Research Article

Multivariate Dynamic Sneak-Out Inequalities on Time Scales

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In this study, we extend some “sneak-out” inequalities on time scales for a function depending on more than one parameter. The results are proved by using the induction principle and time scale version of Minkowski inequalities. In seeking applications, these inequalities are discussed in classical, discrete, and quantum calculus.

1. Introduction

Bennett and Grosse-Erdmann [1] introduce the “sneak-out” principle concerned with the equivalence of two series. Bohner and Saker [2] extended the sneak-out principle on time scales and proved some new dynamic sneak-out inequalities and their converses on time scales which, as special cases, with $\mathbb{T} = \mathbb{N}$, contain the discrete inequalities obtained by Bennett and Grosse-Erdmann (Section 6 in [1]). However, the sneak-out principle on time scales can be applied to formulate the corresponding integral inequalities by choosing $\mathbb{T} = \mathbb{R}$. The paper aims to extend the work given by Bohner and Saker in [2] for functions depending on more than one parameter. Some other inequalities, such as Hardy-type, Hardy-Copson, and Copson-Leindler-type inequalities, are also studied for functions of more than one parameter [3–5] via time scales’ calculus. Some literature concerning with time scale can be seen in [6–13].

The paper is organized as follows. Section 2 provides some basics from time scales’ calculus. Section 3 features two dynamic inequalities of the Copson type, which are needed to prove further results. In Section 4, we present sneak-out inequalities on time scales for functions depending on more than one parameter.

2. Preliminaries

A time scale $\mathbb{T}$ as well as close set in $\mathbb{R}$ are nonempty [14, 15]. Some examples of time scales are $\mathbb{Z}$, $\mathbb{R}$, and Cantor set. Assume that $\inf \mathbb{T} = \phi$, where $\phi$ is empty set and $\sup \mathbb{T} = \infty$. A time-scale interval is denoted by $[t_0, \infty) = (t_0, \infty) \cap \mathbb{T}$, for $t_0 \in \mathbb{T}$.

The operators $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ defined by $\sigma(t) = \inf \{b \in \mathbb{T}; b > l \}$ and $\rho: \mathbb{T} \rightarrow \mathbb{T}$ defined by $\rho(t) = \sup \{b \in \mathbb{T}; b < l \}$ are forward as well as backward jump operators, respectively, for $l \in \mathbb{T}$. The point $l \in \mathbb{T}$ is right-scattered if it satisfies $\sigma(t) > l$, and left-scattered if $\rho(t) < l$. The points which are at the same time left-scattered as well as right-scattered are called isolated. Furthermore, the point $l \in \mathbb{T}$ is right-dense if it satisfies $l < \sup \mathbb{T}$ and $\sigma(l) = l$, and left-dense if it satisfies $l > \inf \mathbb{T}$ and $\sigma(l) = l$; furthermore, the point is called dense if it is left-dense as well as right-dense at the same time. A function $\mu: \mathbb{T} \rightarrow [0, \infty)$, defined by $\mu(l) = \sigma(l) - l$, is called the graininess function.

If a function $g: \mathbb{T} \rightarrow \mathbb{R}$ is continuous at all right-dense points, the left-hand limits exist and are finite at left-dense points in $\mathbb{T}$; then, it is right-dense continuous (rd-continuous) on $\mathbb{T}$. The set denoted by $C_{rd}(\mathbb{T})$ contain all rd-continuous functions on $\mathbb{T}$.
Consider a function $\beta: \mathbb{T} \rightarrow \mathbb{R}$, and define the number $\beta^\lambda (c)$ if it exists with the property that, for given $\varepsilon > 0$, there is a neighborhood $U$ of $\zeta$ which satisfies
\[
|\beta(\sigma(c)) - \beta(r) - \beta^\lambda (c)(\beta(c) - r)| \leq \varepsilon|\sigma(c) - r|, \quad \forall r \in U,
\]
then $\beta^\lambda (c)$ is delta derivative of function $\beta(c)$ at $c \in \mathbb{T}$.

Notation: $\zeta^\lambda (c) = \zeta(\sigma(c))$ for any function $\zeta: \mathbb{T} \rightarrow \mathbb{R}$.

(1) Product and quotient rule for delta derivative (Theorem 1.20 in [14]): assume $\zeta, \eta: \mathbb{T} \rightarrow \mathbb{R}$ are differentiable; then,

(i) $(\zeta \eta)^\Delta = \zeta^\Delta \eta + \zeta \eta^\Delta$,

(ii) $\left( \begin{array}{c} \zeta^\Delta \\ \eta^\Delta \end{array} \right) = \left( \begin{array}{c} \zeta^\Delta (\zeta) \eta(\zeta) - \zeta(\eta)^\Delta (\zeta) \\ \eta(\zeta) \eta^\Delta (\zeta) \end{array} \right)$,

if $\zeta(\zeta) \neq 0, \zeta, \eta \in \mathbb{T}$.

\[
\left\{ \int_a^b |h(t)||f(t)|^p \right\}^{(1/p)} \leq \left\{ \int_a^b |h(t)||f(t)|^p \Delta t \right\}^{(1/p)} + \left\{ \int_a^b |h(t)||g(t)|^p \Delta t \right\}^{(1/p)},
\]

where $p > 1$ and $a, b, t \in \mathbb{T}$.

(4) Fubini’s theorem [16]: let there exist two time scales’ measure spaces $(\nu, M, \phi_\lambda)$ and $(\psi, N, \varphi_\lambda)$ which have finite dimensions. If $\eta: \nu \rightarrow \mathbb{R}$ is a $\phi_\lambda$-integrable function and the function $\psi_i(l) = \int_{\nu} \eta(l,m) \Delta m$ exists for almost every $l \in \nu$, then $\psi_i$ is $\varphi_\lambda$ integrable on $\nu$, $\psi_2$ is $\phi_\lambda$ integrable on $\nu$, and
\[
\int_{\nu} \Delta l \int_{\nu} \eta(l,m) \Delta m = \int_{\nu} \Delta m \int_{\nu} \eta(l,m) \Delta l.
\]

Notation:
\[
\frac{\partial}{\Delta \tau_k} g(t_1, \ldots, t_k, \ldots, t_n) = g^\Delta (t_1, \ldots, t_k, \ldots, t_n), \quad 1 \leq k \leq n.
\]

Some preliminary inequalities [2]: suppose $g: \mathbb{T} \rightarrow \mathbb{R}$ is differentiable. Let $\beta \in \mathbb{R}$, if $g^\beta$ is monotone, i.e., either always negative or always positive, then
\[
\beta g^\Delta (g^{\beta-1})^\sigma \leq (g^\beta)^\lambda \leq \beta g^\Delta (g^{\beta-1})^\sigma, \quad \text{if } 0 \leq \beta \leq 1,
\]
\[
\beta g^\Delta (g^{\beta-1}) \leq (g^\beta)^\lambda \leq \beta g^\Delta (g^{\beta-1})^\sigma, \quad \text{if } \beta \geq 1,
\]
and if $g^{\Delta}$ is positive, then
(2) Integration by parts formula (Theorem 1.77 in [14]): for two delta differentiable functions $g, h: \mathbb{T} \rightarrow \mathbb{R}$, and $\zeta, a, m \in \mathbb{T}$, we have
\[
\int_a^m g(\zeta) h^\lambda (\zeta) \Delta \zeta = g(\zeta) h^{\lambda} (\zeta)_{a}^{m} - \int_a^m g^\Delta (\zeta) h^{\Delta} (\zeta) \Delta \zeta.
\]

(3) Minkowski inequality (Theorem 6.16 in [14]): for three rd-continuous functions $f: \mathbb{T} \rightarrow \mathbb{R}$, $g: \mathbb{T} \rightarrow \mathbb{R}$, and $h: \mathbb{T} \rightarrow \mathbb{R}$, we have

\[
(g^{\beta})^\Delta \leq g^\Delta (g^{\beta-1})^\sigma, \quad \text{if } 0 \leq \beta \leq 1,
\]
\[
(g^{\beta})^\Delta \leq g^\Delta (g^{\beta-1})^\sigma, \quad \text{if } \beta \geq 1.
\]

3. Dynamic Copson-Type Inequalities for Finite Numbers of Parameters
We assume throughout that all the functions are nonnegative and the integrals considered exist. For $h \in \mathbb{N}$, $i \in \{1, 2, \ldots, h\}$, let $\mathbb{T}_i$ be time scales.

