First Order Alternation

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Abstract. We introduce first order alternating automata, a generalization of boolean alternating automata, in which transition rules are described by multisorted first order formulae, with states and internal variables given by uninterpreted predicate terms. The model is closed under union, intersection and complement, and its emptiness problem is undecidable, even for the simplest data theory of equality. To cope with this limitation, we develop an abstraction refinement semi-algorithm based on lazy annotation of the symbolic execution paths with interpolants, obtained by applying (i) quantifier elimination with witness term generation and (ii) Lyndon interpolation in the quantifier-free data theory with uninterpreted predicate symbols. This provides a method for checking inclusion of timed and finite-memory register automata, and emptiness of quantified predicate automata, previously used in the verification of parameterized concurrent programs, composed of replicated threads, with a shared-memory communication model.

1 Introduction

Many results in formal language theory rely on the assumption that languages are defined over finite alphabets. In practice, this assumption is problematic when attempting to use automata as models of real-time systems or even simple programs, whose input and observable output requires taking into account data values, ranging over very large domains, better viewed as infinite mathematical abstractions.

Alternating automata are a generalization of nondeterministic automata with universal transitions, that create several copies of the automaton, which synchronize on the same input word. Alternating automata are appealing for verification because they allow encoding of problems such as temporal logic model checking in linear time, as opposed to the exponential time required by nondeterministic automata [26]. A finite-alphabet alternating automaton is typically described by a set of transition rules \( q \xrightarrow{a} \phi \), where \( q \) is a state, \( a \) is an input symbol and \( \phi \) is a positive boolean combinations of states, viewed as propositional variables.

Here we introduce a generalized alternating automata model in which states are predicate symbols \( q(y_1,\ldots,y_k) \), the input has associated data variables \( x_1,\ldots,x_n \), ranging over an infinite domain and transitions are of the form \( q(y_1,\ldots,y_k) \xrightarrow{a(x_1,\ldots,x_n)} \phi \), where \( \phi \) is any formula in the first-order theory of the data domain, in which each state predicate occurs under an even number of negations. In this model, the arguments of a predicate atom \( q(y_1,\ldots,y_k) \) track the values of the internal variables associated with the state. Together with the input values \( x_1,\ldots,x_n \), these values are used to compute the successor states and are invisible in the input sequence.
Previous attempts to generalize classical Rabin-Scott automata to infinite alphabets, such as timed automata [1] and finite-memory (register) automata [14] face the complement closure problem: there exist automata for which the complement language cannot be recognized by an automaton in the same class. This excludes the possibility of encoding a language inclusion problem \( \mathcal{L}(A) \subseteq \mathcal{L}(B) \) as the emptiness of an automaton recognizing the language \( \mathcal{L}(A) \cap \mathcal{L}^c(B) \), where \( \mathcal{L}^c(B) \) denotes the complement of \( \mathcal{L}(B) \).

The solution we adopt here is a tight coupling of internal variables to control states, using uninterpreted predicate symbols. As we show, this allows for linear-time complementation just as in the case of boolean alternating automata. Complementation is, moreover, possible when the transition formulae contain first-order quantifiers, generating infinitely-branching execution trees. The price to be paid for this expressivity is that emptiness of first-order alternating automata is undecidable, even for the simplest data theory of equality [4].

The main contribution of this paper is an effective emptiness checking semi-algorithm for first-order alternating automata, in the spirit of the IMPACT procedure, originally developed for checking safety of nondeterministic integer programs [18]. However, checking emptiness of first-order alternating automata by lazy annotation with interpolants faces two problems:

1. Quantified transition rules make it hard, or even impossible, to decide if a given symbolic trace is spurious. This is mainly because adding uninterpreted predicate symbols to decidable first-order theories, such as Presburger arithmetic, results in undecidability [8]. To deal with this problem, we assume that the first order data theory, without uninterpreted predicate symbols, has a quantifier elimination procedure, that instantiates quantifiers with effectively computable witness terms.

2. The interpolants that prove the spuriousness of a symbolic path are not local, as they may refer to input values encountered in the past. However, the future executions are oblivious to when these values have been seen in the past and depend only on the data constraints between the values. We use this fact to define a labeling of nodes, visited by the lazy annotation procedure, with conjunctions of existentially quantified interpolants combining predicate atoms with data constraints.

As applications of first order alternating automata, we identified several undecidable problems for which no semi-algorithmic methods exist: inclusion between recognizable timed languages [1], languages recognized by finite-memory automata [14] and emptiness of predicate automata, a subclass of first-order alternating automata used to check safety and liveness properties of parameterized concurrent programs [4,5].

For reasons of space, all proofs of technical results in this paper are given in [12].

Related Work The first order alternating automata model presented in this paper stems from our previous work on boolean alternating automata extended with variables ranging over infinite data [11]. There we considered states to be propositional variables, as in the classical textbook alternating automata model, and all variables of the automaton to be observable in the input. The model in this paper overcomes this latter restriction by allowing for internal variables, whose variables are not visible in the language.

This solves an older language inclusion problem \( \bigcap_{i=1}^n \mathcal{L}(A_i) \subseteq \mathcal{L}(B) \), between finite-state automata with data variables, whose languages are alternating sequences of input events and variable valuations [10]. There, we assumed that all variables of the observer
automaton $B$ must be declared in the automata $A_1, \ldots, A_n$ that model the concurrent components of the system under check. Using first-order alternating automata allows to bypass this limitation of our previous work.

The work probably closest to the one reported here concerns the model of predicate automata (PA) [4,5,15], applied to the verification of parameterized concurrent programs with shared memory. In this model, the alphabet consists of pairs of program statements and thread identifiers, thus being infinite because the number of threads is unbounded. Because thread identifiers can only be compared for (dis-)equality, the data theory in PA is the theory of equality. Even with this simplification, the emptiness problem is undecidable when either the predicates have arity greater than one [4] or quantified transition rules [15]. Checking emptiness of quantifier free PA is possible semi-algorithmically, by explicitly enumerating reachable configurations and checking coverage by looking for permutations of argument values. However, no semi-algorithm is given for quantified PA. Dealing with quantified transition rules is one of the contributions of the work reported in this paper.

2 Preliminaries

For two integers $0 \leq i \leq j$, we denote by $[i,j]$ the set $\{i,i+1,\ldots,j\}$ and by $[i]$ the set $\{0,i\}$. We consider two sorts $\mathbb{D}$ and $\mathbb{B}$, where $\mathbb{D}$ is an infinite domain and $\mathbb{B} = \{\top, \bot\}$ is the set of boolean values true ($\top$) and false ($\bot$), respectively. The $\mathbb{D}$ sort is equipped with finitely many function symbols $f : \mathbb{D}^{#(f)} \to \mathbb{D}$, where $#(f) \geq 0$ denotes the number of arguments (arity) of $f$. When $#(f) = 0$, we say that $f$ is a constant. A predicate is a function symbol $p : \mathbb{D}^{#(p)} \to \mathbb{B}$, denoting a relation of arity $#(p)$ and we write $\text{Pred}$ for the set of predicates.

In the following, we shall consider that the interpretation of all function symbols $f : \mathbb{D}^{#(f)} \to \mathbb{D}$ that are not predicates is fixed by the interpretation of the $\mathbb{D}$ sort, e.g. if $\mathbb{D}$ is the set of integers $\mathbb{Z}$, the function symbols are zero, the successor function and the arithmetic operations of addition and multiplication. For simplicity, we further blur the notational distinction between function symbols and their interpretations.

Let $\text{Var} = \{x,y,z,\ldots\}$ be an infinite countable set of variables, ranging over $\mathbb{D}$. Terms are either constants of sort $\mathbb{D}$, variables or function applications $f(t_1,\ldots,t_{#(f)})$, where $t_1,\ldots,t_{#(f)}$ are terms. The set of first order formulae is defined by the syntax below:

$$\phi ::= t \equiv s \mid p(t_1,\ldots,t_{#(p)}) \mid \neg \phi_1 \mid \phi_1 \land \phi_2 \mid \exists x . \phi_1$$

where $t,s,t_1,\ldots,t_{#(p)}$ denote terms. We write $\phi_1 \lor \phi_2$, $\phi_1 \rightarrow \phi_2$ and $\forall x . \phi_1$ for $\neg(\neg \phi_1 \land \neg \phi_2)$, $\neg \phi_1 \lor \phi_2$ and $\neg \exists x . \neg \phi_1$, respectively. We denote by $\text{FV}(\phi)$ the set of free variables in $\phi$. The size $|\phi|$ of a formula $\phi$ is the number of symbols needed to write it down.

A sentence is a formula $\phi$ in which each variable occurs under the scope of a quantifier, i.e. $\text{FV}(\phi) = \emptyset$. A formula is positive if each predicate symbol occurs under an even number of negations and we denote by $\text{Form}^+(Q,X)$ the set of positive formulae with predicates from the set $Q \subseteq \text{Pred}$ and free variables from the set $X \subseteq \text{Var}$.

A formula is in prenex form if it is of the form $\varphi = Q_1x_1\ldots Q_nx_n . \phi$, where $\phi$ has no quantifiers. In this case we call $\phi$ the matrix of $\varphi$. Every first order formula can be
written in prenex form, by renaming each quantified variable to a unique name and moving the quantifiers upfront.

An interpretation \( I \) maps each predicate \( p \) into a set \( p^I \subseteq D^{#(p)} \), if \( #(p) > 0 \), or into an element of \( D \) if \( #(p) = 0 \). A valuation \( \nu \) maps each variable \( x \) into an element of \( D \). Given a term \( t \), we denote by \( t' \) the value obtained by replacing each variable \( x \) by the value \( \nu(x) \) and evaluating each function application. For a formula \( \phi \), we define the forcing relation \( I, \nu \models \phi \) recursively on the structure of \( \phi \), as usual.

\[
\begin{align*}
I, \nu \models t = s & \iff t' = s' \\
I, \nu \models p(t_1, \ldots, t_{#(p)}) & \iff \langle t'_1, \ldots, t'_{#(p)} \rangle \in p^I \\
I, \nu \models \neg \phi_1 & \iff I, \nu \not\models \phi_1 \\
I, \nu \models \phi_1 \land \phi_2 & \iff I, \nu \models \phi_1, \text{ for all } i = 1, 2 \\
I, \nu \models \exists x. \phi_1 & \iff I, \nu[x \leftarrow d] \models \phi_1, \text{ for some } d \in D
\end{align*}
\]

where \( \nu[x \leftarrow d] \) is the valuation which assigns \( d \) to \( x \) and behaves like \( \nu \) elsewhere. For a formula \( \phi \) and a valuation \( \nu \), we define \( [[\phi]]_\nu \overset{def}{=} \{ I \mid I, \nu \models \phi \} \) and drop the \( \nu \) subscript for sentences. A sentence \( \phi \) is satisfiable (unsatisfiable) if \( [[\phi]] \neq \emptyset \) (\( [[\phi]] = \emptyset \)). An element of \( [[\phi]] \) is called a model of \( \phi \). A formula \( \phi \) is valid if \( I, \nu \models \phi \) for every interpretation \( I \) and every valuation \( \nu \). For two formulae \( \phi \) and \( \psi \) we write \( \phi \models \psi \) for \( [[\phi]] \subseteq [[\psi]] \), in which case we say that \( \phi \) entails \( \psi \).

Interpretations are partially ordered by the pointwise subset order, defined as \( I_1 \subseteq I_2 \) if and only if \( p^{I_1} \subseteq p^{I_2} \) for each predicate \( p \in \text{Pred} \). Given a set \( S \) of interpretations, a minimal element \( I \in S \) is an interpretation such that for no other interpretation \( I' \in S \setminus \{I\} \) do we have \( I' \subseteq I \). For a formula \( \phi \) and a valuation \( \nu \), we denote by \( [[\phi]]_\nu^m \) and \( [[\phi]]^m \) the set of minimal interpretations from \( [[\phi]] \), and \( [[\phi]] \), respectively.

### 3 First Order Alternating Automata

Let \( \Sigma \) be a finite alphabet \( \Sigma \) of input events. Given a finite set of variables \( X \subseteq \text{Var} \), we denote by \( X \mapsto D \) the set of valuations of the variables \( X \) and \( \Sigma[X] = \Sigma \times (X \mapsto D) \) be the possibly infinite set of data symbols \((a, \nu)\), where \( a \) is an input symbol and \( \nu \) is a valuation. A data word (simply called word in the following) is a finite sequence \( w = (a_1, \nu_1)(a_2, \nu_2)\ldots(a_n, \nu_n) \) of data symbols. Given a word \( w \), we denote by \( w_{\Sigma} \overset{def}{=} a_1 \ldots a_n \) its sequence of input events and by \( w_{\Sigma} \) the valuation associating each time-stamped variable \( x^{(i)} \) the value \( \nu_i(x) \), for all \( x \in \text{Var} \) and \( i \in [1, n] \). We denote by \( \varepsilon \) the empty sequence, by \( \Sigma^* \) the set of finite sequences of input events and by \( \Sigma[X]^* \) the set of data words over the variables \( X \).

Formally, a first order alternating automaton is a tuple \( \mathcal{A} = (\Sigma, X, Q, \iota, F, \Delta) \), where \( \Sigma \) is a finite set of input events, \( X \) is a finite set of input variables, \( Q \) is a finite set of predicates denoting control states, \( \iota \in \text{Form}^+(Q, \emptyset) \) is a sentence defining initial configurations, \( F \subseteq Q \) is the set of predicates denoting final states, and \( \Delta \) is a set of transition rules of the form \( q(y_1, \ldots, y_{#(q)}) \xrightarrow{w(X)} \psi \), where \( q \in Q \) is a predicate, \( a \in \Sigma \) is an input event and \( \psi \in \text{Form}^+(Q \cup X \cup \{y_1, \ldots, y_{#(q)}\}) \) is a positive formula, where \( X \cap \{y_1, \ldots, y_{#(q)}\} = \emptyset \). The quantifiers occurring in the right-hand side formula of a transition rule are referred to as transition quantifiers. The size of \( \mathcal{A} \) is defined as \( |\mathcal{A}| = |\iota| + \sum_{\phi \in \Delta} \omega(\phi) \).

