Longitudinal and transverse fermion-boson vertex in QED at finite temperature in the HTL approximation

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We evaluate the fermion-photon vertex in QED at the one loop level in Hard Thermal Loop approximation and write it in covariant form. The complete vertex can be expanded in terms of 32 basis vectors. As is well known, the fermion-photon vertex and the fermion propagator are related through a Ward-Takahashi Identity (WTI). This relation splits the vertex into two parts: longitudinal ($\Gamma_L$) and transverse ($\Gamma_T$). $\Gamma_L$ is fixed by the WTI. The description of the longitudinal part consumes 8 of the basis vectors. The remaining piece $\Gamma_T$ is then written in terms of 24 spin amplitudes. Extending the work of Ball and Chiu and Kızılersü et. al., we propose a set of basis vectors $T^{\mu}_{i}(P_1, P_2)$ at finite temperature such that each of these is transverse to the photon four-momentum and also satisfies $T^{\mu}_{i}(P, P) = 0$, in accordance with the Ward Identity, with their corresponding coefficients being free of kinematic singularities. This basis reduces to the form proposed by Kızılersü et. al. at zero temperature. We also evaluate explicitly the coefficient of each of these vectors at the above-mentioned level of approximation.

I. INTRODUCTION

Schwinger-Dyson equations provide a natural starting point to study the non-perturbative aspects of gauge field theories, in particular in connection with dynamical symmetry breaking. As is well known, knowledge of the three-point vertex is crucial in such studies. Numerous works exist at zero temperature with a gradual improvement on the choice of the full vertex and other underlying assumptions, rendering the technique increasingly powerful. At finite temperature, the complexity of the equations and the number of the unknown functions to be evaluated grow enormously. For example, in QED at zero temperature, the three-point vertex can be written in terms of 12 spin amplitudes, whereas, at finite temperature, this number becomes 32, almost a three-fold increase in the number of unknowns involved. As a result, most of the research is restricted to the use of the bare vertex [1–7].

However, the Ward-Takahashi Identity (WTI) inspired vertex has more recently been employed to study the dynamical breakdown of chiral symmetry [8]. The WTI relates the three-point vertex to the fermion propagator. This identity permits us to decompose the full vertex into two parts, the longitudinal part, which is fixed by the WTI [9], and the transverse part, which vanishes on contraction with the boson momentum and hence remains undetermined by the WTI. For instance in reference [8], Strickland uses the longitudinal part of the vertex to study, among other things, the dynamically generated mass as a function of temperature. He finds out that the gauge dependence of the critical temperature for which chiral symmetry is restored reduces considerably as compared to the results obtained earlier using the bare vertex.

The lessons learned from similar studies at zero temperature suggest that the complete gauge independence of the critical temperature relies crucially on the choice of the transverse vertex. In this context, perturbation theory is an important point of reference as it is natural to believe that physically meaningful solutions of the Schwinger-Dyson equations must agree with perturbative results in the weak coupling regime. This realization has been exploited in Refs. [10–13] to derive constraints on the fermion propagator and the three-point vertex.

In this paper, we extend the works in Refs. [9,14–16] from zero to finite temperature, evaluating the one-loop transverse vertex in the Hard Thermal Loop (HTL) approximation and writing it in covariant form. The procedure we use is as follows: By evaluating the fermion propagator to a given order, it is possible to determine the longitudinal vertex to the same order. A subtraction of the longitudinal part then yields the transverse part, the one which is not fixed by the WTI. According to the choice of Ball and Chiu, which was later modified by Kızılersü et. al. [9], the transverse vertex can be expanded out, at zero temperature, in terms of 8 independent spin structures. We find out that at finite temperature the transverse vertex requires 24 spin structures to be expressed. Following a systematic transition from zero to finite temperature, we construct a basis of 24 transverse vectors. We use this basis to write out
the fermion-photon vertex in the HTL approximation [17], which is known to be a gauge independent scheme valid for temperatures much larger than the fermion mass. The results allow us to make useful comments on the possible non-perturbative structures of the vertex at finite temperature. The work is organized as follows: In Sect. II we briefly recall the expressions for the fermion self-energy and fermion-photon vertex at finite temperature in the HTL approximation. In Sect. III we construct the longitudinal and transverse vertices defining a set of 24 spin structures to serve as the transverse basis. In Sect. IV we find the explicit expressions for the coefficients of the transverse vertex. We finally summarize and conclude in Sect. V. A short appendix collects some of the integrals that are involved in the computations in Sect. IV.

II. FERMION PROPAGATOR AND FERMION-PHOTON VERTEX IN THE HTL APPROXIMATION

We start by expanding out the fermion propagator $S(P)$ in its most general form at finite temperature. This can be obtained by noticing that the only available Lorentz structures are $1, P \bar{\Psi}, \bar{\Psi}$. Therefore, we can write (hereafter, we use capital letters to refer to four-vectors, whereas lower case letters are used to refer to their components)

$$S(P)^{-1} = (1 - a) P - b \bar{\Psi} + c P \bar{\Psi} + d$$

$$= P - \Sigma(P) + c P \bar{\Psi} + d,$$  \hspace{1cm} (2.1)

where

$$\Sigma(P) = a P + b \bar{\Psi}.$$  \hspace{1cm} (2.2)

The Lorentz invariant functions $a, b, c$ and $d$ will in general depend on two Lorentz scalars $P^2$ and $U \cdot P$, where $U^\mu$ is the four-velocity of the heat bath as seen from a general frame. We can choose these scalars to be

$$p_0 \equiv P^\mu U_\mu$$

$$p \equiv [(P^\mu U_\mu)^2 - P^2]^{1/2}.$$  \hspace{1cm} (2.3)

When the temperature $T$ is high enough, all the $T = 0$ masses can safely be neglected in comparison with the scale $g T$ —where $g$ is the fermion-gauge boson coupling constant. In this regime, the dominant contributions to n-point functions at one loop come from the situation in which the loop momentum is hard, i.e. of the order of $T$ [17]. This is the so called HTL approximation which is known to render the results gauge independent when applied, for example, to the computation of damping rates and energy losses of particles propagating through hot plasmas [18]. For QED, in this approximation, the temperature dependent $\Sigma(P)$ is given in Minkowski space explicitly by

$$\Sigma(P) = m_f^2 \int \frac{d \Omega}{4\pi} \frac{\hat{K}}{P \cdot \hat{K}},$$  \hspace{1cm} (2.4)

where $m_f^2 = g^2 T^2 / 8$ is the square of the fermion thermal mass and $\hat{K}^\mu = (-1, \hat{k})$. Evaluation of Eq. (2.4) yields the functions $a$ and $b$ defined in Eq. (2.2), [19]

$$a(p_0, p) = - \frac{m_f^2}{p^2} |1 - p_0 L(P)| \equiv m_f^2 a(p_0, p)$$

$$b(p_0, p) = \frac{m_f^2}{p^2} \left[p_0 + (p^2 - p_0^2) L(P)\right] \equiv m_f^2 b(p_0, p),$$  \hspace{1cm} (2.5)

where

$$L(P) \equiv - \int \frac{d \Omega}{4\pi} \frac{1}{P \cdot \hat{K}} = \frac{1}{2p} \ln \left[\frac{p_0 + p}{p_0 - p}\right].$$  \hspace{1cm} (2.6)

Equations (2.1), (2.2), (2.3) and (2.4) together with $c = d = 0$ constitute the QED fermion propagator at one loop level in the HTL approximation. $d = 0$ corresponds to the case of exact chiral symmetry whereas $c = 0$ is required for a parity conserving theory [19].

