Effectively Closed Infinite-Genus Surfaces and the String Coupling

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Abstract. The class of effectively closed infinite-genus surfaces, defining the completion of the domain of string perturbation theory, can be included in the category $O_G$, which is characterized by the vanishing capacity of the ideal boundary. The cardinality of the maximal set of endpoints is shown to be $2^N$. The product of the coefficient of the genus-$g$ superstring amplitude in four dimensions by $2^g$ in the $g \to \infty$ limit is an exponential function of the genus with a base comparable in magnitude to the unified gauge coupling. The value of the string coupling is consistent with the characteristics of configurations which provide a dominant contribution to a finite vacuum amplitude.
1. Introduction.

The vanishing of multi-loop superstring vacuum amplitudes at finite order in perturbation theory implies that there must be an additional contribution to the inner product

\[ \langle 0_{\text{out}} | S | 0_{\text{in}} \rangle = \langle 0_{\text{in}} | S^\dagger S | 0_{\text{in}} \rangle = 1 \]  

given a unitary S-matrix. The contribution of disconnected bubble diagrams, which is nevertheless factored from the vacuum amplitude, also would be zero, since the contour arguments in the non-renormalization theorems for superstrings continue to be valid.

Additional contributions to the amplitude could arise from surfaces with boundaries with non-zero linear measure and punctures. However, these boundaries occur either when the theory contains both open and closed strings or through non-perturbative effects, whereas punctures are associated with additional asymptotic string states.

It follows that the domain of closed string perturbation theory should be extended to infinite-genus super-Riemann surfaces to calculate the vacuum amplitude. The geometrical conditions imposed on this space will be defined first with respect to the ordinary part of the supermanifold, since there are no additional discrete modular transformations in the odd Grassmann coordinates. As the interaction region in a scattering process has finite volume, the surfaces should be effectively closed, and they share the property that they can be uniformized by Schottky groups, except that the number of generators may be infinite. Specifically, effectively closed surfaces will have handles with thickness decreasing at a rate \( \frac{1}{n^q} \), \( q > \frac{1}{2} \), so that they are conformally equivalent to spheres with handles accumulating to a point even in the intrinsic hyperbolic metric. These surfaces would be required to satisfy the property of conformal rigidity, since it should not be possible to map them to manifolds with a boundary at infinity having non-zero linear measure, which could not contribute to the perturbative expansion of the S-matrix element representing the amplitude for the \( n_1 \) in-states and \( n_2 \) out-states for fixed \( n_1, n_2 [1] \).

While the bosonic Polyakov action is known as the energy functional for the harmonic map \( z \rightarrow X^\mu(z) \) from a Riemann surface to the target space, its extremal value is the area in the intrinsic metric and not the total energy of the scattering process. Infinite-genus surfaces can be included in the perturbative expansion of the S-matrix because their infinite area in the hyperbolic metric is associated with a vanishing weighting factor compensated by an integral over an infinite-dimensional moduli space.
The structure of $M_\infty$ within the category of effectively closed infinite-genus surfaces is described. After determining the appropriate normalization of the superstring amplitudes, it is verified that the volume of the supermoduli space integral increases exponentially with the genus. The functional dependence on the string coupling $\kappa_{\text{str}}$ can be estimated once accurate bounds are placed on the volume integral. In the Schottky parameterization, the primitive-element products and the determinant factor depend exponentially on the genus, whereas, in the Fuchsian parameterization, the integrand is expressed in terms of Selberg zeta functions, with the dominant behaviour given by the lowest eigenvalues of the Laplacian operator.

The ends of an infinite-genus surface can be identified generally with the points in a Cantor set. Defining the cardinality of the set of ends to be $\text{card } E$, finiteness of the infinite-genus contribution to the superstring amplitude implies the condition

$$(\text{card } E)\lim_{g \to \infty} c(\kappa_{\text{str}})^g < \infty$$

where $c(\kappa_{\text{str}})^g$ is the exponential genus-dependence of the supermoduli space integral. This relation would determine a specific value for string coupling, which then can be compared to the gauge coupling of supersymmetric grand unified theories.

2. The Domain of the String Path Integral

2.1. Universal Moduli Space

The closure of moduli space is $\bar{M}_g = M_g \cup D_g$ where $D_g$ is the compactification divisor at genus $g$ consisting of degenerate Riemann surfaces. Universal moduli space can be defined to be $\bar{R} = \prod_{g=0}^\infty (\cup_{k=0}^\infty \text{Sym}^k \bar{M}_g)$ or $\bar{R}_\infty = R \times \cup_{k=0}^\infty \text{Sym}^k (\bar{M}_\infty)$, where $\bar{M}_\infty$ is the space of connected, stable, essentially compact surfaces of infinite genus [2], and the Sym symbol is included so that there is no overcounting of complex structures on two separate genus-$g$ components.

Almost all of the handles on infinite-genus surfaces in $\bar{M}_\infty$ will be very small. According to the geometric hypotheses for the definition of Riemann surfaces [3], they can be constructed by applying glueing maps to copies of $\mathbb{C}$ with open, simply connected neighbourhoods $S_r$ around a set of points and an additional compact set $K_j$ deleted.
gluing maps are constrained by the inequalities

\[ R_\mu(j) < \frac{1}{4} \min_{s \in S_{\nu_\mu}} |s - s_\mu(j)| \quad s_\mu(j) \in S_\mu \]

\[ R_\mu(j) < \frac{1}{4} \text{dist}(s_\mu(j), K_{\nu_\mu}(j)) \]

\[ \sum_{j, \mu} \frac{1}{|s_\mu(j)|^{d-4\delta-2}} < \infty \quad (2.1) \]

\[ r_\mu(j) < \frac{1}{|s_\mu(j)|^d} \quad R_\mu(j) > \frac{1}{|s_\mu(j)|^\delta} \]

\[ |s_1(j) - s_2(j)| > \frac{1}{|s_\mu(j)|^\delta} \]

where the region surrounding the neighbourhoods \( S_\nu \) satisfies the inequalities \( r_\mu(j) \leq |z - s_\mu(j)| \leq R_\mu(j) \), the following condition is imposed on the handles. For all \( j \geq g + 1 \), \( A_j \) is the homology class represented by the oriented loop \( \phi(\{ \sqrt{t_j} e^{i\theta}, \sqrt{t_j} e^{-i\theta} \})|0 \leq \theta \leq 2\pi) \), \( \phi_j : H(t_j) \to Y_j \), then that there exists a \( \beta > 0 \) such that \( \sum_{j \geq g+1} t_j^\beta < \infty \).

The surfaces with nodes should be dense in \( \bar{M}_\infty \) since the addition or subtraction of a handle is a small effect. If the radii of the bases of the handles in the Euclidean metric of the embedding space \( r_n \) decrease sufficiently quickly, then the distance in the moduli space based on the intrinsic metric will tend to zero, and the original surface would be arbitrarily close to a surface with nodes. The condition \( \sum_n r_n^2 < \infty \) implies that \( r_n \sim \frac{1}{n^{\frac{2g-2}{2g-2+\epsilon}}} \) in the Euclidean metric. Given that a surface of genus \( g \) has a hyperbolic metric with curvature -1, so that the total area increases linearly, the cross-sectional areas of the handles \( \pi r_n^2 \) will be multiplied by a factor proportional to \( n \). In the intrinsic metric, therefore, \( r_n^{\text{intr.}} \sim \frac{1}{n^2} \), and the distance in moduli space between a surface which contains the \( n^{th} \) handle and a surface for which \( r_n^{\text{intr.}} \to 0 \) will be arbitrarily small. The property of the denseness of surfaces with nodes in infinite-genus moduli space, is valid, therefore, since the handles on effectively closed infinite-genus surfaces satisfy the above constraint on the cross-sectional areas.

The formulation of string theory in terms of the analytic geometry of universal moduli space is feasible since the process of pinching the surface along an \( A_j \)-homology cycle and removing the nodes allows for a continuous path from the boundary of \( M_g \) to the moduli space of genus \( g - 1 \) surfaces \( M_{g-1} \) [3]. The limiting value of the partition function then exists, so that the factorization condition, implying that the partition function with nodes equals the product of the partition functions of the disconnected components, is well-
defined. Functional derivatives of the partition function, such as the energy-momentum tensors, also exist.

### 2.2. On the Classification of Riemann Surfaces

There does not exist a Green function for the scalar field with a single delta function source on a compact surface of finite surface of finite genus or an $O_G$ surface of infinite genus. The equation

$$
\Delta G(z, z') = \frac{1}{\sqrt{h(z)}} \delta(z, z')
$$

(2.2)

as $\int_{\partial \Sigma} \sqrt{h(z)} \partial_i G(z, z') dz^i = 0$ if the boundary $\partial \Sigma$ has zero linear measure.

Surfaces of $O_{HD}$ type have the property that there exists no non-constant harmonic function $f(z, \bar{z})$ with finite Dirichlet norm [4]

$$
\int_{\Sigma} d^2z \sqrt{h(z)} |f'(z)|^2
$$

(2.3)

Over a locally flat coordinate patch, $\sqrt{h(z)}$ can be set equal to 1, and the coordinates in the overlapping neighbourhoods may be chosen so that this factor equals 1 over the entire domain of the integral. For a general choice of coordinates, however, it is necessary to include the $\sqrt{h(z)}$.

The class of $O_{AD}$ surfaces are defined by the absence of non-constant analytic functions with finite Dirichlet norm. If no Green function exists, then it is not feasible to construct an analytic function by the integral

$$
f(z) = \int_{\Sigma} d^2z' G(z, z') f(z')
$$

(2.4)

Then there is a set of inclusions $O_G \subset O_{HD} \subset O_{AD}$.

A method for identifying the classification type of the surface is the modulus test. If $\Sigma_n$ is an exhaustive covering of $\Sigma$, then the surface belongs to the class $O_{AD}$ if $\prod_{n=1}^{\infty} \mu_n = \infty$ where there is a harmonic function $f$ such that $f|_{\alpha_n i} = 0$, $f|_{\beta_n i} = \log \mu_n i$ and $\int_{\alpha_n i} \ast df = 2\pi$, $\alpha_n i = E_{ni} \cap \partial \Sigma_n$, $\beta_n i = E_{ni} \cap \partial \Sigma_{n+1}$, and $\Sigma_{n+1} - \Sigma_n$ consists of finite number of relatively compact regions [5]. The modulus $\mu_n$ is defined to be $\min_i \mu_n i$. An example is the Schottky covering of a compact genus-$g$ Riemann surface.
2.3. Universal Grassmannian

Consider the action for a free scalar field theory on a Riemann surface

\[ S = \int_{\Sigma} d^2 z \sqrt{h(z)} \partial \bar{\partial} X \] (2.5)

The path integral \( \int_{\Sigma/D\Sigma} D[X] e^{-S} \) can be performed on the complement of a simply connected compact subset of \( \Sigma \) to obtain a functional of the space of field values on the boundary which is homeomorphic to \( S^1 \). This functional \( \phi_{g,\Sigma}(X|_{bdy(D\Sigma)}) \) can be identified with a state \( |\phi_{g,\Sigma}\rangle \) in the space of Hilbert states on the circle [6].

For each Riemann surface \( \Sigma \) of genus \( g \), the path integral produces a state \( |\phi_{g,\Sigma}\rangle \). Consequently, there is a direct correspondence between states in the Hilbert space and compact Riemann surfaces. Since every compact surface corresponds to a particular state through the map

\[ \phi : \Sigma \longrightarrow |\phi_{g,\Sigma}\rangle \] (2.6)

there is an inclusion

\[ \phi : \prod_{g \text{ finite}} M_g \rightarrow H \] (2.7)

where \( H \) is the Hilbert space of states of the free field theory on \( S^1 \).

This inclusion is not an isomorphism, as there exist states in the Hilbert space which are not images of Riemann surfaces under the map \( \phi \). This can be expected since \( M_g \) is homeomorphic to a subset of \( \mathbb{C}^{3g-3} \) and therefore there are linear transformations do not preserve this set, whereas, the Hilbert space is invariant under these transformations by definition.