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The intuition of a transition rule $q(y_1,\ldots,y_{\#(q)}) \xrightarrow{a\phi} \psi$ is the following: $a$ is the input event and $X$ are the input data values that trigger the transition, whereas $q$ and $y_1,\ldots,y_{\#(q)}$ are the current control state and data values in that state, respectively. Without loss of generality, we consider, for each predicate $q \in Q$ and each input event $a \in \Sigma$, at most one such rule, as two or more rules can be joined using disjunction.

The execution semantics of automata is given in close analogy with the case of boolean alternating automata, with transition rules of the form $q \rightarrow \phi$, where $q$ is a boolean constant and $\phi$ a positive boolean combination of such constants. For instance, $q_0 \rightarrow q_1 \land q_2 \lor q_3$ means that the automaton can choose to transition in either both $q_1$ and $q_2$ or in $q_3$ alone. This intuition leads to saying that the steps of the automaton are defined by the minimal boolean models of the transition formulae. In this case, both $\{q_1 \leftarrow \top, q_2 \leftarrow \top, q_3 \leftarrow \bot\}$ and $\{q_1 \leftarrow \bot, q_2 \leftarrow \bot, q_3 \leftarrow \top\}$ are minimal models, however $\{q_1 \leftarrow \top, q_2 \leftarrow \bot, q_3 \leftarrow \top\}$ is a model but is not minimal. The original definition of alternating finite-state automata works around this problem by considering boolean valuations (models) instead of formulae. However, describing first-order alternating automata using interpretations instead of formulae would be rather hard to follow.

Given a predicate $q \in Q$ and a tuple of data values $d_1,\ldots,d_{\#(q)}$, the tuple $q(d_1,\ldots,d_{\#(q)})$ is called a configuration.\footnote{Note that a configuration is not a logical term since data values cannot be written in logic.} To formalize the execution semantics of automata, we relate sets of configurations to models of first order sentences, as follows. Each first-order interpretation $\mathcal{I}$ corresponds to a set of configurations $\mathcal{C}(\mathcal{I}) = \{q(d_1,\ldots,d_{\#(q)}) \mid q \in Q, (d_1,\ldots,d_{\#(q)}) \in q^\mathcal{I}\}$, called a cube. For a set $\mathcal{S}$ of interpretations, we define $\mathcal{C}(\mathcal{S}) = \{\mathcal{C}(\mathcal{I}) \mid \mathcal{I} \in \mathcal{S}\}$.

**Definition 1.** Given a word $w = (a_1,\nu_1)\ldots(a_n,\nu_n) \in \Sigma[X]^*$ and a cube $c$, an execution of $\mathbf{A} = (\Sigma, \mathcal{X}, Q, i, F, A)$ over $w$, starting with $c$, is a (possibly infinite) forest $\mathcal{T} = \{T_1, T_2,\ldots\}$, where each $T_i$ is a tree labeled with configurations, such that:
1. $c = (\mathcal{T}(\varepsilon) \mid T \in \mathcal{T})$ is the set of configurations labeling the roots of $T_1, T_2,\ldots$ and
2. if $q(d_1,\ldots,d_{\#(q)})$ labels a node on the level $j \in [n-1]$ in $T_i$, then the labels of its children form a cube from $\mathcal{C}(\mathcal{I}(\mathcal{I}_{\mathcal{I}})^w)$, where $\eta = \nu_{j+1}[y_1 \leftarrow d_1,\ldots,y_{\#(q)} \leftarrow d_{\#(q)}]$ and $q(y_1,\ldots,y_{\#(q)}) \xrightarrow{a\eta_j(\phi)} \psi \in \Delta$ is a transition rule of $\mathbf{A}$.

**Definition 2.** An execution $\mathcal{T}$ over $w$, starting with $c$, is accepting if and only if
- all paths in $\mathcal{T}$ have the same length $n$, and
- the frontier of each tree $T \in \mathcal{T}$ is labeled with final configurations $q(d_1,\ldots,d_{\#(q)})$, where $q \in F$.

If $\mathbf{A}$ has an accepting execution over $w$ starting with a cube $c \in \mathcal{C}(\mathcal{I})^w$, then $\mathbf{A}$ accepts $w$ and let $L(\mathbf{A})$ be the set of words accepted by $\mathbf{A}$.

In this paper, we address the following questions:
1. boolean closure: given automata $\mathbf{A}_i = (\Sigma, \mathcal{X}, Q_i, i, F_i, \mathcal{A}_i)$, for $i = 1, 2$, do there exist automata $\mathbf{A}_{\gamma}, \mathbf{A}_{\cup}$ and $\mathbf{A}_{\cap}$ such that $L(\mathbf{A}_{\gamma}) = L(\mathbf{A}_1) \cap L(\mathbf{A}_2)$, $L(\mathbf{A}_{\cup}) = L(\mathbf{A}_1) \cup L(\mathbf{A}_2)$ and $L(\mathbf{A}_{\cap}) = \Sigma[X]^* \setminus L(\mathbf{A}_1)$?
2. emptiness: given an automaton $\mathbf{A}$, is $L(\mathbf{A}) = \emptyset$?
3.1 Symbolic Execution

In the upcoming developments it is sometimes more convenient to work with logical formulae defining executions of automata, than with low-level execution forests. For this reason, we first introduce path formulae \( \Theta(\alpha) \), which are formulae defining the executions of an automaton, over words that share a given sequence \( \alpha \) of input events. Second, we restrict a path formula \( \Theta(\alpha) \) to an acceptance formula \( \Upsilon(\alpha) \), which defines only accepting executions over words that share a given input sequence. Otherwise stated, \( \Upsilon(\alpha) \) is satisfiable if and only if the automaton accepts a word \( w \) such that \( w_\Sigma = \alpha \).

Let \( \mathcal{A} = (\Sigma, X, Q, q, \delta, \lambda, F, A) \) be an automaton for the rest of this section. For any \( i \in \mathbb{N} \), we denote by \( Q^i = \{ q^i | q \in Q \} \) and \( X^i = \{ x^i | x \in X \} \) the sets of time-stamped predicates and variables, respectively. As a shorthand, we write \( Q^{\Sigma} \) (resp. \( X^{\Sigma} \)) for the set \( \{ q^i | q \in Q, i \in \mathbb{N} \} \) (resp. \( \{ x^i | x \in X, i \in \mathbb{N} \} \)). For a formula \( \psi \) and \( i \in \mathbb{N} \), we define \( \psi^i \triangleq \psi[X^0/X, Q^0/Q] \) the formula in which all input variables and state predicates (and only those symbols) are replaced by their time-stamped counterparts. As a shorthand, we shall write \( q(y) \) for \( q(y_1, \ldots, y_{\#(q)}) \), when no confusion arises.

Given a sequence of input events \( \alpha = a_1 \ldots a_n \in \Sigma^* \), the path formula of \( \alpha \) is:

\[
\Theta(\alpha) \overset{\text{def}}{=} \iota^{(0)} \wedge \bigwedge_{i=1}^{n} \bigwedge_{q \in X} q(y) \rightarrow \psi_i
\]

The automaton \( \mathcal{A} \), to which \( \Theta(\alpha) \) refers, will always be clear from the context. To formalize the relation between the low-level configuration-based execution semantics and the symbolic path formulae, consider a word \( w = (a_1, q_1) \ldots (a_n, q_n) \in \Sigma[X]^* \). Any execution forest \( \mathcal{T} \) of \( \mathcal{A} \) over \( w \) is associated an interpretation \( I_\mathcal{T} \) of the set of time-stamped predicates \( Q^{\Sigma} \), defined as:

\[
I_\mathcal{T}(q^i) \overset{\text{def}}{=} \{ (d_1, \ldots, d_{\#(q)}) | q(d_1, \ldots, d_{\#(q)}) \text{ labels a node on level } i \text{ in } \mathcal{T} \}, \forall q \in Q \forall i \in \mathbb{N}
\]

Lemma 1. Given an automaton \( \mathcal{A} = (\Sigma, X, Q, q, \delta, \lambda, F, A) \), for any word \( w = (a_1, q_1) \ldots (a_n, q_n) \), we have \( [\Theta(w)]^\mathcal{A}_{w_\Sigma} = [I_\mathcal{T} | \mathcal{T} \text{ is an execution of } \mathcal{A} \text{ over } w] \).

Proof: “\( \subseteq \)” Let \( I \) be a minimal interpretation such that \( I, w_\Sigma \models \Theta(w_\Sigma) \). We show that there exists an execution \( \mathcal{T} \) of \( \mathcal{A} \) over \( w \) such that \( I = I_\mathcal{T} \), by induction on \( n \geq 0 \). For \( n = 0 \), we have \( w = \epsilon \) and \( \Theta(w_\Sigma) = \iota^{(0)} \). Because \( \iota \) is a sentence, the valuation \( w_\Sigma \) is not important in \( I, w_\Sigma \models \iota^{(0)} \) and, moreover, since \( I \) is minimal, we have \( I \in \{ \iota^{(0)} \}^I \).

We define the interpretation \( J(q) = I(q^{10}) \), for all \( q \in Q \). Then \( c(J) \) is an execution of \( \mathcal{A} \) over \( \epsilon \) and \( I = I_{c(J)} \) is immediate. For the inductive case \( n > 0 \), we assume that \( w = u \cdot (a_n, q_n) \) for a word \( u \). Let \( c(J) \) be the interpretation defined as \( I \) for all \( q^i \), with \( q \in Q \) and \( i \in [n-1] \), and \( \emptyset \) everywhere else. Then \( J, w_\Sigma \models \Theta(w_\Sigma) \) and \( \mathcal{J} \) is moreover minimal. By the induction hypothesis, there exists an execution \( \mathcal{G} \) of \( \mathcal{A} \) over \( u \), such that \( \mathcal{J} = I_{c(G)} \). Consider a leaf of a tree \( T \in \mathcal{G} \), labeled with a configuration \( q(d_1, \ldots, d_{\#(q)}) \) and let \( \forall y_1 \ldots \forall y_{\#(q)} \cdot q^{\#(q)}(y) \rightarrow \psi^{\#(q)} \) be the subformula of \( \Theta(w_\Sigma) \) corresponding to the application(s) of the transition rule \( q(y) \overset{a}{\rightarrow} \psi \) at the \( (n-1) \)-th step. Let \( \nu = w_\Sigma[y_1 \leftarrow d_1, \ldots, y_{\#(q)} \leftarrow d_{\#(q)}] \). Because \( I, w_\Sigma \models \forall y_1 \ldots \forall y_{\#(q)} \cdot q^{\#(q)}(y) \rightarrow \psi^{\#(q)} \), we have \( I \in \{ \psi^{\#(q)} \}^I \), and let \( K \) be one of the minimal interpretations such that \( K \subseteq I \) and \( K \in \{ \psi^{\#(q)} \}^I \). It is
not hard to see that \( \mathcal{K} \) exists and is unique, otherwise we could take the pointwise intersection of two or more such interpretations. We define the interpretation \( \overline{\mathcal{K}}(q) = \overline{\mathcal{K}}(q^n) \) for all \( q \in Q \). We have that \( \mathcal{K} \in [\psi]\nu \) — if \( \overline{\mathcal{K}} \) was not minimal, \( \mathcal{K} \) was not minimal to start with, contradiction. Then we extend the execution \( \mathcal{G} \) by appending to each node labeled with a configuration \( q(d_1, \ldots, d_{\#(q)}) \) the cube \( \mathcal{C}(\overline{\mathcal{K}}) \). By repeating this step for all leaves of a tree in \( \mathcal{G} \), we obtain an execution of \( \mathcal{A} \) over \( w \).

“\( \exists \)” Let \( \mathcal{T} \) be an execution of \( \mathcal{A} \) over \( w \). We show that \( \overline{\mathcal{I}}_\mathcal{T} \) is a minimal interpretation such that \( \overline{\mathcal{I}}_\mathcal{T}, w_2 \models \Theta(w_2) \), by induction on \( n \geq 0 \). For \( n = 0 \), \( \mathcal{T} \) is a cube from \( \mathcal{C}([\nu]\nu) \), by definition. Then \( \overline{\mathcal{I}}_\mathcal{T} \models \iota(0) \) and moreover, it is a minimal such interpretation. For the inductive case \( n > 0 \), let \( w = u \cdot (a_n, v_n) \) for a word \( u \). Let \( \mathcal{G} \) be the restriction of \( \mathcal{T} \) to \( u \). Consequently, \( \overline{\mathcal{I}}_\mathcal{G} \) is the restriction of \( \overline{\mathcal{I}}_\mathcal{T} \) to \( \mathcal{G}^{(n-1)} \). By the inductive hypothesis, \( \overline{\mathcal{I}}_\mathcal{G} \) is a minimal interpretation such that \( \overline{\mathcal{I}}_\mathcal{G}, w_2 \models \Theta(w_2) \). Since \( \overline{\mathcal{I}}_\mathcal{T}(\nu^n) = \{ (d_1, \ldots, d_{\#(q)}) \mid q(d_1, \ldots, d_{\#(q)}) \text{ labels a node on the } n\text{-th level in } \mathcal{T} \} \), we have \( \overline{\mathcal{I}}_\mathcal{T}, w_2 \models \varphi \), for each subformula \( \varphi = \forall y_1 \ldots \forall y_{\#(q)} \cdot q^{\nu^{(i)}}(y) \rightarrow \psi^{(0)} \) of \( \Theta(w_2) \), by the execution semantics of \( \mathcal{A} \). This is the case because the children of each node labeled with \( q(d_1, \ldots, d_{\#(q)}) \) on the \( (n-1)\)-th level of \( \mathcal{T} \) form a cube from \( \mathcal{C}([\nu]\nu) \), where \( \nu \) is a valuation that assigns each \( y_i \) the value \( d_i \) and behaves like \( w_2 \), otherwise. Now suppose, for a contradiction, that \( \overline{\mathcal{I}}_\mathcal{T} \) is not minimal and let \( \mathcal{J} \subsetneq \overline{\mathcal{I}}_\mathcal{T} \) be an interpretation such that \( \mathcal{J}, w_2 \models \Theta(w_2) \). First, we show that the restriction \( \mathcal{J}' \) of \( \mathcal{J} \) to \( \bigcup_{e=0}^{n-1} Q^e \) must coincide with \( \overline{\mathcal{I}}_\mathcal{G} \). Assuming this is not the case, i.e. \( \mathcal{J}' \subsetneq \overline{\mathcal{I}}_\mathcal{G} \), contradicts the minimality of \( \overline{\mathcal{I}}_\mathcal{G} \). Then the only possibility is that \( \mathcal{J}(\nu^n) \subsetneq \overline{\mathcal{I}}_\mathcal{T}(\nu^n) \), for some \( q \in Q \). Let \( p_1(y_1, \ldots, y_{\#(p_1)}) \xrightarrow{\alpha_k} y_1, \ldots, p_k(y_1, \ldots, y_{\#(p_k)}) \xrightarrow{\alpha_k} q_k \) be the set of transition rules in which the predicate symbol \( q \) occurs on the right-hand side. Then it must be the case that, for some node on the \( (n-1)\)-th level of \( \mathcal{G} \), labeled with a configuration \( p_i(d_1, \ldots, d_{\#(p_i)}) \), the set of children does not form a minimal cube from \( \mathcal{C}([\nu]^{(i)}\nu) \), which contradicts the execution semantics of \( \mathcal{A} \).

