SOME CHARACTERIZATIONS OF THREE-DIMENSIONAL $f$-KENMOTSU RICCI SOLITONS

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Abstract. The aim of the present paper is to give some characterizations of $f$-Kenmotsu Ricci soliton with a supporting example.

Keywords: $f$-Kenmotsu manifold; Ricci almost soliton; gradient Ricci soliton.

1. Introduction

The revolutionary concept of Ricci flow was introduced by Hamilton [5] in order to solve Poincare conjecture. The conjecture was fully solved by Perelman [11] using Hamilton’s Ricci flow technique. After the work of Perelman, the study of Ricci flow has become an important topic in differential geometry. A Ricci flow is a weak parabolic heat type partial differential equation of the following form

\[
\frac{\partial g_{ij}}{\partial t} = -2S_{ij},
\]

\[
g(0) = g_0.
\]

Here $g_{ij}$ denotes the components of Riemannian metric $g$ and $S_{ij}$ denotes the components of Ricci tensor $S$. A Ricci soliton is a solution of the above equation which is constant up to diffeomorphism and scaling. A Ricci soliton on a Riemannian manifold is characterized by the equation

\[
(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0.
\]

Here $\lambda$ is a constant, called soliton constant and the vector field $V$ is called soliton vector field. A Ricci soliton is called expanding, shrinking or steady while $\lambda$ is positive, negative or zero. A Ricci soliton is called Ricci almost soliton if $\lambda$ is...
A. Sarkar and P. Bhakta considered as a function instead of a constant [12]. A Ricci soliton is called gradient Ricci soliton if the soliton vector field is gradient of a potential function [13]. The study of Ricci solitons on almost contact manifolds was first initiated by Ramesh Sharma [16]. The Ricci solitons on almost contact manifolds have been studied by several authors ([4], [13], [15]). Ricci soliton on \((\kappa, \mu)\) contact metric manifold has been studied by the present authors in [14].

The notion of Kenmotsu manifold was introduced by K. Kenmotsu and was subsequently generalized to \(f\)-Kenmotsu manifolds. For details we refer to [8] and [9]. Ricci solitons on Kenmotsu manifold have been studied in [6]. The notion of \(\phi\)-Ricci symmetric manifolds was introduced by U. C. De and A. Sarkar [2]. The notion of \(\phi\)-symmetric manifolds was introduced by T. Takahashi [17]. Later several authors studied \(\phi\)-symmetric manifolds. Three dimensional quasi-Sasakian manifolds with cyclic parallel and \(\eta\)-parallel Ricci tensor have been studied by U. C. De and A. Sarkar [3].

The objective of the present paper is to give some characterizations of \(f\)-Kenmotsu manifolds with Ricci solitons and hence establish the relations between such manifolds with locally \(\phi\)-symmetric manifolds and manifolds with cyclic parallel and \(\eta\)-parallel Ricci tensors.

The present paper is organised as follows: After the introduction, we give required preliminaries in Section 2. In Section 3, we will study three dimensional \(f\)-Kenmotsu manifolds admitting Ricci soliton. Section 4 contains a supporting example.

2. Preliminaries

An odd dimensional smooth manifold \(M\) is said to be an almost contact metric manifold, if there exists a \((1,1)\) tensor field \(\phi\), a vector field \(\xi\), a 1-form \(\eta\), and a Riemannian metric \(g\) on \(M\) such that [1]

\[
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta(\phi(X)) = 0.
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\]

for any vector fields \(X, Y \in \chi(M)\). Such a manifold of dimension \((2n+1)\) is denoted by \(M^{2n+1}_n(\phi, \xi, \eta, g)\). Also \(M^{2n+1}(\phi, \xi, \eta, g)\) is called an \(f\)-Kenmotsu manifold if the covariant differentiation of \(\phi\) satisfies

\[
(\nabla_X \phi) Y = f(g(\phi X, Y)\xi - \eta(Y)\phi X),
\]

where \(f \in C^\infty(M)\) is such that \(df \wedge \eta = 0\) ([8], [9]). If \(f = \beta\) is nonzero constant, then the manifold is a \(\beta\)-Kenmotsu manifold [7]. If \(f = 0\), then the manifold is cosymplectic [7]. An \(f\)-Kenmotsu manifold is said to be regular if \(f^2 + f' \neq 0\), where \(f' = \xi f\). For an \(f\)-Kenmotsu manifold, it follows from (2.3)

\[
\nabla_X \xi = f(X - \eta(X)\xi).
\]
The condition $df \wedge \eta = 0$ holds only for $\dim M \geq 5$ [10]. In a three dimensional $f$-Kenmotsu manifold, we have

\begin{align*}
R(X,Y)Z &= \left( \frac{r}{2} + 2f^2 + 2f' \right)(X \wedge Y)Z \\
&\quad - \left( \frac{r}{2} + 3f^2 + 3f' \right)\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z \}, \tag{2.5}
\end{align*}

