Stability analysis of a round cross-section rod from Shape Memory Alloys at reverse phase transition

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Abstract. The mechanisms that cause loss of stability of Shape Memory Alloy rods during the reverse phase transformation under a constant load for various methods of preparing the rods before starting the reverse phase transition were considered. The problem was solved for non-, singly- and twice- coupled statements and for the concepts of fixed and varied external loads taking into account the different resistance under tension and compression.

1. Introduction
1.1. Shape Memory Alloys
Shape Memory Alloys (SMAs) are functional materials. They can change their thermomechanical performance in accordance with external conditions such as temperature and stress levels. The unique properties of SMAs are associated with occurring phase transition and structural transformation. Due to the Shape Memory Effect that consist to recovery certain shape with help heat through prespecified temperature interval, SMAs find commercial application in different fields [1, 2, 3, 4, 5].

1.2. Motivation
Experiments [6] show that rods of SMAs lose their stability under loads being several times less than the critical force obtained for an elasticity rod with martensitic (minimum) modulus of elasticity. The Euler’s solution for the elasticity rod can be used as the least upper bound for a set of absolute values of critical forces or critical lengths being under consideration. In order to obtain correct solution of the problem of stability, we should use a nonlinear system of constitutive equations [7, 8] describing processes of phase-structural deformation. Besides, the phenomena of different resistance of SMAs under tension and compression [9] and the bulk effect of phase transition also take into account. To describe a model of structural deformation, an analog of the plastic flow theory offered in [10] is used.

1.3. Stability
In this work, the Euler quasi-static method is regarded as a stability loss criterion. It consists in considering the form of equilibrium adjacent to the initial (linear) one. If a nontrivial form of equilibrium exists, this means that the rod buckles or loses its stability. Such an approach allows the problem to be solved in the linearization formulation with respect to the little variation of deflections.
1.4. Conceptions
Let us consider two conceptions: fixed load (FL) and variated load (VL). In the conception VL, let us assume that load variation and curvature increment associated with rod buckling have one and the same order of smallness. In this case, critical force is defined as the least of all the forces for which at least at one value of the load variation a loss of stability occurs. In the conception FL with the appearance of adjacent form equilibrium, the cross section may be divided into two single-connected areas where stress variations have different signs. In the conception VL, one area occupies the whole cross section and sign of stress variation within the whole cross section is constant.

1.5. Statements
The problem is investigated in three statements depending on the variables that can be variated. Phase composition parameter in the non-coupled statement is not changed during buckling and its increment is equal to zero. Solution that was obtained in the non-coupled statement coincides with the Euler critical force for elastic rod with martensitic (minimum) modulus of elasticity. As stated below, the elastic solution is the precise upper bound of the set of all critical forces for the considered statements. In the singly-coupled statement, the volume fraction of the martensitic phase depends on stress increment and is changed in case of loss of stability but the temperature remains constant. The twice-coupled statement includes the singly-coupled one and takes into account absorption of latent heat and heat emission related to dissipative terms during buckling.

2. Constitutive equations

\[ d\varepsilon = d\varepsilon^e + d\varepsilon^{phst} \quad d\varepsilon^{phst} = \varepsilon^0 dq + d\varepsilon^{ph'} + d\varepsilon^{st} \]  \hspace{1cm} (1)

\[ \varepsilon^e = \frac{\sigma}{E(q)} \quad \frac{1}{E(q)} = \frac{q}{E_M} + \frac{1-q}{E_A}, \quad 0 \leq q \leq 1 \]  \hspace{1cm} (2)

\[ d\varepsilon^{ph'} = \omega dq, \quad \omega = \frac{\varepsilon^{phst}'}{q} \]  \hspace{1cm} (3)

\[ d\varepsilon^{st} = \phi_j(q_{st}, \sigma) d\sigma \]  \hspace{1cm} (4)

\[ \chi(S_{ij}) = \left[ \frac{\varepsilon^{phst}_{ij}}{\rho D(\mu_e)} \right]_{max}; \quad \chi(0) = 0; \quad \chi' (x) > 0, \quad x > 0; \quad \chi'' (x) > 0, \quad x > 0 \]  \hspace{1cm} (5)

