp}^o(f_V)$ eventually and $\rho_W f_V(\alpha_n) \to \rho_W f_V(\alpha)$. Suppose $\alpha \notin \text{supp}^o(f_V)$. So $\rho_W f_V(\alpha) = 0$ and

$$|\rho_W f_V(\alpha_n)| = |\rho_W(\alpha_n)||f_V(\alpha_n)| \leq |f_V(\alpha_n)| \to 0.$$

Thus $\rho_W f_V$ is continuous on $V$. Since $\text{supp}^o(\rho_W f_V)^c \subseteq \text{supp}^o(f_V)^c$ is compact, item (1) follows.

For item (2), fix $W \in H$ with $W \neq V$. Then $\rho_W f_V$ is continuous on $U_W$. Let $(\alpha_n)$ be a sequence in $W$ such that $\alpha_n \to \alpha \in W$. We consider three cases. For the first case, suppose $\alpha \in \text{supp}^o(f_V)$. Then $\alpha_n \in \text{supp}^o(f_V)$ eventually and $\rho_W f_V(\alpha_n) \to \rho_W f_V(\alpha)$. For the second case, suppose $\alpha \in V \setminus \text{supp}^o(f_V)$. So $\rho_W f_V(\alpha) = 0$ and

$$|\rho_W f_V(\alpha_n)| = |\rho_W(\alpha_n)||f_V(\alpha_n)| \leq |f_V(\alpha_n)| \to 0.$$

For the third case, suppose $\alpha \in W \setminus V$. If $\alpha \notin \text{supp}^o(f_V)$ eventually, then

$$\rho_W f_V(\alpha_n) = 0 = \rho_W f_V(\alpha)$$

eventually. So we may assume there is a subsequence such that every $\alpha_{n_j} \in \text{supp}^o(f_V)$. Since the closure of the support of $f_V$ in $V$ is a compact normal subspace of $V$, it is
complete and hence there exists \( \alpha_{n_j} \in V \) such that \( \alpha_{n_j} \to \alpha \). Since \( W \) is Hausdorff and \( \alpha \in W \), \( \alpha \notin W \) and hence \( \rho_W(\alpha_V) = 0 \). Now applying item (1) gives
\[
\rho_W f_V(\alpha_{n_j}) \to \rho_W f_V(\alpha_V) = \rho_W(\alpha_V) f_V(\alpha_V) = 0 = \rho_W f_V(\alpha).
\]
So \( \rho_W f_V \) is continuous on \( W \). We have \( \text{supp}^a(\rho_W f_V)^W \subseteq \text{supp}^a(f_V)^W \) which is compact proving item (2).

Now
\[
f = \rho_V f_V + \sum_{W \in H, W \neq V} f_W + \rho_W f_V
\]
with \( \text{supp}^a(\rho_V f_V) \subseteq O \). To prove the lemma, it suffices to check that
\[
\text{supp}^a(\sum_{W \in H, W \neq V} f_W + \rho_W f_V) \subseteq O
\]
for then the inductive hypothesis applies. By way of contradiction, suppose there exists \( \gamma \in G \setminus O \) such that \( \sum_{W \in H, W \neq V} f_W + \rho_W f_V(\gamma) \neq 0 \). Then
\[
f(\gamma) = \rho_V f_V(\gamma) + \sum_{W \in H, W \neq V} f_W + \rho_W f_W(\gamma) = \sum_{W \in H, W \neq V} f_W + \rho_W f_W(\gamma) \neq 0
\]
which is a contradiction. \( \square \)

**Lemma 6.3.** Let \( G \) be a second countable, locally compact étale groupoid with \( G^{(0)} \) Hausdorff. Suppose \( f \in \mathcal{C}(G) \) such that \( \text{supp}^a(f) \subseteq K \subseteq B \) for some compact set \( K \) and open bisection \( B \). Then \( f \in C_c(B) \).

