TRANSVERSAL LATTICES

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Abstract. A flat of a matroid is cyclic if it is a union of circuits; such flats form a lattice under inclusion and, up to isomorphism, all lattices can be obtained this way. A lattice is a Tr-lattice if all matroids whose lattices of cyclic flats are isomorphic to it are transversal. We investigate some sufficient conditions for a lattice to be a Tr-lattice; a corollary is that distributive lattices of dimension at most two are Tr-lattices. We give a necessary condition: each element in a Tr-lattice has at most two covers. We also give constructions that produce new Tr-lattices from known Tr-lattices.

1. Introduction

A flat $X$ of a matroid $M$ is cyclic if the restriction $M|X$ has no isthmuses. Ordered by inclusion, the cyclic flats form a lattice, which we denote by $Z(M)$.

Every lattice is isomorphic to the lattice of cyclic flats of some (bi-transversal) matroid [4] [8]. (All lattices considered in this paper are finite.) For certain lattices $L$, it is shown in [1, 2] that if $Z(M)$ is isomorphic to $L$, then the matroid $M$ is transversal; lattices with this property are transversal lattices or Tr-lattices. In [4], lattices of width at most two are shown to be Tr-lattices. In this paper we treat a more general sufficient condition for a lattice to be a Tr-lattice, we prove a necessary condition, and we show that the class of Tr-lattices is closed under certain lattice operations.

Following a section of background, Section 3 introduces MI-lattices and shows they are Tr-lattices. Special cases (e.g., distributive lattices of dimension at most two) are also treated. Section 4 shows that each element of a Tr-lattice has at most two covers. Section 5 gives ways to construct new MI-lattices (resp., Tr-lattices) from known MI-lattices (resp., Tr-lattices). Some open problems suggested by this work are mentioned in the concluding section.

2. Background

We assume familiarity with basic matroid theory. Our notation and terminology for matroid theory follow [7]; for ordered sets we mostly follow [10]. For a collection $\mathcal{F}$ of sets, we write $\bigcap \mathcal{F}$ for the intersection $\bigcap_{X \in \mathcal{F}} X$ and $\bigcup \mathcal{F}$ for $\bigcup_{X \in \mathcal{F}} X$.

Recall that every ordered set $P$ can be embedded in a product of chains; the dimension of $P$ is the least number of chains for which there is such an embedding. The lattices of dimension 2 are the planar lattices: their Hasse diagrams can be drawn in the plane without crossings (see, e.g., [10 Chapter 3, Theorem 5.1]). An antichain in an ordered set is a collection of mutually incomparable elements. The width of an ordered set is the maximal cardinality among its antichains. We say $y$ is...
a cover of $x$ in an ordered set $P$ if $x < y$ and there is no $z$ in $P$ with $x < z < y$. The least and greatest elements in a lattice are denoted $0$ and $\hat{1}$, respectively. The atoms of a lattice are the elements that cover $0$; dually, the coatoms are the elements that $\hat{1}$ covers. An ideal in an ordered set $P$ is a subset $I$ of $P$ such that if $x \in I$ and $y \leq x$, then $y \in I$. Dually, a filter in $P$ is a subset $F$ such that if $x \in F$ and $y \geq x$, then $y \in F$.

It is well known and easy to see that while nonisomorphic matroids can have the same cyclic flats, a matroid on a given set is determined by its collection of cyclic flats along with their ranks. In some cases we will want to ignore the cyclic flats and instead focus on the ranks assigned to the elements of an abstract lattice; this is justified by the following special case of [8, Theorem 1].

**Proposition 2.1.** Let $L$ be a lattice. Given $\rho : L \to \mathbb{Z}$ with

(a) $\rho(0) = 0$,
(b) $\rho(x) < \rho(y)$ whenever $x < y$, and
c) $\rho(x \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y)$ whenever $x$ and $y$ are incomparable,

there is a matroid $M$ and an isomorphism $\phi : L \to \mathcal{Z}(M)$ with $\rho(x) = \rho(\phi(x))$.

A key result we use to prove that certain lattices are (or are not) Tr-lattices is the following characterization of transversal matroids, which was first formulated by Mason using cyclic sets and later refined to cyclic flats by Ingleton [5]. (The statement in [5] uses all nonempty collections of cyclic flats, but an elementary argument shows that it suffices to consider nonempty antichains of cyclic flats; see the discussion after [4, Lemma 5.6].)

**Proposition 2.2.** A matroid $M$ is transversal if and only if for every nonempty antichain $A$ in $\mathcal{Z}(M)$,

$$r(\bigcap A) \leq \sum_{F \subseteq A} (-1)^{|F|+1} r(\bigcup F).$$

The join in $\mathcal{Z}(M)$ (as in the lattice of flats) is given by $A \vee B = \text{cl}(A \cup B)$, so one can replace the alternating sum in inequality (MI) by the corresponding alternating sum of ranks of joins of cyclic flats.

