An explicit classical strategy for winning a CHSH\textsubscript{$q$} game

Matej Pivoluska\textsuperscript{1} and Martin Plesch\textsuperscript{1,2}

\textsuperscript{1} Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic
\textsuperscript{2} Institute of Physics, Slovak Academy of Sciences, Bratislava, Slovakia

E-mail: mpivoluska@mail.muni.cz

Keywords: device independence, CHSH game, classical-quantum gap, bell Inequalities

Abstract

A CHSH\textsubscript{$q$} game is a generalization of the standard two player CHSH game, with $q$ different input and output options. In contrast to the binary game, the best classical and quantum winning strategies are not known exactly. In this paper we provide a constructive classical strategy for winning a CHSH\textsubscript{$q$} game, with $q$ being a prime. Our construction achieves a winning probability better than $\frac{1}{2^2} q^{-2}$, which is in contrast with the previously known constructive strategies achieving only the winning probability of $O(q^{-1})$.

1. Introduction

Non-locality is one of the defining features of quantum mechanics qualitatively differentiating it from classical physics\textsuperscript{[5]}. Apart from its foundational importance, scientists have recently realized that quantum non-locality is also an extremely valuable resource enabling various tasks, such as quantum key distribution\textsuperscript{[1,18]} or randomness expansion and amplification\textsuperscript{[4,7,9,13–17]}. All these applications use an unifying feature of quantum mechanics—namely its possibility of providing experimentalist results that exhibit super-classical correlations. Measurements on distant parts of a quantum system can, if performed in a specific way, produce results that are not reproducible by any classical system, even with the help of pre-shared information. Since the seminal work of Bell\textsuperscript{[3]}, who first realized this fact, a long line of research was devoted both to experimental realization of different tests of quantumness (including the recent loophole-free Bell experiment\textsuperscript{[10]}) and its theoretical implications.

One of the recent utilizations of quantum super-correlations is the idea of Device Independence. As quantum devices are capable of producing a different flavour of correlations then purely classical ones, the existence of these kind of correlations (a. k. a. violating some kind of Bell inequality) certifies a quantum nature of the experiment performed. Thus by observing the output data of an experiment and relating it to its input, one is in principle able to conclude quantum nature of the devices, without any need of knowing or testing the inner workings of the devices. And as quantum measurements providing super-classical correlations are inevitably connected with randomness of the outcomes, an experiment can simultaneously check ‘quantumness’ of the devices and provide randomness. This approach is called device independence\textsuperscript{[5]}, and stands in the spotlight of recent research in the area of quantum information.

Arguably the simplest and most studied generalization of the original Bell setting is the Clauser–Horne–Shimony–Holt (CHSH) setting\textsuperscript{[6]}, where two experimentalists choose one out of two possible binary measurements on their part of the system. The setting can be rephrased into a language of games, where two non-communicating players, Alice and Bob, both receive a uniformly chosen single bit input $x$ and $y$ respectively and their goal is to produce single bit outputs $a$ and $b$, such that $a + b \equiv xy \mod 2$ (see figure 1(a)).

It is well known that classical players can win this game with a probability no more than 75%. The strategy achieving this is trivial, consisting of outputting a 0 by both Alice and Bob, irrespectively on the inputs. Utilizing quantum mechanics, players can share a maximally entangled state of two qubits and perform a suitable measurement (dependent on the input) on their respective qubit. In such a way they can increase the probability of winning the game up to $\frac{1 + \sqrt{2}}{2} \approx 85\%$. This fact can be utilized to perform device-independent experiments.
With the standard CHSH setting, in a single round of the protocol only two bits are produced, where only one of them can be utilized due to the correlation with the other output bit. Therefore there appears a natural question if and how one might produce more bits in a single experimental run. This can be easily achieved by allowing Alice and Bob to receive an input from a higher alphabet and also producing a more complicated result. A straightforward generalization is a CHSH$_q$ game, where the dimensionality of both inputs and outputs is limited to a prime $q$ (see figure 1(b)). In this case, the winning condition states $a + b \equiv xy \mod q$. However, to be useful for device independent experiments, the probability of winning the game with a quantum strategy must be higher than the probability with purely classical systems. Therefore, bounds for these probabilities are of utmost importance for its possible use. In this paper we provide a constructive lower bound for the probability of winning a CHSH$_q$ game using purely classical systems.

The paper is organized as follows. In the second section we formally define the CHSH$_q$ game and review the existing bounds for both classical and quantum strategies. In the third section we relate the problem of finding classical strategies to CHSH$_q$ games to solving the problem of point-line incidences. In section four we introduce our classical strategy and prove its efficiency, whereas in the last section we conclude by discussing the results obtained.

