Trace formulas for the modified Mathieu equation

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To my friend Ari Laptev on the occasion of his 70th birthday

Abstract

For the radial and one-dimensional Schrödinger operator $H$ with growing potential $q(x)$ we outline a method of obtaining the trace identities — an asymptotic expansion of the Fredholm determinant $\det_F (H - \lambda I)$ as $\lambda \to -\infty$. As an illustrating example, we consider Schrödinger operator with the potential $q(x) = 2 \cosh 2x$, associated with the modified Mathieu equation.

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1 Introduction

Mathieu functions, solutions of the Mathieu equation and modified Mathieu equation, have many applications in mathematics and theoretical physics. In particular, the modified Mathieu differential equation

$$\frac{d^2 \psi}{dx^2} + (\lambda - 2 \cosh 2x) \psi = 0$$

on the real axis $-\infty < x < \infty$, is a Schrödinger equation

$$H \psi = \lambda \psi$$

with the Hamiltonian

$$H = -\frac{d^2}{dx^2} + 2 \cosh 2x \quad (1.1)$$
in the Hilbert space $L^2(\mathbb{R})$.

A positive, self-adjoint operator $H$ given by (1.1) is a Hamiltonian of the quantum two-particle periodic Toda chain (after separation of the center of mass), and has a purely discrete spectrum. As its classical analog, the quantum $N$-particle periodic Toda chain is also completely integrable, as was shown by M. Gutzwiller [1] for $N = 2, 3$ and by E.K. Sklyanin [2] for general $N$. For $N = 2$ the exact quantization condition in [1] was obtained by applying Floquet theory to the potential $q(x) = 2 \cosh 2x$ with pure imaginary period, and is formulated in terms of infinite Hill determinants (see [3, Ch. XIX]). Sklyanin’s method of quantum separation of variables was developed further by Q. Pasquier and M. Gaudin [4], and by S. Kharchev and D. Lebedev [5].

Recently, N.A. Nekrasov and S.L. Shatashvili [6] found a remarkable connection between quantum integrable systems and $\mathcal{N} = 2$ four-dimensional supersymmetric Yang-Mills theories in the special $\Omega$-background. Among other interesting results they showed, at the physical level of rigor, that exact quantization conditions for the quantum $N$-particle periodic Toda chain can be written in terms of the so-called effective twisted superpotential for the low energy effective two-dimensional gauge theory. In particular, Nekrasov and Shatashvili gave a new interpretation of Gutzwiller and Kharchev and Lebedev quantization conditions.

Replacing the kinetic term $P^2$ in the Hamiltonian $H$ by $2 \cosh P$, where $P = -i \frac{d}{dx}$ is the quantum-mechanical momentum operator, one gets the following functional-difference operator on $L^2(\mathbb{R})$,

$$\hat{H} = 2 \cosh P + 2 \cosh 2Q,$$

where $Q$ is the quantum-mechanical position operator. The operator $\hat{H}$ is a positive self-adjoint operator on $L^2(\mathbb{R})$ with a purely discrete spectrum, and $\hat{H}^{-1}$ is a trace class operator. Operators of such type naturally appear in the theory of topological strings as a quantization of mirror curves of non-compact Calabi-Yau threefolds. These operators have a purely discrete spectrum, and their Fredholm determinants

$$\det_F (\hat{H} - \lambda I) = \frac{\det (\hat{H} - \lambda I)}{\det \hat{H}} = \prod_{n=1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right),$$

where $\lambda_n$ are the eigenvalues, are believed to encode deep relations in the enumerative geometry of the corresponding Calabi-Yau manifolds (see M. Mariño survey [7] for the exact formulation and references). Though in certain special cases corresponding conjectures can be confirmed by numerical computations, their mathematical derivation seems to be out of reach. Nevertheless, it is quite remarkable that operators of the type (1.2) can be rigorously studied by the method developed by Ari Laptev more than twenty years ago in [8]. Namely, it was proved in [9] (see also [10]) that these functional-difference operators have a purely discrete spectrum and their eigenvalue counting function satisfies the Weyl’s law.
The logarithmic derivative of the Fredholm determinant $a(\lambda) = \det_F (\hat{H} - \lambda I)$ is

$$\frac{d}{d\lambda} \log a(\lambda) = \text{Tr}(\hat{H} - \lambda I)^{-1} = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n}.$$ 

As $\lambda \to -\infty$, it admits the asymptotic expansion

$$\frac{d}{d\lambda} \log a(\lambda) = \alpha_0(\nu) + \sum_{n=1}^{\infty} \frac{c_n}{\nu^n} + O(|\nu|^{-\infty}),$$

where $\nu = \sqrt{-\lambda}$ and the function $\alpha_0(\nu)$ is determined by the asymptotic of the eigenvalues (cf. [11, Theorem 3.1]). The coefficients $c_{2k}$ represent regularized divergent series $\sum_{n=1}^{\infty} \lambda_n^k$ and it is expected that they are expressed in terms of the potential of the functional-difference operator $\hat{H}$. Relations of such type are called trace identities.

