HILBERT’S 14-TH PROBLEM AND COX RINGS

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Abstract. Our main result is the description of generators of the total coordinate ring of the blow-up of $\mathbb{P}^n$ in any number of points that lie on a rational normal curve. As a corollary we show that the algebra of invariants of the action of a two-dimensional vector group introduced by Nagata is finitely generated by certain explicit determinants. We also prove the finite generation of the algebras of invariants of actions of vector groups related to T-shaped Dynkin diagrams introduced by Mukai.

1. Introduction

Hilbert’s 14-th Problem that we discuss is the following question: if an algebraic group $G$ acts linearly on a polynomial algebra $S$, is the algebra of invariants $S^G$ finitely generated? The answer is known to be affirmative if $G$ is reductive (Hilbert [H]) and if $G$ is the simplest nonreductive group $G_a$ (Weitzenböck [We]). However, in general the answer is negative – the first counterexample was found by Nagata in 1958. Let

$$G = G_a^m \subset G_a^r$$

be a general linear subspace of codimension at least 3. Consider the following linear action of $G_a^m$ on $S := \mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_r]$: an element $(t_1, \ldots, t_r) \in G_a^r$ acts by:

$$x_i \mapsto x_i, \quad y_i \mapsto y_i + t_i x_i, \quad 1 \leq i \leq r.$$ The induced action of $G$ on $S$ is called the Nagata action. The algebra of invariants $S^G$ is not finitely generated if $g = 13$ (Nagata [Na]), $g = 6$ (Steinberg [St]), and finally $g = 3, r = 9$ (Mukai [MI]). Thus, Hilbert’s 14-th Problem has a negative answer for $G_a^3$. In [MI], Mukai asks what happens if $g = 2$.

Theorem 1.1. Assume without loss of generality that $G = G_a^2 \subset G_a^{n+3}$ is a linear subspace spanned by rows of the matrix

$$
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
a_1 & a_2 & \ldots & a_{n+3}
\end{bmatrix}
$$

where $a_1, \ldots, a_{n+3}$ are general numbers. Then $S^G$ is generated by $2^{n+2}$ invariants

$$F_I =
\begin{bmatrix}
x_{i_1} & x_{i_2} & \ldots & x_{i_{2k+1}} \\
a_{i_1} x_{j_1} & a_{i_2} x_{j_1} & \ldots & a_{i_{2k+1}} x_{j_{2k+1}} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_k}^k x_{j_1} & a_{i_k}^k x_{j_1} & \ldots & a_{i_k}^k x_{j_{2k+1}} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_1} y_{j_1} & a_{i_2} y_{j_1} & \ldots & a_{i_{2k+1}} y_{j_{2k+1}} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_k}^{k-1} y_{j_1} & a_{i_k}^{k-1} y_{j_1} & \ldots & a_{i_k}^{k-1} y_{j_{2k+1}}
\end{bmatrix},
$$

where $I = \{i_1, \ldots, i_{2k+1}\} \subset \{1, \ldots, n+3\}$ is any subset of odd cardinality $2k + 1$. 

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Of course, it is possible that the algebra of invariants of $G_a^2$ is not finitely generated for actions more complicated than Nagata actions.

The ingenious insight of Nagata was to relate $S^G$ to a Cox ring. Let $X$ be a projective algebraic variety over $\C$. Assume that divisors $D_1, \ldots, D_r$ freely generate the Picard group $\text{Pic}(X)$. Then the Cox ring of $X$ is the multigraded ring

$$\text{Cox}(X) = \bigoplus_{(m_1, \ldots, m_r) \in \mathbb{Z}^r} \mathcal{H}^0(X, m_1 D_1 + \cdots + m_r D_r)$$

(the basis is necessary to introduce multiplication in a canonical way). This definition is a generalization of the total coordinate ring of a toric variety introduced by Cox [Co]. In fact, Cox definition is a generalization of the total coordinate ring of a toric variety introduced (the basis is necessary to introduce multiplication in a canonical way). This definition is a generalization of the total coordinate ring of a toric variety introduced by Cox [Co]. In fact, Cox

For an arbitrary variety $X$, Hu and Keel [HK] proved that $\text{Cox}(X)$ is finitely generated if and only if $X$ is a Mori variety, which is as in (1.1), by [M1], one has

$$S^G \cong \text{Cox}(\text{Bl}_r \mathbb{P}^{r-g-1})$$

where $\text{Bl}_r \mathbb{P}^{r-g-1}$ is the blow-up of $\mathbb{P}^{r-g-1}$ at $r$ distinct points. Using this isomorphism, Theorem 1.1 is equivalent to describing the Cox ring of a blow up of $\mathbb{P}^n$ at $n+3$ points. It is a well-known fact that there is a unique rational normal curve $C$ of degree $n$ in $\mathbb{P}^n$ passing through $n+3$ points in general position. We generalize Theorem 1.1 as follows:

**Theorem 1.2.** Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree $n$ and let $p_1, \ldots, p_r$ be distinct points on $C$, $r \geq n+3$. Let $X = \text{Bl}_{p_1, \ldots, p_r} \mathbb{P}^n$. Then $\text{Cox}(X)$ is finitely generated by unique (up to scalar) global sections of exceptional divisors $E_1, \ldots, E_r$ and divisors

$$E = kH - k \sum_{i \in I} E_i - (k-1) \sum_{i \in I^c} E_i$$

for each subset $I \subset \{1, \ldots, r\}$, $|I| = n+2-kq$, $1 \leq k \leq 1+n/2$. Here $H$ is the pull-back of the hyperplane class in $\mathbb{P}^n$.

Geometrically, the divisors $E_i$ are proper transforms of the following hypersurfaces in $\mathbb{P}^n$ [H] Ex. 9.6. If $I$ is empty then $E_0$ is the $(n/2)$-secant variety of $C$. More generally, if $\pi_I : \mathbb{P}^n \to \mathbb{P}^{2k} \to \mathbb{P}^{2k}$ is the projection from the linear subspace spanned by the points $p_i$, $i \in I$ and $C' = \pi_I(C)$, then $C'$ is a rational normal curve of degree $2k-2$ and $E_0$ is the cone over the $(k-1)$-secant variety of $C'$.

An obvious generalization of Theorem 1.2 would be to consider the Cox ring of the iterated blow up of $\mathbb{P}^n$ along points, lines connecting them, 2-planes, etc. A special case of this construction is $\mathcal{M}_{0,n}$, the Grothendieck–Knudsen moduli space of stable $n$-pointed rational curves. If the Cox ring of $\mathcal{M}_{0,n}$ is finitely generated, then results of Keel–Hu [HK] and Keel–McKernan [KM] almost imply the “Fulton conjecture” for $\mathcal{M}_{0,n}$ and therefore the description of the Mori cone of $\mathcal{M}_{g,n}$ (Gibney–Keel–Morrison [GKM]).
Following Mukai \cite{M2}, we also generalize Theorem \ref{thm:main} in a different direction. Let $T_{a,b,c}$ be the $T$-shaped tree with legs of length $a$, $b$, and $c$ with $a + b + c - 2$ vertices. We assume that $a, c \geq 2$ and if $c = 2$ then $a > 2$. Let
\[
X_{a,b,c} = \text{Bl}_{b+c}((\mathbb{P}^{c-1})^{a-1})
\]
be the blow-up of $(\mathbb{P}^{c-1})^{a-1}$ in $r = b + c$ points in general position. The effective cone $\text{Eff}(X_{a,b,c})$ is the set of effective divisors in $\text{Pic}(X_{a,b,c})$. Mukai proves in \cite{M2} that if $T_{a,b,c}$ is not a Dynkin diagram of a finite root system then $\text{Eff}(X_{a,b,c})$ is not a finitely generated semigroup and therefore $\text{Cox}(X_{a,b,c})$ is not a finitely generated algebra. Mukai also shows in \cite{M2} that the Cox algebra of any $X_{a,b,c}$ is isomorphic to the algebra of invariants of a certain “extended Nagata action”. We deduce from Theorem \ref{thm:main} using a trick from commutative algebra, the following

**Theorem 1.3.** The following statements are equivalent:

- $\text{Cox}(X_{a,b,c})$ is a finitely generated algebra.
- $\text{Eff}(X_{a,b,c})$ is a finitely generated semigroup.
- $T_{a,b,c}$ is a Dynkin diagram of a finite root system.
- $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$.

Moreover, in these cases consider $Z = \text{Proj}(\text{Cox}(X))$ with respect to the natural $Z$-grading of $\text{Cox}(X)$ defined in \cite{M2}. Then $Z$ is a locally factorial, Cohen–Macaulay, and Gorenstein scheme with rational singularities. The Picard group $\text{Pic}(Z) = \mathbb{Z}$ is generated by $\mathcal{O}_Z(1)$ and the anti-canonical class is $-K_Z = \mathcal{O}_Z(d)$, where
\[
d = abc \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \right) > 0.
\]

The proof of the “moreover” part is exactly the same as Popov’s proof \cite{P} of the analogous statement for Del Pezzo surfaces (or $X_{2,3,3,3}$ in our notation). We only sketch it for the reader’s convenience.

Explicitly, Theorem \ref{thm:main} includes the following cases. Mukai \cite{M2} shows that $X_{a,b,c}$ is a small modification of $X_{c,b,a}$, so we assume that $a \leq c$ (if $X'$ is a small modification of $X$ then of course $\text{Pic}(X) \cong \text{Pic}(X')$, $\text{Eff}(X) \cong \text{Eff}(X')$, and $\text{Cox}(X) \cong \text{Cox}(X')$).

- $X_{2,2,n+1} = \text{Bl}_{n+3}(\mathbb{P}^n)$.
- $X_{2,3,4} = \text{Bl}_1\mathbb{P}^1$, $X_{2,3,5} = \text{Bl}_8\mathbb{P}^1$.
- $X_{3,2,3} = \text{Bl}_3(\mathbb{P}^2)^2$, $X_{3,2,4} = \text{Bl}_0(\mathbb{P}^3)^2$, $X_{3,3,5} = \text{Bl}_7(\mathbb{P}^4)^2$.
- $X_{s+1,1,n+1} = \text{Bl}_{n+2}(\mathbb{P}^n)^s$. This case is well-known, see Remark \ref{rem:case}.
- Del Pezzo surfaces $X_{2,2,3,3} = \text{Bl}_4\mathbb{P}^2$, $s = 4, 5, 6, 7, 8$. In this case the finite generation of the Cox ring was proved by Batyrev and Popov \cite{BP}.

