$K_{l4}$ decays

Bing An Li

Department of Physics and Astronomy, University of Kentucky
Lexington, KY 40506, USA

Abstract

An effective theory of large $N_C$ QCD of mesons has been used to study six $K_{l4}$ decay modes. It has been found that the matrix elements of the axial-vector current dominate the $K_{l4}$ decays. PCAC is satisfied. A relationship between three form factors of axial-vector current has been found. Non-zero phase shifts are originated in $\rho \to \pi\pi$. The decay rates are calculated in the chiral limit. In this study there is no adjustable parameter.
1 Introduction

There is rich physics in kaon decays. Study on rare kaon decays are still active. The theoretical study of $K_{l4}$ decays has a long history\cite{1,2}.

In Ref.\cite{3} we have proposed an effective theory of large $N_C$ QCD\cite{4} of mesons. In this theory the diagrams at the tree level are at the leading order in large $N_C$ expansion and the loop diagrams of mesons are at higher orders. This theory is phenomenologically successful\cite{5,6,7}. We have used this theory to study $K_{l3}$\cite{3}, $K \rightarrow e\nu\gamma$\cite{5,8}, kaon form factors\cite{7}, and $\pi K$ scattering\cite{6} in the chiral limit. Theoretical results agree well with data. In these studies VMD and PCAC are satisfied. There are five parameters in this theory: three current quark masses, a parameter related to the quark condensate, and a universal coupling constant $g$ which is determined to be 0.39 by fitting $\rho \rightarrow ee^+$. All parameters have been fixed by previous studies.

In this paper we use this theory of pseudoscalar, vector, and axial vector mesons\cite{3} to study $K^- \rightarrow \pi^+\pi^-\nu$, $\pi^0\pi^0\nu$, and $K_L \rightarrow \pi^\pm\pi^0\nu$. In the study of $K_{l4}$ there is no adjustable parameter.

The Lagrangian of the theory of Ref. 3Y is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma \cdot \partial + \gamma \cdot v + \gamma \cdot a\gamma_5 - mu(x))\psi(x) + \frac{1}{2}m_1^2(\rho_\mu^\nu\rho_{\mu\nu} + \omega^\mu\omega_\mu + a_\mu^\nu a_{\mu\nu} + f^\mu f_\mu)$$

$$+ \frac{1}{2}m_2^2(K^{*a}_{\mu}K^{*a}_{\mu} + K^\mu K_{1\mu}) + \frac{1}{2}m_3^2(\phi_\mu\phi^\mu + f_s^\mu f_{s\mu})$$
\[ +\bar{\psi}(x)\gamma \cdot W\psi(x) + L_W + L_{\text{lepton}} - \bar{\psi}M\psi, \quad (1) \]

where \( a_\mu = \tau_i a^i_\mu + \lambda_a K^a_{1_\mu} + \left( \frac{2}{3} + \frac{1}{\sqrt{3}} \lambda_8 \right) f_\mu + \left( \frac{1}{3} - \frac{1}{\sqrt{3}} \lambda_8 \right) f_{s_\mu} (i = 1, 2, 3 \text{ and } a = 4, 5, 6, 7), \]

\[ v_\mu = \tau_i p^i_\mu + \lambda_a K^a_\mu + \left( \frac{2}{3} + \frac{1}{\sqrt{3}} \lambda_8 \right) \omega_\mu + \left( \frac{1}{3} - \frac{1}{\sqrt{3}} \lambda_8 \right) \phi_\mu, \]

\( W^i_\mu \) is the W boson, and \( u = \exp\{\gamma_5 i (\tau_i \pi_i + \lambda_a K^a + \eta + \eta')\} \), \( m \) is a parameter, and \( M \) is the mass matrix of u, d, s quarks, The masses \( m_1^2, m_2^2, \) and \( m_3^2 \) have been determined theoretically.

Using the notations of Ref.[1], we have

\[ <\pi^i \pi^j | A_\mu | K > = \frac{i}{m_K} \{ (p_1 + p_2)_\mu F^{ij} + (p_1 - p_2)_\mu G^{ij} + q_\mu R^{ij} \}, \]

\[ <\pi^i \pi^j | V_\mu | K > = \frac{H^{ij}}{m_K^2} \varepsilon^{\mu\nu\lambda\rho} p_\nu (p_1 + p_2)_\lambda (p_1 - p_2)_\rho, \quad (2) \]

where \( p_1, p_2, p \) are momenta of two pions and kaon respectively, \( q = p - p_1 - p_2 \), and \( i, j = +, -, 0 \). We define

\[ q_1^2 = (p - p_1)^2, \quad q_2^2 = (p - p_2)^2, \quad q_3^2 = (p_1 + p_2)^2. \]

