Broer-Kaup System with Corrections via Inverse Scattering Transform

Xueru Wang and Junyi Zhu

School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, Zhengzhou, Henan 450001, China

Correspondence should be addressed to Junyi Zhu; jyzhu@zzu.edu.cn

Received 18 July 2020; Accepted 3 September 2020; Published 15 September 2020

Academic Editor: Ivan Giorgio

Copyright © 2020 Xueru Wang and Junyi Zhu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The Broer-Kaup system with corrections is considered. Based on the inverse scattering transform, we extend the perturbation theory to discuss the adiabatic approximate solution and ε-order approximate solution of one soliton to the Broer-Kaup system with corrections.

1. Introduction

The coupled integrable system or the Broer-Kaup (BK) system [1–5]

\[ t_i = \frac{1}{2} (v^2 + 2\omega - v_x)_x, \]
\[ \omega_i = \left( v\omega + \frac{1}{2} \omega_x \right)_x, \]

is related to one of the Boussinesq systems by variable transformation. The BK system is used to simulate the bidirectional propagation of long waves in shallow water. Here, we assume that \( v \) and \( \omega \) decay rapidly as \( |x| \to \infty \).

Kaup first proposed the perturbation theory based upon the inverse scattering transform [6]. In addition to the perturbation theory based upon the inverse scattering transform [7–15], there are many other methods, such as the direct soliton perturbation theory [16–30] and the normal form method [31–34]. For more detail about the perturbation theory, see [11, 35] and references therein.

In this paper, we consider the BK system with corrections through perturbation theory based upon the inverse scattering transform [15] and investigate the adiabatic approximate solution and ε-order approximate solution of one soliton to the BK system with corrections. In order to carry out the inverse scattering transform to discuss the BK system with corrections, we keep the first Lax equation but give up the second one. For this reason, the analyticity and asymptotic behaviors of Jost functions are the same as the BK system. In this case, the scattering data are all dependent on time, and the time evolution of scattering data is discussed in detail.

This paper organized as follows. In Section 2, we discuss the spectral analysis and construct the Riemann-Hilbert problem of the BK system. In Section 3, we present the BK system with corrections. In Section 4, we consider the time evolution of the scattering coefficients, the discrete spectrum, and the normalization factors. In Section 5, the conservation laws of the Broer-Kaup system without corrections and its perturbation corrections are given. In Section 6, we obtain the adiabatic approximate solution of one soliton, find the slow variations of spectral parameters, and discuss the ε-order approximate solution of one soliton. In Section 7, we give some short conclusions.

2. Spectral Analysis and Riemann-Hilbert Problem

Equation (1) is the compatibility condition of the following Lax pair:

\[ F_x = LF, \quad F_t = MF, \]
with
\[ L(x, \lambda) = -i\lambda \sigma_3 + U(x), \quad M(x, \lambda) = \lambda^2 \sigma_3 + V(x), \]
\[ U(x) = \begin{pmatrix} \frac{\nu}{2} & 1 \\ -\omega & -\frac{\nu}{2} \end{pmatrix}, \]
\[ V(x) = \begin{pmatrix} -\frac{1}{4} (\nu_x - \nu^2) & i\lambda + \frac{\nu}{2} \\ -i\lambda\omega - \frac{1}{4} (\omega_x + \omega\nu) & \frac{1}{4} (\nu_x - \nu^2) \end{pmatrix}. \]  
(4)

Now we introduce the Jost functions \( \Phi(x, \lambda) \) and \( \psi(x, \lambda) \), which satisfy the following boundary problem:
\[ \begin{align*}
\partial_x \Phi(x, \lambda) &= L(x, \lambda) \Phi(x, \lambda), \\
\Phi(x, \lambda) &\to A e^{-ikx}, \quad x \to -\infty, \\
\partial_x \psi(x, \lambda) &= L(x, \lambda) \psi(x, \lambda), \\
\psi(x, \lambda) &\to A e^{-ikx}, \quad x \to \infty,
\end{align*} \]
(5)
where
\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]
(6)

For simplicity, let the Jost functions have the following form:
\[ \Phi(x, \lambda) = \left( \phi(x, \lambda), \tilde{\phi}(x, \lambda) \right), \quad \psi(x, \lambda) = \left( \psi(x, \lambda), \psi(x, \lambda) \right). \]
(7)

There exists a scattering matrix \( T(\lambda) \) which satisfies
\[ \Phi(x, \lambda) = \Psi(x, \lambda) T(\lambda), \quad T(\lambda) = \begin{pmatrix} a(\lambda) & -\tilde{b}(\lambda) \\ b(\lambda) & \tilde{a}(\lambda) \end{pmatrix}, \]
(8)
then we obtain
\[ a(\lambda) = W[\phi(x, \lambda), \psi(x, \lambda)], \quad b(\lambda) = W[\tilde{\phi}(x, \lambda), \phi(x, \lambda)], \]
(9)
\[ \tilde{a}(\lambda) = W[\tilde{\psi}(x, \lambda), \phi(x, \lambda)], \quad \tilde{b}(\lambda) = W[\tilde{\phi}(x, \lambda), \psi(x, \lambda)], \]
(10)
where \( W(\cdot, \cdot) \) denotes the Wronskian. With the help of the Neumann series, we find that \( \phi(x, \lambda) \) and \( \psi(x, \lambda) \) are analytic in \( D^* = \{ \lambda \in \mathbb{C} \mid \text{Im}\lambda > 0 \} \) and that \( \tilde{\phi}(x, \lambda) \) and \( \tilde{\psi}(x, \lambda) \) are analytic in \( D^* = \{ \lambda \in \mathbb{C} \mid \text{Im}\lambda < 0 \} \). The asymptotic behaviors of the Jost functions \( \Phi(x, \lambda) \) and \( \Psi(x, \lambda) \) take the following form:
\[ \Phi(x, \lambda) e^{ikx} = \begin{pmatrix} e^{\theta} \left( 1 + \frac{1}{2\lambda^2} \int_{-\infty}^{\infty} \omega(y) dy \right) & \frac{1}{2\lambda} e^{-\nu} \\ \frac{1}{2\lambda} e^{\nu} & e^{\nu} \left( 1 - \frac{1}{2\lambda^2} \int_{-\infty}^{\infty} \omega(y) dy \right) \end{pmatrix} + O\left( \frac{1}{\lambda^2} \right), \]
(11)
\[ \psi(x, \lambda) e^{ikx} = \begin{pmatrix} e^{\theta} \left( 1 + \frac{1}{2\lambda^2} \int_{-\infty}^{\infty} \omega(y) dy \right) & \frac{1}{2\lambda} e^{-\nu} \\ \frac{1}{2\lambda} e^{\nu} & e^{\nu} \left( 1 - \frac{1}{2\lambda^2} \int_{-\infty}^{\infty} \omega(y) dy \right) \end{pmatrix} + O\left( \frac{1}{\lambda^2} \right), \]
(12)
where
\[ \eta_y = \int_{\infty}^{x} \nu(y) dy. \]
(13)

