On the estimation of a rational approximation to the functions of linear self-adjoint operator pencils

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Abstract. We suggest an approximate method for evaluating functions of a linear pencil $L(\lambda) = \lambda A - B$ with self-adjoint operator coefficients $A$ and $B$. It is based on the calculation of a special rational function of $L$. We assume that the operator $A$ is bounded and positive definite, and the operator $B$ can be unbounded. The estimates of the approximation error are obtained. As an application, an approximate method for calculating the impulse response of a linear differential equation $Ax' = Bx + g(t)$ is given. The suggested approach can be used in remodeling of the complicated systems.

1. Introduction
In a Hilbert space we consider a linear operator pencil $L(\lambda) = \lambda A - B$ with self-adjoint coefficients. We construct a functional calculus

$$
\tilde{\Psi}(f) = \int_{\sigma(L)} f(\xi) d\tilde{E}(\xi),
$$

where $\sigma(L)$ is the spectrum of the pencil $L$ and $\tilde{E}$ is a special spectral decomposition.

In the present paper we suggest an approximate method to the calculation of $\tilde{\Psi}(f)$ by means of the rational approximation of $f$. The main results are Theorems 4 and 5, where the estimates of the relative and the absolute approximation errors are obtained.

An important example of $\tilde{\Psi}(f)$ is the function

$$
U(t) = \tilde{\Psi}(\exp_t) = \int_{\sigma(L)} \exp_t(\xi) d\tilde{E}(\xi),
$$

which is an operator impulse response of the linear differential equation

$$
Ax'(t) - Bx(t) = g(t). \tag{1}
$$

Thus, applying the suggested method for the approximate calculation of $\tilde{\Psi}(\exp_t)$ we can obtain an approximate solution of Equation (1). In Theorems 6 and 7 the estimates of the relative and the absolute approximation errors for the solution of Equation (1) are given. A close approach was earlier used in the finite-dimensional case in [1], a similar method for the approximate solution of Equation (1) in the case of $A = 1$ was developed in [2], and for special equations of the second order a close approach is discussed in [3].
In this case, one can define the adjoint operator $T^*$ with domain $D(T^*)$ by the formula

$$T^*\psi = \chi.$$
An operator $T$ is called self-adjoint if $T = T^*$. In particular, for a self-adjoint operator we have $D(T^*) = D(T)$.

An operator $T$ is called closed if its graph $\{(\varphi, T\varphi) : \varphi \in D(T)\}$ is a closed subspace of the Cartesian product $\mathbb{H} \times \mathbb{H}$. The adjoint operator is closed [8, Theorem 13.9]. In particular, a self-adjoint operator is always closed. A self-adjoint operator $T : D(T) \subset \mathbb{H} \to \mathbb{H}$ is called positive definite if

$$\langle T\varphi, \varphi \rangle > 0, \quad \varphi \in D(T), \quad \varphi \neq 0,$$

and negative semidefinite if

$$\langle T\varphi, \varphi \rangle \leq 0, \quad \varphi \in D(T), \quad \varphi \neq 0.$$

Let $A : \mathbb{H} \to \mathbb{H}$ be a self-adjoint positive definite operator. We define $A$-multiplication of operators by the formula

$$T \circ S = TAS.$$  \hfill (2)

By $\tilde{B}_A(\mathbb{H})$ we denote a Banach algebra of all bounded linear operators acting in $\mathbb{H}$, with $A$-multiplication (2). We take the function

$$\|T\|_\circ = \|\sqrt{AT} \sqrt{A}\|$$ \hfill (3)

as a norm in $\tilde{B}_A(\mathbb{H})$. It is easy to verify that the algebra $\tilde{B}_A(\mathbb{H})$ has a unit, the unit is the operator $I = A^{-1}$, and $\|I\|_\circ = \|A^{-1}\|_\circ = 1$.

