The universal Hopf algebra associated with a Hopf-Lie-Rinehart algebra

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Abstract

We introduce a notion of Hopf-Lie-Rinehart algebra and show that the universal algebra of a Hopf-Lie-Rinehart algebra acquires an ordinary Hopf algebra structure.

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1 Introduction

For a Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U\mathfrak{g}$ is well known to acquire a cocommutative Hopf algebra structure having $\mathfrak{g}$ as its module of primitive generators. In this paper we will explore diagonal maps for the universal algebra of a general Lie-Rinehart algebra.

Within the framework of standard homological algebra, the diagonal map on the universal enveloping algebra of an ordinary Lie algebra induces the ring structure in Lie algebra cohomology. The ring structure can be described entirely in terms of the Maurer-Cartan algebra of the Lie algebra (the differential graded algebra of alternating forms on the Lie algebra). The terminology Maurer-Cartan algebra goes back at least to [14] and was prompted by the development of the subject in the 1930’s. The terminology Chevalley-Eilenberg algebra is nowadays common as well. As a side remark we note that the familiar Maurer-Cartan equation lives in a variant of the Maurer-Cartan algebra; this equation, in turn, has recently resurged as an interesting topic in its own, in particular in the theory of deformations and as the master equation in physics. It is also worthwhile
recalling that in view of a classical result, in characteristic zero, any cocommutative Hopf algebra is the enveloping algebra of its Lie algebra of primitive elements, and this fact generalizes even to the differential graded setting.

The question under what circumstances the universal algebra $U(A, L)$ of a general Lie-Rinehart algebra $(A, L)$ has a diagonal map turning this universal algebra into a Hopf algebra has hardly been explored in the literature, though. When $(A, L)$ is the Lie-Rinehart algebra of smooth functions $A = C^\infty(N)$ and smooth vector fields $L = \text{Vect}(N)$ on a smooth manifold $N$, the algebra $U(A, L)$ is the algebra of (globally defined) differential operators on $N$. The de Rham cohomology of $N$ then amounts to the appropriate Ext-functors over $U(A, L)$ [13]. For a general smooth manifold $N$, there is no obvious way to put a diagonal map on $U(A, L)$, and the ring structure in cohomology is defined directly in terms of the corresponding Maurer-Cartan algebra (the differential graded algebra of de Rham forms on the manifold). As a side remark we note that this Maurer-Cartan algebra is the starting point for higher homotopies generalizations of the structure, worked out in [8]. This kind of generalization occurs in nature, e. g. in the theory of foliations.

When $A$ is an ordinary Hopf algebra and when $\mathfrak{g}$ is an ordinary Lie algebra acting on $A$ by derivations compatibly with the diagonal, the crossed product algebra $A \odot U\mathfrak{g}$ inherits a Hopf algebra structure in an obvious manner. This situation arises e. g. when $A$ is the Hopf algebra of algebraic functions on an algebraic group $H$ (e.g. a compact Lie group) and when $\mathfrak{g}$ is the Lie algebra of an algebraic group $G$ acting on $H$ by group automorphisms. The resulting Hopf algebra is no longer cocommutative, though, unless $H$ is abelian.

Prompted by a recent posting [12], we decided to communicate some structural insight for the case of a general Lie-Rinehart algebra which we have been familiar with for many years. We have already noted that the universal enveloping algebra of an ordinary Lie algebra acquires a Hopf algebra structure in an obvious way. The point we wish to emphasize here is that in order for the naive extension of the fact just quoted to be valid for a general Lie-Rinehart algebra $(A, L)$, a Hopf-algebra structure on $A$ is needed. This is certainly consistent with the case of an ordinary Lie algebra, the ground ring or ground field being endowed with its obvious Hopf algebra structure (where all structure maps come down to the identity).

We will introduce what we will refer to as a bi-Lie-Rinehart algebra and we will, furthermore, refine this notion to that of a Hopf-Lie-Rinehart algebra, the crossed product algebra relative to a Lie group being a special case. We shall show that the universal algebra of a bi- and, likewise, that of a Hopf-Lie-Rinehart algebra, acquires a comultiplication and a counit which turn the universal algebra into a bialgebra or Hopf algebra as appropriate.

