The Physical Interpretation of the Lanczos Tensor.

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Abstract

The field equations of general relativity can be written as first order differential equations in the Weyl tensor, the Weyl tensor in turn can be written as a first order differential equation in a three index tensor called the Lanczos tensor. Similarly in electro-magnetic theory Maxwell’s equations can be written as first order differential equations in the field tensor $F_{ab}$ and this in can be written as a first order differential equation in the vector potential $A_a$; thus the Lanczos tensor plays a similar role in general relativity to that of the vector potential in electro-magnetic theory. The Aharonov-Bohm effect shows that when quantum mechanics is applied to electro-magnetic theory the vector potential is dynamically significant, even when the electro-magnetic field tensor $F_{ab}$ vanishes. Here it is assumed that in the quantum realm the Lanczos tensor is dynamically significant, and this leads to an attempt to quantize the gravitational field by pursuing the analogy between the vector field and the Lanczos tensor.

1 Introduction

The field equations of general relativity are usually written in terms of the Einstein tensor and the stress tensor; however there is an alternative expression, called Jordan’s formulation of the field equations, in which the field equations are expressed as first order equations in the Weyl tensor [1]. The Weyl tensor can be expressed, using first order equations, in terms of a three index tensor [2] [3] [4] [5] [6] called the Lanczos tensor. This tensor can be used to produce gravitational energy tensors of the correct dimension [3], and these can be used to measure the speed of gravitational waves [6]. In Jordan’s formulation the field equations are of a similar form to the Maxwell equations in terms of the electro-magnetic field tensor. The electro-magnetic field tensor can be expressed as a first order differential equation in the vector potential, and thus the Lanczos tensor in analogous to the vector potential in electro-magnetic theory. The Lanczos tensor is not the only tensor that can be thought of as being analogous to the vector potential; because the field equations are second order in the metric it is possible also to think of the metric (or the difference between the metric and the Minkowski metric) as being analogous to the the vector potential. There is also the Ashtekar potential in the theory of Ashtekar variables [7]; this potential is not the same object as the Lanczos tensor, because the equation for the Weyl tensor involves cross terms in the Ashtekar potential, unlike the Lanczos tensor in which the expression for the Weyl tensor is linear. The differential equa-
tions involving the Lanczos tensor which govern the dynamics of the field equations do not have a Lagrangian formulation, thus traditional methods of quantization cannot be applied to the field equations in this form.

In electro-magnetic theory the vector potential was first introduced in order to express the equations of classical electrodynamics in simpler form. In classical physics the only physical effect of an electro-magnetic field on a charge is the Lorentz force, and this only exists in regions where the electric or magnetic field in non-vanishing. The Aharonov-Bohm effect [8][9][10] demonstrates that this is not so in quantum mechanics; physical effects occur in regions where the electric and magnetic fields both vanish, but where the vector potential does not vanish. It has been experimentally confirmed [11].

In general relativity the existence of the Lanczos tensor might be just a technical curiosity, or it might have fundamental significance in the way that the vector potential does in electro-magnetic theory. The object of the present paper is to determine a thought experiment, similar to the Aharonov-Bohm experiment, which would in principle determine whether the Lanczos tensor effects the dynamics and so is physically significant. At the quantum level the vector potential enters the Schrödinger equation through the application of the electro-magnetic covariant derivative. The main problem in our approach is what should correspond to this covariant derivative; after all there is already a covariant derivative in general relativity constructed from the Christoffel symbol. Here it is postulated that in the quantum realm a covariant derivative involving the Lanczos tensor plays a role, and that it is the correct covariant derivative to apply to the Schrödinger equation in analogs of the Aharonov-Bohm effect. Now the main problem becomes how precisely should this covariant derivative be constructed; this cannot be known a-priori and so it is necessary to construct an example which will motivate a suitable definition of a new covariant derivative. If such a covariant derivative could be constructed it is possible to anticipate several difficulties. Firstly why does the classical theory have no use for a Lanczos covariant derivative? Secondly in the quantum realm the Christoffel covariant derivative is still necessary to connect the Lanczos tensor to the Weyl tensor, why have two covariant derivatives? Thirdly, in the Aharonov-Bohm effect the electro-magnetic field is not quantized, only the test particles are treated in a quantum mechanical manner; in our case would only the test particles be treated quantum mechanically, or would the Lanczos tensor or the metric or both be quantum fields?

