In the present article, we will explore the propagation of a scalar degree of freedom on a one-dimensional lattice. The lattice is constant in time, but the geometric information about the lattice – encoded in a metric-like function – is allowed to vary spatially. In fact, we are most interested in the case where the geometry fluctuates in a stochastic way along the lattice. We call this a random lattice. The dynamics we consider for the scalar field is that of a massless Klein-Gordon field, discretized on a lattice with variable lattice spacing. Thus, in the limit of the lattice spacings going to zero, one expects to obtain the well-known continuum model. However, we are interested in the model with finite lattice spacings, but for excitations of the field that are on very large scales compared to the average lattice spacing. In this regime, the system should show propagating waves, governed by a dispersion relation. The question we pose to ourselves is how this dispersion relation depends on the microscopic parameters of the theory. This is very much in analogy to the question of how to determine the speed of sound, and other macroscopic properties of a (possibly amorphous) solid, from the microscopic properties of its constituents. We are, however, not motivated by solid state physics, but by quantum gravity.

The incorporation of the quantum nature of gravity is likely to change the structure of space-time on small length-scales. Different such structures have been investigated, some motivated by full-fledged attempts to quantize gravity, some as toy models or as effective descriptions. One paradigm is that spacetime is fundamentally discrete, such as in the causal set approach, or in semiclassical considerations of loop quantum gravity. Another paradigm, pioneered by Wheeler, has spacetime smooth, save for localized defects. Our model here is an instance of the former, but could maybe also be read as a model of space-time defects in the limit where their density becomes very high. The model is very similar to the ones considered in loop quantum gravity and references therein (and some intermediate results of the present work were already contained in). There, dispersion relations were also calculated, but the calculations involved simplifications, effectively replacing a random lattice by a regular one, the parameters of which were obtained by averaging. In the present article we go beyond this, by obtaining exact results. We should point out that because we are working with a free massless field in two dimensions, the corresponding continuum model would be symmetric under the exchange of space and time coordinates. Therefore, our model is mathematically equivalent to one in which spatial geometry is smooth and homogeneous, but time is a discrete variable. For simplicity, we will work in the “discrete space, continuous time” picture, but we will also give the results for the opposite case in an appendix.

Since we are working with random lattices, our results are stochastic in nature. So while we said that we would like to compute the coefficients in the dispersion relation, what we actually compute are expectation values (and in one case the variance) of such coefficients.

Fields on random lattices have become a valuable tool in lattice gauge theory, starting from the pioneering works. We are, however, not aware of explicit results on the dispersion relations of these fields at finite lattice spacing. This may have to do with the fact that in this context, the random lattices are a tool to obtain statements on the continuum limit of the theories.

Effects of coupling to random fields on the dispersion relation have already been studied in detail in and . In these works there is no discreteness of space-time. Rather, the random fields are an effective description of the effects of the CPT anomaly, caused by a nontrivial space-time topology on small length scales. While these models are vastly more physical, and can hence be used to obtain bounds on small-scale space-time structure, one can nevertheless compare them to our model on a mathematical level. The results bear some intriguing similarities but also some differences. These are discussed in a bit more detail at the end of this article.

It should be said that our model is rather unphysical, in that it is 1+1 dimensional, and in that the field content is not realistic at all. What we hope to have accomplished, is to show how a calculation of phenomenologically interesting data from a model involving discrete, randomly fluctuating space-time can be accomplished. Our calculations here...
can almost certainly be generalized to more realistic models (using, for example, the Voronoi-construction \cite{11} to obtain a random lattice in arbitrary dimension). They would merely be more cumbersome.

The structure of the article is as follows: In the next section we will specify the details of the model. Section \( \text{III} \) explains the calculation of the dispersion relation. We end with a discussion of further prospects in section \( \text{IV} \). Two appendices contain some of the longer calculations, and the last appendix contains the dispersion relation for the case of continuous space and discrete time.

II. THE MODEL

We consider a bosonic field propagating on discrete space and continuous time. (The case of discrete time and continuous space is briefly discussed in appendix C.) To keep things simple, we work in 1+1 dimensions. The field is a function on the space \( \mathbb{R} \times \mathbb{Z} \). It will be denoted by \( \phi_n(t) \). The geometry of space is encoded in a time-independent, positive function \( g_n \). Often it is also useful to re-express \( g_n \) as \( g_n = l_n^{-2} \). At this point we will make no further assumptions on \( g_n \), but we will later assume that it is random in a certain specific sense.

The action for our model reads

\[
S = \int dt \sum_{n \in \mathbb{Z}} \frac{1}{2} \left[ \dot{\phi}_n^2 - g_n (\partial^+ \phi)_n^2 \right].
\]  

(1)

Here \( \partial^+ \phi \) is the forward discrete derivative, \( (\partial^+ \phi)_n = \phi_{n+1} - \phi_n \). It was used for simplicity. A more symmetric derivative could be used in its place. We will also use its adjoint \( \partial^- \) with respect to the sum over \( \mathbb{Z} \). \( S \) can be viewed as a discretization of the action for a free massless scalar in the continuum, but ultimately an action of this form may be derived from a theory of quantum gravity, and thus regarded as fundamental.

