Current fluctuations in non-interacting run-and-tumble particles in one-dimension

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We present a general framework to study the distribution of the flux through the origin up to time \( t \), in a non-interacting one-dimensional system of particles with a step initial condition with a fixed density \( \rho \) of particles to the left of the origin. We focus principally on two cases: (i) when the particles undergo diffusive dynamics (passive case) and (ii) run-and-tumble dynamics for each particle (active case). In analogy with disordered systems, we consider the flux distribution both for the annealed and the quenched initial conditions, for the passive and active particles. In the annealed case, we show that, for arbitrary particle dynamics, the flux distribution is a Poissonian with a mean \( \mu(t) \) that we compute exactly in terms of the Green’s function of the single particle dynamics. For the quenched case, we show that, for the run-and-tumble dynamics, the quenched flux distribution takes an anomalous large deviation form at large times \( P_{\text{Q}}(Q,t) \sim \exp\left(-\rho v_0 \gamma t^2 \psi_{\text{RTP}}\left(\frac{Q}{v_0}\right)\right) \), where \( \gamma \) is the rate of tumbling and \( v_0 \) is the ballistic speed between two successive tumblings. In this paper, we compute the rate function \( \psi_{\text{RTP}}(q) \) and show that it is nontrivial. Our method also gives access to the probability of the rare event that, at time \( t \), there is no particle to the right of the origin. For diffusive and run-and-tumble dynamics, we find that this probability decays with time as a stretched exponential, \( \sim \exp(-ct\sqrt{t}) \) where the constant \( c \) can be computed exactly. We verify our results for these large deviations by using an importance sampling Monte-Carlo method.

I. INTRODUCTION

Current fluctuations in non-equilibrium open systems has been a major area of research in statistical physics over the last few years [1–9]. The probability distribution of the current of particles across a given point in space typically admits a large deviation form and the corresponding large deviation function (rate function) is often interpreted as an analogue of a free energy in non-equilibrium systems. For example, it was shown in [1] that for a large one-dimensional system in contact with reservoirs at unequal densities at the two ends, there exists an additivity principle obeyed by the large deviation function, much like the free energy in equilibrium systems. This additivity principle has been exploited further to compute cumulants of the current distribution [1–8]. Several theoretical tools have been developed to study current fluctuations, notable among them are the Macroscopic Fluctuation Theory [1–3,6,7] and the Bethe Ansatz [10,7]. Most of these studies have focused on driven diffusive systems, both interacting (as in the simple symmetric exclusion process) and non-interacting, typically in a one-dimensional setting.

Another inherently out-of-equilibrium system, much studied recently, is the so-called active run-and-tumble particle (RTP) [10–12], a new incarnation of the persistent random walk [14,15]. Such motion has been observed in certain bacteria such as E. Coli where the bacterium moves in straight runs, undergoes tumbling at the end of a run and chooses randomly a new direction for the next run [16,17]. These motions are inherently out-of-equilibrium since they consume energy directly from the environment and self-propel themselves without any external force. There has been an enormous amount of work concerning the collective properties of an assembly of such RTPs [16,21]. Even at the single-particle level, RTP displays interesting behaviour and several single-particle observables have been studied recently. These include the position distribution for a free RTP [12,14,21], non-Boltzmann stationary states for an RTP in a confining potential [14,22,25], effects of disordered potentials [26], first-passage properties [27,32], the distribution of the time at which an RTP reaches its maximum displacement [33] and RTP subjected to stochastic resetting [34–36].

However, as far as we are aware, current fluctuations, even in a system of noninteracting RTPs have not been systematically studied. The purpose of this paper is to study the current fluctuations in the simplest setup where RTPs are noninteracting and initially confined on one-half of the real line (i.e. step-function initial condition). Such a setup was used before by Derrida and Gerschenfeld for noninteracting diffusive particles [5] and they were able to compute the large-deviation form of the current or flux \( Q_t \) of particles through the origin up to time \( t \). In this paper, we use exactly the same setup, but for a more general class of noninteracting particles, which includes both diffusive as well as run-and-tumble particles, and compute analytically the flux distribution up to time \( t \).

Thus the main observable of our interest is the flux \( Q_t \) defined as the number of particles that crossed the origin (either from left or right) up to time \( t \), starting from the step initial condition where the particles are uniformly distributed over only the left side of the origin. Let us denote its probability distribution by \( P(Q,t) = \text{Prob}(Q_t = Q) \). Clearly \( Q_t \) is a history dependent quantity, since it involves counting of all the crossings of the origin up to time \( t \). Our
exact results rely on a simple but crucial observation, valid for this special step initial condition: each particle, starting from the left side of the origin, that crosses the origin an even number of times up to time $t$ does not contribute to the flux $Q_t$. But if it crosses the origin an odd number of times, it contributes unity to the flux $Q_t$. Hence, the flux $Q_t$ is exactly equal to the number $N_t^+$ of particles present on the right side of the origin at time $t$, i.e. $Q_t = N_t^+$. Thus the history dependent observable $Q_t$ gets related, via this observation, to $N_t^+$ which is an instantaneous observable at time $t$. As we will see later, it is much easier to compute the probability distribution $P(N^+, t) = \text{Prob}(N_t^+ = N^+)$, rather than the distribution of $Q_t$ directly. Hence, knowing the distribution of $N_t^+$, we can compute the flux distribution from $P(Q, t) = \text{Prob}(N_t^+ = Q)$. Note that this equivalence holds for arbitrary dynamics of the particles, for example it holds both for diffusive as well as RTP dynamics of the particles.

Based on this connection $Q_t = N_t^+$, we can apply our results for the flux distribution to another interesting problem. Consider for instance an ideal gas of noninteracting particles in a box. Imagine that the box is divided into two halves by a removable wall. Initially, all the particles are on the left half of the box and at $t = 0$ we lift the wall and let the particles explore the full box freely. At time $t$, we take a snapshot of the system and observe the locations of the particles. Of course, on an average, one expects that, at long times, the particles will be uniformly distributed throughout the box. We can ask: what is the probability that, at time $t$, all the particles are again back to the left half of the box? Clearly this is an extremely rare event but what is the probability of this event? How does it decay with time? But note that this is exactly the probability $\text{Prob}(N_t^+ = 0) = P(Q_t = 0, t)$. Hence, our computation of the flux distribution with step initial condition gives access to the probability of this rare event (corresponding to all the particles coming back to the left half of the box at time $t$), in a one-dimensional setting.

The rest of the paper is organised as follows. In section II we discuss the model and present a summary of our main results. Then in Sec. III we introduce the general setting and show how the single-particle Green’s function plays the central role in the analysis. Next in Secs. IV and V we calculate the annealed and quenched averages, respectively, of the probability distribution of the flux for both diffusive and run-and-tumble particles. Then in Sec. VI we give the numerical verifications of our results and finally in Sec. VII we summarize and conclude.

II. THE MODEL AND THE MAIN RESULTS

We consider a set of $N$ noninteracting particles initially distributed uniformly with a density $\rho$ on the negative real axis, as in Fig. 1. Without loss of generality, we label the particles $i = 1, 2, \ldots, N$ with $x_i(t)$ denoting the position of the $i$-th particle at time $t$. Each $x_i(t)$ evolves independently by a stochastic (or deterministic) evolution rule (the same law of evolution for each particle). For example, each particle can undergo independent Brownian motion. Alternatively, each particle can undergo independent RTP dynamics in one-dimension. This RTP dynamics for a single particle is defined as follows.

\[ \frac{dx}{dt} = v_0 \sigma(t) \]  

where $v_0$ is the intrinsic speed during a run and $\sigma(t) = \pm 1$ is a dichotomous telegraphic noise that flips from one state to another with a constant rate $\gamma$. The effective noise $\xi(t) = v_0 \sigma(t)$ is coloured which is simply seen by computing

**FIG. 1:** Schematic representation of an initial realization with all particles on the left of an arbitrary origin ($x = 0$) on an infinite line $L \to \infty$. After time $t$ each particle undergoes some displacement depending on the dynamics. The quantity of interest in our case is the number of particles on the right of the origin at time $t$.
its autocorrelation function

\[ \langle \xi(t) \xi(t') \rangle = v_0^2 e^{-2\gamma|t-t'|}. \] (2)

The time scale \( \gamma^{-1} \) is the ‘persistence’ time of a run that encodes the memory of the noise. In the limit \( \gamma \to \infty, v_0 \to \infty \) but keeping the ratio \( D_{\text{eff}} = v_0^2/2\gamma \) fixed, the noise \( \xi(t) \) reduces to a white noise since

\[ \langle \xi(t) \xi(t') \rangle = \frac{v_0^2}{\gamma} \left[ \gamma e^{-2\gamma|t-t'|} \right] \to 2D_{\text{eff}} \delta(t-t'). \] (3)

Thus in this so called ‘diffusive limit’, the persistent random walker \( x(t) \) reduces to an ordinary Brownian motion, which we refer to as a “passive” motion. In the diffusive limit, the active particle dynamics reduces to an ordinary Brownian motion, which we refer to as a “passive” motion.

