The joint extraction of $m_s$ and $V_{us}$ from hadronic $\tau$ decay data

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We discuss the simultaneous determination of $m_s$ and $V_{us}$ from flavor-breaking hadronic $\tau$ decay sum rules, focussing on cut-off choices designed to better control problems associated with the slow convergence of the relevant integrated $D = 2$ OPE series. The results are found to display improved stability and consistency relative to those of conventional analyses based on the “($k, 0$) spectral weights”. The results for $m_s$ agree well with those of recent strange scalar sum rule and strange pseudoscalar sum rule and lattice analyses. Results for $V_{us}$ agree within errors with those of lattice-input-based $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ and $K_{\mu 3}$ analyses. Very significant error reductions are shown to be expected, especially for $V_{us}$, once improved strange spectral data from the B factory experiments becomes available.

1. BACKGROUND

Measurements of the inclusive hadronic $\tau$ decay distributions for processes mediated by the flavor $ij = ud, us$ vector (V) or axial vector (A) currents yield kinematically weighted linear combinations of the spectral functions, $\rho_{V/A;ij}^{(J)}$, of the spin $J = 0$ and 1 parts of the relevant current-current correlators, $\Pi_{V/A;ij}^{(J)}$. With $R_{V/A;ij} = \Gamma[\tau^- \to \nu_\tau \text{ hadrons}_{ij}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]$, one has, explicitly [1],

$$
\frac{R_{V/A;ij}}{[2\pi^2|V_{ij}|^2 S_{EW}]} = \int_0^1 dy_\tau (1 - y_\tau)^2 \left[ (1 + 2y_\tau) \rho_{V/A;ij}^{(0+1)}(s) - 2y_\tau \rho_{V/A;ij}^{(0)}(s) \right]
$$

(1)

where $y_\tau = s/m_\tau^2$, $V_{ij}$ is the flavor $ij$ CKM matrix element, $S_{EW}$ is a short-distance electroweak correction, and the superscript $(0+1)$ denotes the sum of $J = 0$ and $J = 1$ contributions. The absence of kinematic singularities in the correlators corresponding to the spectral function combinations in Eq. (1) allows the spectral integrals to be re-written using the basic finite energy sum rule (FESR) relation. For such correlators $\Pi$, with associated spectral functions $\rho$, and $w(s)$ analytic, this relation has the form

$$
\int_0^{s_0} ds w(s)\rho(s) = \frac{-1}{2\pi i} \oint_{|s|=s_0} ds w(s)\Pi(s). 
$$

(2)

Analogous FESR’s, with spectral integral sides denoted $R_{V/A;ij}^{(k,m)}$, obtained from $R_{V/A;ij}$ by rescaling the integrand with $(1 - y_\tau)^k y_\tau^m$ before integration, are called the “($k, m$) spectral weight sum rules”. Similar spectral integrals and sum rules can be constructed for general non-spectral weights $w(s)$, for either $\Pi_{V/A;ij}^{(0+1)}(s)$ or $s\Pi_{V/A;ij}^{(0)}(s)$, and for any $s_0 < m_\tau^2$. The corresponding spectral integrals are denoted generically by $R_{ij}^{w}(s_0)$ in what follows. For “inclusive” sum rules (those with both $J = 0$ and $J = 0 + 1$ spectral contributions) the purely $J = 0$ contribution will be referred to as “longitudinal”.

$V_{us}$ and/or $m_s$ are then to be extracted using flavor-breaking differences, $\delta R_{ij}^{w}(s_0)$, defined by

$$
\delta R_{ij}^{w}(s_0) = \frac{R_{ud}^{w}(s_0)}{|V_{ud}|^2} - \frac{R_{us}^{w}(s_0)}{|V_{us}|^2}.
$$

(3)