As for the temperature dependent fermion-photon vertex $\Gamma_\mu(P_1, P_2)$, its corresponding HTL expression in Minkowski space is given by [20]

$$\Gamma_\mu(P_1, P_2) = m_f^2 G_{\mu \nu}(P_1, P_2) \gamma^\nu,$$  \hspace{1cm} (2.7)
where

\[ G_{\mu\nu}(P_1, P_2) = \int \frac{d\Omega}{4\pi} \frac{\tilde{K}_\mu \tilde{K}_\nu}{(P_1 \cdot K)(P_2 \cdot K)} . \]  

(2.8)

### III. LONGITUDINAL AND TRANSVERSE VERTEX

#### A. Longitudinal vertex

The WTI relates the fermion propagator and the fermion-photon vertex by

\[ Q^\mu \Gamma_\mu(P_1, P_2) = S^{-1}(P_1) - S^{-1}(P_2) \]

\[ = [1 - a(P_1)] P_1 [1 - a(P_2)] P_2 - b(P_1) \bar{\psi} + b(P_2) \psi \]

\[ + c(P_1) P_1 \bar{\psi} - c(P_2) P_2 \bar{\psi} + d(P_1) - d(P_2) , \]

(3.1)

with \( Q = P_1 - P_2 \). This relation allows us to decompose the vertex into longitudinal \( \Gamma^L_\mu(P_1, P_2) \) and transverse \( \Gamma^T_\mu(P_1, P_2) \) parts

\[ \Gamma_\mu(P_1, P_2) = \Gamma^L_\mu(P_1, P_2) + \Gamma^T_\mu(P_1, P_2) , \]

(3.2)

where the transverse part satisfies

\[ Q^\mu \Gamma^T_\mu(P_1, P_2) = 0 \quad \text{and} \quad \Gamma^T_\mu(P, P) = 0 , \]

(3.3)

and hence remains undetermined by WTI. In order to construct the longitudinal vertex along the lines of the calculation of Ball and Chiu [9], we start from the Ward-Identity

\[ \Gamma^L_\mu(P, P) = \frac{\partial}{\partial P^\mu} S^{-1}(P) , \]

(3.4)

which is the limiting form of WTI when \( P_1 = P_2 = P \). Making use of Eq. (2.1), the above relation yields

\[ \Gamma^L_\mu(P, P) = [1 - a(P)] \gamma^\mu - 2 P^\mu P \frac{\partial}{\partial P^2} a(P) - 2 P^\mu \bar{\psi} \frac{\partial}{\partial P^2} b(P) \]

\[ + \gamma^\mu \bar{\psi} c(P) + 2 P^\mu P \bar{\psi} \frac{\partial}{\partial P^2} c(P) + 2 P^\mu \frac{\partial}{\partial P^2} d(P) . \]

(3.5)

We now symmetrize this expression under the exchange of \( P_1 \leftrightarrow P_2 \) in the following manner

\[ P_\mu \rightarrow \frac{1}{2} (P_\mu + P_{2\mu}) \]

\[ P \rightarrow \frac{1}{2} (P_1 + P_2) \]

\[ \frac{\partial}{\partial P^2} f(P) \rightarrow \frac{f(P_1) - f(P_2)}{P_1^2 - P_2^2} , \]

(3.6)

where \( f(P) \) is a generic function standing for \( a(P) \), \( b(P) \), \( c(P) \) and \( d(P) \). We thus arrive at the non-perturbative expression for the longitudinal vertex

\[ \Gamma^L_\mu(P_1, P_2) = \gamma^\mu - \frac{1}{2} [a(P_1) + a(P_2)] \gamma^\mu - \frac{a(P_1) - a(P_2)}{2(P_1^2 - P_2^2)} (P_\mu + P_{2\mu}) (P_1 + P_2) - \frac{b(P_1) - b(P_2)}{(P_1^2 - P_2^2)} (P_\mu + P_{2\mu}) \bar{\psi} \]

\[ + \frac{1}{2} [c(P_1) + c(P_2)] \gamma^\mu \bar{\psi} + \frac{c(P_1) - c(P_2)}{2(P_1^2 - P_2^2)} (P_\mu + P_{2\mu}) (P_1 + P_2) \bar{\psi} + \frac{d(P_1) - d(P_2)}{(P_1^2 - P_2^2)} (P_\mu + P_{2\mu}) . \]

(3.7)

Note that the coefficients of two spin structures corresponding to \( P_\mu P_{2\mu} \sigma^{\mu\nu} \) and \( \bar{\psi} P_\mu P_{2\mu} \sigma^{\mu\nu} \) will be zero because these do not appear on the right hand side of Eq. (3.1). By construction, the longitudinal part of the vertex is free
This observation would be of help to construct the transverse vertex later in this section. In the HTL approximation, at finite temperature, the available vectors to expand the full vertex $\Gamma$ of kinematical singularities. An interesting and useful comparison with the work of Ball and Chiu [9] reveals that for every structure $V_\mu$ at zero temperature, there exists an additional structure $V_\mu \gamma$ at finite temperature. Therefore,

\[
\text{zero temperature} \rightarrow \text{finite temperature} \quad (P_{1\mu} + P_{2\mu}) (P_1 + P_2) \quad \rightarrow \quad (P_{1\mu} + P_{2\mu}) (P_1 + P_2) \quad \gamma_\mu \gamma \quad \rightarrow \quad (P_{1\mu} + P_{2\mu}) \gamma \gamma \quad \rightarrow \quad (P_{1\mu} + P_{2\mu}) \gamma \gamma . \quad (3.8)
\]