The equations of motion are

\[
\ddot{\phi}_n + \Delta_g \phi_n \equiv \ddot{\phi}_n - g_n \phi_{n+1} - g_n \phi_{n-1} + (g_{n-1} + g_{n+1}) \phi_n = 0
\]  

(2)

where we have introduced the discrete, positive definite Laplacian

\[
\Delta_g = \partial^- g_n \partial^+.
\]  

(3)

For completeness, we also state the Hamiltonian

\[
H = \frac{1}{2} \sum_n \pi_n^2 + g_n (\partial^+ \phi)_n^2.
\]  

(4)

It is of the form that is obtained under certain assumptions in loop quantum gravity \cite{3, 4}.

The form of the function \( g_n \) is obviously vital for the definition of the model. For our consideration, three cases are of interest:

1. \( g_n \) is constant
2. \( g_n \) is periodic, with a certain period \( N \) in \( \mathbb{Z} \).
3. \( g_n \) is obtained as an instance of a random process.

Case 3 is the one we want to consider. The simplest situation would be that the value of \( g_n \), would, for any given \( n \), be determined by the sampling of a certain random variable \( g \). Another way to state this is that there are independent, equally distributed random variables \( g_n \) and the function \( g_n \) is a sampling of these. More complicated situations (for example correlations between the \( g_n \)) are also conceivable.

As we will see, it will be important to have information about the distribution of values \( \{g_n\} \) in the sampling of \( \{g_n\} \). Of particular importance for us will be the mean of the \( l_n \) as well as some related quantities. Such information may in principle be obtained through the law of large numbers, or a central limit theorem.

To be concrete let us fully specify one simple model: To make the description simple, we will not specify the distribution of the variables \( g_n \), but those of the \( l_n \). We denote expectation values by angular brackets \( \langle \cdot \rangle \). Let all the random variables \( l_n \) be independent Gaussian distributions with first and second moments

\[
\langle l_n \rangle = l, \quad \langle (l_n - l)^2 \rangle = d^2.
\]  

(5)

Returning to the list of cases from above, the first case is that of an equidistant lattice. It is immediately solvable. The second case is still solvable and we will use it in order to analyze the case we are really interested in. We will see that we can obtain the lowest order terms of the dispersion relation for case 3 in a limit of case 2. This will be explained in the following section.
The problem of wave propagation on a random lattice is certainly a complicated one. In fact, we will see indications that it is not even always well defined, i.e. that there is not always a long wavelength limit in which something resembling plane waves propagates on the lattice. The question is how to identify the cases in which the problem is well defined, and how to extract characteristic long wavelength quantities. We will not address these problems in all generality. But we will obtain a formula that, in a certain limit, gives the first few terms in the dispersion relation for the field. This limit is by no means well defined for all random lattices. If it is ill-defined, this is a strong hint that there is no regime in which the lattice supports propagating waves.

To start our quantitative discussion, we consider the case of a regular lattice. If all the $l_i$ are equal (to $l$, say), it is easy to solve the equations of motion of (2). The solutions are “plane waves”

$$
\phi_n(t, k) = e^{i(ktn - \omega(k)t)} , \quad \omega^2(k) = \frac{2}{l^2} (1 - \cos(kl)) = k^2 - \frac{l^2}{12} k^4 + O(k^6).
$$

(6)

For the general case on the other hand, with generically all $l_i$ different, it is not possible to explicitly write down any solution to the equations of motion. The analysis we are aiming at in the present section lies somewhere in-between these two extreme cases. We will make an assumption on the $l_i$ under which we are able to treat the system analytically and try to remove it at the end of the analysis: Let us assume that the system is periodic with $N \in \mathbb{N}$ the length of period. More precisely we assume that $g_{n+N} = g_n$ for all $n \in \mathbb{Z}$. We introduce the notation $\phi_n^{(z)} = \phi_{n+zn}$ with $n \in \{0, 1, \ldots, N-1 \}$ and make the Ansatz

$$
\phi_n^{(z)}(t) = c_n \exp i(Lz k - \omega t), \quad L = \sum_{n=0}^{N-1} l_n.
$$

(7)

This Ansatz turns the equations of motion (2) into an eigenvalue problem for $c$ and $\omega$: (7) is a solution iff

$$
M c = \omega^2 c \quad \text{where} \quad M = \begin{pmatrix}
& g_{N-1} + g_0 & -g_0 & 0 & \cdots & 0 & -g_{N-1} e^{ikL} \\
& -g_0 & g_0 + g_1 & -g_1 & 0 & \cdots & 0 \\
& 0 & -g_1 & g_1 + g_2 & -g_2 & 0 & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
& -g_{N-1} e^{-ikL} & 0 & \cdots & 0 & -g_{N-2} & g_{N-2} + g_{N-1}
\end{pmatrix}.
$$

