On the Extreme Flights of One-Sided Lévy Processes*

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Abstract

We explore the statistical behavior of the order statistics of the flights of One-sided Lévy Processes (OLPs). We begin with the study of the extreme flights of general OLPs, and then focus on the class of selfsimilar processes, investigating the following issues: (i) the inner hierarchy of the extreme flights - for example: how big is the 7th largest flight relative to the 2nd largest one?; and, (ii) the relative contribution of the extreme flights to the entire ‘flight aggregate’ - for example: how big is the 3rd largest flight relative to the OLP’s value?. Furthermore, we show that all ‘hierarchical’ results obtained - but not the ‘aggregate’ results - are explicitly extendable to the class of OLPs with arbitrary power-law flight tails (which is far larger than the selfsimilar class).

Keywords: Lévy flights, selfsimilar Lévy processes, order statistics, extreme value theory, Fréchet distribution.

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1 Introduction

Lévy processes - random motions with stationary and independent increments - constitute one of the most important and fundamental family of stochastic processes. Special examples of the Lévy family include Brownian motion (Wiener process), Poisson processes, and Compound Poisson processes. Since their introduction in the 1930s, by the French mathematician Paul Lévy [1]-[3], Lévy processes were studied extensively by both theoreticians and applied scientists. The literature on Lévy processes is vast, and their range of applications encompasses numerous fields of science and engineering. See [4]-[12] for the theory of Lévy processes, and [11]-[17] and references therein for their applications.

Amongst the family of Lévy processes, the class of One-sided Lévy Processes (OLPs), also referred to as Lévy Subordinators, is of special importance. OLPs satisfy the additional requirement of non-negativity of their increments, rendering them monotone non-decreasing and non-negative valued. OLPs are natural models for random flow of positive valued quantities. Examples include: mass, energy, and time in physical systems; work and costumers in queueing systems; data in communication systems; claims in insurance; etc.

Unlike Brownian motion - whose sample paths are continuous, the sample paths of OLPs - regardless of their statistics - are always purely discontinuous. The propagation of OLPs is conducted only by flights (jumps), and not by any sort of continuous motion. This is due to the fact that OLPs are continuum superpositions, or aggregates, of Poisson processes.

Since the structure of OLPs is that of a ‘flight aggregate’, questions of the following type arise naturally: how big is the largest flight? how big is the $n^{th}$ largest flight? how big is the $m^{th}$ largest flight relative to the $n^{th}$ largest flight ($m > n$)? how big is the largest flight relative to the entire aggregate? how big is the combined contribution of the $n^{th}$ largest flights relative to the entire aggregate? - that is, questions regarding the order statistics of the flights of OLPs.

The study of the order statistics of series of Independent and Identically Distributed (IID) random variables is a well established field of probability called Extreme Value Theory (EVT). This theory originated, in the 1920s and 30s, with the pioneering works of von Bortkiewicz [18], Fréchet [19], Fisher & Tippett [20], von Mises [21], Weibull, and Gumbel [22] (the theory’s three possible types of limiting distributions are named after Gumbel (“type 1”), Fréchet (“type 2”), and Weibull (“type 3”)). A rigorous theoretical framework was presented in 1943 by Gnedenko [23]. The statistical analysis of extreme values is of major importance in the analysis of rare and ‘catastrophic’ events such as floods in hydrology, large claims in insurance, crashes in finance, material failure in corrosion analysis, etc. See [24]-[26] and references therein for both the theory and applications of modern EVT. See also [27] for a recent application to the study of complex networks.

OLPs are the continuous time counterparts of non-negative IID random sequences (viewed as discrete time stochastic processes). When passing from
discrete time to continuous time the question “what was the largest observation amongst the first $N$ observations?” transforms to “what was the largest flight that occurred up to time $T$?” More generally, the analogue of the order statistics of a discrete sequence (largest observation, second largest observation, etc) is the order statistics of the flights of a continuous time process (largest flight, second largest flight, etc). Hence, the study of the extreme flights of OLPs is also a natural sequel to the EVT of IID random variables.

Furthermore, the only possible non-trivial scaling limits of non-negative IID random sequences are the, so called, selfsimilar OLPs. This special class of motions, which occupies a predominant role in both the theory and applications of stochastic processes, is composed of all OLPs which are invariant under changes of scale. Since EVT is, in essence, an asymptotic theory (providing limiting results as the total number of observations $N$ tends to infinity) we shall devote our main emphasis in this work to the exploration of this special class of OLPs.

The paper is organized as follows; We begin, in section 2, with an investigation of the extreme flights of general OLPs (following a short review of these processes). In sections 3 and 4, we focus on the class of selfsimilar OLPs. Section 3 studies of the internal hierarchy of the extreme flights - dealing with questions of the type “how big are the extreme flights relative to each other?”. Section 4, on the other hand, studies the magnitude of the extreme flights relative to the entire process - the ‘flight aggregate’. Furthermore, we explain why (and how) the results of section 3 are extendable to the entire class of OLPs with power-law flight tails, and, on the other hand, why the results of section 4 do not extend.

Throughout the manuscript: $P(\cdot) = \text{Probability}$; and, $E[\cdot] = \text{Expectation}$.

2 OLPs and their extreme flights

In this section we give a short review of OLPs, establish a few preliminary results regarding the distribution of their extreme flights, and conclude with the special class of selfsimilar OLPs and its connection to the Fréchet (“type 2”) extreme value distribution [25].

2.1 Lévy processes and OLPs

A stochastic process is said to be Lévy if it is continuous in probability and has stationary and independent increments. A OLP is a Lévy process with non-negative increments.

Characterization of OLPs

The celebrated Lévy-Khinchin formula (see, for example, [8] or [9]) asserts that Lévy processes can be decomposed into two independent parts: (i) a purely continuous part, which is a Brownian motion; and, (ii) a purely discontinuous part, which is a superposition of Poisson processes. Since Brownian motion
is symmetric, OLPs cannot include a continuous part and are hence purely discontinuous (pure jump) processes.

