An improved error bound for linear complementarity problems for $B$-matrices

Lei Gao and Chaoqian Li

Abstract

A new error bound for the linear complementarity problem when the matrix involved is a $B$-matrix is presented, which improves the corresponding result in (Li et al. in Electron. J. Linear Algebra 31(1):476-484, 2016). In addition some sufficient conditions such that the new bound is sharper than that in (García-Esnaola and Peña in Appl. Math. Lett. 22(7):1071-1075, 2009) are provided.

MSC: 90C33; 60G50; 65F35

Keywords: error bound; linear complementarity problem; $B$-matrix

1 Introduction

Given an $n \times n$ real matrix $M$ and $q \in \mathbb{R}^n$, the linear complementarity problem (LCP) is to find a vector $x \in \mathbb{R}^n$ satisfying

\[ x \geq 0, \quad Mx + q \geq 0, \quad (Mx + q)^T x = 0 \tag{1} \]

or to show that no such vector $x$ exists. We denote this problem (1) by LCP($M,q$). The LCP($M,q$) arises in many applications such as finding Nash equilibrium point of a bimatrix game, the network equilibrium problem, the contact problem and the free boundary problem for journal bearing etc.; for details, see [3–5].

It is well known that the LCP($M,q$) has a unique solution for any vector $q \in \mathbb{R}^n$ if and only if $M$ is a $P$-matrix [4]. Here a matrix $M$ is called a $P$-matrix if all its principal minors are positive. For the LCP($M,q$), one of the interesting problems is to estimate

\[ \max_{d \in [0,1]^n} \left\| (I - D + DM)^{-1} \right\|_\infty, \tag{2} \]

which can be used to bound the error $\|x - x^*\|_\infty$ [6], that is,

\[ \|x - x^*\|_\infty \leq \max_{d \in [0,1]^n} \left\| (I - D + DM)^{-1} \right\|_\infty \|r(x)\|_\infty, \]

where $x^*$ is the solution of the LCP($M,q$), $r(x) = \min\{x, Mx + q\}$, $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$ for each $i \in N$, $d = [d_1, d_2, \ldots, d_n]^T \in [0,1]^n$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors.
When the matrix $M$ for the LCP($M, q$) belongs to $P$-matrices or some subclass of $P$-matrices, various bounds for (2) were proposed; e.g., see [2, 6–15] and the references therein. Recently, García-Esnaola and Peña in [2] provided an upper bound for (2) when $M$ is a $B$-matrix as a subclass of $P$-matrices. Here, a matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ is called a $B$-matrix [16] if for each $i \in N = \{1, 2, \ldots, n\}$,

$$\sum_{k \in N} m_{ik} > 0, \quad \text{and} \quad \frac{1}{n} \left( \sum_{k \in N} m_{ik} \right) > m_{ij} \quad \text{for any} \ j \in N, \ j \neq i.$$

**Theorem 1** ([2], Theorem 2.2) *Let $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ be a B-matrix with the form*

$$M = B^* + C,$$

*where*

$$B^* = [b_{ij}] = \begin{bmatrix} m_{11} - r_1^* & \cdots & m_{1n} - r_1^* \\ \vdots & \ddots & \vdots \\ m_{n1} - r_n^* & \cdots & m_{nn} - r_n^* \end{bmatrix}, \quad C = \begin{bmatrix} r_1^* & \cdots & r_1^* \\ \vdots & \ddots & \vdots \\ r_n^* & \cdots & r_n^* \end{bmatrix}, \quad (4)$$

*and $r_i^* = \max\{0, m_{ij}[j \neq i]\}$. Then*

$$\max_{d \in [0,1]} \| (I - D + DM)^{-1} \|_\infty \leq \frac{n - 1}{\min\{\beta, 1\}}, \quad (5)$$

*where $\beta = \min_{i \in N} \{\beta_i\}$ and $\beta_i = b_{ii} - \sum_{j \neq i} |b_{ij}|$. It is not difficult to see that the bound (5) will be inaccurate when the matrix $M$ has very small value of $\min_{i \in N} \{b_{ii} - \sum_{j \neq i} |b_{ij}|\}$; for details, see [17, 18]. To conquer this problem, Li et al., in [1] gave the following bound for (2) when $M$ is a $B$-matrix, which improves those provided by Li and Li in [17, 18].*

