Geometry and moduli of polarised varieties
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Abstract. In this paper we investigate the geometry of projective varieties polarised by ample and more generally nef and big Weil divisors. First we study birational boundedness of linear systems. We show that if $X$ is a projective variety of dimension $d$ with $\epsilon$-lc singularities for $\epsilon > 0$, and if $N$ is a nef and big Weil divisor on $X$ such that $N - K_X$ is pseudo-effective, then the linear system $|mN|$ defines a birational map for some natural number $m$ depending only on $d, \epsilon$. This is key to proving various other results. For example, it implies that if $N$ is a big Weil divisor (not necessarily nef) on a klt Calabi-Yau variety of dimension $d$, then the linear system $|mN|$ defines a birational map for some natural number $m$ depending only on $d$. It also gives new proofs of some known results, for example, if $X$ is an $\epsilon$-lc Fano variety of dimension $d$ then taking $N = -K_X$ we recover birationality of $|-mK_X|$ for bounded $m$.

We prove similar birational boundedness results for nef and big Weil divisors $N$ on projective klt varieties $X$ when both $K_X$ and $N - K_X$ are pseudo-effective (here $X$ is not assumed $\epsilon$-lc).

Using the above we show boundedness of polarised varieties under some natural conditions. We extend these to boundedness of semi-log canonical Calabi-Yau pairs polarised by effective ample Weil divisors not containing lc centres. Combining this with the moduli theory of stable pairs we show that polarised Calabi-Yau pairs and polarised Fano pairs admit projective coarse moduli spaces.

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1. Introduction

We work over an algebraically closed field $k$ of characteristic zero unless stated otherwise. By integral divisor we will mean a Weil divisor with integer coefficients which is not necessarily Cartier.

Assume that $X$ is a normal projective variety and $N$ is an integral divisor on $X$. For each natural number $m$ we have the linear system $|mN|$. In algebraic geometry it is often a central theme to understand such linear systems and their associated maps $\phi_{|mN|}: X \to \mathbb{P}^{h-1}$ defined by a basis of $H^0(mN)$ where $h = h^0(mN)$. When $N$ is ample (or just nef and big) a main problem is to estimate those $m$ for which the linear system $|mN|$ defines a birational map, ideally for $m$ bounded in terms of some basic invariants of $X$. Already in dimension one $m$ depends not only on the genus of $X$ but also on the degree of $N$. We need to impose some conditions to get reasonable results and in practice this means we should somehow get the canonical divisor of $X$ involved.

Indeed there has been extensive studies in the literature when $N = K_X$ or $N = -K_X$ is ample. If $X$ has log canonical (lc) singularities and $N = K_X$ is ample, then $|mN|$ defines a birational map for some $m$ depending on $\dim X$ [23]. On the other hand, when $X$ has $\epsilon$-log canonical ($\epsilon$-lc) singularities with $\epsilon > 0$ and $N = -K_X$ is ample, then $|mN|$ defines a birational map for some $m$ depending on $\dim X, \epsilon$ [9, theorem 1.2]; here the $\epsilon$-lc condition cannot be removed.
In this paper we study the linear systems $|mN|$ in a rather general context when $N$ is nef and big (we will also consider the non-nef and non-big cases in some places). As before we would like to see when there is an $m$ depending only on dimension of $X$ and some other data so that $|mN|$ defines a birational map. It turns out that if $X$ has $\epsilon$-lc singularities and if $N-K_X$ is pseudo-effective, then $|mN|$ defines a birational map for some $m$ depending only on $\dim X, \epsilon$ (see Theorem 1.1). The result in particular can be applied to varieties with terminal and canonical singularities.

On the other hand, we show that if $X$ has klt singularities and if both $K_X$ and $N-K_X$ are pseudo-effective, then $|mN|$ defines a birational map for some $m$ depending only on $\dim X$ (see Theorem 1.3). A corollary of both these results is that when $X$ is klt Calabi-Yau, i.e. $K_X \equiv 0$, then $|mN|$ defines a birational map for some $m$ depending only on $\dim X$; in this case we do not even need to assume $N$ to be nef but only big (see Corollary 1.4).

Applying the results of the previous paragraph we prove boundedness of varieties under certain conditions. Let $d$ be a natural number and $\epsilon, \nu$ be positive rational numbers. If $X$ is a projective variety of dimension $d$ with $\epsilon$-lc singularities, $K_X$ is nef, and $N$ is a nef and big integral divisor with volume $\vol(K_X+N) \leq \nu$, then $X$ belongs to a bounded family (see Theorem 1.5). In particular, if $X$ is a klt Calabi-Yau variety of dimension $d$ and $N$ is a nef and big integral divisor with $\vol(N) \leq \nu$, then $X$ belongs to a bounded family (see Corollary 1.6).

In the Calabi-Yau case we can further prove boundedness in the semi-log canonical (slc) case. Slc spaces are higher dimensional analogues of nodal curves which may not be normal nor irreducible. If $X$ is an slc Calabi-Yau of dimension $d$ and $N \geq 0$ is an ample integral divisor such that $(X,uN)$ is slc for some $u > 0$ and if $\vol(N) = \nu$, then $X$ belongs to a bounded family (see Corollary 1.8). Such $X$ are called polarised Calabi-Yau. Similar boundedness holds for slc Calabi-Yau pairs $(X,B)$ which can then be used to deduce boundedness of polarised slc Fanos.

The boundedness results just mentioned provide an important ingredient for constructing moduli spaces. In general, Calabi-Yau varieties do not carry any “canonical” polarisation. To form moduli spaces one needs to take ample divisors with certain properties, eg fixed volume and bounded Cartier index. However, to get a compact moduli space one needs to consider limits of such polarised varieties and this causes problems. One issue is that the limiting space may not be irreducible any more, that is, one has to consider slc spaces. Another problem is that we need these limiting spaces to be bounded in order to get a finite dimensional moduli space. The boundedness mentioned in the last paragraph is exactly what we need. Thus using the moduli theory of stable pairs we can construct projective coarse moduli spaces for polarised Calabi-Yau pairs with slc singularities (see Theorem 1.10). Similarly we can prove existence of projective coarse moduli spaces for polarised Fano pairs with slc singularities (see Theorem 1.12).

In the rest of this introduction we will state the results mentioned above in more general forms. We actually prove even more general versions of many of them later in the paper.

**Birational boundedness for nef and big integral divisors.** The first main result of this paper is the following.
Theorem 1.1. Let $d$ be a natural number and $\epsilon$ be a positive real number. Then there exists a natural number $m$ depending only on $d, \epsilon$ satisfying the following. Assume that

- $X$ is a projective $\epsilon$-lc variety of dimension $d$,
- $N$ is a nef and big integral divisor on $X$, and
- $N - K_X$ is pseudo-effective.

Then $|m'N + L|$ and $|K_X + m'N + L|$ define birational maps for any natural number $m' \geq m$ and any integral pseudo-effective divisor $L$.

We actually prove a more general statement in which we replace the assumption of $N$ being integral with assuming $N = E + R$ where $E$ is integral and pseudo-effective and $R \geq 0$ is an $\mathbb{R}$-divisor whose non-zero coefficients are $\geq \delta$ for some fixed $\delta > 0$ (see Theorem 4.2). Similarly we will prove more general forms of many of the results below.

Note that if $-K_X$ is pseudo-effective, then $N - K_X$ is automatically pseudo-effective. This is in particular useful on Calabi-Yau pairs as we will see later. Also note that instead of $X$ being $\epsilon$-lc we can assume $(X,B)$ is $\epsilon$-lc for some boundary $B$ because we can apply the theorem on a $\mathbb{Q}$-factorialisation of $X$.

The theorem in particular applies well to the following three cases:

1. when $K_X$ is nef and big and $N = K_X$,
2. when $-K_X$ is nef and big and $N = -K_X$, and
3. when $K_X \equiv 0$ and $N$ is nef and big.

Cases (1) and (2) are well-known by Hacon-McKernan-Xu [23] (see also [21][45][46]) and Birkar [9], respectively; however, we reprove these results as we only rely on some of the ideas and constructions of [23] and [9]. In case (1) our proof is essentially the same as the proof in [23]. But in case (2) we get a new proof which is in some sense quite different from the proof in [9] despite similarities of the two proofs because here we do not use boundedness of complements in dimension $d$ (see 4.9 for more details). Case (3) is new which we will state below more precisely in 1.4.

Corollary 1.2. Let $d$ be a natural number and $\epsilon$ be a positive real number. Then there exist natural numbers $m,l$ depending only on $d, \epsilon$ satisfying the following. Assume that

- $X$ is a projective $\epsilon$-lc variety of dimension $d$, and
- $N$ is a nef and big integral divisor on $X$.

Then $|m'K_X + l'N + L|$ defines a birational map for any natural numbers $m' \geq m$ and $l' \geq lm'$ and any pseudo-effective integral divisor $L$.

In the above results we cannot drop the $\epsilon$-lc assumption, see Example 8.2. However, we can replace it with some other conditions as in the next result.

Theorem 1.3. Let $d$ be a natural number and $\Phi \subset [0,1]$ be a DCC set of rational numbers. Then there is a natural number $m$ depending only on $d, \Phi$ satisfying the following. Assume

- $(X,B)$ is a klt projective pair of dimension $d$,
- the coefficients of $B$ are in $\Phi$,
- $N$ is a nef and big integral divisor, and
- $N - (K_X + B)$ and $K_X + B$ are pseudo-effective.
Then \(|m'N + L|\) and \(|K_X + m'N + L|\) define birational maps for any natural number \(m' \geq m\) and any integral pseudo-effective divisor \(L\).

The theorem does not hold if we replace the klt property of \((X, B)\) with lc. Indeed any klt Fano variety \(X\) of dimension \(d\) admits an lc \(n\)-complement \(K_X + B\) for some \(n\) depending only on \(d\) [9] (so \(n(K_X + B) \sim 0\) but taking \(N = -K_X\) there is no bounded \(m\) so that \(|mN|\) defines a birational map as Example 8.2 shows.

Birational boundedness for big integral divisors on Calabi-Yau pairs. A consequence of both 1.1 and 1.3 is a birational boundedness statement regarding Calabi-Yau pairs. A Calabi-Yau pair \((X, B)\) is a pair with \(K_X + B \sim Q_0\); we do not assume vanishing of \(h^i(O_X)\) for \(0 < i < d\) as is customary in some other contexts.

**Corollary 1.4.** Let \(d\) be a natural number and \(\Phi \subset [0, 1]\) be a DCC set of rational numbers. Then there is a natural number \(m\) depending only on \(d, \Phi\) satisfying the following. Assume

- \((X, B)\) is a klt Calabi-Yau pair of dimension \(d\),
- the coefficients of \(B\) are in \(\Phi\), and
- \(N\) is a big integral divisor on \(X\).

Then \(|m'N + L|\) and \(|K_X + m'N + L|\) define birational maps for any natural number \(m' \geq m\) and any integral pseudo-effective divisor \(L\). In particular, the volume \(\text{vol}(N) \geq \frac{1}{md}\).

A yet special case of this is when \(B = 0\), say when \(\Phi = \{0\}\), in which case \(m\) depends only on \(d\).

Note that the corollary only assumes klt singularities rather than \(\epsilon\)-lc. In fact, we will see that such \(X\) automatically have \(\epsilon\)-lc singularities for some \(\epsilon > 0\) depending only on \(d\) (this follows from [23]), so we can apply Theorem 1.1 immediately after taking a minimal model of \(N\). Alternatively, we can simply apply Theorem 1.3.

The corollary was proved by Jiang [26] in dimension 3 when \(X\) is a Calabi-Yau with terminal singularities. His proof is entirely different as it relies on the Riemann-Roch theorem for 3-folds with terminal singularities. More special cases for 3-folds were obtained earlier by Fukuda [18] and Oguiso-Peternell [42]. The smooth surface case goes back to Reider [43]. Also see [27] for recent relevant results on irreducible symplectic varieties.

**Boundedness of polarised \(\epsilon\)-log canonical nef pairs.** Given a projective variety \(X\) (or more generally pair) polarised by a nef and big integral divisor \(N\), we would like to find conditions which guarantee that \(X\) belongs to a bounded family. This is often achieved by controlling positivity and singularities. For example, if \(X\) is \(\epsilon\)-lc and \(N = -K_X\) is nef and big, then \(X\) is bounded [8]. On the other hand, if \(X\) is \(\epsilon\)-lc and \(N = K_X\) is ample with volume bounded from above, then \(X\) is bounded (this follows from the results of [23]). The next result deals with the case when \(K_X\) (and more generally \(K_X + B\)) is nef.

**Theorem 1.5.** Let \(d\) be a natural number and \(\epsilon, \delta, v\) be positive real numbers. Consider pairs \((X, B)\) and divisors \(N\) on \(X\) such that

- \((X, B)\) is projective \(\epsilon\)-lc of dimension \(d\),
- the coefficients of \(B\) are in \(\{0\} \cup [\delta, \infty)\),
- \(K_X + B\) is nef,
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- $N$ is nef and big and integral, and
- $\text{vol}(K_X + B + N) \leq v$.

Then the set of such $(X, \text{Supp} B)$ forms a bounded family. If in addition $N \geq 0$, then the set of such $(X, \text{Supp}(B + N))$ forms a bounded family.

A consequence of this is the following.

**Corollary 1.6.** Let $d$ be a natural number, $v$ be a positive real number, and $\Phi \subset [0, 1]$ be a DCC set of rational numbers. Consider pairs $(X, B)$ and divisors $N$ on $X$ satisfying the following:

- $(X, B)$ is a projective klt Calabi-Yau pair of dimension $d$,
- the coefficients of $B$ are in $\Phi$, and
- $N$ is nef and big and integral, and
- $\text{vol}(N) \leq v$.

Then the set of such $(X, \text{Supp} B)$ forms a bounded family. If in addition $N \geq 0$, then the set of such $(X, \text{Supp}(B + N))$ forms a bounded family.

The point is that $(X, B)$ is automatically $\epsilon$-lc for some $\epsilon > 0$ depending only on $d, \Phi$, so we can apply Theorem 1.5. Note that if we relax the nef and big property of $N$ to only big, then $X$ is birationally bounded as we can apply the corollary to the minimal model of $N$.

**Boundedness of polarised semi-log canonical Calabi-Yau pairs.** The above boundedness statements are not enough for construction of moduli spaces. The problem is that limits of families of $\epsilon$-lc varieties are not necessarily $\epsilon$-lc. In fact the limit may not even be irreducible. We want to addresses this problem by showing boundedness of appropriate classes of Calabi-Yau pairs. An slc Calabi-Yau pair is an slc pair $(X, B)$ such that $K_X + B \sim_R 0$. The desired boundedness is a consequence of the next result on lc thresholds.

**Theorem 1.7.** Let $d$ be a natural number, $v$ be a positive real number, and $\Phi \subset [0, 1]$ be a DCC set of real numbers. Then there is a positive real number $t$ depending only on $d, v, \Phi$ satisfying the following. Assume that

- $(X, B)$ is a projective slc Calabi-Yau pair of dimension $d$,
- the coefficients of $B$ are in $\Phi$,
- $N \geq 0$ is a nef integral divisor on $X$,
- $(X, B + uN)$ is slc for some real number $u > 0$, and
- for each irreducible component $S$ of $X$, $N|_S$ is big with $\text{vol}(N|_S) \leq v$.

Then $(X, B + tN)$ is slc.

The key point is that $t$ does not depend on $u$.

A polarised slc Calabi-Yau pair consists of a connected projective slc Calabi-Yau pair $(X, B)$ and an ample integral divisor $N \geq 0$ such that $(X, B + uN)$ is slc for some real number $u > 0$. We refer to such a pair by saying $(X, B), N$ is a polarised slc Calabi-Yau pair.

**Corollary 1.8.** Let $d$ be a natural number, $v$ be a positive real number, and $\Phi \subset [0, 1]$ be a DCC set of rational numbers. Consider $(X, B)$ and $N$ such that

- $(X, B), N$ is a polarised slc Calabi-Yau pair of dimension $d$,
- the coefficients of $B$ are in $\Phi$, and
- $\text{vol}(N) = v$. 


Then the set of such \((X, \text{Supp}(B + N))\) forms a bounded family.

This is a consequence of Theorem 1.7 and the main result of Hacon-McKernan-Xu [22] as we can pick a rational number \(t > 0\) depending only on \(d, v, \Phi\) so that \((X, B + tN)\) is a stable pair with
\[
\text{vol}(K_X + B + tN) = t^d v
\]
(see below for discussion on stable pairs).

**Moduli of polarised Calabi-Yau pairs.** Combining the above boundedness results (1.7, 1.8) with the moduli theory of stable pairs shows that there exists a projective coarse moduli space for Calabi-Yau pairs polarised by effective ample divisors, fixing appropriate invariants. The moduli theory of stable pairs has a long history, see for example [14][36][3][2][1][31](also [47]). We will follow Kollár [31] for the necessary definitions and existence results, see Section 7 for more details.

Recall that a stable pair is a connected projective slc pair \((X, \Delta)\) with \(K_X + \Delta\) ample. It takes more work to define stable families \((X, \Delta) \to S\) where for simplicity we only consider reduced base schemes \(S\).

Now we define families of polarised Calabi-Yau pairs and their moduli functor following [1][20][40][5][16][37]. Fix a natural number \(d\) and positive rational numbers \(c, v\). A \((d, c, v)\)-polarised slc Calabi-Yau pair is a polarised slc Calabi-Yau pair \((X, B)\), \(N\) as defined above such that \(\dim X = d\), \(B = cD\) for some integral divisor \(D\), and \(\text{vol}(N) = v\). More generally we define polarised Calabi-Yau families.

**Definition 1.9.** Let \(S\) be a reduced scheme over \(k\). A \((d, c, v)\)-polarised Calabi-Yau family over \(S\) consists of a projective morphism \(f : X \to S\) of schemes, and a \(\mathbb{Q}\)-divisor \(B\) and an integral divisor \(N\) on \(X\) such that
- \((X, B + uN) \to S\) is a stable family for some rational number \(u > 0\) with fibres of pure dimension \(d\),
- \(B = cD\) where \(D \geq 0\) is a relative Mumford divisor,
- \(N \geq 0\) is a relative Mumford divisor,
- \(K_{X/S} + B \sim_{\mathbb{Q}} 0/S\), and
- for any fibre \(X_s\) of \(f\), \(\text{vol}(N|_{X_s}) = v\).

For definition of relative Mumford divisors, see 7.1 (2). From the above definition we see that \(D, N\) are integral divisors and \(N\) is \(\mathbb{Q}\)-Cartier and ample over \(S\). Note that \(u\) is not fixed and a priori depends on the family.

Now define the moduli functor \(\mathcal{PCY}_{d,c,v}\) of \((d, c, v)\)-polarised Calabi-Yau pairs on the category of reduced \(k\)-schemes by setting
\[
\mathcal{PCY}_{d,c,v}(S) = \{(d, c, v)\}-\text{polarised Calabi-Yau families over } S, \text{ up to isomorphism over } S\}.
\]

Recently Kollár-Xu [37] argued, by a limiting process inside the moduli space of stable pairs, that there is a moduli space for \(\mathcal{PCY}_{d,c,v}\) (not necessarily of finite type) and that its irreducible components are projective [37, Theorem 2].

**Theorem 1.10.** The functor \(\mathcal{PCY}_{d,c,v}\) of \((d, c, v)\)-polarised Calabi-Yau pairs has a projective coarse moduli space.

This confirms [37, Conjecture 1]. We will use a more direct approach to prove the theorem. Instead of working on the moduli space of stable pairs we work essentially on the level of Hilbert schemes and parametrising spaces of divisors, that is, we work
with the pairs embedded into some fixed projective space and follow by now standard constructions in moduli theory. We use 1.7 to find a fixed rational number \( t > 0 \) so that if \((X, B), N\) is any \((d, c, v)\)-polarised Calabi-Yau pair, then \((X, B + tN)\) is a stable pair. In particular, by 1.8, we can embed all such pairs into projective spaces \( \mathbb{P}^n \), a finite number of \( n \). Then using moduli spaces of embedded locally stable pairs [31][30] (which are constructed via Hilbert schemes and parameter spaces of divisors), we show that there is a moduli space for the embedded \((d, c, v)\)-polarised Calabi-Yau pairs \((X, B)\), \( N \). Taking quotient by the action of \( \text{PGL}_{n+1}(k) \), for all the \( n \), we get an algebraic space which is a coarse moduli space for \( \mathcal{PCY}_{d,c,v} \). Then we point out that this moduli space is proper as a family of \((d, c, v)\)-polarised Calabi-Yau pairs over a smooth curve can be extended to a family over the compactification of the curve, after a finite base change [37]. Finally the moduli space is projective by recent work on projectivity of moduli spaces [17][39][35].

Restricting the polarised pairs parametrised by \( \mathcal{PCY}_{d,c,v} \) one gets new functors and existence of their moduli spaces in various interesting settings. Related work in the literature include Alexeev [1] on moduli of abelian varieties and [5][4] on moduli of polarised K3 surfaces.

**Moduli of polarised Fano pairs.** A particular case of polarised Calabi-Yaus pairs is that of Fano pairs polarised by effective anti-pluri-log-canonical divisors. Fix a natural number \( d \) and positive rational numbers \( c, a, v \). A \((d, c, a, v)\)-polarised slc Fano pair consists of a connected projective slc pair \((X, B)\) and an ample effective integral divisor \( N \) such that \((X, B + (a + u)N)\) is slc for some \( u > 0 \), \( \dim X = d \), \( B = cD \) for some integral divisor \( D \geq 0 \), \( K_X + B + aN \sim_{\mathbb{Q}} 0 \), and \( \text{vol}(N) = v \). Since \(-(K_X + B) \sim_{\mathbb{Q}} aN\), the pair \((X, B)\) is Fano which is polarised by \( N \). We define polarised Fano families.

**Definition 1.11.** Let \( S \) be a reduced scheme over \( k \). A \((d, c, a, v)\)-polarised Fano family over \( S \) consists of a projective morphism \( f : X \to S \) of schemes, and a \( \mathbb{Q} \)-divisor \( B \) and an integral divisor \( N \) on \( X \) such that

- \((X, B + uN) \to S\) is a stable family for some rational number \( u > a \) with fibres of pure dimension \( d \),
- \( B = cD \) where \( D \geq 0 \) is a relative Mumford divisor,
- \( N \geq 0 \) is a relative Mumford divisor,
- \( K_{X/S} + B + aN \sim_{\mathbb{Q}} 0/S \), and
- for any fibre \( X_s \) of \( f \), \( \text{vol}(N|_{X_s}) = v \).

We define the moduli functor \( \mathcal{PF}_{d,c,a,v} \) from the category of reduced \( k \)-schemes to the category of sets by setting

\[
\mathcal{PF}_{d,c,a,v}(S) = \{ (d, c, a, v)\text{-polarised Fano families over } S, \text{ up to isomorphism over } S \}.
\]

**Theorem 1.12.** The functor \( \mathcal{PF}_{d,c,a,v} \) of \((d, c, a, v)\)-polarised Fano pairs has a projective coarse moduli space.

Restricting the functor \( \mathcal{PF}_{d,c,a,v} \) to certain subfamilies gives moduli spaces in special situations. For example, we can consider only those Fanos which deform to a fixed Fano variety. Such moduli spaces were constructed by Hacking [20] for those smoothable to \( \mathbb{P}^2 \), by Deopurkar-Han for those smoothable to \( \mathbb{P}^1 \times \mathbb{P}^1 \) [15], and by
Plan of the paper. In section 2 we collect some preliminary definitions and results. In section 3 we study non-klt centres and adjunction on such centres in depth establishing results that are key to the subsequent sections. In section 4 we treat birational boundedness of linear systems on $\varepsilon$-lc varieties which will form the basis for subsequent sections, in particular, we prove more general forms of 1.1 and 1.2 together with 1.4. In section 5 we treat birational boundedness on pseudo-effective pairs proving a more general form of 1.3. In section 6 we establish boundedness results including more general forms of 1.5 and 1.7 together with 1.8. In section 7 we discuss moduli and give the proofs of 1.10 and 1.12. Finally in section 8 we present some examples, remarks, and conjectures.
over $Z$, we can decrease its coefficients slightly so that the resulting divisor, say $L'$, is $\mathbb{Q}$-Cartier and also big over $Z$. Take $n \in \mathbb{N}$ so that $nL'$ is Cartier. Then for any $m \in \mathbb{N}$ we have
\[ h^0(mnL'|_{F'}) \geq h^0(mnL'|_{G'}) \]
by the upper-semi-continuity of cohomology, where the second $h^0$ is dimension over the function field of $Z$.

Since $L'$ is big over $Z$, $h^0(mnL'|_{G'})$ grows like $m^{\dim G'}$, hence $h^0(mnL'|_{F'})$ grows like $m^{\dim F'}$ which implies that $L'|_{F'}$ is big. Since the support of $M' - L' \geq 0$ does not contain $F'$, we see that $M'|_{F'}$ is big. This in turn implies that $M|_F$ is also big where $F$ is the fibre of $f$ corresponding to $F'$.

\[ \Box \]

**Lemma 2.4.** Let $X$ be a normal variety and $M$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. Then there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A$ such that $\text{Supp} \ A = \text{Supp} \ M$ and $M - A$ has arbitrarily small coefficients.

