Realizations of automorphism groups of metric graphs induced by rational maps

Song JuAe *

Abstract

For a rational map $\phi$ from a metric graph $\Gamma$ to a tropical projective space $TP^n$ defined by a ratio of rational functions $f_1, \ldots, f_{n+1}$, an automorphism $\sigma$ of $\Gamma$ induces a permutation of the coordinates of $TP^n$ if $\{f_1, \ldots, f_{n+1}\}$ is $\langle \sigma \rangle$-invariant. Through this description, we can realize the automorphism group of $\Gamma$ as ambient automorphism group such as tropical projective general linear group, tropical general linear group and $\mathbb{Z}$-linear transformation group of Euclidean space.

keywords: metric graphs, automorphism groups of metric graphs, rational maps, linear systems

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1 Introduction

A metric graph $\Gamma$ is the underlying metric space of the pair of a graph $G$ and a length function $l : E(G) \to \mathbb{R}_{>0}$. Here, a graph means an unweighted, undirected, finite, connected nontrivial multigraph and we allow the existence of loops and $E(G)$ denotes the set of edges of $G$. In this paper, we give a way to realize (finite) automorphism groups of metric graphs as ambient automorphism groups such as tropical projective linear groups, tropical linear groups and $\mathbb{Z}$-linear transformation groups of Euclidean spaces. We can simultaneously get these realizations by choosing one suitable set of rational functions, which is easy to find. These realizations can be given by permutation matrices.

*Tokyo Metropolitan University 1-1 Minami-Ohsawa, Hachioji, Tokyo, 192-0397, Japan. E-mail: song-jaue@ed.tmu.ac.jp
For any graph $G$, let $1$ be the length function mapping all edge to one. Then for the metric graph $\Gamma$ obtained from the pair $(G,1)$, we have a natural inclusion $\text{Aut}(G) \hookrightarrow \text{Aut}(\Gamma)$, where $\text{Aut}(G)$ and $\text{Aut}(\Gamma)$ denote the automorphism groups of $G$ and $\Gamma$, respectively. $\text{Aut}(\Gamma)$ corresponds with the isometry transformation group of $\Gamma$ (cf. [1] for the definition of (finite harmonic) morphism between metric graphs). Thus, we can also have realizations of automorphism groups of graphs by our constructions.

Let $K$ be a complete algebraically closed non-Archimedean field with non-trivial valuation, and let $X, X'$ be smooth, proper, connected curves over $K$. If $\varphi : (X', V' \cup D') \to (X, V \cup D)$ is a tame covering of triangulated punctured curve $(X, V \cup D)$ $(V, V')$ are semistable vertex sets of $X, X'$ respectively and $D \subset X(K), D' \subset X'(K)$ punctures; see Definitions 3.8, 3.9, 4.25, 4.31 in [1] and the skeleton $\Sigma$ obtained from $(X, V \cup D)$ (see Subsection 3.7 of [1]) has no loops, then the natural group homomorphism $\psi : \text{Aut}_X(X') \to \text{Aut}_{\Sigma}(\Sigma')$ is injective by Theorem 7.4 (1) and Remark 7.5 in [1]. Here, $\Sigma'$ denote the skeleton obtained from $(X', V' \cup D')$ and $\text{Aut}_X(X'), \text{Aut}_{\Sigma}(\Sigma')$ the automorphism groups of $\varphi$ and $\varphi_{|\Sigma}$, respectively. (More precisely, see [1] and [2].) $\text{Aut}_X(X')$ is a subset of the automorphism group of $(X', V' \cup D')$, so the automorphism group $\text{Aut}(X')$ of $X'$. And there is a natural group homomorphism $\psi'$ from $\text{Aut}_{\Sigma}(\Sigma')$ to the automorphism group $\text{Aut}(\Gamma')$ of the underlying metric graph $\Gamma'$ (which may not be injective). Therefore we can realize subgroups of $\text{Aut}(X')$ of the form $\text{Aut}_X(X')$ as the image of $\psi' \circ \psi \subset \text{Aut}(\Gamma')$, and so as subgroup of our three groups.

We make an (injective) group homomorphism $\Psi$ from the automorphism group of a metric graph $\Gamma$ to the $\mathbb{Z}$-linear transformation group $\mathbb{Z}\text{-lin}(\mathbb{R}^n)$ of $\mathbb{R}^n$ such that each automorphism of $\Gamma$ and the image by $\Psi$ are commutative with a rational map $\Gamma \to TP^n \subseteq \mathbb{R}^n$, where $T$ is the tropical semifield $(\mathbb{R} \cup \{-\infty\}, \max, +), TP^n$ is the $n$-dimensional tropical projective space and $i$ is the inclusion $i : \mathbb{R}^n \to TP^n; (X_1, \ldots, X_n) \mapsto (X_1 : \cdots : X_n : 0)$. Note that we mean this inclusion $i$ whenever we write $\mathbb{R}^n \subset TP^n$. Concurrently, we also make other two realizations. To make the group homomorphism, the following simple proposition is important.