I am indebted to Jim Stasheff for a number of comments which helped improve the exposition.

2 Lie-Rinehart algebras

Let $R$ be a commutative ring, fixed throughout; the unadorned tensor product symbol $\otimes$ will always refer to the tensor product over $R$. Further, let $A$ be a commutative $R$-
algebra, let \( L \) be an \( A \)-module (the action being written as \((a \otimes \alpha) \mapsto a\alpha\)) which is also an \( R \)-Lie algebra, and suppose that \( L \) acts on the left of \( A \) by derivations (the action being written as \((\alpha \otimes a) \mapsto a\alpha\)). Following Rinehart [13], we will refer to \( L \) as an \((R, A)\)-Lie algebra provided suitable compatibility conditions are satisfied which generalize standard properties of the Lie algebra of vector fields on a smooth manifold viewed as a module over its ring of functions; these conditions read

\[
(a\alpha)(b) = a(\alpha(b)), \quad \alpha \in L, \ a, b \in A, \tag{2.1}
\]

\[
[\alpha, a\beta] = a[\alpha, \beta] + \alpha(a)\beta, \quad \alpha, \beta \in L, \ a \in A. \tag{2.2}
\]

Occasionally we will spell out the \( L \)-action on \( A \) explicitly in the form \( \omega: L \to \text{Der}(A) \), so that \((\omega(\alpha))(a) = \alpha(a)\). When the emphasis is on the pair \((A, L)\), with the mutual structure of interaction, we refer to \((A, L)\) as a Lie-Rinehart algebra. Given two Lie-Rinehart algebras \((A, L)\) and \((A', L')\), a morphism \((\varphi, \psi): (A, L) \to (A', L')\) of Lie-Rinehart algebras is the obvious thing, that is \( \varphi: A \to A' \) is a morphism of \( R \)-algebras, \( \psi: L \to L' \) is a morphism of \( R \)-Lie algebras, and these morphisms are compatible with the additional structure. More precisely, the obvious diagrams

\[
\begin{array}{ccc}
A \otimes L & \longrightarrow & A' \otimes L' \\
\downarrow & & \downarrow \\
L & \longrightarrow & L'
\end{array}
\quad
\begin{array}{ccc}
L \otimes A & \longrightarrow & L' \otimes A' \\
\downarrow & & \downarrow \\
A & \longrightarrow & A'
\end{array}
\]

are commutative, the unlabelled vertical arrows being the corresponding structure maps.

With this notion of morphism, Lie-Rinehart algebras constitute a category. Apart from the example of smooth functions and smooth vector fields on a smooth manifold, a related (but more general) example is the pair consisting of a commutative algebra \( A \), the example of smooth functions and smooth vector fields on a smooth manifold, a related \( R \)-module \( \text{Der}(A) \), and the \( R \)-module \( \text{Der}(A) \) of derivations of \( A \) with the obvious \( A \)-module structure; here the commutativity of \( A \) is crucial.

Given an \((R, A)\)-Lie algebra \( L \), its universal algebra \((U(A, L), \iota_L, \iota_A)\) is an \( R \)-algebra \( U(A, L) \) together with a morphism \( \iota_A: A \to U(A, L) \) of \( R \)-algebras and a morphism \( \iota_L: L \to U(A, L) \) of Lie algebras over \( R \) having the properties

\[
\iota_A(a)\iota_L(\alpha) = \iota_L(a\alpha), \quad \iota_L(\alpha)a - \iota_A(a)\iota_L(\alpha) = \iota_A(\alpha(a)),
\]

and \((U(A, L), \iota_L, \iota_A)\) is universal among triples \((B, \varphi_L, \varphi_A)\) having these properties. For example, when \( A \) is the algebra of smooth functions on a smooth manifold \( N \) and \( L \) the Lie algebra of smooth vector fields on \( N \), then \( U(A, L) \) is the algebra of (globally defined) differential operators on \( N \). An explicit construction for the \( R \)-algebra \( U(A, L) \) is given in [13]. See our paper [2] for an alternate construction which employs the Massey-Peterson [11] algebra.

