Distributed Variational Bayesian Algorithms for Extended Object Tracking

Junhao Hua, Chunguang Li, Senior Member, IEEE.

Abstract—This paper is concerned with the problem of distributed extended object tracking, which aims to collaboratively estimate the state and extension of an object by a network of nodes. In traditional tracking applications, most approaches consider an object as a point source of measurements due to limited sensor resolution capabilities. Recently, some studies consider the extended objects, which are spatially structured, i.e., multiple resolution cells are occupied by an object. In this setting, multiple measurements are generated by each object per time step. In this paper, we present a Bayesian model for extended object tracking problem in a sensor network. In this model, the object extension is represented by a symmetric positive definite random matrix, and we assume that the measurement noise exists but is unknown. Using this Bayesian model, we first propose a novel centralized algorithm for extended object tracking based on variational Bayesian methods. Then, we extend it to the distributed scenario based on the alternating direction method of multipliers (ADMM) technique. The proposed algorithms can simultaneously estimate the extended object state (the kinematic state and extension) and the measurement noise covariance. Simulations on both extended object tracking and group target tracking are given to verify the effectiveness of the proposed model and algorithms.

Index Terms—Distributed tracking, extended objects, sensor networks, variational Bayes, random matrices, ADMM.

I. INTRODUCTION

TRACKING a moving target is crucial for many applications such as robotics, surveillance, monitoring, and security [1]–[5]. In such scenarios, a sensor network can be deployed in order to increase the size of the surveillance area and cooperatively track targets. Sensor networks consist of an amount of spatially distributed nodes that have limited communication capabilities due to energy and bandwidth constraints. In recent years, the problem of distributed tracking over sensor networks, where nodes gather sensor data about one or multiple targets and then cooperatively estimate their current and future states using only local noisy measurements and information obtained from one-hop neighbors, has attracted a lot of attention [6]–[9]. Compared with the traditional centralized approach, distributed approach does not need a powerful fusion center, so it is more flexible and provides robustness to node and/or link failures. Due to these merits, distributed tracking has been used in a wide range of fields, including security and surveillance [10], [11], environmental monitoring (tracking of weather patterns and pollutants) [12], and biology (tracking of populations or individual animals) [13].

Distributed algorithms for cooperative tracking of target states in sensor networks using multiple sensor measurements have been extensively studied in recent literature. The most common approach for distributed tracking is using linear Kalman filter based on average consensus [14]–[16], and diffusion strategies [7], [17]. Besides, considering the case that the underlying states and/or sensor observation models are nonlinear and/or non-Gaussian, the distributed particle filters are developed based on the sequential Monte Carlo [18]–[22]. Some authors also improve the algorithms by considering some realistic network. In [23], Kalman filter is designed over a packet-dropping network through a probabilistic approach. In [5], a distributed optimal consensus filter is developed for heterogeneous sensor networks.

Previous studies on distributed tracking problems are focused on the scenario where a target is considered as a point source of measurements and can only give rise to at most a single measurement per time step at each node. They assume that the extension of an object is neglectable in comparison with sensor resolution and error. However, since model applications require more and more detailed physical information about the objects for detection, tracking, classification and identification, there is an increasing need for recognizing extended objects as individual units [24], [25]. In this case, the traditional approach does not applicable as it has a significant loss of information. With increased resolution capabilities of modern sensors, multiple measurements from an object can be obtained for each sensor/node, and a better approach is to treat an object as an extended object (EO) with both the kinematic state (e.g., position, velocity, and acceleration) and physical extension (e.g., size, shape and orientation). Also, a group of closely spaced targets can be considered as an extended object. The aim of extended object tracking (EOT) is to estimate both the kinematic state and extension. In the literature, there are some studies on extended object tracking problem, including multiple hypothesis tracking framework [26], [27] random finite sets approaches [28], [29], and random matrix framework [24], [25], [30]–[33], etc. However, to the best of our knowledge, the problem of distributed extended object tracking in a sensor network has not been studied yet.

In this paper, we consider the problem of distributed extended object tracking in sensor networks. We formulate a distributed extended object model in a Bayesian framework. The object extension is modelled as an ellipse represented by a positive definite matrix, called extension matrix [24]. The measurements gathered at each time scan are assumed
to be distributed over the object with a Gaussian distribution whose covariance is related to the extension matrix. Since in many cases statistical sensor error (noise) cannot be neglected when compared to object extension \[25\], we also consider the unknown measurement noise. For tractability, we use latent variables to represent the underlying noise-free measurements and build a complete measurement likelihood. Moreover, to obtain the recursive Bayesian filter, we choose the conjugate priors of the extended object state (kinematic state vector and extension matrix) and noise covariance for this complete measurement likelihood.

Based on this model, we derive a novel distributed variational Bayesian algorithm for simultaneously estimating the extended object state and the measurement noise covariance in sensor networks. In the prediction step, each node updates the predicted distribution of the extended object state locally. In the measurement update step, each node infers the posteriors of these posteriors due to their intractability, and the alternating direction method of multipliers (ADMM) technique is applied to achieve distributed consensus. To evaluate the effectiveness of the proposed algorithm, we empirically demonstrate the superior performance of the proposed algorithm on both extended object tracking and group target tracking scenarios.

The main contributions of this paper are summarized as follows.

- We formulate a Bayesian model for distributed tracking of the extended object with both the kinematic state and extension in sensor networks. We consider the case that the measurement noise exists and is unknown.
- We derive a novel centralized algorithm to simultaneously estimate the extended object state and measurement noise covariance based on variational Bayesian methods.
- We propose a distributed algorithm for extended object tracking in sensor networks by integrating the ADMM technique into the VB iterative procedure.

The rest of the paper is organized as follows. In Section II, we formulate the problem of distributed Bayesian extended objects tracking, and present the measurement model and dynamical model. In Section III, a variational Bayesian algorithm is derived for iteratively estimating the extended object state and measurement noise covariance. In Section IV, a distributed algorithm for extended object tracking is proposed based on the ADMM. Numerical simulations are presented in Section V. Finally, conclusion is drawn in Section VI.

**Notations:** The superscript transposition (·)\(^T\) denotes transposition, and \([i, j]_\{m \times n\}\) denotes the \(i\)-th column and \(j\)-th row of a matrix \(m \times n\). The operator \(\mathbb{E}\{\cdot\}\) or \(\cdot\) denotes the expectation, \(\mathbb{E}\{\cdot\}\) denotes the trace operator, \(\mathcal{N}\{\cdot\}\) is the Gaussian distribution, \(\mathcal{W}\{\cdot\}\) is the Wishart distribution, and \(\mathcal{I}\mathcal{W}\{\cdot\}\) is the inverse Wishart distribution. The definition of the Wishart and inverse Wishart distributions are given in Appendix A. Other notations will be given if necessary.

II. NETWORK MODEL AND BAYESIAN FORMULATION OF EXTENDED OBJECT TRACKING

Let us consider a connected sensor network consisting of \(N\) nodes distributed over a geographic region. We use graphs to represent networks. The considered undirected graph \(G = (\mathcal{V}, \mathcal{E})\) consists of a set of nodes \(\mathcal{V} = \{1, 2, ..., N\}\) and a set of edges \(\mathcal{E}\), where each edge \((k, l) \in \mathcal{E}\) connects an unordered pair of distinct nodes. For each node \(k \in \mathcal{V}\), let \(\mathcal{N}_k = \{l|(k, l) \in \mathcal{E}, k \neq l\}\) be the set of its neighboring nodes (excluding node \(k\) itself). At each time scan \(t\), the node \(k\) collects a set of \(n_{k,t}\) measurements \(Y_{k,t} = \{y_{k,t}^i\}_{i=1}^{n_{k,t}}\) with each individual measurement described by the vector \(y_{k,t}^i \in \mathbb{R}^d\). Let \(Y_t = \{Y_{k,t}\}_{k \in \mathcal{V}}\) denote measurements received from all nodes at time \(t\). The accumulated sensor data of node \(k\) is denoted as \(Y_t = \{Y_{k,t}\}_{i=1}^t\), and the accumulated sensor data from all nodes is denoted as \(Y_t = \{Y_{k,t}\}_{k \in \mathcal{V}}\).

Traditional target tracking approaches only consider a target as a point source of measurements. In this paper, we suppose that a target has some object shape, which is described by a spatial model. We use the random matrix model \[24\] as the spatial model. It models the kinematic target state by a random vector \(x_t\) and the object extension by a symmetric positive definite (SPD) random matrix \(X_t \in \mathbb{S}_d\), where \(d\) is the dimension of the object. The vector \(x_t\) represents the position of an object and its motion properties, such as velocity, acceleration and turn-rate. The SPD matrix \(X_t\) implies that the object shape is approximated by an ellipse. The ellipse shape may seem limiting, however this model is applicable to many real scenarios, such as pedestrian tracking using LIDAR \[31\], and tracking of boats and ships using marine radar \[30\], \[37\].

