Friction-controlled entropy-stability competition in granular systems

Xulai Sun,1 Walter Kob,1,2 Raphael Blumenfeld,3 Hua Tong,1 Yujie Wang,1 and Jie Zhang1,4,*

1School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China
2Laboratoire Charles Coulomb, University of Montpellier and CNRS, F34095 Montpellier, France
3Gonville & Caius College and Cavendish Laboratory, University of Cambridge, Cambridge CB2 1TA, UK
4Institute of Natural Sciences, Shanghai Jiao Tong University, Shanghai 200240, China

(Dated: August 3, 2020)

Using cyclic shear to drive a two dimensional granular system, we determine the structural characteristics for different inter-particle friction coefficients. These characteristics are the result of a competition between mechanical stability and entropy, with the latter's effect increasing with friction. We show that a parameter-free maximum-entropy argument alone predicts an exponential cell order distribution, with excellent agreement with the experimental observation. We show that friction only tunes the mean cell order and, consequently, the exponential decay rate and the packing fraction. We further show that cells, which can be very large in such systems, are short-lived, implying that our system is liquid-like rather than glassy.

Dense granular materials show a highly complex response when subjected to repeated cycles of shear, mainly due to the strongly dissipative, hysteretic, and nonlinear interactions at the frictional contacts. The structures of such systems self-organize dynamically and show characteristics that on large scales appear to be universal [1]. Previous works focused on the motion on the particle scale [2, 3] and phase behavior [4–6], the slow relaxation of stress and density [7–9], and the complex spatio-temporal dynamics [10–12]. This dynamics is relevant in a wide range of fields, including the aging and memory of glasses [13], fatigue of materials [14], catastrophic collapse of soils and sand in civil and geotechnical engineering [15], as well as in geological processes, such as earthquakes and landslides [16]. Despite this multitude of investigations, the nature and role of the contact network, a key quantity of granular systems, are far from being understood and in particular the relation between the properties of this network and the friction between the particles is currently not known. Intuitively, one expects higher friction to give rise to looser structures and, therefore, to overall larger cells in the contact network. Edward proposed that the structural characteristics of granular packs can be understood from entropic considerations [17]. In turn, the entropy should depend on the driving process and friction. To quantify this relation, we have created a set-up that allows to analyze entropy of contact networks for widely different friction coefficients.

We consider a two-dimensional (2D) system and focus on its cells, defined as the smallest (aka irreducible) closed loops of contacts. For our purposes, contacts are only those carrying forces [1, 18, 19]. Other than their importance for the mechanical properties of the system, the properties of cells affect heat conduction in granular assemblies and, in three dimensions, the permeability to fluid flow. Cells are closely related to quadrons [1, 19, 20], which are the smallest volume elements of granular assemblies and play a fundamental role in granular statistical mechanics [17, 20–22]. During cyclic shear contacts are continually broken and made, leading to cells being created and annihilated. Here, we investigate the effect of friction on the structure and dynamics. We show that: (i) entropy dominates in high-friction systems, allowing us to derive the cell order distribution from a maximum entropic argument alone; (ii) structural characteristics of quasi-static dynamics are a result of a competition between mechanical stability and entropy, which the friction tunes; (iii) high-friction systems are liquid-like rather than glassy.

The experimental setup, known as the stadium shear...
device [23], is sketched in Fig. 1(a). We used a stepping motor to drive periodically two stainless steel sprockets, connected to each other by a rubber belt which was corrugated on the inside to ensure no-slip between it and the particles. The particle assembly was sheared between the two parallel sections of the belt in the central region of the device and recirculate under the two sprockets. We observed the particles within the blue shaded area, shown in Fig. 1(a), which thus mimics simple shear between two infinite parallel boundaries [23]. We applied a cyclic strain, whose maximum varied from 3% to 10%. In comparison, the yielding strain of this system is around 3% [24]. More details on the experiment are given in the Supplemental Material (SM).

