ENHANCED ADJOINT ACTION AND THEIR ORBITS FOR THE GENERAL LINEAR GROUP

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Abstract. We studied an enhanced adjoint action of the general linear group on a product of its Lie algebra and a vector space consisting of several copies of defining representations and their duals. We determined regular semisimple orbits (i.e., closed orbits of maximal dimension) and the structure of enhanced null cone, including its irreducible components and their dimensions.

Introduction

Let $G$ be a reductive algebraic group over the complex number field $\mathbb{C}$, and $\mathfrak{g}$ its Lie algebra. The adjoint action of $G$ on $\mathfrak{g}$ is a basic tool for many aspects of representation theory, and also useful for invariant theory, theory of singularities, and so on.

In [AH08], [AHJ11], [Joh10], Achar-Henderson and Johnson considered an enhanced version of nilpotent varieties and classified the nilpotent orbits (there are only finitely many of them). Also, in [Kat09], Kato considered an “exotic” nilpotent cone and give the Deligne-Langlands theory for those exotic nilpotent orbits. There are many related works based on algebraic geometry, combinatorial theory, and theory of character sheaves ([Tra09], [FGT09], [HT12], [FN16], [Ros12]).

In these papers, enhancement of the nilpotent cone is only “one-sided” to get a criterion of finiteness of orbits. However, from the viewpoint of symmetric spaces, it seems better to enhance all the adjoint orbits in two-sided directions. In this respect, we have already had two results ([Oht08], [Nis14]), which relates the orbit structure of two enhanced actions. But, so far, we have not known the explicit orbit structures of individual enhanced adjoint actions.

In this paper, we begin to study (two-sided) “enhanced adjoint action” of $G$ for $G = \text{GL}_n(\mathbb{C})$ (type A). The big difference from those one-sided enhanced (or exotic) ones is that there appear infinitely many nilpotent orbits. So the analysis becomes more difficult, but involving less combinatorics. In easiest cases, we can describe enhanced adjoint orbits fairly explicitly, but in general, we have obtained coarser structures, like regular orbits of maximal possible dimensions, structure of invariants, irreducible components of nilpotent variety.

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To state the main results more explicitly, let us introduce some notations. Let $V = \mathbb{C}^n$ be a vector space of dimension $n$. We consider a natural action of $G = \text{GL}(V) = \text{GL}_n(\mathbb{C})$ on

$$W = (\mathbb{C}^n)^{\oplus p} \oplus (\mathbb{C}^*)^{\oplus q} \oplus M_n = M_{n,p} \oplus M_{q,n} \oplus M_n,$$

with the action of $g \in G$ given by

$$g \cdot (B,C,A) = (gB,Cg^{-1}, \text{Ad}(g)A) \quad \text{for} \quad (B,C,A) \in M_{n,p} \oplus M_{q,n} \oplus M_n.$$

Thus, the part $M_n$ is considered to be $\mathfrak{gl}_n(\mathbb{C})$ and the action is the adjoint action. For the other parts, $M_{n,p}$ is a $p$-copy of natural representations and $M_{q,n}$ is a $q$-copy of its dual. So the space $W$ is the full enhanced adjoint representation we explained.

There are obvious invariants for the action. We put

$$\tau_k := \text{trace } A^k \quad (1 \leq k \leq n - 1),$$

$$\gamma_{i,j}^\ell := (CA^\ell B)_{i,j} \quad (0 \leq \ell \leq n - 1, 1 \leq i \leq q, 1 \leq j \leq p).$$

These invariants are generators of the whole invariant ring $\mathbb{C}[W]^G$, which seems to be known for experts in various forms including quiver theory (see Theorem 1.1). Thus, we can define a quotient map $\pi_W : W \to \mathbb{C}^n \times (M_{q,p})^n$ using these invariants (see Eq. (2.3)).

If $p = 1$ or $q = 1$, the quotient map has a very good property. Namely, we get

**Theorem 0.1** (Theorem 2.1 (2)). If $p = 1$ or $q = 1$, the map $\pi_W : W \to \mathbb{C}^n \times (M_{q,p})^n$ is an affine categorical quotient map (note that $M_{q,p} = \mathbb{C}^p$ or $\mathbb{C}^q$). In particular, the quotient map $\pi_W$ is coregular, and $\mathbb{C}[W]^G$ is a polynomial ring generated by the fundamental invariants listed above.

For general $p \geq 1$ and $q \geq 1$, the following theorem gives a generic structure of enhanced adjoint orbits.

**Theorem 0.2** (Theorem 2.1 and Corollary 2.2). The dimension of the image $\dim \text{Im } \pi_W$ is equal to $n(p+q)$, and a general fiber of $\pi_W$ is a single $G$-orbit of dimension $n^2$. This implies that general orbits for the enhanced adjoint action are closed of dimension $n^2$.

These orbits are called regular semisimple orbits. Another extreme cases are nilpotent orbits. We investigate the null cone $\mathfrak{N}(W) \subset W$ in §3, and get the following results.