2.4. Universal Teichmüller Space

A homeomorphism \( w : D \rightarrow w(D) \) between domains in \( \hat{C} \) is quasiconformal if and only if \( w \) has locally integrable generalized derivatives satisfying almost everywhere on \( D \) the Beltrami equation

\[ w_{z\bar{z}}(z) = \mu(z) w_{\bar{z}}(z) \] (2.8)
for some measurable complex function µ on D called the Beltrami differential with
\[ \text{ess sup}_{z \in D} |\mu(z)| = ||\mu|| < 1 \] (2.9)

By applying the existence and uniqueness theorem to the Beltrami differential which is µ on Δ and is extended to Δ* by reflection \( \tilde{\mu}(\frac{1}{z}) = \bar{\mu}(z) \frac{z^2}{\bar{z}} \) for \( z \in \Delta \) one obtains the quasiconformal homeomorphism \( w_\mu \) of \( \mathbb{C} \) which is µ-conformal in Δ, fixes ±1, and keeps Δ and Δ* invariant. If the Beltrami differential is G-equivariant,
\[ \mu(\gamma z) \bar{\gamma}'(z) = \mu(z) \] (2.10)

Let \( L^\infty(G) = \{ G - \text{compatible differentials satisfying the Beltrami equation} \} \) [7]. Then the Teichmüller space for the Fuchsian group G is \( T(G) = L^\infty(G)/\sim \) where \( \mu \sim \nu \) if and only if \( w_\mu = w_\nu \) on \( \partial \Delta = S^1 \).

The universal Teichmüller space is obtained when \( G = 1 \), and \( T(G) \subset T(1) \) for all \( G \neq 1 \). Other definitions are \( T(1) = L^\infty(\Delta)_1/\sim \) and \( QS(S^1)/SL(2,\mathbb{R}) \). A complex analytic model of T(1) is the set of univalent functions in \( \Delta^* \) of the form \( F_\mu(z) = z + b_1 z + b_2 z^2 + b_3 z^3 + \ldots \) corresponding to the conformal equivalence class of the Beltrami differential \( [\mu] \). The coefficients \( b_n \) are coordinates on \( T(1) \) and \( b_n = O(n^{-c}) \), \( c = 0.509... \) for large \( n \) [8].

Any two Beltrami coefficients can be paired so that
\[ g(\mu, \nu) = -\frac{ia}{3\pi^2} \int \int _\Delta \int \int _\Delta \frac{\mu(z)\nu(\bar{\zeta})}{(1-z\bar{\zeta})^4} d\xi d\zeta \cdot dx dy \] (2.11)
whereas the Weil-Petersson inner product on \( T(G) \) is
\[ g(\mu, \nu) = -\frac{ia}{3\pi^2} \int \int _{\Delta/G} \int \int _\Delta \frac{\mu(z)\nu(\bar{\zeta})}{(1-z\bar{\zeta})^4} d\xi d\zeta \cdot dx dy \] (2.12)

The integration region \( \Delta/G \) is the fundamental domain of the Fuchsian group. In terms of the smooth vector fields on \( S^1 \), \( v = \sum_m v_m L_m \) and \( w = \sum_m w_m L_m \),
\[ g(v, w) = -2ia \text{ Re} \sum_{m=2}^{\infty} v_m \bar{w}_m (m^3 - m) \] (2.13)
where the infinite series converges if \( v \) and \( w \) are \( \mathbb{C}^{3+\epsilon} \) on \( S^1 \) [9].
Each map \( f \in T(G) \) determines a tessellation \( T_f \) of \( U \) which is invariant under \( f \circ G \circ f^{-1} \). Paramaterizing \( T_f \) by a set of glide coefficients \( \{ k_f^i \} \), \( i \in I \), where \( I \) is the index set for the sides of \( T \) factored by \( G \), these coordinates can be used as moduli for Teichmüller spaces of finitely and infinitely generated groups and universal Teichmüller space when \( G = 1 \) [10].

By considering genus-zero surfaces, a presentation can be obtained for \( B \), a universal modular group. It can be formulated in terms of a finite number of relations amongst four operators, the Dehn twist \( t \), the braiding \( \pi \), the order 4 rotation \( \alpha \) and the order 3 rotation \( \beta \). Glueing a once-punctured torus \( \Sigma_{1,1} \) to each cylinder in \( \Sigma_{0,\infty} \) defines an infinite-genus surface. Combining the two glueing maps \( t_s \) with \( \pi, t, \alpha, \beta \) one obtains generators of the universal modular group of an infinite-genus surface \( \Sigma_{\infty,\infty} \), which contains the mapping class groups of surfaces of genus \( g \) with \( n \) boundary components where \( g \geq n \) [11]. Factoring universal Teichmüller space by this universal modular group gives rise to a space which includes the infinite-genus component of universal moduli space.

### 2.5. Extended Schottky Spaces

The extended Teichmüller space \( \bar{T}_g \) is \( \{ (X, \tau) | X \text{ is a stable Riemann surface of genus } g \text{ and } \tau \text{ is a standard set of generators of } \pi_1(X) \} \) [12]. The quotient of a neighbourhood \( U_x \) by the isotropy group \( G^0_x \) of \( x \) in the modular group \( G_g \) is homeomorphic to an open submanifold of \( \bar{S}_g \), the extended Schottky space, so that \( \bar{T}_g \) only can be given the structure of a complex ringed space.

The extended Schottky space \( \bar{S}_g \) which is defined to be \( \bar{S}_g = \{ (X, \sigma) | X \text{ stable Riemann surface of genus } g, \sigma : \pi_1(\Sigma) \to \Gamma \text{ a homomorphism, induced by a Schottky structure on } \Sigma \} / \sim \) where \( (X, \sigma) \sim (X', \sigma') \) if there is an analytic isomorphism \( h : X \to X' \) such that \( \sigma = \pi_1(h) \circ \sigma' \) up to inner automorphisms of \( \Gamma \), the free group generated by the \( \Lambda \)-cycles [13]. The extended Schottky space is a fine moduli space and a complex manifold of dimension \( 3g-3 \). Summation over the genus-\( g \) scattering amplitudes could then be expressed as an integration over the union of extended Schottky spaces \( \prod_{g=0}^{\infty} \bigcup_{k=0}^{\infty} \bar{S}_g \), which can be mapped to the universal moduli space through the map \( \varphi_g : \bar{S}_g \to \bar{M}_g \).

If \( \hat{\tau} \in S_g \), the image under the map \( S_g \to \mathbb{C}^{3g-3} \) is

\[
\tau = (K_1, ..., K_g, q_2, p_3, q_3, ..., p_g, q_g) \in \mathbb{C}^{3g-3}
\]

\[
p_i = \xi_1 i \quad q_i = \xi_2 i
\]  

(2.14)
For dividing nodes, new coordinates \([14]\) must be introduced \(t_1, \ldots, t_g, \rho_1, \ldots, \rho_{2g-3}\), \(t_i = \frac{1}{\kappa_i}, k(j), l(j), m(j), n(j)\) and for \(j = 1, \ldots, g\), \(T_k(p_{k(j)}) = 0\) and \(T_k(q_{k(j)}) = \infty\) while for \(j = g+1, \ldots, 2g-3\), \(T_j(p_{k(j)}) = 0\) \(T_j(p_{l(j)}) = \infty\), \(T_j(p_{m(j)}) = 1\) \(\rho_j = T_j(p_{n(j)})\). New coordinates also would be introduced at the boundary of super-Schottky space, except that a distinction must be made between spin nodes of the Ramond and Neveu-Schwarz type.

The necessity for a change of the coordinates implies that the holomorphic measure defined by the Schottky group parameters cannot be extended over \(\bar{M}_g\). The supermoduli space is known to be non-split, and a singular transformation \([15]\) is required for the integrand to maintain the form \(|F(y)|^2 [\det \text{Im} T]^{-5}\) under super-Schottky transformations defined by

\[
\frac{TZ - Z_{1n}}{TZ - Z_{2n}} = K_n \frac{Z - Z_{1n}}{Z - Z_{2n}}
\]

where \(K_n\) is the multiplier and \(Z_{1m} = (\xi_{1m}, \theta_{1m})\), \(Z_{2m} = (\xi_{2m}, \theta_{2m})\) are the super-fixed points. The integral over the variables \(\{K_n, Z_{1m}, Z_{2m}\}\) is valid over supermoduli space except in the neighbourhood of the compactification divisor.

3. Physical Criteria for the Category of Riemann Surfaces

3.1. Constraints on the Class of Riemann Surfaces

The connections between the infinite-genus surfaces and the domain of string perturbation theory can be analyzed using the concept of flux through the ideal boundary. The invariance of the action under the transformation \(X \rightarrow X + \epsilon X_n\) gives rise to an infinite number of charges \(Q_n\) such that

\[
Q_n|\phi, \Sigma\rangle = 0 \quad n = 1, 2, \ldots
\]

with integrability condition \([Q_m, Q_n] = 0\).

At infinite genus, an extra condition on the state is

\[
\int_{\text{ideal bdy}} j_n|\phi, \Sigma\rangle = 0
\]

Either

(i) The ideal boundary has zero harmonic measure.
(ii) The current \(j_n\) vanishes at the boundary.
(iii) If the ideal boundary is homeomorphic to a circle, the curl of the current \(j_n\) vanishes.
Let \( X_n(t) = \int^t \left[ \eta_n - \pi A_{nI}(I \tau)_{IJ}(\omega - \bar{\omega})_J \right] \) where \( \eta_n \) is a meromorphic differential with only a single pole of order \( n + 1 \) at \( t = 0 \) and residue \( (-n) \) and zero \( A \)-periods [16] and \( A_{nI} \) is defined by

\[
\omega_I = \sum_{n=1}^{\infty} A_{nI} t^{n-1} \quad \eta_n = -\frac{1}{(n-1)!} \partial_t \partial_{\bar{y}}^n \log E(t,y)|_{y=0} dt
\]  

(3.3)

The current is \( j_n = X^h_n \partial X + X^a_n \partial X \) where

\[
X^h_n = t^{-n} - \sum_{m=1}^{\infty} \left[ 2Q_{mn} + \pi A_n (I m \tau)^{-1} \bar{A}_m \right] \frac{\bar{t}^m}{m} \\
X^a_n = \pi \sum_{m=1}^{\infty} A_n (I m \tau)^{-1} \bar{A}_m \frac{\bar{t}^m}{m} \\
Q_{nm} = \frac{1}{2(m-1)!(n-1)!} \partial_t^m \partial_{\bar{y}}^n \log \frac{E(t,y)}{(t-y)^{b_n}} \bigg|_{t=y=0} \\
b_n = \left[ \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(n)_m}{m!} \right]^{-1}
\]

(3.4)

Since \( \partial \bar{\partial} j_n = 0 \) and the ideal boundary has zero harmonic measure, the integral condition is immediately satisfied for \( O_G \) surfaces.

If the ideal boundary does not have zero harmonic measure, the values of the fields \( X_n \), \( X \) must tend to zero as \( t \to \infty \) for the current to vanish. Then \( X^h_n \to 0 \) if and only if \( 2Q_{nm} = -A_{nI}(I m \tau)^{-1} \bar{A}_m \), and, indeed, \( X^h_n \) cannot vanish as \( t \to \infty \) unless \( A_{nI} \), \( I m \tau \) satisfy this condition. Similarly, \( X^a_n \to 0 \) if and only if \( A_{nI} = 0 \). However, this is not possible because the abelian differentials \( \omega_I = \sum_{n=1}^{\infty} A_{nI} t^{n-1} \) do not vanish. Therefore, \( X^a_n \) also does not vanish as \( t \to \infty \). Consequently, it is necessary to impose the harmonic condition \( \partial \bar{\partial} X = 0 \) and \( X(z, \bar{z}) \to 0 \) as \( z \) tends to the ideal boundary.

