Approximate Noether Symmetries and Collineations for Regular Perturbative Lagrangians

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Abstract

Regular perturbative Lagrangians that admit approximate Noether symmetries and approximate conservation laws are studied. Specifically, we investigate the connection between approximate Noether symmetries and collineations of the underlying manifold. In particular we determine the generic Noether symmetry conditions for the approximate point symmetries and we find that for a class of perturbed Lagrangians, Noether symmetries are related to the elements of the Homothetic algebra of the metric which is defined by the unperturbed Lagrangian. Moreover, we discuss how exact symmetries become approximate symmetries. Finally, some applications are presented.

Keywords: Approximate symmetries; Noether symmetries; Collineations.

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1. Introduction

Symmetries play an important role in the study of differential equations. The existence of a symmetry vector implies the existence of a transformation which reduces the order of the differential equation (for ordinary differential equations) or the number of dependent variables (for partial differential equations). However, the existence of a symmetry vector indicates that there exists a curve in the phase space of the dynamical system which constrains the solution of the differential equation. This specific curve is a conservation law for the differential equation.

There are various ways to construct conservation laws, for instance see [1, 2, 3, 4]. One of the most well-known, and simplest methods for the determination of conservation laws is the application of Noether’s theorems [5]. In particular, the first Noether’s theorem states that, if the Lagrangian function...
which describes a dynamical system changes under the action of a point transformation such that the Action integral is invariant, the dynamical system is also invariant under the action of the same point transformation. Moreover, a conservation law corresponds to this point transformation according to Noether’s second theorem.

Usually, when we refer to symmetries, we consider the exact symmetries. However, for perturbative dynamical systems the context of symmetries is extended and the so-called approximate symmetries are defined. In this work we are interested in the application of Noether’s theorem for approximate symmetries on some regular perturbative Lagrangians. Approximate Noether symmetries provide approximate first integrals, functions which can be used as conservation laws until a specific step in numerical integrations. This kind of approximate conservation laws have played an important role for the study of chaotic systems - for an extended discussion we refer the reader to an application in Galactic dynamics.

Whilst recently, the advent of automated software algorithms has made light work of calculating symmetries. Such programs are often limited by models involving many variables or higher-order perturbations. This problem, in part, has fueled the need to write this paper. Here, we take a compound problem, whereby scientists have previously relied on numerical techniques for analysis, and instead frame it in the context of an analytical scheme. We present a set of conditions that may be specialized for appropriate Lagrangian functions that necessarily contains a perturbation. Inspired by the approach of Tsamparlis and Paliathanasis, we show how those conditions can be solved with the use of some theorems from differential geometry. Indeed, geometric based theories have far reaching applications.

Specifically, in this paper, the Noether conditions, or symmetry determining system of equations, are formulated by contemplating point transformations in ascending order of the perturbation parameter $\varepsilon$. To illustrate the advantages of such a formulation, the general conditions are applied to the perturbations of oscillator type equations corresponding to $n$ dimensions. Moreover, we discuss the admitted approximate conserved quantities for symmetries of first-order up to $n^{th}$-order.

This paper assumes familiarity with symmetry-based methods and so does not add to the volume of the work by recapitulating the well established theory. However, in stating this we recommend the interested reader to consult the books, which contain various aspects concerning Lie and Noether symmetries.

The objective of this paper is two-fold. First, we utilize a generalized Lagrangian to formulate Noether symmetry conditions, for symmetries of higher-order perturbations. Thereafter, the latter is used to establish the corresponding approximate conservation laws, also of higher-order perturbations, via Noether’s theorem. In this regard, to cope with the complexity of our derivation, we employ the Einstein summation convention, and require that indices enclosed in parentheses indicate symmetrization,
for instance, $F_{ij} = \frac{1}{2}(F_{ij} + F_{ji})$.

The family of perturbed Lagrangians that we assume are

$$L(t, x^k, \dot{x}^k, \varepsilon) = L_0(t, x^k, \dot{x}^k) + \varepsilon L_1(t, x^k, \dot{x}^k) + O(\varepsilon^2),$$

where we stipulate that the exact and approximate terms are defined by the regular Lagrangians

$$L_0(t, x^k, \dot{x}^k) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V_0(t, x^k),$$

$$L_1(t, x^k, \dot{x}^k) = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j - V_1(t, x^k),$$

respectively, where $g_{ij} = g_{ij}(x^k), h_{ij} = h_{ij}(x^k)$ and dot denotes total derivative with respect to the independent parameter $t$, i.e. $\dot{x}^i = \frac{dx^i}{dt}$. Such Lagrangians are not limited by any specific assumptions in order to preserve generality. The plan of the paper is as follows.

Section 2 is the main core of our analysis where we derive the approximate Noether symmetry conditions for regular Lagrangians of order $(\varepsilon^1)$ and $(\varepsilon^n)$ in general. Moreover, from the conditions we see that there exists a link between the approximate symmetries and the Homothetic vector fields of the metric tensor $g_{ij}$. In Section 3 we demonstrate our results by applying our approach to various examples of maximally symmetric systems, and also on a $sl(2, R)$ exact invariant system. For each case we derive the approximate Noether symmetries and the corresponding approximate conservation laws. Finally, in Section 4 we discuss our results and draw our conclusions.