**Theorem 0.3** (Theorem 3.3). The null cone $\mathfrak{N}(W)$ is reducible and it has $n+1$ irreducible components $C_k \subset \mathfrak{N}(W)$ $(0 \leq k \leq n)$ given in Lemma 3.2. The dimension of the null cone is $n^2 - n + n \cdot \max\{p,q\}$ and $\mathfrak{N}(W)$ is equi-dimensional if and only if $p = q$.

Finally, we get the structure of general (enhanced) nilpotent orbits contained in each component $C_k$ in Theorem 3.4.

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1. Setting

Let $V = \mathbb{C}^n$ be a vector space of dimension $n$. We consider a natural action of $G = \text{GL}(V)$ on

$$W = W(p, q; r) := V^p \oplus (V^*)^q \oplus (V \otimes V^*)^r$$

in the obvious manner. In explicit matrix form, we can identify

$$W = (\mathbb{C}^n)^p \oplus (\mathbb{C}^n)^q \oplus (\mathbb{M}_n)^r = M_{n,p} \oplus M_{q,n} \oplus M_r,$$

with the action of $g \in G$ on

$$(B, C, (A_1, \ldots, A_r)) \in M_{n,p} \oplus M_{q,n} \oplus M_r$$

given by

$$g \cdot (B, C, (A_1, \ldots, A_r)) = (gB, Cg^{-1}, (\text{Ad}(g)A_i)_{i=1}^r).$$

There are obvious invariants, which we list below. For a multi-index $I = (i_1, i_2, \ldots, i_r)$ (1 \leq i_k \leq r), let us write $A_I = A_{i_1}A_{i_2}\cdots A_{i_r}$. We denote $[n] = \{1, 2, \ldots, n\}$ as usual, then the multi-index $I$ above is an element in $[r]^\ell$. We put

$$\tau_I := \text{trace}(A_I) \quad (I \in [r]^{\ell}),$$

$$\gamma_{i,j}^K := (CA_KB)_{i,j} \quad (K \in [r]^\ell, 1 \leq i \leq q, 1 \leq j \leq p),$$

where we allow $\ell = 0$ for $K$, which means $A_K = 1_n$ (identity matrix). These invariants are generators of the whole invariant ring, which is essentially due to a more general result of Le Bruyn and Procesi \cite[§ 3, Theorem 1]{LBP90} (see also \cite{LBPS7, Ito13}).

**Theorem 1.1.**

$$\mathbb{C}[W]^G = \langle \tau_I, \gamma_{i,j}^K \mid I, K \in [r]^{\ell}, \ell \geq 0, i \in [q], j \in [p] \rangle / \text{alg.}$$

**Proof.** In this proof, we will largely follow the notation of \cite{LBP90}. Let us denote a connected quiver by $Q$ and by $\alpha$ its dimension vector. For a representation space $R(Q, \alpha)$ of $Q$, Theorem 1 in \cite{LBP90} states that the invariant ring $\mathbb{C}[Q, \alpha]^{\text{GL}(\alpha)}$ is generated by traces of oriented cycles. So we will consider a quiver $Q$ of two vertices $Q_0 = \{1, 2\}$ with arrows

$$Q_1 = \{a_i \mid 1 \leq i \leq r\} \cup \{b_i \mid 1 \leq i \leq p\} \cup \{c_i \mid 1 \leq i \leq q\},$$

where $a_i$'s are loops connecting 1 and itself (i.e., $h(a_i) = t(a_i) = 1$); $b_i$'s are arrows from 2 to 1 ($h(b_i) = 2, t(b_i) = 1$); and $c_i$'s are arrows from 1 to 2 ($h(c_i) = 1, t(c_i) = 2$). Take a dimension vector $\alpha = (\alpha(1), \alpha(2)) = (n, 1)$, so that $V(1) = \mathbb{C}^n$ and $V(2) = \mathbb{C}$. Then our $W = W(p, q, r)$ coincides with the representation space $R(Q, \alpha)$.

The invariants are considered with respect to the action of $G(\alpha) = \text{GL}_n \times \text{GL}_1$. However, the representation image of $G(\alpha)$ on $W = R(Q, \alpha)$ and that of $\text{GL}_n$ are the same because the action of the torus $\text{GL}_1$ on $V(2) = \mathbb{C}$ can be recaptured by the center of $\text{GL}_n$. So the both invariant rings for $G(\alpha)$ and $\text{GL}_n$ are the same.

Let us consider any closed cycles. Since we take traces, we can start from any vertices contained in the cycle. If it only contains the vertex 1, the traces are $\tau_I$'s If it contains the vertex 2, we will start from 2 which necessarily ends in 2. Decompose the cycle into several cycles which start from 2 and end in 2. Since $V(2) = \mathbb{C}$ is 1-dimensional, a decomposed
cycle starting from 2 represents a scalar being equal to its trace. Thus the trace of the cycle which we are considering is a product of various $\gamma^K_{i,j}$’s.

