2-WALKS IN 2-TOUGH 2K₂-FREE GRAPHS

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Abstract. We prove that every 2-tough 2K₂-free graph admits a 2-walk.

1. Introduction

Let G be a graph with vertex set V(G) and edge set E(G). For any v ∈ V(G), let N_G(v) denote the set of neighbors of v in G. A k-walk of G is a spanning closed walk of G visiting each vertex at most k times. An 1-walk of G is a Hamiltonian cycle in G. For an integer m, denote by m * G the multigraph obtained from G by taking each edge m times. Obviously, a k-walk in G, is also a subgraph of (2k) * G. Let Ω(G) denote the number of connected components of G. The following terminology, due to V. Chvátal [4], turned out to be very important in the research of Hamiltonicity.

Definition 1. G is β-tough, for a positive real β, if Ω(G − S) > 1 implies |S| ≥ β · Ω(G − S) for each S ⊂ G.

That is, G is β-tough if G cannot be split into k (with k > 1) components by removing less than kβ vertices. The toughness of G, denoted τ(G), is the maximum value of β for which G is β-tough. Clearly, if g is Hamiltonian, then G is 1-tough, however, the converse is not true. A famous conjecture of V. Chvátal [4], which is still open, claims that the converse holds at least in an approximate sense.

Conjecture 1. There exists a constant β such that every β-tough graph is Hamiltonian.

The concept of a k-walk is a generalization of the concept of a Hamiltonian cycle; in [6] B. Jackson and N.C. Wormald investigated k-walks and obtained the following results.

Theorem 1. Every 1/(k − 2)-tough graph has a k-walk. In particular every 1-tough graph has a 3-walk. □

In [5], it is proved that

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**Theorem 2.** Every 4-tough graph has a 2-walk. □

The following well-known conjecture related to $k$-walks appeared in [6].

**Conjecture 2.** Every $1/(k-1)$-tough graph has a $k$-walk.

Results just mentioned do not apply to the case $k=1$. For some classes of graphs, there are strong results connected toughness and Hamiltonicity (recall that a 1-walk is a Hamiltonian cycle). E.g. in [2], G. Chen, M.S. Jackson, A.E. Kezdy, and J. Lehel proved

**Theorem 3.** Every 18-tough chordal graph is Hamiltonian. □

**Definition 2.** $G$ is said to be split if $V(G)$ can be partitioned into an independent set $I$ and a clique $C$.

For split graphs, we have many beautiful results. E.g. in [7] the following is proved.

**Theorem 4.** Every $3/2$-tough split graph on at least three vertices is Hamiltonian, and this is best possible in the sense that there is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor and $\tau(G_n) \to 3/2$. □

Let us consider a superclass of split graphs, named $2K_2$-free graphs. These are graphs which do not contain an induced copy of $2K_2$, the graph on four vertices consisting of two vertex disjoint edges. Obviously, every split graph is a $2K_2$-free graph. What is more, every co-chordal graph, i.e. the complement of a chordal graph, is also a $2K_2$-free graph. That means that the class of $2K_2$-free graphs is as rich as the class of chordal graphs.

Recently, in [1], H. Broersma, V. Patel and A. Pyatkin proved the following.

**Theorem 5.** Every 25-tough $2K_2$-free graph on at least three vertices is Hamiltonian. □

In this paper, we prove

**Theorem 6.** Every 2-tough $2K_2$-free graph admits a 2-walk.

2. On $2K_2$-free graphs

We present several structural properties of $2K_2$-free graphs which turn out to be very useful in the proof of the main theorem. For a subset $A \subset V(G)$, let $\text{Dom}(X)$ denote the set of vertices dominated by $A$, i.e. $\text{Dom}(A) = A \cup \{y \in V(G); \text{there exists } x \in A \text{ such that } xy \in E(G)\}$. 


The set $A \subset V(G)$ is said to be dominating if $\text{Dom}(A) = V(G)$. A dominating clique of a graph $G$ is a dominating set which induces a complete subgraph in $G$. The following theorem comes from [3].