The form factors, \( F^{ij}, G^{ij}, R^{ij} \) and \( H^{ij} \) are functions of \( q^2, q_1^2, q_2^2, \) and \( q_3^2 \). These four variables satisfy

\[ q_1^2 + q_2^2 + q_3^2 = m_K^2 + 2m_\pi^2 + q^2. \]

The paper is organized as: 1) introduction; 2) isospin relation; 3) form factors of vector current; 4) \( K^* \rightarrow K\pi\pi \) decay; 5) form factors of axial-vector current; 6) decay rates; 7) conclusions.
2 Isospin relation

For the decay modes $K^- \rightarrow \pi^+\pi^-l\nu, \pi^0\pi^0l\nu$ and $\bar{K}^0 \rightarrow \pi^+\pi^0l\nu$ there are isospin relations between the form factors denoted as $A^{ij}$. We take $-\pi^+, \pi^0,$ and $\pi^-$ as isospin triplet and $-\bar{K}^0$ and $K^-$ as isospin doublet. The isospin relation is obtained as

$$A^{+-} = A^{00} - \frac{1}{\sqrt{2}} A^{+0},$$

where $A^{ij} = F^{ij}, G^{ij}, R^{ij}, H^{ij}$ respectively.

3 Form factors of vector current

The Vector Meson Dominance(VMD) is a natural result of this theory[3]. The coupling between the W bosons and the bosonized vector current($\Delta s = 1$) has been derived as[5]

$$\mathcal{L}^V = \frac{g_W}{4} sin\theta_C g \left\{ - \frac{1}{2} (\partial_\mu W^+_{\nu} - \partial_\nu W^+_{\mu}) (\partial_\mu K^*-_{\nu} - \partial_\nu K^*_{\mu}) - \frac{1}{2} (\partial_\mu W^-_{\nu} - \partial_\nu W^-_{\mu}) (\partial_\mu K^+_{\nu}) \right\} + W^+_{\mu} j^+_{\mu} + W^-_{\mu} j^-_{\mu},$$

where $j^\pm_{\mu}$ is obtained by substituting

$$K^\pm_{\mu} \rightarrow \frac{g_W}{4} sin\theta_C g W^\pm_{\mu}$$

into the vertex in which $K^\mu$ field is involved.

The matrix elements of the vector current of $K_{l4}$ are resulted in anomalous vertices of mesons. The two subprocesses are shown in Fig.1(a,b). There is contact term. Three kinds
of vertices are involved: the contact term $\mathcal{L}_{K^* K \pi}$, $\mathcal{L}_{K^* K \pi}$ and $\mathcal{L}_{K^* K \pi}$, and $\mathcal{L}_{K^* K \rho}$ and $\mathcal{L}_{\rho \pi \pi}$.

In the chiral limit, $m_q \to 0$, all these vertices have been derived from the Lagrangian (1) [3] and are listed below

\[
\mathcal{L}_{K^* K \pi} = -\frac{N_C}{\pi^2 g^2 f_\pi} \varepsilon^{\mu \nu \alpha \beta} d_{a c i} K^\alpha_\mu \partial_\nu K^c_\beta \partial_\beta \pi^i,
\]

\[
\mathcal{L}_{K^* K} = \frac{2}{g} f(q^2) f_{a b i} K^a_\mu (\partial_\mu \pi^i K^b_\rho - \pi^i \partial_\mu K^b_\rho),
\]

\[
f(q^2) = 1 + \frac{q^2}{2\pi^2 f_\pi^2}[(1 - \frac{2c}{g})^2 - 4\pi^2 c^2],
\]

\[
c = \frac{f_\pi^2}{2gm^2_\rho},
\]

\[
\mathcal{L}_{K^* \rho K} = -\frac{N_C}{\pi^2 g^2 f_\pi^2} \varepsilon^{\mu \nu \alpha \beta} d_{a b i} K^a_\mu \partial_\nu K^b_\alpha \partial_\beta K^b_\rho,
\]

\[
\mathcal{L}_{\rho \pi \pi} = \frac{2}{g} f(q^2) \varepsilon_{i j k} \rho^i \pi^j \partial_\mu \pi^k,
\]

\[
\mathcal{L}_{K^* K \pi \pi} = \frac{2}{g\pi^2 f_\pi^3} (1 - \frac{6c}{g} + \frac{6c^2}{g^2}) d_{a b c} f_{c d e} \varepsilon^{\mu \nu \alpha \beta} K^a_\mu \partial_\nu \pi^b \partial_\alpha \pi^c \partial_\beta \pi^d.
\]

