A Unified Framework for Nonmonotonic Reasoning with Vagueness and Uncertainty

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Abstract

An answer set programming paradigm is proposed that supports nonmonotonic reasoning with vague and uncertain information. The system can represent and reason with prioritized rules, rules with exceptions. An iterative method for answer set computation is proposed. The terminating conditions are identified for a class of logic programs using the notion of difference equations. In order to obtain the difference equations, the set of rules are depicted by a signal-flow-graph like structure.

1 Introduction:

Answer set programming (ASP) is a declarative problem solving paradigm for nonmonotonic reasoning. ASP allows intuitive representation of combinatorial search and optimization problems and is widely used for knowledge representation and reasoning in various applications like plan generation, natural language processing etc [14, 15]. But ASP can not deal with fuzzy information, where attributes and truth degrees lie in a continuous range of values. Fuzzy Answer Set Programming (FASP) is proposed as an extension of ASP that allows graded truth values from the interval [0,1]. Theoretical advancement of FASP is remarkable [18, 32, 9, 22, 23]. However, this approach performs reasoning with absolutely certain but vague information and doesn’t involve reasoning with uncertain information. Though often
vagueness and uncertainty are mistakenly considered to be same but vagueness results from gradualness of attribute values whereas uncertainty comes from incomplete information. To deal with uncertainty, possibilistic and probabilistic versions of ASP has been proposed. But these approaches are based on bivalent Boolean logic and hence are incapable of modeling the vagueness of information.

It is clear that a single assignment is not sufficient for representing both fuzziness and uncertainty. In the authors have attempted to salvage the problem by merging possibilistic ASP with FASP in the form of Possibilistic Fuzzy Answer Set Programming (PFASP). In PFASP two separate degrees, from the range [0, 1], are assigned to assert the degree of fuzziness and the degree of certainty of a proposition. But the proposed semantics is only for positive logic programs without incorporating negation, neither classical negation nor negation-as-failure (not). Moreover, assigning two independent numbers as the degree of vagueness and the degree of uncertainty of a piece of information is not always intuitive. Because in human commonsense reasoning our assessment of the degree of truth of a proposition is somewhat dependent on the certainty about that proposition.

Interval-valued fuzzy sets (IVFSs) represent vagueness and uncertainty simultaneously in an intuitive manner by replacing the crisp [0, 1]-valued membership degree by sub-intervals of [0, 1]. The intuition is that the actual membership would be a value within this interval. The intervals can be ordered with respect to their degree of truth as well as their degree of certainty by means of a bilattice-based algebraic structure, namely Bilattice-based Triangle. Hence bilattice-based triangle is suitable for reasoning with vague and uncertain information. However, it is investigated that bilattice-based triangle suffers from some limitations when reasoning is performed in non-monotonic environment, where, conclusions are drawn in absence of complete information and later those conclusions may have to be withdrawn with addition of further information. As an alternative, Preorder-based triangle is proposed, which is suitable for dealing with nonmonotonic fuzzy reasoning with uncertainty.

Thus IVFSs, associated with some dually ordered algebraic structure, e.g., Preorder-based Triangle, can be used to replace the Boolean {0, 1} truth value space of ASP to enhance it for dealing with vagueness and uncertainty simultaneously. But this seemingly promising concoction of IVFS and ASP is yet unexplored. Some approaches to probabilistic ASP assign intervals of probabilities to represent the underlying uncertainty; but these approaches don’t exploit the intuitive interaction of the degree of truth and the degree of certainty of a proposition, which is unavoidable especially in
presence of negation.

In this paper, we attempt to enhance ASP by combining it with IVFSs as representation of truth values of propositions. The proposed modified Answer Set Programming will be called Unified Answer Set Programming (UnASP), since it captures both fuzziness and uncertainty of information along with nonmonotonic reasoning in a single framework.

1.1 Motivation:

Uncertainty and incompleteness of information is unavoidable in real life reasoning. Moreover, in many practical applications reasoning is performed based on incomplete information, rules of thumb or rules with exceptions. These types of reasoning falls under non-monotonic reasoning. Nowadays rule-based expert systems, decision support systems (DSS) are being widely used in various application domains. Such systems involves a knowledge base, which consists of a rule base and inference mechanism. For human like reasoning, which is the purpose of these systems, the knowledge base must be capable of representing and reasoning with imprecise and uncertain information under nonmonotonic situations. A particular application area is discussed here.

Artificial intelligence-based systems are being developed for helping doctors making clinical decisions. This can be extremely helpful in dealing with large number for patients in the emergency units of medical institutions. An automated Triage system can be helpful in emergency conditions in accident sites [16]. Apart from that, various knowledge-based clinical decision support systems (CDSS) [8, 30, 27, 17, 20] are being developed to assist doctors in clinical decision making, where the aim is to build a computer program that could simulate human thinking. A CDSS mainly consists of three parts; 1. a knowledge base, 2. an inference engine and 3. mechanism to communicate with the user. The knowledge base captures the doctor’s knowledge and opinions in the form of ‘IF-THEN’ rules, and the inference engine deduces conclusions using this knowledge base and the specific data presented by the patient.

Proper logical framework must be chosen in order to appropriately represent the doctor’s beliefs and to draw appropriate decisions. Some of the parameters concerned with a patient’s medical condition are qualitative, which have innate imprecision. Moreover, the natural way to express doctors’ knowledge is by means of using linguistic variables which express the imprecision (or vagueness). For instance, it is more natural for a doctor to say ”An older patient with severe stomach ache has a more serious condi-
tion than a young patient”; rather than ”A patient with age > 40, with a stomach ache of intensity 6 or more in a scale of 0-10 is 30% more serious than that of a patient with age < 15”. These linguistic variables and qualitative attributes can only be captured in Fuzzy Logic. Many Fuzzy Logic based DSS are proposed. Moreover, some decisions are complex and depend on too many parameters and incur some uncertainty; e.g., ”A round, opacified area seen in the lungs on a chest radiograph is probably Pneumonia” [8]. This rule doesn’t involve any imprecise attributes, rather a decision which is precise but uncertain. Hence, possibilistic or probabilistic logic may be used for capturing this uncertainty. In [5] the authors have demonstrated the application of possibilistic answer set programming to design an intelligent triage system for emergency cases at accident sites.

However there is another aspect that must be incorporated in the inference mechanism. Medical decision making is often nonmonotonic. Suppose in a scenario a patient arrives at the emergency section with critical condition and his diagnosis requires some medical test which is time consuming or that testing facility may not be available at that instant. In that case, based on the condition of the patient, medical decision (say, assignment of Emergency Severity Index (ESI) level) has to be done without that test result, based on other symptoms and rules of thumb. Later, when the test result arrives, the prior decision may be found to be wrong. For instance, A patient having abdominal pain may be initially ascribed with ESI level 3, but the detail diagnosis may reveal more risky condition and the assignment may have to be revised to ESI level 2. Hence in the inference system there must be a provision to update or revise decisions without invalidating the rules used to deduce the conclusion. Therefore, nonmonotonicity is also an inseparable part of a CDSS.

The above discussion, though focused on a specific application domain, points out the necessity of a unified nonmonotonic reasoning framework for handling both vague information and uncertain information.

In this work we attempt to develop such a framework. The main contributions of this work are as follows:

• A modified version of Answer set Programming, named the Unified Answer Set Programming (UnASP) is developed for nonmonotonic reasoning with vague and uncertain information.

• The set $I(L)$ is ordered with respect to degree of truth and degree of knowledge based on a preorder-based triangle [28]. This algebraic structure has been proved to be an enhancement of bilattice-based triangle, which has been used in earlier approaches of ASP to order truth values based on their degree of truth and certainty. Hence, the use of preorder-based triangle
provides some additional advantages and more intuitive results for practical applications.

- Here, negation-as-failure operator has been defined in a more intuitive way and this modified definition proves to be helpful in computation of the answer set.
- In the semantics knowledge aggregator operator is introduced, that compares the certainty level of positive and negative evidences for a particular piece of information. This operator makes this nonmonotonic reasoning approach more intuitive and interesting.
- An iterative approach for computation of answer set is presented, for a logic program which satisfies certain constraints. This kind of iterative evaluation, which uses the notion of signal flow graphs in order to mathematically analyze the nature of iterations, is novel.

2 Intervals as degree of belief

Selection of a proper set of truth-values, i.e., the set of values and their ordering is an important factor on which the appropriateness, performance and usability of an inference system depends. The unit interval, i.e., \([0, 1]\), is usually used in expert systems to describe membership degree of a vague attributes. The same is also used to describe experts’ degree of belief in their statements. In absence of complete knowledge or in a multi-agent system when different experts have different degrees of belief it becomes impossible, to have a specific degree of membership for some fuzzy attribute or to assert a precise degree of truth to a proposition. In such a scenario ascribing an interval of possible values taken from the unit interval is the natural solution [24]. Intervals are appropriate to describe simultaneously underlying imprecision and uncertainty of a piece of information. Exact intervals, i.e. intervals of the form \([x, x]\), may be used to characterize fuzzy and completely certain propositions. Assigning these intervals as epistemic state of propositions is as good as assigning a fuzzy truth value, i.e. a number from \([0, 1]\). Intervals with non-zero length are used to express underlying uncertainty. Longer is the interval more is the uncertainty. The interval \([0, 1]\) is used as the epistemic state of a proposition for which we have no knowledge at all, it can be true or false or something in between.

Let \(L = \{L, \leq\}\), where \(L = [0, 1]\) and the set of intervals of \(L\) is defined as \(I(L) = \{[x_1, x_2]|x_1, x_2 \in L\text{ and } x_1 \leq x_2\}\). This notation \(I(L)\) will be used throughout this paper to denote the set of all subsets of \([0, 1]\).
2.1 Ordering of intervals based on degree of truth and certainty:

The intervals can be ordered using their degree of vagueness (truth ordering, \( \leq_t \)) and their degree of certainty (knowledge ordering, \( \leq_k \)) by a bilattice-based triangle as follows:

\[
[x_1, x_2] \leq_t [y_1, y_2] \iff x_1 \leq y_1 \text{ and } x_2 \leq y_2.
\]

\[
[x_1, x_2] \leq_k [y_1, y_2] \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2.
\]

for every \([x_1, x_2]\) and \([y_1, y_2] \in I(L)\).

It is demonstrated in [28] that in a nonmonotonic reasoning environment, where inferences are made without complete knowledge about the scenario and new information may invalidate previously drawn conclusions, the knowledge ordering \( \leq_k \) cannot be used to handle the repetitive revision of epistemic states. The truth ordering \( \leq_t \), used in bilattice-based triangle, does not order the overlapping intervals intuitively. It is also demonstrated, in several practical applications involving belief revision and decision making based on comparing the degree of vagueness and certainty of some statements, traditional truth ordering \( \leq_t \) and knowledge ordering \( \leq_k \) do not reflect our intuitive judgement or our commonsense. To salvage the problem some modifications were incurred to the knowledge and truth ordering [28].

For every \([x_1, x_2]\) and \([y_1, y_2] \in I(L)\) the modified truth and knowledge orderings are respectively as follows:

\[
[x_1, x_2] \leq_{tp} [y_1, y_2] \iff x_{m} \leq y_{m}.
\]

\[
[x_1, x_2] \leq_{kp} [y_1, y_2] \iff x_{w} \geq y_{w}.
\]

where, for the interval \(x, x_{m}\) is the midpoint of the interval \(x\) given by \((x_1 + x_2)/2\) and \(x_{w}\) is the length of the interval given by \((x_2 - x_1)\).

The algebraic structure that put together these two modified orderings is \textbf{preorder-based triangle} (\(P(L)\)), which is proved to be an enhanced version of bilattice-based triangle [28].

Figure 1 demonstrates all the intervals formed from the lattice \(L = \{0, 0.25, 0.5, 0.75, 1\}, \leq\) and ordered with \(\leq_{tp}\) and \(\leq_{kp}\). The dashed lines demonstrate the connections that were not imposed by \(\leq_t\) and \(\leq_k\). Thus from the figure it is clear that the modified orders introduce additional comparability between intervals.

From Figure 1 it is clear that the modified knowledge ordering \(\leq_{kp}\) differs from \(\leq_k\) when comparing disjoint or partially overlapped intervals.
In such cases knowledge ordering (or information ordering) can be imposed using $\leq_{kp}$ based on their lengths, which reflects the amount of uncertainty, and this additional comparability is essential for belief revision in nonmonotonic reasoning [28]. In cases where one interval lies completely within the other one (i.e. one is a proper sub-interval of the other), $\leq_t$ fails to order them; but using $\leq_{tp}$ the intervals can be ordered and the intuition behind this ordering is: "$x \leq_{tp} y$ iff the probability that $\hat{x} \leq \hat{y}$ is larger than $\hat{x} \geq \hat{y}$", where $\hat{x}$ and $\hat{y}$ are the actual truth values approximated by the intervals $x$ and $y$. Consideration of this probabilistic aspect may play a crucial role in certain real-life decision making scenarios [28].

The notation $x <_{tp} y$ means $x \leq_{tp} y$ and $x_m \neq y_m$; $x =_{tp} y$ would be used if $x_m = y_m$. Similarly $x <_{kp}$ and $x =_{kp} y$ are interpreted.

In this work, a preorder-based triangle over $I(L)$ is used as the truth-value set, based on which the modified answer set programming framework is developed.

Though the enhanced comparability and mutual independence make the modified truth and knowledge ordering attractive, there is a well-developed theory of constructing conjunctors (t-norms) and disjunctors (t-conorms) [28] for the traditional orders $\leq_t$ and $\leq_k$. Hence while building the unified answer set programming framework we will exploit both pairs of orders appropriately to have the best of both worlds.
2.2 Logical Operators

The traditional logical operations defined over $[0, 1]$ in Fuzzy Answer Set Programming like, classical negation, negation-as-failure, conjunctions, disjunctions can be generalized to define logical operations on $I(L)$. In this subsection the logical operators used in this work are discussed.

2.2.1 Negation

An involutive negator, namely the standard negator (also known as Lukasiewicz Negator), $N$, which is defined as a mapping $[0, 1] → [0, 1] : N(x) = 1 - x$, can be used to define an involutive negator on $I(L)$ as follows:

$$N([x_1, x_2]) = [N(x_2), N(x_1)] = [1 - x_2, 1 - x_1].$$

Thus, it can be seen that strong negation doesn’t alter the degree of certainty but reflects the interval around the central line of the preorder-based triangle. In the rest of the paper, standard negator will be represented using the symbol ‘\(\neg\)’.

2.2.2 Negation-as-failure(not)

In nonmonotonic reasoning, specially in logic programming, ‘not’ is used to draw inferences in absence of complete knowledge. The classical negation is different from ‘not’ in the way they deal with incompleteness of information. In FASP, some monotonically decreasing function, or the Lukasiewicz negator is used to model negation-as-failure [19]. This function holds good there since FASP only deals with imprecision of information where a specific membership value can be given from $[0, 1]$. However the situation is different when the uncertainty or lack of knowledge about the precise membership value is explicitly represented by using intervals of values. Here an intuitive alternative definition of negation-as-failure is developed as described below.