Presume 1:
\[
H_1 = \begin{cases} 
\text{Sup} \mathbb{T}_i = \infty, & b_i \in (0, \infty), \\
\nu_i: \mathbb{T}_i \rightarrow \mathbb{R}^+ \text{ is rd-continuous}, \\
A_i(t_i) = \int_{b_i}^{t_i} \nu(s_i) \Delta s_i, \quad \text{for } t_i \in \mathbb{T}_i.
\end{cases}
\]

Theorem 1. Assume $H_1$. Suppose $g: \mathbb{T}_1 \times \cdots \times \mathbb{T}_h \rightarrow \mathbb{R}^+$ is such that
\[
\phi(t_1, \ldots, t_h) = \int_{t_1}^{\infty} \cdots \int_{t_h}^{\infty} \prod_{i=1}^{h} \nu_i(s_i) g(s_1, \ldots, s_h) \Delta s_h \cdots \Delta s_1,
\]
is well defined and $m \geq 1$. Then,
\[
\int_{b_1}^{\infty} \cdots \int_{b_h}^{\infty} \prod_{i=1}^{h} v_i(\tau_i) \phi^m(\tau_1, \ldots, \tau_h) \Delta \tau_h \cdots \Delta \tau_1 \leq (m)_{1}^\infty \int_{b_1}^{\infty} \cdots \int_{b_h}^{\infty} \prod_{i=1}^{h} v_i(\tau_i) g^m(\tau_1, \ldots, \tau_h) \Delta \tau_h \cdots \Delta \tau_1.
\] (12)

**Theorem 2.** Assume \( H_1 \). Suppose \( g: \mathbb{T}_1 \times \cdots \times \mathbb{T}_h \longrightarrow \mathbb{R} \) is such that

\[
\phi(\xi_1, \ldots, \xi_h) = \int_{c_1}^{\infty} \cdots \int_{c_h}^{\infty} \prod_{i=1}^{h} v_i(s_i) g(s_1, \ldots, s_h) \Delta s_h \cdots \Delta s_1, \quad (\xi_1, \ldots, \xi_h) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_h, \] (13)

is well defined. Let \( m \geq 1 \) and \( 0 \leq c_i < 1 \). Then,

\[
\int_{a_1}^{\infty} \cdots \int_{a_h}^{\infty} \prod_{i=1}^{h} A_i^m(\sigma_i(s_i)) \Delta \xi_i \cdots \Delta \xi_1 \leq \prod_{i=1}^{h} \left( \frac{m}{1 - c_i} \right)^{m} \int_{a_1}^{\infty} \cdots \int_{a_h}^{\infty} \prod_{i=1}^{h} v_i(\xi_i) A_i^{m-c_i}(\sigma_i(\xi_i)) g^m(\xi_1, \ldots, \xi_h) \Delta \xi_h \cdots \Delta \xi_1. \] (14)

Proofs of Theorem 1 and Theorem 2 are after sneak-out inequalities.

**4. Dynamic Sneak-Out Inequalities for Finite Numbers of Parameters**

Let \( i, j, r \in \{1, \ldots, h\} \) and \((i_1, \ldots, i_h) = (j_1, \ldots, j_h) = (1, \ldots, h)\).

Presume 2:

\[
\begin{aligned}
H_2 &= \left\{ \begin{array}{l}
x: \mathbb{T}_1 \times \cdots \times \mathbb{T}_h \longrightarrow \mathbb{R} \text{ is rd \dash continuous}, \\
y(\tau_1, \ldots, \tau_h) = \int_{\tau_1}^{\infty} \cdots \int_{\tau_h}^{\infty} x(s_1, \ldots, s_h) \Delta s_h \cdots \Delta s_1, \\
\psi(\tau_1, \ldots, \tau_h) = \int_{\tau_1}^{\infty} \cdots \int_{\tau_h}^{\infty} \prod_{i=1}^{h} A_i^{\sigma_i(s_i)} x(s_1, \ldots, s_h) \Delta s_h \cdots \Delta s_1.
\end{array} \right.
\end{aligned}
\] (15)

**Lemma 1.** Let \( \mathbb{T}_i \) be the time scales for \( i \in \{1, 2, \ldots, h\} \), under \( H_1 \) and \( H_2 \), and we have

\[
\psi(\tau_i, \ldots, \tau_h) \leq \sum_{1 \leq h < \cdots < j < h} \left[ \left( \prod_{m=1}^{r} A_{\tau_m}^{\sigma_m(s_m)} \right) \prod_{m=1}^{r} \prod_{t_{k_m} < \cdots < t_{k_1} \leq s_m} \frac{\Delta}{\prod_{t_{k_1} \leq \cdots \leq t_{k_r} < \cdots < t_{k_1}}} x(s_m) \right],
\] (16)

Proof. For \( h = 1 \), (16) is true by Theorems 4.1 in [2], i.e.,

\[
\psi(\tau_1) \leq A_1^{\sigma_1(\tau_1)} y(\tau_1) + \alpha_1 \int_{\tau_1}^{\infty} \prod_{m=1}^{r} A_{\tau_m}^{\sigma_m(s_m)} \Delta s_m.
\] (17)
Suppose (16) is true for $1 \leq h \leq p$. To prove for $h = p + 1$, by using $H_2$, we have defined as

$$
\psi(t_1, \ldots, t_{p+1}) = \int_{t_1}^{\infty} \cdots \int_{t_p}^{\infty} \prod_{k=1}^{p} A_k^a_1\left(\sigma_k(s_k)\right) \times \left\{ \int_{t_{p+1}}^{\infty} A_{p+1}^a_1 \left(\sigma_{p+1}(s_{p+1})\right) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_1 \right\} \Delta s_p \ldots \Delta s_1. \quad (18)
$$

Denote

$$
Z_{p+1} = \int_{t_{p+1}}^{\infty} A_{p+1}^a_1 \left(\sigma_{p+1}(s_{p+1})\right) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_{p+1}.
$$

(19)

Use integration by parts' formula (3) in (19) to obtain

$$
Z_{p+1} = -A_{p+1}^a_1(s_{p+1}) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \bigg|_{t_{p+1}}^{\infty} - \int_{t_{p+1}}^{\infty} \frac{\partial}{\partial \Delta t_{p+1}} \left(A_{p+1}^a_1(s_{p+1})\right) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_{p+1}.
$$

(20)

Use the right-hand side part of inequality (8) with $A_{p+1}^a_1 \leq A_{p+1}^a_1$ in (20)

$$
Z_{p+1} \leq A_{p+1}^a_1(s_{p+1}) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}, t_{p+1}) + \alpha_{p+1} \int_{t_{p+1}}^{\infty} v_{p+1}(s_{p+1}) A_{p+1}^a_1(s_{p+1}) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_{p+1}.
$$

(21)

