Infinite sequence on 2-element alphabet as coordinates of points, determines by unimodal maps

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Abstract

We study in this work different \((0,1)\)-codings of points from the unit interval \([0,1]\) in the relation with the treatment of continuous unimodal maps.

1 Introduction

The final aim of the theory of the motions of dynamical systems must be directed toward the qualitative determination of all possible types of motions and of the interrelation of these motions [1, p. 189]. Topological conjugation is the classical tool to divide the dynamical systems to the classes of equivalence, where all the trajectories are the same in a certain sense.

We will call a continuous map \(g : [0,1] \to [0,1]\) unimodal, if it can be written in the form

\[
g(x) = \begin{cases} 
g_l(x), & 0 \leq x \leq v, 
g_r(x), & v \leq x \leq 1, \end{cases}
\]  

(1.1)

where \(v \in (0,1)\) is a parameter, the function \(g_l\) is increasing, the function \(g_r\) is decreasing, and \(g(0) = g(1) = 1 - g(v) = 0\).

If a homeomorphism \(h\) satisfies the functional equation

\[
h \circ g_1 = g_2 \circ h,
\]

where \(g_1\) and \(g_2\) are unimodal maps, then we will say that \(h\) is a conjugation from \(g_1\) to \(g_2\).

For a fixed unimodal map \(g : [0,1] \to [0,1]\) and for each point \(x \in [0,1]\) we will construct different \((0,1)\) sequences, and each of them will be considered as coordinates of \(x\) in some sense. These sequences will be invariant with respect to topological conjugateness of unimodal maps, and we will use them to construct the explicit formula of the topological conjugacy. The usefulness of such approach is well known in the one-dimensional dynamics. It comes from [2]

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Let $g$ be unimodal map. For any $t \in [0, 1]$ write $\varepsilon(t) = 1$ if $t < v$, $\varepsilon(v) = 0$, and $\varepsilon(t) = -1$ if $t > v$. Now for any $x \in [0, 1]$ denote

$$\theta_n = \prod_{i=0}^{n} \varepsilon(g^n(x)).$$

Following [2], the sequence

$$x \leftrightarrow \theta(x) = \theta_0 \theta_1 \ldots \theta_n \ldots$$

is called the invariant coordinates of $x$ (with respect to $g$).

**Lemma 1.1.** [2, p. 478, Lemma 3.1] The map $x \mapsto \theta(x)$ is increasing, where the natural lexicographical order is defined on the set of sequences $\{(\theta_i)_{i \geq 0}\}$ of elements $\{-1; 0; 1\}$.

The classical example of the unimodal maps is the tent map

$$f : x \mapsto 1 - |1 - 2x|.$$ 

**Theorem 1.** [3, p. 53] A unimodal map $g$ is topologically conjugated to the tent $f$ map if and only if the complete pre-image of 0 under the action of $g$ is dense in $[0, 1]$.

Recall, that the set $g^{-\infty}(a) = \bigcup_{n \geq 1} g^{-n}(a)$, where $g^{-n}(a) = \{x \in [0, 1] : g^n(x) = a\}$ for all $n \geq 1$, is called the complete pre-image of $a$ (under the action of the map $g$). We have shown in details in [4] that the original proof of Theorem 1 uses the next construction.

**Notation 1.2.** [4, Notation 2.1] Let $g$ be unimodal map. Then the set $g^{-n}(0)$ consists of $2^{n-1}+1$ points. Thus, for every $n \geq 1$ denote $\{\mu_{n,k}(g), 0 \leq k \leq 2^{n-1}\}$ such that $g^n(\mu_{n,k}(g)) = 0$ and $\mu_{n,k}(g) < \mu_{n,k+1}(g)$ for all $k$.

**Lemma 1.3.** [4, Lemma 3] Suppose that $g_1, g_2 : [0, 1] \rightarrow [0, 1]$ are unimodal maps, and $h$ is the conjugacy from $g_1$ to $g_2$. Then

$$h(\mu_{n,k}(g_1)) = \mu_{n,k}(g_2)$$

for all $n \geq 1$ and $k$, $0 \leq k \leq 2^{n-1}$.

We have studied the properties of the sequence $\mu_{n,k}(g)$ in our [4].

