Integrated Information and Metastability in Systems of Coupled Oscillators

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It has been shown that sets of oscillators in a modular network can exhibit a rich variety of metastable chimera states, in which synchronisation and desynchronisation coexist. Independently, under the guise of integrated information theory, researchers have attempted to quantify the extent to which a complex dynamical system presents a balance of integrated and segregated activity. In this paper we bring these two areas of research together by showing that the system of oscillators in question exhibits a critical peak of integrated information that coincides with peaks in other measures such as metastability and coalition entropy.

Keywords: Synchronisation, chimera states, metastability, integrated information

I. INTRODUCTION

Systems of coupled oscillators are ubiquitous both in nature and in the human-engineered environment, making them of considerable scientific interest [1]. A variety of mathematical models of such systems have been devised and their synchronisation properties have been the subject of much study. Typical studies of this sort, such as the classic work of Kuramoto [2], examine the conditions under which the system converges on a stable state of either full synchronisation or desynchronisation, perhaps identifying an order parameter that determines a critical phase transition from one state to the other. The collection of known attractors of such systems was enlarged with the discovery of so-called chimera states, in which the system of oscillators partitions into two stable subsets, one of which is fully synchronised while the other remains permanently desynchronised [3].

Although systems of coupled oscillators that converge on stable states are both mathematically interesting and scientifically relevant, they are by no means representative of all real-world synchronisation phenomena. For example, the brain exhibits synchronous rhythmic activity on multiple spatial and temporal scales, but never settles into a stable state. Although it enters chimera-like states of high partial synchronisation, these are only temporary. A system of coupled oscillators that continually moves from one highly synchronised state to another under its own intrinsic dynamics is said to be metastable. In [4], it was shown that a modular network of phase-lagged Kuramoto oscillators will exhibit metastable chimera states under certain conditions. Variants of this model have since been used to replicate the statistics of the brain under a variety of conditions, including the resting state [5], cognitive control [6], and anaesthesia [7].

In a separate line of enquiry, a number of researchers have attempted to pin down the notion of dynamical complexity if it exists a balance of integrated and segregated activity, where a system’s activity is integrated to the extent that its parts influence each other and segregated to the extent that its parts act independently [8]. Prominent among these attempts is the Integrated Information Theory (IIT), originally proposed by Balduzzi and Tononi [9], but extended by multiple authors ever since [10–12].

In the present paper, we connect these two lines of enquiry by demonstrating that modular networks of coupled oscillators of the sort described in [4] not only exhibit metastable chimera states, but also have high dynamical complexity. Moreover, we show that measures of both phenomena peak in the narrow critical region of the parameter space wherein the system is poised between order and disorder, and IIT offers a rich picture of dynamical complexity in this critical regime. To our knowledge, this is the first description of a dynamical system in which the three major complexity indicators of criticality, metastability, and integrated information all appear.

II. METHODS

We examine a system of coupled Kuramoto oscillators, extensively used to study non-linear dynamics and synchronisation processes [13]. We build upon the work of [4] with a community-structured network of oscillators. The network is composed of 8 communities of 32 oscillators each, with every oscillator being coupled to all other oscillators in its community with probability 1 and to each oscillator in the rest of the network with probability 1/32. The state of each oscillator i is captured by its phase \( \theta_i \), the evolution of which is governed by the equation

\[
\frac{d\theta_i}{dt} = \omega + \frac{1}{\kappa + 1} \sum_j K_{ij} \sin(\theta_j - \theta_i - \alpha),
\]

where \( \omega \) is the natural frequency of each oscillator, \( \kappa \) is the average degree of the network, \( K \) is the connectivity matrix and \( \alpha \) is a global phase lag. We set \( \omega = 1 \) and \( \kappa = 63 \). To reflect the community structure, the coupling between two oscillators \( i,j \) is \( K_{ij} = 0.6 \) if they are in the same community or \( K_{ij} = 0.4 \) otherwise. We tune the system by modifying the value of the phase lag, parametrised by \( \beta = \pi/2 - \alpha \). We note that the system is fully deterministic, i.e. there is no noise injected in the dynamical equations.
A. Metastability

In this section we review the basic concepts behind metastability and how it is quantified, following a similar description to that of \[14\].

