Tests for zero-inflation and overdispersion

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Abstract

We propose a new methodology to detect zero-inflation and overdispersion based on the comparison of the expected sample extremes among convexly ordered distributions. The method is very flexible and includes tests for the proportion of structural zeros in zero-inflated models, tests to distinguish between two ordered parametric families and a new general test to detect overdispersion. The performance of the proposed tests is evaluated via some simulation studies. For the well-known fetal lamb data, we conclude that the zero-inflated Poisson model should be rejected against other more disperse models, but we cannot reject the negative binomial model.

Keywords: Zero-inflated Poisson distribution; binomial distribution; negative binomial distribution; hypothesis testing; convex order; parametric bootstrap.

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1 Introduction

The Poisson distribution is the standard model for the analysis of count data. However, in many situations this type of observations exhibit a substantially larger proportion of zeros than what is expected for the Poisson model (see Gupta et al. (1996)). For instance, this is often the case with count data coming from medical and public health research (see Böhning et al. (1999) and Campbell et al. (1991)). This phenomenon usually arises when the distribution generating the data is a mixture of two populations, the first of which yields Poisson-distributed counts whereas the second one always contributes with a zero.

One natural model to describe the above situation is the so-called zero-inflated Poisson (ZIP) model. We say that the random variable $Y(\theta, p)$ has a ZIP distribution with parameters $\theta$ and $p$ ($\theta > 0$ and $0 \leq p < 1$) if

$$
\Pr(Y(\theta, p) = k) = \begin{cases} 
  p + (1 - p)e^{-\theta/(1-p)}, & \text{if } k = 0 \\
  e^{-\theta/(1-p)} \frac{\theta^k}{k!(1-p)^{k-1}}, & \text{if } k = 1, 2, \ldots 
\end{cases}
$$

(1)

Therefore, $Y(\theta, p)$ is a mixture of a degenerate-at-zero distribution (with weight $p$) and a Poisson distribution of mean $\theta/(1-p)$ (with weight $1-p$). In particular, $Y(\theta, 0)$ is the classical Poisson variable with mean $\theta$. The ZIP distribution has been used in diverse areas such as medicine (Böhning et al. (1992, 1999) and van den Broek (1995)) or biology (Nie et al. (2006)), among others.

The expected value of the ZIP distribution is $E(Y(\theta, p)) = \theta$ and the variance $\text{Var}(Y(\theta, p)) = \theta + \theta^2 p/(1-p)$ increases as $p$ increases. The zeros coming from the degenerate variable are called structural zeros and those from the Poisson model sampling zeros. It should be observed at this point that, to keep the mean fixed for different values of $p$, we do not follow the usual notation for the ZIP models.

If the proportion of atypical zero observations remains undetected, the variability of the population is underestimated and the properties of standard inference techniques are, to some extent, deteriorated. For this reason, in the recent literature
there are different proposals to determine whether the Poisson model fits a data set well enough or, alternatively, we should choose a ZIP model that allows for an extra proportion of zero counts. A clear and concise review of several of these tests can be found in Xie et al. (2001). A popular and simple choice with good properties is the score test proposed by van den Broek (1995).

Of course, as pointed out by El-Shaarawi (1985) and Thas and Rayner (2005), the rejection of the Poisson model does not imply that the ZIP distribution is the most appropriate model to fit the data. It may happen that an alternative model that accounts for the observed dispersion could fit the data better. The negative binomial and the zero-inflated negative binomial distributions are examples of reasonable alternatives.

In this work we introduce a new procedure to detect zero-inflation and overdispersion. The key idea is to link the notion of overdispersion with the concept of variability stochastic order. These orders arrange distributions according to their variability (see Section 3 of Shaked and Shanthikumar (2006)). Therefore, it is natural to suppose that the observed overdispersion is due to the data actually coming from a different model that dominates the initially assumed distribution in a variability order. The most important variability order is the so-called convex order. We use the properties of this order to derive suitable discrepancy measures for tests in which “overdispersion” is understood as “convex domination”.

The method we propose is flexible and easy to implement. It is based on the empirical comparison of the expected sample extremes of two ordered models. An important feature is that the main ideas can be readily adapted to cover several different testing problems: tests for the proportion of structural zeros in zero-inflated models; procedures for testing if a parametric model is appropriate against another one with more variability; and a new general test to detect overdispersion. We illustrate in detail the application of the methodology to the case of the ZIP models, but the technique can be analogously applied in other situations.

The definitions and relevant results on stochastic convex dominance are briefly
reviewed in Section 2. These results supply the necessary theoretical background for the rest of the paper. In Section 3, we provide a general framework to detect overdispersion in ZIP models, but we note that the proposed method is very general and can be adapted to many other similar scenarios. We find discrepancy measures for tests on the proportion of structural zeros and discuss whether the Poisson model is appropriate or we should opt for a different model with more dispersion. In Section 4 we establish the relationships, in terms of the convex order, for some zero-inflated models usually considered in the literature: the zero-inflated binomial, Poisson and negative binomial model. These results allow to extend the previous ideas to these important discrete models. Section 5 analyzes the performance of the proposed tests via some Monte Carlo studies. Our proposals are very competitive against the well-known score test in the cases in which the latter can be applied. In Section 6, we analyze the fetal lamb data from Leroux and Puterman (1992) using our new procedures. For this data set we conclude that the ZIP distribution should be rejected against other models with more variability. This result is consistent with the previous work by Thas and Rayner (2005). Moreover, we show that the negative binomial model cannot be rejected. Finally, the proofs of the main results are collected in the appendix.

2 The convex order and overdispersion

In this section, we link the overdispersion phenomenon described in the introduction with the convex stochastic order. Given two integrable random variables $X$ and $Y$, it is said that $X$ is less or equal to $Y$ in the convex order, and we denote it by $X \leq_{cx} Y$, if $E(\phi(X)) \leq E(\phi(Y))$ for every convex function $\phi$ for which the previous expectations are well defined. Notice that, by considering the convex functions $\phi(x) = \pm x$, the condition $X \leq_{cx} Y$ implies that $EX = EY$. Furthermore, if the variables have finite second moment, applying the definition of the convex order with $\phi(x) = (x - EX)^2$, we conclude that $\text{Var}(X) \leq \text{Var}(Y)$. Of course, establishing the relation $X \leq_{cx} Y$ is much more informative than just knowing $\text{Var}(X) \leq \text{Var}(Y)$. 