**Notation 1.4.** [4, Notation 4.4] For every $n \geq 1$ denote by $k_n$ the maximal $k \in \{0, \ldots, 2^{n-1}-1\}$ such that $x \in [\mu_{n+1,k}, \mu_{n+1,k+1}]$. For every $n \geq 1$ and $k$, $0 \leq k \leq 2^{n-1} - 1$ denote $L_{n,k} = \mu_{n,k+1} - \mu_{n,k}$. Write $\hat{x}_n = \mu_{n+1,k_n}$. 

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by J. Milnor and W. Thurston (1988), and is widely used in the Kneading theory, created by them.
Lemma 1.5. [4, Lemma 7] For every $x \in [0,1]$ there exists a sequence $(\beta_i)_{i \geq 0}$ of $\beta_i \in \{0; 1\}$ with $\beta_0 = 0$, such that $k_n = \sum_{i=1}^{n} \beta_i 2^{n-i}$ for all $n \geq 1$.

We call the sequence $(\beta_i)_{i \geq 0}$ the $g$-decomposition of $x$.

Remark 1.6. If an unimodal map $g$ is the tent map $f$, then the $f$-decomposition of any $x \in [0,1]$ equals the classical binary decomposition of $x$.

Another example of the use of $(0,1)$ sequences as coordinates of points, determined by unimodal maps, appeared in [5] in the study of the properties of the topological conjugacy of unimodal maps in the case, when graphs of both $g_l$ and $g_r$ in (1.1) are line segments, i.e. $g$ is of the form

$$f_v(x) = \begin{cases} \frac{x}{v}, & \text{if } 0 \leq x \leq v, \\ 1 - \frac{x}{1-v}, & \text{if } v \leq x \leq 1. \end{cases}$$

(1.2)

The topological conjugation of maps of the form (1.2) are studied since early 1980-th, see [6], [7] and [5]. Following [7] and [5] we will call (1.2) skew tent map. It was proved in [6] that for every $v_1, v_2 \in (0,1)$ the maps $f_{v_1}$ and $f_{v_2}$ are topologically conjugated. For any $x \in [0,1]$ and unimodal map $g$ denote

$$\varrho_n(g,x) = \begin{cases} 0 & \text{if } g^{n-1}(x) \leq \mu_2(g), \\ 1 & \text{otherwise} \end{cases}$$

for all $n \geq 1$. Following [5], we will call the sequence $(\varrho_n(g,x))_{n \geq 1}$ the $g$-digit sequence of $x$, with respect to $g$. Notice, that the elements of $g$-digit sequence are denoted by $\varepsilon$-s in [5], but we have changed this notation to avoid the confusion with $\varepsilon$ in the construction of Minor-Thurston’s invariant coordinates. The next theorem was stated in [5].

Theorem 2. [5, Theorem 2] Let $x \in [0,1]$, let $v_1, v_2 \in (0,1)$ be arbitrary, and $h$ be the conjugacy from $f_{v_1}$ to $f_{v_2}$. Denote by $(\varrho_n)_{n \geq 1}$ the $f_{v_1}$-digit sequence of $x$. Then:

(1) The equality

$$x = \varrho_1 + \sum_{n=2}^{\infty} \varrho_n \cdot v_1^{n-1} \cdot \left( \frac{v_1 - 1}{v_1} \right)^{\sum_{j=1}^{n-1} e_j}$$

holds.

(2) The $f_{v_2}$-digit sequence of $h(x)$ is $(\varrho_n)_{n \geq 1}$.

(3) The equality

$$h(x) = \varrho_1 + \sum_{n=2}^{\infty} \varrho_n \cdot v_2^{n-1} \cdot \left( \frac{v_2 - 1}{v_2} \right)^{\sum_{j=1}^{n-1} e_j}$$

holds.
Notice, that (2) of Theorem 2 follows from the basic properties of the topological conjugacy, and (3) follows from (1) and (2). In other words, the unique non-trivial part of Theorem 2 is (1). We will generalize Theorem 2. We will prove

**Theorem 3.** Let \( g \) be unimodal map and let \((\hat{\rho}_i)_{i\geq 1}\) be the \( g \)-digit sequence of a point \( x \in [0, 1] \). For any \( n \geq 1 \) denote

\[
a_n = \hat{x}_n + \beta_{n+1} \cdot L_{n+1,k_n}.
\]

Then

\[
a_n = \hat{\rho}_1 + \sum_{i=2}^{n+1} (-1)^{\sum_{j=1}^{i-1} \hat{\rho}_j} \cdot \hat{\rho}_i \cdot L_{i,k_{i-1}}.
\]

**Theorem 4.** For any \( v \in (0, 1) \) and a point \( x \in [0, 1] \) with \( f_v \)-digit sequence \((\hat{\rho}_i)_{i\geq 1}\) we have that

\[
L_{n+1,k_n} = (1 - v) \sum_{i=1}^{n} \hat{\rho}_i \cdot v^{n - \sum_{i=1}^{n} \hat{\rho}_i}.
\]

Now Theorem 2 follows from Theorems 3 and 4.

