Four conjectures in Nonlinear Analysis

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Abstract. In this chapter, I formulate four challenging conjectures in Nonlinear Analysis. More precisely: a conjecture on the Monge-Ampère equation; a conjecture on an eigenvalue problem; a conjecture on a non-local problem; a conjecture on disconnectedness versus infinitely many solutions.

In this chapter, I intend to formulate four challenging conjectures in Nonlinear Analysis which have their roots in certain results that I have obtained in the past years.

1. A conjecture on the Monge-Ampère equation

CONJECTURE 1.1. - Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a non-empty open bounded set and let \( h : \Omega \to \mathbb{R} \) be a non-negative continuous function.
Then, each \( u \in C^2(\Omega) \cap C^1(\Omega) \) satisfying in \( \Omega \) the Monge-Ampère equation
\[
\det(D^2u) = h
\]
has the following property:
\[
\nabla(\Omega) \subseteq \text{conv}(\nabla(\partial\Omega)).
\]

This conjecture is motivated by [26] where I proved that it is true for \( n = 2 \). I am going to produce such a proof here.

In what follows, \( \Omega \) is a non-empty relatively compact and open set in a topological space \( E \), with \( \partial\Omega \neq \emptyset \), and \( Y \) is a real locally convex Hausdorff topological vector space. \( \overline{\Omega} \) and \( \partial\Omega \) denote the closure and the boundary of \( \Omega \), respectively. Since \( \overline{\Omega} \) is compact, \( \partial\Omega \), being closed, is compact too.

Let us first recall some well-known definitions.

Let \( S \) be a subset of \( Y \) and let \( y_0 \in S \). As usual, we say that \( S \) is supported at \( y_0 \) if there exists \( \varphi \in Y^* \setminus \{0\} \) such that \( \varphi(y_0) \leq \varphi(y) \) for all \( y \in S \). If this happens, of course \( y_0 \in \partial S \).

Further, extending a maximum principle definition for real-valued functions, a continuous function \( f : \overline{\Omega} \to Y \) is said to satisfy the convex hull property in \( \overline{\Omega} \) (see [7], [13] and references therein) if
\[
f(\Omega) \subseteq \text{conv}(f(\partial\Omega)),
\]
\( \text{conv}(f(\partial\Omega)) \) being the closed convex hull of \( f(\partial\Omega) \).

When \( \dim(Y) < \infty \), since \( f(\partial\Omega) \) is compact, \( \text{conv}(f(\partial\Omega)) \) is compact too and so \( \text{conv}(f(\partial\Omega)) = \text{conv}(f(\partial\Omega)) \).

A function \( \psi : Y \to \mathbb{R} \) is said to be quasi-convex if, for each \( r \in \mathbb{R} \), the set \( \psi^{-1}([-\infty, r]) \) is convex.

Notice the following proposition:

PROPOSITION 1.1. - For each pair \( A, B \) of non-empty subsets of \( Y \), the following assertions are equivalent:
\((a_1)\) \( A \subseteq \text{conv}(B) \).
\((a_2)\) For every continuous and quasi-convex function \( \psi : Y \to \mathbb{R} \), one has
\[
\sup_A \psi \leq \sup_B \psi.
\]
PROOF. Let \((a_1)\) hold. Fix any continuous and quasi-convex function \(\psi : Y \to \mathbb{R}\). Fix \(\tilde{y} \in A\). Then, there is a net \(\{y_\alpha\}\) in \(\text{conv}(B)\) converging to \(\tilde{y}\). So, for each \(\alpha\), we have 
\[y_\alpha = \sum_{i=1}^{k} \lambda_i z_i,\]
where \(z_i \in B\), \(\lambda_i \in [0, 1]\) and \(\sum_{i=1}^{k} \lambda_i = 1\). By quasi-convexity, we have 
\[
\psi(y_\alpha) = \psi\left(\sum_{i=1}^{k} \lambda_i z_i\right) \leq \max_{1 \leq i \leq k} \psi(z_i) \leq \sup_{B} \psi
\]
and so, by continuity,
\[
\psi(\tilde{y}) = \lim_{\alpha} \psi(y_\alpha) \leq \sup_{B} \psi
\]
which yields \((a_2)\).

Now, let \((a_2)\) hold. Let \(x_0 \in A\). If \(x_0 \notin \text{conv}(B)\), by the standard separation theorem, there would be \(\psi \in Y^* \setminus \{0\}\) such that \(\sup_{\text{conv}(B)} \psi < \psi(x_0)\), against \((a_2)\). So, \((a_1)\) holds. \(\triangle\)

Clearly, applying Proposition 1.1, we obtain the following one:

PROPOSITION 1.2. - For any continuous function \(f : \Omega \to Y\), the following assertions are equivalent:

\((b_1)\) \(f\) satisfies the convex hull property in \(\Omega\).

\((b_2)\) For every continuous and quasi-convex function \(\psi : Y \to \mathbb{R}\), one has 
\[
\sup_{x \in \Omega} \psi(f(x)) = \sup_{x \in \partial \Omega} \psi(f(x)).
\]

In view of Proposition 1.2, we now introduce the notion of convex hull-like property for functions defined in \(\Omega\) only.

DEFINITION 1.1. - A continuous function \(f : \Omega \to Y\) is said to satisfy the convex hull-like property in \(\Omega\) if, for every continuous and quasi-convex function \(\psi : Y \to \mathbb{R}\), there exists \(x^* \in \partial \Omega\) such that 
\[
\limsup_{x \to x^*} \psi(f(x)) = \sup_{x \in \Omega} \psi(f(x)).
\]

We have

PROPOSITION 1.3. - Let \(g : \overline{\Omega} \to Y\) be a continuous function and let \(f = g|_{\Omega}\).
Then, the following assertions are equivalent:

\((c_1)\) \(f\) satisfies the convex hull-like property in \(\Omega\).

\((c_2)\) \(g\) satisfies the convex hull property in \(\overline{\Omega}\).

PROOF. Let \((c_1)\) hold. Let \(\psi : Y \to \mathbb{R}\) be any continuous and quasi-convex function. Then, by Definition 1.1, there exists \(x^* \in \partial \Omega\) such that 
\[
\limsup_{x \to x^*} \psi(f(x)) = \sup_{x \in \Omega} \psi(f(x)).
\]
But 
\[
\limsup_{x \to x^*} \psi(f(x)) = \psi(g(x^*))
\]
and hence 
\[
\sup_{x \in \partial \Omega} \psi(g(x)) = \sup_{x \in \Omega} \psi(g(x)).
\]
So, by Proposition 1.2, \((c_2)\) holds.

Now, let \((c_2)\) hold. Let \(\psi : Y \to \mathbb{R}\) be any continuous and quasi-convex function. Then, by Proposition 1.2, one has 
\[
\sup_{x \in \partial \Omega} \psi(g(x)) = \sup_{x \in \Omega} \psi(g(x)).
\]
Since \( \partial \Omega \) is compact and \( \psi \circ g \) is continuous, there exists \( x^* \in \partial \Omega \) such that
\[
\psi(g(x^*)) = \sup_{x \in \partial \Omega} \psi(g(x))
\]
But
\[
\psi(g(x^*)) = \lim_{x \to x^*} \psi(f(x))
\]
and, by continuity again,
\[
\sup_{x \in \Omega} \psi(g(x)) = \sup_{x \in \Omega} \psi(g(x))
\]
and so
\[
\lim_{x \to x^*} \psi(f(x)) = \sup_{x \in \Omega} \psi(f(x))
\]
which yields \((c_1)\).

\[\triangle\]

The central result is as follows:

**Theorem 1.1.** For any continuous function \( f : \Omega \to Y \), at least one of the following assertions holds:

(i) \( f \) satisfies the convex hull-like property in \( \Omega \).
(ii) There exists a non-empty open set \( X \subseteq \Omega \), with \( \overline{X} \subseteq \Omega \), satisfying the following property: for every continuous function \( g : \Omega \to Y \), there exists \( \lambda \geq 0 \) such that, for each \( \lambda > \lambda \), the set \( (g + \lambda f)(X) \) is supported at one of its points.

**Proof.** Assume that (i) does not hold. So, we are assuming that there exists a continuous and quasi-convex function \( \psi : Y \to \mathbb{R} \) such that
\[
\limsup_{x \to z} \psi(f(x)) < \sup_{x \in \Omega} \psi(f(x)) \quad (1.1)
\]
for all \( z \in \partial \Omega \).