The universal algebra \( U(A, L) \) admits an obvious filtered algebra structure

\[
U_{-1} \subseteq U_0 \subseteq U_1 \subseteq \ldots , \tag{2.3}
\]

cf. [13], where \( U_{-1}(A, L) = 0 \) and where, for \( p \geq 0 \), \( U_p(A, L) \) is the left \( A \)-submodule of \( U(A, L) \) generated by products of at most \( p \) elements of the image \( T \) of \( L \) in \( U(A, L) \),
and the associated graded object \( E^0(U(A, L)) \) inherits a commutative graded \( A \)-algebra structure. We will refer to the filtration \([2, 3]\) as the Poincaré-Birkhoff-Witt filtration. The Poincaré-Birkhoff-Witt Theorem for \( U(A, L) \) then takes the following form where \( S_A[L] \) denotes the symmetric \( A \)-algebra on \( L \), cf. (3.1) of \([13]\).

**Theorem 2.1** (Rinehart). For an \((R, A)\)-Lie algebra \( L \) which is projective as an \( A \)-module, the canonical \( A \)-epimorphism \( S_A[L] \to E^0(U(A, L)) \) onto the associated graded \( A \)-algebra \( E^0(U(A, L)) \) is an isomorphism of \( A \)-algebras.

Consequently, for an \((R, A)\)-Lie algebra \( L \) which is projective as an \( A \)-module, the morphism \( \iota_L : L \to U(A, L) \) is injective, and we can refer to \( U(A, L) \) as an *enveloping algebra*.

It is worthwhile noting that, for an ordinary Lie algebra \( g \) over a field of characteristic zero, one way to prove the Poincaré-Birkhoff-Witt Theorem consists in noting that the coalgebra structure which underlies the obvious Hopf algebra structure of \( Ug \) is that of the symmetric coalgebra cogenerated by \( g \). This kind of reasoning breaks down for a general Lie-Rinehart algebra \((A, L)\) since there is in general no obvious way to endow the universal algebra \( U(A, L) \) with a coalgebra structure having the correct features. Indeed, in this paper, we will spell out the requisite additional Lie-Rinehart structure to arrive at the desired coalgebra structure on \( U(A, L) \).

### 3 Induced Lie-Rinehart structures

The notion of induced Lie-Rinehart structure been introduced in \([2]\). A geometric analogue thereof can be found in \([1]\). For intelligibility, we will now recall these induced structures:

Let \( A \) and \( A' \) be commutative \( R \)-algebras, let \( L \) be an \((R, A)\)-Lie algebra, with structure maps \( \omega : L \to \text{Der}(A) \) and \([ \cdot, \cdot ] : L \otimes_R L \to L \), let \( \tilde{\omega} : L \to \text{Der}(A') \) be an action of \( L \), viewed as an \( R \)-Lie algebra, on \( A' \) by derivations (but \( L \) is not assumed to admit an \( A' \)-module structure), and let \( \varphi : A \to A' \) be a morphism of algebras which is also a morphism of \( L \)-modules. Under these circumstances, let \( L' = A' \otimes_A L \), the induced \( A' \)-module. Consider the obvious pairings

\[
A' \otimes_R L \otimes_R A' \otimes_R L \to A' \otimes_A L, \tag{3.1}
\]

given by \( u \otimes \alpha \otimes v \otimes \beta \mapsto uv \otimes [\alpha, \beta] - (v\beta(u)) \otimes \alpha + (u\alpha(v)) \otimes \beta \), where \( u, v \in A' \), \( \alpha, \beta \in L \), and

\[A' \otimes_R L \otimes_R A' \to A', \quad u \otimes \alpha \otimes v \mapsto u \cdot \alpha(v), \quad u, v \in A', \alpha \in L. \tag{3.2}\]

A straightforward verification establishes the following, cf. Proposition 1.16 in \([2]\).