We prove Theorem 4 at the end of Section 4, and we prove Theorem 3 at the end of Section 5.

We state the relation between invariant coordinates, \( g \)-decomposition and \( g \)-digit sequences in Section 3.

### 2 Preliminaries

We have introduced the notations from this section in our [4] in generalization of the original proof of Theorem 1.

**Notation 2.1.** [4] 1. For every \( n \geq 0 \) let \( k_n, 0 \leq k_n < 2^{n-1} \) be minimal such that \( x \in [\mu_{n+1,k_n}, \mu_{n+1,k_n+1}] \). Denote \( \hat{x}_n = \mu_{n+1,k_n}, \hat{x}_n^+ = \mu_{n+1,k_n+1}, \hat{x}_n^\pm = \mu_{n+2,2k_n+1}, \hat{x}_n^- = \mu_{n+1,k_n-1} \) and \( \hat{x}_n^\mp = \mu_{n+2,2k_n-1} \).

2. For any \( n \geq 1 \) and \( k, 0 \leq k < 2^{n-1} \) denote \( I_{n,k} = (\mu_{n,k}, \mu_{n,k+1}) \) and \( L_{n,k} = \mu_{n,k+1} - \mu_{n,k} \).

**Notation 2.2.** For every \( i, j : 0 \leq i < j \) denote \( k_{i,j} = \sum_{t=1}^{j} \beta_t 2^{j-t} \). Remark that \( k_n = k_{0,n} = k_{1,n} \).

**Remark 2.3.** Notice, that, by Notation 2.1

\[
I_{n+1,k_n} = (\hat{x}_n, \hat{x}_n^+) \quad \text{and} \quad I_{n+2,2k_n} = (\hat{x}_n, \hat{x}_n^\pm).
\]

**Notation 2.4.** For any \( n \geq 1 \) and \( k, 0 \leq k < 2^n \) denote

\[
\delta_{n,k} = \frac{\mu_{n+1,2k+1} - \mu_{n,k}}{\mu_{n,k+1} - \mu_{n,k}}.
\]
If $I$ is an interval of the form $I = I_{n,k}$ then write $\delta(I)$ for $\delta_{n,k}$. This will be convenient, for example, for the expression $\delta(g(I_{n,k}))$.

**Notation 2.5.**
1. For every $t \in [0,1]$ denote $\mathcal{R}(t) = 1 - t$.
2. For every $n \geq 1$ and $t \in \{0,\ldots,2^n - 1\}$ denote $\mathcal{R}_n(t) = 2^n - t - 1$.

**Remark 2.6.**
*Rem. 2.14* For any $x_1, x_2 \in \{0; 1\}$ and $t \in [0,1]$ we have that
\[
\mathcal{R}^{\mathcal{R}^1(x_2)}(t) = \mathcal{R}^{x_1 + x_2}(t).
\]
The next follows directly from Remark 2.6.

**Remark 2.7.**
For every $x \in \{0; 1\}$ and $t \in [0,1]$ we have
\[
\mathcal{R}^{\mathcal{R}^x(0)}(t) = \mathcal{R}^x(t).
\]

**Remark 2.8.**
It follows from Notations 2.1 and 2.4 that
\[
\begin{align*}
(i) \quad & \hat{x}_n^\pm = \hat{x}_n + \delta_{n+1,k_n} \cdot L_{n+1,k_n}, \\
(ii) \quad & L_{n+2,k_{n+1}} = \mathcal{R}^{x_{n+1}}(\delta_{n+1,k_n}) \cdot L_{n+1,k_n}, \\
(iii) \quad & \delta_{n+1,k_n} \cdot L_{n+1,k_n} = L_{n+2,k_{n+1}} + \beta_{n+1} \cdot (1 - 2 \cdot L_{n+2,k_{n+1}}).
\end{align*}
\]