**Proposition 1.** Let $\Gamma$ be a metric graph and $f_1, \ldots, f_{n+1}$ distinct rational functions on $\Gamma$ other than the constant $-\infty$ function. Let $\phi : \Gamma \to TP^n; x \mapsto (f_1(x) : \cdots : f_{n+1}(x))$ be the induced rational map. For $\sigma \in \text{Aut}(\Gamma)$, if $\{f_1, \ldots, f_{n+1}\}$ is $\langle \sigma \rangle$-invariant, then $\sigma$ extends to a $\mathbb{Z}$-linear transformation of $\mathbb{R}^n \subset TP^n$ through $\phi$, i.e., there is a regular $(n+1) \times (n+1)$ matrix $A_\sigma$
whose all coefficients are integers such that \(i^{-1}(\phi(x))) = \sigma(i^{-1}(\phi(x)))\) holds for any \(x \in \Gamma\).

Here, \("\{f_1, \ldots, f_{n+1}\}\) is \(\langle \sigma \rangle\)-invariant" means that for any \(k\), there exists a unique \(l\) such that \(f_k \circ \sigma = f_l\). The following two propositions, other two cases we want, clearly hold since each permutation matrix is regular (see Subsection 2.1).

**Proposition 2.** Let \(\Gamma\) be a metric graph and \(f_1, \ldots, f_{n+1}\) distinct rational functions on \(\Gamma\) other than the constant \(-\infty\) function. Let \(\phi : \Gamma \to TP^n; x \mapsto (f_1(x) : \cdots : f_{n+1}(x))\) be the induced rational map. For \(\sigma \in \text{Aut}(\Gamma)\), if \(\{f_1, \ldots, f_{n+1}\}\) is \(\langle \sigma \rangle\)-invariant, then \(\sigma\) extends to a permutation matrix in the tropical projective linear group \(PGL_{\text{trop}}(n+1, T)\).

**Proposition 3.** Let \(\Gamma\) be a metric graph and \(f_1, \ldots, f_{n+1}\) distinct rational functions on \(\Gamma\) other than the constant \(-\infty\) function. Let \(\phi : \Gamma \to TP^n; x \mapsto (f_1(x), \ldots, f_n(x))\) be the induced rational map. For \(\sigma \in \text{Aut}(\Gamma)\), if \(\{f_1, \ldots, f_n\}\) is \(\langle \sigma \rangle\)-invariant, then \(\sigma\) extends to a permutation matrix in the tropical linear group \(GL_{\text{trop}}(n, T)\).

\(TP^n\) denotes the \(n\)-dimensional tropical affine space and see Subsection 2.1 for the definitions of \(PGL_{\text{trop}}(n+1, T)\) and \(GL_{\text{trop}}(n, T)\).

By these propositions, our next goal is to find a way to get such rational functions. As an answer, we use a complete linear system; from Proposition 1 we have the following corollary, which is the case that the rational map is induced by a complete linear system.

**Corollary 4.** Let \(\Gamma\) be a metric graph, \(D\) a divisor on \(\Gamma\). For \(\sigma \in \text{Aut}(\Gamma)\), if the \(\langle \sigma \rangle\)-invariant linear system \(|D|^{\langle \sigma \rangle}\) is not empty, there exists a minimal generating set of \(R(D)\) such that \(\sigma\) extends to a \(\mathbb{Z}\)-linear transformation of \(R^n \subset TP^n\) through the induced rational map \(\Gamma \to TP^n\).

Here, \(R(D)\) denotes the set of rational functions corresponding to elements of the complete linear system \(|D|\) together with the constant \(-\infty\) function and \(|D|^{\langle \sigma \rangle}\) is \(\{D' \in |D| \mid \forall x \in \Gamma, D'(\sigma(x)) = D'(x)\}\) (which becomes a linear system; see Theorem 3.17 in [1]). From Propositions 2 and 3, we have the following two corollaries:

**Corollary 5.** Let \(\Gamma\) be a metric graph and \(D\) a divisor on \(\Gamma\). For \(\sigma \in \text{Aut}(\Gamma)\), if the \(\langle \sigma \rangle\)-invariant linear system \(|D|^{\langle \sigma \rangle}\) is not empty, there exists a minimal generating set of \(R(D)\) such that \(\sigma\) extends to a permutation matrix in \(PGL_{\text{trop}}(n+1, T)\) through the induced rational map \(\Gamma \to TP^n\).
Corollary 6. Let $\Gamma$ be a metric graph and $D$ a divisor on $\Gamma$. For $\sigma \in \text{Aut}(\Gamma)$, if the $\langle \sigma \rangle$-invariant linear system $|D|^{\langle \sigma \rangle}$ contains an element $D'$, there exists a $\langle \sigma \rangle$-invariant minimal generating set of $R(D')$ such that $\sigma$ extends to a permutation matrix in $\text{GL}_{\text{trop}}(n, T)$ through the induced rational map $\Gamma \to T^n$.

By these corollaries, we can realize subgroups of automorphism groups of metric graphs which is generated by one element. Next, we prove each finite subgroup case; by Corollary 4, we prove the following theorem:

Theorem 7. Let $\Gamma$ be a metric graph and $D$ a divisor on $\Gamma$. Assume that the complete linear system $|D|$ induces an injective rational map $\Gamma \hookrightarrow T^P_n$. For a finite subgroup $G$ of $\text{Aut}(\Gamma)$, if the $G$-invariant linear system $|D|^G$ is not empty, then there exists a minimal generating set of $R(D)$ which induces an injective group homomorphism from $G$ to $\mathbb{Z}$-linear transformation group of $R^n \subset T^P_n$ such that each element of $G$ and the image are commutative with the induced rational map (which may not be the original one).

