Topological gravity in dimensions two and four

Jack Morava

Abstract. Recent work on gravity in two dimensions generalizes naturally to four dimensions. This is a version of a talk at the 1999 operads conference in Utrecht; the bulk of the paper [§2] is examples.

1. Basic definitions

1.1 The (symmetric monoidal) two-category

\[
(\text{Gravity})_{d+1}
\]

has objects: compact oriented \(d\)-manifolds, with

- morphisms \(V_0 \to V_1\) : \((d+1)\)-manifolds \(W\) with \(\partial W \cong V_0^{op} \sqcup V_1\), and
- diffeomorphisms \(\tilde{W} \to W\) as two-morphisms.

The category \(\text{Mor}(V_0, V_1)\) with cobordisms from \(V_0\) to \(V_1\) as objects and diffeomorphisms (equal to the identity on the boundary) as morphisms, is a hom-object in this two-category. Disjoint union defines the monoidal structure, and the category has an orientation-reversing adjoint equivalence with its opposite.

1.2 The topological category

\[(\text{Gravity})_{d+1}\]

has compact Riemannian \(d\)-manifolds as objects, and the spaces

\[
\text{Mor}(V_0, V_1) := \bigsqcup_{V_0^{op} \sqcup V_1 \cong \partial W} \text{(Metrics/Diff)}(W)
\]

as its hom-objects. Alternately: a morphism is a \((d+1)\)-dimensional cobordism, together with (the equivalence class of) a Riemannian metric on it.

The group of diffeomorphisms which fix a frame at a point acts freely on the space of Riemannian metrics on a complete manifold, so the morphism spaces of \((\text{Gravity})_{d+1}\) are roughly just the classifying spaces \([12]\) of the morphism categories of \(\text{(Gravity)}_{d+1}\).

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When $d = 1$ the group of diffeomorphisms of a Riemann surface (of genus $> 1$) has **contractible** components, and its mapping-class group $\Gamma = \pi_0(\text{Diff})$ acts with **finite** isotropy on Teichmüller space, defining a **rational** homology isomorphism

$$BDiff \sim B\Gamma \sim \text{Teich} \times_\Gamma ET \to \text{Teich} \times_\Gamma \text{pt} = \mathcal{M}.$$ 

(Gravity)$_{1+1}$ is thus very similar to the category Segal constructed to define conformal field theory.

### 1.4

A monoidal functor from the topological gravity category to some simpler monoidal category, such as Hilbert spaces and trace-class maps, or modules over a ring spectrum, defines a **theory of topological gravity**. Replacing the Hom-objects in this category with their sets of components leads to TFT’s in the sense of Atiyah, and replacing the Hom-objects with their rational homotopy types leads to cohomological field theories.

## 2. Some examples

### 2.1

The ‘intersection homology’ of a connected surface

$$\Sigma \mapsto \ker[H^1(\Sigma, \mathbb{C}) \to H^1(\partial\Sigma, \mathbb{C})]$$

is a simple example: if $\Sigma \circ \Sigma'$ is the composition of two surfaces along a boundary component, then the induced map

$$\tau : B\text{Diff}(\Sigma) \to BU$$

fits in the commutative diagram

$$
\begin{array}{ccc}
B\text{Diff}(\Sigma) \times B\text{Diff}(\Sigma') & \longrightarrow & B\text{Diff}(\Sigma \circ \Sigma') \\
\tau \times \tau & & \tau \\
BU \times BU & \oplus & BU
\end{array}
$$

This defines a theory of topological gravity with values in a monoidal topological category having one object, with the $H$-space $BU$ of morphisms. [If we want to fuse along more than one boundary component, though, we need to be more careful.]

This functor has more structure: it takes values in symplectic lattices. The composition

$$B\text{Sp}(\mathbb{Z}) \to B\text{Sp}(\mathbb{R}) \sim BU \to B(U/\text{Sp}) \sim SO/U$$

is a rational homology isomorphism, so $\tau$ lifts to a map

$$\tau_T : B\text{Diff}(\Sigma) \to SO/U$$

which sends a surface to its harmonic one-forms with the complex structure defined by the Hodge $\ast$-operator. This is a form of the Abel-Jacobi-Torelli map

$$\Sigma \mapsto \text{SP}^\infty \Sigma : \mathcal{M} \to \mathcal{A}$$

which sends a surface to its Jacobian; note that the union of $\Sigma$ with its opposite defines a quaternionic object, which $\tau_T$ maps to zero.

