Nonlinear dynamics of the heavy gyro-rotor with two skew rotating axes

Katica (Stevanović) Hedrih¹ and Ljiljana Veljović²

¹. Faculty of Mechanical Engineering University of Niš, Mathematical Institute SANU, ul. Vojvode Tankosić 3/V/22, 18000- Niš, Serbia.
Email: katica@masfak.ni.ac.yu, khedrih@eunet.yu

². Faculty of Mechanical Engineering University of Kragujevac, ul. Sestre Janić 6/34, 34000 - Kragujevac, Serbia.
Email: katica@masfak.ni.ac.yu

Abstract. The rotors are the basic working parts in many machines so that the problem of rotor vibration has existed for a long time. In this paper the rotor is analyzed as a shaft-disc system. The disc is eccentric and shaft is supported on both sides with rigid bearings. Here we present special case when the support shaft is vertical and the rotor shaft is horizontal but they are without intersection. A system of nonlinear differential equations is determined. When the angular velocity of shaft axis is constant, the motion character analysis is performed by means of phase trajectories and that is done for different cases of eccentricity and angle of skew.

Some numerical analysis of obtained analytical solutions is performed through Math Cad.

1. Introduction

Many parts in engineering rotate around fixed axes. But, there are some elements that rotate around moveable axes. The dynamics of motion of such elements is a very old engineering problem but it is actual nowadays. That is because rotors are the basic elements in numerous of machines.

There are many research results and discoveries of new nonlinear phenomena and of stationary and non stationary vibration regimes with different kinetic parameters of the dynamical system. But, many researches pay attention to this problem again.(see Refs. [6-10], [11-12] and [14-18]). There are new numerical and experimental methods that help us discover the properties of nonlinear dynamics.

In References [1-3] applications of basic methods of theory of Liapunov on stability of nonlinear oscillations that especially talk about the method of phase plane and phase trajectories and singularities in researching qualitative properties of nonlinear dynamics are presented. It may be considered phase portrait with singularity structure that gives nonlinear dynamics properties and phenomena of system in phase plane. Therefore it is very important to study the structure of phase portraits, their stability and their transformations and transformations of phase trajectories. In References [6-10], [12] and [14-18] nonlinear dynamics of rotors, coupled rotors, like planetary reductor and gyro-rotors were studied.

¹ To whom any correspondence should be addressed.
First author defines the theorem of trigger of coupled singularities and homoclinic orbits shaped by number eight in References [5] and [13]. Some researchers [5-18] implicated these theorems and the existence of homoclinic orbits and their transformation shaped by number eight. Phase portraits are constructed here and considered the phenomena of transformations of homoclinic orbits, their disintegration, appearance and disappearance of homoclinic orbits shaped by number eight, like trigger of coupled singularities. The influence of mass disbalances of a car at nonlinear dynamic properties are presented in [8]. The Control in Nonlinear Dynamical Systems with Triggers of a Coupled Singularities is presented in Reference [4].

2. The model of the gyro-rotor system and basic equations

Here we presented eccentric disc (eccentricity is e), with mass $m$ and radius $r$ which is inclined to the axes of its own rotation by the angle $\beta$. The angle of own rotation around moveable axis oriented by the unit vector $\mathbf{n}_1$ is $\varphi_1$ and the angular velocity is $\omega_1$. The angle of rotation around the shaft support axis oriented by the unit vector $\mathbf{n}_2$ is $\varphi_2$ and the angular velocity is $\omega_2$ (see Figure 1 a*). When the support shaft is vertical and the rotor shaft is horizontal and when they are without intersection we obtain that angular velocity of rotor is: $\omega_1 = \omega_1 + \omega_2 \mathbf{n}_2 = \dot{\varphi}_1 \mathbf{n}_1 + \dot{\varphi}_2 \mathbf{n}_2$