Next, we give a logical characterization of acceptance, relative to a given sequence of input events \( \alpha \in \Sigma^* \). To this end, we constrain the path formula \( \Theta(\alpha) \) by requiring that only final states of \( \mathcal{A} \) occur on the last level of the execution. The result is the acceptance formula for \( \alpha \):

\[
\mathcal{T}(\alpha) \overset{\text{def}}{=} \Theta(\alpha) \land \bigwedge_{q \in Q \cap F} \forall y_1 \ldots \forall y_{\#(q)} \cdot q^{\nu^{(i)}}(y) \rightarrow \bot \tag{2}
\]

The top-level universal quantifiers from a subformula \( \forall y_1 \ldots \forall y_{\#(q)} \cdot q^{\nu^{(i)}}(y) \rightarrow \psi \) of \( \mathcal{T}(\alpha) \) will be referred to as path quantifiers, in the following. Notice that path quantifiers are distinct from the transition quantifiers that occur within a formula \( \psi \) of a transition rule \( q(y_1, \ldots, y_{\#(q)}) \xrightarrow{\alpha} \psi \) of \( \mathcal{A} \).

The acceptance formula \( \mathcal{T}(\mathcal{A}) \) is false in every interpretation of the predicates that assigns a non-empty set to a non-final predicate occurring on the last level in the execution forest. The relation between the words accepted by \( \mathcal{A} \) and the acceptance formula above, is formally captured by the following lemma:

**Lemma 2.** Given an automaton \( \mathcal{A} = (\Sigma, X, Q, \iota, F, \Delta) \), for every word \( w \in \Sigma[X]^* \), the following are equivalent:
1. there exists an interpretation $I$ such that $I, w_D \models T(w_\Sigma)$.

2. $w \in L(A)$.

Proof: “(1) $\Rightarrow$ (2)” Let $I$ be an interpretation such that $I, w_D \models T(w_\Sigma)$. By Lemma 1, $A$ has an execution $T$ over $w$ such that $I = I_T$. To prove that $T$ is accepting, we show that (i) all paths in $T$ have length $n$ and that (ii) the frontier of $T$ is labeled with final configurations only. First, assume that (i) there exists a path in $T$ of length $0 \leq m < n$. Then there exists a node on the $m$-th level, labeled with some configuration $q(d_1, \ldots, q_{\#(q)})$, that has no children. By the definition of the execution semantics of $A$, we have $c(\models [\psi]_T^\alpha) = 0$, where $q(y)^{d_{a+1}(x)} \models \psi$ is the transition rule of $A$ that applies for $q$ and $d_{a+1}$ and $\eta = w_D[y_1 \leftarrow d_1, \ldots, y_{\#(q)} \leftarrow d_{\#(q)}]$. Hence $\models [\psi]_\eta = 0$, and because $I, w_D \models T(\alpha)$, we obtain that $I, \eta \models q(y) \rightarrow \psi^{(a+1)}$, thus $\langle d_1, \ldots, d_{\#(q)} \rangle \notin I(q)$. However, this contradicts the fact that $I = I_T$ and that $q(d_1, \ldots, q_{\#(q)})$ labels a node of $T$. Second, assume that (ii), there exists a frontier node of $T$ labeled with some configuration $q(d_1, \ldots, d_{\#(q)})$ such that $q \notin Q \backslash F$. Since $I, w_D \models \forall y_1 \ldots \forall y_{\#(q)}. q(y) \rightarrow \bot$, by a similar reasoning as in the above case, we obtain that $\langle d_1, \ldots, d_{\#(q)} \rangle \notin I(q)$, contradiction.

“(2) $\Rightarrow$ (1)” Let $T$ be an accepting execution of $A$ over $w$. We prove that $I_T, w_D \models T(w_\Sigma)$. By Lemma 1 we obtain $I_T, w_D \models \Theta(w_\Sigma)$. Since every path in $T$ is of length $n$ and all nodes on the $n$-th level of $T$ are labeled by final configurations, we obtain that $I_T, w_D \models \forall q \in Q \backslash F. \forall y_1 \ldots \forall y_{\#(q)}. q(y) \rightarrow \bot$, trivially. \hfill $\Box$

As an immediate consequence, one can decide whether $A$ accepts some word $w$ with a given input sequence $w_\Sigma = \alpha$, by checking whether $T(\alpha)$ is satisfiable. However, unlike non-alternating infinite-state models of computation, such as counter automata (nondeterministic programs with integer variables), the satisfiability query for an acceptance (path) formula falls outside of known decidable theories, supported by standard SMT solvers. There are basically two reasons for this, namely (i) the presence of predicate symbols, and (ii) the non-trivial alternation of quantifiers. To understand this point, consider for example, the decidable theory of Presburger arithmetic \cite{23}. Adding even only one monadic predicate symbol to it yields undecidability in the presence of non-trivial quantifier alternation \cite{8}. However the quantifier-free fragment of Presburger arithmetic extended with predicate symbols can be shown to be decidable, using a Nelson-Oppen style congruence closure argument \cite{20}. To tackle this problem, we start from the observation that acceptance formulae have a particular form, which allows the elimination of path quantifiers and of predicates, by a couple of satisfiability-preserving transformations. The result of applying these transformations is a formula with no predicate symbols, whose only quantifiers are those introduced by the transition rules of the automaton, referred to as transition quantifiers. We shall further assume (§4) that the first order theory of the data sort $D$ has quantifier elimination, which allows to effectively decide the satisfiability of such formulae.

For the time being, let us formally define the elimination of transition quantifiers and predicates, respectively. Consider a given sequence of input events $\alpha = a_1 \ldots a_n$ and denote by $a_i$ the prefix $a_1 \ldots a_i$ of $\alpha$, for $i \in [n]$, where $a_0 = \epsilon$. 

\hfill

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Proof: (1) Trivial, since every subformula of the transition quantifiers in \( \Theta \) occurs in \( \Theta_{(\alpha_{i})} \) and, for all \( i \in [1, n] \):

\[
\Theta(\alpha_{i}) \equiv \Theta(\alpha_{i-1}) \land \bigwedge_{q(\alpha_{i-1}) \lor q(\alpha_{i})} q^{(i)}(t_{1}, \ldots, t_{\#(q)}) \rightarrow \psi_{0}[t_{1}/y_{1}, \ldots, t_{\#(q)}/y_{\#(q)}]
\]

We write \( \Theta(\alpha) \) for the prenex normal form of the formula:

\[
\Theta(\alpha_{n}) \land \bigwedge_{q(\alpha_{n}) \lor q(\alpha_{n+1})} q^{(n)}(t_{1}, \ldots, t_{\#(q)}) \rightarrow \perp.
\]

Observe that \( \Theta(\alpha) \) contains no path quantifiers, as required. On the other hand, the scope of the transition quantifiers in \( \Theta(\alpha) \) exceeds the right-hand side formulae from the transition rules, as shown by the following example.

**Example 1.** Consider the automaton \( \mathcal{A} = \langle [a_{1}, a_{2}], [x, q_{f}], i, \{q_{f}\}, \Delta \rangle \), where:

\[
i = \exists z \cdot z \geq 0 \land q(\alpha) + q(z)
\]

\[
\Delta = [q(y) \xrightarrow{a_{2}(\alpha)} x \geq 0 \land \forall z \cdot z \leq y \rightarrow q(x+z), q(y) \xrightarrow{a_{2}(\alpha)} y < 0 \land q_{f}(x+y)]
\]

For the input event sequence \( \alpha = a_{1}a_{2} \), the acceptance formula is:

\[
\Theta(\alpha) = \exists z \cdot z \geq 0 \land q^{(0)}(z) \land \\
\forall y \cdot q^{(0)}(0) \rightarrow [x^{(1)} \geq 0 \land \forall z \cdot z \geq y \rightarrow q^{(1)}(x^{(1)} + z)] \land \\
\forall y \cdot q^{(1)}(0) \rightarrow [y < 0 \land q^{(2)}(x^{(2)} + y)]
\]

The result of eliminating the path quantifiers, in prenex normal form, is shown below:

\[
\Theta(\alpha) = \exists z_{1} \forall z_{2} \cdot z_{1} \geq 0 \land q^{(0)}(z_{1}) \land \\
[q^{(0)}(z_{1}) \rightarrow x^{(1)} \geq 0 \land (z_{2} \geq z_{1} \rightarrow q^{(1)}(x^{(1)} + z_{2}))] \land \\
[q^{(1)}(x^{(1)} + z_{2}) \rightarrow x^{(1)} + z_{2} < 0 \land q^{(2)}(x^{(2)} + x^{(1)} + z_{2})]
\]

The next lemma establishes a formal relation between the satisfiability of an acceptance formula \( \Theta(\alpha) \) and that of the formula \( \Theta(\alpha) \), obtained by eliminating the path quantifiers from \( \Theta(\alpha) \).

**Lemma 3.** For any input event sequence \( \alpha = a_{1} \ldots a_{n} \) and each valuation \( \nu : X^{(\#(\alpha))} \rightarrow \mathbb{D} \), the following hold:

1. for all interpretations \( I \), if \( I, \nu \models \Theta(\alpha) \) then \( I, \nu \models \Theta(\alpha) \).
2. if there exists an interpretation \( I \) such that \( I, \nu \models \Theta(\alpha) \) then there exists an interpretation \( J \subseteq I \) such that \( J, \nu \models \Theta(\alpha) \).

**Proof:** (1) Trivial, since every subformula \( q(t_{1}, \ldots, t_{\#(q)}) \rightarrow \psi[t_{1}/y_{1}, \ldots, t_{\#(q)}/y_{\#(q)}] \) of \( \Theta(\alpha) \) is entailed by a subformula \( \forall y_{1} \ldots \forall y_{\#(q)} \cdot q(y_{1}, \ldots, y_{\#(q)}) \rightarrow \psi \) of \( \Theta(\alpha) \).

(2) By repeated applications of the following fact:
**Fact 1** Given formulae $\phi$ and $\psi$, such that no predicate atom with predicate symbol $q$ occurs in $\psi$, for each valuation $v$, if there exists an interpretation $I$ such that $I, v \models \phi \wedge \bigwedge_{q(t_1, \ldots, t_{n(q)}) \text{ occurs in } \phi} q(t_1, \ldots, t_{n(q)}) \rightarrow \psi[y_1, \ldots, y_{n(q)}]$ then there exists a valuation $J$ such that $J(q) \subseteq I(q)$ and $J(q') = I(q')$ for all $q' \in Q \setminus \{q\}$ and $J, v \models \phi \wedge \bigwedge_{q(t_1, \ldots, t_{n(q)}) \text{ occurs in } \phi} q(t_1, \ldots, t_{n(q)}) \rightarrow \psi$.

**Proof:** Assume w.l.o.g. that $\phi$ is quantifier free. The proof can be easily generalized to the case $\phi$ has quantifiers. Let $J(q) = \{(t'_1, \ldots, t'_{n(q)}) \in I(q) \mid q(t_1, \ldots, t_{n(q)}) \text{ occurs in } \phi\}$ and $J(q') = I(q')$ for all $q' \in Q \setminus \{q\}$. Since $I, v \models \phi$, we obtain that also $J, v \models \phi$ because the tuples of values in $I(q) \setminus J(q)$ are not interpretations of terms that occur within subformulae $q(t_1, \ldots, t_{n(q)})$ of $\phi$. Moreover, $\bigwedge_{q(t_1, \ldots, t_{n(q)}) \text{ occurs in } \phi} q(t_1, \ldots, t_{n(q)}) \rightarrow \psi[y_1, \ldots, y_{n(q)}]$ and $\bigwedge_{q(t_1, \ldots, t_{n(q)}) \text{ occurs in } \phi} q(t_1, \ldots, t_{n(q)})$ are equivalent under $J$, thus $J, v \models \psi[y_1, \ldots, y_{n(q)}] \rightarrow \psi$, as required. $\square$

This concludes the proof.

We proceed with the elimination of predicate atoms from $\overline{T}(\alpha)$, defined below.

**Definition 4.** Let $\overline{\Theta}(\alpha_0), \ldots, \overline{\Theta}(\alpha_n)$ be the sequence of formulae defined by $\overline{\Theta}(\alpha_0) \overset{\text{def}}{=} \iota^{(0)}$ and, for all $i \in [1, n]$, $\overline{\Theta}(\alpha_i)$ is obtained by replacing each occurrence of a predicate atom $q^{(i-1)}(t_1, \ldots, t_{n(q)})$ in $\overline{\Theta}(\alpha_{i-1})$ with the formula $\psi^{(i)}[t_1/y_1, \ldots, t_{n(q)}/y_{n(q)}]$, where $q(y) \overset{\alpha_i^{(x)}}{\rightarrow} \psi \in A$. We write $\overline{T}(\alpha)$ for the formula obtained by replacing, in $\overline{\Theta}(\alpha)$, each occurrence of a predicate $q^{(n)}$, such that $q \in Q \setminus F$ (resp. $q \in F$), by $\bot$ (resp. $\top$).

**Example 2 (Cond. from Example[7].** The result of the elimination of predicate atoms from the acceptance formula in Example[1] is shown below:

$$\overline{T}(\alpha) = \exists z_1 \forall z_2 . z_1 \geq 0 \wedge (z_2 + z_1 \rightarrow x^{(1)} + z_2 < 0)$$

Since this formula is unsatisfiable, by Lemma[5] below, no word $w$ with input event sequence $w_{\Sigma^2} = a_1a_2$ is accepted by the automaton $A$ from Example[1].