$|D|^G$ is the set $\{D' \in |D| | \forall x \in \Gamma, \forall \sigma \in G, D'(\sigma(x)) = D'(x)\}$ (and becomes a linear system by Theorem 3.17 in [7] again). Since for a metric graph $\Gamma$ which is not homeomorphic to a circle, $\text{Aut}(\Gamma)$ is finite, by this theorem, we can realize it as a subgroup of $\mathbb{Z}$-lin$(R^n)$. Other two cases are as follows:

Theorem 8. Let $\Gamma$ be a metric graph and $D$ a divisor on $\Gamma$. Assume that the complete linear system $|D|$ induces an injective rational map $\Gamma \hookrightarrow T^P_n$. For a finite subgroup $G$ of $\text{Aut}(\Gamma)$, if the $G$-invariant linear system $|D|^G$ is not empty, then there exists a minimal generating set of $R(D)$ which induces an injective group homomorphism from $G$ to $\text{PGL}_{\text{trop}}(n+1, T)$ such that the image consists only of permutation matrices and each element of $G$ and the image are commutative with the induced rational map (which may not be the original one).

Theorem 9. Let $\Gamma$ be a metric graph and $D$ a divisor on $\Gamma$. Assume that the complete linear system $|D|$ induces an injective rational map $\Gamma \hookrightarrow T^P_n$. For a finite subgroup $G$ of $\text{Aut}(\Gamma)$, if the $G$-invariant linear system $|D|^G$ contains an element $D'$, then there exists a $G$-invariant minimal generating set of $R(D')$ which induces an injective group homomorphism from $G$ to $\text{GL}_{\text{trop}}(n+1, T)$ such that the image consists only of permutation matrices and each element of $G$ and the image are commutative with the induced rational map $\Gamma \to T^{n+1}$.
“A minimal generating set of $R(D)$ is $G$-invariant” means that it is $\langle \sigma \rangle$-invariant for any $\sigma \in G$. Since canonically $R^n \subset T^n$ and each $n \times n$ permutation matrix is in $Z$-$\text{lin}(R^n)$, we have the following from Theorem 9.

**Theorem 10.** Let $\Gamma$ be a metric graph and $D$ a divisor on $\Gamma$. Assume that the complete linear system $|D|$ induces an injective rational map $\Gamma \hookrightarrow TP^n$. For a finite subgroup $G$ of $\text{Aut}(\Gamma)$, if the $G$-invariant linear system $|D|^G$ contains an element $D'$, then there exists a $G$-invariant minimal generating set of $R(D')$ which induces an injective group homomorphism from $G$ to $Z$-$\text{lin}(R^{n+1})$ such that the image consists only of permutation matrices and each element of $G$ and the image are commutative with the induced rational map $\Gamma \to R^{n+1}$.

One advantage of Theorem 7 compared to Theorem 10 is that the dimension of the Euclidean space in Theorem 7 is that in Theorem 10 minus one. One disadvantage of Theorem 7 compared to Theorem 10 is that the image of group homomorphism in Theorem 10 consists only of permutation matrices but not in Theorem 7.

This paper is organized as follows. Section 2 briefly reviews some basics of tropical algebra and of metric graphs including how to make rational maps induced by (complete) linear systems, which were given in [5]. Proofs of Proposition 1, Corollaries 4, 5, 6, Theorems 7, 8, 9 are given in Section 3. The section includes one corollary of Theorems 7, 8, 9, 10 and three examples of low genus metric graph cases.

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# 2 Preliminaries

In this section, we recall some basic facts of tropical algebra and of metric graphs and some results in [5] which we need later.

## 2.1 Tropical algebra

Let $T$ be the algebraic system $(R \cup \{-\infty\}, \max, +)$. We write the maximum operation max as $\oplus$, the ordinary addition + as $\odot$, respectively. $T$ becomes
a semifield with these two operations and is called tropical semifield. As in the conventional algebra, we extend these two operations to matrices and vectors. By the $n + 1$ dimensional tropical (affine) space $T^{n+1}$ and tropical scalar multiplication by $T^\times = \mathbb{R}$, we can define the $n$ dimensional tropical projective space $T\mathbb{P}^n$ as $T^{n+1}/T^\times$ as in the conventional algebra. $I \in T^{n \times n}$ denotes the identity matrix. A tropical matrix $A \in T^{n \times n}$ is regular or invertible if there exists a tropical matrix $B \in T^{n \times n}$ such that $A \circ B = B \circ A = I$. [3] and [4] show that the only tropical regular matrices are generalized permutation matrices. Here, a permutation matrix is a matrix obtained by permuting the rows and/or the columns of the identity matrix and a generalized permutation matrix is the product of a diagonal matrix and a permutation matrix. The tropical general linear group $\text{GL}_{\text{trop}}(n, T)$ is defined to be the set of all tropical regular square matrices of order $n$. The tropical projective linear group $\text{PGL}_{\text{trop}}(n, T)$ is defined to be $\text{GL}_{\text{trop}}(n, T)$ modulo tropical scalar multiplication by $T^\times$.

