Clustering of boundary interfaces for an inhomogeneous Allen–Cahn equation on a smooth bounded domain

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Abstract
We consider the inhomogeneous Allen–Cahn equation

$$\epsilon^2 \Delta u + V(y)(1 - u^2) u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$ and $V(x)$ is a positive smooth function, $\epsilon > 0$ is a small parameter, $\nu$ denotes the unit outward normal of $\partial \Omega$. For any fixed integer $N \geq 2$, we will show the existence of a clustered solution $u_{\epsilon}$ with $N$-transition layers near $\partial \Omega$ with mutual distance $O(\epsilon |\ln \epsilon|)$, provided that the generalized mean curvature $\mathcal{H}$ of $\partial \Omega$ is positive and $\epsilon$ stays away from a discrete set of values at which resonance occurs. Our result is an extension of those (with dimension two) by Malchiodi et al. (Pac. J. Math. 229(2):447–468, 2007) and Malchiodi et al. (J. Fixed Point Theory Appl. 1(2):305–336, 2007).

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1 Introduction

We consider the inhomogeneous Allen–Cahn equation
\[ \epsilon^2 \Delta u + V(y)(1-u^2)u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \]  
(1.1)

where \( \Omega \) is a smooth and bounded domain in \( \mathbb{R}^d \), \( \epsilon \) is a small positive parameter, \( v \) is the unit outer normal to \( \partial \Omega \), \( V(y) \) is a positive smooth function on \( \Omega \). The non-constant function \( V \) represents the spatial inhomogeneity. The function \( u \) represents a continuous realization of the phase present in a material confined to the region \( \Omega \) at the point \( y \) which, except for a narrow region, is expected to take values close to +1 or −1. A component of the set \( \{ y \in \Omega : u(y) = 0 \} \) is called an interface or a phase transition layer of \( u \).

Here we mention another type inhomogeneous Allen–Cahn equation (called Fife-Greenlee mode)

\[ \epsilon^2 \Delta u + (u-a(y))(1-u^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega. \]  
(1.2)

Problem (1.2) has been studied extensively in recent years. See [1,2,5,7–11,13,16–18,21,23,24,26,30,43–45] for backgrounds and references. The case \( V(y) \equiv 1 \) or \( a(y) = 0 \) corresponds to the Allen–Cahn equation [6]

\[ \epsilon^2 \Delta u + u(1-u^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \]  
(1.3)

for which extensive literature on transition layer solution is available, see for instance [3,4,25,27,28,31–33,36–42], and the references therein.

For the inhomogeneous Allen–Cahn Eq. (1.1), there are also some results recently. In the case of dimension \( N = 1 \), it is shown in [35] that problem (1.1) has interior layer solutions, and the transition layers can appear only near the local minimum and local maximum points of the coefficient \( V \) and that at most one single layer can appear near each local minimum point of \( V \).

For the interior phase transition phenomena (away from \( \partial \Omega \)) to problem (1.1) with inhomogeneity \( V \) on higher dimensional domain, if \( \Gamma \) is a closed curve in \( \Omega \subset \mathbb{R}^2 \) satisfying the stationary and non-degeneracy conditions with respect to \( 
\int_{\Gamma} V \, \nu \), Z. Du and C. Gui [19] constructed a solution with a layer near \( \Gamma \), see also [29]. Later on, J. Yang and X. Yang [47] constructed clustered interior phase transition layers, see also [20]. On the other hand, X. Fan, B. Xu and J. Yang [22] constructed a solution with single interior phase transition layer connecting \( \partial \Omega \) near a curve \( \hat{\Gamma} \), which connects perpendicularly the boundary \( \partial \Omega \) and is also stationary and non-degenerate with respect to \( 
\int_{\hat{\Gamma}} V \, \nu \). Clustered interior phase transition layers connecting \( \partial \Omega \) can be found in the paper by S. Wei and J. Yang [46].

For the boundary interface phenomena, here we will mention some works on the problem (1.3). If \( \Omega \subset \mathbb{R}^d \) is a unit ball, A. Malchiodi, W.-M. Ni and J. Wei [31] constructed a radially symmetric solution \( u_\epsilon \) having \( N \) interfaces \( \{(r,\Theta) : u_\epsilon(r) = 0\} = \bigcup_{j = 1}^N \{(r,\Theta) : r = r_j^\epsilon\} \) such that

\[ 1 = r_0^\epsilon > r_1^\epsilon > r_2^\epsilon > \cdots > r_N^\epsilon, \quad r_j^\epsilon - r_{j-1}^\epsilon = O(\epsilon |\ln \epsilon|), \quad \forall j = 1, \ldots, N. \]

Here \( (r,\Theta) \) are the sphere coordinates in \( \mathbb{R}^d \). For the non-radial case, A. Malchiodi and J. Wei in [32] showed the existence of a single boundary interface under the condition that the mean curvature of \( \partial \Omega \) is positive and \( \epsilon \) stays away from a discrete set of values at which resonance occurs.

In the present paper, we will make an extension of the results for problem (1.3) in [31,32], and consider the existence of clustering boundary transition layers for problem (1.1). It will be shown that the inhomogeneity term \( V(y) \) as well as the boundary of \( \Omega \) will play an
important role in the construction of solutions, see Remark 1.2. To avoid too many tedious computations, we here only consider the two-dimensional case, i.e., $d = 2$ in (1.1).

For simplicity of setting, let $\partial \Omega$ be a simple closed curve in $\mathbb{R}^2$ and $\ell = |\partial \Omega|$ be the total length of the boundary $\partial \Omega$. We consider the natural parameterization $\gamma(\theta)$ of $\partial \Omega$ with positive orientation, where $\theta$ denotes the arc-length parameter measured from a fixed point of $\partial \Omega$. For sufficiently small $\delta_0$, points $y$ near $\partial \Omega$ in $\mathbb{R}^2$ can be represented in the form

$$y = \gamma'(\theta) - t \nu(\theta), \quad |t| < \delta_0, \quad \theta \in [0, \ell),$$

where $\nu(\theta)$ denotes the unit outer normal to $\partial \Omega$. It is well known that $H(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ is the unique heteroclinic solution of the problem

$$H'' + (1 - H^2)H = 0 \quad \text{in } \mathbb{R}, \quad H(\pm\infty) = \pm 1, \quad H(0) = 0.$$  

In the local coordinates $(t, \theta)$ as in (1.4), the main theorem reads:

**Theorem 1.1** Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^2$ and $V(y)$ be a positive smooth function in $\bar{\Omega}$. Assume that the generalized mean curvature of $\partial \Omega$ is positive, i.e.

$$H(\theta) := k(\theta) - \frac{V_1(0, \theta)}{2V(0, \theta)} > 0 \quad \text{on } \partial \Omega,$$

where $k(\theta)$ is the curvature of $\partial \Omega$. Then for any fixed positive integer $N$, there exists a sequence $\{\epsilon_i : i = 1, 2, \ldots\}$ of $\epsilon$ such that problem (1.1) has clustered solutions $u_{\epsilon_i}$ with $N$-phase transition layers at mutual distance $O(\epsilon_i \ln \epsilon_i)$. Near the boundary $\partial \Omega$, $u_{\epsilon_i}$ has the form of

$$u_{\epsilon_i}(y) = \sum_{j=1}^{N} (-1)^j H\left(V^\frac{1}{2}(0, \theta)\frac{t - \epsilon_i f_j(\theta)}{\epsilon_i}\right) + \frac{1}{2}(-1)^N + 1 + o(1),$$

and, away from $\partial \Omega$,

$$u_{\epsilon_i}(y) \to (-1)^N \quad \text{as} \quad \epsilon_i \to 0.$$  

The functions $\{f_j : j = 1, \ldots, N\}$ satisfy

$$f_1(\theta) = \frac{1}{2\sqrt{2}V^\frac{1}{2}(0, \theta)}\left[\ln\frac{1}{N\epsilon_i} - \ln \mathcal{H}(\theta) + \ln\left(9\gamma_1 V(0, \theta)\right)\right] + O(\epsilon^{1/2}),$$

and, for $j = 2, \ldots, N$,

$$f_j(\theta) - f_{j-1}(\theta) = \frac{1}{\sqrt{2}V^\frac{1}{2}(0, \theta)}\left[\ln\frac{1}{(N + 1 - j)\epsilon_i} - \ln \mathcal{H}(\theta) + \ln\left(9\gamma_1 V(0, \theta)\right)\right] + O(\epsilon^{1/2}),$$

where the constant $\gamma_1$ is defined in (3.4).  

Some words are in order to explain the above results.

**Remark 1.2** Let us consider a Riemannian manifold $M$ of $n$ dimension with volume element $dV_0$ and its $n-1$ dimensional submanifold $\mathcal{N}$ with mean curvature $\mathbb{H}$. By the comments of F. Morgan (Page 835 in [34]), in density manifold $M$ with volume element $dV = e^\Psi dV_0$, Springer
M. Gromov first introduced the generalization of mean curvature $\mathcal{H}_\psi$ of submanifold $\mathcal{N}$ in the form

$$\mathcal{H}_\psi = \mathbb{H} - \frac{1}{n-1} \frac{\partial \psi}{\partial \bar{\nu}},$$

where $\bar{\nu}$ is the normal of $\mathcal{N}$. We now consider $\mathbb{R}^2$ as a Riemannian manifold with the metric $g = V(y) \left(dy_1^2 + dy_2^2\right)$, in which the volume density is $e^\psi$ with $\psi = \ln V^{1/2}$. The curve $\partial \Omega$ is a submanifold of the density manifold $\mathbb{R}^2$ and then its generalized mean curvature is $\mathcal{H}$ in (1.6). If $V \equiv 1$, then $\mathcal{H} = k$ and (1.6) is exactly the requirement of positive curvature of $\partial \Omega$ in [32]. It is then obvious that Theorem 1.1 is a natural extension of the results of dimension two in [31] and [32].

As a submanifold of the density Riemannian manifold $\mathbb{R}^2$, $\partial \Omega$ has similar effects to the interfaces as described in the Introduction section of [32] due to the homogeneous Neumann boundary condition. Whence, we will also encounter resonance phenomena in the procedure of constructing solutions in Theorem 1.1. However, there is not any resonance phenomenon in the radially symmetric case for the clustering of multiple interfaces in [31]. This is the reason that we shall do much more analysis than [31] to get more accurate asymptotical expressions of the parameters $f_1, \ldots, f_N$ as well as the formula of the approximate solution $u_3$ in (3.117).

The asymptotic formulae in (1.9)–(1.10) will be formally derived in Sect. 3.1. The behaviors in (1.9)–(1.10) show that the boundary layers squeeze more and more tightly toward $\partial \Omega$. The fact is due to the effect of $\partial \Omega$ toward the boundary layers caused by the homogeneous Neumann boundary condition in (1.1). This is quite different from the results in previous papers (such as [14,15] on clustering interfaces for (1.3), and also [46,47] on interior clustering interfaces for (1.1)) in which multiple layers in the cluster distribute evenly along the limit set (The distances between neighbouring layers are almost the same in the main order).

As we have mentioned that, in order to handle the resonance, we shall first construct a good approximate solution in such a way that it will locally solve problem (1.1) very well. This can be carried out by adjusting functions $f_1, \ldots, f_N$ and then choosing correction terms step by step in Sect. 3. In fact, we will set

$$f_j(\theta) = \hat{f}_j(\theta) + \tilde{f}_j(\theta) + \tilde{f}_j(\theta) + \tilde{f}_j(\theta), \quad \forall \ j = 1, \ldots, N. \quad (1.11)$$

The readers can refer to (3.7)–(3.8), (3.68)–(3.70), (3.94) for $\hat{f}_j(\theta)$, $\tilde{f}_j(\theta)$, $\tilde{f}_j(\theta)$. On the other hand, all functions $\hat{f}_1, \ldots, \hat{f}_N$ with constraints in (3.95) will be found by the reduction method in Sects. 5–6. Please note that they will solve a small perturbation form of the following system,

$$- \epsilon^2 \gamma_0 \beta \hat{f}_j'' + \epsilon 6 \sqrt{2} \beta^2 \gamma_{1,j} \kappa_j \left[e^{-\sqrt{2} \beta (\hat{f}_j - \hat{f}_{j-1})} - 1\right]$$

$$- \epsilon 6 \sqrt{2} \beta^2 \gamma_{2,j} \kappa_{j+1} \left[e^{-\sqrt{2} \beta (\hat{f}_{j+1} - \hat{f}_j)} - 1\right] \approx 0, \quad j = 1, \ldots, N, \quad (1.12)$$

for a universal constant $\gamma_0$ in (3.4). The functions $\beta, \gamma_{1,j}, \gamma_{2,j}, \kappa_j$ are given in (3.6), (3.47), (3.48), (3.96). Therefore, the asymptotical behaviors of the components of $f_1, \ldots, f_N$ in (1.11) will imply (1.9)–(1.10).

The remaining part of this paper is devoted to the proof of Theorem 1.1, which will be organized as follows:

1. In Sect. 2, we will write down the equations in local forms.
2. In Sect. 3, we shall construct the first approximation to a real solution in (3.12) and then compute the error. To improve the approximation, suitable correction terms will be added step by step and a good approximate solution \( u_3 \) will be given in (3.117). These tedious analyses will constitute the main part of the present paper.

3. Next, we set up the inner-outer gluing scheme [12] in Sect. 4, so that we can deduce the projected problem (4.15)–(4.18) and give the existence of its solutions in Proposition 4.1.

4. In order to get a real solution, the well-known infinite-dimensional reduction method will be needed in Sects. 5, 6. The final step is to adjust the parameters \( \tilde{f}_1, \ldots, \tilde{f}_N \), which satisfy a nonlinear coupled system of second-order differential equations, see (6.1)–(6.2). This will be done in Sect. 6. Note that we also need suitable analysis from [13,15,44] to deal with the resonance phenomena in Lemma 6.1.

5. Some tedious computations and a useful lemma will be given in “Appendices A–B”. For the convenience of the readers, a collection of notation and conventions will be provided at the beginning of Sect. 3.

### 2 Local forms of the problem

For the convenience of expressions, by the rescaling \( y = \epsilon \tilde{y} \), problem (1.1) can be rewritten as

\[
\Delta u + V(\epsilon \tilde{y})(1-u^2)u = 0 \quad \text{in} \quad \Omega_\epsilon, \quad \frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \partial \Omega_\epsilon, \tag{2.1}
\]

where \( \Omega_\epsilon = \Omega/\epsilon, \partial \Omega_\epsilon = \partial \Omega/\epsilon \).

Recall the coordinates \((t, \theta)\) given in (1.4) and let

\[
(s, z) = \frac{1}{\epsilon} (t, \theta), \quad z \in \left[0, \frac{\ell}{\epsilon}\right), \quad s \in \left(0, \frac{\delta_0}{\epsilon}\right), \tag{2.2}
\]

be the natural stretched coordinates associated to the curve \( \partial \Omega_\epsilon = \partial \Omega/\epsilon \). Any \( \tilde{y} \) near \( \partial \Omega_\epsilon \) in \( \mathbb{R}^2 \) can be represented in the form

\[
\tilde{y} = \frac{1}{\epsilon} \gamma(\epsilon z) - s v(\epsilon z). \tag{2.3}
\]

We then get

\[
\frac{\partial \tilde{y}}{\partial s} = -v, \quad \frac{\partial \tilde{y}}{\partial z} = \gamma_\theta - \epsilon sv_\theta. \tag{2.4}
\]

Since

\[
|\gamma_\theta| = 1, \quad |v| = 1, \quad \text{and} \quad v_\theta = k \gamma_\theta, \quad \gamma_\theta \cdot v = 0,
\]

the metric matrix is

\[
g = \begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & (1 - \epsilon ks)^2
\end{pmatrix}.
\]

So the determinant of the metric matrix is

\[
det g = (1 - \epsilon ks)^2,
\]

and

\[
g^{-1} = \begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & (1 - \epsilon ks)^{-2}
\end{pmatrix}.
\]
The Laplacian–Beltrami operator has the form
\[
\Delta_g = \frac{1}{1 - \epsilon ks} \frac{\partial}{\partial s} \left[ (1 - \epsilon ks) \frac{\partial}{\partial s} \right] + \frac{1}{1 - \epsilon ks} \frac{\partial}{\partial z} \left[ (1 - \epsilon ks)^{-1} \frac{\partial}{\partial z} \right].
\] (2.5)

In the coordinates \((s, z)\), we will give the computations of \(\nabla u\) as follows
\[
\nabla u = g^{ij} \frac{\partial u}{\partial \tilde{y}_i} \frac{\partial \tilde{y}_j}{\partial s} = \frac{\partial u}{\partial s} \frac{\partial \tilde{y}}{\partial s} + (1 - \epsilon sk) - 2 \frac{\partial u}{\partial z} \frac{\partial \tilde{y}}{\partial z}.
\] (2.6)

Hence, problem (2.1) can be locally recast as follows
\[
u_{ss} + u_{zz} + B_1(u) + V(\epsilon s, \epsilon z)F(u) = 0, \quad \forall (s, z) \in (0, \delta_0/\epsilon) \times (0, \ell/\epsilon),
\] (2.7)
with boundary condition
\[
D(u) = 0, \quad \forall z \in [0, \ell/\epsilon).
\] (2.8)

In the above, the boundary operator is given by
\[
D(u) := -\frac{\partial u(s, z)}{\partial s}{\big|}_{s=0},
\] (2.9)
and the nonlinear term is
\[
F(u) = u - u^3.
\] (2.10)

Moreover, the linear differential operator \(B_1\) has the form
\[
B_1(u) = -\frac{\epsilon ku_2}{1 - \epsilon ks} + \left[ \frac{1}{(1 - \epsilon ks)^2} - 1 \right] u_{zz} + \frac{\epsilon^2 sk}{(1 - \epsilon ks)^3} u_z.
\]

Furthermore, we can expand the operator \(B_1\) as
\[
B_1(u) = -(\epsilon k + \epsilon^2 sk^2) u_s + B_0(u),
\] (2.11)
with
\[
B_0(u) = \epsilon^2 sa_1 u_z + \epsilon sa_2 u_{zz} + \epsilon^3 s^2 a_3 u_z,
\] (2.12)
for certain smooth functions \(a_j(t, \theta), \quad j = 1, 2, 3\). In the neighborhood of \(\partial \Omega_\epsilon\), by taking the Taylor expansion, we also expand
\[
V(\epsilon s, \epsilon z) = V(0, \epsilon z) + \beta_1(\epsilon z) \epsilon s + \frac{1}{2} \beta_2(\epsilon z) \epsilon^2 s^2 + a_4(\epsilon s, \epsilon z) \epsilon^3 s^3,
\] (2.13)
for a smooth function \(a_4(t, \theta)\) and
\[
\beta_1(\theta) = V_t(0, \theta), \quad \beta_2(\theta) = V_{tt}(0, \theta).
\] (2.14)

Then \(u\) solves (2.7) if and only if
\[
S(u) := u_{ss} + u_{zz} + V(0, \epsilon z)F(u) + B_2(u) + B_3(u) + B_4(u) = 0,
\] (2.15)
for \((s, z) \in (0, \delta_0/\epsilon) \times (0, \ell/\epsilon)\), where

\[
B_2(u) = B_0(u) + a_4(\epsilon s, \epsilon z)\epsilon^3 s^3 u(1 - u^2),
\]
\[
B_3(u) = -(\epsilon k + \epsilon^2 sk^2)u_s,
\]
\[
B_4(u) = [\epsilon s \beta_1 + \frac{1}{2} \epsilon^2 s^2 \beta_2](u - u^3).
\]

3 Local approximate solutions

The main objective of this section is to construct a suitable approximate solution in local coordinates \((s, z)\) near \(\partial \Omega_1 / \epsilon\) and then evaluate its error terms, see (3.117) and (3.122). This can be fulfilled step by step to cancel the big error terms by adding correction terms to the approximate solutions.

In the following, we will use \(H(x)\) in (1.5) as the basic block to construct solutions with phase transition layers. It is well known that \(H\) is odd and enjoys the following behaviors

\[
H(x) = \pm \left(1 - 2e^{-\sqrt{2}|x|}\right) + O\left(e^{-2\sqrt{2}|x|}\right), \quad \text{as } x \to \pm \infty,
\]
\[
H'(x) = 2\sqrt{2}e^{-\sqrt{2}|x|} + O\left(e^{-2\sqrt{2}|x|}\right), \quad \text{as } |x| \to +\infty.
\]

From Eq. (1.5), it is easy to derive that

\[
1 - H^2(x) = \sqrt{2} H_x(x), \quad \int_{\mathbb{R}} H_x^2 \, dx = \frac{2\sqrt{2}}{3}.
\]

Integrating by parts, we have

\[
2 \int_{\mathbb{R}} x H_x H_{xx} \, dx = -\int_{\mathbb{R}} H_x^2 \, dx = -\frac{2\sqrt{2}}{3}.
\]

From (1.5), it is also trivial to derive that

\[
3 \int_{\mathbb{R}} (1 - H^2)e^{-\sqrt{2}x} H_x \, dx = -\int_{\mathbb{R}} (H_{xxx} - 2H_x)e^{-\sqrt{2}x} \, dx = 8.
\]

Notation We pause here to set the conventions

\[
\gamma_0 = \int_{\mathbb{R}} H_x^2 \, dx = \frac{2\sqrt{2}}{3}, \quad \gamma_1 = \int_{\mathbb{R}} e^{-\sqrt{2}x} H_x^2 \, dx = \int_{\mathbb{R}} e^{\sqrt{2}x} H_x^2 \, dx = \frac{8}{3\sqrt{2}}.
\]

The set \(\mathcal{S}\) represents the strip in \(\mathbb{R}^2\) of the form

\[
\mathcal{S} = \left\{ (s, z) : s \in (0, +\infty), \quad 0 < z < \ell/\epsilon \right\}.
\]

The functions \(\beta, \beta_1, \beta_2\) are defined in (3.6), (2.14).