**Proposition.** Norm (3) has the properties

$$\|T \circ S\|_\circ \leq \|T\|_\circ \|S\|_\circ,$$

$$\|T^*\|_\circ = \|T\|_\circ,$$

$$\|T^* \circ T\|_\circ = \|T\|_\circ^2.$$

Thus, the algebra $\tilde{B}_A(\mathbb{H})$ is a $C^*$-algebra with the same involution.

**Proof.** We have

$$\|T \circ S\|_\circ = \|TAS\|_\circ = \|\sqrt{AT} \sqrt{A} \sqrt{AS} \sqrt{A}\| \leq \|\sqrt{AT} \sqrt{A}\| \|\sqrt{AS} \sqrt{A}\| = \|T\|_\circ \|S\|_\circ.$$

Since for every operator $Q \in \mathcal{B}(\mathbb{H})$ the original norm $\| \cdot \|$ in $\mathcal{B}(\mathbb{H})$ satisfies the relations

$$\|Q^*\| = \|Q\|,$$

$$\|Q^* \cdot Q\| = \|Q\|^2,$$

then

$$\|T^*\|_\circ = \|\sqrt{AT} \sqrt{A}\| = \|(\sqrt{AT} \sqrt{A})^*\| = \|\sqrt{AT} \sqrt{A}\| = \|T\|_\circ$$

and

$$\|T^* \circ T\|_\circ = \|\sqrt{AT} \sqrt{A} \sqrt{AT} \sqrt{A}\| = \|(\sqrt{AT} \sqrt{A})^* \sqrt{AT} \sqrt{A}\| = \|\sqrt{AT} \sqrt{A}\|^2 = \|T\|_\circ^2.$$

The proof of the proposition is complete.

We call an operator $P \in \tilde{B}_A(\mathbb{H})$ an $A$-projection if $P \circ P = P$.

Let $\Omega$ be a Borel subset of the complex plane $\mathbb{C}$. Let $\Sigma$ be the $\sigma$-algebra of all Borel subsets of $\Omega$. We define a resolution of identity in the algebra $\mathcal{B}_A(\mathbb{H})$ as a mapping $\tilde{E} : \Sigma \to \tilde{B}_A(\mathbb{H})$ with the following properties.

(I) $\tilde{E}(\emptyset) = 0$, $\tilde{E}(\Omega) = I$.

(II) For all $\omega \in \Sigma$ the operator $\tilde{E}(\omega)$ is self-adjoint.
(III) For all $\omega \in \Sigma$ the operator $\tilde{E}(\omega)$ is $A$-projection, i.e. $\tilde{E}(\omega) \circ \tilde{E}(\omega) = \tilde{E}(\omega)$.

(IV) The relation $\tilde{E}(\omega' \cap \omega'') = \tilde{E}(\omega') \circ \tilde{E}(\omega'')$ holds for all $\omega', \omega'' \in \Sigma$.

(V) The relation $\tilde{E}(\omega' \cup \omega'') = \tilde{E}(\omega') + \tilde{E}(\omega'')$ holds for all $\omega', \omega'' \in \Sigma$, $\omega' \cap \omega'' = \emptyset$.

(VI) For arbitrary $\varphi, \psi \in \mathcal{H}$ the function $\tilde{E}_\varphi,\psi(\omega) = \langle \tilde{E}(\omega)\varphi, \psi \rangle$ is a complex measure on $\Sigma$.

We call the **essential range with respect to** $\tilde{E}$ of a Borel measurable function $f: \Omega \to \mathbb{C}$ the intersection of sets $[f(\omega)]$ for all $\omega \in \Sigma$ such that $\tilde{E}(\omega) = 1$, where $[\cdot]$ means the closure of a set. We say that a function $f$ is **essentially bounded with respect to** $\tilde{E}$ if its essential range with respect to $\tilde{E}$ is bounded. In this case, the supremum of the essential range with respect to $\tilde{E}$ of $|f|$ is called the **essential supremum** $\|f\|_{\tilde{E}}$ with respect to $\tilde{E}$ of $f$.

