Analysis of Morphological Characteristics of Rotating Flow Field Using Proper Orthogonal Decomposition

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Abstract. We use discrete proper orthogonal decomposition (POD) to investigate three-dimensional rotating flows to better understand the physical mechanism for the details of rotating flow. The results show that use the POD method to study the complicated engineering rotation flow is feasible, especially for the morphological characteristics of trail vortices in such modal spaces. Additionally, the logical architecture of this implementation using POD method can also be applied to other complicated engineering problems.

Keywords: complicated rotating flow, POD method, modal analysis, spectrum characteristics, characteristic plane.

1. Introduction

For the study of complex rotational flow in engineering applications, the traditional CFD method lacks multiple levels of further analysis capabilities. The POD method can not only perform multi-level analysis in orthogonal space, but also quantify its frequency characteristics. Therefore, it is very necessary to use the POD method to further analyze the complex rotation flow problem. In general [1], the POD method can be divided into two categories: continuous POD and discrete POD. For most practical applications, an analytical solution is difficult to obtain, so continuous POD is seldom used in engineering research. Thus, the discrete POD has already been widely applied for model reduction of large-scale CFD problems [2]. Sirovich had made a significant modification of discrete POD to improve its numerical stability and efficiency [3]. Ruelle et al. [4] found that the low-order model formed from the POD basis function is more accurate than when formed by basic functions of other types; for instance, trigonometric functions, Chebyshev polynomials, etc. Kou et al. [5] discussed the transonic buffeting of OAT15A airfoil based on POD and dynamic mode decomposition. The results show that, when using POD to reconstruct flow fields, the overall error is relatively small and the dynamic mode decomposition (DMD) method can more precisely characterize shock discontinuities.

Although POD is already widely used for conventional flow problems, its application to rotating flow is still quite limited. Christensen et al. [6] used POD to study unsteady rotating flow by using a rotating cap inside a cylindrical vessel. The results show that the first transition from steady to oscillatory motion is a supercritical Hopf bifurcation. The numerical solution obtained by using POD is consistent with experimental data. Sanghi et al. [7] investigated the thermally driven two-dimensional flow of air...
in a horizontal rotating cylinder and showed that, with less than 20 modes, low-dimensional models accurately reproduce the system dynamics both qualitatively and quantitatively. However, both above-mentioned investigations did not apply POD to the flow pattern in the rotating state and related research on applying POD to rotating flows is sparse.

Thus, we discuss herein the application of POD to modal analysis for the complex engineering problem, in which the related spectrum features are also analyzed. The relevant theories and the application of POD are introduced in Sections 2, and the implementation POD for rotating flows using CFD is given in Section 3. Section 4 presents a 3D numerical example of an impeller of a wind turbine to demonstrate the application of POD. Finally, Section 5 gives the conclusions.

2. Discrete proper orthogonal decomposition for rotating-flow field

For the discrete POD used herein, the sample vectors of the flow parameters for transient solutions at time $t_m$ (i.e., a snapshot) are

$$f(x, t_m) = [f(x_1, t_m) \ f(x_2, t_m) \ \cdots \ f(x_n, t_m) \ \cdots \ f(x_N, t_m)]^T,$$

Where $N$ is the total number of discrete points in the spatial domain.

The important sample matrix for applying POD to a rotating-flow field is comprised of several appropriate sample vectors,

$$F = [f(x, t_1) \ f(x, t_2) \ \cdots \ f(x, t_m) \ \cdots \ f(x, t_M)].$$

Where $F$ denotes the sample matrices of flow parameters, which could be velocities, densities, pressures, etc. The quantity $M$ is the total number of instantaneous times; in other words, the total number of snapshots.