Let us denote $\pi = \pi_W : W \rightarrow W//G$, an affine quotient map by the action above. As a set, the quotient $W//G$ corresponds to the set of closed $G$-orbits in $W$. It is known that these closed orbits are precisely the set of equivalence classes of completely reducible representations of a quiver corresponding to $W$.

The followings are our subjects studied in this article.

**Problem 1.2.** (1) Let $\mathfrak{N}(W) = \pi_W^{-1}(\pi_W(0))$ be the nilpotent variety, which consists of the nilpotent elements $x$ with the property $G \cdot x \ni 0$. Investigate detailed structures of $\mathfrak{N}(W)$. In particular, we are interested in $\dim \mathfrak{N}(W)$; irreducible components; orbit structure; if it is reduced or not.

(2) Are generic fibers of $\pi$ a single orbit? If this is affirmative, we know generic orbits are closed. What are their dimensions?

(3) In general, determine the orbit space structure of $W$.

(4) Analyze the singularities of the quotient space $W//G$.

Some comments are in order.

The nilpotent variety $\mathfrak{N}(W)$ is the “worst” fiber. So we are strongly interested in its structure. At the same time, we are also interested in the coinvariants (or the harmonics), which are the “functions” on the nilpotent variety. To study it, the structure of irreducible components of $\mathfrak{N}(W)$ or its reducedness is very important.

On the other hand, general fibers are supposed to have “best” properties we can expect. So this will be helpful to study the quotient space, at least its smooth part. It would be too ambitious to expect getting very explicit orbit structure of the whole space $W$. Also it seems to be a difficult problem to clarify the structure of the singularities of the quotient space.

2. **Enhanced adjoint action**

In the following, we restrict ourselves to the case $r = 1$, so that $W = M_{n,p} \oplus M_{q,n} \oplus M_n$ on which $G = \text{GL}_n$ acts. In the matrix form, $g \in \text{GL}_n$ acts on $(B, C, A) \in M_{n,p} \oplus M_{q,n} \oplus M_n$ via $g \cdot (B, C, A) = (gB, Cg^{-1}, \text{Ad}(g)A)$. We call this action the *enhanced adjoint action*.

Now Theorem 1.1 gives a set of generators of $G$-invariants:

\[
\tau_k := \text{trace}(A^k) \quad (1 \leq k \leq n),
\]

\[
\gamma^k_{i,j} := (CA^kB)_{i,j} \quad (0 \leq k \leq n - 1, \quad 1 \leq i \leq q, \quad 1 \leq j \leq p).
\]

Note that $A^n$ is a linear combination of $A^k$’s ($0 \leq k \leq n - 1$) thanks to Cayley-Hamilton’s formula, so that we don’t need higher powers of $A$ in $\tau_k$ or $\gamma^k_{i,j}$. Let us denote the affine
quotient map by

\[ \pi : W \rightarrow \mathbb{C}^n \oplus (M_{q,p})^n \]

\[ (A, B, C) \mapsto \left( (\tau_k)_{k=1}^n; ((\gamma^k_{i,j})_{i,j})_{k=0}^{n-1} \right) = \left( (\tau_k)_{k=1}^n; (CA^kB)_{k=0}^{n-1} \right) \quad (2.3) \]

By the general theory of quotients, we know the image Im \( \pi_W \) is a closed subvariety of \( \mathbb{C}^n \oplus (M_{q,p})^n \). Let us denote by \( \text{Det}_r(M_{q,p}) \) the determinantal variety consisting of matrices in \( M_{q,p} \) of rank less than or equal to \( r \). Clearly \( \text{Im} \pi_W \) is contained in \( \mathbb{C}^n \times \text{Det}_n(M_{q,p})^n \).

**Theorem 2.1.** Under the setting above, the image Im \( \pi_W \) is isomorphic to the affine quotient \( W//G = \text{Spec} \left( \mathbb{C}[W]^G \right) \). We have:

1. There is a dominant map

   \[ \Psi : \mathbb{C}^n \times (\text{Det}_1(M_{q,p}))^n \rightarrow \text{Im} \pi_W, \]

   whose restriction to a dense open subset of \( \mathbb{C}^n \times (\text{Det}_1(M_{q,p}))^n \) gives an affine quotient map under the natural action of \( S_n \) to a dense open subset of \( \text{Im} \pi_W \). Consequently, we get \( \text{dim } W//G = \text{dim } \text{Im} \pi_W = n(p + q) \), and a general fiber of \( \pi_W \) is of dimension \( n^2 \).

2. If \( p = 1 \) or \( q = 1 \), the quotient map \( \pi_W \) is surjective, and \( \text{Im} \pi_W = \mathbb{C}^n \oplus (M_{q,p})^{\oplus n} \) is an affine space. In particular, the quotient map \( \pi_W \) is coregular, and \( \mathbb{C}[W]^G \) is a polynomial ring of the fundamental invariants listed in (2.1) and (2.2).