For unbounded operators $T: D(T) \subset \mathcal{H} \to \mathcal{H}$ and $S: D(S) \subset \mathcal{H} \to \mathcal{H}$ we define $A$-multiplication (2) as an operator $T \circ S$ such that the relation

$$(T \circ S)\varphi = T(A\varphi), \quad \varphi \in D(T \circ S) = \{\psi \in D(S): A\varphi \psi \in D(T)\},$$

holds.

In Theorem 1 a functional calculus corresponding to the resolution of identity in the algebra $\tilde{B}_A(\mathcal{H})$ is constructed and its properties are given.

**Theorem 1.** Let $\tilde{E}$ be a resolution of identity in the algebra $\tilde{B}_A(\mathcal{H})$. Then the formula

$$\Psi(f) = \int f(\xi) d\tilde{E}_{\varphi,\psi}(\xi), \quad f \in D(\Psi(f)), \psi \in \mathcal{H}, \tag{4}$$

takes each Borel measurable function $f: \Omega \to \mathbb{C}$ to the densely defined closed operator

$$\tilde{\Psi}(f): D(\tilde{\Psi}(f)) \subset \mathcal{H} \to \mathcal{H}$$

with domain

$$D(\tilde{\Psi}(f)) = \{\varphi \in \mathcal{H}: \int |f|^2 d\tilde{E}_{\varphi,\varphi} < \infty\}.$$ 

In addition, the mapping $\tilde{\Psi}$ has the following properties.

(a) For the adjoint operator the relations

$$(\tilde{\Psi}(f))^* = \Psi(\bar{f}), \quad (\tilde{\Psi}(f) \circ (\tilde{\Psi}(f))^*) = \tilde{\Psi}(|f|^2) = (\tilde{\Psi}(f))^* \circ \tilde{\Psi}(f)$$

hold. In particular, for $f: \Omega \to \mathbb{R}$ the operator $\tilde{\Psi}(f)$ is self-adjoint.

(b) The relation

$$\|\sqrt{A}\tilde{\Psi}(f)\varphi\|^2 = \|A\tilde{\Psi}(f)\varphi, \tilde{\Psi}(f)\varphi\|^2 = \int |f|^2 d\tilde{E}_{\varphi,\varphi}$$

holds for all $\varphi \in D(\tilde{\Psi}(f))$.

(c) If $f: \Omega \to \mathbb{C}$ is essentially bounded function with respect to $\tilde{E}$, then $\tilde{\Psi}(f)$ is bounded operator, and its norm satisfies the equality

$$\|\tilde{\Psi}(f)\|_\infty = \|f\|_{\tilde{E}}.$$

(d) The inclusions

$$\tilde{\Psi}(f) + \tilde{\Psi}(g) \subset \tilde{\Psi}(f + g), \quad \tilde{\Psi}(f) \circ \tilde{\Psi}(g) \subset \tilde{\Psi}(fg)$$

hold for arbitrary Borel measurable functions $f, g: \Omega \to \mathbb{C}$. In particular, if $g$ is a bounded function, then

$$\tilde{\Psi}(f) + \tilde{\Psi}(g) = \tilde{\Psi}(f + g), \quad \tilde{\Psi}(f) \circ \tilde{\Psi}(g) = \tilde{\Psi}(fg).$$
The spectrum $\sigma(\tilde{\Psi}(f))$ of the operator $\tilde{\Psi}(f)$ is the essential range of the function $f$ with respect to $E$.

(g) The function $u(\xi) = 1$ is mapped by $\tilde{\Psi}$ to the operator $I$.

Proof. The proof is analogous to that of the theorem on functional calculus in $\mathcal{B}(H)$ [8, Theorem 13.24].

2.2. Spectral decomposition of self-adjoint pencils

Let $A: H \to H$ be a bounded linear operator, and $B: D(B) \subset H \to H$ be an unbounded linear operator. The operator function

$$L(\lambda) = \lambda A - B, \quad \lambda \in \mathbb{C},$$

is called a linear pencil.

