Two Standard Decks of Playing Cards are Sufficient for a ZKP for Sudoku

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Abstract

Sudoku is a logic puzzle with an objective to fill a number between 1 and 9 in each empty cell of a 9 × 9 grid such that every number appears exactly once in each row, each column, and each 3 × 3 block. In 2020, Sasaki et al. proposed a physical zero-knowledge proof (ZKP) protocol for Sudoku using 90 cards, which allows a prover to physically show that he/she knows a solution without revealing it. However, their protocol requires nine identical copies of some cards, which cannot be found in a standard deck of playing cards. Therefore, nine decks of cards are actually required in order to perform that protocol. In this paper, we propose a new ZKP protocol for Sudoku that can be performed using only two standard decks of playing cards. In general, we develop the first ZKP protocol for an n × n Sudoku that can be performed using a deck of all different cards.

Keywords: zero-knowledge proof, card-based cryptography, Sudoku, puzzle

1 Introduction

Sudoku is one of the world’s most popular logic puzzles. A Sudoku puzzle consists of a 9 × 9 grid divided into nine blocks of size 3 × 3. Some of the cells in the grid are already filled with numbers between 1 and 9. The player has to fill a number into each empty cell such that every number from 1 to 9 appears exactly once in each row, each column, and each 3 × 3 block [13] (see Fig. 1). There is also a generalized version of Sudoku where the grid has size n × n and is divided into n blocks of size √n × √n, where n is a perfect square. The generalized Sudoku is known to be NP-complete [24].

1.1 Zero-Knowledge Proof

We want to construct a zero-knowledge proof (ZKP) for Sudoku, which allows a prover P to convince a verifier V that he/she knows a solution of the puzzle without revealing any information about it. Formally, a ZKP is an interactive proof between P and V where both of them are given a computational problem x, but only P knows a solution w. A ZKP with perfect completeness and perfect soundness must satisfy the following properties.

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1. **Perfect Completeness:** If $P$ knows $w$, then $P$ can convince $V$.

2. **Perfect Soundness:** If $P$ does not know $w$, then $P$ cannot convince $V$.

3. **Zero-knowledge:** $V$ does not obtain any information about $w$. Formally, there exists a probabilistic polynomial time algorithm $S$ (called a simulator) that does not know $w$, and the outputs of $S$ follow the same probability distribution as the outputs of the actual protocol.

The concept of a ZKP was first introduced by Goldwasser et al. [5] Recently, many results have been focusing on constructing physical ZKPs using objects found in everyday life such as a deck of cards. These protocols have a benefit that they do not require electronic devices, and also have didactic values since they are easy to understand and verify the correctness, even for non-experts in cryptography.

## 2 Previous Protocols

The first ZKP protocols for Sudoku were developed by Gradwohlet al. [6] in 2009. However, each of their six proposed protocols either has a nonzero soundness error or requires special tools such as scratch-off cards and a machine to seal the cards. In 2020, Sasaki et al. [23] proposed the improved ZKP protocols for Sudoku that have perfect soundness without using special tools.

### 2.1 Uniqueness Verification Protocol

Before showing the protocol of Sasaki et al., we first explain the following subprotocol, which was also developed by the same authors [23]. This protocol allows the prover $P$ to convince the verifier $V$ that a sequence of $n$ cards is a permutation of cards $a_1, a_2, ..., a_n$ in some order, without revealing their orders. It also preserves the orders of the cards in the sequence (so that the sequence can be later used in other protocols).

Let $x_1, x_2, ..., x_n$ be another set of $n$ different cards. $P$ performs the following steps.

1. Publicly place cards $x_1, x_2, ..., x_n$ below the sequence in this order from left to right to form a $2 \times n$ matrix of cards (see Fig. 2).
2. Rearrange all columns of the matrix by a uniformly random permutation. This can be performed in real world by putting both cards in each column into an envelope and scrambling all envelopes together.

3. Turn over all cards in the top row. $V$ verifies that the sequence is a permutation of $a_1, a_2, ..., a_n$. Otherwise, $V$ rejects.

4. Turn over all face-up cards. Rearrange all columns of the matrix by a uniformly random permutation.

5. Turn over all cards in the bottom row. Rearrange the columns such that the cards in the bottom rows are $x_1, x_2, ..., x_n$ in this order from left to right. The sequence in the top row now returns to its original state.