Unlike in the lattice of flats, the meet operation in $\mathcal{Z}(M)$ might not be intersection: $X \wedge Y$ is the union of the circuits that are contained in $X \cap Y$.

Since the complements of the flats of a matroid are the unions of its cocircuits, $X$ is a cyclic flat of $M$ if and only if $E(M) - X$ is a cyclic flat of the dual, $M^*$. Thus, $\mathcal{Z}(M^*)$ is isomorphic to the order dual of $\mathcal{Z}(M)$.

Let $S$ and $E$ be the least and greatest cyclic flats of $M$. Note that for $X \in \mathcal{Z}(M)$, the lattice $\mathcal{Z}(M|X)$ is the interval $[S, X]$ in $\mathcal{Z}(M)$ and, dually, the lattice $\mathcal{Z}(M/X)$ is isomorphic to the interval $[X, E]$ in $\mathcal{Z}(M)$ via the isomorphism $Y \mapsto Y \cup X$. (The lattices of cyclic flats of other minors are not as simple to describe.)

3. **Sufficient conditions for a lattice to be a Tr-lattices**

To convey the spirit of the main result of this section (Theorem 3.3) before defining the technical condition involved, we cite the following theorem, which, as we will show, is implied by the main result.

**Theorem 3.1.** If (a) $\mathcal{Z}(M)$ has dimension at most two and (b) for each antichain $A$ of $\mathcal{Z}(M)$, the sublattice of $\mathcal{Z}(M)$ generated by $A$ is distributive, then $M$ and all of its minors, as well as their duals, are transversal.
Corollary 3.2. If \( Z(M) \) is distributive and has dimension at most two, then \( M \) and all of its minors, as well as their duals, are transversal.

The main result of this section uses the following notions.

Definition 3.3. An MI-ordering of an antichain \( A \) in a lattice \( L \) is a permutation \( a_1, a_2, \ldots, a_t \) of \( A \) so that

(i) \( a_i \lor a_{i+1} \lor \cdots \lor a_k = a_i \lor a_k \) for \( 1 \leq i < k \leq t \) and
(ii) \( (a_1 \land a_2 \land \cdots \land a_k) \lor a_{k+1} = a_k \lor a_{k+1} \) for \( 1 < k < t \).

An antichain is MI-orderable if it has an MI-ordering. A lattice is MI-orderable, or is an MI-lattice, if each of its antichains is MI-orderable.

Theorem 3.4. Let \( M \) be a matroid.

(i) Each MI-orderable antichain in \( Z(M) \) satisfies inequality (MI).
(ii) If \( Z(M) \) is MI-orderable, then \( M \) and all of its minors are transversal.

Corollary 3.5. MI-lattices are Tr-lattices.

Before proving Theorem 3.4 we note a subtlety that explains the approach we take to prove part (ii): if \( N \) is a minor of \( M \) and \( Z(M) \) is MI-orderable, then \( Z(N) \) may or may not be MI-orderable. Indeed, \( Z(N) \) may not even be a Tr-lattice, and this is so even for deletions of \( M \). (Recall that the class of transversal matroids is closed under deletions but not under contractions, so one might expect deletions to be somewhat more tame.) For example, for the matroid \( M \) in Figure 1, \( Z(M) \) is MI-orderable. Since this lattice is isomorphic to its order dual, \( Z(M^*) \) is also MI-orderable. The lattice \( Z(M/x) \) is also shown; by checking directly or applying Theorem 5.1 we have that \( Z(M^*/x) \) is MI-orderable. However, by Theorem 3.4 its order dual, which is \( Z(M^*\!\setminus\!x) \), is not a Tr-lattice. This example also shows that the minor-closed, dual-closed class of matroids described in Theorem 3.1 is not determined by lattice-theoretic properties that apply to the lattices of cyclic flats of all matroids in the class.

We prove Theorem 3.4 via a sequence of lemmas. The first lemma gives a rank inequality associated with each MI-orderable antichain of \( Z(M) \). Note that for two-element antichains, this inequality is the semimodular inequality. (The meet and join operations in this and other results are in \( Z(M) \).