From these vertices the form factors $h_{ij}$ are found

\[
H^{+-} = \frac{m^3_{K^*} m^2_{K^*}}{q^2 - m^2_{K^*}} \left\{ \frac{1}{\pi^2 f_\pi^2} (1 - \frac{6c}{g} + \frac{6c^2}{g^2}) - \frac{N_C}{g^2 \pi^2 f_\pi q^2_\pi - m^2_{K^*}} \frac{f(q^2_2)}{f(q^2_2)} \right\},
\]

\[
H^{00} = -\frac{m^3_{K^*} m^2_{K^*}}{q^2 - m^2_{K^*}} \frac{N_C}{2g^2 \pi^2 f_\pi^2} \left\{ \frac{f(q^2_2)}{q^2_2 - m^2_{K^*}} - \frac{f^2(q^2_1)}{q^2_1 - m^2_{K^*}} \right\},
\]

\[
H^{+0} = \frac{1}{\sqrt{2}} \frac{m^3_{K^*} m^2_{K^*}}{q^2 - m^2_{K^*}} \left\{ -\frac{2}{\pi^2 f_\pi^2} (1 - \frac{6c}{g} + \frac{6c^2}{g^2}) + \frac{N_C}{\pi^2 g^2 f_\pi} \frac{f(q_1)}{q^2_1 - m^2_{K^*}} + \frac{f(q_2)}{q^2_2 - m^2_{K^*}} \right\},
\]

\[
+\frac{m^2_{K^*}}{q^2_3 - m^2_\rho + i\sqrt{q^2_3 \Gamma_\rho(q^2_3)}},
\]

5
where $\Gamma_\rho$ is the decay width of $\rho$ meson

$$\Gamma_\rho(q_3^2) = \frac{\sqrt{q_3^2} f^2(q_3^2)}{12 g^2 \pi} (1 - \frac{4m_\pi^2}{q_3^2})^3. \quad (11)$$

The equations (8-10) show that the isospin relation (3) is satisfied.

### 4 $K^* \rightarrow K \pi \pi$ decay

The vertices (6,7) are responsible for the decay of $K^* \rightarrow K \pi \pi$. As a test the decay widths of $K^* \rightarrow K \pi \pi$ are calculated

$$\Gamma(K^{*-} \rightarrow K^- \pi^+ \pi^-) = \frac{1}{96(2\pi)^3 m_{K^*}} \int dk_1^2 dk_2^2 \{p_1^2 p_2^2 - (\vec{p}_1 \cdot \vec{p}_2)^2\} |A|^2 = 0.29 \times 10^{-5} GeV \quad (12)$$

which is less than the experimental upper limit [9], where $A$ is the amplitude

$$A = \frac{4}{g \pi^2 f_\pi^2} \frac{6c}{g^2} - \frac{4N_c}{g^3 \pi^2 f_\pi} \frac{f(k_2^2)}{k_2^2 - m_{K^*}^2 + i\sqrt{k_2^2 \Gamma_{K^*}(k_2^2)}} \frac{f(k_3^2)}{k_3^2 - m_{K^-}^2 + i\sqrt{k_3^2 \Gamma_{K^-}(k_3^2)}} \quad (13)$$

where $k_1^2 = (p + p_1)^2$, $k_2^2 = (p + p_2)^2$, $k_3^2 = (p_1 + p_2)^2$, and $p_1, p_2, p$ are momenta of $\pi^+,$ $\pi^-$ and $K^-$ respectively, $\Gamma_{K^*}$ is the decay width of $K^*$

$$\Gamma_{K^*}(k_2^2) = \frac{f^2(k_2^2)}{2\pi g^2 k_2^2} \left\{ \frac{1}{4k_2^2} (k_2^2 + m_K^2 - m_{\pi^0}^2)^2 - m_{K^-}^2 \right\}^{\frac{3}{2}}. \quad (14)$$

$$\Gamma(K^{*-} \rightarrow K^- \pi^0 \pi^0) = \frac{1}{192(2\pi)^3 m_{K^*}} \int dk_1^2 dk_2^2 \{p_1^2 p_2^2 - (\vec{p}_1 \cdot \vec{p}_2)^2\}$$
\[
\frac{36}{\pi^4 g^6 f_\pi^2} \left\{ \frac{f(k_1)}{k_1^2 - m_{K^*}^2 + i\sqrt{k_1^2 \Gamma_{K^*}(k_1^2)}} - \frac{f(k_2)}{k_2^2 - m_{K^*}^2 + i\sqrt{k_2^2 \Gamma_{K^*}(k_2^2)}} \right\}^2
\]

