Abstract

Recently the issue of EPR-like correlations in the mutual probability of detecting neutrino together with accompanying charged lepton has received a new impetus. In this paper we describe this effect using the propagators of the particles involved in the Schwinger’s parametric integral representation. We find this description more simple and more suitable to the purpose than the usual momentum-space analysis. We consider the cases of monochromatic neutrino source, wave packet source, and neutrino creation in a localised space-time region. In the latter case we note that the space-time oscillation amplitude depends on the values of the neutrino masses, and becomes rather small for large relative mass differences (mass hierarchy). We obtain the expressions for the oscillation and coherence lengths in various circumstances. In the region of overlap our results confirm those of Dolgov et al. [1]

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1 Introduction

Space-time oscillation of neutrino flavour [2, 3] is considered to be the most promising effect observation of which might indirectly establish nonzero neutrino mass. By its very nature it requires spatio-temporal description of the processes of neutrino creation, propagation and detection, and of the similar processes that occur with the accompanying particles. Such a description has been performed in [4] and further developed in [5, 6] without ambiguities that sometimes accompany noncritical use of neutrino flavour eigenstates.

Recently [7, 1] the issue of EPR-like correlations in the mutual probability of detecting both neutrino and an accompanying charged lepton has received a new impetus. (It was previously considered in [4].) In order to simplify derivation of the basic effects the authors of [1] combined together descriptions in the configuration space and in the momentum space, of the same relevant processes, using simultaneously such mutually exclusive notions as sharp wave packets in momentum space and definite space-time \textit{a posteriori} trajectories of particles. EPR-like experiments of the same type involving neutral kaon and $B$ meson oscillations were considered in [8]. In this paper the authors also adopted a simple approach using the action values on the particle classical trajectories to evaluate the relevant phase factors in the probability amplitude. Although the results obtained in such a simplified approach are correct, they also might call for a more careful derivation. This will be the aim of the present paper.

In this paper we try to analyse the phenomenon in a consistent way using the propagators of the particles involved in the Schwinger’s parametric integral representation (see Eq. (3) below). We find this description more simple and more suitable to the purpose than the usual one which employs propagators in momentum space representation. This latter involves rather complicated momentum integrations (see, e.g., [5, 9]) and, it seems, frequently obscures the physical picture of the phenomenon. Our treatment will be general and will contain the analysis of the EPR-like experiments of detecting neutrino together with the accompanying charged lepton, as well as the standard textbook examples of neutrino flavour space-time oscillations.

After preliminaries in the following section, in Sec. 3 we consider the case of a monochromatic neutrino source and the probability of mutual detection of the neutrino and of the accompanying charged lepton. In Sec. 4 the effect of wave packet neutrino source is analysed. We obtain the expressions for the oscillation and coherence lengths in various circumstances. The case of neutrino source in a strongly localised space-time region will then be considered in Sec. 5. In this case the space-time oscillation amplitude depends rather strongly on the values of the neutrino masses, and becomes rather small for large relative mass differences (neutrino mass hierarchy). We summarise our results in Sec. 6. In the Appendix we provide an alternative derivation of the probability amplitude for the case of monochromatic neutrino source, in order to elucidate the difference between this case and the case of neutrino source strongly localised in space-time.
2 Preliminaries

Throughout this paper we consider a process in which neutrino is created together with accompanying charged lepton, and afterwards both particles are detected. The charged weak currents \( \mathcal{O}_\alpha \nu \) are involved in the description of this process, where \( \mathcal{O}_\alpha = \gamma_\alpha (1 + \gamma_5) \), and \( \gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3 \). The amplitude of creation of a \( l - \nu \) pair at space-time point \( x \) is proportional to \( \mathcal{O}_\alpha J_\alpha S(x) \), with \( J_\alpha S(x) \) being the source current responsible for this process. In the case of pion we would have \( J_\alpha S(x) \propto \partial^\alpha \phi_\pi(x) \), where \( \phi_\pi(x) \) is the pion wave function. The charged lepton produced at space-time point \( x_c \) in a flavour state \( a \) can propagate to space-time point \( x_l \), and the neutrino to space-time point \( x_n \), at which points these particles may be detected.

At the space-time point \( x_n \) one may detect neutrino-induced charged lepton production of flavour \( b \). The amplitude of such a process in which neutrino and the corresponding antilepton are created and subsequently detected will contain the factor

\[
\mathcal{O}_\beta J_\beta D \sum_j U^\dagger_j b U_{ja} \int dx_c S_j(x_n, x_c) O_\alpha J_\alpha S(x_c) S_a(x_c, x_l), \tag{1}
\]

where \( U_{ja} \) is the unitary matrix of neutrino mass–flavour mixing amplitudes, \( U^\dagger_j \) is its Hermitian conjugate, \( a \) and \( b \) numerate flavours, \( j \) numerates the neutrino mass eigenstates, \( S_j \) and \( S_a \) are, correspondingly, the Feynman propagators of neutrino mass specie \( j \) with mass \( m_j \) and of charged lepton flavour \( a \), and \( J_\beta D \) is the current involved in the neutrino detection process, localised around the space-time point \( x_n \).

The total amplitude that will describe the detection of charged lepton and neutrino events will contain, besides the factor (1), also positive and negative energy wave functions of different finite particles involved in the detection process. These factors are of particular nature, they do not affect the dependence of the amplitude on the space-time coordinates \( x_l \) and \( x_n \), hence they will be omitted as irrelevant to the main topic of this paper. Due to these factors, however, as well as, in typical cases, due to the positive-frequency character of the source, the integration region over \( x_c \) in (1) will be effectively restricted to the causal past of both points \( x_n \) and \( x_l \).

The Feynman propagator \( S(x, y) \equiv S_m(x - y) \) for the Dirac field of mass \( m \) has the form

\[
S_m(x) = \left( i\gamma^\alpha \partial_\alpha + m \right) D_m(x), \tag{2}
\]

where \( D_m(x) \) is the Feynman propagator for the Klein-Gordon field of mass \( m \). This propagator has the parametric integral representation (first considered by Schwinger, Dyson and Feynman in the papers collected in [10])

\[
D_m(x) = -\frac{1}{8\pi^2} \lim_{\epsilon \to 0^+} \int_0^\infty d\lambda \exp \left[ -\frac{i}{2} \left( \lambda x^2 + \frac{1}{\lambda} \left[ m^2 - i\epsilon \right] \right) \right], \tag{3}
\]

where \( x^2 = x \cdot x = x^\alpha x_\alpha \) is the Lorentz interval squared. The factors of type \( O_\alpha \) in the amplitude Eq. (1) will have an effect that in the neutrino propagator the term proportional to a unit matrix will not contribute, and only that proportional to the Dirac gamma matrices
will remain. This general property is due to the equality \( O_\alpha O_\beta = 0 \). Thus in (4) we can replace \( S_j(x_n, x_c) \) by \( \tilde{S}_j(x_n, x_c) \equiv \tilde{S}_{m_j}(x_n - x_c) \), where
\[
\tilde{S}_m(x) = i\gamma^\alpha \partial_\alpha D_m(x) .
\]