3.1 Formal derivation of the interfaces

For any positive integer \(N\), we assume that the locations of the phase transition layers are characterized by functions \(s = f_j(\epsilon z), \quad j = 1, \ldots, N\) such that

\[
f_j : (0, \ell) \to \mathbb{R}, \quad 0 < f_1 < f_2 < \cdots < f_N.
\]
Recalling $H$ given in (1.5) and setting
\[ \beta(\theta) = V^{1/2}(0, \theta), \] (3.6)
we set the profile of a solution
\[ v(s, z) \equiv \sum_{j=0}^{N} (-1)^j H(\beta(\epsilon z) (s - f_j(\epsilon z))) - \frac{(-1)^N - 1}{2}, \quad s > 0, \ z \in (0, \ell/\epsilon), \]
where $f_0 = -f_1$. The requirement that $f_0 = -f_1$ will imply that $H(\beta(\epsilon z) (s - f_0(\epsilon z)))$ is an even extension of the function $-H(\beta(\epsilon z) (s - f_1(\epsilon z)))$ with respect to $\partial \Omega_\epsilon$ such that $v(s, z)$ can approximately satisfy the homogeneous Neumann boundary condition. The error is
\[
S(v) = -\epsilon k \beta \sum_{j=1}^{N} (-1)^j H'(\beta(s - f_j)) + \epsilon \beta \sum_{j=1}^{N} (H_j - H_j^3) \\
- \beta^2 \left( v^3 - \sum_{j=1}^{N} H_j^3 \right) + o(\epsilon),
\]
where the notation $H_j$ is given in (3.10)

We shall adjust $f_0, \ldots, f_N$ with
\[ f_0 < 0 < f_1 < \cdots < f_N \quad \text{and} \quad f_0 = -f_1, \]
such that $S(v)$ is approximately orthogonal to $H'(\beta(s - f_j))$, $j = 1, \ldots, N$. It is formally equivalent to solving
\[
e^{-\sqrt{2} \beta (f_1 - f_0)} - e^{-\sqrt{2} \beta (f_2 - f_1)} - \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right] \approx 0, \\
e^{-\sqrt{2} \beta (f_2 - f_1)} - e^{-\sqrt{2} \beta (f_3 - f_2)} - \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right] \approx 0, \\
\cdots \\
e^{-\sqrt{2} \beta (f_{N-1} - f_{N-2})} - e^{-\sqrt{2} \beta (f_N - f_{N-1})} - \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right] \approx 0, \\
e^{-\sqrt{2} \beta (f_N - f_{N-1})} - \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right] \approx 0.
\]
These imply that
\[
e^{-\sqrt{2} \beta (f_1 - f_0)} \approx N \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right], \\
e^{-\sqrt{2} \beta (f_2 - f_1)} \approx (N - 1) \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right], \\
\cdots \\
e^{-\sqrt{2} \beta (f_{N-1} - f_{N-2})} \approx 2 \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right].\]
\[
e^{-\sqrt{2} \beta (f_N - f_{N-1})} \approx \epsilon \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right].
\]

Whence, we have
\[
f_1 \approx \frac{1}{2\sqrt{2} \beta} \left\{ \ln \frac{1}{N \epsilon} - \ln \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right] \right\}, \quad f_0 = -f_1,
\]
\[
f_j - f_{j-1} \approx \frac{1}{2\sqrt{2} \beta} \left\{ \ln \frac{1}{(N - j + 1) \epsilon} - \ln \frac{\sqrt{2}}{12 \beta} \left[ k - \frac{\beta_1}{2 \beta^2} \right] \right\}, \quad j = 2, \ldots, N.
\]

These asymptotic formulae will guide us to set up the parameters \( f_1, \ldots, f_N \) in the sequel.

### 3.2 The first approximation

According to the previous formal calculations, we first set the parameters as the following
\[
f_0(\theta) = \hat{f}_0(\theta) + f_0(\theta) \quad \text{with} \quad \hat{f}_0(\theta) = -\frac{1}{2\sqrt{2} \beta(\theta)} \ln \frac{1}{N \epsilon},
\]
\[
f_1(\theta) = \hat{f}_1(\theta) + f_1(\theta) \quad \text{with} \quad \hat{f}_1(\theta) = \frac{1}{2\sqrt{2} \beta(\theta)} \ln \frac{1}{N \epsilon},
\]
and for \( j = 2, \ldots, N, \)
\[
f_j(\theta) = f_j(\theta) + f_j(\theta) \quad \text{with} \quad \hat{f}_j(\theta) = \frac{1}{2\sqrt{2} \beta(\theta)} \ln \frac{1}{N \epsilon} + \frac{1}{2\sqrt{2} \beta(\theta)} \ln \frac{(N - j)!}{(N - 1)! \epsilon^{j-1}}.
\]

Note that \( f_0(\theta), \ldots, f_N(\theta) \) are of order \( O(1) \) with respect to \( \epsilon \) and will be set up step by step in the rest of the present section, see (3.43) and (3.92). For the convenience of the notation, we will also set
\[
f_{N+1}(\theta) = +\infty, \quad f_0 = -f_1, \quad f_{-1} = -f_2.
\]

By recalling the definition of \( H \) given in (1.5), we let
\[
H_j(x_j) = (-1)^j H(x_j) \quad \text{with} \quad x_j = \beta(\epsilon z)(s - f_j(\epsilon z)), \quad j = 1, \ldots, N
\]
\[
\tilde{H}_j(\tilde{x}_j) = (-1)^j+1 H(\tilde{x}_j) \quad \text{with} \quad \tilde{x}_j = \beta(\epsilon z)(s + f_j(\epsilon z)),
\]
and define the first approximate solution to (2.15) in the form
\[
u_1(s, z) = \sum_{j=1}^{N} H_j(x_j) + \frac{1}{2} \left( (-1)^N + 1 \right) + \sum_{j=1}^{N} \tilde{H}_j(\tilde{x}_j) - \frac{1}{2} \left( (-1)^{N+1} + 1 \right)
\]
\[
= \sum_{j=1}^{N} H_j(x_j) + \sum_{j=1}^{N} \tilde{H}_j(\tilde{x}_j) + (-1)^N, \quad \forall (s, z) \in \mathcal{S}.
\]

Note that \( u_1(s, z) \approx H_j(x_j) \) when \( |s - f_j| \) is not too large.

The error of the first approximate solution \( u_1(s, z) \) is
\[
S(u_1) = u_{1,ss} + u_{1,zz} + \beta^2 F(u_1) + B_2(u_1) + B_3(u_1) + B_4(u_1)
\]
\[
= u_{1,zz} + \beta^2 F(u_1) - \beta^2 \sum_{j=1}^{N} \left[ F(H_j) + F(\tilde{H}_j) \right] + B_2(u_1) + B_3(u_1) + B_4(u_1).
\]
where we have used the fact that

\[ u_{1,ss} + \beta^2 \sum_{j=1}^{N} \left[ F(H_j) + F(\widetilde{H}_j) \right] = 0. \]

### 3.2.1 The computations of error terms

It is easy to get that

\[ (\beta(s - f_j))_z = \epsilon \left[ \beta'(s - f_j) \right] = \epsilon \left[ \frac{\beta'}{\beta} x_j - \beta f_j' \right], \]

\[ (\beta(s + f_j))_z = \epsilon \left[ \beta'(s + f_j) + \beta f_j' \right] = \epsilon \left[ \frac{\beta'}{\beta} \bar{x}_j + \beta f_j' \right], \]

and

\[ (\beta(s - f_j))_{zz} = \epsilon^2 \left[ \beta''(s - f_j) - \beta f_j'' \right] = \epsilon^2 \left[ \frac{\beta''}{\beta} x_j - 2\beta f_j' - \beta f_j'' \right], \]

\[ (\beta(s + f_j))_{zz} = \epsilon^2 \left[ \beta''(s + f_j) + \beta f_j'' \right] = \epsilon^2 \left[ \frac{\beta''}{\beta} \bar{x}_j + 2\beta f_j' + \beta f_j'' \right]. \]

The computations of all components in (3.13) can be done in the following way.

(1). The definition of \( u_1 \) will imply that the first term in (3.13) can be rewritten as follows

\[
\begin{align*}
\mathcal{z} = & \sum_{j=1}^{N} \left[ H_{j,x_j} \right] \left( \beta(s - f_j) \right)_z^2 + \sum_{j=1}^{N} H_{j,x_j} \left( \beta(s - f_j) \right)_{zz} \\
& + \sum_{j=1}^{N} \bar{H}_{j,\bar{x}_j} \left( \beta(s + f_j) \right)_z^2 + \sum_{j=1}^{N} \bar{H}_{j,\bar{x}_j} \left( \beta(s + f_j) \right)_{zz} \\
= & \epsilon^2 \sum_{j=1}^{N} H_{j,x_j} \left[ \left( \frac{\beta'}{\beta} \right)^2 x_j^2 - 2\beta f_j' x_j + \beta^2 f_j'' \right] \\
& + \epsilon^2 \sum_{j=1}^{N} H_{j,x_j} \left[ \frac{\beta''}{\beta} x_j - 2\beta f_j' - \beta f_j'' \right] \\
& + \epsilon^2 \sum_{j=1}^{N} \bar{H}_{j,\bar{x}_j} \left[ \left( \frac{\beta'}{\beta} \right)^2 \bar{x}_j^2 + 2\beta f_j' \bar{x}_j + \beta^2 f_j'' \right] \\
& + \epsilon^2 \sum_{j=1}^{N} \bar{H}_{j,\bar{x}_j} \left[ \frac{\beta''}{\beta} \bar{x}_j + 2\beta f_j' + \beta f_j'' \right].
\end{align*}
\]
(2). Recalling the expression of \( B_3 \) in (2.17), we obtain that

\[
B_3(u_1) = -\left(\epsilon k + \epsilon^2 sk^2\right)u_{1,s}
\]

\[
= -\epsilon k \beta \sum_{j=1}^{N} H_{j,x_j} - \epsilon^2 sk^2 \beta \sum_{j=1}^{N} H_{j,x_j} - \epsilon k \beta \sum_{j=1}^{N} \tilde{H}_{j,x_j} - \epsilon^2 sk^2 \beta \sum_{j=1}^{N} \tilde{H}_{j,x_j}
\]

\[
= -\epsilon k \beta \sum_{j=1}^{N} H_{j,x_j} - \epsilon^2 sk^2 \beta \sum_{j=1}^{N} x_j H_{j,x_j} - \epsilon^2 k^2 \beta \sum_{j=1}^{N} f_j H_{j,x_j}
\]

\[
- \epsilon k \beta \sum_{j=1}^{N} \tilde{H}_{j,x_j} - \epsilon^2 sk^2 \beta \sum_{j=1}^{N} \tilde{x}_j \tilde{H}_{j,x_j} + \epsilon^2 k^2 \beta \sum_{j=1}^{N} f_j \tilde{H}_{j,x_j}.
\]

(3.15)

(3). Recalling the definition of \( B_2 \) as in (2.16), it is easy to derive that

\[
B_2(u_1) = o(\epsilon^{2+\sigma}).
\]

This is due to the setting of \( f_1, \ldots, f_N \) in (3.8) and also the asymptotic behavior of \( H \) in (3.1).

(4). According to the expression of \( B_4 \) as in (2.18), it is derived that

\[
B_4(u_1) = \left[ \epsilon s \beta_1 + \frac{1}{2} \epsilon^2 s^2 \beta_2 \right] F(u_1)
\]

\[
= \epsilon s \beta_1 \sum_{j=1}^{N} \left( F(H_j) + F(\tilde{H}_j) \right) + \epsilon s \beta_1 \left[ F(u_1) - \sum_{j=1}^{N} \left( F(H_j) + F(\tilde{H}_j) \right) \right]
\]

\[
+ \frac{1}{2} \epsilon^2 s^2 \beta_2 \sum_{j=1}^{N} \left( F(H_j) + F(\tilde{H}_j) \right)
\]

\[
+ \frac{1}{2} \epsilon^2 s^2 \beta_2 \left[ F(u_1) - \sum_{j=1}^{N} \left( F(H_j) + F(\tilde{H}_j) \right) \right]
\]

\[
= \epsilon \beta_1 \beta^{-1} \sum_{j=1}^{N} x_j F(H_j) + \epsilon \beta_1 \sum_{j=1}^{N} f_j F(H_j) + \epsilon s \beta_1 \sum_{j=1}^{N} F(\tilde{H}_j)
\]

\[
+ \frac{1}{2} \epsilon^2 \beta_2 \beta^{-2} \sum_{j=1}^{N} x_j^2 F(H_j) + \epsilon^2 \beta_2 \beta^{-1} \sum_{j=1}^{N} x_j f_j F(H_j)
\]

\[
+ \frac{1}{2} \epsilon^2 \beta_2 \sum_{j=1}^{N} f_j^2 F(H_j)
\]

\[
+ \frac{1}{2} \epsilon^2 s^2 \beta_2 \sum_{j=1}^{N} F(\tilde{H}_j) + \epsilon \beta_1 s \left[ F(u_1) - \sum_{j=1}^{N} \left( F(H_j) + F(\tilde{H}_j) \right) \right]
\]

\[
+ \frac{1}{2} \epsilon^2 \beta_2 s^2 \left[ F(u_1) - \sum_{j=1}^{N} \left( F(H_j) + F(\tilde{H}_j) \right) \right].
\]

(3.16)
(5). Now, we focus on the estimates of the nonlinear error terms

$$F(u_1) - \sum_{j=1}^{N} [F(H_j) + F(\bar{H}_j)].$$

(3.17)

We denote the following sets:

$$\mathcal{A}_1 = \left\{ (s, z) \in \left(0, \frac{\delta_0}{\epsilon}\right) \times \left[0, \frac{\epsilon}{\epsilon}\right] : 0 \leq s \leq \frac{f_1(\epsilon z) + f_2(\epsilon z)}{2} \right\},$$

(3.18)

and for $n = 2, \ldots, N$

$$\mathcal{A}_n = \left\{ (s, z) \in \left(0, \frac{\delta_0}{\epsilon}\right) \times \left[0, \frac{\epsilon}{\epsilon}\right] : \frac{f_{n-1}(\epsilon z) + f_n(\epsilon z)}{2} \leq s \leq \frac{f_n(\epsilon z) + f_{n+1}(\epsilon z)}{2} \right\}.$$  

(3.19)

Let us fix $n$ with $2 \leq n \leq N - 1$. Since

$$H(x) = \pm \left(1 - 2e^{-\sqrt{2}|x|}\right) + O\left(e^{-2\sqrt{2}|x|}\right) \quad \text{as } x \to \pm \infty,$$

there hold that for $j < n$,

$$H(\beta(s - f_j)) - 1 = -2e^{-\sqrt{2}\beta(f_n - f_j)}e^{-\sqrt{2}x_n} + O\left(e^{-2\sqrt{2}|x_n+\beta(f_n - f_j)|}\right), \quad \forall (s, z) \in \mathcal{A}_n,$$

(3.20)

while for $j > n$,

$$H(\beta(s - f_j)) + 1 = 2e^{-\sqrt{2}\beta(f_j - f_n)}e^{\sqrt{2}x_n} + O\left(e^{-2\sqrt{2}|x_n+\beta(f_n - f_j)|}\right), \quad \forall (s, z) \in \mathcal{A}_n.$$  

(3.21)

We derive that for any $j$ with $|j - n| \geq 3$

$$|F(H_j)| \leq Ce^{-\sqrt{2}|x_n+\beta(f_n - f_j)|} \leq e^{2+\sigma} e^{-\sqrt{2}|x_n|}, \quad \forall (s, z) \in \mathcal{A}_n,$$

(3.22)

for some $\sigma > 0$ small, due to the fact

$$|f_n - f_j| = \sum_{l=j}^{n-1} \frac{1}{\sqrt{2}\beta} \ln \frac{1}{(N - l)\epsilon} + O(1).$$

On the other hand, we have, for $(s, z) \in \mathcal{A}_n$

$$F(\beta(s - f_{n-1})) = F(H(x_n + \beta(f_n - f_{n-1})))$$

$$= F'(1) b_{1n} + \frac{1}{2} F''(1 + \beta_1 b_{1n}) b_{1n}^2,$$

(3.23)

$$F(\beta(s - f_{n-2})) = F(H(x_n + \beta(f_n - f_{n-2})))$$

$$= F'(1) b_{2n} + \frac{1}{2} F''(1 + \beta_2 b_{2n}) b_{2n}^2,$$

(3.24)

$$F(\beta(s - f_{n+1})) = F(H(x_n + \beta(f_n - f_{n+1})))$$

$$= F'(1) b_{3n} + \frac{1}{2} F''(1 + \beta_3 b_{3n}) b_{3n}^2,$$

(3.25)

$$F(\beta(s - f_{n+2})) = F(H(x_n + \beta(f_n - f_{n+2})))$$

$$= F'(1) b_{4n} + \frac{1}{2} F''(1 + \beta_4 b_{4n}) b_{4n}^2,$$

(3.26)
for certain constants \(\lambda_1, \ldots, \lambda_4 \in (0, 1)\). Here we have denoted
\[
\begin{align*}
b_{1n} &:= H(x_n + \beta(f_n - f_{n-1})) - 1, \\
b_{2n} &:= H(x_n + \beta(f_n - f_{n-2})) - 1, \\
b_{3n} &:= H(x_n + \beta(f_n - f_{n+1})) + 1, \\
b_{4n} &:= H(x_n + \beta(f_n - f_{n+2})) + 1.
\end{align*}
\]

For convenience purpose, we will denote that
\[
\begin{align*}
A_{1n} &:= F''(1 + \lambda_1 b_{1n}), \\
A_{2n} &:= F''(1 + \lambda_2 b_{2n}), \\
A_{3n} &:= F''(1 - \lambda_3 b_{3n}), \\
A_{4n} &:= F''(1 - \lambda_4 b_{4n}).
\end{align*}
\] (3.27)

Note that, for \((s, \varepsilon) \in \mathfrak{A}_n\) with \(n = 1, \ldots, N\),
\[
(-1)^n u_1 = H(\beta(s - f_{n})) - b_{1n} + b_{2n} - b_{3n} + b_{4n} + O(e^{2+\varepsilon}).
\] (3.29)

According to the asymptotic terms given in (3.30)–(3.31), we obtain
\[
\begin{align*}
b_{1n} - b_{2n} + b_{3n} - b_{4n} &
= -2\varepsilon(N - n + 1)e^{-\sqrt{2}x_n}e^{-\sqrt{2}\beta(l_{n+1} - l_n)} + O(e^{-2\sqrt{2}|s_n + \beta(f_n - f_{n-1})|}) \\
&+ 2\varepsilon(N - n)e^{\sqrt{2}x_n}e^{-\sqrt{2}\beta(l_{n+1} - l_n)} + O(e^{-2\sqrt{2}|s_n + \beta(f_n - f_{n-1})|}) \\
&+ 2\varepsilon(N - n + 1)N - n + 2|e^{-\sqrt{2}x_n}e^{-\sqrt{2}\beta(l_{n+1} - l_n - 2)} + O(e^{-2\sqrt{2}|s_n + \beta(f_n - f_{n-2})|}) \\
&- 2\varepsilon(N - n)N - n + 1|e^{\sqrt{2}x_n}e^{-\sqrt{2}\beta(l_{n+1} - l_n)} + O(e^{-2\sqrt{2}|s_n + \beta(f_n - f_{n+2})|}).
\end{align*}
\] (3.30)

Since we have set the notation \(f_0 = -f_1, f_{-1} = -f_2\) in (3.9), (3.30) also holds for \(n = 1, 2, N - 1, N\). For more details, we refer the readers to the computations in “Appendix A”.

Thus for some \(\lambda_5 \in (0, 1)\),
\[
\begin{align*}
(-1)^n F(u_1) &:= F((-1)^n u_1) \\
&= F\left(H(\beta(s - f_{n}))\right) - F'(\lambda_5 (b_{1n} - b_{2n} + b_{3n} - b_{4n})) \\
&+ \frac{1}{2}A_{5n}\left[b_{1n} - b_{2n} + b_{3n} - b_{4n}\right]^2 + O(e^{2+\varepsilon}),
\end{align*}
\] (3.31)

where
\[
A_{5n} := F''\left(H(\beta(s - f_{n})) - \lambda_5 (b_{1n} - b_{2n} + b_{3n} - b_{4n})\right).
\] (3.32)

Combining relations (3.22)–(3.31) and using the facts
\[
F'(1) - F'(H) = -3(1 - H^2), \quad |b_{1n}| + |b_{2n}| + |b_{3n}| + |b_{4n}| = O(\varepsilon e^{-\sqrt{2}|s_n|}),
\]
we obtain, for \((s, \varepsilon) \in \mathfrak{A}_n\) with \(n = 1, \ldots, N\),
\[
\begin{align*}
(-1)^n &\left[F(u_1) - \sum_{j=1}^{N} (F(H_j) + F(H_j))\right] \\
&= -3(1 - H^2)\left(b_{1n} - b_{2n} + b_{3n} - b_{4n}\right) + \frac{1}{2}A_{1n} b_{1n}^2 - \frac{1}{2}A_{2n} b_{2n}^2 \\
&+ \frac{1}{2}A_{3n} b_{3n}^2 - \frac{1}{2}A_{4n} b_{4n}^2 + \frac{1}{2}A_{5n}\left[b_{1n} - b_{2n} + b_{3n} - b_{4n}\right]^2 + O(e^{2+\varepsilon} e^{-\sigma |s_n|}).
\end{align*}
\] (3.33)

where the expressions of \(b_{1n} - b_{2n} + b_{3n} - b_{4n}\) are given in (3.30), \(A_{1n}, \ldots, A_{4n}\) are defined in (3.27)–(3.28).
We now conclude for the computations of terms in (3.17). For fixed $n$ with $2 \leq n \leq N$, let the sets $\mathcal{B}_n$ be in the form

$$\mathcal{B}_n = \left\{ (s, z) \in \left(0, \frac{\delta_0}{\epsilon}\right) \times \left[0, \frac{\ell}{\epsilon}\right] : \frac{f_{n-1}(\epsilon z) + f_n(\epsilon z)}{2} - \tilde{\delta} \leq s \leq \frac{f_n(\epsilon z) + f_{n+1}(\epsilon z)}{2} + \tilde{\delta} \right\},$$

and

$$\mathcal{B}_1 = \left\{ (s, z) \in \left(-\tilde{\delta}, \frac{\delta_0}{\epsilon}\right) \times \left[0, \frac{\ell}{\epsilon}\right] : -\tilde{\delta} \leq s \leq \frac{f_1(\epsilon z) + f_2(\epsilon z)}{2} + \tilde{\delta} \right\},$$

with constant $\tilde{\delta} > 0$ sufficient small. We denote a smooth partition of unity $\{\chi_n\}$ subordinate to the open sets $\{\mathcal{B}_n : n = 1, \ldots, N\}$, satisfying

$$\chi_n(s, z) = 1 \text{ when } (s, z) \in \mathcal{B}_n, \quad 0 \leq \chi_n(s, z) \leq 1,$$

$$\chi_n(s, z) \in C^\infty_0(\mathcal{B}_n) \quad \text{and} \quad \sum_{n=1}^N \chi_n(s, z) = 1 \text{ on } \mathcal{S}. \quad (3.34)$$

Combining all the terms in the above, we know that in the whole strip $\mathcal{S}$

$$F(u_1) - \sum_{j=1}^N \left[ F(H_j) + F(\tilde{H}_j) \right] = \sum_{n=1}^N \chi_n \left[ F(u_1) - \sum_{j=1}^N (F(H_j) + F(\tilde{H}_j)) \right]$$

$$= \sum_{n=1}^N (-1)^n \chi_n \left\{ -3(1 - H_j^2) (b_{1n} - b_{2n} + b_{3n} - b_{4n}) + \frac{1}{2} A_{1n} b_{1n}^2 - \frac{1}{2} A_{2n} b_{2n}^2 + \frac{1}{2} A_{3n} b_{3n}^2 - \frac{1}{2} A_{4n} b_{4n}^2 + \frac{1}{2} A_{5n} \left[b_{1n} - b_{2n} + b_{3n} - b_{4n}\right]^2 \right\} + O(\epsilon^{2+\sigma}). \quad (3.35)$$

Note that (3.35) will give the estimates of $\beta^2 F(u_1) - \beta^2 \sum_{j=1}^N \left[ F(H_j) + F(\tilde{H}_j) \right]$ in (3.13) and also the estimates for the corresponding components in (3.16).

