The Moduli Space of BPS Monopoles

Kimyeong Lee

Physics Department, Columbia University, New York, NY 10027, USA

ABSTRACT

We review a progress in our understanding of the moduli space for an arbitrary number of BPS monopoles in a gauge theory with a group $G$ of rank $r$ that is maximally broken to $U(1)^r$. The derivation of the moduli space metric has been obtained from studying the low energy dynamics of well-separated dyons.
1 Introduction

Among the many remarkable features of the Bogomol’nyi-Prasad-Sommerfield (BPS) limit [1] is the existence of families of degenerate static multimonopole solutions. For any given topological charge, the space of such solutions, with gauge equivalent configurations identified, forms a finite dimensional moduli space. A metric for this space is defined in a natural way by the kinetic energy terms of the Yang-Mills-Higgs Lagrangian. As was first pointed out by Manton [2], a knowledge of this metric is sufficient for determining the low energy dynamics of a set of monopoles and dyons.

The electromagnetic duality proposed by Montonen and Olive [3] in Yang-Mills Higgs systems can be realized in the $N = 4$ supersymmetric Yang-Mills-Higgs systems, which have the BPS magnetic monopoles as solitons. The duality implies that when the coupling constant is large there is a dual theory where the role of monopoles and elementary charged quanta is exchanged. This means an exact match between the spectrum of charged particles and that of magnetic monopoles. This in turn implies the existence of various threshold bound states of fundamental magnetic monopoles. To see such bound states of zero bound energy, one needs to understand the low energy dynamics of BPS monopoles, which can be approached by the moduli space approximation. (For the recent reviews of the electromagnetic duality, see Ref.[4].)

Thus it would be interesting to understand the moduli space of the BPS magnetic monopoles better. For the case of an $SU(2)$ gauge symmetry spontaneously broken to $U(1)$, the moduli space $\mathcal{M}$ of solutions carrying $n$ units of magnetic charge has $4n$ dimensions. This naturally suggests that these solutions should be interpreted as configurations of $n$ monopoles, each of which is specified by three position coordinates and a $U(1)$ phase angle. (Recall that time variation of this phase angle gives an electrically charged dyon.) The metric for the two-monopole moduli space was determined by Atiyah and Hitchin [5]. By using this metric, the threshold dyonic bound states of two identical monopoles have been found by Sen and others[6]. For three or more monopoles, the moduli space metric is still unknown. An asymptotic form, valid in the regions of $\mathcal{M}$ corresponding to widely separated monopoles, has been found by Gibbons and Manton [7], but this develops singularities if any of the intermonopole distances becomes too small, and hence cannot be exact. The dyonic bound states of $n$ identical monopoles have been discussed also [7].

In this talk, which is based on our work [10, 11], we consider in detail the moduli space of the BPS monopoles in the theory with an arbitrary gauge group $G$ of rank $r \geq 2$ in the maximal symmetry breaking (MSB) case where the unbroken gauge symmetry will be a purely abelian subgroup $U(1)^r$. Hopefully this talk is complementary to Erick Weinberg’s talk where he draw more general pictures of the moduli space, duality and unbroken gauge symmetry. In this MSB case, there are $r$ independent topological charges $n_a$ and, correspondingly, $r$ fundamental monopoles, each of which carries a single unit of one of these charges [4]. The moduli spaces corresponding to any combination of $n$ fundamental monopoles are $4n$-dimensional, just as in $SU(2)$. 
The moduli space metric for all two-monopole solutions has been found[10], which generalized the $SU(3)$ result outlined by us in [11] and also found by Gauntlett and Lowe[12] and by Connell [13]. In addition, the asymptotic form of the metric for any number of monopoles is also found [10]. In the case where all monopoles are distinct from each other this asymptotic form has been argued to be the exact metric over the entire moduli space[10, 14, 15].

It might seem odd to try to obtain the moduli space metric for three or more monopoles with a larger gauge group when this cannot even be done for $SU(2)$. The reason that it might actually be easier to work with a larger group is that in a number of such cases the moduli space possesses greater symmetry. Specifically, consider a multimonopole solution with $r$ fundamental monopoles, each corresponding to a different topological charge. The corresponding $U(1)$ factors of the unbroken gauge group act nontrivially on this solution, and this action generates an isometry on the moduli space. Furthermore, for any such collection of topological charges there is a solution that, although it is composite, is spherically symmetric. On the moduli space, such a solution corresponds to a fixed point under the isometries that correspond to overall rotation of the monopole configuration. In $SU(2)$ there are no spherically symmetric solutions with multiple magnetic charge [16], and hence no such fixed points.

