HIGH ORDER ADJUSTED BLOCK-WISE EMPIRICAL LIKELIHOOD FOR WEAKLY DEPENDENT DATA

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ABSTRACT

The upper limit on the coverage probability of the empirical likelihood ratio confidence region severely hampers its application in statistical inferences. The root cause of this upper limit is the convex hull of the estimating functions that is used in the construction of the profile empirical likelihood. For i.i.d data, various methods have been proposed to solve this issue by modifying the convex hull, but it is not clear how well these methods perform when the data is no longer independent. In this paper, we consider weakly dependent multivariate data, and we combine the block-wise empirical likelihood with the adjusted empirical likelihood to tackle data dependency and the convex hull constraint simultaneously. We show that our method not only preserves the much celebrated asymptotic \(\chi^2\) distribution, but also improves the coverage probability by removing the upper limit. Further, we show that our method is also Bartlett correctable, thus is able to achieve high order asymptotic coverage accuracy.

KEYWORDS: Empirical Likelihood Method, Convex Hull Constraint, Confidence Region Coverage Accuracy, Bartlett Correction, Weakly Dependent Data.

1. INTRODUCTION

Empirical likelihood methods have been studied extensively in the past three decades as a reliable and flexible alternative to the parametric likelihood. Among its numerous attractive properties, the ones that are most celebrated are the asymptotic \(\chi^2\) distribution of the empirical likelihood ratio and the ability to use Bartlett correction to improve the corresponding confidence region coverage accuracy. However, despite these desirable properties that are at least parallel to the parametric likelihood methods, there is a serious drawback, where the empirical likelihood confidence region has an under-coverage problem in small sample or high dimensional settings. This undesirable feature was noticed in the early works, for example by Owen (1988) and Tsao (2004). For independent data, various methods have been proposed to address this issue. They can be divided into roughly two main areas. One is to improve
the approximation to the limiting distribution of the log empirical likelihood ratio. For this approach, among others, Owen (1988) proposed to use a bootstrap calibration and DiCiccio et al. (1991) showed that by scaling the empirical likelihood ratio with a Bartlett factor, which can be estimated from the data, the limiting coverage accuracy can be improved from \( O(n^{-1}) \) to \( O(n^{-2}) \). Another approach is to tackle the convex hull constraint, which was first studied in Tsao (2004). There are three major methods in this approach, namely the penalized empirical likelihood by Bartolucci (2007), the adjusted empirical likelihood by Chen et al. (2008), and the extended empirical likelihood by Tsao (2013). These three methods then have been extended and refined by subsequent research. To mention a few that are related to this paper, Zhang and Shao (2016) extended the penalized empirical likelihood to fix-b block-wise method to apply on weakly dependent data. Emerson and Owen (2009) proposed to modify the placement of the extra point in the adjusted empirical likelihood in order to remove the upper bound of the adjusted likelihood ratio statistic, which if not removed will cause a confidence region to cover the whole space in some situations. Liu and Chen (2010) showed that by choosing the tuning parameter in the adjusted empirical likelihood in a specific way, it is possible to achieve the Bartlett corrected coverage error rate. Chen and Huang (2013) studied the finite sample properties of the adjusted empirical likelihood and discussed a generalized version of the method proposed in Emerson and Owen (2009). It is worth pointing out that most of the existing work have focused on independent data, and the aforementioned Zhang and Shao (2016) was the first paper to address the convex hull constraint for weakly dependent data with penalized empirical likelihood under the block-wise framework, which was introduced to empirical likelihood by Kitamura (1997). Recently Piyadi Gamage et al. (2017) studied the adjusted empirical likelihood for time series models under the frequency domain empirical likelihood framework, which was introduced by Nordman and Lahiri (2006). In this paper, we extend the adjusted empirical likelihood to weakly dependent data under the the block-wise empirical likelihood framework. Hereafter, we call it the adjusted block-wise empirical likelihood (ABEL). Compared to the non-standard pivotal asymptotic distribution obtained in Zhang and Shao (2016), we show that the ABEL preserves the much celebrated asymptotic \( \chi^2 \) distribution. In addition, we show that the tuning parameter can be selected such that the ABEL achieves the Bartlett corrected coverage error rate with weakly dependent data.

This paper is organized as the following. Section 2 gives a brief introduction to the empirical likelihood method and its convex hull constraint problem. Basic notations used throughout the paper are also established in this section. Section 3 introduces the ABEL along with its asymptotic properties. In section 4, we show that the tuning parameter associated with the adjustment can be used to achieve Bartlett corrected error rate for weakly dependent data. In section 5, we demonstrate the performance of the ABEL method through a simulation study and discuss possible ways to calculate the tuning parameter. Proofs of the theorems are presented in section 7.
2. EMPIRICAL LIKELIHOOD AND THE CONVEX HULL CONSTRAINT

In this section, we establish notations used in this paper by presenting a brief review of the empirical likelihood methods, the adjusted empirical likelihood and the block-wise empirical likelihood. For a comprehensive review of the empirical likelihood methodology, we refer to Owen (2001). Let \( x_1, \ldots, x_n \in \mathbb{R}^m \) be i.i.d random samples from an unknown distribution \( F(x) \). \( \theta \in \mathbb{R}^p \) is the parameter of interest. Let \( g(x; \theta) : \mathbb{R}^{m+p} \mapsto \mathbb{R}^q \) be a \( q \)-dimensional estimating function, such that \( \mathbb{E}[g(x; \theta_0)] = 0 \), where \( \theta_0 \) is the true parameter. One of the advantages of the empirical likelihood is that more information about the parameter can be incorporated through the estimating equations. In other words, we can have \( q \geq p \). The profile empirical likelihood about \( \theta \) is defined as

\[
EL_n(\theta) = \sup_{p_i} \left\{ \prod_{i=1}^{n} p_i : \sum_{i=1}^{n} p_i = 1, p_i \geq 0, \sum_{i=1}^{n} p_i g(x_i; \theta) = 0 \right\}. \tag{1}
\]

Then by a standard Lagrange argument, we have

\[
EL_n(\theta) = \prod_{i=1}^{n} \frac{1}{n} \frac{1}{1 + \lambda(\theta)^T g(x_i, \theta)},
\]

where \( \lambda(\theta) \) is the Lagrange multiplier that satisfies the equation

\[
0 = \sum_{i=1}^{n} \frac{g(x_i, \theta)}{1 + \lambda(\theta)^T g(x_i, \theta)}.
\]

In the rest of this paper, we write \( \lambda \) in place of \( \lambda(\theta) \) unless the dependency on \( \theta \) needs to be explicitly stressed. The profile empirical likelihood ratio is defined as

\[
ELR_n(\theta) = n^n EL_n(\theta).
\]

Under regularity conditions, for example in Qin and Lawless (1994), it can be shown that

\[
-2 \log ELR_n(\theta_0) \to_d \chi^2_q \text{ as } n \to \infty. \tag{2}
\]

Then an asymptotic \((1 - \alpha)\)% empirical likelihood confidence region for \( \theta \) can be found as

\[
\{ \theta : -2 \log ELR_n(\theta) < \chi^2_{q,1-\alpha} \}. \tag{3}
\]

(2) and (3) are the most celebrated properties of the empirical likelihood, which are parallel to its parametric counterpart. Despite these advantages, it has been noted early on by Owen (1988), that the empirical likelihood confidence region constantly under covers. Tsao (2004) studied the least upper bounds on the coverage probabilities, where it focused on the fact that the \( EL_n(\theta) \) is finite if and only if 0 is in the convex hull constructed by \( g(x_i; \theta) \), and then showed that the empirical likelihood confidence region coverage probability
is upper bounded by the probability of the origin being in the convex hull of \( g(x_i; \theta) \). Further, Tsao (2004) demonstrated that this upper bound is affected by sample size and parameter dimension in such a way that if the parameter dimension is comparable to the sample size, then the upper bound goes to 0 as the sample size goes to infinity. This not only explains the root cause of the under-coverage issue, but also shows the severity of the upper bound problem when the finite sample size is small compared to the parameter dimension. Since then, various researchers tried to address the convex hull constraint directly in order to improve the coverage probability. As mentioned in the introduction, three major approaches have been proposed, and in this paper we will focus on the adjusted empirical likelihood by Chen et al. (2008).

The idea of the adjusted empirical likelihood can be most easily demonstrated and understood by considering the two dimensional population mean. That is we have \( p = q = 2 \), and \( \theta = \mu \in \mathbb{R}^2 \). The estimating function \( g(x_i; \theta) \) then becomes \( x_i - \mu \). In this set up, (1) simplifies to

\[
EL_n(\mu) = \sup_{p_i} \left\{ \prod_{i=1}^{n} p_i : \sum_{i=1}^{n} p_i = 1, p_i \geq 0, \sum_{i=1}^{n} p_i x_i = \mu \right\}. \tag{4}
\]

Notice that \( EL_n(\mu) \) in (4) is defined if and only if \( \mu \) is in the convex hull of \( x_i \). If the hypothesised \( \mu_0 \) is not in the convex hull, then there is no solution to (4), and by convention, \( EL_n(\mu) \) is set to \(-\infty\). As a result, even though when \( \mu_0 \) is the true population mean, it will not be included in the empirical likelihood confidence region because \(-2 \log ELR_n(\mu_0) = \infty \geq \chi^2_{q,1-\alpha} \) for any level \( \alpha \). The first plot in Figure 1 shows 15 sample points whose population mean is \((0,0)\), which is represented as the red dot. For this sample, using the
empirical likelihood defined in (4), $EL_n((0,0)')$ is set to $-\infty$. Even though $(0,0)'$ is the true population mean and it is very close to the convex hull, setting $EL_n((0,0)') = -\infty$ provides no information about the plausibility of $(0,0)'$. In other words, one cannot compare the points $(2,2)'$ and $(0,0)'$ using the non-adjusted empirical likelihood because their likelihood ratio will both be $-\infty$, even though $(0,0)'$ is much closer to the convex hull than $(2,2)'$ is.

To mitigate this problem, Chen et al. (2008) proposed to add an extra point $x_{n+1} = -a(\bar{x} - \mu)$, where $a > 0$, to the original data, and then use the $n+1$ data points to construct the empirical likelihood. They called this the adjusted empirical likelihood,

$$AEEL_{n+1} = \sup_{\mathbf{p}_i} \left\{ \prod_{i=1}^{n+1} p_i : \sum_{i=1}^{n+1} p_i = 1, p_i \geq 0, \sum_{i=1}^{n+1} p_i x_i = \mu \right\}.$$  

The intuition of the adjustment can be seen from the plot on the right of Figure 1, where the adjusted convex hull will always contain the origin by design; thus the situation of forcing $EL_n(\mu) = -\infty$ is avoided. Moreover, it has been shown in Chen et al. (2008) that if the hypothesised parameter is close to or in the convex hull, then the adjustment will alter the empirical likelihood by a negligible amount. Thus, at the true population mean, the asymptotic $\chi^2$ distribution still holds, and a confidence region can be constructed accordingly.