### 3.2.2 Rearrangements of the error terms

In this part, by substituting all the above computations into (3.13), we will rearrange the error components in $S(u_1)$ according to the order of $\epsilon$.

According to the asymptotic behavior of $f_j$, $j = 1, \ldots, N$, as in (3.7)–(3.8), we know that the largest terms in $S(u_1)$ are of order $\epsilon \ln \epsilon$, which can be written as the following

$$\epsilon \ln \epsilon |E_{01}| := \epsilon \beta_1 \sum_{j=1}^N f_j \Xi_{0,j},$$

where we have denoted

$$\Xi_{0,j} = F(H_j) = H_j - \tilde{H}_j^3, \quad j = 1, \ldots, N. \quad (3.36)$$

Recalling the expressions of $b_{1j}$ and $b_{3j}$ as in (A.1) and (A.3), we can select the terms of order $\epsilon$ in $S(u_1)$ and then define

$$\epsilon \sum_{j=1}^N \Xi_{1,j} := \epsilon \sum_{j=1}^N \left[ -k \beta H_{j,xj} + \beta^{-1} \beta_1 x_j F(H_j) \right],$$
and
\[
\epsilon E_{02} = -3\beta^2 \sum_{j=1}^{N} (-1)^j \chi_j (1 - H_j^2) (b_{1j} + b_{3j}).
\]

The terms of order \(\epsilon^2 |\ln \epsilon|^2\) in \(S(u_1)\) can be collected in the following

\[
\epsilon^2 |\ln \epsilon|^2 E_{03} := \epsilon^2 \beta^2 \sum_{j=1}^{N} f_j^2 H_{j,x_jx_j} + \frac{1}{2} \epsilon^2 \beta \sum_{j=1}^{N} f_j^2 F(H_j).
\]

(3.37)

This is due to the definitions of \(f_1, \ldots, f_N\) in (3.8).

Similarly, the \(\epsilon^2 |\ln \epsilon|^2\) terms in \(S(u_1)\) can be rearranged in the following way

\[
\epsilon^2 |\ln \epsilon|^2 E_{04} := \epsilon^2 \sum_{j=1}^{N} \left[ (-2\beta' f_j' - \beta f_j'' H_{j,x_j}) - 2 \beta' f_j' x_j H_{j,x_jx_j} \right]
\]

\[+ \epsilon^2 \beta \sum_{j=1}^{N} f_j x_j F(H_j) - \epsilon^2 \beta \sum_{j=1}^{N} f_j H_{j,x_j},\]

and also

\[
\epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \sum_{k} \epsilon_k \beta \sum_{j=1}^{N} \tilde{H}_{j,s_j} - \epsilon_\beta s \sum_{j=1}^{N} F(H_j).
\]

(3.38)

For terms of order \(\epsilon^2\), we set

\[
\epsilon^2 E_{06} := \epsilon^2 \sum_{j=1}^{N} \left[ \left( \frac{\beta'}{\beta} \right)^2 x_j^2 H_{j,x_jx_j} + \frac{\beta''}{\beta} x_j H_{j,x_j} \right]
\]

\[
+ \frac{1}{2} \epsilon^2 \beta \sum_{j=1}^{N} x_j F(H_j) - \epsilon^2 \beta \sum_{j=1}^{N} x_j H_{j,x_j},
\]

and also

\[
E_{07} := \beta^2 \sum_{j=1}^{N} (-1)^j \chi_j \left[ 3(1 - H_j^2)(b_{2j} + b_{4j}) + \frac{1}{2} A_{1j} b_{1j}^2 \right]
\]

\[
+ \frac{1}{2} A_{3j} b_{3j}^2 + \frac{1}{2} A_{5j} [b_{1j} + b_{3j}]^2 \right] \]

\[+ \epsilon \beta_1 \beta^2 \sum_{j=1}^{N} (-1)^j \chi_j \left( \frac{x_j}{\beta} + f_j \right) \left[ -3(1 - H_j^2) (b_{1j} + b_{3j}) \right].
\]

(3.41)

From the definition of \(\tilde{x}_j = \beta(s + f_j)\) and the asymptotic behavior of \(H\) in (3.1), it is easy to derive that

\[
E_{08} := \epsilon^2 \sum_{j=1}^{N} H_{j,\tilde{x}_j} \left[ \left( \frac{\beta'}{\beta} \right)^2 \tilde{x}_j^2 + 2 \beta' \tilde{x}_j + \beta^2 (f_j')^2 \right]
\]

\[+ \epsilon^2 \sum_{j=1}^{N} \tilde{H}_{j,\tilde{x}_j} \left( \frac{\beta''}{\beta} \tilde{x}_j + 2 \beta' f_j + \beta f_j'' \right].
\]
\[
- \epsilon^2 k^2 \sum_{j=1}^{N} \bar{x}_j \tilde{H}_j \bar{x}_j + \epsilon^2 k^2 \beta \sum_{j=1}^{N} f_j \tilde{H}_j \bar{x}_j + \frac{1}{2} \epsilon^2 s^2 \beta_2 \sum_{j=1}^{N} F(\tilde{H}_j) \\
+ \frac{1}{2} \epsilon^2 \beta s^2 \left\{ F(u_1) - \sum_{j=1}^{N} \left[ F(H_j) + F(\tilde{H}_j) \right] \right\} + B_2(u_1) = O(\epsilon^{2+\sigma}).
\]

Combining the above expressions of components for \( S(u_1) \), we can rewrite the error for \((s, z) \in \mathcal{G}\) in the form

\[
S(u_1) = \epsilon \beta_1 \sum_{j=1}^{N} (\tilde{f}_j + f_j) \Xi_{0,j} + \epsilon \sum_{j=1}^{N} \Xi_{1,j} + \epsilon E_{02} + \epsilon^2 |\ln \epsilon|^2 E_{03} \\
+ \epsilon^2 |\ln \epsilon| E_{04} + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \Xi_{2,j} + \epsilon^2 E_{06} + E_{07} + O(\epsilon^{2+\sigma}).
\]

By the symmetry, we have the boundary error of \( u_1 \)

\[
\mathcal{D}(u_1) = -\frac{\partial u_1}{\partial s} \bigg|_{s=0} = -\beta \sum_{j=1}^{N} \left[ H_j \bar{x}_j + \tilde{H}_j \bar{x}_j \right] \bigg|_{s=0}
\]

\[
= -\beta \sum_{j=1}^{N} (-1)^j \left[ H_j'(\beta(s - f_j)) - H_j'(\beta(s + f_j)) \right] \bigg|_{s=0} = 0.
\]

### 3.2.3 More precise estimates of the error \( S(u_1) \)

For more precise estimates of the error terms, we will set

\[
f_j(\theta) = \tilde{f}_j(\theta) + \hat{f}_j(\theta), \quad \forall j = 1, \ldots, N,
\]

and

\[
\tilde{f}_0(\theta) = -\tilde{f}_1(\theta), \quad \hat{f}_0(\theta) = -\hat{f}_1(\theta).
\]

Here \( \tilde{f}_1, \ldots, \tilde{f}_N \) are of order \( O(1) \) and will be determined by solving system (3.63), see (3.68)–(3.70). On the other hand, \( \hat{f}_j \)'s will be chosen with the constraints (c.f. (3.92)–(3.95))

\[
\| \hat{f}_j \|_{\infty} \leq C \epsilon^{1/2}, \quad \forall j = 1, \ldots, N.
\]

Recalling the expressions of \( b_{1,j}, b_{3,j} \) in (A.1) and (A.3), we can rewrite \( \epsilon E_{02} \) as following

\[
\epsilon E_{02} := -3\beta^2 \sum_{j=1}^{N} (-1)^j \chi_j (1 - H_j^2) (b_{1,j} + b_{3,j}) := \epsilon E_{02}^1 + \epsilon^2 E_{02}^2,
\]

where

\[
\epsilon E_{02}^1 := -6\sqrt{2} \epsilon \beta^2 \sum_{j=1}^{N} \chi_j H_j x_j \left[ -(N - j + 1) e^{-\sqrt{2} x_j} e^{-\sqrt{2} \beta (f_j - f_{j-1})} \right. \\
+ (N - j) e^{\sqrt{2} x_j} e^{-\sqrt{2} \beta (f_{j+1} - f_j)} \right].
\]
and

\[
\epsilon^2 E_{02}^2 := -6 \sqrt{2} \epsilon^2 \beta^2 \sum_{j=1}^{N} \chi_j H_{j,x_j} \left[ (N - j + 1)^2 e^{-2\sqrt{2} x_j} e^{-2\sqrt{2} \beta (\bar{f}_j - \bar{f}_{j-1})} - (N - j)^2 e^{2\sqrt{2} x_j} e^{-2\sqrt{2} \beta (\bar{f}_{j+1} - \bar{f}_j)} \right] + O(\epsilon^2 + \sigma).
\]

We now decompose \( \epsilon E_{02}^1 \). It is easy to derive the following relations, for all \( j = 1, \ldots, N \),

\[
\gamma_{0,j} = \gamma_0 + O(\epsilon), \quad \gamma_{1,j} = \gamma_1 + O(\sqrt{\epsilon}), \quad i = 1, 2,
\]

where \( \gamma_0, \gamma_1 \) are defined as in (3.4) and for \( j = 1, \ldots, N \)

\[
\gamma_{0,j} := \int_{\mathbb{R}} \chi_j H_{j,x_j} dx_j,
\]

\[
\gamma_{1,j} := \int_{\mathbb{R}} \chi_j H_{j,x_j} e^{-\sqrt{2} x_j} dx_j,
\]

\[
\gamma_{2,j} := \int_{\mathbb{R}} \chi_j H_{j,x_j} e^{\sqrt{2} x_j} dx_j.
\]

For \( j = 1, \ldots, N + 1 \), we set the functions

\[
d_j(\theta) := \begin{cases} 
(N - j + 1) e^{-\sqrt{2} \beta(\theta)} (\bar{f}_j - \bar{f}_{j-1}(\theta)) & \text{when } j = 1, \ldots, N, \\
0 & \text{when } j = N + 1.
\end{cases}
\]

Therefore, by recalling \( \bar{f}_j \)'s in (3.43), we can further decompose \( \epsilon E_{02}^1 \) as the following

\[
\epsilon E_{02}^1 = -6 \sqrt{2} \epsilon^2 \beta^2 \sum_{j=1}^{N} \chi_j H_{j,x_j} \left[ -e^{-\sqrt{2} x_j} d_j e^{-\sqrt{2} \beta (\bar{f}_j - \bar{f}_{j-1})} + e^{\sqrt{2} x_j} d_{j+1} e^{-\sqrt{2} \beta (\bar{f}_{j+1} - \bar{f}_j)} \right]
\]

\[
= \epsilon \sum_{j=1}^{N} \chi_j \Xi_{3,j} + \epsilon \sum_{j=1}^{N} \chi_j \Xi_{4,j} + \epsilon \sum_{j=1}^{N} \chi_j \Xi_{5,j}.
\]

In the above, we have denoted

\[
\Xi_{3,j}(x_j, z) := -6 \sqrt{2} \beta^2 H_{j,x_j} \left[ -d_j e^{-\sqrt{2} \beta (\bar{f}_j - \bar{f}_{j-1})} e^{-\sqrt{2} x_j} + d_{j+1} e^{-\sqrt{2} \beta (\bar{f}_{j+1} - \bar{f}_j)} e^{\sqrt{2} x_j} \right]
\]

\[
+ 6 \sqrt{2} \beta^2 H_{j,x_j} \left[ -\frac{\gamma_{1,j}}{\gamma_0,j} d_j e^{-\sqrt{2} \beta (\bar{f}_j - \bar{f}_{j-1})} + \frac{\gamma_{2,j}}{\gamma_0,j} d_{j+1} e^{-\sqrt{2} \beta (\bar{f}_{j+1} - \bar{f}_j)} \right],
\]

in such a way that

\[
\int_{\mathbb{R}} \chi_j \Xi_{3,j}(x_j, z) H_{j,x_j} dx_j = 0,
\]

due to the definitions of \( \gamma_{0,j}, \gamma_{1,j}, \gamma_{2,j} \) in (3.46), (3.47), (3.48). Similarly, we have also set

\[
-6 \sqrt{2} \beta^2 H_{j,x_j} \left[ -\frac{\gamma_{1,j}}{\gamma_0,j} d_j e^{-\sqrt{2} \beta (\bar{f}_j - \bar{f}_{j-1})} + \frac{\gamma_{2,j}}{\gamma_0,j} d_{j+1} e^{-\sqrt{2} \beta (\bar{f}_{j+1} - \bar{f}_j)} \right] = \Xi_{4,j} + \Xi_{5,j},
\]

with

\[
\Xi_{4,j}(x_j, z) := -\frac{1}{\gamma_0,j} 6 \sqrt{2} \beta^2 H_{j,x_j} \left( -\gamma_{1,j} d_j + \gamma_{2,j} d_{j+1} \right),
\]
Indeed, it is easy to see that
\[
\Xi_{5,j}(x_j, z) := -\frac{1}{\gamma_{0,j}} 6\sqrt{2}\beta^2 H_{j,x_j} \left[ -\gamma_{1,j} d_j \left( e^{-\sqrt{2}\beta (f_j - f_{j-1})} - 1 \right) \\
+ \gamma_{2,j} d_{j+1} \left( e^{-\sqrt{2}\beta (f_{j+1} - f_j)} - 1 \right) \right].
\] (3.53)

On the other hand, we can also rewrite \(\epsilon^2 E_{02}^2\) as follows
\[
\epsilon^2 E_{02}^2 = -6\sqrt{2}\epsilon^2 \beta^2 \sum_{j=1}^{N} \chi_j H_{j,x_j} \left[ d_j^2 e^{-2\sqrt{2}\epsilon_j} e^{-2\sqrt{2}\beta (f_j - f_{j-1})} - d_{j+1}^2 e^{2\sqrt{2}\epsilon_j} e^{-2\sqrt{2}\beta (f_{j+1} - f_j)} \right] + O(\epsilon^{2+\sigma})
\]
\[
:= \epsilon^2 \ln \epsilon \sum_{j=1}^{N} \chi_j \Xi_{5,j} + O(\epsilon^{2+\sigma}).
\] (3.54)

From the above expressions of all components, we see that
\[
S(u_1) = \epsilon \beta_1 \sum_{j=1}^{N} (\dot{f}_j + \ddot{f}_j + \dddot{f}_j) \Xi_{0,j} + \epsilon \sum_{j=1}^{N} \Xi_{1,j} + \epsilon \sum_{j=1}^{N} \chi_j \Xi_{3,j}
\]
\[
+ \epsilon \sum_{j=1}^{N} \chi_j \Xi_{4,j} + \epsilon \sum_{j=1}^{N} \chi_j \Xi_{5,j} + \epsilon^2 \ln \epsilon \sum_{j=1}^{N} \Xi_{2,j} + \epsilon^2 \ln \epsilon \sum_{j=1}^{N} \chi_j \Xi_{6,j}
\]
\[
+ \epsilon^2 \ln \epsilon \left( E_{03} + \epsilon^2 \ln \epsilon \right) E_{04} + \epsilon^2 E_{06} + E_{07} + O(\epsilon^{2+\sigma}).
\] (3.55)

### 3.3 The second approximation

By adding correction terms, we now want to construct a further approximation to a solution which eliminates the terms of orders \(O(\epsilon \ln \epsilon)\) and \(O(\epsilon)\), and then compute the new error.

#### 3.3.1 Choosing correction terms

To cancel the terms in \(\epsilon \beta_1 \sum_{j=1}^{N} \dot{f}_j \Xi_{0,j}\), we can choose the correction term as
\[
\epsilon \ln \epsilon |\phi_{11}(s, z) := \epsilon \frac{\beta_1}{\beta_2} \sum_{j=1}^{N} \dot{f}_j \psi_j(x_j) + \epsilon \frac{\beta_1}{\beta_2} \sum_{j=1}^{N} \dot{f}_j \psi_j(-x_j - 2\beta f_j)
\]
\[
= \epsilon \frac{\beta_1}{\beta_2} \sum_{j=1}^{N} \dot{f}_j \psi_j(\beta - f_j) + \epsilon \frac{\beta_1}{\beta_2} \sum_{j=1}^{N} \dot{f}_j \psi_j(-\beta (s + f_j)),
\] (3.56)

where \(\psi_j(x_j) = (-1)^j \psi(x_j)\) and the function \(\psi(x)\) is the unique, odd solution to the problem
\[
\psi_{xx} + (1 - 3H^2) \psi = -(H - H^3) \text{ in } \mathbb{R}, \quad \psi(\pm \infty) = 0, \quad \int_{\mathbb{R}} \psi(x) H_x \, dx = 0.
\] (3.57)

Indeed, it is easy to see that
\[
\psi(x) = \frac{1}{2} x H_x.
\]
Since the Neumann boundary condition is required, we have added the terms of $\psi_j(-\beta(s + f_j))$ in (3.56) so that
\[
\epsilon \ln \epsilon \left| \frac{\partial \phi_{11}(s, z)}{\partial s} \right|_{s=0} = 0.
\]

To cancel the terms in $\epsilon \sum_{j=1}^{N} \chi_j \Xi_{3,j}$, we will choose
\[
\epsilon \phi_{12}(s, z) := \epsilon \sum_{j=1}^{N} \psi_j^*(x_j, z) + \epsilon \sum_{j=1}^{N} \psi_j^*(-x_j - 2\beta f_j, z)
\]
\[
= \epsilon \sum_{j=1}^{N} \psi_j^*(\beta(s - f_j), z) + \epsilon \sum_{j=1}^{N} \psi_j^*(-\beta(s + f_j), z),
\]
where the function $\psi_j^*(x_j, z)$ is the unique solution to the following problem
\[
\psi_j^* + \beta^2 \left[ \psi_j^* x_j x_j + F'(H_{j}) \psi_j^* \right] = -\chi_j \Xi_{3,j}(x_j, z) \quad \text{in} \ \mathbb{R} \times \left[ 0, \frac{\ell}{\epsilon} \right),
\]
\[
\psi_j^* (x_j, 0) = \psi_j^* (x_j, \frac{\ell}{\epsilon}), \quad \psi_j^* (x_j, 0) = \psi_j^* (x_j, \frac{\ell}{\epsilon}),
\]
\[
\int_{\mathbb{R}} \psi_j^* H_j x_j \, dx_j = 0, \quad 0 < z < \frac{\ell}{\epsilon}.
\]

As a direct consequence of Lemma B.2 and (3.51), there exists a unique solution $\psi_j^*(x_j, z)$ to the problem (3.59)–(3.61). Moreover, we can get
\[
\epsilon \left| \frac{\partial \phi_{12}(s, z)}{\partial s} \right|_{s=0} = 0.
\]

On the other hand, if the terms
\[
\epsilon \beta_1 f_j \Xi_{0,j} + \epsilon \Xi_{1,j} + \epsilon \chi_j \Xi_{4,j}, \quad \forall \ j = 1, \ldots, N,
\]
can be cancelled in a similar way, we shall choose $\bar{f}_j$’s such that
\[
\int_{\mathbb{R}} \left[ \epsilon \beta_1 \bar{f}_j \Xi_{0,j} + \epsilon \Xi_{1,j} + \epsilon \chi_j \Xi_{4,j} \right] H_j x_j \, dx_j = 0, \quad \forall \ j = 1, \ldots, N.
\]

In fact, the equalities in (3.62) are equivalent to
\[
\frac{2\sqrt{2}}{3} k - \frac{\sqrt{2}}{3} \frac{\beta_1}{\beta^2} = 6\sqrt{2} \beta^2 \left[ (N - j + 1) \gamma_{1,j} e^{-\sqrt{2} \beta (\bar{f}_j - \bar{f}_{j-1} - f_j)} 
\right.
\]
\[
\left. - (N - j) \gamma_{2,j} e^{-\sqrt{2} \beta (\bar{f}_{j+1} - \bar{f}_j)} \right]
\]
for $j = 1, \ldots, N$, where $\gamma_{i,j}, i = 1, 2$ are defined in (3.47)–(3.48). The relations in (3.63) can be rewritten in the forms
\[
\frac{2\sqrt{2}}{3} k - \frac{\sqrt{2}}{3} \frac{\beta_1}{\beta^2} = 6\sqrt{2} \beta^2 \left[ N \gamma_{1,1} e^{-\sqrt{2} \beta (\bar{f}_1 - \bar{f}_0)} - (N - 1) \gamma_{2,1} e^{-\sqrt{2} \beta (\bar{f}_2 - \bar{f}_1)} \right],
\]
\[
\frac{2\sqrt{2}}{3} k - \frac{\sqrt{2}}{3} \frac{\beta_1}{\beta^2} = 6\sqrt{2} \beta^2 \left[ (N - 1) \gamma_{1,2} e^{-\sqrt{2} \beta (\bar{f}_2 - \bar{f}_1)} - (N - 2) \gamma_{2,2} e^{-\sqrt{2} \beta (\bar{f}_3 - \bar{f}_2)} \right],
\]
\[\ldots \quad \ldots \quad \ldots \]
\[
\frac{2\sqrt{2}}{3} k - \frac{\sqrt{2}}{3} \frac{\beta_1}{\beta^2} = 6\sqrt{2} \beta^2 \left[ 2\gamma_{1,N-1} e^{-\sqrt{2} \beta (\bar{f}_{N-1} - \bar{f}_{N-2})} - \gamma_{2,N-1} e^{-\sqrt{2} \beta (\bar{f}_N - \bar{f}_{N-1})} \right].
\]
\[
\frac{2\sqrt{2}}{3} k - \frac{\sqrt{2}}{3} \beta_1 \beta_2^2 = 6\sqrt{2} \beta^2 \gamma_{1,N} e^{-\sqrt{2} \beta (\bar{f}_N - \bar{f}_{N-1})}.
\] (3.67)

Combining the assumption in (1.6), the constraint \( \bar{f}_0 = -\bar{f}_1 \) and the relations (3.45), there exists a unique solution \( \bar{f} = (\bar{f}_1, \ldots, \bar{f}_N)^T \) to the nonlinear algebraic system (3.64)–(3.67). Moreover, we can conclude that
\[
\bar{f}_0 = -\bar{f}_1 \approx \frac{1}{2\sqrt{2} \beta} \left[ \ln \left( k - \frac{\beta_1}{2\beta^2} \right) - \ln(9 \gamma_1 \beta^2) \right],
\] (3.68)
\[
\bar{f}_2 \approx -\frac{3}{2\sqrt{2} \beta} \left[ \ln \left( k - \frac{\beta_1}{2\beta^2} \right) - \ln(9 \gamma_1 \beta^2) \right],
\] (3.69)
\[
\bar{f}_N \approx -\frac{2N - 1}{2\sqrt{2} \beta} \left[ \ln \left( k - \frac{\beta_1}{2\beta^2} \right) - \ln(9 \gamma_1 \beta^2) \right],
\] (3.70)
where \( \gamma_1 \) is the constant defined in (3.4).

