Optimal efficiency and power and their trade-off in three-terminal quantum thermoelectric engines with two output electric currents

Jincheng Lu,¹ Yefeng Liu,¹ Rongqian Wang,¹ Chen Wang,² and Jian-Hua Jiang¹,∗

¹School of physical science and technology & Collaborative Innovation Center of Suzhou Nano Science and Technology, Soochow University, Suzhou 215006, China.
²Department of Physics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China
(Dated: February 1, 2022)

We establish a theory of optimal efficiency and power for three-terminal thermoelectric engines which have two independent output electric currents and one input heat current. This set-up goes beyond the conventional heat engines with only one output electric current. For such a set-up, we derive the optimal efficiency and power and their trade-off for three-terminal heat engines with and without time-reversal symmetry. The formalism goes beyond the known optimal efficiency and power for systems with or without time-reversal symmetry, showing interesting features that have not been revealed before. A concrete example of quantum-dot heat engine is studied to show that the current set-up can have much improved efficiency and power compared with previous set-ups with only one output electric current. Our analytical results also apply for thermoelectric heat engines with multiple output electric currents, providing an alternative scheme toward future high-performance thermoelectric materials.

I. INTRODUCTION

Thermoelectric phenomena have attracted lots of research attention because of their relevance to fundamental physics and the state-of-art energy applications [1–6]. The understanding of fundamental thermodynamic constraints on the efficiency and power of nanoscale thermoelectric devices is a subject of wide-spread interest in the past decades [7–17]. Recent theoretical [7–35] and experimental [36–41] studies on thermoelectric phenomena in mesoscopic systems have renewed the fundamental understanding on thermoelectric transport and energy conversion. Several concepts, such as reversal thermoelectric energy conversion [42, 43], inelastic thermoelectric transport [5, 18–25], fundamental bounds on the optimal efficiency and power [7–13, 44], universal fluctuations of energy efficiency [14–17], cooperative effects [8, 45, 46], and nonlinear effects [47] were proposed. In particular, with the seminal works by Benenti et al. [48] and later by Brandner et al. [49] mesoscopic thermoelectric heat engines with broken time-reversal symmetry have gained much interest, particularly in multi-terminal transport configurations [6, 27, 50–53] where thermoelectric engines with asymmetric Onsager transport coefficients are studied in the set-up with one heat current input and one electric current output.

In the linear-response regime, the transport property of a thermoelectric engine is described by the following equation,

\[
(I_e) = \begin{pmatrix} G & L_1 \\ L_2 & K \end{pmatrix} \begin{pmatrix} V \\ \frac{T_h - T_c}{T_h} \end{pmatrix},
\]

(1)

where \(I_e\) and \(I_Q\) are the charge and heat currents, \(G\) and \(K\) are the charge and heat conductivity, respectively. \(L_1\) and \(L_2\) describe the Seebeck and Peltier effects, respectively. \(V\) is the voltage bias across the device, \(T_h\) and \(T_c\) are the temperatures of the hot and cold reservoirs, respectively. In time-reversal broken multi-terminal systems the two coefficients \(L_1\) and \(L_2\) can be different [48], though they are often identical for time-reversal symmetric thermoelectric devices. Thermoelectric efficiency is defined as \(\eta = -I_eV/I_Q\) with \(I_eV < 0\) (power output) and \(I_Q > 0\) (heat consumption). As shown in Ref. [48],
for a thermoelectric heat engine described by the above equation, the maximal efficiency and efficiency at maximal power of the thermoelectric heat engine are given by
\[
\eta_{\text{max}} = \frac{\eta_Cr_{12}}{ZT + 1}, \quad \eta(W_{\text{max}}) = \frac{\eta_Cr_{12}ZT}{2(2 + ZT)},
\]
respectively, where \(\eta_C = \frac{T_h - T_c}{T_h}\) is the Carnot efficiency and
\[
ZT = \frac{L_1L_2}{\kappa G}, \quad r_{12} = \frac{L_1}{L_2},
\]
are the thermoelectric figure-of-merit and the partition ratio between the two off-diagonal elements. For a time-reversal symmetric macroscopic system (length \(l\) and cross-section area \(A\), the above equations comes back to the more familiar form, \(r_{12} = 1\) and the figure-of-merit \(ZT = \frac{\sigma S^2}{\kappa}\) where \(\sigma = GL/A\) is the conductivity, \(S = L/(TG)\) is the Seebeck coefficient, \(\kappa = (K - L_1L_2/G)/(AT)\) is the thermal conductivity. Eq. (2) also gives guidance to exceed the so-called Curzon-Ahlborn limit \([54]\) \(\eta_{CA}\) for the efficiency at maximal power (in the linear-response regime \(\eta_{CA} = \frac{\eta_C}{2}\)).

However, the existing studies on thermoelectric energy conversion in time-reversal broken systems are restricted to the situation with only one output electric current \([52, 53]\). Even in multi-terminal systems, other electric currents are suppressed by tuning the electrochemical potentials and temperatures \([27]\). Such artificial constraints limit the study of thermoelectric energy conversion in generic multi-terminal mesoscopic systems. In this work, we go beyond such constraints by studying multi-terminal mesoscopic systems connected with two heat baths while there can be multiple output electric currents using multiple electrodes. For concreteness, we study a three-terminal thermoelectric heat engine with two output electric currents. We find that by going beyond previous limitation of only one output electric current, the efficiency and power can be significantly improved. We derive the analytical expressions for the optimal efficiency and power for the set-up with multiple output electric currents and find their trade-off relations \([8, 10–12, 52]\) in the linear-response regime. Our study shows that multi-terminal mesoscopic systems have the potential of achieving higher energy efficiency and larger output power than two-terminal systems, particularly in the set-ups with multiple output electric currents.

