Abstract—We study a relay feedback system (RFS) having an ideal relay element and a linear, time-invariant, second order plant. We model the relay element using the signum function. And we model the second order plant with a transfer function that: (i) is Hurwitz stable, (ii) is proper, (iii) has a positive real zero, and (iv) has a positive DC gain.

We analyze this RFS using a state space description, with closed form expressions for the state trajectory from one switching time to the next. We prove that the state transformation from one switching time to the next: (a) has a Schur stable linearization, (b) is a contraction mapping, and (c) maps points of large magnitudes to points with lesser magnitudes. Then using the Banach contraction mapping theorem, we prove that every trajectory of this RFS converges asymptotically to an unique limit cycle. This limit cycle is symmetric, and is unimodal as it has exactly two relay switches per period. This result helps understand the behaviour of the relay autotuning method, when applied to second order plants with no time delay.

We also derive the asymptotic behaviour of the RFS if the plant has no finite zero, or if it has exactly one zero and that is negative.

I. INTRODUCTION

Fig. 1: Relay feedback system

FOR what plant transfer functions does the relay feedback system (RFS) of Figure 1 converge to a globally asymptotically stable limit cycle? In this paper we give a sufficient condition, applicable to second order plants.

A symmetric limit cycle is one whose locus in state space is symmetric w.r.t. the origin. If a limit cycle of the RFS is such that the relay element’s output changes sign exactly two times per period, then we call it an unimodal limit cycle. We say the RFS self-oscillates if it has a limit cycle to which all trajectories converge asymptotically.

If the loop of Figure 1 goes into a symmetrical, limit cycle, then the parameters of this limit cycle can be used to estimate details of the plant’s Nyquist locus. Using this, the Relay autotuning method [1], [2] tunes a PID controller for the plant. This motivates the study of relay oscillations.

A. Previous results

1) For plants of first, second order: The classic study of Andronov, Vitt and Khaikin [4], [5] contains an approach for analyzing individual second order systems, based on properties of ordinary differential equations on the plane. For this approach, the authors give the name the method of point transformations, because the method focuses on the transformations of the system state from one switching instant to the next. Using this method they proved the stability of some second order valve oscillators, which were significant at that time for Radio engineering.

The doctoral thesis of Holmberg [3] contains detailed studies of relay oscillations in first and second order RFS. One significant result is its Theorem 5.1 on self-oscillations in second order RFS. This Theorem states that if the system possesses an unique limit cycle, then all trajectories converge globally asymptotically to it.

2) For plants of arbitrary order: Frequency domain tools such as the Hamel locus [6] and the Tsypkin locus [7] give necessary conditions for limit cycles. But these tools give no information about stability.

A state-space necessary condition for a symmetrical, periodic orbit is given in Aström [8]. This paper also gives a linearized analysis for local stability in the neighbourhood of a limit cycle. Goncalves et al. [9] show how a Lyapunov analysis for global stability can be carried out.

Bliman and Krasnosel’skii [10], as well as Varigonda and Georgiou [11] mention the possibility of using fixed point theorems to prove convergence to a limit cycle. Johansson and Rantzer [12] treat the case of a third order plant with no zeros, and with three distinct, real, stable poles. They show that trajectories obey an ‘area contraction’ property.

Megretski [13] gives a graphical template for the step response of a plant, for it to self-oscillate under relay feedback. The template closely resembles that of a stable, nonminimum phase, second order plant. The author proves that if a plant transfer function of arbitrary order resembles this template, then self-oscillations are guaranteed if a discrete time iteration of inter-switching times has an unique stationary solution. And such a unique stationary solution exists if a related discrete time linearized iteration satisfies an $l^1$-norm condition.

3) Status quo for second order RFS: Of all the publications reported, two stand out: [13], [3]. The paper of Megretski [13] contains a powerful toolset to analyze a given RFS. But it is not clear whether this helps to guarantee self-oscillations for a wide class of second order RFS.

Holmberg’s thesis [3] has the theorem that promises self oscillations for any second order RFS, if it has an unique limit cycle. One can speculate about combining this theorem with known properties of stable, nonminimum phase second order RFS. In specific, if the second order plant (a) is stable, (b) nonminimum phase, and (c) has positive DC gain, then the RFS has (i) bounded trajectories, (ii) no equilibrium point, and (iii) no chattering point. Because of these constraints and because trajectories lie on the plane and cannot cross each other, any trajectory faces three mutually exclusive choices: (i) be a limit cycle, (ii) curve inwards, or (iii) curve outwards.
Then is it not inevitable that this RFS must have an unique limit cycle, so that Holmberg’s theorem kicks in?

Not as such. In principle, it is possible that the RFS has a nested sequence of limit cycles, with attracting limit cycles alternating with repelling limit cycles. One has to explicitly rule out such possibilities.

B. Our contributions

We work with a state space realization of the RFS. One can express the trajectory between switching points as an explicit expression. Because the plant is second order, this can be done ‘by hand.’ But we apply a short-cut, made possible because the literature gives an explicit similarity transformation for putting a companion matrix into its Jordan canonical form.

With the closed form trajectory, and basic Calculus we study the function that maps the state at one switching instant to the state at the next switching instant. We prove that this switching point transformation function has a stable linearization.

We prove that self-oscillations are guaranteed for every RFS with a second order plant that is stable, nonminimum phase, and has a positive DC gain. We show that the convergent limit cycle is symmetric and unimodal.

We prove that if the second order plant is stable, has positive DC gain, and has no finite zero (numerator of transfer function is a constant), then the RFS converges asymptotically to the origin.

We prove that if the second order plant is stable, has positive DC gain, and has exactly one zero which is negative, then the RFS converges asymptotically to the chattering set.

II. Preliminary definitions

Let the plant transfer function \( b(s)/a(s) \) be rational and strictly proper. Let the linear time-invariant system \((A, B, C)\) be a minimal realization of it. Then the relay feedback system evolves as per:

\[
\frac{dx}{dt} = Ax - B \text{sign}(Cx), x \in \mathbb{R}^2, \tag{1}
\]

where \( \text{sign}(\cdot) \) is the signum function.

**Definition II.1.** The switching plane is the hyperplane

\[ S \triangleq \{ x \in \mathbb{R}^2 : Cx = 0 \}, \]

and it is precisely that set where the relay switches sign.

**Definition II.2.** The first exit time from positive sign (FET+ for short) is the non-negative function:

\[ \tau_+ : \mathbb{R}^2 \to \mathbb{R} \text{ such that } \tau_+(x) = \inf \{ t > 0 : x(t) < 0 \}, \]

where \( \dot{x} = Ax - B, \) and \( x(0) = x. \)

**Definition II.3.** The first exit time from negative sign (FET- for short) is the non-negative function:

\[ \tau_- : \mathbb{R}^2 \to \mathbb{R} \text{ such that } \tau_-(x) = \inf \{ t > 0 : x(t) > 0 \}, \]

where \( \dot{x} = Ax + B, \) and \( x(0) = x. \)

**Definition II.4.** The first exit map from positive sign (FEM+ for short) is defined at every point for which, the trajectory starting there crosses the switching plane in finite time:

\[ \psi_+ : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } \psi_+(x) = x(\tau_+), \]

where \( \dot{x} = Ax - B, \) and \( x(0) = x. \)

**Definition II.5.** The first exit map from negative sign (FEM- for short) is defined at every point for which, the trajectory starting there crosses the switching plane in finite time:

\[ \psi_- : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } \psi_-(x) = x(\tau_-), \]

where \( \dot{x} = Ax + B, \) and \( x(0) = x. \)

Figure 2 illustrates the above definitions by plotting the first exit time etc. for two examples of second order plants.

**Symmetry w.r.t. the origin of the state space:** The following identities hold: for every \( x \in \mathbb{R}^2, \)

\[ \tau_-(x) = \tau_+(-x), \quad \text{and, } \psi_-(x) = -\psi_+(-x). \]

III. Theorems guaranteeing self-oscillations

We consider plant transfer functions of the form

\[
\frac{-\kappa s^2 + \gamma}{(s - p_1)(s - p_2)}, \tag{2}
\]

where the real parameters \( \kappa, \gamma \) are both positive, and the poles \( p_1, p_2 \) are both stable. For this transfer function, the
Because the poles are all nonzero, the matrix $A$ is invertible, and so we can rewrite the evolution equation (1) as:

$$\begin{align*}
\frac{d}{dt} (x - A^{-1}B) &= A (x - A^{-1}B), \quad x_2 \geq 0, \\
\frac{d}{dt} (x + A^{-1}B) &= A (x + A^{-1}B), \quad x_2 \leq 0.
\end{align*}$$

A. The switching point transformation function

In Section II we defined the first exit map from positive sign. Here we shall define a function with a similar meaning, but for the second order realization (4):

$$A = \begin{bmatrix} 0 & -p_1 p_2 \\ 1 & (p_1 + p_2) \end{bmatrix}, \quad B = \begin{bmatrix} \gamma \\ -\kappa \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Because the poles are all nonzero, the matrix $A$ is invertible, and so we can rewrite the evolution equation (1) as:

$$\frac{d}{dt} (x - A^{-1}B) = A (x - A^{-1}B), \quad x_2 \geq 0,$$

and since $\gamma/p_1 p_2 > 0$, it follows that the sink $A^{-1}B$ lies below the $x_1$-axis, as shown in Figure 3. Hence it is inevitable that in some finite time, the trajectory that started at the point $(x_1, x_2)$ shall cross the $x_1$-axis as it moves towards the sink $A^{-1}B$. Thus the first exit time $\tau_+ (\cdot)$ is defined everywhere on the switching line. Since we have:

$$f_+ (\xi) = (1 \ 0) \psi_+ (\xi, 0) = (1 \ 0) \left( A^{-1}B + e^{A \tau_+ (\xi, 0)} \left( \left( \frac{\xi}{0} \right) - A^{-1}B \right) \right),$$

it follows that $f_+ (\cdot)$ is defined for all real values. \hfill $\blacksquare$

B. Banach’s fixed point theorem

The following two items are tailored to functions on $\mathbb{R}$.