**Proposition 3.1.** Suppose that the action \( \tilde{\omega} \) of \( L \) on \( A' \) is a morphism of \( A \)-modules where \( \text{Der}(A') \) is turned into an \( A \)-module via \( \varphi \). Then \((3.1)\) induces an \( R \)-Lie algebra structure \([ \cdot, \cdot ] : L' \otimes_R L' \to L' \) on \( L' \), \((3.2)\) induces an action \( \omega' : L' \to \text{Der}(A') \) of \( L' \) on \( A' \) by derivations, and \([ \cdot, \cdot ]' \) and \( \omega' \) endow \( L' \) with an \((R, A')\)-Lie algebra structure in such a way that

\[(\varphi, \varphi \otimes \text{Id}) : (A, L) \to (A', L') \]

is a morphism of Lie-Rinehart algebras.
In the situation of Proposition 3.1, we refer to \((L', [\cdot, \cdot]'', \omega')\) as the \((R, A')\)-Lie algebra induced from \(\varphi\); often we will then write \(L'\) rather than \((L', [\cdot, \cdot]'', \omega')\).

Example 3.2. Let \(A\) be a commutative \(R\)-algebra, let \(g\) be an \(R\)-Lie algebra, and let \(\omega: g \rightarrow \text{Der}(A)\) be an action of \(g\) on \(A\) by derivations. Then with the obvious change in notation, (3.1) and (3.2) endow \(A \otimes g\) with an \((R, A)\)-Lie algebra structure, referred to as the crossed product of \(A\) and \(g\), cf. [10], and we will use the notation \(A \otimes g\) for the crossed product \((R, A)\)-Lie algebra. In particular, over the reals \(\mathbb{R}\) as ground ring, given a real algebraic Lie group \(G\), with Lie algebra \(g\), let \(A\) be the algebra of algebraic functions on \(G\), endowed with the obvious \(g\)-action. Then the crossed product \((\mathbb{R}, A)\)-Lie algebra \(A \otimes g\) amounts to the Lie algebra of algebraic vector fields on \(G\). The smooth analogue of this construction yields the ordinary Lie algebra of smooth vector fields on \(G\).

4 Bi-Lie-Rinehart algebras

Let \(A\) be a commutative \(R\)-bialgebra, not necessarily cocommutative, with comultiplication \(\Delta: A \rightarrow A \otimes A\) and counit \(\varepsilon: A \rightarrow R\), and let \(L\) be an \((R, A)\)-Lie algebra, with structure maps \(\omega: L \rightarrow \text{Der}(A)\) and \([\cdot, \cdot]: L \otimes_R L \rightarrow L\). Let \(\omega^\circ: L \rightarrow \text{Der}(A \otimes A)\) be an action of \(L\), viewed as an \(R\)-Lie algebra, on the tensor product algebra \(A \otimes A\) by derivations. To simplify the notation, we will then write

\[
\alpha(a \otimes b) = (\omega^\circ(\alpha))(a \otimes b) .
\]

Thus, letting \(A' = A \otimes A\), \(\varphi = \Delta\), and \(\tilde{\omega} = \omega^\circ\), we have the data needed to endow the induced \((A \otimes A)\)-module \(L^\circ = (A \otimes A) \otimes_A L\) with an \((R, A \otimes A)\)-Lie algebra structure. In view of Proposition 3.1 the following is immediate.

Proposition 4.1. Suppose that, for every \(a, b, c \in A\) and for every \(\alpha \in L\),

\[
(aa)(b \otimes c) = (\Delta(\alpha))(a \otimes b) .
\]

Then (3.1) induces an \(R\)-Lie algebra structure \([\cdot, \cdot]^\circ: L^\circ \otimes_R L^\circ \rightarrow L^\circ\) on \(L^\circ\), (3.2) induces an action \(\omega^\circ: L^\circ \rightarrow \text{Der}(A \otimes A)\) of \(L^\circ\) on the tensor product algebra \(A \otimes A\) by derivations, and \([\cdot, \cdot]^\circ\) and \(\omega^\circ\) endow \(L^\circ\) with an \((R, A \otimes A)\)-Lie algebra structure in such a way that

\[
(\Delta, \Delta \otimes \text{Id}): (A, L) \rightarrow (A \otimes A, L^\circ)
\]

is a morphism of Lie-Rinehart algebras.

