$L^p$-estimates for Riesz transforms on forms in the Poincaré space

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Abstract

Using hyperbolic form convolution with doubly isometry-invariant kernels, the explicit expression of the inverse of the de Rham laplacian $\Delta$ acting on $m$-forms in the Poincaré space $\mathbb{H}^n$ is found. Also, by means of some estimates for hyperbolic singular integrals, $L^p$-estimates for the Riesz transforms $\nabla^i \Delta^{-1}, i \leq 2$, in a range of $p$ depending on $m, n$ are obtained. Finally, using these, it is shown that $\Delta$ defines topological isomorphisms in a scale of Sobolev spaces $H^{s}_{m,p}(\mathbb{H}^n)$ in case $m \neq \frac{n+1}{2}$.

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1 Statement of results and preliminaries

1.1. The main object of study in this paper is the Hodge-de Rham Laplacian $\Delta$ acting on $m$-forms in the Poincaré hyperbolic space $(\mathbb{H}^n, g)$. The aim is to prove that $\Delta$ defines topological isomorphisms in a range $H^{s}_{m,p}(\mathbb{H}^n)$ of Sobolev spaces of forms defined as follows. For $0 \leq m \leq n$, $1 \leq p < \infty$ and $s \in \mathbb{N}$, the Sobolev space $H^{s}_{m,p}(\mathbb{H}^n)$ is the completion of the space $\mathcal{D}_m(\mathbb{H}^n)$ of smooth $m$-forms with compact support with respect the norm

$$
\|\eta\|_{p,s} = \sum_{i=0}^{s} \|\nabla^{(i)} \eta\|_p .
$$

Here $\nabla^{(i)}$ means the $i$-th covariant differential of $\eta$, and for a covariant tensor $\alpha$

$$
\|\alpha\|_p = \left( \int_{\mathbb{H}^n} |\alpha(x)|^p d\mu(x) \right)^{\frac{1}{p}},
$$

$|\alpha|$ being the pointwise norm of $\alpha$ with respect the metric $g$ and $d\mu$ the volume-invariant measure on $\mathbb{H}^n$ given by $g$. The space $H^{s}_{m,p}(\mathbb{H}^n)$ can be alternatively defined in terms of weak derivatives.

The main result of this paper is:

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Theorem A. \( \Delta \) is a topological isomorphism from \( H^{s+2}_{m,p}(\mathbb{H}^n) \) to \( H^s_{m,p} \) for \( p \in (p_1, p_2) \) with

\[
p_1 = \frac{2(n-1)}{n-2 + |n-2m|} \quad \frac{1}{p_1} + \frac{1}{p_2} = 1
\]

in case \( m \neq \frac{n\pm1}{2}, \frac{n}{2} \).

In the exceptional case \( m = \frac{n\pm1}{2} \), \( \Delta \) is one to one but is not a topological isomorphism for any \( p \). For this case we obtain as well some weighted estimates. If \( m = \frac{n}{2} \), \( \Delta \) is known to have a non-trivial kernel. Of course, Sobolev spaces \( H^{s}_{m,p} \) can be considered for non integer \( s \) as well, and the same results holds by interpolation.

Notice that the Hodge star operator \( * \) establishes an isometry from \( H^s_{m,p}(\mathbb{H}^n) \) to \( H^s_{n,m,p}(\mathbb{H}^n) \) which commutes with \( \Delta \), and this is why the range \( (p_1, p_2) \) depends only on \( |n-2m| \). Notice too that the range \( (p_1, p_2) \) always contains \( p = 2 \) in the non-critical case \( |n-2m| > 1 \) and that for functions \( (m = 0) \), the range of \( p \) is \( (1, \infty) \) (see comments below). We point out that the range \( (p_1, p_2) \) equals \( \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{\mu}{n-1} \), where \( \mu \) denotes the greatest lower bound for the spectrum of \( \Delta \) in \( H^0_{m,2}(\mathbb{H}^n) \), whose value (\( \mu \)) is \( \mu = \frac{(n-1-2m)^2}{4} \) (for \( m < \frac{n}{2} \)).

For the Sobolev spaces for \( p = 2 \), \( H^s_{m,2}(\mathbb{H}^n) \), another proof of the theorem, based in energy methods and valid for an arbitrary complete hyperbolic manifold, is given in [BG]. The motivation for the theorem, as with [BG], comes from mathematical physics, where most operators exhibit \( \Delta \) as their principal part and results like the above become essential to establish existence and uniqueness theorems.

Our method of proof is simply to construct an explicit inverse \( L \) for \( \Delta \) on \( D_m(\mathbb{H}^n) \) and show that there is a gain of two covariant derivatives

\[
\|L\eta\|_{p,s+2} \leq \text{const} \|\eta\|_{p,s}.
\]

Thus \( L\eta \) plays the role of the classical Riesz transform in the Euclidean setting. The most delicate part is of course

\[
\|\nabla^{(2)}L\eta\|_p \leq \text{const} \|\eta\|_p, \quad p_1 < p < p_2, \quad \eta \in D_m(\mathbb{H}^n).
\]

Riesz-type operators such as \( \nabla \Delta^{-\frac{1}{2}}, \nabla^{(2)} \Delta^{-1} \) have extensively been studied in different contexts, for the case of functions. On symmetric spaces, they are bounded in \( L^p, 1 < p < \infty \) and of weak type \( (1, 1) \). This was shown in [AL] for the first order ones in some spaces, and later extended to all symmetric spaces in [A]. The \( L^p \)-boundedness holds as well for higher order Riesz transforms in symmetric spaces, but not generally the weak type \( (1, 1) \) estimate. In more general contexts, this has been shown in [L1], [L2], [L3], among others. In case of \( m \)-forms, \( 0 < m < n \), as far as we know, there are much less known results, and is for those that our result is new. In [P1], [P2] some aspects of harmonic analysis of forms are developed; in particular, the exact expression for the heat kernel is given, and it is very likely that from it one can get as well an explicit expression for \( \Delta^{-1} \). Strictly speaking, to prove the result an exact expression of \( \Delta^{-1} \) is not needed, it is enough
having estimates for the resolvent both local and at infinity. In \[L3\], estimates of this kind are obtained and applied to Sobolev-type inequalities for forms, and they might work for this purpose too. However, we feel that our approach, that we next describe, is more elementary and might be interesting in itself.

The de Rham Laplacian \(\Delta\) is invariant by all isometries \(\varphi\) of \(H^n\). These form a a group that we denote here by \(\text{Iso}(H^n)\). Denoting by \(\varphi^*(\eta)(x) = \eta(\varphi(x))\) the pull-back of a form \(\eta\) by \(\varphi\), this means that \(\Delta\) and \(\varphi^*\) commute, for all \(\varphi \in \text{Iso}(H^n)\). Among all isometries of \(H^n\), the hyperbolic translations \(\text{Tr}(H^n)\) constitute a (noncommutative) subgroup, in one to one correspondence with \(H^n\) itself. In section 2 we do some harmonic analysis for forms in \(H^n\) and introduce hyperbolic convolution of forms to describe all operators acting on \(m\)-forms and commuting with \(\text{Tr}(H^n)\). In a second step (subsection 2.2) we characterize the hyperbolic convolution kernels \(k(x, y)\) corresponding to operators commuting with the full group \(\text{Iso}(H^n)\).

Once the general expression of an operator commuting with \(\text{Iso}(H^n)\) has been found, we look for our \(L\) among these. This corresponds to \(L\) having a kernel \(k(x, y)\) which is a fundamental solution of \(\Delta\) in a certain sense, and having the best decay of infinity. This kernel turns out to be unique for \(m \neq \frac{n \pm 1}{2}\), \(\frac{n}{2}\), we call it the Riesz kernel for \(m\)-forms in \(H^n\), it is found in subsection 3.1 and estimated in subsection 3.2. Section 4 is devoted to the proof of the \(L^p\)-estimates. Here we use standard techniques in real analysis (Haussdorf-Young inequalities, Schur’s lemma, etc.). For the second-order Riesz transform, to show its boundedness in the specified range \((p_1, p_2)\) needs considering some notion of “hyperbolic singular integral”. There exist some references dealing with this, e.g. \[L4\], \[I\], and giving some criteria for \(L^p\)-boundedness that might apply; however, as the singular integral arises locally we have found easier and more elementary to treat it with the classical Euclidean Calderón-Zygmund theory as a local model, and patch it in a suitable way to infinity.

1.2. We collect here several notations and known facts about \(H^n\). We will use both the unit ball model \(B^n\) with metric \(g = 4(1 - |x|^2)^{-2} \sum_i dx^i dx^i\) and the half-space model \(\mathbb{R}^n_+ = \{x_n > 0\}\) with metric \(g = x_n^{-2} \sum_i dx^i dx^i\). Both models are connected via the Cayley transform \(\psi: \mathbb{R}^n_+ \to B^n\) given in coordinates by

\[
\begin{align*}
y_i &= \frac{2x_i}{\sum_{i=1}^{n-1} x_i^2 + (x_n + 1)^2}, \quad i = 1, \ldots, n-1; \\
y_n &= \frac{\sum_{i=1}^{n-1} x_i^2 - 1}{\sum_{i=1}^{n-1} x_i^2 + (x_n + 1)^2}.
\end{align*}
\]

We denote by \(e \in \mathbb{H}^n\) the point \((0, 0, \ldots, 1) \in \mathbb{R}^n_+\) or \(0 \in B^n\).

The metric \(g\) defines a pointwise inner product \((\alpha, \beta)(x)\) between forms at \(x\), for every \(x \in \mathbb{H}^n\), and a volume measure \(d\mu\). In the ball model \(d\mu\) is written
\[ d\mu(x) = 2^n (1 - |x|^2)^{-n} dx^1 \ldots dx^n, \] and \( d\mu(x) = x^{-n} dx^1 \ldots dx^n \) in the half-space model. We denote by \( \langle \cdot, \cdot \rangle \) the pairing between forms that makes \( H^*_0(H^n) \) a Hilbert space

\[ \langle \alpha, \beta \rangle = \int_{H^n} (\alpha, \beta)(x) d\mu(x). \]

We write \( |\alpha| \) and \( \|\alpha\| \) for the pointwise and global norms, respectively, of the form \( \alpha \). In terms of the Hodge star operator \( * \) the inner product can be written too

\[ \langle \alpha, \beta \rangle = \int_{H^n} \alpha \wedge * \beta. \]

The group \( \text{Tr}(H^n) \) of hyperbolic translations is in one to one correspondence \( x \mapsto T_x \) with \( H^n \) through the equation \( T_x(e) = x \). The equations of \( z = T_x y \) are better described in the half-space model by

\[ z_i = x_n y_i + x_i, \quad i = 1, \ldots, n - 1; \quad z_n = x_n y_n. \]

It is easily checked that indeed \( \text{Tr}(H^n) \) is a (non-commutative) group. The inverse transformation of \( T_x \) will be denoted \( S_x \). Another explicit isometry \( \varphi_x \) mapping \( e \) to \( x \), satisfying \( \varphi_x^{-1} = \varphi_x \) is given in the ball model by

\[ \varphi_x(y) = \frac{(|x|^2 - 1)y + (|y|^2 - 2xy + 1)x}{|x|^2|y|^2 - 2xy + 1}. \]

Since the isotropy group of 0 is the orthogonal group \( O(n) \), the general expression of \( \varphi \in \text{Iso}(H^n) \) is \( \varphi = \varphi_x \circ U \), with \( x = \varphi(0) \).

The hyperbolic (or geodesic) distance between \( x, y \in H^n \) is written \( d(x, y) \). We will rather use the \textit{pseudohyperbolic distance} \( r = r(x, y) \), related to \( d \) by the formula \( d(x, y) = 2 \text{arctanh} r(x, y) \).

