CO-ADDITION FOR FREE NON-ASSOCIATIVE ALGEBRAS AND THE HAUSDORFF SERIES

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Abstract. Generalizations of the series exp and log to noncommutative non-
associative and other types of algebras were considered by M. Lazard, and recently
by V. Drensky and L. Gerritzen. There is a unique power series exp(x) in one
non-associative variable x such that exp(x) exp(x) = exp(2x), exp′(0) = 1.

We call the unique series \( H = H(x, y) \) in two non-associative variables satisfying
\( \exp(H) = \exp(x) \exp(y) \) the non-associative Hausdorff series, and we show that the
homogeneous components \( H_n \) of \( H \) are primitive elements with respect to the co-
addition for non-associative variables. We describe the space of primitive elements
for the co-addition in non-associative variables using Taylor expansion and a pro-
jector onto the algebra \( A_0 \) of constants for the partial derivations. By a theorem of
Kurosh, \( A_0 \) is a free algebra. We describe a procedure to construct a free algebra
basis consisting of primitive elements.

In this article we are studying the co-addition \( \Delta : K\{X\} \to K\{X\} \otimes K\{X\} \), where
\( K \) is a field of characteristic 0, and where \( K\{X\} \) denotes the free unitary algebra over
\( K \) with one binary operation (a non-associative, noncommutative multiplication),
also called free magma algebra generated by \( X \).

The space \( \text{Prim}(K\{X\}) = \{ f \in K\{X\} : \Delta(f) = f \otimes 1 + 1 \otimes f \} \) of primitive elements
for the co-addition is considered. The first Lazard-cohomology group of the \( \otimes \)-Kurosh
analyzer is given by the primitive elements for the co-addition, see [10], and they
are also called pseudo-linear, following [La]. The smallest space that contains the
variables and is closed under commutators \([f_1, f_2] := f_1 f_2 - f_2 f_1 \) and associators
\((f_1, f_2, f_3) := (f_1 f_2) f_3 - f_1 (f_2 f_3) \) is contained in \( \text{Prim}(K\{X\}) \), and is strictly smaller
than \( \text{Prim}(K\{X\}) \). It does not contain the primitive element \( x^2 x^2 - 2x(x^2 x) + x(x x^2) \),
for example.

The algebra \( (K\{X\})_0 \) of constants relative to all partial derivations \( \frac{d}{dx_i} \), \( x_i \in X \),
contains all primitive elements of order \( \geq 2 \). It is expected that \( (K\{X\})_0 \) is freely
generated by homogeneous primitive polynomials.

For the algebra \( K\{x\} \) in one variable \( x \), we describe a construction of such an
algebra basis for the algebra of constants, in which Lazard-cohomology is used. The
first element of the free algebra basis consisting of primitive elements is \( x x^2 - x^2 x \),
which is the only generator in degree 3. In degree \( n \geq 4 \), there are \( 3(c_{n-1} - c_{n-2}) \)
generators, with Catalan number \( c_n := \frac{(2(n-1))!}{n!(n-1)!} \).

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The basic method is based on the concept of Taylor expansion in \( K\{X\} \), which provides a projector onto the algebra \((K\{X\})_0\) of constants.

The classical Hausdorff series
\[
\log(e^x e^y) = H(x, y) = \sum_{n=1}^{\infty} H_n, \quad H_n \text{ homogeneous of degree } n,
\]
is a series in associative variables with rational coefficients. The famous Campbell-Baker-Hausdorff formula states that all \( H_n \) are Lie polynomials. By a Theorem of Friedrichs, Lie polynomials are characterized as the primitive elements of the free associative co-addition Hopf algebra, cf. [R], p. 20.

Generalizations of the series \exp and \log to noncommutative non-associative and other types of algebras were considered by M. Lazard, see [La], and recently by V. Drensky and L. Gerritzen, see [DG]. There is a unique power series \( \exp(x) \) in one non-associative variable such that \( \exp(x) \exp(x) = \exp(2x) \), \( \exp'(0) = 1 \), and it holds that \( \exp'(x) = \exp(x) \). There exists a unique series \( \hat{H} = \hat{H}(x, y) \) without constant term in two non-associative variables satisfying \( \exp(\hat{H}) = \exp(x) \exp(y) \). We suggest to call \( \hat{H} \) the non-associative Hausdorff series.

We show that the homogeneous components \( H_n \) of \( \hat{H} \) are primitive elements with respect to the co-addition for non-associative variables. We obtain a recursive formula for the coefficients \( c(\tau) \) of \( \hat{H} \), see section 6. Each magma monomial \( \tau \) is a word with parenthesis and can be identified with an \( X \)-labeled planar binary rooted tree. Each coefficient \( \bar{c}(w) \), \( w \) an associative word in \( x \) and \( y \), of the classical Hausdorff series is obtained as a sum \( \sum c(\tau) \) over magma monomials \( \tau \) for which the foliage is equal to \( w \) (see [R], p.84). Several formulas for the coefficients in the classical case are given in [R] §3.3. There is no analogue of the given formula, though.

In section 1 we show that the co-addition is cocommutative, coassociative, and that there are left and right antipodes, which however are not anti-homomorphisms. We get a "non-associative Hopf algebra structure" on the free magma algebra. If we identify the monomials with \( X \)-labeled planar binary trees, the grafting of trees occurs as the multiplication, see also [Ho2], Remark (3.7).

The Hopf algebras on planar binary trees described in [LR] or [BF] are Hopf algebra structures on free associative algebras. The comultiplications can be described by cuts that are given by subsets of the set of vertices.

For the co-addition Hopf algebra of section 1, the image \( \Delta(\tau) \) of a magma monomial under \( \Delta \) is described in terms of contractions of the tree \( \tau \) onto subsets of the set of leaves, see Lemma (1.9).

Polynomials in one variable \( x \) are considered in section 2. The space of constants of degree \( n \) homogeneous elements has dimension \( c_n - c_{n-1} \). We study Taylor expansions
\[
f = \sum_{j=0}^{\infty} x^j a_j
\]
of polynomials $f \in K\{x\}$ and power series $f \in K\{\{x\}\}$. All $a_j = a_j(f)$ are uniquely defined constants for $\frac{d}{dx}$. We define an integral
\[
\int f \, dx = \sum_{j=0}^{\infty} \frac{1}{(j+1)} x^{j+1} a_j(f)
\]
and obtain a projector
\[
\Phi : K\{\{x\}\}_0 \rightarrow (K\{\{x\}\})_0
\]
which maps $f$ onto the Taylor coefficient $a_0(f)$. In the main result of this section, see Proposition (2.11), a system of homogeneous free algebra generators for $(K\{\{x\}\})_0$ is constructed. In section 4, this system is modified into a homogeneous set of primitive polynomials, see Theorem (4.4). Thus we get a co-addition Hopf algebra over a countably infinite set of variables.

Taylor expansions for polynomials in several variables are considered in section 3. We construct a $K$-basis for the algebra of constants in two variables, see Proposition (3.3). In section 5 we show that the composition $f(g_1, \ldots, g_n)$ of pseudo-linear (i.e. primitive) polynomials $f, g_1, \ldots, g_n$ is again primitive, and that all primitive elements are obtained by insertion of primitive elements into multi-linear ones. For example, the primitive element $x^2x^2 - 2x(x^2x) + x(xx^2)$ mentioned above can be realized as composition $f(x, x, x, x)$, where $f(x_1, x_2, x_3, x_4)$ is the multi-linear primitive element
\[
(x_1x_2)(x_3x_4) - x_1(x_2(x_3x_4)) - x_4 \cdot (x_1, x_2, x_3) - x_3 \cdot (x_1, x_2, x_4).
\]
Furthermore, e.g. $f(x_1, [x_2, x_3], (x_4, x_5, x_6), ([x_7, x_8], x_9, x_{10})]$ is also primitive.

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1. Co-Addition.

Let $K$ be a field of characteristic 0, and let $X = \{x_1, x_2, \ldots\}$ be a finite or countable set of variables. We equip each variable $x_i$ with a non-negative degree $d_i = d(x_i)$ such that $d_{i+1} \geq d_i$.