A OLP \( L = (L(t))_{t \geq 0} \) is characterized by its Laplace transform (its ‘spectral representation’ in Laplace space):

\[
E[\exp\{-\omega L(t)\}] = \exp\{-\Psi(\omega) \cdot t\} ; \quad \omega \geq 0 .
\] (1)

The function \( \Psi \) is called the Lévy characteristic of the OLP \( L \) (in the literature, \( \Psi \) is also referred to as the spectral characteristic, or symbol, of \( L \)).

**The Lévy measure**

If \( L \) is a Poisson process with flights of size \( x_0 \) (\( x_0 > 0 \)) and rate \( \lambda_0 \), then its Lévy characteristic is \( \Psi(\omega) = (1 - \exp\{-\omega x_0\}) \lambda_0 \). If \( L \) is a superposition of \( N \) independent Poisson processes - process \( n, n = 1, 2, \cdots, N \), having flights of size \( x_n \) (\( x_n > 0 \)) and rate \( \lambda_n \) - then its Lévy characteristic is given by

\[
\Psi(\omega) = \sum_{n=1}^{N} (1 - \exp\{-\omega x_n\}) \lambda_n .
\] (2)

Hence, passing from (2) to a continuum limit, where flights of size \( x \) (\( x > 0 \)) occur at rate \( \lambda(dx) \), we arrive at

\[
\Psi(\omega) = \int_{x=0}^{\infty} (1 - \exp\{-\omega x\}) \lambda(dx) .
\] (3)

Equation (3) is the Lévy-Khinchin representations (in Laplace space) for OLPs. The rate \( \lambda(\cdot) \) - the Lévy measure of \( L \) - is a measure on the non-negative half line \((0, \infty)\) satisfying the integrability condition \( \int_{x=0}^{\infty} \min\{x, 1\} \lambda(dx) \). The total rate \( \int_{x=0}^{\infty} \lambda(dx) \) could certainly be infinite - not due to non-integrability at \( x = \infty \) but, rather, due to possible non-integrability at \( x = 0 \) (intuitively, large flights can occur only rarely, but tiny flights may occur very frequently). A OLP is Compound Poisson if and only if its Lévy measure has finite total mass.

Given a Lévy measure \( \lambda(\cdot) \) it is natural to introduce its ‘cumulative distribution function’. However, since \( \lambda(\cdot) \) might have infinite total mass (due to possible non-integrability at \( x = 0 \)) we need to define the ‘cumulative distribution function’ by integrating (backwards) from \( x = \infty \) rather than (forward) from \( x = 0 \):

\[
\Lambda(x) = \int_{x}^{\infty} \lambda(dy) ; \quad x > 0 .
\] (4)

We henceforth refer to \( \Lambda \) as the OLP’s flight tail. The meaning of the flight tail is straightforward: \( \Lambda(x) \) is the rate at which flights of size \( > x \) occur.

**Examples**

We give a few examples of classes of OLPs. The first two have finite Lévy measure, whereas the other have infinite Lévy measure:
1. Compound Poisson with exponentially-distributed jumps \((a > 0)\):
\[
\lambda(dx) = a \exp\{-ax\} dx \quad \left( \Lambda(x) = \exp\{-ax\} \right),
\]
\[
\Psi(\omega) = \frac{\omega}{a + \omega}.
\]

2. Compound Poisson with Gamma-distributed jumps \((a, p > 0)\):
\[
\lambda(dx) = \exp\{-ax\} x^{p-1} dx,
\]
\[
\Psi(\omega) = \frac{1}{ap} - \frac{1}{(a + \omega)^p}.
\]

3. Gamma processes - OLPs with Gamma-distributed increments \((a > 0)\):
\[
\lambda(dx) = \frac{\exp\{-ax\}}{x} dx,
\]
\[
\Psi(\omega) = \ln \left( 1 + \frac{\omega}{a} \right).
\]

4. Selfsimilar ('fractal') OLPs \((0 < \alpha < 1)\):
\[
\lambda(dx) = \frac{\alpha}{x^{1+\alpha}} dx \quad \left( \Lambda(x) = \frac{1}{x^\alpha} \right),
\]
\[
\Psi(\omega) = \Gamma(1 - \alpha)\omega^\alpha.
\]

2.2 The extreme flights of OLPs

Given an OLP \(L = (L(t))_{t \geq 0}\) with Lévy measure \(\lambda(\cdot)\) (flight tail \(\Lambda(\cdot)\)), we set
\[
X_1(t) > X_2(t) > X_3(t) > \cdots
\]
(5) to be the order sequence of the flights of the OLP \(L\), i.e; \(X_n(t)\) denotes the \(n^{th}\) largest flight that occurred during the time interval \([0, t]\). If the OLP has finite Lévy measure \(\Lambda(0) < \infty\) (i.e; if it is Compound Poisson) then, up to time \(t\), there will have occurred only a finite number of flights (Poisson with rate \(t\Lambda(0)\), to be exact) and the order sequence will hence be finite. On the other hand, if the OLP has infinite Lévy measure \(\Lambda(0) = \infty\) then the order sequence will be infinite.

Since the Poisson distribution will appear time and again in the sequel, we introduce the following shorthand notation for its probability frequencies:
\[
P_n(\mu) = \frac{\mu^n}{n!} \exp(-\mu).
\]
(6)
That is; \(P_n(\mu)\) denotes the probability that a Poisson random variable with rate \(\mu\) \((\mu > 0)\) equals the integer value \(n\) \((n = 0, 1, 2, \cdots)\).
The joint distribution of the flights of a OLP is of major importance. Given a time \( t (t > 0) \) and an interval \( I \subset (0, \infty) \), let \( \Pi(t; I) \) denote the number of flights of \( L \), during the time period \([0, t]\), whose size laid in the interval \( I \). Due to the ‘Poissonian-superposition’ structure of \( L \) the random variable \( \Pi(t; I) \) is Poisson-distributed with rate \( t \int_I \lambda(dx) \). Moreover, if \( I_1, \cdots, I_K \) are disjoint intervals then \( \Pi(t; I_1), \cdots, \Pi(t; I_K) \) are independent random variables and hence

\[
P (\Pi(t; I_k) = n_k ; k = 1, \cdots, K) = \prod_{k=1}^K P_{n_k} \left( t \int_{I_k} \lambda(dx) \right), \quad (7)
\]

\( \forall n_1, \cdots, n_K \in \{0, 1, 2, \cdots\} \). For further details see [9].