However generally the gauge symmetry might be partially broken to $K \times U(1)^{r-k}$ with unbroken nonabelian group $K$ is a semisimple group of rank $k < r$. In this case there are many oddities and one can define the moduli space of massive and massless monopoles consistently only when total magnetic charge is purely abelian. The the detail aspects of this case and its implications in the generalization of the electromagnetic duality are discussed in Erick’s talk and Ref.[17].

The plan of the talk is as follows. First, we recall some properties of BPS monopoles in general gauge group $G$ in Sec. 2. Here we also consider some characteristics of the moduli space metric. We then expose our detail analysis of the moduli spaces in Sec. 3. Here, following the approach that Gibbons and Manton [8] used for $SU(2)$, we use our knowledge of the interaction between widely separated dyons to infer the asymptotic form of the moduli space metric, which turns out to be smooth everywhere when the monopoles are distinct. Section 4 contains some concluding remarks.

## 2 BPS Monopoles

We begin by recalling some properties of the BPS solutions [1] in an $SU(2)$ theory spontaneously broken to $U(1)$. We fix the normalization of the electric and magnetic charges $g$ and $q$ by writing the large distance behavior of the electromagnetic field
strength (in radial gauge) as

\[ B_i = \frac{\hat{r}_i \hat{r}_a q}{4\pi r^2}, \quad E_i^a = \frac{\hat{r}_i \hat{r}_a q}{4\pi r^2}. \]  

(1)

The dyon solution carrying one unit of magnetic charge (i.e., \( g = 4\pi/e \), with \( e \) being the gauge coupling) may be written as

\[
\Phi^a(r) = \hat{r}^a K(r; v), \\
A_i^a(r) = \epsilon_{iak} \hat{r}^k \left( A(r; v) - \frac{1}{e r} \right), \\
A_0^a(r) = \frac{q}{\sqrt{g^2 + q^2}} \Phi^a, 
\]

(2)

where \( v \) is the asymptotic magnitude of the Higgs field and

\[
K(r; v) = v \coth(ev \eta) - \frac{1}{ev \eta}, \\
A(r; v) = \frac{v \eta}{\sinh(ev \eta)},
\]

(3)

with \( \eta = g/\sqrt{g^2 + q^2} \). The mass of this dyon is \( \tilde{m} = v\sqrt{g^2 + q^2} \). After quantization, the electric charge \( q \) should be an integer multiplet of \( e \).

For an arbitrary gauge group \( G \) of the rank \( r \), its generators can be chosen to be the \( r \) Cartan subalgebra generators \( H_i \) and the lowering and raising operators, \( E_{\alpha} \) one for each root \( \alpha \). They are normalized so that

\[ \text{tr} \, H_i H_j = \delta_{ij}, \quad \text{tr} \, E_{-\alpha} E_{\beta} = \delta_{\alpha\beta} \]

(4)

in a, say, adjoint representation. The roots \( \alpha \) may be viewed as vectors forming a lattice in a \( k \)-dimensional Euclidean space. It is always possible to choose a basis of \( k \) simple roots for this lattice in such a way that all other roots are linear combinations of the simple roots with integer coefficients all of the same sign; a root is called positive or negative according to this sign. A set of simple roots of particular importance is defined as follows. Let \( \Phi_0 \) be the asymptotic value of the Higgs field in some fixed direction (say, the positive \( z \)-axis). We may choose \( \Phi_0 \) to lie in the Cartan subalgebra and then define a vector \( \mathbf{h} \) by

\[ \Phi_0 = \mathbf{h} \cdot \mathbf{H}. \]

(5)

Here we are first concerned with the case of maximal symmetry breaking, with the gauge group spontaneously broken to \( U(1)^k \). This is achieved if and only if \( \mathbf{h} \) has a nonzero inner product with all of the roots. When this is so, there is a unique set of simple roots \( \beta_a \) that satisfy the requirement that \( \mathbf{h} \cdot \beta_a \) be positive for all \( a \); we shall use this basis for the remainder of this talk.
Asymptotically, the magnetic field must commute with the Higgs field. Hence, in the direction chosen to define $\Phi_0$, it must be of the form
\[
B = \frac{\hat{r}}{4\pi r^2} g \cdot H.
\] (6)

Topological arguments lead to the quantization condition [18]
\[
g = g \sum_{a=1}^{k} n_a \beta_a^*.
\] (7)

where again $g = 4\pi/e$ and
\[
\beta_a^* = \frac{\beta_a}{\beta_a^2}.
\] (8)

are the duals of the simple roots and the integers $n_a$ are the topologically conserved charges corresponding to the homotopy class of the scalar field at spatial infinity. For the BPS monopole configurations, the magnetic charge is given so that $n_a \geq 0$.