Then, by (3.62) we can choose
\[
\epsilon \phi_{13}(s, z) := \epsilon \sum_{j=1}^{N} \psi_j^{**}(x_j, z) + \epsilon \sum_{j=1}^{N} \psi_j^{**}(-x_j - 2\beta f_j, z)
= \epsilon \sum_{j=1}^{N} \psi_j^{**}(\beta(s - f_j), z) + \epsilon \sum_{j=1}^{N} \psi_j^{**}(\beta(s + f_j), z),
\] (3.71)
where \( \psi_j^{**}(x_j, z) \) satisfies
\[
\psi_j^{**}(x_j, z) + \beta^2 \left[ \psi_j^{**}(x_j, x_j) + F'(H_j)\psi_j^{**} \right] = -\left[ \beta_1 \beta_j \Xi_{0,j} + \Xi_{1,j} + x_j \Xi_{4,j} \right] \text{ in } \mathbb{R} \times \left[ 0, \frac{\ell}{\epsilon} \right),
\] (3.72)
\[
\psi_j^{**}(x_j, 0) = \psi_{j, z}(x_j, \frac{\ell}{\epsilon}), \quad \psi_j^{**}(x_j, 0) = \psi_{j, z}(x_j, \frac{\ell}{\epsilon}),
\] (3.73)
\[
\int_{\mathbb{R}} \psi_j^{**} H_{j, x_j} \, dx_j = 0, \quad 0 < z < \frac{\ell}{\epsilon}.
\] (3.74)

It can be checked that
\[
\epsilon \frac{\partial \phi_{13}(s, z)}{\partial s} \bigg|_{s=0} = 0.
\]

Finally, we define the first correction as
\[
\phi_1(s, z) = \epsilon \ln \epsilon | \phi_{11}(s, z) + \epsilon \phi_{12}(s, z) + \epsilon \phi_{13}(s, z)
= \epsilon \beta^{-2} \beta_1 \sum_{j=1}^{N} \left[ \hat{f}_j \psi_j(x_j) + \hat{f}_j \psi_j(\tau_j) \right] + \epsilon \sum_{j=1}^{N} \left[ \psi_j^{*}(x_j, z) + \psi_j^{*}(\tau_j, z) \right]
\]
\[
+ \epsilon \sum_{j=1}^{N} \left[ \psi_j^{**}(x_j, z) + \psi_j^{**}(\tau_j, z) \right],
\] (3.75)
where we have denoted
\[
\tau_j(s, z) = -x_j - 2\beta f_j = -\beta(s + f_j).
\] (3.76)
Then we can get easily that \( \phi_1 \) satisfies

\[
D(\phi_1) = 0.
\] (3.77)

We take

\[
u_2(s, z) := u_1(s, z) + \phi_1(s, z)
\] (3.78)
as the second approximate solution.

### 3.3.2 The error to the second approximation

The new error can be computed as the following

\[
S(u_2) = S(u_1) + L_{u_1}(\phi_1) + N_{u_1}(\phi_1) + B_3(\phi_1)
+ B_2(u_1 + \phi_1) - B_2(u_1) + B_4(u_1 + \phi_1) - B_4(u_1),
\] (3.79)

with

\[
L_{u_1}(\phi_1) = \phi_{1,zz} + \phi_{1,ss} + \beta^2 (1 - 3u_1^2) \phi_1, \quad N_{u_1}(\phi_1) = \beta^2 (-3 u_1 \phi_1^2 - \phi_1^3).
\] (3.80)

We will give the details of computations for the new error components in the sequel.

First, recalling the definition of \( \tau_j \) as in (3.76), it is easy to obtain that

\[
\tau_{j,zz}(s, z) = ( - \beta s - \beta f_j)_z = - \epsilon \left[ \frac{\beta'}{\beta} x_j + \beta f_j' + 2 \beta' f_j \right],
\] (3.81)

\[
\tau_{j,zzz}(s, z) = ( - \beta s - \beta f_j)_{zz} = - \epsilon^2 \left[ \frac{\beta''}{\beta} x_j + 2 \beta'' f_j' + \beta f_j'' + 2 \beta' f_j' \right].
\] (3.82)

Here we declare that we will use the symbols \( \psi_j, \psi_j^*, \psi_j^{**} \) to denote \( \psi_j(x_j), \psi_j^*(x_j, z), \psi_j^{**}(x_j, z) \) in the sequel for simplicity.

According to the definition of \( \phi_1 \) as in (3.75), we have

\[
L_{u_1}(\phi_1) = \epsilon \sum_{j=1}^{N} \beta_1 \hat{f}_j \left[ \psi_{j,x_jx_j} + (1 - 3H_j^2) \psi_j \right]
+ \epsilon \sum_{j=1}^{N} \left\{ \psi_{j,zz}^* + \beta^2 [\psi_{j,x_jx_j}^* + (1 - 3H_j^2) \psi_j^*] \right\}
+ 2 \epsilon^2 \sum_{j=1}^{N} \left[ \psi_{j,zx_j}^* \left\{ \frac{\beta'}{\beta} x_j - \beta f_j' \right\} \right]
+ \epsilon \sum_{j=1}^{N} \left\{ \psi_{j,zzz}^* + \beta^2 [\psi_{j,x_jx_j}^{**} + (1 - 3H_j^2) \psi_j^{**}] \right\}
+ 2 \epsilon^2 \sum_{j=1}^{N} \left[ \psi_{j,zx_j}^{**} \left\{ \frac{\beta'}{\beta} x_j - \beta f_j' \right\} + \epsilon \beta_1 \sum_{j=1}^{N} (3H_j^2 - 3u_1^2) \hat{f}_j \psi_j \right.
\]
\[ + \sum_{j=1}^{N} \left\{ \epsilon \beta_1 \hat{f}_j \left[ \psi_{j,x_jx_j} + (1 - 3u_1^2) \psi_j \right] + \epsilon \beta^2 \psi_{j,x_jx_j} + (1 - 3u_1^2) \psi_j \right] \]
\[ + \epsilon \psi_{j,zz}^*(\tau_j, z) + 2\epsilon \psi_{j,\tau_j}(\tau_j, z) \tau_{j,z} \\
+ \epsilon^{\beta^2} \left[ \psi_{j,\tau_j}^*(\tau_j, z) + (1 - 3u_1^2)\psi_j^{**}(\tau_j, z) \right] \\
+ \epsilon \psi_{j,zz}^*(\tau_j, z) + 2\epsilon \psi_{j,\tau_j}(\tau_j, z) \tau_{j,z} \right\} \\
+ \epsilon^{\beta^2} \sum_{j=1}^{N} (\beta \sum_{j=1}^{N} (3H_j^2 - 3u_1^2)(\psi_j^* + \psi_j^{**}) + O(\epsilon^{2+\sigma}) \\
= - \epsilon^2 \beta \sum_{j=1}^{N} \left[ \psi_{j,xz}(x_j, z) + \psi_{j,xx}(x_j, z) \right], \\
\epsilon^2 \psi_{8,j} := 2\epsilon \left[ \psi_{j,xx}(x_j, z) + \psi_{j,zz}(x_j, z) \right] \\
- 2\epsilon^2 \left[ \psi_{j,xx}(x_j, z) + \psi_{j,zz}(x_j, z) \right], \\
\text{and} \\
\mathcal{K}_1 := \sum_{j=1}^{N} \left\{ \epsilon \psi_{j,\tau_j}(\tau_j) + (1 - 3u_1^2)\psi_j^{**}(\tau_j) \right\} + \epsilon^2 \psi_{j,zz}(\tau_j, z) \\
+ \epsilon^{\beta^2} \left[ \psi_{j,\tau_j}^*(\tau_j, z) + (1 - 3u_1^2)\psi_j^{**}(\tau_j, z) \right] + \epsilon \psi_{j,zz}^*(\tau_j, z) \\
+ \epsilon^{\beta^2} \left[ \psi_{j,\tau_j}^*(\tau_j, z) + (1 - 3u_1^2)\psi_j^{**}(\tau_j, z) \right] + \epsilon \psi_{j,zz}^*(\tau_j, z) \right\}. \\
\text{According to the definition of } B_3 \text{ in (2.17), we have} \\
B_3(\phi_1) = - \epsilon^2 \beta^{-1} k \sum_{j=1}^{N} \left[ \beta \psi_{j,xj}(x_j, z) + \beta \psi_{j,xx}(x_j, z) \right] \\
+ \epsilon^2 \sum_{j=1}^{N} \left[ \beta^{-1} k \psi_{j,\tau_j}(\tau_j) + \beta k (\psi_{j,\tau_j}^*(\tau_j, z) + \psi_{j,\tau_j}^{**}(\tau_j, z)) \right] + O(\epsilon^{2+\sigma}) \\
= \epsilon^2 \ln \epsilon \sum_{j=1}^{N} \mathcal{E}_{9,j} + \mathcal{K}_2 + O(\epsilon^{2+\sigma}), \quad (3.86)
with
\[
\varepsilon^2 \ln |\varepsilon| \Xi_{0,j} := - \varepsilon^2 \beta^{-1} k \beta_1 \hat{f}_j \psi_{j,x_j}(x_j) - \varepsilon^2 \beta k \left[ \psi^*_{j,x_j}(x_j, z) + \psi^{**}_{j,x_j}(x_j, z) \right],
\] (3.87)
\[
\mathcal{K}_2 := \varepsilon^2 \sum_{j=1}^{N} \left[ \beta^{-1} k \beta_1 \hat{f}_j \psi_{j,t_j}(\tau_j) + \beta k \left( \psi^*_j(\tau_j, z) + \psi^{**}_j(\tau_j, z) \right) \right].
\] (3.88)

Moreover, we have
\[
B_2(u_1 + \phi_1) - B_2(u_1) = O(\varepsilon^{2+\sigma}).
\]

Recalling the definition of the function $B_4(u_1)$ in (3.16) and together with (3.43)–(3.44), by calculating, we can obtain
\[
B_4(u_1 + \phi_1) - B_4(u_1)
\]
\[
= \left[ \varepsilon s \beta_1 + \frac{1}{2} \varepsilon^2 s^2 \beta_2 \right] \sum_{j=1}^{N} (1 - 3 H_j^2) \phi_1
\]
\[
+ \left[ \varepsilon s \beta_1 + \frac{1}{2} \varepsilon^2 s^2 \beta_2 \right] \sum_{j=1}^{N} (3 H_j^2 - 3 u_1^2) \phi_1 - 3 u_1 \phi_1^2 - \phi_1^3
\]
\[
= \varepsilon^2 \beta_1 \sum_{j=1}^{N} \left( \frac{x_j}{\beta} + f_j \right) (1 - 3 H_j^2) \left[ \beta_1 \hat{f}_j \beta^{-2} \psi_j(x_j)
\]
\[
+ \psi^{*}_j(x_j, z) + \psi^{**}_j(x_j, z) \right] + O(\varepsilon^{2+\sigma})
\]
\[
= \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} \Xi_{10,j} + O(\varepsilon^{2+\sigma}),
\] (3.89)

with
\[
\varepsilon^2 |\ln \varepsilon| \Xi_{10,j} := \varepsilon^2 \beta_1 \left( \frac{x_j}{\beta} + \hat{f}_j + \tilde{f}_j \right) (1 - 3 H_j^2) \left[ \beta_1 \hat{f}_j \beta^{-2} \psi_j(x_j) + \psi^{*}_j(x_j, z) + \psi^{**}_j(x_j, z) \right].
\] (3.90)

For the nonlinear term $N_{u_1}(\phi_1)$, we have that
\[
N_{u_1}(\phi_1) = \beta^2 (-3 u_1 \phi_1^2 - \phi_1^3) = -3 \beta^2 u_1 \phi_1^2 + O(\varepsilon^{2+\sigma}).
\]

Combining (3.55), (3.79), (3.86), (3.89), the new error can be expressed as the following
\[
S(u_2) = \varepsilon \beta_1 \sum_{j=1}^{N} \hat{f}_j \Xi_{0,j} + \varepsilon \sum_{j=1}^{N} x_j \Xi_{5,j} + 3 \beta^2 \varepsilon \sum_{j=1}^{N} (H_j^2 - u_1^2) (\beta_1 \beta^{-2}) \hat{f}_j \psi_j + \psi^{*}_j + \psi^{**}_j
\]
\[
+ \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} \Xi_{2,j} + \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} x_j \Xi_{6,j} + \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} \Xi_{7,j} + \varepsilon^2 \sum_{j=1}^{N} \Xi_{8,j}
\]
\[
+ \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} \Xi_{9,j} + \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} \Xi_{10,j} + \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} \Xi_{10,j} + \varepsilon^2 |\ln \varepsilon| \sum_{j=1}^{N} \Xi_{10,j} + \varepsilon^2 |\ln \varepsilon|^2 E_{53}
\]
\[
+ \varepsilon^2 E_{60} + E_{67} + \mathcal{K}_1 + \mathcal{K}_2 + N_{u_1}(\phi_1) + O(\varepsilon^{2+\sigma}).
\] (3.91)

### 3.3.3 More precise estimates of the error $S(u_2)$

In order to get more precise estimates of the error, we set
\[
\hat{f}_j(\theta) = \tilde{f}_j(\theta) + \tilde{f}_j(\theta), \quad \forall j = 1, \ldots, N,
\] (3.92)

and in particular, let
\[
\tilde{f}_0 = -\tilde{f}_1, \quad \tilde{f}_0 = -\tilde{f}_1.
\] (3.93)
The parameters $\tilde{f}_1, \ldots, \tilde{f}_n$ will be found by solving an algebraic system in (3.111) so that

$$\|\tilde{f}_j\|_{L^\infty(0,\ell)} \leq C\epsilon^{1/2}, \quad j = 1, \ldots, N.$$  \hfill (3.94)

In all what follows, we will make the following assumption on the parameter $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N)^T$

$$\|\tilde{f}\| = \epsilon\|\tilde{f}'\|_{L^2(0,\ell)} + \epsilon^{1/2}\|\tilde{f}\|_{L^2(0,\ell)} + \|\tilde{f}\|_{L^\infty(0,\ell)} = O(\epsilon^{1/2+\sigma}),$$  \hfill (3.95)

which will be determined in Sects. 5 and 6, by using the reduction method.

Similar to (3.49), we set the functions

$$k_j(\theta) := \begin{cases} d_j(\theta)e^{-\sqrt{2}\beta(\theta)(\tilde{f}_j(\theta)-\tilde{f}_{j-1}(\theta))}, & \text{when } j = 1, \ldots, N, \\ 0, & \text{when } j = N + 1. \end{cases}$$  \hfill (3.96)

Therefore, for $j = 1, \ldots, N$, we can decompose the term $\Xi_{5,j}$ (c.f. (3.53)) in the form

$$\Xi_{5,j} = \Xi^1_{5,j} + \Xi^2_{5,j}$$  \hfill (3.97)

where

$$\Xi^1_{5,j} := -6\sqrt{2} \frac{1}{\gamma_0 j} \beta^2 H_j x_j \left[ -\gamma_{1,j} (k_j - d_j) + \gamma_{2,j} (k_{j+1} - d_{j+1}) \right],$$  \hfill (3.98)

and

$$\Xi^2_{5,j} := -6\sqrt{2} \frac{1}{\gamma_0 j} \beta^2 H_j x_j \left[ -\gamma_{1,j} k_j (e^{-\sqrt{2}\beta(\tilde{f}_j-\tilde{f}_{j-1})} - 1) + \gamma_{2,j} k_{j+1} (e^{-\sqrt{2}\beta(\tilde{f}_{j+1}-\tilde{f}_j)} - 1) \right].$$  \hfill (3.99)

Recalling the definition of $E_{07}$ in (3.41) and the expressions of $b_{1,j}$, $b_{3,j}$ as in (A.1) and (A.3), we can rewrite $E_{07}$ as

$$E_{07} = E^1_{07} + E^2_{07}.$$  \hfill (3.100)

Here

$$E^1_{07} = \beta^2 \sum_{j=1}^N (-1)^j \chi_j \left[ \frac{1}{2} A_{1,j} b_{1,j}^2 + \frac{1}{2} A_{3,j} b_{3,j}^2 + \frac{1}{2} A_{5,j} b_{1,j}^2 + \frac{1}{2} A_{5,j} b_{3,j}^2 \right]$$

$$= 2\epsilon^2 \beta^2 \sum_{j=1}^N (-1)^j \chi_j \left[ (A_{1,j} + A_{5,j})k_j^2 e^{-2\sqrt{2}x_j} e^{-2\sqrt{2}\beta(\tilde{f}_j-\tilde{f}_{j-1})} + (A_{3,j} + A_{5,j})k_{j+1}^2 e^{2\sqrt{2}x_j} e^{-2\sqrt{2}\beta(\tilde{f}_{j+1}-\tilde{f}_j)} \right] + O(\epsilon^{2+\sigma})$$

$$= \epsilon^2 \sum_{j=1}^N \chi_j \Xi_{11,j} + O(\epsilon^{2+\sigma}),$$  \hfill (3.101)

with

$$\epsilon^2 \Xi_{11,j} := 2\epsilon^2 \beta^2 (-1)^j \left[ (A_{1,j} + A_{5,j})k_j^2 e^{-2\sqrt{2}x_j} + (A_{3,j} + A_{5,j})k_{j+1}^2 e^{2\sqrt{2}x_j} \right].$$  \hfill (3.102)

On the other hand, we can rewrite $E^2_{07}$ as the following

$$E^2_{07} = \beta^2 \sum_{j=1}^N (-1)^j \chi_j 3(1 - H_j^2)(b_{2,j} + b_{4,j}) + \beta^2 \sum_{j=1}^N (-1)^j \chi_j A_{5,j} b_{1,j} b_{3,j}$$
\[-\epsilon \beta_1 \beta^2 \sum_{j=1}^{N} (-1)^j \chi_j \left(\frac{X_j}{\beta} + f_j\right) 3(1 - H_j^2) (b_{1j} + b_{3j}) \]

\[= 6 \sqrt{2} \epsilon \beta^2 \sum_{j=1}^{N} \chi_j H_{j,x_j} \left[ -k_j k_{j-1} e^{-\sqrt{2} \beta} (f_j - \bar{f}_{j-2}) + k_{j+1} k_{j+2} e^{\sqrt{2} \beta} (f_{j+2} - \bar{f}_j) \right] \]

\[= 6 \sqrt{2} \epsilon \beta^2 \sum_{j=1}^{N} \chi_j \left(\frac{X_j}{\beta} + \bar{f}_j + \bar{f}_j\right) H_{j,x_j} \]

\[\times \left[ -k_j e^{-\sqrt{2} \beta} (\bar{f}_j - \bar{f}_{j-1}) + k_{j+1} e^{\sqrt{2} \beta} (\bar{f}_{j+1} - \bar{f}_j) \right] \]

\[-4 \epsilon \beta^2 \sum_{j=1}^{N} (-1)^j \chi_j A_{5j} k_j k_{j+1} e^{-\sqrt{2} \beta} (\bar{f}_{j+1} - \bar{f}_{j-1}) + O(\epsilon^{2+\sigma}) \]

\[= \epsilon^2 \ln \epsilon \sum_{j=1}^{N} \chi_j \Xi_{12,j} + O(\epsilon^{2+\sigma}), \]

with

\[\epsilon^2 \ln \epsilon \Xi_{12,j} := 6 \sqrt{2} \epsilon \beta^2 H_{j,x_j} \left[ -k_j k_{j-1} e^{-\sqrt{2} \beta} + k_{j+1} k_{j+2} e^{\sqrt{2} \beta} \right] \]

\[= 6 \sqrt{2} \epsilon \beta^2 \left(\frac{X_j}{\beta} + \bar{f}_j\right) H_{j,x_j} \left[ -k_j e^{-\sqrt{2} \beta} + k_{j+1} e^{\sqrt{2} \beta} \right] \]

\[-4 \epsilon \beta^2 (-1)^j A_{5j} k_j k_{j+1}. \quad (3.103)\]

Thanks to the constraint of $\bar{f}$ as in (3.95) and Taylor expansion, the terms $e^{-\sqrt{2} \beta (\bar{f}_{j+1} - \bar{f}_j)} - 1$, $j = 1, \ldots, N$, are of order $\epsilon^{\frac{1}{2}+\sigma}$.