To relax the independence assumption on the data, modifications need to be made to the empirical likelihood method in (1). There are roughly two major approaches. One is the block-based methods in the time domain and the other is the periodogram-based methods in the frequency domain. For a review of these methods, we refer to Nordman and Lahiri (2014) and the references therein. In this paper, we use the block-based method introduced by Kitamura (1997) to work with weakly dependent data, where we assume that $x_i, i = 1, \ldots, n$ is a sample from a stationary stochastic process $\{X_i\}$ that satisfies the strong mixing condition,

$$\alpha_X(k) \to 0 \text{ as } k \to \infty, \quad (5)$$

where $\alpha_X(k) = \sup_{A,B} |P(A \cap B) - P(A)P(B)|, A \in \mathcal{F}^0_{-\infty}, B \in \mathcal{F}^\infty_k$, and $\mathcal{F}^m = \sigma(X_i, m < i < n)$ denotes the $\sigma$-algebra generated by $X_i$, $m \leq i \leq n$. Further, assume

$$\sum_{k=1}^\infty \alpha_X(k)^{1-1/c} < \infty, \quad (6)$$

for some constant $c > 1$. The reason that the empirical likelihood in (1) is inadequate for weakly dependent data is also easily seen by considering the population mean as in (4). The asymptotic $\chi^2$-distribution for $-2\log(ELR_n(\mu))$ is derived by the approximation $-2\log(ELR_n(\mu)) \approx n(\bar{x} - \mu)\Sigma^{-1}(\bar{x} - \mu)$, where $\Sigma := 1/n \sum (x_i - \mu)(x_i - \mu)'$. For i.i.d data, $\Sigma$ provides a proper scale to the score $\sqrt{n}(\bar{x} - \mu)$, so that $-2\log(ELR_n(\mu_0))$ is asymptotically $\chi^2$ distributed. However, for dependent data, $\Sigma$ is inadequate to scale the score because it does not take the auto-correlations among the data into account. As a remedy, Kitamura (1997)
proposed to use blocks of data in place of individual data points. To review this blocking method, let $M$, $L$, and $Q = \lfloor (n - M)/L \rfloor + 1$ be the block length, the gap between block starting points, and the number of blocks respectively, where $M \to \infty$, $M = O(n^{1/2})$, $L = O(M)$, and $L \leq M$, as $n \to \infty$. Define the block-wise estimating equations as

$$T_i(\theta) := \frac{1}{M} \sum_{k=1}^{M} g(x_{(i-1)L+k}; \theta).$$

Then the block-wise empirical likelihood is defined as

$$BEL_n(\theta) = \sup_{p_i} \left\{ \prod_{i=1}^{Q} p_i : p_i \geq 0, \sum_{i=1}^{Q} p_i = 1, \sum_{i=1}^{Q} p_i T_i(\theta) = 0 \right\}. \quad (7)$$

And the log block-wise empirical likelihood ratio is defined as

$$BELR_n(\theta) = \log[Q^Q BL_n(\theta) = - \sum_{i=1}^{Q} \log[1 + \lambda'_i T_i]].$$

Under assumptions A.1-A8 in section 3, it can be shown that

$$-2 \frac{n}{MQ} BEL_n(\theta_0) \to_d \chi^2_q. \quad (8)$$

The proof of the above result (8) can be found in Kitamura (1997) and Owen (2001). It should be noted that the choice of the block length $M$ is important to the performance of the block-wise empirical likelihood method. Various authors have studied and proposed ways to select $M$ with their respective advantages and limitations. For examples on selecting $M$ we refer to Nordman and Lahiri (2014), Nordman et al. (2013), Kim et al. (2013), and Zhang and Shao (2014). The study of the optimal block choice is however beyond the scope of this paper.

### 3. ADJUSTED BLOCK-WISE EMPIRICAL LIKELIHOOD

It is apparent that the block-wise EL method (7) also suffers from the convex hull constraint, which will impede proper coverage probability for finite sample. In this section, we propose to adjust the block-wise empirical likelihood and examine its effectiveness in improving the coverage probability for weakly dependent data. The theoretical appeal of the adjusted empirical likelihood for the i.i.d data is that it preserves the asymptotic $\chi^2$—distribution and at the same time breaks the convex hull constraint. Moreover, Liu and Chen (2010) showed that for i.i.d data the adjusted empirical likelihood confidence region coverage probability error can be reduced from $O(n^{-1})$ to $O(n^{-2})$. Furthermore, simulation studies in Chen et al. (2008), Emerson and Owen (2009), and Liu and Chen (2010) showed that the adjusted empirical likelihood provides significant improvements over the original empirical likelihood in terms of coverage probability. In the rest of this section, we show that all of the desirable
properties of the adjusted empirical likelihood method mentioned above are preserved under
the adjusted block-wise empirical likelihood for weakly dependent data.

Since the convex hull used in the block-wise empirical likelihood is formed by using the
block-wise estimating functions $T_1(\theta), \ldots, T_Q(\theta)$, the extra estimating function used for the
adjustment will naturally be constructed from the $T_i(\theta)$’s in contrast to the individual data
points used in the i.i.d setting. With this we define the adjustment as

$$T_{Q+1}(\theta) := -a\overline{T}(\theta),$$

(9)

where $\overline{T}(\theta) = \frac{1}{Q} \sum_{i=1}^Q T_i(\theta)$ and $a > 0$. Now we construct the adjusted block-wise empir-
ical likelihood with $T_1, \ldots, T_Q, T_{Q+1}$ as the following

$$ABEL_Q(\theta) = \sup_{p_i} \left\{ \frac{Q+1}{Q} \prod_{i=1}^{Q+1} p_i : p_i \geq 0, \sum_{i=1}^{Q+1} p_i = 1, \sum_{i=1}^{Q+1} p_i T_i(\theta) = 0 \right\},$$

(10)

and the log adjusted empirical likelihood ratio is then

$$ABELR_Q(\theta) = \log \frac{ABEL_Q(\theta)}{(Q+1)^{(Q+1)}} = -\sum_{i=1}^{Q+1} \log[1 + \lambda_i^T T_i(\theta)],$$

(11)

where $\lambda \in \mathbb{R}^q$ is the vector of Lagrange multiplier that satisfies

$$0 = \sum_{i=1}^{Q+1} \frac{T_i(\theta)}{1 + \lambda_i^T T_i(\theta)}.$$

Before stating the asymptotic distribution of $ABELR_Q(\theta_0)$ in (11), we first list the
regularity conditions needed. A detailed explanation of these assumptions can be found in
Kitamura (1997). They are generalizations from the assumptions used in the i.i.d setting,
for example in Qin and Lawless (1994), to the weakly dependent setting. The main points of
these assumptions are on the continuity and differentiability of the estimating function $g(\ldots)$
around the true parameter of interest; so that the remainder terms in the Taylor expansion
of the log empirical likelihood ratio are controlled, and that the dominating term converges
to a $\chi^2$–distribution.

A.1 The parameter space $\Theta \subset \mathbb{R}^p$ is compact.

A.2 $\theta_0$ is the unique root of $\mathbb{E}g(x_i; \theta) = 0$.

A.3 For sufficiently small $\delta > 0$ and $\eta > 0$, $\mathbb{E} \sup_{\theta \in \Gamma(\theta, \delta)} \|g(x_i; \theta^*)\|^{2(1+\eta)} < \infty, \forall \theta \in \Theta,$
where $\Gamma(\theta, \delta)$ is a small ball around $\theta$, with radius $\delta$.

A.4 If a sequence $\{\theta_j : j = 1, 2, \ldots\}$ converges to some $\theta \in \Theta$ as $j \to \infty$, then $g(x, \theta_j)$
converges to $g(x, \theta), \forall x$ except on a null set, which may vary with $\theta$.

A.5 $\theta_0$ is an interior point of $\Theta$ and $g(x; \theta)$ is twice continuously differentiable at $\theta_0$. 

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A.6 \( \text{Var}\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g(x_i; \theta_0) \right) \rightarrow S \in \mathbb{R}^{q \times q}, S > 0, \) as \( n \rightarrow \infty. \)

A.7 \( \mathbb{E} \| g(x; \theta_0) \|^c < \infty \) for \( c > 1 \) defined in the strong mixing condition.

\[ \mathbb{E} \sup_{\theta^* \in \Gamma(\theta_0, \delta)} \| g(x; \theta^*) \|^{2+c} < K, M = o(n^{1/2-1/(2+\epsilon)}), \epsilon > 0. \]

\[ \mathbb{E} \sup_{\theta^* \in \Gamma(\theta_0, \delta)} \left\| \frac{\partial g(x; \theta^*)}{\partial \theta} \right\| < K, \] and \( \mathbb{E} \sup_{\theta^* \in \Gamma(\theta_0, \delta)} \left\| \frac{\partial^2 g(x; \theta^*)}{\partial \theta \partial \theta} \right\| < K, \) where \( K < \infty \) and \( g_j(x; \theta) \) is the \( j \)th component of \( g(x; \theta). \)

A.8 \( D = \mathbb{E} \frac{\partial g(x; \theta_0)}{\partial \theta} \) is of full rank.

With these assumptions, the following theorem then shows that the adjusted empirical likelihood ratio has asymptotic \( \chi^2 \) distribution.

\textbf{Theorem 1.} Assume assumptions A.1-A.8 hold, under the strong mixing conditions (5) and (6), if \( a = o(n^{-1/2}) \), then

\[ -2nMQ^M \text{ABELR}_Q(\theta_0) \rightarrow \chi^2_q, \] as \( n \rightarrow \infty. \]

where \( \theta_0 \) is the true parameter.

Similar to the non-adjusted BEL in Kitamura (1997), the factor \( \frac{n}{QM} \) is to account for the overlap between blocks. If the blocks do not overlap, then \( \frac{n}{QM} = 1. \) The \( a = o(n^{-1/2}) \) in theorem 1 shows that the size of the tuning parameter is controlled by the block length \( M \) and sample size \( n \). This should be expected since \( M \) is allowed to grow with \( n \) and the asymptotic result is obtained under this growing block length setting. Moreover, in the block-wise empirical likelihood, we usually have \( \frac{n}{M} \approx Q \), where \( Q \) is the number of blocks; therefore, \( a = o\left( \frac{n}{M} \right) \approx o(Q). \) The intuition is that the adjustment is made on the blocked estimating equations, thus the size of the tuning parameter should be controlled by the number of blocks instead of only by the sample size. By theorem 1, a \((1 - \alpha)\)% asymptotic confidence region based on the ABELR can be constructed as:

\[ CR_{1-\alpha} = \left\{ \theta \mid -2nMQ^M \text{ABELR}_Q(\theta) \leq \chi^2_{q,1-\alpha} \right\}. \] (12)

By the design of the extra point in (9), it is clear that the \( \text{ABEL}_Q(\theta) \) is well defined for any \( \theta. \) As a consequence, there is no upper bound imposed by the convex hull on the coverage probability of (12).