Monopole solutions carrying a single unit of topological charge can be obtained by simple embeddings of the $SU(2)$ solution. Each simple root $\beta_a$ defines an $SU(2)$ subgroup with generators
\[
t^1 = \frac{1}{\sqrt{2} \beta_a^2} \left( E_{\beta_a} + E_{-\beta_a} \right),
\]
\[
t^2 = -\frac{i}{\sqrt{2} \beta_a^2} \left( E_{\beta_a} - E_{-\beta_a} \right),
\]
\[
t^3 = \beta_a^* \cdot H.
\] (9)

If $\Phi^s(r; v)$ and $A_i^s(r; v)$ ($s = 1, 2, 3$) is the $SU(2)$ solution corresponding to a Higgs expectation value $v$, then
\[
A_i(r) = \sum_{s=1}^{3} A_i^s(r; h \cdot \beta_a) t^s,
\]
\[
\Phi(r) = \sum_{s=1}^{3} \Phi^s(r; h \cdot \beta_a) t^s + (h - h \cdot \beta_a^* \beta_a) \cdot H,
\] (10)

is a solution with topological charges
\[
n_b = \delta_{ab},
\] (11)

and mass
\[
m_a = gh \cdot \beta_a^*.
\] (12)
We will refer to such solutions as **fundamental monopoles** [3]. As in the $SU(2)$ case, there are dyon solutions corresponding to these fundamental monopoles such that asymptotically

$$E^{(a)} = \frac{q_{a} \hat{\beta}}{4\pi \epsilon_{a}^{2}} \beta_{a}^{*} \cdot H$$

(13)

After quantization, the electric charge $q_{a}$ is an integer multiple of the unit $e |\beta_{a}^{*}|^{2}$. From Eqs. (2) and (10), we get the asymptotic form of the Higgs field for such a dyon as $\Phi = \Phi_{0} + \Phi^{(a)}$ where

$$\Phi^{(a)} = -\frac{\beta_{a}^{*} \cdot H}{4\pi \epsilon_{a}} \sqrt{g^{2} + q_{a}^{2}}$$

(14)

There are a few differences between the $SU(2)$ and general cases in the several BPS monopole solutions. A notable difference, which will be of importance later, concerns the symmetry of the two-monopole solutions. In the $SU(2)$ case these failed to be axially symmetric because any gauge transformation that compensated the effects of a spatial rotation on one of the monopoles shifted the phase of the other monopole in the wrong direction. By contrast, if the two are different fundamental monopoles, it is always possible to find two gauge transformations such that each gives the required phase shift on one of the monopoles and leaves the other invariant. As a result, the superposition construction yields solutions with exact axial symmetry [19]. This leads to an additional $U(1)$ symmetry in this case.

A second difference from the $SU(2)$ case is the existence of relatively simple, spherically symmetric solutions that can be interpreted as superpositions of several fundamental monopoles at the same point. These are given by Eqs. (3) and (10), but with a composite root $\alpha$, rather than a simple root $\beta_{a}$, defining the $SU(2)$ subgroup. The coefficients $n_{a}$ in the expansion

$$\alpha^{*} = \sum_{a=1}^{k} n_{a} \beta_{a}^{*}$$

(15)

are the topological charges of the solution, while the mass is

$$m = \sum_{a} n_{a} m_{a}.$$ 

(16)

Although the mass and topological charge of these solutions are consistent with their interpretation as superpositions of several noninteracting monopoles, one might still ask why these spherically symmetric solutions should be viewed on such a different basis than those obtained from the simple roots. An answer is obtained by counting the normalizable zero modes about these solutions. After gauge fixing, the number of zero modes about an arbitrary BPS solution of magnetic charge (7) is [3]

$$N = 4 \sum_{a=1}^{k} n_{a}.$$ 

(17)
Thus, each of the fundamental monopoles has four zero modes, three corresponding to spatial translations and the fourth to a global $U(1)$ gauge rotation. By contrast, the solutions based on composite roots all have additional zero modes, with the number precisely that expected if these are in fact superpositions of several fundamental monopoles.

