On High Energy Scattering in Extra Dimensions

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Abstract

We analyze the behaviour of the high-energy scattering amplitude within the brane world scenario in extra dimensions. We argue that contrary to the popular opinion based on the Kaluza-Klein approach, the cross-section does not increase with energy, but changes the slope close to the compactification scale and then decreases like in the 4-dimensional theory. A particular example of the quark-antiquark scattering due to the gluon exchange in the bulk is considered.

1 Introduction

Extra dimensional theories \cite{1} have attracted considerable attention in recent years. Various brane world models provide wide possibilities for phenomenological applications (For review see, e.g. \cite{2}). However, the lack of a consistent field theory in extra dimensions compels one to stick to the Kaluza-Klein approach at the tree level and assume that the string theory cures the problems with divergences at high energy.

One of the immediate consequences of the K-K approach is the increase in the scattering cross section with energy due to the exchange of an infinite tower of K-K modes \cite{3}. To get a finite result, one usually needs to introduce some cutoff, thus making the amplitude essentially cutoff dependent. The cutoff may well have a physical meaning, for instance, representing the width of the brane, but this does not reduce the amount of arbitrariness. Assuming that in the full theory this dependence disappears one can try to look for a qualitative picture.

Yet the increase in the total cross section being attractive from the point of view of observations cannot last forever due to the breaking of unitarity, so at some energy scale this behaviour has to be changed.

We try to analyze this situation following the conception of the low-energy effective theory based on the Wilson approach to the renormalization group \cite{4}. Though it does not allow one to get a fully consistent theory, it has an advantage of possession of the effective low-energy theory without cutoff and has some remarkable properties at the
fixed point [5, 6]. We argue that taking into account the radiative corrections in extra dimensions the behaviour of the cross section is changed and finally leads to decreasing dependence on energy. However, this theory is also valid only up to some energy scale.

To illustrate the main features of an effective theory and to compare it with the K-K approach, we consider below the quark-antiquark scattering due to gluon exchange in the bulk.

2 Quark-antiquark scattering in the brane world

Assume now that the quark fields are localized on the 3-dimensional brane of zero width while gluons can propagate in the bulk of $d$ extra dimensions. Schematically, the situation is shown in Fig.1.

![Figure 1: Quark-antiquark scattering due to gluon exchange in the bulk](image)

The interaction term on the brane looks like

$$g_{4+d} \int d^4 x d^d y \bar{\psi}(x) T^a \gamma^\mu A^a_\mu(x,y) \psi(x) \delta^d(y),$$

where the coupling $g_{4+d} \sim 1/M_c^{d/2}$, $M_c$ being some characteristic scale (scale of compactification of extra dimensions, localization scale, etc.).

At the tree level the corresponding scattering amplitude in the $s$-channel can be written as

$$\text{Amp} \sim \bar{u}(p_1) \gamma^\mu T^a u(p_2) g_{4+d}^2 \frac{(-)^{d/2}}{(2\pi)^d} \int d^d p \frac{1}{s - p^2 + i\varepsilon} \bar{u}(p_3) \gamma^\mu T^a u(p_4),$$

(1)

The Euclidean integral over additional momenta in eq.(1) may be discrete if extra dimensions are compactified. Then one has an infinite sum over the Kaluza-Klein modes with increasing mass.

If $d > 1$, then the integral (or a sum) in eq.(1) is divergent. Usually, one introduces an ultraviolet cutoff $\Lambda$ which may be understood as inclusion of the nonzero width of the brane or some other way. This leads to arbitrariness of predictions and depends on unknown details of localization.
With the cutoff the integral takes the form

\[ I_d = \frac{g^2}{M_c^2} \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \left[ i\pi + \log\left(\frac{s}{\mu^2}\right) \left(\frac{s}{M_c^2}\right)^{d/2-1} + \sum_{k=1}^{\Lambda} \left(\frac{\pi}{k}\right) \left(\frac{\Lambda^2}{M_c^2}\right)^{d/2-1} \right], \tag{2} \]

where we have substituted \( g^2_{4+d} = g^2/M_c^d \) and \( \Omega_d = 2\pi^{d/2}/\Gamma(d/2) \). Equation (2) replaces the usual \( g^2/s \) behaviour in 4 dimensions.

