A precise formulation of the strong Equivalence Principle is essential to the understanding of the relationship between gravitation and quantum mechanics. The relevant aspects are reviewed in a context including General Relativity but allowing for the presence of torsion. For the sake of brevity, a concise statement is proposed for the Principle: An ideal observer immersed in a gravitational field can choose a reference frame in which gravitation goes unnoticed. This statement is given a clear mathematical meaning through an accurate discussion of its terms. It holds for ideal observers (time-like smooth non-intersecting curves), but not for real, spatially extended observers. Analogous results hold for gauge fields. The difference between gravitation and the other fundamental interactions comes from their distinct roles in the equation of force.

1. INTRODUCTION

Despite Synge's forty–years old injunction to the effect that the “midwife be now buried with appropriate honours”, the Principle of Equivalence came back to the fore in the last few years. Such resurgence is largely due to the difficulties in reconciling “local” General Relativity with “non–local” Quantum Mechanics, but also to possible violating couplings coming from String Theory. A certain elusiveness in the language, however, seems even today to cast a shadow on what the Principle really states. We shall here try to dispel at least some of that vagueness by proposing a precise formulation of the Principle which highlights its significance and limitations. It should be clear that our concern is the so-called Strong Principle and not any of its weaker versions.

In one or another of its forms, the Principle is introduced in every basic text, and hardly a theme has been the subject of more intense scrutiny. In fact, so much has been said about it that we seem to be reduced to looking for what the Principle can state. Decades of discussion have brought to light many significant points, which we intend to bring together into a unified view. The danger is worth to be faced that the presentation may seem a pompous listing of commonplace statements which “everybody knows”. This risk is evident in the version of the Principle we shall use which, put in a nutshell, reads:

An ideal observer immersed in a gravitational field can choose a reference frame in which gravitation goes unnoticed.

The apparent simplicity has one merit: it reduces the whole question to giving clear meanings to the three expressions in italics.

The key notion is that of ideal observer, which is a time-like curve, a world-line. Such a curve represents locally, in well-chosen coordinates, a point-like object in 3-space evolving in the timelike 4-th direction. To represent an extended object — in particular, a real observer — a bunch of world-lines is necessary, one line for each of its points in 3-space. A real observer can know whether (s)he is accelerated or not by making experiments with extended objects like accelerometers and gyroscopes. The gravitational field (that is, the Riemann curvature) will be given away by the deviations of the world-lines. We shall see why and how the Principle holds only for ideal — point-like observers. It will be seen also that reference frames are better conceived in terms of fields of vector bases. Such tetrad fields include coordinate systems as particular (holonomic) cases. General, non-holonomic tetrads provide an extension of the Principle to cases with torsion. And it is not difficult to guess in which sense can gravity “go unnoticed”: of all the actors with a part in the play, only can be made to vanish from the scene that one which is non-tensorial, the connection. This raises another question. What has been called “principle of equivalence” in most recent discussions is a mathematical property, the vanishing of the Levi-Civita (or Christoffel) connection at a point or along a curve. Reduced to that aspect, however, the Principle would not be a distinctive character of gravitation. The aptitude to vanish at a point or along a curve by a convenient choice of “gauge” is a general attribute of connections, shared by gauge potentials. The meaning and specificity of the Principle is, consequently, involved with the question of why and how gravitation differs from the other fundamental interactions.
Spacetime is necessarily conceived as a differentiable manifold: otherwise we could not even take derivatives, or write differential equations. A “geometry” consists of structures added to that initial, basic structure. The Principle is naturally inbuilt in the standard geometrical setup of General Relativity. The use of the geometrical structure underlying General Relativity is surely essential for the proper treatment of the subject, but its axiomatic presentation as surely obscures the experimental evidence for the existence of that same structure. The Principle reduces to a theorem. Studying it from a more physical point of view helps understanding why and how gravitation is related to geometry in the way it is. It is frequently said that universality “geometrizes” the gravitational interaction. Universality is absent in the other fundamental interactions, which have nevertheless also a deep geometrical connotation. In which way does the geometrodynamics of gravitation differ from the geometrodynamics of gauge fields? Of course, gravitation engenders forces of inertial type while the other known fundamental interactions do not, but why are so different the geometries related to inertial and to non-inertial forces? Understanding the Principle, with its relationship to universality, is crucial to the understanding of these questions.

The geodesic equation has use only for the symmetric part of the linear connection involved, which coincides with the whole connection in the Levi–Civita case. The vanishing of the symmetric part of a linear connection at a point can be achieved by a choice of coordinates. Notice that this “pointwise” aspect comes from Special Relativity, more precisely from the Locality Hypothesis: an accelerated observer is supposed to be equivalent, at each point of its trajectory, to an inertial observer with the same velocity. An interesting development has been the proof that any linear connection can be made to vanish at a previously chosen point by a choice of tetrad field or, as it is sometimes phrased, by a choice of non–holonomic coordinates (or still, “normal frames”). Such normal coordinates and/or frames were known to exist at a point and along smooth non-intersecting curves for symmetric linear connections. Much stronger results, concerning general derivatives on tensor algebras, have been found along the nineties by Iliev. For the special cases of covariant derivatives, they encompass both linear connections — with torsion or not — on frame bundles and gauge potentials on general bundles.

A simple derivation of these results is given below (section 3), whose advantage is that it can be immediately adapted to gauge theories. We start with a résumé on tetrad fields, the Levi-Civita connection and the Weitzenböck connection of a tetrad. We exhibit then an explicit tetrad field making a previously given connection equal to zero at a prescribed point. In the presence of torsion, that frame is necessarily non-holonomic. It is then shown why a tetrad field which is parallel-transported by a connection along any prescribed curve produces the vanishing of that connection along the curve. A generic tetrad is not parallel-transported along a geodesic of the Levi-Civita connection. For an ideal observer, however, there will be a tetrad which is. At each point $P$ this inertial, free falling frame differs from any other by a Lorentz transformation which depends on $P$. These Lorentz transformations satisfy an interesting equation: the rows of the Lorentz matrix, seen as vectors, are parallel-transported.

The procedure is then applied to gauge theories (section 3), abelian and non-abelian. A gauge can be chosen in which the corresponding connection, the gauge potential, vanishes. The difference is obvious: in gravitation a linear connection is at work, which belongs to the very structure of spacetime. In gauge fields, the gauge potential is an “internal” connection, acting on the multiplets of the gauge group. Dynamically, the difference turns up in the force equation (section 2): the gauge field strength appears explicitly in the Lorentz force, while the gravitational field strength, the curvature, is absent.

We shall use the notations $\{e_\alpha, e^\alpha\}$ for generic tetrads, and $\{H_a, H^a\}$ for tetrads leading to the vanishing of a connection. The symbols $\mu \nu \rho \ldots$ and $\mu \nu \rho \ldots$ will indicate symmetrization and antisymmetrization of included indices in general, as in $\Gamma^\lambda_{\mu \nu} = \frac{1}{2} \left[ \Gamma^\lambda_{\mu \nu} + \Gamma^\lambda_{\nu \mu} \right]$ and $\Gamma^\lambda_{[\mu \nu]} = \frac{1}{2} \left[ \Gamma^\lambda_{\mu \nu} - \Gamma^\lambda_{\nu \mu} \right]$. The “ball” notation will indicate the Levi-Civita connection $\Gamma$ and objects related to it. Letter $u$ will be used for the parameter of a curve whose tangent field is $U$, so that $U^\lambda = \frac{dx^\lambda}{du}$ and $U = \frac{d}{du} U^\lambda \partial_\lambda$; letter $v$ will be used for the parameter if the tangent field is $V$, and so on. Also the standard notation $\nabla_U = \sum \nabla_\lambda$ will be used.

2. METRICS, FRAMES AND CONNECTIONS

General Relativity is a metric theory; it takes metrics as fundamental fields, thereby taking a great distance with respect to the other theories describing fundamental interactions — which have connections as the basic fields. Given a metric as starting notion, there are a class of tetrad fields and two connections which have a special significance.

1. Frame fields

A coordinate system $\{x^\mu\}$ determines, on its domain of definition, a base $\left\{ \partial_{x^\mu} \right\}$ for the tangent vector fields and a base $\{dx^\mu\}$ for the covector fields (1-forms). These bases are dual, in the sense that $dx^\mu \left( \frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu$. Such “holonomic”
bases, related to coordinates, are very particular. Any set of four linearly independent fields \( \{e_a\} \) will form another base, with a dual \( \{e^a\} \) such that \( e^a(e_b) = \delta^a_b \). These “tetrads fields” are the general linear bases on the spacetime differentiable manifold. Their set can be made into another smooth manifold, the bundle of linear frames. On the domains both are defined, the members of a base can be written in terms of the other: \( e_a = e_a^\mu \partial_\mu \), \( e^a = e^a_\mu dx^\mu \) and conversely. The transformations taking \( \{e_a\} \) into any other tetrad \( \{e'_a\} \) constitute the linear group \( GL(4, \mathbb{R}) \) of all real \( 4 \times 4 \) invertible matrices. These frames, with their bundle, are constitutive parts of spacetime, automatically present as soon as it is supposed to be a differentiable manifold.

We call linear connections those related to some subgroup of the linear group \( GL(4, \mathbb{R}) \). They are 1–forms with values in the Lie algebra of that subgroup. The Levi–Civita connections of General Relativity are the most important examples of Lorentzian connections, which have values in the Lie algebra of the Lorentz group. Such connections are “external”, related to groups acting on spacetime itself or its tangent spaces, in contraposition to the “internal” connections, the vector potentials turning up in gauge theories. These have values in the Lie algebra of the gauge group, which acts on “internal” spaces. A connection defines parallelism through a covariant derivative. A vector field is parallel–transported by a linear connection along a curve if its covariant derivative vanishes all along the curve.