2. General CHSH$_q$ games

Formally, with a non-local game $G$, we associate two values: a classical probability of winning $\omega(G)$ and a quantum probability of winning $\omega^*(G)$. The non-local properties of quantum theory are demonstrated by the fact that $\omega^*(G) > \omega(G)$. In case of the standard binary CHSH game we have $\omega(CHSH) = 0.75$ and $\omega^*(CHSH) = \frac{2 + \sqrt{2}}{2} \approx 85\%$. Both these values are known exactly and for both quantum and classical case there exist a constructive strategy that achieves this bound and is efficient to calculate. In fact, the classical strategy is fully trivial with a constant output, whereas the quantum strategy consists of selecting a proper measurement setting given by the binary input and providing the measurement result as the output.

The binary CHSH game can be generalized in the following way. Both Alice and Bob receive inputs $x, y \in \mathbb{F}_q$, i.e. a finite field which exist for any $q$ being a prime power. Their goal is to produce outputs $a, b \in \mathbb{F}_q$ such that $a + b \equiv xy$, where both sum and product are operations of the corresponding field. We will denote a game with inputs in $\mathbb{F}_q$ as CHSH$_q$.

With the standard CHSH setting, in a single round of the protocol only two bits are produced, where only one of them can be utilized due to the correlation with the other output bit. Therefore there appears a natural question if and how one might produce more bits in a single experimental run. This can be easily achieved by allowing Alice and Bob to receive an input from a higher alphabet and also producing a more complicated result. A straightforward generalization is a CHSH$_q$ game, where the dimensionality of both inputs and outputs is limited to a prime $q$ (see figure 1(b)). In this case, the winning condition states $a + b \equiv xy \mod q$. However, to be useful for device independent experiments, the probability of winning the game with a quantum strategy must be higher than the probability with purely classical systems. Therefore, bounds for these probabilities are of utmost importance for its possible use. In this paper we provide a constructive lower bound for the probability of winning a CHSH$_q$ game using purely classical systems.

The paper is organized as follows. In the second section we formally define the CHSH$_q$ game and review the existing bounds for both classical and quantum strategies. In the third section we relate the problem of finding classical strategies to CHSH$_q$ games to solving the problem of point-line incidences. In section four we introduce our classical strategy and prove its efficiency, whereas in the last section we conclude by discussing the results obtained.

2.1. Quantum bound

Contrary to the binary CHSH game, neither the exact value $\omega^*(CHSH_q)$, nor a strategy obtaining the optimal value is known. The only existing result due to [2] introduces an upper bound for the quantum probability...
\[ \omega^*(\text{CHSH}_q) \leq \frac{1}{q^2} + q - \frac{1}{q} = \frac{1}{\sqrt{q}} + \frac{1}{q} - \frac{1}{q\sqrt{q}}. \]

This fact has two important consequences. The first is that being it an upper bound, we will not be able to show \( \omega^*(\text{CHSH}_q) > \omega(\text{CHSH}_q) \) and thus the usefulness of CHSH for device independent experiments. The second consequence is that even the upper bound decreases with \( \frac{1}{q^2} \) in the leading order with large \( q \). Thus, even if the tightness of this bound and a classical-quantum gap could be shown in the future, the statistics of successful outcomes would be decreasing with \( q \) and many experimental runs would be needed.

### 2.2. Classical bounds

With classical bounds the situation is slightly better. As shown by Bavarian and Shor [2], an upper bound exists in the form

\[ \omega(\text{CHSH}_q) = O\left(q^{-\frac{1}{2}}\right) \quad \text{for} \quad q = p^{2k+1}, \]

where \( p \) is a prime, \( k \geq 1 \) and \( \varepsilon > 0 \) is a constant. It is only valid for the case of an odd prime power, but still could serve for a proof of a classical-quantum gap if the quantum bound would be proven tight.

A set of lower bounds also exist (also proven in [2]) in the form

\[ \omega(\text{CHSH}_q) = \begin{cases} \Omega\left(q^{-\frac{1}{2}}\right) & \text{for} \quad q = p^{2k} \\ \Omega\left(q^{-\frac{1}{2}}\right) & \text{for} \quad q = p^{2k+1} \end{cases}. \]

We see that for \( q \) being an even power prime the lower bound is higher than for odd powers and thus for all values of \( q \) there is a significant gap between the lower or upper (partly non-existent) bounds.