**Lemma 2.9.**
*Rem. 2.11* For any $n \geq 1$ we have that:
\[
\begin{align*}
(i) \quad & \hat{x}_{n+1} = \hat{x}_n + \beta_{n+1} \cdot \delta_{n+1,k_n} \cdot L_{n+1,k_n}, \\
(ii) \quad & \hat{x}_{n+1}^+ = \hat{x}_n^+ - (1 - \beta_{n+1}) \cdot (1 - \delta_{n+1,k_n}) \cdot L_{n+1,k_n}.
\end{align*}
\]

**Lemma 2.10.**
For any $n \geq 1$ we have that:
\[
\hat{x}_{n+1}^\pm = \hat{x}_n^\pm + (-1)^{1+\beta_{n+1}} \cdot \mathcal{R}^{1+\beta_{n+1}}(\delta_{n+2,k_{n+1}}) \cdot L_{n+2,k_{n+1}}.
\]

**Proof.** By (i) of Remark 2.8
\[
\hat{x}_n = \hat{x}_n^\pm - \delta_{n+1,k_n} \cdot L_{n+1,k_n}. \quad (2.1)
\]
Also, plug $n + 1$ into (i) of 2.8 whence
\[
\begin{align*}
\hat{x}_{n+1}^\pm &= \hat{x}_n^\pm + \delta_{n+2,k_{n+1}} \cdot L_{n+2,k_{n+1}} \quad \text{(i) of Lem. 2.9} \\
&= \hat{x}_n^\pm + \beta_{n+1} \cdot \delta_{n+1,k_n} \cdot L_{n+1,k_n} + \delta_{n+2,k_{n+1}} \cdot L_{n+2,k_{n+1}} \quad \text{by (2.1)} \\
&= \hat{x}_n^\pm - \delta_{n+1,k_n} \cdot L_{n+1,k_n} + \beta_{n+1} \cdot \delta_{n+1,k_n} \cdot L_{n+1,k_n} + \delta_{n+2,k_{n+1}} \cdot L_{n+2,k_{n+1}} \quad \text{=} \\
&= \hat{x}_n^\pm + (\beta_{n+1} - 1) \cdot \delta_{n+1,k_n} \cdot L_{n+1,k_n} + \delta_{n+2,k_{n+1}} \cdot L_{n+2,k_{n+1}} \quad \text{by (iii) of Rem. 2.8} \\
&= \hat{x}_n^\pm + (\beta_{n+1} - 1) \cdot (L_{n+2,k_{n+1}} + \beta_{n+1} \cdot (1 - 2 \cdot L_{n+2,k_{n+1}})) + \delta_{n+2,k_{n+1}} \cdot L_{n+2,k_{n+1}}.
\end{align*}
\]
If $\beta_{n+1} = 0$, then
\[ \hat{x}_{n+1}^\pm = \hat{x}_n^\pm - L_{n+2,k_{n+1}} + \delta_{n+2,k_{n+1}} \cdot L_{n+2,k_{n+1}} = \hat{x}_n^\pm - \mathcal{R}(\delta_{n+2,k_{n+1}}) \cdot L_{n+2,k_{n+1}}. \]

If $\beta_{n+1} = 1$, then
\[ \hat{x}_{n+1}^\pm = \hat{x}_n^\pm + \delta_{n+2,k_{n+1}} \cdot L_{n+2,k_{n+1}}. \]

In each of these the lemma follows. \qed

**Remark 2.11.** [8, Rem. 2.17] For any $n \geq 1$ and all $i, 1 \leq i \leq n$, one have $g^i(I_{n+1,k_n}) = I_{n+1-i,\mathcal{R}_{n-i}^{\beta_i}(k_{i+1,n})}$.

We have obtained the next fact duffing the proof of [8, Remark 2.19].

**Remark 2.12.** For every $n \geq 1$ and for all $i \leq n$ we have
\[ \delta_{n+1,k} = \mathcal{R}^{\beta_i}(\delta_{n+1-i,\mathcal{R}_{n-i+1}^{\beta_i}(k_{i,n})}). \]

### 3 Invariant coordinates, $g$-sequences and $g$-decompositions

Notice, that there is strong relation between the $g$-expansion $(\beta_i)_{i \geq 0}$ and digit sequence.