By $K\{X\} = F_{\text{MagAlg}}(X)$ we denote the free $K$-algebra with unit 1 generated by $X$, called the free magma algebra over $X$. Here a (magma) algebra over $K$ is just a vector space $V$ together with one binary operation $\cdot : V \times V \rightarrow V$, called the multiplication (which is not required to be associative).

As a vector space, $K\{X\}$ has the set of all planar binary trees with leaves labeled by the letters $x_i$ as a basis.

The algebra $K\{X\}$ is naturally graded, $K\{X\} = \bigoplus_n K\{X\}_n$, where $K\{X\}_n$ is the vector subspace generated by total degree $n$ monomials. The set of non-trivial monomials is given by the free magma $M = M(X) = F_{\text{Mag}}(X)$ without unit generated by $X$, and the total degree $d$-deg is the unique morphism $M \rightarrow \mathbb{N}$ extending $d$. 
Remark 1.1. We can embed $K\{X\}$ into its completion $K\{\{X\}\} = \prod_{n=0}^{\infty} K\{X\}_n$, the free complete magma algebra generated by $X$. We have the canonical degree and order functions $\deg$ and $\ord$ (for $d \equiv 1$).

Definition 1.2. Let $\Delta$ be the algebra homomorphism defined on $K\{X\}$ (or $K\{\{X\}\}$) by $x_i \mapsto x_i \otimes 1 + 1 \otimes x_i \in K\{X\} \otimes K\{X\} \subset K\{\{X\}\} \otimes K\{\{X\}\}$.

The map $\Delta$ is called the co-addition.

Proposition 1.3. The map $\Delta$ is coassociative, $(\Delta \otimes \id)\Delta = (\id \otimes \Delta)\Delta$. It is also cocommutative, i.e. $\tau \circ \Delta = \Delta$, where $\tau$ is the permutation of tensor factors.

There is a unique $K$-linear map $\sigma$, the (left-)antipode, such that $\sigma(w) + w + \sum \sigma(w_{<1>})w_{<2>} = 0$, $\deg w \geq 1$, where $\Delta(w) = w \otimes 1 + 1 \otimes w + \sum w_{<1>}w_{<2>}$.

If $\bar{t}$ denotes the involution induced by $\bar{t}_1t_2 = \bar{t}_2\bar{t}_1$, then $\bar{\sigma}$ given by $\bar{\sigma}(\bar{t}) = \sigma(t)$ fulfills $\bar{\sigma}(w) + w + \sum w_{<1>}\sigma(w_{<2>}) = 0$ ($\bar{\sigma}$ is the right-antipode).

Proof. To check that $(\Delta \otimes \id)\Delta = (\id \otimes \Delta)\Delta$, we note that both homomorphisms map $x_i$ on $x_i \otimes 1 \otimes 1 + 1 \otimes x_i \otimes 1 + 1 \otimes x_i$. The antipode is recursively defined. Its properties are easy to check. \hfill \square

Remark 1.4. The algebra $K\{X\}$ (or $K\{\{X\}\}$) together with $\Delta$ will be called the free (graded or complete) non-associative Hopf algebra with co-addition. Non-associative Hopf algebras can be defined similarly to usual Hopf algebras. (While the comultiplication is required to be coassociative, the multiplication is not required to be associative, and left and right antipodes do not have to coincide). Then the co-addition is a special comultiplication, namely the one which is dual to addition (compare the usage in [BH], where an abelian group valued functor leads to a co-addition).

The notion of a non-associative Hopf algebra should not be confused with the notion of a non-associative coalgebra in [ACM], [Gr], and [Z], which means a not necessarily coassociative coalgebra. The graded linear dual of $(K\{X\}, \Delta)$ is such a coalgebra. Its non-coassociative comultiplication is given by $t \mapsto t \otimes 1 + 1 \otimes t + t_1 \otimes t_2$, if $t = t_1t_2$ with $\deg(t_i) \geq 1$.

Remark 1.5. Let $f = f(x_1, x_2, \ldots)$ be a homogeneous (of total degree $\geq 1$, say) polynomial. For $X'$ a copy of $X$, with bijection $x_i \mapsto x_i'$, the substitution $x_i \mapsto x_i + x_i'$ defines a polynomial $f(x_1 + x_1', x_2 + x_2', \ldots) - f(x_1, x_2, \ldots) - f(x_1', x_2', \ldots)$, which is zero iff $f$ is linear.

If we only require that $f(x_1 + x_1', x_2 + x_2', \ldots) - f(x_1, x_2, \ldots) - f(x_1', x_2', \ldots) = 0$ holds in the case where all variables from $X'$ commute with the variables from $X$, we get the following weaker version of linearity.

Definition 1.6. An element $f$ of the free magma algebra $A = K\{X\}$ (or its completion $K\{\{X\}\}$) is called ($\otimes$-)pseudo-linear, iff $f$ is a primitive element in the free non-associative co-addition Hopf algebra, i.e. iff $\Delta(f) = f \otimes 1 + 1 \otimes f$.

The kernel of the $K$-module homomorphism given by $\partial_i := \Delta - \iota_1 - \iota_2$, where $\iota_1(x_i) = x_i \otimes 1$, $\iota_2(x_i) = 1 \otimes x_i$, is denoted by $\text{Prim}(A)$.
Remark 1.7. (i) For pseudo-linear $f$, $\sigma(f) = -f$. It is worth to note that $\sigma$ is not the anti-homomorphism induced by $x_i \mapsto -x_i$.

For example, let $A = K\{x\}$. While $\sigma(x) = -x$, $\sigma(x^2) = x^2$, we recursively compute that $\sigma(xx^2) = 2xx^2 - 3x^2x$, $\sigma(x^2x) = 3xx^2 - 4x^2x$. The order of $\sigma$ is infinite.

(ii) The notion pseudo-linearity follows [La]. The (tuples of) pseudo-linear elements of $\langle I \rangle$ form the first cohomology group $H^1(\otimes - \hat{\mathcal{K}})$ of $\mathcal{H}^1$.

Definition 1.8. Let $D_i$, $i \in \mathbb{N}$, be the unique $K$-linear derivation defined on $A = K\{\{X\}\}$ such that $D(x_i) = 1$, and $D(x_j) = 0$ else.

For $s$ a tree monomial, and $f \in K\{\{X\}\}$, we define $\Delta_s(f)$ by $\Delta(f) = \sum s \Delta_s(f) \otimes s$.

For $t = t_1t_2$ a tree monomial, let $I(t) = I(t_1) \cup I(t_2)$ be the set of leaves of $t$. For any subset $I$ of $I(t)$ the contraction $t|I$ is the tree with set of leaves $I$ recursively defined by $t_1t_2|I = (t_1|I \cap I(t_1)) \cdot (t_2|I \cap I(t_2))$, where $t|\emptyset = 1$. The neutral element 1 for the grafting multiplication in $M$ is the empty tree.

Informally the contraction $t|I$ is obtained by removing all leaves not in $I$ followed by the necessary edge-contractions to get a binary tree.

Let $\mu_s(t)$ be the number of subsets $I$ of $I(t)$ which yield $s$ by contraction, i.e. for which $t|I = s$.

If $t|I = s$, we call $s' = t|I^c$ a complement of $s$ in $t$, where $I^c$ is the complement of $I$ in $I(t)$.

Lemma 1.9. The following formulas hold:

(i) For $t$ a tree, $\Delta(t) = \sum_{I \subseteq I(t)} (t|I)\otimes (t|I^c)$; and $\Delta_s(t)$ is given by a sum $\sum s' \mu_s(t)s'$ over trees $s'$ which are complements of $s$ in $t$.

(ii) $\Delta_s = D_i$.

(iii) $\Delta_s \circ \Delta_t = \Delta_t \circ \Delta_s$, for all $s, t \in M$.

(iv) $\Delta_s(fg) = \Delta_s(f)g + f\Delta_s(g) + \Delta_{s_1}(f)\Delta_{s_2}(g)$, for $s = s_1s_2$ of degree > 1.