Consider the metric \( g \) which has components \( g_{\mu\nu} \) in some holonomic base \( \{ \frac{\partial}{\partial x^a} \} \):

\[
g = g_{\mu\nu} dx^\mu \otimes dx^\nu = g_{\mu\nu} dx^\mu dx^\nu.
\]

There is then a tetrad field \( \{ h_a = h_a^\mu \frac{\partial}{\partial x^\mu} \} \) which relates \( g \) to the Lorentz metric \( \eta \) by

\[
\eta_{ab} = g_{\mu\nu} h_a^\mu h_b^\nu.
\]

The \( h_a \)'s are, by this expression, particular linear frames which are (pseudo-)orthogonal by the metric \( g \). Each basis \( \{ h_a \} \) constitutes a Lie algebra, fixed either by the vector fields commutation table

\[
[h_a, h_b] = f_{abc} h_c,
\]

or by its dual expression for the covectors \( \{ h^a = h^a_\nu dx^\nu \} \), Cartan’s structure equation

\[
dh^c = -\frac{1}{2} f^{c}_{ab} h^a \wedge h^b = \frac{1}{2} \left( \partial_\nu h^c_\mu - \partial_\mu h^c_\nu \right) dx^\mu \wedge dx^\nu.
\]

The \( f^{c}_{ab} \)'s are the structure coefficients (or anholonomy coefficients),

\[
f^{c}_{ab} = h^c_\mu h^a_\nu (\partial_\nu h^c_\mu - \partial_\mu h^c_\nu) = h^c_\mu [h_a(h^a_\nu) - h_b(h^b_\nu)] = h^c([h_a, h_b]).
\]

The basis will be anholonomic — unrelated to any coordinate system — if some of the structure coefficients are non-vanishing, \( f^{c}_{ab} \neq 0 \) for some \( a, b, c \). The frame \( \{ \frac{\partial}{\partial x^a} \} \) has already been presented as holonomic precisely because their members commute with each other. If \( f^{c}_{ab} = 0 \), then \( dh^c = 0 \) implies the local existence of functions (coordinates) \( y^c \) such that \( h^c = dy^c \). In that case, the \( g_{\mu\nu} \)'s would be simply the components of the Lorentz metric \( \eta \) in the coordinate system \( \{ x^\nu \} \).

The components of basis \( \{ h_a, h^b \} \) members with respect to the base \( \{ \partial_\mu, dx^\nu \} \) satisfy

\[
h^a_\mu h^a_\nu = \delta^\nu_\mu; \quad h^a_\mu h^b_\mu = \delta^b_a;
\]

\[
g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu.
\]

Indices can be raised and lowered in a consistent way. Thus, for example,

\[
g^{\mu\nu} = \eta^{ab} h_a^\mu h_b^\nu; \quad \eta^{ab} = g^{\mu\nu} h_a^\mu h_b^\nu; \quad g_{\mu\nu} = h_a h_a^\nu, \quad \text{etc.}
\]

A tensor with components \( T^{\lambda_1\lambda_2\lambda_3...}_{\mu_1\nu_2\nu_3...} \) in the holonomic base will have, seen from a tetrad frame \( \{ h_a \} \), components given by contractions with the tetrad components. In particular, Eq. (1) tells us that the metric \( g \), seen from the tetrad frame, is just the Lorentz metric. This does not mean that the frame is inertial, because the metric derivatives — which appear in the forces and accelerations — are not tensorial. To define derivatives which are covariant, it is essential to add connections \( \Gamma^\lambda_{\mu\nu} \), whose non-tensorial behavior in the first two indices compensate the non-tensoriality of the usual derivatives. Connections obey in consequence a special law, given below [for example, in Eqs. (20), (24) and (48)]. Furthermore, Eq. (1) holds for other tetrad fields. In effect, another set \( \{ h'_a \} \) can be given such that \( g_{\mu\nu} = \eta_{ab} h'_a h'_b \nu = \eta_{c'd'} h'_c h'_d \nu \). Contracting both sides of this expression with \( h'_a h'_b \nu \), we arrive at

\[
\eta_{ab} = \eta_{c'd'} (h'_c h'_d (h'_e h'_f)).
\]
This equation says that the matrices with entries
\[ A^a_b = h^{a}_a h^b, \]
which are such that \( h^{a}_b = \Lambda^a_{cb} h^c_b \), satisfy
\[ \eta_{c d} \Lambda^c_a \Lambda^{d}_b = \eta_{ab}. \]

This is just the condition that a matrix \( \Lambda \) must satisfy in order to belong to the Lorentz group. Therefore, basis \( \{ h_a \} \) is far from being unique. At each point of the Riemannian manifold, it is determined by \( g \) only up to Lorentz transformations in the tetrad indices \( a, b, c, \ldots \). Tetrads provide matrix representations of the Lorentz group, but with a special characteristic: they are invertible. A group element taking some member of the representation into another can in consequence be written as in (11), in terms the initial and final members. This establishes a deep difference with respect to the other fundamental interactions, described by gauge theories. There are matrix representations in gauge theories, like the adjoint representation, but their members are not invertible. We recall that General Relativity can be entirely written in terms of tetrads. (12)

### 2. Connections

Linear connections have a great degree of “intimacy” with spacetime itself precisely because they are defined on the bundle of linear frames which, as a constitutive part of spacetime, has some specific properties, not shared by the bundles related to gauge theories. In particular, it exhibits soldering, which leads to torsion. Linear connections have torsion while gauge potentials have not. Soldering comes from the existence of a “canonical”, or “solder” form, \( \omega \), a 1-form taking vectors on the bundle into the typical tangent fiber, Minkowski space. Each tetrad field takes this torsion while gauge potentials have not. Soldering comes from the existence of a “canonical”, or “solder” form, \( \omega \), which is at the origin of the well-known cyclic symmetry of the Riemann tensor components in General Relativity. In a holonomic base the torsion components are proportional to the antisymmetric part of the connection components. Summing up,

\[ T^\lambda_{\mu \nu} = \Gamma^\lambda_{\nu \mu} - \Gamma^\lambda_{\mu \nu} = -2 \Gamma^\lambda_{[\mu \nu]} = h^a_a [\partial_\mu h^a_\nu - \partial_\nu h^a_\mu + \omega^a_{\mu \nu} h^b_\nu - \omega^a_{\nu \mu} h^b_\mu]. \]

The right-hand side exhibits the mentioned covariant derivative of the tetrad field, the \( \omega^a_{\nu \mu} \)'s being the connection components as seen from the tetrad frame. The decomposition into symmetric and antisymmetric parts is

\[ \Gamma^\lambda_{\mu \nu} = \Gamma^\lambda_{(\mu \nu)} + \Gamma^\lambda_{[\mu \nu]} = \Gamma^\lambda_{(\mu \nu)} - \frac{1}{2} T^\lambda_{\mu \nu}. \]

This fact already tells us that, in the presence of torsion, it will be impossible to have all the components \( \Gamma^\lambda_{\mu \nu} \) equal to zero in a holonomic base.

When a metric is present, the condition of metric compatibility is that the metric be everywhere parallel–transported by the connection, that is, that the covariant derivative of the metric be zero: \( \nabla_\lambda g_{\mu \nu} = \partial_\lambda g_{\mu \nu} - \Gamma^\rho_{\lambda \mu} g_{\rho \nu} - \Gamma^\rho_{\nu \lambda} g_{\rho \mu} = 0 \), or

\[ \partial_\lambda g_{\mu \nu} = 2 \Gamma_{(\mu \nu)\lambda}. \]

A metric defines a Levi-Civita connection \( \Gamma \), which is that unique metric-compatible connection which has zero torsion. The components are the well-known Christoffel symbols

\[ \Gamma^\lambda_{\mu \nu} = \frac{1}{2} g^{\lambda \rho} [\partial_\mu g_{\rho \nu} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu}]. \]

The symmetry of \( \Gamma \) in its last two indices says that \( \Gamma = 0 \). If \( \Gamma \) preserves a metric and is not its Levi-Civita connection, it will forcibly have \( T \neq 0 \).

The difference between two connections is a tensor. The contorsion tensor \( K \) of the connection \( \Gamma \) is defined by

\[ K^\lambda_{\mu \nu} = \Gamma^\lambda_{\mu \nu} - \Gamma^\lambda_{\mu \nu}. \]
Metric compatibility (14) implies that contorsion is fixed by the torsion tensor. Inserting into (10) the expression for $\Gamma$ obtained from (14) for the three terms in (15), we obtain

$$K^\lambda_{\mu\nu} = \frac{1}{2} \left[ T^\lambda_{\mu\nu} + T_{\mu\nu}^\lambda + T_{\nu\mu}^\lambda \right].$$

(17)

Decompositions (13) and (18) are not the same. Compared with (13), the two last terms in (17) give an extra symmetric contribution of torsion to $\Gamma$.

The geodesic equation

$$\nabla_U U^\lambda \equiv \frac{\nabla}{\nabla u} U^\lambda = \frac{d}{du} U^\lambda + \Gamma^\lambda_{\mu\nu} U^\mu U^\nu = 0$$

defines a self-parallel curve, whose velocity field $U$ is parallel-transported by $\Gamma$ along the curve itself. Seen from the tetrad frame $\{h_a\}$, (18) takes the form

$$\frac{d}{du} U^a + \omega^a_{\ bc} U^b U^c = 0,$$

(19)

where

$$\omega^a_{\ bc} = h^a_{\ c} \left[ h_b (h_c^\lambda) + h^a_{\ c} \Gamma^\lambda_{\mu\nu} h_b^\mu h_c^\nu \right] = h^a_{\ c} \nabla_c (h_b^\lambda)$$

(20)

is the connection as seen from the tetrad frame. This transformation law ensures the tensorial behavior of the covariant derivative: $\nabla_a V^\lambda = h^a_{\ c} h_b^\lambda \nabla_a V^b = h^a_{\ c} h^b_{\ d} \nabla_a V^d$ and $\nabla_a V^b = \Lambda^a_{\ c} \Lambda^b_{\ d} \nabla_a V^d$. We shall frequently use (20) in the alternative form

$$\partial_\nu h_b^\lambda + \Gamma^\lambda_{\mu\nu} h_b^\mu = h_c^\lambda \omega^a_{\ bc}.$$

(21)

The velocity vector will follow the law valid for tensors: its components are related by

$$U^a = h^a_{\ c} U^c.$$

(22)

This means that $U^a$ is, in general, an anholonomic velocity, analogous to the angular velocity of a rigid body: there exists no coordinate $x^a$ such that $U^a = \frac{d}{du} x^a$. A vector field, given by components $U^\lambda$, can be seen as the directional derivative $U = U^\lambda \partial_\lambda = \frac{d}{du}$, just the derivative along its own (local) integral curve $\gamma$ with parameter $u$. For time-like curves $u$ can be seen as the proper time and $U$ can be interpreted as the four-velocity along $\gamma$. Each connection defines an acceleration as its the covariant derivative with respect to proper time,

$$a^\mu = \nabla_U U^\mu \equiv \frac{d}{du} U^\mu + \Gamma^\mu_{\nu\lambda} U^\nu U^\lambda.$$  

(23)

For every metric-preserving connection the acceleration is, as in Special Relativity, orthogonal to the velocity. If $\Gamma$ is the Levi-Civita connection, (23) is the force equation to which a test particle submits in General Relativity.