To obtain the POD modes for the rotating-flow field, the covariance matrix is first calculated as follows [3]:

$$C = F^T F.$$  \hspace{1cm} (3)

Note that $C$ is a real symmetric matrix and has non-negative eigenvalues. Therefore, the eigenproblem of such a matrix has the form

$$CG^{[r]} = \lambda_r G^{[r]},$$  \hspace{1cm} (4)

Where $G^{[r]}$ is the r-th eigenvector for the specific flow parameter, in other words, the vector of time-varying coefficients of the rotating-flow modes. The quantity $\lambda_r$ is the eigenvalue corresponding to the r-th-order POD mode of the specific flow parameter:

$$G^{[r]} = [g'(t_1) \ g'(t_2) \ \cdots \ g'(t_m) \ \cdots \ g'(t_M)]^T.$$  \hspace{1cm} (5)

The rth-order POD mode of the swirling flow field can be defined by

$$f_r(x) = \frac{1}{M \lambda_r} \sum_{m=1}^{M} G^{[r]}(t_m) f(x, t_m),$$  \hspace{1cm} (6)
Where all eigenvalues can be listed in descending order according to the information-content contribution to the flow field. Therefore, the dominant flow modes in the rotating-flow field can be extracted by using the first several POD modes, which describe the essential features of the flow.

### 3. Implement proper orthogonal decomposition for rotating flows

The process of the implementation of POD in a rotating-flow field for complex engineering problems contains the following steps:

1. Carry out a CFD simulation of unsteady flow and assemble the CFD results of snapshots in a time-ordered sequence in the following format,

$$
\begin{bmatrix}
  x_1(t_m) & y_1(t_m) & z_1(t_m) & f_1^1(t_m) & f_1^2(t_m) & \cdots & f_1^D(t_m) \\
  x_2(t_m) & y_2(t_m) & z_2(t_m) & f_2^1(t_m) & f_2^2(t_m) & \cdots & f_2^D(t_m) \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
  x_i(t_m) & y_i(t_m) & z_i(t_m) & f_i^1(t_m) & f_i^2(t_m) & \cdots & f_i^D(t_m) \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
  x_N(t_m) & y_N(t_m) & z_N(t_m) & f_N^1(t_m) & f_N^2(t_m) & \cdots & f_N^D(t_m)
\end{bmatrix},
$$

(7)

Where the $D$ refers to the number of parameter types considered in the POD analysis.

2. To reduce the computational cost of POD, use the flow-parameter field on the characteristic plane to form the equivalent field instead of the original snapshots. The reduced format of the substituted snapshots is given by

$$
\begin{bmatrix}
  x'_1(t_m) & y'_1(t_m) & z'_1(t_m) & f'_1^1(t_m) & f'_1^2(t_m) & \cdots & f'_1^D(t_m) \\
  x'_2(t_m) & y'_2(t_m) & z'_2(t_m) & f'_2^1(t_m) & f'_2^2(t_m) & \cdots & f'_2^D(t_m) \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
  x'_i(t_m) & y'_i(t_m) & z'_i(t_m) & f'_i^1(t_m) & f'_i^2(t_m) & \cdots & f'_i^D(t_m) \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
  x'_{N'}(t_m) & y'_{N'}(t_m) & z'_{N'}(t_m) & f'_{N'}^1(t_m) & f'_{N'}^2(t_m) & \cdots & f'_{N'}^D(t_m)
\end{bmatrix},
$$

(8)

Where $N'$ is the number of discrete points on the characteristic plane, with $N' \ll N$. Once assembled on the characteristic plane, these discrete points should convey sufficient physical information. The flow field is visualized on the characteristic plane by projection using linear interpolation.

3. From the substituted snapshots, generate the sample matrix, which is called the subsample matrix, and calculate the POD modes and corresponding time coefficients by using Eq. (5). In addition, obtain the spectrum characteristics of such time coefficients using the fast Fourier transform (FFT).