In the leading order one has

\[ I_d = \frac{g^2}{(4\pi)^{d/2} \Gamma(d/2)} \frac{1}{M_c^2} \left[ i\pi + \log\left(\frac{s}{\mu^2}\right) \left(\frac{s}{M_c^2}\right)^{d/2-1} \right]. \tag{3} \]

To glue it to the low energy 4-dimensional theory at \( s = M_c^2 \), we adjust the scale \( \mu \) and finally get

\[ I_d = \frac{g^2}{(4\pi)^{d/2} \Gamma(d/2)} \frac{1}{M_c^2} \left[ \log\left(\frac{s}{\mu^2}\right) + (4\pi)^{d/2} \Gamma(d/2) \right] \left(\frac{s}{M_c^2}\right)^{d/2-1}. \tag{4} \]

It is easy to see that expression (4) leads to increasing cross-section. Indeed, the differential cross-section, which in the 4-dimensional case looks like

\[ \frac{d\sigma^{(4)}}{dt} = \frac{g^4}{4\pi^2 s^2} \left\{ t^2 + u^2 + u^2 - \frac{2u^2}{3ts} \right\}, \tag{5} \]

now takes the form (for \( s, |t| \geq M_c^2 \))

\[ \frac{d\sigma^{(4+d)}}{dt} = \frac{g^4}{4\pi^2 s^2} \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \left\{ \left[ \frac{t^2 + u^2}{M_c^4} \right]^{d/2-2} \log^2\left(\frac{s}{M_c^2}\right) + \frac{2u^2}{3M_c^4} \left[ \frac{s|t|}{M_c^4} \right]^{d/2-1} \log\left(\frac{s}{M_c^2}\right) \log\left(\frac{|t|}{M_c^2}\right) \right\}, \tag{6} \]

where \( M_c^2 \equiv M_c^2 \exp[-(4\pi)^{d/2} \Gamma(d/2)] \) and we have taken into account the sign of \( t \). In the c.m. frame one has \( t = -s/2(1-\cos \theta), u = -s/2(1+\cos \theta) \) and, respectively

\[ \frac{d\sigma^{(4)}}{d\cos \theta} = \frac{g^4}{4\pi^2 s^3} \frac{1}{50s} \left\{ 35 + 8\cos \theta + 10\cos^2 \theta - 8\cos^3 \theta + 3\cos^4 \theta \right\}, \tag{7} \]

\[ \frac{d\sigma^{(4+d)}}{d\cos \theta} = \frac{g^4}{4\pi^2 s^3} \left\{ \frac{(1-\cos \theta)^2}{2} + \frac{(1+\cos \theta)^2}{2} \right\} \log^2\left(\frac{s}{M_c^2}\right) + \frac{(1+\cos \theta)^2}{2} \left[ (1-\cos \theta)^2 \right]^{d/2-2} \log^2\left(\frac{s(1-\cos \theta)}{2M_c^2}\right) \tag{8} \]

\[ + \frac{(1-\cos \theta)^2}{3} \left[ (1-\cos \theta)^2 \right]^{d/2-1} \log\left(\frac{s}{M_c^2}\right) \log\left(\frac{s(1-\cos \theta)}{2M_c^2}\right) \right\} \left(\frac{s}{M_c^2}\right)^d. \]
One can see that the total cross section in the K-K approach increases with energy and has a different angular dependence compared to the 4-dimensional case. Note, however, that eq. (7) is true only in the energy region $M_c < E < \Lambda$ and has to be replaced further by some decreasing function.

3 The low-energy effective theory in extra dimensions

We discuss now the low energy effective theory in $D > 4$. We follow the so-called Wilson renormalization group approach [4].

Consider first the usual gauge theory in $D$ dimensions

$$\mathcal{L} = -\frac{1}{4} Tr F_{\mu\nu}^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu].$$  \hspace{1cm} (8)

The fields and the coupling have the following canonical dimensions:

$$[A] = \frac{D - 2}{2}, \quad [F] = \frac{D}{2}, \quad [g] = 2 - \frac{D}{2}.$$  

Thus, for $D > 4$ the coupling has a negative dimension, which defines some intrinsic scale $M$, and the theory is nonrenormalizable in the usual sense. This means that the higher order operators will inevitably be generated in loop expansion.