This observation would be of help to construct the transverse vertex later in this section. In the HTL approximation, the one loop longitudinal vertex reduces to

\[
\Gamma^L_\mu(P_1, P_2) = -\frac{1}{2} [a(P_1) + a(P_2)] \gamma_\mu - \frac{a(P_1) - a(P_2)}{2(P_1^2 - P_2^2)} (P_{1\mu} + P_{2\mu}) (P_1 + P_2)
- \frac{b(P_1) - b(P_2)}{(P_1^2 - P_2^2)} (P_{1\mu} + P_{2\mu}) \gamma \gamma . \quad (3.9)
\]

**B. Transverse Vertex**

At finite temperature, the available vectors to expand the full vertex $\Gamma_\mu(P_1, P_2)$ are $\gamma_\mu, P_{1\mu}, P_2, P_2 \gamma$, and $P_1 \gamma$. Correspondingly there are 8 Lorentz scalars 1, $P_1, P_2, \gamma P_1, P_2, P_1 \gamma, P_2 \gamma, P_1 P_2 \gamma$. Therefore, there are 32 spin amplitudes in total:

\[
V_{1\mu} = P_{1\mu} P_1 , \quad V_{2\mu} = P_{2\mu} P_2 , \quad V_{3\mu} = P_{1\mu} P_2 , \quad V_{4\mu} = P_{2\mu} P_1 ,
V_{5\mu} = \gamma_\mu P_1 P_2 , \quad V_{6\mu} = \gamma_\mu P_2 , \quad V_{7\mu} = P_1 \gamma , \quad V_{8\mu} = P_2 \gamma ,
V_{9\mu} = P_{1\mu} P_1 P_2 , \quad V_{10\mu} = P_{2\mu} P_1 P_2 , \quad V_{11\mu} = \gamma_\mu P_1 , \quad V_{12\mu} = \gamma_\mu P_2 ,
V_{13\mu} = V_{1\mu} \gamma , \quad V_{14\mu} = V_{2\mu} \gamma , \quad V_{15\mu} = V_{3\mu} \gamma , \quad V_{16\mu} = V_{4\mu} \gamma ,
V_{17\mu} = V_{5\mu} \gamma , \quad V_{18\mu} = V_{6\mu} \gamma , \quad V_{19\mu} = V_{7\mu} \gamma , \quad V_{20\mu} = V_{8\mu} \gamma ,
V_{21\mu} = V_{9\mu} \gamma , \quad V_{22\mu} = V_{10\mu} \gamma , \quad V_{23\mu} = V_{11\mu} \gamma , \quad V_{24\mu} = V_{12\mu} \gamma ,
V_{25\mu} = U_{\mu} , \quad V_{26\mu} = U_\mu \gamma , \quad V_{27\mu} = U_{\mu} P_1 , \quad V_{28\mu} = U_\mu P_2 ,
V_{29\mu} = U_{\mu} P_1 \gamma , \quad V_{30\mu} = U_\mu P_2 \gamma , \quad V_{31\mu} = U_{\mu} P_1 P_2 , \quad V_{32\mu} = U_{\mu} P_1 P_2 \gamma . \quad (3.10)
\]

In a construction parallel to that of Ball and Chiu, Eq. (3.7) consumes 8 of these 32 structures. The transverse vertex $\Gamma^T_\mu(P_1, P_2)$ can then be written in terms of the remaining 24 basis vectors as

\[
\Gamma^T_\mu(P_1, P_2) = m^2 T \sum_{i=1}^{24} \tau^i (P_1^2, P_2^2, Q^2) T^i_\mu(P_1, P_2) , \quad (3.11)
\]

where the coefficients $\tau^i$ are to be found from the explicit expression for $\Gamma_\mu$ and the choice of the transverse basis of tensors $\{T^i_\mu\}$, which will be a linear combination of the vectors in Eq. (3.10). The procedure for their selection is as follows:

- Motivated from the success of the basis proposed by Kızılersı et. al., which is a simple modification of the one proposed by Ball and Chiu, we choose 8 of the basis vectors the same.
- Let us now recall the observation made in connection with the construction of the longitudinal vertex. We noticed that for every structure $V_\mu$ at zero temperature, there exists an additional structure $V_\mu \gamma$ at finite temperature. Making use of this observation, we select 8 of the basis vectors, which are a simple multiplication of the previous 8 vectors by $\gamma$. Note that a mere multiplication with $\gamma$ does not change the much desired transversality properties of the vectors and the fact that they vanish for $P_1 = P_2$.
- The remaining 8 transverse vectors must contain the vector $U_\mu$ not used so far. Now the transversality of the new vectors can be guaranteed if they have the form

\[
(Q \cdot U) R_i V_{i\mu} + SU_\mu
\]
with $R_i$ and $S$ scalar functions and $V_{i\mu} = \gamma_{\mu}, P_{1\mu}, P_{2\mu}$ such that
\[(Q \cdot U) R_i Q \cdot V_i + SQ \cdot U = 0 \ .
\]

Making use of this condition we construct the remaining vectors as a simple extension of the ones proposed by Kizilersü et al.. We list all the vectors below:

\[
\begin{align*}
T^1_\mu &= [P_{2\mu}(P_1 \cdot Q) - P_{1\mu}(P_2 \cdot Q)] \\
T^2_\mu &= T^1_\mu (P_1 + P_2) \\
T^3_\mu &= Q^2 \gamma_\mu - Q_\mu Q \\
T^4_\mu &= Q^2 [\gamma_\mu (P_1 + P_2) - P_{1\mu} - P_{2\mu}] - 2(P_1 - P_2)_\mu P_1^\lambda P_2^\nu \sigma_{\lambda\nu} \\
T^5_\mu &= Q^\nu \sigma_{\nu\mu} \\
T^6_\mu &= -\gamma_\mu (P_1^2 - P_2^2) + (P_1 + P_2)_\mu Q \\
T^7_\mu &= -\frac{1}{2} (P_1^2 - P_2^2) [\gamma_\mu (P_1 + P_2) - P_{1\mu} - P_{2\mu}] + (P_1 + P_2)_\mu P_1^\lambda P_2^\nu \sigma_{\lambda\nu} \\
T^8_\mu &= -\gamma_\mu P_1^\rho P_2^\lambda \sigma_{\rho\lambda} + P_{1\mu} P_2 - P_{2\mu} P_1 \\
T^i_\mu &= T^{i-8}_\mu \quad i = 9, 16 \\
T^1_{17} &= (Q \cdot U) [P_{2\mu}(P_1 \cdot Q) + P_{1\mu}(P_2 \cdot Q)] - 2(P_1 \cdot Q) (P_2 \cdot Q) U_{\mu} \\
T^1_{18} &= T^{17}_\mu (P_1 + P_2) U_{\mu} \\
T^1_{19} &= [(Q \cdot U) (Q^2 \gamma_\mu + Q_\mu Q) - 2Q^2 Q_\mu U_{\mu}] U_{\mu} \\
T^2_{20} &= (Q \cdot U) [Q^2 \gamma_\mu (P_1 + P_2) + P_{1\mu} + P_{2\mu} + 2Q_\mu P_1^\lambda P_2^\nu \sigma_{\lambda\nu}] U_{\mu} \\
&- U_{\mu} [Q^2 \gamma_\mu (P_1 + P_2) + P_{1\mu} + P_{2\mu} + 2Q^2 P_1^\lambda P_2^\nu \sigma_{\lambda\nu}] U_{\mu} \\
T^2_{21} &= [(Q \cdot U) Q_\mu - Q^2 U_{\mu}] U_{\mu} \\
T^2_{22} &= (Q \cdot U) \gamma_\mu (P_1^2 - P_2^2) + (P_1 + P_2)_\mu Q - 2(P_1^2 - P_2^2) Q U_\mu \\
T^2_{23} &= (Q \cdot U) \left[\frac{1}{2} (P_1^2 - P_2^2) \gamma_\mu (P_1 + P_2) + P_{1\mu} + P_{2\mu} + (P_1 + P_2)_\mu P_1^\lambda P_2^\nu \sigma_{\lambda\nu}\right] \\
&- U_{\mu} \left[\frac{1}{2} (P_1^2 - P_2^2) \gamma_\mu (P_1 + P_2) + P_{1\mu} + P_{2\mu} + (P_1 + P_2)_\mu P_1^\lambda P_2^\nu \sigma_{\lambda\nu}\right] \\
T^2_{24} &= (Q \cdot U) \left[P_{2\mu} P_1 - P_{1\mu} P_2 - \gamma_\mu P_1^\lambda P_2^\nu \sigma_{\lambda\nu}\right] \\
&- U_{\mu} [(Q \cdot P_1) P_2 - (Q \cdot P_2) P_1 - Q_\mu P_1^\lambda P_2^\nu \sigma_{\lambda\nu}] ,
\end{align*}
\]

where we define $\sigma_{\mu\nu}$ as
\[
\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu].
\]

**IV. COVARIANT HTL VERTEX AND COEFFICIENTS OF THE TRANSVERSE BASIS**

With the choice of the transverse basis given in Eq. (3.13), the computation of the coefficients $\tau^i$ is a straightforward though lengthy exercise. The first step consists of finding the explicit covariant expression for $\Gamma_{\mu}$. Then a subtraction of the longitudinal vertex $\Gamma_{\mu}^L$, given by Eq. (3.7), yields the transverse vertex.

Computation of $\Gamma_{\mu}$ requires finding the different components of the function $G_{\mu\nu}$, defined in Eq. (2.8). These have already been found in terms of a projection onto a particular choice of vector and tensor structures in Ref. [21]. However, in order to explicitly preserve the symmetric properties of the different elements that make up the vertex $\Gamma_{\mu}$ during the computation, we find it convenient to modify the choice of vector and tensor structures with respect to that of Ref. [21]. In this section, we list the results for the components of the function $G_{\mu\nu}$, leaving a brief sketch of their computation for the appendix. Let us start with $G_{00}$ which is explicitly given by
The coefficient $A$ is found by contracting Eq. (4.3) with $\hat{n}^i$ whereas $B^\pm$ are found upon contraction with $\hat{l}^\pm$, respectively. They are explicitly given by

$$A = 0, \quad B^\pm = \frac{1}{\sqrt{2\delta^\pm}} \left[ \left( \frac{p_{10} \pm p_{20}}{p_1} \right) M(P_1, P_2) - \left( \frac{L(P_2) \pm L(P_1)}{p_2} \right) \right].$$

Finally, $G_{ij}$ can be written as

$$G_{ij}(P_1, P_2) = X \hat{n}_i \hat{n}_j + \hat{Y}^+ \hat{l}_i^+ \hat{l}_j^- + \hat{Y}^- \hat{l}_i^- \hat{l}_j^+ + Z (\hat{l}_i^+ \hat{l}_j^- + \hat{l}_i^- \hat{l}_j^+),$$

where the coefficients $\hat{Y}^\pm$ and $Z$ are found by contracting Eq. (4.7) with $\hat{l}_i^+ \hat{l}_j^-$ and $\hat{l}_i^- \hat{l}_j^+$, respectively, whereas $X$ can be found by taking the trace of Eq. (4.7). They are explicitly given by

$$X = \frac{1}{p_1^2 p_2^2} \left[ \Delta M(P_1, P_2) + p_1 \cdot (p_{20} p_1 - p_{10} p_2) L(P_1) + p_2 \cdot (p_{10} p_2 - p_{20} p_1) L(P_2) \right],$$

$$\hat{Y}^\pm = \frac{1}{2\delta^\pm} \left[ \frac{p_{10}}{p_1} \pm \frac{p_{20}}{p_2} \right] \left[ \left( \frac{p_{10}}{p_1} \pm \frac{p_{20}}{p_2} \right) M(P_1, P_2) - \left( \frac{L(P_2) \pm L(P_1)}{p_2} \right) \right],$$

$$Z = \frac{1}{2\sqrt{\delta}} \left[ \left( \frac{p_{10}^2}{p_1^2} - \frac{p_{20}^2}{p_2^2} \right) M(P_1, P_2) - \left( \frac{p_{10} + p_{20} p_1 \cdot p_2}{p_1^2} \right) L(P_2) \right.$$ \n\n$$\left. + \left( \frac{p_{20} + p_{10} p_1 \cdot p_2}{p_2^2} \right) L(P_1) \right].$$

In order to obtain the explicit vectors in terms of which the vertex $\Gamma_\mu$ is expressed, we first look at the space-like part, namely $\Gamma_\mu = m^2 G_{\mu\nu} \gamma^\nu$ and add the necessary time-like pieces to form the corresponding four-vectors. We then

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1We find a sign difference for the term $\Delta M(P_1, P_2)$ in the expression for the coefficient $X$, as compared to the corresponding expression in Ref. [2].
subtract the above added pieces which, together with the time-like part of the vertex, namely \( \Gamma_0 = m_f^2 G_{0\nu \gamma^\nu} \), become the terms proportional to the vector \( U_\mu \). We finally obtain

\[
\Gamma_\mu(P_1, P_2) = -m_f^2 \{ X \gamma_\mu + C_1 P_1 \mu + C_2 P_2 \mu + DU_\mu \},
\]