At this point, we prove the formal relation between the satisfiability of the formulae $\overline{T}(\alpha)$ and $\overline{T}(\alpha)$. Since there are no occurrences of predicates in $\overline{T}(\alpha)$, for each valuation $v : X^{(\infty)} \rightarrow \mathbb{D}$, there exists an interpretation $I$ such that $I, v \models \overline{T}(\alpha)$ if and only if $J, v \models \overline{T}(\alpha)$, for every interpretation $J$. In this case we omit $I$ and simply write $v \models \overline{T}(\alpha)$.

**Lemma 4.** For any input event sequence $\alpha = a_1 \ldots a_n$ and each valuation $v : X^{(\infty)} \rightarrow \mathbb{D}$, there exists a valuation $I$ such that $I, v \models \overline{T}(\alpha)$ if and only if $v \models \overline{T}(\alpha)$.

**Proof:** By induction on $n \geq 0$. The base case $n = 0$ is trivial, since $\overline{T}(\alpha) = \overline{T}(\alpha) = \iota^{(0)}$.

For the induction step, we rely on the following fact:

**Fact 2** Given formulae $\phi$ and $\psi$, such that $\phi$ is positive $q(t_1, \ldots, t_{n(q)})$ is the only one occurrence of the predicate symbol $q$ in $\phi$ and no predicate atom with predicate symbol $q$ occurs in $\psi(y_1, \ldots, y_{n(q)})$, for each interpretation $I$ and each valuation $v$, we have:

$$I, v \models \phi \wedge \bigwedge_{q(t_1, \ldots, t_{n(q)}) \text{ occurs in } \phi} q(t_1, \ldots, t_{n(q)}) \rightarrow \psi[y_1, \ldots, y_{n(q)}] \Leftrightarrow$$

$$v \models \phi[\psi[t_1/y_1, \ldots, t_{n(q)}/y_{n(q)}]/q(t_1, \ldots, t_{n(q)})]$$

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Proof: We assume w.l.o.g. that φ is quantifier-free. The proof can be easily generalized

to the case φ has quantifiers.

"⇒" We distinguish two cases:

- if \( \langle t'_1, \ldots, t'_{\#(q)} \rangle \in I(q) \) then \( I, \nu \models \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \). Since φ is positive, replacing

  \( q(t_1, \ldots, t_{\#(q)}) \) with \( \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \) does not change the truth value of φ

  under \( \nu \), thus \( \nu \models \phi[\psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}]/q(t_1, \ldots, t_{\#(q)})] \).

- else, \( \langle t'_1, \ldots, t'_{\#(q)} \rangle \notin I(q) \), thus \( \nu \models \phi[\bot/q(t_1, \ldots, t_{\#(q)})] \). Since φ is positive and \( \bot \) en-
tails \( \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \), we obtain \( \nu \models \phi[\psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}]/q(t_1, \ldots, t_{\#(q)})] \)

  by monotonicity.

"⇐" Let \( I \) be any interpretation such that \( I(q) = \{ \langle t'_1, \ldots, t'_{\#(q)} \rangle | \nu \models \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \} \).

We distinguish two cases:

- if \( I(q) \neq \emptyset \) then \( I, \nu \models q(t_1, \ldots, t_{\#(q)}) \) and \( \nu \models \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \). Thus replacing

  \( \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \) by \( q(t_1, \ldots, t_{\#(q)}) \) does not change the truth value of φ under

  \( I \) and \( \nu \), and we obtain \( I, \nu \models \phi \). Moreover, \( I, \nu \models \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \) implies

  \( I, \nu \models q(t_1, \ldots, t_{\#(q)}) \rightarrow \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \).

- else \( I(q) = \emptyset \), hence \( \nu \nmid \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \), thus \( \nu \models \phi[\bot/q(t_1, \ldots, t_{\#(q)})] \). Be-

  cause φ is positive, we obtain \( I, \nu \models \phi \) by monotonicity. But \( I, \nu \models q(t_1, \ldots, t_{\#(q)}) \rightarrow \psi[t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)}] \) trivially, because \( I, \nu \nmid q(t_1, \ldots, t_{\#(q)}) \).

This concludes the proof.

Finally, we define the acceptance of a word with a given input event sequence by
means of a formula in which no predicate atom occurs. As previously discussed, several
decidable theories, such as Presburger arithmetic, become undecidable if predicate
atoms are added to them. Therefore, the result below makes a step forward towards de-

ciding whether the automaton accepts a word with a given input sequence, by reducing
this problem to the satisfiability of a quantified formula without predicates.

Lemma 5. Given an automaton \( \mathcal{A} = (\Sigma, X, \mathcal{Q}, \mathcal{F}, \mathcal{I}, \mathcal{D}) \), for every word \( w \in \Sigma[X]^* \), we have \( w_0 \models \overline{T}(w_2) \) if and only if \( w \in \mathcal{L}(\mathcal{A}) \).

Proof: By Lemma[2] \( w \in \mathcal{L}(\mathcal{A}) \) if and only if \( I, w_0 \models T(w_2) \), for some interpretation
\( I \). By Lemma[3] there exists an interpretation \( I \) such that \( I, w_0 \models T(w_2) \) if and only if
there exists an interpretation \( \mathcal{I} \) such that \( \mathcal{I}, v \models \overline{T}(w_2) \). By Lemma[4] there exists an
interpretation \( \mathcal{I} \) such that \( \mathcal{I}, v \models T(w_2) \) if and only if \( v \models \overline{T}(w_2) \).

3.2 Closure Properties

Given a positive formula φ, we define the dual formula \( \phi^\sim \) recursively as follows:

\[
(\phi_1 \lor \phi_2)^\sim = \phi_1^\sim \land \phi_2^\sim \quad (\phi_1 \land \phi_2)^\sim = \phi_1^\sim \lor \phi_2^\sim \quad (t \approx s)^\sim = \neg (t \approx s)
\]

\[
(\exists x \ . \ \phi_1)^\sim = \forall x \ . \ \phi_1^\sim \quad (\forall x \ . \ \phi_1)^\sim = \exists x \ . \ \phi_1^\sim \quad (\neg (t \approx s))^\sim = t \approx s
\]

\[
(q(x_1, \ldots, x_{\#(q)}))^\sim = q(x_1, \ldots, x_{\#(q)})
\]

Observe that, because predicate atoms do not occur negated in φ, there is no need to
define dualization for formulae of the form \( \neg q(x_1, \ldots, x_{\#(q)}) \). The following theorem shows closure of automata under all boolean operations:
Theorem 1. Given automata $\mathcal{A}_i = (\Sigma, X, Q_i, t_i, F_i, A_i)$, for $i = 1, 2$, such that $Q_1 \cap Q_2 = \emptyset$, the following hold:

1. $L(\overline{\mathcal{A}}_i) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$, where $\overline{\mathcal{A}}_i = (\Sigma, X, Q_i, t_i \lor t_2, F_1 \lor F_2, A_1 \lor A_2)$.
2. $L(\mathcal{A}_i) = \Sigma[X]^* \setminus L(\mathcal{A}_i)$, where $\mathcal{A}_i = (\Sigma, X, Q_i, t_i \setminus F_i, A_i \setminus t_i)$ and, for $i = 1, 2$:

$$A_i = \{q(y) \overset{aX}{\longrightarrow} \psi \mid q(y) \overset{aX}{\longrightarrow} \psi \in A_i\}.$$ 
Moreover, $|\mathcal{A}_i| = O(|\mathcal{A}_1| + |\mathcal{A}_2|)$ and $|\overline{\mathcal{A}}_i| = O(|\mathcal{A}_1|)$, for all $i = 1, 2$.

Proof: \(1\) “$\subseteq$” Let $w \in L(\mathcal{A}_i)$ be a word and $T$ be an execution of $\mathcal{A}_i$ over $w$. Since $Q_1 \cap Q_2 = \emptyset$, it is possible to partition $T$ into $T_1$ and $T_2$ such that the roots of $T_i$ form a cube from $c(\{1_i\})^\mu$, for all $i = 1, 2$. Because $A_1 \cap A_2 = \emptyset$, by induction on $|w| \geq 0$, one shows that $T_i$ is an execution of $\mathcal{A}_i$ over $w$, for all $i = 1, 2$. Finally, because $T$ is accepting, we obtain that $T_1$ and $T_2$ are accepting, respectively, hence $w \in L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$. “$\supseteq$” Let $w \in L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ and let $T_i$ an accepting execution of $\mathcal{A}_i$ over $w$, for all $i = 1, 2$. We show that $T_1 \cup T_2$ is an execution of $\mathcal{A}_i$ over $w$, by induction on $|w| \geq 0$. For the base case $|w| = 0$, we have $T_i \in c(\{1_i\})^\mu$ for all $i = 1, 2$ and since $Q_1 \cap Q_2 = \emptyset$, we have $T_1 \cup T_2 \in c(\{1_1 \lor 1_2\})^\mu$. The induction step follows as a consequence of the fact that $A_1 \cup A_2$ is the set of transition rules of $\mathcal{A}_i$. Finally, since both $T_1$ and $T_2$ are accepting, $T_1 \cup T_2$ is accepting as well. Moreover, we have:

$$|\mathcal{A}_i| = |t_1 \lor t_2| + \sum_{q(y) \overset{aX}{\longrightarrow} \psi \in A_1 \cup A_2} |\psi| = 1 + |t_1| + |t_2| + \sum_{q(y) \overset{aX}{\longrightarrow} \psi \in A_1} |\psi| + \sum_{q(y) \overset{aX}{\longrightarrow} \psi \in A_2} |\psi|.$$ 

\(2\) Let $w \in \Sigma[X]^*$ be a word. We denote by $\overline{T}_{\mathcal{A}_1}(w_2)$ and $\overline{T}_{\mathcal{A}_1}(w_2)$ [resp. $\overline{T}_{\mathcal{A}_1}(w_2)$ and $\overline{T}_{\mathcal{A}_1}(w_2)$] the formulae $T(w_2)$ and $T(w_2)$ for $\mathcal{A}_1$ and $\overline{\mathcal{A}}_1$, respectively. It is enough to show that $\overline{T}_{\mathcal{A}_1}(w_2) = \neg \overline{T}_{\mathcal{A}_1}(w_2)$ and apply Lemma\(5\) to prove that $w \in L(\mathcal{A}_1) \Leftrightarrow w \notin L(\overline{\mathcal{A}}_1)$. Since the choice of $w$ was arbitrary, this proves $L(\overline{\mathcal{A}}_1) = \Sigma[X]^* \setminus L(\mathcal{A}_1)$. By induction on the number of predicate atoms in $\overline{T}_{\mathcal{A}_1}(w_2)$ that are replaced during the generation of $\overline{T}_{\mathcal{A}_1}(w_2)$. The proof relies on the following fact:

Fact 3 Let $\phi$ be a positive formula and let $q(t_1, \ldots, t_{\#(q)})$ be the only occurrence of a predicate symbol within $\phi$. Then, every formula $\phi$ with no predicate occurrences: $\neg \phi[q(t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)})/q(t_1, \ldots, t_{\#(q)})] \equiv \phi '$ \(\neg \phi[q(t_1/y_1, \ldots, t_{\#(q)}/y_{\#(q)})/q(t_1, \ldots, t_{\#(q)})]\).

Proof: By induction on the structure of $\phi$. 

\[\square\]

4 The Emptiness Problem

The problem of checking emptiness of a given automaton is undecidable, even for automata with predicates of arity two, whose transition rules use only equalities and disequalities, having no transition quantifiers\(4\). Since even such simple classes of alternating automata have no general decision procedure for emptiness, we use an abstraction-refinement semi-algorithm based on lazy annotation\(18\),\(19\). 

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In a nutshell, a lazy annotation procedure systematically explores the set of execution paths (in our case, sequences of input events) in search of an accepting execution. Each path has a corresponding path formula that defines all words accepted along that path. If the path formula is satisfiable, the automaton accepts a word. Otherwise, the path is said to be spurious. When a spurious path is encountered, the search backtracks and the path is annotated with a set of learned facts, that marks this path as infeasible. The semi-algorithm uses moreover a coverage relation between paths, ensuring that the continuations of already covered paths are never explored. Sometimes this coverage relation provides a sound termination argument, when the automaton is empty.

We check emptiness of first order alternating automata using a version of the IMPACT lazy annotation semi-algorithm [18]. An analogous procedure is given in [11], for a simpler model of alternating automata, that uses only predicates or arity zero (booleans) and no transition quantifiers. For simplicity, we do not present the details of this algorithm and shall content ourselves of several high-level definitions.

Given a finite input event alphabet \( \Sigma \), for two sequences \( \alpha, \beta \in \Sigma^* \), we say that \( \alpha \) is a prefix of \( \beta \), written \( \alpha \preceq \beta \), if \( \alpha = \beta \gamma \) for some sequence \( \gamma \in \Sigma^* \). A set \( S \) of sequences is:

1. **prefix-closed** if for each \( \alpha \in S \), if \( \beta \preceq \alpha \) then \( \beta \in S \), and
2. **complete** if for each \( \alpha \in S \), there exists \( a \in \Sigma \) such that \( \alpha a \in S \) if and only if \( \alpha b \in S \) for all \( b \in \Sigma \).

Observe that a prefix-closed set is the backbone of a tree whose edges are labeled with input events. If the set is, moreover, complete, then every node of the tree has either zero successors, in which case it is called a leaf, or it has a successor edge labeled with \( a \) for each input event \( a \in \Sigma \).

**Definition 5.** An **unfolding** of an automaton \( \mathcal{A} = (\Sigma, X, Q, t, F, \Delta) \) is a finite partial mapping \( U : \Sigma^* \rightarrow \text{Form}^+ (Q, \emptyset) \), such that:

1. \( \text{dom}(U) \) is a finite prefix-closed complete set,
2. \( U(\varepsilon) = \iota \), and
3. for each sequence \( \alpha a \in \text{dom}(U) \), such that \( \alpha \in \Sigma^* \) and \( a \in \Sigma \):

\[
U(\alpha)^{(0)} \land \bigwedge_{q(y) \in (X)} y_1 \ldots y_q \cdot q^{(0)}(y) \rightarrow \psi^{(1)} \models U(\alpha a)^{(1)}
\]

Moreover, \( U \) is **safe** if for each \( \alpha \in \text{dom}(U) \), the formula \( U(\alpha) \land \bigwedge_{q \in Q \setminus F} y_1 \ldots y_{q(a)} \cdot q(y) \rightarrow \bot \) is unsatisfiable.