Given the stochastic dynamics of the individual particles, starting from the step initial condition, our main object of interest is the flux \( Q_t \) of particles through the origin up to time \( t \). If a trajectory crosses the origin from left to right, this will contribute a +1 to the net current while if it crosses from right to left, its contribution is −1. The flux \( Q_t \) is thus the net contribution to the current up to time \( t \). Let us denote by \( P(Q, t, \{x_i\}) \) the probability distribution \( \text{Prob}(Q_t = Q) \) for a given initial condition where \( x_i \)’s denote the initial positions of the particles at time \( t = 0 \). Following Derrida and Gerschenfeld, the effect of the initial condition on the distribution can be studied in two alternative ways, in analogy with the disordered systems where the realisation of a disorder plays an analogous role as the initial condition in our problem. Instead of considering the distribution \( P(Q, t, \{x_i\}) \) directly, it turns out to be convenient to consider its generating function \( \langle e^{-pQ}\rangle_{\{x_i\}} \), where the angular brackets \( \langle \cdots \rangle_{\{x_i\}} \) denote an average over the history, but with fixed initial condition \( x_i \). The annealed and quenched averages are now defined as follows:

\[ \sum_{Q=0}^{\infty} e^{-pQ} P_{\text{an}}(Q, t) = \langle e^{-pQ}\rangle_{\{x_i\}}, \] (4)

\[ \sum_{Q=0}^{\infty} e^{-pQ} P_{\text{qu}}(Q, t) = \exp \left[ \ln\langle e^{-pQ}\rangle_{\{x_i\}} \right], \] (5)

where \( \overline{\cdots} \) denotes an average over the initial conditions. Note that in this problem \( Q_t \) is always an integer. As mentioned in the introduction, for the step initial condition, we can compute both \( P_{\text{an}}(Q, t) \) and \( P_{\text{qu}}(Q, t) \) for arbitrary dynamics of the particles by using the identity \( Q_t = N_t^+ \), where \( N_t^+ \) is the number of particles on the right side of the origin at time \( t \). Indeed, the only quantity that enters the computation for independent particles is the single particle Green’s function \( G(x, x_0, t) \) denoting the probability density of finding the particle at position \( x \) at time \( t \), starting from \( x_0 \) at \( t = 0 \). Let us first define a central object, that will appear in all our formulas

\[ U(z, t) = \int_0^{\infty} G(x, -z, t) \, dx, \quad z \geq 0, \] (6)

obtained by integrating the Green’s function over the final position, with the initial position fixed at \( x_0 = -z \leq 0 \). If we can compute \( U(z, t) \) for a given dynamics, we can express \( P_{\text{an}}(Q, t) \) and \( P_{\text{qu}}(Q, t) \) in terms of this central function \( U(z, t) \). Our main results can now be summarised as follows.

**Annealed case:** In this case, we show that \( P_{\text{an}}(Q, t) \) is always a Poisson distribution

\[ P_{\text{an}}(Q = n, t) = e^{-\mu(t)} \frac{\mu(t)^n}{n!}, \quad n = 0, 1, 2, \ldots, \] (7)

where

\[ \mu(t) = \rho \int_0^{\infty} dz \, U(z, t). \] (8)

The mean and the variance of \( Q_t \) are both given by \( \mu(t) \), which can be explicitly evaluated for different types of particle motion. For example, for a Brownian motion with diffusion constant \( D \), using \( G(x, x_0, t) = e^{-(x-x_0)^2/(4Dt)}/\sqrt{4\pi Dt} \) in Eq. \( (6) \) we get

\[ U(z, t) = \frac{1}{2} \text{erfc} \left( \frac{z}{\sqrt{4Dt}} \right), \quad \text{and} \quad \mu(t) = \rho \sqrt{Dt/\pi}, \] (9)
In addition, expanding the right hand side (rhs) in powers of $e$ mean and the variance of $Q$ in principle obtain $P$, $Q$

The quenched cumulants of $Q$ can then be expressed in terms of the central function $U$

This result is valid for all time $t$,

$$\mu(t) = \frac{1}{2} \rho v_0 t e^{-\gamma t} [I_0(\gamma t) + I_1(\gamma t)], \quad (10)$$

where $I_0(z)$ and $I_1(z)$ are modified Bessel functions of the first kind. Its asymptotic behaviours are given by

$$\mu(t) \approx \begin{cases} \frac{\rho v_0}{2} t, & \text{as } t \to 0, \\ \rho \sqrt{\frac{D_{\text{eff}} t}{\pi}}, & \text{as } t \to \infty, \end{cases} \quad (11)$$

where $D_{\text{eff}} = \frac{v_0^2}{2}\gamma$. Thus at late times, the RTP behaves like a diffusive particle with an effective diffusion constant $D_{\text{eff}}$.

Note that the Poisson distribution in Eq. (7) in the limit $Q \to \infty$, $\mu(t) \to \infty$, keeping the ratio $Q/\mu(t)$ fixed, can be written in a large deviation form (using simply Stirling’s formula)

$$P_{an}(Q, t) \sim \exp \left[ -\mu(t) \Psi_{an} \left( \frac{Q}{\mu(t)} \right) \right], \quad (12)$$

where the rate function $\Psi_{an}(q)$ is universal, i.e., independent of the particle dynamics, and is given by

$$\Psi_{an}(q) = q \ln q - q - 1, \quad q \geq 0. \quad (13)$$

It has the asymptotic behaviours

$$\Psi_{an}(q) \approx \begin{cases} 1, & \text{as } q \to 0 \\ \frac{1}{2}(q - 1)^2, & \text{as } q \to 1 \\ q \ln q, & \text{as } q \to \infty. \end{cases} \quad (14)$$

The quadratic behavior near the minimum at $q = 1$ indicates typical Gaussian fluctuations for $Q$, with mean and variance both equal to $\mu(t)$. Note that the dependence on the particle dynamics in Eq. (12) enters only through the parameter $\mu(t)$ but the function $\Psi_{an}(q)$ is universal.

We also note that, from our general result in Eq. (7), it follows that

$$\left. \text{Prob.}(N_t^+ = 0) \right|_{an} = P_{an}(Q = 0, t) = e^{-\mu(t)}. \quad (15)$$

This result is valid for all $t$ and gives the probability of the rare event that all the particles are back on the left side of the origin at time $t$, as discussed in the introduction.

**Quenched case:** In this case, the generating function in Eq. (9), for arbitrary single particle dynamics, can again be expressed in terms of the central function $U(z, t)$ (8) as follows

$$\sum_{Q=0}^{\infty} P_{qu}(Q, t)e^{-\rho Q} = \exp \left\{ \rho \int_0^\infty dq \ln \left[ 1 - (1 - e^{-q})U(z, t) \right] \right\}. \quad (16)$$

The quenched cumulants of $Q$ can then be extracted and expressed in terms of $U(z, t)$. For instance, the quenched mean and the variance of $Q$ are given by

$$\langle Q \rangle_{qu} = \rho \int_0^\infty U(z, t) dz, \quad (17)$$

$$\sigma^2_{qu} = \langle Q^2 \rangle_{qu} - \langle Q \rangle_{qu}^2 = \rho \int_0^\infty U(z, t)(1 - U(z, t)) dz. \quad (18)$$

In addition, expanding the right hand side (rhs) in powers of $e^{-p}$ and matching with the left hand side (lhs), one can in principle obtain $P_{qu}(Q, t)$ for any integer $Q$ as a functional of $U(z, t)$. For example,

$$\left. \text{Prob.}(N_t^+ = 0) \right|_{qu} = P_{qu}(Q = 0, t) = \exp \left\{ \rho \int_0^\infty \ln \left[ 1 - U(z, t) \right] dz \right\}. \quad (19)$$
which is valid for all times \( t \geq 0 \). However, the formula gets more complicated for higher values of \( Q \).

For the full quenched distribution, we first consider the diffusive motion of the particles. In this case, we obtain, in the scaling limit \( Q \to \infty, t \to \infty \) keeping the ratio \( Q/\sqrt{t} \) fixed, the same large deviation form as Derrida and Gerschenfeld\(^6\),

\[
P_{\text{qu}}(Q,t) \sim \exp \left[ -\rho \sqrt{Dt} \Psi_{\text{diff}} \left( \frac{Q}{\rho \sqrt{Dt}} \right) \right],
\]

where the rate function \( \Psi_{\text{diff}}(q) \) has the following precise asymptotics

\[
\Psi_{\text{diff}}(q) \approx \begin{cases} 
\overline{\alpha} - q + q \ln(q/\overline{\beta}) , & \text{as } q \to 0 \\
\sqrt{\frac{\pi}{2}} \left( q - \frac{2q^2}{\sqrt{\pi}} \right) , & \text{as } q \to 1/\sqrt{\pi}
\end{cases} \quad \overline{\alpha} = 2 \int_0^\infty dz \ln \left( 1 - \frac{1}{2} \text{erfc}(z) \right) = 0.675336 \ldots
\]

with the two constants \( \overline{\alpha} \) and \( \overline{\beta} \) given explicitly by

\[
\overline{\alpha} = 2 \int_0^\infty dz \ln \left( 1 - \frac{1}{2} \text{erfc}(z) \right) = 0.675336 \ldots
\]

\[
\overline{\beta} = \int_0^\infty dz \frac{\text{erfc}(z)}{1 - \frac{1}{2} \text{erfc}(z)} = 0.828581 \ldots
\]