\[= 0.61 \times 10^{-6} GeV. \tag{15}\]

\[
\Gamma(K^{*-} \to \bar{K}^0 \pi^- \pi^0) = \frac{1}{96(2\pi)^3 m_{K^*}} \int dk_1^2 dk_2^2 \{p_1^2 p_2^2 - (\vec{p}_1 \cdot \vec{p}_2)^2\} |B|^2 = 0.38 \times 10^{-4} GeV, \tag{16}\]

where

\[
B = -\frac{8}{\sqrt{2} g f_\pi} \left(1 - \frac{6c}{g} + \frac{6c^2}{g^2}\right) + \frac{12}{\sqrt{2} \pi^2 g f_\pi} \left\{ \frac{f(k_1)}{k_1^2 - m_{K^*}^2 + i\sqrt{k_1^2 \Gamma_{K^*}(k_1^2)}} \right. \\
+ \left. \frac{f(k_2)}{k_2^2 - m_{K^*}^2 + i\sqrt{k_2^2 \Gamma_{K^*}(k_2^2)}} \right\} + \frac{f(k_3)}{k_3^2 - m_{\rho}^2 + i\sqrt{k_3^2 \Gamma_{\rho}(k_3^2)}}. \tag{17}\]

Eq. (16) is compatible with the data 9Y.

## 5 Form factors of axial-vector current

In the chiral limit, the axial-vector part of the interaction between W-boson and mesons is expressed as[5]

\[
\mathcal{L}^{As} = \frac{g_w}{4} f_a \sin \theta_C \{ -\frac{1}{2} (\partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu)(\partial^\mu K^\mp_\nu - \partial^\nu K^\pm_\mu) + W^\pm_\mu j^\mp_\mu \}
+ \frac{g_w}{4} \sin \theta_C \Delta m^2 f_a W^\pm_\mu K^\mp_\mu + \frac{g_w}{4} \sin \theta_C f_K W^\pm_\mu \partial^\mu K^\mp,
\tag{18}\]

where \(j^\pm_\mu\) are obtained by substituting \(K^\pm_\alpha \rightarrow g_w f_a \sin \theta_C W^\pm_\mu\) into the vertex in which \(K\) fields are involved,

\[
f_a = g^{-1} \left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}}, \tag{19}\]
\[ \Delta m^2 = 6m^2g^2 = f_\pi^2 \left(1 - \frac{f_\pi^2}{g^2m_\rho^2}\right)^{-1}, \]  
\[ c = \frac{f_\pi^2}{2gm_\rho^2}. \]

The mass of \( K_1 \) meson is determined by

\[ (1 - \frac{1}{2\pi^2g^2})m_{K_1}^2 = 6m^2 + m_{K^*}^2. \]

The numerical value is \( m_{K_1} = 1.322 \text{GeV} \) which is compatible with the data \([9]\).

Two subprocesses contribute to the matrix element of the axial-vector current. They are shown in Fig.2(a,b). The vertices of mesons involved in these processes are \( \mathcal{L}_{K_1K^*\pi}, \mathcal{L}_{K^*K\pi} \) and \( \mathcal{L}_{K_1\rho K}, \mathcal{L}_{\rho\pi\pi} \). There is a contact term \( \mathcal{L}_{K_1K\pi\pi} \) too. However, the calculation shows that the contribution of the contact term is very small and negligible. In the chiral limit, these vertices have been derived from the Lagrangian(1)

\[ \mathcal{L}_{K_1K^*\pi} = f_{abi} \left\{ A(p^2)K_{1\mu}^aK_{\mu}^{*b}\pi^i - BK_{1\mu}^aK_{\nu}^{*b}\partial_{\mu\nu}\pi^i + DK_{1\mu}^a\partial^\mu(K_{\nu}^{*b}\partial^\nu\pi^i) \right\} \]  
\[ \mathcal{L}_{K_1\rho K} = -f_{abi} \left\{ A(p^2)K_{1\mu}^a\rho_{\mu}^iK^b - BK_{1\mu}^a\rho_{\nu}^i\partial_{\mu\nu}K^b + DK_{1\mu}^a\partial^\mu(\rho_{\nu}^i\partial^\nu K^b) \right\}, \]

where

\[ A(p^2) = \frac{2}{f_\pi}g_f a \left\{ \frac{F^2}{g^2} + p^2 \left[ \frac{2c}{g} + \frac{3}{4\pi^2g^2}(1 - \frac{2c}{g}) \right] \right\} + q^2 \left[ \frac{1}{2\pi^2g^2} - \frac{2c}{g} - \frac{3}{4\pi^2g^2}(1 - \frac{2c}{g}) \right], \]

\[ F^2 = f_\pi^2 \left(1 - \frac{2c}{g}\right)^{-1}, \]
\[ B = -\frac{2}{f_\pi} g f_a \frac{1}{2\pi^2 g^2} (1 - \frac{2c}{g}), \]
\[ D = -\frac{2}{f_\pi} f_a \{2c + \frac{3}{2\pi^2 g} (1 - \frac{2c}{g})\}, \]

where \( q \) and \( p \) are the momentum of \( K_1 \) and the vector meson respectively.