We prove Theorems 1.1–1.3 in reverse order. In Section 2 we describe the effective cone of $X_{a,b,c}$. In Section 3 we prove Theorem \ref{thm:main} (the finite generation of $\text{Cox}(X_{a,b,c})$) in all cases, except for $X_{2,3,4}$ and $X_{2,3,5}$, for which the proof relies on the cases $n = 3$ and $n = 4$ of Theorem \ref{thm:main}. The latter is proved in Section 4 which is the main section of the paper and is independent of the previous sections. Theorem \ref{thm:main} is proved in the concluding Section 5. In particular, we prove the finite generation of $\text{Cox}(\text{Bl}_{n+3}(\mathbb{P}^n))$ twice. First, we give a simple proof in the framework of Theorem \ref{thm:main}. Second, we give an independent proof of the much stronger Theorem \ref{thm:main} that gives explicit generators for this ring. It is crucial for our proof to consider any number of points on a rational normal curve. For example, finding generators for $\text{Cox}(\text{Bl}_{n+3}(\mathbb{P}^n))$ relies on finding generators for the Cox ring of the blow-up of $\mathbb{P}^{n-1}$ in $n + 3$ points lying on a rational normal curve, etc, up to the blow-up of $\mathbb{P}^2$ in $n + 3$ points lying on a conic. Our proof of Theorem \ref{thm:main} was inspired by the “whole-genome shotgun” \cite{VAM} method of genome sequencing that involves
breaking the genome up into very small pieces, sequencing the pieces, and reassembling the pieces into the full genome sequence. This method has some advantages (and disadvantages) over the “clone-by-clone” approach that involves breaking the genome up into relatively large chunks.

During the final stages of the preparation of this paper, Professor Shigeru Mukai sent us his preprint [M3], where he proves that the Cox ring of $X_{a,b,c}$ is finitely generated when $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$ by using a completely different approach based on results of S. Bauer about parabolic bundles on curves.

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2. Root Systems and Effective Cones

From now on we assume that $T_{a,b,c}$ is a Dynkin diagram of a finite root system. It is well-known that this is equivalent to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$. Let 

$$X = X_{a,b,c}.$$ 

The Picard group $\text{Pic}(X)$ is a free $\mathbb{Z}$-module of rank $a + b + c - 1$ with a basis 

$$H_1, \ldots, H_{a-1}, \text{ and } E_1, \ldots, E_r,$$

where $H_i$ is the pull-back of the hyperplane class from the $i$-th factor of $(\mathbb{P}^c)^{a-1}$ and $E_j$ is the class of the exceptional divisor over $p_j$, for $j = 1, \ldots, r$, $r = b + c$. We call this basis tautological. If $a = 2$ then we write $H$ instead of $H_1$ and make the appropriate modifications in all notations. The anticanonical class of $X$ is 

$$-K = c(H_1 + \ldots + H_{a-1}) - (ac - a - c)(E_1 + \ldots + E_r).$$

Following [M2], we define a symmetric bilinear form on $\text{Pic}(X)$ as follows:

$$(H_i, E_j) = 0, \quad (H_i, H_j) = (c - 1) - \delta_{i,j}, \quad (E_i, E_j) = -\delta_{i,j}. \quad (2.1)$$

The following lemma is a straightforward calculation.

**Lemma 2.1.** [M2] $\text{Pic}(X)$ has another $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_{a+r-2}, E_r$, where 

$$\alpha_1 = E_1 - E_2, \quad \ldots, \quad \alpha_{r-1} = E_{r-1} - E_r,$$

$$\alpha_r = H_1 - E_1 - \ldots - E_c,$$

$$\alpha_{r+1} = H_1 - H_2, \quad \ldots, \quad \alpha_{a+r-2} = H_{a-2} - H_{a-1}.$$

Moreover, $\alpha_1, \ldots, \alpha_{a+r-2}$ is a $\mathbb{Z}$-basis of the orthogonal complement $K^\perp$ and a system of simple roots of a finite root system with a Dynkin diagram $T_{a,b,c}$.

Let $W$ be the Weyl group generated by orthogonal reflections with respect to $\alpha_1, \ldots, \alpha_{a+r-2}$. Then $K$ is $W$-invariant. Mukai calls $D \subset X$ a $(-1)$-divisor if there is a small modification $X \rightarrow X'$ such that $D$ is the exceptional divisor for a blow-up $X' \rightarrow Y$ at a smooth point. Note that any $(-1)$-divisor must appear in any set of generators of $\text{Eff}(X)$.

**Lemma 2.2.** [M2] For each transformation $w : \text{Pic}(X) \rightarrow \text{Pic}(X)$ of $W$, there is a small modification $X \rightarrow X_w$ with the following property. $X_w$ is also a blow-up of $(\mathbb{P}^c)^{a-1}$ in $r = b + c$ points $q_1, \ldots, q_r$ in general position and the pull-back of the tautological basis of $X_w$ coincides with the transformation of the tautological basis of $X$ by $w$. In particular, every divisor $E \in W \cdot E_r$ is a $(-1)$-divisor and $H^0(X, E)$ is spanned by a single section $x_E$. 

The proof is an application of Cremona transformations. The case \( a = 2 \) appeared in [Dr] (where it is attributed to Coble). The case \( a = 2, c = 3 \) is well-known from the theory of marked Del Pezzo surfaces.

**Lemma 2.3.** The action of \( W \) on \( \text{Pic}(X) \) preserves \( \text{Eff}(X) \).

**Proof.** Let \( D \in \text{Pic}(X) \) and \( w \in W \). We claim that \( H^0(X, D) \cong H^0(X, w \cdot D) \). We have

\[
D = d_1 H_1 + \ldots + d_{a-1} H_{a-1} - m_1 E_1 - \ldots - m_r E_r.
\]

\( H^0(X, D) \) can be identified with the subspace of polynomial functions on \( (\mathbb{C}^a)^{a-1} \) of multidegree \((d_1, \ldots, d_{a-1})\) vanishing to the order at most \( m_i \) at the point \( p_i \). By Lemma 2.5, \( H^0(X, w \cdot D) \) has the same interpretation for another choice of general points \( q_1, \ldots, q_r \). Now the claim follows from semi-continuity if the points \( p_1, \ldots, p_r \) are sufficiently general.

Let \( \text{Eff}_\mathbb{R}(X) \subset \text{Pic}(X) \otimes \mathbb{R} \) be the cone spanned by \( \text{Eff}(X) \). Let \( N_1(X) \) be the group generated over \( \mathbb{Z} \) by 1-cycles on \( X \) modulo rational equivalence. Intersection of cycles gives a non-degenerate pairing \( \text{Pic}(X) \times N_1(X) \to \mathbb{Z} \). For \( i = 1, \ldots, a-1 \), let \( l_i \in N_1(X) \) be the class of the proper transform of a general line in the \( i \)-th copy of \( \mathbb{P}^{a-1} \). For \( i = 1, \ldots, r \), let \( e_i \in N_1(X) \) be the class of a general line in \( E_i \). Then it is easy to check that

\[
H_i \cdot l_j = \delta_{i,j}, \quad H_i \cdot e_j = 0, \quad E_i \cdot e_j = \delta_{i,j}.
\]

Since the intersection pairing is non-degenerate, it follows that \( N_1(X) \) is generated over \( \mathbb{Z} \) by the classes \( l_1, \ldots, l_{a-1}, e_1, \ldots, e_r \). The action of \( W \) on \( \text{Pic}(X) \) induces an action on \( N_1(X) \).

A class \( \gamma \) in \( N_1(X) \) is called nef if for any effective divisor \( D \) on \( X \), \( D \cdot \gamma \geq 0 \).

**Lemma 2.4.** The classes \( l_1, l_1 + \ldots + l_{a-1} - e_i \) are nef, for all \( i = 1, \ldots, r \).

**Proof.** Note that if a family of curves with class \( f \) covers \( X \) (i.e., through a general point of \( X \) there is an irreducible curve in the family that passes through it), then \( f \) is a nef class: if \( D \) is an effective divisor, there is an irreducible curve in the family that is not contained in \( D \), therefore, \( D \cdot f \geq 0 \). This is obviously the case if \( f = l_i \). If \( f = l_1 + \ldots + l_{a-1} - e_i \), then \( f \) is the proper transform in \( X \) of a curve of multidegree \((1, \ldots, 1)\) in \((\mathbb{P}^{a-1})^{a-1}\) that passes through the point \( p_i \). This family contains an irreducible curve by Bertini’s theorem and we can use the 2-transitive action of \((\text{PGL}_a)^{a-1}\) on \((\mathbb{P}^{a-1})^{a-1}\) to find a curve through any point.

**Definition 2.5.** Define the degree of \( D \in \text{Pic}(X) \) as an integer

\[
\deg(D) = \frac{1}{ac-a-c}(D, -K).
\]

Clearly, \( \deg D \) is \( W \)-invariant and any divisor in the orbit \( W \cdot E_r \) has degree 1.

**Definition 2.6.** Let \( \mathfrak{g}_{a,b,c} \) be a semisimple Lie algebra with the Dynkin diagram \( T_{a,b,c} \). Let \( \Lambda \subset K^\perp \otimes \mathbb{Q} \) be the weight lattice spanned by fundamental weights \( \omega_1, \ldots, \omega_{a+r-2} \) defined by \((\omega_i, \alpha_j) = \delta_{i,j} \). For any \( \omega \in \Lambda \), let \( L_\omega \) be an irreducible \( \mathfrak{g}_{a,b,c} \)-module with the highest weight \( \omega \), see for example [Yo]. \( L_\omega \) is called minuscule if weights \( W \cdot \omega \) are its only weights. Let \( \pi : \text{Pic}(X) \to K^\perp \otimes \mathbb{Q} \) denote the orthogonal projection.