**Proof.** Since $M$ is $\mathbb{R}$-Cartier, we can write $M = \sum r_i M_i$ where $r_i$ are real numbers and $M_i$ are Cartier divisors. Assume that $l$ is minimal. Then the $r_i$ are $\mathbb{Q}$-linearly independent: if not, then say $r_1 = \sum \alpha_i r_i$ where $\alpha_i$ are rational numbers, so
\[ M = \sum_{i=2}^{l} r_i (\alpha_i M_1 + M_i), \]
hence we can write $M = \sum_2 r'_i M'_i$ where $r'_i$ are real numbers and $M'_i$ are Cartier divisors, contradicting the minimality of $l$. Then $\text{Supp} \ M_i \subseteq \text{Supp} \ M$ for every $i$: indeed otherwise there is a prime divisor $D$ which is not a component of $M$ but it is a component of some $M_i$ which gives $0 = \mu_D M = \sum \mu_D M_i$ producing a $\mathbb{Q}$-linear dependence of the $r_i$, a contradiction. Now a small perturbation of the $r_i$ gives the desired $\mathbb{Q}$-divisor $A$.

\[ \Box \]

**Lemma 2.5.** Let $f : X \to Z$ be a contraction of normal varieties and $M$ an $\mathbb{R}$-divisor on $X$. Let $\phi : X' \to X$ be a log resolution of $(X, \text{Supp} \ M)$ and let $M'$ be the sum of the birational transform of $M$ and the reduced exceptional divisor of $\phi$. Let $F$ be a general fibre of $f$ and $F'$ the corresponding fibre of $X' \to Z$. Then $M'|_{F'}$ is the birational transform of $M|_F$ plus the reduced exceptional divisor of $\psi : F' \to F$.

**Proof.** Since $F'$ is a general fibre of $X' \to Z$, $\text{Supp} \ M'$ does not contain $F'$, so $M'|_{F'}$ is well-defined as a divisor. However, $M$ may not be $\mathbb{R}$-Cartier but it can be defined as follows. Let $U$ be the smooth locus of $X$. Then the complement of $F \cap U$ has codimension $\geq 2$ in $F$, so $M|_F$ is well-defined on $F \cap U$ and then we let $M|_F$ be its closure in $F$.

By assumption $M' = M^\sim + E$ where $M^\sim$ is the birational transform of $M$ and $E$ is the reduced exceptional divisor of $\phi$. The exceptional locus of $\psi$ is $|E|_{F'}$, so $M'|_{F'}$ is the sum of $M^\sim|_{F'}$ and the reduced exceptional divisor of $\psi$. On the other hand, letting $U'$ be the inverse image of $U$, there is an exceptional $G$ such that $\phi^* M|_U = M^\sim|_{U'} + G|_{U'}$. Then $M^\sim|_{F \cap U'} + G|_{F \cap U'}$ is the pullback of $M|_{F \cap U}$. This implies that the pushdown of $M^\sim|_{F'}$ to $F$ is $M|_F$. Since no component of $M^\sim|_{F'}$ is exceptional over $F$, $M^\sim|_{F'}$ is the birational transform of $M|_F$, so the claim follows.

\[ \Box \]
2.6. **Linear systems.** Let $X$ be a normal variety and let $M$ be an $\mathbb{R}$-divisor on $X$. The round down $\lfloor M \rfloor$ determines a reflexive sheaf $\mathcal{O}_X(\lfloor M \rfloor)$. We usually write $H^i(M)$ instead of $H^i(X, \mathcal{O}_X(\lfloor M \rfloor))$ and write $h^i(M)$ for $\dim_k H^i(M)$. We can describe $H^0(M)$ in terms of rational functions on $X$ as

$$H^0(M) = \{0 \neq \alpha \in K \mid \text{Div}(\alpha) + M \geq 0\} \cup \{0\}$$

where $K$ is the function field of $X$ and Div$(\alpha)$ is the divisor associated to $\alpha$.

Assume $h^0(M) \neq 0$. The **linear system** $|M|$ is defined as

$$|M| = \{N \mid 0 \leq N \sim M\} = \{\text{Div}(\alpha) + M \mid 0 \neq \alpha \in H^0(M)\}.$$ 

Note that $|M|$ is not equal to $\lfloor |M| \rfloor$ unless $M$ is integral. The **fixed part** of $|M|$ is the $\mathbb{R}$-divisor $F$ with the property: if $G \geq 0$ is an $\mathbb{R}$-divisor and $G \leq N$ for every $N \in \lfloor |M| \rfloor$, then $G \leq F$. In particular, $F \geq 0$. We then define the **movable part** of $|M|$ to be $M - F$ which is defined up to linear equivalence. If $\langle M \rangle := M - \lfloor |M| \rfloor$, then the fixed part of $|M|$ is equal to $\langle M \rangle$ plus the fixed part of $\lfloor |M| \rfloor$. Moreover, if $0 \leq G \leq F$, then the fixed and movable parts of $|M - G|$ are $F - G$ and $M - F$, respectively.

Note that it is clear from the definition that the movable part of $|M|$ is an integral divisor but the fixed part is only an $\mathbb{R}$-divisor.

2.7. **Pairs and singularities.** A **sub-pair** $(X, B)$ consists of a normal quasi-projective variety $X$ and an $\mathbb{R}$-divisor $B$ such that $K_X + B$ is $\mathbb{R}$-Cartier. If $B \geq 0$, we call $(X, B)$ a **pair** and if the coefficients of $B$ are in $[0, 1]$ we call $B$ a boundary.

Let $\phi : W \to X$ be a log resolution of a sub-pair $(X, B)$. Let $K_W + B_W$ be the pullback of $K_X + B$. The **log discrepancy** of a prime divisor $D$ on $W$ with respect to $(X, B)$ is defined as

$$a(D, X, B) := 1 - \mu_D B_W.$$ 

We say $(X, B)$ is **sub-lc** (resp. **sub-klt**) (resp. **sub-$\epsilon$-lc**) if $a(D, X, B) \geq 0$ (resp. $> 0$) (resp. $\geq \epsilon$) for every $D$. This means that every coefficient of $B_W$ is $\leq 1$ (resp. $< 1$) (resp. $\leq 1 - \epsilon$). If $(X, B)$ is a pair, we remove the sub and just say it is lc (resp. klt) (resp. $\epsilon$-lc). Note that since $a(D, X, B) = 1$ for most prime divisors, we necessarily have $\epsilon \leq 1$.

Let $(X, B)$ be a sub-pair. A **non-klt place** of $(X, B)$ is a prime divisor $D$ over $X$, that is, on birational models of $X$, such that $a(D, X, B) \leq 0$, and a **non-klt centre** is the image of such a $D$ on $X$. An **lc place** of $(X, B)$ is a prime divisor $D$ over $X$ such that $a(D, X, B) = 0$, and an **lc centre** is the image on $X$ of an lc place. When $(X, B)$ is lc, then non-lc places and centres are the same as lc centres and places.

A **log smooth** sub-pair is a sub-pair $(X, B)$ where $X$ is smooth and $\text{Supp } B$ has simple normal crossing singularities. Assume $(X, B)$ is a log smooth pair and assume $B = \sum B_i$ is reduced where $B_i$ are the irreducible components of $B$. A **stratum** of $(X, B)$ is a component of $\bigcap_{i \in I} B_i$ for some $I \subseteq \{1, \ldots, r\}$. Since $B$ is reduced, a stratum is nothing but an lc centre of $(X, B)$.

2.8. **Semi-log canonical pairs.** A **semi-log canonical (slc) pair** $(X, B)$ over a field $K$ of characteristic zero (not necessarily algebraically closed) consists of a reduced pure dimensional quasi-projective scheme $X$ over $K$ and an $\mathbb{R}$-divisor $B \geq 0$ on $X$ satisfying the following conditions:

- $X$ is $S_2$ with nodal codimension one singularities,
- no component of Supp $B$ is contained in the singular locus of $X$,
- $K_X + B$ is $\mathbb{R}$-Cartier, and
- if $\pi: X^\nu \to X$ is the normalisation of $X$ and $B^\nu$ is the sum of the birational transform of $B$ and the conductor divisor of $\pi$, then $(X^\nu, B^\nu)$ is lc.

By $(X^\nu, B^\nu)$ being lc we mean after passing to the algebraic closure of $K$, $(X^\nu, B^\nu)$ is an lc pair on each of its irreducible components (we can also define being lc over $K$ directly using discrepancies as in 2.7). The conductor divisor of $\pi$ is the sum of the prime divisors on $X^\nu$ whose images on $X$ are contained in the singular locus of $X$. It turns out that $K_{X^\nu} + B^\nu = \pi^*(K_X + B)$ for a suitable choice of $K_{X^\nu}$ in its linear equivalence class: to see this note that $X$ is Gorenstein outside a codimension $\geq 2$ closed subset, so shrinking $X$ we can assume it is Gorenstein and that $X^\nu$ is regular; in this case $B$ is $\mathbb{R}$-Cartier so we can remove it in which case the equality follows from [32, 5.7]. See [32, Chapter 5] for more on slc pairs.

2.9. b-divisors. A $b$-$\mathbb{R}$-Cartier $b$-divisor over a variety $X$ is the choice of a projective birational morphism $Y \to X$ from a normal variety and an $\mathbb{R}$-Cartier divisor $M$ on $Y$ up to the following equivalence: another projective birational morphism $Y' \to X$ from a normal variety and an $\mathbb{R}$-Cartier divisor $M'$ defines the same $b$-$\mathbb{R}$-Cartier $b$-divisor if there is a common resolution $W \to Y$ and $W \to Y'$ on which the pullbacks of $M$ and $M'$ coincide.

A $b$-$\mathbb{R}$-Cartier $b$-divisor represented by some $Y \to X$ and $M$ is $b$-Cartier if $M$ is $b$-Cartier, i.e. its pullback to some resolution is Cartier.

2.10. Generalised pairs. A generalised pair consists of

- a normal variety $X$ equipped with a projective morphism $X \to Z$,
- an $\mathbb{R}$-divisor $B \geq 0$ on $X$, and
- a $b$-$\mathbb{R}$-Cartier $b$-divisor over $X$ represented by some projective birational morphism $X' \xrightarrow{\phi} X$ and $\mathbb{R}$-Cartier divisor $M'$ on $X$ such that $M'$ is nef/Z and $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := \phi_* M'$.

We refer to $M'$ as the nef part of the pair. Since a $b$-$\mathbb{R}$-Cartier $b$-divisor is defined birationally, in practice we will often replace $X'$ with a resolution and replace $M'$ with its pullback. When $Z$ is a point we drop it but say the pair is projective.

Now we define generalised singularities. Replacing $X'$ we can assume $\phi$ is a log resolution of $(X, B)$. We can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

for some uniquely determined $B'$. For a prime divisor $D$ on $X'$ the generalised log discrepancy $a(D, X, B + M)$ is defined to be $1 - \mu_D B'$.

We say $(X, B + M)$ is generalised lc (resp. generalised klt) (resp. generalised $\epsilon$-lc) if for each $D$ the generalised log discrepancy $a(D, X, B + M)$ is $\geq 0$ (resp. $> 0$) (resp. $\geq \epsilon$).

For the basic theory of generalised pairs see [13, Section 4].

2.11. Minimal models, Mori fibre spaces, and MMP. Let $X \to Z$ be a projective morphism of normal varieties and $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. Let $Y$ be a normal variety projective over $Z$ and $\phi: X \dasharrow Y/Z$ be a birational map whose inverse does not contract any divisor. Assume $D_Y := \phi_* D$ is also $\mathbb{R}$-Cartier and that there is a common resolution $g: W \to X$ and $h: W \to Y$ such that $E := g^* D - h^* D_Y$ is effective and exceptional/Y, and $\text{Supp}_g E$ contains all the exceptional divisors of $\phi$. 

Under the above assumptions we call \( Y \) a minimal model of \( D \) over \( Z \) if \( D_Y \) is nef/\( Z \). On the other hand, we call \( Y \) a Mori fibre space of \( D \) over \( Z \) if there is an extremal contraction \( Y \to T/\mathbb{Z} \) with \(-D_Y \) ample/\( T \) and \( \dim Y > \dim T \).

If one can run a minimal model program (MMP) on \( D \) over \( Z \) which terminates with a model \( Y \), then \( Y \) is either a minimal model or a Mori fibre space of \( D \) over \( Z \). If \( X \) is a Mori dream space, eg if \( X \) is of Fano type over \( Z \), then such an MMP always exists by [12].

2.12. Potentially birational divisors. Let \( X \) be a normal projective variety and let \( D \) be a big \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( X \). We say that \( D \) is potentially birational [23, Definition 3.5.3] if for any pair \( x \) and \( y \) of general closed points of \( X \), possibly switching \( x \) and \( y \), we can find \( 0 \leq \Delta \sim_{\mathbb{Q}} (1 - \epsilon) D \) for some \( 0 < \epsilon < 1 \) such that \((X, \Delta)\) is not klt at \( y \) but \((X, \Delta)\) is lc at \( x \) and \( \{x\} \) is a non-klt centre.

A useful property of potentially birational divisors is that if \( D \) is potentially birational, then \(|K_X + [D]|\) defines a birational map [24, Lemma 2.3.4].

2.13. Bounded families of pairs. We say a set \( \mathcal{Q} \) of normal projective varieties is birationally bounded (resp. bounded) if there exist finitely many projective morphisms \( V^i \to T^i \) of varieties such that for each \( X \in \mathcal{Q} \) there exist an \( i \), a closed point \( t \in T^i \), and a birational isomorphism (resp. isomorphism) \( \phi: V^i_t \to X \) where \( V^i_t \) is the fibre of \( V^i \to T^i \) over \( t \).

Next we will define boundedness for couples. A couple \((X, S)\) consists of a normal projective variety \( X \) and a divisor \( S \) on \( X \) whose coefficients are all equal to 1, i.e. \( S \) is a reduced divisor. We use the term couple instead of pair because \( K_X + S \) is not assumed \( \mathbb{Q} \)-Cartier and \((X, S)\) is not assumed to have good singularities.

We say that a set \( \mathcal{P} \) of couples is birationally bounded if there exist finitely many projective morphisms \( V^i \to T^i \) of varieties and reduced divisors \( C^i \) on \( V^i \) such that for each \((X, S) \in \mathcal{P} \) there exist an \( i \), a closed point \( t \in T^i \), and a birational isomorphism \( \phi: V^i_t \to X \) such that \((V^i_t, C^i_t)\) is a couple and \( E \leq C^i_t \) where \( V^i_t \) and \( C^i_t \) are the fibres over \( t \) of the morphisms \( V^i \to T^i \) and \( C^i \to T^i \), respectively, and \( E \) is the sum of the birational transform of \( S \) and the reduced exceptional divisor of \( \phi \). We say \( \mathcal{P} \) is bounded if we can choose \( \phi \) to be an isomorphism.

A set \( \mathcal{R} \) of projective pairs \((X, B)\) is said to be log birationally bounded (resp. log bounded) if the set of the corresponding couples \((X, \text{Supp } B)\) is birationally bounded (resp. bounded). Note that this does not put any condition on the coefficients of \( B \), eg we are not requiring the coefficients of \( B \) to be in a finite set.

2.14. Families of subvarieties. Let \( X \) be a normal projective variety. A bounded family \( \mathcal{G} \) of subvarieties of \( X \) is a family of (closed) subvarieties such that there are finitely many morphisms \( V^i \to T^i \) of projective varieties together with morphisms \( V^i \to X \) such that \( V^i \to X \) embeds in \( X \) the fibres of \( V^i \to T^i \) over closed points, and each member of the family \( \mathcal{G} \) is isomorphic to a fibre of some \( V^i \to T^i \) over some closed point. Note that we can replace the \( V^i \to T^i \) so that we can assume the set of points of \( T^i \) corresponding to members of \( \mathcal{G} \) is dense in \( T^i \). We say the family \( \mathcal{G} \) is a covering family of subvarieties of \( X \) if the union of its members contains some non-empty open subset of \( X \). In particular, this means \( V^i \to X \) is surjective for at least one \( i \). When we say \( G \) is a general member of \( \mathcal{G} \) we mean there is \( i \) such that \( V^i \to X \) is surjective, the set \( A \) of points of \( T^i \) corresponding to members of \( \mathcal{G} \) is dense in \( T^i \), and \( G \) is the fibre of \( V^i \to T^i \) over a general point of \( A \) (in particular,
G is among the general fibres of \( V^i \to T^i \). Note that the definition of a bounded family here is compatible with 2.13.

2.15. **Creating non-klt centres.** In this subsection we make some preparations on non-klt centres.

(1) First we need the following lemma.

**Lemma 2.16.** Assume that

- \((X, B)\) is a projective pair where \( B \) is a \( \mathbb{Q} \)-divisor,
- \( \Delta \geq 0 \) is a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor and \( H \) is an ample \( \mathbb{Q} \)-divisor,
- \( x, y \in X \) are closed points,
- \((X, B)\) is klt near \( x \) and \((X, B + \Delta)\) is lc near \( x \) with a non-klt centre \( G \) containing \( x \) but \((X, B + \Delta)\) is not klt near \( y \),
- \( G \) is minimal among the non-klt centres of \((X, B + \Delta)\) containing \( x \), and
- either \( y \in G \) or \((X, B + \Delta)\) has a non-klt centre containing \( y \) but not \( x \).

Then there exist rational numbers \( 0 < t \ll s \leq 1 \) and a \( \mathbb{Q} \)-divisor \( 0 \leq E \sim_{\mathbb{Q}} tH \) such that \((X, B + s\Delta + E)\) is not klt near \( y \) but it is lc near \( x \) with a unique non-klt place whose centre contains \( x \), and the centre of this non-klt place is \( G \).

**Proof.** Pick \( 0 \leq M \sim_{\mathbb{Q}} H \) whose support contains \( G \) so that for any \( s < 1 \) any non-klt centre of \((X, B + s\Delta + M)\) passing through \( x \) is contained in \( G \): this is possible by taking a general member \( mM \) of the sublinear system of \(|mH|\) consisting of those elements containing \( G \), for some sufficiently large natural number \( m \) (however, \( M \) may contain other non-klt centres not passing through \( x \)). On the other hand, pick \( 0 \leq N \sim_{\mathbb{Q}} H \) such that \( x \notin \text{Supp} N \). If \( y \notin G \), then pick \( N \) so that \( N \) contains a non-klt centre \( P \) of \((X, B + \Delta)\) with \( y \in P \) but \( x \notin P \): this is possible by the last condition of the lemma.

Let \( \phi : W \to X \) be a log resolution of \((X, B + \Delta + M + N)\).

Then \( \phi^*H \sim_{\mathbb{Q}} A + C \) where \( A \geq 0 \) is ample and \( C \geq 0 \). Let \( C' = \phi_*C \) and \( H' = \phi_*(A + C) \). Replacing \( X \) with a higher resolution we can assume \( \phi \) is a log resolution of \((X, B + \Delta + M + N + C')\);

note that here we pull back \( A, C \) to the new resolution, so \( A \) may no longer be ample but it is nef and big, hence perturbing coefficients in the exceptional components we can make \( A \) ample again. Changing \( A \) up to \( \mathbb{Q} \)-linear equivalence we can assume \( A \) is general, so \( \phi \) is a log resolution of \((X, B + \Delta + M + N + H')\).

Write

\[ K_W + \Gamma_{a,b,c,d} = \phi^*(K_X + B + a\Delta + bM + cN + dH') \]

Let \( T \) be the sum of the components of \( \left[ \Gamma_{a,b,c,d} \right] \) whose image on \( X \) is \( G \) where \( \Gamma_{a,b,c,d} \) denotes the effective part of \( \Gamma_{a,b,c,d} \). We can assume \( T \neq 0 \) since \( G \) is a non-klt centre of \((X, B + \Delta)\).

Pick a rational number \( 0 < b \ll 1 \) and let \( a \) be the lc threshold of \( \Delta \) with respect to \((X, B + bM)\) near \( x \). Then \( a \) is sufficiently close to \( 1 \) but not equal to \( 1 \). Moreover,

\[ \left[ \Gamma_{a,b,0,0} \right] \subseteq \left[ \Gamma_{1,b,0,0} \right] = \left[ \Gamma_{1,0,0,0} \right] \]
By our choice of $M$ and by the minimality of $G$, the only possible non-klt centre of $(X, B + a\Delta + bM)$ through $x$ is $G$, so any component of $\left[ \Gamma^0_{a,b,0,0} \right]$ whose image contains $x$, is a component of $T$.

Now pick a rational number $0 < d \ll b$ and let $\lambda$ be the lc threshold of $a\Delta + bM$ with respect to $(X, B + dH')$ near $x$. Then $\lambda$ is sufficiently close to 1 but not equal to 1. Moreover,

$$\left[ \Gamma^0_{\lambda a, \lambda b,0,0} \right] \subseteq \left[ \Gamma^0_{a,b,0,0} \right] \subseteq \left[ \Gamma^0_{a,b,0,0} \right],$$

so any component of $\left[ \Gamma^0_{\lambda a, \lambda b,0,d} \right]$ whose image contains $x$ is a component of $T$. Thus $G$ is the only non-klt centre of

$$(X, B + \lambda a\Delta + \lambda bM + dH')$$

passing through $x$. In particular, there is a component $S$ of $\left[ \Gamma^0_{\lambda a, \lambda b,0,d} \right]$ whose image is $G$. Now noting that $\phi^*H' = A + C$ and that

$$\Gamma_{\lambda a, \lambda b,0,d} = \Gamma_{\lambda a, \lambda b,0} + dA + dC,$$

possibly after perturbing the coefficients of $C$ and replacing $A$ accordingly, we can assume that $S$ is the only component of $\left[ \Gamma^0_{\lambda a, \lambda b,0,d} \right]$ whose image contains $x$.

Let $s = \lambda a$, $t = \lambda b + d$, and $E = \lambda bM + dH'$. By construction, $(X, B + s\Delta + E)$ is lc near $x$ with a unique non-klt place whose centre contains $x$, and the centre of this non-klt place is $G$. If $(X, B + s\Delta + E)$ is not klt near $y$, then we are done. In particular, this is the case if $y \in G$. Thus we can assume $(X, B + s\Delta + E)$ is klt near $y$ and that $y \notin G$. So by assumption, there is a non-klt centre $P$ of $(X, B + \Delta)$ containing $y$ but not $x$. By our choice of $N$, $P \subset \text{Supp} N$ but $x \notin \text{Supp} N$. Let $c$ be the lc threshold of $N$ with respect to $(X, B + s\Delta + E)$ near $y$. Since $s$ is sufficiently close to 1, $c$ is sufficiently small. Now replace $E$ with $E + cN$ and replace $t$ with $t + c$.

(2) Let $X$ be a normal projective variety of dimension $d$ and $D$ a nef and big $\mathbb{Q}$-divisor. Assume $\text{vol}(D) > (2d)^d$. Then there is a bounded family of subvarieties of $X$ such that for each pair $x, y \in X$ of general closed points, there is a member $G$ of the family and there is $0 \leq \Delta \sim_{\mathbb{Q}} D$ such that $(X, \Delta)$ is lc near $x$ with a unique non-klt place whose centre contains $x$, that centre is $G$, and $(X, \Delta)$ is not klt at $y$ [23, Lemma 7.1].

If in addition we are also given a $\mathbb{Q}$-divisor $0 \leq M \sim_{\mathbb{Q}} D$, then we can assume $\text{Supp} M \subset \text{Supp} \Delta$ simply by adding a small multiple of $M$ to $\Delta$. Since $x, y$ are general, they are not contained in $\text{Supp} M$.

(3) Now under the setting of (2) assume that $A$ is a nef and big $\mathbb{Q}$-divisor. Let $\Delta$ and $G$ be chosen for a pair $x, y \in X$ of general closed points and assume $\dim G > 0$ and $\text{vol}(A|_G) > d^d$. We claim that, possibly after switching $x, y$, there exist a $\mathbb{Q}$-divisor

$$0 \leq \Delta^{(1)} \sim_{\mathbb{Q}} D + 2A$$

and a proper subvariety $G^{(1)} \subsetneq G$ such that $(X, \Delta^{(1)})$ is lc near $x$ with a unique non-klt place whose centre contains $x$, that centre is $G^{(1)}$, and $(X, \Delta^{(1)})$ is not klt at $y$ (compare with [24, Theorem 2.3.5]).

Proof of the claim: First note that since we are concerned with general points of $X$, to prove the claim we can replace $X$ with a resolution and replace $D, \Delta, A$
with their pullbacks and replace $G$ with its birational transform, hence assume $X$ is smooth. First we treat the case when $A$ is ample. Assume that there exist a rational number $0 < c < 2$, and a $\mathbb{Q}$-divisor $0 \leq L \sim_{\mathbb{Q}} cA$ such that $(X, \Delta + L)$ is not klt near $y$ but it is lc near $x$ with a minimal non-klt centre $G' \subsetneq G$ through $x$, and either $y \in G'$ or $(X, \Delta + L)$ has a non-klt centre containing $y$ but not $x$. By Lemma 2.16, there exist rational numbers $0 < t \ll s < 1$ and a $\mathbb{Q}$-divisor $0 \leq L' \sim_{\mathbb{Q}} t A$ such that $(X, s \Delta + sL + L')$ is not klt near $y$ but it is lc near $x$ with a unique non-klt place whose centre contains $x$, and the centre of this non-klt place is $G'$. We then let

$$\Delta^{(1)} = s\Delta + sL + L' + (1 - s)M + (2 - t - sc)A \sim_{\mathbb{Q}} \Delta + 2A$$

and let $G^{(1)} = G'$.

We will find $c, L$, possibly after switching $x, y$. By [33, 6.8.1 and its proof], there exist a rational number $0 < e < 1$ and a $\mathbb{Q}$-divisor $0 \leq N \sim_{\mathbb{Q}} eA$ such that $(X, \Delta + N)$ is lc near $x$ but has a non-klt centre through $x$ other than $G$. Since intersection of non-klt centres near $x$ are union of such centres, the minimal non-klt centre of the pair at $x$ is a proper subvariety $G' \subset G$.

If $(X, \Delta + N)$ is not lc at $y$, then we let $c = e, L = N$ noting that in this case $(X, \Delta + N)$ has a non-klt centre containing $y$ but not $x$. If $(X, \Delta + N)$ is lc at $y$ but has a non-klt centre at $y$ not containing $x$, then adding to $N$ appropriately again we can assume $(X, \Delta + N)$ is not lc at $y$ and proceed as before. Thus we can assume $(X, \Delta + N)$ is lc at $y$ and that any non-klt centre at $y$ also contains $x$, hence $(X, \Delta)$ is lc at $y$ with $G$ the only non-klt centre of $(X, \Delta)$ containing $y$.