Lemma 3.6. Let \( A_1, A_2, \ldots, A_t \) be an antichain of cyclic flats in a matroid \( M \) such that \( (A_1 \land A_2 \land \cdots \land A_k) \lor A_{k+1} = A_k \lor A_{k+1} \) whenever \( 1 \leq k < t \). Then for
Adding this inequality to inequality (1) gives

$$\sum_{i=1}^{k} r(A_i) - \sum_{i=1}^{k-1} r(A_i \cup A_{i+1}).$$

Proof. We prove the inequality by induction on \( k \). Equality holds for \( k = 1 \). Assume the result holds in case \( k \). Semimodularity gives

$$r(A_1 \cap A_2 \cap \cdots \cap A_{k+1}) + r((A_1 \cap A_2 \cap \cdots \cap A_k) \cup A_{k+1}) - r(A_1 \cap A_2 \cap \cdots \cap A_k) \leq r(A_{k+1}).$$

Adding this inequality to inequality (1) gives

$$r(A_1 \cap A_2 \cap \cdots \cap A_{k+1}) + r((A_1 \cap A_2 \cap \cdots \cap A_k) \cup A_{k+1}) \leq \sum_{i=1}^{k+1} r(A_i) - \sum_{i=1}^{k-1} r(A_i \cup A_{i+1}),$$

so if we show \( r((A_1 \cap A_2 \cap \cdots \cap A_k) \cup A_{k+1}) = r(A_{k+1}) \), then the inequality we want follows. This equality holds since \( A_1 \cap A_2 \cap \cdots \cap A_k \subseteq A_1 \cup A_2 \cap \cdots \cap A_k \subseteq A_k \) and \( (A_1 \cap A_2 \cap \cdots \cap A_k) \cup A_{k+1} = A_k \cap A_{k+1} \). □

Lemma 3.7. If an antichain \( A \) in \( \mathcal{Z}(M) \) can be ordered as \( A_1, A_2, \ldots, A_t \) so that

(i) \( A_i \cup A_{i+1} \cup \cdots \cup A_k = A_i \cup A_k \) whenever \( 1 \leq i < k \leq t \) and

(ii) \( r(A_1 \cap A_2 \cap \cdots \cap A_t) \leq \sum_{i=1}^{t} r(A_i) - \sum_{i=1}^{t-1} r(A_i \cup A_{i+1}) \),

then \( A \) satisfies inequality (MI).

Proof. Assume properties (i) and (ii) hold. For \( 1 \leq i \leq j \leq t \), set

**A**_{i,j} = \{ F : F \subseteq A, i = \min(k : A_k \in F), \text{ and } j = \max(k : A_k \in F) \}.

Thus, if \( F \in A_{i,j} \), then \( cl(F) = A_i \cup A_j \). If \( j > i + 1 \), then the terms on the right side of inequality (MI) that arise from the sets in **A**_{i,j} cancel since there is a parity-switching involution \( \phi \) of **A**_{i,j}: fix \( k \) with \( i < k < j \) and let

$$\phi(F) = \begin{cases} 
F \cup \{ A_k \}, & \text{if } A_k \notin F; \\
F \setminus \{ A_k \}, & \text{if } A_k \in F.
\end{cases}$$

Thus, inequality (MI) reduces to the inequality that is assumed in property (ii). □

The previous two lemmas show that MI-lattices are Tr-lattices. To prove the stronger assertion in part (ii) of Theorem 3.4, we show that if the antichains in \( \mathcal{Z}(M) \) satisfy the hypotheses of Lemma 3.7, then so do the antichains of single-element deletions and single-element contractions of \( M \). (Note that unlike the hypotheses of Theorem 3.4, condition (ii) in Lemma 3.7 is not a lattice-theoretic property.) We start with a lemma about the cyclic flats of such minors.

Lemma 3.8. For an element \( x \) of \( M \) and a cyclic flat \( A \) of either \( M \backslash x \) or \( M/x \), the flat \( \bar{A} = cl_M(A) \) of \( M \) is cyclic; furthermore, \( \bar{A} \) is either \( A \) or \( A \cup x \), so \( \bar{A} - x = A \).

Proof. For a cyclic flat \( A \) of \( M \backslash x \), the assertions are transparent. Let \( A \) be a cyclic flat of \( M/x \) and let \( S \) be the ground set of \( M/x \). Thus, \( S \leftarrow A \) is a cyclic flat of the dual of \( M/x \), that is, of \( M^* \backslash x \), so \( cl_{M^*}(S \leftarrow A) \), which is either \( S \leftarrow A \) or \( (S \leftarrow A) \cup x \), is a cyclic flat of \( M^* \). Therefore either \( A \cup x \) or \( A \) is a cyclic flat of \( M \), from which the result follows. □

Lemma 3.9. If each antichain in \( \mathcal{Z}(M) \) can be ordered so that properties (i) and (ii) of Lemma 3.7 hold, then the same is true for each antichain in \( \mathcal{Z}(M \backslash x) \) and each antichain in \( \mathcal{Z}(M/x) \).
Proof. The proofs for $\mathcal{Z}(M\setminus x)$ and $\mathcal{Z}(M/x)$ are similar and, since each deletion of a transversal matroid is transversal, only the result about contractions is needed to prove Theorem 3.4, so we treat only $\mathcal{Z}(M/x)$. We use the notation $\tilde{A}$ of Lemma 3.8.