**Theorem 2.7.** \( \text{Eff}(X) \) is generated as a semigroup by divisors of degree 1. \( \text{Eff}_\mathbb{R}(X) \) is generated as a cone by \( D \in \mathcal{W} \cdot E_r \). Projection \( \pi \) induces a bijection between divisors of degree 1 and weights of \( L_{\omega_{r-1}} \) such that divisors in \( \mathcal{W} \cdot E_r \) correspond to weights in \( \mathcal{W} \cdot \omega_{r-1} \). In particular, \( L_{\omega_{r-1}} \) is minuscule if and only if the only effective divisors of degree 1 are \( D \in \mathcal{W} \cdot E_r \).
Remark 2.8. The classification of minuscule representations is well-known. The only arising cases are
\[ \text{Bl}_{n+3}(\mathbb{P}^n), \text{ Bl}_{n+2}(\mathbb{P}^n)^*, \text{ and } \text{Bl}_n\mathbb{P}^2 \ (s = 4, 5, 6, 7). \]
If \( X = \text{Bl}_{n+3}(\mathbb{P}^n) \) then \( L_{\omega_{r-1}} \) is a half-spinor representation of \( \mathfrak{so}_{2n+6} \). Here’s another example: let \( X = X_{2,3,3} \) be the blow-up of \( \mathbb{P}^2 \) in 6 general points, i.e. a smooth cubic surface. Divisors of degree 1 are the 27 lines. The corresponding minuscule representation \( L_{\omega_{r-1}} \) is the 27-dimensional representation of \( E_6 \) as a Lie algebra of infinitesimal norm similarities of the exceptional Jordan algebra.

Proof of Theorem 2.7. Let \( \Gamma_k \) be the intersection of the convex hull of \( W \cdot (kE_r) \) with \( \text{Eff}(X) \) and let \( \Gamma \subseteq \text{Pic}(X) \otimes \mathbb{R} \) be the cone spanned by \( \Gamma_1 \). Since \( \pi(E_r) = \omega_{r-1} \) and any element of \( K^\perp \) is an integral combination of roots, it follows from the basic representation theory of semisimple Lie algebras [VO] that \( \pi(\Gamma_k) \) is the set of weights of an irreducible \( \mathfrak{g} \)-module \( L_{k\omega_{r-1}} \) with the highest weight \( k\omega_{r-1} \). Since \( L_{k\omega_{r-1}} \subset L_{2\omega_{r-1}}^k \) \( (L_{k\omega_{r-1}}) \) is the so-called Cartan component of \( L_{\omega_{r-1}} \), any weight \( \pi(\Gamma_k) \) is a sum of \( k \) weights from \( \pi(\Gamma_1) \), and therefore any divisor in \( \Gamma_k \) is a sum of \( k \) divisors from \( \Gamma_1 \). It follows that \( \text{Eff}(X) \cap \Gamma \) is generated by \( \Gamma_1 \) as a semigroup.

It remains to show that \( \text{Eff}(X)_{\mathbb{R}} \subseteq \Gamma \). We will find all faces of \( \Gamma \) and show that the inequalities that define them are satisfied by any effective divisor.

By Lemma 2.3 it suffices to find faces of \( \Gamma \) adjacent to the ray spanned by \( E_r \) up to the action of the stabilizer of \( E_r \) in \( W \). The algorithm for finding faces of these so-called Coxeter polytopes is explained, for example, in [Ca], p. 9. They are in one-to-one correspondence with connected maximal subdiagrams of \( T_{a,b,c} \) that contain the support of the highest weight, i.e., the node that corresponds to the simple root \( \alpha_{r-1} \) in our case. There are two types of such diagrams given by roots:

1) \( \alpha_2, \alpha_3, \ldots, \alpha_{a+r-2} \) and 2) \( \alpha_1, \alpha_2, \ldots, \alpha_{a+r-3} \).

For each subdiagram, the linear span of the corresponding face is spanned by simple roots in the subdiagram and by \( E_r \).

Using formulas (2.2), any face of \( \Gamma \) is given (up to the action of \( W \)) by inequality
\[ D \cdot f \geq 0, \tag{2.3} \]
where
1) \( f = l_1 + \ldots + l_{a-1} - e_1 \) and 2) \( f = l_{a-1} \).

By Lemma 2.3 the class \( f \) is nef. Hence, for any \( D \) effective, \( D \cdot f \geq 0 \). We conclude that (2.3) is, in fact, satisfied by any effective divisor and \( \text{Eff}_{\mathbb{R}}(X) = \Gamma. \)

3. Proof of Theorem 1.3

The following is a direct generalization from [BP] (Prop. 4.4):

Proposition 3.1. Let \( \pi : X \to X' \) be the blow up of a smooth point. Let \( E \subset X \) be an exceptional divisor, and let \( x_E \in H^0(X,E) \subset \text{Cox}(X) \) be the corresponding section. Then there is an isomorphism of rings
\[ \text{Cox}(X)_{x_E} \cong \text{Cox}(X')[T,T^{-1}]. \]

Proof. Any divisor \( D \in \text{Pic}(X) \) can be uniquely written as \( D = D_0 - mE \), where \( D_0 \in \pi^*\text{Pic}(X'), m \in \mathbb{Z} \). We identify \( \text{Pic}(X') \) with \( \pi^*\text{Pic}(X') \subset \text{Pic}(X) \) and \( \text{Cox}(X') \) with \( \pi^*\text{Cox}(X') \subset \text{Cox}(X) \). The latter embedding extends to a ring homomorphism
\[ \text{Cox}(X')[T,T^{-1}] \to \text{Cox}(X)_{x_E} \]
by sending \( T \) to \( x_E \). We show that this is an isomorphism by constructing an inverse to it.
If \( m \geq 0 \) and \( s \) is a section in \( H^0(X,D) \), then let \( s_0 = s \cdot x_E^m \in H^0(X',D_0) \). Define a map
\[
H^0(X,D) \to H^0(X',D_0)T^{-m}, \quad s \mapsto s_0 T^{-m}.
\]
If \( m < 0 \) then the canonical inclusion \( H^0(X,D_0) \hookrightarrow H^0(X,D) \) is an isomorphism. To see this, note that for any \( i \geq 0 \) there is an exact sequence
\[
0 \to H^0(X,D_0 + iE) \to H^0(X,D_0 + (i + 1)E) \to H^0(E,(D_0 + (i + 1)E)|_E) = 0,
\]
where the last equality follows from
\[
O(D_0)|_E = O_E \quad \text{and} \quad O(E)|_E = O_E(-1).
\]
Define a map \( H^0(X,D) \to H^0(X',D_0)T^{-m} \) in the same way, by sending \( s \) to \( s_0 \cdot T^{-m} \), where \( s \in H^0(X,D) \) is the image of a section \( s_0 \in H^0(X,D_0) \). This gives a map \( \text{Cox}(X) \to \text{Cox}(X')[T,T^{-1}] \) which maps \( x_E \) to \( T \). One can check directly that this is a ring homomorphism. The induced map \( \text{Cox}(X)_{x_E} \to \text{Cox}(X')[T,T^{-1}] \) is the desired inverse.

**Notation 3.2.** In this section,
\[
X = X_{a,b,c}.
\]

**Proposition 3.3.** \( \text{Cox}(X) \) is a unique factorization domain.

**Proof.** The Cox ring of a normal projective variety is known to be a UFD [EKW]. We can also use a simple observation: the ring of invariants of an UFD with respect to the action of a connected algebraic group without nontrivial characters is a UFD, see [PV]. By [M2], \( \text{Cox}(X) \) is a ring of invariants of an extended Nagata action.

**Definition 3.4.** We define a \( \mathbb{Z} \)-grading of \( \text{Cox}(X) \) by \( \deg(s) = \deg(D) \) for any \( s \in H^0(X,D) \). In particular, \( \deg(x_E) = 1 \) for any \( E \in W \cdot E_r \).

**Definition 3.5.** Let \( \text{Cox}'(X) \subset \text{Cox}(X) \) be a subalgebra generated by sections \( x_E \), for \( E \in W \cdot E_r \). We say that \( \text{Cox}(X) \) is minuscule if \( \text{Cox}(X) = \text{Cox}'(X) \).

**Definition 3.6.** Let \( \mathbb{P}(X) = \text{Proj}(\text{Cox}(X)) \), \( \mathbb{A}(X) = \text{Spec}(\text{Cox}(X)) \), and \( Z = \text{Proj}(\text{Cox}'(X)) \), where \( \text{Cox}(X) \) and \( \text{Cox}'(X) \) are considered with their \( \mathbb{Z} \)-grading as in Definition 3.3 (we will show that in fact \( Z \cong \mathbb{P}(X) \)).

Inspecting the list of all possible \( X_{a,b,c} \) given in the introduction, we see that \( X_{a,b-1,c} \) is contained in the following list:
- \( X_{s+1,1,n+1} = \text{Bl}_{s+2}(\mathbb{P}^n)^s \). This variety is minuscule, see Remark 3.3.
- Del Pezzo surfaces \( X_{2,3,3} = \text{Bl}_4\mathbb{P}^2 \), \( s = 4, 5, 6, 7 \). In this case \( \text{Cox}(X) \) is minuscule by a theorem of Batyrev and Popov [BP].
- \( X_{2,2,4} = \text{Bl}_4(\mathbb{P}^4), X_{2,2,5} = \text{Bl}_7(\mathbb{P}^4) \). These varieties are also minuscule by our Theorem 1.1 (which will be proved later).

Therefore, \( X_{a,b-1,c} \) is minuscule in all cases.

Let \( R = \text{Cox}(X), R' = \text{Cox}'(X), R_0 = \text{Cox}(X_{a,b-1,c}) \). Let \( Q \) be the field of fractions of \( R \). We claim that \( R \) is contained in all the localizations \( R'_{x_E} \subset Q \). By Lemma 2.2 there is a small modification \( \tilde{X} \) of \( X \) isomorphic to \( \text{Bl}_r(\mathbb{P}^{c-1})^{a-1} \), the blow-up of \( (\mathbb{P}^{c-1})^{a-1} \) in \( r = b + c \) points \( q_1, \ldots, q_r \) in general position, such that the pullback of \( E \) is contracted to \( q_r \). By Proposition 3.1 \( R' \subset (R_0)_{x_E} \). It remains to notice that \( R_0 \subset R' \) because \( R_0 \) is minuscule.