In view of (1.1), for each \( z \in \partial \Omega \), there exists an open neighbourhood \( U_z \) of \( z \) such that
\[
\sup_{x \in U_z \cap \Omega} \psi(f(x)) < \sup_{x \in \Omega} \psi(f(x))
\]
Since \( \partial \Omega \) is compact, there are finitely many \( z_1, \ldots, z_k \in \partial \Omega \) such that
\[
\partial \Omega \subseteq \bigcup_{i=1}^{k} U_{z_i} \quad (1.2)
\]
Put
\[
U = \bigcup_{i=1}^{k} U_{z_i}
\]
Hence
\[
\sup_{x \in \Omega \cap \Omega} \psi(f(x)) = \max_{1 \leq i \leq k} \sup_{x \in U_{z_i} \cap \Omega} \psi(f(x)) < \sup_{x \in \Omega} \psi(f(x))
\]
Now, fix a number \( r \) so that
\[
\sup_{x \in U \cap \Omega} \psi(f(x)) < r < \sup_{x \in \Omega} \psi(f(x)) \quad (1.3)
\]
and set
\[
K = \{ x \in \Omega : \psi(f(x)) \geq r \}
\]
Since \( f, \psi \) are continuous, \( K \) is closed in \( \Omega \). But, since \( K \cap U = \emptyset \) and \( U \) is open, in view of (1.2), \( K \) is closed in \( E \). Hence, \( K \) is compact since \( \overline{\Omega} \) is so. By (1.3), we can fix \( \bar{x} \in \Omega \) such that \( \psi(f(\bar{x})) > r \). Notice that
the set $\psi^{-1}([-\infty, r])$ is closed and convex. So, thanks to the standard separation theorem, there exists a non-zero continuous linear functional $\varphi : Y \to \mathbb{R}$ such that

$$\varphi(f(\bar{x})) < \inf_{y \in \psi^{-1}([-\infty, r])} \varphi(y).$$

(1.4)

Then, from (1.4), it follows

$$\varphi(f(\bar{x})) < \inf_{x \in \Omega \setminus K} \varphi(f(x)).$$

Now, choose $\rho$ so that

$$\varphi(f(\bar{x})) < \rho < \inf_{x \in \Omega \setminus K} \varphi(f(x))$$

and set

$$X = \{x \in \Omega : \varphi(f(x)) < \rho\}.$$ 

Clearly, $X$ is a non-empty open set contained in $K$. Now, let $g : \Omega \to Y$ be any continuous function. Set

$$\tilde{\lambda} = \inf_{x \in X} \frac{\varphi(g(x)) - \inf_{z \in K} \varphi(g(z))}{\rho - \varphi(f(x))}.$$ 

Fix $\lambda > \tilde{\lambda}$. So, there is $x_0 \in X$ such that

$$\frac{\varphi(g(x_0)) - \inf_{z \in K} \varphi(g(z))}{\rho - \varphi(f(x_0))} < \lambda.$$ 

From this, we get

$$\varphi(g(x_0)) + \lambda \varphi(f(x_0)) < \lambda \rho + \inf_{z \in K} \varphi(g(z)).$$

(1.5)

By continuity and compactness, there exists $\hat{x} \in K$ such that

$$\varphi(g(\hat{x}) + \lambda f(\hat{x})) \leq \varphi(g(x)) + \lambda f(x)$$

(1.6)

for all $x \in K$. Let us prove that $\hat{x} \in X$. Arguing by contradiction, assume that $\varphi(f(\hat{x})) \geq \rho$. Then, taking (1.5) into account, we would have

$$\varphi(g(x_0)) + \lambda \varphi(f(x_0)) < \lambda \rho + \inf_{z \in K} \varphi(g(z)) + \varphi(g(\hat{x}))$$

contradicting (1.6). So, it is true that $\hat{x} \in X$, and, by (1.6), the set $(g + \lambda f)(X)$ is supported at its point $g(\hat{x}) + \lambda f(\hat{x})$. 

\[\triangle\]

An application of Theorem 1.1 shows a strongly bifurcating behaviour of certain equations in $\mathbb{R}^n$.

**THEOREM 1.2.** - Let $\Omega$ be a non-empty bounded open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ a continuous function.

Then, at least one of the following assertions holds:

\((d_1)\) $f$ satisfies the convex hull-like property in $\Omega$.

\((d_2)\) There exists a non-empty open set $X \subseteq \Omega$, with $\overline{X} \subseteq \Omega$, satisfying the following property: for every continuous function $g : \Omega \to \mathbb{R}^n$, there exists $\tilde{\lambda} \geq 0$ such that, for each $\lambda > \tilde{\lambda}$, there exist $\hat{x} \in X$ and two sequences $\{y_k\}, \{z_k\}$ in $\mathbb{R}^n$, with

$$\lim_{k \to \infty} y_k = \lim_{k \to \infty} z_k = g(\hat{x}) + \lambda f(\hat{x}),$$

such that, for each $k \in \mathbb{N}$, one has

\((j)\) the equation

$$g(x) + \lambda f(x) = y_k$$

has no solution in $X$ ;
(jj) the equation

\[ g(x) + \lambda f(x) = z_k \]

has two distinct solutions \( u_k, v_k \) in \( X \) such that

\[ \lim_{k \to \infty} u_k = \lim_{k \to \infty} v_k = \hat{x} . \]

**PROOF.** Apply Theorem 1.1 with \( E = Y = \mathbb{R}^n \). Assume that \((d_1)\) does not hold. Let \( X \subseteq \Omega \) be an open set as in \((ii)\) of Theorem 1.1. Fix any continuous function \( g : \Omega \to \mathbb{R}^n \). Then, there is some \( \lambda > 0 \) such that, for each \( \lambda > \hat{\lambda} \), there exists \( \hat{x} \in X \) such that the set \( (g + \lambda f)(X) \) is supported at \( g(\hat{x}) + \lambda f(\hat{x}) \). As we observed at the beginning, this implies that \( g(\hat{x}) + \lambda f(\hat{x}) \) lies in the boundary of \( (g + \lambda f)(X) \). Therefore, we can find a sequence \( \{y_k\} \) in \( \mathbb{R}^n \setminus (g + \lambda f)(X) \) converging to \( g(\hat{x}) + \lambda f(\hat{x}) \). So, such a sequence satisfies \((j)\).

For each \( k \in \mathbb{N} \), denote by \( B_k \) the open ball of radius \( \frac{1}{k} \) centered at \( \hat{x} \). Let \( k \) be such that \( B_k \subseteq X \). The set \((g + \lambda f)(B_k)\) is not open since its boundary contains the point \( g(\hat{x}) + \lambda f(\hat{x}) \). Consequently, by the invariance of domain theorem ([29], p. 705), the function \( g + \lambda f \) is not injective in \( B_k \). So, there are \( u_k, v_k \in B_k \), with \( u_k \neq v_k \) such that

\[ g(u_k) + \lambda f(u_k) = g(v_k) + \lambda f(v_k) . \]

Hence, if we take

\[ z_k = g(u_k) + \lambda f(u_k) , \]

the sequences \( \{u_k\}, \{v_k\}, \{z_k\} \) satisfy \((jj)\) and the proof is complete. \( \triangle \)

**REMARK 1.1.** Notice that, in general, Theorem 1.2 is no longer true when \( f : \Omega \to \mathbb{R}^m \) with \( m > n \). In this connection, consider the case \( n = 1, m = 2, \Omega = [0, \pi[ \) and \( f(\theta) = (\cos \theta, \sin \theta) \) for \( \theta \in [0, \pi[ \). So, for each \( \lambda > 0 \), on the one hand, the function \( \lambda f \) is injective, while, on the other hand, \( \lambda f([0, \pi[ \) is not contained in \( \text{conv}(\{f(0), f(\pi)\}) \).

If \( S \subseteq \mathbb{R}^n \) is a non-empty open set, \( x \in S \) and \( h : S \to \mathbb{R}^n \) is a \( C^1 \) function, we denote by \( \det(J_h(x)) \) the Jacobian determinant of \( h \) at \( x \).

A very recent and important result by J. Saint Raymond ([27]) states what follows (for anything concerning the topological dimension we refer to [8]):

**THEOREM 1.A ([27], Theorem 10).** - Let \( A \subseteq \mathbb{R}^n \) be a non-empty open set and \( \varphi : A \to \mathbb{R}^n \) a \( C^1 \) function such that the topological dimension of the set

\[ \{ x \in A : \det(J_{\varphi}(x)) = 0 \} \]

is not positive.

