Exact solutions of dilaton gravity with (anti)-de Sitter asymptotics

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Abstract

We present a technique for obtaining spherically symmetric asymptotically (anti)-de Sitter black hole solutions of dilaton gravity with generic coupling to Maxwell field, starting from exact asymptotically flat solutions and adding a suitable dilaton potential to the action.

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1. Introduction

Charged dilatonic black holes have been largely studied in the context of the low-energy limit of string theories [1]. The main characteristic of the dilatonic models is the presence in the effective action of an exponential coupling between the dilaton and the Maxwell field. It results that this coupling gives rise to nontrivial scalar hair for black holes, with a scalar charge independent from the other parameters of the solution.

Dilatonic black holes have also been investigated in the case of more general dilaton-Maxwell couplings [2], or when a cosmological constant or a scalar potential of Liouville type are added to the action [3]. Also the possibility of multiple exponentially coupled scalar fields has been considered [4].

On the other hand, there has been a renewed interest in the study of black-hole solutions in theories with a cosmological constant, especially in the context of the AdS/CFT correspondence [5]. In particular, the study of anti-de Sitter black holes can give new insight in the nonperturbative structure of some conformal models. Moreover, it has been shown that, even in the case of minimally coupled scalar fields, black holes with anti-de Sitter asymptotics admit scalar hair, for a class of scalar potentials [6]. This opens the possibility of phase transitions and breaking of gauge symmetry near black hole horizons [7].

However, finding exact black hole solutions to dilatonic models with nonvanishing cosmological constant is not trivial. In ref. [3] it was shown that asymptotically anti-de Sitter solutions exist in the case of negative $\lambda$, but their analytical form is not known. Moreover, in the $\lambda > 0$ case, or when an exponential dilatonic potential is added to the action, no asymptotically (anti)-de Sitter black hole solutions at all exist. Nevertheless, the situation can change if more general dilatonic potential are considered.

In fact, recently Gao and Zhang [8] have proposed a method that, starting from an exact spherically symmetric asymptotically flat solution of the Einstein-Maxwell-dilaton system, permits to obtain exact asymptotically de Sitter or anti-de Sitter solutions of dilaton gravity exponentially coupled to the Maxwell field, by the addition to the action
of a suitable scalar potential. Unfortunately, their derivation is rather involved and it is not clear why it can give rise to consistent results.

In this letter we give a much simpler derivation of that result, which clarifies this point, and generalize the method to the case of a generic coupling to the Maxwell field. We then give some examples of application of the formalism to the solutions found by Monni and Cadoni (MC) [2] in the asymptotically flat case for a hyperbolic dilaton-Maxwell coupling.

2. Dilatonic (anti)-de Sitter black holes.

The proposal of [1] can be restated and generalized to arbitrary dilaton-Maxwell coupling $W(\phi)$, as follows: consider the action

\[ I = \int \sqrt{-g} \, d^4x \left[ R - 2(\nabla \phi)^2 - W(\phi)F^2 \right], \quad (1) \]

with field equations

\[ R_{\mu \nu} = 2\nabla_\mu \phi \nabla_\nu \phi + 2W(\phi) \left( F^\rho_\mu F_{\rho \nu} - \frac{1}{4} F^2 g_{\mu \nu} \right) = 0, \]

\[ \nabla^2 \phi = \frac{1}{4} \frac{dW}{d\phi} F^2, \quad \nabla_\mu [W(\phi)F^{\mu \nu}] = 0, \quad (2) \]

and look for spherically symmetric solutions of the form

\[ ds^2 = -U(r) \, dt^2 + U^{-1}(r) \, dr^2 + R^2(r) \, d\Omega^2, \quad \phi = \phi(r), \quad (3) \]

with magnetic field

\[ F = Q \sin \theta \, d\theta \wedge d\phi. \quad (4) \]

The field equations can then be written as

\[ \frac{R''}{R} = -\phi'^2, \quad (5) \]

\[ \frac{1}{R^2}(URR')' = \frac{1}{R^2} - W(\phi) \frac{Q^2}{R^4}, \quad (6) \]

\[ \frac{1}{R^2}(R^2 U \phi')' = \frac{1}{2} \frac{dW(\phi)}{d\phi} \frac{Q^2}{R^4}, \quad (7) \]
and admit asymptotically flat solutions. In particular, in the case $W(\phi) = e^{-2\phi}$, the exact solutions are well known, and read [1]

$$U(r) = 1 - \frac{2M}{r}, \quad R^2(r) = r(r - 2D), \quad e^{-2\phi} = e^{-2\phi_0}\left(1 - \frac{2D}{r}\right).$$