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Roughly speaking, since convex functions take larger values when its argument is large, if \( X \leq_{cx} Y \) holds, then \( Y \) is more likely to take "extreme values" than \( X \). This idea is clear from the following proposition. The result is a consequence of Corollary 4.A.16 and Theorem 4.A.50 in Shaked and Shanthikumar (2006), regarding the expected value of the extreme order statistics of two ordered variables. For \( k \geq 1 \), if \((X_1, \ldots, X_k)\) is a random sample of size \( k \) from \( X \), we denote by \( X_{i;k} \) the \( i \)-th order statistic of the sample, \( i = 1, \ldots, k \). Therefore, \( X_{1;k} \) and \( X_{k;k} \) stand for the minimum and maximum of the sample.

**Proposition 1.** Let \( X \) and \( Y \) be integrable random variables such that \( X \leq_{cx} Y \).

(a) For all \( k \geq 1 \), \( EY_{1;k} \leq EX_{1;k} \) and \( EX_{k;k} \leq EY_{k;k} \).

(b) If for some \( k \geq 2 \) \( EX_{1;k} = EY_{1;k} \) or \( EX_{k;k} = EY_{k;k} \), then \( X \) and \( Y \) have the same distribution.

For instance, for the ZIP variables defined as in (1), we can prove (see Section 7) that

\[
Y(\theta, p_1) \leq_{cx} Y(\theta, p_2), \quad 0 \leq p_1 < p_2 < 1, \quad \theta > 0.
\]

(2)

Hence, Proposition 1 jointly with (2) imply that the ZIP variable \( Y(\theta, p_2) \) is expected to take strictly larger extreme values than \( Y(\theta, p_1) \) whenever \( p_1 < p_2 \).

### 3 Tests for overdispersion in ZIP models

In this section we exploit Proposition 1 to derive discrepancy measures useful to test for overdispersion in ZIP models. We emphasize that the same technique, with the obvious modifications, can be applied in a similar way for the zero-inflated binomial and negative binomial models (see Section 4) or, in general, for any pair of ordered distributions.

The discrepancies introduced in this section are defined in terms of the empirical counterparts of the expected extreme order statistics. Therefore, our goal is to
detect (significant) differences between the estimates of the expected extremes of two distributions.

Actually, we deal with two different problems. In Subsection 3.1 we propose statistical tests to analyze the proportion of structural zeros in ZIP models. In other situations, we may want to check if the ZIP model cannot account for the dispersion of the data. Then it is adequate to apply the nonparametric procedure of Subsection 3.2.

3.1 Tests for the proportion of structural zeros

Given a random sample $Y_1, \ldots, Y_n$ from a variable $Y(\theta, p)$ with the ZIP distribution \(^1\), we are interested in testing $H_0 : p \leq p_0$ against $H_1 : p > p_0$, where $p_0$ is fixed and belongs to $[0, 1)$ (the left unilateral and bilateral tests may be studied by similar arguments). There are several works in the literature devoted to this testing problem with $p_0 = 0$ (see e.g. van den Broek (1995), Xie et al. (2001), Jansakul and Hinde (2002) and He et al. (2003)). This particular case is important since it is equivalent to testing the Poisson model against a ZIP model with a positive proportion of structural zeros. However, as far as we know, there are no references in the literature including tests for values of $p_0 \in (0, 1)$.

The method we propose is based on the following simple idea: \(^2\) states that $Y(\theta, p_1) \leq_{\text{ex}} Y(\theta, p_2)$ whenever $0 \leq p_1 < p_2$ and hence according to Proposition \(^1\) the variable $Y(\theta, p)$ is expected to take strictly larger extreme values under $H_1$ than under $H_0$. Using the information in $Y_1, \ldots, Y_n$, we can estimate the expectation of the maximum (or minimum) in a generic subsample of size $k \geq 2$ from $Y(\theta, p)$ and $Y(\theta, p_0)$. Then, we reject $H_0$ whenever the difference between the two estimates is too large.

More precisely, we denote by $E_{\theta, p}(Y_{1:k})$ and $E_{\theta, p}(Y_{1:k})$ the expected values of the maximum and minimum of $k$ independent copies of $Y(\theta, p)$, respectively. Given the random sample $Y_1, \ldots, Y_n$ from $Y(\theta, p)$, the maximum likelihood estimates of the
parameters $\theta$ and $p$ in the ZIP model satisfy (see Johnson et al. 2005)

$$\hat{\theta} = \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \quad \text{and} \quad \hat{p} = 1 - \frac{1 - n_0/n}{1 - \exp(-\hat{\theta}/(1 - \hat{p}))}, \quad (3)$$

where $n_0$ is the number of zero-counts in the sample. Then, for $k \geq 2$, we compute the discrepancy measures:

$$\Delta_{k:k} = E_{\hat{\theta},\hat{p}}(Y_{k:k}) - E_{\hat{\theta},p_0}(Y_{k:k}) \quad \text{and} \quad \Delta_{1:k} = E_{\hat{\theta},p_0}(Y_{1:k}) - E_{\hat{\theta},\hat{p}}(Y_{1:k}) \quad (4)$$

and reject $H_0$ either if $\Delta_{k:k}$ or $\Delta_{1:k}$ is too large. Observe that, from the equalities $E(Y(\hat{\theta}, p_0)) = E(Y(\hat{\theta}, \hat{p}))$ and $E(X_{1:2}) + E(X_{2:2}) = 2 E(X)$ (which holds for any integrable random variable $X$), it is readily checked that $\Delta_{2:2} = \Delta_{1:2}$.

If we denote by $F_{\theta,p}$ the distribution function of $Y(\theta, p)$, the discrepancies in (4) can be rewritten as:

$$\Delta_{k:k} = \sum_{i=0}^{\infty} \left[ F_{\hat{\theta},p_0}(i) - F_{\hat{\theta},\hat{p}}(i) \right]^k, \quad (5)$$

$$\Delta_{1:k} = \sum_{i=0}^{\infty} \left[ (1 - F_{\hat{\theta},p_0}(i) - (1 - F_{\hat{\theta},\hat{p}}(i)) \right]^k. \quad (5)$$

In practice, we can always truncate the above series to approximate their value.

