A discrete divergence free weak Galerkin finite element method for the Stokes equations

Lin Mu*, Junping Wang†, and Xiu Ye‡

Abstract. A discrete divergence free weak Galerkin finite element method is developed for the Stokes equations based on a weak Galerkin (WG) method introduced in [15]. Discrete divergence free bases are constructed explicitly for the lowest order weak Galerkin elements in two and three dimensional spaces. These basis functions can be derived on general meshes of arbitrary shape of polygons and polyhedrons. With the divergence free basis derived, the discrete divergence free WG scheme can eliminate pressure variable from the system and reduces a saddle point problem to a symmetric and positive definite system with many fewer unknowns. Numerical results are presented to demonstrate the robustness and accuracy of this discrete divergence free WG method.

Key words. Weak Galerkin, finite element methods, the Stokes equations, divergence free.

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1. Introduction. The Stokes problem seeks unknown functions $u$ and $p$ satisfying

\begin{align}
-\nabla \cdot A \nabla u + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

where $\Omega$ is a polygonal domain in $\mathbb{R}^d$ with $d = 2, 3$ and $A$ is a symmetric and positive definite $d \times d$ matrix-valued function in $\Omega$. For the nonhomogeneous boundary condition

$$u = g \quad \text{on } \partial \Omega,$$

one can use the standard procedure by letting $u = u_0 + u_g$, $u_g$ is a known function satisfying $u_g = g$ on $\partial \Omega$ and $u_0$ is zero at $\partial \Omega$ and satisfies (1.1)-(1.2) with different right hand sides.

The weak form in the primary velocity-pressure formulation for the Stokes problem (1.1)-(1.3) seeks $u \in [H^1_0(\Omega)]^d$ and $p \in L^2_0(\Omega)$ satisfying

\begin{align}
(A \nabla u, \nabla v) - (\nabla \cdot v, p) &= (f, v), \quad \forall v \in [H^1_0(\Omega)]^d \\
(\nabla \cdot u, q) &= 0, \quad \forall q \in L^2_0(\Omega).
\end{align}
In the standard finite element methods for the Stokes and the Navier-Stokes equations, both pressure and velocity are approximated simultaneously. The primitive system is a large saddle point problem. Numerical solvers for such indefinite systems are usually less effective and robust than solvers for definite systems. On the other hand, the divergence-free finite element method, discrete or exact, computes numerical solution of velocity by solving a symmetric positive definite system in a divergence-free subspace. It eliminates the pressure from the coupled equations and hence significantly reduces the size of the system. The divergence-free method is particularly attractive in the cases where the velocity is the primary variable of interest, for example, the groundwater flow calculation. The main tasks in the implementation of the divergence-free method are to understand divergence-free subspaces, weakly or exactly, and to construct bases for them.

Many finite element methods, continuous \cite{2,3,8} and discontinuous \cite{1,4,11,12,16}, have been developed and analyzed for the Stokes and the Navier-Stokes equations. Divergence-free basis for different finite element methods have been constructed \cite{5,6,7,9,10,17,18,19}.

A weak Galerkin finite element method was introduced in \cite{15} for the Stokes equations in the primal velocity-pressure formulation. This method is designed by using discontinuous piecewise polynomials on finite element partitions with arbitrary shape of polygons/polyhedra. Weak Galerkin methods were first introduced in \cite{13,14} for second order elliptic equations. In general, weak Galerkin finite element formulations for partial differential equations can be derived naturally by replacing usual derivatives by weakly-defined derivatives in the corresponding variational forms, with an option of adding a stabilization term to enforce a weak continuity of the approximating functions. Therefore the weak Galerkin method developed in \cite{15} for the Stokes equations naturally has the form: find $u_h \in V_h$ and $p_h \in W_h$ satisfying

\begin{align}
(A \nabla_w u_h, \nabla_w v) + s(u_h, v) - (\nabla_w \cdot v, p_h) &= (f, v), \\
(\nabla_w \cdot u_h, q) &= 0
\end{align}

for all the test functions $v \in V_h$ and $q \in W_h$ where $V_h$ and $W_h$ will be defined later. The stabilizer $s(u_h, v)$ in (1.6) is parameter independent.

Let $D_h$ be a discrete divergence free subspace of $V_h$ such that $(\nabla_w \cdot v, q) = 0$ for all $q \in W_h$. Then the discrete divergence free WG formulation is to find $u_h \in D_h$ satisfying

\begin{align}
(A \nabla_w u_h, \nabla_w v) + s(u_h, v) &= (f, v), \\
&\forall v \in D_h.
\end{align}

System (1.8) is symmetric and positive definite with many fewer unknowns. The main purpose of this paper is to construct bases for $D_h$ in two and three dimensional spaces. A unique feature of these divergence free basis functions is that they can be obtained on general meshes such as hybrid meshes or meshes with hanging nodes. Numerical examples in two dimensional space are provided to confirm the theory. Although the Stokes equations is considered, the divergence free basis can be used for solving the Navier-Stokes equations.

2. A Weak Galerkin Finite Element Method. In this section, we will review the WG method for the Stokes equations introduced in \cite{15} with $k = 1$.