To study the effects of friction, we used: gear-shaped nylon particles (friction coefficient $\mu \to \infty$) [25], photoelastic particles ($\mu \approx 0.7$) [26], stainless-steel particles ($\mu \approx 0.4$), and combinations of these. To minimize crystallization, we used a 50%-50% binary mixture of particles of size ratio 1:1.4. The diameter of the small particles was 1.0 cm, except for gear particles, whose small particles had a pitch diameter of 1.6 cm and a tooth height of 0.36 cm. The total number of particles was about 2000 in each system. Each experimental run was started by depositing particles randomly inside the stadium and applying a cyclic shear until the system reached a steady state. Below we show that fewer than 100 cycles suffice to de-correlate the system from its initial state. Halting the quasi-static process at maximum negative strain every cycle, we monitored and analyzed the static structure stroboscopically, while taking snapshots during each cycle for tracking the particles. The following results have been obtained by sampling data over approximately 4000 cycles of the gears and mixed particles systems, and over approximately 400 cycles for the system of photoelastic particles. We present the results for a strain amplitude of 5% and note that the results for other values are qualitatively the same. For each given configuration of particles, we first identified the contacts and then constructed the cells, see Fig. 1(b-c) [1, 19]. We comment in passing that our results should be insensitive to the exact definition of cells [27, 28].

The cell order $n$ is defined as the number of contacts around it, see Fig. 1(b) and (c) and the SM. Figure 2 shows the cell order distribution (COD), $p(n)$, for three systems with different friction coefficients. The graph demonstrates that $p(n)$ is described very well by an exponential function, $p(n) \propto e^{-n/y}$. We find that the exponential form is independent of the friction coefficient, maximum strain, particles stiffness, and whether we applied a simple or pure cyclic shear [24]. As expected, the higher the friction the larger the occurrence frequency of high-order cells - up to $n = 30$ in the gear systems, enabling us to determine the distribution function to high accuracy. Our CODs are somewhat different from those found in numerical studies of granular systems, in which the sample was gradually tilted [28] or isotropically compressed [1, 19], indicating that CODs: (i) are sensitive to driving protocol and (ii) differ between the quenched and quasi-static steady-states.

We can understand this result in the context of the granular statistical mechanics approach [17, 20, 21, 29]. It is plausible that the structure self-organises via a competition between entropy, i.e. increasing disorder with the largest possible cell configurations, and a constraint of mechanical equilibrium, which excludes unstable configurations. Expecting higher friction to enable more stable configurations and hence to increase the importance of entropy, we neglect the stability constraint in our systems and use a maximum-entropy argument to derive the steady-state COD, $p(n)$. We impose two constraints: that $p(n)$ is normalized and that it has a well-defined mean, $\langle n \rangle$, represented by two Lagrangian multipliers, $x$ and $1/y$. Maximising the Gibbs entropy,

$$ S = - \sum_{n=3}^{\infty} p(n) \log p(n) + x \left[ \sum_{n=3}^{\infty} p(n) - 1 \right] $$

$$ - \frac{1}{y} \sum_{n=3}^{\infty} np(n) - \langle n \rangle , $$

yields

$$ p(n) = \frac{1 - e^{-1/y}}{e^{-3/y}} e^{-n/y} , $$

with $y = \left( \log \frac{\langle n \rangle - 2}{\langle n \rangle - 3} \right)^{-1}$ the typical decay of $p(n)$. Using Euler’s relation for planar graphs [30] (see SM), we also
have \( y = \left( \log \frac{4}{5z} \right)^{-1} \), with \( \langle z \rangle > 2 \) the mean coordination number. To test this result, we determined for each system the value of \( \langle n \rangle \) and used it to get the corresponding value of \( y \). In Fig. 2, we include these theoretical predictions for \( p(n) \) from Eq. (2) and find that, apart a minute difference in the likelihood of \( n = 3 \), which is lower in the gears than in the photoelastic particles, the PDFs collapse on top of one another almost perfectly without any fitting parameter!