As with any method that involves a tuning parameter, the choice of \( a \) in practice is delicate, and it may depend on the statistical task that one wishes to tackle. In the i.i.d setting, Liu and Chen (2010) studied the choice of \( a \) through an Edgeworth expansion of the adjusted empirical likelihood ratio, and they found that if \( a \) is specified in relation to the Bartlett correction factor, then the adjusted empirical likelihood confidence region can achieve the Bartlett error rate. In the next section, we will show that Bartlett correction is also possible for the adjusted block-wise empirical likelihood with weakly dependent data.
4. TUNING PARAMETER FOR BARTLETT CORRECTED ERROR RATE

Being Bartlett correctable is an important feature of the parametric likelihood ratio confidence region, where the coverage probability error can be decreased from $O(n^{-1})$ to $O(n^{-2})$. Like its parametric counterpart, DiCiccio et al. (1991) showed that the empirical likelihood for smooth function model is also Bartlett correctable. Further, Chen and Cui (2007) showed that this property also holds for the empirical likelihood with general estimating equations. For weakly dependent data, Kitamura (1997) showed that the block-wise empirical likelihood for smooth function model is Bartlett correctable, where the coverage probability error can be improved from $O(n^{-2/3})$ to $O(n^{-5/6})$. The errors being larger than the ones for i.i.d data is due to the data blocking method, which is used to deal with the weakly dependent data structure. In this section, we show that through an Edgeworth expansion of the adjusted block-wise empirical likelihood ratio, a tuning parameter $a$ can be found such that the adjusted empirical likelihood confidence region coverage error is $O(n^{-5/6})$ for general estimating equations. Here we assume the non-overlapping blocking scheme. In other words, $M = L$ and $n \approx MQ$. In addition to the mixing conditions (5) and (6), we also assume that $\alpha_x(m) \leq ce^{-dm} \forall m$, where $\alpha_x(m)$ and $c$ are defined in (5) and (6) and $d$ is a positive constant. We also need to assume the validity of the Edgeworth expansion of sums of dependent data, which Götze and Hipp (1983) has shown by assuming the existence of more moments, a conditional Cramer condition, and that the random processes are approximated by other exponentially strong mixing processes with exponentially decaying mixing coefficients that satisfy a Markov type condition. For more details on these assumptions, we refer to Kitamura (1997), Bhattacharya and Ghosh (1980), and Götze and Hipp (1983).

To simplify the notations in deriving the tuning parameter, assume that

$$V := M\mathbb{E}[T_i(\theta_0)T_i(\theta_0)^T] = I,$$

where $I$ is the identity matrix, otherwise we can replace $T_i$ by $V^{-1/2}T_i$. Let $T_j$ denote the jth component of $T$. For $j_k \in \{1, \ldots, q\}, k = 1, \ldots, v$, define

$$\alpha^{j_1\cdots j_v} := M^{v-1}\mathbb{E}[T_{j_1} \cdots T_{j_v}].$$

(13)

Notice that $\alpha^{rs}$ is the Kronecker $\delta_{rs}$, where $\alpha^{rr} = 1$ and $\alpha^{rs} = 0$, for $r \neq s$. Further, we let

$$A^{j_1\cdots j_v} = \frac{M^{v-1}}{Q} \sum_{i=1}^{Q} T_{i j_1} \cdots T_{i j_v} - \alpha^{j_1\cdots j_v}.$$

With the above notation, it can be shown by following the calculations in Liu and Chen (2010) and DiCiccio et al. (1988) that

$$-2\frac{n}{MQ}BELR(\theta_0) = n(R_1 + R_2 + R_3)^4(R_1 + R_2 + R_3) + O_p(n^{-1}),$$

(14)
where, for $r, s, t = 1, \ldots, q$

\[
\begin{align*}
R_1^r &= A^r \\
R_2^r &= \frac{1}{3} \alpha^{rst} A^s A^t - \frac{1}{2} A^{rs} A^s \\
R_3^r &= \frac{3}{8} A^{rs} A^s A^t - \frac{5}{12} \alpha^{rst} A^{tu} A^s A^u - \frac{5}{12} \alpha^{stu} A^{rs} A^t A^u \\
&+ \frac{4}{9} \alpha^{rst} A^{tu} A^s A^v + \frac{1}{3} A^{rs} A^t A^u - \frac{1}{4} \alpha^{rst} A^s A^t A^u.
\end{align*}
\]

Here the summation over repeated index is used. Equation (14) is the so called signed-root decomposition of $BELR(\theta_0)$. Since we add an extra blocked estimating equation (9) in the adjusted block-wise empirical likelihood, the signed-root decomposition of $ABELR_Q(\theta_0)$ will be slightly affected by the adjustment. And this is exactly where we can leverage the tuning parameter to achieve Bartlett corrected coverage error rate. By the derivation shown in section 7, the signed-root decomposition of $ABELR$ is

\[
-2 \frac{n}{MQ} \sum_{i=1}^{Q+1} \log(1 + \lambda_i T_i) = n(R_1 + R_2 + R_3 - \frac{a}{Q} R_1)'(R_1 + R_2 + R_3 - \frac{a}{Q} R_1) + O_p(n^{-1}).
\]  

(15)

With $R_1, R_2, R_3$ defined above, in order to derive the tuning parameter $a$ in equation (15) that will yield the Bartlett error rate, we define the counterpart of (13) under dependency as the following: for a sequence of $d$ integers $0 < k(1) < \ldots < k(d) = k$, $k \geq 3$,

\[
\tilde{\alpha}_{j_1 \ldots j_{k(1)} j_{k(1)+1} \ldots j_{k(d)}} := Q^{-1} \sum_{1 \leq i(1), \ldots, i(d) \leq Q} E \left\{ M^{-1} \left( M^{k(1)} T^{j_{i(1)}}_{i(1)} \ldots T^{j_{k(1)}}_{i(1)} \right) \times \left( M^{k(2)-k(1)} T^{j_{i(2)}}_{i(2)} \ldots T^{j_{k(2)}}_{i(2)} \right) \times \ldots \left( M^{k(d)-k(d-1)} T^{j_{i(d)}}_{i(d)} \ldots T^{j_{k(d)}}_{i(d)} \right) \right\} \times I_{\{\max_{p,q<d} |i(p)-i(q)|\leq k-2\}}.
\]

Now if we define $a$ as follows, then the next theorem will show that the adjusted block-wise empirical likelihood confidence region (12) achieves the Bartlett corrected coverage error rate.

Let

\[
a := \frac{1}{2} \frac{Q}{n q} a_{ii},
\]

(16)
where, for \( r, i = 1, \ldots, q \),
\[
a_{ri} = \frac{1}{q} (t_{1a}[2] + t_{1b}[2] + t_{1c} + t_{2a}[2] + t_{2b} + t_{3a}[2] + t_{3b}[2] + t_{3c} ),
\]
with \( t_{ja}[2] = t_{ja} + t_{ja'}, j = 1, 2, 3 \), and similarly for \( t_{jb}[2] \). The quantities \( t_{ja}, t_{jb}, t_{jc}, j = 1, 2, 3 \) are defined as

\[
t_{1a} = \alpha^{rkl} \tilde{\alpha}^{i,k,l},
\]
\[
t_{1b} = \frac{3}{8} \tilde{\alpha}^{rkl} \tilde{\alpha}^{i,l,k} - \frac{5}{6} \alpha^{rkl} \tilde{\alpha}^{i,k,l} - \frac{5}{6} \tilde{\alpha}^{rkl} \tilde{\alpha}^{i,k,l} + \frac{8}{9} \alpha^{rkl} \tilde{\alpha}^{i,k,l},
\]
\[
t_{1c} = \frac{1}{4} \alpha^{rkl} \tilde{\alpha}^{i,l,k} - \frac{2}{3} \tilde{\alpha}^{rkl} \tilde{\alpha}^{i,k,l} + \frac{2}{9} \alpha^{rkl} \tilde{\alpha}^{i,k,l},
\]
\[
t_{2a} = \frac{3}{8} \alpha^{rkl} \tilde{\alpha}^{i,k,l} - \frac{5}{12} \tilde{\alpha}^{rkl} \tilde{\alpha}^{i,k,l} + \frac{4}{9} \alpha^{rkl} \tilde{\alpha}^{i,k,l} - \frac{5}{12} \alpha^{rkl} \tilde{\alpha}^{i,k,l},
\]
\[
t_{2b} = \frac{1}{4} \alpha^{rkl} \tilde{\alpha}^{i,l,k} - \frac{1}{3} \tilde{\alpha}^{rkl} \tilde{\alpha}^{i,l,k} + \frac{1}{9} \alpha^{rkl} \tilde{\alpha}^{i,l,k},
\]
\[
t_{3a} = -\frac{1}{2} \tilde{\alpha}^{r,k,i},
\]
\[
t_{3b} = \frac{3}{8} \tilde{\alpha}^{r,k,i} + \tilde{\alpha}^{i,l,k} - \frac{3}{4} \alpha^{r,k,i},
\]
\[
t_{3c} = \frac{1}{4} \tilde{\alpha}^{r,k,i}.
\]

The \( t_{ja'} \) and \( t_{jb'} \), \( j = 1, 2, 3 \) are the same as \( t_{ja} \) and \( t_{jb} \), \( j = 1, 2, 3 \) correspondingly, except that the superscript \( r \) and \( i \) are exchanged, for example \( t_{1a'} = \alpha^{i,k,l} \tilde{\alpha}^{r,k,l} \).

