A general approach to the exact localized transition points of 1D mosaic disorder models

Yanxia Liu1,*

1School of Physics and Astronomy, Yunnan University, Kunming 650091, PR China

(Dated: August 17, 2022)

In this paper, we present a general correspondence between the mosaic and non-mosaic models, which can be used to obtain the exact solution for the mosaic ones. This relation holds not only for the quasicrystal models, but also for the Anderson models. Despite the different localization properties of the specific models, this relationship shares a unified form. Applying our method to the mosaic Anderson models, we find that there is a discrete set of extended states. At last, we also give the general analytical mobility edge for the mosaic slowly varying potential models and the mosaic Ganeshan-Pixley-Das Sarma models.

I. INTRODUCTION

The Anderson localization (AL) plays a fundamental role in condensed matter physics1, which is conventionally discussed in disordered systems. There are roughly two types of disordered potentials: random disorders and quasi-periodic potentials. The random disorders are also referred as Anderson models, where all states are exponentially localized in one and two dimensions with infinitesimal disorder strength, and the localized and extended states are separated by energy-dependent mobility edge (ME) in three dimensions2–5. The one-dimensional(1D) quasi-periodic models have various forms, such as short-range (long-range) hopping processes6–12, modified quasiperiodic potentials13–15, and some various extensions of AA models16–24. These models support localization transitions or MEs, and some of them can be obtained exactly. Localization transition in quasiperiodic systems has attracted increasing interest both theoretically and experimentally in recent years24–30. Some of these models have been realized in experimental platforms such as ultracold gases.

Recently, some 1D mosaic quasicrystals with exact MEs are devised15,31–33, whose MEs cannot be exactly solved via traditional methods. Luckily, the Avila’s global theory34,35 provides us with a mathematically rigorous tool to solve this kind of models exactly. However, there are two setbacks in the application of Avila’s global theory: one is the models must satisfy extra restrictions while the other one is Avila’s global theory is very hard for many physicists without the specific knowledge in this mathematical area. Meanwhile, Avila’s global theory is powerless in dealing with random disorder systems.

There are already many results of mosaic disorder systems have been obtained15,31–33. However, our knowledge of the mosaic models is still far from comprehensive. Whether there exists any relevance between the 1D non-mosaic models and mosaic models, and whether there exist any new states only in mosaic models, still remains unanswered and will be explored in the following sections. In this paper, we discovered a general correspondence between the non-mosaic models and mosaic models which provides us a simple and elegant tool solving the mosaic models exactly when the corresponding non-mosaic models can be solved exactly.

This paper is organized as follows. In section II, we first introduce the mosaic models and present the general correspondence between the non-mosaic models and mosaic models. In section III, we apply our methodology to some specific cases. In the subsection A, we verify our method with two type models: the mosaic AAH models and the mosaic Wannier-Stark models, which have been studied by Avila’s global theory. Then, in subsection B, we analytically study the localization properties of the mosaic Anderson models and find that there exists a set of discrete eigenenergies for extended states, which is independent of the disorder strength. In subsection C, we also take the 1D mosaic slowly varying potential models as a example and give the general MEs for arbitrary κ. In subsection D, we give the analytical MEs of the mosaic Ganeshan-Pixley-Das Sarma models. Final section is a summary.

II. CORRESPONDENCE BETWEEN MOSAIC AND NON-MOSAIC MODELS

The general 1D mosaic models can be written as

\[ H = t \sum_j (c_j^\dagger c_{j+1} + H.c.) + \sum_j V_j n_j, \]

with potential

\[ V_j = \begin{cases} \lambda \Delta_m, & j = m\kappa, \\ 0, & \text{otherwise}, \end{cases} \]