Even more importantly and perhaps surprisingly, these lower bounds are not connected with any concrete strategy. Quantum strategies existing so far are limited to different heuristics (e.g. trying to maximize the winning probability over all measurements of the maximally entangled bipartite state), random searches and numerics [11, 12]. Best known classical strategies so far obtained only \( \omega(\text{CHSH}_q) = \Omega\left(\frac{1}{q}\right) \) [12], which corresponds to a trivial strategy (both Alice and Bob outputs 0 irrespective of their input and win if either \( x = 0 \) or \( y = 0 \), thus in \( 2q - 1 \) out of \( q^2 \) cases).

In this paper we present the first constructive classical strategy for the CHSH game with the probability of winning \( \Omega(q^{-\frac{1}{2}}) \) for \( q \) being a prime. With this strategy we close the gap between constructive strategies and existence bounds. To be able to present details of the proof, we first relate the problem of classical CHSH game strategies to a well-known problem of point-line incidences.

### 3. Point-line incidences and classical strategies for the CHSH game

Every classical strategy of CHSH can be written as a convex combination of deterministic strategies, which can be written down as two functions—\( a: \mathbb{F}_q \to \mathbb{F}_q \) representing the strategy of Alice and \( b: \mathbb{F}_q \to \mathbb{F}_q \) representing Bob’s strategy.

The winning condition now states

\[ a(x) + b(y) = xy, \]

which can be rewritten into a form

\[ a(x) = xy - b(y), \]

where all additions and multiplications are operations of the finite field \( \mathbb{F}_q \). Note that in this form Alice’s strategy can be seen as a set of points \( P = \{(x, a(x)) \in \mathbb{F}_q^2\} \) and Bob’s strategy can be seen as a set of lines \( L = \{(y, -b(y)) \subseteq \mathbb{F}_q^2\} \), where a line \( l = \{(y, -b(y)) \} \) contains all points \((g, h) \in \mathbb{F}_q^2\) such that \( h = yg - b(y) \). Note that the strategy of Alice and Bob is successful for input \( x, y \) if the point specified by a vector \((x, a(x))\) lies on the line specified by \((y, -b(y))\). Assuming uniform choice of the input pairs, the strategy of Alice and Bob is the more successful, the more of the points of \( P \) lie on the lines in \( L \). Thus one can reformulate the problem of the best strategy for Alice and Bob to a problem of finding \( q \) points and \( q \) lines with the highest number of incidences.

This is a well known and hard problem, even for general sets of points and lines [8]. However, in order to be able to map a set of points and lines to a classical strategy for CHSH two more conditions need to be fulfilled:

- No two points lie on the same vertical line (have the same \( x \));
- No two lines have the same slope \( y \).
Violation of these conditions would make the strategy ambiguous, since it would assign more than one possible output to some inputs \(x, y\).

Let us label the number of point-line incidences by \(I\). The fraction of inputs for which Alice and Bob can produce a correct outcome is given by \(\frac{I}{q}\), which, with an assumption of uniform choice of input pairs \((x, y)\), also gives the probability of winning the CHSH\(_q\) game.

This reduction has already been used in [2], where the authors obtained the bounds on classical values of CHSH\(_q\) introduced in the previous section.

4. Strategy

In this section we construct a strategy for Alice and Bob to win the CHSH\(_q\) game for prime \(q\). We do so by showing an explicit construction for \(q\) points and \(q\) lines with \(I = \frac{1}{12}q^{1/3}\) and thus a fraction of correct outcomes \(\frac{1}{12}q^{-2/3}\). We achieve this by selecting a specific set of points and lines not obeying the unambiguity conditions stated before, but having a large number of mutual incidences. Then we perform a transformation that will remove the ambiguities at the cost of removing a portion of the lines and points we started with. In what follows we will use the letter \(p\) instead of \(q\) to stress that the sums and products are being performed in a field \(\mathbb{F}_p\) of prime order \(p\). We will also use the symbol \(\equiv\) in equations valid modulo \(p\) (unless explicitly a different modulo is stated) and symbol = in standard integer/rational equations.

4.1. Selection of points and lines

We define the following quantities

\[
p_1 = 2 \left\lfloor \frac{p^{1/3}}{2} \right\rfloor,
\]

\[
p_2 = 2 \left\lfloor \frac{p}{2p_1} \right\rfloor.
\]

We see that both \(p_1\) and \(p_2\) are even. Further, we can write \(p_1 = p^{1/3} - \zeta\), where \(0 < \zeta < 2\). Then we have \(p_2 = 2 \left\lfloor \frac{1}{2p_1}(p_1 + \zeta) \right\rfloor > 2 \left\lfloor \frac{1}{2p_1}(p_1^2 + 3p_1\zeta) \right\rfloor\). As \(p_1\) is an even number, we can take out the \(\frac{p_1}{2}\) from the brackets, as this is a natural number, and get \(p_2 > p_1^2 + 2 \frac{p_1}{2p_1}\zeta\). From this we get:

\[
p_1^2 < p_2.<p> (4)
\]