(8)

The eigenvalues $\omega_0, \ldots, \omega_{N-1}$ represent the different branches of the dispersion relation. We will presently see that there is one branch, denoted $\omega_{ac}$ in the following, with $\omega_{ac}(k) \to 0$ for $k \to 0$. Following the custom of condensed matter physics, we call this branch acoustic in contrast to the optical branches nonzero at $k = 0$. The situation is sketched in figure 1. As we are interested in the low energy (i.e. small $\omega$) behavior of the field, the acoustic branch is the relevant one for our purpose and we will compute its small $k$ behavior in the following. Let us start by making the Ansatz

$$
\omega_{ac}^2(k) = w(1) |k| + w(2) k^2 + w(3) k^3 + \ldots
$$

(9)

explicitly forcing $\omega_{ac}(0)$ to be zero. Accordingly, we expand $\det(M - \omega^2 I)$:

$$
\det(M - \omega^2 I) = \sum_{i=0}^{N-1} \omega^{2i} \sum_{j=0}^{\infty} w(i, j) |k|^j.
$$

(10)
We want to determine the coefficients \( w(i) \) of (9). The coefficients \( w(i, j) \) of the expansion (10) on the other hand can be explicitly calculated. The calculation is cumbersome, so we have relegated it to appendix A. We find

**Proposition III.1.** For \( M \) of the form (8),

\[
\det(M - \omega^2 I) = -2g_0 \ldots g_{N-1}(1 - \cos Lk) + \omega^2 Ng_0 \ldots g_{N-1} \sum_{i=0}^{N-1} g_i^{-1} + \omega^4 g_0 \ldots g_{N-1} \sum_{0 \leq i < j \leq N-1} (j - i)[N - (j - i)]g_i^{-1}g_j^{-1} + O(\omega^6).
\]

From this, the coefficients \( w(i, j) \) can be read off. By solving the eigenvalue equation \( \det(M - \omega^2 I) \) order by order, we then obtain the low order coefficients of the dispersion relation (9). We use the shorthands \( c_{ij} \equiv (j - i)[N - (j - i)] \)

and

\[
\mathcal{T} \doteq 1 \sum_{i=0}^{N-1} f_n
\]

for the average of some quantity over the period of the lattice. Then we have

\[
\omega_{nc}^2(k) = \frac{L^2}{N^2} \frac{1}{g_1^{-1}} k^2 + \left( \frac{L^4}{N^6} \sum_{i < j} c_{ij} g_i^{-1} g_j^{-1} - \frac{1}{12N^2} \frac{L^4}{g_1^{-1}} \right) k^4 + O(k^6)
\]

\[
= \frac{\ell^2}{l^2} |k|^2 + \left( \frac{1}{12L^2} \sum_{i < j} c_{ij} \ell_i^2 \ell_j^2 - \frac{L^2 T^2}{12 l^2} \right) |k|^4 + O(|k|^6).
\]

This is a remarkable formula, and one of the main results of the present work. It is an exact result for the lowest orders of the dispersion relation of the field \( \phi \) propagating on a periodic lattice. As such, it reproduces the elementary result (10) for the case of the regular lattice, by setting \( l_n = l \) for all \( n \) and using

\[
\sum_{0 \leq i < j \leq N-1} c_{ij} = \frac{1}{12} N^2(N^2 - 1).
\]

We note that the square of the velocity of propagation,

\[
c^2 := \frac{\ell^2}{l^2}
\]

is neatly expressed in terms of averages over lattice spacings. The formula for the next order coefficient,

\[
\ell^2 := \frac{1}{L^2} \frac{\ell^6}{(l^2)^3} \sum_{i < j} c_{ij} \ell_i^2 \ell_j^2 - \frac{L^2 T^2}{12 l^2}
\]

is certainly more complicated, but also given in terms of such averages. This is more than just aesthetically pleasing: It suggests that (13) may survive the large-\( N \) limit. Moreover, that when interpreting the \( l_n \) as given by a sampling of some random process, such averages may be expressible by averages under this process, through some ergodic-type results.

Let us come back to our original motivation: We were interested in the case of a periodic lattice, because, for the \( l_n \) determined by some random process, the limit \( N \to \infty \), with the average length \( \langle l_k \rangle = l \) held fixed, gives an infinite random lattice. Thus we would like to consider the dispersion relation (13) in that limit. We see, however, no possibility to discuss this limit without fixing a distribution for the \( l_n \). Central limit theorems would make statements about the expectation value of the mean \( \overline{T} \) and functions of it in the limit \( N \to \infty \) for large classes of distributions. They can similarly be used to obtain statements about the mean of the squares, \( \overline{T^2} \), in this limit. But in our case, we need to investigate a function of both, \( \overline{T} \) and \( \overline{T^2} \) in the case of the first term in the expansion (13). The second term in (13) additionally depends on \( N \) and on a curious linear combination of \( \ell_i^2 \ell_j^2 \), thus further complicating the situation.
FIG. 2: Theory and numerical simulation for the coefficient $c^2$: For $l = 1$, we plot the theoretical result (18) as a function of $d$ (continuous curve) together with two numerical simulations ($N = 100$: small dots, $N = 1000$: large dots). Convergence is nicely visible.