If \( \nu \) and \( \omega \) are confined to be real functions, we obtain the symmetry condition \( T^* (-\lambda^*) = T(\lambda) \), which implies
\[ a^*(-\lambda^*) = a(\lambda), \quad \tilde{a}^*(-\lambda^*) = \tilde{a}(\lambda), \]
(14)
where the star denotes the complex conjugate. Since \( a(\lambda) \) is analytic in \( D^* \), if \( \lambda_n \in D^* \) is a zero of \( a(\lambda) \), then \( -\lambda^*_n \) is also a zero. Similarly, \( \tilde{a}(\lambda) \) has the simple zeroes \( \tilde{\lambda}_n, -\tilde{\lambda}^*_n \in D^* \). There are two cases for the eigenvalues: \( \lambda_n = \tilde{\lambda}_n \) and \( \lambda_n \neq -\tilde{\lambda}^*_n \).

In this paper, we only consider the first case. Assume that \( a(\lambda) \) has \( N \) simple zeros \( \lambda_n = i\nu_n \) (\( \nu_n > 0 \)), \( \tilde{a}(\lambda) \) has \( N \) simple zeros \( \tilde{\lambda}_n = i\nu_n \) (\( \nu_n < 0 \)).

Next, introducing the first sectional matrices
\[ M^+(x, t, \lambda) = \begin{pmatrix} \phi(x, t, \lambda) e^{ikx} & \psi(x, t, \lambda)e^{-ikx} \\ a(\lambda) & a(\lambda) \end{pmatrix}, \]
(15)
\[ M^-(x, t, \lambda) = \begin{pmatrix} \tilde{\phi}(x, t, \lambda) e^{ikx} & \tilde{\psi}(x, t, \lambda)e^{-ikx} \\ \tilde{a}(\lambda) & \tilde{a}(\lambda) \end{pmatrix}, \]
(16)
we get the jump condition
\[ M^-(x, t, \lambda) = M^+(x, t, \lambda) (I - H(x, t, \lambda)), \quad \lambda \in \mathbb{R}, \]
(17)
where the matrix \( H(x, t, \lambda) \) is
\[ H(x, t, \lambda) = \begin{pmatrix} 0 & \tilde{b}(\lambda) e^{-2ikx} \\ b(\lambda) e^{2ikx} & 0 \end{pmatrix}. \]
(18)

From the previous asymptotic behaviors, we find that \( M(x, t, \lambda) \) admits the following normalization condition:
\[ M^+(x, t, \lambda) = B + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty, \]
\[ B = \begin{pmatrix} \epsilon^{l_1} & 0 \\ 0 & \epsilon^{-n_2} \end{pmatrix}. \]

Introducing the second sectional matrices
\[ \tilde{M}^+(x, t, \lambda) = \left( \phi(x, t, \lambda) \epsilon^{l_1}, \psi(x, t, \lambda) \epsilon^{-l_2} \right), \]
\[ \tilde{M}^-(x, t, \lambda) = \left( \frac{\psi(x, t, \lambda) \epsilon^{l_1}}{\tilde{a}(\lambda)}, \frac{\phi(x, t, \lambda) \epsilon^{-l_2}}{\tilde{a}(\lambda)} \right), \]
we get another jump condition
\[ \tilde{M}^+(x, t, \lambda) = \tilde{M}^-(x, t, \lambda) \left( I + \tilde{H}(x, t, \lambda) \right), \quad \lambda \in \mathbb{R}, \]
where the matrix \( \tilde{H}(x, t, \lambda) \) is
\[ \tilde{H}(x, t, \lambda) = \begin{pmatrix} 0 & \tilde{b}(\lambda) \epsilon^{-2l_2} \\ \tilde{b}(\lambda) \epsilon^{2l_1} & 0 \end{pmatrix}. \]

In a similar way, \( \tilde{M}^+(x, t, \lambda) \) has the following normalization condition:
\[ \tilde{M}^+(x, t, \lambda) = \tilde{B} + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty, \]
\[ \tilde{B} = \begin{pmatrix} \epsilon^{l_1} & 0 \\ 0 & \epsilon^{-n_2} \end{pmatrix}. \]

From (11), (15), and (20), we get
\[ M^+(x, t, \lambda) = B + \frac{1}{2i\lambda} B^{(1)} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \infty, \]
\[ \tilde{M}^+(x, t, \lambda) = \tilde{B} + \frac{1}{2i\lambda} \tilde{B}^{(1)} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \to \infty, \]
\[ B^{(1)} = \begin{pmatrix} \epsilon^{l_1} \int_{-\infty}^{x} \omega(y) dy & \epsilon^{-n_2} \\ \epsilon^{-n_2} & -\epsilon^{l_1} \int_{-\infty}^{x} \omega(y) dy \end{pmatrix}, \]
\[ \tilde{B}^{(1)} = \begin{pmatrix} \epsilon^{l_1} \int_{-\infty}^{x} \omega(y) dy & \epsilon^{-n_2} \\ \epsilon^{-n_2} & -\epsilon^{l_1} \int_{-\infty}^{x} \omega(y) dy \end{pmatrix}. \]

Solving the above Riemann-Hilbert problem, we have
\[ M^+(x, t, \lambda) = B + \sum_{n=1}^{N} \frac{\text{Res}[M^+, \lambda_n]}{\lambda - \lambda_n} \left( M^+ H \right) + P_\pm(M^+ H), \]
\[ = \tilde{B} + \sum_{n=1}^{N} \frac{\text{Res} \left[ \tilde{M}^-, \lambda_n \right]}{\lambda - \lambda_n} + P_\pm(M^- H), \]
where
\[ \text{Res}[M^+, \lambda_n] = \left( \frac{\phi(x, t, \lambda_n) \epsilon^{l_1}}{\tilde{a}(\lambda_n)}, \frac{\psi(x, t, \lambda_n) \epsilon^{-l_2}}{\tilde{a}(\lambda_n)} \right), \]
\[ \text{Res} \left[ \tilde{M}^-, \lambda_n \right] = \left( \frac{\psi(x, t, \lambda_n) \epsilon^{l_1}}{\tilde{a}(\lambda_n)}, -\frac{\phi(x, t, \lambda_n) \epsilon^{-l_2}}{\tilde{a}(\lambda_n)} \right), \]
and the dot denotes the derivative with respect to \( \epsilon \). Here, we have used the Cauchy projectors \( P_\pm \) over the real axis
\[ P_\pm[f](k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(l)}{l - (k \pm i0)} dl. \]

### 3. BK System with Corrections

Consider the BK system with corrections
\[ \nu_1 - \frac{1}{2} \left( \nu^2 + 2\omega - \nu_x \right) x = \epsilon q(x), \]
\[ \nu_2 - \frac{\nu_2 + \omega}{2} \frac{x}{x} = \epsilon p(x), \]
where \( q(x) \) and \( p(x) \) are functionals of \( v(x) \) and \( \omega(x) \), and \( \epsilon \) is a real parameter. When \( \epsilon \to 0 \), (33) reduces to the BK system without corrections.