Substitute (21) in (18):

$$
\psi(t_1, \ldots, t_{p+1}) = \int_{t_1}^{\infty} \cdots \int_{t_p}^{\infty} \prod_{k=1}^{p} A_k^a_1\left(\sigma_k(s_k)\right) \times \left\{ \int_{t_{p+1}}^{\infty} A_{p+1}^a_1 \left(\sigma_{p+1}(s_{p+1})\right) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_1 \right\} \Delta s_p \ldots \Delta s_1
$$

$$
+ \alpha_{p+1} \int_{t_{p+1}}^{\infty} \cdots \int_{t_p}^{\infty} \prod_{k=1}^{p} A_k^a_1\left(\sigma_k(s_k)\right) \times \left\{ \int_{t_{p+1}}^{\infty} v_{p+1}(s_{p+1}) A_{p+1}^a_1(s_{p+1}) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_1 \right\} \Delta s_p \ldots \Delta s_1.
$$

(22)

Use (5) "p times" in second term of (22):

$$
\psi(t_1, \ldots, t_{p+1}) \leq A_{p+1}^a_1(s_{p+1}) \left(\sigma_{p+1}(s_{p+1})\right) \int_{t_1}^{\infty} \cdots \int_{t_p}^{\infty} \prod_{k=1}^{p} A_k^a_1\left(\sigma_k(s_k)\right) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_1 \ldots \Delta s_1
$$

$$
+ \alpha_{p+1} \int_{t_{p+1}}^{\infty} \cdots \int_{t_p}^{\infty} \prod_{k=1}^{p} A_k^a_1\left(\sigma_k(s_k)\right) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_1 \ldots \Delta s_1
$$

$$
\times \left\{ \int_{t_{p+1}}^{\infty} v_{p+1}(s_{p+1}) A_{p+1}^a_1(s_{p+1}) y^{\Delta_1-\Delta_{p+1}}(s_1, \ldots, s_{p+1}) \Delta s_1 \right\} \Delta s_p \ldots \Delta s_1.
$$

(23)
Use induction hypothesis for $\psi(\tau_{i_1}, \ldots, \tau_{i_p})$ with fix $\tau_{ip_1}, s_{ip_1} \in T_{ip_1}$ (instead for $\psi(\tau_1, ..., \tau_j)$) in (23) and (25) to obtain

$$
\psi(\tau_1, ..., \tau_{ip_1}) \leq A^{a_{ip_1}}(\sigma_{ip_1}(\tau_{ip_1}))
$$

$$
\times \left[ \prod_{k=1}^{p} A^{a_k}(\sigma_k(\tau_{ik})) y(\tau_{i_1}, ..., \tau_{ip_1}) + \sum_{1 \leq j_i < < j_p \leq r} \left( \prod_{m=1}^{r} \alpha_{jm} \right) \prod_{i_u=1}^{p-r} A^{a_{i_u}}(\sigma_{i_u}(\tau_{i_u})) \right]
$$

$$
\times \int_{\cap_{1 \leq j_i < < j_p \leq r} \Delta s_{ip_1}} \prod_{m=1}^{r} A^{a_{jm}}(\sigma_{jm}(s_{jm})) u_{jm}(s_{jm}) y(\tau_{i_1}, ..., \tau_{ip_1}, s_{j_1}, ..., s_{j_p}) 
$$

$$
\times \int_{\cap_{1 \leq j_i < < j_p \leq r} \Delta s_{ip_1}} \prod_{m=1}^{r} A^{a_{jm}}(\sigma_{jm}(s_{jm})) u_{jm}(s_{jm}) y(\tau_{i_1}, ..., \tau_{ip_1}, s_{j_1}, ..., s_{j_p}) \Delta s_{ip_1}.
$$

(25)

By applying (5) "p times" on (25) and making simplification, we obtain

$$
= \prod_{k=1}^{p} A^{a_k}(\sigma_k(\tau_{ik}))
$$

$$
\times \left[ A^{a_{ip_1}}(\sigma_{ip_1}(\tau_{ip_1})) y(\tau_{i_1}, ..., \tau_{ip_1}) + \alpha_{ip_1} \int_{\tau_{ip_1}}^{\Delta s_{ip_1}} y(\tau_{i_1}, ..., \tau_{ip_1}, s_{ip_1}) \Delta s_{ip_1} \right]
$$

$$
+ \sum_{1 \leq j_i < < j_p \leq r} \left( \prod_{m=1}^{r} \alpha_{jm} \right) \prod_{i_u=1}^{p-r} A^{a_{i_u}}(\sigma_{i_u}(\tau_{i_u})) \int_{\cap_{1 \leq j_i < < j_p \leq r} \Delta s_{ip_1}} \prod_{m=1}^{r} A^{a_{jm}}(\sigma_{jm}(s_{jm})) u_{jm}(s_{jm})
$$

$$
\times \left[ A^{a_{ip_1}}(\sigma_{ip_1}(\tau_{ip_1})) y(\tau_{i_1}, ..., \tau_{ip_1}, s_{j_i}, ..., s_{j_p}) + \alpha_{ip_1} \int_{\tau_{ip_1}}^{\Delta s_{ip_1}} u_{ip_1}(s_{ip_1}) A^{a_{ip_1}}(\sigma_{ip_1}(s_{ip_1})) y(\tau_{i_1}, ..., \tau_{ip_1}, s_{j_1}, ..., s_{j_p}) \Delta s_{ip_1} \right]
$$

(26)
Hence, by using (17) for \( \tau_{p+1} \in \Theta_{p+1} \), we obtain

\[
\psi(\tau_1, \ldots, \tau_{p+1}) \leq \prod_{k=1}^{p+1} A_n^{\alpha_k}(\sigma_k(\tau_k)) \frac{\psi}{\tau_1, \ldots, \tau_{p+1}} + \sum_{1 \leq j_1, \ldots, j_p \leq p+1} \left[ \prod_{m=1}^r \alpha_{m, j_m} \right] \prod_{m=1}^{p+1-r} A_n^{\alpha_m}(\sigma_m(\tau_m)) \times \prod_{1 \leq i_1 < \cdots < i_{p+1}, j_m \leq p+1} \Delta \psi_{j_m, i_m}.
\]

(27)

Thus, by mathematical induction, (16) holds for all \( h \in \mathbb{N} \), which completes the proof. \( \square \)

**Remark.** If we chose \( h = 3 \) in Lemma 1, then (16) becomes the following inequality:

\[
\psi(\tau_1, \tau_2, \tau_3) \leq A_n^{\alpha_1}(\sigma_1(\tau_1)) A_n^{\alpha_2}(\sigma_2(\tau_2)) A_n^{\alpha_3}(\sigma_3(\tau_3)) y(\tau_1, \tau_2, \tau_3) \\
+ a_1 A_n^{\alpha_2}(\sigma_2(\tau_2)) A_n^{\alpha_3}(\sigma_3(\tau_3)) \int_{\tau_2}^{\infty} A_n^{\alpha_3-1}(\sigma(\tau_1)) a_1(\tau_1) y(\tau_1, \tau_2, \tau_3) \Delta s_1 \\
+ a_2 A_n^{\alpha_1}(\sigma_1(\tau_1)) A_n^{\alpha_3}(\sigma_3(\tau_3)) \int_{\tau_3}^{\infty} A_n^{\alpha_3-1}(\sigma(\tau_2)) a_2(\tau_2) y(\tau_1, \tau_2, \tau_3) \Delta s_2 \\
+ a_3 A_n^{\alpha_1}(\sigma_1(\tau_1)) A_n^{\alpha_2}(\sigma_2(\tau_2)) \int_{\tau_2}^{\infty} A_n^{\alpha_2-1}(\sigma(\tau_3)) a_3(\tau_3) y(\tau_1, \tau_2, \tau_3) \Delta s_3 \\
+ a_1 a_2 A_n^{\alpha_1}(\sigma_1(\tau_1)) A_n^{\alpha_2}(\sigma_2(\tau_2)) \int_{\tau_1}^{\infty} A_n^{\alpha_2-1}(\sigma(\tau_3)) a_2(\tau_2) a_1(\tau_1) y(\tau_1, \tau_2, \tau_3) \Delta s_1 \Delta s_2 \\
+ a_1 a_3 A_n^{\alpha_1}(\sigma_1(\tau_1)) A_n^{\alpha_3}(\sigma_3(\tau_3)) \int_{\tau_2}^{\infty} A_n^{\alpha_3-1}(\sigma(\tau_1)) a_3(\tau_3) a_1(\tau_1) y(\tau_1, \tau_2, \tau_3) \Delta s_1 \Delta s_3 \\
+ a_2 a_3 A_n^{\alpha_2}(\sigma_2(\tau_2)) A_n^{\alpha_3}(\sigma_3(\tau_3)) \int_{\tau_1}^{\infty} A_n^{\alpha_3-1}(\sigma(\tau_2)) a_3(\tau_3) a_2(\tau_2) y(\tau_1, \tau_2, \tau_3) \Delta s_2 \Delta s_3 \\
+ a_1 a_2 a_3 A_n^{\alpha_1}(\sigma_1(\tau_1)) A_n^{\alpha_2}(\sigma_2(\tau_2)) A_n^{\alpha_3}(\sigma_3(\tau_3)) \int_{\tau_1}^{\infty} A_n^{\alpha_3-1}(\sigma(\tau_2)) a_3(\tau_3) a_2(\tau_2) a_1(\tau_1) y(\tau_1, \tau_2, \tau_3) \Delta s_1 \Delta s_2 \Delta s_3.
\]

