On the Sample Complexity of Representation Learning in Multi-Task Bandits with Global and Local Structure

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Abstract
We investigate the sample complexity of learning the optimal arm for multi-task bandit problems. Arms consist of two components: one that is shared across tasks (that we call representation) and one that is task-specific (that we call predictor). The objective is to learn the optimal (representation, predictor)-pair for each task, under the assumption that the optimal representation is common to all tasks. Within this framework, efficient learning algorithms should transfer knowledge across tasks. We consider the best-arm identification problem with fixed confidence, where, in each round, the learner actively selects both a task, and an arm, and observes the corresponding reward. We derive instance-specific sample complexity lower bounds, which apply to any algorithm that identifies the best representation, and the best predictor for a task, with prescribed confidence levels. We devise an algorithm, OSRL-SC, that can learn the optimal representation, and the optimal predictors, separately, and whose sample complexity approaches the lower bound. Theoretical and numerical results demonstrate that OSRL-SC achieves a better scaling with respect to the number of tasks compared to the classical best-arm identification algorithm. The code can be found here https://github.com/rssalesio/OSRL-SC.

Introduction
Learning from previous tasks and transferring this knowledge may significantly improve the process of learning new tasks. This idea, at the core of transfer learning (Pan and Yang 2009; Skinner 1965; Woodworth and Thorndike 1901), lifelong learning (Thrun 1996) and multi-task learning (Baxter et al. 2000; Caruana 1995, 1997), has recently triggered considerable research efforts with applications in both supervised and reinforcement learning. Previous work on transfer and multi-task learning has mostly focused on batch learning problems (Lazaric 2012; Pan and Yang 2009), where when a task needs to be solved, a training dataset is directly provided. Online learning problems, where samples for a given task are presented to the learner sequentially, have been much less studied (Taylor and Stone 2009; Zhan and Taylor 2015).

In this paper, we consider a multi-task Multi-Armed Bandit (MAB) problem, where the objective is to find the optimal arm for each task using the fewest number of samples, while allowing to transfer knowledge across tasks. The problem is modelled as follows: in each round, the learner actively selects a task, and then selects an arm from a finite set of arms. Upon selecting an arm, the learner observes a random reward from an unknown distribution that represents the performance of her action in that particular task. To allow the transfer of knowledge across the various tasks, we study the problem for a simple, albeit useful model. We assume that the arms available to the learner consist of two components: one that is shared across tasks (that we call representation) and one that is task-specific (that we call predictor). Importantly, the optimal arms for the various tasks share the same representation. The benefit of using this model is that we can study the sample complexity of learning the best shared representation across tasks while learning the task-specific best action.

Contribution-wise, in this work we derive instance-specific sample complexity lower bounds satisfied by any $(\delta_G, \delta_H)$-PAC algorithm (such an algorithm identifies the best representation with probability at least $1 - \delta_G$, and the best predictors with probability at least $1 - \delta_H$). Again, our lower bounds can be decomposed into two components, one for learning the representation, and one for learning the predictors. We devise an algorithm, OSRL-SC, which can learn the optimal representation, and predictors, separately, and whose sample complexity approaches the lower bound, scaling at most as $H(G \log(1/\delta_G) + X \log(1/\delta_H))$. Technically, we also show a novel regularization technique to obtain unique allocations in best-arm identification problems with fixed confidence. Finally, we present numerical experiments to illustrate the gains in terms of sample complexity one may achieve by transferring knowledge across tasks. To the best of our knowledge, this paper is the first to study how tasks should be scheduled toward a sample-optimal instance-specific best-arm identification algorithm.

Related work. Multi-task learning has been investigated under different assumptions on the way the learner interacts with tasks. One setting concerns batch learning (often referred to as learning-to-learn), where the training datasets for all tasks are available at the beginning (Baxter et al. 2000;
Maurer 2005, 2009; Maurer, Pontil, and Romera-Paredes 2013). The so-called batch-within-online setting considers that tasks arrive sequentially, but as soon as a task arrives, all its training samples are available (Balcan, Blum, and Vempala 2015; Pentina and Ben-David 2015; Pentina and Urner 2016; Alquier, Mai, and Pontil 2017).

Next, in the online multi-task learning (Agarwal, Rakhlin, and Bartlett 2008; Abernethy, Bartlett, and Rakhlin 2007; Dekel, Long, and Singer 2007; Cavallanti, Cesa-Bianchi, and Gentile 2010; Saha et al. 2011; Lugosi, Papaspiropoulos, and Stoltz 2009; Murugesan et al. 2016; Yang et al. 2020), in each round, the learner observes a new sample for each task, which, in some cases, this may not be possible. Our framework is different as in each round the learner can only select a single task. This framework has also been considered in (Lazaric, Brunskill et al. 2013; Soare et al. 2014; Soare 2015; Alquier, Mai, and Pontil 2017; Wu, Wang, and Lu 2019), but typically there, the learner faces the same task for numerous consecutive rounds, and she cannot select the sequence of tasks. Also, note that the structure tying the reward functions of the various tasks together is different from ours. The structure tying the rewards of actions for various contexts is typically linear, and it is commonly assumed that there exists a common low-dimensional representation, or latent structure, to be exploited (Soare et al. 2014; Soare 2015; Deshmukh, Dogan, and Scott 2017; Kveton et al. 2017; Hao, Lattimore, and Szepesvari 2020; Lale et al. 2019; Yang et al. 2020; Lu, Meisami, and Tewari 2021), or that the reward is smooth across tasks and/or arms (Magureanu, Combes, and Proutière 2014; Slivkins 2014). The aforementioned papers address scenarios where the context sequence is not controlled and investigate regret. Meta-learning is also closely connected to meta-learning (Cella, Lazaric, and Pontil 2020; Kveton et al. 2021; Azizi et al. 2022). In (Azizi et al. 2022) the authors investigate the problem of simple regret minimization in a fixed horizon setting when tasks are sampled i.i.d. from some unknown distribution.

**Model and Assumptions**

In this section, we describe the analytical model of the multi-task MAB problem considered, and describe the framework of best-arm identification for this class of multi-task MAB models.