To carry out the inverse scattering transform to the BK system with corrections, we keep the first Lax equation
\[ \partial_x \phi(x, \lambda) = L(x, \lambda) \phi(x, \lambda). \]

In this way, the analyticity and the asymptotic behavior of Jost functions are the same as those of the BK system without corrections.

Next, we give up the second Lax equation
\[ \partial_x \phi(x, \lambda) = M(x, \lambda) \phi(x, \lambda). \]

And introduce the new functions
\[ \phi'(x, \lambda) = \{ \partial_t - M \} \phi(x, \lambda), \]
\[ \tilde{\phi}(x, \lambda) = \{ \partial_t - M \} \tilde{H}(\lambda) \phi(x, \lambda). \]
Here, functions $h(\lambda)$ and $\tilde{h}(\lambda)$ satisfy

$$h_t(\lambda) = \lambda^2 h(\lambda), \quad \tilde{h}_t(\lambda) = -\lambda^2 \tilde{h}(\lambda),$$

(39)

and $\lambda_t \neq 0$. Computing $\{\partial_t - L\} \hat{\phi}(x, \lambda)$ and $\{\partial_t - L\} \tilde{\phi}(x, \lambda)$ and using (33), we get

$$\{\partial_t - L\} \hat{\phi}(x, \lambda) = G(x, \lambda) h(\lambda) \phi(x, \lambda),$$

(40)

$$\{\partial_t - L\} \tilde{\phi}(x, \lambda) = G(x, \lambda) \tilde{h}(\lambda) \tilde{\phi}(x, \lambda),$$

(41)

where

$$G(x, \lambda) = -i \lambda \sigma_3 + \varepsilon Q(x),$$

(42)

$$Q(x) = \begin{pmatrix} \frac{q(x)}{2} & 0 \\ -p(x) & -\frac{q(x)}{2} \end{pmatrix}.$$  

(43)

We note that the BK system with corrections (33) is equivalent to (42). We note that if $v(x)$ and $\omega(x)$ satisfy the BK system with corrections (33), then (42) will determine the associated time evolution of scattering data. Conversely, if the time evolution of scattering data is determined from (42), the solution of the BK system with corrections can be rebuilt. Here and after, we will start from (3.7) to establish the perturbation theory of the BK system with corrections.

4. Evolution of Scattering Data

4.1. Time Dependence of $a(\lambda)$ and $\tilde{a}(\lambda)$. In this section, we discuss the time evolution of the scattering data from (42). For the decay potential, we find, from (37), that

$$\dot{\phi}(x, \lambda) \longrightarrow \{\partial_t - \lambda^2 \} h(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i \lambda x}, \quad x \longrightarrow -\infty,$$

$$\tilde{\phi}(x, \lambda) \longrightarrow \{\partial_t + \lambda^2 \} h(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i \lambda x}, \quad x \longrightarrow -\infty.$$  

(44)

We note that, for the BK system with corrections, $a(\lambda)$, $\tilde{a}(\lambda)$, $b(\lambda)$, and $\tilde{b}(\lambda)$ are dependent on $t$. One may find that functions $\phi(x, \lambda)$ and $\tilde{\phi}(x, \lambda)$ in (37) take the following asymptotic behaviors as $x \longrightarrow \infty$:

$$\phi(x, \lambda) \longrightarrow h(\lambda) a_t(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i \lambda x},$$

$$\tilde{\phi}(x, \lambda) \longrightarrow \{\partial_t \} h(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i \lambda x},$$

(45)

$$+ h(\lambda) \{b_t(\lambda) + \lambda^2 b(\lambda)\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i \lambda x}.$$  

$$\hat{\phi}(x, \lambda) \longrightarrow h(\lambda) \{\tilde{b}_t(\lambda) - \lambda^2 \tilde{b}(\lambda)\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i \lambda x},$$

$$+ \tilde{h}(\lambda) \tilde{a}_t(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i \lambda x}.$$  

(46)

It is remarked that (40) are nonhomogeneous equations about $\phi(x, \lambda)$ and $\tilde{\phi}(x, \lambda)$; thus, they can be expressed as the linear combination of corresponding homogeneous solutions. So, we have the following expressions:

$$\dot{\phi}(x, \lambda) = a(x, \lambda) \psi(x, \lambda) + b(x, \lambda) \phi(x, \lambda),$$

(47)

$$\tilde{\phi}(x, \lambda) = \tilde{a}(x, \lambda) \tilde{\psi}(x, \lambda) + \tilde{b}(x, \lambda) \tilde{\phi}(x, \lambda),$$

(48)

where the coefficients $a$ and $b$ and $\tilde{a}$ and $\tilde{b}$ are defined by the following equations:

$$a(x, \lambda) = \frac{i}{\alpha(\lambda)} \int_{-\infty}^{\infty} \psi(y, \lambda) G_{2}(y, \lambda) \phi(y, \lambda) dy,$$

(49)

$$b(x, \lambda) = \frac{-i}{\alpha(\lambda)} \int_{-\infty}^{\infty} \psi(y, \lambda) G_{2}(y, \lambda) \phi(y, \lambda) dy,$$

(50)

$$\tilde{a}(x, \lambda) = \frac{i}{\alpha(\lambda)} \int_{-\infty}^{\infty} \tilde{\psi}(y, \lambda) G_{2}(y, \lambda) \tilde{\phi}(y, \lambda) dy,$$

(51)

$$\tilde{b}(x, \lambda) = \frac{i}{\alpha(\lambda)} \int_{-\infty}^{\infty} \tilde{\psi}(y, \lambda) G_{2}(y, \lambda) \tilde{\phi}(y, \lambda) dy.$$  

(52)

As $x \longrightarrow \infty$, (47) and (48) become

$$\dot{\phi}(x, \lambda) \longrightarrow h(\infty, \lambda) a(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i \lambda x},$$

$$+ \{a(\infty, \lambda) + b(\infty, \lambda) b(\lambda)\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i \lambda x},$$

(53)

$$\tilde{\phi}(x, \lambda) \longrightarrow \{a(\infty, \lambda) - b(\infty, \lambda) b(\lambda)\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i \lambda x},$$

$$+ b(\infty, \lambda) \tilde{a}(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i \lambda x}.$$  

(54)

Comparing (45) with (53), we find

$$h(\lambda) a_t(\lambda) = b(\infty, \lambda) a(\lambda),$$

(55)

$$h(\lambda) \{b_t(\lambda) + \lambda^2 b(\lambda)\} = a(\infty, \lambda) + b(\infty, \lambda) b(\lambda),$$