Then, the function \( \varphi \) is open.

A joint application of Theorem 1.1 and Theorem 1.A gives

**THEOREM 1.3.** - Let \( f : \Omega \to \mathbb{R}^n \) be a \( C^1 \) function.

Then, at least one of the following assertions holds:

(a1) \( f \) satisfies the convex hull-like property in \( \Omega \).

(a2) There exists a non-empty open set \( X \subseteq \Omega \), with \( \text{cl}(X) \subseteq \Omega \), satisfying the following property: for every continuous function \( g : \Omega \to \mathbb{R}^n \) which is \( C^1 \) in \( X \), there exists \( \lambda \geq 0 \) such that, for each \( \lambda > \tilde{\lambda} \), the topological dimension of the set

\[ \{ x \in X : \det(J_{g+\lambda f}(x)) = 0 \} \]

is greater than or equal 1.

**PROOF.** Assume that \((a1)\) does not hold. Let \( X \) be an open set as in \((ii)\) of Theorem 1.1. Let \( g : \Omega \to \mathbb{R}^n \) be a continuous function which is \( C^1 \) in \( X \). Then, there is some \( \tilde{\lambda} \geq 0 \) such that, for each \( \lambda > \tilde{\lambda} \), there exists \( \hat{x} \in X \) such that the set \((g + \lambda f)(X)\) is supported at \( g(\hat{x}) + \lambda f(\hat{x}) \). As already remarked, this implies that \( g(\hat{x}) + \lambda f(\hat{x}) \in \partial(g + \lambda f)(X) \) and so \((g + \lambda f)(X)\) is not open. Now, \((a2)\) is a direct consequence of Theorem 1.A. \( \triangle \)
In turn, here is a consequence of Theorem 1.3 when \( n = 2 \).

**THEOREM 1.4.** - Let \( \Omega \) be a non-empty bounded open set of \( \mathbb{R}^2 \), let \( h : \Omega \to \mathbb{R} \) be a continuous function and let \( \alpha, \beta : \Omega \to \mathbb{R} \) be two \( C^1 \) functions such that \( |\alpha_x \beta_y - \alpha_y \beta_x| + |h| > 0 \) and \( (\alpha_x \beta_y - \alpha_y \beta_x) h \geq 0 \) in \( \Omega \). Then, any \( C^1 \) solution \((u, v)\) in \( \Omega \) of the system

\[
\begin{align*}
  u v_y - u_y v_x &= h \\
  \beta_y u_x - \beta_x u_y - \alpha_y v_x + \alpha_x v_y &= 0
\end{align*}
\]  

(1.7)

satisfies the convex hull-like property in \( \Omega \).

**PROOF.** Arguing by contradiction, assume that \((u, v)\) does not satisfy the convex hull-like property in \( \Omega \). Then, by Theorem 1.3, applied taking \( f = (u, v) \) and \( g = (\alpha, \beta) \), there exist \( \lambda > 0 \) and \((\hat{x}, \hat{y}) \in \Omega \) such that

\[
\det(J_{g+\lambda f}(\hat{x}, \hat{y})) = 0
\]

On the other hand, for each \((x, y) \in \Omega \), we have

\[
\det(J_{g+\lambda f}(x, y)) = (u v_y - u_y v_x)(x, y)\lambda^2 + (\beta_y u_x - \beta_x u_y - \alpha_y v_x + \alpha_x v_y)(x, y)\lambda + (\alpha_x \beta_y - \alpha_y \beta_x)(x, y)
\]

and hence

\[
h(\hat{x}, \hat{y})\lambda^2 + (\alpha_x \beta_y - \alpha_y \beta_x)(\hat{x}, \hat{y}) = 0
\]

which is impossible in view of our assumptions. \( \triangle \)

Finally, taking Proposition 1.3 in mind, here is the proof of Conjecture 1.1 when \( n = 2 \):

**THEOREM 1.5.** - Let \( \Omega \) be a non-empty bounded open subset of \( \mathbb{R}^2 \), let \( h : \Omega \to \mathbb{R} \) be a continuous non-negative function and let \( w \in C^2(\Omega) \) be a function satisfying in \( \Omega \) the Monge-Ampère equation

\[
w_{xx}w_{yy} - w_{xy}^2 = h.
\]

Then, the gradient of \( w \) satisfies the convex hull-like property in \( \Omega \).

**PROOF.** It is enough to observe that \((w_x, w_y)\) is a \( C^1 \) solution in \( \Omega \) of the system (1.7) with \( \alpha(x, y) = -y \) and \( \beta(x, y) = x \) and that such \( \alpha, \beta \) satisfy the assumptions of Theorem 1.4. \( \triangle \)

2. A conjecture on an eigenvalue problem

**CONJECTURE 2.1.** - Let \( n \geq 2 \) and let \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \), with \( 0 < a < b \).

Then, there exists \( \lambda > 0 \) such that the problem

\[
\begin{align*}
  \Delta u &= \lambda \sin u \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

has at least one non-zero classical solution.

The above conjecture has its roots in Pohozaev identity ([19]). Let me recall it.

So, let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain, and let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function. Put

\[
F(\xi) = \int_{0}^{\xi} f(t)dt
\]

for all \( \xi \in \mathbb{R} \). For \( \lambda > 0 \), consider the problem

\[
\begin{align*}
  -\Delta u &= \lambda f(u) \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(\( P_{\lambda f} \))
In the sequel, a classical solution of problem \((P_{\lambda f})\) is any \(u \in C^2(\Omega) \cap C^1(\overline{\Omega}),\) zero on \(\partial \Omega,\) satisfying the equation pointwise in \(\Omega.\) Set

\[
\Lambda_f = \{ \lambda > 0 : (P_{\lambda f}) \text{ has a non-zero classical solution} \}.
\]

When \(n \geq 2,\) the Pohozaev identity tells us that, if \(u\) is a classical solution of \((P_{\lambda f})\), then one has

\[
\frac{2-n}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx + n \lambda \int_{\Omega} F(u(x)) \, dx = \frac{1}{2} \int_{\partial \Omega} |\nabla u(x)|^2 x \cdot \nu(x) \, ds
\]

\((2.1)\)

where \(\nu\) denotes the unit outward normal to \(\partial \Omega.\)

From \((2.1),\) in particular, it follows that, if \(\Omega\) is star-shaped with respect to 0 (so \(x \cdot \nu(x) \geq 0\) on \(\partial \Omega),\) then the set \(\Lambda_f\) is empty in the two following cases:

\((a)\) \(f(\xi) = |\xi|^{p-2} \xi\) with \(n \geq 3\) and \(p \geq \frac{2n}{n-2};\)

\((b)\) \(\sup_{\xi \in \mathbb{R}} F(\xi) = 0.\)

A natural question arises: what about problem \((P_{\lambda f})\) in cases \((a)\) and \((b)\) when \(\Omega\) is not star-shaped?

It is very surprising to realize that, while a great amount of research has been produced on case \((a)\) (see, for instance, \([1]-[5],[12],[14],[17],[18]\)), apparently the only papers dealing with case \((b)\) are \([9]-[11],[23].\)

In \([11],\) the following result has been pointed out:

**THEOREM 2.1.** - Let \(n \geq 2\) and \(\Omega = \{ x \in \mathbb{R}^n : a < |x| < b \}\) with \(0 < a < b,\)

Then for every continuous function \(f : \mathbb{R} \rightarrow \mathbb{R},\) with \(\sup_{\xi \in \mathbb{R}} F(\xi) = 0,\) and every \(\lambda > 0,\) problem \((P_{\lambda f})\) has no radially symmetric non-zero classical solutions.