The greatest flight

The distribution of the maximal flight \( X_1(t) \) is given by:

\[
P (X_1(t) \leq x) = \exp \{-t\Lambda(x)\}. \quad (8)
\]

The explanation of (8) is given by the following straightforward deduction:

\[
P (X_1(t) \leq x)
\]

\[
= P (\Pi(t; (x, \infty)) = 0)
\]

\[
= P_0 \left( t \int_x^\infty \lambda(ds) \right)
\]

\[
= \exp \{-t\Lambda(x)\}.
\]

Equation (8) enables the ‘shock tolerance’ design of systems. We explain;

Assume that \( L = (L(t))_{t \geq 0} \) is the inflow to a system, which cannot absorb flights (i.e; inflow surges) of size greater than a tolerance level \( l \). The following natural engineering question arises: what should the tolerance level \( l \) be so that to ensure that the system would withstand the time period \([0, T]\) with no failure, with probability greater than \( 1 - \delta \) (\( 0 < \delta < 1 \))? The answer, deduced by a simple inversion of (8) is

\[
l = \Lambda^{-1} \left( \frac{1}{T} \ln \left( \frac{1}{1 - \delta} \right) \right).
\]

The maximal flight, viewed as a stochastic process \( (X_1(t))_{t \geq 0} \), is Markovian. Indeed, when at level \( x \) \( (x > 0) \): (i) the process has to wait an exponential time, with rate \( \Lambda(x) \), till transition; and then, (ii) the process will transit to the level \( y (y > x) \) with probability \( \lambda(dy) / \Lambda(x) \). In other words, the infinitesimal generator of the process \( (X_1(t))_{t \geq 0} \) is given by the integral operator:

\[
(L_1 \varphi) (x) = \int_x^\infty (\varphi(y) - \varphi(x)) \lambda(dy).
\]

The \( n \)th runner-up
The distribution of the \( n \)th largest-order flight \( X_n(t) \) is given by:

\[
P(X_n(t) \leq x) = \sum_{k=0}^{n-1} P_k (t\Lambda(x)) .
\]

(9)

The explanation of (9) is analogous to the deduction of (8):

\[
P(X_n(t) \leq x)
= P(\Pi(t; (x, \infty)) \leq n - 1)
= \sum_{k=0}^{n-1} P_k (t \int_x^\infty \lambda (ds))
= \sum_{k=0}^{n-1} P_k (t\Lambda(x)) .
\]

The order sequence

The entire order sequence can be constructed/simulated sequentially, according to the iterative scheme \((y < x)\):

\[
P(X_{n+1}(t) \leq y \mid X_n(t) = x) = \exp\{-t (\Lambda(y) - \Lambda(x))\} ,
\]

(10)

The derivation of (11) is analogous to the derivation of (8)-(10).

2.3 Self-similarity and the Fréchet distribution

In the proceeding sections we shall focus on selfsimilar OLPs, i.e; OLPs with flight tail \( \Lambda(x) = 1/x^\alpha \) \((0 < \alpha < 1)\). For these processes the distribution of the maximal flight \( X_1(t) \) is given by (due to (8)):

\[
P(X_1(t) \leq x) = \exp\{-t/x^\alpha\} .
\]

(12)
The Fréchet distribution

The probability law (12) is known as the Fréchet distribution, or as the “type 2” extreme value distribution [25]. It emerges as the asymptotic probability law of the maximum of Independent and Identically Distributed (IID) heavy tailed random variables. We explain;

Let $Y_1, \ldots, Y_N$ be a sequence of IID, non-negative, random variables with heavy (‘fat’) tails: $P(Y > y) \sim a/y^\alpha$ as $y \to \infty$ (for some $a, \alpha > 0$). Then, the appropriately scaled sequence maximum converges, in law, to the Fréchet distribution:

$$P\left(\frac{1}{N^{1/\alpha}} \max\{Y_1, \cdots, Y_N\} \leq y\right) \xrightarrow{N \to \infty} \exp\left\{-\frac{a}{y^{\alpha}}\right\}. \quad (13)$$

Indeed;

$$P\left(\frac{1}{N^{1/\alpha}} \max\{Y_1, \cdots, Y_N\} \leq y\right)
\quad = (1 - P(Y > N^{1/\alpha}y))^N
\quad \sim \left(1 - \frac{a/y^\alpha}{N}\right)^N
\quad \sim \exp\{-a/y^\alpha\}.$$

Note, however, the discrepancy between (12) and (13): in (12) we are restricted to the parameter range $0 < \alpha < 1$, whereas in (13) the exponent $\alpha$ may admit any positive value. This discrepancy stems from the following reason;

Consider the IID sequence $Y_1, \cdots, Y_N$ introduced above. The scaling limit of its maximum is given by (13), for all $\alpha > 0$. However, in the scaling limit of their sum (aggregate) $S_N = Y_1 + \cdots + Y_N$ a phase transition occurs when passing from $\alpha < 1$ to $\alpha > 1$: (i) for $\alpha < 1$ the appropriate scaling is $S_N/N^{1/\alpha}$ and the limiting distribution is a selfsimilar Lévy law of order $\alpha$; but, (ii) for $\alpha > 1$ the scaling is the universal Law of Large Numbers scaling - $S_N/N$ - leading to the deterministic limit $E[Y]$. Hence, scaling limits of non-negative IID sums - yielding the selfsimilar OLPs - are restricted to the ‘Lévy range’ $0 < \alpha < 1$, whereas scaling limits of non-negative IID maxima admit the entire ‘Fréchet range’ $\alpha > 0$.

