A NOTE ON BOOLE POLYNOMIALS WITH q-PARAMETER

DAE SAN KIM, YU SEON JANG, TAEKYUN KIM, AND SEOG-HOON RIM,

Abstract. Recently, Boole polynomials have been studied by Kim and Kim over the p-adic number field. In this paper, we consider a q-extension of Boole polynomials by using the fermionic p-adic integrals on \( \mathbb{Z}_p \) and give some new identities related to those polynomials.

1. Introduction

Let \( p \) be a fixed odd prime number. Throughout this paper, \( \mathbb{Z}_p, \mathbb{Q}_p \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic numbers and the completion of algebraic closure of \( \mathbb{Q}_p \). The \( p \)-adic norm \( | \cdot |_p \) is normalized as \( |p|_p = 1/p \). Let \( C(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \). For \( f \in C(\mathbb{Z}_p) \), the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by Kim to be

\[
I(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see [11]}).
\]

From (1), we have

\[
I_{-1}(f_n) = 2 \sum_{x=0}^{n-1} (-1)^{n-1-a} f(a) + (-1)^n I_{-1}(f).
\]

Let us assume that \( q \) is an indeterminate in \( \mathbb{C}_p \) with \( |1-q|_p < p^{-1/(p-1)} \). The Stirling number of the first kind is defined by

\[
(x)_n = \sum_{\ell=0}^{n} S_1(n, \ell)x^{\ell}, \quad (n \geq 0),
\]

and the Stirling number of the second kind is given by

\[
x^n = \sum_{\ell=0}^{n} S_2(n, \ell)x^{\ell}, \quad (n \geq 0), \quad (\text{see [9, 16]}).
\]

The Boole polynomials are defined by the generating function to be
2 D. S. KIM, Y. S. JANG, T. KIM, AND S.-H. RIM, 

\[ \sum_{n=0}^{\infty} B_{l_n}(x|\lambda) \frac{t^n}{n!} = \frac{1}{1 + (1 + t)^\lambda} (1 + t)^x, \quad \text{(see [9, 16]).} \]

In [9], Kim and Kim gave a Witt-type formula for \( B_{l_n}(x|\lambda) \) over the \( p \)-adic number field as follows:

\[ \int_{\mathbb{Z}_p} (x + \lambda y)_n d\mu_{-1}(y) = 2B_{l_n}(x|\lambda), \quad (n \geq 0), \]

where \( \lambda \in \mathbb{Z}_p \) and \( (x)_n = x(x - 1) \cdots (x - n + 1) \).

Let us define the \( q \)-product of \( x \) as follows:

\[ (x)_{n,q} = x(x - q)(x - 2q) \cdots (x - (n - 1)q), \quad (n \geq 0). \]

As is known, the Euler polynomials are defined by

\[ \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1 - 20]).} \]

When \( x = 0, E_n = E_n(0) \) are called the Euler numbers. From (2), we have

\[ \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \]

In this paper, we consider a \( q \)-extension of Boole polynomials by using the fermionic \( p \)-adic integrals on \( \mathbb{Z}_p \) and give some new identities of those polynomials.

2. Boole polynomials with \( q \)-parameter

In this section, we assume that \( t \in \mathbb{C}_p \) with \( |t|_p < p^{-1/(p-1)}|q|_p \) and \( \lambda \in \mathbb{Z}_p \). Now, we consider the Boole polynomials with \( q \)-parameter as follows:

\[ B_{l_n,q}(x|\lambda) = \int_{\mathbb{Z}_p} (x + \lambda y)_{n,q} d\mu_{-1}(y), \quad (n \geq 0). \]

Thus, by (10), we get

\[ B_{l_n,q}(x|\lambda) = \sum_{\ell=0}^{n} S_1(n, \ell) q^{n-\ell} \lambda^\ell \int_{\mathbb{Z}_p} \left( \frac{x}{\lambda} + y \right)^\ell d\mu_{-1}(y) \]

\[ = \sum_{\ell=0}^{n} S_1(n, \ell) q^{n-\ell} \lambda^\ell E_{\ell} \left( \frac{x}{\lambda} \right). \]