**Theorem 2** ([1], Theorem 2.4) *Let $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ be a B-matrix with the form $M = B^* + C$, where $B^* = [b_{ij}]$ is the matrix of (4). Then*

$$\max_{d \in [0,1]} \| (I - D + DM)^{-1} \|_\infty \leq \frac{n - 1}{\min\{\beta, 1\}} \prod_{j=1}^{i-1} \frac{b_{ij}}{\bar{\beta}_j}, \quad (6)$$

*where $\bar{\beta}_i = b_{ii} - \sum_{j=i+1}^{n} |b_{ij}| l_k(B^*)$ with $l_k(B^*) = \max_{1 \leq k \leq n} \frac{1}{|b_{ii}|} \sum_{j \neq i} |b_{ij}|$, and $\prod_{j=1}^{i-1} \frac{b_{ij}}{\bar{\beta}_j} = 1$ if $i = 1$. In this paper, we further improve error bounds on the LCP($M, q$) when $M$ belongs to $B$-matrices. The rest of this paper is organized as follows: In Section 2 we present a new error bound for (2), and then prove that this bound is better than those in Theorems 1 and 2. In Section 3, some numerical examples are given to illustrate our theoretical results obtained.*
2 Main result

In this section, an upper bound for (2) is provided when $M$ is a $B$-matrix. Firstly, some definitions, notation and lemmas which will be used later are given as follows.

A matrix $A = [a_{ij}] \in C^{n \times n}$ is called a strictly diagonally dominant (SDD) matrix if $|a_{ii}| > \sum_{j \neq i}^{n} |a_{ij}|$ for all $i = 1, 2, \ldots, n$. A matrix $A = [a_{ij}] \in R^{n \times n}$ is called a nonsingular $M$-matrix if its inverse is nonnegative and all its off-diagonal entries are nonpositive [3]. In [16] it was proved that a $B$-matrix has positive diagonal elements, and a real matrix $A$ is a $B$-matrix if and only if it can be written in the form (3) with $B^*$ being a SDD matrix. Given a matrix $A = [a_{ij}] \in C^{n \times n}$, let

$$
|a_{ij}| + \sum_{k \neq i}^{n} |a_{ik}|w_k(A)
$$

$$
m_{ij}(A) = \frac{|a_{ij}| + \sum_{k \neq i}^{n} |a_{ik}|w_k(A)}{|a_{ii}|}, \quad i \neq j.
$$

Lemma 1 ([19], Theorem 14) Let $A = [a_{ij}]$ be an $n \times n$ row strictly diagonally dominant $M$-matrix. Then

$$\|A^{-1}\|_{\infty} \leq \sum_{i=1}^{n} \left( \frac{1}{a_{ii} - \sum_{j \neq i}^{n} |a_{ij}|m_{ij}(A)} \prod_{j=1}^{i-1} \frac{1}{1 - u_i(A)l_j(A)} \right),
$$

where $u_i(A) = \frac{1}{|a_{ii}|} \sum_{j \neq i}^{n} |a_{ij}|$, $l_i(A) = \max_{k \leq i} \left( \frac{1}{|a_{ii}|} \sum_{j \neq i}^{n} |a_{ij}| \right)$, $\prod_{j=1}^{i-1} \frac{1}{1 - u_i(A)l_j(A)} = 1$ if $i = 1$, and $m_{ij}(A)$ is defined as in (7).

Lemma 2 ([17], Lemma 3) Let $\gamma > 0$ and $\eta \geq 0$. Then, for any $x \in [0, 1]$,

$$
\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min[\gamma, 1]}
$$

and

$$
\frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}.
$$

Lemma 3 ([18], Lemma 5) Let $A = [a_{ij}]$ with $a_{ii} > \sum_{j \neq i}^{n} |a_{ij}|$ for each $i \in N$. Then, for any $x_i \in [0, 1]$,

$$
\frac{1 - x_i + a_{ii}x_i}{1 - x_i + a_{ii}x_i - \sum_{j \neq i}^{n} |a_{ij}|x_i} \leq \frac{a_{ii}}{a_{ii} - \sum_{j \neq i}^{n} |a_{ij}|}.
$$

Lemmas 2 and 3 will be used in the proofs of the following lemma and Theorem 3.

