Survey of Hopf Fibrations and Rotation Conventions in Mathematics and Physics

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Abstract. We present a unifying framework for understanding several different versions of the Hopf fibration, and use this framework to reconcile two methods of representing rotations of 3-space by unitary matrices—the mathematician’s convention based on quaternion algebra, and the physicist’s convention based on the Bloch sphere.
1. Introduction

Sophus Lie made a profound contribution to mathematics and physics in the late 19th century by developing a theory based on his observation that solutions to certain problems in mechanics must be invariant under rigid motions of space, and that the structure in symmetry groups can be exploited to solve differential equations [1]. Although Lie theory is a rare find in the undergraduate curriculum, one of its topics—the special orthogonal group $SO(3)$ of rotations of space—is impossible to miss in courses such as linear algebra, differential equations and quantum mechanics.

In theory and in practical computations, mathematicians and physicists use $2 \times 2$ unitary matrices as a replacement for $3 \times 3$ real orthogonal matrices. How this is done, and more important, why this is natural, are main points of this article. The explanation rests on the Hopf fibration. Our secondary aim is to reconcile the differences between math and physics conventions in the use of unitary matrices to represent rotations. This is accomplished by comparing different versions of the Hopf fibration.

The exposition presented here requires no special background beyond university level vector calculus, linear algebra, and an introduction to group theory. Definitions for those few objects which may exceed this minimum background—projective space, higher dimensional spheres, commutative diagrams and quaternions—are given in the Appendix.

2. A Survey of Hopf Fibrations

Heinz Hopf defined a mapping in his 1931 paper [2] that we now call the Hopf fibration. It was a landmark discovery in the young subject of algebraic topology that has since been recognized in many guises in mathematics and physics with applications including magnetic monopoles, rigid body mechanics, and quantum information theory [6].

The heart of the Hopf map is the canonical projection

$$\mathbb{C}^2 \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^1$$

that sends the complex vector $(z, w)$ to its equivalence class $[z, w]$ in projective space. We interpret this as a map $S^3 \to S^2$ by identifying $S^3$ as the subset of norm 1 vectors in $\mathbb{R}^4 = \mathbb{C}^2$, and by identifying $\mathbb{P}^1$ with $S^2$. The latter identification is a two-step procedure. First identify $\mathbb{P}^1$ with the extended complex plane $\mathbb{C}^+ = \mathbb{C} \cup \{\infty\}$. One way to do this is the map

$\text{chart} : \mathbb{P}^1 \to \mathbb{C}^+$

given by $[z_0, z_1] \mapsto z_0/z_1$ ("chart" is for "coordinate chart"). Second, identify $\mathbb{C}^+$ with $S^2$ using some version of stereographic projection. We will use two stereographic projections, $\text{stereo}_j : S^2 \to \mathbb{C}^+$ for $j = 1, 3$, given by

$$\text{stereo}_1(x, y, z) = \frac{y}{1-x} + i \frac{z}{1-x}$$

$$\text{stereo}_3(x, y, z) = \frac{x}{1-z} + i \frac{y}{1-z}$$

and $\text{stereo}_1(1, 0, 0) = \infty = \text{stereo}_3(0, 0, 1)$. We put these maps together to form a template for the generic Hopf map. Here and in diagrams that follow, we highlight the core map [1] with a frame.

$$S^3 \xrightarrow{\text{inclusion}} \mathbb{C}^2 \setminus \{0\} \xrightarrow{x} \mathbb{P}^1 \xrightarrow{\text{chart}} \mathbb{C}^+ \xrightarrow{\text{stereo}_j^{-1}} S^2$$
H. Hopf’s original map \[2\] arises from this template by choosing \( j = 3 \). One obtains variations by altering the identifications with \( S^3 \) on the left and with \( S^2 \) on the right, for example, by using alternative coordinate charts on \( \mathbb{P}^1 \) and by choosing different basepoints for stereographic projection. These variations are motivated by the desire to adapt coordinates to fit particular interpretations.