The antisymmetric part of $\omega^a_{\ bc}$ in the last two indices can be computed by using Eqs. (12) and (5). The result shows that torsion, seen from the anholonomic frame, includes the anholonomy:

$$T^a_{\ bc} = - f^a_{\ bc} - (\omega^a_{\ bc} - \omega^a_{\ cb}).$$

(24)

There is a constraint on the first two indices of $\omega^a_{\ bc}$ if $\Gamma$ preserves the metric. In effect, Eqs. (14) and (2) lead to

$$\omega_{abc} = - \omega_{bac}.$$  

(25)

This antisymmetry in the first two indices, after lowering with the Lorentz metric, says that $\omega$ is a Lorentz connection. This is to say that it is of the form

$$\omega = \frac{1}{2} J^a_{\ bc} \omega^b_{\ ac} h^c,$$  

(26)

with $J^a_{\ bc}$ the Lorentz generators written in an appropriate representation. Therefore, any connection preserving the metric is, when seen from the tetrad frame, a Lorentz-algebra-valued 1-form. If we use Eq. (10) and the inverse $(\Lambda^{-1})^a_{\ b'} = h^a_{\ \mu} h_{b'}^\mu = \eta_{b'c'} \eta^{ad}\Lambda^c_{\ d} = \Lambda_b^a$, we find how the components change under tetrad transformation:

$$\omega^a_{\ b'} = \Lambda^a_{\ c} \omega^c_{\ d'} (\Lambda^{-1})^d_{\ b'} + \Lambda^a_{\ c} \partial_b (\Lambda^{-1})^c_{\ b'}.$$  

(27)
Under change of tetrad, the connection $\omega$ (which is a metric-preserving $\Gamma$ seen from any tetrad) transforms as a Lorentz connection.

The Riemannian metric $g = (g_{\mu\nu})$ is a Lorentz invariant. For what concerns $g$, any two tetrad fields $\{h_a\}$ and $\{h'_a\}$ as above are equivalent. A metric corresponds to an equivalence class of tetrad fields, the quotient of the set of all tetrad by the Lorentz group. The sixteen fields $h^a{}_{\mu}$ correspond, from the field-theoretical point of view, to ten degrees of freedom—like the metric—once the equivalence under the six-parameter Lorentz group is taken into account. In simple words, all tetrad fields related by Lorentz transformations determine the same metric, which differs from the Lorentz metric if and only if the tetrad is anholonomic.

3. The Levi-Civita connection

The tetrad field $h_a$ takes the components (15) of the Levi-Civita connection,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} \left\{ h_b{}^\lambda (\partial_\nu h_a{}^b + \partial_\mu h_b{}^\nu) + g^{\lambda\rho} [h_{a\nu}(\partial_\rho h_a{}^\mu) + h_{a\mu}(\partial_\rho h_a{}^\nu) - \partial_\mu h_b{}^\nu h_a{}^\rho + \partial_\nu h_b{}^\rho h_a{}^\mu] \right\},$$

(28)

into those of the so-called spin-connection

$$\omega^a{}_{\mu\nu} = h^a{}_{\lambda} \Gamma^\lambda{}_{\mu\nu} h_b{}^\mu + h^a{}_{\rho} \partial_\nu h_b{}^\rho,$$

(29)

which is simply $\Gamma$ as seen from the frame defined by the tetrad $\{h_a\}$. The expression above can be rewritten as in (21),

$$\partial_\nu h_b{}^\lambda + \Gamma^\lambda{}_{\mu\nu} h_b{}^\mu = h_a{}^\lambda \omega^a{}_{\nu\mu}.$$

(30)

If the tetrad were parallel-transported everywhere, the left-hand side would vanish. The metric would in that case reduce to the Lorentz metric. Indeed, from the ensuing expression

$$\Gamma^\lambda{}_{\mu\nu} = h_b{}^\lambda \partial_\nu h_a{}^b,$$

(31)

it follows that the curvature tensor vanishes. The spin-connection consequently measures how much the tetrad field $\{h_a\}$ deviates from parallelism, and how much the metric differs from that of Lorentz. We shall see below [see Eq.(88) and the comments around it] that equation (30) actually encodes the Equivalence Principle. Combined with (31), (29) leads to

$$\omega^a{}_{bc} - \omega^a{}_{cb} = f^a{}_{cb}.$$

(32)

The force equation is now

$$\frac{d}{du} U^a + \omega^a{}_{bc} U^b U^c = h^a{}_{\mu} a^\mu = \tilde{a}^a.$$

(33)

The term $\omega^a{}_{bc} U^b U^c$ is responsible for the inertial force, seen from the frame itself. The weak Principle is implied, of course, by the total absence of the mass in the “force” equation.

Equation (30) gives, for any $U$,

$$\nabla_U h_a{}^\lambda = h_c{}^\lambda \omega^c{}_{ab} U^b.$$

(34)

The timelike member $h_0$ of a set $\{h_a\}$ of vector fields constituting a tetrad will define, for each set of initial conditions, an integral curve $\gamma$. For that curve, $h_0 = U$, $h_0{}^\lambda = U^\lambda$ and $U^a = h_a{}^\lambda h_0{}^\lambda = \delta^a_0$. The force law will say whether the frame, as it is carried along that timelike curve, is inertial or not: (34) gives, for that curve, the acceleration

$$\tilde{a}^\lambda = \nabla_U h_0{}^\lambda = h_c{}^\lambda \omega^c{}_{00}. $$

(35)

The Fermi-Walker derivative will be

$$\nabla_U^{(FW)} h_a{}^\lambda = \nabla_U h_a{}^\lambda + \tilde{a}^\lambda U^a - U_a \tilde{a}^\lambda = h_c{}^\lambda \omega^c{}_{ab} U^b + \tilde{a}^\lambda U^a - U_a \tilde{a}^\lambda. $$

(36)
The particular case

$$\nabla_{U}^{(FW)} h_0^\lambda = \nabla_{U} h_0^\lambda - \overset{\circ}{\alpha}^\lambda = 0$$  \hspace{1cm} (37)

implies that $h_0 = U$ is kept tangent along the curve. Equation (33) can be obtained, alternatively, by contracting $U^\nu$ with Eq. (22) written for $h_0$. The expression for the frame acceleration seen from the frame itself,

$$\overset{\circ}{\alpha}^\nu = \omega^\nu_{00},$$  \hspace{1cm} (38)

follows also from Eq. (33) by inserting $U^a = h_0^a \lambda h_0^\lambda = \delta^a_0$, but with one great advantage: the external acceleration appears then clearly as equal to the inertial acceleration. We see here in which sense a frame satisfying Eq. (2) is equivalent to the gravitational field defined by the metric $g_{\mu\nu}$: it is accelerated, and its acceleration is just the inertial gravitational acceleration.

Of course, $\overset{\circ}{\alpha}^a_0 = \omega^a_{00} = 0$, and only the space components are nonvanishing. Seen from another frame $h_{a'} = \Lambda^b_a h_b$, the velocity will be $U^{a'} = \Lambda^a_{a'} U^b = \Lambda^a_{a'0}$ and

$$a'^\nu = h'^\nu_{\rho} a^\rho = \Lambda^a_{a'0} h_{a'} (\Lambda^c_{a'0}) + \omega^c_{a'b'} \Lambda^a_{a'0} \Lambda^b_{b'0} = \Lambda^a_{a'} a^c.$$  \hspace{1cm} (39)

### 4. The Weitzenböck connection

Each tetrad field $\{h_a\}$ defines a very particular connection, the Weitzenböck connection whose components are

$$\Gamma^\lambda_{\mu\nu} = h_0^a \partial_{\nu} h_a^\mu.$$  \hspace{1cm} (40)

This connection vanishes when seen from frame $\{h_a\}$ itself:

$$\omega^a_{b\nu} = h_0^a \partial_{\nu} h_b^\mu = h_0^a \nabla_{\nu} h_b^\mu = 0.$$  \hspace{1cm} (41)

This means that it parallel-transports each vector of the tetrad $\{h_a\}$ everywhere: $\nabla_\nu h_0^a = 0$ (which justifies the name “teleparallelism” given to the approach to gravity based on this connection). In consequence, it preserves also the metric $g$: $\nabla_\lambda g_{\mu\nu} = 0$. It has vanishing Riemann curvature tensor: $R^a_{\mu\nu\lambda} = 0$ and $R^a_{b\nu\lambda} = 0$. It has, however, a non-vanishing torsion $\overset{\circ}{\kappa}$, actually a mere version of the structure coefficients of the tetrad frame seen from the holonomic base:

$$\overset{\circ}{\kappa}^\lambda_{\nu\mu} = h_{a} \partial_{\nu} h^a_{\mu} - \partial_{\mu} h_a^\nu = h_0^a f^c_{ab} h_a^\mu h^b_{\nu}.$$  \hspace{1cm} (42)

As $\overset{\circ}{\omega}^a_{bc} \equiv 0$, a geodesic of the Weitzenböck connection has always the form

$$\nabla_{\nabla U} U^a = \frac{d}{du} U^a = 0$$  \hspace{1cm} (43)

when seen from the tetrad frame. The geodesic equation (18) for $\overset{\circ}{\kappa}$, however, acquires in terms of $\overset{\circ}{\kappa}$ an aspect of force equation:

$$\frac{d}{du} U^\lambda + \overset{\circ}{\kappa}^\lambda_{\mu\nu} U^\mu U^\nu = - \overset{\circ}{K}^\lambda_{\mu\nu} U^\mu U^\nu = \frac{1}{2} \left[ \overset{\circ}{T}_{\mu}^\lambda_{\nu} + \overset{\circ}{T}^\lambda_{\nu\mu} \right] U^\mu U^\nu.$$  \hspace{1cm} (44)

Using (10) for the Weitzenböck connection, we identify its contorsion as

$$\overset{\circ}{K}^\lambda_{\mu\nu} = h_a^\lambda \overset{\circ}{\omega}^a_{b\nu} h_b^\mu.$$  \hspace{1cm} (45)

This means that $\overset{\circ}{\omega}$ is the Weitzenböck contorsion seen from the tetrad frame $\{h_a\}$: $\overset{\circ}{\omega}^a_{bc} = \overset{\circ}{\kappa}^a_{bc}$. And using (7) with (23) in this expression gives

$$\overset{\circ}{\omega}^a_{bc} = - \frac{1}{2} \left[ f^a_{bc} + f_{bc}^a + f_{cb}^a \right],$$  \hspace{1cm} (46)

where the indices are raised and lowered with the Lorentz metric. The trivial property $f_c(ab) = 0$ implies (23) for $\overset{\circ}{\omega}$. Quite consistently, $\overset{\circ}{\omega}^a_{bc}$ will vanish if the base $\{h_a\}$ is holonomic.