(56)

$$\tilde{h}(\lambda) \tilde{a}_t(\lambda) = b(\infty, \lambda) \tilde{a}(\lambda),$$

(57)
\[-\hbar(\lambda) \left\{ \tilde{b}_i(\lambda) - \lambda^2 \tilde{b}(\lambda) \right\} = \alpha^-(\infty, \lambda) - \beta^- (\infty, \lambda) \tilde{b}(\lambda). \]  \hfill (58)

Substituting (49) into (55), we have
\[
a_i(\lambda) = -i \int_{-\infty}^{\infty} \psi^T(x, \lambda) \sigma_2 G(x, \lambda) \phi(x, \lambda) dx, \]  \hfill (59)

\[
b_i(\lambda) + \lambda^2 b(\lambda) = -ia^{-1}(\lambda) \int_{-\infty}^{\infty} \{ \phi(x, \lambda) - b(\lambda) \psi(x, \lambda) \}^T \sigma_2 G \cdot (x, \lambda) \phi(x, \lambda) dx, \]  \hfill (60)

\[
a_t(\lambda) = i \int_{-\infty}^{\infty} \psi^T(x, \lambda) \sigma_2 G(x, \lambda) \tilde{\phi}(x, \lambda) dx, \]  \hfill (61)

\[
b_t(\lambda) - \lambda^2 \tilde{b}(\lambda) = ia^{-1}(\lambda) \int_{-\infty}^{\infty} \left\{ \tilde{\phi}(x, \lambda) + \tilde{b}(\lambda) \tilde{\psi}(x, \lambda) \right\}^T \sigma_2 G \cdot (x, \lambda) \tilde{\phi}(x, \lambda) dx. \]  \hfill (62)

These conditions mean that the corrections are too small to change the bound state solution of the scattering problem. In other words, the soliton solutions of the nonlinear problem can still be found. Let \( \lambda \rightarrow \lambda_n \) in (59), using (67) and (42), we have
\[
2\lambda_{t,n} \int_{-\infty}^{\infty} \phi_1(x, \lambda_n) \phi_2(x, \lambda_n) dx + \epsilon \int_{-\infty}^{\infty} \phi^T(x, \lambda_n) \sigma_2 \tilde{Q}(x) \phi(x, \lambda_n) dx = 0, \]  \hfill (68)

which can be further reduced to
\[
\hat{a}(\lambda_n) \lambda_{t,n} = -ed_{t,n}^{-1}(t) \int_{-\infty}^{\infty} q(x) \phi_1(x, \lambda_n) \phi_2(x, \lambda_n) dx + p(x) \phi^T_1(x, \lambda_n) dx, \]  \hfill (69)

\[
\hat{a}(\lambda_n) \lambda_{t,n} = \epsilon d_{t,n}^{-1}(t) \int_{-\infty}^{\infty} q(x) \tilde{\phi}_1(x, \lambda_n) \tilde{\phi}_2(x, \lambda_n) dx + p(x) \phi^T_1(x, \lambda_n) dx, \]  \hfill (70)

in terms of (9). These equations give the time evolution of parameter \( \lambda_n \) for the BK system with correction terms. When \( \epsilon \rightarrow 0 \), (69) become \( \lambda_{t,n} = 0 \) and \( \hat{\lambda}_{t,n} = 0 \); these are the results of the BK system.

4.3. Time Dependence of \( d_n(t) \) and \( \tilde{d}_n(t) \). To consider the time evolution of the normalization factors, we introduce another set of functions
\[
\psi(x, \lambda) = \{ \partial_t - M \} h^{-1}(\lambda) \psi(x, \lambda), \]  \hfill (71)

\[
\tilde{\psi}(x, \lambda) = \{ \partial_t - M \} h^{-1}(\lambda) \tilde{\psi}(x, \lambda), \]  \hfill (72)

then the two sets of functions satisfy the following equation:
\[
\tilde{\phi}(x, \lambda_n) = \tilde{h}_n^2(\lambda_n) \tilde{d}_n(t) \psi(x, \lambda_n), \]  \hfill (73)

\[
\hat{\phi}(x, \lambda_n) = \hat{h}_n^2(\lambda_n) \hat{d}_n(t) \tilde{\psi}(x, \lambda_n). \]  \hfill (74)
In view of the analyticity of $\phi(x, \lambda)$ and $\psi(x, \lambda)$, let $\lambda = \lambda_n$ in (47) and $\lambda = \tilde{\lambda}_n$ in (48); we have
\[ \phi'(x, \lambda_n) = \xi(x, \lambda_n)\phi(x, \lambda_n), \tag{74} \]
\[ \tilde{\phi}(x, \tilde{\lambda}_n) = \xi(x, \tilde{\lambda}_n)\tilde{\phi}(x, \tilde{\lambda}_n), \tag{75} \]
in terms of (66).
Substituting the above equation into (42), we get
\[ \zeta(x, \lambda_n) = i\frac{h(\lambda_n)}{a(\lambda_n)d_n(t)} \int_{-\infty}^{x} \left\{ \phi(y, \lambda_n) - d_n(t)\psi(y, \lambda_n) \right\} \sigma_2 G(y, \lambda_n) \phi(y, \lambda_n) dy, \tag{78} \]
where
\[ \gamma(x, \lambda_n) = -i \frac{1}{\tilde{a}(\tilde{\lambda}_n)\tilde{h}(\tilde{\lambda}_n)} \int_{-\infty}^{x} \left\{ \tilde{\phi}(y, \tilde{\lambda}_n) - \tilde{d}_n(t)\tilde{\psi}(y, \tilde{\lambda}_n) \right\} \sigma_2 \tilde{G}(y, \tilde{\lambda}_n) \tilde{\phi}(y, \tilde{\lambda}_n) dy. \tag{84} \]
It is verified that $\psi'(x, \lambda_n)$ and $\psi'(x, \tilde{\lambda}_n)$ admit
\[ \{\partial_x - L(x, \lambda_n)\} \psi'(x, \lambda_n) = G(x, \lambda_n)h^{-1}(\lambda_n)\psi(x, \lambda_n), \tag{80} \]
\[ \{\partial_x - L(x, \tilde{\lambda}_n)\} \tilde{\psi}'(x, \tilde{\lambda}_n) = G(x, \tilde{\lambda}_n)h^{-1}(\tilde{\lambda}_n)\tilde{\psi}(x, \tilde{\lambda}_n). \tag{81} \]
\[ \psi'(x, \lambda_n) = \gamma(x, \lambda_n)\psi(x, \lambda_n), \tag{82} \]
\[ \tilde{\psi}'(x, \tilde{\lambda}_n) = \tilde{\gamma}(x, \tilde{\lambda}_n)\tilde{\psi}(x, \tilde{\lambda}_n), \tag{83} \]
Substituting (74) and (78) and (82) and (84) into (72), we get the evolution of $d_n(t)$ and $\tilde{d}_n(t)$
\[ d_{n,t}(t) + 2\lambda_n^2 d_n(t) = i \frac{\lambda_{nt}}{a(\lambda_n)d_n(t)} \int_{-\infty}^{x} \left\{ \phi(x, \lambda_n) - d_n(t)\psi(x, \lambda_n) \right\} \sigma_1 \phi(x, \lambda_n) dx + i \frac{\varepsilon}{a(\lambda_n)} \int_{-\infty}^{x} \left\{ \phi(x, \lambda_n) - d_n(t)\psi(x, \lambda_n) \right\} \sigma_2 Q(x) \phi(x, \lambda_n) dx, \tag{86} \]
\[ \tilde{d}_{n,t}(t) - 2\tilde{\lambda}_n^2 \tilde{d}_n(t) = \frac{i}{\tilde{a}(\tilde{\lambda}_n)\tilde{d}_n(t)} \int_{-\infty}^{x} \left\{ \tilde{\phi}(x, \tilde{\lambda}_n) - \tilde{d}_n(t)\tilde{\psi}(x, \tilde{\lambda}_n) \right\} \sigma_1 \tilde{\phi}(x, \tilde{\lambda}_n) dx - i \frac{\varepsilon}{\tilde{a}(\tilde{\lambda}_n)} \int_{-\infty}^{x} \left\{ \tilde{\phi}(x, \tilde{\lambda}_n) - \tilde{d}_n(t)\tilde{\psi}(x, \tilde{\lambda}_n) \right\} \sigma_2 \tilde{Q}(x) \tilde{\phi}(x, \tilde{\lambda}_n) dx, \tag{87} \]
which reduce to

