SECOND ORDER ESTIMATES FOR A CLASS OF COMPLEX HESSIAN EQUATIONS ON HERMITIAN MANIFOLDS

WEISONG DONG

ABSTRACT. In this paper, we derive an a priori second order estimate for solutions which are in \( \Gamma_{k+1} \) cone to a class of complex Hessian equations with both sides of the equation depending on the gradient on compact Hermitian manifolds.

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1. Introduction

Let \((M, \omega)\) be a compact complex manifold of complex dimension \( n \geq 2 \) with Hermitian metric \( \omega \). For any smooth function \( u \in C^\infty(M) \), let \( \chi(z, u) \) be a smooth real \((1,1)\) form on \( M \) and \( \psi(z, v, u) \in C^\infty \left( (T^{1,0}(M))^s \times \mathbb{R} \right) \) be a positive function, where \( T^{1,0}(M) \) is the holomorphic tangent bundle. Given any smooth \((1,0)\)-form \( a \) on \( M \), we obtain a new real \((1,1)\)-form

\[
g = \chi(z, u) + \sqrt{-1}a \wedge \bar{u} - \sqrt{-1}a \wedge \partial u + \sqrt{-1}i\partial \bar{u}.
\]

Consider the following complex Hessian equation

\[
Q(\lambda) = \sum_{s=1}^{k} \alpha_s \sigma_s(\lambda) = \psi(z, Du, u), \quad \text{for} \ 1 \leq k \leq n,
\]

where \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_{k+1} \) are the eigenvalues of \( g \) with respect to \( \omega \) and \( \alpha_1, \cdots, \alpha_{k-1} \) are non-negative constants and \( \alpha_k \) is a positive constant. For \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \), \( \sigma_s(\lambda) \) is the \( s \)-th elementary symmetric function defined by

\[
\sigma_s(\lambda) = \sum \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_s}, \quad \text{where the sum is over} \ \{1 \leq i_1 < \cdots < i_s \leq n\}.
\]

We will also sometimes use the convention \( \sigma_0(\lambda) = 1 \) and \( \sigma_s(\lambda) = 0 \) if \( s > n \) or \( s < 0 \). The cone \( \Gamma_s \) is defined by

\[
\Gamma_s := \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \cdots, s \}.
\]

Apparently, \( \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n \). It was shown in [2] that \( \Gamma_s \) is an open convex symmetric cone in \( \mathbb{R}^n \) with vertex at the origin and the equation is elliptic if \( g \in \Gamma_k(M) \) (see Section 2 for the notation).

If the coefficients \( \alpha_1 = \cdots = \alpha_{k-1} = 0 \) in (1.2), it is just the \( k \)-Hessian equation, and the second order estimate was established by Phong-Picard-Zhang [27] and Dong-Li [10]. The \( k \)-Hessian equations in both real and complex case are related to many geometric problems and have been studied extensively. Equations of a similar form as (1.2) also emerge frequently. In the Calibrated geometry, the special Lagrangian equations introduced by Harvey and Lawson [19] can be written as the alternative combinations of elementary symmetric functions. A complex analogue
of the special Lagrangian equation appeared naturally from the study of Mirror Symmetry, see [22, 5, 8]. Another important example is the Fu-Yau [13, 14] equation arising from the study of Hull-Strominger system. There have been a lot of works on this topic recently, see [27, 28, 3, 29, 4]. Such type of equations also originate in the study of $J$-equation on toric varieties by Collins-Székelyhidi [7]. The real analogous equation of (1.2) was studied by Li-Ren-Wang [23], and see Guan-Zhang [18] for interesting related work.

A typical example of the complex Hessian equation involving a linear gradient term on the left hand side arising from the Gauduchon conjecture was studied by Székelyhidi-Tosatti-Weinkove [31]. See also Guan-Nie [16] for a work on related results. The Monge-Ampère equation with an additional linear gradient term inside the determinant can be found in the study of the Calabi-Yau equation on certain symplectic non-Kähler 4-manifolds by Fino-Li-Salamon-Vezzoni [12]. Many progresses have been made recently for general complex Hessian equations involving a linear gradient term on the left hand sides, see Tosatti-Weinkove [33], Feng-Ge-Zheng [11] and Yuan [34, 35]. In the Fu-Yau equation, the right hand side function $\psi$ of a particular structure depends on the gradient $Du$. Phong-Picard-Zhang [27] first investigated the complex $k$-Hessian equation with general right hand side $\psi(z, Du, u)$ on Kähler manifolds. The complex $k$-Hessian equation with gradient terms on both sides was studied by Li and the author [10] on Hermitian manifolds, and the second order estimate was derived for $g \in \Gamma_{k+1}(M)$. In this paper, we concentrate on the second order estimate for equation (1.2) with the condition $g \in \Gamma_{k+1}(M)$. We need to assume that $Q$ satisfies the so called quotient concavity property introduced by Li-Ren-Wang [23].

**Definition 1.1** ([23]). Suppose that $k - 1$ polynomials $S_1, \cdots, S_{k-1}$ are defined by

$$S_l(\lambda) = \sigma_l(\lambda) + \sum_{s=0}^{l-1} \beta^l_s \sigma_s(\lambda), \text{ for } 1 \leq l \leq k - 1,$$

where $\beta^l_s$ are constants depending on indices $s$ and $l$. If the function $Q(\lambda) = \sum_{s=1}^{k-1} \alpha_s \sigma_s(\lambda)$ satisfies that $(Q/S_l)^{1/(k-l)}$ are concave functions with respect to $\lambda = (\lambda_1, \cdots, \lambda_n)$, we call that the operator $Q$ is quotient concave.

We prove the following main result.

**Theorem 1.2.** Let $(M, \omega)$ be a compact Hermitian manifold of complex dimension $n$ and $u \in C^\infty(M)$. Suppose $g$ defined in (1.1) is in $\Gamma_{k+1}(M)$ and satisfies (1.2). Assume $\chi(z, u) \geq \varepsilon \omega$ and $Q$ is quotient concave. Then we have the uniform second order derivative estimate

$$|D^2 u|_\omega \leq C,$$

where $C$ is a uniform constant depending only on $(M, \omega)$, $\varepsilon$, $n$, $k$, $\chi$, $\psi$, $a$, $\sup_M |u|$, $\sup_M |Du|$. 

**Remark 1.3.** For a non-negative constant $\alpha$, $Q = \alpha \sigma_{k-1}(\lambda) + \sigma_k(\lambda)$ satisfies the quotient concavity, see [23]. In general, $Q$ may not be quotient concave. A sufficient condition for $Q$ to be quotient concave is given by Theorem 3 in [23].

**Remark 1.4.** The quotient concavity is crucial to our estimate since it will give us more good third order terms by Lemma 2.3 and $g \in \Gamma_{k+1}(M)$, see [36, 23] and (3.31).
Remark 1.5. There are no further assumptions on $\psi$ besides that $\psi > 0$. If $k = n$ in (1.2), our method works for $g \in \Gamma_n(M)$ which is a natural elliptic condition. For $1 < k < n$, the more natural assumption to derive the estimate is that $g \in \Gamma_k(M)$, which is still open.

If the right hand side function $\psi$ does not depend on $Du$, a second order estimate for complex $k$-Hessian equations on compact Kähler manifolds was first derived by Hou-Ma-Wu [20]. The particular form of their estimate was used to establish gradient estimates by a blowup argument and Liouville type theorem due to Dinew-Kolodziej [9], where the complex $k$-Hessian equations were solved. On Hermitian manifolds, the problem was settled by Székelyhidi [30] and Zhang [36]. See Collins-Picard [6] for the Dirichlet problem. We remark that Zhang [37] proved a uniform gradient estimate for solutions in $\Gamma_{k+1}$ cone to the complex $k$-Hessian equations involving gradient terms on the left hand sides.

The function $\psi$ depending on the gradient $Du$ creates substantial new difficulties due to the difference between the terms $|DDu|^2$ and $|D\overline{Du}|^2$. A consequence of this is that we cannot control the bad third order terms directly as in Li [25] or [20]. But, in the real case, if $\psi$ is convex with respect to $Du$, one can establish the second order estimate as Guan-Jiao [13]. Without the convexity assumption on $\psi$, the desired estimate was first derived for solutions in $\Gamma_n$ cone to the real $k$-Hessian equations by Guan-Ren-Wang [17] and then for solutions in $\Gamma_{k+1}$ cone by Li-Ren-Wang [24]. For the real counterpart of equation (1.2), the second order estimate was proved by Li-Ren-Wang [23].

For the complex $k$-Hessian equations, Phong-Picard-Zhang [26] first generalized the result in [17] to Kähler manifolds. It is more difficult to control the negative third order terms due to complex conjugacy, since we get only half as many useful terms “$B$” and “$D$” (see Section 3) as in the real case. These are the main difficulties that have been overcome in [20]. On Hermitian manifolds, there will be more bad third order terms of the form $T \ast D^3u$, where $T$ is the torsion of $\omega$. Furthermore, the linear gradient terms in $g$ also bring some bad terms, see “$H$” and “$I$” in Section 3. The method in [20] cannot be used to overcome these new difficulties. Therefore, we apply the maximum principle to the test function used by the author in [10]. This gives us a little more good third order terms which are sufficient to push the argument through.