The angles $\varphi_1$ and $\varphi_2$ are generalized coordinates in case when we investigate system with two degrees of freedom. In this case $\varphi_1$ is generalized coordinate. The second angle $\varphi_2$ is a rheonomic coordinate which is defined by a time function. The tensor matrix of mass inertia moments of eccentric disc in a relation to the point $O$, (this point is cross section of rotor support axis and a plane which contains the support shaft axis and that is perpendicular to rotor shaft axis as it can be seen in Fig.1a*), in the system coordinate axes coupled with support:

$$\mathbf{J} = \begin{bmatrix} J_{uu} & -J_{uv} & -J_{uw} \\ -J_{vu} & J_{vv} & -J_{vw} \\ -J_{wu} & -J_{wv} & J_{ww} \end{bmatrix},$$

where $J_{ij}$ are the mass inertia moments of disc for moveable axes and $O_{u,v,w}$ is the coordinate system.

The angular momentum of the system is $\mathbf{L}_O = \mathbf{J}_O \dot{\omega}$. Using the theorem of angular momentum derivative we obtain the differential vector equation of the rotation around the rotor shaft in case when the support shaft is vertical and the rotor shaft is horizontal and when they are without intersection in a form:

$$\frac{d\mathbf{L}_O}{dt} = \dot{\mathbf{M}}_O \left( \mathbf{F}_i \right) + m \mathbf{P}_c \times \left( \mathbf{v}_O \times \dot{\mathbf{a}}_O - \dot{\omega}_O \right).$$

(2)
where \( \vec{r}_C \) - the position vector of the mass center of rotor (\( \vec{r}_C = \overrightarrow{OC} \)), \( \vec{v}_O \) - the velocity of pole O (\( \vec{v}_O = \vec{\omega}_h \times \vec{O}_h \)), \( \vec{a}_O \) - the acceleration of pole O (\( \vec{a}_O = \vec{\omega}_2 \times \vec{O}_2 + \vec{\omega}_2 \times \vec{v}_O \)). The angular velocity of the rotor in moveable system \( O_Ov_O \), which is rigidly connected with the moving shaft of the gyro-rotor, is:

\[
\vec{\omega}_2 = \omega_2 \sin \varphi_1 \vec{u}_1 + \omega_2 \cos \varphi_1 \vec{v}_1 + \omega_1 \vec{u}_1
\]

The differential equation of the rotation around the rotor shaft is in the form:

\[
\dot{\omega}_1 - \frac{J_m}{J_n} \dot{\omega}_1 \sin \varphi_1 - \frac{J_v - J_u}{J_u} \omega_2^2 \sin \varphi_1 \cos \varphi_1 - \frac{2m \ell \omega_2^2}{J_u} \sin \beta \cos \varphi_1 + \frac{mg \sin \beta}{J_u} \sin \varphi_1 = 0
\]

(4)

where \( \ell = O_Ov \) and \( g \) - gravitational acceleration. When the angular velocity of shaft support axis is constant, that is \( \varphi_2 = \omega_1t + \varphi_{20}, \ \dot{\varphi}_2 = \omega_2 = \text{const} \), \( \dot{\varphi}_2 = 0 \) (in this case the angle \( \varphi_2 \) is a rheonomic coordinate defined by previous time dependent function), the differential equation of the rotation can be written in a form:

\[
\dot{\omega}_1 + \Omega^2 \left( \lambda - \cos \varphi_1 \right) \sin \varphi_1 - \Omega^2 \Psi \cos \varphi_1 = 0
\]

(5)

We use the following notation:

\[
\Omega^2 = \frac{J_v - J_u}{J_u} \omega_2^2, \quad \lambda = \frac{m \ell \omega_2^2}{(J_v - J_u) \omega_2^2} \quad \text{and} \quad \Psi = \frac{2m \ell}{(J_v - J_u)} \sin \beta
\]

(6)

