On counterterms in cosmological perturbation theory

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Cosmological perturbation theory is the theory of fluctuations (scalar as well as tensor) around the inflationary cosmological background solution. It is important to understand the details of the process of renormalization in this theory. In more familiar applications of quantum field theory, the dependence on the external momenta of the dimensionally regulated expression of the one-loop contribution to a correlator determines the number of counter terms (and their forms) required to renormalize it. In this work, it is pointed out that in cosmological perturbation theory, though this still happens, it happens in a completely different way such that in the late time limit, the information about the number and forms of counter terms required gets erased. This is to be compared with what happens in spontaneous symmetry breaking where the use of fluctuation fields around a chosen vacuum seems to suggest that more counter terms shall be needed to renormalize the theory than are actually required. We also comment on how the field strength of curvature perturbation, \( \zeta \), could get renormalized.

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I. INTRODUCTION

The methods of quantum field theory (QFT) are some of the most universally applicable techniques in all of physics. The same techniques apply to (among other things) elementary particle physics \(^1\), to statistical and condensed matter physics \(^2\), to the non-linear theory of structure formation in the universe \(^3\), to the theory of turbulence in fluids and plasmas \(^4\), to the calculation of spectrum of gravitational waves from inspiralling binaries \(^5\), to quantum optics \(^6\), to black hole thermodynamics \(^7\), as well as to the calculation of correlations of primordial metric perturbations in inflationary cosmology \(^8\). In this light, it is not surprising that the details of calculations in these varied scenarios, though similar, are not exactly identical. Thus, it is important to find out which ideas apply universally to all these problems and which ones are specific to the applications we are most familiar with (e.g. scattering problems). To aid the discussion, in this work, the familiar version of QFT shall be referred to as “usual” QFT.

It is a well known fact that inflationary cosmological perturbation theory is almost like a QFT of fluctuations around a time-dependent background solution \(^10\). Consequently, cosmological perturbation theory derives most of its calculational machinery from the usual QFT. Still, there are many differences, and in this work, we look at some of these differences. The interactions of metric perturbations cause corrections to cosmological correlations calculated in linear perturbation theory. This can cause not only primordial non-Gaussianity \(^10\), but also, higher order corrections to e.g. the two-point function \(^11\).

Just like in the usual QFT, in cosmological perturbation theory too, while calculating correlations at sufficiently high order, we encounter expressions which are ultraviolet (UV) divergent. In usual QFT, we cancel these infinities by (i) collecting the divergent terms with similar dependence on external momenta, (ii) finding which terms (allowed by the symmetries of the theory) in the action can lead to the terms in the correlations with the said dependence on external momenta, and finally, (iii) adding these counterterms to the old action (which is now called the renormalized action while the sum of the renormalized action and the counterterm is the bare action) in order to get correlations which are UV finite at every step of the calculation.

Does the same procedure work in cosmological perturbation theory too? As the analysis of this work illustrates, there are subtleties associated with this, one thus has to be extremely careful (in this context, see \(^12\) \(^14\)). In cosmological perturbation theory, one is studying fluctuations around a time dependent background solution, thus, the action of these fluctuations is not Lorentz invariant. Thus, we can not simply list all the possible terms in the action (moreover, since the typical interactions are irrelevant, infinite counterterms shall be required). All this ensures that it is difficult to spot the counterterms in cosmological perturbation theory. Given this situation, one could ask, given a correlator (e.g. the two-point function), which counterterms shall we need to renormalize it at one loop?

From our experience in usual QFT, we are used to spotting the counterterms by looking at the expression for dimensionally regulated correlators. This is because the expression for the correlator (with external lines amputated) is of the form \( \sum_n c_n(k^n) \) (where, \( k \) is the Lorentzian momentum of the external line) i.e., a polynomial in the external momenta with divergent coefficients.