**Definition III.3.** Let $\Omega \subseteq \mathbb{R}$. Then a function $f : \Omega \rightarrow \mathbb{R}$ is a contraction mapping if there is a $\rho$ such that $0 \leq \rho < 1$, and

$$|f(\xi) - f(\zeta)| \leq \rho |\xi - \zeta|, \quad \forall \xi, \zeta \in \Omega.$$

**Theorem III.4** (Banach’s fixed point theorem [14]). Let $\Omega \subset \mathbb{R}$, and let it be non-empty and compact. Let

1. $f$ be a contraction mapping on $\Omega$, and,
2. $f (\Omega) \subseteq \Omega$.

Then $f$ has an unique fixed point in $\Omega$ and, for every $\xi \in \Omega$, the infinite sequence of iterates:

$$\{ f (\xi), f (f (\xi)), f (f (f (\xi))), \ldots \}$$

converges to the said fixed point.

C. Properties of the switching point transformation function that guarantee self-oscillations

**Theorem III.5.** Consider the RFS with the second order plant transfer function $(-\kappa s + \gamma)/(s - p_1) (s - p_2)$. Suppose that the following four assumptions hold:

A 1. the parameters $\kappa, \gamma$ are positive,
A 2. the poles $p_1, p_2$ are stable,
and the ODE: $\dot{x} = Ax - B$ derived from the transfer function via (4) is such that the resulting switching point transformation function $f_+ (\cdot)$ obeys the next two conditions:

A 3. $f_+ (\cdot)$ is a contraction mapping on every finite interval of the form $[0, \theta]$, with $0 < \theta < +\infty$, and,
A 4. there exists a positive real number $\eta$ such that

$$-\xi \leq f_+ (\xi) \forall \xi \geq \eta.$$

Then all trajectories of the RFS converge globally asymptotically to a symmetric, unimodal limit cycle.

**Proof.** We investigate the generic trajectory of the RFS by investigating the sequence of its switching points. We shall show that the assumptions of the theorem imply that this sequence is never ending, and converges to a discrete time periodic orbit. We then show that this corresponds to the RFS converging to a limit cycle.

Denote the first switching point by $(\xi_0, 0)$. Without loss of generality, we can assume that this point lies to the right of the point $(-\kappa, 0)$. The sequence of switching points can then be described as:

$$\Upsilon (\xi_0) = \{ (\xi_0, 0), (f_+ (\xi_0), 0), (f_+ (- f_+ (\xi_0)), 0), (f_+ (- f_+ (f_+ (\xi_0))), 0), \ldots \}$$

Fig. 3: Vector field at the switching line under $\dot{x} = Ax - B$. 

observer realization takes the form:

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where

$$A = \begin{bmatrix} 0 & -p_1 p_2 \\ 1 & (p_1 + p_2) \end{bmatrix}, \quad B = \begin{bmatrix} \gamma \\ -\kappa \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

In Section II we defined the first exit map from positive sign. Here we shall define a function with a similar meaning, but

$$f_+ (\xi) = (1 \ 0) \psi_+ (\xi, 0), \quad \forall \xi \in \mathbb{R},$$

where $\psi_+ (\cdot)$ is the first exit map under plus sign for the second order realization (4).

**Definition III.1.** The switching point transformation function is:

$$f_+ (\xi) \triangleq (1 \ 0) \psi_+ (\xi, 0), \quad \forall \xi \in \mathbb{R},$$

where $\psi_+ (\cdot)$ is the first exit map under plus sign for the second order RFS (4), (5), (6).

It follows that the function $f_+ (\cdot)$ is defined at the real number $\xi$ if the first exit time $\tau_+ (\xi, 0)$ is finite, and it also follows that $f_+ (\cdot)$ is undefined otherwise.

**Lemma III.2.** Consider the RFS with the second order plant transfer function $(-\kappa s + \gamma)/(s - p_1) (s - p_2)$. If this transfer function: (i) is Hurwitz stable, and (ii) has a positive DC gain, then the first exit time and the switching point transformation function are defined everywhere on the switching line.

**Proof.** We prove by showing that trajectories cannot forever postpone crossing the switching line, which in this case is the $x_1$-axis. Consider the point $(x_1, x_2)$ where $x_2 \geq 0$. Under the flow of the ODE (5), the trajectories shall asymptotically converge to the sink $A^{-1}B$. Since

$$A^{-1}B = \begin{bmatrix} \gamma + \kappa p_1 p_2 \\ p_1 + p_2 \end{bmatrix},$$

we have

$$f_+ (\xi) = (1 \ 0) \psi_+ (\xi, 0), \quad \forall \xi \in \mathbb{R},$$

where $\psi_+ (\cdot)$ is the first exit map under plus sign for the second order realization (4).

In Section II we defined the first exit map from positive sign. Here we shall define a function with a similar meaning, but

$$f_+ (\xi) = (1 \ 0) \psi_+ (\xi, 0), \quad \forall \xi \in \mathbb{R},$$

where $\psi_+ (\cdot)$ is the first exit map under plus sign for the second order realization (4).

In Section II we defined the first exit map from positive sign. Here we shall define a function with a similar meaning, but

$$f_+ (\xi) = (1 \ 0) \psi_+ (\xi, 0), \quad \forall \xi \in \mathbb{R},$$

where $\psi_+ (\cdot)$ is the first exit map under plus sign for the second order realization (4).
Moreover we have:

\[ \xi < \xi' < \xi, \quad \xi > \xi' > \xi \]

Thus, for \( \xi > 0 \) we get the following set inclusion relations:

\[ -f_+([0, \xi]) \subseteq [-f_+(0), \max \{\xi, -f_+(\eta)\}], \]

\[ \subseteq [0, \max \{\xi, -f_+(\eta)\}] \]

By Lemma III.2, every element of this sequence is well defined. The following sequence of real numbers:

\[ \Xi(\xi_0) = \{-\xi_0, -f_+(-\xi_0), -f_+(-f_+(-\xi_0)), \]

\[ -f_+(-f_+(-f_+(-\xi_0)))\}, \]

\[ \text{can be exactly constructed from the sequence } \Upsilon(\xi_0), \text{ which in turn can be exactly constructed from } \Xi(\xi_0). \]

Since the function \(-f_+(-)\) is defined everywhere on its domain, the sequence \(\Xi(\xi_0)\) is never-ending.

We now consider \(\xi, \xi' \in (\kappa, +\infty)\) such that \(\xi < \xi'\). Under the flow of the ODE \(\dot{x} = Ax - B\), the two trajectories starting from the two points: \((\xi, 0), (\xi', 0)\), must satisfy the following:

1. the two trajectories do not cross each other,
2. the two trajectories lie in the half space \(\{x_1, x_2 : x_2 \geq 0\}\).
3. the two trajectories start at points on the switching line that are to the right of the point \((-\kappa, 0)\), and end at points on the switching plane that are to the left of the point \((-\kappa, 0)\).

Therefore we can conclude that

\[ f_+ (\xi) < f_+ (\xi'), \text{ if } \xi, \xi' \in (\kappa, +\infty) \text{ and } \xi < \xi'. \tag{8} \]

Figure 4 shows the flow of the ODE \(\dot{x} = Ax - B\) at the switching line. It shows that if \(\xi < -\kappa\), then the trajectory starting at the point \((\xi, 0)\) must instantaneously cross into the open half space \(\{x_1, x_2 : x_2 < 0\}\). Hence if \(\xi < -\kappa\), then \(\tau_+ (\xi, 0) = 0\), and \(f_+ (\xi) = \xi\).

From the above discussions we can see that the second and later elements of the sequence \(\Xi(\xi_0)\) are strictly monotone. Moreover we have:

\[ -f_+ (\xi) \leq -f_+ (\eta) \quad \text{if } -\kappa < \xi \leq \eta \quad \text{(inequality S)} \]

\[ -f_+ (\xi) \leq -\xi \quad \text{if } \eta \leq \xi \quad \text{(assumption A 4)} \]

From these we can conclude that

\[ -f_+ (\xi) \leq \max \{\xi, -f_+ (\eta)\} \quad \text{if } -\kappa < \xi \]

Thus, for \(\xi > 0\) we get the following set inclusion relations:

\[ -f_+([0, \xi]) \subseteq [-f_+(0), \max \{\xi, -f_+(\eta)\}], \]

\[ \subseteq [0, \max \{\xi, -f_+(\eta)\}] \]

Thus \(-f_+(-)\) satisfies the following that are all the prerequisites needed for applying Banach’s fixed point theorem:

1. the interval \([0, \max \{\xi_2, \eta\}]\) that contains the second element \(\xi_2\) of \(\Xi(\xi_0)\) is closed and bounded,
2. the function \(-f_+(-)\) maps this interval into itself, and,
3. the function \(-f_+(-)\) is a contraction mapping by assumption A 3.