3 Monochromatic source

In this section we investigate the case of a monochromatic source current \( J_\alpha^S(x) \) that can arise, for instance, in the process of pion decay. Let
\[
J_\alpha^S(x) \propto e^{-ip \cdot x} ,
\]
with constant four-momentum \( p \). In the case of pion we would have \( J_\alpha^S(x) \propto \partial^\alpha \phi_\pi(x) \propto p^\alpha \exp(-ip \cdot x) \), where \( \phi_\pi(x) \) is the pion wave function. We make notation
\[
x_{nc} = x_n - x_c , \quad x_{lc} = x_l - x_c , \quad x_{nl} = x_n - x_l .
\]
In the amplitude (1) we represent the propagators using (2)–(4), first perform integration over \( x_c \), then over the parameters \( \lambda_l \) and \( \lambda_n \) that appear in the representation (3), respectively, for charged lepton and neutrino propagators. The integral over \( x_c \) is Gaussian, hence it can be evaluated exactly, pre-exponential factors can be obtained after integration over \( x_c \) by taking partial derivatives with respect to \( x_l \) and \( x_n \) according to (2). The remaining integral over \( \lambda_l, \lambda_n \) will be afterwards evaluated in the stationary phase approximation.

Consider the integral over \( x_c \) of one of the terms in the sum of (1). The phase in the exponent of the integrand will stem from the expression (3) for propagators, and from the source current in (1). It will be given by
\[
\phi = -\frac{1}{2} \lambda_l x_{lc}^2 - \frac{1}{2} \lambda_n x_{nc}^2 - p \cdot x_c .
\]
Its extremal point \( x_c = x_c(\lambda_l, \lambda_n) \) is determined from the equation
\[
\frac{\partial \phi}{\partial x_c} \equiv \lambda_l x_{lc} + \lambda_n x_{nc} - p = 0 .
\]
We also have for the matrix of the second derivatives
\[
\frac{\partial^2 \phi}{\partial x_c^\alpha \partial x_c^\beta} = -(\lambda_l + \lambda_n) g_{\alpha\beta} ,
\]
so that integration over \( x_c \) will produce a factor
\[
\int dx_c e^{i\phi} = \frac{4i\pi^2}{(\lambda_l + \lambda_n)^2} e^{i\phi_*} ,
\]
where \( \phi_* \) is the value of the phase \( \phi \) at the extremal point:
\[
\phi_* = -\frac{\lambda_l \lambda_n x_{nl}^2 - m^2 + 2p \cdot (\lambda_l x_l + \lambda_n x_n)}{2(\lambda_l + \lambda_n)} .
\]
Now consider the integral over the \( \lambda \)'s. It will be evaluated in the stationary phase approximation. The phase of the integrand is given by

\[
\Phi = \phi_\ast - \frac{1}{2} \left( \frac{m_l^2}{\lambda_l} + \frac{m_n^2}{\lambda_n} \right),
\]

(12)

where \( m_l \) and \( m_n \) are the masses, respectively, of charged lepton and of neutrino. The stationary point is determined by differentiating \( \Phi \) using \( \Pi \), or by the equivalent conditions in the convenient form obtained using \( \Pi \) and \( \Phi_\ast \),

\[
\frac{\partial \Phi}{\partial \lambda_l} \equiv -\frac{1}{2}x_{lc}^2 + \frac{m_l^2}{2\lambda_l^2} = 0, \quad \frac{\partial \Phi}{\partial \lambda_n} \equiv -\frac{1}{2}x_{nc}^2 + \frac{m_n^2}{2\lambda_n^2} = 0.
\]

(13)

In these equations \( x_c = x_c(\lambda_l, \lambda_n) \) is the solution of Eq. \( \Pi \). From \( \Pi \) we have the relation

\[
\lambda_l x_{lc} = p_l, \quad \lambda_n x_{nc} = p_n,
\]

(14)

satisfied by the extremal values of \( \lambda \)'s, where \( p_l \) and \( p_n \) are the four-momenta that the charged lepton and the neutrino respectively would have were they free classical particles moving from the space-time creation point \( x_c \) respectively to the registration points \( x_l \) and \( x_n \). Then Eq. \( \Pi \) expresses the energy-momentum conservation law, the condition from which the extremal point \( x_c \) with extremal \( \lambda \)'s can be found most easily.

We need also the matrix of the second derivatives of \( \Phi \) over \( \lambda \)'s at the extremal point. Differentiating the identity \( \Pi \) we find

\[
\frac{\partial x_c}{\partial \lambda_l} = \frac{x_{lc}}{\lambda_l + \lambda_n}, \quad \frac{\partial x_c}{\partial \lambda_n} = \frac{x_{nc}}{\lambda_l + \lambda_n},
\]

(15)

and, differentiating \( \Pi \),

\[
\frac{\partial^2 \Phi}{\partial \lambda_l^2} = -\frac{x_{lc}^2}{\lambda_l + \lambda_n} - \frac{m_l^2}{\lambda_l^2} = -\frac{m_l^2}{\lambda_l^2} \frac{2\lambda_l + \lambda_n}{\lambda_l + \lambda_n}.
\]

(16)

\[
\frac{\partial^2 \Phi}{\partial \lambda_n^2} = -\frac{x_{nc}^2}{\lambda_l + \lambda_n} - \frac{m_n^2}{\lambda_n^2} = -\frac{m_n^2}{\lambda_n^2} \frac{2\lambda_n + \lambda_l}{\lambda_l + \lambda_n},
\]

(17)

\[
\frac{\partial^2 \Phi}{\partial \lambda_l \partial \lambda_n} = -\frac{x_{lc} \cdot x_{nc}}{\lambda_l + \lambda_n} = -\frac{p_l \cdot p_n}{\lambda_l \lambda_n (\lambda_l + \lambda_n)}.
\]

(18)

Let

\[
m = \sqrt{p \cdot p},
\]

(19)

be the effective mass of the source. In the case of pion decay this will be equal to the pion mass \( m_\pi \). In a realistic case

\[
m - m_l \gg m_n, \quad m_l \gg m_n.
\]

(20)