The trace identities for the Sturm-Liouville operators were derived in the classic papers by I.M. Gelfand and B.M. Levitan [12] and L.A. Dikii [11], for the radial Schrödinger operator with rapidly decaying potential $q(x)$ — by V.S. Buslaev and L.D. Faddeev in [13], and for the one-dimensional Schrödinger operator with rapidly decaying potential $q(x)$ — by L.D. Faddeev and V.E. Zakharov in [14] (see also [15]).

To the best of our knowledge, the trace identities for growing as $|\nu| \to \infty$ potential $\nu$ have not been considered in the literature. Instead one studies the relative trace identities — the asymptotic expansion of

$$\text{Tr} \left\{ \left( -\frac{d^2}{dx^2} + q(x) + r(x) - \lambda I \right)^{-1} - \left( -\frac{d^2}{dx^2} + q(x) - \lambda I \right)^{-1} \right\}$$

as $\lambda \to -\infty$, where potential $q(x)$ grows as $|x| \to \infty$, and function $r(x)$ is rapidly decaying. The interesting example $q(x) = x^2$ was considered in [16].

In the present paper we outline a simple method for the derivation of the trace identities for the Schrödinger operator with the potential $q(x)$ that rapidly grows to infinity as $|x| \to \infty$, and illustrate it by explicit formulas for the case $q(x) = 2 \cosh 2x$, the modified Mathieu operator (1.1). In Sect. 2.1 we briefly review the Liouville-Green method, which is used to obtain the trace identities for the radial Schrödinger operator in Sect. 2.2.1, and for the one-dimensional Schrödinger operator in Sect. 2.2.2. In Section 3 we illustrate our method on two examples: a well-known radial Schrödinger operator with potential $q(x) = e^{2x}$ in Sect. 3.1, and modified Mathieu operator (1.1) in Sect. 3.2. Our main result — the trace identities for the modified Mathieu operator — is presented in Theorem 1. Finally, in Section 4 we prove that modified Mathieu differential equation has a solution that behaves like the modified Bessel function of the second kind as $x \to \infty$.

## 2 General case

Here we consider the differential equation

$$-\psi'' + q(x)\psi = \lambda\psi \quad \text{(2.1)}$$
one the half-line $0 < x < \infty$ and on the real line $\infty < x < \infty$, where $q(x)$ is a positive smooth function that grows to infinity as $x \to \infty$ (or $|x| \to \infty$).

### 2.1 The Liouville-Green method

When $\lambda < 0$, decaying as $x \to \infty$ solution of (2.1) has the following double asymptotic

$$
\psi_1(x, \lambda) = \frac{C_1(\lambda)}{\sqrt[4]{q(x)} - \lambda} e^{-\int_0^x \sqrt{q(s) - \lambda} \, ds} (1 + \epsilon_1(x, \lambda)),
$$

(2.2)

where in may cases

$$
|\epsilon_1(x, \lambda)| \leq \frac{\tau(x)}{|\lambda|}, \quad \tau(x) \to 0 \quad \text{as} \quad x \to \infty.
$$

Similarly, solution of (2.1) that grows as $x \to \infty$ has the asymptotic

$$
\psi_2(x, \lambda) = \frac{C_2(\lambda)}{\sqrt[4]{q(x)} - \lambda} e^{\int_0^x \sqrt{q(s) - \lambda} \, ds} (1 + \epsilon_2(x, \lambda)).
$$

(2.3)

This follows from the classical Liouville-Green method, developed by F. Olver [17] (see also [18]).

Put $Q(x, \lambda) = q(x) - \lambda$ and introduce

$$
\chi(x, \lambda) = \log \psi_1(x, \lambda) + \frac{1}{4} \log Q(x, \lambda) + \int_0^x \sqrt{Q(s, \lambda)} \, ds - \log C_1(\lambda)
$$

(2.4)

and

$$
\sigma(x, \lambda) = \chi'(x, \lambda) = \frac{\psi_1'(x, \lambda)}{\psi_1(x, \lambda)} + \frac{Q'(x, \lambda)}{4Q(x, \lambda)} + \sqrt{Q(x, \lambda)},
$$

(2.5)

where prime stands for the $x$-derivative. Denoting for brevity $\sigma = \sigma(x, \lambda)$ and $Q = Q(x, \lambda)$, we see that the function $\sigma$ satisfies the Riccati equation

$$
\sigma' = \frac{\psi_1''}{\psi_1} - \left(\frac{\psi_1'}{\psi_1}\right)^2 + \frac{Q''}{4Q} - \frac{1}{4} \left(\frac{Q'}{Q}\right)^2 + \frac{Q'}{2\sqrt{Q}},
$$

or

$$
\sigma' = -\sigma^2 + 2\sqrt{Q}\sigma + \frac{Q'}{2Q}\sigma + \frac{Q''}{4Q} = \frac{5}{16} \left(\frac{Q'}{Q}\right)^2.
$$

(2.6)

### 2.2 Trace identities

The Riccati equation (2.6) can be used to obtain an asymptotic expansion of $\sigma(x, \lambda)$ as $\lambda \to -\infty$ and get the trace identities for the Schrödinger operator with growing as $x \to \infty$ potential $q(x)$ on $L^2(0, \infty)$ and on $L^2(\mathbb{R})$. In the latter case potential $q(x)$ is assumed to be an even function.
2.2.1 Radial Schrödinger operator

Here we consider the Schrödinger operator $H$ on $L^2(0, \infty)$ with a smooth positive potential $q(x)$ satisfying
\[
\int_0^\infty \frac{dx}{\sqrt{q(x)}} < \infty, \tag{2.7}
\]
supplemented with the boundary condition $\psi(0) = 0$. Restricting further, we consider potentials satisfying the inequality $q(x) \geq Cx^{2+\varepsilon}$ for some $\varepsilon > 0$. It follows from the Weyl’s law that $H^{-1}$ is a trace class operator, and its Fredholm determinant is an entire function of order $1/2$.