Summarizing what we have seen in this section: there is a functional sixfold infinity of tetrad fields related to a given metric as in Eq. (7). These tetrad fields differ by point-dependent (that is, local) Lorentz transformations (wherefrom the functional sixfoldedness). Holonomic tetrads correspond to the Lorentz metric itself. Each tetrad field defines a Cartan-Weitzenböck flat connection. This connection is a “vacuum” of every other connection. When alone, the force law reduces to that of Special Relativity. Its interest to the Equivalence Principle is consequently evident. We shall actually see that, given a connection $\Gamma$, the free-falling frame along a curve will be a tetrad field whose Weitzenböck connection coincides with $\Gamma$ along the curve.
3. THE CASE OF A LINEAR CONNECTION

Consider a general linear connection $\Gamma$ defined on a manifold $M$. Choose a point $P \in M$, and around it a coordinate system $\{x^\mu\}$ such that $x^\mu(P) = 0$. Such a system will cover a neighborhood $N$ of $P$ (its coordinate neighborhood), and will provide dual holonomic bases $\{\frac{\partial}{\partial x^\mu}, dx^\mu\}$ for vector and covector fields on $N$. Any other base will be given by the components of its members in terms of such initial holonomic bases, as $\{e_a = e_a^\mu(x)\frac{\partial}{\partial x^\mu}, e^a = e^a_\lambda(x)dx^\lambda\}$. It will be enough for our purposes to consider inside $N$ a non-empty sub-domain $N'$, small enough to ensure that only terms up to first order in the $x^\mu$'s can be retained in the calculations.

Let us indicate by $\Gamma^\lambda_{\mu\nu}(x)$ the components of $\Gamma$ in the holonomic base, and by $\omega^a_{\mu\nu}(x)$ the components of $\Gamma$ referred to a generic base $\{e_a, e^a\}$. Such components will be related by

$$\Gamma^\lambda_{\mu\nu}(x) = e_a^\lambda(x) \omega^a_{\mu\nu}(x) e^b_{\mu}(x) + e_c^\lambda(x) \partial_c e^b_{\mu}(x);$$

$$\omega^a_{\mu\nu}(x) = e^c_\lambda(x) \Gamma^\lambda_{\mu\nu}(x) e^b_{\mu}(x) + e^a_\lambda(x) \partial_c e^b_{\mu}(x).$$

The piece $e_c^\lambda(x) \partial_c e^c_\mu(x)$ in (47) is the Weitzenböck connection of the tetrad field $\{e_a\}$.

1. At a point

Let $\gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}(P)$ be the value of the holonomic components of the connection at the point $P$. On the small domain $N'$, the connection components will be approximated by

$$\Gamma^\lambda_{\mu\nu}(x) = \gamma^\lambda_{\mu\nu} + x^\rho \left[ \partial_\rho \Gamma^\lambda_{\mu\nu}\right]_P$$

to first order in the coordinates $x^\mu$. We shall simply exhibit a particular base in which the connection components vanish at $P$. Indeed, choose on $N'$ the tetrad field $\{H_a, H^a\}$ whose components are

$$H_a^\lambda = \delta_a^\lambda - \delta^a_\sigma \gamma^\lambda_{\rho\nu} x^\rho; \quad H^a_\mu = \delta^a_\mu + \delta^a_\lambda \gamma_{\mu\nu} x^\nu.$$  

(50)

The Cartan structure equation

$$dH^a = -\frac{1}{2} c^a_{bc} H^b \wedge H^c$$

(51)

for the 1–forms $\{H^a = H^a_\mu dx^\mu\}$ will, in the present case, be

$$dH^a = \frac{1}{2} \delta^a_\lambda \left[ \gamma^\lambda_{\mu\nu} - \gamma^\lambda_{\nu\mu}\right] dx^\nu \wedge dx^\mu = \delta^a_\lambda \gamma^\lambda_{[\mu\nu]} dx^{[\nu} \wedge dx^{\mu]}.$$  

(52)

We shall eventually replace indices fixed by $\delta^a_\lambda$ for notational convenience. With this convention the structure coefficients are, to order zero,

$$c^a_{bc} = \gamma^a_{cb} - \gamma^a_{bc} = -2 \gamma^a_{[bc]}.$$  

(53)

Keeping always expressions only up to first order in the $x^\mu$'s, we find

$$\omega^a_{\mu\nu}(x) = H^a_{\lambda}(x) \Gamma^\lambda_{\mu\nu}(x) H_{\rho}(x) + H^a_{\rho}(x) \partial_\nu H_{\rho}(x)$$

$$= x^\rho \left[ \partial_\rho \Gamma^\lambda_{\mu\nu}(P) - \gamma^\rho_{\sigma\nu} \gamma^\lambda_{\mu\sigma}\right].$$

(54)

We see that at the point $\{x^\mu = 0\}$ the connection components in base $\{H_a\}$ vanish: $\omega^a_{\mu\nu}(P) = 0$. The tetrad $\{H_a\}$ is such that its Weitzenböck connection

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \gamma^\lambda_{\mu\nu} - \gamma^\lambda_{\sigma\mu} \gamma^\sigma_{\nu\rho} x^\rho$$

(55)

coincides with $\Gamma$ at $P$. The curvature and the torsion tensors at $P$ are

$$R^\lambda_{\sigma\mu\nu}(P) = \partial_\mu \Gamma^\lambda_{\sigma\nu}(P) - \partial_\nu \Gamma^\lambda_{\sigma\mu}(P) + \gamma^\lambda_{\rho\mu} \gamma^\rho_{\sigma\nu} - \gamma^\lambda_{\rho\nu} \gamma^\rho_{\sigma\mu},$$

$$T^\lambda_{\mu\nu}(P) = \gamma^\lambda_{\nu\mu} - \gamma^\lambda_{\mu\nu}.$$  

(56)
To see what happens to the anholonomy of base \( \{ H_a \} \), we calculate

\[
[H_a, H_b] = \{ 2\gamma^c_{[ab]} + [2\gamma^d_{[ab]}\gamma^c_{d\rho} + \gamma^c_{bd}\gamma^d_{a\rho} - \gamma^c_{ad}\gamma^d_{b\rho}] \} H_c
\]  

(58)

The commutator vanishes at \( P \) if the connection is symmetric. Otherwise, the anholonomy is minus the torsion of \( \Gamma \) at \( P \):

\[
[H_a, H_b] = 2\gamma^c_{[ab]} H_c.
\]  

(59)

Define a Lorentz transformation like that of Eq.(10):

\[
\Lambda_b(x) = H^a_\lambda h_b^\lambda.
\]  

(60)

The result of contracting Eq.(21) with \( H^a_\lambda \) is

\[
H^a_\lambda \partial_\nu h^\nu_b^\lambda + \Gamma^\lambda_{\mu\nu} H^a_\lambda h^\mu_b^\nu = h^\nu_c H^a_\lambda \omega^c_{b\nu},
\]  

(61)

from which follows

\[
\partial_\nu H^a_\lambda - h^\nu_b^\lambda (\partial_\nu H^a_\lambda - \Gamma^\mu_{\lambda\nu} H^a_\mu) = \partial_\nu \Lambda^a_b - h^\nu_b^\lambda \nabla_\nu H^a_\lambda = \Lambda^a_c \omega^c_{b\nu}.
\]  

(62)

At the point \( P, \nabla_\nu H^a_\lambda = 0 \), so that

\[
(\Lambda^{-1})^a_\nu \partial_\nu \Lambda^a_b = \omega^a_{b\nu}.
\]  

(63)

The connection \( \omega^a_{b\nu} \), which is \( \Gamma \) seen from the frame \( \{ h_a \} \), reduces to a gauge vacuum. If another frame \( H^a_\mu = \Lambda^a_b h^b_\mu \) satisfies the above condition, then also \((\Lambda^{-1})^a_\nu \partial_\nu \Lambda^a_b = \omega^a_{b\nu} \). Equating both expressions shows that the Lorentz transformation \( \Lambda = \Lambda^a_b \) is a gauge transformation. Conversely, if \( H^a_\mu \) satisfies the above condition, then so does \( H^a_\mu = \Lambda^a_b H^b_\mu \), with \( \Lambda^a_b \) constant. There is actually a sixfold infinity of tetrads satisfying the above conditions, which differ from each other by point-independent Lorentz transformations.

Using the tetrad \( H_a \), the metric is seen to have, to first order, components

\[
g_{\mu\nu} = \eta_{ab} H^a_\mu H^b_\nu = \eta_{\mu\nu} + 2\gamma_{(\mu\nu)\sigma} x^\sigma,
\]  

(64)

with derivatives

\[
\partial_\lambda g_{\mu\nu} = 2\gamma_{(\mu\nu)\lambda}.
\]  

(65)

This equality, compared with (14), tells us that the metric is, as expected, parallel-transported by \( \Gamma \) at \( P \).