To obtain the rejection region of the tests we need to find the distribution of $\Delta_{k:k}$ or $\Delta_{1:k}$ for $k \geq 2$ under $H_0$. In Theorem 1 we obtain the asymptotic distribution of $\Delta_{2:2}$ when $p_0 = 0$. However, in general, the distribution of these quantities is rather involved and a simple parametric bootstrap schema can be used instead. The following procedure is described for the discrepancy $\Delta_{k:k}$ but the corresponding one for $\Delta_{1:k}$ is analogous:

(a) Find the estimate $\hat{\theta} = \bar{Y}$.

(b) Extract $B$ parametric bootstrap samples of size $n$, $Y_{1:b}^*, \ldots, Y_{n:b}^*$, for $b = 1, \ldots, B$, from the distribution of $Y(\hat{\theta}, p_0)$.

(c) For each sample $Y_{1:b}^*, \ldots, Y_{n:b}^*$, obtain the estimates $\hat{\theta}_b^*$ and $\hat{p}_b^*$ using (3).
(d) Compute the discrepancies $\Delta_{k}^{*,b} = E_{\hat{\theta}_b, p_b}(Y_{k:k}) - E_{\hat{\theta}_0, p_0}(Y_{k:k}), b = 1, \ldots, B$.

(e) For a significance level $\alpha$, find $Q_{k:k}^{*}(\alpha)$, the $(1 - \alpha)$-quantile of the values $\{\Delta_{k}^{*,b}, b = 1, \ldots, B\}$.

The rejection region for the test $H_0 : p \leq p_0$ versus $H_1 : p > p_0$, at significance level $\alpha$, is approximated by

$$R_{\alpha} = \{\Delta_{k} > Q_{k:k}^{*}(\alpha)\}.$$  \hspace{1cm} (6)

As it was mentioned before, the case $p_0 = 0$ corresponds to testing the Poisson model against a ZIP model with $p > 0$. The simulation studies in Subsection 5.2 show that $\Delta_{2:2}$ has a good behavior. The use of $\Delta_{2:2}$ means that we compare what we expect to obtain for the maximum (or minimum) of two independent Poisson variables with that of two ZIP variables with $p > 0$. In this case, there is a closed-form expression for $E_{\theta,0}(Y_{2:2})$ (see Johnson et al. (2005), p. 166):

$$M_2(\hat{\theta}) := E_{\theta,0}(Y_{2:2}) = \theta + \theta e^{-2\hat{\theta}} \left( I_0(2\hat{\theta}) + I_1(2\hat{\theta}) \right),$$ \hspace{1cm} (7)

where $I_0$ and $I_1$ are modified Bessel functions of the first kind (see e.g. Abramowitz and Stegun (1965)). Using (7) we can rewrite the discrepancy $\Delta_{2:2}$ given in (5) with $p_0 = 0$ as

$$\Delta_{2:2} = 2\hat{\theta} + (1 - \hat{\theta})^2 M_2(\hat{\theta}/(1 - \hat{\theta})) - M_2(\hat{\theta}).$$ \hspace{1cm} (8)

This enables us to obtain the asymptotic distribution of $\Delta_{2:2}$ under $H_0 : p = 0$ (Poissonness). In the following theorem the symbol $\rightarrow_d$ stands for “convergence in distribution” and $N(0, 1)$ is a standard normal variable.

**Theorem 1.** Under $H_0 : p = 0$, it holds that

$$\sqrt{n} \frac{\Delta_{2:2}}{\sigma(\hat{\theta})} \rightarrow_d N(0, 1), \quad n \rightarrow \infty,$$

where

$$\sigma^2(\hat{\theta}) := \frac{\hat{\theta}^2 \left[ (1 + \hat{\theta}) I_0(2\hat{\theta}) - I_1(2\hat{\theta}) + \hat{\theta} I_2(2\hat{\theta}) \right]^2}{e^{\hat{\theta} - 1 - \hat{\theta}}},$$ \hspace{1cm} (9)

and $I_0$, $I_1$ and $I_2$ are modified Bessel functions of the first kind.
As an immediate consequence of Theorem 1 a critical region with asymptotic
significance level \( \alpha \) for \( H_0 : p = 0 \) against \( H_1 : p > 0 \) is

\[
R_\alpha = \left\{ \sqrt{n} \frac{\Delta_{2:2}}{\sigma(\hat{\theta})} > z_\alpha \right\},
\]

(10)

with \( z_\alpha \) being the \( (1 - \alpha) \)-quantile of the standard normal distribution. We remark
that this test is very simple and easy to implement since the Bessel functions appearing in \( \Delta_{2:2} \) and \( \sigma(\hat{\theta}) \) can be evaluated by any standard mathematical software
package.

3.2 A general test to detect overdispersion

Here, we deal with the problem of detecting if a data set comes from a Poisson
distribution or there is dispersion that the Poisson model cannot take into account.
The same procedure works for the more general ZIP model or the distributions
considered in Section 4 but we illustrate the ideas with the Poisson distribution for
the sake of simplicity.

Let us consider the family \( \mathcal{P} := \{ Y(\theta) : \theta > 0 \} \), where \( Y(\theta) \) is a Poisson variable
with mean \( \theta \). We denote by \( \mathcal{P}_{cx} \) the set of all integrable random variables, not
having the Poisson distribution, that dominate in the convex order a variable in \( \mathcal{P} \).
Therefore, \( \mathcal{P}_{cx} \) includes distributions with strictly more dispersion than the Poisson
variables. In particular, according to (2) and Proposition 3 in Section 4 all the ZIP
(with \( p > 0 \)) and the (zero-inflated) negative binomial distributions are included in
\( \mathcal{P}_{cx} \). Given a random sample \( Y_1, \ldots, Y_n \) from \( Y \), we want to test \( H_0 : Y \in \mathcal{P} \) against
\( H_1 : Y \in \mathcal{P}_{cx} \).