Then it follows from Theorem III.4 that the function \(-f_+(-)\) has an unique fixed point in the interval \([0, \max \{\xi_2, \eta\}]\) and that the sequence \(\Xi(\xi_0)\) converges to this fixed point. Denote this fixed point by \(\xi^\circ\) (pronounced as xi cycle). Then we get:

\[ \psi_+ (\xi^\circ, 0) = (-\xi^\circ, 0), \]

\[ -\psi_+ (-\xi^\circ, 0) = (\xi^\circ, 0). \]

In other words, the symmetric points \((\xi^\circ, 0), (-\xi^\circ, 0)\) are the two switching points of a symmetric unimodal limit cycle.

Because solutions of any linear ODE depend continuously on the initial condition and time, we can conclude that every trajectory approaches this limit cycle, as the trajectory’s switching points approach those of the limit cycle.

Next we analyze three cases of stable, nonminimum phase, second order transfer functions: (a) having distinct real poles, (b) having a repeated real pole, and, (c) having a pair of complex conjugate poles. In each case we show that the above theorem holds.

D. Main result

**Theorem III.6.** Suppose the relay feedback system has a second order linear time invariant plant whose transfer function: (a) is proper, (b) is stable, (c) is nonminimum phase, and (d) has a positive DC gain. Then there is a symmetric, unimodal limit cycle to which every trajectory converges asymptotically.

**Proof.** It is enough to show that Theorem III.5 holds for the above listed three cases of stable, nonminimum phase, second order systems. And to show that this theorem holds, it is enough to show that the switching point transformation function satisfies both the contraction mapping property, and assumption A 4 in the statement of Theorem III.4. We show precisely this, in the coming three sections where we deal individually with each of the above listed three cases.

IV. Case of Two Distinct, Stable, Real Poles

Consider the case where the transfer function takes the form:

\[ \frac{-\kappa s + \gamma}{(s + \alpha)(s + \beta)}, \]

where the real parameters \(\kappa, \gamma, \alpha, \beta\) are all positive, and \(\alpha > \beta\). Then the observer realizes the form:

\[ \dot{x} = Ax + Bu, \quad y = Cx, \]

\[ A = \begin{bmatrix} 0 & -\alpha \beta \\ 1 & -\alpha \beta + \beta \end{bmatrix}, \quad B = \begin{bmatrix} \gamma \\ -\kappa \end{bmatrix}, \quad C = [0 \ 1]. \]

**A. The trajectory over the duration of a first exit time**

Consider the point \((\xi_0, 0)\) where \(\xi > -\kappa\). Let \((p, q)\) denote the state of the RFS after \(t\) seconds from starting at \((\xi_0, 0)\). Then for time \(t\) such that \(0 \leq t \leq \tau_+ (\xi, 0)\):

\[ \begin{bmatrix} p \\ q \end{bmatrix} = A^{-1}B + e^{At} \begin{bmatrix} \xi \\ 0 \end{bmatrix} - A^{-1}B. \tag{9} \]

The RHS of the above equation can be put in closed form if the matrix exponential in it can be put in closed form.
1) Matrix exponential via the Jordan diagonal form: Since \( A \) is a companion matrix with distinct eigenvalues, we can get its Jordan diagonal form by applying a special similarity transformation involving a Vandermonde matrix and its inverse [15]:

\[
VAV^{-1} = \text{diag}(-\alpha, -\beta), \quad \text{where} \\
V = \begin{bmatrix} 1 & -\alpha \\ 1 & -\beta \end{bmatrix}, \quad V^{-1} = \frac{1}{\alpha - \beta} \begin{bmatrix} -\beta & \alpha \\ 1 & 1 \end{bmatrix}.
\]

Then Equation (10) reduces to

\[
\begin{align*}
&\left( \begin{array}{c}
\psi \\
\phi
\end{array} \right) = A^{-1}B + V^{-1}Ve^{At}V^{-1}V \left( \begin{array}{c}
\xi \\
0
\end{array} \right) - A^{-1}B, \\
&= A^{-1}B + V^{-1}e^{VAV^{-1}t}V \left( \begin{array}{c}
\xi \\
0
\end{array} \right) - A^{-1}B,
\end{align*}
\]

\[
= A^{-1}B + V^{-1}\text{diag}\{e^{-\alpha t}, e^{-\beta t}\}V \left( \begin{array}{c}
\xi \\
0
\end{array} \right) - A^{-1}B,
\]

\[
= \left( \frac{\alpha + \beta}{\alpha \beta} \gamma - \kappa \right) \xi + \frac{1}{\alpha - \beta} \left( \alpha \nu_\alpha (\xi) e^{-\beta t} - \beta \mu_\alpha (\xi) e^{-\alpha t} \right) + \mu_\alpha (\xi) \frac{\alpha + \beta}{\alpha \beta} \gamma + \frac{\alpha + \beta}{\alpha \beta} \gamma / \alpha, \quad \text{and} \quad \nu_\beta (\xi) \frac{\alpha + \beta}{\alpha \beta} \gamma + \nu_\beta (\xi) \frac{\alpha + \beta}{\alpha \beta} \gamma / \beta.
\]

B. Expression for the derivative \( f_*'(\xi) \)

We can set \( t = \tau_+ (\xi, 0) \) to get \( q = 0 \), and \( p = (1 \ 0) \psi_+ (\xi, 0) \). Thus the function \( f_*'(\cdot) \) is defined implicitly by Equation (10) and in general we cannot get a closed form expression for it.

But we do get a closed form expression for the derivative of this function. Indeed, by differentiating the component scalar equations in the vector Equation (10) we get:

\[
\begin{align*}
\frac{\partial \tau_+ (\xi, 0)}{\partial \xi} &= \frac{e^{-\alpha \tau_+ (\xi, 0)} - e^{-\beta \tau_+ (\xi, 0)}}{\alpha \mu_\alpha (\xi) e^{-\alpha \tau_+ (\xi, 0)} - \beta \nu_\beta (\xi) e^{-\beta \tau_+ (\xi, 0)}}, \\
f_*'(\xi) &= \frac{\alpha \mu_\alpha (\xi) e^{\alpha \tau_+ (\xi, 0)} - \beta \nu_\beta (\xi) e^{\beta \tau_+ (\xi, 0)}}{\alpha \mu_\alpha (\xi) e^{\alpha \tau_+ (\xi, 0)} - \beta \nu_\beta (\xi) e^{\beta \tau_+ (\xi, 0)}}.
\end{align*}
\]

C. The derivative \( f_*'(\xi) \) has magnitude less than one

We shall bound the magnitude of \( f_*'(\xi) \) by studying the properties of the functional form of \( f_*'(\cdot) \). So let

\[
\phi (t) \triangleq \frac{\alpha \mu_\alpha - \beta \nu_\beta}{\alpha \mu_\alpha e^{\beta \tau_+} - \beta \nu_\beta e^{\alpha \tau_+}}, \quad \text{for} \ t \in [0, \tau_+]
\]

where, for convenience we have abbreviated the symbols \( \tau_+ (\xi, 0), \mu_\alpha (\xi, 0), \nu_\beta (\xi, 0) \) as \( \tau_+, \mu_\alpha, \nu_\beta \) respectively.

1) Relationship between the functions \( q(\cdot), \phi (\cdot) \) : Recall that \( q(t) \) is the \( x_2 \)-coordinate as a function of time, as given by Equation (10). We shall study the behaviour of the function \( q(t) \) on \([0, \tau_+]\) together with the behaviour of the function \( \phi (\cdot) \) on the same interval. If we describe by \( \chi_1, \chi_2, \chi_3 \) the terms that are independent of \( t \) in the equations for \( q(t), \phi(t) \), then:

\[
q (t) = \chi_1 + \chi_2 (\nu_\beta e^{-\beta t} - \mu_\alpha e^{-\alpha t}),
\]

\[
1/\phi (t) = (1/\chi_3) \left( \alpha \mu_\alpha e^{\beta t} - \beta \nu_\beta e^{\alpha t} \right),
\]

\[
= (1/\chi_3) e^{(\alpha + \beta) t} \frac{d}{dt} \left( \nu_\beta e^{-\beta t} - \mu_\alpha e^{-\alpha t} \right).
\]

This implies that \( q'(t) = 0 \) if and only if \( 1/\phi(t) = 0 \). The equation \( q'(t) = 0 \) has exactly one root because:

\[
q'(t)/\chi_2 = \alpha \mu_\alpha e^{-\alpha t} - \beta \nu_\beta e^{-\beta t},
\]

and (a) the function \( e^{-\alpha t} \) decays faster than the function \( e^{-\beta t} \), and (b) the coefficient \( \alpha \mu_\alpha \) is bigger than the coefficient \( \beta \nu_\beta \). Denote by \( \tau^* \) the common root of the equations: \( q'(t) = 0, 1/\phi(t) = 0 \).