(28)

**Theorem 3.** Assume \( H_1, H_2 \), and \( l, \alpha_i \geq 1, \forall i \in \{1, 2, \ldots, h\}, h \in \mathbb{N} \). Then,

\[
\int_{\prod_{m=1}^h b_{j_m} \leq j_m \leq j_m} \prod_{m=1}^h \int_{\tau_{j_m}}^{\infty} \prod_{l=1}^{m-1} \alpha_{j_l} \Delta \tau_{j_m} \psi(\tau_{j_1}, \ldots, \tau_{j_m}) \Delta \tau_{j_m} \leq \left( 1 + \sum_{1 \leq j_1 < \cdots < j_h \leq h} \left( \prod_{m=1}^r \alpha_{j_m} \right)^l \right)
\]

(29)

Using (27) in (30) for \( h = p + 1 \),

\[
\int_{\prod_{m=1}^{p+1} b_{j_m} \leq j_m \leq j_m} \prod_{m=1}^{p+1} \int_{\tau_{j_m}}^{\infty} \prod_{l=1}^{m-1} \alpha_{j_l} \Delta \tau_{j_m} \psi(\tau_{j_1}, \ldots, \tau_{j_{p+1}}) \Delta \tau_{j_m}.
\]

(30)
Apply Minkowski's inequality (4) on (31) to obtain

$$\left\{ \int_{\prod_{j_{m}=1}^{p+1} \prod_{j_{n}=1}^{\infty}} v_{j_{m}}(\tau_{j_{m}}) \psi\left(\tau_{j_{1}}, \ldots, \tau_{r_{j_{p+1}}}, \Delta \tau_{j_{m}}\right) \right\}_{(1)} \leq \left\{ \int_{\prod_{j_{m}=1}^{p+1} \prod_{j_{n}=1}^{\infty}} v_{j_{m}}(\tau_{j_{m}}) \int_{\prod_{j_{m}=1}^{p+1} \prod_{j_{n}=1}^{\infty}} \Delta \tau_{j_{m}} \right\}_{(1)}$$

(31)

where

$$I_{p+1} = \prod_{k=1}^{p+1} A_{j_{k}}^{m_{k}}(\sigma_{j_{k}}(\tau_{j_{k}})) y(\tau_{j_{1}}, \ldots, \tau_{r_{j_{p+1}}}) + \sum_{1 \leq j_{1} < \ldots < j_{r_{j_{p+1}}}} \left( \int_{\prod_{j_{m}=1}^{p+1} \prod_{j_{n}=1}^{\infty}} r_{j_{m}} A_{j_{m}}^{m_{j_{m}}-1}(\sigma_{j_{m}}(s_{j_{m}})) v_{j_{m}}(s_{j_{m}}) y(\tau_{j_{1}}, \ldots, \tau_{r_{j_{p+1}}}, s_{j_{1}}, \ldots, s_{j_{r_{j_{p+1}}}}) \right)_{(32)}$$

and one has that

$$\bar{T}_{p+1} = \prod_{m=1}^{p+1} A_{j_{m}}^{m_{j_{m}}}(\sigma_{j_{m}}(\tau_{j_{m}})) \times \int_{\prod_{j_{m}=1}^{p+1} \prod_{j_{n}=1}^{\infty}} r_{j_{m}} A_{j_{m}}^{m_{j_{m}}-1}(\sigma_{j_{m}}(s_{j_{m}})) v_{j_{m}}(s_{j_{m}}) y(\tau_{j_{1}}, \ldots, \tau_{r_{j_{p+1}}}, s_{j_{1}}, \ldots, s_{j_{r_{j_{p+1}}}}) \Delta s$$

(34)

Denote

$$W_{p+1} = \int_{\prod_{j_{m}=1}^{p+1} \prod_{j_{n}=1}^{\infty}} v_{j_{m}}(\tau_{j_{m}}) \bar{T}_{p+1} \Delta \tau_{j_{m}}$$

(35)
\[ W_{p+1} = \int_{x_n \in B_{x_m}} \prod_{m=1}^{p+1} \prod_{h=1}^{n} v_j(\tau_{j,m}) \prod_{p+1-r}^{p+1} A_{j,m}^{\alpha}(\sigma_{j,m}(\tau_{j,m})) \]

\[ \times \left\{ \int_{x_n \in B_{x_m}} \prod_{m=1}^{r} v_j(\tau_{j,m}) A_{j,m}^{\alpha}(\sigma_{j,m}(\tau_{j,m})) y(\tau_{i_1}, \ldots, \tau_{i_{p+1}}, s_{j_1}, \ldots, s_{j_r}) \Delta s \right\} \times \Delta \prod_{m=1}^{p+1} \tau_{j,m}. \]  

(36)

Use Theorem 1 in (36) by taking \( g(s_{j_1}, \ldots, s_{j_r}) = \prod_{j=1}^r A_{j,m}^{\alpha}(\sigma_{j,m}(s_{j,m})) y(\tau_{i_1}, \ldots, \tau_{i_{p+1}}, s_{j_1}, \ldots, s_{j_r}) \) to obtain

\[ W_{p+1} \leq (l) \int_{x_n \in B_{x_m}} \prod_{m=1}^{p+1} v_j(\tau_{j,m}) A_{j,m}^{\alpha}(\sigma_{j,m}(\tau_{j,m})) y(\tau_{j_1}, \ldots, \tau_{j_r}) \Delta \prod_{m=1}^{p+1} \tau_{j,m}. \]  

(37)

Substitute (37) in (33) and take power \( l \) on both sides to obtain

\[ \int_{x_n \in B_{x_m}} \prod_{m=1}^{p+1} v_j(\tau_{j,m}) \psi(\tau_{j_1}, \ldots, \tau_{j_r}) \Delta \prod_{m=1}^{p+1} v_j(\tau_{j,m}) A_{j,m}^{\alpha}(\sigma_{j,m}(\tau_{j,m})) y(\tau_{j_1}, \ldots, \tau_{j_r}) \Delta \prod_{m=1}^{p+1} \tau_{j,m}. \]  

(38)

Thus, by mathematical induction, (29) holds for all \( \forall j \in \mathbb{N} \).