In nonmonotonic logic programming, for any atomic statement $p$, ‘not $p$’ is true if $p$ cannot be proved to be true or information about $p$ is absent. The statement $p$, being completely unknown, the epistemic state assigned to it is $[0,1]$. In absence of any information regarding statement $p$, not $p$ is inferred to be true (i.e. the assigned epistemic state is $[1,1]$). Thus from an intuitive perspective, it can be said not $[0,1] = [1,1]$. Whereas, in case of standard negation ($\neg$), we have $\neg[0,1] = [1 - 1, 1 - 0] = [0,1]$; i.e, if no information is available about a statement then nothing can be said about its negation.
The value of \textit{not} \( p \) for an atomic statement \( p \) would depend on how much knowledge about \( p \) is available, as well as, on how much the experts believe in the truth of \( p \). When \( p \) represents absolutely certain but fuzzy attribute, i.e. has an exact interval as its epistemic state, then negation-as-failure would behave in the same way as classical negation. Information content, i.e. certainty level, of \( \neg p \) is same as that of \( p \). Thus if the interval is of the form \([x, x]\),

\[
\text{not}[x, x] = \neg[x, x] = [1 - x, 1 - x].
\]

The interval assigned to \textit{not} \( p \) depends only on the epistemic state of \( p \), which expresses the experts’ degree of belief on \( p \). Once epistemic state of \( p \) is obtained, \textit{not} \( p \) can be evaluated from it and hence the assignment for \textit{not} \( p \) does not directly come from the experts’ opinions, rather it’s a meta level assignment based on the epistemic state of \( p \), which is already asserted. Thus there would be no question of uncertainty while determining the epistemic state of \textit{not} \( p \) once information about \( p \) is at hand. Therefore epistemic state of \textit{not} \( p \) will be exact intervals, solely depending on the epistemic state of \( p \).

Hence, for an interval \([x_1, x_2] \in I(L)\)

\[
\text{not}[x_1, x_2] = [1 - x_1, 1 - x_1].
\]

In Figure 2 the negation-as-failure is shown. From this definition it is clear that the operation ’\textit{not}’ is not \textit{involutive} when applied on an in-exact interval and this is a significant difference between this definition and the way ’\textit{not}’ is defined in FASP.
2.2.3 Conjunctors and disjunctors

T-representable t-norms and t-conorms are used as conjuctors and disjunc-
tors for the set of intervals.

**Definition 1** A t-norm $T$ on $I(L)$ is called t-representable if there exist
t-norms $T_1$ and $T_2$ on $[0,1]$, such that $T_1 \leq T_2$ and such that $T$
can be represented as, for all $[x_1, x_2], [y_1, y_2] \in I(L)$:

$$T([x_1, x_2], [y_1, y_2]) = [T_1(x_1, y_1), T_2(x_2, y_2)].$$

$T_1$ and $T_2$ are called representants of $T$. In this particular case $T_1$ and $T_2$
are identical.

Similarly a t-representable t-conorm can be defined on $I(L)$ using two
t-conorms on $[0,1]$.

In this work, the Product t-(co)norm is chosen as representant for con-
structing the t-representable t-(co)norm on $I(L)$. For two intervals $[x_1, x_2], [y_1, y_2] \in I(L)$, their t-(co)norm $\wedge$ and $\vee$ are defined respectively as follows:

$$[x_1, x_2] \wedge [y_1, y_2] = [x_1 \cdot x_2, y_1 \cdot y_2]$$
$$[x_1, x_2] \vee [y_1, y_2] = [x_1 + x_2 - x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2]$$

3 Syntax

The constructed UnASP language has infinitely many variables, finitely
many constants and predicate symbols, including comparative predicates
like equality, less-than, greater-than etc. No function symbol is allowed.
A term is a variable or a constant. An atom is an expression of the form
$p(t_1, t_2, ..., t_n)$, where $p$ is a predicate symbol of arity $n$ and $t_1, t_2, ..., t_n$
are terms or a sub-interval from $[0,1]$. An atom is grounded if it contains no
variables. A literal is a positive atom $p$ or its negation of the form $\neg p$. A
literal of the form $\neg l$ is a naf-literal.

A rule is of the form:

$$r : a \rightarrow b_1, b_2, ..., b_m, \neg b_{m+1}, ..., \neg b_n$$

where, $a$ and $b_i$ are positive or negated literals or elements of $I(L)$. The
literal $a$ is the head of rule. The conjunction of $b_1, b_2, ..., b_m, \neg b_{m+1}, ..., \neg b_n$
is the body of the rule. The label ‘$r$’ is used to refer to this rule in other
rules. $\alpha_r$ is an interval of the form $[x_1, x_2]$ that specifies the weight of the
rule i.e. the degree of truth and certainty of the rule. In other words, the
weight denotes what would be the epistemic state of the consequent (or
head) of the rule when the antecedent is absolutely true, i.e. assigned with
[1, 1]. The epistemic state (or truth status) of the body of a rule is combined with the weight by means of the product t-norm (\(\land\)) and then it propagates to the head.

All rules are taken to be universally quantified and hence the quantification (\(\forall x\)) is not specified explicitly.

A rule is said to be a constraint if its head \(a\) is an element of \(I(L)\). A rule is a fact if \(b_i, 1 \leq i \leq n\) are elements of \(I(L)\).

A Unified Answer Set Program (UnASP) is a set of weighted rules as defined above.

A program is called positive if for all rules in the program \(n = m\), i.e. no naf literal is present in rule body and \(a, b_1, b_2, ..., b_m\) are positive atoms. A program is said to be general or normal if the literals in the heads and bodies of all rules are positive atoms, i.e. don’t contain classical negation \(\neg\). Programs where naf literals as well as negated literals are allowed in the rules are syntactically referred to as extended programs.

For a program \(P\), \(P_l\) denotes the subset of \(P\) consisting of the rules in \(P\) whose head is the literal \(l\).

**Significance of the rule weight \(\alpha_r\):**

The unified reasoning approach presented here is aimed to be a generalised framework suitable for reasoning with classical, vague as well as uncertain information. All these aspects can be captured by appropriately choosing the weight of the rules from the set of intervals \(I(L)\).

- To represent statements free from any uncertainty and vagueness, as are dealt with in classical ASP, the weights are chosen to be \([1, 1]\). For instance 'Birds gives egg' can be expressed as:

\[
r : \text{GivesEgg}(x) \quad [1, 1] \leftarrow \text{Bird}(x).
\]

This rule states that, if 'x is a Bird' is True, i.e. assigned \([1, 1]\), then 'x gives egg' is also True and is assigned the interval \([1, 1]\).

- The weight \(\alpha_r\) can be used capture the vagueness, where even if the body of the rule is satisfied the head may be partially true, having a degree of truth between \([0, 0]\) and \([1, 1]\). The proposition 'Small cars are moderately safe' can be formally written as:

\[
r : \text{Safe}(x) \quad [0.7, 0.7] \leftarrow \text{SmallCar}(x).
\]

Here, the weight \(\alpha\) being an exact interval, \([0.7, 0.7]\), depicts that the rule is certain but vague, i.e. the degree of safety varies over a range of \([0,1]\).
• When statements from different sources are considered they may have various degrees of reliabilities. Thus, wider intervals, as weights, would imply lesser importance or reliability of the rule.

• The weight $\alpha_r$ helps to represent default statements which are assumed to be true under normal conditions but have exceptional cases. Default statements or dispositions \[33\] are used to represent commonsense knowledge or the commonplace statement of facts. Some examples are:

  i. Birds can fly.
  ii. Glue is sticky.
  iii. Glass is fragile.
  iv. Where there is smoke there is fire.
  v. Swedes are taller than Italians.

In a default rule, the conclusion is a plausible inference, drawn in absence of complete information. Hence even if the body of the default rule is satisfied the conclusion is not certain and can be inferred with some level of uncertainty. In this scenario, the weight of the rule will become the epistemic state of the head and hence play the role of a marker, designating that the corresponding information is attained in presence of some uncertainty and is subject to change when more concrete evidence comes into account. For instance, in this framework the prototypical example of nonmonotonic reasoning concerning flying capability of birds can be modeled as follows:

$r1 : Fly(x) \quad [0.7, 1] \leftarrow Bird(x), notPenguin(x)$

$r2 : ¬Fly(x) \quad [1, 1] \leftarrow Penguin(x)$

$f1 : Bird(Tweety) \quad [1, 1].$

Now, when performing reasoning about flying ability of Tweety, it can be seen that we have no information about whether Tweety is a Penguin or not. Hence, $not\ Penguin(Tweety)$ is True, i.e. $[1, 1]$. From rule $r1$, $Fly(Tweety)$ is ascribed $[0.7, 1]$. Here lies the significant difference between using weighted rules and un-weighted rules as in classical ASP. In classical ASP, $Fly(Tweety)$ would become True and hence bearing no trace that the information has some underlying uncertainty because of having incomplete information about Tweety. On the other hand, in case of UnASP, the epistemic state $[0.7, 1]$ attached to $Fly(Tweety)$ will signify that it has some uncertainty.

• Another advantage of using weights, though not explored in this work, is variable weights, of the form $[\alpha, 1]$, can be attached to dispositions, such that the value $\alpha$ varies with the number of exceptions and thus providing additional control and flexibility of knowledge representation.
Fly(x) \overset{[\alpha,1]}{\leftarrow} Bird(x)

In general, as the uncertainty regarding the default rule increases the weight \(\alpha_r\) becomes a wider interval. Thus, using intervals of different widths, default rules can also be prioritized.

4 Declarative Semantics

The Atom Base (\(B_P\)) for a program \(P\) is the set of all grounded atoms and \(Lit_P\) is the set of all grounded literals i.e. \(Lit_P = \{a | a \in B_P\} \cup \{\neg a | a \in B_P\}\). When no specific program is in the context of discussion the only \(B\) and \(Lit\) is used to represent the atom base and set of literals of any arbitrary program.

For a UnASP program \(P\) an interpretation is a set of the form \(\{a : [x_1, x_2] | a \in Lit_P\} \cup \{\neg a : [y_1, y_2] \in I(L)\}\). In other words, an interpretation is a mapping from the set of grounded literals to the set of intervals \(I(L)\).

Cardinality of an interpretation, \(|\mathcal{I}|\), denoted by \(|\mathcal{I}|\), is the number of literals specified in \(\mathcal{I}\). An interpretation is partial if it specifies the epistemic states of some of the literals of \(Lit_P\) and doesn’t specify the others; i.e. if \(|\mathcal{I}| < |Lit_P|\). Otherwise, if \(|\mathcal{I}| = |Lit_P|\), then \(\mathcal{I}\) is said to be a total interpretation.

If \(a : [x_1, x_2] \in \mathcal{I}\), \(\mathcal{I}(a) = [x_1, x_2]\). For an interpretation \(\mathcal{I}\), \(\mathcal{I}^{\neg Lit} = \{l | l : \hat{\mathcal{I}} \in \mathcal{I} for some \hat{\mathcal{I}} \in I(L)\}\).

**Definition 2** An interpretation is said to be consistent if for any pair of complementary literals \(a : [x_1, x_2] \in \mathcal{I}\) and \(\neg a : [y_1, y_2] \in \mathcal{I}\),

i) \((x_2 - x_1) \neq (y_2 - y_1)\) or

ii) \((x_2 - x_1) = (y_2 - y_1)\) and \(x_1 + y_2 = 1\). In this case the corresponding interpretation is called strictly consistent interpretation.

Thus, for a strictly consistent interpretation \(\mathcal{I}\), if \(\mathcal{I}(a) = [x_1, x_2]\) then \(\mathcal{I}(\neg a) = [1 - x_2, 1 - x_1]\). Therefore for a strictly consistent interpretations of a program \(P\) it is sufficient to mention the truth status of positive atoms from \(B_P\) and values for the corresponding negated literals would be obtained by negating the values of corresponding atoms using strong negation; i.e. a strictly consistent total interpretation can be uniquely specified by its positive subset, that is the partial interpretation containing the positive atoms.

For any consistent interpretation which is not strictly consistent, any pair of literals \(a, \neg a\) are assigned epistemic states with different certainty degrees. In such a scenario \(a\) and \(\neg a\) are supposed to be completely independent of each other, i.e. two separate piece of information coming from
different sources. But since both assertions are concerned about a, whether positive or negative way, the assigned epistemic states must be merged into a single value; specifically for the purpose of query answering. Hence, strictly consistent interpretations are of practical importance and intuitively more attractive.

The knowledge ($\leq k_p$) ordering, that are defined over the set of intervals, can also be imposed on the set of interpretations to order them.

For two interpretations $\mathcal{I}$ and $\mathcal{I}^*$, $\mathcal{I} \leq_{k_p} \mathcal{I}^*$ iff $\forall a \in \text{Lit}, \mathcal{I}(a) \leq_{k_p} \mathcal{I}^*(a)$.

**Definition 3**

An interpretation $\mathcal{I}$ is said to be $k$-minimal of a set of interpretations $\Gamma$ if there is no other interpretation $\mathcal{I}^* \in \Gamma$ such that $\mathcal{I}^* \leq_{k_p} \mathcal{I}$. If the $k$-minimal interpretation is unique then it is called the $k$-least interpretation of $\Gamma$.

### 4.1 Satisfaction and Model:

**Definition 4**

Let $P$ be any program, $\rho$ be any rule in $P$ and $\mathcal{I}$ be an interpretation of $P$. Let $r: a \overset{\rho}{\leftarrow} b$ be a ground instance of $\rho$, then:

1. $\mathcal{I}$ satisfies $r$ iff i) $\mathcal{I}(a) \geq_{k_p} (\mathcal{I}(b) \land \alpha_r)$ and $\mathcal{I}(a) \geq_{t_p} (\mathcal{I}(b) \land \alpha_r)$ or $\mathcal{I}(a) >_{k_p} (\mathcal{I}(b) \land \alpha_r)$ or $\mathcal{I}(a) >_{t_p} (\mathcal{I}(b) \land \alpha_r)$, where, $\mathcal{I}(b)$ is obtained using the epistemic states of the components of $b$ as assigned by $\mathcal{I}$.

2. $\models_\mathcal{I} \rho$ iff $\mathcal{I}$ satisfies every ground instance of $\rho$.

3. $\mathcal{I}$ satisfies $P$, $\models_\mathcal{I} P$, i.e. $\mathcal{I}$ is a model of $P$, iff $\mathcal{I}$ satisfies every rule in $P$.

Hence, in simple words, a grounded rule is satisfied by an interpretation if the epistemic state of the head is at least as true and has equal level of certainty or strictly more certain as compared to the product of the epistemic state of the body and the weight of the rule.

A logic program may have several models; among which a unique model or a set of models are chosen as the preferred one.

**Definition 5**

A model $\mathcal{I}_P$ of a program $P$ is called supported iff:

i. For every grounded rule $r: a \overset{\rho}{\leftarrow} b$, such that $a$ doesn’t occur in the head of any other rule, $\mathcal{I}_P(a) = \mathcal{I}_P(b)$.

ii. For grounded rules $\{a \overset{\rho_1}{\leftarrow} b_1, a \overset{\rho_2}{\leftarrow} b_2, ..., a \overset{\rho_n}{\leftarrow} b_n\} \in P$ having same head $a$, $\mathcal{I}_P(a) = (\mathcal{I}_P(b_1) \land \alpha_1) \lor ... \lor (\mathcal{I}_P(b_n) \land \alpha_n)$.

iii. For literal $l \in \text{lit}$, and grounded rules $r_l: l \leftarrow b_l$, and $r_{-l}: -l \leftarrow b_{-l}$, in $P$, $\mathcal{I}_P(l) = \max_k \{\mathcal{I}_P(b_l), -\mathcal{I}_P(b_{-l})\}$ and $\max_k \{\mathcal{I}_P(b_l), -\mathcal{I}_P(b_{-l})\}$ exists in $I(L)$. Where, for any $x, y \in I(L) \max_k \{x, y\} = x$ if $y \leq_{k_p} x$. 