Lazy annotation semi-algorithms [18,19] build unfoldings of automata trying to discover counterexamples for emptiness. If the automaton \( \mathcal{A} \) in question is non-empty, a systematic enumeration of the input event sequences \( \Sigma^* \) will suffice to discover a word \( w \in \mathcal{L}(\mathcal{A}) \), provided that the first order theory of the data domain \( \mathbb{D} \) is decidable (Lemma 2). However, if \( \mathcal{L}(\mathcal{A}) = \emptyset \), the enumeration of input event sequences may, in principle, run forever. The typical way of fighting this divergence problem is to define a coverage relation between the nodes of the unfolding tree.

\[\text{For instance, using breadth-first search.}\]
Definition 6. Given an unfolding $U$ of an automaton $A = \langle \Sigma, X, Q, \tau, F, A \rangle$ a node $\alpha \in \text{dom}(U)$ is covered by another node $\beta \in \text{dom}(U)$, denoted $\alpha \subseteq \beta$, if and only if there exists a node $\alpha' \preceq \alpha$ such that $U(\alpha') \models U(\beta)$. Moreover, $U$ is closed if and only if every leaf from $\text{dom}(U)$ is covered by an uncovered node.

A lazy annotation semi-algorithm will stop and report emptiness provided that it succeeds in building a closed and safe unfolding of the automaton. Notice that, by Definition 6 for any three nodes of an unfolding $U$, say $\alpha, \beta, \gamma \in \text{dom}(U)$, if $\alpha < \beta$ and $\alpha \subseteq \gamma$, then $\beta \subseteq \gamma$ as well. As we show next (Theorem 2), there is no need to expand covered nodes, because, intuitively, there exists a word $w \in \mathcal{L}(A)$ such that $\alpha \preceq w_2$ and $\alpha \subseteq \gamma$ only if there exists another word $u \in \mathcal{L}(A)$ such that $\gamma \preceq w_2$. Hence, exploring only those input event sequences that are continuations of $\gamma$ (and ignoring those of $\alpha$) suffices in order to find a counterexample for emptiness, if one exists.

An unfolding node $\alpha \in \text{dom}(U)$ is said to be spurious if and only if $\mathcal{Y}(\alpha)$ is unsatisfiable. In this case, we change (refine) the labels of (some of the) prefixes of $\alpha$ (and that of $\gamma$), such that $U(\alpha)$ becomes $\bot$, thus indicating that there is no real execution of the automaton along that input event sequence. As a result of the change of labels, if a node $\gamma \preceq \alpha$ used to cover another node from $\text{dom}(U)$, it might not cover it with the new label. Therefore, the coverage relation has to be recomputed after each refinement of the labeling. The semi-algorithm stops when (and if) a safe complete unfolding has been found. For a detailed presentation of the emptiness procedure, we refer to [11].

Theorem 2. If an automaton $A$ has a nonempty safe closed unfolding then $\mathcal{L}(A) = \emptyset$.

Proof: Let $U$ be a safe and complete unfolding of $A$, such that $\text{dom}(U) \neq \emptyset$. Suppose, by contradiction, that there exists a word $w \in \mathcal{L}(A)$ and let $\alpha \equiv w_2$. Since $w \in \mathcal{L}(A)$, by Lemma 2 there exists an interpretation $I$ such that $I, w_2 \models \mathcal{Y}(\alpha)$. Assume first that $\alpha \in \text{dom}(U)$. In this case, one can show, by induction on the length $n \geq 0$ of $w$, that $\Theta(\alpha) \models U(\alpha)^{(n)}$, thus $I, w_2 \models U(\alpha)^{(n)}$. Since $I, w_2 \models \mathcal{Y}(\alpha)$, we have $I, w_2 \models \bigwedge_{q \in Q} \forall y_1 \ldots \forall y_{\#(q)} . q^{(n)}(y) \rightarrow \bot$, hence $U(\alpha)^{(n)} \wedge \bigwedge_{q \in Q} \forall y_1 \ldots \forall y_{\#(q)} . q^{(n)}(y) \rightarrow \bot$. By renaming $q^{(n)}$ with $q$ in the previous formula, we obtain $U(\alpha) \wedge \forall y_1 \ldots \forall y_{\#(q)} . q(y) \rightarrow \bot$ is satisfiable, thus $U$ is not safe, contradiction.

We proceed thus under the assumption that $\alpha \notin \text{dom}(U)$. Since $\text{dom}(U)$ is a nonempty prefix-closed set, there exists a strict prefix $\alpha'$ of $\alpha$ that is a leaf of $\text{dom}(U)$. Since $U$ is closed, the leaf $\alpha'$ must be covered and let $\alpha_1 \preceq \alpha' \preceq \alpha$ be a node such that $U(\alpha_1) \models U(\beta_1)$, for some uncovered node $\beta_1 \in \text{dom}(U)$. Let $\gamma_1$ be the unique sequence such that $\alpha_1 \gamma_1 = \alpha$. By Definition 6 since $\alpha_1 \subseteq \beta_1$ and $w_2 = \alpha_1 \gamma_1 \in \mathcal{L}(A)$, there exists a word $w_1$ and a cube $c_1 \in \mathcal{C}[[U(\alpha_1)]] \subseteq \mathcal{C}[[U(\beta_1)]]$, such that $w_1 \#(c) = \gamma_1$ and $A$ accepts $w_1$ starting with $c_1$. If $\beta_1 \gamma_1 \in \text{dom}(U)$, we obtain a contradiction by a similar argument as above. Hence $\beta_1 \gamma_1 \notin \text{dom}(U)$ and there exists a leaf of $\text{dom}(U)$ which is also a prefix of $\beta_1 \gamma_1$. Since $U$ is closed, this leaf is covered by an uncovered node $\beta_2 \in \text{dom}(U)$ and let $\alpha_2 \in \text{dom}(U)$ be the minimal (in the prefix partial order) node such that $\beta_1 \preceq \alpha_2 \preceq \beta_1 \gamma_1 \gamma_2$. Let $\gamma_2$ be the unique sequence such that $\alpha_2 \gamma_2 = \beta_1 \gamma_1$. Since $\beta_1$ is uncovered, we have $\beta_1 \neq \alpha_2$ and thus $|\gamma_1| > |\gamma_2|$. By repeating the above rea-
soning for $\alpha_2, \beta_2$ and $\gamma_2$, we obtain an infinite sequence $|\gamma_1| > |\gamma_2| > \ldots$, which is again a contradiction. □

As mentioned above, we check emptiness of first order alternating automata using the same method previously used to check emptiness of a simpler model of alternating automata, which uses boolean constants for control states and whose transition rules have no quantifiers [11]. The higher complexity of the automata model considered here, manifests itself within the interpolant generation procedure, used to refine the labeling of the unfolding. We discuss generation of interpolants in the next section.

5 Interpolant Generation

Typically, when checking the unreachability of a set of program configurations [18], the interpolants used to annotate the unfolded control structure are assertions about the values of the program variables in a given control state, at a certain step of an execution. However, in an alternating model of computation, it is useful to distinguish between (i) locality of interpolants w.r.t. a given control state (control locality) and (ii) locality w.r.t. a given time stamp (time locality). In logical terms, control-local interpolants are defined by formulae involving a single predicate symbol, whereas time-local interpolants involve only predicates $q^0$ and variables $x^i$, for a single $i \geq 0$.

Remark When considering an alternating model of computation, control-local interpolants are not always enough to prove emptiness, because of the synchronization of several branches of the computation on the same sequence of input values. Consider, for instance, an automaton with the following transition rules and final state $q_f$:

- $q_0(y) \xrightarrow{a,k_1} q_1(y + x) \land q_2(y - x)$
- $q_1(y) \xrightarrow{a,k_1} y + x > 0 \land q_f$
- $q_2(y) \xrightarrow{a,k_1} y - x > 0 \land q_f$

Started in an initial configuration $q_0(0)$ with an input word $(a, v_1 \ldots (a, v_{n-1})(a, v_n)$, such that $v_i(x) = k_i$, the automaton executes as follows:

- $q_0(0) \xrightarrow{(a,x_1)} \{q_1(k_1), q_2(-k_1)\} \ldots \xrightarrow{(a,x_{n-1})} \{q_1(\sum_{i=1}^{n-1} k_i), q_2(-\sum_{i=1}^{n-1} k_i)\} \xrightarrow{(a,x_n)} \emptyset$

An overapproximation of the set of cubes generated after one or more steps is defined by the formula: $\exists x_1 \exists x_2 \ldots q_1(x_1) \land q_2(x_2) \land x_1 + x_2 \approx 0$. Observe that a control-local formula using one occurrence of a predicate would give a too rough overapproximation of this set, unable to prove the emptiness of the automaton. □

First, let us give the formal definition of the class of interpolants we shall work with. Given a formula $\phi$, the vocabulary of $\phi$, denoted $V(\phi)$ is the set of predicate symbols $q \in Q^0$ and variables $x \in X^0$, occurring in $\phi$, for some $i \geq 0$. For a term $t$, its vocabulary $V(t)$ is the set of variables that occur in $t$. Observe that quantified variables and the interpreted function symbols of the data theory [11] do not belong to the vocabulary of a formula. By $P^*(\phi) [P^*(\phi)]$ we denote the set of predicate symbols that occur in $\phi$ under an even [odd] number of negations.

3 E.g., the arithmetic operators of addition and multiplication, when $\mathbb{D}$ is the set of integers.
Definition 7 (L7). Given formulae φ and ψ such that φ ∧ ψ is unsatisfiable, a Lyndon interpolant is a formula I such that φ ⊨ I, the formula I ∧ ψ is unsatisfiable, V(I) ⊆ V(φ) ∩ V(ψ), P⁺(I) ⊆ P⁺(φ) ∩ P⁺(ψ) and P⁻(I) ⊆ P⁻(φ) ∩ P⁻(ψ).

In the rest of this section, let us fix an automaton A = (Σ, X, Q, t, F, Δ). Due to the above observation, none of the interpolants considered will be control-local and we shall use the term local to denote time-local interpolants, with no free variables.

Definition 8. Given a non-empty sequence of input events α = α₁...αₙ ∈ Σ*, a generalized Lyndon interpolant (GLI) is a sequence (I₀,...,Iₙ) of formulae such that, for all k ∈ [n−1]:
1. P⁻(Iₖ) = ∅,
2. t⁽ⁱ⁾(I₀) I₀ ∧ Iₖ \big( \bigwedge_{q ∈ Q,F} q⁻(y) \bigwedge_{φ ∈ d} \forall y \exists yiq(φ) . q⁽ⁱ⁾(y) → φ⁽ⁱ⁾(y) \big) ⊨ Iₖ₊₁,
3. Iₙ \bigwedge_{q ∈ Q,F} q⁻(y) \bigwedge_{φ ∈ d} \exists y \forall yiq(φ) . q⁽ⁿ⁾(y) is unsatisfiable.
Moreover, the GLI is local if and only if V(Iₖ) ⊆ Q⁽ⁱ⁾, for all k ∈ [n].

The following proposition states the existence of local GLI for the theories in which Lyndon’s Interpolation Theorem holds.

Proposition 1. If there exists a Lyndon interpolant for any two formulae φ and ψ, such that φ ∧ ψ is unsatisfiable, then any sequence of input events α = α₁...αₙ ∈ Σ*, such that T(α) is unsatisfiable, has a local GLI (I₀,...,Iₙ).

Proof: By definition, T(α) is the formula:

\[ t⁽ⁱ⁾(α) ∧ \bigwedge_{i=1}^{n} \bigwedge_{q(y) ∈ d} q⁽⁽ⁱ⁾⁻(y)} \bigwedge_{φ ∈ d} \forall y \exists yiq(φ) . q⁽⁽ⁱ⁾⁺(y) \rightarrow \psi⁽ⁱ⁾(y) \bigwedge_{q ∈ Q,F} q⁻(y) \rightarrow \bot \]

We define the formulae:

\[ \varphi_i \stackrel{\text{def}}{=} \bigwedge_{q(y) ∈ d} q⁽⁽ⁱ⁾⁻(y)} \bigwedge_{φ ∈ d} \forall y \exists yiq(φ) . q⁽⁽ⁱ⁾⁺(y) \rightarrow \psi⁽ⁱ⁾(y) \text{, for all } i \in [1,n] \]

\[ \psi \stackrel{\text{def}}{=} \bigwedge_{q ∈ Q,F} q⁻(y) \rightarrow \bot \]

Observe that V(t⁽ⁱ⁾(α) ⊆ Q⁽ⁿ⁾, V(φᵢ) ⊆ Q⁽⁽ⁱ⁾⁻∪Q⁽ⁱ⁾⁺∪X⁽ⁱ⁾, for all i ∈ [1,n], and V(ψ) ⊆ Q⁽ⁿ⁾. We apply Lyndon’s Interpolation Theorem for the formulæ t⁽ⁱ⁾(α) and \bigwedge_{i=1}^{n} \varphi_i ∧ ψ and obtain a formula I₀, such that t⁽ⁱ⁾(α) \models I₀, I₀ \land \bigwedge_{i=1}^{n} \varphi_i ∧ ψ is unsatisfiable, V(I₀) ⊆ V(t⁽ⁱ⁾(α)) \cap (\bigcup_{i=1}^{n} V(φ_i) \cup V(ψ)) ⊆ Q⁽ⁿ⁾ and P⁻(I₀) ⊆ P⁻(t⁽ⁱ⁾(α)) \cap (\bigcup_{i=1}^{n} P⁻(φ_i) \cup P⁻(ψ)) = ∅. Repeating the reasoning for the formulæ I₀ ∧ φ₁ and \bigwedge_{i=2}^{n} \varphi_i ∧ ψ, we obtain I₁, such that I₀ ∧ φ₁ \models I₁, I₁ \land \bigwedge_{i=2}^{n} \varphi_i ∧ ψ is unsatisfiable, V(I₁) \subseteq (V(I₀) \cup V(φ₁)) \cap (\bigcup_{i=2}^{n} V(φ_i) \cup V(ψ)) ⊆ Q⁽ⁿ⁾ and P⁻(I₁) \subseteq (P⁻(I₀) \cup P⁻(φ₁)) \cap (\bigcup_{i=2}^{n} P⁻(φ_i) \cup P⁻(ψ)) = ∅. Continuing in this way, we obtain formulæ I₀, I₁,...,Iₙ as required.