However, for an \( O_{HD} \) surface, there is no non-constant harmonic function which has finite Dirichlet norm

\[
\int_\Sigma |f'(z)|^2 \, d\Sigma d\bar{z} < \infty
\]

(3.5)

a functional condition which is equivalent to finiteness of the action for \( X(z) \)

\[
\int_\Sigma d^2z \, \partial X \bar{\partial} X < \infty
\]

(3.6)
If \( X(z) \sim \frac{1}{|z|^\varepsilon}, \varepsilon > -1 \), and \( \partial X(z) \sim \frac{1}{|z|^1} \), then
\[
\begin{align*}
\lim_{N(\partial \Sigma) \to \partial \Sigma} \int_{N(\partial \Sigma)} \, d^2 \! z \, \partial X \bar{\partial} X & \sim \lim_{\varepsilon \to \infty} kr_{idl.bdy} \int d|z| \frac{|z|}{|z|^{2+2\varepsilon}} \\
& = \lim_{\varepsilon \to \infty} kr_{idl.bdy} \int \frac{d|z|}{|z|^{1+2\varepsilon}} = -\lim_{\varepsilon \to \infty} kr_{idl.bdy} \frac{|z|^{2\varepsilon}}{2\varepsilon} = 0 \tag{3.7}
\end{align*}
\]

Thus, there are no harmonic functions \( X(z) \) satisfying the fall-off condition \( X \sim \frac{1}{|z|^\varepsilon}, \varepsilon > -1 \), as they would have finite Dirichlet norm.

If \( \Sigma \notin O_{HD} \), the existence of a harmonic function which vanishes at the ideal boundary is consistent with the flux condition.

When the ideal boundary is homeomorphic to a circle,
\[
\int_{\text{ideal bdy}} j_n \, dz = \int_{\Sigma} d j_n \, d^2 z \tag{3.8}
\]
and
\[
d j_n = (\partial + \bar{\partial})(X^h_n \bar{\partial} X + X^a_n \partial X) \]
\[
= \partial X^h_n \bar{\partial} X + X^h_n \partial \bar{\partial} X + \partial X^a_n \partial X + X^a_n \partial^2 X \\
+ \bar{\partial} X^h_n X^a_n \bar{\partial} \partial X + \bar{\partial} X^a_n \partial X + X^a \partial \bar{\partial} X \tag{3.9}
\]
vanishes, in particular, if \( \partial X = 0 \) and \( \bar{\partial} X = 0 \) so that \( X = \text{constant} \) and \( d j_n = 0 \) everywhere on the surface. The general solution to the constraint \( d j_n = 0 \) can be obtained by \( X \) equal to \( \sum_{k,\ell} a_{k,\ell} z^k \bar{z}^\ell \) with the coefficients satisfying the recursion relation
\[
- n(\ell + 1) a_{k+n+1,\ell+1} - (\ell + 1) \sum_{m=1}^\infty [2Q_{mn} + \pi A_n(Im \tau)^{-1}] a_{k+1-m,\ell+1} \\
+ (\ell + 1)(\ell + 2) a_{k+n,\ell+2} - (\ell + 1)(\ell + 2) \sum_{m=1}^\infty \frac{1}{m} [2Q_{mn} + \pi A_n(Im \tau)^{-1}] a_{k-m,\ell+2} \\
+ (k + 1)(k + 2) \pi \sum_{m=1}^\infty \frac{1}{m} A_n(Im \tau)^{-1} \bar{A}_m a_{k+2,\ell-m} \\
+ (k + 1) \pi \sum_{m=1}^\infty A_n(Im \tau)^{-1} \bar{A}_m a_{k+1,\ell+1-m} = 0 \tag{3.10}
\]
Surfaces which have a boundary at infinity with positive line measure would introduce an additional string state and therefore are excluded from the S-matrix expansion, which is defined only for a fixed number of asymptotic states. The domain of string perturbation theory then should be restricted to a set of effectively closed surfaces. Finiteness of the size of the interaction region implies that the infinite-genus surfaces must be conformally equivalent to spheres of bounded volume with an infinite number of handles of diminishing size. A surface of this kind also can be uniformized by an infinitely-generated Schottky group. When there is more than one accumulation point for the handles, the countable sequence of generators should be partitioned into subsequences, each corresponding to a set of isometric circles with radius decreasing to zero.

For a planar region with the boundary $\delta$ having capacity $c_\delta = e^{-k_\delta}$, the Green function with pole at $\infty$ is $G(z, \infty) = \log|z| + k_\beta$. Then the Robin constant $r(\delta) = k_\delta$ is shifted to $r(\delta') = r(\delta) + \log a$ under the conformal transformation $z = f^{-1}(z')$ which equals $az' + b + \frac{c}{z'} + \frac{d}{z'^2} + \ldots$ [5]. Since an orientable Riemann surface can be conformally mapped to the unit disk, it follows that the property of $r(\beta) = \infty$ or equivalently zero capacity of the ideal boundary $\beta$ is invariant under conformal transformations. It may be noted that a Stoilăuideal boundary point of a noncompact surface, included in the set $\Sigma^* - \Sigma$, where $\Sigma^*$ is the compactification of $\Sigma$, is stable if $c_\gamma = 0$ and mapped to a nondegenerate continuum if $c_\gamma > 0$ [5]. The identification of the class of effectively closed surfaces with the category $O_G$ then would be conformally invariant.

3.2. String Amplitudes defined by Effectively Closed Surfaces of Infinite Genus

Since string scattering amplitudes includes an integral over the locations of the vertex operators, they will depend on the Green functions for fields of various spin on the Riemann surface. As every closed surface can be uniformized by a Schottky group, it can be represented as $\Sigma \sim D/\Gamma$ where $\Gamma$ is the free group $\langle T_1, \ldots, T_g \rangle$ with $g$ generators characterized by multipliers $K_n$ and fixed points $\xi_{1n}, \xi_{2n}$ through the relations $\frac{T_n z - \xi_{1n}}{T_n z - \xi_{2n}} = K_n \frac{z - \xi_{1n}}{z - \xi_{2n}}$, $n = 1, 2, \ldots, g$ and $D$ is the set of ordinary points points of $\Gamma$ in the extended complex plane. When the vertex operators represent tachyons or dilatons, their correlators are expressed
in terms of products of exponentials of scalar field Green functions with two sources

\[ G_{QS}(P,R) = \sum_{\alpha \neq I} \ln \left| \frac{z_P - V_\alpha z_R}{z_P - V_\alpha z_S} \right| \frac{z_Q - V_\alpha z_R}{z_Q - V_\alpha z_S} \]

\[ - \frac{1}{2\pi} \sum_{m,n} \text{Re} \{ v_m(z_P) - v_m(z_Q) \} (\text{Im } \tau)^{-1}_{mn} \text{Re} \{ v_n(z_R) - v_n(z_S) \} \]

\[ v_n(z) = \sum_{\alpha \neq I} ^{(n)} \ln \left( \frac{z - V_\alpha \xi_{1n}}{z - V_\alpha \xi_{2n}} \right) \]

\[ v_n(z) - v_n(T_m z) = 2\pi i \tau_{mn} \]  \hspace{1cm} (3.11)

with \( V_\alpha \) being an arbitrary element of the group \( \Gamma \) and \( \sum_{\alpha \neq I} ^{(n)} \) being a sum over elements \( V_\alpha \) with the left-most member of the product of generators not being either \( T_n \) or \( T_n^{-1} \).

The first term is

\[ \sum_{\alpha \neq I} \ln \left[ 1 + \frac{V_\alpha z_S - V_\alpha z_R}{z_P - V_\alpha z_S} \right] \left[ 1 + \frac{V_\alpha z_R - V_\alpha z_S}{z_Q - V_\alpha z_R} \right] \]  \hspace{1cm} (3.12)

and since \( |V_\alpha z_S - V_\alpha z_R| = |\gamma_\alpha z_S + \delta_\alpha|^{-1} |\gamma_\alpha z_R + \delta_\alpha|^{-1} |z_S - z_R| \), finiteness of the sum follows from convergence of the Poincare series \( \sum_{\alpha \neq I} |\gamma_\alpha|^{-2} \). If \( r_n \sim \frac{1}{n^q}, |K_n|^\frac{2}{q} = (c_1 n^q + c_2)^{-1}, |\xi_{1n} - \xi_{2n}| < c', c < \left| \frac{\xi_{2n} - \xi_{1n}}{\xi_{1n} - \xi_{2n}} \right| \), \( n = 1, 2, ... \) then

\[ \sum_{\alpha \neq I} |\gamma_\alpha|^{-2} < c^2 c'^2 \left[ \frac{2}{c_1^2 c_2^2} \sum_{n=1}^\infty \frac{1}{n^{2q}} + \left( \frac{2}{c_1^2 c_2} \sum_{n=1}^\infty \frac{1}{n^{2q}} \right)^2 + ... \right] \]  \hspace{1cm} (3.13)

which converges only if \( q > \frac{1}{2} \) [1].

The existence of an inverse of the imaginary part of the period matrix, \( \text{Im } \tau \), follows from the period relations for a compact genus-\( g \) surface

\[ (\omega*, \sigma) = \sum_{k=1}^g \left[ \int_{A_k} \omega \int_{B_k} \sigma - \int_{A_k} \sigma \int_{B_k} \omega \right] \]  \hspace{1cm} (3.14)

where \( \omega, \sigma \) are harmonic differentials and the cycles \( A_k, B_k \) represent a canonical homology basis. These relations can be generalized to open manifolds with exhaustion \( \{ \Omega_n \} \) defined so that there is a sequence of cycles \( A_1, B_1, A_2, B_2, ..., A_{p(n)}, B_{p(n)}, ... \) such that \( A_1, B_1, ..., A_{p(n)}, B_{p(n)} \) form a basis modulo the dividing cycles of \( \Omega_n \)

\[ (\omega, \sigma^*) = \sum_{k=1}^\infty \left[ \int_{A_k} \omega \int_{B_k} \sigma - \int_{A_k} \sigma \int_{B_k} \omega \right] + \lim_{n \to \infty} \int_{\partial \Omega_n} \omega \sigma \]  \hspace{1cm} (3.15)
where \( u \) is a function on \( \partial \Omega_n \), such that \( u(p) = \int_{p_0}^p \omega \) with \( p_0 \) being a fixed point on a contour \( \alpha \) of \( \partial \Omega_n \) and integration taken to be in the positive sense of \( \alpha \) [17]. The relations reduce to the usual form for surfaces in the class \( OG \) as the ideal boundary has zero harmonic measure.

Holomorphic one-forms and period matrices can be defined generally for infinite-genus surfaces uniformized by Schottky groups [18] as

\[
\omega_j^{(g)} = \sum_{\alpha}^{(j)} \left( \frac{1}{z - V_\alpha \xi_{1j}} - \frac{1}{z - V_\alpha \xi_{2j}} \right) dz
\]

\[
\int_{a_i} \omega_j^{(g)} = 2\pi i \delta_{ij}
\]

\[
lim_{g \to \infty} \tau_{ij}^{(g)} = \frac{1}{2\pi i} \lim_{g \to \infty} \int_{b_i} \omega_j^{(g)}
\]

The period matrix is

\[
\tau_{mn} = \frac{1}{2\pi i} \left[ \ln K_m \delta_{mn} + \sum_{\alpha}^{(m,n)} \ln \left( \frac{\xi_{1m} - V_\alpha \xi_{1n}}{\xi_{1m} - V_\alpha \xi_{2n}} \right) \right]
\]

where \( \sum_{\alpha}^{(m,n)} \) is a sum over all \( V_\alpha \) which do not have \( T_m, T_n^{-1} \) as a left-most member and \( T_n, T_{n^{-1}} \) as a right-most member, and if \( |K_n| \simeq (c_1 n^q + c_2)^{-2} \), then

\[
\text{Im} \, \tau_{nn} = -\frac{1}{2\pi} \left[ -2q \ln n - 2\ln c_1 - \frac{2c_2}{c_1 n^q} \right] = \frac{q}{\pi} \ln n + \frac{1}{\pi} \ln c_1 + \frac{c_2}{\pi c_1 n^q}
\]

Given the harmonic functions \( v(z) = \int_{\omega_n} \omega_n \) [19],

\[
\text{Re} \, v_n(z) = \sum_{\alpha}^{(n)} \ln \left| 1 + \frac{(\xi_{2n} - \xi_{1n}) \gamma_{\alpha}^{-2}}{(\xi_{1n} + \frac{\delta_{\alpha}}{\gamma_{\alpha}})(\xi_{2n} + \frac{\delta_{\alpha}}{\gamma_{\alpha}})(z - V_\alpha \xi_{2n})} \right|
\]

\[
\equiv \sum_{\alpha}^{(n)} \text{Re} \, v_{\alpha n}(z)
\]

have the following dependence on \( n \), given that \( z \) is a bounded distance \( d(z, I_{T_{n_0}}) \) from \( I_{T_{n_0}} \) for finite \( n_0 \):