Let $T_h$ be a partition of the domain $\Omega$ consisting mix of polygons satisfying a set of conditions specified in \cite{14}. In addition, we assume that all the elements $T \in T_h$ are convex. Denote by $F_h$ the set of all edges in 2D or faces in 3D in $T_h$, and let $F^n_h = F_h \setminus \partial \Omega$ be the set of all interior edges or faces.
We define a weak Galerkin finite element space for the velocity as follows
\[ V_h = \{ \mathbf{v} = (\mathbf{v}_0, \mathbf{v}_b) : \{ \mathbf{v}_0, \mathbf{v}_b \} |_{T} \in [P_1(T)]^d \times [P_0(e)]^d, \ e \subset \partial T, \ \mathbf{v}_b = 0 \text{ on } \partial \Omega \} . \]

We would like to emphasize that there is only a single value \( \mathbf{v}_b \) defined on each edge in 2D and face in 3D. For the pressure variable, we have the following finite element space
\[ W_h = \{ q : q \in L_0^2(\Omega), q|_T \in P_0(T) \} . \]

For a given \( \mathbf{v} \in V_h \), a weak gradient and a weak divergence are defined locally on each \( T \in T_h \) as follows.

**Definition 2.1.** A weak gradient, denoted by \( \nabla_w \), is defined as the unique polynomial \( (\nabla_w \mathbf{v}) \in [P_0(T)]^{d \times d} \) for \( \mathbf{v} \in V_h \) satisfying the following equation,
\[ (\nabla_w \mathbf{v}, q)_T = - (\mathbf{v}_0, \nabla \cdot q)_T + (\mathbf{v}_b, q \cdot \mathbf{n})_{\partial T}, \quad \forall q \in [P_0(T)]^{d \times d}, \]

**Definition 2.2.** A weak divergence, denoted by \( (\nabla_w \cdot \cdot \cdot) \), is defined as the unique polynomial \( (\nabla_w \cdot \cdot \cdot) \in [P_0(T)]^d \) for \( \mathbf{v} \in V_h \) that satisfies the following equation
\[ (\nabla_w \cdot \cdot \cdot, \cdot \cdot \cdot)_T = -(\mathbf{v}_0, \nabla \cdot \cdot \cdot) + (\mathbf{v}_b \cdot \cdot \cdot, \mathbf{n})_{\partial T}, \quad \forall \cdot \cdot \cdot \in P_0(T). \]

Denote by \( Q_0 \) the \( L^2 \) projection operator from \( [L^2(T)]^d \) onto \([P_1(T)]^d \) and denote by \( Q_b \) the \( L^2 \) projection from \([L^2(e)]^d \) onto \([P_0(e)]^d \). Let \( (\mathbf{u}; p) \) be the solution of \( (1.1)-(1.3) \). Define \( Q_h \mathbf{u} = \{ Q_0 \mathbf{u}, Q_b \mathbf{u} \} \in V_h \). Let \( Q_h \) be the local \( L^2 \) projections onto \( P_0(T) \).

We introduce three bilinear forms as follows
\[ s(\mathbf{v}, \mathbf{w}) = \sum_{T \in T_h} h_T^{-1} \langle Q_h \mathbf{v}_0 - \mathbf{v}_b, \ Q_b \mathbf{w}_0 - \mathbf{w}_b \rangle_{\partial T}, \]
\[ a(\mathbf{v}, \mathbf{w}) = \sum_{T \in T_h} (A \nabla_w \mathbf{v}, \ \nabla_w \mathbf{w})_T + s(\mathbf{v}, \mathbf{w}), \]
\[ b(\mathbf{v}, q) = \sum_{T \in T_h} (\nabla_w \cdot \mathbf{v}, q)_T. \]

**Algorithm 1.** A numerical approximation for \( (1.1)-(1.3) \) can be obtained by seeking \( \mathbf{u}_h = \{ \mathbf{u}_0, \mathbf{u}_h \} \in V_h \) and \( p_h \in W_h \) such that
\[ a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{u}, p_h) = (f, \mathbf{v}), \quad \forall \mathbf{v} \in V_h, \]
\[ b(\mathbf{u}_h, q) = 0, \quad \forall q \in W_h. \]

Define
\[ \| \mathbf{v} \|^2 = a(\mathbf{v}, \mathbf{v}). \]

The following optimal error estimates have been derived in [15].

**Theorem 2.1.** Let \( (\mathbf{u}; p) \in [H^1_0(\Omega) \cap H^2(\Omega)]^d \times (L_0^2(\Omega) \cap H^1(\Omega)) \) and \( (\mathbf{u}_h; p_h) \in V_h \times W_h \) be the solution of \( (1.1)-(1.3) \) and \( (2.3)-(2.4) \), respectively. Then, the following error estimates hold true
\[ \| Q_h \mathbf{u} - \mathbf{u}_h \|^2 + \| Q_h p - p_h \| \leq C h \| \mathbf{u} \|_2 + \| p \|_1, \]
\[ \| Q_0 \mathbf{u} - \mathbf{u}_0 \| \leq C h^2 (\| \mathbf{u} \|_2 + \| p \|_1). \]
Define a discrete divergence free subspace $D_h$ of $V_h$ by

\[(2.8)\quad D_h = \{ \mathbf{v} \in V_h; \quad b(\mathbf{v}, q) = 0, \quad \forall q \in W_h \}.
\]

By taking the test functions from $D_h$, the weak Galerkin formulation \((2.3)-(2.4)\) is equivalent to the following divergence-free weak Galerkin finite element scheme.