(4.9)

where the functions \( C_1, C_2 \) and \( D \) are given by

\[
C_1 = -\frac{\gamma_0}{p_1} \left( \frac{B^+}{\sqrt{2\delta_+}} + \frac{B^-}{\sqrt{2\delta_-}} \right) p_1 \cdot \gamma + \frac{p_2 \cdot \gamma}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} + \frac{Y^-}{2\delta_-} + \frac{Z}{\sqrt{\delta}} \right) \right) + p_1 \cdot \gamma \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} \right)
\]

\[
C_2 = -\frac{\gamma_0}{p_2} \left( \frac{B^+}{\sqrt{2\delta_+}} - \frac{B^-}{\sqrt{2\delta_-}} \right) p_2 \cdot \gamma + \frac{p_1 \cdot \gamma}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} - \frac{Z}{\sqrt{\delta}} \right) + p_2 \cdot \gamma \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} \right)
\]

\[
D = -(M + X)\gamma_0 + \frac{p_1 \cdot \gamma}{p_1} \left( \frac{B^+}{\sqrt{2\delta_+}} + \frac{B^-}{\sqrt{2\delta_-}} \right) p_2 \cdot \gamma \left( \frac{B^+}{\sqrt{2\delta_+}} - \frac{B^-}{\sqrt{2\delta_-}} \right) - C_1 p_{10} - C_2 p_{20},
\]

(4.10)

and we have defined \( Y^\pm = \tilde{Y}^\pm - X \). In order to obtain the spin structures out of Eqs. (4.9) and (4.10), we notice that, working in a general frame, we can write

\[
\gamma_0 = \psi
\]

\[
p \cdot \gamma = p_0 \psi - P.
\]

(4.11)

Making use of these identities and of Eq. (4.10), we can write out Eq. (4.9) in terms of the basis vectors given in Eq. (3.10):

\[
\Gamma_\mu(P_1, P_2) = -m_f^2 \left[ \tilde{h}_1 V_{1\mu} + \tilde{h}_2 V_{2\mu} + \tilde{h}_3 V_{3\mu} + \tilde{h}_4 V_{4\mu} + \tilde{h}_6 V_{6\mu} + \tilde{h}_{19} V_{19\mu} + \tilde{h}_{20} V_{20\mu} + \tilde{h}_{26} V_{26\mu} + \tilde{h}_{27} V_{27\mu} + \tilde{h}_{28} V_{28\mu} \right],
\]

(4.12)

where

\[
\tilde{h}_1 = -\frac{1}{p_1^2} \left( \frac{Y^+}{2\delta_+} + \frac{Y^-}{2\delta_-} + \frac{Z}{\sqrt{\delta}} \right)
\]

\[
\tilde{h}_2 = -\frac{1}{p_2^2} \left( \frac{Y^+}{2\delta_+} + \frac{Y^-}{2\delta_-} - \frac{Z}{\sqrt{\delta}} \right)
\]

\[
\tilde{h}_3 = \tilde{h}_4 = -\frac{1}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} + \frac{Y^-}{2\delta_-} \right)
\]

\[
\tilde{h}_6 = X
\]

\[
\tilde{h}_{19} = -\frac{1}{p_1} \left( \frac{B^+}{\sqrt{2\delta_+}} + \frac{B^-}{\sqrt{2\delta_-}} \right) + \frac{p_{10}}{p_1^2} \left( \frac{Y^+}{2\delta_+} + \frac{Y^-}{2\delta_-} + \frac{Z}{\sqrt{\delta}} \right) + \frac{p_{20}}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} \right)
\]

\[
\tilde{h}_{20} = -\frac{1}{p_2} \left( \frac{B^+}{\sqrt{2\delta_+}} - \frac{B^-}{\sqrt{2\delta_-}} \right) + \frac{p_{20}}{p_2^2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} - \frac{Z}{\sqrt{\delta}} \right) + \frac{p_{10}}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} \right)
\]

\[
\tilde{h}_{26} = M - 3X \frac{p_{10}^2}{p_1^2} \left( \frac{Y^+}{2\delta_+} + \frac{Y^-}{2\delta_-} + \frac{Z}{\sqrt{\delta}} \right) - \frac{p_{20}^2}{p_2^2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} - \frac{Z}{\sqrt{\delta}} \right) - \frac{2p_{10} p_{20}}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} \right)
\]

\[
\tilde{h}_{27} = -\frac{1}{p_1^2} \left( \frac{B^+}{\sqrt{2\delta_+}} + \frac{B^-}{\sqrt{2\delta_-}} \right) + \frac{p_{10}}{p_1^2} \left( \frac{Y^+}{2\delta_+} + \frac{Y^-}{2\delta_-} + \frac{Z}{\sqrt{\delta}} \right) + \frac{p_{20}}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} \right)
\]

\[
\tilde{h}_{28} = -\frac{1}{p_2^2} \left( \frac{B^+}{\sqrt{2\delta_+}} - \frac{B^-}{\sqrt{2\delta_-}} \right) + \frac{p_{20}}{p_2^2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} - \frac{Z}{\sqrt{\delta}} \right) + \frac{p_{10}}{p_1 p_2} \left( \frac{Y^+}{2\delta_+} - \frac{Y^-}{2\delta_-} \right).
\]

(4.13)

Eqs. (4.12) and (4.13) complete the covariantization of the full QED vertex at one loop in HTL approximation. The coefficients \( \tilde{h}_5, \tilde{h}_{7,18}, \tilde{h}_{21,25} \) and \( \tilde{h}_{29,32} \) of the vectors \( V_{5\mu}, V_{7\mu,18\mu}, V_{21\mu,25\mu} \) and \( V_{29\mu,32\mu} \) respectively are zero at this order. Subtracting the longitudinal vertex, Eq. (3.9), the transverse vertex \( \Gamma^T_\mu \) comes out to be
\[ \Gamma^T_\mu = \Gamma_\mu - \Gamma^L_\mu \]
\[ = -m_f^2 \left\{ h_1V_{1\mu} + h_2V_{2\mu} + h_3V_{3\mu} + h_4V_{4\mu} + h_5V_{5\mu} + h_{19}V_{19\mu} + h_{20}V_{20\mu} + h_{26}V_{26\mu} + h_{27}V_{27\mu} + h_{28}V_{28\mu} \right\}, \]
where the coefficients \( h_i \) are given by

\[ h_1 = \tilde{h}_1 - \frac{[\hat{a}(P_1) - \hat{a}(P_2)]}{2(P_1^2 - P_2^2)} \]
\[ h_2 = \tilde{h}_2 - \frac{[\hat{a}(P_1) - \hat{a}(P_2)]}{2(P_1^2 - P_2^2)} \]
\[ h_3 = \tilde{h}_3 - \frac{[\hat{a}(P_1) - \hat{a}(P_2)]}{2(P_1^2 - P_2^2)} \]
\[ h_6 = \tilde{h}_6 - \frac{2}{2} \]
\[ h_{19} = \tilde{h}_{19} - \frac{[\hat{b}(P_1) - \hat{b}(P_2)]}{P_1^2 - P_2^2} \]
\[ h_{20} = \tilde{h}_{20} - \frac{[\hat{b}(P_1) - \hat{b}(P_2)]}{(P_1^2 - P_2^2)} \]
\[ h_{26} = \tilde{h}_{26} \]
\[ h_{27} = \tilde{h}_{27} \]
\[ h_{28} = \tilde{h}_{28} . \]