Let us therefore consider the situation described in (5) in which the $l_i$ are independently Gaussian distributed, with average $\langle l_i \rangle = l$ and variance $\Gamma^2 - \Gamma^2 = d^2$. What we want to do is to compute expectation values of the coefficients (13) under this distribution, in the limit $N \to \infty$. Let us first consider the coefficient $c_2$ of the $|k|^2$-term in (13). In appendix B we show that while we cannot compute its expectation value directly, we can expand it in powers of $d/l$, compute the expectation values of the first few terms, and take $N$ to infinity. The result

$$\lim_{N \to \infty} \langle \Gamma^2 \rangle = 1 - \frac{d^2}{l^2} + \frac{d^4}{l^4} - \frac{d^6}{l^6} + \ldots$$

(17)

strongly suggests, that

$$\lim_{N \to \infty} \langle \Gamma^2 \rangle = 1 - \frac{d^2}{l^2} = \frac{\langle \Gamma^2 \rangle}{\langle \Gamma^2 \rangle}.$$  

(18)

For independent confirmation we have also checked this result on a computer, by calculating instances of $c_2^2$ for randomly generated sets $\{l_i\}$. Formula (18) describes the numerical results very well. An example is given in figure 2. Furthermore, we are able to calculate the variance of $c_2^2$, again as a series in powers of $d/l$, and take the $N \to \infty$ limit term-wise. It turns out that the variance vanishes in this limit:

$$\lim_{N \to \infty} \langle c^4 \rangle - \lim_{N \to \infty} \langle c^2 \rangle^2 = 0.$$  

(19)

Details are again given in appendix B.

The situation is more complicated with the term proportional to $|k|^4$. Expansion in $d/l$ (see appendix) gives

$$\langle \ell^2 \rangle = -\frac{l^2}{12} + \frac{l^2}{12N}(-4N^2 + N - 6) \left( \frac{d}{l} \right)^2 + O \left( \left( \frac{d}{l} \right)^3 \right)$$

(20)

which does not converge term by term in the limit $N \to \infty$. But this does not say that the limit does not exist. In
FIG. 3: Theory and numerical simulation for the coefficient $\ell^2$: For $l = 1$, we plot the theoretical result (21) as a function of $d$ (continuous curve) together with two numerical simulations ($N = 100$: small dots, $N = 1000$: large dots). Correspondence between theory and numerics is visible for very small values of $d$ only, and convergence apparently gets worse for larger $N$.

In fact, if one would extrapolate from the result for the coefficient $c^2$,

$$\lim_{N \to \infty} \langle \ell^2 \rangle = \lim_{N \to \infty} \left( \frac{1}{L^2} \frac{1}{(P^2)^3} \sum_{i<j} c_{ij} \langle l^2 i^2 j \rangle - \lim_{N \to \infty} \frac{L^2}{12} \frac{1}{P^2} \right)$$

$$= \frac{1}{N^2} \frac{1}{P^3} \sum_{i<j} c_{ij} \langle l^2 i^2 j \rangle - \frac{N^2}{12} \frac{1}{P^2}$$

$$= \frac{1}{12} l^4 + d^2 (N^2 - 1) - \frac{1}{12} l^4 + d^2 N^2$$

$$= - \frac{1}{12} l^2 \frac{1}{1 + \frac{d^2}{l^2}}$$

one expects the result to be finite. Numerical results for small $d/l$ indeed point towards convergence (figure 3), but for larger $d/l$, convergence in the limit $N \to \infty$ could not be established. It may be there but too slow to be seen. For completeness, we have also considered the case of discrete time and continuous space in appendix C. The resulting dispersion relation is

$$\omega^2(k) = \frac{1}{c^2} k^2 - \ell^2 |k|^4 + \ldots$$

(22)

However, we have not considered the expectation values of the coefficients in any detail.

**IV. DISCUSSION AND OUTLOOK**

In the present paper we have calculated the (expectation values of the) first coefficients in the dispersion relation

$$\omega^2(k) = c^2 |k|^2 + \ell^2 |k|^4 + \ldots$$

(23)

for a scalar field propagating on what one could call a Gaussian random lattice in 1+1 dimensions. The first term, also setting the phase velocity, is unitless in our convention, thus the natural value is 1. We have seen

$$\lim_{N \to \infty} \langle c^2 \rangle = \frac{1}{1 + \frac{d^2}{l^2}}$$

(24)

i.e. it is in fact very close to 1, as long as the variance of the lattice spacing is small. We have also presented very strong evidence that its variance is strictly zero for an infinitely extended lattice, thus one does not even have to
take into account that in principle it has a probabilistic nature. In more realistic models, this coefficient would be very interesting for phenomenology: As was demonstrated in [3], experiments and observations are very sensitive to differences of propagation speed of the fields in nature. The second term had a more complicated structure, and convergence in the desired limit could not be demonstrated to complete satisfaction. We have conjectured the limit

$$\lim_{N \to \infty} \langle \ell^2 \rangle = -\frac{1}{12} \frac{l^2}{1 + \frac{d^2}{l^2}}.$$  \hspace{1cm} (25)$$