The rest of the paper is organized as follows. In Section 2, we introduce some useful notations, properties of the $k$-th elementary symmetric functions, and some preliminary calculations and estimates. In Section 3, we prove Theorem 1.2.

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2. Preliminaries

First, we introduce some notations. Let $A^{1,1}(M)$ be the space of smooth real $(1,1)$-forms on $(M, \omega)$. For any $h \in A^{1,1}(M)$, written in local coordinates as $h = \sqrt{-1}h_{ij}dz^i \wedge \overline{dz}^j$, we say

$$h \in \Gamma_k(M)$$

if the vector of eigenvalues of the Hermitian endomorphism $h_{ij} = \omega^{ik}h_{jk}$ lies in the $\Gamma_k$ cone at each point. With the above notation, in local coordinates, (1.2) can be
rewritten as follows:

\(Q(g^j) = Q\left(\omega^{ik}(\chi_j u + a_j u + a_k u_j)\right) = \psi(z, Du, u).\)

In local complex coordinates \((z_1, \ldots, z_n)\), the subscripts of a function \(u\) always denote the covariant derivatives of \(u\) with respect to the Chern connection of \(\omega\) in the directions of the local frame \((\partial/\partial z^1, \ldots, \partial/\partial z^n)\). Namely,

\[u_i = D_i u = D_{\partial/\partial z^i} u, \quad u_i = D_{\partial/\partial z^i} D_{\partial/\partial z^i} u, \quad u_i = D_{\partial/\partial z^i} D_{\partial/\partial z^i} D_{\partial/\partial z^i} u.\]

We have the following commutation formula on Hermitian manifolds (see [32] for more details):

\[u_i = u_i - u_p R_{ij}^p, \quad u_p = u_p - T_{mj}^q u_p, \quad u_i = u_i - T_{il}^p u_i.\]

\[u_i = u_i + u_p R_{ijk}^p - u_p R_{ij}^p - T_{ml}^q u_i - T_{nl}^q u_i - T_{il}^q u_i.\]

As in [6] and [29], we define the tensor

\[\sigma^p_{ij} = \frac{\partial \sigma_k}{\partial g^p} \omega^{ij} \text{ and } \sigma^p_{ij} = \frac{\partial^2 \sigma_k}{\partial g^p \partial g^q} \omega^{ij}.\]

Then, we introduce the following notations

\[Q^p_q = \sum_{s=1}^k \alpha_s \sigma^p_{qs}, \quad Q^p_{qs} = \sum_{s=1}^k \alpha_s \sigma^p_{qs}, \quad Q = \sum_p Q^p_{qs} \omega_q.\]

and the following notations as in [20]

\[(DDu)^2_Q = Q^{p}_{qs} \omega^{mp} u_{mp} \eta_{qs}; \quad |D^2 Du|^2_Q = Q^{p}_{qs} \omega^{mp} u_{mp} \eta_{qs}; \quad |\eta|^2_Q = Q^{p}_{qs} \eta_p \eta_q.\]

for any 1-form \(\eta\).

We use \(\sigma_k(\lambda|i)\) to denote the \(k\)-th elementary symmetric function with \(\lambda_i = 0\) and \(\sigma_k(\lambda|i)\) the \(k\)-th elementary function with \(\lambda_i = 0\). We list here some properties of the \(k\)-th elementary symmetric function.

**Lemma 2.1.** For \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n\) and \(k = 1, \ldots, n\), we have

1. \(\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|\bar{i})\), \(\forall 1 \leq i \leq n;\)
2. \(\sum_{i=1}^n \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda);\)
3. For \(\lambda \in \Gamma_k\) and \(\lambda_1 \geq \cdots \geq \lambda_n\), we have \(\lambda_1 \sigma_{k-1}(\lambda|\bar{1}) \geq \frac{k}{n} \sigma_k(\lambda);\)
4. If \(\lambda \in \Gamma_k\), we have \(\lambda i j \in \Gamma_{k-1}\) for all \(1 \leq i \leq n;\)
5. If \(\lambda_1 - \lambda_j\), we have \(\sigma_{k-1}(\lambda|i) \leq \sigma_{k-1}(\lambda|\bar{i});\)
6. For \(\lambda \in \Gamma_k\) and \(\lambda_1 \geq \cdots \geq \lambda_n\), we have \(\sigma_k(\lambda) \leq C\lambda_1 \cdots \lambda_k.\)

**Proof.** For (1) (2), it is trivial. For (3) see [20]. For (4) see [21]. For (5) and (6), see [23].

**Lemma 2.2.** For \(\lambda \in \Gamma_{k+1}\) with ordering \(\lambda_1 \geq \cdots \geq \lambda_n\), we have \(\sigma_k \geq \lambda_1 \cdots \lambda_k.\)

If \(\lambda\) satisfies \(Q(\lambda) = \psi\) in addition, we have \(\lambda_i + K_0 \geq 0\) for a uniform positive constant \(K_0\) depending on \(\psi\) and \(\alpha_k.\)

For the proof of the above lemma, see [24]. We also need the following lemma.
Lemma 2.3 ([17]). Suppose $1 \leq \ell < k \leq n$, and let $\beta = 1/(k-\ell)$. Let $W = (w_{pq})$ be a Hermitian tensor in the $\Gamma_k$ cone. $Q$ given in ([12]) satisfies the quotient concavity. Then for any $\theta > 0$,

\begin{equation}
- Q^{\ell \sigma} (W) w_{pq} w_{s\sigma} + \left( 1 - \beta + \frac{\beta}{\theta} \right) \frac{|D_\ell Q(W)|^2}{Q(W)} \geq Q(W)(\beta + 1 - \beta \theta) \frac{|D_\ell S_\ell(W)|^2}{S_\ell(W)} - \frac{Q}{S_\ell(W)} S^{\ell \sigma} (W) w_{pq} w_{s\sigma}.
\end{equation}

The proof is similar to the proof of Lemma 2.2 in [17] since $(Q/S_\ell)^{1/(k-\ell)}$ are concave functions.

Lemma 2.4 ([1]). Suppose that $F(A) = f(\lambda_1, \ldots, \lambda_n)$ is a symmetric function of the eigenvalues of a Hermitian matrix $A = (h^i_j)$, then at a diagonal matrix $A$ with distinct eigenvalues, we have,

\begin{equation}
\frac{\partial F}{\partial h^i_j} = \delta_{ij} f_i,
\end{equation}

\begin{equation}
\frac{\partial^2 F}{\partial h^i_j \partial h^{r_s}} T_i T_j s = \sum f_{ij} T_i T_j + \sum_{p \neq q} f_p - f_q |T_p|^2,
\end{equation}

where $T$ is an arbitrary Hermitian matrix.

Note that these formulæ make sense even when the eigenvalues are not distinct, since it can be interpreted as a limit.

Now we do some basic calculations which will be used in the next Section. In the following, $C$ will be a uniform constant depending on the known data as in Theorem 1.2 but may change from line to line.

Our calculations are carried out at a point $p_0$ on the manifold $M$, and we use coordinates such that at this point $\omega = \sqrt{-1} \sum \delta_{\ell k} dz^k \wedge d\bar{z}^\ell$ and $g_{ij}$ is diagonal. Also, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $g^i_j$, with the ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Note that $(Q^{ij})$ is diagonal at the point $p_0$ by Lemma 2.4. Moreover, we see

\begin{equation}
Q^{\ell \sigma} = \frac{\partial Q}{\partial \lambda_p} = \sum_{s=1}^k \alpha_s \sigma_{s-1} (\lambda | p) \quad \text{and} \quad Q^{\ell \sigma} = \frac{\partial^2 Q}{\partial \lambda_p \partial \lambda_q} = \sum_{s=1}^k \alpha_s \sigma_{s-2} (\lambda | pq),
\end{equation}

since $\sigma^{\ell \sigma} = \frac{\partial \sigma}{\partial \lambda_p} = \sigma_{s-1} (\lambda | p)$ and $\sigma^{\ell \sigma} = \frac{\partial^2 \sigma}{\partial \lambda_p \partial \lambda_q} = \alpha_s \sigma_{s-2} (\lambda | pq)$. Using (2.7), one can obtain the well-known identity

\begin{equation}
- Q^{\ell \sigma} D_\ell g_{ij} D_\ell g_{ij} = -Q^{\ell \sigma} D_\ell g_{ij} D_\ell g_{ij} + Q^{\ell \sigma} |D_\ell g_{ij}|^2.
\end{equation}