Let $\mathcal{A}$ be an antichain in $\mathcal{Z}(M/x)$. Note that $\{\tilde{A} : A \in \mathcal{A}\}$ is an antichain in $\mathcal{Z}(M)$. By hypothesis, there is an ordering $A_1, A_2, \ldots, A_t$ of $\mathcal{A}$ so that in $M$ and $\mathcal{Z}(M)$,

\[ \tilde{A}_i \vee \tilde{A}_{i+1} \vee \cdots \vee \tilde{A}_k = \tilde{A}_i \vee \tilde{A}_k, \quad \text{for } 1 \leq i < k \leq t, \]

and

\[ r_M(\tilde{A}_1 \cap \tilde{A}_2 \cap \cdots \cap \tilde{A}_t) + \sum_{i=1}^{t-1} r_M(\tilde{A}_i \cup \tilde{A}_{i+1}) \leq \sum_{i=1}^t r_M(\tilde{A}_i). \]

Since $\tilde{A}_j = \text{cl}_M(A_j)$ and since $A \vee B$ in $\mathcal{Z}(M)$ is $\text{cl}_M(A \cup B)$, by equation (2), $A_i \cup A_{i+1} \cup \cdots \cup A_k$ and $A_i \cup A_k$ have the same closure in $M$, and so in $M/x$; thus, as needed, $A_i \vee A_{i+1} \vee \cdots \vee A_k = A_i \vee A_k$ in $\mathcal{Z}(M/x)$. The rank inequality in $M/x$ is immediate if $x$ is a loop of $M$, so assume this is not the case. Assume $x$ is in exactly $h$ of the cyclic flats $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_t$ of $M$. Thus,

\[ h + \sum_{i=1}^t r_{M/x}(A_i) = \sum_{i=1}^t r_M(\tilde{A}_i). \]

Also, since $x$ is in at least $h$ of the sets $\tilde{A}_1 \cap \tilde{A}_2 \cap \cdots \cap \tilde{A}_t$ and $\tilde{A}_i \cup \tilde{A}_{i+1}$, we have

\[ h + r_{M/x}(A_1 \cap A_2 \cap \cdots \cap A_t) + \sum_{i=1}^{t-1} r_{M/x}(A_i \cup A_{i+1}) \]

\[ \leq r_M(\tilde{A}_1 \cap \tilde{A}_2 \cap \cdots \cap \tilde{A}_t) + \sum_{i=1}^{t-1} r_M(\tilde{A}_i \cup \tilde{A}_{i+1}). \]

The last two conclusions and inequality (3) give

\[ r_{M/x}(A_1 \cap A_2 \cap \cdots \cap A_t) + \sum_{i=1}^{t-1} r_{M/x}(A_i \cup A_{i+1}) \leq \sum_{i=1}^t r_{M/x}(A_i), \]

as needed. □

The lemmas above complete the proof of Theorem 3.4. We now show that $\mathcal{Z}(M^*)$ is isomorphic to the order dual of $\mathcal{Z}(M)$, so $M^*$ satisfies the hypotheses of Theorem 3.1 if and only if $M$ does. Thus, the next lemma suffices to prove Theorem 3.4.

Lemma 3.10. If a lattice $L$ has dimension at most two and each of its antichains generates a distributive sublattice, then $L$ is MI-orderable.

Proof. We may assume $L$ is a suborder of $\mathbb{N}^2$. List the elements of an antichain as $a_1, a_2, \ldots, a_t$ where $a_i = (x_i, y_i)$ with $x_1 > x_2 > \cdots > x_t$; thus, $y_1 < y_2 < \cdots < y_t$. Clearly $a_i \vee a_{i+1} \vee \cdots \vee a_k \geq a_i \vee a_k$. Let $a_i \vee a_k$ be $(p, q)$. Thus, $p \geq x_i$ and $q \geq y_k$, so $(p, q) \geq (x_j, y_j)$ for $i \leq j \leq k$, and so $a_i \vee a_{i+1} \vee \cdots \vee a_k = a_i \vee a_k$. One gets $a_i \wedge a_{i+1} \wedge \cdots \wedge a_k = a_i \wedge a_k$ similarly, so property (ii) of Definition 3.3 can be rewritten as $(a_1 \wedge a_k) \vee a_{k+1} = a_k \vee a_{k+1}$, or, by the distributive law, $(a_1 \vee a_{k+1}) \wedge (a_k \vee a_{k+1}) = a_k \vee a_{k+1}$, that is, $a_1 \vee a_{k+1} \geq a_k \vee a_{k+1}$. This property holds since $a_1 \vee a_{k+1} = a_1 \vee a_2 \vee \cdots \vee a_k \vee a_{k+1} \geq a_k \vee a_{k+1}$. □
Antichains of at most two elements are trivially MI-orderable, so Theorem 3.4 has the following corollary (as do Theorem 3.1 and Proposition 2.2).