2) The first exit time \( \tau_+ \) is greater than \( \tau^* \), and \( \phi (\cdot) \) grows in magnitude on \([\tau^*, \tau_+]\) : We shall now study the rise and fall of \( q(t) \) over \([0, \tau_+]\). Because of these four facts: (i) \( q(\cdot) \) is continuously differentiable on \([0, +\infty)\), (ii) \( q(0) = 0 \), and as \( t \to +\infty, q(t) \to -\gamma / \alpha \beta \), (iii) \( q(0) = \alpha \mu_\alpha - \beta \nu_\beta = (\alpha - \beta)(\xi + \kappa) > 0 \), and, (iv) the only critical point of \( q(\cdot) \) is at the time \( \tau^* \), we can make the two inferences: (a) \([0, \tau^*] \) is an interval of ascent where the function \( q(\cdot) \) rises from \( q(0) = 0 \) to its peak \( q(\tau^*) \), (b) \([\tau^*, +\infty) \) is an interval of descent where the function \( q(\cdot) \) falls from its peak \( q(\tau^*) \) to \(-\gamma / \alpha \beta \).

Therefore there is exactly one positive time instant (namely \( \tau_+ \)) when \( q(\cdot) \) equals its initial value \( q(0) = 0 \). And the instant time \( \tau_+ \) must satisfy:

\[
\tau^* < \tau_+, \quad \text{if} \ \xi > -\kappa.
\]

Note that the denominator of \( \phi (\cdot) \) has a magnitude that is an increasing function of \( t \) for \( t > \tau^* \). Since the numerator of \( \phi (\cdot) \) is independent of \( t \), it follows that \( \phi (\cdot) \) has a magnitude that is a decreasing function of \( t \) for \( t > \tau^* \).

3) Growth of \( q(\cdot) \) on \([0, \tau^*] \) and on \([\tau^*, \tau_+]\) : We shall now compare the growth of \( q(\cdot) \) on the two finite intervals: \([0, \tau^*], [\tau^*, \tau_+]\). Let the time \( \delta \) be chosen such that \( 0 \leq \delta \leq \tau^* \). Since \( \alpha \mu_\alpha e^{-\alpha \tau^*} = \beta \nu_\beta e^{-\beta \tau^*} \) we get:

\[
q'(\tau^* - \delta) = \alpha \mu_\alpha e^{-\alpha \tau^*} \times (\gamma / (\alpha - \beta)) \times (e^{\alpha \delta} - e^{\beta \delta}),
\]

\[
q'(\tau^* + \delta) = \alpha \mu_\alpha e^{-\alpha \tau^*} \times (\gamma / (\alpha - \beta)) \times (e^{-\alpha \delta} - e^{-\beta \delta}),
\]
that $0 < \beta < \alpha$, but that of the second equation is negative. And this is because

\[
q'((\tau^*-\delta) - q'((\tau^*+\delta)) = 2\alpha\mu_\alpha \gamma e^{-\alpha\tau^*} (\cosh\alpha\delta - \cosh\beta\delta),
\]

\[
q'((\tau^*+\delta) - q'((\tau^*+\delta)) = 2\alpha\mu_\alpha \gamma e^{-\alpha\tau^*} (\sinh\alpha\delta - \sinh\beta\delta).
\]

The right hand sides of both of the above equations are positive because $\alpha > \beta > 0$. And this implies that $|q'((\tau^*-\delta)) - \alpha\mu_\alpha e^{\alpha\tau} + \beta\nu_\beta e^{\beta\tau}| > |q'((\tau^*+\delta)) + \alpha\mu_\alpha e^{\alpha\tau} + \beta\nu_\beta e^{\beta\tau}|$ for $0 \leq \delta \leq \tau^*$. Moreover, this inequality in turn implies that the rise in magnitude of $q(\cdot)$ over the interval $[0, \tau^*]$ is greater than the fall in magnitude of $q(\cdot)$ over the interval $[\tau^*, 2\tau^*]$. Indeed,

\[
q((\tau^*)) = q((\tau^*)) - q(0) = \int_0^{\tau^*} q'((\tau^*+\delta)) d\delta = \int_0^{\tau^*} q'((\tau^*+\delta)) d\delta.
\]

Hence $q((2\tau^*)) > 0$. And this has the important consequence:

\[\tau^* + 2\tau^*, \text{ for } \xi > -\kappa. \tag{13}\]

4) Growth of $\phi(\cdot)$ on $[0, \tau^*]$ and on $[\tau^*, \tau_+]$ : We shall now do a similar comparison of the growth of $\phi(\cdot)$ on the two finite intervals: $[0, \tau^*], [\tau^*, \tau_+]$. Let the time $\delta$ be chosen such that $0 \leq \delta \leq \tau^*$. Since $\alpha\mu_\alpha e^{\alpha\tau \cdot r} = \beta\nu_\beta e^{\beta\tau \cdot r}$.

\[
\chi_3/|\phi((\tau^*+\delta))| < \chi_3/|\phi((\tau^*+\delta))| \text{ for } 0 \leq \delta \leq \tau^*.
\]

Setting $\delta = \tau^*$ we get:

\[
|\phi((2\tau^*))| < |\phi(0)| = 1.
\]

And since the magnitude of $\phi(\cdot)$ decreases monotonically on the interval $[\tau^*, \tau_+]$, we can now say that if $\xi > -\kappa$, then

\[
|\phi((\tau_+))| < |\phi((\tau^*))| < 1. \tag{14}
\]

D. The function $f_+(\xi)$ is strictly convex on a semi-infinite interval

To show that $f_+(\xi)$ is strictly convex on the interval $(-\kappa + \gamma/ \beta, +\infty)$ it is enough to show that $f''_+(\xi) > 0$ on this interval. Using Equations (11), (12) we get:

\[
f''_+(\xi) = \frac{G_0 - G_{\text{plus}} e^{-(\alpha-\beta)\tau} - G_{\text{minus}} e^{-(\alpha-\beta)\tau}}{e^{-(\alpha-\beta)\tau} d\tau} \left(\alpha\mu_\alpha e^{\alpha\tau} + \beta\nu_\beta e^{\beta\tau}\right)^2, \tag{15}
\]

where the coefficients of the numerator are given by:

\[
G_0 = 2(\alpha - \beta) (\alpha\beta + \kappa)^2 - \gamma^2, \quad G_{\text{plus}} = (\alpha - \beta) (\alpha\beta + \kappa)^2 + \gamma (\alpha - \beta) (\xi + \kappa) - \gamma^2, \quad G_{\text{minus}} = \gamma (\alpha - \beta) (\alpha\beta + \kappa)^2 - \gamma (\alpha - \beta) (\xi + \kappa) - \gamma^2.
\]

The denominator in the RHS of Equation (15) is negative, because $\tau^* - \tau_+$. We shall show that the numerator too is negative, by studying its functional form. So let

\[
g(X) \triangleq G_0 - G_{\text{plus}}X - G_{\text{minus}}X, \text{ for } X \in (0, +\infty).
\]

The function $g(\cdot)$ is continuously differentiable on $(0, +\infty)$. On this interval it has exactly one critical point given by:

\[
G_{\text{plus}} + G_{\text{minus}}/(X^*)^2 = 0 \implies X^* = \sqrt{G_{\text{minus}}/G_{\text{plus}}},
\]

g(\cdot) is monotonically decreasing on $(X^*, +\infty)$, because $g(X) < 0$ for $X > X^*$. Note that (a) $G_0, G_{\text{plus}}, G_{\text{minus}} > 0$ if $\xi > -\kappa + \gamma/ \beta$, and (b) $G_{\text{plus}} - G_{\text{minus}} = 2\gamma (\xi + \kappa)(\alpha - \beta)^2 > 0$. Hence the critical point $\sqrt{G_{\text{minus}}/G_{\text{plus}}} < 1$. Hence $g(X) < g(1)$, if $X > 1$. But $g(1) = 0$. Hence $g(X) < 0$, if $X > 1$, and thus for every $\xi > -\kappa + \gamma/ \beta$ we have:

\[
f''_+(\xi) = \frac{g((\tau_+)(\xi))}{e^{-(\alpha-\beta)\tau} d\tau} \left(\alpha\mu_\alpha e^{\alpha(\tau_+)\tau} + \beta\nu_\beta e^{\beta(\tau_+)\tau}\right)^2 > 0.
\]

E. The derivative $f_+'(\cdot)$ is continuous on $[0, +\infty)$

The first exit time $\tau_+(\xi, 0)$ is implicitly defined by Equation (25). Both the LHS and the RHS of this equation are continuously differentiable in $\xi$. The implicit function theorem states that the implicitly defined function inherits the differentiability properties of the defining functions. Thus if $\tau_+(\xi, 0)$ is defined at some $\xi$, then there is a small neighbourhood
including \( \xi \) where \( \tau_+ (\xi, 0) \) is also continuously differentiable in \( \xi \).

The derivative \( f_+ ' (\xi) \) is a fraction whose numerator and denominator are continuously differentiable functions of \( \xi \). And the derivative is defined everywhere on the interval \([0, +\infty)\). Hence the derivative is continuous on this interval \([0, +\infty)\).

