Tri-bimaximal Neutrino Mixing from $A_4$ and $\theta_{13} \sim \theta_C$

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Abstract

It is a common believe that, if the Tri-bimaximal mixing (TBM) pattern is explained by vacuum alignment in an $A_4$ model, only a very small reactor angle, say $\theta_{13} \sim O(\lambda_C^2)$ being $\lambda_C \equiv \theta_C$ the Cabibbo angle, can be accommodated. This statement is based on the assumption that all the flavon fields acquire VEVs at a very similar scale and the departures from exact TBM arise at the same perturbation level. From the experimental point of view, however, a relatively large value $\theta_{13} \sim O(\lambda_C)$ is not yet excluded by present data. In this paper, we propose a Seesaw $A_4$ model in which the previous assumption can naturally be evaded. The aim is to describe a $\theta_{13} \sim O(\lambda_C)$ without conflicting with the TBM prediction for $\theta_{12}$ which is rather close to the observed value (at $\lambda_C^2$ level). In our model the deviation of the atmospheric angle from maximal is subject to the sum-rule: $\sin^2 \theta_{23} \approx 1/2 + \sqrt{2}/2 \cos \delta \sin \theta_{13}$ which is a next-to-leading order prediction of our model.

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1 Introduction

The present data [1], at 1σ, on solar and atmospherical angles:

θ_{12} = (34.5 ± 1.4)° , \quad θ_{23} = (42.3^{+4.4}_{−3.5})° , \quad (1)

are fully compatible with the TBM matrix:

\begin{equation}
U_{\text{TB}} = \begin{pmatrix}
\sqrt{2/3} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2}
\end{pmatrix}, \quad (2)
\end{equation}

which corresponds to \( \sin^2 \theta_{12} = 1/3 \) (\( \theta_{12} = 35.3° \)) and \( \sin^2 \theta_{23} = 1/2 \). The TBM pattern also predicts \( \theta_{13} = 0 \), however, recent analysis based on global fits of the available data leads to hints for \( \theta_{13} > 0 \) [2, 3]. Differently from solar or atmospherical mixing angles, the reactor one is less constrained and its value can still be relatively large, even at 1σ level, say \( \sim 10^{-2} \):

\[ \sin^2 \theta_{13} = 0.016 \pm 0.010 \quad 2, \quad \sin^2 \theta_{13} = 0.010^{+0.016}_{−0.011} \quad 3. \]

For this reason the future experimental sensitivity on the reactor angle is fundamental for theoretical understanding of the TBM, which, due to its highly symmetric structure, strongly suggests an underlying non-abelian flavour symmetry. A natural and economical class of models based on \( A_4 \) flavour symmetry [4, 5, 6, 7, 8] has been proposed in describing TBM pattern. There are also more involved models based on other discrete groups [9, 10] or continuous flavour symmetries [11]. However, if the TBM pattern results from a spontaneously broken flavour symmetry, higher order corrections should introduce deviation from exact TBM, generally all of the same order. Since the experimental departures of \( \theta_{12} \) from its Tri-bimaximal value are at most of order \( \lambda^2 \), a future observed value of \( \theta_{13} \) near its present upper bound should impose severe constraints on model buildings [12]. If this is the case, one apparently has to renounce the nice symmetric nature of \( \theta_{12} \) and simply imagines that its Tri-bimaximal value may be completely accidental [13].

In this paper we will show that the TBM pattern can be explained by \( A_4 \) symmetry without necessarily implying a very small value for \( \theta_{13} \). The fundamental new ingredient of our construction is to allow a moderate hierarchy, of order \( \lambda_C \), between the VEVs of flavon fields of the charged lepton and the neutrino sectors. The paper is organized as follows. In the next section, we characterize some general conditions under which a hierarchy between VEVs of flavon fields of different sectors can be accommodated without fine tuning. In the section 3, we introduce the field content of our Seesaw model based on \( A_4 \times Z_3 \times Z_4 \) flavour group with great emphasis on vacuum alignment and its stability. In section 4, we explain how the charged lepton hierarchy can be reproduced by a particular symmetry breaking pattern of \( A_4 \). In section 5, we describe the neutrino mass at leading order by Seesaw mechanism and obtain an exact TBM at this level. Then in section 6 we include all subleading corrections up to terms suppressed by \( 1/\Lambda^2 \) to our model and analyze possible deviation from TBM. In the end, in section 7, we comment on other possible phenomenological consequences of our model and conclude.
2 General Consideration

The difficulty in the standard formulation of $A_4$ models \cite{4,5} in generating a relatively large value of $\theta_{13}$ is related to the vacuum alignment problem which plays a fundamental role in order to naturally describe the TBM pattern from spontaneously broken flavour symmetries. The group $A_4$ (see Appendix A) has two important subgroups: $G_S$, which is a reflection subgroup generated by $S$ and $G_T$, which is the group generated by $T$, isomorphic to $Z_3$. $A_4$ can be spontaneously broken by VEVs of two sets of flavon fields, $\Phi$ for the neutrino sector and $\Phi'$ for the charged lepton sector. The direction of $\langle \Phi \rangle$ should leave the subgroup $G_S$ unbroken leading to the TBM. However one generally has two options for the the alignment of $\Phi'$. $\langle \Phi' \rangle$ can be such that $G_T$ is preserved leading to diagonal charged lepton masses but their hierarchy is usually generated by an independent Froggatt-Nielsen (FN) mechanism \cite{14}. The second option is to consider a vacuum alignment of $\Phi'$ which entirely breaks $A_4$ and in this case the mass hierarchy is directly related to $\langle \Phi' \rangle/\Lambda$, being $\Lambda$ the cut-off scale, without an extra FN component \cite{6,7,8,15}. A natural mechanism for the vacuum alignment of $\Phi$ and $\Phi'$ in different directions requires the existence of an Abelian factor $G_A$ in addition to $A_4$. The aim of $G_A$ is to guarantee the following decomposition of the scalar potential as:

$$V(\Phi, \Phi') = V_\nu(\Phi) + V_\nu(\Phi') + V^{NLO}(\Phi, \Phi') + \cdots \quad (3)$$