Lemma 4 Let $M = [m_{ij}] \in R^{n \times n}$ be a $B$-matrix with the form $M = B^* + C$, where $B^* = [b_{ij}]$ is the matrix of (4). And let $B_D^* = I - D + DB^* = [b_{ij}]$ where $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$. Then

$$
w_i(B_D^*) \leq \max_{i \neq j} \left( \frac{|b_{ij}|}{b_{ii} - \sum_{k \neq i}^{n} |b_{ik}|} \right).
$$
and

\[ m_{ij}(B_D^*) \leq v_{ij}(B^*) < 1, \]

where \( w_i(B_D^*) \), \( m_{ij}(B_D^*) \) are defined as in (7), and

\[ v_{ij}(B^*) = \frac{1}{b_{ii}} \left( |b_{ij}| + \sum_{k \neq i}^{n} \left( |b_{ik}| \cdot \max_{\ell \neq k} \left\{ \frac{|b_{\ell k}|}{b_{kk} - \sum_{l \neq k}^{n} |b_{kl}|} \right\} \right) \right). \]

Proof. Note that

\[ \left[ B_D^* \right]_{ij} = \tilde{b}_{ij} = \begin{cases} 1 - d_i + d_i b_{ij}, & \text{if } i = j, \\ d_i b_{ij}, & \text{if } i \neq j. \end{cases} \]

Since \( B^* \) is SDD, \( b_{ii} - \sum_{k \neq i}^{n} |b_{ik}| > |b_{ij}| \) for each \( i \neq j \). Hence, by Lemma 2 and (7), it follows that

\[ w_i(B_D^*) = \max_{j \neq i} \left\{ \tilde{b}_{ij} - \sum_{k \neq i}^{n} |b_{ik}| \right\} < 1. \tag{8} \]

Furthermore, it follows from (7), (8) and Lemma 2 that for each \( i \neq j \) (\( j < i \leq n \))

\[
\begin{align*}
m_{ij}(B_D^*) &= \frac{|b_{ij}| \cdot d_i + \sum_{k \neq j}^{n} |b_{ik}| \cdot d_i \cdot w_k(B_D^*)}{1 - d_i + b_{ii} \cdot d_i} \\
&\leq \frac{1}{b_{ii}} \cdot \left( |b_{ij}| + \sum_{k \neq j}^{n} |b_{ik}| \cdot w_k(B_D^*) \right) \\
&\leq \frac{1}{b_{ii}} \left( |b_{ij}| + \sum_{k \neq j}^{n} \left( |b_{ik}| \cdot \max_{\ell \neq k} \left\{ \frac{|b_{\ell k}|}{b_{kk} - \sum_{l \neq k}^{n} |b_{kl}|} \right\} \right) \right) \\
&= v_{ij}(B^*) \\
&< \frac{1}{b_{ii}} \left( |b_{ij}| + \sum_{k \neq j}^{n} |b_{ik}| \right) < 1.
\end{align*}
\]

The proof is completed. \( \square \)

By Lemmas 1, 2, 3 and 4, we give the following bound for (2) when \( M \) is a \( B \)-matrix.

**Theorem 3** Let \( M = [m_{ij}] \in R^{n \times n} \) be a \( B \)-matrix with the form \( M = B^* + C \), where \( B^* = [b_{ij}] \) is the matrix of (4). Then

\[
\max_{d \in [0,1]^n} \| (I - D + DM)^{-1} \|_\infty \leq \frac{n - 1}{\min\{\beta_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{ij}}{\beta_j}, \tag{9}
\]
where \( \beta_i = b_{ij} - \sum_{k=i+1}^{n} |b_{ik}| \cdot v_{ki}(B^*) \) with \( v_{ki}(B^*) \) is defined in Lemma 4, \( \beta_i \) is defined in Theorem 2, and \( \prod_{i=1}^{n} \frac{b_{ij}}{\beta_i} = 1 \) if \( i = 1 \).