\[ \mathcal{L}_{K^*K\pi} = \frac{2}{g} f_{ab} f (p^2) K^a \mu_{\nu} \pi^i \partial_{\mu} \pi^i, \]
\[ \mathcal{L}_{\rho\pi\pi} = \frac{2}{g} \epsilon_{ijk} f (p^2) \rho_{\mu}^i \pi^j \partial_{\mu} \pi^k, \]
\[ f(p^2) = 1 + \frac{p^2}{2\pi f_\pi^2} [(1 - \frac{2c}{g})^2 - 4\pi^2 g^2], \]

where \( p \) is the momentum of the vector meson.

By using Eqs.(18,23,24), we obtain

\[ < \pi^+ \pi^- | A_\mu | K^- > = \frac{1}{\sqrt{2}} \left( \frac{g_{\mu\nu}}{q^2} - g_{\mu\nu} \right) \frac{g^2 f_a m_{K^*}^2}{q^2 - m_{K^*}^2} < \pi^+ \pi^- | \left\{ A(p_{K^*}) K^0_{\nu} \pi^- - B K^0_{\lambda} \partial_{\lambda\nu} \pi^- \right\} \]
\[ -\frac{1}{\sqrt{2}} \left\{ A(p_\rho) \rho_{\nu}^0 K^- - B \rho_{\lambda}^0 \partial_{\lambda\nu} K^- \right\} | K^- >. \]

In the chiral limit PCAC is satisfied. The reason is that the Lagrangian(1) is chiral symmetric in the limit \( m_q \to 0 \). On the other hand, the satisfaction of PCAC is resulted in the cancellations between the four terms of Eq.(18). The Eq.(18) shows that the axial-vector current has more complicated structure than the vector current does(5). Because of the PCAC the form factor R(2) is not an independent quantity and determined as

\[ R = -\frac{1}{q^2} \{ q \cdot (p_1 + p_2) F + q \cdot (p_1 - p_2) G \}. \]
Substituting the vertices (29,30) into Eq. (32), the three form factors are obtained

\[ F^{+-} = \frac{gf_a m^2_{K^*} m_K}{q^2 - m^2_{K^*}} \left\{ \frac{f(q_1^2)}{q_1^2 - m^2_{K^*}} - \frac{3}{2} A(q_2^2) + \frac{1}{2} B p_1 \cdot (p + p_2) \right\} \]
\[ + \frac{f(q_3^2)}{q_3^2 - m^2_\rho + i \sqrt{q^2_3} \Gamma_\rho(q_3^2)} B p \cdot (p_2 - p_1), \] 
(34)

\[ G^{+-} = \frac{gf_a m^2_{K^*} m_K}{q^2 - m^2_{K^*}} \left\{ \frac{f(q_1^2)}{q_1^2 - m^2_{K^*}} + \frac{1}{2} A(q_2^2) + \frac{1}{2} B p_1 \cdot (p + p_2) \right\} \]
\[ - \frac{f(q_3^2)}{q_3^2 - m^2_\rho + i \sqrt{q^2_3} \Gamma_\rho(q_3^2)} A(q_3^2). \] 
(35)

In the same way the form factors of other two decay modes are obtained

\[ F^{00} = \frac{1}{2} \frac{gf_a m^2_{K^*} m_K}{q^2 - m^2_{K^*}} \left\{ \frac{f(q_1^2)}{q_1^2 - m^2_{K^*}} - \frac{3}{2} A(q_2^2) + \frac{1}{2} B p_2 \cdot (p + p_2) \right\} \]
\[ + \frac{f(q_3^2)}{q_3^2 - m^2_\rho + i \sqrt{q^2_3} \Gamma_\rho(q_3^2)} B p_1 \cdot (p_1 + p_2), \] 
(36)

\[ G^{00} = \frac{1}{2} \frac{gf_a m^2_{K^*} m_K}{q^2 - m^2_{K^*}} \left\{ \frac{f(q_1^2)}{q_1^2 - m^2_{K^*}} - \frac{1}{2} A(q_2^2) + \frac{1}{2} B (p_1 \cdot p + p_1 \cdot p_2) \right\} \]
\[ + \frac{f(q_3^2)}{q_3^2 - m^2_\rho + i \sqrt{q^2_3} \Gamma_\rho(q_3^2)} B (p_2 \cdot p + p_2 \cdot p_1), \] 
(37)