where we see that the interaction term between $\Phi$ and $\Phi'$ appears from next-to-leading order (NLO). We will refer this situation as a “partial” separation in the scalar potential which is tightly related on the fact that only one of the sets $\Phi$ and $\Phi'$ is charged under $G_A$, a standard choice in the literature \cite{4,5,8}. At leading order, the two scalar sectors are actually separated, however, the vacuum alignments are affected by NLO corrections encoded in $V^{NLO}(\Phi, \Phi')$. The order of magnitude of the corrections to the VEVs $\langle \Phi \rangle$ and $\langle \Phi' \rangle$ depends on $\langle \Phi \rangle/\Lambda$ and $\langle \Phi' \rangle/\Lambda$ and they are subject to some conditions. First of all, the corrections to the Tri-bimaximal value of $\theta_{12}$ are at most of order $\lambda_C^2$. Furthermore, the corrections to $\langle \Phi' \rangle$ are required to be smaller than $m_\mu/m_\tau \sim O(\lambda_C^2)$ or more restrictively smaller than $m_\tau/m_\mu \sim O(\lambda_C^2)$; otherwise, the generated charged lepton hierarchy should not be stable. These conditions shall translate to upper bounds on the scale of flavour symmetry breaking with respect of the cut-off scale:

$$\langle \Phi \rangle/\Lambda, \langle \Phi' \rangle/\Lambda \lesssim \lambda_C^2 \quad (4)$$

$\langle \Phi' \rangle/\Lambda \lesssim \lambda_C^2$. In conclusion, a value of $\theta_{13}$ near its present experimental bound can not be described if the scalar potential is “partially” separated as quoted in (3).

In this paper we will exploit the possibility of a “fully” separated scalar potential which corresponds to (3) with $V^{NLO}(\Phi, \Phi') = V^{NLO}(\Phi)$ or $V^{NLO}(\Phi, \Phi') = V^{NLO}(\Phi')$. The “fully” separated scalar potential can be obtained if $G_A$ is a direct product of two Abelian factors $G_A^\nu$ and $G_A^e$ which separately acts on $\Phi$ and $\Phi'$. In this case, since $V_\nu(\Phi)$ and $V_\nu(\Phi')$ can be minimized in a completely independent way, even including NLO corrections, we are not necessarily subject to the strict condition (4). In fact, it is possible to construct a completely natural model for TBM based on the $A_4$ symmetry in which $\langle \Phi' \rangle/\Lambda \sim O(\lambda_C^2)$ and $\langle \Phi \rangle/\Lambda \sim O(\lambda_C)$ can be compatible with all experimental constraints. The model belongs the constrained $A_4$ models considered in \cite{6,7} in which the leading order neutrino
3 Field Content and Vacuum Alignment

In this section we introduce the field content of the model and analyze the most general scalar potential which is invariant under the flavour symmetry $A_4 \times Z_3 \times Z_4$. The lepton SU(2) doublets $l_i$ ($i = e, \mu, \tau$) are assigned to the triplet $A_4$ representation, while the lepton singlets $e^c, \mu^c$ and $\tau^c$ are all invariant under $A_4$. The neutrino sector is described by Seesaw mechanism with 3 heavy right-handed neutrinos $\nu^c_i$ which also form an $A_4$ triplet. The symmetry breaking sector consists of the scalar fields neutral under the SM gauge group, divided in two sets as advanced before: $\Phi = \{\varphi_S, \xi, \zeta\}$ and $\Phi' = \{\varphi_T, \xi', \zeta\}$. As anticipated before, in addition to $A_4$, we also have an Abelian symmetry $G_A = Z_3 \times Z_4$ which is a distinguishing feature of our construction. All the fields of the model, together with their transformation properties under the flavour group, are listed in Table 1. We observe that $\Phi$ is charged under $Z_3$ while $\Phi'$ is charged under $Z_3'$. The vacuum alignment problem of the model can be solved by the supersymmetric driving field method introduced in [5]. This approach exploits the continuous $U(1)_R$ symmetry in the superpotential $w$ under which matter fields have $R = +1$, while Higgses and flavons have $R = 0$. The spontaneous breaking of $A_4$ can be employed by adding to fields already present in Table 1 a new set of multiplets, called driving fields, with $R = 2$. We introduce a driving field $\xi_0$, fully invariant under $A_4$, and two driving fields $\varphi_0^T$ and $\varphi_0^S$, triplet of $A_4$. The driving fields $\xi_0$ and $\varphi_0^S$, which are responsible for the alignment of $\varphi_S$, have a charge $\omega$ under $Z_3$ and are invariant under $Z_4$. $\varphi_0^T$ has a charged $-1$ under $Z_4$, invariant under $Z_3$, and drives a non-trivial VEV of $\varphi_T$. The most general driving superpotential $w_d$ invariant under $A_4 \times G_A$ with $R = 2$ is a sum of two independent parts $w_d = w_d^\nu(\xi_0, \varphi_0^S, \Phi) + w_d^c(\varphi_0^T, \Phi')$ where

\begin{align}
w_d^\nu &= g_1\varphi_0^S\varphi_S^2 + g_2\xi(\varphi_0^S\varphi_S) + g_3\xi_0(\varphi_S\varphi_S) + g_4\xi_0^2\xi_0 + g_5\xi_0\xi_0^2 + g_6\xi_0\xi_0^2 + g_7M_\xi\xi_0\xi_0 \\
w_d^c &= h_1\xi(\varphi_0^T\varphi_T)^2 + h_2(\varphi_0^T\varphi_T\varphi_T).
\end{align}