Now if $y \notin G'$, then again we let $c = e, L = N$. So assume $y \notin G'$. Let $G''$ be the minimal non-klt centre of $(X, \Delta + N)$ at $y$. Then $G'' \subsetneq G$. If $G''$ is a proper subvariety of $G$, then we switch $x, y$ and switch $G', G''$ and again let $c = e, L = N$. Thus assume $G'' = G$ which means $(X, \Delta + N)$ has no other non-klt centre at $y$. Then $(X, \Delta + (1 + \epsilon)N)$ is lc at $y$ but not lc at $x$ for a small $\epsilon > 0$. Here we apply [33, 6.8.1 and its proof] to find a rational number $0 < e' < 1$ and a $\mathbb{Q}$-divisor $0 \leq N' \sim_{\mathbb{Q}} e'A$ such that $(X, \Delta + N + N')$ is lc at $y$ with a minimal non-klt centre $G'' \subsetneq G$ at $y$ but not lc at $x$. This time we switch $x, y$ and switch $G', G''$ and let $c = e + e', L = N + N'$.

Finally it remains to treat the case when $A$ is nef and big. Write $A \sim_{\mathbb{Q}} H + P$ independent of $x, y$ where $H \geq 0$ is ample and $P \geq 0$ and these are $\mathbb{Q}$-divisors. Since $x, y$ are general, they are not contained in $P$. Moreover, for any rational number $t \in (0, 1)$, we have

$$A \sim_{\mathbb{Q}} (1 - t)A + tH + tP$$

where $(1 - t)A + tH$ is ample. Now taking $t$ small enough we have

$$\text{vol}((1 - t)A + tH) > d^d,$$

so we can apply the above arguments to $(1 - t)A + tH$ to construct $\Delta^{(1)}$ and $G^{(1)}$ and then add $2tP$ to $\Delta^{(1)}$.

3. Geometry of non-klt centres

In this section we establish some results around the geometry of non-klt centres which are crucial for latter sections.
3.1. Definition of adjunction. First we recall the definition of adjunction which was introduced in [24]. We follow the presentation in [9]. Assume the following setting:

- \((X, B)\) is a projective klt pair,
- \(G \subset X\) is a subvariety with normalisation \(F\),
- \(X\) is \(\mathbb{Q}\)-factorial near the generic point of \(G\),
- \(\Delta \geq 0\) is an \(\mathbb{R}\)-Cartier divisor on \(X\), and
- \((X, B + \Delta)\) is lc near the generic point of \(G\), and there is a unique non-klt place of this pair whose centre is \(G\).

We will define an \(\mathbb{R}\)-divisor \(\Theta_F\) on \(F\) with coefficients in \([0, 1]\) giving an adjunction formula

\[ K_F + \Theta_F + P_F \sim_{\mathbb{R}} (K_X + B + \Delta)|_F \]

where in general \(P_F\) is determined only up to \(\mathbb{R}\)-linear equivalence. Moreover, we will see that if the coefficients of \(B\) are contained in a fixed DCC set \(\Phi\), then the coefficients of \(\Theta_F\) are also contained in a fixed DCC set \(\Psi\) depending only on \(\text{dim} \; X\) and \(\Phi\) [23, Theorem 4.2].

Let \(\Gamma\) be the sum of \((B + \Delta)^{<1}\) and the support of \((B + \Delta)^{\geq 1}\). Put \(N = B + \Delta - \Gamma\) which is supported in \([\Gamma]\). Let \(\phi: W \to X\) be a log resolution of \((X, B + \Delta)\) and let \(\Gamma_W\) be the sum of the reduced exceptional divisor of \(\phi\) and the birational transform of \(\Gamma\). Let

\[ N_W = \phi^*(K_X + B + \Delta) - (K_W + \Gamma_W). \]

Then \(\phi_*N_W = N \geq 0\) and \(N_W\) is supported in \([\Gamma_W]\). Now run an MMP/\(X\) on \(K_W + \Gamma_W\) with scaling of some ample divisor. We reach a model \(Y\) on which \(K_Y + \Gamma_Y\) is a limit of movable/\(X\) \(\mathbb{R}\)-divisors. Applying the general negativity lemma (cf. [11, Lemma 3.3]), we deduce \(N_Y \geq 0\). In particular, if \(U \subset X\) is the largest open subset where \((X, B + \Delta)\) is lc, then \(N_Y = 0\) over \(U\) and \((Y, \Gamma_Y)\) is a \(\mathbb{Q}\)-factorial dlt model of \((X, B + \Delta)\) over \(U\). By assumption, \((X, B + \Delta)\) is lc but not klt at the generic point of \(G\). By [9, Lemma 2.33], no non-klt centre of the pair contains \(G\) apart from \(G\) itself, hence we can assume there is a unique component \(S\) of \([\Gamma_Y]\) whose image on \(X\) contains \(G\), and that this image is \(G\). Moreover, \(G\) is not inside the image of \(N_Y\).

Let \(h: S \to F\) be the morphism induced by \(S \to G\). By [9, Lemma 2.33], \(h\) is a contraction. By divisorial adjunction we can write

\[ K_S + \Gamma_S + N_S = (K_Y + \Gamma_Y + N_Y)|_S \sim_{\mathbb{R}} 0/F \]

where \(N_S = N_Y|_S\) is vertical over \(F\). If \(S\) is exceptional over \(X\), then let \(\Sigma_Y\) be the sum of the exceptional/\(X\) divisors on \(Y\) plus the birational transform of \(B\). Otherwise let \(\Sigma_Y\) be the sum of the exceptional/\(X\) divisors on \(Y\) plus the birational transform of \(B\) plus \((1 - \mu_G B)|_S\). In any case, \(S\) is a component of \([\Sigma_Y]\) and \(\Sigma_Y \leq \Gamma_Y\). Applying adjunction again we get \(K_S + \Sigma_S = (K_Y + \Sigma_Y)|_S\). Obviously \(\Sigma_S \leq \Gamma_S\).

Now we define \(\Theta_F\): for each prime divisor \(D\) on \(F\), let \(t\) be the lc threshold of \(h^*D\) with respect to \((S, \Sigma_S)\) over the generic point of \(D\), and then let \(\mu_D \Theta_F := 1 - t\). Note that \(h^*D\) is defined only over the generic point of \(D\) as \(D\) may not be \(\mathbb{Q}\)-Cartier.

Note that if the coefficients of \(B\) are contained in a fixed DCC set \(\Phi\) including 1, then the coefficients of \(\Gamma_Y\) are contained in \(\Phi\) which in turn implies that the coefficients of \(\Gamma_S\) are contained in a fixed DCC set \(\Psi\). Applying the ACC for lc thresholds [23, Theorem 1.1] shows that the coefficients of \(\Theta_F\) are also contained in
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a fixed DCC set $\Psi$ depending only on $\dim X$ and $\Phi$. Moreover, it turns out that $P_F$ is pseudo-effective (see [23, Theorem 4.2] and [9, Theorem 3.10]).

3.2. Boundedness of singularities on non-klt centres. The next result is a generalisation of [9, Proposition 4.6] which puts strong restrictions on singularities that can appear on non-klt centres under suitable assumptions. We will not need the proposition in its full generality in this paper but it likely be useful elsewhere.

Proposition 3.3. Let $d, v$ be natural numbers and $\epsilon, \epsilon'$ be positive real numbers with $\epsilon' < \epsilon$. Then there exists a positive real number $t$ depending only on $d, v, \epsilon, \epsilon'$ satisfying the following. Assume $X, C, M, S, \Delta, G, F, \Theta, P_F$ are as follows:

1. $(X, C)$ is a projective $\epsilon$-lc pair of dimension $d$,
2. $K_X$ is $\mathbb{Q}$-Cartier,
3. $M \geq 0$ is a nef $\mathbb{Q}$-divisor on $X$ such that $|M|$ defines a birational map,
4. $S \geq 0$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$,
5. the coefficients of $C + S$ are in $\{0\} \cup [\epsilon, \infty)$ and the coefficients of $M + C + S$ are in $[1, \infty)$,
6. $G$ is a general member of a covering family of subvarieties of $X$, with normalisation $F$,
7. $\Delta \geq 0$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$,
8. there is a unique non-klt place of $(X, \Delta)$ whose centre is $G$,
9. the adjunction formula
   $$K_F + \Theta_F + P_F \sim_R (K_X + \Delta)|_F$$
   is as in 3.1 assuming $P_F \geq 0$,
10. $\text{vol}(M|_F) \leq v$,
11. $K_X + C + \Delta$ is nef,
12. $M - (K_X + C + S + \Delta)$ is big, and
13. $tM - \Delta - S$ is big.

Then for any $0 \leq L_F \sim_R M|_F$, the pair

$$(F, \Theta_F + P_F + C|_F + tL_F)$$

is $\epsilon'$-lc.

Proof. Let $P$ be the set of couples and $c$ the number given by [9, Proposition 4.4], for $d, v, \epsilon$. Let $\delta > 0$ be the number given by [9, Proposition 4.2] for $P, \epsilon'$. Let $l \in \mathbb{N}$ be the smallest number such that $\frac{\delta}{l} > \frac{\epsilon}{\epsilon'}$; such $l$ exists because $\frac{\epsilon'}{\epsilon} < 1$. Let $t = \frac{\delta}{2lc}$. We will show that this $t$ satisfies the proposition. Note that $t$ depends on $\delta, l, c$ which in turn depend on $P, c, \epsilon, \epsilon'$ and these in turn depend on $d, v, \epsilon, \epsilon'$.

Step 1. In this step we introduce some basic notation. We will assume $\dim G > 0$ otherwise the statement is vacuous. Note that we are assuming that $C, M, S$ are independent of $G$ so $G$ being general it is not contained in $\text{Supp}(M + C + S)$ (however, $\Delta$ depends on $G$ whose support contains $G$). On the other hand, by [9, Lemma 2.6], there is a log resolution $\phi: W \to X$ of $(X, \text{Supp}(M + C + S))$ such that we can write

$$M_W := \phi^* M = A_W + R_W$$

where $A_W$ is the movable part of $|M_W|$, $|A_W|$ is based point free defining a birational contraction, and $R_W \geq 0$ is the fixed part.
Step 2. In this step we have a closer look at the adjunction formula given in the statement, and the related divisors. First note that since $G$ is a general member of a covering family, $X$ is smooth near the generic point of $G$. As pointed out in 3.1, the coefficients of $\Theta_F$ are in a fixed DCC set $\Psi$ depending only on $d$.

By [23, Theorem 4.2] [9, Lemma 3.12] (by taking $B = 0$), we can write $K_F + \Lambda_F = K_X|_F$ where $(F, \Lambda_F)$ is sub-klt and $\Lambda_F \leq \Theta_F$. On the other hand, since $G$ is not contained in $\text{Supp} C$, the unique non-klt place of $(X, \Delta)$ whose centre is $G$ is also a unique non-klt place of $(X, C + \Delta)$ whose centre is $G$. Thus applying [9, Lemma 3.12] once more (this time by taking $B = C$), we can write $K_F + \tilde{C}_F = (K_X + C)|_F$ where $(F, \tilde{C}_F)$ is sub-$\epsilon$-lc. Note that

$$\tilde{C}_F = \Lambda_F + C|_F \leq \Theta_F + C|_F.$$  

Step 3. Let $M_F := M|_F$, $C_F := C|_F$, and $S_F := S|_F$. In this step we show

$$(F, \text{Supp}(\Theta_F + C_F + S_F + M_F))$$

is log birationally bounded using [9, Proposition 4.4]. Since $G$ is a general member of a covering family, we can choose a log resolution $F' \to F$ of the above pair such that we have an induced morphism $F' \to W$ and that $|A_{F'}|$ defines a birational contraction where $A_{F'} := A_W|_{F'}$. Thus $|A_F|$ defines a birational map where $A_F$ is the pushdown of $A_{F'}$. This in turn implies $|M_F|$ defines a birational map because $A_F \leq M_F$. Moreover,

$$K_F + C_F + \Theta_F + P_F \sim_R (K_X + C + \Delta)|_F$$

is nef. In addition,

$$M_F - (K_F + C_F + S_F + \Theta_F + P_F) \sim_R (M - (K_X + C + S + \Delta))|_F$$

is big by the generality of $G$ and by Lemma 2.3. This in turn implies

$$M_F - (K_F + C_F + S_F + \Theta_F)$$

is big as well.

On the other hand, by [9, Lemma 3.11],

$$\mu_D(\Theta_F + C_F + S_F + M_F) \geq 1$$

for any component $D$ of $C_F + S_F + M_F$ because each non-zero coefficient of $C + S + M$ is $\geq 1$ (note that although the latter divisor may not be a $\mathbb{Q}$-divisor but the lemma still applies as its proof works for $\mathbb{R}$-divisors as well). Similarly, applying the lemma again,

$$\mu_D(\Theta_F + \frac{1}{\epsilon}(C_F + S_F)) \geq 1$$

for any component $D$ of $C_F + S_F$ because each non-zero coefficient of $\frac{1}{\epsilon}(C + S)$ is $\geq 1$. In particular, replacing $\epsilon$ with the minimum of $\Psi^0 \cup \{\epsilon\}$, we can assume that the non-zero coefficients of $\Theta_F + C_F + S_F$ are $\geq \epsilon$.

Now applying [9, Proposition 4.4] to $F, B_F := \Theta_F + C_F + S_F, M_F$, there is a projective log smooth couple $(\mathcal{T}, \Sigma_T) \in \mathcal{P}$ and a birational map $\mathcal{T} \dasharrow F$ satisfying:

- $\Sigma_T$ contains the exceptional divisor of $\mathcal{T} \dasharrow F$ and the birational transform of $\text{Supp}(\Theta_F + C_F + S_F + M_F)$, and
- if $f: F' \to F$ and $g: F' \to \mathcal{T}$ is a common resolution and $M_F$ is the pushdown of $M_F|_{F'}$, then each coefficient of $M_T$ is at most $c$.  

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Step 4. In this step we compare log divisors on $F$ and $\overline{F}$. First define $\Gamma_{\overline{F}} := (1 - \epsilon)\Sigma_{\overline{F}}$. Let $K_F + \tilde{C}_F$ be the pushdown of $K_F + \bar{C}_F$ and let $K_{\overline{F}} + \tilde{C}_{\overline{F}}$ be the pushdown of $K_{\overline{F}} + \bar{C}_{\overline{F}}$ to $\overline{F}$. We claim that $\tilde{C}_{\overline{F}} \leq \Gamma_{\overline{F}}$. If $\tilde{C}_{\overline{F}} \leq 0$, then the claim holds trivially. Assume $\tilde{C}_{\overline{F}}$ has a component $D$ with positive coefficient. Then $D$ is either exceptional$/F$ or is a component of the birational transform of $\overline{C}_F$ with positive coefficient. In the former case, $D$ is a component of $\Sigma_{\overline{F}}$ because $\Sigma_{\overline{F}}$ contains the exceptional divisor of $\overline{F} \dasharrow F$. In the latter case, $D$ is a component of the birational transform of $\overline{C}_F$ because $\overline{C}_F \leq \Theta_F + C_F$ by Step 2, hence again $D$ is a component of $\Sigma_{\overline{F}}$ as it contains the birational transform of $\text{Supp}(\Theta_F + C_F)$. Moreover, since $(F, \tilde{C}_F)$ is sub-$\epsilon$-lc, the coefficient of $D$ in $\tilde{C}_{\overline{F}}$ is at most $1 - \epsilon$, hence $\mu_D \tilde{C}_{\overline{F}} \leq \mu_D \Gamma_{\overline{F}}$. We have then proved the claim $\tilde{C}_{\overline{F}} \leq \Gamma_{\overline{F}}$.

Step 5. In this step we define a divisor $I_F$ and compare singularities on $F$ and $\overline{F}$. Let

$$I_F := \Theta_F + P_F - \Lambda_F.$$ 

By Step 2, $I_F \geq 0$. First note

$$I_F = \Theta_F + P_F - \Lambda_F = K_F + \Theta_F + P_F - K_F - \Lambda_F$$

$$\sim_{\mathbb{R}} (K_X + \Delta)|_F - K_X|_F \sim_{\mathbb{R}} \Delta|_F.$$ 

Pick $0 \leq L_F \sim_{\mathbb{R}} M_F$. Recalling $\tilde{C}_F = C_F + \Delta_F$ from Step 2, we see that

$$K_F + \tilde{C}_F + I_F + tL_F = K_F + C_F + \Delta_F + \Theta_F + P_F - \Lambda_F + tL_F$$

$$= K_F + C_F + \Theta_F + P_F + tL_F \sim_{\mathbb{R}} (K_X + C + \Delta + tM)|_F$$

is nef.

Let $I_{\overline{F}} = g_s f^* I_F$, $S_{\overline{F}} = g_s f^* S_F$, and $L_{\overline{F}} = g_s f^* L_F$. Then by the previous paragraph and by the negativity lemma,

$$f^*(K_F + \tilde{C}_F + I_F + tL_F) \leq g^*(K_{\overline{F}} + \tilde{C}_{\overline{F}} + I_{\overline{F}} + tL_{\overline{F}})$$

which implies that

$$(F, \tilde{C}_F + I_F + tL_F)$$

is sub-$\epsilon$-lc if

$$(\overline{F}, \tilde{C}_{\overline{F}} + I_{\overline{F}} + tL_{\overline{F}})$$

is sub-$\epsilon$-lc.

Step 6. In this step we finish the proof using [9, Proposition 4.2]. Note

$$2tM_F - (I_F + S_F + tL_F) \sim_{\mathbb{R}} 2tM_F - \Delta|_F - S|_F - tM_F$$

$$\sim_{\mathbb{R}} (tM - \Delta - S)|_F$$

is big by the generality of $G$ and by Lemma 2.3. Thus there is

$$0 \leq J_{\overline{F}} \sim_{\mathbb{R}} 2tM_{\overline{F}} - (I_{\overline{F}} + S_{\overline{F}} + tL_{\overline{F}}).$$

The coefficients of $2tM_{\overline{F}}$ are bounded from above by $2ltc$, hence they are bounded by $\delta$. Thus by our choice of $\delta$ and by [9, Proposition 4.2], we deduce that

$$(\overline{F}, \Gamma_{\overline{F}} + tI_{\overline{F}} + tS_{\overline{F}} + tJ_{\overline{F}} + tL_{\overline{F}})$$

is klt. So

$$(\overline{F}, \Gamma_{\overline{F}} + I_{\overline{F}} + tL_{\overline{F}})$$
is $\epsilon'$-lc by [8, Lemma 2.3] because
\[
\Gamma_{\mathcal{F}} + I_{\mathcal{F}} + tL_{\mathcal{F}} = \left(\frac{l-1}{l}\right) \Gamma_{\mathcal{F}} + \frac{1}{t} (\Gamma_{\mathcal{F}} + tI_{\mathcal{F}} + tL_{\mathcal{F}})
\]
and because \( (\frac{l-1}{l})\epsilon > \epsilon' \). This then implies that
\[
(\mathcal{F}, \tilde{C}_{\mathcal{F}} + I_{\mathcal{F}} + tL_{\mathcal{F}})
\]
is sub-$\epsilon'$-lc as $\tilde{C}_{\mathcal{F}} \leq \Gamma_{\mathcal{F}}$ by Step 4. Therefore, by Step 5,
\[
(\mathcal{F}, \tilde{C}_{\mathcal{F}} + I_{\mathcal{F}} + tL_{\mathcal{F}})
\]
is also sub-$\epsilon'$-lc. In other words,
\[
(\mathcal{F}, C_{\mathcal{F}} + \Theta_{\mathcal{F}} + P_{\mathcal{F}} + tL_{\mathcal{F}})
\]
is $\epsilon'$-lc because
\[
\tilde{C}_{\mathcal{F}} + I_{\mathcal{F}} + tL_{\mathcal{F}} = C_{\mathcal{F}} + \Theta_{\mathcal{F}} + P_{\mathcal{F}} + tL_{\mathcal{F}}
\]
as we saw in Step 5.

It is clear from the proof that if we replace the nef condition of $K_X + C + \Delta$ with $K_X + C + S + \Delta$ being nef, then we can deduce that
\[
(\mathcal{F}, C_{\mathcal{F}} + S_{\mathcal{F}} + \Theta_{\mathcal{F}} + P_{\mathcal{F}} + tL_{\mathcal{F}})
\]
is $\epsilon'$-lc.

### 3.4. Descent of divisor coefficients along fibrations.

The adjunction formula in the next lemma is adjunction for fibrations, cf. [28][6][9, 3.4].

**Lemma 3.5.** Let $\epsilon$ be a positive real number. Then there is a natural number $l$ depending only on $\epsilon$ satisfying the following. Let $(X,B)$ be a klt pair and $f: X \to Z$ be a contraction. Assume that

- $K_X + B \sim_{\mathbb{R}} 0/Z$,
- $K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$ is the adjunction formula for fibrations,
- $E = f^*L$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ on $Z$,
- $\Phi \subset \mathbb{R}$ is a subset closed under multiplication with elements of $\mathbb{N}$,
- the coefficients of $E$ are in $\Phi$, and
- any component $D$ of $L$ has coefficient $\leq 1 - \epsilon$ in $B_Z$.

Then the coefficients of $lL$ are in $\Phi$.

**Proof.** We can assume $L \neq 0$. Let $D$ be a component of $L$ and let $u$ be its coefficient in $L$. Shrinking $Z$ around the generic of $D$ we can assume that $D$ is the only component of $L$. Let $t$ be the lc threshold of $f^*D$ with respect to $(X, B)$ over the generic point of $D$. By definition of adjunction for fibrations, the coefficient of $D$ in $B_Z$ is $1 - t$. Since $1 - t \leq 1 - \epsilon$ by assumption, $t \geq \epsilon$. So $(X, B + \epsilon f^*D)$ is lc over the generic point of $D$. Shrinking $Z$ around the generic point of $D$ again we can assume that $(X, B + \epsilon f^*D)$ is lc everywhere and that $f^*D$ is an integral divisor. Thus the coefficients of $f^*D$ are bounded from above by $v := \lceil \frac{1}{\epsilon} \rceil$. Now by assumption, $E = f^*L = uf^*D$ has coefficients in $\Phi$. If $C$ is a component of $E$ with coefficient $e$ and if $h$ is its coefficient in $f^*D$, then $e = uh \in \Phi$ where $h \leq v$. Therefore, letting $l = v!$ we see that $ul \in \Phi$ as $ul$ is a multiple of $e$ and $\Phi$ is closed under multiplication with elements of $\mathbb{N}$.

\[\square\]
3.6. **Descent of divisor coefficients to non-klt centres.** Next we will show that coefficients of divisors to non-klt centres behave well under suitable conditions, as in the next proposition. The first half of the proof of the proposition is similar to that of [9, Proposition 3.15] but the second half is very different. The proposition is one of the key elements which will allow us to consider birational boundedness of divisors in a vastly more general setting than that considered in [9] which treated only anti-canonical divisors of Fano varieties.

**Proposition 3.7.** Let $d$ be a natural number and $\epsilon$ be a positive real number. Then there is a natural number $q$ depending only on $d, \epsilon$ satisfying the following. Let $(X, B), \Delta, G, \Theta_F, P_F$ be as in 3.1. Assume in addition that

- $X$ is of dimension $d$,
- $P_F$ is big and for any choice of $P_F \geq 0$ in its $\mathbb{R}$-linear equivalence class the pair $(F, \Theta_F + P_F)$ is $\epsilon$-lc,
- $\Phi \subset \mathbb{R}$ is a subset closed under addition,
- $E$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$ with coefficients in $\Phi$, and
- $G \not\subseteq \text{Supp } E$.

Then $qE\vert_F$ has coefficients in $\Phi$.

**Proof.** Step 1. In this step we show $G$ is an isolated non-klt centre of $(X, B + \Delta)$. We use the notation of 3.1. Remember that $(X, B + \Delta)$ is lc near the generic point of $G$. Also recall that $\Gamma + N = B + \Delta$,

$$K_Y + \Gamma_Y + N_Y = \pi^*(K_X + \Gamma + N),$$

and $[\Gamma_Y]$ has a unique component $S$ mapping onto $G$ where $\pi$ denotes $Y \rightarrow X$.

Assume $G$ is not an isolated non-klt centre. Then some non-klt centre $H \neq G$ of $(X, \Gamma + N)$ intersects $G$. Applying [9, Lemma 3.14(2)], we can choose $P_F \geq 0$ in its $\mathbb{R}$-linear equivalence class such that the pair $(F, \Theta_F + P_F)$ is not $\epsilon$-lc, a contradiction. Therefore, $(Y, \Gamma_Y + N_Y)$ is plt near $S$ because it is plt over the generic point of $G$ and any non-klt centre other than $S$ would map to a non-klt centre on $X$ other than $G$. In particular, no component of $[\Gamma_Y] - S$ intersects $S$ which implies that any prime exceptional divisor $J \neq S$ of $\pi$ is disjoint from $S$ because all the prime exceptional divisors of $\pi$ are components of $[\Gamma_Y]$.

Step 2. In this step we show that there is a natural number $p$ depending only on $\epsilon$ such that $pE_S$ has coefficients in $\Phi$ where $E_S = E_Y\vert_S$ and $E_Y = \pi^*E$. Note that $E_S$ and $E\vert_F$ are well-defined as $\mathbb{Q}$-Weil divisors as $\text{Supp } E$ does not contain $G$. Moreover, $S$ is not a component of $E_Y$ and near $S$, $E_Y$ is just the birational transform of $E$ because no exceptional divisor $J \neq S$ of $\pi$ intersects $S$, by Step 1. Thus the coefficients of $E_Y$ near $S$ belong to $\Phi$.

