AN ALGORITHM FOR MAP ENUMERATION

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ABSTRACT. Bauer and Itzykson showed that associated to each labeled map embedded on an oriented Riemann surface there was a group generated by a pair of permutations. From this result an algorithm may be constructed for enumerating labeled maps, and this construction is easily augmented to bin the numbers by the genus of the surface the map is embedded in. The results agree with the calculations of Harer and Zagier of 1-vertex maps; with those of Bessis, Itzykson, and Zuber of 4-valent maps; and with those of Ercolani, McLaughlin, and Pierce for 2\nu-valent maps.

We then modify this algorithm to one which counts unoriented maps or Mobius graphs. The results in this case agree with the calculation of Goulden and Jackson on 1-vertex unoriented maps.

1. Introduction

The recent work of Ercolani, McLaughlin, and Pierce [6] analyzed the fine structure of families of probability measures on the space of \(N \times N\) Hermitian matrices. The principle measures analyzed in [6] are of the form

\[
d\mu_t = \frac{1}{Z_N} \exp \left(-N \text{Tr} [V_t(M)]\right) dM,
\]

where \(dM\) is the product measure

\[
dM = \left( \prod_{1 \leq i < j \leq N} d\text{Re}(M_{ij})d\text{Im}(M_{ij}) \right) \left( \prod_{1 \leq i \leq N} dM_{ii} \right),
\]

the potential is

\[V_t(\lambda) = \frac{1}{2} \lambda^2 + t \lambda^{2\nu},\]

and where \(Z_N\) is the normalization of the measure. The question of interest in that paper was the asymptotic structure of \(\log(Z_N)\) for large values of \(N\), and its analytic dependence on \(t\).

More generally, the partition function of random matrices, introduced above as the normalization factor of the probability measure (1.1), is defined as

\[
Z_N(t_3, t_4, \ldots, t_{2\nu}) = \int_{\mathcal{H}_N} \exp \left(-N \text{Tr} \left(\frac{1}{2}M^2 + t_3 M^3 + t_4 M^4 + \cdots + t_{2\nu} M^{2\nu}\right)\right) dM,
\]

where the integral is taken over the space of \(N \times N\) Hermitian matrices. The partition function has been used extensively as a model for the partition function of 2D-Quantum Gravity [11], [4], [3].

Derivatives of \(Z_N(t_3, t_4, \ldots, t_{2\nu})\) evaluated at \(t_j = 0\) capture the Gaussian moments \(\int_{\mathcal{H}_N} \text{Tr}(M^j) d\mu_0(M)\). These moments are evaluated with the Wick lemma in terms of pair correlators of the matrix entries. In analogy to Feynman diagrams, there is a correspondence between labeled maps and the terms in the Wick expansion. This fact led to the conjecture [2] that the logarithm of the partition function (1.2) has an asymptotic expansion of the form

\[
\log \frac{Z_N(t_3, t_4, \ldots, t_{2\nu})}{Z_N(0)} = N^2 e_0(t_3, t_4, \ldots, t_{2\nu}) + e_1(t_3, t_4, \ldots, t_{2\nu}) + \ldots
\]

where the coefficients \(e_g(t_3, t_4, \ldots, t_{2\nu})\) are generating functions for counting labeled, connected, oriented maps embedded on a Riemann surface of genus \(g\).

A map \(D\) on a compact, oriented connected surface \(X\) is a pair \(D = (K(D), [\iota])\) where

(1) \(K(D)\) is a connected 1-complex;

(2) \([\iota]\) is an isotopical class of inclusions \(\iota : K(D) \rightarrow X\);

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(3) the complement of $K(D)$ in $X$ is a disjoint union of open cells (faces);

(4) the complement of $K_0(D)$ (vertices) in $K(D)$ is a disjoint union of open segments (edges).

The number of edges connected to a vertex $z \in K_0(D)$ is called the degree of $z$. We may represent maps as fat or ribbon graphs embedded on an oriented connected surface $X$. See also the theory of dessins d’enfants.

Ercolani and McLaughlin [5] rigorously showed that $\log(Z_N)$ possesses an asymptotic expansion of the form

$$\frac{1}{N^2} \log \left( \frac{Z_N(t_3, t_4, \ldots, t_{2\nu})}{Z_N(0)} \right) = e_0(t_3, t_4, \ldots, t_{2\nu}) + \frac{1}{N^2} e_1(t_3, t_4, \ldots, t_{2\nu}) + \ldots$$

in a non-trivial $(t_3, t_4, \ldots, t_{2\nu})$ domain. The functions $e_g(t_3, t_4, \ldots, t_{2\nu})$ are analytic in a neighborhood of 0 and are counting functions for labeled maps.