This gap between the ‘Lévy range’ and the ‘Fréchet range’ can be bridged by taking scaling limits of OLPs with power-law flight tails;

Scaling limits

Let $L$ be a OLPs whose flight tail asymptotics (as $x \to \infty$) are $\Lambda(x) \sim 1/x^\alpha$, for some exponent $\alpha > 0$. Introduce the scaled order sequence, $X_1^{(c)}(t) > X_2^{(c)}(t) > X_3^{(c)}(t) > \cdots$, given by

$$X_n^{(c)}(t) = \frac{1}{c} X_n(c^\alpha t)$$

1Selfsimilar OSLMs with exponent $\alpha \geq 1$ are trivial: $L(t) \equiv \text{const} \cdot t$ if $\alpha = 1$, and $L(t) \equiv 0$ if $\alpha > 1$.  

8
(where \( c > 0 \) is a scaling factor), and consider the \textit{scaling limit} of the order sequence, i.e; the limit, in law, as \( c \to \infty \), of the scaled sequence: \( X_1^{(\infty)}(t) > X_2^{(\infty)}(t) > X_3^{(\infty)}(t) > \cdots \). For the scaling limit \( X_1^{(\infty)}(t) \) we have:

\[
P\left(X_1^{(\infty)}(t) \leq x\right) = \exp\left(-t/x^\alpha\right),
\]

and now the exponent \( \alpha \) is \textit{not} (!) restricted to the ‘Lévy range’ \( 0 < \alpha < 1 \) - as it is in \( 12 \).

In fact, we obtain the following counterpart of \( 11 \):

\[
P\left(a_k < X_k^{(\infty)}(t) < b_k ; k = 1, \cdots, K\right)
= \exp\left\{ -t/\left(b_k/b_k\right)^\alpha \right\} \cdot t^{K-1} \prod_{k=1}^{K-1} \left(\frac{1}{(a_k)^\alpha} - \frac{1}{(b_k)^\alpha}\right),
\]

which holds for the entire ‘Fréchet range’ \( \alpha > 0 \). The deduction of \( 14 \) stems from \( 11 \) combined with the scaling procedure described above. Indeed, for \( c \gg 1 \) we have:

\[
P\left(a_k < X_k^{(\infty)}(t) < b_k ; k = 1, \cdots, K\right)
= \exp\left\{ -(c^\alpha t)\Lambda(c b_k)\right\} \cdot \left(c^\alpha t\right)^{K-1} \prod_{k=1}^{K-1} \left(\Lambda(c a_k) - \Lambda(c b_k)\right)
\sim \exp\left\{ -t/\left(b_k/b_k\right)^\alpha \right\} \cdot t^{K-1} \prod_{k=1}^{K-1} \left(\frac{1}{(a_k)^\alpha} - \frac{1}{(b_k)^\alpha}\right).
\]

3 The hierarchy of extreme flights

In this section we explore the \textit{hierarchical} structure of the order sequence of selfsimilar OLPs. We assume, throughout this section, that the OLP \( L \) is \( \alpha \)-selfsimilar, i.e; that its flight tail is \( \Lambda(x) = 1/x^\alpha \).

The \textit{Beta distribution} will emerge naturally and play a key role in this section \( 9 \). We denote by \( B_{n,m}(x), x \in [0, 1], \) the cumulative distribution function of a \textit{Beta}(\( n, m \)) distribution \( (n, m > 0)\):

\[
B_{n,m}(x) = \int_0^x \frac{\Gamma(n + m)}{\Gamma(n)\Gamma(m)} u^{n-1}(1 - u)^{m-1} du.
\]

Let us begin with the investigation of the pairwise hierarchy;

3.1 Pairwise hierarchy

In this subsection we study the statistics of the ratio \( X_{n+m}(t)/X_n(t) \), i.e; the size of the \((n + m)\)th largest-order flight \textit{relative} to the size of the \(n\)th largest-order flight. Clearly, this ratio admits values ranging in the unit interval. We
assert that the cumulative distribution function of this ratio is independent of time $t$ and is given, $\forall u \in [0, 1]$, by:

$$P\left(\frac{X_{n+m}(t)}{X_n(t)} \leq u\right) = B_{n,m}(u^\alpha). \quad (16)$$

Equation (16) is a particular case of proposition 1 which will be presented in the following subsection.

Moreover, the moments of the ratio $X_{n+m}(t)/X_n(t)$ are derived from (16); $\forall p > 0$ we have:

$$E\left[\left(\frac{X_{n+m}(t)}{X_n(t)}\right)^p\right] = \prod_{k=n}^{n+m-1} \frac{k}{k + p/\alpha}. \quad (17)$$

The proof of (17) is brought in the appendix.

We point out two special cases:

‘Consecutive relativity’

The statistics of the ratio $X_{n+1}(t)/X_n(t)$ are given by:

$$P\left(\frac{X_{n+1}(t)}{X_n(t)} \leq u\right) = u^{\alpha n},$$

and

$$E\left[\left(\frac{X_{n+1}(t)}{X_n(t)}\right)^p\right] = \frac{n}{n + p/\alpha}.$$  

‘Maximal relativity’

The statistics of the ratio $X_{1+m}(t)/X_1(t)$ are given by:

$$P\left(\frac{X_{1+m}(t)}{X_1(t)} \leq u\right) = 1 - (1 - u^\alpha)^m,$$

and

$$E\left[\left(\frac{X_{1+m}(t)}{X_1(t)}\right)^p\right] = \prod_{k=1}^{m} \frac{k}{k + p/\alpha}.$$  

3.2 Multidimensional hierarchy

In the previous subsection we investigated the pairwise hierarchy of the extreme flights. Now we turn to study the multidimensional hierarchical structure of the order sequence: given an increasing sequence of integers $n_1 < \cdots < n_K < n_{K+1}$ we wish to compute the joint distribution of the vector of ratios

$$\left(\frac{X_{n_2}(t)}{X_{n_1}(t)}, \frac{X_{n_3}(t)}{X_{n_2}(t)}, \cdots, \frac{X_{n_{K+1}}(t)}{X_{n_K}(t)}\right). \quad (18)$$

To that end we have;
Proposition 1 The multidimensional cumulative distribution function of the vector of ratios \( \{18\} \) is independent of time \( t \) and is given, \( \forall u_1, \cdots, u_K \in [0, 1] \), by
\[
P \left( \frac{X_{n_k+1}(t)}{X_{n_k}(t)} \leq u_k ; k = 1, \cdots, K \right) = \prod_{k=1}^{K} B_{n_k, n_{k+1}-n_k}((u_k)\alpha) . \tag{19}
\]

The proof of proposition \( \text{\textbullet} \) is brought in the appendix. This proposition can be re-stated as follows:

The ratios in \( \{18\} \) are independent and are distributed according to \( \{16\} \).