Differentiating (2.1) yields

\begin{equation}
Q^{\ell \sigma} D_\ell g_{ij} = D_\ell \psi.
\end{equation}

Differentiating the equation a second time gives

\begin{equation}
Q^{\ell \sigma} D_\ell D_\ell g_{ij} + Q^{\ell \sigma} D_\ell g_{ij} D_\ell g_{ij} = D_\ell D_\ell \psi
\end{equation}

\begin{equation}
\geq -C(1 + |DDu|^2 + |DDu|^2) + \sum_\ell \psi_{ij} u_{\ell j} + \sum_\ell \psi_{ij} u_{\ell j} - C\lambda_1.
\end{equation}
Using the Cauchy inequality, we have
\begin{equation}
(2.13)
Q^\varphi\eta D^2\eta D^\varphi D^2\eta D^\eta u + Q^\varphi\eta D^2\eta D^\varphi D^2\eta D^\eta (a_\nu u_\gamma + a_\gamma u_\nu) - C\lambda_1 \chi_1 \eta.
\end{equation}
By (2.22) and (2.23), commuting derivatives yields that
\begin{equation}
(2.12)
\begin{aligned}
& u_{ijpq} = D^2\eta D^i u_\beta - D^2\eta D^i (a_\nu u_\beta + a_\beta u_\nu) - D^2\eta D^i \chi_1 \eta \\
& - R_{ijpq} a_\nu u_\alpha + R_{pqgi} a_\nu u_\alpha - T_{pq} a_\nu u_\alpha - T_{pq} a_\nu u_\alpha.
\end{aligned}
\end{equation}
Combining (2.10), (2.11) and (2.12), we have
\begin{equation}
(2.17)
Q^\varphi\eta D^\varphi D^\eta g_{ij} \leq -Q^\varphi\eta D^\varphi D^\eta g_{ij} + \sum \psi_{ij} g_{ij} + \sum \psi_{ij} g_{ij} - Q^\varphi\eta (T_{ij}^\alpha a_\beta u_\alpha)
\end{equation}
By (2.22) direct calculation gives
\begin{equation}
(2.15)
Q^\varphi\eta\left|Du\right|_{\eta}^2
= Q^\varphi\eta\left(u_{m\nu}\eta D^m u + u_{m\nu}\eta \omega^m \varphi\right) + |Du|_\eta^2 + |Du|_{\eta}^2
= Q^\varphi\eta D_m (g_{\pi\rho} - \chi_{\pi\rho}) D^m u - Q^\varphi\eta T_{pm} u_\pi D^m u
+ Q^\varphi\eta u_{\nu m} T_{q}^m u_\pi + |Du|_\eta^2 + |Du|_{\eta}^2
\end{equation}
Using the Cauchy inequality, we have
\begin{equation}
(2.14)
Q^\varphi\eta (a_\nu u_\rho + a_\rho u_\nu - a_\nu u_\rho - a_\nu u_\rho) \leq \frac{1}{4} |Du|_\eta^2 + C\eta.
\end{equation}
Using the Cauchy inequality, we have
\begin{align*}
& |Q^\varphi\eta (T_{pm}^\alpha u_\pi D^m u) + |Q^\varphi\eta u_{m\nu} \omega^m T_{q}^\alpha u_\pi| \\
& + |Q^\varphi\eta D_m (a_\nu u_\rho + a_\rho u_\nu) D^m u| + |Q^\varphi\eta D_\varphi (a_\nu u_\rho + a_\rho u_\nu) u_{m\nu} \omega^m | \\
& \leq \frac{1}{2} |Du|_\eta^2 + \frac{1}{2} |Du|_{\eta}^2 + C\eta.
\end{align*}
Substituting the above inequality into (2.15), we get
\begin{equation}
(2.16)
Q^\varphi\eta\left|Du\right|_{\eta}^2 \geq D_m (Q) u_{m\nu} \omega^m + D_\varphi (Q) u_{m\nu} \omega^m + \frac{1}{2} |Du|_\eta^2 + \frac{1}{2} |Du|_{\eta}^2 - C\eta.
\end{equation}
Using the differential equation (2.22), we obtain
\begin{equation}
(2.17)
Q^\varphi\eta\left|Du\right|_{\eta}^2 \geq 2 \text{Re} \left\{ \sum p, m (D_p D_m u D_{\eta}^2 u + D_p u D_{\varphi} D_m u) \psi_{vm} \right\} \\
- C - C\eta + \frac{1}{2} |Du|_\eta^2 + \frac{1}{2} |Du|_{\eta}^2.
\end{equation}
Hence,
\[ g \psi \leq k \psi. \]

Henceforth, we assume that
\[ \gamma < 0 \]
and the desired estimate follows in turn from \( g \in \Gamma_2 \subset \{ \lambda | \sum \lambda_i + \cdots + \lambda_{n-1} \geq 0 \} \). Henceforth, we assume that \( k \geq 2 \).

Since \( g \in \Gamma_{k+1} \), by Lemma 2.2 there exists a positive constant \( K_0 \) depending on \( \psi, \sup_M |u|, \sup_M |Du| \) and \( \alpha_k \) such that \( \lambda_i(g) + K_0 \geq 0 \) for any \( 1 \leq i \leq n \). We apply the maximum principle to the following test function:
\[ G = \log P_m + \varphi(|Du|^2) + \phi(u), \]
where \( P_m = \sum_j \kappa_j^m \) and \( \kappa_j = \lambda_j(g) + K_0 \). Here, \( \varphi \) and \( \phi \) are positive functions to be determined later, which satisfy the following assumptions
\[ \varphi'' - 2\phi''(\frac{\phi'}{\phi})^2 \geq 0, \varphi' > 0, \phi' < 0, \phi'' > 0. \]

We may assume that the maximum of \( G \) is achieved at some point \( p_0 \in M \). We choose the coordinate system centered at \( p_0 \) such that \( \omega = \sqrt{-1} \sum \delta_i dz^k \wedge d\bar{z}^k, g_{\bar{z}z} \)

is diagonal, and the eigenvalues of the Hermitian endomorphism \( g_{\bar{z}z} \) are ordered as \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \).

Differentiating \( G \), we first obtain the critical equation
\[ \frac{DP_m}{P_m} + \varphi' D|Du|^2 + \phi' Du = 0. \]

Differentiating \( G \) a second time, using (2.7) and contracting with \( Q^\nu \)
yields
\[ 0 \geq \frac{m}{P_m} \sum_j \kappa_j^{m-1} Q^\nu Dp g_{j\bar{j}} + \frac{mQ^\nu}{P_m} (m-1) \sum_j \kappa_j^{m-2} |D_p g_{j\bar{j}}|^2 \]
\[ \quad + \frac{mQ^\nu}{P_m} \sum_{j \neq j} \frac{\kappa_i^{m-1} - \kappa_j^{m-1}}{\kappa_i - \kappa_j} |D_p g_{j\bar{j}}|^2 + Q^\nu (\varphi'' D_p u D_{\bar{\nu}} + \phi' u_{\bar{\nu}}) \]
\[ \quad + Q^\nu (\varphi'' D_p |Du|^2 D_{\bar{\nu}} |Du|^2 + \varphi' |Du|^2) - \frac{|DP_m|^2 Q}{P_m}. \]

Here we used the notation introduced in Section 2.

Using the critical equation (3.3), we obtain
\[ D_p u D_{\bar{\nu}} u \geq \frac{1}{2|\phi'|^2} \frac{|D_p P_m|^2}{P_m^2} - \frac{(\varphi'')^2 |D_p| |Du|^2}{2Q}. \]
Substituting (3.5) into (3.4) and by (2.13), (2.14), (2.17), (2.18)

\[
0 \geq - \frac{C}{P_m} \sum_j \kappa_j^{m-1} (1 + |DDu|^2 + |D\overline{D}u|^2 + (1 + \lambda_1) Q + \lambda_1) \\
+ \frac{\sum_j \kappa_j^{m-1}}{\kappa_i - \kappa_j} \left( - Q^{\overline{\sigma}} C_\overline{j} g_j \overline{D}_j g_{\overline{j}} + \sum \psi_{v_i} g_{\bar{j} \ell} + \sum \psi_{r_i} g_{\overline{j} \bar{\ell}} \right) \\
- Q^{\overline{\sigma}} (T_{p_j} u_{\sigma \overline{j}} + \overline{T}_{p_j} u_{\sigma \overline{j}}) \right) \\
+ \frac{m - 1}{P_m} \sum_j \kappa_j^{m-2} Q^{\overline{\sigma}} |D_p g_j|^2 \\
+ \frac{Q^{\overline{\sigma}}}{P_m} \sum_{i \neq j} \frac{\kappa_i^{m-1} - \kappa_j^{m-1}}{\kappa_i - \kappa_j} |D_p g_j|^2 \\
+ \left( 1 - \frac{\phi''}{2(\phi')^2} \right) \frac{2P_m^2}{m} \\
\geq C \frac{\phi'}{m},
\]

as long as \( \frac{\phi'}{m} \geq \frac{C}{\kappa_1} \). Here we used \( \phi' > 0 \) and \( \phi' < 0 \) in (3.2).