The main part of the paper is organized as follows. In Sec. II, we introduce the mesoscopic transport model. In Sec. III, we obtain the optimal efficiency and power, and derive the relations between the maximum efficiency, maximum power, efficiency at maximal power and power at maximal efficiency in the linear-response regime. In Sec. IV, we deduce the bounds for the optimal efficiency and power in the linear-response regime. In Sec. V, we analyze the efficiency and power of a triple-quantum-dot three-terminal mesoscopic system. We conclude and remark for future studies in Sec. VI.

**II. THEORETICAL MODEL AND FORMULATION**

As shown in Fig. 1, we consider a nanoscale thermoelectric device consisting of three quantum dots (QDs) coupled to three electrodes. This is a minimal model...
to demonstrate the set-up with two output electric currents. Although this model has been studied before [27, 50, 51, 53], the configuration with two output electric currents has never been studied in the time-reversal symmetry broken regime. This model is valid when the Coulomb interaction in the QDs can be neglected [55]. Each QD is coupled to the nearby reservoir when the Coulomb interaction in the QDs can be neglected [55]. Each QD is coupled to the nearby reservoir and we thus employ the indices 1/2/3 to label the leads $L/R/P$, respectively [16].

Hoppings between QDs are affected by the magnetic flux $\Phi$ piercing through the device at the center with the phase $\phi/3$ assigned to each of the hoppings ($\phi = 2\pi\Phi/\Phi_0$ where $\Phi_0$ is flux quantum). The system is described by the Hamiltonian [50]

$$\hat{H} = \hat{H}_{qd} + \hat{H}_{lead} + \hat{H}_{tun},$$

where

$$\hat{H}_{qd} = \sum_{i=1,2,3} E_i d_i^\dagger d_i + (t e^{i\phi/3} d_{i+1}^{\dagger} + \text{H.c.}),$$

$$\hat{H}_{lead} = \sum_{i=1,2,3} \sum_{k} \varepsilon_{ik} c_i^\dagger c_k,$$

$$\hat{H}_{tun} = \sum_{i,k} V_{ik} d_i^\dagger c_k + \text{H.c.}.$$

Here, $d_i^\dagger$ and $d_i$ create and annihilate an electron in the $i$th QD with an energy $E_i$, respectively, $t$ is the tunneling amplitude between the QDs. $c_i^\dagger$ and $c_i$ create and annihilate an electron in the $i$-th electrode with the energy $E_i$ ($i = 1, 2, 3$).

The chemical potential and temperature of three reservoirs are denoted by $\mu_i$ and $T_i$ ($i = L, R, P$), respectively. For each reservoir, there are an electric and a heat currents flowing out of the reservoir. In total there are six currents. However, only four of them are independent, due to charge and energy conservation [50]. We choose the charge and heat currents flowing out of the $L$ and $P$ reservoirs as the independent currents which are denoted as $I_e^L$ and $I_Q^P$ ($i = L, P$), respectively. The corresponding thermodynamic forces are

$$F_e^i = \frac{\mu_i - \mu_R}{e}, \quad F_Q^i = \frac{T_i - T_R}{T_i} \quad (i = L, P).$$

where $e < 0$ is the electronic charge. We focus on the set-up where $L$ reservoir is connected to the hot bath and the $R$ and $P$ reservoirs are connected to the cold bath, i.e., $T_L = T_h$ and $T_P = T_R = T_c$. There are two independent output electric currents, $I_e^L$ and $I_e^P$ (i.e., the charge currents flowing out of the $L$ and $P$ reservoirs), whereas there is only one input heat current $I_Q = I_Q^L$ (i.e., the heat current flowing out of the hot reservoir $L$) with the corresponding force $F_Q = F_Q^L$.

With such a set-up, the phenomenological Onsager transport equation is written in the linear-response regime as

$$\begin{pmatrix} \vec{I}_e \\ I_Q \end{pmatrix} = \begin{pmatrix} \hat{M}_{ee} & \hat{M}_{eQ} \\ \hat{M}_{QQ} & \hat{M}_{QQ} \end{pmatrix} \begin{pmatrix} \vec{F}_e \\ F_Q \end{pmatrix},$$

where the symbols $e$ and $Q$ are used to abbreviate the indices of forces and currents for charge and heat, respectively (i.e., $\vec{I}_e = (I_e^L, I_e^P)^T$, $\vec{F}_e = (F_e^L, F_e^P)^T$, $I_Q = I_Q^L$, and $F_Q = F_Q^L$.); here the superscript $T$ stands for vector/matrix transpose). $\hat{M}_{ee}$ denotes the $2 \times 2$ charge conductivity tensor, the $2 \times 1$ matrix $\hat{M}_{eQ}$ describes the Seebeck effect, while the matrix $\hat{M}_{QQ}$ describes the Peltier effect. The $1 \times 1$ matrix (scalar) $\hat{M}_{QQ}$ represents the heat conductivity. For systems with time-reversal symmetry (e.g., $\phi = 0, \pi$), Onsager’s reciprocal relation gives $\hat{M}_{eQ} = \hat{M}_{QQ}^T$. In contrast, for time-reversal broken systems, they are not equal to each other.