Further, \(p_1p_2 = p_1^2 + 2 \frac{p}{2p_1} = p\) and \(p_1p_2 = p_2^2 + 2 \frac{p}{2p_2} > 2p_2^2(p_1 - 1) = p - 2p_1\), thus

\[
p = 2p_1 < p_1p_2 < p. \quad (5)
\]

Now we define a set of \(p_1, p_2\) points by all points with coordinates \((x, a)\) and

\[
x \in (0, p_1)
\]

\[
a \in (0, p_2). \quad (6)
\]

We also define \(\frac{p_1p_2}{4}\) lines in a following way: instead of using the standard notation for a line \(y = ax + b\) we will use the form \(\zeta = a \tau + b\) to mimic the notation for CHSH. Thus each line is defined by a pair of values \((y, b)\) and we define the set of lines by defining the set of values by

\[
y \in \left(0, \frac{p_1}{2}\right)
\]

\[
b \in \left(0, \frac{p_2}{2}\right). \quad (7)
\]

For \(\tau \in (0, p_1)\), \(\zeta\) will be non-negative and bounded from above by \(\frac{p_1}{2} - 1 + \left(\frac{p_2}{2} - 1\right)\left(p_1 - 1\right) < p_2 - 1\) for each line from (7). Thus for each of these lines there will be \(p_1\) points from (6) that lie on this line (points with all different values of \(x\) and corresponding \(a\)). Consequently the number of incidences within this set is exactly

\[
I = \frac{p_1^2p_2}{4},
\]

which is roughly \(\frac{p^{1/3}}{4}\).
4.2. Transformation

Now we perform the following transformation of both points and lines:

\[
(x, a) \rightarrow \left(1 + \frac{2a}{p_2x - a}, \frac{2p_2b}{p_2 - y'} + y\right),
\]

where all sums and products are performed in \( \mathbb{F}_p \) and division is understood as multiplication by the inverse element. Transformation is well defined for all the points but \((0, 0)\) and all the lines. With a bit of technical exercise one can see that the transformed points lie on a transformed line if and only if only if the original points did. It is also easy to see that we have successfully removed all the ambiguity in points, as \( p_2x - a \) is different for all pairs of \((x, a)\) satisfying (6) so is the inverse element. Therefore, we have a new set of \((p_1p_2 - 1)\) points that all have different \(x\) coordinates.

The situation of lines is much different. The slope is defined by the fraction \( \frac{2p_2b}{p_2 - y'} \) for which it is not easy to see how many different values it can acquire in \( \mathbb{F}_p \). Here we will show that among the \( \frac{p_1p_2}{4} \) lines transformed according to (8) there will be at least \( \frac{p_1p_2}{20} \) with different slopes.

4.3. Identifying ambiguities

In order to prove the result, we will sum up all the lines that share a slope with another line and show that there aren’t too many of them. In fact we could leave one of the lines sharing a slope with another line and remove all the rest, but instead we will remove all of them. This makes the procedure redundant, but easier to tackle and does not influence the final results by more than a constant.

We will work with the equation \( k' = \frac{2p_2b}{p_2 - y'} \), which, after the substitution \( k' = \frac{2p_2}{k} \), is equivalent to the equation

\[
k'b \equiv p_2 - y. \tag{9}
\]

We will search for values of \( k \) for which there exists more than one solution of \( y \) and \( b \) within the given range (7). To do so, we can visualize the situation as follows: we start from the element 0 on the left-hand side of (9) and make steps of length \( p \) (corresponding to \( b = 0 \) on the left-hand side of (9)) and are seeking for cases when we ‘visit’ the interval

\[
\left(p_2, p_2 \right), \tag{10}
\]

more than once, as this is the interval of values the right-hand side of (9) can acquire. This can happen in two principally different cases:

- \( k < \frac{p_1}{2} \) or \( k > p - \frac{p_1}{2} \) and thus the interval could be repeatedly visited within subsequent steps
- \( k > 2p_1 \) and the interval is visited after one or more cycles within the field.

For \( \frac{p_1}{2} \leq k \leq 2p_1 \), the size of the step is larger than the interval we are trying to hit, therefore we cannot visit the interval twice without at least one cycle in the field, yet the step is too short to finish a single cycle within the field. Thus for \( \frac{p_1}{2} \leq k \leq 2p_1 \) there cannot exist more than one solution of (9).

