Value distribution theory of Nevanlinna

S Rajeshwari
Department of Mathematics, School of Engineering, Presidency University, Yelahanka,
Bangalore-560 064, INDIA
E-mail: <rajeshwari.s@presidencyuniversity.in, rajeshwaripreetham@gmail.com>

Abstract.
In the field of research, a century old value distribution theory of Nevanlinna is still alive. It contains a broad range of implementations inside and outside a function theory. It is the study to speculate different values by complex function. As you are aware that each single non-constant variable polynomial upon C has has not more than one root which is complex. This builds polynomials with real coefficients, because upon seeing the real number as a complex number with its imaginary part equal to zero up to constant multiple. This article express mainly the classical version for complex analytic map to the Riemann sphere of single variable, with priority on mero-morphic functions in the z-plane.

1. Introduction
It is well known that finding an answer to certain hypothetical or experimental problems mainly based on analysing the nature of the origins of an equalization such like

\[ f(z) = a \]  

(1)

here \( f(z) \) is holo-morphic function \( \mathbb{C} \) or except for poles, a function is holo-morphic in \( D \) as well \( 'a' \) is a complex value. It is particularly essential to explore the distribution of \( n(r,a,f) \) of the roots of (1) in the region of a plane bounded by a circle \( |z| \leq r \), according to its multiplicity each root being counted . It was the experiment on such theme that rained the curtain of values of entire or mero-morphic functions on the theory of distribution.

In rearmost century, the prominent mathematician E.Picard acquired the innovative result: whichever function having a range that includes more than one value of entire function \( f(z) \) should take each finite complex value to an infinite extent multiple times, with at most one exception. Thereafter, Borel E., by instigating the idea of order of an entire functions, more precise formulation gave the above outcome as follows. An integral function \( f(z) \) of order \( \lambda (0 < \lambda < \infty) \) persuade

\[ \limsup_{r \to \infty} \frac{\log n(r,a,f)}{\log r} = \lambda \]

for all limited complex value \( a \), with at most one exception. Obtained outcome, normally called as Picard – Borel theorem, set the establishment for the value distribution theory and on view of this it has been the origin of multiple position papers on this course.

At the origination of decennium, Montel P. instigate to one of two theorems about families of holo-morphic functions give conditions under which a family of holo-morphic functions is normal.
It was R. Nevanlinna who build the effective improvement in enlargement of the theory of value distribution. Previously, the mechanism of the theory and principal objects were the kind of maximum modulus and the entire function, commonly. It was Nevanlinna who upraised the theory of single valued function to new stage by establishing the characteristic function $T(r, f)$ for a single valued function $f(z)$ in the range $|z| < R(R \leq \infty, R > r > 0)$, like a systematic device.

In the current description we present a short statement of Nevanlinna theory also owned application to mixture of field. for a higher exhaustive analysis, the reader is familiar to Hayman’s popular book [6]. Being greater distance growth into disparate direction one can read to the books by Cherry and Ye [4], Chung and Yang [3], Gross [5], Lo [10], Nevanlinna [12], Ru [13], Rubel [14], Zhang [20] and Iipo Laine [7].

2. Maximum Moduli of Functions
Consider a domain $D$ which is bounded in $\mathbb{C}$ and assume $\partial D$ indicates the terminal of $D$ so that $D = D \cup \partial D$ is bounded and closed domain in the complex plane.

Initially, we satisfying the mean value property by establishing the maximum modulus theorem for extended complex functions. This is more extensive group of function and comprise analytic function.

To manifest the main solution, we require the following premises.

**Theorem 1.** A non constant continuous complex function $f(\zeta)$ is defined on the boundary $D$ as well as satisfies the mean value property then $|f|$ does not attain its highest value in $D$ unless it is a stable.

**Proof.** Presume $|f|$ gain its maximum value at $a$, particularly

$$|f(\zeta)| \leq |f(a)| \quad \forall \ a \in D \tag{2}$$

latterly, we demonstrate $f$ is steady in $D$. Since $D$ is open, $\exists \ r > 0 \ni \zeta = \{ \zeta : |\zeta - a| = r \} \subset D$.