The building block of the dynamical quantities we study in this article is the instantaneous synchronisation \( R_c(t) \), that quantifies the dispersion in \( \theta \)-space of a given set of oscillators. In general, we denote as \( R_c(t) \) the instantaneous synchronisation of a community \( c \) of oscillators at time \( t \), given by

\[
R_c(t) = |\langle e^{i\theta_j(t)} \rangle_{j\in c}|. \tag{2}
\]

To quantify metastability, we use the metastability index \( \lambda \), which is defined as the average temporal variance of the synchrony of each community \( c \), i.e.

\[
\lambda_c = \text{var}_t R_c(t) \quad \text{and} \quad \lambda = \langle \lambda_c \rangle_c. \tag{3a}
\]

Last, we also define global synchrony \( \xi \) as the average across time and space of instantaneous synchrony,

\[
\xi = |\langle R_c(t) \rangle_{t,c}|. \tag{4}
\]

According to Eq. (2), \( R_c \) (and therefore \( \xi \)) is bounded in the \([0, 1]\) interval. \( R_c(t) \) will be 1 if all oscillators in \( c \) have the same phase at time \( t \), and will be 0 if they are maximally spread across the unit circle. This \([0, 1]\) bound on \( R \) allows us to place an upper bound on \( \lambda \) – assuming a unimodal synchrony distribution, the maximum possible value of \( \lambda \) is \( \lambda_{\text{max}} = 1/9 \).

As defined in Eq. (3a), \( \lambda_c \) represents the size of the fluctuations in the internal synchrony of a community. A system that is either hypersynchronised or completely desynchronised will have a very small \( \lambda_c \), whereas one whose elements fluctuate in and out of synchrony will have a high \( \lambda_c \). In other words, a system of oscillators exhibits metastability if its elements remain in the vicinity of a synchronised state without falling into such a state permanently.

B. Integrated Information

Although other information-theoretic quantities have also been linked to complexity in a neuroscience context \[7, 14\], we take integrated information \( \Phi \) as the main informational measure of complexity in our study \[9\]. There are more modern accounts of the theory \[15, 16\], but the latest versions have not been as thoroughly studied and are not amenable to easy estimation from time series data. For these reasons, we focus on the methods in \[10\] to empirically estimate \( \Phi \) from an observed time series.

The building block of integrated information is effective information, \( \varphi \). Effective information quantifies how much better a system \( X \) is at predicting its own future (or decoding its own past) after a time \( \tau \) when it is considered as a whole compared to when it is considered as the sum of two subsystems \( M^{(1,2)} \). We refer to \( \tau \) as the integration timescale. In other words, \( \varphi \) evaluates how much information is generated by the system but not by the two subsystems alone. For a specific bipartition \( B = \{M^1, M^2\} \), the effective information of the system beyond \( B \) is

\[
\varphi[X; \tau, B] = I(X_{t-\tau}, X_t) - \sum_{k=1}^2 I(M^k_{t-\tau}, M^k_t). \tag{5}
\]

The main idea behind the computation of \( \Phi \) is to exhaustively search all possible partitions of the system and calculate the effective information of each of them. Among those we select the partition with lowest \( \varphi \) (under some considerations, see below), termed the Minimum Information Bipartition (MIB). Then, the integrated information of the system is the effective information beyond its MIB. Given the above expression for \( \varphi \), \( \Phi \) is defined as

\[
\Phi[X, \tau] = \varphi[X; \tau, B^{\text{MIB}}] \tag{6a}
\]

\[
B^{\text{MIB}} = \arg\min_B \varphi[X; \tau, B] \tag{6b}
\]

\[
K(B) = \min \left\{ H(M^1), H(M^2) \right\}, \tag{6c}
\]

where \( K \) is a normalisation factor to avoid biasing \( \Phi \) to excessively unbalanced bipartitions. Defined this way, \( \Phi \) can be understood as the minimum information loss incurred by splitting the system into two subsystems. It quantifies the collective emergent behaviour that is present in the whole system but not in any bipartition.

III. RESULTS

We ran 1500 simulations with values of \( \beta \) distributed uniformly at random in the range \([0, 2\pi]\) using RK4 with a stepsize of 0.05 for numerical integration. Each simulation was run for \( 5 \times 10^6 \) timesteps, of which the first \( 10^4 \) are discarded to avoid effects from transient states. All information-theoretic measures are reported in bits.

We first study the system from a purely dynamical perspective, following the analysis in \[4\]. Global synchrony and metastability are shown in Fig. 1. The first characteristic we observe is that there are two well differentiated dynamical regimes – one of hypersynchronisation and one of complete desynchronisation, with strong metastability appearing in the narrow transition bands between one and the other.

It is in this transition region where the oscillators operate in a critical regime poised between order and disorder and complex phenomena appear. As the system moves from desynchronisation to full synchronisation there is a sharp increase in metastability, followed by a smoother decrease as the system becomes hypersynchronised. In the region \( 0 < \beta < \pi/8 \), the system remains in a complex equilibrium between an ordered and a disordered phase.
FIG. 1. Global synchrony and metastability for different phase lags $\beta$ for the whole $[0, 2\pi]$ range (bottom) and around the critical transition region (top). Rapid increase of metastability marks the onset of the phase transition. Note the different $\beta$ ranges in both plots.