Below we will see [cf. Eq. \( \Pi \)] that in the limit \( m_n \rightarrow 0 \) the extremal values of \( \lambda \)'s remain finite. Thus we can approximate the determinant of the matrix \( \partial^2 \Phi/\partial \lambda_i \partial \lambda_j \) by its limit as \( m_n \rightarrow 0 \). The result is

\[
\det \left( \frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_j} \right) \approx -\left( \frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_j} \right)^2 = -\left( \frac{p_l \cdot p_n}{\lambda_l \lambda_n (\lambda_l + \lambda_n)} \right)^2.
\]

(21)
Therefore the integral over \( \lambda \)'s will produce a factor
\[
\frac{2\pi \lambda_l \lambda_n (\lambda_l + \lambda_n)}{p_l \cdot p_n} e^{i\Phi_*},
\]  
where \( \Phi_* \) is the extremal value of the phase \( \Phi \), which is given by
\[
\Phi_* = -m_l \sqrt{x_{lc}^2 - m_n^2} - p \cdot x_c,
\] 
with \( x_c \) being the extremal point of \( \phi \) at extremal values of \( \lambda \)'s – the solution to (8), (13). Using the extremality conditions (8), (13) we easily find
\[
\frac{\partial \Phi_*}{\partial x_l} = -p_l, \quad \frac{\partial \Phi_*}{\partial x_n} = -p_n,
\] 
and
\[
\Phi_* = -p_l \cdot x_l - p_n \cdot x_n = -p \cdot x_l - p_n \cdot x_{nl},
\] 
Note that the four-momenta \( p_l \) and \( p_n \) lie in the plane formed by the four-vectors \( p \) and \( x_{nl} \), and are determined by energy-momentum conservation.

Combining together the factors calculated (10) and (22), dropping the resulting overall numerical constant \( i/8\pi \), and using (25) we obtain the expression for the amplitude (1) in the case of monochromatic source
\[
e^{-i p \cdot x_l} \sum_j \frac{\lambda_l \lambda_n}{(\lambda_l + \lambda_n) p_l \cdot p_n} O_D \gamma_\alpha P^\alpha_{\gamma} O_S \left(m_l - \gamma_\beta P^\beta_{\gamma}\right) U^\dagger_{bj} U_{ja} e^{-i p_n \cdot x_{nl}},
\] 
where \( O_D = O_\alpha J^\alpha_D \), \( O_S = O_\alpha J^\alpha_S \). Note that the coordinate-dependence of the current \( J_S(x) \) has transformed to the phase of (26). Also note that the extremal values of \( \lambda \)'s as well as the four-momenta \( p_l \) and \( p_n \) under the sum (26) depend on the neutrino specie \( j \). However, the prefactors in our expression (26), as well as in (30) below, are calculated only up to terms proportional to \( m_n^2 \), with this precision they can be taken in the limit \( m_n = 0 \).

The extremal values of \( \lambda_l \) and \( \lambda_n \) determined by the system of equations (8), (13) can be easily obtained from the kinematics of the problem. Let us denote by \( t \) and by \( d \) correspondingly the time difference and the absolute spatial distance between the events \( x_n \) and \( x_l \) in the rest frame of the source [in which \( p^\alpha = (m, 0) \)], and by \( v_l \) and \( v_n \) the velocities, respectively, of the charged lepton and of the neutrino in this frame (see Fig. 1). In the rest frame of the source one has
\[
t_{lc} = \frac{d - v_n t}{v_l + v_n}, \quad d_l := |\mathbf{x}_{lc}| = v_l t_{lc}, \quad t_{nc} = \frac{d + v_l t}{v_l + v_n}, \quad d_n := |\mathbf{x}_{nc}| = v_n t_{nc},
\] 
where also \( d_l \) denotes the spatial distance in the source rest frame between the point \( x_l \) of charged lepton detection and the extremal point \( x_c \), and \( d_n \) has the same meaning for neutrino. Then
\[
x_{lc}^2 = \left(\frac{d - v_n t}{v_l + v_n}\right)^2 \left(1 - v_l^2\right), \quad x_{nc}^2 = \left(\frac{d + v_l t}{v_l + v_n}\right)^2 \left(1 - v_n^2\right),
\]
and, using (13), we obtain
\[ \lambda_l = \frac{m_l}{\sqrt{x_l^2}} = \frac{v_l E_l}{d_l}, \quad \lambda_n = \frac{m_n}{\sqrt{x_n^2}} = \frac{v_n E_n}{d_n}, \] (29)

where \( E_l \) and \( E_n \) are the energies, respectively, of the charged lepton and of the neutrino in the rest frame of the source. In this notation and in the approximation of \( m_n = 0 \) for the prefactors (but not for the phase) the amplitude (26) will acquire the form
\[ e^{-i p \cdot x_l} O_D \gamma_\alpha P^\alpha D S \left( m_l - \gamma_\beta P^\beta \right) \sum_j U_{bi}^\dagger U_{ja} e^{-i p_n \cdot x_{nl}}. \] (30)

By the way, from the expressions (29) it is clear that the extremal values of \( \lambda \)'s remain finite in the limit of \( m_n \to 0 \), as was stated above.

![Figure 1: Time and length definitions in the neutrino source rest frame.](image)

We shall now estimate the applicability limits of the stationary phase approximation used. Our approximation will be good when the extremal values of \( \lambda \)'s are much larger than their dispersions determined by the matrix \( \partial^2 \Phi / \partial \lambda_i \partial \lambda_j \). Using (29) and (16)–(18) we obtain after straightforward analysis the conditions
\[ d_l \leq d_n \ll m_l^2 \quad \text{or} \quad d_n \leq d_l \ll m_n^2, \] (31)
under which our approximation is valid. They imply also the condition
\[ m d \gg 1, \] (32)
which is quite reasonable.

In the limit of (20) the energy-momenta \( p_l \) and \( p_n \) change relatively very slightly with the neutrino mass specie \( j \), and, we remember, our prefactors in (30) were actually calculated
in this limit. In this case the space-time behaviour of the $l$-$\nu$ pair detection probability is given by

$$
P_{ba}(x_n, x_l) \propto \frac{1}{d^2} \left[ \sum_j |U_{b_j}^* U_{j_a}|^2 + \sum_{j \neq k} |U_{b_j}^* U_{j_a} U_{k_b}^* U_{k_b}| \cos \left(p_{jk} \cdot x_{nl} + \varphi_{jk}^{ab}\right) \right], \quad (33)
$$

where $p_j$ and $p_k$ are neutrino four-momenta of mass species, correspondingly, $j$ and $k$, $p_{jk} = p_j - p_k$, and $\varphi_{jk}^{ab}$ are constant phases that stem from the product of matrices $U$ and $U^\dagger$. The second sum in the square brackets of (33) describes space-time oscillations of the probability.