**Proof.** Let \( M_D = I - D + DM \). Then

\[
M_D = I - D + DM = I - D + D(B^* + C) = B_D^* + C_D,
\]

where \( B_D^* = I - D + DB^* = \tilde{B}_D \) and \( C_D = DC \). Similarly to the proof of Theorem 2.2 in [2], we find that \( B_D^* \) is an SDDM-matrix with positive diagonal elements and that

\[
\|M_D^{-1}\|_{\infty} \leq \|(I + (B_D^*)^{-1}C_D)^{-1}\|_{\infty} \|(B_D^*)^{-1}\|_{\infty} \leq (n-1)\|(B_D^*)^{-1}\|_{\infty}.
\]

(10)

Next, we give an upper bound for \( \|(B_D^*)^{-1}\|_{\infty} \). By Lemma 1, we have

\[
\|(B_D^*)^{-1}\|_{\infty} \leq \sum_{i=1}^{n} \left( \frac{1}{1 - d_i + b_{ij}d_i} \right) \sum_{k=1}^{n} \frac{|b_{ik}| \cdot d_i \cdot m_{ki}(B_D^*)}{1 - u_i(B_D^*)l_i(B_D^*)},
\]

(11)

where

\[
u_i(B_D^*) = \sum_{j=1}^{n} \frac{|b_{ij}|d_j}{1 - d_j + b_{ij}d_j}, \quad l_i(B_D^*) = \max_{k \leq n} \left\{ \frac{\sum_{j=1}^{n} |b_{jk}|d_j}{1 - d_j + b_{ij}d_i} \right\},
\]

and

\[m_{ki}(B_D^*) = \frac{|b_{ik}| \cdot d_k + \sum_{j=1}^{n} |b_{ij}| \cdot d_k \cdot w_j(B_D^*)}{1 - d_k + b_{ik} \cdot d_k},\]

with \( w_j(B_D^*) = \max_{k \neq j} \left( \frac{|b_{ij}|d_j}{1 - d_j + b_{ij}d_i} \right) \).

By Lemmas 2 and 4, we can easily see that, for each \( i \in N \),

\[
\frac{1}{1 - d_i + b_{ij}d_i} - \sum_{k=1}^{n} |b_{ik}| \cdot d_i \cdot m_{ki}(B_D^*) \leq \frac{1}{\min[b_{ii} - \sum_{k=1}^{n} |b_{ik}| \cdot m_{ki}(B_D^*)]} \leq \frac{1}{\min[b_{ii} - \sum_{k=1}^{n} |b_{ik}| \cdot v_{ki}(B^*)]} = \frac{1}{\min[\beta_i]}.
\]

(12)

and that, for each \( k \in N \),

\[
l_i(B_D^*) = \max_{k \leq n} \left\{ \frac{\sum_{j=1}^{n} |b_{ij}|d_j}{1 - d_j + b_{ij}d_i} \right\} = \max_{k \leq n} \left\{ \frac{1}{\beta_i} \sum_{j=1}^{n} |b_{ij}| \right\} = l_i(B^*) < 1.
\]

(13)

Furthermore, according to Lemma 3 and (13), it follows that, for each \( j \in N \),

\[
\frac{1}{1 - u_j(B_D^*)l_j(B_D^*)} = \frac{1 - d_j + b_{ij}d_j}{1 - d_j + b_{ij}d_i - \sum_{k=j+1}^{n} |b_{ik}| \cdot d_j \cdot l_j(B_D^*)} \leq \frac{b_{ij}}{\beta_j},
\]

(14)
By (11), (12) and (14), we have
\[
\| (B'_i)^{-1} \|_\infty \leq \frac{1}{\min(\hat{\beta}_i, 1)} + \sum_{i=2}^{n} \left( \frac{1}{\min(\hat{\beta}_i, 1)} \prod_{j=1}^{i-1} \frac{b_{ij}^j}{\hat{\beta}_j} \right).
\] (15)

The conclusion follows from (10) and (15). □

The comparisons of the bounds in Theorems 2 and 3 are established as follows.