From (2) and (10), we can derive the generating function of \( B_{l_n,q}(x|\lambda) \) as follows:
A NOTE ON BOOLE POLYNOMIALS WITH $q$-PARAMETER

\[
\sum_{n=0}^{\infty} B_{n,q}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x + \lambda y)_n q d\mu_{-1}(y) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \left( \frac{x + \lambda y}{q} \right) d\mu_{-1}(y) t^n
\]
\[
= \int_{\mathbb{Z}_p} (1 + qt)^{x + \lambda y} d\mu_{-1}(y)
\]
\[
= (1 + qt)^{x + \lambda y} \left( \frac{2}{(1 + qt)^{\lambda/q} + 1} \right).
\]

Therefore, by (12), we obtain the following theorem.

**Theorem 2.1.** Let $F(t, x|\lambda) = \sum_{n=0}^{\infty} B_{n,q}(x|\lambda) \frac{t^n}{n!}$. Then we have

\[
F(t, x|\lambda) = \frac{2}{(1 + qt)^{\lambda/q} + 1} (1 + qt)^{x/q}.
\]

By replacing $t$ by $(e^t - 1)/q$ in (12), we get

\[
\sum_{n=0}^{\infty} B_{n,q}(x|\lambda) q^{-n} \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n \left( \frac{x}{\lambda} \right) \left( \frac{\lambda}{q} \right)^n \frac{t^n}{n!}
\]

and

\[
\sum_{n=0}^{\infty} B_{n,q}(x|\lambda) q^{-n} \frac{(e^t - 1)^n}{n!} = \sum_{n=0}^{\infty} B_{n,q}(x|\lambda) q^{-n} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!}
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \left( \sum_{\ell=0}^{m} B_{\ell,q}(x|\lambda) S_2(m, \ell) q^n \right) \frac{t^m}{m!} \right).
\]

Therefore, by (13) and (14), we obtain the following theorem.

**Theorem 2.2.** For $m \geq 0$, we have

\[
\lambda^m E_m \left( \frac{x}{\lambda} \right) = \sum_{n=0}^{m} B_{n,q}(x|\lambda) q^{m-n} S_2(m, n),
\]

and

\[
B_{m,q}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m, \ell) q^{m-n} \lambda^\ell E_\ell \left( \frac{x}{\lambda} \right).
\]

Note that $\lim_{q \rightarrow 1} B_{n,q}(x|\lambda) = 2B_{n}(x|\lambda)$, $(n \geq 0)$. When $x = 0$, $B_{n,q}(\lambda) = B_{n,q}(0|\lambda)$ are called the $q$-Boole numbers.

Now, we consider the $q$-Boole polynomials of the second kind as follows:
\( \bar{B}_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} (-\lambda y + x)_{n,q} d\mu_{-1}(y), \ (n \geq 0). \)

Thus, by (15), we get
\[
\bar{B}_{n,q}(x|\lambda) = q^n \int_{\mathbb{Z}_p} \left( \frac{-\lambda y + x}{q} \right)_n d\mu_{-1}(y)
\]
\[
= q^n \int_{\mathbb{Z}_p} \sum_{\ell=0}^n \frac{\lambda^\ell S_1(n,\ell)}{q^\ell} (-1)^\ell (y - \frac{x}{\lambda})^\ell d\mu_{-1}(y)
\]
\[
= \sum_{\ell=0}^n S_1(n,\ell) q^{n-\ell} \lambda^\ell (-1)^\ell E_\ell \left( -\frac{x}{\lambda} \right).
\]

From (8), we have
\[
\sum_{n=0}^\infty E_n \left( -\frac{x}{\lambda} \right) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{(-\frac{x}{\lambda})t}
\]
\[
= \frac{2}{1 + e^{-t}} e^{-(1 + \frac{x}{\lambda})t}
\]
\[
= \sum_{n=0}^\infty (-1)^n E_n \left( 1 + \frac{x}{\lambda} \right) \frac{t^n}{n!}.
\]