From the critical equation (3.3), we obtain

\[
\frac{1}{P_m} \sum_j \kappa_j^{m-1} \left( \psi_{v_i} D_j g_{\overline{j}} + \psi_{r_i} D_j g_{\overline{j}} \right) \\
+ 2 \frac{\phi'}{m} \left( \sum_{p, m} (u_{m p} u_{\overline{p}} + u_p u_{m \overline{p}}) \psi_{v_m} \right) \\
= - \frac{\phi'}{m} \sum \psi_{v_i} u_{\overline{f} \ell} - \frac{2 \phi'}{m} \left( \sum \psi_{v_m} T_{p m}^k u_k u_{\overline{p}} \right) \\
\geq C \frac{\phi'}{m},
\]

and

\[
2 \frac{\phi'}{m} Q^{\overline{\sigma}} \text{Re} \{a_p u_{\overline{p}}\} = \left| 2 \frac{Q^{\overline{\sigma}}}{m} \text{Re} \left\{ a_p \left( \frac{D_p P_m}{P_m} + \phi' D_{\overline{\sigma}} |Du|^2 \right) \right\} \right| \\
\leq \frac{\phi''}{4 \phi'^2} \frac{|DP_m|^2}{m} + \frac{Q^{\overline{\sigma}}}{m} \frac{\phi''}{\phi'^2} \left| D_p |Du|^2 \right|^2 + C \frac{\phi'^2}{\phi'^2} Q.
\]
With the above calculations and \(2.18\), we have

\[
0 \geq \frac{1}{P_m} \sum_j \kappa_j^{m-1} \left( -Q^\varphi P\varphi D_j g_{\varphi \bar{\varphi}} D_j g_{\varphi \bar{\varphi}} + Q^\varphi P\varphi |D_j g_{\varphi \bar{\varphi}}|^2 \right) \\
- 2Q^\varphi P\varphi \sum_j \kappa_j^{m-1} \text{Re} \left( \overline{T_{j,p}} u_{j\varphi} \right) + \frac{(m-1)Q^\varphi P\varphi}{P_m} \sum_j \kappa_j^{m-2} |D_j g_{\varphi \bar{\varphi}}|^2 \\
+ \frac{Q^\varphi P\varphi}{P_m} \sum_{i \neq j} \frac{\kappa_i^{m-1} - \kappa_j^{m-1}}{\kappa_i - \kappa_j} |D_p g_{j\bar{j}}|^2 - \left( 1 - \frac{\phi''}{4(\phi')^2} \right) \frac{|D_P|^2 Q}{mP_m^2} \\
+ \frac{\phi'}{4m} \left( |DD_u|^2 + |D\overline{D}u|^2 \right) + C \frac{\phi'}{m} - C \frac{\phi''}{m} - C \\
+ \left( - \frac{\phi'}{m^2} - C \frac{\phi'}{m} - C \frac{\phi''}{m} \right) Q - \frac{C}{\lambda_1} \left( |DD_u|^2 + |D\overline{D}u|^2 \right) \\
+ \frac{Q^\varphi P\varphi}{P_m} \sum_j \kappa_j^{m-1} \left( a_j u_{j\varphi \bar{\varphi}} + a_j u_{j\bar{\varphi} \varphi} - a_j u_{j\bar{\varphi} \bar{\varphi}} - a_j u_{j\varphi \varphi} \right),
\]

where we used \(\phi'' - 2\phi' \frac{\phi''}{\phi'} \geq 0\) in \(3.2\) and \(\kappa_1 \gg 1\).

Let

\[
\tilde{A}_j = \frac{1}{P_m} \kappa_j^{m-1} \sum_{p,q} Q^\varphi P\varphi D_j g_{\varphi \bar{\varphi}} D_j g_{\varphi \bar{\varphi}}, \quad \tilde{B}_q = \frac{1}{P_m} \sum_{j,p} \kappa_j^{m-1} Q^\varphi P\varphi |D_j g_{\varphi \bar{\varphi}}|^2, \\
C_p = \frac{m-1}{P_m} \sum_{j,p} \kappa_j^{m-2} |D_j g_{\varphi \bar{\varphi}}|^2, \quad \tilde{D}_p = \frac{Q^\varphi P\varphi}{P_m} \sum_{j \neq i} \frac{\kappa_i^{m-1} - \kappa_j^{m-1}}{\kappa_i - \kappa_j} |D_j g_{\varphi \bar{\varphi}}|^2, \\
E_i = \frac{m-1}{P_m} \sum_p \kappa_p^{m-1} |D_i g_{\varphi \bar{\varphi}}|^2, \quad H_p = \frac{2Q^\varphi P\varphi}{P_m} \sum_{j \neq i} \kappa_j^{m-1} \text{Re}(\overline{T_{j,p}} u_{j\varphi}), \\
\text{and}
\]

\[
I_p = \frac{Q^\varphi P\varphi}{P_m} \sum_j \kappa_j^{m-1} \left( a_j u_{j\varphi \bar{\varphi}} + a_j u_{j\bar{\varphi} \varphi} - a_j u_{j\bar{\varphi} \bar{\varphi}} - a_j u_{j\varphi \varphi} \right).
\]

Then \(3.9\) becomes

\[
0 \geq - \sum_j \tilde{A}_j + \sum_q \tilde{B}_q + \sum_p C_p + \sum_p \tilde{D}_p - \sum_p H_p + \sum_p I_p \\
- \left( 1 - \frac{\phi''}{4(\phi')^2} \right) \sum_i E_i + \frac{\phi'}{4m} \left( |DD_u|^2 + |D\overline{D}u|^2 \right) \\
+ \left( - \frac{\phi'}{m^2} - C \frac{\phi'}{m} - C \frac{\phi''}{m} \right) Q - C \left( - \frac{\phi'}{m} + \frac{\phi'}{m} + 1 \right) \\
- \frac{C}{\lambda_1} \left( |DD_u|^2 + |D\overline{D}u|^2 \right).
\]
We first deal with the torsion term $H_p$. For any $0 < \tau < 1$, by (2.2) we can estimate
\[
H_p \leq \frac{2Q^p}{P_m} \sum_{j,s} \kappa_j^{m-1} |T^s_p D_p g |^2 + CQ^p \frac{C}{\kappa_1} \sum_j |u_{jp}|^2
\]
\[
\leq \frac{\tau Q^p}{2 T_m} \sum_{j,s} \kappa_j^{m-2} |D_p g|^2 + \frac{C}{\tau} Q^p \frac{C}{\kappa_1} \sum_j |u_{jp}|^2.
\]
Similarly, we can estimate
\[
|I_p| \leq \frac{\tau Q^p}{4 P_m} \sum_j \kappa_j^{m-2} (|D_p g|^2 + |D_p g_j|^2) + \frac{C}{\tau} Q^p \frac{C}{\kappa_1} \sum_j |u_{jp}|^2
\]
\[
\leq \frac{\tau Q^p}{2 P_m} \sum_{j,s} \kappa_j^{m-2} |D_p g|^2 + \frac{C}{\tau} Q^p \frac{C}{\kappa_1} \sum_j |u_{jp}|^2.
\]
By direct computation, we have
\[
(3.11) \quad \tilde{D}_p = \frac{Q^p}{P_m} \sum_{j \neq i} \kappa_j^{m-2-s} \kappa_j^s |D_p g_j|^2 \geq \frac{Q^p}{P_m} \sum_{j \neq i} \kappa_j^{m-2} |D_p g_j|^2.
\]
Now we have
\[
(3.12) \quad 0 \geq (1 - \tau) \sum_p \tilde{D}_p + (1 - \tau) \sum_p C_p - \frac{C}{\tau} Q - \frac{C}{\kappa_1} |DDu|^2_Q,
\]
as $m > 2$ will be chosen large enough. Substituting (3.12) into (3.10) yields
\[
(3.13) \quad 0 \geq - \sum_j \tilde{A}_j + \sum_q \tilde{B}_q + (1 - \tau) \left( \sum_p C_p + \sum_p \tilde{D}_p \right)
\]
\[
- \left( 1 - \frac{\phi''}{4(\phi')^2} \right) \sum_i E_i + \frac{\phi' \phi^2}{4m} \left( |DDu|^2_Q + |D\tilde{B}u|^2_Q \right)
\]
\[
+ \left( -\frac{\phi'}{m} - C \frac{\phi'}{m} - \frac{C}{\tau} - \frac{\phi^2}{\phi''} \right) Q - C(\phi' - \phi' + 1)
\]
\[
- \frac{C}{\kappa_1} \left( |DDu|^2 + |D\tilde{B}u|^2 \right).
\]
Now we shall denote
\[
A_j = \frac{1}{P_m} \kappa_j^{m-1} \left( K |D_j Q|^2 - Q^p \phi^2 \sum_{j \neq i} |D_j g |^2 \right),
\]
\[
B_q = \frac{1}{P_m} \sum_{p \neq i} \kappa_p^{m-1} |Q^p \phi^2 |^2 |D_p g|^2,
\]
\[
D_i = \frac{1}{P_m} \sum_{p \neq i} Q^{p} \left( \kappa_p^{m-1} - \kappa_i^{m-1} \right) |D_p g|^2.
\]
Taking $\ell = 1$ and $\theta = 1/2$ in Lemma 2.3, we see that $A_j \geq 0$ for $1 \leq j \leq n$ for $K > (1 + \beta)(\inf \psi)^{-1}$ if $2 \leq k \leq n$. Define
\[
H_j g = D_j (\chi_j g + a_j u_j + a_j w_j) - D_p (\chi_j g + a_j u_j + a_j w_j).
\]
Note that $|H_{ij}| \leq C + C\lambda_1$, since $u_{pj} - u_{jp} = T_{pj}^k u_k \leq C$. For any constant $0 < \tau < 1$, we can estimate