For every $\lambda$ equation (2.1) for $0 < x < \infty$ has a unique solution $\psi(x, \lambda)$ with the asymptotic
\[
\psi(x, \lambda) = \frac{1}{\sqrt{q(x)}} e^{-\int_0^x \sqrt{q(s)}\,ds} (1 + o(1)) \quad \text{as} \quad x \to \infty. \tag{2.8}
\]
For fixed $x$ solution $\psi(x, \lambda)$ is an entire function of $\lambda$ of order $1/2$ and $\psi(0, \lambda)$ is proportional to the Fredholm determinant $a(\lambda)$ of the operator $H$,
\[
a(\lambda) = \frac{\psi(0, \lambda)}{\psi(0, 0)}.
\]
Comparing equations (2.2) and (2.8) for $\lambda < 0$, we see that for such $\lambda$ solution $\psi(x, \lambda)$ has asymptotic (2.2) with
\[
C_1(\lambda) = e^{\int_0^\infty (\sqrt{Q(x,\lambda)}-\sqrt{q(x)})\,dx}.
\]
It follows from (2.7) that the integral in this formula is convergent, so $C_1(\lambda)$ is well-defined.

Putting in formula (2.4) $\lambda = -\nu^2$, we have
\[
\chi(x, \lambda) = \log \psi(x, \lambda) + \frac{1}{4} \log(q(x) + \nu^2) + \int_0^x \sqrt{q(s) + \nu^2} \, ds
\]
\[
- \int_0^\infty \left( \sqrt{q(s) + \nu^2} - \sqrt{q(s)} \right) ds.
\]
From here we obtain
\[
\chi(0, \lambda) = \log a(\lambda) - a_0(\lambda),
\]
\[
\lim_{x \to \infty} \chi(x, \lambda) = 0,
\]
where
\[
a_0(\lambda) = -\frac{1}{4} \log(q(0) + \nu^2) + \int_0^\infty \left( \sqrt{q(x) + \nu^2} - \sqrt{q(x)} \right) dx - \log \psi(0, 0).
\]
It is not difficult to show that $a_0(\lambda)$ admits an asymptotic expansion as $\lambda = -\nu^2 \to \infty$,

$$a_0(\lambda) = \alpha(\nu) + \sum_{n=1}^{\infty} \frac{a_n}{\nu^n} + O(\nu^{-\infty}), \quad (2.9)$$

where the leading term — a function $\alpha(\nu)$ — is determined by the asymptotic of the eigenvalues and can be computed explicitly (see examples in Section 3).

The function $\sigma(x, \lambda) = \chi'(x, \lambda)$ satisfies the Riccati equation (2.6), which admits the asymptotic solution

$$\sigma(x, \lambda) = \sum_{n=1}^{\infty} \frac{c_n(x)}{\nu^n} + O(\nu^{-\infty}), \quad (2.10)$$

where the coefficients $c_n(x)$ are determined recursively in terms of the potential $q(x)$ and its derivatives. Since

$$\log \chi(0, \lambda) = -\int_{0}^{\infty} \sigma(x, \lambda) dx, \quad (2.11)$$

we have

$$\log a(\lambda) = \alpha_0(\lambda) - \int_{0}^{\infty} \sigma(x, \lambda) dx. \quad (2.12)$$

Using the asymptotic expansion (2.10), we obtain the trace identities

$$\log a(\lambda) = \alpha(\nu) + \sum_{n=1}^{\infty} \frac{c_n}{\nu^n} + O(\nu^{-\infty}), \quad \text{where} \quad c_n = \alpha_n - \int_{0}^{\infty} c_n(x) dx. \quad (2.13)$$

### 2.2.2 One-dimensional Schrödinger operator

Here we assume that the potential $q(x)$ is a smooth even function satisfying (2.7). A fundamental system of solutions of the differential equation (2.1) is given by solutions $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$ with the following asymptotic as $x \to \infty$,

$$\psi_1(x, \lambda) = \frac{1}{\sqrt{q(x)}} e^{-\int_{0}^{x} \sqrt{q(s)} ds} (1 + o(1)), \quad (2.14)$$

$$\psi_2(x, \lambda) = \frac{1}{\sqrt{q(x)}} e^{\int_{0}^{x} \sqrt{q(s)} ds} (1 + o(1)). \quad (2.15)$$

Assuming that

$$\lim_{x \to \infty} \frac{q'(x)}{(\sqrt{q(x)})^3} = 0,$$

we get from (2.14) and (2.15) that

$$W(\psi_1, \psi_2) = 2,$$
where \( W(f, g) = fg' - f'g \) is the Wronskian of two functions. The functions \( \psi_1(x, \lambda) \) and \( \psi_2(x, \lambda) \) satisfy asymptotic formulas (2.2)–(2.3), where

\[
C_1(\lambda) = e^{\int_0^\infty \left( \sqrt{Q(x, \lambda)} - \sqrt{q(x)} \right) dx} \quad \text{and} \quad C_2(\lambda) = e^{-\int_0^\infty \left( \sqrt{Q(x, \lambda)} - \sqrt{q(x)} \right) dx}.
\]