The projection \( \mathbb{P}^1 \) comes to life when we view it in terms of group action. In general, when a group \( G \) acts on a set \( X \), we have a bijection

\[
G/I_x \leftrightarrow O_x
\]

given by \( gI_x \leftrightarrow gx \) for each \( x \in X \), where \( I_x = \{ g \in G : gx = x \} \) is the isotropy subgroup for \( x \) and \( O_x = \{ gx : g \in G \} \) is the orbit of \( x \). We now apply this fact twice, where the group is \( G = SU(2) \) and the actions arise from the natural action of \( G \) on \( \mathbb{C}^2 \). First, let \( X \) be the set of norm 1 vectors in \( \mathbb{C}^2 \). The action on \( X \) is transitive, and the isotropy subgroup of every point is trivial, so we have the identification

\[
SU(2) \sim \to S^3
\]

given by \( g \leftrightarrow (1, 0) \). Second, let \( X = \mathbb{P}^1 \). The action of \( G \) on \( X \) is transitive, and the isotropy subgroup of the point \([1,0]\) is the torus

\[
T = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} : \theta \in \mathbb{R} \right\},
\]

so we have the identification

\[
SU(2)/T \sim \to \mathbb{P}^1
\]

given by \( gT \leftrightarrow g[1,0] \). Now we can rephrase the heart of the Hopf map \( \mathbb{P}^1 \) as the map

\[
SU(2) \to \mathbb{P}^1
\]

given by \( g \mapsto g[1,0] \), where “rephrase” means the following diagram commutes.

\[
\begin{array}{ccc}
SU(2) & \text{act on } [1,0] & \mathbb{P}^1 \\
\text{act on } (1,0) & \downarrow & \downarrow \\
\mathbb{C}^2 \setminus \{0\} & \pi & \mathbb{P}^1
\end{array}
\]

Now we are ready to define and compare several versions of the Hopf fibration in terms of \( \mathbb{P}^1 \) and \( \mathbb{H} \). We begin with a Hopf fibration expressed in the language of quaternion algebra.

We identify the quaternions \( \mathbb{H} \) with \( \mathbb{R}^4 \) and \( \mathbb{C}^2 \) via

\[
x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \leftrightarrow (x_0, x_1, x_2, x_3) \leftrightarrow (x_0 + ix_1, x_2 + ix_3)
\]

and regard \( \mathbb{H} \) as a real vector space with canonical basis \( \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \) and also as a complex vector space with canonical basis \( \{1, \mathbf{j}\} \). We identify \( \mathbb{R}^3 \) with the pure quaternions, that is, the subspace of \( \mathbb{R}^4 \) consisting of points with zero in the first coordinate. Under this identification, the name \( p \) for point \( p = (x, y, z) \) in \( \mathbb{R}^3 \) shall

\[\ddagger\] Under the right conditions, when \( G \) and \( X \) are manifolds, the bijection \( \mathbb{P}^1 \) is a diffeomorphism \( \mathbb{P}^1 \). It is not necessary to know about manifolds or diffeomorphisms to follow our presentation.
also denote the quaternion \( p = xi + yj + zk \). We identify the unit length quaternions with \( S^3 \subset \mathbb{R}^4 \). The 2-sphere \( S^2 \subset \mathbb{R}^3 \) is identified with the “equator” of \( S^3 \) which is the set of unit length pure quaternions.