When the rotor is eccentrically positioned disc, the coefficients in differential equation (5) are:

\[
\Omega^2 = \frac{\varepsilon \sin^2 \beta - 1}{\varepsilon \sin^2 \beta + 1} \omega_2^2, \quad \lambda = \frac{g(\varepsilon - 1) \sin \beta}{(\varepsilon \sin^2 \beta - \varepsilon) \omega_2^2}, \quad \Psi = \frac{8 \ell}{(\varepsilon \sin^2 \beta - 1) r^2 \sin \beta}, \quad \varepsilon = 1 + 4 \left( \frac{e}{r} \right)^2
\]

(7)

All parameters are in function of eccentricity \( e \) and angle \( \beta \). So, it was very interesting to analyze the influence of these parameters on nonlinear dynamics behavior of system.

As we know that \( \dot{\omega}_1 \) is equal with \( \dot{\varphi}_1 \) so the differential equation of the rotation around the rotor shaft is:

\[
\dot{\varphi}_1 + \Omega^2 \left( \lambda - \cos \varphi_1 \right) \sin \varphi_1 - \Omega^2 \Psi \cos \varphi_1 = 0
\]

(8)

The solution-first integral of these differential equation with the initial conditions: \( t_0 = 0, \ \varphi_1(t_0) = \varphi_{10}, \ \dot{\varphi}_1(t_0) = \dot{\varphi}_{10} \) is in a form:

\[
\dot{\varphi}_1^2 = \dot{\varphi}_{10}^2 + 2\Omega^2 \left[ \lambda (\cos \varphi_1 - \cos \varphi_{10}) + \frac{1}{2} (\cos^2 \varphi_{10} - \cos^2 \varphi_1) + \Psi (\sin \varphi_1 - \sin \varphi_{10}) \right]
\]

(9)

and it is the energy integral because the conservative system is analyzed [19-20]. The potential energy exchange curves for different values of the system parameters (the eccentricity \( e \) and the angle \( \beta \)) are given on Figure 2 b*. Phase trajectories for different system parameters are given in Figure 1 b* and c*.
The influence of parameters of system (eccentricity and angle) on parameter $\Psi$ of equation;

$$\psi(\beta)$$

$$\psi'(\beta)$$

$$\psi''(\beta)$$

The dynamic relative equilibrium positions of the gyro-rotor can be determined from equation:

$$(\lambda - \cos \varphi_1) \sin \varphi_1 - \Psi \cos \varphi_1 = 0$$  \hspace{1cm} (10)

The graphical solution is performed through MathCad. The equilibrium position depends on parameters $\Psi$ and $\lambda$ and for $\Psi = -0.717$ and $\lambda = -0.73$ we obtain that the dynamic relative equilibrium positions of the gyro-rotor are for: $\varphi_{01} = 25^\circ 30'$ and $\varphi_{02} = -114^\circ 8'$

3. **Numerical experiment and graphical presentation**

We study small oscillations around relative equilibrium position by using linearization of the differential equation (8). When we take that approximations are: $\sin \varphi_1 \approx \sin \varphi_{10} + (\varphi_1 - \varphi_{10}) \cos \varphi_{10}$ and $\cos \varphi_1 \approx \cos \varphi_{10} - (\varphi_1 - \varphi_{10}) \sin \varphi_{10}$, we get the knew form of equation:
\[ \dot{\phi}_i + \Omega^2 \left( \lambda - \cos \varphi_{10} - \varphi_i \sin \varphi_{10} - \varphi_i \sin \varphi_{10} \right) \left( \sin \varphi_{10} + \varphi_i \cos \varphi_{10} - \varphi_i \cos \varphi_{10} \right) - \\
\Omega^2 \Psi \left( \cos \varphi_{10} - \varphi_i \sin \varphi_{10} + \varphi_i \sin \varphi_{10} \right) = 0 \]  

(7)