$$\sum_q \tilde{B}_q \geq \frac{1}{P_m} \sum_{j,q} \kappa_j^{m-1} Q^{\tilde{j},q\tilde{q}} |D_q g_{qj}|^2$$

$$= \frac{1}{P_m} \sum_{j,q} \kappa_j^{m-1} Q^{\tilde{j},q\tilde{q}} |D_q g_{qj} - T_{aq}^a u_{aqj} + H_{aqj}|^2$$

$$\geq \frac{1}{P_m} \sum_{j,q} \kappa_j^{m-1} Q^{\tilde{j},q\tilde{q}} \left( (1 - \tau) |D_q g_{qj}|^2 - \frac{1}{\tau} |H_{aqj} - T_{aqj}^a u_{aqj}|^2 \right)$$

$$= (1 - \tau) \sum_q B_q - \frac{1}{\tau P_m} \sum_{q,j} \kappa_j^{m-1} Q^{\tilde{j},q\tilde{q}} |H_{aqj} - T_{aqj}^a u_{aqj}|^2.$$
Note that $\frac{e^{m-1}}{P_m}|D_j Q|^2 \leq \frac{C}{\kappa_1} (|DDu|^2 + |D\overline{Du}|^2)$. Then (3.13) becomes

$$0 \geq (1 - \tau)^2 \sum_i \left( A_i + B_i + C_i + D_i \right) - \left( 1 - \frac{\phi'}{4(\phi')^2} \right) \sum_i E_i$$

(3.14)

$$+ \frac{\phi'}{4m} \left( |DDu|^2 + |D\overline{Du}|^2 \right) \frac{C(K)}{\kappa_1} (|DDu|^2 + |D\overline{Du}|^2)
+ \left( -\frac{\phi'}{m} \varepsilon - C \frac{\phi'}{m} \frac{\phi'}{m} - C \frac{\phi'}{m} \frac{\phi'}{m} \right) Q - C(\phi' - \phi' + \frac{1}{\tau}),$$

when $\lambda_1$ is sufficiently large.

Now we choose $\phi$ and $\phi$ to satisfy (3.2). Let $\varphi(t) = e^{Nt}$ and $\phi(s) = e^{M(-s+L)}$ where $L \geq |u|_{C^1} + 1$ is a constant. Then, we see

$$\varphi'' - 2\frac{\varphi'}{\varphi'} = N^2 e^{Nt} - 2 \frac{N^2 e^{2Nt}}{e^{M(-s+L)}} > 0, \quad \varphi' > 0, \quad \varphi' < 0, \quad \varphi'' > 0,$$

when $M \gg N > 1$, which shows the assumption (3.2) is satisfied. Choosing

$$2\tau = \frac{\phi''}{4(\phi')^2} = \frac{1}{4e^{M(-u(p)+L)}},$$

we obtain that $(1 - \tau)^2 \geq 1 - \frac{\phi'}{\varepsilon \phi'},$ By Lemma 2.1 (3) and (5), we have

$$Q^\tau \geq Q^{\tau} \geq \frac{Q}{n\lambda_1} \geq \frac{1}{C\lambda_1}$$

for any fixed $i$, where $C$ depends on $\inf \psi > 0$ and other known data. We can estimate

$$|DDu|^2 + |D\overline{Du}|^2 \geq \frac{1}{C\lambda_1} (|DDu|^2 + |D\overline{Du}|^2) \geq \frac{1}{C\lambda_1} |DDu|^2 + \frac{\lambda_1}{C}.$$

Now (3.14) becomes

$$0 \geq (1 - \tau)^2 \sum_i \left( A_i + B_i + C_i + D_i - E_i \right)$$

(3.15)

$$+ \left( \frac{\varphi'}{mC} - C(K) \right) \lambda_1 + \frac{1}{\lambda_1} \left( \frac{\varphi'}{mC} - C(K) \right) |DDu|^2
+ \left( -\frac{\phi'}{m} \varepsilon - C \frac{\phi'}{m} \frac{\phi'}{m} - C \frac{\phi'}{m} \frac{\phi'}{m} \right) Q - C(\phi' - \phi' + \frac{1}{\tau}).$$

Taking $N$ large enough, we can ensure that $\frac{\varphi'}{mC} - C(K) > 0$. For fixed $N$, it follows that

$$-\frac{\phi'}{m} \varepsilon - C \frac{\phi'}{m} \frac{\phi'}{m} - C \frac{\phi'}{m} \frac{\phi'}{m} = \frac{M}{m} \varepsilon \phi - C \frac{N}{m} \varphi - C \phi > 0$$

when $M \gg N$.

Claim: For sufficiently large $m$, we may assume

$$A_i + B_i + C_i + D_i - E_i \geq 0, \quad \forall i = 1, \ldots, n.$$

This leads to

$$0 \geq \left( \frac{\varphi'}{mC} - C(K) \right) \lambda_1 - C(\phi' - \phi' + \phi),$$

which finally implies an upper bound of $\lambda_1$.

We now prove the claim to finish the proof of Theorem 1.2.
**Lemma 3.1.** For sufficiently large $m$, the following estimates hold:

\[(3.16)\quad P_m^2(B_i + C_i + D_i - E_i) \geq P_m \kappa_1^{m-2} \sum_{j \neq i} Q^{ij} |D_i g_{jj}|^2 - Q^{i1} \kappa_1^{m-2} |D_i g_{ii}|^2,\]

and for any index $i \neq 1$,

\[(3.17)\quad P_m^2(B_i + C_i + D_i - E_i) \geq 0.\]

**Proof.** For any index $i$, we compute that

\[P_m(B_i + D_i) = \sum_{j \neq i} Q^{ji} \kappa_j^{m-1} |D_i g_{jj}|^2 + \sum_{j \neq i} Q^{jj} \sum_{l=0}^{m-2} \kappa_i^{m-2-l} \kappa_j^{l} |D_i g_{jj}|^2.\]

Note that $\kappa_j Q^{ji} i q + Q^{ij} \geq Q^{ii}$. To see this, we compute that

\[
\kappa_j \sigma_s^{ji} + \sigma_s^{ji} = \lambda_j \sigma_s - 2(\lambda jj) + K_0 \sigma_s - 2(\lambda ij) + \sigma_s - 2(\lambda j) = \sigma_s - 2(\lambda ij) + K_0 \sigma_s - 2(\lambda ij) + \sigma_s - 2(\lambda j) \geq \sigma_s - 2(\lambda i),
\]

for every $1 \leq s \leq k$. We therefore have

\[
P_m(B_i + D_i) \geq \sum_{j \neq i} Q^{ji} \kappa_j^{m-2} |D_i g_{jj}|^2 + \sum_{j \neq i} Q^{jj} \sum_{l=0}^{m-3} \kappa_i^{m-2-l} \kappa_j^{l} |D_i g_{jj}|^2.
\]

It follows that

\[
P_m(B_i + C_i + D_i) \geq m Q^{ii} \sum_{j \neq i} \kappa_j^{2m-2} |D_i g_{jj}|^2 + (m-1) Q^{ii} \kappa_i^{2m-2} |D_i g_{ii}|^2 + \sum_{j \neq i} Q^{jj} \sum_{l=0}^{m-3} \kappa_i^{m-2-l} \kappa_j^{l} |D_i g_{jj}|^2.
\]

For $E_i$, we have

\[
P_m^2 E_i = m Q^{ii} \sum_{j \neq i} \kappa_j^{2m-2} |D_i g_{jj}|^2 + m Q^{ii} \kappa_i^{2m-2} |D_i g_{ii}|^2 + m Q^{ii} \sum_{p \neq i} \sum_{q \neq p, i} \kappa_p^{m-1} \kappa_q^{m-1} D_i g_{pp} \overline{D_i g_{qq}} + 2m Q^{ii} \text{Re} \sum_{j \neq i} \kappa_j^{m-1} D_i g_{ji} \overline{\kappa_j^{m-1} D_i g_{jj}}.
\]

By Cauchy-Schwarz inequality, we know

\[
\sum_{p \neq i} \sum_{q \neq p, i} \kappa_p^{m-1} \kappa_q^{m-1} D_i g_{pp} \overline{D_i g_{qq}} \leq \sum_{p \neq i} \sum_{q \neq p, i} \frac{1}{2} \left( \kappa_p^{m-2} \kappa_q |D_i g_{pp}|^2 + \kappa_q^{m-2} \kappa_p |D_i g_{qq}|^2 \right).
\]