The main problem with the local GLI construction described in the proof of Proposition 1 is that the existence of Lyndon interpolants (Definition 7) is guaranteed in principle, but the proof is non-constructive. Building an interpolant for an unsatisfiable conjunction of formulæ φ ∧ ψ is typically the job of the decision procedure that proves
the unsatisfiability and, in general, there is no such procedure, when \( \phi \) and \( \psi \) contain predicates and have non-trivial quantifier alternation. In this case, some provers use instantiation heuristics for the universal quantifiers that are sufficient for proving unsatisfiability, however these heuristics are not always suitable for interpolant generation. Consequently, from now on, we assume the existence of an effective Lyndon interpolation procedure only for decidable theories, such as the quantifier-free linear (integer) arithmetic with uninterpreted functions (UFLIA, UFLRA, etc.) [23].

This is where the predicate-free path formulae (Definition 4) come into play. For a given event sequence \( \alpha \), the automaton \( \mathcal{A} \) accepts a word \( w \) such that \( w_2 = \alpha \) if and only if \( \overline{T}(\alpha) \) is satisfiable. Assuming further that the equality atoms in the transition rules of \( \mathcal{A} \) are written in the language of a decidable first order theory, such as Presburger arithmetic, Lemma 5 gives us an effective way of checking emptiness of \( \mathcal{A} \), relative to a given event sequence. However, this method does not cope well with lazy annotation, because there is no way to extract, from the unsatisfiability proof of \( \overline{T}(\alpha) \), the interpolants needed to annotate \( \alpha \). This is because (i) the formula \( \overline{T}(\alpha) \), obtained by repeated substitutions (Definition 4) loses track of the steps of the execution, and (ii) quantifiers that occur nested in \( \overline{T}(\alpha) \) make it difficult to write \( \overline{T}(\alpha) \) as an unsatisfiable conjunction of formulae from which interpolants are extracted (Definition 7).

The solution we adopt for the first issue (i) consists in partially recovering the timestamped structure of the acceptance formula \( T(\alpha) \) using the formula \( \overline{T}(\alpha) \), in which only transition quantifiers occur. The second issue (ii) is solved under the additional assumption that the theory of the data domain \( D \) has witness-producing quantifier elimination. More precisely, we assume that, for each formula \( \exists x . \phi(x) \), there exists an effectively computable term \( \tau \), in which \( x \) does not occur, such that \( \exists x . \phi[\tau/x] \) is equisatisfiable. These terms, called witness terms in the following, are actual definitions of the Skolem function symbols from the following folklore theorem:

**Theorem 3** [21]. Given \( Q_1 x_1 \ldots Q_n x_n . \phi \) a first order sentence, where \( Q_1, \ldots, Q_n \in \{ \exists, \forall \} \) and \( \phi \) is quantifier-free, let \( \eta_i \overset{\text{def}}{=} f_i(y_1, \ldots, y_{k_i}) \) if \( Q_i = \exists \) and \( \eta_i \overset{\text{def}}{=} x_i \) if \( Q_i = \forall \), where \( f_i \) is a fresh function symbol and \( \{ y_1, \ldots, y_{k_i} \} = \{ x_j \mid j < i, Q_j = \exists \} \). Then the entailment \( Q_1 x_1 \ldots Q_n x_n . \phi \models \forall \phi[\eta_1/x_1, \ldots, \eta_n/x_n] \) holds.

**Proof**: See [2] Theorem 2.1.8] and [2] Lemma 2.1.9.  

Examples of witness-producing quantifier elimination procedures can be found in the literature for e.g. linear integer (real) arithmetic (LIA, LRA), Presburger arithmetic and boolean algebra of sets and Presburger cardinality constraints (BAPA) [16].

Under the assumption that witness terms can be effectively built, let us describe the generation of a non-local GLI for a given input event sequence \( \alpha = a_1 \ldots a_n \). First, we generate successively the acceptance formula \( T(\alpha) \) and its equisatisfiable forms \( T(\alpha) = Q_1 x_1 \ldots Q_m x_m . \Phi \) and \( \overline{T}(\alpha) = Q_1 x_1 \ldots Q_m x_m . \overline{\Phi} \), both written in prenex form, with matrices \( \Phi \) and \( \overline{\Phi} \), respectively. Because we assumed that the first order theory of \( D \) has quantifier elimination, the satisfiability problem for \( \overline{T}(\alpha) \) is decidable. If \( \overline{T}(\alpha) \) is satisfiable, we build a counterexample for emptiness \( w \) such that \( w_2 = \alpha \) and \( w_2 \) is a satisfying assignment for \( T(\alpha) \). Otherwise, \( \overline{T}(\alpha) \) is unsatisfiable and there exist witness terms \( \tau_{i_1} \ldots \tau_{i_\ell} \), where \( \{ i_1, \ldots, i_\ell \} = \{ j \in [1, m] \mid Q_j = \forall \} \), such that \( \forall \tau_{i_1/x_{i_1}} \ldots \tau_{i_\ell/x_{i_\ell}} \)
is unsatisfiable (Theorem 3). Then it turns out that the formula $\overline{\Phi}[\tau_i/x_i, \ldots, \tau_l/x_l]$, obtained analogously from the matrix of $\overline{T}(\alpha)$, is unsatisfiable as well (Lemma 6). Because this latter formula is structured as a conjunction of formulae $\varphi^0 \land \Phi_1 \land \ldots \land \Phi_m \land \psi$, where $\forall \varphi \in \Phi = Q^{\leq 0} \cup Q^{\leq k}$ and $\forall (\psi) \in Q^{\leq 0} \cup Q^{\leq k}$, it is now possible to use an existing interpolation procedure for the quantifier-free theory of $\mathbb{D}$, extended with uninterpreted function symbols, to compute a sequence of non-local GLI $(I_0, \ldots, I_n)$ such that $\forall (I_k) \in Q^{\leq 0} \cup Q^{\leq k}$, for all $k \in [n]$.

Example 3 (Condt. from Examples 7 and 2). The formula $\overline{T}(\alpha)$ (Example 2) is unsatisfiable and let $\tau_2 = z_1$ be the witness term for the universally quantified variable $z_2$. Replacing $z_2$ with $\tau_2$ in the matrix of $\overline{T}(\alpha)$ (Example 1) yields the unsatisfiable conjunction:

\[
z_1 \geq 0 \land q^{y_0}(z_1) \land q^{y_0}(z_1) \rightarrow x^{(1)} \geq 0 \land (z_1 \geq z_1 \rightarrow q^{(1)}(x^{(1)} + z_1)) \land q^{(1)}(x^{(1)} + z_1) \rightarrow x^{(1)} + z_1 < 0 \land q^{y_1}(x^{(1)} + z_1 + z_1)
\]

A non-local GLI for the above is $(q^{y_0}(z_1) \land z_1 \geq 0, x^{(1)} \geq 0 \land q^{y_0}(x^{(1)} + z_1) \land z_1 \geq 0, \perp)$. ■

A function $\xi : \mathbb{N} \rightarrow \mathbb{N}$ is [strictly] monotonic iff for each $n < m$ we have $\xi(n) \leq \xi(m)$ if $\xi(n) < \xi(m)$ and finite-range iff for each $n \in \mathbb{N}$ the set $\{m \mid \xi(m) = n\}$ is finite. If $\xi$ is finite-range, we denote by $\xi^{-1}(n) \in \mathbb{N}$ the maximal value $m$ such that $\xi(m) = n$. The lemma below gives the proof of correctness for the construction of non-local GLI.

Lemma 6. Given a non-empty input event sequence $\alpha = a_1 \ldots a_n \in \Sigma^*$, such that $T(\alpha)$ is unsatisfiable, let $Q_1 x_1 \ldots Q_m x_m \cdot \overline{\Phi}$ be a prenex form of $T(\alpha)$ and let $\xi : [1, m] \rightarrow [n]$ be a monotonic function mapping each transition quantifier to the minimal index from the sequence $\Theta(\alpha_1), \ldots, \Theta(\alpha_n)$ where it occurs. Then one can effectively build:

1. witness terms $\tau_{i_1}, \ldots, \tau_{i_\ell}$ where $\{i_1, \ldots, i_\ell\} = \{j \in [1, m] \mid Q_j = \forall\}$ and $\forall (\tau_{i_\ell}) \subseteq X^{y_0(i_\ell)} \cup \{x_k \mid k < i_\ell, Q_k = \exists\}$, $\forall j \in [1, \ell]$ such that $\overline{\Phi}[\tau_{i_1}/x_{i_1}, \ldots, \tau_{i_\ell}/x_{i_\ell}]$ is unsatisfiable, and
2. a GLI $(I_0, \ldots, I_n)$ for $\alpha$, such that $\forall (I_k) \subseteq Q^{y_0} \cup X^{y_0} \cup \{x_j \mid j < \xi^{-1}(k), Q_j = \exists\}$, for all $k \in [n]$.

Proof: If $T(\alpha)$ is unsatisfiable, by Lemmas 3 and 4 we obtain that, successively $\overline{T}(\alpha)$ and $\overline{T}(\alpha)$ are unsatisfiable. Let $Q_1 x_1 \ldots Q_m x_m \cdot \overline{\Phi}$ and $Q_1 x_1 \ldots Q_m x_m \cdot \overline{\Phi}$ be prenex forms for $T(\alpha)$ and $\overline{T}(\alpha)$, respectively. Since we assumed that the first order theory of the data domain has witness-producing quantifier elimination, using Theorem 3 one can effectively build witness terms $\tau_{i_1}, \ldots, \tau_{i_\ell}$, where $\{i_1, \ldots, i_\ell\} = \{i \in [1, m] \mid Q_i = \forall\}$ and:

- $\forall (\tau_{i_\ell}) \subseteq X^{y_0(i_\ell)} \cup \{x_k \mid k < i_\ell, Q_k = \exists\}$, for all $j \in [1, \ell]$ and
- $\overline{\Phi}[\tau_{i_1}/x_{i_1}, \ldots, \tau_{i_\ell}/x_{i_\ell}]$ is unsatisfiable.

Let $\Phi_0, \ldots, \Phi_n$ be the sequence of quantifier-free formulae, defined as follows:

- $\Phi_0$ is the matrix of some prenex form of $\varphi^0$,
- for all $i = 1, \ldots, n$, let $\Phi_i$ be the matrix of some prenex form of:

\[
\Phi_i \equiv \Phi_{i-1} \land \bigwedge_{q^{(i-1)}(t_1, \ldots, t_{#(\alpha)}) \text{ occurs in } \Phi_{i-1}} q^{(i-1)}(t_1, \ldots, t_{#(\alpha)}) \rightarrow \psi^{(i)}[\tau_1/y_1, \ldots, t_{#(\alpha)}/y_{#(\alpha)}] \land q(y_1, \ldots, y_{#(\alpha)}) \rightarrow \psi \in \mathbb{D}
\]

\[
\Phi_i \equiv \Phi_{i-1} \land \bigwedge_{q^{(i-1)}(t_1, \ldots, t_{#(\alpha)}) \text{ occurs in } \Phi_{i-1}} q^{(i-1)}(t_1, \ldots, t_{#(\alpha)}) \rightarrow \psi^{(i)}[\tau_1/y_1, \ldots, t_{#(\alpha)}/y_{#(\alpha)}] \land q(y_1, \ldots, y_{#(\alpha)}) \rightarrow \psi \in \mathbb{D}
\]
It is easy to see that \( \hat{\Phi} \) is the matrix of some prenex form of:

\[
\hat{\Phi}_n \land \bigwedge_{q \in Q \setminus F} q^{n}(t_1, \ldots, t_{|q|}) \Rightarrow \bot
\]

Applying the equivalence from Fact \(^2\) in the proof of Lemma \(^4\), we obtain a sequence of quantifier-free formulae \( \overline{\Phi}_0, \ldots, \overline{\Phi}_n \) such that \( \overline{\Phi}_i \equiv \hat{\Phi}_i \), for all \( i \in [n] \) and \( \overline{\Phi} \) is obtained from \( \overline{\Phi}_n \) by replacing each occurrence of a predicate atom \( q(t_1, \ldots, t_{|q|}) \) in \( \overline{\Phi}_n \) by \( \bot \) if \( q \in Q \setminus F \) and by \( \top \) if \( q \in F \). Clearly \( \overline{\Phi} \equiv \hat{\Phi} \), thus \( \overline{\Phi}[t_1/x_1, \ldots, t_i/x_i] \equiv \hat{\Phi}[t_1/x_1, \ldots, t_i/x_i] \equiv \bot \).

\((\text{2})\) With the notation introduced at point \((1)\), we have \( \hat{\Phi} = \hat{\Phi}_0 \land \bigwedge_{i=1}^n \phi_i \land \psi \). Consider the sequence of witness terms \( t_1, \ldots, t_i \), whose existence is proved by point \((1)\). Because \( V(t_i) \subseteq X^{(i)} \cup \{ x_j \mid k < i, Q_k = \exists \} \), for all \( j \in [1, l] \), and moreover \( \varepsilon^{-1} \) is strictly monotonic, we obtain:

- \( V(\hat{\Phi}_0)[t_1/x_1, \ldots, t_i/x_i] \subseteq Q^{(0)} \cup X^{(0)} \cup \{ x_j \mid j < \varepsilon_{\max}(0), Q_j = \exists \} \),
- \( V(\phi_i)[t_1/x_1, \ldots, t_i/x_i] \subseteq Q^{(i)} \cup X^{(i)} \cup \{ x_j \mid j < \varepsilon_{\max}(i), Q_j = \exists \} \), for all \( i \in [1, n] \),
- \( V(\psi)[t_1/x_1, \ldots, t_i/x_i] \subseteq Q^{(0)} \cup X^{(0)} \cup \{ x_j \mid j \in [1, m], Q_j = \exists \} \).

By repeatedly applying Lyndon’s Interpolation Theorem, we obtain a sequence of formulae \( (I_0, \ldots, I_n) \) such that:

- \( \hat{\Phi}_0[\tau_1/x_1, \ldots, \tau_i/x_i] \models I_0 \) and \( V(I_0) \subseteq Q^{(0)} \cup X^{(0)} \cup \{ x_j \mid j < \varepsilon_{\max}(0), Q_j = \exists \} \),
- \( I_{k-1} \land \phi_i[\tau_1/x_1, \ldots, \tau_i/x_i] \models I_k \) and \( V(I_k) \subseteq Q^{(k)} \cup X^{(k)} \cup \{ x_j \mid j < \varepsilon_{\max}(k), Q_j = \exists \} \),
- \( I_n \land \psi[\tau_1/x_1, \ldots, \tau_i/x_i] \) is unsatisfiable.

