Complete open Kähler manifolds with non-negative bisectional curvature and non-maximal volume growth

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Abstract. It is still an open problem that a complete open Kähler manifold with positive bisectional curvature is Stein. This paper partially resolve the problem by putting a restriction to volume growth condition. The partial solution here improves the observation in ([8], page 341). The improvement is based on assuming a weaker volume growth condition that is is not sufficiently maximal.

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1. Introduction

One of the most useful tool in studying structures of nonnegatively curved complete open manifolds is the Busemann function. The spherical Busemann function is defined as

\[ b_p(x) = \lim_{r \to \infty} \{r - d(x, \partial(B(p, r)))\}, \]

In the process of proving soul theorem in [3], Cheeger and Gromoll proved that a complete open Riemaniann manifold with nonnegative sectional curvature admits a convex and exhaustion Busemann function, \( b_p \).

It still remains unknown whether complete open manifolds with non-negative Ricci curvature admit exhaustion Busemann function, except with the restriction to maximum volume growth as was proved by Shen in [7]. In this paper we drop the maximum volume growth condition and adapt a weaker condition.
Let $S_p M \subset T_p M$ be a unit tangent sphere in the tangent space $T_p M$ for a point $p \in M$. For any subset $N \subset S_p M$, define

$$C(N) = \{q \in M \mid \text{there is a minimizing geodesic } \gamma \text{ from } p \text{ to } q \text{ such that } \gamma'(0) \in N\}$$

to be the cone over $N$. The restriction of a geodesic ball of radius $r$ centered at $p$ to $C(N)$ is denoted by

$$B_N(p, r) = B(p, r) \cap C(N)$$

Let $\Sigma = \{v \subset S_p M \mid \exp_p(rv) : [0, \infty) \to M \text{ is a ray}\}$. A cone of rays is defined by $C(\Sigma)$. Consequently,

$$B_\Sigma(p, r) = B(p, r) \cap C(\Sigma).$$

From lemma 4 in [5] we have

**Lemma 1.1 ([5] Ordway-Stephens-Yang]).** Let $M^n$ be a complete open manifold with $\text{Ric}_M \geq 0$. Suppose that $M$ has a maximum volume growth i.e

$$\lim_{r \to \infty} r^{-n} \text{Vol}(B(p, r)) = \alpha_M, \ \alpha_M > 0$$

then

$$\lim_{r \to \infty} r^{-n} \text{Vol}(B_\Sigma(p, r)) = \alpha_M$$

By limit properties, we obtain the following corollary

**Corollary 1.2.** Let $M^n$ be a complete open manifold with $\text{Ric}_M \geq 0$. Suppose that $M$ has a maximum volume growth. Then

$$\text{Vol}(B_\Sigma(p, r)) \sim \text{Vol}(B(p, r)),$$

$\sim$ means asymptotic

It is essential to note that nonnegative Ricci curvature ensures that the volume growth condition in corollary 1.2 above is independent of the base point: let $p, q \in M^n$ and $d = d(p, q)$. Then it is clear that $B(p, r) \subset B(q, r+d)$ and $B(q, r) \subset B(p, r+d)$. By Bishop-Gromov volume comparison theorem,

$$\lim_{r \to \infty} \inf \frac{\text{Vol}(B_\Sigma(p, r))}{\text{Vol}(B(p_1, r))} \geq \lim_{r \to \infty} \inf \left\{ \left[ \frac{r}{r+d} \right]^n \frac{\text{Vol}(B_\Sigma(p, r+d))}{\text{Vol}(B(p_1, r))} \right\}$$

$$\geq \lim_{r \to \infty} \inf \left\{ \left[ \frac{r}{r+d} \right]^n \frac{\text{Vol}(B_\Sigma(q, r))}{\text{Vol}(B(p_1, r))} \right\}$$

$$\geq \lim_{r \to \infty} \left[ \frac{r}{r+d} \right]^n \lim_{r \to \infty} \inf \frac{\text{Vol}(B_\Sigma(q, r))}{\text{Vol}(B(p_1, r))}$$

$$= \lim_{r \to \infty} \inf \frac{\text{Vol}(B_\Sigma(q, r))}{\text{Vol}(B(p_1, r))}$$
Likewise
\[
\lim_{r \to \infty} \inf \frac{\text{Vol}(B_\Sigma(p, r))}{\text{Vol}(B(p_1, r))} \leq \lim_{r \to \infty} \inf \frac{\text{Vol}(B_\Sigma(q, r))}{\text{Vol}(B(p_1, r))}
\]

**Lemma 1.3.** Let \( M^n \) be a complete open manifold with \( \text{Ric}_M \geq 0 \). For a fixed \( p_1 \in M \), the volume growth
\[
\lim_{r \to \infty} \inf \left[ \frac{\text{Vol}(B_\Sigma(p, r))}{\text{Vol}(B(p_1, r))} \right]^{-1} = \alpha(n)
\]
is independent of the base point \( p \in M \).