**Theorem 2.** Assume that conditions A.1-A.8 in section 3 hold with non-overlapping blocking scheme. And such that the bounds on \( \alpha_a(m) \), and the assumptions for the Edgeworth expansion for sums of dependent data mentioned in the beginning of this section hold, if \( a \) is as in (16), then as \( n \to \infty \),

\[
P \left( -2 \frac{n}{M \bar{Q}} \sum_{i=1}^{Q+1} \log(1 + \lambda^T_i T_i) \leq x \right) = P(\chi^2_q \leq x) + O(n^{-5/6}).
\]

In practice, the unknown population quantity \( V = ME[T_i(\theta_0)T_i(\theta_0)'] \) can be replaced by \( \hat{V} = M \frac{1}{Q} \sum_{i=1}^{Q} T_i(\hat{\theta})T_i(\hat{\theta})' \), where \( \hat{\theta} \) is the maximum block-wise empirical likelihood estimator of \( \theta_0 \). The quantity \( a_{ii} \) is composed of various population moments, which can be replaced by their corresponding sample moments to obtain an estimator of \( a_{ii} \). Moreover, the estimated \( a_{ii} \) may be positive or negative. If it is positive, then the convex hull constructed with the
extra point will always contain the origin. However, if \(a_{ii}\) is negative, then \(a = \frac{1}{2} \frac{Q}{n} a_{ii}\) is also negative. As a result, the convex hull with the new point will not contain the origin if the original convex hull does not. To avoid the second situation, if \(a_{ii} < 0\), then we add two extra points \(T_{n+1} = -a_1 T\) and \(T_{n+2} = -a_2 T\), such that \(a = a_1 + a_2\). We can let \(a_1 = 2a = n \frac{Q}{2} a_{ii} < 0\) and \(a_2 = -a = -n \frac{Q}{2} a_{ii} > 0\), such that \(T_{n+2}\) will guarantee that the origin is in the new convex hull. Moreover, since \(a = a_1 + a_2\), adding \(T_{n+1}\) and \(T_{n+2}\) will have the same effect as adding \(T_{n+1}\) with tuning parameter \(a\) in terms of obtaining the Bartlett coverage probability.

5. SIMULATION

In this section, we examine the numerical properties of the adjusted block-wise empirical likelihood through a simulation study. We compare the confidence regions constructed by the adjusted block-wise empirical likelihood (10) with several tuning parameters to the one constructed by the non-adjusted block-wise empirical likelihood (7). The data \(x_i, i = 1, \ldots, n\) are simulated from an AR(1) model

\[
x_{i+1} = \text{diag}(\rho) x_i + \epsilon_{i+1}, \quad i = 1, \ldots, n,
\]

where \(\epsilon_i\) are i.i.d \(d\)-dimensional multivariate standard normal random variables and \(\text{diag}(\rho)\) is a diagonal matrix with \(\rho\) on the diagonal. The parameter of interest is the population mean \(E(x_i) =: \mu\). In order to see how the data dependencies affect the performance of the methods, we simulate the data with a range of \(\rho\)'s. In particular, we look at \(\rho = -0.8, -0.5, -0.2, 0.2, 0.5, 0.8\). We also simulate \(x_i \in \mathbb{R}^d\) for dimension \(d = 2, 3, 4, 5, 10\) to see how the parameter dimension affects the performances. We look at two sample sizes \(n = 100\) and \(400\). For each scenario, we calculate the block-wise empirical likelihood ratio at block lengths ranging from 2 to 16 in order to examine the effects of block choices. In addition, we also use the progressive blocking method proposed by Kim et al. (2013), which does not require to fix a block-length. For each scenario, we simulate 1000 data sets and calculate the likelihood ratio for each data set at the true mean. The coverage probability is then calculated as the number of times the likelihood ratio is less than the theoretical \(\chi^2\) quantile at levels \(\alpha = 0.1, 0.05, 0.01\) divided by 1000. The likelihood ratios are calculated by the block-wise empirical likelihood without adjustment (BEL), adjusted block-wise empirical likelihood with \(a = \log(n)/2\) (ABEL_log), \(a = 0.5\) (ABEL_0.5), \(a = 0.8\) (ABEL_0.8), \(a = 1\) (ABEL_1), and \(a\) given in (16) (ABEL_bart). The Bartlett tuning parameter (16) is estimated by the plug-in estimator, which is then bias corrected by a block-wise bootstrap. The full simulation results are shown in Table 2 in the appendix. Table 1 shows a snapshot of Table 2, where the AR(1) coefficients \(\rho = -0.2, 0.2, 0.5, 0.8\). \(M\) is the block length that gives the best coverage rates of each particular method, where \(M = \text{pro}\) indicates that the progressive block method gives the best result. It can be seen that for negative \(\rho\), BEL performed well and at least one of the adjusted BEL matched or surpassed
the BEL performance. As $\rho$ becomes positive, the BEL starts to show its vulnerability of under-coverage and this becomes worse as dimension increases. In contrast, the adjusted BEL still provides adequate coverage. This phenomenon where the BEL does not suffer as severe under-coverage for negative $\rho$ as it does for positive $\rho$ exemplifies the fact that the coverage probability is upper bounded by the probability of the convex hull containing the origin. For when $\rho$ is negative, the consecutive points are likely to be on the opposite sides in relation to the origin, therefore the resulting convex hull is likely to contain the origin and does not impose an upper bound on the coverage probability. Whereas, for positive $\rho$, especially when it is close to 1, the consecutive points are likely to be close to each other; thus, the probability that the resulting convex hull contains the origin is small.

Table 1: Comparison of Coverage Probabilities, $M$ is the block length, $M = pro$ means progressive blocking method is used.

| $\rho$ | d | Methods     | n= 100       | n= 400       |
|-------|---|-------------|--------------|--------------|
| -0.2  | 3 | BEL         | 3 0.90 0.94 0.98 | 6 0.89 0.94 0.99 |
| -0.2  | 3 | ABEL_log    | 3 0.94 0.97 0.99 | pro 0.90 0.96 1.00 |
| -0.2  | 3 | ABEL_0.8    | 3 0.91 0.95 0.98 | 7 0.90 0.94 0.99 |
| -0.2  | 3 | ABEL_1      | 14 0.90 0.95 0.99 | 7 0.90 0.95 0.99 |
| -0.2  | 3 | ABEL_bart   | 14 0.91 0.94 0.97 | pro 0.90 0.95 0.99 |
| 0.2   | 3 | BEL         | 3 0.82 0.89 0.95 | 9 0.88 0.93 0.98 |
| 0.2   | 3 | ABEL_log    | 4 0.89 0.95 1.00 | 6 0.90 0.95 0.99 |
| 0.2   | 3 | ABEL_0.8    | 3 0.83 0.90 0.96 | 9 0.88 0.94 0.98 |
| 0.2   | 3 | ABEL_1      | 14 0.88 0.95 0.99 | 9 0.88 0.94 0.99 |
| 0.2   | 3 | ABEL_bart   | 12 0.93 0.96 0.98 | 8 0.90 0.95 1.00 |
| 0.5   | 3 | BEL         | 5 0.68 0.77 0.89 | 10 0.82 0.87 0.95 |
| 0.5   | 3 | ABEL_log    | 5 0.89 0.97 1.00 | 13 0.91 0.96 1.00 |
| 0.5   | 3 | ABEL_0.8    | 16 0.74 0.88 0.97 | 10 0.83 0.89 0.96 |
| 0.5   | 3 | ABEL_1      | 14 0.87 0.95 0.99 | 10 0.83 0.89 0.96 |
| 0.5   | 3 | ABEL_bart   | 14 0.92 0.95 0.97 | pro 0.90 0.96 0.99 |
| 0.5   | 4 | BEL         | 4 0.64 0.72 0.85 | 9 0.77 0.85 0.95 |
| 0.5   | 4 | ABEL_log    | 16 0.92 0.94 0.97 | 11 0.88 0.95 1.00 |
| 0.5   | 4 | ABEL_0.8    | 14 0.72 0.86 0.95 | 9 0.79 0.87 0.96 |
| 0.5   | 4 | ABEL_1      | 14 0.86 0.92 0.97 | 9 0.79 0.87 0.96 |
| 0.5   | 4 | ABEL_bart   | 13 0.91 0.94 0.96 | pro 0.91 0.97 0.99 |
| 0.8   | 2 | BEL         | 9 0.58 0.67 0.76 | 16 0.77 0.85 0.94 |
| 0.8   | 2 | ABEL_log    | 7 0.87 0.98 1.00 | 16 0.88 0.95 1.00 |
| 0.8   | 2 | ABEL_0.8    | 16 0.72 0.86 0.98 | 16 0.80 0.86 0.94 |
| 0.8   | 2 | ABEL_1      | 16 0.85 0.95 0.99 | 16 0.80 0.87 0.95 |
6. CONCLUSION

Originally proposed to improve the coverage probability of the empirical likelihood confidence region coverage probability for i.i.d data, the adjusted empirical likelihood in this paper is shown to be effective in improving the coverage probability when combined with the blocking method in dealing with weakly dependent data. In particular, we have shown that the ABEL possesses the asymptotic $\chi^2$ property similar to its non-adjusted counterpart. Moreover, we have shown that the adjustment tuning parameter can be used to achieve the asymptotic Bartlett corrected coverage error rate of $O(n^{-5/6})$. This tuning parameter that gives the Bartlett corrected rate involves higher moments that needs to be estimated in practice. How to best estimate this tuning parameter needs to be further studied. In the simulation study, we used a block-wise bootstrap to correct the bias in estimating the tuning parameter by plugging in the sample moments. The results show that the adjusted BEL performs comparable to the non-adjusted BEL when the non-adjusted BEL performs well, and it outperforms the non-adjusted BEL when the non-adjusted BEL suffers from the under-coverage issue. Our bootstrap bias corrected tuning parameter performs well most of the time, but sometimes it is outperformed by other choices of the tuning parameter. As mentioned above, the optimal way to estimate the tuning parameter will be addressed in future studies.

7. PROOFS

Proof of Theorem 1. The first step in proving theorem 1 is to show that the Lagrange multiplier $\lambda_a$ is $O_p(n^{-1/2}M)$, where we use the subscript $a$ to emphasis that this is the Lagrange multiplier for the adjusted empirical likelihood. First, we note that $\lambda_a$ solves the following equation

$$\sum_{i=1}^{Q+1} \frac{T_i}{1 + \lambda^T_i} = 0. \quad (18)$$

Now, define $\hat{\lambda}_a := \lambda_a / \rho$, where $\rho := \|\lambda_a\|$. Multiply $\hat{\lambda}_a / Q$ on both sides of equation (18), and recall that $T_{Q+1} = -a \bar{T}$. Then we have
0 = \hat{\lambda}_a T_i 
= \hat{\lambda}_a \sum_{i=1}^{Q+1} \frac{T_i}{1 + \lambda_i^a T_i} 
\leq \hat{\lambda}_a \mathbb{T}(1 - a) - \frac{\rho}{1 + \rho T^*} \sum_{i=1}^{Q} (\hat{\lambda}_a T_i)^2 
= \hat{\lambda}_a \mathbb{T} - \frac{\rho}{1 + \rho T^*} \hat{\lambda}_a \frac{1}{Q} \sum_{i=1}^{Q} T_i T^*_i \lambda_a + O_p(n^{-1/2}Q^{-1}a) \quad (19)

where \( T^* := \max_{1 \leq i \leq Q} \|T_i\| \). By law of large numbers and the central limit theorem, and the argument in Owen (1990) and Kitamura (1997), it can be shown that \( T^* = o_p(n^{1/2}M^{-1}) \) a.s. It has also been shown in Kitamura (1997) that \( \bar{T} = O_p(n^{-1/2}) \) and \( M/Q \sum_{i=1}^{Q} T_i^2 T^*_i \rightarrow_p S \). Then, we can deduce from (19) that

\[
0 \leq \hat{\lambda}_a \mathbb{T} - \frac{\rho}{M(1 + \rho T^*)} (1 - \epsilon) \sigma_1^2 + O_p(n^{-1/2}Q^{-1}a).
\]

where \( 0 < \epsilon < 1 \) and \( \sigma_1 \) is the smallest eigenvalue of \( S \). Then

\[
\frac{\rho}{M(1 + \rho T^*)} = O_p(n^{-1/2}Q^{-1}a) + O_p(n^{-1/2}) \Rightarrow \frac{\rho}{1 + \rho T^*} = O_p(n^{-1/2}Q^{-1}Ma) + O_p(n^{-1/2}M).
\]

By the assumption that \( a = o(n/M) \), we have

\[
\frac{\rho}{1 + \rho T^*} = O_p(n^{-1/2}M) \quad \Rightarrow \quad \rho = O_p(n^{-1/2}M).
\]

Therefore, \( \lambda_a = O_p(n^{-1/2}M) \), which in particular means that \( \lambda^T T^* = o_p(1) \).