To show that \( (I_0, \ldots, I_n) \) is a GLI for \( a_1 \ldots a_n \), it is sufficient to notice that

\[
\bigwedge_{q(\mathbf{y})} \forall y_1 \ldots \forall y_{|q|} : q^{(i)}(\mathbf{y}) \Rightarrow \psi^{(k+1)} \models \phi_k
\]

for all \( k \in [1, n] \). Consequently, we obtain:

- \( I^{(0)} = \emptyset \models I_0 \), by Theorem \(^3\)
- \( I_{k-1} \land \bigwedge_{q(\mathbf{y})} \forall y_1 \ldots \forall y_{|q|} : q^{(i)}(\mathbf{y}) \Rightarrow \psi^{(k)} \models I_{k-1} \land \phi_k \models I_k \), and
- \( I_n \land \bigwedge_{q \in Q \setminus F} \forall y_1 \ldots \forall y_{|q|} : q(\mathbf{y}) \Rightarrow \bot \models I_n \land \psi \models \bot \),

as required by Definition \(^8\). \( \square \)

In conclusion, under two assumptions about the first order theory of the data domain, namely the(i) witness-producing quantifier elimination, and (ii) Lyndon interpolation for the quantifier-free fragment with uninterpreted functions, we developed a rather generic method that produces generalized Lyndon interpolants for unfeasible input event sequences. Moreover, each formula \( I_k \) in the interpolant refers only to the current predicate symbols \( Q^{(k)} \), the current and past input variables \( X^{(k)} \) and the existentially quantified transition variables introduced at the previous steps \( \{ x_j \mid j < l \} \).
ξ max (k), Q j = ⊤]. The remaining question is how to use such non-local interpolants to label the unfolding of an automaton (Definition 5) and to compute the coverage between nodes of the unfolding (Definition 6).

5.1 Unfolding with Non-local Interpolants

As required by Definition 5, the unfolding U of an automaton \( \mathcal{A} = \langle \Sigma, X, Q, t, F, A \rangle \) is labeled by formulae \( U(\alpha) \in \text{Form}^+(Q, \emptyset) \), with no free symbols, other than predicate symbols, such that the labeling is compatible with the transition relation of the automaton, according to the point 3 of Definition 5. The following lemma describes the refinement of the labeling of an input sequence \( \alpha \) of length \( n \) by a non-local GLI \( (I_0, \ldots, I_n) \), such that \( V(I_k) \subseteq Q^0 \cup X^0 \cup X_k \), where \( X_k \) are the existentially quantified variables from the prenex normal form of \( T(\alpha_k) \).

**Lemma 7.** Let \( U \) be an unfolding of an automaton \( \mathcal{A} = \langle \Sigma, X, Q, t, F, A \rangle \) such that \( \alpha = a_1 \ldots a_n \in \text{dom}(U) \) and \( (I_0, \ldots, I_n) \) be a GLI for \( \alpha \). The mapping \( U' : \text{dom}(U) \rightarrow \text{Form}^+(Q, \emptyset) \) defined as:
- \( U'(\alpha_k) = U(\alpha_k) \land J_k \), for all \( k \in [n] \), where \( J_k \) is the formula obtained from \( I_k \) by replacing each time-stamped predicate symbol \( q^k \) by \( q \) and existentially quantifying each free variable in \( I_k \),
- \( U'(\beta) = U(\beta) \) if \( \beta \in \text{dom}(U) \) and \( \beta \neq \alpha \),

is an unfolding of \( \mathcal{A} \).

**Proof:** The new set of formulae \( U'(\alpha_0), \ldots, U'(\alpha_n) \) complies with Definition 5 because:
- \( U'(\alpha_0) \equiv t \), since, by point 2 of Definition 8, we have \( t^0 \models I_0 \), thus \( t \models J_0 \) and \( U'(\alpha_0) = U(\alpha_0) \land J_0 \equiv t \land J_0 \equiv t \), and
- by Definition 8(3) we have, for all \( k \in [n - 1] \):

\[
I_k \land \bigwedge_{q^k(y) \in q^0(X)} \forall y_1 \ldots \forall y_{\#(q)} . q^k(y) \rightarrow \psi^{k+1} \models I_{k+1}
\]

We write \( I_k^{(i)} \) for the formula in which each predicate symbol \( q^k \) is replaced by \( q^{(i)} \). Then the following entailment holds:

\[
I_k^{(0)} \land \bigwedge_{q^k(y) \in q^0(X)} \forall y_1 \ldots \forall y_{\#(q)} . q^{(0)}(y) \rightarrow \psi^{(1)} \models I_{k+1}^{(1)}
\]

Because \( J_k \) is obtained by removing the time stamps from the predicate symbols and existentially quantifying all the free variables of \( I_k \), we also obtain, applying Fact 4 below:

\[
J_k^{(0)} \land \bigwedge_{q^k(y) \in q^0(X)} \forall y_1 \ldots \forall y_{\#(q)} . q^{(0)}(y) \rightarrow \psi^{(1)} \models J_{k+1}^{(1)}
\]

Since \( U \) satisfies the labeling condition of Definition 5 (3) and \( U'(\alpha_k) = U(\alpha_k) \land J_k \), we obtain, as required:

\[
U'(\alpha_k)^{(0)} \land \bigwedge_{q^k(y) \in q^0(X)} \forall y_1 \ldots \forall y_{\#(q)} . q^{(0)}(y) \rightarrow \psi^{(1)} \models U'(\alpha_{k+1})^{(1)}
\]
Fact 4  Given formulae \( \phi(x,y) \) and \( \psi(x) \) such that \( \phi(x,y) \models \psi(x) \), we also have \( \exists x . \phi(x,y) \models \exists x . \psi(x) \).

Proof: For each choice of a valuation for the existentially quantified variables on the left-hand side, we chose the same valuation for the variables on the right-hand side.  

Observe that, by Lemma 6 (2), the set of free variables of a GLI formula \( I_k \) consists of (i) variables \( X^{< k} \) keeping track of data values seen in the input at some earlier moment in time, and (ii) variables that track past choices made within the transition rules. Basically, it is not important when exactly in the past a certain input has been read or when a choice has been made, as only the value of the variable determines the future behavior. Intuitively, existential quantification of these variables does the job of ignoring when in the past these values have been seen.

The last ingredient of the lazy annotation semi-algorithm based on unfoldings consist in the implementation of the coverage check, when the unfolding of an automaton is labeled with conjunctions of existentially quantified formulae with predicate symbols, obtained from interpolation. By Definition 6, checking whether a given node \( \alpha \in \text{dom}(U) \) is covered amounts to finding a prefix \( \alpha' \preceq \alpha \) and a node \( \beta \in \text{dom}(U) \) such that \( U(\alpha') \models U(\beta) \), or equivalently, the formula \( U(\alpha') \land \neg U(\beta) \) is unsatisfiable. However, the latter formula, in prenex form, has quantifier prefix in the language \( \exists^* \forall^* \) and, as previously mentioned, the satisfiability problem for such formulae becomes undecidable when the data theory subsumes Presburger arithmetic [8].

Nevertheless, if we require just a yes/no answer (i.e. not an interpolant) recently developed quantifier instantiation heuristics [24] perform rather well in answering a large number of queries in this class. Observe, moreover, that coverage does not need to rely on a complete decision procedure. If the prover fails in answering the above satisfiability query, then the semi-algorithm assumes that the node is not covered and continues exploring its successors. Failure to compute complete coverage may lead to divergence (non-termination) and ultimately, to failure to prove emptiness, but does not affect the soundness of the semi-algorithm (real counterexamples will still be found).

6  Applications

The main application of first order alternating automata is checking inclusion between various classes of automata extended with variables ranging over infinite domains that recognize languages over infinite alphabets. The most widely known such classes are \textit{timed automata} [1] and \textit{finite-memory (register) automata} [14]. In both cases, complementation is not possible inside the class and inclusion is undecidable. Our contribution is providing a systematic semi-algorithm for these decision problems. In addition, the method described in §4 can extend our previous \textit{generic register automata} [10] inclusion checking framework, by allowing monitor (right-hand side) automata to have local variables, that are not visible in the language.
Another application is checking safety (mutual exclusion, absence of deadlocks, etc.) and liveness (termination, lack of starvation, etc.) properties of parameterized concurrent programs, consisting of an unbounded number of replicated threads that communicate via a fixed set of global variables (locks, counters, etc.). The verification of parametric programs has been reduced to checking the emptiness of a (possibly infinite) sequence of first order alternating automata, called predicate automata \[^{[4,5]} \]
encoding the inclusion of the set of traces of a parametric concurrent program into increasingly general proof spaces, obtained by generalization of counterexamples. The program and the proof spaces are first order alternating automata over the infinite alphabet of pairs consisting of program statements and thread identifiers.

### 6.1 Timed Automata

The standard definition of a finite timed word is a sequence of pairs \((a_1, \tau_1), \ldots, (a_n, \tau_n) \in (\Sigma \times \mathbb{R})^n\), where \(\mathbb{R}\) is the set of real numbers, such that \(0 \leq \tau_i < \tau_{i+1}\) for all \(i \in [1,n-1]\). Intuitively, \(\tau_i\) is the moment in time where the input event \(a_i\) occurs. Given a set \(C\) of clocks, the set \(\Phi(C)\) of clock constraints is defined inductively as the set of formulae \(x \leq c, x \geq c, -\delta, \delta \land -\delta, \text{where } x \in C, c \in \mathbb{Q}\) is a rational constant and \(\delta, \delta_1, \delta_2 \in \Phi(X)\).

A timed automaton is a tuple \(T = (\Sigma, S, S_0, F, C, E)\), where: \(\Sigma\) is a finite set of input events, \(S\) is a finite set of states, \(S_0, F \subseteq S\) are sets of initial and final states, respectively, \(C\) is a finite set of clocks and \(E \subseteq S \times \Sigma \times S \times 2^C \times \Phi(C)\) is the set of transitions \((s, a, s', \lambda, \delta)\) from state \(s\) to state \(s'\) with symbol \(a\), \(\lambda\) is the set of clocks to be reset and \(\delta\) is a clock constraint. A run of \(T\) over a timed word \(w = (a_1, \tau_1) \ldots (a_n, \tau_n)\) is a sequence \((s_0, \gamma_0) \ldots (s_n, \gamma_n)\), where \(s_i \in S, \gamma_i : C \rightarrow \mathbb{R}\) are clock valuations, for all \(i \in [n]\), and:

- \(s_0 \in S_0\) and \(\gamma_0(x) = 0\) for all \(x \in C\),
- for all \(i \in [n]\), there exists a transition \((s_i, a_i, s_{i+1}, \lambda_i, \delta_i) \in E\) such that \(\gamma_i + \tau_{i+1} - \tau_i \models \delta_i\), and for all \(x \in C\), \(\gamma_i(x) = 0\) if \(x \not\in \lambda_i\) and \(\gamma_{i+1}(x) = \gamma_i(x) + \tau_{i+1} - \tau_i\), otherwise.

Here \(\tau_0 \overset{0}{=} 0\) and \(\gamma_i + \tau_{i+1} - \tau_i\) is the valuation mapping each \(x \in C\) to \(\gamma_i(x) + \tau_{i+1} - \tau_i\).

The run is accepting iff \(s_n \in F\), in which case \(T\) accepts \(w\). As usual, we denote by \(L(T)\) the set of finite words accepted by \(T\). It is well-known that, in general, there is no timed automaton accepting the complement language \((\Sigma \times \mathbb{R})^* \backslash L(T)\) and, moreover, the language inclusion problem is undecidable \[^{[1]}\).

Given a timed automaton \(T = (\Sigma, S, S_0, F, C, E)\), we define a first order alternating automaton \(A_T = (\Sigma, [t], Q_T, \iota_T, F_T, \Delta_T)\), with a single input variable \(t\), ranging over \(\mathbb{R}\), such that each timed word \(w = (a_1, \tau_1) \ldots (a_n, \tau_n)\) corresponds to a unique data word \(d(w) = (a_1, \nu_1) \ldots (a_n, \nu_n)\) such that \(\nu_i(t) = \tau_i\) for all \(i \in [1,n]\) and \(L(A_T) = \{d(w) \mid w \in L(T)\}\). The only difficulty here is capturing the fact that all the clocks of \(T\) evolve at the same pace, which is easily done using a technique from \[^{[7]}\], which replaces each clock \(x_i\) of \(T\) by a variable \(y_i\) tracking the difference between the values of \(t\) and \(x_i\).

Formally, if \(C = \{x_1, \ldots, x_k\}\) and \(S = \{s_1, \ldots, s_m\}\), we define \(Q_T = \{q_1, \ldots, q_m\}\), where \(#(q_i) = k + 1\) for all \(i \in [1,m]\), \(\iota_T = \bigvee_{s_i \in S_0} q_i(0, \ldots, 0)\), \(F_T = \{q_i \mid s_i \in F\}\) and, for each transition \((s_i, a, s_j, \lambda, \delta) \in E\), \(\Delta_T\) contains the rule:

\[
q_i(y_1, \ldots, y_k, z) \overset{a(t)}{\rightarrow} t > z \land \delta(z-y_1, \ldots, z-y_k) \land q_j(y_1', \ldots, y_k', t)
\]
where $y'_i$ stands for $z$ if $x_i \in \lambda$ and for $y_i$, otherwise. Moreover, nothing else is in $\Delta_P$. We establish the following connection between a timed automaton and its corresponding first order alternating automaton.

**Proposition 2.** Given a timed automaton $T = (\Sigma, S_0, F, C, E)$, the first order alternating automaton $A_T = (\Sigma, (t), Q_T, t_T, F_T, A_T)$ recognizes the language $L(A_T) = \{d(w) \mid w \in L(T)\}$.