**Proof.** Let us fix a generic diagonal matrix \( A = t = \text{diag}(t_1, \ldots, t_n) \), where \( t_i \neq t_j \) (\( i \neq j \)). For \( 1 \leq r \leq n \), put

\[ X^{(r)} = \begin{pmatrix} c_{1,r} \\ c_{2,r} \\ \vdots \\ c_{p,r} \end{pmatrix} (b_{r,1}, b_{r,2}, \ldots, b_{r,q}) \in \text{Det}_1(M_{q,p}), \]

where \( c_{i,j} \) denotes the \((i, j)\)-element of the matrix \( C \in M_{q,n} \) and similarly \( b_{i,j} \) for \( B \in M_{n,p} \). We get

\[ CA^kB = (\gamma^k_{i,j})_{i,j} = \left( \sum_{r=1}^n c_{i,r} t_r^k b_{r,j} \right)_{i,j} = \sum_{r=1}^n t_r^k X^{(r)} =: \Gamma^{(k)}. \quad (2.4) \]

Thus, in the matrix form, it holds

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
t_1 & t_2 & \cdots & t_n \\
\vdots & \vdots & \ddots & \vdots \\
t_1^{n-1} & t_2^{n-1} & \cdots & t_n^{n-1}
\end{pmatrix}
\begin{pmatrix}
X^{(1)} \\
X^{(2)} \\
\vdots \\
X^{(n)}
\end{pmatrix}
= 
\begin{pmatrix}
\Gamma^{(0)} \\
\Gamma^{(1)} \\
\vdots \\
\Gamma^{(n-1)}
\end{pmatrix}, \quad (2.5)
\]
where $D(t) = (t_i^{j-1})_{i,j}$ denotes the Vandermonde matrix in the former equation (2.5). We define a map $\Psi$ from $U = C^n \times (\text{Det}_1(M_{q,p}))^n$ to $\text{Im} \pi_W$ by

$$\Psi : U \to \text{Im} \pi_W$$

$$t; (X^{(k)})_{k=1}^n \mapsto \left( \left( \sum_{i=1}^n t_{i}^{k} \right)^n, \left( \Gamma^{(k)} \right)_{k=1}^n = D(t)(X^{(k)})_{k=1}^n \right)$$  (2.7)

The map $\Psi$ is generically an $n!$-fold covering map, and it is invariant under $S_n$ which acts on $U$ by the diagonal coordinate permutation on the both factor.

Take $(\tau; \Gamma(k)) \in \text{Im} \pi_W$ for which $\tau$ is in an image of regular semisimple $A$. Those elements consist an open dense set $(\text{Im} \pi_W)' \subset \text{Im} \pi_W$. Then we can recover $X(k)$’s via the formula (2.6), if we pick a $t$ from the fiber of $\tau$ so that $t_i \neq t_j$ $(i \neq j)$ hold. Thus we have a surjective covering map from an open dense subset of $U$ to $(\text{Im} \pi_W)'$. Consequently, the image $\text{Im} \Psi$ is contained in $\text{Im} \pi_W$. Thus we conclude $\Psi : U \to \text{Im} \pi_W$ is dominant, and

$$\dim \text{Im} \pi_W = \dim U = n + n(p + q - 1) = n(p + q),$$

where we used $\dim \text{Det}_1(M_{q,p}) = p + q - 1$. Comparing the dimension, we know the dimension of a generic fiber of $\pi_W$ is $n^2 = \dim W - \dim \text{Im} \pi_W$.

Now let us assume $p = 1$ or $q = 1$. Then $M_{q,p} = C^q$ or $C^p$, and get $\dim (C^n \oplus (M_{q,p})^{\oplus n}) = n(p + q) = \dim \text{Im} \pi_W$ (the last equality follows from (1)). Since the image $\text{Im} \pi_W$ is closed in $C^n \oplus (M_{q,p})^{\oplus n}$, we have a surjective quotient map $\pi_W : W \to C^n \oplus (M_{q,p})^{\oplus n}$ so that $W//G \simeq C^n \oplus (M_{q,p})^{\oplus n}$, an affine space. This means the invariants are algebraically independent and $C[W]^G$ is a polynomial ring. \hfill \Box

**Corollary 2.2.** Let us denote the quotient map by $\pi_W : W \to C^n \oplus (M_{q,p})^n$ as in (2.3). And assume that $(\tau; \Gamma) = (\tau; \left( \Gamma^{(k)} \right)_{k=1}^n) \in C^n \oplus (M_{q,p})^n$ satisfies the following conditions (i) and (ii).

(i) There exists a regular diagonal matrix $t$ with $\tau = (\tau_k(t))_{k=1}^n$, i.e., $\tau \in C^n$ with the $k$-th coordinate being $\tau_k = \sum_{i=1}^n t_i^k$, where $t_i \neq t_j$ $(i \neq j)$.

(ii) $\Gamma^{(k)}$ $(0 \leq k \leq n-1)$ corresponds to $X^{(k)}$ via (2.6), which are of rank 1.

Then $(\tau; \Gamma)$ is in the image $\text{Im} \pi_W$ and the $\dim \pi_W^{-1}(\tau; \Gamma) = n^2$, i.e., the fiber of $(\tau; \Gamma)$ is generic and of dimension $n^2$. Moreover, it is a single closed $G$-orbit.