Another fundamental system of solutions is given by the functions \( \psi_1(-x, \lambda) \) and \( \psi_2(-x, \lambda) \), and we have

\[
\psi_1(x, \lambda) = t_{11}(\lambda)\psi_1(-x, \lambda) + t_{12}(\lambda)\psi_2(-x, \lambda),
\]

where

\[
t_{12}(\lambda) = \frac{1}{2} W(\psi_1(x, \lambda), \psi_1(-x, \lambda)).
\]

For fixed \( x \) the functions \( \psi_1(x, \lambda) \) and \( \psi_2(x, \lambda) \) are entire functions of order 1/2, as it can be shown using condition (2.7). Therefore, \( t_{12}(\lambda) \) is an entire function of order 1/2 with zeros at the eigenvalues of the operator \( H \). As in the previous section, for the Fredholm determinant \( a(\lambda) \) of the operator \( H \) we obtain

\[
a(\lambda) = \frac{t_{12}(\lambda)}{t_{12}(0)}.
\]

It follows from (2.2) that the function \( \chi(x, \lambda) \), defined by (2.4), satisfies

\[
\lim_{x \to \infty} \chi(x, \lambda) = 0.
\]

To investigate its behavior as \( x \to -\infty \) we observe that it follows from (2.14)–(2.15) that as \( x \to -\infty \) the first term in (2.16) is exponentially small with respect to the second term, so

\[
\lim_{x \to -\infty} \left( \log \psi_1(x, \lambda) - \log \psi_2(-x, \lambda) \right) = \log t_{12}(\lambda).
\]

Since the antiderivative \( \int_0^x \sqrt{Q(s, \lambda)} ds \) is an odd function of \( x \), we obtain

\[
\lim_{x \to -\infty} \chi(x, \lambda) = \log a(\lambda) - a_0(\lambda),
\]

where now

\[
a_0(\lambda) = \int_{-\infty}^0 \left( \sqrt{q(x)} + \nu^2 - \sqrt{q(x)} \right) dx = \log t_{12}(0).
\]

As in Sect. 2.2.1, the function \( a_0(\lambda) \) admits an asymptotic expansion of the type (2.9). The corresponding Riccati equation (2.6) has an asymptotic solution

\[
\sigma(x, \lambda) = \sum_{n=1}^\infty \frac{c_n(x)}{\nu^n} + O(\nu^{-\infty}),
\]

where the coefficients \( c_n(x) \) are determined recursively and are expressed in terms of the potential \( q(x) \) and its derivatives. Thus we obtain the trace identities

\[
\log a(\lambda) = \alpha(\nu) + \sum_{n=1}^\infty \frac{c_n}{\nu^n} + O(\nu^{-\infty}), \quad \text{where} \quad c_n = \alpha_n - \int_{-\infty}^\infty c_n(x) dx.
\]
Remark. The interesting example of the harmonic oscillator — the potential $q(x) = x^2$ — does not satisfy condition (2.7). However, one can modify the outlined here method by exploiting the substitution $x = \sqrt{-\lambda} t$.\footnote{This is the substitution used in \cite{17} to obtain uniform asymptotic for Weber functions (parabolic cylinder functions).} In this case the Fredholm determinant is

$$a(\lambda) = \frac{2^{-\frac{\lambda}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-\lambda}{2}\right)}$$

and corresponding trace identities give an alternate derivation of the Stirling asymptotic expansion for the Euler gamma-function. We plan to discuss it in a separate publication.

3 Examples

3.1 Potential $q(x) = e^{2x}$ on the half-line

It is well-known that the differential equation

$$-\psi'' + e^{2x} \psi = \lambda \psi$$

has two linearly independent solutions, the modified Bessel function of the first kind $I_{\nu}(e^x)$, and the modified Bessel function of the second kind $K_{\nu}(e^x)$, where $\nu = \sqrt{-\lambda}$. For fixed $\nu$ and real $y \to \infty$ we have the asymptotic

$$K_{\nu}(y) = \frac{\sqrt{\pi}}{2y} e^{-y} \left(1 + O(y^{-1})\right), \quad (3.1)$$

so the eigenvalues determined by the boundary condition $\psi(0) = 0$ are the zeros $\{\lambda_n\}_{n=1}^{\infty}$ of an entire function $K_{\nu}(1)$, where $\lambda = -\nu^2$. The eigenvalues $\lambda_n$ are positive, simple and accumulate to infinity, and

$$a(\lambda) = \frac{K_{\nu}(1)}{K_{0}(1)}. \quad (3.2)$$

Remark. It is well-known (see, e.g., \cite{19}) that the total number of zeros of $K_{\nu k}(1)$ with $0 < k < T$ is

$$\frac{T}{\pi} \log \frac{2T}{e} + O(1),$$

so for the eigenvalue counting function $N(\lambda)$ we obtain

$$N(\lambda) = \frac{\sqrt{\lambda}}{\pi} \log \frac{2\sqrt{\lambda}}{e} + O(1).$$