The set $\rho(L)$ of all $\lambda \in \mathbb{C}$ such that the operator $L(\lambda) = \lambda A - B$ has a bounded inverse is called the resolvent set of the pencil $L$, and the function $\lambda \mapsto R_\lambda = (\lambda A - B)^{-1}$ is called the resolvent. The complement $\sigma(L)$ of the resolvent set $\rho(L)$ is called the spectrum of the pencil. The pencil is called regular if its resolvent set is not empty. In this article only regular pencils are considered.

With the pencil $L(\lambda) = \lambda A - B$ we consider the auxiliary operator

$$G = IBI, \quad D(G) = \{ \varphi \in H : I\varphi \in D(B) \}.$$ 

It is easy to verify that the operator $G$ is self-adjoint.

We call the resolvent set of the operator $G$ in the algebra $\tilde{B}_A(H)$ the set $\rho_G(G)$ of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - G$ is invertible in $\tilde{B}_A(H)$, i.e. there exists a bounded operator $(\lambda I - G)^{-1}: H \to D(G)$ such that

$$(\lambda I - G) \circ (\lambda I - G)^{-1} \varphi = I\varphi, \quad \varphi \in H,$$

$$(\lambda I - G)^{-1} \circ (\lambda I - G)\psi = I\psi, \quad \psi \in D(G).$$

The function $\lambda \mapsto (\lambda I - G)^{-1}$ is called the resolvent of the operator $G$ in the algebra $\tilde{B}_A(H)$. The complement $\sigma_G(G)$ of $\rho_G(G)$ is called the spectrum of the operator $G$ in the algebra $\tilde{B}_A(H)$.

**Theorem 2.** The resolvent $R_\lambda = (\lambda A - B)^{-1}$ of the pencil $L$ is the resolvent of the operator $G$ in the algebra $\tilde{B}_A(H)$, i.e.

$$(\lambda I - G) \circ (\lambda A - B)^{-1} \varphi = I\varphi, \quad \varphi \in H,$$

$$(\lambda A - B)^{-1} \circ (\lambda I - G)\psi = I\psi, \quad \psi \in D(G),$$

and therefore $\sigma(L) = \sigma_G(G)$.

Proof. For example, we prove the first equality. Obviously, it is equivalent to

$$(\lambda I - G)A = I(\lambda A - B)$$

or

$$\lambda IA - IBA = \lambda IA - IB,$$

which is true by $IA = I$. The second equality can be proved similarly. The proof of the theorem is complete.

Theorem 3 for a pencil $L$ is an analogue of the spectral theorem [8, 9] for self-adjoint operators in the sense that in it for the pencil $L$ the resolution of identity in the algebra $\tilde{B}_A(H)$ is
constructed. The resolution of identity from this theorem is called the spectral decomposition of the pencil $L$.

**Theorem 3.** Let $A$ and $B$ be a self-adjoint operators, and the operator $A$ is bounded and positive definite. Then for the pencil $L(\lambda) = \lambda A - B$ there exists a unique resolution of identity $\tilde{E}$ in the algebra $\tilde{B}_A(H)$, defined on the Borel subsets of the spectrum $\sigma(L)$ of the pencil $L$, such that the relation

$$
\langle R_\lambda \varphi, \psi \rangle = \langle (\lambda A - B)^{-1} \varphi, \psi \rangle = \int_{\sigma(L)} \frac{1}{\lambda - \xi} d\tilde{E}_{\varphi, \psi}(\xi), \quad \lambda \in \rho(L); \varphi, \psi \in H,
$$

holds.