**Proof.** Using Lemma 6.2 write \( f = \sum_{U \in F} f_U \) such that for each \( U \in F \), \( f_U \in C_c(U) \) and \( \text{supp}^a(f_U) \subseteq B \). We show for each \( U \in F \) that \( f_U \in C_c(B) \). Fix \( U \in F \) and let \( (\alpha_n) \subseteq B \) be a sequence such that \( \alpha_n \to \alpha \in B \). We show \( f_U(\alpha_n) \to f_U(\alpha) \). Observe that
\[
B = (\text{supp}^a(f_U) \cap B) \cup (B \setminus \text{supp}^a(f_U))
\]
so we consider two cases. First suppose \( \alpha \) is in the open set \( \text{supp}^a(f_U) \cap B \). Then \( \alpha_n \) is in \( \text{supp}^a(f_U) \cap B \) eventually and \( f_U(\alpha_n) \to f_U(\alpha) \) by the continuity of \( f_U \) in \( U \).

For the second case, suppose \( \alpha \in B \setminus \text{supp}^a(f_U) \). Then \( f_U(\alpha) = 0 \). We show \( f_U(\alpha_n) \to 0 \). If \( \alpha_n \in B \setminus \text{supp}^a(f_U) \) eventually, then \( f_U(\alpha_n) = 0 \) eventually and we are done. So suppose there exists a maximal subsequence such that every \( \alpha_{n_j} \in \text{supp}^a(f_U) \). By maximal we mean that the \( \alpha_n \) that do not appear in the subsequence are outside \( \text{supp}^a(f_U) \). Since the closure of the support of \( f_U \) in \( U \) is a complete subspace of \( U \), there exists \( \alpha_U \in U \) such that \( \alpha_{n_j} \to \alpha_U \) and hence \( f_U(\alpha_{n_j}) \to f_U(\alpha_U) \). If \( \alpha_U = \alpha \), we are done. Otherwise, since \( B \) is Hausdorff and \( \alpha \in B \), \( \alpha \notin B \). So \( f_U(\alpha_U) = 0 \) and hence \( f_U(\alpha_{n_j}) \to 0 \), which implies \( f_U(\alpha_n) \to 0 \) by maximality.

Thus \( f \in C(B) \). Notice \( \text{supp}^a(f)^B \subseteq K \) and hence \( f \in C_c(B) \). \( \square \)

**Lemma 6.4.** Let \( G \) be a second countable, locally compact étale groupoid with \( G^{(0)} \) Hausdorff. Let \( \tilde{B}_1, \ldots, \tilde{B}_k \) and \( B_1, \ldots, B_k \) be open bisections and \( K, K_1, \ldots, K_k \) are compact sets such that for each \( 1 \leq i \leq k \) we have
\[
\tilde{B}_i \subseteq K_i \subseteq B_i.
\]
Suppose \( f \in \mathcal{C}(G) \) such that \( \text{supp}^a(f) \subseteq K \subseteq \bigcup_{i=1}^k \tilde{B}_i \). Then for each \( 1 \leq i \leq k \), there exists \( f_i \in C_c(B_i) \) such that \( f = \sum_{i=1}^k f_i \).
Proof. We do a proof by induction on $k$ using the strategy inspired by the proof of [21, Proposition 4.1]. The base case $k = 1$ follows from Lemma 6.3. For the inductive step, suppose for every $h \in \mathcal{C}(G)$ with supp$(h)$ inside some $K \subseteq \bigcup_{i=1}^{k} \tilde{B}_i$ with $k \geq 1$ satisfying the hypotheses of the lemma, there exists $h_i \in C_{c}(B_i)$ for each $1 \leq i \leq k$ such that $h = \sum_{i=1}^{k} h_i$. Fix $f \in \mathcal{C}(G)$ such that $f = \sum_{U \in H} f_U$ and supp$(f) \subseteq K \subseteq \bigcup_{i=1}^{k+1} \tilde{B}_i$ as in the hypotheses of the lemma.