**PROOF.** Let \(f : \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function, with \(\sup_{\xi \in \mathbb{R}} F(\xi) = 0,\) let \(\lambda > 0,\) and let \(u\) be a radially symmetric classical solution of \((P_{\lambda f}).\) Then

\[
\begin{cases}
-(r^{n-1} u'(r))' = \lambda r^{n-1} f(u(r)) \quad \text{for } r \in (a, b) \\
u(a) = u(b) = 0,
\end{cases}
\]

that is

\[
\begin{cases}
u''(r) + \frac{n-1}{r} u'(r) + \lambda f(u(r)) = 0 \quad \text{for } r \in (a, b) \\
u(a) = u(b) = 0.
\end{cases}
\]

Multiplying both sides of the equation in \((2.2)\) by \(u',\) we have

\[
u''(r) u'(r) + \frac{n-1}{r} (u'(r))^2 + \lambda f(u(r)) u'(r) = 0
\]

\((2.3)\)

for all \(r \in (a, b).\) Let \(r_1 \in (a, b)\) be such that \(u'(r_1) = 0.\) Define

\[
I_{r_1}(r) = \frac{1}{2} |u'(r)|^2 + (n-1) \int_{r_1}^{r} \frac{(u'(t))^2}{t} \, dt + \lambda F(u(r))
\]

for all \(r \in [a, b].\) Then \((2.3)\) shows that \(I_{r_1}(r) = 0\) for all \(r \in (a, b)\) and so, for some \(c \in \mathbb{R},\) one has

\[
I_{r_1}(r) = c
\]

for all \(r \in [a, b].\) Since

\[
I_{r_1}(r_1) = 0 + 0 + \lambda F(u(r_1)) \leq 0,
\]

we have \(c \leq 0.\) On the other hand, since

\[
I_{r_1}(b) = \frac{1}{2} |u'(b)|^2 + (n-1) \int_{r_1}^{b} \frac{(u'(t))^2}{t} \, dt + 0 \geq 0,
\]

we have \(c \geq 0.\) Since

\[
I_{r_1}(r_1) = c - c = 0,
\]

we have \(c = 0.\) Therefore, for all \(r \in (a, b),\)

\[
I_{r_1}(r) = 0.
\]
have \( c \geq 0 \), and so \( c = 0 \). In particular \( I_{r_1}(b) = 0 \), which implies \( u'(b) = 0 \), and consequently \( u(r) = 0 \) for all \( r \in [a,b] \), as claimed. \( \triangle \)

REMARK 2.1. - It is important to note the drastic difference between cases (a) and (b) enlightened by Theorem 2.1 when \( \Omega \) is an annulus. Actually, in this case, it was remarked in [13] that the problem

\[
\begin{cases}
-\Delta u = \lambda |u|^{p-2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

has radially symmetric non-zero classical solutions for \( p \geq \frac{2n}{n-2} \) \((n \geq 3)\), and \( \lambda > 0 \).

Now, I recall a very general result proved in [23].

For any real Hilbert space \( X \), denote by \( A_X \) the set of all \( C^1 \) functionals \( I : X \to \mathbb{R} \) such that 0 is a global maximum of \( I \) and \( I' \) is Lipschitzian with Lipschitz constant less than 1. Set

\[
\gamma_X = \inf_{I \in A_X} \inf \{ \lambda > 0 : x = \lambda I'(x) \text{ for some } x \neq 0 \} .
\]

We have:

THEOREM 2.2. - For any real Hilbert space \((X, \langle \cdot, \cdot \rangle)\), with \( X \neq \{0\} \), one has

\[
\gamma_X = 3 .
\]

We first prove

PROPOSITION 2.1. - One has

\[
\gamma_R = 3 .
\]

PROOF. Let \( I_0 \in A_R \) and let \( L < 1 \) be the Lipschitz constant of \( I_0' \). Set

\[
I = I_0 - I_0(0) .
\]

Fix \( \lambda \in [0,3] \). Let us prove that 0 is the only solution of the equation

\[
x = \lambda I'(x) .
\]

Arguing by contradiction, assume that

\[
x_0 = \lambda I'(x_0)
\]

for some \( x_0 \neq 0 \). It is not restrictive to assume that \( x_0 > 0 \) (otherwise, we would work with \( I'(-x) \)). Consider now the function \( g : \mathbb{R} \to \mathbb{R} \) defined by

\[
g(x) = \begin{cases}
-\frac{x^2}{2} & \text{if } x < \frac{x_0}{3} \\
\frac{x^2}{2} - \frac{2x_0x}{3} + \frac{x_0^2}{9} & \text{if } \frac{x_0}{3} \leq x \leq x_0 \\
-\frac{x^2}{2} + \frac{4x_0x}{3} - \frac{8x_0^2}{9} & \text{if } x_0 > x .
\end{cases}
\]

Clearly, \( g \in C^1(\mathbb{R}) \). Let \( x > 0 \) with \( x \neq x_0 \). Let us prove that

\[
g'(x) < I'(x) .
\]

We distinguish two cases. If \( 0 < x \leq \frac{x_0}{3} \), We have

\[
g'(x) = -x < -Lx \leq I'(x) .
\]

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If $x > \frac{x_0}{3}$, We have
\[ g'(x) = \frac{x_0}{3} - |x - x_0| < \frac{x_0}{3} - L|x - x_0| = \frac{\lambda I'(x_0)}{3} - L|x - x_0| \leq I'(x_0) - L|x - x_0| \leq I'(x) . \]
So, in particular, we get
\[ I\left(\frac{4x_0}{3}\right) = \int_0^{\frac{4x_0}{3}} I'(x)dx > \int_0^{\frac{4x_0}{3}} g'(x)dx = g\left(\frac{4x_0}{3}\right) = 0 \]
which contradicts the fact that the function $I$ is non-positive, since 0 is a global maximum of $I_0$. From what we have just proven, it clearly follows that
\[ 3 \leq \gamma_R . \]
Now, fix any $\mu > 1$. Continue to consider the function $g$ defined above (for a fixed $x_0 > 0$). Clearly, the function $\frac{1}{\mu}g$ belongs to $A_R$ and
\[ x_0 = 3\mu \frac{g(x_0)}{\mu} . \]
Of course, from this we infer that
\[ \gamma_R \leq 3\mu \]
and the conclusion clearly follows. \triangle

**Proof of Theorem 2.2.** First, let us prove that
\[ \gamma_X \leq 3 \]. \hspace{1cm} (2.4)
To this end, fix any $\varphi \in A_R$ and any $\lambda > 0$ such that
\[ t = \lambda \varphi'(t) \]
for some $t \neq 0$. Fix also $u \in X$, with $\|u\| = 1$, and consider the functional $I$ defined by putting
\[ I(x) = \varphi(\langle u, x \rangle) \]
for all $x \in X$. It is readily seen that $I \in A_X$. In particular, note that
\[ I'(x) = \varphi'(\langle u, x \rangle)u . \]
Finally, set
\[ \dot{x} = \lambda \varphi(t)u . \]
Of course, $\dot{x} \neq 0$. Since
\[ \langle u, \dot{x} \rangle = \lambda \varphi'(t) \]
we also have
\[ \langle u, \dot{x} \rangle = t \]
and so
\[ \dot{x} = \lambda I'(\dot{x}) . \]
From this, it clearly follows that
\[ \gamma_X \leq \gamma_R \]
and so (2.4) follows now from Proposition 2.1.
Now, let us prove that
\[ 3 \leq \gamma_X . \] \hspace{1cm} (2.5)
To this end, fix $I \in \mathcal{A}_X$, $\lambda > 0$ and $x \in X \setminus \{0\}$ such that

$$x = \lambda I'(x).$$

(2.6)

Then, consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = I\left(\frac{tx}{\|x\|}\right)$$

for all $t \in \mathbb{R}$. Clearly, 0 is a global maximum for $\varphi$. Moreover, $\varphi \in C^1(\mathbb{R})$ and one has

$$\varphi'(t) = \left\langle I'(\frac{tx}{\|x\|}) \frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle.$$  

Therefore, if $L$ is the Lipschitz constant of $I'$, for each $t, s \in \mathbb{R}$, we have

$$|\varphi'(t) - \varphi'(s)| = \left|\left\langle I'(\frac{tx}{\|x\|}) - I'(\frac{sx}{\|x\|}), \frac{x}{\|x\|}\right\rangle\right|$$

$$\leq \left\| I'(\frac{tx}{\|x\|}) - I'(\frac{sx}{\|x\|}) \right\| \leq L|t - s|.$$  

This shows that $\varphi'$ is a contraction, and so $\varphi \in \mathcal{A}_R$. Now, from (2.6), we get

$$\|x\| = \lambda \left\langle I'(x), \frac{x}{\|x\|}\right\rangle$$

that is

$$\|x\| = \lambda \varphi'(|x|).$$

From this, we infer that

$$\gamma_R \leq \gamma_X.$$  

So (2.5) follows from Proposition 2.1, and the proof is complete. \hfill \triangle

Now, for each $L > 0$, denote by $\mathcal{C}_L$ the class of all Lipschitzian functions $f : \mathbb{R} \to \mathbb{R}$, with Lipschitz constant $L$, such that $f(0) = 0$ and $\sup_{\xi \in \mathbb{R}} F(\xi) = 0$. Also denote by $\lambda_1$ the first eigenvalue of the problem

$$\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega.
\end{cases}$$

From Theorem 2.2, it directly follows that

$$\inf_{f \in \mathcal{C}_L} \inf \Lambda_f \geq \frac{3\lambda_1}{L}.$$  

In [10], X. L. Fan obtained the finer inequality

$$\inf_{f \in \mathcal{C}_L} \inf \Lambda_f \geq \frac{3\lambda_1}{L}.$$  

Conjecture 2.1 says that $\Lambda_f \neq \emptyset$ for $f(\xi) = -\sin \xi$, $\Omega$ being an annulus. Due to what precedes, if Conjecture 2.1 is true, then $\lambda$ must necessarily be larger than $3\lambda_1$. 

