The Lie algebra of the fundamental group of a surface as a symplectic module

Simion Filip

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Abstract

This note provides a formula for the character of the Lie algebra of the fundamental group of a surface, viewed as a module over the symplectic group.

1 Introduction

To an arbitrary group $G$ one canonically associates a Lie algebra by considering the lower central series and its associated graded. This Lie algebra often carries a lot of information about the group itself and its symmetries. This note provides an explicit description of the symplectic symmetries of the Lie algebra corresponding to the fundamental group of $\Sigma_g$ - a closed surface of genus $g$.

More concretely, denote this group by $G$. Its lower central series is defined recursively by

$$G^{(i+1)} := [G^{(i)}, G] \quad \text{and} \quad G^{(1)} := G$$

In other words, the $i^{th}$ group is the group generated by elements which are $(i - 1)$-fold commutators. Next, form the associated graded quotients

$$\mathfrak{g}_i := G^{(i)}/G^{(i+1)} \quad \forall i \geq 1 \quad (1.1)$$

These quotients are abelian groups and if we consider

$$\mathfrak{g} := \bigoplus_{i \geq 1} \mathfrak{g}_i$$

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then we obtain a graded Lie algebra, since the bracket \([a, b] := aba^{-1}b^{-1}\) descends to the level of the quotients.

Since all the constructions were natural, symmetries of \(G\) will act on the Lie algebra \(g\) (and preserve the grading). Note that conjugating by an element \(x \in G\) induces the trivial automorphism on \(g\), so we can consider the action of the mapping class group \(Mod(\Sigma_g)\) on \(g\). Observe further that if \(\phi \in Mod(\Sigma_g)\) acts trivially on \(g_1\), then \(\phi\) acts trivially on all of \(g\).

Indeed, the first assertion is equivalent to \(x\phi(x)^{-1} \in G^{(2)}\) for any \(x \in G\). Using this and the identity \([ab, c] = a[b, c]a^{-1}[a, c]\) one can check directly that \(x\phi(x)^{-1} \in G^{(i+1)}\) for any \(x \in G^{(i)}\).

In our present case, since \(G\) is the fundamental group of a surface \(\Sigma_g\) of genus \(g\), it is well-known that \(g_1 \cong H_1(\Sigma_g, \mathbb{Z})\) (see the monograph of Farb and Margalit [FM12, Chapter 5] for this result and more context). Further, the action of \(Mod(\Sigma_g)\) factors through \(\text{Sp}_g(\mathbb{Z})\), the group of symplectic automorphisms of this \(2g\)-dimensional space.

In the case when \(G\) is a free group, one obtains the free Lie algebra with an action of \(\text{GL}_n\) (\(g_1\) is then the standard module). The representation which occur in this case are classical, see the monograph of Reutenauer [Reu93] for a comprehensive collection of results. This note describes the representation of \(\text{Sp}_g(\mathbb{Z})\) on the graded piece \(g_N\) as follows.

**Theorem 1.1.** The character of the representation of \(\text{Sp}_g(\mathbb{Z})\) on \(g_N\), denoted \(\chi_N\), is given by the formula

\[
\chi_N = \frac{1}{N} \sum_{d|N} \mu\left(\frac{N}{d}\right) d \sum_{k \geq 0} \chi^{d-2k,\left(\frac{N}{d}\right)} \frac{1}{d-k} \binom{d-k}{k} (-1)^k
\]

Here \(\mu\) denotes the Möbius function and \(\chi^x\) denotes the character of \(\text{Sp}_g\) on \(H_1(\Sigma_g)\). The expression \(\chi^{x,(y)}\) denotes the function on \(\text{Sp}_g\) given by \(\chi^{x,(y)}(M) = \chi(M^y)^x\).

This formula is proved in Proposition 4.4 where another equivalent formulation is presented.