Example 1. Let \( T_j = \mathbb{N} \) and \( b_{j,m} = 1, \forall j, m \in \{1, \ldots, h\} \) and \( n_{j,m}, h \in \mathbb{N} \). In this case, (29) in Theorem 3 takes the form

\[ \sum_{k_{j_1}=1}^{\infty} \cdots \sum_{k_{j_r}=1}^{\infty} \prod_{m=1}^{h} v_j(k_{j,m}) \left( \sum_{n_{j_1}=n_{j_1}} \cdots \sum_{n_{j_r}=n_{j_r}} \prod_{m=1}^{h} A_{j,m}^{\alpha}(n_{j,m}+1) x(n_{j_1}, \ldots, n_{j_r}) \right)^l \]

\[ \leq \left( 1 + \sum_{1 \leq j_1 < \cdots < j_r \leq p+1} f(\prod_{m=1}^{h} A_{j,m}^{\alpha}(k_{j,m})) \right)^l \sum_{k_{j_1}=1}^{\infty} \cdots \sum_{k_{j_r}=1}^{\infty} v_j(k_{j,m}) A_{j,m}^{\alpha}(k_{j,m}+1) \]

(39)

Note that (39) is extension of Example 4.4 in [2].

Example 2. Let \( T_j = \mathbb{R} \) \( \forall j, m \in \{1, \ldots, h\} \) in Theorem 3. In this case, (29) takes the form

\[ A_{j,m}(k_{j,m}) = \sum_{n_{j,m}=1}^{k_{j,m}+1} v_j(n_{j,m}), \quad k_{j,m} \in \mathbb{N}. \]  

(40)
\begin{align}
\int_0^{\infty} \prod_{m=1}^{b} h_j \left( \int_0^{\infty} \prod_{m=1}^{b} A_{j_m}^{a_{j_m}}(s_{j_m}) x(s_{j_m}, \ldots, s_{j_m}) d s_{j_m} \right) \prod_{m=1}^{b} \tau_{j_m} \\
\leq \left( 1 + \sum_{1 \leq j_{1} \leq \cdots \leq j_{r} \leq \tau_{1}+1} \right) \int_0^{\infty} \prod_{m=1}^{b} h_j \left( \int_0^{\infty} \prod_{m=1}^{b} v_{j_m}(\tau_{j_m}) A_{j_m}^{a_{j_m}}(\tau_{j_m}) x(s_{j_m}, \ldots, s_{j_m}) d s_{j_m} \right) \prod_{m=1}^{b} \tau_{j_m},
\end{align}

(41)

where

\begin{align}
A_{j_m}(\tau_{j_m}) &= \int_{b_j}^{\tau_{j_m}} v_{j_m}(s_{j_m}) \prod_{m=1}^{b} s_{j_m}, \quad \tau_{j_m} \in \mathbb{R}.
\end{align}

Example 3. Let \( T_i = q_{i_1}^{h_i}, q_{i_r}^{h_i} > 1 \), and \( \forall j, m \in \{1, \ldots, h\} \), in Theorem 3. In this case, (29) takes the form

\begin{align}
\int_0^{\infty} \prod_{m=1}^{b} h_j \left( \int_0^{\infty} \prod_{m=1}^{b} A_{j_m}^{a_{j_m}}(s_{j_m}) x(s_{j_m}, \ldots, s_{j_m}) d s_{j_m} \right) \prod_{m=1}^{b} \tau_{j_m} \\
\leq \left( 1 + \sum_{1 \leq j_{1} \leq \cdots \leq j_{r} \leq \tau_{1}+1} \right) \int_0^{\infty} \prod_{m=1}^{b} h_j \left( \int_0^{\infty} \prod_{m=1}^{b} v_{j_m}(\tau_{j_m}) A_{j_m}^{a_{j_m}}(\tau_{j_m}) x(s_{j_m}, \ldots, s_{j_m}) d s_{j_m} \right) \prod_{m=1}^{b} \tau_{j_m},
\end{align}

(43)

where

\begin{align}
A_{j_m}(q_{j_m}^{m}) &= \sum_{n_{j_m}=1}^{k_{j_m}-1} v_{j_m}(q_{j_m}^{m}) q_{j_m}^{n_{j_m}}(q_{j_m}^{m}-1), \quad k_{j_m} \in \mathbb{N}_0.
\end{align}

(44)

\begin{align}
\gamma(\tau_{i_1}, \ldots, \tau_{i_r}) &\leq \int_{k=1}^{h} A_{j_k}^{a_{j_k}}(\sigma_{j_k}(\tau_{i_k})) \psi(\tau_{i_1}, \ldots, \tau_{i_r}) + \sum_{1 \leq j_{1} \leq \cdots \leq j_{r} \leq h} \left( \int_{m=1}^{h} A_{j_m}^{a_{j_m}}(\sigma_{j_m}(\tau_{i_m})) \right) \\
&\times \prod_{1 \leq j_{1} \leq \cdots \leq j_{r}} \prod_{m=1}^{r} A_{j_m}^{a_{j_m}}(\sigma_{j_m}(s_{j_m})) v_{j_m}(s_{j_m}) \psi(\tau_{i_1}, \ldots, \tau_{i_r}, s_{j_1}, \ldots, s_{j_r}) \prod_{1 \leq j_{1} \leq \cdots \leq j_{r}} \Delta s_{j_m},
\end{align}

(45)
Proof. For \( h = 1 \), (45) is true by Theorems 4.6 in [2], i.e.,
\[
y(\tau_1, \ldots, \tau_p) = \int_{\tau_1}^{\infty} \cdots \int_{\tau_p}^{\infty} \prod_{k=1}^{p} A_{\Delta, k}^{-\alpha_k, \alpha_k}(s_k) \times \left\{ \int_{\tau_p}^{\infty} A_{\Delta, p}^{-\alpha_p, \alpha_p}(s_p) \psi^{1-\Delta, p}(s_1, \ldots, s_p) \Delta s_p \right\} \Delta s_{p-1} \cdots \Delta s_1.
\] (46)

Denote
\[
Z_{\tau_p} = \int_{\tau_p}^{\infty} A_{\Delta, p}^{-\alpha_p, \alpha_p}(s_p) \psi^{1-\Delta, p}(s_1, \ldots, s_p) \Delta s_p.
\] (47)

Use integration by parts formula (3) in (48) to obtain
\[
Z_{\tau_p} = \int_{\tau_p}^{\infty} A_{\Delta, p}^{-\alpha_p, \alpha_p}(s_p) \psi^{1-\Delta, p}(s_1, \ldots, s_p) \Delta s_p - \int_{\tau_p}^{\infty} \frac{\partial}{\partial s_p} \left( A_{\Delta, p}^{-\alpha_p, \alpha_p}(s_p) \right) \psi^{1-\Delta, p}(s_1, \ldots, s_p) \Delta s_p.
\] (49)

Use \( \psi(s_1, \ldots, s_p, \infty) = 0 \) and the right-hand side part of inequality (8) with \( A_{\Delta, p}^{-\alpha_p, \alpha_p} \) in (49)
\[
Z_{\tau_p} \leq A_{\Delta, p}^{-\alpha_p, \alpha_p}(s_p) \psi^{1-\Delta, p}(s_1, \ldots, s_p, \tau_p) + \int_{\tau_p}^{\infty} \psi^{1-\Delta, p}(s_p) A_{\Delta, p}^{-\alpha_p, \alpha_p-1}(s_p) \psi^{1-\Delta, p}(s_1, \ldots, s_p, \tau_p) \Delta s_p.
\] (50)

Substitute (50) in (47)
\[
y(\tau_1, \ldots, \tau_p) \leq \int_{\tau_1}^{\infty} \cdots \int_{\tau_p}^{\infty} \prod_{k=1}^{p} A_{\Delta, k}^{-\alpha_k, \alpha_k}(s_k) \times \left\{ \int_{\tau_p}^{\infty} A_{\Delta, p}^{-\alpha_p, \alpha_p-1}(s_p) \psi^{1-\Delta, p}(s_1, \ldots, s_p, \tau_p) \right\}
\] \times \Delta s_p \cdots \Delta s_1 + \int_{\tau_1}^{\infty} \cdots \int_{\tau_p}^{\infty} \prod_{k=1}^{p} A_{\Delta, k}^{-\alpha_k, \alpha_k}(s_k) \times \left\{ \int_{\tau_p}^{\infty} \psi^{1-\Delta, p}(s_p) A_{\Delta, p}^{-\alpha_p, \alpha_p-1}(s_p) \psi^{1-\Delta, p}(s_1, \ldots, s_p, \tau_p) \right\}
\] \times \Delta s_p \cdots \Delta s_1.
\] (51)