FIG. 2. Integrated information $\Phi$ and coalition entropy $H_c$ in the phase transition. Within the broad region between order and disorder in which $H_c$ rises there is a narrower band in which complex spatiotemporal patterns generate high $\Phi$.

Both $\Phi$ and $H_c$ peak precisely at the same point. Although both measures depend on the chosen coalition threshold $\gamma$, the results are qualitatively the same for a wide range of thresholds.

A. Information-theoretic analysis

One feature of $\Phi$ (and of any other information-theoretic measure), compared to $\lambda$, is that it does not need to be calculated directly from the state of the system. According to its definition, $\Phi$ is substrate-agnostic, meaning that the relevant quantity for the calculation of $\Phi$ is not the physical state of the system, but some informational state – the configuration of the system that we consider to contain information. Therefore, we must define an informational state mapping, that extracts the information-bearing symbols from the physical state of the system.

Although calculating $\Phi$ on the real-valued phases is possible, for simplicity we choose the coalition configuration of the system as the informational state, defined as the set of communities that are highly internally synchronised. To calculate the coalition configuration at time $t$ we calculate $R_c(t)$ of each community and threshold it, such that

$$X^c_t = \begin{cases} 
1 & \text{if } R_c(t) > \gamma \\
0 & \text{otherwise.}
\end{cases}$$

We refer to $\gamma$ as the coalition threshold. After calculating the coalitions, the history of the system is reduced to a time series with 8 binary variables. Having a multivariate discrete time series, it is now tractable to compute $\Phi$. By default, we use $\gamma = 0.8$ in all our analyses shown here.

As depicted in Fig. 2 $\Phi$ shows a similar behaviour to $\lambda$ – it peaks in the transition regions and shrinks in the fully ordered and the fully disordered regimes. We also compare $\Phi$ with arguably the simplest information-theoretic measure – entropy $H$. The entropy of the state of the network calculated on the coalitions $X_t$ forms the coalition entropy $H_c$.

Both $\Phi$ and $H_c$ peak precisely at the same point. Although both measures depend on the chosen coalition threshold $\gamma$, the results are qualitatively the same for a wide range of thresholds.

Although it peaks in the same region as $\lambda$ and $H_c$, we note that $\Phi$ reveals new properties of the system by virtue of explicitly incorporating temporal information in its definition. That is, to have a high $\Phi$ a system must exhibit complex spatial and temporal patterns. We can explicitly verify this by performing a random time-shuffle on the time series. This shuffling leaves $\lambda$ and $H_c$ unaltered, as they don’t explicitly depend on time correlations, but has a high impact on $\Phi$, which shrinks to zero. This indicates that $\Phi$ is sensitive to properties of the system that are not reflected by other measures.

Furthermore, $\Phi$ can be used to investigate the behaviour of the system at multiple timescales. Figure 3 shows the behaviour of $\Phi$ for several values of $\tau$, and compares it with standard time-delayed mutual information (TDMI) $I(X_{t-\tau}, X_t)$.

The first thing we note is that $\Phi$ and TDMI have opposite trends with $\tau$. TDMI decreases for longer timescales while $\Phi$ increases. At short timescales the system is highly predictable – thus the high TDMI – but this short-term evolution does not involve any system-wide interaction – thus the low $\Phi$. Furthermore, at these timescales $\Phi$ is negative, which can be interpreted as an indication of redundancy [17] in the evolution of the system: the parts share some information, such that they separately contain more information about their past than the whole system about its past. For larger $\tau$ TDMI decreases, as the evolution of the system becomes harder to track using the coalition configuration. Simultaneously, $\Phi$ becomes higher, indicating that the remaining TDMI has a stronger integrated component that is not accounted for by the TDMI of the partitions of the system. Overall, we see a
clear trend of TDMI diminishing at longer timescales but becoming progressively more integrated in nature.

It might seem counterintuitive to the reader that both TDMI and $\Phi$ converge to a non-zero value for arbitrarily high $\tau$. However, recall that the system is deterministic, so there is no reason to expect TDMI to vanish even in the $\tau \rightarrow \infty$ limit. Perfect knowledge of all of the oscillators’ phases at any given time step is enough to reconstruct the whole history of the system. The reason why we see a decreasing TDMI is because the informational state we chose (the coalition configuration) is not descriptive enough to capture the evolution of the system perfectly. That is, this is not a result of stochasticity, but of degeneracy.