By (16) and (17), we get
\[
\bar{B}_{n,q}(x|\lambda) = \sum_{\ell=0}^n \lambda^\ell |S_1(n,\ell)| q^{n-\ell} E_\ell \left( 1 + \frac{x}{\lambda} \right).
\]

From (15), we can derive the generating function of the Boole polynomials of the second kind as follows:
\[
\sum_{n=0}^\infty \bar{B}_{n,q}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^\infty q^n \int_{\mathbb{Z}_p} \left( \frac{-\lambda y + x}{q} \right) d\mu_{-1}(y) t^n
\]
\[
= \int_{\mathbb{Z}_p} (1 + qt) \frac{-\lambda y + x}{q} d\mu_{-1}(y)
\]
\[
= (1 + qt) \frac{x}{q} \int_{\mathbb{Z}_p} (1 + qt) \frac{-\lambda y}{q} d\mu_{-1}(y)
\]
\[
= (1 + qt)^\frac{\lambda}{q} \frac{2}{(1 + qt)^{\lambda/q} + 1}.
\]
By replacing $t$ by $(e^t - 1)/q$ in (19), we get
\[
\sum_{n=0}^{\infty} \hat{B}_{l,n,q}(x|\lambda) q^n \left( e^t - 1 \right)^n = e^{\frac{1}{q}(x+\lambda)t} \frac{2}{e^t + 1}
\]
(20)
and
\[
\sum_{n=0}^{\infty} \hat{B}_{l,n,q}(x|\lambda) q^{-n} \left( e^t - 1 \right)^n = \sum_{n=0}^{\infty} \hat{B}_{l,n,q}(x|\lambda) q^{-n} \sum_{m=n}^{\infty} S_2(m,n) t^m - \frac{m}{m!}
\]
(21)

Therefore, by (18), (20) and (21), we obtain the following theorem.

**Theorem 2.3.** For $m \geq 0$, we have
\[
\sum_{n=0}^{m} \hat{B}_{l,n,q}(x|\lambda) q^{m-n} S_2(m,n) = \lambda^m E_m \left( 1 + \frac{x}{\lambda} \right)
\]

and
\[
\hat{B}_{l,m,q}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m,\ell) q^{m-\ell} \lambda^\ell E_\ell \left( 1 + \frac{x}{\lambda} \right).
\]

For $\alpha \in \mathbb{N}$, let us consider $q$-Boole polynomials of the first kind with order $\alpha$ as follows:

(22)

\[
\hat{B}_{l,n,q}^{(\alpha)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_\alpha + x)_{n,q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_\alpha)
\]

\[
= q^n \sum_{\ell=0}^{n} S_1(n,\ell) \frac{1}{q^{\ell}} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_\alpha + x)^\ell d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_\alpha)
\]

\[
= \sum_{\ell=0}^{n} S_1(n,\ell) q^{n-\ell} \lambda^\ell E_\ell^{(\alpha)} \left( \frac{x}{\lambda} \right),
\]

where $E_n^{(\alpha)}(x)$ are the Euler polynomials of order $\alpha$ which are defined by

\[
\left( \frac{2}{e^t + 1} \right)^\alpha e^t = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}.
\]
From (22), we note that the generating function of $Bl_{n,q}^{(\alpha)}(x|\lambda)$ are given by
\[
\sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x|\lambda) \frac{x^n}{n!} = \sum_{n=0}^{\infty} q^n \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} \left( \frac{\lambda x_1 + \cdots + \lambda x_{\alpha} + x}{q} \right) d\mu_1(x_1) \cdots d\mu_{-1}(x_{\alpha}) t^n
\]
(23)
\[
= \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} (1 + qt)^{\frac{\lambda x_1 + \cdots + \lambda x_{\alpha} + x}{q} - 1} d\mu_1(x_1) \cdots d\mu_{-1}(x_{\alpha})
\]
\[
= (1 + qt)^{\frac{2}{\lambda}} \left( \frac{2}{(1 + qt)^{\frac{1}{\lambda}} + 1} \right)^\alpha.
\]