By expressing Eq. (4.14) in terms of the set of 32 elemental spin structures and, on the other hand, comparing this with the expansion of the transverse vertex given by Eq. (3.11), we find that the coefficients \( \tau_i, i = 1 \ldots 24 \), satisfy the following systems of linear equations

\[ (P_2 \cdot Q)\tau_2 + \tau_3 - \tau_6 - (Q \cdot U)\tau_{22} = h_1 \]
\[ -(P_1 \cdot Q)\tau_2 + \tau_3 + \tau_6 + (Q \cdot U)\tau_{22} = h_2 \]
\[ (P_2 \cdot Q)\tau_2 - \tau_3 + \tau_6 - \tau_8 + (Q \cdot U)\tau_{22} + (Q \cdot U)\tau_{24} = h_3 \]
\[ -(P_1 \cdot Q)\tau_2 - \tau_3 - \tau_6 + \tau_8 - (Q \cdot U)\tau_{22} - (Q \cdot U)\tau_{24} = h_3 \]
\[ \tau_8 + (Q \cdot U)\tau_{24} = 0 \]
\[ -Q^2\tau_3 + (P_1^2 - P_2^2)\tau_6 - (P_1 \cdot P_2)\tau_8 - (Q \cdot U)(P_1^2 - P_2^2)\tau_{22} - (Q \cdot U)(P_1 \cdot P_2)\tau_{24} = h_6 \]
\[ 2(P_1^2 - P_2^2)\tau_{22} + 2(Q \cdot P_2)\tau_{24} = h_{27} \]
\[ -2(P_1^2 - P_2^2)\tau_{22} - 2(Q \cdot P_1)\tau_{24} = h_{28} , \]

\[ -(P_2 \cdot Q)\tau_1 - (Q^2 - 2P_1 \cdot P_2)\tau_4 + \tau_5 + \frac{1}{2}(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_7 + \]
\[ (Q \cdot U)(P_2 \cdot Q)\tau_{17} + \frac{1}{2}(Q \cdot U)(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_{23} = 0 \]
\[ (P_1 \cdot Q)\tau_1 - (Q^2 + 2P_1 \cdot P_2)\tau_4 - \tau_5 + \frac{1}{2}(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_7 + \]
\[ (Q \cdot U)(P_1 \cdot Q)\tau_{17} + \frac{1}{2}(Q \cdot U)(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_{23} = 0 \]
\[ -2\tau_4 + \tau_7 + (Q \cdot U)\tau_{23} = 0 \]
\[ 2\tau_4 + \tau_7 + (Q \cdot U)\tau_{23} = 0 \]
\[ Q^2\tau_4 - \tau_5 - \frac{1}{2}(P_1^2 - P_2^2)\tau_7 + \frac{1}{2}(Q \cdot U)(P_1^2 - P_2^2)\tau_{23} = 0 \]
\[ Q^2\tau_4 + \tau_5 - \frac{1}{2}(P_1^2 - P_2^2)\tau_7 + \frac{1}{2}(Q \cdot U)(P_1^2 - P_2^2)\tau_{23} = 0 \]
\[ -2(P_1 \cdot Q)(P_2 \cdot Q)\tau_{17} - (P_1^2 - P_2^2)(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_{23} = 0 \]
\[ 2(P_1^2 - P_2^2)\tau_{23} = 0 , \]
Equations (4.16)-(4.19) constitute 4 sets, each consisting of 8 linear algebraic equations for the 24 unknowns \( \tau_i \). We would like to emphasize the following points:

This means that not all of the equations are independent and therefore, there must be some relations among the several inhomogeneous terms. It is also a straightforward exercise to verify that indeed this is the case and that such relations are

\[
(P_1 \cdot Q)\tau_9 + (Q^2 - 2P_1 \cdot P_2)\tau_{12} - \tau_{13} - \frac{1}{2}(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_{15} - (Q \cdot U)(Q^2 - 2P_1 \cdot P_2)\tau_{20} - (Q \cdot U)\tau_{21} = h_{19}
\]

\[
(P_1 \cdot Q)\tau_9 + (Q^2 + 2P_1 \cdot P_2)\tau_{12} + \tau_{13} - \frac{1}{2}(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_{15} - (Q \cdot U)(Q^2 + 2P_1 \cdot P_2)\tau_{20} + (Q \cdot U)\tau_{21} = h_{20}
\]

\[
2Q^2(P_1^2 - P_2^2 - 2P_1 \cdot P_2)\tau_{20} + Q^2\tau_{21} = h_{26}
\]

\[
4Q^2\tau_{20} = 0.
\]

Equations (4.16)-(4.19) constitute 4 sets, each consisting of 8 linear algebraic equations for the 24 unknowns \( \tau_i \). This means that not all of the equations are independent and therefore, there must be some relations among the several inhomogeneous terms. It is also a straightforward exercise to verify that indeed this is the case and that such relations are

\[
(P_1 \cdot Q)h_{19} + (P_2 \cdot Q)h_{20} + (Q \cdot U)h_{26} = 0
\]

\[
(P_1 \cdot Q)h_1 - (P_2 \cdot Q)h_2 - Q^2h_3 + 2h_6 + (Q \cdot U)(h_{27} - h_{28}) = 0
\]

\[
(P_1 \cdot Q)h_1 + (P_2 \cdot Q)h_2 + (P_1^2 - P_2^2)h_3 + (Q \cdot U)(h_{27} + h_{28}) = 0.
\]

The non-vanishing coefficients which solve the sets of Eqs. (4.16)-(4.19) are given by

\[
\tau_2 = -\frac{1}{2Q^2}(h_1 + h_2 + 2h_3)
\]

\[
\tau_3 = \frac{1}{4}(h_1 + h_2 - 2h_3)
\]

\[
\tau_6 = -\frac{1}{4}(h_1 - h_2) + \frac{(Q \cdot U)}{2Q^2(P_1^2 - P_2^2)} [(P_2 \cdot Q)h_{27} + (P_1 \cdot Q)h_{28}]
\]

\[
\tau_8 = \frac{(Q \cdot U)}{Q^2} (h_{27} + h_{28})
\]

\[
\tau_9 = -\frac{1}{Q^2}(h_{19} + h_{29})
\]

\[
\tau_{21} = \frac{1}{Q^2}h_{26}
\]

\[
\tau_{22} = \frac{1}{2Q^2(P_1^2 - P_2^2)} [(P_1 \cdot Q)h_{27} + (P_2 \cdot Q)h_{28}]
\]

\[
\tau_{24} = -\frac{1}{2Q^2}(h_{27} + h_{28}).
\]
• The $\tau_i$ in Eqs. (4.21) are independent of the gauge parameter $\xi$, since they are obtained from the HTL vertex, which is known to be gauge fixing independent.