where \( c_j^\dagger \) \((c_j)\) and \( n_j = c_j^\dagger c_j \) are the creation (annihilation) operator and the number operator at site \( j \), \( t \) is the hopping coefficient (for convenience, we set \( t = 1 \) as the energy unit), \( \lambda \Delta_m \) is the potential, which takes place at every \( \kappa \) sites. The explicit forms \( \Delta_m \) can make it disordered or not. We take the wave function as \( \Psi = \sum \psi_j c_j^\dagger |0\rangle \). The schrödinger equation \( H \Psi = E \Psi \) then takes the form of
\[
\begin{aligned}
\psi_{\kappa(m-1)} + \psi_{\kappa(m-1)+2} &= E \psi_{\kappa(m-1)+1}, \\
\cdots, \\
\psi_{\kappa m-2} + \psi_{\kappa m} &= E \psi_{\kappa m-1}, \\
\end{aligned}
\]
and
\[
\psi_{\kappa m-1} + \psi_{\kappa m+1} + \lambda \Delta_m \psi_{\kappa m} = E \psi_{\kappa m},
\]
Here we demand \(m > 1\) and ignore the boundary conditions. Eqs. (3) can give us
\[
\psi_{\kappa m-1} = \frac{1}{a_\kappa} \psi_{\kappa(m-1)} + \frac{a_{\kappa-1}}{a_\kappa} \psi_{\kappa m},
\]
and taking \(m \to m + 1\) in Eqs. (3) give us
\[
\psi_{\kappa m+1} = \frac{1}{a_\kappa} \psi_{\kappa(m+1)} + \frac{a_{\kappa-1}}{a_\kappa} \psi_{\kappa m},
\]
where
\[
a_\kappa(E) = \frac{1}{\sqrt{E^2 - 4}} \left( \frac{E + \sqrt{E^2 - 4}}{2} \right)^\kappa - \left( \frac{E - \sqrt{E^2 - 4}}{2} \right)^\kappa.
\]
The detailed derivation of Eq. (5) and Eq. (6) is given in the Appendix. We then substitute Eq. (5) and Eq. (6) into Eq. (4)
\[
\psi_{\kappa(m-1)} + \psi_{\kappa(m+1)} + \lambda a_\kappa \Delta_m \psi_{\kappa m} = (a_\kappa E - 2a_{\kappa-1}) \psi_{\kappa m},
\]
which is a difference equation only including the wave function at sites \(j = \kappa m\). When \(\kappa = 1\), Eq. (8) is the Schrödinger equation of the non-mosaic model, where the localization-delocalization transitions can occur for disordered \(\Delta_m\). Normally, the regions of extended states are determined by
\[
f(\lambda, E) < 1, \quad \text{or} \quad f(\lambda, E) = 1.
\]
The explicit choice depends on the model considered. The \(f(\lambda, E)\) is a function of \(\lambda\) and \(E\), whose forms depend on the concrete potential \(\Delta_m\). When \(\kappa \neq 1\), the extended states only appear at
\[
f(a_\kappa \lambda, a_\kappa E - 2a_{\kappa-1}) < 1,
\]
or
\[
f(a_\kappa \lambda, a_\kappa E - 2a_{\kappa-1}) = 1.
\]
That means once the localization-delocalization transition points of the models with \(\kappa = 1\) is known, the localization-delocalization transition points of the models with \(\kappa \neq 1\) can be obtained by simple substitution \(\lambda \to a_\kappa \lambda; E \to a_\kappa E - 2a_{\kappa-1}\). This provides a straightforward way solving the mosaic models and show the mechanism of multiple mobility edges in the mosaic models.

Since \(a_\kappa(E)\) is a function of \(E\) to the \(\kappa - 1\) degree, \(a_\kappa E - 2a_{\kappa-1}\), which appears at the "energy" position of (8), is a \(\kappa\)th order polynomial of \(E\). If \(a_\kappa E - 2a_{\kappa-1}\) has \(L\) solutions, the number of eigenenergies \(E\) for Hamiltonian (1) are \(N = \kappa L\), where \(N\) is the lattice length and \(L\) is the quasi-cell number. Eq. (8) thus gives a complete set of solutions of Hamiltonian (1). The forms of Eq. (8) with different \(\kappa\) are essentially the same, as the \(a_\kappa(E)\) is independent of the explicit forms of potentials. Thus, the eigenstates of the systems with different \(\kappa\) have the same structure. So, the cases \(\kappa > 1\) will not exhibit new states to \(\kappa = 1\). We can also see that if the non-mosaic models have the same transition points, the corresponding mosaic models will have same transition points, regardless the non-mosaic models being disorder or not.