Since $\delta R_{ij}^{w}(s_0)$ vanishes in the $SU(3)_F$ limit, its OPE representation, $\delta R_{ij}^{OPE}(s_0)$, begins at dimension $D = 2$. The $D = 2$ term is proportional to $m_s^2$. Experimental determinations of
δR(w)(s0) over a range of s0 and w(s) allow a fit for mω and/or Vus to be performed, so long as s0 is large enough that the OPE representation can be reliably employed \[23,11,57\]. The approach is especially well-suited to the determination of Vus \[57\] since the smallness of mω makes the integrated D = 2 OPE contributions at scales \(\sim 2 - 3\) GeV\(^2\), and hence also the corresponding flavor-breaking spectral integral differences, much smaller than the individual flavor ud and us spectral integral terms (typically at the few to several percent level). Explicitly, since Eq. (3) implies

\[
|V_{us}| = \sqrt{\frac{R^w(u,s)(s_0)}{V_{ud}(s_0)^2 - \delta R^w \text{OPE}(s_0)}},
\]

it follows that an uncertainty \(\Delta (\delta R^w \text{OPE}(s_0))\) in \(\delta R^w \text{OPE}(s_0)\) produces a fractional uncertainty \(\sim \Delta (\delta R^w \text{OPE}(s_0)) / 2 R^w \text{OPE}(s_0)\) in |Vus| which is much smaller than that on \(\delta R^w \text{OPE}(s_0)\) itself \[5\]. Moderate precision for \(\delta R^w \text{OPE}(s_0)\) thus suffices for high precision on |Vus|, provided experimental errors can be brought under control.

We report here on a combined extraction of mω and |Vus| employing existing spectral data. The ud data, especially the V+A sum, are already quite precise \[8,9,10,11\]. The us data \[10,12,13,14\], however, suffer from low statistics and have very sizeable errors above the \(K^+\). Ongoing analyses at BABAR and BELLE will very significantly reduce the us errors in the near future. The focus here will be on bringing uncertainties on the theoretical (OPE) side of the analysis under better control, in particular those associated with the slower-than-previously-anticipated convergence of the relevant D = 2 OPE series \[15\].

2. OPE COMPLICATIONS

The first major problem on the OPE side is the very bad behavior of the integrated longitudinal \(D = 2\) OPE series which shows no sign of converging at any kinematically accessible scale \[15,16\], and, even worse, for all truncation schemes used in the literature, badly violates spectral-positivity-based constraints \[16,17\]. Inclusive analyses employing the longitudinal OPE representation are thus untenable.

Fortunately, for a combination of chiral and kinematic reasons, the longitudinal \(D = 2\) OPE problem is easily handled phenomenologically. Apart from the \(\pi\) and \(K\) pole terms, longitudinal spectral contributions vanish in the \(SU(3)_f\) limit and are doubly-chirally suppressed away from it. This double chiral suppression is preserved in the ratio of integrated non-pole to pole contributions because the longitudinal kinematic weight in Eq. (1) has essentially the same value at the \(K\) pole as in the region of excited strange scalar and pseudoscalar (PS) resonances. The small residual non-pole us PS and scalar contributions can, moreover, be well-constrained phenomenologically \[3\], the former via a sum rule analysis of the us PS channel \[18\], the latter by dispersive single- or coupled-channel \(K\pi\)-scattering-data-based dispersive analyses \[19,20\]. With the dominant \(\pi\) and \(K\) pole contributions already accurately known, a bin-by-bin subtraction of the longitudinal contributions to the experimental distribution, and hence a determination of the \((0 + 1)\) spectral function, can be performed, allowing FESR’s not afflicted by the longitudinal \(D = 2\) OPE problem to be constructed.

Since the us scalar and PS spectral “models” used for the longitudinal subtraction correspond, via scalar and PS sum rules \[18,20,21\], to values of mω in excellent agreement with recent \(N_f = 2 + 1\) lattice results \[22\], significantly larger non-pole contributions are ruled out. The residual non-pole longitudinal subtractions are thus certainly small, and very well under control at the level required for \((0 + 1)\) FESR analyses. We thus focus, in what follows, on sum rules involving the flavor-breaking combination

\[
\Delta \Pi(s) \equiv \Pi_{V+A;ud}^{(0+1)}(s) - \Pi_{V+A;us}^{(0+1)}(s).
\]