• Given that, at finite temperature in the HTL approximation, the explicit expression for the vertex $\Gamma_\mu(P_1, P_2)$, Eqs. (2.7) and (2.8), is symmetric under the exchange of $P_1$ and $P_2$, the coefficients of (anti)symmetric vectors $T_\mu$ are also (anti)symmetric.

Note that $\tau_6$ and $\tau_{22}$ have a kinematic singularity when $P_1^2 \to P_2^2$. In this limit, these vectors point along the same direction

$$
T_6^\mu = -\gamma_\mu (P_1^2 - P_2^2) + (P_1 + P_2)_\mu \mathbf{Q},
$$

$$
T_{22}^\mu = (Q \cdot U) \left[ \gamma_\mu (P_1^2 - P_2^2) + (P_1 + P_2)_\mu \mathbf{Q} \right] - 2(P_1^2 - P_2^2) \mathbf{Q} \mathbf{U}_\mu,
$$

(4.22)

Therefore, they can be added up and the kinematic singularities are canceled out in the sum, as expected. This observation helps us redefine one of these vectors in such a way that the coefficients of both of them are independently free of the singularities.

$$
T_{22}^\mu \to T_{22}^{\prime \mu} = \frac{1}{2} \frac{P_1^2 - P_2^2}{P_1^2 - P_2^2} \left[ T_{22}^\mu - (Q \cdot U) T_6^\mu \right]
$$

$$
= (Q \cdot U) \gamma_\mu - \mathbf{Q} \mathbf{U}_\mu.
$$

(4.23)

With this redefinition, the new coefficients $\tau_6'$ and $\tau_{22}'$ are

$$
\tau_6' = \tau_6 + (Q \cdot U) \tau_{22}
$$

$$
= -\frac{1}{4} (h_1 - h_2) + \frac{(Q \cdot U)}{2Q^2} \left[ h_{27} + h_{28} \right]
$$

$$
\tau_{22}' = 2(P_1^2 - P_2^2) \tau_{22}
$$

$$
= \frac{1}{Q^2} \left[ (P_1 \cdot Q) h_{27} + (P_2 \cdot Q) h_{28} \right],
$$

(4.24)

which are explicitly free of kinematic singularities.

V. CONCLUSIONS

In this paper, we present the one loop calculation of the fermion-boson vertex in QED at finite temperature in the HTL approximation. In the most general form, the vertex can be written in terms of 32 independent Lorentz vectors. Following the procedure outlined by Ball and Chiu, 8 of the 32 vectors define the longitudinal vertex which satisfies the WTI relating it to the fermion propagator. The transverse vertex is written in terms of the remaining 24 vectors. The choice of these basis vectors is not unique but is a natural and straightforward extension of the $T = 0$ basis given in Ref. [14]. We have evaluated the coefficients of the basis vectors. Given that in this scheme, a kinematic singularity appears in two of the coefficients, we have modified the choice of one of the transverse vectors to simultaneously remove both singularities. Any non-perturbative vertex ansatz should reproduce Eqs. (4.21) and (4.24) in the weak coupling and high temperature regime. Therefore, Eqs. (4.21) and (4.24) should serve as a guide to construct a non-perturbative vertex in QED at finite, albeit large, temperature. Finally, it is important to also note that the present analysis is valid for any SU(N) gauge theory, (for instance QCD) provided that the thermal fermion mass includes the corresponding group factor [20].

A reliable non-perturbative vertex is essential for the study of dynamical symmetry breaking through the corresponding Schwinger-Dyson Equations. One may also be able to make useful predictions for the gauge theory contributions to the top-quark condensate scheme of electroweak symmetry breaking at finite temperature [23], leading to a better insight into the electroweak phase transition. All this is for the future.
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APPENDIX

Here, we explicitly compute the integrals involved in the calculations of Sect. IV. Capital letters are used throughout to denote four-momentum vectors. $K^\mu = (-1, \mathbf{k})$. We start with the function $G_{00}$

$$G_{00}(P_1, P_2) = \int \frac{d\Omega}{4\pi} \frac{1}{(P_1 \cdot \mathbf{K})(P_2 \cdot \mathbf{K})}. \quad (5.1)$$

Introducing the Feynman parameter $x$, we can write

$$G_{00}(P_1, P_2) = \int_0^1 dx \int_0^{4\pi} \frac{d\Omega}{4\pi} \frac{1}{(xP_1 + (1-x)P_2) \cdot \mathbf{K}^2} = \frac{1}{2} \int_0^1 dx \int_{-1}^1 d(cos \theta) \frac{1}{|u + v cos \theta|^2}, \quad (5.2)$$

where we have defined the functions $u$ and $v$ by

$$u \equiv xP_{10} + (1-x)p_{20}$$
$$v \equiv |xP_1 + (1-x)P_2|. \quad (5.3)$$

After integration over the polar angle $\theta$, we obtain

$$G_{00}(P_1, P_2) = \int_0^1 \frac{dx}{u^2 - v^2} = \int_0^1 \frac{dx}{|xP_1 + (1-x)P_2|^2}, \quad (5.4)$$

from where the result in Eq. (4.1) follows. Next, for the computation of the function $G_{0i}$, we require to compute integrals of the type

$$\int \frac{d\Omega}{4\pi} \frac{P_1 \cdot \mathbf{k}}{(P_1 \cdot \mathbf{K})(P_2 \cdot \mathbf{K})}. \quad (5.5)$$