F. The function \( f_+ (\xi) \) is a contraction mapping on \((0, +\infty)\)

We have established that on the interval \((-\kappa, +\infty)\) the derivative \( f_+ ' (\xi) \) is negative, and that its magnitude is less than one. We now derive a tighter bound on its magnitude on a subinterval \((0, +\infty)\), using the strict convexity of \( f_+ (\xi) \) on \((-\kappa + \gamma/\beta, +\infty)\).

The magnitude of \( f_+ ' (\xi) \) is a decreasing function on \((-\kappa + \gamma/\beta, +\infty)\) because
\[
\frac{d}{d\xi} (f_+ ' (\xi))^2 = 2f_+ ' (\xi) f_+ '' (\xi),
\]
and this is,
\[
< 0, \text{ if } \xi > -\kappa + \gamma/\beta.
\]
Thus \( -f_+ ' (-\kappa + \gamma/\beta) \) is a tight upper bound for the magnitude of \( f_+ ' (\xi) \) on the semi-infinite interval \([-\kappa + \gamma/\beta, +\infty)\).

We now find a tight upper bound on the finite interval \([0, -\kappa + \gamma/\beta] \). The function \( f_+ ' (\xi) \) is continuous on this interval. By the Weierstrass theorem on extreme values of continuous functions, it follows that the magnitude of \( f_+ ' (\xi) \) attains its maximum on this closed and bounded interval. Denote this maximum value by \( \overline{\xi}_0 \).

Since \( \overline{\xi}_0 \geq -f_+ ' (-\kappa + \gamma/\beta) \), it follows that there is an upper bound for the magnitude of \( f_+ ' (\xi) \) on the semi-infinite interval \([0, +\infty)\). Clearly
\[
\overline{\xi}_0 < 1, \text{ because } |f_+ ' (\xi)| < 1, \forall \xi \in (-\kappa, +\infty).
\]

And so we have the following useful inequality:
\[
|f_+ ' (\xi)| < \overline{\xi}_0 < 1, \forall \xi > 0. \quad (16)
\]
The contraction mapping property is proved next. For \( \xi, \xi' \geq 0 \)
\[
|f_+ (\xi) - f_+ (\xi')| = \left| \int_{\min \{\xi, \xi'\}}^{\max \{\xi, \xi'\}} f_+ ' (\tilde{\xi}) d\tilde{\xi} \right|,
\]
\[
\leq \int_{\min \{\xi, \xi'\}}^{\max \{\xi, \xi'\}} |f_+ ' (\tilde{\xi})| d\tilde{\xi},
\]
\[
\leq \overline{\xi}_0 \int_{\min \{\xi, \xi'\}}^{\max \{\xi, \xi'\}} \tilde{\xi} d\tilde{\xi},
\]
\[
= \overline{\xi}_0 |\xi - \xi'|. \quad (17)
\]

Since \( f_+ (\xi) \) is a contraction mapping on \([0, \infty)\), it follows that it is a contraction mapping on every finite interval of the form: \([0, \theta]\) with \( 0 < \theta < \infty \).

G. The function \( f_+ (\xi) \) satisfies assumption A 4 of Theorem IIC.

Since \( f_+ (\xi) \) is strictly convex on \((-\kappa + \gamma/\beta, +\infty)\), its graph on this interval lies above or equal to the tangent drawn at any point on this interval. Fix \( \xi_{cvx} \in (-\kappa + \gamma/\beta, +\infty) \). Then for every \( \xi \in (-\kappa + \gamma/\beta, +\infty) \):
\[
f_+ (\xi) \geq f_+ ' (\xi_{cvx}) (\xi - \xi_{cvx}) + f_+ (\xi_{cvx}).
\]
The two straight lines in \((\xi, \tilde{\xi})\)-space given by:
\[
\tilde{\xi} = f_+ ' (\xi_{cvx}) (\xi - \xi_{cvx}) - \xi_{ikn} + f_+ (\xi_{cvx}),
\]
\[
\tilde{\xi} = -\xi,
\]
have exactly one intersection point because they have different slopes. Let this intersection happen at \( \xi = \xi_{ikn} \). If \( \xi_{ikn} < \xi_{cvx} \), then for \( \xi \geq \xi_{cvx} \):
\[
f_+ (\xi) \geq f_+ ' (\xi_{cvx}) (\xi - \xi_{cvx}) + f_+ (\xi_{cvx}),
\]
\[
= f_+ ' (\xi_{cvx}) (\xi - \xi_{cvx}) - \xi_{ikn} + f_+ ' (\xi_{cvx}) (\xi_{cvx} - \xi_{ikn}),
\]
\[
= f_+ ' (\xi_{cvx}) (\xi - \xi_{cvx}) - \xi_{ikn} \leq -\eta (\xi - \xi_{cvx}) - \xi_{ikn} \tilde{\xi}
\]
\[
\geq -\xi.
\]
Similarly, if \( \xi_{ikn} \geq \xi_{cvx} \), then for \( \xi \geq \xi_{cvx} \):
\[
f_+ (\xi) \geq f_+ ' (\xi_{cvx}) (\xi - \xi_{cvx}) + f_+ (\xi_{cvx}),
\]
\[
= f_+ ' (\xi_{cvx}) (\xi - \xi_{cvx}) - \xi_{ikn} \leq -\eta (\xi - \xi_{cvx}) - \xi_{ikn} \tilde{\xi}
\]
\[
\geq -\xi.
\]
Then we can write:
\[
f_+ (\xi) \geq -\xi, \text{ if } \xi \geq \max \{\xi_{cvx}, \xi_{ikn}\}. \quad (18)
\]

V. Case of a stable, repeated, real pole

We carry out an entirely similar set of calculations for the case where the transfer function takes the form:
\[
\frac{-\kappa s + \gamma}{(s + \alpha)^2},
\]
where the real parameters \( \kappa, \gamma, \alpha \) are all positive. Because the calculations are similar, the presentation below is somewhat abbreviated when compared to the case of distinct real poles.

The observer realization takes the form:
\[
\hat{x} = Ax + Bu, \quad y = Cx,
\]
where,
\[
A = \begin{bmatrix} 0 & -\alpha^2 \\ 1 & -2\alpha \end{bmatrix}, \quad B = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

A. The trajectory over the duration of a first exit time

Consider the starting point \((\xi_0, 0)\) where \( \xi > \kappa \). Let \((p, q)\) denote the state of the RFS after a duration of \( t \) seconds.

1) Matrix exponential via the Jordan canonical form: Since \( A \) is a companion matrix with repeated eigenvalues, we get its Jordan canonical form via a similarity transformation involving a confluent Vandermonde matrix and its inverse [15]:
\[
\tilde{V} A \tilde{V}^{-1} = \begin{bmatrix} -\alpha & 0 \\ 1 & -\alpha \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}, \quad \tilde{V}^{-1} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.
\]
The matrix exponential of \( \tilde{V} A \tilde{V}^{-1} \) can be computed by expressing it as a sum of two commuting matrices, as below:
\[
\exp \left[ \begin{bmatrix} -\alpha & 0 \\ 1 & -\alpha \end{bmatrix} t \right] = \exp \left[ \begin{bmatrix} -\alpha^2 t + 0 & 0 \\ 0 & 1 \end{bmatrix} t \right],
\]
\[
= \begin{bmatrix} e^{-\alpha^2 t} & 0 \\ 0 & e^{-\alpha t} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
\[
= \begin{bmatrix} e^{-\alpha t} & 0 \\ 0 & e^{-\alpha t} \end{bmatrix}.
\]
Then for time \( t \) such that \( 0 \leq t \leq \tau_+ (\xi_0) \):
\[
\begin{bmatrix} p \\ q \end{bmatrix} = A^{-1} B + \tilde{V} A \tilde{V}^{-1} t \tilde{V} \begin{bmatrix} \xi_0 \\ 0 \end{bmatrix} - A^{-1} B,
\]
\[
= - \begin{bmatrix} \tilde{\alpha} + \kappa \\ \kappa/\alpha \end{bmatrix} + \begin{bmatrix} \tilde{\mu}_0 + \gamma/\alpha + \alpha \tilde{\mu}_0 t e^{-\alpha t} \\ \tilde{\mu}_0 t + \gamma/\alpha^2 e^{-\alpha t} \end{bmatrix}, \quad (19)
\]
where \( \tilde{\mu}_0 (\xi) = \xi + \kappa + \gamma/\alpha \).
B. Expression for the derivative $f_+(\xi)$

Differentiating the component scalar equations in (19) gives:

$$\frac{\partial \tau_+(\xi, 0)}{\partial \xi} = -\frac{\tau_+}{\xi + \kappa - \alpha \tau_+(\xi + \kappa + \gamma/\alpha)}.$$  \hspace{1cm} (20)

$$f_+(\xi) = \frac{e^{-\alpha \tau_+}}{(\xi + \kappa - \alpha \tau_+(\xi + \kappa + \gamma/\alpha))},$$  \hspace{1cm} (21)

where, for convenience we have abbreviated the symbol $\tau_+(\xi, 0)$ as $\tau_+$.