For the rule \( a \leftarrow [1, 1] [0, 1] \) and any interval \( \tau \in I(L) \), the strictly consistent interpretation \( \mathcal{I} = \{ a : \tau \} \) is a model for the rule. But the unique supported model is \( \{ a : [0, 1], \neg a : [0, 1] \} \).

For the rule \( a \leftarrow [1, 1] [0, 0] \) and any interval \( \tau \in I(L) \), the strictly consistent interpretation \( \mathcal{I} = \{ a : \tau \} \) is a model for the rule. But the unique supported model is \( \{ a : [0, 0], \neg a : [1, 1] \} \).

Supportedness of a model ensures that the epistemic state of the head of a rule is no more true than the epistemic state of the body and the degree of certainty of the head of a rule is not more than the certainty of the body. Thus while drawing inferences we do not infer truer or surer knowledge than is necessary to satisfy the rule.

Now a program may have more than one supported models.

**Proposition:** For a program, with no naf-literal, the unique \( k \)-minimal supported model is the unified answer set.

**Example 1:**
Let \( P_1 \) be

\[
\begin{align*}
  a & \leftarrow [1, 1] b \\
  b & \leftarrow [1, 1] a
\end{align*}
\]

Any strictly consistent interpretation of the form \( \mathcal{I} = \{ a : x, b : x \} \); such that \( x \in I(L) \), is a supported model of \( P_1 \). However, the unique answer set is \( \{ a : [0, 1], b : [0, 1], \neg a : [0, 1], \neg b : [0, 1] \} \), which is the unique \( k \)-minimal supported model.

**Example 2:**
Let the following rules comprise the program \( P_2 \):

\[
\begin{align*}
  r_1 & : a \leftarrow [0.7, 1] b \\
  r_2 & : \neg a \leftarrow [1, 1] c \\
  r_3 & : a \leftarrow [1, 1] [0.3, 0.5] \\
  f_1 & : c \leftarrow [1, 1] [1, 1] \\
  f_2 & : b \leftarrow [1, 1] \end{align*}
\]

This example is of particular importance to demonstrate the interaction of the epistemic state of an atom and its negative literal. Rules \( r_1,r_3 \) and fact \( f_2 \) evaluate the epistemic state of atom \( a \), which is obtained to be \( a : [0.7, 1] \). Rule \( r_2 \) and fact \( f_1 \) evaluate the epistemic state of the literal \( \neg a \), which happens to be \( \neg a : [1, 1] \). The second assertion for \( \neg a \) is more certain than the epistemic state for \( a \). Therefore the unique answer set (i.e. the \( k \)-minimal supported model) for the program is \( \{ a : [0, 0], \neg a : [1, 1], b : [1, 1], \neg b : [0, 0], c : [1, 1], \neg c : [0, 0] \} \).

**Definition 6** The \textbf{reduct} of a rule \( r : a \leftarrow b_1, b_2, ..., b_m, \not b_{m+1}, ..., \not b_n \) with respect to an interpretation \( \mathcal{I} \) is \( r^\mathcal{I} \) and is defined as:
\( r^3 : a \leftarrow b_1, b_2, ..., b_m, \exists (b_{m+1}), ..., \exists (b_n) \).

i.e. \( r^3 \) doesn’t contain any naf-literal in it.

The reduct of a program \( P \) (\( P^3 \)) is defined as:

\[
P^3 = \{ r^3 | r \in P \}.
\]

**Definition 7** For a UnASP program \( P \) an interpretation \( \exists \) is said to be its answer set if \( \exists \) is the k-minimal supported model of \( P^3 \).

A UnASP program may have zero, one or more answer sets.

**Example 3:**
Let program \( P_3 \) be:

\[
p \leftarrow \text{not } p
\]

The only answer set for \( P_3 \) is \( \exists_{P_3} = \{ p : [0, 0.5], \text{not } p : [0.5, 0.5] \} \), because the reduct of \( P^3_{P_3} \) becomes \( p \leftarrow [0.5, 0.5] \). Intuitively this means that \( p \) is absolutely in the midway of the [0, 1] scale, i.e. \( p \) is neither true nor false; which has the similar essence of its unique well-founded model assigning \( p \) unknown.

**Example 4:**
Let \( P_4 \) be:

\[
a \leftarrow \text{not } b
\]

\[
b \leftarrow \text{not } a
\]

For any \([x, x] \in I(L)\), such that \( x \in [0, 1] \) the strictly consistent interpretation \( \exists_{P_4} = \{ a : [x, x], b : [1-x, 1-x] \} \) is an answer set of \( P_4 \); since, \( \exists_{P_4} \) is unique k-minimal supported model of the reduct \( P^3_{P_4} = \{ a \leftarrow [1, 1], b \leftarrow [1, 1], [1-x, 1-x] \} \). No interval of the form \([x, y]\) where \( x \neq y \) is an answer set of \( P_4 \).

**Example 5:**
Program \( P_5 \) is as follows:

\[
r1 : a \leftarrow [1, 1], [0.9, 1]
\]

\[
r2 : \text{not } a \leftarrow [1, 1]
\]
No strictly consistent interpretation can satisfy both the rules (since $x_{1a} + x_{2a} = 0.9 + 1 = 1.9 \neq 1$). Thus $P_5$ has no answer set. Intuitively the program is inconsistent since it assigns intervals with very high truth degree to both $a$ and $\neg a$, which are complementary.

**Theorem 1**
Any normal UnASP program, i.e., program that does not contain any classically negated literal, is consistent and has at least one answer set.

**Proof:** It is to prove that for any UnASP program, without any classically negated literal, there is at least one interpretation which is k-minimal supported model of the program.

Let’s assume that $P_{pos}$ is a UnASP program containing no classically negated literal. Also assume that $P_{pos}$ doesn’t have supported model, i.e. for each model one of the three conditions of definition [4.1] fails.

For any rule $r_P$ in $P_{pos}$, for which the head atom, say ‘$a$’, doesn’t occur in any other rule of $P_{pos}$, absence of a supported model means, for every interpretation $\mathfrak{I}$, $\mathfrak{I}(r_{P-Body}) \notin I(L)$. The main connective of the rule body is t-norm ($\wedge$), and $\wedge$ is closed over $I(L)$. So for any chosen intervals of $I(L)$ chosen as epistemic states of the literals of the rule body, their t-norm gives an interval from $I(L)$. Therefore, there has to be some $x \in I(L)$ such that $\mathfrak{I}(r_{P-Body}) = x$ and $\mathfrak{I}$ can be the supported model for rule $r_P$ by making $\mathfrak{I}(a) = x$.

Now suppose, there are m rules $r_1, r_2, \ldots, r_m$ having same head $a_m$. From the above line of reasoning it can be said that using some interpretation $\mathfrak{I}$ the epistemic states of the bodies of each of the rules $r_1, r_2, \ldots, r_m$ can be obtained. Let these values are $x_1, x_2, \ldots, x_m$. Then $\mathfrak{I}$ would be supported model of $r_1, r_2, \ldots, r_m$ if the $\mathfrak{I}(a_m) = x_1 \vee x_2 \vee \ldots \vee x_m$. As $\vee$ is closed over $I(L)$ there always be an interval in $I(L)$ which is equal to $x_1 \vee x_2 \vee \ldots \vee x_m$ that can be assigned as epistemic state of $a_m$.

Now since there isn’t any classically negated literal in $P_{pos}$ there is no question of violating condition (iii) of definition [4.1] fails.

Hence our initial assumption that $P_{pos}$ doesn’t have any supported model is incorrect and $P_{pos}$ must have at least one supported model. If $P_{pos}$ has a unique supported model then it is the answer set. If more than one supported model is there, then there must be a k-least model or more than one k-minimal model. Thus, the program must have at least one answer set.

4.2 **Constraints:**

Some programs containing classical negation (¬) as well as negation-as-failure (not) give rise to unintuitive answer sets by assigning some specific
values to atoms which do not occur in the head of any rule, and hence ideally should have epistemic state $[0, 1]$. For instance, consider the following program:

$$P = \{ r_1 : a \leftarrow \text{not } b, r_2 : \neg a \leftarrow [0.6, 0.6] \}.$$  

The only supported interpretation satisfying the program is $\{ a : [0.4, 0.4], b : [0.6, 0.6] \}$. But this assignment of an epistemic value of high certainty to $b$ is unintuitive, since there is no rule with atom $b$ as its head, i.e., there is no source of information about $b$ is available. Intuitively the epistemic state of $b$ should be $[0, 1]$ and hence the program $P$ should have no answer set. This is ensured by introducing a constraint of the form

$$r_3 : b \leftarrow [0, 1]$$

to $P$, where rule $r_3$ stops assignment of any arbitrary epistemic state to $b$. Thus this type of constraints are added to the program for every atom that does not occur in the head of any rule.

5 Iterative approach to answer sets:

This section discusses an iterative approach for constructing the answer set of a restricted class of UnASP program.

5.1 The program transformation:

The first step of developing the iterative computation of the answer set of a program is to transform it in such a way that each atom appears in the head of at most one rule. For any program $P$ the corresponding transformed program is $P^*$.

**Definition 8**

For a program $P$ and a literal $l$ the $r$-join operator for $l$ is defined as:

$$r\text{-}join(l, P) = \bigvee\{ r_{\text{body}} \land \alpha_r | r \in P \text{ and } r_{\text{head}} = l \}$$

For an atom $a \in B_P$ the transformed rule in $P^*$ corresponding to $a$ becomes:

$$r^*(a) = r_{\text{-}join}(a, P) \otimes_k \neg r_{\text{-}join}(\neg a, P).$$

where, $\otimes_k$ is the knowledge aggregator operator which takes into account the interaction of epistemic states of an atom and its corresponding negated literal based on their certainty levels. $\otimes_k$ accounts for representing the nonmonotonic relation between an atom and its negation and is defined as follows:

**Definition 9**
For two intervals $x = [x_1, x_2]$ and $y = [y_1, y_2]$ in $I(L)$;

$$x \otimes_k y = \max_k \{x, y\}$$ if $(x_2 - x_1) \neq (y_2 - y_1)$ or $x = y$;

and $x \otimes_k y = [\xi, \xi]$ otherwise; where $\xi$ is a large negative number.

When for some epistemic states of $x$ and $y$, $x \otimes_k y$ is undefined, any arbitrary large negative value $\xi$ is assigned to $x \otimes_k y$, so that occurrence of this value would denote inconsistency.

Now, consider the transformation of a program $P$ containing the following rules:

- $r_1 : a \leftarrow^\alpha b \land d$
- $r_2 : a \leftarrow^\beta c \land e$
- $r_3 : \neg a \leftarrow^\gamma f \land g$
- $r_4 : \neg a \leftarrow^\delta k$

The rules are conjoined to form a single rule in the transformed program $P^*$ as follows:

$$r^* : a \leftarrow [(b \land d) \land \alpha] \lor [(c \land e) \land \beta] \otimes_k \neg [(f \land g) \land \gamma] \lor (k \land \delta)].$$

which can be further decomposed and simplified as combination of standard DNF and CNF as follows:

$$r^* : a \leftarrow [(b \land d \land \alpha) \lor (c \land e \land \beta)] \otimes_k [(\neg f \lor \neg g \lor \neg \gamma) \land (\neg k \lor \neg \delta)].$$

Each rule, $r^*$ in the transformed program, $P^*$, combines all those rules of the original program $P$ that have either a particular atom, say $a$ (by means of $r\_join(a)$) or its negated literal $\neg a$ in their heads(by means $r\_join(\neg a)$). One important point to note that the transformed rules in $P^*$ are not weighted rules any more, as a transformed rule collects weights of all associated rules of the original program $P$ in its body.

For each atom $a$ that does not occur in the head of any rule an extra rule of the form $a \leftarrow [0, 1]$ is added to the transformed program.

The notion of satisfaction of rules and model of a transformed program is defined in the same way as in Definition 4. Since there are no two rules containing same atom in their heads, condition (i) of the Definition 5 is sufficient for a model to be a supported model of the transformed program. The rest of the conditions of Definition 5 are taken care of in the construction of transformed rules. The reduct and answer set of a transformed program can be defined following Definition 6 and 7 respectively.

**Theorem 2**

Any interpretation that is a supported model of a UnASP program is also a supported model of the transformed program corresponding to that UnASP program.
Proof: For an interpretation to be a supported model of a UnASP program, it has to satisfy the three conditions of supportedness as given in Definition 4.1. The rules in the transformed program capture those three conditions in transformed rules. Now rules in the transformed program are not weighted and no two rules have the same literal as their heads. If an interpretation $\mathcal{I}$ is a supported model of the transformed program then for any rule $r^*_P$, $\mathcal{I}(r^*_P-Head) = \mathcal{I}(r^*_P-Body)$. Now according to the formation of the transformed program the this assignment of epistemic states satisfies all the conditions of supportedness, thus it is the supported model of the original program as well. Each supported model of the transformed program is also a supported model of the original program, vice versa.

Corollary: The answer sets of a UnASP program and its transformed program are the same, provided the program is consistent.

5.2 Iterative Computation of Answer Sets:

The iterative computation of answer sets is done on the transformed program.

5.2.1 Monotonic Iteration Stage (MI):

The first stage of computing the answer set is the monotonic iteration stage. Here an immediate consequence operator ($\Gamma$) is used, which asserts an epistemic state to an atom $a$, if there is a rule $r^*$ in $P^*$ with $a$ in its head and the body of $r^*$ is already evaluated.

In the transformed program each atom appears in the head of at most one rule. The iteration in the MI stage starts with the 'explicit' information in the program which is specified in terms of facts. As the iteration progresses, any atom, say $a$, will either remain unevaluated or will be assigned an epistemic state using the facts already known; and once an epistemic state is ascribed, it won’t change throughout the iterations. That means, once a piece of information is attained it is used to deduce further information and that knowledge is never retracted or withdrawn. Hence, the reasoning process is monotonic. Hence the stage is named as Monotonic Iteration Stage.

The monotonic iteration is based on an immediate consequence operator, $\Gamma$, which is a mapping from set of partial interpretations to set of partial interpretations and can be defined as:

Definition 10 For a partial interpretation $\mathcal{I}$ and a transformed program $P^*$,
\[ \Gamma_{P^*}(\mathcal{I}) = \mathcal{I} \cup \{ \hat{c} \mid a \in B_{P^*} \text{ and } \hat{c} \in I(L) \cup [\xi, \xi] \} \]

such that there is a rule \( r \in P^* \) and \( r_{\text{head}} = a \) and \( \forall l \in r_{\text{body}}, l \in \mathcal{I}^{\text{lit}} \) and \( \hat{c} = \mathcal{I}(r_{\text{body}}) \).