The Aharonov-Bohm effect depends crucially on the existence of a choice of the vector potential which is well-defined and continuous everywhere. For
example, as discussed in [10], it is possible to choose a gauge in which the vector potential vanishes outside the solenoid and claim that there should be no effect; in fact, as experiments [11] vindicate, a gauge should be chosen in which the vector potential is continuous everywhere. It is assumed that a similar criteria on continuity exists for the present examples, so that it is possible to fix the gauge and then calculate the global effect of having a non-vanishing Lanczos tensor in the exterior region of the space-time. In the examples presented here it is found that the Lanczos tensor is either continuous or not depending on whether the derivative of the metric is continuous or not, irrespective of the choice of gauge. Thus there is no criteria to inform us which is the correct gauge, and hence the analogy cannot be carried through completely. This might be because the present examples are so geometrically simple. The Aharonov-Bohm effect depends on a current in a solenoid and it is not clear what is the general relativistic analog of a current in a solenoid. The simple cylindrical space-time used here might just be analogous to a line of charges, for which the Aharonov-Bohm effect does not work. Perhaps the correct analogy is a fluid in a pipe, i.e. a cylindrical space-time with a perfect fluid moving along the axis; however such an exact solution is not known.

The example discussed here use a simple cylindrically symmetric space-time. This space-time is general enough to include the simplest cosmic string [12] [13]. The approach used here relies on the Aharonov-Bohm effect being a quantum mechanical effect, and should not be confused with classical analogs of the Aharonov-Bohm effect which exist in cosmic string and some other space-times [14].

In section 2 the elementary properties of the Lanczos tensor are expounded. In section 3 the Lanczos tensor is produced for a simple cylindrically symmetric space-time. In section 4 the construction of covariant derivatives involving the Lanczos tensor are discussed.

2 The Lanczos Tensor.

The field equations of general relativity can be re-written in Jordan’s form [1]

\[ C_{abcd} = J_{abc}; \]

\[ J_{abc} = R_{ca;b} - R_{cb;a} + \frac{1}{6} g_{cb} R_{;a} - \frac{1}{6} g_{ca} R_{;b}, \]  

(1)
which is analogous to Maxwell’s equations

$$\mathbf{F}_{a;b} = \mathbf{J}_a.$$  

(2)

The Weyl tensor can be expressed in terms of the Lanczos tensor:

$$C_{abcd} = H_{abc;d} - H_{abcd;c} + H_{cdab} - H_{cdba}$$

$$- (g_{ac}(H_{bd} + H_{db}) - g_{ad}(H_{bc} + H_{cb}) + g_{bd}(H_{ac} + H_{ca}) - g_{bc}(H_{ad} + H_{da}))/2$$

$$+ 2H^{e,f}_{c:e} (g_{ac}g_{bd} - g_{ad}g_{bc})/3,$$  

(3)

where the Lanczos tensor has the symmetries

$$H_{abc} + H_{bac} = 0,$$

$$H_{abc} + H_{bca} + H_{cab} = 0,$$

(4)  

(5)

and where $H_{bd}$ is defined by

$$H_{bd} \equiv H_{b,d;e}^e - H_{b,e;d}^e.$$  

(6)

Equation (3) is invariant under the algebraic gauge transformation

$$H_{abc} \rightarrow H'_{abc} = H_{abc} + \chi_ag_{bc} - \chi_bg_{ac},$$

(7)

where $\chi_a$ is an arbitrary four vector.