This also easily follows from the Weyl’s law. Thus as $n \to \infty$ for the $n$-th eigenvalue $\lambda_n$ we have

$$\lambda_n \simeq \left(\frac{2\pi n}{e W\left(\frac{2\pi n}{e}\right)}\right)^2,$$

where $W(x)$ is the Lambert function, $W(x)e^{W(x)} = x$ for $x > 0$.\footnote{This is the substitution used in \cite{17} to obtain uniform asymptotic for Weber functions (parabolic cylinder functions).}
The asymptotic expansion of $\log a(\lambda)$ as $\lambda = -\nu^2 \to -\infty$ can be easily obtained from the well-known asymptotic of the modified Bessel function of the second kind. However, it can also be derived directly, using the method outlined in Sect. 2.2.1.

Namely, we have
\[
\int_0^x \sqrt{e^{2x} + y^2} \, ds = \sqrt{\nu^2 + e^{2x} + \nu x - \nu \log \left( \nu + \sqrt{\nu^2 + e^{2x}} \right)} - \sqrt{\nu^2 + 1 + \nu \log \left( \nu + \sqrt{\nu^2 + 1} \right)}.
\]

Since
\[
\sqrt{\nu^2 + e^{2x} + \nu x - \nu \log \left( \nu + \sqrt{\nu^2 + e^{2x}} \right)} = e^x + O(e^{-x}) \quad \text{as} \quad x \to \infty,
\]
comparison with (2.2) gives $K_{\nu}(e^x) = \psi_1(x, \lambda)$, where
\[
C_1(\lambda) = \sqrt{\pi} e^{-\nu^2 + 1 - \nu \log(\nu^2 + 1)}. \tag{3.3}
\]

Of course, this is a well-known asymptotic of the modified Bessel function of the second kind,
\[
K_{\nu}(e^x) = \sqrt{\pi} e^{-\nu^2 + 1 - \nu \log(\nu^2 + 1)} \left( 1 + O(\nu^{-1}) \right), \tag{3.4}
\]
which is uniform for $-\infty < x < \infty$ (see [17, Ch. 10, §7]).

Now the function $\chi(x, \lambda)$, defined in (2.4), satisfies
\[
\lim_{x \to \infty} \chi(x, \lambda) = 0,
\]
\[
\chi(0, \lambda) = \log a(\lambda) - a_0(\lambda),
\]
where
\[
a_0(\lambda) = \sqrt{\nu^2 + 1 - \nu \log(\nu^2 + 1)} - \frac{1}{4} \log(\nu^2 + 1) + \frac{1}{2} \log \frac{\pi}{2} - \log K_0(1).
\]

The function $a_0(\lambda)$ admits an asymptotic expansion (2.9), where
\[
\alpha_0(\lambda) = \nu \log(2\nu) - \nu - \frac{1}{2} \log \nu + \frac{1}{2} \log \frac{\pi}{2} - \log K_0(1).
\]

This formula agrees with the asymptotic
\[
K_{\nu}(e^x) = \sqrt{\pi} \left( \frac{e^{x+1}}{2\nu} \right)^{\nu} \left( 1 + O(\nu^{-1}) \right) \tag{3.4}
\]

One needs to put $z = 1/\nu$ and $p = \nu/\sqrt{1 + \nu^2}$ in formula (7.17) in [17, Ch. 10, §7].
for fixed $x$ and $\nu \to \infty$, and with the Weyl’s law.

The function $\sigma(x, \lambda)$, defined in (2.5), satisfies the Riccati equation (2.6). Its asymptotic expansion (2.10) can be easily obtained by using the substitution $\tau(t, \lambda) = \sigma(t + \log \nu, \lambda)$, which transforms (2.6) into the equation

$$\tau' = -\tau^2 + 2\nu \sqrt{1 + e^{2t} \tau} + \frac{e^{2t}}{1 + e^{2t} \tau} + \frac{4e^{2t} - e^{4t}}{4(1 + e^{2t})^2}. \quad (3.5)$$

We have

$$\tau(t, \nu) = \sum_{n=1}^{\infty} \frac{c_n(t)}{\nu^n},$$

where

$$c_1(t) = \frac{e^{4t} - 4e^{2t}}{8(1 + e^{2t})^{3/2}}$$

and recursively

$$2\sqrt{1 + e^{2t}} c_{n+1}(t) = c'_n(t) - \frac{e^{2t}}{1 + e^{2t}} c_n(t) + \sum_{k=1}^{n} c_k(t) c_{n-k}(t).$$

Plugging this into (2.12), we obtain

$$\log a(\lambda) = \nu \log(2\nu) - \nu - \frac{1}{2} \log \nu + \frac{1}{2} \log \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{\alpha_n - c_n(\nu)}{\nu^n} + O(\nu^{-\infty}),$$

where

$$c_n(\nu) = -\int_{-\log \nu}^{\infty} \sigma_n(t) dt.$$

Moreover, it can be shown (cf. [17, Ch.10, §7]) that the coefficients $c_n(\nu)$ are polynomials in $\nu/\sqrt{\nu^2 + 1}$, which gives the asymptotic expansion of $\log a(\lambda)$.