The aim of target tracking in a sensor network is to collaboratively estimate the extended object state using the measurements collected by a network of nodes. In the following, we model the measurements and the state transition in a Bayesian methodology, and propose a variational Bayesian approach to estimate the extended object states in next section.

A. Measurement Modelling

The measurement model at node \(k\) is assumed as

\[
y_{k,t} = (H_t \otimes I_d) x_t + \nu_{k,t}, i = 1, \ldots, n_{k,t},
\]

where \(\otimes\) stands for the Kronecker product, \(H_t = [1 \ 0 \ 0] \in \mathbb{R}^{1 \times 3}\) is the measurement matrix in one-dimensional space (assuming that only positions are measured and the state in one-dimensional space is \([position, velocity, acceleration]^T\) ), \(I_d\) is an identity matrix, and \(\nu_{k,t}^i\) is a white Gaussian noise. The matrix \((H_t \otimes I_d)\) picks out the Cartesian position from the kinematic vector \(x_t\). As pointed out in \[25\], when the measurement noise is not negligible compared with the extension \(X_t\), it is better to consider both the measurement noise and the uncertainties in \(X_t\) simultaneously. Therefore, we assume

\[
\nu_{k,t}^i \sim \mathcal{N}(0, sX_t + R),
\]

where \(s\) is a real scalar describing the effect of \(X_t\), and \(R \in \mathbb{S}_d\) is an unknown measurement noise covariance. Note that
it is assumed that the covariance of measurement noise $R$ is unchanged with the time scan $t$.

From (1) and (2), it is concluded that the measurements \{y_{k,t}\} are independent and identically Gaussian distributed (given the extended object state $x_t, X_t$, and measurement noise covariance $R$) as,

$$P(y_{k,t}|x_t, X_t, R) = N(y_{k,t}; (H_t \otimes I_d)x_t, sX_t + R).$$

(3)

The scaling factor $s$ can be used to describe different types of EOs. For example, as justified by [25], $s = 1/4$ indicates that the scattering centers within an ellipse are uniformly distributed. If the point targets in the group are uniformly distributed spatially, $s = 1/4$ also can be used.

Due to the measurement noise covariance in (3), the exact analytical solution for the extended object state can not be obtained using the measurement likelihood (3). Therefore, following the approach in [30], we rewrite (3) as

$$P(y_{k,t}|x_t, X_t, R) = \int P(y_{k,t}|z_{k,t}, X_t, R)d\mathbf{z}_{k,t},$$

(4)

where the latent variable $z_{k,t} \in \mathbb{R}^d$ is introduced to represent an underlying noise-free measurement corresponding to the noisy measurement $y_{k,t}$. Thus, we have the complete measurement likelihood

$$P(y_{k,t}, z_{k,t}|x_t, X_t, R) = P(y_{k,t}|z_{k,t}, R)P(z_{k,t}|x_t, X_t) = N(y_{k,t}; z_{k,t}, R)N(z_{k,t}; (H_t \otimes I_d)x_t, sX_t).$$

(5)

For notation simplicity, let $Z_{k,t} = \{z_{k,t}^{i}\}_{i=1}^{n_{k,t}}$ denote the set of latent variables at node $k$, and $Z_t = \{Z_{k,t}\}_{k \in \mathcal{V}}$ denote a set of latent variables from all nodes at time scan $t$.

B. Object State Dynamics Modeling

The object dynamic model describes how the object state evolves over time. In the context of the extended object tracking, this involves the descriptions of how the target kinematic state and the extension changed over time. We assume that the evolution of the extension is independent of the kinematic state, and thus the transition density can be expressed as [24].

$$P(x_t, X_t|x_{t-1}, X_{t-1}) = P(x_t|x_{t-1}, X_{t-1})P(X_t|X_{t-1}).$$

(6)

The object kinematic state has a linear Gaussian transition density

$$P(x_t|X_{t-1}, x_{t-1}) = N(x_t; (F_t \otimes I_d)x_{t-1}, Q_t \otimes X_t),$$

(7)

where $(F_t \otimes I_d) \in \mathbb{R}^{3d \times 3d}$ is the evolution matrix with $F_t \in \mathbb{R}^{3 \times 3}$ being the dynamic matrix in the one-dimensional physical space, and $(Q_t \otimes X_t) \in \mathbb{R}^{3d \times 3d}$ is the dynamics noise covariance. We use the dynamic matrix $F_t$ given by [24]

$$F_t = \begin{pmatrix}
1 & \Delta_t & \frac{1}{2}\Delta_t^2 \\
0 & 1 & \Delta_t \\
0 & 0 & e^{-\Delta_t/\theta}
\end{pmatrix},$$

(8)

and the matrix $Q_t$ is given by van Keuk’s model [38]

$$Q_t = \Sigma^2 (1 - e^{-2\Delta_t/\theta}) \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},$$

(9)

with the scan time $\Delta_t$, the scalar acceleration rms value $\Sigma$ and the maneuver correlation time constant $\theta$.

We adopt the same assumption as [24] that the extension evolution has a Wishart transition density

$$P(X_t|X_{t-1}) = \mathcal{W}(X_t; \eta_t, X_{t-1}/\eta_t),$$

(10)

where the parameter $\eta_t > 0$ governs the noise level of the prediction: the smaller $\eta_t$ is, the higher the process noise.

C. Prior of the Extended object State

In the recursive Bayesian filtering, the prior of the extended object state is the predicted distribution $P(x_t, X_t|Y_{t-1})$, and we typically want the prior and the posterior to be the same functional form. To ensure this, we should choose the conjugate prior for the complete measurement likelihood [5]. Since the measurement likelihood [5] is a product of two Gaussian distributions with unknown mean and unknown covariance, the conjugate prior of the extended object state is Gaussian inverse Wishart (GIW) distribution and the conjugate prior of the measurement noise covariance is inverse Wishart distribution.

Therefore, we first assume that the posterior of the extended object state at the time scan $t-1$ is Gaussian inverse Wishart (GIW) distributed,

$$P(x_{t-1}, X_{t-1}|Y_{t-1}) = P(x_{t-1}|X_{t-1}, Y_{t-1})P(X_{t-1}|Y_{t-1})$$

$$= \mathcal{N}(x_{t-1}; m_{t-1|t-1}, P_{t-1|t-1} \otimes X_{t-1}) \times \mathcal{IW}(X_{t-1}; \nu_{t-1|t-1}, V_{t-1|t-1}),$$

(11)

and then we set the predicted distribution $P(x_t, X_t|Y_{t-1})$ as Gaussian inverse Wishart distributed presented in next section. As for the measurement noise covariance $R$, we assume that it follows an inverse Wishart distribution at time scan $t-1$, i.e.,

$$P(R|Y_{t-1}) = \mathcal{IW}(R; v_{t-1}, U_{t-1}).$$

(12)

To capture the conditional dependence structure between random variables above, the graphical representation of the Bayesian extended object model is depicted in Fig.1.
III. VARIATIONAL BAYESIAN APPROACH TO EXTENDED OBJECT TRACKING

For clarity, before presenting the distributed algorithm to estimate the extended object state in the considered network, we would like to first present the corresponding centralized algorithm, in which at every time scan the measurements from all nodes can be gathered in a fusion center and the computation is performed in the fusion center. After that, we propose the distributed extended object tracking algorithm in next section. Note that in the case when there is only a single node or measurements are collected at a central node, the centralized algorithm can be used.

The posterior of the extended object state at time scan \(t\) can be recursively computed in two steps: the time update and the measurement update. In the time update step, we predict the distribution of the extended target state based on the prior and the complete predicted distribution as a prior, we update the posterior of the extended object state based on the prior and the complete measurement likelihood [5].