Note that the large \( q \) behavior \( \Psi_{\text{diff}}(q) \approx q^3/12 \) coincides with the result of Derrida and Gerschenfeld\(^6\) obtained by a different method. The small \( q \) behavior was not investigated in Ref.\(^6\). Taking the \( q \to 0 \) limit in Eq.\(^20\) and using the small \( q \) behavior in the first line of Eq.\(^21\) implies that for large \( t \) \( P_{\text{qu}}(Q = 0, t) \sim \exp \left[ -\overline{\alpha} \rho \sqrt{D t} \right] \)

where the constant \( \overline{\alpha} \) is given in Eq.\(^22\). In fact, this result is valid not just at large time but at all times. Indeed, substituting \( U(z, t) = (1/2)\text{erfc}(z/\sqrt{4Dt}) \) in our general formula\(^19\), it follows that at all times \( t \geq 0 \),

\[
P_{\text{qu}}(Q = 0, t) \bigg|_{\text{RTP}} \sim \exp \left[ -\overline{\alpha} \rho \sqrt{D_{\text{eff}} t} \right],
\]

where \( \overline{\alpha} \) is the same constant as in Eq.\(^22\) and \( D_{\text{eff}} = \gamma v_0^2/(2\gamma) \). One of the main results of this analysis is to find a new scaling limit \( Q \to \infty, t \to \infty \), keeping the ratio \( Q/(\rho v_0 t) \) fixed where the quenched distribution admits a large deviation form \( \text{[quite different from the diffusive case in Eq.}\(^20\)] \)

\[
P_{\text{qu}}(Q, t) \sim \exp \left[ -\rho v_0 \gamma t^2 \Psi_{\text{RTP}} \left( \frac{Q}{\rho v_0 t} \right) \right],
\]

where the rate function \( \Psi_{\text{RTP}}(q) \) is given explicitly by

\[
\Psi_{\text{RTP}}(q) = q - \frac{q}{2} \sqrt{1 - q^2} - \sin^{-1} \left[ \sqrt{\frac{1}{2} - q^2} \right], \quad 0 \leq q \leq 1.
\]

The rate function has the asymptotic behavior

\[
\Psi_{\text{RTP}}(q) \approx \begin{cases} 
\frac{q^3}{6} , & q \to 0 \\
1 - \frac{\pi}{4} , & q = 1.
\end{cases}
\]
One consequence of our result is the prediction of the probability of the rare event that the flux $Q$ up to time $t$ achieves its maximum possible value, namely $Q = \rho v_0 t$ – this corresponds to the case where all the particles move ballistically to the right up to time $t$. We find that the probability of this rare event is given by

$$P_{qu}(Q = \rho v_0 t, t) \approx \exp \left[ - \left( 1 - \frac{\pi}{4} \right) \rho v_0 \gamma t^2 \right]. \quad (29)$$

Such a faster than exponential decay for the probability of this rare event is a nontrivial prediction of our theory.

## III. THE GENERAL SETTING AND THE SINGLE-PARTICLE GREEN’S FUNCTION

We start with a step initial condition where $N$ particles are initially located on the negative half line at positions $\{x_1, x_2, \cdots, x_N\}$ where all $x_i < 0$. As stated before, for this step initial condition, the flux $Q_t$ up to time $t$ is identical in law to the number of particles $N_i^+$ to the right of the origin at time $t$. Let us introduce an indicator function $I_i(t)$ such that $I_i(t) = 1$ if the $i$th particle is to the right of the origin at time $t$, else $I_i(t) = 0$. Hence we have

$$N_i^+ = \sum_{i=1}^{N} I_i(t). \quad (30)$$

For fixed $x_i$’s the flux distribution is then given by

$$P(Q, t, \{x_i\}) = \text{Prob.}(N_i^+ = Q) = \left\langle \delta \left[ Q - \sum_{i=1}^{N} I_i(t) \right] \right\rangle_{\{x_i\}}, \quad (31)$$

where the angular brackets $\langle \cdots \rangle_{\{x_i\}}$ denote an average over the history, but with fixed initial condition $x_i$. Taking the Laplace transform on both sides of Eq. (31) gives

$$\sum_{Q=0}^{\infty} e^{-pQ} P(Q, t, \{x_i\}) = \langle e^{-pQ} \rangle_{\{x_i\}} = \left\langle \exp[-p \sum_{i=1}^{N} I_i(t)] \right\rangle_{\{x_i\}}. \quad (32)$$

Since the $I_i$ can only take the values 0 or 1, one has the identity $e^{-pI_i} = 1 - (1 - e^{-p})I_i$. Inserting this identity in Eq. (32) and using the independence of the random variables $I_i$’s we get

$$\langle e^{-pQ} \rangle_{\{x_i\}} = \prod_{i=1}^{N} \left[ 1 - (1 - e^{-p})\langle I_i(t) \rangle \right], \quad (33)$$

where the right hand side (r.h.s.) implicitly depends on the $x_i$’s. The average $\langle I_i(t) \rangle$ is just the probability that the $i$th particle is to the right of the origin at time $t$, starting initially at $x_i$ and hence we have

$$\langle I_i(t) \rangle = \int_{0}^{\infty} G(x, x_i, t) dx = U(-x_i, t), \quad x_i < 0 , \quad (34)$$

where $G(x, x_i, t)$ is the single-particle Green’s function, i.e., the propagator for a particle to reach $x$ at time $t$, starting initially at $x_i < 0$. Note that $U(z, t)$ is defined in Eq. (5) and corresponds to the probability that a particle is on the positive side of the origin at time $t$, starting initially at $-z < 0$. Inserting Eq. (34) into Eq. (33), one obtains

$$\langle e^{-pQ} \rangle_{\{x_i\}} = \prod_{i=1}^{N} \left[ 1 - (1 - e^{-p})U(-x_i, t) \right], \quad x_i < 0 , \quad \forall i = 1, \cdots, N. \quad (35)$$

This Eq. (35) is general, i.e., valid for any set of non-interacting particles undergoing a common dynamics in one-dimension. The information about the dynamics is entirely encoded in the function $U(z, t)$.

For instance, for simple diffusion, the single-particle Green’s function is given by

$$G(x, x_i, t) = \frac{1}{\sqrt{4\pi Dt}} \exp \left[ - \frac{(x - x_i)^2}{4Dt} \right], \quad (36)$$
which gives \( U(z, t) = (1/2)\text{erfc}(z/\sqrt{4Dt}) \). For the RTP, on the other hand, the Green’s function is known explicitly \[ G(x, x_i, t) = \frac{e^{-\gamma t}}{2} \left\{ \delta(x - x_i - v_0 t) + \delta(x - x_i + v_0 t) + \frac{\gamma}{v_0} \left[ \text{I}_0(\omega) + \frac{\gamma t \text{I}_1(\omega)}{\rho} \right] \Theta(v_0 t - |x - x_i|) \right\}, \] (37)
where \( \omega \) is given by
\[
\omega = \frac{\gamma}{v_0} \sqrt{v_0^2 t^2 - (x - x_i)^2}.
\] (38)

In Eq. (37), \( \Theta(z) \) is the Heaviside Theta function, and \( \text{I}_0(\omega) \) and \( \text{I}_1(\omega) \) are modified Bessel functions. Computing \( U(z, t) = \int_0^\infty G(x, -z, t) \, dx \) explicitly using Eq. (37) is complicated. It is however much more useful, as we will see later, to work with the Laplace transform of \( G(x, x_i, t) \) with respect to \( t \), which has a much simpler expression, namely
\[
\tilde{G}(x, x_i, s) = \int_0^\infty dt \, e^{-st} G(x, x_i, t) = \frac{\lambda(s)}{2s} e^{\lambda(s)|x-x_i|}, \quad \lambda(s) = \frac{\sqrt{s(s + 2\gamma)}}{v_0}.
\] (39)

The relation in Eq. (35) is the central result of this section and we will analyse the annealed and the quenched cases separately in the next two sections.

IV. FLUX DISTRIBUTION IN THE ANNEALED CASE

The annealed distribution \( P_{\text{an}}(Q, t) \) is defined in Eq. (4) where the \( \cdots \) denotes an average over the initial conditions. Performing this average in Eq. (35) gives
\[
\langle e^{-pQ} \rangle_{\{x_i\}} = \prod_{i=1}^N \left[ 1 - (1 - e^{-p}) \overline{U}(-x_i, t) \right], \quad \text{(40)}
\]
where \( \overline{U}(-x_i, t) \) is defined in Eq. (34). To perform the average over the initial conditions with a fixed uniform density \( \rho \), we assume that each of the \( N \) particles is distributed independently and uniformly over a box \([-L, 0]\) and then eventually take the limit \( N \to \infty, L \to \infty \) keeping the density \( \rho = N/L \) fixed. For this uniform measure, each \( x_i \) is uniformly distributed in the box \([-L, 0]\]. Using the independence of the \( x_i \)’s we then get
\[
\langle e^{-pQ} \rangle_{\{x_i\}} = \prod_{i=1}^N \left[ 1 - (1 - e^{-p}) \int_{-L}^0 U(-x_i, t) \frac{dx_i}{L} \right] = \left[ 1 - \frac{1}{L} (1 - e^{-p}) \int_0^L U(z, t) \, dz \right]^N, \quad \text{(41)}
\]
where, in the last equality, we made the change of variable \( z = -x_i \). Taking now the limit \( N \to \infty, L \to \infty \) keeping \( \rho = N/L \) fixed gives
\[
\sum_{Q=0}^\infty e^{-pQ} P_{\text{an}}(Q, t) = \langle e^{-pQ} \rangle_{\{x_i\}} = \exp \left[ -\mu(t) (1 - e^{-p}) \right], \quad \text{where} \quad \mu(t) = \rho \int_0^\infty dz \, U(z, t).
\] (42)

By expanding \( \exp [-\mu(t) (1 - e^{-p})] \) in powers of \( e^{-p} \) and comparing to the left hand side, we see that \( Q \) can take only integer values \( Q = n = 0, 1, 2, \cdots \) and the probability distribution is simply a Poisson distribution with mean \( \mu(t) \) as given in Eqs. (4) and (5).