### 2. Approximate Noether Conditions

In the interest of clarity and completeness we have decided to present the immediate work that follows in great detail, dividing the procedure into important steps. Following Govinder et al. [14], under a point transformation

$$\bar{x}^i = x^i + a \left( \eta^i_0 + \varepsilon \eta^i_1 + O(\varepsilon^2) \right),$$

$$\bar{t} = t + a \left( \xi_0 + \varepsilon \xi_1 + O(\varepsilon^2) \right),$$

with $\xi_A = \xi_A(t, x^k)$, $\eta_A^i = \eta_A^i(t, x^k)$, $A = 0, 1$ and “$a, \varepsilon$” as two infinitesimal parameters, we have the generator

$$X = X_0 + \varepsilon X_1 + O(\varepsilon^2),$$

where $X_A = \xi_A \partial_t + \eta_A^i \partial_i$. Here, $X$ is a first-order approximate vector field composed of an exact ($X_0$) and an approximate ($X_1$) part. The generator [6] is a Noether point symmetry if the following equation is satisfied

$$X^{[1]} L + L \frac{d \xi}{dt} = \frac{df}{dt}.$$

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or equivalently
\[
(X_0^{[1]} + \varepsilon X_1^{[1]}) (L_0 + \varepsilon L_1) + (L_0 + \varepsilon L_1) \frac{d}{dt} (\xi_0 + \varepsilon \xi_1) - \frac{d}{dt} (f_0 + \varepsilon f_1) = O (\varepsilon^2),
\]
where \(f_A = f_A (t, x^i)\) and the term \(X_0^{[1]} A\) is the first prolongation expressed as
\[
X_0^{[1]} A = \xi_A \partial_t + \eta^A_i \partial_i + \left( \eta^i_A - \dot{x}^i A \right) \partial_x^i.
\]
Inserting (2) and (3) into the left-hand-side of the Noether condition (8), we find
\[
X_0^{[1]} (f_0) = \left( \xi_0 \partial_t + \eta^0_i \partial_i + \left( \eta^i_0 - \dot{x}^i 0 \right) \partial_x^i \right) \left( \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V_0 (t, x^k) \right)
\]
\[
= \frac{1}{2} g_{ij,k} \eta^0_k \dot{x}^i \dot{x}^j + g_{ij} \left( \eta^i_0 - \dot{x}^i 0 \right) \dot{x}^j - \xi_0 V_{0,t} - \eta^k_0 V_{0,i}
\]
\[
= \frac{1}{2} g_{ij,k} \eta^k \dot{x}^i \dot{x}^j + g_{ij} \left( \eta^i_0 - \xi_0 \right) \dot{x}^j + g_{ij,0} \dot{x}^i \dot{x}^j - g_{ij} \xi_0 \dot{x}^i \dot{x}^j - \dot{x}^k \xi_0 V_{0,t} - \eta^k_0 V_{0,i}
\]
\[-\xi_0 V_{0,t} - \eta^k_0 V_{0,i}.
\]
Similarly,
\[
\varepsilon X_0^{[1]} L_1 = \varepsilon \left( \frac{1}{2} h_{ij,k} \eta^0_k \dot{x}^i \dot{x}^j + h_{ij} \left( \eta^i_0 - \xi_0 \right) \dot{x}^j + h_{ij,0} \dot{x}^i \dot{x}^j - h_{ij} \xi_0 \dot{x}^i \dot{x}^j - \xi_0 V_{1,t} - \eta^k_0 V_{1,i} \right).
\]
\[
\varepsilon X_1^{[1]} L_0 = \varepsilon \left( \frac{1}{2} g_{ij,k} \eta^k \dot{x}^i \dot{x}^j + g_{ij} \left( \eta^i_1 - \xi_1 \right) \dot{x}^j + g_{ij,1} \dot{x}^i \dot{x}^j - g_{ij} \xi_1 \dot{x}^i \dot{x}^j - \xi_1 V_{0,t} - \eta^k_1 V_{0,i} \right).
\]
\[
\varepsilon^2 X_1^{[1]} L_1 = \varepsilon^2 \left( \frac{1}{2} h_{ij,k} \eta^k \dot{x}^i \dot{x}^j + h_{ij} \left( \eta^i_1 - \xi_1 \right) \dot{x}^j + h_{ij,1} \dot{x}^i \dot{x}^j - h_{ij} \xi_1 \dot{x}^i \dot{x}^j - \xi_1 V_{1,t} - \eta^k_1 V_{1,i} \right).
\]
For the middle terms, we have the relation
\[
(L_0 + \varepsilon L_1) \frac{d}{dt} (\xi_0 + \varepsilon \xi_1) = \dot{\xi}_0 L_0 + \varepsilon L_1 \dot{\xi}_0 + \varepsilon \dot{\xi}_1 L_0 + \varepsilon^2 \dot{\xi}_1 L_1,
\]
and therefore
\[
\dot{\xi}_0 L_0 = (\dot{\xi}_0 + \dot{\xi}_0 \dot{x}^k) \left( \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V_0 (t, x^k) \right)
\]
\[
= \frac{1}{2} \dot{\xi}_0 g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \xi_0 g_{ij} \dot{x}^i \dot{x}^j - \dot{\xi}_0 V_0 - V_0 \dot{\xi}_0 \dot{x}^k.
\]
Similarly, the remaining middle terms are