Its scale is set by the average lattice spacing squared, $l^2$, as long as the variance of the lattice spacing is small. We have also briefly discussed a model in which the roles of space and time are switched, i.e., time is discrete and space is continuous. There we see potential problems with causality, with phase and group velocity increasing due to the lattice effects.

An interesting observation was made by a referee of a draft version of this article: The first two terms of the dispersion relation for the field on a regular lattice (6) can be turned into the first two terms of the dispersion relation on the random lattice, by rescaling $k \to kl$ and dividing by $\langle \ell^2 \rangle$ in (6). It is very well possible, that this procedure gives the correct dispersion relation on the random lattice to all orders. This does not detract from the value of the results presented in the article: While there may be a shortcut, it is still necessary to show that the shortcut indeed provides the right results. In fact, one big motivation for the article was that several different “shortcuts” were used in the literature on quantum gravity phenomenology, with different results. We should also point out that, despite the fact that the dispersion relation of the random lattice seems to be obtainable from that of a regular lattice by the above scaling operations, it is not that of such a regular lattice.

Similar, but much more detailed, calculations were carried out in [9, 10] for random fields coupled to scalar and electromagnetic fields in physical space-time dimensions. Since these were continuum models, one expects differences in the results: By reformulation our model in terms of a continuum field, one can show that in the equations of motion, there are correction terms of arbitrary derivative order, compared to the equations of motion for the free scalar field. In contrast, the fields of [9, 10] receive only corrections in terms of first derivatives of the field. Still the results on the dispersion relation are similar: The phase velocity receives a correction downward, and the coefficient of $k^4$ is negative, thus ensuring causal propagation. Moreover, the mathematical structure is similar: In both cases, the second moment of the random field enters the phase velocity, and the coefficient of $k^4$ is given by moments the autocorrelation of the random field, the first moment in case of [9, 10], the second moment (of $l_n^2$) in the present model.

Besides finishing the discussion begun in the present work, by showing convergence or divergence of the second coefficient $\ell^2$, and studying the complete model for more generic distributions for the random lattice, it would be interesting to study the nature of the eigenvectors $\ell$ at least in low order in $k$. For example: Does the one for the acoustic branch really look like a plane wave, at least for large $N$? Other further goals are the extension of the formalism to vector and fermionic fields and to physical dimension. At that point, one would be in a position to compare the models with experiment and put bounds on parameters of, for example, loop quantum gravity.

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Appendix A: Proof of proposition III.1

It is an elementary combinatorial fact that

$$\det(M - \omega^2 I) = \sum_{i=0}^{N-1} (-1)^i \omega^{2i} \left( \text{sum over all } (N - i) \times (N - i) \times \text{principal sub determinants of } M \right).$$ \hspace{1cm} (A1)
Therefore the proof of the theorem reduces to the calculation of numerous sub-determinants of $M$. These calculations are tedious but elementary. As a preparation, we observe that for $n$ in $\{0, 1, \ldots, N - 1\}$

$$
\begin{vmatrix}
  g_0 + g_1 & -g_1 & 0 \\
  -g_1 & g_1 + g_2 & -g_2 \\
  \cdot & \cdot & \cdot \\
  0 & -g_n & g_n \\
\end{vmatrix} = g_0 \cdots g_n
$$

(A2)

by repeatedly adding all other columns to the first one and pulling out factors. Repeated use of the linearity of the determinant in the last column of (A2) yields

$$
\begin{vmatrix}
  g_0 + g_1 & -g_1 & 0 \\
  -g_1 & g_1 + g_2 & -g_2 \\
  \cdot & \cdot & \cdot \\
  0 & -g_{n-1} & g_n \\
\end{vmatrix} = g_0 \cdots g_n \sum_{i=0}^{n} g_i^{-1},
$$

(A3)

another identity which will be used frequently. Finally we introduce the abbreviation $q = \exp(-ik)$.

We turn now to the calculation of the lowest order coefficients in (A1).