Unlike in usual QFT, in cosmological perturbation theory, the dimensionally regulated logarithmically divergent two-point correlator (for external momentum \( k \)) is
momentum integrals at non-zero of counterterms also determines the number of observa-
gets rid of UV divergences gets erased. Since the number
needed and the number of counterterms needed to
integrals, the information about the forms of countert-
taken before regularizing the UV divergent momentum
in the
\[ \eta \]
theory. During inflation, we are typically interested
form (for illustrative purposes, we work with Euclidean
\[ I_Q = \int \frac{d^d \ell}{(\ell^2 + \Delta)} , \] (2)
which, on dimensional regularization gives
\[ I_Q = \Delta \left( \frac{\mu^2}{\Delta} \right)^{\delta/2} F(\delta) , \]
where, \( \delta = 4 - d, \mu \) is the fake renor-
malization scale (which inevitably gets introduced while
performing dimensional regularization) and \( F(\delta) \) is a di-
mensionless function of \( \delta \) which contains poles of \( \delta \). If \( F \)
has a simple pole and its Laurent series expansion of is
\[ F = F_{\text{LP}} \delta^{-1} + F_0 + F \delta + \cdots , \]
then,
\[ I_Q = \Delta \left( \frac{F_{\text{LP}}}{\epsilon} + \frac{F_0 - 1}{2} \log \left( \frac{\mu^2}{\Delta} \right) + F_0 + \cdots \right) . \] (3)
Typically, \( \Delta \) is a polynomial in the external momentum
\[ \frac{k^2}{m^2} = -E^2 + \vec{p}^2 \]
or the masses (and often, the Feynman parameters). The presence of \( \Delta \) in the above expression
causes the dimensionally regulated UV divergent integral
to be a sum of two parts; the first part is polynomial in
the external momentum with divergent coefficients and
the second one is a finite function of the external mo-
menta, the masses and the fake renormalization scale \( \mu \).
E.g. while evaluating the two-point function in \( \phi^4 \) the-
ory in \( d = 4 \), the corresponding \( \Delta \) turns out to be simply
\( m^2 \), so this is a trivial example. On the other hand, while
renormalizing the two-point function for \( \phi^4 \) in \( d = 6 \), we
find that the Fourier transform of the two-point function
is given by an expression of the form
\[ G(k^2) = G_{\text{Free}}(k^2) + G_{\text{Free}}(k^2) [\Pi(k^2)] G_{\text{Free}}(k^2) + \cdots , \] (4)
where \( G_{\text{Free}}(k^2) \) is the Feynman propagator. \( \Pi(k^2) \) is
the contribution of the one-loop diagram (with external
lines amputated) and is given by a quadratically diver-
gent integral and the expression for dimensionally regu-
lated \( \Pi(k^2) \) is of the form
\[ \Pi(k^2) = A_2 \left( \frac{1}{\epsilon} \right) k^2 + A_0 \left( \frac{1}{\epsilon} \right) m^2 + F(k^2, m, \mu) . \] (5)
The following points are important to notice
1. \( \Pi(k^2) \) has two kinds of contributions: (1) a poly-
nomial in the external momentum \( k^2 \) (the \( m^2 \) term
  can be thought of as the term \( k^2 \delta^0 \)), (2) another
  function \( F \), which depends on, among other things, the
  fake scale \( \mu \).
2. The coefficients of the polynomial in the external
  momentum \( k^2 \), the \( A_i \), are divergent, they are func-
tions of \( 1/\epsilon \), on the other hand, the function \( F \) is
  finite.
3. The forms of the terms in the polynomial i.e. how
  they depend on the external momenta, dictate the form
  and number of counter-terms needed to be
  introduced in the Lagrangian in order to cancel the

\[ \sum_n c_n(|k|)^n \] (1)
where, \( k \) is the 3-momentum of the external line and
\( \eta \) is the conformal time at which the correlator is evalu-
ated. We thus have an additional factor involving a poly-
nomial in \( -k \eta \); it is this extra piece which determines the
forms and number of counter terms needed to renormalize
the theory. During inflation, we are typically interested
in cosmological perturbation theory. Their main focus
is on the non-trivial logarithmic running which may
appear (see, e.g., [12]). E.g. Senatore and Zaldarriaga
[14] have studied renormalization of two-point function
in cosmological perturbation theory. Their main focus
has been on the nature of logarithmic running. In con-
trast, in the present work, we look at the actual process
of renormalization and the associated subtleties.