The function

$$e_g(t_3, t_4, \ldots, t_{2\nu}) = \sum_{j_3, j_4, \ldots, j_{2\nu} = 0}^{\infty} \kappa_g(j_3, j_4, \ldots, j_{2\nu}) t_3^{j_3} t_4^{j_4} \ldots t_{2\nu}^{j_{2\nu}},$$

where $\kappa_g(j_3, j_4, \ldots, j_{2\nu})$ is the number of labeled, genus $g$, maps with $K_0(D)$ containing: $j_3$ vertices of degree 3, $j_4$ vertices of degree 4, ..., $j_{2\nu}$ vertices of degree $2\nu$.

Harer and Zagier [8] solved this enumeration problem in the special case of 1-vertex maps (or monopoles). Let $G_n(N)$ be the generating function for oriented maps with a single vertex, that is

$$G_n(N) = N^{1+n} \sum_{g \geq 0} a_{n,g} N^{-2g},$$

where $a_{n,g}$ is the number of oriented maps with 1 vertex, and $n$ edges, which are embedded into a genus $g$ surface. This generating function is a finite series in $N$ and is given by the matrix integral

$$(1.3) \quad G_n(N) = A_N \int_{\mathcal{H}_N} \text{Tr}(M^{2n}) \exp \left[ -\frac{1}{2} \text{Tr}(M^2) \right] dM,$$

where the integral is taken over the space of $N \times N$ Hermitian matrices, and

$$A_N = \left( \int_{\mathcal{H}_N} \exp \left[ -\frac{1}{2} \text{Tr}(M^2) \right] dM \right)^{-1}.$$

**Theorem 1.1** (Harer-Zagier [8]). Harer and Zagier explicitly evaluated (1.3), to find that

$$G_n(N) = \frac{(2n)!}{2^{2n} n!} \sum_{k=0}^{n} 2^k \binom{n}{k} \frac{N}{k+1}.$$

Bauer and Itzykson [1] show that the data of a labeled map is equivalent to a pair of permutations satisfying some compatibility conditions. This result gives an efficient method to determine the genus of a map given by its pair of permutations. What emerges is an algorithm for counting the number of genus $g$ maps with a particular structure on $K_0(D)$. The idea is to count the number of pairs of permutations satisfying the characterization which produce a genus $g$ map. Zvonkin [13] expanded on this idea.

The particular partition function $Z_N(0, 0, \ldots, t_{2\nu})$ has been the subject of an extensive calculation [6]. The conclusion of this calculation is a prescription for calculating $e_g(0, 0, \ldots, t_{2\nu})$. Explicit formulas for these functions are found for low values of $g$ ($g = 0, 1, 2, 3$) in terms of an auxiliary function. In the case of $g = 0$ this result produces a closed form expression for the Taylor coefficients $\kappa_0(2\nu)(0, 0, \ldots, j_{2\nu})$. In the other cases worked out explicitly this result is used to find the first few Taylor coefficients of each of the functions.

A purely combinatorial method for producing an expression for $e_0(t_3, t_4, \ldots, t_{2\nu})$ in terms of generating functions of fundamental objects is given in [8].

The algorithm detailed in this paper has been developed as an independent method for checking the detailed work on the fine structure of $e_g(0, 0, \ldots, t_{2\nu})$ carried out in [6] and of deriving the constants of integration for the formulas derived in that work, and in the author’s doctoral thesis. As of the writing of this paper the calculation has been done for the Taylor coefficients satisfying $\nu j \leq 20$ and we have found exact agreement for all of these terms.

We will conclude this paper with a modification of our method for counting the number of unoriented maps or Möbius graphs. These are maps which are embedded into an unoriented surface. To be more precise...
the objects we will count are Moebius graphs, ribbon graphs where the edges are allowed to twist. This is the setting of interest if the partition function (1.2) is replaced with

\begin{equation}
Z_N^{(1)}(t_3, t_4, \ldots, t_{2\nu}) = \int_{S_N} \exp \left[ -N \text{Tr} \left( \frac{1}{4} M^2 + t_3 M^3 + t_4 M^4 + \ldots + t_{2\nu} M^{2\nu} \right) \right] dM,
\end{equation}

where the integral is over \( N \times N \) symmetric matrices. This function is called the partition function of the Gaussian orthogonal ensemble (GOE).

In this case the hypothesis is that

\begin{equation}
\frac{1}{N^2} \log \left( \frac{Z_N^{(1)}(t_3, t_4, \ldots, t_{2\nu})}{Z_N(0)} \right) = E_0(t_3, t_4, \ldots, t_{2\nu}) + \frac{1}{N} E_1(t_3, t_4, \ldots, t_{2\nu}) + \frac{1}{N^2} E_2(t_1, t_2, \ldots, t_{2\nu} + \ldots)
\end{equation}

where (at least formally)

\[ E_\chi(t_3, t_4, \ldots, t_{2\nu}) = \sum_{j_3, j_4, \ldots, j_{2\nu} = 0} ^\infty \kappa_\chi(j_3, j_4, \ldots, j_{2\nu}) t_3^{j_3} t_4^{j_4} \ldots t_{2\nu}^{j_{2\nu}}, \]

where \( \kappa_\chi(j_3, j_4, \ldots, j_{2\nu}) \) is the number of labeled, Euler characteristic \( \chi \) unoriented maps with \( K_0(D) \) containing: \( j_3 \) vertices of degree 3, \( j_4 \) vertices of degree 4, \ldots, \( j_{2\nu} \) vertices of degree \( 2\nu \). The justification of this enumeration problem is given in [7] and [9].