2.2 Protocol of Sasaki et al.

Sasaki et al. [23] proposed three protocols to verify a solution of an $n \times n$ Sudoku puzzle. Here we will show only the first protocol, which is the one using the least number of cards.

Each card used in this protocol has a positive number on the front side: 1, 2, ..., $n$; all cards have identical back sides. On each cell already having a number $j$, $P$ publicly places a $j$. On each empty cell having a number $j$ is $P$’s solution, $P$ secretly places a $j$.

$P$ then applies the uniqueness verification protocol to verify that every row, column, and block contains a permutation of 1, 2, ..., $n$.

In total, this protocol uses $n^2 + n$ cards: $n$ identical copies of 1, 2, ..., $n$ (to encode the numbers in the grid), and another set of $n$ different cards (to use in the uniqueness verification protocol). For a standard $9 \times 9$ puzzle, the protocol uses 90 cards, which is less than the number of cards in two standard decks; however, it requires nine identical copies of 1, 2, ..., 9. As a standard deck consists of 54 different cards (including two different jokers), nine identical decks are actually required in order to perform this protocol, which are too many to be practical. Another choice is to use a different kind of deck (e.g. cards from board games) that includes several identical copies of some cards, but these decks are more difficult to find in everyday life.

Considering a drawback of this protocol, we aim to develop a more practical ZKP protocol for a $9 \times 9$ Sudoku that can be performed using only two standard decks of playing cards.

2.3 Related Work

After the discovery of the physical ZKP protocols for Sudoku, physical ZKP protocols for other popular logic puzzles have been proposed as well, including Nonogram [3], Akari [1],
| Protocol          | Standard Deck? | #Cards | #Shuffles |
|-------------------|----------------|--------|-----------|
| Sasaki et al. [23] | no             | 90     | 45        |
| Ours (§5.1)       | yes            | 120    | 108       |
| Ours (§5.2)       | yes            | 108    | 322       |

Table 1: The number of required cards and shuffles for each protocol for a $9 \times 9$ Sudoku

Takuzu [14], Kakuro [15], KenKen [1], Makaro [2], Norinori [4], Slitherlink [12], Juosan [14], Numberlink [21], Suguru [20], Ripple Effect [22], Nurikabe [19], and Hitori [19].

Besides verifying solutions of logic puzzles, card-based protocols have also been extensively studied in secure multi-party computation, a setting where multiple parties want to jointly compute a function of their secret inputs without revealing the input of any party. The vast majority of work in this area, however, also uses identical copies of $♣$ and $♥$ in the protocols. The only exceptions are [9, 11, 16, 17] which introduced AND, XOR, and copy protocols using a standard deck, and [13] which introduced a Yao’s millionaire protocol using a standard deck. In [11], the authors posed a challenging open problem to develop ZKP protocols for logic puzzles using a standard deck.

Pratically, a standard deck of playing cards consists of 54 different cards (including two different jokers). Theoretically, it is also a challenging problem to develop a protocol that uses a deck of all different cards, so we also study the setting that the deck consists of $1, 2, ..., $ where each card can have an arbitrarily large number on it.

### 3 Our Contribution

In this paper, we propose a new ZKP protocol for a generalized $n \times n$ Sudoku puzzle with perfect completeness and soundness using a set of all different cards.

There are two slightly different methods to implement our protocol. The first one uses $n^2 + n\sqrt{n} + n + \sqrt{n}$ cards and $4n\sqrt{n}$ shuffles. The second one uses $n^2 + 2n + 3\sqrt{n}$ cards and at most $2n^2(\sqrt{n} - 1) + 2$ shuffles (see Tables 1 and 2).

In particular, for a standard $9 \times 9$ Sudoku puzzle, our protocol (with the second method of implementation) uses 108 cards and can be performed using two standard decks of playing cards, regardless of whether the two decks are the same or different types.

Theoretically, this work is also a breakthrough step in card-based cryptography as it is the first ZKP protocol for any logic puzzle that can be performed using a deck of all different cards.