Under the circumstances spelled out just before the previous proposition, we will say that \((A, \Delta, L, \omega^\circ)\) is a bi-Lie-Rinehart algebra provided \((A, \Delta, L, \omega^\circ)\) satisfies the requirement (4.2) and, furthermore,

\[
(\varepsilon, 0): (A, L) \rightarrow (R, 0)
\]

is a morphism of Lie-Rinehart algebras. We use the terminology bi-Lie-Rinehart algebra to avoid conflict with the notion of Lie-Rinehart bialgebra which, in turn, has been introduced
in [7] as an abstraction from the notion of Lie bialgebroid. We shall come back to Lie-Rinehart bialgebras in the last section.

Given a bialgebra \( A \) and an \( A \)-module \( L \), the symmetric \( A \)-algebra \( S_A[L] \) acquires an \( R \)-bialgebra structure in an obvious way, the submodule \( L \) being primitive.

**Theorem 4.2.** The universal algebra \( U(A, L) \) of a bi-Lie-Rinehart algebra acquires a comultiplication

\[
\Delta : U(A, L) \longrightarrow U(A, L) \otimes U(A, L) \tag{4.4}
\]

and counit

\[
\varepsilon : U(A, L) \longrightarrow R \tag{4.5}
\]

turning \( U(A, L) \) into an \( R \)-bialgebra. Furthermore, the graded algebra \( E^0(U(A, L)) \) associated with the Poincaré-Birkhoff-Witt filtration acquires an obvious \( R \)-bialgebra structure and, when \( L \) is projective as an \( A \)-module, the canonical \( A \)-epimorphism

\[
S_A[L] \longrightarrow E^0(U(A, L))
\]

onto \( E^0(U(A, L)) \) is an isomorphism of \( R \)-bialgebras.

**Proof.** The direct sum \( L \otimes A \oplus A \otimes L \) acquires an obvious \((R, A \otimes A)\)-Lie algebra structure and the standard \( R \)-Lie algebra diagonal map \( L \to L \oplus L \) which assigns the pair \((x, x)\) to \( x \in L \) induces a morphism

\[
L^\circ \longrightarrow L \otimes A \oplus A \otimes L \tag{4.6}
\]

of \((R, A \otimes A)\)-Lie algebras. Thus the composite of (4.3) and (4.6) yields a morphism

\[
(\Delta, \Delta) : (A, L) \longrightarrow (A \otimes A, L \otimes A \oplus A \otimes L) \tag{4.7}
\]

of Lie-Rinehart algebras. This morphism, in turn, induces the morphism

\[
U(\Delta, \Delta) : U(A, L) \longrightarrow U(A \otimes A, L \otimes A \oplus A \otimes L) \tag{4.8}
\]

between the universal algebras. It remains to show that the algebra \( U(A, L) \otimes U(A, L) \) is canonically isomorphic to the universal algebra \( U(A \otimes A, L \otimes A \oplus A \otimes L) \). In order to justify this claim, it suffices to note that \( U(A, L) \otimes U(A, L) \) satisfies the corresponding universal property. Indeed, relative to the obvious morphism

\[
\iota_{A \otimes A} = \iota_A \otimes \iota_A : A \otimes A \longrightarrow U(A, L) \otimes U(A, L)
\]

of \( R \)-algebras and relative to the obvious morphism

\[
\iota_{L \otimes A \oplus A \otimes L} = \iota_L \otimes \iota_A \oplus \iota_A \otimes \iota_L : L \otimes A \oplus A \otimes L \longrightarrow U(A, L) \otimes U(A, L)
\]

of Lie algebras over \( R \), the algebra \( U(A, L) \otimes U(A, L) \) plainly satisfies the universal property which characterizes the universal algebra \( U(A \otimes A, L \otimes A \oplus A \otimes L) \).