The independence of the theoretical PDF of higher \( n \) moments is non-trivial and reminiscent of the Boltzmann distribution, which depends only on the mean energy. In thermal statistical mechanics this is because the energy is defined up to an arbitrary constant [31]. For the energy to be a proper extensive macro-quantity this constant must cancel on calculating its higher moments, which is only possible if the distribution is independent of the higher moments. In contrast, there is no physical reason why adding an arbitrary constant, \( n \to n + n_0 \), should not change the higher moments of \( p(n) \). To test whether or not higher moments need to be included via additional Lagrange multipliers, we compare the experimentally computed variance of the COD, for different strain amplitudes, to that calculated from Eq. (2). The inset of Fig. 2 shows that the two coincide perfectly, establishing that the variance of the COD depends only on \( \langle n \rangle \) and higher moments need not be included.

In this derivation we also neglect spatial cell-order correlations and to test this assumption we measured the conditional probabilities that a cell of order \( n \) neighbors a cell of order \( m \), \( p(n|m) \). In Fig. 3 we superpose these conditional PDFs for different values of \( m \) for the gear and photoelastic particle systems. For both systems all the PDFs collapse onto a master curve, establishing that \( p(n|m) \) is independent of \( m \) and that hence the cell orders of neighboring cells are not correlated. This observation supports the basic assumption that the structural characteristics are dominated by local entropic effects. The good prediction of \( p(n) \) supports strongly our hypothesis that entropy dominates the cell structures in systems, with a negligible effect of the stability constraint. Below, we provide further support from observations of cell shapes and volumes distributions, all of which should also be sensitive to the entropy-stability competition.

A further important quantity characterizing the geometry of the cells are the distribution of the internal angles, \( \theta \). In Fig. 4, we plot the conditional probability density function of \( \theta \), given the cell order, \( q(\theta|n) \), for the gear and photoelastic systems. Up to small details the two sets of PDFs are similar, showing that friction has little qualitative effect. The small difference at large \( \theta \) for large \( n \) stems from the gears ability to support more cell shapes with reflex angles. The distinct features for \( n \leq 5 \) are due to geometric constraints when the number of angles is small: given the particle radii and their sequence around a cell, only \( n - 3 \) internal angles are variable, limiting the explorable configuration space. These features blur progressively as \( n \) increases and with it the configurations space. The tails at large \( \theta \) for large \( n \) indeed indicate a plethora of shapes and hence an increase in disorder.

The entropy-stability competition affects also the cell aspect ratios. The number of configurations increases roughly exponentially with \( n \), with ever more tortuous and elongated cells appear as \( n \) increases. However, the stability constraint excludes such cells or shortens their life spans [19], as we show below. We probed this effect via the ratio of the mean volume of \( n \)-cells (detailed in the SM) to that of the equivalent regular \( n \)-polygon, \( \gamma(n) \equiv V_c(n)/V_{rp}(n) \). The smaller the fraction of elongated cells the closer is \( \gamma \) to 1. Thus, increasing entropy reduces \( \gamma \) while the stability constraints this trend. Figure 5 shows that \( \gamma \) is a monotonically decreasing function of \( n \), suggesting that entropy dominates over the stability constraint. This explains why the entropy maximisation
The ratio $\gamma = V_c(n)/V_{rp}(n)$ increases to 1 as cells are more round and vice versa. Increasing entropy reduces $\gamma$, opposing the effect of mechanical stability. The monotonic decrease of $\gamma(n)$ in our experiments indicates that entropy dominates over stability. For comparison, we include, with permission, measurements of $\gamma$ in a different process from [19] for two friction coefficients, showing the opposite effect: stability dominates for large $n$. The calculation is such an excellent predictor of $p(n)$ in our processes. This is unlike in the compressive processes studied in [19], where $\gamma$ increases for large $n$, as shown in Fig. 5, indicating dearth of elongated cells.

An important aspect of the dynamic structural self-organization is the lifetime of the cells. Cells appear and disappear as contacts form and break [32]. We define $S(t)$ - the probability that a cell existing at time $t_0$ neither merges with another cell nor splits until time $t_0 + t$, with time measured in units of cycles. As shown in Fig. 6, $S(t) \sim e^{-t/\tau}$ with $\tau$ decreasing strongly with $n$ (see insets). This trend and the exponential form are consistent with the assumption that merging and splitting of cells is a local uncorrelated process. The relaxation times are short: $\tau \leq 0.84$ and $\leq 1.43$ for the gear and photoelastic particles, respectively, supporting our observation that the steady state is reached within at most a few dozen of cycles. The decreasing survivability with $n$ is mainly due to their weaker mechanical stability and is another fingerprint of the interplay between process- and friction-governed entropy and stability. Although the decay of $S(t)$ is faster for small $n$ in the photoelastic particles than in the gears, this trend is reversed for large $n$, consistent with the expected increased stability of high-order cells with friction. In passing, we show in Fig. 6 the survival probabilities of the rattlers in both systems, which reveals that rattlers keep being disconnected from the force-bearing network for times longer than $\tau$. This suggests that, to lowest order, their effect on the structural organization can be neglected, at least in this type of dynamics.