A. Prediction

The predicted distribution at time scan \(t\) is defined as

\[
P(x_t, X_t|Y_{t-1}) = \int P(x_t, X_t|x_{t-1}, X_{t-1})P(x_{t-1}, X_{t-1}|Y_{t-1})dx_{t-1}dX_{t-1}.
\]

Using [6] and [11], the above predicted distribution can be factorized as

\[
P(x_t, X_t|Y_{t-1}) = P(x_t|X_t, Y_{t-1})P(X_t|Y_{t-1}),
\]

where

\[
P(x_t|X_t, Y_{t-1}) = \int P(x_t|X_t, x_{t-1})P(x_{t-1}, X_{t-1}|Y_{t-1})dx_{t-1},
\]

and

\[
P(X_t|Y_{t-1}) = \int P(X_t|X_{t-1})P(X_{t-1}|Y_{t-1})dX_{t-1}.
\]

1) Kinematical State Part: We adopt the assumption that \(P(x_{t-1}|X_t, Y_{t-1})\) follows a Gaussian distribution with the same parameters as \(P(x_{t-1}|X_{t-1}, Y_{t-1})\). Since \(P(x_{t-1}|X_{t-1}, Y_{t-1})\) and the transition density \(P(x_t|X_t, x_{t-1})\) are both Gaussian, the predicted distribution \(P(x_t|X_t, Y_{t-1})\) in [15] is also Gaussian. Using the usual rules for Kronecker products, we obtain

\[
P(x_t|X_t, Y_{t-1}) = N(x_t; m_{t|t-1}, P_{t|t-1} \otimes X_t),
\]

where

\[
m_{t|t-1} = (F_t \otimes I_d)m_{t-1|t-1},
\]

\[
P_{t|t-1} = F_tP_{t-1|t-1}F_t^T + Q_t.
\]

2) Object Extension Part: Since \(P(X_t|X_{t-1})\) is Wishart distributed and \(P(X_{t-1}|Y_{t-1})\) is inverse Wishart distributed, the predicted density \(P(X_t|Y_{t-1})\) in [16] is given by a “Generalized Beta Type II” density [24]. However, as we have mentioned in Section II-C, we typically want the predicted density (prior) \(P(X_t|Y_{t-1})\) to be the same functional form of its posterior. Therefore, alternatively, we assume the predicted density \(P(X_t|Y_{t-1})\) is also inverse Wishart distributed, and we use a heuristic approach to set its parameters. In fact, according to [24], if there is a sufficient number of sensor measurements, the prediction part of the tracking process is unimportant compared with the gain obtained in the measurement update step.

Following [25], we postulate that the expectation of the extension is unchanged in the prediction, i.e., \(E[X_t] = E[X_{t-1}]\), and gradually decrease its degrees of freedom. Thus, the predicted density [16] is heuristically approximated by

\[
P(X_t|Y_{t-1}) \approx \mathcal{IW}(X_t; V_{t|t-1}, V_{t|t-1}),
\]

where

\[
V_{t|t-1} = \frac{V_{t-1}}{\nu_{t-1|t-1}} - d - 1,
\]

\[
V_{t|t-1} = \frac{V_{t-1}}{\nu_{t-1|t-1}} - d - 1V_{t-1|t-1},
\]

with the temporal decay constant \(\tau\) and scan time \(\Delta_t\).

B. Variational Bayesian Measurement Update

In a fully Bayesian context, given the priors [12] and [14], the aim of the measurement update is to compute the entire posterior distribution of the unobserved variables (extended object state \(x_t, X_t\), the measurement noise covariance \(R\), and the latent variables \(Z_t\)). The posterior is given by Bayes’ theorem,

\[
P(x_t, X_t, Z_t, R|Y_t) = \frac{P(Y_t, Z_t|x_t, X_t, R)P(x_t, X_t, R|Y_{t-1})}{P(Y_t|Y_{t-1})},
\]

where its denominator is the model evidence defined as

\[
P(Y_t|Y_{t-1}) = \int P(Y_t, Z_t|x_t, X_t, R)P(x_t, X_t, R|Y_{t-1})dx_t dX_t dR dZ_t.
\]

However, the computation of exact posterior [21] is intractable. We apply an approximation technique, called variational Bayesian (VB) method [34], for solving this problem. The VB is to approximate the posterior \(P(x_t, X_t, Z_t, R|Y_t)\) by a tractable distribution \(Q(x_t, X_t, Z_t, R)\). It is found by minimizing the Kullback-Leibler (KL) divergence between these two distributions [34].

\[
\text{KL}(Q || P) = - \mathcal{L}(Q) + \ln P(Y_t|Y_{t-1}),
\]

where \(\mathcal{L}(Q)\) is the evidence lower bound of the marginal log-likelihood of the measurements \(\ln P(Y_t|Y_{t-1})\), and it can be written as

\[
\mathcal{L}(Q) = \mathbb{E}_{Q(x_t, X_t, Z_t, R)} \left[ \ln \frac{P(x_t, X_t, Z_t, R|Y_t)}{Q(x_t, X_t, Z_t, R)} \right].
\]
Since the log evidence \( \ln P(Y_t|Y_{t-1}) \) is fixed with respect to \( Q \), the minimization of \( \mathcal{L}(Q) \) is equivalent to maximizing the lower bound \( \mathcal{L}(Q) \). To obtain an analytical approximate solution, we make the mean field assumption \([34]\) and factorize the joint variational distribution of \( x_t, X_t, Z_t \) and \( R \) as

\[
Q(x_t, X_t, Z_t, R) = q(x_t, X_t)q(Z_t)q(R).
\]

Following \([35]\), the global lower bound \( \mathcal{L}(Q) \) can be replaced by an average of the local lower bounds,

\[
\mathcal{L}(Q) = \mathbb{E}_q \left[ \ln \frac{P(x_t, X_t, Z_t, R|Y_t)}{q(x_t, X_t)q(R)} \prod_{k=1}^N q(Z_{k,t}) \right] = \frac{1}{N} \sum_{k=1}^N \mathcal{L}_k(q),
\]

where \( \mathcal{L}_k(q) \) is the local lower bound of each node \( k \), defined as

\[
\mathcal{L}_k(q) \triangleq \mathbb{E}_q \left[ \ln \frac{P(x_t, X_t, Z_{k,t}|Y_{t-1}) P(R|Y_{t-1})}{q(x_t, X_t)q(R)} \right] + \frac{N}{2} \mathbb{E}_q \left[ \ln P(Y_{k,t}|Z_{k,t}, R) P(Z_{k,t}|x_t, X_t) \right].
\]

Note that this local objective function \( \mathcal{L}_k(q) \) only contains the local measurements \( Y_{k,t} \) and latent variables \( Z_{k,t} \), and thus can be optimized locally at each node \( k \). To obtain “best” variational distribution of \( Q \), the VB alternates between maximizing the lower bound \( \mathcal{L}_k(q) \) with respect to the variational distributions of the latent variables \( \{Z_{k,t}\} \) and that of the parameters \( x_t, X_t, R \), consisting of three iterative steps:

1) posterior of latent variables: The optimal variational distribution \( q^*(Z_{k,t}) \) has the form

\[
q^*(Z_{k,t}) = \arg \max_{q(Z_{k,t})} \mathcal{L}_k(q^*(x_t, X_t), q(Z_{k,t}), q^*(R)),
\]

\[
q^*(x_t, X_t) = \arg \max_{q(x_t, X_t)} \frac{1}{N} \sum_{k=1}^N \mathcal{L}_k(q(x_t, X_t), q^*(Z_{k,t}), q^*(R)),
\]

\[
q^*(R) = \arg \max_{q(R)} \frac{1}{N} \sum_{k=1}^N \mathcal{L}_k(-, q^*(Z_{k,t}), q^*(R)).
\]

Based on the VB theory \([34]\), we can obtain the analytical solutions for variational distributions \( q^*(Z_{k,t}), q^*(x_t, X_t) \) and \( q^*(R) \). In the following, we give these solutions respectively.

\section*{1) Posterior of latent variables:}

The optimal variational distribution \( q^*(z_{k,t}) \) follows the Gaussian distribution,

\[
q^*(z_{k,t}) = \mathcal{N}(\hat{z}_{k,t}^i; \hat{\mu}_{k,t}^i, \Sigma_{k,t}^i),
\]

with the parameters given by

\[
\hat{\mu}_{k,t}^i = \hat{\Sigma}_{k,t}^i((R^{-1})y_{k,t}^i + \frac{1}{s}(X_t^{-1})(H_t \otimes I_d)(x_t)),
\]

\[
\hat{\Sigma}_{k,t}^i = ((R^{-1}) + \frac{1}{s}(X_t^{-1}))^{-1}.
\]

The expected sufficient statistics for updating other variational distributions are as follows

\[
\langle z_{k,t} \rangle = \hat{\mu}_{k,t}^i,
\]

\[
\langle z_{k,t}^i z_{k,t}^j \rangle = \hat{\Sigma}_{k,t}^i + \hat{\mu}_{k,t}^i \hat{\mu}_{k,t}^j,
\]

\[
\langle y_{k,t} - z_{k,t} \rangle (y_{k,t} - z_{k,t}^j)^T = \langle y_{k,t} - \hat{\mu}_{k,t}^i \rangle (y_{k,t} - \hat{\mu}_{k,t}^j)^T + \hat{\Sigma}_{k,t}^i.
\]

2) Posterior of the extended object state:

The optimal variational distribution \( q^*(x_t, X_t) \) can be expressed as

\[
q^*(x_t, X_t) = \sum_{k=1}^N \mathbb{E}_{Z_{k,t}}[\ln P(Z_{k,t}|x_t, X_t)]
\]

\[
+ \ln P(x_t|X_t, Y_{t-1}) + \ln P(X_t|Y_{t-1}) + c_x
\]

\[
= -\frac{1}{2} \sum_{k=1}^N \sum_{i=1}^{n_{k,t}} ((z_{k,t} - (H_t \otimes I_d)x_t)^T
\]

\[
\times (sX_t^{-1})(z_{k,t} - (H_t \otimes I_d)x_t)) - \frac{1}{2} \ln |sX_t| - \frac{1}{2} \ln |P_{t_{|t-1}} \otimes X_t|
\]

\[
- \frac{1}{2} \left( (x_t - m_{t_{|t-1}})^T (P_{t_{|t-1}} \otimes X_t)^{-1} (x_t - m_{t_{|t-1}}) - \nu_{t_{|t-1}} \right) + d + 1
\]

\[
\frac{1}{2} \ln |X_t| - \frac{1}{2} \text{tr}(X_t^{-1}V_{t_{|t-1}}) + c_x,
\]

where \( c_x \) denotes a constant term with respect to \( x_t \) and \( X_t \). In the Appendix \([33]\) we proved that \( \mathcal{L}(Q) \) can be rewritten as

\[
q^*(x_t, X_t) = q^*(x_t|X_t)q^*(X_t) = \mathcal{N}(x_t; \tilde{m}_t, \tilde{P}_t \otimes X_t) \mathcal{I}^W(X_t; \tilde{v}_t, \tilde{V}_t),
\]

where the parameters are given by

\[
\tilde{m}_t = m_{t_{|t-1}} + (w_t \otimes I_d)(\tilde{z}_t - (H_t \otimes I_d)m_{t_{|t-1}}),
\]

\[
\tilde{P}_t = P_{t_{|t-1}} - b_tw_tw_t^T,
\]

\[
\tilde{v}_t = v_{t_{|t-1}} + N_t,
\]

\[
\tilde{V}_t = V_{t_{|t-1}} + \frac{N_t}{s} S_t + K_t.
\]

In \( \mathcal{L}(Q) \), \( b_t \) is a scalar innovation factor, \( w_t \) is a gain vector defined as

\[
b_t \triangleq \frac{N_t}{s} + H_t P_{t_{|t-1}} H_t^T,
\]

\[
w_t \triangleq P_{t_{|t-1}} H_t^T b_t^{-1},
\]

and the corresponding statistics are given by

\[
N_t \triangleq \sum_{k=1}^N n_{k,t}, \quad \tilde{z}_t \triangleq \frac{1}{N_t} \sum_{k=1}^N n_{k,t} \langle z_{k,t} \rangle,
\]

\[
S_t \triangleq \frac{1}{N_t} \sum_{k=1}^N \sum_{i=1}^{n_{k,t}} (z_{k,t}^i (z_{k,t}^i)^T) - \tilde{z}_t \tilde{z}_t^T,
\]

\[
K_t \triangleq b_t^{-1}(\tilde{z}_t - (H_t \otimes I_d)m_{t_{|t-1}})(\tilde{z}_t - (H_t \otimes I_d)m_{t_{|t-1}})^T.
\]
The expected sufficient statistics of \( q^*(x_t, X_t) \) are

\[
\langle x_t \rangle = \bar{m}_t, \quad (38a)
\]

\[
\langle X_t \rangle = \frac{\bar{V}_t}{\bar{v}_t - d - 1}, (X_t^{-1}) = \bar{v}_t \bar{V}_t^{-1}. \quad (38b)
\]

3) Posterior of the measurement noise covariance: The optimal variational posterior \( q^*(R) \) has the form

\[
\ln q^*(R) = \sum_{k=1}^N \mathbb{E}_{Z_k,t} [\ln P(Y_k,t|Z_k,t,R)] + \ln P(R|Y_{t-1}) + c_R
\]

\[
= -\frac{(N_t + v_t-1) + d + 1}{2} \ln |R|
\]

\[
- \frac{1}{2} \text{tr}(R^{-1} \sum_{k=1}^{n_k,t} (y^t_k - z^t_k)(y^t_k - z^t_k)^T + U_t)) + c_R,
\]

where \( c_R \) denotes a constant term with respect to \( R \). From (39), we conclude that \( q^*(R) \) follows an inverse Wishart distribution,

\[
q^*(R) = \mathcal{IW}(R; \hat{v}_t, \hat{U}_t), \quad (40)
\]

with the parameters given by

\[
\hat{v}_t = v_{t-1} + N_t, \quad (41a)
\]

\[
\hat{U}_t = U_{t-1} + \sum_{k=1}^{n_k,t} \sum_{i=1}^{n_i,t} (y^t_k - z^t_k)(y^t_k - z^t_k)^T. \quad (41b)
\]

The expected sufficient statistics of \( q^*(R) \) are

\[
\langle R \rangle = \frac{\hat{U}_t}{\hat{v}_t - d - 1}, \langle R^{-1} \rangle = \hat{v}_t \hat{U}_t^{-1}. \quad (42)
\]

Using a superscript \((n)\) to denote the iteration number of the VB, starting from some initial parameters, the VB alternates between the VBE step and VBM step. In the VBE step, the VB computes \( q^{(n)}(Z_{k,t}) \) using the expected sufficient statistics of \( q^{(n-1)}(x_t, X_t) \) and \( q^{(n-1)}(R) \). In the VBM step, the VB computes \( q^{(n)}(x_t, X_t) \) and \( q^{(n)}(R) \) using the expected sufficient statistics of \( q^{(n)}(Z_{k,t}) \). We set the initial parameters as:

\[
\bar{y}_t = \frac{1}{N} \sum_{k=1}^{n_k,t} \sum_{i=1}^{n_i,t} y^t_k, \quad m^{(0)} = (H_t^T \otimes I_d) \bar{y}_t, \quad P^{(0)} = P_{t|t-1}, V^{(0)} = 0.1 I_d, \quad \nu^{(0)} = d+1+0.1, \quad U^{(0)} = U_{t-1}, \quad v_t^{(0)} = v_{t-1}.
\]

For clarity, the centralized variational Bayesian algorithm for extended object tracking (cVBEOT) is summarized in Algorithm 1. Note that computing a distribution means computing its parameters. The inner loop terminates when the VB algorithm converges or a predefined stopping criterion (e.g., a maximum iteration number) is satisfied.

### IV. DISTRIBUTED VARIATIONAL BAYESIAN ALGORITHM FOR EXTENDED OBJECT TRACKING

In this section, we extend the centralized algorithm to the distributed scenario where every node in the network only communicates with its one-hop neighboring nodes. The main difficulty in doing this is that the updating of the parameters \( x_t, X_t \) and \( R \) needs all expected sufficient statistics of latent variables \( \{Z_{k,t}\}_{k \in V} \). To collect all these statistics, a possible solution is to find a cyclic path through all the nodes. However, it is not suitable for a low-cost networked system, since the communication resources are limited and exploring the network topology is hard and expensive. To solve this problem, we present a fully distributed VB algorithm for the considered network. Consider replacing the common extended object states \( x_t, X_t \) and measurement noise covariances \( R \) with a set of per node variables \( \{x_{k,t}, X_{k,t}, R_{k}\} \), we solve this problem by making an agreement on these parameters \( \{x_{k,t}, X_{k,t}, R_{k}\} \) among all nodes. The proposed algorithm consists of two steps: the local prediction and the distributed VB measurement update. In the following, we present these two steps respectively.

#### A. Local Prediction

Since the dynamic matrices \( F_t \) and \( Q_t \) are the same among all nodes, the prediction step can be performed locally. The predicted distribution of extended object state at node \( k \) is the same as its centralized counterpart in (7) and (9), namely

\[
P(x_{k,t}, X_{k,t}|Y_{t-1}) = \mathcal{N}(x_{k,t}; m_{k,t|t-1}, P_{k,t|t-1} \otimes X_{k,t}) \times \mathcal{I}(X_{k,t}; \nu_{k,t|t-1}, V_{k,t|t-1}), \quad (43)
\]

where the parameters are given by

\[
m_{k,t|t-1} = (F_t \otimes I_d)m_{k,t-1|t-1}, \quad (44a)
\]

\[
P_{k,t|t-1} = F_t P_{k,t-1|t-1} F_t^T + Q_t, \quad (44b)
\]

\[
\nu_{k,t|t-1} = d + 3 + e^{-\Delta t/\gamma}(\nu_{k,t-1|t-1} - d - 3), \quad (44c)
\]

\[
V_{k,t|t-1} = \frac{\nu_{k,t-1|t-1} - d - 1}{\nu_{k,t-1|t-1} - d - 1} V_{k,t-1|t-1}, \quad (44d)
\]

#### B. Distributed Variational Bayesian Measurement Update

After performing the local prediction step, each node collaboratively updates the kinematic state \( x_{k,t}, \) extension \( X_{k,t} \)}
and measurement noise covariance $R_k$ using its local measurements $\{y_{k,t}\}$ collected at time scan $t$ and the information obtained from its neighboring nodes. The distributed variational Bayesian measurement update consists of three iterative steps: the VBE step, the consensus step, and the VBM step. In the following, we present these three steps respectively.

