Rayleigh Taylor instability of two superposed compressible fluids in un-magnetized plasma

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Abstract. The linear Rayleigh Taylor instability of two superposed compressible Newtonian fluids is discussed with the effect of surface tension which can play important roles in space plasma. As in both the superposed Newtonian fluids, the system is stable for potentially stable case and unstable for potentially unstable case in the present problem also. The equations of the problem are solved by normal mode method and a dispersion relation is obtained for such a system. The behaviour of growth rate is examined in the presence of surface tension and it is found that the surface tension has stabilizing influence on the Rayleigh Taylor instability of two superposed compressible fluids. Numerical analysis is performed to show the effect of sound velocity and surface tension on the growth rate of Rayleigh Taylor instability. It is found that both parameters have stabilizing influence on the growth rate of Rayleigh Taylor instability.

1. Introduction
Rayleigh Taylor instability is a phenomenon that is arised when heavier fluid is supported by lighter fluid in the presence of gravitational force [1]. Its occurrence in space and astrophysical phenomenon’s, e.g. supernova explosion, implosion, crab nebula interplanetary medium, intergalactic medium, hydrodynamic phenomenon’s e.g. pouring of oil into water, inverting glass of water, geophysics and ICF (inertial confinement fusion) intends to discuss it in various system [2, 3]. Chandrasekhar [4] studied Rayleigh Taylor instability of incompressible fluids comprehensively. Sharma et al. [5] carried out the effect of surface tension and rotation on Rayleigh Taylor instability of two superposed fluids with suspended particles. Ghasemizad [6] et al. have studied the growth rate of Rayleigh Taylor instability in inertial confinement fusion. Hoshoudy [7] studied the Rayleigh Taylor instability in quantum magnetized viscous plasma. Sharma et al. [8] have discussed the Rayleigh-Taylor (R-T) and Kelvin-Helmholtz (K-H) instabilities of a plasma in the incompressible porous medium with horizontal magnetic field and suspended dust particles.

Compressibility is an important aspects of fluid dynamics. In this context Vandervroot [9] investigated the character of the equilibrium of compressible fluid of varying density. Verma and Verma [10] have discussed the Rayleigh Taylor instability for an interface separating two compressible media the presence of magnetic field. Furthermore, Ariel [11] studied the effect of vertical magnetic field on the Rayleigh Taylor instability of compressible fluid of stratified variable density by using variational principle. Bhatia [12] has discussed the Rayleigh Taylor instability of
viscous compressible fluid. More recently Diaz et al. [13] have reported the Rayleigh Taylor instability in partially ionized compressible plasmas.

Thus, for the better insight we have considered the effect of surface tension on the Rayleigh Taylor instability of two superposed compressible fluids in un-magnetized plasma where magnetic field is negligible. Several authors have discussed Rayleigh Taylor instability of incompressible fluids. However the study of Rayleigh Taylor instability for compressible fluids is more realistic.

2. Formulation of problem

We assume that two homogenous compressible nonviscous superposed fluids are separated by a plane interface $z = 0$ in presence of gravitational force $g (0, 0, g)$. The region $z < 0$ and $z > 0$ are denoted for fluid 1 and 2 respectively. The effect of surface tension is also considered on the interface of superposed compressible fluids. The physical quantities of system are taken as density of fluid $\rho$, pressure $p$, and velocity of fluid $U$.

In order to investigate the stability or instability of the system, we have used the normal mode method. First we expect the basic fluid equation, the adiabatic equation and the equation of state which concern the flow of compressible fluids in the equilibrium configuration. A small perturbation is applied to the system, therefore the physical quantities of system becomes as

$$U = U_0 + u, \quad \rho = \rho_0 + \delta \rho, \quad p = p_0 + \delta p, \quad z_s = z_s + \delta z$$

(1)

Here $U_0, \rho_0, \rho_0$ indicates the fluid velocity, fluid density and pressure respectively in the equilibrium and $u (u_x, u_y, u_z), \delta \rho$ indicates the perturbed fluid velocity, fluid density and pressure respectively. The governing equations of the problem are linearized. To seek solutions into normal modes whose dependence on $x, y, t$ is given by

$$\exp(ik_x x + ik_y y + int)$$

(2)

Where $in$ is growth rate and $k_x, k_y$ are the wave numbers.