**Lemma 3.1.** Let $\varrho_i$ be $g$-digit sequence of $x$. Then
\[ \begin{aligned} (i) \quad & \varrho_i = \mathcal{R}^{\beta_{i-1}}(\beta_i), \text{ and} \\ (ii) \quad & \beta_i = \mathcal{R}\sum_{j=1}^i \varrho_j(0) \text{ for } i \geq 1. \end{aligned} \]

**Proof.** Part (i) follow from Remark 2.11

By definitions write
\[ \beta_1 = \varrho_1. \]

Now, by (i), obtain
\[ \beta_2 = \mathcal{R}^{\varrho_2}(\varrho_1) = \mathcal{R}^{\varrho_2+\varrho_1}(0). \]

Thus, part (ii) follows from (i) by induction. \qed

**Lemma 3.2.** There is the correspondence between $g$-expansions and invariant coordinates, precisely:

1. The rule of the construction of $\theta_{i-1}$ by $\beta_i$ is as below:
   a. If $\beta_i = 0$, but not all $\{\beta_k, k > i\}$ are zero, then $\theta_{i-1} = -1$;
   b. If $\beta_k = 0$ for all $k \geq i$, then $\theta_{i-1} = 0$.
   c. If $\beta_i = 1$, then $\theta_{i-1} = 1$.

2. The rule of the construction of $\beta_{i+1}$ by $\theta_i$ is as below: if $\theta_i < 1$, then $\beta_{i+1} = 0$, otherwise $\beta_{i+1} = 1$.  

6
4 Skew tent maps

We will specify in this section the results about \( g \)-expansion of carcass maps to the case of skew carcass map. Till the end of this section let fix arbitrary number \( v \in (0, 1) \), skew tent map \( f_v \), and \( x \in [0, 1] \) with \( f_v \)-decomposition \((\beta_i)_{i \geq 0}\). We will show in this section that some our results of [4], which we have obtained about skew tent maps, are partial case of more general reasonings about unimodal maps.

**Lemma 4.1.** [4, Lema 8] For every \( n \geq 1 \) we have that

\[ \delta_k = \mathcal{R}^{\beta_n}(v). \]

**Proof.** Plug \( i = n \) into Remark 2.12 and obtain

\[ \delta(I_{n+1,k}) = \mathcal{R}^{\beta_n}(\delta(I_{1,R_1^{\beta_n}(\beta_n)})) = \mathcal{R}^{\beta_n}(v) \]

and we are done. \( \square \)

**Remark 4.2.** Notice that Lemma 4.1 (i.e. Lemma 8 in [4]) was formulated in [4] as follows: for each \( n \geq 1 \) the equality \( \hat{x}^+_n - \hat{x}_n = (\hat{x}^+_n - \hat{x}_n) \cdot \varphi(\beta_n, 0) \) holds, where

\[ \varphi(a, b) = \begin{cases} v & \text{if } a = b, \\ 1 - v & \text{if } a \neq b \end{cases} \]

for all \( a, b \in \{0, 1\} \).

**Remark 4.3.** [4, Lemma 11] For every \( n \geq 1 \) we have

\[ L_{n+1,k} = L_{n,k_{n-1}} \cdot \mathcal{R}^{\beta_n+\beta_{n+1}}(v). \]

**Proof.** By (ii) of Remark 2.8 write

\[ L_{n+1,k} = L_{n,k_{n-1}} \cdot \mathcal{R}^{\beta_n}(\delta_{k_{n-1}})^\text{Lem. 4.1} = L_{n,k_{n-1}} \cdot \mathcal{R}^{\beta_n+\beta_{n+1}}(v). \]

\( \square \)

**Remark 4.4.** Notice that Lemma 4.3 (i.e. Lemma 11 in [4]) was formulated in [4] as follows: for every \( n \geq 1 \) we have \( \hat{x}^+_n - \hat{x}_{n+1} = (\hat{x}^+_n - \hat{x}_n) \cdot \varphi(\beta_n, \beta_{n+1}) \), where \( \varphi \) means the same as in Remark 4.2.

We will need the next fact for our further computations.

**Lemma 4.5.** For every \( n \geq 1 \) we have

\[ L_{n+1,k} = \prod_{i=0}^{n-1} \mathcal{R}^{\beta_i+\beta_{i+1}}(v). \]
Proof. It follows from Lemma 4.3 by induction on \( n \) that \( L_{n+1,k_n} = \prod_{i=1}^{n} R_{\beta_{n+1-i} + \beta_{n-i}}(v) \). Now, change \( i \) to \( n - i \), and we are done. \( \square \)

We are ready now to prove Theorem 4.