### 2.2

Kontsevich-Witten theory is a much deeper example, with the (rationalized) complex cobordism ring-spectrum as target object. A toy version is easy to construct:
Suppose $\partial \Sigma$ has at most one component; then capping it off defines the closed surface $\Sigma_D := \Sigma \circ D$. The cobordism class of the bundle

$$[\Sigma_D] : \Sigma_D \times_{\text{Diff}(\Sigma)} E \text{Diff}(\Sigma) \to B \text{Diff}(\Sigma)$$

is primitive, in the sense that it behaves additively under composition: the pull-back

$$\mu^* [\Sigma \circ \Sigma'] \in MU^{-2}(B \text{Diff}(\Sigma) \times B \text{Diff}(\Sigma'))$$

under the composition

$$\mu : \text{Diff}(\Sigma) \times \text{Diff}(\Sigma') \to \text{Diff}(\Sigma \circ \Sigma')$$

is the sum

$$\epsilon^* [\Sigma_D] \otimes 1 + 1 \otimes \epsilon'^* [\Sigma'_D],$$

where

$$\epsilon : \text{Diff}(\Sigma) \to \text{Diff}(\Sigma_D)$$

extends the diffeomorphism by the identity [9].

This says we can pull $\Sigma \circ \Sigma'$ apart as if it were made of taffy: the standard family of quadratic cones in $\mathbb{R}^3$ glued to $(\Sigma \circ \Sigma') \times I$ defines a $\text{Diff}(\Sigma) \times \text{Diff}(\Sigma')$-equivariant cobordism from $\Sigma \circ \Sigma'$ to $\Sigma_D \sqcup \Sigma'_D$.

If we tensor with $\mathbb{Q}$ [and supress the grading] then the map $\tau_{kw}$ representing

$$\exp(\Sigma) \in MU_*(B \text{Diff}(\Sigma))$$

fits in the homotopy-commutative diagram

$$\begin{array}{ccc}
BDiff(\Sigma) & \mathbb{Q} & BDiff(\Sigma') \\
\tau_{kw} & \tau_{kw} & \tau_{kw} \\
MU_\mathbb{Q} & MU_\mathbb{Q} & MU_\mathbb{Q} \\
\end{array}$$

In both these examples, a surface is sent to some kind of configuration space of points: such constructions take unions to products. Proper Kontsevich-Witten theory involves much more complicated configuration spaces.

These constructions capture much of what’s known about the stable cohomology of moduli spaces. Mumford’s conjecture, for example, is equivalent to the assertion that $\tau_{kw}$ defines an isomorphism on rational cohomology.

2.3 The Floer homology $HF^*(Y)$ of a compact 3-manifold $Y$ (e.g. a homology sphere) is defined by the Chern-Simons functional on the space of connections $\mathcal{C}(Y)$ mod gauge equivalence on a (trivial) $G$-bundle over $Y$. It is periodically graded, and has a kind of Poincaré pairing. Following Cohen, Jones, and Segal [3], I assume it is defined by an underlying spectrum $\mathbf{HF}(Y)$. Because the space of connections satisfies

$$\mathcal{C}(Y_0 \sqcup Y_1) = \mathcal{C}(Y_0) \times \mathcal{C}(Y_1),$$

it follows that

$$\mathbf{HF}(Y_0 \sqcup Y_1) = \mathbf{HF}(Y_0) \wedge \mathbf{HF}(Y_1).$$

Atiyah [1] saw that Floer homology is a topological field theory: when $Y$ bounds $Z$, the space $A(Z)$ of Yang-Mills instantons on $Z$ defines (by restriction to $\partial Z$) a kind of Lagrangian cycle

$$[A(Z) \to \mathcal{C}(Y)] \in HF^*(Y),$$
and if
\[ \partial Z = Y^{op} \sqcup Y', \quad \partial Z' = Y'^{op} \sqcup Y'' \]
then \([A(Z)] \wedge [A(Z')]\) should map to \([A(\partial Z) \cup Y Z']\) under the pairing
\[ HF(Y^{op} \sqcup Y') \wedge HF(Y'^{op} \sqcup Y'') \rightarrow HF(Y^{op} \sqcup Y'') . \]

In fact Yang-Mills on \(Z\) presupposes a Riemannian metric, and there is a family
\[ A(Z) \times_{\text{Diff}(Z)} EDiff(Z) \rightarrow \mathcal{C}(\partial Z) \times BDiff(Z) \]
of Lagrangian cycles; its hypothetical class
\[ \tau_A : BDiff(Z)^+ \rightarrow HF(\partial Z) \]
should define a theory of topological gravity.