\[ F^{+0} = \frac{1}{\sqrt{2}} \frac{gf_a m^2_{K^*} m_K}{q^2 - m^2_{K^*}} \left\{ \frac{f(q_1^2)}{q_1^2 - m^2_{K^*}} - \frac{3}{2} A(q_2^2) + \frac{1}{2} B p_2 \cdot (p + p_2) \right\} \]
\[ - \frac{f(q_3^2)}{q_3^2 - m^2_\rho + i \sqrt{q^2_3} \Gamma_\rho(q_3^2)} B (p_1 + p_2), \] 
(38)

\[ G^{+0} = \frac{1}{\sqrt{2}} \frac{gf_a M^2_{K^*} m_K}{q^2 - m^2_{K^*}} \left\{ \frac{f(q_1^2)}{q_1^2 - m^2_{K^*}} - \frac{1}{2} A(q_2^2) + \frac{1}{2} B (p_1 \cdot p + p_1 \cdot p_2) \right\} \]
\[ + \frac{2f(q_3^2)}{q_3^2 - m^2_\rho + i \sqrt{q^2_3} \Gamma_\rho(q_3^2)} A(q_3^2). \] 
(39)

The isospin relations (3) between these form factors are satisfied.
The partial wave analysis of these form factors can be done. The decay channel $\rho \to \pi\pi$ contributes to the decay modes of $\pi^+\pi^-$ and $\pi^+\pi^0$. The range of the variable $q_3^2$ is $4m_\pi^2 < q_3^2 < (m_K - m_\pi)^2$ in which the decay width $\Gamma_\rho(q_3^2)$ is not zero. The form factors, $A^{+-}$ and $A^{+0}$ are complex functions of $q_3^2$. The $\rho \to \pi\pi$ doesn’t contribute to $\pi^0\pi^0$ mode. Therefore, $F^{00}$ and $G^{00}$ are real. $K_{14}$ are decays at low energies. s- and p- waves are major partial waves. The $q_1^2$ and $q_2^2$ variables are expressed as

$$q_1^2 = \frac{1}{2}(m_K^2 + 2m_\pi^2 + q^2 - q_3^2) + (1 - \frac{4m_\pi^2}{q_3^2})^\frac{1}{2}X\cos\theta_\pi, \quad (40)$$

$$q_2^2 = \frac{1}{2}(m_K^2 + 2m_\pi^2 + q^2 - q_3^2) - (1 - \frac{4m_\pi^2}{q_3^2})^\frac{1}{2}X\cos\theta_\pi, \quad (41)$$

where $X = \{\frac{1}{4}(m_K^2 - q^2 - q_3^2)^2 - q^2q_3^2\}^{\frac{1}{2}}$ and $\theta_\pi$ is the angle between $\mathbf{p}_1$ and $\mathbf{p}$ in the rest frame of the two pions.

The s- and p- wave amplitudes are obtained from Eqs.(34-39)

1. $F^{+-}_{s}$ is real. Only Fig.2(a) contributes to it. $F^{+-}_{p}$ is a complex function of $q_3^2$ resulted by $\rho \to \pi\pi$. $F^{+-}_{p}$ has a phase shift.

$$F^{+-} = F^{+-}_{s} + |F^{+-}_{p}|e^{i\delta^{+-}_{p}}(1 - \frac{4m_\pi^2}{q_3^2})^\frac{1}{2}X\frac{m_K^2}{m_\pi^2}\cos\theta_\pi. \quad (42)$$

2. $G^{+-}_{s}$ is complex and has a phase shift. $G^{+-}_{p}$ is real.

$$G^{+-} = |G^{+-}_{s}|e^{i\delta^{+-}_{s}} + G^{+-}_{p}(1 - \frac{4m_\pi^2}{q_3^2})^\frac{1}{2}X\frac{m_K^2}{m_\pi^2}\cos\theta_\pi. \quad (43)$$
3. Both $G_{s}^{00}$ and $G_{p}^{00}$ are real.

$$
F_{s}^{00} = F_{s}^{00},
$$

$$
G_{s}^{00} = |G_{s}^{00}|\left(1 - \frac{4m_{\pi}^{2}}{q_{3}^{2}}\right)^{\frac{1}{2}} \frac{X}{m_{K}} \cos \theta_{\pi}. \tag{44}
$$