Our observations have another implication. A number of works in the literature model granular systems as glasses, especially in the quasi-static regime, where dynamic processes are slow. Yet, the short relaxation times we observe suggest that, in our process, the medium behaves rather as a liquid. Whether this conclusion extends more generally to quasi-static dynamics and how it depends on the density remain open questions. Intriguingly, our high-friction systems contain a non-negligible fraction of particle strands with $z = 2$ (see SM). Similar strands, of dynamically cross-linked particles, have been observed in what is known as ‘empty liquids’ [33–36]. Whether this similarity provides a route to study empty liquids via macroscopic high-friction granular systems remains to be explored.

Our work shows that the value of the interparticle friction coefficient does not affect the fundamental structural characteristics of quasi-static dynamic granular systems qualitatively. Rather, it modulates the entropy-stability competition, expanding the cell configurations space as it increases. This tunes the COD and affects the packing fraction and the cell survival time distributions. Finally,
being able to understand the cell structural characteristics from entropy considerations supports the premise of Edwards and collaborators that granular systems can be described by entropy-based statistical mechanics [17][20–22].

XLS and JZ thank I. Procaccia for valuable discussions. WK is member of the Institut Universitaire de France. This work is supported by the NSFC (No.11774221 and No.11974238).

---

*jiezhang2012@sjtu.edu.cn

[1] T. Matsushima and R. Blumenfeld, Phys. Rev. Lett. 112, 098003 (2014).

[2] B. Kou, Y. Cao, J. Li, C. Xia, Z. Li, H. Dong, A. Zhang, J. Zhang, W. Kob, and Y. Wang, Nature 551, 360 (2017).

[3] M. Mailman, M. Harrington, M. Girvan, and W. Losert, Phys. Rev. Lett. 112, 228001 (2014).

[4] J. R. Royer and P. M. Chaikin, Proc. Natl. Acad. Sci. U.S.A. 112, 49 (2015).

[5] J. Ren, J. A. Dijksman, and R. P. Behringer, Granular Matter 12, 159 (2010).

[6] J. Zhang, T. S. Majmudar, A. Tordesillas, and R. P. Behringer, Phys. Rev. Lett. 107, 010603 (2011).

[7] J. R. Royer and P. M. Chaikin, Proc. Natl. Acad. Sci. U.S.A. 112, 49 (2015).

[8] A. Peshkov, M. Girvan, D. C. Richardson, and W. Losert, Phys. Rev. E 100, 042905 (2019).

[9] N. C. Keim and S. R. Nagel, Rev. Mod. Phys. 91, 035002 (2019).

[10] C. C. Wanjura, P. Gago, T. Matsushima, and R. Blumenfeld, “Structural evolution of granular systems: Theory,” (2019), arXiv:1904.06549 [cond-mat.soft].

[11] T. Matsushima and R. Blumenfeld, Phys. Rev. E 95, 032905 (2017).

[12] R. Blumenfeld and S. F. Edwards, Phys. Rev. Lett. 121, 248001 (2018).

[13] S. Papanikolaou, C. S. O’Hern, and M. D. Shattuck, Phys. Rev. Lett. 110, 198002 (2013).

[14] A. Tordesillas, J. Zhang, and R. Behringer, Geom. Geoengin. 4, 3 (2009).

[15] A. Tordesillas, D. M. Walker, and Q. Lin, Phys. Rev. E 81, 011302 (2010).

[16] A. G. Smart and J. M. Ottino, Phys. Rev. E 77, 041307 (2008).

