Spivey’s Bell Number Formula Revisited

Mahid M. Mangontarum
Department of Mathematics
Mindanao State University—Main Campus
Marawi City 9700
Philippines
mmangontarum@yahoo.com
mangontarum.mahid@msumain.edu.ph

Abstract

This paper introduces an alternative form of the derivation of Spivey’s Bell number formula, which involves the $q$-Boson operators $a$ and $a^\dagger$. Furthermore, a similar formula for the case of the $(q, r)$-Dowling polynomials is obtained, and is shown to produce a generalization of the latter.

1 Introduction

Consider the Stirling numbers of the second kind, denoted by $\{m\}$, which appear as coefficients in the expansion of

$$t^n = \sum_{k=0}^{n} \{n\} \binom{n}{k} (t)_k,$$

where $(t)_k = t(t-1)(t-2)\cdots(t-k+1)$. The Bell numbers, denoted by $B_n$, are defined by

$$B_n = \sum_{j=0}^{n} \{n\} \binom{n}{j}$$

and are known to satisfy the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

In 2008, Spivey [13] obtained a remarkable formula which unifies the defining relation in (2) and the identity (3). The said formula is given by

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \binom{m}{j} \binom{n}{k} B_k$$

and is popularly known as “Spivey’s Bell number formula”. Equation (4) was proved in [13] using a combinatorial approach involving partition of sets. Different proofs and extensions
of (4) were later on studied by several authors. For instance, a proof which made use of generating functions was done by Gould and Quaintance [5] which was then generalized by Xu [14] using Hsu and Shuie’s [6] generalized Stirling numbers. Belbachir and Mihoubi [2] presented a proof that involves decomposition of the Bell polynomials into a certain polynomial basis. Mező [12] obtained a generalization of the Spivey’s formula in terms of the $r$-Bell polynomials via combinatorial approach. The notion of dual of (4) was also presented in the same paper. On the other hand, the work of Katriel [7] involved the use of the operator $X$ satisfying
\[ DX - qXD = 1, \]
where $D$ is the $q$-derivative defined by
\[ Df(x) = \frac{f(qx) - f(x)}{x(q - 1)}. \]

For the sake of clarity and brevity, this method will be referred to as “Katriel’s proof”.

Now, aside from being implicitly implied in Katriel’s proof, none of the previously-mentioned studies considered establishing $q$-analogues. It is, henceforth, the main purpose of this paper to obtain a generalized $q$-analogue of Spivey’s Bell number formula.

## 2 Alternative form of “Katriel’s proof”

We direct our attention to the $q$-Boson operators $a$ and $a^\dagger$ satisfying the commutation relation
\[ [a, a^\dagger]_q = aa^\dagger - qa^\dagger a = 1 \]  
(see [1]). We define the Fock space (or Fock states) by the basis $\{|s\rangle \}$ so that the relations $a |s\rangle = \sqrt{[s]_q} |s - 1\rangle$ and $a^\dagger |s\rangle = \sqrt{[s + 1]_q} |s + 1\rangle$ form a representation that satisfies [7]. The operators $a^\dagger a$ and $(a^\dagger)^k a^k$, when acting on $|s\rangle$, yield
\[ a^\dagger a |s\rangle = [s]_q |s\rangle \]
and
\[ (a^\dagger)^k a^k |s\rangle = [s]_{q,k} |s\rangle, \]
respectively, where $[s]_q = \frac{q^s - 1}{q - 1}$ and $[s]_{q,k} = [s]_q[s - 1]_q[s - 2]_q \cdots [s - k + 1]_q$. Hence, the $q$-Stirling numbers of the second kind $\left\{ \begin{array}{c} n \\ k \end{array} \right\}_q$ can be defined alternatively as
\[ (a^\dagger a)^n = \sum_{k=1}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_q (a^\dagger)^k a^k. \]
(10)