**Proof.** Using arguments similar to those given in the proof of the spectral theorem [8, Theorem 13.33] for an unbounded self-adjoint operator, we obtain that for the operator $G$ there exists a unique resolution of identity $\tilde{E}$ in the algebra $\tilde{B}_A(H)$ defined on Borel subsets of its spectrum $\sigma(G) = \sigma(L)$ such that the relation

$$
\langle G \varphi, \psi \rangle = \int_{\sigma(G)} \xi d\tilde{E}_{\varphi, \psi}(\xi), \quad \varphi, \psi \in H,
$$

holds. For all $\lambda \in \rho(G) = \rho(L)$ we have

$$
\langle (\lambda A - B)^{-1} \varphi, \psi \rangle = \langle (\lambda I - G)^{-1} \varphi, \psi \rangle = \int_{\sigma(L)} \frac{1}{\lambda - \xi} d\tilde{E}_{\varphi, \psi}(\xi), \quad \varphi, \psi \in H.
$$

The proof of the theorem is complete.

The spectral decomposition $\tilde{E}$ of the pencil $L$ allows one to construct a functional calculus

$$
\langle \tilde{\Psi}(f) \varphi, \psi \rangle = \int_{\sigma(L)} f(\xi) d\tilde{E}_{\varphi, \psi}(\xi), \quad \varphi \in D(\tilde{\Psi}(f)), \psi \in H.
$$

**Corollary.** Let $\tilde{E}$ be a spectral decomposition of the pencil $L$ and $\tilde{\Psi}$ be a corresponding functional calculus. Then

(a) the function $u(\xi) = \xi$ is mapped by $\tilde{\Psi}$ to the operator $G = IBI$;
(b) if the function $f: \sigma(L) \to \mathbb{C}$ is essentially bounded with respect to $\tilde{E}$ then the operator $\tilde{\Psi}(f)$ is bounded and

$$
\|\tilde{\Psi}(f)\| \leq \sup_{\xi \in \sigma(L)} |f(\xi)|.
$$

**Proof.** The proof of (a) is obvious, (b) follows from $\|f\|_{\infty, \tilde{E}} \leq \sup_{\xi \in \sigma(L)} |f(\xi)|$.

2.3. Rational approximation to the functions of the pencil

We consider the case of the function $f$ where the exact calculation of $\tilde{\Psi}(f)$ is impossible. For the approximate evaluation of $\tilde{\Psi}(f)$ we replace on $\sigma(L)$ the function $f$ by a rational function

$$
r(\xi) = \sum_{k=1}^{K} \frac{p_k}{\xi - q_k}
$$

(5)

where the roots of denominators lie outside $\sigma(L)$. For the approximation of $\tilde{\Psi}(f)$ we take the function

$$
\tilde{\Psi}_{\text{approx}}(f, r) = \tilde{\Psi}(r) = \int_{\sigma(L)} r(\xi) d\tilde{E}(\xi).
$$
To compute $\tilde{\Psi}_{\text{approx}}(f, r)$ it is convenient to use the representation
\[ \tilde{\Psi}_{\text{approx}}(f, r) = \sum_{k=1}^{K} p_k (B - q_k A)^{-1}, \]
which follows from Theorem 3.

As a rational function $r$ one can use the Padé approximation [10] or the rational functions of the best approximation [11].

**Remark.** In applications it is often necessary to find not the operator $\tilde{\Psi}(f)$ but only the vectors $\tilde{\Psi}(f)b$ for some $b$. In this case as an approximation for $\tilde{\Psi}(f)b$, obviously, one can take the vector
\[ \tilde{\Psi}_{\text{approx}}(f, r)b = \sum_{k=1}^{K} p_k (B - q_k A)^{-1}b. \] (6)

To calculate vectors of the kind
\[ w = (B - q_k A)^{-1}b \]
from (6) it is sufficient [2] to solve (exactly or approximately) the equations
\[ Bw - q_k Aw = b. \]

3. Results and discussion

3.1. Estimates of the approximation error

In the following two theorems, we estimate the accuracy of the approximation. First, we present the estimates of the relative error of the approximation.