**Model.** We consider multi-task MAB problems with a finite set \( \mathcal{X} \) of \( X \) tasks. In each round \( t \geq 1 \), the learner chooses a task \( x \in \mathcal{X} \) and an arm \( (g, h) \in \mathcal{G} \times \mathcal{H} \). The components \( g \) and \( h \) are, respectively, referred to as the representation and the predictor. When in round \( t \), \( x(t) = x \) and the learner selects \( (g, h) \), she collects a binary reward \( Z_t(x, g, h) \) of mean \( \mu(x, g, h) \) (for the sake of the analysis we only analyze the binary case, although it can be extended to the Gaussian case as in (Garivier and Kaufmann 2016)). The rewards are i.i.d. across rounds, tasks, and arms. Consequently, the system is characterized by \( \mu = (\mu(x, g, h))_{x, g, h} \), which is unknown to the learner.

The main assumption made throughout the paper is that tasks share a common optimal representation \( g^* \): for any task \( x \in \mathcal{X} \), there is a predictor \( h^*_x \) such that \( (g^*, h^*_x) \) yields an optimal reward. Formally,

\[
\forall x \in \mathcal{X}, \quad \mu(x, g^*, h^*_x) > \max_{(g, h) \neq (g^*, h^*_x)} \mu(x, g, h). \tag{1}
\]

Moreover, note that there is no assumption on the smoothness of \( \mu \) with respect to \( (x, g, h) \).

This type of model represents the case where a learner can actively choose the task to execute (as if a generative model is available to the learner), and in this way maximize the learning efficiency by accurately picking tasks to reduce the sample complexity. Since the model is quite generic, it can be applied to a variety of problems where a collection of tasks have a local component, and a shared global component: (i) influence mechanisms with global/local groups; (ii) hyperparameters learning across multiple tasks; (iii) for advanced clinical trials, where, depending on the group of patients (tasks, that vary according to factors such as age, severity of the disease, etc.), different drugs and dosages can be used for inoculation \( (g, h) \).

**Sample complexity.** The objective of the learner is to devise a policy that learns the best arms \( (g^*, h^*_1, \ldots, h^*_\mathcal{X}) \) with the least number of samples. Here, a policy \( \pi \) is defined as follows. Let \( \mathcal{F}_t^\pi \) denote the \( \sigma \)-algebra generated by the observations made under \( \pi \) up to and including round \( t \). Then \( \pi \) consists of (i) a sampling rule: in each round \( t \), \( \pi \) selects a \( \mathcal{F}_{t-1} \)-measurable task \( x^\pi(t) \) and an arm \( (g^\pi(t), h^\pi(t)) \); (ii) a stopping rule defined by \( \tau \), which is a stopping time w.r.t. the filtration \( (\mathcal{F}_t)_{t \geq 1} \); (iii) a decision rule returning a \( \mathcal{F}_\tau \)-measurable estimated best arm for each task \( (g, h_1, \ldots, h_\mathcal{X}) \). Then, the performance of a policy \( \pi \) is assessed through its PAC guarantees and its expected sample complexity \( \mathbb{E}[\tau] \). PAC guarantees concern both learning \( g^* \) and \( (h^*_1, \ldots, h^*_\mathcal{X}) \). Denote by \( \mathcal{M} = \{\mu : \mathbb{E}(g^*, h^*_1, \ldots, h^*_\mathcal{X}) : \text{Eq. (1) holds}\} \) the set of systems where tasks share a common optimal representation. Then, we say
that \( \pi \) is \((\delta_G, \delta_H)\)-PAC if for all \( \mu \in \mathcal{M} \),
\[
\Pr_{\mu}(\tau < \infty) = 1, \quad \Pr_{\mu}(\hat{g} \neq g^*) \leq \delta_G; \quad \text{and} \quad (2)
\]
\[
\Pr_{\mu}(\hat{h}_x \neq h_x^*, \hat{g} = g^*) \leq \delta_H, \quad \forall x \in \mathcal{X}. \quad (3)
\]

Sample Complexity Lower Bound and the OSRL-SC Algorithm

This section is devoted to the best-arm identification problem for the model considered in this work. We first derive a lower bound for the expected sample complexity of any \((\delta_G, \delta_H)\)-PAC algorithm, and then present an algorithm approaching this limit. In what follows, we let \( \delta = (\delta_G, \delta_H) \).

Sample Complexity Lower Bound

We begin by illustrating the intuition behind the sample complexity lower bound, and then state the lower bound theorem. To identify the optimal representation \( g^* \) in a task as quickly as possible, an algorithm should be able to perform some sort of information refactoring, i.e., be able to use all the available information across tasks to estimate \( g^* \).

To illustrate this concept, we use the model illustrated in Fig. 1. In this model, there are only 2 tasks \( x_1, x_2 \), and 3 arms in \( \mathcal{G} \) (and only 1 arm in \( \mathcal{H} \), thus it can be ignored). For this model, to learn that \( g_3 \) is suboptimal, we should mainly sample task \( x_1 \), since the gap between the rewards of \( g^* \) and \( g_3 \) is the largest. Similarly, to learn that \( g_2 \) is suboptimal, we should mainly choose task \( x_2 \). Using the same task, to infer that \( g_2 \) and \( g_3 \) are suboptimal, would be less efficient. This observation also motivates why it is inefficient to consider tasks separately, even in the case where \( \mu \) is highly non-smooth with respect to \( (x, g, h) \), and also motivates the expression of the sample complexity lower bound that we now present.

Sample complexity lower bound. Computing the sample complexity lower bound amounts to finding the lower bound of a statistical hypothesis testing problem, which is usually done by finding what is the most confusing model. In this case, the lower bound is given by the solution of the following optimization problem.

**Theorem 1.** The sample complexity \( \tau_\delta \) of any \( \delta \)-PAC algorithm satisfies: \( \mathbb{E}_\mu[\tau_\delta] \geq K^*(\mu, \delta) \) for any \( \mu \in \mathcal{M} \), where \( K^*(\mu, \delta) \) is the value of the optimization problem\( ^{2} \):
\[
\min_{\eta} \sum_{x, g, h} \eta(x, g, h) \quad (4)
\]
\[
\text{s.t.} \quad \min_{h \neq h_x^*} f_\mu(\eta, x, h) \geq \text{kl}(\delta_H, 1 - \delta_H) \quad \forall x, \quad (5)
\]
\[
\min_{g \neq g^*} \sum_{h_x} \min_{\hat{h}_x} \eta(\hat{g}, \hat{h}_x) \geq \text{kl}(\delta_G, 1 - \delta_G), \quad (6)
\]

where \( \text{kl}(a, b) \) is the KL divergence between two Bernoulli distributions of respective means \( a \) and \( b \).