1. Without torsion

The choice of basis \( [51] \) produces the vanishing of the connection at \( P \) even in the presence of torsion. To make contact with the standard \( T = 0 \) case, let us remark that

\[
d H^a = -\delta^a_\lambda \gamma^\lambda_{[\mu\nu]} dx^\mu \land dx^\nu = \frac{1}{2} \delta^a_\lambda T^\lambda_{\mu\nu} dx^\mu \land dx^\nu,
\]  

(66)

so that \( \{ H_a \} \) is holonomic at \( P \) when \( T^\lambda_{\mu\nu}(P) = 0 \). To recover the usual prescription for the vanishing of a symmetric connection at a point it is enough to choose the coordinates

\[
y^a = \delta^a_\mu x^\mu + \frac{1}{2} \delta^a_\lambda (\gamma_{\mu\nu}) x^\nu x^\mu.
\]  

(67)

A non-holonomic base is necessary in the presence of torsion, but a coordinate base suffices in its absence — which is, of course, the standard result. Now:

\[
\frac{d}{du} y^a = \delta^a_\mu U^\mu + \delta^a_\lambda (\gamma_{\mu\nu}) U^\nu U_x^\mu;
\]  

(68)

\[
\frac{d^2}{du^2} y^a = \delta^a_\lambda \left[ \frac{d}{du} U^\lambda + \gamma_{\lambda(\mu\nu)} U^\nu U_x^\mu \right] + \delta^a_\lambda (\gamma_{\mu\nu}) x^\mu \left[ \frac{d}{du} U^\nu \right].
\]  

(69)
Consequently, at \( P \),
\[
\frac{dy^a}{du} = \delta^a_\mu \ U^\mu (P) ; \quad \frac{d^2y^a}{du^2} = \delta^a_\lambda a^\lambda (P).
\] (70)
Suppose a self-parallel curve goes through \( P \) in \( N \) with velocity \( U^\mu \). Then,
\[
\frac{d^2}{du^2} y^a = - \delta^a_\lambda \gamma^\lambda (\mu \nu) \gamma^\nu (\rho \sigma) U^\mu U^\sigma x^\mu.
\] (71)
The Lorentz transformation (80) reduces in the present case to
\[
\Lambda^a_b (x) = H^a_\lambda h_b^\lambda = h_b (y^a).
\] (72)

Seen from the frame \( \{ H_a \} \), the velocity is
\[
U^a = (\delta^a_\mu + \delta^a_\lambda \gamma^\lambda \nu \mu U^\nu) \frac{dx^\mu}{du} = \frac{d}{du} y^a + \frac{1}{2} \delta^a_\lambda \gamma^\lambda \nu \mu (x^\nu U^\mu - x^\mu U^\nu).
\] (73)
Notice that \( U^a = H^a (\frac{d}{du}) \) is a non-holonomic velocity if \( T \neq 0 \). If \( T = 0 \), \( U^a = \frac{d}{du} y^a \).

In this standard general-relativistic case, given a geodesic \( \gamma \) going through a point \( P \) \((\gamma (0) = P)\), there is always a very special system of coordinates (riemannian normal coordinates) or geodesic coordinates \( \{ H_a \} \) in a neighborhood \( N \) of \( P \) such that the components of the Levi-Civita connection vanish at the point \( P \). The geodesic is a straight line \( y^a = c^a \). As long as \( \gamma \) traverses \( N \), the ideal observer will not feel gravitation: the geodesic equation reduces to the forceless equation \( \frac{dc^a}{dy^a} = \frac{dx^a}{ds} = 0 \). This is an inertial observer in the absence of external forces. Coupling to source fields and test particles is given by the minimal coupling prescription: derivatives go into covariant derivatives. If the components of the connection vanish, covariant derivatives reduce to usual derivatives. All laws of Physics reduce to the expressions they have in Special Relativity.

Actually, \( du = \sqrt{\eta_{\mu \nu} dx^\mu dx^\nu} \) has been taken as the curve parameter, and the curve stands on the Riemann space. For Special Relativity to hold, it would be necessary to change to the special relativistic proper time \( ds = \sqrt{\eta_{\mu \nu} dx^\mu dx^\nu} \) as a parameter. This change is achieved by a rescaling \( dx^a \rightarrow dx^a \frac{du}{ds} \) in all the coordinates. This would correspond to transforming \( H^a_\mu \) to the tetrad \( \tilde{H}^a_\mu = \frac{\partial x^a}{\partial u} \frac{du}{ds} \).

There is actually more, as will be seen in (72). Given any smooth curve, it is possible to find a coordinate system, defined on a domain \( U \), in which the components of the Levi-Civita connection vanish along the curve. And still more: along any differentiable curve and any linear connection, it is possible to find a local frame, defined on a domain \( N \), in which the connection components vanish along the curve. Gravitation seems to be absent. Using that frame an ideal observer, accelerated or not, can employ Special Relativity. A real observer will see more. Only the connection appears in the geodesic equation, but curvature (and, eventually, torsion) makes itself present in the Jacobi equation. Before going into that, let us say a few words on what happens in the presence of torsion.

2. With torsion

Torsion will not add a force, by the last expression in Eq.\((13)\), which leads to
\[
a^\lambda = \frac{d}{du} U^\lambda + \Gamma^\lambda_{\mu \nu} U^\mu U^\nu = \frac{d}{du} U^\lambda + \Gamma^\lambda_{\mu \nu} U^\mu U^\nu.
\] (74)
In what concerns the force, only the symmetric part of the connection contributes and a choice of coordinate system is enough to make the effect of torsion to vanish. All the geodesics are consequently fixed by the symmetric part of the connection, which is by itself a connection. Adding a torsion to a symmetric connection does not change the geodesics. Nevertheless, these are not the geodesics of \( \tilde{\Gamma}^\lambda_{\mu \nu} \) if \( \Gamma^\lambda_{\mu \nu} \) is metric preserving. The difference comes from the symmetric part added by torsion to \( \tilde{\Gamma} \), as shown in Eq.\((17)\):
\[
a^\lambda = \frac{d}{du} U^\lambda + \Gamma^\lambda_{\mu \nu} U^\mu U^\nu = \frac{d}{du} U^\lambda + \tilde{\Gamma}^\lambda_{\mu \nu} U^\mu U^\nu + \frac{1}{2} \left[ T^\lambda_{\mu \nu} + T^\lambda_{\nu \mu} \right] U^\mu U^\nu.
\] (75)
There are consequently two distinct coordinate systems:

(i) one makes the “force” term coming from the Levi–Civita piece to vanish:
\[
\tilde{y}^a = x^a + \frac{1}{2} \tilde{\gamma}^a_{\mu \nu} x^\mu x^\nu;
\]
(ii) the other causes a metric–preserving \( \tilde{\omega}^a_{bc} \) to vanish:
\[
\tilde{y}^a = x^a + \frac{1}{2} \left[ \tilde{\gamma}^a_{\mu \nu} + T^a_{\mu \nu}(P) + T^a_{\nu \mu}(P) \right] x^\mu x^\nu.
\]
2. Along a curve

The previous results hold for any tetrad whose covariant derivative vanishes at \( P \). Indeed, expression (27) says that

\[
\partial_\nu H_b^\lambda(x) + \Gamma^\lambda_{\mu\nu}(x) H_b^\mu(x) = H_a^\lambda(x) \omega^a_{\mu\nu}(x).
\]

The condition \( \omega^a_{\mu\nu}(P) = 0 \) is the same as

\[
\nabla_\nu H_b^\lambda(P) = \partial_\nu H_b^\lambda(P) + \Gamma^\lambda_{\mu\nu}(P) H_b^\mu(P) = 0.
\]

Expressions (54) provide a first-order solution of

\[
H_b^\lambda(x) \partial_\nu H_b^\mu(x) = \Gamma^\lambda_{\mu\nu}(x)
\]

in the neighborhood \( N' \) of \( P \). The left-hand side is just the Weitzenböck connection \( \bar{\Gamma}^{(H)}_{\lambda\mu\nu} \) of the tetrad field \( \{H_a\} \).

The next natural question is: can the same be done along a curve? Take a differentiable curve \( \gamma \) which is an integral curve of a field \( U \), with \( U^\mu = \frac{dx^\mu}{du} = \frac{dv^\mu(u)}{du} \). The condition for the connection to vanish along \( \gamma \), \( \omega^a_{\mu\nu}(\gamma(u)) = 0 \), will be

\[
U^\nu \partial_\nu H_a^\lambda(\gamma(u)) + \Gamma^\lambda_{\mu\nu}(\gamma(u)) U^\mu H_a^\mu(\gamma(u)) = 0,
\]

that is,

\[
\frac{d}{du} H_a^\lambda(\gamma(u)) + \Gamma^\lambda_{\mu\nu}(\gamma(u)) U^\nu H_a^\mu(\gamma(u)) = 0.
\]

This is simply the requirement that the tetrad (each of the four members \( H_a \)) be parallel-transported along \( \gamma \). Thus, if the connection vanishes at an end-point of the curve in a frame, it will vanish when seen from the parallel displacements of that frame. For the dual base \( \{H^a\} \), the above formula reads

\[
\frac{d}{du} H^a_\mu(\gamma(u)) - \Gamma^p_{\mu\nu}(\gamma(u)) U^\nu H^a_\mu(\gamma(u)) = 0.
\]

Any differentiable curve \( \gamma \) defines a mapping between tangent spaces by parallel displacement; this means that, given such a curve and a linear connection, any vector field can be parallel-transported along \( \gamma \).