1) VBE step: In the VBE step, the posteriors of the local latent variables $\{z_{k,t}\}$ at node $k$ are computed using the expected sufficient statistics of the local extended object state $x_{k,t},X_{k,t}$ and local noise covariance $R_k$. Following (30), the variational distribution $q^* (z_{k,t})$, $\forall i = 1, \ldots, n_{k,t}$, is

$$q^* (z_{k,t}) = \mathcal{N}(\hat{z}_{k,t}^{i}, \hat{\Sigma}_{k,t}^{i}),$$

(45)

with the parameters given by

$$\hat{\mu}_{k,t}^{i} = \hat{\Sigma}_{k,t}^{i} (\hat{\hat{\nu}}_{k,t} U_{k,t}^{-1} y_{k,t} + \bar{\nu}_{k,t} \hat{V}_{k,t}^{-1} (H_t \otimes I_d) \hat{m}_{k,t}),$$

(46a)

$$\hat{\Sigma}_{k,t}^{i} = (\hat{\hat{\nu}}_{k,t} U_{k,t}^{-1} + \bar{\nu}_{k,t} \hat{V}_{k,t}^{-1})^{-1}.$$  

(46b)

2) Consensus step: We observe that the computation of variational distributions of the global variables $x_{k,t}, X_{k,t}, R_k$ only needs an average of all expected sufficient statistics of latent variables $\{z_{k,t}\}$. In detail, let us define a set of expected sufficient statistics of local latent variables as

$$\omega_{k,t}^{i} \triangleq [\langle \omega_{k,t}^{1} \rangle^T, \langle \omega_{k,t}^{2} \rangle^T, \langle \omega_{k,t}^{3} \rangle^T]^T, \forall k \in \mathcal{V},$$

(47)

where

$$\omega_{k,t}^{1} \triangleq \frac{1}{N_t} \sum_{i=1}^{n_{k,t}} (z_{k,t}^{i} - \hat{z}_{k,t}^{i})^T,$$

$$\omega_{k,t}^{2} \triangleq \frac{1}{N_t} \sum_{i=1}^{n_{k,t}} (y_{k,t}^{i} - \hat{y}_{k,t}^{i})^T,$$

$$\omega_{k,t}^{3} \triangleq \frac{1}{n_{k,t}} \sum_{i=1}^{n_{k,t}} ((y_{k,t}^{i} - \hat{y}_{k,t}^{i})(y_{k,t}^{i} - \hat{y}_{k,t}^{i})^T).$$

(48)

The computation of variational distributions $q^* (x_{k,t}, X_{k,t})$ and $q^* (R_k)$ then only needs an average of all local expected sufficient statistics, i.e.,

$$\bar{\phi}_t \triangleq \frac{1}{N} \sum_{k=1}^{N} \omega_{k,t}.$$  

(49)

Our aim now becomes to compute (49) in a distributed manner, which results in a distributed averaging problem [14], [35]. We use the ADMM technique [39] to solve this problem. Let us define a set of per node intermediate quantities $\{\phi_{k,t}\}_{k=1}^{N}$ and add consensus constraints to force these variables to agree across neighboring nodes. Thus, we could obtain a consensus-based optimization problem,

$$\min_{\{\phi_{k,t}\}_{k=1}^{N}, \{\nu_{k,t}\}_{k=1}^{N}} \frac{1}{2} \sum_{k=1}^{N} ||\phi_{k,t} - \omega_{k,t}||^2,$$

s.t. $\phi_{k,t} = \varphi_{k,j}, \nu_{k,t} = \phi_{j,t}, \forall k \in \mathcal{V}, j \in N_k$, whose optimal value is equal to (49). In (50), the auxiliary variable $\varphi_{k,j}$ decouples local variable $\phi_{k,t}$ at node $k$ from those of its neighbors $j \in N_k$. Let $\lambda_{kj1}$ ($\lambda_{j2}$) denote the Lagrange multiplier corresponding to the constraint $\phi_{k,t} = \varphi_{k,j}$ (respectively $\varphi_{k,j} = \phi_{j,t}$), and we construct the augmented Lagrangian function for the problem (50) as follows,

$$\mathcal{L}_\rho(\{\phi_{k,t}\}_{k=1}^{N}, \{\varphi_{k,j}\}_{k=1}^{N}, \{\lambda_{kj}\}_{k=1}^{N}) =$$

$$\frac{1}{2} \sum_{k=1}^{N} ||\phi_{k,t} - \omega_{k,t}||^2 + \rho \sum_{j \in N_k} ||\phi_{k,t} - \varphi_{k,j} + \lambda_{kj}||^2,$$

$$+ \rho \sum_{j \in N_k} ||\varphi_{k,j} - \phi_{j,t} + \lambda_{kj2}||^2,$$

(51)

where $\rho > 0$ is a penalty parameter. The ADMM cyclically minimizes $\mathcal{L}_\rho$ with respect to the local variables $\{\phi_{k,t}\}$ and auxiliary variables $\{\varphi_{k,j}\}$, followed by a gradient ascent step over the dual variables $\{\lambda_{kj1}, \lambda_{kj2}\}$. Initializing all the Lagrange multipliers to zeros, the auxiliary variables $\{\hat{\varphi}_{k,j}\}$ can be expressed by $\{\hat{\varphi}_{k,t}\}$, and we can obtain the iterations required by per node $k$ for solving (50),

$$\phi_{k,t}^{(l)} = \omega_{k,t} - 2\lambda_{k}^{(l-1)} + \rho \sum_{j \in N_k} (\phi_{k,t}^{(l-1)} + \phi_{j,t}^{(l-1)}),$$

$$\lambda_{k}^{(l)} = \lambda_{k}^{(l-1)} + \rho/2 \sum_{j \in N_k} (\phi_{k,t}^{(l)} - \phi_{j,t}^{(l)}),$$

(52a)

(52b)

where $l > 0$ is an iteration step. $|N_k|$ is the number of the neighboring nodes of node $k$, and $\lambda_{k}^{(l)} := \sum_{j \in N_k} \lambda_{kj}^{(l)}$, $\forall k \in \mathcal{V}$, are the scaled local aggregate Lagrange multipliers. As proved in [40], for any $\phi_{k,t}^{(0)} \in \mathbb{R}^{d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}}$ and $\lambda_{k}^{(0)} = 0$, the iterations (52a) and (52b) yield that $\phi_{k,t}^{(l)} \rightarrow \bar{\phi}_t$.

Note that the computation of (52) at node $k$ only relies on the local information $\omega_{k,t}$ and the quantities $\{\hat{\phi}_{j,t}\} \in N_k$, from its neighboring nodes. Alternating between (52a) and (52b) for all node $k \in \mathcal{V}$, every node can obtain the average $\bar{\phi}_t$ in a distributed manner. To save the communication costs, the iterations (52a) and (52b) could be early stopped before it converges, and a relatively small number of the maximum iteration $L$ can be set. After the consensus step, each node $k$ can use the quantity $\phi_{k,t}^{(L)}$ to update the variational distributions of the global variables $x_{k,t}, X_{k,t}$ and $R_k$.

3) VBM step: Let us decompose the quantity $\phi_{k,t}^{(L)}$ as three parts: $[\phi_{k,t}^{(L)}]^T, [\phi_{k,t}^{(L)}]^T, [\phi_{k,t}^{(L)}]^T = \phi_{k,t}^{(L)}$. These three parts $\phi_{k,t}^{1}, \phi_{k,t}^{2}, \phi_{k,t}^{3}$ are consensus results corresponding to the true sufficient statistics $\omega_{k,t}^{1}, \omega_{k,t}^{2}, \omega_{k,t}^{3}$, respectively. Once obtaining the quantities $\phi_{k,t}^{1}, \phi_{k,t}^{2}, \phi_{k,t}^{3}$, each node can update the posterior $q^* (x_{k,t}, X_{k,t})$ and $q^* (R_k)$.

a) Posterior of the extended object state: Using the quantities $\phi_{k,t}^{1}, \phi_{k,t}^{2}$, each node $k$ can update its variational distribution $q^* (x_{k,t}, X_{k,t})$ as

$$q^* (x_{k,t}, X_{k,t}) = q^* (x_{k,t}, X_{k,t}) q^* (X_{k,t})$$

$$= \mathcal{N}(x_{k,t}; \hat{m}_{k,t}, \hat{P}_{k,t} \otimes \hat{X}_{k,t}) ZW(X_{k,t}; \hat{y}_{k,t}, \hat{V}_{k,t}),$$

(53)

with the parameters given by

$$\hat{m}_{k,t} = m_{k,t}^{(t+1)} + (w_t \otimes I_d) e_{k,t},$$

$$\hat{P}_{k,t} = P_{k,t}^{(t)} - b_t w_t w_t^T,$$

$$\hat{y}_{k,t} = \nu_{k,t}^{(t)} + N_t,$$

$$\hat{V}_{k,t} = V_{k,t}^{(t)} + \frac{N_t}{s} (\phi_{k,t}^{1} (\phi_{k,t}^{1})^T) + b_t^{-1} e_{k,t} e_{k,t}^T.$$  

(54a)

(54b)

(54c)

(54d)
Algorithm 2 Distributed VB Measurement Update

Initialization: $\bar{y}_{k,t} = \frac{1}{m} \sum_{i=1}^{n_{k,t}} y_{k,t}^i$, $m_{k,t}^{(0)} = (H^T \otimes I_d)\bar{y}_{k,t}$, $P_{k,t}^{(0)} = \sigma_{k,t-1}^v \bar{y}_{k,t}$, $V_{k,t}^{(0)} = 0.1I_d$, $\nu_{t}^{(0)} = d + 1 + 0.1$, $U_{k,t}^{(0)} = U_{k,t-1}$.

for $n \leftarrow 0, 1, 2, \ldots$ do

for $k \in \mathcal{V}$ do

Update $q^{(n)}(z_{k,t}^i) = \mathcal{N}(\mu_{k,t}^{(n)}, \Sigma_{k,t}^{(n)}), \forall i$ via (45).