Use (5) “\( p \) times” on (51):
\[
y(\tau_1, \ldots, \tau_p) \leq A_{\Delta, p}^{-\alpha_p, \alpha_p}(s_p) \int_{\tau_p}^{\infty} \cdots \int_{\tau_p}^{\infty} \prod_{k=1}^{p} A_{\Delta, k}^{-\alpha_k, \alpha_k}(s_k) \psi^{1-\Delta, p}(s_1, \ldots, s_p, \tau_p) \Delta s_p \cdots \Delta s_1.
\] (52)
Use induction hypothesis for $y(\tau_{i_1}, \ldots, \tau_{i_p})$ with fix $\tau_{i_p}, \delta_{i_p} \in T_{i_p}$ (instead for $y(\tau_{i_1}, \ldots, \tau_{i_p})$) in (52) to obtain

$$
\begin{align*}
y(\tau_{i_1}, \ldots, \tau_{i_p}) & \leq A_{i_p}^{-\alpha_{i_p}}(\sigma_{i_p}(\tau_{i_p})) \times \left[ \prod_{k=1}^{p} A_{i_k}^{-\alpha_{i_k}}(\sigma_{i_k}(\tau_{i_k})) \psi(\tau_{i_1}, \ldots, \tau_{i_p}) + \sum_{1 \leq j_1 < \cdots < j_r \leq p} \prod_{m=1}^{r-p} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) \right] \\
& \times \left[ \int_{\bar{T}_{i_p}} \prod_{m=1}^{r} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) (\sigma_{j_m}(\delta_{j_m})) v_{j_m}(\delta_{j_m}) \psi(\tau_{i_1}, \ldots, \tau_{i_{p-1}}, \delta_{j_1}, \ldots, \delta_{j_r}) \Delta \prod_{1 \leq j_1 < \cdots < j_r \leq p} \delta_{j_m} \right] \\
& + \int_{T_{i_p}} \prod_{m=1}^{r} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) (\sigma_{j_m}(\delta_{j_m})) v_{j_m}(\delta_{j_m}) \psi(\tau_{i_1}, \ldots, \tau_{i_{p-1}}, \delta_{j_1}, \ldots, \delta_{j_r}) \Delta \prod_{1 \leq j_1 < \cdots < j_r \leq p} \delta_{j_m} \\
& + \prod_{k=1}^{p} A_{i_k}^{-\alpha_{i_k}}(\sigma_{i_k}(\tau_{i_k})) \times \left[ \int_{\bar{T}_{i_p}} \prod_{m=1}^{r} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) (\sigma_{j_m}(\delta_{j_m})) v_{j_m}(\delta_{j_m}) \psi(\tau_{i_1}, \ldots, \tau_{i_{p-1}}, \delta_{j_1}, \ldots, \delta_{j_r}) \Delta \prod_{1 \leq j_1 < \cdots < j_r \leq p} \delta_{j_m} ight] \\
& \times \left[ \int_{T_{i_p}} \prod_{m=1}^{r} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) (\sigma_{j_m}(\delta_{j_m})) v_{j_m}(\delta_{j_m}) \psi(\tau_{i_1}, \ldots, \tau_{i_{p-1}}, \delta_{j_1}, \ldots, \delta_{j_r}) \Delta \prod_{1 \leq j_1 < \cdots < j_r \leq p} \delta_{j_m} \right]
\end{align*}
$$

Hence, by using (46) for $\tau_{i_p}$, we obtain

$$
\begin{align*}
y(\tau_{i_1}, \ldots, \tau_{i_p}) & \leq \prod_{k=1}^{p} A_{i_k}^{-\alpha_{i_k}}(\sigma_{i_k}(\tau_{i_k})) \psi(\tau_{i_1}, \ldots, \tau_{i_p}) \\
& + \sum_{1 \leq j_1 < \cdots < j_r \leq p+1} \prod_{m=1}^{r-p} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) \times \left[ \int_{\bar{T}_{i_p}} \prod_{m=1}^{r} A_{i_m}^{-\alpha_{i_m}}(\sigma_{i_m}(\tau_{i_m})) (\sigma_{j_m}(\delta_{j_m})) v_{j_m}(\delta_{j_m}) \psi(\tau_{i_1}, \ldots, \tau_{i_{p-1}}, \delta_{j_1}, \ldots, \delta_{j_r}) \Delta \prod_{1 \leq j_1 < \cdots < j_r \leq p} \delta_{j_m} \right]
\end{align*}
$$

Thus, by mathematical induction, (54) holds for all $h \in \mathbb{N}$, which completes the proof.

**Theorem 4.** Assume $H_1$, $H_2$, and $l, \alpha_i \geq 1$. for $i \in \{1, 2, \ldots, h\}$, $h \in \mathbb{N}$. Then,
\[ \int_{0}^{\infty} \prod_{k=1}^{p} \prod_{m=1}^{h} u_{jm}(\tau_{jm})A_{jm}^{la_{jm}}(\sigma_{jm}(\tau_{jm}))^{(1/\rho)} \prod_{m=1}^{p} \tau_{jm} \geq \sum_{1 \leq \delta < \gamma < \eta \leq \nu} \left( \frac{1 + \delta}{1 + \rho} \prod_{m=1}^{\nu} \alpha_{jm} + \rho \right)^{l} \]

**Proof.** We prove the result by using mathematical induction. For \( h = 1 \), statement is true by Theorems 4.6 in [2]. Assume for \( 1 \leq h \leq \rho \), (55) holds. To prove the result for \( h = \rho + 1 \), take L.H.S of (55) with \( h = \rho + 1 \) in the following form:

\[ \int_{0}^{\infty} \prod_{k=1}^{p+1} \prod_{m=1}^{h} u_{jm}(\tau_{jm})A_{jm}^{la_{jm}}(\sigma_{jm}(\tau_{jm}))^{(1/\rho)} \prod_{m=1}^{p+1} \tau_{jm} \]

\[ \leq \left( \int_{0}^{\infty} \prod_{k=1}^{p+1} \prod_{m=1}^{h} u_{jm}(\tau_{jm})A_{jm}^{la_{jm}}(\sigma_{jm}(\tau_{jm})) \prod_{m=1}^{p+1} \tau_{jm} \right)^{(1/\rho)} \]

Using (27) in (56) for \( h = \rho + 1 \):

\[ \int_{0}^{\infty} \prod_{k=1}^{p+1} \prod_{m=1}^{h} u_{jm}(\tau_{jm})A_{jm}^{la_{jm}}(\sigma_{jm}(\tau_{jm}))^{(1/\rho)} \prod_{m=1}^{p+1} \tau_{jm} \]

\[ \leq \left( \int_{0}^{\infty} \prod_{k=1}^{p+1} \prod_{m=1}^{h} u_{jm}(\tau_{jm})A_{jm}^{la_{jm}}(\sigma_{jm}(\tau_{jm})) \prod_{m=1}^{p+1} \tau_{jm} \right)^{(1/\rho)} \]

where

\[ I_{p+1} = \prod_{k=1}^{p+1} A_{ik}^{-a_{ik}}(\sigma_{ik}(\tau_{ik}))^{(1/\rho)} + \sum_{1 \leq j_{1} < \cdots < j_{\rho+1} \leq p+1} \prod_{m=1}^{\rho+1} A_{jm}^{-a_{jm}}(\sigma_{jm}(\tau_{jm})) \]