The output power and energy efficiency of the thermoelectric heat engine are written respectively as

$$W = -\vec{F}_e^T \vec{F}_e = -(\vec{F}_e^T \hat{M}_{ee} \vec{F}_e + \vec{F}_Q^T \hat{M}_{eQ} F_Q) > 0,$$

and

$$\eta = \frac{W}{I_Q} = \frac{\vec{F}_e^T \hat{M}_{ee} \vec{F}_e + \vec{F}_Q^T \hat{M}_{eQ} F_Q}{\hat{M}_{QQ} \vec{F}_e + \hat{M}_{QQ} F_Q} \leq \eta_C.$$

Here $\eta_C = 1 - T_c/T_h = F_Q$ is the Carnot efficiency which is the absolute upper bound for the attainable energy efficiency due to the second-law of thermodynamics of thermodynamics.
III. MAXIMAL EFFICIENCY AND POWER FOR TIME-REVERSAL BROKEN SYSTEMS

We note that in the linear-response regime the energy efficiency is invariant under the scaling transformation \( \vec{F}_e \rightarrow a \vec{F}_e \) and \( F_Q \rightarrow a F_Q \) with \( a \) being an arbitrary constant. In comparison, the output power scales as \( W \rightarrow a^2 W \). We can then fix \( F_Q \) and obtain the maximal energy efficiency by solving the following differential equation,

\[
\frac{\partial \eta}{\partial \vec{F}_e} = 0. \tag{12}
\]

We obtain that

\[
\vec{F}_e = -\frac{1}{2} \left[ \eta_{\text{max}} \left( \tilde{M}_{ee} \right)^{-1} \tilde{M}_{Qe} F_Q + \left( \tilde{M}_{ee} \right)^{-1} \tilde{M}_{eQ} F_Q \right]. \tag{13}
\]

Here we define

\[
\tilde{M}_{ee} \equiv \frac{1}{2} (\tilde{M}_{ee} + \tilde{M}_{ee}^T) \tag{14}
\]

as the symmetric charge conductivity tensor. Inserting Eq. (13) into Eq. (11), we arrive at

\[
\eta_{\text{max}} = \eta_C \frac{\lambda_1 - \lambda_2 (\eta_{\text{max}}/\eta_C)^2}{4 - 2(\lambda_2 (\eta_{\text{max}}/\eta_C) + \lambda_3)}. \tag{15}
\]

Solving the above quadratic equation, we obtain the maximal efficiency as

\[
\eta_{\text{max}} = \eta_C \frac{2 - \lambda_3 - \sqrt{(2 - \lambda_3)^2 - 4 \lambda_1 \lambda_2}}{\lambda_2}. \tag{16}
\]

Here,

\[
\lambda_1 \equiv \tilde{M}_{Qe} \left( \tilde{M}_{ee} \right)^{-1} \tilde{M}_{eQ} \tilde{M}_{QQ}^{-1}, \tag{17a}
\]

\[
\lambda_2 \equiv \tilde{M}_{Qe} \left( \tilde{M}_{ee} \right)^{-1} \tilde{M}_{Qe}^T \tilde{M}_{QQ}^{-1}, \tag{17b}
\]

\[
\lambda_3 \equiv \tilde{M}_{Qe} \left( \tilde{M}_{ee} \right)^{-1} \tilde{M}_{eQ} \tilde{M}_{QQ}^{-1}. \tag{17c}
\]

are three dimensionless parameters that characterize the thermoelectric transport properties of the system. The output power at maximum efficiency is

\[
W(\eta_{\text{max}}) = W_0 \left[ \lambda_1 - \lambda_2 \left( \frac{\eta_{\text{max}}}{\eta_C} \right)^2 \right], \quad W_0 \equiv \frac{1}{4} \tilde{M}_{QQ} F_Q^2. \tag{18}
\]

Similarly, we can obtain the maximal output power with fixed \( F_Q \) by solving the following equation

\[
\frac{\partial W}{\partial F_e} = 0, \tag{19}
\]

which yields

\[
W_{\text{max}} = \lambda_1 W_0. \tag{20}
\]

Meanwhile, the efficiency at maximum output power is \[56, 57]\]

\[
\eta(W_{\text{max}}) = \eta_C \frac{\lambda_1}{4 - 2\lambda_3}. \tag{21}
\]

Comparing the energy efficiency and output power for the above two optimization schemes, we find that

\[
\frac{\eta_{\text{max}}}{\eta(W_{\text{max}})} = 1 + \frac{\lambda_2}{\lambda_1} \left( \frac{\eta_{\text{max}}}{\eta_C} \right)^2, \tag{22}
\]

and

\[
\frac{W(\eta_{\text{max}})}{W_{\text{max}}} = 1 - \frac{\lambda_2}{\lambda_1} \left( \frac{\eta_{\text{max}}}{\eta_C} \right)^2. \tag{23}
\]

The above trade-off relations between the optimization of the efficiency and power is presented graphically in Fig. 2. These relations also reveal two important properties: First, the performance of the thermoelectric engine is better when \( \lambda_2 < \lambda_1 \) compared with the situation with \( \lambda_2 > \lambda_1 \). In addition, when \( \lambda_2 < \lambda_1 \), the efficiency at maximal output power can possibly exceed the Curzon-Ahlborn limit \[54\] in the linear-response \( \eta_{CA} = \eta_C/2 \). Second, for \( \lambda_2 < \lambda_1 \), the second-law of thermodynamics does not forbid the Carnot efficiency at finite output power. Although there have been many debates on such a possibility \[48, 58–62\], our study here opens a regime for further investigation of such an issue in quantum heat engines without the limitation of having only one electric and one heat currents.

