Finite-Element Simulation of an Accelerator Magnet: an Exercise

H. De Gersem, I. Kulchytska-Ruchka and S. Schöps*
Institute for Accelerator Science and Electromagnetic Fields (TEMF), TU Darmstadt, Germany

Abstract
This note describes an extended exercise on the finite-element (FE) simulation of an accelerator magnet. The students construct and simulate a magnet model using the FEMM freeware. They get the opportunity to exercise on the theory of FEs, including Maxwell equations, magnetoquasistatic formulation, weighted residual approach, choice of appropriate FE shape functions and algebraic system of equations, thereby guided by fill-in sheets. They are invited to implement the most crucial parts of a simple FE solver. Finally, the own software is used to simulate the magnet once more and to develop some problem-dedicated post-processing routines. The exercise educates students in accelerator physics and electrical engineering on the construction and simulation of accurate and manageable FE models, the algorithms behind a standard FE solver and some ideas to extend a FE solver for own purposes. All necessary files to carry out the exercise are freely available.

Keywords
Accelerator magnets; finite-element method; electromagnetic field simulation.

1 Introduction
Finite-element (FE) simulation is indispensable in design procedures for normal-conducting and superconducting magnets. Many tools are available, as a commercial product or as freeware. The use of FE simulation, however, requires specific skills, which need to be educated consciously. Within that context, it makes sense that students get an insight in the interior of a FE solver. Moreover, customizing or even extending a FE solver for specific purposes can boost technological progress, e.g., in the field of accelerator magnets. This note describes an exercise aiming at these goals.

The note is structured as follows. In Section II, the structure and content of the exercise is described. Section III gives an overview of the provided material and hints for its usage. Section IV reports about experience gained by tutors during the past years while guiding BSc- and MSc-students through the exercise. Section V provides a summary and conclusion.

2 Structure of the exercise
The exercise is divided in 5 parts, which are preferably carried out at one exercise day, and in the prescribed order, but which also can be carried out individually and in different sessions. The exercise is accompanied by a self-explanatory guide (see Figs. 1–27). An introduction or explanation by a tutor before the start is not foreseen. It is assumed that the students attended the preceding lecture [1]. The worksheets try to address the students, invite them to carry out the tasks (numbered from 1 up to 41), give hints which strategy is the most appropriate one and indicate how to interpret the results.

*The exercise partially originate in the collaboration project Simulation of Transient Effects in Accelerator Magnets (STEAM, https://espace.cern.ch/steam) between CERN and TEMF. This exercise has been carried out during lecture series at the RWTH University Aachen, at the TU Darmstadt and at the CERN Accelerator School. This work has been supported by the Excellence Initiative of the German Federal and State Governments and the Graduate School of Computational Engineering at TU Darmstadt.

Available online at https://cas.web.cern.ch/previous-schools
The first 4 worksheets (Figs. 2–5) describe the superconducting dipole magnet intended for bending the heavy-ion beam in the SIS-100 synchrotron of the Facility for Antiproton and Ion Research (FAIR, www.fair.de [2]), which nears completion at the Helmholtzzentrum fAjr Schwerionenforschung (GSI, Facility for Heavy Ion Research, www.gsi.de [3]) in Darmstadt, Germany.

2.1 Part 1: Model construction
The worksheets "Geometry", "FEMM", "Materials" and "Boundary Conditions" (Figs. 6–9) describe how to construct the magnet model from scratch using the graphical user interface of FEMM [4]. For some students, this may be the first time confronted with a FE modeller. The model construction typically takes half an hour. When limited in time, one can decide to skip Part 1 and directly turn to Part 2. For that option, an already completed model is provided as well.

2.2 Part 2: FE solution
The worksheet with title "Solve" (Fig. 10) guides the students through the solution process, thereby giving a bit of background information. The students are recommended to spend some time to the interpretation of the obtained magnetic flux distribution by a set of questions at the bottom of the worksheet. Here, the involvement of a tutor is known to improve the understanding. He/she should invite the students to play around with the software, thereby deliberately changing some parameters of the FE model and checking their impact on the solution. This part of the exercise takes only a quarter of an hour.

2.3 Part 3: Mathematical derivation
The worksheets "Own implementation" and "Routines" (Figs. 11 and 12) already show a road map for the implementation of an own FE solver. The students should have a look in the FEMM data files where they will recognize the data they have inserted for constructing the model and the mesh generated by the built-in Triangle mesh generator [5]. Thereafter, they get an overview about the provided Matlab/Octave [6, 7] routines with which they will build an own FE solver. At this point, most probably, the student will have no idea how to start the implementation task. For that reason, the following worksheets offer the students the possibility to repeat the theory about magnetostatic FE simulation, given during a preceding lecture [1], on their own.

The worksheet "Formulation" (Fig. 13) guides the students from the Maxwell equations to the magnetostatic formulation as a function of the magnetic vector potential. In the worksheets "Discretization" (Figs. 14–15), the weighted residual approach and the FE shape functions are introduced. The worksheet entitled "Reduction to 2D" (Fig. 16) instantiates the discrete formulation for the 2D cartesian case. The worksheets "Compute Coefficients" (Figs. 17–18) allow the students to calculate the entries of the 3-by-3 elementary matrices and 3-by-1 elementary vector which after assembly form the FE system of equations. These derivations form the tasks 10 to 27. While this part of the exercise is traditionally experienced as difficult, this part has been divided in a large number of small tasks. This part takes 0.5 to 1.5 hours of time.