The Lanczos tensor with the above symmetries has 20 degrees of freedom, but the Weyl tensor has 10. Lanczos reduced the degrees of freedom to 10 by choosing the Lanczos algebraic gauge

$$3\chi_a = H_{a,b}^b = 0,$$

(8)

and the Lanczos differential gauge

$$L_{ab} = H_{ab;c}^c = 0.$$  

(9)

These gauge choices are in some ways different than those in electro-magnetic theory. The algebraic gauge is different because it is algebraic and not differential in nature. The differential gauge is different because a differential gauge transformation alters components which do not participate in constructing the Weyl tensor; in electro-magnetic theory a gauge transformation alters components in the vector potential all of which participate in
constructing the electro-magnetic tensor. These difference are well illustrated by the example in the next section.

When the Lanczos tensor happens to be the gradient of an anti-symmetric tensor of the second order

$$H_{abc} = F_{abc},$$

and if the Lanczos tensor is in the algebraic gauge $\mathcal{S}, \mathcal{S},$ and $\mathcal{I}$ imply that $F_{ab}$ obeys Maxwell’s equations. It is not possible to introduce a source $J_a$ to $\mathcal{I}$ without it having an un-natural constraint by virtue of the identity $\mathcal{I}$.

In the case of weak gravity

$$g_{ab} = \eta_{ab} + h_{ab},$$

where $\eta_{ab}$ is the Minkowski metric and $h_{ab}$ and its derivatives are small, the Lanczos tensor can be written as

$$4H_{abc} = \partial_b h_{ac} - \partial_a h_{bc} + \frac{1}{6} h_{a\eta} - \frac{1}{6} h_{b\eta} + h_{ac\eta},$$

where $h = h^a_a$.

3 The Lanczos Tensor for a Simple Cylindrically Symmetric Space-time.

In this section we find the Lanczos tensor for a simple static cylindrically symmetric space-time with line element

$$ds^2 = -dt^2 + dr^2 + X d\phi^2 + dz^2.$$  

(13)

The non-vanishing Christoffel symbols are

$$\Gamma_{\phi\phi}^r = -\frac{1}{2} X_r, \quad \Gamma_{\phi r}^\phi = X_r/2X.$$  

(14)

The Riemann, Ricci, Einstein, and Weyl tensors are conveniently expressed in terms of the Ricci scalar

$$R = 2R_{\phi\phi} = 2R_{\phi r} = -2G^t_t = -2G^z_z = 2R_{\phi r\phi r} = 2R_{\phi r\phi r}/X,$$

$$R = -6C_{tztz} = 6C_{\phi r\phi r}/X = 12C_{t\phi t\phi}/X = -12C_{z\phi z\phi}/X = 12C_{t\phi t\phi} = -12C_{trtr},$$

(15)
[note added 1999 which $C_{t\phi t\phi}$ is correct] where

$$R = -rrX/X + X_r^2/2X = -X^{-\frac{3}{2}}(X'X^{-\frac{1}{2}})' \quad \text{(17)}$$

This space-time is general enough to include the simple cosmic string for which the metric is [12] [13]

$$ds^2 = -dt^2 + dr^2 + \rho^* \sin^2(\rho/\rho^*)d\phi^2 + dz^2 \quad \text{(18)}$$

where $\rho^* = (8\pi\epsilon)^{\frac{1}{2}}$ and $\epsilon$ is the density of the string. The Ricci scalar is

$$R = 4\pi\epsilon \quad \text{(19)}$$

At the join $r = r_0$, $\rho = \rho_0$ the interior metric is attached to the exterior metric

$$ds^2 = -dt^2 + dr^2 + a^2r^2d\phi^2 + dz^2 \quad \text{(20)}$$

where $a$ is given by

$$a = 1 - 4\mu,$$

$$\mu = \int_0^{\rho_0} \int_0^{2\pi} \epsilon\rho^* \sin(\rho/\rho^*) d\phi d\rho = 2\pi\rho^*^2 (1 - \cos(\rho/\rho^*)) \quad \text{(21)}$$

and $\mu$ is called the linear energy density. The requirement that the metric is continuous at the join is

$$ar_0 = \rho^* \sin(\rho/\rho^*) \quad \text{(22)}$$

The derivative of the metric is continuous at the join, as this is required for there to be no surface stress present; this requirement gives $a$ in [21] otherwise $a$ would be simply absorbed into the line element.