Compute $\{\omega_{k,t}^{(n)}\}$ via (47).

end for

for $l \leftarrow 1, 2, \ldots, L$ do

Consensus step

Compute $\hat{\phi}_{k,t}^{(l)}$ via (52a), $\forall k \in \mathcal{V}$.

Broadcast $\hat{\phi}_{k,t}^{(l)}$ to its neighbors in $\mathcal{N}_k$, $\forall k \in \mathcal{V}$.

Compute $X_k^{(l)}$ via (52b), $\forall k \in \mathcal{V}$.

end for

for $k \in \mathcal{V}$ do

VBM step

Update $q^{(n)}(x_{k,t}, X_{k,t}) = \mathcal{N}(x_{k,t}; m_{k,t}^{(n)}, P_{k,t}^{(n)} \otimes X_{k,t}) \times \mathcal{T}W(X_{k,t}; v_{k,t}^{(n)}, V_{k,t}^{(n)})$ via (53).

Update $q^{(n)}(R_k) = \mathcal{T}W(R_k; \nu_{k,t}^{(n)}, U_{k,t}^{(n)})$ via (55).

end for

end for

Set $m_{k,t} = m_{k,t}^{(n)}$, $P_{k,t} = P_{k,t}^{(n)}, \forall k$.

Set $\nu_{k,t} = \nu_{k,t}^{(n)}$, $V_{k,t} = V_{k,t}^{(n)}, \forall k$.

Set $v_{k,t} = v_{k,t}^{(n)}$, $U_{k,t} = U_{k,t}^{(n)}, \forall k$.

Output:

$P(x_{k,t}|X_{k,t}, \gamma_i) = \mathcal{N}(x_{k,t}; m_{k,t}, P_{k,t} \otimes X_{k,t})$.

$P(X_{k,t}|\gamma_i) = \mathcal{T}W(X_{k,t}; v_{k,t}, V_{k,t})$.

$P(R_k|\gamma_i) = \mathcal{T}W(R_k; \nu_{k,t}, U_{k,t})$.

where $c_{k,t} = \phi_{k,t}^{-1} - (H_t \otimes I_d)m_{k,t-1}$.

b) Posterior of the measurement noise covariance:

Using the quantity $\phi_{k,t}^3$, each node $k$ updates the variational distribution $q^*(R_k)$ as

$q^*(R_k) = \mathcal{T}W(R_k; \hat{\nu}_{k,t}, \hat{U}_{k,t})$, \hspace{1cm} (55)

with the parameters given by

$\hat{\nu}_{k,t} = v_{k,t-1} + N_t$, \hspace{1cm} (56a)

$\hat{U}_{k,t} = U_{k,t-1} + N_t \phi_{k,t}^3$, \hspace{1cm} (56b)

For clarity, the distributed variational Bayesian measurement update is presented in Algorithm 2 and the distributed VB algorithm for extended object tracking (dVBEOT) is summarized in Algorithm 3.

Remark 1 (The case of known measurement noise covariance).

If the true measurement noise covariance $R_{true}$ is known in advance, the proposed algorithm can be easily adapted to this case and makes use of this prior information. To do this, we only need to replace the statistic $\langle R_k^{-1} \rangle = \hat{\nu}_{k,t}\hat{U}_{k,t}^{-1}$ by $R_{true}^{-1}$ in (46a) and (46b) to update $q^*(z_{k,t}^i)$, and eliminate the variational distribution $q^*(R_k)$ in the VBM step. Then, the distributed VB algorithm for extended object tracking with known measurement noise covariance is presented. We examine this algorithm in the simulation.

Remark 2 (The case of neglecting the sensor error). If sensor error is negligible compared to the object extension, the proposed algorithm can also be modified by setting $R = 0_d$. In detail, we eliminate the variational distribution $q^*(R_k)$ and replace $\langle R_k^{-1} \rangle = \hat{\nu}_{k,t}\hat{U}_{k,t}^{-1}$ by an extreme large value $\infty I_d$. Then the updates in (46a) and (46b) reduce to $\hat{\mu}_{k,t} = \hat{y}_{k,t}$ and $\hat{\Sigma}_{k,t} = 0_d$, i.e., the latent variable $z_{k,t}$ is equal to the measurement $y_{k,t}$. Therefore, latent variables are unchanged with the VB iteration, which means we only need to perform the VB step once. After this modification, we found that the VB measurement update becomes the same as the measurement update in Koch’s approach [24]. Therefore, as a special case of the proposed algorithm, the modified algorithm can be treated as a distributed implementation of the Koch’s approach.

Remark 3 (Group target tracking). As we mentioned in Introduction, a group of closely spaced targets can be considered as an extended object. The proposed distributed algorithm can also be used for group target tracking.

V. SIMULATIONS

In this section, the performance of the proposed distributed extended object tracking is evaluated via numerical simulations.

Our numerical simulations consider a randomly generated sensor network with $N = 20$ nodes. The nodes are randomly placed in a $2.5 \times 2.5$ square, and the communication distance is taken as $0.8$, as shown in Fig. Two scenarios were simulated: S1 for extended object tracking (EOT), and S2 for group target tracking (GTT).

For comparison, we also simulate the centralized VB algorithm for EOT (cVBEOT), in which all data is available in a fusion center as presented in Algorithm 1 and the corresponding non-cooperative algorithm (non-coopVBEOT), in which each node $k$ does not perform a consensus step and estimate the extended object state independently using local quantity $\omega_{k,t}$. We also simulate the distributed VB algorithm for EOT with true measurement noise covariance (dVBEOT-without-$R_{true}$) as presented in Remark 1 and the algorithm neglecting the sensor error (dVBEOT-without-R) as presented in Remark 3.
Besides, the Koch’s approach [24], which can be treated as a non-cooperative VBEOT without considering the sensor error, is also simulated.

A. Extended Object Tracking Scenario

Extended object tracking have been applied in many different scenarios and have been evaluated using data from many different sensors such as LIDAR, camera, radar, RGB-depth sensors, and unattended ground sensors [41]. For example, in harbours, there are many vessels share the water, from small boats to large ships. To keep track of where the vessels are, marine X-band radar can be used.

In S1, we simulate the scenario of [25] for maneuvering extended object tracking. The extended object (EO) is an ellipse with diameters of 340m and 80m (about the size of an aircraft carrier of the Nimitz-class [25]) in the (x,y)-plane (thus, d = 2). The trajectory is shown in Fig.3 where the speed was assumed constant at 27 knots (about 50km/h), and the formation went through a 45° and two 90° turns. We assumed that scattering centers are uniformly distributed over the extension X_t while measurements are subject to a zero-mean Gaussian noise with variance R. The true measurement noise is R_{true} = diag(\{\sigma_x^2, \sigma_y^2\}), where \sigma_x = 50m, \sigma_y = 50m. The number of measurements of each node at each time scan is Poisson distributed with mean 20. The scan time of each senor is \Delta_t = 10s. We set the scaling factor s = 1/4 for this scenario.

The parameters of the kinematical evolution model (7) are chosen as \Sigma = 1g, \theta = 40s (according to [24]). The temporal decay constant in (20a) is chosen as \tau = \Delta_t. The penalty parameter of the ADMM is set as \rho = 0.5. Assuming that no further information about the initial state and extension is given, we use the mean of the first measurements of each node k to initialize the position state and use \mathbb{E}[X_{k,1}] = I_d \ (with \ V_{k,1} = 0.1I_d, \ \nu_{k,1} = d+1+0.1)\) representing a circle with a radius of 1km to initialize the extension.