The existence of the expansion (1.5) is not rigorously shown to date. In particular a proof is needed of the analyticity of \( E_\chi \). We will give a heuristic argument for the counting property of \( E_\chi \).

Goulden and Jackson [7] generalized the calculation of Harer and Zagier [8] for 1-vertex maps (monopoles). Let \( F_n(N) \) be the generating function for unoriented maps with a single vertex, that is

\[ F_n(N) = N^{n-1} \sum_{\chi \leq 2} f_{n,\chi} N^\chi, \]

where \( f_{n,\chi} \) is the number of unoriented maps with 1 vertex, and \( n \) edges, which are embedded into a genus \( g \) surface. This generating function is a finite series in \( N \) and is given by the matrix integral

\begin{equation}
F_n(N) = B_N \int_{S_N} \text{Tr}(M^{2n}) \exp \left[ -\frac{1}{4} \text{Tr}(M^2) \right] dM,
\end{equation}

where the integral is taken over the space of \( N \times N \) symmetric matrices, and

\[ B_N = \left[ \int_{S_N} \exp \left[ -\frac{1}{4} \text{Tr}(M^2) \right] dM \right]^{-1}. \]

**Theorem 1.2** (Goulden-Jackson [7]). Goulden and Jackson explicitly evaluated (1.2), to find that

\[ F_n(N) = n! \sum_{k=0}^{n} \sum_{r=0}^{n} \binom{n-1/2}{n-r} \binom{k+r-1}{k} \binom{(N-1)/2}{r} + \frac{(2n)!}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} \binom{N-1}{k+1}. \]

Our algorithm agrees with the numbers given by \( F_n(N) \) for all cases checked. What we do not have is a method for generalizing Theorem 1.2 to larger families of unoriented maps.

In section 2 we outline the heuristic argument which motivates using the terms of the asymptotic expansion of the partition function (1.2) as the generating functions of the number of maps of genus \( g \). In section 3 we illustrate the result of 1 which gives a one-to-one correspondence between maps and pairs of permutations, of a restricted type. So the group structure provides an algorithm for counting the number of maps of a specified type. Section 4 is a collection of tables summarizing some of the calculations we have done with this algorithm. It is provided as a means of checking with other known results such as the 1-face maps [8], [12] and those in [2] and [6]. In addition the tables contain all the numbers which were needed to evaluate the constants of integration found in [6]. In section 5 we redo the heuristic argument of section 2 in the setting of an integral over the symmetric matrices rather than the Hermitian matrices. We show that the equivalent counting problem we are concerned with is that of counting maps embedded into an unoriented surface. In section 6 we generalize the results of section 3 and associate to each unoriented map a triple of permutations. This gives a one-to-one correspondence between unoriented maps and triples of permutations of the given type.

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2. The Wick lemma and Hermitian matrix integrals

We will first outline the heuristic argument that the $e_g(t_3, t_4, \ldots, t_{2n})$ are counting functions of labeled genus $g$ maps (see [2] and [12]). This calculation centers around the Wick lemma for Gaussian expectations:

**Lemma 2.1.** If $\langle \cdot \rangle$ is a Gaussian expectation, and $l_1, l_2, \ldots, l_{2n}$ are homogeneous linear functions then

$$\langle l_1 l_2 \cdots l_{2n} \rangle = \sum \langle l_{i_1} l_{j_1} \rangle \langle l_{i_2} l_{j_2} \rangle \cdots \langle l_{i_n} l_{j_n} \rangle,$$

where the sum is taken over the set $1 = i_1 < i_2 < \cdots < i_n < 2n$ and $i_k < j_k$.