In the MSB phase there are as many charged vector bosons as the number of positive roots, whose number is much larger than that of simple roots if the gauge group is bigger than $SU(2)$. On the other hand we just argued that classically the number of fundamental monopoles is identical to that of simple roots. This seems to contradict with the duality hypothesis that the spectrum of charged particles should match exactly that of magnetic monopoles. However the duality is an intrinsically quantum mechanical statement: one has to study quantum mechanical spectrum of magnetic monopoles, which raises a possible existence of the quantum mechanical bound state of fundamental monopoles for each composite root. As the BPS mass formula is expected to be exact in the $N = 4$ supersymmetric theory, the bound energy of these composite states would be zero. This compels us to understand the low energy dynamics of monopoles better.

The BPS configurations for a given magnetic charge, and so the same energy, are parameterized by the $N$ collective coordinates $z_\alpha, \alpha = 1 \ldots N$, or moduli up to local gauge transformations. For a given magnetic charge, the space of gauge-inequivalent BPS configurations is called the moduli space. A typical configuration is

$$A_\mu(x; z_\alpha) = (A(x; z_\alpha), \Phi(x, z_\alpha))$$

with $A_4 = \Phi$. The $4N$ zero modes $\delta_\alpha A_\mu = \partial A_\mu / \partial z_\alpha - D_\mu \epsilon_\alpha$ satisfy the linearized BPS equation and are chosen to satisfy the background gauge $D_\mu \delta_\alpha A_\mu = 0$ with $\partial_4 = 0$.

When we consider fluctuations around the BPS magnetic monopole configurations, there are massless modes and massive modes. If the initial energy is arbitrary small, the dynamics of BPS monopoles may be approximated by that of moduli. The initial field configuration at a given time will be characterized by $A_\mu(x, z_\alpha(t))$ and its time derivative, $\dot{z}_\alpha \delta_\alpha A_\mu$ in the $A_0 = 0$ gauge. The Gauss law constraint on the initial configuration is exactly the background gauge.

Since there is no force between monopoles at rest, one expects the low energy dynamics is given by the kinetic part of the Yang-Mills-Higgs Lagrangian. In the $A_0 = 0$ gauge, this becomes

$$\mathcal{L} = \frac{1}{2} G_{\alpha\beta}(z_\alpha) \dot{z}_\alpha \dot{z}_\beta$$

where $G_{\alpha\beta}(z_\alpha) = \int d^3 x \ tr \delta_\alpha A_\mu \delta_\beta A_\mu$. While one can study some characteristics of this metric, it is hard to obtain directly from the BPS field configurations which themselves are not known in general. However some formal characteristics of the metric can be deduced from this. The important property of the metric is that it is hyper-Kähler. When we consider the supersymmetric Yang-Mills theory, the original supersymmetry which should be incorporated by supersymmetrizing the above Lagrangian.
A 4n-dimensional manifold with a metric is hyper-Kähler if it possesses three covariantly constant complex structures \( J^{(k)} \), \( k = 1, 2, 3 \) that also form a quarternionic structure and if the metric is pointwise Hermitian with respect to each \( J^{(k)} \). If we recast the zero modes \( A_\mu \) into a spinor \( \Psi = \delta A_4 + i \tau_i \delta A_i \) where \( \tau_i \)'s are the Pauli matrices, the zero mode equation is manifestly invariant under right multiplications by the \( i \tau_k \)'s \([9] \), and this induces the almost quarternionic structure on the moduli space. Detailed arguments that the moduli space is hyper-Kähler can be found in Refs. \([5, 21] \).

### 3 Distinct Fundamental Monopoles

The metric on the moduli space determines the motions of slowly moving monopoles. Conversely, the form of the moduli space metric can be inferred from a knowledge the interactions between monopoles. These become quite complicated when several monopoles approach one another as their nonabelian cores overlap. However, for finding the metric in the regions of the moduli space corresponding to large inter-monopole distances, it is sufficient to examine the long-range pairwise interactions between widely separated monopoles. In large separation of monopoles, the electric charge of individual monopole, the momentum for the phase zero mode, is conserved and the interaction between \( n \) monopoles of \( 4n \) coordinates becomes that between dyons of \( 3n \) coordinates. This analysis has been carried out previously for \( SU(2) \) \([8] \) and generalized to larger gauge groups in Ref. \([10] \).