\begin{align*}
S(X,Y) &= \left( \frac{r}{2} + f^2 + f' \right)g(X,Y) - \left( \frac{r}{2} + 3f^2 + 3f' \right)\eta(X)\eta(Y), \tag{2.6}
\end{align*}

\begin{align*}
QX &= \left( \frac{r}{2} + f^2 + f' \right)X - \left( \frac{r}{2} + 3f^2 + 3f' \right)\eta(X)\xi, \tag{2.7}
\end{align*}

where $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$, also $R$, $S$ and $r$ are Riemannian curvature tensor, Ricci curvature tensor and scalar curvature on $M$ respectively [9]. From (2.5) and (2.6) we get

\begin{align*}
R(X,Y)\xi &= -(f^2 + f')(\eta(Y)X - \eta(X)Y), \tag{2.8}
\end{align*}

\begin{align*}
S(X,\xi) &= -2(f^2 + f')\eta(X), \tag{2.9}
\end{align*}

\begin{align*}
S(\xi,\xi) &= -2(f^2 + f'), \tag{2.10}
\end{align*}

\begin{align*}
Q\xi &= -2(f^2 + f')\xi. \tag{2.11}
\end{align*}

As a consequence of (2.4), we also have

\begin{align*}
(\nabla_X \eta)(Y) &= fg(\phi X, \phi Y). \tag{2.12}
\end{align*}

Also from (2.9) it follows that

\begin{align*}
S(\phi X, \phi Y) &= S(X,Y) + 2(f^2 + f')\eta(X)\eta(Y), \tag{2.13}
\end{align*}

for all vector fields $X, Y \in \chi(M)$.

An $f$-Kenmotsu manifold $M^{(2n+1)} (\phi, \xi, \eta, g)$ is said to be $\phi$-symmetric if its curvature tensor $R$ bears the condition

\begin{align*}
\phi^2(\nabla_X R)(Y, Z)W = 0, \tag{2.14}
\end{align*}

for all vector fields $X, Y, Z, W \in \chi(M)$ [17]. In particular, if $X, Y, Z, W$ are orthogonal to $\xi$, then $M^{(2n+1)} (\phi, \xi, \eta, g)$ is said to be locally $\phi$-symmetric. An $f$-Kenmotsu manifold $M^{(2n+1)} (\phi, \xi, \eta, g)$ is said to be $\phi$-Ricci symmetric if its Ricci operator $Q$ bears the condition

\begin{align*}
\phi^2(\nabla_X Q)Y = 0 \tag{2.15}
\end{align*}

for all vector fields $X, Y \in \chi(M)$. If $X$ and $Y$ are orthogonal to $\xi$, then $M^{(2n+1)} (\phi, \xi, \eta, g)$ is said to be locally $\phi$-Ricci symmetric. It may be noted that $\phi$-symmetric implies $\phi$-Ricci symmetric, but the converse is not valid in general.

Ricci tensor $S$ of a Riemannian manifold $(M, g)$ is called $\eta$-parallel if

\begin{align*}
g((\nabla_X S)Y, Z) = 0
\end{align*}
for all vector fields $X, Y, Z$ tangent to $M$ and orthogonal to $\xi$ where $g$ and $\nabla$ denote Riemannian metric and Riemannian connection respectively.

Ricci tensor $S$ of a Riemannian manifold $(M, g)$ is called cyclic-parallel if
\begin{equation}
(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0
\end{equation}
for all vector fields $X, Y, Z$ tangent to $M$. Here $\nabla$ denotes Riemannian connection.

3. Three-dimensional $f$-Kenmotsu manifolds with Ricci soliton

In this section we prove the following:

**Theorem 3.1.** In a three-dimensional $f$ Kenmotsu Ricci soliton, if $f$ is constant and the soliton vector field is Killing, then the soliton is expanding.