This equation (7) is well known as simple oscillation: \( \dot{\phi}_i + \omega^2 \varphi_i = h \), where are:

\[ \omega^2 = \Omega^2 \left( \lambda - \cos \varphi_{10} - \varphi_i \sin 2 \varphi_{10} - \varphi_i \sin \varphi_{10} \right) \]

and

\[ h = \Omega^2 \Psi \left( \cos \varphi_{10} - \varphi_i \sin \varphi_{10} \right) - \Omega^2 \left( \lambda - \cos \varphi_{10} - \varphi_i \sin \varphi_{10} \right) \left( \sin \varphi_{10} - \varphi_i \cos \varphi_{10} \right) \]

It is nonhomogenous second order linear differential equation, “own” – eigen frequency \( \omega^2 \) and “intensity of force” \( h \), which involve such motion depend on system characteristics and on initial conditions [10-20]. Here, the influence of initial conditions on frequency is presented. \( \omega^2 \). The frequency \( \omega^2 \) can be positive, but it can be negative, too. For some values of parameters \( \lambda \) and \( \Psi \) it is evident that there are intervals where \( \omega^2 > 0 \). Then we have oscillatory motion that is periodically motion around the dynamic relative equilibrium position. Such position is stable but it is stable only for the given initial conditions. On the other hand, for parameters which give \( \omega^2 < 0 \), the motion around the dynamic relative equilibrium position is not periodic and it is unstable position.

**Figure 3 a** The influence of initial conditions on “own” frequency; **b** The oscillations around the stable relative equilibrium position; **c** The oscillations around the unstable relative equilibrium position

In the case when the dynamic relative equilibrium position is defined by \( \varphi_{101} \) the relative equilibrium position is stable and the motion around that position may be defined as the sum of homogenous and particular solution: \( \varphi_i = \varphi_{i0} + \varphi_{ip} \) or \( \varphi_i = C \sin(\omega t + \gamma) + \frac{h}{\omega^2} \), where

\[ C = \sqrt{\varphi_{i0}^2 + \left( \varphi_{i0} - \frac{h}{\omega^2} \right)^2} \] and \( \gamma = \arctg \frac{\omega \varphi_{10} - h}{\varphi_{10}^2} \). We have some graphic solutions of it. In the case when the dynamic relative equilibrium position is defined by \( \varphi_{102} \) the relative equilibrium position is unstable and the motion around that position may be defined as the sum of homogenous and particular solution but as it can be seen the motion is not periodic.

4. **Concluding remarks**
In this paper research results of influence of eccentricity and angle of inclination at nonlinear dynamics of gyro-rotor are presented. Also, by using equations of phase trajectories some properties of nonlinearity are investigated. We were analyzed homoclinic orbits and their transformation shaped by number eight, their appearance and disappearance by changing some parameters of system. We used MathCad program for drawing families of phase portraits.

Acknowledgment:
Parts of this research were supported by the Ministry of Sciences and Environmental Protection of Republic of Serbia through Mathematical Institute SANU Belgrade Grant ON144002 “Theoretical and Applied Mechanics of Rigid and Solid Body. Mechanics of Materials” and Faculty of Mechanical Engineering University of Niš.