For performance evaluation of extended object estimates with ellipsoidal extents, we use the Gaussian Wasserstein Distance (GWD) metric as the measure of performance, which is best choice for comparing elliptic shapes [42]. The squared Wasserstein distance between two multivariate Gaussian is defined as [43]

\[
d_{GW}(\mathcal{N}_y,\mathcal{N}_\hat{y})^2 = ||\mu_y - \mu_\hat{y}||^2 + \text{tr} \left( \Sigma_y + \Sigma_\hat{y} - 2(\Sigma_y^{\frac{1}{2}} \Sigma_\hat{y}^{\frac{1}{2}}) \right) ,
\]

where \(\mathcal{N}_y(\mu_y, \Sigma_y)\) is the groundtruth ellipse and \(\mathcal{N}_\hat{y}(\mu_\hat{y}, \Sigma_\hat{y})\) is the estimated ellipse. The comparison results are the root Gaussian Wasserstein error (RGWE) over \(N_s\) independent Monte Carlo runs with randomly generate samples, calculated as follows:

\[
\text{RGWE}_{k,t} = \left( \frac{1}{N_s} \sum_{i=1}^{N_s} d_{GW}(\mathcal{N}_{k,t}, \mathcal{N}_i) \right)^{\frac{1}{2}},
\]

where \(\mathcal{N}_i((H_t \otimes I_d)x_{t}, sX_t)\) is the groundtruth ellipse and \(\mathcal{N}_{k,t,\hat{}}((H_t \otimes I_d)\hat{x}_{k,t}, s\hat{X}_{k,t})\) is the estimated ellipse at time scan t at node k. For the distributed algorithms, the final results are averaged over all nodes.

1) Convergence Study: We check the convergence of the proposed dVBEOT algorithm. We first evaluated the root Gaussian Wasserstein error (RGWE) of the dVBEOT with the VB iterations at different time scans, and four of them are shown in Fig.4. The number of the ADMM iterations in the consensus step is set as \(L = 50\). We observed that the proposed VB algorithm converges after only 10 iterations at all time scans. We then evaluated the RGWE performance with different number of the inner iterations L. As shown in Fig.5, we set the scaling factor s = 1/4 for this scenario.
the RGWE is getting smaller and smaller with the increasing of the number of the ADMM iterations $L$. When $L > 30$, the RGWE is almost unchanged, which means all nodes have achieved consensus on the expected sufficient statistics $\{\phi_{k,t}\}$ in every VB step. In the following simulations, we set $L = 30$ and run the VB step 20 times.

2) Tracking Performance: We compared the proposed dVBEOT algorithm with Koch’s approach, the non-coopVBEOT, the cVBEOT, the dVBEOT-without-$R$ and the dVBEOT-with-$R_{true}$. Fig 6 shows the tracking results of all six algorithms at a randomly selected node $k$. The estimated extension is represented by 90%-confidence ellipse. In Fig 7, we summarized the RGWE results of each algorithm over $N_s = 100$ Monte Carlo runs. As shown in Fig 6 and Fig 7, both Koch’s approach and the dVBEOT-without-$R$ overestimated the extension and have highest estimation error, since they do not consider the actual measurement noise. The non-coopVBEOT has a better RGWE performance than that of Koch’s approach and that of the dVBEOT-without-$R$, but it is still not so good. The proposed dVBEOT performs much better than the above three mentioned algorithms, and is almost as good as the corresponding centralized algorithm (cVBEOT), which utilizes all measurements in a fusion center. This shows that the proposed distributed algorithm can make use of the information from all nodes, which verifies the effectiveness of the proposed algorithm. The dVBEOT-with-$R_{true}$, which utilizes the prior information about the true measurement noise, has the best tracking performance and the lowest RGW error, and its estimated position and extension are almost same as the ground truth except the case when the formation of the extended object went turns (at time scan $t = 40, 80, 110$). Nevertheless, after the formation went turns, the proposed algorithm can amend its extension very quickly and still achieve very low RGW error.

B. Group target Tracking Scenario

In S2, we test the performance of the proposed algorithm on the group target tracking problem. In this scenario, a group of uniformly spaced five targets moves in a plane with the trajectories shown in Fig 8. Five individual targets fly with constant speed $v = 300$ m/s in the $(x,y)$-plane (thus, $d = 2$). The targets were arranged in a line with 500m distance between neighboring targets, where the formation first went through a 45° and two 90° turns (with radial accelerations 2g, 2g, 1g, respectively) before performing a split-off maneuver. The scan time of each sensor is $\Delta t = 10s$. The true measurement noise is $R_{true} = diag([\sigma_x^2, \sigma_y^2])$, where $\sigma_x = 500m$, $\sigma_y = 100$ m. We assume that each node has a probability of detection $P_d = 80\%$ for each target at each time scan $t$. We set $L = 50$, run the VB step 80 times and set other parameters the same as S1.

For ease of comparison, Fig 9 shows the tracking results of three algorithms: Koch’s approach, the dVBEOT and the dVBEOT-with-$R_{true}$. It is shown that the Koch’s approach often has large errors on the centroid and extension due to its sensitive to missed detections. While, the dVBEOT has a significant improvement on extended state estimation, especially on extension estimation. This is because the nodes in sensor networks can gather more measurements than a single node to give a more complete description on the extended object, and the proposed distributed algorithm can obtain this information through the cooperation among nodes. Besides, the dVBEOT can avoid overestimating the object size to some extent by estimating the sensor errors. Moreover, it is shown that the dVBEOT-with-$R_{true}$ can effectively utilize the prior information about the true measurement noise, and has best extended object state estimation performance. The comparison results demonstrate the effectiveness of the proposed distributed algorithms to the group target tracking.

VI. CONCLUSION

Consider the case that an object is spatially structured, we propose a distributed tracking algorithm for extended objects in sensor networks. We formulate a distributed Bayesian model for extended object tracking with unknown measurement noise. Based on the variational Bayesian methods, we derive a new measurement update for the estimation of the kinematic state and extension as well as the measurement noise covariance. Then, using the ADMM technique, we derive the corresponding distributed Bayesian tracking algorithm. Numerical simulations on both extended object tracking and group target tracking demonstrate that the proposed dVBEOT algorithm has superior performance for cases where sensor errors cannot be neglected any more in comparison with object extension. The proposed distributed algorithm performs as good as the corresponding centralized algorithm. Moreover, when the true measurement noise covariance is available, the dVBEOT can utilize this prior information and achieve much better performance.

APPENDIX

A. Wishart and Inverse Wishart Distributions

1) Wishart density: $W(X;n,W)$ denotes a Wishart density defined over the matrix $X \in \mathbb{S}_+^d$ with scalar degrees of
freedom $n > d - 1$ and parameter matrix $W \in S_{++}^d$, 

$$W_d(X; n, W) = \frac{|W|^{-\frac{d}{2}} |X|^{-\frac{n-d-1}{2}}}{2^{n/2} \Gamma_d\left(\frac{n}{2}\right)} \text{etr} \left(-\frac{1}{2} W^{-1} X \right),$$ (59)

where etr(·) = exp(tr(·)) is exponential of the matrix trace, and $\Gamma_d(·)$ is the multivariate gamma function. The expectation of $X$ is given by $E[X] = nW$.

2) Inverse Wishart density: If $X \sim W_d(X; n, W)$, then the random variance matrix $\Sigma = X^{-1}$ has an inverse Wishart distribution, denoted by 

$$\mathcal{IW}_d(\Sigma; n, W) = \frac{|W|^{\frac{d}{2}} |\Sigma|^{-\frac{n+d+1}{2}}}{2^{n/2} \Gamma_d\left(\frac{n}{2}\right)} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} W \right),$$ (60)

with scalar degrees of freedom $n > d - 1$ and parameter matrix $W \in S_{++}^d$. The expectation of $\Sigma$ is given by $E[\Sigma] = W/(n - d - 1)$ when $n > d + 1$. 

Fig. 6. Tracking results of all six algorithms for the sensor data of Fig. 3 at a randomly selected node ($k = 2$). The estimated extension is represented by 90%-confidence ellipse. Shown are, for each time scan $t$, the true centroid (block point), the groundtruth ellipse (black dotted line), the estimated centroid (red +), and the estimated extension (blue thick line).

Fig. 7. The root Gaussian Wasserstein error (RGWE) of all six algorithms for extended object tracking in S1.