**Corollary 3.11.** [14 Theorem 5.7.] *Lattices of width at most two are Tr-lattices.*

Section 5 includes examples of lattices to which Theorem 3.4 but not Theorem 3.1 applies, as well as Tr-lattices that are not MI-lattices.

4. A NECESSARY CONDITION FOR A LATTICE TO BE A TR-LATTICE

Condition (ii) of Definition 3.3 is violated by any three covers of a given element, so each element of an MI-lattice has at most two covers. In this section, we show that the same is true of any Tr-lattice. (The examples in the next section show there is no bound on the number of elements that an element in a Tr-lattice covers.)

**Theorem 4.1.** *Each element of a Tr-lattice has at most two covers.*

**Proof.** Let the element \( x \) of a lattice \( L \) have at least three covers. We prove that \( L \) is not a Tr-lattice by defining a function \( \rho : L \to \mathbb{Z} \) so that properties (a)–(c) in Proposition 2.1 hold and inequality (MI) fails. For \( y \in L \), let \( F_y \) be the principal filter \( \{ u : u \geq y \} \) in \( L \). Thus, the sublattice \( F_y \) of \( L \) has at least three atoms.

Define \( \rho' : L \to \mathbb{Z} \) by \( \rho'(y) = |L - F_y| \). It follows easily that \( \rho' \) satisfies properties (a)–(c) in Proposition 2.1. For \( u, v, w \in F_x - \{ x \} \), let

\[
m(u, v, w) = \rho'(u) + \rho'(v) + \rho'(w) - \rho'(u \lor v) - \rho'(u \lor w) - \rho'(v \lor w) + \rho'(u \lor v \lor w) - \rho'(x).
\]

By inclusion-exclusion, \( m(u, v, w) = |F_x - (F_u \cup F_v \cup F_w)| \). Set

\[
k = \min\{m(u, v, w) : u, v, w > x\} = |F_x| - \max\{|F_u \cup F_v \cup F_w| : u, v, w > x\}.
\]

Thus, \( k \) is the minimal size of the complement, in \( F_x \), of the union of three proper principal filters in \( F_x \). Note that if \( k = m(u, v, w) \), then \( u, v, w \) are distinct covers of \( x \). Define \( \rho : L \to \mathbb{Z} \) by

\[
\rho(y) = \begin{cases} 
\rho'(y), & \text{if } y \leq x, \\
\rho'(x) - k - 1, & \text{otherwise}.
\end{cases}
\]

Clearly \( \rho \) satisfies property (a) of Proposition 2.1. Properties (b) and (c) for \( \rho' \) follow from these properties for \( \rho' \) except in two cases, which we address below:

(i) \( \rho(y) < \rho(z) \) if \( y < z \), \( y \leq x \), and \( z \notin x \), and

(ii) \( \rho(y) + \rho(z) \geq \rho(y \lor z) + \rho(y \land z) \) if \( y \not\leq x \), \( z \notin x \), and \( y \land z \leq x \).

Assume \( y < z \), \( y \leq x \), and \( z \notin x \). Thus, \( F_x \subseteq F_y \). The inequality in statement (i) reduces to \( k + 2 \leq \rho'(z) - \rho'(y) = |F_y - F_z| \). Note that \( F_z \cap F_x \) is the principal filter \( F_{z \lor x} \), which, since \( z \notin x \), is properly contained in \( F_z \); thus, there are at least \( k + 2 \) elements in \( F_z - F_x \), and so in \( F_y - F_z \), which proves statement (i).

Now assume \( y \not\leq x \), \( z \notin x \), and \( y \land z \leq x \). The inequality in statement (ii) is

\[
|L| - |F_y| - k - 1 + |L| - |F_z| - k - 1 \geq |L| - |F_{y \lor z}| - k - 1 + |L| - |F_{y \land z}|,
\]

that is, \( |F_{y \land z} - (F_y \cup F_z)| \geq k + 1 \). Note that \( F_x \subseteq F_{y \land z} \) and

\[
(F_y \cup F_z) \cap F_x = (F_y \cap F_x) \cup (F_z \cap F_x) = F_{y \lor z} \cup F_{z \lor x},
\]

which is the union of two principal filters that are properly contained in \( F_x \); thus, there are at least \( k + 1 \) elements in \( F_x - (F_y \cup F_z) \) and so in \( F_{y \land z} - (F_y \cup F_z) \), which proves statement (ii).
Figure 2. The lattices Acketa considered.