In particular, proposition \( \text{\textbullet} \) implies that:
\[
P \left( \frac{X_{k+1}(t)}{X_k(t)} \leq u_k ; k = 1, \cdots, K \right) = \prod_{k=1}^{K} (u_k)^{\alpha_k} , \tag{20}
\]
\( \forall u_1, \cdots, u_K \in [0, 1] \).

**Multidimensional ‘maximal relativity’**

The distribution of the truncated order sequence relative to the *maximal flight*
\[
\left( \frac{X_2(t)}{X_1(t)} , \frac{X_3(t)}{X_1(t)} , \cdots , \frac{X_{K+1}(t)}{X_1(t)} \right) , \tag{21}
\]
rather than its multidimensional ‘consecutive relativity’ counterpart \( \{20\} \), is given, for a decreasing sequence \( 1 \geq a_1 > \cdots > a_K > a_{K+1} = 0 \), by the formula:
\[
P \left( a_{k+1} < \frac{X_{k+1}(t)}{X_k(t)} < a_k ; k = 1, \cdots, K \right) = \Gamma(K)(a_K)^{\alpha K} \prod_{k=1}^{K-1} \left( \frac{1}{(a_{k+1})^\alpha} - \frac{1}{(a_k)^\alpha} \right) . \tag{22}
\]
The proof of \( \{22\} \) is brought in the appendix.

**A ‘Fréchet remark’**

All the results of this section may be extended from the restricted self-similar ‘Lévy range’ \( 0 < \alpha < 1 \) to the entire ‘Fréchet range’ \( \alpha > 0 \) following the scaling procedure described in subsection \( 2.3 \) replace the order sequence \( X_1(t) > X_2(t) > X_3(t) > \cdots \) of an \( \alpha \)-selfsimilar OLP \( 0 < \alpha < 1 \) by the scaling limit \( X_1(\infty)(t) > X_2(\infty)(t) > X_3(\infty)(t) > \cdots \) of the order sequence of a OLP whose flight tail asymptotics are \( \Lambda(x) \sim 1/x^\alpha \) \( \alpha > 0 \).
4 The extreme flights vs the aggregate

In the previous section we studied the behavior of the extreme flights relative to each other. We now turn to study the extreme flights relative to the entire process \( L \). Since \( L \) is a (continuum) superposition of Poisson processes, it is, in fact, a ‘flight aggregate’ (where flights of size \( x \) occur with rate \( \lambda(dx) \)). Hence, we are interested in investigating the relative contribution of the largest flights, up to time \( t \), to the entire ‘flight aggregate’ \( L(t) \). As in the previous section, we assume in this section that the OLP \( L \) is \( \alpha \)-selfsimilar.

Before we begin, let us introduce the family of functions \( \{G_\alpha\}_{0<\alpha<1} \), defined on the non-negative half line (\( \omega \geq 0 \)), and given by the integral formula

\[
G_\alpha(\omega) = \alpha \int_0^1 \frac{1 - \exp\{-\omega u\}}{u^{1+\alpha}} du ,
\]

or, equivalently, by the power series

\[
G_\alpha(\omega) = -\alpha \sum_{m=1}^{\infty} \frac{(-1)^m}{m - \alpha} \cdot \frac{\omega^m}{m!} .
\]

4.1 The greatest flight vs the aggregate

Let \( \xi(t) \) denote the magnitude of the “entire flight aggregate, up to time \( t \), except for the maximal flight”, relative to the maximal flight:

\[
\xi(t) = \frac{L(t) - X_1(t)}{X_1(t)} .
\]

We assert that the ratio \( \xi(t) \) is independent of time \( t \) and that its Laplace transform is given by:

\[
E[\exp\{-\omega \xi(t)\}] = \frac{1}{1 + G_\alpha(\omega)} .
\]

Equation (25) is a particular case of proposition 2 which will be presented in the following subsection. Let us denote by \( G_\alpha \) the probability law whose Laplace transform is given by the right hand side of (25).

Equation (25) implies that the mean and variance of \( \xi(t) \) are independent of time \( t \), and are given, respectively, by

\[
E[\xi(t)] = \frac{\alpha}{1 - \alpha} ,
\]

\[
\text{Var}(\xi(t)) = \frac{\alpha}{(1 - \alpha)^2(2 - \alpha)} .
\]
The derivation of (26)-(27) is obtained by differentiating (25) and using the power expansion (24). Combining (26)-(27) together we also have

$$\text{Var} \left( \frac{\xi(t)}{E[\xi(t)]} \right) = \frac{1}{\alpha(2-\alpha)} .$$

(28)

From (26)-(28) we see that:

When $\alpha \rightarrow 1$ the underlying OLP $L$ converges to the degenerate deterministic linear limit $\equiv t$, and hence all flights tend to zero. Indeed, from (26) we obtain that $E[\xi(t)] \rightarrow \infty$ as $\alpha \rightarrow 1$. That is, the size of the aggregate, relative to the greatest flight, tends to infinity. Moreover, the variance of normalized ratio $\xi(t)/E[\xi(t)]$ converges, as $\alpha \rightarrow 1$, to 1.

On the other hand, when $\alpha \rightarrow 0$ the tails of the OLP become infinitely ‘fat’ and hence the greatest flight should dominate. And indeed, we obtain that $E[\xi(t)] \rightarrow 0$ as $\alpha \rightarrow 0$. That is, the size of the aggregate, relative to the greatest flight, tends to zero. In fact, the dominance of the largest flight is so great that even $\text{Var}(\xi(t)) \rightarrow 0$ as $\alpha \rightarrow 0$.