**Proof of Theorem 4** Keeping in mind Lemma 4.5, notice that \( R_{\beta_i + \beta_{i+1}}(v) \) in the product \( \prod_{i=0}^{n-1} R_{\beta_i + \beta_{i+1}}(v) \) equals either \( v \), or \( 1 - v \) and, moreover the exponent of \( (1 - v) \) in the product is \( \sum_{i=0}^{n-1} |\beta_{i+1} - \beta_i| \). Now, simplify

\[
\sum_{i=0}^{n-1} |\beta_{i+1} - \beta_i| = \sum_{i=0}^{n-1} R_{\beta_i} \beta_{i+1} \cdot \beta_i + 1 \cdot R_{\beta_i} \cdot L_{i,k_i-1},
\]

and we are done by Lemma 4.5. \( \square \)

## 5 Explicit formulas for the conjugacy

**Lemma 5.1.** Let \( (\beta_i)_{i\geq 0} \) be the \( g \)-decomposition of a number \( x \in [0,1] \). Then for every \( n \geq 1 \) we have that:

(i) \( \hat{x}_n = \sum_{i=1}^{n} \beta_i \cdot \delta_{k_{i-1}} \cdot L_{i,k_{i-1}} \)

(ii) \( \hat{x}_n^\pm = 1 - \sum_{i=1}^{n} (1 - \beta_i) \cdot (1 - \delta_{k_{i-1}}) \cdot L_{i,k_{i-1}} \)

(iii) \( \hat{x}_n^\pm = v + \sum_{i=1}^{n} (-1)^{1+\beta_i} \cdot R_{1+\beta_i} \delta_{k_i} \cdot L_{i+1,k_i} \)

Proof. Notice that \( \hat{x}_0 = 0; \hat{x}_0^\pm = v \) and \( \hat{x}_0^+ = 1 \). Then, the lemma follows from Lemmas 2.9 and 2.10. \( \square \)

**Lemma 5.2.** For every \( n \geq 1 \) we have

\[
\hat{x}_n = \beta_1 \cdot v + \sum_{n=2}^{\infty} \beta_n \cdot R_{\beta_{n-1}}(v) \cdot \prod_{i=0}^{n-2} R_{\beta_i + \beta_{i+1}}(v).
\]

Proof. By (i) of Lemma 5.1

\[
\hat{x}_n = \sum_{i=1}^{n} \beta_i \cdot \delta_{k_{i-1}} \cdot L_{i,k_{i-1}}(x) \overset{\text{Lem. 4.1}}{=} \beta_1 \cdot v + \sum_{i=2}^{n} \beta_i \cdot R_{\beta_{i-1}}(v) \cdot L_{i,k_{i-1}}(x),
\]

and the lemma follows from Lemma 4.5. \( \square \)

**Lemma 5.3.** For any \( v \in (0,1) \), any \( x \in (0,1) \) with \( f_v \)-digit sequence \( (\varrho_i)_{i\geq 1} \) the equality

\[
\hat{x}_n = \varrho_1 \cdot v + \sum_{i=2}^{n} R_{\sum_{j=1}^{i-1} \varrho_j} (0) \cdot R_{\sum_{j=1}^{i-1} \varrho_j} (v) \cdot (1 - v) \sum_{j=1}^{i-1} \varrho_j \cdot v^{i-1 - \sum_{j=1}^{i-1} \varrho_j}
\]

holds.
Proof. By (i) of Lemma 4.1
\[
\hat{x}_n = \sum_{i=1}^{n} \beta_i \cdot \delta_{k_i-1} \cdot L_i,k_{i-1}(x) = \beta_1 \cdot v + \sum_{i=2}^{n} \beta_i \cdot R^{\beta_{i-1}}(v) \cdot L_i,k_{i-1}(x) \tag{5.1}
\]
\[
= \varrho_1 \cdot v + \sum_{i=2}^{n} \sum_{j=1}^{i-1} \varrho_j(0) \cdot R^{\sum_{j=1}^{i-1} \varrho_j(0)}(v) \cdot L_i,k_{i-1}(x) \tag{5.2}
\]
\[
= \varrho_1 \cdot v + \sum_{i=2}^{n} \sum_{j=1}^{i-1} \varrho_j(0) \cdot R^{\sum_{j=1}^{i-1} \varrho_j(0)}(v) \cdot L_i,k_{i-1}(x),
\]
and we are done by Theorem 4.