(8)

The solutions depend on three independent parameters. To simplify the following discussion, we choose as independent parameters the mass $M$, the dilatonic charge $D$ and the asymptotic value $\phi_0$ of the dilaton. Of course, from a physical point of view, the magnetic charge $Q$ is more relevant and could be chosen instead of $\phi_0$. The magnetic charge is related to the other parameters by $Q^2 = 2MD e^{-2\phi_0}$.

The solution displays a curvature singularity at $r = 2D$ and a horizon at $r = 2M$ and hence describes black holes if $M > D > 0$. An independent scalar hair is present, provided the magnetic charge is nonvanishing.

Define now a new metric given by

$$ds^2 = -\left[U(r) - \frac{\lambda}{3}R^2(r)\right]dt^2 + \left[U(r) - \frac{\lambda}{3}R^2(r)\right]^{-1}dr^2 + R^2(r)d\Omega^2,$$

(9)

where $U(r)$ and $R(r)$ are given by (8). Under the conditions $M > D > 0$, this is an asymptotically (anti)-de Sitter black hole with cosmological constant $\lambda$. More precisely, for $\lambda < 0$, one has an asymptotically anti-de Sitter black hole with a single horizon, while for $\lambda > 0$, provided $\lambda D^2 < 3/4$, it results an asymptotically de Sitter black hole with one cosmological horizon and one event horizon.

In this case, Gao and Zhang found that it is possible to add a scalar potential

$$V(\phi) = \frac{2}{3}\lambda(2 + \cosh[2(\phi - \phi_0)]),$$

(10)

to the action (1), so that (9) is a solution of the ensuing field equations.

A drawback of this formalism is that $V(\phi)$ depends explicitly on the asymptotic value $\phi_0$ of the scalar field, and hence no longer is an independent parameter of the solution. It follows that for a given potential only two among the charges $M$, $D$ and $Q$ (or $\phi_0$) are independent, and in particular the scalar charge is a function of $M$ and $Q$.

More generally, one can define the action

$$I = \int \sqrt{-g} \, d^4x[R - 2(\nabla \phi)^2 - V(\phi) - W(\phi)F^2].$$

(11)
With the ansatz (3)-(4), the field equations take the form

\[ \frac{R''}{R} = -\phi'^2, \]  

(12)

\[ \frac{1}{R^2} (URR')' = \frac{1}{R^2} - W(\phi) \frac{Q^2}{R^4} - V(\phi). \]  

(13)

\[ \frac{1}{R^2} (R^2 U\phi')' = \frac{1}{2} \frac{dW(\phi)}{d\phi} \frac{Q^2}{R^4} + \frac{1}{4} \frac{dV(\phi)}{d\phi}, \]  

(14)

Defining

\[ U(r) = \bar{U}(r) - \frac{\lambda}{3} R^2(r), \]  

(15)

where \( \bar{U}(r) \) is a solution of (5)-(7), and exploiting the linearity of (13), (14) in \( U \), it is easy to see that the field equations (13), (14) reduce to

\[ \frac{\lambda}{3} \frac{1}{R^2} (R^3 R')' = \frac{1}{2} V(\phi), \]  

(16)

\[ \frac{\lambda}{3} \frac{1}{R^2} (R^4 \phi')' = -\frac{1}{4} \frac{dV(\phi)}{d\phi}, \]  

(17)

while the field equation (12) is unchanged. Therefore, the solution for the scalar field is identical to the asymptotically flat one. Then, inverting \( \phi(r) \), one can substitute in (16) or (17) and obtain \( V \) as a function of \( \phi \). Of course, one must show that the two equations are compatible. In fact, substituting (12) in (16) and taking the derivative,

\[ \frac{\lambda}{3} \left(12 RR'' + 4RR'\phi'^2 + 4R^2 \phi' \phi''\right) = -\frac{dV}{d\phi} \phi', \]  

(18)

and substituting again (12), one obtains (17).