Let $V$ be a prime divisor on $S$. If $V$ is horizontal over $F$, then $V$ is not a component of $E_S$ because $E_S$ is vertical over $F$. Assume $V$ is vertical over $F$. We claim that if $T \neq S$ is any prime divisor on $Y$, then $pT$ is Cartier near the generic point of $V$, for some natural number $p$ depending only on $\epsilon$. Let $l$ be the Cartier index of $K_Y + S$ near the generic point of $V$. Then $\mu_V(\Gamma_S + N_S) \geq 1 - \frac{1}{l}$ and the Cartier index of $T$ near the generic point of $V$ divides $l$, by [44, Proposition 3.9]. Assume $\frac{1}{l} < \epsilon$. Then $\mu_V(\Gamma_S + N_S) > 1 - \epsilon$, hence $(S, \Gamma_S + N_S)$ is not $\epsilon$-lc along $V$. But then since $V$ is vertical over $F$, by [9, Lemma 3.14(1)], we can choose $P_F \geq 0$ so that $(F, \Theta_F + P_F)$
is not $\epsilon$-lc, a contradiction. Therefore, $\frac{1}{\epsilon} \geq \gamma$, so $\frac{1}{\epsilon} \geq l$. Now let $p = \lfloor\frac{1}{\epsilon}\rfloor!$. Then $pT$ is Cartier near the generic point of $V$ as claimed.

Let $E_1, \ldots, E_r$ be the irreducible components of $E_Y$ which intersect $S$ and let $e_1, \ldots, e_r$ be their coefficients. Then

$$\mu_V(pE_S) = \mu_V \left( \sum p e_i E_i | S \right) = \sum e_i \left( \mu_V pE_i | S \right).$$

Since $\mu_V pE_i | S$ are non-negative integers, since $e_i \in \Phi$, and since $\Phi$ is closed under addition, we see that $\mu_V(pE_S) \in \Phi$.

**Step 3.** In this step we consider adjunction for $(S, \Gamma_S + N_S)$ over $F$. Recall that $S \to F$ is denoted by $h$ which is a contraction. Since $(Y, \Gamma_Y + N_Y)$ is plt near $S$, $(S, \Gamma_S + N_S)$ is klt. As $K_S + \Gamma_S + N_S \sim_\mathbb{R} 0/F$, we have the adjunction formula

$$K_S + \Gamma_S + N_S \sim_\mathbb{R} h^*(K_F + \Omega_S + M_F)$$

where $\Omega_S$ is the discriminant divisor and $M_F$ is the moduli divisor which is pseudo-effective (cf. [9, 3.4]). Recall from 3.1 that $\Theta_F$ is defined by taking lc thresholds with respect to some boundary $\Sigma_S \leq \Gamma_S + N_S$. Then $\Theta_F \leq \Omega_F$ by definition of adjunction.

We claim that the coefficients of $\Omega_F$ do not exceed $1 - \epsilon$. Assume not. Pick a small real number $t > 0$ such that some coefficient of

$$\Xi_F := \Theta_F + (1 - t)(\Omega_F - \Theta_F) = (1 - t)\Omega_F + t\Theta_F$$

exceeds $1 - \epsilon$. By construction,

$$K_F + \Omega_F + M_F \sim_\mathbb{R} (K_X + \Gamma + N)|_F = (K_X + B + \Delta)|_F \sim_\mathbb{R} K_F + \Theta_F + P_F,$$

hence

$$\Omega_F + M_F \sim_\mathbb{R} \Theta_F + P_F.$$ 

Since $M_F$ is pseudo-effective and $P_F$ is big, we can find

$$0 \leq R_F \sim_\mathbb{R} (1 - t)M_F + tP_F.$$

Then

$$\Xi_F + R_F \sim_\mathbb{R} (1 - t)\Omega_F + t\Theta_F + (1 - t)M_F + tP_F = (1 - t)(\Omega_F + M_F) + t(\Theta_F + P_F) \sim_\mathbb{R} \Theta_F + P_F$$

which in particular means $K_F + \Xi_F + R_F$ is $\mathbb{R}$-Cartier.

Now $(F, \Xi_F + R_F)$ is not $\epsilon$-lc by our choice of $\Xi_F$. On the other hand, since

$$\Xi_F - \Theta_F = (1 - t)(\Omega_F - \Theta_F) \geq 0,$$

choosing

$$0 \leq P_F := \Xi_F - \Theta_F + R_F$$

we see that

$$(F, \Theta_F + P_F) = (F, \Xi_F + R_F)$$

is not $\epsilon$-lc, a contradiction. This proves the claim that the coefficients of $\Omega_F$ do not exceed $1 - \epsilon$.

**Step 4.** In this step we finish the proof. By Step 2, the coefficients of $pE_S$ are in $\Phi$. Moreover, by construction,

$$pE_S = h^*(pE|_F).$$

In addition, since $\Phi$ is closed under addition, it is closed under multiplication with elements of $\mathbb{N}$. Thus applying Lemma 3.5 to $(S, \Gamma_S + N_S) \to F$ and $pE_S$, there is a
natural number $l$ depending only on $\epsilon$ such that the coefficients of $lpE|_F$ are in $\Phi$. Now let $q = lp$.

$\square$

In practice examples of $\Phi$ include $\Phi = \mathbb{Z}$ and $\Phi = \{ r \in \mathbb{R} \mid r \geq \delta \}$ for some $\delta$.

4. Birational boundedness on $\epsilon$-lc varieties

In this section we treat birational boundedness of linear systems of nef and big divisors on $\epsilon$-lc varieties which serves as the basis for the subsequent sections.

4.1. Main result. The following theorem is a more general form of 1.1 and the main result of this section.

**Theorem 4.2.** Let $d$ be a natural number and $\epsilon, \delta$ be positive real numbers. Then there exists a natural number $m$ depending only on $d, \epsilon, \delta$ satisfying the following. Assume
- $X$ is a projective $\epsilon$-lc variety of dimension $d$,
- $N$ is a nef and big $\mathbb{R}$-divisor on $X$,
- $N - K_X$ is pseudo-effective, and
- $N = E + R$ where $E$ is integral and pseudo-effective and $R \geq 0$ with coefficients in $\{0\} \cup [\delta, \infty)$.

Then $|m'N + L|$ and $|K_X + m'N + L|$ define birational maps for any integral pseudo-effective divisor $L$ and for any natural number $m' \geq m$.

Special cases of the theorem are when $R = 0$ which is the statement of 1.1, and when $E = 0$ in which case $N$ is effective with coefficients $\geq \delta$. The theorem does not hold if we drop the pseudo-effectivity condition of $E$: indeed, one can easily find counter-examples by considering $\epsilon$-lc Fano varieties $X$, $\mathbb{Q}$-divisors $0 \leq B \sim_{\mathbb{Q}} -K_X$ with coefficients $\geq \delta$ and then letting $E = K_X$ and $R = B + tB$ with $t$ arbitrarily small.

The theorem implies a more general form of 1.2.

**Corollary 4.3.** Let $d$ be a natural number and $\epsilon, \delta$ be positive real numbers. Then there exist natural numbers $m, l$ depending only on $d, \epsilon, \delta$ satisfying the following. Assume that
- $X$ is a projective $\epsilon$-lc variety of dimension $d$,
- $N$ is a nef and big $\mathbb{R}$-divisor on $X$, and
- $N = E + R$ where $E$ is integral and pseudo-effective and $R \geq 0$ with coefficients in $\{0\} \cup [\delta, \infty)$.

Then $|m'K_X + l'N + L|$ defines a birational map for any natural numbers $m' \geq m$ and $l' \geq ml$ and any pseudo-effective integral divisor $L$.

4.4. Birational or volume boundedness. In this and the next subsection we aim to derive a special case of Theorem 4.2, where we add the condition that $K_X + N$ is also big, from the theorem in lower dimension. Later in the section we will remove this condition using the BAB [8, Theorem 1.1] in dimension $d$. We begin with a birationality statement in which we either bound the birationality index or the volume of the relevant divisor.
Proposition 4.5. Let $d$ be a natural number and $\epsilon, \delta$ be positive real numbers. Assume that Theorem 4.2 holds in dimension $\leq d - 1$. Then there exists a natural number $v$ depending only on $d, \epsilon, \delta$ satisfying the following. Assume

- $X$ is a projective $\epsilon$-lc variety of dimension $d$,
- $N$ is a nef and big $\mathbb{Q}$-divisor on $X$,
- $N - K_X$ and $N + K_X$ are big, and
- $N = E + R$ where $E$ is integral and pseudo-effective and $R \geq 0$ with coefficients in $\{0\} \cup [\delta, \infty)$.

If $m$ is the smallest natural number such that $\lceil mN \rceil$ defines a birational map, then either $m \leq v$ or $\text{vol}(mN) \leq v$.

Proof. We follow the proof of [9, Proposition 4.8].

Step 1. In this step we setup basic notation and make some reductions. If the proposition does not hold, then there is a sequence $X_i, N_i$ of varieties and divisors as in the proposition such that if $m_i$ is the smallest natural number so that $|m_iN_i|$ defines a birational map, then both the numbers $m_i$ and the volumes $\text{vol}(m_iN_i)$ form increasing sequences approaching $\infty$. Let $n_i \in \mathbb{N}$ be a natural number so that $\text{vol}(n_iN_i) > (2d)^d$. Taking a $\mathbb{Q}$-factorialisation we can assume $X_i$ are $\mathbb{Q}$-factorial.

Assume that $\frac{n_i}{m_i}$ is always (i.e. for any choice of $n_i$ as above) bounded from above. Then letting $n_i \in \mathbb{N}$ be the smallest number so that $\text{vol}(n_iN_i) > (2d)^d$, either $n_i = 1$ which immediately implies $m_i$ is bounded from above, or $n_i > 1$ in which case we have

$$\text{vol}(m_iN_i) = (\frac{m_i}{n_i} - 1)^d \text{vol}((n_i - 1)N_i) \leq (\frac{m_i}{n_i} - 1)^d(2d)^d$$

so $\text{vol}(m_iN_i)$ is bounded from above.

Therefore, it is enough to show that $\frac{n_i}{m_i}$ is always bounded from above where $n_i$ is arbitrary as in the first paragraph. Assume otherwise. We can then assume that the numbers $\frac{n_i}{m_i}$ form an increasing sequence approaching $\infty$. We will derive a contradiction.

Step 2. In this step we fix $i$ and create a covering family of non-klt centres on $X_i$. Replacing $n_i$ with $n_i + 1$ we can assume $\text{vol}((n_i - 1)N_i) > (2d)^d$. Thus applying 2.15(2), there is a covering family of subvarieties of $X_i$ such that for any two general closed points $x_i, y_i \in X_i$ we can choose a member $G_i$ of the family and choose a $\mathbb{Q}$-divisor

$$0 \leq \tilde{\Delta}_i \sim_{\mathbb{Q}} (n_i - 1)N_i$$

so that $(X_i, \tilde{\Delta}_i)$ is lc near $x_i$ with a unique non-klt place whose centre contains $x_i$, that centre is $G_i$, and $(X_i, \tilde{\Delta}_i)$ is not klt near $y_i$.

On the other hand, by assumption $N_i - K_{X_i}$ is big. Fix some

$$0 \leq Q_i \sim_{\mathbb{Q}} N_i - K_{X_i},$$

independent of the choice of $x_i, y_i$; we can assume $x_i, y_i$ are not contained in $\text{Supp} Q_i$ as $x_i, y_i$ are general. Then we get

$$0 \leq \Delta_i := \tilde{\Delta}_i + Q_i \sim_{\mathbb{Q}} n_iN_i - K_{X_i}$$

such that $(X_i, \Delta_i)$ is lc near $x_i$ with a unique non-klt place whose centre contains $x_i$, that centre is $G_i$, and $(X_i, \Delta_i)$ is not klt near $y_i$.

Since $x_i, y_i$ are general, we can assume $G_i$ is a general member of the above covering family of subvarieties. Recall from 2.14 that this means that the family is given
by finitely many morphisms $V^j \to T^j$ of projective varieties with accompanying surjective morphisms $V^j \to X$ and that each $G_i$ is a general fibre of one of the morphisms $V^j \to T^j$. Moreover, we can assume the points of $T^j$ corresponding to such $G_i$ are dense in $T^j$.

Let

$$d_i := \max\{\dim V^j - \dim T^j\}.$$ 

Assume $d_i = 0$, that is, $\dim G_i = 0$ for all the $G_i$. Then $n_i N_i - K_{X_j}$ is potentially birational. Let $r \in \mathbb{N}$ be the smallest number such that $r \delta \geq 1$. By assumption, $N_i = E_i + R_i$ with $E_i$ integral and pseudo-effective and $R_i \geq 0$ having non-zero coefficients $\geq \delta$. Thus the fractional part of $(n_i + r)N_i - K_{X_j}$, say $R'_i$, is supported in $R_i$ and the coefficients of $r R_i$ are $\geq 1$. So

$$[(n_i + r)N_i - K_{X_j}] = (n_i + r)N_i - K_{X_j} - R'_i = n_i N_i - K_{X_j} + r E_i + r R_i - R'_i$$

where $rR_i - R'_i \geq 0$, hence $[(n_i + r)N_i - K_{X_j}]$ is potentially birational. Therefore,

$$|[n_i + r)N_i]| = |K_{X_j} + [(n_i + r)N_i - K_{X_j}]|$$

defines a birational map by [24, Lemma 2.3.4] which in turn implies $|(n_i + r)N_i|$ defines a birational map; this means $m_i \leq n_i + r$ giving a contradiction as we can assume $m_i/n_i \gg 0$. Thus we can assume $d_i > 0$, hence $\dim G_i > 0$ for all the $G_i$ appearing as general fibres of $V^j \to T^j$ for some $j$.

**Step 3.** In this step we find a sub-family of the $G_i$ so that $\text{vol}(m_i N_i|_{G_i})$ is bounded from above, independent of $i$. For each $i$ let $l_i \in \mathbb{N}$ be the smallest number so that $\text{vol}(l_i N_i|_{G_i}) > d^d$ for all the $G_i$ with positive dimension. Assume $\frac{l_i}{n_i}$ is bounded from above by some natural number $a$. Then for each $i$ and each positive dimensional $G_i$, we have

$$d^d < \text{vol}(l_i N_i|_{G_i}) \leq \text{vol}(a N_i|_{G_i}).$$

Thus applying 2.15(3) and replacing $n_i$ with $3a n_i$ we can modify $\Delta_i, G_i$, and possibly switch $x_i, y_i$, so that we decrease the number $d_i$. Repeating the process we either get to the situation in which $d_i = 0$ which yields a contradiction as in Step 2, or we can assume $\frac{l_i}{n_i}$ is an increasing sequence approaching $\infty$.

On the other hand, if $\frac{l_i}{n_i}$ is not bounded from above, then we can assume $\frac{l_i}{n_i}$ is an increasing sequence approaching $\infty$, hence we can replace $n_i$ with $l_i$ (by adding appropriately to $\Delta_i$) in which case $\frac{l_i}{n_i}$ is bounded so we can argue as in the previous paragraph by decreasing $d_i$. So we can assume $\frac{l_i}{n_i}$ is bounded from above.

Now for each $i$, there is $j$ so that if $G_i$ is a general fibre of $V^j \to T^j$, then $G_i$ is positive dimensional and

$$\text{vol}((l_i - 1)N_i|_{G_i}) \leq d^d,$$

by definition of $l_i$. In order to get a contradiction in the following steps it suffices to consider only such $G_i$. From now on when we mention $G_i$ we assume it is positive dimensional and that it satisfies the inequality just stated. In particular,

$$\text{vol}(m_i N_i|_{G_i}) = \left(\frac{m_i}{l_i - 1}\right)^{\dim G_i} \text{vol}((l_i - 1)N_i|_{G_i}) \leq \left(\frac{m_i}{l_i - 1}\right)^d d^d$$

is bounded from above, so $\text{vol}(m_i N_i|_{G_i}) < v$ for some natural number $v$ independent of $i$. 

Step 4. Let $F_i$ be the normalisation of $G_i$. In this step we fix $i$ and apply adjunction by restricting to $F_i$. Since $G_i$ is a general member of a covering family, $X_i$ is smooth near the generic point of $G_i$. By 3.1 (taking $B = 0$ and $\Delta = \Delta_i$), we can write

$$K_{F_i} + \Delta_{F_i} := K_{F_i} + \Theta_{F_i} + P_{F_i} \sim_{\mathbb{R}} (K_{X_i} + \Delta_i)|_{F_i} \sim_{\mathbb{R}} n_i N_i|_{F_i}$$

where $\Theta_{F_i} \geq 0$ with coefficients in some fixed DCC set $\Psi$ independent of $i$, and $P_{F_i}$ is pseudo-effective. Since $x_i$ is a general point, we can pick $0 \leq \tilde{N}_i \sim_{\mathbb{Q}} N_i$ not containing $x_i$. By definition of $\Theta_{F_i}$, adding $\tilde{N}_i$ to $\Delta_i$ does not change $\Theta_{F_i}$, but changes $P_{F_i}$ to $P_{F_i} + \tilde{N}_i|_{F_i}$. Thus replacing $n_i$ with $n_i + 1$ and changing $P_{F_i}$ up to $\mathbb{R}$-linear equivalence we can assume $P_{F_i}$ is effective and big.

On the other hand, by [23, Theorem 4.2] and [9, Lemma 3.12], we can write

$$K_{F_i} + \Lambda_{F_i} = K_{X_i}|_{F_i}$$

where $(F_i, \Lambda_{F_i})$ is sub-$\epsilon$-lc and $\Lambda_{F_i} \leq \Theta_{F_i}$.

Step 5. In this step we reduce to the situation in which $(F_i, \Delta_{F_i})$ is $\epsilon'$-lc for every $i$ where $\epsilon' = \frac{\delta}{2}$. Pick $0 \leq M_i \sim m_i N_i$, independent of $x_i, y_i$. Since $G_i$ is general, we can assume $\text{Supp} \ M_i$ does not contain $G_i$. Let $M_{F_i} := M_i|_{F_i}$. Let $t$ be the number given by Proposition 3.3 for the data $d, v, \epsilon, \epsilon'$. Let $r \in \mathbb{N}$ be the smallest number such that $r \geq \frac{1}{3}$. We can assume that $\frac{n_i + 1 + r}{m_i} < t$ for every $i$. Let $C_i = 0$ and let $S_i = r R_i$. We want to apply 3.3 to

$$X_i, C_i, M_i, S_i, \Delta_i, G_i, F_i, \Theta_{F_i}, P_{F_i}$$

where $P_{F_i}$ can be any effective divisor in its $\mathbb{R}$-linear equivalence class. Conditions (1)-(4) of 3.3 are obviously satisfied. Condition (5) is satisfied because the non-zero coefficients of $S_i$ are $\geq 1$ and similarly the non-zero coefficients of $M_i + S_i$ are $\geq 1$ as the fractional part of $M_i$ is supported in $\text{Supp} \ R_i = \text{Supp} \ S_i$; the latter claim follows from $M_i \sim m_i E_i + m_i R_i$. Conditions (6)-(9) are satisfied by construction, and condition (10) is ensured by the end of Step 3. Condition (11) follows from $K_{X_i} + \Delta_i \sim_{\mathbb{Q}} n_i N_i$, and (12) follows from bigness of

$$M_i - (K_{X_i} + S_i + \Delta_i) \sim_{\mathbb{Q}} M_i - (K_{X_i} + \Delta_i) - S_i$$

$$\sim_{\mathbb{Q}} m_i N_i - n_i N_i - S_i = (m_i - n_i)N_i - r R_i = (m_i - n_i - r)N_i + r E_i$$

as we can assume $m_i > n_i + r$. Finally, condition (13) is satisfied because from $\frac{n_i + 1 + r}{m_i} < t$, we have

$$tm_i - n_i - 1 - r > 0,$$

hence

$$tM_i - \Delta_i - S_i \sim_{\mathbb{R}} tm_i N_i - n_i N_i + K_{X_i} - S_i = (tm_i - n_i - 1 - r)N_i + N_i + K_{X_i} + r N_i - S_i$$

is big as $N_i + K_{X_i}$ is big by assumption and $r N_i - S_i = R E_i$ is pseudo-effective.

Now by Proposition 3.3, $(F_i, \Delta_{F_i} + t M_{F_i})$ is $\epsilon'$-lc for every $i$ which in particular means $(F_i, \Delta_{F_i})$ is $\epsilon'$-lc.

Step 6. In this step we finish the proof. Let $E_{F_i} = E_i|_{F_i}$, $R_{F_i} = R_i|_{F_i}$, and $N_{F_i} = N_i|_{F_i}$ which are all well-defined as $\mathbb{Q}$-Weil divisors as we can assume $G_i$ is not contained in $\text{Supp} \ E_i + \text{Supp} \ R_i$. Applying Proposition 3.7, by taking $\Phi = \mathbb{Z}$, there is a natural number $q$ depending only on $d, \epsilon'$ such that $q E_{F_i}$ is an integral divisor. Applying the proposition again, this time taking $\Phi = \mathbb{R}^\geq \delta$, and replacing $q$ we can assume that the non-zero coefficients of $q R_{F_i}$ are $\geq \delta$. Thus $q N_{F_i} = q E_{F_i} + q R_{F_i}$ is the
sum of a pseudo-effective integral divisor and an effective \( \mathbb{Q} \)-divisor whose non-zero coefficients are \( \geq \delta \).

On the other hand, by construction,

\[
q_n i N_F - K_F = (q - 1)n_i N_F + n_i N_F - K_F \sim_{\mathbb{R}} (q - 1)n_i N_F + \Delta_F
\]

is big. Thus since we are assuming Theorem 4.2 in dimension \( \leq d - 1 \), we deduce that there is a natural number \( p \) depending only on \( d, \epsilon', \delta \) such that \( |pqn_i N_F| \) defines a birational map (note that \( K_F \) may not be \( \mathbb{Q} \)-Cartier but we can apply the theorem to a small \( \mathbb{Q} \)-factorialisation of \( F_i \)). Then \( \text{vol}(pqn_i N_F) \geq 1 \), hence

\[
\text{vol}(2dqn_i N_F|_{G_i}) = \text{vol}(2dpqn_i N_F) \geq (2d)^d.
\]

But then by Step 3, we have \( l_i - 1 < 2dpqn_i \) which is a contradiction since we assumed \( \frac{l_i}{m} \) is an unbounded sequence.

4.6. Birational boundedness. In this subsection we strengthen 4.5 by showing that we can actually take \( m \) itself to be bounded.

**Lemma 4.7.** Let \( d, v \) be natural numbers, \( \epsilon \) be a positive real number, and \( \Phi \subset [0, 1] \) be a finite set of rational numbers. Assume Theorem 4.2 holds in dimension \( \leq d - 1 \). Assume that

- \( (X, B) \) is a projective \( \epsilon \)-lc pair of dimension \( d \),
- the coefficients of \( B \) are in \( \Phi \),
- \( K_X + B \) is ample with \( \text{vol}(K_X + B) \leq v \), and
- \( 2K_X + B \) is big.

Then such \( (X, \text{Supp } B) \) form a bounded family.

**Proof.** There is a natural number \( l > 1 \) depending only on \( \Phi \) such that \( N := l(K_X + B) \) is integral. Moreover,

\[
N - K_X = (l - 1)(K_X + B) + B
\]

and

\[
N + K_X = (l - 1)(K_X + B) + 2K_X + B
\]

are both big. Now applying Proposition 4.5 to a \( \mathbb{Q} \)-factorialisation of \( X \) we deduce that there is a natural number \( v' \) depending only on \( d, \epsilon \) such that if \( m \in \mathbb{N} \) is the smallest number such that \( |mN| \) defines a birational map, then either \( m \leq v' \) or \( \text{vol}(mN) \leq v' \). If \( m \leq v' \), then

\[
\text{vol}(mN) = \text{vol}(ml(K_X + B)) \leq (ml)^d v \leq (v')^d v,
\]

hence in any case \( \text{vol}(mN) \) is bounded from above.

Pick \( 0 \leq M \sim mN \). Then, by [9, Proposition 4.4], \( (X, \text{Supp } (B + M)) \) is birationally bounded. Finally apply [23, Theorem 1.6].

**Proposition 4.8.** Let \( d \) be a natural number and \( \epsilon, \delta \) be positive real numbers. Assume that Theorem 4.2 holds in dimension \( \leq d - 1 \). Then there exists a natural number \( m \) depending only on \( d, \epsilon, \delta \) satisfying the following. Assume

- \( X \) is a projective \( \epsilon \)-lc variety of dimension \( d \),
- \( N \) is a nef and big \( \mathbb{Q} \)-divisor on \( X \),
- \( N - K_X \) and \( N + K_X \) are big, and
\( N = E + R \) where \( E \) is integral and pseudo-effective and \( R \geq 0 \) with coefficients in \( \{0\} \cup [\delta, \infty) \).

Then \(|mN|\) defines a birational map.

**Proof. Step 1.** In this step we apply 4.5 and introduce some notation. Taking a small \( \mathbb{Q} \)-factorialisation, we will assume \( X \) is \( \mathbb{Q} \)-factorial. Let \( v \in \mathbb{N} \) be the number given by Proposition 4.5 for the data \( d, \epsilon, \delta \). Then there is \( m \in \mathbb{N} \) such that \(|mN|\) defines a birational map and either \( m \leq v \) or \( \text{vol}(mN) \leq v \). In the former case we are done by replacing \( m \) with \( v \!), hence we can assume the latter holds and that \( m \) is sufficiently large.

Since \( N - K_X \) is big, we can find

\[
0 \leq \Delta \sim_\mathbb{Q} N - K_X.
\]

Pick \( 0 \leq M \sim mN \).

**Step 2.** In this step we show that \((X, \text{Supp} M)\) is birationally bounded. Let \( B = \frac{1}{\delta} R \). Then

\[
M - (K_X + B) \sim mN - K_X - \frac{1}{\delta} R = (m - \frac{1}{\delta} - 1)N + N - K_X + \frac{1}{\delta} N - \frac{1}{\delta} R
\]

is big because \( m - \frac{1}{\delta} - 1 \geq 0 \), \( N - K_X \) is big, and \( \frac{1}{\delta} N - \frac{1}{\delta} R = \frac{1}{\delta} E \) is pseudo-effective. On the other hand, since \( M \sim mE + mR \), the fractional part of \( M \) is supported in \( B \), so for any component \( D \) of \( M \) we have \( \mu_D(B + M) \geq 1 \). Therefore, by [9, Proposition 4.4], there exist a bounded set of couples \( P \) and a natural number \( c \) depending only on \( d, v \) such that we can find a projective log smooth couple \((\overline{X}, \Sigma) \in P\) and a birational map \( \overline{X} \dashrightarrow X \) such that

- \( \text{Supp} \Sigma \) contains the exceptional divisors of \( \overline{X} \dashrightarrow X \) and the birational transform of \( \text{Supp}(B + M) \);
- if \( \phi : X' \rightarrow X \) and \( \psi : X' \rightarrow \overline{X} \) is a common resolution and \( \overline{M} \) is the pushdown of \( M' := M|_{X'} \), then each coefficient of \( \overline{M} \) is at most \( c \);
- we can choose \( X' \) so that \( M' \sim A' + R' \) where \( A' \) is big, \( |A'| \) is base point free, \( R' \geq 0 \), and \( A' \sim 0/\overline{X} \).