\[ d_{n,t}(t) + 2\lambda_n^2 d_n(t) = i \frac{\varepsilon}{\delta(\lambda_n)} \int_{-\infty}^{\infty} \{ \tilde{\phi}(x, \lambda_n) - \tilde{d}_n(t) \tilde{\psi}(x, \lambda_n) \} \sigma_2 Q(x) \phi(x, \lambda_n) dx, \]

\[ \tilde{d}_{n,t}(t) - 2\lambda_n^2 \tilde{d}_n(t) = -i \frac{\varepsilon}{\tilde{\alpha}(\lambda_n)} \int_{-\infty}^{\infty} \{ \tilde{\phi}(x, \lambda_n) - \tilde{d}_n(t) \tilde{\psi}(x, \lambda_n) \} \sigma_2 Q(x) \tilde{\phi}(x, \lambda_n) dx. \]

When \( \varepsilon \rightarrow 0 \), (87) are the results of the BK system without corrections.

5. Perturbation Corrections of the Conservation Laws

The BK system has an infinite number of conservation laws, which can be derived by considering the linear spectral problem in (3), that is,

\[ \phi_{1,x} = \left(-i\lambda + \frac{\nu}{2}\right) \phi_1 + \phi_2, \]  

(89)

\[ \phi_{2,x} = -\omega \phi_1 + \left(i\lambda - \frac{\nu}{2}\right) \phi_2. \]  

(90)

To this end, we eliminate \( \phi_2 \) from system (89) and introduce a new function \( \tilde{\phi} \) by

\[ \phi_1 = e^{-i\lambda \nu \eta + \nu \lambda} \]  

(91)

then we get a Riccati equation

\[ \tilde{\phi}_{xx} + \tilde{\phi}_x^2 - 2i\lambda \tilde{\phi}_x + \nu \tilde{\phi}_x + \omega = 0. \]  

(92)

Assume that \( \tilde{\phi} \) has the following expansion:

\[ \tilde{\phi}(x, \lambda) = \sum_{j=0}^{\infty} \frac{I_j}{(2i\lambda)^j}. \]  

(93)

then by substituting (93) into (92), we find

\[ \tilde{\phi}_1 = \int_{-\infty}^{\infty} \omega(y) dy, \quad \tilde{\phi}_2 = \int_{-\infty}^{\infty} \omega(y) \nu(y) dy, \]  

(94)

\[ \tilde{\phi}_3 = \int_{-\infty}^{\infty} \omega_2(y) + \omega^2(y) + \nu^2(y) \omega(y) + (\nu \omega)_x(y) + \nu(y) \omega_y(y) dy. \]  

For the BK system, \( a(\lambda) \) and \( b(\lambda) \) are independent of \( t \), then we obtain the following conserved densities:

\[ I_1 = \int_{-\infty}^{\infty} \omega(x) dx, \quad I_2 = \int_{-\infty}^{\infty} \nu(x) \omega(x) dx, \]  

(95)

\[ I_3 = \int_{-\infty}^{\infty} \omega^2(x) + \nu^2(x) \omega(x) + \nu(x) \omega_2(x) dx, \]  

\[ \frac{d}{dt} I_j = 0, I_j = \int_{-\infty}^{\infty} T_j(x) dx. \]  

(96)

Hence, there exists an associated flux \( X_j(x) \) that satisfies

\[ \partial_x T_j(x) + \partial_x X_j(x) = 0, \]  

(98)

and the BK system (1); we find

\[ T_1 = \omega, X_1(x) = -\left( \nu \omega + \frac{1}{2} \omega_x \right). \]  

(99)

For the BK system with corrections, we still have

\[ \tilde{\phi}(x, \lambda) = \sum_{j=0}^{\infty} \frac{I_j}{(2i\lambda)^j}. \]  

(100)

Since, \( a(\lambda) \) depends on \( t \), (97) is not valid. In fact, the evolution of densities \( I_j \) is an \( \varepsilon \)-order term which denotes perturbation correction of the conservation laws. To find the evolution, we rewrite the BK system with corrections

\[ \nu_t = S + \varepsilon q(x), \quad \omega_t = R + \varepsilon p(x) \]  

\[ S = \frac{1}{2} \left( \nu^2 + 2\omega - \nu_x \right)_x, \quad R = \left( \nu \omega + \frac{1}{2} \omega_x \right)_x. \]  

(101)

Making use of the functional derivative, we get

\[ \frac{d}{dt} I_j = \int_{-\infty}^{\infty} \frac{\partial \delta I_j}{\partial \nu(x)} \frac{d}{dt} \nu(x) + \frac{\partial \delta I_j}{\partial \omega(x)} \frac{d}{dt} \omega(x) dx, \]  

(102)

\[ \frac{d}{dt} I_j = \int_{-\infty}^{\infty} \frac{\partial \delta I_j}{\partial \nu(x)} - \frac{\partial \delta I_j}{\partial \omega(x)} R dx + \frac{\partial \delta I_j}{\partial q(x)} q(x) dx + \frac{\partial \delta I_j}{\partial p(x)} p(x) dx. \]  