(i) \( d(I_{V_\alpha}, I_{T_{n_0}}), d(I_{V_{\alpha^{-1}}}, I_{T_{n_0}}) \) are bounded

\[
\text{Re} \, v_{\alpha n}(z) = \mathcal{O} \left( \frac{1}{n^2} \right) |\gamma_{\alpha}|^{-2}
\]
(ii) $d(I_{V\alpha}, I_{T_0}), d(I_{V^{-1}}, I_{T_N})$ are bounded

$$\text{Re } v_{n\alpha}(z) = \mathcal{O}\left(\frac{1}{n}\right) |\gamma_\alpha|^{-2}$$

(iii) $d(I_{V\alpha}, I_{T_n}), d(I_{V^{-1}}^{-1}, I_{T_{n_0}})$ are bounded

$$\text{Re } v_{n\alpha}(z) = \mathcal{O}(1) |\gamma_\alpha|^{-2}$$

(iv) $d(I_{V\alpha}, I_{T_n}), d(I_{V^{-1}}, I_{T_n})$ are bounded

$$\text{Re } v_{n\alpha}(z) = \mathcal{O}\left(\frac{1}{n}\right) |\gamma_\alpha|^{-2}$$

when the distances between the isometric circles are bounded. Given that $|\gamma_n| \sim \frac{1}{n^q}$, the Poincare series is bounded by $2e^{c^2} \zeta(2q,n) \rightarrow n \rightarrow \infty \frac{1}{2q-1} n^{1-2q} + \mathcal{O}\left(\frac{1}{n^{2q}}\right)$ and the sum over the elements in categories (iii) and (iv) will decrease at least as $\frac{1}{n^{2q-1}}$. Together with the contributions of the elements in categories (i) and (ii), which decrease as $\mathcal{O}\left(\frac{1}{n^{2q}}\right)$ and $\mathcal{O}\left(\frac{1}{n^{2q}}\right)$, the sum over $V\alpha$ gives $\text{Re } v_n = \mathcal{O}\left(\frac{1}{n^{2q}}\right)$ and $\text{Re } \{v_n(z_P) - v_n(z_Q)\} < \frac{v_{PQ}}{\pi \ln n}$ for some constant $v_{PQ}$. Similarly, the dependence of the entries $(\text{Im} \tau)_{mn}, m \neq n$ is $\text{Im } \tau_{mn} = \mathcal{O}\left(\frac{1}{|m-n|^{2q-1}}\right)$ so that the eigenvalues of $(\text{Im} \tau)^{-1}$ for large $n$ are approximately $\lambda_n \simeq \frac{\pi}{q \ln n}$. Then

$$\frac{1}{2\pi} \sum_{m,n=1}^{\infty} \text{Re} \{v_m(z_P) - v_m(z_Q)\} (\text{Im} \tau)^{-1}_{mn} \text{Re} \{v_N(z_R) - v_N(z_S)\}$$

$$< \frac{1}{2q} v_{PQ} v_{RS} \sum_{n=1}^{\infty} \frac{1}{n^{4q-2} \ln n}$$

which is finite for $q > \frac{3}{4}$.

The integration over the locations of the vertex operators is followed by a moduli space integral so that the bosonic N-point $g$-loop amplitude [20] is

$$A_g = \frac{2\pi}{(4\pi(8\pi^2)^{13})^g} \left(\frac{K}{\pi}\right)^{N+2g-2} \int \prod_{n=1}^{g} \frac{d^2 K_n}{|K_n|^4} |1 - K_m|^4 \int \prod_{m=1}^{g} \frac{d^2 \xi_{1m} d^2 \xi_{2m}}{|\xi_{1m} - \xi_{2m}|^4}$$

$$\cdot \prod_{\alpha} |1 - K_\alpha|^{-4} \prod_{\alpha} \prod_{p=1}^{\infty} |1 - K_p^\alpha|^{-48}$$

$$\int \prod_{s=1}^{N} \frac{d^2 z_s}{\text{Vol}(SL(2,\mathbb{C}))} \langle V_s(z_s) \rangle$$

(3.21)
The generalization of the Polyakov measure to the space of surfaces that can be uniformized by infinitely-generated groups of Schottky type can be achieved by extending the upper limits of the products over the multipliers and fixed points to infinity.

Infinitely-generated Schottky groups only uniformize a specific category of Riemann surfaces. If $\Gamma$ is a group of Schottky type with either a finite or infinite number of generators with non-overlapping isometric circles, and $D$ is the set of ordinary points in the extended complex plane, then $\frac{D}{\Gamma}$ is a Riemann surface in the class $O_G$. The proof is based on a demonstration of the divergence of the Poincaré series for the Fuchsian group uniformizing the surface [1]. Given the existence of a Schottky covering and a universal covering of the surface by the unit disk, there are conformal equivalences $\Sigma \cong D/\Gamma \cong \tilde{D}/G$, so that there exists coordinate neighbourhoods $N_S$ of $z_S \in D$ and $N_U$ of $z_U \in \tilde{D}$ such that there exist homeomorphisms $\Phi_0 : N_U \rightarrow N_S$ and $\Phi_\alpha : \tilde{T}_\alpha N_U \rightarrow T N_S$, where $T \in \Gamma$ is the image of $T_\alpha \in G$ obtained by mapping the homology generators $A_1, \ldots, A_g, \ldots$ to the identity.

It follows that the Poincaré series for the Fuchsian group is

$$\sum_{T \in G} |\tilde{T}'(z_U)| = \sum_{T \in \Gamma} \sum_{\alpha} |\tilde{T}'(z_U)| = \sum_{T \in \Gamma} \sum_{\alpha} |\Phi_\alpha^{-1}(T\Phi_0(z_U))\Phi_0'(z_U)||T'(z_S)|| \quad (3.22)$$

Let $\tilde{T}_\alpha = A_{i_1} B_{j_1} \ldots A_{i_{N_2}} B_{j_{N_1}}$ where

$$A_{i_1} z_U = \frac{\alpha_{i_1} z_U + \beta_{i_1}}{\gamma_{i_1} z_U + \delta_{i_1}} \quad B_{j_1} z_U = \frac{\alpha_{j_1} z_U + \beta_{j_1}}{\gamma_{j_1} z_U + \delta_{j_1}} \quad (3.23)$$

It follows that

$$|\gamma_{A_{i_1} B_{j_1}}|^{-2} = |\gamma_{i_1}|^{-2}_{\text{hyp.}} |\gamma_{j_1}|^{-2}_{\text{hyp.}} \left| \frac{\alpha_{j_1}}{\gamma_{j_1}} + \frac{\delta_{i_1}}{\gamma_{i_1}} \right|^{-2} \quad (3.24)$$

and

$$|\gamma_\alpha|^{-2} = |\gamma_{i_1}|^{-2}_{\text{hyp.}} |\gamma_{j_1}|^{-2}_{\text{hyp.}} \ldots |\gamma_{i_{N_2}}|^{-2}_{\text{hyp.}} |\gamma_{j_{N_1}}|^{-2}_{\text{hyp.}} \cdot \left| \frac{\alpha_{i_1}}{\gamma_{i_1}} + \frac{\delta_{i_{N_2}}}{\gamma_{i_{N_2}}} \right|^{-2} \left| \frac{\alpha_{j_{N_1}}}{\gamma_{j_{N_1}}} + \frac{\delta_{j_{N_1}}}{\gamma_{j_{N_1}}} \right|^{-2} \quad (3.25)$$

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Since \( |T'(z_S)| = |\gamma|^{-2} z_S + \frac{\delta}{\gamma_{\text{Euc.}}} \) and

\[
|\gamma|^{-2} = |\gamma_{j_1}|^{-2}_{\text{Euc.}} ... |\gamma_{j_{N_1}}|^{-2}_{\text{Euc.}} \left[ \frac{\alpha_{j_{N_1}-1}}{\gamma_{j_{N_1}-1}} + \frac{\delta_{j_{N_1}}}{\gamma_{j_{N_1}}} \right]^{-2} ... \left[ \frac{\alpha_{j_1}}{\gamma_{j_1}} + \frac{\delta_{B_{j_2}...B_{j_{N_1}}}}{\gamma_{B_{j_2}...B_{j_{N_1}}}} \right]^{-2}
\]

\[
|\tilde{T}_\alpha(z_U)| = \left| \frac{\alpha_{i_{N_2}}}{\gamma_{i_{N_2}}} + \frac{\delta_{j_{N_1}}}{\gamma_{j_{N_1}}} \right|^{-2}_{\text{hyp.}} ... \left[ \frac{\alpha_{i_1}}{\gamma_{i_1}} + \frac{\delta_{B_{j_1}...A_{i_{N_2}}B_{j_{N_1}}}}{\gamma_{B_{j_1}...A_{i_{N_2}}B_{j_{N_1}}}} \right]^{-2}_{\text{hyp.}} \left[ \frac{\alpha_{j_{N_1}-1}}{\gamma_{j_{N_1}-1}} + \frac{\delta_{j_{N_1}}}{\gamma_{j_{N_1}}} \right]^{-2}_{\text{Euc.}} ... \left[ \frac{\alpha_{j_1}}{\gamma_{j_1}} + \frac{\delta_{B_{j_2}...B_{j_{N_1}}}}{\gamma_{B_{j_2}...B_{j_{N_1}}}} \right]^{-2}_{\text{Euc.}} |T'(z_S)|
\]

(3.26)

For a given \( T \in \Gamma \), represented by a word of length \( N_1 \), a genus \( g \) can be defined as the number of handles of the smallest compact subset of the Riemann surface whose fundamental group projects to the minimal subgroup of \( \Gamma \) containing \( T \). The number of irreducible words of length \( N_1 + N_2 \) in \( G \) which project to \( T \) is \( 2g(2g - 1)^{N_2 - 1} \frac{(N_1 + N_2)!}{N_1!N_2!} \)

and since \( f(z_U, z_S, \gamma_{i_1}, ... \gamma_{i_{N_2}}, \gamma_{j_1}, \gamma_{j_{N_1}}) \prod_{k_1=1}^{N_1+N_2-1} d_{k_1}^{-2} \prod_{k_2=1}^{N_1-1} d_{k_2}^{-2} |T'(z_S)| = |\tilde{T}_\alpha(z_U)| \),

where \( f \) is a function of the radii of the arcs and isometric circles for \( G \) and \( \Gamma \) respectively, \( \{d_{k_1}\} \) are the hyperbolic distances between the centers of the arcs corresponding to the homology generators \( A_{i_{k_1}}, k_1 = 1, ..., N_2 \), \( B_{j_{N_1}+N_2-k_1}, k_1 = N_2 + 1, ..., N_1 + N_2 - 1 \) and the center of the arc for \( B_{j_{N_1}} \) in the unit disk and \( \{d_{k_2}\} \) are the Euclidean distances between the centers of the isometric circles,

\[
\sum_{T \in G} |\tilde{T}'(z_U)| > \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \left( \frac{2g}{2g - 1} \right) \frac{2g}{2g - 1}^{N_2} f(z_U, z_S, \gamma_{i_1}, ... \gamma_{i_{N_2}}, \gamma_{j_1}, \gamma_{j_{N_1}}) \prod_{k_1=1}^{N_1+N_2-1} d_{k_1}^{-2} \prod_{k_2=1}^{N_1-1} d_{k_2}^{-2} \frac{(N_1 + N_2)!}{N_1!N_2!} \sum_{T \in \Gamma_{N_1}} |T'(z_S)|
\]

(3.28)

with \( \Gamma_{N_1} \) denoting the elements of \( \Gamma \) consisting of the product of \( N_1 \) generators or their inverses. The Euclidean radii of the arcs must tend to zero at least as fast as \( \frac{1}{n^\alpha} \), \( \alpha > 1 \) so that the fundamental domain of the infinitely generated group \( G \) lies within the unit disk. Since even the hyperbolic radii of the some of the arcs may be arbitrarily small, it follows that the hyperbolic distances \( d_{k_1} \) could tend to zero, leading immediately to a divergent product \( \prod_{k_1=1}^{N_1+N_2-1} d_{k_1}^{-2} \). The product of the distances \( \{d_{k_2}\} \) does not affect the divergence of the sum over \( N_2 \). By definition of the parameters \( \gamma_{j_1}, ..., \gamma_{j_{N_2}} \) and the dilation
of the distances in the hyperbolic metric, the ratios \( \frac{|\gamma_{i_1}|_{Euc.}}{|\gamma_{j_1}|_{hyp.}}, ... , \frac{|\gamma_{N_2}|_{Euc.}}{|\gamma_{N_2}|_{hyp.}} \) can be set equal to one. Although the radii \( |\gamma_{i_1}|_{hyp.}, ..., |\gamma_{N_2}|_{hyp.} \) can decrease to zero, the distances \( d_{k_1} \) simultaneously will tend to zero so that the product of the radii will be compensated by the product \( \prod_{k_1=1}^{N_1+N_2-1} d_{k_1} \) if the final element \( B_{jN_2} \) has an arc with an infinitesimally small radius. Even when the distances \( d_{k_1} \) do not tend to zero, \( (2g-1)N_2 \prod_{n=1}^{N_2} \frac{1}{n^\alpha} \) diverges for fixed \( \alpha > 1 \) as \( N_2 \to \infty \).