From [3] and [15] we have from the $C_{tztz}$ or $C_{\phi r \phi r}$ component

$$R = 2\partial_r H_{trt} - 2\partial_r H_{\phi r \phi} / X + X_r H_{trt} / X - X_{-r} H_{\phi r \phi} / X^2 \quad \text{(23)}$$

from the $C_{t\phi t\phi}$ or $C_{trtr}$ component

$$R = -4\partial_r H_{trt} - 8\partial_r H_{\phi r \phi} / X + 4X_r H_{trt} / X + 2X_r H_{\phi r \phi} / X^2 \quad \text{(24)}$$

from the $C_{\phi z \phi}$ or $C_{trtr}$ component

$$R = 8\partial_r H_{trt} + 4\partial_r H_{\phi r \phi} / X - 2X_r H_{trt} / X - 4X_r H_{\phi r \phi} / X^2 \quad \text{(25)}$$
Subtracting 24 from 23 or 25 from 24 we have

\[ 0 = 2\partial_r H_{trt} + 2\partial_r H_{zrz} - X_r H_{trt}/X - X_r H_{zrz}/X \]

\[ = 2\sqrt{X}\partial_r (H_{trt}X^{-\frac{1}{2}}) + 2\sqrt{X}\partial_r (H_{zrz}X^{-\frac{1}{2}}), \]

which integrates to give

\[ H_{trt} = k\sqrt{X} - H_{zrz}, \]

where \( k \) is a constant. From 27 and 23 or 24 or 25 we have

\[ R = 2k X_r X^{-\frac{1}{2}} - 4\partial_r H_{zrz} + 4\partial_r H_{\phi r\phi}/X - 2X_r H_{zrz}/X - 2X_r H_{\phi r\phi}/X^2. \]  

Here 8 the Lanczos algebraic condition is \(-H_{trt} + H_{zrz} + H_{\phi r\phi}/X = 0\), it gives

\[ H_{zrz} = k\sqrt{X}/2 - H_{\phi r\phi}/2X. \]

Equations 17, 28, and 29 give

\[ R = -3X_r H_{r\phi r}/X + 6\partial_r H_{\phi r\phi}/X = -X_{rr}/X + X_r^2/2X^2, \]

integrating

\[ H_{\phi r\phi} = -X_r/6 + l\sqrt{X}/6, \]

where \( l \) is a constant. From ref eq 27, 28 and 31 and inserting the gauge vector we have

\[ H = \frac{k\sqrt{X}}{2} + \frac{l}{12\sqrt{X}} - \frac{X_r}{12X} + \chi_r, \]

\[ H = \frac{k\sqrt{X}}{2} - \frac{l}{12\sqrt{X}} + \frac{X_r}{12X} - \chi_r, \]

\[ H = \frac{l\sqrt{X}}{6} - \frac{X_r}{6} - X\chi_r. \]

the result is in the Lanczos algebraic gauge when \( \chi_r = 0 \). The derivative of the metric \( X_r \) appears in each term. No matter what the choice of algebraic gauge (i.e. choice of \( \chi_r \)) we cannot remove it. Thus the continuity or otherwise of the Lanczos tensor depends on the continuity or otherwise of the derivative of the metric. We can make any possible discontinuity in any
single component, or even a whole component vanish by means of a suitable algebraic gauge. For example choosing

$$\chi_r = \frac{-k\sqrt{X}}{2} - \frac{l}{12\sqrt{X}} + \frac{X_r}{12X},$$  \hspace{1cm} (33)$$
gives

$$H_{trt} = 0,$$
$$H_{xrx} = k\sqrt{X},$$
$$H_{\phi r\phi} = \frac{kX\sqrt{X}}{2} + \frac{l\sqrt{X}}{4} - \frac{X_r}{4}.$$  \hspace{1cm} (34)$$