In the first constraint, \( f_\mu(\eta, x, h) = (\eta(x, g^*, h_x^*) + \eta(x, g^*, h) )I_{x, g^*, h_x^*}(\mu(x, g^*, h_x^*) \mu(x, g^*, h)) \) accounts for the difficulty of learning the best predictor \( h_x^* \) for each task \( x \). The term \( \alpha_{x, g, h} = \eta(x, g^*, h_x^*)/\eta(x, g^*, h_x^*) + \eta(x, g, h) \) represents the proportion of time \((x, g^*, h_x^*)\) is picked over \((x, g, h)\), while \( L_\alpha(\mu_1, \mu_2) \), \( \alpha \in [0, 1] \), is a generalization of the Jensen-Shannon divergence, defined as
\[
L_\alpha(\mu_1, \mu_2) := \text{okl}(\mu_1, d_\alpha(\mu_1, \mu_2)) + (1 - \alpha)\text{kl}(\mu_2, d_\alpha(\mu_1, \mu_2)),
\]
with \( d_\alpha(\mu_1, \mu_2) := \alpha \mu_1 + (1 - \alpha) \mu_2 \).

**In the second constraint,** \( \ell_\mu(\eta, \hat{g}, \hat{h}_x) \) accounts for the difficulty of learning the optimal \( g^* \). To define it, let the average reward over some subset of arms \( C \subseteq \mathcal{G} \times \mathcal{H} \) for a task \( x \) and allocation \( \eta \) be defined as
\[
m(x, \eta, C) := \sum_{(g, h) \in C} \eta(x, g, h) \mu(x, g, h),
\]

Then, \( \ell_\mu(\eta, \hat{g}, \hat{h}_x) \) is defined as:
\[
\ell_\mu(\eta, \hat{g}, \hat{h}_x) := \sum_{(g, h) \in C} \eta(x, g, h) \text{kl}(\mu(x, g, h), m(x, \eta, U^\mu_{x, \hat{g}, \hat{h}_x})),
\]
where \( U^\mu_{x, \hat{g}, \hat{h}_x} \) is the set of confusing arms for a task \( x \) and \( m(x, \eta, U^\mu_{x, \hat{g}, \hat{h}_x}) \) represents the average reward of the confusing model (when \( (\hat{g}, \hat{h}_x) \) is optimal for task \( x \) in the confusing model).

Confusing arms. The set \( U^\mu_{x, \hat{g}, \hat{h}_x} \) is defined through \( N^\mu_{x, g, h, \hat{g}, \hat{h}_x} \), which is the set of arms whose mean is bigger than \( \mu(x, g, h) \) and that also include \((\hat{g}, \hat{h}_x)\), which is
\[
N^\mu_{x, g, h, \hat{g}, \hat{h}_x} := \{(g', h') : \mu(x, g', h') \geq \mu(x, g, h) \} \cup \{(\hat{g}, \hat{h}_x)\}.
\]

Then, the set of confusing arms \( U^\mu_{x, \hat{g}, \hat{h}_x} \) is defined as:
\[
U^\mu_{x, \hat{g}, \hat{h}_x} = \{ (g, h) : \mu(x, g, h) \geq m(x, \eta, N^\mu_{x, g, h, \hat{g}, \hat{h}_x}) \} \cup \{(\hat{g}, \hat{h}_x)\}.
\]

The set \( U^\mu_{x, \hat{g}, \hat{h}_x} \) can be computed recursively. We start with a set that only contains \((g^*, h_x^*)\) and \((\hat{g}, \hat{h}_x)\). We compute the corresponding value of \( m(x, \eta, U^\mu_{x, \hat{g}, \hat{h}_x}) \), and we add to the set \( U^\mu_{x, \hat{g}, \hat{h}_x} \) the arm \((g, h)\) with the highest mean not already in the set. We iterate until convergence. Fig. 2 provides an illustration of the set \( U^\mu_{x, \hat{g}, \hat{h}_x} \).

We now provide the proof of Theorem 1.

**Proof of Theorem 1.** The proof relies on classical change-of-measure arguments, as those used in the classical MAB (Kaufmann, Cappé, and Garivier 2016). To simplify the notation, let \( \tau = \tau_\delta \), and further let \( \mathbb{E}_\mu[N_{\tau}(x, g, h)] = \mathbb{E}_{\mu}[N_{\tau}(x, g, h)] \) at the stopping time \( \tau \), thus \( \mathbb{E}_{\mu}[\tau] = \sum_{x, g, h} \eta(x, g, h) \). For any model \( \mu \in \mathcal{M} \) we denote the optimal representation of \( \mu \) by \( g^*(\mu) \) and the optimal set of predictors (associated to \( g^*(\mu) \)) by \( h^*_x(\mu) = (h^*_1, \ldots, h^*_N)(\mu) \).

Whenever possible, we write \( g^* = g^*(\mu) \) (similarly for \( h^*_x = h^*_x(\mu) \), \( \forall x \in \mathcal{X} \)).

We define the set of confusing problems as
\[
\Lambda(\mu) := \{ \lambda \in \mathcal{M} : (g^1, h^1_x, \ldots, h^N_x)(\lambda) \neq (g^*, h^*_1, \ldots, h^*_N)(\mu) \}.
\]
We split the analysis by considering two subsets of $\Lambda$, defined as follows:

$$
\Lambda^1_{x} := \{ \lambda \in \Lambda : g^*(\lambda) = g^*(\mu) \},
\Lambda^2_{x} := \{ \lambda \in \Lambda : g^*(\lambda) \neq g^*(\mu) \}.
$$

We now focus on the first subset $\Lambda^1_{x}$, from which we derive the first constraint (5) of Theorem 1. Then, we focus on the second subset $\Lambda^2_{x}$, from which follows the second constraint (6).