The explicit expression of \( r(x, y)^2 \) in the \( R^n_+ \) model and the \( B^n \) model is respectively

\[ r^2 = \frac{|x - y|^2}{|x - y|^2 + 4x_n y_n}, \quad x, y \in R^n_+, \]

\[ r^2 = |\varphi_x(y)|^2 = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2) + |x - y|^2}, \quad x, y \in B^n. \]

Associated to the group of translations we have the basis of orthonormal translation-invariant vector fields \( X_i(x) = (T_x)_* (X_i(e)) \), such that \( X_i(e) = \frac{\partial}{\partial x_i} \). They satisfy \( X_i(u \circ T_x) = (X_i u) \circ T_x \) for every smooth function \( u \). We will denote by \( w^i(x) \) the dual basis of \( X_i \), which accordingly is orthonormal and translation invariant too: \( T_x w^i = w^i \). Their expression in the \( R^n_+ \) model is simply

\[ X_i(x) = x_n \frac{\partial}{\partial x_i}, \quad w^i(x) = x_n^{-1} dx^i, \quad i = 1, \ldots, n. \]

Because of their translation-invariance property, the \( (X_i, w^i) \) are more suitable than the \( (X_i, \eta^i) \) defined in the ball model \( B^n \) by

\[ Y_i(x) = \frac{(1 - |x|^2)}{2} \frac{\partial}{\partial x_i}, \quad \eta^i(x) = 2(1 - |x|^2)^{-1} dx^i. \]
For an increasing multindex $I$ of length $|I| = m$ we write $w^I = w^{i_1} \wedge w^{i_2} \wedge \cdots \wedge w^{i_m}$, and similarly $dx^I$ or $\eta^I$. The $\{w^I\}_I$ is an orthonormal translation-invariant basis of $m$-forms.

Recall that the de Rham Laplacian is defined $\Delta = d\delta + \delta d$, where $\delta$ is the adjoint of $d$ with respect to $\langle \cdot, \cdot \rangle$. Although strictly speaking not needed, the following expression of $\Delta$ in $w^I$-coordinates will simplify the analysis at some points. If $\alpha = \sum_I \alpha_I w^I$, a computation shows that in case $n \not\in J$

$$\Delta J \alpha = \Delta J \alpha + 2 \sum_{k \in J} X_k \alpha_{Jk} - p(n - p - 1) \alpha_J.$$  

(3.1)

Here $Jk$ means the multindex obtained replacing $k$ by $n$. In case $n \in J$,

$$\Delta J \alpha = \Delta J \alpha - 2 \sum_{l \not\in J} X_l \alpha_{Jl} - (1 - p)(p - n) \alpha_J$$  

(3.2)

where $lJ$ means the multiindex obtained replacing $n$ by $l$. For a function $f$

$$\Delta f = - \sum_{i=1}^n X_i^2 f + (n - 1) X_n f.$$  

In the ball model, with usual coordinates,

$$\Delta f = -\frac{1}{4} (1 - |x|^2)^2 \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} + (1 - \frac{n}{2}) (1 - |x|^2) \sum x_i \frac{\partial f}{\partial x_i}$$  

(3.3)

2 Translation invariant and isometry invariant operators on forms

2.1. We are interested in finding the general expression of an operator acting on $m$-forms, and isometry-invariant. In a first step we consider translation-invariant operators acting on $m$-forms; these are described by what we might call hyperbolic convolution as follows. Let $k(x,y)$ be a double $m$-form in $x,y$ and define

$$(C_k \alpha)(x) = \int_{\mathbb{H}^n} \alpha(y) \wedge *_y k(x,y) = \langle \alpha, k(x, \cdot) \rangle, \quad \alpha \in \mathcal{D}_m(\mathbb{H}^n).$$

If $T_z$ is a translation with inverse $S_z$

$$C_k(T_z^* \alpha)(x) = \int_{\mathbb{H}^n} (T_z^* \alpha)(y) \wedge *_y k(x,y) = \int_{\mathbb{H}^n} \alpha(T_zy) \wedge *_y k(x,y)$$

$$= \int_{\mathbb{H}^n} \alpha(y) \wedge *_y k(x, S_zy)$$

$$T_z^*(C_k \alpha)(x) = C_k \alpha(T_z x) = \int_{\mathbb{H}^n} \alpha(y) \wedge *_y k(T_z x, y).$$

Therefore $C_k$ is translation invariant if $k$ is doubly translation invariant in the sense that

$$k(x,y) = k(S_z x, S_z y) \quad \forall S_z.$$
Using the translation-invariant basis of $m$-forms $w^I$ we see that the general expression of $k$ is

$$k(x, y) = \sum_{I,J} k_{I,J}(x, y) w^I(x) \otimes w^J(y)$$

with $k_I(x, y)$ doubly-invariant functions, that is, of the form $k_{I,J}(x, y) = a_{I,J}(S_y x)$ for some function (or distribution) $a_{I,J}$. If $\delta_0$ denotes the Delta-mass at $e$ and

$$\delta(x, y) = \sum_{I,J} \delta_0(S_y x) w^I(x) \otimes w^J(y)$$

then formally

$$\alpha(x) = \int_{\mathbb{H}^n} \alpha(y) \wedge \ast_y \delta(x, y).$$

If $P$ is an operator on $m$-forms commuting with the $T_y, S_y$, we will thus have

$$P\alpha(x) = \int_{\mathbb{H}^n} \alpha(y) \wedge \ast_y P_x(\delta(x, y))$$

and indeed $k(x, y) = P_x(\delta(x, y))$ is formally doubly-invariant. This shows, in loose terms, that the operator $C_k$ of convolution with a doubly translation invariant kernel $k$ gives the general translation-invariant operator acting on $m$-forms. If

$$k(x, y) = \sum_{I,J} a_{I,J}(S_y x) w^I(x) \otimes w^J(y)$$

and $\alpha(x) = \sum \alpha_I(x) w^I(x)$, then $C_k \alpha$ has in the basis $w^I(x)$ coefficients given by

$$(C_k \alpha)_I(x) = \int_{\mathbb{H}^n} a_{I,J}(S_y x) \alpha_J(y) \, d\mu(y).$$

Thus in the basis $w^I$ everything reduces of course to convolution of functions. For a function convolution kernel $a(S_y x)$ and a test function $u \in \mathcal{D}(\mathbb{H}^n)$ we may think in

$$C_a u(x) = \int_{\mathbb{H}^n} u(y) a(S_y x) \, d\mu(y)$$

as an infinite linear combination of inverse translates $a(S_y x)$ of $a(x)$. Since the vector fields $X_i$ commute with translations, it follows that whenever everything makes sense

$$X_i(C_a u) = C_{X_i a} u. \quad (4)$$

We point out that this convolution is not commutative, $C_a u$ is in general different from $C_a a$. Correspondingly, $X_i C_a u - C_a X_i u$ is in general not zero; in fact one can easily show ([12G, lemma 3.1]) that these commutators are linear combinations of other convolution operators built from $a(S_y x)$. 

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2.2. Let $P$ be a generic translation-invariant operator acting on $m$-forms. We have seen in the previous subsection that we can associate to $P$ a doubly-translation invariant kernel $k(x, y)$ so that $P = C_k$. By the same argument as before, $P$ will be isometry invariant if and only if $k(\varphi x, \varphi y) = k(x, y)$ \(\forall \varphi \in \text{Iso}(\mathbb{H}^n)\) in which case we say that $k$ is doubly isometry-invariant. Working in the ball model and since every $\varphi \in \text{Iso}(\mathbb{H}^n)$ is the composition of a translation with some $U \in O(n)$, the additional requirement on the kernel

\[ k(x, y) = \sum_{I,J} a_{I,J}(S_y x) w^I(x) \otimes w^J(y) \]

amounts to $k(Ux, U0) = k(x, 0)$, that is,

\[ \sum_{I,J} a_{I,J}(Ux) U^* w^I(x) \otimes U^* w^J(0) = \sum_{I,J} a_{I,J}(x) w^I(x) \otimes w^J(0), \forall U \]

Thus we are interested in describing those $k(x, 0)$ —which is a $m$-form at 0 whose coefficients are $m$-forms in $x$— that are doubly invariant by all $U \in O(n)$ in the sense above. Once the $k(x, 0)$ having this property are known, $k(x, y) = k(S_y x, 0)$ defines the general doubly isometry invariant $m$-form. For $m = 0$ the $k(x, 0)$ are simply the radial functions $a(|x|)$, and $a(|S_y x|) = a(|\varphi x|)$ is the general doubly isometry invariant function. For $m \neq 0$ their general expression is not so simple.

We find it more convenient to use the usual basis $dx^I$ so we look at $k(x, 0)$ in the form

\[ k(x, 0) = \sum_{|I|=|J|=m} b_{I,J}(x) dx^I \otimes dx^J(0) \]  

and we must impose

\[ \sum_{I,J} b_{I,J}(Ux)d(Ux)^I \otimes d(Ux)^J(0) = k(x, 0) \forall U. \]  

For instance

\[ \gamma(x, 0) = \sum_{i=1}^n dx^i \otimes dx^i(0) \]

is easily seen to be doubly $O(n)$-invariant, and so is

\[ \gamma_m = \frac{1}{m!} \gamma \wedge \cdots \wedge \gamma = \sum_{|I|=m} dx^I \otimes dx^I(0) \]

(here we use the symbol $\wedge$ to denote as well the exterior product of double forms defined by $(\alpha_1 \otimes \beta_1) \wedge (\alpha_2 \otimes \beta_2) = (\alpha_1 \wedge \alpha_2) \otimes (\beta_1 \wedge \beta_2)$). Another doubly $O(n)$-invariant 1-form is

\[ \tau(x, 0) = \left( \sum_{i=1}^n x_i dx^i \right) \otimes \left( \sum_{i=1}^n x_i dx^i(0) \right). \]

**Lemma 2.1.** The double forms $\gamma$ and $\tau$ generate all doubly $O(n)$-invariant $k(x, 0)$. More precisely, their general expression in the ball model is

\[ k(x, 0) = A_1(|x|) \gamma_m + A_2(|x|) \tau \wedge \gamma_{m-1} \]

\[ k(x, 0) = A(|x|) \gamma_m, \quad m = 0, n \]  

\[ 0 < m < n \]
Proof. First we prove by induction the following statement \( S(n) \): if \( k(x,0) \) is a doubly invariant \((p,q)\)-form \( \sum_{|I|=p,|J|=q} c_{I,J} \, dx^I \otimes dx^J(0) \) with constant coefficients, then \( k \equiv 0 \) if \( p \neq q \), or \( k \) is diagonal i.e. \( k(x,0) = c \sum_{|I|=p} dx^I \otimes dx^I(0) = c_\gamma p \) if \( p = q \). Of course \( S(1) \) is obvious; assuming \( S(n-1) \), let us break \( k(x,0) \) in four pieces, depending on whether \( i_1, j_1 = 1 \) or not:

\[
\begin{align*}
k &= \sum_{i_1=j_1=1} c_{I,J} \, dx^I \otimes dx^J(0) + \sum_{i_1\neq j_1=1} + \sum_{i_1=1,j_1\neq 1} + \sum_{i_1\neq 1,j_1\neq 1} \text{def} k_1 + k_2 + k_3 + k_4.
\end{align*}
\]