In this new test the alternative hypothesis is not completely specified in the sense
that it is not given by a parametric family. However, to handle this problem we can
use similar ideas to those in Subsection 3.1. We first estimate the parameter \( \theta, \hat{\theta} = \bar{Y} \).
Then, we compute the expectation of the maximum or minimum of \( k \) independent
copies of \( Y(\hat{\theta}) \), \( E_{\hat{\theta}}(Y_{k:k}) \) and \( E_{\hat{\theta}}(Y_{1:k}) \), as before in Subsection 3.1. On the other hand,
since there is no parametric restriction under \( H_1 \), we estimate \( EY_{k:k} \) and \( EY_{1:k} \) by
means of the following nonparametric plug-in estimators:

$$E_{F_n}(Y_{k:k}) := \sum_{i=1}^{n} \left[ \left( \frac{i}{n} \right)^k - \left( \frac{i-1}{n} \right)^k \right] Y_{i:n},$$

$$E_{F_n}(Y_{1:k}) := \sum_{i=1}^{n} \left[ \left( 1 - \frac{i-1}{n} \right)^k - \left( 1 - \frac{i}{n} \right)^k \right] Y_{i:n},$$

where $F_n$ is the empirical distribution function of the sample $Y_1, \ldots, Y_n$. Hence, for $k \geq 2$, we consider the discrepancies

$$\Lambda_{k,k} := E_{F_n}(Y_{k:k}) - E_{\hat{\theta}}(Y_{k:k}) \quad \text{and} \quad \Lambda_{1,k} := E_{\hat{\theta}}(Y_{1:k}) - E_{F_n}(Y_{1:k}). \quad (11)$$

Under $H_0$ these discrepancies are close to 0 whereas, if $H_1$ holds, then $\Lambda_{1,k}$ and $\Lambda_{k,k}$ are (strictly) positive for $n$ large enough. Therefore, we reject $H_0$ whenever $\Lambda_{1,k}$ or $\Lambda_{k,k}$ are too large. The rejection region of these tests can be derived by using a parametric bootstrap approach similar to the one described in Subsection 3.1.

We finally note that we actually have a different test for each discrepancy. The power of the test may depend on the selection of the statistic. The choice of a test with good power is addressed in Subsection 5.1.

4 Extensions to other models

The application of the methodology described in the previous section relies on verifying the convex domination of the involved variables. In this section, we establish all the relationships, according to the convex order, among the zero-inflated versions of some commonly used models for count data: the Poisson, the binomial and the negative binomial models. For these important discrete models, these relationships allow to extend straightaway the ideas developed in the previous section.

We first note that, given a data set, it is sensible to assume that the models that could fit the data have the same mean. Hence, all the parametric distributions considered in this section are selected to have the same expectation $\theta$.

For $m \geq 1$, $0 \leq p < 1$ and $0 < \theta \leq m(1-p)$, let us consider the random variable $X(m, \theta, p)$ which is the mixture between the degenerate-at-zero variable with weight
$p$ and a binomial variable of parameters $m$ and $\theta/[m(1-p)]$ with weight $1-p$. In other words, $X(m, \theta, p)$ has the zero-inflated binomial (ZIB) distribution with probabilities

\[
\Pr(X(m, \theta, p) = k) = \begin{cases} 
  p + \left(1 - \frac{\theta}{m(1-p)}\right)^m, & \text{if } k = 0 \\
  (1-p)\binom{m}{k} \left(\frac{\theta}{m(1-p)}\right)^k \left(1 - \frac{\theta}{m(1-p)}\right)^{m-k}, & \text{if } 1 \leq k \leq m.
\end{cases}
\]

Furthermore, we also consider the variable $Z(t, \theta)$ with negative binomial (NB) distribution of parameters $1/t$ and $t\theta$ ($t > 0$ and $\theta > 0$), i.e.,

\[
\Pr(Z(t, \theta) = k) = \binom{k + 1/t - 1}{k} \frac{(\theta t)^k}{(1 + \theta t)^{k+1/t}}, \quad k \geq 0.
\]

Among the different parametrizations of the NB distribution, we have chosen the unique one, $Z(t, \theta)$, with mean $\theta$ (for all $t$) and increasing in $t$ for the convex order, that is, satisfying $Z(t_1, \theta) \leq_{\text{cx}} Z(t_2, \theta)$ whenever $0 < t_1 < t_2$ (see Proposition 3 (d) below).

However, there are infinitely many possibilities to inflate with zeros the variable $Z(t, \theta)$ preserving the mean $\theta$. Among them, we only consider the most representative two. On the one hand, for $t, \theta > 0$ and $0 \leq p < 1$, let $Z_1(t, \theta, p)$ be the mixture between the degenerate-at-zero variable with weight $p$ and the variable $Z(t(1-p), \theta/(1-p))$ with weight $1-p$. On the other hand, for $t, \theta > 0$ and $0 \leq p < 1$ let $Z_2(t, \theta, p)$ be the mixture between the degenerate-at-zero variable with weight $p$ and the variable $Z(t, \theta/(1-p))$ with weight $1-p$. We refer to these two models as the zero-inflated negative binomial (ZINB) models.

In order to clarify the notation, Table 1 summarizes the relevant information about the models considered throughout this section. We note that all the variables have a fixed mean $\theta$ and a proportion $p$ of structural zeros.

The variance of all the zero-inflated variables described before is an increasing function of $p \in [0,1)$. Actually, the next proposition shows that they are convexly ordered for different values of $p$. 

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Table 1: Summary of the considered models.

| Model | Notation | Variance |
|--------|----------|----------|
| ZIB    | $X(m, \theta, p)$ | $\theta + \frac{\theta^2 p}{1-p} - \frac{\theta^2}{m(1-p)}$ |
| ZIP    | $Y(\theta, p)$ | $\theta + \frac{\theta^2 p}{1-p}$ |
| ZINB(1)| $Z_1(t, \theta, p)$ | $\theta + \frac{\theta^2 p}{1-p} + \theta^2 t$ |
| ZINB(2)| $Z_2(t, \theta, p)$ | $\theta + \frac{\theta^2 p}{1-p} + \frac{\theta^2 t}{1-p}$ |

Proposition 2. Let $X(m, \theta, p), Y(\theta, p)$ and $Z_i(t, \theta, p)$ ($i = 1, 2$) be variables with the ZIB, ZIP and ZINB distributions described above. If $0 \leq p_1 < p_2 < 1$, then

(a) $X(m, \theta, p_1) \leq_{cx} X(m, \theta, p_2)$, for all $m \geq 1$ and $0 < \theta \leq m(1-p_2)$.