**Theorem 7.** If $G$ is 2$K_2$-free and the maximum size of cliques $\omega(G) \geq 3$, then $G$ has a dominating clique of size $\omega(G)$. □

Let us consider a generalization of dominating set. The set $A \subset V(G)$ is said to be weakly-dominating if for any edge $v_1v_2 \in G$, we have $v_1 \in \text{Dom}(A)$ or $v_2 \in \text{Dom}(A)$. And, a weakly-dominating clique of a graph $G$ is a weakly-dominating set which induces a complete subgraph in $G$. We say a clique $Q_j$ of $G$ is weakly-dominated by a clique $Q_i$ of $G$, if for any pair $(v_1, v_2)$ of vertices in $Q_j$ one has that at least one of $v_1$ and $v_2$ is adjacent to a vertex in $Q_i$.

Obviously, a dominating set is a weakly-dominating set. Then, similarly to Theorem 7, we get:

**Theorem 8.** If $G$ is 2$K_2$-free and the maximum size of clique $\omega(G) \geq 2$, then $G$ has a weakly-dominating clique of size $\omega(G)$.

**Proof.** If $\omega(G) \geq 3$, by Theorem 7 we get a dominating clique, thus it is also a weakly-dominating clique. If $\omega(G) = 2$, for any clique of size 2, say $Q_0 = v_1v_2$ and any edge $v_3v_4 \in G$, if $v_1v_3, v_1v_4, v_2v_3, v_2v_4 \not\in E(G)$, then $v_1v_2$ and $v_3v_4$ form a 2$K_2$. □

In fact, in a 2$K_2$-free graph $G$, any edge is weakly-dominating. The following observation from [1] is very useful.

**Lemma 1.** A graph $G = (V, E)$ is 2$K_2$-free if and only if for every $A \subset V$ at most one component of the graph $G - A$ contains edges. □

Using these two properties as tools, we can look at 2$K_2$-free graphs more closely.

Given a 2$K_2$-free graph $G$, by Theorem 7 we can find one of its maximum weakly-dominating clique, namely $Q_1$. Obviously, any induced subgraph of a 2$K_2$-free graph is again 2$K_2$-free. Then $G - Q_1$ is also a 2$K_2$-free graph. By Lemma 1, $G - Q_1$ is made up by two parts, one is an independent subset (possibly empty) of $G - Q_1$, denoted by $D_1$, another part is a non-trivial component (possibly empty), denoted by $G_1$, which is also 2$K_2$-free. For the same reason, we can find a maximum weakly-dominating clique in $G_1$, namely $Q_2$, and a non-trivial component in $G_1 - Q_2$, namely $G_2$ and an independent subset in $G_1 - Q_2$, namely $D_2$. Repeating this process, we get

**Theorem 9.** For a 2$K_2$-free graph $G = G_0$, we can find a sequence of cliques $\{Q_i; i = 1, \ldots, k\}$, where $|Q_i| \geq 2$ and $|Q_i| \geq |Q_{i+1}|$, such
that $Q_1$ is a maximum weakly-dominating clique in $G_0$ and $Q_{i+1}$ is a maximum weakly-dominating clique in $G_i \subset G_{i-1} - Q_i, (i = 1, \ldots, k - 1)$. Additionally, $G_i$ is the only non-trivial component in $G_{i-1} - Q_i$. The subset $D_i = G_{i-1} - Q_i - G_i$ is an independent set. We call vertices in $D = \bigcup_{i=1}^{k} D_i$ the first class vertices.

In addition, we get

**Theorem 10.** A $2K_2$-free graph $G$ can be divided into two parts, a "clique tower" $Q = \bigcup_{i=1}^{k} Q_i$ and an independent set $D = \bigcup_{i=1}^{k} D_i$, where $V(G) = V(Q) \cup V(D)$ and $V(Q) \cap V(D) = \emptyset$. Note that we allow any of these sets to be empty.

3. The proof of the main result

In short, the proof is divided into two parts. In the first part, we construct an auxiliary graph $\Gamma$, which is an Eulerian (multi)graph. There are two kinds of edges in $\Gamma$, namely blue edges and red edges. In the second part, with the help of $\Gamma$, we find a subgraph $H$ of $2 \ast G$, which is Eulerian and with all vertices of degree 2 or 4, where $2 \ast G$ denotes the multigraph obtained by doubling each edge of $G$ into a pair of parallel edges. $H$ is the 2-walk we want. In $H$, there are three kinds of edges, namely first-class edges, second-class-edges and third-class edges. The first-class edges are corresponding to the blue edges in $\Gamma$, the second-class edges are corresponding to red edges in $\Gamma$, and the third-class edges are used to make sure $H$ is connected and to adjust the vertex degrees to 2 or 4.