- The procedure is then very simple: take, in the way previously discussed, a point \( P \) on the curve and find the corresponding “nullifying” tetrad \( \{H_a\} \); then, parallel-transport it along the curve.
- The components \( \omega^a_{\mu\nu}(x) \) vanish along the curve; this means that the components \( \Gamma^\lambda_{\mu\nu}(x) \) reduce to those of the Weitzenböck connection of the tetrad \( \{H_a\} \) along the curve; the problem is equivalent to finding a tetrad field whose Weitzenböck connection coincides with \( \Gamma \) along the curve.
- Applying a point-independent Lorentz transformation to the fixed point equation (72) yields the same equation; applied to a solution, it gives another solution; thus, such solutions are defined up to fixed-point Lorentz transformations; along a curve, as it will be seen below, the Lorentz transformations will be distinct at each point.
- Take (80) in the form

\[
H_b^\mu(x) \frac{d}{du} H_b^\lambda(x) = -\bar{\Gamma}^{(H)}_{\lambda\mu\nu}(x) U^\nu = -\Gamma^\lambda_{\mu\nu}(x) U^\nu
\]

and contract it with \( U^\mu \):

\[
U^\mu H_b^\mu(x) \frac{d}{du} H_b^\lambda(x) = -\Gamma^\lambda_{\mu\nu}(x) U^\mu U^\nu.
\]

Seen from the tetrad, the tangent field will be \( U^b = H_b^\mu U^\mu \), and the above formula is

\[
U^\mu \frac{d}{du} H_b^\lambda(x) + \Gamma^\lambda_{\mu\nu}(x) U^\mu U^\nu = 0,
\]

or

\[
H_b^\lambda(x) \frac{d}{du} U^b = \frac{d}{du} U^\lambda(x) + \Gamma^\lambda_{\mu\nu}(x) U^\mu U^\nu = \nabla_U U^\lambda(x).
\]
• Seen from the frame \( \{ H_a = H_a^\lambda \frac{\partial}{\partial x^\lambda} \} \), the equation for a self-parallel curve will be

\[
\frac{d}{du} U^a = 0. \tag{86}
\]

• If \( U^a = \frac{dy^a}{d\tau} \), for \( y^a \) some coordinates on Minkowski space and \( d\tau^2 = \eta_{ab} dy^a dy^b \), then \( \frac{d}{du} U^a = \frac{d\tau}{du} \frac{du^a}{d\tau} \) with \( \frac{d\tau}{du} \neq 0 \), and \( \frac{d}{du} U^a = 0 \iff \frac{d^2\tau^a}{d\tau^2} = 0. \)

• If an external force is present, then \( m \nabla a \nabla u U^\lambda(x) = F^\lambda \) and \( m \frac{d}{du} U^a = F^a. \)

• The point solution given above is a first-order local solution; there can be a global unique solution only if the connection is flat.

Summing up: given (i) given a piecewise-differentiable curve \( \gamma \), (ii) a point \( P \) on \( \gamma \) with a coordinate chart \( (N, x^\mu) \) around it, and (iii) the components \( \Gamma^\lambda_{\mu\nu}(x) \) of a linear connection in the holonomic base defined by the coordinate system \( \{ x^\mu \} \), then there exists a tetrad frame \( \{ H_a^\lambda \} \) at \( P \), parallel-transported along \( \gamma \), in which the connection components vanish along the curve, as long as it traverses a small enough neighborhood \( N' \subset N. \)

A generic tetrad \( h_a \) is not parallel-transported along a Levi-Civita geodesic of velocity \( U \). Its deviation from parallelism is measured by the spin connection. Indeed, from Eq.(30),

\[
\frac{d}{du} h^\lambda b + \Gamma^\lambda_{\mu\nu} U^\nu h^\mu = h^\lambda a \odot \omega_{ab} b U^\nu. \tag{87}
\]

Let us write it in still another form,

\[
\nabla_U h_b = \frac{d}{du} h_b + \odot \left( \frac{d}{du} \right) h_b = \odot h_b (\frac{d}{du}) h_a. \tag{88}
\]

This holds for any curve with tangent vector \( U = (\frac{d}{du}) \). It means that the frame \( \{ h_b \} \) can be parallel-transported along no curve. The spin connection forbids it, and gives the rate of change with respect to parallel transport.

### 3. Free-falling frame

The Principle says that it is possible to choose a frame in which the connection vanishes. Let us see now how to obtain such a free falling frame \( \{ H_a \} \) from an arbitrary initial frame. Contracting \( [21] \) with \( H^a_\lambda \),

\[
H^a_\lambda \frac{d}{du} h^\lambda b + \Gamma^\lambda_{\mu\nu} U^\nu H^a_\lambda h^\mu = H^a_\lambda h^\nu c \omega^c_{\nu\beta} U^\nu, \tag{89}
\]

which is the same as

\[
\frac{d}{du} \left( H^a_\lambda h^\lambda b \right) - h^\mu b \left( d^a_\mu H^\alpha_\mu - \Gamma^\lambda_{\mu\nu} U^\nu H^a_\lambda \right) = H^a_\lambda h^\nu c \omega^c_{\nu\beta} U^\nu. \tag{90}
\]

The second term in the left-hand side vanishes by Eq.(81), so that we remain with

\[
\frac{d}{du} \left( H^a_\lambda h^\lambda b \right) - (H^a_\lambda h^\nu c \omega^c_{\nu\beta} U^\nu) = 0. \tag{91}
\]

What appears in both parentheses is a point-dependent Lorentz transformation \( [22] \) relating the tetrad \( h_a \) to the frame \( H_a \) in which \( \Gamma \) vanishes:

\[
H^a_\lambda = \Lambda^a_\beta h^\beta_\lambda, \quad H^a = \Lambda^a_\beta h^\beta. \tag{92}
\]

This means

\[
\Lambda^a_\beta = H^a_\lambda h^\lambda b. \tag{93}
\]

and holds on the common definition domain of the tetrad fields. We have, of course,

\[
\nabla_U h_\lambda = H_\lambda^a \nabla_U \Lambda^a_\alpha. \tag{94}
\]
For the particular case of the Levi-Civita connection $\Gamma^a$, taking (103) into (101), we arrive at a relationship which holds on the intersection of that domain with a geodesic: 
\[
\frac{d}{du} \Lambda^a_{\ b} - \Lambda^a_{\ c} \omega^c_{\ bd} U^d = 0. \tag{95}
\]
This equation gives the change, along the metric geodesic, of the Lorentz transformation taking a tetrad $h_a$ into the frame $H_a$. The vector formed by each row of the Lorentz matrix is parallel-transported along the line. Contracting with the inverse Lorentz transformation $(\Lambda^{-1})^a_{\ b} = h^a_{\ \rho} H_b^\rho$ the expression above gives
\[
\omega^a_{\ bd} U^d = (\Lambda^{-1})^a_{\ c} \frac{d}{du} \Lambda^c_{\ b} = (\Lambda^{-1} \frac{d}{du} \Lambda)^a_{\ b}. \tag{96}
\]
This is, in the language of differential forms,
\[
\omega^a_{\ bd} h^d \left( \frac{d}{du} \right) = (\Lambda^{-1} d\Lambda)^a_{\ b} \left( \frac{d}{du} \right). \tag{97}
\]
Thus on the points of the curve, the connection has the form of a gauge Lorentz vacuum:
\[
\omega^a_{\ b} = (\Lambda^{-1} d\Lambda)^a_{\ b}. \tag{98}
\]
This expression deserves further attention. A Lorentz transformation, as a member of the Lorentz group, has the form
\[
\Lambda = e^W = e^\frac{1}{2} J_{ab} \alpha^{ab}. \tag{99}
\]
Matrix $W$ belongs to the group Lie algebra, $J_{ab}$ are the generators and $\alpha^{ab}$ are parameters specifying the transformation ($\alpha^{ij}$ for rotation angles, $\alpha^{a0}$ for boosts parameters). Clearly
\[
\omega^a_{\ b} = (\Lambda^{-1} d\Lambda)^a_{\ b} = (d \ln \Lambda)^a_{\ b} = (dW)^a_{\ b} = dW^a_{\ b}. \tag{100}
\]

The Lorentz group can be obtained in a standard way. Introduce first the canonical basis $\{ \Delta_a^b \}$ for the generators of the linear group $GL(4, \mathbb{R})$. These matrices have entries $(\Delta_a^b)^c_d = \delta^c_a \delta^d_b$. Define new matrices with indices lowered by the Lorentz metric, $\{ \Delta_{ab} \} = \eta_{bc} \Delta_a^c$. Then, the set of matrices $\{ J_{ab} = \Delta_{ab} - \Delta_{ba} \}$ provide a representation for the generators of the Lorentz group. Their entries are $(J_{cd})^a_{\ b} = \eta_{db} \delta^c_a - \eta_{cb} \delta^d_a$. Consequently,
\[
W^a_{\ b} = \frac{1}{2} (J_{cd})^a_{\ b} \alpha^{cd} = \frac{1}{2} (\eta_{db} \delta^c_a - \eta_{cb} \delta^d_a) \alpha^{cd} = \frac{1}{2} (\alpha^{ac} - \alpha^{ca}) = \alpha^{ab}. \tag{101}
\]
Therefore, on the points of the curve,
\[
\omega^a_{\ b} = \partial_b \alpha^{ab}. \tag{102}
\]

The spin connection, seen from the tetrad $h$, is given by the derivatives of the parameters of the Lorentz transformation taking $h$ into the locally inertial tetrad $H$. Once we have learned about the relationship of the spin connection to Lorentz transformations, we can go back to
\[
\nabla_U h^\lambda = \frac{\nabla}{\nabla u} h^\lambda = \frac{d}{du} h^\lambda + \Gamma^\lambda_{\ mu} U^u h^\mu = h_a^\lambda \omega^a_{\ b} U^b = h_a^\lambda \frac{d}{du} \alpha^a_{\ b}. \tag{103}
\]
If we choose a frame such that $h_a^\lambda = U^\lambda$, the acceleration will be
\[
\ddot{\alpha}^\lambda = \frac{d}{du} U^\lambda + \Gamma^\lambda_{\ mu} U^u U^\nu = h_a^\lambda \omega^a_{\ 0b} U^b = h_0^\lambda \omega^0_{\ 0b} U^b + h_j^\lambda \omega^0_{\ j0} U^b
\]
\[
= h_j^\lambda \omega^0_{\ j0} U^b = h_k^\lambda \frac{d}{du} \alpha^k_{\ 0}. \tag{104}
\]
Only boosts turn up. This means that, on a geodesic, the spin connection in some tetrad field $\{ h_a \}$ is determined by the special Lorentz transformation (103) taking $\{ h_a \}$ into that tetrad field $\{ H_a \}$ in which it vanishes. The spin connection is a vacuum along each geodesic. Boost parameters (essentially frame velocities) and rotation angles will
continuously change along the curve to transform $h_a$ into $H_a$ and keep the connection, as seen from $\{H_a\}$, equal to zero. At each point of a metric-geodesic observer, the velocity $U$ differs by a Lorentz transformation from the velocity $\bar{U}$ satisfying the forceless equation \[ \frac{\partial}{\partial u} U^\alpha + \Lambda^{-1} \frac{\partial}{\partial u} \Lambda^a_{\ b} U^b = 0 \] (105)

which, once multiplied on the left by $\Lambda$, gives Eq.\[ \bar{U}^c = 0. \] (106)