The “fully” separated superpotential is guaranteed by $G_A = Z_3 \times Z_4$. [5] and [6] gives two decoupled sets of F-terms for driving fields which characterize the supersymmetric

| Field | $l_i$ | $e^c$ | $\mu^c$ | $\tau^c$ | $\nu^c$ | $h_u$ | $h_d$ | $\varphi_T$ | $\xi'$ | $\varphi_S$ | $\xi$ | $\zeta$ |
|-------|-------|-------|-------|-------|-------|-------|-------|----------------|-------|----------------|-------|-------|
| $A_4$ | 3     | 1     | 1     | 1     | 3     | 1     | 1     | $3$            | $1'$  | 3              | 1     | 1    |
| $Z_3$ | 1     | 1     | 1     | 1     | $\omega$ | 1     | 1     | 1              | 1     | $\omega$       | $\omega$ | $\omega^2$ |
| $Z_4$ | 1     | $-1$  | $-i$  | 1     | 1     | $-i$  | $i$   | 1              | 1     | 1              | 1     | 1    |

Table 1: The transformation properties of leptons, electroweak Higgs doublets and flavons under $A_4 \times Z_3 \times Z_4$. TBM and the charged lepton mass hierarchy are simultaneously reproduced by the vacuum alignment. Our choice for $G_A$ in order to guarantee a “fully” separated scalar potential is given by $Z_3 \times Z_4$. We are particularly interested in analyzing the possibility to have a relatively large value of $\theta_{13}$ without fine-tuning. We will show indeed that $\theta_{13}$ can be of order $\lambda_C$ while $\theta_{12}$ is corrected by subleading effects arising at order $\lambda_C^2$. Furthermore, deviations from TBM can be more intriguing since they obey a definite sum-rule which can be in principle tested.
minimum. In other words, \( w_d^\nu \) and \( w_d^{\nu} \) independently determine the vacuum alignment of \( \Phi \) and \( \Phi' \), respectively. From (5) we have:

\[
\begin{align*}
\frac{\partial w}{\partial \varphi_{01}} &= g_2 \tilde{\xi} \varphi s_1 + 2g_1(\varphi s_1^2 - \varphi s_2 \varphi s_3) = 0 \\
\frac{\partial w}{\partial \varphi_{02}} &= g_2 \tilde{\xi} \varphi s_3 + 2g_1(\varphi s_2^2 - \varphi s_1 \varphi s_3) = 0 \\
\frac{\partial w}{\partial \varphi_{03}} &= g_2 \tilde{\xi} \varphi s_2 + 2g_1(\varphi s_3^2 - \varphi s_1 \varphi s_2) = 0 \\
\frac{\partial w}{\partial \xi_0} &= g_4 \xi^2 + g_5 \xi \tilde{\xi} + g_6 \tilde{\xi}^2 + g_7 M_\zeta \zeta + g_3(\varphi s_1^2 + 2\varphi s_2 \varphi s_3) = 0.
\end{align*}
\]

(7)

In a finite portion of the parameter space, we find the following stable solution

\[
\langle \tilde{\xi} \rangle = 0, \quad \langle \xi \rangle = u, \quad \langle \zeta \rangle = v,
\]

\[
\langle \varphi_S \rangle = (v_s, v_s, v_s), \quad v_s^2 = -\frac{g_4 u^2 + g_7 M_\zeta v}{3g_3},
\]

(8)

with \( u \) and \( v \) undetermined. Since \( \langle \tilde{\xi} \rangle = 0 \), we have ignored the existence of \( \tilde{\xi} \) in the rest of the paper. Setting to zero the F-terms from Eq. (6), we obtain:

\[
\begin{align*}
\frac{\partial w}{\partial \varphi_{01}^T} &= h_1 \xi' \varphi_{T3} + 2h_2(\varphi_{T1}^2 - \varphi_{T2} \varphi_{T3}) = 0 \\
\frac{\partial w}{\partial \varphi_{02}^T} &= h_1 \xi' \varphi_{T2} + 2h_2(\varphi_{T2}^2 - \varphi_{T1} \varphi_{T3}) = 0 \\
\frac{\partial w}{\partial \varphi_{03}^T} &= h_1 \xi' \varphi_{T1} + 2h_2(\varphi_{T3}^2 - \varphi_{T1} \varphi_{T2}) = 0
\end{align*}
\]

and the stable solution to these four equations is:

\[
\langle \xi' \rangle = u' \neq 0, \quad \langle \varphi_T \rangle = (0, v_T, 0), \quad v_T = -\frac{h_1 u'}{2h_2},
\]

(9)

with \( u' \) undetermined. The flat directions can be removed by the interplay of radiative corrections to the scalar potential and soft SUSY breaking terms. It is worth to observe that, thanks to \( G_A \), the VEV alignments (8) and (9) are independent even at NLO.

Since the VEVs of the scalar fields in \( \Phi' (\Phi) \) are related each other by adimensional constants of order one, we should expect that they have a common scale indicated by \( \langle \Phi \rangle (\langle \Phi' \rangle) \). However \( \langle \Phi \rangle / \Lambda \) and \( \langle \Phi' \rangle / \Lambda \) can be in principle different and they are subject to phenomenological constraints. As we will see in the next section, \( \langle \Phi' \rangle \) is responsible for charged lepton hierarchy so we have to require

\[
\frac{m_e}{m_\mu} \sim \lambda_C^3 \lesssim \frac{\langle \Phi' \rangle}{\Lambda} \lesssim \lambda_C^2 \sim \frac{m_\mu}{m_\tau}.
\]

\(^1\)Since there is no fundamental distinction between the singlets \( \xi \) and \( \tilde{\xi} \) we have defined \( \tilde{\xi} \) as the combination that couples to \( \langle \varphi_S^0 \varphi_S \rangle \) in the superpotential \( w_d \). The introduction of an additional singlet is essential to recover a non-trivial solution.
The superpotential $w''_\Phi$ is affected by non-renormalizable terms (see Appendix B for the detail) from the neutrino sector $\Phi$ suppressed by $1/\Lambda^2$. Requiring that the sub-leading corrections to $\langle \Phi' \rangle$ are smaller than $m_\mu/m_\tau \sim O(\lambda_C^2)$, we obtain the condition
\[
\frac{\langle \Phi \rangle}{\Lambda} \lesssim \lambda_C .
\]
The vacuum alignment with a “fully” separated scalar potential allows a hierarchy between the VEVs of the scalars in different sectors $\langle \Phi' \rangle \ll \langle \Phi \rangle$.