3. The generalization of MC solutions.

As an example of application of this formalism, we consider the exact solutions obtained by Monni and Cadoni [2] for the coupling \( W(\phi) = \cosh(2\alpha \phi) \), when \( \alpha = 1 \) or \( \alpha = \sqrt{3} \). This coupling is invariant under the S-duality symmetry [9] \( \phi \rightarrow -\phi \), and therefore may be of interest for string theory.

In the case \( \alpha = 1 \), the solution reads [2]

\[ \bar{U}(r) = \frac{[r - (M - D)r^2 - q^2]}{r(r + 2D)}, \quad R^2 = r(r + 2D), \quad e^{-2\phi} = e^{-2\phi_0} \left(1 + \frac{2D}{r}\right), \]  

(19)
where
\[ q^2 = M^2 + D^2 - 2MD \coth 2\phi_0. \]  
(20)

The solution depends on three independent parameters: again we choose \( M, D \) and \( \phi_0 \), in terms of which the magnetic charge reads
\[ Q^2 = \frac{2MD}{\sinh 2\phi_0}. \]  
(21)

It describes black holes if
\[ M > D > 0, \quad \phi_0 > 0, \quad M^2 + D^2 - 2MD \cosh 2\phi_0 \geq 0. \]  
(22)

(Solutions with \( D < 0, \phi_0 < 0 \) are related by duality to the previous ones). In this case

a curvature singularity is located at the origin and is shielded by two horizons located at \( r_{\pm} = M - D \pm q \).

A metric function of the form (15) reads explicitly
\[ U = -\frac{\lambda}{3} r^4 - \frac{4\lambda}{3} Dr^3 + (1 - \frac{4\lambda}{3} D^2)r^2 - 2(M - D)r - 2MD(1 - \coth 2\phi_0) \quad \frac{r(r + 2D)}{r(r + 2D)}, \]  
(23)

and applying the algorithm described above one can check that it is a solution of the field

equations for a potential
\[ V(\phi) = \frac{2}{3}\lambda(2 + \cosh[2(\phi - \phi_0)]). \]  
(24)

Curiously, the potential has the same form as in the GHS case (10). Unfortunately, it

breaks the duality invariance \( \phi \to -\phi \).

The solution is singular at \( r = 0 \). If \( \lambda < 0 \), it is asymptotically anti-de Sitter and, under the conditions (22) it always exhibits two horizons. If \( \lambda > 0 \), it is asymptotically de Sitter and for \( \lambda D^2 < 3/4 \) it displays three horizons, otherwise a naked singularity emerges. To the previous conditions, one must add the request of positivity of the discriminant of the quartic polynomial in the numerator of (23), in order for its roots to be real. The expression of this condition is awkward and will not be reported here, but its effect is to impose a limitation on the range of allowed values of the mass in function of the other
parameters. The properties of the solution are analogous to those of the (anti)-de Sitter-Reissner-Nordström black holes, which are recovered in the limit $D = \phi_0 = 0$, with $Q \neq 0$.

In the case $\alpha = \sqrt{3}$, the MC solution reads [2]

$$U(r) = \frac{(r - M)^2 - q_0^2}{\sqrt{P_1 P_2}}, \quad R^2(r) = \sqrt{P_1 P_2},$$

$$e^{-2\sqrt{3}\phi} = e^{-2\sqrt{3}\phi_0} \left( \frac{P_1}{P_2} \right)^{3/2},$$

where

$$q_0^2 = \frac{(D + M)^3 F^2 + (D - M)^3}{(D + M)^2 + (D - M)^2} \geq 0,$$

and

$$P_1 = (r - D)^2 - q_1^2, \quad P_2 = (r + D)^2 - q_2^2.$$  \hspace{1cm} (25)

Also this solution depends on three independent parameters, which we have chosen to be the mass $M$, the scalar charge $\frac{D}{\sqrt{3}}$ and $F = e^{2\sqrt{3}\phi_0}$, with $\phi_0$ the asymptotic value of the scalar field. The magnetic charge $Q$ is then related to the other charges by the constraint

$$Q^2 = \frac{4D^2(D^2 - M^2)F}{(D + M)^2 + (D - M)^2}.$$ \hspace{1cm} (26)