2.4 Part 4: Implementation
Task 28 asks for a first implementation step. In the routines curlcurl_11.m, edgemass_11.m and current_Pstr.m, the students find places marked by the comment "IMPLEMENTATION POINT" where they should code the matrix entries as calculated before. This task is crucial in the whole exercise. The tutor should make sure that all students reach this task and accomplish it successfully. The worksheets "Magnetic Flux Density" (Fig. 19) and "Magnetic Energy" (Fig. 20) require similar derivations and implementations. Up to this point, no own FE solution is required. The implemented routines
are tested by post-processing the magnetic energy and the magnetic flux density from the solution for the magnetic vector potential obtained by FEMM.

The worksheet ”Boundary Conditions” (Fig. 21) gives basic information about inserting Dirichlet boundary conditions. The treatment of the boundary conditions is given as a ready implementation (routines shrink.m and inflate.m). After inserting the boundary conditions, the system of equations can be solved and a solution close to the one of FEMM should come out. For the complete part 4, students typically need 1 to 2 hours.

2.5 Part 5: Post-processing

Part 5 of the exercise comprises a post-processing step dedicated to accelerator-magnet design, i.e., the calculation of the harmonic components and the skew harmonic components of the magnet’s aperture field. The theory is written down on the worksheet ”Aperture Field (1/2)” (Fig. 22), whereas the guidance for implementation is organized as three tasks on the next worksheet (Fig. 23).

The last part of the exercise extends the own FE solver to deal with nonlinear magnetic materials. Possible linearization techniques are described in the worksheet ”Nonlinear Material Properties” (Fig. 24). The successive-substitution and Newton algorithms are explained in the three last worksheets (Figs. 25–27), together with indication for implementation of the algorithm in the main file of the own software. The outcome is a renewed calculation of the (skew) harmonic components of the aperture field, now accounting for the inevitable saturation of the magnet yoke. The results gathered in the table of Fig. 27 should show a decrease of the magnitudes of the higher-harmonic components compared to the results collected in the table of Fig. 23. This illustrates that the SIS-100 magnet has been optimized such that the aperture field is maximally homogeneous for the maximal field [8].

Part 5 takes a total of 1 up to 2 hours for completion.

3 Exercise material

The material needed to carry out the exercise can be found in [9]. The worksheets supporting the exercise are also contained within this note, see Figs. 1–27. The figure captions give a few didactical hints, which come from the tutors who guided the exercise during the past years. Together with that, the students can download the software. The material is divided in 5 independent parts, which allows the students to enter the exercise at any point, e.g., skipping the construction of the model (part 1) and directly start calculating (part 2), or skipping the mathematical part (part 3) and directly passing from the calculation using FEMM (part 2) to the implementation of an own FE solver (part 4). The material also included the slides of the preceding lecture.

The version of the software provided for part 4 of the exercise is deliberately made incomplete. At clearly marked so-called ”IMPLEMENTATION POINTS”, the students are invited to complete the code. The code fragments are chosen in order to support insight in the key parts of a FE solver, thereby avoiding all parts which are solely intended for bookkeeping (handling mesh data, boundary conditions, material distributions, excitations). The software provided for part 5 is, however, complete. Uninterested student could decide to copy-paste the corresponding code fragments – from which they would not learn that much. The completeness of the provided material, however, makes sure that all students return home with a fully functional 2D nonlinear magnetostatic FE solver.
4 Discussion

The exercise has been used as part of several lecture series at several universities. By that, experience has been gathered, which is shared in this section.

Upon invitation to work in groups, the students typically form groups of two or three, but do not divide the work. Instead, they work on one computer through the worksheets from start to finish. This is beneficial for guarding against errors and improving understanding, but requires maximal working time, amounting to 4 hours for a team of skilled and motivated students up to 12 hours for unexperienced BSc students. When time is limited, the tutors should actively suggest working in parallel.

When the students create the FE model themselves, a lot of modelling errors occur. The most common ones are wrong units, wrong assignment of material properties, wrong assignment of exciting currents and wrong assignment of boundary conditions (of course, this would be slightly different when another software would be used). Often, the students need a little help to find the error. Mostly, it is sufficient to force them to look at the (wrong) magnetic flux distribution, point them explicitly to the erroneous region and ask them what could be the origin thereof. If the students are proficient in FE modelling, the first part of the exercise can be skipped.

The students hesitate to start the implementation part (part 3), which is particularly caused by the size of the FE software (some dozens of MATLAB/Octave routines) and the lack of programming experience. Here, the tutor should recommend to start with the implementation tasks one by one and in the given order. The implementation points in the software are well marked and stick closely to the mathematical derivation. This should keep the hurdle as low as possible. A related observation is made when the students finish the implementation part. Then, they are surprised to be successful and get more confident in FE modelling and simulation. Some students ask for hints for further implementation of the more advanced methods discussed in the lecture. Other students try to adapt the software for own purposes, e.g., for solving an example related to their BSc- or MSc-thesis work.

The implementation of the post-processing routine for calculating the harmonic and skew-harmonic components of the aperture field is tough. Students with a pronounced interest in accelerator physics remain motivated. However, for students with focus on studying magnetic field simulation by FE methods, this part feels superfluous. The tutor may invite the latter to proceed to the treatment of the nonlinear case and use the magnetic energy as a reference quantity of interest instead of the harmonic and skew-harmonic components.

5 Summary and conclusions

This paper describes an extended exercise where students in physics, mathematics and engineering are invited to set up a 2D finite-element (FE) model of a superconducting accelerator magnet. Furthermore, the exercise briefly recalls standard FE theory and enables the students to bring this knowledge into praxis by implementing a few parts in a given FE software framework. The exercise ends by adding a post-processing tool to derive the harmonic and skew-harmonic field components from the FE solution for a superconducting magnet model. The exercise takes 4 hours for skilled and motivated MSc-students up to 12 for unexperienced BSc-students. The exercise can be carried out in parts. The main learning achievement is the fact that students get convinced about the capabilities of FE simulation and get confident for future tasks involving FE simulation.