Differently from $w''_\Phi$, $w'_d$ receives NLO corrections which are suppressed only by $1/\Lambda$ but don’t depend on the charged lepton sector $\Phi'$:
\[
\delta w'_d = \frac{1}{\Lambda} \left[ (\varphi_0^S \varphi_S) \zeta^2 + \xi_0 \xi \zeta^2 \right] .
\]
One may wonder if a large VEV of $\Phi$ with $\langle \Phi \rangle/\Lambda \sim \lambda_C$ could introduce a too large correction to the leading order vacuum alignment destroying the stability of the TBM prediction. Fortunately, this is not the case. Since there is no fundamental distinction between $\zeta^2$ and $\xi$ the NLO correction $\delta w'_d$ should induce terms which have the same form of those already present in $w''_\Phi$. In fact, including $\delta w'_d$ in the minimization, one easily find that the $\langle \varphi_S \rangle$ receives only a small shift in the same direction of the leading order alignment. For this reason we will no longer consider VEV shifts of $\varphi_S$ in the following.

### 4 Charged Lepton Hierarchy

In the present section, we illustrate how a fully broken $A_4$ symmetry can generate the charged lepton hierarchy. The key ingredient is the alignment $\langle \varphi_T \rangle \sim (0, 1, 0)$. Such a VEV breaks the permutation symmetry of the second and third generation of neutrinos in a maximal way in the sense that
\[
\langle \varphi_T \rangle^T S_{2-3} \langle \varphi_T \rangle = 0 ,
\]
where
\[
S_{2-3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

The $A_4$ group is fully broken in the charged lepton sector by $\Phi'$ with the vacuum structure quoted in and only the tau mass is generated at leading order. The muon and electron masses are generated respectively by $\langle \varphi_T \rangle^2 \sim (0, 0, 1)$ and $\langle \varphi_T \rangle^3 \sim (1, 0, 0)$. Then the correct hierarchy between the charged lepton masses $m_e \ll m_\mu \ll m_\tau$ is reproduced if we assume $\lambda_C^2 \lesssim \langle \Phi' \rangle/\Lambda \lesssim \lambda_C^3$.

Since $\Phi'$ carries a charge $i$ under $Z_4$ we have to assign different $Z_4$ charges for lepton singlets. Considering only insertions of $\Phi'$, the charged lepton masses are described by $w_e$, given by, up to $1/\Lambda^3$:
\[
w_e = \alpha_1 \tau^c (l \varphi_T) h_d/\Lambda \\
+ \beta_1 \mu^c \xi^c (l \varphi_T) h_d/\Lambda^2 \\
+ \gamma_1 \epsilon^c (\xi^c)^2 (l \varphi_T) h_d/\Lambda^3 + \gamma_2 \epsilon^c (l \varphi_T h_d)/\Lambda^3 + \gamma_3 \epsilon^c (l \varphi_T \varphi_T h_d)/\Lambda^3
\]

\footnote{Similarly as explained in [6, 8], a residual symmetry $A_4 \times Z_3$ from $A_4 \times Z_4$ survives in the charged lepton sector guaranteeing the stability of the vacuum alignment.}
After electroweak symmetry breaking, \( \langle h_{u,d} \rangle = v_{u,d} \), given the specific orientation of \( \langle \varphi_T \rangle \propto (0,1,0) \), \( w_e \) give rise to diagonal and hierarchical mass terms for charged leptons. Defining the expansion parameter \( \nu_T/\Lambda \equiv \lambda^2 \ll 1 \) (it is not restrictive to consider \( \nu_T \) to be positive) and the Yukawa couplings \( y_l \) \((l = e, \mu, \tau)\) as

\[
\begin{align*}
y_{\tau} &= |\alpha_1| , \\
y_{\mu} &= |\beta_1 u'/\nu_T + 2\beta_2| \lambda^2 , \\
y_{e} &= |\gamma_1 (u'/\nu_T)^2 - \gamma_2 u'/\nu_T - 2\gamma_3| ,
\end{align*}
\]

the charged lepton masses are given by

\[
m_l = y_l \lambda^2 v_d \quad (l = e, \mu, \tau) .
\] (10)

As already pointed out in the previous section and analyzed in detail in Appendix B, the vacuum alignment for \( \varphi_T \) receives correction of order \( \langle \Phi \rangle^2/\Lambda^2 \sim \lambda_C^2 \), different for each component:

\[
\varphi_T = (\delta_{T1}, \nu_T + \delta_{T2}, \delta_{T3}) .
\]

Including correction to the vacuum alignment for \( \varphi_T \), the diagonal form of the charged lepton mass should slightly change and small off-diagonal entries appear:

\[
m_e = \begin{pmatrix} m_e & m_e O(\lambda_C^2) & m_e O(\lambda_C^2) \\ m_\mu O(\lambda_C^2) & m_\mu & m_\mu O(\lambda_C^2) \\ m_\tau O(\lambda_C^2) & m_\tau & m_\tau \end{pmatrix} .
\] (11)

The transformation needed to diagonalize \( m_e \) is \( V_e^T m_e U_e = \text{diag}(m_e, m_\mu, m_\tau) \) and the unitary matrix \( U_e \) is given by

\[
U_e = \begin{pmatrix} 1 & O(\lambda_C^2) & O(\lambda_C^2) \\ O(\lambda_C^2) & 1 & O(\lambda_C^2) \\ O(\lambda_C^2) & O(\lambda_C^2) & 1 \end{pmatrix} .
\] (12)

Another source of off-diagonal correction to charged leptons comes from the interaction with the neutrino sector. In fact, the products \( \zeta \zeta \) and \( \varphi_S \zeta \) are invariant combination under \( G_A \) and we can include them on top of each term in \( w_e \). However, we find that the introduction of these additional terms changes the charged lepton mass \( m_e \) exactly in the same way as the corrections induced by VEV shifts of \( \varphi_T \), i.e. (11). Then (12) is the most general structure of the charged lepton contribution to TB mixing.