\[ S_{ij} d\sigma'_{ij} > 0, \quad S_{ij} = \sigma'_{ij} - r_{ij} \]  \hspace{1cm} (6)

\[ dr_{ij} = g d\varepsilon^{st}_{ij} \]  \hspace{1cm} (7)

\[ q = \frac{1}{2} \left( 1 - \cos(\pi t) \right) \]  \hspace{1cm} (8)

\[ t = 1 - \frac{T - A_f^s}{A_f^s - A_s^f}, \quad A_s^f = A_s^0 + \frac{\omega + Z(\sigma) + \varepsilon^0}{\Delta S}, \quad A_f^0 - A_s^0 = A_f^0 - A_s^0, \quad Z(\sigma) = \frac{E_A - E_M}{2E_A E_M} \sigma^2 \]  \hspace{1cm} (9)

\[ k_q \Delta T = C_\sigma \dot{T} + T \theta \dot{\sigma} - \left( \Delta U + \varepsilon^0 \sigma + \omega \sigma + Z(\sigma) \right) \dot{q} - \sigma \dot{\varepsilon}^{st} \]  \hspace{1cm} (9)

\( \varepsilon, \varepsilon^e, \varepsilon^{ph}, \varepsilon^{st} \) are axial components of total, elastic, phase and structural strains; sign "\( \varepsilon^0 \)" mark deviator part of tensor; \( \varepsilon^0 \) is the linear part of the deformation bulk effect of phase transition; \( \sigma \) an axis component of stress; \( q \) is the volume fraction of the martensitic phase (phase composition parameter) which is defined from the equation (8), \( q_{st} \) is part of volume that
undergoes structural transformation; $E(q)$, $E_A$, $E_M$ is the modulus of elasticity and its values for austenitic and martensitic phase states accordingly; $\rho_D(\mu_\varepsilon)$ is the material function returning the maximum value of structural deformation corresponding to totally detwinned martensite that depends on a type of deformed state, $\mu_\varepsilon$ is parameter defining type of deformed state, where $\rho_D = \rho_D(-1)$, $\rho_D = \rho_D(1)$; $\sigma_{ij}$ is stress tensor; $\varphi(S_i)$ is the material function which defines a type of surface loading, $S_{ij}$ is active stress, $S_{ij} = \sqrt{\frac{3}{2}}S_{ij}S_{ij}$ is active stress intensity, $r_{ij}$ is translation stress; $A_s^0$, $A_f^0$, $A_s^\sigma$ are temperatures of start and finish of reverse phase transition for unloading state and temperature of finish of reverse transition under stress $\sigma$, $\Delta S$ is difference between entropy volume densities for austenite and martensite phases; $k_q$, $C_\sigma$, $\theta$, $\Delta U$ are heat conductivity, heat capacity per unit volume under constant stress, thermal expansion coefficient and volume density of latent heat of phase transition; $g$ is the material constant defining slope of line approximating the segment of the reverse loading diagram with only translational strain hardening.

For equiatomic titanium nickelide: $\Delta U = 15000 \div 30000 \ [J/kg]$, $\Delta S = 50 \div 100 \ [J/(kg \cdot K)]$, $C_\sigma = 500 \ [J/(kg \cdot K)]$, density $\rho_{NiTi} = 6440 \ [kg/m^3]$. The following parameter values are obtained from the experiment: $\rho_D(1) = \rho_D(1) = 2\rho_D(1) = 0.043$, $g = 3066.4 \ [MPa]$.

3. Initial phase-structural strain

This section describes a method for obtaining initial strain as a result of reverse loading before starting the reverse phase transition.

Let the rod in martensitic phase be undeformed, i.e. it consists of the twinned martensite. These states are conveniently described by points on a diagram with axises $(\varepsilon_{phst}'$, $\sigma)$. For example, the point $(0,0)$ in this diagram is corresponding to the state of chaotic (twinned) martensite.