From (7), it is clear that
\[ [a, (a^\dagger)^k]_{q,k} = [a, (a^\dagger)^{k-1}]_{q^{k-1}} a^\dagger + q^{k-1}(a^\dagger)^{k-1} [a, a^\dagger]_q, \]
and by induction on $k$, we have
\[ [a, (a^\dagger)^k]_{q,k} = [k]_q (a^\dagger)^{k-1}. \]
(12)
Since \( a\ket{0} = 0 \), then by (12),
\[
a(a^\dagger)^\ell \ket{0} = [a, (a^\dagger)^\ell]_{q^\ell} \ket{0} = [\ell]_{q^\ell} (a^\dagger)^{\ell-1} \ket{0}.
\]
Moreover,
\[
a^k(a^\dagger)^\ell \ket{0} = \frac{[\ell]_{q^\ell}^k}{[\ell-k]_{q^\ell}} (a^\dagger)^{\ell-k} \ket{0},
\]
for \( k \leq \ell \) and
\[
a^k(a^\dagger)^\ell \ket{0} = 0,
\]
for \( k > \ell \). Finally,
\[
a^k e_q(xa^\dagger) \ket{0} = x^k e_q(xa^\dagger) \ket{0},
\]
where \( e_q(xa^\dagger) \) is the \( q \)-exponential function defined by
\[
e_q(t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{[\ell]_{q^\ell}^\ell}.
\]
Applying (15) to (10) yields
\[
(a^\dagger a)^n e_q(t a^\dagger) \ket{0} = B_{n,q}(t a^\dagger) e_q(t a^\dagger) \ket{0},
\]
where \( B_{n,q}(t a^\dagger) \) denotes the \( q \)-Bell polynomials defined by
\[
B_{n,q}(t) = \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\}_q t^k.
\]
Let \( x = t a^\dagger \) so that
\[
(a^\dagger a)^n e_q(x) \ket{0} = B_{n,q}(x) e_q(x) \ket{0}.
\]
Before proceeding, note that by definition,
\[
[a, (a^\dagger)^k]_{q^k} = a(a^\dagger)^k - q^k (a^\dagger)^k a.
\]
By (12),
\[
a(a^\dagger)^k - q^k (a^\dagger)^k a = [k]_{q^k} (a^\dagger)^{k-1} \]
\[
a(a^\dagger)^k = q^k (a^\dagger)^k a + [k]_{q^k} (a^\dagger)^{k-1}.
\]
This can be further expressed as
\[
(a^\dagger a)(a^\dagger)^k = (a^\dagger)^k ([k]_{q^k} + q^k (a^\dagger a)).
\]
Now, we have
\[
(a^\dagger a)^{n+m} = (a^\dagger a)^n \sum_{j=0}^{m} \left\{ \frac{m}{j} \right\}_q (a^\dagger)^j a^j
\]
\[
= \sum_{j=0}^{m} \left\{ \frac{m}{j} \right\}_q (a^\dagger)^j ([j]_{q^j} + q^j (a^\dagger a))^n a^j
\]
\[
= \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ \frac{m}{j} \right\}_q \left( \frac{n}{k} \right) [j]_{q^j}^{n-k} q^j (a^\dagger)^j (a^\dagger a)^k a^j.
\]
Multiplying both sides with $e_q(x) |0\rangle$ makes the left-hand side
\[(a^\dagger a)^{n+m}e_q(x) |0\rangle = B_{n+m,q}(x)e_q(x) |0\rangle, \tag{22}\]
while the right-hand side becomes
\[
\sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} q \binom{n}{k} [j]_q^{n-k} q^j (a^\dagger a)^j e_q(x) |0\rangle a^j = \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} q \binom{n}{k} [j]_q^{n-k} q^j B_{k,q}(x)e_q(x) |0\rangle (a^\dagger)^j a^j.
\]

Dividing both sides by $e_q(x) |0\rangle$ and using (22) gives
\[
B_{n+m,q}(x) = \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} q \binom{n}{k} [j]_q^{n-k} q^j B_{k,q}(x)[x]_{q,j}, \tag{23}\]

As $q \to 1$, we obtain a polynomial version of Spivey’s Bell number formula which, in return, reduces to (4) when we set $x = 1$.