The \(D = 2\) OPE contribution to \(\Delta \Pi\) is known up to \(O(\alpha_s^3)\) and given by \[15\]

\[
[\Delta \Pi(Q^2)]_{D=2}^{\text{OPE}} = \frac{3}{2\pi^2} m_s \left[1 + 2.333 \bar{a} + 19.933 \bar{a}^2 + 208.746 \bar{a}^3 + (2378 \pm 200) \bar{a}^4\right]
\]

where \(\bar{a} = \alpha_s(Q^2)/\pi\) and \(m_s = m_s(Q^2)\), with \(\alpha_s(Q^2)\) and \(m_s(Q^2)\) the running coupling and strange quark mass in the \(\overline{MS}\) scheme. The
$O(\bar{a}^4)$ coefficient is an estimate obtained using approaches which accurately predicted the $O(\bar{a}^3)$ coefficient in Eq. (6) and $n_f$-dependent $O(\bar{a}^3 m_s^2)$ coefficients of the electromagnetic current correlator in advance of their explicit calculation [23]. Since independent high-scale determinations of $\alpha_s(M_Z)$ [24] correspond, after 4-loop running and matching [25,29], to $\bar{a}(m_s^2) \simeq 0.10 - 0.11$, the convergence of this series, at the spacelike point on the contour $|s| = s_0$, is marginal at best, even at the highest scales accessible in hadronic $\tau$ decay. Although $|\alpha_s(Q^2)|$ does decrease as one moves around the contour away from the spacelike point, allowing improvement in the convergence through judicious choices of weight, $w(s)$, one must expect to find very slow convergence of the integrated $D = 2$ series for those $w(s)$ not chosen specifically with this improvement criterion in mind. The $(k, 0)$ spectral weights, $w^{(k,0)}(y) = (1 + 2y)(1 - y)^{k+2}$, with $y = s/s_0$, are highly nonoptimal in this regard, since $|1 - y| = 2|\sin(\phi/2)|$ (with $\phi$ the angular position measured counterclockwise from the timelike point) is peaked precisely in the spacelike direction. Slow convergence, deteriorating with increasing $k$, is thus expected for the integrated $D = 2$ series of the $(k, 0)$ spectral weights. The results of Table I of Ref. [15] bear out this expectation [24].

Several estimates of the integrated $(0+1)$ $D = 2$ OPE truncation uncertainty have been considered in the literature: the size of the last term kept, the level of residual scale dependence, and Adler function evaluations, and various combinations thereof. The slow convergence of the integrated series can make it difficult to be sufficiently conservative. E.g., the quadrature sum of the last term size plus residual scale dependence version of the $O(\bar{a}^3)$ Adler function truncation uncertainty [5], yields a result $\sim 2.5$ times smaller than the actual difference between the $O(\bar{a}^3)$-truncated Adler function and $O(\bar{a}^4)$-truncated correlator results [4].

Since, due to the growth of $\alpha_s$ with decreasing scale, higher order terms are relatively more important at lower scales, premature truncation of a slowly converging series will induce an unphysical $s_0$-dependence in extracted, nominally $s_0$-independent quantities. With polynomial weights, $w(y) = \sum_m c_m y^m$ (for which integrated $D = 2N + 2$ OPE contributions not suppressed by additional powers of $\alpha_s$ scale as $c_N/s_0^N$) such unphysical $s_0$-dependence can also result if higher $D$ contributions which might in principle be present are incorrectly assumed negligible and omitted from the analysis. The absence of phenomenological input for the values of the relevant $D > 6$ condensates makes such omission most dangerous for those $w(y)$ having large coefficients $c_m$, with $m > 2$, where such unknown $D > 6$ contributions are potentially enhanced. The $(2,0), (3,0)$ and $(4,0)$ spectral weights, $w^{(2,0)}(y) = 1 - 2y - 2y^2 + 8y^3 - 7y^4 + 2y^5$, $w^{(3,0)}(y) = 1 - 3y + 10y^3 - 15y^4 + 9y^5 - 2y^6$, and $w^{(4,0)}(y) = 1 - 4y + 3y^2 + 10y^3 - 25y^4 + 24y^5 - 11y^6 + 2y^7$ provide examples of weights having such large higher order coefficients.

In view of the above discussion, $s_0$-stability tests are crucial to establishing the reliability of any theoretical error estimate. Failure to find a stability window in $s_0$ or, if not a stability window, then at least a window within which the observed instability is safely smaller than the estimated theoretical uncertainty, is a clear sign of an insufficiently conservative error.

3. RESULTS AND DISCUSSION

We restrict our attention, in what follows, to the V+A spectral combination, to weights satisfying $w(s = s_0) = 0$, and to scales $s_0 > 2$ GeV$^2$, all of which serve to strongly suppress possible residual OPE breakdown effects [5,28,29].