From Eq. (33) one obtains the oscillation length and oscillation time of the probability considered. Using the energy-momentum conservation in the source rest frame of reference one has

$$
p_{jk}^0 = \frac{\Delta_{jk}}{2m}, \quad |p|_{jk} \equiv |p_j| - |p_k| \approx -\frac{\Delta_{jk}}{2v_l m} = -\frac{\Delta_{jk} E_l}{2v_n m E_n}, \quad (34)
$$

where approximation uses the assumption (20) that neutrino has very small mass, and

$$
\Delta_{jk} = m_j^2 - m_k^2. \quad (35)
$$

Thus, oscillation length $L_{osc}$ and oscillation time $T_{osc}$ of the $(jk)$-component of (33) in this frame of reference are given, respectively, by (we use the limit of $v_n = 1$)

$$
L_{osc} = \frac{m}{E_l} L, \quad T_{osc} = \frac{m}{E_n} L, \quad (36)
$$

where

$$
L = \frac{2m}{|\Delta_{jk}|} \quad (37)
$$

is the standard expression. To proceed to any other reference frame what one has to do is to transform the four-vector components $p_{jk}^0$ obtained in (34) to this new frame. The expressions (36) and the relevant expressions in the laboratory frame of reference have been obtained in [1].

If one of the particles, charged lepton or neutrino, is not observed, then the probability of detecting the other one is uniform in space and time. This is quite obvious without any calculations and is due to the fact that the $l$-$\nu$ pair creation probability for a monochromatic source is homogeneous in space and time. If neutrino is not detected, oscillations in the charged lepton detection probability disappear also due to orthogonality of neutrino mass eigenstates. This last cause will operate with any type of source, not necessarily monochromatic. Specifically, it is the necessity of summing the probability over the neutrino flavour index $b$ that will eliminate the oscillatory terms in this case. A detailed discussion of these issues is presented in [1].

4 Wave-packet source

First of all let us analyse in a little more detail the effective region of integration over $x_c$ in (1) in the case of monochromatic source considered in the previous section. In other words, it
is the region of constructive interference, from which most of the contribution to the integral in (1) comes. The extension of this region in space-time around the extremal point \( x_c \) is determined by the covariance matrix (3) for given values of \( \lambda 's \), and by the variation of \( \lambda 's \) that are determined by the covariance matrix \( \partial^2 \Phi / \partial \lambda_i \partial \lambda_j \) with the components (16)–(18).

First, using (9) and (29) we obtain the estimate of the linear dimensions \( \delta x \) of the effective region of integration for fixed extremal values of \( \lambda 's \) as

\[
\delta x \simeq (\lambda_l + \lambda_n)^{-1/2} = \sqrt{\frac{d d_n}{E_n}} \leq \sqrt{\frac{d}{E_n}}.
\]  (38)

Next, we must estimate the linear dimensions \( \delta x_c \) of the spread of the extremal value \( x_c(\lambda_l, \lambda_n) \) caused by the spread \( \delta \lambda \) of the values of \( \lambda 's \). This latter spread can be estimated using (21) as

\[
\delta \lambda \simeq \left| \det \left( \frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_j} \right) \right|^{-1/4} \approx \sqrt{\frac{\lambda_l \lambda_n (\lambda_l + \lambda_n)}{p_l \cdot p_n}}.
\]  (39)

Then using (15), (27), (29) and the condition \( \delta \lambda \ll \lambda_l, \lambda_n \) provided by Eq. (31), we will have the estimate

\[
\delta x_c \simeq \frac{d \delta \lambda}{\lambda_l + \lambda_n} \approx d \sqrt{\frac{\lambda_l \lambda_n}{p_l \cdot p_n (\lambda_l + \lambda_n)}} \approx \sqrt{\frac{d}{m}}.
\]  (40)

Approximations “\( \approx \)” in (38) and (40) use smallness of the neutrino mass and become equalities in the limit \( m_n \to 0 \). The last value in (38) is larger than that in (40) hence the dimension of the region of constructive interference will be estimated by (38).

In realistic situations the source of neutrinos can often be approximated by a wave packet with sharp energy and momentum distribution (therefore small relative energy-momentum spread). Let us suppose that in the rest frame of the source the spread in the coordinate space is \( \sigma_x \), in momentum space \( \sigma_p \), so that \( \sigma_x \sigma_p \sim 1 \). The source has also finite coherence time, in the case of pion this will be determined by its lifetime, or by its collision time with the environment. Typically, however, the time spread \( \sigma_t \) of the wave packet is much larger than its spatial spread and the latter will determine most of the interesting effects.

If the source is to a high precision monochromatic so that its spatial and temporal spread is sufficiently large, namely, if

\[
\delta x \ll \sigma_x, \sigma_t,
\]  (41)

then we can use the expressions from the previous section for the detection probability amplitude whenever the effective region of integration over \( x_c \) (region of constructive interference) lies well within the source wave packet. Due to Eq. (38) the conditions (41) essentially imply

\[
d \ll E_n \times \min \left( \sigma_x^2, \sigma_t^2 \right).
\]  (42)

Spatio-temporal oscillations in the mutual detection probability can be observed only up to certain relative distances between the detection points of charged lepton and neutrino. We are going to determine such maximal distances, called coherence lengths beyond which oscillations cease to occur. The reason for such distances to exist is that for different neutrino masses \( m_j \) and \( m_k \) the corresponding centres (extremal points) \( x_j \) and \( x_k \) of the integration
region in $x_c$ are different. If they become sufficiently separated in space-time they may no
longer be able to lie simultaneously within the wave packet of the source, thus components in the probability amplitude that correspond to different neutrino mass species will not be able to interfere.

Consider this effect quantitatively. The shift four-vector $x_{jk} = x_j - x_k$ lies in the plane
of the four-momenta $p_l$ and $p_n$, and can be decomposed into components $x_{jk}^{(l)}$ and $x_{jk}^{(n)}$ that go, respectively, along the directions of $p_l$ and $p_n$. These components can be easily estimated. Using Eq. (27) we obtain for the time components (approximation uses $v_n \approx 1$ and $\Delta v_l, \Delta v_n \ll v_l, v_n$)

\[
t_{jk}^{(l)} \approx \frac{\Delta v_n}{(v_l + v_n)^2} (d + v_l t) \approx -\frac{\Delta_{jk} E_l}{2mE_n^2} d_n ,
\]

\[
t_{jk}^{(n)} \approx \frac{\Delta v_l}{(v_l + v_n)^2} (d - v_n t) \approx -\frac{\Delta_{jk} m_l^2}{2m^2E_n^2} d_l ,
\]

where $\Delta_{jk}$ is given by (35). The distances $d_l$ and $d_n$ then change as

\[
\Delta d_l = -\Delta d_n = v_n t_{jk}^{(n)} - v_l t_{jk}^{(l)} \approx \frac{-\Delta_{jk} m_l^2}{2m^2E_n^2} d_l + \frac{\Delta_{jk}}{2mE_n} d_n .
\]

For large enough absolute values of $t_{jk} = t_{jk}^{(l)} + t_{jk}^{(n)}$ and/or $\Delta d_l$ the centres $x_j$ and $x_k$ will not be able to lie simultaneously within the source wave packet. Components in the probability amplitude which correspond to different neutrino mass species will not be able to interfere, and probability will cease to oscillate. This will determine coherence lengths of the detection probability oscillations in various cases.