[17] D. Asenjo, F. Paillussou, and D. Frenkel, Phys. Rev. Lett. 112, 098002 (2014).

[18] H. S. M. Coxeter, Regular Polytopes (Courier Corporation, 1973).

[19] R. P. Feynman, R. B. Leighton, and M. Sands, The Feynman lectures on physics, Vol. I: mainly mechanics, radiation, and heat, Vol. 1 (Pearson P T R, 1970) pp. 72–89.

[20] K. Terzaghi, R. B. Peck, and G. Mesri, Soil mechanics in engineering practice (John Wiley & Sons, 1996).

[21] C. H. Scholz, The mechanics of earthquakes and faulting (Cambridge university press, 1998).

[22] K. Terzaghi, R. B. Peck, and G. Mesri, Soil mechanics in engineering practice (John Wiley & Sons, 1996).

[23] S. F. Edwards and R. Oakeshott, Physica A 157, 1080 (1989).

[24] L. Papadopoulos, M. A. Porter, K. E. Daniels, and D. S. Bassett, J. Comp. Net. 6, 485 (2018).

[25] T. Matsushima and R. Blumenfeld, Phys. Rev. E 95, 032905 (2017).

[26] R. Blumenfeld and S. F. Edwards, Phys. Rev. Lett. 90, 114303 (2003).

[27] A. Melita and S. F. Edwards, Physica A 157, 1091 (1989).

[28] A. Baule, F. Morone, H. J. Herrmann, and H. A. Makse, Rev. Mod. Phys. 90, 015006 (2018).

[29] T. Miller, P. Rognon, and I. Einav, AIP Conf Proc 1542, 483 (2013).
Supplementary Material:

Friction-controlled entropy-stability competition in granular systems

Xulai Sun, Walter Kob, Raphael Blumenfeld, Hua Tong, Yujie Wang, and Jie Zhang

In this Supplementary material we give additional information on 1) Experimental procedure, 2) Details on the calculation of the cell volume, 3) The Euler relation, 4) Effects of boundary conditions on the structure, 5) Distribution of the coordination numbers.

1 Experimental details
During the experiment, particles were cyclically sheared within the stadium shear device driven by a stepping motor. In our system, we defined strain as the ratio between the displacement of lateral boundary and the width of the shear device, i.e. the distance between the two lateral boundaries. The strain was varied from 3% to 10%. This maximum strain ensured that not too many particles would leave the camera window during a cycle, which could affect the statistics. The shearing was halted every cycle at maximum negative strain for two seconds and two snapshots were taken to improve the accuracy of the image analysis. The granular assembly was sheared by the shearing belt, which was kept under a constant confining pressure of 12.5N/m. We estimate the inertial number of the particle motion at $O(10^{-4})$, allowing us to regard the process as quasi-static [37]. A snapshot of the system was taken every second to allow tracking of all the particle trajectories during each cycles.

At our snapshots resolution, the pitch radii of large and small gear particles were 43 and 34 pixels, respectively. In comparison, the radii of the large and small photoelastic particles were 30 and 22 pixels, respectively. This could introduce uncertainty in interparticle contact identification. To minimise it, we checked the effect of varying the contact criterion on the cell order distributions. We found that its exponential form, Fig. 2 of the main text, was not affected and only the decay rate changed at most by 12%.

Following the initial random placing of particles, several hundred shear cycles were applied; Fig. 6 of the main text shows that this is sufficiently long to decorrelate the system and to reach a steady state. Each data set presented in the main text and here was obtained in steady state from snapshots over approximately 4000 cycles in the gear and mixed particles systems, and 400 cycles in the photoelastic system. The latter number of cycles is smaller than the former because of the small, but finite, probability of the photoelastic particles to buckle away from the plane, invalidating a large number of runs.

At steady state, the packing fractions of gear and photoelastic systems were found to be 0.74 ± 0.01 and 0.81 ± 0.01, respectively. These values, which are considerably smaller than the typical values for planar marginally rigid systems of frictionless particles, affirm the expectation that friction and shear under constant confining pressure result in reduced packing density.