Furthermore, $\mu_s(t) = \mu_s(t_1) + \mu_s(t_2) + \mu_{s_1}(t_1)\mu_{s_2}(t_2)$, for $t = t_1t_2$ a tree in $M$.

(v) Let $M_n(\{x_i\})$ be the set of elements of degree $n$ in $M(\{x_i\})$.

Then $\sum_{s \in M_n(\{x_i\})} \Delta_s = \frac{1}{n!} D^n_i$.

Proof. 1) The formula in (i) follows by a direct inspection of the co-addition map. Then (ii) and (iii) are easy consequences of (i). Since $\Delta$ is an algebra homomorphism (and cocommutative, coassociative), we get (iv).

2) To show (v), we proceed by induction on $\deg(t_1), \deg(t_2)$:

The sum $n! \cdot \sum_{s \in M_n(\{x_i\})} \Delta_s(t_1t_2) = \sum_k \sum_{s_1 \in M_k(\{x_i\}), s_2 \in M_{n-k}(\{x_i\})} \Delta_{s_1}(t_1)\Delta_{s_2}(t_2)$ is then equal to $\sum_k \binom{n}{k} D^n_i D^{n-k}_i (t_1t_2) = D^n_i (t_1t_2)$. Hence (v) follows.  \( \square \)
Lemma 1.10. For $f$ homogeneous of degree $n$, $f$ is pseudo-linear if and only if 
$\Delta_s(f) = 0$ for all $s \in M$ with $1 \leq \deg s < \frac{n+1}{2}$.

Proof. Using cocommutativity, the criterion follows. \qed

2. Algebra of Constants and Taylor Expansion in one variable.

Let $X = \{x\}$, and let $A = K\{X\}$ (or $A = K\{\{X\}\}$). Let $A_0$ be the subalgebra of elements $a \in A$ with $D(a) = 0$. The algebra $A_0$ is called algebra of constants.

By Lemma (1.10), the pseudo-linear elements of order $\geq 2$ form a subspace of $A_0$. We are going to describe $A_0$ next.

Lemma 2.1. Let $L : A \to A$ be the left multiplication by $x$, given by $L(f) := x \cdot f$.

It holds that

$D \circ L^k - L^k \circ D = kL^{k-1}, k \geq 1$

$D^k \circ L - L \circ D^k = kD^{k-1}, k \geq 1$.

Proof. As in the classical proof for associative variables, the equation $D \circ L - L \circ D = \text{id}$ follows from the Leibniz rule. The generalization for $k \geq 1$ follows by induction on $k$. \qed

Remark 2.2. In Proposition (2.3), we are going to use the following ordering on $M$ induced by the order $x_1 < x_2 < \ldots$ of variables.

Let us first order by increasing total degree. Then, if $s = s_1 s_2 \ldots$ and $t = t_1 t_2 \ldots$ are of the same degree, we set $s < t$ if $s_1 = t_1$ and $s_2 < t_2$, or if $s_1 < t_1$. We will call the maximal monomial of a homogeneous polynomial $f$ the leading term, denoted by $f^\prec$.

We write $x^j f$ for $L^j f$, and $x^j := L^j(1)$.

We denote $Df$ by $\frac{d}{dx} f$.

Proposition 2.3. (i) There is a unique Taylor expansion $f = \sum_{j=0}^{\infty} x^j a_j$ for every $f \in A$, with $a_j = a_j(f) \in A_0$. Moreover, the elements $a_j$ are homogeneous of degree $n - j$ if $f$ is homogeneous of degree $n$.

(ii) If $f = \sum_{j=0}^{\infty} x^j a_j$, then the Taylor expansion of $x f$ is given by $\sum_{j=1}^{\infty} x^j a_{j-1}$.

(iii) The operator $\Phi$ given by $f \mapsto a_0(f)$ is a projector onto $A_0$ with $\ker \Phi = \text{im} L$.

Proof. 1) We note that $D \sum_{j=0}^{\infty} x^j a_j = \sum_{j=0}^{\infty} (j+1)x^j a_{j+1}$, by (2.1), and since $D(a_j) = 0$ by construction.

We prove the uniqueness and existence of the Taylor expansion (i) by induction on the degree of $f$. The case $n = \deg f = 0$ is trivial. For $n \geq 1$, let $\frac{d}{dx} f$ be given by the unique Taylor expansion $\sum_{j=0}^{\infty} x^j b_j$. Let $g$ be given by $\sum_{j=0}^{\infty} x^{j+1} a_{j+1}$, where $a_{j+1} := b_j \frac{1}{j+1}$. Then $\frac{d}{dx} (f-g) = 0$, thus $a_0 := f-g \in A_0$. \qed
Now $\sum_{j=0}^{\infty} x^j a_j$ is the desired Taylor expansion of $f$. For homogeneous $f$, we get homogeneous $a_j$.

2) Assertion (ii) follows directly from (i). For (iii), it remains to show that $\ker \Phi \subseteq \text{im} L$. Let us assume that $f \in \ker \Phi$ is not in $\text{im} L$. Then with respect to the ordering we use, the leading term of $f \in \ker \Phi$ cannot be of the form $x.h$. But now we easily get the contradiction $\Phi(f)^c = f^c \neq 0$.

\[\square\]

**Definition 2.4.** We define the integral $\int f \, dx$ by $\sum_{j=0}^{\infty} \frac{1}{(j+1)!} x^{j+1} a_j(f)$.

**Remark 2.5.** Clearly the expressions for $f$ and $\int (\frac{d}{dx} f) \, dx$ differ exactly by $a_0(f)$.

Let us describe an algorithm to obtain the Taylor coefficients. For $f$ a polynomial, and $n$ maximal such that $D^n f \neq 0$, let $a_n = \frac{1}{n!} D^n f$. Then $\tilde{f} := f - x^n a_n$ can be used to obtain the coefficients $a_j$, $j < n$, and we have that $D^n \tilde{f} = 0$. Repeating the step, the coefficient $a_{n-i}$ can be obtained by applying $\frac{1}{(n-i)!} D^{n-i} \tilde{f}$ to $(\text{id} - \frac{1}{(n-i+1)!} L^{n-i+1} D^{n-i+1}) \cdots (\text{id} - \frac{1}{n!} L^n D^n) f$.

**Proposition 2.6.** For $f \in A$ homogeneous of degree $n$, we have

\[\int f \, dx = \sum_{k=0}^{n+1} \frac{(-1)^{k-1}}{k!} L^k (D^{k-1} f)\]

This sum is finite for homogeneous $f$, and we get a continuous operator $\int f \, dx : K\{\{X\}\} \to K\{\{X\}\}$.

**Proof.** Applying $D$ to the operator in question, we get

\[\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k!} D L^k D^{k-1} f = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} L^k \circ D^k f + \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{(k-1)!} L^{k-1} D^{k-1} f\]

\[= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k!} L^k (D^{k-1} f) - \sum_{k=0}^{n} \frac{(-1)^{k-1}}{k!} L^k D^k f = f.\]

Since $\sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k!} L^k (D^{k-1} f)$ is of the form $f + x.h$, its Taylor expansion has the same constant term as the Taylor expansion of $\int f$, by Proposition (2.3)(ii).

\[\square\]

**Example 2.7.** Let $f = x^2 x$. Then 3 is the maximal $n$ with $D^n f \neq 0$. Thus we set $a_3 = \frac{1}{3} D^3 f = 1$.

For $\tilde{f} := f - x^3 a_3 = f - xx^2$ we repeat the step and find that $\tilde{f} \in A_0$. Thus the Taylor expansion of $f = x^2 x$ is given by $a_0 = a_0(x^2 x) = x^2 x - xx^2$, $a_3 = 1$, $a_j = 0$ else.

While there are no nonzero elements in $(A_0)_2$, the space $(A_0)_3$ is one-dimensional with generator $a_0(x^2 x)$. 
Definition 2.8. Let $\Gamma := M - (xM \cup \{x\})$. It is a sub-magma of $M$.

Proposition 2.9. A vector space basis of $A_0$ is given by the homogeneous polynomials $\Phi(s)$, $s \in \Gamma$. The dimension of $(A_0)_n$ is given by $c_n - c_{n-1}$, where the $c_n := \frac{(2(n-1))!}{n!(n-1)!}$ are the Catalan numbers.