For the term $K_1$ (c.f. (3.85)), we can decompose it as follows

\[K_1 = \epsilon^2 K_3 + K_4, \]

where

\[\epsilon^2 K_3 = \epsilon \beta_1 f_1 \left[ \psi_{1,\bar{f}_1} (\bar{f}_1) + (1 - 3 H_1^2) \psi_1 (\bar{f}_1) \right] \]

\[+ \epsilon \beta^2 \left[ \psi_{1,\bar{f}_1} (\bar{f}_1, z) + (1 - 3 H_1^2) \psi_1^* (\bar{f}_1, z) \right] + \epsilon \psi_{1,zz} (\bar{f}_1, z) \]

\[+ \epsilon \beta^2 \left[ \psi_{1,\bar{f}_1}^* (\bar{f}_1, z) + (1 - 3 H_1^2) \psi_1^{**} (\bar{f}_1, z) \right] + \epsilon \psi_{1,zz}^* (\bar{f}_1, z). \quad (3.104)\]

\[K_4 = \sum_{j=2}^{N} \left[ \epsilon \beta_1 f_j \left[ \psi_{j,\bar{f}_j} (\bar{f}_j) + (1 - 3 u_1^2) \psi_j (\bar{f}_j) \right] \right. \]

\[+ \epsilon \beta^2 \left[ \psi_{j,\bar{f}_j} (\bar{f}_j, z) + (1 - 3 u_1^2) \psi_j^* (\bar{f}_j, z) \right] + \epsilon \psi_{j,zz} (\bar{f}_j, z) \]

\[+ \epsilon \beta^2 \left[ \psi_{1,\bar{f}_j}^* (\bar{f}_j, z) + (1 - 3 u_1^2) \psi_{j,zz}^* (\bar{f}_j, z) \right] + \epsilon \psi_{j,zz}^* (\bar{f}_j, z) \]

\[+ \epsilon \beta_1 f_1 \left[ \psi_{1,\bar{f}_1} (\bar{f}_1) - \psi_{1,\bar{f}_1} (\bar{f}_1) + (1 - 3 H_1^2) \psi_1 (\bar{f}_1) - \psi_1 (\bar{f}_1) \right] \]

\[+ \epsilon \left[ \psi_{1,zz} (\bar{f}_1, z) - \psi_{1,zz}^* (\bar{f}_1, z) \right] + \epsilon \left[ \psi_{1,zz}^* (\bar{f}_1, z) - \psi_{1,zz} (\bar{f}_1, z) \right] \]

\[+ \epsilon \beta^2 \left[ \psi_{1,\bar{f}_1} (\bar{f}_1, z) - \psi_{1,\bar{f}_1}^* (\bar{f}_1, z) + (1 - 3 H_1^2) (\psi_1 (\bar{f}_1) - \psi_1 (\bar{f}_1)) \right] \]
\[ + \epsilon \beta^2 \left[ \psi_{1,1\tilde{r}_1}(\tau_1, z) - \psi_{1,\tilde{r}_1}(\tilde{r}_1, z) + (1 - 3H_1^2)(\psi_{1*}(\tau_1, z) - \psi_{1*}(\tilde{r}_1, z)) \right]. \]  

(3.105)

with

\[ \tilde{r}_1 = -x_1 - 2\beta(f_1 + \tilde{f}_1 + \tilde{f}_1). \]  

(3.106)

Then the new error can be rewritten as the following

\[ S(u_2) = \epsilon \beta^1 \sum_{j=1}^{N} (f_j + \tilde{f}_j) \Xi_{0,j} + \epsilon \sum_{j=1}^{N} \chi_j \Xi_{5,j}^1 + \epsilon \sum_{j=1}^{N} \chi_j \Xi_{11,j} \]

\[ + 3\beta^2 \sum_{j=1}^{N} (H_1^2 - u_1^2) \left[ \beta \beta^{-2} \tilde{f}_j \psi_j + \psi_j^* + \psi_j^{**} \right] + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \Xi_{2,j} \]

\[ + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \chi_j \Xi_{6,j} + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \Xi_{7,j} + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \Xi_{8,j} + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \Xi_{9,j} \]

\[ + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \Xi_{10,j} + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \chi_j \Xi_{11,j} + \epsilon^2 |\ln \epsilon| \sum_{j=1}^{N} \Xi_{12,j} + \epsilon^2 |\ln \epsilon| |E_{04} \]

\[ + \epsilon^2 |\ln \epsilon|^2 E_{03} + \epsilon^2 E_{06} + K_2 + \epsilon^3 K_3 + K_4 + N_{u_1}(\phi_1) + O(\epsilon^{2+\sigma}). \]  

(3.107)

3.4 The third approximation

It turns out that, in the expression of \( S(u_2) \), the terms of order \( \epsilon^{3/2} \) are

\[ \epsilon \beta^1 \sum_{j=1}^{N} \tilde{f}_j \Xi_{0,j}, \quad \epsilon \frac{3}{2} K_3, \quad \epsilon \sum_{j=1}^{N} \chi_j \Xi_{5,j}^1 \quad \text{and} \quad \epsilon \frac{3}{2} \sum_{j=1}^{N} \chi_j \Xi_{11,j}. \]

Since these terms are too large for our purpose, we add a correction to the approximate solution to cancel these terms and then compute the corresponding error.

3.4.1 New correction terms

In order to cancel the terms

\[ \epsilon \beta^1 \sum_{j=1}^{N} \tilde{f}_j \Xi_{0,j} + \epsilon \frac{3}{2} K_3 + \epsilon \sum_{j=1}^{N} \chi_j \Xi_{5,j}^1 + \epsilon \frac{3}{2} \sum_{j=1}^{N} \chi_j \Xi_{11,j}, \]

for any \( j = 1, \ldots, N \), we shall choose \( \tilde{f}_j \) such that

\[ \int_{\mathbb{R}} \left[ \epsilon \beta^1 \tilde{f}_j \Xi_{0,j} + \epsilon \frac{3}{2} K_3 + \epsilon \chi_j \Xi_{5,j}^1 + \epsilon \frac{3}{2} \chi_j \Xi_{11,j} \right] H_j, x_j \, dx_j = 0. \]  

(3.108)

Here we note that, by using the comparison principle argument and combining the effect of the cut-off functions \( \chi_j \), there exists a universal constant \( C \) such that

\[ |\psi_j(x_j)| \leq Ce^{-\sqrt{2} \beta (s-f_j)}, \quad |\psi_{j*}(x_j, z)| \leq Ce^{-\sqrt{2} \beta (s-f_j)}, \]  

(3.109)
Then, we will choose the second correction term as

\[ |\psi^*(x_j, z)| \leq \begin{cases} 
C, & \text{when } s \in \{s \in (0, +\infty) : (s, z) \in \mathcal{B}_j\}, \\
Ce^{-\sqrt{2}b(s-f_j)}, & \text{when } s \notin \{s \in (0, +\infty) : (s, z) \in \mathcal{B}_j\},
\end{cases} \tag{3.110} \]

where \( \mathcal{B}_j \) is the support of \( \chi_j \). Recall the expression of \( \varepsilon^{\frac{3}{2}}k_3 \) in (3.104) and the definition of \( \bar{t}_1 \) in (3.106), we can obtain

\[ \varepsilon^\frac{3}{2} \int_{\mathbb{R}} K_3 H_{j,x_j} \, dx_j = \varepsilon^\frac{3}{2} \mathcal{X}_0(\hat{f}_j, \tilde{f}_j, \bar{f}_j) \delta_{j1} + O(\varepsilon^{2+\sigma}), \]

where function \( \mathcal{X}_0(\hat{f}_j, \tilde{f}_j, \bar{f}_j) \) is of order \( O(1) \) and we can show easily the Lipschitz dependence on \( \bar{f}_j \)

\[ |\mathcal{X}_0(\hat{f}_j, \tilde{f}_j, \bar{f}_j^{(1)}) - \mathcal{X}_0(\hat{f}_j, \tilde{f}_j, \bar{f}_j^{(2)})| \leq C|\bar{f}_j^{(1)} - \bar{f}_j^{(2)}|. \]

Thus the equalities in (3.108) give that, for \( j = 1, \ldots, N \),

\[ 6\sqrt{2} \varepsilon \beta^2 \left[ -\gamma_{1,j} d_j (e^{-\sqrt{2}b(f_j-f_j-1)} - 1) + \gamma_{2,j} d_{j+1} (e^{-\sqrt{2}b(f_j+1-f_j)} - 1) \right] \]

\[ + \varepsilon^\frac{3}{2} \mathcal{X}_0(\hat{f}_j, \tilde{f}_j, \bar{f}_j) \delta_{j1} = 2\varepsilon^\frac{3}{2} \beta^2 \left( (A_{1,j} + A_{5,j}) \gamma_{3,j} k_j^2 \right) + O(\varepsilon^{2+\sigma}), \tag{3.111} \]

where

\[ \varepsilon^{-1} \gamma_{3,j} = (-1)^j \int_{\mathbb{R}} \chi_j e^{-\sqrt{2}x_j} H_{j,x_j} \, dx_j, \]

\[ \varepsilon^{-1} \gamma_{4,j} = (-1)^j \int_{\mathbb{R}} \chi_j e^{\sqrt{2}x_j} H_{j,x_j} \, dx_j, \]

and \( \gamma_{3,j}, \gamma_{4,j} \) are positive functions of order \( O(1) \), functions \( d_j, k_j \) are defined in (3.49) and (3.96). By using the similar method in solving system (3.63) and combining with Contraction Mapping Principle, we can find solution \( \hat{f}(\theta) = (\hat{f}_1(\theta), \ldots, \hat{f}_N(\theta))^T \) of (3.111) with the constraints in (3.94).

Therefore, combining Lemma B.2 and orthogonality conditions (3.108), we can obtain that there exists a unique solution \( \omega^*_j(x_j, z) \) satisfying

\[ \omega^*_j, zz + \beta^2 \left[ \omega^*_j, x_j, x_j + F'(H_j) \omega^*_j \right] = -\varepsilon^\frac{3}{2} \left[ \varepsilon^b_1 \hat{f}_j \mathcal{X}_{0,j} + \varepsilon^\frac{3}{2} k_3 \right] \]

\[ + \varepsilon \chi_j \mathcal{X}_{5,j}^1 + \varepsilon^\frac{3}{2} \chi_j \mathcal{X}_{11,j}, \] \quad \text{in } \mathbb{R} \times \left[ 0, \frac{\ell}{\varepsilon} \right), \tag{3.112} \]

\[ \omega^*_j(x_j, 0) = \omega^*_j \left( x_j, \frac{\ell}{\varepsilon} \right), \quad \omega^*_j, z(x_j, 0) = \omega^*_j \left( x_j, \frac{\ell}{\varepsilon} \right), \] \quad \text{for } x_j \geq 0, \tag{3.113} \]

\[ \int_{\mathbb{R}} \omega^*_j H_{j,x_j} \, dx_j = 0, \quad 0 < z < \frac{\ell}{\varepsilon}. \tag{3.114} \]

Then, we will choose the second correction term as

\[ \varepsilon^\frac{3}{2} \phi_2(s, z) := \varepsilon^\frac{3}{2} \sum_{j=1}^{N} (\omega^*_j(x_j, z) + \sum_{j=1}^{N} \omega^*_j(-x_j - 2\beta f_j, z)) \]

\[ = \varepsilon^\frac{3}{2} \sum_{j=1}^{N} \omega^*_j(\beta(s + f_j), z) + \varepsilon^\frac{3}{2} \sum_{j=1}^{N} \omega^*_j(-\beta(s + f_j), z). \tag{3.115} \]

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Using the symmetry, we can get the boundary error for $\phi_2$

$$D(\phi_2) = 0. \quad (3.116)$$

### 3.4.2 The third approximate solution and its error

We define the **third approximate solution** as

$$u_3(s, z) := u_2(s, z) + \epsilon^\frac{3}{2} \phi_2(s, z). \quad (3.117)$$

The error is now expressed as

$$S(u_3) = S(u_2) + L_{u_2}(\epsilon^\frac{3}{2} \phi_2) + N_{u_2}(\epsilon^\frac{3}{2} \phi_2) + B_3(\epsilon^\frac{3}{2} \phi_2)$$

$$+ B_2(u_2 + \epsilon^\frac{3}{2} \phi_2) - B_2(u_2) + B_4(u_2 + \epsilon^\frac{3}{2} \phi_2) - B_4(u_2), \quad (3.118)$$

where

$$L_{u_2}(\epsilon^\frac{2}{2} \phi_2) = \epsilon^\frac{2}{3} \phi_{2,zz} + \epsilon^\frac{2}{3} \phi_{2,ss} + \epsilon^\frac{2}{3} \beta^2 (1 - 3u_2^2) \phi_2,$$

and

$$N_{u_2}(\epsilon^\frac{3}{2} \phi_2) = -\beta^2 \left[3u_2(\epsilon^\frac{3}{2} \phi_2)^2 + (\epsilon^\frac{3}{2} \phi_2)^3\right] = O(\epsilon^{2+\sigma}).$$

According to the expression of $\epsilon^\frac{3}{2} \phi_2$ as in (3.115), we obtain that

$$L_{u_2}(\epsilon^\frac{2}{2} \phi_2) = \epsilon^\frac{2}{3} \sum_{j=1}^{N} \left\{ \omega^*_{j,zz} + \beta^2 [\omega^*_{j,xj} + (1 - 3H_j^2)\omega^*_j] \right\} + B_5(\epsilon^\frac{3}{2} \phi_2) + K_5 + K_6$$

$$= \epsilon^\frac{2}{3} \beta_1 \sum_{j=1}^{N} \tilde{f}_j \Xi_{0,j} - \epsilon^\frac{3}{2} K_3 - \epsilon \sum_{j=1}^{N} \chi_j \Xi_{5,j}$$

$$- \epsilon^\frac{3}{2} \sum_{j=1}^{N} \chi_j \Xi_{11,j} + B_5(\epsilon^\frac{3}{2} \phi_2) + K_5 + K_6,$$

where

$$B_5(\epsilon^\frac{3}{2} \phi_2) := \epsilon^\frac{2}{3} \beta^2 \sum_{j=1}^{N} 3(H_j^2 - u_2^2)\omega^*_j + \epsilon^\frac{2}{3} \sum_{j=1}^{N} \omega^*_{j,xj} \left[ \frac{\beta'}{\beta} x_j - \beta f'_j \right]^2$$

$$+ \epsilon^\frac{2}{3} \sum_{j=1}^{N} \omega^*_{j,xx} \left[ \frac{\beta''}{\beta} x_j - 2\beta f'_j + \beta f''_j \right] + \epsilon^\frac{2}{3} \sum_{j=1}^{N} \omega^*_{j,xz} \left[ \frac{\beta'}{\beta} x_j - \beta f'_j \right].$$

$$K_5 := \epsilon^\frac{2}{3} \sum_{j=1}^{N} \left\{ \omega^*_{1,zz}(\tau_j, z) + \beta^2 [\omega^*_1,\tau_j,\tau_j(\tau_j, z) + (1 - 3H_j^2)\omega^*_1(\tau_j, z)] \right\},$$

$$K_6 := \epsilon^\frac{2}{3} \sum_{j=1}^{N} \left\{ \omega^*_{1,\tau_j,\tau_j}(\tau_j, z) \tau_j^2 + 2\omega^*_{1,\tau_j,\tau_j}(\tau_j, z) \tau_j + \omega^*_{1,\tau_j}(\tau_j, z) \tau_j \right\} = O(\epsilon^{2+\sigma}).$$

We can refer to (3.76), (3.81), (3.82) for the expressions of $\tau_j, \tau_j, \tau_j, \tau_j, \tau_j, \tau_j, \tau_j$. Recalling the definitions of operators $B_2, B_3, B_4$ in (2.16)–(2.18), we obtain that

$$B_3(\epsilon^\frac{3}{2} \phi_2) + B_5(\epsilon^\frac{3}{2} \phi_2) + B_2(u_2 + \epsilon^\frac{3}{2} \phi_2) - B_2(u_2) + B_4(u_2 + \epsilon^\frac{3}{2} \phi_2) - B_4(u_2) = O(\epsilon^{2+\sigma}).$$
According to the definition of $\tilde{f}_j(\theta)$ as in (3.92) and the expression of $\epsilon^2 \ln \epsilon \vert E_{04}$ in (3.38), we can decompose $\epsilon^2 \ln \epsilon \vert E_{04}$ as follows

$$
\epsilon^2 \ln \epsilon \vert E_{04} = \epsilon^2 \ln \epsilon \sum_{j=1}^{N} \Xi_{13,j} + \epsilon^2 \sum_{j=1}^{N} \Xi_{14,j},
$$

with

$$
\epsilon^2 \ln \epsilon \vert \Xi_{13,j} := \epsilon^2 \left[ -2\beta' (\tilde{f}_j' + \tilde{f}_j + \tilde{f}_j'') - \beta (\tilde{f}_j'' + \tilde{f}_j''') H_{j,x_j} - 2\epsilon^2 k^2 \beta (\tilde{f}_j + \tilde{f}_j + \tilde{f}_j) H_{j,x_j} + \epsilon^2 \beta^{-1} \beta_2 (\tilde{f}_j + \tilde{f}_j + \tilde{f}_j) F(H_j), \right.
$$

(3.119)

and

$$
\epsilon^2 \Xi_{14,j} := \epsilon^2 \left( -2\beta' \tilde{f}_j' - \beta \tilde{f}_j'' \right) H_{j,x_j} - 2\epsilon^2 \beta' \tilde{f}_j' x_j H_{j,x_j,x_j} - e^{2} k^2 \beta (\tilde{f}_j + \tilde{f}_j + \tilde{f}_j) H_{j,x_j} + \epsilon^2 \beta^{-1} \beta_2 \tilde{f}_j x_j F(H_j).
$$

(3.120)

According to the expression of $u_1$ in (3.29), we can obtain, for $(s, z) \in \mathbb{A}_j, j = 1, \ldots, N$

$$
H_j^2 - u_1^2 = H_j^2 - \left[ (-1)^j u_1 \right]^2
$$

$$
= H_j^2 - \left[ H (\beta (s - f_j)) - b_{1j} + b_{2j} - b_{3j} + b_{4j} + O(\epsilon^{2+\sigma}) \right]^2
$$

$$
= 2H(x_j) \left[ b_{1j} - b_{2j} + b_{3j} - b_{4j} \right] + O(\epsilon^{1+\sigma}).
$$

Therefore, we can write it as follows

$$
3 \epsilon \beta^2 \sum_{j=1}^{N} (H_j^2 - u_1^2) \left[ \beta_1 \beta^{-2} \tilde{f}_j \psi_j + \psi_j^* + \psi_j^{**} \right]
$$

$$
= 6 \epsilon \beta^2 \sum_{j=1}^{N} H(x_j) (b_{1j} + b_{3j}) \left[ \beta_1 \beta^{-2} \tilde{f}_j \psi_j + \psi_j^* + \psi_j^{**} \right] + O(\epsilon^{2+\sigma})
$$

$$
= 12 \epsilon \beta^2 \sum_{j=1}^{N} H(x_j) \left[ -k_j e^{-\sqrt{2} x_j} e^{-\sqrt{2} \beta (\tilde{f}_j - \tilde{f}_j - 1)} + k_{j+1} e^{\sqrt{2} x_j} e^{-2\sqrt{2} \beta (\tilde{f}_j - \tilde{f}_j)} \right]
$$

$$
\times \left[ \beta_1 \tilde{f}_j \beta^{-2} \psi_j + \psi_j^* + \psi_j^{**} \right] + O(\epsilon^{2+\sigma})
$$

$$
= 12 \epsilon \beta^2 \sum_{j=1}^{N} H(x_j) \left[ -k_j e^{-\sqrt{2} x_j} + k_{j+1} e^{\sqrt{2} x_j} \right] \times \left[ \beta_1 \tilde{f}_j \beta^{-2} \psi_j + \psi_j^* + \psi_j^{**} \right]
$$

$$
+ 12 \epsilon \beta^2 \sum_{j=1}^{N} H(x_j) \left[ -k_j e^{-\sqrt{2} x_j} \left[ e^{-\sqrt{2} \beta (\tilde{f}_j - \tilde{f}_j - 1)} - 1 \right] + k_{j+1} e^{\sqrt{2} x_j} \left[ e^{-2\sqrt{2} \beta (\tilde{f}_j - \tilde{f}_j)} - 1 \right] \right]
$$

$$
\times \left[ \beta_1 \tilde{f}_j \beta^{-2} \psi_j + \psi_j^* + \psi_j^{**} \right] + O(\epsilon^{2+\sigma})
$$

$$
= \epsilon^2 \ln \epsilon \sum_{j=1}^{N} \Xi_{15,j} + O(\epsilon^{2+\sigma}).
$$
with
\[ \epsilon^2 | \ln \epsilon | \Xi_{15,j} := 12 \epsilon^2 \beta^2 \sum_{j=1}^{N} H(x_j) \left( -k_j e^{-\sqrt{2}x_j} + k_{j+1} e^{\sqrt{2}x_j} \right) \left[ \beta_1 f_j \beta^{-2} \psi_j + \psi^*_j + \psi^{**}_j \right]. \quad (3.121) \]

Combining with (3.107), (3.118), and the expressions for
\[ L_{u_2}(\epsilon^2 \phi_2), \quad \epsilon^2 | \ln \epsilon | E_{04}, \quad 3 \epsilon \beta^2 \sum_{j=1}^{N} (H_j^2 - u_j^2) \left( \beta_1 f_j \beta^{-2} \psi_j + \psi^*_j + \psi^{**}_j \right) \]

we can obtain
\[ S(u_3) = 3 \epsilon \beta \sum_{j=1}^{N} f_j \Xi_{0,j} + \epsilon \sum_{j=1}^{N} \epsilon_j \Xi_{5,j} + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \epsilon_j \Xi_{6,j} \]
\[ + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \Xi_{7,j} + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \Xi_{8,j} + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \Xi_{9,j} + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \Xi_{10,j} \]
\[ + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \epsilon_j \Xi_{12,j} + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \Xi_{13,j} \]
\[ + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \epsilon_j \Xi_{14,j} + \epsilon^2 | \ln \epsilon | \sum_{j=1}^{N} \Xi_{15,j} \]
\[ + \epsilon^2 | \ln \epsilon | E_{03} + \epsilon^2 E_{06} + N_{u_1}(\phi_1) + Q_1 + \tilde{K}, \quad (3.122) \]

with
\[ \tilde{K} := K_2 + K_4 + K_5 + K_6, \quad (3.123) \]

and
\[ Q_1 := B_5(\epsilon^2 \phi_2) + N_{u_2}(\epsilon^2 \phi_2) + B_3(\epsilon^2 \phi_2) + B_2(u_2 + \epsilon^2 \phi_2) - B_2(u_2) + B_4(u_2 + \epsilon^2 \phi_2) - B_4(u_2) + O(\epsilon^{2+\sigma}) \quad (3.124) \]
\[ = O(\epsilon^{2+\sigma}). \]

In the last of this part, we will give the Neumann boundary error of the third approximate solution. By combining the results (3.42), (3.77), (3.116), we can get
\[ D(u_3) = 0. \quad (3.125) \]

4 The gluing procedure

Recall the set \( \mathcal{S} \) in (3.5) and the coordinates \((s, z)\) in (2.3). The gluing method from [12] will be used to transform problem (2.1) in \( \Omega_\epsilon \) into a projected problem on the infinite strip \( \mathcal{S} \), in which we can use the coordinates \((s, z)\). We define a smooth cut-off function \( \eta_3^\delta(s) = \eta_3^\delta(|s|) \), where \( \eta_3^\delta(t) \) is given by
\[ \eta_3^\delta(t) = 1 \quad \text{for} \quad 0 \leq t \leq \delta \quad \text{and} \quad \eta_3^\delta(t) = 0 \quad \text{for} \quad t > 2\delta, \]

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for any fixed number $\delta < \delta_0/100$ with small $\delta_0$ given in (1.4). For a solution in Theorem 1.1, the global approximation $H(\tilde{y})$ can be defined simply in the form

$$H(\tilde{y}) = \eta_{38}^\epsilon(s)\left[\eta_3(s, z) + (-1)^N + 1\right] + (-1)^N, \quad \forall \tilde{y} \in \Omega_\epsilon,$$

(4.1)

where $\eta_3$ is defined as in (3.117) and the relation between $\tilde{y}$ and $(s, z)$ is given in (2.3).