First constraint (5). We restrict our attention to $\Lambda^1_{x}$. Define the set of confusing problems for task $x \in X$ as

$$
\Lambda^1_{x}(x) := \{ \lambda \in \Lambda^1_{x} : h^*_x(\lambda) = h^*_y(\mu), \forall y \in X \setminus \{x\} \}.
$$

Now, consider a $(\delta_G, \delta_H)$-PAC algorithm, and for a specific task $x \in X$ define the event $\mathcal{E} = \{ \hat{h}_x \neq h^*_x, \hat{g} = g^* \}$, where $\hat{h}_x$ and $\hat{g}$ denote respectively the estimated predictor for task $x$ and the estimated optimal representation at the stopping time $\tau$. Let then $\lambda \in \Lambda^1_{x}(x)$, be an alternative bandit model: the expected log-likelihood ratio $L_x$ at the stopping time $\tau$ of the observations under the two models $\mu$ and $\lambda$ is given by

$$
E_{\mu}[L_x] = \sum_{(y,g,h) \in X \times G \times H} \eta(y,g,h) kl_{\mu | \lambda}(y,g,h),
$$

and in view of the transportation Lemma 1 in (Kaufmann, Cappé, and Garivier 2016) and the definition of $(\delta_G, \delta_H)$-PAC algorithm, we can lower bound the previous quantity at the stopping time $\tau$:

$$
E_{\mu}[L_x] \geq \text{kl}(P_\mu(\mathcal{E}), P_\lambda(\mathcal{E})) = \text{kl}(\delta_G, 1 - \delta_G).
$$

We can get a tight lower bound by considering the worst possible model $\lambda$. To do so, first introduce the set $\Lambda^1_{x}(x, \hat{h}) := \{ \lambda \in \Lambda^1_{x}(x) : \lambda(x, g^*, h^*_x) > \lambda(x, g^*, \hat{h}) \}$, which is the set of confusing problems where the predictor $h^*_x$ is not optimal in task $x$. Observe that one can write $\Lambda^1_{x}(x) = \cup_{h \neq h^*_x} \Lambda^1_{x}(x, h)$. This rewriting allows us to derive the first constraint as follows:

$$
\text{kl}(\delta_H, 1 - \delta_H) \leq \inf_{\lambda \in \Lambda^1_{x}(x)} \sum_{y,g,h} \eta(y,g,h) kl_{\mu | \lambda}(y,g,h),
$$

where (a) follows from the fact that in $\lambda$ we are changing the predictor of only one task $x$.

Then, the last term is equal to $\min_{h \neq h^*_x} \inf_{\lambda \in \Lambda^1_{x}(x)} \{ \eta(x,g^*,h^*_x) kl_{\mu | \lambda}(x,g^*,h^*_x) + \eta(x,g^*,h) kl_{\mu | \lambda}(x,g^*,h) \}$.

Second constraint (6). Similarly to the previous case, consider a $(\delta_G, \delta_H)$-PAC algorithm and define the event $\mathcal{E} = \{ \hat{g} \neq g^* \}$; then, we can apply the transportation Lemma 1 (Kaufmann, Cappé, and Garivier 2016) at the stopping time $\tau$ to obtain: for every $\lambda \in \Lambda^2_{x}$

$$
\text{kl}(\delta_G, 1 - \delta_G) \leq \sum_{x,y,h} \eta(x,g,h) kl_{\mu | \lambda}(x,g,h).
$$

We consider subsets of $\Lambda^2_{x}$ defined as follows: for every $\tilde{g} \in \mathcal{G}$ such that $\tilde{g} \neq g^*$ we define

$$
\Lambda^2_{x}(\tilde{g}) := \{ \lambda \in \Lambda^2_{x} : g^*(\lambda) = \tilde{g} \}.
$$

Similarly, for all $(\tilde{g}, \tilde{h}) \in (\mathcal{G} \setminus \{g^*\}) \times H^X$, where $(\tilde{g}, \tilde{h}) = (\tilde{g}, h_1, \ldots, h_X)$, we also define

$$
\Lambda^2_{x}(\tilde{g}, \tilde{h}) := \{ \lambda \in \Lambda^2_{x}(\tilde{g}) : (g^*, h_1, \ldots, h_X)(\lambda) = (\tilde{g}, \tilde{h}) \}.
$$
Thus, we observe

$$
\Lambda^\mu = \bigcup_{\bar{g} \neq g^*} \Lambda^\mu_2(g) = \bigcup_{\bar{g} \neq g^*} \bigcup_{\bar{h} \in \mathcal{H}} \Lambda^\mu_2(\bar{g}, \bar{h}).
$$

In conclusion, by minimizing the r.h.s. over the set of confusing models, i.e.,

$$
\text{kl}(\delta_G, 1 - \delta_G) \leq \inf_{\lambda \in \Lambda^\mu} \sum_{x,g,h} \eta(x,g,h) \text{kl}_{\mu|\lambda}(x,g,h),
$$

which stems from the fact that $$(\bar{g}, \bar{h})$$ is fixed, for all the tasks. We conclude by observing that the right-hand side of the last inequality can be rewritten using Lemma (1) in the appendix.

Discussion of Theorem 1. In Theorem 1 $$\eta(x,g,h)$$ can be interpreted as the minimal expected number of times any $$\delta$$-PAC algorithm should select $$(x,g,h)$$. Eq. (5) corresponds to the constraints on $$\eta$$ one has to impose so that the algorithm learns the optimal predictor $$h^*_x$$ for all $$x$$, while Eq. (6) is needed to ensure that the algorithm identifies the best representation $$g^*$$ across all tasks. Both Eq. (5) and Eq. (6) define two convex sets in terms of $$\eta$$, and hence $$K^*(\mu, \delta)$$ is the value of a convex program.

We wonder if it is possible to learn only the optimal representation $$g^*$$, without learning the optimal predictors. In fact, observe that the constraints in Eq. (5) and Eq. (6) share the components of $$\eta$$ that concern $$g^*$$ only. We believe that actually separating the problem into two problems, one for each constraint, as formulated in the proposition below, yields a tight upper bound of $$K^*(\mu, \delta)$$.

Proposition 1. We have $$K^*(\mu, \delta) \leq K^*_H(\mu, \delta_H) + K^*_\Lambda(\mu, \delta_G)$$, where $$K^*_H(\mu, \delta_H)$$ (resp. $$K^*_\Lambda(\mu, \delta_G)$$) is the value of the optimization problem: $$\min_{\eta \geq 0} \sum_{x,g,h} \eta(x,g,h)$$ subject to (Eq. (5)) (resp. (Eq. (6))).