Using (1) and by mean value property of $f(\zeta)$, we notice that

$$|f(\zeta)| \leq \frac{1}{2\pi i} \int_{\zeta}^{2\pi} f(a + re^{i\varphi})d\varphi$$

$$\leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(a + re^{i\varphi})|d\varphi = |f(a)|$$

So,

$$\frac{1}{2\pi} \int_{0}^{2\pi} [|f(a)| - |f(a + re^{i\varphi})|]d\varphi = 0$$

further, in view of the Lemma, we have $|f(a + re^{i\varphi})| = |f(a)|$ for $0 \leq \varphi \leq 2\pi$. This equation holds for all circles $|\zeta - a| = \zeta, 0 < \zeta \leq r$, and thus $|f(\zeta)|$ is consistent on $|\zeta - a| \leq r$. So far, we have shown that $f(\zeta)$ is consistent within as well on circles centred at $a$ and continued in $D$. Now we demonstrate that on the full domain $f$ is constant.

Take an arbitrary point $b$ in $D$. Since $D$ is open and connected, $\exists$ a continuous function $\Upsilon$ defined on $[a, 1]$ so that $\Upsilon(a) = a, \Upsilon(1) = b, \Upsilon(t) \in D$ as shown in fig 1.

From the composite function $g(t) = f(\Upsilon(t))$, clearly, $g(0) = f(\Upsilon(0)) = f(a)$ also $g(1) = f(\Upsilon(1)) = f(b)$ so, $g$ is a continuous function connecting $f(a)$ and $f(b)$.

Consider

$$\mathcal{E} = \{ t : f(\Upsilon(t)) = f(a) \}, \quad t \in [a, 1]$$

Consider $t_0$ be the supremum of $\mathcal{E}$. Then $f(\Upsilon(t_0)) = f(a)$ is obvious by the following reasoning.
Suppose let us take \( f \) and the above analysis, we can find a circle \( \text{CI} \) such that \( f(\text{CI}) = 1 \) and \( \text{CI} \) is arbitrary, \( f \) is constant.

Thus, at \( \text{CI} \) the function has maximum value \( |f(a)| \), again, by utilizing the mean value property and the above analysis, we can find a circle \( |3 - \text{CI}| < \delta, \delta > 0 \) in which the function remains constant, i.e., \( f(a) = f(\text{CI}) = f(3) \) \( \forall |3 - \text{CI}| < \delta \). This way \( \text{CI} \) is continuous, we have \( f(\text{CI}) = 1 \) also \( f(\text{CI}) = f(a) \) \( \forall a \). Since \( \text{CI} \neq 1 \), i.e., \( t_0 < 1 \), \( \exists t' < 1 \) and \( f(t') = f(\text{CI}) \). Then, \( f(t') = f(a) \). This implies that \( t' \) is supremum of \( \text{CI} \), a contradiction. Therefore \( t_0 = 1 \) also \( f(a) = f(\text{CI}) = f(b) \). Since \( b \) is arbitrary, \( f \) is constant.

\[ \square \]

## 3. Poisson Integral Formula

Suppose let us take \( f(3) \), an analytic function on the closed disc \(|3| \leq \mathbb{R} \) also \( \text{CI} = \mathbb{R} e^{i\varphi}(0 < \tau < \mathbb{R}) \) (i.e., any point of the domain \(|3| \leq \mathbb{R} \)). So, we get

\[
f(\text{CI}) = \frac{1}{2\pi} \int_{|3|=\mathbb{R}} f(3) \, d3 \]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} f(\mathbb{R} e^{i\varphi}) \, d(\mathbb{R} e^{i\varphi}) \\
= \frac{1}{2\pi} \int_{0}^{2\pi} f(\mathbb{R} e^{i\varphi}) \, d\varphi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} f(\mathbb{R} e^{i\varphi}) \, d\varphi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R} \, d\varphi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R}^2 - r^2 - 2\mathbb{R} r \cos(\varphi - \phi) + r^2 \, d\phi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R}^2 - r^2 + r^2 - 2\mathbb{R} r \cos(\varphi - \phi) + r^2 \, d\phi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R}^2 - r^2 \, d\phi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R}^2 - r^2 + r^2 - 2\mathbb{R} r e^{-i(\varphi - \phi)} + r^2 \, d\phi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R}^2 - r^2 - 2\mathbb{R} r e^{-i(\varphi - \phi)} + r^2 \, d\phi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R}^2 - r^2 - 2\mathbb{R} r e^{-i(\varphi - \phi)} + r^2 \, d\phi \tag{4}
\]