**Outline of the approach.** The first step is to find the Euler-Poincaré series of the universal enveloping algebra \(U(g)\). This is possible due to the algebraic results of Labute, explained in Section 2. The actual computation for \(U(g)\) is done in Section 3. The calculation of the character \(\chi_N\) of the \(N\)th graded piece \(g_N\) is contained in Section 4.
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Notation and Remarks. Throughout the text, the following notations and conventions are used:

- If $\mathfrak{h}$ is a Lie algebra, its universal enveloping algebra is denoted $\mathcal{U}(\mathfrak{h})$.
- Universal enveloping algebras have a natural filtration, and $\text{gr}^\bullet \mathcal{U}(\mathfrak{h})$ denotes the associated graded with respect to this filtration.
- If $M$ is a $\mathbb{Z}$-module, the free Lie algebra on this module is denoted $\mathcal{F}(M)$.
- If the $\mathbb{Z}$-module $M$ has an extra structure (e.g., grading or $\text{Sp}_g$-module structure), the free Lie algebra $\mathcal{F}(M)$ inherits it.
- A closed surface of genus $g$ is fixed and denoted $\Sigma_g$, its homology is denoted $V := H_1(\Sigma_g, \mathbb{Z})$ and its fundamental group is denoted $G$.
- The Lie algebra $\mathfrak{g}$ is the one defined by Equation 1.1 and its graded pieces are $\mathfrak{g}_i$.
- For any module $M$, its symmetric algebra is denoted $\text{Sym}^\bullet (M) := \bigoplus_{i \geq 0} \text{Sym}^i (M)$.
- Any extra structure on $M$ (grading or $\text{Sp}_g$-module) is inherited by $\text{Sym}^\bullet (M)$.
- In all the situations below, modules will start in homogeneous degree at least 1 and will be finitely generated in each homogeneous component. These two properties together are stable under taking symmetric algebras or universal enveloping algebras.
- A key tool at many steps will be the Poincaré-Birkhoff-Witt theorem. It states that for any Lie algebra $\mathfrak{h}$ the natural map
  $$\text{gr}^\bullet \mathcal{U}(\mathfrak{h}) \to \text{Sym}^\bullet (\mathfrak{h})$$
  is an isomorphism. Moreover, this isomorphism is compatible with any extra gradings or $\text{Sp}_g$-module structures on $\mathfrak{h}$.
This section collects purely algebraic results that will be used when comput-
ing the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.

**Setup.** Recall the surjection $\mathbb{F}_{2g} \twoheadrightarrow G$, where $\mathbb{F}_{2g}$ is the free group on $2g$ generators. To fix notation, the group $G$ is a group freely generated by elements $a_1, b_1, \ldots, a_g, b_g$ with the single relation $[a_1, b_1] \cdots [a_g, b_g] = 1$.

We make use of the results of Labute from [Lab70]. The main theorem of the paper implies that $\mathfrak{g}$ is a free Lie algebra with a single relation. The results can be summarized as follows.

(i) Let $V_{2g} := H_1(\Sigma_g, \mathbb{Z})$ be the $\text{Sp}_{g}$-module which is also canonically identified with the abelianization of $\pi_1$, and let $\mathbb{F}(V_{2g})$ be the free Lie algebra on this module. The Lie sub-algebra $\mathfrak{r} \subset \mathbb{F}(V_{2g})$ is defined by the following canonical short exact sequence.

\[
0 \rightarrow \mathfrak{r} \twoheadrightarrow \mathbb{F}(V_{2g}) \twoheadrightarrow \mathfrak{g} \rightarrow 0 \tag{2.1}
\]

(ii) Consider the element $\rho := [a_1, b_1] \cdots [a_g, b_g] \in \mathbb{F}_{2g}$ and let $\bar{\rho} \in \mathbb{F}(V_{2g})$ be its image in the free Lie algebra. Then $\bar{\rho} \in \mathfrak{r}$ and moreover, $\mathfrak{r}$ is the Lie ideal generated by this element.

The following observations will be needed later.