The infinite factor multiplying the non-vanishing Poincare series for the restricted Schottky group \( \sum_{T \in \Gamma_{N_1}} |T'(z\mathbf{g})| \) causes the Poincare series for the Fuchsian group to diverge. Since the uniformizing group is a Fuchsian group of the first kind, the surface belongs to the class \( O_G \).

For infinite-genus hyperelliptic surfaces, given a diagonal matrix \( T = diag(t_1,t_2,...) \) with \( t_j > 0, \ j \geq 1 \) and real numbers \( p > 1, C > 0 \), there is a sequence \( \Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 < ... \to \infty \} \) such that the period matrix \( \tau \) of \( \Sigma(\lambda) \) is purely imaginary and satisfies \( |(\tau - it)_{ij}| \leq \frac{C}{t_i t_j} \) for all \( i,j \leq 1 \) [21]. Given a sequence of ramification points \( \lambda_{2i-1}, \lambda_{2i} \geq C i^2, \ i \geq 1 \), let

\[
\begin{align*}
    a_1 &= 2 \int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\left( \lambda \left( \frac{\lambda}{\lambda_1} - 1 \right) \left( 1 - \frac{\lambda}{\lambda_2} \right) \right)^{3/2}}, \\
    a_i &= 2 \int_{\lambda_{2i-1}}^{\lambda_{2i}} \frac{d\lambda}{\left( \frac{\lambda}{\lambda_{2i-2}} - 1 \right) \left( \frac{\lambda}{\lambda_{2i-1}} - 1 \right) \left( 1 - \frac{\lambda}{\lambda_{2i}} \right)^{3/2}} \quad i > 1 \\
    R(\lambda) &= \left( -\lambda \prod_{i=1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_{2i-1}} \right) \right)^{3/2}, \\
    \varphi_i(\lambda) &= a_i^{-1} \prod_{1 \leq k \leq \infty} \left( 1 - \frac{\lambda}{\mu_k} \right) \quad \mu_i = \sqrt{\lambda_{2i} \lambda_{2i-1}}
\end{align*}
\]

The holomorphic differentials \( \phi_i = \varphi_i(\lambda) R(\lambda) \) with norm \( \| \phi_i \| = \left( \frac{i}{2} \int_{C} \left| \frac{\varphi_i(\lambda)}{R(\lambda)} \right|^2 d\lambda \wedge d\bar{\lambda} \right)^{1/2} \) form a basis of smooth, closed square-integrable one-forms on \( \Sigma \). If \( \omega, \sigma \) are linear combinations of \( \phi_i \), then

\[
\int_{\Sigma} \omega \wedge \bar{\sigma} = \sum_{i=1}^{\infty} \left[ \int_{A_i} \omega \int_{B_i} \bar{\sigma} - \int_{A_i} \bar{\sigma} \int_{B_i} \omega \right]
\]

If \( A = (A_{ij})_{i,j=1}^{\infty}, \ A_{ij} = \int_{A_i} \phi_j \), then there exists a bounded inverse \( A^{-1} \) such that \( |(A^{-1})_{ij}| \leq \delta_{ij} + \frac{C}{t^P} \rho(i,j) \) and \( \psi_j = \sum_{k=1}^{\infty} (A^{-1})_{kj} \phi_k \) for each \( j \in \mathbb{N} \), the series converges.
in $L^2$ to a square integrable one-form $\psi_j$[21]. The period matrix $\tau_{ij} = \int_{B_j} \psi_j$ is symmetric and $Im \, \tau$ is positive-definite, implying the existence of an inverse.

The determinant factors in the bosonic string measure on the space of hyperelliptic surfaces [22] parameterized by the branch points $\{a_i\}$ are

\[
\begin{align*}
\det' \partial_0 &= \prod_{i<j} (a_i - a_j)^{\frac{1}{4}} \\
\det' \partial_2 &= \prod_{i<j} (a_i - a_j)^{\frac{5}{4}}
\end{align*}
\] (3.31)

so that $\mu_g = (\det' \partial_0)^{-13}(\det' \partial_2) = \prod_{i<j} (a_i - a_j)^{-2}$ and

\[
Z_g^{hyp.} \sim \int \prod_k d^2 a_k \prod_{i<j} |a_i - a_j|^{-4}(\det Im \, \tau)^{-13}
\] (3.32)

The exclusion of surfaces which have fused handles at points other than the base is apparent in the hyperelliptic representation. Although hyperelliptic surfaces have the property that they possess an involution, so that the branch points all lie on the real axis, for example, if the $2g + 2$ branch points are allowed to have an arbitrary location in the extended complex plane, such that the lines joining $a_{2i}$ and $a_{2i+1}$ do not intersect for any $i$, then all orientable closed Riemann surfaces with a finite number of disjoint handles can be obtained.

The generalization of the moduli space measure to superstrings introduces a dependence on the spin structure, which is the choice of square root of the cotangent bundle and can be viewed as the selection of signs for a fermion as it traverses around the homology cycles. The holomorphic part of the measure for the Neveu-Schwarz sector [23], consisting of even spin structures, is

\[
\frac{1}{dV_{ABC}} \prod_{n=1}^g \frac{dK_n}{K_n^{\frac{3}{4}}} \frac{dZ_{1n}dZ_{2n}}{K_n^{\frac{3}{4}}} \left( \frac{1 - K_n}{1 - (-1)^{B_n} K_n^{\frac{3}{4}}} \right)^2 (\det Im \, T)^{-5}
\] (3.33)

\[
\cdot \prod_{\alpha} \prod_{p=1}^\infty \left( \frac{1 - (-1)^{N_{ap}^B} K_\alpha^{p-\frac{1}{2}}}{1 - K_\alpha^{p-\frac{1}{2}}} \right)^{10} \prod_{\alpha} \prod_{p=2}^\infty \left( \frac{1 - K_\alpha^p}{1 - (-1)^{N_{ap}^B} K_\alpha^{p-\frac{1}{2}}} \right)^2
\] (3.34)

with

\[
dV_{ABC} = \frac{dZ_A dZ_B dZ_C}{[(Z_A - Z_B)(Z_C - Z_A)(Z_B - Z_C)]^{\frac{5}{2}}} \cdot \frac{1}{d\Theta_{ABC}}
\] (3.35)
\[
\Theta_{ABC} = \frac{\theta_A(Z_B - Z_C) + \theta_B(Z_C - Z_A) + \theta_C(Z_A - Z_B) + \theta_A\theta_B\theta_C}{[(Z_A - Z_B)(Z_C - Z_A)(Z_B - Z_C)]^{1/2}}
\] (3.36)

and the super-period matrix is

\[
T_{mn} = \frac{1}{2\pi i} \left[ \ln K_n \delta_{mn} + \sum_{\alpha} (m,n) \ln \left[ \frac{Z_{1m} - V_{\alpha}Z_{1n}}{Z_{1m} - V_{\alpha}Z_{2n}} \right] \right]
\] (3.37)

where \(Z_{1n}, Z_{2n}\) are the superfixed points of the generators of the super-Schottky group. The primitive-element products in the measure can be derived from the superdeterminants of differential operators

\[
\begin{align*}
sdet' \bar{D}_0 &= \prod_{\alpha} \prod_{p=1}^{\infty} \left( \frac{1 - K_\alpha^p}{1 - K_\alpha^{p-\frac{1}{2}}} \right)^2 \\
sdet' \bar{D}_{-1} &= \prod_{\alpha} \prod_{p=2}^{\infty} \left( \frac{1 - K_\alpha^p}{1 - K_\alpha^{p-\frac{1}{2}}} \right)^2
\end{align*}
\] (3.38)

where \(D_Z = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}\).

The superstring amplitude in terms of Fuchsian parameters [24] is

\[
Z_g = \int_{sM_g} d\mu_{sWP} \left[ sdet'(-\triangle_0^2) \right]^{-\frac{5}{2}} \left[ sdet(-\triangle_2^2) \right]^{\frac{1}{2}}
= \left( \frac{1}{2\pi^4} \right)^{g-1} \int_{sM_g} d\mu_{sWP} \left( -1 \right)^{\frac{5}{2} \triangle_{00}^0} \left( \frac{Z_0(1)}{Z_1(\frac{1}{2})} \right)^5 \frac{Z_0(2)}{Z_1(\frac{3}{2})} \left( \frac{Z_1(1)}{Z_1(0)} \right)^2
\] (3.39)

where the super-Selberg zeta functions are

\[
Z_q(s) = \prod_{\{\gamma\}_p} \prod_{k=0}^{\infty} \left[ 1 - \chi_\gamma^q e^{-(s+k)l_\gamma} \right] \quad \text{Re } s > 1
\] (3.40)

with \(q = 0, 1, \chi_\gamma\) denoting the spin structure, \(l_\gamma\) being the length of the geodesic corresponding to the primitive element \(\gamma\), and \(\triangle_{00}^0\) equaling the number of even zero-modes minus the number of odd zero modes of the Laplace-Dirac operator.

By analogy with the bosonic string partition function,

\[
Z_g = \int_{M_g} d\mu_{WP} Z(2)Z'(1)^{-13}
\] (3.41)
where $Z(s)$ is the ordinary Selberg zeta function [25], an expression for the integrand of the superstring partition function in terms of branch points has been obtained for the set of hyperelliptic surfaces [26].

A proof of the equivalence of the bosonic string measure in the Fuchsian and Schottky parameterizations can be given by deriving both expressions from the path integral

$$Z_g = \int \frac{[Dh][DX]}{Vol(Diff(\Sigma)) \times Vol(Weyl(\Sigma))} e^{-S(g,X)} \quad (3.42)$$

where $S$ is the Polyakov action. The evaluation of the determinants of the Laplacians on genus-$g$ surfaces in the Fuchsian representation gives rise to the Selberg zeta functions. In the Schottky parameterization, the determinants of the differential operators are obtained on a sphere with with $g+1$ disks, with the string configurations specified on the boundaries of the sphere, and then this surface is glued to its Schottky double [27]. The equivalence has been demonstrated for manifolds which do not have fused handles, which would follow from the hyperbolicity of products of hyperbolic generators of the Schottky group $\Gamma$, implying the non-overlap of the isometric circles.

### 3.3. Dirichlet Boundaries and Non-Perturbative Effects

Attaching a disk to the Riemann surface, it follows that the non-perturbative effect of adding a Dirichlet boundary is weighted by a factor of $exp(-S_{disk})$. Since $S_{disk} \sim c_{str} \kappa$, the contribution of the Dirichlet boundary is weighted by $e^{-c_{str} \kappa}$ [28].

Since gravitational diagrams can be regarded as the square of Yang-Mills diagrams in $N = 8$ supergravity, the coupling constants can be related as $\kappa_{grav} \sim \kappa_{YM}^2$ [29]. The low-energy limit of the non-perturbative contribution to the string amplitude is therefore consistent with the weighting of non-perturbative effects in non-abelian gauge field theories, $e^{-\frac{c_{str} \kappa}{\kappa_{YM}^2}}$.