**Proof.** For a three-dimensional $f$-Kenmotsu manifold, from (2.7), we get
\begin{equation}
QX = \left(\frac{r}{2} + f^2 + f'\right)X - \left(\frac{r}{2} + 3f^2 + 3f'\right)\eta(X)\xi.
\end{equation}
Differentiating covariantly along $Y$ and using (2.4) and (2.12) we obtain
\begin{align}
(\nabla_Y Q)X &= \left(\frac{dr(Y)}{2} + 2f df(Y) + df'(Y)\right)X + \left(\frac{r}{2} + f^2 + f'\right)\nabla_Y X \\
&- \left(\frac{dr(Y)}{2} + 6f df(Y) + 3df'(Y)\right)\eta(X)\xi \\
&- \left(\frac{r}{2} + 3f^2 + 3f'\right)f g(\phi X, \phi Y)\xi - \left(\frac{r}{2} + 3f^2 + 3f'\right) \eta(X) f(Y - \eta(Y)\xi).
\end{align}
(3.2)
Taking inner product of (3.2) with $Y$ we have
\begin{align}
g((\nabla_Y Q)X, Y) &= \left(\frac{dr(Y)}{2} + 2f df(Y) + df'(Y)\right)g(X, Y) \\
&+ \left(\frac{r}{2} + f^2 + f'\right)g(\nabla_Y X, Y) \\
&- \left(\frac{dr(Y)}{2} + 6f df(Y) + 3df'(Y)\right)\eta(X)\eta(Y) \\
&- \left(\frac{r}{2} + 3f^2 + 3f'\right)f g(\phi X, \phi Y)\eta(Y) \\
&- \left(\frac{r}{2} + 3f^2 + 3f'\right)\eta(X)g(Y, Y) f \\
&+ \left(\frac{r}{2} + 3f^2 + 3f'\right)\eta(X)(\eta(Y))^2 f.
\end{align}
(3.3)
Let $\{e_1, e_2, \xi\}$ be an orthonormal $\phi$-basis at any point of a tangent space. It is known that
\begin{equation}
\text{div}(Q)X = g((\nabla_{e_1} Q)X, e_1) + g((\nabla_{e_2} Q)X, e_2) + g((\nabla_{e_3} Q)X, e_3).
\end{equation}
(3.4)
Using (3.3) in (3.4) we get

\[
\text{div}(Q)X = \left( \frac{dr(e_1)}{2} + 2f df(e_1) + df'(e_1) \right)g(X, e_1) + \left( \frac{r}{2} + f^2 + f' \right)g(\nabla_{e_1}X, e_1) - \left( \frac{dr(e_2)}{2} + 6f df(e_2) + 3df'(e_2) \right)g(X, e_2) + \left( \frac{r}{2} + 3f^2 + 3f' \right)g(\nabla_{e_2}X, e_2) + \left( \frac{dr(\xi)}{2} + 2f df(\xi) + df'(\xi) \right)g(X, \xi) + \left( \frac{r}{2} + f^2 + f' \right)g(\nabla_{\xi}X, \xi) - \left( \frac{dr(\xi)}{2} + 2f df(\xi) + df'(\xi) \right)g(X, \xi). 
\]

(3.5)

We know that \( \text{div}(Q)X = \frac{1}{2}dr(X). \) Putting \( X = \xi \) in (3.5) we obtain

\[
\frac{1}{2}dr\xi = 2\left( \frac{r}{2} + f^2 + f' \right)f - 4f df(\xi) - 2df'(\xi).
\]

(3.6)

If \( f \)-Kenmotsu manifold admits Ricci soliton then

\[
S(X, Y) = -\frac{1}{2}((\mathcal{L}_V g)(X, Y) - \lambda g(X, Y)).
\]

(3.7)

If \( V \) is a Killing vector field, from (3.7) we get \( r = -3\lambda = \text{constant}. \) Therefore, from (3.6)

\[
\left( \frac{r}{2} + f^2 + f' \right)f = 2f df(\xi) - df'(\xi).
\]

(3.8)

If \( f \) is a non-zero constant then

\[
r = -2f^2.
\]

(3.9)

Consequently, \( \lambda = \frac{2}{3}f^2. \) This completes the proof. \( \square \)

We know from [6] that a three-dimensional non cosymplectic \( f \)-Kenmotsu manifold \( M^3(\phi, \xi, \eta, g) \) with \( f \) being constant, is locally \( \phi \)-Ricci symmetric if and only if the scalar curvature is constant. So we get the following corollary

**Corollary 3.1.** If a three-dimensional \( f \)-Kenmotsu manifold with constant \( f \) admits a Ricci soliton with Killing soliton vector field, then it is \( \phi \)-Ricci symmetric, and hence \( \phi \)-symmetric.

Again we know from [6] that in a three-dimensional non cosymplectic \( f \)-Kenmotsu manifold \( M^3(\phi, \xi, \eta, g) \) with \( f \) being constant, the Ricci tensor is \( \eta \)-parallel if and only if the scalar curvature is constant. Hence we get
**Corollary 3.2.** If a three-dimensional $f$-Kenmotsu manifold with constant $f$ admits Ricci soliton with Killing soliton vector field, then its Ricci tensor is $\eta$-parallel.

From [6] we know that a three-dimensional non cosymplectic $f$-Kenmotsu manifold $M^3(\phi, \xi, \eta, g)$ with $f$ being constant, satisfies cyclic parallel Ricci tensor if and only if the scalar curvature is constant. So, we can state the following:

**Corollary 3.3.** If a three-dimensional $f$-Kenmotsu manifold with constant $f$ admits Ricci soliton with Killing soliton vector field, then its Ricci tensor is cyclic parallel.