The form of the known $D = 4, 6$ contributions to $[\Delta \Pi(Q^2)]_{OPE}$ may be found in Ref. [1]. Standard values for the required OPE input parameters are employed (for details, see Ref. [31]). Our central $D = 2$ determinations employ the contour-improved (CIPT) prescription [31] for the RG-improved correlator difference $[\Delta \Pi]^{OPE}_{D=2}$. An alternate CIPT evaluation, employing the truncated, RG-improved Adler function, provides one measure of the truncation uncertainty. (The two versions are equal to all orders but differ at $O(\bar{a}^{N+1})$ and higher when both are truncated at $O(\bar{a}^N)$.) Exact solutions corresponding to the 4-
loop-truncated $\beta$ and $\gamma$ functions [26] are used for the running of $\bar{a}$ and $\tilde{m}_s$.

For the spectral integrals we employ the ALEPH $ud$ [9] and $us$ [12] data, for which both data and covariance matrices are publicly available. A small global renormalization of the $ud$ data is required to reflect minor changes in the $e$, $\mu$ and total strange branching fractions since the original ALEPH publication. Global mode-by-mode rescalings of the strange exclusive distributions have also been performed to bring the original ALEPH branching fractions into agreement with current world average (PDG06) values [32]. Errors on the $K$ and $\pi$ pole contributions have also been reduced by using the more precise values implied by $\Gamma[\pi\mu2]$ and $\Gamma[K\nu2]$. The resulting errors on the $ud$ and $us$ spectral integrals are at the $\sim 0.5\%$ and $\sim 3 - 4\%$ levels, respectively. The $us$ errors will be drastically reduced by results from BABAR and BELLE.

3.1. The $(k, 0)$ Spectral Weight Analyses

As argued above, very slow convergence is expected for the integrated $D = 2$, $J = 0 + 1$ $(k, 0)$ spectral weight OPE series. Such slow convergence is seen explicitly in the results reported in Refs. [7,15]. The situation for $k = 0$ is of particular practical interest since the $s_0 = m_s^2$, $(0, 0)$ $us$ spectral integral is fixed by the total strangeness branching fraction. Spectral integral errors can thus be reduced through improvements to the various exclusive strange branching fractions. Such improvements are much less experimentally challenging than would be an improved determination of the full $us$ spectral distribution, which analyses employing other $s_0$ and/or other weights would require. The very small normalization of the $(0, 0)$ $D = 2$ OPE integral, which makes for a rather weak dependence of $V_{us}$ on $m_s$, is another favorable feature of this weight [5]. Unfortunately, these positive features must be weighed against the very poor convergence of the integrated $D = 2$ OPE series, and the concomitant difficulty in obtaining a reliable estimate of the truncation uncertainty. In the most recent version of this analysis (see the last of Refs. [5]) $V_{us}$ is obtained using external input for $m_s$, and $s_0 = m_s^2$ only. The quoted combined OPE-induced uncertainty is $\pm 0.0011$. As seen in Ref. [7], however, the $(0, 0)$ sum rule displays rather poor $s_0$-stability. $V_{us}$, e.g., changes by 0.0021 even over the rather restricted range $m_s^2 - 0.4$ GeV$^2 < s_0 < m_s^2$. An alternate estimate of the $D = 2$ truncation uncertainty, obtained by taking twice the quadrature sum of the last term kept and the correlator-minus-Adler-function difference yields a result, $\pm 0.0022$, much more in keeping with the size of the observed $s_0$ instability. Unfortunately, since a reduction in the truncation uncertainty does not appear likely, this more conservative estimate puts a sub-1% $V_{us}$ determination out of reach of the $(0, 0)$ spectral weight analysis. Figure 1 shows the OPE and spectral integral differences, as a function of $s_0$, for various input $m_s = m_s(2$ GeV), with $V_{us}$ in each case obtained by fitting to the spectral integral difference at $s_0 = m_s^2$. The very different $s_0$-dependences of the two curves is the source of the $s_0$-instability in the extracted $V_{us}$ values. The figure makes clear that the instability in $V_{us}$ found in Ref. [7] is not specific to the $m_s$ employed as input in that analysis.

A final illustration of the problematic features of the $(k, 0)$ spectral weight analyses is provided in Figure 4. Given any pair of $(k, 0)$ weights, $m_s$ and $V_{us}$ can be obtained by a simultaneous fit using the two $s_0 = m_s^2$ spectral integrals as input. The figure shows the $1\sigma$ contours for a series of such fits. It is evident that no good common fit region for $m_s$ and $V_{us}$ exists, further strengthening the conclusion that the OPE representations for the $(k, 0)$ spectral weights are not under sufficiently good control to allow a reliable determination of $m_s$ and $V_{us}$.