We may write \( k_1 = (dx^1 \otimes dx^1(0)) \wedge \tilde{k}_1, k_2 = (dx^1 \otimes 1) \wedge \tilde{k}_2, k_3 = (1 \otimes dx^1(0)) \wedge \tilde{k}_3, \) with \( \tilde{k}_1, \tilde{k}_2, \tilde{k}_3, k_4 \) double forms in the \( dx^2, \ldots, dx^n, dx^2(0), \ldots, dx^n(0) \) of bidegrees \((p-1,q-1), (p-1,q), (p,q-1) \) and \((p,q)\), respectively. Imposing that \( k \) is doubly invariant by \( U \) of the type

\[
U = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & 0 & U_1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & & & 0
\end{pmatrix}, \quad U_1 \in \text{O}(n-1)
\]

we see that \( \tilde{k}_1, \tilde{k}_2, \tilde{k}_3 \) and \( k_4 \) are \( \text{O}(n-1) \)-invariant. We apply the induction hypothesis: if \( p = q \)

\[
\tilde{k}_2 = \tilde{k}_3 = 0, \text{ and } \tilde{k}_1, k_4 \text{ are diagonal, i.e.}
\]

\[
k = c_1 \sum_{i_1=1} dx^I \otimes dx^I(0) + c_2 \sum_{i_1\neq 1} dx^I \otimes dx^I(0).
\]

If we use now \( U \in \text{O}(n) \) permuting the first two axes we see that \( c_1 = c_2 \) and hence \( k \) is diagonal, establishing \( S(n) \) in case \( p = q \). If \( |p-q| > 1 \) everything is 0. Finally if \( |p-q| = 1, \) say \( p = q + 1 \), then \( \tilde{k}_2 \) is diagonal and all others are zero

\[
k = c(dx^1 \otimes 1) \wedge \sum_{|J|=q} dx^J \otimes dx^J(0)
\]

where \( x' = (x_2, \ldots x_n) \). If we impose the invariance under the permutation of the first two axes as before, it is clear that \( k \) must be zero.

Having proved that \( S(n) \) holds for all \( n \), let now \( k(x,0) \) be as in \( 5 \) doubly \( \text{O}(n) \)-invariant. Clearly \( k(x,0) \) is then determined by its values \( k(\vec{r},0) \), where \( \vec{r} = (r,0,0,\ldots,0) \). Fixed \( r \), \( k(\vec{r},0) \) may be regarded as a double \((m,m)\)-form with constant coefficients, which is invariant by all \( U \in \text{O}(n) \) fixing \( \vec{r} \), that is, of type \( 7 \). We write now the decomposition of \( k(\vec{r},0) \) in terms of \( \tilde{k}_1(r,0), \tilde{k}_2(r,0), \tilde{k}_3(r,0) \) and \( k_4(r,0) \) as before, and applying \( S(n) \) we get \( 8 \)

\[
k(\vec{r},0) = c_1(r) \sum_{i_1=1} dx^I(\vec{r}) \otimes dx^I(0) + c_2(r) \sum_{i_1 \neq 1} dx^I(\vec{r}) \otimes dx^I(0)
\]
(if $m = n$ the last term is zero and the first is $\gamma_m$) which we write

\[
=c_1(r) - c_2(r) \sum_{|\mathbf{i}| = m} dx^I(\mathbf{r}) \otimes dx^I(0) + c_2(r) \sum_{|\mathbf{i}| = m} dx^I(\mathbf{r}) \otimes dx^I(0)
\]

\[
=(c_1(r) - c_2(r))dx^1(\mathbf{r}) \otimes dx^1(0) \wedge \sum_{|\mathbf{i}| = m-1} dx^I(\mathbf{r}) \otimes dx^I(0) + c_2(r) \sum_{|\mathbf{i}| = m} dx^I(\mathbf{r}) \otimes dx^I(0)
\]

\[
=(c_1(r) - c_2(r))r^{-2} \tau(\mathbf{r}, 0) \gamma_{m-1}(\mathbf{r}, 0) + c_2(r) \gamma_m(\mathbf{r}, 0).
\]

Finally, fixed $x$ we choose $U$ such that $Ux = \mathbf{r}$, $r = |x|$, and use the invariance of $k$, $\tau$, $\gamma$ to find $\square$ with $A_1(r) = c_2(r)$, $A_2(r) = r^{-2}(c_1(r) - c_2(r))$. 

To find the general expression of a doubly isometry-invariant kernel $k(x, y)$ we must translate $k(x, 0)$ to an arbitrary point: $k(x, y) = k(S_y x, S_y y)$. We may use any isometry mapping $y$ to 0, for instance we may use $\varphi_y$ given by (1) instead of $S_y$. We introduce the basic forms $\alpha, \beta, \tau$ and $\gamma$

\[
\alpha = \alpha(x, y) = \sum_i \varphi^i_y(x) d\varphi^i_y(x), \quad \beta = \sum_i \varphi^i_y(x) d\varphi^i_y(y) = -\sum_i \varphi^i_y(x) \frac{dy^i}{1 - |y|^2}
\]

\[
\tau = \alpha \otimes \beta; \quad \gamma(x, y) = \sum_{i=1}^n d\varphi^i_y(x) \otimes d\varphi^i_y(y) = \frac{-1}{1 - |y|^2} \sum_{i=1}^n d\varphi^i_y(x) \otimes dy^i = dx \beta.
\]

The lemma gives part (a) of the following theorem. Part (b) gives other equivalent general expressions, which are intrinsic, that is, independent of the model of $\mathbb{H}^n$ at use.

**Theorem 2.2.** (a) The general expression of an $(m, m)$-form $k(x, y)$ doubly isometry-invariant in $\mathbb{H}^n$, in the ball model, is

\[
k(x, y) = A_1(|\varphi_y x|) \gamma_m(x, y) + A_2(|\varphi_y x|) \tau(x, y) \wedge \gamma_{m-1}(x, y), \quad 0 < m < n
\]

\[
k(x, y) = A(|\varphi_y x|) \gamma_m(x, y) \quad m = 0, n.
\]

(b) Another equivalent expression for $0 < m < n$ is

\[
k(x, y) = B_1(D)(dx dy D)^m + B_2(D)(dx D \otimes dy D) \wedge (dx dy D)^{m-1} =
\]

\[
= (C_1(D)dx dy D + C_2(D)dx D \otimes dy D)^m
\]

where $D$ denotes an arbitrary function of the geodesic distance $d(x, y)$.

(c) All such $k(x, y)$ are symmetric in $x, y \in \mathbb{H}^n$

**Proof.** Part (a) has been already proved. For (b) note first that it is enough to consider one function of $d$: we choose $D = r(x, y)^2$ which in the ball model equals $|\varphi_y(x)|^2$. Then $dx D = 2\alpha$, and using (1), (2) one finds

\[
dy D = 2(1 - D) \sum_i \varphi^i_y(x) \frac{dy}{1 - |y|^2} = -2(1 - D) \beta.
\]
This gives \( \tau = \alpha \otimes \beta = -\frac{1}{4} 1_D d_x D \otimes d_y D \), and
\[
d_x d_y D = +2 d_x D \otimes \beta - 2(1 - D)d_x \beta = +4 \tau - 2(1 - D)\gamma.
\]
Therefore \((d_x d_y D)^{m-1} \) and \(2^{m-1}(1 - D)^{m-1} \gamma_{m-1}\) differ in a term containing \(\tau\), and so (b) follows. Part (c) is a consequence of (b).

We will need the expression of the generators \(\tau, \gamma\) in terms of the invariant basis \(w^i\). We obtain these using formula (2) for \(r^2(x, y)\) in the half-space model. First
\[
\alpha = \frac{d_x r^2}{2} = \frac{1 - r^2}{2(|x - y|^2 + 4x_n y_n)} \left( \sum_{i=1}^{n-1} x_n(x_i - y_i)w^i(x) + (2x_n(x_n - y_n) - |x - y|^2)w^n(x) \right)
\]
\[
\beta = \frac{d_y r^2}{2(r^2 - 1)} = \frac{-1}{2(|x - y|^2 + 4x_n y_n)} \left( \sum_{j=1}^n y_n(y_j - x_j)w^j(y) + (2y_n(y_n - x_n) - |x - y|^2)w^n(y) \right).
\]
In the following we write \(w^{ij} = w^i(x) \otimes w^j(y)\). We have
\[
\tau = \alpha \otimes \beta = \frac{1}{4} \frac{1 - r^2}{(|x - y|^2 + 4x_n y_n)^2} \sum_{i,j} P_{i,j}(x,y) w^{i,j}
\]
where the \(P_{i,j}(x,y)\) are certain homogeneous polynomials. As we know, everything can be written in terms of \(z = S_y x\): for instance
\[
1 - r^2 = \frac{4x_n y_n}{|x - y|^2 + 4x_n y_n} = \frac{4z_n}{|z|^2 + 2z_n + 1}
\]
and say for \(i,j < n\)
\[
P_{i,j} = \frac{x_n y_n(x_i - y_i)(x_j - y_j)}{(|x - y|^2 + 4x_n y_n)^2} = \frac{z_n z_i z_j}{(|z|^2 + 2z_n + 1)^2}.
\]
Therefore we may write
\[
\tau = \frac{1 - r^2}{(|z|^2 + 2z_n + 1)^2} \sum_{i,j} P_{i,j}(z) w^{i,j} \tag{9}
\]
For \(\gamma = d_x \beta\) we obtain a similar expression
\[
\frac{4}{1 - r^2} \gamma = \sum_{i,j=1}^{n-1} \left( \delta_{ij} - \frac{2(x_i - y_i)(x_j - y_j)}{|x - y|^2 + 4x_n y_n} \right) w^{i,j} +
\]
\[
+ \left( 1 - \frac{2 \sum_{i=1}^{n-1} |x_i - y_i|^2}{|x - y|^2 + 4x_n y_n} \right) w^{n,n} + \sum_{i=1}^{n-1} \frac{2(x_i - y_i)(x_n - y_n)}{|x - y|^2 + 4x_n y_n} (w^{i,n} - w^{n,i}).
\]

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Again this can be written
\[\gamma = \frac{1 - r^2}{4(|x - y|^2 + 4x_ny_n)} \sum_{i,j} Q_{ij}(x, y) w^{i,j} = \frac{1 - r^2}{(|z|^2 + 2z_n + 1)} \sum q_{ij}(z) w^{i,j}.\] (10)

Notice that
\[\frac{p_{ij}(z)}{(|z|^2 + 2z_n + 1)^2} = O(1), \quad \frac{q_{ij}(z)}{(|z|^2 + 2z_n + 1)} = O(1)\]
and hence
\[|\tau(x, y)| = O(1 - r^2), \quad |\gamma(x, y)| = O(1 - r^2).\] (11)

3 Riesz forms and Riesz form-potentials in \(\mathbb{H}^n\)

3.1. Our next objective is now to find an explicit left-inverse \(L\) for \(\Delta\) on \(D_m(\mathbb{H}^n)\). Since \(\Delta\) is invariant by all isometries, \(L\) should be too. By what has been discussed in section 2, \(L\) should have a kernel \(k_m(x, y)\),
\[L\eta(x) = \int_{\mathbb{H}^n} \eta(y) \wedge * y k_m(x, y)\]
doubly invariant by all isometries. Alternatively, notice that if \(k\) is some kernel such that
\[\eta(x) = \int_{\mathbb{H}^n} \Delta \eta(y) \wedge * y k(x, y), \quad \eta \in D_m(\mathbb{H}^n)\] (12)
(which formally exists because \(\Delta \eta = 0, \eta \in D_m(\mathbb{H}^n)\) imply \(\eta = 0\)) then its average over the unitary group \(O(n)\) with respect the normalized left-invariant measure \(d\mu(U)\),
\[k_1(x, y) = \int_{O(n)} k_0(Ux, Uy) d\mu(U)\]
still satisfies (12) and it is doubly invariant by \(O(n)\). If \(\varphi_x\) is an isometry mapping \(x\) to 0, \(k_2(x, y) = k_1(\varphi_xx, \varphi_xy)\) is independent of \(\varphi_x\), satisfies (12) and is doubly invariant by all isometries.