\[\square\]

Lemma 5.4. Let \( x \in [0, 1] \) has \( f_v \)-digits \((\varrho_i)_{i \geq 1}\). Then
\[
\hat{x}_n^+ = 1 - (1 - \beta_1) \cdot (1 - v) - \sum_{i=2}^{n} (1 - \beta_i) \cdot R^{1+\beta_{i-1}}(v) \cdot \prod_{j=0}^{i-2} R^{\beta_j + \beta_{j+1}}(v)
\]

Proof. By (ii) of Lemma 5.1
\[
\hat{x}_n^+ = 1 - \sum_{i=1}^{n} (1 - \beta_i) \cdot (1 - \delta_{k_i-1}) \cdot L_i,k_{i-1} \tag{5.3}
\]
\[
= 1 - \sum_{i=1}^{n} (1 - \beta_i) \cdot (1 - R^{\beta_{i-1}}(v)) \cdot L_i,k_{i-1} \tag{5.4}
\]
\[
= 1 - (1 - \beta_1) \cdot (1 - v) - \sum_{i=2}^{n} (1 - \beta_i) \cdot R^{1+\beta_{i-1}}(v) \cdot \prod_{j=0}^{i-2} R^{\beta_j + \beta_{j+1}}(v)
\]
and we are done.

\[\square\]

Lemma 5.5. Let \( x \in [0, 1] \) has \( f_v \)-digits \((\varrho_i)_{i \geq 1}\). Then
\[
\hat{x}_n^+ = 1 - (1 - \varrho_1) \cdot (1 - v) - \sum_{i=2}^{n} R^{1+\sum_{j=1}^{i} \varrho_j(0)}(v) \cdot R^{\sum_{j=1}^{i} \varrho_j(0)}(v) \cdot (1 - v) \sum_{i=1}^{n-1} \varrho_i \cdot v + n - 1 - \sum_{i=1}^{n-1} \varrho_i .
\]

Proof. By (ii) of Lemma 5.1
\[
\hat{x}_n^+ = 1 - \sum_{i=1}^{n} (1 - \beta_i) \cdot (1 - \delta_{k_i-1}) \cdot L_i,k_{i-1} \tag{5.5}
\]
\[
= 1 - \sum_{i=1}^{n} (1 - \beta_i) \cdot (1 - R^{\beta_{i-1}}(v)) \cdot L_i,k_{i-1} \tag{5.6}
\]
\[
= 1 - (1 - \beta_1) \cdot (1 - v) - \sum_{i=2}^{n} (1 - \beta_i) \cdot R^{1+\beta_{i-1}}(v) \cdot (1 - v) \sum_{i=1}^{n-1} \varrho_i \cdot v + n - 1 - \sum_{i=1}^{n-1} \varrho_i \tag{5.7}
\]
\[
= 1 - (1 - \varrho_1) \cdot (1 - v) - \sum_{i=2}^{n} R^{1+\sum_{j=1}^{i} \varrho_j(0)}(v) \cdot R^{\sum_{j=1}^{i} \varrho_j(0)}(v) \cdot (1 - v) \sum_{i=1}^{n-1} \varrho_i \cdot v + n - 1 - \sum_{i=1}^{n-1} \varrho_i .
\]
which is necessary.

\[\square\]
Lemma 5.6. Let \( x \in [0, 1] \) has \( f_v \)-digits \((\xi_i)_{i \geq 1}\). Then

\[
\hat{x}_n^+ = v + \sum_{i=1}^{n} (-1)^{\beta_i + 1} \cdot R_{\beta_i + 1}(v) \cdot \prod_{i=0}^{n-1} R_{\beta_i + \beta_{i+1}}(v)
\]

Proof. By (iii) of Lemma 5.1

\[
\hat{x}_n^+ = v + \sum_{i=1}^{n} (-1)^{\beta_i + 1} \cdot R_{\beta_i + 1}(\delta_{i+1,k_i}) \cdot L_{i+1} \quad \text{by Lem. 4.1}
\]

\[
= v + \sum_{i=1}^{n} (-1)^{\beta_i + 1} \cdot R_{\beta_i + 1}(v) \cdot L_{i+1}
\]

and the lemma follows from Lemma 4.5. \(\square\)