We begin in Sec. II by recalling how counterterms are
found in usual QFT. Then, in sec. III after introducing
the particular regime of effective field theory of inflation
for which we present the arguments about the UV diver-
genent two-point correlator, we shall describe how the case
of cosmological perturbation theory is so different from
the usual QFT in so peculiar a way. We summarize the
results in Sec. IV We have set \( \hbar = c = 1 \).

II. COUNTERTERMS IN USUAL QFT:

In usual QFT, LSZ reduction formula ensures that the
most relevant quantity to evaluate is the vacuum expecta-
tion value of time ordered product of the Heisenberg
picture fields. While evaluating the Fourier transforms
of such correlators, at sufficiently high order in pertur-
bation theory one encounters UV divergences. E.g. one
could encounter quadratically divergent integrals of the
form (for illustrative purposes, we work with Euclidean
integrals)
\[ I_Q = \int \frac{d^d \ell}{(\ell^2 + \Delta)} , \]

Homogeneity of the inflationary background implies that all such correlators shall be invariant under translations in space at a fixed time, which implies that the Fourier transform of the above correlator shall be of the form

\[ F(\eta, k_1, \cdots, k_n) = (2\pi)^3 \delta^3(\mathbf{k}_1 + \cdots + \mathbf{k}_n) \tilde{G}(\eta, k_1, \cdots, k_n) . \]  

It is worth mentioning that whenever we talk about the usual QFT, we shall be dealing with four dimensional Lorentzian momenta while whenever we talk about cosmological perturbation theory, we shall be dealing with three dimensional Euclidean momenta. At sufficiently high order in perturbation theory, one expects to encounter Feynman diagrams with loops. This issue, in cosmological perturbation theory has been studied in great detail in the last few years. Beginning with [12], there was a debate about whether these loop corrections to the cosmological correlations shall freeze at late times. Recently (see [11, 19]), it is claimed to be shown that this shall surely happen at all loops. The familiar primordial power spectrum \( \Delta^2(k) \) is defined by the Eq.

\[ \lim_{\eta \to 0^-} \langle \zeta(\eta, \mathbf{k}) \zeta(\eta, \mathbf{k}') \rangle = \frac{2\pi^2}{k^3} \delta^3(\mathbf{k} + \mathbf{k}') \Delta^2(k) , \]  

for slow roll inflation with a single (canonical) scalar field, the lowest order (i.e. tree level) contributions to power spectrum, assuming Bunch-Davies vacuum (in the limit \( \eta \to 0^- \)), is given by (the classic result)

\[ \Delta^2(k)_{\text{tree}} = \frac{1}{2\epsilon(\eta_k)} \left( \frac{H(\eta_k)}{2\pi M_{\text{Pl}}} \right)^2 , \]  

where \( \eta_k \) is the conformal time when the mode in question crosses Hubble radius (i.e. when \( k = aH \)). Notice that, on comparing with Eq (10), it becomes clear that for two-point function,

\[ \lim_{\eta \to 0^-} \tilde{G}(\eta, \mathbf{k}) = \frac{1}{4\pi k^3} \Delta^2(k) = \frac{1}{32\pi^3 k^3} \left( \frac{H^2}{\epsilon M_{\text{Pl}}^2} \right) . \]  

Since the result in Eq (11) is obtained in linear theory, it corresponds to a free theory calculation. A well known fact is that Eq (11) implies that, the dimensionless Primordial Power spectrum is a power law

\[ \Delta^2_{\zeta, \text{Free}}(k) = A_s \left( \frac{k}{k_0} \right)^{n_s - 1} , \]  

where \( A_s \) is the spectral index while \( n_s \) is the spectral amplitude and \( k_0 \) is a pivot scale, see [4]. This implies that

\[ \tilde{G}_{\text{Free}}(k) = \frac{A_s}{4\pi k^3} \left( \frac{k}{k_0} \right)^{n_s - 4} . \]  