In the case

\[d_n = \frac{m_l^2}{mE_n} d_l ,\]

from (45) it follows that $\Delta d_l = 0$, and the shift $x_{jk}$ is in the temporal direction in the source rest frame. For sufficiently large value of $d = d_l + d_n$ the absolute value of the shift $t_{jk}$ becomes larger than the extension $\sigma_t$ of the source wave packet in the temporal direction. This determines coherence length $L_c$ – the largest value of $d$ at which oscillations can be observed. In the case considered (46) it is determined by using (43), (44) and (46) as

\[L_c = \sigma_t E_n L \left( 1 + \frac{mE_n}{m_l^2} \right) ,\]

with $L$ given by (37). The condition (42) necessary for our approximation, will imply that the formula (47) is valid for

\[\frac{\sigma_t^2}{\sigma_t} \gg L \left( 1 + \frac{mE_n}{m_l^2} \right) .\]

Equation (46) implies rather special experimental coincidence conditions. In a less special case

\[d_n > \frac{m_l^2}{mE_n} d_l ,\]
the second term in the last expression of (45) dominates. In this case the coherence length will be determined by the condition that either $|t_{jk}|$ becomes larger than $\sigma_t$, or/and $|\Delta d_l|$ becomes larger than $\sigma_x$. Using (43) and (45) we obtain in this case

$$L_c = L m \times \min \left( \sigma_x, \sigma_t \frac{E_n}{E_l} \right). \quad (50)$$

The condition (42) will determine the validity limit of (50) to be [see Eq. (36)]

$$\sigma_x, \sigma_t^2 \gg \frac{m}{E_n} L = T_{osc}; \quad \sigma_t, \sigma_t^2 \gg \frac{m}{E_l} L = L_{osc}. \quad (51)$$

The case

$$d_n < \frac{m^2}{m E_n} d_l \quad (52)$$

can describe the situation when the momentum of the charged lepton is measured to a good accuracy by measuring its time of flight. Its analysis is quite similar to that of the previous cases. The oscillations in the probability will disappear at the length

$$L_c = \frac{m^2 E_n}{m_l^2} L \times \min(\sigma_x, \sigma_t), \quad (53)$$

and the analysis is valid as long as the estimate (42) is fulfilled, what gives

$$\sigma_x, \sigma_t, \sigma_t^2, \sigma_t^2 \gg \frac{m^2}{m_l^2} L. \quad (54)$$

If the source wave packet is sufficiently broad in space and if one of the particles, charged lepton or neutrino, is not observed, the probability of observing the other one will not oscillate in space and time. This is because after integrating the probability (33) over one of the variables $\{x_l, x_n\}$ the oscillatory terms are averaged to approximately zero. However, with the source wave packet sufficiently narrow in space (but still such that the condition (41) or its equivalent (42) holds), namely,

$$\sigma_x \ll L, \quad (55)$$

even if the accompanying charged lepton is not observed the neutrino flavour oscillations can be observed relative to the source spatial position. Indeed, when integrating the detection probability $P_{ba}(x_n, x_l)$ over $x_l$ we notice that only in a small region of $x_l$ the probability will be nonzero and will be given by (33). This will be the region for which the extremal values of $x_n$ lie within the wave packet of the source. The linear dimensions of this region are similar to $\sigma_x m/E_l$, and Eq. (55) follows from the condition that these dimensions be much smaller than the oscillation length $L_{osc}$ which is given by (36). Then, after integration over $x_l$ the phases of the oscillatory terms in (33) will be fixed and their dependence on the value of $x_n$ will remain. Detection of neutrino alone is the case most frequently discussed in the literature.
Consider this effect more thoroughly. The phase \( p_{jk} \cdot x_{nl} \) in the oscillating term in the probability (33) can be written in terms of the distances \( d_n \) and \( d_l \) introduced in (27). Choosing the z-axis in the direction of \( p_n \) in the source rest frame one will have

\[
p_{jk} = \left( p_{jk}^0, 0, 0, |p|_{jk} \right), \quad x_{nl} = (t, 0, 0, d). \tag{56}
\]

Taking into account the values \( p_{jk}^0 = \Delta_{jk}/2m, \) \( |p|_{jk} \approx -\Delta_{jk} E_l/2v_n m E_n \) [see Eq. (34)] and using (27), one obtains the standard expression for the phase

\[
p_{jk} \cdot x_{nl} = \frac{\Delta_{jk}}{2v_n m} \left( d_n - \frac{E_l}{E_n} d_l \right) + \frac{\Delta_{jk} E_l}{2v_n m E_n} (d_n + d_l) = \frac{d_n}{v_n L} = \frac{t_{nc}}{L}. \tag{57}
\]

From this expression it is again clear that if the distance \( d_n \) is fixed with accuracy better than \( L \) by the position of the source wave packet relative to the neutrino detector, the phases of the probability oscillations will remain fixed even after integration of (33) over the unobserved point \( x_l \), and the oscillations will be observed with respect to the value of \( d_n \). This condition again leads to the estimate (55).

Remarkably, the phase (57) does not depend on \( d_l \). This fact can also be explained as follows. If \( d_n \) is fixed and \( d_l \) is changing, this means that the four-vector \( x_{nl} \) changes (say, by amount \( \Delta x_{nl} \)) in the direction of the charged lepton’s four-momentum \( p_l \) (this is clear from Fig. 1). Then the oscillation phase change is

\[
p_{jk} \cdot \Delta x_{nl} \propto p_{jk} \cdot p_l = -\Delta p_l \cdot p_l = -\Delta \left( p_l^2 \right)/2 = 0.
\]

The expression (57) for the phase coincides with that derived in [1] (see Eqs. (47) and (54) of this reference).

As noted already at the end of the previous section, and as it was discussed in [1], if neutrino is not detected oscillations in the charged lepton detection probability disappear in any case due to orthogonality of neutrino mass eigenstates. Specifically, it is the summation of (33) over the neutrino flavour index \( b \) that will eliminate the oscillatory terms.