By assumption, \((\varepsilon, 0) : (A, L) \longrightarrow (R, 0)\) is a morphism of Lie-Rinehart algebras. The induced morphism

\[
\varepsilon : U(A, L) \longrightarrow U(R, 0) = R
\]

of algebras yields the requisite counit where the notation \( \varepsilon \) is abused somewhat. This establishes the theorem.
5 Hopf-Lie-Rinehart algebras

Let \((A, L)\) be a Lie-Rinehart algebra. Let \(L^-\) be the \(R\)-Lie algebra opposite to \(L\), that is, as an \(R\)-module, \(L^-\) coincides with \(L\) whereas the bracket on \(L^-\) is the negative of the bracket on \(L\).

The \(L\)-action on \(A\) by derivations being written as \(\omega : L \to \operatorname{Der}(A)\), let
\[
\omega^- = -\omega : L^- \to \operatorname{Der}(A).
\]

This is plainly an \(L^-\)-action on \(A\) by derivations and, together with the \(A\)-module structure on \(L^-\) (which, as an \(R\)-module, coincides with \(L\)), the \(A\)-module \(L^-\) thus acquires an \((R, A)\)-Lie algebra structure. In other words, \((A, L^-)\) is a Lie-Rinehart algebra, which we will refer to as the Lie-Rinehart algebra opposite to \((A, L)\). We remind the reader that, given an \(R\)-algebra \(U\), the opposite algebra \(U^{\text{opp}}\) has the same underlying \(R\)-module as \(U\), with multiplication given by \(x^{\text{opp}}y^{\text{opp}} = (yx)^{\text{opp}} \ (x, y \in U)\). The canonical \(R\)-Lie algebra morphism from \(L\) to \(U(A, L)\) is as well an \(R\)-Lie algebra morphism from \(L^-\) to \(U(A, L)^{\text{opp}}\) and, indeed, \(U(A, L)^{\text{opp}}\) satisfies the corresponding universal property so that \(U(A, L)^{\text{opp}},\) together with the obvious morphisms \(L^- \to U(A, L)^{\text{opp}}\) (which, as a morphism of \(A\)-modules, is just the morphism \(L \to U(A, L)\) associated with the universal algebra for \((A, L))\) and \(A \to U(A, L)^{\text{opp}},\) yields the universal algebra \(U(A, L^-)\).

Let \((\Delta, \varepsilon)\) be a bialgebra structure on \(A\), and suppose that \((A, \Delta, L, \omega^{\otimes})\) is a bi-Lie-Rinehart algebra. Furthermore, let \(S : A \to S\) be an antipode turning \((A, \Delta, \varepsilon)\) into a Hopf-algebra. We will refer to \((A, \Delta, S, L, \omega^{\otimes})\) as a Hopf-Lie-Rinehart algebra provided
\[
(S, -) : (A, L) \longrightarrow (A, L^-)
\]
is a morphism of Lie-Rinehart algebras; here \(- : L \to L^-\) refers to the \(A\)-linear map which sends \(x \in L\) to \(-x\), the \(L^-\) underlying \(A\)-module being the same as that underlying \(L\).

**Theorem 5.1.** The universal algebra \(U(A, L)\) of a Hopf-Lie-Rinehart algebra acquires a comultiplication
\[
\Delta : U(A, L) \longrightarrow U(A, L) \otimes U(A, L),
\]
counit
\[
\varepsilon : U(A, L) \longrightarrow R,
\]
and antipode
\[
S : U(A, L) \longrightarrow U(A, L)
\]
turning \(U(A, L)\) into an \(R\)-Hopf algebra. Furthermore, the associated graded algebra \(E^0(U(A, L))\) acquires an obvious \(R\)-Hopf algebra structure and, when \(L\) is projective as an \(A\)-module, the canonical \(A\)-epimorphism \(S_A[L] \longrightarrow E^0(U(A, L))\) onto \(E^0(U(A, L))\) is an isomorphism of \(R\)-Hopf algebras.