C. The derivative $f_+(\xi)$ has magnitude less than one

We shall bound the magnitude of $f_+(\xi)$ by studying the properties of the functional form of $f_+(-\xi)$. So let

$$\bar{\phi}(t) = \frac{e^{-\alpha t}}{(\xi + \kappa - \alpha t) e^{-\alpha t}}, \text{ for } t \in [0, \tau_+].$$

1) Relationship between the functions $q()$, $\bar{\phi}()$: Consider the behavior of the function $q()$ on $[0, \tau_+]$ together with the behavior of the function $\bar{\phi}(t)$ on the same interval. We have:

$$q(t) = \frac{\gamma}{\alpha^2} + \tau(t) e^{-\alpha t},$$

$$\bar{\phi}(t) = \frac{\xi + \kappa - \alpha t}{(\xi + \kappa - \alpha t) e^{-\alpha t}},$$

$$\frac{d}{dt} \left\{ \tau(t) e^{-\alpha t} \right\}.$$  \hspace{1cm} (23)

This implies that $q'(t) = 0$ if and only if $t = 0$. The equation $q'(0) = 0$ has exactly one root because $q'(t)$ is a product of two factors, one exponential in $t$ and another linear in $t$ - the root comes from the linear factor. Denote by $\tau^*$ the common root of the equations: $q(t) = 0$, $\bar{\phi} = 0$.

2) The first exit time $\tau^*$ is greater than $\tau^*$, and $\phi(t)$ grows in magnitude on $[\tau^*, \tau_+]$: We shall now study the rise and fall of $q(t)$ over $[0, \tau_+]$. Because of these three facts: (i) $q(t)$ is continuously differentiable on $[0, \tau_+]$, (ii) $q(0) = 0$, and (iii) $q'(0) = \xi + \kappa > 0$, and, (iv) the only critical point of $q(t)$ is at the time $\tau^*$, we can make the two inferences: (a) the interval $[0, \tau^*]$ is an interval of ascent where the function $q(t)$ rises from $q(0) = 0$ to its peak $q(\tau^*)$, (b) the interval $[\tau^*, \tau_+]$ is an interval of descent where the function $q(t)$ falls from its peak $q(\tau^*)$ to $-\gamma/\alpha\beta$.

Therefore there is exactly one positive time instant (namely $\tau^*$) when $q(t)$ equals its initial value $q(0) = 0$. And the time instant $\tau^*$ must satisfy:

$$\tau^* < \tau^*, \text{ if } \xi > -\kappa.$$  \hspace{1cm} (24)

Note that the denominator of $\bar{\phi}(\xi)$ has a magnitude that is an increasing function of $t$ for $t < \tau^*$. Since the numerator of $\bar{\phi}(\xi)$ is positive, and decreasing in $t$, it follows that $\phi(t)$ has a magnitude that is a decreasing function of $t$ for $t > \tau^*$.

3) Growth of $q(t)$ on $[0, \tau^*]$ and on $[\tau^*, \tau_+]$: We shall now compare the growth of $q(t)$ on the two finite intervals: $[0, \tau^*], [\tau^*, \tau_+]$. Let the time $\delta$ be chosen such that $0 \leq \delta \leq \tau^*$. Since $\xi + \kappa = \alpha (\xi + \kappa + \gamma/\alpha) \tau^*$, we get:

$$q(\tau^* - \delta) = \frac{e^{+\alpha \delta} \times \alpha \delta (\xi + \kappa + \gamma/\alpha) e^{-\alpha \tau^*}}{\xi + \kappa - \alpha \tau^*(\xi + \kappa + \gamma/\alpha)},$$

$$q(\tau^* + \delta) = \frac{-e^{-\alpha \delta} \times \alpha \delta (\xi + \kappa + \gamma/\alpha) e^{-\alpha \tau^*}}{\xi + \kappa - \alpha \tau^*(\xi + \kappa + \gamma/\alpha)}.$$  \hspace{1cm} (25)

Clearly, $|q(\tau^* - \delta)| > |q(\tau^* + \delta)|$ if $0 \leq \delta \leq \tau^*$. The derivative $q'(\xi)$ is positive over the interval $[0, \tau^*]$, and negative over the interval $[\tau^*, \infty)$. In other words, the derivative changes sign only at the point $\tau^*$. Therefore we can infer that the magnitude of rise in the value of the function over the interval $[0, \tau^*]$ is greater than the magnitude of fall over the interval $[\tau^*, 2\tau^*]$. Indeed, just at we derived in the last part of Section V-C3, we get:

$$q(\tau^*) = \int_0^{\tau^*} q'(\tau^* - \delta) d\delta = \int_0^{\tau^*} |q'(\tau^* - \delta)| d\delta,$$

$$> \int_0^{\tau^*} |q'(\tau^* + \delta)| d\delta = \int_0^{\tau^*} -q'(\tau^* + \delta) d\delta,$$

$$= q(\tau^*) - q(2\tau^*).$$

Hence $q(2\tau^*) > 0$. And this has the important consequence:

$$\tau^* > 2\tau^*, \text{ for } \xi < -\kappa.$$  \hspace{1cm} (26)

4) Growth of $\bar{\phi}(\xi)$ on $[0, \tau^*]$ and on $[\tau^*, \tau_+]$: Just as we did in Section IV-C4 we shall now compare the growth of $\bar{\phi}()$ on the two finite intervals: $[0, \tau^*], [\tau^*, \tau_+]$. Let the time $\delta$ be chosen such that $0 \leq \delta \leq \tau^*$. Since $\xi + \kappa = \alpha (\xi + \kappa + \gamma/\alpha) \tau^*$, we get:

$$\frac{(\xi + \kappa + \gamma/\alpha) e^{+\alpha \tau^*}}{\alpha t},$$

$$\frac{(\xi + \kappa + \gamma/\alpha) e^{+\alpha \tau^*}}{\alpha t},$$

Clearly, $|q'(\tau^* - \delta)| < |q'(\tau^* + \delta)|$ if $0 \leq \delta \leq \tau^*$. And so $|\xi + \kappa + \gamma/\alpha| < |\xi + \kappa + \gamma/\alpha|$. Hence $\bar{\phi}(2\tau^*) < 1$. And since the magnitude of $\bar{\phi}(\xi)$ is a decreasing function on $[\tau^*, +\infty)$ we get:

$$|f_+(\xi)| = |\bar{\phi}(\tau_+)| < |\bar{\phi}(2\tau^*)| < 1.$$  \hspace{1cm} (27)

D. The function $f_+(\xi)$ is strictly convex on a semi-infinite interval

To show that $f_+(\xi)$ is strictly convex on the interval $(-\kappa + \gamma/\beta, +\infty)$ it is enough to show that $f''_+(\xi) > 0$ on this interval. Using Equations (20), (21) we get:

$$f''_+(\xi) = \frac{-\alpha^2 (\xi + \kappa + \gamma/\alpha) - 2 \gamma (\xi + \kappa + \gamma/\alpha) + \xi + \kappa - \alpha \tau_+(\xi + \kappa + \gamma/\alpha)}{e^{+\alpha \tau_+}(\xi + \kappa + \gamma/\alpha)}.$$  \hspace{1cm} (28)

If $\xi + \kappa > \gamma/\alpha$, then $f''_+(\xi) > 0$.

E. The derivative $f_+(\xi)$ is continuous on $[0, +\infty)$

Exactly like in the case of distinct real poles (Section IV-E), in this case too is $f_+(\xi)$ a continuous function on $[0, +\infty)$.

F. Contraction mapping property, and assumption A 4 of Theorem III.2

Sections IV-E and IV-G give arguments showing that the contraction mapping property and assumption A 4 hold in the case where the plant has distinct real poles. Those arguments depend on the convexity of $f_+(\xi)$ on the semi-infinite interval $(-\kappa + \gamma/\alpha, +\infty)$.

VI. Case of a complex conjugate pair of poles

Consider the case where the transfer function takes the form:

$$\frac{-\kappa s + \gamma}{(s + \sigma)^2 + \omega^2},$$

where the real parameters $\kappa, \gamma, \eta, \omega$ are all positive.

The observer realization takes the form:

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where,

$$A = \begin{bmatrix} 0 & -\sigma^2 - \omega^2 \\ 1 & -2\sigma \end{bmatrix}, \quad B = \begin{bmatrix} \gamma \\ -\kappa \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$
A. The trajectory over the duration of a first exit time
Consider the starting point \((ξ_0, 0)\) where \(ξ > -κ\). Let \((p, q)\) denote the state of the RPS after a duration of \(t\) seconds.