5 Source in a localised space-time region

First consider a hypothetical process in which neutrino, together with a charged lepton, is created at a fixed space-time point \( x_c \). Note that in this case the energy and momentum of the neutrino created is totally undetermined. The probability amplitude of detecting neutrino-induced charged lepton production of flavour \( b \) at the space-time point \( x_n \) will contain the factor

\[
A_{ba}(x_n, x_c) = \sum_j U^\dagger_{bj} U_{ja} \tilde{S}_j(x_{nc}), \tag{58}
\]

if the charged lepton created together with the neutrino at point \( x_c \) is of flavour \( a \).

Again, as was already discussed in Sec. 2, the total amplitude that will describe the detection of neutrino event will contain, besides the factor (58), also those related to the processes of creation, propagation and detection of other particles involved. These factors, however, are of particular nature, and do not affect the dependence of the probability amplitude of neutrino detection on the space-time points \( x_n \) and \( x_c \), hence they will be omitted.
The propagator $S_m(x)$ of Eq. (2) has the leading asymptotic behaviour (see, e.g., [1])

$$S_m(x) \sim \left( \frac{e^{3\pi i/4}}{4\sqrt{2\pi^{3/2}}} \right) \frac{m^{3/2}}{(x^2)^{3/4}} \left( 1 + \frac{\gamma_\alpha x^\alpha}{\sqrt{x^2}} \right) \exp \left( -i m \sqrt{x^2} \right), \quad \text{for } m \sqrt{x^2} \gg 1,$$

$$S_m(x) \sim \frac{\gamma_\alpha x^\alpha}{2\pi^2 (x^2)^2}, \quad \text{for } m \sqrt{x^2} \ll 1,$$  \hspace{1cm}(59)  \hspace{1cm}(60)

Hence, oscillations in the neutrino detection probabilities can develop in space and time around $x_n$ only when $m_j \sqrt{x_{nc}^2} \gg 1$ at least for the largest of the neutrino masses, since $\tilde{S}_j(x_{nc})$ do not differ for different $j$ in the opposite limit $m_j \sqrt{x_{nc}^2} \ll 1$. Consider, therefore, the case of $m_j \sqrt{x_{nc}^2} \gg 1$ for all $j$. In this limit the amplitude (58) up to one and the same factor will be given by

$$A_{ba}(x_n, x_c) \propto \sum_j m_j^{3/2} U_{bj}^\dagger U_{ja} \exp \left( -i m_j \sqrt{x_{nc}^2} \right).$$  \hspace{1cm}(61)

Note the mass-dependence of the coefficients in the last equation. The space-time variation of the probabilities of the corresponding processes will be given by

$$P_{ba}(x_n, x_c) = \text{tr} \left[ \cdots A_{ba}^\dagger(x_n, x_c) \cdots A_{ba}(x_n, x_c) \cdots \right] \propto \sum_j m_j^3 |U_{bj}\rangle |U_{ja}\rangle|^2 + \sum_{j \neq k} (m_j m_k)^{3/2} |U_{bj}\rangle |U_{ja}\rangle |U_{ak}\rangle |U_{kb}\rangle \cos \left( m_j \sqrt{x_{nc}^2} + \varphi_{jk}^{ab} \right),$$  \hspace{1cm}(62)

where

$$m_{jk} = m_j - m_k,$$  \hspace{1cm}(63)

and $\varphi_{jk}^{ab}$ are constant phases that stem from the product of matrices $U$ and $U^\dagger$. The last term in the equation (62) describes space-time oscillations of the probabilities.

As an illustration consider mixing between two mass eigenstates $\nu_1$ and $\nu_2$, with two flavour eigenstates $\nu_\mu$ and $\nu_e$. Let, for definiteness, $m_1 > m_2$,

$$\nu_\mu = \nu_1 \cos \theta + \nu_2 \sin \theta, \quad \nu_e = -\nu_1 \sin \theta + \nu_2 \cos \theta,$$  \hspace{1cm}(64)

and let neutrino be created in a flavour eigenstate $\nu_\mu$. The corresponding probabilities of detecting muon and electron events at $x_n$ will be

$$P_{\mu}(x_n, x_c) \propto m_1^3 \cos^4 \theta + m_2^3 \sin^4 \theta + 2 (m_1 m_2)^{3/2} \sin^2 \theta \cos^2 \theta \cos \left( \Delta m \sqrt{x_{nc}^2} \right),$$  \hspace{1cm}(65)

$$P_{e}(x_n, x_c) \propto \left[ m_1^3 + m_2^3 - 2 (m_1 m_2)^{3/2} \cos \left( \Delta m \sqrt{x_{nc}^2} \right) \right] \sin^2 \theta \cos^2 \theta,$$  \hspace{1cm}(66)

where $\Delta m = m_1 - m_2$. The last terms in the square brackets in the equations (59) and (61) describe space-time oscillations of the probabilities. Due to mass-dependence of the coefficients in these expressions the amplitude of oscillations will be suppressed if $m_2 \ll m_1$, and $\cos \theta$ is not too small.

The conditions $m_j \sqrt{x_{nc}^2} \gg 1$, together with the asymptotic form (59) of the propagator, imply that all the mass eigenstate neutrinos $\nu_j$ arrive at the space-time point $x_n$ practically...
as on-mass-shell particles, with four-velocity \( u = x_{nc}/\sqrt{x_{nc}^2} \). In the opposite limit \( m\sqrt{x^2} \ll 1 \) the Feynman propagator has the asymptotic behaviour (64) independent of mass. Then, for instance, in our example of two neutrinos, in the region \( m_2\sqrt{x_{nc}^2} \ll 1 \ll m_1\sqrt{x_{nc}^2} \), which exists in the case of large relative mass difference, \( m_2 \ll m_1 \), one obtains

\[
P_\mu(x_n, x_c) \propto \sin^4 \theta + \frac{\pi}{8} \zeta^3 \cos^4 \theta + \sqrt{\frac{\pi}{2}} \zeta^{3/2} \sin^2 \theta \cos^2 \theta \cos (\zeta - 3\pi/4) ,
\]

\[
P_\epsilon(x_n, x_c) \propto \left( 1 + \frac{\pi}{8} \zeta^3 - \sqrt{\frac{\pi}{2}} \zeta^{3/2} \cos (\zeta - 3\pi/4) \right) \sin^2 \theta \cos^2 \theta ,
\]

where

\[
\zeta = m_1\sqrt{x_{nc}^2}.
\]

It is important to stress the difference between the cases of sufficiently extended wave packet source and the fixed space-time point source. In the first case the probability of detecting neutrino is given by Eq. (33) with the phase given by Eq. (57), in the second case the probability is given by Eq. (62). The origin of this difference lies in the fact that in the former case the amplitude involves integration over \( x_c \), whereas in the latter case the point \( x_c \) is fixed. In the asymptotic limit in which Eq. (59) is valid neutrino propagators have strong pre-exponential mass dependence that results in the peculiar mass-dependence of the probability (62). With propagators in the asymptotic limit it can be explicitly demonstrated that integration of the amplitude over \( x_c \) in the case of monochromatic source produces neutrino-mass-dependent factors that cancel out such pre-exponential neutrino-mass-dependence of the probability amplitude and also modify the phase of the probability amplitude, leading to Eq. (30). In view of the derivation of Eq. (30) presented in Sec. 3 such a demonstration in the main text would be redundant. Therefore, in order to make this point clear, we perform it in the Appendix.