The leading interactions of the metric fluctuations are typically due to cubic operators, so, it is expected that
the loop correction to \( \hat{G} \) shall be of the form (notice that the mass dimension of \( \hat{G} \) is \(-3\))

\[
\hat{G}_{1\text{-loop}} = k^3 (\hat{G}_{\text{tree}})^2 \times \text{(factors)}.
\]

(15)

The factors on the RHS can involve various non-trivial logarithmic running e.g. \( \log a \sim Ht \) or \( \log kL \) or \( \log (H/\mu) \) while a running of the form \( \log k/\mu \) is not possible since this shall not leave \( \hat{G} \) invariant under the transformation \( x \rightarrow \Lambda x, k \rightarrow k/\Lambda \) and \( a \rightarrow a/\Lambda \).

\section{A. The \( \hat{\pi}^3 \) theory}

While evaluating correlations for an interacting quantum field on an accelerating universe, one can encounter, apart from the familiar UV divergences, relatively unfamiliar divergences too. To avoid having to deal with these unfamiliar divergences, and still deal with a realistic model of cosmological perturbations which has simple interactions, we work with “the theory of large \( \hat{\pi}^3 \) interactions [14].” The Effective Field Theory (EFT) of inflation provides the most general framework for systematically studying the dynamics of fluctuations around an inflationary background solution. The action of the theory of fluctuations can be expanded in powers of the relevant fluctuation field (and also in powers of the slow-roll parameters such as the Hubble flow functions). The theory is first formulated in unitary slicing of the perturbed spacetime in which \( \Delta \phi \) vanishes (and all the dynamics lies in the metric) and then general covariance is restored by introducing the Stueckelberg field (denoted by \( \pi \) in the following). It turns out that if one chooses to ignore \( \mathcal{O}(\epsilon^2) \) terms in the action (which also corresponds to the interaction terms which give rise to primordial non-Gaussianity for a canonical scalar field, see [10]) and one chooses to fix the sound speed of fluctuations to unity (i.e. we wish to only consider the cases in which \( c_s \rightarrow 1 \)), \( 1 \) the leading order interactions for the Stueckelberg field shall be captured by terms of the form (ignoring \( \mathcal{O}(\pi^5) \) terms, see [22])

\[
\frac{M_4^3}{6} (g^{00} + 1)^3 = -\frac{2}{3} M_3^3 \left[ 2\hat{\pi}^3 + 3\hat{\pi}^4 - \frac{3}{a^2} \hat{\pi}^2 (\partial_i \pi)^2 \right].
\]

(16)

It is important to recognize that, thanks to the EFT formulation, it is very easy to identify a regime in which the Stueckelberg field has non-negligible self-interactions without violating the slow-roll nature of the background solution. Since the symmetry arguments can not fix the value (or sign) of the coefficient \( M_4 \), (which should be determined from observations, see [22] for the latest limits), one can write \( M_4^3(t) = -c_3(t) M^4 \) where \( M \) is a mass scale characterizing the interaction. An extra shift symmetry can be imposed requiring that the time dependence of \( c_3 \) is negligibly weak.

In this regime, the action of the \( \pi \) field becomes (see [14])

\[
S = \int d^4x \ a^3 \left[ -\dot{H} M_{Pl}^2 \left( \hat{\pi}^2 - \frac{1}{a^2} (\partial_i \pi)^2 \right) + \frac{2}{3} c_3 M^4 \left( 2\hat{\pi}^3 + 3\hat{\pi}^4 - \frac{3}{a^2} \hat{\pi}^2 (\partial_i \pi)^2 \right) \right],
\]

(17)

This is perhaps the simplest possible interacting cosmological perturbation theory. It is also observationally interesting (see [23]). This is the most general kind of interactions which are not slow roll suppressed and which are leading order when we impose the requirement that \( c_s \rightarrow 1 \). To connect to the usual perturbation variables, one can make a gauge transformation to comoving gauge (see [24]) and find that \( \zeta = -H \pi + \mathcal{O}(\epsilon^2) \).