**Algorithm 2.** A discrete divergence free WG approximation for \((1.1)-(1.3)\) is to find $\mathbf{u}_h = \{ \mathbf{u}_0, \mathbf{u}_b \} \in D_h$ such that

\[(2.9)\quad a(\mathbf{u}_h, \mathbf{v}) = (f, \mathbf{v}_0), \quad \forall \mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} \in D_h.
\]

System \((2.9)\) is symmetric and positive definite with many fewer unknowns. It can be solved effectively by many existing solvers.

The main task of this paper is to construct basis for $D_h$. In the next two sections, discrete divergence free bases will be constructed explicitly for two and three dimensional spaces.

**3. Construction of Discrete Divergence Free Basis for Two Dimensional Space.** For a given partition $\mathcal{T}_h$, let $V_h^0$ be the set of all interior vertices. Let $N_F = \text{card}(\mathcal{F}_h^0)$, $N_V = \text{card}(V_h^0)$ and $N_K = \text{card}(\mathcal{T}_h)$. It is known based on the Euler formula that for a partition consisting of convex polygons, then

\[(3.1)\quad N_F + 1 = N_V + N_K.
\]

For a mesh $\mathcal{T}_h$ with hanging nodes, the relation in \((3.1)\) is still true if we treat the hanging nodes as vertices.

First we need to derive a basis for $V_h$. For each $T \in \mathcal{T}_h$ and any $\mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} \in V_h$, $\mathbf{v}_0$ is a vector function with two components and each component is a linear function. Therefore there are six linearly independent linear functions $\Phi_{j+1}, \Phi_{j+2}, \ldots, \Phi_{j+6}$ in $V_h$ such that they are nonzero only at the interior of element $T$. For each $e_i \in \mathcal{F}_h^0$, $\mathbf{v}_b$ is a constant vector function. Thus, there are two linearly independent constant functions $\Psi_{i,1}$ and $\Psi_{i,2}$ in $V_h$ which take nonzero value only on $e_i$. Then it is easy to see that

\[(3.2)\quad V_h = \text{span}\{ \Phi_{1,1}, \ldots, \Phi_{6N_K}, \Psi_{1,1}, \Psi_{1,2}, \ldots, \Psi_{N_F,1}, \Psi_{N_F,2} \}.
\]

For a given function $\mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} \in V_h$, it is easy to see that $\mathbf{v}_0$ can be spanned by the basis functions $\Phi_i$ and $\mathbf{v}_b$ by the basis functions $\Psi_{j,k}$.

Next, we will find the dimension of $D_h$. Since the dimension for pressure space $W_h$ is $N_K - 1$, it follows from \((3.1)\) that

\[(3.3)\quad \dim(D_h) = \dim(V_h) - \dim(W_h) = 6N_K + 2N_F - N_K + 1 = 6N_K + N_F + N_V.
\]

**Lemma 3.1.** The basis functions $\Phi_{1,1}, \ldots, \Phi_{6N_K}$ of $V_h$ in \((3.2)\) are in $D_h$ and linearly independent.

**Proof.** Let $\Phi_{i} = \{ \Phi_{i,0}, \Phi_{i,b} \}$. The definition of $\Phi_{i}$ implies $\Phi_{i,b} = 0$. For any $q \in W_h$, it follows from \((2.2)\)

\[\begin{align*}
b(\Phi_i, q) &= \sum_{T \in \mathcal{T}_h} (\nabla w \cdot \Phi_i, q)_T \\
&= \sum_{T \in \mathcal{T}_h} (\nabla w, q)_T = 0,
\end{align*}
\]

\[\nabla q = 0\] and $\Phi_{i,b} = 0$,
which proves the lemma since the linear independence of \( \Phi_1, \cdots, \Phi_{6N_K} \) is obvious. \( \square \)

For any \( e_i \in \mathcal{F}_h^0 \), let \( \psi_{i,1} \) and \( \psi_{i,2} \) be two basis functions of \( V_h \) associated with \( e_i \). Let \( n_{e_i} \) and \( t_{e_i} \) be a normal vector and a tangential vector to \( e_i \) respectively. Define \( \Upsilon_i = C_1 \psi_{i,1} + C_2 \psi_{i,2} \) such that \( \Upsilon_i \mid_{e_i} = t_{e_i} \). Obviously \( \Upsilon_i \in V_h \) is only nonzero on \( e_i \).

**Lemma 3.2.** Functions \( \Upsilon_1, \cdots, \Upsilon_{N_F} \in V_h \) are in \( D_h \) and linearly independent.

**Proof.** Let \( \Upsilon_i = \{ \Upsilon_{i,0}, \Upsilon_{i,b} \} \). For any \( q \in W_h \), it follows from (2.2) and \( \nabla q = 0 \),

\[
b(\Upsilon_i, q) = \sum_{T \in \mathcal{T}_h} \langle \nabla_w \cdot \Upsilon_i, q \rangle_T = \sum_{T \in \mathcal{T}_h} (-\langle \Upsilon_{i,0}, \nabla q \rangle_T + \langle \Upsilon_{i,b} \cdot n, q \rangle_{\partial T}) = \sum_{T \in \mathcal{T}_h} \langle \Upsilon_{i,b} \cdot n, q \rangle_{\partial T} = 0,
\]

where we use the fact \( t_{e_i} \cdot n = 0 \). Since \( \Upsilon_i \) is only nonzero on \( e_i \), \( \Upsilon_1, \cdots, \Upsilon_{N_F} \) are linearly independent. We completed the proof. \( \square \)