Anyway, we look for a doubly isometry-invariant kernel \(k_m\) for which (12) holds, and then consider the operator \(L\) defined by \(k_m\) as above. Taking for granted by now that this operator \(L\) is well defined on \(D_m(\mathbb{H}^n)\) and maps \(D_m(\mathbb{H}^n)\) into locally integrable \(m\)-forms, notice that (12) and the symmetry of \(k_m\) together imply that \(L\) is a right-inverse too, that is, \(\Delta L \alpha = \alpha\) for \(\alpha \in D_m(\mathbb{H}^n)\) in the weak sense:
\[\langle \Delta L \alpha, \eta \rangle = \langle L \alpha, \Delta \eta \rangle = \int_x L \alpha(x) \wedge * \Delta \eta(x) = \int_x \left\{ \int_y \alpha(y) \wedge * y k_m(x, y) \right\} \wedge * \Delta \eta(x) = \int_y \alpha(y) \wedge * \left\{ \int_x k_m(x, y) \wedge * \Delta \eta(x) \right\} = \langle \alpha, \eta \rangle .\]
We work in the ball model. By Theorem 2.2, \( k_m(x, y) \) is of type

\[
k_m(x, y) = A(|\varphi_x y|)\gamma_m, \quad m = 0, n
\]

\[
k_m(x, y) = A_1(|\varphi_x y|)\gamma_m + A_2(|\varphi_x y|)\tau \wedge \gamma_{m-1}, \quad 0 < m < n
\]

where \( \gamma = \sum_i d\varphi_x^i(x) \otimes d\varphi_x^i(y) \), \( \tau = \alpha \otimes \beta \) with \( \alpha = \sum_i \varphi_x^i(y)d\varphi_x^i(x) \), \( \beta = \sum_i \varphi_x^i(y)d\varphi_x^i(y) \) (notice that we are exchanging \( x, y \), using (c) in Theorem 2.2). Condition (12) implies \( \Delta_y k_m(x, y) = 0 \) in \( y \neq x \) (while \( \Delta Lw = w \) implies \( \Delta_x k_m(x, y) = 0 \) in \( x \neq y \)). In fact, (12) amounts to requiring \( \Delta_y k_m(x, y) = \delta_x \) in a sense to be described below.

3.2. In a first step we look for conditions on the \( A_1, A_2 \), so that \( \Delta_y k_m(x, y) = 0 \) in \( y \neq x \). A lengthy computation will show that the general harmonic \( k_m \) depends on four parameters. By the invariance of \( k_m \), we may assume \( x = 0 \), in which case, writing \( r = |y| \),

\[
k_m(x, y) = A(r)\gamma_m, \quad m = 0, n
\]

\[
k_m(0, y) = A_1(r)\gamma_m + A_2(r)\tau \wedge \gamma_{m-1}
\]

with \( \gamma = \sum_i dx^i(0) \otimes dy^i \), \( \tau = \alpha \otimes \beta \), \( \alpha = \sum y^i dx^i(0) \), \( \beta = rdr \). Since \( \ast_x \ast_y k_m(x, y) \) is again doubly invariant, it must have an analogous expression with \( m \) replaced by \( n - m \). Indeed, it is easily checked that

\[
\ast_x \ast_y \gamma_m = \frac{m!}{(n-m)!}(1 - r^2)^{2m-n} \gamma_{n-m}
\]

\[
\ast_x \ast_y (\tau \wedge \gamma_{m-1}) = (m-1)!(1 - r^2)^{2m-n} \left( \frac{r^2 \gamma_{n-m}}{(n-m)!} - \frac{\tau \wedge \gamma_{n-m-1}}{(n-m-1)!} \right)
\]

whence

\[
\ast_x \ast_y k_m(x, y) = \frac{m!}{(n-m)!}(1 - r^2)^{2m-n} \gamma_{n-m} \quad \text{for} \quad m = 0, n \quad \text{and} \quad 0 < m < n,
\]

\[
\ast_x \ast_y k_m(0, y) = \frac{(m-1)!(1 - r^2)^{2m-n}}{(n-m)!} \left[ (mA_1 + r^2A_2)\gamma_{n-m} - (n-m)A_2 \tau \wedge \gamma_{n-m-1} \right]. \quad (13)
\]

Moreover, since \( \ast \) commutes with \( \Delta \), it is natural to require as well that \( \ast_x \ast_y k_m = k_{n-m} \), that is, we may assume from now on that \( 0 \leq m \leq n/2 \).

For \( m = 0 \), using (5.3) we find

\[
\Delta(A(r)) = \frac{1}{4}(1 - r^2) \left[ -1(1 - r^2)A'' + ((3 - n)r + r^{-1}(1 - n))A' \right]
\]

from which it follows that \( A'(r) = c_0(1 - r^2)^{-n+1}r^{1-n} \) and

\[
A(r) = c_1 - c_0 \int_r^1 (1 - s^2)^{n-2}s^{1-n} ds.
\]
We start now computing $\Delta y k_m(0, y)$ for $0 < m \leq n/2$ using that on $m$-forms $\Delta$ equals $(-1)^{m+1}(*d* d + (-1)^n d * d*)$. The double form $\Delta y k_m(x, y)$ is also doubly invariant, and therefore it must have the same expression as $k_m$ with $A_1, A_2$ replaced by other functions $B_1, B_2$ to be found. In the computations we will use besides (13) the equations

$$d_y \alpha = \gamma, \quad d_y (\tau \wedge \gamma_{m-1}) = -r dr \wedge \gamma_m = -\beta \wedge \gamma_m$$

$$*_x *y dr \wedge \gamma_m = (-1)^m \frac{m!}{(n-m-1)!} (1 - r^2)^{2m+2-n} r^{-1} \alpha \wedge \gamma_{n-m-1}$$

which are easily checked as well. First, $d_y k_m(0, y) = (A'_1 - r A_2) dr \wedge \gamma_m$, so by the equations above

$$*_x *y d_y k_m(0, y) = (-1)^m \frac{m!}{(n-m-1)!} (1 - r^2)^{2m+2-n} (A'_1 - r A_2)r^{-1} \alpha \wedge \gamma_{n-m-1} = \left(\begin{array}{c}
\text{(14)}
\end{array}\right)$$

$$\int_{x} \int_{y} d_y k_m(0, y) = (-1)^{m(n-m-1)} *y \int_{x} (A_3 \gamma_{n-m} + A'_3 r^{-1} \tau \wedge \gamma_{n-m-1})$$

$$*y d_y *y d_y k_m(0, y) = (-1)^{m(n-m)} *y \int_{x} (A_3 \gamma_{n-m} + A'_3 r^{-1} \tau \wedge \gamma_{n-m-1}) =$$

$$= (-1)^{m(n-m-1)} \frac{m!}{(n-m-1)!} (1 - r^2)^{n-2m} \left( A_3 \frac{(n-m)!}{m!} \gamma_m + A'_3 r \frac{(n-m-1)!}{m!} \gamma_m \right)$$

$$- A'_3 r^{-1} \frac{(n-m-1)!}{(m-1)!} r^{-1} \wedge \gamma_{n-m-1}$$

$$= (-1)^{m(n-m+1)} (1 - r^2)^{n-2m} \left[ ((n-m)A_3 + A'_3 r) \gamma_m - mA'_3 r^{-1} \tau \wedge \gamma_{n-m-1} \right].$$

By the analogous computation, applying $d_y$ to (13)

$$*_x *y d_y k_m(0, y) = \frac{(m-1)!}{(n-m)!} \left[ \left( (m A_1 + r^2 A_2)(1 - r^2)^{n-2m} \right)' + (n-m) r A_2 (1 - r^2)^{n-2m} \right] dr \wedge \gamma_{n-m}$$

$$*y d_y *y d_y k_m(0, y) = (-1)^{(m+1)(n-m)} (1 - r^2)^{2m+2-n} r^{-1}$$

$$= \left( (m A_1 + r^2 A_2)(1 - r^2)^{n-2m} \right)' + (n-m) r A_2 (1 - r^2)^{n-2m} \right) \alpha \wedge \gamma_{m-1} = \left(\begin{array}{c}
\text{(15)}
\end{array}\right)$$

$$\int_{x} \int_{y} d_y k_m(0, y) = (-1)^{(n-m)(m+1)} (A'_4 r^{-1} \tau \wedge \gamma_{m-1} + A_4 \gamma_m)$$

It follows finally that $\Delta = (-1)^{m+1}(*d* d + (-1)^n d * d*)$ on $k_m$ equals

$$\Delta y k_m(0, y) = B_1 \gamma_m + B_2 \tau \wedge \gamma_{m-1}$$
with
\[ B_1 = -A_4 - (1 - r^2)^{n-2m}((n - m)A_3 + A'_3) \]
\[ B_2 = -A_4 r^{-1} + m(1 - r^2)^{n-2m}A'_3 r^{-1}. \]

Therefore, \( \Delta_y k(0, y) = 0 \) is equivalent to the system \( B_1 = 0, B_2 = 0 \). It easily follows from this that \( A_3 \) satisfies the equation
\[ r(1 - r^2)A''_3 + [(n + 1) - r^2(3n + 1 - 4m)] A'_3 - 2(n - 2m)(n - m) r A_3 = 0. \]

Replacing in the equation \( B_1 = 0, A_4 \) by its expression in terms of \( A_1 \) and \( A_2 \), and then \( A_2 \) by its expression in terms of \( A_1 \) and \( A_3 \), we find that \( A_1 \) satisfies the inhomogeneous equation
\[ r(1 - r^2)A''_1 + [(n + 1) + (n - 1 - 4m)r^2] A'_1 + 2m(n - 2m) r A_1 = 0. \]

The change of variables \( A_1(r) = G(x), A_3(r) = H(x), x = r^2 \), transforms these into the hypergeometric equations
\[ x(1 - x)H''(x) + \left[ \frac{n}{2} + 1 - \left( \frac{3}{2}n + 1 - 2m \right)x \right] H'(x) - \left( \frac{n}{2} - m \right)(n - m) H = 0 \]
\[ x(1 - x)G''(x) + \left[ \frac{n}{2} + 1 - (2m + 1 - \frac{n}{2})x \right] G'(x) - m \left( \frac{n}{2} - m \right) G = \]
\[ = \frac{1}{2} (1 + x) (1 - x)^{n-2m-2} H(x) \]
\[ = f(x) \]

This system is equivalent to \( \Delta_y k_m(x, y) = 0 \) in \( y \neq x \), whence the general doubly-invariant \( k_m \) harmonic in \( y \neq x \) depends on four parameters. Note that for \( m = \frac{n}{2} \) the homogeneous equations are the same and can be solved explicitly: the general solution is \( H = a s^{-\frac{n}{2}} + b \) and
\[ G(x) = cx^{-\frac{n}{2}} + d + \frac{1}{2} \int_{1/2}^x t^{-\frac{n}{2}-1} \left\{ \int_0^t s^{n/2}(1 + s)(1 - s)^{-3}(as^{-\frac{n}{2}} + b) \, ds \right\} \, dt. \]

For \( m < \frac{n}{2} \) a fundamental family for the first equation in (16) is given by
\[ u_1(x) = x^{-\frac{n}{2}} F\left(-m, \frac{n}{2} - m, 1 - \frac{n}{2}, x\right), \]
\[ u_2(x) = F\left(\frac{n}{2} - m, n - m, \frac{n}{2} + 1, x\right) \]