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3. A conjecture on a non-local problem

**Conjecture 3.1.** Let $a \geq 0$, $b > 0$ and let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, with $n > 4$. Then, for each $\lambda > 0$ large enough and for each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains $H^1_0(\Omega)$, there exists $v^* \in C$ such that the problem

\[
\begin{aligned}
&- \left( a + b \int_{\Omega} |\nabla u(x)|^2 \, dx \right) \Delta u = |u|^{n-2}u + \lambda(u - v^*(x)) \\
&u = 0
\end{aligned} \quad \text{in } \Omega
\]

\[
\text{on } \partial \Omega
\]

has at least three weak solutions, two of which are global minima in $H^1_0(\Omega)$ of the functional

\[
u \rightarrow \frac{a}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right)^2 - \frac{n-2}{2n} \int_{\Omega} |u(x)|^{\frac{2n}{n-2}} \, dx - \frac{\lambda}{2} \int_{\Omega} |u(x) - v^*(x)|^2 \, dx.
\]

Conjecture 3.1 comes from the results I have obtained in [25]. I am going to reproduce them here. Let $a, b, \Omega$ be as in Conjecture 3.1.

On the Sobolev space $H^1_0(\Omega)$, we consider the scalar product

\[
\langle u, v \rangle = \int_{\Omega} \nabla u(x) \nabla v(x) \, dx
\]

and the induced norm

\[
\|u\| = \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

Denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

\[
\sup_{(x, \xi) \in \Omega \times \mathbb{R}} \frac{|f(x, \xi)|}{1 + |\xi|^p} < +\infty
\]

for some $p \in \left]0, \frac{n+2}{n-2}\right]$. Moreover, denote by $\tilde{\mathcal{A}}$ the class of all Carathéodory functions $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

\[
\sup_{(x, \xi) \in \Omega \times \mathbb{R}} \frac{|g(x, \xi)|}{1 + |\xi|^q} < +\infty
\]

for some $q \in \left]0, \frac{2n}{n-2}\right]$. Furthermore, denote by $\mathcal{A}$ the class of all functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ of the type

\[
h(x, \xi) = f(x, \xi) + \alpha(x)g(x, \xi)
\]

with $f \in \mathcal{A}, g \in \tilde{\mathcal{A}}$ and $\alpha \in L^2(\Omega)$. For each $h \in \mathcal{A}$, define the functional $I_h : H^1_0(\Omega) \to \mathbb{R}$, by putting

\[
I_h(u) = \int_{\Omega} H(x, u(x)) \, dx
\]

for all $u \in H^1_0(\Omega)$, where

\[
H(x, \xi) = \int_0^\xi h(x, t) \, dt
\]

for all $(x, \xi) \in \Omega \times \mathbb{R}$. 

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By classical results (involving the Sobolev embedding theorem), the functional $I_h$ turns out to be sequentially weakly continuous, of class $C^1$, with compact derivative given by

$$I_h'(u)(w) = \int_{\Omega} h(x, u(x))w(x)dx$$

for all $u, w \in H^1_0(\Omega)$.

Now, recall that, given $h \in \hat{A}$, a weak solution of the problem

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = h(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

is any $u \in H^1_0(\Omega)$ such that

$$\left(a + b \int_{\Omega} |\nabla u(x)|^2 dx\right) \int_{\Omega} \nabla u(x) \nabla w(x)dx = \int_{\Omega} h(x, u(x))w(x)$$

for all $w \in H^1_0(\Omega)$. Let $\Phi : H^1_0(\Omega) \to \mathbb{R}$ be the functional defined by

$$\Phi(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4$$

for all $u \in H^1_0(\Omega)$.

Hence, the weak solutions of the problem are precisely the critical points in $H^1_0(\Omega)$ of the functional $\Phi - I_h$ which is said to be the energy functional of the problem.

The central result is as follows:

**THEOREM 3.1.** - Let $n \geq 4$, let $f \in A$ and let $g \in \tilde{A}$ be such that the set

$$\left\{ x \in \Omega : \sup_{\xi \in \mathbb{R}} |g(x, \xi)| > 0 \right\}$$

has a positive measure.

Then, there exist $\lambda^* > 0$ such that, for each $\lambda > \lambda^*$ and each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains the set $\{G(\cdot, u(\cdot)) : u \in H^1_0(\Omega)\}$, there exists $v^* \in C$ such that the problem

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = f(x, u) + \lambda(G(x, u) - v^*(x))g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H^1_0(\Omega)$ of the functional

$$u \to \frac{a}{2} \int_{\Omega} |\nabla u(x)|^2 dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^2 - \int_{\Omega} F(x, u(x))dx - \frac{\lambda}{2} \int_{\Omega} |G(x, u(x)) - v^*(x)|^2 dx .$$

If, in addition, the functional

$$u \to \frac{a}{2} \int_{\Omega} |\nabla u(x)|^2 dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^2 - \int_{\Omega} F(x, u(x))dx$$

has at least two global minima in $H^1_0(\Omega)$ and the function $G(x, \cdot)$ is strictly monotone for all $x \in \Omega$, then $\lambda^* = 0$.

The main tool we use to prove Theorem 3.1 is Theorem 3.C below which, in turn, is a direct consequence of two other results recently established in [24].
To state Theorem 3.C in a compact form, we now introduce some notations.

Here and in what follows, $X$ is a non-empty set, $V, Y$ are two topological spaces, $y_0$ is a point in $Y$.

We denote by $\mathcal{G}$ the family of all lower semicontinuous functions $\varphi : Y \to [0, +\infty[$, with $\varphi^{-1}(0) = \{y_0\}$, such that, for each neighbourhood $U$ of $y_0$, one has

$$\inf_{Y \setminus U} \varphi > 0.$$  

Moreover, denote by $\mathcal{H}$ the family of all functions $\Psi : X \times V \to Y$ such that, for each $x \in X$, $\Psi(x, \cdot)$ is continuous, injective, open, takes the value $y_0$ at a point $v_x$ and the function $x \to v_x$ is not constant. Furthermore, denote by $\mathcal{M}$ the family of all functions $J : X \to \mathbb{R}$ whose set of all global minima (noted by $M_J$) is non-empty.

Finally, for each $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$, put

$$\theta(\varphi, \Psi, J) = \inf \left\{ \frac{J(x) - J(u)}{\varphi(\Psi(x, v_u))} : (u, x) \in M_J \times X \text{ with } v_x \neq v_u \right\} .$$

When $X$ is a topological space, a function $\psi : X \to \mathbb{R}$ is said to be inf-compact if $\psi^{-1}([-\infty, r])$ is compact for all $r \in \mathbb{R}$.

**THEOREM 3.A ([24], Theorem 3.1).** - Let $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$.

Then, for each $\lambda > \theta(\varphi, \Psi, J)$, one has

$$\sup_{v \in V} \inf_{x \in X} (J(x) - \lambda \varphi(\Psi(x, v))) < \inf_{x \in X} \sup_{z \in X} (J(x) - \lambda \varphi(\Psi(x, v))) .$$

**THEOREM 3.B ([24], Theorem 3.2).** - Let $X$ be a topological space, $E$ a real Hausdorff topological vector space, $C \subseteq E$ a convex set, $f : X \times C \to \mathbb{R}$ a function which is lower semicontinuous and inf-compact in $X$, and upper semicontinuous and concave in $C$. Assume also that

$$\sup_{v \in C} \inf_{x \in X} f(x, v) < \inf_{x \in X} \sup_{v \in C} f(x, v) .$$

Then, there exists $v^* \in C$ such that the function $f(\cdot, v^*)$ has at least two global minima.