4. Real observers

It is important to stress that the above results hold only along a curve — a one-dimensional domain — so that curvature is not probed. Curvature, the real gravitational field strength, only manifests itself on two-dimensional domains. If curvature is nonvanishing, no vector field can be parallel-transported along two distinct lines. In effect, consider two curves with tangent fields $U$ and $V$, with parameters $u$ and $v$, so that

\[ \nabla_U H^\lambda = U^\nu \nabla_\mu H^\lambda = U^\nu [\partial_\nu H^\lambda + \Gamma^\lambda_{\ \mu \rho} U^\rho H^\mu]; \] (107)

\[ \nabla_V H^\lambda = V^\mu \nabla_\mu H^\lambda = V^\mu [\partial_\mu H^\lambda + \Gamma^\lambda_{\ \mu \rho} V^\rho H^\mu]. \] (108)

Then, one finds

\[ [\nabla_U \nabla_V - \nabla_V \nabla_U] H^\lambda = R^\lambda_{\mu \rho \nu} H^\mu U^\rho V^\nu. \] (109)

This shows that, if $\nabla_U H^\lambda = 0$, then forcibly $\nabla_V H^\lambda = R^\lambda_{\mu \rho \nu} H^\mu U^\rho V^\nu \neq 0$. The apparent “turning-off” of gravitation is an effect of the one-dimensional character of the curve representing the ideal observer. A real observer will have spatial extension and will be, consequently, represented by a bunch of neighboring timelike curves. Such curves will deviate from each other in a way which depends of the curvature (and torsion, if present). The deviation $X^\lambda$ will, in effect, respect Jacobi’s equation for a general curve. The Jacobi equation for a general connection in a holonomic frame\[ 2 \] is

\[ \nabla_U \nabla_U X^\lambda + R^\lambda_{\mu \rho \nu} U^\rho X^\nu U^\mu + U^\nu \nabla_U (T^\lambda_{\mu \nu} X^\mu) = 0. \] (110)

When the curve is self-parallel ($\nabla_U U^\nu = 0$), it gives the geodesic deviation:

\[ \nabla_U \nabla_U X^\lambda + R^\lambda_{\mu \rho \nu} U^\rho X^\nu U^\mu + \nabla_U (T^\lambda_{\mu \nu} X^\mu U^\nu) = 0. \] (111)

For a metric geodesic in the tetrad frame, the equation for the deviation $X^a$ will be

\[ \nabla_U \nabla_U X^a + R^\alpha_{\ bcd} U^b X^c U^d = 0. \] (112)

The Equivalence Principle is not stated for real observers. These are extended objects in the 3-dimensional space sections and, by observing nearby curves, will always be able to detect the curvature. Actually, curvature will be detectable on domains of 2 or more dimensions. Let us profit to make clear what should be meant by the word “local”, by which some authors understand “at a point in spacetime”, others “in a neighborhood of a point in spacetime”, and others “on a piece of trajectory through a point in spacetime”. This issue has been definitively clarified by Iliev:\[ 4 \] in what concerns the Principle, only the last meaning applies.

We have seen that:

1. given any connection $\Gamma$ and any differentiable curve $\gamma$, there exists a tetrad field $H$ which is parallel-transported by $\Gamma$ along $\gamma$;
2. along $\gamma$, any other tetrad field is taken into $H$ by a point-dependent Lorentz transformation;
3. if $\gamma$ is a geodesic of $\Gamma$, then $H$ is in free fall along $\gamma$; seen from $H$, $\gamma$ is a straight line;
4. if $\gamma$ is a timelike geodesic of $\Gamma$, then $H$ can be assimilated to a local inertial frame, or to an inertial ideal observer, which will see the world as described by Special Relativity;
5. this holds only on the points of the 1-dimensional domain $\gamma$; a real observer, composed of bunch of curves around $\gamma$, will sense the gravitational field.
4. THE CASE OF A GAUGE CONNECTION

Reduced to the statement that the connection can be made to vanish at a point, the Equivalence Principle is not specific of gravitation. In effect, it has been clearly shown by Iliev\(^7\) that also in a gauge theory it is possible to choose a gauge to make the connection components to vanish at a point and along a curve. As all the main points are already present in the abelian case, we shall present in some detail the case of electrodynamics and only indicate the generalization to the non–abelian case.

1. Electrodynamics

Consider a potential \(A_\mu(x)\). Take a point \(P\) and a coordinate neighborhood around \(P\). For the sake of simplicity, choose coordinates \(x^\lambda\) with origin at \(P\), that is, such that \(x^\lambda(P) = 0\). Suppose further that the field \(F_{\mu\nu}\) is well-defined, that is, the derivatives of \(A_\mu(x)\) are finite in some neighborhood \(N\) around \(P\). We can take \(N\) as a member of the implicit differentiable atlas. Call \(\gamma_\mu\) the value of \(A_\mu(x)\) at the point \(P\), \(\gamma_\mu = A_\mu(P)\). There will be a domain \(N' \subset N\) around \(P\), small enough for \(A_\mu(x)\) to be approximated as

\[
A_\mu(x) = \gamma_\mu + x^\lambda (\partial_\lambda A_\mu)(P) = \gamma_\mu + x^\lambda \gamma_\mu^\lambda.
\]  

What we shall do is to choose, in the general expression for a gauge transformation

\[
A'_\mu(x) = A_\mu(x) + \partial_\mu \phi(x),
\]

a very particular \(\phi(x)\). As the two constants \(\gamma_\mu\) and \(\gamma_\mu^\lambda\) are given, we take on \(N'\) the function

\[
\phi(x) = -\gamma_\mu x^\mu - \gamma_\mu^\lambda x^\lambda.
\]

Then, \(\partial_\mu \phi(x) = -\gamma_\mu - \gamma_\mu^\lambda x^\lambda - \gamma_\mu^\lambda x^\lambda\) and, always on \(N'\),

\[
A'_\mu(x) = A_\mu(x) + \partial_\mu \phi(x) = - x^\lambda \gamma_\mu^\lambda,
\]

which gives \(A'_\mu(P) = 0\). As \(\partial_\mu A'_\mu(x) = -\gamma_\mu^\mu\), \(F_{\mu\nu}(P) = \gamma_\nu^\mu - \gamma_\mu^\nu\) as it should be. Thus, given any electromagnetic potential and a point \(P\), it is always possible to choose a gauge in which the potential vanishes at \(P\). It is possible to go a bit beyond the above approximations, by taking for \(A_\mu(x)\) a Taylor expansion around \(P\):

\[
A_\mu(x) = A_\mu(P) + \sum_{j=1}^{\infty} \frac{x^{\lambda_1}x^{\lambda_2}...x^{\lambda_j}}{j!} [\partial_{\lambda_1} \partial_{\lambda_2} ... \partial_{\lambda_j} A_\mu]_P.
\]

Choose then

\[
\phi(x) = -x^\rho A_\rho(P) - x^\rho \sum_{j=1}^{\infty} \frac{x^{\lambda_1}x^{\lambda_2}...x^{\lambda_j}}{(j+1)!} [\partial_{\lambda_1} \partial_{\lambda_2} ... \partial_{\lambda_j} A_\rho]_P,
\]

so that

\[
\partial_\mu \phi(x) = -A_\mu(P) - \sum_{j=1}^{\infty} \frac{x^{\lambda_1}x^{\lambda_2}...x^{\lambda_j}}{(j+1)!} [\partial_{\lambda_1} \partial_{\lambda_2} ... \partial_{\lambda_j} A_\mu]_P - \sum_{j=1}^{\infty} \frac{x^{\lambda_1}x^{\lambda_2}...x^{\lambda_{j-1}}}{(j+1)!} [\partial_{\lambda_1} \partial_{\lambda_2} ... \partial_{\lambda_{j-1}} \partial_\mu A_\rho]_P.
\]

We have then

\[
A'_\mu(x) = A_\mu(x) + \partial_\mu \phi(x) = \frac{1}{2} x^\rho F_{\rho\mu}(P) + \sum_{j=2}^{\infty} \frac{x^{\lambda_1}x^{\lambda_2}...x^{\lambda_j}}{(j+1)!} [\partial_{\lambda_1} \partial_{\lambda_2} ... \partial_{\lambda_{j-1}} F_{\lambda_j\mu}]_P.
\]

From which again \(A'_\mu(P) = 0\), but further \([\partial_\mu A'_\mu - \partial_\mu A'_\mu]_P = F_{\rho\mu}(P)\).
2. Non–abelian case

Considerations analogous to those above will keep holding in what concerns coordinates and neighborhoods. The generic gauge transformation is now

\[ A'_\mu(x) = g^{-1}(x)A_\mu(x)g(x) + g^{-1}(x)\partial_\mu g(x) = g^{-1}(x)A_\mu(x)g(x) + \partial_\mu \ln g(x). \]  

(121)

The main difference comes from the fact that now matrices \((A, g, \gamma, F)\) are at work. Choose in some \(N'\) the same form of \((13)\), \(A_\mu(x) = \gamma_\mu + x^\lambda \gamma_{\lambda \mu} \). Take

\[ \ln g(x) = -\gamma_\mu x^\mu - \gamma_{\rho \sigma} x^\rho x^\sigma. \]

(122)

It follows that

\[ g(x) \approx I - \gamma_\mu x^\mu - \gamma_{\rho \sigma} x^\rho x^\sigma + \frac{1}{2} \gamma_\mu x^\rho \gamma_{\rho \sigma} x^\sigma; \quad g^{-1}(x) \approx I + \gamma_\mu x^\mu + \gamma_{\rho \sigma} x^\rho x^\sigma + \frac{1}{2} \gamma_\mu x^\rho \gamma_{\rho \sigma} x^\sigma, \]

(123)

\[ A'_\mu(x) = -\gamma'_{\lambda \mu} x^\lambda + [\gamma_{\lambda \mu}, \gamma_\mu] x^\lambda. \]

(124)

This gives back \((16)\) in the abelian case. We see that the gauge transformed \(A'_\mu(P) = 0\), so that the original \(A_\mu(P)\) “touches” a vacuum at the point \(P\), \(A_\mu(P) = g \partial_\mu g^{-1}\). The covariant field strength remains what it should be:

\[ F_{\mu \nu}(P) = \gamma'_{\mu \nu} - \gamma'_{\nu \mu} + \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu. \]

(125)

We insist that the vanishing of \(A'_\mu\) takes place at one point. If \(A_\mu(x) = g \partial_\mu g^{-1}\) in an open domain, however small, then \(F_{\mu \nu}(x) = 0\) in the domain. To obtain the vanishing of the gauge potential along a curve, it is necessary to proceed by infinitesimal steps. The procedure would be as follows:

- start at a point \(P\) on the curve \(\gamma\) with \(A^a_\mu = 0\);
- take a neighboring point \(Q\) on \(\gamma\), separated from \(P\) by a variation \(du\) in the parameter; at \(Q\),

\[ A'^{a}_\mu \frac{dx^\mu}{du} du = A^a_\mu \frac{dx^\mu}{du} du - \nabla_\mu W^a \frac{dx^\mu}{du} du = 0 - \frac{\nabla}{du} W^a du, \]

or \(A'^{a}_\mu = -\nabla_\mu W^a\), a pure gauge;
- start now from \(Q\), and repeat the procedure, taking another point \(Q'\) on \(\gamma\); and so on.