We would like to emphasize again that for the inflationary background caused by canonical scalar fields, the interaction terms in the action of \( \zeta \) are \( \mathcal{O}(\epsilon^2) \) so that the interactions we are dealing with are different from those. Moreover, as is obvious from Eq (17), in this regime the interaction terms are much simpler (and fewer) as compared to e.g. those in [10]. Thus, if we wish to try anything new e.g. loop corrections in cosmological perturbation theory and we want a regime which is realistic but which is also simple, this “theory of large \( \hat{\pi}^3 \) interactions” is the best possible choice [14].

Given the action (Eq (17)), the Hamiltonian can be readily worked out and then one can use the in-in formalism (see [12, 22] and references therein) to evaluate the two-point function at one-loop. In the rest of the present work, we shall focus on only the \( \hat{\pi}^3 \) term in the action given by Eq. (17). This interaction term leads to two contributions to the two-point function only one of which is UV divergent (see [14]). On canonical normalization, it becomes apparent that the \( \hat{\pi}^3 \) interaction is of mass dimension +6. In the action, a dimension six operator is expected to be accompanied with a factor of \( 1/\Lambda_U^2 \), where \( \Lambda_U \) is the “unitarity bound” of the theory. This is what happens, on canonical normalization,

\[
\pi_c = \sqrt{-2\hat{H} M_{Pl}^2 \pi}, \quad \text{and the coefficient of } \hat{\pi}_c^3 \text{ operator turns out to be}
\]

\[
\frac{4}{3} c_3 \sqrt{2/3} \left[ H^3 M_{Pl}^4 \right] = \frac{4 c_3}{3} \frac{1}{\Lambda_U^2}
\]

(18)

where \( \Lambda_U \) is the energy scale at which this (non-renormalizable) theory becomes strongly coupled (i.e. the perturbative calculations are valid only at energy scales much smaller than this scale). We have,

\[
\frac{1}{\Lambda_U^2} = \frac{M^4}{(2\epsilon)^{3/2} H^3 M_{Pl}^4}
\]

(19)

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1 Here, the decoupling limit has already been taken and so the terms in the action of EFT which cause Stueckelberg field to mix with gravity already vanish, see [22] for details.
In correlators, it is expected that the (three-line) interaction vertex of \( \hat{\pi}^3 \) theory is always going to be accompanied with factors of the form \( \frac{4}{3} \frac{H^2}{\Lambda^2} \), since \( H \) is the energy scale of the inflationary “experiment.” This suggests that for \( \hat{\pi}^3 \) theory,

\[
\tilde{G}_{1-\text{loop}} = k^3 (\tilde{G}_{\text{tree}})^2 \times \left( \frac{H^2}{\Lambda^2} \right)^2 \text{(factors)} . \tag{20}
\]

Dimensional analysis and homogeneity of the background suggest that the correlator \( \langle \zeta_k(\eta) \zeta_{\bar{k}}(\eta) \rangle \) shall be of the form

\[
\langle \zeta_k(\eta) \zeta_{\bar{k}}(\eta) \rangle = \frac{\delta^3(\vec{k} + \vec{\bar{k}})}{k^3} \times \text{(rest)} , \tag{21}
\]

where the rest terms have to be dimensionless. In the one loop calculation, the rest terms shall contain UV divergent momentum integral. Thus, apart from a few numerical factors and a factor of \( (c_3^2 M^8/e^4 M_p^8) \), the rest terms shall be of the form

\[
\int d^3 \vec{k}_1 d^3 \vec{k}_2 \delta^3(\vec{k} + \vec{k}_1 + \vec{k}_2) f(\eta, H, \vec{k}, \vec{k}_1, \vec{k}_2) . \tag{22}
\]