We are working with the Gaussian Expectation

$$\langle f(M) \rangle = C_N \int_{H_N} f(M) \exp \left[ -N \frac{1}{2} \text{Tr} (M^2) \right] dM,$$

where the integral is taken over the space of $N \times N$ Hermitian matrices, and $C_N$ is a normalizing constant. The quadratic expectations of this probability are

$$\langle M_{ij} M_{kl} \rangle = \frac{1}{N} \delta_{il} \delta_{jk}.$$  

Consider the partition function

$$Z_N(t_4) = C_N \int_{H_N} \exp \left[ -N \text{Tr} \left( \frac{1}{2} M^2 + t_4 M^4 \right) \right] dM.$$  

We may expand the $t_4$ expression as a Taylor Series and (formally at least) commute this sum with the integration to write

$$Z_N(t_4) = \sum_{n=0}^{\infty} \frac{(-t_4)^n}{n!} N^n \langle [\text{Tr}(M^4)]^n \rangle.$$  

Write (2.2) as a function of the entries of $M$:

$$Z_N(t_4) = \sum_{n=0}^{\infty} \frac{(-t_4)^n}{n!} \sum_{i,j,k,l} \langle M_{ij} M_{kl} M_{jk} M_{li} \rangle + \cdots$$

If we use Wick’s lemma on the $n = 1$ term of (2.4), we find that the contribution is given by

$$\langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle + \langle M_{ij} M_{kl} \rangle \langle M_{jk} M_{li} \rangle + \langle M_{ij} M_{li} \rangle \langle M_{jk} M_{kl} \rangle$$

Each of these terms is non-zero if $i, j, k, \text{ and } l$ satisfy the conditions for the corresponding quadratic expectations to be non-zero. In this example we find the conditions are respectively $i = k, i = l = k = j,$ and $j = l$. In an analogy to Feynman diagrams these reductions are cataloged by families of maps (see for example Figure 1).

Moreover the number of relations between the indices is the number of faces of the corresponding map, while the contribution to the expectation $\langle \exp \left[ -N \text{Tr} (M^4) \right] \rangle$ is given by the Euler Characteristic of the map.

The fact that the maps organizing the contributions to the Hermitian matrix integral (2.2) are oriented is because of (2.1).

The final step is to take a logarithm of $Z_N(t_4)$ to restrict our consideration to connected maps. This argument easily generalizes to include other times.

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Figure 1. The three ribbon graphs representing the maps associated to the $n = 1$ term of $\mathcal{D}$. 

3. Group data associated to each map $[D]$

We will now outline the group data associated to a map, the data comes in the form of a pair of permutations. These two permutations provide a mechanism for counting the number of maps by computing the number of permutations satisfying some conditions. Conveniently this permutation structure also provides an efficient method of computing the structure of the faces of the map and therefore of calculating the Euler Characteristic of the map.

Let $E$ be the number of edges of $K(D)$. We will view the permutations of $S_{2E}$ as acting on the set of darts of $K(D)$, which is represented by

\[ \Omega = \Omega_1 \cup \Omega_2, \]

where

\[ \Omega_1 = \{(v, e) : v \text{ is a vertex and } e \text{ is an edge with two distinct vertices one of them being } v \}, \]

and

\[ \Omega_2 = \{(v, e, \pm) : v \text{ is a vertex and } e \text{ is an edge with a single vertex } v \}. \]

The element $(v, e) \in \Omega_1$ represents the dart based at $v$ and going along $e$. The element $(v, e, \pm) \in \Omega_2$ represents the dart based at $v$ going along $e$ in the counterclockwise (resp. clockwise) orientation. For each edge there are two darts therefore $|\Omega| = 2E$ and we can think of $S_{2E}$ as acting by permutations on the set $\Omega$.

Given a map, $(K(D), [\iota])$, we define a subgroup of $S_{2E}$ generated by two permutations $\langle \sigma, \tau \rangle$. The orientation on $X$ induces (via $[\iota]$) a cyclic ordering on the darts attached to each vertex; the first permutation $\sigma$ is given by this action. Explicitly, $\sigma$ maps the element $(v, e) \in \Omega_1$ to the element $(v, \tilde{e}) \in \Omega_1$ or $(v, \tilde{e}, \pm) \in \Omega_2$ where $\tilde{e}$ is the edge counter clockwise in the orientation at $v$ from $e$. Likewise, $\sigma$ maps the element $(v, e, \pm) \in \Omega_2$ to the element $(v, \tilde{e}) \in \Omega_1$ or $(v, \tilde{e}, \pm) \in \Omega_2$ where $\tilde{e}$ is the edge counter clockwise in the orientation at $v$ from $e$. The second permutation $\tau$, is given explicitly as the permutation which acts on $\Omega_1$ by sending $(v, e)$ to $(\tilde{v}, e)$ where $\tilde{v}$ is the other endpoint of $e$; and $\tau$ acts on $\Omega_2$ by sending $(v, e, \pm)$ to $(v, e, \mp)$.

One sees that $\sigma$ is a product of independent cycles; the length of the cycles in $\sigma$ correspond to the degrees of the vertices in $K_0(D)$. The permutation $\tau$ is a product of $E$ disjoint 2-cycles.

The orbit of $\langle \sigma, \tau \rangle$ (the group generated by $\sigma$ and $\tau$) of any element in $\Omega$, is all of $\Omega$ because $K(D)$ is connected.