For a perturbation term $\Phi(\tilde{y}) = \eta_{38}^\epsilon \tilde{\phi} + \tilde{\psi}$ defined in $\Omega_\epsilon$, the function $u(\tilde{y}) = H(\tilde{y}) + \Phi(\tilde{y})$ satisfies (2.1) if the pair $(\tilde{\phi}, \tilde{\psi})$ satisfies the following coupled system:

$$\eta_{38}^\epsilon L(\tilde{\phi}) = \eta_6^\epsilon \left[ -\mathcal{E} + N(\eta_{38}^\epsilon \tilde{\phi} + \tilde{\psi}) - 3V(1 - H^2) \tilde{\psi} \right] \text{ in } \Omega_\epsilon,$$

(4.2)

$$\frac{\partial \tilde{\phi}}{\partial v} + \frac{\partial H}{\partial v} = 0 \text{ on } \partial \Omega_\epsilon,$$

(4.3)

and

$$\Delta \tilde{\psi} - 2V \tilde{\psi} + 3V(1 - \eta_6^\epsilon)(1 - H^2) \tilde{\psi} = -\epsilon^2(\Delta \eta_{38}^\epsilon) \tilde{\phi} - 2\epsilon \eta_{38}^\epsilon \cdot \nabla \tilde{\phi} - (1 - \eta_6^\epsilon)\mathcal{E} + (1 - \eta_6^\epsilon)N(\eta_{38}^\epsilon \tilde{\phi} + \tilde{\psi}) \text{ in } \Omega_\epsilon,$$

(4.4)

$$\frac{\partial \tilde{\psi}}{\partial v} = 0 \text{ on } \partial \Omega_\epsilon,$$

(4.5)

where

$$L(\tilde{\phi}) = \Delta \tilde{\phi} + V \left[1 - 3H^2\right] \tilde{\phi},$$

$$\mathcal{E} = \Delta H + V H(1 - H^2), \quad N(\tilde{\phi}) = V \tilde{\phi}^3 + 3V H \tilde{\phi}^2.$$}

Note that the nonlinear operator $N$ has a power-like behavior with power greater than one. For any given small $\tilde{\phi}$ satisfying the following decay property

$$|\nabla \tilde{\phi}| + |\tilde{\phi}| \leq e^{-v_1 \delta/\epsilon} \text{ for } s > \delta/\epsilon,$$

(4.6)

where $v_1$ is a very small constant, a direct application of Contraction Mapping Principle yields that (4.4)–(4.5) has a unique (small) solution $\tilde{\psi} = \tilde{\psi}(\tilde{\phi})$ with

$$\|\tilde{\psi}(\tilde{\phi})\|_{L^\infty} \leq C \epsilon \left[\|\tilde{\phi}\|_{L^\infty(s > \delta/\epsilon)} + \|\nabla \tilde{\phi}\|_{L^\infty(s > \delta/\epsilon)}\right] + e^{-v_2 \delta/\epsilon},$$

(4.7)

where $s > \delta/\epsilon$ denotes the complement of $\delta/\epsilon$-neighborhood of $\partial \Omega_\epsilon$ and $v_2$ denotes some small constant. Moreover, the nonlinear operator $\tilde{\psi}$ satisfies a Lipschitz condition of the form

$$\|\tilde{\psi}(\tilde{\phi}_1) - \tilde{\psi}(\tilde{\phi}_2)\|_{L^\infty} \leq C \epsilon \left[\|\tilde{\phi}_1 - \tilde{\phi}_2\|_{L^\infty(s > \delta/\epsilon)} + \|\nabla \tilde{\phi}_1 - \nabla \tilde{\phi}_2\|_{L^\infty(s > \delta/\epsilon)}\right].$$

(4.8)

Therefore, after solving (4.4)–(4.5), we can concern (4.2)–(4.3) as a local nonlinear problem involving $\tilde{\psi} = \tilde{\psi}(\tilde{\phi})$, which can be expressed in local coordinates $(s, z)$. Here is the setting-up.

1. Note first that $\frac{\partial H}{\partial v} = 0$ due to (3.125). The boundary condition in (4.3) has the form

$$\phi_s(0, z) = 0, \quad 0 < z < \frac{\ell}{\epsilon}.$$

(4.9)

2. We make an extension and consider the function $\tilde{\phi}$ defined in the strip $S$. In terms of $(s, \zeta)$, the operator $L$ can also be extended to a new one, say $L$, so that they coincide on the region $0 < s < \frac{10\delta}{\epsilon}$. In fact, by using the computations in Sect. 2, we set

$$L(\tilde{\phi}) = L_0(\tilde{\phi}) + \eta_{10\delta}^\epsilon(s)\left[B_1(\tilde{\phi}) + B_0(\tilde{\phi})\right],$$

(4.10)
where
\[
\mathcal{L}_0(\tilde{\phi}) = \tilde{\phi}_{xx} + \tilde{\phi}_{zz} + \beta^2 (1 - 3H^2) \tilde{\phi},
\]
\[
B_6(\tilde{\phi}) = \left[ \epsilon \beta_1 s + \frac{\epsilon^2}{2} \beta_2 s^2 + a_4(\epsilon s, \epsilon x) \epsilon^3 s^3 \right] (1 - 3H^2) \tilde{\phi},
\]
and \(B_1, a_4\) are given in (2.11) and (2.13).

(3). For the local form of the nonlinear part, we denote it by the notation
\[
\mathcal{N}(\tilde{\phi}) = \mathbf{N}(\eta_3^s \tilde{\phi} + \tilde{\psi}(\tilde{\phi})) - 3V(1 - H^2)\tilde{\psi}(\tilde{\phi}).
\]

(4). The term \(E\) can be locally recast in \((s, z)\) coordinate system by the relation
\[
\eta_3^s(s) E = \eta_3^s(s) \mathbf{S}(u_3),
\]
where for the clear expression of \(\mathbf{S}(u_3)\), the reader can refer to (3.122).

Note that the approximate solution \(u_3\) has unknown parameters \(\tilde{f}_1, \ldots, \tilde{f}_N, \) see (3.92) and (3.95). We shall deal with the following projected problem: for \(\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N)^T\), finding functions \(\tilde{\phi} \in H^2(\mathbb{S}), \ c = (c_1, \ldots, c_N)\) with \(c_j \in L^2(0, \ell)\), such that
\[
\mathcal{L}(\tilde{\phi}) = \eta_3^s(s) \left[ - E + \mathcal{N}(\tilde{\phi}) \right] + \sum_{j=1}^N c_j(\epsilon z) \chi_j(s, z) H'(\beta(s - f_j)) \quad \text{in} \, \mathbb{S},
\]
\[
\tilde{\phi}(s, 0) = \tilde{\phi}(s, \frac{\ell}{\epsilon}), \quad \tilde{\phi}_z(s, 0) = \tilde{\psi}(s, \frac{\ell}{\epsilon}), \quad 0 < s < +\infty,
\]
\[
\tilde{\phi}_z(0, z) = 0, \quad 0 < z < \frac{\ell}{\epsilon},
\]
\[
\int_0^{+\infty} \tilde{\phi}(s, z) \chi_j(s, z) H'(\beta(s - f_j)) ds = 0, \quad 0 < z < \frac{\ell}{\epsilon}, \quad j = 1, \ldots, N.
\]

The smooth cut-off functions \(\chi_1, \ldots, \chi_N\) are defined by (3.34). The resolution theory for \(\tilde{\phi}\) with the constraint (4.6) can be provided in the following:

**Proposition 4.1** There exist numbers \(D > 0, \nu_3 > 0\) such that for all sufficiently small \(\epsilon\) and all \(\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N)^T\) satisfying (3.95), problem (4.15)–(4.18) has a unique solution \(\tilde{\phi} = \tilde{\phi}(\tilde{f})\) which satisfies
\[
\|\tilde{\phi}\|_{H^2(\mathbb{S})} \leq D\epsilon^{\frac{3}{2}} |\ln \epsilon|^2,
\]
\[
\|\tilde{\phi}\|_{L^\infty(\mathbb{S})} + \|\nabla \tilde{\phi}\|_{L^\infty(\mathbb{S})} \leq e^{-\nu_3 \delta/\epsilon}.
\]

Besides \(\tilde{\phi}\) is a Lipschitz function of \(\tilde{f}\), and for given \(\tilde{f}_1, \tilde{f}_2 : (0, \ell) \to \mathbb{R}^N\) with constraint in (3.95) there holds
\[
\|\tilde{\phi}(\tilde{f}_1) - \tilde{\phi}(\tilde{f}_2)\|_{H^2(\mathbb{S})} \leq C\epsilon \|\tilde{f}_1 - \tilde{f}_2\|_{H^2(0, \ell)}.
\]

**Proof** Based on linearized theory developed in [13] and Contraction Mapping Theorem, we can prove Proposition 4.1 as done for Proposition 5.1 in [13]. The details of the proof will be omitted here. \(\Box\)
5 The deriving of the reduced equations

As a standard step in the reduction method to make the Lagrange multiplier $\epsilon$ vanish in (4.15), we will set up a system of differential equations involving $\tilde{f}$ with constraints in (3.95) such that

$$\epsilon(\tilde{f}) = 0.$$  

(5.1)

These equations are obtained by simply integrating the Eq. (4.15) (only in $s$) against the functions

$$H_{n,x_n} = (-1)^n H'(\beta(s - f_n)), \quad n = 1, \ldots, N.$$  

It is easy to derive that (5.1) is equivalent to the following relations

$$\int_{0}^{+\infty} \left[ \eta^\delta(s) \mathcal{E}(s, z) - \eta^\delta(s) \mathcal{N}(\phi) + \mathcal{L}(\phi) \right] H_{n,x_n} \, ds = 0 \quad \text{for} \quad n = 1, \ldots, N. \quad \text{(5.2)}$$

We will give the details of computations for the terms in (5.2) in the sequel.

To start the computations for the first term in (5.2), by recalling the definitions of $\mathfrak{A}_n$ in (3.18)-(3.19), for each $n$ with $n = 1, \ldots, N,$ we introduce the notation

$$\mathfrak{A}_n = \left\{ s \in (0, +\infty) : (s, z) \in \mathfrak{A}_n \right\}. \quad \text{(5.3)}$$

Then, we will consider the integrals for $n = 1, \ldots, N,\,$

$$\int_{0}^{+\infty} \eta^\delta(s) \mathcal{E}(s, z) \, H_{n,x_n} \, ds = \left\{ \int_{\mathfrak{A}_n} + \int_{\mathbb{R}^+ \setminus \mathfrak{A}_n} \right\} \eta^\delta(s) \mathcal{E}(s, z) \, H_{n,x_n} \, ds$$

$$:= \mathbb{E}_{n1} + \mathbb{E}_{n2}. \quad \text{(5.4)}$$

Note that, if $(s, z) \in \mathfrak{A}_n,$ we have

$$\eta^\delta(s) \mathcal{E}(s, z) = \mathcal{S}(u_3).$$

By recalling $x_n = \beta(s - f_n),$ then we obtain that

$$\mathbb{E}_{n1} = \epsilon \sum_{j=1}^{N} \int_{\mathfrak{A}_n} \beta_1 \tilde{f}_j \mathcal{E}_0, j H_{n,x_n} \, ds + \epsilon \sum_{j=1}^{N} \int_{\mathfrak{A}_n} \chi_j \mathcal{E}_2, j H_{n,x_n} \, ds$$

$$+ \epsilon^2 \sum_{j=1}^{N} \int_{\mathfrak{A}_n} \mathcal{E}_{14}, j H_{n,x_n} \, ds$$

$$+ \epsilon^2 \left\{ \int_{\mathfrak{A}_n} \eta^\delta(s) \mathcal{E}_2, j + \epsilon^2 | \ln \epsilon | \mathcal{Z}_2, j + \epsilon^2 | \ln \epsilon | \mathcal{Z}_6, j \right\}$$

$$+ \epsilon^2 | \ln \epsilon | \mathcal{Z}_7, j + \epsilon^2 | \mathcal{Z}_8, j + \epsilon^2 | \ln \epsilon | \mathcal{E}_9, j$$

$$+ \epsilon^2 | \ln \epsilon | \mathcal{E}_{10}, j + \epsilon^2 | \ln \epsilon | \mathcal{E}_{12}, j + \epsilon^2 | \ln \epsilon | \mathcal{E}_{13}, j + \epsilon^2 | \ln \epsilon | \mathcal{E}_{15}, j \right\} H_{n,x_n} \, ds$$

$$+ \epsilon^2 | \ln \epsilon | ^2 \int_{\mathfrak{A}_n} \mathcal{E}_{03} H_{n,x_n} \, ds + \epsilon^2 \int_{\mathfrak{A}_n} \mathcal{E}_{06} H_{n,x_n} \, ds$$

$$+ \int_{\mathfrak{A}_n} \mathcal{N}_{u_1}(\phi_1) H_{n,x_n} \, ds + \int_{\mathfrak{A}_n} \tilde{K} H_{n,x_n} \, ds + \int_{\mathfrak{A}_n} \mathcal{Q}_1 H_{n,x_n} \, ds$$

$$= I_0^n + I_1^n + I_2^n + I_3^n + I_4^n + I_5^n + I_6^n + I_7^n + I_8^n.$$
For \( \tilde{f} \) satisfying (3.95), we denote uniformly bounded continuous functions of the forms
\[
\tilde{f}^n_0(\tilde{f}, \tilde{f}', \tilde{f}'') \quad \text{and} \quad \tilde{g}^j_n(\tilde{f})(\theta), \quad \text{for } j = 1, \ldots, 8, \ n = 1, \ldots, N,
\]
and uniformly bounded continuous functions independent of \( \tilde{f} \) in the forms
\[
G^j_n(\theta), \quad \forall \ j = 1, \ldots, 7, \ n = 1, \ldots, N.
\]
In the sequel, we will give the estimates of the above terms one by one.

By using the constraint of \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N)^T \) in (3.95) and \( \Xi_{0,j} \) in (3.36), it is easy to obtain that for \( n = 1, \ldots, N \)
\[
I^n_0 = \epsilon \sum_{j=1}^N \int_{\mathcal{R}_n} \beta_1 \tilde{f}_j H_j (1 - H_j^2) H_{n,x_n} \ dx
\]
\[
= \epsilon \int_{\mathcal{R}_n} \beta_1 \tilde{f}_n H_n (1 - H_n^2) H_{n,x_n} \ dx + \epsilon \sum_{j \neq n} \int_{\mathcal{R}_n} \beta_1 \tilde{f}_j H_j (1 - H_j^2) H_{n,x_n} \ dx
\]
\[
= O(\epsilon^2 |\ln \epsilon|) \sum_{j=1}^N |\tilde{f}_j(\theta)|. \tag{5.5}
\]

According to the expression of \( \Xi^2_{5,j} \) as in (3.99), we can derive that
\[
I^n_1 = -6 \sqrt{2} \epsilon \frac{1}{\gamma_{0,n}} \beta^2 \left[ -\gamma_{1,n} \ k_n \left( e^{-\sqrt{2} \beta (\tilde{f}_n - \tilde{f}_{n-1})} - 1 \right) \right.
\]
\[
+ \gamma_{2,n} k_{n+1} \left( e^{-\sqrt{2} \beta (\tilde{f}_{n+1} - \tilde{f}_n)} - 1 \right) \left( e^{2 \beta (\tilde{f}_{n+1} - \tilde{f}_n)} \right)
\]
\[
= 6 \sqrt{2} \epsilon \beta \gamma_{1,n} k_n \left[ e^{-\sqrt{2} \beta (\tilde{f}_n - \tilde{f}_{n-1})} - 1 \right]
\]
\[
- 6 \sqrt{2} \epsilon \beta \gamma_{2,n} k_{n+1} \left[ e^{-\sqrt{2} \beta (\tilde{f}_{n+1} - \tilde{f}_n)} - 1 \right] + O(\epsilon^2 |\ln \epsilon|) \sum_{j=1}^N |\tilde{f}_j(\theta)|, \tag{5.6}
\]

where \( \gamma_{1,n}, \gamma_{2,n} \) are functions defined in (3.47)–(3.48).

By the definition of \( \Xi_{14,j} \) in (3.120), it follows that
\[
I^n_2 = \epsilon^2 \left( -2 \beta' \tilde{f}'_n - \beta \tilde{f}''_n \right) \int_{\mathcal{R}_n} H_{n,x_n}^2 \ dx - 2 \epsilon^2 \beta' \tilde{f}'_n \int_{\mathcal{R}_n} \ x_n H_{n,x_n} H_{n,x_n} \ dx
\]
\[
+ 2\epsilon^2 \beta^{-1} \beta_2 \tilde{f}_n \int_{\mathcal{R}_n} x_n F(\tilde{H}_n) H_{n,x_n} \ dx - 2 \epsilon^2 k^2 \beta \tilde{f}_n \int_{\mathcal{R}_n} H_{n,x_n}^2 \ dx
\]
\[
= -2 \epsilon^2 \gamma_0 \tilde{f}''_n - 2 \epsilon^2 \gamma_0 \beta' \beta^{-1} \tilde{f}'_n + 2 \epsilon^2 \gamma_0 \left( \frac{1}{2} \beta_2 \beta^{-1} \right) \tilde{f}'_n
\]
\[
+ 2 \epsilon^2 + \sigma^2 \int_{\mathcal{R}_n} \tilde{f}^n_0(\tilde{f}, \tilde{f}', \tilde{f}'')(\theta), \tag{5.7}
\]
where we have used the constraint of \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N)^T \) in (3.95) and the fact
\[
\gamma_0 = \int_{\mathcal{R}} H_{x}^2 \ dx, \quad 2 \int_{\mathcal{R}} x H_{x} H_{xx} \ dx = -\int_{\mathcal{R}} H_{x}^2 \ dx.
\]

When \( s \in \mathcal{R}_n \), it is easy to obtain that
\[
\tilde{x}_j = \beta(s + f_j) \in \left( \frac{\beta f_{n-1} + \beta f_n}{2} + \beta f_j, \frac{\beta f_{n+1} + \beta f_n}{2} + \beta f_j \right).
\]
Then, recalling the expression of $\epsilon H$ and the asymptotic behavior of $f_j$ as in (3.1) and the expressions of $f_j$ ($j = 1, \ldots, N$) in (3.7)–(3.8), it is easy to obtain that

$$\int_{\mathcal{G}_n} \tilde{H}_j, x_j H_{n,x_n} \, ds = |\ln \epsilon| O(e^{-\sqrt{2}\beta |f_j + f_n|}), \quad \forall j, n = 1, \ldots, N.$$ 

Then, recalling the expression of $\epsilon^2 |\ln \epsilon| \Xi_{2,j}$ in (3.39), we can derive that

$$e^2 |\ln \epsilon| \sum_{j=1}^N \int_{\mathcal{G}_n} \Xi_{2,j} H_{n,x_n} \, ds$$

$$= -\epsilon k \beta \sum_{j=1}^N \int_{\mathcal{G}_n} \tilde{H}_j, x_j H_{n,x_n} \, ds - \epsilon \beta_1 \sum_{j=1}^N \int_{\mathcal{G}_n} \left( \frac{\tilde{x}_j}{\beta} - f_j \right) F(\tilde{H}_j) H_{n,x_n} \, ds$$

$$= e^2 |\ln \epsilon| G^1_n(\theta) + O(e^2 |\ln \epsilon|) \sum_{j=1}^N |\tilde{f}_j(\theta)| + e^{2+\sigma} \beta^1_n(\bar{f})(\theta). \quad (5.8)$$

Recalling the definition of $e^2 |\ln \epsilon| \chi_j \Xi_{6,j}$ as in (3.54), it is easy to check that

$$\sum_{j=1}^N \int_{\mathcal{G}_n} e^2 |\ln \epsilon| \chi_j \Xi_{6,j} H_{n,x_n} \, ds$$

$$= -6 \sqrt{2} e^2 \beta^2 \sum_{j=1}^N k_j e^{-2\sqrt{2}\beta (f_j + f_{j-1})} \int_{\mathcal{G}_n} \chi_j H_{j,x_j} e^{-2\sqrt{2}\beta} \, H_{n,x_n} \, ds$$

$$+ 6 \sqrt{2} e^2 \beta^2 \sum_{j=1}^N k_j e^{-2\sqrt{2}\beta (f_{j+1} - f_j)} \int_{\mathcal{G}_n} \chi_j H_{j,x_j} e^{2\sqrt{2}\beta} \, H_{n,x_n} \, ds$$

$$= e^2 |\ln \epsilon| G^2_n(\theta) + O(e^2 |\ln \epsilon|) \sum_{j=1}^N |\tilde{f}_j|.$$ 

According to the definitions of $e^2 |\ln \epsilon| \Xi_{7,j}$, $e^2 \Xi_{8,j}$, $e^2 |\ln \epsilon| \Xi_{9,j}$, $e^2 |\ln \epsilon| \Xi_{10,j}$ as in (3.33), (3.34), (3.37), (3.90), it is easy to check that

$$\sum_{j=1}^N \int_{\mathcal{G}_n} \left[ e^2 |\ln \epsilon| \Xi_{7,j} + e^2 \Xi_{8,j} + e^2 |\ln \epsilon| \Xi_{9,j} + e^2 |\ln \epsilon| \Xi_{10,j} \right] H_{n,x_n} \, ds$$

$$= 2e^2 \beta \sum_{j=1}^N \int_{\mathcal{G}_n} \left[ (\tilde{f}_j' + \tilde{f}_j) \psi^*_{j,xj}(x_j, z) - (f_j' + f_j) \psi^{**}_{j,xj}(x_j, z) \right] H_{n,x_n} \, ds$$

$$+ 2e^2 \sum_{j=1}^N \int_{\mathcal{G}_n} \left[ \frac{\beta'}{\beta} x_j \left( \psi^*_{j,xj}(x_j, z) + \psi^{**}_{j,xj}(x_j, z) \right) - \left( \psi^*_{j',x_j}(\tau_j, z) + \psi^{**}_{j',x_j}(\tau_j, z) \right) \right] H_{n,x_n} \, ds$$

$$+ e^2 \beta_1 \sum_{j=1}^N \int_{\mathcal{G}_n} \left( \frac{x_j}{\beta} + \tilde{f}_j + \tilde{f}_j \right) (1 - 3H_j^2) (\beta_1 \tilde{f}_j \beta^{-2} \psi_j + \psi_j^* + \psi_j^{**}) H_{n,x_n} \, ds$$.
\[
\epsilon^2 \sum_{j=1}^{N} \int_{\mathcal{G}_n} \left\{ -\beta^{-1}k\beta_1 \hat{f}_j \psi_{j,x_j}(x_j) - \beta k \left[ \psi_{j,x_j}^*(x_j) + \psi_{j,x_j}^{**}(x_j) \right] \right\} H_{n,x_n} \, ds \\
= \epsilon^2 \ln |\epsilon| G^3_n(\theta) + \epsilon^{2+\sigma} 3^2_n(\tilde{f})(\theta).
\]