Notice that the Lanczos tensor does not necessarily vanish for flat space-time. For example in the Lanczos algebraic gauge or in the gauge \[33\] the choice

$$k = 0, \quad l = 2,$$  \hspace{1cm} (35)$$
gives flat space-time with vanishing Lanczos tensor; however, for example, in the gauge $\chi_r = -X_r/6X$, the Lanczos tensor cannot be made to vanish for flat space-time. From \[32\] the metric can be expressed in terms of the Lanczos tensor

$$X = \frac{(H_{t,1}^1 - H_{r,2}^2)^2}{k^2}. \hspace{1cm} (36)$$

The differential gauge \[3\] is

$$L_{tr} = -\partial_r H_{trt} + X_r (H_{\phi r\phi}/X - H_{rtr})/2X,$$
$$L_{r\phi} = \partial_r H_{t\phi r} + X_r H_{tr\phi}/2X,$$
$$L_{t2} = \partial_r H_{t2r} + X_r H_{tr2r}/2X,$$
$$L_{r\phi} = \partial_r H_{r\phi r},$$
$$L_{r2} = \partial_r H_{r2r} + X_r (-H_{\phi 2}\phi/X - H_{r2r})/2X,$$
$$L_{\phi 2} = \partial_r H_{\phi 2r} + X_r H_{r2\phi}/2X.$$  \hspace{1cm} (37)$$

For the Lanczos differential gauge condition \[3, 37\] can be expressed in the integral form

$$H_{r\phi r} = a_1,$$
\[ H_{zt} = \frac{a_2}{\sqrt{X}}, \]
\[ H_{\phi z} = \frac{a_3}{\sqrt{X}}, \]
\[ H_{rz} = \frac{1}{\sqrt{X}}(a_4 + \int X_r X^{-3/2} H_{\phi t z} dr), \]
\[ H_{rz} = \frac{1}{\sqrt{X}}(a_5 + \int X_r X^{-3/2} H_{\phi x z} dr), \]
\[ H_{z\phi} = \frac{1}{\sqrt{X}}(a_6 + \int X_r X^{-3/2} H_{r\phi z} dr), \]

(38)

where \( a_1 \ldots a_6 \) are constants. This illustrates a property of the differential gauge alluded to in section II; the components of the Lanczos tensor in (38) do not coincide with any of those in (32), thus these components do not participate in the construction of the Weyl tensor.

Here the Lanczos tensor cannot be expressed as the gradient of an anti-symmetric tensor of the second order. Using that the space-time is only \( r \) dependent and that the Christoffel symbols are \( \Gamma \), (11) gives

\[ F_{t r; t} = F_{z r; z} = F_{\phi t; \phi} = F_{r t; t} = F_{r z; z} = F_{r\phi; \phi} = 0. \]

(39)

Any added current to (10) would have component \( J_z \), and then the cyclic identity (3) would fail.

4 The Covariant Derivative.

In the Aharonov-Bohm effect, the electro-magnetic field alters the dynamics of test particles because the electro-magnetic covariant derivative replaces the partial derivative in the test particles Schrödinger’s equation. In this section we list 15 possible covariant derivatives involving the Lanczos tensor. None of the possibilities can be used to complete our analogy with the Aharonov-Bohm effect. This is because the criteria of continuity cannot be used to fix the algebraic gauge in the example in the last section, and all the possibilities give different results depending on algebraic gauge. We denote the covariant derivative by \( D_a \) and the coupling constant by \( c \).

\[ i) \quad \partial_a \rightarrow D_a = +cH_{a, b}^b. \]

(40)
From refq:7 and 8 we see immediately that this choice depends on the algebraic gauge. The choice

\[ ii) \quad \partial_a \rightarrow D_a = +cH^b_{ab}, \]

is the same as 40 with the sign of \( c \) reversed, by virtue of the symmetry 3. The choice

\[ iii) \quad \partial_a \rightarrow D_a = +cH^b_{ba}, \]

gives that the covariant and partial derivative are identical by 4.

\[ iv) \quad \partial_a \rightarrow D_a = +cH^B_{aB}, \]

where \( B \) is a fixed component not summed. Covariant derivatives of this type have all the disadvantages of 40 to 42, with the added disadvantage of picking out one component.