The hypergeometric function in \( u_1 \) is a polynomial in \( x \) of degree \( m \) with positive coefficients, \( 1 + x \) if \( m = 1 \). A fundamental family for the second is given by
\[ u_3(x) = x^{-\frac{n}{2}} F\left(m - n, m - \frac{n}{2}, 1 - \frac{n}{2}, x\right) = x^{-\frac{n}{2}} (1 - x)^{n+1-2m} F\left(\frac{n}{2}, 1 - m, 1 - m, 1 - \frac{n}{2}, x\right), \]
\[ u_4(x) = F\left(m, m - \frac{n}{2}, 1 + \frac{n}{2}, x\right) \]
The hypergeometric function in \( u_3 \) is a polynomial of degree \( m - 1 \) with positive coefficients (see [1] for all these facts). The wronskian \( w(x) \) for this second equation is, by Liouville’s formula

\[
W(x) = W(x_0) \exp \left( \int_{x_0}^{x} \frac{\frac{n}{2} + 1 - (2m + 1 - \frac{n}{2})t}{t(1-t)} dt \right) = c_{mn} x^{\frac{n}{2} - 1/2} (1 - x^{2m - n})
\]

It follows from this that the parametrization for \( G \) is given by

\[
G(x) = c(x)u_3(x) + d(x)u_4(x)
\]

where \( c(x), d(x) \) satisfy, with \( H(x) = au_1(x) + bu_2(x) \),

\[
c'(x) = \frac{u_4(x)f(x)}{x(1-x)W(x)} = \frac{1}{2} c_{mn} H(x)(1+x)x^{\frac{n}{2}} (1-x)^{-3} u_4(x)
\]

\[
d'(x) = -\frac{u_3(x)f(x)}{x(1-x)W(x)} = -\frac{1}{2} c_{mn} H(x)(1+x) x^{\frac{n}{2}} (1-x)^{-3} u_3(x).
\]

Once \( A_1(r) = G(r^2) \) and \( A_3(r) = H(r^2) \) are known, the kernel \( k_{m}(x, y) \) is completely known, because by the definition of \( A_3 \) in [14], \( A_2(r) = - (1-r^2)^{m-2} A_3(r) + r^{-1} A_1'(r) = - (1-r^2)^{m-2} H(x) + 2G'(x) \).

The choice \( a = 0, c(0) = 0 \) (\( a = c = 0 \) in the parametrization [17]) for \( m = \frac{n}{2} \) gives all doubly invariant \( k_{m}(x, y) \) which are \textit{globally} harmonic, with no singularity, and they are therefore spanned by the forms corresponding to the choice \( G = u_4 \) and to the choice \( a = 0, b = 1, c(0) = 0, d(0) = 0 \),

\[
G(x) = \left\{ \int_0^x (1 + t)(1-t)^{-3} t^{n/2} u_2(t) u_4(t) dt \right\} u_3(x)
\]

\[
- \left\{ \int_0^x (1 + t)(1-t)^{-3} t^{n/2} u_2(t) u_3(t) dt \right\} u_4(x).
\]

As a particular case, note that for \( m = \frac{n}{2}, \gamma_m \) is harmonic in \( \mathbb{H}^{2m} \), and it is the simplest example of a non-zero harmonic \( m \)-form in \( L^2(\mathbb{H}^{2m}) \).

3.3. Besides being harmonic in \( y \neq x \), the singularity at \( y = x \) must be such that [12] holds. Again, we may assume \( x = 0 \); we check this property using \textit{second’s Green identity}, whose version for general forms we recall now.

The operator \( \delta \) being the adjoint of \( d \), one has for a smooth domain \( \Omega \subset \mathbb{H}^n \) and \( \alpha, \beta \) smooth forms on \( \Omega \) with degree\( \alpha = \text{degree} \beta - 1 \)

\[
\int_{\partial \Omega} \alpha \wedge \ast \beta = \int_{\Omega} d\alpha \wedge \ast \beta - \int_{\Omega} \alpha \wedge \delta \beta .
\]

Given two \( m \)-forms \( \eta, \omega \), applying this with \( \alpha = \delta \eta, \beta = \omega \), next with \( \alpha = \omega, \beta = d\eta \) and substracting one gets \textit{the first Green’s identity for \( m \)-forms}

\[
\int_{\partial \Omega} (\delta \eta \wedge \ast \omega - \omega \wedge \ast d\eta) = \int_{\Omega} (\Delta \eta \wedge \ast \omega - \delta \eta \wedge \ast \delta \omega - d\eta \wedge \ast d\omega)
\]

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Permuting $\omega, \eta$ and substracting again gives the second Green's identity

$$\int_{\partial \Omega} (\delta \eta \wedge * \omega - \omega \wedge * d\eta - \delta \omega \wedge * \eta + \eta \wedge * d\omega) = \int_{\Omega} (\Delta \eta \wedge * \omega - \Delta \omega \wedge * \eta).$$

We apply this to $\Omega = B(0, R) - B(0, \varepsilon)$ $0 < \varepsilon < R < 1$, $\eta \in D_m(\mathbb{H}^n)$ and our $k_m(0, y)$ to get

$$\int_{|y| \geq \varepsilon} \Delta \eta \wedge * y k_m(0, y) = \int_{|y| = \varepsilon} (k_m \wedge * d\eta + \delta_y k_m \wedge * \eta - \delta \eta \wedge * y k_m - \eta \wedge * d k_m). \quad (19)$$

In case $m = 0$, the terms in $\delta k_m, \delta \eta$ are of course zero; to get a term in $\eta(0)$ on the right when $\varepsilon \to 0$ we need $d k_m$ of the order of $\varepsilon^{1-n}$ and $k_m$ of the order of $\varepsilon^{2-n}$ in $|y| = \varepsilon$. That makes $k_m$ locally integrable too, and (12) is obtained letting $\varepsilon \to 0$. This means that for $m = 0$ $k$ is unique and is given by the well-known Green's function

$$A(r) = c_n \int_0^1 (1 - s^2)^{n-2} s^{1-n} ds \quad (20)$$

for an appropriate choice of $c_n$. In case $m > 0$, again we need $|k_m(0, y)| = o(r^{1-n})$ as $r \to 0$, so that the first and third terms on the right have limit 0 as $\varepsilon \to 0$; then $k_m$ is integrable in $y$ and the integral on the left converges to $\int \Delta \eta \wedge * k_m$. Using the expression for $* d k_m$ in (19), we find

$$\int_{|y| = \varepsilon} \eta \wedge * d y k_m = \frac{(-1)^m (n-m+1)!}{(n-m-1)!} A_3(\varepsilon) * x \int_{|y| = \varepsilon} \eta \wedge \alpha \wedge \gamma_{n-m-1}.$$

By Stoke's theorem, and since $\alpha = O(r)$, the last integral equals

$$(-1)^m \int_{|y| < \varepsilon} \eta \wedge \gamma_{n-m} + O(\varepsilon).$$

If $A_3(\varepsilon) = a_0 \varepsilon^{-n} + \ldots$, we see that

$$\lim_{\varepsilon \to 0} \int_{|y| = \varepsilon} \eta \wedge * d y k_m = c_n (n-m) m! a_0 \eta(0).$$

Using (15) for $\delta k_m = (-1)^{n(m+1)} * d * $, and proceeding in the same way,

$$\int_{|y| = \varepsilon} \delta_y k_m \wedge * \eta = -A_4(\varepsilon) \int_{|y| = \varepsilon} \alpha \wedge \gamma_{m-1} \wedge * \eta = -A_4(\varepsilon) \int_{|y| < \varepsilon} (\gamma_m \wedge * \eta + O(\varepsilon)).$$

But by the equation $B_1 = 0$, $A_4(\varepsilon) = -(1 - \varepsilon^2)^{n-2m}((n-m)A_3(\varepsilon) + \varepsilon A'_3(\varepsilon)) = a_0 m \varepsilon^{-n} + O(\varepsilon^{1-n})$, and hence the limit of the above expression is $-c_n m! a_0 m \eta(0)$. Altogether, we conclude that if $A_3(\varepsilon) = a_0 \varepsilon^{-n} + o(\varepsilon^{1-n})$ and $k_m(0, y) = o(r^{1-n})$, one has

$$\int \Delta \eta \wedge * y k_m(0, y) = -c_n m! a_0 \eta(0)$$
so (12) will hold for an appropriate choice of \( a_0 \). Taking into account the definition of \( A_3 \) in (14) and that \( |k_m| \simeq |A_1| + r^2|A_2| \), we see from (17) that if \( m = \frac{n}{2} \) this is accomplished by the choice \( c = 0, a = a_0 \); then \( G(x) \sim \log x, A_1(r) \sim \log r, A'_1(r) = O(1/r), A_2(r) = O(r^2) \) if \( n = 2 \); if \( n > 2 \), \( A_1(r) \sim r^{2-n} \) and \( A_2 = O(r^{-n}) \). For \( 0 < m < \frac{n}{2} \), in terms of the functions \( H, G \) introduced before, this translates to \( H(x) \sim a_0 x^{-\frac{2}{x}}, G(x) \sim x^{-\frac{2}{x}} \). Now look at the general expression of \( H, G \) in (18). The condition \( H(x) \sim c_0 x^{-\frac{2}{x}} \) fixes \( a = a_0 \); then near \( x = 0 \) \( c'(x) \) is bounded and \( d'(x) \) behaves like \( x^{-\frac{2}{x}} \). Since \( u_4(x) \) is bounded, the term \( d(x)u_4(x) \) behaves like \( x^{-\frac{2}{x}} \). So, we must normalize \( c(x) \) by \( c(0) = 0 \), so that \( c(x) = O(x) \) and the other term \( c(x)u_3(x) \) will behave like \( x^{-\frac{2}{x}} \).

In conclusion, all this discussion shows that the doubly invariant kernels \( k_m(x, y) \) satisfying (12) constitute a two parameter family described by \( H = a_0 u_1(x) + b u_2(x), c(0) = 0 \). The two parameters are \( b \) and the constant of integration for \( d(x) \) in (18). Equivalently, they are obtained by adding to the form corresponding to \( H = a_0 u_1(x), c(0) = 0 \) and say \( d(\frac{1}{2}) = 0 \) the general globally smooth one described before.

### 3.4.
In order to produce the best estimates, in a sense we need to choose the best of the kernels \( k_m \). Naturally enough, we choose the \( k_m \) having the best behaviour at infinity, \( x = 1 \), that is, so that \( G, H \) have the best decrease in size as \( x \to 1 \). In case \( m = \frac{n}{2} \), where we already have the normalization \( c = 0, a = a_0 \), the choice \( b = -a \) gives the best growth \( H(x) = O(1 - x) \) and \( G(x) = O(\log(1 - x)) \).