The addition of these Dirichlet boundaries can be viewed as a separate contribution to the string path integral. This formulation of string theory, then, would be based on a sum over closed finite-genus surface, effectively closed infinite-genus surfaces and surfaces with Dirichlet boundaries inserted. The first two sums arise within the domain of string perturbation theory, with only orientable surfaces for closed strings but non-orientable surfaces also included for open strings. The last sum which represents non-perturbative
effects [30] is only well-defined if the perturbative sum is convergent or can be obtained from analytic continuation from the domain of convergence.

4. Infinite-Genus Surfaces and the String Coupling

4.1. The Contribution of Infinite-Genus Surfaces to the Scattering Amplitude

If an infinite-genus surface has a single end, there is an exhaustion by a sequence of compact surfaces. In general, the ends may be identified with the points in a Cantor set defined by the division of the interval. Defining the cardinality of the set of ends to be $\text{card } E$, the contribution of infinite-genus surfaces to the $N$-point scattering amplitude

$$
\sum_{n_e=1}^{\text{card } E} \sum_{i=1}^{n_e} \lim_{g \to \infty} \kappa_{\text{str}}^{2g-2} \int_{sM_\infty,(i)} d\mu_{\infty,(i),(L,L')} \cdot \prod_{r=1}^{N} dt_r d\tilde{t}_r V_r(t_r, \tilde{t}_r)_{L,L'} \quad \text{(4.1)}
$$

where $(L, L')$ denotes the spin structure, $(t_r, \tilde{t}_r)$ represent the local coordinates $(z_r, \theta_r, \bar{z}_r, \bar{\theta}_r)$ on the super-Riemann surface and $\kappa_{\text{str}}$ is the string coupling.

The $N$-point $g$-loop graviton Type II superstring amplitude can be determined in the limit $K_1 \sim K_2 \sim \ldots \sim K_g \sim 0$ to have the property of exponentiation in impact space if the coefficient of the integral is

$$
C_g^{II} = \left( \frac{1}{4} \right)^g (g_D)^{2g-2} \left( \frac{1}{2\pi} \right)^{\frac{Dg}{2}+3g-3} (\alpha')^{-2+\frac{g(4-D)}{2}}
$$

where the D-dimensional string coupling is $g_D^2 = \frac{2\pi^2}{\alpha'}$, with $\kappa_{D=4}^2 = 8\pi G_N$ [31]. The same limit implies that the coefficient in the bosonic string amplitude is

$$
C_g^{bos.} = 4g C_g^{II}
$$

For open strings, the tension is given by $T = \frac{1}{2\pi \alpha' [\ell_P]}$ where the factor of $[\ell_P]$ is required for the correct units. More generally, the tension is $\frac{m_P c^2}{\ell_P}$. The mass of the closed string can be found by assuming uniform density in a circular string with radius equal to the length of an open string so that $m_P \to 2\pi m_P$. Because of the oscillatory motion of the string, the effective length of half of the closed string then would be $2\ell_P$. The quantity defining
the tension is then transformed to \( \frac{2\pi m \rho c^2}{16\pi T_{\text{open}}} \) so that the value of \( G_N = \frac{1}{|T|} \) is shifted from \( 2\pi \alpha' \) to \( 16\alpha' \). Then \( g_4^2 = \frac{16\pi G_N}{\alpha'} = 256\pi \), \( g_4 = 16\sqrt{\pi} \), so that

\[
C_g^{II} = \left( \frac{1}{4} \right)^g g_4^{2g-2} \left( \frac{1}{2\pi} \right)^{5g-3} \left( \alpha' \right)^{-2} = \frac{1}{(16\sqrt{\pi})^2(2\pi)^3\alpha'^2} \left( \frac{2}{\pi^4} \right)^g \tag{4.4}
\]

The physical considerations of §3 imply that the infinite-genus moduli space \( \bar{M}_\infty \) should be identified with the category of \( O_G \) surfaces. Since the boundary of the image of a surface conformally mapped to the unit disk is a subset of the circle, the class \( O_G \) can be defined by constraints on integers \( \{p_n\} \), which denote Cantor sets \( E(p_1p_2...) \) obtained by successively dividing the unit interval into \( p_n \) parts and deleting the central subinterval. The generalized Cantor set \( E(p_1p_2...) = \bigcap_{\nu=1}^{\infty} E(p_1...p_n) \) has length \( \prod_{n=1}^{\infty} \left( 1 - \frac{1}{p_n} \right) \) so that the condition of zero linear measure is equivalent to \( \sum_{\nu=1}^{\infty} \frac{1}{p_\nu} = \infty \) [5].

It has been established that if \( r_{n\nu} \) is the Robin constant of \( E(p_{\nu}...p_n) \), then \( r_n = r_{n1} \) satisfies the inequalities [5]

\[
\frac{\log 4}{2^n} + \log \left[ \prod_{\nu=1}^{n} \left( 1 - \frac{1}{p_\nu} \right)^{-2^{-\nu}} \right] + \left( 1 - \frac{1}{2^n} \right) \log 2 \leq r_n \leq \frac{\log 4}{2^n} + \log \left[ \prod_{\nu=1}^{n} \left( 1 - \frac{1}{p_\nu} \right)^{-2^{-\nu}} \right] + \left( 1 - \frac{1}{2^n} \right) \log 2 + \sum_{\nu=1}^{n} \frac{\log p_\nu}{2^\nu} \tag{4.5}
\]

As the condition for the Cantor set \( E(p_1p_2...) \) to belong to the class \( O_G \) is \( \lim_{n\to\infty} r_n = \infty \), the integers \( \{p_n\} \) would be constrained by the relation \( \prod_{\nu=1}^{\infty} \left( 1 - \frac{1}{p_\nu} \right)^{2^{-\nu}} = 0 \) if \( \sum_{\nu=1}^{\infty} \frac{\log p_\nu}{2^\nu} \) is finite. However, since the product equals \( \left( 1 - \frac{1}{p} \right)^{\sum_{\nu=1}^{\infty} 2^{-\nu}} = 1 - \frac{1}{p} \) when \( p_n = p \) for all \( n \), there exists no sequence of positive integers \( \{p_n\} \) with \( p_n \leq p_{n+1} \) such that \( \prod_{\nu=1}^{\infty} \left( 1 - \frac{1}{p_\nu} \right)^{2^{-\nu}} = 0 \). However, when \( \langle p_\nu \rangle \) is allowed to tend to 1 as \( \nu \to \infty \) at a rate of \( \frac{1}{1-k(\nu)} \), where \( \langle p_\nu \rangle \) defines an average number of deletions over the subintervals of \( E(p_1...p_{\nu-1}) \), then a generalized Cantor set of zero capacity would be obtained if \( k(\nu) < e^{-\eta^\nu}, \eta > 2 \).

An average of the form \( \langle p_\nu \rangle = \frac{1}{1-e^{-\eta^\nu}} \) can be obtained if all except one of the subintervals in each of the of the partitions at the \( (\nu - 1)^{\text{th}} \) level into \( \left[ e^{\eta^\nu} \right]_{\text{odd}}, n \geq 3 \) parts are deleted, including the central section, with \( [\alpha]_{\text{odd}} \) denoting the largest odd integer less than \( \alpha \). This partitioning would yield \( \lim_{\nu\to\infty} 2^\nu \) endpoints. Different partitionings give rise to endpoint sets which are equivalent under a conformal transformation.
Given that the cardinality of the endpoint set is \( \text{card } E = 2^N \), the infinite-genus contribution to the scattering amplitude will be multiplied by a factor of \( \lim_{g \to \infty} 2^g \). The coefficient of the infinite-genus amplitude then becomes \( \lim_{g \to \infty} 2^g \left( \frac{2}{\pi^2} \right)^g = \lim_{g \to \infty} \left( \frac{4}{\pi^2} \right)^g \), which implies that the coupling is approximately \( \frac{4}{\pi} \simeq \frac{1}{24.3523} \), a value which is comparable to the unified gauge coupling \( \alpha_{\text{GUT}} \simeq \frac{2\pi}{153} = \frac{1}{24.3507} \) \([32][33]\).

### 4.2. The Volume of the Moduli Space of Effectively Closed Infinite-Genus Surfaces

The symplectic volume \([34]\) of the fundamental domain of the modular group acting on the Siegel upper half space \( H_g \) is

\[
V_g = 2^{g+1} \pi^{g+1} \prod_{k=1}^{g} \left[ \frac{(k-1)!}{(2k)!} B_{2k} \right] = 2 \prod_{k=1}^{g} (k-1)! \pi^{-k} \zeta(2k) \tag{4.6}
\]

The projection of the fundamental domain from \( H_g \) to Teichmüller space leads to a change in the exponent in the first two factors in the formula for \( V_g \). Lowering the dimension of the space from \( \frac{1}{2}g(g+1) \) to \( 3g-3 \) implies that \( \prod_{k=1}^{g} (k-1)! \to \prod_{k=3}^{g} (k-2) = 3^{g-2}(g-2)! \), \( \prod_{k=1}^{g} \pi^{-k} \to \pi^{3-3g} \), and \( V_g \to 2 \cdot 3^{g-2}(g-2)! \pi^{3-3g} \prod_{k=1}^{g} \zeta(2k) \).

The differential of the diagonal element of the period matrix is

\[
d\tau_{nn} = \frac{1}{2\pi i} \left[ \frac{dK_n}{K_n} + \sum_{\alpha} \langle (m,n) \rangle \text{ln} \left( \frac{\xi_{1m} - V_{\alpha} \xi_{2n} \xi_{2m} - V_{\alpha} \xi_{2n}}{\xi_{1m} - V_{\alpha} \xi_{2n} \xi_{2m} - V_{\alpha} \xi_{1n}} \right) dK_m \right.
\]

\[
+ \sum_{\alpha} \langle (n,m) \rangle \text{ln} \left( \frac{\xi_{1n} - V_{\alpha} \xi_{1m} \xi_{2m} - V_{\alpha} \xi_{1n}}{\xi_{1n} - V_{\alpha} \xi_{2m} \xi_{2m} - V_{\alpha} \xi_{1n}} \right) d\xi_{2n}
\]

\[
+ \sum_{\alpha} \langle (n,m) \rangle \text{ln} \left( \frac{\xi_{1n} - V_{\alpha} \xi_{1m} \xi_{2m} - V_{\alpha} \xi_{2n}}{\xi_{1n} - V_{\alpha} \xi_{2m} \xi_{2m} - V_{\alpha} \xi_{1n}} \right) d\xi_{1m} \tag{4.7}
\]

and since

\[
\frac{d}{dK_m} T_{m11} = \frac{(\xi_{1n} - \xi_{2m})(\xi_{1m} - \xi_{2m})(\xi_{1n} - \xi_{1m})}{[(\xi_{1n} - \xi_{2m}) - K_m(\xi_{1n} - \xi_{1m})]^2} \tag{4.8}
\]

the coefficient of \( dK_n \) is of order \( O(1) \). The integral

\[
\int_0^\infty \frac{dx}{(\text{ln} \left( \frac{1}{x} \right))^{g+1}} = (-1)^{g+1} \frac{1}{g!} \text{li}(x) - (-1)^{g+1} x \left[ \frac{1}{g!} \left( \text{ln} \ x \right)^0 + \frac{1}{g!} \left( \text{ln} \ x \right)^2 + \ldots + \frac{1}{g!} \left( \text{ln} \ x \right)^g \right] \tag{4.9}
\]

contains a term of maximal magnitude

\[
\frac{1}{g(g-1)\ldots(g-k)} \left( \text{ln} \left( \frac{1}{x} \right) \right)^{g-k} \tag{4.10}
\]
when
\[
\frac{d}{dk} \left[ \frac{1}{g(g-1)\ldots(g-k)} \frac{1}{(\ln \left( \frac{1}{e} \right))^{g-k}} \right] \approx 0
\] (4.11)
or equivalently \( \psi(g - k + 1) - \ln (\ln (\frac{1}{e})) = 0 \) which has solution \( k \approx g - \ln (\frac{1}{e}). \) The magnitude of the dominant term, when the integrand includes \( [det \text{Im } \tau]^{-5} \) instead of \( [det \text{Im } \tau]^{g+1} \), is altered by a factor of \( (\frac{g}{4})^{g} (\ln (\frac{1}{e}))^{g} \) after integration over all \( g \) multipliers \( K_n. \)

Since the symplectic form
\[
\frac{1}{2} g^{(g+1)} \frac{\partial \tau_{ij}}{\partial \bar{\tau}_{ij}} \prod_{(i,j)} [det \text{Im } \tau]_{ij}^{g+1}
\] (4.12)
yields a volume element containing
\[
\frac{d^2 K_1 \, d^2 K_2 \, d^2 K_g}{|K_1|^2 \, |K_2|^2 \cdots |K_g|^2} \frac{1}{[det \text{Im } \tau]^{g+1}}
\] (4.13)
finiteness in the \( |K_n| \rightarrow 0 \) limit follows for both measures. The change in the integral over \( |K_n| \) from \( \int \frac{d|K_n|}{|K_n|(\ln (\frac{1}{|K_n|}))^{g+1}} \) to \( \int \frac{d|K_n|}{|K_n|(\ln (\frac{1}{|K_n|}))^{g+1}} \) provides an extra factor of \( (\frac{g}{4})^{g} (\ln (\epsilon'))^{g(g-4)}. \) The presence of additional terms in the symplectic measure implies that there is no direct correspondence between the genus-dependence of the symplectic volume and the superstring integral.