The next step is to express \( \lambda_a \) in terms of \( \mathbb{T} \). Notice that equation (18) can be written as the sum of two parts

\[
0 = \frac{1}{Q} \sum_{i=1}^{Q+1} \frac{T_i}{1 + \lambda_i^a T_i} = \frac{1}{Q} \sum_{i=1}^{Q} \frac{T_i}{1 + \lambda_i^a T_i} + \frac{1}{Q} - a \mathbb{T} \quad (20)
\]

where the first part on the right hand side can be written as
\[
\frac{1}{Q} \sum_{i=1}^{Q} T_i \left[ 1 - \lambda^i_a T_i + \frac{(\lambda^i_a T_i)^2}{1 + \lambda^i_a T_i} \right] = \frac{1}{Q} \sum_{i=1}^{Q} T_i - \frac{1}{Q} \sum_{i=1}^{Q} T_i \lambda^i_a T_i \\
+ \frac{1}{Q} \sum_{i=1}^{Q} T_i \frac{(\lambda^i_a T_i)^2}{1 + \lambda^i_a T_i} \\
= T - \frac{1}{Q} \sum_{i=1}^{Q} T_i T_i^T \lambda_a + o_p(n^{-1/2}).
\]

The last part in (20) is

\[
\frac{1}{Q} \frac{a^T}{1 - a \lambda^T_i} = \frac{o_p(\frac{n}{MQ}) O_p(n^{-1/2})}{1 - o_p(n/M) O_p(n^{-1/M})} \\
= \frac{o_p(n^{1/2} M^{-1} Q^{-1})}{1 - o_p(1)} \\
= o_p(n^{-1/2}) \text{ since } n \leq MQ.
\]

Now, we have the relationship

\[
\lambda_a = MS^{-1} T + o_p(n^{-1/2} M).
\]  

(21)

The final step is done through Taylor expansion of the adjusted block-wise empirical likelihood ratio \(2 \frac{n}{MQ} \sum_{i=1}^{Q+1} \log(1 + \lambda^i_a T_i)\). The adjusted block-wise empirical likelihood ratio can be written in two parts as

\[
2 \frac{n}{MQ} \sum_{i=1}^{Q+1} \log(1 + \lambda^i_a T_i) = 2 \frac{n}{MQ} \sum_{i=1}^{Q} \log(1 + \lambda^i_a T_i) + 2 \frac{n}{MQ} \log(1 + \lambda^i_{aQ+1} T_{Q+1})
\]

where the second part \(2 \frac{n}{MQ} \log(1 + \lambda^i_{aQ+1} T_{Q+1}) = o_p(1)\). This can be seen through a Taylor expansion, \(\log(1 + \lambda^i_{aQ+1} T_{Q+1}) = \lambda^i_{aQ+1} T_{Q+1} - \frac{1}{2} (\lambda^i_{aQ+1} T_{Q+1})^2 + \eta\), where for some finite \(B, P(\|\eta\| \leq B \|\lambda^i_{aQ+1} T_{Q+1}\|) \rightarrow 1\). \(T_{Q+1}\) is defined as \(-a^T\), and from the first step of the proof, we know that \(\lambda_a = o_p(n^{-1/2} M)\). Therefore, \(\lambda^i_{aQ+1} T_{Q+1} = o_p(n^{-1/2} M) o_p(n/M) O_p(n^{-1/2}) = o_p(1)\) and \(\eta = o_p(1)\).

Now, a Taylor expansion of the first term gives
\[
2 \frac{n}{MQ} \sum_{i=1}^{Q} \log(1 + \lambda_a^i T_i) = 2 \frac{n}{MQ} \sum_{i=1}^{Q} \left[ \lambda_a^i T_i - \frac{1}{2}(\lambda_a^i T_i)^2 + \eta_i \right]
\]
\[
= 2nM^{-1}\lambda_a^iT - nM^{-1}\lambda_a^iSM^{-1}\lambda_a + 2 \frac{n}{MQ} \sum_{i=1}^{Q} \eta_i
\]
\[
= 2nT^dS^{-1}T - nT^dS^{-1}SS^{-1}T + 2 \frac{n}{MQ} \sum_{i=1}^{Q} \eta_i + o_p(1)
\]
\[
= nT^dS^{-1}T + o_p(1) \rightarrow_d \chi^2_q.
\]

**Proof of Theorem 2.** The foundation of the second order coverage probability of the empirical likelihood ratio confidence region was laid out by DiCiccio et al. (1988). The major steps include the signed-root decomposition of the log empirical likelihood ratio, measuring the size of the third and fourth joint cumulant of the signed-root, and the Edgeworth expansion of the density of the signed-root. In the i.i.d setting, Liu and Chen (2010) exploited the fact that the tuning parameter features in the signed-root of the adjusted empirical likelihood ratio and used it as a leverage to eliminate the large error terms to achieve a Bartlett corrected error rate. Here we will use the same technique to eliminate the intermediate error term in order to achieve the \(O(n^{-5/6})\) error rate for weakly dependent data.

First, we relate the Lagrange multiplier \(\lambda_a\) for the adjusted block-wise empirical likelihood (10) with the Lagrange multiplier \(\lambda\) for the non-adjusted block-wise empirical likelihood (7). Let

\[
f(\zeta) := \frac{1}{Q} \sum_{i=1}^{Q} \frac{T_i}{1 + \zeta^i T_i}.
\]

Then, by definition, \(f(\lambda) = 0\). Similarly, the adjusted Lagrange multiplier \(\lambda_a\) satisfies

\[
0 = \frac{1}{Q} \sum_{i=1}^{Q+1} \frac{T_i}{1 + \lambda_a T_i} = \frac{1}{Q} \sum_{i=1}^{Q} \frac{T_i}{1 + \lambda_a^i T_i} + \frac{1}{Q} \frac{-a_n T}{1 - \lambda_a^i a_n T}
\]
\[
= f(\lambda_a) - \frac{1}{Q} a_n T
\]
\[
= f(\lambda_a) - \frac{1}{Q} [a + O_p(n^{-1/2})] T
\]
\[
= f(\lambda_a) - \frac{1}{Q} a T + O_p(n^{-1}Q^{-1}).
\]

Then, by Taylor expansion, we have
\[ f(\lambda_a) = f(\lambda) + \frac{\partial f(\lambda)}{\partial \lambda} (\lambda_a - \lambda) + O_p((\lambda_a - \lambda)^2), \]

which gives

\[ \lambda_a - \lambda = \left( \frac{\partial f(\lambda)}{\partial \lambda} \right)^{-1} f(\lambda_a) + O_p((\lambda_a - \lambda)^2) \]

\[ = \frac{a}{Q} \left( \frac{\partial f(\lambda)}{\partial \lambda} \right)^{-1} \mathcal{T}(\theta_0) + O_p(n^{-1}Q^{-1}), \tag{22} \]

where

\[ \frac{\partial f(\lambda)}{\partial \lambda} = -\frac{1}{Q} \sum_{i=1}^{Q} \frac{T_i T_i^t}{[1 + \lambda^t T_i]^2} \]

\[ = -\frac{1}{Q} \sum_{i=1}^{Q} T_i T_i^t \left[ 1 - \lambda^t T_i + \frac{(\lambda^t T_i)^2}{1 + \lambda^t T_i} \right]^2 \]

\[ = -\frac{1}{Q} \sum_{i=1}^{Q} T_i T_i^t \left[ 1 + (\lambda^t T_i)^2 + \frac{(\lambda^t T_i)^4}{(1 + \lambda^t T_i)^2} \right. \]

\[ -2\lambda^t T_i + 2 \left( \frac{(\lambda^t T_i)^2}{1 + \lambda^t T_i} - \frac{(\lambda^t T_i)^3}{1 + \lambda^t T_i} \right) \]

\[ = -\frac{1}{Q} \sum_{i=1}^{Q} T_i T_i^t + O_p(n^{-1/2}M) \]

\[ = -\tilde{S} + O_p(n^{-1/2}M), \text{ where } \tilde{S} := \frac{1}{Q} \sum_{i=1}^{Q} T_i T_i^t. \]

Now plug \( \frac{\partial f(\lambda)}{\partial \lambda} \) into (22) to get

\[ \lambda_a - \lambda = -\frac{a}{Q} \tilde{S}^{-1} \mathcal{T} + O_p(n^{-1}). \tag{23} \]

Since \( M \tilde{S} = I + o_p(1) \) and \( \mathcal{T} = O_p(n^{-1/2}) \), equation (23) becomes

\[ \lambda_a - \lambda = -\frac{a}{Q} \tilde{S}^{-1} \mathcal{T} + O_p(n^{-1}) \]

\[ = -\frac{a}{Q} M \mathcal{T} + \frac{a}{Q} o_p(1) O_p(n^{-1/2}) + O_p(n^{-1}) \]

\[ = -\frac{a}{Q} M \mathcal{T} + O_p(n^{-5/6}) \tag{24} \]

By the assumption of existence of sufficient amount of moments, using similar argument
as in the proof of Theorem 1, it can be shown that $M\overline{T} = \lambda + O_p(n^{-1}M)$. With this, equation (24) becomes

$$\lambda_a = \left(1 - \frac{a}{Q}\right) \lambda + O_p(n^{-5/6}).$$

(25)