It is important to emphasize that this is not a new proof, but an alternative form of Katriel’s proof, since the operators $a$, $a^\dagger$ and the operators $X$, $D$ generate isomorphic algebras.

### 3 A generalization of Spivey’s Bell number formula

The main result of this paper is the following identity:
\[
D_{m,r,q}(n + \ell, x) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} m^j W_{m,r,q}(\ell, j) \binom{n}{k} (m[k]_q + r)^{n-k} q^j D_{m,0,q}(k, x)[x]_{q,j}. \tag{24}\]

Here, $D_{m,r,q}(n, x)$ is a $(q, r)$-Dowling polynomial defined previously by the author and Katriel [9] as
\[
D_{m,r,q}(n, x) = \sum_{k=0}^{n} W_{m,r,q}(n, k)x^k, \tag{25}\]
where $W_{m,r,q}(n, k)$ is the $(q, r)$-Whitney numbers of the second kind. Several properties of $D_{m,r,q}(n, x)$ can be seen in [3, 9].

To derive (24), we first multiply both sides of (21) by $m$ and then add $r(a^\dagger)^k$ to yield
\[
(ma^\dagger a + r)(a^\dagger)^k = (a^\dagger)^k (m[k]_q + r + mq^k a^\dagger a). \tag{26}\]

Also, multiplying both sides of the defining relation in [9, Equation 16] by $e_q(ta^\dagger) |0\rangle$ and applying (15) yields
\[
(ma^\dagger a + r)^n e_q(ta^\dagger) |0\rangle = \sum_{k=0}^{n} m^k W_{m,r,q}(n, k)(a^\dagger)^k a^k e_q(ta^\dagger) |0\rangle
= \sum_{k=0}^{n} m^k W_{m,r,q}(n, k)(a^\dagger)^k t^k e_q(ta^\dagger) |0\rangle
= D_{m,r,q}(n, mta^\dagger)e_q(ta^\dagger) |0\rangle.
\]
Now, by (26),

\[(ma^+a + r)^{n+\ell} = \sum_{j=0}^{\ell} m^j W_{m,r,q}(\ell, j)(ma^+a + r)^n(a^+)^ja^j\]

\[= \sum_{j=0}^{\ell} m^j W_{m,r,q}(\ell, j)(a^+)^ja^j(m[j]_q + r + mq^j a^+a)^n a^j\]

\[= \sum_{j=0}^{\ell} \sum_{k=0}^{n} m^{j+k} W_{m,r,q}(\ell, j) \binom{n}{k} (a^+)^j(m[j]_q + r)^{n-k} q^k (a^+a)^k a^j.\]

Applying this expression to the operator identity \(e_q(ta^+) |0\rangle\), combining with the previous equation, using (9), (19) and \(W_{m,0,q}(k, i) = m^k i^k q\) (see [9, Equation 18]), and then dividing both sides of the resulting identity by \(e_q(ta^+) |0\rangle\) completes the derivation.

4 Remarks

Since \(W_{1,0,q}(\ell, j) = \binom{n}{j} q^j\), then by setting \(x = 1, m = 1\) and \(r = 0\), we have

\[D_{1,0,q}(n + \ell, 1) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} \binom{\ell}{j} \binom{n}{k} q^j B_{k,q}, \tag{27}\]

where \(B_{k,q} := B_{k,q}(1)\). This is a \(q\)-analogue of (1) which was first obtained by Katriel [7]. On the other hand, setting \(x = 1\) and then taking the limit of (24) as \(q \to 1\) provides a generalization of Spivey’s Bell number formula in terms of the \(r\)-Whitney numbers of the second kind, denoted by \(W_{m,r}(\ell, j)\), and the \(r\)-Dowling numbers, denoted by \(D_{m,r}(n)\), (see [4, 11]), given by

\[D_{m,r}(n + \ell) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} m^j W_{m,r}(\ell, j) \binom{n}{k} (m[j]_q + r)^{n-k} D_{m,0}(k). \tag{28}\]