### III. APPLICATIONS

#### A. The mosaic AAH models and mosaic Wannier-Stark models

To show the validity of our method, we first consider the mosaic AAH models with \(\Delta_m = 2\cos(2\pi a_m)\), which host multiple MEs and has been studied in ref.\(^{25,36}\). Here \(\alpha = (\sqrt{5} - 1)/2\) is an irrational number. The MEs have been exactly solved by applying the Avila’s global theory. The Avila’s global theory is an important mathematical result, but is not friendly to the physics researcher who is not a mathematician and doesn’t know much specific mathematical knowledge. Now we deal this model with a much simpler method, which does not involve complex mathematics.

When \(\kappa = 1\), Eq. (8) is the simplest quasiperiodic model (AAH model) and exhibits self-duality symmetry. Thus the localization-delocalization transition points of this model can be exactly determined by a self-duality condition\(^{25,36}\), which can be expressed as \(\alpha = 1\) and the extended region is \(|\lambda| < 1\), i.e. \(f(\lambda, E) < 1\) with \(f(\lambda, E) = |\lambda|\). Substituting \(\lambda\) with \(a_\kappa \lambda\), the localization-delocalization points for the cases \(\kappa > 1\) read
\[
|a_\kappa \lambda| = 1,
\]
which is in agreement with the result from the Avila’s global theory\(^{15}\).

This method can also be applied to the mosaic Wannier-Stark models, which can be described by Eqs. (1) and (2) with \(\Delta_m = m\) (a tilted potential). For \(\kappa = 1\), all the eigenstates are extended when \(|\lambda| < 2\), i.e., \(f(\lambda, E) < 1\) with \(f(\lambda, E) = |\lambda| - 1\). This model has the same characteristics with AAH model: the transition points are independent of the energy(no ME). Do the same thing, substituting \(\lambda\) with \(a_\kappa \lambda\), then we get that for the cases \(\kappa > 0\), extended states exist when
\[
|a_\kappa \lambda| > 2,
\]
which is also verified with Avila’s global theory\(^{31}\).
B. The 1D mosaic Anderson models

Here we use our method to study 1D mosaic Anderson models, which can be described by Eqs. (1) and (2) with \( \Delta_m = w_m \), where \( w_m \in [-1, 1] \) is uniformly distributed random variable. For the case \( \kappa = 1 \), the infinitesimal disorder strength \( \lambda \) makes all eigenstates localized, that is to say, only when \( \lambda = 0 \), the states are extended, and it is shown in Eq. (9) with \( f(\lambda, E) = \lambda + 1 = 1 \), which is independent of \( E \). Now, we substitute \( \lambda \) with \( a_\kappa \lambda \) for the case \( \kappa \neq 1 \), where the extended states appear at \( a_\kappa \lambda = 0 \). In non-trivial case \( \lambda \neq 0 \), the extended states are located at

\[
a_\kappa(E) = 0. \tag{14}
\]

The solutions of Eq. (14) are discrete points, which are independent of the \( \lambda \). Before solving the Eq. (14), we define \( E = 2 \cos \theta \) in Eq. (7) for simplicity. After some algebra, we can get \( a_\kappa(2 \cos \theta) = \sin(\kappa \theta)/\sin \theta \). The solutions of \( a_\kappa(2 \cos \theta) = 0 \) are given by \( \theta = \pm l \pi / \kappa \) with \( l = 1, 2, \ldots, \kappa - 1 \). There are \( 2(\kappa - 1) \) different solutions. The solutions \( \theta \) and \(-\theta \) correspond to the same \( E \), so there are \( \kappa - 1 \) independent roots of \( a_\kappa(E) = 0 \), which can be written as

\[
E = 2 \cos(l \pi / \kappa), \quad l = 1, 2, \ldots, \kappa - 1. \tag{15}
\]

For \( \kappa = 2 \), the extended states appear at energy \( E = 0 \), while for \( \kappa = 3 \) and \( \kappa = 4 \), the extended states appear at energy \( E = \pm 1 \) and \( E = \pm \sqrt{2}, 0 \), which is independent of \( \lambda \). We can find that if \( \kappa \neq 0 \), there exist some extended states with discrete energies, which do not exist in Anderson models or quasiperiodic lattice models.