Based on the confining pressure, particle sizes and masses, and the friction coefficient between nylon particles and the acrylic support, the support-particles friction forces are two orders of magnitude weaker than the interparticle forces and therefore negligible for our purposes.

2 Effect of the stability-entropy competition on cell volume
Structural properties of cells, such as order, shape and volume distributions, which determine the packing fraction, are governed by the competition between the entropy and the mechanical stability. The shapes can be quantified not only by the internal angles distribution, as done in the main text, Fig. 4, but also by the ratio of cell volumes to that of the equivalent regular n-polygon, Fig. 5 in the main text. Note that cell edges can be defined in two ways, both of which give the same cell identity and order, but different volumes.

Definition 1: the edges extend between neighboring particle centers around the smallest (aka irreducible) void loops, e.g. the brown regions in Fig. 5(b). This definition is useful mainly for disks and particles with aspect ratio close to one.

Definition 2: the edges extend between contact points around such loops, e.g. the blue regions in Fig. 5(b). This definition yields smaller cells and it has two main advantages: it is not sensitive to the particle shape and it describes directly the contact network. Since the latter underlies stress transmission during the dynamics of granular materials, as well as in the static states of granular materials, Definition 2 has been shown to be useful for the analysis of granular structures [19 20] in that it allows one to define the smallest possible volume element in granular systems - the ‘quadrons’ [20]. A quadron is based on the contact network and both a particle and one of its neighboring cells contribute to its volume. To construct a quadron (see also Fig. 5(b)): (i) determine the centroid of a particle as the mean position of its contacts, $\vec{\rho}_q$; (ii) determine the centroid of one of its neighboring cells as the mean position of the contacts around it, $\vec{\rho}_c$; (iii) draw a vector between the two contacts of the particle that are shared with this cell, $\vec{r}_{cq}$; (iv) the quadron is the quadrilateral, whose diagonals are $\vec{r}_{cq}$ and $\vec{\rho}_c - \vec{\rho}_q$. An example of a quadron, shaded grey, is shown in Fig. 5(b). The quadrons tessellate the system perfectly, except for wildly non-convex particles, in which case a small percentage of them may overlap [38].

To analyze the cell volumes, we define the following quantities: $V_c$, the volume of a specific cell as given by definition 2; $V_q$, the sum over the volumes of the quadrons.
Figure S1. (a) Cells (shaded brown) defined by connecting particle centers. (b) Cells (shaded blue) defined by connecting particle contacts. The grey shaded region is an example of a quadron - the smallest volume element of the structure [20]. (c) Cell and quadron volumes for the gear system: the main panel displays the average volumes of cells per cell edge (circles) and of quadrons (square) as a function of $n$ with a logarithmic scale. Inset: $n$-dependence (on a linear scale) of the volumes in the main panel, multiplied by $n$, and the the difference between them, $V_T(n)$ (triangles). (d) As in (c) for the photoelastic particles.

To test the dependence of the volumes on the cell order $n$, we plot the cell volume per contact, $V_c/n$, and the mean quadron volumes, $V_q/n$ against $n$ for the gear (Fig. S1(c)) and photoelastic (Fig. S1(d)) systems. In these plots, volumes are measured in units of $\pi \bar{R}^2$, with $\bar{R} \equiv (R_s + R_l)/2$, with $R_s$ and $R_l$ the radii of the small and large particles, respectively. The mean of $V_T$ increases linearly with $n$, see insets, which is to be expected with this distribution of particle sizes.

An apparent reasonable fit for $V_c(n)$ is

$$V_c(n) = n[A \ln(n) - B]$$

with $A$ and $B$ constants, although we know of no theoretical model that predicts this form. The trend in Figs. S1 can be understood as follows. At low values of $n$, geometry and mechanical stability constrain the mean volume of cells to approximate the equivalent regular polygons, $V_c(n) \approx V_{rp}(n) = n\bar{R}^2 \cos(\pi/n)/\sin(\pi/n)$, albeit with some stretching along the shear-driven tilted principal stress. As $n$ increases, more elongated cells can be realized, which suppresses the mean volume increase rate. We compare the $n$-dependence of $V_c(n)$ with that of $V_{rp}(n)$ in Fig. S2. While $V_{rp} \sim n^2$ for large $n$, we observe that $V_c$ increases almost linearly with $n$. The monotonic increase in the difference between the two curves with $n$ indicates that the configurations space increases rapidly in spite of the mechanical equilibrium constraint, supporting our conclusion that entropy dominates over stability.