### 3.2 Potential $q(x) = 2 \cosh 2x$ on the line

Here we consider the modified Mathieu differential equation

$$-\psi'' + 2 \cosh 2x \psi = \lambda \psi, \quad -\infty < x < \infty. \quad (3.6)$$

It is well-known that the antiderivative of the function $\sqrt{2 \cosh 2x - \lambda}$ is expressed in terms of elliptic integrals. Namely, let

$$F(\varphi, k) = \int_{0}^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{and} \quad E(\varphi, k) = \int_{0}^{\varphi} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$
be, correspondingly, elliptic integrals of the first and second kinds, where \(0 < k^2 < 1\) and \(-\pi/2 < \varphi < \pi/2\). Putting \(\lambda = 2 - \nu^2\), where \(\nu > 0\), we have for \(\nu > 2\) (see [20, formula (22) on p. 136])

\[
\int_0^x \sqrt{\nu^2 - 2 + 2 \cosh 2s} \, ds = \nu(F(\varphi, k) - E(\varphi, k)) + \tanh \frac{x}{2} \sqrt{\nu^2 - 2 + 2 \cosh 2x}, \tag{3.7}
\]

where

\[
\varphi = \sin^{-1} \left( \tanh \frac{x}{2} \right) \quad \text{and} \quad k^2 = \frac{\nu^2 - 4}{\nu^2}. \tag{3.8}
\]

The corresponding solutions \(\psi_1(x, \lambda)\) and \(\psi_2(x, \lambda)\) of the differential equation (3.6) with asymptotics (2.2)–(2.3) are

\[
\psi_1(x, \lambda) \approx \frac{C_1(\lambda)}{\sqrt{2 \cosh 2x + \nu^2}} e^{\nu(F(\varphi, k) - E(\varphi, k)) + \tanh \frac{x}{2} \sqrt{\nu^2 - 2 + 2 \cosh 2x}}, \tag{3.9}
\]

\[
\psi_2(x, \lambda) \approx \frac{C_2(\lambda)}{\sqrt{2 \cosh 2x + \nu^2}} e^{-\nu(F(\varphi, k) - E(\varphi, k)) - \tanh \frac{x}{2} \sqrt{\nu^2 - 2 + 2 \cosh 2x}}, \tag{3.10}
\]

where \(\lambda = 2 - \nu^2 \to -\infty\) (cf. [21]). Since \(\varphi \to \pi/2\) as \(x \to \infty\), we have

\[
\int_0^x \sqrt{\nu^2 - 2 + 2 \cosh 2s} \, ds = \nu(K(k) - E(k))) + e^x + O(e^{-x}),
\]

where \(K(k)\) and \(E(k)\) are, respectively, complete elliptic integrals of the first and second kinds. Choosing the constants in (3.9)–(3.10) as

\[
C_1(\lambda) = \sqrt{\frac{\pi}{2}} e^{-\nu(K(k) - E(k))} \quad \text{and} \quad C_2(\lambda) = \sqrt{\frac{1}{2\pi}} e^{\nu(K(k) - E(k))}, \tag{3.11}
\]

we get a solution \(\psi_1(x, \lambda)\) with the same asymptotic as \(x \to \infty\) as the modified Bessel function of the second kind \(K_{i\sqrt{\lambda}}(e^x)\),

\[
\psi_1(x, \lambda) = \sqrt{\frac{\pi}{2e^x}} e^{-e^x} (1 + O(e^{-x})) \quad \text{as} \quad x \to \infty,
\]

and a solution \(\psi_2(x, \lambda)\) with the same asymptotic as \(x \to \infty\) as the modified Bessel function of the first kind \(I_{i\sqrt{\lambda}}(e^x)\),

\[
\psi_2(x, \lambda) = \sqrt{\frac{1}{2\pi e^x}} e^{e^x} (1 + O(e^{-x})) \quad \text{as} \quad x \to \infty.
\]

We have \(W(\psi_1, \psi_2) = 1\) and

\[
\psi_1(x, \lambda) = t_{11}(\lambda)\psi_1(-x, \lambda) + t_{12}(\lambda)\psi_2(-x, \lambda), \tag{3.12}
\]

\[
\psi_2(x, \lambda) = t_{21}(\lambda)\psi_1(-x, \lambda) + t_{22}(\lambda)\psi_2(-x, \lambda).
\]
where
\[ t_{12}(\lambda) = W(\psi_1(x, \lambda), \psi_1(-x, \lambda)) \]  
(3.13)
is an entire function of order 1/2 with zeros — the eigenvalues \( \lambda_n \) of the Schrödinger operator (1.1). Corresponding Fredholm determinant is

\[ a(\lambda) = \frac{t_{12}(\lambda)}{t_{12}(0)}. \]

Let \( \chi(x, \lambda) \) be the function defined in (2.4). Since the antiderivative (3.7)–(3.8) is an odd function of \( x \), as in Sect. 2.2.2 we get

\[ \lim_{x \to \infty} \chi(x, \lambda) = 0, \]  
(3.14)\[ \lim_{x \to -\infty} \chi(x, \lambda) = \log a(\lambda) + \log \pi + \log t_{12}(0) - 2\nu(K(k) - E(k)). \]  
(3.15)

We state the main result of this section.