Let $M$ be a matroid arising from $L$ and $\rho$ as in Proposition 2.1. Fix $u, v, w$ with $k = m(u, v, w)$ and let $U$, $V$, and $W$ be the corresponding cyclic flats of $M$. The definitions of $m$ and $\rho$ give
\[
r(U) + r(V) + r(W) - r(U \cup V) - r(U \cup W) - r(V \cup W) + r(U \cup V \cup W) = r(X) - 1.
\]
Since $r(X) \leq r(U \cap V \cap W)$, it follows that the antichain $\{U, V, W\}$ of $Z(M)$ does not satisfy inequality (MI). Thus, $M$ is not transversal, so $L$ is not a Tr-lattice.

A matroid $M$ is nested if $Z(M)$ is a chain. These matroids have arisen many times in a variety of contexts (see [3, Section 4] for more information). That $Z(M \oplus N)$ is isomorphic to the product $Z(M) \times Z(N)$ gives the following corollary.

**Corollary 4.2.** If $Z(M)$ is a Tr-lattice, then the matroid obtained from $M$ by deleting all loops and isthmuses is either a direct sum of at most two nested matroids or it is connected.

5. Examples and constructions

This section gives examples of MI-lattices to which Theorem 3.1 does not apply and Tr-lattices that are not MI-lattices. We also show how to construct new MI-lattices from given MI-lattices, and likewise for Tr-lattices.

Acketa [1, 2] proved that chains and the lattices $L_1$, $L_2$, and $L_3$ of Figure 2 are Tr-lattices (Corollary 3.11 applies); he noted that $L_4$ is not a Tr-lattice; he conjectured that $L_5$, $L_6$, and $L_7$ are Tr-lattices (Corollary 3.11 applies to $L_5$ and $L_6$; Theorem 4.1 shows that $L_7$ is not a Tr-lattice); he proved that $L_8$ is a Tr-lattice; he also showed that $L_9$ (the dual of $L_8$) is not a Tr-lattice. We note that $L_8$ is in an infinite family of MI-lattices; Figure 3a gives another such lattice. The defining properties of these lattices are that the interval between $\hat{0}$ and any coatom is a chain, and for one of these chains (e.g., the left-most chain in Figure 3a), all other such chains intersect it in different initial segments.

Sublattices of MI-lattices are clearly MI-lattices. The next result gives another simple construction for MI-lattices. (See Figure 3b.)

**Theorem 5.1.** For any ideal $I$ in an MI-lattice $L$, the lattice $L_1$ induced on the set $I \cup \{1\}$ by the same order is MI-orderable.

**Proof.** Each antichain $A$ of $L_1$ is an antichain of $L$; order $A$ so that properties (i) and (ii) of Definition 3.3 hold in $L$. Let $z$ be the join of $\{a_i, a_i+1, \ldots, a_k\}$ and of
Figure 3. (a): A generic lattice like $L_8$. (b): A lattice $L_I$ obtained from an ideal in a product of two three-element chains.

Figure 4. Three nonplanar Tr-lattices; only $D^d$ is MI-orderable.

\[ \{a_i, a_k\} \] in $L$. If $z \in I$, then $z$ is the join of each of these sets in $L_I$, otherwise both sets have join $\hat{1}$ in $L_I$. Thus, property (i) holds in $L_I$. The same ideas show that property (ii) holds in $L_I$ since the meet operations are identical in $L$ and $L_I$. \hfill $\square$

Recall that the linear sum (or ordinal sum) of partial orders $P$ and $Q$, where $P$ and $Q$ are disjoint, is the order on $P \cup Q$ in which the restriction to $P$ is the order on $P$, the restriction to $Q$ is the order on $Q$, and every element of $P$ is less than every element of $Q$. The following result is immediate.

**Theorem 5.2.** The class of MI-lattices is closed under linear sums.

The same result holds for the closely-related operation that, given lattices $L$ and $L'$, forms the Hasse diagram of the new lattice from those of $L$ and $L'$ by identifying the greatest element of $L$ with the least element of $L'$. It follows from Theorem 5.3 below that the same two results hold for Tr-lattices. By Theorem 4.1, the class of MI-lattices and the class of Tr-lattices are not closed under direct products.

We next treat three particular Tr-lattices of dimension 3, only one of which is MI-orderable. These lattices, which are shown in Figure 4, are among the forbidden sublattices for planar lattices (see [6]). (No other forbidden sublattices for planar lattices satisfy the necessary condition for Tr-lattices given in Theorem 4.1.)