4. The isospin of the two pions of the $\pi^{+}\pi^{0}$ mode is one. Because of Bose statistics $F^{+0}$ only has p-wave which is complex and has phase shift. $G^{+0}$ has s wave only. $G_{s}^{+0}$ is complex and it has phase shift.

$$
F^{+0} = |F_{s}^{+0}|e^{i\delta^{+0}} \left(1 - \frac{4m_{\pi}^{2}}{q_{3}^{2}}\right)^{\frac{1}{2}} \frac{X}{m_{K}} \cos \theta_{\pi},
$$

$$
G^{+0} = |G_{s}^{+0}|e^{i\delta_{s}^{+0}}. \tag{45}
$$

All the phase shifts are caused by the decay $\rho \rightarrow \pi\pi$ and functions of $q^{2}$ and $q_{3}^{2}$.

6 Decay rates

The decay rates of the three modes of $K_{e4}$ and $K_{\mu4}$ are calculated. As mentioned above, all the form factors are derived in the chiral limit. Therefore, only the leading terms of the masses of kaon and pions are kept in the calculation of the decay rates.

Ignoring $m_{e}$, only the form factors $F$, $G$, and $H$ contribute to the decay rates of $K_{e4}$. By using the formula of Ref.[1] we obtain

$$
\Gamma(K^{-} \rightarrow \pi^{+}\pi^{-}e\nu) = 2.06 \times 10^{-21}GeV, \quad B = 3.87 \times 10^{-5}.
$$
\[ \Gamma(K^- \rightarrow \pi^0\pi^0\nu) = 0.221 \times 10^{-21} GeV, \quad B = 0.42 \times 10^{-5}. \]
\[ \Gamma(K^- \rightarrow \pi^+\pi^0\nu) = 3.24 \times 10^{-21} GeV, \quad B = 2.55 \times 10^{-4}. \]

The experimental data are

\[ B(\pi^+\pi^-) = (3.91 \pm 0.17) \times 10^{-5}[10], \]
\[ B(\pi^0\pi^0) = (2.54 \pm 0.89) \times 10^{-5}(10 \text{ events})[11], \]
\[ B(\pi^-\pi^0) = (5.16 \pm 0.20 \pm 0.22) \times 10^{-5}[12], \]
\[ B(\pi^-\pi^0) = (6.2 \pm 2.0) \times 10^{-5}[13], \]
\[ B(\pi^-\pi^0) < 200 \times 10^{-5}[14]. \]

Theoretical result of \( \pi^+\pi^- \) mode agrees well with the data.

The form factors of the vector current are determined by anomalous vertices. The numerical calculation shows that the contribution of the form factor H is only 0.5\% of the total decay rate of \( K^- \rightarrow \pi^+\pi^-\nu \). Therefore, the axial-vector current dominates the \( K_{l4} \) decays.

As shown in Fig.2(a,b) there are two channels in \( K_{l4} \) decays. Numerical calculation of \( K^- \rightarrow \pi^+\pi^-\nu \) shows that the contribution of \( \rho \rightarrow \pi\pi \) (Fig.2(b)) is twice of the process, \( K^* \rightarrow K\pi \), (Fig.2(a)). Only the process(Fig.2(a)) contribute to \( K^- \rightarrow \pi^0\pi^0\nu \). Because of Bose statistics there is an additional factor of \( \frac{1}{2} \) in the formula of the decay rate of this mode. Therefore, this theory predicts smaller decay rate for this decay mode. On the other
hand, the numerical calculation shows that the process (Fig. 2(b)) is the major contributor of the decay $\bar{K}^0 \to \pi^+\pi^0 e\nu$. The theory predicts a larger branching ratio for $\bar{K}^0 \to \pi^+\pi^0 e\nu$.

All the form factors contribute to $K_{\mu 4}$ decays. Eq.(33) shows that in the chiral limit PCAC predicts that the form factor R is determined by other two form factors, F and G. The branching ratio of $K_{\mu 4}$ provides a test on this prediction. The numerical results are

$$\Gamma(K^- \to \pi^+\pi^-\mu\nu) = 0.634 \times 10^{-21} GeV, \quad B = 1.19 \times 10^{-5}.$$  

$$\Gamma(K^- \to \pi^0\pi^0\mu\nu) = 0.673 \times 10^{-22} GeV, \quad B = 0.126 \times 10^{-5}.$$  

$$\Gamma(\bar{K}^0 \to \pi^+\pi^0\mu\nu) = 1.01 \times 10^{-21} GeV, \quad B = 0.793 \times 10^{-4}.$$  

The experimental data[9] is

$$B(K^- \to \pi^+\pi^-\mu\nu) = (1.4 \pm 0.9) \times 10^{-5}.$$  

Theory agrees with the data well.