The monotonic iteration takes the empty interpretation, \( \mathcal{I}_{MI_0} = \Phi \) as the starting point and an upward iteration is performed on the grounded transformed program \( P^* \). However as the upward iteration progresses the program is not held constant, but it is modified or reduced in size using the information derived at each step (i.e., using those atoms that are assigned with epistemic states). Using the operator \( \Gamma \), more and more information is deduced and the set of facts grows larger. The iteration proceeds as follows:

1. \( \mathcal{I}_{MI_0} = \Phi \) and \( P_0 = P^* \);
2. If for some \( n > 0 \) there is some atom \( a \in B_{P_n} \) such that \( a : [\xi, \xi] \in \mathcal{I}_n \) then the iteration proceeds no further. Otherwise, \( \mathcal{I}_{MI_{n+1}} = \Gamma_{P_{n+1}}(\mathcal{I}_{MI_n}) \), where, \( P_{n+1} \) is obtained from \( P_n \) by modifying it based on \( \mathcal{I}_{MI_n} \) as follows:
   i. In \( P_n \), if each literal of the body of a rule \( r : a \leftarrow r_{\text{body}} \) can be evaluated using \( \mathcal{I}_{MI_n} \) then effectively rule \( r \) becomes of the form \( r : a \leftarrow \hat{c}, \) where \( \hat{c} \) is an element of \( I(L) \) then \( \mathcal{I}_{MI_{n+1}} = \mathcal{I}_{MI_n} \cup \{ a : \mathcal{I}_{MI_n}(r_{\text{body}}) \} \) and \( P_{n+1} = P_n \setminus r \).
   ii. If a literal \( l \) occurs in the body of a rule \( r \) of \( P_n \) and epistemic state of \( l \) can be evaluated using \( \mathcal{I}_{MI_n} \), then \( P_{n+1} = \{ P_n \setminus r \} \cup \{ r(l/\mathcal{I}_{MI_n}(l)) \} \), where, the notation \( r(l/\hat{c}) \) means the literal \( l \) in \( r \) is replaced be \( \hat{c} \).
   iii. Any rule of the form \( r : a \leftarrow b_1 \land \ldots \land b_n \land [1, 1] \) is replaced with \( r : a \leftarrow b_1 \land \ldots \land b_n \land [0, 0] \) is removed and \( a : [0, 0] \) is added to \( \mathcal{I}_{MI_{n+1}} \).

3. The iterations proceed until come to a fixpoint (The following theorem guarantees the existence of a fixpoint).

**Theorem 3**

For a consistent program the iterations in the MI stage terminates at a least fixpoint of \( \Gamma \).

**Proof:** The monotonic iteration starts with the empty interpretation (partial). As iteration progresses evaluation of some rules assign some epistemic states to some of the atoms, which are used for drawing further inferences. Once an epistemic state is assigned to some atom that epistemic state remains unaltered throughout the MI stage (since, each atom occurs in the head of at most one rule). Hence to investigate the monotonicity of the iterations we can just take into account \( \mathcal{I}_{Mi}^{\text{lit}} \) at each stage of iteration.
Initially $\mathcal{L}_{M_{[0]}} = \phi$. As iteration progresses, more and more atoms are added to $\mathcal{M}_{M_{[n]}}$. At any iteration stage $\mathcal{M}_{M_{[n]}}$ is a subset of atom base $B_{P^*}$.

From this perspective $\Gamma$ can be thought of a mapping from $2^{B_{P^*}}$ to $2^{B_{P^*}}$. Set of all subsets of $B_{P^*}$ forms a complete lattice under subset operation, $\subseteq$. Moreover it can be seen that $\mathcal{M}_{M_{[n]}}$ is monotonically increasing with $n$, i.e., for two interpretations $\mathcal{M}_{1}, \mathcal{M}_{2}$, for $\mathcal{M}_{1}^{\text{Lit}} \subseteq \mathcal{M}_{2}^{\text{Lit}}$, then for some consistent transformed program $P^{*}; \{\Gamma_{P^{*}}(\mathcal{M}_{1})\}^{\text{Lit}} \subseteq \{\Gamma_{P^{*}}(\mathcal{M}_{2})\}^{\text{Lit}}$.

Therefore, from the above line of reasoning $\Gamma$ can be viewed as a mapping from the complete lattice $(2^{B_{P^*}}, \subseteq)$ to itself and the operator $\Gamma$ is monotonic.

Moreover, $\Gamma$ is continuous, i.e. for any chain $\mathcal{M}_{1} \subseteq \mathcal{M}_{2} \subseteq \ldots \subseteq \mathcal{M}_{n}$, $\bigcup_{i=1,n} \Gamma(\mathcal{M}_{i}) = \Gamma(\bigcup_{i=1,n} \mathcal{M}_{i})$. It is evident since $(\bigcup_{i=1,n} \mathcal{M}_{i}) = \mathcal{M}_{n}$ and the monotonicity of $\Gamma$ leads to $\Gamma(\mathcal{M}_{1}) \subseteq \Gamma(\mathcal{M}_{2}) \subseteq \ldots \subseteq \Gamma(\mathcal{M}_{n})$ and thus $\bigcup_{i=1,n} \Gamma(\mathcal{M}_{i}) = \Gamma(\mathcal{M}_{n})$.

Hence from Knaster and Tarski’s fixpoint theorem $\Gamma$ will have a least fixpoint and the iteration in the monotonic iteration stage would halt at this least fixpoint. The final interpretation $\mathcal{M}_{M_{\infty}}$ and the corresponding final program $P_{M}$ are the output of the Monotonic Iteration stage and they are subjected to further modifications in next stage.

**Example 6:** Let $P$ be a grounded logic program (the propositional atoms are actually short-hand representation of grounded predicate atoms):

$$\{r1 : p \leftarrow [0.7,1], q, \neg s, \quad r2 : p \leftarrow [0.3,0.3], r, \neg t, \quad r3 : p \leftarrow [1,1], s, \quad r4 : p \leftarrow [1,1], t, \quad r5 : m \leftarrow [0.6,0.8], n, \quad r6 : s \leftarrow m, \quad r7 : a \leftarrow [1,1], b, p, \quad r8 : b \leftarrow [1,1], s, \quad r9 : b \leftarrow [1,1], g, \quad r10 : d \leftarrow [1,1], \neg g, c, \quad r11 : e \leftarrow [1,1], d, w, \quad r12 : f \leftarrow [1,1], \neg e, \quad r13 : g \leftarrow [1,1], c, f, \quad r14 : c \leftarrow [1,1], h, \quad r15 : h \leftarrow [0.7,1], k, \quad r16 : i \leftarrow [1,1], \neg h, \quad r17 : k \leftarrow [1,1], \neg j, \quad r18 : j \leftarrow [0.8,0.8], i, \quad r19 : j \leftarrow [1,1], s, \quad r20 : x \leftarrow [0.2,0.3], v, \quad r21 : v \leftarrow [1,1], x, \quad r22 : v \leftarrow [1,1], \neg u, \quad r23 : w \leftarrow [1,1], x, \quad r24 : u \leftarrow [0.5,0.8], w, \quad r25 : y \leftarrow [1,1], \neg z, \quad r26 : z \leftarrow [1,1], \neg y, \quad r27 : l \leftarrow [0,0.4,0.6], z, \quad f1 : q \leftarrow [0.7,0.7], \quad f2 : r \leftarrow [0.5,0.5], \quad f3 : n \leftarrow [0.7,0.9]\}$$

The corresponding transformed program is $P^{*}$ as follows:

$$\{r1^{*} : p \leftarrow ((q \land \neg s \land [0.7,1]) \lor (r \land \neg t \land [0.3,0.3])) \otimes k (\neg s \land \neg t), \quad r2^{*} : m \leftarrow n \land [0.6,0.8], \quad r3^{*} : s \leftarrow m, \quad r4^{*} : a \leftarrow b \land p, \quad r5^{*} : b \leftarrow \neg a \land g, \quad r6^{*} : d \leftarrow \neg g \land a, \quad r7^{*} : e \leftarrow d \land w, \quad r8^{*} : f \leftarrow \neg e, \quad r9^{*} : g \leftarrow \neg c \land f, \quad r10^{*} : c \leftarrow h, \quad r11^{*} : p \leftarrow \neg c, \quad r12^{*} : q \leftarrow \neg t, \quad r13^{*} : r \leftarrow \neg s, \quad r14^{*} : s \leftarrow r, \quad r15^{*} : t \leftarrow s, \quad r16^{*} : u \leftarrow t, \quad r17^{*} : v \leftarrow u, \quad r18^{*} : w \leftarrow v, \quad r19^{*} : x \leftarrow w, \quad r20^{*} : y \leftarrow x, \quad r21^{*} : z \leftarrow y, \quad r22^{*} : a \leftarrow z, \quad r23^{*} : b \leftarrow a, \quad r24^{*} : c \leftarrow b, \quad r25^{*} : d \leftarrow c, \quad r26^{*} : e \leftarrow d, \quad r27^{*} : f \leftarrow e, \quad r28^{*} : g \leftarrow f, \quad r29^{*} : h \leftarrow g\}$$
r11* : \( h \leftarrow k \wedge [0.7, 1] \),  
\[ r12^* : i \leftarrow \neg h, \quad r13^* : k \leftarrow \neg j, \]
\[ r14^* : j \leftarrow ([0.8, 0.8] \wedge i \wedge \neg s) \otimes_k \neg s, \]
\[ r15^* : x \leftarrow v \wedge [0.2, 0.3], \quad r16^* : v \leftarrow x \vee \neg u, \quad r17^* : w \leftarrow x, \]
\[ r18^* : u \leftarrow w \wedge [0.5, 0.8], \quad r19^* : y \leftarrow \neg z, \quad r20^* : z \leftarrow \neg y, \]
\[ r21^* : l \leftarrow z \wedge [0.4, 0.6], \]
\[ f1^* : q \leftarrow [0.7, 0.7], \quad f2^* : r \leftarrow [0.5, 0.5], \quad f3^* : n \leftarrow [0.7, 0.9], \]
\[ f4^* : t \leftarrow [0, 1]. \]

The monotonic Iterations progress on the transformed program \( P^* \) as follows:

1. \( \mathcal{Z}_{M_0} = \Phi, P_0 = P^* \)

2. \( \mathcal{Z}_{M_1} = \Gamma(\mathcal{Z}_{M_0}) \)
\[ = \{ q : [0.7, 0.7], r : [0.5, 0.5], n : [0.7, 0.9], t : [0, 1] \}. \]
\[ P_1 = P^*. \]

3. \( \mathcal{Z}_{M_2} = \Gamma(\mathcal{Z}_{M_1}) \)
\[ = \mathcal{Z}_{M_1} \cup \{ m : [0.42, 0.56] \}. \]
\[ P_2 = P_1 \setminus \{ r1^*, r2^*, r3^* \} \cup \{ r1^*(q/[0.7, 0.7], r/[0.5, 0.5], t/[0, 1]), r3^*(m/[0.42, 0.56]) \}. \]
where,
\[ r1^*(r/[1, 1], t/[0, 1]) : p \leftarrow ((([0.49, 0.7] \wedge \neg s) \lor [0.15, 0.15]) \otimes_k (s \lor [0, 1]), \]
\[ r3^*(m/[0.42, 0.56]) : s \leftarrow [0.42, 0.56]. \]

4. \( \mathcal{Z}_{M_3} = \Gamma(\mathcal{Z}_{M_2}) \)
\[ = \mathcal{Z}_{M_2} \cup \{ s : [0.42, 0.56] \}. \]
\[ P_3 = P_2 \setminus \{ r1^*, r3^*, r14^* \} \cup \{ r1^*(s/[0.42, 0.56]), r14^*(s/[0.42, 0.56]) \}. \]
where,
\[ r1^*(s/[0.42, 0.56]) : p \leftarrow ((([0.49, 0.7] \wedge [0.58, 0.58]) \lor [0.15, 0.15]) \otimes_k \
(0.42, 0.56] \lor [0, 1]), \]
\[ r14^*(s/[0.42, 0.56]) : j \leftarrow ([0.8, 0.8] \wedge i \wedge [0.58, 0.58]) \otimes_k [0.44, 0.58]. \]

5. \( \mathcal{Z}_{M_4} = \Gamma(\mathcal{Z}_{M_3}) \)
\[ = \mathcal{Z}_{M_3} \cup \{ p : [0.3916, 0.495] \}. \]
\[ P_4 = P_3 \setminus \{ r1^*, r4^* \} \cup \{ r4^*(p/[0.3916, 0.495]) \}. \]
where,
\[ r4^*(p/[0.928, 0.941]) : a \leftarrow b \wedge [0.3916, 0.495]. \]

6. \( \mathcal{Z}_{M_5} = \Gamma(\mathcal{Z}_{M_4}) = \mathcal{Z}_{M_4} \)
The MI stage stops here, because further iterations doesn’t modify the program, nor the deduced set of facts is modified.

\[ \mathcal{M}_{I\infty} = \mathcal{M}_{I4} = \{ q : [0.7, 0.7], r : [1, 1], n : [0.7, 0.9], t : [0, 1], m : [0.42, 0.56], s : [0.42, 0.56], p : [0.3916, 0.495] \} \]

\[ P_{MI} = P_{4} = \{ r_{1MI} : a \leftarrow b \land [0.3916, 0.495], \quad r_{2MI} : b \leftarrow \text{not } a \lor g, \]
\[ r_{3MI} : d \leftarrow \neg g \land a, \quad r_{4MI} : e \leftarrow d \land w, \]
\[ r_{5MI} : f \leftarrow \text{not } e, \quad r_{6MI} : g \leftarrow \neg c \land f, \]
\[ r_{7MI} : c \leftarrow h, \quad r_{8MI} : h \leftarrow k \land [0.7, 1], \]
\[ r_{9MI} : i \leftarrow \text{not } h, \quad r_{10MI} : k \leftarrow \neg j, \]
\[ r_{11MI} : j \leftarrow ([0.464, 0.464] \land i) \otimes k [0.44, 0.58], \]
\[ r_{12MI} : x \leftarrow v \land [0.2, 0.3], \]
\[ r_{13MI} : v \leftarrow x \lor \neg u, \quad r_{14MI} : w \leftarrow x, \]
\[ r_{15MI} : u \leftarrow w \land [0.5, 0.8], \quad r_{16MI} : y \leftarrow \text{not } z, \]
\[ r_{17MI} : z \leftarrow \text{not } y, \quad r_{18MI} : l \leftarrow z \land [0.4, 0.6] \}. \]

The output of the Montonic Iteration is a reduced program, where, the rules already evaluated are absent. This reduced program \( P_{MI} \) consists of rules where the epistemic state of any atom is by some way self dependent or dependent on epistemic state of some self assessing atoms. So to evaluate the epistemic states of atoms in the head of such rules 'Guess and Check' approach is taken. An initial guess for the epistemic states is made and then those epistemic states are put in the bodies of rules and the heads are evaluated. If the evaluated epistemic states match with the guessed epistemic states then the guess was a stable valuation. This stable valuation is to be obtained by using iterations starting from an initial guess, \([0, 1]\), which signifies that nothing is known about the corresponding atom. This iteration continues until the evaluated epistemic state is "sufficiently close" to the guessed interval.

Next subsections describe the iteration method, its analysis and sufficient condition for the convergence of iterations to a fixed-point.

### 5.2.2 Programs represented by Dependency Graph and Program Splitting:

To extract information about the interrelation of rules in \( P_{MI} \), detection of self assertive set of rules (i.e. cycles) and their connections, Dependency
Graph (DG) is constructed from $P_{MI}$. It is a directed graph with weighted edges.

**Definition 11**
A dependency Graph (DG) for a transformed program $P$ is defined as a triplet,

$$DG = < V, E, W >$$

where,

$V$ is set of vertices; $V = \mathbf{B}_P \cup I(L) \cup \{\land, \lor, \otimes_k\}$

$E = \{< p, q > | p, q \in V \}$. An edge can be between two literals or between a literal and a logical connective (i.e. $\lor, \land, \otimes_k$) depending on the rules in the program $P$.