(103)
6. Adiabatic Approximate Solutions

In this section, we consider the adiabatic approximate solution of the BK system with corrections. Since the linear spectral problem is valid, from (15) to (30) and in the case of reflectionless potentials, we still have

\[
\psi_1(x, \lambda) = e^{\eta_1 e^{\frac{i}{\lambda_1}} x} + \sum_{j=1}^{N} \frac{i c_j}{\lambda - \lambda_j} \psi_2(x, \lambda) e^{\frac{i}{\lambda_j} x},
\]

\[
\psi_2(x, \lambda) = \sum_{j=1}^{N} \frac{i \tilde{c}_j}{\lambda - \lambda_j} \psi_2(x, \lambda) e^{\frac{i}{\lambda_j} x}.
\]

where

\[
\Delta = 1 + f_1 \tilde{f}_1, \quad f_1 = \frac{i c_1}{\lambda_1 - \lambda_1} e^{2\frac{i}{\lambda_1} x}, \quad \tilde{f}_1 = \frac{i \tilde{c}_1}{\lambda_1 - \lambda_1} e^{-2\frac{i}{\lambda_1} x}.
\]

Due to

\[
a(\lambda) = e^{\theta} \frac{\lambda - \lambda_1}{\lambda - \lambda_1}, \quad \tilde{a}(\lambda) = e^{\theta} \frac{\lambda - \tilde{\lambda}_1}{\lambda - \tilde{\lambda}_1},
\]

where

\[
\eta = \eta - \eta_+ = \frac{1}{2} \int_{-\infty}^{\infty} v(x) dx.
\]

Hence,

\[
f_1 = -d_1(t) e^{-\eta_+ e^{-2\lambda_1 t} x} \tilde{f}_1 = \tilde{d}_1(t) e^{-\eta_+ e^{-2\lambda_1 t} x}.
\]

For the BK system and \( N = 1, \lambda_1 = iv_1 \) and \( \tilde{\lambda}_1 = iv_1, \) using \( d_{1,t}(t) + 2\lambda_1^2 d_1(t) = 0 \) and \( d_{1,t}(t) - 2\lambda_1^2 d_1(t) = 0, \)

\[
f_1 = e^{-\frac{i}{x}} \tilde{f}_1 = e^{-\frac{i}{\lambda_1 t}},
\]

where

\[
Z = \eta + 2v_1 x - 2v_1^2 t + Z_{10},
\]

\[
\tilde{Z} = -\eta - 2v_1 x - 2v_1^2 t + \tilde{Z}_{10}.
\]

Comparing (25) and (29), we obtain

\[
e^{\theta_1} \omega = -2c_1 \psi_2(x, \lambda_1) e^{\frac{i}{\lambda_1} x},
\]

\[
e^{-\eta_+} = -2c_1 \tilde{\psi}_1(x, \tilde{\lambda}_1) e^{\frac{i}{\lambda_1} x},
\]

which gives rise to the solution of the BK system

\[
v = -(v_1 + \tilde{v}_1) + \tanh (F),
\]

\[
\omega = (v_1 - \tilde{v}_1)^2 \text{sech}^2 (F),
\]

where

\[
F = \frac{1}{2} (Z + \tilde{Z}).
\]
It is important to note that the form of the adiabatic approximate solutions is similar to one soliton solution of the BK system without correction terms, while the scattering data follow the time evolution discussed in Section 4. Hence, \((e^{i\omega t})^a\) and \((e^{-\eta t})^a\) are neither the solution of the BK system nor the solution of the BK system with correction terms; they give a part of the \(\varepsilon\)-order approximate solutions to the BK system with correction terms. In Section 8, we will discuss the other part of the \(\varepsilon\)-order approximate solutions.

7. Slow Variations of the Spectral Parameters

To discuss the slow variations of the spectral parameters, we rewrite \(Z\) and \(\tilde{Z}\) in (117) as

\[
Z = \eta + 2\nu_1 x + \delta, \quad \frac{d\delta}{dt} = -2\nu_1, \quad (124)
\]

\[
\tilde{Z} = -\eta - 2\nu_1 x + \delta, \quad \frac{d\delta}{dt} = -2\nu_1. \quad (125)
\]

In this case,

\[
d_1(t) = e^{\delta}, \quad \tilde{d}_1(t) = e^{\tilde{\delta}}. \quad (126)
\]

In addition, for reflectionless case and \(N = 1\), the lost functions take the following form:

\[
\psi_1(x, \lambda) = \frac{1}{2} \frac{\lambda_1 - \lambda}{\lambda - \lambda_1} e^{i\nu t} e^{i\kappa f_1 e^F} \text{sech} (F), \quad (127)
\]

\[
\psi_2(x, \lambda) = \frac{e^{i\kappa x e^{-\eta t}}}{\lambda - \lambda_1} \left[ \lambda - \frac{1}{2} \left( \lambda_1 + \lambda_1 \right) \right] + \frac{1}{2} \left( \lambda_1 - \lambda_1 \right) \tanh (F), \quad (128)
\]

\[
\phi_1(x, \lambda) = \frac{e^{-i\kappa x e^{-\eta t}}}{\lambda - \lambda_1} \left[ \lambda - \frac{1}{2} \left( \lambda_1 + \lambda_1 \right) \right] - \frac{1}{2} \left( \lambda_1 - \lambda_1 \right) \tanh (F), \quad (129)
\]

\[
\psi_2(x, \lambda) = \frac{1}{2} \frac{\lambda_1 - \lambda_1}{\lambda - \lambda_1} e^{-\eta t} e^{-i\kappa f_1 e^F} \text{sech} (F), \quad (130)
\]

\[
\phi_1(x, \lambda) = \frac{e^{-i\kappa x e^{-\eta t}}}{\lambda - \lambda_1} \left[ \lambda - \frac{1}{2} \left( \lambda_1 + \lambda_1 \right) \right] - \frac{1}{2} \left( \lambda_1 - \lambda_1 \right) \tanh (F), \quad (131)
\]

\[
\psi_2(x, \lambda) = \frac{1}{2} \frac{\lambda_1 - \lambda_1}{\lambda - \lambda_1} e^{\eta t} e^{-i\kappa f_1 e^F} \text{sech} (F), \quad (132)
\]

\[
\phi_1(x, \lambda) = \frac{1}{2} \frac{\lambda_1 - \lambda_1}{\lambda - \lambda_1} e^{-\eta t} e^{i\kappa f_1 e^F} \text{sech} (F), \quad (133)
\]

\[
\phi_2(x, \lambda) = \frac{1}{2} \frac{\lambda_1 - \lambda_1}{\lambda - \lambda_1} e^{\eta t} e^{i\kappa f_1 e^F} \text{sech} (F), \quad (134)
\]