At the beginning, the rod is accumulating a positive strain $\varepsilon^{phst}_1$ under stresses monotonically increasing from zero to $\sigma_1$. After that, the rod is completely unloaded. The further loading path consists of steps following each other which we call Cases. Then, monotonic loading takes place in the field of compressive stresses. The initial strain $\varepsilon^* = \varepsilon^0 + \varepsilon(\varepsilon^{phst}_1, \sigma)$, where the deviator component $\varepsilon$ belongs to the following intervals:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{loading_path.png}
\caption{Loading path}
\end{figure}
Case III. Elastic unloading until the exit to Case II, which is located between the points \((\varepsilon_1^{\text{phst}'}, 0)\) and \((\varepsilon_1^{\text{phst}'}, \sigma_s)\).

Case II. Purely translational hardening up to the exit to section I concluded between the points \((\varepsilon_1^{\text{phst}'}, \sigma_s)\) and \((−\frac{\rho_D}{\rho_{D_1}}\varepsilon_1^{\text{phst}'}, \sigma_3)\).

Case I. Combined hardening when the point moves along the martensitic non-elasticity diagram in the direction of decreasing stresses until the beginning of plastic deformation.

\(\sigma_s\) is the stress at which condition (5) is satisfied for the first time. \(\sigma_3\) is stress at which the point \((\varepsilon, \sigma_3)\) goes from the curve corresponding to the kinematic hardening (Case II) to the diagram of martensitic non-elasticity corresponding to the combined hardening due to the monotonic decrease in stresses. In such a case, the following condition is fulfilled 

\[−\frac{\varepsilon}{\rho_{D_2}} = \varepsilon_1^{\text{phst}'}.\]

From the conditions (5) and (6) follows that additional structural transformation associated to negative increment of stress occurs if point \((\varepsilon, \sigma)\) belongs to curve II and \(\sigma > \sigma_3\) or if such a point belongs to curve I and \(\sigma < \sigma_3\). In Case III, additional structural deformation does not occur.

Thus, the possibility of additional structural deformation depends on the initial tensile strain acting at the time of the reverse phase transition of stress and on the sign of its increment.

In various Cases, functions \(\phi_j\) depending on location of point \((\varepsilon, \sigma)\) that determines stress-strain state are used for analytical recording of the increment of the structural component of strain.

\[
\begin{align*}
\phi_1(q_{st}, \sigma) &= \rho_{D_2}q_{st}\varphi_2(|\sigma|) \\
\phi_2(q_{st}) &= \frac{\varphi_2}{\varphi} \\
\phi_3 &\equiv 0
\end{align*}
\]

Note that \(q_{st} = 0\), when \(\delta\sigma > 0\). The subscript \(j\) correspond to considering Case.

\(\varphi_2(|\sigma|)\) – material function approximate diagram martensitic non-elasticity. In [11], it was proposed to use the gamma distribution to approximate the martensitic non-elasticity diagram. In this paper, a comparison was made between the Weibull distribution (old approach) and gamma distributions. It is established that for large strain values \(\varepsilon_1^{\text{phst}' >} \frac{d}{\rho_{D_2}}\) the best approximation in terms of statistics corresponds to the gamma distribution. Thus

\[
\varphi_2(x) = \gamma(x, d, \alpha) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1}e^{-t}dt
\]

with the scale parameter \(d = 34.66 \, [MPa]\) and shape parameter \(\alpha = 7.9\).

4. Phase-structural strain before loss of stability

At the beginning, the rod loaded by a constant force is uniformly heated from the temperature of start of reverse phase transition \(A^p_\sigma\). It is expected that this process is sufficiently slow to ensure the uniform distribution of temperature inside the rod. Thus, the volume fraction of the martensitic phase and the axis component of phase-structural strain are calculated with the help of equations (8) and (3). It should be noted that structural deformation does not occur in the case when stresses are constant and hence \(d\sigma = 0\). Solving equation (3) with the initial condition \(\varepsilon_1^{\text{phst}'}(q = 1, \sigma) = \varepsilon\), we obtain phase-structural component strain \(\varepsilon_1^{\text{phst}} = \varepsilon^*q\).