This Poisson distribution, in the annealed case, is thus universal, i.e., holds for any dynamics. The details of the dynamics is encoded in the single parameter \( \mu(t) \) which can be computed explicitly for different types of dynamics. For example, for diffusing particles, using the explicit expression for the Brownian propagator, we get \( U(z, t) = (1/2)\text{erfc}(z/\sqrt{4Dt}) \) and, hence, \( \mu(t) = \rho \sqrt{Dt}/\pi \) as mentioned in Eq. (9). In contrast, for the RTP dynamics, \( \mu(t) \) is nontrivial. As discussed earlier, computing \( U(z, t) \) from the Green’s function in Eq. (37) is difficult. Consequently, calculating \( \mu(t) = \rho \int_0^\infty U(z, t) \, dz \) is also hard. However, it turns out that its Laplace transform is much easier to manipulate, due to the simple nature of the formula in Eq. (39). The Laplace transform of \( \mu(t) \) is given by
\[
\tilde{\mu}(s) = \int_0^\infty dt \, e^{-st} \mu(t) = \rho \int_0^\infty dz \, \tilde{U}(z, s), \quad \text{with} \quad \tilde{U}(z, s) = \int_0^\infty dt \, e^{-st} U(z, t),
\] (43)
where we have used the relation \( \mu(t) = \rho \int_0^\infty U(z, t) \, dz \). The Laplace transform of \( U(z, t) \) can be computed as follows

\[
\tilde{U}(z, s) = \int_0^\infty dt \, e^{-st} U(z, t) = \int_0^\infty dt \, e^{-st} \int_0^\infty G(x, -z, t) \, dx .
\]  

(44)

Exchanging the integrals over \( z \) and \( t \), and using the relation in Eq. (39) and integrating over \( x \) we get

\[
\tilde{U}(z, s) = e^{-
\frac{\lambda(s)z}{2s}}, \quad \text{where} \quad \lambda(s) = \sqrt{\frac{s(s + 2\gamma)}{v_0}} .
\]

(45)

Inserting this relation in Eq. (43) and performing the integral over \( z \), we get

\[
\tilde{\mu}(s) = \frac{1}{2s} \frac{v_0}{\lambda(s)} = \frac{v_0}{2s \sqrt{s(s + 2\gamma)}} .
\]

(46)

This Laplace transform can be explicitly inverted, yielding the result in Eq. (10).

V. FLUX DISTRIBUTION IN THE QUENCHED CASE

As stated in Section II, the quenched flux distribution is defined as

\[
\sum_{Q=0}^\infty P_{q}(Q, t) e^{-pQ} = \exp \left[ \ln \left( \langle e^{-pQ} \rangle_{\{x_i\}} \right) \right] .
\]

(47)

where \( \langle \cdot \rangle \) once again represents an average over the initial positions \( \{x_i\} \). Our starting point is again Eq. (55). Taking the logarithm on both sides of (55) gives

\[
\ln \left( \langle e^{-pQ} \rangle_{\{x_i\}} \right) = \sum_{i=1}^N \ln \left[ 1 - (1 - e^{-p})U(-x_i, t) \right] .
\]

(48)

We now perform the average over the initial positions, as in the annealed case, i.e., choosing each \( x_i \) independently and uniformly from the box \([-L, 0]\) and finally taking the limit \( N \to \infty, L \to \infty \) keeping \( \rho = N/L \) fixed. This gives

\[
\log \left( \langle e^{-pQ} \rangle_{\{x_i\}} \right) = \frac{N}{L} \int_{-L}^{0} dx_i \ln \left[ 1 - (1 - e^{-p})U(-x_i, t) \right] \rightarrow \rho \int_0^\infty dz \ln \left[ 1 - (1 - e^{-p})U(z, t) \right] .
\]

(49)

Therefore the Laplace transform of the quenched flux distribution is given by

\[
\sum_{Q=0}^\infty P_{q}(Q, t) e^{-pQ} = \exp \left[ I(p, t) \right] ,
\]

(50)

where

\[
I(p, t) = \rho \int_0^\infty dz \ln \left[ 1 - (1 - e^{-p})U(z, t) \right] .
\]

(51)

Before extracting the full distribution \( P_{q}(Q, t) \) from this Laplace transform, it is useful to study first the asymptotic behaviors of \( I(p, t) \) in the two limits : (i) \( p \to 0 \) and (ii) \( p \to \infty \).

• \( p \to 0 \) limit: Expanding \( e^{-p} \) in powers of \( p \) in Eq. (51), we get

\[
I(p, t) = -p \rho \int_0^\infty dz \, U(z, t) + \frac{p^2}{2} \rho \int_0^\infty dz \, U(z, t) [1 - U(z, t)] + O(p^3) .
\]

(52)

Substituting this in Eq. (50) and expanding both sides in powers of \( p \) we immediately get the mean and the variance of the flux \( Q_i \) for the quenched case as stated in Eqs. (17) and (18) respectively.
• $p \to \infty$ limit: In this case we expand $I(p, t)$ in Eq. (51) in powers of $e^{-p}$. The two leading terms are given by

$$I(p, t) = A(t) + B(t)e^{-p} + O(e^{-2p}) ,$$

where

$$A(t) = \rho \int_0^\infty \ln[1 - U(z, t)]dz$$

$$B(t) = \rho \int_0^\infty \frac{U(z, t)}{1 - U(z, t)}dz .$$

Substituting this expansion (53) on the rhs of Eq. (50) and matching the powers of $e^{-p}$ on both sides of Eq. (50) immediately gives

$$P_{qu}(Q = 0, t) = e^{A(t)} = \exp \left[ \rho \int_0^\infty \ln(1 - U(z, t))dz \right]$$

$$P_{qu}(Q = 1, t) = B(t)e^{A(t)} .$$

The first line yields the general result mentioned in Eq. (19).

These results so far are quite general, i.e., they hold for any dynamic $s$ – the dependence on the dynamics comes only through the function $U(z, t)$. In the following, we focus on two interesting dynamics, namely the diffusive and the RTP and extract the large time behavior of $P_{qu}(Q, t)$ using Eqs. (50) and (51).

A. $P_{qu}(Q, t)$ for simple diffusion

In this case, using the explicit expression $U(z, t) = (1/2)erfc(z/\sqrt{4Dt})$, we get from Eq. (51)

$$I(p, t) = \rho \sqrt{4Dt} \int_0^\infty dz \ln \left[ 1 - \frac{1}{2} (1 - e^{-p})erfc(z) \right] = -\rho \sqrt{Dt} \phi(p),$$

where

$$\phi(p) = -2 \int_0^\infty dz \ln \left[ 1 - \frac{1}{2} (1 - e^{-p})erfc(z) \right] .$$

Therefore Eq. (50) reads for all time $t$

$$\sum_{Q=0}^{\infty} e^{-pQ} P_{qu}(Q, t) = \exp \left[ -\rho \sqrt{Dt} \phi(p) \right] .$$

In the long time limit $t \to \infty$, we anticipate, and verify a posteriori, that $P_{qu}(Q, t)$ takes a large deviation form in the limit where $Q \to \infty$, $t \to \infty$ but with the dimensionless ratio $q = Q/(\rho \sqrt{Dt})$ fixed

$$P_{qu}(Q, t) \sim \exp \left[ -\rho \sqrt{Dt} \Psi_{diff} \left( \frac{Q}{\rho \sqrt{Dt}} \right) \right] ,$$

where $\Psi_{diff}(q)$ is a rate function that we wish to compute. Substituting this large deviation form (61) on the left hand side (lhs) of Eq. (60) and replacing the discrete sum over $Q$ by an integral (which is valid for large $Q \sim \sqrt{t}$), we get

$$\int_0^\infty e^{-pQ} P_{qu}(Q, t)dQ \sim \rho \sqrt{Dt} \int_0^\infty e^{-\rho \sqrt{Dt}[pq + \Psi_{diff}(q)]}dq .$$

For large $t$, we can now evaluate the integral over $q$ in Eq. (62) by a saddle point method, which gives

$$\int_0^\infty e^{-pQ} P_{qu}(Q, t)dQ \sim \exp \left[ -\rho \sqrt{Dt} \min_q[pq + \Psi_{diff}(q)] \right] .$$
Comparing this with the rhs of Eq. (60) we get
\[ \min_q [p q + \Psi_{\text{diff}}(q)] = \phi(p) . \] (64)

Inverting this Legendre transform one gets
\[ \Psi_{\text{diff}}(q) = \max_p [\phi(p) - p q] , \] (65)

where \( \phi(p) \) is given in Eq. (59).