**Calculation of $\text{det } M$:**

$$
\text{det } M =
\begin{vmatrix}
  g_{N-1}(2 - q - q^{-1}) & g_{N-1}(1 - q^{-1}) \\
  g_0 + g_1 & -g_1 \\
  -g_1 & g_1 + g_2 & -g_2 \\
  \cdot & \cdot & \cdot \\
  g_{N-1}(1 - q) & -g_{N-2} & g_{N-2} + g_{N-1} \\
\end{vmatrix}
$$

by adding all columns to the first column and subsequently all rows to the first one,

$$
\begin{vmatrix}
  g_0 + g_1 & -g_1 & 0 \\
  -g_1 & g_1 + g_2 & -g_2 \\
  \cdot & \cdot & \cdot \\
  0 & -g_{N-2} & g_{N-2} \\
\end{vmatrix} = g_{N-1}(2 - q - q^{-1})
$$

by pulling out a factor and eliminating the entry in the upper right and lower left corners,

$$
\begin{vmatrix}
  g_1 + g_2 & -g_2 \\
  -g_2 & g_2 + g_3 & -g_3 \\
  \cdot & \cdot & \cdot \\
  0 & -g_{N-2} & g_{N-2} \\
\end{vmatrix} = g_{N-1}(2 - q - q^{-1})g_0
$$

by adding all columns to the first column and subsequently all rows to the first one and pulling out a factor,

$$
= g_0 \cdots g_{N-1}(2 - q - q^{-1})
$$

by applying (A2).
Calculation of the \((N - 1) \times (N - 1)\) sub-determinants:

Let \(0 < n < N - 1\). We consider computing the sub-determinant of \(M\) where row and column \(n\) are deleted.

\[
\det M'_{n,n} = \begin{vmatrix}
    g_{N-1} + g_0 & -g_0 & -g_{N-1}q^{-1} \\
    \vdots & \ddots & \ddots \\
    g_{N-1} + g_n & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & g_{n+1} + g_{n+2} \\
    \vdots & \ddots & \ddots \\
    -g_{N-1}q & -g_{N-2} & g_{N-2} + g_{N-1} \\
    g_{N-1} + g_0 & -g_0 & -g_{N-1} \\
    -g_0 & \ddots & \ddots \\
    -q^{-1} g_{N-1} & 1 & -q g_{N-1} \\
    0 & \ddots & 0 \\
    0 & 0 & -g_{N-2} \\
    g_0 + g_1 & -g_1 & 0 \\
    -g_1 & \ddots & \ddots \\
    -g_{N-1}' & 1 & 0 \\
    0 & \ddots & -g_{N-3} \\
    0 & 0 & -g_{N-3} \\
\end{vmatrix}
\]

by expanding in the first and last column of the matrix. It is not hard to see that the determinants in the terms proportional to \(q\) and \(q^{-1}\) vanish: The corresponding matrices can be brought to a form where they contain a zero column by simple column operations. The remaining determinants can be treated using \([A3]\):

\[
\begin{align*}
&= g_{N-1} g_0 \cdots g_n \left( g_{N-1}^{-1} + \sum_{i=0}^{n} g_i^{-1} \right) g_{n+1} \cdots g_{N-1} \left( \sum_{i=n+1}^{N-1} g_i^{-1} \right) \\
&\quad - g_{N-1}^2 g_0 \cdots g_n \left( \sum_{i=0}^{n} g_i^{-1} \right) g_{n+1} \cdots g_{N-2} \left( \sum_{i=n+1}^{N-2} g_i^{-1} \right) \\
&\quad = g_0 \cdots g_{N-1} \sum_{i=0}^{N-1} g_i^{-1}
\end{align*}
\]

The cases where \(n = 0\) and \(n = N - 1\) have to be treated separately, either by an explicit calculation or by appealing to the symmetry of the problem under cyclic permutations of \(g_0 \ldots g_{N-1}\). They yield the same result.

Calculation of the \((N - 2) \times (N - 2)\) sub-determinants:

The calculation of the \((N - 2) \times (N - 2)\) sub-determinants proceeds analogously to that in the last paragraph. Let \(0 < n < m < N - 1\) and consider the sub-determinant of \(M\) where row and column \(m\) and \(n\) are deleted. Again we start by expanding linearly in the first and the last column:

\[
\det M'_{m,m} =
\]
Again, the case where the first or the last row and column get deleted have to be considered in a separate calculation. The same result is obtained in these cases. We start with the lowest order coefficient. We want to calculate

$$\langle e^2 \rangle \equiv \left\langle \frac{1}{F} \right\rangle .$$

(A4)

**Appendix B: Calculation of expectation values**

In the present appendix we collect some calculations regarding expectation values of the coefficients of the dispersion relation. To make the notation more simple, we will not distinguish between the random variables \( \{ l_i \} \) and a sample \( \{ l_i \} \) of the random process anymore. The context will hopefully make the distinction clear. We start with the lowest order coefficient. We want to calculate

$$\langle e^2 \rangle \equiv \left\langle \frac{1}{F} \right\rangle .$$

(B1)
It is convenient to go over to new random variables
\[ \lambda_n := \frac{1_n - l}{d} . \]