**Claim 3.7.** \( R \) is integral over \( R' \)

**Proof.** This is a standard proof, see for example [Ha] p. 123. Let \( z \in R \) be a homogeneous element of a positive degree. To show that \( z \) is integral over \( R' \), it suffices to find a faithful \( R'[z] \)-module \( M \) finitely generated as an \( R' \)-module. Let \( M \) be the set of elements in \( R' \) of degree greater than \( N \), where \( N \) has to be chosen
adequately. Obviously, $M$ is an $R'[z]$-module if $z M \subset R'$. So choose $N$ to be $kn+1$, where $k$ is the number of generators $x_i$ in $R'$, and $n$ is the maximum of integers $n_i$ such that $zx_i^{n_i} \in R'$. Clearly, $M$ is a finitely generated $R'$-module. Since $R$ is a domain, $M$ is of course a faithful $R'[z]$-module.

It follows that $R$ is integral over $R'$ and, therefore, $R$ is finitely generated.

Now we prove the “moreover” part of the theorem following Popov’s proof \[Po\] of the analogous statement for Del Pezzo surfaces.

For each $E \in W \cdot E_r$ consider the open chart $U_E(X) \subset P(X)$ be a chart given by $x_E \neq 0$. Since $R$ is integral over $R'$, it is easy to see that the radical of the ideal of $R$ generated by $x_E$’s is the irrelevant ideal. It follows that charts $U_E'(X)$ cover $P(X)$. Since $R \subset \cap E R'_{x_E}$, we have $R_{x_E}^r = R_{x_E}$ for any $E \in W \cdot E_r$. It follows that, in fact, $U_E(X) \simeq U_E'(X)$, the inclusion $R' \subset R$ induces an isomorphism $\phi : P(X) \to Z$, and $x E O_Z(m) \cong O_{P(X)}(m)$. Moreover, it is true in general that if a graded ring $R$ is a UFD and the irrelevant ideal is the radical of the ideal generated by degree 1 elements, then the Picard group of $\text{Proj}(R)$ is $Z$ and it is generated by $O(1)$.

It follows from Proposition 5.1 that $U_E(X) \cong A(X_{a,b-1,c})$ is factorial by Proposition 5.3 Therefore, $Z$ is locally factorial and, in particular, $Z$ is normal.

Arguing by induction on $k$, we can assume that all statements of Theorem 1.3 are satisfied for $Y = X_{a,b-1,c}$. Let $W = \mathbb{P}(Y)$. Thus $W$ is a Cohen–Macaulay and Gorenstein scheme with rational singularities, $\text{Pic}(W) = Z$ is generated by $O_W(1)$ and the anti-canonical line bundle $\omega_W$ is ample.

**Lemma 3.8.** $H^i(W, O(k)) = 0$ for $i \geq 1, k \geq 0$.

**Proof.** Notice that $O(k) = \omega_W \otimes L$ with $L$ ample. Let $\pi : \hat{W} \to W$ be a resolution of singularities. Then $H^i(\hat{W}, \omega_{\hat{W}} \otimes \pi^*(L)) = 0$ by Kodaira vanishing because $\pi^*(L)$ is big and nef. Now use the Leray spectral sequence and the definition of rational singularities ($R^i \pi_*, \omega_{\hat{W}} = 0$ for $i > 0$) to conclude that $H^i(W, O(k)) = 0$. □

Since $Y$ is minuscule, $W$ is projectively normal in the projective embedding given by $O_W(1)$. Note that $U_E(X) \cong A(Y)$ is an affine cone over $Y$. It follows that $A(Y)$ has rational singularities by [KH, Theorem 1] and therefore is Cohen–Macaulay [KK]. Since $A(Y)$ is factorial and Cohen–Macaulay, it is Gorenstein [Ea, Ex. 21.21].

It remains to calculate the anticanonical class of $P(X)$. By [HK], $X$ is the GIT quotient of $A(X)$ for the action of the torus $\text{Hom}(\text{Pic}(X), G_m) = G_{m}^{r+1}$. Moreover, $X$ is the GIT quotient of $P(X)$ for the induced action of $G_{m}^{r}$. Let $U$ be the semi-stable locus in $P(X)$. Note that there are no strictly semi-stable points [HK Prop. 2.9]. It is easy to see by induction using charts $U_E(X)$ that $G_{m}^{r}$ acts on $P(X)$ with connected stabilizers. By Luna’s étale slice theorem [MFK, p. 199], this implies that $\pi : U \to X$ is a principal étale fibre bundle. In particular, $U$ is smooth. By the general theory of Cox varieties [HK, Prop. 2.9], $P(X) \setminus U$ has codimension at least 2 in $P(X)$, and therefore $\text{Pic}(U) \cong Z(\mathcal{O}(1))$. By the GIT, the pull-back map $\pi^*$ between the Picard groups is the map given by degree: $\pi^*(D) = \deg(D)$.

It is enough to prove that $K_U = O_U(-d)$. Let $\mathcal{T}_X$ (resp. $\mathcal{T}_U$) be the tangent sheaf of $X$ (resp. $U$). There is an exact sequence of locally free sheaves:

$$0 \to \mathcal{O}_U \to \mathcal{T}_U \to \pi^* \mathcal{T}_X \to 0$$

(the relative tangent sheaf of a principal étale bundle is canonically a trivial bundle with fiber isomorphic to the Lie algebra of $G_{m}^{r}$). Taking Chern classes, it follows that $c_1(\mathcal{T}_U) = \pi^*(c_1(\mathcal{T}_X))$; hence, $-K_U = \pi^*(-K_X) = O(d)$, where

$$d = \deg(-K_X) = abc \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \right).$$
Remark 3.9. Here we consider the case of \( X = X_{s+1,1,n+1} = \text{Bl}_{n+1}(\mathbb{P}^n)^{s+1} \). Then it is well-known and easy to check that \( X \) is the GIT quotient of the Grassmannian \( G(s+1, n+s+2) \). It follows from [HK] that \( \text{Cox}(X) \) is isomorphic to the total coordinate ring of \( G(s+1, n+s+2) \) which is generated by the \( \binom{n+s+2}{s+1} \) Plücker coordinates. On the other hand, the orbit \( \mathcal{W} \cdot E \) in this case consists of precisely \( \binom{n+s+2}{s+1} \) divisors, the dimension of the minuscule representation of \( \mathfrak{g}_{s+1,1,n+1} = \mathfrak{s} l_{n+1} \) in \( L_{\omega_n} = \Lambda^{s+1}C^{n+s+2} \). It follows that \( \text{Cox}(X) \) is minuscule.

4. Proof of Theorem \ref{thm:main}

Notation 4.1. Let \( X = \text{Bl}_r \mathbb{P}^n \) be the blow-up of \( \mathbb{P}^n \) at \( r \) distinct points \( p_1, \ldots, p_r \) \( (r \geq n+3) \) that lie on a rational normal curve \( C \) of degree \( n \). Let \( E_1, \ldots, E_r \) be the exceptional divisors and \( H \) the hyperplane class. Let \( \alpha = r - n - 2 \).

Let \( \tilde{C} \) be the proper transform of \( C \) on \( X \).

Lemma 4.2. Let \( D \subset \mathbb{P}^n \) be a hypersurface of degree \( d \) that contains \( C \) with multiplicity \( m \). If \( D \) has multiplicity \( m_i \) at \( p_i \), \( i = 1, \ldots, r \), then one has:

\[
m \geq \sum_{i=1}^r m_i - nd.
\]

Proof. Recall that the multiplicity of a divisor along a curve is the multiplicity at a general point of a curve. Let \( \tilde{D} \) be the proper transform of \( D \) on \( X \). Let \( \pi' : X' \rightarrow X \) be the blow-up of \( X \) along \( \tilde{C} \) and let \( E \) be the exceptional divisor. Then \( E = \mathbb{P}(N_{C/X}) \), where \( N_{C/X} \) is the normal bundle of \( \tilde{C} \) in \( X \). One has

\[
N_{C|\mathbb{P}^n} \cong \mathcal{O}(n+2)^{\oplus(n-1)},
\]

and therefore

\[
N_{C/X} \cong \pi^*N_{C|\mathbb{P}^n} \otimes \mathcal{O}_X(-E_1 - \cdots - E_r) \cong \mathcal{O}(n+2)^{\oplus(n-1)} \otimes \mathcal{O}(-r) \cong \mathcal{O}(-\alpha)^{\oplus(n-1)}.
\]

It follows that \( E \cong \mathbb{P}^1 \times \mathbb{P}^{n-2} \). Let

\[
q_1 : \mathbb{P}^1 \times \mathbb{P}^{n-2} \rightarrow \mathbb{P}^1, \quad q_2 : \mathbb{P}^1 \times \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-2}
\]

be the two projections. Then \( \mathcal{O}(E)|_E \cong q_1^*\mathcal{O}(-\alpha) \otimes q_2^*\mathcal{O}(-1) \).

Let \( D' \) be the proper transform of \( \tilde{D} \) on \( X' \). Then \( \pi'^*D = D' + mE \). Denote

\[
a = -\tilde{D} \cdot \tilde{C} = \sum_{i=1}^r m_i - nd.
\]

Note that \( \pi'^*\mathcal{O}_X(D) \big|_E = q_1^*\mathcal{O}(-a) \). Since \( \mathcal{O}_X(D') \big|_E = q_1^*\mathcal{O}(-a + ma) \otimes q_2^*\mathcal{O}(m) \) is an effective divisor on \( E \), it follows that \(-a + ma \geq 0 \). Hence, \( m \geq a/\alpha \).

Lemma 4.3. Consider the divisor \( \lfloor r \rfloor \) on \( X \). Then \( E \) is the proper transform of a unique hypersurface of degree \( k \) in \( \mathbb{P}^n \) that has multiplicity \( k \) at any \( p_i \) with \( i \in I \) and \( k - 1 \) at all other points of \( C \). In particular, \( H^0(X, E) \cong \mathbb{C} \) and \( E - E_i \) is not effective for any \( i = 1, \ldots, r \).