Set

$$F = \tilde{B}_{k+1} \setminus \bigcup_{i=1}^{k} \tilde{B}_i.$$ 

Then $F$ is Hausdorff and closed in $\tilde{B}_{k+1}$. We claim that $f|_F$ is continuous on $F$. To see this, it suffices to show each $f_U|_F$ is continuous on $F$. Fix $U \in H$. We can assume that supp$(f_U) \subseteq \bigcup_{i=1}^{k+1} \tilde{B}_i$ by Lemma 6.2. Let $(\alpha_n) \subseteq F$ such that $\alpha_n \to \alpha \in F$. If $\alpha \in$ supp$(f_U)$ which is open, then $\alpha_n \in$ supp$(f_U)$ eventually and we get $f_U|_F(\alpha_n) \to f_U|_F(\alpha)$ by continuity in $U$. So assume $\alpha \notin$ supp$(f_U)$. We consider two cases: First suppose $\alpha_n \in F \setminus$ supp$(f_U)$ eventually. Then $f_U|_F(\alpha_n) = 0 = f_U|_F(\alpha)$ eventually. For the second cases, suppose there exists a maximal subsequence such that for every $n_j$, $\alpha_{n_j} \in$ supp$(f_U) \cap F$. Then if $\alpha_{n_j}$ is not included in the subsequence, $f_U|_F(\alpha_{n_j}) = 0$. Since the closure of supp$(f_U)$ is a complete subspace of $U$, there exists $\alpha_U \in U$ such that $\alpha_{n_j} \to \alpha_U$ and $f_U(\alpha_{n_j}) \to f_U(\alpha_U)$. If $\alpha_U \in F$, then $\alpha_U = \alpha$ because $F$ is Hausdorff. Finally, suppose $\alpha_U \notin F$. We claim that $\alpha_U \notin$ supp$(f_U)$ which suffice for then $f_U|_F(\alpha_n) \to f_U|_F(\alpha)$. By way of contradiction, suppose $\alpha_U \in$ supp$(f_U)$. So there exists $\tilde{B}_i$ with $i \neq k + 1$ such that $\alpha_U \in \tilde{B}_i$. Since $\tilde{B}_i$ is open, there exists $\alpha_{n_j}$ such that $\alpha_{n_j} \in \tilde{B}_i$. But by assumption every $\alpha_{n_j} \in F$ and $F \cap \tilde{B}_i = \emptyset$, which is a contradiction. Thus we have shown $f_U|_F$ is continuous on $F$ and hence so is $f|_F$.

Notice $f|_F$ is 0 outside

$$K' = K \setminus \bigcup_{i=1}^{k} \tilde{B}_i$$

which is closed in $K$ and hence compact. Since $K' \subseteq F$, we have $f|_F \in C_c(F)$. Now we apply Tietze’s extension theorem to get a function $f_{k+1} \in C_c(\tilde{B}_{k+1}) \subseteq C_c(B_{k+1})$ such that $f_{k+1}|_F = f|_F$. Now $f = (f - f_{k+1}) + f_{k+1}$. Notice that $f - f_{k+1} \in \mathcal{C}(G)$ and is 0 on $F$. In fact

$$\text{supp}^\alpha(f - f_{k+1}) \subseteq \bigcup_{i=1}^{k} \tilde{B}_i.$$ 

Now apply the inductive hypothesis to $f - f_{k+1}$ to get the remaining $f_i$’s. 

Lemma 6.5. Let $G$ be a second countable, locally compact étale groupoid with $G(0)$ Hausdorff. Let $B_1, \ldots, B_k$ be open bisections. Suppose $f \in \mathcal{C}(G)$ such that $f = \sum_{i=1}^{k} f_i^0$ where each $f_i^0 \in C_c(B_i)$. Let $\epsilon > 0$. Then for each $i$ there exists $f_i \in C_c(B_i)$ such that

1. $\|f_i\|_\infty \leq 2\|f\|_\infty + \epsilon$ and
2. $f = \sum_{i=1}^{k} f_i$.