(i) The middle and last groups in the sequence (2.1) are naturally $\text{Sp}_{g}$-modules, and moreover the maps are $\text{Sp}_{g}$-equivariant. Since the free Lie algebra $\mathbb{F}(V_{2g})$ can be expressed by tensor operations in terms of the module $V_{2g}$, it is semisimple as an $\text{Sp}_{g}$-module.

(ii) The previous remark implies that $\mathfrak{g}$ and $\mathfrak{r}$ are also semisimple $\text{Sp}_{g}$-modules and their components are derived from the module $V_{2g}$ by tensor operations.

(iii) For the same reasons as before, the exact sequence (2.1) splits as a sequence of $\text{Sp}_{g}$-modules.

The last observation yields the following result.
Proposition 2.1. Given the short exact sequence (2.1), we have an isomorphism of the associated graded of the universal enveloping algebras:

\[ \text{gr}^\bullet \mathcal{U}(\mathbb{F}(V_{2g})) \cong \text{gr}^\bullet \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{Z}} \text{gr}^\bullet \mathcal{U}(\mathfrak{r}) \]  

Moreover, this isomorphism is compatible with the Sp\(_g\)-structure on all the terms involved.

Proof. By the Poincaré-Birkhoff-Witt theorem (see Equation 1.2) the claim reduces to the identity

\[ \text{Sym}^\bullet (\mathbb{F}(V_{2g})) \cong \text{Sym}^\bullet (\mathfrak{g}) \otimes_{\mathbb{Z}} \text{Sym}^\bullet (\mathfrak{r}) \]  

Because the short exact sequence (2.1) splits as a sequence of Sp\(_g\)-modules, the above isomorphism is implied by the generally valid identity

\[ \text{Sym}^\bullet (A \oplus B) = \text{Sym}^\bullet (A) \otimes \text{Sym}^\bullet (B) \]  

Moreover, this last identity respects any extra gradings or Sp\(_g\)-module structures that \( A \) or \( B \) may have. \hfill \Box

The next task is to understand the Lie algebra \( \mathfrak{r} \), or rather its universal enveloping algebra \( \mathcal{U}(\mathfrak{r}) \). Consider the adjoint action of \( \mathfrak{g} \) on \( \mathfrak{r} / [\mathfrak{r}, \mathfrak{r}] \). It is well-defined and extends to give \( \mathfrak{r} / [\mathfrak{r}, \mathfrak{r}] \) the structure of a \( \mathcal{U}(\mathfrak{g}) \)-module.

The structure of \( \mathfrak{r} / [\mathfrak{r}, \mathfrak{r}] \) is given by Theorem 1 in [Lab67], which states the following.

Theorem 2.2 ([Lab67]). As a \( \mathcal{U}(\mathfrak{g}) \)-module, \( \mathfrak{r} / [\mathfrak{r}, \mathfrak{r}] \) is free on the generator \( \overline{p} \). Here, \( p = [a_1, b_1] \cdots [a_g, b_g] \in \mathbb{F}_{2g} \) and \( \overline{p} \) denotes its image in \( \mathfrak{r} \), and by abuse of notation in \( \mathfrak{r} / [\mathfrak{r}, \mathfrak{r}] \).

To proceed, note that \( \mathfrak{r} \) is a Lie subalgebra of a free Lie algebra, thus it is itself free (see [Sir53]). Later calculations require a basis on which \( \mathfrak{r} \) is free. This is provided by the next result, which follows from Proposition 2, Section 3 of [Lab67] (see also [Lab67, sec. 4]).

Proposition 2.3. Consider the map \( \mathfrak{r} \rightarrow \mathfrak{r} / [\mathfrak{r}, \mathfrak{r}] \), a map of graded Lie algebras and also Sp\(_g\)-modules. For each graded component of \( \mathfrak{r} / [\mathfrak{r}, \mathfrak{r}] \), take a lift to \( \mathfrak{r} \) of a homogeneous basis (respecting the Sp\(_g\)-module structure). Then \( \mathfrak{r} \) is a free Lie algebra on these lifts.