### 4 Preliminaries

At first, we assume that all cards used in our protocols have different front sides and identical back sides (although we will later show that some pairs of cards can have identical front sides or different back sides and our protocol still works correctly).
| Protocol         | Standard Deck? | #Cards                  | #Shuffles                  |
|------------------|----------------|-------------------------|----------------------------|
| Sasaki et al. [23] | no             | \(n^2 + n\)             | 5n                         |
| Ours (§5.1)      | yes            | \(n^2 + n\sqrt{n} + n + \sqrt{n}\) | 4n\(\sqrt{n}\)          |
| Ours (§5.2)      | yes            | \(n^2 + 2n + 3\sqrt{n}\) | 2n\(^2\)(\(\sqrt{n} - 1\)) for even \(n\)  
|                  |                |                         | 2n\(^2\)(\(\sqrt{n} - 1\)) + 2 for odd \(n > 3\) |

Table 2: The number of required cards and shuffles for each protocol for an \(n \times n\) Sudoku

| Column | 0 | 1 | 2 | 3 | 4 | 5 |
|--------|---|---|---|---|---|---|
| 0      | ??| ??| ??| ??| ??| ??|
| 1      | p1| ??| ??| ??| ??| ??|
| Row 2  | p2| ??| ??| ??| ??| ??|
| 3      | p3| ??| ??| ??| ??| ??|
| 4      | p4| ??| ??| ??| ??| ??|

Figure 3: An example of a \(4 \times 5\) marked matrix

4.1 Marked Matrix

Suppose we have a \(k \times \ell\) matrix of face-down cards (we call these cards *encoding cards*). Let Row \(i\) denote an \(i\)-th topmost row and let Column \(j\) denote a \(j\)-th leftmost column. To the left of Column 1, publicly place face-down cards \(p_1, p_2, ..., p_k\) in this order from top to bottom; this new column is called Column 0. Analogously, above Row 1, publicly place face-down cards \(q_1, q_2, ..., q_\ell\) in this order from left to right; this new row is called Row 0.

We call this structure a \(k \times \ell\) marked matrix (see Fig 3), and we call the cards in Row 0 and Column 0 *marking cards*.

4.2 Shuffle Operations

Suppose we have a \(k \times \ell\) marked matrix. For a set \(S \subseteq \{1, 2, ..., k\}\), an operation \(\text{row\_shuf}(S)\) rearranges the rows in \(S\) (including marking cards in Column 0) by a uniformly random permutation. For example, \(\text{row\_shuf}\{1, 3, 4\}\) rearranges Row 1, Row 3, and Row 4 of the matrix by a uniformly random permutation. This can be performed in real world by putting all cards in each row in \(S\) into an envelope and scrambling all envelopes together.

Analogously, for a set \(S \subseteq \{1, 2, ..., \ell\}\), an operation \(\text{col\_shuf}(S)\) rearranges the columns in \(S\) (including marking cards in Row 0) by a uniformly random permutation.

4.3 Rearrangement Protocol

After applying some shuffle operations to a marked matrix, a rearrangement protocol reverts the matrix back to its original state. Slightly different variants of this protocol with the same idea has been used in previous work [2, 7, 8, 21, 22, 23].
Suppose we have a $k \times \ell$ marked matrix $M$ with marking cards $p_1, p_2, \ldots, p_k$ in Column 0 and $q_1, q_2, \ldots, q_\ell$ in Row 0. We perform the following steps.

1. Apply \texttt{row\_shuffle}\{1,2,\ldots,k\} and \texttt{col\_shuffle}\{1,2,\ldots,\ell\} to $M$.

2. Turn over all marking cards in Column 0 and Row 0. Rearrange the rows of $M$ such that the marking cards in Column 0 are $p_1, p_2, \ldots, p_k$ in this order from top to bottom. Rearrange the columns of $M$ such that the marking cards in Row 0 are $q_1, q_2, \ldots, q_\ell$ in this order from left to right.

### 4.4 Standard Deck Chosen Cut Protocol

Given a $k \times \ell$ marked matrix $M$, this protocol allows the prover $P$ to choose a card located at Row $i$ and Column $j$ of $M$ he/she wants without revealing $i$ or $j$. It was modified from an original chosen cut protocol of Koch and Walzer \cite{KochWalzer2007} so that it can be performed using a standard deck. $P$ performs the following steps.