\[ v) \quad \partial_a \rightarrow D_a = \partial_a + c(3H^t_{a,t} - H^b_{a,b}), \]

for our example this is invariant under the choice of algebraic gauge, however by 8

\[ H^b_{a,b} = 3x_r, \]

thus this choice amounts to no more than an arbitrary choice of component in the Lanczos algebraic gauge.

\[ vi) \quad \partial_a \rightarrow D_a = \partial_a + c\epsilon^{abcd}H^d_{bcd}, \]

in our example, or more generally in any space-time which can be expressed with a diagonal metric, components of the Lanczos tensor with identical adjacent indices plays an essential role, but they would not effect this covariant derivative.

\[ vii) \quad \partial_a \rightarrow D_a = \partial_a + cH^b_{abc}p^d_{bc}, \]

where \( p^d_{bc} \) is the stress tensor of the ”test” particle. This coupling is unusual as the test particles own stress contributes, it is no longer just a test particle. By changing the algebraic gauge the contribution of \( p^d_{bc} \) changes in an arbitrary manner and thus this choice is un-useable.

\[ viii) \quad \text{Require that the covariant derivative coincides with the weak field covariant derivative when the fields are weak. The weak field covariant derivative is} \]

\[ 2\eta_{ad}W^d_{bc} = \partial_b h_{ac} + \partial_c h_{ab} - \partial_a h_{bc}. \]
Using the algebraic gauge
\[ \chi_a = -\frac{1}{6} h_a, \] (49)
\[ \text{Eq. 42 becomes} \]
\[ 4H_{abc} = \partial_b h_{ca} - \partial_a h_{bc}, \] (50)
using 4 this is
\[ \partial_a h_{bc} = -2H_{abc}. \] (51)
Substituting 50 into 47 and using 4 and 5 gives
\[ 2\eta_{ad} W_{b,c}^d = 4H_{abc} + 2H_{acb}, \] (52)
now
\[ \eta_{ad} W_{c,b}^d = \eta_{ad} W_{b,c}^d, \] (53)
thus from 51 and 52
\[ H_{abc} = H_{acb}, \] (54)
using 4 and 5, 54 gives
\[ H_{bca} = 0. \] (55)
Thus the weak field covariant derivative cannot be expressed in terms of the weak field expression for \( H_{abc} \), and we cannot require that the covariant derivative coincides with the weak field covariant derivative when the fields are weak.

ix) Apply \( H_{abc} = F_{abc} \) and then use the electro-magnetic covariant derivative. This is too restrictive, there is no \( F_{ab} \) satisfying this in our example.

x) \( \partial_a \rightarrow D_a = \partial_a + cH_a \), (56)
where
\[ H_a = H_{abc} H^{bc}_-, \] (57)
and \( H^{bc}_- \) is given by 6. This has the disadvantage that it involves products and derivatives of the Lanczos tensor; the electro-magnetic covariant derivative has no products and derivatives of the vector potential. For the example of the previous section in the Lanczos algebraic gauge
\[ H_r = -\frac{k^2 X_r}{2} - \frac{X_r X_{rr}}{24 X^2} + \frac{5X^3}{144 X^3} + \frac{l^2 X_r}{72 X^2} - \frac{7lX^2}{144 X^{3/2}} + \frac{lX_{rr}}{24 X^{3/2}}, \] (58)
in the \( \chi_r = X_r/12X \) gauge
\[ H_r = -\frac{k^2 X_r}{2} - \frac{X_r X_{rr}}{16 X^2} + \frac{X^3}{32 X^3} + \frac{l^2 X_r}{72 X^2} - \frac{lX^2}{24 X^{3/2}} + \frac{lX_{rr}}{24 X^{3/2}}, \] (59)
in the $\chi_r = -X_r/6X$ gauge

$$H_r = -\frac{k^2 X_r}{2} - \frac{X_r X_{rr}}{8X^2} + \frac{X_r^3}{16X^3} + \frac{l^2 X_r}{72X^2} - \frac{IX_r^2}{X^{3/2}} + \frac{lX_{rr}}{24X^{3/2}},$$  \hspace{1cm} (60)

illustrating the algebraic gauge dependence of $H_a$.

xi) Replacing ref eq:57 by

$$H_a = H_{bac}H_{bc}^{-1},$$  \hspace{1cm} (61)

is the same as 57 with the sign of the coupling constant reversed,

xii) replacing 57 by

$$H_a = H_{bac}H_{bc}^{-1},$$  \hspace{1cm} (62)

we have that the indices $b$ and $c$ will always be identical for any space-time where the metric is in diagonal form and thus $H_a = 0$.