The governing linearized equations are in scalar form as

$$\rho \nu u_x = -i k_x \delta \rho$$

$$\rho \nu u_y = -i k_y \delta \rho$$

$$\rho \nu u_z = -i D \rho - g \delta \rho - \sum T_i \left(k_{ix}^2 + k_{iy}^2\right) \delta (z - z_i) u_i$$

$$in \delta \rho + u \rho + \rho (ik_x u_x + ik_y u_y + Du_z) = 0$$

$$in \delta \rho - igu_y = V^2 [in \delta \rho + u \rho]$$

(5)

(6)

(7)

On using Eq.(2) in Eqs (3-7) and eliminating some variables we finally obtain an equation in the $z$ component of velocity $(u_z)$ as

$$\left( \frac{in}{Q_j} \right)^2 D \rho u_z \left( \frac{Q_j}{Q_j'} \right) - \left( \frac{in}{Q_j} \right)^2 g D \rho u_z \left( \frac{Q_j}{Q_j'} \right) + gu \rho \rho u_z + \left( \frac{in}{Q_j} \right)^2 gu \rho u_z - \left( \frac{in}{Q_j} \right)^2 \rho u_z + \frac{g^2 k^2 \rho u_z}{Q_j^2} - k^2 \sum T_i u_i \delta (z - z_i) = 0$$

(8)

Here $Q_j = \left( \frac{in}{V_j} \right)^2 + k^2$

3. Dispersion relation

We consider that two superposed compressible fluids are separated by plane interface $z = 0$, and bounded by $z = \pm \infty$. We shall now derive the dispersion relation of compressible fluid from Eq.(8) for the constant density and sound velocity, we get

$$D^2 - q^2 = 0$$

$$q_j^2 = \left[ \frac{in}{V_j} \right]^2 + k^2 - \frac{g^2 k^2}{V_j}$$

(9)

(10)

The general solution of Eq. (9) written as
\[ u_z = A e^{q_1 z} + A e^{-q_2 z} \]  \hfill (11)

Here A is an arbitrary constant. The appropriate boundary conditions to solve the problem are [1] Velocity is continuous at the interface and satisfied as \( u_{z_1} = u_{z_2} = u_{z_0} \).

[2] The total pressure is continuous at the interface \( z = 0 \).

By using boundary conditions and assuming \( \omega = (in) \), the dispersion relation for the two superposed compressible fluids can be written as,

\[
\begin{align*}
\omega^2 \rho_2 &\left[ \frac{\omega^2}{V_2^2} + k^2 - \frac{g^2 k^2}{\omega^2 V_2^2} \right]^{\frac{1}{2}} \left( \frac{\omega^2}{V_2^2} + k^2 \right)^{-\frac{1}{2}} + \omega^2 \rho_1 \left[ \frac{\omega^2}{V_1^2} + k^2 - \frac{g^2 k^2}{\omega^2 V_1^2} \right]^{\frac{1}{2}} \left( \frac{\omega^2}{V_1^2} + k^2 \right)^{-\frac{1}{2}} \\
+ k^2 T &= g \rho_2 k^2 \left( \frac{\omega^2}{V_2^2} + k^2 \right)^{\frac{1}{2}} - g \rho_1 k^2 \left( \frac{\omega^2}{V_1^2} + k^2 \right)^{\frac{1}{2}} \tag{12}
\end{align*}
\]

Eq. (12) is dispersion relation for the Rayleigh Taylor instability of two superposed compressible fluids in presence of surface tension. The dispersion relation (12) is similar with Vandervroot [7] and Verma and Verma [8] in absence of surface tension. In case \( V_1 \to \infty \) and \( V_2 \to \infty \), the dispersion relation (12) is reduced to

\[
\omega^2 = \frac{g k (\rho_2 - \rho_1)}{\rho_2 + \rho_1} - \frac{k^3 T}{\rho_2 + \rho_1} \tag{13}
\]

The dispersion relation (13) represents the effect of surface tension on the Rayleigh Taylor instability of two incompressible fluids and it is identical to Sharma et al. [5].