**Proof:** “$\subseteq$” Let $w = (a_1, v_1)\ldots(a_n, v_n) \in L(A_T)$ be a word. We show the existence of a timed word $(a_1, \tau_1)\ldots(a_n, \tau_n) \in L(T)$ such that $v_i(0) = \tau_i$, for all $i \in [1, n]$, by induction on $n \geq 0$. In fact we shall prove the following stronger statements:

1. each execution of $A_T$ over $w$ starting with a cube $c \in \alpha([1_T]^\mu)$ is a linear tree, in which each node has at most one child.
2. for each execution $q_0, d_1^1, \ldots, d_k^1, \tau_0, \ldots, q_n, d_1^n, \ldots, d_k^n, \tau_n$ of $A_T$. $T$ has an execution $(s_0, 0) \ldots (s_n, 0)$ over the timed word $(a_1, \tau_1)\ldots(a_n, \tau_n)$, such that, for all $i \in [1, n]$ and each $\ell \in [1, k]$, we have $\gamma_i(x_\ell) = \tau_{\ell-1} - d_\ell^1$.

The first point above is by inspection of $i_T = \sqrt{\cup_{s \in S_0} q_0(0, \ldots, 0)}$ and of the rules from $\Delta_T$. Indeed, each minimal model of $i_T$ corresponds to a cube $q(0, \ldots, 0)$ and each rule has exactly one predicate atom on its right-hand side, thus each node of the execution will have at most one successor. The second point is by induction on $n \geq 0$.

“$\supseteq$” Let $w = (a_1, \tau_1)\ldots(a_n, \tau_n) \in L(T)$ be a timed word. By induction on $n \geq 0$, we show that for each run $(s_0, 0) \ldots (s_n, 0)$ of $T$ over $w$, $A_T$ has a linear execution $(d_1^0, \ldots, d_k^0, \tau_0, \ldots, d_1^n, \ldots, d_k^n, \tau_n)$ such that, for all $i \in [1, n]$ and each $\ell \in [1, k]$, we have $\gamma_i(x_\ell) = \tau_{\ell-1} - d_\ell^i$. □

An easy consequence is that the timed language inclusion problem “given timed automata $T_1$ and $T_2$, does $L(T_1) \subseteq L(T_2)$?” is reduced in polynomial time to the emptiness problem $L(A_{T_1}) \cap \overline{L(A_{T_2})} = \emptyset$, for which [21] provides a semi-algorithm. Observe, moreover, that no transition quantifiers are needed to encode timed automata as first order alternating automata.

### 6.2 Register Automata

Finite-memory automata, most commonly referred to as register automata [14] are among the first attempts at lifting the finite alphabet restriction of classical Rabin-Scott automata. In a nutshell, a register automaton is a finite-state automaton equipped with a finite set of registers $x_1, \ldots, x_r$ able to copy input values and compare them with subsequent input. Consequently, basic results from classical automata theory, such as the pumping lemma or the closure under complement do not hold in this model and, moreover, inclusion of languages recognized by register automata is undecidable [21].

Let $\Sigma$ be an infinite alphabet, $\#$ be a symbol not in $\Sigma$ and $r > 0$ be an integer constant, denoting the number of registers. An assignment is a word $v = v_1 \ldots v_r$ such that if $v_i = v_j$ and $i \neq j$ then $v_i = $, for all $i, j \in [1, r]$. We write $[v]$ for the set $\{v_i \mid i \in [1, r]\}$ of values in the assignment $v$. A finite-memory (register) automaton is a tuple $R = (S, q_0, u, \rho, \mu, F)$, where $S$ is a finite set of states, $q_0 \in S$ is the initial state, $u = u_1 \ldots u_r$ is the initial assignment, $\rho : S \rightarrow [1, r]$ is the reassignment partial function, $\mu \subseteq S \times [1, r] \times S$ is the
transition relation and $F \subseteq S$ is the set of final states. A run of $A$ over an input word $a_1 \ldots a_n \in \Sigma^*$ is a sequence $(s_0, v_0) \ldots (s_n, v_n)$ such that $v_0 = u$ and, for all $i \in [1, n]$, exactly one of the following holds:
- if there exists $k \in [1, r]$ such that $a_i = (v_{i-1})_k$ then $v_i = v_{i-1}$ and $(s_{i-1}, k, s_i) \in \mu$,
- otherwise $a_i \notin [v_{i-1}], \rho(s_{i-1})$ is defined, $(v_i)_{\rho(s_{i-1})} = a_i$, for each $k \in [1, r] \setminus \{\rho(s_{i-1})\}$, we have $(v_i)_k = (v_{i-1})_k$ and $(s_{i-1}, \rho(s_{i-1}), s_i) \in \mu$.

Intuitively, if the input symbol is already stored in some register, the automaton moves to the next state if, moreover, the transition relation allows it, otherwise it copies the input to the register indicated by the reassignment, erasing its the previous value, and moves according to the transition relation.

The translation of register automata to first order alternating automata is quite natural, because registers can be encoded as arguments of predicate atoms. Formally, given a register automaton $R = \langle S, s_0, u, \rho, \mu, F \rangle$, such that $S = \{s_0, \ldots, s_m\}$, we define the alternating automaton $A_R = \langle \alpha, \{x\}, Q_R, \iota_R, F_R, A_R \rangle$, where $\alpha \notin \Sigma$, $Q_R \defeq \{q_0, \ldots, q_m\}$ and $\#(q_i) = r$ for all $i \in [m]$, $\iota_R \defeq q_0(u)$, $F_R \defeq \{q_i | s_i \in F\}$ and, for each transition $(s_i, k, s_j) \in \mu$, $A_T$ contains the rule:

$$q_i(y_1, \ldots, y_r) \xrightarrow{a_i} y_k = x \land q_j(y_1, \ldots, y_r) \lor \bigwedge_{i=1}^r x \neq y_i \land q_j(y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots, y_r)$$

Moreover, nothing else is in $A_R$. The connection between register automata and first order alternating automata is stated below.

**Proposition 3.** Given a register automaton $R = \langle S, s_0, u, \rho, \mu, F \rangle$ over an infinite alphabet $\Sigma$, the first order alternating automaton $A_R = \langle \alpha, Q_R, \iota_R, F_R, A_R \rangle$ recognizes the language $L(A_R) = \{q_0(v_0), \ldots, q_n(v_0) | a_1 \ldots a_n \in L(R)\}$.

**Proof:** “$\subseteq$” Let $w = (\alpha, a_1) \ldots (\alpha, a_n) \in L(A_R)$. First, it is easy to show that each execution of $A_R$, that starts in some cube $c \in \mathcal{C}(\{v_i\})^n$, is a linear tree with labels $q_0(v_0), \ldots, q_n(v_0)$ such that $v_0 = u$. Second by induction on $n \geq 0$, we prove that $A_R$ has a run as above over $w$ only if $R$ has a run $(q_0(v_0), \ldots, q_n(v_n))$ over $a_1 \ldots a_n$. “$\supseteq$” Let $w = a_1 \ldots a_n \in L(R)$ and $q_0(v_0), \ldots, q_n(v_n)$ be a run of $R$ over $w$, such that $v_0 = u$. By induction on $n \geq 0$, we can build an execution of $A_R$ over $a_1 \ldots a_n$ that is a linear tree with labels $q_0(v_0), \ldots, q_n(v_n)$.

Consequently, the language inclusion problem “given register automata $R_1$ and $R_2$, does $L(R_1) \subseteq L(R_2)$?” is reduced in polynomial time to emptiness problem $L(A_{R_1}) \cap \overline{L(A_{R_2})} = \emptyset$, for which (4) provides a semi-algorithm. Notice further that the encoding of register automata as first order alternating automata uses no transition quantifiers.

### 6.3 Predicate Automata

The model of predicate automata \cite{45} has emerged recently as a tool for checking safety and liveness properties of parameterized concurrent programs, in which there is an unbounded number of replicated threads that communicate via global variables. Predicate automata recognize finite sequences of actions that are pairs $(\sigma, i)$, where $\sigma$
is from a finite set $\Sigma$ of program statements and $i \in \mathbb{N}$ ranges over an unbounded set of thread identifiers. To avoid clutter, we shall view a pair $(\sigma, i)$ as a data symbol $(\sigma, \nu)$ where $\nu(x) = i$, for a designated input variable $x$.

Since thread identifiers can only be compared for equality, the data theory of predicate automata is the first order theory of equality. Moreover, transition quantifiers are only needed for checking termination and, generally, liveness properties. However, the execution semantics of predicate automata differs from that of first order automata with respect to the following detail: initial configurations and successors of predicate automata are defined using the entire sets of models of the initial sentence and transition rules, not just the minimal ones, as in our case.

Formally, a run of a predicate automaton $P = (\Sigma, \{x\}, Q, t, F, A)$ over a word $(a_1, \nu_1) \ldots (a_n, \nu_n)$ is a sequence of interpretations $I_0, \ldots, I_n$ such that $I_0 \in [[t]]$ and for each $i \in [1, n]$, each $q \in Q$ and each tuple $\langle d_1, \ldots, d_{\#(q)} \rangle \in I_{i-1}(q)$, we have $I_i \in [[\psi]]_{\nu_i}$ for each rule $q(y_1, \ldots, y_{\#(q)}) \rightarrow \psi \in A$, where $\nu = \nu_1[d_1, \ldots, y_{\#(q)} \Leftarrow d_{\#(q)}]$. The run is accepting if and only if $I(q) = \emptyset$ for all $q \in Q \setminus F$.

In fact, as shown next, this more simple execution semantics is equivalent, from the language point of view, with the semantics given by Definitions 1 and 2. We believe that the semantics of first order alternating automata based on minimal models is important for its relation to the textbook semantics of boolean alternating automata.

### Proposition 4

Given a predicate automaton $P = (\Sigma, \{x\}, Q, t, F, A)$, let $A_P$ be the first order alternating automaton that has the same description as $P$. Then $L(P) = L(A_P)$.

**Proof:** “$\subseteq$” Let $w = (a_1, \nu_1) \ldots (a_n, \nu_n) \in L(P)$ be a word and $I_0, \ldots, I_n$ be an accepting execution of $P$ over $w$. Let $I_j^{(0)}$ be the interpretation that associates each predicate $q^{(0)}$ the set $I_j(q)$, for $i, j \in [n]$. Then one builds, by induction on $n \geq 0$, an execution $T$ of $A_P$ such that $I_T \subseteq \bigcup_{i=0}^n I_i^{(0)}$, where $I_T$ is the unique interpretation associated with $T$. Since $I_0, \ldots, I_n$ is accepting, we have $I_j^{(0)}(q^{(0)}) = \emptyset$, for all $q \in Q \setminus F$ and hence $I_T(q^{(0)}) = \emptyset$, for all $q \in Q \setminus F$ and, consequently $w \in L(A_P)$. “$\supseteq$” Let $w = (a_1, \nu_1) \ldots (a_n, \nu_n) \in L(A_P)$ be a word and $T$ be an accepting execution of $A_P$ over $w$. We define the sequence of interpretations $I_0, \ldots, I_n$ as $I_i(q) = I_T(q^{(0)})$, for each $i \in [n]$ and each $q \in Q$. By induction on $n \geq 0$ one shows that $I_0, \ldots, I_n$ is an execution $P$. Moreover, since $T$ is accepting, we have $I_n(q) = I_T(q^{(0)}) = \emptyset$, for each $q \in Q \setminus F$, thus $w \in L(P)$. □

As before, this result enables using the semi-algorithm from for checking emptiness of predicate automata. We point out that, although quantifier-free predicate automata with predicates of arity one are decidable for emptiness, currently there is no method for checking emptiness of predicate automata with predicates of arity greater than one, other than the explicit enumeration of cubes. Moreover, no method for dealing with emptiness in the presence of transition quantifiers is known to exist.

### 7 Experimental Results

We have implemented a version of the IMPACT semi-algorithm in a prototype tool called FOADA, which is available online. The tool is written in Java and uses the Z3 SMT solver, via the JavaSMT interface. for spuriousness and coverage queries...
and also for interpolant generation. The experiments were carried out on a MacOS x64 - 1.3 GHz Intel Core i5 - 8 GB 1867 MHz LPDDR3 machine.

The experimental results, reported in Table 1, come from several sources, namely predicate automata models (*.pa) [4,5] available online [22], timed automata inclusion problems (abp.ada, train.ada, rr-crossing.foada), array logic entailments (array_rotation.ada, array_simple.ada, array_shift.ada) and hardware circuit verification (hw1.ada, hw2.ada), initially considered in [10]. The train-simpleN.foada and fischer-mutexN.foada examples are parametric verification problems in which one checks inclusions of the form $\bigcap_{i=1}^{N} L(A_i) \subseteq L(B)$, where $A_i$ is the $i$-th copy of the same template automaton.

The advantage of using FOADA over the INCLUDER [9] tool from [10] is the possibility of having infinite alphabet automata with hidden (local) variables, whose values are not visible in the input. In particular, this is essential for checking inclusion of timed automata that use internal clocks to control the computation.

| Example            | Size (bytes) | Inclusion | Nodes Expanded | Nodes Visited | Time (ms) |
|--------------------|--------------|-----------|----------------|---------------|-----------|
| incdec.pa          | 499          | no        | 21             | 17            | 779       |
| localdec.pa        | 678          | no        | 49             | 35            | 1814      |
| ticket.pa          | 4250         | no        | 229            | 91            | 9543      |
| count_thread0.ada  | 9769         | no        | 154            | 128           | 8255      |
| count_thread1.ada  | 10235        | no        | 766            | 692           | 76771     |
| loc0.ada           | 10595        | no        | 73             | 27            | 1431      |
| local1.ada         | 11385        | no        | 1135           | 858           | 101042    |
| array_rotation.ada | 1834         | yes       | 9              | 8             | 1543      |
| array_simple.ada   | 32409        | yes       | 11             | 10            | 6587      |
| array_shift.ada    | 874          | yes       | 6              | 5             | 413       |
| abp.ada            | 6909         | no        | 52             | 47            | 4788      |
| train.ada          | 1825         | yes       | 68             | 67            | 7319      |
| hw1.ada            | 322          | Solver Error | 9             | 7             | /         |
| hw2.ada            | 674          | yes       | 20             | 22            | 4974      |
| rr-crossing.foada  | 1780         | yes       | 67             | 67            | 7574      |
| train-simple1.foada| 5421         | yes       | 43             | 44            | 2893      |
| train-simple2.foada| 10177        | yes       | 111            | 113           | 8386      |
| train-simple3.foada| 19561        | yes       | 196            | 200           | 15041     |
| fischer-mutex1.foada| 3000         | yes       | 23             | 23            | 808       |
| fischer-mutex2.foada| 4452         | yes       | 33             | 33            | 1154      |

Table 1. Experiments with First Order Alternating Automata

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