\[ \varepsilon L_1 \xi_0 = \varepsilon \left( \frac{1}{2} \xi_{0,t} h_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \xi_{0,k} h_{ij} \dot{x}^i \dot{x}^j \dot{x}^k - \xi_{0,t} V_1 - V_1 \xi_{0,k} \dot{x}^k \right), \]

\[ \varepsilon \xi_1 L_0 = \varepsilon \left( \frac{1}{2} \xi_{1,t} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \xi_{1,k} g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k - \xi_{1,t} V_0 - V_0 \xi_{1,k} \dot{x}^k \right), \]

\[ \varepsilon^2 \xi_1 L_1 = \varepsilon \left( \frac{1}{2} \xi_{1,t} h_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \xi_{1,k} h_{ij} \dot{x}^i \dot{x}^j \dot{x}^k - \xi_{1,t} V_1 - V_1 \xi_{1,k} \dot{x}^k \right). \]

Lastly, the end terms produce the expression

\[ f_0 + \varepsilon f_1 = (f_{0,t} + f_{0,k} \dot{x}^k) + \varepsilon (f_{1,t} + f_{1,k} \dot{x}^k). \]

Next, in collating terms sans \( \varepsilon \) we obtain the expression

\[ \varepsilon^0 = \frac{1}{2} g_{ij,k} \eta_0^k \dot{x}^i \dot{x}^j + g_{ij} \left( \eta_{0,k}^i - \xi_{0,t} \right) \dot{x}^i \dot{x}^j + g_{ij} \eta_0^i \dot{x}^j + g_{ij} \xi_{0,k} \dot{x}^i \dot{x}^j \dot{x}^k - \xi_0 V_{0,t} \]

\[ - \eta_0^k V_{0,i} + \left( \frac{1}{2} \xi_{0,t} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \xi_{0,k} g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k - \xi_{0,t} V_0 - V_0 \xi_{0,k} \dot{x}^k \right) \]

\[ - (f_{0,t} + f_{0,k} \dot{x}^k). \]

By collecting terms of equal powers of \((\dot{x})^K\), we find the following determining system of equations

\[ \xi_{0,k} g_{ij} = 0, \quad (10) \]

\[ \frac{1}{2} g_{ij,k} \eta_0^k + g_{k(i} \left( \eta_{0,j}^i - \frac{1}{2} \xi_{0,t} \right) \dot{x}^i \dot{x}^j + h_{ij} \eta_0^i \dot{x}^j + h_{ij} \eta_0^j \dot{x}^i - \xi_{0,t} V_1 - \eta_0^k V_{1,i} = 0, \quad (11) \]

\[ \xi_{0} V_{0,t} + \eta_0^i V_{0,i} + \xi_{0,i} V_0 + f_{0,t} = 0, \quad (12) \]

which reveals that \( \xi_0 = \xi_0 (t) \).

Terms involving \( \varepsilon^1 \) are

\[ \varepsilon^1 = \frac{1}{2} h_{ij,k} \eta_1^k \dot{x}^i \dot{x}^j + h_{ij} \left( \eta_{1,k}^i - \xi_{1,t} \right) \dot{x}^i \dot{x}^j + h_{ij} \eta_1^i \dot{x}^j + h_{ij} \eta_1^j \dot{x}^i - \xi_{1,t} V_1 - \eta_0^k V_{1,i} \]

\[ + \frac{1}{2} g_{ij,k} \eta_1^k \dot{x}^i \dot{x}^j + g_{ij} \left( \eta_{1,k}^i - \xi_{1,t} \right) \dot{x}^i \dot{x}^j + g_{ij} \eta_1^i \dot{x}^j + g_{ij} \eta_1^j \dot{x}^i - g_{ij} \xi_{0,k} \dot{x}^i \dot{x}^j \dot{x}^k - \xi_1 V_{0,t} \]

\[ - \eta_0^k V_{0,i} \left( \frac{1}{2} \xi_{0,t} h_{ij,k} \dot{x}^i \dot{x}^j - \xi_{0,t} V_1 - V_1 \xi_{0,k} \dot{x}^k \right) \]

\[ + \left( \frac{1}{2} \xi_{1,t} g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \xi_{1,k} g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k - \xi_{1,t} V_0 - V_0 \xi_{1,k} \dot{x}^k \right) - (f_{1,t} + f_{1,k} \dot{x}^k). \]