Remark 2.4. The results of Labute refer to a situation without an Sp\(_g\) action. However, that result combined with the semisimplicity of the Sp\(_g\)-module \( \mathfrak{r} \) and provided the lifts respect the decomposition, yields the statement of the proposition.
3 Identifying $\mathcal{U}(\mathfrak{g})$

The calculations in this section identify the Euler-Poincaré series of $\mathcal{U}(\mathfrak{g})$ as an $\text{Sp}_g$-module.

**Notation.** For a graded $\text{Sp}_g$-module $M$ with graded pieces $M_i$, i.e. such that $M = \oplus M_i$, define the formal sum

$$h(M; t) := \sum_i t^i \chi_i$$

Here $\chi_i : \text{Sp}_g(\mathbb{Z}) \to \mathbb{Z}$ is the character of the $\text{Sp}_g$-representation on $M_i$. The generating function $h(M; t)$ is viewed as a formal combination of functions on the group. Most often, the formal variable $t$ will be omitted from the expression for $M$.

**Proposition 3.1.** (i) If $A$, $B$ and $C$ are graded $\text{Sp}_g$-modules and $C \cong A \otimes B$ (as graded $\text{Sp}_g$-modules) then

$$h(C; t) = h(A; t) \cdot h(B; t)$$

(ii) Suppose $\mathfrak{h}$ is a free Lie algebra on a $\text{Sp}_g$-module $M$, with elements of $M$ in homogeneous degree 2. If $\mathcal{U}(\mathfrak{h})$ is its enveloping algebra (with the induced $\text{Sp}_g$-action) then

$$h(\mathcal{U}(\mathfrak{h}); t) = \frac{1}{1 - h(M; t)}$$

**Remark 3.2.** With the above notation, one has $h(M; t) = t^2 \cdot \chi_M$, where $\chi_M$ is the character of the $\text{Sp}_g$-module $M$.

**Proof.** The first part follows because characters multiply upon taking tensor products. For the second part, observe that $\mathcal{U}(\mathfrak{h})$ is the free tensor algebra on $M$, therefore

$$h(\mathcal{U}(\mathfrak{h}); t) = 1 + h(M; t) + h(M; t)^2 + \cdots$$

This is exactly the statement in the second part. \qed
Corollary 3.3. In the notation of Section 2, the Euler-Poincaré series of $\mathcal{U}(\mathfrak{r})$ is

$$h(\mathcal{U}(\mathfrak{r})) = \frac{1}{1 - h(\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}])} \quad (3.1)$$

Proof. This follows from the second part of the previous proposition, combined with Proposition 2.3. □

Proposition 3.4. With the notation from Section 2, the following formula holds

$$h(\mathcal{U}(\mathfrak{g}); t) = \frac{1}{1 - t \cdot \chi_V + t^2} \quad (3.2)$$

The character $\chi_V$ is that of the standard representation of $\text{Sp}_g(\mathbb{Z})$ on $V$, a $2g$-dimensional symplectic $\mathbb{Z}$-module.

Proof. Proposition 2.1 and the first part of Proposition 3.1 imply the identity

$$h(\mathcal{U}(\mathfrak{g}); t) = h(\mathcal{U}(\mathfrak{r}) \cdot h(\mathcal{U}(\mathfrak{g})) \quad (3.3)$$

The second part of Proposition 3.1 implies that

$$h(\mathcal{U}(\mathfrak{g}); t) = \frac{1}{1 - t \cdot \chi_V} \quad (3.4)$$

Because $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free $\mathcal{U}(\mathfrak{g})$-module (Theorem 2.2), with generator in homogeneous degree two, we have

$$h(\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]; t) = t^2 h(\mathcal{U}(\mathfrak{g})) \quad (3.5)$$

Combining equations (3.1), (3.4) and (3.5), we find the identity

$$\frac{1}{1 - t \cdot \chi_V} = \frac{h(\mathcal{U}(\mathfrak{g}))}{1 - t^2 h(\mathcal{U}(\mathfrak{g}))}$$

Rearranging, we find

$$h(\mathcal{U}(\mathfrak{g}); t) = \frac{1}{1 - t \cdot \chi_V + t^2}$$

This is exactly the desired expression. □
4 Computing the characters of the representation

Outline. Equation 3.2 gives an expression for the character of the universal enveloping algebra. This section contains the calculation of the character for the Lie algebra $\mathfrak{g}$ itself. The approach requires determining the character of the representation $\mathfrak{g}_i$ from knowing the character of $\text{Sym}^\bullet \mathfrak{g}_i$. This, in turn, is based on the Möbius inversion formula.