**Proof.** In view of Theorem 4.2 it remains to establish the existence of the antipode. However, this is straightforward: By definition, the antipode \(S\) fits into a morphism of Lie-Rinehart algebras of the kind \((5.1)\) and this morphism in turn, induces the morphism
\[
U(S, -) : U(A, L) \longrightarrow U(A, L^-)
\]
of \(R\)-algebras. However, as \(R\)-modules, \(U(A, L^-)\) and \(U(A, L)\) coincide whence \(U(S, -)\) yields the requisite antipode. We leave the details to the reader. \(\square\)
Example 5.2. Let \((A, L)\) be a Lie-Rinehart algebra and suppose that, for some \(R\)-Lie algebra \(g\) acting on \(A\) by derivations, the \((R, A)\)-Lie algebra \(L\) can be written as the crossed product \((R, A)\)-Lie algebra \(A \odot g\). Furthermore, suppose that \(\Delta : A \to A \otimes A\) and \(\varepsilon : A \to R\) constitute a coalgebra structure turning \(A\) into a Hopf algebra such that, relative to the obvious \(g\)-action on \(A \otimes A\), \(\Delta\) and \(\varepsilon\) are compatible with the \(g\)-actions. Then the crossed product \((A \otimes A) \odot g\) is an \((R, A \otimes A)\)-Lie algebra, and the comultiplication \(\Delta\) and counit \(\varepsilon\) induce morphisms
\[
(A, A \odot g) \longrightarrow (A \otimes A, (A \otimes A) \odot g)
\]
and
\[
(\varepsilon, 0) : (A, L) \longrightarrow (R, 0)
\]
of Lie-Rinehart algebras. This yields the requisite data to apply Theorem 5.1, and the universal algebra \(U(A, L)\) thus acquires a Hopf algebra structure.

The case mentioned earlier where \(A\) is the coordinate ring of an algebraic group \(H\) and where \(g\) is the Lie algebra of an algebraic group \(G\) acting on \(H\) by group automorphisms is an example for this situation. More generally, \(A\) could be a general commutative Hopf algebra and \(g\) the Lie algebra of an algebraic group \(G\) acting on \(A\) by Hopf algebra automorphisms.

Likewise, let \(V\) be a rational representation of an algebraic group \(G\) over the ground field \(k\). Addition in \(V\) induces a Hopf algebra diagonal on the affine coordinate ring \(k[V]\) of \(V\) that is compatible with the \(G\)-actions. Consequently the familiar algebra of differential operators associated with the representation then acquires a Hopf algebra structure over the ground field.

Thus, in the case at hand, we obtain a genuine Hopf algebra structure on the universal algebra of the Lie-Rinehart algebra under discussion. It would be interesting to construct examples which are more general than the above crossed products. Such examples arise, perhaps, in differential Galois theory.

In the papers [9] and [12], for a general Lie-Rinehart algebra \((A, L)\) defined over a field of characteristic zero, it is shown that the classical construction of the diagonal map for the universal enveloping algebra of an ordinary Lie algebra extends to an \(A\)-linear morphism from \(U(A, L)\) to \(U(A, L) \otimes_A U(A, L)\) which turns \(U(A, L)\) into an \(A\)-coalgebra. However this does not yield a genuine bialgebra structure unless \(L\) is an ordinary \(A\)-Lie algebra so that \(U(A, L)\) is then the ordinary universal algebra associated with the \(A\)-Lie algebra \(L\), and only a suitably defined subalgebra \(U(A, L) \odot_A U(A, L)\) of \(U(A, L) \otimes_A U(A, L)\) acquires an algebra structure. The precise structure which \(U(A, L)\) carries is that of an \(A\)-bialgebra, cf. [9] and the references there. For intelligibility we recall that, given the commutative algebra \(A\) over the field \(k\), a left \(A\)-bialgebra \(H\) over \(k\) consists of an algebra \(H\) (not necessarily commutative) containing \(A\), a morphism \(\varepsilon : H \to A\) of left \(A\)-modules which satisfies the identity
\[
\varepsilon(uv) = \varepsilon(u \cdot \varepsilon(v))
\]
together with a morphism \(\Delta : H \to H \odot_A H\) of algebras into a suitably defined submodule \(H \odot_A H\) of the ordinary left \(A\)-module tensor product \(H \odot_A H\), the submodule \(H \odot_A H\) being an algebra under ordinary tensor product multiplication.
6 Concluding remarks related with Lie-Rinehart bialgebras