1) Matrix exponential via the Jordan canonical form: We have the following similarity transformation:

\[
TAT^{-1} = \begin{pmatrix} -σ & 0 \\ ω & ω \end{pmatrix}, \quad \text{where} \quad T = \frac{1}{\sqrt{ω}} \begin{pmatrix} 1 & -σ \\ 0 & ω \end{pmatrix}, \quad T^{-1} = \frac{1}{\sqrt{ω}} \begin{pmatrix} ω & σ \\ 0 & 1 \end{pmatrix}.
\]

The matrix exponential of \(TAT^{-1}\) \(t\) can be computed by expressing it as a sum of two commuting matrices, as below:

\[
e^{-σt} \begin{pmatrix} -σ & 0 \\ ω & ω \end{pmatrix} = \begin{pmatrix} e^{-σt} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -σt \\ 0 & e^{σt} \end{pmatrix},
\]

\[
e^{-t} \begin{pmatrix} -σ & 0 \\ ω & ω \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} 1 & -σ \sin t \\ 0 & \cos t \end{pmatrix}.
\]

Then for time \(t\) such that \(0 ≤ t ≤ τ_+ (ξ, 0)\):

\[
\begin{pmatrix} p \\ q \end{pmatrix} = A^{-1}B + T^{-1}e^{TAT^{-1}t}T \begin{pmatrix} ξ(t) \\ 0 \end{pmatrix} - A^{-1}B.
\]

Next we find a relationship which we use to establish that \(1/\tilde{φ}(\cdot)\) vanishes with the derivative \(q'(t)\):

\[
(ωµ_1 - σν_0)e^{-σt}/\tilde{φ}(t) = (ωµ_1 - σν_0)\cos wt - (σµ_1 + νω_0)\sin wt,
\]

\[
= ωe^{+σt} x \frac{d}{dt}\{q(t)\}.
\]

This implies that \(q'(t) = 0\) if an only if \(1/\tilde{φ}(t) = 0\). Since \(q'(t)\) is a product of two factors, one exponential in \(t\) and another trigonometric in \(t\), it follows that the roots come from the trigonometric factor. Denote by \(τ^*\) the smallest positive common root of the equations: \(q'(t) = 0, 1/\tilde{φ}(t) = 0\).

2) \(q'(\cdot)\) has exactly one root in \([0, π/ω]\): The derivative \(q'(t)\) vanishes if and only if

\[
(ωµ_1 - σν_0)\cos wt = (σµ_1 + νω_0)\sin wt.
\]

Now we show that the last equation has positive coefficients for both \(\cos wt\) and \(\sin wt\). It is clear that \(ν_0 > 0\). For every \(ξ > -κ\), we have: \(µ_1 > 0\), and \(ωµ_1 - σν_0 = ξ(ξ + κ) > 0\). Hence it follows that

\[
(ωµ_1 - σν_0, σµ_1 + νω_0) > 0, \ \forall ξ > -κ.
\]

On the interval \([0, π/(2ω)]\) the function \(\cos wt\) falls monotonically from 1 to 0, while the function \(\sin wt\) rises monotonically from 0 to 1. Hence there is exactly one time instant \(t\) from \([0, π/(2ω)]\) such that

\[
(ωµ_1 - σν_0)\cos wt = (σµ_1 + νω_0)\sin wt.
\]

On the interval \([π/(2ω), π/ω]\) the functions \(\cos wt, \sin wt\) have opposite signs. Hence there is no time instant \(t\) from \([π/(2ω), π/ω]\) where \(q'(t)\) vanishes.

Thus there is exactly one time instant (namely \(τ^*\)) from the interval \([0, π/ω]\) where \(q'(t)\) vanishes.

3) The first exit time \(τ_+\) is greater than \(τ^*\): We shall now study the rise and fall of \(q(t)\) over \([0, τ_+]\). Because of these four facts: (i) \(q(\cdot)\) is continuously differentiable on \([0, +∞]\), (ii) \(q(0) = 0\), and, (iii) \(q(π/(2ω)) = -1 + e^{-σ/ω}ν_0\cos τ_+ < 0\), (iii) \(q'(0) = ξ + κ > 0\), and, (iv) the only critical point of \(q(\cdot)\) is at the time \(τ^*\), we can make these two inferences: (a) the interval \([0, τ^*]\) is an interval of ascent where the function \(q(\cdot)\) rises from \(q(0) = 0\) to its peak \(q(τ^*)\), (b) the interval \([τ^*\), \(π/ω]\) is an interval of descent where \(q(\cdot)\) falls from \(q(τ^*)\) to \(-1 + e^{-σ/ω}ν_0/ω\).

Therefore there is exactly one positive time instant (namely \(τ_+\)) when \(q(\cdot)\) equals its initial value \(q(0) = 0\). And the time instant \(τ_+\) must satisfy:

\[
τ^* < τ_+ < π/ω, \ \text{if} \ ξ > -κ.
\]

4) Growth of \(q(\cdot)\) on \([0, τ^*]\) and on \([τ^*, π/ω]\) : We shall now compare the growth of \(q(\cdot)\) on the two finite intervals: \([0, τ^*]\), \([τ^*, π/ω]\).

\[
q'(t) = \frac{e^{-σt}}{ω} [(ωµ_1 - σν_0)\cos wt - (σµ_1 + νω_0)\sin wt],
\]

\[
= \frac{χ_4}{υ_0} e^{-σt} \cos [(ωµ_1 - σν_0)\sin wt + (σµ_1 + νω_0)\cos wt].
\]

where \(χ_4 = \sqrt{(ωµ_1 - σν_0)^2 + (σµ_1 + νω_0)^2}\). The cosine function is an odd function w.r.t. any of its roots. This is:

\[
\cos (t^* + t) = -\cos (t^* - t), \ \forall t, \ \text{if} \ \cos t^* = 0.
\]
The magnitude of rise in the value of the function over the interval \([\tau^*, \tau^* + \delta]\) Clearly, \(|\tau^* - \delta| < 0\). Let the time \(\tau\) be chosen such that \(0 \leq \tau \leq \tau^*\). Then because of the odd symmetry we can write:

\[
\frac{d}{dt} \left( \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right) e^{-\sigma \delta} \cos \left( -\omega \delta + \omega \tau^* + \arccos \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right) \\
< e^{+\sigma \delta} \cos \left( -\omega \delta + \omega \tau^* + \arccos \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right),
\]

\[
\frac{d}{dt} \left( \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right) e^{-\sigma \delta} \cos \left( -\omega \delta + \omega \tau^* + \arccos \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right) \\
= e^{+\sigma \delta} \cos \left( +\omega \delta + \omega \tau^* + \arccos \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right),
\]

\[
\frac{d}{dt} \left( \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right) e^{-\sigma \delta} \cos \left( -\omega \delta + \omega \tau^* + \arccos \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right) \\
\leq e^{+\sigma \delta} \cos \left( +\omega \delta + \omega \tau^* + \arccos \frac{\omega \mu_1 - \sigma \nu_0}{\lambda_4} \right).
\]

Hence we conclude that \(\hat{\phi}(\tau^*) < \hat{\phi}(2\tau^*)\).

7) The derivative \(f_+(\xi)\) is less than one if \(\xi > -\kappa\): From the preceding, it follows that:

\[
\left| f_+(\tau^* + \delta) \right| < \left| \hat{\phi}(\tau^*) \right| < \left| \hat{\phi}(2\tau^*) \right| < 1.
\]

D. The derivative \(f_+(\cdot)\) is continuous on \([0, +\infty)\)

Exactly like in the case of distinct real poles (Section IV-C), in this case too is \(f_+(\cdot)\) a continuous function on \([0, +\infty)\).

E. The function \(f_+(\xi)\) is a contraction mapping on every finite interval of the form: \([0, \theta]\)

We shall find a tight upper bound for the magnitude of the derivative \(f_+(\cdot)\) on finite intervals of the form: \([0, \theta]\) with \(0 < \theta < +\infty\). Just like in Section IV-C we use the continuity of \(f_+(\xi)\).

The magnitude of \(f_+(\xi)\) is continuous on \([0, \theta]\). By the Weierstrass theorem on extreme values of continuous functions, it follows that the magnitude of \(f_+(\cdot)\) attains its maximum on this closed and bounded interval. Denote this maximum value by \(\overline{\theta}\). And so we have:

\[
\left| f_+(\xi) \right| \leq \overline{\theta} < 1, \forall \xi \in [0, \theta].
\]
because of: (i) $\tau_+$ is bounded above, (ii) $\nu_0$ is independent of $\xi$, and (iii) if $\xi \to \infty$ then $\mu_1 \to \infty$. And since
$$\lim_{\xi \to \infty} \tau_+ (\xi, 0) \leq \frac{\pi}{\omega}, \quad \sin \left( \omega \lim_{\xi \to \infty} \tau_+ (\xi, 0) \right) = 0,$$

it follows that
$$\lim_{\xi \to \infty} \tau_+ (\xi, 0) = \frac{\pi}{\omega}.$$

2) If $\xi \to +\infty$, then $|f_+ (\xi)| \to e^{-\sigma\pi/\omega}$: From Equation (33) we get:
$$f_+ (\xi) = -\kappa - \frac{2\sigma}{(\sigma^2 + \omega^2)} \gamma + \frac{1}{\omega}(\omega\mu_1 + \sigma\nu_0) \cos \omega \tau_+ e^{-\sigma\tau_+} + \frac{1}{\omega}(\sigma\mu_1 - \omega\nu_0) \sin \omega \tau_+ e^{-\sigma\tau_+}.$$

Hence we can do the straightforward derivation below:
$$\lim_{\xi \to \infty} \left| \frac{f_+ (\xi)}{\xi} \right| = \lim_{\xi \to \infty} \kappa + \frac{2\gamma/\sigma^2 + \omega^2}{\xi} + \lim_{\xi \to \infty} \frac{(\omega\mu_1 + \sigma\nu_0) \cos \omega \tau_+ e^{-\sigma\tau_+}/\omega}{\xi} + \lim_{\xi \to \infty} \frac{(\sigma\mu_1 - \omega\nu_0) \sin \omega \tau_+ e^{-\sigma\tau_+}/\omega}{\xi} = 0 - e^{-\sigma\pi/\omega} + 0 = e^{-\sigma\pi/\omega} < 1.$$

Hence for all sufficiently large $\xi$ we get the inequality $|f_+ (\xi)| < |\xi|$. Since $f_+ (\xi)$ is negative, it follows that assumption A 4 holds in this case also.

