Minimal genus four manifolds
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Abstract

In 2018, M. Chu and S. Tillmann gave a lower bound for the trisection genus of a closed 4-manifold in terms of the Euler characteristic of $M$ and the rank of its fundamental group. We show that given a group $G$, there exist a 4-manifold $M$ with fundamental group $G$ with trisection genus achieving Chu-Tillmann’s lower bound.

1 Introduction

Let $M$ be a smooth, oriented, closed 4-manifold. D. Gay and R. Kirby showed in [2] that $M$ can be written as the union of three 4-dimensional 1-handlebodies of genus $k_1$, $k_2$ and $k_3$, respectively; with pairwise intersections 3-dimensional connected handlebodies and triple intersection a connected surface of genus $g$. This is called a $(g; k_1, k_2, k_3)$–trisection of $M$. The trisection genus of a 4-manifold $M$, denoted by $g(M)$, is the smallest $g$ such that $M$ admits a $(g; *, *, *)$–trisection.

Let $\tau$ be a trisection of $M$; i.e. $M = X_1 \cup X_2 \cup X_3$. Each 1-handlebody $X_i$ can be used to span the fundamental group and the first homology of $M$. In [1], M. Chu and S. Tillmann used this to give a lower bound to the trisection genus of a closed 4-manifold. They showed that if $M$ admits a $(g; k_1, k_2, k_3)$–trisection then

$$g \geq \chi(M) - 2 + 3\text{rk}(\pi_1(M)).$$

**Question 1** (From [1]). Given any finitely presented group $G$, is there a smooth, oriented closed 4-manifold $M$ with $\pi_1(M) = G$ such that $g(M) = \chi(M) - 2 + 3\text{rk}(G)$?

In this note, we will give a positive answer of Question 1.

Fix a finitely presented group $G$. We divide the answer in two steps as follows: (1) We state an equivalent version of Question 1 in terms of Kirby diagrams of closed 4-manifolds, which translates into a knot theory problem; and then (2) we show the existence of a special type of links.

Along this note, all 4-manifolds will be smooth, oriented and compact. For a link $L$ in a 3-manifold $Y$, $t_Y(L)$ will denote the tunnel number of $L$ in $Y$. We will omit the sub-index $Y$ if there is no confusion of the ambient manifold.

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1A 1-handlebody of dimension four is a 4-ball with 1-handles attached along its boundary; i.e. $\natural_k S^1 \times B^3$. 

2 The proof

Let $M$ be a closed 4-manifold and $\tau$ be a $(g; k_1, k_2, k_3)$-trisection of $M$. In [1], M. Chu and S. Tillmann showed the inequality

$$g \geq \chi(M) - 2 + 3\text{rk}(\pi_1(M))$$

(1)

Since $\chi(M) = 2 + g - (k_1 + k_2 + k_3)$ and $k_i \geq \text{rk}(\pi_1(M))$ for $i = 1, 2, 3$, equality of Equation (1) is equivalent to

$$k_i = \text{rk}(\pi_1(M)) \quad i = 1, 2, 3$$

(2)

The following lemma is an application of Lemmas 13 and 14 of [2]. It shows that Question 1 is equivalent to finding links with the correct homotopy class and “small” tunnel number.

Lemma 2.1. Let $G \neq 1$ be a finitely presented group of rank $n$. The following are equivalent:

(a) There is a smooth, oriented closed 4-manifold $M$ with $\pi_1(M) = G$ such that

$$g(M) = \chi(M) - 2 + 3\text{rk}(G)$$

(b) There is a smooth, oriented closed 4-manifold $M$ with $\pi_1(M) = G$ having a handle decomposition with one 0-handle, $n$ 1-handles, $j$ 2-handles, $n$ 3-handles and one 4-handle such that the attaching region of the 2-handles is a link in $\#_n S^1 \times S^2$ of tunnel number $n + j - 1$.

(c) There is a link $L$ in $\#_n S^1 \times S^2$ of tunnel number at most $n + |L| - 1$ such that the link $L$, thought of as a set of homotopy classes of loops based at a point, gives relations for a rank $n$ group presentation for $G$.

Proof. (a) $\Rightarrow$ (b). Let $M$ be a 4-manifold with $\pi_1(M) = G$ such that $g(M) = \chi(M) - 2 + 3\text{rk}(G)$. Let $\tau$ be a $(g; k_1, k_2, k_3)$-trisection with $g = g(M)$. In particular, $\tau$ satisfies Equation (2); i.e., $\tau$ is a $(g; n, n, n)$-trisection of $M$. Lemma 13 of [2] asserts that $M$ admits a handle decomposition with one 0-handle, $n$ 1-handles, $g - n$ 2-handles, $n$ 3-handles and one 4-handle such that the attaching region of the 2-handles is a framed link $L$ contained in the core of one of the handlebodies of a genus $g$ Heegaard splitting for $\#_n S^1 \times S^2$. In particular, $t(L) \leq g - 1$. The latter must be an equality since Proposition 4.2 of [4] states that $M$ admits a new $(t(L) + 1; n, t(L) + 1 - |L|, n)$-trisection and $t(L) < g - 1$ will give us $t(L) + 1 - |L| < (g - 1) + 1 - (g - n) = n$; contradicting the fact that $k_2 \geq n$. Hence $t(L) = g - 1$ and, (b) holds taking $j = g - n$.

(b) $\Rightarrow$ (a). Let $M$ be a 4-manifold satisfying (b) with the given handlebody decomposition. By taking the tubular neighborhood of the attaching region of the 2-handles and the tunnels, we obtain a Heegaard surface for $\#_n S^1 \times S^2$ of genus $g = n - j$ satisfying the assumptions of Lemma 14 of [2]. By the lemma, $M$ admits a $(g; n, n, n)$-trisection. In particular, Equation (2) holds and Equation (1) becomes an equality, thus (a).
(b) ⇒ (c). Take \( L \) to be the attaching region of the 2-handles.