1. Secretly stack a face-down card $x_1$ on a card located at Row $i$ and Column $j$.

2. On each of the remaining $k\ell - 1$ cards in the matrix, secret stack each of face-down cards $x_2, x_3, \ldots, x_{k\ell}$ in a uniformly random order. The cards $x_1, x_2, \ldots, x_{k\ell}$ are called \textit{helper cards}.

3. Apply \texttt{row\_shuffle}\{1,2,\ldots,k\} and \texttt{col\_shuffle}\{1,2,\ldots,\ell\} to $M$.

4. Turn over all helper cards. Locate the position of $x_1$. The encoding card in that stack is the one originally located at Row $i$ and Column $j$ as desired.

5. Remove all helper cards. Apply the rearrangement protocol to revert $M$ to its original state.

This protocol will be implicitly used in our main protocol, with Step 3 being replaced by equivalent operations.

### 5 Main Protocol

For simplicity, we will show a protocol for a standard $9 \times 9$ Sudoku puzzle. Our protocol can be straightforwardly generalized to an $n \times n$ puzzle.

We use the following cards in our protocol.

- encoding cards $a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j, i_j$ ($j = 1, 2, \ldots, 9$)
- marking cards $p_j$ ($j = 1, 2, 3$) and $q_j$ ($j = 1, 2, \ldots, 9$)
- helper cards $x_j, y_j, z_j$ ($j = 1, 2, \ldots, 9$)

Suppose the grid is divided into blocks $A, B, \ldots, I$ (see Fig. 4). We use a card $a_j$ ($j = 1, 2, \ldots, 9$) to encode a number $j$ in Block $A$. Analogously, we use cards $b_j, c_j, \ldots, i_j$ ($j = 1, 2, \ldots, 9$) to encode numbers $j$ in blocks $B, C, \ldots, I$, respectively.
On each cell already having a number, $P$ publicly places a face-down corresponding card (e.g. places a card $b_3$ on a cell with a number $3$ in Block $B$). On each empty cell, $P$ secretly places a face-down corresponding card according to his/her solution.

Apply the uniqueness verification protocol in Section 2.1 to verify that Block $A$ consists of cards $a_1, a_2, ..., a_9$ in some order. Do the same for Blocks $B, C, ..., I$. Now $V$ is convinced that every number from 1 to 9 appears exactly once in each block.

Next, we will show two methods to verify that every number from 1 to 9 appears exactly once in each row and column.

### 5.1 Method A
First, $P$ performs the following steps to verify that a number 1 appears exactly once in each of the three topmost rows.

1. Take the cards from the three topmost rows to form a $3 \times 9$ matrix and publicly place marking cards $p_1, p_2, p_3$ in Column 0 and $q_1, q_2, ..., q_9$ in Row 0 to create a marked matrix $M$.
2. Secretly stack face-down cards $x_1, y_1, z_1$ on $a_1, b_1, c_1$, respectively.
3. On each of the remaining 8 cards in Block $A$, secret stack each of face-down cards $x_2, x_3, ..., x_9$ in a uniformly random order. Do the same for cards $y_2, y_3, ..., y_9$ in Block $B$ and $z_2, z_3, ..., z_9$ in Block $C$.
4. Apply $\text{row\_shuffle}([1,2,3])$, $\text{col\_shuffle}([1,2,3])$, $\text{col\_shuffle}([4,5,6])$, and $\text{col\_shuffle}([7,8,9])$ to $M$.
5. Turn over all helper cards. Locate the positions of $x_1, y_1, z_1$. Turn over the encoding cards in these three stacks to show that they are $a_1, b_1$, and $c_1$, respectively, and that they are all located at different rows. Otherwise, $V$ rejects.
6. Remove all helper cards and turn all encoding cards face-down. Apply the rearrangement protocol in Section 4.3 to revert $M$ to its original state.

Note that Steps 2 to 6 are equivalent to applying the standard deck chosen cut protocol in Section 4.4 to Blocks $A$, $B$, and $C$, simultaneously. These steps ensure that the three 1s
in Blocks $A$, $B$, and $C$ are all located at different rows. Since it has already been shown that each block contains exactly one 1, this implies there is exactly one 1 in each of the three topmost rows.