Proof. From the CI formula we have

\[ f(\text{CI}) = \frac{1}{2\pi} \int_{0}^{2\pi} \mathbb{R}^2 - r^2 + r^2 - 2\mathbb{R} r e^{-i(\varphi - \phi)} + r^2 \, d\phi \tag{3} \]

Figure 1.
from (3) and (4) it is enough to prove that

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - R e^{-i(\varphi - \phi)} |f(Re^{i\phi})|}{R^2 - 2R \cos(\varphi - \phi) + r^2} d\phi = 1 \frac{1}{2\pi} \int_0^{2\pi} \frac{|r - Re^{-i(\varphi - \phi)}|}{R^2 + r^2 - 2R \cos(\varphi - \phi)} d\phi = 0
\]

Let \( \delta_1 \) be inverse of \( \delta_0 \) with respect to \( |\delta| = R \), then \( \delta_1 = \frac{\delta_0}{R} \), \( \delta_0 = \frac{R}{\delta} \) lies outside the circle \( |\delta| = R \). (Since, \( \delta_1 = \frac{\delta_0 e^{i\varphi}}{R} \implies |\delta_1| = \frac{\delta_0 e^{i\varphi}}{R} \), \( \delta_0 > \frac{\delta_0}{R} = R(\cdot |e^{i\varphi} = 1 \text{ and } r < R) \) i.e., \( |\delta_1| > R \implies \delta_1 \) lies outside the disk \( |\delta| \leq R \). Hence \( \delta_1 \) is coherent on \( |\delta| \leq R \), then from the Cauchy’s theorem, we retain

\[
\int_{|\delta| = R} \frac{f(\delta)}{\delta - \delta_1} d\delta = 0
\]

\[
\implies \int_0^{2\pi} \frac{f(Re^{i\phi})Re^{i\phi}}{Re^{i\phi} - \frac{R^2}{\sqrt{r^2 + \epsilon^2}} |e^{i\phi}} d\phi = 0
\]

\[
\implies \int_0^{2\pi} \frac{r|r - Re^{i(\varphi - \phi)}| |f(Re^{i\phi})|}{|r - Re^{i(\varphi - \phi)}||r - Re^{-i(\varphi - \phi)}|} d\phi = 0
\]

\[
\implies \int_0^{2\pi} \frac{r|r - Re^{-i(\varphi - \phi)}| |f(Re^{i\phi})|}{R^2 + r^2 - 2R \cos(\varphi - \phi)} d\phi = 0
\]

from (4) and (5), we get

\[
f(\delta_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) |f(Re^{i\phi})|}{R^2 - 2R \cos(\varphi - \phi) + r^2} d\phi
\]

(6)

4. The Poisson-Jenson Formula

Suppose in a path-connected domain \( D \), \( f(\delta) \) is complex differentiable containing \( |\delta| \leq R \), \( \delta > 0 \), except for the poles \( b_1, b_2, ..., b_n \) within \( |\delta| < R \). The roots of \( f(\delta) \) are \( a_1, a_2, ..., a_m \) in \( |\delta| < R \). Then, for \( \delta \neq a_1, \delta \neq b_j \), retain

\[
\log |f(\delta)| = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) \log |f(Re^{i\phi})| d\phi}{R^2 - 2R \cos(\varphi - \phi) + r^2} + \sum_{j=1}^{n} \log \left| \frac{R^2 - \delta_j}{(\delta - \delta_j)} \right| + \sum_{i=1}^{m} \log \left| \frac{R^2 - \overline{a_i}\delta}{(\delta - \overline{a_i})} \right|
\]

Proof. Let

\[
\Psi(\delta) = \frac{(\delta - b_1) ... (\delta - b_n)}{(\delta - a_1) ... (\delta - a_m)} f(\delta)
\]

(7)

Then, \( \Psi(\delta) \) has no zeros and no poles and \( \Psi \) is differentiable on \( D \). We recollect that “If \( f \) is complex differentiable in the area of expertise \( D \) and \( f(\delta) \neq 0 \) in \( D \). Then \( \exists \) a holo-morphic branch of \( \log f \) in \( D \).”