Proof. Let $M_n$ be the subset of $M$ consisting of degree $n$ monomials, and let $\Gamma_n$ be the subset of $\Gamma$ of elements of degree $n$. Then the number of elements in $\Gamma_n$ is given by $\#\Gamma_n = \#M - \#(xM \cup \{x\}) = c_n - c_{n-1}$.

By Proposition (2.3)(iii), we get a vector space basis $\Phi(s)$, $s \in \Gamma$.

□

Proposition 2.10. (i) The magma $\Gamma$ has the following set as a free generating set:

$\Omega = \{s_1 \cdot s_2 : s_1 \in M - \{x\}, s_2 \in M, \text{s.t. } s_1 \not\in \Gamma \text{ or } s_2 \not\in \Gamma\}$

$= \{(x^i \cdot v_1) \cdot (x^j \cdot v_2) : v_1, v_2 \in \Gamma \cup \{1\}, i + j \geq 1, v_1 \neq 1 \text{ if } i \leq 1, v_2 \neq 1 \text{ if } j = 0\}$

(ii) Let $\Omega_n := \{\omega \in \Omega : \deg(\omega) = n\}$. We have that $\Omega_1 = \Omega_2 = \emptyset$, $\Omega_3 = \{x^2x\}$.

For $n \geq 4$, $\#\Omega_n = 3(c_{n-1} - c_{n-2})$.

Proof. 1) Clearly, $\Gamma - \Gamma \cdot \Gamma$ is a generating set. Since the restriction of the multiplication $M \times M \to M$ on $\Gamma$ is injective as well, the generating set is free.

2) The set $\Omega_n$ is the union $\Omega'_n \cup \Omega''_n$ of $\Omega'_n = \{(x \cdot t_1)t_2 : t_1, t_2 \in M, \deg(t_1) + \deg(t_2) = n - 1\}$ and $\Omega''_n = \{t_1(x \cdot t_2), t_1 \in M - \{x\}, t_2 \in M \cup \{1\}, \deg(t_1t_2) = n - 1\}$.

The number of elements of $\Omega'_n$ is given by $c_{n-1}$, since the pairs $(t_1, t_2)$ can be identified with trees $t_1t_2$ of degree $n - 1$. For $\Omega''_n$, one similarly counts $c_{n-1}$ elements of the form $t_1 \cdot x$ plus $c_{n-1} - c_{n-2}$ elements of the form $t_1(x \cdot t_2)$ with $t_1, t_2 \in M, t_1 \neq x$.

The number of elements of $\Omega'_n \cap \Omega''_n = \{(xt_1)(xt_2) : t_1, t_2 \in M, \deg(t_1t_2) = n - 2\} \cup \{(xt_1)x : t_1 \in M, \deg(t_1) = n - 2\}$ is given by $2c_{n-2}$. Thus assertion (ii) follows.

□

Proposition 2.11. Let $E = \{\Phi(\omega) : \omega \in \Omega\}$. Then $E$ is a sequence

$y_{3,1} = a_0(x^2x), \ y_{4,1}, \ldots, y_{4,3c_3-3c_2}, \ y_{5,1}, \ldots, y_{5,3c_4-3c_3}, \ldots$

of elements in $A_0$, ordered by the leading monomials as in (2.3), that freely generates the algebra $A_0$.

Proof. Since $\Phi$ does not change the leading term, the assertion follows from Proposition (2.3)(iii) and Proposition (2.10).
Example 2.12. The tree \((x^2x)^2 \in \Gamma\) is not an element of \(\Omega\). It is the smallest element of \(\Gamma\) that is not contained in \(\Omega\).

To determine the Taylor expansions of the trees \(t = t_1t_2\) of degree \(\geq 4\) in \(\Omega\), let us recall that by (1.3)(iv) and (v), \(\mu_s(t) = \mu_s(t_1) + \mu_s(t_2) + \mu_s(t_1)\mu_s(t_2)\) is the coefficient of \(s\) in \(\frac{1}{n!}D^n\).

For \(s = x^2x\), the formula reads \(\mu_{x^2x}(t) = \mu_{x^2x}(t_1) + \mu_{x^2x}(t_2) + \left(\text{deg } t_1\right) \text{deg } t_2\).

While \(a_n = 1, a_{n-1} = a_{n-2} = 0\), the Taylor coefficient \(a_{n-3} = \frac{1}{(n-3)!}D^{n-3}(f - x^n)\) can be determined by this formula, because it has to be of the form \(\alpha \cdot y_{3,1}\) with \(\alpha \in \mathbb{Q}\). It is simply given by \(\mu_{x^2x}(t)\), as \((D^{n-3}x^n)^< = x^3\).

In degree 4, the Taylor expansions are given by coefficients \(a_4 = 1, a_3 = a_2 = 0\), and \(a_1, a_0\) as follows:

For \(x^2x^2\), \(a_1 = 2y_{3,1}, a_0 = x^2x^2 - 2x.y_1 - x^4.1 = x^2x^2 - 2x(x^2x) + x^4\).

For \(x^3x\), \(a_1 = 3y_{3,1}, a_0 = x^3x - 3x.y_1 - x^4 = x^3x - 3x(x^2x) + 2x^4\).

For \((x^2x)x\), \(a_1 = 4y_{3,1}, a_0 = ((x^2)x)x - 4x.y_1 - x^4 = ((x^2)x)x - 4x(x^2x) + 3x^4\).

The elements \(y_{4,1} = a_0(x^2x^2), y_{4,2} = a_0(x^3x), y_{4,3} = a_0((x^2)x)x\) form a basis of \((A_0)_4\).

Remark 2.13. Similarly, for \(t\) a tree of degree \(n\), and \(1 \leq r \leq n - 3\), the Taylor coefficient \(a_r(t)\) is of the form \(\sum \alpha_{n-r,i} \cdot y_{n-r,i}\), with \(\alpha_{n-r,i} = \mu_{y_n^{<,r}}(t) \in \mathbb{N}\).

Example 2.14. For \(s = x^3x^{r-l}\) and \(t = x^kx^{n-k}\) elements of \(\Omega\) with \(r = \text{deg } s \geq 3\), \(n = \text{deg } t \geq 3\), and such that \(2 \leq l \leq r - 1, 2 \leq k \leq n - 1\), one easily shows that \(\mu_s(t) = \binom{k}{l} \binom{n-k}{r-l}\).

Thus \(z_{n,k} := a_0(x^kx^{n-k})\) is given by \(x^kx^{n-k} - x^n - \sum_{r-l} \binom{k}{l} \binom{n-k}{r-l} x^{n-r} \cdot z_{r,l}\), where the sum ranges over all elements \(z_{r,l}\) with leading terms \(x^j x^{r-l}\).

E.g., for \(x^3x^2\), we get \(a_2 = 3y_{1,1}, a_1 = 3y_{1,2}, z_{5,2} = x^2x^3 - x^5 - 3x^2.z_{3,2} = 3x.z_{4,2}\).

In degree 5, \((A_0)_5\) is 9-dimensional. To obtain a basis we take \(z_{5,2}, z_{5,3}, z_{5,4}\), and we take the \(a_0\)-terms of the following six trees, the Taylor coefficients of which we give here: The tree \(x^2(x^2x)\), with coefficients \(a_2 = 4y_{3,1}, a_1 = 3y_{1,1}\); the tree \((x^2x)x^2\), with \(a_2 = 7y_{3,1}, a_1 = 3y_{1,1} + 2y_{1,3}\); the tree \((x^2x)x\), with \(a_2 = 7y_{3,1}, a_1 = 3y_{4,2} + y_{4,3}\); the tree \((x^2x)x\), with \(a_2 = 7y_{3,1}, a_1 = y_{4,1} + 2y_{4,2} + 2y_{4,3}\); the tree \((x^3)x)x\), with \(a_2 = 3y_{1,1}\), \(a_1 = 2y_{4,2}\); and the tree \(((x^2)x)x)x\), with \(a_2 = 10y_{3,1}, a_1 = 5y_{4,3}\).