For a given interior vertex \( P_i \in \mathcal{V}_h^0 \), assume that there are \( r \) elements having \( P_i \) as a vertex which form a hull \( \mathcal{H}_{P_i} \) as shown in Figure 3.1. Then there are \( r \) interior edges \( e_j \) \( (j = 1, \cdots, r) \) associated with \( \mathcal{H}_{P_i} \). Let \( n_{e_j} \) be a normal vector on \( e_j \) such that normal vectors \( n_{e_j}, j = 1, \cdots, r \) are counterclockwise around vertex \( P_i \) as shown in Figure 3.1. For each \( e_j \), let \( \Theta_{j,1} \) and \( \Theta_{j,2} \) be the two basis functions of \( V_h \) which is only nonzero on \( e_j \). Define \( \Theta_j = C_1 \Theta_{j,1} + C_2 \Theta_{j,2} \in V_h \) such that \( \Theta_j \mid_{e_j} = n_{e_j} \). Define \( \Lambda_i = \sum_{j=1}^{r} \frac{1}{|e_j|} \Theta_j \).

**Lemma 3.3.** Functions \( \Lambda_1, \cdots, \Lambda_{N_V} \in V_h \) are in \( D_h \) and linearly independent.

**Proof.** Suppose that there exist constants \( c_1, \cdots, c_{N_V} \) such that \( \sum_{i=1}^{N_V} c_i \Lambda_i = 0 \). Let \( \Lambda_i \) be associated with a hull \( \mathcal{H}_{P_i} \) such that there exists \( e_m \) as one of the interior edges of \( \mathcal{H}_{P_i} \) and edge \( e_m \) has a boundary node as one of its end points. Since \( \Lambda_i = 0 \) on \( e_m \) for \( i \neq l \), we have

\[
0 = \sum_{i=1}^{N_V} \int_{e_m} c_i \Lambda_i = \int_{e_m} c_l \Lambda_l = \int_{e_m} c_l \Theta_m = c_l n_{e_m},
\]

which implies \( c_l = 0 \). By this way, we can prove that all \( c_i = 0 \) and \( \Lambda_1, \cdots, \Lambda_{N_V} \) are linearly independent. Next, we will show that \( b(\Lambda_i, q) = 0 \) for all \( q \in W_h \). Let
\( q_j \in W_h \) such that \( q_j = 1 \) on \( T_j \in \mathcal{T}_h \) and \( q_j = 0 \) otherwise. So we only need to show that
\[
(3.4) \quad b(\Lambda_i, q_j) = 0, \quad \forall j = 1, \ldots, N_K.
\]
Let \( \Lambda_i \) and \( q_j \) be associated with hull \( \mathcal{H}_P \) and element \( T_j \) respectively. If \( T_j \in \mathcal{T}_h \) is not in \( \mathcal{H}_P \), we easily have \( b(\Lambda_i, q_j) = 0 \). If \( T_j \in \mathcal{H}_P \), let \( e_s \) and \( e_{s+1} \) be its two edges in \( \mathcal{H}_P \) shown in Figure 3.3.

\[
b(\Lambda_i, q_j) = \sum_{T \in \mathcal{T}_h} (\nabla_w \cdot \Lambda_i, q_j)_T
\]
\[
= (\nabla_w \cdot \Lambda_i, q_j)_{T_j}
\]
\[
= -(\Lambda_i, 0, \nabla q)_{T_j} + (\Lambda_i, \hat{n}, q)_{\partial T_j}
\]
\[
= \int_{e_s} \frac{1}{|e_s|} n_{e_s} \cdot \hat{n} + \int_{e_{s+1}} \frac{1}{|e_{s+1}|} n_{e_{s+1}} \cdot \hat{n}
\]
\[
= 0.
\]

We proved \( \Lambda_i \in D_h \) for \( i = 1, \ldots, NV \). \( \Box \)

**Theorem 3.4.** Let \( D_h \) be defined in (2.8). Then for two dimensional space, \( D_h \) is spanned by the following basis functions,
\[
(3.5) \quad D_h = \text{Span}\{\Phi_1, \ldots, \Phi_{6N_K}, \Upsilon_1, \ldots, \Upsilon_{NV}, \Lambda_1, \ldots, \Lambda_{NV}\}.
\]

**Proof.** The number of the functions in the right hand side of (3.5) is \( 6N_K + NV \) which is equal to \( \dim(D_h) \) due to (3.3). Next, we prove that
\[
\Phi_1, \ldots, \Phi_{6N_K}, \Upsilon_1, \ldots, \Upsilon_{NV}, \Lambda_1, \ldots, \Lambda_{NV}
\]
are linearly independent. Since \( \Phi_i \) take zero value on all \( f \in \mathcal{F}_h \), \( \Phi_i \) will be linearly independent to all \( \Upsilon_i \) and \( \Lambda_m \). Suppose
\[
(3.6) \quad C_1 \Upsilon_1 + \cdots + C_{NV} \Upsilon_{NV} + C_{N_F+1} \Lambda_1 + \cdots + C_{N_F+NV} \Lambda_{NV} = 0.
\]