Acknowledgements

We thank Wolfgang Ackermann, Markus Borkowski, Thorben Casper, Idoia Cortes Garcia, Laura A.M. D’Angelo, Erion Gjonaj, Dimitrios Loukrezis, Nicolas Marsic, Andreas Pels and Erik Schnaubelt for their contributions to this field of research and for the development and maintenance of the software. Moreover, we thank Gregor Bavendiek, Christian Kräijtgen, Fabian Mäijller, Peter Offermann, Aryanti Putri and Dries Vanoost for their dedication as tutors in the exercise classes. We thank all students of
TU Darmstadt, RWTH Aachen, KU Leuven, Université de Lille and all participants of the CERN Accelerator School for their suggestions for improvement of the worksheets. Last but not least, we thank David Meeker and Jonathan Shewchuk for putting FEMM [4] and Triangle [5] as freeware on the world wide web.

References
[1] H. De Gersem, I. Cortes Garcia, and S. Schüps, “Magnetodynamic finite-element simulation of accelerator magnets.” CERN Accelerator School, 2019.
[2] FAIR, “Facility for Antiproton and Ion Research in Europe GmbH,” Nov. 2016.
[3] GSI, “Helmholtzzentrum fÄîjr Schwerionenforschung GSI,” Nov. 2016.
[4] D. Meeker, Finite Element Method Magnetics, version 4.2 (09nov2010 build) ed., 2010. User’s Manual.
[5] J. R. Shewchuk, “Triangle: Engineering a 2D quality mesh generator and Delaunay triangulator,” in Applied Computational Geometry: Towards Geometric Engineering (M. C. Lin and D. Manocha, eds.), vol. 1148 of Lecture Notes in Computer Science, pp. 203–222, Springer-Verlag, May 1996.
[6] Mathworks, MATLAB Getting Started Guide, 2009.
[7] J. W. Eaton, D. Bateman, S. Hauberg, and R. Wehbring, GNU Octave 4.2 Reference Manual. Samurai Media Limited, May 2017.
[8] A. Kovalenko, A. Kalimov, H. Khodzhibagiyani, G. Moritz, and C. MÄîjhle, “Optimization of a superferric nuclotron type dipole for the GSI fast pulsed synchrotron,” IEEE Trans. Appl. Super., vol. 12, pp. 161–165, Mar. 2002.
[9] H. De Gersem, I. Kulchytska-Ruchka, and S. Schüps, ”Magnetostatic simulation of an accelerator magnet: Exercise worksheets, https://github.com/degersem/exercise_accelerator_magnet.git.”
Magnetostatic Simulation of an Accelerator Magnet

make your own finite-element solver

CERN Accelerator School
11-23 November 2018
Thessaloniki, Greece

Prof. Dr.-Ing. Herbert De Gersem
Iryna Kulchytska-Ruchka, M.Sc.
Prof. Dr. rer. nat. Sebastian Schüps

Institute for Accelerator Science and Electromagnetic Fields (TEMF)
TU Darmstadt
Germany

Fig. 1: Title sheet.
what we will do ...

In this exercise, we start from the Maxwell equations and finish at the magnetic field in an accelerator magnet. Except for the construction of the geometry and the definition of the materials, we will implement all steps ourselves.

make sure that you bring with you ...

1. your laptop,
2. a recent version of FEMM (www.femm.info/),
3. Matlab (www.mathworks.com) or Octave (http://www.gnu.org/software/octave/)
4. the slides of the previous lectures.

work together ...

1. in small groups (2 or 3 persons).
2. divide the work,
3. but keep informed about the what and how the other is doing.
4. adapt a careful attitude, double check everything, in numerical simulation, a small error is sufficient to blow up the universe.

the goal of an exercise is to learn, therefore ...

1. ask everything you do not know or you are not sure of.
2. repeat an exercise (maybe with slightly different parameters) until you understand its solution thoroughly.

enjoy ...

Fig. 2: Introductory sheet: summary, software installation, goal, strategy. This sheet should attract the students’ attention and motivate them to start with the exercise.
This is an accelerator magnet. Many of them are installed in synchrotrons in order to keep accelerated charged particles on a more or less circular track.

The force on an ion with mass $m$ and charge $q$, moving at a velocity $\vec{v}$ in a magnetic field $\vec{B}$ is $\vec{F}_B = q\vec{v} \times \vec{B}$. When no further forces are present, this force is a centripetal force keeping the particle with centripetal acceleration $\vec{a}_c$ at a circular track with radius $r$:

$$\sum_j \vec{F}_j = m\vec{a}_c \implies qvB = m\frac{v^2}{r}.$$ 

The radius of the particle’s trajectory is

$$r = \frac{mv}{qB}.$$ 

(a) Building smaller synchrotrons (small $r$) or
(b) achieving higher velocities (large $v$) necessitates designing for higher magnetic fields (large $B$).

[1] (figures and basic information) R.A. Serway, J.W. Jewett, Physics for Scientists and Engineers, 6th ed, Thomson, 2004.
[2] (further reading) M. Wilson, Superconducting Magnets, Clarendon, Oxford, 2002.
[3] (further reading) S. Russenschuck, Field Computation for Accelerator Magnets: Analytical and Numerical Methods for Electromagnetic Design and Optimization, Wiley, Berlin, 2010.
The magnet is known as the Nuclotron magnet (Dubna, Russia)

Currently, the design is called "SIS-100" and has been modified for use in the Facility for Antiproton and Ion Research (FAIR, www.fair.de), which is currently under construction at the Helmholtzzentrum fÃ¼r Schwerionenforschung (GSI, Facility for Heavy Ion Research, www.gsi.de) in Darmstadt, Germany.