3. Taylor Expansion for Several Variables.

Let \(X = \{x_1, x_2, \ldots\}\) and \(A = K\{X\}\) (or \(A = K\{\{X\}\}\)). Let \(L_i\) denote the left multiplication by \(x_i\). For \(j\) a tuple \((j_1, \ldots, j_r)\) with entries in \(\mathbb{N}\), we set \(x_j.f := (L_{j_1}^\circ \ldots \circ L_{j_r}^\circ)(f)\).
Proposition 3.1. (i) Let \( B \) be a graded magma algebra with unit, and let \( x \in B \). Let a derivation \( D : B \to B \) be given, such that \( D \) is a nilpotent operator with \( D(x) = 1 \). Then there is a unique Taylor expansion \( f = \sum_{j=0}^{\infty} x^j b_j \) for every \( f \in B \), with \( D(b_j) = 0 \) for all \( j \).

(ii) For each \( x_i \in X \), there is a unique Taylor expansion \( f = \sum_{j=0}^{\infty} x_i^j b_j \) (with respect to one variable) for every \( f \in A \), with \( \frac{d}{dx_i} b_j = 0 \) for all \( j \).

If \( \Gamma(X, i) \) denotes the sub-magma of \( M(X) \) given by \( M(X) - (x_i M(X) \cup \{ x_i \}) \), then the elements \( b_0(s), s \in \Gamma(X, i) \), form a vector space basis of the algebra of constants with respect to \( D_i \).

(iii) There is a unique (total) Taylor expansion \( f = \sum_j x_i^j a_j \) for every \( f \in A \), such that all \( a_j \) are in \( A_0 := \{ f \in A : \frac{d}{dx_i} f = 0, \text{ all } i \} \).

Proof. Assertion (i) is proven as in Proposition (2.3), where the degree of \( f \) is now replaced by the smallest \( n \) such that \( D^n f = 0 \).

Using (i) we can prove assertion (ii) as in the case of one variable.

Having obtained the Taylor expansion with respect to the variable \( x_1 \), we can expand all coefficients again with respect to the next variable. The set \( \bar{X} := \Omega(\Gamma(X, 1)) := \Gamma(X, 1) - \Gamma(X, 1) \cdot \Gamma(X, 1) \) is a free generating set for the algebra of constants with \( \frac{d}{dx_1} \). Iterating the process, we get as the result an expansion \( f = \sum_j x_i^j \cdot (x_2^j \cdot (\ldots \cdot a_j \ldots)) = \sum_j x_i^j a_j \). The elements \( a_j \) are constants with respect to all \( \frac{d}{dx_i} \).

Remark 3.2. If \( \Phi_i \) is given by \( \Phi_i(f) = f - \int D_i f dx_i \), then
\[
\Phi_i = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L^k_i \circ D^k_i.
\]

Let \( \Phi : f \mapsto a_{(0, \ldots, 0)} \) be the projector \( A \to A_0 \) obtained by the above Taylor expansion of several variables.

Proposition 3.3. For \( X = \{ x, y \} \), a vector space basis of \( A_0 = \text{im} \Phi = \text{im}(\Phi_2 \circ \Phi_1) \) is given by all \( \Phi(s), s \in \Gamma \), where
\[
\Gamma = \{ s_1 s_2 \in M(x, y) : \deg s_1 \geq 2, \deg s_2 \geq 1 \} \cup \{ y \cdot (x \cdot t) \in M(x, y) : \deg t \geq 0 \}.
\]

Proof. 1) First we extend the basis of \( \text{ker} \Phi_1 = \{ x \cdot h : h \in A \} \), see Proposition (2.3), to a basis of \( \text{ker} (\Phi_2 \circ \Phi_1) \). To obtain the desired basis elements, we can without loss of generality look at the space of all \( f \in A \) with \( f = \Phi_1(f) = y \cdot g \) for some \( g \in A \). Then necessarily \( \frac{d}{dx} (g) = 0 \). A basis for this space is the set \( \{ y \Phi_1(t) : x \neq t \in M - x M \} \).

2) To get a basis for \( \text{im} \Phi \), we can include all \( \Phi(s) \) with \( s = s_1 s_2 \in M(x, y) \) with \( \deg s_1 \geq 2 \). When we furthermore take elements \( \Phi_2(\Phi_1(s)) \) we have to exclude the trees \( s \) with \( \Phi_1(s) \preceq y \Phi_1(t) \preceq, \text{ by 1} \). For \( t \not\in x(M(x, y) \cup \{ 1 \}) \), \( y \Phi_1(t) \preceq = \Phi_1(yt) \preceq \), and the assertion follows.
Example 3.4. Let $X = \{x, y\}$. The monomials $xy$ and $y(x)$ are not in $\Gamma$.

The monomial $t = xy$ is an element of $\Gamma$. Its Taylor expansion $\sum_{j=0}^{\infty} x^j b_j$ with respect to $x$ is given by $x.b_1 + b_0$ with $b_1 = y$, $b_0 = xy - xy$. Applying Taylor expansion with respect to $y$ on $b_1, b_0$, we get the (total) Taylor expansion $t = x^{(1,1)}(1 + x^{(0,0)}, a_{(0,0)}(y))$ with $a_{(0,0)}(y) = xy - xy$.

The Taylor expansion of $y(xy) \in \Gamma$ with respect to $x$ is given by $b_1 = y^2, b_0 = y(xy) - xy^2$. Its total Taylor expansion is $t = x^{(2,1)}(1 + x^{(0,1)}, a_{(0,0)}(xy) + a_{(0,0)}(t))$ with $a_{(0,0)}(t) = 2y(xy) - xy^2 - y(y(x))$. We note that the coefficient of $y(xy)$ is not 1 in $a_{(0,0)}(y(xy))$, and that $y(xy)$ is not the leading monomial.

For $t = y(y(xy)) \in \Gamma$, we get $a_{(0,0)}(t) = y(y(xy)) - 3y(y(xy)) + 3y(xy) - x(yy^2)$.

Definition 3.5. For $X = \{x_1, x_2, \ldots\}$ a set of variables, let $x_j \in K\{X\} \otimes \cdots \otimes K\{X\}$ denote $1 \otimes \cdots \otimes 1 \otimes x_j \otimes 1 \otimes \cdots \otimes 1$.

We regard the tensors $x_j$ as variables (and note that some associativity and commutativity occurs among them).

Let $X = \{x\}$, let $n \geq 2$, and let $j \in \mathbb{N}^n$.

We will need the following variation of Proposition (3.1) in order to express the formula in Proposition (3.1). Here $x^j, f$ denotes the image of $f \in A^\otimes n$ under $L_1 \otimes \cdots \otimes L_n : A^\otimes n \to A^\otimes n$.

Proposition 3.6. (i) There is a unique Taylor expansion $f = \sum_j x^j a_j$ for every $f \in A^\otimes n$, with $a_j \in A_0^\otimes n$.

(ii) The operator $A^\otimes n \to A_0^\otimes n, f \mapsto a_0, \ldots, 0$, is a projector with $\ker \Phi = \bigcup \text{im} L_i$ and equal to $\Phi^\otimes n$, where $\Phi : K\{x\} \to K\{x\}_0, f \mapsto a_0(f)$.

Proof. The Taylor expansion with respect to variables $x^{(p)}, 1 \leq p \leq n$, is defined completely analogously to Proposition (3.1).

Having expanded with respect to $x^{(p)}, \text{the resulting expansion has coefficients in } A \otimes \cdots \otimes A_0 \otimes A \otimes \cdots \otimes A (A_0 \text{ in } p\text{-th position}).$

We can expand the coefficients again by the remaining $x^{(p)}$, and we finally get a Taylor expansion $f = \sum_j x^j a_j$. The difference to the situation in Proposition (3.1) is that the expansion is independent of the chosen order of steps. The coefficients $a_j$ are constants with respect to all $\frac{d}{dx^{(p)}}$, and the projector given by $f \mapsto a_0, \ldots, 0$ is equal to $\Phi^\otimes n$. □
4. Primitive elements in the case of one variable.

Let \( X = \{ x \} \) and \( A_0 \) be the algebra of constants.