We now make two important remarks. First, the above results are valid for the situation with multiple output electric currents. This can be readily verified through the vectorial (matrix) formulation used in the above discussions. Second, the second-law of thermodynamics imposes the following constraints on the dimensionless parameters,

\[
\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_1 + \lambda_2 + 2\lambda_3 \leq 4. \tag{24}
\]

The derivation of the above constraints goes as follows. The entropy production rate associated with the thermo-
FIG. 2. (Color online) Trade-off relations for optimal efficiency and power, Eqs. (22) and (23). (a) $\eta_{\text{max}}/\eta_C (W_{\text{max}})$ and (b) $W(\eta_{\text{max}})/W_{\text{max}}$ as functions of $\eta_{\text{max}}/\eta_C$ for various $\lambda_2/\lambda_1$.

Electric transport is [7]

$$T_R \dot{S} = \vec{I}_T \vec{F}_e + I_Q F_Q = (\vec{F}_e F_Q) \begin{pmatrix} \hat{M}_{ee} & \hat{M}_{eQ} \\ \hat{M}_{Qe} & \hat{M}_{QQ} \end{pmatrix} (\vec{F}_e F_Q).$$

(25)

The second-law of thermodynamics requires $\dot{S} \geq 0$ for all values of $\vec{F}_e$ and $F_Q$, which is equivalent to require the following matrix to be positive semi-definite,

$$
\begin{pmatrix}
\hat{M}_{ee} & \hat{M}_{eQ} \\
\hat{M}_{Qe} & \hat{M}_{QQ}
\end{pmatrix}
$$

(26)

Therefore, $M_{QQ} \geq 0$ and the matrix $\hat{M}_{ee}$ is positive semi-definite. In addition, the determinant of the above matrix is positive semi-definite which yields

$$
|M_{ee}| M_{QQ} - \frac{\hat{M}_{eQ}^T + \hat{M}_{Qe}^T}{2} (M_{ee})^{-1} \frac{\hat{M}_{Qe}^T + \hat{M}_{eQ}^T}{2} \geq 0,
$$

(27)

where $||$ is the determinant of the matrix. From these positive semi-definite properties, one can deduce Eq. (24) straightforwardly.

We now compare our results with previous studies on thermoelectric energy conversion in time-reversal broken mesoscopic systems. In all previous studies, the charge and heat currents flowing out of the $P$ terminal are tuned to vanish by adjusting the chemical potential and temperature at the $P$ terminal (often called as a probe-terminal in mesoscopic physics). Under such constraints, there are effectively only one heat current and one electric current in the system. Thermoelectric transport is then described by a $2 \times 2$ Onsager matrix [6, 27, 50–53]. In this limit, the matrices $\hat{M}_{ee}$, $\hat{M}_{eQ}$ and $\hat{M}_{Qe}$ become scalar quantities. From the definition in Eq. (17), one finds that for such a set-up

$$\lambda_3^2 = \lambda_1 \lambda_2.$$

(28)

The above constraint is the main limitation of previous studies, which is overcome in this work. As a consequence, the maximum efficiency in our set-up can exceed that in previous set-ups, as shown in Fig. 3(a). In the figure, the black dot represent the limit (28) considered in previous studies. It is seen that the maximum efficiency can be improved by going beyond such a limit when $\lambda_1 < \lambda_2$. Because of the power-efficiency trade-off, the higher efficiency is achieved at lower output power, as shown in Fig. 3(b). Figs. 3(c) and 3(d) show the maximal efficiency and the output power at such an efficiency. It is seen that large efficiency and power can be simultaneous obtained when $\lambda_1 > \lambda_2$.

FIG. 3. (Color online) (a) $\eta_{\text{max}}/\eta_C$ and (b) $W(\eta_{\text{max}})$ as functions of $\lambda_3$ for different $\lambda_1$, where $\lambda_2 = 1$. The black dots represent the limit given by Eq. (28). (c) $\eta_{\text{max}}$ and (d) $\eta(W_{\text{max}})$ as functions of $\lambda_1$ and $\lambda_2$ for $\lambda_3 = 1$. The white region is forbidden by the second-law of thermodynamics. The unit of the output power is $W_0$. 


IV. UPPER BOUNDS FOR EFFICIENCY AND POWER

The bounds for the maximal efficiency $\eta_{\text{max}}$ and efficiency at the maximum power $\eta(W_{\text{max}})$ are reached at the reversible limit with $\lambda_1 + \lambda_2 + 2\lambda_3 = 4$, leading to

$$\eta_{\text{max}} \mid \text{bound} = \begin{cases} \eta_C \frac{\lambda_1}{\lambda_2}, & \text{if } \lambda_1 < \lambda_2, \\ \eta_C, & \text{if } \lambda_1 \geq \lambda_2. \end{cases}$$

(29)

The above results are presented graphically in Fig. 4(a) for various $\lambda_1$ and $\lambda_2$. The upper bound for the efficiency at the maximum power is

$$\eta(W_{\text{max}}) \mid \text{bound} = \eta_C \frac{\lambda_1}{\lambda_1 + \lambda_2}. \quad (30)$$

From the above, the Curzon-Ahlborn limit [54, 63] for the energy efficiency at maximum power $\eta = \eta_C/2$ can in principle be overcome for $\lambda_1 > \lambda_2$. A particularly interesting regime is when $\lambda_1 \gg \lambda_2$, where both the maximal efficiency and the efficiency at maximum power are bounded by the Carnot efficiency.