Proof. Let \( J \subset I \) be any subset with \( |J| = 2k + 1 \). The divisor

\[
E' = kH - \sum_{i \in I} E_i - (k - 1) \sum_{i \in J} E_i
\]

is an effective divisor of degree 1 on the blow-up \( \text{Bl}_{n+3} \mathbb{P}^n \) of \( \mathbb{P}^n \) along the points \( p_i \) for \( i \in I \cup J \). It follows that \( H^0(X, E') = 1 \) and, for any \( i \in I \cup J \), the divisor \( E - E_i \),
is not effective. It follows that $E$ is the proper transform of a unique hypersurface $Z$ of degree $k$ in $\mathbb{P}^n$ such that
\[
\text{mult}_p Z = k \quad (i \in I) \quad \text{and} \quad \text{mult}_p Z = k - 1 \quad (i \in J).
\]
Since $Z$ is the image of $E$, and therefore does not depend on the choice of $J$, we have $\text{mult}_p Z = k - 1$ for any $i \in I$. If $p$ is a point on $C$ different than $p_1, \ldots, p_r$, consider the variety $Bl_r \mathbb{P}^n$ that is the blow-up of $X$ at $p$. Let $E_{r+1}$ be the exceptional divisor. By applying the same argument to the divisor $E - (k - 1)E_{r+1}$ on $Bl_{r+1}\mathbb{P}^n$, it follows that the multiplicity of $Z$ at $p$ is exactly $k - 1$.

**Definition 4.4.** We call the divisors $E$ in $\mathbb{P}^n$ minimal divisors on $Bl_r\mathbb{P}^n$. We call an element in $\text{Cox}(X)$ a distinguished section if it is a monomial in the sections $x_E \in H^0(X,E)$, where $E$ is either a minimal divisor on $X$ or an exceptional divisor $E_r$. The ring $\text{Cox}(X)$ is minuscule if it is generated by distinguished sections.

We prove that $\text{Cox}(X)$ is minuscule by induction on $n$ and $r$. Theorem 4.23 proves this for $n = 2$. Assume from now on that $n \geq 3$.

**Definition 4.5.** Let
\[
D = dH - \sum_{i=1}^r m_i E_i
\]
be any divisor on $X$. We call $d$ the $H$-degree of $D$, denoted by $\text{hdeg}(D)$.

**Notation 4.6.** Consider the projection $\pi_1 : \mathbb{P}^n - \to \mathbb{P}^{n-1}$ from $p_1$ and let $q_i = \pi(p_i)$ for $i = 2, \ldots, r$. Note that $q_2, \ldots, q_r$ lie on a rational normal curve $\pi_1(C)$ of degree $n - 1$ in $\mathbb{P}^{n-1}$. Let $Y = Bl_{r-1}\mathbb{P}^{n-1}$ be the blow-up of $\mathbb{P}^{n-1}$ at $q_2, \ldots, q_r$. Let $E_2, \ldots, E_r$ be the exceptional divisors on $Y$ and $H$ the hyperplane class. Consider the linear map $\text{Pic}(X) \to \text{Pic}(Y)$ that maps $[I]$ to
\[
\bar{D} = m_1 H - \sum_{i=2}^r (m_1 + m_i - d) E_i.
\]

**Lemma 4.7.** If $\text{hdeg}(D) = \text{hdeg}(D')$ and $\bar{D} = \bar{D}'$ then $D = D'$.

**Proof.** This is because by (4.2), $\bar{\Delta} = 0$ implies that $\Delta = \epsilon(H - \sum_{i=1}^r E_i)$, for some $\epsilon \in \mathbb{Z}$. Hence, if the $H$-degree of $\Delta$ is 0, then $\Delta = 0$.

**Lemma 4.8.** There is a map $r$ that makes the following diagram commutative:
\[
\begin{array}{ccc}
H^0(X,D) & \xrightarrow{r} & H^0(Y,\bar{D}) \\
\downarrow r' & & \downarrow i \\
H^0(E_1,D_{1,E_1}) & & H^0(\mathbb{P}^{n-1},\mathcal{O}(m_1))
\end{array}
\]
Here $r'$ is the restriction map and $i$ is the canonical injective map given by push-forward. For any divisors $D_1, D_2$ on $X$ and $s_1 \in H^0(X,D_1)$, $s_2 \in H^0(X,D_2)$, if $D = D_1 + D_2$, then
\[
\bar{D} = \bar{D}_1 + \bar{D}_2, \quad r(s_1 s_2) = r(s_1) r(s_2).
\]

**Proof.** We can identify $E_1$ with the image of the projection $\pi_1$ and view $r'$ as a map
\[
r' : H^0(X,D) \to H^0(\mathbb{P}^{n-1},\mathcal{O}(m_1)) = H^0(Y,m_1 H).
\]
Let $x_{E_1}$ be a generator for $H^0(Y,E_1) \cong \mathbb{C}$. Note that if for some $i = 2, \ldots, r$ one has $m_1 + m_i - d > 0$, then the image of $r'$ lies in the linear subsystem
\[
[m_1 H - (m_1 + m_i - d) E_i] \subset [m_1 H]_{m_1 H}
\]
and therefore \( r'(s) \) is divisible by \( x_{\overline{E}_i}^{d+m_1+m_i} \) for any \( s \in H^0(X, D) \). It follows that we can formally define

\[
    r(s) = r'(s) \prod_{i=2}^{r} x_{\overline{E}_i}^{d-m_1-m_i}.
\]

The last statement of the lemma is clear. \( \square \)

**Remark 4.9.** The geometric interpretation for the map \( r \) is as follows. Let \( l_{i,j} \) be the proper transform on \( X \) of the line in \( \mathbb{P}^n \) joining the points \( p_i \) and \( p_j \). Then \( q_2, \ldots, q_r \) are the points on \( E_1 \cong \mathbb{P}^{n-1} \) where \( l_{1,2}, l_{1,3}, \ldots, l_{1,n} \) intersect \( E_1 \). Let \( \tilde{X} \) be the blow up of \( X \) along \( l_{1,2}, \ldots, l_{1,n} \) and let \( E_{1,2}, \ldots, E_{1,n} \) be the exceptional divisors. The normal bundle \( N_{l_{i,j}}|X \) of \( l_{i,j} \cong \mathbb{P}^1 \) in \( X \) is \( \mathcal{O}(-1)^{\oplus (n-1)} \). The exceptional divisors \( E_{1,j} \) are given by:

\[
    E_{1,i} \cong \mathbb{P}(N_{l_{i,j}}|X) \cong l_{i,j} \times \mathbb{P}^{n-2} \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}.
\]

For any \( n \geq 3 \), there is morphism \( \tilde{X} \to X' \) that contracts all the divisors \( E_{1,i} \), using the projection onto \( \mathbb{P}^{n-2} \). There is an induced rational map \( \psi : X \dashrightarrow X' \) that is an isomorphism in codimension 1. Let \( E'_1 = \psi(E_1) \). Then \( E'_1 \cong Y \). In fact, the rational map \( X \dashrightarrow Y \) is resolved by this flip and induces a regular map \( X' \to Y \) that is a \( \mathbb{P}^1 \)-bundle, with \( E'_1 \) as a section. If \( D \) is a divisor on \( X \), let \( D' = \psi(D) \). Using geometric arguments, one checks that on \( E'_1 \cong Y \) one has \( D'|_{E'_1} = D \) when \( D = H, H - E_1, E_i \), for \( i = 2, \ldots, r \). Hence, the formula holds in general by linearity. Then \( r \) is the composition of the isomorphism \( H^0(X, D) \cong H^0(X', D') \) with the restriction map \( H^0(X, D') \to H^0(E'_1, D'|_{E'_1}) \).

**Notation 4.10.** Let \( q = \tilde{C} \cap E_1 \). Obviously, \( q \in \pi_1(C) \). Let \( Y' = \text{Bl}_q \mathbb{P}^{n-1} \) be the blow-up of \( Y \) at \( q \) and let \( E_q \) be the exceptional divisor.

**Lemma 4.11.** Let \( E \) be a minimal divisor on \( X \) of \( H \)-degree \( k \). Then \( E \cdot (l - e_1) \) is either 0 or 1. In the first case, \( E \) is a minimal divisor on \( Y \). In the second case, the divisor \( E' = E - (k - 1)E_q \) is minimal on \( Y' \), except when \( k = 1 \). In the latter case, one has:

\[
    E = H - \sum_{i \in I} E_i, \quad E' = \sum_{i \in I'} \overline{E}_i, \quad I \subset \{2, \ldots, r\}, \quad |I| = n, \quad |I'| = \alpha + 1. \tag{4.3}
\]

**Proof.** In the first case:

\[
    E = kH - kE_1 - k \sum_{i \in I} E_i - (k - 1) \sum_{i \in I'} E_i, \tag{4.4}
\]

where \( I \subset \{2, \ldots, r\} \), \( |I| = n + 1 - 2k \), and

\[
    E' = kH - k \sum_{i \in I} \overline{E}_i - (k - 1) \sum_{i \in I'} \overline{E}_i. \tag{4.5}
\]

In the second case:

\[
    E = kH - (k - 1)E_1 - k \sum_{i \in I} E_i - (k - 1) \sum_{i \in I'} E_i, \tag{4.6}
\]

where \( I \subset \{2, \ldots, r\} \), \( |I| = n + 2 - 2k \), and

\[
    E' = (k - 1)H - (k - 1) \sum_{i \in I} \overline{E}_i - (k - 2) \sum_{i \in I'} \overline{E}_i. \tag{4.6}
\]

Let \( s \in H^0(Y, \tilde{E}) \) be the image of the section \( x_E \) via the map \( r \) of (4.2). Let \( Z \) be the zero-locus of \( s \). By Lemma 4.4, the divisor \( E \) has multiplicity \( k - 1 \) along \( \tilde{C} \). Therefore,

\[
    \text{mult}_q Z = \text{mult}_q E \cap E_1 \geq \text{mult}_q E \geq \text{mult}_q E = k - 1.
\]
It follows that the image of $r$ is in each case contained in the push-forward of the linear system $|E'|$ on $Y'$. Except in Case 2) when $k = 1$, $E'$ is minimal on $Y'$.

We prove that $H^0(X, D)$ is generated by distinguished sections for any effective divisor $D$.