(5.10)

Recalling the expression of \(\epsilon^2 |\ln \epsilon| \chi_j \Xi_{12,j}\) as in (3.103), direct computation leads to

\[
\sum_{j=1}^{N} \int_{\mathcal{G}_n} \epsilon^2 |\ln \epsilon| \chi_j \Xi_{12,j} H_{n,x_n} \, ds \\
= 6\sqrt{2} \epsilon^2 \beta^2 \sum_{j=1}^{N} \chi_j H_{j,x_j} \left[ -k_j k_{j-1} e^{-\sqrt{2}x_j} + k_{j+1} k_{j+2} e^{\sqrt{2}x_j} \right] H_{n,x_n} \, ds \\
- 6\sqrt{2} \epsilon^2 \beta^2 \sum_{j=1}^{N} \int_{\mathcal{G}_n} \chi_j \left( \frac{x_j}{\beta} + \hat{f}_j + \tilde{f}_j \right) H_{n,x_n} \, ds \\
\times \left[ -e^{-\sqrt{2}x_j} k_j + e^{\sqrt{2}x_j} k_{j+1} \right] H_{n,x_n} \, ds \\
- 4\epsilon^2 \beta^2 \sum_{j=1}^{N} \int_{\mathcal{G}_n} (-1)^j A_{5,j} k_j k_{j+1} H_{n,x_n} \, ds \\
= \epsilon^2 |\ln \epsilon| G^4_n(\theta) + \epsilon^{2+\sigma} 3^4_n(\tilde{f})(\theta).
\]

(5.11)

For \(j = 1, \ldots, N\), \(\hat{f}_j\) (c.f.(3.7)–(3.8)) are of order \(|\ln \epsilon|\). According to the expression of \(\epsilon^2 |\ln \epsilon| \Xi_{13,j}\) as in (3.119), the direct computation leads to

\[
\sum_{j=1}^{N} \int_{\mathcal{G}_n} \epsilon^2 |\ln \epsilon| \Xi_{13,j} H_{n,x_n} \, ds \\
= \epsilon^2 \sum_{j=1}^{N} \left[ -2\beta' (\hat{f}_j'' + \tilde{f}_j'' + \check{f}_j') - \beta (\hat{f}_j'' + \bar{f}_j'' + \check{f}_j') \right] \int_{\mathcal{G}_n} H_{j,x_j} H_{n,x_n} \, ds \\
- 2\epsilon^2 \sum_{j=1}^{N} \beta' (\hat{f}_j' + \bar{f}_j' + \check{f}_j') \int_{\mathcal{G}_n} x_j H_{j,x_j,x_j} H_{n,x_n} \, ds \\
- \epsilon^2 k^2 \beta \sum_{j=1}^{N} (\hat{f}_j + \check{f}_j + \bar{f}_j) \int_{\mathcal{G}_n} H_{j,x_j} H_{n,x_n} \, ds \\
+ \epsilon^2 \beta^{-1} \beta_2 (\hat{f}_j + \tilde{f}_j + \check{f}_j) \sum_{j=1}^{N} \int_{\mathcal{G}_n} x_j F(H_j) H_{n,x_n} \, ds \\
= \epsilon^2 |\ln \epsilon| G^5_n(\theta) + \epsilon^{2+\sigma} 3^5_n(\tilde{f})(\theta).
\]

(5.12)

From the expression of \(\epsilon^2 |\ln \epsilon| \Xi_{15,j}\) in (3.121), it can be easily verified that

\[
\sum_{j=1}^{N} \int_{\mathcal{G}_n} \epsilon^2 |\ln \epsilon| \Xi_{15,j} H_{n,x_n} \, ds
\]
\[ I_3^n = \sum_{j=1}^{N} \int_{\mathcal{G}_n} \left\{ e^2 | \ln \epsilon | \mathcal{E}_{2,j} + e^2 | \ln \epsilon | \partial_j \mathcal{E}_{6,j} + e^2 | \ln \epsilon | \mathcal{E}_{7,j} + e^2 \mathcal{E}_{8,j} + e^2 | \ln \epsilon | \mathcal{E}_{9,j} \right\} + e^2 | \ln \epsilon | \mathcal{E}_{10,j} + e^2 | \ln \epsilon | \partial_j \mathcal{E}_{12,j} + e^2 | \ln \epsilon | \mathcal{E}_{13,j} + e^2 | \ln \epsilon | \mathcal{E}_{15,j} \right\} H_{n,x_n} ds,
\]

Using the definition of \( \tilde{K} \) as in \( (3.123) \), we can obtain that

\[ I_3^n = \int_{\mathcal{G}_n} \tilde{K} H_{n,x_n} ds = e^2 G_7^n(\theta) + e^{2+\sigma} \tilde{J}_n^8(\tilde{\Phi})(\theta). \]  

(5.15)

In fact, when \( s \in \mathcal{R}_n \), it is easy to obtain that

\[ \tau_j = -\beta(s + f_j) \in \left( -\frac{\beta f_{n+1} + \beta f_n}{2}, -\frac{\beta f_{n-1} + \beta f_n}{2} - \beta f_j \right). \]

Recall the asymptotic estimates of \( \psi_j, \psi_j^* \) in \( (3.109), (3.110) \), and that \( \psi_j^{**}, \omega_j^* \) have similar asymptotic estimates of \( \psi_j^* \). Then the proof of \( (5.15) \) is very straightforward. In the following, the term \( e^2 G_7^n(\theta) \) will be absorbed in \( e^2 | \ln \epsilon | \mathcal{D}_n(\theta) \).

Next, recalling the expressions of \( \mathcal{Q}_1 \) as in \( (3.124) \), we can get

\[ I_8^n = \int_{\mathcal{G}_n} \mathcal{Q}_1 H_{n,x_n} ds = e^{2+\sigma} \tilde{J}_n^8(\tilde{\Phi})(\theta). \]

(5.15)
Recall that for \( s \in \mathbb{R}^+ \setminus \mathbb{R}_n \), we have

\[
H_{n,x_n} = (-1)^n H'(\beta(s - f_n)) = \max_{j \neq n} O(e^{-\frac{\sqrt{2}}{2}} |f_j - f_n|) = O(\varepsilon^\frac{1}{2}).
\]

Combining above fact and the assumption \((3.95)\), we can get easily

\[
\mathbb{E}_{n2} = O(\varepsilon^\frac{1}{2}) \sum_{i=0}^{8} I_i^n.
\]

This finishes the estimates of the first term in \((5.2)\).

Finally, we turn to the estimates of the other two terms in \((5.2)\). Denote

\[
\mathcal{W}_n(\varepsilon z) := - \int_0^{+\infty} e^{-\sqrt{2}/2} \eta_{\beta}(s) N(\tilde{\phi}) H_{n,x_n} ds.
\]

Combining the estimate of \( \tilde{\phi} \) in \((4.19)\) and \((4.20)\), we can conclude that

\[
\|\mathcal{W}_n\|_{L^2(0, \ell)} \leq C \varepsilon^{2+\sigma}.
\]

The last term in \((5.2)\) can be rewritten as

\[
\mathcal{Q}_n(\varepsilon z) := \int_0^{+\infty} \mathcal{L}(\tilde{\phi}) H_{n,x_n} ds
\]

\[
= \int_0^{+\infty} \phi_{zz} H_{n,x_n} ds + 3 \beta^2 \int_0^{+\infty} \tilde{\phi} \left[ H^2(\beta(s - f_n)) - H^2 \right] H_{n,x_n} ds
\]

\[
+ \int_0^{+\infty} \eta_{10}(s) \left[ B_1(\tilde{\phi}) + B_6(\tilde{\phi}) \right] H_{n,x_n} ds,
\]

\[(5.16)\]

A similar estimate holds

\[
\|\mathcal{Q}_n\|_{L^2(0, \ell)} \leq C \varepsilon^{2+\sigma}.
\]

By the notation of

\[
\mathbb{M}_n(\theta, \tilde{f}, \tilde{f}', \tilde{f}'') = e^{-2 \gamma_0} \beta' \beta^{-1} \tilde{f}_n - e^{2 \gamma_0} \left[ \frac{1}{2} \beta_2 \beta^{-1} - k^2 \beta \right] \beta^{-1} \tilde{f}_n
\]

\[- e^{2+\sigma} \mathcal{Y}_n(\theta, \tilde{f}', \tilde{f}'')(\theta) - e^{2+\sigma} \sum_{i=1}^{8} \mathcal{Z}_i(\tilde{f})(\theta)
\]

\[- \mathcal{Q}_n(\theta) - \mathcal{W}_n(\theta) + O(\varepsilon^2 |\ln \varepsilon|) \sum_{j=1}^{N} |\tilde{f}_j|,
\]

we can get that the relations in \((5.2)\) are equivalent to the following system of differential equations, for \( n = 1, \ldots, N, \)

\[
- e^{2} \gamma_0 \tilde{f}_n'' + 6 \sqrt{2} \varepsilon \beta G_{1,n} k_n \left[ e^{-\sqrt{2}/2} (\tilde{f}_n - \tilde{f}_{n-1}) - 1 \right]
\]

\[- 6 \sqrt{2} \varepsilon \beta G_{2,n} k_{n+1} \left[ e^{-\sqrt{2}/2} (\tilde{f}_{n+1} - \tilde{f}_n) - 1 \right] = - \varepsilon^2 |\ln \varepsilon| \mathcal{D}_n + \mathbb{M}_n(\theta, \tilde{f}, \tilde{f}', \tilde{f}'').
\]

Therefore, we can draw a conclusion in the following proposition
Proposition 5.1  For the validity of (5.1), there should hold the following equations, for \( n = 1, \ldots, N \),

\[
-\varepsilon^2 \gamma_0 \tilde{f}''_n = -6\sqrt{2} \varepsilon \beta \gamma_{1,n} k_n \left[ e^{-\sqrt{2} \beta (\tilde{f}'_n - \tilde{f}'_{n-1})} - 1 \right] + 6\sqrt{2} \varepsilon \beta \gamma_{2,n} k_{n+1} \left[ e^{-\sqrt{2} \beta (\tilde{f}'_{n+1} - \tilde{f}'_n)} - 1 \right] - \varepsilon^2 |\ln \varepsilon| \Omega_n + M_n(\theta, \tilde{f}, \tilde{f}', \tilde{f}'').
\]

(5.18)

Moreover, for any \( n = 1, \ldots, N \), the operator \( M_n(\theta, \tilde{f}, \tilde{f}', \tilde{f}'') \) can be decomposed in the following way

\[
M_n(\theta, \tilde{f}, \tilde{f}', \tilde{f}'') = M_{n1}(\theta, \tilde{f}, \tilde{f}') + M_{n2}(\theta, \tilde{f}, \tilde{f}', \tilde{f}'')
\]

where \( M_{n1}(\theta, \tilde{f}, \tilde{f}') \) and \( M_{n2}(\theta, \tilde{f}, \tilde{f}', \tilde{f}'') \) are continuous of their arguments and also satisfy

\[
\|M_{n1}(\theta, \tilde{f}, \tilde{f}')\|_{L^2(0,\ell)} \leq C \varepsilon^{2+\sigma},
\]

(5.19)

\[
\|M_{n2}(\theta, \tilde{f}, \tilde{f}', \tilde{f}'')\|_{L^2(0,\ell)} \leq C \varepsilon^{2+\sigma},
\]

(5.20)

\[
\|M_{n2}(\theta, \tilde{f}^{(1)}, \tilde{f}^{(1)'}, \tilde{f}^{(1)''}) - M_{n2}(\theta, \tilde{f}^{(2)}, \tilde{f}^{(2)'}, \tilde{f}^{(2)''})\|_{L^2(0,\ell)} \leq C \varepsilon^{1+\sigma} \|\tilde{f}^{(1)} - \tilde{f}^{(2)}\|.
\]

(5.21)

\( \square \)

We omit the details of the proof of this proposition. The reader can refer to [13] and [14].

6 Solve the system of reduced equations

Note that the parameters \( \tilde{f}_1, \ldots, \tilde{f}_N \) will determine the locations of phase transition layers and play an important role in the description of the interaction between neighboring layers in the clustering phenomena. As a consequence of Proposition 5.1, to find \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N) \), we have to deal with the following system, for \( n = 1, \ldots, N \),

\[
-\varepsilon \gamma_0 \tilde{f}''_n - 12 \beta^2 \gamma_{1,n} k_n (\tilde{f}'_n - \tilde{f}'_{n-1}) + 12 \beta^2 \gamma_{2,n} k_{n+1} (\tilde{f}'_{n+1} - \tilde{f}'_n) = J_n(\tilde{f}) - \varepsilon |\ln \varepsilon| \Omega_n - \varepsilon^{-1} M_n(\theta, \tilde{f}, \tilde{f}', \tilde{f}''),
\]

(6.1)

with boundary conditions

\[
\tilde{f}(0) = \tilde{f}(\ell), \quad \tilde{f}'(0) = \tilde{f}'(\ell).
\]

(6.2)

Here, the nonlinear terms \( J_n \) for \( n = 1, \ldots, N \) are given by

\[
J_n(\tilde{f}) = -6\sqrt{2} \beta \gamma_{1,n} k_n \left[ e^{-\sqrt{2} \beta (\tilde{f}'_n - \tilde{f}'_{n-1})} - 1 \right] + 6\sqrt{2} \beta \gamma_{2,n} k_{n+1} \left[ e^{-\sqrt{2} \beta (\tilde{f}'_{n+1} - \tilde{f}'_n)} - 1 \right].
\]

(6.3)

In the above, for \( n = 1, \ldots, N \), the terms \( \gamma_{1,n}, \gamma_{2,n}, k_n \) are given in (3.47), (3.48), (3.96).

Step 1: The first try is to simplify the above system. We will denote

\[
\epsilon_n(\theta) := 12 \beta^2 \gamma_{1,n} k_n(\theta), \quad \delta_n(\theta) := 12 \beta^2 \gamma_{2,n} k_{n+1}(\theta),
\]

\[
J(\tilde{f})(\theta) := (J_1(\tilde{f})(\theta), \ldots, J_N(\tilde{f})(\theta))^T, \quad \Omega(\theta) := (\Omega_1(\theta), \ldots, \Omega_N(\theta))^T,
\]

and

\[
\gamma(\tilde{f})(\theta) := \left( -\varepsilon^{-1} M_1(\tilde{f})(\theta), \ldots, -\varepsilon^{-1} M_N(\tilde{f})(\theta) \right)^T.
\]
Then system (6.1) becomes:

\[-\epsilon\gamma_0 \ddot{\bar{f}} = B\dot{\bar{f}} + \mathcal{J}(\bar{f}) - \epsilon|\ln \epsilon|\mathcal{D} + \gamma(\bar{f}),\]

(6.4)

where \( B \) can be expressed as

\[
B = \begin{pmatrix}
2c_1 + d_1 & -d_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-c_2 & (c_2 + d_2) & -d_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -c_{N-1} \left( c_{N-1} + d_{N-1} \right) & -d_{N-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -c_N & c_N
\end{pmatrix}.
\]

We have the relations, for \( n = 1, \ldots, N \),

\[
c_n = a_{n-1} + O(\epsilon^{\frac{1}{2}}), \quad \vartheta_{n-1} = a_{n-1} + O(\epsilon^{\frac{1}{2}}),
\]

where \( a_{n-1} \) can be expressed as

\[
a_{n-1}(\theta) = 12\beta^2(\gamma_1) k_n(\theta).
\]

For \( n = 1, \ldots, N \), recalling the expressions of \( d_n \) and \( k_n \) in (3.49), (3.96), we get

\[
a_{n-1} = 12\beta^2(\gamma_1) k_n(\theta) = 12\beta^2(\gamma_1) d_n e^{-\sqrt{2}(\gamma_n-\gamma_{n-1})}
\]

\[
= 12\beta^2(\gamma_1) \frac{(N-n+1)}{9\beta^2(\gamma_1)} (H + O(\epsilon^{\frac{1}{2}})) e^{-\sqrt{2}(\gamma_n-\gamma_{n-1})} > 0, \quad \forall \theta \in [0, \ell],
\]

(6.5)

where \( H \) is given in (1.6). It is obvious that \( B \) is a perturbation of a symmetric matrix \( A \) defined in the form

\[
A = \begin{pmatrix}
2a_0 + a_1 & -a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-a_1 & (a_1 + a_2) & -a_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & (a_{N-2} + a_{N-1}) & -a_{N-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -a_{N-1} & a_{N-1}
\end{pmatrix}.
\]

Using elementary matrix operations, it is easy to prove that there exists an invertible matrix \( Q \) independent of \( \theta \) such that

\[
QAQ^T = \text{diag}(2a_0, a_1, a_2, \ldots, a_{N-1}).
\]

Since \( a_0(\theta), a_1(\theta), \ldots, a_{N-1}(\theta) \) are positive functions defined in (6.5), we can assume that all eigenvalues of the matrix \( A \) are

\[
\rho_1(\theta) \geq \cdots \geq \rho_{N-1}(\theta) \geq \rho_N(\theta) > 0.
\]

Similarly, it is easy to prove that there exists an invertible matrix \( \hat{Q} \) such that

\[
\hat{Q}B\hat{Q}^T = \text{diag}\left( 2a_0 + O(\epsilon^{\frac{1}{2}}), a_1 + O(\epsilon^{\frac{1}{2}}), \ldots, a_{N-1} + O(\epsilon^{\frac{1}{2}}) \right).
\]

We also assume that all eigenvalues of the matrix \( B \) are \( \rho_1^\varepsilon(\theta), \ldots, \rho_N^\varepsilon(\theta) \). Naturally, we have

\[
\rho_n^\varepsilon(\theta) = \rho_n(\theta) + O(\epsilon^{\frac{1}{2}}), \quad \forall n = 1, \ldots, N.
\]
Moreover, there exists another invertible matrix $P$ in the form
\[
P^T P = I, \quad P = \begin{pmatrix}
p_{11} & \cdots & p_{1N-1} & p_{1N} \\
p_{21} & \cdots & p_{2N-1} & p_{2N} \\
\vdots & & \vdots & \vdots \\
p_{N1} & \cdots & p_{NN-1} & p_{NN}
\end{pmatrix},
\]
(6.6)
such that
\[
P^T BP = \text{diag}(\rho_1^\epsilon, \ldots, \rho_N^\epsilon).
\]

Now, we define new vectors
\[
u := (u_1, \ldots, u_N)^T = P^T \tilde{f}, \quad \tilde{\mathcal{D}} := (\tilde{\mathcal{D}}_1, \ldots, \tilde{\mathcal{D}}_N)^T = P^T \mathcal{D},
\]
and
\[
\begin{align*}
\tilde{J}(u) &:= (\tilde{J}_1(u), \ldots, \tilde{J}_N(u))^T = P^T J(\tilde{f}), \\
W(u) &:= (W_1(u), \ldots, W_N(u))^T = P^T P'u' + P^T P'\nu', \\
\tilde{\Upsilon}(u) &:= (\tilde{\Upsilon}_1(u), \ldots, \tilde{\Upsilon}_N(u))^T = P^T \Upsilon(\tilde{f}) = P^T \Upsilon(Pu).
\end{align*}
\]

Then the system (6.4) can be rewritten as
\[
-\epsilon \gamma_0 u'' - \text{diag}(\rho_1^\epsilon, \ldots, \rho_N^\epsilon)u = \tilde{J}(u) - \epsilon |\ln \epsilon| \tilde{\mathcal{D}} + \epsilon \gamma_0 W(u) + \tilde{\Upsilon}(u).
\]
(6.7)

In order to cancel the term $-\epsilon |\ln \epsilon| \tilde{\mathcal{D}}$ in (6.7), we further set,
\[
u_n(\theta) = \hat{\nu}_n(\theta) + \frac{\epsilon |\ln \epsilon| \hat{\mathcal{D}}_n(\theta)}{\rho_n^\epsilon(\theta)}, \quad \forall n = 1, \ldots, N.
\]
Then (6.7) becomes a system of $\hat{\nu}(\theta) = (\hat{\nu}_1(\theta), \ldots, \hat{\nu}_N(\theta))^T$,
\[
-\epsilon \gamma_0 \hat{\nu}'' - \text{diag}(\rho_1^\epsilon, \ldots, \rho_N^\epsilon)\hat{\nu} = \tilde{J}(\hat{\nu} + \epsilon |\ln \epsilon| \mathcal{E}) + \epsilon \gamma_0 W(\hat{\nu} + \epsilon |\ln \epsilon| \mathcal{E})
+ \epsilon^2 |\ln \epsilon| \gamma_0 \mathcal{E}'' + \tilde{\Upsilon}(\hat{\nu} + \epsilon |\ln \epsilon| \mathcal{E}),
\]
(6.8)
where we have denoted
\[
\mathcal{E}(\theta) = (\mathcal{E}_1(\theta), \ldots, \mathcal{E}_N(\theta))^T = \left(\frac{\hat{\mathcal{D}}_1(\theta)}{\rho_1^\epsilon(\theta)}, \ldots, \frac{\hat{\mathcal{D}}_N(\theta)}{\rho_N^\epsilon(\theta)}\right)^T.
\]
(6.9)

It is easy to check that $\hat{\nu}$ satisfies the boundary conditions
\[
\hat{\nu}(0) = \hat{\nu}(\ell), \quad \hat{\nu}'(0) = \hat{\nu}'(\ell).
\]
(6.10)

**Step 2:** For the solvability of (6.8), we pause here to consider the linear operators
\[
\mathbb{L}_n^\epsilon v_n := -\epsilon \gamma_0 v_n'' - \rho_n^\epsilon(\theta) v_n, \quad \forall n = 1, \ldots, N,
\]
and then provide the resolution theory for the linear problems.