**Theorem 5.3.** The lattice $D^d$ is MI-orderable. The lattices $F_0$ and $C$ are Tr-lattices that are not MI-orderable.
Proof. The sublattice of $D^d$ formed by removing $b$ is the linear sum of the lattice in Figure 3b and a single-element lattice. Since this linear sum is MI-orderable, we need only check that the antichains in $D^d$ that contain $b$ are MI-orderable. Antichains of two elements are automatically MI-orderable; the only larger antichain in $D^d$ that includes $b$ is $\{a, b, c\}$, for which $b, a, c$ is an MI-ordering.

In $F_0$, the antichains of more than two elements are $\{A, S, X\}$, $\{X, T, D\}$, and $\{X, A, D\}$. The first two are MI-orderable (ordered as written), so we need only show that in any matroid $M$ for which $Z(M)$ is isomorphic to $F_0$, the flats corresponding to $X, A, D$ (for which we use the same notation) satisfy inequality (MI), which in this case is $r(X) + r(A) + r(D) - r(R) - r(E) \geq r(X \cap A \cap D)$. By semimodularity,

$$r(A) + r(S) \geq r(E) + r(A \cap S).$$

The inclusions $T \subseteq A \cap S \subseteq S$ give $cl((A \cap S) \cup X) = R$, so

$$r(A \cap S) + r(X) \geq r(R) + r(A \cap S \cap X).$$

The inclusions $U \subseteq A \cap S \cap X \subseteq S$ give $cl((A \cap S \cap X) \cup D) = S$, so

$$r(A \cap S \cap X) + r(D) \geq r(S) + r(A \cap S \cap X \cap D).$$

Note that $A \cap S \cap X \cap D$ is $A \cap X \cap D$. Adding the three inequalities and simplifying yields the desired inequality.

A similar argument applies to the lattice $C$, for which it suffices to consider the antichains $\{A, B, Y\}$, $\{A, W, Y\}$, $\{B, V, Y\}$, and $\{V, W, Y\}$. The last three are listed in MI-orderings. For $\{A, B, Y\}$, apply semimodularity to the pairs $\{A, S\}$, $\{A \cap S, B\}$, and $\{A \cap S \cap B, Y\}$; the inclusions $V \subseteq A \cap S \subseteq S$ and $T \subseteq A \cap S \cap B \subseteq S$ give $cl((A \cap S) \cup B) = E$ and $cl((A \cap S \cap B) \cup Y) = S$; add the resulting inequalities and cancel the common terms to get inequality (MI) for $\{A, B, Y\}$. \hfill $\square$

We now consider two operations for producing new Tr-lattices. Given lattices $L_1$ and $L_2$, let $L_1 \ast L_2$ be the lattice on $\{L_1 \cup L_2 \cup \{0, 1\} \} - \{\hat{1}_{L_1}, \hat{1}_{L_2}\}$ with $x \leq y$ if and only if (i) $y = 1$, or (ii) $x = 0$, or (iii) for some $i \in \{1, 2\}$, both $x$ and $y$ are in $L_i$ and $x \leq y$ in $L_i$. Figure 3a illustrates this operation; note that the unique four-element antichain in this lattice is not MI-orderable.

**Theorem 5.4.** If $L_1$ and $L_2$ are Tr-lattices, then so is $L_1 \ast L_2$.

The proofs of Theorems 5.4 and 5.5 are similar, so we prove only the second result, which concerns lexicographic sums [10 Section 1.10]. Let $L$ be a lattice and let $\mathcal{L} = (L_x : x \in L)$ be a family of lattices that is indexed by the elements of $L$. The lexicographic sum $L \oplus \mathcal{L}$ is defined on the set $\{(x, a) : x \in L, a \in L_x\}$; the order is given by $(x, a) \leq (y, b)$ if and only if either (i) $x < y$ in $L$ or (ii) $x = y$ and $a \leq b$ in $L_x$. Figure 3b illustrates this operation. It is easy to see that $L \oplus \mathcal{L}$ is not necessarily MI-orderable even if all of the constituent lattices are.

**Theorem 5.5.** If $L$ has width at most two and if $\mathcal{L} = (L_x : x \in L)$ is a family of Tr-lattices, then $L \oplus \mathcal{L}$ is a Tr-lattice.

**Proof.** Let $\phi : Z(M) \to L \oplus \mathcal{L}$ be an isomorphism. We must show that any antichain $\mathcal{A}$ in $Z(M)$ satisfies inequality (MI).

For $F \in Z(M)$, let $\phi_1(F)$ be the first component of $\phi(F)$; thus, $\phi_1(F) \in L$. For $x \in \phi_1(A)$, set $\mathcal{A}_x = \{F : F \in \mathcal{A}, \phi_1(F) = x\}$. Since $L$ has width at most two and $\mathcal{A}$ is an antichain in $Z(M)$, there are at most two such sets; these sets partition $\mathcal{A}$. 