By replacing $t$ by $(e^t - 1)/q$, we get
\[
\sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x|\lambda) \frac{1}{q^n} \frac{(e^t - 1)^n}{n!} = e^{\frac{2t}{\lambda}} \left( \frac{2}{e^{\frac{2t}{\lambda}} + 1} \right)^\alpha
\]
(24)
\[
= \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)} \left( \frac{x}{\lambda} \right) \frac{\lambda^n x^n}{q^n}
\]
and
\[
\sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x|\lambda) \frac{1}{q^n} \frac{(e^t - 1)^n}{n!} = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x|\lambda) \frac{1}{q^n} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!}
\]
(25)
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} B_{n,q}^{(\alpha)}(x|\lambda) S_2(m, n) \right) \frac{t^m}{m!}
\]
Therefore, by (24) and (25), we obtain the following theorem.

**Theorem 2.4.** For $m \geq 0$, we have
\[
\lambda^m E_{m,q}^{(\alpha)} \left( \frac{x}{\lambda} \right) = \sum_{n=0}^{m} q^{m-n} B_{n,q}^{(\alpha)}(x|\lambda) S_2(m, n)
\]
and
\[
B_{l,q}^{(\alpha)}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m, \ell) q^{m-\ell} \lambda^\ell E_{\ell,q}^{(\alpha)} \left( \frac{x}{\lambda} \right).
\]

We consider the $q$-Boole polynomials of the second kind with order $\alpha$ as follows:

(26) $B_{l,q}^{(\alpha)}(x|\lambda) = \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} (-\lambda x_1 - \cdots - \lambda x_{\alpha} + x)_{n,q} d\mu_1(x_1) \cdots d\mu_{-1}(x_{\alpha})$.

Then, by (26), we get
(27) $B_{l,q}^{(\alpha)}(x|\lambda) = q^n \sum_{\ell=0}^{n} S_1(n, \ell) (-1)^\ell q^{n-\ell} \lambda^\ell E_{\ell,q}^{(\alpha)} (-\frac{x}{\lambda}) = \sum_{\ell=0}^{n} S_1(n, \ell) (-1)^\ell q^{n-\ell} \lambda^\ell E_{\ell,q}^{(\alpha)} (-\frac{x}{\lambda})$. 
From the definition of the higher-order Euler polynomials, we note that
\[
\sum_{n=0}^{\infty} E_n^{(\alpha)} \left( -\frac{x}{\lambda} \right) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^\alpha e^{-\frac{x}{\lambda}t}
\]
(28)
\[
= \left( \frac{2}{1 + e^{-t}} \right)^\alpha e^{-\left( \frac{x}{\lambda} + \alpha \right)t}
\]
\[
= \sum_{n=0}^{\infty} (-1)^n E_n^{(\alpha)} \left( \frac{x}{\lambda} + \alpha \right) \frac{t^n}{n!}.
\]
Thus, by (27) and (28), we get
\[
\hat{B}_n^{(\alpha)}(x|\lambda) = \sum_{\ell=0}^{n} S_1(n, \ell) q^{n-\ell} \lambda^\ell E_\ell^{(\alpha)} \left( \frac{x}{\lambda} + \alpha \right).
\]
(29)
From (26), we have
\[
\sum_{n=0}^{\infty} \hat{B}_n^{(\alpha)}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} q^n \prod_{\ell=p}^{\infty} \int_{z_p} \frac{-\lambda x_1 + \cdots + -\lambda x_n + x}{n} \frac{q}{n} \cdot \prod_{\ell=1}^{n} \frac{d\mu(x_1) \cdots d\mu(x_n) t^n}{n!}.
\]
(30)
By replacing \( t \) by \( (e^t - 1)/q \), we get
\[
\sum_{n=0}^{\infty} \hat{B}_n^{(\alpha)}(x|\lambda) \frac{(e^t - 1)^n}{n!} = e^{\frac{x + \alpha}{\lambda}} \left( \frac{2}{e^t + 1} \right)^\alpha
\]
(31)
and
\[
\sum_{n=0}^{\infty} \hat{B}_n^{(\alpha)}(x|\lambda) \frac{1}{q^n} \frac{1}{n!} (e^t - 1)^n = \sum_{n=0}^{\infty} \hat{B}_n^{(\alpha)}(x|\lambda) \frac{1}{q^n} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \]
(32)
Therefore, by (31) and (32), we obtain the following theorem.