The converse to corollary 1.2 above is not true. In other words, the volume growth condition
\[
\text{Vol}(B_\Sigma(p, r)) \sim \text{Vol}(B(p, r)), \quad (1.1)
\]
does not necessarily imply maximum volume growth.

For example, the vertex 0 of a paraboloid \( M \subset \mathbb{R}^{n+1} \) has an empty cut locus. Thus volume growth condition (1.1) holds at 0 and extends to other points by lemma 1.3 above. On the other hand, as a special case of lemma 4.1 in [6], the paraboloid \( M \) in \( \mathbb{R}^{n+1} \) defined by
\[
M = \{(x_1, x_2, \ldots, x_n, z) : z = x_1^2 + x_2^2 + \cdots + x_n^2\}
\]
has a volume growth of at most \( r^{n+1} \) which is not maximal. Furthermore, we can create a non-empty cut locus of the point 0 at the same time maintaining positive curvature and manifesting volume growth conditions like that of (1.1).

**Example 1.4.** Consider \( M = \{(x_1, x_2, \ldots, x_n, z) : z = x_1^2 + x_2^2 + \cdots + x_n^2\} \subset \mathbb{R}^{n+1} \). \((M, ds_M^2)\) is a complete open manifold with positive Ricci curvature \((n > 2)\). Here, \( ds_M^2 \) is an induced Euclidean metric. For \( 0 \neq q \in M \), let \( D_l(q) \) be a geodesic ball of radius \( l \) centered at \( q \). Consider a smooth function \( f : D_l(q) \to \mathbb{R} \). For a small neighborhood \( U \) of \( D_l(q) \), there exists a smooth function \( h : M \to \mathbb{R} \) such that \( h|_{D_l(q)} = f \) and \( \text{supp } h \subset U \). For \( \varepsilon > 0 \), denote \( M_\varepsilon = (M, ds_M^2 + \varepsilon h ds_M^2) \). We can choose \( \varepsilon \) small enough such that the Ricci curvature remains positive throughout \( M \) and an extension \( \gamma : [0, \infty) \to M_\varepsilon \) of a minimizing geodesic from 0 to \( q \) leaves \( D_l(q) \) and intersect a ray at a point. It follows that the cut locus of the point 0 is no longer empty. Since only rays intersecting and neighboring \( D_l(q) \) are affected in this new manifold, for a fixed \( a \approx 1, a \leq 1 \), we can choose \( l > 0 \) and \( \varepsilon > 0 \) small enough such that
\[
\lim_{r \to \infty} \left[ \frac{\text{Vol}(B_\Sigma(p, r))}{\text{Vol}(B(p, r))} \right]^{-1} = a
\]

Given two real valued functions \( f, g : \mathbb{R} \to \mathbb{R} \). Denote the limit
\[
\lim_{r \to \infty} f(r)[g(r)]^{-1} = a, \quad a > 0
\]
if it exists by \( f \sim_a g \).

Now we extend the result by Shen in [7] by replacing the maximum volume growth condition with a weaker volume growth condition.
Lemma 1.5. Let $M$ be a complete open manifold with $\text{Ric}_M \geq 0$. Let $[9^n - 1]9^{-n} < a \leq 1$ where $n = \text{dim}_\mathbb{R} M$. If 
\[
\lim_{r \to \infty} \inf \left[\text{Vol}(B_\Sigma(p, r))]\text{Vol}(B(p, r))\right]^{-1} = a,
\]
then for any $t \in \mathbb{R}$, $b_p^{-1}(t)$ is compact.

The following theorem is the main result in this paper:

Theorem 1.6. Let $M$ be a complete open Kähler manifold with nonnegative bisectional curvature. Then $M$ is a Stein manifold if the followings holds
(a) The bisectional curvature is positive outside a compact set
(b) \[
\text{Vol}(B_\Sigma(p, r)) \sim_a \text{Vol}(B(p, r)),
\]
where $[9^{2n} - 1]9^{-2n} < a \leq 1$ and $n = \text{dim}_\mathbb{C} M$

2. Proofs

We will prove Lemma 1.5 first then Theorem 1.6.

Proof of Lemma 1.5. Proving by contradiction, we assume that $b^{-1}(t)$ is non-compact and then show that the assumed volume growth condition doesn’t hold.