As before, these expressions are substituted back into the Noether condition to check monomials in \((\dot{x})^K\), and we have the determining system of equations

\[ g_{ij} \xi_{1,k} = 0, \quad (13) \]

which immediately implies that \( \xi_1 = \xi_1 (t) \), and additionally

\[ \left[ \frac{1}{2} h_{ij,k} \eta_1^k + h_{k(j} \left( \eta_{1,k}^i - \frac{1}{2} \xi_{1,t} \right) \right] + \left[ \frac{1}{2} g_{ij,k} \eta_1^k + g_{k(j} \left( \eta_{1,i}^i - \frac{1}{2} \xi_{1,t} \right) \right] = 0, \quad (14) \]

\[ h_{ij} \eta_{0,k}^i + g_{ij} \eta_1^i = f_{1,j} = 0, \quad (15) \]

\[ \xi_0 V_{1,t} + \eta_0^i V_{1,i} + \xi_1 V_{0,t} + \eta_1^i V_{0,i} + \xi_{0,i} V_1 + \xi_{1,t} V_0 + f_{1,t} = 0. \quad (16) \]
Note that we impose the restriction that terms involving $\varepsilon^2$ are taken to be zero.

Finally, the system of equations (10)-(12) and (13)-(16) provide the approximate Noether symmetry conditions for the perturbed Lagrangian (11) together with equations (2), (3) and symmetry generator (6), viz.

$(\varepsilon)^0$:

\[
L_{\eta_0} g_{ij} = 2 \left( \frac{1}{2} \xi_{0,t} \right) g_{ij}, \quad (17)
\]

\[
g_{ij} \eta_{0,t}^{\alpha} = f_{0,j}, \quad (18)
\]

\[
L_{\eta_0} V_0 + \xi_{0,t} V_0 + \xi_0 V_{0,t} + f_{0,t} = 0, \quad (19)
\]

$(\varepsilon)^1$:

\[
L_{\eta_0} h_{ij} + L_{\eta_1} g_{ij} - 2 \left( \frac{1}{2} \xi_{0,t} \right) h_{ij} - 2 \left( \frac{1}{2} \xi_{1,t} \right) g_{ij} = 0, \quad (20)
\]

\[
h_{ij} \eta_{0,t}^{\alpha} + g_{ij} \eta_{1,t}^{\alpha} = f_{1,j}, \quad (21)
\]

\[
L_{\eta_0} V_1 + \xi_{0,t} V_1 + \xi_0 V_{1,t} + \xi_1 V_{0,t} + \xi_{1,t} V_0 = - f_{1,t}, \quad (22)
\]

where $L_{\eta_j}$ is the geometric derivative, that is, the Lie derivative operator along $\eta_j$.

2.1. A Special Case of $L_1$

Suppose that $L_1 \left( t, x^k, \dot{x}^k \right) = L_1 \left( t, x^k \right) = - V_1 \left( t, x^k \right)$. Then the Noether symmetry conditions (17)-(19) and (20)-(22) transform to

$(\varepsilon)^0$:

\[
L_{\eta_0} g_{ij} = 2 \left( \frac{1}{2} \xi_{0,t} \right) g_{ij}, \quad (23)
\]

\[
g_{ij} \eta_{0,t}^{\alpha} = f_{0,j}, \quad (24)
\]

\[
L_{\eta_0} V_0 + \xi_{0,t} V_0 + \xi_0 V_{0,t} + f_{0,t} = 0, \quad (25)
\]

$(\varepsilon)^1$:

\[
L_{\eta_0} g_{ij} = 2 \left( \frac{1}{2} \xi_{1,t} \right) g_{ij}, \quad (26)
\]

\[
g_{ij} \eta_{1,t}^{\alpha} = f_{1,j}, \quad (27)
\]

\[
L_{\eta_0} V_1 + L_{\eta_1} V_0 + \xi_0 V_{1,t} + \xi_{0,t} V_1 + \xi_1 V_{0,t} + \xi_{1,t} V_0 = - f_{1,t}, \quad (28)
\]

respectively.

In fact, with the use of equation (23) and (26), $\eta_A^i$ can be expressed as $\eta_A^i = T_A^i \left( t \right) Y_A^i \left( x^k \right)$ where the $Y_A^i$ are Homothetic vectors (HVs) or Killing vectors (KVs), of the metric $g_{ij}$ with conformal factor $\psi$, and therefore $T_A L_{Y_A^i} g_{ij} = 2 T_A \psi_A g_{ij} = 2 \left( \frac{1}{2} \xi_{A,t} \right) g_{ij}$, so that we may specify the relation $\xi_{A,t} = 2 \psi_A T_A$. Consequently, we have detected that the approximate Noether symmetry conditions are

respectively.
\((\varepsilon)^0:\)

\[
L_{\eta_0}g_{ij} = 2\psi_0g_{ij}, \quad \tag{29}
\]
\[
g_{ij}\eta_{0,t} = f_{0,j}, \quad \tag{30}
\]
\[
L_{\eta_0}V_0 + 2\psi_0T_0V_0 + \xi_0V_0,t + f_{0,t} = 0, \quad \tag{31}
\]
\[
\xi_{0,t} = 2\psi_0T_0, \quad \tag{32}
\]