## 4. Example

**Example 4.1.** Let $M = \{(u, v, w) \in R^3 : u, v, w(\neq 0) \in R\}$ be a Riemannian manifold, where $(u, v, w)$ denotes the standard coordinates of a point in $R^3$. Let us suppose that

\begin{equation}
(\text{4.1})
\begin{align*}
e_1 &= 3w \frac{\partial}{\partial u}, \\
e_2 &= 3w \frac{\partial}{\partial v}, \\
e_3 &= -3w \frac{\partial}{\partial w}
\end{align*}
\end{equation}

are three linearly independent vector fields at each point of $M$ and therefore it forms a basis for the tangent space $\chi(M)$. We also define the Riemannian metric $g$ of the manifold $M$ given by

\begin{equation}
(\text{4.2})
g = \frac{1}{w^2}[du \odot du + dv \odot dv + dw \odot dw].
\end{equation}

Let $\eta$ be the one form satisfying

\begin{equation}
(\text{4.3})
\eta(U) = g(U, e_3)
\end{equation}

for any $U \in \chi(M)$ and let $\phi$ be the $(1, 1)$ tensor field defined by $\phi e_1 = -e_2$, $\phi e_2 = e_1$, $\phi e_3 = 0$. By the linear properties of $\phi$ and $g$, we can easily verify the following relations

\begin{equation}
(\text{4.4})
\eta(e_3) = 1, \quad \phi^2(U) = -U + \eta(U)e_3
\end{equation}

\begin{equation}
(\text{4.5})
g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)
\end{equation}

for arbitrary vector fields $U, V \in \chi(M)$. This shows that $\xi = e_3$ the structure $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$. If $\nabla$ is the Livi-Civita connection with respect to the Riemannian metric $g$, then with the help of above, we can easily calculate that

\begin{equation}
(\text{4.6})
[e_1, e_2] = 0, \quad [e_1, e_3] = 3e_1, \quad [e_2, e_3] = 3e_2.
\end{equation}

Now we recall Koszul’s formula as

\[2g(\nabla_U V, W) = U(g(V, W)) + V(g(W, X)) - W(g(U, V)) - g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V])\]
for arbitrary vector fields $U, V, W \in \chi(M)$. Making use of Koszul’s formula, we get the following:

\begin{align*}
\nabla_{e_2} e_3 &= 3e_2 & \nabla_{e_2} e_2 &= 3e_3 & \nabla_{e_2} e_1 &= 0 \\
\nabla_{e_3} e_3 &= 0 & \nabla_{e_3} e_2 &= 0 & \nabla_{e_3} e_1 &= 0 \\
\nabla_{e_1} e_3 &= 3e_1 & \nabla_{e_1} e_2 &= 0 & \nabla_{e_1} e_1 &= 3e_3.
\end{align*}

From the above calculation, it is clear that $M$ satisfies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where $f = 3$ is a non-zero constant. Thus we conclude that $M$ leads to an $f$-Kenmotsu manifold. Also $f^2 + f'$ is non-zero. This implies that $M$ is a three-dimensional regular $f$-Kenmotsu manifold. We find the components of curvature tensor and Ricci tensor as follows:

\begin{align*}
R(e_2, e_3)e_3 &= -3e_2, & R(e_3, e_2)e_2 &= -3e_3, \\
R(e_1, e_3)e_3 &= -3e_1, & R(e_3, e_1)e_1 &= -3e_3, \\
R(e_1, e_2)e_2 &= -3e_1, & R(e_1, e_2)e_3 &= 0, \\
R(e_2, e_1)e_1 &= -3e_2, & R(e_3, e_1)e_2 &= 0, \\
S(e_1, e_1) &= -6, & S(e_2, e_2) &= -6, & S(e_3, e_3) &= -6, \\
S(\phi e_1, \phi e_1) &= -6, & S(\phi e_2, \phi e_2) &= -6, & S(\phi e_3, \phi e_3) &= 0,
\end{align*}

$S(\phi e_i, \phi e_j) = 0$ for all $i, j = 1, 2, 3(i \neq j)$. From the above consequence, it is clear that $\phi^2\{\nabla_U Q(V)\} = 0$ for all vector fields $U, V \in \chi(M)$. Hence $M$ is locally $\phi$-Ricci symmetric. From above we get $r = -18$, this implies the scalar curvature is constant. Moreover, $(\nabla_X S)(\phi e_i, \phi e_j) = 0$ for $X \in \chi(M)i, j = 1, 2, 3$. So $M$ is $\eta$-parallel, cyclic parallel. This example is also satisfying the Ricci soliton equation if $\lambda = 6$. Hence $\lambda = \frac{2}{3}f^2$ is verified. So the soliton is expanding. Thus, Theorem 3.1 and the associated corollaries are verified by this example.
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