Multiplying (3.6) by \( \Upsilon_i \) and integrating over \( e_i \), we have
\[
C_i |e_i| = 0,
\]
where we use the fact \( t_{e_i} \cdot n_{e_i} = 0 \). Thus we can obtain \( C_i = 0 \) for \( i = 1, \ldots, NF \). By Lemma 3.3 we can prove \( C_i = 0 \) for \( i = NF + 1, \ldots, NF + NV \). The proof of the lemma is completed. \( \Box \)

**4. Construction of Discrete Divergence Free Basis for Three Dimensional Space.** Let \( \mathcal{T}_h \) be a partition of \( \Omega \subset \mathbb{R}^3 \) consisting polyhedrons without hanging nodes. Recall \( NF = \text{card}(\mathcal{F}_h^0) \), \( NV = \text{card}(\mathcal{V}_h^0) \) and \( NK = \text{card}(\mathcal{T}_h) \). Denote by \( \mathcal{E}_h \) all the edges in \( \mathcal{T}_h \) and let \( \mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega \). Let \( NE = \text{card}(\mathcal{E}_h^0) \).

It is known based on the Euler formula that for a partition consisting of convex polyhedrons, then
\[
(4.1) \quad NV + NF + 1 = NE + NK.
\]

For each \( T \in \mathcal{T}_h \) and any \( v = \{v_0,v_b\} \in V_h \), \( v_0 \) is a vector function with three components and each component is a linear function. Therefore there are twelve
linearly independent linear functions $\Phi_{j+1}, \Phi_{j+2}, \ldots, \Phi_{j+12}$ in $V_h$ such that they are nonzero only at the interior of element $T$. For each face $f_i \in \mathcal{F}_h^n$, $v_h$ is a constant vector function with three component. Thus there are three linearly independent constant vector functions $\Psi_{i,1}, \Psi_{i,2}$ and $\Psi_{i,3}$ in $V_h$ which take nonzero value only on the face $f_i$. Then it is easy to see that

$$V_h = \text{span}\{\Phi_1, \ldots, \Phi_{12N_K}, \Psi_{1,1}, \Psi_{1,2}, \Psi_{1,3}, \cdots, \Psi_{N_F,1}, \Psi_{N_F,2}, \Psi_{N_F,3}\}.$$  

Since the dimension for pressure space $W_h$ is $N_K - 1$, (4.1) implies

$$\dim(D_h) = \dim(V_h) - \dim(W_h) = 12N_K + 3N_F - N_K + 1 = 12N_K + 2N_F + N_E - N_V.$$  

**Lemma 4.1.** The functions $\Phi_1, \ldots, \Phi_{12N_K}$ in (4.2) are in $D_h$ and linearly independent.

**Proof.** Let $\Phi_i = \{\Phi_{i,0}, \Phi_{i,b}\}$. The definition of $\Phi_i$ implies $\Phi_{i,b} = 0$. For any $q \in W_h$, it follows from (2.2), $\nabla q = 0$ and $\Phi_{i,b} = 0$ that for any $q \in W_h$,

$$b(\Phi_i, q) = \sum_{T \in T_h} (\nabla_w \cdot \Phi_i, q)_T$$

$$= \sum_{T \in T_h} (-\langle \Phi_{i,0}, \nabla q \rangle_T + \langle \Phi_{i,b}, n \rangle_{\partial T})$$

$$= 0,$$

which finished the proof of the lemma since the linear independence of $\Phi_1, \ldots, \Phi_{12N_K}$ is obvious. □

For any face $f_i \in \mathcal{F}_h^n$, let $n$ be a unit normal vector of $f_i$ and let $t_1$ and $t_2$ be two linearly independent unit tangential vectors to the face $f_i$. Define $\Upsilon_{i,1} = C_{1,1}\Psi_{i,1} + C_{1,2}\Psi_{i,2} + C_{1,3}\Psi_{i,3}$ and $\Upsilon_{i,2} = C_{2,1}\Psi_{i,1} + C_{2,2}\Psi_{i,2} + C_{2,3}\Psi_{i,3}$ such that $\Upsilon_{i,1}|_{f_i} = t_1$ and $\Upsilon_{i,1}|_{f_i} = t_2$ respectively. Obviously $\Upsilon_{i,1}$ and $\Upsilon_{i,2}$ are in $V_h$ and only nonzero on $f_i$.

**Lemma 4.2.** Functions $\Upsilon_{1,1}, \Upsilon_{1,2}, \cdots, \Upsilon_{N_F,1}, \Upsilon_{N_F,2} \in V_h$ are in $D_h$ and linearly independent.