We have constructed the generators of \( \bigoplus_{i=1}^{5} (A_{0_i}) \) in a way that they are all pseudo-linear. The elements \( \Phi(t), t \in \Omega_{\geq 6} \), are no longer pseudo-linear in general, though. We are going to show, that \( A_0 \) is a (cocommutative) Hopf sub-algebra of the co-addition Hopf algebra.

**Proposition 4.1.** The map \( \Delta \) restricts to an algebra homomorphism \( A_0 \to A_0 \otimes A_0 \).

Furthermore, for \( t \) a tree monomial,

\[
\Delta(a_0(t)) = a_{0,0}(\Delta(t)) = a_0(t) \otimes 1 + 1 \otimes a_0(t) + \sum_{\emptyset \neq I \subseteq I(t)} a_0(t|I) \otimes a_0(t|I^c).
\]

**Proof.**

1) By Lemma \( (1.9) \), (ii),(iii), we get that

\[
\Delta \circ D = (D \otimes \text{id}) \circ \Delta = (\text{id} \otimes D) \circ \Delta.
\]

Hence if \( Df = 0 \) then \( \Delta(f) \in (\ker D \otimes A) \cap (A \otimes \ker D) \).

2) For \( f = \sum x^k.a_k, \Delta(f) = \sum \sum (k) (x^i \otimes x^{k-i}).\Delta(a_k). \)

By Proposition \( (3.6) \), \( a_{0,0}(\Delta(f)) = \Phi^{\otimes 2}(\Delta(f)) = \Phi(\Delta(a_0)) \). Now \( \Phi(\Delta(a_0)) = \Delta(a_0) \), as \( \Delta(a_0) \in A_0 \otimes A_0 \).

\( \square \)

**Remark 4.2.** Lazard-Lie theory (see \[La\]), in the generalized setting presented in \[Ho\] (involving the tensor product of not necessarily commutative or associative algebras), can be used to show that \( A_0 \) is again the free non-associative Hopf algebra with co-addition (over an infinite set of generators).

We fix some notation first.

**Definition 4.3.** Let \( X = \{ x_1, x_2, ... \} \) be a set of variables, and let \( x^{(p)}_j \) be defined as in Definition \( (3.3) \).

Let \( |\otimes - \text{MagAlg}(X)|^n \) denote the subalgebra without unit generated by all variables \( x^{(p)}_j, 1 \leq p \leq n, \) in \( K\{X\}^\otimes_n \). For \( 1 \leq p \leq n \), let \( \partial_p \) be the \( K \)-linear map \( \underbrace{\text{id} \otimes ... \otimes \text{id} \otimes \partial_i \otimes \text{id} \otimes ... \otimes \text{id}}_{p-1} : |\otimes - \text{MagAlg}(X)|^n \to |\otimes - \text{MagAlg}(X)|^{n+1} \).

Generalizing the \( n = 1 \) case, given \( 1 \leq p \leq n, \) an element \( f \) of \( |\otimes - \text{MagAlg}(X)|^n \) is called pseudo-linear with respect to the \( p \)-th tensor argument, iff \( \partial_p(f) = 0 \).

**Theorem 4.4.** Let \( K \) be a field of characteristic 0. The algebra \( A_0 \) of constants is a free algebra generated by a set of variables \( Y = \{ y_1, y_2, ... \} \) with non-negative degrees \( d_i \) such that \( d_{i+1} \geq d_i, d_1 = 3 \). Let \( \Delta' : K\{Y\} \to K\{Y\} \otimes K\{Y\} \) be given by \( \Delta|A_0 \).

Then \( (K\{Y\}, \Delta') \) is isomorphic to the free non-associative co-addition Hopf algebra \( (K\{Y\}, \Delta) \). The isomorphism \( \varphi \) with \( \Delta = (\varphi \otimes \varphi) \circ \Delta' \circ \varphi^{-1} \) is strict in the sense that its linear part is the identity on the vector space generated by all \( y_i \).
Proposition 4.5. The algebra freely generated by the elements $z_{n,k} = a_0(x^k x^{n-k})$, $n \geq 3$, $2 \leq k \leq n - 1$, is a Hopf subalgebra of $A_0$. The image $\Delta(z_{n,k})$ is given by

$$z_{n,k} \otimes 1 + 1 \otimes z_{n,k} + \sum_{m=2}^{k-2} \sum_{m+1 \leq l \leq m+n-k-1} \binom{k}{m} \binom{n-k}{l-m} z_{l,m} \otimes z_{n-l,k-m}.$$ 

Especially, $z_{n,n-1} = a_0(x^{n-1} \cdot x)$, $z_{n,2} = a_0(x^2 \cdot x^{n-2})$, and $z_{n,3} = a_0(x^3 \cdot x^{n-3})$ are always pseudo-linear.
\textbf{Proof.} By Proposition (4.4), \( \Delta(z_{n,k}) = \Delta(a_0(x^k x^{n-k})) \) is given by \( \sum_I a_0(x^k x^{n-k}|I) \otimes a_0(x^k x^{n-k}|I') \), which is a sum \( \sum \mu_{x_1 x_2}(x^k x^{n-k}) z_{l_1 l_2 l_3} \otimes z_{n-(l_1+l_2), k-l_3} \) over all possible \( l_1 \leq k, l_2 \leq n-l_1 \).

By induction, one verifies that \( \mu x^n x^m = \binom{m}{n} x^{n-m} \) and that
\[
\mu x_1 x_2(x^k x^{l}) = \binom{k_1}{l_1} \binom{k_2}{l_2},
\]
if \( l_1, k_1 \geq 2 \), and \( l_2, k_2 \geq 1 \).

\( \square \)

\textbf{Example 4.6.} Let us illustrate the modification process of theorem (1.4) in the first nontrivial case, i.e. we look at the generators \( y_i \) with \( d_i = 6 \).

By Proposition (4.3), the elements \( z_{6,5}, z_{6,3} \) and \( z_{6,2} \) of \( (A_0)_6 \) are pseudo-linear. It is also easy to see that \( \Phi(x^3(x^2 x)) \) and \( \Phi((x^2 x)x^3) \) (given by the remaining trees in \( \Omega \) that are a product of two deg 3-monomials) are pseudo-linear.

For \( z_{6,4} = \Phi(x^4 x^2) \),
\[
\Delta(z_{6,4}) = z_{6,4} \otimes 1 + 1 \otimes z_{6,4} + \binom{4}{2} z_{6,2} \otimes z_{3,2} = z_{6,4} \otimes 1 + 1 \otimes z_{6,4} + 12a_0(x^2 x) \otimes a_0(x^2 x).
\]

Now \( z_{6,4} - 6a_0(x^2 x) \cdot a_0(x^2 x) \) is pseudo-linear, because \( \Delta((a_0(x^2 x))^2) = (\Delta(a_0))^2 = a_0 \otimes 1 + 1 \otimes a_0 + 2a_0 \otimes a_0 \).

For \( t = (((x^2 x)x)x)x \), the element \( \Phi(t) \) is not pseudo-linear, as \( \Delta(a_0(t)) = t \otimes 1 + 1 \otimes t + \binom{6}{3} a_0(x^2 x) \otimes a_0(x^2 x) = t \otimes 1 + 1 \otimes t + 20a_0(x^2 x) \otimes a_0(x^2 x) \). But \( \Phi(t) - 10(x^2 x - x^3)^2 \) is pseudo-linear.

Similarly we can handle the remaining four trees \( t_1 \cdot x^2, \text{deg } t_1 = 4 \), the twelve trees of the form \( t_1 \cdot x, t_1 \neq x^5 \), and the four trees \( x^2 \cdot t_2, t_2 \neq x^4 \).

5. Primitive elements for Several Variables.

Let the set \( X \) of variables have \( n \geq 1 \) or countably many elements.

\textbf{Lemma 5.1.} The following elements (and their \( K \)-linear combinations) are pseudo-linear (primitive for the co-addition):

(i) variables \( x_i \in X \)

(ii) the commutators \( [f_1, f_2] := f_1 f_2 - f_2 f_1 \) of pseudo-linear \( f_i \), \( i = 1, 2 \)

(iii) the associators \( (f_1, f_2, f_3) := (f_1 f_2) f_3 - f_1 (f_2 f_3) \) of pseudo-linear \( f_i \), \( i = 1, 2, 3 \).