Following the Wilson approach one has to write down the renormalization group equation which in principle involves an infinite number of operators of type

$$F^2, \quad g^2(DF)^2, \quad g^4(DDF)^2, \ldots,$$

where the coupling $g \sim 1/M^{d/2}$ is the coupling in the 4+d dimensional theory. Their influence on a full theory is crucial, since they generate an infinite series of counterterms with arbitrary coefficients.

Not pretending to resolve this problem at high energy we go now in the infrared direction, i.e., take $E \ll M$. Then the situation is somewhat simplified. One finds that all the higher dimensional operators are irrelevant being suppressed by powers of $g \sim 1/M^{d/2}$ and may be ignored at the tree level [4, 7]. The only gauge invariant operator which is left is the one of the lowest dimension, namely, $F^2_{\mu\nu}$.

When going to the loop expansion the situation is more tricky. Formally, all the UV divergences are absorbed into the redefinition of the higher order operators which are irrelevant at low energy. As for the finite contributions of these operators to the Green functions one expects that they are also suppressed compared to the leading contribution from the lowest dimensional operator [7].

Having all these in mind and leaving a single operator $F^2_{\mu\nu}$ one can write down the RG equation for the numerical coupling. Since it is dimensionful it is useful to take the

\footnote{In the same way one is left with the $\phi^4$ operator in case of a scalar theory of critical phenomena [4].}
\footnote{We have to admit that we have not seen any rigorous proof and/or any explicit demonstration of this fact in the literature.}
dimensionless combination

\[ \tilde{g} \equiv g\mu^{D/2-2} \Rightarrow [\tilde{g}] = 0, \]

where \( \mu \) is some scale.

In what follows we proceed according to the so-called \( \varepsilon \)-expansion procedure and consider a theory in \( 4 + d \) dimensions performing renormalization in the vicinity of the critical dimension \( D = 4 \). Then the RG equation has a simple form

\[ \frac{d}{d\mu} \tilde{g} = \beta(\tilde{g}, \ldots) = \tilde{g}(\frac{D}{2} - 2 + \gamma_g(\tilde{g})), \] (9)

where the dots stand for all those irrelevant operators that we ignore at low energy. Here \( \gamma_g \) is the usual anomalous dimension calculated within some renormalization scheme. In general \( \gamma_g(g) \), and hence \( \beta(g) \), may depend on \( D \) being finite while \( D \to 4 \). However, in the MS-scheme this dependence is absent and \( \gamma_g \) can be calculated directly in the critical dimension \( D = 4 \).

Equation (9) has a nontrivial fixed point \( \tilde{g} = g^* \). It is the so-called Wilson-Fisher fixed point \[ \mathbb{R} \] usual in scalar theories for \( D < 4 \). However, since in gauge theories, contrary to the scalar case, the beta function in critical dimension is negative, the fixed point of this kind exists for \( D > 4 \) \[ \mathbb{R} \mathbb{R} \]. It is ultraviolet stable, the coupling approaches it when momentum increases.

At the fixed point the theory possesses no scale: the effective coupling becomes dimensionless \[ \mathbb{R} \]. Though the value of the coupling is unknown, the value of the anomalous dimension is known exactly: \( \gamma_g(g^*) = 2 - D/2 \). It is not small but is an integer and includes all the radiative corrections.

In the vicinity of the fixed point the Green functions exhibit a power-like behaviour. In particular the gluon propagator in \( D \) dimensions behaves like

\[ G(p) \sim \frac{1}{(p^2)^{1-\gamma_A}} \] (10)

In the background gauge, which is appropriate for our case since the beta function here is completely defined by the propagator, \( \gamma_A = \gamma_g \). The latter can be calculated by perturbation theory or found from eq.(9) at the fixed point.

At the same time, the low energy effective theory may be used in the limited energy interval \( M_c < E < M \), where \( M_c \) is the compactification scale and \( M > M_c \) is the intrinsic scale of a higher dimensional theory. For higher energies it should be replaced by a full theory.