The quasiperiodic lattices are some intermediates between periodic and random lattice, which can realize the crossover of physical property from periodic to random lattices by change the strength of the quasiperiodic potential. Actually, the mosaic Anderson models can be also seen as some intermediates between periodic and random lattices. It becomes the standard Anderson models when \( \kappa = 1 \), and becomes the periodic model when \( \kappa = \infty \) (when the lattice site \( N \) is finite, \( \kappa > N \)). The mechanism of the crossover of this model is that the extended states come into play with increasing \( \kappa \), which are unaffected by \( \lambda \).

To conceptualize this, we demonstrate numerical results of Lyapunov exponent (LE). The LE can be numerically calculated via

\[
\gamma(E) = \ln \left( \max \left( \theta_1^+, \theta_1^- \right) \right), \tag{16}
\]

where \( \theta_i^+ \in \mathbb{R} \) denote eigenvalues of the matrix

\[
\Theta = \left( T_N^\dagger T_N \right)^{1/(2N)}. \tag{17}
\]

with the transfer matrix

\[
T_N(E) = \prod_{j=1}^{N} T_j \prod_{j=1}^{N} \left( \begin{array}{cc} E - V_j & -1 \\ 1 & 0 \end{array} \right). \tag{18}
\]

where \( N = \infty \) as required by the definition of LE. The LE is a non-negative number (\( \gamma \geq 0 \)) which characterize the localization properties of the eigenstates. A localized state of disorder systems can be expressed as

\[
|\psi_j| \propto e^{-\gamma|j-j_0|},
\]

where \( j_0 \) is the localization center. It is clear that \( \psi_n \) is a localized state with \( \gamma > 0 \) and an extended state with \( \gamma = 0 \).

In Figs. 1(a1)-(a3), we display the LE of all eigenenergies with \( (a1)-(a3) \) \( \kappa = 1 \), lattice length is \( N = 2000 \), \( (b1)-(b3) \) \( \kappa = 2 \), \( N = 2000 \), and \( (c1)-(c3) \) \( \kappa = 3 \) \( N = 2001 \), respectively. The value of parameter \( \lambda \) in each column is the same: \( (a1)-(c1) \) \( \lambda = 1 \), \( (a2)-(c2) \) \( \lambda = 2 \), and \( (a3)-(c3) \) \( \lambda = 4 \), respectively. All results are the average of 200 random samples.
are the extended regions. When \( \kappa = 2 \) with size \( N = 2 \times 233 \) and \( \kappa = 3 \) with size \( N = 3 \times 233 \). Without loss of generality, we set \( \theta = 0 \). The black dashed lines represent the MEs given in Eq. (26) and (27).

C. The 1D mosaic slowly varying potential models

In this part, we extend our method to the mosaic slowly varying potential model, which is another famous quasiperiodic models described by Eqs. (1) and (2) with \( \Delta_m = \cos(2\pi am^\nu) \), \( 0 < \nu < 1 \). For the case \( \kappa = 1 \), the extended states are clustered in \( |E| < 2 - \lambda \) with \( \lambda < 2 \), which can be obtained by asymptotic semiclassical WKB-type theory\(^{13,14} \). When \( \lambda > 2 \), all states are localized. Applying Eq. (9), the extended region is \( f(\lambda, E) < 1 \) with \( f(\lambda, E) = |E| - 1 + \lambda \). The Eq. (10) provides us the condition for extended states with \( \kappa > 1 \). Substituting \( \lambda \) with \( a_k \lambda \) and \( E \) with \( a_k E - 2a_k - 1 \), we get

\[
f(a_k \lambda, a_k E - 2a_k - 1),
\]

with \( a_k \lambda < 2 \). When \( \kappa = 2 \), \( a_2 = E \); \( a_1 = 1 \), then we get

\[
f(a_2 \lambda, a_2 E - 2a_2) = |E^2 - 2| - 1 + E \lambda < 1 \) and \( E \lambda < 2 \) are the extended regions. When \( \kappa = 3 \), \( a_3 = E^2 - 1 \),

\[
f(a_3 \lambda, a_3 E - 2a_3) = |E^3 - 3E - 1 + (E^2 - 1)\lambda| < 1 \) and \( (E^2 - 1)\lambda < 2 \) are the extended regions. The results of the case \( \kappa = 2 \) and \( \kappa = 3 \) are agreement with the result in\(^{32} \) (\[^32\] doesn’t give a general result for arbitrary \( \kappa \)).