The First reduction. At the level of Euler-Poincaré series, $\mathcal{U}(\mathfrak{g})$ and $\text{gr}^\bullet \mathcal{U}(\mathfrak{g})$ coincide because the filtration is compatible with the $\text{Sp}_g$-module structure. Using the Poincaré-Birkhoff-Witt theorem, we find

$$\text{gr}^\bullet \mathcal{U}(\mathfrak{g}) = \text{Sym}^\bullet (\mathfrak{g}) = \bigotimes_{i \geq 1} \text{Sym}^\bullet (\mathfrak{g}_i)$$

The second equality above follows from formula 2.3. Reading this equality at the level of Euler-Poincaré series, combined with equation 3.2 and part (i) of Proposition 3.1, we find

$$1 \cdot \chi_{V} + t = \prod_{i \geq 1} h(\text{Sym}^\bullet (\mathfrak{g}_i); t)$$

Note that $\mathfrak{g}_i$ sits in homogeneous degree $i$, so the above product makes sense as a formal power series in $t$, and moreover both sides are of the form

$$1 + (\ldots)t + (\ldots)t^2 + \cdots$$

This means we can take formally a log of both power series to find

$$-\log(1 - t \cdot \chi_{V} + t^2) = \sum_{i \geq 1} \log h(\text{Sym}^\bullet (\mathfrak{g}_i); t)$$

(4.1)

The following result relates the character of a module and its symmetric powers.

**Proposition 4.1.** Let $\text{GL}_k(\mathbb{Z})$ act naturally on the free $\mathbb{Z}$-module $\mathbb{Z}^k =: M$. Assign homogeneous degree $i$ to the module $M$ and form the graded module $\text{Sym}^\bullet (M)$. Then the following equality of formal power series in $t$ holds:

$$\log h(\text{Sym}^\bullet (M); t) = \sum_{d \geq 1} \frac{\chi(d)}{d} \cdot t^{d}$$
The function $\chi^{(d)} : \text{GL}_k(\mathbb{Z}) \to \mathbb{Z}$ is defined by $\chi^{(d)}(A) = \chi(A^d)$.

Proof. The statement of the proposition is that of an equality between coefficients of two formal power series. These coefficients are polynomials with rational coefficients in the entries of matrices in $\text{GL}_k(\mathbb{Z})$. To check such an equality, it suffices to check it for the group $\text{GL}_k(\mathbb{C})$, and moreover the dependence in $t$ can be viewed as an analytic function in a small disk around the origin. To check this last identity, it suffices to check an equality between two analytic functions on $D \times \text{GL}_k(\mathbb{C})$ (where $D$ is a small disk). It now suffices to check this for diagonalizable matrices in $\text{GL}_k(\mathbb{C})$, since this set is dense.

Take one such diagonalizable $A$ and suppose its eigenvalues on $M \otimes_{\mathbb{Z}} \mathbb{C}$ are $\lambda_1, \ldots, \lambda_k$. Then the sum of the eigenvalues on $\text{Sym}^\bullet(M)$ are, after arranging them by homogeneity using the variable $t$, given by

$$\prod_{j=1}^{k} \frac{1}{1 - t^j \lambda_j}$$

Taking logs of the above, we see that the function $A \mapsto \log h(\text{Sym}^\bullet(M); t)(A)$ is given by

$$A \mapsto \sum_{n \geq 1} \sum_{d \geq 1} \frac{\lambda_1^d + \ldots + \lambda_k^d}{d} \cdot t^{i \cdot d}$$