References
[1] Andronov AA, Vitt AA and Haykin SE 1981 Teoriya kolebaniy (Nauka, Moskva.)
[2] Gerard I and Daniel J 1980 Elementary Stability and BIfurcation Theory (Springer Verlag)
[3] Guckenheimer J and Holmes Ph 1983 Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Fields (Springer-Verlag)
[4] Hedrih (Stevanović) K 2007 The Control in Nonlinear Dynamical Systems with Triggers of a Coupled Singularities, Invited Participation, The 14th International Workshop on Dynamics & Control to be held on May 28–June 2, 2007, in Zvenigorod – Moscow, Russia, Abstracts, 2007. p. 40. Institute for Problems in Mechanics of the the Russian Academy of Sciences and Steklov Mathematical Institute of the Russian Academy of Sciences
[5] Hedrih (Stevanović) K 2001 Trigger of Coupled Singularities (invited plenary lecture), Dynamical Systems-Theory and Applications, Edited by J. Awrejcewicz and all, Lodz pp. 51-78
[6] Hedrih (Stevanović) K 2000 Nonlinear Dynamics of a Gyrorotor, and Sensitive Dependence on initial Conditions of a Heavy Gyrorotor Forced Vibration/Rotation Motion, Semi-Plenary Invited Lecture, Proceedings: COC 2000, Edited by F.L. Chernousko and A.I. Fradkov, IEEE, CSS, IUTAM, SPICS, St. Petersburg, Inst. for Problems of Mech. Eng. of RAS, 2 pp. 259-266
[7] Hedrih (Stevanović) K 2005 Nonlinear Dynamics of a Heavy Material Particle Along Circle which Rotates and Optimal Control, Chaotic Dynamics and Control of Systems and Processes in Mechanics (Eds: G. Rega, and F. Vestroni), p. 37-45. IUTAM Book, in Series Solid Mechanics and Its Applications, Edited by G.M.L. Gladwell, Springer XXVI 504
[8] Hedrih (Stevanović) K 2004 Contribution to the coupled rotor nonlinear dynamics, Advances in nonlinear Sciences, Monograph, Belgrade, Academy of Nonlinear Sciences, 229-259. (engleski)
[9] Hedrih (Stevanović) K 2004 Phase Portraits and Homoclinic Orbits Visualization of Nonlinear Dynamics of Multiple Step Reductor/Multiplier, Proceedings, Volume 2, The eleventh world congress in Mechanism and machine Sciences, IFToMM, China Machine press, Tianjin, China, April 1-4, 2004, pp. 1508-1512. http://www.iftom2003.com
[10] Hedrih (Stevanović) K 2002 On Rheonomic Systems with Equivalent Holonomic Conservative System, Int. Conf. ICNM-IV, Edited by W. Chien and all. Shanghai, T. Nonlinear dynamics. pp. 1046-1054.
[11] Hedrih (Stevanović) K 2001 The Vector Method of the Heavy Rotor Kinetic Parameter Analysis and Nonlinear Dynamics, Monograph, University of Niš, pp. 252
[12] Hedrih (Stevanović) K 2004 A Trigger of Coupled Singularities, MECCANICA 39 295 DOI: 10.1023/B:MECC.0000022994.81090.5f
[13] Hedrih (Stevanović) K 2004 On Rheonomic Systems with Equivalent Holonomic Conservative Systems Applied to the Nonlinear Dynamics of the Watt’s Regulator, Proceedings, Volume 2, The eleventh world congress in Mechanism and machine sciences, IFToMM, China Machine press, Tianjin, China, April 1-4, 2004, pp. 1475-1479
[14] Hedrih (Stevanović) K Knežević and Cvetković, R. 2001 Int. J. Nonlinear Sci. 2 265
[15] Katica (Stevanović) Hedrih and Goran Janevski 2000 Nonlinear dynamics of a gyro-disc-rotor and structural dependence of a phase portrait on the initial conditions, Plenary Lecture, Proceedings of Dynamics of Machines 2000, Institute of Thermomechanics, Czech Committee of the European Mechanics Society, Prague, 8 - 9 pp. 81-88.
[16] Hedrih (Stevanović) K and Simonović J 2002 Phase Portraits and Homoclinic Orbits – Visualization of Nonlinear Dynamics of Reductor, Journal of Politechnica University Timisoara, Romania, Mechanical Vibrations, Transaction on Mechanical Engineering, Tom 47(61), Supplement, Editura Politehnica., pp. 76-86.
[17] Hedrih (Stevanović) K and Veljović Lj 2004 Facta Universitatis, Series Mechanics, Automatic Control and Robotics 4 55
[18] Stoker J J 1950 Nonlinear Vibrations (New York: Interscience Publishers)
[19] Rašković D 1965 Teoirija oscilacija (Theory of Oscillations) Naučna knjiga, Beograd