Lemma 5.7. Let \( x \in [0, 1] \) has \( f_v \)-digits \((\xi_i)_{i \geq 1}\). Then

\[
\hat{x}_n^+ = v + \sum_{i=1}^{n} (v - 1)^{1+\sum_{j=1}^{i} \xi_j} \cdot v^{i-\sum_{j=1}^{i} \xi_j}
\]

Proof. By (iii) of Lemma 5.1

\[
\hat{x}_n^+ = v + \sum_{i=1}^{n} (-1)^{\beta_i + 1} \cdot R_{\beta_i + 1}(\delta_{i+1,k_i}) \cdot L_{i+1} \quad \text{by Lem. 4.1}
\]

\[
= v + \sum_{i=1}^{n} (-1)^{\beta_i + 1} \cdot R_{\beta_i + 1}(v) \cdot L_{i+1} \quad \text{by Th. 4}
\]

\[
= v + \sum_{i=1}^{n} (-1)^{\beta_i + 1} \cdot (1 - v) \cdot (1 - v)^{\sum_{j=1}^{i} \xi_j} \cdot v^{i-\sum_{j=1}^{i} \xi_j} \quad \text{by Lem. 3.1}
\]

\[
= v + \sum_{i=1}^{n} (-1)^{1+\sum_{j=1}^{i} \xi_j} \cdot (0) \cdot (1 - v)^{\sum_{j=1}^{i} \xi_j} \cdot v^{i-\sum_{j=1}^{i} \xi_j} =
\]

\[
= v + \sum_{i=1}^{n} (v - 1)^{1+\sum_{j=1}^{i} \xi_j} \cdot v^{i-\sum_{j=1}^{i} \xi_j} =
\]

and we are done. \(\square\)

Proof of Theorem 5. Denote \( a_n = \hat{x}_n + \beta_{n+1} \cdot L_{n+1,k_n} \).

\[
a_{n+1} = \hat{x}_{n+1} + \beta_{n+2} \cdot L_{n+2,k_{n+1}} \quad \text{(i) of Lem. 2.9}
\]

\[
= \hat{x}_n + \beta_{n+1} \cdot \delta_{n+1,k_n} \cdot L_{n+1,k_n} + \beta_{n+2} \cdot L_{n+2,k_{n+1}} \quad \text{(ii) of Rem. 2.8}
\]

\[
= \hat{x}_n + \beta_{n+1} \cdot \delta_{n+1,k_n} \cdot L_{n+1,k_n} + \beta_{n+2} \cdot R_{\beta_{n+1}}(\delta_{n+1,k_n}) \cdot L_{n+1,k_n} =
\]

\[
a_n + (\beta_{n+1} \cdot \delta_{n+1,k_n} - \beta_{n+1} + \beta_{n+2} \cdot R_{\beta_{n+1}}(\delta_{n+1,k_n})) \cdot L_{n+1,k_n}
\]
If $\beta_{n+1} = 0$, then
\[ a_{n+1} = a_n + \beta_{n+2} \cdot \delta_{n+1,k_n} \cdot L_{n+1,k_n} \]

If $\beta_{n+1} = 1$, then
\[ a_{n+1} = a_n + (\delta_{n+1,k_n} - 1 + \beta_{n+2} \cdot \mathcal{R}(\delta_{n+1,k_n})) \cdot L_{n+1,k_n} = a_n + (\beta_{n+2} \cdot \mathcal{R}(\delta_{n+1,k_n})) \cdot L_{n+1,k_n} = a_n - \mathcal{R}(\beta_{n+2}) \cdot \mathcal{R}(\delta_{n+1,k_n}) \cdot L_{n+1,k_n} \]

In general,
\[ a_{n+1} = a_n + (-1)^{\beta_{n+1}} \cdot \mathcal{R}^{\beta_{n+1}}(\beta_{n+2}) \cdot \mathcal{R}^{\beta_{n+1}}(\delta_{n+1,k_n}) \cdot L_{n+1,k_n} \]

Since $a_0 = \beta_1$, then for any $n \geq 1$ we have
\[ a_n = \beta_1 + \sum_{i=1}^{n} (-1)^{\beta_i} \cdot \mathcal{R}^{\beta_i}(\beta_{i+1}) \cdot L_{i+1,k_i} \]

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