(b) $Y(\theta, p_1) \leq_{cx} Y(\theta, p_2)$, for all $\theta > 0$.

(c) $Z_i(t, \theta, p_1) \leq_{cx} Z_i(t, \theta, p_2)$, for all $t > 0$, $\theta > 0$ and $i = 1, 2$.

The limiting distribution of $X(m, \theta, p)$ (as $m \uparrow \infty$) and of $Z_i(t, \theta, p)$ (as $t \downarrow 0$) for $i = 1, 2$ is the ZIP variable $Y(\theta, p)$. The smaller $m$ is, the more the ZIB variable differs from the ZIP one. Also, the larger $t$ is, the more the ZINB variables differ from the ZIP one.

For a fixed proportion of structural zeros, the next proposition presents the relationships among these four discrete models.

Proposition 3. For a fixed $p \in [0, 1)$, we have:

(a) $X(m, \theta, p) \leq_{cx} X(m + 1, \theta, p)$, for all $m \geq 1$ and $0 < \theta \leq m(1-p)$.

(b) $X(m, \theta, p) \leq_{cx} Y(\theta, p)$, for all $m \geq 1$ and $0 < \theta \leq m(1-p)$.  

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Proposition 2 allows to test on the proportion of structural zeros in all the models of this section. Further, Proposition 3 makes possible the comparison of these parametric families. The nonparametric tests described in Subsection 3.2 can also be adapted to these models. An example of the application of these tests can be found in Section 6.

5 Simulations

We have carried out a Monte Carlo study to check the performance of the tests described above. The simulations also give insight into the choice of the suitable test statistic. The significance level in all cases is fixed as $\alpha = 0.05$.

5.1 The choice of the discrepancy measure

The approach discussed in Section 3 generates a family of discrepancies for the addressed testing problems. We actually have a different test if we select the maximum or minimum in the discrepancy: $\Delta_{k:k}$ or $\Delta_{1:k}$ in the tests of Subsection 3.1 and $\Lambda_{k:k}$ or $\Lambda_{1:k}$ in the nonparametric case of Subsection 3.2. Moreover, the test statistics also differ for each $k \geq 2$. Hence, the question of finding a test with good power arises.

Regarding the tests on the proportion of structural zeros discussed in Subsection 3.1, observe that both hypotheses assume that the observations follow a parametric (ZIP) distribution. The tests mainly rely on the estimation of the parameters of the model, and the choice of the discrepancy is of secondary importance. Some preliminary simulations showed that different discrepancies and values of $k$ yield similar powers. Therefore, in this situation we opt for the simplest one $\Delta_{2:2} = \Delta_{1:2}$ defined in (8), which has computational advantages over the others with larger $k$’s.

We now turn to the test for overdispersion of Subsection 3.2. $H_0$ is given by a parametric model whereas $H_1$ includes all the distributions that strictly dominate an
element of the initial family. Hence, \( H_1 \) is not specified by any parametric family. In this case, the power of the tests strongly depends both on the distribution generating the data and on the parametric family assumed in \( H_0 \). For a fixed discrepancy, different alternatives could lead to very different powers. Therefore, it is advantageous to have a family of discrepancies since this provides flexibility to select a good test in each situation.

Let us briefly explain how the coefficient of variation (CV) of the discrepancy is useful to choose a test with good properties. Under \( H_1 \), an adequate discrepancy to detect deviations from \( H_0 \) should have a large mean and low variance, that is, a low CV. The CV of the discrepancy describes well how the corresponding test behaves. In general, under \( H_1 \), a low CV is paralleled by a high power. This is clearly reflected in Figure 1 where, for 1000 Monte Carlo samples, we plot the power of the test for overdispersion for the Poisson family and the inverse of the CV of the discrepancy \( \Lambda_{1:k} \) defined in (11), for different values of \( k \). In Figure 1(a), the observations are generated from a ZIP distribution \( Y(3, 0.05) \), while in Figure 1(b) they are drawn from the NB distribution \( Z(0.05, 3) \). In the first case, a value of \( k \) around 20 is a good choice, but in the second case \( k = 2 \) is clearly the best one. Therefore we use these two values of \( k \) in the simulations of Subsection 5.3.

We finally note that when analyzing only one data set, it also becomes possible to choose a suitable discrepancy by estimating its CV via bootstrap (see Section 6 for details).

### 5.2 Simulations for the test on the proportion of structural zeros

We consider the test on the proportion of structural zeros in a ZIP model (Subsection 3.1). As argued in Subsection 5.1 we select \( k = 2 \). For the case \( p_0 = 0 \) (\( H_0 \) represents the Poisson distribution), we compare the performance of the score test (van den Broek 1995) and the test methodology that rejects \( H_0 \) if the discrepancy \( \Delta_{1:2} = \Delta_{2:2} \) in (8) is too large. The rejection region for the latter method is chosen in two ways: via bootstrap as in (6) and also using the asymptotic distribution of \( \Delta_{2:2} \).
as in (10). The number of bootstrap samples is $B = 5000$.

In Table 2 we record the proportion of times that $H_0 : p = 0$ is rejected. For each combination of $p$ and $\theta$ in the table, we generate 5000 Monte Carlo samples of sizes $n = 50, 100$ and 200 from $Y(\theta, p)$. Note that our proposed procedure has a very competitive performance in comparison to the score test. This is specially apparent for the lowest values of $\theta$, where, when $p > 0$, in general our procedure yields a higher power than the score test.

In Table 3 the results for the test $H_0 : p \leq 0.2$ against $H_1 : p > 0.2$ are displayed. In this case we only use the procedure based on $\Delta_{2,2}$ with rejection region (6). The number of Monte Carlo samples is again 5000.