5 A Seesaw realization of the constrained \( A_4 \) model

The masses of light neutrinos of our model is described by Seesaw superpotential with 3 heavy right-handed neutrinos \( \nu^c \), triplet of \( A_4 \). Terms in the superpotential which contain \( \nu^c \) invariant under the flavour group are given by:

\[
w_\nu = y(\nu^c l) \zeta h^u/\Lambda + x_a \xi (\nu^c \nu^c) + x_b (\varphi_S \nu^c \nu^c) + h.c. + \cdots
\] (13)

In the heavy neutrino sector \( A_4 \times Z_3 \) is broken by \( \langle \varphi_S \rangle = (v_S, v_S, v_S) \) and \( \langle \xi \rangle = u \) down to \( G_S \) (with \( Z_4 \) unbroken) with an accidental extra \( G_{2-3} \) symmetry. Then the residual
symmetry of the right-handed neutrino masses is \( G_{TB} = G_S \times G_{2-3} \). \( G_{TB} \) can be transferred to the light neutrino sector if the Dirac neutrino mass commute its generators. This is in fact the case. After electroweak and \( A_4 \) symmetry breaking from (13) we obtain the following leading contribution to the Dirac and Majorana masses:

\[
m_0^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} y v_u \frac{v}{\Lambda}, \quad M = \begin{pmatrix} a + 2b & -b & -b \\ -b & 2b & a - b \\ -b & a - b & 2b \end{pmatrix} u ,
\]

where

\[
a \equiv x_a , \quad b \equiv x_b \frac{v_S}{u} .
\]

We immediately see that \([m_0^D, S] = 0 \). The leading order lepton mixing matrix is entirely encoded in the right-handed neutrino mass matrix \( M \) which is diagonalized by the transformation:

\[
U_0^T M U_0^* = \text{diag}(|a + 3b|, |a|, |a - 3b|) u ,
\]

with \( U_0 = U_{TB} \Omega \) where \( \Omega = \text{diag}(e^{i\phi_1/2}, e^{i\phi_2/2}, e^{i\phi_3/2}) \) and \( \phi_1, \phi_2, \phi_3 \) are respectively phases of \( a + 3b, a, a - 3b \). Naturally \( \phi_1 \) and \( \phi_3 \) depend on \( \phi_2 \) and \( \Delta \), the relative phase between \( a \) and \( b \). The light neutrino masses are given by the type I Seesaw mechanism: \( m_\nu = (m_0^D)^T M^{-1} m_0^D \) which is invariant under \( G_{TB} \) and then also diagonalized by \( U_0 \). Denoting the physical masses of \( \nu_i \) as \( M_1 = |a + 3b|, M_2 = |a| \) and \( M_3 = |a - 3b| \), we obtain

\[
U_0^T m_0^D U_0 = \left[ \frac{y v_u v}{\Lambda} \right]^2 \text{diag}\{ \frac{1}{M_1}, \frac{1}{M_2}, \frac{1}{M_3} \} = \text{diag}\{ m_1, m_2, m_3 \} .
\]

\( m_2 > m_1 \) implies \( t \equiv |3b|/|a| > -2 \cos \Delta \) and in principle both normal and inverted hierarchies in the neutrino spectrum can be reproduced. The normal hierarchy is realized for \( t/2 \leq \cos \Delta \leq 1 \) whereas an inverted spectrum requires \( -t/2 < \cos \Delta \leq 0 \). The ratio \( r = \Delta m^2_{\text{sun}}/\Delta m^2_{\text{atm}} \) (where \( \Delta m^2_{\text{sun}} = m_2^2 - m_1^2 \) and \( \Delta m^2_{\text{atm}} = |m_3^2 - m_1^2| \)) is given in our model by:

\[
r = \frac{(t + 2 \cos \Delta)(1 + t^2 - 2t \cos \Delta)}{4 \cos \Delta} .
\]

One can show that for the normal hierarchy, a small value of \( r \approx 1/30 \) can be reproduced only for \( \cos \Delta \approx t \approx 1 \). In particular, a normal ordered spectrum can never be degenerate. Then we can expand \( t = 1 + \delta t \) with \( \delta t \ll t \) obtaining the following approximate spectrum:

\[
m_1 \approx \sqrt{\Delta m^2_{\text{sun}}/3} , \quad m_2 \approx 2m_1 , \quad m_3 \approx \sqrt{\Delta m^2_{\text{atm}} - \Delta m^2_{\text{sun}}/3} .
\]

The inverted hierarchy can be realized only for \( t \approx -2 \cos \Delta \) and in this case we can expand \( \cos \Delta = -t/2 + \delta t' \) with \( \delta t' \ll t \). Expressing \( \delta t \) in function of \( r \) we obtain

\[
m_1^2 = \Delta m^2_{\text{atm}} \left[ 1 + \frac{1}{2t^2} + \left( \frac{1}{t^2} - \frac{1}{1 + 2t^2} \right) r \right]
\]

\[
m_2^2 = \Delta m^2_{\text{atm}} \left[ 1 + \frac{1}{2t^2} + \left( \frac{1}{t^2} - \frac{1}{1 + 2t^2} \right) r \right]
\]

\[
m_3^2 = \Delta m^2_{\text{atm}} \left[ \frac{1}{2t^2} + \left( \frac{1}{t^2} - \frac{1}{1 + 2t^2} \right) r \right] .
\]

\( ^3 \)The overall phase appearing in the Dirac neutrino mass \( m_0^D \) can be absorbed by the redefinition of \( \phi_2 \) and there are only two independent Majorana phases.
In principle, the previous expansion is valid also for a degenerate spectrum realized by $t \ll 1$ which is, however, parametrically fine tuned\textsuperscript{4} in our model.

6 Deviation from TBM and $\theta_{13} \sim \lambda_C$

In this section we show how a relatively large reactor angle, say $\theta_{13} \sim \theta_C$, can naturally arise in our model, without conflicting with the precise value of $\theta_{12}$ predicted by TBM. The neutrino mass described in the previous section predicts an exact TBM. Including sub-leading contributions dictated by higher-dimensional operators, the leading order lepton mixing matrix should be modified. As we shall see in a moment, not all deviations from TBM arise at the same perturbation level, this is one of the most important feature of the model. We find that the NLO corrections generate a non-vanishing reactor angle which is correlated with deviation of atmospheric angle from maximal. While the corrections to solar angle appear only at next-to-next to leading order (NNLO).