This amounts to a step-by-step definition of an ordered product and says that, along any differentiable curve traversing a domain \(N'\), there is a continuous choice of gauges in which the potential vanishes. Notice that this only holds for a theory which keeps on gauge invariance. Except for the abelian electromagnetic sector, gauge symmetry is broken in electroweak theory. A gauge is chosen once and for all by the extra scalar field, so as to provide the boson masses. No choice of gauge is left, and the boson fields cannot be annulled.

5. THE FORCE EQUATION

A massive particle without additional structure (for instance, supposing that the effect of its spin is negligible, or zero) will follow the geodesic equation. A charged test particle in a gravitational and an electromagnetic field will respect instead the Lorentz law of force in the presence of a gravitational field

\[ mc \nabla_U U^\lambda = mc \left( \frac{d}{du} U^\lambda + \Gamma^\lambda_{\mu \rho} U^\mu U^\rho \right) = \frac{e}{c} F^\lambda_{\mu} U^\mu. \]

(126)

Gravitation, represented by the term in \(\Gamma\), is an inertial force, because that term can be made to vanish at each point by a choice of reference system. The same holds if we think of a test field, for example a vector field \(\phi^a\). It couples to gravitation through the \(\Gamma\) term in the minimal coupling prescription \(\partial_\mu \phi^a \rightarrow \partial_\mu \phi^a + \Gamma^a_{\rho \beta} \phi^\rho \phi^\beta\). It decouples at a point, by the same choice of reference system. This can, as seen above, be generalized to a curve.

To fix the ideas, let us consider the paradigm of a gauge theory, the original Yang–Mills \(SU(2)\) model. A test field \(\phi^a\) carrying isospin will belong to some representation of \(SU(2)\) with generators \(T_a\), and will feel the presence
of a Yang–Mills field through the minimal coupling prescription $\partial_\mu \phi^a = \partial_\mu \phi^a + A^a_{\mu} \, T_\alpha \phi^\alpha$. Matrices $g = \exp[\omega^a T_a]$, representing $SU(2)$ transformations will act on the (internal) vector space (or module) $V$ to which $\phi^a$ belongs. A gauge transformation can be conceived in the usual, active way, as a change in $\phi^a$. But it can also be conceived in the passive way, as a change of frame in $V$. In that case, a choice of gauge corresponds to a choice of frame. And we have seen that, fixed a point $P$ and a gauge potential $A_\mu$, it is possible to choose a frame in $V$ such that $\phi^a$ decouples from $A_\mu$ at $P$. It is doubtful that we can use the word “inertia” in this internal case, but the possibility of zeroing the connection is a common feature of all gauge interactions. As already remarked, this is not true for the broken-symmetry electroweak interactions.

The motion of a test particle of mass $m$ submitted to an $SU(2)$ Yang–Mills field is described by (i) its spacetime coordinates, and (ii) an “internal” vector $\mathbf{I} = \{I_\alpha\}$ giving its state in isotopic space. The corresponding dynamic equations are (i) the generalised Lorentz force law

\[
mc \nabla U^\lambda = \frac{1}{c} \, I_\alpha F^{\alpha \lambda}_\nu \, U^\nu = \frac{1}{c} \, I_\alpha (\partial^\lambda A^\alpha_{\nu} - \partial_\nu A^{\alpha \lambda} + c_\rho \, A^{\nu \lambda} A^\rho_{\nu}) \, U^\nu, \tag{127}
\]

and (ii) the charge–precession, or Wong equation,

\[
\nabla U \, \mathbf{I} + A_\mu \, U^\mu \times \mathbf{I} = 0. \tag{128}
\]

This equation says that internal motion is a parallel–transport by the internal connection and, furthermore, a precession (because $I^2$ is conserved). For unbroken models it is possible to choose, at each point, a gauge in which there is no charge–precession. The force, however, does not change. The vanishing of $A$ along a curve would mean a gauge $A_\mu U^\mu(s) = 0$, which would generalize to general spaces the well-known gauge $A_\mu n^\mu = 0$ used in Minkowski space. That would, however, lead only to absence of precession, not of force (though only the abelian, derivative part of $F$ would contribute). Inertial and non-inertial forces are clearly distinguished. Gauges are internal contrivances, while frames participate in the very structure of the habitat of each physical object and the scene of physical process, spacetime. Furthermore, Physics is not invariant under changes of frames, unless they are related by Lorentz transformations and consequently induce the same metric. An ideal observer in a gravitational field is locally equivalent to an ideal observer in the absence of gravitation, while an ideal observer in a gauge field will always feel its presence. At least two ideal observers are needed to detect gravitation, but only one is enough to detect an electromagnetic field. In this sense gauge fields are local, and gravitation is not.

Concerning the Quantum Mechanics of a system immersed in a gravitational background, an ideal observer—a point in 3-space—is indeterminate. Quantum Mechanics in 3-space will always probe a 3-dimensional domain, intersecting a bunch of curves in spacetime and, consequently, will always be aware of a gravitational field, however small its effect may be.

**Acknowledgments**

The authors would like to thank FAPESP-Brazil and CNPq-Brazil for financial support.

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1. J. L. Synge, *Relativity: The General Theory*, in the Preface (J. Wiley, New York, 1960).
2. C. Lämmerzahl, *Gen.Rel.Grav.* 28 1043, 1996; “Minimal coupling and the equivalence principle in quantum mechanics” (arXiv:gr-qc/9807072).
3. P. D. Mannheim, “The equivalence principle in classical mechanics and quantum mechanics” (arXiv:gr-qc/9810087).
4. S. Manoff and B. Dimitrov, “On the existence of a gyroscope in spaces with affine connections and metrics” (arXiv:gr-qc/0012011).
5. D. V. Ahluwalia, “Principle of equivalence and wave–particle duality in quantum gravity” (arXiv:gr-qc/0009033).
6. T. Damour, “Questioning the equivalence principle” (arXiv:gr-qc/0109063).
7. B. Z. Iliev, *Journal of Geometry and Physics* 24 209, 1998.
8. It is probably to Synge that should be given the credit for stressing the concept of observer, by the emphasis on the “world function” he has put in his treatise. Even so, despite the statement made in Fermi’s *Collected Papers* Editor’s comment, true Fermi coordinates deserve only a footnote mention (p. 84). Cosmologists made large use of fundamental observers. Hawking and Ellis discuss “timelike curves” in detail, but have no use for the Principle, implicit in the supposed geometrical framework.
9. E. Fermi, *Collected Papers* (U. Chicago Press/Accad.Nat. Lincei, Chicago, 1962).
10. G.F.R. Ellis, “Standard and inflationary cosmologies”, Lectures at the Summer Research Institute on Gravitation, Banff Centre, Banff, Alberta, Canada, August 12-25, 1990. SISSA-Trieste preprint 1990; G.F.R. Ellis and H. van Elst, “Cosmological models”, Cargèse Lectures 1998, arXiv:gr-qc/9812046 v4.
The vanishing of the Levi-Civita connection at a point, a standard geometrical result, is found in most classical texts on General Relativity. Recently Iliev has extended it to general derivatives, furthermore in the presence of torsion.

S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, 2nd edition (Interscience, New York, 1966).

C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1966); C. W. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973); R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984); I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University Press, Princeton, 1995).

E. Fermi, *Atti R. Accad. Lincei Rend, Cl. Fis. Mat. Nat.* 31 21-23, 51-52, 101-103 (1922). *Collected Papers*, reference above, vol. I, ps. 17-23.

B.Z. Iliev, *J. Phys.* A 29 6895, 1996; *J. Phys.* A 30 4327, 1997; *J. Phys.* A 31 1287, 1998.

B.Z. Iliev, “Normal frames for linear connections in vector bundles and the equivalence principle in classical gauge theories” (arXiv:hep-th/010109).

B. Mashhoon, “Measurement theory and general relativity” (arXiv:gr-qc/0003014); “Relativity and nonlocality” (arXiv:gr-qc/0011013).

B.Z. Iliev, not only references 8 and 16 above, but also their early versions: “Special bases for derivations of tensor algebras; I. Cases in a neighborhood and at a point”, JINR Communication E5-92-507, Dubna, 1992; “Special bases for derivations of tensor algebras; II. Case along paths”, JINR Communication E5-92-508, Dubna, 1992; “Special bases for derivations of tensor algebras; III. Case along smooth maps with separable points of selfintersection”, JINR Communication E5-92-543, Dubna, 1992.

D. Hartley, *Class. Quantum Grav.* 12 L103, 1995.

R. Aldrovandi and J.G. Pereira, *An Introduction to Geometrical Physics* (World Scientific, Singapore, 1995).

S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1992).

L. Landau and E. Lifshitz, *Théorie des champs*, 4th. edition (MIR, Moscow, 1989).

J.L. Synge and A. Schild, *Tensor Calculus* (Dover, New York, 1978).

S.K. Wong, *Nuovo Cimento* 65A 689, 1970; W. Drechsler and A. Rosenblum, *Phys.Lett.* 106B 81, 1981.