$W$ is a *partial mapping* from the set of edges $E$ to the set $\{-1, \neg\}$ such that,

- $W(e) = -1$ if $e \in E$ and $e = < p, q >$ such that $e$ connects a literal $\neg p$ to another vertex $q$;
- $W(e) = \neg$ if $e \in E$ and $e = < p, q >$ such that $e$ connects a literal $\neg p$ to another vertex $q$;
- $W(e)$ is unspecified otherwise.

The dependency graph of the program $P_{MI}$ is shown in figure 3. Let we call this graph $G$ for further references.

On the dependency graph $G$ Kosaraju’s algorithm is applied to detect the *Strongly-Connected Components* in it. Then the Component Graph ($G^{SCC}$) is constructed by merging all strongly connected nodes into a single node. This $G^{SCC}$ is a Directed Acyclic Graph (DAG). The nodes in $G^{SCC}$ are sorted using a Topological Sort, which actually determines the interdependencies of rules in the program $P_{MI}$ and determines a sequence in which the calculation of epistemic states of atoms would proceed.

From DG $G$ the component graph $G^{SCC}$ is shown in figure 4. One of the many possible topologically sorted version of $G^{SCC}$ is

$$hijk, uvwx, c, gbadf, yz, l$$

ignoring the constant nodes (e.g. [.44, .58]) and nodes for operators (e.g. $\land$). Each node in the topologically sorted list is either a singleton, i.e., a constant or connective, or an SCC with multiple nodes. As can be seen from figure 3 that each non-singleton SCC may be composed of a simple cycle or may contain many interconnected simple cycles.
Figure 3: Dependency graph of the program $P_{MI}$
5.2.3 Nonmonotonic iterations

In the Monotonic Iteration stage epistemic states of some of the atoms (and literals) are evaluated using the given facts. Epistemic states of the rest of the atoms have to be evaluated by iterations, which will be referred to as Nonmonotonic Iteration (NMI), starting from the initial epistemic value \([0, 1]\). This iterative evaluation progresses according to the topologically sorted sequence of "program segments", represented by individual SCCs.

Program segment represented by a particular SCC is denoted by \(P_{scc}\). Instead of initializing all the atoms in \(P_{scc}\) with \([0, 1]\) and comparing their evaluated values for checking convergence, some 'special' atoms are chosen from the set of atoms in \(B_{P_{scc}}\) to assign with \([0, 1]\) and the iterations are observed from the perspective of these 'special' atoms only. Once stable valuation for these atoms are attained, the stable epistemic states of the rest of the atoms in \(B_{P_{scc}}\) can be calculated. The set of these chosen 'special' atoms is called the Assumption Set.

**Definition 12**

An Assumption Set for a set of rules or program \(P\), denoted by \(AsP\), is the set of atoms such that the epistemic states of all other atoms in the set of rules can be uniquely determined if the epistemic state of the elements of the assumption set is specified. Each element in the assumption set is called a chosen element or a chosen atom.
The assumption set is the set of atoms using which the interconnected cycles of the SCC can be 'unfurled' into linear forward paths, along which the computation progresses and epistemic states of the chosen elements propagates through the SCC and other atoms are evaluated. This unfurled graph obtained from the SCC is called the value-propagation graph (vpg). This vpg provides a way to mathematically analyse the nature of NMI iterations and investigate the terminating conditions. The assumption set has to be chosen in such a way so that the resulting vpg is cycle-free; wrong choice may lead to backward paths in the vpg, which is unwanted.

Rules for constructing the assumption set of a program given its dependency graph:

Given a program segment which corresponds to a non-singleton SCC in the dependency graph, Johnson's Algorithm is applied to detect simple cycles in that SCC. The set of simple cycles in the SCC as obtained by Johnson’s algorithm is denoted by \( J_{scc} \). With the set of atoms in the SCC, i.e. \( B_{P_{SCC}} \), and the set of simple cycles, \( J_{scc} \), an intersection table is constructed with elements of \( B_{P_{SCC}} \) as column heads and elements of \( J_{scc} \) as row heads and putting a tick (✓) in the cell \((a, C)\), if the atom \(a\) occurs in the cycle \(C\). If an atom, occurring in the \(i^{th}\) column is chosen then all the cycles that have a ✓ in the \(i^{th}\) column will be unfurled in the vpg. So, in order to construct the assumption set the set of atoms is picked up based on fulfillment of the following criteria:

1. the set of chosen atoms has to be such that all the cycles from \( J_{scc} \) are unfurled in vpg.
2. for each chosen element, there has to be a cyclic path in the SCC starting and ending in the chosen atom and that does not include any other chosen element.
3. the assumption set has to be as small as possible set. (the justification for minimality is explained in section 5.2.5)
4. as will be demonstrated in the next subsection that the choice of assumption set influences the mathematical treatment of the NMI iteration. Therefore assumption set will have to be chosen in such a way that the sufficient condition of convergence of iterations is met. This criteria is fully discussed in section 5.2.5.

Condition 2, mentioned above, ensures that for each chosen element, say \(a\), there is a path from \(a_{n-1}\) to \(a_{n}\) in the vpg that is obtained by unfurling all the cycles from the SCC that start and end at \(a\) and doesn’t have any other chosen element on it. Such a path is called value-propagation path (vpp) of \(a\).

Clearly, the assumption set for any program may not be unique.
The dependency graph and concept of studying iterations through elements of the assumption set offers a new perspective to study the nature and convergence conditions of iterations.

**NMI Iteration:**

Based on the concept of assumption set for a set of rules or program segment $P$, the steps of NMI iterations are as follows:

1. The iteration starts with an interpretation $\mathcal{I}^0_{NMI} = \{a : [0,1]|a \in As_P\}$.

2. Having $\mathcal{I}^n_{NMI}$, $\mathcal{I}^{n+1}_{NMI}$ is obtained in two steps.
   - i. $P$ is modified to construct $P(\mathcal{I}^n_{NMI})$, such that for any rule $r \in P$ and for any $a \in As_P$ if $a \in r_{Body}$ it is replaced with $\mathcal{I}^n_{NMI}(a)$.
   - ii. Monotonic iterations are performed on $P(\mathcal{I}^n_{NMI})$ using the immediate consequence operator $\Gamma$ (Definition 5.2.1); with $\mathcal{I}_0 = \Phi$; $\mathcal{I}_m = \Gamma_P(\mathcal{I}^n_{NMI})(\mathcal{I}_{m-1})$, for some $m \in \mathbb{N}$ until a fixpoint $\mathcal{I}_\infty$ is obtained.

$$\mathcal{I}^{n+1}_{NMI} = \{a : \mathcal{I}_\infty(a)|a \in As_P\}.$$

3. For each element $a = [a_1,a_2] \in AS_P$, $|\mathcal{I}^n_{NMI}(a) - \mathcal{I}^{n+1}_{NMI}(a)|$ can be represented by a vector of $D_a = [|a_{1n+1} - a_{1n}| \ |a_{2n+1} - a_{2n}|]$. Step 2 is repeated until $\forall a \in AS_P; ||D_a||_\infty < \epsilon$ for some pre-decided $\epsilon > 0$, i.e. difference in magnitude of both of the upper bound and the lower bound is less than $\epsilon$.

**Example 7:** Let us consider the program segment, whose dependency graph and the corresponding intersection table are shown in Figure 5. The corresponding set of rules are:

$P_{Ex.7} = \{a \leftarrow c \land [0.6,0.8];
\quad c \leftarrow b;
\quad b \leftarrow \lnot a \lor g;
\quad d \leftarrow \lnot g \land a;
\quad e \leftarrow d \land [0.9,1];
\quad f \leftarrow \lnot e;
\quad g \leftarrow f \land [0.3,0.7]\}.$

Say the chosen assumption set is $As_{P_{Ex.7}} = \{a, g\}$. It is clear from the intersection table that these two chosen elements cover all the cycles present,
**Simple Cycles |**

|        | a | b | c | d | e | f | g |
|--------|---|---|---|---|---|---|---|
| abc    | ✓ |   | ✓ |   |   |   |   |
| adefgbca| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| gdefg   | ✓ | ✓ | ✓ |   |   |   |   |

Figure 5: Dependency graph of $P_{Ex.2}$ and the Intersection table

as can be seen in the vpg in Figure 6. The vpg shows how the epistemic states of the chosen elements at the $n^{th}$ iteration can be obtained from the epistemic states of the $(n - 1)^{th}$ iteration. Say $\epsilon$ is chosen to be 0.009

1. $\mathcal{NMI}_0 = \{a : [0, 1], g : [0, 1]\}$;
2. $P_1(\mathcal{NMI}_0) = \{a \leftarrow c \land [0.6, 0.8];
   e \leftarrow b;\}
   b \leftarrow \text{not} [0, 1] \lor [0, 1];
   d \leftarrow \neg [0, 1] \land [0, 1];
   e \leftarrow d \land [0.9, 1];
   f \leftarrow \neg e;\}
   g \leftarrow f \land [0.3, 0.7];\}$

The monotonic iteration using $\Gamma$ goes as follows:

$\mathcal{NMI}_{0_1} = \Gamma_{P_{Ex.2}(\mathcal{NMI})}(\Phi) = \{b : [1, 1], d : [0, 1]\}$

$\mathcal{NMI}_{0_2} = \Gamma_{P_{Ex.2}(\mathcal{NMI})}(\mathcal{NMI}_{0_1}) = \mathcal{NMI}_{0_1} \cup \{c : [1, 1], e : [0, 1]\}$

$\mathcal{NMI}_{0_3} = \Gamma_{P_{Ex.2}(\mathcal{NMI})}(\mathcal{NMI}_{0_2}) = \mathcal{NMI}_{0_2} \cup \{a : [0.6, 0.8], f : [0, 1]\}$

$\mathcal{NMI}_{0_4} = \Gamma_{P_{Ex.2}(\mathcal{NMI})}(\mathcal{NMI}_{0_3}) = \mathcal{NMI}_{0_3} \cup \{g : [0, 0.7]\}$

$\mathcal{NMI}_{0_\infty} = \mathcal{NMI}_{0_4}$.

$\mathcal{NMI}_{NMI} = \{a : [0.6, 0.8], g : [0, 0.7]\}$.
Figure 6: Value Propagation Graph for $P_{Ex.2}$

Since $||D_a||_\infty = max\{0.6, 0.2\} = 0.6 > \epsilon$ and $||D_g||_\infty = max\{0, 0.3\} = 0.3 > \epsilon$; the iteration goes on.

3. The immediate consequence operator $\Gamma$ iterated on $P_1(\mathcal{S}_{NMI}^0)$ proceeds as follows:

$$\mathcal{S}_{11} = \Gamma_{P_{Ex.2}(\mathcal{S}_{NMI}^1)}(\Phi) = \{b : [0.4, 0.82], d : [0.18, 0.8]\};$$

$$\mathcal{S}_{12} = \Gamma_{P_{Ex.2}(\mathcal{S}_{NMI}^1)}(\mathcal{S}_{11}) = \mathcal{S}_{11} \cup \{c : [0.4, 0.82], e : [0.162, 0.8]\};$$

$$\mathcal{S}_{13} = \Gamma_{P_{Ex.2}(\mathcal{S}_{NMI}^1)}(\mathcal{S}_{12}) = \mathcal{S}_{12} \cup \{a : [0.24, 0.656], f : [0.2, 0.838]\};$$

$$\mathcal{S}_{14} = \Gamma_{P_{Ex.2}(\mathcal{S}_{NMI}^1)}(\mathcal{S}_{13}) = \mathcal{S}_{13} \cup \{g : [0.06, 0.5866]\};$$

$$\mathcal{S}_{1\infty}^1 = \mathcal{S}_{14}. $$

$$\mathcal{S}_{NMI}^2 = \{a : [0.24, 0.656], g : [0.06, 0.5866]\}; \text{ and } ||D_a||_\infty = max\{0.36, 0.144\} = 0.36 > \epsilon \text{ and } ||D_g||_\infty = max\{0.06, 0.1134\} = 0.1134 > \epsilon$$

The modification of the assumption set after each stage of NMI iteration, until the termination condition is satisfied, is as follows:

$$\mathcal{S}_{NMI}^3 = \{a : [0.46464, 0.72063], g : [0.11501, 0.63749]\};$$

$$\mathcal{S}_{NMI}^4 = \{a : [0.35328, 0.66525], g : [0.10868, 0.59389]\};$$

$$\mathcal{S}_{NMI}^5 = \{a : [0.41107, 0.68522], g : [0.12211, 0.60961]\};$$

$$\mathcal{S}_{NMI}^6 = \{a : [0.38348, 0.67162], g : [0.11954, 0.5989]\};$$

$$\mathcal{S}_{NMI}^7 = \{a : [0.39742, 0.67695], g : [0.1226, 0.6031]\};$$

$$\mathcal{S}_{NMI}^8 = \{a : [0.39078, 0.67381], g : [0.12181, 0.60063]\};$$

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The iteration is terminated here. The termination of the iterations is decided solely by the intended accuracy of computation. The choice of \( \epsilon = 0.009 \) the accuracy up to 2\(^{nd} \) decimal point is ensured. The monotonic iteration of \( \Gamma \) on \( P_{SCC}(\mathcal{I}_N^{MNI}) \) which gives the epistemic states of all other atoms proceeds as follows:

\[
\begin{align*}
\mathcal{I}_1 &= \Gamma_{P_{E.x.2}(\mathcal{I}_N^{MNI})}(\Phi) = \{b : [0.65682, 0.84393], d : [0.15607, 0.59173]\}; \\
\mathcal{I}_2 &= \Gamma_{P_{E.x.2}(\mathcal{I}_N^{MNI})}(\mathcal{I}_1) = \mathcal{I}_1 \cup \{c : [0.65682, 0.84393], e : [0.140463, 0.59173]\}; \\
\mathcal{I}_3 &= \Gamma_{P_{E.x.2}(\mathcal{I}_N^{MNI})}(\mathcal{I}_2) = \mathcal{I}_1 \cup \{a : [0.39409, 0.67514], f : [0.40827, 0.85954]\}; \\
\mathcal{I}_4 &= \Gamma_{P_{E.x.2}(\mathcal{I}_N^{MNI})}(\mathcal{I}_3) = \mathcal{I}_1 \cup \{g : [0.12248, 0.60168]\}; \\
\mathcal{I}_\infty &= \mathcal{I}_4.
\end{align*}
\]

Therefore, considering up to 2\(^{nd} \) decimal place and rounding off the values the solution becomes \( \{a : [0.39, 0.67], b : [0.66, 0.84], c : [0.66, 0.84], d : [0.16, 0.52], e : [0.14, 0.52], f : [0.48, 0.86], g : [0.12, 0.60]\} \), which is the intended answer set.

Thus NMI iteration comprises of several stages of monotonic iterations using the immediate consequence operator \( \Gamma \).

### 5.2.4 Mathematical analysis of NMI iterations:

A proper mathematical model is needed to study the convergence of iterations described in previous subsection. The assumption set for a particular program segment (represented by an SCC) gives a way to "unfurl" the interconnected cycles into linear paths along which the computation progresses and the epistemic states of atoms in the assumption set propagates from one iteration step to the next one. The value propagation graph enables us to describe the NMI iteration process by means of a system of difference equations in terms of the atoms in \( As_{P_{E.x.2}} \).