Proof. In what follows, we adjust these functions so that (2) still holds in order to ensure (1) holds as well. Fix $\epsilon > 0$ and fix $1 \leq i \leq k$ such that $\|f_i^0\|_\infty > 2\|f\|_\infty + \epsilon$. If no such $i$
exists, we are finished. Set

$$O = \{ \gamma \in B_i : |f_i^0(\gamma)| < \|f\|_\infty + \frac{\epsilon}{2} \}.$$ 

Then $O$ is open. Since $\|f_i^0\|_\infty > \frac{\|f^0\|_\infty}{2} > \|f\|_\infty + \frac{\epsilon}{2}$, the set $B_i \setminus O$ is nonempty. Further $B_i \setminus O$ is closed in $B_i$ and contained in the compact support of $f_i^0$ (also inside $B_i$) and hence $B_i \setminus O$ is compact.

Fix $\gamma \in B_i \setminus O$. Then $|f_i^0(\gamma)| > \|f\|_\infty$ so in particular $f(\gamma) \neq f_i^0(\gamma)$ and hence there exists a maximal nonempty

$$S_\gamma \subseteq \{1, ..., i - 1, i + 1, ..., k\}$$

such that

$$\gamma \in B_i \cap \left( \bigcap_{j \in S_\gamma} B_j \right) \text{ and } f(\gamma) = f_i^0(\gamma) + \sum_{j \in S_\gamma} f_j^0(\gamma).$$

Choose an open bisection $C_{\gamma,S_\gamma} \subseteq B_i \cap \left( \bigcap_{j \in S_\gamma} B_j \right)$ such that

- $\gamma \in C_{\gamma,S_\gamma}$,
- the restriction of $(f_i^0 + \sum_{j \in S_\gamma} f_j^0)$ to $C_{\gamma,S_\gamma}$ is continuous and
- for each $\alpha \in C_{\gamma,S_\gamma}$ we have

$$(6.1) \quad |(f_i^0(\alpha) + \sum_{j \in S_\gamma} f_j^0(\alpha)) - f(\gamma)| \leq \frac{\epsilon}{2}.$$ 

Then the collection $\{C_{\gamma,S_\gamma} \}_{\gamma \in B_i \setminus O}$ covers $B_i \setminus O$ so there exists a finite subcover $\{C_{\gamma_p,S_p}\}_{p=1}^s$.

Now, we adjust the functions. For notational convenience, let $C_{\gamma_0,S_0} = O$. Choose a partition of unity $\{\rho_p\}_{p=0}^s$ in $C(B_i)$ subordinate to the cover $\{C_{\gamma_p,S_p}\}_{p=0}^s$. Define

$$f_i = f_i^0 + \sum_{j=1}^k \sum_{\{p : j \in S_p\}} \rho_p f_j^0 \text{ (with pointwise multiplication)}.$$ 

To ensure (2) still holds, for $j \neq i$ define

$$f_j = f_j^0 - \sum_{\{p : j \in S_p\}} \rho_p f_j^0 \text{ (with pointwise multiplication)}.$$ 

Since the compactly supported functions are an ideal in the ring of continuous functions, we have $f_i \in C_c(B_i)$ and for each $j$, $f_j \in C_c(B_j)$.
Notice for \( j \neq i \) we have \( \|f_j\|_\infty \leq \|f_i\|_\infty \). We claim that \( \|f_i\|_\infty \leq 2\|f\|_\infty + \epsilon \). To prove the claim, fix \( \alpha \in G \). It suffices to show \( |f_i(\alpha)| \leq 2\|f\|_\infty + \epsilon \). We compute

\[
|f_i(\alpha)| = |f_i^0(\alpha) + \sum_{j=1}^{k} \sum_{\{p,j\} \in S_p} \rho_p(\alpha)f_j^0(\alpha)|
\]