The current technological challenges are
1. superconducting coils,
2. cooling (liquid helium),
3. reducing losses (eddy currents, hysteresis, ...) and
4. guaranteeing a homogeneous dipole magnetic field for all operation modes.

**Fig. 4:** Background of the SIS-100 superconducting dipole magnet. This background knowledge and the provided links invites the interested students to have a look to the larger context.
Our Goal

what we want to achieve …

1. validating the magnitude and the homogeneity of the aperture field,
2. checking the saturation of the iron parts and
3. possibly improving the design on these points.

how to achieve this goal …
by 2D magnetostatic field simulations based on

1. the geometry of the magnet’s cross section,
2. the currents through the magnet windings and
3. the given material characteristics.

with which means …

1. a freeware package for 2D FE modelling (femm) and
2. an own implementation of the nonlinear, magnetostatic formulation and the 2D FE discretisation (Matlab or Octave).

Fig. 5: Goals of the exercise and strategy to accomplish these goals. This worksheet provides a task description. The goals range beyond the purpose of this exercise (which are getting familiar with the FE method for magnetic field simulation) but trigger students with a clear interest in accelerator physics.
all dimensions are in millimeter

- enter this geometry in FEMM;
- work bottom-up (first points, then lines/arcs, then surfaces);
- a quick guide of FEMM is on the next page.

Fig. 6: Geometry and dimensions of the SIS-100 superconducting dipole magnet. This worksheet enables the students to build up the model themselves. Nonetheless, a ready model can be offered to bring down the extend of the exercise or to serve as a backup. The steps to be carried out by the students are marked by 1 up to 41. This allows the tutor to monitor the progress of the students or the student groups.
quick guide

- use femm by applying the buttons directly under the menu, one-by-one, from left to right, including [ ].
- adding model items is done by the left mouse button or the tab-key.
- editing model items is done by the right mouse button followed by the space-bar.
- geometry: femm is a bottom-up modeller:
  - first define points,
  - then connect them to line or arc segments,
  - and finally, indicate block labels.
  - the responsibility for avoiding crossing lines and arcs is at you.
- define materials and boundaries (menu-item ”Properties”) and then assign them to the blocks and the edges, respectively.
- save your model and have a look in the .fem file with the editor of your choice, you will recognise your inputs.
- for further information, use the manual (menu-item Help, Help Topics) or carry out one of the tutorials.

Fig. 7: Overview of the most frequently used parts of FEMM. For further information, the students are referred to the user manual or the tutorials.
– define material properties and excitations (each of the windings carries 6045.76 A).

– for the first simulation, use a linear material for the iron parts (relative permeability: 1000).

– define boundary conditions (see below).

– define problem settings (the magnet is 3 m long).

– assign materials to the blocks of the geometry.

– assign boundary conditions to some of the line and arc segments of the geometry.

Fig. 8: Material: nonlinear iron for the magnet’s yoke. It is recommended to carry out a first simulation with a linear material and a second one with the true nonlinear material. One should find that the field quality is better for the true nonlinear case than for the linear case.

---

| $B$ (T) | $H$ (A/m) |
|--------|-----------|
| 0.010  | 9.9997048e+000 |
| 0.020  | 1.9999410e+001 |
| 0.035  | 2.9999114e+001 |
| 0.050  | 3.9998819e+001 |
| 0.070  | 4.9998524e+001 |
| 0.107  | 5.9998229e+001 |
| 0.140  | 6.9997933e+001 |
| 0.200  | 7.9997638e+001 |
| 0.330  | 9.9997048e+001 |
| 1.020  | 1.9999410e+002 |
| 1.250  | 2.9999114e+002 |
| 1.350  | 3.9998819e+002 |
| 1.400  | 4.9998524e+002 |
| 1.430  | 5.9998229e+002 |
| 1.450  | 6.9997933e+002 |
| 1.470  | 7.9997638e+002 |
| 1.480  | 8.9997343e+002 |
| 1.495  | 9.9997048e+002 |
| 1.580  | 1.9999410e+003 |
| 1.640  | 2.9999114e+003 |
| 1.675  | 3.9998819e+003 |
| 1.710  | 4.9998524e+003 |
| 1.730  | 5.9998229e+003 |
| 1.760  | 6.9997933e+003 |
| 1.780  | 7.9997638e+003 |
| 1.795  | 8.9997343e+003 |
| 1.810  | 9.9997048e+003 |
| 1.950  | 1.9999410e+004 |
| 2.030  | 2.9999114e+004 |
| 2.070  | 3.9788735e+004 |
| 2.180  | 7.9577469e+004 |
| 2.250  | 1.1140846e+005 |
The presence of the other parts outside the computational domain is faked by boundary conditions (BCs).

Magnetic flux lines stay parallel to boundaries with electric BCs.

Magnetic flux lines cross boundaries with magnetic BCs perpendicularly.

Magnetic flux lines that leave the model under an angle different from $0$ or $90^\circ$, indicate a Robin’s (or combined) BC.

For a magnetic-vector-potential formulation, electric BCs correspond to Dirichlet BCs (constraint on the magnetic vector potential), whereas magnetic BCs correspond to Neumann BCs (constraint on the magnetic field strength).

are you uncertain ... try the model with different BCs and have a careful look to the magnetic-flux-line plot.

Fig. 9: Boundary conditions: explanation about the different types of boundary conditions, implementation of the boundary conditions. Boundary conditions typically raise many questions. The tutor may invite the student to try different boundary conditions and interpret the resulting magnetic flux distributions.
Solve

let FEMM do it for you ...