D. The mosaic Ganeshan-Pixley-Das Sarma models

Now, we consider the mosaic Ganeshan-Pixley-Das Sarma model

\[
\Delta_m = \frac{2 \cos(2\pi am + \theta)}{1 - b \cos(2\pi am + \theta)},
\]

where \( b \in (-1, 1) \) and \( \theta \) is the phase offset. The case with \( \kappa = 1 \) is the first quasiperiodic model where the analytic form of mobility edges are obtained by looking for the self-dual point\(^8 \). The extended region is

\[
2\text{sgn}(\lambda)(1 - |\lambda|) < bE,
\]

where \( \text{sgn}(\ ) \) is sign function, then we get

\[
f(\lambda, E) = 2\text{sgn}(\lambda)(1 - |\lambda|) - bE + 1 < 0.
\]

When \( \kappa \neq 1 \), substituting \( \lambda \) with \( a_k \lambda \) and \( E \) with \( a_k E - 2a_k - 1 \), we get the extended regions:

\[
f(a_k \lambda, a_k E - 2a_k - 1) = 2\text{sgn}(a_k \lambda)(1 - |a_k \lambda|) - b(a_k E - 2a_k) + 1 < 0,
\]

that is, the MEs are

\[
2\text{sgn}(a_k \lambda)(1 - |a_k \lambda|) = b(a_k E - 2a_k),
\]

It is obvious that Eq. (24) with \( b = 0 \) is identical to the MEs of the mosaic AA model:

\[
|\lambda a_k| = 1.
\]

And the LE (24) with \( \kappa = 2 \) and \( \kappa = 3 \) can lead to

\[
\lambda = \frac{(2 - E^2)b \pm 2}{2E},
\]

and

\[
\lambda = \frac{b(3E - E^3) \pm 2}{E^2 - 1}.
\]

Equations (26) and (27) shows the energy boundary of localized and extended states. To characterize the ME, we introduce the fractal dimension of an eigenstate\(^9-^{11} \), which is defined as

\[
\Gamma = -\lim_{N \to \infty} \frac{\ln(\text{IPR})}{\ln N},
\]

where the inverse participation ratio \( \text{IPR} = \sum_j |\psi_j|^4 \). The fractal dimension \( \Gamma \) approaches 0 for a localized state and approaches 1 for an extended state. Figure 2 shows the distribution of the spectra versus \( \lambda \) with \( b = 0.7 \) for \( \kappa = 2 \) and 3, respectively. The different color of each spectrum represents the fractal dimension \( \Gamma \). The dashed lines in the Fig. 2 (a) and (b) represent the MEs (26) and (27) for \( b = 0.7 \). The eigenstates corresponding to the spectra surrounded by dashed lines are extended, and outside the four (\( \kappa = 2 \)) or six (\( \kappa = 3 \)) dashed lines are localized. It is shown that the analytical MEs agrees well with numerical results from fractal dimension \( \Gamma \) and spectra. Moreover, we can also obtain the emergence of multiple mobility edges in these models, which can be regarded as \( f(a_k \lambda, a_k E - 2a_k - 1) \) for \( \kappa > 1 \) being a power function of \( E \).

For \( \lambda \to \infty \), the extended eigenstates are concentrated at \( E = 0 \) for \( \kappa = 2 \) and \( E = \pm 1 \) for \( \kappa = 3 \), which is independent of \( b \). Something interesting is, when \( \lambda \to \infty \), the concentration of spectral of extended states is identical to that of Anderson model, which is irrelevant of \( \lambda \).
IV. SUMMARY

In this paper, we have proposed a new way to deal with the mosaic disorder systems via the decoupling the Schrödinger equations. Such method not only establishes a general correspondence between 1D non-mosaic disorder models and mosaic disorder models, but also can give exact solutions of MEs to a variety of models, such as the mosaic slowly varying potential models and the mosaic Ganeshan-Pixley-Das Sarma models. With this method, we strictly demonstrate that there is a discrete set of extended states in 1D mosaic Anderson models and revealed the mechanism of the emergence of multiple mobility edges.

Acknowledgments

We thank Y.-S. Cao for useful discussion.