2.2 Metric graphs and related basic facts

Let $\Gamma$ be a metric graph. The genus $g(\Gamma)$ of $\Gamma$ is its first Betti number. We have the equality $g(\Gamma) = \#E(G) - \#V(G) + 1$, where $V(G)$ is the set of vertices of $G$, respectively, for any underlying graph $G$ of $\Gamma$.

Let $\text{Div}(\Gamma)$ be the free abelian group generated by all points of $\Gamma$, i.e., $\text{Div}(\Gamma) := \oplus_{x \in \Gamma} \mathbb{Z}x$. An element of $\text{Div}(\Gamma)$ is a divisor on $\Gamma$. When $D$ is a divisor on $\Gamma$, the sum of all coefficients of $D$ is called the degree of $D$. For a point $x \in \Gamma$, the degree of $D$ at $x$ is denoted by $D(x)$. $D$ is effective, written by $D \geq 0$, if all coefficients of $D$ are nonnegative. The set of all points of $\Gamma$ where the coefficients of $D$ are not zero is called the support of $D$.

Let $f: \Gamma \to \mathbb{R} \cup \{-\infty\}$ be a continuous map. $f$ is a rational function on $\Gamma$ if $f \equiv -\infty$ or $f$ is a piecewise $\mathbb{Z}$-affine function. Let $\text{Rat}(\Gamma)$ denote the set of all rational functions on $\Gamma$. For $f, g \in \text{Rat}(\Gamma)$ and $a \in T$, we define tropical sum of $f$ and $g$, and tropical scalar multiplication of $f$ by $a$ as pointwise tropical operations, i.e., $(f \oplus g)(x) := \max\{f(x), g(x)\}, (a \odot f)(x) := a + f(x)$ for any $x \in \Gamma$. By these operations, $\text{Rat}(\Gamma)$ becomes a tropical semimodule over $T$. Note that in fact we can define tropical multiplication on $\text{Rat}(\Gamma)$ and this makes $\text{Rat}(\Gamma)$ a tropical semiring over $T$. However, we need not this fact in this paper.

For $f \in \text{Rat}(\Gamma)^\times = \text{Rat}(\Gamma) \setminus \{-\infty\}$ and $x \in \Gamma$, let $\text{ord}_x(f)$ denote the sum of the outgoing slopes of $f$ at $x$. The principal divisor $\text{div}(f)$ defined
by \( f \) is \( \sum_{x \in \Gamma} \text{ord}_x(f) \cdot x \). We define a relation \( \sim \) on \( \text{Div}(\Gamma) \) as follows. For \( D_1, D_2 \in \text{Div}(\Gamma) \), \( D_1 \sim D_2 \) if there exists \( f \in \text{Rat}(\Gamma)^\times \) such that \( \text{div}(f) = D_1 - D_2 \). This relation \( \sim \) becomes an equivalence relation, which is called linear equivalence. By the linear equivalence \( \sim \), for a divisor \( D \) on \( \Gamma \), the complete linear system \( |D| \) associated to \( D \) is defined as the set of all effective divisors linearly equivalent to \( D \). Corresponding to the complete linear system \( |D| \), we write \( R(D) \) as the union \( \{ f \in \text{Rat}(\Gamma)^\times \mid D + \text{div}(f) \geq 0 \} \cup \{-\infty\} \). Then \( R(D) \) becomes a tropical subsemimodule over \( T \) of \( \text{Rat}(\Gamma) \) with the tropical sum and scalar multiplication ([5, Lemma 4]).

It is not clear that \( R(D) \) is finitely generated, however, in fact it is true. In [5], the authors proved that \( R(D) \) is generated by the extremals and the set of all extremals is unique and finite up to the tropical scalar multiplication and a complete system of representatives is minimal ([5, Corollary 9]). Here, \( f \in R(D) \) is called extremal if \( g, h \in R(D) \), \( f = g \oplus h \) implies \( f = g \) or \( f = h \). Extremals are characterized in the language of subgraphs:

**Lemma 11 ([5, Lemma 5])**. A rational function \( f \) is an extremal of \( R(D) \) if and only if there are not two proper subgraphs \( \Gamma_k \) (i.e. \( \Gamma_k \not= \Gamma, \emptyset \)) covering \( \Gamma \) (i.e. \( \Gamma_1 \cup \Gamma_2 = \Gamma \)) such that each can fire on \( D + \text{div}(f) \).

Here, a subgraph of \( \Gamma \) means a compact subset of \( \Gamma \) with a finite number of connected components and a subgraph \( \Gamma' \) of \( \Gamma \) can fire on a divisor \( D \) if for any its boundary point \( x_0 \), the outdegree of \( \Gamma' \) at \( x_0 \) in \( \Gamma \) is not greater than the coefficient of \( D \) at \( x_0 \). Note that by Lemma 11, we can find all extremals of \( R(D) \) when \( \Gamma \) and \( D \) are given concretely. Especially, it suffices that we look into only subgraphs whose all boundary points are in the support of \( D + \text{div}(f) \) to check whether a rational function \( f \) is an extremal of \( R(D) \).