7 Conclusion

All the four form factors of $K_{\mu 4}$ have been derived from an effective theory of large $N_C$ QCD in the chiral limit. It has been found that the contribution of the vector current is negligible and the axial-vector current is dominant in $K_{\mu 4}$ decays. PCAC is revealed from the theory. In the chiral limit it has been predicted that the form factor R is determined by the form
factors F and G. The prediction has been tested by \( K^- \rightarrow \pi^+ \pi^- \mu \nu \). Theory agrees with the data. The partial wave analysis has been done. Non-zero phase shifts originate in the decay \( \rho \rightarrow \pi \pi \). The process \( K_1 \rightarrow \rho K \) and \( \rho \rightarrow \pi \pi \) (Fig.2(b)) plays important role in \( K_{l4} \) decays. Because of this channel the theory predicts larger branching ratio for \( K^- \rightarrow \pi^+ \pi^- e \nu \) and \( \bar{K}^0 \rightarrow \pi^+ \pi^0 e \nu \). The former agrees well with the data. \( \rho \) resonance doesn’t contribute to \( K^- \rightarrow \pi^0 \pi^0 e \nu \). Therefore, the branching ratio of this decay mode is predicted to be smaller.

This research was partially supported by DOE Grant No. DE-91ER75661.

References

[1] A.Pais and S.B. Treiman, Phys. Rev. 168, 168(1968); L.M.Choumet, J.M.Gallard, and M.K.Gaillard, Phys. Rep. C4, 199(1972). Due to convention the form factors defined in Eq.(2) are the form factors defined in [1] multiplied by \( \frac{1}{\sqrt{2}} \).

[2] N.Cabibbo and A. Maksymowicz, Phys.Rev. 137, B 438(1965); S. Weinberg Phys.Rev.Lett. 17, 336(1966); J. Bijnens, Nucl. Phys., B337, 635(1990); C. Riggenbach et al., Phys. Rev., D43, 127(1991); G.Colangelo, Phys.Lett., B336, 543(1994).

[3] B.A.Li, Phys.Rev., D52, 5165(1995), 5184(1995).

[4] G.’t Hooft, Nucl. Phys. B72, 461(1974); B75, 461(1974); E.Witten, Nucl. Phys. B160, 57(1979).
[5] B.A. Li, Phys. Rev. D 55, 1436 (1997); 55 1425 (1997).

[6] D.N. Gao, B.A. Li, and M.L. Yan, Phys. Rev. D 56, 4115 (1997); B.A. Li, D.N. Gao, and M.L. Yan, Phys. Rev. D 58, 094031 (1998).

[7] J.Gao and Bing An Li, hep-ph/9911438

[8] Bing An Li, hep-ph/9810311.

[9] Particle Data Group, Euro. Phys. J. C3,1(1998).

[10] L.Rosselet et al., Phys. Rev., D15, 574(1977).

[11] V.V. Barmin et al., SJNP 55, 547(1988).

[12] G.Makoff et al., Phys.Rev.Lett. 70,1591(1993).

[13] A.S.Carroll et al., Phys. Lett., 96B, 407(1980).

[14] G.Doaldson, Thesis SLAC-0184.
Figure Captions

Fig.1 Feynman Diagrams of vector current

Fig.2 Feynman diagrams of axial-vector current

Fig.3 Phase shifts

Fig.4 Phase shifts Fig.5 Phase shifts

Fig.5 Phase shifts

Fig.6 Phase shifts

Fig.7 Form factors od $\pi^+\pi^-$ mode

Fig.8 Form factors od $\pi^+\pi^-$ mode

Fig.9 Form factors od $\pi^+\pi^0$ mode

Fig.10 Form factors od $\pi^+\pi^0$ mode

Fig.11 Form factors od $\pi^0\pi^0$ mode

Fig.12 Form factors od $\pi^0\pi^0$ mode
Fig. 1
Fig. 2
\[ q^2 = 0.14 \text{(GeV)}^2 \]

Fig. 3
Fig. 4
$q^2 = 0.0168 (\text{GeV})^2$

Fig. 5
\[ q^2 (\text{GeV})^2 = 0.14(\text{GeV})^2 \]

Fig. 6

23
$q^2 = 0.0168 \text{(GeV)}^2$
Fig. 8

\[ q_s^2 = 0.14 \text{(GeV)}^2 \]
Fig. 9
q^2 = 0.0168(\text{GeV})^2

Fig. 10
Fig. 11
Fig. 12

$q_0^2 = 0.14 (\text{GeV})^2$