Appendix A: Decoupling difference equation

The potentials of the mosaic quasi-periodic models take place with fixed site interval \( n = \kappa m \). As one can infer from Eqs. (3) and (4), the wave functions for the lattice sites \( n = \kappa m \) can be decoupled from the wave functions at other sites \( n \neq \kappa m \). The difference equation involving potential is

\[
\psi_{\kappa m-1} + \psi_{\kappa m+1} + \lambda \Delta_m \psi_{\kappa m} = E \psi_{\kappa m},
\]

(A1)

In order to obtain a difference equation only with sites \( n = \kappa m \), \( \psi_{\kappa m-1} \) and \( \psi_{\kappa m+1} \) need to be replaced by \( \psi_{\kappa (m+1)} \) and \( \psi_{\kappa m} \), somehow. Rewrite Eqs. (3) with transfer matrix \( T \),

\[
\begin{pmatrix}
\psi_{\kappa (m+1)} \\
\psi_{\kappa (m+1)-1}
\end{pmatrix} = T^{\kappa} \begin{pmatrix}
\psi_{\kappa m-1} \\
\psi_{\kappa m}
\end{pmatrix},
\]

(A2)

and \( m \to m + 1 \),

\[
\begin{pmatrix}
\psi_{\kappa (m+1)} \\
\psi_{\kappa (m+1)-1}
\end{pmatrix} = T^{\kappa-1} \begin{pmatrix}
\psi_{\kappa m+1} \\
\psi_{\kappa m}
\end{pmatrix},
\]

(A3)

where \( T \) is the transfer matrix and

\[
T^{\kappa-1} = \left( \begin{array}{cc}
E & -1 \\
1 & 0
\end{array} \right)^{\kappa-1} = \left( \begin{array}{cc}
a_{\kappa} & -a_{\kappa-1} \\
a_{\kappa-1} & -a_{\kappa-2}
\end{array} \right),
\]

(A4)

with

\[
a_{\kappa} = \frac{1}{\sqrt{E^2-4}} \left( \frac{E + \sqrt{E^2-4}}{2} \right)^{\kappa} - \left( \frac{E - \sqrt{E^2-4}}{2} \right)^{\kappa}.
\]

From Eq. (A2) and Eq. (A3), we get

\[
\psi_{\kappa (m-1)} = a_{\kappa} \psi_{\kappa m-1} - a_{\kappa-1} \psi_{\kappa m};
\]

(A5)

\[
\psi_{\kappa (m+1)} = a_{\kappa} \psi_{\kappa m+1} - a_{\kappa-1} \psi_{\kappa m}.
\]

(A6)

The wave functions \( \psi_{\kappa m-1} \) and \( \psi_{\kappa m+1} \) thus can be represented as

\[
\psi_{\kappa m-1} = \frac{1}{a_{\kappa}} \psi_{\kappa (m-1)} + \frac{a_{\kappa-1}}{a_{\kappa}} \psi_{\kappa m},
\]

(A7)

\[
\psi_{\kappa m+1} = \frac{1}{a_{\kappa}} \psi_{\kappa (m+1)} + \frac{a_{\kappa-1}}{a_{\kappa}} \psi_{\kappa m}.
\]

(A8)

Substituting Eq. (A7) and Eq. (A8) into Eq. (A9) yields

\[
\psi_{\kappa (m-1)} + \psi_{\kappa (m+1)} + \lambda a_{\kappa} \Delta_m \psi_{\kappa m} = (E - 2a_{\kappa-1}) \psi_{\kappa m},
\]

(A9)

which is a decoupled difference equation at sites \( n = \kappa m \).
tials, Phys. Rev. B 101, 064203 (2020).

12. X. Deng, S. Ray, S. Sinha, G. V. Shlyapnikov, and L. Santos, One-Dimensional Quasicrystals with Power-Law Hopping, Phys. Rev. Lett. 123, 025301 (2019).

13. S. Das Sarma, S. He, and X. C. Xie, Mobility edge in a model one-dimensional potential, Phys. Rev. Lett. 61, 2144 (1988).

14. S. Das Sarma, S. He, and X. C. Xie, Localization, mobility edges, and metal-insulator transition in a class of one-dimensional slowly varying deterministic potentials, Phys. Rev. B 41, 5544 (1990).