Since \( \Psi(\delta) \neq 0 \), then \( \exists \) a holo-morphic branch of \( \log \Psi(\delta) \). Hence, by the Poisson’s integral formula, we have

\[
\log \Psi(\delta) = \frac{1}{2\pi} \int_0^{2\pi} \beta \log |\Psi(Re^{i\phi})| d\phi
\]

(8)

where

\[
\beta = \frac{R^2 - r^2}{R^2 - 2R \cos(\varphi - \phi) + r^2}
\]
We recall that we have $\log_3 = \log|z| + i\phi$

$\implies$ Real of $\log_3 = \text{Real of } (\log|z| + i\phi)$

$\implies$ Real of $\log_3 = \log|z|$. Taking the real part of both sides in (8), we retrieve

$$\log|\Psi(\bar{z})| = \frac{1}{2\pi} \int_0^{2\pi} \beta |\log|f(e^{i\phi})|| d\phi$$

(9)

from (7), we get

$$\log|\Psi(\bar{z})| = \log|f(z)| + \sum_{i=1}^n \log|z - b_j| - \sum_{i=1}^m \log|z - a_i|$$

(10)

Taking the real part on both sides of (10), we get

$$\log|\Psi(\bar{z})| = \log|f(z)| + \sum_{i=1}^n \log|z - b_j| - \sum_{i=1}^m \log|z - a_i|$$

(11)

Replacing the value of $\log|\Psi(\bar{z})|$ from equation (11) in LHS and RHS of equation (9), we get

$$\log|f(z)| + \sum_{i=1}^n \log|z - b_j| - \sum_{i=1}^m \log|z - a_i| = \frac{1}{2\pi} \int_0^{2\pi} \beta |\log|f(e^{i\phi})|| d\phi$$

(12)

$$+ \sum_{i=1}^n \log|Re^{i\phi} - b_j| - \sum_{i=1}^m \log|Re^{i\phi} - a_i|$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \beta |\log|f(e^{i\phi})|| d\phi + \int_0^{2\pi} \beta \sum_{i=1}^m \log|Re^{i\phi} - b_j| d\phi$$

$$- \sum_{i=1}^m \int_0^{2\pi} \beta |\log|Re^{i\phi} - a_i| d\phi$$

Since

$$|Re^{i\phi} - a_i| = |e^{i\phi}(R - \frac{a_i}{e^{i\phi}})| = |e^{i\phi}| |R - a_i e^{-i\phi}|$$

$$= |R - a_i e^{-i\phi}| (\because |e^{i\phi} = 1|)$$

$$= |R - \bar{a_i} e^{-i\phi}|$$

$$\therefore |Re^{i\phi} - a_i| = |R - \bar{a_i} e^{i\phi}|$$

(13)

Similarly,

$$|Re^{i\phi} - b_j| = |R - b_j e^{i\phi}|$$

(14)

from (12), (13) and (14), we have

$$\log|f(z)| + \sum_{i=1}^n \log|z - b_j| - \sum_{i=1}^m \log|z - a_i| = \frac{1}{2\pi} \int_0^{2\pi} \beta |\log|f(e^{i\phi})|| d\phi$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \beta \sum_{i=1}^m \log|Re^{i\phi} - b_j| d\phi$$

$$- \sum_{i=1}^m \int_0^{2\pi} \beta |\log|Re^{i\phi} - a_i| d\phi$$

Now, since on $|z| < R$, the zeros of $f(z)$ are $a_i$'s, we get $|a_i| < R$ and hence

$$\frac{|\bar{a}_i|}{R} = \frac{|a_i|}{R} < R$$

$$\implies \frac{|\bar{a}_i}{R} < R$$

$$\implies R - \frac{|\bar{a}_i|}{R} \neq 0, \forall \bar{z} \in |z| < R$$

5
f multiplicities, and again considering on f (mean) proximity function. Above one is acclaimed as the

Similarly, we obtain

Taking the real part of both sides in (15), we obtain

Similarly, we obtain

from (14), (15) and (16), we acquire

\[
\log|f(\tilde{z})| + \sum_{i=1}^{n} \log|\tilde{z} - b_i| - \sum_{i=1}^{m} \log|\tilde{z} - a_i| = \frac{1}{2\pi} \int_{0}^{2\pi} \beta \log(f(Re^{i\phi}))d\phi
\]

\[= \frac{1}{2\pi} \int_{0}^{2\pi} \beta \log(f(Re^{i\phi}))d\phi + \sum_{j=1}^{n} \log|R - \overline{\tilde{b}_j}|R(\tilde{z} - b_j) - \sum_{i=1}^{m} \log|R - \overline{\tilde{a}_i}|R(\tilde{z} - a_i)\]

Therefore

\[\log|f(\tilde{z})| = \frac{1}{2\pi} \int_{0}^{2\pi} (R^2 - r^2)\log|f(Re^{i\phi})|d\phi + \sum_{j=1}^{n} \log|R^2 - \overline{\tilde{b}_j}|R(\tilde{z} - b_j) - \sum_{i=1}^{m} \log|R^2 - \overline{\tilde{a}_i}|R(\tilde{z} - a_i)\]

Hence the proof.