$P$ performs these steps analogously for numbers $2, 3, ..., 9$. Now $V$ is convinced that every number appears exactly once in each of the three topmost rows.

$P$ then does the same for Blocks $D$, $E$, and $F$ and for Blocks $G$, $H$, and $I$ to verify the rest of the rows. The verification for columns works analogously ($P$ takes the cards from Blocks $A$, $D$, and $G$, from Blocks $B$, $E$, and $H$, and from Blocks $C$, $F$, and $I$, and just transposes the matrix).

This method uses 81 encoding cards, 12 marking cards, and 27 helper cards, resulting in the total of 120 cards, slightly more than the number of cards in two standard decks, and uses 342 shuffles.\footnote{The number of shuffles can be reduced to 108 after optimization. See Appendix A}

We aim to further reduce the number of required cards as a trade-off between the number of cards and shuffles.

### 5.2 Method B

In Method A, we verify that the three 1s in Blocks $A$, $B$, and $C$ are all located at different rows by verifying these three blocks at the same time, which requires a lot of marking and helper cards. Instead, we can first verify that the two 1s in Blocks $A$ and $B$ are located at different rows, then do the same for Blocks $A$ and $C$, and for Blocks $B$ and $C$. This leads to the same conclusion that the three 1s in Blocks $A$, $B$, and $C$ are all located at different rows. The formal steps for verifying Blocks $A$ and $B$ are shown below.

1. Take the cards from blocks $A$ and $B$ to form a $3 \times 6$ matrix and publicly place marking cards $p_1, p_2, p_3$ in Column 0 and $q_1, q_2, ..., q_6$ in Row 0 to create a marked matrix $M$.
2. Secretly stack face-down cards $x_1$ and $y_1$ on $a_1$ and $b_1$, respectively.
3. On each of the remaining 8 cards in Block $A$, secret stack each of face-down cards $x_2, x_3, ..., x_9$ in a uniformly random order. Do the same for cards $y_2, y_3, ..., y_9$ in Block $B$.
4. Apply \texttt{row\_shuffle}($\{1, 2, 3\}$), \texttt{col\_shuffle}($\{1, 2, 3\}$), and \texttt{col\_shuffle}($\{4, 5, 6\}$) to $M$.
5. Turn over all helper cards. Locate the positions of $x_1$ and $y_1$. Turn over the encoding cards in both stacks to show that they are $a_1$ and $b_1$, respectively, and that they are located at different rows. Otherwise, $V$ rejects.
6. Remove all helper cards and turn all encoding cards face-down. Apply the rearrangement protocol in Section 4.3 to revert $M$ to its original state.

We say that two cards are from the same set if they are denoted by the same letter with different indices (e.g. $d_2$ and $d_5$ are from the same set). Notice that in both methods, cards from different sets never get mixed together. Therefore, cards from different sets can have identical front sides or different back sides (or even different sizes) and our protocol
still works correctly. The only requirement is that all cards from the same set must have
different front sides and identical back sides.

This method uses 81 encoding cards, nine marking cards, and 18 helper cards, resulting
in the total of 108 cards, which is exactly the number of cards from two standard decks
(including jokers), and uses 828 shuffles. We can, for example, use 54 cards from the first
deck in the sets $a_j, b_j, \ldots, f_j$ and 54 cards from the second deck in the remaining sets. The
protocol works correctly regardless of whether the two decks are identical or different, since
it allows cards from different sets to have identical front sides (in case of identical decks)
or different back sides or sizes (in case of different decks). Note that in some decks, the
two jokers are identical; in that case, we just need to make sure that the two jokers are in
different sets.

5.3 Generalization

Our protocol can be straightforwardly generalized to an $n \times n$ puzzle.

Method A uses $n^2$ encoding cards, $n + \sqrt{n}$ marking cards, and $n\sqrt{n}$ helper cards,
resulting in the total of $n^2 + n\sqrt{n} + n + \sqrt{n}$ cards. It uses $4n\sqrt{n}$ shuffles (after the optimization
in Appendix A).

Method B uses $n^2$ encoding cards, $3\sqrt{n}$ marking cards, and $2n$ helper cards, resulting
in the total of $n^2 + 2n + 3\sqrt{n}$ cards. It uses at most $2n^2(\sqrt{n} - 1) + 2$ shuffles (after the
optimization in Appendix A).