**Proof.** Let $\Upsilon_{i,j} = \{\Upsilon_{i,j,0}, \Upsilon_{i,j,b}\}$ with $j = 1, 2$. For any $q \in W_h$, it follows from (2.2) and $\nabla q = 0$ for $j = 1, 2$,

$$b(\Upsilon_{i,j}, q) = \sum_{T \in T_h} (\nabla_w \cdot \Upsilon_{i,j}, q)_T$$

$$= \sum_{T \in T_h} (-\langle \Upsilon_{i,j,0}, \nabla q \rangle_T + \langle \Upsilon_{i,j,b}, n \rangle_{\partial T})$$

$$= \sum_{T \in T_h} \langle \Upsilon_{i,j,b}, n \rangle_{\partial T}$$

$$= 0,$$

where we use the fact $t_j \cdot n = 0$ with $j = 1, 2$. Since $\Upsilon_{i,j}$ is only nonzero on $f_i$, $\Upsilon_{1,1}, \Upsilon_{1,2}, \cdots, \Upsilon_{N_F,1}, \Upsilon_{N_F,2}$ are linearly independent. We completed the proof. □

For a given interior edge $E_i \in E_h$, assume there are $r$ elements having $E_i$ as one of their edges which form a solid denoted by $\mathcal{S}_{E_i}$ shown in Figure 4.1. Then there are
r interior faces $f_j$ $(j=1,\cdots,r)$ in $S_{E_i}$. Let $n_j$ be a unit normal vector on the face $f_j$ such that normal vectors $n_j$ for $j=1,\cdots,r$ form oriented loop around the interior edge $E_i$ shown in Figure 4.1. For each $f_j$, let $\Psi_{j,1}$, $\Psi_{j,2}$ and $\Psi_{j,3}$ be the three basis functions of $V_h$ which are only nonzero on $f_j$. Define $\Theta_j = C_1 \Psi_{j,1} + C_2 \Psi_{j,2} + C_3 \Psi_{j,3} \in V_h$ such that $\Theta_j|_{f_j} = n_j$. Define $\Lambda_i = \sum_{j=1}^r (\Psi_{j,1} \Psi_{j,2} \Psi_{j,3})$.

**Lemma 4.3.** Functions $\Lambda_1, \cdots, \Lambda_N \in V_h$ are in $D_h$.

**Proof.** Let $\Lambda_i$ and $q_j$ be associated with hull $S_{E_i}$ and element $T_j$ respectively. If $T_j \not\in S_{E_i}$, we easily have $b(\Lambda_i, q_j) = 0$. If $T_j \in S_{E_i}$, let faces $f_s$ and $f_{s+1}$ be its two faces in $S_{E_i}$ shown in Figure 4.1.

$$b(\Lambda_i, q_j) = \sum_{T \in T_h} (\nabla_w \Lambda_i, q_j)_T$$
$$= (\nabla_w \Lambda_i, q_j)_{T_j}$$
$$= -\langle \Lambda_i, \nabla q \rangle_{T_j} + (\Lambda_i \nabla q)_{\partial T_j}$$
$$= \int_{f_s} \frac{1}{|f_s|} n_{f_s} \cdot n + \int_{f_{s+1}} \frac{1}{|f_{s+1}|} n_{f_{s+1}} \cdot n$$
$$= 0$$

We proved $\Lambda_i \in D_h$ for $i=1,\cdots,N_E$.

Unfortunately, $\Lambda_1, \cdots, \Lambda_N$ are linearly dependent. Let $P_i$ be an interior vertex in $T_h$ and $G_{P_i}$ be a hull formed by the elements $T \in T_h$ sharing $P_i$. Let $e_j \in E_{P_i}^0$, $j=1,\cdots,t$ with $P_i$ as one of its end point and $\Lambda_j$, $j=1,\cdots,t$ be the discrete divergence free functions associated with $e_j$. With appropriate choosing $n_j$ in defining $\Lambda_j$, one can prove that $\sum_{j=1}^t \Lambda_j = 0$. However, if we eliminate one function from $\{\Lambda_1, \cdots, \Lambda_t\}$ randomly, say $\Lambda_1$, we will prove that $\{\Lambda_2, \cdots, \Lambda_t\}$ are linearly independent in the following lemma.

**Lemma 4.4.** Functions $\Lambda_2, \cdots, \Lambda_t$ are linearly independent.

**Proof.** Let $f \in F_{h}^0$ be an interior face in $G_{P_i}$ with $e_1$ and $e_2$ as its two edges in $G_{P_i}$. The definition of $\Lambda_j$ implies that only $\Lambda_1$ and $\Lambda_2$ are nonzero on $f$. Suppose
that there exist constants \(c_2, \ldots, c_t\) such that \(\sum_{i=2}^{t} c_i \Lambda_i = 0\). Then we have
\[
0 = \sum_{i=2}^{t} \int f c_i \Lambda_i = \int f c_2 \Lambda_2 = \int f c_2 \Theta_f = c_2 n_f,
\]
which implies \(c_2 = 0\). By this way, we can prove that all \(c_i = 0\) and \(\Lambda_2, \ldots, \Lambda_t\) are linearly independent.

We start with \(\{\Lambda_1, \ldots, \Lambda_{N_E}\}\) and eliminate one function for each \(G_P\) for \(i = 1, \ldots, N_V\). With renumbering the functions, we end up with \(N_E - N_V\) discrete divergence free functions: \(\{\Lambda_1, \ldots, \Lambda_{N_E-N_V}\}\).