Require that rather than a covariant derivative we need a change of phase

$$xiii) \quad \alpha \rightarrow \alpha' = \alpha + cH_{abc}H_{abc},$$  \hspace{1cm} (63)

This has the same difficulties as the example of the preceding paragraph, and also we have a difficulty in how to integrate over the different trajectories.

xiv) Another possibility is to start with the Klein-Gordon equation or the Dirac equation in the space-time 13. In the non-relativistic limit the Schrödinger equation can be recovered from the Klein-Gordon and Dirac equations and this might give information on the correct covariant derivative.

To investigate this we begin by showing that there are no solutions with non-vanishing gravitational and Klein-Gordon field with line element 13. The equations for an Einstein-Klein-Gordon field are

$$R_{ab} = 2\phi_a\phi_b + 2g_{ab}m^2\phi^2,$$  \hspace{1cm} (64)

and

$$m^2\phi = \Box \phi = \frac{1}{\sqrt{-g}}(\sqrt{-g}g_{ab}\phi_a)_{b}.$$  \hspace{1cm} (65)

Thus for the line element 13, using $ph_{\phi} = 0,$

$$R'_{r} = 2\partial'^2 + 2m^2\phi^2, \quad R'_{\phi} = 2m^2\phi^2,$$  \hspace{1cm} (66)
Now \( \mathbf{13} \) gives \( R^r_r = R^\phi_\phi \) therefore \( \phi' = 0 \), integrating

\[
\phi = \sigma, \quad (67)
\]

where \( \sigma \) is a constant. Equation \( \mathbf{13} \) becomes

\[
m^2 \phi = \frac{1}{\sqrt{X}} (\sqrt{X} \phi')', \quad (68)
\]

which vanishes by \( \mathbf{67} \). Therefore \( m^2 = 0 \) and \( \phi = \sigma \) or \( \phi = 0 \). Thus there are no solutions of \( \mathbf{14} \) with line element \( \mathbf{13} \) and both \( m \) and \( \phi \) non-vanishing. Similarly, using \( \mathbf{13} \) it can be shown that there are no solutions of the Einstein-Dirac equations with line element \( \mathbf{13} \) and both gravitational and Dirac fields non-vanishing.

Instead of considering coupled systems we could consider the Klein-Gordon or Dirac fields as test fields which do not contribute to the stress of the space-time. Using \( \mathbf{37, 65, 68} \) the Klein-Gordon equation can be expressed as

\[
m^2 \phi = \nabla_a \partial^a = \phi'' + (\ln(H_{r,r} - H_{z,z}))' \phi', \quad (69)
\]

suggesting the covariant derivative

\[
xv \rightarrow D_a = \partial_a + (\ln(H_{r,r} - H_{z,z}))'. \quad (70)
\]

This covariant derivative has the disadvantage of artificially picking out a component and involves neither the gauge vector \( \chi_a \) or the constants \( k \) and \( l \). The analogy with the Aharonov-Bohm effect suggests that the exterior region should be Minkowski space-time, where \( X = r^2 \), and that the criteria of continuity should fix the gauge vector \( \chi_a \) and the constants \( k \) and \( l \); but this cannot be done with covariant derivative \( \mathbf{70} \).

5 Conclusion

An attempt was made to test in principle whether the Lanczos tensor is microscopically dynamically significant in the quantum realm in a similar manner to the vector potential. So far the results have not produced a definitive result: however whether it is possible to quantize by this method should become clearer upon the discovery of a suitable exact solution to the general relativity field equations which can produce a closer analogy to the Aharonov-Bohm experiment.
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