\[
= |\sum_{p=0}^{s} \rho_p(\alpha)f_i^0(\alpha) + \sum_{p=1}^{s} \sum_{j \in S_p} \rho_p(\alpha)f_j^0(\alpha)|
\]

\[
= |\rho_0(\alpha)f_i^0(\alpha) + \sum_{p=1}^{s} \rho_p(\alpha)(f_i^0(\alpha) + \sum_{j \in S_p} f_j^0(\alpha))|
\]

\[
\leq (\|f\|_\infty + \frac{\epsilon}{2}) + \sum_{p=1}^{s} \rho_p(\alpha)|f_i^0(\alpha) + \sum_{j \in S_p} f_j^0(\alpha)|
\]

\[
\leq (\|f\|_\infty + \frac{\epsilon}{2}) + \sum_{p=1}^{s} \rho_p(\alpha)(\|f\|_\infty + \frac{\epsilon}{2}) \text{ by (6.1)}
\]

\[
\leq 2\|f\|_\infty + \epsilon.
\]

Now check if there are any \( j \) such that \( \|f_j\|_\infty > 2\|f\|_\infty + \epsilon \). If so, pick one such \( j \) and repeat the above process to get a new collection of functions continuing until no such \( j \) exists and hence (1) holds.

\( \square \)

**Proof of Theorem 6.1.** Fix a \(::C(B)(G) \to B(H)\). Suppose \( f_n \to f \) uniformly and \( \text{supp}^0(f_n) \subseteq K \) eventually for some compact \( K \). We claim that there are open bisections \( \tilde{B}_1, ..., \tilde{B}_k \) and \( B_1, ..., B_k \) and compact sets \( K_1, ..., K_k \) such that for each \( i, 1 \leq i \leq k \) we have

\[
\tilde{B}_i \subseteq K_i \subseteq B_i \quad \text{and} \quad K \subseteq \bigcup_{i=1}^{k} \tilde{B}_i.
\]

To see this, for each \( \gamma \in K \), let \( B_\gamma \) be an open bisection containing \( \gamma \). By local compactness, \( \gamma \) has a neighborhood base of compact sets, so we can find \( K_\gamma \) and \( \tilde{B}_\gamma \) such that

\[
\gamma \in \tilde{B}_\gamma \subseteq K_\gamma \subseteq B_\gamma.
\]

Then the \( \tilde{B}_\gamma \)'s cover \( K \) and we can find a finite subcover as needed.

Fix \( \epsilon > 0 \). Since \( \text{supp}^0(f_n - f) \subseteq K \) eventually, we can apply Lemma 6.4 and then Lemma 6.5 to eventually write each \( f_n - f = \sum_{i=1}^{k} f_{n,i} \) such that each \( f_{n,i} \in C_c(B_i) \) with \( \text{supp}^0(f_{n,i}) \subseteq B_i \) and

\[
\|f_{n,i}\|_\infty \leq 2\|f_n - f\|_\infty + \frac{\epsilon}{2k^2}.
\]

Choose \( N \) so that if \( n \geq N \), then

\[
\|f_n - f\|_\infty \leq \frac{\epsilon}{4k}.
\]
Now we compute for $n \geq N$, applying [17 Proposition 3.14] at the second inequality,

$$\|\pi(f_n - f)\| \leq \sum_{i=1}^k \|\pi(f_{n,i})\| \leq \sum_{i=1}^k \|f_{n,i}\| \leq k(\|f_n - f\|_\infty + \frac{\epsilon}{2k}) \leq \epsilon.$$ 

\[ \square \]

The following corollary follows immediately from Theorem 6.1 and the disintegration theorem [13 Theorem 7.8].

**Corollary 6.6.** Let $G$ be a second countable, locally compact étale groupoid with $G^{(0)}$ Hausdorff. Suppose $\pi : \mathcal{C}(G) \to B(\mathcal{H})$ is a *-homomorphism for some Hilbert space $\mathcal{H}$. Then $\pi$ is bounded with respect to the $I$-norm on $\mathcal{C}(G)$.