By the symmetry of $p$ and $q$ in the above inequality, we obtain

\[
\sum_{p \neq i} \sum_{q \neq p, i} \kappa_p^{m-1} \kappa_q^{m-1} D_i g_{pp} \overline{D_i g_{qq}} \leq \sum_{p \neq i} \sum_{q \neq p, i} \kappa_p^{m-2} \kappa_q^{m-1} |D_i g_{pp}|^2.
\]
Therefore,
(3.18)
\[ P_m^2 (B_i + C_i + D_i - E_i) \]
\[ \geq Q^{ii} \sum_{j \neq i} |mP_m - m\kappa_j^{m-2}|D_t^jg_{jj}|^2 + P_m \sum_{j \neq i} Q^{ij} \sum_{l=0}^{m-1} \kappa_j^{m-2-l} \kappa_j^l |D_t^jg_{jj}|^2 \]
\[ + [(m-1)P_m - m\kappa_k^m]Q^{ij} \kappa_j^{m-2}|D_t^jg_{ii}|^2 - mQ^{ii} \sum_{j \neq i} \sum_{l \neq j} \kappa_j^{m-2} \kappa_l^m |D_t^jg_{jj}|^2 \]
\[ - 2mQ^{ii} \text{Re} \sum_{j \neq i} \kappa_j^{m-1} D_t^jg_{ii} \kappa_j^{m-1} \overline{D_t^jg_{jj}}. \]

By calculating, we see \( mP_m - m\kappa_j^m - m \sum_{q \neq j} \kappa_q^m = m\kappa_j^m \), and we arrive at
(3.19)
\[ P_m^2 (B_i + C_i + D_i - E_i) \]
\[ \geq mQ^{ii} \sum_{j \neq i} \kappa_j^{m-2} |D_t^jg_{jj}|^2 + [(m-1)P_m - m\kappa_k^m]Q^{ij} \kappa_j^{m-2}|D_t^jg_{ii}|^2 \]
\[ + P_m \sum_{j \neq i} Q^{ij} \sum_{l=0}^{m-3} \kappa_j^{m-2-l} \kappa_j^l |D_t^jg_{jj}|^2 - 2mQ^{ii} \text{Re} \sum_{j \neq i} \kappa_j^{m-1} D_t^jg_{ii} \kappa_j^{m-1} \overline{D_t^jg_{jj}}. \]

We now estimate the third term in the right hand side of the above inequality.

Case A: \( \lambda_i \geq \lambda_j \). Then \( \kappa_i \geq \kappa_j \) and \( Q^{ij} \geq Q^{ii} \). Hence
\[ P_m Q^{ij} \sum_{l=1}^{m-3} \kappa_i^{m-2-l} \kappa_j^l \geq \kappa_i^m Q^{ii} \sum_{l=1}^{m-3} \kappa_i^{m-2-l} \kappa_j^l \geq (m-3)Q^{ii} \kappa_i^m \kappa_j^{m-2}. \]

Case B: \( \lambda_i \leq \lambda_j \). We further divide into two subcases. If \( \lambda_i \geq K_0 \),
\[ P_m Q^{ij} \sum_{l=1}^{m-3} \kappa_i^{m-2-l} \kappa_j^l \geq \kappa_i^m \sum_{s=1}^{k} \alpha_s (\lambda_i \sigma_s^{i,j} + \sigma_{s-1}(\lambda|i)) \sum_{l=1}^{m-3} \kappa_i^{m-2-l} \kappa_j^l \]
\[ \geq \frac{1}{2} \kappa_i^m \sum_{s=1}^{k} \alpha_s (\lambda_i \sigma_s^{i,j} + 2\sigma_{s-1}(\lambda|i)) \sum_{l=1}^{m-3} \kappa_i^{m-2-l} \kappa_j^l \]
\[ \geq \frac{1}{2} \kappa_i^m \sum_{s=1}^{k} \alpha_s (\lambda_i \sigma_s^{i,j} + 2\sigma_{s-1}(\lambda|i)) \sum_{l=1}^{m-3} \kappa_i^{m-1-l} \kappa_j^{l-1} \]
\[ \geq \frac{1}{2} \kappa_i^m \sum_{s=1}^{k} \alpha_s (\lambda_i \sigma_s^{i,j} + \sigma_{s-1}(\lambda|i)) \sum_{l=1}^{m-3} \kappa_i^{m-1-l} \kappa_j^{l-1} \]
\[ \geq \frac{1}{2} (m-3)Q^{ii} \kappa_i^m \kappa_j^{m-2}, \]
where we used \( \sigma_{s-1}(\lambda|i) > 0 \) for \( 1 \leq s \leq k \) by our assumption \( \lambda \in \Gamma_{k+1} \) in the third inequality. If \( \lambda_i \leq K_0 \), for \( k \leq l \leq \left\lceil \frac{m-3}{2} \right\rceil \), since \( \lambda_1 \sigma_{s-1}(\lambda|1) \geq \frac{\psi}{n}\sigma_s(\lambda) \), we know
\[ \kappa_i^{l+1} Q^{ij} = \kappa_i^{l+1} \sum_{s=1}^{k} \alpha_s \sigma_s^{i,j} \geq \frac{\psi}{n} \sum_{s=1}^{k} \alpha_s \sigma_s \geq \inf_{n} \psi \kappa_i^l, \]
from which we obtain that $\kappa_1 Q^j j \geq Q^i i$ when $\kappa_1$ is sufficiently large since $Q^i i \leq C\lambda_1^{k-1}$. We then obtain
\[
P_m Q^j j \sum_{l=1}^{m-3} \kappa_i^{m-2-l} \kappa_j^l \geq \sum_{l=k}^{m-3} Q^i i \kappa_1^{m-l-1} \kappa_i^{m-2-l} \kappa_j^l \geq \frac{m-3}{2} Q^i i \kappa_i^m \kappa_j^{m-2}
\]
where in the last inequality we used $\kappa_1 \kappa_i^{m-2-l} \geq \kappa_i^m$ when $\kappa_1$ is sufficiently large.

Combining both cases, we have
\[
P_m Q^j j \sum_{l=0}^{m-3} \kappa_i^{m-2-l} \kappa_j^l |D_1 g_{jj}|^2 \geq \left( \frac{m-3}{2} Q^i i \kappa_i^m \kappa_j^{m-2} + P_m Q^j j \kappa_i^{m-2} \right) |D_1 g_{ij}|^2.
\]
Using Cauchy-Schwarz inequality again, we see
\[
2m \kappa_i^{m-1} D_i g_{ii} \kappa_j^{m-1} D_i g_{jj} \leq \frac{3m-3}{2} \kappa_i^{m} \kappa_j^{m-2} |D_1 g_{jj}|^2 + (m-2) \kappa_i^{m-2} \kappa_i^m |D_1 g_{ii}|^2,
\]
where we also used $m^2 \leq \frac{3m-3}{2} (m-2)$ when $m$ is sufficiently large. Substituting the above two inequalities into (3.19), we arrive at
\[
P_m^2 (B_1 + C_1 + D_1 - E_1)
\]
\[
\geq P_m \kappa_i^{m-2} \sum_{j \neq i} Q^j j |D_1 g_{jj}|^2 - (m-2)Q^i i \sum_{j \neq i} \kappa_i^{m-2} \kappa_j^m |D_1 g_{ii}|^2
\]
\[
+ [(m-1)P_m - m \kappa_i^m] Q^i i \kappa_i^{m-2} |D_1 g_{ii}|^2
\]
\[
\geq [(m-1)P_m - m \kappa_i^m + (m-2)(P_m - \kappa_i^m)] Q^i i \kappa_i^{m-2} |D_1 g_{ii}|^2 \geq 0,
\]
where we used $i \neq 1$ in the last inequality. For the case $i = 1$, by the first inequality of (3.20), we have
\[
P_m^2 (B_1 + C_1 + D_1 - E_1)
\]
\[
\geq P_m \kappa_i^{m-2} \sum_{j \neq 1} Q^j j |D_1 g_{jj}|^2 - \kappa_1^m Q^1 1 \kappa_1^{m-2} |D_1 g_{11}|^2.
\]

\[\square\]

**Lemma 3.2.** Suppose there exists $0 < \delta < 1$ such that $\lambda_\mu \geq \delta \lambda_1$ for some $1 \leq \mu \leq k - 1$. There exists a sufficiently small positive constant $\delta'$ such that if $\lambda_\mu + 1 \leq \delta' \lambda_1$, then
\[
A_1 + B_1 + C_1 + D_1 - E_1 \geq 0.
\]

**Remark 3.3.** With the above hypothesis, if for some $1 \leq s \leq \mu$, $\alpha_s \neq 0$ in $Q$, we have from the equation that
\[
\psi \geq \alpha_s \sigma_s \geq \alpha_s \lambda_1 \cdots \lambda_s \geq \alpha_s \delta^{s-1} \lambda_1^s
\]
from which we can obtain a upper bound for $\lambda_1$. Henceforth, we may assume in the following proof that $\alpha_1 = \alpha_2 = \cdots = \alpha_\mu = 0$ and $Q = \sum_{s=1}^{k} \alpha_s \sigma_s$.