**THEOREM 3.C.** - Let $\varphi \in \mathcal{G}$, $\Psi \in \mathcal{H}$ and $J \in \mathcal{M}$. Moreover, assume that $X$ is a topological space, that $V$ is a real Hausdorff topological vector space and that $\varphi(\Psi(x, \cdot))$ is convex and continuous for each $x \in X$.

Finally, let $\lambda > \theta(\varphi, \Psi, J)$ and let $C \subseteq V$ a convex set, with $\{v_x : x \in X\} \subseteq \overline{C}$, such that the function $x \to J(x) - \lambda \varphi(\Psi(x, v))$ is lower semicontinuous and inf-compact in $X$ for all $v \in C$.

Under such hypotheses, there exist $v^* \in C$ such that the function $x \to J(x) - \lambda \varphi(\Psi(x, v^*))$ has at least two global minima in $X$.

**PROOF.** Set

$$D = \{v_x : x \in X\}$$

and, for each $(x, v) \in X \times V$, put

$$f(x, v) = J(x) - \lambda \varphi(\Psi(x, v)).$$

Theorem 3.A ensures that

$$\sup_{v \in V} \inf_{x \in X} f(x, v) < \inf_{x \in X} \sup_{v \in D} f(x, v) . \quad (3.3)$$

But, since $f(x, \cdot)$ is continuous and $D \subseteq \overline{C}$, we have

$$\sup_{v \in D} f(x, v) = \sup_{v \in \overline{C}} f(x, v) \leq \sup_{v \in C} f(x, v) = \sup_{v \in C} f(x, v)$$

for all $x \in X$, and hence, from (3.3), it follows that

$$\sup_{v \in C} \inf_{x \in X} f(x, v) \leq \inf_{x \in X} \sup_{v \in C} f(x, v) .$$
At this point, the conclusion follows applying Theorem 3.1 to the restriction of the function \( f \) to \( X \times C. \triangle \)

**Proof of Theorem 3.1.** For each \( \lambda \geq 0, \ v \in L^2(\Omega) \), consider the function \( h_{\lambda,v} : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
h_{\lambda,v}(x,\xi) = f(x,\xi) + \lambda(G(x,\xi) - v(x))g(x,\xi)
\]

for all \((x,\xi) \in \Omega \times \mathbb{R}\). Clearly, the function \( h_{\lambda,v} \) lies in \( \hat{A} \) and

\[
H_{\lambda,v}(x,\xi) = F(x,\xi) + \frac{\lambda}{2} \left( |G(x,\xi) - v(x)|^2 - |v(x)|^2 \right).
\]

So, the weak solutions of the problem are precisely the critical points in \( H^1_0(\Omega) \) of the functional \( \Phi - I_{h_{\lambda,v}} \). Moreover, if \( p \in \left[0, \frac{n+2}{n-2}\right] \) and \( q \in \left[0, \frac{2(n-2)}{n-2}\right] \) are such that (3.1) and (3.2) hold, for some constant \( c_{\lambda,v} \), we have

\[
\int_\Omega |H_{\lambda,v}(x,u(x))|dx \leq c_{\lambda,v} \left( \int_\Omega |u(x)|^{p+1} + \int_\Omega |u(x)|^{2(q+1)}dx + 1 \right)
\]

for all \( u \in H^1_0(\Omega) \). Therefore, by the Sobolev embedding theorem, for a constant \( \tilde{c}_{\lambda,v} \), we have

\[
\Phi(u) - I_{h_{\lambda,v}}(u) \geq \frac{b}{4} \|u\|^4 - \tilde{c}_{\lambda,v}(\|u\|^{p+1} + \|u\|^{2(q+1)} + 1)
\]

for all \( u \in H^1_0(\Omega) \). On the other hand, since \( n \geq 4 \), one has

\[
\max\{p+1,2(q+1)\} < \frac{2n}{n-2} \leq 4.
\]

Consequently, from (3.4), we infer that

\[
\lim_{\|u\| \rightarrow +\infty} (\Phi(u) - I_{h_{\lambda,v}}(u)) = +\infty.
\]  

Since the functional \( \Phi - I_{h_{\lambda,v}} \) is sequentially weakly lower semicontinuous, by the Eberlein-Šmulian theorem and by (3.5), it follows that it is inf-weakly compact.

Now, we are going to apply Theorem 3.C taking \( X = H^1_0(\Omega) \) with the weak topology and \( V = Y = L^2(\Omega) \) with the strong topology, and \( y_0 = 0 \). Also, we take

\[
\varphi(w) = \frac{1}{2} \int_\Omega |w(x)|^2dx
\]

for all \( w \in L^2(\Omega) \). Clearly, \( \varphi \in \mathcal{G} \). Furthermore, we take

\[
\Psi(u,v)(x) = G(x,u(x)) - v(x)
\]

for all \( u \in H^1_0(\Omega), v \in L^2(\Omega), x \in \Omega \). Clearly, \( \Psi(u,v) \in L^2(\Omega) \), \( \Psi(u, \cdot) \) is a homeomorphism, and we have

\[
v_u(x) = G(x,u(x)).
\]

We show that the map \( u \rightarrow v_u \) is not constant in \( H^1_0(\Omega) \). For each \( x \in \Omega \), set

\[
\alpha(x) = \inf_{\xi \in \mathbb{R}} G(x,\xi)
\]

and

\[
\beta(x) = \sup_{\xi \in \mathbb{R}} G(x,\xi).
\]  

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Since $G$ is a Carathéodory continuous, we have

$$\alpha(x) = \inf_{\xi \in \mathcal{Q}} G(x, \xi)$$

and

$$\beta(x) = \sup_{\xi \in \mathcal{Q}} G(x, \xi) ,$$

and so the functions $\alpha, \beta$ are measurable. Set

$$A = \{ x \in \Omega : \alpha(x) < \beta(x) \} .$$

Clearly, we have

$$A = \left\{ x \in \Omega : \sup_{\xi \in \mathcal{R}} |g(x, \xi)| > 0 \right\} .$$

Hence, by assumption, $\text{meas}(A) > 0$. Then, by the classical Scorza-Dragoni theorem ([6], Theorem 2.5.19), there exists a compact set $K \subset A$, of positive measure, such that the restriction of $G$ to $K \times \mathbb{R}$ is continuous. Fix a point $\bar{x} \in K$ such that the intersection of $K$ and any ball centered at $\bar{x}$ has a positive measure. Next, fix $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$G(\bar{x}, \xi_1) < G(\bar{x}, \xi_2) .$$

By continuity, there is a closed ball $B(\bar{x}, r)$ such that

$$G(x, \xi_1) < G(x, \xi_2)$$

for all $x \in K \cap B(\bar{x}, r)$. Finally, consider two functions $u_1, u_2 \in H^1_0(\Omega)$ which are constant in $K \cap B(\bar{x}, r)$. So, we have

$$G(x, u_1(x)) < G(x, u_2(x))$$

for all $x \in K \cap B(\bar{x}, r)$. Hence, as $\text{meas}(K \cap B(\bar{x}, r)) > 0$, we infer that $v_{u_1} \neq v_{u_2}$, as claimed. As a consequence, $\Psi \in \mathcal{H}$. Of course, $\varphi(\Psi(u, \cdot))$ is continuous and convex for all $u \in X$. Finally, take

$$J = \Phi - I_f .$$

Clearly, $J \in \mathcal{M}$. So, for what seen above, all the assumptions of Theorem 3.C are satisfied. Consequently, if we take

$$\lambda^* = \theta(\varphi, \Psi, J) \quad (3.6)$$

and fix $\lambda > \lambda^*$ and a convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains the set $\{ G(\cdot, u(\cdot)) : u \in H^1_0(\Omega) \}$, there exists $v^* \in C$ such that the functional $\Phi - I_{h_{\lambda, v^*}}$ has at least two global minima in $H^1_0(\Omega)$ which are, therefore, weak solutions of the problem. To guarantee the existence of a third solution, denote by $k$ the inverse of the restriction of the function $at + bt^3$ to $[0, +\infty]$. Let $T : X \rightarrow X$ be the operator defined by

$$T(w) = \begin{cases} k(\|w\|)w & \text{if } w \neq 0 \\ 0 & \text{if } w = 0 \end{cases} ,$$

Since $k$ is continuous and $k(0) = 0$, the operator $T$ is continuous in $X$. For each $u \in X \setminus \{0\}$, we have

$$T(\Phi'(u)) = T((a + b\|u\|^2)u) = \frac{k((a + b\|u\|^2)\|u\|)}{(a + b\|u\|^2)\|u\|}(a + b\|u\|^2)u = \frac{\|u\|}{(a + b\|u\|^2)\|u\|}(a + b\|u\|^2)u = u .$$