Note that $$K^*_H(\mu, \delta_H)$$ scales as $$H \text{X} \mathcal{KL}(\delta_H, 1 - \delta_H)$$ (since the corresponding optimization problem is obtained in a regular bandit problem for each task (Garivier and Kaufmann 2016), which scales as $$H \text{kl}(\delta_H, 1 - \delta_H)$$ for each task). Now, to know how $$K^*_\Lambda(\mu, \delta_G)$$ scales, we can further derive an upper bound of $$K^*_\Lambda(\mu, \delta_G).

Proposition 2. We have $$K^*_\Lambda(\mu, \delta_G) \leq L^*_G(\mu, \delta_G)$$, where $$L^*_G(\mu, \delta_G)$$ is the value of the optimization problem: $$\min_{\eta \geq 0} \sum_{x,g,h} \eta(x,g,h)$$ subject to $$\eta \geq 0$$ satisfying for all $$\bar{g} \neq g^*$$, $$\max_x \min_{\bar{h}} \eta(x,g^*, h^*_x + \eta(x, \bar{g}, \bar{h}_x)) I_{u_x \neq g^*, \bar{h}_x} (\mu(x, g^*, h^*_x), \mu(x, \bar{g}, \bar{h}_x)) \geq \text{kl}(\delta_G, 1 - \delta_G)).$$

One can show that $$L^*_G(\mu, \delta_G)$$ scales as $$GH \text{kl}(\delta_G, 1 - \delta_G)$$ (since, even with one task, we need to sample all $$GH$$ arms to identify $$g^*$$). To summarize, we have shown that the lower bound $$K^*(\mu, \delta)$$ scales at most as $$H(G \text{kl}(\delta_G, 1 - \delta_G) + \mathcal{X} \text{kl}(\delta_H, 1 - \delta_H))$$. The latter scaling indicates the gain in terms of sample complexity one can expect when exploiting the structure of $$\mathcal{M}$$, i.e., a common optimal representation.

Without exploiting this structure, identifying the best arm for each task would result in a scaling of $$GH \text{X} \text{kl}(\delta, 1 - \delta)$$ for $$\delta = \delta_G + \delta_H$$.

Differences with classical best-arm identification. To better understand the lower bound in Theorem 1 it is instructive to compare it with a classical MAB problem.

Consider the best-identification problem in MAB with $$K$$ arms. Then, the set of confusing problems is $$\Lambda(\mu) = \{\lambda \in [0,1]^K : a^*(\lambda) \neq a^*(\mu)\}$$, where $$a^*(\mu)$$ denotes the optimal arm under $$\mu$$, i.e., $$a^*(\mu) = \arg \max_{a \in [0,1]^K} \mu(a)$$. The sample complexity lower bound derived for such these models (Kaufmann, Cappé, and Garivier 2016; Garivier and Kaufmann 2016) exploits the fact that the set $$\Lambda(\mu)$$ can be written as $$\Lambda(\mu) = \bigcup_{\lambda \neq a^*(\mu)} \Lambda_a(\mu)$$, where

$$\Lambda_a(\mu) := \{\lambda \in [0,1]^K : \lambda_a > \lambda_{a^*(\mu)}\}.$$

Unfortunately, this way of rewriting the set of confusing problems cannot be used in our problem setting. The reason is that the constraint in $$\Lambda_a(\mu)$$ does not account for the model structure, i.e., the optimal representation $$g^*$$ needs to be the same across all the tasks (which is equivalent to imposing that $$a$$ is optimal for all tasks). This fact also explains why the lower bound in Theorem 1 appears more complex than the one in (Garivier and Kaufmann 2016). In the appendix, we show how to account for this kind of structure.

Because of this difference, with the model specified in Eq. (1) the confusing parameter $$\lambda$$ differs from $$\mu$$ for more than 2 arms (i.e., we need to consider all arms in the set $$\mathcal{U}_{x,G,h}$$, see also Lemma 1 in the appendix), whereas in classical MAB problems to learn that $$a$$ is suboptimal, the confusing parameter $$\lambda \in \Lambda_a(\mu)$$ differs from $$\mu$$ only for arms $$a$$ and $$a^*(\mu)$$. In fact, in our model to identify that $$(\bar{g}, \bar{h}_x)$$ is suboptimal, we need to consider an alternative model where only the average reward of the arms in the set $$\mathcal{U}_{x,G,h}$$ changes.

Algorithm

We now present OSRL-SC (Algorithm 1), a $$\delta$$-PAC algorithm whose expected sample complexity is asymptotically upper bounded by $$K^*_G(\mu, \delta_G) + K^*_\Lambda(\mu, \delta_G)$$. The algorithm proceeds in two phases: a first phase aimed at learning $$g^*$$, and a second phase devoted to learning the optimal predictor for each task. At the end of the first phase, we have an estimate $$\tilde{g}$$ of the best representation. In the second phase, for each task $$x$$, we use the optimal track-and-stop algorithm (Garivier and Kaufmann 2016) to identify $$\tilde{h}_x$$, the best predictor associated to $$\tilde{g}$$. In the remaining part of the section, we just describe the first phase.

A track-and-stop algorithm to learn $$g^*$$. The lower bound describes the minimal expected numbers $$\eta$$ of observations of the various tasks needed to learn $$g^*$$. These numbers minimize $$\sum_{x,g,h} \eta(x,g,h)$$ over $$\eta \geq 0$$ satisfying (Eq. (6)). In other words, the sampling budget should be allocated according to the following distribution: $$q^*(\mu) \in \Sigma$$, solving: $$\max_{\rho \in \Sigma} \rho(q, \mu)$$, where

$$\rho(q, \mu) = \min_{\bar{g} \neq g^*} \sum_{x \neq \bar{g}} \min_{\bar{h}_x} \ell_\mu(q, \bar{g}, \bar{h}_x),$$

and $$\Sigma$$ denotes the set of distributions over $$\mathcal{X} \times \mathcal{G} \times \mathcal{H}$$. We design an algorithm tracking this optimal allocation: it consists of (i) a sampling rule, (ii) a stopping rule, and (iii)
Algorithm 1: OSRL-SC $(\delta_G, \delta_H, \sigma)$