\textbf{Proof.} Since the elements \( f_i \) are pseudo-linear, it is clear that computing the deviation to the co-additive part we can assume that for each \( i \), the leaves \( x \) of \( f_i \) either all have to be substituted by the corresponding \( x^{(2)} \) or all by \( x^{(1)} \). The associativity and commutativity relations for the variables \( x_j^{(1)}, x_j^{(2)} \) force the resulting expressions to be zero. \( \square \)

\textbf{Remark 5.2.} For \( X = \{x\} \), the space \( (A_0)_4 \) is 3-dimensional, and \( [x, (x, x, x)] = a_0(x^3 x) - a_0((x^2 x)x) \) is the only pseudo-linear element which can be generated by the process of (5.1).
Lemma 5.3. (i) For \( \#X \geq n \), let \( f \) be a pseudo-linear element in \( n \) variables. Then for all pseudo-linear \( g_1, \ldots, g_n \), the composition \( \eta_{g_1, \ldots, g_n}(f) = f(g_1, \ldots, g_n) \) is again pseudo-linear.

(ii) If \( \#X \geq 2 \), each Lie polynomial \( f = \sum x_{i_1} \cdot \ldots \cdot x_{i_r} \) over \( X \) in associative variables induces a pseudo-linear element \( f = \sum x_{i_1} (x_{i_2} \cdot (\ldots x_{i_r}) \ldots) \) with right normed brackets inserted.

Proof. The algebra homomorphism \( K\{X\} \rightarrow K\{X'\} \) given on the generators by \( x_i \mapsto g_i \) is a homomorphism for the co-addition, if the elements \( g_i \) are pseudo-linear. Thus (i) follows. Assertion (ii) is an easy observation. \( \square \)

Example 5.4. The Lie polynomial \([y, [y, x]] = yyx + xy^2 - 2yxy\) in associative variables \( x \) and \( y \) leads to the pseudo-linear element \( y(xy) + xy^2 - 2y(xy) \), which is equal to \(-a_{0,0}(y(xy))\) by Example 5.3. Similarly, \([x, [x, y]]\) leads to \(yx^2 + x(xy) - 2x(xy)\), which is equal to \(a_{0,0}(y^2x)\).

Using Proposition 5.3, for \( X = \{x, y\} \) a basis of the space of pseudo-linear elements in degree 3, can be given by

\[
\begin{align*}
a_{0,0}(xx^2) &= a_{0,0}(yy^2) = a_{0,0}(yx^2) = a_{0,0}(y(xy)), \\
a_{0,0}(xy) &= (xy)y - xy^2 = (x, y, y), \\
a_{0,0}((yx)y) &= (yx)y + y(xy) - y(xy) - xy^2 = (y, x, y) + a_{0,0}(y(xy)), \\
a_{0,0}(y^2x) &= y^2x - 2y(xy) + 2y(xy) - xy^2 = (y, y, x) + a_{0,0}(y(xy)), \\
a_{0,0}((xy)x) &= (xy)x - x(xy) = (y, x, x), \\
a_{0,0}(xy)x &= (xy)x - x(xy) = (x, y, x), \\
a_{0,0}(x^2y) &= x^2y - x(xy) = (x, x, y).
\end{align*}
\]

Example 5.5. The multi-linear element \( f(x_1, x_2, x_3, x_4) \) given by

\[
a_0((x_1x_2)(x_3x_4)) = (x_1x_2)(x_3x_4) - x_4 \cdot (x_1, x_2, x_3) - x_3 \cdot (x_1, x_2, x_4) - x_1 \cdot (x_2(x_3x_4))
\]

is pseudo-linear. Using Proposition 5.3(i), we see that \( f(x, x, x, x) \) is pseudo-linear, too. It is equal to \( a_0(x^2x^2) \), see (2.12), and is linearly independent of \([x, (x, x, x)]\).

Remark 5.6. The co-addition \( \Delta \) respects the multi-degree.

Let a homogeneous pseudo-linear element \( f \) be given which is not multi-linear. For pseudo-linear \( g_1, \ldots, g_n \), e.g. sums of variables, not only \( f(g_1, \ldots, g_n) \) but also all its multi-degree components are pseudo-linear. In this way we can obtain a pseudo-linear multi-linear element \( \tilde{f} \) that yields \( f \) (up to a constant factor) by evaluation on the original set of variables, cf. [La], §2. Hence all pseudo-linear elements can be obtained by evaluation of multi-linear ones.

Thus the construction of multi-linear primitive elements is crucial, and we will study this problem in a subsequent paper.
6. The Hausdorff Series.

Let \( e^x = \exp x = 1 + \sum_{t \in M(x)} a(t)t \in K\{x\} \) be the unique series with constant term 1 such that \( \exp(x)\exp(x) = \exp(2x) \), \( \exp'(x) = \exp(x) \), see [DG] .

The composition inverse of \( \exp x - 1 \) is the series \( \log(1 + x) = \sum_{t \in M(x)} b(t)t \).

We write \( \exp(x)\exp(y) = 1 + \sum_{s \in M(x,y)} d(s)s \). The Hausdorff series \( H(x,y) = \sum_{\tau \in M(x,y)} c(\tau)\tau \) is defined by \( H(x,y) = \log(\exp(x)\exp(y)) \).

Let \( H = \sum_{n=1}^{\infty} H_n \), \( H_n \) homogeneous of degree \( n \). Then \( H_1 = x + y \), \( H_2 = \frac{1}{2}(xy - yx) \).

Similar to the classical case of associative variables, the Hausdorff series has the following property.

**Theorem 6.1.** The components \( H_n \) of the non-associative Hausdorff series \( H(x,y) \) are primitive elements for the co-addition.

**Proof.**

1) We first show, that \( e^{x \otimes 1 + 1 \otimes x} = e^x \otimes e^x \).

Let \( f(x) = e^{x \otimes 1 + 1 \otimes x} \). Substitution of \( x \otimes 1 + 1 \otimes x \) for \( x \) in \( e^{2x} = e^x e^x \) yields \( f(2x) = e^{x \otimes 1 + 1 \otimes x} e^{x \otimes 1 + 1 \otimes x} = f(x)f(x) \). Let \( g(x) = e^x \otimes e^x \). Clearly \( g(x)g(x) = g(2x) \).

Writing \( f = \sum f_n \) and \( g = \sum g_n \) as sums over homogeneous elements, one has \( f_0 = g_0 = 1 \otimes 1 \), \( f_1 = g_1 = x \otimes 1 + 1 \otimes x \). Then \( f_n \), and similarly \( g_n \), are uniquely determined by the equation \( 2^n - 2 \) \( f_n(x) = \sum_{a i is \leq n} f_i(x) \cdot f_{n-i}(x) \), compare also [DG]. Thus \( f = g \).

It follows that \( \Delta(e^x e^y) = \Delta(e^x)\Delta(e^y) = e^{\Delta(x)}e^{\Delta(y)} = e^x e^y \otimes e^x e^y \).

2) The rest of the proof goes along the well-known line, cf. [R] §3.

To \( \Delta(H(x,y)) = H(x,y) \otimes 1 + 1 \otimes H(x,y) \) we can apply \( \exp(x) - 1 \), to get the equivalent equation

\[
\exp(\Delta(\log(\exp(x)\exp(y)))) - 1 = \exp(H \otimes 1 + 1 \otimes H) - 1.
\]

Writing \( Z(x,y) \) for the series \( \exp(x)\exp(y) - 1 \) without constant term, the left-hand-side is given by \( e^{\Delta(\log(1 + Z(x,y)))} - 1 = \Delta(e^{\log(1 + Z(x,y))} - 1) = \Delta(Z(x,y)) = \Delta(e^x e^y) - 1 \). Since \( e^{H \otimes 1 + 1 \otimes H} - 1 = e^x e^y \otimes e^x e^y - 1 = \Delta(e^x e^y) - 1 \) by 1), the equation is true.

\[\square\]

**Definition 6.2.** Let \( | \cdot | : M(x,y) \to M(z) \) be the homomorphism given by \( x \mapsto z, y \mapsto z \). We call \( |\tau| \) the underlying (unlabeled) tree of \( \tau \in M(x,y) \). Let \( t \in M(z) \) be of degree \( n \), and let \( s_1, \ldots, s_n \in M(x,y) \). Then \( t(s_1, \ldots, s_n) \) denotes the result of grafting each \( s_i \) to the \( i \)-th leaf of \( t \).