Combining Eq. (29) and Eq. (30), we find the ratio between those bounds for energy efficiency,

$$\frac{\eta_{\text{max}}}{\eta(W_{\text{max}}) \mid \text{bound}} = \begin{cases} 1 + \frac{\lambda_1}{\lambda_2}, & \text{if } \lambda_1 < \lambda_2, \\ 1 + \frac{\lambda_2}{\lambda_1}, & \text{if } \lambda_1 \geq \lambda_2. \end{cases}$$

(31)

Meanwhile, the ratio between those bounds for output power is given by

$$\frac{W(\eta_{\text{max}})}{W_{\text{max}} \mid \text{bound}} = \begin{cases} 1 - \frac{\lambda_1}{\lambda_2}, & \text{if } \lambda_1 < \lambda_2, \\ 1 - \frac{\lambda_2}{\lambda_1}, & \text{if } \lambda_1 \geq \lambda_2. \end{cases}$$

(32)

As presented graphically in Figs. 4(c) and 4(d) for various $\lambda_1$ and $\lambda_2$, the trade-off between the optimal efficiency and power is significantly reduced when $\lambda_1 \gg \lambda_2$, which implies that in this regime, large energy efficiency and power can be obtained simultaneously.

V. LINEAR-RESPONSE COEFFICIENTS IN A NONINTERACTING QD SYSTEM

We now investigate the optimal efficiency and power with a concrete model. The model adopted here is the three QDs model illustrated in Fig. 1, which has been studied extensively for the situations with only one electric and one heat currents. By releasing such a constraint, the charge and heat transport are described by the following equation,

$$\begin{pmatrix} I^L_e \\ I^P_e \\ I_Q \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} F^L_e \\ F^P_e \\ F_Q \end{pmatrix},$$

(33)

The coherent flow of charge and heat through a noninteracting QD system can be described using the Landauer-Büttiker theory. The charge and heat currents flowing out of the left reservoir are given by [1, 64]

$$I^L_e = \frac{2e}{h} \int dE \sum_i [\mathcal{T}_{Li}(E)f_L(E) - \mathcal{T}_{Li}(E)f_R(E)],$$

(34a)

$$I_Q = \frac{2}{h} \int dE \sum_i (E - \mu_L)[\mathcal{T}_{Li}(E)f_L(E) - \mathcal{T}_{Li}(E)f_R(E)],$$

(34b)

where $f_i(E) = \{\exp[(E - \mu_i)/k_B T_i] + 1\}^{-1}$ is the Fermi function and $T_{ij}$ is the transmission probability from terminal $j$ to terminal $i$, $h$ is the Planck constant. The factor of two comes from the spin degeneracy of electrons.
Analogous expression can be written for $I_e^P$, provided the label $L$ is substituted by $P$ in (34a).

The Onsager coefficients $M_{ij}$ are obtained from the linear expansion of the electronic currents $I_e^i$ ($i = L, P$) and the heat current $I_Q$ in terms of the thermodynamic forces [1, 64],

$$
M_{11} = \frac{2e^2}{\hbar k_B T} \int_{-\infty}^{\infty} dE \sum_{i \neq L} T_{Li}(E) F(E),
$$

$$
M_{12} = -\frac{2e^2}{\hbar k_B T} \int_{-\infty}^{\infty} dE T_{LP}(E) F(E),
$$

$$
M_{13} = M_{31} = \frac{2e}{\hbar k_B T} \int_{-\infty}^{\infty} dE (E - \mu) \sum_{i \neq L} T_{Li}(E) F(E),
$$

$$
M_{21} = -\frac{2e^2}{\hbar k_B T} \int_{-\infty}^{\infty} dE T_{PL}(E) F(E),
$$

$$
M_{22} = \frac{2e^2}{\hbar k_B T} \int_{-\infty}^{\infty} dE \sum_{i \neq P} T_{Pi}(E) F(E),
$$

$$
M_{23} = -\frac{2e}{\hbar k_B T} \int_{-\infty}^{\infty} dE (E - \mu) T_{PL}(E) F(E),
$$

$$
M_{32} = -\frac{2e}{\hbar k_B T} \int_{-\infty}^{\infty} dE (E - \mu) T_{LP}(E) F(E),
$$

$$
M_{33} = \frac{2}{\hbar k_B T} \int_{-\infty}^{\infty} dE (E - \mu)^2 \sum_{i \neq L} T_{Li}(E) F(E).
$$

where $F(E) \equiv \{4 \cosh^2[(E - \mu)/k_B T]\}^{-1}$.

The transmission probability $T_{ij}(E)$ is calculated as [51]

$$
T_{ij} = \text{Tr}[\Gamma_i(E) G(E) \Gamma_j(E) G^\dagger(E)],
$$

where the (retarded) Green’s function for the QD system is $G(E) \equiv (E - H_{qd} - i\Gamma/2)^{-1}$, the damping rate $\Gamma = 2\pi \sum_k |V_{ik}|^2 \delta(\omega - \epsilon_{ik})$ is assumed to be a constant for all three leads.

When an external magnetic field $\Phi$ is applied to the system, the laws of physics remain unchanged if time $t$ is replaced by $-t$, provided that simultaneously the magnetic field $\Phi$ is replaced by $-\Phi$. In this case, the transport coefficients meet the Onsager-Casimir relations [65]

$$
M_{ij}(\phi) = M_{ji}(-\phi).
$$

It is seen from Fig. 5 that the optimal efficiency and power vary strongly with the QD energy $E_1$ and the magnetic flux $\phi$. For these cases, the dependence on the QD energy is stronger than that on the magnetic flux. The efficiency and power are large when $E_1 \approx 2k_B T$. The maximum efficiency $\eta_{\text{max}}$ can reach $0.6\eta_C$. The results here reveal that a small external magnetic field can improve both the power and efficiency, when starting from the time-reversal limit of $\phi = \pi$.