– create a mesh (FEMM). Behind the FEMM package (by David Meeker), there is the Triangle package (by Jonathan Shewchuk) for meshing (in this case, triangularisation). Meshing is a tedious task which has to be carried out in exact arithmetic in order to avoid the accumulation of geometric errors due to floating point errors.

– solve the problem (FEMM). At first, we solve a magnetostatic problem (put the frequency to 0). This is the most time-consuming part of a simulation. However, for a small 2D model like this one, the computation time is fully acceptable. For large 3D models, however, this may become prohibitive.

– look to the solution (FEMM). Beside a magnetic-flux-line plot, you can ask for a colour plot or an arrow plot of the magnetic flux density.

– compare the results to what you would expect.

can you ”read” the magnetic-flux-line plot?

– where is the highest magnetic flux density?
– where is the lowest magnetic flux density?
– which fact is giving evidence of the homogeneity of the aperture field?
– what is the function of the negative shimming and the air inclusion of the yoke?
– what would happen to the magnetic flux lines when the current changes sign?
– where do you expect ferromagnetic saturation?
– what happens when you interchange Dirichlet and Neumann boundary conditions?

Fig. 10: Procedure for simulation the magnet FE model. A bit of background information is given about the mesher and the solver. It is important that the students are persuaded to carefully look at the resulting magnetic flux distributions. The tutor should point to the magnetic flux lines refracting at material interfaces, to the magnetic flux lines encircling the wires, to the magnetic flux lines touching the boundary normally or tangentially and to regions with high saturation.
Own Implementation

- formulation
- discretisation
- implementation in Matlab

1. 2D, linear, magnetostatic solver
   - getting information out of FEMM data files
   - check this information (e.g. using figures)
   - constructing the system of equations
   - applying boundary conditions
   - solving the system
   - post-processing
   - write the solution to a FEMM data file

2. 2D, nonlinear, magnetostatic solver
   - reading a nonlinear material characteristic
   - finding elements belonging to the iron parts
   - setting up a nonlinear iteration by successive substitution and/or Newton

- take a look to the FEMM data files (extensions .fem and .ans). Figure out which information is needed for setting up an own finite-element solver.

- some of the auxiliary routines are provided.
- write the main routines yourself!

**Fig. 11:** Road map to come to an own implementation of a FE solver. This worksheet points to the third phase of the exercise, i.e., writing an own FE solver for carrying out the same simulation starts. The students are invited to have a look in the FEMM data file, in which they should find all data related to the FE model, e.g., the mesh. The implementation makes use of auxiliary routines, which are provided.
Fig. 12: Overview of the provided MATLAB routines. Most of the routines are given in full. In the underlined routines, one or several implementation points are inserted, at which the students should add own coding. Complete routines are provided as well, such that the students can check their implementation or skip the implementation step. At this point, most students feel overburdened. For that reason, in an intermediate step starting from the following worksheet, they are invited to rehearse the theory.

- more detailed information can be obtained by typing "help filename" in MATLAB.
- files in which you should add some implementation are underlined. You have to insert some implementation at places indicated by "IMPLEMENTATION POINT".
Formulation

Maxwell’s equations

\[ \nabla \cdot \vec{D} = \rho \]
\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

where we turn the Maxwell equations into the magnetoquasistatic formulation as a function of the magnetic vector potential ...

material laws

\[ \vec{D} = \varepsilon \vec{E} \]
\[ \vec{B} = \mu \vec{H} = \frac{1}{\nu} \vec{H} \]
\[ \vec{J} = \sigma \vec{E} \]

neglect electrical energy w.r.t. magnetic energy and losses

integrate in space and define magnetic vector potential

integrate in space and define electric scalar potential

combine

magnetoquasistatic formulation

Fig. 13: Worksheet for deriving the magnetoquasistatic formulation from the Maxwell equations. The worksheet allows to repeat theory step by step. This is preferably carried out using the suggestions provided on the worksheet but can also be accomplished by copying from the slides shown during the lecture.
Discretisation (1/2)

from the continuous to the discrete level ...

**strong** formulation
\[ \nabla \times \left( \nu \nabla \times \vec{A} \right) + \sigma \frac{\partial \vec{A}}{\partial t} = \vec{J}_s \]

14 apply the weighted residual method, i.e., (a) multiply with test functions \( \vec{w}_i(\vec{r}) \); (b) integrate over the computational domain \( V \).

15 apply the vector-calculus formula
\[ (\nabla \times \vec{v}) \cdot \vec{w} = \nabla \cdot (\vec{v} \times \vec{w}) + \vec{v} \cdot (\nabla \times \vec{w}) . \]

16 apply Gauss theorem \( \int_V \nabla \cdot \vec{v} \, dV = \int_{\partial V} \vec{v} \cdot d\vec{A} \).

17 assume that only (a) Dirichlet boundary conditions and (b) homogeneous Neumann boundary conditions are applied.

---

**Fig. 14:** Worksheet for deriving the weak form of the magnetoquasistatic formulation. Again, the students can follow the provided suggestions or can copy the appropriate parts of the lecture.

---

19
and end up with a system of equations ...

\[ \int_{V} \nu \nabla \times \vec{A} \cdot \nabla \times \vec{w}_i \, dV + \int_{V} \sigma \frac{\partial \vec{A}}{\partial t} \cdot \vec{w}_i \, dV = \int_{V} \vec{J}_s \cdot \vec{w}_i \, dV \]

insert the discretisation for the magnetic vector potential \( \vec{A}(\vec{r}) = \sum_{j} u_j \vec{w}_j(\vec{r}) \).

find expressions for the coefficients.

\[ K_{\nu,ij} = \]

\[ M_{\sigma,ij} = \]

\[ f_{s,i} = \]

write down the resulting algebraic system of equations.