In other words, $T$ is a continuous inverse of $\Phi'$. Then, since $I'_{h_{\lambda, v^*}}$ is compact, the functional $\Phi - I_{h_{\lambda, v^*}}$ satisfies the Palais-Smale condition ([30], Example 38.25) and hence the existence of a third critical point of the same functional is assured by Corollary 1 of [20].
Finally, assume that the functional \( \Phi - I_f \) has at least two global minima, say \( \hat{u}_1, \hat{u}_2 \). Then, the set 
\[ D := \{ x \in \Omega : \hat{u}_1(x) \neq \hat{u}_2(x) \} \]
has at least three solutions, two of which are global minima in \( \lambda \)
and so \( \lambda > 0 \) in view of \( (3.6) \). 

\[ G(x, \hat{u}_1(x)) \neq G(x, \hat{u}_2(x)) \] 

for all \( x \in D \), and so \( v_{\hat{u}_1} \neq v_{\hat{u}_2} \). Then, by definition, we have 
\[ 0 \leq \theta(\varphi, \Psi, J) \leq \frac{J(\hat{u}_1) - J(\hat{u}_2)}{\varphi(\Psi(\hat{u}_1, v_{\hat{u}_2}))} = 0 \]
and so \( \lambda^* = 0 \) in view of \( (3.6) \). 

Notice the following corollary of Theorem 3.1:

**COROLLARY 3.1.** - Let \( n \geq 4 \), let \( \nu \in \mathbb{R} \) and let \( p \in \left[ 0, \frac{n+2}{n-2} \right] \).

Then, for each \( \lambda > 0 \) large enough and for each convex set \( C \subseteq L^2(\Omega) \) whose closure in \( L^2(\Omega) \) contains \( H_0^1(\Omega) \), there exists \( v^* \in C \) such that the problem
\[
\begin{cases}
- \left( a + b \int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \nu |u|^{p-1}u + \lambda(u - v^*)(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has at least three solutions, two of which are global minima in \( H_0^1(\Omega) \) of the functional
\[ u \rightarrow \frac{a}{2} \int_{\Omega} |\nabla u(x)|^2 dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^2 - \frac{\nu}{p+1} \int_{\Omega} |u(x)|^{p+1} dx - \frac{\lambda}{2} \int_{\Omega} |u(x) - v^*(x)|^2 dx . \]

**PROOF.** Apply Theorem 3.1 taking \( f(x, \xi) = |\xi|^{p-1} \xi \) and \( g(x, \xi) = 1 \). 

\[ \square \]

**REMARK 3.1.** - In Theorem 3.1, the assumption made on \( g \) (besides \( g \in \mathcal{A} \)) is essential. Indeed, if \( g = 0 \), for \( f = 0 \) (which is an allowed choice), the problem would have the zero solution only.

**REMARK 3.2.** - The assumption \( n \geq 4 \) is likewise essential. Indeed, Corollary 3.1 does not hold if \( n = 3 \). To see this, take \( p = 4 \) (which, when \( n = 3 \), is compatible with the condition \( p < \frac{n+2}{n-2} \)) and observe that the corresponding energy functional is unbounded below.

Besides Corollary 3.1, among the consequences of Theorem 3.1, we highlight the following

**THEOREM 3.2.** - Let \( n \geq 4 \), let \( f \in \mathcal{A} \) and let \( g \in \mathcal{A} \) be such the set
\[ \left\{ x \in \Omega : \sup_{\xi \in \mathbb{R}} F(x, \xi) > 0 \right\} \]
has a positive measure. Moreover, assume that, for each \( x \in \Omega \), \( f(x, \cdot) \) is odd, \( g(x, \cdot) \) is even and \( G(x, \cdot) \) is strictly monotone.

Then, for each \( \lambda > 0 \), there exists \( \mu^* > 0 \) such that, for each \( \mu > \mu^* \) and for each convex set \( C \subseteq L^2(\Omega) \) whose closure in \( L^2(\Omega) \) contains the set \( \{ G(\cdot, u(\cdot)) : u \in H_0^1(\Omega) \} \), there exists \( v^* \in C \) such that the problem
\[
\begin{cases}
- \left( a + b \int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \mu f(x, u) - \lambda v^*(x)g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has at least three weak solutions, two of which are global minima in \( H_0^1(\Omega) \) of the functional
\[ u \rightarrow \frac{a}{2} \int_{\Omega} |\nabla u(x)|^2 dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^2 - \mu \int_{\Omega} F(x, u(x)) dx + \lambda \int_{\Omega} v^*(x)G(x, u(x)) dx . \]
PROOF. Set

\[ D = \left\{ x \in \Omega : \sup_{\xi \in \mathbb{R}} F(x, \xi) > 0 \right\}. \]

By assumption, \( \text{meas}(D) > 0 \). Then, by the Scorza-Dragoni theorem, there exists a compact set \( K \subset D \), of positive measure, such that the restriction of \( F \) to \( K \times \mathbb{R} \) is continuous. Fix a point \( \hat{x} \in K \) such that the intersection of \( K \) and any ball centered at \( \hat{x} \) has a positive measure. Choose \( \xi \in \mathbb{R} \) so that \( F(\hat{x}, \xi) > 0 \). By continuity, there is \( r > 0 \) such that \( F(x, \hat{\xi}) > 0 \) for all \( x \in K \cap B(\hat{x}, r) \).

Set

\[ M = \sup \{ F(x, \xi) : x, \xi \in \Omega \times [-|\xi|, |\xi|] \}. \]

Since \( f \in \mathcal{A} \), we have \( M < +\infty \). Next, choose an open set \( \tilde{\Omega} \) such that \( K \cap B(\hat{x}, r) \subset \tilde{\Omega} \subset \Omega \) and

\[ \text{meas}(\tilde{\Omega} \setminus (K \cap B(\hat{x}, r))) < \frac{\int_{K \cap B(\hat{x}, r)} F(x, \hat{\xi}) dx}{M}. \]

Finally, choose a function \( \tilde{u} \in H^1_0(\Omega) \) such that

\[ \tilde{u}(x) = \hat{\xi} \]

for all \( x \in K \cap B(x, r) \),

\[ \tilde{u}(x) = 0 \]

for all \( x \in \Omega \setminus \tilde{\Omega} \) and

\[ |\tilde{u}(x)| \leq |\hat{\xi}| \]

for all \( x \in \Omega \). Thus, we have

\[ \int_{\Omega} F(x, \tilde{u}(x)) dx = \int_{K \cap B(\hat{x}, r)} F(x, \hat{\xi}) dx + \int_{\tilde{\Omega} \setminus (K \cap B(\hat{x}, r))} F(x, \tilde{u}(x)) dx \]

\[ > \int_{K \cap B(\hat{x}, r)} F(x, \hat{\xi}) dx - M \text{meas}(\tilde{\Omega} \setminus (K \cap B(\hat{x}, r))) > 0. \]

Now, fix any \( \lambda > 0 \) and set

\[ \mu^* \overset{\Delta}{=} \frac{\Phi(\tilde{u}) + \lambda I_G(\tilde{u})}{I_f(\tilde{u})}. \]

Fix \( \mu > \mu^* \). Hence

\[ \Phi(\tilde{u}) - \mu I_f(\tilde{u}) + \lambda I_G(\tilde{u}) < 0. \]

From this, we infer that the functional \( \Phi - \mu I_f + \lambda I_G \) possesses at least to global minima since it is even. At this point, we can apply Theorem 3.1 to the functions \( g \) and \( \mu f - \lambda Gg \). Our current conclusion follows from the one of Theorem 3.1 since we have \( \lambda^* = 0 \) and hence we can take the same fixed \( \lambda > 0 \).

4. A conjecture on disconnectedness versus infinitely many solutions

**CONJECTURE 4.1.** - Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain, with \( n \geq 3 \). Let \( \tau \) be the strongest vector topology on \( H^1_0(\Omega) \).
Then, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$, with
\[
\sup_{\xi \in \mathbb{R}} \frac{|f(\xi)|}{1 + |\xi|^2} < +\infty ,
\]
such that the set
\[
\left\{(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) : \int_{\Omega} \nabla u(x) \nabla v(x) dx - \int_{\Omega} f(u(x)) v(x) dx = 1 \right\}
\]
is disconnected in $(H^1_0(\Omega), \tau) \times (H^1_0(\Omega), \tau)$.