4.3.1. Small steps

If \( k \leq \frac{p_1}{2} \), the analysis is very simple. We can upper bound the number of solutions for each \( k \) by \( \frac{p_1}{2} \) and thus the number of repeated solutions by \( R_{\text{small}} = \frac{p_1^2}{4} \). This bound is in fact very loose, but for large \( p \) is fully satisfactory. The case \( k > p - \frac{p_1}{2} \) is even simpler, as even after the maximum number of steps \( \frac{p_1}{2} \) we are not able to reach the desired interval, as \( \frac{p_1}{2} > \frac{p_1p_2}{4} - \frac{p_1}{2} \equiv p \).

4.3.2. Large steps

The second case is more complicated. Here we know that the left hand side of equation (9) is 0 for \( b = 0 \). Let \( b_1 \) be the smallest \( b \) such that (9) holds for a given \( k \) and let \( b_2 \) be the next \( b \) for which (9) holds. Let

\[
d = k(b_2 - b_1) \equiv p. \tag{11}
\]
We now define

\[ \delta = \begin{cases} 
   d & \text{for } d < p/2 \\
   d - p & \text{for } d > p/2. 
\end{cases} \tag{12} \]

Here \( \delta \) is an integer, thus \( \delta \) can acquire both positive and negative values and therefore \(|\delta|\) is the standard absolute value. It is easy to see that

\[ |\delta| < \frac{p}{2}, \tag{13} \]

due to the limited width of the interval (10). This condition means that before visiting the interval \( \left( p_2 - \frac{p_1}{2}, p_2 \right) \) twice, we have to visit also the interval \( \left( -\frac{p_1}{2}, \frac{p_1}{2} \right) \) in point \( d = \delta \) once again after staring from 0.

Let us now define

\[ l \equiv \frac{d}{k}, \tag{14} \]

This means that after \( l \) steps of length \( k \) we visit the point \( d = \delta \). Switching back to integers, this means that there exists a positive \( s \) such that

\[ kl = sp + \delta. \tag{15} \]

As \( k < p \) and \( d < p \), clearly \( 0 \leq s \leq l \) and as \( k > 2p_2, s > 0 \). We can also write

\[ k = \frac{sp + \delta}{l}. \tag{16} \]

Now it is easy to see that the points in the field visited in the \( r^{th} \) step \((0 < r < l)\) have the form

\[ rk = r\frac{sp + \delta}{l}. \tag{17} \]

Let us now define a set of rational numbers

\[ Q = \left\{ \frac{p + \delta}{l} \mid 0 < q < l \right\}. \tag{18} \]

For the specific case \( s = 1 \), elements of \( Q \) are natural numbers and for each \( q \) they exactly define elements of \( \mathbb{P}_p \) visited by the \( q^{th} \) step of length \( k \). In all the other cases we want to relate the \( r^{th} \) visited element of the field with a specific element of \( Q \). We do it as follows — the \( r^{th} \) visited point is associated with element of \( Q \) defined by

\[ q(r) := rs \mod l. \tag{19} \]

Therefore the element of \( \mathbb{P}_p r\frac{p + \delta}{l} \) is associated with (see also figure 2)

\[ q(r)\frac{p + \delta}{l} = (rs \mod l)\frac{p + \delta}{l}. \tag{20} \]

We can rewrite \( q(r)\frac{p + \delta}{l} \) to \( \frac{p + \delta}{l} \) \( p + \delta \) \( \leq \frac{l-1}{l} \left( p + \frac{p_1}{2} \right) < p - \frac{p_1}{l} + \frac{p_1}{2} \leq p - 2\frac{p_2}{l} + \frac{p_1}{2} < p \), where we used (5), thus \( q(r)\frac{p + \delta}{l} < p \). What is not trivial to see is that the relation between \( q(r) \) and \( r \) is a bijection: in order to show a contradiction, consider \( r_1 < r_2 \) for which \( q(r_1) = q(r_2) = q \). Then \( l \) divides \( s(r_2 - r_1) \) and the step \( r_2 - r_1 \) points to

\[ (r_2 - r_1)k = \frac{(r_2 - r_1)sp + (r_2 - r_1)\delta}{l} \equiv \frac{\delta r_2 - r_1}{l}. \tag{21} \]

Note that since the left hand side of the equation is an integer, so is it’s right hand side. Additionally, since \( r_2 - r_1 < l \), we also have \(|\delta| \geq \frac{\delta p_2}{l} \). This would mean that before reaching the point \( \delta \) in \( l \) steps, we would reach a point \( \delta \frac{q - s}{l} \), which is closer to zero than \( \delta \), in less than \( l \) steps, which is a contradiction with the definition of \( \delta \). Thus we can conclude that there is a one-to-one correspondence between \( r \) and \( q(r) \).