To compute the genus of a map from $\langle \sigma, \tau \rangle$ we compute the genus of the associated graph from the Euler Characteristic

\[ \chi = V - E + F; \]

where $V$, the number of vertices, is the number of cycles in $\sigma$; where $E$, the number of edges, is $1/2$ the number of elements acted upon by $\sigma$ and $\tau$, or is the number of 2-cycles in $\tau$; and where $F$, the number of faces, is the number of cycles of the permutation $\sigma \circ \tau$ (this is merely computing each face by finding all the edges which border it).
This leads to the theorem:

**Theorem 3.1** (Bauer and Itzykson [1]). There is a one-to-one correspondence between connected maps $D = (K(D), [i])$ and pairs of permutations $(\sigma, \tau)$, where $\sigma$ is a product of disjoint cycles, and $\tau$ a product of $E$ disjoint 2-cycles, which satisfy connectedness.

To complete the proof we show that given $(\sigma, \tau)$ as above one may determine $D$. The condition of connectedness is that the orbit of the group $<\sigma, \tau>$ · 1 is all $2E$ letters. The following steps produce a map:

1. Let $K_0(D)$ be the set of cycles of $\sigma$;
2. Let $K_1(D)$ be the set of 2-cycles of $\tau$;
3. Each 2-cycle in $\tau$ ties together two points of $K_0(D)$ identifying the two endpoints of that element of $K_1(D)$;
4. The Euler Characteristic $\chi$ gives the genus of $X$;
5. Each cycle in $\sigma$ induces the orientation on $X$ in a neighborhood of $\iota(K_0(D))$, this is sufficient to determine both $X$ and $\iota$ up to isotopy (see section 6).

The punchline is that what we now have is an algorithm for computing $\kappa_g(j_3, j_4, \ldots, j_{2\nu})$ for finite $g, j_3, j_4, \ldots, j_{2\nu}$. Let

$$2E = 3j_3 + 4j_4 + \cdots + 2\nu j_{2\nu}.$$ 

Fix $\sigma$ to be a permutation formed by $j_3$ cycles of length 3, $j_4$ cycles of length 4, $\ldots, j_{2\nu}$ cycles of length $2\nu$ formed from a permutation of $n$ letters. Then we choose each product of $E$ disjoint 2-cycles in $S_{2E}$. Check if $(\sigma, \tau)$ is connected (by verifying that the orbit of $<\sigma, \tau>$ · 1 is all $2E$ letters). If $(\sigma, \tau)$ is connected compute the genus as above with $\chi = 2-2g$ and bin the result.

3.1. **Example**. As an example we will show how this algorithm works by computing $\kappa_g(j_3 = 2, j_4 = 1)$. Fix $\sigma = (1 2 3)(4 5 6)(7 8 9 10)$. Then choose each $\tau$ which is a product of 2-cycles of $\{1, 2, \ldots, 10\}$.

For example, we might choose $\tau = (1 2)(3 4)(5 6)(7 8)(9 10)$; for this choice of $\tau$ the orbit of $<\sigma, \tau>$ forms two disjoint sets $\{1, 2, \ldots, 6\}$ and $\{7, 8, 9, 10\}$.

Choose $\tau = (1 2)(3 4)(5 6)(7 8)(9 10)$; for this choice of $\tau$ the orbit of $<\sigma, \tau>$ is the set $\{1, 2, \ldots, 10\}$, therefore $(\sigma, \tau)$ represents a map. We compute the genus of this map by computing the number of distinct orbits of $\sigma \cdot \tau$. We find that $\sigma \cdot \tau = (1 3 5 9 7 4)(2)(6 8)(10)$, therefore the Euler characteristic is $3 - 5 + 4 = 2$ and the genus is 0.

Choose $\tau = (1 2)(3 4)(5 7)(6 9)(8 10)$; for this choice of $\tau$ the orbit of $<\sigma, \tau>$ is the set $\{1, 2, \ldots, 10\}$, therefore $(\sigma, \tau)$ represents a map. We find that $\sigma \cdot \tau = (1 3 5 8 7 6 10 9 4)(2)$, therefore the Euler characteristic is $3 - 5 + 2 = 0$ and the genus is 1.

4. **Results**

In tables [1][1] we present a selection of the results of this calculation, included are the numbers which were needed for the calculations in [6].

The algorithm can also be used to compute the number of maps of mixed type. For the example from section 3.1 of 2 vertices of degree 3 and one of degree 4 we find 432 genus 0 maps and 468 genus 1 maps. For the number of maps with one vertex of degree 3, one of degree 4, and one of degree 5: 2160 (genus 0), 6480 (genus 1), and 1440 (genus 2).

5. **Wick lemma and symmetric matrix integrals**

This section is motivated by the partition function $Z_N^{(1)}$ given by [1][1].