1: **Initialization.**
2: $N_t(x, g, h), \mu_t(x, g, h) \leftarrow 0, \forall (x, g, h) \in \mathcal{X} \times G \times H$
3: $t \leftarrow 1$
4: **Phase 1. Learning $g^*$**
5: while $\max_{x \in G} \Psi_t(g) \leq \beta_t(\delta_G)$ do
6:   if $U_t = \emptyset$ and $\tilde{\mu}_t \in \mathcal{M}$ then
7:     $(x(t), g(t), h(t)) \leftarrow \arg\max_{x, g, h} L^p_t(x, g, h; \tilde{\mu}_t) - N_t(x, g, h)$
8:   else
9:     $(x(t), g(t), h(t)) \leftarrow \arg\min_{x, g, h} N_t(x, g, h)$
10:   end if
11: end while
12: return $\hat{g} = \arg\max_{g} \mu_{T_\tau}(g)$
13: **Phase 2. Learning $(h_1^*, \ldots, h_N^*)$**
14: For all task $x$, $\tilde{h}_x \leftarrow$ (track-and-stop (Garivier and Kaufmann 2016)) with arms $(\hat{g}, h)_{h \in H}$ and confidence $\delta_H$
15: return $(\hat{g}, h_1^*, \ldots, h_N^*)$

**(i) Sampling rule.** We adapt the D-tracking rule introduced in (Garivier and Kaufmann 2016). The idea is to track $q^*(\mu)$, the optimal proportion of times we should sample each (task, arm) pair. One important issue is that the solution to max$_{\mu \in \Sigma} \rho(\mu)$ is not unique (this happens for example when two tasks are identical).

To solve this problem, we employ the following novel idea: we propose to regularize the optimization problem by tracking $q^*_\sigma(\mu)$, the unique solution of

$$(P_\sigma): \max_{q \in \Sigma} \rho(q, \mu) - \frac{1}{2\sigma} \|q\|_2^2, \quad \sigma > 0. \quad (10)$$

When $\sigma$ is large, Berge’s Maximum theorem (Berge 1963) implies that $q^*_\sigma(\mu)$ approaches the set of solutions of max$_{\mu \in \Sigma} \rho(q, \mu)$, and that the value $C_{\sigma}(\mu) = \rho(q^*_\sigma(\mu), \mu)$ converges to $K^*_G(\mu, \delta_G)/k(\delta_G, 1 - \delta_G)$. In what follows, we let $K^*_G(\mu, \delta_G) := C_{\sigma}(\mu)k(\delta_G, 1 - \delta_G)$.

Our D-tracking rule targets $q^*_\sigma(\mu_t)$, which is the unique maximizer of max$_{\mu \in \Sigma} \rho(q, \mu_t) - \frac{1}{2\sigma} \|q\|_2^2$, where $\rho(q, \mu_t)$ for any $\mu_t \in \mathcal{M}$ is defined as

$$\rho(q, \mu_t) = \min_{\hat{g} \neq g^*} \sum_{x, h} \min_{h_x} \ell_{\mu_t}(q, \hat{g}, \tilde{h}_x). \quad (11)$$

More precisely, if the set of under-sampled tasks and arms $U_t = \{(x, g, h) \in \mathcal{X} \times G \times H : N_t(x, g, h) < \sqrt{-G \mathcal{X}/2}\}$ is not empty, or when $\mu_t \notin \mathcal{M}$, we select the least sampled (task, arm) pair. Otherwise, we track $q^*_\sigma(\mu_t)$, and select arg max$_{x, g, h} L^p_t(x, g, h; \mu_t) - N_t(x, g, h)$.

**(ii) Stopping rule.** We use Chernoff’s stopping rule, which is formulated as a Generalized Likelihood Ratio Test (Chernoff 1959). The derivation of this stopping rule is detailed in appendix B. The stopping condition is max$_{\mu} \Psi_t(g) > \beta_t(\delta_G)$, where the exploration threshold $\beta_t(\delta_G)$ needs to be appropriately chosen, and where

$$\Psi_t(g) = \min_{\hat{g} \neq g^*} \sum_{h_x} \min_{h_x} \ell_{\mu_t}(N_t, \hat{g}, \tilde{h}_x).$$

**Decision rule.** The first phase ends at time $\tau_G$, and $\hat{g}$ is chosen as the best empirical representation: $\hat{g} = \arg\max_{g} \mu_{T_\tau}(g)$.

**PAC and Sample Complexity Analysis.**

We now present the sample complexity upper bound for Algorithm 1. First, we outline the stopping rule used by the algorithm. Following (Kaufmann and Koolen 2018), we define $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ as $\phi(x) = 2\bar{\beta}_3/2 \left(\frac{x - (1 + x)^2}{\ln(2(x))}\right)$, where $\zeta(s) = \sum_{n \geq n} n^{-s}, p(u) = u - \ln(u)$ for $u \geq 1$ and for any $z \in [1, e]$ and $x \geq 0$:

$$\hat{\beta}_t(x) = \begin{cases} e^{1/p-1}(x)p^{-1}(x) & \text{if } x \geq p^{-1}(1/\ln z), \\ z(x - \ln z) & \text{otherwise.} \end{cases}$$

Then, the following theorem states that with a carefully chosen exploration threshold, the first phase of OSRL-SC returns the optimal representation w.p. greater than $1 - \delta_G$.

**Theorem 2.** Let $\delta_G \in (0, 1)$: for any sampling rule, Chernoff’s stopping rule with threshold $\beta_t(\delta_G) = \beta_1(t) + \beta_2(\delta_G)$, where $\beta_1(t) = 3\sum_{x, g, h} \ln(1 + \ln(N_t(x, g, h)))$, and $\beta_2(\delta_G) = GGH \phi(\ln(-\delta_G)/C^G)$, ensures that for all $\mu \in \mathcal{M}$, $\mathbb{P}_\mu(\tau_G < \infty, \hat{g} \neq g^*) \leq \delta_G$.