**Remark 12.** Let \( D \sim D' \). Then \( n = m \) and \( \{ D + \text{div}(f_1), \ldots, D + \text{div}(f_n) \} = \{ D' + \text{div}(g_1), \ldots, D' + \text{div}(g_n) \} \) for any minimal generating sets \( \{ f_1, \ldots, f_n \} \) of \( R(D) \) and \( \{ g_1, \ldots, g_m \} \) of \( R(D') \). In fact, as \( D \sim D' \), there exists a rational function \( h \in \text{Rat}(\Gamma)^\times \) such that \( D' = D + \text{div}(f) \). Therefore \( R(D) \) is isomorphic to \( R(D') \) via \( R(D) \to R(D'); h \mapsto h - f \) (the inverse correspondence is given by \( R(D') \to R(D); h \mapsto h + f \)) and we have \( D + \text{div}(f_k) = D' - \text{div}(f) + \text{div}(f_k) = D' + \text{div}(f_k - f) \) for any \( k \). Since \( f_k \) is an extremal of \( R(D) \), by Lemma 11 there are not two proper subgraphs covering \( \Gamma \) such that each can fire on \( D + \text{div}(f) = D' + \text{div}(f_k - f) \). Thus \( f_k - f \) is an extremal of \( R(D') \), and this means conclusions we wanted above.
Remark 13. For \( \sigma \in \text{Aut}(\Gamma) \), if \( D \) is \( (\sigma) \)-invariant (i.e. for any \( x \in \Gamma \), \( D(\sigma(x)) = D(x) \) holds), then an extremal \( f \) of \( R(D) \) is mapped by \( \sigma \) to another extremal (possibly \( f \) itself) of \( R(D) \). In fact, \( f \circ \sigma \) is in \( R(D) \) since
\[
0 \leq (D + \text{div}(f))(\sigma(x)) = D(\sigma(x)) + (\text{div}(f))(\sigma(x)) = D(x) + (\text{div}(f \circ \sigma))(x)
\]
hold for any \( x \in \Gamma \). If \( f \circ \sigma \) is not an extremal of \( R(D) \), then by Lemma 11, there are two proper subgraphs \( \Gamma_1 \) and \( \Gamma_2 \) covering \( \Gamma \) such that each can fire on \( D + \text{div}(f \circ \sigma) \). The proper subgraphs \( \sigma^{-1}(\Gamma_1) \) and \( \sigma^{-1}(\Gamma_2) \) cover \( \Gamma \) and each can fire on \( D + \text{div}(f) \), and this means that \( f \) is not an extremal of \( R(D) \) by Lemma 11 again.

Remark 14. For a finite subgroup \( G \) of \( \text{Aut}(\Gamma) \), if \( D \) is \( G \)-invariant (i.e. for any \( \sigma \in G \), \( x \in \Gamma \), \( D(\sigma(x)) = D(x) \) holds), then there exists a \( G \)-invariant minimal generating set \( \{f_1, \ldots, f_n\} \) of \( R(D) \). In fact, it is enough to choose each \( f_k \) as the maximum value is zero. For any \( \sigma \in G \), for any \( k \), there exists a unique \( l \) such that \( f_k \circ \sigma = f_l \) since \( f_k \) and \( f_l \) have the same maximum value zero and by Remark 13. Section 2 of [5] is also helpful to understand this argument. Since every rational function is an (ordinary) sum of chip firing moves plus a constant by Lemma 2 of [5], choosing the maximum value of \( f_k \) as zero corresponds to choosing this constant as zero.

For a divisor \( D \) on \( \Gamma \), there is a natural one-to-one correspondence between the complete linear system \( |D| \) and the projection of \( R(D) \), i.e., \( PR(D) = (R(D) \setminus \{-\infty\})/T^\times \). Thus \( |D| \) has a structure of finitely generated tropical projective space and induces a rational map from \( \Gamma \) to a tropical projective space. Concretely, for a minimal generating set \( \{f_1, \ldots, f_{n+1}\} \) of \( R(D) \), which all are extremals of \( R(D) \), the rational map \( \phi_{|D|} : \Gamma \to TP^n \) induced by \( |D| \) is given by the correspondence \( x \mapsto (f_1(x) : \cdots : f_{n+1}(x)) \) for any \( x \in \Gamma \). Note that we use the ratio in tropical meaning and there is an arbitrariness of the choice of a minimal generating set \( \{f_1, \ldots, f_{n+1}\} \) of \( R(D) \). Exchanging \( \{f_1, \ldots, f_{n+1}\} \) to another minimal generating set of \( R(D) \) induces a (classical) parallel translation of the image and a renumbering. In other word, \( |D| \) define a rational map up to the action of \( \text{PGL}_{\text{trop}}(n+1, T) \) on \( TP^n \). We can always find a divisor whose complete linear system induces an injective rational map (cf. [3 Theorem 45]). We can define a distance function on the image of a rational map and with this distance function, an injective rational map \textit{induced} by a complete linear system always becomes an isometry (see [8]), but in this paper, we need not this fact.
3 Main results

In this section, we give proofs of our main results, their corollaries and some examples.

First, we give our proof of Proposition 1.