6 Proof of Correctness and Security

We will prove the perfect completeness, perfect soundness, and zero-knowledge properties
of our protocol.

**Lemma 1** (Perfect Completeness). If $P$ knows a solution of the Sudoku puzzle, then $V$
always accepts.

**Proof.** Suppose $P$ knows a solution and places cards on the grid accordingly. Every number
from 1 to 9 will appear exactly once in each row, each column, and each block. Hence,
the uniqueness verification protocol will pass for every block. Also, the same numbers from
different blocks are always located at different rows and columns, so both Methods A and
B will pass. Therefore, $V$ always accepts.

**Lemma 2** (Perfect Soundness). If $P$ does not know a solution of the Sudoku puzzle, then $V$
always rejects.

**Proof.** Suppose $P$ does not know a solution. There will be a number that appears at least
twice in the same row, column, or block. If it appears twice in a block, the uniqueness
verification protocol for that block will fail. If it appears twice in different blocks in the
same row (resp. column), Method A will fail when verifying the three blocks containing
that row (resp. column); also, method B will fail when verifying the two blocks where these
two numbers appear. Therefore, $V$ always rejects.

**Lemma 3** (Zero-Knowledge). During the verification, $V$ learns nothing about $P$’s solution.

\footnote{The number of shuffles can be reduced to 322 after optimization. See Appendix A}
Proof. It is sufficient to show that all distribution of cards that are turned face-up can be simulated by a simulator $S$ that does not know $P$’s solution.

- In Steps 3 and 5 of the uniqueness verification protocol, the orders of the $n$ cards are uniformly distributed among all $n!$ permutations. Hence, it can be simulated by $S$.

- In Step 2 of the rearrangement protocol, the orders of $p_1, p_2, ..., p_k$ and $q_1, q_2, ..., q_\ell$ are uniformly distributed among all $k!$ permutations and $\ell!$ permutations, respectively. Hence, it can be simulated by $S$.

- In Step 5 of Method A, the rows where $x_1$, $y_1$, and $z_1$ are located are uniformly distributed among all $3! = 6$ permutations of the first three rows; the columns where they are located are uniformly distributed among all $3^3 = 27$ combinations of three columns from Blocks A, B, and C. Also, the orders of $x_2, x_3, ..., x_9$ are uniformly distributed among all $8!$ permutations of the remaining cards in Block A; the same goes for $y_2, y_3, ..., y_9$ in Block B and $z_2, z_3, ..., z_9$ in Block C. Hence, it can be simulated by $S$.

- In Step 5 of Method B, the rows where $x_1$ and $y_1$ are located are uniformly distributed among all $\frac{3!}{2!} = 6$ 2-permutations of the first three rows; the columns where they are located are uniformly distributed among all $3^2 = 9$ combinations of two columns from Blocks A and B. Also, the orders of $x_2, x_3, ..., x_9$ are uniformly distributed among all $8!$ permutations of the remaining cards in Block A; the same goes for $y_2, y_3, ..., y_9$ in Block B. Hence, it can be simulated by $S$.

7 Future Work

We developed the first ZKP protocol for Sudoku, and also the first one for any logic puzzle, that uses a deck of all different cards. Our protocol for a standard $9 \times 9$ Sudoku can be performed using two standard decks of playing cards. However, the drawback of our protocol is that it uses a large number of shuffles, which makes it not very practical. A possible future work is to develop a ZKP protocol for Sudoku that uses asymptotically less number of shuffles. Other challenging future work includes developing ZKP protocols for other logic puzzles (e.g. Kakuro, Numberlink) that uses a deck of all different cards.

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A Optimization of the Number of Shuffles

A.1 Method A

- When verifying that every number appears exactly once in each block, we can verify three blocks at a time using cards $x_j, y_j, z_j$ ($j = 1, 2, ..., 9$) as 27 additional cards in the uniqueness verification protocol (with a condition that all cards in the sets $x_j, y_j,$ and $z_j$ must have different front sides and identical back sides). This reduces the number of shuffles by 12 to 330.

- We do not need to verify that a number 9 appears exactly once in each row and column. Since we have already verified that each of the numbers 1, 2, ..., 8 appears exactly once in each row (resp. column), the only remaining position in each row (resp. column) must contain a 9. This reduces the number of shuffles by 36 to 294.