---

Fig. 15: Worksheet for deriving the algebraic system of equations after applying the FE method to the weak form of the magnetoquasistatic formulation. The tutor can indicate that the arising system of equations enables the implementation of the FE method on a computer, thereby making a link to the third part of the exercise.
Reduction to 2D

- when the geometry/excitation/boundary conditions remain the same along the axis of the device,
- when the current is perpendicular to the cross-section,
- then the flux is aligned with the cross-section,

- then \( \vec{A}(x, y, z) = (\ , \ , \ ) \)

- and \( \vec{B}(x, y, z) = (\ , \ , \ ) \)

- apply the edge functions

\[
\vec{w}_j = \frac{N_j(x, y)}{\ell_z} \vec{e}_z
\]

both as test and trial functions,

- then \( K_{\nu,ij} = \)

- and \( M_{\sigma,ij} = \)

- and \( f_{s,i} = \)

---

**Fig. 16:** Worksheet for adapting the formulation to the 2D case. The instantiation of the general 3D FE method to the cartesian 2D case should make the students more familiar with the FE method.
We use linear finite-element "hat" functions for $N_i(x, y)$. In each element $A_e$ incident to node $i$, these functions are linear and attain the value 1 at $(x_i, y_i)$ and 0 at all other nodes:

$$N_{e,i}(x, y) = \frac{a_{e,i} + b_{e,i}x + c_{e,i}y}{2S_e},$$

where

$$a_{e,i} = x_j y_k - x_k y_j$$
$$b_{e,i} = y_j - y_k$$
$$c_{e,i} = x_k - x_j$$
$$S_e = \text{element area}.$$

A helpful rule is

$$\int_{V_e} N_{e,i}^\alpha N_{e,j}^\beta N_{e,k}^\gamma \, dV = 2\ell泽 S_e \frac{\alpha!\beta!\gamma!}{(\alpha + \beta + \gamma + 2)!}.$$ 

The integration of the coefficients is organised element-by-element, i.e., by summing up the local matrices of the individual elements ($V_e$ denotes the prism with triangular cross section $S_e$ and length $\ell泽$).

Remark: Notice the fact that (a) true edge functions with unit 1/m are defined and (b) the coefficients are computed by volume integration. This is rather uncommon but has many advantages, e.g., for circuit coupling [1]. More widespread is, however, using $\vec{w}_j(x, y) = N_j(x, y)\vec{e}_z$ and integrating over the cross sections [2,3,4].

Fig. 17: Worksheet for explaining the computation of coefficients for hat FE functions. The general integration rule for lowest-order 2D hat functions allows to calculate all elementary matrices asked for in the next worksheets.

[1] J. Gyselinck, Twee-dimensionale dynamische eindige-elementenmodellering van statische en roterende elektromagnetische energieomzetters, PhD, Universiteit Gent, Belgium 2000.
[2] A. Kost, Numerische Methoden in der Berechnung elektromagnetischer Felder, Springer, Berlin, 1996.
[3] P.P. Silvester, R.L. Ferrari, Finite Elements for Electrical Engineers, 2nd ed, Cambridge UP, 1996.
[4] J.P. Bastos, N. Sadowski, Electromagnetic Modeling by Finite Element Methods, Marcel Dekker Ltd, 2003.
Compute the local coefficient matrices and vectors:

\[ K^{(e)}_{\nu,ij} = \]

\[ M^{(e)}_{\sigma,ij} = \]

\[ f^{(e)}_{s,i} = \]

– In the case of triangular elements, two 3-by-3 matrices and one 3-by-1 vector are found for each element.
– The assembling process is the procedure bringing such local contributions together in an overall algebraic system of equations.

– Think how assembling can be organised in an efficient way when using Matlab.
– Implement these formulae in `curlcurl_ll.m`, `edgemass_ll.m` and `current_Pstr.m`.

Fig. 18: Worksheet for calculating the coefficients of the algebraic system of equations. Here, a bit of calculus is required to come up with expressions for the matrix coefficients and right-hand-side contributions. Task requires a first implementation action at the implementation points in three routines and thereby marks the start of the third part of the exercise. Here, the students may need some support when coding the first lines.
Magnetic Flux Density

– The magnetic flux density is

\[ B_x = \frac{\partial A_z}{\partial y} \]
\[ B_y = -\frac{\partial A_z}{\partial x} \]

– The magnetic flux density in element \( e \) is expressed in function of the degrees of freedom \( u \) and the shape functions by

\[ B_x = \]
\[ B_y = \]

Remark: Notice the inversion of coordinates and the minus sign in the above formulae. This is typical for the curl-curl case. In case of an electrostatic formulation in terms of the electric scalar potentials, the formulae for \( E_x \) and \( E_y \) would look completely different.

– Derive the coefficients for \( B_x \) and \( B_y \).
– Implement these in \texttt{curl.m}.
– Compute the element-wise magnitude of the magnetic flux density (use the function \texttt{pyth.m}).
– Search for the element with the highest magnetic flux density.
– Compute the magnetic energy relying upon these values for the magnetic flux density.

\textbf{Fig. 19:} Worksheet for calculating the magnetic flux density from the FE solution for the magnetic vector potential.
The magnetic energy equals (only in the linear case)

\[ W_{magn} = \frac{1}{2} u^T K_\nu u, \]

where \( u \) is the vector of degrees of freedom and \( K_\nu \) is the curl-reluctance-curl matrix.

Post-process for the magnetic energy using FEMM

\[ W_{magn} = \]

Compute the magnetic energy based on the FEMM solution and the reluctance matrix computed by your own.

(The FEMM solution is the third column in prb.node)
(The values in prb.node are line-integrated magnetic vector potentials (in Wb), despite of the fact that FEMM uses and stores magnetic vector potentials (in Tm) in the .ans file.)