## 4 Quark-antiquark scattering in the low-energy effective theory approach

Consider now the cross-section of the quark-antiquark scattering in the ET approach. It is given by the same diagrams, but now one has to use the modified gluon propagator due to eq.(10). At high energy the behaviour of the gluon propagator is affected by the
fixed point since it is UV attractive. However, here we have a kind of a controversy: our ET is a low-energy one since we ignore all the higher order operators and we move in the UV direction where their influence is essential.

Not being able to go beyond the scale $M$ within the ET approach we can still solve the RG equations in the vicinity of the fixed point. If the initial value of the coupling is close enough to its fixed point value, the influence of the fixed point is essential even at finite energy.

We proceed in two steps. First we consider the coupling equal to its fixed point value. The advantage is that the anomalous dimension of the gluon propagator is known exactly to all orders of perturbation theory and is equal to $\gamma_A(g^*) = -d/2$. Then the integral (11) becomes

$$I_d = g_{4+d}^2 \frac{(-)^{d/2}}{(2\pi)^d} \int d^dp \frac{(\mu^2)^{d/2}}{(s - p^2 + i\varepsilon)^{1+d/2}}.$$  

(11)

It is convergent now for any $d$ and equals

$$I_d = \frac{g_{4+d}^2(\mu^2)^{d/2}}{(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(1+d/2)} \frac{1}{s}. \quad \text{(12)}$$

Note that for $d = 0$ we have the standard answer $I_0 = g^2/s$.

At the fixed point

$$g_{4+d}^2(\mu^2)^{d/2} \equiv \tilde{g}^2 = (g^*)^2,$$

and we have

$$I_d = \frac{(g^*)^2}{(4\pi)^{d/2}\Gamma(1+d/2)} \frac{1}{s} \quad \text{(13)}$$

This means that we have the same behaviour of the cross section as in 4 dimensions! The only difference is the coefficient that depends on the number of extra dimensions and the coupling at the fixed point. The latter is unknown but can be calculated in perturbation theory.

The result is not surprising since it is given essentially by dimensional analysis: at the fixed point the theory is scale invariant and a naive power counting is valid.

Consider now the initial conditions when the coupling is smaller but close to the fixed point. We have no all loop information now, but bearing in mind that we are still in perturbative regime, we take the one-loop approximation. One has

$$\gamma_A(\tilde{g}^2) = -b\tilde{g}^2, \quad \beta(\tilde{g}^2) = d/2\tilde{g}^2 - b\tilde{g}^4.$$  

Then the solutions of the RG equations are

$$\frac{1}{\tilde{g}^2} = \frac{1}{\tilde{g}_0^2} \left( \frac{p^2}{p_0^2} \right)^{-d/2} - \frac{2b}{d} \left[ \left( \frac{p^2}{p_0^2} \right)^{-d/2} - 1 \right], \quad \text{(14)}$$

$$G_A(p^2) = \frac{1}{p^2} \left[ 1 + \frac{2b}{d} \frac{\tilde{g}_0^2}{\left( \frac{p^2}{p_0^2} \right)^{d/2} - 1} \right]. \quad \text{(15)}$$
where \( p \) is a \((4 + d)\)-dimensional momentum. Replacing now the propagator in eq. (11) by (13) one gets

\[
I_d = g_{4+d}^2 \frac{(-)^{d/2}}{(2\pi)^d} \int d^dp \frac{1}{s - p^2 + i\varepsilon} \frac{1}{1 + \frac{2b}{d} \tilde{g}^2 \left[ \left( \frac{s - p^2}{\mu^2} \right)^{d/2} - 1 \right]}.
\]

This integral is again convergent for any \( d \). To get an explicit expression we calculate it for \( d = 2 \). The result is

\[
I_d = \frac{1}{\mu^2} \frac{\tilde{g}^2}{4\pi(1 - b\tilde{g}^2)} \log\left(1 - \frac{1 - 1/b\tilde{g}^2}{s/\mu^2}\right).
\]

In the limit \( s \to \infty \) one gets

\[
\frac{1}{4\pi b} \frac{1}{s},
\]

which coincides with (13) for \( \tilde{g}^2 = 1/b \) and \( d = 2 \).