\((\varepsilon)^1:\)

\[
L_{\eta_1}g_{ij} = 2\psi_1g_{ij}, \quad \tag{33}
\]
\[
g_{ij}\eta_{1,t} = f_{1,j}, \quad \tag{34}
\]
\[
L_{\eta_0}V_1 + L_{\eta_1}V_0 + \xi_0V_1, t + 2\psi_0T_0V_1 + \xi_1V_0,t + 2\psi_1T_1V_0 = -f_{1,t}, \quad \tag{35}
\]
\[
\xi_{1,t} = 2\psi_1T_1, \quad \tag{36}
\]

which leads to the following theorem:

**Theorem 1.** The approximate Noether symmetries of the perturbed Lagrangian

\[
L(t, x^k, \dot{x}^k, \varepsilon) = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j - V_0(t, x^k) - \varepsilon V_1(t, x^k) + O(\varepsilon^2),
\]

are generated from the HV algebra of the metric \(g_{ij}\).

This is an important result which tells us the maximum number of approximate symmetries that can be constructed. Moreover, it implies that if the metric tensor \(g_{ij}\) does not admit any Homothetic/Killing vector fields, then we will not be able to determine any approximate conservation laws for the dynamical system.

2.1.1. Approximate Noether Symmetries of \((\varepsilon^n)\).

For completeness we extend our analysis to the case of approximate symmetries of order \(\varepsilon^n\). Suppose we have a point transformation of the form

\[
\bar{x}^i = x^i + a \left( \eta_{(0)} + \sum_{\gamma=1}^{n} \varepsilon^\gamma \eta_{(\gamma)} + O(\varepsilon^{\gamma+1}) \right), \quad \tag{37}
\]
\[
\bar{t} = t + a \left( \xi_{(0)} + \sum_{\gamma=1}^{n} \varepsilon^\gamma \xi_{(\gamma)} + O(\varepsilon^{\gamma+1}) \right), \quad \tag{38}
\]

with symmetry generator

\[
X = X_0 + \sum_{\gamma=1}^{n} \varepsilon^\gamma X_\gamma + O(\varepsilon^{\gamma+1}). \quad \tag{39}
\]
Hence, we find the generalized Noether condition

\[
\left( X_0^{[1]} + \sum_{\gamma=1}^{n} \varepsilon^\gamma X_1^{[1]} \right) (L_0 + \varepsilon L_1) + (L_0 + \varepsilon L_1) \frac{d}{dt} \left( \xi_0 + \sum_{\gamma=1}^{n} \varepsilon^\gamma \xi_1 \right)
\]

\[
- \frac{d}{dt} \left( f_0 + \sum_{\gamma=1}^{n} \varepsilon^\gamma f_\gamma \right) = O \left( \varepsilon^{\gamma+1} \right).
\]

The derivation of the Noether symmetry conditions is similar to that of the previous section. For this reason, and for the sake of brevity we shall not present further details, but merely state the relevant formulae. To this end, the Noether symmetry conditions are

\[(\varepsilon)^0:\]

\[
L_{\eta_0} g_{ij} = 2 \left( \frac{1}{2} \xi_{0,t} \right) g_{ij},
\]

\[
g_{ij} \eta_{0,t} = f_{0,j},
\]

\[
L_{\eta_0} V_0 + \xi_{0,t} V_0 + \xi_0 V_{0,t} + f_{0,t} = 0,
\]

\[(\varepsilon)^\gamma, \gamma = 1 \ldots n:\]

\[
L_{\eta_{\gamma-1}} h_{ij} + L_{\eta_\gamma} g_{ij} - 2 \left( \frac{1}{2} \xi_{\gamma-1,t} \right) h_{ij} - 2 \left( \frac{1}{2} \xi_{\gamma,t} \right) g_{ij} = 0,
\]

\[
h_{ij} \eta_{\gamma-1,t}^i + g_{ij} \eta_{\gamma,t}^i = f_{\gamma,j},
\]

\[
L_{\eta_{\gamma-1}} V_1 + L_{\eta_{\gamma}} V_0 + \xi_{\gamma-1,t} V_1 + \xi_{\gamma-1,t} V_1 + \xi_{\gamma,t} V_0 + \xi_{\gamma,t} V_0 = -f_{\gamma,t}.
\]

An important observation is that Theorem 1 is still valid.

3. Applications

The primary objective of our work was the development of explicit approximate Noether symmetry conditions. Having accomplished this, we are now in a position to apply the theory to some notorious examples from the literature. Tracing the same progression so far, we start with some benign 1-dimensional cases, followed by 2-dimensional cases and finally, tackle the \(n\)-dimensional case. In every example, we have solved the Noether conditions, but due to the economy of space, these calculations will not be displayed here. Instead the Noether symmetries and some of the integrals are simply listed below. The formulae for the derivation of the approximate conservation laws are given in Appendix A.