- In Step 4 of Method A, we can apply $\text{col_shuffle}([1, 2, ..., 9])$ instead of $\text{col_shuffle}([1, 2, 3])$, $\text{col_shuffle}([4, 5, 6])$, and $\text{col_shuffle}([7, 8, 9])$ to $M$ (with a condition that all cards in the sets $x_j, y_j,$ and $z_j$ must have different front sides, so that we can tell different blocks apart after turning over helper cards). This reduces the number of shuffles by 96 to 198.
• After verifying that numbers $j$ ($j = 1, 2, ..., 7$) in three selected blocks are all located at different rows (resp. columns), we do not have to revert $M$ back to its original state. Since $P$ knows exactly where the number $j + 1$ in each block is, $P$ can immediately start the next round by performing the chosen cut protocol to find the number $j + 1$. This reduces the number of shuffles by 84 to 114.

• After verifying that numbers 8 in three selected blocks are all located at different columns, we do not have to revert $M$ back to its original state since the cards in these three blocks will not be used anymore. This reduces the number of shuffles by 6 to 108.

For an $n \times n$ puzzle, originally this method uses $2n + (\sqrt{n} + 3)(n)(2\sqrt{n}) = 2n^2 + 6n\sqrt{n} + 2n$ shuffles. After the optimization, it uses $2\sqrt{n} + (2(n - 1) + 2)(2\sqrt{n}) - 2\sqrt{n} = 4n\sqrt{n}$ shuffles.

A.2 Method B

• When verifying that every number appears exactly once in each block, we can verify three blocks at a time using cards $x_j, y_j$ ($j = 1, 2, ..., 9$), $p_j$ ($j = 1, 2, 3$), and $q_j$ ($j = 1, 2, ..., 6$) as 27 additional cards in the uniqueness verification protocol (with a condition that all cards in the sets $x_j, y_j, p_j$ and $q_j$ must have different front sides and identical back sides). This reduces the number of shuffles by 12 to 816.

• We do not need to verify that a number 9 appears exactly once in each row and column. Since we have already verified that each of the numbers 1, 2, ..., 8 appears exactly once in each row (resp. column), the only remaining position in each row (resp. column) must contain a 9. This reduces the number of shuffles by 90 to 726.

• In Step 4 of Method B, we can apply $\text{col\_shuffle}\{\{1, 2, ..., 6\}\}$ instead of $\text{col\_shuffle}\{\{1, 2, 3\}\}$ and $\text{col\_shuffle}\{\{4, 5, 6\}\}$ to $M$ (with a condition that all cards in the sets $x_j$ and $y_j$ must have different front sides, so that we can tell different blocks apart after turning over helper cards). This reduces the number of shuffles by 144 to 582.

• After verifying that numbers $j$ ($j = 1, 2, ..., 7$) in two selected blocks are located at different rows (resp. columns), we do not have to revert $M$ back to its original state. Since $P$ knows exactly where the number $j + 1$ in each block is, $P$ can immediately start the next round by performing the chosen cut protocol to find the number $j + 1$. This reduces the number of shuffles by 252 to 330.

• We set the order of verification such that the last four pairs of blocks we verify are Blocks $A$ and $B$, Blocks $D$ and $E$, Blocks $G$ and $H$, and Blocks $C$ and $F$. For these four pairs of blocks, after verifying that numbers 8 in the two blocks are located at different rows or columns, we do not have to revert $M$ back to its original state since the cards in these two blocks will not be used anymore. This reduces the number of shuffles by 252 to 330.

For an $n \times n$ puzzle, originally this method uses $2n + 5n(\sqrt{n} + 1)(2\sqrt{n}) = 5n^2(\sqrt{n} - 1) + 2n$ shuffles. After the optimization, it uses $2\left\lceil \frac{n}{2} \right\rceil + (2(n - 1) + 2)(\sqrt{n} - 2\left\lceil \frac{n}{2} \right\rceil - 2\left\lfloor \frac{n}{2} \right\rfloor) = 2n^2(\sqrt{n} - 1)$ shuffles for an even $n$ and $2n^2(\sqrt{n} - 1) + 2$ shuffles for an odd $n > 3$. 

13