### 7. Reconciling our construction and those done previously

**The reduced $C^*$-algebra** The restriction of the left regular representation $\pi$ of $\mathcal{C}(G)$ to $A(G)$ is precisely the left regular representation of $A(G)$. So the reduced $C^*$-algebra we have constructed $\overline{\pi(A(G))}$ is contained in the usual one $\overline{\mathcal{C}(G)}$. It suffices to show $\pi(A(G))$ is dense in $\overline{\mathcal{C}(G)}$. Fix $\pi(f) \in \overline{\mathcal{C}(G)}$. Lemma 5.5 says $A(G)$ is dense in $\mathcal{C}(G)$ in the inductive limit topology so there exists a sequence $(f_n) \subseteq A(G)$ such that $f_n \to f$ in the inductive limit topology. From Proposition 5.2, $\pi$ is continuous with respect to the inductive limit topology so $\pi(f_n) \to \pi(f)$ as needed.

**The full $C^*$-algebra**

**Theorem 7.1.** Let $G$ be a locally compact étale groupoid with Hausdorff unit space. Consider the following 4 norms where the first is on $A(G)$ and the other three are on $\mathcal{C}(G)$:

\[
\begin{align*}
\|f\|_1 &= \sup\{\|\pi(f)\| : \pi : A(G) \to B(\mathcal{H}) \text{ is a } \ast\text{-hm for some } \mathcal{H}\} \quad \text{(Theorem 4.7)} \\
\|f\|_2 &= \sup\{\|\pi(f)\| : \pi : \mathcal{C}(G) \to B(\mathcal{H}) \text{ is a } \ast\text{-hm for some } \mathcal{H}\} \quad \text{(for example, see [17 Def. 3.17])} \\
\|f\|_3 &= \sup\{\|\pi(f)\| : \pi : \mathcal{C}(G) \to B(\mathcal{H}) \text{ is a } \ast\text{-hm that is continuous with respect to the inductive limit topology for some } \mathcal{H}\} \quad \text{(for example, see [18 Def. II.1.3])} \\
\|f\|_4 &= \sup\{\|\pi(f)\| : \pi : \mathcal{C}(G) \to B(\mathcal{H}) \text{ is an } I\text{-norm bounded } \ast\text{-hm for some } \mathcal{H}\} \\
&\quad \text{(for example, see [16 Page 101])}
\end{align*}
\]

1. If $G$ is ample, then our $C^*$-algebra $\overline{A(G)}_{\|\cdot\|_i}$ (see Def. 4.8) is the same as $\overline{\mathcal{C}(G)}_{\|\cdot\|}$, for $i = 2, 3$.
2. If $G$ is a second countable, then $\overline{\mathcal{C}(G)}_{\|\cdot\|}$, for $i = 2, 3, 4$ are all the same.

**Remark 7.2.** Typically $\|\cdot\|_2$ only appears in the Hausdorff setting where the existence of the “inductive limit topology” on $C_c(G)$ is guaranteed, see for example, [17 Proposition D.1]. When $G$ is not Hausdorff, this definition of $\|\cdot\|_2$ still makes sense but is less helpful.

**Proof.** For item 1, suppose $G$ is ample. Note that every *-homomorphism $\pi : \mathcal{C}(G) \to B(\mathcal{H})$ restricts to a *-homomorphism on $A(G)$. By Proposition 5.2 and Lemma 5.5 any *-homomorphism on $A(G)$ extends to a *-homomorphism on $\mathcal{C}(G)$ that is continuous with
respect to the inductive limit topology. Putting this all together gives \( \|f\|_1 = \|f\|_2 = \|f\|_3 \) for \( f \in A(G) \). Thus we have
\[
A(G)^{\|\cdot\|_1} = A(G)^{\|\cdot\|_2} = A(G)^{\|\cdot\|_3} \subseteq C(G)^{\|\cdot\|_i}
\]
for \( i = 2, 3 \). Lemma 5.5 also implies \( C(G) \subseteq A(G)^{\|\cdot\|_3} \) and item 1 follows.