The value $\alpha = 1/2$ turns out to be the “break-even” point where the mean of ratio $\xi(t)$ equals 1. That is, the entire aggregate splits, on average, half-half between the largest flight and “all the rest”. As $\alpha$ ‘moves up’ towards 1 the weight of “all the rest” takes the lead and eventually dominates, and vice-versa as $\alpha$ ‘moves down’ towards 0.

Another immediate consequences of (25), after noting that $X_1(t)/L(t) = 1/(1 + \xi(t))$, is that the cumulative distribution function of the ratio $X_1(t)/L(t)$ is independent of time $t$ and is given, $\forall u \in [0, 1]$, by

$$P \left( \frac{X_1(t)}{L(t)} \leq u \right) = P \left( \xi \geq \frac{1}{u} - 1 \right) ,$$

(29)

where $\xi$ is $G_{\alpha}$-distributed.

### 4.2 The top-$n$ vs the aggregate

We now proceed to study the contribution of the ‘top-$n$’ flights, $X_1(t), X_2(t), \cdots, X_n(t)$, relative to the entire aggregate $L(t)$. The key result is:

**Proposition 2** The random vector

$$\left( \frac{L(t) - (X_1(t) + \cdots + X_{n+1}(t))}{X_{n+1}(t)} ; \frac{X_1(t)}{X_{n+1}(t)} , \cdots , \frac{X_n(t)}{X_{n+1}(t)} \right)$$

(30)

is independent of time $t$ and equal, in law, to the vector

$$(\xi_1 + \cdots + \xi_{n+1} ; Y_{(1)}, \cdots, Y_{(n)})$$

where: (i) $\xi_1, \cdots, \xi_{n+1}, Y_1, \cdots, Y_n$ are independent random variables; (ii) $\xi_1, \cdots, \xi_{n+1}$ are $G_{\alpha}$-distributed; and, (iii) $Y_{(1)}, \cdots, Y_{(n)}$ are the order statistics of $Y_1, \cdots, Y_n$ which, in turn, are Pareto($\alpha$)-distributed: $P(Y > y) = 1/y^\alpha$, $y \geq 1$. 

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The proof of proposition 2 is brought in the appendix.

Noting that $Y(1) + \cdots + Y(n) = Y_1 + \cdots + Y_n$, proposition 2 implies the following pair of corollaries:

(i) The ratio $X_{n+1}(t)/L(t)$ - the contribution of the $(n+1)^{th}$ largest flight, relative to the entire aggregate - is equal, in law, to

\[ (1 + \xi_1 + \cdots + \xi_{n+1} + Y_1 + \cdots + Y_n)^{-1}. \]

Hence, $\forall u \in [0, 1]$ we have:

\[ \mathbb{P}\left( \frac{X_{n+1}(t)}{L(t)} \leq u \right) = \mathbb{P}\left( \xi_1 + \cdots + \xi_{n+1} + Y_1 + \cdots + Y_n \geq \frac{1}{u} - 1 \right). \quad (31) \]

(ii) The ratio $(X_1(t) + \cdots + X_n(t))/L(t)$ - the combined contribution of the $n$ largest flights, relative to the entire aggregate - is equal, in law, to:

\[ \left( 1 + \frac{1 + \xi_1 + \cdots + \xi_{n+1}}{Y_1 + \cdots + Y_n} \right)^{-1}. \]

Hence, $\forall u \in [0, 1]$ we have:

\[ \mathbb{P}\left( \frac{X_1(t) + \cdots + X_n(t)}{L(t)} \leq u \right) = \mathbb{P}\left( \frac{1 + \xi_1 + \cdots + \xi_{n+1}}{Y_1 + \cdots + Y_n} \geq \frac{1}{u} - 1 \right). \quad (32) \]

A Fréchet remark

The results of this section are not extendable from the restricted selfsimilar ‘Lévy range’ $0 < \alpha < 1$ to the entire ‘Fréchet range’ $\alpha > 0$. This is since the scaling procedure described in subsection 2.3, which holds for any $\alpha > 0$ when regarding the extremes - fails to hold for the aggregate when $\alpha > 1$ (since, as explained in subsection 2.3, the limiting behavior of the aggregate undergoes a phase transition when passing from the ‘Lévy kingdom’ $\alpha < 1$ to the ‘Law of Large Numbers kingdom’ $\alpha > 1$).

5 Appendix: proofs

5.1 Equation (17): moments of the pairwise hierarchy

Proof. Combining (15) and (16), the probability density function of the ratio $X_{n+m}(t)/X_n(t)$ is given by

\[ f_{n,m}(u) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}(u^\alpha)^{n-1}(1 - u^\alpha)^{m-1} \cdot u^{\alpha-1}. \]
Hence, changing the integration variable to $x = u^\alpha$ and using Beta integrals, we obtain

$$
E \left[ \left( \frac{X_{n+m}(t)}{X_n(t)} \right)^p \right] \\
= \int_0^1 u^p \cdot f_{n,m}(u) du \\
= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 x^{(n+p)/\alpha - 1} (1 - x)^{m-1} dx \\
= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(n+m+p/\alpha)} \\
= \prod_{k=n}^{n+m-1} \frac{k}{k+p/\alpha}.
$$

5.2 proposition

We begin with a lemma;

**Lemma 3** Let $\Lambda(x) = 1/x^\alpha$, $x > 0$, and $n,m > 0$. Then, $\forall x > 0$ and $\forall 0 < u < 1$ we have

$$
\frac{1}{\Gamma(n)\Gamma(m)} \int_{x/u}^\infty \Lambda(y)^{n-1} (\Lambda(x) - \Lambda(y))^{m-1} \lambda(dy) = \frac{\Lambda(x)^{n+m-1}}{\Gamma(n+m)} B_{n,m}(u^\alpha).
$$

(33)

**Proof.** First, note that we can re-write the left hand side of (33) as follows

$$
\frac{\Lambda(x)^{n+m-1}}{\Gamma(n)\Gamma(m)} \int_{x/u}^\infty \left( \frac{\Lambda(y)}{\Lambda(x)} \right)^{n-1} \left( 1 - \frac{\Lambda(y)}{\Lambda(x)} \right)^{m-1} \lambda(dy) = \frac{\Lambda(x)^{n+m-1}}{\Gamma(n+m)} B_{n,m}(u^\alpha).
$$

(34)

Now, using the change of variable $s = \Lambda(y)/\Lambda(x)$, the integral part of (34) equals

$$
\int_0^{u^\alpha} s^{n-1} (1 - s)^{m-1} ds = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} B_{n,m}(u^\alpha).
$$

(35)

Substituting (35) back into (34) concludes the proof.