Proof of Proposition 1. Since \( \{f_1, \ldots, f_{n+1}\} \) is \( \langle \sigma \rangle \)-invariant, \( \sigma \) induces a permutation of \( \{1, \ldots, n+1\} \). There is a number \( s \) in \( \{1, \ldots, n+1\} \) such that \( \sigma(s) = n+1 \). Let \( A_\sigma = (a_{k,l})_{1 \leq k,l \leq n} \) be the \( n \times n \) matrix given by

\[
a_{k,l} := \begin{cases} 
1 & \text{if } k \neq s \text{ and } l = \sigma(k), \\
-1 & \text{if } l = \sigma(n+1), \text{ and} \\
0 & \text{otherwise.} 
\end{cases}
\]

Then, \( \sigma \) and \( A_\sigma \) are commutative with \( \phi \). \( \square \)

Remark 15. In the construction of the \( n \times n \) matrix \( A_\sigma \) in the above proof, we can see a peculiar phenomenon in the tropical world that we can make \( A_\sigma \) as a \( \mathbb{Z} \)-linear transformation of \( \mathbb{R}^n \) unlike classical case since the tropical division is the usual subtraction.

We specify here that the proof of Proposition 1 was inspired by that of Corollary 7.5 in [6] and thank the authors for their great works.

Proof of Corollary 4. By the assumption, there is an element \( D' \in |D|^{\langle \sigma \rangle} \), and thus there is a rational function \( f \in R(D) \setminus \{-\infty\} \) such that \( D' = D + \text{div}(f) \). By Remark 13 and tropical scalar multiplication, there exists a \( \langle \sigma \rangle \)-invariant minimal generating set \( \{g_1, \ldots, g_{n+1}\} \) of \( R(D') \). By Proposition 1 there is an \( n \times n \) matrix \( A_\sigma \) whose all coefficients are integers, and which and \( \sigma \) are commutative with the induced rational map \( \Gamma \to \mathbb{T}P^n; x \mapsto (g_1(x): \cdots: g_{n+1}(x)) \). For each \( k \), let \( f_k := g_k + f \). By Remark 12 \( \{f_1, \ldots, f_{n+1}\} \) is a minimal generating set of \( R(D) \). Since \( (g_1(x): \cdots: g_{n+1}(x)) = (f_1(x)-f(x): \cdots: f_{n+1}(x)-f(x)) = (f_1(x): \cdots: f_{n+1}(x)) \) hold for any \( x \in \Gamma \), \( j(\phi(\sigma(x))) = \langle A_\sigma \rangle^t(j(\phi(x))) \) holds with \( \phi : \Gamma \to \mathbb{T}P^n; x \mapsto (f_1(x): \cdots: f_{n+1}(x)) \) and \( j : \text{Im}(i) \hookrightarrow \mathbb{R}^n; (X_1: \cdots: X_{n+1}) \mapsto (X_1 - X_{n+1}, \ldots, X_n - X_{n+1}) \).

Using \( \{f_1, \ldots, f_{n+1}\} \) (resp. \( \{g_1, \ldots, g_{n+1}\} \)) in the above proof and Proposition 2 (resp. Proposition 3), we have Corollary 5 (resp. Corollary 6).

Proof of Theorem 7. By Corollary 1, Remark 14 and the injectivity of induced rational map, we have the conclusion. \( \square \)
Proof of Theorem 8. By Corollary 5, Remark 14 and the injectivity of induced rational map, we have the conclusion. □

Proof of Theorem 9. By Remark 14, there exists a $G$-invariant generating set $\{g_1, \ldots, g_{n+1}\}$ of $R(D')$. Since $|D|$ induces an injective rational map $\Gamma \hookrightarrow TP^n, \phi : \Gamma \to T^{n+1}; x \mapsto (g_1(x), \ldots, g_{n+1}(x))$ is also injective. In fact, if $\phi(x) = \phi(y)$, then $g_k(x) = g_k(y)$ for all $k$, and so $(g_1(x) : \cdots : g_{n+1}(x)) = (g_1(y) : \cdots : g_{n+1}(y))$ holds. Thus, we have $x = y$. By Corollary 6 and the injectivity, we have the conclusion. □

Remark 16. Except genus one leafless metric graph case, since in other cases all metric graphs have finite automorphism groups, we can always find a divisor satisfying the conditions of Theorems 7, 8, 9 for $G = Aut(\Gamma)$.

Remark 17. In the proofs of Propositions 1, 2, 3, Corollaries 4, 5, 6 and Theorems 7, 8, 9 essentially we only use the $\langle \sigma \rangle$-invariance (or $G$-invariance) of the rational function set $\{f_1, \ldots, f_{n+1}\}$ defining rational map and the injectivity of rational map. Thus, in this case, we use not a complete linear system but a linear subsystem. Moreover, actually we need not take $\{f_1, \ldots, f_{n+1}\}$ as a subset of a minimal generating set of $R(D)$. However, this construction is very practical since a minimal generating set of $R(D)$ for a suitable divisor $D$ always has the two properties and we can easily find such $D$.

A metric graph is hyperelliptic if it has a divisor of degree two and of rank one. Here, for a divisor $D$ on a metric graph, its rank is defined to be the minimum integer $s$ such that for some effective divisor $E$ of degree $s+1$, the complete linear system associated to $D - E$ is empty. The canonical divisor $K_\Gamma$ of a metric graph $\Gamma$ is the divisor on $\Gamma$ whose coefficient at each point $x$ is the valency of $x$ minus two, where the valency of $x$ is the number of connected components of $U \setminus \{x\}$ for any sufficiently small connected neighborhood $U$ of $x$.