We begin by considering the interactions between a single pair of fundamental dyons, with positions \( x_a \) and velocities \( v_a \) and carrying magnetic charges \( gb_\alpha^* \) and electric charges \( q_a \beta_\alpha^* \) (\( a = 1, 2 \)). If the separation between the dyons is much larger than the radius of a monopole core, the electromagnetic interactions between them can be well approximated by the standard results for a pair of moving point charges. There is also a long-range scalar force that is manifested as a position-dependent shift in the dyon mass. Recall that the mass of an isolated dyon is \( \tilde{m}_a = h \cdot \beta_a^* \sqrt{g^2 + q_a^2} \). When there is a second dyon present, its correction to the scalar field must be added to \( h \) in this formula.

The electric and scalar fields for dyon 2 at the origin are given by Eqs. (13) and (14). The magnetic field for dyon 2 at the origin is \( B^{(0)} \) obtained from Eq. (13) with only \( n_a = 1 \) nonvanishing. The four potential \( A^{(2)} \) and \( A_{0}^{(2)} \) for dyon 2 can be obtained from the electromagnetic field. The dual four potential \( \tilde{A}^{(2)} \) and \( \tilde{A}_{0}^{(2)} \) is defined so that \( E = - \nabla \times \tilde{A} \) and \( B = - \nabla \tilde{A}_0 - \partial \tilde{A} / \partial t \). The scalar field and four-vector potentials can be generalized to the Lienard-Wiechart forms by taking into account the motion of dyon 2 \([8] \).

The motion of dyon 1 in the background of the fields generated by dyon 2 is described by the Lagrangian

\[
L^{(1)} = \sqrt{g^2 + q_1^2} \text{tr} \left\{ \beta_1^* \cdot H[\Phi_0 + \Phi^{(2)}(x_1)] \right\} \sqrt{1 - v_1^2}
\]
Substituting the Lienard-Wiechart form of the potentials and scalar field into Eq. (20) and keeping only terms of up to second order in $q_j$ or $v_j$, we obtain

\[
L^{(1)} = -m_1 \left( 1 - \frac{1}{2} v_1^2 + \frac{q_1^2}{2 g^2} \right) - \frac{g}{8 \pi r_{12}} \beta_1^* \cdot \beta_2^* \left[ (v_1 - v_2)^2 + \frac{(q_1 - q_2)^2}{g^2} \right] - \frac{g}{4 \pi} \beta_1^* \cdot \beta_2^* (v_1 - v_2) \cdot w_{12},
\]

where $m_1 = g h \cdot \beta_1^*$, $r_{12} = |x_1 - x_2|$ and $w_{12} \equiv w(x_1 - x_2)$ is a Dirac monopole potential, defined so that \[
\nabla \times w(x) = -\frac{x}{|x|^3}.
\]

The Lagrangian describing the dynamics of slowly moving two dyons at large separation follows if one symmetrizes the above Lagrangian by adding the noninteracting part for dyon 1.

The extension to an arbitrary number of fundamental dyons is straightforward, provided that their mutual separations are all large. The Lagrangian obtained by adding all the pairwise interactions can be written as

\[
L = \frac{1}{2} M_{ab} \left( v_a \cdot v_b - \frac{q_a q_b}{g^2} \right) + \frac{g}{4 \pi} q_a W_{ab} \cdot v_b,
\]

where

\[
M_{aa} = m_a - \sum_{c \neq a} \frac{g^2 \beta_a^* \cdot \beta_c^*}{4 \pi r_{ac}}, \quad M_{ab} = \frac{g^2 \beta_a^* \cdot \beta_b^*}{4 \pi r_{ab}} \quad \text{if } a \neq b,
\]

with $m_a = g \beta_a^* \cdot h$, and

\[
W_{aa} = -\sum_{c \neq a} \beta_a^* \cdot \beta_c^* w_{ac}, \quad W_{ab} = \beta_a^* \cdot \beta_b^* w_{ab} \quad \text{if } a \neq b.
\]

with $w_{ab}$ being value at $x_a$ of the Dirac potential due to the $b$th monopole. The monopole rest masses have been omitted.