The group \( SU(2) \) is isomorphic with the group of unit quaternions via the map

\[
\begin{bmatrix}
  z & w \\
  -\overline{w} & \overline{z}
\end{bmatrix} \leftrightarrow z + wj
\]

where \((z, w)\) is a unit length vector in \( \mathbb{C}^2 \). The group of unit quaternions is also naturally identified with \( S^3 \) via \( \mathbb{C}^2 \). The matrix \( \begin{bmatrix}
  z & w \\
  -\overline{w} & \overline{z}
\end{bmatrix} \) identifies with \((z, w)\) by \( \mathbb{C}^2 \) and identifies with \((z, -\overline{w})\) by \( \mathbb{C}^2 \). We will denote by \( T \) the map

\[
T: S^3 \xrightarrow{\mathbb{C}^2} SU(2) \xrightarrow{\mathbb{C}^2} S^3
\]
given by \((z, w) \mapsto (z, -\overline{w})\) that arises from combining these identifications. We call it \( T \) for “transpose” because this is the map you get when you interpret the quaternion as a matrix by \( \mathbb{C}^2 \), transpose it, then reinterpret as a point in \( \mathbb{C}^2 \).

The group of unit quaternions acts naturally on the subspace of pure quaternions (where we interpret the pure quaternions as \( \mathbb{R}^3 \), see \( \mathbb{C}^2 \) and \( \mathbb{C}^2 \) for details) via

\[
S^3 \times \mathbb{R}^3 \to \mathbb{R}^3
\]
given by \((g, p) \mapsto gpg^*\), where \( p \) is a pure quaternion, \( g \) is a unit quaternion and \( g^* \) is the conjugate of \( g \) (the conjugate of \( x_0 + x_1i + x_2j + x_3k \) is \( x_0 - x_1i - x_2j - x_3k \) and is what you get if you take the hermitian (conjugate transpose) of \( g \) viewed as a matrix via \( \mathbb{C}^2 \)). This action preserves the Euclidean length of \( p \), and so restricts to an action on \( S^3 \).

\[
S^3 \times S^2 \to S^2
\]

We choose the basepoint \( p_0 = (1, 0, 0) = i \) and define a map \( S^3 \to S^2 \) by

\[
g \mapsto g1g^*.
\]

The action \( \mathbb{C}^2 \) is transitive and the isotropy subgroup of \( p_0 \) is \( \{e^{i\theta}\} \). As matrices, this isotropy subgroup is the same as the torus \( T \). Thus the map \( \mathbb{C}^2 \) identifies with the Hopf fibration \( \mathbb{C}^2 \) as shown in the following commutative diagram. The correspondence on the bottom row of the diagram is given by \( g[1, 0] \leftrightarrow g1g^* \).

Another Hopf map (although it is rarely if ever identified as such) arises from a coordinate system on \( S^2 \) called the Bloch sphere. It is defined as follows: given \((a, b)\)

\[\text{§ Named after Felix Bloch, recipient of the 1952 Nobel Prize in physics.}\]
in \( \mathbb{C}^2 \) with \( a \) real, the equations \( a = \cos \theta / 2 \) and let \( b = e^{i\phi} \sin \theta / 2 \) determine spherical coordinates \((\theta, \phi)\) for the point 
\[
(\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)
\]
on \( S^2 \). This is equivalent to the following.

\[
\text{Bloch}(a, b) = \text{stereo}^{-1}_{3}(a/b) \quad (12)
\]

We will take the map “Bloch” to be given by \((12)\) whether or not \( a \) is real. Here is a comparison diagram that shows how the quaternion action and the Bloch coordinate projection fit into the generic scheme \((2)\). From now on, we will use the labels “HopfClassic”, “QuatHopf”, and “Bloch” to refer to the Hopf’s original map, the map \((11)\), and \((12)\), respectively.