**Theorem 4** Let \( M = [m_{ij}] \in \mathbb{R}^{n \times n} \) be a \( B \)-matrix with the form \( M = B^* + C \), where \( B^* = [b_{ij}] \) is the matrix of (4). Let \( \hat{\beta}_i \) and \( \bar{\beta}_i \) be defined in Theorems 2 and 3, respectively. Then
\[
\sum_{i=1}^{n} \frac{n-1}{\min(\hat{\beta}_i, 1)} \prod_{j=1}^{i-1} \frac{b_{ij}}{\hat{\beta}_j} \leq \sum_{i=1}^{n} \frac{n-1}{\min(\bar{\beta}_i, 1)} \prod_{j=1}^{i-1} \frac{b_{ij}}{\bar{\beta}_j}.
\]

**Proof** Note that
\[
\hat{\beta}_i = b_{ii} - \sum_{j=i+1}^{n} |b_{ij}| l_i(B^*), \quad \bar{\beta}_i = b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| v_k(B^*),
\]
and \( B^* \) is a SDD matrix, it follows that for each \( i \neq j \) \((i < i \leq n)\)
\[
v_i(B^*) = \frac{1}{b_{ii}} \left( |b_{ij}| + \sum_{k=i+1}^{n} \left( |b_{ik}| \cdot \max_{k \neq k} \left\{ \frac{|b_{kk}|}{b_{kk} - \sum_{l=i+1}^{n} |b_{kl}|} \right\} \right) \right)
\leq \frac{1}{b_{ii}} \sum_{k=i+1}^{n} |b_{ik}|
\leq \max_{j \leq n} \left\{ \frac{1}{b_{ii}} \sum_{k=j}^{n} |b_{ik}| \right\} = l_i(B^*).
\]

Hence, for each \( i \in N \)
\[
\hat{\beta}_i = b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| v_k(B^*) > b_{ii} - \sum_{k=i+1}^{n} |b_{ik}| l_i(B^*) = \bar{\beta}_i,
\]
which implies that
\[
\frac{1}{\min(\hat{\beta}_i, 1)} \leq \frac{1}{\min(\bar{\beta}_i, 1)}.
\]

This completes the proof. □

Remark here that, when \( \hat{\beta}_i < 1 \) for all \( i \in N \), then
\[
\frac{1}{\min(\hat{\beta}_i, 1)} < \frac{1}{\min(\bar{\beta}_i, 1)}.
\]
which yields

\[ \sum_{i=1}^{n} \frac{n-1}{\min(\beta_i, 1)} \prod_{j=1}^{i-1} \frac{b_{ij}}{\beta_j} < \sum_{i=1}^{n} \frac{n-1}{\min(\beta_i, 1)} \prod_{j=1}^{i-1} \frac{b_{ij}}{\beta_j}. \]

Next it is proved that the bound (9) given in Theorem 3 can improve the bound (5) in Theorem 1 (Theorem 2.2 in [2]) in some cases.

**Theorem 5** Let \( M = [m_{ij}] \in \mathbb{R}^{n \times n} \) be a B-matrix with the form \( M = B^* + C \), where \( B^* = [b_{ij}] \) is the matrix of (4). Let \( \beta, \tilde{\beta}, \hat{\beta} \) be defined in Theorems 1, 2 and 3, respectively, and let \( \alpha = 1 + \sum_{i=1}^{n} \prod_{j=1}^{i-1} \frac{b_{ij}}{\beta_j} \) and \( \tilde{\beta} = \min_{i \in \mathbb{N}} \{ \tilde{\beta}_i \} \). If one of the following conditions holds:

(i) \( \hat{\beta} > 1 \) and \( \alpha < \frac{1}{\beta} \),
(ii) \( \hat{\beta} < 1 \) and \( \alpha \beta < \hat{\beta} \),

then

\[ \sum_{i=1}^{n} \frac{n-1}{\min(\beta_i, 1)} \prod_{j=1}^{i-1} \frac{b_{ij}}{\beta_j} < \frac{n-1}{\min(\beta, 1)}. \]