3.1. One-dimensional Perturbed Lagrangians

Table 1 summarizes the results of three different Lagrangians of dimension one. The appropriate application of (17)–(22) leads to the derivation of both exact and approximate Noether symmetries...
Table 1: Approximate symmetries for one-dimensional Perturbed Lagrangians

| Case | $L(t, x^k, \dot{x}^k, \varepsilon)$ | Noether Symmetry |
|------|-------------------------------------|------------------|
| (I)  | $L_0 = \frac{1}{2} \dot{x}^2 + \omega_0 \cos x$, $L_1 = \cos(\omega t) \cos x$ | $1^{st}$-order ($X = X_0 + \varepsilon X_1$): $Z^I = \varepsilon \partial_t$ $2^{nd}$-order ($X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2$): $Z^I = \varepsilon^2 \partial_t$ |
| (II) | $L_0 = \frac{1}{2} \dot{x}^2 + V_0$, $L_1 = \frac{V_1}{2 \omega^2} x^2$ | $Z_1^{II} = \varepsilon (2 t \partial_t + x \partial_x)$ $Z_2^{II} = \frac{1}{V_1} \partial_t + \varepsilon (2 \ln (t) \partial_t + x \partial_x)$ $Z_3^{II} = \varepsilon (t^2 \partial_t + tx \partial_x)$ $Z_4^{II} = -\frac{1}{2 V_1^2} t^2 \partial_t - \frac{1}{2 V_1} t x \partial_x$ $+ \varepsilon (t^2 \ln (t) - \frac{t}{2}) \partial_t + t \ln (t) \partial_x$ $Z_5^{II} = 2 t \partial_t + x \partial_x$ $Z_6^{II} = \varepsilon \partial_t$ |
| (III) | $L_0 = \frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2$, $L_1 = \frac{1}{2} e^{\omega t} x^2$ | $Z_1^{III} = \left( \frac{\omega}{\omega^2 + 1} + \omega \right) \partial_t + \varepsilon \left( \frac{\omega}{\omega^2 + 1} e^{\omega t} \partial_t + c_1 e^{\omega t} x \partial_x \right)$ $Z_2^{III} = \varepsilon (\cos (2 t) \partial_t + \sin (2 t) x \partial_x)$ $Z_3^{III} = \varepsilon ((x \cos (2 t) + \sin (2 t)) \partial_x)$ $Z_4^{III} = \sin (2 t) \partial_t + 2 \omega \cos (2 t) x \partial_x$ $+ \varepsilon \left( e^{\omega t} \frac{2 \omega \sin (2 t) - 2 \cos (2 t)}{\omega^2 + 4} \partial_t + e^{\omega t} \sin (2 t) \partial_x \right)$ $Z_5^{III} = \cos (2 t) \partial_t - 2 \omega \sin (2 t) x \partial_x$ $+ \varepsilon \left( e^{\omega t} \frac{2 \omega \cos (2 t) + 2 \sin (2 t)}{\omega^2 + 4} \partial_t + e^{\omega t} \cos (2 t) \partial_x \right)$ $Z_6^{III} = \varepsilon (\sin (t) \partial_x)$ $Z_7^{III} = \varepsilon (\cos (t) \partial_x)$ $Z_8^{III} = \omega \left( \omega \sin (t) + 2 \cos (t) \right) \partial_y + \varepsilon \left( e^{\omega t} \sin (t) \partial_x \right)$ $Z_9^{III} = \omega \left( \omega \cos (t) - 2 \sin (t) \right) \partial_y + \varepsilon \left( e^{\omega t} \cos (t) \partial_x \right)$ $Z_{10}^{III} = \varepsilon \left( \partial_t \right)$ |
up-to first-order, whilst via conditions (41) - (46), we find the approximate Noether symmetries up to second-order.

Case (I) corresponds to the Hamiltonian

\[ H_0 = \frac{1}{2} \dot{x}^2 - \omega_0 \cos x, \quad H_1 = -\varepsilon \cos(\omega t) \cos x, \]

while Case (II) possesses the Hamiltonian

\[ H_0 = \frac{1}{2} \dot{x}^2 - \frac{V_0}{x^2}, \quad H_1 = -\varepsilon \frac{V_1}{2t^2} x^2, \]

and has the corresponding nonzero boundary terms: \( f(Z_{II}^{1}) = \varepsilon \frac{-x^2}{2t^2} \), \( f(Z_{II}^{2}) = \varepsilon \frac{x^2}{2} \), \( f(Z_{II}^{3}) = -\frac{x^2}{4} + \varepsilon \left( \frac{x^2}{2} \ln(t) + 1 \right) \).