In reality, creation of neutrino cannot occur at a fixed space-time point. However, possible creation region might happen to be sufficiently localised by the nature of the source or by the experimenter, so that the phase differences between components of (71) will be well fixed. Thus the equations of this section will apply to the situation when neutrino creation region is localised in space and time in such a way that

\[
\delta \left[ m_{jk} \sqrt{(x_n - x_c)^2} \right] \approx 1,
\]

where by \( \delta[f(x_c)] \) we signify characteristic variation of \( f(x_c) \) due to variation of \( x_c \) over the creation region. As the probability oscillation phases \( m_{jk} \sqrt{(x_n - x_c)^2} \) are symmetric with respect to \( x_n \leftrightarrow x_c \) interchange, this means that the creation region is to be restricted in space and time by the oscillation time and length scales in the vicinity of the point \( x_n \). Since for small variations

\[
\delta \left[ m_{jk} \sqrt{(x_n - x_c)^2} \right] \approx - \frac{m_{jk} x_{nc} \cdot \delta x_c}{\sqrt{x_{nc}^2}} ,
\]

these oscillation scales will be determined by the four-momentum differences \( p_{jk} = m_{jk} x_{nc}/\sqrt{x_{nc}^2} \), with fixed four-vector \( x_{nc}/\sqrt{x_{nc}^2} \) which is the neutrino four-velocity \( u \) at the detection event \( x_n \).
Within the model of two neutrinos considered above we will have

\[ p_1 - p_2 = u \Delta m = p_1 \frac{\Delta m}{m_1}, \]  

(72)

where \( p_1 \) is the four-momentum of the heavier neutrino mass specie at the detection point. In the case of close neutrino masses, \( \Delta m \ll m_1 \), the oscillation length and time will be given by

\[ L_{osc} \simeq T_{osc} \simeq \frac{m_1^2}{E_n \Delta m} \approx \frac{m_1^2}{E_n^2} L, \]  

(73)

where \( L \) is given by Eq. (37) and \( E_n \) is the neutrino energy at the detection point. Then the equations (65), (66) will be valid as long as

\[ \sigma_x, \sigma_t < \frac{m_1^2}{E_n^2} L. \]  

(74)

In the case of mass hierarchy, \( m_1 \gg m_2 \) we have

\[ p_1 - p_2 \simeq p_1, \]  

(75)

so that

\[ L_{osc} \simeq T_{osc} \simeq E_1^{-1}, \]  

(76)

where \( E_1 \) is the energy of the heavier neutrino at the detection point. The equations for the probabilities in this case are valid for

\[ \sigma_x, \sigma_t < E_1^{-1}. \]  

(77)

Due to incoherent distribution of the sources in realistic situations (in a supernova or in the Sun) the probabilities (65)–(68) will be averaged over the sources and the oscillatory terms are likely to disappear. Whenever this is the case, the probabilities will describe the so-called global effects of appearance-disappearance of neutrino flavour specie, and will look like

\[ P_\mu(x_n, x_c) \propto m_1^3 \cos^4 \theta + m_2^3 \sin^4 \theta, \]

(78)

\[ P_e(x_n, x_c) \propto \left( m_1^3 + m_2^3 \right) \sin^2 \theta \cos^2 \theta, \]

(79)

in the case \( m_j \sqrt{x_{nc}} \gg 1 \), and

\[ P_\mu(x_n, x_c) \propto \sin^4 \theta + \frac{\pi}{8} \zeta^3 \cos^4 \theta, \]

(80)

\[ P_e(x_n, x_c) \propto \left( 1 + \frac{\pi}{8} \zeta^3 \right) \sin^2 \theta \cos^2 \theta \approx \frac{\pi}{8} \zeta^3 \sin^2 \theta \cos^2 \theta, \]

(81)

in the case \( m_2 \sqrt{x_{nc}} \ll 1 \ll m_1 \sqrt{x_{nc}} \) where \( \zeta \) is given by (69).

To see whether the assumption of a well-localised neutrino source is realistic, let us make some estimates for the cases of solar and supernova neutrinos. We take all the data from
In all these cases one has $\sigma_x \ll \sigma_t$, and it is the value of $\sigma_t$ which will be important. In the case of solar neutrinos

$$\sigma_t \sim 10^{-7} \, \text{cm}, \quad E_n \sim 10 \, \text{MeV}. \quad (82)$$

With $\Delta_{12} = m_1^2 - m_2^2 \sim 10^{-4} \, \text{eV}^2$ in the case of close neutrino masses the condition (74), under which the formulae of this section will apply, will read

$$m_1 \gtrsim 3 \, \text{eV}. \quad (83)$$

Since $E_n^{-1} \sim 10^{-12} \, \text{cm}$, in the case of mass hierarchy the formulae of this section will not work.

For supernova neutrinos from the core

$$\sigma_t \sim 10^{-14} \div 10^{-13} \, \text{cm}, \quad E_n \sim 100 \, \text{MeV}, \quad (84)$$

so that $E_n^{-1} \sim 10^{-13} \, \text{cm}$, the condition (77) will be on the edge of fulfillment and the formulae of this section might be applicable.

For supernova neutrinos from the neutrino sphere

$$\sigma_t \sim 10^{-9} \, \text{cm}, \quad E_n \sim 10 \, \text{MeV}, \quad (85)$$

in the case of mass hierarchy the expressions of this section will not be applicable. In the case of close neutrino masses for $\Delta_{12} = m_1^2 - m_2^2 \sim 10^{-4} \, \text{eV}^2$ we will have the condition

$$m_1 \gtrsim 0.3 \, \text{eV}, \quad (86)$$

for which the relevant expressions of this section will apply.

In connection with the above examples we must note that the probabilities (65)–(68) will refer to neutrinos as they appear from the source. Subsequent neutrino scattering off the particles of the solar or supernova media will result in the well-known Mikheyev-Smirnov-Wolfenstein effect (see [2, 3]), which is not considered in this paper.