**Theorem 2.5.** For \( m \geq 0 \), we have
\[
\lambda^m E_m^{(\alpha)} \left( \frac{x + \alpha}{\lambda} \right) = \sum_{n=0}^{m} q^{m-n} \hat{B}_n^{(\alpha)}(x|\lambda) S_2(m, n)
\]
(33)
and

$$\overline{B}_{m,q}^{(\alpha)}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m, \ell) q^{m-\ell} \lambda^\ell E_\ell^{(\alpha)} \left( \frac{x + \alpha}{\lambda} \right).$$

Remark. When $x = 0$, $\overline{B}_{n,q}(\lambda) = \overline{B}_{n,q}(0|\lambda)$ are called the $q$-Boole numbers of the second kind.

Now, we observe that

$$\begin{align*}
\frac{\overline{B}_{n,q}(\lambda)}{n!} &= \frac{1}{n!} \int_{\mathbb{Z}_p} (-\lambda y)_{n,q} d\mu_{n-1}(y) \\
&= q^n \int_{\mathbb{Z}_p} \left( \frac{\lambda y}{n} \right) d\mu_{n-1}(y) \\
&= (-1)^n q^n \int_{\mathbb{Z}_p} \left( \frac{\lambda y + n - 1}{n} \right) d\mu_{n-1}(y) \\
&= (-1)^n q^n \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{1}{(\ell - 1)!} \int_{\mathbb{Z}_p} \left( \frac{\lambda y}{q} \right)_{\ell} d\mu_{n-1}(y) \\
&= (-1)^n q^n \sum_{\ell=0}^{n} \binom{n-1}{\ell-1} \frac{B_{\ell,q}(\lambda)}{\ell! q^\ell}.
\end{align*}$$

Therefore, by (35), we obtain the following theorem.

**Theorem 2.6.** For $n \geq 0$, we have

$$\frac{(-1)^n}{q^n} \frac{\overline{B}_{n,q}(\lambda)}{n!} = \sum_{\ell=0}^{n} \binom{n-1}{\ell-1} \frac{B_{\ell,q}(\lambda)}{\ell! q^\ell}.$$

**References**

[1] S. Araci and M. Acikgoz, A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials, *Adv. Stud. Contemp. Math.* 22 (2012), no. 3, 399–406.

[2] A. Bayad and J. Chikhi, Apostol-Euler polynomials and asymptotics for negative binomial reciprocals, *Adv. Stud. Contemp. Math.* 24 (2014), no. 1, 33–37.

[3] M. Can, M. Cenkci, V. Kurt, and Y. Simsek, Twisted Dedekind type sums associated with Barnes’ type multiple Frobenius-Euler $\ell$-functions, *Adv. Stud. Contemp. Math.* 18 (2009), no. 2, 135–160.

[4] J. Choi, D. S. Kim, T. Kim, and Y. H. Kim, Some arithmetic identities on Bernoulli and Euler numbers arising from the $p$-adic integrals on $\mathbb{Z}_p$, *Adv. Stud. Contemp. Math.* 22 (2012), no. 2, 239–247.

[5] D. Ding and J. Yang, Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials, *Adv. Stud. Contemp. Math.* 20 (2010), no. 1, 7–21.