Corollary 18. Let $\Gamma$ be a metric graph of genus at least two. If $\Gamma$ is not hyperelliptic, then the canonical linear system $|K_\Gamma|$ induces an injective rational map $\phi : \Gamma \to TP^n$ and an injective group homomorphism $\Psi : Aut(\Gamma) \hookrightarrow Z\text{-lin}(R^n)$ such that $\phi$ commutes with each element $\sigma$ of $Aut(\Gamma)$ and $\Psi(\sigma)$, where $n$ is the number of elements of a minimal generating set of $R(K_\Gamma)$ minus one.
Proof. As $\Gamma$ has genus at least two, $\text{Aut}(\Gamma)$ is finite. By Theorem 49 of [5], the canonical map $\phi_{|K_{\Gamma}|}$ is injective. Since $K_{\Gamma}$ is $\text{Aut}(\Gamma)$-invariant, by Theorem 7, we get our conclusion.

By the same proof of Corollary 18 using Theorems 8, 9, 10 instead of Theorem 7, respectively, we have the following three corollaries:

**Corollary 19.** Let $\Gamma$ be a metric graph of genus at least two. If $\Gamma$ is not hyperelliptic, then the canonical linear system $|K_{\Gamma}|$ induces an injective rational map $\phi : \Gamma \rightarrow TP^n$ and an injective group homomorphism $\Psi : \text{Aut}(\Gamma) \rightarrow PGL_{\text{trop}}(n, T)$ such that $\phi$ commutes with each element $\sigma$ of $\text{Aut}(\Gamma)$ and $\Psi(\sigma)$, where $n$ is the number of elements of a minimal generating set of $R(K_{\Gamma})$ minus one.

**Corollary 20.** Let $\Gamma$ be a metric graph of genus at least two. If $\Gamma$ is not hyperelliptic, then the canonical linear system $|K_{\Gamma}|$ induces an injective rational map $\phi : \Gamma \rightarrow T^n$ and an injective homomorphism $\Psi : \text{Aut}(\Gamma) \rightarrow GL_{\text{trop}}(n, T)$ such that $\phi$ commutes with each element $\sigma$ of $\text{Aut}(\Gamma)$ and $\Psi(\sigma)$, where $n$ is the number of elements of a minimal generating set of $R(K_{\Gamma})$.

**Corollary 21.** Let $\Gamma$ be a metric graph of genus at least two. If $\Gamma$ is not hyperelliptic, then the canonical linear system $|K_{\Gamma}|$ induces an injective rational map $\phi : \Gamma \rightarrow R^n$ and an injective homomorphism $\Psi : \text{Aut}(\Gamma) \rightarrow \text{Z-lin}(R^n)$ such that $\phi$ commutes with each element $\sigma$ of $\text{Aut}(\Gamma)$ and $\Psi(\sigma)$, where $n$ is the number of elements of a minimal generating set of $R(K_{\Gamma})$.

**Remark 22.** We can also make the projective space $RP^n$ (in the usual sense) versions in the same arguments as up until now since $R^{n+1} \subset RP^n$ and by the definition of projective linear group. Moreover, for a topological space $X$ (plus some additional structures) with its automorphism group $\text{Aut}(X)$ (for a definition of automorphism of $X$), if $X$ contains $R^n$ and $\text{Aut}(X)$ contains all permutation matrices or elements of the form $A_{\sigma}$ in the proof of Proposition 11 (or corresponding automorphisms), then we have the same conclusions for $X$.

**Example 23.** Let $\Gamma$ be the closed interval $[0, 1]$. We call the point 0 (resp. 1, 1/2) as $x$ (resp. $y$, $z$). Let $\iota$ be the unique nontrivial automorphism of $\Gamma$, i.e., $\iota$ is an isometry $\Gamma \rightarrow \Gamma$ such that $\iota(x) = y$ holds. We have $\text{Aut}(\Gamma) = \langle \iota \rangle$. Let $D = x$. Then the rational function $f_1$ with slope one on $\Gamma$ and setting $f_1(x) := 1$ and $f_1(y) := 0$ and the constant zero function $f_2$ on $\Gamma$ generate
$R(D)$. The image of $\Gamma \to TP^1 \supset R^1; x \mapsto (f_1(x) : f_2(x))$ is the closed interval $[0,1] \subset R^1$ and so $t$ induces a $Z$-affine transformation of $R^1$ but not a $Z$-linear transformation of $R^1$. Since $t$ fixes $z$ and the $\langle t \rangle$-invariant linear system $\langle D \rangle$ contains the divisor $z =: D'$, we can find an $\langle t \rangle$-invariant generating set $\{f'_1, f'_2\}$ of $R(D')$ such that $f'_1|_{[x,z]} \equiv 0$, $f'_1(y) := -1/2$, $f'_1$ has slope one on $[z,y]$ and $\langle f'_1 \rangle = f'_2$ holds. Then the induced rational map $\phi' := (f'_1: f'_2) : \Gamma \to TP^1 \supset R^1$ has the image $[-1/2, 1/2] \subset R^1$ and $t$ induces the square matrix $A_t = (-1)$. Finally, we have the injective group homomorphism $\text{Aut}(\Gamma) \hookrightarrow Z\text{-lin}(R^1); \text{id}_R \mapsto (1), t \mapsto (-1)$, where $\text{id}_R$ denotes the identity map of $\Gamma$. Also $\phi'$ induces $\text{Aut}(\Gamma) \hookrightarrow \text{PGL}_{\text{trop}}(2,T); \text{id}_R \mapsto \left(\begin{array}{ccc} -\infty & 0 & 0 \\ 0 & -\infty & 0 \\ 0 & 0 & -\infty \end{array}\right)$, $t \mapsto \left(\begin{array}{ccc} -\infty & 0 & 0 \\ 0 & -\infty & 0 \\ 0 & 0 & -\infty \end{array}\right)$ and $\Gamma \hookrightarrow T^2; x \mapsto (f_1(x), f_2(x))$ induces $\text{Aut}(\Gamma) \hookrightarrow \text{GL}_{\text{trop}}(2,T); \text{id}_R \mapsto \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$, $t \mapsto \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$.