**Proof** When \( \hat{\beta} > 1 \) and \( \alpha < \frac{1}{\beta} \), we can easily get

\[ \sum_{i=1}^{n} \frac{n-1}{\min(\beta_i, 1)} \prod_{j=1}^{i-1} \frac{b_{ij}}{\beta_j} < \frac{n-1}{\min(\beta, 1)} \left( \sum_{i=1}^{n} \prod_{j=1}^{i-1} \frac{b_{ij}}{\beta_j} \right) = (n-1)\alpha < \frac{n-1}{\beta} \leq \frac{n-1}{\min(\beta, 1)}. \]

Similarly, for \( \hat{\beta} < 1 \) and \( \alpha \beta < \hat{\beta} \), the conclusion can be proved directly. \( \square \)

**3 Numerical examples**

Two examples are given to show that the bound in Theorem 3 is sharper than those in Theorems 1 and 2.

**Example 1** Consider the family of B-matrices in [17]:

\[ M_k = \begin{bmatrix} 1.5 & 0.5 & 0.4 & 0.5 \\ -0.1 & 1.7 & 0.7 & 0.6 \\ 0.8 & -0.1 \frac{k}{\pi} & 1.8 & 0.7 \\ 0 & 0.7 & 0.8 & 1.8 \end{bmatrix}, \]

where \( k \geq 1 \). Then \( M_k = B_k^* + C_k \), where

\[ B_k^* = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ -0.8 & 1 & 0 & -0.1 \\ 0 & -0.1 \frac{k}{\pi} & -0.8 & 1 & -0.1 \\ -0.8 & -0.1 & 0 & 1 \end{bmatrix}. \]

By computations, we have \( \beta = \frac{1}{10(k+1)}, \ \tilde{\beta}_1 = \tilde{\beta}_2 = \frac{902+91}{1000+100}, \ \hat{\beta}_3 = 0.99, \ \hat{\beta}_4 = 1, \ \beta_4 = 820k+828 \), \( \hat{\beta}_2 = 0.99, \ \hat{\beta}_3 = 1 \) and \( \hat{\beta}_4 = 1 \). Then it is easy to verify that \( M_k \) satisfies the condition (ii) of
Theorem 5. Hence, by Theorem 1 (Theorem 2.2 in [2]), we have

$$\max_{d \in [0,1]} \left\| (I - D + DM_k)^{-1} \right\|_\infty \leq \frac{4 - 1}{\min[\beta, 1]} = 30(k + 1).$$

It is obvious that

$$30(k + 1) \to +\infty, \quad \text{when} \quad k \to +\infty.$$ 

By Theorem 2, we find that, for any $k \geq 1$,

$$\max_{d \in [0,1]} \left\| (I - D + DM_k)^{-1} \right\|_\infty \leq 3 \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \cdot \frac{1}{\beta_1} + \frac{1}{\beta_1 \beta_2} + \frac{1}{\beta_1 \beta_2 \beta_3} \right)$$

$$= 3 \left( \frac{100k + 100}{90k + 91} + \frac{(100k + 100)^2}{(90k + 91)^2} + \frac{2(100k + 100)^2}{0.99(90k + 91)^2} \right) < 14.5193.$$ 

By Theorem 3, we find that, for any $k \geq 1$,

$$\max_{d \in [0,1]} \left\| (I - D + DM_k)^{-1} \right\|_\infty \leq 3 \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \cdot \frac{1}{\beta_1} + \frac{1}{\beta_1 \beta_2} + \frac{1}{\beta_1 \beta_2 \beta_3} \right)$$

$$= 3 \left( \frac{900k + 900}{820k + 828} + \frac{(100k + 100)}{0.99(90k + 91)} + \frac{1.99(100k + 100)^2}{0.99(90k + 91)^2} \right)$$

$$< 3 \left( \frac{100k + 100}{90k + 91} + \frac{(100k + 100)^2}{(90k + 91)^2} + \frac{2(100k + 100)^2}{0.99(90k + 91)^2} \right).$$