**Definition 13**

A difference equation of order \((k + 1)\) is an equation of the form

\[
x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), n = 0, 1, ...
\]

where \( f \) is a continuous function from \( D^{k+1} \rightarrow D \), for some domain \( D \).

**Example 7 (contd)** From the value propagation graph shown in Figure 6 the following equations can be written.

\[
\begin{align*}
[a_{1_n}, a_{2_n}] &= [0.6, 0.8] \land [c_{1_n}, c_{2_n}] = [0.6c_{1_n}, 0.8c_{2_n}] ; \\
[c_{1_n}, c_{2_n}] &= [b_{1_n}, b_{2_n}] ; \\
[b_{1_n}, b_{2_n}] &= not [a_{1(n-1)}, a_{2(n-1)}] \lor [g_{1(n-1)}, g_{2(n-1)}] ; \\
\end{align*}
\]

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The assumption set is important in order to attain some well-behaved iteration on the assumption set, is not unique for an SCC. Therefore, proper selection of iteration function then $\bar{a}$, has reached to a point $\bar{a}$, linear difference equations.

One point should be noted that, the iteration function, being dependent system of first order non-linear difference equations. Hence, $NMI$ iterations can be described by a system of first order non-linear difference equations. Termination of $NMI$ iterations means that the iteration has reached to a point $\bar{x} \in [0,1]^4$ such that

$$\bar{x} = \bar{F}^{n}(\bar{x}).$$

Thus, the problem of studying the convergence of the $NMI$ iteration reduces to the problem of finding the iterative fixed-point for the system of difference equations.

On a generalized setting, suppose for some program $P$, $AsP = \{a_1, a_2, ..., a_m\}$, then $\tilde{a} = [a_1, a_{12} ... a_m]^{T}$ and $\bar{F}$ is a mapping from $[0,1]^{2m} \rightarrow [0,1]^{2m}$, given by $\bar{F} = [f_1(\tilde{a}) f_2(\tilde{a}) ... f_{2m}(\tilde{a})]^{T}$ and $\tilde{a}^{n+1} = \bar{F}(\tilde{a}^n)$.

The mapping $\bar{F}$, which is the set of functions describing the iteration, is called the iteration function of $P$.

One point should be noted that, the iteration function, being dependent on the assumption set, is not unique for an SCC. Therefore, proper selection of assumption set is important in order to attain some well-behaved iteration function that is convergent.
5.2.5 Convergence conditions:

Sufficient conditions for convergence of the iterative functions for any starting epistemic states are investigated in this subsection.

1. Programs without not and $\otimes_k$:

Lemma 1

If a program segment $P$ doesn’t contain negation-as-failure and the knowledge aggregator operation ($\otimes_k$) then for any atom $a = [a_1, a_2] \in AS_P$ for any $n > 0$ at the $n^{th}$ and $(n+1)^{th}$ steps of NMI iteration $a_1 \leq a_{1(n+1)} \leq a_{2(n+1)} \leq a_2$, with $a_0 = [a_1^0, a_2^0] = [0, 1]$.

Proof: This would be proved using Induction.

Base Case: $n = 0$.

Elements of assumption set are initiated to $[0, 1]$. Hence, for any atom $a = [a_1, a_2] \in AS_P$ $a_0 = [a_1^0, a_2^0] = [0, 1]$. Each of the initial epistemic states iterates over the program and the initial assigned epistemic state propagates through paths of the value propagation graph and gets modified by conjunction and disjunction with some values in $I(L)$ and gets negated by classical negation. For any operation in the propagation path, it is evident $0 \leq a_{11} \leq a_{21} \leq 1$, i.e., $a_{10} \leq a_{11} \leq a_{21} \leq a_{20}$.

Induction Hypothesis: Suppose for the $(m-1)^{th}$ and $m^{th}$ iterations, with $m > 1$, for any atom $a \in AS_P$,

$$a_{1(m-1)} \leq a_1 \leq a_{2m} \leq a_{2(m-1)} \ldots$$

Then it is to be proved, $\forall a \in AS_P, a_{1m} \leq a_{1(m+1)} \leq a_{2(m+1)} \leq a_{2m}$.

For any atom $a \in AS_P$, $a_{m+1}$ is obtained from $a_m$ by passing through the path for $a$ in the value propagation graph. Typically any such path would look like as shown in Figure 7.

where, $[x_1, x_2]$ and $[y_1, y_2]$ (and all other conjuncts or disjuncts not shown in the figure) can be fixed intervals from $I(L)$, which remain constant for all iterations, or they can be some atoms from $AS_P$ with their value at the $m^{th}$ iteration. So, in any case, generally, it can be written from the induction hypothesis.
\[ x_{1,m-1} \leq x_{1,m} \leq x_{2,m} \leq x_{2,m-1}, \quad y_{1,m-1} \leq y_{1,m} \leq y_{2,m} \leq y_{2,m-1}, \ldots \] (ii)

In other words, in terms of ordering intervals based on the Bilattice-based triangle [1], it can be said from equations (i) and (ii) that:

\[ a_{m-1} \leq_k a_m, \quad x_{m-1} \leq x_m \quad \text{and} \quad y_{m-1} \leq y_m, \ldots \] (iii)

Now, all the logical operators involved in the value propagation path of \( a \), i.e., \( \land \), \( \lor \) and \( \neg \) are monotonic with respect to \( \leq_k \). That is to say, from equation (iii) the following can be said:

\[ a_{m-1} \land x_{m-1} \leq_k a_m \land x_m \Rightarrow b_m \leq_k b_{m+1}; \]

Moreover,

\[ b_{m-1} \lor y_{m-1} \leq_k b_m \lor y_m \Rightarrow c_m \leq_k c_{m+1}; \]

Also, \( \neg c_m \leq_k \neg c_{m+1}. \)

Proceeding this way along the value propagation path of \( a \) it is obtained that, \( a_m \leq_k a_{m+1}. \)

Therefore, following the principle of induction the theorem is proved for any atom \( a \in As_P \) for any \( n > 0. \)

**Theorem 4**

If a program segment doesn’t contain negation-as-failure and the knowledge aggregator operation \(( \otimes_k \)) then NMI iteration, started from the initial epistemic state \([0, 1]\), terminates and gives the answer set.

**Proof:** From Lemma 1, it can be seen that NMI iterations are monotonic with respect to the knowledge ordering \( \leq_k \) of the epistemic states of the atoms in the assumption set. Since, for any \( x, y \in I(L) \), \( x \leq_k y \Rightarrow x \leq_k p \), NMI iterations are monotonic with respect to \( \leq_k \) as well;

i.e. \( \mathcal{NMI}_n \leq_k \mathcal{NMI}_{n+1}. \)

All the operations involved are continuous and since, \( \otimes_k \) does not occur in the program the program is consistent. So NMI iterations are continuous as well. Hence, iterations terminate at the least fixpoint.

2. Program segment whose dependency graph is a simple cycle without any \( \otimes_k \):

**Definition 14**

Suppose the dependency graph of a program segment there is a simple cycle \( S \) connected with some constant intervals from \( I(L) \) as conjuncts and disjuncts. Then the gain of the cycle as observed from a particular node
a ∈ B_S (i.e. a is in the assumption set) is given by the \( \|G_S\|_\infty \), where \( G_S = [G_1 \ G_2] \), such that at any iteration \([a_{1n}, a_{2n}] = [X + G_1 \times a_{i_{n-1}}, Y + G_2 \times a_{j_{n-1}}] \), where, \( X, Y \) are constants and \( i, j \in \{1, 2\} \) and depend on the \( \sim \) or \( \text{not} \) present in the cycle.

Consider a simple cycle with some atom \( a \) as the chosen element. Say, there are \( N \) nodes, \( n_1, n_2, ...n_N \), in the simple cycle apart from the node containing \( a \). The set of nodes is expanded to \( n_0, n_1, n_2, ...n_N, n_{N+1} \), where, \( n_0 = n_{N+1} = a \). The intervals \([x_1, y_1], [x_2, y_2], ..., [x_n, y_n] \) from \( I(L) \) are connected to the conjunctive (\( \land \)) nodes of the simple cycle and \([u_1, v_1], [u_2, v_2], ..., [u_m, v_m] \) from \( I(L) \) are connected to the disjunctive (\( \lor \)) nodes of the simple cycle.

**Algorithm 1:** Calculating Gain of a Simple Cycle

**Initialization:** flag = 0, \( G_1 = 1, G_2 = 0, G = 0 \);

for \( i \leftarrow 1 \) to \( N \) do
  if \( w(e(n_i, n_{i-1})) = \sim \) and flag == 1 then
    \( G_1 \leftarrow G_2 ; \)
    \( G_2 \leftarrow G_1 ; \)
  else if \( w(e(n_i, n_{i-1})) = -1 \) and flag == 1 then
    \( G_2 \leftarrow G_1 ; \)
  else
    \( ; \)
  end
  if \( n_i = \land \) and \( e(n_i, [x_i, y_i]) \in E \) then
    flag ← 1 ;
    \( G_1 \leftarrow G_1 \times x_i ; \)
    \( G_2 \leftarrow G_2 \times y_i ; \)
  else if \( n_i = \lor \) and \( e(n_i, [u_i, v_i]) \in E \) then
    \( G_1 \leftarrow G_1 \times (1 - u_i) ; \)
    \( G_2 \leftarrow G_2 \times (1 - v_i) ; \)
  else
    \( ; \)
  end
end

\( G = \max(\|G_1\|, \|G_2\|) ; \)

**Example 8:** Consider a program segment whose corresponding dependency graph is a simple cycle and for any chosen atom the corresponding value propagation graph is as shown in figure 8. Now,

\[
[a_{(m+1),1}, a_{(m+1),2}] = [x_4(1-x_3)(1-x_2) + x_4x_2y_1(1-x_3)a_{m_2}, y_4(1-x_3)(1-x_2) + x_4x_2y_1(1-x_3)a_{m_1}]
\]
Following Algorithm 1 we have $G = [y_1 x_2 x_4 (1-x_3) \ y_1 x_2 y_4 (1-x_3)]$; and the Gain is $\|G\|_\infty = \max(\|y_1 x_2 x_4 (1-x_3)\|, \|y_1 x_2 y_4 (1-x_3)\|)$.

**Definition 15**

A mapping $\bar{F} : [0, 1]^n \rightarrow [0, 1]^n$ is nonexpansive on $[0, 1]^n$ if for any $\bar{x}, \bar{y} \in [0, 1]^n$

$$\|\bar{F}(\bar{x}) - \bar{F}(\bar{y})\| \leq \|x - y\|.$$  

A mapping $\bar{F} : [0, 1]^n \rightarrow [0, 1]^n$ is a contraction mapping on $[0, 1]^n$ (or simply a contraction) if there is a $\alpha < 1$ such that $\|\bar{F}(\bar{x}) - \bar{F}(\bar{y})\| \leq \alpha \|x - y\|$ for any $\bar{x}, \bar{y} \in [0, 1]^n$. Such a $\alpha$ is called a contraction modulus or Lipschitz Constant.

**Theorem 5**

Contraction mapping theorem: Let $\bar{F} : D \subset R^n \rightarrow R^n$ maps a closed set $D_0 \subset D$ into itself and the components of $\bar{F}$ are continuously differentiable at all points of $D_0$ and further assume $\max_{\bar{x} \in D_0} \|J_F(\bar{x})\|_\infty < 1$; where, $\bar{F} = [f_1, f_2, ..., f_n]^T$ and $J_F(\bar{x})$ is the $n \times n$ Jacobian matrix at some point $\bar{x} \in D_0$ with element $J_F(\bar{x})_{ij} = \frac{\partial f_i(\bar{x})}{\partial x_j}$, $i, j = 1, ..., n$. Then,

1. $\bar{x} = \bar{F}(\bar{x})$ has a unique solution $\bar{\alpha} \in D_0$.
2. For any initial point $\bar{x}_0 \in D_0$, the iteration $\bar{x}_k = \bar{F}(\bar{x}_{k-1})$, $k = 1, 2, ...$ will converge in $D_0$ to $\bar{\alpha}$.

**Theorem 7**

Let $P$ be a transformed program, whose dependency graph contains exactly one simple cycle, such that none of its nodes is $\otimes_k$ and all the...
Figure 9: Basic building Blocks of value propagation path

| Logical Operation                      | Realisation using building Blocks |
|----------------------------------------|-----------------------------------|
| $a_n = [x, y] \land \text{not } a_{n-1}$ | $a_{n-1} \quad \checkmark \quad \land \quad [0, 0]$ |
| $a_n = \neg a_{n-1}$                   | $a_{n-1} \quad \checkmark \quad \land \quad [0, 0]$ |
| $a_n = ([x_1, x_2] \land a_{n-1}) \lor [y_1, y_2]$ | $a_{n-1} \quad \checkmark \quad \land \quad \checkmark \quad [0, 0] [x_1, x_2] [y_1, y_2]$ |

Table 1: Realisation of Logical Operation Using Building Blocks

conjuncts and disjuncts to the simple cycle are constant intervals from $I(L)$. The NMI iteration for such a program converges, for any initial epistemic state of the chosen elements, if the gain of the cycle is $< 1$.

**Proof:** The dependency graph of $P$ being a simple cycle there is only one element (say $a$) in $As_P$ and the corresponding value propagation graph will be composed of just a single path from $a_{n-1}$ to $a_n$. Therefore the system of difference equations corresponding to the NMI iterations will be of the form:

$$a^n = \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} f_{a_1}(a_{n-1}, a_{n-2}) \\ f_{a_2}(a_{n-1}, a_{n-2}) \end{bmatrix} = \bar{F}_{sc}(a^{n-1}).$$
Any such value propagation path for $a$ can be constructed by composition of the suitably modified building blocks of Figure 9. Some examples of constructing logical operations in the value propagation path by means of the two basic building blocks are shown in Table 1. Thus suppose the value propagation path is constructed using blocks $b_1, b_2, ..., b_n$ and the functions corresponding to them are $\bar{F}_b, \bar{F}_{b2}, ..., \bar{F}_{bn}$ respectively. Then,

$$\bar{F}_{sc} = \bar{F}_{b1} \circ \bar{F}_{b2} \circ ... \circ \bar{F}_{bn}.$$  

**Claim 1:** The function $\bar{F}_{bb_1} : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ corresponding to the building block 1 (Figure 9 a) is a non-expansive mapping. $\bar{F}_{bb_1}$ would be a contraction iff the path gain from $bb_1_{in}$ to $bb_1_{out}$, $\|G_{bb_1}\| < 1$.

**Claim 2:** The function $\bar{F}_{bb_2} : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ corresponding to the building block 2 (Figure 9 b) is a non-expansive mapping. $\bar{F}_{bb_2}$ would be a contraction iff the path gain from $bb_2_{in}$ to $bb_2_{out}$, $\|G_{bb_2}\| < 1$.

**Claim 3:** Composition of non-expansive mappings is a non-expansive mapping. Composition of a non-expansive mapping with a contraction mapping is a contraction.