The hypergeometric function \( u_3 \) behaves like \( (1 - x)^{n+1-2m} \) near \( x = 1 \) while \( u_4(x) = F(m, m - \frac{n}{2}, 1 + \frac{n}{2}, x) \) is bounded because \( 1 + \frac{n}{2} - m - (m - \frac{n}{2}) = 1 + n - 2m > 0 \). Similarly, \( u_4 \) is bounded near \( x = 1 \); for \( u_2(x) = F\left(\frac{n}{2} - m, n - m, \frac{n}{2} + 1, x\right) \) we have \( \frac{n}{2} + 1 - (\frac{n}{2} - m) - (n - m) = 2m + 1 - n \) and hence it behaves like \( (1 - x)^{2m+1-n} \) if \( 2m < n - 1 \) and like \( \log(1 - x) \) if \( 2m = n - 1 \). We use equations (18)

\[
c(x) = c_{m,n} \int_0^x H(t)(1 + t)^{\frac{n}{2}}(1 - t)^{-3} u_4(t) dt
\]

\[
d(x) = -c_{m,n} \int_0^x H(t)(1 + t)^{\frac{n}{2}}(1 - t)^{-3} u_3(t) dt + d_0.
\]

If \( b \neq 0 \), then \( H(t) = a_0 u_1(t) + b u_2(t) \) behaves like \( (1 - t)^{2m+1-n} \) if \( 2m < n - 1 \) and like \( \log(1 - t) \) if \( 2m = n - 1 \), resulting in \( c(x) = O(1 - x)^{2m-n-1}, d(x) = O(\log(1 - x)) \) if \( 2m < n - 1 \) and \( c(x) = O((1 - x)^{-2}\log(1 - x)), d(x) = O((1 - x)^{-1}\log(1 - x)) \) if \( 2m = n - 1 \). So if \( b \neq 0 \) one has \( G(x) = O(\log(1 - x)) \) if \( 2m < n - 1 \) and \( G(x) = O((1 - x)^{-1}\log(1 - x)) \) if \( 2m = n - 1 \). If \( b = 0 \), then \( H \) is bounded, giving \( c(x) = O((1 - x)^{-2}) \) and \( d(x) = O(1) \) for \( 2m < n - 1 \), \( d(x) = O(\log(1 - x)) \) for \( 2m = n - 1 \). In case \( 2m < n - 1 \), however, we can choose the constant \( d_0 \) so that \( d(1) = 0 \), and then \( d(x) = O(1 - x)^{n-2m-1} \). This choice gives \( G(x) = O((1 - x)^{n-2m-1}) \) for \( 2m < n - 1 \). For \( 2m = n - 1 \), no choice of \( d_0 \) can improve the bound \( G(x) = O(\log(1 - x)) \).

It remains to estimate the growth of \( A_2(r) \) near \( r = 1 \). Recall that the definition (14) of \( A_3 \) translates to \( A_2(r) = 2G'(x) - (1 - x)^{n-2m-2}H(x) \). Both terms grow like \( (1 - x)^{n-2m-2} \), but a
cancellation occurs. The functions $u_1, u_3$ are $C^\infty$ at 1 and have developments

$$u_3(x) = A(1-x)^{n+1-2m} + O(1-x)^{n+2-2m}$$

$$u_3'(x) = -A(n+1-2m)(1-x)^{n-2m} + O(1-x)^{n+1-2m}$$

$$H(x) = a_0 u_1(x) = B + O(1-x).$$

In $u_4(x) = F(m, m - \frac{n}{2}, 1 + \frac{n}{2}, x), 1 + \frac{n}{2} - m - (m - \frac{n}{2}) = n + 1 - 2m \geq 2$, whence $u_4$ has a finite derivative at 1 and a development

$$u_4(x) = C + D(1-x) + O(1-x)^{1+\epsilon} \quad \forall \epsilon < 1, \quad u_4'(x) = O(1).$$

Then $W(x) = u_3'u_4 - u_3u_4' = CA(2m - n - 1)(1-x)^{n-2m} + \ldots$, and so the constant $c_{mn}$ in (18) is $CA(2m - n - 1)$. Then from (18)

$$c'(x) = \frac{B(1-x)^{-3}}{A(2m - n - 1)} + O(1-x)^{-2}$$

$$d'(x) = -\frac{B(1-x)^{n-2m-2}}{C(2m - n - 1)} + O(1-x)^{n-2m-1}$$

which gives

$$c(x) = \frac{1}{2} \frac{B}{2(2m - n - 1)}(1-x)^{-2} + O(1-x)^{-1}$$

$$d(x) = O(1-x)^{n-2m-1}, \quad 2m < n - 1, \quad O(\log(1-x)), \quad 2m = n - 1.$$ 

But $G' = c(x)u_3'(x) + d(x)u_4'(x); \quad$ the second term $d(x)u_4'(x)$ satisfies the required bound while the first $c(x)u_3'(x)$ has a development

$$c(x)u_3'(x) = \frac{1}{2} \frac{B}{A(2m - n - 1)}(n+1-2m)(1-x)^{n-2m-2} + O(1-x)^{n-2m-1}$$

$$= \frac{B}{2}(1-x)^{n-2m-2} + O(1-x)^{n-2m-1}.$$ 

As $(1-x)^{n-2m-2}H(x) = B(1-x)^{n-2m-2} + O(1-x)^{n-2m-1}$ the bound for $A_2$ follows for $2m \leq n - 1$. However, for $m = \frac{n}{2}$, this no longer holds. Indeed, from (17), where $c = 0, \quad a = a_0, \quad b - a)$

$$2G'(x) = x^{-\frac{n}{2}-1} \int_0^x s^{n/2}(1+s)(1-s)^{-3}a(s^{\frac{n}{2}} - 1) \, ds$$

has development

$$2G'(x) = na(1-x)^{-1} + O(\log(1-x))$$

while

$$(1-x)^{-2}a(x^{-\frac{n}{2}} - 1) = \frac{n}{2} a(1-x)^{-1} + \ldots$$

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We point out that all this can be obtained, in loose terms, working directly with the hypergeometric equations relating $G, H$

$$x(1-x)G''(x) + \left[\frac{n}{2} + 1 - (2m + 1 - \frac{n}{2})x\right]G'(x) - m(m - \frac{n}{2})G = \frac{1}{2}(1 + x)(1 - x)^{n-2m-2}H(x)$$

and using asymptotic developments. If $H(x) = h_0 + h_1(1 - x) + \ldots$ and $G(x) = g_j(1 - x)^j + \ldots$, identifying the lower order terms in both sides gives,

$$g_j(j - 1 - n + 2m)(1 - x)^{j-1} = h_0(1 - x)^{n-2m-2}.$$ 

When $H \equiv 0$, one must have either $j = 0$ (corresponding to $u_1$) or $j = n - 2m + 1$ (corresponding to $u_3$). For the inhomogeneous equation, if $j \neq 0$, $j \neq n + 1 + 2m$ (that is $G$ contains no contribution from $u_3, u_4$) one finds $j = n - 2m - 1$ if $2m < n - 1$ and $g_j = -\frac{h_0}{j}$. Then $2G''(x) = h_0(1 - x)^{n-2m-2} + \ldots, (1 - x)^{n-2m-2}H(x) = h_0(1 - x)^{n-2m-2}$, showing cancellation. An analogous argument works if $2m = n - 1$, but not for $2m = n$.

We summarize the results in this and the previous subsections:

**Theorem 3.1.** For $|n - 2m| > 1$, there is a unique doubly invariant kernel

$$k_m(x, y) = A_1(|\varphi_x y|)\gamma_m + A_2(|\varphi_x y|)\tau \wedge \gamma_{m-1}, m \neq 0; k_m(x, y) = A(|\varphi_x y|)\gamma_m, m = 0, n$$

for which (12) holds, and satisfying moreover

$$|A_i(r)| = O(1 - r^2)^{|n-2m|-1} \quad \text{as} \quad r \to 1.$$ 

For $m = \frac{n+1}{2}$, there is a one-parameter family of such kernels satisfying

$$|A_i(r)| = O(\log(1 - r^2)).$$ 

For $m = \frac{n}{2}$, there is a one-parameter family of such kernels satisfying

$$|A_i(r)| = O(1 - r^2)^{-1}.$$ 

In all cases $A_1(r) \sim r^{2-n}, A_2(r) \sim r^{-n}$ as $r \to 0$.

For $|n - 2m| > 1$, we call $k_m(x, y)$ the Riesz kernel for $m$-forms in $\mathbb{H}^n$, and

$$L\eta(x) = \int_{\mathbb{H}^n} \eta(y) \wedge *_y k_m(x, y)$$

the Riesz potential of $\eta$, whenever this is defined. From (14) we see that

$$|k_m(x, y)| = O(1 - r^2)^{n-m-1}. \quad (21)$$

With the notations used before, the function $A_3(r) = H(r^2)$ is bounded with bounded derivatives near $r = 1$. Then (14) and symmetry imply

$$|d_x k_m(x, y)|, |d_y k_m(x, y)| = O(1 - r^2)^{n-m-1} \quad (22)$$
too. The growth of $A_3$ also implies $A_4 = O(1 - r^2)^{n-2m}$ because $B_1 \equiv 0$, and then (15) gives as well
\[ |\delta_x k_m(x,y)|, |\delta_y k_m(x,y)| = O(1 - r^2)^{n-m-1}. \]  
(23)

By construction, one has $L \Delta \eta = \eta$ for $\eta \in D_m(\mathbb{H}^m)$. We will need the following generalization of this fact.

**Proposition 3.2.** If $\eta$ is a smooth form in $\mathbb{H}^n$ such that
\[ |\eta(y)|, |\nabla \eta(y)| = o(1 - |y|^2)^m, \quad y \in \mathbb{H}^n \]
then $L \Delta \eta = \eta$.

**Proof.** In (19) we would get an extra term
\[ \int_{|y|=R} (k_m \wedge * d\eta + \delta k_m \wedge * - \delta \eta \wedge * k_m - \eta \wedge * dk_m). \]
Estimates (21), (22) and (23) imply that with $x$ fixed and $|y| = R \gg 1$
\[ |k_m|, |\delta k_m|, |dk_m| = O(1 - R^2)^{n-m-1}. \]
Inserting $|\eta(y)|, |\nabla \eta(y)| = o(1 - |y|^2)^m$ we see that this extra term vanishes as $R \gg 1$.

\[ 4 \quad \text{Proof of the main theorem} \]

4.1. Once the Riesz form $k_m(x,y)$ has been found, our aim is now to prove that the corresponding convolution
\[ L_m \eta(x) = \int_{\mathbb{H}^n} \eta(y) \wedge * y k_m(x,y) \]
satisfies
\[ ||L_m \eta||_{p,s+2} \leq c||\eta||_{p,s} \]  
(24)
for $m \neq \frac{n+1}{2}, \frac{n}{2}$, and $p$ in the range $p_1(m) = \frac{n-1}{n-1-m} < p < \frac{n-1}{m} = p_2(m)$, and for a compactly supported $m$-form $\eta$ (recall that we are assuming without loss of generality that $m \leq \frac{n}{2}$). Since these are dense in the Sobolev spaces and we already know that $\Delta L_m \eta = L_m \Delta \eta = \eta$, this will prove the theorem for $m \neq \frac{n+1}{2}, \frac{n}{2}$. The case $m = \frac{n+1}{2}$ will be commented later.

We work in the translation invariant basis $w^I$. Taking into account formulas (9) and (10) for $\gamma, \tau$, the Riesz- form is written in the $\mathbb{R}_+^n$ model
\[ k_m(x,y) = \sum_{|I|=|J|=m} a_{I,J}(S_y x) w^I(x) \otimes w^J(y) \]
where each coefficient $a_{I,J}$ has an expression, with $z = S_y x$
\[ a_{I,J}(z) = \Psi_{I,J}(r) \frac{p_{I,J}(z)}{(|z|^2 + 2z_n + 1)^{2m}}, \quad r^2 = \frac{1 + |z|^2 - 2z_n}{1 + |z|^2 + 2z_n} = \frac{|x-y|^2}{|x-y|^2 + 4x_n y_n}. \]
Here \( p_{i,j}(z) \) is a certain polynomial in \( z_1, \ldots, z_n \), \( \Psi_{i,j} \) is \( C^\infty \) in \((0,1)\) with \( \Psi_{i,j}(r) \sim c_0 r^{2-n} \) as \( r \searrow 0 \), \( \Psi_{i,j}(r) = O(1 - r^2)^{n-m-1} \) as \( r \nearrow 1 \). The term \( q_{i,j}(z) = \frac{p_{i,j}(z)}{(|z|^2 + 2z_n + 1)^{2m}} \) is bounded.