2.4 In honest Kontsevich-Witten theory [6,13] the analogue \(\tau_{kw}\) of \(\exp(\Sigma)\) is the class
\[ \sum_{n \geq 0} \langle \overline{M}_g^n \rangle / n! \in MU^*_Q(\overline{M}_g) , \]
where \(\langle \overline{M}_g^n \rangle\) is the cobordism class of a forgetful map to the Deligne-Mumford space of stable algebraic curves of genus \(g\), from a compactification of the space of smooth curves marked with \(n\) distinct points. Its characteristic number polynomial is
\[ \Phi_*(m_{tot}(-\nu_{fake}) \in H^* (\overline{M}_g, MU_Q) , \]
where \(\Phi\) is the forgetful map from the Deligne-Mumford-Knudsen space \(\overline{M}_g\), \(m_{tot}\) is the characteristic class defined by the total monomial symmetric function, and \(-\nu_{fake}\) is the sum of the tangent line bundles to the modular curve at its marked points. I’m indebted to Gorbounov, Manin, and Zograf for correcting my mistaken assertion [10] that \(\nu_{fake}\) is the formal normal bundle of \(\Phi\) : above the divisor \(\overline{M}_g^{g-k+1} \times \overline{M}_q^{1-k} \) on \(\overline{M}_d\) defined by curves with two irreducible components, one of genus zero, these two bundles differ stably by the sum of the pair of tangent lines at the double point.

I claim that \(\tau_{kw}\) respects a monoidal structure defined by Knudsen’s gluing map \(\mu\); in the simplest case this means that
\[ \overline{M}_g^{1+} \wedge \overline{M}_h^{1+} \rightarrow \overline{M}_g^{1+} \]
\[ MU_Q \wedge_Q MU_Q \rightarrow MU_Q \]
commutes, or equivalently that
\[ \mu^*(\overline{M}_g^{1+}) = \sum_{p+q=n} \langle \overline{M}_g^{p+1} \rangle \times \langle \overline{M}_h^{q+1} \rangle , \]
where $\langle \mathcal{M}_g^{r} \rangle \in MU^*_Q(\mathcal{M}_g)$ is defined by the partially forgetful morphism $\Phi_r : \mathcal{M}_g^{r} \to \mathcal{M}_g$. This follows because the diagram

$$
\bigsqcup_{p+q=a} \mathcal{M}_g^{p+1} \times \mathcal{M}_h^{q+1} \xrightarrow{\Phi \times \Phi} \mathcal{M}_g^{p+1} \times \mathcal{M}_h^{q+1} \xrightarrow{\mu} \mathcal{M}_g^{p+1} \times \mathcal{M}_h^{q+1}
$$

is a pullback in the category of smooth stacks, so

$$
\mu^* \Phi_* = (\tilde{\Phi}_1 \times \tilde{\Phi}_1)_* \tilde{\mu}^*
$$

by the base-change theorem [8] of Moerdijk and Pronk. Now $\tilde{\mu}^*$ pulls back tangent lines at marked points to tangent lines at marked points, so

$$
\mu^* (\nu^{(n)}_{fake}) = \nu^{(p)}_{fake} \otimes 1 + 1 \otimes \nu^{(q)}_{fake},
$$

and $m_{tot}$ is multiplicative, so

$$
\mu^* (\mathcal{M}_g^{n+}) = \mu^* \Phi_* m_{tot}(-\nu^{(n)}_{fake})
$$

is a sum of terms of the form

$$
(\tilde{\Phi}_1 \times \tilde{\Phi}_1)_* (m_{tot}(-\nu^{(p)}_{fake}) \times m_{tot}(-\nu^{(q)}_{fake})),
$$

QED.