Knowing \( \phi(p) \) explicitly, one can plot the large deviation function \( \Psi_{\text{diff}}(q) \) using Eq. (64) – see Fig. 2. Clearly, \( \Psi_{\text{diff}}(q) \) has a concave shape with a minimum at \( q = q_{\min} \), the value of \( q_{\min} \) will be computed shortly. The asymptotic behaviors of the rate function \( \Psi_{\text{diff}}(q) \) can also be extracted in the limits \( q \to q_{\min} \), \( q \to 0 \) and \( q \to \infty \) by analysing \( \phi(p) \) respectively in the limits \( p \to 0 \), \( p \to +\infty \) and \( p \to -\infty \) (where \( \phi(p) \) in Eq. (59) has to be continued analytically to negative \( p \)). The results are summarised in Eqs. (21), (22) and (23) in section II. Below we provide the derivation of these results.

**FIG. 2:** Large deviation function \( \Psi_{\text{diff}}(q) \) vs \( q \) for the diffusive case with quenched initial conditions. On both panels, the dashed black lines correspond to evaluating via Mathematica \( \Psi_{\text{diff}}(q) \) from Eq. (65) with \( \phi(p) \) given in Eq. (64). **Left:** Solid yellow curve corresponds to the small \( q \) asymptotic behavior of \( \Psi_{\text{diff}}(q) \) in Eq. (76) and the dotted-dashed green curve corresponds to the quadratic behavior in Eq. (69). **Right:** We zoom in on the large \( q \) tail. The solid violet curve corresponds to the leading asymptotic behavior \( \Psi_{\text{diff}}(q) \approx q^3/12 \), while the dotted-dashed green curve is the quadratic behavior as in Eq. (69). The solid and the dashed curves clearly demonstrate the non-Gaussian tail of \( \Psi_{\text{diff}}(q) \).

1. Typical Fluctuations : \( Q \sim \langle Q \rangle_{\text{qu}} \)

In order to derive the result for \( \Psi_{\text{diff}}(q \to q_{\min}) \), we need to analyze \( \phi(p) \) for \( p \to 0 \). We expand \( \phi(p) \) in Eq. (59) up to order \( p^2 \) and get
\[ \phi(p) = \alpha p - \beta p^2 + \mathcal{O}(p^3) , \] (66)

where
\[ \alpha = \int_0^\infty \text{erfc}(z) dz = \sqrt{\frac{1}{\pi}} , \] (67)
\[ \beta = \frac{1}{4} \int_0^\infty (2 \text{erfc}(z) - \text{erfc}^2(z)) dz = \frac{1}{\sqrt{8\pi}} . \] (68)

Substituting \( \phi(p) = \alpha p - \beta p^2 \) in Eq. (65) and maximising with respect to \( p \) gives a quadratic form for the rate function
\[ \Psi_{\text{diff}}(q) \sim \frac{(q - \alpha)^2}{4\beta} = \frac{\pi}{2} \left( q - \sqrt{\frac{1}{\pi}} \right)^2 . \] (69)
This form holds for \( q \) close to \( q_{\text{min}} = \alpha = 1/\sqrt{\pi} \) and gives the result in the second line in Eq. (21). Substituting this quadratic behavior in the large deviation form in Eq. (61) predicts a Gaussian form for the quenched flux distribution for \( q \) close to \( q_{\text{min}} \)

\[
P_{\text{qu}}(Q,t) \sim \exp \left[ -\frac{(Q - \langle Q \rangle_{\text{qu}})^2}{2\sigma^2_{\text{qu}}} \right]
\]

where the mean the variance are given by

\[
\langle Q \rangle_{\text{qu}} = \rho \sqrt{\frac{Dt}{\pi}} \quad \text{(71)}
\]

\[
\sigma^2_{\text{qu}} = \rho \sqrt{\frac{Dt}{2\pi}} \quad \text{(72)}
\]

Notice that these expressions for the mean and the variance, though derived here for large \( t \), actually hold for all \( t \), as one can verify directly from the formulae in Eqs. (17) and (18) with \( U(z,t) = (1/2)\text{erfc}(z/\sqrt{4Dt}) \). Comparing with the annealed case, while the means in both cases are identical, both given by \( \mu(t) = \rho \sqrt{Dt/\pi} \), their variances and higher moments differ. For example, the variance in the annealed case is \( \mu(t) = \rho \sqrt{Dt/\pi} \) which differs by a factor \( 1/\sqrt{2} \) from the quenched case in Eq. (72). These results agree with those obtained in [6]. This typical quadratic behavior is shown by the solid green curve in Fig. 2.

2. Atypical fluctuations on the left of the mean: \( Q \ll \langle Q \rangle_{\text{qu}} \)

In order to infer about the fluctuations of \( P_{\text{qu}}(Q,t) \) around \( Q \to 0 \), we need to evaluate how \( \Psi_{\text{diff}}(q) \) behaves when \( q \to 0 \). This corresponds to the limit \( p \to \infty \) for \( \phi(p) \) from Eq. (63). We use the large \( p \) expansion in Eq. (53) and evaluate \( A(t) \) and \( B(t) \) from Eq. (64) using \( U(z,t) = (1/2)\text{erfc}(z/\sqrt{4Dt}) \). This gives

\[
A(t) = -\overline{\alpha} \rho \sqrt{4Dt}, \quad \text{where} \quad \overline{\alpha} = -2 \int_0^\infty \ln \left[ 1 - \frac{1}{2} \text{erfc}(z) \right] dz = 0.675336 \ldots \quad \text{(73)}
\]

\[
B(t) = \overline{\beta} \rho \sqrt{4Dt}, \quad \text{where} \quad \overline{\beta} = \int_0^\infty \frac{\text{erfc}(z)}{1 - \frac{1}{2} \text{erfc}(z)} dz = 0.828582 \ldots . \quad \text{(74)}
\]

From Eqs. (58) and (59) we get the two leading terms of \( \phi(p) \) for large \( p > 0 \)

\[
\phi(p) = -\frac{I(p,t)}{\rho \sqrt{4Dt}} \approx \overline{\alpha} - \overline{\beta} \ e^{-p}. \quad \text{(75)}
\]

Plugging this result for \( \phi(p) \) in Eq. (65) and maximizing with respect to \( p \), we get the leading small \( q \) behavior of \( \Psi_{\text{diff}}(q) \)

\[
\Psi_{\text{diff}}(q) \approx \overline{\alpha} - q + q \ln \left( \frac{q}{\overline{\beta}} \right). \quad \text{(76)}
\]

This reproduces the first line of Eq. (21). In particular, for \( q = 0 \), using \( \Psi_{\text{diff}}(q = 0) = \overline{\alpha} \) in Eq. (61), we obtain \( P_{\text{qu}}(Q = 0, t) \sim \exp (-\overline{\alpha} \rho \sqrt{4Dt}) \) as announced in Eq. (21). The small \( q \) behavior of \( \Psi_{\text{diff}}(q) \) is shown by the solid yellow curve in the left panel of Fig. 2.

3. Atypical fluctuations on the right of the mean: \( Q \gg \langle Q \rangle_{\text{qu}} \)

In order to derive the large \( q \) asymptotics of \( \Psi_{\text{diff}}(q) \) from Eq. (65), we first need to continue \( \phi(p) \) in Eq. (59) analytically to negative \( p \) and use its asymptotics in the limit \( p \to -\infty \). For this, it is convenient to write first \( p = -u \) where \( u = |p| \). We write

\[
\phi(p = -u) = \tilde{\phi}(u) = -2 \int_0^\infty dz \ln \left[ 1 + \frac{(e^u - 1)}{2} \text{erfc}(z) \right] \approx -2 \int_0^\infty dz \ln \left[ 1 + \frac{e^u}{2} \text{erfc}(z) \right]. \quad \text{(77)}
\]
To extract the large \( u \) behavior of \( \tilde{\phi}(u) \) from the integral on the rhs, it is convenient to take the derivative with respect to \( u \)

\[
\tilde{\phi}'(u) \approx -2 \int_0^\infty dz \frac{e^z \text{erfc}(z)}{1 + \frac{z}{u} \text{erfc}(z)}.
\]  

(78)

For large \( u \) the dominant contribution to this integral comes from large \( z \) where \( \text{erfc}(z) \approx e^{-z^2/(z \sqrt{\pi})} \). Hence we see that, for \( z > \sqrt{u} \) the integrand is essentially 0 as \( u \to \infty \), while, for \( z < \sqrt{u} \), the integrand is 1 as \( u \to \infty \). Hence, the integrand can be approximated by a Fermi function

\[
\tilde{\phi}'(u) \approx -2 \int_0^{\sqrt{u}} dz = -2 \sqrt{u}.
\]

(79)

Integrating it back, we get the leading order behavior for \( \tilde{\phi}(u) \) for large \( u \)

\[
\tilde{\phi}(u) \approx -\frac{4}{3} u^{\frac{3}{2}}.
\]

(80)

Therefore \( \phi(p) \approx -(4/3)(-p)^{3/2} \) as \( p \to -\infty \). Substituting this behavior in Eq. (65) and maximizing with respect to \( p \) one gets \( \Psi_{\text{diff}}(q) \approx q^3/12 \) as \( q \to \infty \). This then gives the last line of the result in Eq. (21). As mentioned earlier, this leading large \( q \) asymptotic behavior of \( \Psi_{\text{diff}}(q) \) coincides with the result of Ref. [6] obtained by a different method. The large \( q \) behavior of \( \Psi_{\text{diff}}(q) \) is shown by the solid violet curve in the right panel of Fig. 2.