3.2. Non-Spectral Weight Analyses

To reduce the problems associated with the slow convergence of the $D = 2$, $J = 0 + 1$ OPE series, we shift to FESR’s based on three nonspectral weights, $w_{10}$, $w_{12}$, and $w_{20}$, constructed in Ref. [3]. By design, these weights simultaneously (i) improve $D = 2$ convergence; (ii) suppress spectral contributions above 1 GeV$^2$ (where $us$ errors are large); and (iii) control weight coefficients $c_m$, $m > 2$ (which might otherwise enhance $D > 6$ OPE contributions). Ref. [3] and
Table I of Ref. [7] show explicitly the much improved $D = 2$ convergence which results. Table II of Ref. [7] (the left, “ACO”, half corresponding to the $us$ data treatment used here) also shows the much improved stability of $V_{us}$ with respect to $s_0$ obtained for these weights, at least for the PDG04 input $m_s(2 \text{ GeV}) = 105 \pm 25 \text{ MeV}$ employed there. Figures 2 and 3 show the $m_s$-dependent OPE vs. spectral integral comparisons, analogous to those shown for the $(0,0)$ spectral weight in Figure 1, for $w_{20}$ and $w_{10}$, respectively. The results for $w_{10}$, which are similar, have been omitted for brevity, but may be found in Ref. [30]. The existence of a window of $m_s$ values over which improved $s_0$-stability for $V_{us}$ will be obtained for all three non-spectral weights is clear. Figure 5 shows the $1\sigma$ contours for the various pairwise fits and combined 3-fold fit involving these weights. A good common fit region exists, in sharp contrast to the situation for the $(k,0)$ spectral weights. This strongly suggests that the lack of a good common fit region for the $(k,0)$ spectral weights is a result of the poor convergence behavior of the relevant integrated OPE series.

The results of Figure 5 correspond to $V_{us} = 0.2202 \pm 0.0046$ and $m_s = m_s(2 \text{ GeV}) = 89 \pm 25 \text{ MeV}$. A modest reduction of the combined fit values can be achieved by adding a new non-spectral weight, $w_s(y)$, to the analysis [30]. An excellent common fit region remains. The central fit values and (somewhat) reduced errors are then [30]

$$V_{us} = 0.2213 \pm 0.0039, \quad m_s = 97 \pm 19 \text{ MeV} \ . \ (7)$$

The result for $V_{us}$ is compatible, within errors, with both (i) the ICHEP06 $K_{\ell3}$ review update [33], $V_{us} = 0.2232 \pm 0.0006 \ (0.2249 \pm 0.0019)$ if the most recent lattice input for $f_+(0)$ [34] is replaced by the old Leutwyler-Roos value [35], and (ii) the recent MILC update of the $\Gamma[K_{\pi2}]/\Gamma[\pi\mu2]$ determination [36], $V_{us} = 0.2223^{+0.0026}_{-0.0013}$. The result for $m_s$ is in excellent agreement with recent strange scalar and PS sumrule, and $N_f = 2 + 1$ lattice, results.

To see how a reduction in the $us$ spectral errors might impact the errors on $V_{us}$ and $m_s$, we consider a scenario in which the $us$ spectral function central values remain unchanged but the errors (covariances) are reduced by a factor of 3 (9). The combined fit errors on $V_{us}$ and $m_s$ are reduced to $\pm 0.0015$ and $\pm 13$ MeV, respectively. This, of course, provides only a rough guide, since non-trivial shifts in the central $us$ spectral function values are certainly to be expected. Nonetheless, the exercise makes clear that very significant reductions in the $V_{us}$ error should be anticipated once B factory data becomes available. A more modest reduction is observed for the $m_s$ errors.

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Figure 1. $s_0$-stability plots for the $(0,0)$ spectral weight

Figure 2. $s_0$-stability plots for $w_{20}$
Figure 3. $s_0$-stability plots for $w_{10}$

OPE vs Spectral Integral (ACO; $w_{10}$)

$V_{m_s} - m_s$ One-Sigma Contours

Figure 4. $(k, 0)$ spectral weight joint fit contours

$V_{m_s} - m_s$ One-Sigma Contours

Figure 5. Non-spectral weight joint fit contours
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