2.5i Topological gravity coupled to the quantum cohomology of a smooth projective variety $V$ is a (largely) conjectural theory defined, in the terms suggested here, by

$$
\sum_{n \geq 0} \langle \mathcal{M}_g^{r}(V) \rangle / n! \in MU^*_Q(M_g^{r} \wedge V^{k+}).
$$

The configuration spaces are now suitable moduli spaces of stable maps [2] from curves to $V$, graded by degree (in $H_2(V, \mathbb{Z})$); but this grading will be supressed here. The representing morphism

$$
\tau_{kW}(V) : \mathcal{M}_g^{k+} \to [V^{k+}, MU_Q]
$$

hypothetically defines a functor to a monoidal category with objects $\mathbb{N}$, morphisms

$$
\text{mor}(j, k) = [V^{j+k+}, MU_Q],
$$

and compositions defined by a Poincaré trace

$$
[V^+ \wedge V^+, MU_Q] \to MU_Q;
$$

the superscript $+$'s indicate the addition of a disjoint basepoint, according to the conventions of homotopy theory. There is a natural monoidal functor $n \mapsto [V^{n+}, MU_Q]$ to the usual category of $MU_Q$-module spectra; this is the first two-dimensional case in which the target category for a topological gravity theory has not been slightly degenerate.

The simplest case of the monoidal axiom asserts the commutativity of

$$
\mathcal{M}_g^{r} \wedge \mathcal{M}_h^{r} \xrightarrow{\mu} \mathcal{M}_g^{r} \wedge \mathcal{M}_h^{r} \xrightarrow{\mu} \mathcal{M}_g^{r} \wedge \mathcal{M}_h^{r} \xrightarrow{\mu} \mathcal{M}_g^{r} \wedge \mathcal{M}_h^{r}
$$

$[V, MU_Q] \wedge MU_Q [V, MU_Q] \to MU_Q$.
the map from $\overline{M}_0^{g+3}$ to $[V^{3+}, \text{MU}_Q]$ then defines a new (quantum) product. When $V$ is a point, this agrees with the usual product on $\text{MU}_Q$, after a renormalization which shifts the identity element: the simplified model of §2.2 multiplies the identity by $q = \exp(CP_1)$, while in proper Kontsevich-Witten theory the analogous $q$ is a generating function for the cobordism classes of the spaces $\overline{M}_0^{g+3}$. Here the word renormalization is meant quite literally: $q$ accounts for the cloud of virtual particles familiar in other quantum contexts.

A relative version of this construction uses Gromov-Witten classes

$$\langle \overline{M}_g^{n;k}(V) \rangle \in [\overline{M}_g^{k+}(V) \wedge V^{n+}, B\text{BT}^{n+} \wedge \text{MU}_Q]_*$$

which record the tangent lines at the marked points; summing over $n$-fold cap products with a cycle $z \in H_*(V, H^*(\text{BT}))$ defines a theory based on configuration spaces with marked points restricted to lie on $z$. This recovers the Gromov-Witten potential and the WDVV family of quantum multiplications.

2.5ii Here is a sketch of a symplectic analogue, restricted for simplicity to surfaces with at most one boundary component. Let $\text{Acx}$ be the (contractible) space of almost-complex structures compatible with the symplectic structure on $(\Sigma, \omega)$. A smooth map $u$ from a closed $\Sigma$ to $V$ will be called \textbf{symplectic} if $u^*\omega$ is nondegenerate (and hence symplectic) on $\Sigma$; such a map induces a monomorphism of tangent bundles. Let $\text{Smpl}(\Sigma; V)$ be the manifold of these maps, and let $\text{Smpl}^*(\Sigma; V)$ be the bundle over $\text{Smpl}(\Sigma; V) \times \text{Acx}$ with fiber at $(u, J_V)$ the (contractible) space of almost-complex structures on $\Sigma$ compatible with $u^*\omega$. $\text{Diff}(\Sigma)$ acts freely on this space, with quotient the bundle of spaces of pseudoholomorphic maps from $\Sigma$ to $V$ over $\text{Acx}$. More generally, if $D \subset \Sigma$ is a smooth disk in the surface, and $x$ is a point in its interior, let $\text{Diff}(\Sigma, D, x)$ be the group of orientation-preserving diffeomorphisms of $\Sigma$ which preserve both $D$ and $x$. There is then a bundle $\text{Smpl}^*(\Sigma, D, x; V)$ with fiber the space of almost-complex structures on $\Sigma - \partial D$ compatible with $u^*\omega$; to be precise, we want almost-complex structures on both sides, which extend continuously to $\partial D$, but which need not agree there. Restriction to $D$ is a Fredholm map from $\text{Smpl}^*(\Sigma, D, x; V)$ to

$$\text{Acx} \times \text{Diff}(\Sigma, D, x) \overline{\text{L}}V,$$

the last term in the product being the universal cover of the free loopspace of $V$.