**Theorem 4.** If rational function (5) approximates the function $f$ on $\sigma(\mathcal{L})$ with a relative error $\varepsilon \geq 0$, i.e.
\[ |r(\xi) - f(\xi)| \leq \varepsilon |f(\xi)|, \quad \xi \in \sigma(\mathcal{L}), \]
then the estimates
\[ \|\tilde{\Psi}_{\text{approx}}(f, r) - \tilde{\Psi}(f)\|_\infty \leq \varepsilon \|\tilde{\Psi}(f)\|_\infty, \]
\[ \|\sqrt{A}(\tilde{\Psi}_{\text{approx}}(f, r) - \tilde{\Psi}(f))\varphi\| \leq \varepsilon \|\sqrt{A}\tilde{\Psi}(f)\varphi\| \]
hold.

**Proof.** We define auxiliary functions
\[ g_1(\xi) = r(\xi) - f(\xi), \quad g_2(\xi) = f(\xi), \quad \xi \in \sigma(\mathcal{L}). \]
We note that
\[ \tilde{\Psi}_{\text{approx}}(f, r) - \tilde{\Psi}(f) = \tilde{\Psi}(r) - \tilde{\Psi}(f) = \tilde{\Psi}(g_1), \quad \tilde{\Psi}(f) = \tilde{\Psi}(g_2). \]

By Theorem 1 (b) it follows that
\[ \|\sqrt{A}\tilde{\Psi}(g_1)\varphi\|^2 = \int_{\sigma(\mathcal{L})} |g_1|^2 dE_{\varphi, \varphi} = \int_{\sigma(\mathcal{L})} |g_1|^2 d\tilde{E}_{\varphi, \varphi} \leq \varepsilon^2 \int_{\sigma(\mathcal{L})} |g_2|^2 d\tilde{E}_{\varphi, \varphi} = \varepsilon^2 \int_{\sigma(\mathcal{L})} |g_2|^2 d\tilde{E}_{\varphi, \varphi} = \varepsilon^2 \|\sqrt{A}\tilde{\Psi}(g_2)\varphi\|^2 = \varepsilon^2 \|\sqrt{A}\tilde{\Psi}(f)\varphi\|^2. \]
We substitute the vector $\sqrt{A}\psi$ into this inequality as $\varphi$, as a result we get
\[ \|\sqrt{A}\tilde{\Psi}(g_1)\sqrt{A}\psi\|^2 \leq \varepsilon^2 \|\sqrt{A}\tilde{\Psi}(f)\sqrt{A}\psi\|^2 \leq \varepsilon^2 \|\sqrt{A}\tilde{\Psi}(f)\sqrt{A}\|^2 \|\psi\|^2. \]
Therefore,
\[
\|\tilde{\Psi}(g_1)\|_\sigma = \|\sqrt{A}\tilde{\Psi}(g_1)\sqrt{A}\| \leq \varepsilon \|\sqrt{A}\tilde{\Psi}(f)\sqrt{A}\| = \varepsilon \|\tilde{\Psi}(f)\|_\sigma.
\]

The proof of the theorem is complete.

**Theorem 5.** If rational function (5) approximates the function \( f \) on \( \sigma(L) \) with an absolute error \( \varepsilon \geq 0 \), i.e.
\[
|r(\xi) - f(\xi)| \leq \varepsilon, \quad \xi \in \sigma(L),
\]
then the estimates
\[
\|\tilde{\Psi}_{approx}(f, r) - \tilde{\Psi}(f)\|_\sigma \leq \varepsilon,
\]
\[
\|\sqrt{A}(\tilde{\Psi}_{approx}(f, r) - \tilde{\Psi}(f))\varphi\| \leq \varepsilon \|\sqrt{A^{-1}}\varphi\|
\]
hold.