Power counting makes it clear that the mass dimension of \( f \) is \(-3\). Since this integral is dimensionless, naively, we’d expect that on dimensional regularization it would give

\[
\left( \frac{k}{\mu} \right)^\delta F(\delta) , \tag{23}
\]

where \( F(\delta) \) is a dimensionless function (which contains poles of \( \delta = D - 3 \)). This result shall not leave \( \tilde{G} \) invariant under the transformation \( x \to \Lambda x \), \( k \to k/\Lambda \) and \( a \to a/\Lambda \). Thus, it is not correct and a detailed calculation (by Senatore and Zaldarriaga \[14\]) shows that in fact the UV divergent momentum integral gives

\[
\left( \frac{k}{\mu} \right)^\delta F(\delta) G(\delta, k, -\eta, H) , \tag{24}
\]

and where \( G \) is another dimensionless function and when \( \eta = 0 \), it is of the form

\[
G = 1 + \delta \log(-cH\eta_k) + \cdots , \tag{25}
\]

where \( c \) is an \( O(1) \) constant. This changes the logarithmic running to \( \log(H/\mu) \) \[14\].

## B. Counterterms

A careful look at the argument presented by Senatore and Zaldarriaga \[14\] (to establish that the logarithmic running is \( \log(H/\mu) \)) also tells that when \( \eta = 0 \), only one kind of divergence is present. This may suggest from the arguments in familiar applications of QFT (Sec. \[11\]) that we need just one counterterm to cancel the UV divergences in the two-point function in this theory. Thus, when \( \eta = 0 \), we have (see \[14\]),

\[
\tilde{G}_{1-\text{loop}} \sim \frac{1}{k^3} \left( \frac{F_{-1}}{\epsilon} + \frac{F_{-1}}{2} \log \left( \frac{H(\eta_k)}{\mu} \right) + F_0 + \cdots \right) , \tag{26}
\]

where a factor of \( (c_3^2 M^8/e^4 M_p^8) \) is understood to sit in the front, apart from some numerical factors. This implies that

\[
\tilde{G}_{1-\text{loop}} \sim k^3 \tilde{G}_{\text{tree}}^2 \left( \frac{F_{-1}}{\epsilon} + \frac{F_{-1}}{2} \log \left( \frac{H(\eta_k)}{\mu} \right) + F_0 + \cdots \right) , \tag{27}
\]

with a factor of \( (c_3^2 M^8/e^4 M_p^8 H^4) \) in the front. This should be compared with Eq. \[3\] and \[5\].

At this point it is worth reminding ourselves that the theory that we are dealing with is a non-renormalizable theory so that the counterterm (CT) needed is not necessarily one of the terms we have already written down in the action Eq. \[17\]. From Eq. \[20\], it is clear that the loop correction shall have a factor of \( (H/\Lambda^2)^3 \). Again, by dimensional analysis, it is clear that a single vertex of dimension +8 operator can give this factor. But since we want the CT to renormalize the two-point function, it better have two external lines. Thus, counterterm shall be a dimension +8 quadratic operator. On canonical normalization, \( \pi_c \) has dimension +1 and shift symmetry forbids any polynomials in \( \pi_c \) to be present in the action. Thus, we can only take derivatives, hence CTS can only be operators of the form \( (\partial^3 \pi_c)^2 \). The derivatives that we can take are either w.r.t. time or w.r.t. space, since we want to write a rotationally invariant action, the only options are the square of \( \partial_t \partial_t \partial_t \pi \), the square of \( \partial_t \partial_t \partial_t \pi \), and \( \partial_t \partial_t \partial_t \partial_t \partial_t \pi \). This means that by dimensional analysis, there are three possible candidates for the CTS. On the other hand, we have only one “kind” of divergent term present in the dimensionally regulated expression for the \( \langle \zeta_k(\eta) \zeta_{\bar{k}}(\eta) \rangle \) when \( \eta = 0 \) since there is just a monomial of \( k^3 \) in the front (see Eq. \[27\] as compared to Eq. \[11\]). In this case, we cannot determine the coefficients of the operators in the CT Lagrangian in any unique way.