On the right-hand side, the operator $A$ is first raised to the $d^{th}$ power and then trace is taken. So we find what we wanted:

$$\log h(\text{Sym}^\bullet(M); t) = \sum_{d \geq 1} \frac{\chi^{(d)}}{d} \cdot t^{i \cdot d}$$

Using the previous proposition, we can rewrite right-hand side of Equation 4.1 as follows

$$\sum_{i \geq 1} \log h(\text{Sym}^\bullet(\mathfrak{g}_i); t) = \sum_{i \geq 1} \sum_{d \geq 1} \frac{\chi_i^{(d)}}{d} \cdot t^{i \cdot d}$$

$$= \sum_{N \geq 1} t^N \sum_{d \cdot i = N} \frac{\chi_i^{(d)}}{d} \tag{4.2}$$

In this formula $\chi_i$ denotes the character of the $i^{th}$ graded piece $\mathfrak{g}_i$. 

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**The Second Reduction.** There are two relatively convenient ways to express the left hand side of Equation 4.1, and we do not want to pick out one yet. Let us therefore introduce the notation

\[- \log(1 - t\chi_V + t^2) = \sum_{N \geq 1} A_N t^N \quad (4.3)\]

With this notation, the next proposition gives an expression for the character \( \chi_N \).

**Proposition 4.2.** For the \( N^{th} \) graded piece \( g_N \), the formula for its character is

\[N \cdot \chi_N = \sum_{d | N} \mu \left( \frac{N}{d} \right) \cdot d \cdot A_{d}^{(\frac{N}{d})} \]

Here, \( \mu \) is the Möbius function and \( A_{d}^{(\frac{N}{d})} \) represents the function \( A_d \), precomposed with the operation of taking the \( \frac{N}{d} \) power of a matrix.

**Proof.** The expression for the right-hand side of Equation 4.1 provided by Equation 4.2 combined with the definition of \( A_n \) yields

\[A_n = \sum_{d | n} \frac{1}{d} \cdot \chi_{\frac{N}{d}}^{(d)} \]

Multiplying by \( n \), this can be rewritten as

\[n \cdot A_n = \sum_{d | n} n \cdot \frac{n}{d} \cdot \chi_{\frac{N}{d}}^{(d)} \quad (4.4)\]

To express a given character \( \chi_N \) using the above relation, take in the above some \( n | N \) and precompose the above identity with taking \( \frac{N}{n} \) powers and multiply by \( \mu(\frac{N}{n}) \) to find

\[n \cdot A_{\frac{N}{n}}^{(\frac{N}{n})} \cdot \mu \left( \frac{N}{n} \right) = \mu \left( \frac{N}{n} \right) \cdot \sum_{d | n} \frac{n}{d} \cdot \chi_{\frac{N}{d}}^{(d,n)}\]
Keep $N$ fixed and sum the above equation over all $n|N$ to find

$$\sum_{n|N} n A_n(n) \mu\left(\frac{N}{n}\right) = \sum_{n|N} \mu\left(\frac{N}{n}\right) \sum_{d|n} n \chi_{\frac{N}{d}}(d) \chi_{\frac{N}{d}}(d) =$$

$$= \sum_{x|N} \sum_{d|\frac{N}{x}} \mu\left(\frac{N}{xd}\right) x\chi_{\frac{N}{x}}(x) =$$

$$= \mu(1) N \chi_N^{(1)} = N \chi_N$$

In passing to the second line above, we changed summation to the variable $x := \frac{n}{d}$. In passing to the last line, we used the standard fact about the Möbius function that $\sum_{d|B} \mu(d) = 0$, unless $B = 1$. This finishes the proof of the proposition.