**Theorem 1.** Fredholm determinant \( a(\lambda) \) of the Schrödinger operator (1.1) admits the following asymptotic expansion as \( \lambda = 2 - \nu^2 \to -\infty \)

\[ \log a(\lambda) = 2\nu(K(k) - E(k)) - \log \pi - \log t_{12}(0) + \sum_{n=1}^{\infty} \frac{c_n}{\nu^n} + O(|\nu|^{-\infty}), \]  
(3.16)

where \( k^2 = 1 - 4\nu^{-2} \) and the coefficients \( c_n \) are determined explicitly.

**Proof.** It follows from (3.14)–(3.15) that

\[ \log a(\lambda) = 2\nu(K(k) - E(k)) - \log \pi - \log t_{12}(0) - \int_{-\infty}^{\infty} \sigma(x, \lambda) dx, \]  
(3.17)

where \( \sigma(x, \lambda) \) is defined in (2.5) and satisfies the Riccati equation (2.6). Using

\[ 2 \cosh 2x - \lambda = \nu^2 + 2 \cosh 2x - 2 = \nu^2 + 4 \sinh^2 x, \]

we can rewrite the equation (2.6) as

\[ \frac{d\sigma}{dx} = -\sigma^2 + 2\sqrt{\nu^2 + 4 \sinh^2 x} \sigma + \frac{2 \sinh 2x}{\nu^2 + 4 \sinh^2 x} \sigma \]

\[ + \frac{8 \cosh 2x}{\nu^2 + 4 \sinh^2 x} - 5 \left( \frac{\sinh 2x}{\nu^2 + 4 \sinh^2 x} \right)^2. \]

It is convenient to change variables by \( \sinh x = \frac{\nu}{2} \sinh y \), so

\[ \nu^2 + 4 \sinh^2 x = \nu^2 \cosh^2 y \]
and
\[
\frac{dx}{dy} = \frac{\nu \cosh y}{2 \cosh x} = \frac{\nu \cosh y}{\sqrt{4 + \nu^2 \sinh^2 y}} = \left(1 - \frac{k^2}{\cosh^2 y}\right)^{-\frac{1}{2}}.
\]

Next, introduce the function \( \tau(y, \lambda) = \sigma \left( \sinh^{-1} \left( \frac{\nu}{2} \sinh y \right), \lambda \right) \), so
\[
\int_{-\infty}^{\infty} \sigma(x, \lambda) \, dx = \int_{-\infty}^{\infty} \tau(y, \lambda) \left(1 - \frac{k^2}{\cosh^2 y}\right)^{-\frac{1}{2}} \, dy. \tag{3.18}
\]

The Riccati equation for \( \tau(y, \lambda) \) takes the form
\[
\left(1 - \frac{k^2}{\cosh^2 y}\right)^{\frac{1}{2}} \frac{d\tau}{dy} = -\tau^2 + 2\nu \cosh y \tau + \tanh y \left(1 - \frac{k^2}{\cosh^2 y}\right)^{\frac{1}{2}} \tau \\
+ 4 \tanh^2 y - \frac{5}{4} \tanh^4 y + \frac{1}{\nu^2 \cosh^2 y} (8 - \frac{5}{4} \tanh^2 y). \tag{3.19}
\]

Now it is straightforward to show that that equation (3.19) admits an asymptotic solution
\[
\tau(y, \lambda) = \sum_{n=1}^{\infty} \frac{\tau_n(y)}{\nu^n},
\]
where
\[
\tau_1(y) = -\frac{1}{2 \cosh y} (4 \tanh^2 y - \frac{5}{4} \tanh^4 y).
\]

The coefficients \( \tau_n(y) \) are obtained recursively using the expansion
\[
\left(1 - \frac{k^2}{\cosh^2 y}\right)^{\frac{1}{2}} = \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{m}\right) \frac{k^{2m}}{\cosh^{2m} y},
\]
and the binomial expansion for \( k^{2m} = (1 - 4\nu^{-2})^m; \) only finitely many terms from these expansions are needed in order to get the \( n \)-th term. Substituting the asymptotic expansion for \( \tau(y, \lambda) \) into (3.18) and (3.17) and using
\[
\left(1 - \frac{k^2}{\cosh^2 y}\right)^{\frac{1}{2}} = \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{m}\right) \frac{k^{2m}}{\cosh^{2m} y},
\]
we get the desired asymptotic expansion (3.16).

\( \square \)

**Remark.** Since
\[
K(k) = \log \frac{4}{\sqrt{1 - k^2}} + o(1) \quad \text{and} \quad E(k) = 1 + o(1) \quad \text{as} \quad k \to 1,
\]
we have as $\nu \to \infty$,

$$\nu(K(k) - E(k)) = \nu \log(2\nu) - 2\nu + o(1).$$

Here the $o(1)$ term — the remainder — is of the form $f_1(\nu^{-1}) + f_2(\nu^{-1}) \log \nu$, where $f_1(x)$ and $f_2(x)$ are convergent for $|x| < 1$ power series in $x^2$ that vanish at $x = 0$ (see [22, p. 54]).

## 4 Existence of the solution $\psi_1(x, \lambda)$

Here for the convenience of the reader we prove that differential equation (3.6) has a solution $\psi_1(x, \lambda)$ which asymptotically as $x \to \infty$ behaves like the modified Bessel function of the second kind. Namely, we have the following result.