5.3 Simulations for the overdispersion test

We test $H_0 : Y \in P$ ($P$ being the Poisson family) against $H_1 : Y \in P_{\text{ex}}$ following the procedure described in Subsection 3.2. The number of Monte Carlo samples is 5000 and the number of bootstrap samples used to compute the rejection region is $B = 5000$. We generate observations with sample sizes $n = 50, 100$ and 200, from a ZIP distribution $Y(\theta, p)$ and apply the nonparametric procedure based on $\Lambda_{1:20}$. Afterwards, we generate samples from the NB distribution $Z(t, \theta)$ and carry out the test with $\Lambda_{1:2}$. Recall that the justification for selecting such discrepancies was detailed in Subsection 5.1. In Tables 4 and 5 we display the proportion of times that $H_0$ is rejected. Observe how close the powers in Table 4 are to those of Table 2. We found this property appealing since in this test for overdispersion no parametric model is specified for the alternative hypothesis.

6 An example with real data

To illustrate the usefulness of the methods proposed throughout the paper, we analyze a data set from Leroux and Puterman (1992). The number of movements by a fetal lamb observed through ultrasound were recorded. We consider one particular
sequence of counts of the number of movements in each of 240 consecutive 5-second intervals (see Table 6).

If we assume that the data follow a Poisson distribution with mean $\theta$, the estimate of $\theta$ is $\hat{\theta} = 0.36$. The differences between the observed and the expected frequencies in Table 6 point out that the Poisson model is unsuitable. Douglas (1994) used the Pearson $\chi^2$ statistics to argue that a ZIP model provides a substantially improved fit. The estimates of the parameters under the ZIP model are $\hat{\theta} = 0.36$ and $\hat{p} = 0.58$. The corresponding expected frequencies are in the fourth row of Table 6. The fit seems indeed better, but we could formalize this statement by testing $H_0 : p = 0$ versus $H_1 : p > 0$. We apply both the asymptotic test (10) and the score test. Both results point out a strong evidence (p-values below 0.0001) against the Poisson model. This leads us to the conclusion that the ZIP distribution fits the data much better than the Poisson one.

Rejecting the Poisson model does not necessarily imply that the ZIP model provides the best fit. Another model could account better for the observed dispersion. Therefore, using the nonparametric test developed in Subsection 3.2 we now test the null hypothesis that the distribution is ZIP against the alternative that the true model has more variability than the ZIP one. In this case, we have to select the appropriate statistics ($\Lambda_{1:k}$ or $\Lambda_{k:k}$) and a suitable value for $k$ (see Subsection 5.1). For that purpose, we obtain bootstrap estimates (based on 500 bootstrap samples) of the inverse of the CV of $\Lambda_{1:k}$ and $\Lambda_{k:k}$, for different $k$’s. The estimates as a function of $k$ are displayed in Figure 2.

According to the results depicted in Figure 2 the test based on $\Lambda_{k:k}$ is preferable. Moreover, for $\Lambda_{k:k}$, there is a wide range of $k$ values (between 50 and 200, say) for which the results are fairly similar. For the tests based on $\Lambda_{k:k}$ with $k = 50, 90, 130$ the p-values are under 0.0005. We conclude that the ZIP model is also clearly rejected so that other distributions with higher dispersion are more appropriate to fit this data set. Other authors have reached the same conclusion by rather different approaches. For instance, Ridout et al. (2001) reject the ZIP against the ZINB using a score
test in the spirit of van den Broek (1995). Thas and Rayner (2005) reject the ZIP against general smooth alternatives in the sense of Neyman. A generalized Poisson distribution to fit this data set has also been proposed by Gupta et al. (1996).

A simpler alternative to model this data is the NB distribution. The estimated parameters are $\hat{\theta} = 0.36$ and $\hat{\tau} = 1.89$, and the corresponding expected frequencies can be found in the fifth row of Table 6. At first sight it seems the fit provided by the NB is slightly better than the one furnished by the ZIP. To confirm this feature, we adapt the nonparametric procedure described in Subsection 3.2 to test the null hypothesis that the data come from a NB distribution against the alternative that the data come from a distribution that dominates the NB in the convex order.

We have used bootstrap estimates of the inverse of the CV to conclude that in this case $\Lambda_{k:k}$ with $k \approx 8$ yields an appropriate test (details are omitted). The p-values of the tests for $k = 4, 6, 8, 10, 12$ are all above 0.33. Therefore, we cannot reject the null hypothesis and conclude that the NB distribution accounts for the dispersion of the data better than the ZIP model.

7 Appendix: Proofs

Proof of Proposition 2

We need to introduce some notation. Given two integrable random variables $X$ and $Y$, it is said that $X$ is smaller than $Y$ in the increasing convex order, written $X \leq_{icx} Y$, if $E(\phi(X)) \leq E(\phi(Y))$, for all increasing and convex function $\phi$, provided the expectations exist. It is easy to see that

$$X \leq_{cx} Y \text{ if and only if } X \leq_{icx} Y \text{ and } EX = EY.$$  \hspace{1cm} (12)

Therefore, since all the variables considered in Proposition 2 have the same expectation $\theta$, if suffices to show that they are ordered for the increasing convex order. Moreover, since the proof of parts (a), (b) and (c) with $i = 1$ are similar, we only consider the case of ZIP variables (part (b) of the proposition).
We first note that the family \( \mathcal{P} := \{ Y(\theta) : \theta \in [0, \infty) \} \), where \( Y(\theta) \) is a Poisson random variable of mean \( \theta \geq 0 \) \( (Y(0) \equiv 0) \) is stochastically increasing and convex (see Example 8.A.2 in Shaked and Shanthikumar (2006)). For \( 0 \leq p_1 < p_2 < 1 \), we define the random variables (independent of the variables in \( \mathcal{P} \)) \( \Theta_i = \frac{\theta}{1-p_i} B(1-p_i) \) \( (i = 1, 2) \), where \( B(1-p_i) \) is a Bernoulli variable of parameter \( 1-p_i \). It is readily checked that \( \Theta_1 \leq_{cx} \Theta_2 \). Therefore, a direct application of Theorem 8.A.14 (p. 362) in Shaked and Shanthikumar (2006) yields \( P(\Theta_1) \leq_{icx} P(\Theta_2) \), and taking into account (12), we conclude \( P(\Theta_1) \leq_{cx} P(\Theta_2) \). Therefore, the proof of part (b) is finished since the ZIP variable \( Y(\theta, p_i) \) has the same distribution as \( P(\Theta_i) \) \( (i = 1, 2) \).