These are Gaussian distributed with expectation zero and spread one. Also introducing
\[ c_1 := \sum_{n=0}^{n-1} \lambda_n, \quad c_2 = \sum_{n=0}^{n-1} \lambda_n^2, \quad \delta = \frac{d}{l} \]
we can rewrite the expectation value as
\[ \frac{\langle c_1^2 \delta^2 + 2Nc_1 \delta + N^2 \rangle}{Nc_2 \delta^2 + 2Nc_1 \delta + N^2} = \frac{\langle c_1^2 \delta^2 + 2Nc_1 \delta + N^2 \rangle}{Nc_2 \delta^2 + 2Nc_1 \delta + N^2} . \]

We were not able to compute this expectation value exactly. We can however expand it in a series that converges for small variance \( \delta \). To that end, we Taylor-expand in \( \delta \), obtaining
\[
\frac{c_2^2 \delta^2 + 2Nc_1 \delta + N^2}{Nc_2 \delta^2 + 2Nc_1 \delta + N^2} = 1 + \left( c_1^2 - c_2 N \right) \left( \frac{\delta}{N} \right)^2 - 2 \left( c_1 - c_1 c_2 N \right) \left( \frac{\delta}{N} \right)^3 \\
+ \left( 4c_1^4 - 5c_1^2 c_2 N + c_2^2 N^2 \right) \left( \frac{\delta}{N} \right)^4 - 4 \left( 2c_1^5 - 3c_1^3 c_2 N + c_1 c_2^2 N^2 \right) \left( \frac{\delta}{N} \right)^5 \\
+ \left( 16c_1^6 - 28c_1^4 c_2 N + 13c_1^2 c_2^2 N^2 - c_2^3 N^3 \right) \left( \frac{\delta}{N} \right)^6 + O \left( \delta^7 \right).
\]

The terms in this series are such that their expectation values can be computed through tedious but straightforward calculations. We use
\[ \langle (\lambda_i)^k \rangle = \begin{cases} 
0 & \text{if } k \text{ odd} \\
\frac{1}{\sqrt{\pi}} 2^k \Gamma \left( \frac{k+1}{2} \right) & \text{if } k \text{ even} , \\
\langle \lambda_i \lambda_j \rangle = 0 & \text{for } i \neq j
\end{cases} \]

\[ (B6) \]
to compute
\[
\langle c_1^2 \rangle = N, \\
\langle c_2 \rangle = N, \\
\langle c_1^4 \rangle = 3N^2, \\
\langle c_1^2 c_2 \rangle = N(N+2), \\
\langle c_2^2 \rangle = N(N+2), \\
\langle c_1^6 \rangle = 15N^3, \\
\langle c_1^4 c_2 \rangle = 3N(N^2 + N + 3), \\
\langle c_2^2 c_2 \rangle = N(N^2 + 6N + 8), \\
\langle c_2^4 \rangle = N(N^2 + 6N + 8).
\]

\[ (B7) \]
Monomials in \( c_1, c_2 \) that contain odd powers of the \( \lambda_i \) vanish. Taking the expectation value of the above Taylor series then gives
\[
\langle c^2 \rangle \equiv \left( \frac{c_2^2 \delta^2 + 2Nc_1 \delta + N^2}{Nc_2 \delta^2 + 2Nc_1 \delta + N^2} \right) = 1 + (N - N^2) \left( \frac{\delta}{N} \right)^2 + (N^4 - 3N^3 + 2N^2) \left( \frac{\delta}{N} \right)^4 \\
+ (-N^6 + 7N^5 - 14N^4 + 260N^3 - 252N^2) \left( \frac{\delta}{N} \right)^6 + O(\delta^8).
\]

\[ (B8) \]
A remarkable aspect of this series expansion of the expectation value is that term by term, the limit \( N \to \infty \) is well defined, and suggests

\[
\lim_{N \to \infty} \left( \frac{\Gamma^2}{\Gamma^2} \right) = 1 - \frac{d^2}{l^2} + \frac{d^4}{l^4} - \frac{d^6}{l^6} + \ldots = \frac{1}{1 + \frac{d^2}{l^2}}. \tag{B9}
\]

While the above calculations do not constitute a proof in the strict sense, we are quite confident that it is correct, since it is backed up both by heuristics and by numerical simulations. This is discussed in the main text.

For this term, we can even easily compute the variance with the same methods. We first have to expand

\[
\left( \frac{c_1^2 \delta^2 + 2N c_1 \delta + N^2}{N c_2 \delta^2 + 2N c_1 \delta + N^2} \right) = 1 + 2 \left( c_1^2 - c_2 N \right) \left( \frac{\delta}{N} \right)^2 + 4 \left( c_1 c_2 N - c_1^3 \right) \left( \frac{\delta}{N} \right)^3 + 3 \left( c_2^2 \delta^2 - 4c_2^2 N^2 + 3c_1^4 \right) \left( \frac{\delta}{N} \right)^4 - 4 \left( 3c_2 c_1 N^2 - 8c_2 c_1^3 N + 5c_1^5 \right) \left( \frac{\delta}{N} \right)^5 + \left( -4c_2^3 N^3 + 42c_2^2 c_1^2 N^2 - 82c_2 c_1^4 N + 44c_1^6 \right) \left( \frac{\delta}{N} \right)^6 + O \left( \delta^7 \right)
\]

again, the expectation value of each of the terms can be evaluated and the limit \( N \to \infty \) taken. We will only give the result:

\[
\lim_{N \to \infty} \left\langle \frac{\Gamma^2}{\Gamma} \right\rangle = 1 - 2 \frac{d^2}{l^2} + 3 \frac{d^4}{l^4} - 4 \frac{d^6}{l^6} + \ldots = \frac{1}{(1 + \frac{d^2}{l^2})^2}. \tag{B11}
\]