If \( \eta = \sum_i \eta_i w^i(y) \), the coefficient \((L\eta)_I(x)\) of \( L\eta \) in the basis \( w^I \) is a finite linear combination of hyperbolic convolutions
\[
(L\eta)_I(x) = \sum_J \int_{\mathbb{R}^n} \Psi_{i,j}(r) q_{i,j}(z) \eta_J(y) \, d\mu(y).
\]
By ellipticity of \( \Delta \), \( L\eta \) is a smooth form. Moreover, since \( \eta \) has compact support, we see from (2) and (21) that, in the ball model,
\[
|L\eta(x)|, |d(L\eta)(x)|, |\delta(L\eta)(x)| = O(1 - |x|^2)^{n-m-1}
\]
which amounts to
\[
|(L\eta)_I(x)|, |X_i(L\eta)_I(x)| = O(1 - |x|^2)^{n-m-1}.
\] (25)
We claim that for second-order derivatives we have too
\[
|X_j X_i(L\eta)_I(x)| = O(1 - |x|^2)^{n-m-1}, \quad \text{i.e.} \quad |\nabla^{(2)}(L\eta)(x)| = O(1 - |x|^2)^{n-m-1}.
\] (26)
Notice that since we already know that \( \Delta L\eta = \eta \), from the expression of \( \Delta \) in the basis \( w^I \) given in (3) it follows that it is enough to show that for \( j < n \). We will see below (equation (30) and invariance of the \( X_i \) that each of the functions \( a(z) = \Psi_{i,j}(r) q_{i,j}(z) \) satisfies
\[
|X_j X_i a(z)| = O(1 - r^2)^{n-m-1}
\]
from which (26) follows as before. In fact, the discussion that follows will show that \( |\nabla^{(k)}(L\eta)(x)| = O(1 - |x|^2)^{n-m-1} \forall k \).

We continue the proof of (24). We claim first that it is enough to prove (21) for \( s = 0 \). For a smooth form \( \eta = \sum \eta_i w^I \) let \( X_i \eta \) denote here the \( m \)-form \( X_i \eta = \sum X_i \eta_i w^I \). It is clear from formulas (3) and the commutation properties
\[
[X_i, X_j] = 0, \quad i, j < n, \quad [X_1, X_i] = X_i, \quad i < n
\]
that for each \( i \) there is an operator \( P_i \) of order two in the \( X_1, \ldots, X_n \) such that
\[
X_i \Delta \eta - \Delta (X_i \eta) = P_i(X) \eta.
\]
Applying this to \( L\eta \), which is smooth by the ellipticity of \( \Delta \), we get \((X_i - \Delta X_i) L\eta = P_i(X) L\eta \). But \( X_i L\eta \) satisfies, by (25) and (26)
\[
|X_i L\eta(x)|, |d(X_i L\eta)(x)|, |\delta(X_i L\eta)(x)| = O(1 - |x|^2)^{n-m-1}
\]
and hence by Proposition 3.2, \( L\Delta = \text{Id} \) on it. We conclude that for all \( \eta \in D_m(\mathbb{H}^n) \)
\[
(LX_i - X_i L)\eta = LP_i(X) L\eta.
\]
Assume that (24) has been proved up to \( s \), so that by density it holds for \( \alpha \in H^s_{m,p}(\mathbb{H}^n) \) too, and let \( \gamma \) be a multiindex of length \( |\gamma| \leq s \). For \( i = 1, \ldots, n \) and \( \eta \in \mathcal{D}_m(\mathbb{H}^n) \),

\[
X^\gamma X_i L\eta = X^\gamma LX_i \eta - X^\gamma LP(X)L\eta
\]

so using twice the induction hypothesis

\[
||X^\gamma X_i L\eta||_p \leq \text{const} (||X_i \eta||_{p,s} + ||P_i(X)L\eta||_{p,s}) \leq \text{const} (||\eta||_{p,s+1} + ||\eta||_{p,s})
\]

proving (24) for \( s + 1 \). Proving (24) for \( s > 0 \) means proving

\[
||(L\eta)_I||_p, ||X_i(L\eta)_I||_p, ||X_jX_i(L\eta)_I||_p \leq \text{const} ||\eta||_p.
\]

As before, using that we already know that \( \Delta L\eta = \eta \) we see that for the second-order derivatives we may assume \( j < n \). In the following we delete the indexes \( I, J \) and denote by \( a(z) = \psi(r)Q(z) \) a convolution kernel with \( \psi, Q \) as above, and proceed to prove that the convolution

\[
(C_\alpha \alpha)(z) = \int_{\mathbb{H}^n} a(S_x \alpha(y)) d\mu(y)
\]

satisfies

\[
||C_\alpha \alpha||_{p_1}, ||X_i(C_\alpha \alpha)||_{p_1}, ||X_jX_i(C_\alpha \alpha)||_{p_1} \leq \text{const} ||\alpha||_{p_1}, \quad p_1 \leq p \leq p_2
\]  

(27)

where in the last case we may assume that \( j < n \). The fields \( X_i \) are invariant, and therefore \( X_iC_\alpha \alpha, X_jX_iC_\alpha \alpha \) are obtained, respectively, by convolution with \( Z_ja, Z_jZ_ia \) (by (11)). Recall that

\[
\psi(r) = O(1 - r^2)^{n-m-1} = O \left( \frac{4z_n}{1+|z|^2+2z_n} \right)^{n-m-1} \text{ as } r \nearrow 1 \text{ and } \psi(r) \sim r^{2-n} \text{ as } r \searrow 0.
\]

In order to estimate \( Z_ja, Z_jZ_ia \), we collect first some auxiliary estimates. We claim that

\[
|Z_iQ| \leq \text{const}, \quad |Z_iZ_iQ| \leq \text{const}
\]

\[
|Z_i r| \leq \text{const} (1 - r^2), \quad |Z_i Z_j r| \leq \text{const} r^{-1} (1 - r^2).
\]  

(28)

The first two are routinely checked, for instance, when differentiating the denominator in \( Q \),

\[
\left| \frac{1}{(1 + |z|^2 + 2z_n)^m} \right| = \left| \frac{4m z_n z_1}{(1 + |z|^2 + 2z_n)^{m+1}} \right| \leq \frac{\text{const}}{(1 + |z|^2 + 2z_n)^{2m}} \quad (i < n)
\]

so that the term \( p_{l,j}(z)Z_l \left[ (1 + |z|^2 + 2z_n)^{-m} \right] \) will still be bounded. All other terms can be treated similarly. Differentiating \( 1 - r^2 = \frac{4z_n}{1+|z|^2+2z_n} \) we get

\[
Z_i r = \frac{1 - r^2}{2} \frac{z_i z_n}{r(1 + |z|^2 + 2z_n)}, \quad Z_n r = -\frac{1 - r^2}{2} \frac{1 + |z|^2 - 2z_n^2}{r(1 + |z|^2 + 2z_n)}
\]

\[
Z_j Z_i r = \frac{1 - r^2}{2r} \left\{ \frac{\delta_{ij} z_n^2}{1 + |z|^2 + 2z_n} - \frac{1 + 5r^2}{2r^2} \frac{z_i z_j z_n^2}{(1 + |z|^2 + 2z_n)^2} \right\}, \quad i, j < n
\]

\[
Z_j Z_n r = \frac{1 - r^2}{r} \left\{ \frac{2z_n^2 z_j (1 + z_n)}{(1 + |z|^2 + 2z_n)^2} + \frac{(1 + r^2)}{4r^2} \frac{z_j z_n (1 + |z|^2 - 2z_n)}{(1 + |z|^2 + 2z_n)^2} \right\}, \quad j < n.
\]
These imply (28) because
\[ |z_i z_n|, 1 + |z|^2 - 2z_n^2 \leq (1 + |z|^2 - 2z_n)^{1/2} (1 + |z| + 2z_n)^{1/2} = r(1 + |z|^2 + 2z_n). \]
Now
\[ Z_i a(z) = \psi'(r) Z_i r Q(z) + \psi(r) (Z_i Q)(z) \]
\[ Z_j Z_i a(z) = \psi''(r) (Z_i r)(Z_j r) Q(z) + \psi'(r) (Z_j Z_i r) Q + \psi(r) Z_j r Z_i Q + \psi'(r) Z_j r Z_i Q. \]
The estimates (28) imply (29) because (27) will be proved for them. As \( b \) is \(-\)-admissible kernels, and so (27) will be proved for them. As \( Z_j Z_i a(z) \) is not an \( m \)-admissible kernel. Notice however from (29) that the last three terms \( \psi'(r) Z_j r Z_i Q, \psi'(r) Z_j r Z_i Q, \psi(r) Z_j Z_i Q \) are indeed \( m \)-admissible. Moreover, the estimate \( |Z_i Q| \leq \text{const} \) implies that \( Q \) is Lipschitz with respect the hyperbolic metric, in particular
\[ Q(z) = Q(e) + O(\log \frac{1+r}{1-r}) = Q(e) + O(r) \]
for small \( r \). This means that replacing \( Q \) by \( Q - Q(e) \) in the first two terms leads to an \( m \)-admissible kernel again. All this leaves us with the kernel
\[ \psi''(r) Z_i r Z_j r + \psi'(r) Z_j Z_i r, \quad j < n. \]
If \( \psi(r) = c_0 r^{2-n} + \ldots \), write \( \phi(r) = c_0 r^{2-n}(1 - r^2)^{n-m-1} \); then the above differs from
\[ \phi''(r) Z_i r Z_j r + \phi'(r) Z_j Z_i r \]
in an \( m \)-admissible kernel. By the same reason, we may replace \( \phi''(r), \phi'(r) \) respectively by
\( (r^{2-n})''(1 - r^2)^{n-m-1}, (r^{2-n})'(1 - r^2)^{n-m-1} \), that is to say we must deal with the convolution kernel
\[ (1 - r^2)^{n-m-1} Z_j Z_i (r^{2-n}). \]

We introduce a class of singular hyperbolic convolution kernels to deal with the later. For this purpose it is more convenient to work in the ball model, so now \( b \) is defined in \( B^n \) and \( r = |z| \). We
replace the integrable singularity \( r^{1-n} \) by a typical Calderón-Zygmund singularity (see e.g. [S]). Thus, we will call \( b \) a \( m \)-Calderón-Zygmund singular kernel if it has the form
\[
b(z) = \Omega(w)r^{n-1}r^{2n-m-1}, \quad z = rw, \quad w \in S^{n-1}
\]
where \( \Omega \) is some Lipschitz function on \( S^{n-1} \) satisfying the cancellation condition
\[
\int_{S^{n-1}} \Omega(w) \, d\sigma(w) = O. \tag{32}
\]
In Theorem 4.2. below we prove that \( m \)-Calderón-Zygmund singular kernels define bounded operators in the same range of \( p \). With the following proposition, applied to \( \phi_2(z) = |z|^{2-n} \) this will end the proof of the main result. The proposition is the analogue of the well-known statement that for \( \phi \) smooth and homogeneous of degree \( 1-n \) in \( \mathbb{R}^n \), \( \frac{\partial \phi}{\partial z_i} \) defines a Calderón-Zygmund kernel (it is homogeneous of degree \( -n \) and the cancellation condition (32) is automatically satisfied, because
\[
\int_{r_1 < |x| < r_2} \frac{\partial \phi}{\partial z_i} \, dV(x) = \left( \int_{|x| = r_2} - \int_{|x| = r_1} \right) \phi(x) \, dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n = 0.
\]

**Proposition 4.1.** If \( \phi_1, \phi_2 \) are homogeneous functions of degree \( 1-n, 2-n \) respectively, the kernels \( (1-r^2)^{n-m-1}Z_i\phi_1, (1-r^2)^{n-m-1}Z_j\phi_2 \) are sum of \((m-1)\)-admissible and \( (m-1) \)-Calderón-Zygmund singular kernels.