Subsequently, the approximate Noether integrals are derived to be

\[
\begin{align*}
I(Z_{II}^{1}) &= \varepsilon (2tH_0 - x\dot{x}), \\
I(Z_{II}^{2}) &= \frac{1}{V_1} H_0 + \varepsilon \left( \left( 2 \ln(t) H_0 - \frac{x}{t} \dot{x} - \frac{1}{2t^2} x^2 \right) + \frac{1}{V_1} H_1 \right), \\
I(Z_{II}^{3}) &= \varepsilon \left( t^2 H_0 - tx \dot{x} + \frac{1}{2} x^2 \right), \\
I(Z_{II}^{4}) &= -\frac{1}{2V_1} t^2 H_0 + \frac{1}{2V_1} tx \dot{x} - \frac{1}{4V_1} x^2 + \\
&
\quad + \varepsilon \left( \left( t^2 \left( \ln(t) - \frac{1}{2} \right) H_0 - t \ln(t) \dot{x} + \frac{1}{2} x^2 \ln(t) + 1 \right) \\
&
\quad - \frac{1}{2V_1} t^2 H_1 + \frac{1}{2V_1} tx \dot{x} \right), \\
I(Z_{II}^{5}) &= 2tH_0 - x\dot{x}, \\
I(Z_{II}^{6}) &= \varepsilon H_0.
\end{align*}
\]

On the other hand, Case (III) admits the Hamiltonian functions

\[ H_0 = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2, \quad H_1 = -\varepsilon \frac{1}{2} t^{\omega t} x^2, \]

\[ ^1 \text{An interesting discussion on the relation between perturbations and the } SL(2, \mathbb{R}) \text{ Lie algebra can be found in } [28]. \]
while the corresponding boundary terms are

\[
\begin{align*}
    f(Z_{111}^{III}) &= \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 \right), \\
    f(Z_{211}^{III}) &= \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 + x^2 \cos (2t) \right), \\
    f(Z_{311}^{III}) &= \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 - x^2 \sin (2t) \right), \\
    f(Z_{411}^{III}) &= -\omega x^2 \sin (2t) + \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 + \frac{1}{2} x^2 e^{\omega t} (\omega \sin (2t) + 2 \cos (2t)) \right), \\
    f(Z_{511}^{III}) &= -\omega x^2 \cos (2t) + \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 + \frac{1}{2} x^2 e^{\omega t} (\omega \cos (2t) - 2 \sin (2t)) \right), \\
    f(Z_{611}^{III}) &= \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 + x \cos (t) \right), \\
    f(Z_{711}^{III}) &= \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 - x \sin (t) \right), \\
    f(Z_{811}^{III}) &= x \omega (\omega \cos (t) + 2 \sin (t)) + \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 + x e^{\omega t} (\omega \sin (t) + \cos (t)) \right), \\
    f(Z_{911}^{III}) &= -x \omega (\omega \sin (t) + 2 \cos (t)) + \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 + x e^{\omega t} (\omega \cos (t) - \sin (t)) \right), \\
    f(Z_{1011}^{III}) &= \varepsilon \left( \frac{1}{2} \omega e^{\omega t} x^2 \right).
\end{align*}
\]

### 3.2. Two-Dimensional Perturbed Lagrangians

In Table 2 we consider Case (IV) and (V), which are both two dimensional approximate Lagrangians with the Hamiltonians

\[
\begin{align*}
    H_0 &= \frac{1}{2} (\dot{x}^2 - \dot{y}^2) - \frac{1}{2} (x^2 + y^2), \quad
    H_1 = \varepsilon \left( x^2 y - \frac{y^3}{3} \right), \\
    H_0 &= \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \frac{1}{x^2 + y^2}, \quad
    H_1 = \varepsilon \frac{1}{2} (x^2 + y^2),
\end{align*}
\]

respectively.

For Case (IV) the Illustratively, the exact and approximate Noether integrals of motion are

\[
\begin{align*}
    I_0 &= H_0, \\
    I_t &= \varepsilon H_0, \quad I_{\{a_{-1}\}} = \varepsilon (\dot{y} \dot{x} - x \dot{y}), \\
    I_{1,2} &= \varepsilon \left( i (a_2 e^{-i2t} - a_1 e^{i2t}) E - (a_1 e^{i2t} + a_2 e^{-i2t}) (x \dot{x} + y \dot{y}) \right) \\
           &\quad + i (a_1 e^{i2t} - a_2 e^{-i2t}) (x^2 + y^2), \\
    I_{5,6} &= \varepsilon \left( (a_5 \dot{e}^{i\omega t} + a_6 \dot{e}^{-i\omega t}) (c_1 \dot{x} + c_2 \dot{y}) - i (a_5 \dot{e}^{i\omega t} - a_6 \dot{e}^{-i\omega t}) (c_1 x + c_2 y) \right).
\end{align*}
\]

We omit the derivation of the conservation laws for Case (V). They can be calculated easily from the formulae \([A.2]\) and \([A.3]\).
At this juncture, we are able to state that:

$$V$$ for a general function

3.3. The $n$-Dimensional Perturbed Lagrangian

As a final example, we culminate our results for the derivation of the $n$-dimensional flat case. Consider the approximate Lagrangian

$$L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} (x^2 y) - \frac{v}{x^2 + y^2},$$

for a general function $V_1$.