**Proposition 6.3.** The coefficients \( c(\tau), \tau \in M(x,y) \), of the Hausdorff series are given by

\[
c(\tau) = d(\tau) - \sum_{k=2}^{\deg \tau} c_k(\tau)
\]
Example 6.5. For a

\[ \text{Lemma 6.6.} \]

Proof. Insertion of \( H(x, y) \) into \( \sum_{t \in M(x)} a(t)t \) yields

\[ e^{H(x,y)} = 1 + \sum_{k=1}^{\infty} \sum_{t, s_1, \ldots, s_k} a(t)c(s_1) \cdots c(s_n)t(s_1, \ldots, s_k), \]

where the second sum is over all \( t \in M(z) \) with \( \deg t = k \) and \( s_1, \ldots, s_k \in M(x, y) \). Now we can compare the coefficients of

\[ e^{H(x,y)} = 1 + \sum_{k=1}^{\infty} \sum_{\tau(t_1, \ldots, s_n) = \tau} a(t)c(s_1) \cdots c(s_k), \]

with the coefficients of \( \exp(x)\exp(y) \), which are easily determined to be the values given above. Using \( c(x) = a(x) = 1 \), we get that

\[ c(\tau) = d(\tau) - a(|\tau|) - \sum_{k=2}^{\deg \tau - 1} \sum_{\tau = t(s_1, \ldots, s_k)} a(t)c(s_1) \cdots c(s_k) = d(\tau) - \sum_{k=2}^{\deg \tau} c_k(\tau). \]

Remark 6.4. The \( c(\tau) \) can be recursively computed as stated above, because the coefficients \( a(t) \) of \( \exp(x) \) satisfy the formula

\[ a(t) = \frac{a(t_1)a(t_2)}{2^n - 2}, \ \text{if} \ t = t_1 \cdot t_2, \ n := \deg(t). \]

Example 6.5. For \( \tau \in M(x) \), \( \deg \tau > 1 \), \( c(\tau) = 0 \), as \( d(\tau) = a(|\tau|) \), and because \( c(s_i) = 0 \) for at least one \( i \) in the formula above.

The coefficients \( c(x \cdot t_2) \), for \( t_2 \in M(y) \) of degree \( n - 1 \), are given by \( a(x)a(t_2) - a(|xt_2|) = a(t_2)(1 - \frac{1}{2^n - 2}) = \frac{2^n - 3}{2^n - 2}a(t_2). \)

Lemma 6.6.

(i) There is a unique continuous involution \( * \) given on \( K\{x,y\} \) such that

\[ x^* = y, \ y^* = x. \]

(ii) It holds that \( (\exp(x))^* = \exp(y) \).

(iii) \( H(x, y)^* = H(x, y) \). Furthermore \( H(x, y)^*_n = H(x, y)_n \) for all \( n \).
Proof. For the involution $\ast$, we want to show that $(\exp(x))^{\ast} = \exp(y)$. Then also

$$H(x, y)^{\ast} = \log(\exp(x) \exp(y))^{\ast} = \log(\exp(x) \exp(y)) = H(x, y).$$

Since $(\exp(x))^{\ast} = g(y)$ is a series in $y$ (with constant term 1) satisfying $g(y)g(y) = g(2y)$, we conclude that $g(y) = \exp(y)$.

□

Example 6.7. For $X = \{x, y\}$, the vector space of elements of multi-degree $(2, 1)$ in $A_0$ has $(x, x, x), (y, x, x)$, together with $[x, [x, y]]$ as a basis, see Example (5.4). The elements $(x, x, y)^{\ast} = -(x, y, y), (x, y, x)^{\ast} = -(y, x, y), (y, x, x)^{\ast} = -(y, y, x)$, together with $[x, [x, y]]^{\ast} = -[y, [x, y]]$ form a basis of the multi-degree $(1, 2)$-part.

Further computation shows:

**Proposition 6.8.** The homogeneous part $H_3$ of degree 3 is given by

$$H_3 = \frac{1}{12}([x, [x, y]] + [x, [x, y]])^{\ast} + \frac{1}{12}((y, x, x) + (y, x, x)^{\ast})$$

$$+ \frac{5}{12}((x, x, y) + (x, x, y)^{\ast}) + \frac{1}{4}((x, y, x) + (x, y, x)^{\ast})$$

$$= \frac{1}{3}((x, x, y) + (x, y, x) + \frac{1}{4}([x, [x, y]] + [x^2, y] - x[x, y] - [x, y]x)$$

$$+ \frac{1}{3}((x, x, y) + (x, y, x) + \frac{1}{4}([x, [x, y]] + [x^2, y] - x[x, y] - [x, y]x))^{\ast}. $$

**Remark 6.9.** In this article we have considered the free magma algebra; we dealt with one operation without relations. The free commutative magma algebra is also very interesting. For example, the similarly defined Hausdorff series is given by

$$H^a(x, y) = x + y + \frac{1}{3}((x, x, y) + (x, x, y)^{\ast}) + \text{terms of order } \geq 4.$$ 

It seems also important to study non-associative Hausdorff series in $m > 2$ non-associative variables $x_1, \ldots, x_m$ (cf. [Lo] for the associative case).

**References**

[ACM] J. Anquela, T. Cortés, and F. Montaner, Nonassociative Coalgebras, *Comm. Algebra* 22(1994), 4693–4716.

[BF] C. Brouder and A. Frabetti, QED Hopf algebras on planar binary trees, math.QA/0112043.

[BH] G.M. Bergman and A.O. Hausknecht, ”Cogroups and Co–rings in Categories of Associative Rings,” AMS Math. Surveys and Monographs 45, 1996.

[DG] V. Drensky and L. Gerritzen, Nonassociative Exponential and Logarithm, Preprint 2002.

[Fr] B. Fresse, Algèbre des Descentes et Cogroupes dans les Algèbres sur une Opérade, *Bull. Soc. math. France* 126 (1998), 407–433.

[Ge1] L. Gerritzen, Taylor expansion of noncommutative Polynomials, *Arch. Math.* 71(1998), 279–290.

[Ge2] L. Gerritzen, Taylor expansion of noncommutative Power Series with an Application to the Hausdorff Series, *J. Reine Angew. Math.*, 2002.

[Gr] G. Griffing, The Cofree Nonassociative Coalgebra, *Comm. Algebra* 16(1988), 2387–2414.
[Ho1] R. Holtkamp, A pseudo-analyzer approach to formal group laws not of operad type, *J. Algebra* 237 (2001), 382–405.

[Ho2] R. Holtkamp, Comparison of Hopf algebras on Trees, Preprint 2001, to appear in *Arch. Math.*

[Ku] A. Kurosh, Non-associative free algebras and free products of algebras, *Rec. Math.[Mat. Sbornik] N.S.* 20(62), (1947), 239–262.

[La] M. Lazard, Lois de groupes et analyseurs, *Ann. École Norm.Sup.* 72(1955), 299–400.

[Lo] J.-L. Loday, Série de Hausdorff, idempotents Eulériens et algèbres de Hopf, *Expo. Math* 12(1994), 165–178.

[LR] J.-L. Loday and M. Ronco, Hopf Algebra of the Planar Binary Trees, *Advances in Math.* 139(1998), 299-309.

[R] C. Reutenauer, ”Free Lie Algebras”, London Math. Soc. Monographs, Oxford University Press, New York, 1993.

[Zh] V. Zhelyabin, The Kantor-Koecher-Tits construction for Jordan coalgebras, *Algebra and Logic* 35(1996), no.2, 173-189.

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