**Proof.** By our assumption, we can always assume that $\lambda_\mu > 1$, otherwise we can obtain an upper bound for $\lambda_1$. By Lemma 3.1 we see
\[
P_m^2 (A_1 + B_1 + C_1 + D_1 - E_1)
\]
\[
\geq P_m^2 A_1 + P_m \kappa_i^{m-2} \sum_{j \neq 1} Q^j j |D_1 g_{jj}|^2 - \kappa_1^m Q^1 1 \kappa_1^{m-2} |D_1 g_{11}|^2.
\]
Choosing $\theta = \frac{1}{2}$ in Lemma 2.3 we have for $\mu < k$

$$P_m^2 A_1 \geq \frac{P_m \kappa_1^{m-1} Q}{S_{\mu}^2} \left( (1 + \frac{\beta}{2}) \sum_j |S_{\mu}^{ij} D_1 g_{jj}|^2 - S_{\mu} S_{\mu}^{pp,qq} D_1 g_{pp} D_1 g_{qq} \right)$$

$$\geq \frac{P_m \kappa_1^{m-1} Q}{S_{\mu}^2} \left( (1 + \frac{\beta}{2}) \sum_j |S_{\mu}^{jj} D_1 g_{jj}|^2 + \frac{\beta}{2} \sum_{p \neq q} S_{\mu}^{pp} S_{\mu}^{qq} D_1 g_{pp} D_1 g_{qq} \right)$$

$$+ \sum_{p \neq q} (S_{\mu}^{pp} S_{\mu}^{qq} - S_{\mu} S_{\mu}^{pp,qq}) D_1 g_{pp} D_1 g_{qq} \right).$$

We now estimate $P_m A_1^2$ case by case. For $\mu = 1$, it is easy to see

$$(1 + \frac{\beta}{2}) \sum_j D_1 g_{jj}^2 \geq (1 + \frac{\beta}{2}) \left[ 2 \text{Re} \sum_{a \neq 1} D_1 g_{aa} \overline{D_1 g_{11}} + |D_1 g_{11}|^2 \right]$$

$$\geq (1 + \frac{\beta}{4}) |D_1 g_{11}|^2 - C_\beta \sum_{a \neq 1} |D_1 g_{aa}|^2.$$ 

Then, using the fact $Q \geq \lambda_1 Q^{11}$ for $\lambda \in \Gamma_{k+1}$, we can derive from the first inequality of (3.22) that

$$P_m^2 A_1 \geq \frac{P_m \kappa_1^{m-2} Q^{11}}{S_1^2} |D_1 g_{11}|^2 - C_\beta \frac{P_m \kappa_1^{m-2} Q^{11}}{S_1^2} \sum_{a \neq 1} |D_1 g_{aa}|^2$$

$$\geq \frac{(1 + \frac{\beta}{4}) P_m \kappa_1^{m-2} Q^{11}}{(1 + \sum_{j \neq 1} \lambda_j / \lambda_1 + \beta_0 / \lambda_1)^2} |D_1 g_{11}|^2 - C_\beta \frac{P_m \kappa_1^{m-1} Q}{S_1^2} \sum_{a \neq 1} |D_1 g_{aa}|^2,$$

where in the last inequality we used $1 + \frac{\beta}{4} \geq (1 + (n-1)\delta' + \beta_0 / \lambda_1)^2$ for sufficiently small positive $\delta'$ and sufficiently large $\lambda_1$. For $\mu \geq 2$, we have from Lemma 2.3 with $\theta = 1$ that

$$P_m^2 A_1 \geq \frac{P_m \kappa_1^{m-1} Q}{S_2^2} \left( \sum_j |S_{\mu}^{jj} D_1 g_{jj}|^2 - \sum_{p \neq q} F_{pq} D_1 g_{pp} D_1 g_{qq} \right),$$

where we used the notation $F_{pq} = S_{\mu}^{pp} S_{\mu}^{qq} - S_{\mu} S_{\mu}^{pp,qq}$.

We compute $F_{pq}$. Recall $S_{\mu} = \sigma_{\mu} + \sum_{s=0}^n \beta_s^\mu \sigma_s$, where $\beta_s^\mu \geq 0$ are constants depending on $s$ and $\mu$. As in Section 2, we can similarly define $S_{\mu}^{\alpha\alpha}$ and $S_{\mu}^{\alpha\alpha,\beta\beta}$. Define

$$S_{\mu-1}(\lambda|a) := \sigma_{\mu-1}(\lambda|a) + \sum_{s=0}^{\mu-1} \beta_s^\mu \sigma_{s-1}(\lambda|a),$$

and similarly define $S_{\mu-1}(\lambda|ab)$ and $S_{\mu-2}(\lambda|ab)$, where we use notation that $\sigma_s = 0$ for $s < 0$ and $\sigma_0 = 1$. Note that $S_{\mu-1}(\lambda|a) = S_{\mu}^{\alpha\alpha}$ and $S_{\mu-2}(\lambda|ab) = S_{\mu}^{\alpha\alpha,\beta\beta}$. By Lemma 2.1 (1), it is easy to show

$$F_{pq} = S_{\mu-1}(\lambda|pq)^2 - S_{\mu}(\lambda|pq) S_{\mu-2}(\lambda|pq).$$
We now estimate $F^{pq}$. By the first result of Lemma 2.2 we see

\begin{equation}
S^a_\mu \geq \frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a} \text{ if } a \leq \mu \text{ and } S^{a\bar{a}}_\mu \geq \lambda_1 \cdots \lambda_{\mu-1} \text{ if } a > \mu.
\end{equation}

For $a \leq \mu$ and $b \leq \mu$, by Lemma 2.1 (6), we have

\[\sigma_s(\lambda|ab) \leq C\frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a \lambda_b}, \sigma_{\mu-1}(\lambda|ab) \leq C\frac{\lambda_1 \cdots \lambda_{\mu+1}}{\lambda_a \lambda_b}, \sigma_\mu(\lambda|ab) \leq C\frac{\lambda_1 \cdots \lambda_{\mu+2}}{\lambda_a \lambda_b},\]

where $s \leq \mu - 2$. Hence, we have

\[S_{\mu-1}(\lambda|ab) \leq C\frac{1+\lambda_{\mu+1}}{\lambda_b}S^a_\mu, \quad S_{\mu-2}(\lambda|ab) \leq C\frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a \lambda_b} \leq C\frac{S^{a\bar{a}}_\mu}{\lambda_b},\]

and

\[S_\mu(\lambda|ab) \leq C\frac{1+\lambda_{\mu+1}+\lambda_{\mu+1} \lambda_{\mu+2}}{\lambda_b}S^a_\mu.\]

Now, we can estimate that

\[
\sum_{p,q \leq \mu} |F^{pq}D_1g_{pp}D_1g_{qq}| \leq C\left(\frac{(1+\lambda_{\mu+1})^2}{\delta^2 \lambda_1^2} + \frac{1+\lambda_{\mu+1}+\lambda_{\mu+1} \lambda_{\mu+2}}{\delta^2} \lambda_1^2 \lambda_2^2 \right) \sum_{p \leq \mu} |S^{pq}_\mu D_1g_{pp}|^2 \leq C\frac{(1+\delta^2 \lambda_1^2)}{\lambda_1^2} \sum_{p \leq \mu} |S^{pq}_\mu D_1g_{pp}|^2.
\]

For any undetermined positive constant $\tau$, we obtain

\begin{equation}
\sum_{p,q \leq \mu} |F^{pq}D_1g_{pp}D_1g_{qq}| \leq \tau \sum_{p \leq \mu} |S^{pq}_\mu D_1g_{pp}|^2
\end{equation}

as long as $\delta' \leq \frac{\delta^2}{\delta C}$ and $\frac{1}{\lambda_1} \leq \delta \sqrt{\frac{\mu}{n}}$.