Given that the gain of the cycle is $< 1$, there is at least one conjunct to the cycle $c \in I(L)$ that is not $[1, 1]$ or at least one disjunct $d \in I(L)$ that is not $[0, 0]$. So in the re-construction of the value propagation path of $a$ using building blocks, there is at least one component block to which conjunct $c$ or disjunct $d$ is connected; hence path gain for that block is $< 1$ and that particular component gives a contraction mapping (according to claim 1 and claim 2). Therefore, following claim 3, the overall composition $\bar{F}_{b1} \circ \bar{F}_{b2} \circ ... \circ \bar{F}_{bn}$ is a contraction. Hence, the function $\bar{F}_{sc}$ corresponding to the complete propagation path is a contraction mapping. Moreover $\bar{F}_{sc}$ maps $I(L)$ into itself. Hence, from Banach’s Fixed point theorem it follows that for any starting value $a_0 \in I(L)$ the iteration $\bar{a}^k = \bar{F}_{sc}(\bar{a}^{k-1})$ will reach at the unique fixed point of $\bar{F}_{sc}$ in $I(L)$.

Therefore, for any starting epistemic state $a_0 \in I(L)$ NMI iteration will terminate at the unique fixed point, which is the unique answer set of the program segment under consideration.

Now it is only left to prove the claims used above.

**Proof of Claim 1:**

From Figure 9 a,

- $bb_{1\text{out}} = [d1_1, d1_2] \lor ([c1_1, c1_2] \land \text{not } bb_{1\text{in}})$;
- $bb_{1\text{out_1}, bb_{1\text{out_2}}} = [d1_1, d1_2] \lor ([c1_1, c1_2] \land \text{not } [bb_{1\text{in_1}}, bb_{1\text{in_2}}])$
  = $[d1_1, d1_2] \lor ([c1_1, c1_2] \land [1 - bb_{1\text{in_1}}, 1 - bb_{1\text{in_2}}])$
  = $[d1_1, d1_2] \lor [c1_1 \ast (1 - bb_{1\text{in_1}}), c1_2 \ast (1 - bb_{1\text{in_1}})]$
Proceeding in the same way as in the proof of Claim 1, we get:

\[
bb1_{out1} = (d1_2*1-1) + c1_2*(1-d1_2)\times bb1_{in1};
\]
\[
bb1_{out2} = (d1_2*(1-c1_2)+c1_2) - c2_2*(1-d2_2)\times bb1_{in2}.
\]

Thus, the mapping \(F_{bb1} : [0,1] \times [0,1] \rightarrow [0,1] \times [0,1]\), which corresponds to the building block 1 is \(F_{bb1} = [f_{bb1}, f_{bb2}]^T\).

Now, for any two points \(\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in [0,1]^2\);

\[
\|F_{bb1}(\bar{x}) - F_{bb1}(\bar{y})\|_\infty = \max_{z \in [0,1] \times [0,1]} ||J_{F_{bb1}}(z)||_\infty \|\bar{x} - \bar{y}\|_\infty
\]

where \(J_{F_{bb1}}(z)\) is the Jacobian matrix of \(F_{bb1}\) at \(z \in [0,1] \times [0,1]\) given by:

\[
J_{F_{bb1}}(z) = \begin{pmatrix}
\frac{\partial f_{bb1_1}(z)}{\partial x_1} & \frac{\partial f_{bb1_1}(z)}{\partial x_2} \\
\frac{\partial f_{bb2_1}(z)}{\partial x_1} & \frac{\partial f_{bb2_1}(z)}{\partial x_2}
\end{pmatrix}
\]

\[
||J_{F_{bb1}}(z)||_\infty = \max \left(\left|\frac{\partial f_{bb1_1}}{\partial x_1}(z)\right| + \left|\frac{\partial f_{bb1_1}}{\partial x_2}(z)\right| + \left|\frac{\partial f_{bb2_1}}{\partial x_1}(z)\right| + \left|\frac{\partial f_{bb2_1}}{\partial x_2}(z)\right|\right)
\]

\[
= \max \left(\left|c1_1(1-d1_1)\right| + 0, \left|c1_2(1-d1_2)\right| + 0\right)
\]

\[
\leq 1 \text{ [since } 0 \leq c1_1, c1_2, d1_1, d1_2 \leq 1].
\]

Hence, \(\max_{z \in [0,1] \times [0,1]} ||J_{F_{bb1}}(z)||_\infty \leq 1\) and thus, the mapping \(F_{bb1}\) becomes non-expansive.

The path gain for building block 1, \(\|G_{bb1}\|_\infty = max(c1_1(1-d1_1), c1_2(1-d1_2)) = ||J_{F_{bb1}}(z)||_\infty\) for any \(z \in [0,1] \times [0,1]\). Therefore, if \(\|G_{bb1}\|_\infty < 1\) we have,

\[
\max_{\bar{x} \in [0,1] \times [0,1]} ||J_{F_{bb1}}(\bar{x})||_\infty < 1 \text{ and hence, } ||F_{bb1}(\bar{x}) - F_{bb1}(\bar{y})|| < ||\bar{x} - \bar{y}||
\]

for any \(\bar{x_1}, \bar{x_2} \in [0,1]^{2}; \text{ i.e. } F_{bb1}\) is a contraction mapping.

**Proof of Claim 2:**

Proceeding in the same way as in the proof of Claim 1, we get;

\[
bb2_{out1} = (d2_2*(1-c2_1)+c2_1) - c2_1*(1-d2_1)\times bb2_{in2};
\]
\[
bb2_{out2} = (d2_2*(1-c2_2)+c2_2) - c2_2*(1-d2_2)\times bb2_{in2}.
\]

In the similar fashion, it can be seen;

\[
\max_{z \in [0,1] \times [0,1]} ||J_{F_{bb2}}(z)||_\infty = max(c2_1(1-d2_1), c2_2(1-d2_2)) \leq 1
\]

i.e., \(F_{bb2}\) is a non-expansive mapping. It would become a contraction if \(\|G_{bb1}\|_\infty < 1\).
Proof of Claim 3:
Suppose $F_1 : [0,1]^n \rightarrow [0,1]^n$ and $F_2 : [0,1]^n \rightarrow [0,1]^n$ are two non-expansive mappings. Now, it is to investigate whether the composition $F_2 \circ F_1$ is a non-expansive mapping.

For any $\bar{x}, \bar{y} \in [0,1]^n$, we have
\[
\|F_2 \circ F_1(\bar{x}) - F_2 \circ F_1(\bar{y})\| \\
= \|F_2(F_1(\bar{x})) - F_2(F_1(\bar{y}))\| \\
\leq \gamma_2 \cdot \|F_1(\bar{x}) - F_1(\bar{y})\| \quad [\gamma_2 \leq 1 \text{ as } F_2 \text{ is non-expansive}] \\
\leq \gamma_2 \cdot \gamma_1 \|\bar{x} - \bar{y}\| \quad [\gamma_1 \leq 1 \text{ as } F_1 \text{ is non-expansive}] \\
= \gamma \|\bar{x} - \bar{y}\| \quad [\text{where } \gamma = \gamma_1 \cdot \gamma_2 \leq 1]
\]

Thus, $F_2 \circ F_1$ is a non-expansive mapping.

In particular, if at least one of $F_1$ and $F_2$ is contraction, then $\gamma_1 \cdot \gamma_2 < \gamma < 1$. Hence, $F_2 \circ F_1$ becomes a contraction.

This proof can be extended to composition of any number of functions by using induction.

3. Program segment with 'not' whose dependency graph is a SCC with multiple simple cycles without any $\otimes_k$:
Consider for some SCC corresponding to some program segment, $m$ atoms are chosen to construct the assumption set and $As_{P_{SCC}} = \{a_1, a_2, ..., a_m\}$. The iteration function $\bar{F}$ has $n$ component functions, where $n = 2m$. Say, $\bar{F} = [f_1 f_2 ... f_n]^T$, where, $a_1 = [a_{1,1}, a_{1,2}]$ and $f_1$ corresponds to $a_{1,1}$ and $f_2$ corresponds to $a_{1,2}$ so on. Each row of the Jacobian matrix for $\bar{F}$ is of the form
\[
J_{\bar{F}}(\bar{x})_i = [\frac{\partial f_i(\bar{x})}{\partial a_{1,1}} \frac{\partial f_i(\bar{x})}{\partial a_{1,2}} \ldots \frac{\partial f_i(\bar{x})}{\partial a_{m,1}} \frac{\partial f_i(\bar{x})}{\partial a_{m,2}}]
\]
for some point $\bar{x} \in [0,1]^n$.

So, satisfaction of the sufficient condition for convergence as specified in theorem 6, requires,
\[
\max_{\bar{x} \in D_0} \left( \max_{i \in \{1,2,...,n\}} \|J_{\bar{F}}(\bar{x})_i\|_\infty \right) < 1
\]

i.e., for every $1 \leq i \leq n$, $\|J_{\bar{F}}(\bar{x})_i\|_\infty < 1$ for all $\bar{x} \in [0,1]^n$. From the definition of vector norms,
\[
\|J_{\bar{F}}(\bar{x})_i\|_\infty = \|\frac{\partial f_i(\bar{x})}{\partial a_{1,1}}\| + \|\frac{\partial f_i(\bar{x})}{\partial a_{1,2}}\| + \|\frac{\partial f_i(\bar{x})}{\partial a_{m,1}}\| + \|\frac{\partial f_i(\bar{x})}{\partial a_{m,2}}\|
\]

Increasing the number of chosen element would increase the number of independent variables in each of the component functions of the iteration.
function, thus inserting additional terms inn the rows of the Jacobian ma-
trix. The incorporation of additional terms make it more difficult to satisfy
the sufficient condition for being contraction. Thus, keeping the number of
chosen elements as small as possible is intended.

Lemma 2: Suppose \( f: [0.1]^m \rightarrow [0,1] \) be a logical function (con-
structed using product t-norm, t-conorm and negation operators only) of \( m \)
variables, say \( a_1, a_2, ..., a_m \), for any \( m \geq 2 \). Then
\[
|\frac{\partial f}{\partial a_1}| + |\frac{\partial f}{\partial a_2}| + ... + |\frac{\partial f}{\partial a_m}| \leq m.
\]

Proof: The proof is done using mathematical induction.

Base case: \( m = 2 \).

Case 1. \( f_2 = a_1.a_2 \) (product t-norm)
\[
|\frac{\partial f_2}{\partial a_1}| + |\frac{\partial f_2}{\partial a_2}| = |a_1| + |a_2| \leq 2.
\]

Case 2. \( f_2 = a_1 + a_2 - a_1.a_2 \) (product t-conorm)
\[
|\frac{\partial f_2}{\partial a_1}| + |\frac{\partial f_2}{\partial a_2}| = |1 - a_1| + |1 - a_2| \leq 2.
\]

Case 3. \( f_2 = a_1(1 - a_2) \) (product with negated variable)
\[
|\frac{\partial f_2}{\partial a_1}| = |1 - a_2| + |1 - a_1| \leq 2.
\]

The same can be shown for the rest of the combinations using De Mor-
gan’s Law for negation.

Induction hypothesis: Say \( f_n \) be a logical function of \( n \) variables
satisfying the aforementioned condition, i.e.
\[
|\frac{\partial f_n}{\partial a_1}| + |\frac{\partial f_n}{\partial a_2}| + ... + |\frac{\partial f_n}{\partial a_n}| \leq n.
\]

Now, any logical function with \((n+1)\) logical variable can be constructed
from basic logical operations on \( f_n \) and \( a_{n+1} \).

Case 1: \( f_{n+1} = f_na_{n+1} \) (product t-norm)
\[
|\frac{\partial f_{n+1}}{\partial a_1}| + |\frac{\partial f_{n+1}}{\partial a_2}| + ... + |\frac{\partial f_{n+1}}{\partial a_{n+1}}| = a_{n+1}(|\frac{\partial f_n}{\partial a_1}| + |\frac{\partial f_n}{\partial a_2}| + ... + |\frac{\partial f_n}{\partial a_n}|) + f_n \leq n.a_{n+1} + 1 \leq n + 1.
\]

Case 2: \( f_{n+1} = f_n + a_{n+1} - f_na_{n+1} \) (t-conorm)
or, \( f_{n+1} = a_{n+1} + f_n(1 - a_{n+1}) \)
\[
|\frac{\partial f_{n+1}}{\partial a_1}| + |\frac{\partial f_{n+1}}{\partial a_2}| + ... + |\frac{\partial f_{n+1}}{\partial a_{n+1}}| = (1 - a_{n+1})(|\frac{\partial f_n}{\partial a_1}| + |\frac{\partial f_n}{\partial a_2}| + ... + |\frac{\partial f_n}{\partial a_n}|) + (1 - f_n) \leq (1 - a_{n+1})n + (1 - f_n) \leq n + 1.
\]

Case 3: \( f_{n+1} = (1 - a_{n+1})f_n \) (product with negated variable)
\begin{equation*}
|\frac{\partial f_{n+1}}{\partial a_1}| + |\frac{\partial f_{n+1}}{\partial a_2}| + \ldots + |\frac{\partial f_{n+1}}{\partial a_{n+1}}|
\leq (1 - a_{n+1})(|\frac{\partial f_n}{\partial a_1}| + |\frac{\partial f_n}{\partial a_2}| + \ldots + |\frac{\partial f_n}{\partial a_n}|) + f_n
\leq n + 1.
\end{equation*}

For rest of the combinations the proof can be derived using De Morgan’s Law. Therefore, following the principle of mathematical induction the theorem is proved for any \( m \geq 2 \). Q.E.D

**Theorem 8**

Suppose the value-propagation path for a chosen element \( a \), has only conjunctive nodes (\( \land \)), and no disjunctions and moreover, among the conjuncts, \( k \) conjuncts are not constants but come from value-propagation paths of other chosen elements or from other nodes of the value-propagation path of \( a \). These \( k \) conjuncts, say \( g_1, g_2, \ldots, g_k \), vary over iterations. Then the infinite norm (\( || \cdot ||_{\infty} \)) of the rows corresponding to \( a \) in the Jacobian matrix will be < 1 if the path gain of the vpp of \( a \)(considering only the constant conjuncts) is strictly less that \( \frac{1}{k+2} \).

**Proof:** The path gain of the vpp for atom \( a \) can be calculated by using Algorithm 1 considering using the constant conjuncts only. Since in the vpp of the atom \( a \) no disjunctions are there, the two component functions of the iteration function corresponding to the atom \( a \) will be of the form

\( \bar{f}_a = [f_1, f_2]^T \), where,

\( f_1 = G_1.f_a(a_1, a_2, g_1, \ldots, g_k) \) and \( f_2 = G_2.f_a(a_1, a_2, g_1, \ldots, g_k) \).

\( f_1 \) and \( f_2 \) comprise two rows of the Jacobian matrix of the whole SCC.

The infinite norm of \( \bar{f}_a \) is given by:

\[
\max\left(|\frac{\partial f_1}{\partial a_1}| + |\frac{\partial f_1}{\partial a_2}| + \ldots + |\frac{\partial f_1}{\partial g_k}|, \right.\left. |\frac{\partial f_2}{\partial a_1}| + |\frac{\partial f_2}{\partial a_2}| + \ldots + |\frac{\partial f_2}{\partial g_k}|\right)
\leq \max(G_1, (k + 2), G_2, (k + 2)) \quad \text{[following Lemma 2]}
\leq (k + 2) \cdot \max(G_1, G_2)
\leq 1. \quad \text{[since path gain = ||G||_{\infty} = \max(G_1, G_2) \leq \frac{1}{k+2}] Q.E.D}

If the vpp contains disjunctions then satisfaction of the sufficient condition depends on the chosen assumption set and cannot be controlled by the path gain alone. This is illustrated with the help of the following examples.