6 Summary

In this paper we treated the problem of neutrino flavour oscillations by consistently using space-time description of the relevant processes of particle creation and subsequent detection. We described the EPR-like experiments of detecting neutrino together with the accompanying charged lepton, as well as the standard textbook examples of neutrino flavour space-time oscillations, without invoking a priori the notion of particle trajectories. From our analysis it is also clear why in fact it is possible to use such a notion. The effective integration region

\footnote{Note that in [3] as well as in some other literature $\sigma_x$ usually stands for the emitted neutrino wave packet spread. In this paper both $\sigma_x$ and $\sigma_t$ denote the spread of the neutrino source. Also note that we put the speed of light as well as the Planck constant to unity and measure $\sigma_t$ in units of length.}
(the region of constructive interference) in the probability amplitude \( \Pi \) over the space-time point \( x_c \) of particle creation is localised around the place determined by particle classical trajectories, and the contribution to the phase of the amplitude comes mainly from the action along these trajectories. In the case of wave packet neutrino source our treatment enabled us to obtain in a rather simple way the coherence lengths of the oscillations. We also considered the case of neutrino source strongly localised in space and time and in this case found out dependence of the probability oscillation amplitude on the neutrino masses.

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Appendix: Alternative derivation of the probability amplitude.

In this Appendix we shall derive the expression for the probability amplitude \( \Pi \) in the case of monochromatic source, in the limit \( m_n \sqrt{x_{nc}^2} \gg 1, m_l \sqrt{x_{lc}^2} \gg 1 \), using the asymptotic expression \( \Pi \) for the propagators in the coordinate representation. The integral over \( x_c \) in \( \Pi \) will be evaluated in the stationary phase approximation, according to the assumption that the main contribution comes from the region of stationary phase of the integrand. This phase stems from the propagators and from the source current in \( \Pi \) and is given by the expression

\[
\Phi(x_l, x_n, x_c) = -m_l \sqrt{x_{lc}^2} - m_n \sqrt{x_{nc}^2} - p \cdot x_c , \tag{A1}
\]

and its stationary point is determined from the condition

\[
\frac{\partial \Phi}{\partial x_\alpha} \equiv m_l \hat{x}_l^\alpha + m_n \hat{x}_n^\alpha - p^\alpha = 0 , \tag{A2}
\]

where the notation is used \( \hat{x} = x/\sqrt{x^2} \). Since \( \hat{x}_l \) and \( \hat{x}_n \) are just four-velocities, respectively, of the charged lepton and the neutrino at their respective detection points, the equation \( A2 \) expresses the total energy-momentum conservation. Note that for different neutrino masses \( m_n = m_j \) the value of \( x_c \) determined by Eq. \( A2 \) will be different. Let \( x_j \) be the solution for \( x_c \) of Eq. \( A2 \) with \( m_c = m_j \). For further convenience we make notation \( x_l - x_j = x_{lj} \), \( x_n - x_j = x_{nj} \), \( \hat{x}_{lj} = u_l \), \( \hat{x}_{nj} = u_n \). The phase \( \Phi \) can be developed in powers of \( x = x_c - x_j \) around the stationary point \( x = 0 \) with the result

\[
\Phi(x_l, x_n, x_c) = \frac{1}{2} C_{\alpha\beta} x^\alpha x^\beta + \ldots , \tag{A3}
\]
where
\[ C_{\alpha\beta} = c_l (u_l)_\alpha (u_l)_\beta + c_n (u_n)_\alpha (u_n)_\beta - (c_l + c_n) g_{\alpha\beta}, \]  
\[ c_l \equiv \frac{m_l}{\sqrt{x_{ij}^2}}, \quad c_n \equiv \frac{m_j}{\sqrt{x_{nj}^2}}. \]  
\[ (A4) \]
\[ (A5) \]

Dropping the higher order terms in (A3), denoted by dots, we will be interested in the value of a Gaussian integral
\[ \int d^4 x \exp \left( \frac{i}{2} C_{\alpha\beta} x^\alpha x^\beta \right). \]  
\[ (A6) \]

This, up to a constant factor, is given by \( |\det \{ C_{\alpha\beta} \}|^{-1/2} \). In terms of the velocities \( v_l \) and \( v_n \), respectively, of the charged lepton and the neutrino in the source rest frame the determinant is given by
\[ \det \{ C_{\alpha\beta} \} = c_l c_n (c_l + c_n)^2 \left[ (u_l \cdot u_n)^2 - 1 \right] = c_l c_n (c_l + c_n)^2 \left( \frac{(v_l + v_n)^2}{(1 - v_l^2) (1 - v_n^2)} \right). \]  
\[ (A7) \]

The values of \( c_l \) and \( c_n \) given by \( (A5) \) can be easily seen to coincide with the extremal values, respectively, of \( \lambda_l \) and \( \lambda_n \) determined by the system of equations (8), (13), and given by (29).

Combining all the factors in (A1) together we obtain the expression for the amplitude in the limit of \( m_n \to 0 \) for the prefactors and up to an irrelevant constant as
\[ \sum_j \frac{c_l c_n (c_l + c_n)}{(c_l + c_n)^2} \frac{(1 - v_l^2) (1 - v_n^2)}{v_l + v_n} O_D \gamma_{\alpha\beta} u_n^\alpha O_S \left( 1 - \gamma_{\beta\gamma} u_l^\gamma \right) U^\dagger_{bj} U_{ja} e^{i\Phi_j} \]
\[ = \frac{1}{m_D} O_D \gamma_{\alpha\beta} O_S \left( m_l - \gamma_{\beta\gamma} p_l^\gamma \right) \sum_j U^\dagger_{bj} U_{ja} e^{i\Phi_j}, \]  
\[ (A8) \]

where \( \Phi_j \) is the value of the phase at the stationary point of \( x_c = x_j \) that corresponds to neutrino mass specie \( j \). In view of the expression \( (A1) \) for the phase we see that this value is identical to that of Eq. \( (23) \). Therefore the expression \( (A8) \) for the amplitude coincides with that of Eq. \( (30) \).

Note that the pre-exponential factors in the final expression \( (A8) \) for the amplitude remain finite in the limit of \( m_n = 0 \) in spite of the fact that neutrino propagators \( (59) \) have pre-exponential factors \( m_n^{3/2} \). The reason is that the strong neutrino-mass-dependence of these factors has been counterbalanced by the neutrino-mass-dependence of the value \( x_{nj}^2 \), as well as of the factor \( |\det \{ C_{\alpha\beta} \}|^{-1/2} \), with the determinant given by \( (A7) \). In the case of fixed neutrino creation point there is no integration over \( x_c \) and factors \( m_n^{3/2} \) remain in the probability amplitude, Eq. \( (31) \).

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