The integral over the superstring measure can be evaluated partially by using a parameterization of supermoduli space. The super-Schottky space can be divided into two domains: the first, defined by limits which are consistent with a genus-independent cut-off for the length of closed geodesics on the super-Riemann surface and the second, the remainder of the space, representing a neighbourhood of the compactification divisor.

In another set of coordinates on supermoduli space \( \{K_n, B_m, H_m, \theta_{1i}, \theta_{2i}\} \), valid over a subset of supermoduli space complementary to a neighbourhood of the compactification divisor, the holomorphic part of the integral over the super-fixed points is replaced by
\[
\prod_{m=2}^{g} dB_m \prod_{m=2}^{g-1} dH_m \prod_{i=1}^{g-1} d\theta_{1i} \prod_{i=1}^{g} d\theta_{2i}
\] (4.14)
Since the genus-g superstring measure has the form \([F(y)]^2 [sdet(\phi|\phi)]^{-5}\), where \( \{\phi_i\} \) are super-holomorphic half-differentials [35], modular invariance implies that \( F(y) \sim \frac{1}{y^g} \) independently of the component of the boundary of supermoduli space. The superdeterminant
factor is
\[ \text{sdet} \langle \phi | \phi \rangle \sim \frac{1}{\ln|y|} \]
for an \( A \) - cycle
\[ \text{sdet} \langle \phi | \phi \rangle \sim 1 \]
for a \( B \) - or \( C \) - cycle
\[ (4.15) \]

Under a modular transformation \( \sigma_1 \) of the surface which maps an \( A \)-cycle to a \( B \)-cycle, the dependence is
\[ \text{sdet} \langle \sigma_1^* \phi | \sigma_1^* \phi \rangle \sim \frac{1}{\ln|y|} \]
for the \( A \) - cycle
\[ (4.16) \]

and under the transformation \( \sigma_2 \) from an \( A \)-cycle to a \( C \)-cycle,
\[ \text{sdet} \langle \sigma_2^* \phi | \sigma_2^* \phi \rangle \sim \frac{1}{\ln|y|} \]
for the \( A \) - cycle
\[ (4.17) \]

Applying the modular transformation from an \( A \)-cycle to a \( B \)-cycle, the holomorphic part of the measure, multiplied by the determinant factor, changes from
\[ \prod_{n=1}^{g} \frac{dK_n}{K_n^2} \prod_{m=2}^{g-1} \frac{dB_m}{B_m^2} \prod_{m=2}^{g-1} \frac{dH_m}{H_m^2} \prod_{i=1}^{g} d\theta_{1i} \prod_{i=1}^{g} d\theta_{2i} \left( \frac{1 - K_n}{1 - (-1)^B_n K_n^2} \right)^2 \]
\[ \left[ \text{det Im } T \right]^{-5} \prod_{\alpha}^{\prime} \prod_{p=1}^{\infty} \left( \frac{1 - (-1)^{N^B_n} K^p_{\alpha} - \frac{1}{2}}{1 - K^p_{\alpha}} \right)^{10} \prod_{p=2}^{\infty} \left( \frac{1 - K^p_{\alpha}}{1 - (-1)^{N^B_n} K^p_{\alpha} - \frac{1}{2}} \right)^2 \]
\[ (4.18) \]

to
\[ \prod_{n'=1}^{g} \frac{dK_n}{K_n^2} \prod_{m=2}^{g-1} \frac{dB_m}{B_m^2} \prod_{m=3}^{g-1} \frac{dH_m}{H_m^2} \prod_{i=1}^{g} d\theta_{1i} \prod_{i=1}^{g} d\theta_{2i} \left( \frac{1 - K_n(H_2, K_n, \xi_{1n}, \xi_{2n})}{1 - (-1)^{B_n} K_n^2} \right)^2 \]
\[ \left( \frac{1 - H_2}{1 - (-1)^B_n H_2^2} \right)^2 \left[ \text{det Im } T \right]^{-5} \prod_{\alpha}^{\prime} \prod_{p=1}^{\infty} \left( \frac{1 - (-1)^{N^B_n} K^p_{\alpha} - \frac{1}{2}(H_2, K_n, \xi_{1n}, \xi_{2n})}{1 - K^p_{\alpha}} \right)^{10} \prod_{p=2}^{\infty} \left( \frac{1 - K^p_{\alpha}(H_2, K_n, \xi_{1n}, \xi_{2n})}{1 - (-1)^{N^B_n} K^p_{\alpha} - \frac{1}{2}} \right)^2 \]
\[ (4.19) \]
whereas the modular transformation from an A-cycle to a C-cycle yields

$$
\frac{dK_2}{K_2^3} \frac{dB_2}{B_2^3} \prod_{n=1}^{9} \frac{dK_n}{K_n^3} \prod_{m=3}^{9} \frac{dB_m}{B_m^3} \prod_{n=2}^{9} \frac{dH_m}{H_m^3} \prod_{m=1}^{9} d\theta_1 \prod_{m=1}^{9} d\theta_2 \left( \frac{1 - K_n(B_2, K_n, \xi_{1n}, \xi_{2n})}{1 - (-1)^{B_n} K_n^{\frac{p}{2}}} \right)^2
$$

$$
\left( \frac{1 - B_2}{1 - (-1)^{B_1} B_2^{\frac{p}{2}}} \right)^2 [\det \text{Im } T]^{-5} \prod_{\alpha} \prod_{p=1}^{10} \left( \frac{1 - (-1)^{\frac{N_{A\alpha} B}{2}} (B_2, K_n, \xi_{1n}, \xi_{2n})}{1 - K_{A\alpha}^p} \right)^2
$$

Finiteness of the integrals of these measures, over the integration domains $\sigma_1(F_g)$ and $\sigma_2(F_g)$ respectively, can be verified by evaluating the $|H_m| \to 0$ and $|B_m| \to 0$ limits.

The modular transformation $A_n \to B_n, B_n \to -A_n$, which has the effect of mapping the entry $\tau_{nn}$ of the period matrix to $-\tau_{nn}^{-1}$, interchanges the limit $|K| \to 0$ with $|K| \to 1$. If $K \to 1, |K_{A\alpha}| \to 1$. Since $\lim_{K_{A\alpha} \to 1} \left( \frac{1 - K_{A\alpha}^p}{1 - K_{A\alpha}^{p-\frac{p}{2}}} \right) = \frac{p - \frac{p}{2}}{p}$, the product of the two primitive-element factors for each $\alpha$ in this limit would be $\prod_{p=1}^{10} \left( \frac{p - \frac{p}{2}}{p} \right)^{10} \prod_{p=2}^{10} \left( \frac{p - \frac{p}{2}}{p} \right)^2 = \frac{1}{2} \cdot \prod_{p=2}^{10} \left( \frac{p - \frac{p}{2}}{p} \right)^8 = 0$ when $N_{A\alpha}$ is even. If $N_{A\alpha}$ is odd, the product diverges in the limit $K_{A\alpha} \to 1$, but it is well defined if $\text{arg } K_{A\alpha} \neq 0$. The non-zero value of $\text{arg } K_{A\alpha}$ follows from the formula

$$
K_{T^2} = K_T^2 \left( \frac{\xi_{1T} + \delta_T}{T(\xi_{1T}^2)} + \frac{\xi_{1T} + \delta_T}{\xi_{1T}^2 + \delta_T} \right) \tag{4.21}
$$

as

$$
\text{arg } K_{T^2} = 2 \text{ arg } K_T + 2 \text{ arg } (\xi_{1T} + \delta_T) - \text{ arg } (T(\xi_{1T}^2) + \delta_T) - \text{ arg } (\xi_{1T}^2 + \delta_T)
$$

$$
= 2 \text{ arg } (\xi_{1T} + \delta_T) - \text{ arg } (T(\xi_{1T}^2) + \delta_T) - \text{ arg } (\xi_{1T}^2 + \delta_T) \tag{4.22}
$$

Divergences in the $H_m, B_m$ integrals in the Neveu-Schwarz sector must be cancelled in a sum over all spin structures. While a modular transformation maps the entire $\{K_n, H_m, B_m\}$ fundamental domain to another integration region in this parameter space, the divergences are cancelled amongst each set of coordinates $\{K_n\}, \{H_m\}$ or $\{B_m\}$. To obtain the finite remainder, it is sufficient to consider the effect of the modular transformation on the integrals over each set of coordinates. Divergent $H_m, B_m$ integrals can be mapped to $K_n$
integrals in another sector which will yield a finite contribution to the amplitude upon summation over paired spin structures. The integrals can be evaluated given the range of the multiplier in the fundamental domain and its image under the modular transformation.

Since \( \frac{|\gamma_n|^{-2}}{|\xi_{1n}-\xi_{2n}|^2} = \frac{|K_n|}{|1-K_n|^2} \), it follows that if \( |\gamma_n|^{-2} \) is bounded below, the limits \( |K_n| \to 0 \) and \( |\xi_{1n}-\xi_{2n}| \to 0 \) cannot be taken simultaneously. If \( |\gamma_n|^{-2} \) is allowed to vanish, then the limit \( |K_n| \to 0 \) is consistent with a finite value of \( |\xi_{1n}-\xi_{2n}| \) and the limit \( |\xi_{1n}-\xi_{2n}| \to 0 \) is consistent with a finite value of \( |K_n| \).

At genus 3, for instance, there are sequential pairwise cancellations of divergences in the \( K_n \) integrals in the Neveu-Schwarz sector: \((--++--), (--++--), (--++--), (--++--), (--++--), (--++--), (--++--), (--++--), (--++--), (--++--). The finiteness of the \( K_n \) integrals can be demonstrated for other sectors obtained by modular transformations of the Neveu-Schwarz and Ramond spin structures. The application of the \( A_n \to B_n, B_n \to -A_n \) transformations to the entire set of genus-3 spin structures leads to cancellations of divergences in the \( H_m \) integrals in the region \( \sigma(F_3) \). In this domain, the range of the coordinate \( H_m \) is determined by the limits by its equivalent role to the \( K_m \) variable.