**Example 24.** In the same setting as Example 23, let $E := 2z$. Then $R(E)$ is generated by the three rational functions $g_1, g_2$ and $g_3$, where $g_1|_{[x,z]} \equiv 0$, $g_1(y) := -1$, $g_1$ has slope two on $[z,y]$, and $\langle f_1 \rangle = f_2$ holds, and $g_3(x) := g_3(y) := -1/2$, $g_3(z) := 0$ and $g_3$ has slope one on $[x,z]$ and $[z,y]$. Since the set $\{g_1, g_2, g_3\}$ is $\langle t \rangle$-invariant and the induced rational map $\psi := (g_1: g_2: g_3) : \Gamma \to TP^2 \supset R^2$ is injective, we have the injective group homomorphism $\text{Aut}(\Gamma) \hookrightarrow Z\text{-lin}(R^2); \text{id}_R \mapsto \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$, $t \mapsto \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$.

Also $\psi$ induces $\text{Aut}(\Gamma) \hookrightarrow \text{PGL}_{\text{trop}}(3,T); \text{id}_R \mapsto \left(\begin{array}{ccc} -\infty & 0 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{array}\right)$, $t \mapsto \left(\begin{array}{ccc} -\infty & 0 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{array}\right)$ and $\Gamma \hookrightarrow T^3; x \mapsto (g_1(x), g_2(x), g_3(x))$ induces $\text{Aut}(\Gamma) \hookrightarrow \text{GL}_{\text{trop}}(3,T); \text{id}_R \mapsto \left(\begin{array}{ccc} -\infty & 0 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{array}\right)$, $t \mapsto \left(\begin{array}{ccc} -\infty & 0 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{array}\right)$.

**Example 25.** Let $\Gamma$ be a circle of length four. Fix a point $x \in \Gamma$. Let $\sigma$ be the 180 degrees rotation and $x' := \sigma(x)$. For the divisor $D := x + x'$, we can choose a $\langle \sigma \rangle$-invariant minimal generating set $\{f_1, f_2\}$ of $R(D)$. Concretely, if we call the midpoints of the two paths $P_1$ and $P_2$ between $x$ and $x'$ as $p_1$ and $p_2$ respectively, then for example, we can choose $f_1$ as
$f_1(x) := f_2(x) := 1$, $f_1(p_1) := 0$, $f_1$ has slope one on $[x,p_1] \cap P_1$ and $[x',p_1] \cap P_1$ and $f_1|_{P_1} \equiv 1$, and $f_2 := f_1 \circ \sigma$. Then, $\phi : \Gamma \to TP^1 \supset R^1$ is not injective and the image in $R^1$ is $[-1,1]$. $\phi$ induces the injective group homomorphisms $\langle \sigma \rangle \hookrightarrow Z\text{-lin}(R^1); \text{id}_R \mapsto (1), \sigma \mapsto (-1), \langle \sigma \rangle \hookrightarrow$ PGL$_{\text{trop}}(2,T); \text{id}_R \mapsto \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}, \sigma \mapsto \begin{pmatrix} -\infty & 0 \\ 0 & -\infty \end{pmatrix}$. Also $\Gamma \to T^2; x \mapsto (f_1(x),f_2(x))$ induces the injective group homomorphism $\langle \sigma \rangle \hookrightarrow$ GL$_{\text{trop}}(2,T); \text{id}_R \mapsto \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}, \sigma \mapsto \begin{pmatrix} -\infty & 0 \\ 0 & -\infty \end{pmatrix}$. On the other hand, for the isometry $\iota : \Gamma \to \Gamma$ which maps each point to the line symmetric point with the line $xx'$ as the axis of symmetry, $\{f_1,f_2\}$ is also $\langle \iota \rangle$-invariant, so we have three injective group homomorphisms from $\langle \iota \rangle$ having the same images as above. Thus, $\phi$ and $(f_1,f_2)$ do not induce injective group homomorphisms from $\langle \sigma,\iota \rangle$ to $Z\text{-lin}(R^1)$ or PGL$_{\text{trop}}(2,T)$ or GL$_{\text{trop}}(2,T)$. It comes from the fact that $|D|$ dose not induce an injective rational map.

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