**Lemma 4.5.** Functions \(\{\Lambda_1, \ldots, \Lambda_{N_E-N_V}\}\) are linearly independent.

**Proof.** The proof of the lemma is similar to the proofs of Lemma 3.3 and Lemma 4.4.

**Theorem 4.6.** Let \(D_h\) be defined in (2.8). Then for three dimensional space, \(D_h\) is spanned by the following basis functions,
\[
(4.4) \quad D_h = \text{Span}\{\Phi_1, \ldots, \Phi_{12N_K}, \Upsilon_{1,1}, \Upsilon_{1,2}, \ldots, \Upsilon_{N_F,1}, \Upsilon_{N_F,2}, \Lambda_1, \ldots, \Lambda_{N_E-N_V}\}.
\]

**Proof.** The number of the functions in the right hand side of (4.4) is \(12N_K + 2N_F + N_E - N_V\) which is equal to \(\text{dim}(D_h)\) due to (4.3). Similar to the proof of Theorem 3.4, we can prove that \(\{\Phi_1, \ldots, \Phi_{6N_K}, \Upsilon_{1,1}, \Upsilon_{1,2}, \ldots, \Upsilon_{N_F,1}, \Upsilon_{N_F,2}, \Lambda_1, \ldots, \Lambda_{N_E-N_V}\}\) are linear independent.

**5. Numerical Experiments.** In this section, we shall report several results of numerical examples for two dimensional Stokes equations. The divergence-free finite element scheme introduced in Algorithm 2 is used. The main purpose is to numerically validate the accuracy and efficiency of the WG scheme.

Let \(v_h \in D_h\) and \(q_h \in W_h\), the error for the WG-FEM solution is measured in three norms defined as follows:
\[
\begin{align*}
\|v_h\|_2 &:= \sum_{T \in T_h} \left( \int_T |\nabla_w v_h|^2 dT + h_T^{-1} \int_{\partial T} (v_0 - v_h)^2 ds \right), \quad \text{(A discrete } H^1\text{-norm)}, \\
\|v_0\|_2 &:= \sum_{T \in T_h} \int_T |v_0|^2 dx, \quad \text{(Element-based } L^2\text{-norm}).
\end{align*}
\]

**Table 5.1**

| \(h\) | \(\|u_h - Q_h u\|\) | order | \(\|u_0 - Q_h u\|\) | order |
|---|---|---|---|---|
| 1/4 | 8.1050e-01 | | 2.9957e-01 | |
| 1/8 | 6.9698e-01 | 2.9957e-01 | 9.9634e-02 | 1.5882 |
| 1/16 | 4.4578e-01 | 6.6479e-01 | 3.1031e-02 | 1.6829 |
| 1/32 | 2.4452e-01 | 8.6638e-01 | 8.5507e-03 | 1.8596 |
| 1/64 | 1.2620e-01 | 9.5424e-01 | 2.2131e-03 | 1.9500 |
| 1/128 | 6.3751e-02 | 9.8519e-01 | 5.5968e-04 | 1.9834 |
Fig. 5.1. Example 1: Level 1 of mixed polygonal mesh.

Table 5.2
Test Case 1: Numerical error and convergence rates for the Stokes equation with homogeneous boundary conditions on the mixed polygonal meshes.

| $h$      | $||u_h - Q_h u||$ | order | $||u_0 - Q_0 u||$ | order |
|----------|------------------|-------|------------------|-------|
| 4.1016e-01 | 8.191e-01        |       | 3.0927e-01       |       |
| 2.0508e-01 | 7.0386e-01       |       | 2.1887e-01       | 1.5694|
| 1.0254e-01 | 6.4002e-01       | 3.0579| 1.3478e-01       | 1.6382|
| 5.1270e-02 | 2.5560e-01       | 8.4781| 9.4392e-03       | 1.8265|
| 2.5635e-02 | 1.3230e-01       | 9.5007e-01| 2.4560e-03 | 1.9424 |
| 1.2818e-02 | 6.6890e-02       | 9.8401e-01| 6.2217e-04 | 1.9810 |

5.1. Test case 1. The domain is set as $\Omega = (0, 1) \times (0, 1)$. Let the exact solution $u$ and $p$ as follows,

$$u = \left( 10x^2y(x-1)^2(2y-1)(y-1) -10xy^2(2x-1)(x-1)(y-1)^2 \right)$$

and $p = 10(2x-1)(2y-1)$.

It is easy to check that homogeneous Dirichlet boundary condition is satisfied for this testing. The right hand side function $f$ is given to match the exact solutions.

The first test shall be performed on the uniform rectangular meshes and the mixed polygonal meshes. The uniform rectangular meshes are generated by partition the domain $\Omega$ into $n \times n$ sub-rectangles. The mesh size is denoted by $h = 1/n$. Moreover, the WG divergence free algorithm is also test on the mixed polygonal type meshes. We start with the initial mesh shown as the Figure 5.1, which contains the mixture of triangles and quadrilaterals. The next level of mesh is to refine the previous level of mesh by connecting the mid-point on each edge. The mesh size in this case is also denoted by $h$.

The error profile is reported in Table 5.2 for the rectangular meshes and mixed polygonal meshes, respectively. Both of the tables show the same convergence rate as the theoretical conclusion, which is $O(h)$ in the $H^1$-norm and $O(h^2)$ in the $L^2$-norm.
5.2. Test case 2. The domain is given by $\Omega = (0, 1) \times (0, 1)$. Let the exact solutions $u$ and $p$ as follows,

$$u = \begin{pmatrix} x(1-x)(1-2y) \\ -y(1-y)(1-2x) \end{pmatrix}, \text{ and } p = 2(y-x).$$

The Dirichlet boundary condition and the right hand side function is set to match the above exact solutions. It is easy to check that the exact solution $u$ satisfies the non-homogeneous boundary condition.