The first-order ($X = X_0 + \varepsilon X_1$) approximate Noether symmetries are

$$Z_0 = \varepsilon \partial_t, \quad Z_n = \varepsilon \partial_{t^n},$$

$$Z_1 = \varepsilon (t^2 \partial_t + t \partial_x + t \partial_y),$$

$$Z_2 = -t \partial_t - x \partial_x - y \partial_y,$$

$$Z_3 = \varepsilon (t^2 \partial_t + t x \partial_x + t y \partial_y),$$

$$Z_4 = -t^2 \partial_t - x t \partial_x - y t \partial_y,$$

$$Z_5 = \varepsilon (t^4 \partial_t + x t^3 \partial_x + y t^3 \partial_y).$$

At this juncture, we are able to state that:

- $X_{ij}$ are the $\frac{n(n-1)}{2}$ rotations of the Euclidean space,
- $S_i^j$ are the $n$ gradient KVs of the Euclidean space,
- $H^i$ is the HV of the Euclidean space.
Here, the Noether integrals are in fact only approximate in nature, namely

\[
I_t = \varepsilon H_0, \quad I_{\gamma(n-1)}^{\gamma} = \varepsilon X_{ij} \dot{x}^i \dot{x}^j,
\]

\[
I_{1,2} = \varepsilon \left( i \left( a_2 e^{-i2t} - a_1 e^{i2t} \right) H_0 - \left( a_1 e^{i2t} + a_2 e^{-i2t} \right) H^i \dot{x}^i \right),
\]

\[
I_{5,6} = \varepsilon \left( (a_5 e^{i2t} + a_6 e^{-i2t}) S_{ij} \dot{x}^i - i \left( a_5 e^{i2t} - a_6 e^{-i2t} \right) x_J \right),
\]

where \( H_0 \) is the Hamiltonian of the zero-order Lagrangian. It is important to see that the exact integrals of the unperturbed system become approximate integrals.

4. Discussion

The main scope of this work was to devise approximate Noether symmetry conditions for regular perturbed Lagrangians in a geometric way, to gain understanding into the role of the underlying geometry in the existence of approximate symmetries. The family of Lagrangians that we considered are those that are defined by a Kinetic energy and a potential, which in general describe the motion of a particle in a \( n \)-dimensional space under the action of an autonomous force.

We found a strong connection between the Homothetic algebra of the underlying geometry and the approximated symmetries and in particular, in the scenario where the perturbation terms do not modify the Kinetic energy, approximate symmetries exist if and only if the metric that defines the Kinetic energy admits a nontrivial Homothetic algebra. Finally, in order to demonstrate our results, we determined in a geometric way, the approximate symmetries for various systems of special interest.

In a forthcoming work, we want to extend this analysis and perform various classifications in which the perturbations are invariant under a specific algebra.

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Appendix A. Approximate conservation laws

In this appendix the approximate conservation law are determined as they are given by Noether’s theorem \[14\]. In general it is well known that if vector field \( X = \xi \partial_t + \eta^i \partial_i \), is a symmetry for the Lagrangian \( L = L(t, x^k, \dot{x}^k) \) with a boundary term \( f \), then the following function is a conservation law:

\[
I = \xi H - \frac{\partial L}{\partial \dot{x}^i} \eta^i + f, \quad (A.1)
\]
where $H$ is the Hamiltonian function of $L$.

In a similar way, for any approximate Lagrangian $L = L_0 + \varepsilon L_1$, and an approximate Noether generator $X = X_0 + \varepsilon X_1$, we derive the exact conservation law, $I_0$ and the first order approximate conservation law, $I_1$ as follows:

\[ I_0 = \xi_0 H_0 - \frac{\partial L_0}{\partial \dot{x}^i} \eta_0^i + f_0, \]  
\[ I_1 = \left( H_0 \xi_1 - \frac{\partial L_0}{\partial \dot{x}^i} \eta_1^i + f_1 \right) + \xi_0 H_1 - \frac{\partial L_1}{\partial \dot{x}^i} \eta_0^i. \]  

(A.2)  

(A.3)

A generalization of this idea to the higher-order case of $\varepsilon$, with the symmetry generator

\[ X = X_0 + \sum_{\gamma=1}^{n} \varepsilon^\gamma X_\gamma + O (\varepsilon^{\gamma+1}), \]  

(A.4)

leads us to deduce the formulae for the associated Noether integrals, viz.

\[ I_0 = \xi_0 H_0 - \frac{\partial L_0}{\partial \dot{x}^i} \eta_0^i + f_0, \]  
\[ I_\gamma = \left( H_0 \xi_\gamma - \frac{\partial L_0}{\partial \dot{x}^i} \eta_\gamma^i + f_\gamma \right) + \xi_{\gamma-1} H_1 - \frac{\partial L_1}{\partial \dot{x}^i} \eta_{\gamma-1}^i. \]  

(A.5)  

(A.6)

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