5. Disturbed process

5.1. Variation of phase composition parameter

In order to find the variation of the phase composition parameter, we use equation (8).

\[
\delta q = \frac{\Upsilon_j(q_{st}, \varepsilon^*, \sigma)}{\Theta(q, \varepsilon^*, \sigma)} \delta\sigma
\]
\[ \Upsilon_j(q_{st}, \varepsilon, \sigma) = \varepsilon + B\sigma + \left( \frac{1}{q} - \frac{\Delta S}{C_{\sigma}} \right) \sigma \phi_j(q_{st}, \sigma), \quad B = \frac{E_A - E_M}{E_A E_M} \]

\[ \frac{1}{\Theta(q, \varepsilon, \sigma)} = \frac{\lambda(q)}{\Delta S \left( 1 + \frac{\lambda(q)}{C_{\sigma}} \left( \Delta U + \varepsilon + Z(\sigma) \right) \right)}, \quad \lambda(q) = \frac{\pi \sqrt{q(1-q)}}{A_0} \]

For the singly-coupled statement \( \frac{1}{\Theta} = \lambda(q) \). Function \( \Theta > 0 \) for nitinol and many other SMAs regardless of the values of its variables. It is easily to prove that \( \text{sign}(\delta q) = \text{sign}(\Upsilon_j) \) \( \text{sign}(\delta \sigma) \).

The variation of the phase composition parameter and, therefore, the increment of phase strain indirectly depends on the method of obtaining the initial strain.

5.2. Incrementation of elastic, phase and structural strains

Additional phase transition takes place only if \( \delta q < 0 \). By varying equation (2), we get

\[ \delta \varepsilon = \frac{\delta \sigma}{E(q)} + B\sigma H(-\delta q) \delta q \]

where \( H(x) \) is Heaviside unit step function.

Variation of phase strain is determined from equation (3):

\[ \delta \varepsilon_{ph} = \varepsilon^* H(-\delta q) \delta q \]

Using (4), we get

\[ \delta \varepsilon_{st} = \phi_j(q_{st}, \sigma) H(-\delta \sigma) \delta \sigma \]

5.3. Incrementation of total strain

Let the rod have round cross section with a radius \( h \ll l \), where \( l \) - length of the rod. We enter a rectangular Cartesian coordinate system \( xOy \) in each cross section of the rod. Axises \( Ox \) and \( Oy \) are lines of symmetry. Plane section hypothesis is suitable for investigation the problem because we consider linearized equations in relation to small variation of rod deflection.

\[ \delta \varepsilon = \delta \varepsilon_0 + y \delta \kappa \quad (11) \]

where \( \delta \varepsilon_0 \) is variation of axis strain’s component, \( \delta \kappa \) is variation of curvature.

Combining (1) and (11), we get the equation in increment of stress. By resolving it, we get:

5.4. Variation of stresses

\[ \delta \sigma = \frac{\zeta_0 - \zeta}{\Psi_j} h \delta \kappa, \quad -1 < \zeta < \zeta_0 \]

\[ \delta \sigma = \frac{\zeta_0 - \zeta}{\Psi_2} h \delta \kappa, \quad \zeta_0 < \zeta < 1 \]

\[ \Psi_j = \frac{1}{E(q)} + \frac{\Upsilon_j(\varepsilon^* \sigma)}{\Theta(q, \varepsilon^* \sigma)} (\varepsilon^* + B\sigma) H((-1)^{k+1} \Upsilon_j) \]

where \( \zeta = \frac{y}{h} \in [-1, 1] \) is dimensionless coordinate in cross section; \( \zeta_0 = \frac{y_0}{h} = -\frac{\delta \varepsilon_0}{2\varepsilon} \) is the value of \( \zeta \) in the conception FL determining the location of neutral axis on which \( \delta \sigma = 0 \); superscript to function \( \Psi_j \) corresponds to two areas of cross section where stress increments from the loss of stability have different signs. If \( \delta \sigma < 0 \) then additional loading take place and \( k = 1 \) if this not the case that \( \delta \sigma < 0 \) and \( k = 2 \). It is easy to show that \( H(-\delta q) = -H((-1)^{k+1} \Upsilon_j) \).
6. Solution

6.1. Neutral axis

In the conception FL, variation of critical force is equal to zero. This condition can be written in integral form:

$$\delta P = \int_{F} \delta \sigma dS = 0$$

Which after integration, if $$\Psi_{j}^{1} \neq \Psi_{j}^{2}$$, takes the following form

$$\frac{1}{2}(1 + \frac{1}{2} \zeta_{0}^{2}) \sqrt{1 - \zeta_{0}^{2}} + \frac{1}{2} \left( \arcsin(\zeta_{0}) - \frac{\pi}{2} \frac{\Psi_{j}^{1} + \Psi_{j}^{2}}{\Psi_{j}^{1} - \Psi_{j}^{2}} \right) \zeta_{0} = 0$$

The resulting transcendental equation on $$\zeta_{0}$$ can not be solved analytically. Therefore we solve it using numerical methods. If $$\Psi_{j}^{1} = \Psi_{j}^{2}$$, then $$\zeta_{0} \equiv 0$$.