### 4. Numerical example of three-dimensional impeller of wind turbine

The radius of the overall cylindrical solution domain is 3.5 m, which is five times that of the impeller blade. To better capture the vortex structures, a nonuniformly distributed mixed mesh is used to discretize the solution domain (see Fig. 1). A dense structured prism grid is used for the downstream vortex-dominated region in the center of the solution domain, and a relatively sparse tetrahedral mesh is used for the outer domain. The total number of cells is approximately 3.89 M.
The sliding-mesh method is used to exchange flow-flux data between the body-fitted rotating area of the impeller and the peripheral stationary region. The unsteady turbulent incompressible flows are simulated by using the LES method with the sub-grid model of Smagorinsky and the SIMPLEC algorithm. The velocity-inlet and pressure-outlet boundary conditions are applied on the left and right surfaces of the solution domain. The inlet velocity is 10 m/s along the positive X axis, and the rotational velocity of the impeller is 750 rev/min. The time step in the transient-flow calculation is $2.22 \times 10^{-4}$ s, which corresponds to 1° per step.

When the transient flow reaches a stable periodicity, 60 snapshots for POD analysis are chosen at equal time intervals during a single revolution of the impeller. Next, the reduced-substituted snapshots can be obtained from the characteristic plane of the CFD model at $Z = 0$ position. According to the spiral structure of the tip and central vortices in the downstream zone and of the mesh resolution of the solution domain, the radial velocity is clearly more influenced by vortices than by the velocities in the other two directions. Therefore, these POD modes of the first 6 orders in the characteristic plane of the radial velocities shall be further investigated and are displayed in Fig. 2.

Additionally, the corresponding characteristics of the frequency spectrum of the time coefficient of the first 9 modes are also brought out by using a fast Fourier transform to better reveal the physical mechanism of the rotating flow around the wind turbine, as shown in Fig. 3.
The following observations may be made based on the spatial distributions of the POD modes and the frequency spectrum of the time coefficients:

(1) As shown in Fig. 6, the peak frequency of the first-order mode is zero, indicating the rigid-body displacement or the average value of the rotating flow. The corresponding POD-mode distribution is shown in Fig. 5(a), in which the spatial symmetry of the distribution over a large range is well displayed, except for regions of the central vortex and at the boundaries of the CFD model.

(2) The peak frequencies of the second- and third-order POD modes shown in Figs. 5(b) and 5(c) are both 12.51 Hz, which complies with the component of fundamental frequency of impeller rotation and also shows that the central vortex is the dominant cause of the spatiotemporal periodicity. In these flow
patterns, the spatiotemporal periodicity is well presented mainly for the central vortex along the forward direction of the X axis.

(3) The spatiotemporal periodicity of the fourth- and fifth-order modes shown in Figs. 5(d) and 5(e) is mainly caused by the tip vortex. Because the impeller has three blades, the corresponding peak frequencies of these two modes are both 37.52 Hz, which is the integrated fundamental rotating frequency in the near wake.

(4) In the sixth mode, the tip and central vortices fuse [see Fig. 5(f)], and the spatiotemporal periodicity for vortices is well expressed.

5. Conclusion
This paper presents the theory and implementation of discrete POD for modal analysis in rotating-flow fields. The spectral characteristics of the time coefficient of the flow modality have also been discussed based on the characteristic plane. For unsteady flows, the overwhelming computational cost of POD analysis is incurred in computing snapshots and other calculations of POD analysis are negligible for such complicated wake of wind turbine.

According to the computational results, it is feasible to use the POD method to study the complicated engineering rotation flow. In the low-order flow modes, the displacement of the rigid body of the rotating flow, the fundamental frequency component, the integrated fundamental frequency, the frequency multiplication of the fundamental frequency, etc. have been correspondingly presented, especially for the morphological characteristics of trail vortices in such modal spaces. Therefore, this study can help to further understand the intrinsic physical mechanism of complex rotational flow, and the logical architecture of this implementation using POD method can be applied to other complicated engineering problems, such as turbomachinery, high-energy disc brakes, rotary bioreactors in biomedicine, self-circulation radiators for computers, etc.

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