**Step 3.** In this step we want to show that there is a positive rational number \( t \) depending only on \( P, c, \epsilon \) such that we can assume \((X, tM)\) is \( \frac{\epsilon}{2} \)-lc. We do this by finding \( t \) so that \((X, \Delta + tM)\) is \( \frac{\epsilon}{2} \)-lc. For now let \( t \) be a positive rational number. Then

\[
K_X + \Delta + tM \sim_\mathbb{Q} N + tM
\]

is nef and big. Let \( K_{\overline{X}} + \overline{\Delta} \) be the crepant pullback of \( K_X \) to \( \overline{X} \). Since \( X \) has \( \epsilon \)-lc singularities, the coefficients of \( \overline{\Delta} \) are at most \( 1 - \epsilon \), and its support is contained in \( \Sigma \). On the other hand, since \( N + K_X \) is assumed big, we can find

\[
0 \leq C \sim_\mathbb{Q} N + K_X.
\]

Note that \( \Delta + C \sim_\mathbb{Q} 2N \). Then

\[
(\frac{2}{m} + t)M \sim_\mathbb{Q} 2N + tM \sim_\mathbb{Q} \Delta + C + tM.
\]
Now if $\mathcal{X} = \psi_*\phi^*\Delta$ and $\mathcal{C} = \psi_*\phi^*C$, then

$$\psi_*\phi^*2(\Delta + C + tM) = 2(\mathcal{X} + \mathcal{C} + t\mathcal{M}) \sim_{\mathbb{Q}} (\frac{4}{m} + 2t)\mathcal{M}.$$ 

By Step 2, $\text{Supp}\mathcal{M} \subseteq \mathcal{X}$ and $(\frac{4}{m} + 2t)\mathcal{M}$ has coefficients $\leq (\frac{4}{m} + 2t)c$. Therefore, applying [9, Proposition 4.2], by taking $m$ sufficiently large and taking a fixed $t$ sufficiently small, depending only on $P, c, \epsilon$, we can assume that

$$(\mathcal{X}, (1 - \epsilon)\mathcal{X} + 2\mathcal{X} + t\mathcal{M})$$

is klt. Then

$$(\mathcal{X}, (1 - \epsilon)\mathcal{X} + \mathcal{X} + t\mathcal{M})$$

is $\mathcal{X}$-lc as $(\mathcal{X}, (1 - \epsilon)\mathcal{X})$ is $\epsilon$-lc which in turn implies

$$(\mathcal{X}, \mathcal{X} + \mathcal{X} + t\mathcal{M})$$

is sub-$\mathcal{X}$-lc because $\mathcal{X} \leq (1 - \epsilon)\mathcal{X}$.

Now since $K_X + \Delta + tM$ is nef and since its crepant pullback to $\mathcal{X}$ is just

$$K_{\mathcal{X}} + \mathcal{X} + \mathcal{X} + t\mathcal{M},$$

we deduce that $(X, \Delta + tM)$ is $\ell$-lc because by the negativity lemma

$$\phi^*(K_X + \Delta + tM) \leq \psi^*(K_{\mathcal{X}} + \mathcal{X} + t\mathcal{M}).$$

Therefore, $(X, tM)$ is $\ell$-lc.

**Step 4.** In this step we finish the proof by applying Lemma 4.7. First we show $K_X + t\lfloor M \rfloor$ and $2K_X + t\lfloor M \rfloor$ are big. As noted above, the fractional part of $M$, say $P$, is supported in $R$. Thus $\frac{1}{\delta}R - P \geq 0$. Then since $m$ is sufficiently large and $t$ is fixed, and since $K_X + N$ is big, we can ensure that

$$K_X + t\lfloor M \rfloor \sim_{\mathbb{Q}} K_X + tM - tP \sim_{\mathbb{Q}} K_X + tmN - tP$$

$$\sim_{\mathbb{Q}} K_X + N + (tm - 1 - \frac{t}{\delta})N + \frac{t}{\delta}E + \frac{t}{\delta}R - tP$$

is big. Similar reasoning shows that $2K_X + t\lfloor M \rfloor$ is also big.

Now $(X, t\lfloor M \rfloor)$ has an lc model, say $(Y, t\lfloor M \rfloor)$. By construction, $(Y, t\lfloor M \rfloor)$ is $\ell$-lc, the coefficients of $t\lfloor M \rfloor$ are in a fixed finite set depending only on $t$, and

$$\text{vol}(K_Y + t\lfloor M \rfloor) = \text{vol}(K_X + t\lfloor M \rfloor) < \text{vol}(N + tM) = \text{vol}(\frac{1}{m} + t)M$$

is bounded from above as $\text{vol}(M) \leq v$ by Step 1. Also $2K_Y + t\lfloor M \rfloor$ is big. Therefore, by Lemma 4.7, such such $(Y, \text{Supp} \lfloor M \rfloor)$ form a bounded family. In particular, there is a very ample divisor $A_Y$ on $Y$ such that $A_Y^{d-1}$ is big. $\text{Supp} \lfloor M \rfloor$ is bounded from above (note that the intersection number $A_Y^{d-1} \cdot Q$ is well-defined for any $\mathbb{R}$-divisor even if $Q$ is not $\mathbb{R}$-Cartier). Since the coefficients of $\lfloor M \rfloor$ are bounded from above, $A_Y^{d-1}\lfloor M \rfloor$ is also bounded from above.

Now since

$$\lfloor M \rfloor \sim_{\mathbb{Q}} mNY - PY \sim_{\mathbb{Q}} (m - \frac{1}{\delta})NY + \frac{1}{\delta}E_Y + R_Y - PY,$$

where $\frac{1}{\delta}E_Y + \frac{1}{\delta}R_Y - PY$ is pseudo-effective, we see that $A_Y^{d-1} \cdot (m - \frac{1}{\delta})NY$ is bounded from above. However,

$$A_Y^{d-1} \cdot NY = A_Y^{d-1} \cdot (E_Y + R_Y) \geq \delta.$$
Therefore,
\[ A^{d-1}_Y \cdot (m - \frac{1}{\delta})N_Y \geq (m - \frac{1}{\delta})\delta \]
can get arbitrarily large, a contradiction.

\[ \square \]

4.9. Some remarks. (1) So far in this and the previous sections we have tried to avoid using results of [24][23][9][8] as much as possible because one of our goals (here and in the future) is to get new proofs of some of the main results of those papers. To be more precise, when trying to prove a statement in dimension \( d \) we have tried to minimise relying on results of those papers in dimension \( d \). Although we have used many of the technical auxiliary results and ideas of the first three papers but we have not used their main results in dimension \( d \) with the exception of [24, Theorem 1.8]: indeed we used [23, Theorem 1.6] in the proof of Lemma 4.7, which is a quick consequence of [24, Theorem 1.8].

(2) We will argue that if in Propositions 4.5 and 4.8 we assume \( R \neq 0 \), then we can modify their proofs so that we do not need Lemma 4.7 nor Theorem 4.2 in lower dimension. Indeed we can modify Step 6 of the proof of 4.5 as follows. Since \( R_i \) has non-zero coefficients \( \geq \delta \), adding \( \frac{1}{\delta}N \) to \( \Delta \) and using [9, Lemma 3.14(2)] we can ensure that \((F_i, \Delta F_i)\) is not \( \epsilon'\)-lc, contradicting Step 5.

In 4.8, we can modify Steps 3-4 of the proof as follows. By the previous paragraph, we can find \( m \) such that \( |mN| \) defines a birational map and either \( m \) or \( \text{vol}(mN) \) is bounded from above. Since \( R \neq 0 \) and since its coefficients are \( \geq \delta \), there is \( 0 \leq L \sim_Q \frac{3}{\delta}N = \frac{1}{\delta}N + \frac{2}{\delta}E + \frac{2}{\delta}R \)
such that some coefficient of \( L \) exceeds 1. On the other hand,

\[ L + \Delta + C + tM \sim_Q \left( \frac{3}{m\delta} + \frac{3}{m} + t \right)M, \]

so applying the arguments of Step 3 we can assume \((X, L + tM)\) is \( \frac{2}{\epsilon} \)-lc. This is a contradiction because \((X, L + tM)\) is not lc as some coefficient of \( L \) exceeds 1.

(3) Let \( X \) be an \( \epsilon \)-lc Fano variety of dimension \( d \), for \( \epsilon > 0 \), and let \( N = -2K_X \). Assuming Theorem 4.2 in lower dimension and applying Proposition 4.8, we deduce that there is a natural number \( m \) depending only on \( d, \epsilon \) such that \( | -mK_X | \) defines a birational map. So we get a new proof of [9, Theorem 1.2] which is one of the main results of that paper. The two proofs have obvious similarities but also quite different in some sense. The proof in [9] relies heavily on the theory of complements. Indeed the birational boundedness of \( | -mK_X | \) and the boundedness of complements are proved together in an inductive process.

(4) Now assume that \( X \) is a projective \( \epsilon \)-lc variety of dimension \( d \), for \( \epsilon > 0 \), with \( K_X \) ample and let \( N := 2K_X \). Assuming Theorem 4.2 in lower dimension and applying Proposition 4.8, we deduce that there is a natural number \( m \) depending only on \( d, \epsilon \) such that \( |mK_X| \) defines a birational map. This is a special case of [23, Theorem 1.3]. The proof here is not that different because in this special setting many of the complications of the current proof disappear and the proof simplifies to that of [23, Theorem 1.3] except that towards the end of the proof where we have \( m \) with \( |mK_X| \) birational and \( \text{vol}(mK_X) \) bounded and we want to show \( m \) is bounded, we use different arguments.

(5) Now assume \( X \) is a projective \( \epsilon \)-lc variety of dimension \( d \), for \( \epsilon > 0 \), with \( K_X \equiv 0 \), that is, \( X \) is Calabi-Yau. Assume \( N \) is a nef and big \( \mathbb{Q} \)-divisor on \( X \) such
that $N = E + R$ where $E$ is integral and pseudo-effective and $R \geq 0$ with coefficients in $\{0\} \cup [\delta, \infty)$, for some $\delta > 0$. Then assuming Theorem 4.2 in lower dimension and applying Proposition 4.8, we deduce that there is a natural number $m$ depending only on $d, \epsilon, \delta$ such that $|mN|$ defines a birational map. Note that by (2), if $R \neq 0$, then we do not need to assume Theorem 4.2 in lower dimension. Also note that we can replace the $\epsilon$-lc condition with klt but for this we need to use the global ACC result [23, Theorem 1.5].

4.10. **Pseudo-effective threshold of nef and big divisors.** The following lemma proves to be useful in many places in this paper. Its proof in dimension $d$ relies on the BAB [8, Theorem 1.1] in dimension $d$.

**Lemma 4.11.** Let $d$ be a natural number and $\epsilon, \delta$ be positive real numbers. Then there is a natural number $t$ depending only on $d, \epsilon, \delta$ satisfying the following. Assume

- $X$ is a $\epsilon$-lc variety of dimension $d$,
- $X \to Z$ is a contraction,
- $N$ is an $\mathbb{R}$-divisor on $X$ which is nef and big over $Z$, and
- $N = E + R$ where $E$ is integral and pseudo-effective over $Z$ and $R \geq 0$ with coefficients in $\{0\} \cup [\delta, \infty)$.

Then $K_X + tN$ is big over $Z$.

**Proof.** Taking a $\mathbb{Q}$-factorialisation we can assume $X$ is $\mathbb{Q}$-factorial. Let $t$ be the smallest non-negative real number such that $K_X + tN$ is pseudo-effective over $Z$. It is enough to show $t$ is bounded from above because $K_X + [(t + 1)]N$ is big over $Z$. In particular, we can assume $t > 0$.

Consider $(X, tN)$ as a generalised pair over $Z$ with nef part $tN$. Then the pair is generalised $\epsilon$-lc. Since $tN$ is big over $Z$, we can run an MMP over $K_X + tN$ which ends with a minimal model $X'$ on which $K_{X'} + tN'$ is nef and semi-ample over $Z$ [13, Lemma 4.4]; here $K_{X'} + tN'$ is the pushdown of $K_X + tN$. So $K_{X'} + tN'$ defines a contraction $X' \to V'/Z$ which is non-birational otherwise $K_X + tN$ would be big over $Z$ which means we can decrease $t$ keeping $K_X + tN$ pseudo-effective over $Z$, contradicting the definition of $t$ and the assumption $t > 0$.

Now since $tN'$ is big over $V'$ and $K_{X'} + tN' \equiv 0/V'$, $K_{X'}$ is not pseudo-effective over $V'$, hence we can run an MMP on $K_{X'}$ which ends with a Mori fibre space $X'' \to W''/V'$. Since $(X, tN)$ is generalised $\epsilon$-lc, $(X', tN')$ is generalised $\epsilon$-lc which in turn implies $(X'', tN'')$ is generalised $\epsilon$-lc because $K_{X''} + tN'' \equiv 0/V''$ (note that the nef parts of both $(X', tN')$ and $(X'', tN'')$ are pullbacks of $tN$ to some common resolution of $X, X', X''$). Thus $X''$ is $\epsilon$-lc because for each prime divisor $D$ over $X''$ we have

$$a(D, X'', 0) \geq a(D, X'', B'' + tN'').$$

Let $F''$ be a general fibre of $X'' \to W''$. Then $F''$ is an $\epsilon$-lc Fano variety. Restricting to $F''$ we get

$$K_{F''} + tN_{F''} = (K_{X''} + tN'')|_{F''} \equiv 0$$

where $N_{F''} = N''|_{F''}$. Moreover, $N_{F''} = E_{F''} + R_{F''}$ where $E_{F''} = E''|_{F''}$, $R_{F''} = R''|_{F''}$ and $E'', R''$ are the pushdowns of $E, R$. Then $E_{F''}$ is integral and the coefficients of $R_{F''}$ are in $\{0\} \cup [\delta, \infty)$; it is enough to argue that if $D''$ is a prime divisor on $X''$, then $D''|_{F''}$ is an integral divisor; indeed, the singular locus of $X''$, say $S''$, is of codimension $\geq 2$ and $F''$ being general, $S'' \cap F''$ has codimension $\geq 2$ in $F''$; then $D''|_{F''}$ is Cartier outside $S'' \cap F''$ which shows $D''|_{F''}$ is integral. On the other hand,
Since $E''$ is pseudo-effective over $W''$, $E_{F''}$ is pseudo-effective. Therefore, replacing $X, N, E, R$ with $F'', N_{F''}, E_{F''}, R_{F''}$ and replacing $Z$ with a point, we can assume that $X$ is an $\epsilon$-lc Fano variety and that $K_X + tN \equiv 0$.

Now by [8, Theorem 1.1], $X$ belongs to a bounded family of varieties. Thus there is a very ample divisor $A$ on $X$ with $-K_X \cdot A^{d-1}$ bounded from above. If $R \neq 0$, then $E \cdot A^{d-1} \geq 0$ and $R \cdot A^{d-1} \geq \delta$ because $A^{d-1}$ can be represented by a curve inside the smooth locus of $X$. But if $R = 0$, then $N \cdot A^{d-1} = E \cdot A^{d-1} \geq 1$. Letting $\lambda = \min\{1, \delta\}$, we then have

$$N \cdot A^{d-1} = (E + R) \cdot A^{d-1} \geq \lambda.$$  

Thus

$$t = \frac{-K_X \cdot A^{d-1}}{N \cdot A^{d-1}} \leq \frac{1}{\lambda}K_X \cdot A^{d-1}$$

is bounded from above. Finally note that $\lambda$ depends only on $\delta$ while $-K_X \cdot A^{d-1}$ depends only on $d, \epsilon$.

\[\square\]

4.12. Proofs of 4.2, 1.1, 4.3, 1.2, 1.4.

Proof. (of Theorem 4.2) We apply induction on dimension so assume the theorem holds in lower dimension. Taking a small $\mathbb{Q}$-factorialisation we can assume that $X$ is $\mathbb{Q}$-factorial. Replacing $N$ with $2N$ we can assume $N - K_X$ is big. By Lemma 4.11, there is a natural number $l$ depending only on $d, \epsilon, \delta$ such that $X, N - K_X$ are big. Letting $l$ we can assume that $l \geq \frac{1}{\delta}$. There is an $\mathbb{R}$-divisor $B \leq \frac{1}{\delta}R$ such that $M := K_X + B + 3lN$ is a $\mathbb{Q}$-divisor and $(X, B)$ is $\frac{\epsilon}{2}$. In particular, $(X, B + 3lN)$ is generalised $\frac{1}{\delta}$-lc where the nef part is $3lN$.

Now $[K_X + 2lN]$ is big because if $Q$ is the fractional part of $K_X + 2lN$, then

$$[K_X + 2lN] = K_X + 2lN - Q = K_X + lN + lE + lR - Q$$

where $lR - Q \geq 0$ as $Q$ is supported in Supp $R$ and $lR$ has non-zero coefficients $\geq 1$. Thus letting

$$F := [K_X + 2lN] + lE$$

and $S = B + Q + lR$

we see that $M = F + S$ where $F$ is big and integral and the non-zero coefficients of $S$ are $\geq 1$.

We can run an MMP on $M$ ending with a minimal model $X'$ [13, Lemma 4.4]. Then $(X', B' + 3lN')$ is generalised $\frac{\epsilon}{2}$-lc which implies that $X'$ is $\frac{\epsilon}{2}$-lc. Moreover, under our assumptions, $M'$ is a nef and big $\mathbb{Q}$-divisor, and $M' - K_{X'}$, and $M' + K_{X'}$ are big, and $M' = F' + S'$ where $F'$ is big and integral and the non-zero coefficients of $S'$ are $\geq 1$. Therefore, applying Proposition 4.8, there is a natural number $n$ depending only on $d, \epsilon, \delta$ such that $\lfloor nM' \rfloor$ defines a birational map. Then $\lfloor nM \rfloor$ also defines a birational map.

Let $m = (20dn + 2)l$. Let $L$ be any pseudo-effective integral divisor. We will show that $\lfloor m'N + L \rfloor$ and $\lfloor K_X + m'N + L \rfloor$ define birational maps for any natural number $m' \geq m$. Since $B \leq \frac{1}{\epsilon}R$ and $N$ is big and $E$ is pseudo-effective,

$$N - B = E + R - B$$

is big. Moreover, if $P$ is the fractional part of $m'N + L$, then $P$ is supported in $R$ and $P < lR$. Now since $\lfloor nM \rfloor$ defines a birational map, $4dnM + L$ and $4dnM + L - K_X$ are
potentially birational. Moreover, $lN - K_X$ and $lN - B$ are big and $3lN = M - K_X - B$. Then
\[ m'N + L = m'N + L - P = (20dn + 1)lN + L - P + (m' - m + l)N \]
\[ = 4dn(M - K_X - B) + 8dnN + lN + L - P + (m' - m + l)N \]
\[ = 4dnM + L + 4dn(lN - K_X) + 4dn(lN - B) + lE + lR - P + (m' - m + l)N \]
is potentially birational. Similar reasoning shows that $|mN + L - K_X|$ is potentially birational where we make use of the extra $(m' - m + l)N$ at the end of the above formula and the fact that $N - K_X$ is big.

Therefore, $|K_X + [m'N + L]|$ and $|m'N + L|$ define birational maps by [24, Lemma 2.3.4]. This in turn implies that $|K_X + m'N + L|$ and $|m'N + L|$ define birational maps.

\[ \square \]

**Proof.** (of Theorem 1.1) This follows from Theorem 4.2.

**Proof.** (of Corollary 4.3) By Lemma 4.11, there is a natural number $n > 1$ depending only on $d, \epsilon, \delta$ such that $K_X + nN$ is big. Let $r \in \mathbb{N}$ be the smallest number such that $r\delta \geq 1$. Let $M := K_X + (n + 2r)N$,
\[ F := [K_X + (n + r)N + rE], \]
and $T = M - F$. Then $F$ is big because if $P$ is the fractional part of $K_X + (n + r)N + rE$, then
\[ F = K_X + (n + r)N + rE - P = K_X + nN + 2rE + rR - P \]
where $rR - P \geq 0$ as $P$ is supported in $R$ and the non-zero coefficients of $rR$ are $\geq 1$. Moreover,
\[ T = M - F = K_X + (n + 2r)N - (K_X + (n + r)N + rE - P) = rR + P \]
is supported in $R$ with any non-zero coefficient $\geq 1$.

Considering $(X, (n + 2r)N)$ as a generalised pair with nef part $(n + 2r)N$, running an MMP on $M$ ends with a minimal model $X'$ [13, Lemma 4.4]. Then $M' = F' + T'$ is nef and big where $F'$ is integral and big and $T' \geq 0$ with any non-zero coefficient $\geq 1$, $M' - K_{X'}$ is big, and $X'$ is $\epsilon$-lc. Applying Theorem 4.2, there is a natural number $m$ depending only on $d, \epsilon$ such that $|mM'|$ defines a birational map. Therefore, $|mM|$ also defines a birational map.

Now since $|mM|$ defines a birational map, $3dmM$ is potentially birational. Replacing $m$ with $3dm + 1$, we can assume $mM - K_X$ is potentially birational. Let $l = n + 3r$. Let $L$ be an integral pseudo-effective divisor, and pick natural numbers $m' \geq m$ and $l' \geq lm'$. If $Q$ is the fractional part of $m'K_X + l'N + L - K_X$, then we see that
\[ m'K_X + l'N + L - K_X = m'K_X + l'N + L - K_X - Q \]
\[ = m'M + (l' - lm' + rm')N + L - K_X - Q \]
\[ = mM - K_X + (m' - m)M + (l' - lm')N + L + rm'N - Q \]
is potentially birational because $rm'R - Q \geq 0$ as $Q$ is supported in $R$. Therefore,
\[ |m'K_X + l'N + L| \]
defines a birational map by [23, 2.3.4] which in turn implies that
\[ |m'K_X + l'N + L| \]
defines a birational map.

Proof. (of Corollary 1.2) This is a special case of Corollary 4.3.

Proof. (of Corollary 1.4) Since $K_X + B \equiv 0$ and $N$ is big, there is a minimal model of $N$. Replacing $X$ with the minimal model we can assume $N$ is nef and big. Taking a $\mathbb{Q}$-factorialisation we can assume that $X$ is $\mathbb{Q}$-factorial. On the other hand, there is a positive real number $\epsilon$ depending only on $d, \Phi$ such that $X$ is $\epsilon$-lc: this follows from the global ACC result of [23], see for example [9, Lemma 2.48]. Now $N - K_X \equiv N + B$ is big, so we can apply Theorem 1.1.

Note that in the previous proof if $B = 0$, then $N + K_X$ is big so instead of 1.1 we can apply 4.8 if we assume 4.2 in lower dimension.

5. Birational boundedness on pseudo-effective pairs

In this section we treat birational boundedness of divisors on pairs $(X, B)$ with pseudo-effective $K_X + B$.

5.1. Main result. The main result of this section is the following more general form of Theorem 1.3.

**Theorem 5.2.** Let $d$ be a natural number, $\delta$ be a positive real number, and $\Phi \subset [0, 1]$ be a DCC set of rational numbers. Then there is a natural number $m$ depending only on $d, \delta, \Phi$ satisfying the following. Assume

- $(X, B)$ is a klt projective pair of dimension $d$,
- the coefficients of $B$ are in $\Phi$,
- $N$ is a nef and big $\mathbb{R}$-divisor,
- $N - (K_X + B)$ and $K_X + B$ are pseudo-effective, and
- $N = E + R$ where $E$ is integral and pseudo-effective and $R \geq 0$ with coefficients in $\{0\} \cup [\delta, \infty)$.

Then $|m' N + L|$ and $|K_X + m' N + L|$ define birational maps for any natural number $m' \geq m$ and any integral pseudo-effective divisor $L$.

5.3. Pseudo-effective log divisors. For a real number $b$ and a natural number $l$ let $b_{\lfloor l \rfloor} := \lfloor lb \rfloor$. Similarly for an $\mathbb{R}$-divisor $B$ and a natural number $l$ let $B_{\lfloor l \rfloor} := \lfloor lB \rfloor$. The following statement was proved in [23] when $K_X + B$ is big (it follows from [23, Lemma 7.3]). We extend it to the case when $K_X + B$ is only pseudo-effective.

**Proposition 5.4.** Let $d$ be a natural number and $\Phi \subset [0, 1]$ be a DCC set of rational numbers. Then there is a natural number $l$ depending only on $d, \Phi$ satisfying the following. Assume

- $(X, B)$ is an lc projective pair of dimension $d$,
- the coefficients of $B$ are in $\Phi \cup (\frac{l-1}{l}, 1]$ and
- $K_X + B$ is pseudo-effective.

Then $K_X + B_{\lfloor l \rfloor}$ is pseudo-effective.
Proof. Step 1. In this step we introduce some notation. Adding 1 to $\Phi$ we can assume $1 \in \Phi$. Assume the proposition does not hold. Then for each $l \in \mathbb{N}$ there is a pair $(X^l, B^l)$ such that $(X^l, B^l)$ is lc projective of dimension $d$, the coefficients of $B^l$ are in $\Phi \cup \{\frac{l-1}{l}, 1\}$, and $K_{X^l} + B^l$ is pseudo-effective but such that $K_{X^l} + B^l_{[l]}$ is not pseudo-effective. Replacing $(X^l, B^l)$ with a $\mathbb{Q}$-factorial dlt model we can assume $(X^l, 0)$ is $\mathbb{Q}$-factorial klt. Moreover, increasing coefficients of $B_l$ in $(\frac{l-1}{l}, 1)$ slightly we can assume that $B^l$ is a $\mathbb{Q}$-divisor; note that this does not change $B^l_{[l]}$ because for any $b \in (\frac{l-1}{l}, 1)$, we have $b_{[l]} = \frac{l-1}{l}$.