**Remark 4.3.** Recall that for two sequences $A_n, B_n$, their Dirichlet convolution $C_n = (A \ast B)_n$ is defined by

$$C_n := \sum_{d|n} A_d \cdot B_{\frac{n}{d}}$$

With this in mind, the computation above can be interpreted as follows. Equation 4.4 reads expresses $n A_n$ as a Dirichlet product

$$n A_n = n \chi_n \ast [\bullet]^{(n)}$$

where $[\bullet]^{(n)}$ is the operator taking a matrix to its $n^{th}$ power. Since we are interested in $n \chi_n$ we need the Dirichlet inverse of $[\bullet]^{(n)}$ and this is given by $\mu(n)[\bullet]^{(n)}$.

**The formulas.** The proposition below provides explicit formulas for both $A_N$ and $\chi_N$.

**Proposition 4.4.** Two (equivalent) expressions for $A_N$ are

$$A_N = \sum_{k \geq 0} \chi^N_{\frac{N-2k}{N-k}} \left(\frac{N-k}{k}\right) (-1)^k \left(\frac{N-k}{k}\right) (-1)^k \left(1 \right)^k$$

$$A_N = \frac{1}{N} \left[ \left(\chi_{\frac{\chi_2}{2}} + \chi_{\frac{\chi_2}{2}}^2 - 4 \right)^N + \left(\chi_{\frac{\chi_2}{2}} - \chi_{\frac{\chi_2}{2}}^2 - 4 \right)^N \right]$$
Combined with the formula given in Proposition 4.2, we get two (equivalent) formulas for the characters

\[ \chi_N = \frac{1}{N} \sum_{d|N} \mu \left( \frac{N}{d} \right) d \sum_{k \geq 0} \chi_N^{d-2k} \left( \frac{N}{d} \right) \frac{1}{d-k} \binom{d-k}{k} (-1)^k \]  

(4.7)

\[ \chi_N = \frac{1}{N} \sum_{d|N} \mu \left( \frac{N}{d} \right) \left[ \left( \frac{\chi_N^{(\frac{N}{d})}}{\chi_V} + \sqrt{\frac{\chi_V^{2} - 4}{2}} \right)^d + \left( \frac{\chi_N^{(\frac{N}{d})}}{\chi_V} - \sqrt{\frac{\chi_V^{2} - 4}{2}} \right)^d \right] \]  

(4.8)

In the last two equations, for a character \( \chi \), the expression \( \chi^{x,(y)} \) is defined by \( \chi^{x,(y)}(M) = \chi(M^y)^x \).

**Proof.** Only formulas 4.5 and 4.6 require proof, since they imply the other two directly. Recall the definition of \( A_N \) given by Equation 4.3

\[ \sum_{N \geq 1} A_N t^N = -\log(1 - t \chi_V + t^2) \]

The first formula for \( A_N \) follows from the following rewriting

\[ -\log(1 - t \chi_V + t^2) = -\log (1 - t(\chi_V - t)) = \sum_{N \geq 1} \frac{1}{N} t^N (\chi_V - t)^N \]

Expanding each individual term using the binomial formula, we arrive at the formula given by equation 4.5.

For the second formula, we can factor the quadratic \( 1 - t \chi_V + t^2 \) to find

\[ -\log(1 - t \chi_V + t^2) = -\log \left( \left( 1 - t \cdot \frac{\chi_V - \sqrt{\chi_V^2 - 4}}{2} \right) \cdot \left( 1 - t \cdot \frac{\chi_V + \sqrt{\chi_V^2 - 4}}{2} \right) \right) \]

So we have

\[ \sum_{N \geq 1} A_N t^N = -\log \left( \left( 1 - t \cdot \frac{\chi_V - \sqrt{\chi_V^2 - 4}}{2} \right) \right) - \log \left( 1 - t \cdot \frac{\chi_V + \sqrt{\chi_V^2 - 4}}{2} \right) = \]

\[ = \sum_{N \geq 1} \frac{t^N}{N} \left[ \left( \frac{\chi_V + \sqrt{\chi_V^2 - 4}}{2} \right)^N + \left( \frac{\chi_V - \sqrt{\chi_V^2 - 4}}{2} \right)^N \right] \]

This is exactly the second formula for \( A_N \).
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