For the conception VL, on under the condition that $$0 < |\delta P|/\delta \kappa < \infty$$ and from the definition of critical load given at the beginning of this paper, it follows that the critical force is reached if the sign of the load variation is constant in the entire cross section, i.e. $$\zeta_{0} = \pm 1$$.

6.2. Critical length

From the equation for the variation of the bending moment

$$\delta M = h^{2} \int_{F} \sigma \zeta dS = P \delta w$$

After integration we obtain the differential equation for the variation of deflection

$$\delta w'' + \eta \delta w = 0$$

FL:

$$\eta = 2 \sqrt{\pi s} \Psi_{j}^{1} \Psi_{j}^{2} \left( (\Psi_{j}^{1} - \Psi_{j}^{2}) \left( \frac{1}{2} (2 \zeta_{0}^{2} - 5) \zeta_{0} \sqrt{1 - \zeta_{0}^{2}} - \arcsin(\zeta_{0}) + \frac{\Psi_{j}^{1} + \Psi_{j}^{2}}{\Psi_{j}^{1} - \Psi_{j}^{2}} \right) \right)^{-\frac{1}{2}}$$

VL:

$$\eta = \frac{2 \sqrt{\pi s} \Psi_{j}}{k}, \quad \Psi_{j} = \max_{k} \Psi_{j}^{k}$$

with boundary condition for hinge support $$\delta w(0) = \delta w(l) = \delta w''(0) = \delta w''(l) = 0$$. The reader will have no difficulty in showing that minimal critical length

$$l = \frac{\max \{ \eta(\sigma, \varepsilon, \epsilon) \}}{q}$$

where $$q \in [0, 1]$$.

6.3. Analysis

Non-dimensional variables $$s = -\sigma$$ and $$L = \frac{l}{h}$$ are considering. $$L_{E}$$ - Euler critical length (non-coupled statement); 1, 2 - singly- and twice- coupled statements; I, II, III - Cases.

Figures 2 and 3 shows the relationship between critical length and stress for the value $$\varepsilon_{1}^{phst} = 0.038$$.

Figure 4 shows that the additional loading zone is not less than the additional unloading zone. This is due to the predominance in this region of additional phase transformations in Case III and phase-structural transformations in Case I, II.

Figure 5 shows how the critical value of the phase composition parameter in conception FL changes depending on the acting stress. It should be noted that in case II, the loss of stability occurs to a greater extent due to additional structural transformation (q = 1).

The gaps in all the figures given correspond to the points $$-\sigma$$ and $$-\sigma$$ where transitions between the Cases occur. At the first point, an additional structural strain is added abruptly, and at the second its value suffers a gap associated with the transition from kinematic hardening to combined one.
Critical lengths were obtained for an SMA rod in case of reverse phase transition under a constant load. All of them do not exceed the Euler critical length which corresponds to the non-coupled statement. Three possible options for additional phase-structural deformation are considered. It was established that additional structural deformation can occur only in Cases I and II.

The maximum contribution of structural strain increment to variation of total strain occurred in Case II. In Case III, only phase and elastic strains took place.

Depending on the method of preparation of the rod, different critical lengths can be obtained at the same stress (different Cases can correspond to one stress, for different $\varepsilon_1^{phst}$).

The twice-coupled statement and the conception FL always give somewhat larger values of the critical length (ceteris paribus) in comparison with the singly-coupled statement and the conception VL respectively.

It is also important to note that when considering two rods, one of which was prepared using martensitic non-elasticity, and the second was prepared in accordance with the method described in Case II, the critical length for the second rod will be significantly less than for the first, this
can be clearly seen in fig. 2 and fig. 3 at the point corresponding to the transition from the case of $II$ to the case of $I$.

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