When \( \tilde{G}_{1-\text{loop}} \) is worked out for \( \eta \) which is non-zero but still such that \(-k\eta \ll 1\), then we get an expression of the form (save for some numerical factors)

\[
\tilde{G}_{1-\text{loop}} \sim k^3 \tilde{G}_{\text{tree}}^2 \frac{c_3^2 M^8}{e^4 M_p^8 H^4} \times \left( \frac{F_{-1}}{\epsilon} + \frac{F_{-1}}{2} \log \left( \frac{H(\eta_k)}{\mu} \right) + F_0 + \cdots \right) \times \left( \sum_{i=0}^{n} c_i (-k\eta)^i \right) , \tag{28}
\]

i.e., we get an extra term multiplied to the \( \eta = 0 \) result which is a polynomial, not in the external momentum \( k \) but in \(-k\eta\). Thus, in this case, even for a dimensionless
In the limit $\eta \to 0$, only one divergent term is left and thus in taking this limit, we end up erasing the information about the form of CTs or their number completely. This is similar to what happens in e.g. spontaneous symmetry breaking. If we consider the $Z(2)$ symmetric renormalizable scalar field theory $L = -\frac{1}{2}(\partial \phi)^2 - m^2 \phi^2 - \lambda \phi^4/4$. When $m^2 > 0$, three CTs are enough to absorb all the infinites in the theory. The same is true when $m^2 < 0$, but in that case, we can also write the same theory as $L = -\frac{1}{2}(\partial \rho)^2 - \lambda v^2 \rho^2/6 - \lambda \rho^4/24$ (where $v = (+6)m^2/\lambda)^{1/2}$ and $\rho = \phi - v$) and looking at this Lagrangian, it may appear that we shall need more CTs for the $\rho^4$ term as well as to cancel the tadpole (i.e. divergent one-point function) it shall cause. In this case, the change of variables from $\phi$ to $\rho$ seems to suggest that we shall need more CTs while in reality, we do not. In contrast for the case we are dealing with, the process of taking $\eta \to 0$ limit suggests that we shall need fewer CTs, while in reality we do not.

\section{Remarks on renormalization}

In $\pi^4$ theory, the field is $\pi$ while the various parameters appearing in the action are $\epsilon, H/M_{Pl}$ and $\epsilon^4 M^4$. The parameter $\epsilon^4 M^4$ can be constrained from the observations of Primordial Non-Gaussianity in the CMB sky (and it has been constrained by the Planck collaboration $^{22}$). There also are (upper) limits on the values of $\epsilon$ and $H/M_{Pl}$ $^{24}$. If we knew the actual values of these parameters (rather than just the limits), we expect that we could work in the On-shell (OS) renormalization scheme to perform the actual renormalization of the theory. In usual QFT, while renormalizing the two-point function in OS scheme, we choose the counter-terms such that (1) the divergent part in the loop integral (the $1/\epsilon$ term) gets cancelled, (2) the dependence on the fake renormalization scale gets cancelled, and (3) the rest of the part of the counterterm is chosen such that $\Pi(-m^2) = 0$ and $\Pi'(\eta^2) = 0$, see $^{1}$. These conditions ensure that the parameter $m$ appearing in the Lagrangian is the physical mass and the field strength is normalized (thus, at $k^2 = -m^2$, the propagator has a simple pole with unit residue). In Eq. (27), we can use

$$\log \left( \frac{H(\eta k)}{\mu} \right) = \log \left( \frac{H(\eta k)}{H(\eta k_0)} \right) + \log \left( \frac{H(\eta k_0)}{\mu} \right), (29)$$