To obtain the moduli space metric, we need a Lagrangian that is purely kinetic; i.e., one in which all terms are quadratic in velocities. This can be done by interpreting the $q_b/e$ as conserved momenta conjugate to cyclic angular variables $\xi_j$. Because $q_b/e$ is quantized in integer multiples of $\beta_b^2$, the period of $\xi_j$ must be $2\pi/\beta_b^2$. Thus, if we make the identification

\[
q_a/e = \frac{g^4}{16 \pi^2} (M^{-1})_{ab} (\dot{\xi}_b + W_{bc} \cdot v_c),
\]
the desired Lagrangian $\mathcal{L}$ is the Legendre transform

$$\mathcal{L} = L + \sum_j \dot{\xi}_b q_b / e = \frac{1}{2} M_{ab} v_a \cdot v_b + \frac{g^4}{2 (4\pi)^2} (M^{-1})_{ab} (\ddot{\xi}_a + W_{ac} \cdot v_c) \left( \dot{\xi}_b + W_{bd} \cdot v_d \right). \quad (27)$$

From this we immediately obtain the large separation approximation to the moduli space metric,

$$G = M_{ab} dx_a \cdot dx_b + \frac{g^4}{16\pi^2} (M^{-1})_{ab} (d\xi_a + W_{ac} \cdot dx_c) (d\xi_b + W_{bd} \cdot dx_d). \quad (28)$$

Note that this metric is equipped with a number of $U(1)$ isometries, each of which is generated by the constant shift of one of the $\xi_b$’s.

The fact that this asymptotic metric is hyper-Kähler can be shown trivially, following the argument by Gibbons and Manton[8]. The key ingredient is that $\nabla 1/r = \nabla \times w(r)$. The question is whether the asymptotic metric when it is extended in the interior region is nonsingular.

For the case of $SU(2)$, one can easily see that this approximation to the moduli space metric cannot be exact [8]. First of all, it develops singularities if any of the intermonopole distances becomes too small, whereas the moduli space metric should be nonsingular. Second, for the case of two monopoles the approximate metric is independent of the relative phase angle $\xi_1 - \xi_2$. If this isometry were exact, the two-monopole solutions would be axially symmetric, which we know is not the case.

Neither of these objections arises for the moduli space corresponding to a collection of several distinct fundamental monopoles in a larger group, provided that each corresponds to a different simple root. The metric can be simplified by going to the center of mass frame for $r$ fundamental monopoles of mass $m_a$ for connected simple roots $\beta_a$ of a simple group $G$. (For a given number of the distinct monopoles, we choose for the convenience the minumum size gauge group which allows them.) The center-of-mass position and total charge are given as

$$R = \frac{1}{M} \sum_a m_a x_a \quad (29)$$

$$q_\chi = \frac{1}{e M} \sum_a m_a q_a \quad (30)$$

with the total mass $M = \sum_a m_a$. The conjugate global phase of the total charge $q_\chi$ is $\chi = \sum_a \xi_a$ whose value lies on the real line $R$. We label by $A = 1, \ldots, r - 1$ the $r - 1$ links between the adjacent pairs of simple roots, which can be read from the Dynkin diagram. The relative coordinates and charges between monopoles are defined as

$$r_A = x_a - x_b \quad (31)$$

$$q_A = \frac{\lambda_A}{2e} (q_a - q_b) \quad (32)$$
where \( \beta_a \) and \( \beta_b \) are connected by the rink \( A \) and \( \lambda_A = -2\beta_a^* \cdot \beta_b^* \). The conjugate angular variable \( \psi_A \) for each \( q_A \) has the range \([0, 4\pi]\).

The center-of-mass part of the moduli space metric (28) is the

\[
G_{\text{cm}} = M \left( dR^2 + \frac{g^4}{16\pi^2 M^2} d\chi^2 \right),
\]

and is the metric of a flat four-dimensional manifold. The metric \([11]\) for the relative coordinates obtained from Eq. (28) is

\[
G_{\text{rel}} = C_{AB} dr_A \cdot dr_B + \frac{g^4 \lambda_A \lambda_B}{64\pi^2} (C^{-1})_{AB} D\psi_A D\psi_B
\]

where \( D\psi_A = d\psi_A + w(r_A) \cdot dr_A \) and the \((r - 1) \times (r - 1)\) matrix \( C_{AB} \) is

\[
C_{AB} = \mu_{AB} + \delta_{AB} \frac{g^2 \lambda_A}{8\pi r_A}
\]

with \( r_A = |r_A| \) and the reduced mass matrix \( \mu_{AB} \) can be deduced from the free Lagrangian. The total moduli space is not a simple direct product of the center-of-mass part \( R^4 \) and the relative part. From the translations generated from \( \xi_i \to \xi_i + 4\pi \), the space \( R \times S_{r-1} \) of the \((\chi, \psi_A)\) should be divided by an identification map of integer group \( Z \).