Let $\Psi$ be the union of the coefficients of all the $B^l$. Then $\Psi$ is a DCC set: since $\Phi$ is DCC it is enough to check that $\Psi \setminus \Phi$ is DCC; the latter follows from the fact that if $b_l \in \Psi \setminus \Phi$ is a coefficient of $B_l$, then $b_l \in (\frac{l-1}{l}, 1)$, so any infinite sequence of such coefficients approaches 1, hence cannot be a strictly decreasing sequence.

Step 2. In this step we show certain sets of coefficients are DCC. Suppose that for each $l$ we have a boundary $C^l$ such that $B^l_{[l]} \leq C^l \leq B^l$. We argue that the set of coefficients of all the $C^l$ put together satisfies DCC. Assume not. Then there is an infinite subset $L \subset \mathbb{N}$ of numbers that for each $l \in L$ we can pick a coefficient $c_l$ of $C^l$ such that the $c_l$ form a strictly decreasing sequence, that is, $c_l < c_{l'}$ for any $l, l' \in L$ with $l < l'$. For each $l$, $c_l$ is the coefficient of $C^l$ of some component, say $D^l$. Let $b_l$ be the coefficient of $D^l$ in $B^l$. Replacing $L$ with an infinite subset we can assume that the $b_l$ with $l \in L$ form an increasing sequence approaching a limit $b$. But then the numbers $c_l$ also approach $b$ as $l$ goes to $\infty$ because

$$b - \frac{1}{l} < b_{[l]} \leq c_l \leq b_l$$

where the first inequality follows from $lb_l - 1 < \lfloor lb_l \rfloor$, a contradiction.

Step 3. In this step we run an MMP and reduce to the case in which we have a Mori fibre space structure $X^l \to T^l$ for all but finitely many $l$. For ease of notation in this step we write $\Delta^l := B^l_{[l]}$. Since $K_{X^l} + \Delta^l$ is not pseudo-effective and since $(X^l, \Delta^l)$ is $\mathbb{Q}$-factorial dlt, we can run an MMP on $K_{X^l} + \Delta^l$, with scaling of some ample divisor, ending with a Mori fibre space $Y^l \to T^l$ [12]. Denote the pushdowns of $B^l, \Delta^l$ to $Y^l$ by $B_{Y^l}, \Delta_{Y^l}$. By assumption, $K_{Y^l} + B_{Y^l}$ is pseudo-effective, hence it is nef over $T^l$. We claim that $(Y^l, B_{Y^l})$ is lc for all but finitely many $l$. Assume not. Then there is an infinite subset $L \subset \mathbb{N}$ such that for each $l \in L$, $(Y^l, B_{Y^l})$ is not lc.

Then for each $l \in L$ we have a boundary $C_{Y^l}$ such that

- $\Delta_{Y^l} \leq C_{Y^l} \leq B_{Y^l}$,
- $(Y^l, C_{Y^l})$ is lc,
- some component $D^l$ of $C_{Y^l}$ contains a non-klt centre of $(Y^l, C_{Y^l})$, and
- if $c_l, b_l$ are the coefficients of $D^l$ in $C_{Y^l}$ and $B_{Y^l}$ respectively, then $c_l < b_l$.

By the previous step, the set of the coefficients of all the $C_{Y^l}$ satisfies DCC. Moreover, the set of the $c_l$ is not finite because otherwise replacing $L$ we can assume $c_l$ is fixed and then from $b_l - \frac{1}{l} < c_l < b_l$ we deduce that the $b_l$ approach $c_l$ so the set of the $b_l$ is not DCC, a contradiction. Therefore, replacing $L$ we can assume that the $c_l$ form a strictly increasing sequence. Now $c_l$ is the lc threshold of $D^l$ with respect to the pair $(Y^l, C_{Y^l} - c_l D^l)$ as $D^l$ contains a non-klt centre of $(Y^l, C_{Y^l})$. Moreover, the set of
the coefficients of all the \( C_{l'} - c^l D_l \) satisfies DCC. Therefore, we get a contradiction by the ACC for lc thresholds [23, Theorem 1.1].

For those \( l \) such that \((Y^l, B_{Y^l})\) is lc we replace \((X^l, B_l)\) with \((Y^l, B_{Y^l})\). Therefore, we can assume that for all but finitely many \( l \) we have a Mori fibre space structure \( X^l \to T^l \) such that \( K_{X^l} + B^l_{[l]} \) is anti-ample over \( T^l \).

**Step 4.** In this step we derive a contradiction. For each \( l \) as in the previous paragraph (that is, those for which we have \( X^l \to T^l \)) we can find a boundary \( \Theta^l \) such that \( B^l_{[l]} \leq \Theta^l \leq B^l \) and \( K_{X^l} + \Theta^l \equiv 0/T^l \). By Step 2, the set \( \Omega \) of the coefficients of all such \( \Theta^l \) form a DCC set. Let \( F_l \) be a general fibre of \( X^l \to T^l \) and let \( \Theta_{F^l} = \Theta^l |_{F^l} \) (if \( \dim T^l = 0 \), then \( F^l = X^l \)). Then \((F^l, \Theta_{F^l})\) is lc, \( K_{F^l} + \Theta_{F^l} \equiv 0 \), and the coefficients of \( \Theta_{F^l} \) belong to \( \Omega \) because if \( \Theta^l = \sum a_i^l B_i^l \) where \( B_i^l \) are the irreducible components, then \( \Theta_{F^l} = \sum a_i^l B_i^l |_{F^l} \) where \( B_i^l |_{F^l} \) are reduced with no common components for distinct \( i \). By the global ACC [23, Theorem 1.5], the set of the coefficients of all the \( \Theta_{F^l} \) is finite, hence the set of the horizontal (over \( T^l \)) coefficients of all the \( \Theta^l \) is also finite.

Since \( K_{X^l} + B^l_{[l]} \) is anti-ample over \( T^l \), we can find a horizontal component of \( B^l \) with coefficients \( b^l_{[l]}, a^l, b^l \) in \( B^l_{[l]}, \Theta^l, B^l \), respectively, such that \( b^l_{[l]} \leq a^l \). Since the \( a^l \) belong to a finite set, the \( b^l \) also belong to a finite set otherwise they would not form a DCC set. Thus there is a natural number \( p \) such that \( pb^l \) is integral for all such \( b^l \). But then for every \( l \) divisible by \( p \), we have
\[ b^l_{[l]} = \frac{lb^l}{l} = \frac{lb^l}{T} = b^l, \]
contradicting \( b^l_{[l]} < a^l \leq b^l \).

\[ \square \]

5.5. **Proofs of 5.2 and 1.3.**

**Proof.** (of Theorem 5.2) **Step 1.** In this step we introduce some notation. Let \( l \) be as in Proposition 5.4 for the data \( d, \Phi \). Let \((X, B), N, E, R\) be as in Theorem 5.2. Replacing \( X \) with a \( \mathbb{Q} \)-factorialisation we can assume that \( X \) is \( \mathbb{Q} \)-factorial. Let \( X' \to X \) be the birational map which extracts exactly the exceptional prime divisors \( D \) over \( X \) with log discrepancy
\[ a(D, X, B) < \frac{1}{2l}. \]
Here by exceptional prime divisors over \( X \) we mean prime divisors on birational models of \( X \) which are exceptional over \( X \). If there is no such \( D \), then \( X' \to X \) is the identity morphism. Let \( K_{X'} + B', N', E', R' \) be the pullbacks of \( K_X + B, N, E, R \) to \( X' \), respectively. Note that replacing \( N \) with \( 2N \) we can assume that \( N - (K_X + B) \) is big.

**Step 2.** In this step we study \( \Delta' := B^l_{[l]} \). By construction, \( K_{X'} + B' \) is pseudo-effective and the coefficients of \( B' \) belong to \( \Phi \cup (1 - \frac{1}{2l}, 1) \) because the non-exceptional/X components of \( B' \) have coefficients in \( \Phi \) and the exceptional/X components \( D \) of \( B' \) have coefficients
\[ \mu_D B' = 1 - a(D, X, B) \in (1 - \frac{1}{2l}, 1) \]
Thus by Proposition 5.4, $K_{X'} + \Delta'$ is pseudo-effective. Moreover, for exceptional $D$, $\mu_D \Delta' = 1 - \frac{1}{2}$ by definition of $\Delta'$, hence $\mu_D (B' - \Delta') > \frac{1}{2l}$.

On the other hand, by our choice of $X' \to X$, for any exceptional prime divisor $C$ over $X'$, we have

$$a(C, X', B') = a(C, X, B) \geq \frac{1}{2l}.$$ 

Thus $(X', 0)$ is $\frac{1}{2l}$-lc.

**Step 3.** In this step we introduce some notation. Let $r \in \mathbb{N}$ be the smallest number such that $r \delta \geq 1$. Let

- $S'$ be the sum of all the exceptional $/X$ prime divisors on $X'$,
- $J' = \text{Supp}(S' + R')$,
- $F' = \lfloor 6lN' - 2S' + J' \rfloor$,
- $P'$ be the fractional part of $6lN' - 2S' + J'$,
- $T' = 6lN' - F'$,
- $G' = \lfloor 2rE' \rfloor$,
- $Q'$ be the fractional part of $2rE'$, and
- $V' = 2rR' + Q'$.

Note that $P'$ is supported in $J'$ because $P'$ is also the fractional part of $6lN'$ and because any component of $6lN' = 6lE' + 6lR'$ with non-integral coefficient is either a component of $R'$ or an exceptional component of $E'$ as $E$ is integral. Moreover, $Q'$ is exceptional over $X$ because $2rE$ is integral, hence $Q'$ is supported in $S'$.

**Step 4.** In this step we show that $F' + G'$ is big and that $T' + V'$ is supported in $J'$ whose non-zero coefficients $\geq 1$. Let $A' = N' - (K_{X'} + B')$ which is big as $N - (K_X + B)$ is big by assumption. We then have

$$N' = K_{X'} + B' + A' = K_{X'} + \Delta' + A' + B' - \Delta'$$

where $K_{X'} + \Delta' + A'$ is big as $K_{X'} + \Delta'$ is pseudo-effective by Step 2, and $B' - \Delta' \geq 0$ with coefficients of exceptional components $> \frac{1}{2}$ again by Step 2. Thus $2l(B' - \Delta') - S' \geq 0$, hence $2lN' - S'$ is big.

Now

$$F' + G' = \lfloor 6lN' - 2S' + J' \rfloor + \lfloor 2rE' \rfloor$$

$$= 6lN' - 2S' + J' - P' + 2rE' - Q'$$

$$= 6lN' - 3S' + 2rE' + J' - P' + S' - Q',$$

is big because $6lN' - 3S'$ is big by the previous paragraph, $2rE'$ is pseudo-effective, and $J' - P' + S' - Q' \geq 0$ as $J'$ contains the support of $P'$ and as $S'$ contains the support of $Q'$ by Step 3.

On the other hand,

$$T' + V' = 6lN' - F' + 2rR' + Q'$$

$$= 6lN' - (6lN' - 2S' + J' - P') + 2rR' + Q'$$

$$= 2S' - J' + P' + 2rR' + Q'$$

$$\geq 2S' - J' + 2rR'.$$
where the inequality follows from $P', Q' \geq 0$. In particular, $T' + V'$ is supported in $J'$. Moreover, $S' - J' + rR' \geq 0$ because the non-zero coefficients of $S' + rR'$ are $\geq 1$: this is clear for the exceptional components; and for the non-exceptional components it follows from $r\delta \geq 1$ and the assumption that the non-zero coefficients of $R$ are $\geq \delta$. Therefore,

$$T' + V' \geq 2S' - J' + 2rR' \geq S' - J' + rR' + S' + rR' \geq S' + rR',$$

so $\text{Supp}(T' + V') = J'$, and the non-zero coefficients of $T' + V'$ are $\geq 1$ by the previous sentence.

Step 5. In this step we finish the proof by applying 4.2. From the equalities in Step 4 we see that

$$F' + G' + T' + V'$$

$$= (6lN' - 2S' - J' - P' + 2rE' - Q') + (2S' - J' + P' + 2rR' + Q')$$

$$= 6lN' + 2rE' + 2rR'$$

$$= (6l + 2r)N'.$$

Also recall from Step 2 that $X'$ is $\frac{1}{2}\text{-lc}$. Moreover, $(6l + 2r)N' - K_{X'}$ is big because $N' - K_{X'} = B' + A'$ is big by the first paragraph of Step 4. Therefore, applying Theorem 4.2 to

$$X', (6l + 2r)N' = (6l + 2r)(F' + G') + (6l + 2r)(T' + V'),$$

we deduce that there is a natural number $n$ depending only on $d, \frac{1}{2r}$ such that the linear system $|n(6l + 2r)N'|$ defines a briational map. In particular, $3dn(6l + 2r)N'$ is potentially birational which in turn implies that $3dn(6l + 2r)N'$ is potentially bi- rational. We will show that $m := 3dn(6l + 2r) + r + 2$ satisfies the proposition.

Let $L$ be any pseudo-effective integral divisor on $X$. Pick a natural number $m' \geq m$. Let $I$ be the fractional part of $m'N$. Then

$$|m'N + L| = m'N - I + L$$

$$= 3dn(6l + 2r)N + rE + (m' - m + 2)N + L + rR + I$$

is potentially birational because $rE + (m' - m + 2)N + L$ is big and $rR + I \geq 0$ as $I$ is supported in $R$, $r\delta \geq 1$, and the non-zero coefficients of $R$ are $\geq \delta$. Similar reasoning shows that $|m'N + L - K_{X'}|$ is potentially birational where we make use of the fact that $N - K_{X'}$ is big. Therefore, by [24, Lemma 2.3.4],

$$|m'N + L| \quad \text{and} \quad |K_{X'} + m'N + L|$$

define briational maps which in turn imply that

$$|m'N + L| \quad \text{and} \quad |K_{X'} + m'N + L|$$

define briational maps. Finally note that $m$ depends only on $d, \Phi, \delta$ because $n$ depends only on $d, \frac{1}{2r}$ and because $l, r$ depend only on $d, \Phi, \delta$. \hfill \Box

Proof. (of Theorem 1.3) This follows from Theorem 5.2. \hfill \Box
In this section we treat boundedness of polarised pairs, namely we prove 1.5, 1.6, 1.7, and 1.8. Bounding certain lc thresholds plays a key role in the proofs of all these results. This bounding is achieved through a combination of birational boundedness of linear systems, birational boundedness of pairs, and boundedness of lc thresholds on bounded families.

6.1. Polarised nef $\epsilon$-lc pairs. We begin with proving a more general version of Theorem 1.5.

**Theorem 6.2.** Let $d$ be a natural number and $v, \epsilon, \delta$ be positive real numbers. Consider pairs $(X, B)$ and divisors $N$ on $X$ such that

- $(X, B)$ is projective $\epsilon$-lc of dimension $d$,
- the coefficients of $B$ are in $\{0\} \cup [\delta, \infty)$,
- $K_X + B$ is nef,
- $N$ is a nef and big $\mathbb{R}$-divisor,
- $N = E + R$ where $E$ is integral and pseudo-effective and $R \geq 0$ with coefficients in $\{0\} \cup [\delta, \infty)$, and
- $\text{vol}(K_X + B + N) \leq v$.

Then the set of such $(X, \text{Supp} B)$ forms a bounded family. If in addition $N \geq 0$, then the set of such $(X, \text{Supp}(B + N))$ forms a bounded family.

**Proof.** Step 1. In this step we will define a divisor $M$ and study some of its properties. From here to the end of Step 4 we assume that $K_X + B + N$ is ample. Applying Corollary 4.3 on a small $\mathbb{Q}$-factorialisation of $X$, there exist natural numbers $m, l \geq 1$ depending only on $d, \epsilon, \delta$ such that the linear system $|mK_X + lN + mE|$ defines a birational map. Pick an element $L$ of this linear system and then define $M := mB + mR + L$ which is effective. Then

$$M \sim mB + mR + mK_X + lN + mE = m(K_X + B + N) + lN$$

is ample, and $|M|$ defines a birational map as $M \geq L$. Moreover, for any component $D$ of $M$, we have $\mu_D(B + M) \geq \mu_D M \geq 1$: if $D$ is not a component of the fractional part of $M$, then obviously $\mu_D M \geq 1$; if $D$ is a component of the fractional part of $M$, then $D$ is a component of $B + R$ and we have

$$\mu_D M \geq \mu_D(mB + mR) \geq 1$$

as $m\delta \geq 1$ and as the non-zero coefficients of $B + R$ are $\geq \delta$. In addition,

$$\text{vol}(M) = \text{vol}(m(K_X + B + N) + lN)$$

$$\leq \text{vol}(m(K_X + B + N) + lm(K_X + B) + lmN)$$

$$= \text{vol}((l + 1)m(K_X + B + N)) \leq ((l + 1)m)^d v.$$ 

Also it is clear from the definition of $M$ that $M - (K_X + B)$ is big.

Step 2. In this step we show that $(X, \text{Supp}(B + M))$ is birationally bounded. By Lemma 2.4, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A$ such that $\text{Supp} A = \text{Supp} M$ and $M - A$ has arbitrarily small coefficients. In particular, $A \geq 0$ as $M \geq 0$, and we can assume that $A$ is ample as $M$ is ample with $\text{vol}(A) \leq ((l + 1)m)^d v + 1$. Moreover, we can assume that $2A \geq M$ which in particular means that $|2A|$ defines a birational
map and the coefficients of $2A$ are $\geq 1$ by Step 1. In addition, $2A - (K_X + B)$ is big as $M - (K_X + B)$ is big. Also note that for any component $D$ of $2A$ we have $\mu_D(B + 2A) \geq \mu_D(B + M) \geq 1$.

Now applying [9, Proposition 4.4] to $(X, B), 2A$, we deduce that there exist a bounded set of couples $\mathcal{P}$ and a natural number $c$ depending only on $d, v, l, m, \delta$ such that we can find a projective log smooth couple $(\overline{X}, \Sigma) \in \mathcal{P}$ and a birational map $\overline{X} \to X$ such that

- Supp $\Sigma$ contains the exceptional divisors of $\overline{X} \to X$ and the birational transform of Supp $(B + 2A) = \text{Supp}(B + M)$;
- if $\phi : X' \to X$ and $\psi : X' \to \overline{X}$ is a common resolution, then each coefficient of $\overline{A} := \psi_* \phi^* A$ is at most $c$.

**Step 3.** Next we show that the lc threshold $t$ of $M$ with respect to $(X, B)$ is bounded from below away from zero. Let $K_{\overline{X}} + B = \psi_* \phi^*(K_X + B)$. By definition of $t$, $(X, B + tM)$ is not klt, hence $(X, B + 2tA)$ is also not klt as $2A \geq M$. Then, since $K_X + B + 2tA$ is ample, by the negativity lemma

$$\phi^*(K_X + B + 2tA) \leq \psi^*(K_{\overline{X}} + B + 2tA),$$

which implies that $(\overline{X}, B + 2tA)$ is not sub-klt. Thus $(\overline{X}, (1 - \epsilon)\Sigma + 2tA)$ is not sub-klt as $\overline{B} \leq (1 - \epsilon)\Sigma$ because $(X, B)$ is $\epsilon$-lc. Then by [9, Proposition 4.2], $t$ is bounded from below away from zero depending only on $\epsilon, \mathcal{P}, c$. Thus $t$ depends only on $d, v, l, m, \epsilon, \delta$, so we can assume it depends only on $d, v, \epsilon, \delta$. Also note that $t \leq 1$ because as noted above each component $D$ of $M$ satisfies $\mu_D M \geq 1$.

**Step 4.** In this step we show that $(X, \text{Supp}(B + M))$ is bounded. By the previous step, $(X, B + M)$ is $\frac{\epsilon}{2}$-lc as $(X, B)$ is $\epsilon$-lc. Moreover, $K_X + B + \frac{\epsilon}{2}M$ is ample and the non-zero coefficients of $B + \frac{\epsilon}{2}M$ are $\geq \min\{\delta, \frac{\epsilon}{2}\}$. And by Step 2, $(X, \text{Supp}(B + M))$ is birationally bounded. Therefore, applying [23, Theorem 1.6], we deduce that $(X, \text{Supp}(B + M))$ is bounded.

Now assume $N \geq 0$. By adding a multiple of $N$ to $M$ in Step 1, say by replacing $l$ with $l + 1$, we can assume $M \geq N$. Thus boundedness of $(X, \text{Supp}(B + M))$ implies boundedness of $(X, \text{Supp}(B + N))$.

**Step 5** Now we treat the general case in which $K_X + B + N$ is only nef and big. Since $(X, B)$ is klt, $K_X + B$ is nef, and $N$ is nef and big, we see that $K_X + B + N$ is nef and big and semi-ample by the base point freeness theorem, hence it defines a contraction $X \to Y$. Then $K_X + B \equiv 0/Y$ and $N \equiv 0/Y$ as both $K_X + B, N$ are nef, hence by again applying the base point freeness theorem to $K_X + B$ over $Y$ we have $K_X + B \sim_Y 0/Y$ and $N \sim_Y 0/Y$ which in turn implies $M \sim_Y 0/Y$. By the above steps, $(Y, \text{Supp} B_Y + M_Y)$ is bounded where $B_Y, M_Y$ are pushdowns of $B, M$. Therefore, applying [7, Theorem 1.3] to $(Y, B_Y + \frac{\epsilon}{2}M_Y)$ we deduce that $(X, \text{Supp} B + M)$ is bounded. In particular, $(X, \text{Supp} B + N)$ is bounded if $N \geq 0$ as we can assume $M \geq N$ as in Step 4.

It is worth pointing out that we used [7, Theorem 1.3] in the proof but only when $K_X + B + N$ is not ample.

**Proof.** (of Theorem 1.5) This is a special case of Theorem 6.2.
Proof. (of Corollary 1.6) Since \((X, B)\) is klt and \(K_X + B = 0\), by [9, Lemma 2.48], there is a positive real number \(\epsilon\) depending only on \(d, \Phi\) such that \((X, B)\) has \(\epsilon\)-lc singularities (note that the lemma requires \((X, 0)\) to be klt but we can achieve this on a small \(\mathbb{Q}\)-factorialisation of \(X\)). Moreover, \(\text{vol}(K_X + B + N) = \text{vol}(N) \leq v\). Now apply Theorem 1.5.

6.3. Lc thresholds on slc Calabi-Yau pairs. Next we prove a more general version of Theorem 1.7.

Theorem 6.4. Let \(d\) be a natural number, \(v, \delta\) be positive real numbers, and \(\Phi \subset [0, 1]\) be a DCC set of real numbers. Then there is a positive real number \(t\) depending only on \(d, v, \delta, \Phi\) satisfying the following. Assume that

- \((X, B)\) is a projective slc Calabi-Yau pair of dimension \(d\),
- the coefficients of \(B\) are in \(\Phi\),
- \(N \geq 0\) is a nef \(\mathbb{R}\)-divisor on \(X\) with coefficients \(\geq \delta\),
- \((X, B + uN)\) is slc for some real number \(u > 0\),
- for each irreducible component \(S\) of \(X\), \(N|_S\) is big and \(\text{vol}(N|_S) \leq v\).

Then \((X, B + tN)\) is slc.

Proof. Step 1. In this step we reduce the theorem to the case when \(X\) is normal and irreducible. Let \(X^\nu \to X\) be the normalisation of \(X\) and let \(K_X^\nu + B^\nu\) and \(N^\nu\) be the pullbacks of \(K_X + B\) and \(N\). Since \(K_X + B \sim_{\mathbb{R}} 0\), we get \(K_X^\nu + B^\nu \sim_{\mathbb{R}} 0\). Recall from 2.8 that \(B^\nu\) is the sum of the birational transform of \(B\) and the reduced conductor divisor of \(X^\nu \to X\). So the coefficients of \(B^\nu\) belong to \(\Phi \cup \{1\}\). Replacing \(\Phi\) with \(\Phi \cup \{1\}\) we can assume these coefficients are in \(\Phi\). On the other hand, since \((X, B + uN)\) is slc for some \(u > 0\), \(\text{Supp} N\) does not contain any singular codimension one point of \(X\). Thus if \(U\) is the normal locus of \(X\), then \(N\) is the closure of \(N|_U\).

Similarly if \(U^\nu\) is the inverse image of \(U\), then \(N^\nu\) is the closure of \(N^\nu|_{U^\nu}\), hence \(N^\nu\) is the birational transform of \(N\), so the coefficients of \(N^\nu\) are \(\geq \delta\).

By definition of slc pairs, for any real number \(t \geq 0\), \((X, B + tN)\) is slc iff \((X^\nu, B^\nu + tN^\nu)\) is lc on each irreducible component of \(X^\nu\). By assumption, \(N^\nu\) is nef and big on each irreducible component of \(X^\nu\) with volume at most \(v\). Therefore, replacing \((X, B), N\) with the restriction of \((X^\nu, B^\nu), N^\nu\) to an arbitrary irreducible component of \(X^\nu\) we can assume \(X\) is normal and irreducible. In particular, \((X, B + uN)\) is an lc pair for some \(u > 0\).

Step 2. In this step we reduce the theorem to the case when \(X\) has \(\epsilon\)-lc singularities for some fixed \(\epsilon > 0\) depending only on \(d, \Phi\). Let \((X', B')\) be a \(\mathbb{Q}\)-factorial dlt model of \((X, B)\) and let \(N'\) be the pullback of \(N\). By definition of dlt models, each exceptional \(X\) prime divisor on \(X'\) appears in \(B'\) with coefficient 1. By assumption, \((X, B + uN)\) is lc for some \(u > 0\), hence \((X', B' + uN')\) is lc, so \(\text{Supp} N'\) cannot contain any exceptional divisor which means \(N'\) is just the birational transform of \(N\) so its coefficients are \(\geq \delta\). Replacing \((X, B), N\) with \((X', B'), N'\) we can assume that \((X, 0)\) is \(\mathbb{Q}\)-factorial klt.