**Proof.** We replace the \( Z_j \) by \( Y_j = (1-r^2)\frac{\partial}{\partial z_j} \); we have
\[
Y_i(\phi_1) = (1-r^2)\frac{\partial \phi_1}{\partial z_i} \quad Y_i(\phi_2) = (1-r^2)\frac{\partial \phi_2}{\partial z_i} = (1-r^2)O(r^{1-n})
\]
\[
Y_jY_i(\phi_2) = (1-r^2)\frac{\partial^2 \phi_2}{\partial z_i \partial z_j} - 2(1-r^2)z_j \frac{\partial \phi_2}{\partial z_i} = (1-r^2)\frac{\partial^2 \phi_2}{\partial z_i \partial z_j} + (1-r^2)O(r^{2-n})
\]
so in all cases we get an extra factor \((1-r^2)\). Besides \( \frac{\partial \phi}{\partial z_i}, \frac{\partial^2 \phi}{\partial z_i \partial z_j} \) are, as noted before, homogeneous of degree \(-n\), and satisfy the cancellation condition (32).

4.2. It remains to prove

**Theorem 4.2.** Both \( m \)-admissible and \( m \)-Calderón-Zygmund kernels define by hyperbolic convolution bounded operators in \( L^p(\mathbb{H}^n) \) for \( \frac{n-1}{n-1-m} < p < \frac{n-1}{m}, \ 0 \leq m < \frac{n-1}{2} \).

We will make use of the following well-known Schur’s lemma for boundedness in \( L^p \) of an integral operator with positive kernel.

**Lemma 4.3.** If \( K(x,y) \) is a positive kernel in a measure space \( X \) and \( 1 < p < \infty \), the operator \( Kf(x) = \int_X K(x,y)f(y) \, d\mu(y) \) is bounded in \( L^p(\mu) \) if and only if there exists \( h \geq 0 \) such that
\[
\int_X K(x,y)h(y)^q \, d\mu(y) = O(h(x)^q), \quad x \in X \tag{33}
\]
\[
\int_X K(x,y)h(x)^p \, d\mu(x) = O(h(y)^p), \quad y \in Y. \tag{34}
\]
Here $q$ is the conjugate exponent of $p$, $\frac{1}{p} + \frac{1}{q} = 1$. If $h$ can be taken $\equiv 1$ that is

$$\sup_x \int_X K(x, y) \, d\mu(y), \sup_y \int_X K(x, y) \, d\mu(x) < +\infty$$

then $K$ is bounded in $L^p(\mu)$ for all $p$, $1 \leq p \leq \infty$.

Proof. Let us prove Theorem 4.2. If $b$ is $m$-admissible, $b = b_1 + b_2$ with $b_1(z) = O(r^{1-n})$ for $r \leq 1/2$, $b_1(z) = 0$ for $r > 1/2$, and $b_2(z) = O(1 - r^{n-m-1})$ for all $r$. We apply to $b_1$ the second criteria in Lemma 4.3 working in the ball model (recall that $|S_y X| = |\varphi_y x|$ is symmetric in $x, y$)

$$\int_X b_1(S_y x) \, d\mu(x), \int_X b_1(S_y x) \, d\mu(y) \leq c \int_{|S_y x| \leq 1/2} |S_y x|^{1-n} \, d\mu(x)$$

$$= c \int_{|z| \leq 1/2} |z|^{1-n} \, d\mu(z) = \text{const} \int_0^{1/2} \frac{dr}{(1-r^2)^n} < +\infty.$$

We apply to $(1 - r^2)^{n-m-1}$ the criteria of the first part on Lemma 4.3 working this time for convenience in the half-space model, where the kernel is written

$$K(x, y) = (1 - r^2)^{n-m-1} = \left( \frac{4z_n}{1 + |z|^2 + 2z_n} \right)^{n-m-1} = \left( \frac{4x_n y_n}{|x-y|^2 + 4x_n y_n} \right)^{n-m-1}.$$

We test $h(y) = y_n^\alpha$ in (43) for an exponent $\alpha$ to be chosen, so we need

$$\int_{y_n > 0} \frac{y_n^{-k+1+\alpha} \, dy}{(|x-y|^2 + 4x_n y_n)^{n-k-1}} = O \left( x_n^{\alpha q+k+1-n} \right).$$

We write $|x-y|^2 + 4x_n y_n = |x'-y'|^2 + (x_n+y_n)^2$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and analogously for $y'$, and integrate first in $y'$. One has for $2m < n - 1$

$$\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|x'-y'|^2 + (x_n+y_n)^2)^{n-m-1}} = c \int_0^\infty \frac{s^{n-2}}{(s^2 + (x_n+y_n)^2)^{n-m-1}}$$

$$= O((x_n+y_n)^{2m+1-n})$$

and so the above becomes

$$\int_0^\infty \frac{y_n^{\alpha q-m-1} \, dy_n}{(x_n+y_n)^{n-1-2m}} = O \left( x_n^{\alpha q+m+1-n} \right).$$

By homogeneity ($y_n = x_n t$) this reduces to

$$\int_0^\infty \frac{t^{\alpha q-m-1}}{(1+t)^{n-1-2m}} = O(1)$$

which holds whenever $m < \alpha q < n - 1 - m$. By symmetry, for (44) we need as well $m < \alpha p < n - 1 - m$. Therefore, a choice of $\alpha$ is possible whenever $m \max \left( \frac{1}{p}, \frac{1}{q} \right) < (n-1-m) \min \left( \frac{1}{p}, \frac{1}{q} \right)$ and this gives the range $\frac{n-1}{n-1-m} < p < \frac{n-1}{m}$. 

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Consider now a $m$-Calderón-Zygmund kernel $b(z) = \Omega(w)r^{-n}(1-r^2)^{n-m-1}$. Since $|S_gx| = |\varphi_{x,y}|$, we may replace $z = S_gx$ by $z = \varphi_{x,y}$. Using this is given by
\[
z = \frac{(x-y)(1-|x|^2) + x|x-y|^2}{A}
\]
where we use the notation $A = (1-|x|^2)(1-|y|^2) + |x-y|^2$; note that
\[
(1-|x|^2), (1-|y|^2) \lesssim A^{1/2}
\]
Also recall that $r = |z|$ and $Ar^2 = |x-y|^2$. Hence we can write
\[
\frac{z}{r} - \frac{x-y}{|x-y|} = \frac{x-y}{|x-y|} \left( \frac{1-|x|^2}{\sqrt{A}} - 1 \right) + x \cdot r.
\]
But
\[
\frac{1-|x|^2}{\sqrt{A}} - 1 = \frac{(1-|x|^2)^2 - A}{\sqrt{A}((1-|x|^2)^2 + \sqrt{A})} = \frac{(1-|x|^2)O(|x-y|) + O(|x-y|^2)}{A}
\]
is $O(r)$. Therefore, modulo an $m$-admissible kernel, we may replace $\Omega(w)$ by $\Omega \left( \frac{x-y}{|x-y|} \right)$. This leaves us with the kernel
\[
K = (1-r^2)^{n-m-1}\Omega \left( \frac{x-y}{|x-y|} \right) r^{-n} = (1-r^2)^{n-m-1}|x-y|^{-n} \Omega \left( \frac{x-y}{|x-y|} \right) A^\frac{n}{p}(x,y).
\]
Fix $p, 1 < p < \infty$. Write
\[
A^\frac{n}{p}(x,y) = (1-|x|^2)^\frac{n}{p}(1-|y|^2)^\frac{n}{q} + O \left( |x-y| A^{\frac{n-1}{p}} \right)
\]
Since $|x-y|^{1-n} A^{\frac{n-1}{p}} = r^{1-n}$, the kernel $K$ differs from
\[
(1-r^2)^{n-m-1}|x-y|^{-n} \Omega \left( \frac{x-y}{|x-y|} \right) (1-|x|^2)^\frac{n}{p}(1-|y|^2)^\frac{n}{q}
\]
in a $m$-admissible kernel, so we keep this one. We write it as the sum of
\[
|x-y|^{-n} \Omega \left( \frac{x-y}{|x-y|} \right) (1-|x|^2)^\frac{n}{p}(1-|y|^2)^\frac{n}{q} = K_1(x,y)
\]
and another $K_2(x,y)$ which we estimate by
\[
|K_2(x,y)| = O \left( r^2|x-y|^{-n}(1-|x|^2)^\frac{n}{p}(1-|y|^2)^\frac{n}{q} \right) = O \left( r^{2-n}(1-|x|^2)^\frac{n}{p}(1-|y|^2)^\frac{n}{q} A^{-\frac{n}{p}} \right) .
\]
Write $K_\Omega$ for the (euclidean) Calderón-Zygmund convolution operator with kernel $|x-y|^{-n} \Omega \left( \frac{x-y}{|x-y|} \right)$, which as it is well-known, satisfies an $L^p(dV)$-estimate. Notice that
\[
K_1f(x) = (1-|x|^2)^\frac{n}{p} K_\Omega \left( f(1-|y|^2)^{-\frac{n}{q}} \right)
\]
and therefore, using the $L^p$-boundedness of $K_\Omega$
\[
\int_{\mathbb{R}^n} |K_1f(x)|^p \, d\mu(x) = \int_{\mathbb{R}^n} \left| K_\Omega(f(1-|y|^2)^{-\frac{n}{q}}) \right|^p \, dV(x) \leq \int_{\mathbb{R}^n} |f(x)|^p \, d\mu(y) .
\]
For $K_2$, we can ignore the integrable singularity $r^{2-n}$ and arguing as we just did with $K_1$, we need to show that the integral operator

$$K_3 f(x) = \int_{|y| \leq 1} \frac{1}{(1 - |x| + |x - y|)^n} f(y) \, dV(y)$$

satisfies $L^p(dV)$-estimates for all $p, 1 < p < \infty$. To see this, just check that the criteria in Lemma 4.3 holds with $h(x) = (1 - |x|^2)^{-\frac{1}{m}}$.

Notice that in case $m = 0$ a $m$-Calderón-Zygmund kernel defines a bounded operator in all $L^p(\mathbb{H}^n)$, $1 < p < \infty$: this is the right analogue of the euclidian kernels, because $(1 - r^2)^{n-1}$ is the typical growth at infinity of a weak $L^1(d\mu)$ function in $\mathbb{H}^n$.

4.3. Finally we make some comments, with no proofs, on the critical case $m = \frac{n-1}{2}$ in the main Theorem. In this case the $m$-admissible and $m$-Calderón-Zygmund operators appearing in $X_jX_iC_au$, etc. have $(1 - r^2)^{\frac{n-1}{2}} \log \frac{1}{r}$ instead of $(1 - r^2)^{n-m-1} = (1 - r^2)^{\frac{n-1}{2}}$ as a factor. One can then prove that for $\beta > 0$ and $2 \leq p < 2 + \frac{2\beta}{(n-1)}$,

$$||L_\mu \eta||_{p,2} \leq \text{const} \int_{\mathbb{H}^n} |\eta|^p (1 - |y|^2)^{-\beta} \, d\mu(y).$$

The $L^p$-estimates do not hold in this case for any $p$, because they do not hold for $p = 2$ and $\Delta$ is self-adjoint.

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