For \( u \in L \), let \( Z_u \) and \( E_u \) be the least and greatest flats \( F \in \mathcal{Z}(M) \) with \( \phi_1(F) = u \). Thus, if \( \phi_1(A) = u \) and \( \phi_1(B) = v \) with \( u \neq v \), then \( A \lor B = Z_{u \lor v} \) and \( A \land B = E_{u \land v} \), by the definition of \( L \oplus L \).

Let \( x \) be in \( \phi_1(A) \). Note that \( \mathcal{Z}(M|E_x/Z_x) \) is isomorphic to \( L_x \), so \( M|E_x/Z_x \) is transversal. Thus, by Proposition 2.2,

\[
    r_{M|E_x/Z_x} \left( \bigcap (A_x) - Z_x \right) \leq \sum_{\mathcal{F} \subseteq A_x} (-1)^{|\mathcal{F}| + 1} r_{M|E_x/Z_x} \left( \bigcup (\mathcal{F}) - Z_x \right),
\]

which gives

\[
    r(\bigcap (A_x)) \leq \sum_{\mathcal{F} \subseteq A_x} (-1)^{|\mathcal{F}| + 1} r(\bigcup (\mathcal{F}))
\]

in \( M \). If \( |\phi_1(A)| = 1 \), then the last inequality is the required inequality (MI) for \( A \).

If, instead, \( |\phi_1(A)| = 2 \), let \( \phi_1(A) = \{x, y\} \), so we also have

\[
    r(\bigcap (A_y)) \leq \sum_{\mathcal{F} \subseteq A_y} (-1)^{|\mathcal{F}| + 1} r(\bigcup (\mathcal{F})).
\]

The equality

\[
    \sum_{\mathcal{F} \subseteq A} (-1)^{|\mathcal{F}| + 1} r(\bigcup (\mathcal{F})) = \sum_{\mathcal{F} \subseteq A_x} (-1)^{|\mathcal{F}| + 1} r(\bigcup (\mathcal{F})) + \sum_{\mathcal{F} \subseteq A_y} (-1)^{|\mathcal{F}| + 1} r(\bigcup (\mathcal{F})) + \sum_{\mathcal{F}_x \subseteq A_x, \mathcal{F}_y \subseteq A_y, \mathcal{F}_x \neq \emptyset \land \mathcal{F}_y \neq \emptyset} (-1)^{|\mathcal{F}_x| + |\mathcal{F}_y| + 1} r(\bigcup (\mathcal{F}_x) \cup \bigcup (\mathcal{F}_y))
\]
and that $r(X \cup Y) = r(Z_{x \lor y})$ for any $X \in L_x$ and $Y \in L_y$ give
\[
\sum_{F \subseteq A} (-1)^{|F| + 1} [r(\bigcup (F))] = \sum_{F \subseteq A_x} (-1)^{|F_x| + 1} [r(\bigcup (F))] + \sum_{F \subseteq A_y} (-1)^{|F_y| + 1} [r(\bigcup (F))]
\]
\[
- r(Z_{x \lor y}) \sum_{F_x \subseteq A_x, F_x \neq \emptyset} (-1)^{|F_x|} \sum_{F_y \subseteq A_y, F_y \neq \emptyset} (-1)^{|F_y|}
\]
\[
\geq r(\bigcap (A_x)) + r(\bigcap (A_y)) - r(Z_{x \lor y})
\]
\[
\geq r(\bigcap (A)).
\]
(The last line uses semimodularity.) Thus, inequality (MI) holds, as needed. □

6. Open problems

The results in this paper suggest the following problems.

(1) Is the converse of Theorem 4.1 true?

(a) Find a lattice-theoretic characterization of Tr-lattices, perhaps via a recursive description using operations such as those in Section 5.

(b) Is the converse of Theorem 4.1 true for planar lattices?

(c) If $Z(M)$ has the property of covers in Theorem 4.1, is $M$ a gammoid?

(d) Is every sublattice of a Tr-lattice also a Tr-lattice? Is this true at least for intervals, or upper intervals?

(e) Is the counterpart of Theorem 5.1 true for Tr-lattices?

(2) Are there Tr-lattices, or MI-lattices, of all dimensions?

(3) Can one capture the minor-closed, dual-closed class of transversal matroids described in Theorem 3.1 by special presentations that such matroids have? What are the excluded minors for this class of matroids?

(4) Can one deduce any substantial properties of a matroid $M$ other than being a gammoid (or specializations, such as transversal or nested) from lattice-theoretic properties of $Z(M)$?

(5) If $N$ is a minor of $M$ where $Z(M)$ is a Tr-lattice, must $N$ be transversal?

(6) What can we say about $M$ (more particular than transversal) when $Z(M)$ is a Tr-lattice?

Acknowledgements

I am very grateful for Joseph Kung, whose questions and comments prompted me to pursue more deeply the implications of Proposition 2.2.

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