The previous argument, based on the properties of stochastically increasing and convex families, cannot be used to prove part (c) with \( i = 2 \) since it has not been established yet whether the collection of negative binomial variables is stochastically increasing and convex in its second parameter. We therefore need to introduce another technique inspired in the ideas used to prove Lemma 10 in de la Cal and Cárcamo (2005). Fix \( t > 0, \theta > 0 \) and \( 0 \leq p_1 < p_2 < 1 \) and let \( Z_2(t, \theta, p_i) \) \( (i = 1, 2) \) be the ZINB distributions defined in Section 4. Taking into account Lemma 9 in de la Cal and Cárcamo (2005) and Theorem 3.A.44 (p. 133) in Shaked and Shanthikumar (2006), to prove part (c) (with \( i = 2 \)) it is enough to show that the function

\[
p(k) := \Pr(Z_2(t, \theta, p_1) = k) - \Pr(Z_2(t, \theta, p_2) = k), \quad k \geq 0, \tag{13}
\]

has two changes of sign, being the sign sequence \(-, +, -\). To show this, we first consider the function

\[
\varphi(k) := \frac{\Pr(Z_2(t, \theta, p_1) = k)}{\Pr(Z_2(t, \theta, p_2) = k)}, \quad k \geq 0.
\]

After some simple computations, it is easy to check that the function \( f(p) := \Pr(Z_2(t, \theta, p) = 0) \) is an increasing function of \( p \in [0, 1) \). Therefore, \( \varphi(0) < 1 \). Also, since

\[
\frac{\varphi(k+1)}{\varphi(k)} = \frac{1 - p_2 + \theta t}{1 - p_1 + \theta t} =: c < 1, \quad k \geq 1,
\]

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we have that $\varphi(k) = c^{k-1}\varphi(1)$ ($k \geq 1$) and this entails $\varphi(k) \downarrow 0$ as $1 \leq k \uparrow \infty$. Moreover, the equality $\sum_{k=0}^{\infty} \Pr(Z_2(t, \theta, p_1) = k) = 1 = \sum_{k=0}^{\infty} \Pr(Z_2(t, \theta, p_2) = k)$ yields $\varphi(1) > 1$. This implies the desired result and the proof is complete.

Proof of Proposition 3

In the case $p = 0$, parts (a)-(d) follow from Lemmas 5 and 10 in de la Cal and Cárcamo (2005) and Theorem 3.A.44 (p. 133) in Shaked and Shanthikumar (2006). Therefore, using that the convex order is closed under mixtures (see Theorem 3.A.12 (p. 119) of Shaked and Shanthikumar (2006)), we conclude that for any fixed $0 < p < 1$, (a)-(c) and the first stochastic inequality in (d) are valid. To finish, we observe that the distribution of $Z_1(t, \theta, p)$ is the same as the distribution of $Z_2(t(1-p), \theta, p)$ and applying part (c) of Proposition 3 we get $Z_2(t(1-p), \theta, p) \leq_{cx} Z_2(t, \theta, p)$. This shows that $Z_1(t, \theta, p) \leq_{cx} Z_2(t, \theta, p)$ and the proof is complete.

Proof of Theorem 1

We first note that the discrepancy $\Delta_{2:2} = \Delta_{2:2}(\hat{\theta}, \hat{p})$ given in (8) is a smooth function of the maximum likelihood estimates, $\hat{\theta}$ and $\hat{p}$. Therefore, the desired asymptotic distribution can be obtained combining the classical asymptotic theory for maximum likelihood estimators and the delta method.

According to the the asymptotic theory for maximum likelihood estimators, we have that:

$$\sqrt{n}(\hat{\theta} - \theta, \hat{p} - p)^t \longrightarrow_d N((0, 0)^t, \Sigma), \quad n \to \infty,$$

where $N((0, 0)^t, \Sigma)$ is a bivariate normal distribution centered at the origin with covariance matrix $\Sigma$. The matrix $\Sigma$ is the inverse of the expected Fisher information matrix, that is, $\Sigma^{-1} = -E_{\theta,p}[\ell''(Y; \theta, p)]$, where $\ell''(y; \theta, p)$ is the $2 \times 2$ matrix of second partial derivatives with respect to $\theta$ and $p$ of the log-likelihood function $\ell(y; \theta, p)$. Using this result, after some algebra it is possible to show that, under $H_0 : p = 0$,

$$\sqrt{n}(\hat{\theta} - \theta, \hat{p})^t \longrightarrow_d N((0, 0)^t, \Sigma_0), \quad n \to \infty,$$
where
\[
\Sigma_0 = \begin{pmatrix} \theta & 0 \\ 0 & (e^\theta - 1 - \theta)^{-1} \end{pmatrix}.
\]

Now, let \( \nabla \Delta_{2:2}(\theta, p) \) be the gradient of \( \Delta_{2:2}(\theta, p) \) evaluated at \( (\theta, p) \). Using the delta method (see e.g. van der Vaart (1998), Theorem 3.1., p. 26) we deduce that, under \( H_0 : p = 0 \),
\[
\sqrt{n} \Delta_{2:2} \xrightarrow{d} N(0, \sigma^2(\theta)), \quad n \to \infty,
\]
where \( \sigma^2(\theta) := \nabla \Delta_{2:2}(\theta, 0)^t \cdot \Sigma_0 \cdot \nabla \Delta_{2:2}(\theta, 0) \). Now we observe that
\[
\frac{\partial \Delta_{2:2}(\theta, p)}{\partial \theta} \bigg|_{p=0} = 0, \quad \frac{\partial \Delta_{2:2}(\theta, p)}{\partial p} \bigg|_{p=0} = 2\theta - M_2(\theta) - \theta^2 e^{-\theta} [I_0(2\theta) - I_2(2\theta)],
\]
where the function \( M_2 \) is defined in (7). To obtain the last equality above we use the following properties of the modified Bessel functions of the first kind: \( I_0'(x) = I_1(x) \) and \( I_1'(x) = [I_0(x) + I_2(x)]/2 \) (see Abramowitz and Stegun (1965), properties 9.6.27 and 9.6.29, p. 376). Replacing these partial derivatives and the matrix \( \Sigma_0 \) in the expression \( \nabla \Delta_{2:2}(\theta, 0)^t \cdot \Sigma_0 \cdot \nabla \Delta_{2:2}(\theta, 0) \) yields
\[
\sigma^2(\theta) = \frac{(2\theta - M_2(\theta) - \theta^2 e^{-2\theta} [I_0(2\theta) - I_2(2\theta)])^2}{e^\theta - 1 - \theta} = \frac{\theta^2 (1 - e^{-2\theta} [(1 + \theta)I_0(2\theta) - I_1(2\theta) + \theta I_2(2\theta)])^2}{e^\theta - 1 - \theta}.
\]

Finally, it is obvious that \( \sigma(\hat{\theta}) \) defined in (9) is a consistent estimator of the standard deviation \( \sigma(\theta) \). As a consequence, from (14) we also deduce that the conclusion of Theorem \( \square \) holds.