5. Theory of Nevanlinna

To refashion the study of meromorphic functions, it was good time for Rolf Nevanlinna. Through a chain of publications he did this in 1922-1925 at the span of 26. His primal concept was to use with a slight alteration of Jensen’s formula.

Delineate the existent functions $\log^+(x) = \max(\log x, 0)$. To measure the behaviour of $f$, Nevanlinna then accustomed the logarithmic function to connect real numbers to each member of its domain. Considering Poisson-Jensen formula, he described

\[m(\tau, f) = \frac{1}{2\pi} \int_{\phi}^{2\pi} \log^+|f(Re^{i\phi})|d\phi.\]

Above one is acclaimed as the (mean) proximity function that significantly estimate larger $f$ on $|z| = \tau$. Additionally, we indicates the number of poles of $f$ as $n(t, f) \in |z| \leq t$, counting multiplicities, and again considering $f(0) \neq \infty$. Describe

\[N(\tau, f) = \int_{0}^{\tau} \frac{n(t, f)}{t} dt,\]
as a unintegrable function of Nevanlinna. Inside the disk $|z| < r$, the average of the logarithmic poles of $f$ is clearly counts. We now rephrase Jensen’s formula as to $\log^+$. Observe one $\log^+ = \log^+ - \log\frac{1}{r}$ is truthful for $r > 0$. Thus

$$
\frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\phi})|d\phi = \frac{1}{2\pi} \int_0^{2\pi} \log^+|f(Re^{i\phi})|d\phi - \frac{1}{2\pi} \int_0^{2\pi} \log^+\frac{1}{|f(Re^{i\phi})|}d\phi
$$

$$
= m(\Re, f) - m(\Re, \frac{1}{f})
$$

We can rewrite this in the form

$$
\log|f(0)| = m(\Re, f) - m(\Re, \frac{1}{f}) - N(\Re, \frac{1}{f}) + N(\Re, f)
$$

basically, in the courage of argument earlier, while the mean value of $f$ is measured by $m(\Re, f)$ on the circle, by calculating how close $f$ is to $0$ on average $m(\Re, \frac{1}{f})$ looks at the same condition from another angle. Correspondingly for $N(\Re, f)$ and $N(\Re, \frac{1}{f})$.

6. (Nevanlinna) Characteristic function

$$
\Sigma(f) = m(f) + N(f)
$$

which we can recycled to edit the raised equation in a highly substantial mode

$$
\Sigma(\Re, \frac{1}{f}) = \Sigma(\Re, f) - \log|f(0)|
$$

Given comprehension was expressed by Ahlfors (1976), a scholar of Nevanlinna, as the mark when Nevanlinna theory was born.

**Theorem 2. (Cartan’s Identity):** Consider a holo-morphic function $f(z)$ and a complex number $a \in |z| < \Re(\leq \infty)$. Then we get

$$
\Sigma(r, f) = \frac{1}{2\pi} \int_0^{2\pi} N(r, \frac{1}{f-e^{i\phi}})d\phi + \log^+|f(0)|
$$

(18)

**Proof.** Adopting Jensen’s formula to the operator $f - e^{i\phi}$, we have

$$
\log |f(0) - e^{i\phi}| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})|d\phi + \sum_{j=1}^{m(0)} \log |b_j| - \sum_{i=1}^{m} \log |a_i|
$$

(19)

where the zeros of $f - e^{i\phi}$ are $a_i(j = 1, 2, 3, ..., m)$ within $|z| < r$ and the poles of $f - e^{i\phi}$ are $b_i(i = 1, 2, 3, ..., n)$ within $|z| < r$ but poles of $f - e^{i\phi}$ are poles of $f$. Above equation (19) can also be written as

$$
\log |f(0) - e^{i\phi}| = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\phi})| - e^{i\phi}|d\phi + N(r, f) - \Sigma(r, \frac{1}{f - e^{i\phi}})
$$