Now, substitute $\lambda_a$ from (25) in the $ABR_n(\theta_0)$, we can show that

$$\frac{n}{MQ} ABR_n(\theta_0) = 2 \frac{n}{MQ} \sum_{i=1}^{Q+1} \log(1 + \lambda_a^i T_i)$$

$$= 2 \frac{n}{MQ} \sum_{i=1}^{Q+1} \log \left[1 + \left(1 - \frac{a}{Q}\right) \lambda^i T_i\right] + O_p(n^{-5/6})$$

$$= 2 \frac{n}{MQ} \sum_{i=1}^{Q} \log \left[1 + \left(1 - \frac{a}{Q}\right) \lambda^i T_i\right]$$

$$+ 2 \frac{n}{MQ} \log \left[1 + \left(1 - \frac{a}{Q}\right) \lambda^T_{n+1}\right] + O_p(n^{-5/6}).$$

(26)

(26) can be decomposed into one part that does not involve $a$ and the other that involves $a$ as follows,

$$2 \frac{n}{MQ} \sum_{i=1}^{Q} \log \left[1 + \left(1 - \frac{a}{Q}\right) \lambda^i T_i\right]$$

$$= 2 \frac{n}{MQ} \sum_{i=1}^{Q} \left[\left(1 - \frac{a}{Q}\right) \lambda^i T_i - \left(1 - \frac{a}{Q}\right) \frac{1}{2} (\lambda^i T_i)^2 + \left(1 - \frac{a}{Q}\right) \frac{3}{3} (\lambda^i T_i)^3 - \ldots\right]$$

$$= 2 \frac{n}{MQ} \sum_{i=1}^{Q} \left[\lambda^i T_i - \frac{1}{2} (\lambda^i T_i)^2 + \frac{1}{3} (\lambda^i T_i)^3 - \ldots + \text{terms involving } a\right]$$

$$= 2 \frac{n}{MQ} \sum_{i=1}^{Q} \log(1 + \lambda^i T_i) + \frac{2n}{MQ} \sum_{i=1}^{Q} \text{(terms involving } a).$$

It can be shown that the only terms involving $a$ that are larger than $O_p(n^{-1/2}Q^{-1})$ are

$$- \frac{n}{MQ} \sum_{i=1}^{Q} \frac{a}{Q} \lambda^i T_i \text{ and } \frac{n}{MQ} \sum_{i=1}^{Q} \frac{a}{Q} (\lambda^i T_i)^2,$$

but these two terms cancel each other because

$$- \frac{n}{MQ} \sum_{i=1}^{Q} \frac{a}{Q} \lambda^i T_i = - a \frac{n}{MQ^2} \lambda^i \lambda + O_p(n^{-5/6})$$

and

$$\frac{n}{MQ} \sum_{i=1}^{Q} \frac{a}{Q} (\lambda^i T_i)^2 = a \frac{n}{MQ^2} \lambda^i \lambda.$$
Therefore,

\[
2 \frac{n}{MQ} \sum_{i=1}^{Q} \log \left[ 1 + \left( 1 - \frac{a}{Q} \right) \lambda_i T_i \right] \\
= 2 \frac{n}{MQ} \sum_{i=1}^{Q} \log(1 + \lambda_i T_i) + O_p(n^{-5/6}).
\]

Now for (27), we have

\[
2 \frac{n}{MQ} \log \left[ 1 + \left( 1 - \frac{a}{Q} \right) \lambda T_{n+1} \right] \\
= 2 \frac{n}{MQ} \left[ \lambda T_{n+1} - \frac{a}{Q} \lambda T_{n+1} - \frac{1}{2} \zeta^2 \right], \text{ where } |\zeta| \leq \left| \left( 1 - \frac{a}{Q} \right) \lambda T_{n+1} \right| \\
= -2 \frac{n}{MQ} \lambda T + O_p(n^{-1/2}Q^{-1}) \\
= -2 \frac{n}{MQ} \lambda \tilde{S} + O_p(n^{-5/6}).
\]

As a result, equation (26) plus (27) can be written as

\[
2 \frac{n}{MQ} \sum_{i=1}^{Q+1} \log(1 + \lambda_i T_i) = 2 \frac{n}{MQ} \sum_{i=1}^{Q} \log(1 + \lambda_i T_i) - 2 \frac{n}{MQ} \lambda \tilde{S} + O_p(n^{-5/6}).
\]

Since \(2 \frac{n}{MQ} \sum_{i=1}^{Q} \log(1 + \lambda_i T_i) = n(R_1 + R_2 + R_3)^t(R_1 + R_2 + R_3) + O_p(n^{-1})\), it then can be shown that

\[
2 \frac{n}{MQ} \sum_{i=1}^{Q+1} \log(1 + \lambda_i T_i) = n(R_1 + R_2 + R_3)^t(R_1 + R_2 + R_3 - \frac{a}{Q} R_1) \\
+ O_p(n^{-5/6}),
\]

where the \(2 \frac{n}{MQ} \lambda \tilde{S}\) term is factored in the \(\frac{a}{Q} R_1\) term. The next step is to derive the joint cumulants of \(R := (R_1 + R_2 + R_3 - \frac{a}{Q} R_1)\), where \(R_1, R_2\) and \(R_3\) are defined in section 4. Let \(\kappa^r, \kappa^ri, \kappa^{rij},\) and \(\kappa^{rijk}\) denote the first 4 joint cumulants of \(R\). It has been shown in Kitamura (1997) that \(\kappa^{rij} = O(n^{-5/6})\) and \(\kappa^{rijk} = O(n^{-5/6})\). The first culumant is the first moment, so \(\kappa^r := \sqrt{n} E(R_1^r + R_2^r + R_3^r - \frac{a}{Q} R_1^r)\). Using the notation given in section 4, we can calculate that

\[
E(R_1^r) = 0,
\]
\[
\begin{align*}
\mathbb{E}(R_2^r) &= \frac{1}{3} \alpha^{rs} E(A^s A^t) - \frac{1}{2} E(A^{rs} A^s) \\
&= \frac{1}{3} \alpha^{rs} \delta_{st} n^{-1} - \frac{1}{2} \alpha^{rs,s} Q^{-1} M^{-1} \\
&= \left( \frac{1}{3} \alpha^{rs} - \frac{1}{2} \delta^{rs,s} \right) n^{-1},
\end{align*}
\]
and the first cumulant is then
\[
\kappa_r = \left( \frac{1}{3} \alpha^{rss} - \frac{1}{2} \tilde{\alpha}^{rs,s} \right) n^{-1/2} + O(n^{-1/2} Q^{-1}).
\]

The second cumulant \( \kappa_r \) is more complex, but fortunately the adjusted signed-root \( R \) is different from the non-adjusted signed-root \( R := R_1 + R_2 + R_3 \) by just a factor of \( \frac{a}{Q} R_1 \). This will allow us to utilize the Bartlett factor formula in Kitamura (1997) for \( R = R - \frac{a}{Q} R_1 \). With this notation, the second cumulant can be written in a relatively simple form as

\[
\begin{align*}
\kappa_r := nCov(R^r, R^i) \\
&= nCov(R^r, R^i) - 2nCov \left( R^r, \frac{a}{Q} R^i_1 \right) + nCov \left( \frac{a^2}{Q^2} R^r_1, R^i_1 \right).
\end{align*}
\]

For the first term, we have

\[
\begin{align*}
nCov(R^r, R^i) &= nE(R^r R^i) - nE(R^r) E(R^i) \\
&= nE(R^r_1 R^i_1) + n^{-1} a_{ri} + O(n^{-5/6}) \\
&= \delta_{ri} + n^{-1} a_{ri} - n^{-1} D + O(n^{-5/6}) \\
&= \delta_{ri} + n^{-1} (a_{ri} - D) + O(n^{-5/6}),
\end{align*}
\]

where \( a_{ri} \) is given in (17) and \( D := nE(R^r R^i) + O(n^{-1} Q^{-1}) \). For the second and third term, it can be shown that

\[
\begin{align*}
nCov \left( R^r, \frac{a}{Q} R^i_1 \right) &= \frac{a}{Q} \delta_{ri} + O(n^{-1} Q^{-1})
\end{align*}
\]
and

\[
\begin{align*}
nCov \left( \frac{a}{Q} R^r_1, \frac{a}{Q} R^i_1 \right) &= O(n^{-4/3}).
\end{align*}
\]
As a result,

\[ \kappa^i = \delta_i + \left[ a_{ri} - \delta_{ri} \frac{n}{Q} \right] n^{-1} + O(n^{-5/6}). \]

With the above first 4 joint cumulants of \( R \), the Edgeworth expansion of the density of \( R \) can be obtained as

\[ f_R(x) = \phi(x) + n^{-1/2}W_1(x) + n^{-1}W_2(x) + O(n^{-5/6}), \]

where

\[ W_1(x) = \left( \frac{1}{3} \alpha^{iis} - \frac{1}{2} \tilde{\alpha}^{is,s} x_i \right) \phi(x), \]
\[ W_2(x) = \frac{1}{2} \left( a_{ij} - 2a \delta_{ij} \frac{n}{Q} \right) (x_i x_j - \delta_{ij}) \phi(x). \]

Then

\[ P(nR^t R \leq x) = \int_{z^t z \leq x} \left[ \phi(z) + \sum_{i=1}^{2} n^{-i/2} W_i(z) \right] dz + O(n^{-5/6}). \]

Note that \( W_1 \) is an odd function, thus it integrates to 0 over the region \( z^t z \leq x \). The only term left is \( W_2 \). But for \( i \neq j \), \( W_2 \) is also odd, thus we only need to consider \( W_2(x) = \frac{1}{2} (a_{ii} - 2a q \frac{n}{Q}) (x_i x_i - q) \phi(x) \). That is

\[ \int_{z^t z \leq x} W_2(z) dz = \int_{z^t z \leq x} \frac{1}{2} \left( a_{ij} - 2a \delta_{ij} \frac{n}{Q} \right) (z^t z - \eta_{ij}) \phi(x) dz \]
\[ = \frac{1}{2} \left( a_{ii} - 2a q \frac{n}{Q} \right) \int_{z^t z \leq x} (z^t z - q) \phi(z) dz \]

Therefore, by letting

\[ a = \frac{1}{2} Q \frac{1}{n \cdot q} a_{ii} \]

the \( n^{-1} W_2 = n^{-1} O(M) \) term vanishes as well. As a result,

\[ P(nR^t R \leq x) = P(\chi_q^2 \leq x) + O(n^{-5/6}) \]

\[ \square \]
8. APPENDIX

Table 2: Comparison of Coverage Probabilities, $M$ is the block length, $M = \text{pro}$ means progressive block method is used.