We are ready now to prove the proposition;
Proof. First, we use \( \) in order to compute the multidimensional density function of the vector \( (X_{n_1}(t), \ldots, X_{n_K}(t), X_{n_{K+1}}(t)) \):

\[
P(X_{n_k}(t) \in dx_k : k = 1, \ldots, K + 1)
= \prod_{k=1}^{K+1} P_t(t(\lambda(dx_k))) P_{n_k-n_{k-1}}(t \int \lambda(ds))
= \prod_{k=1}^{K+1} (\lambda(dx_k)) \left( \frac{(t(\Lambda(x_k)-\Lambda(x_{k-1})))^{n_k-n_{k-1}-1}}{\Gamma(n_k-n_{k-1})} \exp \{-t(\Lambda(x_k)-\Lambda(x_{k-1}))\} \right)
= t^{n_{K+1}} \exp \{-t\Lambda(x_{K+1})\} \cdot \prod_{k=1}^{K+1} \frac{(\Lambda(x_k)-\Lambda(x_{k-1}))^{n_k-n_{k-1}-1}}{\Gamma(n_k-n_{k-1})} \lambda(dx_k)
= f(x_1, \ldots, x_K, x_{K+1}) \cdot \lambda(dx_1) \cdots \lambda(dx_K) \cdot \lambda(dx_{K+1})
\]

\( \forall \) decreasing sequence \( \infty = x_0 > x_1 > \cdots > x_K > x_{K+1} > 0 \), and where \( n_0 = 0 \).

Hence

\[
P\left( \frac{X_{n_{k+1}}(t)}{X_{n_k}(t)} \leq u_k : k = 1, \ldots, K \right)
= \int_{x_{K+1}}^{\infty} \lambda(dx_{K+1}) \int_{x_{K+1}/u_k}^{\infty} \lambda(dx_K) \cdots \cdots \int_{x_2/u_2}^{\infty} \lambda(dx_2) \int_{x_1/u_1}^{\infty} \lambda(dx_1) f(x_1, \ldots, x_K, x_{K+1})
\]

(36)

Now, computing the multiple integral (36), using lemma 3 repeatedly, yields

\[
P\left( \frac{X_{n_{k+1}}(t)}{X_{n_k}(t)} \leq u_k : k = 1, \ldots, K \right) = I \cdot \prod_{k=1}^{K} B_{n_k-n_{k-1}-n_k}(u_k)
\]

where

\[
I = \int_0^{\infty} \frac{t^{n_{K+1}}}{\Gamma(n_{K+1})} \exp \{-t\Lambda(x_{K+1})\} \Lambda(x_{K+1})^{n_{K+1}-1} \lambda(dx_{K+1})
\]

which, in turn, must equal 1. \( \blacksquare \)

5.3 Equation (22): multidimensional ‘maximal relativity’

Proof. (Recall that \( \Lambda(x) = 1/x^n \))

Analogously to the proof of proposition 11, we have

\[
P(a_{k+1} < \frac{X_{k+1}(t)}{X_k(t)} < a_k : k = 1, \ldots, K)
= \int_0^{\infty} \lambda(dx_1) \int_{a_2x_1}^{a_1x_1} \lambda(dx_2) \cdots \int_{a_{K+1}x_1}^{a_Kx_1} \lambda(dx_K) \int_0^{a_{K+1}x_1} \lambda(dx_{K+1}) f(x_1, \ldots, x_K, x_{K+1})
\]

(37)
where
\[ f(x_1, \cdots, x_K, x_{K+1}) = t^{K+1} \exp\{-t\Lambda(x_{K+1})\}. \]

Now, \( \forall k = 2, \cdots, K \) we have
\[
\int_{a_{k-1}x_1}^{a_kx_1} \lambda(dx_k) = \Lambda(a_kx_1) - \Lambda(a_{k-1}x_1) \]
\[ = (\Lambda(a_k) - \Lambda(a_{k-1})) \Lambda(x_1). \]  

(38)

And, using the change of variable \( s = \Lambda(x_{K+1}) \),
\[
\int_0^{a_Kx_1} \lambda(dx_{K+1})t \exp\{-t\Lambda(x_{K+1})\}
\]
\[ = \int_{\Lambda(a_K)x_1}^{\infty} t \exp\{-ts\} \, ds \]
\[ = \exp\{-t\Lambda(a_K)x_1\}. \]  

(39)

Substituting (38) and (39) into the multiple integral on the right hand side of (37) yields
\[ I \prod_{k=2}^{K} (\Lambda(a_k) - \Lambda(a_{k-1})) , \]
where
\[
I = t^K \int_{0}^{\infty} \exp\{-t\Lambda(a_K)x_1\} \Lambda(x_1)^{K-1} \, \lambda(dx_1). \]

However, using the change of variable \( s = \Lambda(x_1) \) gives
\[ I = t^K \int_{0}^{\infty} \exp\{-t\Lambda(a_K)s\} s^{K-1} \, ds = \frac{\Gamma(K)}{\Lambda(a_K)^K} \]

(41)

Finally, combining (40) and (41) together implies that the right hand side of (37) equals
\[
\frac{\Gamma(K)}{\Lambda(a_K)^K} \prod_{k=2}^{K} (\Lambda(a_k) - \Lambda(a_{k-1})) = \Gamma(K)(a_K)^{aK} \prod_{k=1}^{K-1} \left( \frac{1}{(a_{k+1})^\alpha} - \frac{1}{(a_k)^\alpha} \right). \]