The metric is rotationally invariant and so there is a conserved angular momentum in the dynamics, which can also be written as a sum of \( J_{\text{cm}} = M R \times \dot{R} \) and

\[
J_{\text{rel}} = \sum_{AB} C_{AB} r_A \times \dot{r}_B + \sum_A q_A \dot{r}_A
\]

While our metric is obtained by considering the monopoles in large separations, this metric \( G_{\text{rel}} \) is smooth everywhere, complete and has the right symmetry. It is not only invariant under the spatial rotation but also under the \( r - 1 \ U(1) \) phase shifts \( \psi_A \to \psi_A + \text{constant} \). The origin where \( r_A = 0 \) is the point where all monopoles come together, which should be spherically symmetric as it can be represented by the \( SU(2) \) embedding solution mentioned earlier with the maximum positive root \( \sum_i \beta \). This leads one to suspect our metric is the right metric everywhere. This conjecture has been argued to be correct lately by Murray and Chalmers \([14, 15]\).

In particular when \( r = 2 \), we can confirm this explicitly. The relative moduli space is a four-dimensional hyper-Kähler space with rotational symmetry. Atiyah-Hitchin\([3]\) showed that such a space is one of the following manifolds:

1) flat \( R^4 \).
2) the Taub-NUT geometry with an \( SU(2) \) rotational isometry.
3) the Atiyah-Hitchin geometry with an \( SO(3) \) rotational isometry.
4) the Eguchi-Hanson gravitational instanton \([23]\).

The only one with the right symmetry and interaction, the asymptotic form and the symmetric point turns out to be the Taub-NUT metric \([1, 2, 3]\).
4 Discussion

In this talk we have reviewed the method to derive the moduli space metric which describes the low energy dynamics of the \( n \) distinct fundamental BPS monopoles in the Yang-Mills-Higgs systems with arbitrary gauge group. The asymptotic metric obtained by studying the low energy dynamics of dyons in large separation seems to be correct everywhere as it has the right isometries and properties, like hyper-Kählerness, smoothness, and completeness. We then focused ourselves to the moduli space metric for two monopoles, whose correctness everywhere can be seen easily.

There are several questions raised from this point. One interesting issue concerns theories for which there is the nonabelian unbroken gauge group. There has been a considerable progress along this direction by us \([17]\), which is summerized in Erick’s talk. As discussed in detail there, the moduli space of monopoles with total magnetic charge purely abelian in this case is the appropriate limit of the moduli space of the identical magnetic charge in the MSB case. The massless monopoles does not correspond to distinct solitons but instead manifest themselves as non-abelian cloud surrounding the massive monopoles. While the zero modes for massive monopoles remain to be interpreted as the positions and abelian phases, the zero modes corresponding to the massless monopoles are transformed into an equal number of nonabelian global gauge orientation and gauge-invariant structure parameters characterizing the nonabelian cloud. The unbroken nonabelian gauge group becomes the part of the isometry group of the moduli space.

The second question is whether one can extend the above work to the many-monopole case where some of monopoles are identical. The asymptotic form for the metric develops a curvature singularity if a pair of identical monopoles are brought too close together. Recently there seems to be some progress for the case where the number of identical monopoles are less than or equal to two by using the Legendre transformation \([15]\). If this is successful, one further expects some simplifications in the cases involving two or more identical monopoles where exist of spherically symmetric solutions. Further progress along this direction would be quite interesting as it may shed some light on how nonabelian cloud structures itself around the sources.

Finally, one of the more compelling motivations for studying the low energy dynamics of monopoles and dyons arises from the Montonen-Olive duality conjecture \([3]\). In certain supersymmetric Yang-Mills theories, duality maps strongly coupled electric theories to weakly coupled magnetic ones, thus enabling one to probe the nonperturbative nature of strongly coupled Yang-Mills theories. A notable example of this is the \( N = 2 \) supersymmetric gauge theories softly broken to \( N = 1 \), where confinement is explicitly realized through magnetic monopole condensation \([24]\).

In order for such a duality to make sense, however, the spectrum of magnetically charged particles must be consistent with that predicted by the duality mapping of the electrically charged ones. In \( N = 4 \) supersymmetric Yang-Mills theories, duality relates each elementary massive vector meson of electric charge \( \gamma \) to a tower of dyons...
of magnetic charge $\gamma^*$ among others, where $\gamma$ is any root of the gauge algebra. The analysis of the classical solutions tells us, however, that a monopole based on a composite root is not a fundamental entity, but rather corresponds to a mere coincidence point on a multi-monopole moduli space. Thus, it is imperative to see if the quantization of the multi-monopole dynamics leads to a bound state that could conceivably be dual to the vector meson whenever $\gamma$ is composite.