First of all we focus on higher order corrections to the right-handed Majorana neutrino mass up to terms suppressed by $1/\Lambda^2$. At NLO, there is only one additional contribution to heavy Majorana mass: $\zeta^2 \nu^c \nu^c / \Lambda$. Since $\zeta^2$ has exactly the same property of $\xi$, this term can be absorbed by a redefinition of $a$. The NNLO contributions arise from adding the products $\xi \zeta$ and $\varphi_S \zeta$, invariant combination under $G_A$, on top of the leading order terms. In this case, not all the corrections have the same structure of the terms already present in $w_\nu$ and consequently cannot be regarded as small shifts of $a$ and $b$, for example $(\nu^c \nu^c)'(\varphi_S \varphi_S)$'' and $(\nu^c \nu^c)'(\varphi_S \varphi_S)'$. However these terms can be absorbed by parameters $y_1$ and $y_2$ in the NLO correction to the Dirac mass $\delta m^D$ as will be clear in a moment.

Now we move to consider the correction to Dirac neutrino mass: $\delta m^D$ beginning with terms suppressed by $1/\Lambda^2$. There are many independent terms of the type $(\nu^c \varphi \varphi)h^u$, with $\varphi \in \{\varphi_S, \xi\}$, invariant of $A_4$ which contribute to $\delta m^D$ at this order:

$$\begin{align*}
\delta w_\nu &= h^u_{\nu_1} \frac{y_1}{\Lambda^2} (\nu^c l)'(\varphi_S \varphi_S)''' + h^u_{\nu_2} \frac{y_2}{\Lambda^2} (\nu^c l)''(\varphi_S \varphi_S)' + h^u_{\nu_3} \frac{y_3}{\Lambda^2} \nu^c (l \varphi_S)_{A} \xi + h^u_{\nu} \frac{y'_1}{\Lambda^2} (\nu^c l)_1 (\varphi_S \varphi_S)_1 + h^u_{\nu} \frac{y''_1}{\Lambda^2} (\nu^c l) \xi^2 + h^u_{\nu} \frac{y'_2}{\Lambda^2} \nu^c (l \varphi_S)_2 \xi .
\end{align*}$$

Observe that the operators with coefficients $y'_1$, $y''_1$, $y'_2$ give contribution to Dirac mass matrix in a form invariant under $G_{TB}$ exactly as right-handed neutrino mass. Then these corrections can be adsorbed into a redefinition of the leading-order coefficients. The relevant correction to the Dirac mass comes from the first three terms in Eq. (19) and has the following form:

$$\delta m^D = \begin{pmatrix}
0 & y_1 + \bar{y}_3 & y_2 - \bar{y}_3 \\
y_1 - \bar{y}_3 & y_2 & \bar{y}_3 \\
y_2 + \bar{y}_3 & \bar{y}_3 & y_1
\end{pmatrix} v_u \frac{v_s^2}{\Lambda^2}, \quad (20)
$$

\textsuperscript{4}The fine-tuning required in order to reproduce a small $r$ becomes more severe if we include in $w_\nu$ also the five-dimensional operator $lh^u lh^u / \Lambda'$ which leads to a mass matrix structure similar to the term $\xi \nu^c \nu^c$. Indeed if the Weinberg operator has a cutoff scale $\Lambda' \sim \Lambda$, its contribution becomes larger than the Seesaw one. This situation is equivalent to go to the limit $a \gg b$ and then it is disfavored. In order to avoid this problem we will assume that the lepton number is violated only by Majorana mass term up to $\Lambda$. In other words, we require $\Lambda' \gg \Lambda$ and a direct five-dimensional operator can be neglected.
where \( y_1, y_2, \tilde{y}_3 \equiv y_3 u/v_s \) are generally complex numbers of order 1. Before discussing the important consequence when we include the NLO correction to the Dirac neutrino mass, we comment possible NNLO effects on \( m_D \). Here the NNLO contributions are suppressed by \( 1/\Lambda^3 \) and they are of the type \( (v^2 \zeta \phi h^u) \). All these terms can be absorbed by a redefinition of \( y_3 \) and \( y'_2 \), then we can forget them in the following analysis.

In order to find the correction to the leading neutrino mixing matrix \( U_0 = U_{\text{TB}} \Omega \), it is convenient to define
\[
\hat{m}^D = U_0^\dagger m^D U_0 ,
\]
where \( m^D = m^D_0 + \delta m^D \). The light neutrino mass is then formally given by
\[
m_\nu = U_0 \hat{m}_\nu U_0^T
\]
where \( \hat{m}_\nu \equiv (\hat{m}^D)^T M^{-1}_{\text{diag}} \hat{m}^D \) with \( M^{-1}_{\text{diag}} = \text{diag}\{1/M_1, 1/M_2, 1/M_3\} \). If \( \hat{m}_\nu \) can be diagonalized by the unitary matrix \( \delta U \sim I \) as
\[
\delta U \hat{m}_\nu \delta U^T = \text{diag}\{\hat{m}_1, \hat{m}_2, \hat{m}_3\} ,
\]
where \( \hat{m}_i \approx m_i \), the full PMNS mixing matrix will be given by
\[
U_{\text{PMNS}} = U_e^\dagger U_0 \delta U.
\]

In our case, the matrix \( \hat{m}^D \) has a very simple expression:
\[
\hat{m}^D \approx \begin{pmatrix}
1 & 0 & e^{i \phi_{31} c_+ \epsilon} \\
0 & 1 & 0 \\
e^{i \phi_{31} c_- \epsilon} & 0 & -1
\end{pmatrix} \frac{y v u}{\Lambda} ,
\]
where \( \phi_{31} = (\phi_3 - \phi_1)/2, c_{+(-)} = i \sqrt{3}/2 (y_2 - y_1 + (-)2 \tilde{y}_3) \) and \( \epsilon = v_3^2/(v \Lambda) \sim \lambda C \). Then we get
\[
\hat{m}_\nu = \begin{pmatrix}
m_1 & 0 & e^{i \phi_{31} (c_+ m_1 + c_- m_3) \epsilon} \\
0 & m_2 & 0 \\
e^{i \phi_{31} (c_+ m_1 + c_- m_3) \epsilon} & 0 & m_3
\end{pmatrix} + O(\epsilon^2) .
\]

This result means that a correction \( (\delta U)_{13} \sim \lambda C \) can be present and we can expect that a deviation of \( \theta_{12} \) from its Tri-bimaximal value arises only at order \( \lambda^2 C \). However, observe that if \( m_1 \approx m_3 \) i.e. the spectrum becomes degenerate, a fine-tuning will be required in order to reproduce a small \( (\delta U)_{13} \). From this viewpoint, a degenerate spectrum is disfavored if we require that the deviation from TBM is naturally small.