The magnetic energy also equals (only in the linear case)

\[ W_{magn} = \frac{1}{2} L I^2, \]

where \( I \) is the the applied current and \( L \) is the inductance

Compute the inductance of the magnet.

\[ L = \]

**Fig. 20:** Worksheet for calculating the magnetic energy and the inductance from the FE solution. The magnetic energy \( W_{magn} \) can be calculated using the own implementation for \( K_\nu \) combined with the solution \( u \) already obtained by FEMM. This allows an early check of the own implementation, i.e., before boundary conditions are applied and the own system of equations is solved. The values for \( W_{magn} \) and for the inductance \( L \) can be compared with the results previously obtained in FEMM.
- The only boundary conditions present are homogeneous Dirichlet boundary conditions.
- Unconstrained nodes (subscript b, index set idxdof) are distinguished from constrained nodes (subscript c, index set idxdir).
- The unconstrained system would be
\[
\begin{bmatrix}
K_{bb} & K_{bc} \\
K_{cb} & K_{cc}
\end{bmatrix}
\begin{bmatrix}
u_b \\
u_c
\end{bmatrix} =
\begin{bmatrix}
f_b \\
f_c
\end{bmatrix}.
\]
- Adding constraints leads to (I_{cq}y_b denotes the boundary-integral term)
\[
\begin{bmatrix}
K_{bb} & K_{bc} & 0 \\
K_{cb} & K_{cc} & I_{cq} \\
0 & I_{qc} & 0
\end{bmatrix}
\begin{bmatrix}
u_b \\
u_c \\
y_q
\end{bmatrix} =
\begin{bmatrix}
f_b \\
f_c \\
u_c
\end{bmatrix}.
\]
- The values for \(u_c\) are known. Shifting \(u_c\) to the right-hand side and eliminating the Lagrange multipliers \(y_q\) leads to the constrained system
\[
K_{bb}u_b = f_b - K_{bc}u_c.
\]
- ”Shrink” the unconstrained system up to the constrained system.
- Solve the system of equations.
- ”Inflate” the solution vector to a full solution vector including the constrained nodes.
- Compute the magnetic energy and compare to previously obtained values.
- Write the solution to a FEMM .ans file.
- (be aware of the fact that we work with line-integrated magnetic vector potentials (Wb), whereas FEMM uses magnetic vector potentials (Tm), the save_femm routine does the necessary conversion, have a look inside.)
- Plot your own solution using FEMM.

Fig. 21: Worksheet for explanation the introduction of boundary conditions in the algebraic system of equations; procedure for solving the constrained system of equations. The introduction of boundary conditions is a comparably difficult part of a FE solver. For that purpose, two routines, i.e. shrink and inflate, hiding all technicalities, are provided. Nevertheless, the worksheet explains the impact of Dirichlet boundary conditions on the algebraic system of equations and explicitly points the students to the lines in the code where the boundary conditions are inserted.
The magnetic vector potential inside a circle with reference radius $r_{\text{ref}}$ in the aperture can be expressed by

$$A_z(r, \varphi) = \sum_{p \in P} (a_p \cos(p\varphi) + b_p \sin(p\varphi)) \left( \frac{r}{r_{\text{ref}}} \right)^p.$$  

Then, the magnetic flux density is

$$B_x(r, \varphi) = \sum_{p \in P_0} \frac{p}{r} (-a_p \sin((p-1)\varphi) + b_p \cos((p-1)\varphi)) \left( \frac{r}{r_{\text{ref}}} \right)^p;$$

$$B_y(r, \varphi) = \sum_{p \in P_0} \frac{p}{r} (-a_p \cos((p-1)\varphi) - b_p \sin((p-1)\varphi)) \left( \frac{r}{r_{\text{ref}}} \right)^p.$$  

The Fourier coefficients of the magnetic flux density evaluated at the reference circle are

$$\mathcal{F}(B_x) = \left( \frac{p}{r_{\text{ref}}} b_p, -\frac{p}{r_{\text{ref}}} a_p \right);$$

$$\mathcal{F}(B_y) = \left( -\frac{p}{r_{\text{ref}}} a_p, -\frac{p}{r_{\text{ref}}} b_p \right).$$  

These coefficients are called harmonic components (here under the assumption of a vertical main dipole field).

**Fig. 22:** Worksheet for explaining the concept of harmonic components and skew-harmonic components of the aperture field of an accelerator magnet.
– Find the nodes in the mesh which lie on the reference circle (use a geometric tolerance!) and order them along the circle.
– Extract the magnetic vector potential at the quarter of the reference circle which is inside the computational domain (notice: our solution consists of line-integrated magnetic vector potentials!).
– Combine this signal with itself until you have a periodic signal.
– Make a Fourier transformation to obtain the coefficients for $A_z$.
– Compute the harmonic and skew harmonic components.
– Make bar plots of both, discard the main dipole component and plot again.
– Fill the most important components in the table.

|  | harmonic component | skew harmonic component |
|---|--------------------|------------------------|
| dipole |                     |                        |
| quadrupole |                |                        |
| sextupole |                   |                        |

The quality of the aperture field of an accelerator magnet is determined by the ratio of the magnitude of the higher harmonic components with respect to the main component (in this case the vertical dipole component). In typical magnets, this ratio is below $10^{-4}$. What about this magnet?