(20)

Integrating above equality with respect to $\phi$ between the limits $\phi = 0$ to $\phi = 2\pi$ and then dividing by $2\pi$, we get

$$
\frac{1}{2\pi} \int_0^{2\pi} \log |f(0) - e^{i\phi}|d\phi = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\phi})| - e^{i\phi}|d\phi \right]d\phi + \frac{1}{2\pi} \int_0^{2\pi} N(r, f)d\phi
$$

$$
- \frac{1}{2\pi} \int_0^{2\pi} N(r, \frac{1}{f - e^{i\phi}})d\phi
$$

(21)
by changing the order of integration, we procure
\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(0) - e^{i\varphi}| d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{i\phi}) - e^{i\varphi}| d\phi \right] d\phi + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}(r, f) d\varphi
\]  
\tag{22}
\]

We have, for any finite complex number \( a \),
\[
\frac{1}{2\pi} \int_0^{2\pi} \log|a - e^{i\varphi}| d\varphi = \log |a|
\]

We acquire
\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(0) - e^{i\varphi}| d\varphi = \log |f(0)|
\]  
\tag{23}
\]

constantly
\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\phi}) - e^{i\varphi}| d\varphi = \log |f(e^{i\phi})|
\]  
\tag{24}
\]

consequently, using (23) and (24) in (22), we procure
\[
\log^+ |f(0)| = \frac{1}{2\pi} \log^+ |f(e^{i\phi})| + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}(r, f) d\varphi - \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}(r, \frac{1}{f - e^{i\varphi}}) d\varphi
\]
\[
= \mathcal{M}(r, f) + \mathcal{M}(r, f) - \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}(r, \frac{1}{f - e^{i\varphi}}) d\varphi
\]
\[
\implies \log^+ |f(0)| = \mathcal{T}(r, f) - \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}(r, \frac{1}{f - e^{i\varphi}}) d\varphi
\]
\[
\implies \mathcal{T}(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}(r, \frac{1}{f - e^{i\varphi}}) d\varphi + \log^+ |f(0)|
\]

\[\square\]

**Applications of Cartan’s Identity.** The following are two important applications in Cartan’s Identity:

1. \( \mathcal{T}(r, f) \) is an increasing function of \( r \).
2. \( \mathcal{T}(r, f) \) is a convex function of \( \log r \).

[Description (Convex function): As \( \log r \), is a convex function of \( \phi(x) \) only when \( x \phi'(x) \) is non-decreasing function.]

7. First Fundamental Theorem

**Theorem 3.** If \( f \) is holo-morphic in \( |z| \leq R \) and \( \mathcal{M}(r, f) = \max |f(z)| : |z| \leq r \), so
\[
\mathcal{T}(r, f) \leq \log^+ \mathcal{M}(r, f) \leq \frac{R + r}{R - r} \mathcal{T}(R, f) \quad (R \geq r > 0).
\]

**Proof.** In \( |z| \leq R, f \) is holomorphic, we have
\[
\mathcal{T}(r, f) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(e^{i\phi})| d\varphi
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \mathcal{M}(r, f) d\varphi
\]
\[
= \log^+ \mathcal{M}(r, f)
\]
inequality in the left hand side is established. If $M(r, f) \leq 1$, then the inequality in the right hand side trivially holds. Hence we take over $M(r, f) > 1$. On $|z| = r$, $z_0$ is a point $\exists f(z_0) = M(r, f)$. Following $f$ doesn’t have poles in $|z| < r$ also $|1 - \frac{M(r, f)}{1 - M(r, f)}| < 1$, section 4 yields that

$$\log^+ M(r, f) = \log |f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\phi})| \frac{g_R^2 - r^2}{g_R^2 - 2Mg_R \cos(\phi - \phi) + r^2} d\phi$$

$$\leq \frac{g_R + r}{g_R - r} \int_0^{2\pi} \log^+ |f(re^{i\phi})| = \frac{g_R + r}{g_R - r} m(R, f)$$

By previous result, $\Xi(r, f) \sim \log^+ M(r, f)$ as $r \to \infty$, we define a mero-morphic function $f$ of order $\rho$ as

$$\rho = \limsup_{r \to \infty} \frac{\log^+ \Xi(r, f)}{\log r}.$$ 

\[\square\]