Forgetting for a moment \( U_e \) which arises only at NNLO, from Eq. (21), one find that
\[
U_{e3} = \sqrt{\frac{2}{3}} e^{i \phi_{13} (\delta U)_{13}} , \quad U_{\mu 3} = -\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{6}} e^{i \phi_{13} (\delta U)_{13}}
\]
and \( U_{l2}, l = \text{e, } \mu, \tau \), remain unchanged. As a result, the solar angle \( \theta_{12} \) remains rather close to its Tri-bimaximal value. However \( (\delta U)_{13} \) simultaneously induces a departure of \( \theta_{13} \) and of \( \theta_{23} - \pi/4 \) from zero. Defining \( \delta' \) as the phase of \( (\delta U)_{13} \), the CP-violating Dirac phase is
given by \(-\delta = \delta' + \phi_{13}\). Since \(\sin \theta_{13} = \sqrt{2/3}|(\delta U)_{13}|\), the deviation of the atmospherical angle from maximal is subject to the following sum-rule:

\[
\sin^2 \theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e 3}|^2} \approx \frac{1}{2} + \frac{\sqrt{2}}{2} \cos \delta \sin \theta_{13} + O(\theta_{13}^2),
\]

(25)

this is a prediction of our model. This is a special feature of the present Seesaw \(A_4\) model. The presence of the abelian factor \(G_A\) in our model, not only allows a relatively large value of \(\theta_{13}\), at \(\theta_C\) level, also strongly suppresses possible higher order contributions giving rise correlation between them.

Independently from the Seesaw sector, TBM and in particular the solar angle receives corrections from charged lepton sector. Adopting the standard parametrization of \(U_{PMNS}\), from (21) and (12) one finds that all the mixing angles receive a correction of order \(\lambda_C^2\). Then we in particular obtain

\[
\sin^2 \theta_{12} = \frac{1}{3} + O(\lambda_C^2).
\]

As claimed in the beginning, \(\theta_{13}\) can be of order \(\lambda_C\) since it arises from corrections at NLO in the neutrino sector while \(\theta_{12}\) receives corrections only of order \(\lambda_C^2\) which are subleading effects at NNLO.

### 7 Conclusion and discussion

In this paper we have addressed one of the most important issues in the \(A_4\) realization of TBM i.e. if a \(\theta_{13} \sim \theta_C\) can be allowed without fine tuning. We have discussed a framework, referred as constrained \(A_4\) model, in which the vacuum alignment is realized by a fully separated scalar potential. The model is based on the \(A_4 \times Z_3 \times Z_4\) flavour symmetry and (Type I) Seesaw mechanism. In the charged lepton sector, the \(A_4\) group is entirely broken by the set of scalar field \(\Phi' = \{\varphi_T, \xi'\}\). The symmetry breaking parameter \(\langle \Phi' \rangle / \Lambda \sim \lambda_C^2\) directly controls the charged lepton mass hierarchy without requiring a \(U(1)_{FN}\) symmetry. In the neutrino sector, the set of scalar fields \(\Phi = \{\varphi_S, \xi, \zeta\}\) breaks the \(A_4\) group to its subgroup \(G_S\) guaranteeing the TBM at leading order. The symmetry breaking parameter \(\langle \Phi \rangle / \Lambda\), however, can be chosen at order of the Cabibbo angle \(\lambda_C\) without altering the required vacuum alignment for \(\Phi'\). Moreover, a non-vanishing \(\theta_{13}\) and a deviation of \(\theta_{23}\) from \(\pi/4\) are simultaneously generated at order \(O(\lambda_C)\) leaving \(\theta_{12}\) unchanged. Subsequently, a deviation of the solar angle from its TBM value is generated at order \(O(\lambda_C^2)\) which just corresponds to its 1\(\sigma\) experimental sensitivity.

The model is called constrained \(A_4\) model because, differently from its standard formulation widely studied in literature, the NLO corrections are also dictated by \(A_4\) symmetry itself. This is another interesting feature of our model. There is, indeed, a correlation between the deviation of \(\theta_{23}\) from maximal and the value of generated \(\theta_{13}\):

\[
\sin^2 \theta_{23} \approx 1/2 + \sqrt{2}/2 \cos \delta \sin \theta_{13} + O(\theta_{13}^2)
\]

which can be in principle tested by future experiments. Concerning the neutrino spectrum, it can be either of normal hierarchy or inverted one. However, a degenerated spectrum is parametrically fine tuned and is disfavored requiring that the deviation from TBM is naturally small. For this reason, we should also expect that the effect of running on mixing angles is negligible. Since the solar
angle has been measured more precisely than the others, its running can be potentially important if the neutrino spectrum were degenerate.