**Theorem 2.** Modified Mathieu equation (3.6) has a solution $\psi_1(x, \lambda)$ with the following asymptotic

$$\psi_+(x, \lambda) = K_{ik}(e^x)(1 + o(1)) \quad \text{as} \quad x \to \infty, \quad \text{where} \quad \lambda = k^2.$$

For fixed $x$ the function $\psi_1(x, \lambda)$ is entire of order $1/2$.

**Proof.** We consider the Schrödinger operator $H = -\frac{d^2}{dx^2} + 2 \cosh 2x$ as a perturbation of the operator $H_0 = -\frac{d^2}{dx^2} + e^{2x}$ by a small as $x \to \infty$ potential $e^{-2x}$. The operator $H_0$ has a simple absolutely continuous spectrum $[0, \infty)$, and for $\lambda \in \mathbb{C} \setminus [0, \infty)$ its resolvent $R_0^0 = (H_0 - \lambda I)^{-1}$ is an integral operator in $L^2(\mathbb{R})$ with the integral kernel (see, e.g., [23, Ch. 4.15])

$$K_\lambda^0(x, y) = \begin{cases} I_{-ik}(e^x)K_{ik}(e^y), & x \leq y, \\ K_{ik}(e^x)I_{-ik}(e^y), & x \geq y. \end{cases}$$

(4.1)

Here $k = \sqrt{\lambda}$ and $\text{Im} \, k > 0$. The operator $H_0$ has a Volterra type Green’s function

$$G(x, y, k) = \begin{cases} I_{-ik}(e^x)K_{ik}(e^y) - K_{ik}(e^x)I_{-ik}(e^y), & x \leq y, \\ 0, & x > y. \end{cases}$$

Since

$$K_\nu(z) = \frac{\pi}{2 \sin \pi \nu} (I_{-\nu}(z) - I_\nu(z)),$$

$G(x, y, k)$ is an even function of $k$ and, therefore, is an entire function of $\lambda$ of order $1/2$.

The function $\psi_1(x, \lambda)$ satisfies the integral equation

$$\psi_1(x, \lambda) = K_{ik}(e^x) + \int_x^\infty G(x, y, k)e^{-2y}\psi_1(y, \lambda)dy,$$

(4.2)
which can be solved by successive approximations. Indeed, put \( f_0(x, k) = K_{ik}(e^x) \) and
\[
f_n(x, k) = \int_x^\infty G(x, y, k)e^{-2y}f_{n-1}(y, k)dy.
\]
Using the estimates
\[
|K_{ik}(e^x)| \leq Ce^{-x}e^{-\frac{1}{2}x} \quad \text{and} \quad |I_{ik}(e^x)| \leq Ce^xe^{-\frac{1}{2}x}
\]
for \( a \leq x < \infty \) (the constant \( C \) depends on \( a \), we have \( f_n(x, -k) = f_n(x, k) \) and
\[
|f_n(x, k)| \leq \frac{2^n C^{2n+1}}{3^n n!} e^{-x} e^{-\frac{6n+1}{2}x},
\]
which can be easily proved by induction. Namely, (4.4) holds for \( n = 0 \), and using the estimates (4.3), we get
\[
|f_{n+1}(x, k)| \leq |I_{-ik}(e^x)| \int_x^\infty e^{-2y}|K_{ik}(e^y)f_n(y, k)|dy
\]
\[
+ |K_{ik}(e^x)| \int_x^\infty e^{-2y}|I_{-ik}(e^y)f_n(y, k)|dy
\]
\[
\leq \frac{2^n C^{2n+3}}{3^n n!} \left[ e^{-x} e^{-\frac{1}{2}x} \int_x^\infty e^{-2y}e^{-2y} e^{-\frac{1}{2}y} e^{-\frac{6n+1}{2}y} dy
\]
\[
+ e^{-x} e^{-\frac{1}{2}x} \int_x^\infty e^{y} e^{-\frac{1}{2}y} e^{-2y} e^{-y} e^{-\frac{6n+1}{2}y} dy \right]
\]
\[
\leq \frac{2^{n+1} C^{2n+3}}{3^{n+1} (n + 1)!} e^{-x} e^{-\frac{6n+3}{2}x}.
\]
Thus
\[
\psi_1(x, \lambda) = \sum_{n=0}^\infty f_n(x, k)
\]
is given by the absolutely convergent series and
\[
|\psi_1(x, \lambda) - K_{ik}(e^x)| \leq Ce^{-x}e^{-\frac{1}{2}x}. \quad \square
\]
As in Sect. 3.2, we have
\[
t_{12}(\lambda) = W(\psi_1(x, \lambda), \psi_1(-x, \lambda)) = -2\psi_1(0, \lambda)\psi_1'(0, \lambda),
\]
and the eigenvalue problem (3.6) is equivalent to two radial eigenvalue problems
\[
-\psi'' + 2 \cosh 2x \psi = k^2 \psi, \quad 0 < x < \infty, \quad \psi(0) = 0,
\]
\[
-\psi'' + 2 \cosh 2x \psi = k^2 \psi, \quad 0 < x < \infty, \quad \psi'(0) = 0.
\]

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