Apply Minkowski’s inequality (4) on (57) to obtain

\[ \left( \int_{0}^{\infty} \prod_{k=1}^{p+1} \prod_{m=1}^{h} u_{jm}(\tau_{jm})A_{jm}^{la_{jm}}(\sigma_{jm}(\tau_{jm}))^{(1/\rho)} \prod_{m=1}^{p+1} \tau_{jm} \right)^{(1/\rho)} \]

\[ \leq \left( \int_{0}^{\infty} \prod_{k=1}^{p+1} \prod_{m=1}^{h} u_{jm}(\tau_{jm})A_{jm}^{la_{jm}}(\sigma_{jm}(\tau_{jm})) \prod_{m=1}^{p+1} \tau_{jm} \right)^{(1/\rho)} \]

where
\[ I_{p+1} = \prod_{m=1}^{m_{l}} A_{m}^{-\alpha_{m}}(\sigma_{m}(\tau_{m})) \times \int_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} \sigma_{j_{m}}(s_{j_{m}}) v_{j_{m}}(s_{j_{m}}) \psi(\tau_{1}, \ldots, \tau_{p+1}, s_{j_{1}}, \ldots, s_{j_{r}}) \Delta s. \]  

(60)

Denote

\[ W_{p+1} = \int_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} \prod_{m=1}^{m_{l}} v_{j_{m}}(\tau_{j_{m}}) A_{j_{m}}^{-\alpha_{j_{m}}}(\sigma_{j_{m}}(\tau_{j_{m}})) \left[ I_{p+1} \right]^{l} \prod_{m=1}^{m_{l}} \tau_{j_{m}}. \]  

(61)

and one has that

\[ W_{p+1} = \int_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} \prod_{m=1}^{m_{l}} v_{j_{m}}(\tau_{j_{m}}) A_{j_{m}}^{-\alpha_{j_{m}}}(\sigma_{j_{m}}(\tau_{j_{m}})) \times \left[ \prod_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} \int_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} v_{j_{m}}(s_{j_{m}}) A_{j_{m}}^{-\alpha_{j_{m}}}(\sigma_{j_{m}}(s_{j_{m}})) \psi(\tau_{1}, \ldots, \tau_{p+1}, s_{j_{1}}, \ldots, s_{j_{r}}) \Delta s \right]^{l} \prod_{m=1}^{m_{l}} \tau_{j_{m}}. \]  

(62)

Use Theorem 2 in (62) by taking \( g(s_{j_{1}}, \ldots, s_{j_{r}}) = \prod_{j_{m}=1}^{m_{l}} A_{j_{m}}^{-\alpha_{j_{m}}}(\sigma_{j_{m}}(s_{j_{m}})) \psi(\tau_{1}, \ldots, \tau_{p+1}, s_{j_{1}}, \ldots, s_{j_{r}}) \) and \( c = (\prod_{m=1}^{m_{l}} \alpha_{j_{m}}) \):

\[ W_{p+1} \leq \left( 1 + p! \prod_{m=1}^{m_{l}} \alpha_{j_{m}} \right)^{l} \int_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} \prod_{m=1}^{m_{l}} v_{j_{m}}(\tau_{j_{m}}) A_{j_{m}}^{-\alpha_{j_{m}}}(\sigma_{j_{m}}(\tau_{j_{m}})) \psi(\tau_{1}, \ldots, \tau_{p+1}) \prod_{m=1}^{m_{l}} \tau_{j_{m}}. \]  

(63)

Substitute (63) in (59) and take power \( l \) on both sides to obtain

\[ \int_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} \prod_{m=1}^{m_{l}} v_{j_{m}}(\tau_{j_{m}}) \psi(\tau_{1}, \ldots, \tau_{p+1}) \prod_{m=1}^{m_{l}} \tau_{j_{m}} \geq \sum_{1 \leq j_{1} < \cdots < j_{r} \leq p+1} \left( \frac{1 + p! \prod_{m=1}^{m_{l}} \alpha_{j_{m}}}{1 + p! \prod_{m=1}^{m_{l}} \alpha_{j_{m}} + p} \right)^{l} \]  

(64)

Thus, by mathematical induction, (55) holds for all \( h \), which completes the proof.

Example 4. Let \( \mathbb{T} = \mathbb{N} \) and \( b_{j_{m}} = 1, \forall j, m \in \{1, \ldots, h\} \) and \( n_{j_{m}}, k_{j_{m}}, \in \mathbb{N} \forall j_{m} \), in Theorem 4. In this case, (55) takes the form
where

\[
A_{jn}(k_{j_{n}}) = \sum_{n_{jm}=1}^{k_{j_{n}}-1} \nu_{jm}(n_{jm}), \quad k_{j_{n}} \in \mathbb{N}. \tag{66}
\]

Note that (65) is extension of Example 4.7 in [2].

Example 5. Let \( T_{j} = \mathbb{R} \forall j, m \in \{1, \ldots, h\} \), in Theorem 4. In this case, (55) takes the form

\[
\int_{b_{jn}}^{c_{jn}} \prod_{m=1}^{h} \nu_{jm}(s_{jm}) \prod_{m=1}^{h} A_{jm}^{n_{jm}}(s_{jm}) x(s_{j_{1}}, \ldots, s_{j_{h}}) \, ds_{jm} \prod_{m=1}^{h} \tau_{jm} \\
\geq \sum_{1 \leq j_{1} < \cdots < j_{p} \leq p+1} \left( \frac{1 + F \prod_{m=1}^{h} \alpha_{jm}}{1 + F \prod_{m=1}^{h} \alpha_{jm}} \right)^{\ell} \int_{b_{jn}}^{c_{jn}} \prod_{m=1}^{h} \nu_{jm}(s_{jm}) A_{jm}^{n_{jm}}(s_{jm}) x(s_{j_{1}}, \ldots, s_{j_{h}}) \, ds_{jm} \prod_{m=1}^{h} \tau_{jm}, \tag{67}
\]

where

\[
A_{jm}(\tau_{jm}) = \int_{b_{jm}}^{c_{jm}} \prod_{m=1}^{h} \nu_{jm}(s_{jm}) d s_{jm}, \quad \tau_{jm} \in \mathbb{R}. \tag{68}
\]

Note that (67) is extension of Example 4.8 in [2].

Example 6. Let \( T_{j} = \mathbb{N} \), \( q_{jm} > 1 \), and \( \forall j, m \in \{1, \ldots, h\} \), in Theorem 4. In this case, (55) takes the form

\[
\int_{b_{jm}}^{c_{jm}} \prod_{m=1}^{h} \nu_{jm}(q_{jm}) \prod_{m=1}^{h} A_{jm}^{n_{jm}}(q_{jm}) x(q_{j_{1}}, \ldots, q_{j_{h}}) \, dq_{jm} \prod_{m=1}^{h} \tau_{jm} \\
\geq \sum_{1 \leq j_{1} < \cdots < j_{p} \leq p+1} \left( \frac{1 + F \prod_{m=1}^{h} \alpha_{jm}}{1 + F \prod_{m=1}^{h} \alpha_{jm}} \right)^{\ell} \times \sum_{k_{j_{1}}=1}^{\infty} \cdots \sum_{k_{j_{h}}=1}^{\infty} \left[ \prod_{m=1}^{h} \nu_{jm}(q_{jm}) A_{jm}^{n_{jm}}(q_{jm}) x(q_{j_{1}}, \ldots, q_{j_{h}}) \right]^{\ell}, \tag{69}
\]

where
Proof. We use mathematical induction to prove the result. For \( h = 1 \), (12) is true by Theorem 3.1 in [2]. Assume for \( 1 \leq h \leq p \), and (12) holds. To prove the result for \( h = p + 1 \), take L.H.S of (12) in the following form:

\[
A_{j_1}(q_{j_m}^{n_m}) = \sum_{n_m=1}^{k_{j_m}-1} v_{j_1}(q_{j_m}^{n_m}) q_{j_m}^{n_m}(q_{j_m} - 1), \quad k_{j_m} \in \mathbb{N}_0.
\]