Consider the section of an SCC as shown in Figure 11 where, \([g_{11}, g_{12}]\) and \([g_{21}, g_{22}]\) are inputs coming from other sections of the SCC and they vary with iterations. Now suppose, each of atoms \( a, b \) and \( c \) is chosen in the assumption set. Then the corresponding iterative expressions are respectively as follows:

\[
\begin{align*}
\bar{a}_{n+1} &= [g_{21}(x_2 + (1 - x_2)x_1a_{n,1}g_{11}), g_{22}(y_2 + (1 - y_2)y_1a_{n,2}g_{12})]; \\
\bar{b}_{n+1} &= [x_2 + (1 - x_2)x_1b_{n,1}g_{21}, y_2 + (1 - y_2)y_1b_{n,1}g_{22}]; \\
\bar{c}_{n+1} &= [x_1x_2g_{21} + x_1(-x_2)c_{n,1}g_{11}g_{21}, y_1y_2g_{22} + y_1(-y_2)c_{n,2}g_{12}g_{22}].
\end{align*}
\]

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The infinite norm of the corresponding rows of the Jacobian matrix are given by:

\[ \|J_a\| = \max(x_2 + x_1(1 - x_2)(g_{11}g_{21} + a_{n,1}g_{21} + a_{n,1}g_{11}), y_2 + y_1(1 - y_2)(g_{12}g_{22} + a_{n,2}g_{22} + a_{n,2}g_{12})) \leq \max(x_2 + 3x_1(1 - x_2), y_2 + 3y_1(1 - y_2)). \]

\[ \|J_b\| = \max(x_1(1 - x_2)(g_{11}g_{21} + b_{n,1}g_{21} + b_{n,1}g_{11}), y_1(1 - y_2)(g_{12}g_{22} + b_{n,2}g_{22} + b_{n,2}g_{12})) \leq \max(3x_1(1 - x_2), 3y_1(1 - y_2)). \]

\[ \|J_c\| = \max(x_1x_2 + x_1(1 - x_2)(g_{11}g_{21} + c_{n,1}g_{21} + c_{n,1}g_{11}), y_1y_2 + y_1(1 - y_2)(g_{12}g_{22} + c_{n,2}g_{22} + c_{n,2}g_{12})) \leq \max(x_1x_2 + 3x_1(1 - x_2), y_1y_2 + 3y_1(1 - y_2)). \]

Clearly atom \( b \) is the best choice for assumption set, since in this case the sufficient condition can be satisfied only by making the path gain sufficiently low.

Note: Since, presence of disjunction in the vpp gives rise to extra term in the differential expression in the Jacobian matrix, choosing the atom that is the predecessor of the disjunctive node (since rules are represented in DNF it is guaranteed that there will be an atom) is better choice, as can be seen form figure 10.

4. Program segments with \( \text{not} \) and \( \otimes_k \)

Theorem 9

Consider a set of rules \( P_C \), whose dependency graph is of the form of an SCC, say \( C \), such that, it has a \( \otimes_k \)-node, which has an incoming edge from a constant interval, say \([x, y] \in I(L)\) and the other incoming edge is from an atom, say \( a \) in the SCC. The output of the \( \otimes_k \) node goes to some node \( b \). Now if we remove the \( \otimes_k \) node, \([x, y]\) node and the connecting edge and directly connect node \( a \) to node \( b \) then the corresponding modified SCC is called \( C^{\sim \otimes_k} \). Now suppose, the NMI iterations for \( C^{\sim \otimes_k} \) gives rise to a
contraction mapping and the iterations terminates at a unique fixed point for the elements in the assumption set of \( C^{-\otimes_k} \) and gives the epistemic state of atom \( a \) to be \([a_1, a_2]\). The epistemic states determined by the NMI iterations of \( C^{-\otimes_k} \) corresponds to the unique answer set of the program segment \( P_C \) if \([a_1, a_2] \geq_{k_p} [x, y]\).

**Proof:** Since, iterations of the elements of the assumption set of \( C^{-\otimes_k} \) corresponds to a contraction mapping, it has a unique fixed point which assigns \([a_1, a_2]\) to atom \( a \) and for any starting value the iteration terminates at the same point.

The assumption sets of \( C \) and \( C^{-\otimes_k} \) are same and NMI iterations for both of them are similar except the fact that, because of the presence of \( \otimes_k \) in \( C \) the epistemic state of atom \( b \) would be depend on the knowledge level of epistemic state of atom \( a \) as well as on \([x, y]\). Say at the \( k^{th} \) iteration the epistemic state of atom \( a \) is \([a_{k,1}, a_{k,2}]\) and \([a_{k,1}, a_{k,2}] \leq_{k_p} [x, y]\). Hence, the atom \( b \) would be assigned with \([x, y]\) instead of \([a_{k,1}, a_{k,2}]\) and the course of iteration would change. Assigning \([x, y]\) to atom \( b \) would ascribe some value to the chosen elements at the beginning of the \((k+1)^{st}\) iteration. Now, if we consider the \((k+1)^{st}\) iteration as the starting point of NMI iteration, it would mean just changing the initial epistemic states of the chosen elements of \( C^{-\otimes_k} \). Now, iterations of \( C^{-\otimes_k} \) being a contraction mapping that iteration would also lead to the same fixed point and atom \( a \) would get the same value \([a_1, a_2]\). The presence of \( \otimes_k \) just alters the starting value of iteration; doesn’t change the nature of iteration and hence doesn’t alter the fixed point.

That fixed point will be stable if \([a_1, a_2] \geq [x, y]\), which won’t be the case if \([a_1, a_2] \leq [x, y]\). \( \Box \)

**Theorem 8** can be applied on more complex case where none of the inputs come from constant interval. Consider the SCC in Figure [III(a)]. Following the rationale presented in the proof of theorem 8 the SCC can be broken into the cases shown in figure [III(b)] and (c). According to the assumption mentioned in theorem the iteration functions for both of the structures are contraction mappings. Thus NMI iterations for both of the SCC will terminate at a fixed point. Fixed point of the modified SCC in figure (b) will be an answer set of the actual SCC if \( v(a) \geq_{k_p} v(f) \), and the fixed point of the modified SCC in figure (c) will be an answer set of the actual SCC if \( v(f) \geq_{k_p} v(a) \), where, \( v(a) \) and \( v(f) \) denotes the epistemic states of atoms \( a \) and \( f \).

**Note:** The conditions stated in the above theorems are sufficient conditions and not necessary conditions. Thus, meeting those conditions guarantees termination to a unique fixed point. But if those conditions are not
satisfaction then also the NMI iteration may terminate to unique fixed point,
as is clear from the Example 7.

5.2.6 Branch and Bound

In the previous subsection it is assumed that for a program segment with
not the values of the constant conjuncts and disjuncts are such that NMI
iterations always terminate at a unique fixed point. But for simple cycles
with unity gain or SCCs with no constant conjunct or disjunct, there may
be more than one stable valuations, not attained by the NMI iterations.

For example, for the set of rules $P = \{a \leftarrow not \ b, \ b \leftarrow not \ a\}$, with
$As_{P} = \{a\}$, any valuation of the form $\{a : [x, x]|x \in [0, 1]\}$ is a stable
valuation and the corresponding answer set is $\{a : [x, x], \ b : [1-x, 1-x]\}$. But NMI iteration, would terminate at a value dependent on the starting
epistemic state. All other answer sets cannot be obtained. In such a scenario,
branch-and-bound is required.

Firstly assumption set is constructed from the SCC following the same
conditions specified in section 5.2.3, except for replacing the condition 4 by
the following:

4'. The chosen atom has to be at the head of an edge having weight
'-1', i.e., the chosen element, say $a$, occurs as $a \leftarrow not \ b$ for some $b$ in the
program segment.

Every possible epistemic state of the form $[x, x]$ can not be checked for

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stability for all $x \in [0, 1]$. Therefore, a predefined positive integer, $N_B$, is chosen, so that at any time the branch-and-bound is called for, only $N_B$ equidistant points are are chosen from $[0, 1]$ and only those epistemic states are checked for stability. For all the elements in the assumption set all combinations of $N_B$ epistemic states are checked for stability.

After assigning each set of values monotonic iteration is performed.

Say, $S_1$ be an SCC to be evaluated using branch-and-bound. An atom, say $a$, is connected to a conjunction node of SCC $S_2$, placed just higher position in the topologically sorted list of SCCs. Thus the course of iteration and the epistemic states of atoms in $S_2$ depend upon the epistemic state of atom $a$ in $S_1$. So for each stable valuation of $a$, a version of $S_2$ is generated with that value of $a$ coming to the conjunctive node of $S_2$. Maximum $N_B$ versions of $S_2$ can be generated if each of the assignments gives stable valuation of $a$. In other words, whenever an SCC is evaluated using branch-and-bound the computation of the next SCCs proceeds in different branches.

Here, it is assumed $S_2$ satisfies the sufficient condition for being a contraction even if the contribution of $a$ is absent.

Therefore, the iterative computation method is not *semantically complete*, in the sense that all possible answer sets can not be obtained.

**Assumption and Nonmonotonic Evaluation:**

From the topologically sorted list of SCCs, obtained from the dependency graph of the reduced program after MI stage, one after another is chosen along with the associated constant nodes.

1. If the SCC has some constant interval connected to some conjunctive ($\wedge$) and disjunctive ($\vee$) nodes, then NMI iteration procedure are done. It is assumed here, that the constant values are such that with proper selection of assumption set the sufficient condition for convergence of iteration is met and the unique fixed point is reached through iteration.

2. If the SCC has no constant interval connected to some conjunctive ($\wedge$) and disjunctive ($\vee$) nodes and no edge is of weight $-1$ and no node is $\otimes$, then also NMI iteration started.

3. If the SCC has no constant interval connected to some conjunctive ($\wedge$) and disjunctive ($\vee$) nodes and at least one edge of weight $-1$ is present, branch-and-bound is called for.

4. With evaluation of each SCC, the SCC placed just higher position in the topologically sorted list gets modified by the evaluated epistemic states of the atoms in the former one.

**Example 6 (contd):**
From topologically sorted list of SCCs, $hijk$, $uvxw$, $c$, $gbadf$, $yz$, $l$, one after one is chosen and epistemic states are evaluated.

1. $SCC_{hijk}$ is chosen. $A_{hijk} = \{ h \}$.

Iteration function for $SCC_{hijk}$ is a contraction mapping (Theorem 7) and for any initial value NMI iteration terminates at a unique epistemic state of atom $h$, which is $h : [0.5557, 0.7938]$. From this epistemic states of other atoms of $SCC_{hijk}$ are calculated as:

$$v(hijk) = \{ h : [0.5557, 0.7938], i : [0.4443, 0.4443], j : [0.2062, 0.2062], k : [0.7938, 0.7938]\}.$$

Now, in $SCC_{hijk}$ $[0.44, 0.58] \leq_{k_p} [0.2062, 0.2062]$; hence $v(hijk)$ gives the answer set of the atoms in $SCC_{hijk}$ (Theorem 8).

2. The epistemic state of atom $c$ is $\{ c : [0.5557, 0.7938] \} = v(h)$.

3. $SCC_{uvxw}$ is chosen. $A_{uvxw} = \{ x \}$.

NMI iterations gives:

$$\{ x : [0.16212, 0.28255], w : [0.16212, 0.28255], u : [0.0811, 0.22604], v : [0.8106, 0.9418]\}.$$

3. $SCC_{gbadf}$ is modified by already calculated epistemic states of atom $c$ and $w$.

$$A_{gbadf} = \{ a, g \}.$$

NMI iteration gives: $\{ a : [0.2967, 0.4116], g : [0.1834, 0.4322], b : [0.7577, 0.8315], d : [0.1685, 0.3361], e : [0.0271, 0.0949], f : [0.9050, 0.9727]\}$.

4. In $SCC_{yz}$, since there is no conjuncts or disjuncts, branch-and-bound evaluation is followed. Say $N_B$ is chosen to be 4 and $A_{yz} = \{ y \}$.

So, Monotonic iterations are done with four initial values, i.e., $y : [0, 0], y : [0.25, 0.25], y : [0.75, 0.75]$ and $y : [1, 1]$. All of them are stable; hence we get four answer sets respectively

$$v_1 = \{ y : [0, 0], z : [1, 1] \}, v_2 = \{ y : [0.25, 0.25], z : [0.75, 0.75] \}, v_3 = \{ y : [0.75, 0.75], z : [0.25, 0.25] \} \text{ and } v_4 = \{ y : [1, 1], z : [0, 0] \}.$$

5. Based on the epistemic states of $z$, $l$ has four possible values corresponding to each of the values of $z$. They are $v_1 = \{ l : [0.4, 0.6] \}$, $v_2 = \{ l : [0.3, 0.45] \}, v_3 = \{ l : [0.1, 0.15] \}$ and $v_4 = \{ l : [0, 0] \}$.

Therefore, for the UnASP program in Example 6 the four computed answer sets are as follows:

Suppose $v = \{ q : [0.7, 0.7], r : [1, 1], n : [0.7, 0.9], t : [0, 1], m : [0.42, 0.56], s : [0.42, 0.56], p : [0.3916, 0.495], h : [0.5557, 0.7938], i : [0.4443, 0.4443], j : [0.2062, 0.2062], k : [0.7938, 0.7938], c : [0.5557, 0.7938], a : [0.2967, 0.4116], g : [0.1834, 0.4322], b : [0.7577, 0.8315], d : [0.1685, 0.3361], e : [0.0271, 0.0949], f : [0.9050, 0.9727]\}$.

Then,

$$Answer \setminus set1 = v \cup \{ y : [0, 0], z : [1, 1], l : [0.4, 0.6] \}.$$
Answer – set2 = v ∪ \{y : [0.25, 0.25], z : [0.75, 0.75], l : [0.3, 0.45]\}.

Answer – set3 = v ∪ \{y : [0.75, 0.75], z : [0.25, 0.25], l : [0.1, 0.15]\}.

Answer – set4 = v ∪ \{y : [1, 1], z : [0, 0], l : [0, 0]\}.

6 Conclusion:

This paper presents a semantics for unified logic program that can handle nonmonotonic reasoning with vague and uncertain information. Truth values are sub-intervals of $[0, 1]$ ordered using an algebraic structure named preorder-based triangle. Weighted rules are used, where rules weights are intervals depicting the degree of uncertainty. Weighted rules can be used to distinguish between propositions and dispositions (propositions having exceptions as per Zadeh), thus allowing us to perform nonmonotonic reasoning. Most of the proposed approaches don’t consider any kind of negation or only a single kind of negation. Here both classical negation and negation-as-failure are considered. A special knowledge aggregator operator is used to take care of the interaction of positive and negative evidences for a piece of information. This operator makes the nonmonotonic reasoning more intuitive. Lastly, an iterative approach for computation of the answer set is presented here, which is influenced by the three stages of computation of classical answer sets. However, the truth-values being real numbers, to guarantee the termination of iterations become difficult. Iterations are mathematically investigated by means of different equations, which are obtained by representing rules as signal-flow graph-like structures, namely value-propagation-graph. This analysis specifies the conditions under which the sufficient condition for convergence (for any starting value) is satisfied. The aim of this analysis is to give a glimpse of such a method. But, whether we can specify the necessary conditions for convergence or whether we can explain the convergence of programs not satisfying the sufficient condition using the value propagation graph is of further study.

This work attempts to propose a realisable fuzzy answer set programming paradigm that enable nonmonotonic reasoning with uncertainty and vagueness.

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