These equations can be derived by taking \(N = 1\) and substituting (108) into (29). It is remarked that \(\psi^a(x, \lambda), \psi^{\bar{a}}(x, \lambda), \phi^a(x, \lambda), \) and \(\phi^{\bar{a}}(x, \lambda)\) have a similar form to those in (127) for the BK system with corrections. In fact, the eigenfunctions in (69) are \(\psi^a(x, \lambda), \psi^{\bar{a}}(x, \lambda), \phi^a(x, \lambda),\) and \(\phi^{\bar{a}}(x, \lambda).\) Substituting (127) into (69), we get

\[
\lambda_{1,2} = -\frac{1}{4} i e \int_{-\infty}^{\infty} \text{sech}^2 \left( q(x) - p(x) e^{2\eta t} e^{-2} \right) dF, \quad (135)
\]

\[
\tilde{\lambda}_{1,2} = -\frac{1}{4} i e \int_{-\infty}^{\infty} \text{sech}^2 \left( q(x) + p(x) e^{2\eta t} e^{2} \right) dF, \quad (136)
\]

Next, we investigate the variation of parameter in \(d_1(t)\) and \(\tilde{d}_1(t).\) To this end, combining (127) and (66), we find

\[
d_1(t) = e^{-2i\kappa x e^{-2} e^{\eta t}}, \quad (137)
\]

\[
\tilde{d}_1(t) = e^{2i\kappa x e^{2} e^{-\eta t}}. \quad (138)
\]

For the BK system with corrections, (126) is not valid and (124) implies

\[
d_{1,2}(t) + 2\lambda_2^2 d_1(t) = \lambda_1(t) \left( \delta_t - 2\nu_1^2 \right), \quad (139)
\]

Using relevant integral equations in (87), we find from (138) that

\[
\delta_t = \frac{2\nu_1^2}{2} + \frac{\nu}{2} \int_{-\infty}^{\infty} x \text{sech}^2 (F) \left( q(x) - p(x) e^{2\eta t} e^{-2} \right)
\]

\[
+ \frac{\text{sech} (F)}{\nu_1 - \nu_1} \left( q(x) \sin (F) - p(x) e^{2\eta t} e^{-2} e^F \right) dF,
\]

\[
\tilde{\delta}_t = -2\nu_1^2 e^{-2} + \frac{\nu}{2} \int_{-\infty}^{\infty} x \text{sech}^2 (F) \left( q(x) + p(x) e^{2\eta t} e^2 \right)
\]

\[
- \frac{\text{sech} (F)}{\nu_1 - \nu_1} \left( q(x) \cos (F) - p(x) e^{2\eta t} e^{2} e^F \right) dF. \quad (140)
\]

Now, for one soliton solution, the formulas of spectral parameters are given.

8. Corrections of the Adiabatic Solutions

In this section, we will discuss the other part of the \(\varepsilon\)-order approximate solutions to the BK system with correction terms. In the case of \(N = 1,\)
\[
\psi_1(x, \lambda) e^{i \lambda x} = e^{i \xi} + \frac{i c_1}{\lambda - \lambda_1} \psi_1(x, \lambda_1) e^{i \lambda_1 x} + J_1(x, \lambda), \quad (141)
\]

\[
\psi_2(x, \lambda) e^{i \lambda x} = \frac{i c_1}{\lambda - \lambda_1} \psi_2(x, \lambda_1) e^{i \lambda_1 x} + J_2(x, \lambda), \quad (142)
\]

\[
\psi_1(x, \lambda) e^{-i \lambda x} = \frac{i c_1}{\lambda - \lambda_1} \psi_1(x, \lambda_1) e^{-i \lambda_1 x} + \bar{J}_1(x, \lambda), \quad (143)
\]

\[
\psi_2(x, \lambda) e^{-i \lambda x} = e^{-\eta_i} + \frac{i c_1}{\lambda - \lambda_1} \psi_2(x, \lambda_1) e^{-i \lambda_1 x} + \bar{J}_2(x, \lambda), \quad (144)
\]

where

\[
J_j(x, \lambda) = \frac{1}{2 i \pi} \int_{-\infty}^{\infty} \frac{\rho(k)}{k - \lambda - i 0} \psi_j(x, k) e^{ikx} dk, \quad j = 1, 2,
\]

\[
\bar{J}_j(x, \lambda) = \frac{1}{2 i \pi} \int_{-\infty}^{\infty} \frac{\tilde{\rho}(k)}{k - \lambda} \tilde{\psi}_j(x, k) e^{ikx} dk, \quad j = 1, 2.
\]

Using Cramer’s rule, we find, from (148), that

\[
\delta \psi_2(x, \lambda_1) = \frac{1}{\Delta} e^{i \lambda_1 x} \left( J_2^\alpha(x, \lambda_1) + J_2^\delta(x, \lambda_1) \right), \quad (154)
\]

\[
\delta \psi_1(x, \lambda_1) = \frac{1}{\Delta} e^{-i \lambda_1 x} \left( J_1^\alpha(x, \lambda_1) + J_1^\delta(x, \lambda_1) \right). \quad (155)
\]

Hence, \( \varepsilon \)-order approximate solutions of the BK system with correction terms can be written as

\[
e^{i \xi} \omega = (e^{i \xi} \omega)^a + \delta (e^{i \xi} \omega), \quad (156)
\]

\[
e^{-\eta_i} = (e^{-\eta_i})^a + \delta (e^{-\eta_i}), \quad (157)
\]

In addition, from (122) and (156), we know that

\[
\delta (e^{i \xi} \omega) = -2i \xi \delta \psi_2(x, \lambda_1) e^{i \lambda_1 x} + \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(k) \psi_2^\alpha(x, k) dk, \quad (158)
\]

\[
\delta (e^{-\eta_i}) = -2i \xi \delta \psi_1(x, \lambda_1) e^{-i \lambda_1 x} - \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{\rho}(k) \tilde{\psi}_1^\alpha(x, k) dk. \quad (159)
\]

Substituting (154) into (158), we get

\[
\delta (e^{i \xi} \omega) = \frac{(e^{i \xi})^a}{\pi} \int_{-\infty}^{\infty} \rho(k) \psi_2^\alpha(x, k) e^{ikx} dk, \quad (160)
\]

\[
\delta (e^{-\eta_i}) = \frac{(e^{-\eta_i})^a}{\pi} \int_{-\infty}^{\infty} \tilde{\rho}(k) \tilde{\psi}_1^\alpha(x, k) e^{ikx} dk. \quad (160)
\]

9. Conclusions and Discussions

In this work, we discussed the perturbation theory of the Broer-Kaup equation with corrections which is a nearly integrable system. The technique based on the inverse scattering transform is extended to obtain the evolutions of the scattering data, including the discrete spectrums, the scattering coefficients, and the normalized factors. These evolution equations of the scattering data are determined by integral representations of corrections and eigenfunctions. Hence, the adiabatic dynamic of one soliton can be figured out by the time-dependent scattering data, if the corrections are given.