15. Y. Wang, X. Xia, L. Zhang, H. Yao, S. Chen, J. You, Q. Zhou, and X. Liu, One dimensional quasiperiodic mosaic lattice with exact mobility edges, Phys. Rev. Lett. 125, 196604 (2020).

16. H. A. Ceccatto, Quasiperiodic Ising Model in a Transverse Field: Analytical Results, Phys. Rev. Lett. 62, 203 (1989).

17. L. Zhou, H. Pu, and W. Zhang, Anderson localization of cold atomic gases with effective spin-orbit interaction in a quasiperiodic optical lattice, Phys. Rev. A 87, 023625 (2013).

18. M. Kohmoto and D. Tobe, Localization problem in a quasiperiodic system with spin-orbit interaction, Phys. Rev. B 77, 134204 (2008).

19. X. Cai, L.-J. Lang, S. Chen, and Y. Wang, Topological superconductor to Anderson localization transition in one-dimensional incommensurate lattices, Phys. Rev. Lett. 110, 176403 (2013).

20. W. DeGottardi, D. Sen, and S. Vishveshwara, Majorana fermions in superconducting 1D systems having periodic, quasiperiodic, and disordered Potentials, Phys. Rev. Lett. 110, 146404 (2013).

21. F. Liu, S. Ghosh, and Y. D. Chong, Localization and Adiabatic Pumping in a Generalized Aubry-Andre-Harper Model, Phys. Rev. B 91, 014108 (2015).

22. A. Chandran and C. R. Laumann, Localization and Symmetry Breaking in the Quantum Quasiperiodic Ising Glass, Phys. Rev. X 7, 031061 (2017).

23. Y.-C. Wang, X.-J. Liu and S. Chen. Properties and applications of one dimensional quasiperiodic lattices. Acta Physica Sinica, 68 040301, (2019).

24. M. Kohmoto, Metal-insulator transition and scaling for incommensurate systems, Phys. Rev. Lett. 26, 1198 (1983).

25. S. Aubry and G. André, Analyticity breaking and Anderson localization in incommensurate lattices, Ann. Israel Phys. Soc. 3, 133 (1980).

26. D. J. Thouless, Localization by a potential with slowly varying period, Phys. Rev. Lett. 61, 2141 (1988).

27. G. Roati, C. DErrico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Anderson localization of a non-interacting bose Einstein condensate, Nature (London) 453, 895 (2008).

28. H. P. Lüschen, S. Scherg, T. Kohlert, M. Schreiber, P. Bordia, X. Li, S. Das Sarma, and I. Bloch, Single-particle mobility edge in a one-dimensional quasiperiodic optical lattice, Phys. Rev. Lett. 120, 160404 (2018).

29. F. A. An, E. J. Meier, and B. Gadway, Engineering a flux-dependent mobility edge in disordered zigzag chains, Phys. Rev. X 8, 031045 (2018).

30. F. A. An, K. Padavić, E. J. Meier, S. Hegde, S. Ganeshan, J. H. Pixley, S. Vishveshwara, and B. Gadway, Observation of tunable mobility edges in generalized Aubry-André lattices, Phys. Rev. Lett. 126, 040603 (2021).

31. D. Dwiputra, F. P. Zen, Single-particle mobility edge with-out disorder, Phys. Rev. B 105, L081110 (2022).

32. L.-Y. Gong, H. Lu, and W.-W. Cheng, Exact Mobility Edges in 1D Mosaic Lattices Inlaid with Slowly Varying Potentials, Adv. Theory Simul. 4, 2100135 (2021).

33. Y. Liu, Y. Wang, X.-J. Liu, Q. Zhou, and S. Chen, Exact mobility edges, PT-symmetry breaking, and skin effect in one-dimensional non-Hermitian quasicrystals, Phys. Rev. B 103, 014203 (2021).

34. A. Avila, Global theory of one-frequency Schröinger operators, Acta. Math. 1, 215, (2015).

35. A. Avila, J. You , Q. Zhou, Sharp phase transitions for the almost Mathieu operator, Duke. Math. J. 14, 166 (2017).

36. S. Y. Jitomirskaya, Metal-insulator transition for the almost mathieu operator, Ann. Math. 3, 150 (1999).