The requirements $q_0^2 \geq 0$, $Q^2 > 0$ yield

$$F^2 \leq \left( \frac{M - D}{M + D} \right)^3.$$ \hspace{1cm} (27)

Under these conditions, $q_1^2 < 0$, $q_2^2 > 0$, and hence $P_1$ has no real roots, while a curvature singularity occurs at the greatest root of $P_2$, namely $r_0 = -D + q_2$. This is shielded by two horizons placed at $r_\pm = M \pm q_0$ iff $M > D > 0$, $F < 1$ (again, solutions with $D < 0$, $F > 1$ are obtained by duality).

Let us now consider a metric of the form (15). The metric function $U(r)$ reads explicitly

$$U = -\frac{\lambda r^4 + (1 + 2\lambda D^2)r^2 - 2[M + \frac{\lambda}{3} D(q_1^2 - q_2^2)]r + M^2 - q_0^2 - \frac{\lambda}{3}(3D^4 - q_1^2 q_2^2)}{R^2(r)}.$$  \hspace{1cm} (28)
Applying the formalism developed above, it is easy to check that $U(r)$ solves the field equations for a potential

$$V = 2\lambda \cosh \frac{2}{\sqrt{3}} (\phi - \phi_0).$$

(32)

Also in this case the duality invariance is broken.

A curvature singularity still occurs at $r_0$, and the solution describes black holes under the same restriction of (30). Moreover, in the anti-de Sitter case, $\lambda < 0$, one must require

$$|\lambda| \geq \frac{3}{4} \frac{M + D - q^2}{q_2^2(q_2 - 2D)},$$

(33)

obtaining a black hole with two horizons.

Analogously, in the de Sitter case, a black hole with three horizon exists if

$$\lambda \leq \frac{1}{2q_2(q_2 - 2D)}.$$

(34)

As in the previous case, one must also impose a condition that ensures the existence of real roots of the polynomial in the numerator of (30), and gives rise to restrictions on the allowed values of the mass. All other values of the parameters lead to naked singularities. Also in this case, the structure of the solution is analogous to that of the (anti)-de Sitter-Reissner-Nordstr"om black holes. These are recovered when $D = 0, F = 1$.

The difference between the properties of the asymptotically (anti)-de Sitter GHS solution compared to the RN or MC ones is due to the degree of the polynomial appearing in the metric function. It appears that the GHS is a degenerate limit case, in which one of the roots of $\bar{U}$ coincides with one root of $R^2$, while the generic case is of Reissner-Nordstr"om type.

Electric charged black holes for the dilaton-Maxwell coupling $W(\phi) = 1/ \cosh 2\alpha \phi$ can be obtained as usual by duality from the magnetic ones. The metric maintains its form, while $\phi \rightarrow -\phi$. The CM solutions are already invariant under this transformation, and hence also the scalar potential is unchanged, except for the sign of $\phi_0$.

Finally we notice that, as in the Reissner-Nordstr"om case, the MC solutions can present degenerate horizons when two of the horizons merge. This happens when the discriminant discussed above vanishes.
4. Conclusions.

We have generalized a method proposed in [8] to obtain spherically symmetric asymptotically (anti)-de Sitter black hole solutions in dilatonic models nonminimally coupled to the Maxwell field, by the addition of a suitable scalar potential. In particular, we have found that in the case of hyperbolic coupling, the dilatonic potential is also hyperbolic. In this case, the metric has the same causal structure as that of the (anti)-de Sitter-Reissner-Nordström solution of general relativity.

It is interesting to notice that, while in the asymptotically flat case the asymptotic value of the dilaton is free, in the (anti)-de Sitter case the presence of a dilaton potential fixes its value, and in general the solutions only depend on two parameters. In this case, the dilatonic charge is therefore determined by the mass and the Maxwell charge. This is in accordance with the fact that, according to well known no hair theorems for minimally coupled anti-de Sitter black holes, the asymptotic value of the scalar field must coincide with an extremum of the potential [10]. It would be interesting to investigate if this fact could lead to no-hair theorems for nonminimally coupled models. In particular, it is not clear if the solutions obtained are unique or if the absence of an independent scalar charge is an artifact of the way in which the solutions have been constructed.

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