The range of this map is homotopy-equivalent to the product of $B\text{Diff}(\Sigma)$ with the Borel construction [with respect to the action of the circle by rotating loops] on $\overline{\text{L}}V$, and its index [11 Ch. 8] is $(1 - g)\text{dim}_C(V) + u^*(c_1(V))|\Sigma|$. The pullback of its cobordism class to the formal normal bundle to the fixed-point set is a kind of Gromov-Witten invariant in $\text{MU}^*(B\text{Diff}(\Sigma)^+ \wedge \text{HF}(V))$, where $\text{HF}(V)$ is a pro-spectrum [as in the appendix to [3]] constructed from the normal bundle of the fixed loops. This is a conjectural analogue, for the Floer homotopy type associated to the area functional on a free loopspace, of the theory sketched in §2.3.

2.6 Finally, I want to mention that the space $H^1(\Sigma, G)$ of principle bundles [with compact Lie structure group $G$] defines a potential Gromov-Witten map

$$H^1(\Sigma, G) \times \text{Diff}(\Sigma) E\text{Diff}(\Sigma) \rightarrow H^1(\partial \Sigma, G) \times B\text{Diff}(\Sigma)$$

which [as far as I know] has not yet been made the basis for a theory of topological gravity . . .
3. General nonsense

Monoidal functors between monoidal categories form a monoid, much as homomorphisms between abelian groups form an abelian group. Manin and Zograf [7] suggest that we think of these families of theories as parametrized by the Picard group of invertible objects. Such objects can be identified with the points of Spec $\text{MU}_\mathbb{Q}$, which are one-dimensional formal group laws; but there is no natural way to compose them. Kontsevich-Witten theory [4] suggests the natural parametrizing object is the Hopf algebra $\mathcal{Q}$ of Schur $Q$-functions: these are Hall-Littlewood symmetric functions of the eigenvalues of a positive-definite matrix $\Lambda$, evaluated at $t = -1$.

The Kontsevich-Witten genus $MU \to \mathcal{Q}$ defines a formal group law with $Q$-function coefficients; its exponential (aside from normalization) is the asymptotic expansion as $\Lambda \to +\infty$ of the Mittag-Leffler exponential

$$\sum_{n \geq 0} \frac{\text{Tr} \Lambda^n}{\Gamma(1 + \frac{1}{2n})}.$$ 

There is a natural twisted charge one action of the Virasoro algebra on the $Q$-functions, and the image under $\tau_{\text{kw}}$ of the fundamental class $\exp(\sum [M_g]) \in H_*(\text{SP}_\infty, \mathbb{Q})$ of the space of not-necessarily-connected curves is an $sl_2$-invariant highest-weight vector, or vacuum state.

Recently Eguchi et al., Dubrovin, Getzler [5], and others have begun to extend this fundamental result of Kontsevich-Witten theory to topological gravity coupled to quantum cohomology. The relevant Virasoro representation appears to be defined on a group of loops on the torus $H^*(V, \mathbb{R}/\mathbb{Z})$, twisted by the endomorphism $\frac{1}{2}H + tX$, where $H, X, Y$ generate the standard $sl_2$ action on the Hodge cohomology of $V$, and $t = c_1(V)/\omega$ depends on the Kähler class.

This twisted torus is mysterious even when $V$ is a point. The adjoint operation on $(\text{Gravity})_{1+1}$ defines an involution on the Picard group of invertible theories, and the analogy with Abel-Jacobi theory suggests that $\mathcal{Q}$ represents its skew-adjoint part. [The Hopf algebra defined by the cohomology of $BU$ represents the Witt ring functor, defined on commutative rings by $A \mapsto \mathbb{W}(A) = (1 + tA[[t]])^\times$; away from two, the involution sending $a(t) \in \mathbb{W}(A)$ to $a(-t)$ splits $\mathbb{W}$ into eigenspaces. The cohomology of $\text{SO}/U$ is the negative eigenspace.]
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Department of Mathematics, Johns Hopkins University, Baltimore, Maryland 21218

E-mail address: jack@math.jhu.edu