In Fig. 6, we compare explicitly the performance of our three-terminal quantum heat engine with the previously studied limit. The latter is illustrated in Fig. 6(a), where the heat and electric currents flowing out of the $P$ terminal vanish by adjusting the chemical potential $\mu_P$ and temperature $T_P$. In this limit (denoted as $P = 0$ briefly) there are only one electric and one heat currents, yielding the relation in Eq. (28). As shown in Figs. 6(b), Fig. 6(c) and Fig. 6(d), the maximal efficiency, the efficiency at maximum power, and the maximum output power can be significantly improved by releasing the limit of $P = 0$. Our quantum heat engine with two output electric currents demonstrate superior efficiency and power for a large range of parameters.
thermoelectric engine with two output electric currents. These results go beyond previous studies with time-reversal symmetry [8], and the time-reversal broken systems [48, 49] with only one electric current, revealing universalities in multi-terminal thermoelectric energy conversion differing from the existing theories. Numerical calculations for a triple-QD thermoelectric engine show that the efficiency and power can be substantially improved for the set-ups with two output electric currents compared with previous set-ups with only one electric current. We also find regimes where the energy efficiency and output power can be optimized at close conditions. Our results offer useful guidelines for the search of high-performance thermoelectric systems in the mesoscopic regime, with particular emphasizes on multi-terminal set-ups with multiple output electric currents.

ACKNOWLEDGMENT

J.L., Y.L., R.W., and J.-H.J. acknowledge support from the National Natural Science Foundation of China (NSFC Grant No. 11675116) and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD). C.W. is supported by the National Natural Science Foundation of China under Grant No. 11704093.

In conclusion, we derived the optimal efficiency and power, and their trade-off relations for a three-terminal thermoelectric engine when the electric current $I^p_0$ and heat current $I^q_0$ vanish (denoted as $P = 0$ briefly). (b)-(d) Comparing the optimal efficiency and power for the $P = 0$ limit and the case with two output electric currents for various QD energy $E_1$ and magnetic flux $\phi$: (b) the maximal efficiency, (c) the efficiency at maximum power, and (d) the maximum output power. The other parameters are $t = -0.2k_B T$, $\mu = 0$, $E_2 = 1.0k_B T$ and $E_3 = 2.0k_B T$.

VI. CONCLUSION

In conclusion, we derived the optimal efficiency and power, and their trade-off relations for a three-terminal thermoelectric engine with two output electric currents. These results go beyond previous studies with time-reversal symmetry [8], and the time-reversal broken systems [48, 49] with only one electric current, revealing universalities in multi-terminal thermoelectric energy conversion differing from the existing theories. Numerical calculations for a triple-QD thermoelectric engine show that the efficiency and power can be substantially improved for the set-ups with two output electric currents compared with previous set-ups with only one electric current. We also find regimes where the energy efficiency and output power can be optimized at close conditions. Our results offer useful guidelines for the search of high-performance thermoelectric systems in the mesoscopic regime, with particular emphasizes on multi-terminal set-ups with multiple output electric currents.

ACKNOWLEDGMENT

J.L., Y.L., R.W., and J.-H.J. acknowledge support from the National Natural Science Foundation of China (NSFC Grant No. 11675116) and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD). C.W. is supported by the National Natural Science Foundation of China under Grant No. 11704093.

In conclusion, we derived the optimal efficiency and power, and their trade-off relations for a three-terminal thermoelectric engine with two output electric currents. These results go beyond previous studies with time-reversal symmetry [8], and the time-reversal broken systems [48, 49] with only one electric current, revealing universalities in multi-terminal thermoelectric energy conversion differing from the existing theories. Numerical calculations for a triple-QD thermoelectric engine show that the efficiency and power can be substantially improved for the set-ups with two output electric currents compared with previous set-ups with only one electric current. We also find regimes where the energy efficiency and output power can be optimized at close conditions. Our results offer useful guidelines for the search of high-performance thermoelectric systems in the mesoscopic regime, with particular emphasizes on multi-terminal set-ups with multiple output electric currents.

VI. CONCLUSION

In conclusion, we derived the optimal efficiency and power, and their trade-off relations for a three-terminal thermoelectric engine when the electric current $I^p_0$ and heat current $I^q_0$ vanish (denoted as $P = 0$ briefly). (b)-(d) Comparing the optimal efficiency and power for the $P = 0$ limit and the case with two output electric currents for various QD energy $E_1$ and magnetic flux $\phi$: (b) the maximal efficiency, (c) the efficiency at maximum power, and (d) the maximum output power. The other parameters are $t = -0.2k_B T$, $\mu = 0$, $E_2 = 1.0k_B T$ and $E_3 = 2.0k_B T$.
[8] Jian-Hua Jiang, “Thermodynamic bounds and general properties of optimal efficiency and power in linear responses,” Phys. Rev. E 90, 042126 (2014).

[9] Michael Bauer, Kay Brandner, and Udo Seifert, “Optimal performance of periodically driven, stochastic heat engines under limited control,” Phys. Rev. E 93, 042112 (2016).

[10] Karel Proesmans, Bart Cleuren, and Christian Van den Broeck, “Power-efficiency-dissipation relations in linear thermodynamics,” Phys. Rev. Lett. 116, 220601 (2016).

[11] Patrick Pietzonka and Udo Seifert, “Universal trade-off between power, efficiency, and constancy in steady-state heat engines,” Phys. Rev. Lett. 120, 190602 (2018).