The following commutative diagram demonstrates identifications among Hopf fibrations appearing vertically in dotted line frames. Hopf’s original map is the second column from the left.

\[
\begin{array}{c}
\text{HopfClassic} \\
S^3 \subseteq \mathbb{C}^2 \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\text{chart}} \mathbb{C}^+ \xrightarrow{\text{stereo}^{-1}_{3}} S^2
\end{array}
\]

\[
\begin{array}{c}
\text{QuatHopf} \\
S^3 \xrightarrow{T} S^3 \subseteq \mathbb{C}^2 \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\text{chart}} \mathbb{C}^+ \xrightarrow{i} \mathbb{C}^+ \xrightarrow{\text{stereo}^{-1}_{3}} S^2
\end{array}
\]

\[
\begin{array}{c}
\text{Bloch} \\
\mathbb{C}^2 \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\text{chart}} \mathbb{C}^+ \xrightarrow{\text{stereo}^{-1}_{3}} S^2
\end{array}
\]

We conclude with one more comparison (by commutative diagram) of Bloch and QuatHopf. The label “reverse” denotes the reflection of \( \mathbb{R}^3 \) that sends \((x, y, z)\) to \((z, y, x)\).

\[
\begin{array}{c}
\mathbb{C}^2 \setminus \{0\} \xleftarrow{T} S^3 \\
\text{Bloch} \quad \text{QuatHopf}
\end{array}
\]

\[
\text{S}^2 \xrightarrow{\text{reverse}} S^2
\]
3. Rotations by Hopf Actions

In the action (9) of the unit quaternions on $\mathbb{R}^3$, the quaternion $q = a + bi + cj + dk$ acts as a rotation by $\theta/2$ radians about the axis specified by the unit length vector $\hat{n} = (n_1, n_2, n_3)$ where $\theta, \hat{n}$ are given by the following equations [5, 6].

\[
\begin{align*}
a &= \cos \theta/2 \\
(b, c, d) &= \sin \theta/2 \hat{n}
\end{align*}
\]

Given a real number $\theta$ and a point $\hat{n}$ on $S^2$, let

\[
g_Q = g_Q(\theta, \hat{n}) = \cos \theta/2 + \sin \theta/2(n_1i + n_2j + n_3k).
\]

We view $g_Q$ both as a quaternion and as the matrix

\[
g_Q = \begin{bmatrix}
\cos \theta/2 + in_1 \sin \theta/2 & \sin \theta/2(n_2 - in_3) \\
\sin \theta/2(\hat{n}_2 + in_3) & \cos \theta/2 - in_1 \sin \theta/2
\end{bmatrix}
\]

associated via (8). Let us denote by $R(\theta, \hat{n}, p)$ the image of $p$ under the rotation by $\theta$ radians about the axis specified by $\hat{n}$. Then we have

\[
g_Q p g_Q^* = R(\theta, \hat{n}, p).
\]

We can also write $R(\theta, \hat{n}, p)$ in terms of the Hopf fibration in the following way. Let $h_Q$ be any preimage of $p$ under $\text{QuatHopf}$. Then we have

\[
\text{QuatHopf}(g_Q h_Q) = R(\theta, \hat{n}, p).
\]

Here is the one-line proof.

\[
\text{QuatHopf}(g_Q h_Q) = (g_Q h_Q) i (g_Q h_Q)^* = g_Q (\text{QuatHopf}(h_Q)) g_Q^* = g_Q p g_Q^*.
\]

There is a corresponding expression in terms of Bloch [4]. Given a real number $\theta$ and a point $\hat{n}$ on $S^2$, let

\[
g_B = g_B(\theta, \hat{n}) = \begin{bmatrix}
\cos \theta/2 - in_3 \sin \theta/2 & \sin \theta/2(n_2 + in_3) \\
\sin \theta/2(\hat{n}_2 - in_1) & \cos \theta/2 + in_3 \sin \theta/2
\end{bmatrix}.
\]

Let $h_B$ be any preimage of $p$ under Bloch. Then we have

\[
\text{Bloch}(g_B h_B) = R(\theta, \hat{n}, p).
\]

The purpose of the remainder of this section is to explain the equality

\[
\text{QuatHopf}(g_Q h_Q) = \text{Bloch}(g_B h_B). \tag{14}
\]