In particular, when $k = 1$,

$$3 \left( \frac{900k + 900}{820k + 828} + \frac{(100k + 100)}{0.99(90k + 91)} + \frac{1.99(100k + 100)^2}{0.99(90k + 91)^2} \right) \approx 13.9878,$$

$$3 \left( \frac{100k + 100}{90k + 91} + \frac{(100k + 100)^2}{(90k + 91)^2} + \frac{2(100k + 100)^2}{0.99(90k + 91)^2} \right) \approx 14.3775,$$

and the bound (5) in Theorem 1 is

$$\frac{4 - 1}{\min[\beta, 1]} = 30(k + 1) = 60.$$ 

When $k = 2$,

$$3 \left( \frac{900k + 900}{820k + 828} + \frac{(100k + 100)}{0.99(90k + 91)} + \frac{1.99(100k + 100)^2}{0.99(90k + 91)^2} \right) \approx 14.0265,$$

$$3 \left( \frac{100k + 100}{90k + 91} + \frac{(100k + 100)^2}{(90k + 91)^2} + \frac{2(100k + 100)^2}{0.99(90k + 91)^2} \right) \approx 14.4246.$$
and the bound (5) in Theorem 1 is
\[
\frac{4 - 1}{\min\{\beta, 1\}} = 30(k + 1) = 90.
\]

**Example 2** Consider the following family of \(B\)-matrices:
\[
M_k = \begin{bmatrix}
\frac{1}{k} & \frac{a}{k} \\
0 & \frac{1}{k}
\end{bmatrix},
\]
where \(\sqrt{\frac{5-1}{2}} < a < 1\) and \(\frac{2-a^2}{1+\alpha} < k < 1\). Then \(M_k = B_k + C\) with \(C\) is the null matrix.

By simple computations, we can get
\[
\beta = 1 - \frac{a}{k}, \quad \bar{\beta}_1 = \frac{1 - a^2}{k}, \quad \bar{\beta}_2 = \frac{1}{k}, \quad \hat{\beta}_1 = \frac{1}{k} \quad \text{and} \quad \hat{\beta}_2 = \frac{1}{k}.
\]

It is not difficult to verify that \(M_k\) satisfies the condition (i) of Theorem 5. Thus, the bound (6) of Theorem 2 (Theorem 2.4 in [1]) is
\[
\sum_{i=1}^{2} \frac{2 - 1}{\min\{\beta_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\beta_j} = k + 1 \leq \frac{2-a^2}{1-a^2},
\]
which is larger than the bound
\[
\frac{1}{\min\{\beta, 1\}} = \frac{k}{1 - a}
\]
given by (5) in Theorem 1 (Theorem 2.2 in [2]). However, by Theorem 3 we can get
\[
\max_{d \in [0,1]^2} \| (I - D + DM_k)^{-1} \|_{\infty} \leq \frac{2 - a^2}{1 - a^2},
\]
which is smaller than the bound (5) in Theorem 1, i.e.,
\[
\frac{2 - a^2}{1 - a^2} < \frac{k}{1 - a}.
\]

In particular, when \(a = \frac{4}{5}\) and \(k = \frac{8}{9}\), the bounds in Theorems 1 and 2 are, respectively,
\[
\frac{1}{\min\{\beta, 1\}} = \frac{k}{1 - a} = \frac{360}{81}
\]
and
\[
\sum_{i=1}^{2} \frac{2 - 1}{\min\{\beta_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\beta_j} = k + 1 \leq \frac{425}{81},
\]
while the bound (9) in Theorem 3 is
\[
\sum_{i=1}^{2} \frac{2 - 1}{\min\{\beta_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\beta_j} = \frac{2-a^2}{1-a^2} = \frac{306}{81}.
\]
These two examples show that the bound in Theorem 3 is sharper than those in Theorems 1 and 2.