**Claim 4.12.** We may assume that $0 < m_1 \leq m_2 \leq \ldots \leq m_r$.

**Proof.** Indeed, if $m_i \leq 0$ for some $i$, then $H^0(X, D) \cong H^0(X, D_0)$, where $D_0 = D + m_i E_i$ is a divisor on $\text{Bl}_{r-1} \mathbb{P}^n$. The ring $\text{Cox}(\text{Bl}_{r-1} \mathbb{P}^n)$ is minuscule: this follows by Remark 3.9 if $r = n + 3$, and by induction if $r > n + 3$. Hence, $H^0(X, D_0)$ is generated by distinguished sections.

**Claim 4.13.** It suffices to prove that any distinguished section in the image of $r : H^0(X, D) \rightarrow H^0(Y, \tilde{D})$ can be lifted to a linear combination of distinguished sections.

**Proof.** Since $\text{Cox}(Y)$ is minuscule by induction and the kernel of $r$ is $H^0(X, D - E_1)$, we are then reduced to show that $H^0(X, D - E_1)$ is generated by distinguished sections. If $D - E_1$ is effective, we may replace $D$ with $D - E_1$ and repeat the process. The process stops only when $D - E_1$ is not effective, in which case $r_{E_1}$ is an isomorphism onto its image. Since for any effective $D$, one has $D.(l - e_1) = d - m_i \geq 0$, for all $i$, the process must stop.

**Notation 4.14.**

$$m = \max \left\{ \left\lfloor \frac{\sum_{i=1}^{r} m_i - nd}{\alpha} \right\rfloor , 0 \right\}$$  (4.7)

**Proposition 4.15.** If $m = 0$, then $r$ surjects onto $H^0(Y, \tilde{D})$ and any distinguished section $s \in H^0(Y, \tilde{D})$ can be lifted to a distinguished section.

**Proof.** The section $s$ is a monomial in the sections corresponding to minimal divisors on $Y$ and sections $x_{E_i}, i = 2, \ldots, r$; hence, it corresponds to a decomposition:

$$\tilde{D} = S + \sum_{i=2}^{r} l_i E_i,$$  (4.8)

where $l_i \geq 0$ and $S$ is a sum of minimal divisors on $Y$. Denote

$$\beta = d - m_1 = D.(l - e_1) \geq 0.$$

**Lemma 4.16.** We have $l_i \leq \beta$ and $\sum_{i=2}^{r} l_i \geq (\alpha + 1)\beta$.

**Proof.** For each $k \geq 0$, let $a_k \geq 0$ be the number of minimal divisors of $H$-degree $k$ that appear in $S$. Since $\tilde{D}$ and $S$ have the same $H$-degree,

$$m_1 = \sum_{k \geq 1} ka_k.$$

By counting the number of $E_i$’s in both sides of (4.8), one has the following formula:

$$\sum_{i=2}^{r} l_i = (\alpha + 1)\beta + (nd - r \sum_{i=1}^{r} m_i) + \alpha(m_1 - \sum_{k \geq 1} a_k)$$  (4.9)

Since $m = 0$ and

$$m_1 = \sum_{k \geq 1} ka_k \geq \sum_{k \geq 1} a_k,$$
it follows that
\[ \sum_{i=2}^{r} l_i \geq (\alpha + 1)\beta. \]
Finally,
\[ d - m_i = \tilde{D}.(l - e_i) = S.(l - e_i) + l_i \geq l_i, \]
and therefore
\[ l_i \leq (d - m_i) \leq (d - m_1) = \beta. \]

We lift the minimal divisors \((4.5)\) on \(Y\) to minimal divisors \((4.3)\) on \(X\) of the same \(H\)-degree. Let \(D_0\) be the divisor on \(X\) equal to the sum of the lifts of all terms of \(S\). Hence, \(S = D_0\) and
\[ \text{hdeg}(D) - \text{hdeg}(D_0) = \text{hdeg}(\tilde{D}) = \beta. \] (4.10)

By Lemma \((1.19)\) and Lemma \((1.17)\) we may lift \(\sum_{i=2}^{r} l_iE_i\) to an effective divisor \(D_1\) on \(X\), \(hdeg(D_1) = \beta\). Let \(D' = D_0 + D_1\). Then \(D'\) has the same \(H\)-degree as \(D\). Since \(\tilde{D}' = \tilde{D}\), it follows from Lemma \((1.7)\) that \(D = D'\). By construction, there is a distinguished section \(t\) in \(H^0(X, D)\) such that \(r(t) = s\). \(\square\)

**Lemma 4.17.** Consider the divisor \(\sum_{i=2}^{r} l_iE_i\) on \(Y\) and assume that
\[ l_i \leq \beta \quad (i = 2, \ldots, r), \quad \sum_{i=2}^{r} l_i \geq (\alpha + 1)\beta. \]

Then we may lift \(\sum_{i=2}^{r} l_iE_i\) to an effective divisor \(D_1\) on \(X\) with \(hdeg(D_1) = \beta\).

Moreover, there is a distinguished section \(t \in H^0(X, D_1)\) such that \(r(t) = \prod_{i=2}^{r} x_{E_i}^{l_i}\).

**Proof.** For all \(i = 2, \ldots, r\), we may write \(l_i = l_i' + l_i''\), for some \(l_i', l_i'' \geq 0\), such that \(0 \leq l_i' \leq \beta\) and \(\sum_{i=2}^{r} l_i' = (\alpha + 1)\beta\). By partitioning \(\sum_{i=2}^{r} l_i' E_i\) into a sum of \((\alpha + 1)\)-tuples of the form \(E_{i_1} + \ldots + E_{i_{\alpha+1}}\) (the precise procedure for the partitioning is explained in the proof of Lemma \((1.24)\)), we may lift \(\sum_{i=2}^{r} l_i'E_i\) using \((1.3)\) to a divisor \(D_1'\) on \(X\) which is a sum of \(\beta\) “hyperplane classes” \(H - \sum E_i\). Hence, \(hdeg(D_1') = \beta\). Moreover, there is a distinguished section \(t' \in H^0(X, D_1')\) such that \(r(t') = \prod_{i=2}^{r} x_{E_i}^{l_i'}\). Let \(D_1 = D_1' + \sum_{i=2}^{r} l_i'' E_i\) and \(t = t' \prod_{i=2}^{r} x_{E_i}^{l_i''}\). Since \(E_i = E_i\) and \(r(x_{E_i}) = x_{E_i}\), for all \(i = 2, \ldots, r\), the Lemma follows. \(\square\)

**Proposition 4.18.** Let \(m > 0\). Then the image of \(r\) is the push-forward of \(H^0(Y', E - mE_q)\), and we may lift any distinguished section \(s \in H^0(Y', E - mE_q)\) to a section \(t\) in the subspace of \(H^0(X, D)\) generated by distinguished sections. By lift, here we mean that \(r(t) = s.x_{E_q}^m\).

**Proof.** By Lemma \((1.2)\) the multiplicity of \(D,E_1\) at \(q\) is at least \(m\). Hence, the map \(r\) has image in \(H^0(Y', E - mE_q)\).

**Lemma 4.19.** If \(E'\) is a minimal divisor on \(Y'\) of \(H\)-degree \(k \geq 1\), then the multiplicity at \(q\) of a push-forward of \(E'\) to \(Y\) is either: 1) \(k - 1\), or 2) \(k\). The push-forward is equal to \(\tilde{E}\), where \(E\) is a minimal divisor on \(X\) of \(H\)-degree \(k\) in Case 1) and \(k + 1\) in Case 2).
Lemma 4.17. \( x \) in Case 1), respectively Case 2) of Lemma 4.19, whose sections

Claim 4.21. We may assume that \( a = 0 \) or \( \sum_{k \geq 1} b_k = 0 \).

Proof. We may lift \( E' \) using (4.15) and (4.10) to a minimal divisor \( E \) on \( X \):

1) \( E = kH - k \sum_{i \in I} E_i - (k - 1) \sum_{i \in I^c} E_i, \quad |I| = n + 2 - 2k, \quad 1 \in I \)

2) \( E = (k + 1)H - (k + 1) \sum_{i \in I} E_i - k \sum_{i \in I^c} E_i, \quad |I| = n - 2k, \quad 1 \in I^c \)

Then \( r(x_E) = x_{E'} \cdot x_{E_q}^{k-1} \) in Case 1) and \( r(x_E) = x_{E'} \cdot x_{E_q}^k \) in Case 2).

Notation 4.20.

\[ \beta = \text{hdeg}(D) - \text{hdeg}(D_0). \]

If \( \beta = 0 \), from Lemma 4.17, \( E_i = \overline{E}_i \) and \( r(x_{E_i}) = x_{\overline{E}_i} \), for all \( i = 2, \ldots, r \), it follows that \( D = D_0 + \sum_{i=2}^r l_i E_i \) and we may lift \( s \) to a distinguished section in \( H^0(X, D) \). For the general case, it is enough to show that, by eventually rewriting \( s \) as a sum of distinguished sections in \( H^0(Y, \overline{D}) \) corresponding to different decompositions of \( \overline{D} - mE_q \), we may reduce to the case when \( l_i = l_i' + l_i'' \), for some \( l_i', l_i'' \geq 0 \), such that \( 0 \leq l_i' \leq \beta \) and \( \sum_{i=2}^r l_i'' = (\alpha + 1)\beta \). Then we can finish the proof by using Lemma 4.17.