VII. ASYMPTOTIC BEHAVIOUR OF RFS FOR OTHER ZERO LOCATIONS

We continue with the RFS for the transfer function:
$$\frac{e^{-\kappa s} + \gamma}{\sigma^2 + a_1 s + a_2},$$

with $\gamma, a_1, a_2 > 0$, but in contrast to previous sections, we consider the coefficient $\kappa$ as a free parameter. We ask how the asymptotic behaviour of the RFS changes as we change the sign of $\kappa$. And we answer via a study of how the switching point transformation function changes as we change the sign of $\kappa$.

A. INFLUENCE OF $\kappa$ ON THE SWITCHING POINT TRANSFORMATION FUNCTION

To emphasize dependence on $\kappa$, let $f_{+*, \kappa} (\cdot)$ denote the Switching point transformation function corresponding to a value of $-\kappa$ for the coefficient of $s$ in the numerator of the plant transfer function. Clearly the function $f_{+*, \kappa} (\cdot)$ is completely determined by the dynamics of the ODE:
$$\frac{d}{dt} \left( x - A^{-1} B \right) = A \left( x - A^{-1} B \right).$$

The only effect of varying $\kappa$ is to vary the $x_1$-coordinate of the sink $A^{-1} B$ (see (7)). Thus, if we vary $\kappa$, then the entire vector field of the above ODE is simply translated along the $x_1$-axis. If $l, \bar{\kappa}$ are two values for the parameter $\kappa$, then
$$f_{+*, \kappa} (\xi) = f_{+*, \bar{\kappa}} (\xi + l - \bar{\kappa}) - l - \bar{\kappa}, \quad \forall \xi \in \mathbb{R},$$

and,
$$f_{+*, \kappa} (\xi) = f'_{+*, \kappa} (\xi + l - \bar{\kappa}), \quad \forall \xi \in \mathbb{R}.$$

From the known properties of $f_{+*, \kappa} (\cdot), f'_{+*, \kappa} (\cdot)$ for positive values of $\bar{\kappa}$, we shall derive properties when the parameter $\kappa$ is either zero or negative.
B. Plant with no finite zero

Let \( \hat{\kappa} \) be positive. Then for the RFS corresponding to \( \kappa = 0 \), the origin is an equilibrium point, and there are no chattering points. Furthermore,

\[
f_{+0}(\xi) = f_{+\hat{\kappa}}(\xi - \hat{\kappa}) + \hat{\kappa}, \quad \forall \xi \in \mathbb{R}, \text{ and so,}
\]

\[
f_{-0}(\xi) = \begin{cases} 
-\xi & \text{if } \xi < 0, \\
\int_0^{\xi} f'_{-\hat{\kappa}}(\zeta - \hat{\kappa}) d\zeta & \text{if } \xi \geq 0.
\end{cases}
\]

We have proved in earlier sections that \(-f_{+\hat{\kappa}}(\xi - \hat{\kappa})\) is monotonically increasing on \([-\hat{\kappa}, \infty)\), and that the magnitude of the derivative \(f'_{+\hat{\kappa}}(\xi - \hat{\kappa})\) is strictly less than one on \((-\hat{\kappa}, \infty)\). Hence we can conclude: (i) that \(-f_{+0}(\xi)\) is monotonically increasing on \([0, \infty)\), and (ii) that

\[-f_{+0}(\xi) < \xi, \quad \text{for } \xi > 0.\]

Hence zero is the only possible fixed point for \(-f_{+0}(\cdot)\). Thus starting with the second element of \(\Xi(\xi)\), the successive iterates of \(-f_{+0}(\cdot)\) decrease monotonically and converge to zero. Therefore we conclude that all trajectories converge to the origin, for the RFS with the following class of plant transfer functions

\[
\frac{\gamma}{s^2 + a_1 s + a_2}, \quad \text{with } \gamma, a_1, a_2 > 0.
\]

This conclusion could not have been reached if we had tried to apply the Circle criterion to this class of plants. We do not yet know whether this conclusion could have been reached by applying the Popov criterion.

C. Plant zero is negative real

Let \( \hat{\kappa} \) be positive. Then for the RFS corresponding to \( \kappa = -\hat{\kappa} \), the set \( \{(x_1, x_2) : x_1 \in [-\hat{\kappa}, \hat{\kappa}], \text{ and } x_2 = 0\} \) is the chattering set, and there is no equilibrium point. Furthermore,

\[
f_{+,-\hat{\kappa}}(\xi) = f_{+\hat{\kappa}}(\xi - 2\hat{\kappa}) + 2\hat{\kappa}, \quad \forall \xi \in \mathbb{R}, \text{ and so,}
\]

\[
f_{-0}(\xi) = \begin{cases} 
-\xi & \text{if } \xi < \kappa, \\
\int_0^{\xi} f'_{+,-\hat{\kappa}}(\zeta - 2\hat{\kappa}) d\zeta & \text{if } \xi \geq \kappa.
\end{cases}
\]

As in Section VII-B we can conclude: (i) that \(-f_{+,-\hat{\kappa}}(\xi)\) is monotonically increasing on \([\kappa, \infty)\), and (ii) that

\[-f_{+,-\hat{\kappa}}(\xi) < \xi - 2\hat{\kappa}, \quad \text{for } \xi > \hat{\kappa}.
\]

Starting with the second element of \(\Xi(\xi)\), under each iteration of \(-f_{+,-\hat{\kappa}}(\cdot)\) the switching point decreases by at least 2\(\hat{\kappa}\), unless it is already in the chattering set. And once the RFS hits a chattering point, it always "stays" there. Hence if the plant transfer function belongs to the class

\[
\frac{\kappa s + \gamma}{s^2 + a_1 s + a_2}, \quad \text{with } \kappa, \gamma, a_1, a_2 > 0,
\]

then all trajectories of the RFS converge asymptotically to the chattering set, which is bounded, and of measure zero.

References

[1] K. J. Aström and T. Hägglund, “Automatic tuning of simple regulators with specifications on phase and amplitude margins,” Automatica, vol. 20, no. 5, pp. 645–651, 1984.

[2] K. J. Aström and T. Hägglund, “Auto-tuners for PID controllers,” in: The Impact of Control Technology, T. Samad and A.M. Annaswamy (eds.), IEEE Control Systems Society, 2nd edition, 2014, available: [http://www.ieeeccss.org/]

[3] U. Holmberg, “Relay Feedback of Simple Systems,” Doctoral dissertation, Department of Automatic Control, Lund Institute of Technology, 1991. [http://portal.research.lu.se/portal/files/4722582/8568298.pdf]

[4] A. A. Andronov, A. A. Vitt, and S. E. Khaitkin, Theory of Oscillations, 2nd ed., ser. International Series of Monographs in Physics, Vol. 4. Oxford: Pergamon press, 1966, Russian edition published as Teoriya kolebanii, Moscow: Gostekhizdat, in 1937.

[5] N. Minorsky, Nonlinear Oscillations, 1st ed. Van Nostrand, 1962.

[6] J.-C. Gille, M. J. Pelegrin, and P. Decaulne, Feedback Control Systems. MacGraw-Hill book company, Inc., 1959.

[7] Y. Z. Tsypkin, Relay control systems. Cambridge: Cambridge University Press, 1984, translated from the Russian by C. Constanda.

[8] K. J. Aström, “Oscillations in systems with relay feedback,” in Adaptive Control, Filtering, and Signal Processing, K. J. Aström, G. C. Goodwin, and P. R. Kumar, Eds. Springer New York, 1995, pp. 1–25.

[9] J. M. Goncalves, A. Megretski, and M. A. Dahleh, “Global stability of relay feedback systems,” IEEE Transactions on Automatic Control, vol. 46, no. 4, pp. 550–562, Apr 2001.

[10] P.-A. Bliman and A. Krasnosel’skii, “Periodic solutions of linear systems coupled with relay,” Nonlinear Analysis: Theory, Methods & Applications, vol. 30, no. 2, pp. 687–696, 1997, proceedings of the Second World Congress of Nonlinear Analysts.

[11] S. Varigonda and T. T. Georgiou, “Dynamics of relay relaxation oscillators,” IEEE Transactions on Automatic Control, vol. 46, no. 1, pp. 65–77, Jan 2001.

[12] Karl H. Johansson, A. Rantzer, “Global Analysis of Third-Order Relay Feedback Systems,” IFAC Proc., vol: 29, no. 1, pp. 1937-1942, 1996.

[13] A. Megretski, “Global Stability of Oscillations Induced by a Relay Feedback,” IFAC Proc., vol: 29, no. 1, pp. 1931-1936, 1996.

[14] R. S. Palais, “A simple proof of the Banach contraction principle,” Jnl. of Fixed Point Theory and Appln., vol. 2, no. 2, pp. 221–223, 2007.

[15] F. Csáki, “Conversion methods from phase-variable to canonical forms,” Periodica polytechnica electrical engineering, Budapest, vol. 18, no. 4, pp. 341–379, 1974. [Online]. Available: https://pp.bme.hu/ce/article/download/4989/4994/