**Lemma 6.1** We consider the following system, for $n = 1, \ldots, N$,
\[
\mathbb{L}_n^\epsilon(v_n) = h_n, \quad v_n(0) = v_n(\ell), \quad v_n'(0) = v_n'(\ell).
\]
(6.11)
There exists a sequence \( \{ \epsilon_l \} : l \in \mathbb{N} \) such that there exists a unique solution \( v = (v_1, \ldots, v_N)^T \) to the system (6.11) with estimates, for all \( n = 1, \ldots, N \),

\[
\epsilon_l \| v''_n \|_{L^2(0, \ell)} + \sqrt{\epsilon_l} \| v'_n \|_{L^2(0, \ell)} + \| v_n \|_{L^\infty(0, \ell)} \leq C \frac{1}{\epsilon_l} \| h_n \|_{L^2(0, \ell)}. \tag{6.12}
\]

Moreover, if \( h_n \in H^2(0, \ell) \) then there hold

\[
\epsilon_l \| v''_n \|_{L^2(0, \ell)} + \| v'_n \|_{L^2(0, \ell)} + \| v_n \|_{L^\infty(0, \ell)} \leq C \| h_n \|_{H^2(0, \ell)}. \tag{6.13}
\]

**Proof** The proof is similar as the one for Lemma 8.1 in [13].

**Step 3:** Giving \( h = (h_1, \ldots, h_N)^T \in L^2(0, \ell) \) with \( \| h \|_{L^2(0, \ell)} \) \leq \( \epsilon^{1+\sigma} \), we consider the nonlinear problem

\[
-\epsilon \gamma_0 \tilde{u}'' - \text{diag}(\rho_1^e, \ldots, \rho_N^e) \tilde{u} = \tilde{J}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}) + \epsilon \gamma_0 W(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})
\]

\[+
\epsilon^2 | \ln \epsilon | \gamma_0 \mathcal{C}'' + h,
\]

\[
\tilde{u}(0) = \tilde{u}(\ell), \quad \tilde{u}'(0) = \tilde{u}'(\ell), \tag{6.14}
\]

where \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N)^T \).

Define the set

\[\mathcal{X} = \left\{ \tilde{u} \in H^2(0, \ell) : \epsilon \| \tilde{u}'' \|_{L^2(0, \ell)} + \sqrt{\epsilon} \| \tilde{u}' \|_{L^2(0, \ell)} + \| \tilde{u} \|_{L^\infty(0, \ell)} \leq \epsilon^{\frac{1}{2}+\sigma} \right\}.
\]

In fact, for any \( \tilde{u} \in \mathcal{X} \), there holds

\[
\tilde{J}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}) = \mathcal{P}^T \begin{pmatrix}
\mathcal{J}_1(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})) \\
\vdots \\
\mathcal{J}_N(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}))
\end{pmatrix},
\]

where \( \mathcal{J}_n \) is defined in (6.3). Therefore, we have

\[
\| \tilde{J}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}) \|_{L^2(0, \ell)} \leq C \sum_{n=1}^N \| \mathcal{J}_n(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})) \|_{L^2(0, \ell)},
\]

and

\[
\mathcal{J}_n(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})) = O\left(\left(\left(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})\right)_n - \left(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})\right)_{n-1}\right)^2\right) + O\left(\left(\left(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})\right)_{n+1} - \left(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})\right)_n\right)^2\right). \tag{6.16}
\]

Recall the definition of \( \mathcal{P} \) in (6.6), we have the following expression

\[
(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}))_n = \sum_{j=1}^N p_{n+1,j} \left(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}_j\right) - \sum_{j=1}^N p_{n,j} \left(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}_j\right).
\]

From the definition of \( \mathcal{C} \) as in (6.9), we obtain that

\[
\| \mathcal{J}_n(\mathcal{P}(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C})) \|_{L^2(0, \ell)} \leq \epsilon^{1+\sigma}. \tag{6.17}
\]

We can also get

\[
\| \epsilon \gamma_0 W(\tilde{u} + \epsilon | \ln \epsilon | \mathcal{C}) \|_{L^2(0, \ell)} \leq \epsilon^{1+\sigma}. \tag{6.18}
\]
\[ \left\| \varepsilon^2 |\ln |\gamma_0| \varepsilon'' \right\|_{L^2(0, \ell)} \leq \varepsilon^{1+\sigma}. \] (6.19)

Lemma 6.1 implies that there exists a sequence \( \{\varepsilon_l : l \in \mathbb{N}\} \) such that all operators \( L_{\varepsilon_l}^l, n = 1, \ldots, N \) are invertible with estimates in (6.12)–(6.13). By using (6.17)–(6.19) and applying the Contraction Mapping Theorem, we find a solution \( \tilde{u} \in \mathcal{C} \) to the problem (6.14)–(6.15) with the following estimate
\[
\varepsilon_l \left\| \tilde{u}'' \right\|_{L^2(0, \ell)} + \sqrt{\varepsilon_l} \left\| \tilde{u}' \right\|_{L^2(0, \ell)} + \left\| \tilde{u} \right\|_{L^\infty(0, \ell)} \leq C \sqrt{\frac{1}{\varepsilon_l}} \left( \frac{1}{\varepsilon_l} \left\| \tilde{u} \right\|_{L^2(0, \ell)}^2 \right) + \sqrt{\frac{1}{\varepsilon_l}} \left\| \tilde{u} \right\|_{L^2(0, \ell)}^2 \leq \varepsilon_l^{1+\sigma} + \left\| \tilde{u} \right\|_{L^2(0, \ell)}.
\]

Here is the conclusion.

**Proposition 6.2** For given \( \mathbf{h} = (h_1, \ldots, h_N)^T \) with \( \| \mathbf{h} \|_{L^2(0, \ell)} \leq \varepsilon^{1+\sigma} \) for some constant \( 0 < \sigma < 1 \), there exists a sequence \( \{\varepsilon_l : l \in \mathbb{N}\} \) approaching 0 such that problem (6.14)–(6.15) admits a solution \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N)^T \) with the estimates
\[
\varepsilon_l \left\| \tilde{u}'' \right\|_{L^2(0, \ell)} + \sqrt{\varepsilon_l} \left\| \tilde{u}' \right\|_{L^2(0, \ell)} + \left\| \tilde{u} \right\|_{L^\infty(0, \ell)} \leq C \varepsilon_l^{-\frac{1}{2}} \| \mathbf{h} \|_{L^2(0, \ell)} + \varepsilon_l^{1+\sigma}. \]

\( \square \)

**Step 4:** We go back to solve (6.8) and (6.10) and finish the proof of Theorem 1.1. By recalling (5.19), (5.21) and Proposition 5.1, combining the assumption (3.95) of \( \hat{f} \) (this will give the constraint of \( \hat{u} \)), it is readily checked the validity of the decomposition
\[ \hat{\mathcal{Y}} (\hat{u} + \varepsilon |\ln |\varepsilon| \mathcal{C} ) = \hat{\mathcal{Y}} (\hat{u}, \hat{u}', \hat{u}'') + \hat{\mathcal{Y}} (\hat{u}, \hat{u}'), \] (6.20)
with properties, for \( n = 1, \ldots, N \),
\[ \left\| \hat{\mathcal{Y}}_n (\hat{u}, \hat{u}', \hat{u}'') \right\|_{L^2(0, \ell)} \leq C \varepsilon^{1+\sigma}, \] (6.21)
\[ \left\| \hat{\mathcal{Y}}_n (\hat{u}^{(1)}, \hat{u}'^{(1)}), \hat{u}''^{(1)}) - \hat{\mathcal{Y}}_n (\hat{u}^{(2)}, \hat{u}'^{(2)}, \hat{u}''^{(2)}) \right\|_{L^2(0, \ell)} \leq C \varepsilon^2 \left\| \hat{u}^{(1)} - \hat{u}^{(2)} \right\|_{H^2(0, \ell)}, \] (6.22)
and
\[ \left\| \hat{\mathcal{Y}}_n (\hat{u}, \hat{u}') \right\|_{L^2(0, \ell)} \leq C \varepsilon^{1+\sigma}. \] (6.23)

We define
\[ \mathbb{D} = \left\{ \mathcal{V} \in H^2(0, \ell) : \varepsilon_l \left\| \mathcal{V}'' \right\|_{L^2(0, \ell)} + \sqrt{\varepsilon_l} \left\| \mathcal{V}' \right\|_{L^2(0, \ell)} + \left\| \mathcal{V} \right\|_{L^\infty(0, \ell)} \leq D_1 \varepsilon_l^{1+\sigma} \right\} \]
with the sequence \( \{\varepsilon_l\} \) given in Lemma 6.1 and a fixed constant \( D_1 \), and then for given \( \mathcal{V} \in \mathbb{D} \) set the right hand side of (6.14) with
\[ h_n (\mathcal{V}) = \hat{\mathcal{Y}}_n (\hat{u}, \hat{u}', \hat{u}'') + \hat{\mathcal{Y}}_n (\mathcal{V}, \mathcal{V}'), \quad n = 1, \ldots, N. \]

Whence, by the facts in (6.22), the theory developed in Proposition 6.2 and the Contraction Mapping Theorem, we find \( \mathcal{V} \) for a fixed \( \mathcal{V} \) in \( \mathbb{D} \). This will give a mapping as
\[ \mathcal{Z} (\tilde{\mathcal{V}}) = \mathcal{V}, \]
and the solution to our problem is simply a fixed point of $\mathcal{Z}$. Continuity of $\tilde{Y}_1, \ldots, \tilde{Y}_N$ with respect to their parameters and standard regularity arguments allows us to conclude that $\mathcal{Z}$ is compact as a mapping from $H^2(0, \ell)$ into itself. The Schauder Theorem applies to yield the existence of a fixed point of $\mathcal{Z}$ as required. This ends the proof of Theorem 1.1. \hfill \Box

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Appendices

A The computations of (3.30)

The main objective in this section is to compute the quantities in (3.30). For the cases $n = 3, \ldots, N$, from the expressions of $b_{1n}, b_{2n}$ as in (3.20)–(3.21), we can obtain that

\begin{align}
b_{1n} &= -2e^{-\sqrt{2}x_n}e^{-\sqrt{2}\beta(f_n-f_{n-1})} + O(e^{-2\sqrt{2}|x_n+\beta(f_n-f_{n-1})|}) \\
&= -2e(N - n + 1)e^{-\sqrt{2}x_n}e^{-\sqrt{2}\beta(f_n-f_{n-1})} + O(e^{-2\sqrt{2}|x_n+\beta(f_n-f_{n-1})|}), \quad \text{(A.1)}
b_{2n} &= -2e^{-\sqrt{2}x_n}e^{-\sqrt{2}\beta(f_n-f_{n-2})} + O(e^{-2\sqrt{2}|x_n+\beta(f_n-f_{n-2})|}) \\
&= -2e^2(N - n + 1)(N - n + 2)e^{-\sqrt{2}x_n}e^{-\sqrt{2}\beta(f_n-f_{n-2})} + O(e^{-2\sqrt{2}|x_n+\beta(f_n-f_{n-2})|}). \quad \text{(A.2)}
\end{align}

Similarly, when $n = 1, 2$, we obtain

\begin{align}
b_{11} &= H(\beta(s + f_1)) - 1 \\
&= -2e^{-\sqrt{2}x_1}e^{-2\sqrt{2}\beta f_1} + O(e^{-2\sqrt{2}|x_1+2\beta f_1|}) \\
&= -2eN e^{-\sqrt{2}x_1}e^{-2\sqrt{2}\beta f_1} + O(e^{-2\sqrt{2}|x_1+2\beta f_1|}), \\
b_{21} &= H(\beta(s + f_2)) - 1 \\
&= -2e^{-\sqrt{2}x_1}e^{-\sqrt{2}\beta(f_2+f_1)} + O(e^{-2\sqrt{2}|x_1+\beta(f_2+f_1)|}) \\
&= -2e^2N(N - 1)e^{-\sqrt{2}x_1}e^{-\sqrt{2}\beta(f_2+f_1)} + O(e^{-2\sqrt{2}|x_1+\beta(f_2+f_1)|}),
\end{align}

and

\begin{align}
b_{12} &= H(\beta(s - f_1)) - 1 \\
&= -2e^{-\sqrt{2}x_2}e^{-\sqrt{2}\beta(f_2-f_1)} + O(e^{-2\sqrt{2}|x_2+\beta(f_2-f_1)|}) \\
&= -2(N - 1)e^{-\sqrt{2}x_2}e^{-\sqrt{2}\beta(f_2-f_1)} + O(e^{-2\sqrt{2}|x_2+\beta(f_2-f_1)|}), \\
b_{22} &= H(\beta(s + f_1)) - 1 \\
&= -2e^{-\sqrt{2}x_2}e^{-\sqrt{2}\beta(f_2+f_1)} + O(e^{-2\sqrt{2}|x_2+\beta(f_2+f_1)|}) \\
&= -2e^2N(N - 1)^2e^{-\sqrt{2}x_2}e^{-\sqrt{2}\beta(f_2+f_1)} + O(e^{-2\sqrt{2}|x_2+2\beta f_2|}).
\end{align}

Similarly, for the cases $n = 1, \ldots, N - 2$, we can also obtain

\begin{align}
b_{3n} &= 2e^{\sqrt{2}x_n}e^{-\sqrt{2}\beta(f_{n+1}-f_n)} + O(e^{-2\sqrt{2}|x_n+\beta(f_{n+1})|}) \\
&= 2e(N - n)e^{\sqrt{2}x_n}e^{-\sqrt{2}\beta(f_{n+1}-f_n)} + O(e^{-2\sqrt{2}|x_n+\beta(f_{n+1})|}), \quad \text{(A.3)}
b_{4n} &= 2e^{\sqrt{2}x_n}e^{-\sqrt{2}\beta(f_{n+2}-f_n)} + O(e^{-2\sqrt{2}|x_n+\beta(f_{n+2})|})
\end{align}
By combining the above formulas, we obtain the following:

**Case 1:** When \( n = 3, \ldots, N - 2 \), (3.30) holds.

**Case 2:** When \( n = 1, 2 \), we obtain that

\[
\begin{align*}
\mathbf{b}_{11} - \mathbf{b}_{21} + \mathbf{b}_{31} - \mathbf{b}_{41} &= -2e N e^{-\sqrt{2} x_1} e^{-2\sqrt{2} \beta \left( f_1 - f_1 \right)} + O(e^{-2\sqrt{2} |x_1|+\beta |f_1|}) \\
&\quad + 2e (N - 1) e^{\sqrt{2} x_1} e^{-\sqrt{2} \beta \left( f_2 - f_1 \right)} + O(e^{-2\sqrt{2} |x_1|+\beta (f_1 - f_2)}) \\
&\quad + 2e^2 (N - 1) e^{-\sqrt{2} x_1} e^{-\sqrt{2} \beta \left( f_2 + f_1 \right)} + O(e^{-2\sqrt{2} |x_1|+\beta (f_2 + f_1)}) \\
&\quad - 2e^2 (N - 2)(N - 1) e^{\sqrt{2} x_1} e^{-\sqrt{2} \beta \left( f_3 - f_1 \right)} + O(e^{-2\sqrt{2} |x_1|+\beta (f_1 - f_3)}),
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{b}_{12} - \mathbf{b}_{22} + \mathbf{b}_{32} - \mathbf{b}_{42} &= -2e (N - 1) e^{-\sqrt{2} x_2} e^{-\sqrt{2} \beta \left( f_2 - f_1 \right)} + O(e^{-2\sqrt{2} |x_2|+\beta \left( f_2 - f_1 \right)}) \\
&\quad + 2e (N - 2) e^{\sqrt{2} x_2} e^{-\sqrt{2} \beta \left( f_3 - f_2 \right)} + O(e^{-2\sqrt{2} |x_2|+\beta (f_2 - f_3)}) \\
&\quad + 2e^2 (N - 1) e^{-\sqrt{2} x_2} e^{-\sqrt{2} \beta \left( f_2 + f_1 \right)} + O(e^{-2\sqrt{2} |x_2|+\beta f_2}) \\
&\quad - 2e^2 (N - 2)(N - 3) e^{\sqrt{2} x_2} e^{-\sqrt{2} \beta \left( f_4 - f_2 \right)} + O(e^{-2\sqrt{2} |x_2|+\beta (f_2 - f_4)}).
\end{align*}
\]

**Case 3:** When \( n = N - 1 \), \( N \), we get that

\[
\begin{align*}
\mathbf{b}_{1N-1} - \mathbf{b}_{2N-1} + \mathbf{b}_{3N-1} - \mathbf{b}_{4N-1} &= -4e N e^{-\sqrt{2} x_{N-1}} e^{-\sqrt{2} \beta \left( f_{N-1} - f_{N-2} \right)} + O(e^{-2\sqrt{2} |x_{N-1}|+\beta (f_{N-1} - f_{N-2})}) \\
&\quad + 2e e^{\sqrt{2} x_{N-1}} e^{-\sqrt{2} \beta \left( f_{N-1} - f_{N-1} \right)} + O(e^{-2\sqrt{2} |x_{N-1}|+\beta (f_{N-1} - f_{N-1})}) \\
&\quad + 12e^2 e^{-\sqrt{2} x_{N-1}} e^{-\sqrt{2} \beta \left( f_{N-1} - f_{N-3} \right)} + O(e^{-2\sqrt{2} |x_{N-1}|+\beta (f_{N-1} - f_{N-3})}),
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{b}_{1N} - \mathbf{b}_{2N} + \mathbf{b}_{3N} - \mathbf{b}_{4N} &= -2e e^{-\sqrt{2} x_{N}} e^{-\sqrt{2} \beta \left( f_{N-1} - f_{N-1} \right)} + O(e^{-2\sqrt{2} |x_{N}|+\beta (f_{N} - f_{N-1})}) \\
&\quad + 4e^2 e^{-\sqrt{2} x_{N}} e^{-\sqrt{2} \beta \left( f_{N-1} - f_{N-2} \right)} + O(e^{-2\sqrt{2} |x_{N}|+\beta (f_{N} - f_{N-2})}).
\end{align*}
\]
B Linear problem

We first set

\[ S = \mathbb{R} \times (0, \ell / \epsilon), \]

and provide the following lemma in [13].

**Lemma B.1** (Lemma 4.2 in [13]) For a given function \( \Phi_\ast(x, z) \in L^2(S) \) with

\[ \int_{\mathbb{R}} \Phi_\ast(x, z) \text{H}_x \, dx = 0, \quad 0 < z < \frac{\ell}{\epsilon}, \]

let us consider the following problem

\[ \phi_{\ast, xx} + \phi_{\ast, zz} + (1 - 3\text{H}^2) \phi_\ast = \Phi_\ast \text{ in } S, \quad (B.1) \]

with the conditions

\[ \phi_\ast(x, 0) = \phi_\ast(x, \ell / \epsilon), \quad \phi_{\ast, z}(x, 0) = \phi_{\ast, z}(x, \ell / \epsilon), \quad x \in \mathbb{R}, \quad (B.2) \]

\[ \int_{\mathbb{R}} \phi_\ast(x, z) \text{H}_x \, dx = 0, \quad 0 < z < \frac{\ell}{\epsilon}. \quad (B.3) \]

The problem (B.1)–(B.3) has a unique solution \( \phi_\ast \in H^2(S) \).

Then, by using the above lemma, we can obtain following result:

**Lemma B.2** For a given function \( \Phi^\ast(x, z) \in L^2(S) \) with

\[ \int_{\mathbb{R}} \Phi^\ast(x, z) \text{H}_x \, dx = 0, \quad 0 < z < \frac{\ell}{\epsilon}, \]

consider the following problem

\[ \phi_{zz}^\ast + \beta^2 \left[ \phi_{xx}^\ast + (1 - 3\text{H}^2) \phi^\ast \right] = \Phi^\ast \text{ in } S, \quad (B.4) \]

with the conditions

\[ \phi^\ast(x, 0) = \phi^\ast(x, \ell / \epsilon), \quad \phi_z^\ast(x, 0) = \phi_z^\ast(x, \ell / \epsilon), \quad x \in \mathbb{R}, \quad (B.5) \]

\[ \int_{\mathbb{R}} \phi^\ast(x, z) \text{H}_x \, dx = 0, \quad 0 < z < \frac{\ell}{\epsilon}. \quad (B.6) \]

There exists a unique solution \( \phi^\ast \in H^2(S) \) to problem (B.4)–(B.6), which satisfies

\[ \| \phi^\ast \|_{H^2(S)} \leq C \| \Phi^\ast \|_{L^2(S)}. \quad (B.7) \]

**Proof** Let

\[ \phi^\ast(x, z) = \tilde{\phi}^\ast(x, \iota(z)), \quad \iota(z) = \epsilon^{-1} \int_{0}^{z} \beta(r) \, dr. \]

Here, the map

\[ \iota : \left[0, \frac{\ell}{\epsilon}\right) \rightarrow \left[0, \frac{\hat{\ell}}{\epsilon}\right), \quad \hat{z} = \iota(z) \]

is a diffeomorphism, where \( \hat{\ell} = \int_{0}^{\ell} \beta(r) \, dr \).

It is easy to derive that

\[ \phi_z^\ast(x, z) = \beta \tilde{\phi}_z^\ast(x, \hat{z}), \quad \phi_{zz}^\ast(x, z) = \beta^2 \tilde{\phi}_{zz}^\ast(x, \hat{z}) + \epsilon \beta' \tilde{\phi}_z^\ast(x, \hat{z}), \]

\( \Box \)
while differentiation in $x$ does not change. Therefore, problem (B.4)–(B.6) can be rewritten as

$$\tilde{\phi}_{zz}^* + \left[ \tilde{\phi}_{xx}^* + (1 - 3H^2)\tilde{\phi}^* \right] = \tilde{\phi}^* - \epsilon\beta^{-2}\beta'\tilde{\phi}_z^* \quad \text{in } \mathbb{R} \times \left[ 0, \frac{\ell}{\epsilon} \right), \quad \text{(B.8)}$$

$$\int_{\mathbb{R}} \tilde{\phi}^* (x, \tilde{z}) H_x \, dx = 0, \quad 0 < \tilde{z} < \frac{\ell}{\epsilon}, \quad \text{(B.9)}$$

with the conditions

$$\tilde{\phi}^* (x, 0) = \phi^* (x, \ell/\epsilon), \quad \tilde{\phi}_z^* (x, 0) = \tilde{\phi}_z^* (x, \ell/\epsilon), \quad x \in \mathbb{R}, \quad \text{(B.10)}$$

$$\int_{\mathbb{R}} \tilde{\phi}^* (x, \tilde{z}) H_x \, dx = 0, \quad 0 < \tilde{z} < \frac{\ell}{\epsilon}. \quad \text{(B.11)}$$

From Lemma B.1, we can know that problem (B.8)–(B.11) has a unique solution $\tilde{\phi}^* (x, \tilde{z})$. The result follows by transforming $\tilde{\phi}^* (x, t(z))$ into $\phi^* (x, z)$ via change of variables. By using the method of sub-supersolutions, we can get the estimate (B.7). This concludes the proof of the lemma. \hfill \Box

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