For each \( k \geq 1 \), let \( a_k \geq 0 \), respectively \( b_k \geq 0 \), be the number of divisors \( E' \) as in Case 1), respectively Case 2) of Lemma 4.19, whose sections \( x_{E'} \) appear in the monomial \( s \) (taken with multiplicities). One has the following relations:

\[ 0 = \text{hdeg}(\overline{D}) - \text{hdeg}(S) = m_1 - \sum_{k \geq 1} ka_k - \sum_{k \geq 1} kb_k. \] (4.12)

\[ \beta = \text{hdeg}(D) - \text{hdeg}(D_0) = d - \sum_{k \geq 1} ka_k - \sum_{k \geq 1} (k + 1)b_k = d - m_1 - \sum_{k \geq 1} b_k. \] (4.13)

Note that by finding the coefficients of \( E_q \) in both sides of the expression in (4.11), one has the following relation:

\[ m + a = \sum_{k \geq 1} (k - 1)a_k + \sum_{k \geq 1} kb_k. \] (4.14)

By counting the number of \( \overline{E}_i \)'s in both sides of (4.11) and using (4.12), (4.13), one has:

\[ \sum_{i=1}^r m_i - nd = (\alpha + 1)\beta + (m + a)\alpha - \sum_{i=2}^r l_i \] (4.15)
Proof. Assume $a > 0$ and $b_k > 0$, for some $k \geq 1$. Then the monomial $s$ contains a section $x_{E'}$, where $E'$ is a minimal divisor of the form:

$$E' = kH - k \sum_{i \in I} E_i - (k - 1) \sum_{j \in J} E_j - kE_q,$$

where $I \subset \{2, \ldots, r\}$, $|I| = n - 2k$. By Lemma 4.22, applied to the divisor $E' + E_q$, we may replace the section $x_{E'}$ with a linear combination of sections of the form $x_{E' + \sum_{j \in J} E_j - (k - 1) E_q}$, where $j \in \{2, \ldots, r\}$ and $E'' = E' + E_q - \sum_{j \in J} E_j$. Then $E''$ is a minimal divisor as in Case 1) of Lemma 4.19. Hence, we may replace $s$ with a linear combination of distinguished sections with smaller $a$ and smaller $\sum_{k \geq 1} b_k$. □

Assume $\sum b_k = 0$.

Then $\beta = d - m_1 \geq 0$. It follows that $\beta \geq l_i \geq 0$, for all $i = 2, \ldots, r$. This is because one has from (4.11):

$$d - m_i = \tilde{D} (l - e_i) = S. (l - e_i) + l_i \geq l_i$$

(4.16)

Hence, $l_i \leq d - m_i \leq \beta$, for all $i = 2, \ldots, r$.

By definition (4.8), one has $0 \leq m_\alpha - (\sum_{i=1}^r m_i - nd)$. From (4.15) it follows that

$$(\alpha + 1) \beta \leq \sum_{i=2}^r l_i.$$ We are done by Lemma 4.17.

Assume $a = 0$.

We show that in this case $\beta \geq 0$. By definition (4.7), one has $0 \leq m_\alpha - (\sum_{i=1}^r m_i - nd) < \alpha$. From (4.15) it follows that

$$0 \leq \sum_{i=2}^r l_i - (\alpha + 1) \beta < \alpha.$$

It follows that $\beta \geq 0$. We find $l'_i$, $l''_i \geq 0$ such that $l_i = l'_i + l''_i$ and $l'_i \leq \beta$, for all $i = 2, \ldots, r$ and $\sum_{i=2}^r l'_i = (\alpha + 1) \beta$. First, randomly choose $l'_i$, $l''_i$ with $l_i = l'_i + l''_i$, $l'_i, l''_i \geq 0$ and $\sum_{i=2}^r l'_i = (\alpha + 1) \beta$. We show that by eventually replacing $s$ with a linear combination of distinguished sections (with smaller $l'_i$), we may reduce to the case when $l'_i \leq \beta$, for all $i$. First take the case when $i \in \{2, \ldots, r\}$ is such that in $S$ there is no minimal divisor $E'$ of the form

$$E' = kH - kE_i - k \sum_{j \in I} E_j - (k - 1) \sum_{j \in J} E_j - kE_q$$

(4.17)

where $I \subset \{2, \ldots, r\}$, $i \notin I$ and $|I| = n - 1 - 2k$, $J = \{2, \ldots, r\} \setminus \{\{i\} \cup I\}$.

We claim that $l_i \leq \beta$. Since in each $E'$ appearing in $S$, the divisor $E_i$ appears with coefficient $-(k - 1)$, one has:

$$d - m_i = \tilde{D} (l - e_i) = S. (l - e_i) + l_i \geq \sum_{k \geq 1} b_k + l_i.$$ It follows that $l_i \leq (d - m_i) - \sum_{k \geq 1} b_k \leq \beta$.

Assume now that $i \in \{2, \ldots, r\}$ is such that $l'_i > \beta$. By the previous observation, $S$ contains at least one minimal divisor $E'$ of the form (4.17). By Lemma 4.22, applied to the divisor $E' + E_i$, we may replace the section $x_{E' + \sum_{j \in J} E_j}$ with a linear combination of sections of the form $x_{E' + \sum_{j \in J} E_j}$, where $j \in J$ and $E'' = E' + E_q - \sum_{j \in J} E_j$ is a minimal divisor on $Y'$. Moreover, we claim that we may choose only indices $j \in J$ with $l'_j < \beta$. Let us call $j \in \{2, \ldots, r\}$ a good index if $l'_j < \beta$. We claim that there are at least $k + 1$ good indices in $J$. Clearly, $|J| = r - n + 2k - 1 \geq k + 1$. Assume there
are at most $k$ good indices in $J$. Then there are at least $(r - n + k - 1) = (\alpha + k + 1)$ indices in $J$ that are not good. Since $l_i \geq \beta$ and $i \notin J$, it follows that:

$$(\alpha + 1)\beta = \sum_{i=2}^{k} l_i > (\alpha + k + 1)\beta + \beta \geq (\alpha + 1)\beta$$

which is a contradiction. Hence, the claim follows. By repeating the process, we end up with $l_i \leq \beta$, for all $i = 2, \ldots, r$, and we are done by Lemma 4.17.

Lemma 4.22. Let $X = Bl_r \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ in $r \geq n+4$ points on a rational normal curve $C$ of degree $n$. For any $1 \leq k \leq (n+1)/2$ and any $I \subset \{1, \ldots, r\}$, $|I| = n + 1 - 2k$, let

$$D = kH - k \sum_{i \in I} E_i - (k-1) \sum_{i \in I^c} E_i$$

Then $h^0(X, D) = k + 1$. For any $i \in I^c$, the divisor $D - E_i$ is minimal and, for any choice of $k + 1$ indices $i \in I^c$, the sections $x_{D-E_i}x_{E_i}$ generate $H^0(X, D)$.

Proof. Consider the exact sequence:

$$0 \to H^0(X, D - E_i) \to H^0(X, D) \to H^0(E_i, D_{|E_i})$$  (4.18)

We argue by induction on $n \geq 2$. If $n = 2$, then $k = 1$ and $D = H - E_j$, for some $j \in \{1, \ldots, r\}$. Clearly, for any $i \neq j$, the divisor $H - E_i - E_j$ is minimal. Since $(H - E_j).E_i = 0$, one has $H^0(E_i, D_{|E_i}) \cong \mathbb{C}$. For any $l \neq i, j$, the section $x_{H-E_j-E_i}x_{E_i}$ has non-zero restriction to $E_l$. Hence, the map $H^0(X, D) \to H^0(E_l, D_{|E_l})$ is surjective and $H^0(X, D)$ is generated by the sections $x_{H-E_j-E_i}x_{E_i}$ and $x_{H-E_j-E_i}x_{E_l}$.

Assume $n \geq 3$. Fix some $i \in I^c$. From Lemma 4.3 the divisor $E = D - E_i$ is a minimal divisor. Let $Y = Bl_{r-1} \mathbb{P}^{n-1}$ be the blow-up of $\mathbb{P}^{n-1}$ in $r - 1$ points corresponding to the projection from $p_i$ and let $Y' = Bl_{r-1} \mathbb{P}^{n-1}$ be the blow-up of $Y$ at the extra point $q$. Then the restriction map in (4.18) factors through the map $r_{E_i} : H^0(X, D) \to H^0(Y, \tilde{D})$, where

$$\tilde{D} = (k - 1)H - (k - 1) \sum_{j \in I} E_j - (k - 2) \sum_{j \in I^c \setminus \{i\}} E_j.$$

Note that by Lemma 4.2, the multiplicity at $q$ of any divisor in the linear system $|D|$ is at least:

$$\frac{k(n + 1 - 2k) + (k - 1)(r + 2k - n - 1) - nk}{r - n - 2} = k + 1 - \frac{1}{r - n - 2}$$

Since $r \geq n+4$, the map $r_{E_i}$ has image in $H^0(Y', D')$, where $D' = \tilde{D} - (k - 1)E_i$. By induction, $H^0(Y', D')$ has dimension $k$ and it is generated by any distinct $k$ sections of the form $x_{E_i}x_{E_j}$, where $E_i = D' - E_j$ and $j \in I^c \setminus \{i\}$. On $X$, the divisor $E = D - E_j$ is minimal. By Lemma 4.17, $r_{E_i}(x_{E_i}) = x_{E_i}x_{E_j}^{k-1}$. Since $r_{E_i}(x_{E_i}) = x_{E_i}x_{E_j}^{k-1}$, it follows that $r_{E_i}(x_{E_i}x_{E_j}) = x_{E_i}x_{E_j}^{k-1}$. Hence, the map $r_{E_i}$ has image $H^0(Y', D')x_{E_j}^{k-1}$. Therefore, $H^0(X, D)$ has dimension $k + 1$ and it is generated by any $k + 1$ sections of the form $x_{E_i}x_{E_j}$, where $i \in I^c$, $E = D - E_i$. □

Theorem 4.23. Let $r \geq 5$ and let $X = Bl_r \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $r$ distinct points $p_1, \ldots, p_r$ that lie on an irreducible conic. Then $\text{Cox}(X)$ is minuscule.

Proof. Let $C$ be the proper transform on $X$ of the conic in $\mathbb{P}^2$ that contains the points $p_1, \ldots, p_r$. Then $C = 2H - \sum_{i=1}^{r} E_i$. For any $i, j \in \{1, \ldots, r\}$ with $i \neq j$, let $L_{i,j}$ be the proper transform on $X$ of the line that passes through $p_i, p_j$. The classes $C$ and $L_{i,j}$ are the minimal divisors on $X$. Let $x_C$, resp. $x_{L_{i,j}}$, be the
corresponding sections. A distinguished section on $X$ is a monomial in $x_C$, $x_{L_{i,j}}$ and $x_{E_i}$, for all $i, j$.

We prove by induction on $r$ that $\text{Cox}(\text{Bl}_r \mathbb{P}^2)$ is generated by distinguished sections. The case $r = 5$ was proved in [FP]. Assume $r \geq 6$.