### B. \( P_{\text{qu}}(Q, t) \) for run-and-tumble particles

In this case, our starting point again are Eqs. (50) and (51), except that the function \( U(z, t) \) for the RTP is more complicated. Its Laplace transform in Eq. (44). As shown in Appendix C of [32] it can be formally inverted to obtain \( U(z, t) \) in real time

\[
U(z, t) = \frac{1}{2} \left[ e^{\frac{-\gamma z^2}{v_0 t}} + \frac{\gamma z}{v_0} \int_{1}^{\infty} dt e^{\frac{-\gamma z^2}{v_0 t} I_1(\frac{2\gamma z}{\sqrt{1 - t}}) / \sqrt{1 - t}} \right] \Theta(v_0 t - z).
\]

(81)

However, it turns out that this expression is not very useful to extract the large deviation function at late times.

Before proceeding to compute the large deviation function at late times, it is useful to discuss the large \( t \) behavior of \( P_{\text{qu}}(Q, t) \) in different regimes of \( Q \). In the following, we will first discuss the \( Q \to 0 \) limit of \( P_{\text{qu}}(Q, t) \), followed by the discussion of the typical fluctuations where \( Q = O(\sqrt{t}) \). In this regime, we will recover the Gaussian fluctuations. When \( Q/(\rho \sqrt{D t}) \gg 1 \), we expect to recover the large deviation regime for the diffusive behaviour discussed in the previous section. This is because, as explained in the introduction, at late times, the RTP motion essentially reduces to that of a diffusive particle with an effective diffusion constant \( D_{\text{eff}} = \rho v_0^2/(2\gamma) \). However, there exists yet another “larger deviations regime” where \( Q \sim O(t) \) where we will show that \( P_{\text{qu}}(Q, t) \) carries the signature of activity and has a novel large deviation form

\[
P_{\text{qu}}(Q, t) \sim \exp \left[ -\rho v_0 \gamma t^2 \Psi_{\text{RTP}} \left( \frac{Q}{\rho v_0 t} \right) \right].
\]

(82)

In the following, we will indeed compute this rate function \( \Psi_{\text{RTP}}(q) \) and show that it is given by Eq. (24).

#### 1. Typical fluctuations: \( Q \sim \langle Q \rangle_{\text{qu}} \)

In order to extract the typical fluctuations of \( Q \) around its mean value for the RTP case, we need to use the small \( p \) expansion of \( I(p, t) \) in Eq. (50). Quite generally, the small \( p \) expansion of \( I(p, t) \) is given in Eq. (52). We use this expansion on the rhs of Eq. (50) and approximate the sum on lhs by an integral. The resulting Laplace transform can be easily inverted and yields a Gaussian form

\[
P_{\text{qu}}(Q, t) \approx \exp \left[ -\frac{(Q - \langle Q \rangle_{\text{qu}})^2}{2 \sigma_{\text{qu}}^2} \right],
\]

(83)
where \( \langle Q \rangle_{\text{qu}} \) and \( \sigma_{\text{qu}}^2 \) are given in Eqs. (17) and (18) respectively where \( U(z,t) \) is given in Eq. (81) – alternatively its Laplace transform is given by the simpler form in Eq. (45). The mean value \( \langle Q \rangle_{\text{qu}} \) can be computed explicitly. Indeed

\[
\langle Q \rangle_{\text{qu}} = \rho \int_0^\infty U(z,t) \, dz = \mu(t) = \frac{\rho v_0}{2} t e^{-\gamma t} \left[ I_0(\gamma t) + I_1(\gamma t) \right],
\]

where the last equality follows from Eq. (10). The variance \( \sigma_{\text{qu}}^2 = \rho \int_0^\infty U(z,t) |1 - U(z,t)| \, dz \) is however difficult to compute explicitly using \( U(z,t) \) from Eq. (81). However, it can be easily evaluated numerically. At large times, \( \langle Q \rangle_{\text{qu}} \) and \( \sigma_{\text{qu}}^2 \) converge to the diffusive limits given in Eqs. (71) and (72) respectively.

2. Atypical fluctuations on the left of the mean: \( Q \ll \langle Q \rangle_{\text{qu}} \)

Exactly at \( Q = 0 \) or \( Q = 1 \), we have an exact expression at all times \( t \) for \( P_{\text{qu}}(Q,t) \) in terms of \( U(z,t) \), as given in Eqs. (56) and (57). The function \( U(z,t) \) for RTP appearing in these expressions is given in Eq. (81). Given this rather complicated expression of \( U(z,t) \), it is hard to obtain explicit formulae valid at all times for \( P_{\text{qu}}(Q,t) \) even for \( Q = 0 \) or \( Q = 1 \). However, at late times, since \( U(z,t) \) converges at late times to that of the diffusive limit in Eq. (9) with an effective diffusion constant \( D_{\text{eff}} = v_0^2/(2\gamma) \), we recover the diffusive results for this extreme left tail of \( P_{\text{qu}}(Q,t) \). For instance \( P_{\text{qu}}(Q = 0,t) \), which represents the probability of having no particle on the right side of the origin at time \( t \), decays at late times as in the diffusive case

\[
P_{\text{qu}}(Q = 0,t) \bigg|_{\text{RTP}} \approx \exp \left[ -\pi \rho \sqrt{D_{\text{eff}}} t \right], \tag{85}
\]

where \( \pi = 0.675336 \ldots \) is given in Eq. (22).

3. Atypical fluctuations on the right of the mean: \( Q \sim O(t) \gg \langle Q \rangle_{\text{qu}} \)

In this section, we derive the result in Eq. (82). We recall that in the diffusive case, the atypical fluctuations of \( Q \) are encoded in the large deviation form in Eq. (61) with \( Q \sim \rho \sqrt{Dt} \). The extreme fluctuations to the right of \( \langle Q \rangle_{\text{qu}} \) in this case are described by the large argument behavior of the large deviation function \( \Psi_{\text{diff}}(q = Q/(\rho \sqrt{Dt})) \), i.e., when \( Q \gg \rho \sqrt{Dt} \). Thus, in the diffusive case, there is a single scale for the fluctuations of \( Q \) at late times, namely \( Q \sim \sqrt{t} \). In contrast, for the RTP, in addition to the scale \( \sqrt{t} \) that describes the moderate large deviations around the mean, there is yet another scale where \( Q \sim t \). This comes from the fact that each particle in time \( t \) can move a maximum distance \( v_0 t \), where \( v_0 \) is the velocity. So for an initial density \( \rho \), the maximum possible flux through the origin is \( Q_{\text{max}} = \rho v_0 t \). Hence \( Q \sim t \) describes the scale of fluctuations at the very right tail of the distribution \( P_{\text{qu}}(Q,t) \).

To extract this extreme right tail, we again start from Eqs. (50) and (51) with \( U(z,t) \), for RTP, given by its Laplace transform in Eq. (45). Before extracting the large deviation form of \( P_{\text{qu}}(Q,t) \), we first analyse \( U(z,t) \) in the limit \( z \sim t \). Inverting the Laplace transform of \( U(z,t) \) in Eq. (45) we get

\[
U(z,t) = \frac{ds}{2\pi i} \exp \left[ t \left( s - \sqrt{s(s + 2\gamma)} \frac{z}{v_0 t} \right) \right], \tag{86}
\]

where \( \Gamma \) represents the Bromwich contour in the complex \( s \)-plane. In the limit \( t \to \infty \), \( z \to \infty \) with the ratio \( z/t \) fixed, the integral can be evaluated by the saddle-point method, which yields (up to pre-exponential factors)

\[
U(z,t) \approx \exp \left[ -\gamma t \left( 1 - \frac{1}{\sqrt{1 - \frac{z^2}{v_0^2 t^2}}} \right) \right] \Theta(v_0 t - z). \tag{87}
\]

We have checked numerically that this approximation (87) works very well, at large times, as one would expect.