The importance of Conjecture 4.1 is shown by Proposition 4.3 below. But, first the relevant theory should be fixed.

The central abstract result, obtained in [21], is as follows (see also [16]):

**THEOREM 4.1.** - Let $X$ be a connected topological space, let $E$ be a real topological vector space, with topological dual $E^*$, and let $A : X \to E^*$ be an operator such that the set
\[
\{ y \in E : x \to \langle A(x), y \rangle \text{ is continuous} \}
\]
is dense in $E$ and the set
\[
\{ (x, y) \in X \times E : \langle A(x), y \rangle = 1 \}
\]
is disconnected.

Then, $A$ does vanish at some point of $X$.

**PROOF.** Denote by $p_X$ the projection from $X \times E$ onto $X$. Moreover, for any $C \subseteq X \times E$, $x \in X$, put
\[
C_x = \{ y \in E : (x, y) \in C \}.
\]
Arguing by contradiction, assume that $A(x) \neq 0$ for all $x \in X$. Denote by $\Gamma$ the set
\[
\{ (x, y) \in X \times E : \langle A(x), y \rangle = 1 \}.
\]
Since $\Gamma$ is disconnected, there are two open sets $\Omega_1, \Omega_2 \subseteq X \times E$ such that
\[
\Omega_1 \cap \Gamma \neq \emptyset, \quad \Omega_2 \cap \Gamma \neq \emptyset, \quad \Omega_1 \cap \Omega_2 \cap \Gamma = \emptyset, \quad \Gamma \subseteq \Omega_1 \cup \Omega_2.
\]
We now prove that $p_X(\Omega_1 \cap \Gamma)$ is open in $X$. So, let $(x_0, y_0) \in \Omega_1 \cap \Gamma$. Since $E$ is locally connected ([28], p.35), there are a neighbourhood $U_0$ of $x_0$ in $X$ and an open connected neighbourhood $V_0$ of $y_0$ in $E$ such that $U_0 \times V_0 \subseteq \Omega_1$. Since $\langle A(x_0), \cdot \rangle$ is a non-null continuous linear functional, it has no local extrema. Consequently, since $\langle A(x_0), y_0 \rangle = 1$, the sets
\[
\{ u \in V_0 : \langle A(x_0), u \rangle < 1 \},
\]
\[
\{ u \in V_0 : \langle A(x_0), u \rangle > 1 \}
\]
are both non-empty and open. Then, thanks to our density assumption, there are $u_1, u_2 \in V_0$ such that the set
\[
\{ x \in U_0 : \langle A(x), u_1 \rangle < 1 < \langle A(x), u_2 \rangle \}
\]
is a neighbourhood of $x_0$. Then, if $x$ is in this set, due to the connectedness of $V_0$, there is some $y \in V_0$ such that $\langle A(x), y \rangle = 1$, and so, $x$ actually lies in $p_X(\Omega_1 \cap \Gamma)$, as desired. Likewise, it is seen that $p_X(\Omega_2 \cap \Gamma)$ is open. Now, observe that, for any $x \in X$, the set $\{ x \} \times \Gamma_x$ is non-empty and connected, and so it is contained either in $\Omega_1$ or in $\Omega_2$. Summarizing, we then have that the sets $p_X(\Omega_1 \cap \Gamma)$ and $p_X(\Omega_2 \cap \Gamma)$ are non-empty, open, disjoint and cover $X$. Hence, $X$ would be disconnected, a contradiction. $\triangle$

Once Theorem 4.1 has been obtained, we can state the following formally more complete result:
**Theorem 4.2.** - Let $X$ be a topological space, let $E$ be a real topological vector space, and let $A : X \to E^*$ be such that the set 
\[ \{ y \in E : x \to \langle A(x), y \rangle \text{ is continuous} \} \]
is dense in $E$.

Then, the following assertions are equivalent:

(i) The set 
\[ \{(x, y) \in X \times E : \langle A(x), y \rangle = 1 \} \]
is disconnected.

(ii) The set $X \setminus A^{-1}(0)$ is disconnected.

**Proof.** Let (i) hold. Since 
\[ \{(x, y) \in X \times E : \langle A(x), y \rangle = 1 \} = \{(x, y) \in (X \setminus A^{-1}(0)) \times E : \langle A(x), y \rangle = 1 \}, \]
if $X \setminus A^{-1}(0)$ were connected, we could apply Theorem 4.1 to $A|_{(X \setminus A^{-1}(0))}$, and so $A$ would vanish at some point of $X \setminus A^{-1}(0)$, which is absurd.

Conversely, if (ii) holds, then (i) follows at once observing that, with the notations of the proof of Theorem 4.1, one has $X \setminus A^{-1}(0) = p_X(\Gamma)$. \hfill $\triangle$

**Remark 4.1.** - When $X$ is a connected topological space, $E$ is an infinite-dimensional real vector space (with algebraic dual $E'$), and $A : X \to E'$ is a $\sigma(E', E)$-continuous operator, one could try to apply Theorem 4.1 endowing $E$ with the strongest vector topology ([15], p.53).

**Remark 4.2.** - In Theorem 4.1, the role of the constant 1 can actually be assumed by any continuous real function on $X$. Precisely, we have the following

**Proposition 4.1.** - Let $X$ be a topological space, let $E$ be a real topological vector space, and let $A : X \to E'$. Assume that, for some continuous function $\alpha : X \to \mathbb{R}$, the set 
\[ \Lambda := \{(x, y) \in X \times E : \langle A(x), y \rangle = \alpha(x) \} \]
is disconnected.

Then, either $A(x) = 0$ for some $x \in X$, or the set 
\[ \Gamma := \{(x, y) \in X \times E : \langle A(x), y \rangle = 1 \} \]
is disconnected.

**Proof.** Assume that $A^{-1}(0) = \emptyset$. So, $p_X(\Gamma) = X$. Consider the function $f : X \times E \to X \times E$ defined by putting $f(x, y) = (x, \alpha(x)y)$ for all $(x, y) \in X \times E$. Of course, $f$ is continuous. Arguing by contradiction, assume that $\Gamma$ is connected. Then, $f(\Gamma)$ is connected too. Now, observe that 
\[ \Lambda = \bigcup_{x \in \alpha^{-1}(0)} (f(\Gamma) \cup \{(x) \times \Lambda_x\}). \]
Furthermore, note that, if $x \in \alpha^{-1}(0)$, then $(x, 0) \in f(\Gamma) \cap (\{x\} \times \Lambda_x)$, and so $f(\Gamma) \cup (\{x\} \times \Lambda_x)$ is connected. In turn, the sets $f(\Gamma) \cup (\{x\} \times \Lambda_x)$ ($x \in \alpha^{-1}(0)$) are clearly pairwise non-disjoint, and hence $\Lambda$ is connected, a contradiction. \hfill $\triangle$

In [22], the following proposition was pointed out:

**Proposition 4.2 ([22], Proposition 3).** - Let $E$ be an infinite-dimensional Hausdorff topological vector space and $K$ a relatively compact subset of $E$.

Then, the set $E \setminus K$ is connected.

Finally, as said, the following proposition shows the importance of Conjecture 4.1:

**Proposition 4.3.** - Let $f$ be a function satisfying Conjecture 4.1.
Then, the problem
\[
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has infinitely many weak solutions.

PROOF. Let \( X = W^{1,2}_0(\Omega) \), with the usual norm \( \|u\| = (\int_\Omega |\nabla u(x)|^2 dx)^{\frac{1}{2}} \). For \( 0 < q \leq \frac{n+2}{n-2} \) and \( f \in A_q \), put
\[
J(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \left( \int_0^{u(x)} f(\xi) d\xi \right) dx
\]
for all \( u \in X \).

So, the functional \( J \) is of class \( C^1 \) on \( X \) and one has
\[
J'(u)(v) = \int_\Omega \nabla u(x) \nabla v(x) dx - \int_\Omega f(x, u(x)) v(x) dx
\]
for all \( u, v \in X \). Hence, the critical points of \( J \) in \( X \) are exactly the weak solutions of the problem. Since \( J \) is of class \( C^1 \), clearly the operator \( J' : X \to X^* \) is \( \tau \)-weakly-star continuous. Hence, by Theorem 4.2, the set \( X \setminus (J')^{-1}(0) \) is \( \tau \)-disconnected. Then, due to Proposition 4.2, the set \( (J')^{-1}(0) \) is not \( \tau \)-relatively compact, and hence is infinite. \( \triangle \)
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