**Illustration:** Let $\Psi(z) = \sum_{t=0}^{n} a_t z^t$ be a $n$ degree polynomial and esteem $f(z) = e^{\Psi(z)}$. preferably we compute $\Xi(r, f)$ as $\Psi(z) = a_n z^n$. Let $a_n = |a_n| e^{i\phi}$, $z = r e^{i\phi}$. Later

$$|f(z)| = |a_n| r^n \cos(\pi + \phi)$$

and so

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |a_n| r^n \cos(n \phi + \phi) d\phi$$

$$= \frac{1}{2\pi} \int_{2\pi + \phi}^{2\pi} \log^+ |a_n| r^n \cos(n \phi + \phi) d\phi$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |a_n| r^n \cos(n \phi) \cos(n \phi + \phi) d\phi = \frac{|a_n| r^n}{2\pi} \int_{0}^{2\pi} \cos \eta \cos \phi d\phi$$

$$= \frac{|a_n| r^n}{2\pi} \int_{0}^{2\pi} \cos \eta d\eta$$

Considering $f$ is an entire function, therefore

$$\Xi(r, f) = \frac{|a_n| r^n}{\pi}.$$

then

$$\Xi(r, f) = \Xi(r, e^{\Psi(z)}) = \Xi(r, e^{a_n z^n}, e^{a_{n-1} z^{n-1}}, \ldots, e^{a_0})$$

also

$$\Xi(r, e^{a_n z^{n-k}}) = \frac{a_{n-k}}{\pi} r^{n-k} = o\left(\frac{|a_n|}{\pi} r^n\right) = o\left(\Xi(r, e^{a_n z^n})\right),$$

it yields the form

$$\Xi(r, f) = \Xi(r, e^{\Psi(z)}) \sim \Xi(r, e^{a_n z^n}) = \frac{|a_n|}{\pi} r^n (r \to \infty).$$
8. Second Fundamental Theorem

Nevanlinna’s 2nd theorem on meromorphic function describe that for a given \( f \) in \( |f| \leq r \) also \( q > 2 \) distinct values \( a_1, a_2, ..., a_q \) of extended plane (complex) we admit the variability

\[
(q - 2) \mathcal{T}(r, f) \leq \sum_{\nu=1}^{q} \mathcal{M}(r, a_\nu) - \mathcal{N}_1(r) + \mathcal{S}(r, f)
\]

Here \( \mathcal{N}_1(r) \) is always non negative and

\[
\mathcal{S}(r, f) = m(r, f) + m(r, \sum_{\nu=1}^{q} \frac{f}{f - a_\nu}) + q \log^+ \frac{3q}{\delta} + \log^2 + \log \frac{1}{|f'(0)|},
\]

if \( |a_\mu - a_\nu| \geq \delta \) as \( 1 \leq \mu < \nu \leq q \), with refinement if \( f(0) = \infty \) or \( f'(0) = 0 \).

The term \( \mathcal{S}(r, f) \) is a small error term will be, in general, insignificant w.r.t \( \mathcal{T}(r, f) \). More exactly

\[
\mathcal{S}(r, f) = O(\log^2 \mathcal{T}(r, f)) + O(\log r)
\]

as \( r \to \infty \) over all values if \( f \) has finite order, and outside an exceptional set of finite measure otherwise. In particular (26) gives

\[
\mathcal{S}(r, f) = o(\mathcal{T}(r, f))
\]

for a function \( f \) outside an exceptional set \( \mathcal{E} \), of finite measure i.e., the error term is a small comparison with characteristic for most values of \( r \).

From (25) and (27) we obtain

\[
(q - 2) \mathcal{T}(r, f) < \left( \sum_{\nu=1}^{q} \mathcal{M}(r, a_\nu) \right) (1 + o(1)).
\]

9. Implementation

Nevanlinna theory is implemented in the theory of analytic function for differential equations and functional equations, minimal surface, holo-morphic dynamics and hyperbolic geometry of complex, which deals to greater dimensions with generalizations of Picard’s theorem.

Another application, is the bond between Diophantine approximations and Nevanlinna theory. In 1987, Paul Vojta specifically expressed the resemblance between Diophantine approximation and Nevanlinna theory. He demonstrate that in Diophantine approximation many functions of Nevanlinna have quantities or alternative functions and moreover that the two main theorems of Nevanlinna are equivalent to new existed theorems.