On the other hand, since \(K_X + B \sim_{\mathbb{R}} 0\) and since the coefficients of \(B\) are in the DCC set \(\Phi\), there is a positive real number \(\epsilon\) depending only on \(d, \Phi\) such that if \(D\) is a prime divisor over \(X\) with log discrepancy \(a(D, X, B) < \epsilon\), then \(a(D, X, B) = 0\).
[9, Lemma 2.48]. In particular, if $D$ is a prime divisor over $X$ with $a(D, X, 0) < \epsilon$, then $a(D, X, B) = 0$.

Assume $X'' \to X$ extracts exactly all the prime divisors $D$ over $X$ with $a(D, X, 0) < \epsilon$. If there is no such $D$, then $X'' \to X$ is the identity morphism. Let $K_{X''} + B''$ be the pullback of $K_X + B$. Then each exceptional prime divisor of $X'' \to X$ has coefficient 1 in $B''$ by the previous paragraph. In particular, the coefficients of $B''$ are in $\Phi$, and $\text{Supp} N$ does not contain the image of such divisors on $X$. Thus if $N''$ is the pullback of $N$, then $N''$ is just the birational transform of $N$, so its coefficients are $\geq \delta$. In addition, by construction, $(X'', 0)$ has $\epsilon$-lc singularities because for any prime divisor $C$ on birational models of $X''$ and exceptional over $X''$, we have

$$a(C, X'', 0) \geq a(C, X, 0) \geq \epsilon$$

by our choice of $X''$. Replacing $(X, B), N$ with $(X'', B''), N''$ we can then assume that $(X, 0)$ has $\epsilon$-lc singularities.

**Step 4.** In this step we show that $(X, \text{Supp} B + N)$ is birationally bounded. We want to apply [9, Proposition 4.4] but since this is stated only for nef $\mathbb{Q}$-divisors, we need to apply it indirectly as follows. Since $X$ is $\epsilon$-lc and $N$ is nef and big with coefficients $\geq \delta$, by Theorem 4.3, there exist natural numbers $m, l$ depending only on $d, \epsilon, \delta$ such that $|mK_X + lN|$ defines a birational map. Pick an element $L$ of this linear system. Replacing $l$ we can assume that $L \geq N + \text{Supp} N$. Let $M := m\Delta + L$ where $0 \leq \Delta \leq N$ is a small $\mathbb{R}$-divisor so that $M$ is a $\mathbb{Q}$-divisor and $(X, \Delta)$ is $\frac{\epsilon}{d}$-lc.

Considering $(X, \Delta + \frac{L}{m} N)$ as a generalised pair with nef part $\frac{1}{m} M$, running an MMP on $K_X + \Delta + \frac{L}{m} N \sim_{\mathbb{Q}} \frac{1}{m} M$ ends with a minimal model, say $Y$. Thus $M_Y$ is a nef and big $\mathbb{Q}$-divisor. Moreover, $M_Y - (K_Y + B_Y) \equiv M_Y$ is big. In addition,

$$\text{vol}(M_Y) = \text{vol}(M) = \text{vol}(-mB + m\Delta + lN) \leq \text{vol}((m + l)N) \leq (m + l)^d v.$$

On the other hand, for any component $D$ of $M_Y$, $\mu_D(B_Y + M_Y) \geq \mu_D M_Y \geq 1$: if $D$ is a not a component of the fractional part of $M_Y$, this is obvious; otherwise $D$ is a component of $N_Y$ in which case the assertion follows from $M_Y \geq L_Y \geq \text{Supp} N_Y$.

Therefore, applying [9, Proposition 4.4] to $(Y, B_Y), M_Y$, there is a positive real number $c$ and a bounded set of couples $\mathcal{P}$ depending only on $d, v, m, l, \Phi$ such that there is a projective log smooth couple $(\overline{X}, \Sigma) \in \mathcal{P}$ and a birational map $\overline{X} \dashrightarrow Y$ such that

- $\text{Supp} \Sigma$ contains the exceptional divisor of $\overline{X} \dashrightarrow Y$ and the birational transform of $\text{Supp}(B_Y + M_Y)$;
- if $\rho: X' \to Y$ and $\psi: X' \to \overline{X}$ is a common resolution and $\overline{M} = \psi_*\rho^* M_Y$, then each coefficient of $\overline{M}$ is at most $c$.

Now $\Sigma$ contains the exceptional divisors of the induced map $\overline{X} \dashrightarrow X$ and the birational transform of $\text{Supp}(B + N)$ because any component of the latter is either exceptional over $Y$ or is the birational transform of some component of $B_Y + M_Y$. Moreover, replacing $X'$ we can assume $\phi: X' \to X$ is also a resolution. But then since $N$ is nef,

$$\overline{N} = \psi_*\phi^* N \leq \psi_*\rho^* N_Y \leq \psi_*\rho^* M_Y = \overline{M},$$

hence $\overline{N}$ is supported in $\Sigma$ with coefficients $\leq c$. 


Step 5. In this step we finish the proof. Let $K_{X'} + B' = \phi^*(K_X + B)$, $N' = \phi^*N$, and $K_X + B = \psi_*\phi^*(K_X + B)$. Then $(X', B')$ is sub-lc and $(\overline{X}, \overline{B})$ is also sub-lc as $\operatorname{Supp} \overline{B} \subset \overline{\Sigma}$. Since $\operatorname{Supp} \overline{N}$ does not contain any non-klt centre of $(X, B)$, $\operatorname{Supp} \overline{N}'$ does not contain any non-klt centre of $(X', B')$, hence no component of $\overline{N}'$ has coefficient 1 in $\overline{B}'$. Thus no component of $\overline{N}$ has coefficient 1 in $\overline{B}$.

On the other hand, by Step 2, no component of $\overline{B}$ has coefficient in $(1 - \epsilon, 1)$ otherwise we would find a prime divisor $D$ over $X$ with $0 < a(D, X, B) < \epsilon$ which is not possible by our choice of $\epsilon$. Thus every component of $\overline{N}$ has coefficient $\leq 1 - \epsilon$ in $\overline{B}$. Also by the previous step the coefficients of $\overline{N}$ are at most $c$. Therefore, letting $t = \frac{\epsilon}{c}$ we see that the coefficients of $\overline{B} + t\overline{N}$ do not exceed 1 because for any prime divisor $D$ either $\mu_D \overline{B} = 1$ and $\mu_D t\overline{N} = 0$, or $\mu_D \overline{B} \leq 1 - \epsilon$ and $\mu_D t\overline{N} \leq \epsilon$. Moreover, since $\operatorname{Supp} \overline{\Sigma}$ contains $\operatorname{Supp} \overline{B} \cup \operatorname{Supp} \overline{N}$,

$$(\overline{X}, \operatorname{Supp} \overline{B} \cup \operatorname{Supp} \overline{N})$$

is log smooth. Therefore, $(\overline{X}, \overline{B} + t\overline{N})$ is sub-lc.

Now since $K_X + B + tN$ is nef, by the negativity lemma, we have

$$\phi^*(K_X + B + tN) \leq \psi^*(K_X + \overline{B} + t\overline{N}),$$

hence we deduce that $(X, B + tN)$ is lc. Note that we can assume that $t$ depends only on $d, v, \delta, \Phi$ because $\epsilon$ depends only on $d, \Phi$, and $m, l$ depend only on $d, \epsilon, \delta$, and $c$ depends only on $d, v, m, l, \Phi$.

\[\square\]

Proof. (of Theorem 1.7) This is a special case of Theorem 6.4.

\[\square\]

6.5. Polarised slc Calabi-Yau pairs.

Proof. (of Corollary 1.8) If $X_i$ are the irreducible components of $X$, then $\operatorname{vol}(N) = \sum \operatorname{vol}(N|_{X_i})$, hence $\operatorname{vol}(N|_{X_i}) \leq v$ for each $i$. Thus by Theorem 1.7, there is a rational number $t > 0$ depending only on $d, v, \delta, \Phi$ such that $(X, B + tN)$ is slc. Since the coefficients of $B$ are in the DCC set $\Phi$ and since $N \geq 0$ is integral, the coefficients of $B + tN$ belong to a DCC set depending only on $\Phi, t$. Moreover, $K_X + B + tN$ is ample with

$$\operatorname{vol}(K_X + B + tN) = \operatorname{vol}(tN) = t^dv.$$

Therefore, we can apply [22, Theorem 1.1] to deduce that $(X, \operatorname{Supp}(B + tN))$ belongs to a bounded family.

\[\square\]

7. Moduli of polarised Calabi-Yau and Fano pairs

In this section we prove the main results on existence of moduli spaces, that is, Theorems 1.10 and 1.12. Recall that $k$ is our fixed algebraically closed ground field. In this section given a morphism $X \to S$ of schemes and a point $s \in S$ we denote the fibre over $s$ by $X_s$. 
7.1. Moduli of embedded marked locally stable pairs. We begin with recalling some definitions and results regarding stable pairs from Kollár [30]. Throughout this subsection we fix natural numbers \( d, n \), a positive rational number \( v \), and a vector \( \alpha = (a_1, \ldots, a_m) \) with positive rational coordinates (we allow the possibility \( m = 0 \) in which case \( \alpha \) is empty).

(1) A \((d, \alpha)\)-marked locally stable pair over a field \( K \) of characteristic zero is a projective geometrically connected slc pair \( (X, \Delta) \) over \( K \) with \( \dim X = d \) and a decomposition \( \Delta = \sum \nolimits a_i D_i \) where \( D_i \geq 0 \) are integral divisors. The \( D_i \) are not assumed to be \( \mathbb{Q} \)-Cartier and they are not necessarily distinct. Two such marked pairs \( (X, \Delta) \) and \((X', \Delta')\) are isomorphic if there is an isomorphism \( X \to X' \) mapping \( D_i \) onto \( D'_i \) hence mapping \( \Delta \) onto \( \Delta' \) preserving the decomposition.

In general, \( \Delta \) can possibly be written as \( \sum a_i D_i \) in several ways, so the underlying slc pair can be marked in several ways giving distinct marked pairs. For example, let \( X = \mathbb{P}^2 \) and let \( L, C \) be a line and a smooth conic, respectively, intersecting transversally, and let \( \Delta = \frac{1}{2} L + \frac{1}{2} C \). Letting \( \alpha = (\frac{1}{4}, \frac{1}{4}) \), we can mark \( \Delta \) by taking \( D_1 = 2L, D_2 = 2C \), or taking \( D_1 = 2C, D_2 = 2L \), or taking \( D_1 = D_2 = L + C \). Thus there are at least three mutually non-isomorphic \((2, \alpha)\)-marked pairs with the same underlying pair \((X, \Delta)\).

A \((d, \alpha, v)\)-marked stable pair over \( K \) is a \((d, \alpha)\)-marked locally stable pair \( (X, \Delta) \) over \( K \) such that \( K_X + \Delta \) is ample with \( \text{vol}(K_X + \Delta) = (K_X + \Delta)^d = v \).

When we are not concerned with the data \( d, \alpha, v \) and the marking we refer to \((X, \Delta)\) just as a stable pair (similarly for the locally stable case).

(2) Let \( f : X \to S \) be a flat morphism of schemes with \( S_2 \) fibres of pure dimension. A closed subscheme \( D \subset X \) is a relative Mumford divisor over \( S \) if there is an open subset \( U \subset X \) such that

- codimension of \( X_s \setminus U_s \) in \( X_s \) is \( \geq 2 \) for every \( s \in S \),
- \( D|_U \) is a relative Cartier divisor,
- \( D \) is the closure of \( D|_U \), and
- \( X \) is smooth at the generic points of \( X_s \cap D \) for every \( s \in S \).

By \( D|_U \) being relative Cartier we mean that \( D|_U \) is a Cartier divisor on \( U \) and that its support does not contain any irreducible component of any fibre \( U_s \).

Given a morphism \( T \to S \) of schemes, pulling back \( D|_U \) to \( T \times_S U \) and taking its closure gives a relative Mumford divisor \( D_T \) on \( T \times_S X \) over \( T \) which we refer to as the divisorial pullback of \( D \). In particular, for each \( s \in S \), we define \( D_s = D|_{X_s} \) to be the closure of \( D|_U \), which is the divisorial pullback of \( D \) to \( X_s \).

Now assume \( f : X \to S \) is a flat projective morphism of schemes with \( S_2 \) fibres of pure dimension. We can define a functor \( \mathcal{M}D\text{iv}(X/S) \) on the category of reduced schemes over \( S \) by setting

\[
\mathcal{M}D\text{iv}(X/S)(T) = \{ \text{relative Mumford divisors on } T \times_S X \text{ over } T \}
\]

for any morphism \( T \to S \) from a reduced scheme and by using divisorial pullback to define

\[
\mathcal{M}D\text{iv}(X/S)(T) \to \mathcal{M}D\text{iv}(X/S)(T')
\]

for a morphism \( T' \to T \) of reduced schemes over \( S \). Since over reduced bases relative Mumford divisors are the same as K-flat divisors [30, Definition 2 and point 6], this functor is represented by a reduced separated scheme \( \mathcal{M}D\text{iv}(X/S) \) over \( S \) [30, Theorem 4]. In particular, since this moduli space is a fine moduli space (as the functor
(3) Next we recall the definition of marked locally stable families. Let $S$ be a reduced scheme over $k$. A \textbf{$(d, \alpha)$-marked locally stable family} $f : (X, \Delta) \to S$ is given by a morphism $f : X \to S$ of schemes and closed subschemes $D_i \subset X$ for $i = 1, \ldots, m$ where

- $f$ is flat and projective with reduced geometrically connected $S_2$ fibres of pure dimension $d$ whose codimension one singularities are nodal,
- $D_i$ are relative Mumford divisors over $S$,
- $\Delta = \sum a_i D_i$,
- $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier, and
- for each $s \in S$, $(X_s, \Delta_s)$ is an slc pair over the residue field $k(s)$ where $\Delta_s = \sum a_i D_{i,s}$.

In particular, the log fibres $(X_s, \Delta_s)$ are $(d, \alpha)$-marked locally stable pairs. Note that in the above setting the dualising sheaf $\omega_{X/S}$ exists and commutes with base change \[31, 2.69\]. Moreover, since codimension one singularities of the fibres of $f$ are nodal, there is an open subset $V \subseteq X$ such that codimension of $X_s \setminus V_s$ in $X_s$ is $\geq 2$ and such that $\omega_{X/S}$ is locally free on $V$; thus $\omega_{X/S}$ corresponds to a canonical divisor class $K_{X/S}$; therefore, given the assumptions it makes sense to say that $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier. For more on dualising sheaves and canonical classes see \[31, 2.69, 2.70, 2.70.7, 3.66\].

(4) Let $S$ be a reduced scheme over $k$. A \textbf{$(d, \alpha, v)$-marked stable family} $f : (X, \Delta) \to S$ is a $(d, \alpha)$-marked locally stable family such that

- for each $s \in S$, $K_{X_s} + \Delta_s$ is ample with $\text{vol}(K_{X_s} + \Delta_s) = v$.

So the log fibres $(X_s, \Delta_s)$ are $(d, \alpha, v)$-marked stable pairs.

(5) We now define embedded versions of the above notions. Given a reduced scheme $S$ over $k$, a \textbf{strongly embedded $(d, \alpha, \mathbb{P}^n)$-marked locally stable family} $f : (X \subset \mathbb{P}^n_S, \Delta) \to S$ is a $(d, \alpha)$-marked locally stable family $f : (X, \Delta) \to S$ together with a closed embedding $g : X \to \mathbb{P}^n_S$ such that

- $f = \pi g$ where $\pi$ denotes the projection $\mathbb{P}^n_S \to S$; and
- letting $\mathcal{L} = g^* \mathcal{O}_{\mathbb{P}^n_S}(1)$, we have $R^q f_* \mathcal{L} \simeq R^q \pi_* \mathcal{O}_{\mathbb{P}^n_S}(1)$ for each $q$.

In particular, $f_* \mathcal{L} \simeq \pi_* \mathcal{O}_{\mathbb{P}^n_S}(1)$ and $R^q f_* \mathcal{L} = 0$ for $q > 0$.

If in addition,

- for each $s \in S$, $K_{X_s} + \Delta_s$ is ample with $\text{vol}(K_{X_s} + \Delta_s) = v$,

then we say the family is a \textbf{strongly embedded $(d, \alpha, v, \mathbb{P}^n)$-marked stable family}.

(6) Consider the moduli functor $\mathcal{E}^a \mathcal{M} \mathcal{L} \mathcal{S} \mathcal{P}_{d, \alpha, \mathbb{P}^n}$ of strongly embedded $(d, \alpha, \mathbb{P}^n)$-marked locally stable pairs from the category of reduced $k$-schemes to the category of sets by setting $\mathcal{E}^a \mathcal{M} \mathcal{L} \mathcal{S} \mathcal{P}_{d, \alpha, \mathbb{P}^n}(S) = \{\text{strongly embedded $(d, \alpha, \mathbb{P}^n)$-locally stable families over } S\}$.
For a morphism $T \to S$ of reduced schemes, the map

$$E^s\text{MLSP}_{d,\alpha,\mathbb{P}^n}(S) \to E^s\text{MLSP}_{d,\alpha,\mathbb{P}^n}(T)$$

is given via base change: given a family

$$(X \subset \mathbb{P}^n_S, \Delta = \sum a_iD_i) \to S,$$

we pull back each $D_i$ to $X_T := X \times_S T$ as in (2) which then determines $\Delta_T$ with an $\alpha$-marking, hence a strongly embedded $(d, \alpha, \mathbb{P}^n)$-marked locally stable family

$$(X_T \subset \mathbb{P}^n_T, \Delta_T) \to T.$$
7.3. Calabi-Yau pairs in families. In this subsection we aim to identify Calabi-Yau pairs that appear in a locally stable family.

Lemma 7.4. Assume that

- $X$ is a normal variety and $B \geq 0$ is an effective $\mathbb{Q}$-divisor,
- $X \to S$ is a contraction onto a variety over $k$,
- $\Pi \subset S$ is a dense set of closed points, and
- for general $s \in \Pi$, $(X_s, B_s)$ is an lc Calabi-Yau pair.

Then there exists a non-empty open set $U \subset S$ such that $(X, B)$ is lc Calabi-Yau over $U$, that is, $K_{X/S} + B \sim_{\mathbb{Q}} 0$ over $U$.

Proof. Shrinking $S$ we will assume $S$ is smooth. By general $s \in \Pi$ we mean points in $\Pi$ belonging to some appropriate open subset of $S$ which we shrink if necessary. In particular, for such $s$, $(X_s, B_s)$ is well-defined as we can assume that $B = cD$ for some relative Mumford divisor $D$ over $S$. Shrinking $S$ we can assume that $(X_s, B_s)$ is an lc Calabi-Yau pair for every $s \in \Pi$.

Let $\phi : X' \to X$ be a log resolution and let $B'_s$ be the sum of the birational transform of $B$ and the prime exceptional divisors of $\phi$ which are horizontal over $S$. Let $s \in \Pi$ and let $X'_s, X_s$ be the fibres of $X' \to S$ and $X \to S$ over $s$. By Lemma 2.5, shrinking $S$ we can assume that $X'_s$ is smooth and $X$, normal, that $\text{Supp} B'_s$ does not contain any irreducible component of any fibre so $B'_s := B'_s|_{X'_s}$ is well-defined, and that $B'_s$ is the sum of the birational transform of $B_s$ and the reduced exceptional divisor of the induced morphism $\psi : X'_s \to X_s$. Moreover, we can assume $(X'_s, B'_s)$ is lc.

By assumption, for $s \in \Pi$, $(X_s, B_s)$ is an lc Calabi-Yau pair, so

$$K_{X'_s} + B'_s = \psi^*(K_{X_s} + B_s) + E_s \sim_{\mathbb{Q}} E_s$$

for some effective and exceptional$/X_s$ divisor $E_s \geq 0$.

Let $A'$ be any ample$/S$ divisor on $X'$ and let $X'_s$ be the generic fibre of $X' \to S$. Pick $s \in \Pi$. Let $r$ be a natural number so that $r(K_{X'_s} + B'_s)$ is integral. Then by the upper-semi-continuity of cohomology, for any natural numbers $m, l$ we have

$$h^0(mr(K_{X'_s} + B'_s) + lA'|_{X'_s}) \geq h^0(mr(K_{X'_s} + B'_s) + lA'|_{X'_s})$$

where $B'_s = B'_s|_{X'_s}$. Since $K_{X'_s} + B'_s \sim_{\mathbb{Q}} E_s$ where $E_s$ is exceptional over $X_s$, fixing $l$ but varying $m$ the left hand side of the inequality is a bounded function of $m$. This in turn implies the right hand side is a bounded function of $m$. Therefore, the numerical Kodaira dimension $\kappa_\sigma(K_{X'_s} + B'_s) = 0$ or $-\infty$.

Assume $\kappa_\sigma(K_{X'_s} + B'_s) = -\infty$. This means that $h^0(mr(K_{X'_s} + B'_s) + lA'|_{X'_s}) = 0$ for any fixed $l$ and for any $m \gg 0$. In turn this is equivalent to saying that $K_{X'_s} + B'_s$ is not pseudo-effective (we can see this by passing to the algebraic closure of $k(\eta)$). Thus $K_{X'_s} + B'_s$ is not pseudo-effective over $S$, hence we can run an MMP$/S$ on $K_{X'_s} + B'_s$ ending with a Mori fibre space which implies that $\kappa_\sigma(K_{X'_s} + B'_s) = -\infty$ for general $t \in \Pi$, contradicting our assumption on $\Pi$.

So we can assume $\kappa_\sigma(K_{X'_s} + B'_s) = 0$. This means that $K_{X'_s} + B'_s$ is pseudo-effective and that for any fixed $l$, $h^0(mr(K_{X'_s} + B'_s) + lA'|_{X'_s})$ is a bounded function of $m$ but it does not uniformly vanish for $m \gg 0$. Passing to the algebraic closure of $k(\eta)$ and applying [19], we deduce that $h^0(mr(K_{X'_s} + B'_s)) \neq 0$ for some $m$. So $K_{X'_s} + B'_s \sim_{\mathbb{Q}} D'/S$ for some $\mathbb{Q}$-divisor $D' \geq 0$.

Shrinking $S$ we can assume $\text{Supp} D'$ does not contain any fibre of $X' \to S$. Now for each $s \in \Pi$ we get $K_{X'_s} + B'_s \sim_{\mathbb{Q}} D'_s \geq 0$. But since $K_{X'_s} + B'_s \sim_{\mathbb{Q}} E_s$ and since
Lemma 7.5. Assume that \((X, B) \to S\) is a locally stable family where \(S\) is a reduced \(k\)-scheme of finite type. Let \(S'\) be the set of (not necessarily closed) points \(s \in S\) such that \((X_s, B_s)\) is an slc Calabi-Yau pair. Then \(S'\) is a locally closed subset of \(S\) (with reduced structure) and the induced family \((X', B') \to S'\) obtained by base change is a family of Calabi-Yau pairs, that is, \(K_{X'/S'} + B' \sim_\mathbb{Q} 0/S'\).

Proof. Recall that we defined locally stable families only in the projective setting, so \(X \to S\) is assumed projective. Since the family is locally stable, \(S'\) is the set of points \(s\) such that \(K_{X_s} + B_s \sim_\mathbb{Q} 0\). Replacing \(S'\) with its closure in \(S\), we can assume that \(S'\) is dense in \(S\). Let \(p\) be a natural number such that \(p(K_{X/S} + B)\) is Cartier: such \(p\) exists as \(X\) is of finite type. Assume that for each \(s \in S\), \(K_{X_s} + B_s \sim_\mathbb{Q} 0\) if and only if \(p(K_{X_s} + B_s) \sim 0\). Then the lemma follows from [47, Lemma 1.19]. On the other hand, to find such a \(p\), it is enough to show that for each irreducible component \(T\) of \(S\) there is an open subset \(U \subset T\) such that \(K_{X/S} + B \sim_\mathbb{Q} 0\) over \(U\) because then we can replace \(S\) with \(S \setminus U\) and apply Noetherian induction. Thus replacing \(S\) with an irreducible component, we can assume that \(S\) is irreducible. We can shrink \(S\) to any non-empty open subset if necessary.

Let \(s \in S'\). Then there is an open subset \(R\) of the closure of \(s\) such that \(R \subset S'\). Since \(S'\) is dense in \(S\), we deduce that there is a subset \(\Pi \subset S'\) of closed points which is dense in \(S\).

Shrinking \(S\) we will assume \(S\) is smooth. Then \((X, B)\) is slc [38, Corollary 9]. Let \(X'\) be the normalisation of \(X\) and let \(K_{X'} + B'\) be the pullback of \(K_X + B\). Pick \(s \in \Pi\) and let \((X_s, B_s)\) and \((X'_s, B'_s)\) be the log fibres of \((X, B)\) and \((X', B')\) over \(s\). Then \(K_{X'_s} + B'_s \sim_\mathbb{Q} 0\) because \(K_{X_s} + B_s \sim_\mathbb{Q} 0\) by assumption, so \((X'_s, B'_s)\) is an lc Calabi-Yau pair on each irreducible component of \(X'_s\). Taking the Stein factorisation of \(X' \to S\) and applying Lemma 7.4 to the irreducible components of \(X'\) and shrinking \(S\) we can assume that \(K_{X'} + B' \sim_\mathbb{Q} 0/S\). Now by the gluing theory of [32] or by [25], \(K_X + B\) is semi-ample over \(S\), hence \(K_X + B \sim_\mathbb{Q} 0/S\) as \(K_{X'} + B' \sim_\mathbb{Q} 0/S\). Since \(S\) is smooth, \(K_{X/S} + B \sim_\mathbb{Q} 0/S\).