Figure S2. The $n$-dependent mean cell volume in the photoelastic and gear systems and the volume of the equivalent regular $n$-polygon. The volumes are measured in units of $\pi \bar{R}^2$.

3 Euler relation and boundary corrections

To check for potential influence of the boundary on our results, we first use Euler’s relation to establish a relation between the particles mean coordination number, $\langle z \rangle$, and the mean cell order, $\langle n \rangle$ [1].

$$\langle n \rangle = \frac{2\langle z \rangle}{\langle z \rangle - 2} + O\left(N^{-1/2}\right)$$

with $A$ and $B$ constants, although we know of no theoretical model that predicts this form.
Here \( N \) is the total number of particles and the second term on the right hand side represents boundary corrections. We then plot in Fig. S3 both relation \((4)\) for \( N \rightarrow \infty \) (solid line) and our measured values of \( \langle z \rangle \) and \( \langle n \rangle \) (symbols). Given sufficient statistics, which we have, any deviation between the experimental data and the curve would be evidence for existing finite boundary corrections. The experimental data falls squarely on the theoretical curve, showing that the boundary corrections to our observations are negligibly small.

![Figure S3](image)

Figure S3. Check of the boundary correction, using Euler’s relation. The symbols represent measurements of \( \langle z \rangle \) and \( \langle n \rangle \), corresponding to different systems (see legend) in different conditions: strain amplitudes, etc. The solid line is Euler’s relation for \( N \rightarrow \infty \), in which limit the boundary correction vanishes. We observe that the experimental data fall nicely on top of the theoretical limit, establishing that the boundary effects are negligible.

4 Homogeneity of the system

In simple shear there is a danger of a density gradient developing between the boundaries. To check the structure for statistical homogeneity, we divided the system into nine slices parallel to the shearing boundaries, as shown in the inset of Fig. S4, and calculated the COD in each slice. Plotting all the PDFs in Fig. S4, we observe that these are all exponential for \( n > 5 \), \( p(n) \sim e^{-n/y} \), with \( y \) depending very weakly on the slice’s position. To a good accuracy, we can thus regard the PDFs as collapsing onto a master curve. The noise for very large values of \( n \) we attribute to the decreasing statistics in this regime. This collapse is consistent with the system being statistically homogeneous.

5 Structural properties of the cells

A key characteristic of the local structure is the distribution of the particles’ coordination numbers, \( C(z) \). Since these distributions are expected to depend on the particle sizes, we show in Fig. S5 the \( C(z) \) for the large and small particles separately.

In the gear system the mean coordination numbers for the small and large particles are \( z = 3.09 \pm 0.07 \) and \( z = 3.51 \pm 0.09 \), respectively, Fig. S5(a). These values are significantly lower than the densest state, \( \langle z \rangle = 6 \) and are close to the marginally rigid limit of an infinite system, which indicates a very dilute structure. Figure S5(a) also shows that for about 10% of the particles \( z \leq 1 \) - these are rattlers that do not transmit forces. The corresponding mean coordination numbers for the photoelastic particles are \( z = 4.20 \pm 0.03 \) and \( z = 4.82 \pm 0.03 \), respectively. These systems are denser than the gear systems, which is to be expected because of the lower inter-particle friction.

The fast structural relaxation of the system discussed in the main text suggests that these systems are liquid-like rather than glassy and because of the low density they resemble an empty liquids \[33–36\]. However, since \( l \) depends on the de-
strands and lower densities may be achievable in other setups, which could then be used as convenient macroscopic models for empty liquids.

Figure S5. The conditional probabilities of the particle coordination numbers, given the size, $C(z | \text{large})$ and $C(z | \text{small})$, in the gear and photoelastic systems at a shear strain of 5%.
Figure S6. (a) Sketch of a strand of length \( l \). (b) The strand length PDFs for the gear and photoelastic systems strain amplitude 5%.