We will now show, using the Wick lemma, that the asymptotic expansion of $\log \left( Z_N^{(1)} \right)$ enumerates unoriented maps. This will be a formal calculation only, the necessary analyticity of the terms of this expansion has not been rigorously shown. We present the calculation here as a motivation for the counting problem we are computing.

We are working with the Gaussian Expectation

$$\langle f(M) \rangle = C_N \int_{S_N} f(M) \exp \left[ -N \frac{1}{4} \text{Tr}(M^2) \right] dM,$$
Table 1. First the number of one vertex maps found by [8] (whose duals are the well studied 1-face maps [12]).

| Degree | Genus | 0     | 1     | 2     | 3     | 4     | 5     |
|--------|-------|-------|-------|-------|-------|-------|-------|
| 4      |       | 2     | 1     |       |       |       |       |
| 6      |       | 5     | 10    |       |       |       |       |
| 8      |       | 14    | 70    | 21    |       |       |       |
| 10     |       | 42    | 420   | 483   |       |       |       |
| 12     |       | 132   | 2310  | 6468  | 1485  |       |       |
| 14     |       | 429   | 12012 | 66066 | 56628 |       |       |
| 16     |       | 1430  | 60060 | 570570| 1169740| 225225|       |
| 18     |       | 4862  | 291720| 4390386| 17454580| 12317877|       |
| 20     |       | 16796 | 1385670| 31039008| 351683046| 59520825|       |

Table 2. The number of two vertex maps

| Degree | Genus | 0     | 1     | 2     | 3     | 4     |
|--------|-------|-------|-------|-------|-------|-------|
| 3      |       | 12    | 3     |       |       |       |
| 4      |       | 36    | 60    |       |       |       |
| 5      |       | 180   | 600   | 165   |       |       |
| 6      |       | 600   | 4800  | 4770  |       |       |
| 7      |       | 2800  | 34300 | 81340 | 16695 |       |
| 8      |       | 9800  | 215600| 1009400| 781200|       |
| 9      |       | 44100 | 1323000| 10478160| 19158300| 3455865|       |
| 10     |       | 158760| 7408800| 94091760| 333774000| 218402730|       |

Table 3. For the case of degree 4 maps we sum up our calculations in the following table (these numbers agree with those found in [21] and [22]).

| Vertices | Genus | 0     | 1     | 2     | 3     |
|----------|-------|-------|-------|-------|-------|
| 1        |       | 2     | 1     |       |       |
| 2        |       | 36    | 60    |       |       |
| 3        |       | 1728  | 6336  | 1440  |       |
| 4        |       | 145152| 964224| 770688|       |
| 5        |       | 17915904| 192098304| 348033024| 58060800|       |

where the integral is taken over the space of $N \times N$ symmetric matrices, and $C_N$ is a normalizing constant. The quadratic expectations of this probability are

\begin{equation}
\langle M_{ij}M_{kl} \rangle = \frac{1}{N} \left[ \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right].
\end{equation}

Consider the partition function

\begin{equation}
Z_N^{(1)}(t_4) = C_N \int_{S_N} \exp \left[ -N \text{Tr} \left( \frac{1}{4} M^2 + t_4 M^4 \right) \right] dM.
\end{equation}

We may expand out the $t_4$ expression as a Taylor Series and (formally at least) commute this sum with the integration to write

\begin{equation}
Z_N^{(1)}(t_4) = \sum_{n=0}^{\infty} \frac{(-t_4)^n}{n!} N^n \langle \text{Tr}(M^4)^n \rangle.
\end{equation}
Write this expression as a function of the entries of $M$:

$$Z_N^{(1)}(t_4) = \sum_{n=0}^{\infty} \frac{(-t_4)^n}{n!} N^n \sum_{i_m, j_m, k_m, l_m} \langle \prod_{m=1}^{n} M_{i_m, j_m} M_{j_m, k_m} M_{k_m, l_m} M_{l_m, i_m} \rangle$$

$$= 1 + (-t_4) \sum_{i,j,k,l} \langle M_{i,j} M_{j,k} M_{k,l} M_{l,i} \rangle + \ldots$$

If we use Wick’s Lemma on the $n = 1$ term we find that the contribution is given by

$$\langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle + \langle M_{ij} M_{kl} \rangle \langle M_{jk} M_{li} \rangle + \langle M_{ij} M_{ki} \rangle \langle M_{jk} M_{kl} \rangle$$

Each of these terms is non-zero if $i, j, k$ and $l$ satisfy the conditions for the corresponding quadratic expectations to be non-zero. As in the case of oriented maps these reductions may be cataloged by families of maps. In this case the maps are allowed to have edges that reverse orientation because of formula (5.1).

Focus on the middle term of (5.2)

$$\langle M_{ij} M_{kl} \rangle \langle M_{jk} M_{li} \rangle.$$  

There are four cases which give that this contribution is non-zero: $i = l = k = j$, $i = k = l = j$, $i = l = k = j$, or $(i = k, j = l)$, these correspond to the four unoriented maps represented by the Mobius graphs in Figure 2.