**Proof.** Obviously, the function
\[
g_1(\xi) = r(\xi) - f(\xi), \quad \xi \in \sigma(L),
\]
is essentially bounded with respect to \( \tilde{E} \). By Corollary (b) we have
\[
\|\tilde{\Psi}_{approx}(f, r) - \tilde{\Psi}(f)\|_\sigma = \|\tilde{\Psi}(g_1)\|_\sigma \leq \sup_{\xi \in \sigma(L)} |g_1(\xi)| \leq \varepsilon.
\]

In addition,
\[
\|\sqrt{A}(\tilde{\Psi}_{approx}(f, r) - \tilde{\Psi}(f))\varphi\|^2 = \|\sqrt{A}\tilde{\Psi}(g_1)\varphi\|^2 = \int_{\sigma(L)} |g_1|^2 d\tilde{E}_{\varphi,\varphi} = \int_{\sigma(L)} |g_1|^2 d\tilde{E}_{\varphi,\varphi} = \varepsilon^2 \int_{\sigma(L)} d\tilde{E}_{\varphi,\varphi} = \varepsilon^2 \|\sqrt{A}\tilde{\Psi}(\varphi)\|^2 = \varepsilon^2 \|\sqrt{A^{-1}}\varphi\|^2,
\]
where \( u(\xi) = 1 \) and \( \tilde{\Psi}(u) = A^{-1} \).

3.2. **Approximate solution of the equation** \( Ax' = Bx + g \)

We consider a linear differential equation
\[
Ax'(t) = Bx(t) + g(t), \quad x(0) = 0,
\]
where the operator \( A \) is bounded and positive definite, and the operator \( B \) is unbounded and negative semidefinite. We note that \( \sigma(L) \subset [0, +\infty) \).

We call an *impulse response* of Equation (7) the operator function
\[
U(t) = \tilde{\Psi}(\exp_t) = \int_{\sigma(L)} \exp_t(\xi) \, d\tilde{E}(\xi),
\]
where \( \exp_t(\xi) = e^{\xi t} \). We call a *generalized solution* of Equation (7) the function
\[
x(t) = \int_0^t U(t - \tau)g(\tau) \, d\tau.
\]

Let us construct an approximate impulse response of Equation (7). We fix \( t > 0 \) and approximate on \( \sigma(L) \subset [0, +\infty) \) the function \( \xi \mapsto \exp_t(\xi) \) by a rational function \( \xi \mapsto r_t(\xi) \) such that the roots of denominators lie outside \( \sigma(L) \). For the approximate impulse response we take the function
\[
\tilde{U}(t) = \tilde{\Psi}_{approx}(\exp_t, r_t) = \int_{\sigma(L)} r_t(\xi) \, d\tilde{E}(\xi),
\]
and for the approximate generalized solution we take the function
\[ \tilde{x}(t) = \int_0^t \tilde{U}(t - \tau) g(\tau) \, d\tau. \]

In the following assertion, we estimate the accuracy of the approximation to the impulse response. Theorem 6(a) allows one to estimate the relative error, and in Theorem 6(b) the estimate of the absolute error is given.

**Theorem 6.**

(a) Let \( \varepsilon : [0, +\infty) \rightarrow [0, +\infty) \) be a given function, such that for each \( t > 0 \) the estimate
\[ |r_t(\xi) - \exp_t(\xi)| \leq \varepsilon(t), \quad \xi \in [0, +\infty) \]
holds. Then the approximate impulse response satisfies the estimate
\[ \|U(t) - \tilde{U}(t)\|_\circ \leq \varepsilon. \]

(b) Let \( \varepsilon : [0, +\infty) \rightarrow [0, +\infty) \) be a given function, such that for each \( t > 0 \) the estimate
\[ |r_t(\xi) - \exp_t(\xi)| \leq \varepsilon(t) \exp_t(\xi), \quad \xi \in [0, +\infty) \]
holds. Then the approximate impulse response satisfies the estimate
\[ \|U(t) - \tilde{U}(t)\|_\circ \leq \varepsilon \|\tilde{U}(t)\|_\circ. \]

**Proof.** The desired assertion follows from Theorems 4 and 5.

If the free term \( g \) in Equation (7) has the form \( g(t) = bv(t) \), where \( b \in \mathbf{H} \) is a given vector, and \( v : [0, +\infty) \rightarrow \mathbf{C} \) is a given continuous function, we can immediately construct an approximation for the generalized solution at a given point \( t > 0 \).