4 Conclusions

In this paper, we give a new error bound for the linear complementarity problem when the matrix involved is a $B$-matrix, which improves those bounds obtained in [2] and [1]. Numerical examples are given to illustrate the corresponding results.

Acknowledgements

This work is partly supported by National Natural Science Foundations of China (11601473, 31600299), Young Talent fund of University Association for Science and Technology in Shaanxi, China (20160234), the Research Foundation of Baoji University of Arts and Sciences (ZK2017021), and CAS ‘Light of West China’ Program.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details

1 School of Mathematics and Information Science, Baoji University of Arts and Sciences, Baoji, Shannxi 721013, P.R. China.
2 School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, P.R. China.

Publisher’s Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 March 2017 Accepted: 2 June 2017 Published online: 20 June 2017

References

1. Li, CQ, Gan, MT, Yang, SR: A new error bound for linear complementarity problems for $B$-matrices. Electron. J. Linear Algebra 31(1), 476-484 (2016)
2. García-Esnaola, M, Peña, JM: Error bounds for linear complementarity problems for $B$-matrices. Appl. Math. Lett. 22(7), 1071-1075 (2009)
3. Berman, A, Plemmons, RJ: Nonnegative Matrix in the Mathematical Sciences. SIAM, Philadelphia (1994)
4. Cottle, RW, Pang, JS, Stone, RE: The Linear Complementarity Problem. Academic Press, San Diego (1992)
5. Murty, KG: Linear Complementarity, Linear and Nonlinear Programming. Heldermann, Berlin (1988)
6. Chen, XJ, Xiang, SH: Perturbation bounds of $P$-matrix linear complementarity problems. SIAM J. Optim. 18(4), 1250-1265 (2007)
7. Chen, TT, Li, W, Wu, X, Vong, S: Error bounds for linear complementarity problems of $M$-matrices. Numer. Algorithms 70(2), 341-356 (2015)
8. Chen, XJ, Xiang, SH: Computation of error bounds for $P$-matrix linear complementarity problems. Math. Program. 106(3), 513-525 (2006)
9. Dai, PF: Error bounds for linear complementarity problems of $D$-$B$-matrices. Linear Algebra Appl. 434(3), 830-840 (2011)
10. Dai, PF, Li, YT, Lu, CJ: Error bounds for linear complementarity problems for $S$-$B$-matrices. Numer. Algorithms 61(1), 121-139 (2012)
11. Dai, PF, Lu, CJ, Li, YT: New error bounds for the linear complementarity problem with an $S$-$B$-matrix. Numer. Algorithms 64(4), 741-757 (2013)
12. García-Esnaola, M, Peña, JM: Error bounds for linear complementarity problems involving $B^\infty$-matrices. Appl. Math. Lett. 25(10), 1379-1383 (2012)
13. García-Esnaola, M, Peña, JM: Error bounds for linear complementarity problem with a $\Sigma$-$S$-$B$-$D$ matrix. Linear Algebra Appl. 438(3), 1339-1346 (2013)
14. García-Esnaola, M, Peña, JM: $B$-Nekrasov matrices and error bounds for linear complementarity problems. Numer. Algorithms 72(2), 435-445 (2016)
15. Li, CQ, Dai, PF, Li, YT: New error bounds for linear complementarity problems of Nekrasov matrices and $B$-Nekrasov matrices. Numer. Algorithms 74(4), 997-1009 (2017)
16. Peña, JM: A class of $P$-matrices with applications to the localization of the eigenvalues of a real matrix. SIAM J. Matrix Anal. Appl. 22(4), 1027-1037 (2001)
17. Li, CQ, Li, YT: Note on error bounds for linear complementarity problems for $B$-matrices. Appl. Math. Lett. 57, 108-113 (2016)
18. Li, CQ, Li, YT: Weakly chained diagonally dominant $B$-matrices and error bounds for linear complementarity problems. Numer. Algorithms 73(4), 985-998 (2016)
19. Yang, Z, Zheng, B, Lian, X: A new upper bound for $\|A^{-1}\|_\infty$ of a strictly $\alpha$-diagonally dominant $M$-matrix. Adv. Numer. Anal. 2013, 980615 (2013)