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1The map $\Psi$ is a priori defined to be one from $U$ to $C^n \oplus (M_{q,p})^n$. However, since $\text{Im} \pi_W$ is closed, we know the image of $\Psi$ is contained in $\text{Im} \pi_W$. See below.

2Unfortunately, $\Psi$ may not be a quotient map. See Remark 2.4.
Proof. By condition (i), we can choose a regular diagonal matrix \( t \) with \( \tau = (\tau_k(t))_{k=1}^{n} \). Thus we can define \( (X^{(k)}) = D(t)^{-1} \Gamma \) via (2.6). If \( X^{(k)} \) is of rank 1, then we can write \( X^{(k)} = c_k t^i b_k \) for certain \( c_k \in \mathbb{C}^q, b_k \in \mathbb{C}^p \). From these vectors, we can restore \( t^i B = (b_1, \ldots, b_n) \) and \( C = (c_1, \ldots, c_n) \). Thus \( (\tau; \Gamma) = \pi_W(t, B, C) \in \text{Im} \pi_W \).

There is not so much choice for the fiber. We know the fiber over \( \tau \) of the adjoint quotient is just the conjugation of \( t \), which is of dimension \( n^2 - n \). For \( B \) and \( C \), since any column of \( B \) and \( C \) is nonzero, we can only multiply scalars column by column, which is of dimension \( n \).

It is now clear that any element in the fiber can be obtained from \( (t, B, C) \) through the action of \( G \). Since the stabilizer of the fiber \( (t, B, C) \) is trivial, we again get the right dimension \( n^2 \).

□

Remark 2.3. Let us assume \( p = 1 \) or \( q = 1 \). In this case, the action of \( G = \text{GL}_n(\mathbb{C}) \) on \( W \) is coregular, i.e., the quotient space is an affine space and the generators listed in (2.1) and (2.2) are algebraically independent.

However, if we consider an action of the simple group \( \text{SL}_n(\mathbb{C}) \) instead of \( \text{GL}_n(\mathbb{C}) \), this action is not coregular (coregular actions are classified for simple groups, see [Sch78], [AG84]).

To see this, let us assume \( p = q = 1 \) for simplicity. Consider two invariants \( D_1, D_2 \) with respect to the action of \( \text{SL}_n \) defined as follows. For \( (u, v, A) \in V \oplus V^* \oplus M_n \) (we consider \( V = \mathbb{C}^n \) as a column vector), we put

\[
D_1(u, v, A) = \det \begin{pmatrix} v \\ vA \\ \vdots \\ vA^{n-1} \end{pmatrix}, \quad D_2(u, v, A) = \det(u, Au, A^2u, \ldots, A^{n-1}u)
\]

Both \( D_1 \) and \( D_2 \) are clearly \( \text{SL}_n \)-invariants, and they are not \( \text{GL}_n \)-invariants so that they cannot be expressible by using \( \tau_k \) and \( \gamma^k \) above\(^3\). However, it is easy to see

\[
D_1 \cdot D_2 = \det(vA^{i+j}u)_{i,j} = \det(\gamma^{i+j})_{i,j}
\]

which gives a relation. This shows that the action of \( \text{SL}_n \) is not coregular.

When \( p > 1 \) or \( q > 1 \), similar arguments lead to the same conclusion.

However, even if it is not coregular, it seems the \( \text{SL}_n \)-orbit structure has good properties. We will discuss it in future.

\(^3\) Note that, since \( p = q = 1 \), we do not need subscription \( i \) and \( j \) for \( \gamma_{i,j} \)
Remark 2.4. Let us consider a toy model for the map (2.7). Assume that $V$ is a vector space and $S_n$ acts on $\mathbb{C}^n \times V^n$ as the diagonal coordinate permutation.

\[
\mathbb{C}^n \times V^n \ni (a_1, \ldots, a_n; v_1, \ldots, v_n) \xrightarrow{\psi} (\sum_{i=1}^n a_i^k, \sum_{i=1}^n a_i v_i) \xrightarrow{\pi} \mathbb{C}^n \times V^n / S_n
\]

Consider a closed set $Z = \{(a; v) \mid a_i v_i = u (1 \leq i \leq n)\}$ for a fixed non-zero vector $u$, which is stable under the $S_n$-action. The image $\psi(Z)$ does not contain an element of the form $(0; w)$, however its closure contains $(0; (nu, 0, \ldots, 0))$. Thus the image $\psi(Z)$ is not closed, hence $\psi$ is not a quotient map.

3. Structure of the null cone

We will study the structure of null cone $\mathcal{N}(W) = \pi_W^{-1}(\pi_W(0))$ in this section. For this, we follow the strategy of Popov [Pop03] and Kraft and Wallach [KW06]. We briefly recall their theory.

3.1. In this subsection, we consider a general situation so that the notation is independent of those in the former subsections.