Define the excess function for two points $p, q$ as 
\[
e_{p,q} = d(p, x) + d(x, q) - d(p, q).
\]

By the triangle inequality, we have that 
\[
e_{p,q}(x) \leq 2h(x) \tag{2.1}
\]

Denote $r_p(x) = d(p, x)$.
Assume that the minimizing geodesic between $p$ and $q$ is part of a ray emanating from $p$. Now, taking the limit of inequality (2.1) as $q$ goes to infinity, we end up with the following inequality
\[
r_p(x) - \lim_{t \to \infty} \{t - d(x, \gamma(t))\} \leq 2h_\gamma(x), \tag{2.2}
\]

where $h_\gamma(x)$ is a distance from $x$ to a ray $\gamma$ emanating from $p$. Since 
\[
r_p(x) - b_p(x) \leq r_p(x) - \lim_{t \to \infty} \{t - d(x, \gamma(t))\},
\]

for each ray $\gamma$ emanating from $p$, inequality (2.2) implies that 
\[
r_p(x) - b_p(x) \leq 2h_\gamma(x) \tag{2.3}
\]

Let $h_p(x) = d(x, Rp)$, where $Rp$ is a union of rays emanating from $p$. Since inequality (2.3) holds for any ray $\gamma$, we have that 
\[
r_p(x) - b_p(x) \leq 2h_p(x) \tag{2.4}
\]

Next, note that 
\[
C(\Sigma) \cap C(\Sigma^c) = \emptyset.
\]
Therefore, for any \( r > 0 \) and \( p \in M \), we have that

\[
B_\Sigma(p, r) \cap B_{\Sigma^c}(p, r) = \emptyset
\]

Observe that \( B(x, h_p(x)) \subset C(\Sigma^c) \). It follows that \( B(x, h_p(x)) \subset B_{\Sigma^c}(p, r_p(x) + h_p(x)) \).

Since \( b_p \) is exhaustion whenever \( h_p \) is bounded, we assume that \( h_p \) is unbounded. Due to noncompactness of \( b_p^{-1}(t) \), we can construct a diverging sequence \( \{x_m\} \subset b_p^{-1}(t) \). Consequently, \( \{h_p(x_m)\} \) is a divergence sequence.

Denote \( h_m = h_p(x_m) \) and \( r_m = r(x_m) \). By Bishop-Gromov volume comparison theorem,

\[
\frac{Vol(B_{\Sigma^c}(p, r_m - h_m))}{Vol(B_{\Sigma^c}(p, r_m + h_m))} \geq \left[ \frac{r_m - h_m}{r_m + h_m} \right]^n
\]

It is easy to verify that

\[
B(x_m, h_m) \subset B_{\Sigma^c}(p, r_m + h_m) \setminus B_{\Sigma^c}(p, r_m - h_m) \quad (2.5)
\]

and that

\[
Vol(B(x_m, h_m)) \leq Vol(B_{\Sigma^c}(p, r_m + h_m)) - Vol(B_{\Sigma^c}(p, r_m - h_m))
\]

\[
\leq \left\{ \left[ \frac{r_m - h_m}{r_m + h_m} \right]^n \right\} Vol(B_{\Sigma^c}(p, r_m + h_m))
\]

\[
\leq Vol(B_{\Sigma^c}(p, 3h_m + a)) \quad (2.6)
\]

Inequality 2.6 is due to the fact that \( h \leq r \) and

\[
r_p(x) - b_p(x) \leq 2h_p(x)
\]

In particular

\[
r_p(x) + h(x) \leq 3h(x) + a, \text{ when } x \in b_p^{-1}(a)
\]

Now, denote \( r_1(x) = d(x_1, x) \). By triangle inequality and (2.4),

\[
\lim_{m \to \infty} \sup_{h_p(x_m)} \frac{r_1(x_m)}{h_p(x_m)} \leq \lim_{m \to \infty} \sup_{h_p(x_m)} \frac{r_1(p)}{h_p(x_m)} + \lim_{l \to \infty} \sup_{h_p(x_m)} \frac{r_p(x_m)}{h_p(x_m)}
\]

\[
\leq 2 \quad (2.7)
\]

Also note that

\[
B(x_1, h_m) \subset B(x_m, h_m + r_1(x_m)) \quad (2.8)
\]