Fig. 23: Procedure for calculating the (skew) harmonic components of the SIS-100 magnet and worksheet for tabulating the results. This part requires a considerable amount of implementation but has a high relevance for the example of an accelerator magnet. At this point, a student group can decide to split up where a first subgroup addresses the aperture field, whereas a second subgroup proceeds with nonlinear materials.
Nonlinear Material Properties

\[ \nu \quad \text{chord reluctivity} \]
\[ \nu_d \quad \text{differential reluctivity} \]
relation: \[ \nu_d = \nu + 2B^2 \frac{d\nu}{dB^2} \]
\[ H_m \quad \text{magnetisation field strength} \]
\[ B_m \quad \text{magnetisation flux density} \]

– The data of a nonlinear material characteristic are kept in a data structure \((\text{nlin in driver.m})\).
– The data can be evaluated given a set of magnetic flux densities.
– Check the results for the magnetic field strength, chord reluctivity, differential reluctivity, derivative of the reluctivity and coercitive field strength.

Remark: The coercitive field strength and the remanence here have another meaning than for the case of a permanent magnet. Here, \(H_m\) and \(B_m\) are defined by the crossing points of the line tangent to the BH-characteristic in the actual working point. Hence, \(H_m\) and \(B_m\) will change when the working point changes. The points \((H_m, 0)\) and \((0, B_m)\) are no valid working points of the nonlinear material.

linearisations in the working point \(P\):
1. successive substitution \(: H = \nu B \)
2. Newton \(: H = H_m + \nu_d B \)

Fig. 24: Definition of quantities involved with the modelling of nonlinear materials. The worksheet introduces the notions of chord and differential reluctivity and puts them into relation to each other. The concept of an operating point at the BH-characteristic is explained. It should become clear to the students that each element of the iron-yoke part of the FE model may be operated at a different nonlinear operating point.
Successive Substitution

Algorithm: \( \nabla \times \left( \nu(\vec{A}_n) \nabla \times \vec{A}^*_n \right) = \vec{J}_s. \)

The convergence of a successive-substitution approach is poor. For that reason, commonly, relaxation is applied. We will use relaxation with a fixed relaxation factor.

Relaxation: \( \vec{A}_{n+1} = \alpha \vec{A}^*_n + (1 - \alpha) \vec{A}_n, \)

with relaxation factor \( \alpha. \)

The convergence of the nonlinear iteration is checked on the basis of a convergence criterion. Monitoring the convergence of the magnetic energy is the most appropriate. Here, we apply a simpler criterion, based on the relative difference between two successively obtained solutions.

Convergence criterion: \( \varepsilon_{nlin} = \frac{\|\vec{A}_{n+1}\| - \|\vec{A}_n\|}{\|\vec{A}_n\|}. \)

- Implement the successive-substitution approach with relaxation and a convergence check.
- Solve the nonlinear problem and compare to results obtained with FEMM.
- What are the main differences between the linear and the nonlinear solution.

---

**Fig. 25:** Worksheet for explaining and implementing the successive-substitution method, combined with a simple relaxation technique. The students are invited to implement the successive-substitution method in the main file of the software.
algorithm 1: \( \nabla \times \left( \nu(d \nabla \times \delta A_{n+1}) \right) = \vec{J}_s - \nabla \times \left( \nu(A_n) \nabla \times \vec{A}_n \right). \)

The above form is commonly used. However, it may be more convenient to think about the Newton method as being similar to the successive-substitution approach, only differing concerning the linearisation of the working point. By introducing
\[
\vec{H} = \vec{H}_m(A_n) + \nu(d \nabla \times A_n) \vec{B}
\]
in the magnetostatic equation, we arrive at
algorithm 2: \( \nabla \times \left( \nu(d \nabla \times A_{n+1}) \right) = \vec{J}_s - \nabla \times \vec{H}_m(A_n). \)

Here, no increments are needed and only a matrix assembly is necessary for the left side of the equation. Notice that the differential reluctivity is a tensor (2-by-2 in the 2D case, 3-by-3 in the 3D case)!

The main challenge is the computation of the differential reluctivity tensor. In an element \( e \) where the previous solution for the magnetic flux density is given by
\[
B^{(e)} = \begin{bmatrix} B_x^{(e)} \\ B_y^{(e)} \end{bmatrix},
\]
the differential reluctivity tensor is
\[
\vec{\nu}_d^{(e)} = \nu^{(e)} \vec{1} + 2B^{(e)} \nu_d^{(e)} B^{(e)T},
\]
where \( \nu^{(e)} \) and \( \nu_d^{(e)} \) are the chord and differential reluctivities obtained by evaluating the material characteristic with input \( B^{(e)} \).

**Fig. 26:** Worksheet for explaining the Newton method. The Newton method is formulated as an alternative linearization method such that algorithm 2 is of the same form as in the linear case.
– Implement the nonlinear system matrix and additional right-hand-side term in curlcurl_ll_nonlinear.m.
– Set up a Newton iteration and check for convergence.
– Write the solution to a FEMM .ans file and compare with results obtained by FEMM.

Remark: The convergence of a Newton approach should be good enough to be convergent without relaxation.

– Compare the convergence of the successive-substitution approach with the convergence of the Newton approach. Try to find an optimal relaxation factor for the successive-substitution approach.

– Compute the harmonic components and the skew harmonic components based on the nonlinear solution.
– Compare to the values obtained for the linear solution.

| harmonic component | skew harmonic component |
|--------------------|-------------------------|
| dipole             |                         |
| quadrupole         |                         |
| sextupole          |                         |

Fig. 27: Worksheet for implementing the Newton method and calculating the (skew) harmonic components in the magnet’s aperture accounting for ferromagnetic saturation of the iron yoke. The implementation of the Newton method is quite involved. Nevertheless, the result obtained by the students themselves is a nonlinear magnetostatic FE solver which is from the algorithmic side close to optimal. The results for the (skew) harmonic components should be compared to the results obtained for the (virtual) linear case. One should notice that the higher-order components in the nonlinear case are considerably lower than the ones for the linear case, which indicates that the magnet has been optimized for providing a possibly high homogeneity at maximal field.