Let $G$ be a connected reductive algebraic group $G$ over $\mathbb{C}$, which acts on a vector space $V$ linearly. Let $\pi : V \to V//G$ be the quotient map, and

\[\mathcal{N}_V := \pi^{-1}(0) = \{v \in V \mid Gv \ni 0\}\]

the null cone. For any one parameter subgroup (abbreviated as “1-PSG”) $\lambda : \mathbb{C}^\times \to G$, we define $V(\lambda) := \{v \in V \mid \lim_{t \to 0} \lambda(t)v = 0\}$. Then $v \in V$ is in the null cone $\mathcal{N}_V$ if and only if $v \in V(\lambda)$ for a suitable 1-PSG $\lambda$ (Hilbert-Mumford criterion).

Let $T \subset G$ be a maximal torus. We fix $T$ once and for all, and denote by $X^*(T)$ the character group of $T$. Then $V$ has the weight space decomposition

\[V = \bigoplus_{\gamma \in X^*(T)} V_\gamma, \quad V_\gamma := \{v \in V \mid tv = \gamma(t)v \ (t \in T)\}.\]

We denote the set of 1-PSGs $\lambda : \mathbb{C}^\times \to T$ by $X_*(T)$. Then there is a natural pairing $\langle -,- \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}$ determined as follows. For $(\lambda, \gamma) \in X_*(T) \times X^*(T)$, $m = \langle \lambda, \gamma \rangle$ if $\gamma(\lambda(t)) = t^m \ (t \in \mathbb{C}^\times)$.

With these notations, for a 1-PSG $\lambda : \mathbb{C}^\times \to T \subset G$, we have

\[V(\lambda) = \bigoplus_{\langle \lambda, \gamma \rangle > 0} V_\gamma.\]
Since every 1-PSG of $G$ is conjugate to a certain $\lambda \in X_*(T)$, we get

$$\mathcal{N}_V = \bigcup_{\lambda \in X_*(T)} G \cdot V(\lambda).$$

In this decomposition, there appear only finitely many different $V(\lambda) \neq 0$. Thus, a maximal $V(\lambda)$ may contribute to an irreducible components of $\mathcal{N}_V$ (but not always). We call such $U = V(\lambda)$ a maximal unstable subspace, and put $\mathcal{X}_U := \{\gamma \in X^*(T) \mid V_{\gamma} \subset U\} = \{\gamma \mid \langle \lambda, \gamma \rangle > 0\}$, a maximal unstable subset of weights. Let $\mathcal{X}_1, \ldots, \mathcal{X}_s$ be a complete set of representatives of maximal unstable subsets of weights up to the conjugation of the Weyl group $W_G(T)$, and $U_i = \bigoplus_{\gamma \in \mathcal{X}_i} V_{\gamma} (1 \leq i \leq s)$ the corresponding maximal unstable subspace.

For a 1-PSG $\lambda$, put

$$P(\lambda) := \{g \in G \mid \text{the limit } \lim_{t \to 0} \text{Ad}(\lambda(t)) g \text{ exists}\}.$$ 

Then $P(\lambda)$ is a parabolic subgroup which leaves $V(\lambda)$ stable (Kempf [Kem78]). If $U = V(\lambda)$ is a maximal unstable subspace, then the stabilizer $\text{Stab}_G(U)$ contains $P(\lambda)$ and hence it is a parabolic subgroup.

Define $P_i := \text{Stab}_G(U_i)$ for each $1 \leq i \leq s$. Thus, we get a natural multiplication map $G \times P_i U_i \to C_i \subset \mathcal{N}_V$, where $C_i = G \cdot U_i$. Since $G/P_i$ is projective, the image $C_i$ is closed and irreducible. Thus we can choose $C_1, \ldots, C_r$ which give irreducible components of $\mathcal{N}_V$ after renumbering if necessary. In this way, we can determine the irreducible decomposition of $\mathcal{N}_V$:

$$\mathcal{N}_V = \bigcup_{k=1}^r C_k. \quad (3.1)$$

Let us apply this theory to our situation of the enhanced adjoint representation.

3.2. Now let us return back to our original notation, so that $G = GL_n(\mathbb{C})$ which acts on $W = M_{n,p} \oplus M_{q,n} \oplus M_n$ as before. It is easy to see that the set of weights of $W$ is given by

$$\Lambda = \Lambda(W) := \{0\} \cup \Delta_n \cup \{\pm \varepsilon_i \mid 1 \leq i \leq n\}, \quad \Delta_n = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}.$$ 

Here, $\Delta_n$ denotes the set of roots of type $A_{n-1}$ and $\varepsilon_i$ denotes the standard basis in $t^*$, where $t$ is the Lie algebra of the diagonal torus $T \subset G$. The multiplicity of $\alpha \in \Delta_n$ is one, while the multiplicity of $\varepsilon_i = 0$ is $n$; that of $\varepsilon_i$ is $p$ and that of $-\varepsilon_i$ is $q$. We describe a family of maximal unstable subsets of weights up to the Weyl group conjugation. Take a standard positive system $\Delta_n^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$ of $\Delta_n$.