For this testing, the WG divergence free algorithm is performed on the triangular grids. The uniform triangular grids are generated by: (1) partition the domain into $n \times n$ sub-rectangles; (2) divide each square element into two triangles by the diagonal line with a negative slope. The mesh size is denoted by $h = 1/n$.

For the calculation of the pressure $p_h$, we shall make use of the basis function $v \in V_h \setminus D_h$. This basis function is corresponding to the velocity $v_b$ related to the normal direction on each edge. Let $v \in V_h \setminus D_h$, the pressure $p_h$ is computed as follows,

$$b(v, p_h) = a(u_h, v) - (f, v_0).$$

Beside testing two norms of the error in velocity, we also measure the $L^2$-error in pressure. The numerical results in Table 5.3 show an $O(h)$ convergence in the $\| \cdot \|$ norm for velocity, $O(h^2)$ convergence in the $L^2$-norm for velocity, and $O(h)$ convergence in the $L^2$-norm for pressure, which are confirmed by Theorem 2.1.

| $h$       | $\|u_h - Q_h u\|$ | $\|u_0 - Q_0 u\|$ | $\|p_h - p\|$ |
|-----------|---------------------|---------------------|---------------------|
| 2.5000e-01| 2.8901e-01          | 4.2990e-02          | 2.2624e-01          |
| 1.2500e-01| 1.4367e-01          | 1.0896e-02          | 1.2246e-01          |
| 6.2500e-02| 7.1997e-02          | 2.7432e-03          | 6.4525e-02          |
| 3.1250e-02| 3.6052e-02          | 6.8773e-04          | 3.3224e-02          |
| 1.5625e-02| 1.8037e-02          | 1.7210e-04          | 1.6871e-02          |
| 7.8125e-03| 9.0202e-03          | 4.3038e-05          | 8.5037e-03          |
| Conv.Rate | 9.9966e-01          | 1.9934              | 9.4871e-01          |

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REFERENCES

[1] B. Cockburn, G. Kanschat, D. Schötzau, and C. Schwab, Local discontinuous Galerkin methods for the Stokes system, SIAM J. Numer. Anal., 40 (2002) 319–343.

[2] M. Crouzeix and P. A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, RAIRO Anal. Numer., 7 (1973) 33–76.

[3] V. Girault and P.A. Raviart, Finite Element Methods for the Navier-Stokes Equations: Theory and Algorithms, Springer-Verlag, Berlin, 1986.

[4] V. Girault, B. Riviére, and M.F. Wheeler, A discontinuous Galerkin method with nonconforming domain decomposition for Stokes and Navier-Stokes problems, Math. Comp., 74 (2004) 53–84.

[5] D. Griffiths, Finite element for incompressible flow, Math. Meth. in Appl. Sci., 1 (1979) 16–31.

[6] D. Griffiths, The construction of approximately divergence-free finite element, The Mathematics of Finite Element and Its Applications III, Ed. J.R. Whiteman, Academic Press, 1979.

[7] D. Griffiths, An approximately divergence-free 9-node velocity element for incompressible flows, Inter. J. Num. Meth. in Fluid, 1 (1981) 323–346.

[8] M. D. Gunzburger, Finite Element Methods for Viscous Incompressible Flows, A Guide to Theory, Practice and Algorithms, Academic, San Diego, 1989.

[9] K. Gustafson and R. Hartman, Divergence-free basis for finite element schemes in hydrodynamics, SIAM J. Numer. Anal., 20 (1983) 697–721.

[10] K. Gustafson and R. Hartman, Graph theory and fluid dynamics, SIAM J. Alg. Disc. Meth., 6 (1985) 643-656.

[11] O. A. Karakashian and W. N. Jureidini, A nonconforming finite element method for the stationary Navier-Stokes equations, SIAM J. Numerical Analysis, 35 (1998) 93–120.

[12] J. Liu and C. Shu, A high order discontinuous Galerkin method for 2D incompressible flows, J. Comput. Phys., 160 (2000) 577–596.

[13] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comp. and Appl. Math, 241 (2013) 103-115.

[14] J. Wang and X. Ye, A Weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp., 83 (2014), 2101-2126.

[15] J. Wang and X. Ye, A Weak Galerkin Finite Element Method for the Stokes Equations, Advances in Computational Mathematics, DOI 10.1007/s10444-015-9415-2, [arXiv:1302.2707]

[16] J. Wang and X. Ye, New finite element methods in computational fluid dynamics by $H(div)$ elements, SIAM J. Numer. Anal., 45 (2007) 1269–1286.

[17] J. Wang, X. Wang and X. Ye, Finite Element Methods for the Navier-Stokes equations by $H(div)$ Elements, Journal of Computational Mathematics, 26 (2008) 1–28.

[18] X. Ye and C. Hall, A Discrete divergence free basis for finite element methods, Numerical Algorithms, 16 (1997) 365–380.

[19] X. Ye and C. Hall, The Construction of null basis for a discrete divergence operator, J. Computational and Applied Mathematics, 58 (1995) 117–133.