![Figure 2](image)

**Figure 2.** Four Mobius graphs corresponding to the unoriented maps whose underlying oriented map is the single, genus one, oriented map with one vertex of degree four.

The final step is to take a logarithm of $Z_N^{(1)}(t_4)$ to restrict to connected unoriented maps.

6. UNORIENTED MAPS

We will lift the algorithm for counting labeled connected oriented maps to count labeled connected unoriented maps. We will first go over the modifications which must be made to the group data to account for the independence that the local orientations of each vertex have.
An unoriented map on a compact unoriented connected surface \( X \) is a pair \( D = (K(D), [,]) \) where

1. \( K(D) \) is a connected 1-complex;
2. \([,]\) is an isotopical class of inclusions \( \iota : K(D) \to X \);
3. the complement of \( K(D) \) in \( X \) is a disjoint union of open cells (faces);
4. the complement of \( K_0(D) \) (vertices) in \( K(D) \) is a disjoint union of open segments (edges).

In the same way that ribbon graphs are associated to an oriented map we associate a Mobius graph (ribbon graph with orientation reversing and orientation preserving edges) to an unoriented map.

Given an unoriented map \( D \), there are three permutations associated to the data contained in \( D \). To begin, for each vertex, \( v \in K_0(D) \) choose a local orientation, \( \vec{n}_v \) induced by the embedding \( \iota \). We will view the four permutations as acting on the set of darts \( \times \mathbb{Z}_2 \), given explicitly as: \( \Omega = \Omega_1 \cup \Omega_2 \), where

\[
\Omega_1 = \{(v, e, \pm \vec{n}_v) : v \text{ is a vertex and } e \text{ is an edge with two distinct vertices one of them being } v \},
\]

and

\[
\Omega_2 = \{(v, e, \pm \vec{n}_v, \pm) : v \text{ is a vertex and } e \text{ is an edge with a single vertex } v \}.
\]

The element \((v, e, \pm \vec{n}_v) \in \Omega_1\) represents the dart based at \( v \) pointing along \( e \) together with the normal vector \( \vec{n}_v \) (resp. \(-\vec{n}_v\)). The element \((v, e, \vec{n}_v; \pm) \in \Omega_2\) represents the dart based at \( v \) which traverses \( e \) in the counter clockwise direction with respect to the local orientation (resp. clockwise direction with respect to the local orientation) together with the normal vector \( \vec{n}_v \). Likewise the element \((v, e, -\vec{n}_v; \pm) \in \Omega_2\) represents the dart based at \( v \) which traverses \( e \) in the counter clockwise direction with respect to the local orientation (resp. clockwise direction with respect to the local orientation) together with the normal vector \(-\vec{n}_v\).

An edge is said to be orientation reversing if \( \vec{n}_v \) at \( v \) on the edge is \(-\vec{n}_\tilde{v}\) after translating to \( \tilde{v} \) along the edge; otherwise we say that the edge is orientation preserving. This is a property of both the map and the choice of local orientations.

We find three permutations \((\phi, \sigma, \tau)\):

On \( \Omega_1 \) we define the action of \((\phi, \sigma, \tau)\) by:

- \( \phi \) acts by sending \((v, e, \pm \vec{n}_v) \) to \((v, e, \mp \vec{n}_v) \).
- \( \sigma \) acts by sending \((v, e, \vec{n}_v) \) (resp. \((v, e, -\vec{n}_v) \)) to the dart which is counter clockwise (resp. clockwise) from \((v, e) \) at \( v \), maintaining the orientation \( \vec{n}_v \) (resp. \(-\vec{n}_v\)).
- \( \tau \) acts by sending \((v, e, \pm \vec{n}_v) \) to \((\tilde{v}, e, \mp \vec{n}_v) \) when \( e \) is orientation preserving or \((\tilde{v}, e, \mp \vec{n}_v) \) when \( e \) is orientation reversing.

On \( \Omega_2 \) we define the action of \((\phi, \sigma, \tau)\) by:

- \( \phi \) acts by sending \((v, e, \pm \vec{n}_v; \pm) \) to \((v, e, \mp \vec{n}_v; \pm) \).
- \( \sigma \) acts by sending \((v, e, \vec{n}_v; \pm) \) (resp. \((v, e, -\vec{n}_v; \pm) \)) to the dart which is counter clockwise (resp. clockwise) from \((v, e, \pm) \) at \( v \), maintaining the orientation \( \vec{n}_v \) (resp. \(-\vec{n}_v\)).
- \( \tau \) acts by sending \((v, e, \vec{n}_v; \pm) \) to \((v, e, \pm \vec{n}_v; \mp) \) (resp. sending \((v, e, -\vec{n}_v; \pm) \) to \((v, e, \mp \vec{n}_v; \mp) \)) depending on whether \( \tau \) is orientation preserving or reversing.