**Lemma 3.1.** For $0 \leq k \leq n$, put $X_k := \Delta_n^+ \cup \{\varepsilon_i \mid 1 \leq i \leq k\} \cup \{-\varepsilon_j \mid k < j \leq n\}$. Then $X_0, X_1, \ldots, X_n$ gives a complete system of representatives of maximal unstable subset of weights up to the conjugation of the Weyl group $W_G(T) = S_n$.

**Proof.** Let $X$ be a maximal unstable subset corresponding to a 1-PSG $\lambda$. Taking conjugation of $\lambda$ by $S_n$, we can assume $\lambda = (\lambda_1, \ldots, \lambda_n), \lambda_1 > \lambda_2 > \cdots > \lambda_n$. Note that, if an equality appears among $\lambda_i$’s, the corresponding unstable subset is not maximal. If $\lambda_k > 0 \geq \lambda_{k+1}$, $X$ is given by $X_k$. \qed
Let \( U_k \subset W \) be the maximal unstable subspace corresponding to \( X_k \) so that
\[
U_k = \bigoplus_{\alpha \in X_k} W_\alpha = \{ (\xi, \eta, v) \in M_{n,p} \oplus M_{q,n} \oplus M_n \mid \xi_{i,j} = 0 \ (i > k), \eta_{j',j} = 0 \ (j' \leq k), v \in n^+ \},
\]
where \( n^+ \) denotes a maximal nilpotent subalgebra consisting of upper triangular matrices with 0’s on the diagonal. It is the Lie algebra of the unipotent radical of a Borel subgroup \( B \) of upper triangular matrices in \( G = \text{GL}_n \). Note that
\[
\xi = \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \ (\xi_1 \in M_{k,p}), \quad \text{while} \quad \eta = (0, \eta_2) \ (\eta_2 \in M_{q,n-k}).
\]

**Lemma 3.2.** Let \( U_k \) (\( 0 \leq k \leq n \)) be a maximal unstable subspace as above. Then the stabilizer \( P_k = \text{Stab}_G(U_k) \) of \( U_k \) is the Borel subgroup \( B \) for any \( k \) and \( \psi_k : G \times_B U_k \to C_k \subset \mathfrak{N}(W) \) is a resolution of singularity. In particular, \( C_k \) is an irreducible closed subvariety in \( \mathfrak{N}(W) \) of dimension \( (n^2 - n) + pk + q(n - k) \).

**Proof.** Since \( P_k \) stabilizes \( n^+ \), it is contained in \( B \). On the other hand, clearly \( B \) stabilizes \( U_k \), hence \( P_k = B \).

Let us show generic fiber of the map \( \psi_k \) is a one point set. Since \( C_k \supset U_k \), we will examine the fiber of \( (\xi, \eta, v) \in U_k \), where \( v \in n^+ \) is a principal nilpotent element. Take an element \( [g, (\xi', \eta', u)] \in \psi_k^{-1}((\xi, \eta, v)) \). Then \( (\xi, \eta, v) = \psi_k([g, (\xi', \eta', u)]) = (g\xi', \eta'g^{-1}, \text{Ad}(g)u) \). In particular, we have \( v = \text{Ad}(g)u \in \text{Ad}(g)b =: b^g \). It is well known that a principal element belongs to a unique Borel subalgebra. Since \( v \in b \), we conclude \( b = b^g \), hence \( g \in B \). Now we know \( [g, (\xi', \eta', u)] \sim [1_n, (\xi, \eta, v)] \), which means the element in the fiber is uniquely determined.

The set of elements \( \{ (\xi, \eta, v) \in C_k \mid v \text{ is principal nilpotent} \} \) is open dense in \( C_k \), so the map \( \psi_k \) is generically one-to-one, hence it is birational. Since \( G \times_N U_k \) is a vector bundle over a projective variety, the map \( \psi_k \) is proper and it is a resolution. \( \square \)

**Theorem 3.3.** Let \( \mathfrak{N}(W) \) be the null cone, and denote \( C_k \subset \mathfrak{N}(W) \) (\( 0 \leq k \leq n \)) as in Lemma 3.2.

1. \( \mathfrak{N}(W) = \bigcup_{k=0}^n C_k \) gives the irreducible decomposition. So the null cone has \( (n+1) \) components, the number of which is independent of \( p \geq 1 \) and \( q \geq 1 \). The dimension of \( \mathfrak{N}(W) \) is \( n^2 - n + n \cdot \max\{p, q\} \).
2. The null cone \( \mathfrak{N}(W) \) is equidimensional if and only if \( p = q \). In this case, the dimension of \( \mathfrak{N}(W) \) is \( n^2 - n + pn \).
3. The dimension of \( \mathfrak{N}(W) \) is \( n^2 \) if and only if \( p = q = 1 \). If this is the case, any fiber \( \pi_W^{-1}((\tau; \Gamma)) \) of \( (\tau; \Gamma) \in \text{Im} \pi_W \) is of dimension \( n^2 \).