[12] Robert S. Whitney, “Most efficient quantum thermoelectric at finite power output,” Phys. Rev. Lett. 112, 130601 (2014).

[13] Robert S. Whitney, “Finding the quantum thermoelectric with maximal efficiency and minimal entropy production at given power output,” Phys. Rev. B 91, 115425 (2015).

[14] Gatien Verley, Massimiliano Esposito, Tim Willaert, and Christian Van den Broeck, “The unlikely carnot efficiency,” Nat. Commun. 5, 4721 (2014).

[15] M. Polettini, G. Verley, and M. Esposito, “Efficiency statistics at all times: Carnot limit at finite power," Phys. Rev. Lett. 114, 050601 (2015).

[16] Jian-Hua Jiang, Bijay Kumar Agarwalla, and Dvira Segal, “Efficiency statistics and bounds for systems with broken time-reversal symmetry,” Phys. Rev. Lett. 115, 040601 (2015).

[17] Karel Proesmans, Yannik Dreher, Mom čilo Gavrilov, John Bechhoefer, and Christian Van den Broeck, “Brownian duet: A novel tale of thermodynamic efficiency,” Phys. Rev. X 6, 041010 (2016).

[18] D. Sánchez and L. Serra, “Thermoelectric transport of mesoscopic conductors coupled to voltage and thermal probes,” Phys. Rev. B 84, 201307 (2011).

[19] Rafael Sánchez and Markus Böttiker, “Optimal energy quanta to current conversion,” Phys. Rev. B 83, 085428 (2011).

[20] Jian-Hua Jiang, Ora Entin-Wohlman, and Yoseph Imry, “Thermoelectric three-terminal hopping transport through one-dimensional nanosystems,” Phys. Rev. B 85, 075412 (2012).

[21] Björn Sothmann, Rafael Sánchez, Andrew N. Jordan, and Markus Böttiker, “Rectification of thermal fluctuations in a chaotic cavity heat engine,” Phys. Rev. B 85, 205301 (2012).

[22] Jian-Hua Jiang, Ora Entin-Wohlman, and Yoseph Imry, “Hopping thermoelectric transport in finite systems: Boundary effects,” Phys. Rev. B 87, 205420 (2013).

[23] Lena Simine and Dvira Segal, “Path-integral simulations with fermionic and bosonic reservoirs: Transport and dissipation in molecular electronic junctions,” J. Chem. Phys. 138, 214111 (2013).

[24] Björn Sothmann, Rafael Sánchez, Andrew N Jordan, and Markus Böttiker, “Powerful energy harvester based on resonant-tunneling quantum wells,” New J. Phys. 15, 095021 (2013).

[25] Jian-Hua Jiang, Ora Entin-Wohlman, and Yoseph Imry, “Three-terminal semiconductor junction thermoelectric devices: improving performance,” New J. Phys. 15, 075021 (2013).

[26] Francesco Mazza, Stefano Valentini, Riccardo Bosisio, Giuliano Benenti, Vittorio Giovannetti, Rosario Fazio, and Fabio Taddei, “Separation of heat and charge currents for boosted thermoelectric conversion,” Phys. Rev. B 91, 245435 (2015).

[27] Kay Brandner and Udo Seifert, “Multi-terminal thermoelectric transport in a magnetic field: bounds on onsager coefficients and efficiency,” New J. Phys. 15, 105003 (2013).

[28] Bijay Kumar Agarwalla, Jian-Hua Jiang, and Dvira Segal, “Full counting statistics of vibrationally assisted electronic conduction: Transport and fluctuations of thermoelectric efficiency,” Phys. Rev. B 92, 245418 (2015).

[29] Lijie Li and Jian-Hua Jiang, “Staircase quantum dots configuration in nanowires for optimized thermoelectric power,” Sci. Rep. 6, 31974 (2016).

[30] Kaoru Yamamoto, Ora Entin-Wohlman, Amnon Aharony, and Naomichi Hatano, “Efficiency bounds on thermoelectric transport in magnetic fields: The role of inelastic processes,” Phys. Rev. B 94, 121402 (2016).

[31] Bijay Kumar Agarwalla, Jian-Hua Jiang, and Dvira Segal, “Quantum efficiency bound for continuous heat engines coupled to noncanonical reservoirs,” Phys. Rev. B 96, 104304 (2017).

[32] Katarzyna Macieszczak, Kay Brandner, and Juan P. Garrahan, “Unified thermodynamic uncertainty relations in linear response,” Phys. Rev. Lett. 121, 130601 (2018).

[33] Jian-Hua Jiang and Yoseph Imry, “Near-field three-terminal thermoelectric heat engine,” Phys. Rev. B 97, 125422 (2018).
[34] Naoto Shiraishi, Keiji Saito, and Hal Tasaki, “Universal trade-off relation between power and efficiency for heat engines,” Phys. Rev. Lett. 117, 190601 (2016).

[35] Jincheng Lu, Rongqian Wang, Jie Ren, Manas Kulkarni, and Jian-Hua Jiang, “Quantum-dot circuit-qed thermoelectric diodes and transistors,” Phys. Rev. B 99, 035129 (2019).

[36] K. D. Petersson, L. W. McFaul, M. D. Schroer, M. Jung, J. M. Taylor, A. A. Houck, and J. R. Petta, “Circuit quantum electrodynamics with a spin qubit,” Nature 490, 380 (2012).

[37] Sun-Yong Hwang, David Sánchez, Minchul Lee, and Rosa López, “Magnetic-field asymmetry of nonlinear thermoelectric and heat transport,” New J. Phys. 15, 105012 (2013).