To extract the large \( Q \sim t \gg \langle Q \rangle_{\text{qu}} \) behavior from Eq. (50) we need to analytically continue \( I(p,t) \) to \( p \) negative and study the limit \( p \to -\infty \), as in the diffusive case. Setting \( p = -u \) with \( u > 0 \), and approximating the discrete sum on the lhs of Eq. (50) by an integral, we get

\[
\int_0^\infty P_{\text{qu}}(Q,t) e^{uQ} \, dQ \approx e^{I(u,t)}, \tag{88}
\]
where

$$\tilde{I}(u, t) = \rho \int_0^\infty dz \ln (1 + (e^u - 1) U(z, t)) \approx \rho \int_0^\infty dz \ln (1 + e^u U(z, t)) .$$  \hfill (89)

To extract the large $u$ behavior of $\tilde{I}(u, t)$ we follow the same procedure as in the diffusive case and take a derivative with respect to $u$. We get

$$\frac{d\tilde{I}(u, t)}{du} \approx \rho \int_0^\infty \frac{dz}{1 + e^{-u - \frac{I(u, t)}{\rho v_0 \gamma t}}} .$$  \hfill (90)

For large $y$, this integral is dominated by the region where $z \sim t$ where we can use the approximate form of $U(z, t)$ given in Eq. (87). Substituting this form for $U(z, t)$ in Eq. (90), we get

$$\frac{d\tilde{I}(u, t)}{du} \approx \rho \int_0^{v_0 t} \frac{dz}{1 + \exp \left[ - \left( u - \gamma t + \frac{z^* u}{v_0} \right) \sqrt{1 - \frac{z^* u}{v_0^2 t^2}} \right]} .$$  \hfill (91)

We now analyse this integral in two different cases, assuming $u \sim t \gg 1$:

• If $u > \gamma t$: in this case as $t \to \infty$ it is clear that the integrand in Eq. (91) is always 1 for any $z$. Hence

$$\frac{d\tilde{I}(u, t)}{du} \approx \rho v_0 t \quad \text{if} \quad u > \gamma t .$$  \hfill (92)

• If $u < \gamma t$: this case is a bit more complicated to analyze. Since $u < \gamma t$ the argument of the exponential in Eq. (91) can be either positive or negative. Accordingly, the integrand will either 0 or 1 for large $u \sim t \gg 1$. The value of $z$ for which the argument of the exponential changes sign is given by

$$z^*(u) = \frac{v_0}{\gamma} \sqrt{u(2\gamma t - u)} .$$  \hfill (93)

Thus, if $z > z^*(u)$ the integrand is 0 while for $z < z^*(u)$ the integrand is 1. Hence this integral over $z$ in (91) gets cut-off at $z = z^*(u)$ (note that $z^*(u) < v_0 t$ for all $u$). Hence we get

$$\frac{d\tilde{I}(u, t)}{du} \approx \rho z^*(u) \sqrt{u(2\gamma t - u)} \quad \text{if} \quad u < \gamma t .$$  \hfill (94)

Integrating it back with respect to $u$ we obtain

$$\tilde{I}(u, t) \approx \rho v_0 \gamma t^2 \psi_{RTP}(\frac{u}{\gamma t}) .$$  \hfill (95)
where

\[
W(x) = \begin{cases} 
\int_0^x \sqrt{y(2-y)}dy & , \quad \text{if } x < 1, \\
\int_0^1 \sqrt{y(2-y)}dy + (x-1) & , \quad \text{if } x > 1.
\end{cases}
\]  

Performing these integrals explicitly, we get

\[
W(x) = \begin{cases} 
\frac{(x-1)}{2} \sqrt{x(2-x)} + \sin^{-1}(\sqrt{x/2}) & , \quad x < 1 \\
\sqrt{x} + x - 1 & , \quad x > 1.
\end{cases}
\]

The function \(W(x)\) is plotted vs \(x\) in Fig. 3 (left). Interestingly, while \(W(x)\) and its first two derivatives are continuous at \(x = 1\), its third derivative is discontinuous. Indeed one has

\[
W'''(x \to 1^-) = -1 \\
W'''(x \to 1^+) = 0.
\]

Using this result for \(W(x)\) in Eq. (97) gives us an expression for \(I(u, t)\) in (95).

Inverting formally this Laplace transform, we obtain

\[
P_{qu}(Q,t) \sim \int \frac{du}{2\pi i} \exp \left[ -uQ + \rho v_0 t^2 W \left( \frac{u}{\gamma t} \right) \right] .
\]

Rescaling \(u/(\gamma t) = z\) we get, up to pre-exponential factors,

\[
P_{qu}(Q,t) \sim \int \frac{dz}{2\pi i} \exp \left[ -\rho v_0 t^2 \left( -W(x) + xQ \right) \right] , \quad \text{where } Q = \frac{Q}{\rho v_0 t} .
\]

where \(\Gamma\) is the Bromwich contour in the complex \(x\)-plane. Performing this integral by using a saddle-point for large \(t\), we get

\[
P_{qu}(Q,t) \sim \exp \left[ -\rho v_0 t^2 \Psi_{RTP} \left( q = \frac{Q}{\rho v_0 t} \right) \right] ,
\]

with the rate function given by

\[
\Psi_{RTP}(q) = \max_x \left[ q x - W(x) \right] ,
\]

where \(W(x)\) is given explicitly in Eq. (97). It is easy to verify that the maximum of the function \(q x - W(x)\) occurs at \(x = x^* = 1 - \sqrt{1 - q^2} < 1\). Since \(x^* < 1\) we use the branch of \(W(x)\) in the first line of Eq. (97). Substituting this value of \(x^*\) in Eq. (103) we get the result in Eq. (97). The asymptotic behaviours of this function \(\Psi_{RTP}(q)\) are given in Eq. (25) and a plot of this function is shown in Fig. 3 (right). Note that for small \(q\), i.e. \(Q \ll \rho v_0 t\), \(\Psi_{RTP}(q)\) behaves as \(\Psi_{RTP}(q) \sim q^3/6\). Substituting this behavior in Eq. (102) gives

\[
P_{qu}(Q,t) \bigg|_{\text{RTP}} \sim \exp \left( -\frac{\gamma Q^3}{6\rho^2 v_0^2 t} \right) \sim \exp \left( -\frac{Q^3}{12\rho^2 D_{\text{eff}} t} \right) , \quad \text{where } D_{\text{eff}} = \frac{v_0^2}{2\gamma} .
\]

On the other hand, for the diffusive case, from Eq. (20), using \(\Psi_{\text{diff}}(q) \approx q^3/12\) for large \(q\), i.e. \(Q \gg \rho \sqrt{D t}\), one gets

\[
P_{qu}(Q,t) \bigg|_{\text{diff}} \sim \exp \left( -\frac{Q^3}{12\rho^2 D t} \right)
\]

Comparing these two tails (104), and (105), one sees that for the RTP, on a scale where \(\rho \sqrt{D_{\text{eff}} t} \ll Q \ll \rho v_0 t\) these two behaviors match perfectly, supporting the expectation that, at large \(t\), even for moderately large fluctuations to the right of the mean, the flux distribution for the RTP and the diffusive case coincide, once one identifies the effective diffusion constant as \(D_{\text{eff}} = \frac{v_0^2}{2\gamma} \).
VI. NUMERICAL RESULTS

In this section, we present the results from our Monte Carlo simulations that we confront to our analytical results and from which we also extract the properties of the physical realizations corresponding to large values of $Q$. 

In order to obtain the tails of $P_{an}(Q,t)$ and $P_{qu}(Q,t)$ we have employed the method of “importance sampling” – see Ref. [39, 40] for details. The main idea is that we sample the realizations of the particle-configurations with an exponential bias $e^{-\theta Q}$. The adjustable parameter $\theta$ allows to explore atypical realizations: a negative $\theta$ favours realizations with large flux $Q$, while a positive $\theta$ favours small flux $Q$. The sampling is done using a standard Metropolis algorithm as discussed in [39, 40]. To proceed, it is convenient to write the actual position $x_i(t)$ at time $t$ as the sum of two contributions: (i) the initial (negative) position $x_i(0) < 0$ and (ii) the total displacement $\Delta x_i(t)$, i.e.,

$$x_i(t) = x_i(0) + \Delta x_i(t), \quad \forall i = 1, \ldots, N. \tag{106}$$

The total displacement $\Delta x_i(t)$ depends on the stochastic process under consideration: for the diffusive particles it is a Gaussian number of zero mean and standard deviation $\sqrt{2D t}$. For RTP, it can be expressed as

$$\Delta x_i(t) = \pm v_0(T_1 - T_2), \tag{107}$$

where $T_1$ is the total time spent moving in the initial direction while $T_2 = t - T_1$ is the time spent in the opposite direction. The signs $+$ or $-$ correspond to the possible initial directions of the velocity and they are chosen with equal probability $1/2$. The times $T_1$ and $T_2$ are determined as follows: the run times $\tau_1, \tau_2, \ldots, \tau_n$ are drawn from an exponential distribution of rate $\gamma$, the last run being the first time interval for which $\sum_{i=1}^n \tau_i > t$. Then $\tau_n$ is replaced by $t - \sum_{i=1}^{n-1} \tau_i$ and $T_1$ (respectively $T_2$) is computed as the sum of the $\tau_i$’s during which the velocity has the same sign (respectively the opposite one) as the initial velocity.

The choice of the initial conditions is the crucial point that makes the annealed case different from the quenched one: in the annealed case, averages are performed over all initial conditions while in the quenched case the initial condition is fixed and corresponds to the case where all the particles are equally spaced [see Eqs. [38, 39]]. We first study the annealed case followed by the quenched case, and in both cases we compare the RTP and the diffusive dynamics.

A. Annealed case

We start with the annealed case where the flux of RTP depends on the evolution of the particles with an initial position $x(0)$ in $[-v_0 t, 0]$. Typically we expect that the average number of particles in this interval is $\sim \rho v_0 t$ but rare
and better as we observe that the agreement between our numerical simulations and the predicted Poissonian tails gets better.

In the right panel of Fig. 4, the difference is that there are many more particles with $\Delta x > 0$ for $\theta = 1$, than for $\theta = 0$ (typical large values of the flux). As we will see now, this mechanism can not work for the quenched case since the initial density profile is always flat in this case.