Therefore we obtain the very simple result that

\[
\lim_{N \to \infty} \langle c^4 \rangle - \lim_{N \to \infty} \left\langle c^2 \right\rangle^2 = 0. \tag{B12}
\]

The situation for the coefficient \( \ell^2 \) of \( |k|^4 \) is more murky. Again we change variables and obtain, employing the same notation as above,

\[
\ell^2 = \frac{1}{L^2} \sum_{i<j} \frac{c_{ij} 1^2 c_{ij}}{(1^2)^3} - \frac{1}{12} \frac{L^2}{\Pi^2} = \frac{\ell^2}{N^3 \left( c_2^2 \delta^2 + 2c_1 \delta + N^3 \right)} \sum_{i<j} c_{ij} \left( \lambda_i^2 \delta^2 + 2 \lambda_i \delta + 1 \right) \left( \lambda_j^2 \delta^2 + 2 \lambda_j \delta + 1 \right) - \frac{1}{12} \frac{L^2}{c_2 \delta^2 + 2c_1 \delta + N^3} \tag{B13}
\]

Again, we try an expansion in \( \delta \). We use

\[
\frac{\left( c_1 \delta + N \right)^4}{\left( c_2 \delta + c_2 \delta^2 + N^3 \right)^3} = N - 2c_1 \delta + \frac{3 \left( 2c_1^2 - c_2 N \right)}{N} \delta^2 + O(\delta^3)
\]

and

\[
\frac{\left( c_1 \delta + N \right)^4}{c_2 \delta^2 + 2c_1 \delta + N} = n^3 + 2c_1 n^2 \delta + \left( 2c_1 n - c_2 n^2 \right) \delta^2
\]

to find after some calculation

\[
\langle \ell^2 \rangle = \frac{\ell^2}{12} + \frac{L^2}{12N} (-4N^2 + N - 6) \delta^2 + O(\delta^3). \tag{B16}
\]

The problem with this expansion is that, at least term by term, the \( N \to \infty \) limit can not be taken: The term proportional to \( \delta^2 \) diverges linearly with \( N \). This is despite some cancellation between the two terms that comprise \( C \). We will discuss the implications of this in the main text.
Appendix C: Discrete time, continuous space

As mentioned in the introduction, we can also apply our calculations to the situation in which time is discrete and space is continuous and homogeneous, by exchanging the role of space and time in all formulas. The action in this case would read

\[ S = \int dx \sum_{n \in \mathbb{Z}} [g_n (\partial^+ \phi)_n - (\phi'_n)^2] \]

with the prime denoting the spatial derivative of the field \( \phi_n(x) \). Determination of the equations of motion proceeds in parallel with the main text. The wave-ansatz is now

\[ \phi_n^{(z)}(x) = c_n \exp (L z \omega - k x), \quad L = \sum_{n=0}^{N-1} l_n. \]  

(C1)

The calculation of the main text then gives the dispersion relation

\[ k^2 (\omega) = c^2 \omega^2 + \ell^2 \omega^4 + \ldots, \]  

(C2)

with \( c^2 \) and \( \ell^2 \) exactly as in (15), and (16). We can invert this series to get to the more familiar form

\[ \omega^2 (k) = \frac{1}{c^2} k^2 - \frac{\ell^2}{c^4} k^4 + \ldots. \]  

(C3)

We have not analyzed the expectation value \( \langle c^{-2} \rangle \) in detail, but one can argue that since the distribution of \( c^2 \) is sharp in the limit \( N \to \infty \) one should have

\[ \lim_{N \to \infty} \langle c^{-2} \rangle = \left( \lim_{N \to \infty} \langle c^2 \rangle \right)^{-1} = \left( 1 + \frac{d^2}{l^2} \right)^2, \]  

(C4)

i.e. we obtain a positive correction to the continuum phase velocity. Also, the coefficient of \( k^4 \) comes with a negative sign, thus presumably leading to an increase in group velocity, as well.

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[13] In fact, as we will explain below, our calculations are immediately applicable also to the reversed situation: a space-time that is spatially homogeneous, but discrete and with fluctuating geometry in the time direction.
[14] The acoustic branch accommodates arbitrarily low frequencies, and involves the field at neighboring sites move approximately in parallel. In solids, these modes can be excited by sound waves, hence their name. The other branches in solids are high frequency oscillations that can be excited by electromagnetic radiation, typically microwaves (see for example [12]).