(70)

\[
\int_{b_1}^{\infty} \cdots \int_{b_p}^{\infty} \prod_{i=1}^{P} \nu_i(\tau_i) \times \left\{ \int_{b_{p+1}}^{\infty} v_{p+1}(\tau_{p+1}) \phi_{p}^m(\tau_1, \ldots, \tau_{p+1}) \Delta \tau_{p+1} \right\} \Delta \tau_{p} \cdots \Delta \tau_1. \tag{71}
\]

Denote

\[
I_{p+1} = \int_{b_{p+1}}^{\infty} v_{p+1}(\tau_{p+1}) \phi_{p}^m(\tau_1, \ldots, \tau_{p+1}) \Delta \tau_{p+1}. \tag{72}
\]

Use Theorem 3.1 in [2] in (72) with respect to \( \tau_{p+1} \in \mathbb{T}_{p+1} \) for fix \((\tau_1, \ldots, \tau_p) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_p \) to obtain

\[
\phi_{p}(\tau_1, \ldots, \tau_{p+1}) = \int_{\tau_1}^{\infty} \int_{\tau_2}^{\tau_1} \cdots \int_{\tau_{p+1}}^{\tau_{p}} \prod_{i=1}^{P} \nu_i(\sigma_i) \Delta \sigma_i \Delta \tau_{p+1} \cdots \Delta \tau_1. \tag{74}
\]

(74)

Substitute (73) in (71) and use (5) "\( p \) times" in resultant inequality to obtain

\[
\int_{b_1}^{\infty} \cdots \int_{b_p}^{\infty} \prod_{i=1}^{P} \nu_i(\tau_i) \phi_{p}^m(\tau_1, \ldots, \tau_{p+1}) \Delta \tau_{p+1} \cdots \Delta \tau_1 \leq m^{P+1} \int_{b_1}^{\infty} \cdots \int_{b_p}^{\infty} \prod_{i=1}^{P} \nu_i(\tau_i) \Delta \tau_{p+1} \cdots \Delta \tau_1. \tag{75}
\]

Use induction hypothesis for \( \phi_{p}(\tau_1 \ldots \tau_{p+1}) \) with fix \( \tau_{p+1} \in \mathbb{T}_{p+1} \), instead for \( \phi_{p}(\tau_1 \ldots \tau_p) \) to obtain

\[
\int_{b_1}^{\infty} \cdots \int_{b_p}^{\infty} \prod_{i=1}^{P} \nu_i(\tau_i) \phi_{p}^m(\tau_1, \ldots, \tau_{p+1}) \Delta \tau_{p+1} \cdots \Delta \tau_1 \leq m^{(p+1)m} \int_{b_1}^{\infty} \cdots \int_{b_p}^{\infty} \prod_{i=1}^{P} \nu_i(\tau_i) \phi_{p}^m(\tau_1, \ldots, \tau_{p+1}) \Delta \tau_{p+1} \cdots \Delta \tau_1. \tag{76}
\]

(76)

Thus, by mathematical induction, (12) holds for all \( h \in \mathbb{N} \). ☐

Proof. We prove the result by using mathematical induction. For \( h = 1 \), statement is true by Theorem 3.3 in [2]. Assume for \( 1 \leq h \leq p \), (14) holds. To prove the result for \( h = p + 1 \), left-hand side of (14) can be written as

\[
\int_{a_1}^{\infty} \cdots \int_{a_p}^{\infty} \prod_{i=1}^{P} \nu_i(\zeta_i) \left\{ \int_{a_{p+1}}^{\infty} \Delta \zeta_{p+1} \right\} \phi_{p}^m(\zeta_1, \ldots, \zeta_{p+1}) \Delta \zeta_{p+1} \cdots \Delta \zeta_1. \tag{77}
\]

(77)
Denote
\[ I_{p+1} = \int_{a_{p+1}}^{\infty} \frac{v_{p+1}(\zeta_{p+1})}{A_p^{m-1}(\sigma_{p+1}(\zeta_{p+1}))} \phi^m(\zeta_{p+1}, \ldots, \zeta_{p+1}) \Delta \zeta_{p+1}. \]
(78)

Use Theorem 3.3 in [2] in (72) with respect to \( \zeta_{p+1} \in \mathbb{T}_{p+1} \) for fix \((\zeta_1, \ldots, \zeta_p) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_p \) to obtain
\[ \left( I_{p+1} \right)^m \leq \left( \frac{m}{1 - \zeta_{p+1}} \right)^m \prod_{i=1}^{p+1} v_i(\zeta_i) \int_{a_{p+1}}^{\infty} \frac{\phi^m(\zeta_{p+1})}{A_p^{m-1}(\sigma_{p+1}(\zeta_{p+1}))} \Delta \zeta_{p+1}, \]
(79)

where
\[ \phi_p(\zeta_1, \ldots, \zeta_{p+1}) \equiv \int_{\zeta_1}^{\infty} \left( \prod_{j=1}^{p} v_j(s_j) g(s_1, \ldots, s_p, \zeta_{p+1}) \Delta s_p \cdots \Delta s_1. \]
(80)

Substitute (79) in (77) and use \( \text{fd5}(5) \) “\( p \) times” in resultant inequality to obtain
\[ \int_{a_{p+1}}^{\infty} \left( \prod_{i=1}^{p} v_i(\zeta_i) \right) \left( \int_{a_{p+1}}^{\infty} \frac{v_{p+1}(\zeta_{p+1})}{A_p^{m-1}(\sigma_{p+1}(\zeta_{p+1}))} \phi^m(\zeta_{p+1}, \ldots, \zeta_{p+1}) \Delta \zeta_{p+1} \right) \Delta \zeta_p \cdots \Delta \zeta_1 \]
\[ \leq \left( \frac{m}{1 - \zeta_{p+1}} \right)^m \prod_{i=1}^{p+1} v_i(\zeta_i) A_p^{m-1}(\sigma_{p+1}(\zeta_{p+1})) \Delta \zeta_{p+1} \times \int_{a_{p+1}}^{\infty} \left( \prod_{i=1}^{p} \frac{v_i(\zeta_i) \phi^m(\zeta_{p+1}, \ldots, \zeta_{p+1})}{A_p^{m-1}(\sigma_{p+1}(\zeta_{p+1}))} \Delta \zeta_{p+1} \cdots \Delta \zeta_1. \right) \]
(81)

Use induction hypothesis for \( \phi_p(\zeta_1, \ldots, \zeta_{p+1}) \) with fix \( \zeta_{p+1} \in \mathbb{T}_{p+1} \), instead for \( \phi_p(\zeta_1, \ldots, \zeta_p) \), to obtain
\[ \int_{a_{p+1}}^{\infty} \left( \prod_{i=1}^{p} v_i(\zeta_i) \right) \frac{\phi^m(\zeta_{p+1}, \ldots, \zeta_{p+1})}{A_p^{m-1}(\sigma_{p+1}(\zeta_{p+1}))} \Delta \zeta_{p+1} \cdots \Delta \zeta_1 \]
\[ \leq \prod_{i=1}^{p+1} \left( \frac{m}{1 - \zeta_i} \right)^{(p+1)m} \int_{a_{p+1}}^{\infty} \left( \prod_{i=1}^{p} v_i(\zeta_i) A_i^{m-1}(\sigma_i(\zeta_i)) g^m(\zeta_1, \ldots, \zeta_{p+1}) \Delta \zeta_{p+1} \cdots \Delta \zeta_1. \right) \]
(82)

Hence, by mathematical induction, (14) is true for all \( h \in \mathbb{N}. \)

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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