We introduce an auxiliary function
\[ \theta_t(\xi) = \int_0^t \exp_{t-\tau}(\xi) v(\tau) \, d\tau = \int_0^t \exp_{\tau}(\xi) v(t - \tau) \, d\tau. \]

We approximate on \( \sigma(\mathcal{L}) \subset [0, +\infty) \) the function \( \xi \mapsto \theta_t(\xi) \) by a rational function \( \xi \mapsto r_t(\xi) \) and for the approximate generalized solution we take the function
\[ \tilde{x}(t) = \tilde{\Psi}_{\text{approx}}(\theta_t, r_t)b. \]

To estimate the error of the obtained approximation one can use Theorem 7.

**Theorem 7.** We fix \( t > 0 \).

(a) Let the rational function \( \xi \mapsto r_t(\xi) \) approximates the function \( \xi \mapsto \theta_t(\xi) \) on \( [0, +\infty) \) with the relative error \( \varepsilon(t) \), i.e.
\[ |r_t(\xi) - \theta_t(\xi)| \leq \varepsilon(t)|\theta_t(\xi)|, \quad \xi \in [0, +\infty). \]

Then the approximate generalized solution satisfies the estimate
\[ \|\sqrt{A}\tilde{x}(t) - \sqrt{A}x(t)\| \leq \varepsilon(t)\|\sqrt{A}x(t)\|. \]

(b) Let the rational function \( \xi \mapsto r_t(\xi) \) approximates the function \( \xi \mapsto \theta_t(\xi) \) on \( [0, +\infty) \) with the absolute error \( \varepsilon(t) \), i.e.
\[ |r_t(\xi) - \theta_t(\xi)| \leq \varepsilon(t), \quad \xi \in [0, +\infty). \]

Then the approximate generalized solution satisfies the estimate
\[ \|\sqrt{A}\tilde{x}(t) - \sqrt{A}x(t)\| \leq \varepsilon(t)\|\sqrt{A}^{-1}b\|. \]

**Proof.** The desired assertion follows from Theorems 4 and 5.
4. Conclusion
The spectral theorem for the linear self-adjoint pencils which is an analogue of the spectral theorem for self-adjoint operators is given. The functional calculus corresponding to the pencil is constructed. An approximate method for evaluating the functions of the pencil with the use of the rational approximation is developed. Theorems for the estimation of a rational approximation to the functions of linear self-adjoint operator pencils are proved. The variant of the suggested method for an approximate solution of the equation \( Ax' = Bx + g \) is described.

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References
[1] Kurbatov V G and Oreshina M N 2004 Interconnect macromodelling and approximation of matrix exponent Analog Integrated Circuits and Signal Processing 40(1) 5–19
[2] Oreshina M N 2017 Approximate solution of a parabolic equation with the use of a rational approximation to the operator exponential Differential Equations 53(3) 398–408
[3] Oreshina M N 2018 J. Phys.: Conf. Series 973 012057
[4] Frommer A and Simoncini V 2008 Matrix functions. Model order reduction: theory, research aspects and applications Mathematics in Industry 13 275–303
[5] Benner P, Gugercin S and Willcox K A survey of projection-based model reduction methods for parametric dynamical systems SIAM Rev. 57(4) 483–531
[6] Petrova A A and Smagin V V 2016 Convergence of the Galyorkin method of approximate solving parabolic equation with weight integral condition Russian Mathematics 60(8) 42–51
[7] Samarskii A A and Gulin A V 2003 Numerical Methods of Mathematical Physics (Moscow: Scientific World)
[8] Rudin W 1991 Functional analysis (New York: McGraw-Hill)
[9] Helmskii A Ya 2005 Lectures and Exercises on Functional Analysis (Providence: American Mathematical Society)
[10] Baker G A Jr and Graves-Morris P 1996 Padé Approximants (Cambridge: Cambridge Univ. Press)
[11] Carpenter A J, Ruttan A, and Varga R S 1983 Extended numerical computations on the 1/9 conjecture in rational approximation theory Rational Approximation and Interpolation, Lecture Notes in Math. 1105 383–411