| $\rho$ | $d$ | Methods   | $n=100$ | $n=400$ |
|-------|-----|-----------|---------|---------|
|       |     | $M$       | 0.90    | 0.95    | 0.99    | 0.90    | 0.95    | 0.99    |
| -0.8  | 2   | BEL       | 6       | 0.91    | 0.95    | 0.98    | 16      | 0.92    | 0.95    | 0.99    |
|       |     | ABEL_log  | 4       | 0.98    | 0.99    | 1.00    | pro     | 0.93    | 0.97    | 1.00    |
| -0.8  | 2   | ABEL_0.8  | 9       | 0.91    | 0.94    | 0.99    | pro     | 0.89    | 0.93    | 0.98    |
| -0.8  | 2   | ABEL_1    | 8       | 0.91    | 0.95    | 0.99    | pro     | 0.89    | 0.93    | 0.98    |
| -0.8  | 2   | ABEL_bart | 15      | 0.92    | 0.96    | 0.98    | pro     | 0.91    | 0.96    | 1.00    |
| -0.8  | 3   | BEL       | 7       | 0.89    | 0.93    | 0.96    | 15      | 0.90    | 0.95    | 0.99    |
| -0.8  | 3   | ABEL_log  | 15      | 0.97    | 0.98    | 0.99    | pro     | 0.92    | 0.97    | 1.00    |
| -0.8  | 3   | ABEL_0.8  | 6       | 0.91    | 0.95    | 0.98    | 16      | 0.91    | 0.95    | 0.98    |
| -0.8  | 3   | ABEL_1    | 16      | 0.89    | 0.95    | 0.98    | 16      | 0.91    | 0.95    | 0.98    |
| -0.8  | 3   | ABEL_bart | 13      | 0.91    | 0.94    | 0.96    | pro     | 0.90    | 0.96    | 1.00    |
| -0.8  | 4   | BEL       | 4       | 0.94    | 0.97    | 0.99    | 13      | 0.92    | 0.95    | 0.99    |
| -0.8  | 4   | ABEL_log  | 16      | 0.91    | 0.93    | 0.95    | pro     | 0.93    | 0.98    | 1.00    |
| -0.8  | 4   | ABEL_0.8  | 7       | 0.88    | 0.92    | 0.97    | 15      | 0.90    | 0.94    | 0.98    |
| -0.8  | 4   | ABEL_1    | 7       | 0.89    | 0.94    | 0.99    | 15      | 0.91    | 0.95    | 0.98    |
| -0.8  | 4   | ABEL_bart | 9       | 0.91    | 0.95    | 0.98    | pro     | 0.90    | 0.95    | 0.99    |
| -0.8  | 5   | BEL       | 5       | 0.92    | 0.95    | 0.98    | 12      | 0.90    | 0.94    | 0.98    |
| -0.8  | 5   | ABEL_log  | 14      | 0.92    | 0.93    | 0.96    | pro     | 0.94    | 0.99    | 1.00    |
| -0.8  | 5   | ABEL_0.8  | 4       | 0.94    | 0.96    | 0.99    | 12      | 0.91    | 0.95    | 0.98    |
| -0.8  | 5   | ABEL_1    | 10      | 0.88    | 0.96    | 0.99    | 14      | 0.90    | 0.95    | 0.98    |
| -0.8  | 5   | ABEL_bart | 5       | 0.92    | 0.96    | 0.99    | pro     | 0.90    | 0.96    | 0.99    |
| -0.5  | 2   | BEL       | 5       | 0.91    | 0.94    | 0.98    | 15      | 0.90    | 0.95    | 0.99    |
| -0.5  | 2   | ABEL_log  | 4       | 0.95    | 0.98    | 1.00    | pro     | 0.91    | 0.96    | 1.00    |
| -0.5  | 2   | ABEL_0.8  | 6       | 0.91    | 0.95    | 0.98    | 16      | 0.90    | 0.95    | 0.99    |
| -0.5  | 2   | ABEL_1    | 7       | 0.90    | 0.95    | 0.98    | 16      | 0.90    | 0.95    | 0.99    |
| -0.5  | 2   | ABEL_bart | 15      | 0.91    | 0.95    | 0.98    | 13      | 0.90    | 0.96    | 0.99    |
| -0.5  | 3   | BEL       | 4       | 0.92    | 0.95    | 0.98    | 9       | 0.91    | 0.95    | 0.99    |
| -0.5  | 3   | ABEL_log  | 15      | 0.95    | 0.96    | 0.98    | pro     | 0.92    | 0.97    | 1.00    |
| -0.5  | 3   | ABEL_0.8  | 5       | 0.92    | 0.95    | 0.98    | 10      | 0.90    | 0.95    | 0.99    |
| -0.5  | 3   | ABEL_1    | 14      | 0.90    | 0.96    | 0.99    | 10      | 0.91    | 0.95    | 0.99    |
| -0.5  | 3   | ABEL_bart | 14      | 0.91    | 0.94    | 0.96    | 8       | 0.91    | 0.95    | 0.99    |
| -0.5  | 4   | BEL       | 4       | 0.89    | 0.94    | 0.97    | 8       | 0.91    | 0.95    | 0.98    |
| -0.5  | 4   | ABEL_log  | 13      | 0.94    | 0.95    | 0.97    | pro     | 0.92    | 0.98    | 1.00    |
| -0.5 | 4 | ABEL_0.8 | 4 | 0.91 | 0.95 | 0.98 | 9 | 0.91 | 0.94 | 0.98 |
| -0.5 | 4 | ABEL_1   | 4 | 0.92 | 0.96 | 0.98 | 9 | 0.91 | 0.94 | 0.98 |
| -0.5 | 4 | ABEL_bart| 14| 0.91 | 0.94 | 0.96 | 9 | 0.91 | 0.96 | 0.99 |
| -0.5 | 5 | BEL      | 3 | 0.93 | 0.96 | 0.99 | 7 | 0.91 | 0.95 | 0.99 |
| -0.5 | 5 | ABEL_log | 14| 0.91 | 0.93 | 0.95 | 8 | 0.94 | 0.97 | 0.99 |
| -0.5 | 5 | ABEL_0.8 | 3 | 0.94 | 0.96 | 0.99 | 8 | 0.91 | 0.95 | 0.98 |
| -0.5 | 5 | ABEL_1   | 4 | 0.88 | 0.92 | 0.97 | 8 | 0.91 | 0.95 | 0.98 |
| -0.5 | 5 | ABEL_bart| 12| 0.93 | 0.94 | 0.96 | pro| 0.90 | 0.95 | 0.99 |
| -0.5 | 10| BEL      | 2 | 0.93 | 0.96 | 0.99 | 5 | 0.92 | 0.95 | 0.99 |
| -0.5 | 10| ABEL_log | 7 | 0.96 | 0.97 | 0.98 | 6 | 0.94 | 0.98 | 1.00 |
| -0.5 | 10| ABEL_0.8 | 2 | 0.94 | 0.97 | 0.99 | 6 | 0.90 | 0.94 | 0.98 |
| -0.5 | 10| ABEL_1   | 2 | 0.95 | 0.97 | 0.99 | 6 | 0.90 | 0.94 | 0.98 |
| -0.5 | 10| ABEL_bart| 3 | 0.94 | 0.97 | 0.99 | 5 | 0.95 | 0.98 | 1.00 |
| -0.2 | 2 | BEL      | 3 | 0.91 | 0.95 | 0.99 | 8 | 0.90 | 0.95 | 0.99 |
| -0.2 | 2 | ABEL_log | 3 | 0.94 | 0.97 | 1.00 | pro| 0.90 | 0.95 | 1.00 |
| -0.2 | 2 | ABEL_0.8 | 4 | 0.91 | 0.95 | 0.99 | 10| 0.90 | 0.95 | 0.99 |
| -0.2 | 2 | ABEL_1   | 5 | 0.90 | 0.95 | 0.98 | 10| 0.91 | 0.95 | 0.99 |
| -0.2 | 2 | ABEL_bart| 3 | 0.91 | 0.96 | 0.99 | pro| 0.90 | 0.95 | 0.99 |
| -0.2 | 3 | BEL      | 3 | 0.90 | 0.94 | 0.98 | 6 | 0.89 | 0.94 | 0.99 |
| -0.2 | 3 | ABEL_log | 3 | 0.94 | 0.97 | 0.99 | pro| 0.90 | 0.96 | 1.00 |
| -0.2 | 3 | ABEL_0.8 | 3 | 0.91 | 0.95 | 0.98 | 7 | 0.90 | 0.94 | 0.99 |
| -0.2 | 3 | ABEL_1   | 14| 0.90 | 0.95 | 0.99 | 7 | 0.90 | 0.95 | 0.99 |
| -0.2 | 3 | ABEL_bart| 14| 0.91 | 0.94 | 0.97 | pro| 0.90 | 0.95 | 0.99 |
| -0.2 | 4 | BEL      | 2 | 0.91 | 0.95 | 0.99 | 7 | 0.90 | 0.94 | 0.99 |
| -0.2 | 4 | ABEL_log | 3 | 0.93 | 0.97 | 1.00 | pro| 0.90 | 0.97 | 1.00 |
| -0.2 | 4 | ABEL_0.8 | 2 | 0.92 | 0.95 | 0.99 | 8 | 0.89 | 0.94 | 0.99 |
| -0.2 | 4 | ABEL_1   | 3 | 0.89 | 0.94 | 0.98 | 8 | 0.90 | 0.95 | 0.99 |
| -0.2 | 4 | ABEL_bart| 2 | 0.92 | 0.96 | 0.99 | 7 | 0.91 | 0.95 | 1.00 |
| -0.2 | 5 | BEL      | 2 | 0.90 | 0.95 | 0.99 | 5 | 0.91 | 0.95 | 0.99 |
| -0.2 | 5 | ABEL_log | 3 | 0.92 | 0.96 | 1.00 | 6 | 0.92 | 0.96 | 1.00 |
| -0.2 | 5 | ABEL_0.8 | 2 | 0.91 | 0.95 | 0.99 | 7 | 0.90 | 0.94 | 0.99 |
| -0.2 | 5 | ABEL_1   | 2 | 0.91 | 0.96 | 0.99 | 7 | 0.90 | 0.95 | 0.99 |
| -0.2 | 5 | ABEL_bart| 2 | 0.91 | 0.97 | 1.00 | 6 | 0.91 | 0.95 | 1.00 |
| -0.2 | 10| BEL      | 2 | 0.82 | 0.87 | 0.96 | 3 | 0.91 | 0.95 | 0.99 |
| -0.2 | 10| ABEL_log | 2 | 0.90 | 0.96 | 1.00 | 6 | 0.89 | 0.95 | 0.99 |