Denoting the modular transformation which switches \( A- \) and \( B- \) cycles on all three handles by \( \sigma \), the transformed spin structures may be listed as

\[
\begin{align*}
\sigma(NS) : & \quad (+ - + - -) \\
& \quad (+ - + - -) \\
& \quad (+ - - + -) \\
& \quad (+ - - + -) \\
& \quad (- + + + -) \\
& \quad (- + + + -) \\
& \quad (- + + + -) \\
& \quad (- + + + -) \\
& \quad (- - - - -) \\
\sigma(R) : & \quad (+ + + + -) \quad (+ + + + +) \\
& \quad (- + + + -) \quad (- + + + +) \\
& \quad (+ - + + +) \quad (+ - + - +) \\
& \quad (- + + + +) \quad (- + + + -)
\end{align*}
\]
\[ \sigma(S_3): \begin{array}{ll}
(++++--) & (++---) \\
++++-- & ++--- \\
(--++) & (++++) \\
++-- & ++-- \\
(---++) & (++++) \\
\end{array} \]

\[ \sigma(S_4): \begin{array}{ll}
(--+-++) & (+++) \\
+-+-++ & +++-+ \\
(+-+-+) & (++++) \\
+-++ & ++-- \\
(---+) & (+++) \\
\end{array} \]

\[ \sigma(S_5): \begin{array}{ll}
(+-+-+--) & (+++) \\
+-+-+++ & ++-- \\
+-+-+ & ++-- \\
+-++ & ++-- \\
(---++) & (+++) \\
\end{array} \]

\[ \sigma(S_6): \begin{array}{ll}
(+-+-+--) & (+++) \\
+-+-+++ & ++-- \\
+-+-+ & ++-- \\
+-++ & ++-- \\
(---++) & (+++) \\
\end{array} \]

\[ \sigma(S_7): \begin{array}{ll}
(+-+-+--) & (+++) \\
+-+-+++ & ++-- \\
+-+-+ & ++-- \\
+-++ & ++-- \\
(---++) & (+++) \\
\end{array} \]

\[ \sigma(S_8): \begin{array}{ll}
(+-+-+--) & (+++) \\
+-+-+++ & ++-- \\
+-+-+ & ++-- \\
+-++ & ++-- \\
(---++) & (+++) \\
\end{array} \]

The cancellations in the \( H_m \) integrals occur between spin structures in different sectors, such as \((+++--+)\) in \( \sigma(R) \) and \((+++--+)\) in \( \sigma(S_3) \).

Modular transformations of the type \( A_n \to B_n, B_n \to -A_n \) could be used to obtain the integrand and domain for the odd spin structures given an expression for the Ramond measure. An odd spin structure can be obtained from an even spin structure with a genus-one component \((--)\) by changing the periodicity of the fermion from \( \psi \to -\psi \) to \( \psi \to \psi \)

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Upon traversal of either the A- or B-cycle, so that the spin structure for this component is then \((++\))
. This transformation maps the spin structures of the Neveu-Schwarz sector
\((-\sigma_{h_1} \ldots -\sigma_{h_g})\), \(\sigma_{h_i} = \pm 1, i = 1, \ldots, g\) to the spin structures of the Ramond sector
\((+(-\sigma_{h_1}) \ldots +(-\sigma_{h_g}))\). This can be viewed as a coordinate change on the super-Riemann surface \(z \to z, \theta \to -\theta\).

Under \(\text{SPL}(2;\mathbb{C})\) transformation \([24]\)
\[
\begin{align*}
z' &= \frac{az + b}{cz + d} + \theta \frac{\alpha z + \beta}{(az + b)^2} \\
\theta' &= \frac{\alpha + \beta z}{cz + d} + \frac{\chi_T \theta}{cz + d}
\end{align*}
\]
\(ad - bc = 1 + \alpha \beta\)

\(s\text{det } T = \chi_T = \pm 1\)

\((z, \theta) \to (z, -\theta)\) if \(a = 1, b = 0, c = 0, d = 1, \alpha = 0, \beta = 0, \chi_T = -1\). Since this transformation has superdeterminant -1, its product with a super-Schottky group transformation cannot be expressed in the form
\[
\frac{\tilde{T}Z - \tilde{Z}_1 n}{\tilde{T}Z - \tilde{Z}_2 n} = \hat{K}_n \frac{Z - \tilde{Z}_1 n}{Z - \tilde{Z}_2 n}
\]

The direct substitution of the variables \((K_n, Z_1 n, Z_2 n) \to (\tilde{K}_n, \tilde{Z}_1 n, \tilde{Z}_2 n)\) cannot be used to define the measure for the spin structure with the opposite signs for periodicity properties of the fermion.

It is therefore preferable to approximate the superstring path integral either by summing over distinct triangulations of genus-\(g\) surfaces or dominant contributions determined by extremizing the action. Triangulations of a Riemann surface lead to the following estimate \([36]\)
\[
\text{Vol}[M(T_a)] \to_{\text{Vol}(\sigma^2) \ll 1} \hat{A}_h(V) \cdot N_2^\frac{\chi(\Sigma)}{2} \cdot \sigma_2
\]
\(\sigma_2\) denotes the 2-simplex, \(\hat{A}_h(V)\) is a constant and \(N_2\) is the number of two-dimensional simplices, which satisfies the constraint
\[
N_0 - N_1 + N_2 = \chi(T)
\]
A typical triangulation of a genus-\(g\) surface is given by replacing the handle by a tetrahedron with the following number of simplices
\[
N_0 = g + 2 \\
N_1 = 12g \\
N_2 = 9g
\]
\(N_0 - N_1 + N_2 = 2 - 2g\)
These values gives rise to a genus-dependence \( \tilde{A}_h(V)(9g)^{\frac{1}{2} - 2g} \), which represents a decreasing portion of the entire moduli space as \( g \to \infty \). With increasing genus, the condition \( V \ln(\sigma^2) \ll 1 \) limits the moduli space to triangulations by simplices which can be included in a sequence of embeddings \( \ldots \subset S^{n-4} \subset S^{n-2} \subset S^n \subset \ldots \) with \( \ln(\sigma^2) \sim \ln(S^2) \leq \ln(S^n) \to 0 \).

The weighting factor in the partition function for two-dimensional Lorentzian gravity, \( e^{-S_{\text{grav.}}} \), has an exponential dependence on the number of handles in the manifold. A similar result would be expected for the bosonic string action which is bounded below by the area of the worldsheet and equals the minimum value when the intrinsic metric is conformal to \( \partial_\alpha X^\mu \partial_\beta X_\mu \). Since the femionic string action \([37]\]

\[
S(E, \Phi) = \frac{1}{2\pi \alpha'} \int d^2 z d^2 \theta e^{E_\alpha} \partial_M \Phi(Z) E^{\alpha N} \partial_N \Phi(Z)
\]

\[
S(h, X, \chi, \psi) = \frac{1}{2\pi \alpha'} \int d^2 z \sqrt{h} \left[ \frac{1}{2} h^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right. \\
\left. + \bar{\chi} \rho^\alpha \rho^a \partial_\alpha X_a + \frac{1}{4} \bar{\psi}^\mu \psi^\nu \bar{\chi} \rho^a \rho^b \chi_b \right]
\]

has equation of motion \( D_\alpha \tilde{D}^\alpha \Phi(Z) = 0, D_\alpha = E^{\alpha M} \partial_M \), for the coordinate superfield \( \Phi(Z) \), the extremal value is given by the embedded area in superspace determined by harmonic maps \( z \to \Phi(Z) \).

It is known that the conformal factor in the metric \( ds^2 = e^{\phi} dz d\bar{z} \) is approximately \( \frac{1}{r^2 (\ln r)^2} \) near a parabolic puncture \([38]\), with \( r \) being the distance parameter. The equation of motion for \( h^{\alpha \beta} \) implies that \( h_{\alpha \beta} = \partial_\alpha X^\mu \partial_\beta X_\mu \) so that \( \partial_\alpha X^\mu \partial_\beta X^\mu = \frac{1}{2r^2 (\ln r)^2} \) defines a configuration which gives a dominant contribution to the string path integral. A solution to this equation is \( X^\mu = X_0^\mu + \frac{1}{2} \ln(\ln r) e^\mu \). The spinor field then will take the form \( \psi_0^\mu - i \rho^\alpha \partial_\alpha X^\mu \epsilon = \psi_0^\mu - i \frac{1}{4r^2 (\ln r)^2} (\rho^i z_i + \rho^i \bar{z}_i) e^\mu \epsilon \). A gauge can be chosen so that \( e_\alpha^a = \delta_\alpha^a, \chi_\alpha = 0 \), implying that only the first two terms of \( S(h, X, \chi, \psi) \) would be non-vanishing.

Substituting the functional dependence of \( X^\mu \) and \( \psi_\mu \) into the action

\[
\sum_i \int_{N(Q_i)} d^2 z_i \sqrt{h} \left[ \frac{1}{2} h^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right] \\
= \sum_i \int_{N(Q_i)} d^2 z_i \left[ \frac{1}{2} \frac{1}{r_i^2 (\ln r_i)^2} - \frac{1}{2} i \left[ \bar{\psi}_0^\mu + i \frac{1}{4r_i^2 (\ln r_i)^2} \bar{\epsilon} e^\mu (\rho^i z_i + \rho^i \bar{z}_i) \right] \rho^\alpha \partial_\alpha \left[ \psi_0^\mu - i \frac{1}{4r_i^2 (\ln r_i)^2} (\rho^i z_i + \rho^i \bar{z}_i) e_\mu \epsilon \right] \right]
\]

(4.29)
where \( Q_i \) denotes the location of the puncture on the surface. Setting the average value of the fermion in the background field \( \langle \psi_\mu \rangle \) equal to zero, the oscillatory factor can be discarded in an evaluation of the magnitude of the path integral leaving

\[
\int D(h, X, \psi) e^{-S(h, X, \chi, \psi)} \sim \sum_{\text{worldsheets}} e^{-\frac{1}{2\alpha'} \sum N \sum_i \int_{\partial (N(Q_i))} dr_i (\ln r_i)^2} = \sum_{\text{worldsheets}} e^{-\frac{1}{2\alpha'} \sum N \sum_i \frac{1}{(\ln r_i)^2}} \tag{4.30}
\]

where the sums are defined by the decomposition of the Riemann surface into \( 2g-2 \) three-punctured spheres [39]. Given that the maximum distance between three punctures on a sphere is \( \frac{2\pi}{3} R \), where \( R \) is the radius of the sphere, the contribution to the string path integral is \( e^{\frac{3}{2\alpha'} (g-1) \cdot (\log \frac{\pi}{3} R)^{-1}} \) multiplied by a combinatorial factor equal to the number of different ways of linking the \( 2g-2 \) spheres with each element of the set \( I_1 = \{ S_2, ..., S_{g-1} \} \) connected to three spheres, the spheres \( S_3, ..., S_{g-2} \) having two adjacent connections in \( I_1 \), and every element of the set \( I_2 = \{ S_1, S_g, ..., S_{2g-2} \} \) linked only to one of the spheres in the set \( I_1 \). The combinatorial factor is then \( \left( \frac{4}{2} \right) \left( \frac{2g-4}{g-2} \right) \rightarrow g \rightarrow \infty \ 6 \cdot 2^{2g-4} \).

Since \( \ell_p \simeq 4R \), \( R \sim (2\pi)^{\frac{1}{4}} \) and the contribution of the infinite-genus surfaces to the vacuum amplitude becomes

\[
\lim_{g \to \infty} \frac{3}{2} \kappa_{\text{str}}^{2g-2} \cdot 8^g \cdot e^{6(g-1) \left( \ln \left( \frac{\pi (2\pi)^{\frac{1}{3}}}{N} \right) \right)^{-1}}
\]

\[
= \lim_{g \to \infty} \frac{3}{2} \kappa_{\text{str}}^{2g-2} \cdot e^{\ln \left( \frac{\pi (2\pi)^{\frac{1}{3}}}{N} \right)^{-1}} \left( 8 \kappa_{\text{str}}^2 \cdot e^{\ln \left( \frac{\pi (2\pi)^{\frac{1}{3}}}{N} \right)^{-1}} \right)^g \tag{4.31}
\]

Setting the base of the exponential equal to 1, it follows that

\[
\kappa_{\text{str}}^2 = \frac{1}{8} e^{\ln N \cdot \frac{6}{(24.3507)^2}} \tag{4.32}
\]

Suppose that \( \kappa_{\text{str}} \simeq \frac{1}{24.3507} \), a condition derived from the equality of \( \kappa_{\text{str}} \) and the unified gauge coupling constant \( \alpha_{\text{GUT}} \). Then \( N = e^{(1.84527 + \ln 0.61447 \cdot \frac{6}{(24.3507)^2})} \simeq 1.57107 \). The integration range for the spherical radius of the neighbourhood of the puncture with metric \( \frac{1}{r^2 (\ln r)^2} dzd\bar{z} \) would extend to \( \frac{\pi}{1.57107} R \) or approximately \( \frac{2\pi}{3} R \).

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