For item 2, when \( G \) is second countable, Theorem 6.1 and Corollary 6.6 imply all the norms on \( C(G) \) are the same. \( \square \)

References

[1] J. Brown, L.O. Clark, C. Farthing, and A. Sims. Simplicity of algebras associated to étale groupoids. *Semigroup Forum*, 88(2):433–452, 2014.

[2] L.O. Clark, A. an Huef, and I. Raeburn. The equivalence relations of local homeomorphisms and Fell algebras. *New York J. Math.*, 19:367–394, 2013.

[3] L.O. Clark, R. Exel, E. Pardo, A. Sims, and C. Starling. Simplicity of algebras associated to non-Hausdorff groupoids. *Trans. Amer. Math. Soc.*, 372(5):3669–3712, 2019.

[4] L.O. Clark, J. Fletcher, and A. an Huef. All classifiable Kirchberg algebras are \( C^* \)-algebras of ample groupoids. *Expo. Math.*, 38(4):559–565, 2020.

[5] A. Connes. Feuilletages et algèbres d’opérateurs. In *Bourbaki Seminar, Vol. 1979/80*, volume 842 of *Lecture Notes in Math.*, pages 139–155. Springer, Berlin-New York, 1981.

[6] A. Connes. A survey of foliations and operator algebras. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 521–628. Amer. Math. Soc., Providence, R.I., 1982.

[7] R. Exel. Inverse semigroups and combinatorial \( C^* \)-algebras. *Bull. Braz. Math. Soc. (N.S.)*, 39:191–313, 2008.

[8] R. Exel. Reconstructing a totally disconnected groupoid from its ample semigroup. *Proc. Amer. Math. Soc.*, 138(8):2991–3001, 2010.

[9] R. Exel and E. Pardo. The tight groupoid of an inverse semigroup. *Semigroup Forum*, 92:274–303, 2016.

[10] R. Exel and E. Pardo. Self-similar graphs, a unified treatment of Katsura and Nekrashevych \( C^* \)-algebras. *Adv. Math.*, 306:1046–1129, 2017.

[11] A. Kumjian and D. Pask. Higher rank graph \( C^* \)-algebras. *New York J. Math.*, 6:1–20, 2000.

[12] A. Kumjian, D. Pask, I. Raeburn, and J. Renault. Graphs, groupoids, and Cuntz-Krieger algebras. *J. Funct. Anal.*, 144(2):505–541, 1997.

[13] P.S. Muhly and D.P. Williams. Renault’s equivalence theorem for groupoid crossed products, volume 3 of *New York Journal of Mathematics. NYJM Monographs*. State University of New York, University at Albany, Albany, NY, 2008.

[14] J.R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [ MR0464128].

[15] G.J. Murphy. *\( C^* \)-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[16] A.L.T. Paterson. *Groupoids, Inverse Semigroups, and their Operator Algebras*, volume 170 of *Progress in Mathematics*. Birkhäuser, Boston, MA, 1999.

[17] I. Raeburn and D. P. Williams. *Morita equivalence and continuous-trace \( C^* \)-algebras*, volume 60 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[18] J. Renault. *A groupoid approach to \( C^* \)-algebras*. Lecture notes in mathematics. Springer, Berlin, 1980.

[19] A. Sims. Hausdorff étale groupoids and their \( C^* \)-algebras. In *Operator algebras and dynamics: groupoids, crossed products and Rokhlin dimension*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser, 2020.

[20] B. Steinberg. A groupoid approach to discrete inverse semigroup algebras. *Adv. Math.*, 223:689–727, 2010.

[21] J.-L. Tu. Non-Hausdorff groupoids, proper actions and \( K \)-theory. *Doc. Math.*, 9:565–597, 2004.