By volume comparison theorem we obtain

\[
Vol(B(x_m, h_m)) \geq \left[ \frac{h_m}{h_m + r_1(x_m)} \right]^n Vol(B(x_m, h_m + r_1(x_m)) \quad (2.9)
\]
Denote \( f_p(r) = Vol(B(p,r)) \) for a fixed \( p \in M \). From (2.7), (2.8), and (2.9), we have

\[
\lim_{m \to \infty} \inf \frac{Vol(B(x_m, h_m))}{f_p(h_m)} \geq \lim_{m \to \infty} \inf \left[ \frac{h_m}{(h_m + r_1(x_m))} \right]^n \frac{Vol(B(x_m, h_m + r_1(x_m)))}{f_p(h_m)} \geq \lim_{m \to \infty} \inf \left[ \frac{h_m}{(h_m + r_1(x_m))} \right]^n \frac{Vol(B(x_1, h_m))}{f_p(h_m)} \geq \lim_{m \to \infty} \inf \left[ \frac{1}{(1 + \frac{r_1(x_m)}{h_m})^n} \right] \lim_{m \to \infty} \inf \frac{Vol(B(x_1, h_m))}{f_p(h_m)} \geq 3^{-n} \]  

(2.10)

The last inequality is due to the fact that the volume growth

\[
\lim_{m \to \infty} \inf [Vol(B(x_1, h_m))] \left[ f_p(h_m) \right]^{-1}
\]

is independent of the base point \( x_1 \).

From inequalities (2.6), (2.10), and the volume comparison theorem, we have

\[
3^{-n} \leq \lim_{m \to \infty} \inf \frac{Vol(B(x_m, h_m))}{f_p(h_m)} \leq \lim_{m \to \infty} \inf \frac{Vol(B_{\Sigma^c}(p, 3h_m + a))}{f_p(h_m)} \leq 3^n \lim_{m \to \infty} \inf \frac{Vol(B_{\Sigma^c}(p, h_m))}{f_p(h_m)} \]  

(2.11)

Which leads to the inequality

\[
\lim_{m \to \infty} \inf [Vol(B_{\Sigma^c}(p, h_m))] \left[ f_p(h_m) \right]^{-1} \geq 9^{-n} \]  

(2.12)

Since

\[
Vol(B(p, r) = Vol(B_{\Sigma^c}(p, r)) + Vol(B_{\Sigma^c}(p, r)),
\]

the volume growth condition assumption implies that

\[
\lim_{r \to \infty} \inf [Vol(B_{\Sigma^c}(p, r))] \left[ f_p(r) \right]^{-1} < 9^{-n} \]  

(2.13)

Evidently, inequality (2.12) contradicts inequality (2.13). Hence \( b_p^{-1}(t) \) must be compact.

\[ \Box \]

**Proof of Theorem 1.6.** The Ricci curvature is nonnegative and positive outside a compact set because the bisectional curvature is assumed. The Busemann function \( b_p \) is a continuous plurisubharmonic exhaustion by lemma 1.5 and a result by H.Wu in [9]. In the same paper (Theorem C [9]), it follows that
there exist a strictly plurisubharmonic exhaustion function. This completes the proof.

3. Applications

Let $H_k(M, \mathbb{Z})$ denote the k-th singular homology group of $M$ with integer coefficients. It is well known that if $M$ is a complete proper Riemannian n-dimensional manifold with $\text{Ric}_M \geq 0$, then using Morse theorem, $M$ has the homotopy type of a CW complex with cells each of dimension $\leq n-2$ and $H_i(M, \mathbb{Z}) = 0$, $i \geq n - 1$. ([6], [4])

As an application of lemma 1.5, we have the following result.

**Corollary 3.1.** Let $(M, g)$ be a complete open manifold with $\text{Ric}_M \geq 0$. If $\text{Vol}(B_S(p, r)) \sim_a \text{Vol}(B(p, r))$, where $[9^n - 1]9^{-n} \leq a < 1$, then $M$ has the homotopy type of a CW complex with cells each of dimension $\leq n - 2$. In particular, $H_i(M, \mathbb{Z}) = 0$, $i \geq n - 1$.

It is also known that if $M$ is a Stein manifold of n-complex dimension, then the homology groups $H_k(M, \mathbb{Z})$ are zero if $k > n$ and $H_n(M, \mathbb{Z})$ is torsion free (theorem 1 [1]), [2]. As an application of theorem 1.6, we have the following result.

**Corollary 3.2.** Let $M$ be a complete open Kähler manifold with nonnegative bisectional curvature. If the followings holds

(a) The bisectional curvature is positive outside a compact set
(b) $\text{Vol}(B_S(p, r)) \sim_a \text{Vol}(B(p, r))$, where $[9^{2n} - 1]9^{-2n} \leq a < 1$ and $n = \text{dim}_C M$

then $H_k(M, \mathbb{Z}) = 0$, for $k > n$ and $H_n(M, \mathbb{Z})$ is torsion free

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