**Proof.** The proof can be seen as multi-task version of Proposition 12 in (Garivier and Kaufmann 2016) with an improved bound from (Kaufmann and Koolen 2018). Let $\gamma^*_t$ be the decision rule at time $t$. Since $\mathbb{P}_\mu(\tau_G < \infty, \hat{g} \neq g^*) = \mathbb{P}_\mu(\exists t \in N : g^*_t \neq g^*, \Psi_t(g^*_t) > \beta_t(\delta_G))$, we can apply a union bound over the set $\mathcal{G} \setminus \{g^*_t\}$ up to upper bound the probability of error as follows:

$$\mathbb{P}_\mu(\tau_G < \infty, \hat{g} \neq g^*) \leq \sum_{\hat{g} \neq g^*} \mathbb{P}_\mu(\exists t : g^*_t = \hat{g}, \Psi_t(g^*_t) > \beta_t(\delta_G)).$$

Then, note that $\ell_{\mu_t}(N_t, \tilde{h}_x)$ lower bounds $\sum_{(x, g, h) \in t^k} N_t(x, g, h)k(\mu_t(x, g, h), \mu(x, g, h))$, with $t^k_{x, g, h} = U^k_{x, g, h}$. Therefore, we derive that

$$\mathbb{P}_\mu(\tau_G < \infty, \hat{g} \neq g^*) \leq \sum_{\hat{g} \neq g^*} \mathbb{P}_\mu\left(\exists t \in N : \sum_{x, g, h} N_t(x, g, h)k(\mu_t(x, g, h), \mu(x, g, h)) > \beta_t(\delta_G)\right).$$

We conclude by applying (Kaufmann and Koolen 2018, Thm. 14) with $x = \ln \left(\frac{e^{-1}}{\sigma^2} \right)$ and $S = \mathcal{X} \times G \times H$. □

From (Garivier and Kaufmann 2016), the second phase of OSRL-SC also returns the optimal predictors for each task w.p. greater than $1 - \delta_H$. Finally, in the next theorem, we show that OSRL-SC stops in finite time a.s., and that its expected sample complexity approaches $K^*_G(\mu, \delta_G) + K^*_H(\mu, \delta_H)$ for sufficiently small values of the risks $\delta_G, \delta_H$, and sufficiently large $\sigma$. 

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Theorem 3. If the exploration threshold of the first phase of OSRL-SC is chosen as in Theorem 2, then we have:  
\[ P_\mu[\tau_G < \infty] = 1 \text{ and } P_\mu[\tau_H < \infty] = 1 \]  
(\tau_H\text{ is the time at which the second phase ends}). In addition, the sampling complexity of OSRL-SC satisfies:  
\[ \limsup_{\delta \to 0} K_{G,\sigma}^* (\mu, \delta_G) + K_H^* (\mu, \delta_H) \leq 1, \]  
where  
\[ K_{G,\sigma}^* (\mu, \delta_G) = C_\sigma (\mu) k(\delta_G, 1 - \delta_G), \]  
with  
\[ C_\sigma (\mu) = \rho (\eta^*(\mu), \mu)^{-1}. \]

Proof. The result follows from Theorem 7 (in the appendix) and Theorem 14 in (Garivier and Kaufmann 2016). The latter result upper bounds the expected duration of the second phase of OSRL-SC if we use, for this phase, a threshold rule \( \beta_1 (\delta_H) \) that is increasing in \( t \) and for which there exists constants \( C, D > 0 \) such that  
\[ \forall t \geq C, \forall \delta_H \in (0, 1), \beta_1 (\delta_H) \leq \ln \left( \frac{Dt}{\delta_H} \right). \]

For example, in the Bernoulli case, one can choose \( \beta_1 (\delta_H) = \log \left( \frac{2(1-\delta_H)}{\delta_H} \right) \). Note that the sample complexity of the second phase can be rewritten as  
\[ E_\mu [\tau_H | \hat{g} = g^*] = E_\mu [\tau_H | \hat{g} = g^*] E_{\mu}(\hat{g} = g^*) + E_\mu [\tau_H | \hat{g} = g^*] E_{\mu}(\hat{g} = g^*) \]  
where  
\[ E_\mu [\tau_H | \hat{g} = g] \]  
denotes the conditional expected sample complexity of the second phase, given that the first phase outputs \( \hat{g} \). From the result of (Garivier and Kaufmann 2016), we know that  
\[ \limsup \sup_{\delta_H \to 0} \frac{E_\mu [\tau_H | \hat{g} = g^*]}{K_H^* (\mu, \delta_H; g^*)} \leq 1, \]

where  
\[ K_H^* (\mu, \delta_H; g^*) := \sum_x T^* (x, g^*; \mu) k(\delta_H, 1 - \delta_H), \]

and  
\[ T^* (x, g^*; \mu) \]  
is defined as  
\[ T^* (x, g^*; \mu)^{-1} = \sup_{g \in \Sigma} \min_{h \neq \hat{h}_x^*} \left( \eta (x, g^*, \hat{h}_x^*) + \eta (x, g^*, h) \right) I_{\alpha_x, g^*, h} (\mu (x, g^*, \hat{h}_x^*), \mu (x, g^*, h)), \]

with  
\[ \hat{h}_x^* = \arg \max_h \mu (x, g, h). \]  

for some constant \( C \) that depends on the threshold rule \( \beta_1 (\delta_H) \). Since the first phase of the algorithm is \( \delta_G \)-PAC, we have:  
\[ \limsup \sup_{\delta_H \to 0} \frac{E_\mu [\tau_H | \hat{g} = g]}{K_H^* (\mu, \delta_H; g^*)} \leq 1. \]

Hence, we obtain the result by letting \( \delta_G \to 0 \):  
\[ \limsup \sup_{\delta_H \to 0} \frac{E_\mu [\tau]}{K_H^* (\mu, \delta_H) + K_H^* (\mu, \delta_H)} \leq 1. \]

\[ \square \]

Corollary 1. Additionally, due to Berge’s theorem, since  
\[ \rho (q, \mu) = \frac{1}{\| q \|_2^2} \]  
is continuous in \( q \) for each \( (\sigma, \mu) \), we have:  
\[ \lim_{\sigma \to \infty} K_{G,\sigma}^* (\mu, \delta_G) = K_{G}^* (\mu, \delta_G). \]

### Numerical Results

We analyze the performance of OSRL-SC, and compare it directly with TRACK AND STOP (TAS) (Garivier and Kaufmann 2016). We are interested in answering the following question: is it easier to learn the best representation by just focusing on one task, or should we consider multiple tasks at the same time?