where $k_0$ is a pivot scale to absorb the $\mu$ dependent term in the CT, but instead of the two conditions (given by Lehmann-Källén form of the exact propagator), in the usual flat spacetime field theory, we shall need three (since we have three CTs) and we have no equivalent of the Lehmann-Källén spectral representation of the propagator. Thus, in cosmological perturbation theory, we seem to have no straightforward way to apply the OS scheme since we can not naturally relate the measured correlations to the physical values of the parameters of the Lagrangian of the theory. At this stage, we can again go back to usual field theory to seek inspiration about how to perform renormalization. In usual QFT, in any chosen renormalization scheme, when we have the expression for e.g. $G(k^2)$ (the notation of Eq. (4)), we find $M$ such that $k^2 = -M^2$, $G(k^2)$ has a simple pole, we realize that $M$ is the physical mass (as opposed to the parameter which merely turns up in the renormalized part of the Lagrangian). At $k^2 = -M^2$, if the residue of $G(k^2)$ is $R$ (and in general, it is not unity), we define the renormalized field as $\phi_{ren} \equiv \phi/\sqrt{R}$, so that the residue of the pole for renormalized field is unity.

In cosmological perturbation theory, we could proceed in the following way: if loop corrected power spectrum is

$$\Delta^2_\xi(k) = A_s \left( \frac{k}{k_0} \right)^n \left[ 1 + g(k, \mu, \epsilon_3 M^4, H) \right], (30)$$

then, in general at $k = k_0$, $\Delta^2_\xi(k) \neq A_s$. But if $\Delta^2_\xi(k_0) = \Delta^2_\xi(k) \neq A_s$.
$R(\mu)$, then, let us redefine $\zeta$ such that

$$\zeta_{\text{ren}}(\mu) \equiv \sqrt{\frac{A_s}{R(\mu)}} \zeta,$$

and this is how field strength renormalization could be done in cosmological perturbation theory. Moreover, one could use the observations of $A_s$, $n_s$, and $dn_s/d\log k$ (the running of the spectral index) to fix the finite parts of the three CTs of this theory. At this stage however, the observational constraints on most parameters: $M_3$, $\epsilon$, $H$ (during inflation), $dn_s/d\log k$ are not good enough to perform this procedure. Notice that had we taken the $\eta \to 0$ limit before renormalizing, we could not have known that we need three CTs and the above would not have been possible.

IV. SUMMARY

In this work, we explored issues of renormalization in cosmological perturbation theory. In the more familiar applications of QFT, a logarithmically divergent loop integral has a trivial polynomial dependence on external momenta e.g.

$$\mathcal{I}_1(k) = \int_0^\infty \frac{d\ell}{\ell + k} = C - \log k,$$

(32)

(where $C$ is divergent), on the other hand, for a quadratically divergent integral $\mathcal{I}_2$, one gets,

$$\mathcal{I}_2(k) = \int_0^\infty \frac{\ell d\ell}{\ell + k} = a + bk + k \log k,$$

(33)

where $a$ and $b$ are divergent. In general, in usual QFT, every diagram with external lines amputated and with no sub-divergences is of the form

$$G = \sum_{i=0}^n A_i (1/e)(k^2)^i + G_{\log}(k^2, m, \mu),$$

(34)

(with $k^2$ being Lorentz invariant) where, for the case of a logarithmically divergent diagram, only the the $i = 0$ term is present. In cosmological perturbation theory, in contrast, even for a logarithmically divergent diagram, one gets, when $-k\eta \ll 1$,

$$G = [A_0(1/e)k^3 + G_{\log}(k, m, \mu)] \left[ \sum_{i=0}^n c_i (-k\eta)^i \right].$$

(35)

We argued that if one intends to perform renormalization, one needs to identify the CTs from the expression of dimensionally regulated correlators. Unlike the case of usual QFT, here, when $-k\eta \ll 1$, the expression for dimensionally regulated correlator is a polynomial in $-k\eta$, and taking the limit $\eta \to 0$ erases information about the forms and number of CTs required to renormalize the theory. We thus realized that one should be very careful in taking the limit in which the external time $\eta$ goes to zero. We also explored how the process of renormalization could be performed in cosmological perturbation theory (e.g. how $\zeta$ shall undergo field strength renormalization). This illustrates the many subtleties and surprises associated with field theoretic aspects of cosmological perturbation theory.

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