To link with the low-energy 4-dimensional theory, we normalize it to \( g^2/s \) at \( s = M_c^2 \). This gives the equation

\[
g^2/M_c^2 = \frac{\tilde{g}^2}{4\pi(1 - b\tilde{g}^2)\mu^2} \log\left(1 - \frac{1 - 1/b\tilde{g}^2}{M_c^2/\mu^2}\right)
\]

This is a single equation for two dimensionless variables: \( \tilde{g}^2 \) and \( \mu^2/M_c^2 \).

Taking, for instance, \( \tilde{g}^2 = 1/2b, b = 7/16\pi^2, g^2 = 2\pi/5 \) (this choice corresponds to QCD with 6 flavours and \( \alpha_s(M_c^2) = 0.10 \)), we get the equation

\[
x = \frac{10}{7} \log(1 + x), \quad x \equiv \mu^2/M_c^2.
\]

The solution is \( x \approx 0.96 \).

The same propagator has to be used in the \( t \)-channel, but again one has to take into account the sign change.

To get the cross-section, one has to replace the propagators in eq. (4) by their expressions in the effective theory (17) when the argument exceeds \( M_c^2 \). The fixed-point cross section is obtained by taking the asymptotic form of the propagator (13).

As an illustration, we show in Fig. 2 the resulting cross section in the case of \( d = 2 \) (or \( D = 6 \)) for \( \theta = \pi/2 \). The compactification scale \( M_c \) is taken to be equal to 1 TeV and the parameters \( b = 7/16\pi^2 \) and \( g^2 = 0.4\pi \) as above. For comparison we show also the 4-dimensional cross section (6) and the K-K one (7).

One can see that the cross section in the effective theory decreases with energy in agreement with unitarity and in contradiction with the K-K approach. It slowly departs from the 4-dimensional cross section starting from \( E = M_c \) by changing the slope and finally approaches the fixed-point solution. Note, however, the range of validity of effective theory.
Figure 2: The cross section of quark-antiquark scattering in pb for $D = 6$ as a function of energy (solid curve, ET). The dashed curve (4D) corresponds to the 4-dimensional case, and the dotted one (FP) is the fixed point limit. The increasing curve is the K-K prediction (K-K).

5 Conclusion

In should be emphasized that the problem we discuss has no solid background due to nonrenormalizability of original higher dimensional theory. Therefore, any attempts to get the low-energy results either within the K-K approach or the ET one inevitably faces the problem of arbitrariness while handling with the UV behaviour. The naive cutoff is not a solution to this problem even if irrelevant operators are suppressed at low energy because the results for various processes essentially depend on the cutoff procedure. From this point of view the effective theory approach based on a fixed point within the Wilsonian renormalization group has an advantage of being universal at low energy.

One can see that the cross-section calculated within the ET approach essentially departs from the one calculated in the leading order of the K-K approach. The reason is that the radiative corrections are now power-like and are not the logs as in 4-dimensional theory. Therefore, their summation drastically changes the behaviour for increasing momenta. Of course, our curves have sense only in the IR region; however, as we have already mentioned, in the vicinity of the fixed point even for small momenta the behaviour is governed by its attraction.

We considered quark-antiquark scattering via gluon exchange as an example to probe the theory at higher dimensions. Apparently the same situation takes place for other reactions and other exchange quanta. This means that the increase in the cross-section with energy which was considered to be the "smoking gun" for extra dimensional theory, probably is not the case. Based on our formulas we conclude that the manifestation of possible extra dimensions is the change of the slope with modified angular dependence,
though the signal is not pronounced and has to be studied in more detail for a particular process.

**Acknowledgements**

We are grateful to V.Rubakov, A.Slavnov, Y.Okada, and O.Teryaev for useful discussions. Financial support from RFBR grant # 02-02-16889 and grant of the Ministry of Science and Technology Policy of the Russian Federation # 2339.2003.2 is kindly acknowledged. D.K.I would like to thank the Theory Group of KEK where this paper was finished for support and hospitality.

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