This program has recently been carried out in the MSB cases. The bound state of two distinct monopoles in the MSB case was found by the authors [11], and also independently by Gauntlett and Lowe [12]. The most massive of the charged vector mesons in this theory is dual to a threshold bound state with composite magnetic charge, which is realized as a unique normalizable harmonic form on the Taub-NUT manifold. The threshold bound states of any number of distinct monopoles have been found recently by Gibbons [22]. In theories with unbroken nonabelian gauge symmetry, the duality implies the existence of many threshold bound states whose magnetic charges are be abelian or nonabelian [17]. Since some candidate of the moduli space metric is known explicitly, it would be a challenge to find such bound states in this case.

**Acknowledgments**

We thank the organizers, especially Peter Tinyakov and Valery Rubakov, of the Quark-96 conference at Yaroslavl, Russia for warm hospitality and pleasant atmosphere. This work was supported in part by the NSF Presidential Young Investigator program.

**References**

[1] E.B. Bogomol’nyi, Sov. J. Nucl. Phys. 24, 449 (1976); M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975); S. Coleman, S. Parke, A. Neveu and C.M. Sommerfield, Phys. Rev. D15, 544 (1977).

[2] N.S. Manton, Phys. Lett. 110B, 54 (1982).

[3] C. Montonen and D. Olive, Phys. Lett. 72B, 117 (1977); H. Osborn, Phys. Lett. 83B, 321 (1979).

[4] D. Olive, *Exact Electromagnetic Duality*, hep-th/9508089; J.A. Harvey, *Magnetic Monopoles, Duality and Supersymmetry*, hep-th/9603986.

[5] M.F. Atiyah and N.J. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton Univ. Press, Princeton (1988); Phys. Lett. 107A, 21 (1985); Phil. Trans. R. Soc. Lon. A315, 459 (1985).
[6] A. Sen, Phys. Lett. B329, 217 (1994); S. Sethi, M. Stern and E. Zaslow, Nucl. Phys. B457, 484 (1995); J.P. Gauntlett and J. Harvey, Nucl. Phys. B463, 287 (1996).

[7] M. Porrati, Phys. Lett. B377, 67 (1996).

[8] N.S. Manton, Phys. Lett. B 154, 397 (1985); G.W. Gibbons and N.S. Manton, Phys. Lett. B356, 32 (1995).

[9] E.J. Weinberg, Nucl. Phys. B167, 500 (1980).

[10] K. Lee, E.J. Weinberg and P. Yi, Phys. Rev. D54, 1633 (1996).

[11] K. Lee, E.J. Weinberg and P. Yi, Phys. Lett. B376, 97 (1996).

[12] J.P. Gauntlett and D.A. Lowe, Nucl. Phys. B 472, 194 (1996).

[13] S.A. Connell, The dynamics of the SU(3) charge (1,1) magnetic monopoles, University of South Australia preprint.

[14] M.K. Murray, A note on the (1,1,...,1) monopole metric, hep-th/9505054, a University of Adelaide preprint.

[15] G. Chalmers, Multi-monopole moduli spaces for SU(N) gauge group, ITP-SB-96-12, [hep-th/9605182].

[16] A.H. Guth and E.J. Weinberg, Phys. Rev. D14, 1660 (1976).

[17] K. Lee, E.J. Weinberg and P. Yi, Massive and Massless Monopoles with Non-abelian Magnetic Charges, CU-TP-55 and [hep-th/9605229].

[18] P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B125, 1 (1977); F. Engert and P. Windey, Phys. Rev. D14, 2728 (1976).

[19] C. Athorne, Commun. Math. Phys. 88, 43 (1983).

[20] J.D. Blum, Phys. Lett. B333 (1994) 92; J. Gauntlett, Nucl. Phys. B411, (1994) 443.

[21] J. Gauntlett, Nucl. Phys. B411, 443 (1994).

[22] G.W. Gibbons, Phys. Lett. B282, 53 (1996).

[23] T. Eguchi and A.J. Hanson, Ann. Phys. 120, 82 (1979).

[24] N. Seiberg and E. Witten, Nucl. Phys. B426, 19 (1994); (E) B430, 485 (1994); ibid. B431, 484 (1994).