First observe that the multiplications $g_Q h_Q$ and $g_B h_B$ are different operations. The binary operation in the expression $g_Q h_Q$ is quaternion multiplication or matrix multiplication, depending on whether you view $g_Q, h_Q$ as quaternions or matrices. The binary operation in $g_B h_B$ is the multiplication of the $2 \times 2$ matrix $g_B$ by the $2 \times 1$ vector $h_B$. To keep track of this distinction, we will write $g_B \odot h_B$ to denote the latter operation. Having pointed out the difference, we now relate the two operations. Let $h_B$ denote the quaternion associated to $h_B$ by [7], that is, if $h_B = (z, w)$, then $h_B = z + wj$. Then we have

\[
g_B \odot h_B = h_B g_B^T \tag{15}
\]
where the operation on the right-hand side is quaternion multiplication and we view \( g_B \) as a quaternion by \( \mathbb{H} \), or the operation is matrix multiplication where we view \( \tilde{h}_B \) as a \( 2 \times 2 \) matrix by \( \mathbb{H} \).

Now we can derive (14). We have

\[
\text{Bloch}(g_B \odot \tilde{h}_B) = \text{Bloch}(\tilde{h}_B^T g_B^T) = \text{reverse}(\text{QuatHopf}(g_B \tilde{h}_B^T)) = g_Q p Q Q_p = g_Q p Q.
\]

The first equality (16) is by (15). The second equality (17) is by (13). Here is a geometric explanation for the final equality (18). Interpret \( \tilde{h}_B^T \) as a QuatHopf lift of \( (z, y, x) \) (by virtue of (13)) and interpret \( g_B \) as a (“quat”) rotation by \(-\theta\) around \( (n_3, n_2, n_1) \), so QuatHopf\((g_B \tilde{h}_B^T)\) calculates \( R(-\theta, \text{reverse}(\hat{n}), \text{reverse}(p)) \). So reversing this result is the same as rotating \( p \) by \( g_Q \). Thus we have completed our goal or reconciling Bloch sphere rotation conventions with the standard quaternion approach. We conclude with a commutative diagram that expresses (14).

4. Appendix

The set \( \mathbb{P}^1 \), called 1-dimensional complex projective space, is the set of equivalence classes in \( \mathbb{C}^2 \setminus \{0\} \), where 0 denotes the zero vector \( \mathbf{0} = (0, 0) \), with respect to the equivalence relation \( \sim \) defined by \( (z, w) \sim (z', w') \) if and only if \( (z, w) = \lambda (z', w') \) for some nonzero complex scalar \( \lambda \).

The set \( S^n \), called the \( n \)-dimensional sphere, or simply the \( n \)-sphere, is the set of points \((x_0, x_1, \ldots, x_n)\) in \( \mathbb{R}^{n+1} \) that satisfy

\[
x_0^2 + x_1^2 + \cdots + x_n^2 = 1.
\]

To say that a diagram of sets and functions commutes means that if there are two different function compositions that start at set \( A \) and end at set \( B \), then those compositions must be equal as functions. For example, to say the following diagram commutes means that \( r \circ t = b \circ f \).

\[
\begin{array}{ccc}
A & \xrightarrow{t} & X \\
\downarrow{\ell} & & \downarrow{r} \\
Y & \xrightarrow{b} & B
\end{array}
\]

The quaternions are the set \( \mathbb{R}^4 \) endowed with a noncommutative multiplication operation, given below. The standard basis vectors

\((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\)
are denoted $1, i, j, k$, respectively, so that the vector $(x_0, x_1, x_2, x_3)$ in $\mathbb{R}^4$ is written $x_0 + x_1i + x_2j + x_3k$ as a quaternion. The multiplication is determined by the relations

\begin{align*}
i^2 = j^2 = k^2 &= -1 \\
ij = kj &= i \\
jk &= ki = j \\
ji &= -k \\
kj &= -i \\
kj &= -j
\end{align*}

and extending linearly.

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