4. Discussion

The dispersion relation (12) is complicated to discuss, therefore, we simplify the dispersion relation by making the assumption that the superposed fluids are highly compressible. For this case the Eq. (10) is solved by using binomial expansion as

\[
q_j = \left[ k + \frac{\omega^2}{2kV_j^2} - \frac{g^2 k}{2\omega^2 V_j^2} \right] \quad j = 1, 2
\]

The dispersion relation becomes as

\[
\begin{align*}
\omega^6 + \omega^4 \left[ k^2 V^{12} (\beta_1 + \beta_2 + 4) + \frac{2 k^3 T}{\rho_1 + \rho_2} \right] + \omega^2 \left[ 2k^4 V^{14} (\beta_1 + \beta_2 + 2) - g^2 k^2 \right] + \\
+ \left[ 2k^2 T V^{14} \left( \frac{\beta_1 + \beta_2 + 2}{\rho_1 + \rho_2} \right) + 2gk^2 V^{14} \left( \beta_1 - \beta_2 \right) - g^2 k^2 V^{12} \left( \beta_1 + \beta_2 \right) \right] &= 0 \tag{14}
\end{align*}
\]

Where \( \beta_1 = \frac{\rho_1}{\rho_2} \), \( \beta_2 = \frac{\rho_2}{\rho_1} \), \( V^{12} = \frac{p}{\rho_2 + \rho_1} \).

4.1 In absence of surface tension

The dispersion relation of the Rayleigh Taylor instability for compressible fluids in absence of surface tension is written as

\[
\begin{align*}
\omega^6 + \omega^4 \left[ k^2 V^{12} (\beta_1 + \beta_2 + 4) \right] + \omega^2 \left[ 2k^4 V^{14} (\beta_1 + \beta_2 + 2) - g^2 k^2 \right] + \\
+ \left[ 2gk^2 V^{14} \left( \beta_1 - \beta_2 \right) - g^2 k^2 V^{12} \left( \beta_1 + \beta_2 \right) \right] &= 0 \tag{15}
\end{align*}
\]

We can able to define the criterion for stability of system through the constant term of Eq. (15) as

\[
k_c = \frac{g}{2V^{12}} \left( \frac{\rho_1^2 + \rho_2^2}{\rho_2^2 - \rho_1^2} \right) \tag{16}
\]

Here \( k_c \) is critical wave number. The system will be stable if \( k > k_c \) and unstable if \( k < k_c \).
4.2 In presence of surface tension

In order to perform numerical calculations on the growth rate of the Rayleigh Taylor instability of two compressible fluids in presence of surface tension, we write the dispersion relation (14) in the dimensionless form as

\[ \omega^* = \frac{\omega}{\sqrt{g k}} \quad V^* = \frac{V}{g} \quad T^* = \frac{k^2 T}{g (\beta_1 + \beta_2)} \quad k^* = k \sqrt{g k} \quad [\omega^*, V^*, T^* \text{ and } k^* \text{ are dimensionless parameters}] \]

\[ \omega^* = \omega^* \left[ \frac{\beta_2 + \beta_1 + 2}{2} + 2T^* \right] + \omega^* \left[ \frac{2k^* V^* \beta_2 (\beta_2 + \beta_1 + 2) + 2T^* k^* V^* (\beta_1 + \beta_2 + 2)}{2T^* k^* V^* (\beta_1 + \beta_2 + 2) - 1} \right] \]

To examine the effect of surface tension, sound velocity, on the growth rate of Rayleigh Taylor instability of two superposed compressible fluids following figures have been drawn. In figure 1, we have potted the growth rate against wave number with increasing sound velocity \( V^* = 4.5, 5.5, 6.3 \) and we find that the growth rate is decreasing on increasing the value of sound velocity of fluid which indicates the stabilizing effect of sound velocity. In this case we have taken constant values of \( \beta_2 = 2, \beta_1 = 0.5, \) and \( T^* = 0.8. \)

The figure 2 illustrates that the growth rate is decreasing on increasing the value of surface tension while the parameters have been taken as \( \beta_2 = 2, \beta_1 = 0.5, \) \( V^* = 4.5, \) \( T^* =0.4, 0.8, 1.2. \) This figure reveals the stabilizing effect of surface tension on the unstable mode of RT instability.

**Figure 1.** The growth rate \((\omega^*)\) verses wave number \((k^*)\) in variation of sound velocity \((V^*)\).

**Figure 2.** The growth rate \((\omega^*)\) verses wave number \((k^*)\) in variation of surface tension \((T^*)\).

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