On some exponential Diophantine equations, we are giving a brief survey. Consider a rational integers \( x_1; y_1; m_1; x_1q \) with \( x_1 > 1; y_1 > 1; m_1 > 2; x_1 \geq 2 \). We entertain

\[
\frac{x_1^{m_1} - 1}{x_1 - 1} = y_1^q
\]

with unknowns \( x_1, y_1, m_1, q_1 \). Ljunggren [9] obtained that (28) along with \( q_1 = 2 \) doesn’t have result other than \( x_1 = 3, y_1 = 11, m_1 = 5 \) and \( x_1 = 7, y_1 = 20, m_1 = 4 \). So, without loss of generality we are accepting that an uneven prime number is \( q_1 \). Saradha and Shorey [15] obtained so that (28) has finitely large solutions at any time \( x_1 \) is a equilateral. Latterly, Y. Bugeaud and M. Mignotte declared that there doesn’t have solution. Now we examine a resembling problem if \( x_1 \) is a cube or a higher power. Especially being \( \mu_1 \) a constant rational integer with \( \mu_1 \geq 3 \) and we entertain (28) along with \( x_1 = z_1^{\mu_1} \) where \( z_1 > 1 \) is a rational integer. Without loss of generality we can assume that an uneven prime number is \( \mu_1 \). This convert us to see (28) in unidentified \( z_1, x_1, y_1, m_1 \).
References

[1] Ahlfors L 1976 Das mathematische Schaffen Rolf Nevanlinnas. (German) Ann. Acad. Sci. Fenn. Ser. A I Math. 2, pp 1–15
[2] Charak K 2009 Value distribution theory of meromorphic functions Mathematics Newsletter 18(4) pp 120
[3] Cherang C T and Yang C C 1990 fix-points and factorization of Meromorphic functions world Scientific publishing Co.Pet, Ltd Singapore
[4] Cherry W and Ye Z 2001 Nevanlinnas theory of value distribution. Springer Monographs in Mathematics Springer-Verlag Berlin
[5] Gross F 1972 factorization of meromorphic functions Mathematics Research Center Naval Research Laboratory Washington pp v+258
[6] Hayman W K 1964 Meromorphic functions Oxford Mathematical Monographs Oxford
[7] Laine I 1993 Nevanlinna theory and differential equations, Walter de Gruyter Berlin NewYork
[8] Lehto O 1982 On the birth of the nevanlinna theory Ann. Acad. Sci. fenn. Ser. AI Math 7 pp 523.
[9] Ljunggren W 1943 Noen setninger om ubestemte likninger av formen $\frac{x^n-1}{x-1} = y^q$ Norsk. Mat., Tidsskr 25(1) pp 17-20
[10] Lo Y 1993 Value distribution theory Springer-Verlag, Berlin, Heidelberg, Newyork and science press Beijing
[11] Nevanlinna R 1925 Zur theorie de merophen finktionen Acta Math. 45 pp 1-99
[12] Nevanlinna R 1970 Analytic functions Springer-Verlag, Newyork Berlin Heidelber
[13] Ru M 2001 Nevanlinna theory and its relation to Diophantine approximation World Scientific Publishing Co., Inc., River Edge pp xiv+323
[14] Rubel and Lee A 1996 Entire and meromorphic functions With the assistance of James E Colliander Universitext Springer-Verlag New York pp viii+187
[15] Saradha N and Shorey T N 1999 The equation $(x^n - 1)/(x - 1) = y^q$ with x square Math. Proc. Cambridge Philos. Soc 125(1) pp 1–19
[16] Stein E M and Shakarchi R 2003 Complex Analysis Princeton Lectures in Analysis II Princeton University Press Princeton
[17] Veena L Pujari 2015 On The Growth Of An E-Valued Meromorphic function And Its Derivative Applied Mathematics E-Notes 15 pp 137-146
[18] Vojta P on Nevanlinna Theory and Diophantine Approximation, http://math.berkeley. edu/ vojta/cime/cime.pdf
[19] Vojta P 1987 Diophantine approximation and value distribution theory Lecture Notes in Math Springer
[20] Zhang G H 1993 Theory of entire and meromorphic function AMS