The corrections beyond the leading order are important not only in describing deviations from TBM, but also give rise to other interesting phenomenology. For example, the same breaking pattern for charged lepton sector can be easily extended to the quark sector. In this case, the $V_{\text{CKM}}$ arises when the correction to the vacuum alignment $\varphi_T$ is taken into account. Then the resulting $V_{\text{CKM}}$ should have the same form of the unitary matrix diagonalizing charged leptons $U_e$ given in (12). The inclusion of the sub leading corrections can also play an important role in explaining the baryon asymmetry of the universe (BAU) through leptogenesis [16]. As pointed out in [7], the generated BAU can be indeed directly triggered by low energy phases appearing $U_{e3}$. Moreover, the structure of $A_4$ symmetry breaking pattern can be revealed by other physical effects [17], not directly related to neutrino properties, such as lepton flavour violating process as well as the anomalous magnetic moments and the electric dipole moments of charged leptons. Such a possibility becomes realistic if there is new physics at a much lower energy scale around $1 - 10$ TeV. All these issues merit a further and more detailed study.

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Appendix A: The group $A_4$

The group $A_4$ has 12 elements and four non-equivalent irreducible representations: one triplet and three independent singlets 1, 1’ and 1”. Elements of $A_4$ are generated by the two generators $S$ and $T$ obeying the relations:

$$S^2 = (ST)^3 = T^3 = 1.$$  \hspace{1cm} (26)

We will consider the following unitary representations of $T$ and $S$:

for $1$: $S = 1$ $T = 1$
for $1'$: $S = 1$ $T = e^{i4\pi/3} = \omega^2$
for $1'': S = 1$ $T = e^{i2\pi/3} = \omega$

and for the triplet representation

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$  \hspace{1cm} (27)

The tensor product of two triplets is given by $3 \times 3 = 1 + 1' + 1'' + 3_S + 3_A$. From (27) and (28), one can easily construct all multiplication rules of $A_4$. In particular, for two triplets
ψ = (ψ1, ψ2, ψ3) and φ = (φ1, φ2, φ3) one has:

\[ \begin{align*}
\psi_1 φ_1 + ψ_2 φ_2 + ψ_3 φ_2 & \sim 1, \\
ψ_3 φ_3 + ψ_1 φ_2 + ψ_2 φ_1 & \sim 1', \\
ψ_2 φ_2 + ψ_3 φ_1 + ψ_1 φ_3 & \sim 1'', \\
\end{align*} \]

\[ \begin{pmatrix}
2ψ_1 φ_1 - ψ_2 φ_2 - ψ_3 φ_2 \\
2ψ_3 φ_3 - ψ_1 φ_2 - ψ_2 φ_1 \\
2ψ_2 φ_2 - ψ_1 φ_3 - ψ_3 φ_1
\end{pmatrix} \sim 3S, \\
\begin{pmatrix}
ψ_2 φ_3 - ψ_3 φ_2 \\
ψ_1 φ_2 - ψ_2 φ_1 \\
ψ_3 φ_1 - ψ_1 φ_3
\end{pmatrix} \sim 3A. \quad (29)

**Appendix B: Correction to Alignment of ϕ_T and ϕ_S**

In this appendix we will study correction to the leading order alignment of ϕ_S and ϕ_T when we include higher dimensionality operators up to the order 1/Λ^2.

In our model, the correction to the driving superpotential for ϕ_S, depends only on Φ at NNLO, then the obtained vacuum alignment ⟨ϕ_S⟩ ∝ (1, 1, 1) is always stable since it preserves the subgroup G_S of A_4. However a relative large ⟨Φ⟩/Λ ∝ λ_C may have some effects on the leading order alignment for ϕ_T ∝ (0, 1, 0). The products ξζ and ϕ_Sζ are invariant combination under G_A, then we can include them on top of each term in w^c_a. With the introduction of these higher dimensionality operators, w^c_a should be modified into w^c_a + δw^c_a where⁤\(^5\)

\[ δw^c_a = \frac{1}{Λ^2} \left[ t_1 ζ ξ' (ϕ^T_0 ϕ_T)'' + t_2 ζ ξ (ϕ^T_0 ϕ_Tϕ_T) + t_3 ζ ξ' (ϕ^T_0 ϕ_Tϕ_S)'' + t_4 ζ (ϕ^T_0 ϕ_S)'(ϕ_Tϕ_T)'' \right]. \]

The alignment for ϕ_T should be shifted (the shift in ξ' is needless) and we can look for a solution that perturbs ⟨ϕ_T⟩ to second order in the 1/Λ expansion:

\[ \langle ξ' \rangle = u', \quad ⟨ϕ_T⟩ = (δ_T1, v_T + δ_T2, δ_T3). \]

The minimum conditions from w^c_a + δw^c_a become equations in the shifts δv_T:

\[ \begin{align*}
-4h_2v_Tδv_T3 & + \left( t_4 - t_3 \frac{4h_2}{h_1} \right) \frac{vv_S}{Λ^2} v_T^2 = 0 \\
2h_2v_Tδv_T2 & + \left( t_4 + t_3 \frac{4h_2}{h_1} \right) \frac{vv_S}{Λ^2} v_T^2 + \left( 2t_2 - t_1 \frac{2h_2}{h_1} \right) \frac{vu}{Λ^2} v_T^2 = 0 \\
-4h_2v_Tδv_T1 & + \left( t_4 + t_3 \frac{4h_2}{h_1} \right) \frac{vv_S}{Λ^2} v_T^2 = 0
\end{align*} \]

These equations are linear in δv_Ti and can be easily solved by:

\[ \begin{align*}
\frac{δv_T3}{v_T} & = \left( t_4 - t_3 \frac{4h_2}{h_1} \right) \frac{vv_S}{Λ^2} \\
\frac{δv_T2}{v_T} & = -\left( \frac{t_1}{2h_2} + \frac{2t_3}{h_1} \right) \frac{vv_S}{Λ^2} + \left( \frac{t_1}{h_1} - \frac{t_2}{h_2} \right) \frac{vu}{Λ^2} \\
\frac{δv_T1}{v_T} & = \left( t_4 + t_3 \frac{4h_2}{h_1} \right) \frac{vv_S}{Λ^2}
\end{align*} \]

⁤\(^5\)Here we omit the term ζ(ξ')^2(ϕ_Sφ^c_0)' since it induces only a small shift of u' and then can be included in the redefinition of u'.
Observe that the shifts in three components are different but all of the same order of magnitude, as claimed in the text:

$$\frac{\delta v_{T_i}}{v_T} \sim O(\lambda_C^2) .$$

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