Put another way, the TDMI of the whole system remains non-zero in the $\tau \rightarrow \infty$ limit because we are dealing with a causally closed system. Again, if we considered the totality of the system’s state (i.e. the phases $\theta_i$) then TDMI would be maximal and constant for any $\tau$. In contrast, when considering the TDMI of any partition $M$ we start dealing with a causally open system, since one partition is affected by the other. This effectively introduces stochasticity in our observations $M_t$, which does cause TDMI of the partition to vanish when $\tau \rightarrow \infty$. This explains that the TDMI of the whole system converges to a non-zero value for large $\tau$ while the TDMI of any partition fades to zero, leaving a positive $\Phi$.

Finally, it is interesting to combine the insights from the dynamical and information-theoretic analyses. Inspecting Figs. 1 and 2 we see that the peak in $\Phi$ is much narrower than the peaks in $\lambda$ and $H_c$. While some values of $\beta$ do give rise to non-trivial dynamics, it is only at the centre of the critical region that these dynamics give rise to integration. A certain degree of internal variability is necessary to establish integrated information, but not all configurations with high internal variability lead to a high $\Phi$. This means that $\Phi$ is sensitive to more complicated dynamic patterns than the other measures considered, and is in that sense more discriminating.

We note that $\lambda$ is a community-local quantity – that is, the calculation of $\lambda$ for each community is independent of the rest. Conversely, $\Phi$ relies exclusively on the irreducible interaction between communities. These two quantities are nevertheless intrinsically related, insofar as internal variability enables the system to visit a larger repertoire of states in which system-wide interaction can take place.

**B. Robustness of $\Phi$ against measurement noise**

We will now consider the impact of measurement noise on $\Phi$, wherein the system runs unchanged but our recording of it is imperfect. For this experiment we run the (deterministic) simulation as presented in the previous section and take the binary time series of coalition configurations. We then emulate the effect of uncorrelated measurement noise by flipping each bit in the time series with probability $p$, yielding the corrupted time series $\hat{X}$. Finally we recalculate $\Phi$ on the corrupted time series, and show the results in Fig. 4. To quantify how fast $\Phi$ changes we calculate the ratio between the corrupted and the original time series,

$$\eta = \frac{\Phi[\hat{X}, \tau]}{\Phi[X, \tau]}.$$  

In order to avoid instabilities as $\Phi[X, \tau]$ gets close to zero, we calculate $\eta$ only in the region within 0.5 rad of the centre of the peak, where $\Phi[X, \tau]$ is large. The inset of Fig. 4 shows the mean and standard deviation of $\eta$ at different noise levels $p$.

**FIG. 4.** Integrated information $\Phi$ for different levels of measurement noise $p$. Inset: (blue) Mean and variance of the ratio $\eta$ between $\Phi$ of the corrupted and the original time series. (red) Exponential fit $\eta = \exp(-p/\ell)$, with $\ell \approx 0.04$. 

**FIG. 3.** Integrated information $\Phi$ and time-delayed mutual information $I(X_{t-\tau}, X_t)$ for several timescales $\tau$. See text for details.
We find that $\Phi$ monotonically decays with $p$, reflecting the gradual loss of the precise spatiotemporal patterns characteristic of the system. The distortion has a greater effect on time series with greater $\Phi$, but preserves the dominant peak in $\beta \approx 0.15$. The inset shows that both the mean and variance of $\eta$ decay as a clean exponential with $p$. $\Phi$ is highly sensitive to noise and undergoes a rapid decline, as a measurement noise of 5% wipes out 70% of the perceived integrated information of the system.

IV. CONCLUSION

We have presented a community-structured network of Kuramoto oscillators and discussed their collective behaviour in terms of metastability \cite{4} and integrated information \cite{9}. We showed that the system undergoes a phase transition whose critical region presents a sharp, clear peak of integrated information $\Phi$ that coincides with a strong increase in metastability. To our knowledge, this is the first description of a dynamical system in which the three major complexity indicators of criticality, high metastability, and high integrated information all appear. The resulting confluence of two major research directions in complexity science suggests that this is a system that merits further study.

In the context of the present model, the high internal variability of the system’s components enables system-wide interaction, which in turn leads to high $\Phi$. As we have seen, the system presents a region of high metastability, but notably it is only within an even narrower band that we find strong integrated information. Moreover, as we have also seen, shuffling the time series data preserves the peak of metastability, despite the fact that the result is meaningless. By contrast, $\Phi$ only peaks when the relevant temporal structure is present in the data. In this way we provide evidence that complex dynamics – as quantified by the metastability index $\lambda$ – are a necessary but not sufficient condition for complex information processing – as quantified by integrated information $\Phi$.

Dynamical and information-theoretic measures provide different lenses through which we can understand a system, and offer complementary views of its behaviour. Our findings support the claim that $\Phi$, despite having some theoretical drawbacks \cite{11}, is a valuable tool for understanding complex spatial and temporal behaviour in dynamical systems, particularly when combined with other analysis techniques.

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