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Table 2: Proportion of times that $H_0 : p = 0$ was rejected.

| $n$ | $\theta$ | $p$    |    |    |
|-----|----------|--------|----|----|
|     |          | 0      | 0.05 | 0.1 |
| 50  | 3        | 0.047  | 0.386 | 0.784 | Bootstrap |
|     |          | 0.036  | 0.313 | 0.722 | Score    |
| 50  | 5        | 0.041  | 0.768 | 0.972 | Bootstrap |
|     |          | 0.044  | 0.779 | 0.978 | Score    |
| 50  | 10       | 0.002  | 0.923 | 0.994 | Bootstrap |
|     |          | 0.002  | 0.923 | 0.994 | Score    |
| 100 | 3        | 0.052  | 0.585 | 0.964 | Bootstrap |
|     |          | 0.049  | 0.494 | 0.943 | Score    |
| 100 | 5        | 0.043  | 0.944 | 0.999 | Bootstrap |
|     |          | 0.045  | 0.945 | 0.999 | Score    |
| 100 | 10       | 0.003  | 0.994 | 1.000 | Bootstrap |
|     |          | 0.003  | 0.994 | 1.000 | Score    |
| 200 | 3        | 0.051  | 0.827 | 0.999 | Bootstrap |
|     |          | 0.048  | 0.762 | 0.999 | Score    |
| 200 | 5        | 0.050  | 0.999 | 1.000 | Bootstrap |
|     |          | 0.043  | 0.999 | 1.000 | Score    |
| 200 | 10       | 0.007  | 1.000 | 1.000 | Bootstrap |
|     |          | 0.007  | 1.000 | 1.000 | Score    |
Table 3: Proportion of times that $H_0 : p \leq 0.2$ was rejected.

| $n$ | $\theta$ | 0.2 | 0.25 | 0.3 |
|-----|----------|-----|------|-----|
| 50  | 3        | 0.069 | 0.272 | 0.581 |
| 50  | 5        | 0.067 | 0.253 | 0.554 |
| 50  | 10       | 0.061 | 0.244 | 0.546 |
| 100 | 3        | 0.062 | 0.366 | 0.781 |
| 100 | 5        | 0.065 | 0.346 | 0.780 |
| 100 | 10       | 0.061 | 0.370 | 0.774 |
| 200 | 3        | 0.061 | 0.536 | 0.953 |
| 200 | 5        | 0.065 | 0.557 | 0.962 |
| 200 | 10       | 0.069 | 0.584 | 0.963 |

Table 4: Proportion of rejections of $H_0 : Y \in \mathcal{P}$ when sampling from a ZIP $Y(\theta, p)$.

| $n$ | $\theta$ | 0 | 0.05 | 0.1 |
|-----|----------|---|------|-----|
| 50  | 3        | 0.043 | 0.358 | 0.780 |
| 50  | 5        | 0.045 | 0.732 | 0.966 |
| 50  | 10       | 0.051 | 0.901 | 0.993 |
| 100 | 3        | 0.047 | 0.576 | 0.952 |
| 100 | 5        | 0.052 | 0.911 | 0.999 |
| 100 | 10       | 0.053 | 0.982 | 1.000 |
| 200 | 3        | 0.054 | 0.839 | 0.999 |
| 200 | 5        | 0.054 | 0.992 | 1.000 |
| 200 | 10       | 0.050 | 0.999 | 1.000 |
Table 5: Proportion of rejections of $H_0 : Y \in \mathcal{P}$ when sampling from a NB $Z(t, \theta)$.

\begin{center}
\begin{tabular}{cccc}
\hline
$t$ & $n$ & $\theta$ & 0.05 & 0.1 \\
\hline
 & 50 & 3 & 0.183 & 0.385 \\
 & 50 & 5 & 0.309 & 0.635 \\
 & 100 & 3 & 0.269 & 0.583 \\
 & 100 & 5 & 0.479 & 0.871 \\
 & 200 & 3 & 0.409 & 0.812 \\
 & 200 & 5 & 0.710 & 0.989 \\
\hline
\end{tabular}
\end{center}

Table 6: Lamb data set and expected frequencies based on Poisson, ZIP and NB distributions.

\begin{center}
\begin{tabular}{cccccccc}
\hline
Outcome & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
Obs. Freq. & 182 & 41 & 12 & 2 & 2 & 0 & 0 & 1 \\
Expect. Freq. (Poisson) & 167.7 & 60.1 & 10.8 & 1.3 & 0.1 & 0.0 & 0.0 & 0.0 \\
Expect. Freq. (ZIP) & 182.0 & 36.9 & 15.6 & 4.4 & 0.9 & 0.2 & 0.0 & 0.0 \\
Expect. Freq. (NB) & 182.5 & 39.0 & 12.0 & 4.1 & 1.5 & 0.5 & 0.2 & 0.1 \\
\hline
\end{tabular}
\end{center}
Figure 1: Power (in black) of the overdispersion test for the Poisson family and 1/CV of the discrepancy (in grey).

(a) n observations from ZIP(3,0.05)

(b) n observations from NB(0.05,3)
Figure 2: Bootstrap estimates of $CV^{-1}$ for $\Lambda_{k:k}$ (solid line) and $\Lambda_{1:k}$ (dashed line) for several values of $k$. 

\[ \text{Figure 2: Bootstrap estimates of } CV^{-1} \text{ for } \Lambda_{k:k} \text{ (solid line) and } \Lambda_{1:k} \text{ (dashed line) for several values of } k. \]