Here we study RTP with $\rho = 1, v_0 = 1, \gamma = 0.05$. At time $t = 400$, the average flux predicted by Eq. (10) is $\mu = 35.4573$, which is very close to the value predicted for diffusive particles with an effective diffusion coefficient $D_{\text{eff}} = v_0^2/(2\gamma)$, namely $Q = \mu = \sqrt{D_{\text{eff}}t/\pi} \approx 35.68$. At large times, one expects that the typical realizations (corresponding to $\theta = 0$) for the RTP and for the diffusive particle are quite similar (and we have corroborated this property through our numerical simulations). However when the bias is applied (e.g. $\theta = -1$) the sampled realizations have a larger flux, $Q \approx 93$ and it is instructive to characterize their statistical properties both for RTP and diffusion.

In Fig. 5 (left) we show the initial profile of particles. While the profile is flat for $\theta = 0$ (typical fluctuations), it displays for $\theta = -1$ (corresponding to an atypically large values of the flux $Q$), an accumulation of particles around the origin. Consequently, the total number of particles in the interval $[-v_0 t, 0]$ is much larger than $\rho v_0 t$. In Fig. 5 (middle) we show the histogram (normalized by the number of realizations) of the displacements of the particles with a positive final position for two different values of the bias $\theta = 0$ (typical fluctuations) and $\theta = -1$ (atypically large values of the current): both distributions have a peak at (more or less) the same location $\Delta x \approx 100$ but the main difference is that there are many more particles with $\Delta x > 0$ for $\theta = -1$ than for $\theta = 0$. This shows that, in the annealed case, larger values of the flux are essentially due to rare fluctuations of the initial conditions that are thus completely insensitive to the nature of the particle motion. It is thus not a surprise that the non-Gaussian tails of the flux are of the same nature, namely Poissonian, both for diffusive and active particles.

In the right panel of Fig. 5 we observe that the agreement between our numerical simulations and the predicted Poissonian tails gets better and better as $L$ increases: this confirms that, in the annealed case, the origin of anomalous fluctuations of the flux lies in the rare realizations where, initially, the concentration of particles close to the origin is large. As we will see now, this mechanism can not work for the quenched case since the initial density profile is always flat in this case.

### B. Quenched case

In the quenched case the initial condition is fixed, i.e. the particles are initially equally spaced, and the number of particles with an initial position in $[-v_0 t, 0]$ is always $\rho v_0 t$. In practice, the position of the first particle is fixed to $x_1$ which is randomly drawn from a uniform distribution between 0 and $-1/(2\rho)$. Subsequently, the positions of all the other $N - 1$ particles are then slaved to $x_1$ according to

$$x_i(0) = x_1(0) - i .$$

We have first checked that our Monte Carlo simulations reproduce the known analytical results for the diffusive case obtained in Ref. [5]. The results are shown in Fig. 5 and they show a very good agreement with the results in Eqs. (20) and (65). For the quenched case of RTPs, the results are shown in Fig. 6. Here also, our numerical results...
and small probabilities, when distributions deviate from the Gaussian (solid green) but they become separable from each other only at extremely large times. The histograms of diffusion and RTP are inseparable from each other, since the $P_{RTP}(Q,t)$ regime where $\delta_{RTP}$ together with Eq. (81). However, this also shows that it is very hard to reach numerically the very large time regime where $P_{RTP}(Q,t)$ takes the large deviation from as in Eqs. (26) and (27).}

agree very well with our predictions in Eqs. (50) and (51) – or equivalently, for large $Q \sim t$, from Eqs. (88) and (89) – together with Eq. (81). However, this figure also shows that it is very hard to reach numerically the very large time regime where $P_{RTP}(Q,t)$ takes the large deviation form as in Eqs. (26) and (27).

It is also interesting to probe the differences between RTP and diffusive particles in the quenched case. In the left panel of Fig. 7, we compare the exact distribution of RTP (red long dashed lines) with $\rho = v_0 = 1$, $\gamma = 0.5$ and $t = 80$ (red triangles) with the diffusive one (black short dashed lines) with $D_{eff} = v_0^2/(2\gamma) = 1$ (circles). Both distributions deviate from the Gaussian (solid green) but they become separable from each other only at extremely small probabilities, when $Q \approx 60 \approx v_0 t$. The importance sampling strategy enables us to explore the non-Gaussian
tails of the distribution but not the extremely rare configurations where the fingerprints of activity are present. Indeed in the right panel of Fig. 7 we can hardly see a difference between normal diffusion and RTP in the histograms of the displacements of the particles with positive final position both for θ = 0 (i.e. typical value of Q) and θ = −3 (atypically large value of Q). This is because the displacements involved are still very small compared to \( v_0 t = 80 \).

We end by showing how the annealed and quenched cases corresponding to the same value of Q can be distinguished from each other. We study diffusive particles. The inherent mechanism is the same for RTPs. In this regard, we plot the histograms of displacement of particles with positive final position for both typical and atypical fluctuations of Q (see Fig. 8). While expectedly the typical flux corresponds to inseparable histograms for the two processes, probing the atypical regimes of both annealed and quenched cases corresponding to the same flux, shows that the quenched case has more particles with larger displacements on average as compared to the annealed case. This is understood from the fact that to have the same atypically large value of Q, from a flat initial profile (in contrast to the annealed case where particles accumulate near the origin), the quenched case must correspond to the situation where more (than annealed) particles travel a longer distance to the right.

VII. CONCLUSION

In this paper, we have presented a general framework to study current fluctuations for non-interacting particles executing a common random dynamics in one dimension and starting from a step initial condition. The probability distribution \( P(Q, t)\) depends on the initial positions of the particles \( \{x_i\} < 0 \). The initial positions are distributed uniformly on the negative axis with a uniform density \( \rho \). There are two different ways to perform the average over the initial positions, namely (i) annealed and (ii) quenched averages, in analogy with disordered systems: here the initial condition plays the role of the disorder. In the annealed case, the distribution \( P(Q, t)\) is averaged directly over the initial positions. In contrast, in the quenched case, one considers the configurations of \( x_i \)'s that lead to the most likely current distribution (i.e., the typical current distribution). In both cases, we have shown that, for noninteracting particles, the distribution can be fully characterized in terms of the single particle Green’s function, which in general will depend on the dynamics of the particles. In this article, we have focused mostly on two different dynamics: a) when the single particle undergoes simple diffusion and b) when the single particle undergoes run-and-tumble dynamics (RTP).

For the annealed case, we have shown that \( P_{an}(Q, t) \), at all times, is given by a Poisson distribution, with parameter \( \mu(t) \) given by the exact formula in Eq. (8). We provide exact formula for \( \mu(t) \) in the RTP case (for the diffusive case this was known already from Ref. [6]). For the quenched diffusive case we show that our formalism correctly recovers the large deviation result obtained in Ref. [6] using a different approach. For the RTP case, we showed that there is a new large deviation regime with \( Q \sim t \), where \( P_{qu}(Q, t)\) \( \sim \exp \left[ -\rho v_0 \gamma t^2 \Psi_{RTP} \left( \frac{Q}{\rho v_0 t} \right) \right] \). One of the main
results of this paper is an explicit computation of the rate function \( \Psi_{\text{RTP}}(q) \) given by

\[
\Psi_{\text{RTP}}(q) = q - \frac{q}{2} \sqrt{1 - q^2} - \sin^{-1}\left(\frac{\sqrt{1 - q^2}}{2}\right), \quad 0 \leq q \leq 1.
\] (109)

Our method gives access to another physical observable, namely the probability of an extremely rare event that there is no particle on the right side of the origin at time \( t \). We have shown that this is just the probability of having zero flux up to time \( t \), i.e., \( P(Q = 0, t) \), both for the annealed and the quenched case. For the annealed case, this is just \( P_{\text{an}}(Q = 0, t) = e^{-\mu(t)} \). For the quenched case, we have that, both for the diffusive and RTP cases, this probability decays at late times as a stretched exponential \( P_{\text{qq}}(Q = 0, t) \sim e^{-\alpha(t) D_{\text{eff}}^{-\gamma} t} \), where we computed the constant \( \alpha = 0.675336 \ldots \) analytically [see Eq. (22)]. For diffusive particles, \( D_{\text{eff}} = D \) while for RTP’s, \( D_{\text{eff}} = v_0^2/(2\gamma) \).

We have also verified our analytical predictions by numerical simulations. Computing numerically the large deviation function is far from trivial. Even for the diffusive case the large deviation function predicted for \( P(Q, t) \) (both annealed and quenched) in Ref. [7] was never verified numerically. In this paper, we used a sophisticated importance sampling method to compute numerically this large deviation function in the diffusive case up to an impressive accuracy of order \( 10^{-200} \). We further used the same technique to compute the large deviation function in the RTP case.

The formalism developed in this paper can be easily generalized in different directions. For instance, one can compute the flux distribution exactly for the case where there are, initially, arbitrary densities \( \rho_{\text{left}} \) and \( \rho_{\text{right}} \) to the left and to the right of the origin respectively, both the diffusive and for the RTP cases. One could also generalise this result in higher dimensions, with step-like initial conditions, where for instance one region of the space is initially occupied by particles with uniform density. For the diffusive case, the flux distribution in the presence of hard-core repulsions between particles was studied in Ref. [7] (for the simple symmetric exclusion process). It would be interesting to see whether our formalism can be generalized to study the flux distribution for RTP’s with hard core repulsions.

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