**Proof.** From Lemma 3.2 the subvariety \( C_k \) is closed and irreducible. The general theory described in \( \S 3.1 \) gives the irreducible decomposition of \( \mathfrak{N}(W) \) (cf. Equation (3.1)). Since \( \dim C_k = (n^2 - n) + pk + q(n - k) \), we have
\[
\dim \mathfrak{N}(W) = \max_{0 \leq k \leq n} \left\{(n^2 - n) + pk + q(n - k)\right\} = n^2 - n + n \cdot \max\{p, q\}.
\]
This proves (1). The claim (2) follows immediately from (1).

Let us prove (3). For any \((\tau; \Gamma) \in \text{Im } \pi_W\), the dimension of the fiber \(\pi_W^{-1}((\tau; \Gamma))\) is greater than or equal to that of a general fiber, which is \(n^2\) by Theorem 2.1. On the other hand, the dimension of the null cone is the greatest among those of the fibers (see [PV94]). This completes the proof.

\[\□\]

3.3. Orbits in the null cone. Let us investigate orbits in an irreducible component \(C_k = G \cdot U_k \subset \mathfrak{N}(W)\) (cf. (3.2)). So pick \(w = (\xi, \eta, v) \in U_k\), where \(v \in \mathfrak{n}^+\) is a principal nilpotent element. We denote the \(G\) orbit through \(w\) by \(O(w)\).

We compute the stabilizer \(Z_G(w)\) of \(w\). Up to \(G\) conjugacy, we can assume

\[
v = e := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}.
\]

By direct calculation, we get

\[
Z_G(e) = \exp \left( \{ \sum_{i=0}^{n-1} \sigma_i e^i \mid \sigma_i \in \mathbb{R} \} \right) \ni \sum_{j=1}^n x_j e^{j-1} =: g.
\] (3.3)

Assume that \(k \geq n - k\), and we denote \(\xi \in M_{n,p}\) and \(\eta \in M_{q,n}\) as

\[
\xi = \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \quad (\xi_1 \in M_{k,p}), \quad \eta = \begin{pmatrix} 0 \\ \eta_1 \end{pmatrix} \quad (\eta_1 \in M_{q,n-k}).
\] (3.4)

Here we take

\[
\xi_1 = (e_k, \xi_1') \quad (\xi_1' \in M_{k,p-1}),
\] (3.5)

where \(e_k \in \mathbb{C}^k\) is the \(k\)-th elementary vector whose \(k\)-th coordinate is 1 and the other coordinates are zero. Then, the element \(g\) in (3.3) stabilizes \(\xi\) and \(\eta\) if and only if \(x_1 = 1, x_2 = \cdots = x_k = 0\). Thus we get \(Z_G(w) = \{ 1_n + \sum_{j=k+1}^n x_j e^{j-1} \}\). In particular, we know \(\text{codim } O(w) = n - k\). For the orbit \(O(w)\), we can take \(\xi_1'\) in (3.5) and \(\eta_1\) in (3.4) freely, and they are uniquely determined by the orbit. So there is a fibration of orbits \(O(w)\) with the base space \(M_{k,p-1} \times M_{q,n-k}\) of dimension

\[
\dim O(w) + \dim M_{k,p-1} \times M_{q,n-k} = n^2 - (n - k) + k(p - 1) + q(n - k)
\]

\[
= n^2 - n + kp + (n - k)q = \dim C_k.
\]

This means the family of orbits \(\{ O(w) \}\) makes up an open dense subset of the irreducible component \(C_k\). Since the orbits of the largest possible dimension constitute an open set, \(\dim O(w) = n^2 - n + k\) is the largest among the orbits in \(C_k\). For the family parametrized by \(M_{k,p-1} \times M_{q,n-k}\), there is no reason to specialize the first column of \(\xi\). So, if the \(k\)-th row of \(\xi\) does not vanish, we can follow the same arguments.

Notice that this construction also applies to the case of \(k \leq n - k\), considering \(\eta\) instead of \(\xi\).
Let us summarize what we have proven here.

**Theorem 3.4.** Let $C_k \subset \mathfrak{N}(W)$ ($0 \leq k \leq n$) be an irreducible component of the null cone $\mathfrak{N}(W)$ (see Lemma 3.2). The largest dimension of the nilpotent orbits in $C_k$ is $n^2 - \min\{k, n - k\}$. Moreover, there exists an open dense subset of $C_k$ which is fibered over an affine space of dimension $kp + q(n - k) - \max\{k, n - k\}$ with the fiber of isomorphic nilpotent orbits $\mathbb{O}$ of the largest dimension.

In particular, an irreducible component $C_k$ contains a nilpotent orbit of dimension $n^2$ if and only if $k = 0$ or $n$.

**Remark 3.5.** Let us consider $w = (\xi, \eta, v) \in U_k$ as above. Even if $v$ is not principal, a $G$-orbit $\mathbb{O}(w)$ through $w$ can attain the largest possible dimension in the irreducible component $C_k$. It seems rather subtle to describe when an orbit $\mathbb{O}(w)$ has the largest dimension.

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