In a recent paper, Mansour et al. [10] obtained the following generalization of Spivey’s Bell number formula:

\[D_{p,q}(a + b; x) = \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{\ell=0}^{j} (mq^j)^{j-\ell} x^{i+\ell} \binom{b}{j} \left([r]_p + m[j]_q\right)^{b-j} W_{p,q}(a, i) S_q(j, \ell). \tag{29}\]

Here, \(D_{p,q}(n; x)\) and \(W_{p,q}(n, k)\) denote the \((p, q)\)-analogues of the \(r\)-Dowling polynomials and the \(r\)-Whitney numbers of the second kind, respectively. The \((p, q)\)-analogues are natural generalizations of \(q\)-analogues. However, since the manner by which the numbers \(W_{m,r,q}(n, k)\) were defined in [9] differs from the work of Mansour et al. [10], the main result of this paper is not generalized by (29).
5 Acknowledgment

The author is very thankful to the editor-in-chief and to the referee(s) for carefully reading the paper. Their comments and suggestions were very helpful. Special thanks also to Dr. Jacob Katriel for his insights on $q$-Boson operators. This paper is dedicated to my family and all other victims of the war in Marawi.

References

[1] M. Arik and D. Coon, Hilbert spaces of analytic functions and generalized coherent states, *J. Math. Phys.* **17** (1976), 524–527.

[2] H. Belbachir and M. Mihoubi, A generalized recurrence for Bell polynomials: An alternate approach to Spivey and Gould–Quaintance formulas, *European J. Combin.* **30** (2009), 1254–1256.

[3] L. Carlitz, $q$-Bernoulli numbers and polynomials, *Duke Math. J.* **15** (1948), 987–1000.

[4] G.-S. Cheon and J.-H. Jung, The $r$-Whitney numbers of Dowling lattices, *Discrete Math.* **312** (2012), 2337–2348.

[5] H. W. Gould and J. Quaintance, Implications of Spivey’s Bell number formula, *J. Integer Sequences* **11** (2008). [Article 08.3.7]

[6] L. Hsu and P. J. Shiue, A unified approach to generalized Stirling numbers, *Advances Appl. Math.* **20** (1998), 366–384.

[7] J. Katriel, On a generalized recurrence for Bell numbers, *J. Integer Sequences* **11** (2008). [Article 08.3.8]

[8] M. M. Mangontarum, Some theorems and applications of the $(q, r)$-Whitney numbers, *J. Integer Sequences* **20** (2017). [Article 17.2.5]

[9] M. M. Mangontarum and J. Katriel, On $q$-boson operators and $q$-analogues of the $r$-Whitney and $r$-Dowling numbers, *J. Integer Sequences* **18** (2015). [Article 15.9.8]

[10] T. Mansour, J. L. Ramirez and M. Shattuck, A generalization of the $r$-Whitney numbers of the second kind, *J. Comb.* **8** (2017), 29–55.

[11] I. Mező, A new formula for the Bernoulli polynomials, *Results Math.* **58** (2010), 329–335.

[12] I. Mező, The dual of Spivey’s Bell number formula, *J. Integer Sequences* **15** (2012). [Article 12.2.4]

[13] M. Z. Spivey, A generalized recurrence for Bell numbers, *J. Integer Sequences* **11** (2008). [Article 08.2.5]

[14] A. Xu, Extensions of Spivey’s Bell number formula, *Electron. J. Combin.* **19** (2012), #P6.