From this definition the following properties are apparent:

1. \( \phi \sigma \phi = \sigma^{-1} \) and \( \phi \tau \phi = \tau \)
2. \( \phi \) and \( \tau \) are fixed point free permutations, each is a disjoint product of 2-cycles. No 2-cycle of \( \tau \) is a 2-cycle of \( \phi \).
3. \( \sigma \) is a disjoint product of \( 2j_3 \) 3-cycles, \( 2j_4 \) 4-cycles, ..., \( 2j_{2n} \) \( 2n \)-cycles.
4. A connected map has the property that \((\phi, \sigma, \tau)\) has a single orbit.
5. \( \tau \) contains the data of how the darts of \( K(D) \) are attached together to form the map; together with \( \phi \), \( \tau \) determines how the local orientations relate globally.
6. The cycles of \( \sigma \circ \tau \) correspond to faces of the map. There are two cycles of equal length for each face, representing the two directions the face can be traced along its edges.

This leads to the theorem:

**Theorem 6.1.** There is a one-to-one correspondence between unoriented maps and triples \((\phi, \sigma, \tau)\) satisfying conditions (1-6).
We will complete the proof by showing that given three permutations \((\phi, \sigma, \tau)\) satisfying conditions (1-6), we can reconstruct (up to the choice of local orientation) the map \((K(D), [\iota])\) and the surface \(X\).

A. Let \(K_0(D)\) be the set of cycle pairs \((\sigma_1, \sigma_2)\) in \(\sigma\) such that \(\phi \sigma_1 \phi = \sigma_2^{-1}\).

B. From each pair \((\sigma_1, \sigma_2)\) choose \(\sigma_1\) to represent the counter clockwise or up orientation.

C. Let \(K_1(D)\) be the set of 2-cycle pairs \((\tau_1, \tau_2)\) in \(\tau\) such that \(\phi \tau_1 \phi = \tau_2\). Each pair \((\tau_1, \tau_2)\) connects two (possibly the same) elements of \(K_0(D)\). If \((\tau_1, \tau_2)\) connects the up orientation to the up orientation this edge is orientation preserving, otherwise it is orientation reversing.

D. To construct the surface \(X\) and the embedding class \([\iota]\) do the following:
   1. For each vertex \((\sigma_1, \sigma_2)\) draw a polygon with sides labeled and oriented by the elements that \(\sigma_1\) acts on.
   2. For an orientation preserving edge connecting two vertices, glue together the corresponding sides of their polygons so that the relative orientations agree at the corresponding side. For an orientation reversing edge connecting two vertices, glue together the corresponding sides of their polygons so that the relative orientations are opposite at the side.

   In conclusion: up to the choice of local orientations at each vertex, the data given by \((\phi, \sigma, \tau)\) produces a map. Likewise, a map, up to the choice of local orientations at each vertex, produces a triple \((\phi, \sigma, \tau)\) satisfying conditions 1-6.

6.1. The algorithm. We now have all the tools necessary to lift our algorithm to the enumeration of unoriented maps.

   Fix \(\phi\) to be a fixed point free disjoint product of 2-cycles. Fix \(\sigma\) to be a product of \(2j_3\) 3-cycles, \(2j_4\) 4-cycles, \ldots, and \(2j\nu\) \(2\nu\)-cycles, such that \(\phi\) connects cycles in \(\sigma\) into pairs \((\sigma_1, \sigma_2)\) of the same cycle type, choose \(\sigma_1\) to represent the \(n_{(\sigma_1, \sigma_2)}\) orientation.

   Cycle through all fixed point free products of disjoint 2-cycles \(\tau\) such that conditions 1-6 are satisfied by \((\phi, \sigma, \tau)\), in particular no 2-cycle in \(\phi\) appears in \(\tau\). If, for this choice of \(\tau\), the corresponding map is connected, then compute the Euler characteristic: the number of vertices is half the number of cycles in \(\sigma\), the number of edges is half the number of 2-cycles in \(\tau\), and the number of faces is half the number of cycles in \(\sigma \cdot \tau\). Bin the results sorted by the Euler characteristic.

7. Conclusion

The algorithm presented here for counting oriented maps was used in [8] and the authors dissertation to evaluate constants of integration appearing in the explicit calculation of \(e_g(t_{2\nu})\) for low genus \((g \leq 3)\). Additionally the algorithm provided a useful method for verifying the validity of the results in [8].

In looking beyond the calculations carried out in [8] it became apparent that it would be useful to have a similar algorithm to compute the number of unoriented maps. The permutation structure associated to oriented maps has a precise analogue in unoriented maps. This method may prove useful in further studies of the GOE and GSE matrix integrals, and associated problems.

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