Fig. 8. Group tracking scenario with 5 targets. (a) The true target position. (b) The measurements of a random selected node.
B. proof of (34)

From (33), grouping terms involving $x_t$, we have

$$
\ln^* q(x_t | X_t)
= -\frac{1}{2} x_t^T (P_{t|t-1}^{-1} \otimes X_t)^{-1}
+ \frac{n_t}{s} (X_t^{-1}) (sX_t^{-1}(H_t \otimes I_d)) x_t
+ (m_{t|t-1}^{-1}(P_{t|t-1}^{-1} \otimes X_t)^{-1}) + \frac{n_t}{s} (sX_t^{-1}(H_t \otimes I_d)) x_t
$$

where the second equation is derived by using the fact that $h$ is a row vector and

$$
(P_{t|t-1}^{-1} \otimes X_t)^{-1} = P_{t|t-1}^{-1} \otimes X_t^{-1},
$$

$$
(H_t \otimes I_d)^T X_t^{-1}(H_t \otimes I_d) = (H_t^T H_t) \otimes X_t^{-1},
$$

$$
X_t^{-1}(H_t \otimes I_d) = H_t \otimes X_t^{-1}.
$$

From (61), we conclude that $q^*(x_t | X_t)$ is Gaussian distributed,

$$
q^*(x_t | X_t) = \mathcal{N}(x_t; \hat{m}_t, \hat{P}_t \otimes X_t),
$$

with the parameters

$$
\hat{P}_t = (P_{t|t-1}^{-1} + \frac{n_t}{s} H_t^T H_t)^{-1},
$$

$$
\hat{m}_t = (\hat{P}_t \otimes X_t)((P_{t|t-1}^{-1} \otimes X_t^{-1})m_{t|t-1} + \frac{n_t}{s} (H_t^T \otimes X_t^{-1})\bar{z}_t).
$$

Furthermore, using the matrix inversion lemma, we can further simply the updates,

$$
\frac{n_t}{s} \hat{P}_t H_t^T = \frac{n_t}{s} (P_{t|t-1}^{-1} + \frac{n_t}{s} H_t^T H_t)^{-1} H_t^T
$$

$$
= \frac{n_t}{s} (I_3 + \frac{n_t}{s} P_{t|t-1}^{-1} H_t^T H_t)^{-1} P_{t|t-1}^{-1} H_t^T H_t
$$

$$
= \frac{n_t}{s} (I_3 - P_{t|t-1}^{-1} H_t^T (\frac{s}{N_t} + H_t P_{t|t-1} H_t^T)^{-1} H_t P_{t|t-1}^{-1} H_t^T)
$$

$$
= \frac{n_t}{s} P_{t|t-1}^{-1} H_t^T (I_3 - \frac{s}{N_t} H_t P_{t|t-1} H_t^T)^{-1} H_t P_{t|t-1} H_t^T
$$

$$
= P_{t|t-1}^{-1} H_t^T (\frac{s}{N_t} + H_t P_{t|t-1} H_t^T)^{-1} H_t P_{t|t-1} H_t^T
$$

$$
= u_t,
$$

and

$$
\hat{P}_t P_{t|t-1}^{-1} = (P_{t|t-1}^{-1} + \frac{n_t}{s} H_t^T H_t)^{-1} P_{t|t-1}^{-1}
$$

$$
= (I_3 + \frac{n_t}{s} P_{t|t-1}^{-1} H_t^T H_t)^{-1} P_{t|t-1}^{-1}
$$

$$
= I_3 - P_{t|t-1}^{-1} H_t^T (\frac{s}{N_t} + H_t P_{t|t-1} H_t^T)^{-1} H_t
$$

$$
= I_3 - u_t H_t.
where we define
\[ w_t \triangleq P_{t|t-1}H_T^T b_{t-1}, \]
\[ b_t \triangleq \frac{s}{N_t} + H_t P_{t|t-1}H_T^T. \]  
(67a, 67b)

Note that \( w_t \in \mathbb{R}^{2 \times 1} \) is a column vector, and \( b_t \in \mathbb{R} \) is a scalar.

Based on the above results, we can rewrite the parameters as
\[
\dot{m}_t = ((I_3 - w_tH_t) \otimes I_d)m_{t|t-1} + (w_t \otimes I_d)\dot{z}_t = ((I_3 - w_t \otimes I_d)(H_t \otimes I_d))m_{t|t-1} + (w_t \otimes I_d)\dot{z}_t
\]
\[ = m_{t|t-1} + (w_t \otimes I_d)(\dot{z}_t - (H_t \otimes I_d)m_{t|t-1}), \]
(68)

and
\[ \dot{P}_t = (I_3 - w_tH_t)P_{t|t-1} = P_{t|t-1} - \dot{w}_t b_t w_T. \]  
(69)

Thus, the variational distribution \( q^*(x_t, X_t) \) in (33) can be rewritten as
\[
\ln q^*(x_t, X_t) = \ln \mathcal{N}(x_t; \dot{m}_t, \dot{P}_t \otimes X_t) + \frac{1}{2} \dot{m}_T(\dot{P}_t \otimes X_t)^{-1}\dot{m}_t
\]
\[ - \frac{1}{2} \text{tr}(X_t^{-1}(N_{ti} - V_{ti(-1)})) - \frac{1}{2} \frac{1}{s} \dot{z}_T X_t^{-1} \dot{z}_t
\]
\[ - \frac{1}{2}\dot{m}_{t|t-1}((P_{t|t-1}^{-1} \otimes X_t^{-1})m_{t|t-1} - (N_t + \nu_t|t-1) + d + 1 \ln |X_t| + c. \]  
(70)

Let us define
\[
\Delta_t \triangleq \frac{M}{s}\dot{z}_T X_t^{-1} \dot{z}_t + \frac{M}{s} \dot{m}_{t|t-1}((P_{t|t-1}^{-1} \otimes X_t^{-1})m_{t|t-1} - (N_t + \nu_t|t-1) + d + 1 \ln |X_t| + c. \]  
(71)

From (61), we have
\[
\dot{m}_T(\dot{P}_t \otimes X_t)^{-1}\dot{m}_t = (m_{t|t-1} + (w_t \otimes I_d)(\dot{z}_t - (H_t \otimes I_d)m_{t|t-1}))^T
\]
\[ \times ((P_{t|t-1}^{-1} \otimes X_t^{-1})m_{t|t-1} + \frac{M}{s}(H_T^T \otimes X_t^{-1})\dot{z}_t). \]  
(72)

Moreover,
\[
(w_t^T \otimes I_d)(P_{t|t-1}^{-1} \otimes X_t^{-1}) = b_{t-1}^1 H_t \otimes X_t^{-1}, \]
\[ (w_t^T \otimes I_d)(H_T^T \otimes X_t^{-1}) = w_t^T H_T^T X_t^{-1}, \]  
(73a, 73b)

\[ H_T^T \otimes X_t^{-1} = (H_T^T \otimes I_d)X_t^{-1}. \]  
(73c)

Therefore, we have
\[
\Delta_t = \frac{M}{s} \dot{z}_t^T X_t^{-1} \dot{z}_t - \frac{M}{s} \dot{z}_t^T X_t^{-1}(H_t \otimes I_d)m_{t|t-1}
\]
\[ - (\dot{z}_t - (H_t \otimes I_d)m_{t|t-1})^T(b_{t-1}^1 H_t \otimes I_d))m_{t|t-1} + \frac{M}{s} w_t^T H_T^T X_t^{-1} \dot{z}_t
\]
\[ = \frac{M}{s} \dot{z}_t^T (1 - w_t^T H_T^T) X_t^{-1} \dot{z}_t
\]
\[ + \frac{M}{s} \dot{z}_t^T b_{t-1}^1 H_t w_t X_t^{-1}(H_t \otimes I_d)m_{t|t-1}
\]
\[ + \frac{M}{s} \dot{m}_{t|t-1}((H_T^T \otimes I_d)b_{t-1}^1 X_t^{-1}(H_t \otimes I_d)m_{t|t-1}. \]  
(74)

Using (67a), we have
\[
\Delta_t = \frac{M}{s}(1 - w_t^T H_T^T) = b_{t-1}^1 \text{ and } - \frac{M}{s} - b_{t-1}^1 + \frac{M}{s} H_t w_t = -2b_{t-1}^1. \]  
Thus,
\[ \Delta_t = (\dot{z}_t - (H_t \otimes I_d)m_{t|t-1})^T(b_{t-1}^1 X_t^{-1}(\dot{z}_t - (H_t \otimes I_d)m_{t|t-1}). \]  
(75)

Let us define
\[ K_t \triangleq b_{t-1}^{-1}(\dot{z}_t - (H_t \otimes I_d)m_{t|t-1})(\dot{z}_t - (H_t \otimes I_d)m_{t|t-1})^T. \]  
(76)

We can rewrite \( q^*(x_t, X_t) \) in (70) as
\[
\ln q^*(x_t, X_t) = \ln \mathcal{N}(x_t; \dot{m}_t, \dot{P}_t \otimes X_t)
\]
\[ - \frac{1}{2} \text{tr}(X_t^{-1}(N_t - V_{ti(-1)} + K_t))
\]
\[ - \frac{1}{2}(N_t + \nu_t|t-1) + d + 1 \ln |X_t| + c. \]  
(77)

where
\[ \dot{V}_t = \frac{M}{s}S_t + K_t + V_{ti(-1)}. \]  
(78a)

Thus, we obtain (54).

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