[6] D. V. Dolgy, T. Kim, B. Lee, and C. S. Ryoo, On the $q$-analogue of Euler measure with weight $\alpha$, *Adv. Stud. Contemp. Math.* 21 (2011), no. 4, 429–435.

[7] S. Gaboury, R. Tremblay, and B.-J. Fugère, Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials, *Proc. Jangjeon Math. Soc.* 17 (2014), no. 1, 115–123.
A NOTE ON BOOLE POLYNOMIALS WITH $q$-PARAMETER

[8] J. H. Jeong, J.-H. Jin, J.-W. Park, and S.-H. Rim, On the twisted weak $q$-Euler numbers and polynomials with weight 0, *Proc. Jangjeon Math. Soc.* 16 (2013), no. 2, 157–163.
[9] D. S. Kim and T. Kim, A note on Boole polynomials, *Integral Transforms Spec. Funct.* 25 (2014), no. 8, 627–633.
[10] T. Kim, D. S. Kim, D. V. Dolgy, and S.-H. Rim, Some identities on the Euler numbers arising from Euler basis polynomials, *Ars Combin.* 109 (2013), 433–446.
[11] T. Kim, Symmetry of power sum polynomials and multivariate fermionic $p$-adic invariant integral on $\mathbb{Z}_p$, *Russ. J. Math. Phys.* 16 (2009), no. 1, 93–96.
[12] T. Kim, Some identities on the $q$-Euler polynomials of higher order and $q$-Stirling numbers by the fermionic $p$-adic integral on $\mathbb{Z}_p$, *Russ. J. Math. Phys.* 16 (2009), no. 4, 484–491.
[13] T. Kim, A study on the $q$-Euler numbers and the fermionic $q$-integral of the product of several type $q$-Bernstein polynomials on $\mathbb{Z}_p$, *Adv. Stud. Contemp. Math.* 23 (2013), no. 1, 5–11.
[14] Q.-M. Luo, $q$-analogues of some results for the Apostol-Euler polynomials, *Adv. Stud. Contemp. Math.* 20 (2010), no. 1, 103–113.
[15] H. Ozden, I. N. Cangul, and Y. Simsek, Remarks on $q$-Bernoulli numbers associated with Duhee numbers, *Adv. Stud. Contemp. Math.* 18 (2009), no. 1, 41–48.
[16] S. Roman, The umbral calculus, Academic Press, Inc., New York, 1984.
[17] C. S. Ryoo, H. Song and R. P. Agarwal, On the roots of the $q$-analogue of Euler-Barnes’ polynomials, *Adv. Stud. Contemp. Math.* 9 (2004), no. 2, 153–163.
[18] Y. Simsek, O. Yurekli and V. Kurt, On interpolation functions of the twisted generalized Frobenius-Euler numbers, *Adv. Stud. Contemp. Math.* 15 (2007), no. 2, 187–194.
[19] E. Sen, Theorems on Apostol-Euler polynomials of higher order arising from Euler basis, *Adv. Stud. Contemp. Math.* 23 (2013), no. 2, 337–345.
[20] Z. Zhang and H. Yang, Some closed formulas for generalized Bernoulli-Euler numbers and polynomials, *Proc. Jangjeon Math. Soc.* 11 (2008), no. 2, 191–198.

D. S. Kim
Department of Mathematics
Sogang University
Seoul, Republic of Korea
e-mail: dskim@sogang.ac.kr

Y. S. Jang
Department of Applied Mathematics
Kangnam University
Yongin 446-702, Republic of Korea
e-mail: ysjang@kangnam.ac.kr

T. Kim
Department of Mathematics
Kwangwoon University
Seoul 139-701, Republic of Korea
e-mail: tkkim@kw.ac.kr

S.-H. Rim
Department of Mathematics Education
KYUNGPOOK NATIONAL UNIVERSITY
TAEGU 702-701, REPUBLIC OF KOREA
e-mail: shrim@knu.ac.kr