Now we calculate the difference between the points defined by \( r \) and by \( q(r) \):

\[ \frac{rs - q(r)\frac{p + \delta}{l}}{l} = \frac{\delta r - q(r)}{l}, \tag{22} \]

where in the second equivalence we used (19). As both \( r \) and \( q(r) \) are non-negative and smaller than \( l \), the absolute value of this distance is smaller than \(|\delta|\).

This has an important consequence. We can conclude that the points visited by walking through the field with \( l \) steps of length \( k \) (defined in (17)) are elements of the field — natural numbers that do not differ by more than \(|\delta|\) from the rational numbers defined in (18) for \( 0 < q < l \). We know that within the \( l \) steps, one (say \( \omega \)) did fall into the interval (10). This was however only possible if one of the newly defined points \( q(\omega) \in Q \) fits
into a larger interval

\[ q(\omega) \frac{p + \delta}{l} \in \left( p_2 - \frac{p_1}{2} - |\delta|, p_2 + |\delta| \right) \]  

(23)

for some value of \( 0 < q(\omega) < l \). The last condition can be rewritten in a form of an inequality

\[ p_2 - \frac{p_1}{2} - |\delta| < q(\omega) \frac{p + \delta}{l} \leq p_2 + |\delta|. \]  

(24)

Now the task is to find for which values of \( l \) there exist a suitable \( q \) that fulfills this inequality. For each of these values of \( l \) we will have to remove the number of possible repeated solutions. These are limited in two ways. Each return to the interval costs at least \( l \) steps, as \( l \) is the minimal number of steps that would allow the return to an already visited point closed by \( \frac{p_1}{2} \), as shown above. As only \( \frac{p_2}{2} \) steps are available, the number of repeated solutions is limited to \( \frac{p_2}{2} \). The other limitation is due to the narrowness of the interval (if the first return to the interval was \( \delta \) away from the first visit, the second return will be \( 2\delta \) away etc). As the interval is only \( \frac{p_2}{2} \) broad, the maximal number of repeated solutions is \( \frac{p_2}{2} \). The overall limitation is given by the minimum of these two values.

We can rephrase the task also in a slightly different way: for each natural \( q < l \), we will find the number of different values of \( l \) that fulfill (24) and for each of them calculate the number of repeated solutions. One very important observation is that (24) can only have solutions for \( q \leq \frac{p_1}{2} \). This is easy to see from the fact that the middle part of (24) needs to be smaller than or equal to \( p_2 + |\delta| \), and as it is always smaller than \( p_1 \), we can limit ourselves to solutions within natural numbers, without taking into account field properties. As \( l \) is limited to \( \frac{p_2}{2} \), using (4) we get \( q \leq \frac{p_1}{2} \).

Let us define \( l_q \) as the largest \( l \) for which (24) is satisfied. Then it holds:

\[ l_q < \frac{q \cdot p + \delta}{p_2 - \frac{p_1}{2} - |\delta|}. \]  

(25)

Let us also define \( l_q - x_q \) as the smallest \( l \) for which (24) is satisfied. For \( l_q - x_q \) to solve the inequality, it must hold

\[ l_q \left( p_2 - \frac{p_1}{2} - |\delta| \right) < (l_q - x_q)(p_2 + |\delta|) \]  

(26)
and thus

\[ x_q < l_q \frac{P_1}{P_2} + 2|\delta| \]

We identify \( x_q \) as the number of different \( ls \) that can solve (24) for a fixed value of \( q \).

### 4.4. Removing ambiguities

Now we can calculate the upper bound of repeated solutions for a specific \( \delta \)

\[
R_\delta = \sum_{q=1}^{P_1/2} x_q \min \left[ \frac{P_1}{2|\delta|} - \frac{P_2}{2q}, \frac{P_1}{2}, \frac{P_2}{2q} \right] < \sum_{q=1}^{P_1/2} l_q \left( \frac{P_1}{2|\delta|} - \frac{P_2}{2q} \right).
\]

We get rid of the minimum in the sum by a simple trick—as \( l_q \) grows with \( q \), we will take the first value \( \frac{P_1}{2|\delta|} \) for small values of \( q \) and the second value \( \frac{P_2}{2q} \) for larger values of \( q \). We choose the breaking point to be \( \frac{P_1}{2|\delta|} \), which is roughly where the transition takes place. Importantly, we do not need to make this decision precise, as a wrong breaking point will only increase the value of the sum and we are interested in an upper bound. The sum now reads

\[
R_\delta < \frac{P_1}{2} + |\delta| \left( \frac{|\delta| - 1}{2} \frac{P_1}{P_2} - |\delta| \frac{P_1}{P_2} + \left( \frac{P_1}{2} - |\delta| \right) \frac{P_2}{2} \right).
\]