[38] J. Matthews, F. Battista, D. Sánchez, P. Samuelsson, and H. Linke, “Experimental verification of reciprocity relations in quantum thermoelectric transport,” Phys. Rev. B 90, 165428 (2014).

[39] H. Thierschmann, R. Sánchez, B. Rothmann, F. Arnold, C. Heyn, W. Hansen, H. Buhmann, and L. W. Molenkamp, “Three-terminal energy harvester with coupled quantum dots,” Nat. Nanotech. 10, 854 (2015).

[40] B. Roche, P. Roulleau, T. Jullien, Y. Jompol, I. Farrer, D. A. Ritchie, and D. C. Glattli, “Harvesting dissipated energy with a mesoscopic ratchet,” Nat. comm. 6, 6738 (2015).

[41] Longji Cui, Ruijiao Miao, Kun Wang, Dakotah Thompson, Linda Angela Zotti, Juan Carlos Cuevas, Edgar Meyhofer, and Pramod Reddy, “Feltier cooling in molecular junctions,” Nat. Nanotech. 13, 122 (2018).

[42] T. E. Humphrey, R. Newbury, R. P. Taylor, and H. Linke, “Reversible quantum brownian heat engines for electrons,” Phys. Rev. Lett. 89, 116801 (2002).

[43] T. E. Humphrey and H. Linke, “Reversible thermoelectric nanomaterials,” Phys. Rev. Lett. 94, 096601 (2005).

[44] I. Iyyappan and M. Pommurugan, “General relations between the power, efficiency, and dissipation for the irreversible heat engines in the nonlinear response regime,” Phys. Rev. E 97, 012141 (2018).

[45] Jian-Hua Jiang, “Enhancing efficiency and power of quantum-dots resonant tunneling thermoelectrics in three-terminal geometry by cooperative effects,” J. Appl. Phys. 116, 194303 (2014).

[46] Jincheng Lu, Rongqian Wang, Yefeng Liu, and Jian-Hua Jiang, “Thermoelectric cooperative effect in three-terminal elastic transport through a quantum dot,” J. Appl. Phys. 122, 044301 (2017).

[47] Jian-Hua Jiang and Joseph Imry, “Enhancing thermoelectric performance using nonlinear transport effects,” Phys. Rev. Applied 7, 064001 (2017).

[48] Giuliano Benenti, Keiji Saito, and Giulio Casati, “Thermodynamic bounds on efficiency for systems with broken time-reversal symmetry,” Phys. Rev. Lett. 106, 230602 (2011).

[49] Kay Brandner, Keiji Saito, and Udo Seifert, “Strong bounds on onsager coefficients and efficiency for three-terminal thermoelectric transport in a magnetic field,” Phys. Rev. Lett. 110, 070603 (2013).

[50] Keiji Saito, Giuliano Benenti, Giulio Casati, and Tomasz Prosen, “Thermopower with broken time-reversal symmetry,” Phys. Rev. B 84, 201306 (2011).

[51] Vinitha Balachandran, Giuliano Benenti, and Giulio Casati, “Efficiency of three-terminal thermoelectric transport under broken time-reversal symmetry,” Phys. Rev. B 87, 165419 (2013).

[52] Kay Brandner and Udo Seifert, “Bound on thermoelectric power in a magnetic field within linear response,” Phys. Rev. E 91, 012121 (2015).

[53] Julian Stark, Kay Brandner, Keiji Saito, and Udo Seifert, “Classical nernst engine,” Phys. Rev. Lett. 112, 140601 (2014).

[54] F. L. Curzon and B. Ahlborn, “Efficiency of a carnot engine at maximum power output,” Am. J. Phys. 43, 22–24 (1975).

[55] M. Buttiker, “Coherent and sequential tunneling in series barriers,” IBM J. Res. Dev. 32, 63–75 (1988).

[56] C. Van den Broeck, “Thermodynamic efficiency at maximum power,” Phys. Rev. Lett. 95, 190602 (2005).

[57] N. Golubeva and A. Imparato, “Efficiency at maximum power of interacting molecular machines,” Phys. Rev. Lett. 109, 190602 (2012).

[58] Naoto Shiraishi, Keiji Saito, and Hal Tasaki, “Universal trade-off relation between power and efficiency for heat engines,” Phys. Rev. Lett. 117, 190601 (2016).

[59] O. Raz, Y. Subaši, and R. Pugatch, “Geometric heat engines featuring power that grows with efficiency,” Phys. Rev. Lett. 116, 160601 (2016).

[60] Armen E. Allahverdyan, Karen V. Hovhannisyan, Alexey V. Melkikh, and Sasun G. Gevorkian, “Carnot cycle at finite power: Attainability of maximal efficiency,” Phys. Rev. Lett. 111, 050601 (2013).
[61] Giuliano Benenti, Giulio Casati, and Jiao Wang, “Conservation laws and thermodynamic efficiencies,” Phys. Rev. Lett. 110, 070604 (2013).

[62] Viktor Holubec and Artem Ryabov, “Cycling tames power fluctuations near optimum efficiency,” Phys. Rev. Lett. 121, 120601 (2018).

[63] Udo Seifert, “Stochastic thermodynamics, fluctuation theorems and molecular machines,” Rep. Prog. Phys. 75, 126001 (2012).

[64] U. Sivan and Y. Imry, “Multichannel landauer formula for thermoelectric transport with application to thermopower near the mobility edge,” Phys. Rev. B 33, 551–558 (1986).

[65] Supriyo Datta, Electronic transport in mesoscopic systems (Cambridge university press, Cambridge, UK, 1995).