Let $L$ and $D$ be $(R,A)$-Lie algebras which, as $A$-modules, are finitely generated and projective, in such a way that, as an $A$-module, $D$ is isomorphic to $L^* = \text{Hom}_A(L, A)$. We say that $L$ and $D$ are in duality. We write $d$ for the differential on $\text{Alt}_A(L, A) \cong \Lambda_A^1$ coming from the Lie-Rinehart structure on $L$ and $d^*$ for the differential on $\text{Alt}_A(D, A) \cong \Lambda_A^1$. The triple $(A, L, D)$ is said to constitute a Lie-Rinehart bialgebra if the differential $d^*$ on $\text{Alt}_A(D, A)$ and the Gerstenhaber bracket $[,]$ on $\Lambda_A^1$ are related by $d^*[x,y] = [d^*x, y] + [x, d^*y]$, $x, y \in L$, or equivalently, if the differential $d$ on $\text{Alt}_A(L, A) \cong \Lambda_A^1$ behaves as a derivation for the Gerstenhaber bracket $[,]^*$ in all degrees, that is to say $d[x, y]^* = [dx, y]^* - (-1)^{|x|} [x, dy]^*$, $x, y \in \Lambda_A^1$.

See [7] for details.

Let $(\Delta, \varepsilon)$ be a bialgebra structure on $A$, suppose that $(A, \Delta, L, \omega^\otimes)$ is a bi-Lie-Rinehart algebra, and let

$$\Delta: U(A, L) \longrightarrow U(A, L) \otimes U(A, L)$$

(6.1)

and

$$\varepsilon: U(A, L) \longrightarrow R$$

(6.2)

be the resulting coalgebra structure on the universal algebra $U(A, L)$ given by Theorem 4.2 above turning the latter into a bialgebra. The diagonal map of $A$ induces a morphism

$$\text{Hom}_A(L, A) \longrightarrow \text{Hom}_A(L, A \otimes A)$$

(6.3)

of $A$-modules.

Suppose that, as an $A$-module, $L$ is finitely generated and projective. Then the canonical morphism

$$\text{Hom}_A(L, A) \otimes \text{Hom}_A(L, A) \longrightarrow \text{Hom}_{A \otimes A}(L \otimes L, A \otimes A)$$

(6.4)

is an isomorphism of $(A \otimes A)$-modules, and so is the canonical morphism

$$\text{Hom}_A(L, A \otimes A) \cong \text{Hom}_{A \otimes A}(L^\otimes, A \otimes A).$$

(6.5)

Let

$$[,] : \text{Hom}_A(L, A) \otimes \text{Hom}_A(L, A) \longrightarrow \text{Hom}_A(L, A)$$

(6.6)
be a Lie bracket on $\text{Hom}_A(L, A)$ turning $(A, L, \text{Hom}_A(L, A))$ into a Lie-Rinehart bialgebra. The composite of (6.6) with the inverse of the isomorphism (6.4) and with the morphism (6.3) yields the morphism

$$\text{Hom}_{A \otimes A}(L \otimes L, A \otimes A) \rightarrow \text{Hom}_A(L, A \otimes A)$$

(6.7)

which, in view of the isomorphism (6.5), takes the form

$$\text{Hom}_{A \otimes A}(L \otimes L, A \otimes A) \rightarrow \text{Hom}_{A \otimes A}(L^\otimes, A \otimes A).$$

(6.8)

The $(A \otimes A)$-dual thereof dual—here we use the hypothesis that, as an $A$-module, $L$ is finitely generated and projective—is a morphism of the kind

$$L^\otimes \rightarrow L \otimes L.$$

The composite thereof with the morphism $L \rightarrow L^\otimes$ which is a constituent of the morphism

$$(\Delta, \Delta \otimes \text{Id}): (A, L) \rightarrow (A \otimes A, L^\otimes)$$

(6.9)

of Lie-Rinehart algebras given above as (4.3) yields a morphism of the kind

$$L \rightarrow L \otimes L.$$  

(6.10)

We conjecture that perturbing the comultiplication (6.1) by means of the morphism (6.10) yields a new comultiplication which, together with the other data, again combines to a bialgebra structure on $U(A, L)$. Since we do not have a particular example yet, we will pursue these issues elsewhere.

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