7.6. Moduli of embedded polarised Calabi-Yau pairs. We now turn our attention to polarised slc Calabi-Yau pairs. Before giving the proof of Theorem 1.10 we treat the moduli space of a related functor, an embedded version of \(\mathcal{PCY}_{d,c,v}\) that was defined in the introduction.

Lemma 7.7. \(d\) be a natural number and \(c, v\) be positive rational numbers. Then there exist a positive rational number \(t\) and a natural number \(r\) such that \(rc, rt \in \mathbb{N}\) satisfying the following. Assume \((X, B), N\) is a \((d, c, v)\)-polarised slc Calabi-Yau pair over a field \(K\) of characteristic zero. Then

- \((X, B + tN)\) is slc,
- \(B + tN\) uniquely determines \(B, N\), and
- \(r(K_X + B + tN)\) is very ample with

\[
h^1(mr(K_X + B + tN)) = 0
\]

for \(m, j > 0\).
Proof. Assume \((X, B), N\) is a \((d, c, v)\)-polarised slc Calabi-Yau pair over \(K\). By definition of such pairs, \(B = cD\) for some integral divisor \(D \geq 0\), so the coefficients of \(B\) belong to a fixed finite set \(\Phi\) depending only on \(c\). Moreover, \(N\) is an ample effective integral divisor with \(\text{vol}(N) = v\).

Assume that there exists \(t\) depending only on \(d, c, v\) such that \((X, B + tN)\) is slc. Then the coefficients of \(N\) are bounded from above, say by \(q\). If we decrease \(t\) so that \(t\) is a sufficiently small depending only on \(\Phi, q\), then the coefficients of \(B + tN\) uniquely determine the coefficients of \(B\): indeed this follows by ensuring that for distinct \(b, b' \in \Phi\) the intervals \([b, b + tr]\) and \([b', b' + tr]\) do not intersect.

To find \(t, r\) satisfying the first and the third assertions of the lemma, it is enough to assume \(K = \mathbb{C}\) by the Lefschetz principle noting that being slc ascends under field extensions in characteristic zero and properties such as Cartier, very ample, vanishing of cohomology, all ascend and descend under field extensions (such extensions are faithfully flat). In the case \(K = \mathbb{C}\), we apply Theorem 1.7 and Corollary 1.8.

\(\square\)

Let \(d, c, v, t, r\) be as in the lemma. Let \(n\) be a natural number. To simplify notation, let \(\Xi = (d, c, v, t, r, \mathbb{P}^n)\). Let \(S\) be a reduced scheme over \(k\). A strongly embedded \(\Xi\)-polarised Calabi-Yau family over \(S\) is a \((d, c, v)\)-polarised Calabi-Yau family \(f: (X, B), N \to S\) (as in 1.9) together with a closed embedding \(g: X \to \mathbb{P}^n_S\) such that

- \((X, B + tN) \to S\) is a stable family,
- \(f = \pi g\) where \(\pi\) denotes the projection \(\mathbb{P}^n_S \to S\),
- letting \(\mathcal{L} := g^* \mathcal{O}_{\mathbb{P}^n}(1)\), we have \(R^q f_* \mathcal{L} \simeq R^q \pi_* \mathcal{O}_{\mathbb{P}^n}(1)\) for all \(q\), and
- for every \(s \in S\), we have \(\mathcal{L}_s \simeq \mathcal{O}_{X_s}(r(K_{X_s} + B_s + tN_s))\).

We denote the family by

\[f: (X \subset \mathbb{P}^n_S, B), N \to S.\]

Define the functor \(E^* \text{PCY}_\Xi\) on the category of reduced \(k\)-schemes by setting

\[E^* \text{PCY}_\Xi(S) = \{\text{strongly embedded } \Xi\text{-polarised slc Calabi-Yau families over } S\}.\]

**Proposition 7.8.** The functor \(E^* \text{PCY}_\Xi\) is represented by a reduced separated \(k\)-scheme \(E^* \text{PCY}_\Xi\) of finite type.

**Proof.** We will follow standard arguments in moduli theory of varieties. Most of the technicalities needed are covered by [31].

**Step 1.** Put \(\alpha := (c, t)\). Recall the functor \(E^* \text{MLSP}_{d, \alpha, \mathbb{P}^n}\) and its fine moduli space \(M := E^* \text{MLSP}_{d, \alpha, \mathbb{P}^n}\) from Theorem 7.2. There is a strongly embedded \((d, \alpha, \mathbb{P}^n)\)-marked locally stable universal family

\[(X \subset \mathbb{P}^n_M, \Delta := cD + tN) \to M.\]

The moduli space \(M\) is very large, we only need parts of it related to the functor \(E^* \text{PCY}_\Xi\). More precisely, we want to identify the points of \(M\) which parametrise strongly embedded \(\Xi\)-polarised slc Calabi-Yau pairs.

**Step 2.** Suppose we are given a strongly embedded \(\Xi\)-polarised Calabi-Yau family

\[f: (X' \subset \mathbb{P}^n_S, B'), N' \to S.\]
The divisor $\Delta' := B' + tN'$ has a natural $\alpha$-marking as $B' = cD'$ for some relative Mumford divisor $D'$ over $S$, hence the family

$$(X' \subset \mathbb{P}^n_S, \Delta') \to S$$

is a strongly embedded $(d, \alpha, \mathbb{P}^n)$-marked locally stable family. Thus there is a unique morphism $S \to M$ so that the latter family is the pullback of the universal family over $M$. Moreover, by assumption, for every $s \in S$, we have

$$\mathcal{O}_{X_s}(1) \simeq \mathcal{O}_{X'_s}(r(K_{X'_s} + \Delta'_s)).$$

Again by assumption,

$$\text{vol}(r(K_{X'_s} + \Delta'_s)) = \text{vol}(rtN'_s) = r^d t^d v.$$ 

In particular, the degrees of $X'_s$ and $\Delta'_s$ with respect to $\mathcal{O}_{X'_s}(1)$ are bounded from above. Therefore, there is an open subscheme $M^{(0)} \subset M$ of finite type over $k$, independent of $S$, such that $S \to M$ factors through $M^{(0)}$.

Denote the pullback of the above universal family over $M$ to a family over $M^{(0)}$, by

$$(X^{(0)} \subset \mathbb{P}^n_{M^{(0)}}, \Delta^{(0)} = cD^{(0)} + tN^{(0)}) \to M^{(0)}.$$ 

To simply notation we will denote the log fibre of this family over a point $s$ by

$$(X_s \subset \mathbb{P}^n_s, \Delta_s = cD_s + tN_t)$$

that is, we drop the superscript $(0)$. We do similarly below.

**Step 3.** We apply the following process to distinguish the $\Xi$-polarised slc Calabi-Yau pairs among the fibres of the above universal family.

(i). First, the points $s$ of $M^{(0)}$ such that $K_{X_s} + \Delta_s$ is ample, is an open subset, say $M^{(1)}$. Pull back the family over $M^{(0)}$ to a family over $M^{(1)}$ using similar notation with $(0)$ replaced by $(1)$.

(ii). The volume $\text{vol}(K_{X_s} + \Delta_s)$ is locally constant on $M^{(1)}$. This can be seen as follows. Since $M^{(1)}$ is of finite type, there is a natural number $q$ such that $q(K_{X^{(1)}_s} + \Delta^{(1)}_s)$ is Cartier. Then the polynomial $\mathcal{A}(mq(K_{X_s} + \Delta_s))$ is locally constant on $M^{(1)}$. Now by the asymptotic Riemann-Roch formula, $\text{vol}(K_{X_s} + \Delta_s)$ is determined by the leading coefficient of the mentioned polynomial. Therefore, $\text{vol}(K_{X_s} + \Delta_s)$ is locally constant on $M^{(1)}$. Then the set $M^{(2)}$ of points $s \in M^{(1)}$ for which we have $\text{vol}(K_{X_s} + \Delta_s) = t^d v$, is an open and closed subset. The induced family

$$(X^{(2)} \subset \mathbb{P}^n_{M^{(2)}}, \Delta^{(2)} = cD^{(2)} + tN^{(2)}) \to M^{(2)}$$

is a family of strongly embedded $(d, \alpha, t^d v, \mathbb{P}^n)$-marked stable pairs.

(iii). By [31, Theorem 4.7], there is a locally closed partial decomposition $M^{(3)} \to M^{(2)}$ satisfying the following: given a morphism $S \to M^{(2)}$ from a reduced scheme, remove $tN^{(2)}$ in the family in (ii) and then pull back

$$(X^{(2)} \subset \mathbb{P}^n_{M^{(2)}}, cD^{(2)}) \to M^{(2)}$$

to a family over $S$, say

$$(X_S \subset \mathbb{P}^n_S, cD_S) \to S;$$

then $(X_S, cD_S) \to S$ is a locally stable family iff $S \to M^{(2)}$ factors through $M^{(3)} \to M^{(2)}$. 
Pull back the family in (ii) to a family over \( M^{(3)} \) to get
\[
(X^{(3)} \subset \mathbb{P}^n_{M^{(3)}}, \Delta^{(3)} = cD^{(3)} + tN^{(3)}) \to M^{(3)}.
\]
Then
\[
(X^{(3)} \subset \mathbb{P}^n_{M^{(3)}}, cD^{(3)}) \to M^{(3)}
\]
is locally stable. In particular, \( K_{X^{(3)/M^{(3)}}} + cD^{(3)} \) is \( \mathbb{Q} \)-Cartier, hence \( N^{(3)} \) is also \( \mathbb{Q} \)-Cartier.

(vi). By Lemma 7.5, the set of points \( s \in M^{(3)} \), say \( M^{(4)} \), over which we have
\[
K_{X_s} + cD_s \sim_\mathbb{Q} 0
\]
is locally closed. Pull back the family over \( M^{(3)} \) to get the family
\[
(X^{(4)} \subset \mathbb{P}^n_{M^{(4)}}, \Delta^{(4)} = cD^{(4)} + tN^{(4)}) \to M^{(4)}.
\]
Then by the lemma,
\[
K_{X^{(4)/M^{(4)}}} + cD^{(4)} \sim_\mathbb{Q} 0/M^{(4)}.
\]
Thus
\[
(X^{(4)}, cD^{(4)}), N^{(4)} \to M^{(4)}
\]
is a \((d, c, v)\)-polarised Calabi-Yau family.

(v). By Lemma 7.7, \( r(K_{X_s} + \Delta_s) \) is Cartier for every \( s \in M^{(4)} \). Then
\[
r(K_{X^{(4)/M^{(4)}}} + \Delta^{(4)})
\]
is Cartier by \([31, 4.36 \text{ and } 2.92]\). Consider the set of points \( s \in M^{(4)} \), say \( M^{(5)} \), over which we have
\[
O_{X_s}(1) \simeq O_{X_s}(r(K_{X_s} + \Delta_s))
\]
where \( O_{X_s}(1) \) is the pullback of \( O_{\mathbb{P}^n_{M^{(4)}}}(1) \) to \( X_s \). Then \( M^{(5)} \) is a locally closed subset of \( M^{(4)} \), by \([47, \text{ Lemma } 1.19]\). Pull back the family over \( M^{(4)} \) to a family over \( M^{(5)} \), using similar notation replacing (4) with (5). The resulting family is a strongly embedded \( \Xi \)-polarised Calabi-Yau family.

**Step 3.** We claim that \( M^{(5)} \) is the moduli space \( \mathbb{P}^s \text{PCY}_\Xi \). It is enough to show that for any strongly embedded \( \Xi \)-polarised Calabi-Yau family
\[
f: (X' \subset \mathbb{P}^n_{S}, B'), N' \to S,
\]
there is a unique morphism \( S \to M^{(5)} \) so that the family \( f \) is the pullback of the family
\[
(X^{(5)} \subset \mathbb{P}^n_{M^{(5)}}, cD^{(5)}), N^{(5)} \to M^{(5)}.
\]
As noted in Step 2, there is a unique morphism \( S \to M \). It is enough to show that \( S \to M \) factors through \( M^{(5)} \).

By Step 2, \( S \to M \) factors through \( M^{(0)} \). Looking at the definitions of \( M^{(i)} \) for \( i = 1, \ldots, 5 \) we can see that for each \( i \), \( S \to M \) factors through \( M^{(i)} \).
7.9. Moduli of polarised Calabi-Yau pairs.

Proof. (of Theorem 1.10) Step 1. Let $t, r$ be the numbers given by Lemma 7.7 for $d, c, v$. Let $f: (X, B), N \to S$ be a $(d, c, v)$-polarised Calabi-Yau family where $S$ is a reduced $k$-scheme. Put $\Delta = B + tN$. By the lemma, for each $s \in S$, $(X_s, \Delta_s)$ is slc, $r(K_{X_s} + \Delta_s)$ is very ample, and

$$h^q(r(K_{X_s} + \Delta_s)) = 0$$

for $q > 0$. Thus $(X, \Delta) \to S$ is a stable family. Moreover, there are finitely many possibilities for the number

$$n := \mathcal{N}(r(K_{X_s} + \Delta_s))$$

depending only on $d, c, v, t, r$: as in the proof of 7.7, this finiteness can be reduced to the case $s = \text{Spec} \mathbb{C}$ in which case we can apply 1.8. By [31, 4.36 and 2.92], $r(K_{X/S} + \Delta)$ is Cartier. Thus $n$ is locally constant on $S$.

Now fix $n$ and let $\mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}$ be the restriction of the functor $\mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v}$ to families $f$ in which $n = h^0(r(K_{X_s} + \Delta_s))$ for every $s$. Put $\Xi = (d, c, v, t, r, \mathbb{P}^n)$.

Step 2. Let $f: (X, B), N \to S$ be a family for the functor $\mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}$. By base change of cohomology,

$$R^q f_* \mathcal{O}_X(r(K_{X/S} + \Delta)) = 0$$

and for each $s \in S$, the natural map

$$R^q f_* \mathcal{O}_X(r(K_{X/S} + \Delta)) \otimes k(s) \to H^q(r(K_{X_s} + \Delta_s))$$

is an isomorphism, for each $q$. In particular,

$$f^* f_* \mathcal{O}_X(r(K_{X/S} + \Delta)) \to \mathcal{O}_X(r(K_{X/S} + \Delta))$$

is surjective, so we get a closed embedding

$$g: X \to \mathbb{P}(f_* \mathcal{O}_X(r(K_{X/S} + \Delta)))$$

over $S$. Therefore, for each point $s \in S$ we can shrink $S$ around $s$ so that we get a closed embedding into $\mathbb{P}_{S}^n$ in which case $f: (X \subset \mathbb{P}_{S}^n, B), N \to S$ is a strongly embedded $\Xi$-polarised Calabi-Yau family. However, the closed embedding is not unique: it is determined up to an automorphism of $\mathbb{P}_{S}^n$ over $S$.

Step 3. Now recall the functor $\mathcal{E}^s\mathcal{P}\mathcal{C}\mathcal{Y}_\Xi$ and its fine moduli space $E^s\mathcal{P}\mathcal{C}\mathcal{Y}_\Xi$ which is reduced, separated and of finite type over $k$, by Proposition 7.8. There is a natural action of $\text{PGL}_{n+1}(k)$ on $E^s\mathcal{P}\mathcal{C}\mathcal{Y}_\Xi$. This action is proper because automorphism groups of stable pairs are finite. The quotient

$$\mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n} := E^s\mathcal{P}\mathcal{C}\mathcal{Y}_\Xi/\text{PGL}_{n+1}(k)$$

is then an algebraic space [29][34] which is a coarse moduli space for the functor $\mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}$. Note that by Steps 1 and 2, given any family $f: (X, B), N \to S$ for the functor $\mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}$, there is an open covering $S = \cup S_i$ and morphisms $S_i \to E^s\mathcal{P}\mathcal{C}\mathcal{Y}_\Xi$ such that the induced morphisms $S_i \to \mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}$ are uniquely determined and they agree on overlaps, hence they determine a unique morphism $S \to \mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}$; moreover, the map

$$\mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}(\text{Spec } k) \to \mathcal{P}\mathcal{C}\mathcal{Y}_{d,c,v,n}(k)$$

is bijective in view of Lemma 7.7 (as $B + tN$ uniquely determines $B, N$).
The moduli space $\text{PCY}_{d,c,v,n}$ is proper because every family for $\text{PCY}_{d,c,v,n}$ over a smooth curve can be extended to a family over the compactification of the curve, after a finite base change, by [37, Lemma 7]. The moduli space is projective by [35][17][39]. Finally, the moduli space $\text{PCY}_{d,c,v}$ for the functor $\text{PCY}_{d,c,v}$ is the disjoint union of $\text{PCY}_{d,c,v,n}$ for the finitely many possible $n$.

7.10. Moduli of polarised Fano pairs.

Proof. (of Theorem 1.12) The proof can be carried out similar to that of Theorem 1.10 with some slight modifications which we outline. First, let $l$ be the common denominator of $c,a$ and let $c' = \frac{1}{l}$. Any $(d,c,a,v)$-polarised Fano family $(X,B + aN), N \to S$ is a $(d,c',v)$-polarised Calabi-Yau family because by definition $B = cD$ for some relative Mumford divisor $D \geq 0$, so $B + aN = c'(lcD + laN)$. Apply Lemma 7.7 to find $t,r$ for $d,c',v$.

Let $\Pi = (d,c,a,v,t,r,P^m)$ and define a functor $\mathcal{E}^s\mathcal{PF}_\Pi$ similar to $\mathcal{E}^s\mathcal{PCY}_\Xi$ in 7.6 which takes $(d,c,a,v)$-polarised Fano families $(X,B + aN), N \to S$ such that $(X,B + (a + t)N) \to S$ is stable, etc. Next show that there is a moduli space $\mathcal{E}^s\mathcal{PF}_\Pi$ for $\mathcal{E}^s\mathcal{PF}_\Pi$ as in 7.8 by modifying some of the arguments as follows: in Steps 1, use $\alpha = (c,a + t)$; in Step 3, (iii) and (iv), we use $cD + aN$ instead of $cD$ so that $K_{X(4)/M(4)} + cD(4) + aN(4) \cong \mathbb{Q}/M(4)$. Step 4 can be written similarly.

Next as in the proof of 1.10, we define a functor $\mathcal{PF}_{d,c,a,v,n}$ and construct a moduli space $\mathcal{PF}_{d,c,a,v,n}$ for it by taking the quotient of $\mathcal{E}^s\mathcal{PF}_\Pi$ by $\text{PGL}_{n+1}$, and then let $\mathcal{PF}_{d,c,a,v}$ be the union of the finitely many $\mathcal{PF}_{d,c,a,v,n}$.

8. Further remarks

We present some examples and remarks related to the some of the results in this paper.

Example 8.1. This example shows that we cannot drop the condition $N - K_X$ being pseudo-effective in Theorem 1.1 in general. Assume $X$ is a smooth projective curve and $N$ is one point on $X$. Let $m$ be the smallest natural number such that $|mN|$ defines a birational map. In general $m$ is not bounded. Assume not, that is, assume $m$ is bounded from above. Then $\text{vol}(mN) = m$ is bounded, hence $X$ is birationally bounded [24, Lemma 2.4.2(2)] which implies that $X$ is bounded, a contradiction as $X$ is an arbitrary smooth projective curve.

Example 8.2. This example shows that the condition $X$ having $\epsilon$-lc singularities in Theorem 1.1 cannot be replaced with just assuming $X$ having klt singularities. Let $X$ be the weighted projective surface $\mathbb{P}(p,q,r)$ where $p,q,r$ are coprime natural numbers. Then $X$ is a toric Fano surface with klt singularities. Let $N = -K_X$. Then

$$\text{vol}(N) = \frac{(p + q + r)^2}{pqr}$$

can get arbitrarily small meaning that there is no positive lower bound on $\text{vol}(N)$ [23, Example 2.1.1]. Thus if $m$ is a natural number such that $|mN|$ defines a birational map, then there is no upper bound on $m$ because $\text{vol}(N) \geq \frac{1}{m^2}$.
**Example 8.3.** This example shows that we cannot drop the nefness of $N$ in Theorem 1.1. Let $X$ be as in Example 8.2 and $N = -K_X$. Take a resolution $\phi: W \to X$ and write $K_W + E = \phi^*K_X$. Write $E = E' - E''$ where $E', E''$ are effective with no common component. Let $N_W = [\phi^*N + E'']$. Then $N_W = \phi^*N + E'' + G$ for some $G \geq 0$. By construction $W$ is smooth, $N_W$ is integral and big, and

$$N_W - K_W = \phi^*N + E'' + G - K_W$$

$$= \phi^*N + E'' + G - \phi^*K_X + E = 2\phi^*N + G + E'$$

is big. If $|mN_W|$ defines a birational map, then $|mN|$ also defines a birational map. But it was noted in 8.2 that $m$ depends on $X$ and in general not bounded.

**Remark 8.4.** If in addition to the assumptions of Corollary 1.2 we assume that the Cartier index of $N$ is bounded by some fixed number $p$, then the corollary essentially follows from the results of [13]. Indeed in this case we can find a bounded natural number $l$ such that $K_X + lN$ is big, so we can apply [13, Theorem 1.3] to deduce that $|m(K_X + lN)|$ defines a birational map for some bounded natural number $m$. In this case we do not need the $\epsilon$-lc condition on $X$.

**Example 8.5.** It is tempting to try to generalise Theorem 1.1 to the case when $N$ is only nef but not necessarily big. More precisely, assume $X$ is a projective variety with $\epsilon$-lc singularity with $\epsilon > 0$, of dimension $d$, and $N$ is a nef integral divisor with $N - K_X$ big. Then one may ask whether there is $m$ depending only on $\epsilon, d$ such that $|mN|$ defines the Iitaka fibration associated to $N$. Such an $m$ does not exist as the following example in dimension two shows. Assume $Y$ is the projective cone over an elliptic curve $T$ defined over $\mathbb{C}$, and $X$ is obtained by blowing up the vertex. Then $X$ is smooth and $Y$ has an lc (non-klt) singularity at the vertex but smooth elsewhere. Then taking $a > 0$ small, $-(K_X + (1 + a)T)$ is positive on both extremal rays of $X$, so it is ample. Thus $-K_X$ is big.

Pick a natural number $n$ and pick a torsion Cartier divisor $G$ on $T$ such that $nG \sim 0$ but $n'G \not\sim 0$ for any natural number $n' < n$ (we find such a $G$ by viewing $T$ as $\mathbb{C}/\Lambda$ for some lattice $\Lambda$). Let $N$ be the pullback of $G$ via $X \to T$. Then $|n'N|$ is empty for every natural number $n' < n$ although $N - K_X$ is big. Since $n$ can be arbitrarily large, there is no bounded $m$ independent of the choice of $N$ such that $|mN|$ defines the Iitaka fibration of $N$.

If we restrict ourselves to Fano type varieties, then at least conjecturally we expect much better behaviour with respect to the above problem.

**Conjecture 8.6.** Let $d$ be a natural number and $\epsilon$ be a positive real number. Then there is a natural number $m$ depending only on $d, \epsilon$ satisfying the following. Assume that

- $X$ is an $\epsilon$-lc projective variety of dimension $d$,
- $X \to Z$ is a contraction,
- $X$ is of Fano type over $Z$, and
- $N$ is an integral divisor which is nef over $Z$.

Let $f: X \to V/Z$ be the contraction defined by $N$. Then there exists an integral divisor $L$ on $V$ such that

- $mN \sim f^*mL$, and
- $|mL|_G$ defines a birational map where $G$ is the generic fibre of $V \to Z$. 

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The latter basically says that $mN$ defines a birational map over $Z$. Note that since $X$ is of Fano type over $Z$ and $N$ is nef over $Z$, $N$ is semi-ample over $Z$ so indeed it defines a contraction. The global case, i.e. when $Z$ is a point, is a generalisation of Theorem 1.1 for nef but not necessarily big divisors.

The conjecture is stronger than it may look at first sight. For example, consider the case when $X$ is a $\mathbb{Q}$-factorial $\epsilon$-lc projective variety of dimension $d$ and $X \to Z$ is a Mori fibre space where $Z$ is a curve. Assuming $N$ is the reduction of a fibre of $X \to Z$ (that is, a fibre with induced reduced structure), the conjecture implies that $mN \sim 0/Z$ for some $m$ depending only on $d, \epsilon$. It is not hard to see that this implies Shokurov’s conjecture on boundedness of singularities in fibrations [10, Conjecture 1.2]. Conversely, Shokurov’s conjecture combined with Theorem 1.1 implies the above conjecture. If $N$ is big over $Z$, then we can apply Theorem 1.1. Assume $N$ is not big over $Z$. Running an MMP on $K_X$ over $V$ and replacing $X$ with the resulting model we can assume we have a Mori fibre space $h: X \to T/Z$ such that $N \sim Q 0/T$. The fibres of $X \to T$ over closed points are $\epsilon$-lc Fano varieties, so they belong to a bounded family [8, Theorem 1.1]. Thus $lN$ is Cartier near the generic fibre of $X \to T$ for some bounded $l \in \mathbb{N}$, so $lN \sim 0$ over the generic point of $T$. Applying Shokurov conjecture to $(X, B) \to T$ for some general $0 \leq B \sim Q -K_X/T$ and then applying Lemma 3.5 and replacing $l$ we can assume $lN \sim h^*D$ for some integral divisor $D$.

Replacing $X, N$ with $T, D$ we can apply induction on dimension.

**Remark 8.7.** One may wonder if in Theorem 1.1 and Conjecture 8.6 (say when $Z$ is a point) we can choose $m$ so that $|mN|$ is base point free. But this is not the case. For example, there exist a log smooth lc pair $(X, S)$ of dimension two and a nef and big divisor $N$ such that $K_X + S \sim 0$ and $S$ is the stable base locus of $N$ [41, §2.3.A] $(X$ can be obtained by blowing up $\mathbb{P}^2$ in 12 suitable points on an elliptic curve; $S$ is then the birational transform of the elliptic curve; $N$ is also constructed using the 12 points). In particular, $N - K_X \sim N + S$ is big but $|mN|$ is not base point free for any $m$.

On the other hand, there are 3-folds $X$ with terminal singularities and $N := K_X$ ample but with arbitrarily large Cartier index, so $|mN|$ cannot be free for bounded $m$.

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