| -0.2 | 10| ABEL_0.8 | 2 | 0.83 | 0.89 | 0.97 | 3 | 0.91 | 0.96 | 0.99 |
| -0.2 | 10| ABEL_1   | 2 | 0.84 | 0.90 | 0.97 | 4 | 0.89 | 0.94 | 0.98 |
| -0.2 | 10| ABEL_bart| 8 | 0.96 | 0.97 | 0.98 | 3 | 0.92 | 0.96 | 0.99 |
| 0.2  | 2 | BEL      | 3 | 0.83 | 0.90 | 0.97 | 9 | 0.89 | 0.95 | 0.98 |
| 0.2  | 2 | ABEL_log | 4 | 0.90 | 0.95 | 0.99 | 5 | 0.90 | 0.96 | 0.99 |
|       | 0.2  | 2    | ABEL_0.8 | 4      | 0.85 | 0.91 | 0.97 | 9      | 0.90 | 0.95 | 0.98 |
|-------|------|------|----------|--------|------|------|------|--------|------|------|------|
|       | 0.2  | 2    | ABEL_1   | 16     | 0.90 | 0.97 | 0.99 | 9      | 0.90 | 0.95 | 0.99 |
|       | 0.2  | 2    | ABEL_bart| 3      | 0.88 | 0.95 | 1.00 | 11     | 0.89 | 0.95 | 0.99 |
|       | 0.2  | 3    | BEL      | 3      | 0.82 | 0.89 | 0.95 | 9      | 0.88 | 0.93 | 0.98 |
|       | 0.2  | 3    | ABEL_log | 4      | 0.89 | 0.97 | 1.00 | 9      | 0.91 | 0.95 | 1.00 |
|       | 0.2  | 3    | ABEL_0.8 | 3      | 0.83 | 0.90 | 0.96 | 9      | 0.88 | 0.94 | 0.98 |
|       | 0.2  | 3    | ABEL_1   | 14     | 0.88 | 0.95 | 0.99 | 9      | 0.88 | 0.94 | 0.99 |
|       | 0.2  | 3    | ABEL_bart| 12     | 0.93 | 0.96 | 0.98 | 8      | 0.90 | 0.95 | 1.00 |
|       | 0.2  | 4    | BEL      | 2      | 0.79 | 0.86 | 0.95 | 5      | 0.86 | 0.91 | 0.97 |
|       | 0.2  | 4    | ABEL_log | 4      | 0.89 | 0.97 | 1.00 | 9      | 0.91 | 0.95 | 1.00 |
|       | 0.2  | 4    | ABEL_0.8 | 3      | 0.81 | 0.87 | 0.95 | 5      | 0.86 | 0.92 | 0.98 |
|       | 0.2  | 4    | ABEL_1   | 14     | 0.85 | 0.91 | 0.97 | 5      | 0.86 | 0.92 | 0.98 |
|       | 0.2  | 4    | ABEL_bart| 12     | 0.93 | 0.95 | 0.96 | 8      | 0.90 | 0.96 | 1.00 |
|       | 0.2  | 5    | BEL      | 2      | 0.73 | 0.82 | 0.94 | 4      | 0.83 | 0.90 | 0.97 |
|       | 0.2  | 5    | ABEL_log | 4      | 0.92 | 0.99 | 1.00 | 9      | 0.89 | 0.96 | 1.00 |
|       | 0.2  | 5    | ABEL_0.8 | 2      | 0.75 | 0.84 | 0.94 | 6      | 0.85 | 0.90 | 0.97 |
|       | 0.2  | 5    | ABEL_1   | 11     | 0.78 | 0.90 | 0.97 | 6      | 0.85 | 0.91 | 0.97 |
|       | 0.2  | 5    | ABEL_bart| 12     | 0.93 | 0.95 | 0.96 | 6      | 0.89 | 0.96 | 1.00 |
|       | 0.2  | 10   | BEL      | 2      | 0.54 | 0.62 | 0.78 | 4      | 0.78 | 0.85 | 0.95 |
|       | 0.2  | 10   | ABEL_log | 3      | 0.91 | 0.99 | 1.00 | 8      | 0.91 | 0.98 | 1.00 |
|       | 0.2  | 10   | ABEL_0.8 | 2      | 0.56 | 0.65 | 0.81 | 4      | 0.79 | 0.86 | 0.95 |
|       | 0.2  | 10   | ABEL_1   | 2      | 0.57 | 0.66 | 0.82 | 4      | 0.79 | 0.86 | 0.95 |
|       | 0.2  | 10   | ABEL_bart| 8      | 0.96 | 0.97 | 0.98 | 2      | 0.89 | 0.97 | 1.00 |
|       | 0.5  | 2    | BEL      | 6      | 0.73 | 0.82 | 0.92 | 12     | 0.86 | 0.92 | 0.97 |
|       | 0.5  | 2    | ABEL_log | 6      | 0.90 | 0.97 | 1.00 | 10     | 0.91 | 0.94 | 0.99 |
|       | 0.5  | 2    | ABEL_0.8 | 16     | 0.76 | 0.90 | 0.99 | 12     | 0.87 | 0.93 | 0.98 |
|       | 0.5  | 2    | ABEL_1   | 16     | 0.88 | 0.96 | 1.00 | 12     | 0.88 | 0.93 | 0.98 |
|       | 0.5  | 2    | ABEL_bart| 3      | 0.89 | 0.97 | 1.00 | 13     | 0.89 | 0.96 | 0.99 |
|       | 0.5  | 3    | BEL      | 5      | 0.68 | 0.77 | 0.89 | 10     | 0.82 | 0.87 | 0.95 |
|       | 0.5  | 3    | ABEL_log | 5      | 0.89 | 0.97 | 1.00 | 13     | 0.91 | 0.96 | 1.00 |
|       | 0.5  | 3    | ABEL_0.8 | 16     | 0.74 | 0.88 | 0.97 | 10     | 0.83 | 0.89 | 0.96 |
|       | 0.5  | 3    | ABEL_1   | 14     | 0.87 | 0.95 | 0.99 | 10     | 0.83 | 0.89 | 0.96 |
|       | 0.5  | 3    | ABEL_bart| 14     | 0.92 | 0.95 | 0.97 | pro    | 0.90 | 0.96 | 0.99 |
|       | 0.5  | 4    | BEL      | 4      | 0.64 | 0.72 | 0.85 | 9      | 0.77 | 0.85 | 0.95 |
|       | 0.5  | 4    | ABEL_log | 16     | 0.92 | 0.94 | 0.97 | 11     | 0.88 | 0.95 | 1.00 |
|       | 0.5  | 4    | ABEL_0.8 | 14     | 0.72 | 0.86 | 0.95 | 9      | 0.79 | 0.87 | 0.96 |
|       | 0.5  | 4    | ABEL_1   | 14     | 0.86 | 0.92 | 0.97 | 9      | 0.79 | 0.87 | 0.96 |
|       | 0.5  | 4    | ABEL_bart| 13     | 0.91 | 0.94 | 0.96 | pro    | 0.91 | 0.97 | 0.99 |
|       | 0.5  | 5    | BEL      | 4      | 0.54 | 0.64 | 0.78 | 10     | 0.76 | 0.83 | 0.94 |
|       | 0.5  | 5    | ABEL_log | 14     | 0.91 | 0.94 | 0.96 | 11     | 0.90 | 0.98 | 1.00 |
| 0.5  | 5   | ABEL_0.8 | 11  | 0.57 | 0.77 | 0.95 | 10  | 0.78 | 0.85 | 0.94 |
|------|-----|---------|-----|------|------|------|-----|------|------|------|
| 0.5  | 5   | ABEL_1  | 11  | 0.79 | 0.90 | 0.97 | 10  | 0.78 | 0.85 | 0.94 |
| 0.5  | 5   | ABEL_bart | 12  | 0.94 | 0.95 | 0.97 | 5   | 0.91 | 0.98 | 1.00 |
| 0.8  | 2   | BEL     | 9   | 0.58 | 0.67 | 0.76 | 16  | 0.77 | 0.85 | 0.94 |
| 0.8  | 2   | ABEL_log| 7   | 0.87 | 0.98 | 1.00 | 16  | 0.88 | 0.95 | 1.00 |
| 0.8  | 2   | ABEL_0.8| 16  | 0.72 | 0.86 | 0.98 | 16  | 0.80 | 0.86 | 0.94 |
| 0.8  | 2   | ABEL_1  | 16  | 0.85 | 0.95 | 0.99 | 16  | 0.80 | 0.87 | 0.95 |
| 0.8  | 2   | ABEL_bart| 4   | 0.91 | 0.94 | 0.97 | 13  | 0.90 | 0.96 | 1.00 |
| 0.8  | 3   | BEL     | 7   | 0.45 | 0.53 | 0.65 | 16  | 0.69 | 0.78 | 0.90 |
| 0.8  | 3   | ABEL_log| 15  | 0.96 | 0.97 | 0.99 | 16  | 0.91 | 0.98 | 1.00 |
| 0.8  | 3   | ABEL_0.8| 16  | 0.70 | 0.84 | 0.94 | 16  | 0.72 | 0.81 | 0.92 |
| 0.8  | 3   | ABEL_1  | 14  | 0.83 | 0.92 | 0.98 | 16  | 0.73 | 0.82 | 0.93 |
| 0.8  | 3   | ABEL_bart| 12  | 0.93 | 0.95 | 0.97 | pro | 0.92 | 0.98 | 1.00 |
| 0.8  | 4   | BEL     | 5   | 0.30 | 0.38 | 0.49 | 16  | 0.63 | 0.73 | 0.85 |
| 0.8  | 4   | ABEL_log| 16  | 0.91 | 0.94 | 0.97 | 14  | 0.89 | 0.98 | 1.00 |
| 0.8  | 4   | ABEL_0.8| 14  | 0.62 | 0.77 | 0.91 | 16  | 0.69 | 0.76 | 0.88 |
| 0.8  | 4   | ABEL_1  | 14  | 0.77 | 0.86 | 0.95 | 16  | 0.69 | 0.77 | 0.89 |
| 0.8  | 4   | ABEL_bart| 14  | 0.89 | 0.92 | 0.95 | 4   | 0.96 | 1.00 | 1.00 |
| 0.8  | 5   | BEL     | 5   | 0.20 | 0.25 | 0.34 | 14  | 0.58 | 0.67 | 0.80 |
| 0.8  | 5   | ABEL_log| 12  | 0.94 | 0.96 | 0.98 | 13  | 0.88 | 0.97 | 1.00 |
| 0.8  | 5   | ABEL_0.8| 14  | 0.50 | 0.66 | 0.80 | 14  | 0.61 | 0.70 | 0.84 |
| 0.8  | 5   | ABEL_1  | 11  | 0.65 | 0.81 | 0.94 | 15  | 0.62 | 0.72 | 0.85 |
| 0.8  | 5   | ABEL_bart| 12  | 0.93 | 0.94 | 0.95 | pro | 0.99 | 0.99 | 1.00 |
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