(c) ⇒ (b). Let \( L \) be such link and let \( x \in \mathbb{Z}^{|L|} \) be any fixed vector. Consider \( N \) to be the smooth 4-manifold with a handle decomposition given by one 0-handle, \( n \) 1-handles and \(|L|\) 2-handles attached along \( L \) with framing \( x \) with respect to the blackboard. Take \( M \) to be the double of \( N \). By assumption, \( \pi_1(N) \cong G \) and so \( \pi_1(M) \cong G \). Turning the handle decomposition of \( N \) upside-down gives \( M \) a handle decomposition with one 0-handle, \( n \) 1-handles, \( 2|L| \) 2-handles, \( n \) 3-handles and one 4-handle, where half of the 2-handles are attached along \( L \) and for each component of \( L \) there is a 0-framed unknot linked once along the given component and unlinked from the rest of the diagram.

Let \( \hat{L} = L \cup L' \) be the attaching region for the 2-handles of \( M \). Notice that

\[
t(\hat{L}) \leq t(L) + |L| = n + |\hat{L}| - 1.
\]

By a version of Lemma 14 of [2] for unbalanced trisections (see Proposition 4.2 of [4]), \( M \) admits a \((t(\hat{L})+1; n, t(\hat{L}) + 1 - |L|, n)\)-trisection. Using Equation (1) with \( k_2 = t(\hat{L}) + 1 - |\hat{L}| \) we obtain

\[
n \leq t(\hat{L}) + 1 - |\hat{L}|
\]

\[
\leq (t(L) + |L|) + 1 - 2|L|
\]

\[
= t(L) + 1 - |L|
\]

\[
\leq n.
\]

Thus, \( t(\hat{L}) = n + |\hat{L}| - 1 \) and \( M \) is the desired 4-manifold. Hence (b).

**Remark 2.2.** We have shown in Lemma 2.1 that to answer Question 1 in the positive, is enough to find a link \( L \) in \( \#_n S^1 \times S^2 \) with tunnel number at most \( n + |L| - 1 \) such that the homotopy classes of the components of \( L \), together, read a rank \( n \) presentation for the given group \( G \). The closed 4-manifold answering Question 1 will be the double of a 4-manifold with a Kirby diagram with \( n \) 1-handles and 2-handles attached along \( L \) with any framing.

The following proposition shows how to build links satisfying (c) in Lemma 2.1.

**Proposition 2.3.** Let \( G \) be a finitely presented group of rank \( n \) with a presentation \( \langle X \mid R \rangle \). There exists an \(|R|\)-component link in \( \#_{\left|X\right|} S^1 \times S^2 \) with tunnel number at most \( |X| + |R| - 1 \) such that the words read by the components of \( L \) in \( \pi_1(\#_{\left|X\right|} S^1 \times S^2, *) \) agree with the words in \( R \).

**Corollary 2.4.** Given a finitely presented group \( G \), there is a closed 4-manifold with fundamental group isomorphic to \( G \) satisfying \( g(M) = \chi(M) - 2 + 3rk(G) \).

**Proof of Proposition 2.3.** Take an unknotted graph in \( \#_n S^1 \times S^2 \) made by \( n \) loops, denoted by \( \Gamma_0 \), generating \( \pi_1(\#_n S^1 \times S^2, *) \) and \(|R|\) unknotted loops \( c_1, \ldots, c_{|R|} \) around a neighborhood of \( * \); see Figure 1. For each relation \( r_i \in R \), take the \( i \)-th unknotted circle \( c_i \) and slide one of its ends along the loops on \( \Gamma_0 \) so that \( c_i \) now reads the word \( r_i \) as an element of \( \pi_1(\#_n S^1 \times S^2, *) \). Each component of \( L \) will be a circle \( c_i \), and the rest of the graph can be homotoped to be a system of tunnels for \( L \).

We have described how to build a link \( L \) in \( \#_n S^1 \times S^2 \) with \( t(L) \leq n + |L| - 1 \).  

\( \square \)
Figure 1: Unknotted graph in $\#_3 S^1 \times S^2$. The red and blue loops are $c_1$ and $c_2$, respectively.

The following figures illustrate the construction of $L$ for the group $G = \langle x, y, z | x^3 y^{-2}, [y, z] \rangle$.

Figure 2: Sliding the end of $c_1$ (red) to produce the word $x^3 y^{-2}$.

Figure 3: Left: The graph after the homotopies, red loop reads $x^3 y^{-2}$ and blue loop reads $[y, z]$. Right: The desired link $L$ for $G = \langle x, y, z | x^3 y^{-2}, [y, z] \rangle$.

Remark 2.5. Note that the knot type of the link $L$ doesn’t change the final 4-manifold built in Corollary 2.4. This happens since, when taking double of the 4-manifold, the 0-framed unknots around each component of $L$ allow us to slide the handlebody 2-handles and so to change the crossings of the link. The same reason explains why we only care about the framing of the components of $L$ modulo 2. In any case, one can consider connected sums of our manifolds with copies of $S^2 \times S^2$ and $\pm CP^2$ to build infinitely many 4-manifolds solving Question 1.
One of the two 4-manifolds appearing when running the construction with finite cyclic groups $\langle x | x^m = 0 \rangle$ is the spun lens space $S_m$ trisected by J. Meier in [3]. One can check this by comparing the Kirby diagram we construct with the one drawn by J. Montesinos in [5].

References

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