**Simulation setup.** We consider 2 tasks \( (x_1, x_2) \), 3 representations \( (g_1, g_2, g_3) \) and 2 predictors \( (h_1, h_2) \). This setting is rather simple, although not trivial. Note that as the number of (task, arm) pairs decreases, we expect the gap between the two algorithms to decrease and thus favor TAS. Hence, considering examples with small numbers of tasks are informative about OSRL abilities to factor information across
Table 1: Analysis of OSRL-SC. Comparison of the optimal allocation vector $q^*_G(\mu)$ and the average proportion of arm pulls $N_{ts} / \tau_G$ at the stopping time.

| Task $x_1$ | $(g_1, h_1)$ | $(g_1, h_2)$ | $(g_2, h_1)$ | $(g_2, h_2)$ | $(g_3, h_1)$ | $(g_3, h_2)$ |
|------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $N_{ts}(x_1)/\tau_G$ | 0.14 | 0.05 | 0.02 | 0.02 | 0.12 | 0.15 |
| $q^*_G(x_1; \mu)$ | 0.18 | 5.10^{-3} | 6.10^{-4} | 7.10^{-4} | 0.13 | 0.18 |

| Task $x_2$ | $(g_1, h_1)$ | $(g_1, h_2)$ | $(g_2, h_1)$ | $(g_2, h_2)$ | $(g_3, h_1)$ | $(g_3, h_2)$ |
|------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $N_{ts}(x_2)/\tau_G$ | 0.14 | 0.05 | 0.12 | 0.16 | 0.02 | 0.02 |
| $q^*_G(x_2; \mu)$ | 0.18 | 5.10^{-3} | 0.13 | 0.18 | 6.10^{-4} | 7.10^{-4} |

We set up tasks $x_1$ and $x_2$ so that they are very similar: actually, the only difference is that the 2nd and 3rd row of the above matrices are swapped. Therefore, it should not matter which task TAS picks, but, on the other hand, OSRL-SC should benefit from this small difference. Hence, for each simulation of TAS, we picked one task uniformly at random. Finally, we averaged results over 1120 runs.

**Algorithms.** We test TAS and OSRL-SC with various risks $\delta \in \{0.01, 0.05, 0.1\}$ (with $\delta = \delta_G = \delta_H$ for OSRL-SC). For TAS, we use the following threshold $\beta_1(\delta_G) = \log \left( \frac{2(\delta G - 1)}{\delta} \right)$. We tried the same threshold as in OSRL-SC, but it yielded worse results. For OSRL-SC, we set $\sigma = 10^3$. For the example considered, one can see that $\arg \max_{\mu \in \Sigma} \mu(q, \mu)$ has a unique maximizer. Therefore, $\sigma$ will not influence the value of the lower bound if $\hat{\mu}_t$ is approximately close to $\mu$, in norm. However, when $\hat{\mu}_t$ is visibly different from $\mu$, then the value of $C_\sigma(\hat{\mu}_t)$ may be affected by the value of $\sigma$. We have not thoroughly explored different values of $\sigma$, but we may suggest that a value of $\sigma > 10^3$ is a safe choice.

We computed $q^*_G(\hat{\mu}_t)$ every 12 rounds (which is equal to $GHX$) to reduce the computational time (this is theoretically motivated in the appendix). Despite that, one needs to keep in mind that tracking a suboptimal, or wrong, reference vector $q^*_G$ may sensibly affect the sample complexity. We can also motivate this period update by the fact that $q^*_G(\hat{\mu}_t)$ in a small time interval does not change much, as shown numerically.

To compute $q^*_G(\hat{\mu}_t)$, in round $t + 1$, we use as initial condition a convex combination of the previous solution and a uniform point in the probability simplex (with a factor 0.5). This is done to speed up the algorithm (for more details, refer to the appendix).

**Comparison of OSRL-SC and TAS.** In Table 1, we report the sample complexity of the two algorithms. In bold, we highlighted results for the first phase of OSRL-SC. Even if the number of representations is higher than the number of predictors, somewhat surprisingly, the first phase OSRL-SC seems, on average, very stable, with a small confidence interval (when compared to Phase 2 or TAS). It is worth observing that with the smallest number possible of tasks ($X = 2$), OSRL-SC manages to reduce the required number of rounds to identify the optimal representation, and the predictors, when compared to TAS. Furthermore, the first phase of OSRL-SC appears stable also when $\delta$ decreases. Between $\delta = 0.1$ and $\delta = 0.05$ there is a relative increase of average sample complexity of roughly 2% for OSRL-SC; between $\delta = 0.05$ and $\delta = 0.01$ we have that the average sample complexity for OSRL-SC has a relative increase of roughly 5%. Overall, these results indicate that OSRL-SC is able to re-factor information efficiently.

**Analysis of OSRL-SC: First phase.** To analyze the convergence of OSRL-SC, we focus on its first phase, specifically on the following quantities: $\hat{\mu}_t$, $q^*_G(\hat{\mu}_t)$ and $C^{-1}_\sigma(\hat{\mu}_t)$.

![Figure 3: Analysis of $C^{-1}_\sigma(\hat{\mu}_t)$ and $\hat{\mu}_t$ under OSRL-SC. (a) Average dynamics of $C^{-1}_\sigma(\hat{\mu}_t)$, (b) Dynamics of $\hat{\mu}_t$. $\|\hat{\mu}_t - \hat{\mu}_{t-1}\|_2$ is normalized by $\sqrt{GHX}$ to show the average change of each component, and low-pass filtered using a 8-th order Butterworth filter with critical frequency $\omega_0 = 0.025$. The shadowed areas represent 97.5% confidence interval.](image-url)
In this work, we analyzed knowledge transfer in stochastic multi-task bandit problems with finite arms, using the framework of best-arm identification with fixed confidence. We proposed OSRL-SC, an algorithm that transfers knowledge across tasks while approaching the sample complexity lower bound. We believe that this paper constitutes a sound starting point to study the transfer of knowledge in more general online multi-task learning problems. The limitation of this work is that we only consider models with a finite number of tasks and arms, which could limit their application in real life. Furthermore, our algorithm converges to an upper bound of the lower bound. Lastly, we think it would be interesting to study different structural assumptions (e.g. linearity) that tie reward functions across tasks together, or extend this work to multi-task reinforcement learning in MDPs.

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