3.1. The proof of Theorem 6. Let $G$ be a 2-tough $2K_2$-free graph. If it has only 2 vertices, this is a trivial case and nothing to prove. So, we assume there are at least three vertices. The sequence of cliques defined in Theorem 9 is $\{Q_1, \ldots, Q_k\}$, where each $Q_i$ is weakly-dominated by $Q_j$, when $j < i$, and $|Q_i| \geq |Q_{i+1}| \geq 2$.

First, if $|Q_1| = 2$, then $G$ is triangle-free, then, by [1, Theorem 4], $G$ is Hamiltonian, and thus $G$ has a 2-walk. Additionally, if there is only one clique $Q_1$ in the sequence, i.e. $k = 1$, then $G$ is split graph. By [7, Theorem 3.3], $G$ is Hamiltonian, and thus has a 2-walk.

Now, we assume $k \geq 2$ and $|Q_1| \geq 3$. The independent set (also called the first-class vertex set) is denoted by $D$, as in Theorem 9. Let $D_0 \subseteq D$. By 2-toughness, the size of the neighbor set $N_G(D_0)$, of $D_0$ in $Q$ is at least $2|D_0|$, i.e. $|N_G(D_0)| \geq 2|D_0|$, otherwise, after deleting $N(D_0)$, there are at least $|D_0|$ components (isolated vertices), since $D_0$ is an independent set. By the polygamous form of Hall theorem, there is a subset $Q'$ of $Q$, $|Q'| = 2|D|$, and each vertex in $D$ is adjacent to
two distinct vertices in $Q'$. That means there is a subset $E' \subset E(G)$ where each $e \in E'$ is connected to a vertex in $D$ and a vertex in $Q'$. Moreover, each vertex in $Q'$ is incident to exactly one edge in $E'$, and each vertex in $D$ is incident to exactly two edges in $E'$. We call the edges in $E'$ the first-class edges in $G$, (also in $H$, we will see that later).

The construction of $\Gamma$. Now, let us construct the auxiliary graph $\Gamma$. First, the vertex set of $\Gamma$, $V(\Gamma) = \{w_1, \ldots, w_k\} \cup D'$, each $w_i$ corresponds to the clique $Q_i$, and each $v_j' \in D'$ corresponds to $v_j \in D$. For each first-class edge $e \in E(G)$, we draw the corresponding edge $e'$ on $\Gamma$, i.e., if $e = v_i v_j \in E(G)$, with $v_i \in D$ and $v_j \in Q_i$, then we have $v_i'w_i \in E(\Gamma)$. We call it a blue edge in $\Gamma$. Note that we allow parallel edges in $\Gamma$, and degrees of vertices count parallel edges with multiplicity.

After drawing the blue edges on $\Gamma$, let us look at the components of $\Gamma$ (some of them may be trivial components, i.e. single points). Obviously, each component has even number of vertices with odd degree.

Case 1. If there is only one component, then pair up all the odd degree vertices by adding edges, that means drawing a maximum matching between these odd degree vertices. And the edges in the matching are called the red edges. Then $\Gamma$ becomes an Eulerian graph, with each vertex incident to at most one red edge. And note that only vertices $\{w_i\}$ are possibly incident to a red edge.