After substituting for \( l_q \), the sums can be solved and yield

\[
R_\delta < \frac{P_1}{2} + |\delta| \left( \frac{|\delta| - 1}{2} \frac{P_1}{P_2} - |\delta| \frac{P_1}{P_2} + \left( \frac{P_1}{2} - |\delta| \right) \frac{P_2}{2} \right).
\]

Using (4) and (5) it is easy to see that \( \frac{P_1}{2} + |\delta| (P_1 - |\delta|) \). Now we are ready to sum all \( R_\delta \) with \( \delta \) in the interval given by (13). As only absolute value of \( \delta \) enters into the formula and \( \delta = 0 \) is not a valid case, we can write:

\[
R_{\text{large}} < 2 \sum_{\delta=1}^{P_1/2} \frac{P_1}{4} (P_1 - \delta) < P_1 \left( \frac{1}{8} + \frac{1}{16} - \frac{1}{48} \right) = P_1^3 \frac{5}{6}.
\]

The total number of repeated solutions is then upper bounded by

\[
R = R_{\text{small}} + R_{\text{large}} = P_1 \left( \frac{1}{2} + \frac{P_1}{6} \right).
\]

As \( \frac{P_1}{6} + \frac{1}{2} \) for \( p_1 > 30 \) (thus for fields larger than 27000) and \( p_2 > P_1^2 \) due to (4), we can upper bound

\[
R < \frac{P_1 P_2}{5}.
\]

So even if we remove all repeated solutions, we will stay with at least \( \frac{P_1 P_2}{4} - \frac{P_1 P_2}{5} = \frac{P_1 P_2}{20} \) lines, each reaching at least \( (P_1 - 1) \) points (as we lost one point during the transformation). This will lead us to

\[
I = \frac{P_1 P_2}{20} (P_1 - 1)
\]

point-line incidences.

Using (5) we can write

\[
I > \frac{p_1 - 2P_1}{20} (P_1 - 1) > \frac{P_1^2}{20} - \frac{P_1}{10} > \frac{P_1}{21}
\]
for \( p_1 > 30 \), as \( pp_1 > 42p_1^2 + 21p \). Further we can show that

\[
I > \frac{p^{4/3}}{22},
\]

as \( p_1 > \frac{21}{22}p^{1/3} \) for \( p_1 > 30 \).

### 4.5. Formulating the strategy

In the previous subsection we have shown that the number of incidences is lower bounded by \( \frac{p^{4/3}}{22} \). Now we are ready to formulate the strategy for both Alice and Bob, which will utilize this fact and lead to a victory in the CHSH\(_q\) game in more than \( \frac{p^{4/3}}{22} \) cases out of \( p^2 \).

For Alice, the situation is rather simple. After getting the input \( x = 0 \), she will compute the inverse element \( x^{-1} \). Then she will find the solution of an equation \( x^{-1} = p_2 x' - a' \) within the range (6). To do that, she will need to calculate the inverse element of \( x \)(which can be done in an efficient way) and find the quotient \((x' - 1)\) and remainder \((p_2 - a')\) after dividing by \( p_2 \). The resulting \( x' \) and \( a' \) will fit into the range (6) (this will happen in \( p_1p_2 \) cases), she will compute the outcome as \( a = 1 + \frac{2x'}{p_2x^2 - a} \), which again involves an efficient computation of an inverse element. In all remaining cases Alice will return 0, mimicking the trivial strategy.

Bob is in a slightly more complicated situation. For a given input \( a \) he will have to find the solution of

\[
a = \frac{2p_1y}{x - y} \text{ within the range (7).}
\]

In the worst case he will have to try \( p_1/2 \) different values for \( y' \) and check whether \( \frac{a}{2} > \frac{y'}{p_1} < \frac{p_1}{2} \). If he finds a solution for \( y' \), he will output \( b = \frac{p_1 + y'}{p_1 - y} \). In this way he will also utilize some of the ambiguous solutions (he will keep the first \( y' \) that satisfies conditions, even if other values might as well)—this will potentially lead to winning the game (if the choice by Alice reflects correctly the solution chosen by Bob), but this chance is not incorporated in the bound. If Bob does not find a solution after trying all possible \( y' \), he will output 0.

Bob will need to calculate the inverse element of \( p_2 \) and 2, which are one-off efforts. Then he will need to perform simple inequality check up to \( p_1/2 \) times and if successful, he will need to calculate one more inverse element. If this would be considered still inefficient, he can adopt the techniques from the previous subsections to find approximate values of \( \frac{y}{p_1} \) for the set of \( 0 \leq y' < \frac{p_1}{2} \) in advance and then test only a minor subset of \( y' \)s.