In general, the perturbations by adding corrections can be dissipative terms.
**Data Availability**

The data will be made available on reasonable request.

**Conflicts of Interest**

The authors declare that there is no conflict of interest regarding the publication of this paper.

**Acknowledgments**

This work was supported by the National Natural Science Foundation of China [Grant number 11471295].

**References**

[1] D. J. Kaup, "A higher-order water-wave equation and the method for solving it," *Progress in Theoretical Physics*, vol. 54, no. 2, pp. 396–408, 1975.

[2] L. J. F. Broer, "Approximate equations for long water waves," *Applied Scientific Research*, vol. 31, no. 5, pp. 377–395, 1975.

[3] Y. Li, W. X. Ma, and J. E. Zhang, "Darboux transformations of classical Boussinesq system and its new solutions," *Physics Letters A*, vol. 275, no. 1-2, pp. 60–66, 2000.

[4] P. V. Nabelek and V. E. Zakharov, "Solutions to the Kaup–Broer system and its (2+1) dimensional integrable generalization via the dressing method," *Physica D*, vol. 409, article 132478, 2020.

[5] Y. Li and J. E. Zhang, "Darboux transformations of classical Boussinesq system and its multi-soliton solutions," *Physics Letters A*, vol. 284, no. 6, pp. 253–258, 2001.

[6] D. J. Kaup, "A perturbation expansion for the Zakharov–Shabat inverse scattering transform," *SIAM Journal on Applied Mathematics*, vol. 31, no. 1, pp. 121–133, 1976.

[7] D. J. Kaup and A. C. Newell, "Solitons as particles, oscillators, and in slowly changing media: a singular perturbation theory," *Proceedings of the Royal Society of London A*, vol. 361, pp. 413–446, 1997.

[8] V. I. Karpman, "Soliton evolution in the presence of perturbation," *Physica Scripta*, vol. 20, no. 3-4, pp. 462–478, 1979.

[9] E. M. Maslov, "Perturbation theory for solitons in the second approximation," *Theoretical and Mathematical Physics*, vol. 42, no. 3, pp. 237–245, 1980.

[10] G. L. Lamb, *Elements of Soliton Theory*, Wiley, New York, 1980.

[11] Y. S. Kivshar and B. A. Malomed, "Dynamics of solitons in nearly integrable systems," *Reviews of Modern Physics*, vol. 61, no. 4, pp. 763–915, 1989.

[12] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[13] P. Deift and X. Zhou, "Perturbation theory for infinite-dimensional integrable systems on the line. A case study," *Acta Mathematica*, vol. 188, no. 2, pp. 163–262, 2002.

[14] V. M. Lashkin, "Perturbation theory for dark solitons: inverse scattering transform approach and radiative effects," *Physical Review E*, vol. 70, no. 6, p. 066620, 2004.

[15] N. N. Huang, *Theory of Solitons and Method of Perturbations*, Shanghai Scientific and Technological Education Publishing House, Shanghai, 1996.

[16] M. B. Fogel, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, "Dynamics of sine-Gordon solitons in the presence of perturbations," *Physical Review B*, vol. 15, no. 3, pp. 1578–1592, 1977.

[17] J. P. Keener and D. W. Mclaughlin, "Solitons under perturbations," *Physical Review A*, vol. 16, no. 2, pp. 777–790, 1977.

[18] D. J. Kaup, "Perturbation theory for solitons in optical fibers," *Physical Review A*, vol. 42, no. 9, pp. 5689–5694, 1990.

[19] R. L. Herman, "A direct approach to studying soliton perturbations," *Journal of Physics A: Mathematical and General*, vol. 23, no. 12, pp. 2327–2362, 1990.

[20] R. Grimshaw and H. Mitsudera, "Slowly-varying solitary wave solutions of the perturbed Korteweg-de Vries equation revisited," *Studies in Applied Mathematics*, vol. 90, no. 1, pp. 75–86, 1993.

[21] Y. Matsuno, "Multisoliton perturbation theory for the Benjamin-Ono equation and its application to real physical systems," *Physical Review E*, vol. 51, no. 2, pp. 1471–1483, 1995.

[22] J. Yan and Y. Tang, "Direct approach to the study of soliton perturbations," *Physical Review E*, vol. 54, no. 6, pp. 6816–6824, 1996.

[23] D. J. Kaup and T. I. Lakoba, "The squared eigenfunctions of the massive Thirring model in laboratory coordinates," *Journal of Mathematical Physics*, vol. 37, no. 1, pp. 308–323, 1996.

[24] T. I. Lakoba and D. J. Kaup, "Perturbation theory for the Manakov soliton and its applications to pulse propagation in randomly birefringent fibers," *Physical Review E*, vol. 56, no. 5, pp. 6147–6165, 1997.

[25] D. J. Kaup, T. I. Lakoba, and Y. Matsuno, "Perturbation theory for the Benjamin-Ono equation," *Inverse Problems*, vol. 15, no. 1, pp. 215–240, 1999.

[26] J. K. Yang, "Multisoliton perturbation theory for the Manakov equations and its applications to nonlinear optics," *Physical Review E*, vol. 59, no. 2, pp. 2393–2405, 1999.

[27] J. K. Yang, "Dynamics of embedded solitons in the extended Korteweg–de Vries equations," *Studies in Applied Mathematics*, vol. 106, no. 3, pp. 337–365, 2001.

[28] J. Yang, "Stable embedded solitons," *Physical Review Letters*, vol. 93, article 194102, 2003.

[29] X.-J. Chen and J. Yang, "Direct perturbation theory for solitons of the derivative nonlinear Schrödinger equation and the modified nonlinear Schrödinger equation," *Physical Review E*, vol. 65, no. 6, article 066608, 2002.

[30] S. M. Hoseini and T. R. Marchant, "Soliton perturbation theory for a higher order Hirota equation," *Mathematics and Computers in Simulation*, vol. 80, no. 4, pp. 770–779, 2009.

[31] Y. Kodama, "Normal forms for weakly dispersive wave equations," *Physics Letters A*, vol. 112, no. 5, pp. 193–196, 1985.

[32] Y. Kodama, "On solitary-wave interaction," *Physics Letters A*, vol. 123, no. 6, pp. 276–282, 1987.

[33] T. Kano, "Normal form of nonlinear Schrödinger equation," *Journal of the Physical Society of Japan*, vol. 58, no. 12, pp. 4322–4328, 1989.

[34] S. M. Hoseini and T. R. Marchant, "Solitary wave interaction and evolution for a higher-order Hirota equation," *Wave Motion*, vol. 44, no. 2, pp. 92–106, 2006.

[35] J. K. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems*, SIAM, Philadelphia, 2010.