Case 2. If there are at least two components, say $C_1, \ldots, C_n$, where $n \geq 2$, let us select some representative vertices in each of them. For any component, say $C_i$, with some odd degree vertices, select two odd degree vertices, denoted by $v^+_i$, $v^-_i$, from them as representative vertices. Note that they are not in $D'$, since all vertices in $D'$ have degree 2. On the other hand, for any component, say $C_j$ with only even degree vertices, select one vertex, denoted by $v_j$, as representative vertex. Note that we can require $v_j \notin D'$, since each component has at least one vertex $w_i$. For convenience, we denote $v^-_j = v^+_j = v_j$ in this case. Then, we draw $v^+_1 v^-_2$, $v^+_2 v^-_3$, $v^+_3 v^-_4$, $v^+_4 v^-_5$, $v^+_5 v^-_6$, $v^+_6 v^-_7$, $v^+_7 v^-_8$, $\ldots$, $v^+_n v^-_1, v^+_1 v^-_2, v^+_2 v^-_1$ on $\Gamma$ (if $n = 2$ and both components have only even degree vertices, this circle is a pair of parallel edges $v^+_1 v^-_2, v^+_2 v^-_1$), and we also call them the red edges. For the odd degree vertices which are not selected as representative vertices, we pair them up within their components, that means drawing a matching on these vertices with all these matching edges do not cross different components. These matching edges are also called the red edges.
Checking **Cases 1 and 2**, we find that for each vertex $w_i$ one of the following holds.

1. $w_i$ is not incident to any red edges.
2. $w_i$ is incident to one red edge as representative vertex.
3. $w_i$ is incident to two red edges as representative vertex.
4. $w_i$ is incident to one red edge for pairing up.

Now, $\Gamma$ is an Eulerian (multi)graph, since it is a connected graph with all vertices of even degree. Note that all red edges are only incident to vertices \{\[w_i\}\}, and each vertex in $D'$ is incident to exactly two blue edges.

**The construction of $H$.** Now with the help of $\Gamma$ we are going to find a spanning subgraph $H$ of $2 \times G$.

First, the vertex set of $H$, is set to be the vertex set of $G$. So, we use the same notation for these vertices in $H$ and $G$. Additionally, when we talk about any $Q_i$ in $G$, we mean the clique $Q_i$, on the other hand, when we talk about $Q_i$ in $H$, we are talking about the subset of vertices.

**Step 1.** Add all the first-class edges connecting $Q'$ and $D$ to $H$. That means the first-class edges in $G$ and the first-class edges in $H$ are exactly the same set.

**Step 2.** (Finding second-class edges in $H$.)

For each red edge $w_iw_j \in \Gamma$, (we assume $i < j$), we want to find an edge in $G$, with one endpoint in $Q_i$ and another in $Q_j$. This is quite an easy task. By Theorem 9, $Q_j$ is weakly-dominated by $Q_i$, that means, in $G$, some vertices in $Q_j$ are adjacent to some vertices in $Q_i$. Now, we can arbitrarily select one edge, with one endpoint in $Q_j$ while another in $Q_i$, from $G$ into $H$. Such edges are called **second-class edges**.

**Step 3.** (Finding third-class edges in $H$.) The final step in the construction is to find some so-called **third-class edges** for $H$. After adding all the first-class edges and all the second-class edges, let us look at each $Q_i$.

In any $Q_i$, the sum of vertex degrees (corresponding to first-class and second-class edges) is obviously even, because $\Gamma$ is an Eulerian graph, and the sum of vertex degrees in $Q_i$ equals the degree of $w_i$ in $\Gamma$. So, in $Q_i$, there must be even number of odd degree vertices, if any.

In $Q_i$ there is an even number (maybe 0) of odd degree (in $H$) vertices, and by properties of $\Gamma$ and by construction of $H$ carried so far at most one of these vertices, say $v_x$, has degree 3, while the rest of them have degree 1. Now, select a maximum matching, of **third-class edges**, on them. Now $v_x$, if it exist at all, has degree 4, and the remaining vertices in $Q_i$ have degree 0 or 2.
Now we connect all the vertices in $Q_i$ (except $v_x$, if it exists), by a circle of **third-class edges**. More precisely, if $v_x$ is present, and $|Q_i| = 2$, we do nothing, and in case we need to connect just two vertices by a circle we use a pair of parallel edges.

After drawing all these first-, second-, and third-class edges on $H$, we observe that:

1. $H$ is connected multigraph.
2. $H$ is a subgraph of $2 \ast G$.
3. each vertex in $H$ has degree 2 or 4, where the degree of parallel edges is counted with multiplicity.

We conclude that $H$ is Eulerian, and $H$ is a 2-walk in $G$. \hfill \qed

We conclude this paper with the following

**Conjecture 3.** Every 2-tough $2K_2$-free graph on at least three vertices has a 2-trail, i.e. a 2-walk with each edge appearing in the walk at most once.

**References**

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