### 5. Conclusion

In this paper we have provided an explicit constructive strategy for winning a generalized CHSH\(_q\) game. The winning probability is lower bounded by \( \frac{p^{4/3}}{22} \), what perfectly mimics the non-constructive existence bound known so far.

This result is useful for potential design of device independent algorithms based on higher alphabet CHSH games in different aspects. First, it closes the gap between existing explicit strategies and proven existence bounds, which helps the understanding of the nature of the problem. Second, and most importantly, the presented result provides the first non-trivial classical strategy for a CHSH game, where Alice and Bob need to act in a way that depends on their input, and their output is a result of a non-trivial calculation.

There is also a set of open questions that remain. The obvious one is, how one could generalize the result presented in this paper for prime power fields. This is not easy, as the nature of the proof relays on the relation between addition and multiplication, which is unique for prime fields. Also the fact that known existence bounds crucially depend on whether they are deployed on even or odd power prime field suggests that any possible generalization will not be straightforward.

More ambitious goals include the aim of finding tight bounds on classical strategies. This might, in accordance with suitable heuristic results for quantum strategies, lead to the possibility of direct use of higher-order CHSH\(_q\) games in experiments. The ultimate goal, naturally, remains to directly prove a gap between classical and quantum strategies.

In a broader sense, open questions remain on understanding the CHSH\(_q\) games in general context of binary games with more inputs and outputs. A glimpse view give us results obtained for the CGLMP game with two inputs and more outputs, that provides better classical-quantum separation, but understanding on why this happens is lacking.
Acknowledgments

This research was supported by EU project RAQUEL, as well as project VEGA 2/0043/15. MPi also acknowledges the support of the Czech Science Foundation GAČR project P202/12/1142 and MPI acknowledges the support of Czech Science Foundation GAČR standard project GA16-22211S.

References

[1] Acín A, Brunner N, Gisin N, Massar S, Pironio S and Scarani V 2007 Device-independent security of quantum cryptography against collective attacks Phys. Rev. Lett. 98 230501
[2] Bavarian M and Shor P W 2015 Information causality, szemerédi–trotter and algebraic variants of chsh Proc. 2015 Conference on Innovations in Theoretical Computer Science, ITCS’15 (New York: ACM) pp 123–32 (arXiv:1311.5186)
[3] John S B 1964 On the Einstein–Podolsky–Rosen paradox Physics 1 195–200
[4] Bouda J, Pawlowski M, Pivoluska M and Plesch M 2014 Device-independent randomness extraction from an arbitrarily weak min-entropy source Phys. Rev. A 90 032313
[5] Brunner N, Cavalcanti D, Pironio S, Scarani V and Wehner S 2014 Bell nonlocality Rev. Mod. Phys. 86 419–78
[6] Clauser J F, Horne M A, Shimony A and Richard A H 1969 Proposed experiment to test local hidden-variable theories Phys. Rev. Lett. 23 880–4
[7] Colbeck R 2009 Quantum and relativistic protocols for secure multi-party computation PhD Thesis (arXiv:0911.3814)
[8] Zeev D 2010 Incidence theorems and their applications Foundations and Trends in Theoretical Computer Science 6 237–393
[9] Gallego R, Masanes L, de la Torre G, Dhara C, Aolita L and Acín A 2013 Full randomness from arbitrarily deterministic events Nat. Commun. 4 2654
[10] Hensen B et al 2015 Experimental loophole-free violation of a Bell inequality using entangled electron spins separated by 1.3 km Nature 526 682–6
[11] Ji S-W, Lee J, Lim J, Nagata K and Lee H-W 2008 Multisetting bell inequality for qudits Phys. Rev. A 78 052103
[12] Liang Y-C, Lim C-W and Deng D-L 2009 Reexamination of a multisetting bell inequality for qudits Phys. Rev. A 80 052116
[13] Pironio S et al 2010 Random numbers certified by Bell’s theorem Nature 464 1021–4
[14] Pivoluska M and Plesch M 2014 Device independent random number generation Acta Phys. Slovaca 64 600–63
[15] Ramanathan R, Brandão F G S L, Horodecki K, Horodecki M, Horodecki P and Wojewódka H 2015 Randomness amplification against no-signaling adversaries using two devices (arXiv:1504.06313)
[16] Vazirani U and Vidick T 2012 Certifiable quantum dice: or, true random number generation secure against quantum adversaries Proc. 44th symposium on Theory of Computing pp 61–76
[17] Vazirani U and Vidick T 2014 Fully device-independent quantum key distribution Phys. Rev. Lett. 113 140501