Adding and subtracting $p_{10}$ in the numerator of the integral in Eq. (5.3), we get

$$\int \frac{d\Omega}{4\pi} \frac{P_1 \cdot \mathbf{k}}{(P_1 \cdot \mathbf{K})(P_2 \cdot \mathbf{K})} = -\int \frac{d\Omega}{4\pi} \left\{ \frac{p_{10} - P_1 \cdot \mathbf{k}}{(P_1 \cdot \mathbf{K})(P_2 \cdot \mathbf{K})} \right\} = -\int \frac{d\Omega}{4\pi} \frac{1}{(P_2 \cdot \mathbf{K})} - \int \frac{d\Omega}{4\pi} \frac{p_{10}}{(P_1 \cdot \mathbf{K})(P_2 \cdot \mathbf{K})}. \quad (5.6)$$

The first of the expressions in the second line of Eq. (5.6) is explicitly given by

$$\int \frac{d\Omega}{4\pi} \frac{1}{(P_2 \cdot \mathbf{K})} = -\frac{1}{2} \int_{-1}^1 d(cos \theta) \frac{1}{p_{20} + p_2cos \theta}$$
$$= -\frac{1}{2p_2} \ln \left( \frac{p_{20} + p_2}{p_{20} - p_2} \right)$$
$$= -L(P_2), \quad (5.7)$$

whereas the second of the terms in the second line of Eq. (5.6) is simply $-p_{10}$ times $G_{00}$. Therefore,
\[
\int \frac{d\Omega}{4\pi} \frac{\mathbf{p}_1 \cdot \hat{k}}{(P_1 \cdot K)(P_2 \cdot K)} = -p_{10}M(P_1, P_2) + L(P_2).
\]

(5.8)

Analogously,

\[
\int \frac{d\Omega}{4\pi} \frac{\mathbf{p}_2 \cdot \hat{k}}{(P_1 \cdot K)(P_2 \cdot K)} = -p_{20}M(P_1, P_2) + L(P_1),
\]

(5.9)

from where the result in Eqs. (4.6) follows. Finally, for the computation of the function \(G_{ij}\), we first compute an integral of the type

\[
\int \frac{d\Omega}{4\pi} \frac{(\mathbf{p}_1 \cdot \hat{k})^2}{(P_1 \cdot K)(P_2 \cdot K)}. \tag{5.10}
\]

Adding and subtracting \(p_{10}\) in one of the factors in the numerator of Eq. (5.10), we obtain

\[
\int \frac{d\Omega}{4\pi} \frac{(\mathbf{p}_1 \cdot \hat{k})^2}{(P_1 \cdot K)(P_2 \cdot K)} = \int \frac{d\Omega}{4\pi} \left\{ \frac{(p_{10} + \mathbf{p}_1 \cdot \hat{k} - p_{10})(\mathbf{p}_1 \cdot \hat{k})}{(P_1 \cdot K)(P_2 \cdot K)} \right\}
= - \int \frac{d\Omega}{4\pi} \frac{\mathbf{p}_1 \cdot \hat{k}}{(P_2 \cdot K) + p_{10}} \int \frac{d\Omega}{4\pi} \frac{1}{(P_2 \cdot K)}
+ p_{10}^2 \int \frac{d\Omega}{4\pi} \frac{1}{(P_1 \cdot K)(P_2 \cdot K)}. \tag{5.11}
\]

The first of the integrals in the second line of Eq. (5.11) is given explicitly by

\[
\int \frac{d\Omega}{4\pi} \frac{\mathbf{p}_1 \cdot \hat{k}}{(P_2 \cdot K)} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{2p_2} \int_{-1}^{1} d(\cos \theta) \frac{\cos \theta}{p_{20} + p_2 \cos \theta}
= -\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{p_2^2} + \left( \frac{p_{20}^2}{p_2^2} \mathbf{p}_1 \cdot \mathbf{p}_2 \right) L(P_2). \tag{5.12}
\]

On the other hand, the second of the terms in the second line of Eq. (5.11) is simply \(-p_{10}\) times \(L(P_2)\), whereas the third term is just \(p_{10}^2\) times \(G_{00}\). Therefore, the integral under scrutiny is given by

\[
\int \frac{d\Omega}{4\pi} \frac{(\mathbf{p}_1 \cdot \hat{k})^2}{(P_1 \cdot K)(P_2 \cdot K)} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{p_2^2}
- \left( \frac{p_{20}^2}{p_2^2} \mathbf{p}_1 \cdot \mathbf{p}_2 \right) L(P_2) - p_{10}L(P_2) + p_{10}^2M(P_1, P_2). \tag{5.13}
\]

Analogously,

\[
\int \frac{d\Omega}{4\pi} \frac{(\mathbf{p}_2 \cdot \hat{k})^2}{(P_1 \cdot K)(P_2 \cdot K)} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{p_1^2}
- \left( \frac{p_{10}^2}{p_1^2} \mathbf{p}_1 \cdot \mathbf{p}_2 \right) L(P_1) - p_{20}L(P_1) + p_{20}^2M(P_1, P_2). \tag{5.14}
\]

The last integral required is given by

\[
\int \frac{d\Omega}{4\pi} \frac{(\mathbf{p}_1 \cdot \hat{k})(\mathbf{p}_2 \cdot \hat{k})}{(P_1 \cdot K)(P_2 \cdot K)}, \tag{5.15}
\]

Adding and subtracting \(p_{10}\) in the first and \(p_{20}\) in the second of the factors in the numerator of the above integral, we get

\[
\int \frac{d\Omega}{4\pi} \frac{(\mathbf{p}_1 \cdot \hat{k})(\mathbf{p}_2 \cdot \hat{k})}{(P_1 \cdot K)(P_2 \cdot K)} = \int \frac{d\Omega}{4\pi} \frac{(p_{10} + \mathbf{p}_1 \cdot \hat{k} - p_{10})(p_{20} + \mathbf{p}_2 \cdot \hat{k}) - p_{20}}{(P_1 \cdot K)(P_2 \cdot K)}
= 1 + p_{10} \int \frac{d\Omega}{4\pi} \frac{1}{(P_1 \cdot K)} + p_{20} \int \frac{d\Omega}{4\pi} \frac{1}{(P_2 \cdot K)}
+ p_{10}p_{20} \int \frac{d\Omega}{4\pi} \frac{1}{(P_1 \cdot K)(P_2 \cdot K)}, \tag{5.16}
\]
which, by using Eqs. (5.1) and (5.7), can be written as

\[ \int \frac{d\Omega}{4\pi} (p_1 \cdot \hat{k})(p_2 \cdot \hat{k}) (P_1 \cdot \hat{K})(P_2 \cdot \hat{K}) = 1 - p_{10}L(P_1) - p_{20}L(P_2) + p_{10}p_{20}M(P_1, P_2), \] (5.17)

from where Eqs. (4.8) follow.