5.4 proposition 2

We use the shorthand notation
\[ (U(t); V(t)) = \left( \frac{L(t) - (X_1(t) + \cdots + X_{n+1}(t))}{X_{n+1}(t)}, \frac{X_1(t)}{X_{n+1}(t)}, \cdots, \frac{X_n(t)}{X_{n+1}(t)} \right). \]

Proof. (Recall that \( \lambda(dx) = \alpha x^{-(1+\alpha)} dx \) and \( \Lambda(x) = 1/x^\alpha \))
Step 1
Due to the ‘Poissonian-superposition’ structure of the OLP $L$, given the event $\{X_{n+1}(t) = x\}$ the vector $(U(t); V(t))$ is equal, in law, to the vector:

$$\left( \frac{S^x(t)}{x}; \frac{J^x_{(1)}}{x}, \cdots, \frac{J^x_{(n)}}{x} \right)$$

where:

(i) $S^x(t)$ is the value, at time $t$, of a OLP with Lévy measure

$$\lambda^x(dy) = \begin{cases} 
\lambda(dy) & 0 < y < x \\
0 & y > x 
\end{cases}$$

(42)

(ii) $J^x_{(1)}, \cdots, J^x_{(n)}$ are the order statistics of $J^x_1, \cdots, J^x_n$ which, in turn, are IID random variables distributed according to the probability law (supported on the half line $(x, \infty)$):

$$P(J^x \in dy) = \begin{cases} 
0 & 0 < y < x \\
\lambda(dy)/\Lambda(x) & y > x 
\end{cases}$$

(43)

(iii) $S^x(t)$ and $J^x_1, \cdots, J^x_n$ are mutually independent.

Hence, $\forall \omega \geq 0$ and $\forall \theta = (\theta_1, \cdots, \theta_n) \geq 0$ we have

$$E[\exp\{-\omega U(t) - \theta V(t)\} \mid X_n(t) = x] = E\left[\exp\left\{-\frac{\omega}{x} S^x(t)\right\}\right] \cdot E\left[\exp\left\{-\frac{\theta}{x} J^x\right\}\right],$$

where $\overrightarrow{J}^x = (J^x_{(1)}, \cdots, J^x_{(n)})$.

Step 2
Since $S^x(t)$ is the value, at time $t$, of a OLP with Lévy measure (42) we have

$$E\left[\exp\left\{-\frac{\omega}{x} S^x(t)\right\}\right] = \exp\left\{-\Psi^x\left(\frac{\omega}{x}\right) t\right\},$$

where

$$\Psi^x\left(\frac{\omega}{x}\right) = \int_0^\infty (1 - \exp\left\{-\frac{\omega}{x} y\right\}) \lambda(dy)$$

$$= \frac{\alpha}{x^\alpha} \int_0^1 \frac{1-\exp\left[-\frac{\omega}{x} u\right]}{u^{1+\alpha}} du$$

$$= \Lambda(x) G_\alpha(\omega).$$

On the other hand, using (43), $\forall y > 1$ we have:

$$P\left(\frac{J^x}{x} > y\right) = \frac{\Lambda(xy)}{\Lambda(x)} = \frac{1}{y^\alpha}.$$
Hence, combining the above together, we conclude that

$$E \left[ \exp \left\{ -\omega U(t) - \theta V(t) \right\} \mid X_n(t) = x \right] = \exp \left\{ -\Lambda(x)G_\alpha(\omega)t \right\} \cdot L(\theta),$$  

(44)

where $L(\theta)$ is the Laplace transform (at the point $\theta = (\theta_1, \cdots, \theta_n)$) of the order statistics of $\{Y_1, \cdots, Y_n\}$ which are IID Pareto($\alpha$)-distributed random variables:

$$P(Y > y) = \frac{1}{y^\alpha}, \quad y \geq 1.$$

**Step 3**

Using (7), the probability density function of $X_{n+1}(t)$ is given by

$$P(X_{n+1}(t) \in dx) = P_1(t \lambda(dx)) \cdot P_n(t \int_x^\infty \lambda(ds))$$

$$= (t \lambda(dx)) \cdot \frac{(t \Lambda(x))^n}{n!} \exp \left\{ -t \Lambda(x) \right\}$$

$$= \frac{t^{n+1}}{n!} \exp \left\{ -t \Lambda(x) \right\} \Lambda(x)^n \lambda(dx).$$

(45)

Combining (44) and (45) together we hence obtain

$$E \left[ \exp \left\{ -\omega U(t) - \theta V(t) \right\} \mid X_n(t) = x \right] P(X_n(t) \in dx) = I \cdot L(\theta),$$

where

$$I = \frac{t^{n+1}}{n!} \int_0^\infty \exp \left\{ -\Lambda(x)G_\alpha(\omega)t \right\} \cdot \exp \left\{ -t \Lambda(x) \right\} \Lambda(x)^n \lambda(dx).$$

However, using the change of variable $s = \Lambda(x)$ gives

$$I = \frac{t^{n+1}}{n!} \int_0^\infty \exp \left\{ -t(1 + G_\alpha(\omega)) \cdot s \right\} s^n ds = \frac{1}{(1 + G_\alpha(\omega))^{n+1}},$$

and hence we can conclude that

$$E \left[ \exp \left\{ -\omega U(t) - \theta V(t) \right\} \right] = \left( \frac{1}{1 + G_\alpha(\omega)} \right)^{n+1} \cdot L(\theta).$$

(46)

Equation (46) implies that $(U(t); V(t)) = (\xi_1 + \cdots + \xi_{n+1}; Y_1, \cdots, Y_n)$ where:

(i) $\xi_1, \cdots, \xi_{n+1}, Y_1, \cdots, Y_n$ are independent random variables;

(ii) $\xi_1, \cdots, \xi_{n+1}$ are $\mathcal{G}_\alpha$-distributed;

(iii) $Y_1, \cdots, Y_n$ are the order statistics of $Y_1, \cdots, Y_n$ which are Pareto($\alpha$)-distributed: $P(Y > y) = 1/y^\alpha, \quad y \geq 1.$
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