Let $D$ be an effective divisor on $X$. If $m_i = D.E_i \leq 0$ for some $i \in \{1, \ldots, r\}$, then $H^0(X, D) \cong H^0(X, D_0)$, where $D_0 = D + m_i E_i$ is a divisor on $\text{Bl}_{r-1} \mathbb{P}^2$ and $H^0(X, D_0)$ is generated by distinguished sections by induction. It follows that $H^0(X, D)$ is generated by distinguished sections (obtained by multiplying sections of $H^0(X, D_0)$ by $x_{E_i}^{-m_i}$). Hence, we may assume that $d, m_i > 0$ and argue by induction on $d$.

From the exact sequence

$$0 \rightarrow H^0(X, D - C) \rightarrow H^0(X, D) \rightarrow H^0(C, D_{C})$$

it follows that if $D.C = -a < 0$, then $H^0(C, D_{C}) = 0$ and $H^0(X, D) \cong H^0(X, D - C)$ is generated by global sections by induction.

Assume now $D.C = 2d - \sum_{i=1}^r m_i \geq 0$ and $m_i > 0$ for all $i = 1, \ldots, r$. Without loss of generality, we may assume $m_i \leq m_i$, for all $i$. Consider the exact sequence:

$$0 \rightarrow H^0(X, D - E_1) \rightarrow H^0(X, D) \rightarrow H^0(E_1, D_{|E_1}).$$

Note $H^0(E_1, D_{|E_1}) = H^0(\mathbb{P}^1, \mathcal{O}(m_1))$. For $i = 2, \ldots, r$, let $q_i = L_{1,i} \cap E_i$. Let $x_i \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ be the section vanishing at $q_i$. The divisor $D_{|E_1}$ has multiplicity at least $m_1 + m_i - d$ at $q_i$. Let $I \subset \{2, \ldots, r\}$ be the set of indices $i$ for which $m_1 + m_i - d \geq 0$. It follows that the image of the restriction map $r : H^0(X, D) \rightarrow H^0(E_1, D_{|E_1})$ (4.19)

lies in the subspace

$$V = \prod_{i \in I} x_i^{m_1 + m_i - d}H^0(\mathbb{P}^1, \mathcal{O}(e)) \subset H^0(\mathbb{P}^1, \mathcal{O}(m_1)),$$

where

$$e = m_1 - \sum_{i \in I} (m_1 + m_i - d)$$ (4.20)

We claim that one may lift any section in $V$ to a section in $H^0(X, D)$ that is generated by distinguished sections. Then we are reduced to show that $H^0(X, D - E_1)$ is generated by distinguished sections. If $D - E_1$ is not effective, we are done; if not, we replace $D$ with $D - E_1$ and repeat the process until either $D - E_1$ is not effective or $D.E_1 \leq 0$.

Clearly, $H^0(\mathbb{P}^1, \mathcal{O}(e))$ is generated by sections $\prod_{i=2}^r x_i^{k_i}$, where $k_i \geq 0$ and $\sum k_i = e$ (of course, we may assume that, for example, $k_4 = k_5 = \ldots = 0$). Note that $r(x_{L_{1,j}}) = x_j$, for all $j = 2, \ldots, r$. Consider the following divisor on $X$:

$$D_0 = \sum_{i \in I} k_i L_{1,i} + \sum_{i \in I} (k_i + m_1 + m_i - d)L_{1,i}$$

$$= m_1 H - m_1 E_1 - \sum_{i \in I} k_i E_i - \sum_{i \in I} (k_i + m_1 + m_i - d)E_i.$$

The restriction map $r$ maps the section

$$t' = \prod_{i=2}^r x_i^{k_i} \prod_{i \in I} x_i^{m_1 + m_i - d} \in H^0(X, D_0)$$

to the section

$$s = \prod_{i=2}^r x_i^{k_i} \prod_{i \in I} x_i^{m_1 + m_i - d} \in H^0(E_1, D_0|_{E_1}) = H^0(\mathbb{P}^1, \mathcal{O}(m_1)).$$
Consider

\[ D - D_0 = (d - m_1)H - \sum_{i \in I^e} (m_i - k_i)E_i - \sum_{i \in I} (d - m_1 - k_i)E_i. \]

Since

\[ d \geq m_1, \quad m_i \geq m_1 \geq e \geq k_i, \quad d - m_1 \geq m_i \geq e \geq k_i \]

and using (4.20) one has:

\[ \sum_{i \in I^e} (m_i - k_i) + \sum_{i \in I} (d - m_1 - k_i) = \sum_{i \in I} m_i + (d - m_1)|I| - e = \sum_{i=1}^r m_i - 2m_1 \leq 2(d - m_1). \]

It follows from Lemma 4.24 that \( D - D_0 \) is an effective divisor on \( X \). Since \( (D - D_0)E_1 = 0 \), the space \( H^0(X, D - D_0) \) is generated by distinguished sections by induction. Let \( t'' \in H^0(X, D - D_0) \) be any distinguished section not zero on \( E_1 \). Then \( t''t' \) is a distinguished section in \( H^0(X, D) \) that maps to \( s \).

Lemma 4.24. Let \( X \) be the blow-up of \( \mathbb{P}^n \) in any \( r \) distinct points. Let \( D = dH - \sum_{i=1}^r m_iE_i \), \( d, m_i \geq 0 \), be a divisor class with \( \sum_{i=1}^r m_i \leq nd \) and \( d \geq m_i \), for all \( i = 1, \ldots, r \). Then \( D \) is an effective divisor.

Proof. We claim that \( D \) is an effective combination of (effective) classes \( H - (E_{i_1} + \ldots + E_{i_l}) \), for \( i_1, \ldots, i_l \in \{1, \ldots, r\} \) and \( 0 \leq l \leq n \). Consider the table with \( n \) rows and \( d \) columns filled with \( E_i \)'s in the following way. Start in the upper left corner and write \( m_1 \) \( E_1 \)'s in the first row. Then write \( m_2 \) \( E_2 \)'s passing to the second row if necessary, and so on. Fill the remaining entries with zeros. In the following example \( n = 3 \) and \( D = 5H - 3E_1 - 3E_2 - 2E_3 - 5E_4 - E_5 \):

\[
\begin{array}{cccccc}
E_1 & E_1 & E_1 & E_2 & E_2 \\
E_2 & E_3 & E_3 & E_4 & E_4 \\
E_4 & E_4 & E_4 & E_5 & 0 \\
\end{array}
\]

Our conditions guarantee that all entries of a given column are different. Therefore \( D \) is the sum of classes \( H - (E_{i_1} + \ldots + E_{i_l}) \), one for each column, where \( E_{i_1}, \ldots, E_{i_l} \) are entries of the column. In the example above,

\[ D = (H - E_1 - E_2 - E_4) + (H - E_1 - E_3 - E_4) + \ldots + (H - E_2 - E_3). \]

\[ \square \]

5. Proof of Theorem 1.1

By \([11]\), there is an isomorphism \( \phi : S^G \to \text{Cox}(X) \) where \( X \) is the blow-up of \( \mathbb{P}^n \) in \( n + 3 \) points \( p_1, \ldots, p_{n+3} \) in general position. By Theorem 1.3 the ring \( \text{Cox}(X) \) is generated by the sections \( x_{E_i} \), for each exceptional divisor \( E_i \), \( i = 1, \ldots, n + 3 \), and the sections \( x_{E_i} \), corresponding to the minimal divisors

\[ E = kH - k \sum_{i \in I} E_i - (k - 1) \sum_{i \in I^e} E_i \quad (5.1) \]

for each subset \( I \subset \{1, \ldots, n + 3\}, |I| = n + 2 - 2k, 1 \leq k \leq 1 + n/2 \). Then \( |I^e| = 2k + 1 \). Note that if \( k = 0 \) in (5.1), then \( E = E_i \).

The polynomials \( F_I \) in (1.2) are clearly invariant (just use the rule of differentiating a determinant). We claim that, for all \( 0 \leq k \leq 1 + n/2 \) one has \( \phi(F_I) = x_{E_i} \), where \( E \) is as in (5.1). It is clear from \([11]\) that \( \phi(x_{E_i}) = x_{E_i} \). Following \([11]\), if \( F_0 = \ldots = F_n = 0 \) are \( n + 1 \) linear equations in \( t_1, \ldots, t_r \) that cut \( G \) in \( \mathbb{G}^{n+3} \), let \( J_0, \ldots, J_n \) be the polynomials in \( S \) generated by \( J_i = F_i(y_1/x_1, \ldots, y_r/x_r)x_1 \ldots x_r \).
Then sections of the divisor $D = dH - \sum_{i=1}^{n+3} m_i E_i$ on $X$, for $d, m_i \geq 0$, correspond by $\phi$ to an invariant polynomial of the form

$$Q = \frac{P(J_0, \ldots, J_n)}{\prod_{i=1}^{n+3} x_i^{m_i}}$$

where $P(z_0, \ldots, z_n)$ is a homogeneous polynomial of degree $d$ in variables $z_0, \ldots, z_n$, such that $P(J_0, \ldots, J_n)$ is divisible by $\prod_{i=1}^{n+3} x_i^{m_i}$. If we let $\deg_x(Q)$, resp. $\deg_y(Q)$, to be the degree of $Q$ in the $x_i$’s, resp. in the $y_i$’s, then

$$\deg_x(Q) = d, \quad \deg_y(Q) = (n+2)d - \sum_{i=1}^{n+3} m_i, \quad \deg(D) = \deg_x(Q) - \deg_y(Q). \quad (5.2)$$

Hence, $\phi(F_{\tau})$ is a section in $H^0(X, D)$, where $D$ is a divisor with $d = k$ and $\deg(D) = 1$. To show that $D = E$, consider the following action of the torus $G^r_m$ on $S$: $(\lambda_1, \ldots, \lambda_r) \in G^r_m$ acts by $x_i \mapsto \lambda_i x_i$, $y_i \mapsto \lambda_i y_i$. The action of $G^r_m$ on $S$ is compatible with the action of $G^s_n$ on $S$. Hence, there is an induced action of $G^r_m$ on $S^0$. Since $(\lambda_1, \ldots, \lambda_r) \in G^r_m$, maps $J_i$ onto $\lambda_1 \ldots \lambda_r J_i$, it follows that $Q$ is mapped to $\prod_{i=1}^{n+3} \lambda_i ^{-m_i}$. Since $F_{\tau}$ is mapped to $\prod_{i=\tau} \lambda_i$, it follows that $D = E$. \hfill $\square$

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