For $a \leq \mu$ and $b > \mu$, by (3.25), we have that

\[S_{\mu-2}(\lambda|ab) \leq C\lambda_1 \cdots \lambda_{\mu-2} \leq C\frac{1}{\lambda_{\mu-1}}S^{b\bar{b}}_\mu \quad \text{and} \quad S_{\mu-1}(\lambda|ab) \leq C\frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a} \leq CS^{a\bar{a}}_\mu \leq C\frac{S^{b\bar{b}}_\mu}{\lambda_a},\]

Note that $\sigma_s(\lambda|ab) \leq C\frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a}$ for $s \leq \mu - 1, a \leq \mu$ and $b > \mu$. So we have

\[S_\mu(\lambda|ab) \leq C \left(\frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a} + \frac{\lambda_1 \cdots \lambda_{\mu+1}}{\lambda_a}\right) \leq (1+\lambda_{\mu+1})S^{a\bar{a}}_\mu.\]

By the above calculations, we have, for any positive constant $\tau$,

\[
\sum_{p \leq \mu, q > \mu} |F^{pq}D_1g_{pp}D_1g_{qq}| \leq (\sum_{p \leq \mu, q > \mu} S_{\mu-1}(\lambda|pq)^2 + S_\mu(\lambda|pq)S_{\mu-2}(\lambda|pq)\|D_1g_{pp}\|D_1g_{qq}| \leq \tau \sum_{p \leq \mu} |S^{pq}_\mu D_1g_{pp}|^2 + C\tau \sum_{q > \mu} |S^{qq}_\mu D_1g_{qq}|^2.
\]

For $a > \mu$ and $b > \mu$, we have

\[S_{\mu-i}(\lambda|ab) \leq C\lambda_1 \cdots \lambda_{\mu-i}, \text{ where } i = 0, 1, 2.\]
So by (3.26) we obtain

$$\sum_{p \neq q, q > \mu} |F^{pq} D_1 g_{pp} D_1 g_{qq}| \leq C \sum_{q > \mu} |S^{qq}_{\mu} D_1 g_{qq}|^2.$$  

Combining (3.26), (3.27), and (3.28), we obtain for any positive constant \(\tau\):

$$\sum_{p \neq q} |F^{pq} D_1 g_{pp} D_1 g_{qq}| \leq 2\tau \sum_{p \leq \mu} |S^{pp}_{\mu} D_1 g_{pp}|^2 + C\tau \sum_{q > \mu} |S^{qq}_{\mu} D_1 g_{qq}|^2. \tag{3.29}$$

Substituting (3.29) into (3.24), we have

$$\sum_{p \leq \mu} |S^{pp}_{\mu} D_1 g_{pp}|^2 \leq \sum_{q > \mu} |S^{qq}_{\mu} D_1 g_{qq}|^2,$$

Substituting (3.23) or (3.32) into the inequality (3.21), we obtain

$$P^2_m A_1 \geq \frac{P_m \kappa_{1}^{m-1} Q}{S_{\mu}^{2}} \sum_{q > \mu} |S^{qq}_{\mu} D_1 g_{qq}|^2.$$

Since \(S_{\mu} \geq \lambda_1 \cdots \lambda_\mu\), we see

$$\frac{\kappa_k S_{\mu}^{11}}{S_{\mu}} \geq \frac{S_{\mu} - S_{\mu}(\lambda|1)}{S_{\mu}} \geq 1 - \frac{C\lambda_{\mu+1}}{\lambda_1} - \frac{C}{\lambda_1},$$

where we used \(\sigma_{\mu}(\lambda|1) \leq C\lambda_2 \cdots \lambda_{\mu+1}\) and \(\sigma_{\nu}(\lambda|1) \leq C\lambda_2 \cdots \lambda_{\mu}\) for \(1 \leq s \leq \mu - 1\), in the second inequality. We can estimate the first term on the right hand side of (3.30) as below

$$\frac{P_m \kappa_{1}^{m-1} Q}{S_{\mu}^{2}} (1 - 2\tau) |S_{\mu}^{11} D_1 g_{11}|^2$$

$$= (1 - 2\tau) P_m \kappa_1^{m-2} Q \left( \frac{\kappa_k S_{\mu}^{11}}{S_{\mu}} \right)^2 |D_1 g_{11}|^2$$

$$\geq (1 - 2\tau)(1 + \delta^m) \kappa_1^{2m-2} Q \left( 1 - C\frac{\lambda_{\mu+1}}{\lambda_1} - \frac{C}{\lambda_1} \right)^2 |D_1 g_{11}|^2$$

$$\geq (1 - 2\tau)(1 - C\delta^2 - \frac{C}{\lambda_1})^2 (1 + \delta^m) \kappa_1^{2m-2} \frac{\lambda_1 Q^{11}}{\kappa_1} |D_1 g_{11}|^2,$$

where in the last inequality we used \(Q \geq \lambda_1 Q^{11}\). For \(\delta^2\) and \(\tau\) small enough and \(\lambda_1\) large enough, we obtain

$$P^2_m A_1 \geq \kappa_1^{2m-2} Q^{11} |D_1 g_{11}|^2 - C\tau \frac{P_m \kappa_{1}^{m-1} Q}{S_{\mu}^{2}} \sum_{q > \mu} |S^{qq}_{\mu} D_1 g_{qq}|^2. \tag{3.32}$$

Substituting (3.28) or (3.32) into the inequality (3.21), we obtain

$$P^2_m (A_1 + B_1 + C_1 + D_1 - E_1)$$

$$\geq P_m \kappa_1^{m-2} \sum_{j > \mu} \left( Q^{jj} - C\tau \frac{\kappa_k Q^{(j)^j}}{S_{\mu}^{2}} \right) |D_1 g_{jj}|^2. \tag{3.33}$$

Assume \(\kappa_1 \leq 2\lambda_1\). We show that \(Q^{jj} - C\tau \frac{\lambda_1 Q^{(j)^j}}{S_{\mu}^{2}} \geq 0\) for \(j > \mu\). For any \(j > \mu\), since \(S^{jj}_{\mu} \leq C\lambda_1 \cdots \lambda_{\mu-1}\) and \(S_{\mu} \geq \lambda_1 \cdots \lambda_{\mu}\), we have

$$C\frac{\lambda_1 (S^{jj}_{\mu})^2}{S_{\mu}^2} \leq C\frac{\lambda_1^2}{\lambda_1^2} \leq \frac{C}{\lambda_1^2}.$$
By the remark below Lemma 3.2, for \( \mu < j \leq k \), we see

\[
Q^{ij} = \sum_{j<s\leq k} \alpha_s \sigma_s^{ij} + \sum_{\mu<s\leq j} \alpha_s \sigma_s^{ij} \\
\geq \sum_{j<s\leq k} \alpha_s \frac{\lambda_1 \cdots \lambda_s}{\lambda_j} + \sum_{\mu<s\leq j} \alpha_s \lambda_1 \cdots \lambda_{s-1} \\
\geq \frac{1}{C \lambda_j} \sum_{j<s\leq k} \alpha_s \sigma_s + \frac{1}{C \lambda_s} \sum_{\mu<s\leq j} \alpha_s \sigma_s \\
\geq \frac{1}{C \delta' \lambda_1} \sum_{\mu<s\leq k} \alpha_s \sigma_s = \frac{Q}{C \delta' \lambda_1}.
\]

and for \( j > k \),

\[
Q^{ij} = \sum_{\mu<s\leq k} \alpha_s \sigma_s^{ij} \geq \sum_{\mu<s\leq k} \alpha_s \lambda_1 \cdots \lambda_{s-1} \\
\geq \sum_{\mu<s\leq k} \alpha_s \frac{\lambda_1 \cdots \lambda_{s-1} \lambda_s}{\lambda_s} \geq \frac{1}{C \delta' \lambda_1} \sum_{\mu<s\leq k} \alpha_s \sigma_s \geq \frac{Q}{C \delta' \lambda_1}.
\]

It follows that for \( \delta' \) sufficiently small, we have

\[
Q^{ij} \geq \frac{Q}{C \delta' \lambda_1} \geq \frac{C \tau Q}{\lambda_1 \delta^2} \geq C \tau \frac{\lambda_1 Q(S^{ij}_s)^2}{S^2_\mu}.
\]

It follows from (3.33) that

\[
P^2_m(A_1 + B_1 + C_1 + D_1 - E_1) \geq 0.
\]

With Lemma 3.1 and Lemma 3.2, we prove that we may assume in (3.15) the claim holds, i.e. for sufficiently large \( m \),

(3.34) \[ A_i + B_i + C_i + D_i - E_i \geq 0, \quad \forall i = 1, \ldots, n. \]

Proof. Set \( \delta_1 = 1 \). If \( \lambda_2 \leq \delta_2 \lambda_1 \) for \( \delta_2 \) small enough, then by Lemma 3.2 we see that (3.34) holds. Otherwise \( \lambda_2 \geq \delta_2 \lambda_1 \). If \( \lambda_3 \leq \delta_3 \lambda_1 \) for \( \delta_3 > 0 \) small enough, then by Lemma 3.2 we see that (3.34) holds. Otherwise, we have \( \lambda_3 \geq \delta_3 \lambda_1 \). Proceeding iteratively, we may arrive at \( \lambda_k \geq \delta_k \lambda_1 \). But in this case, an upper bound for \( \lambda_1 \) follows directly from the equation as

\[
C \geq \sigma_k(\lambda) \geq \lambda_1 \cdots \lambda_k \geq (\delta_k)^{k-1} \lambda_1^k.
\]

Therefore, we may assume (3.34) in (3.15). This proves the claim.

\[ \square \]

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School of Mathematics, Tianjin University, Tianjin, P.R.China, 300354

Email address: dr.dong@tju.edu.cn