DYON ELECTRIC CHARGE AND FERMION FRACTIONALIZATION
IN $N = 2$ GAUGE THEORY

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ABSTRACT

A first principles calculation of the quantum corrections to the electric charge of a
dyon in an $N = 2$ gauge theory with arbitrary gauge group is presented. These corrections
arise from the fermion fields via the mechanism of fermion fractionalization. For a dyon
whose magnetic charge is a non-simple co-root, the correction is a discontinuous function
on the moduli space of vacua and the discontinuities occur precisely on co-dimension one
curves on which the dyon decays. In this way, the complete spectrum of dyons at weak
coupling is found for a theory with any gauge group. It is shown how this spectrum is
consistent with the semi-classical monodromies.
1. Introduction

It has been known for a long time that magnetic monopoles can acquire electric charge. This happens in the presence of a CP violating theta term where the electric charge receives a contribution proportional to the magnetic charge [1]. When there are fermion fields in the theory, it has also been known for a long time that the electric charge of a monopole can receive additional contributions from these fields at the quantum level due to a phenomenon known as fermion fractionalization (see [2,3] and the review article [4]). At a microscopic level, the Dirac vacuum of the fermion sector in the background of a monopole has a non-trivial fermion number and since the fermion carry electric charge, the vacuum has a non-trivial electric charge which expresses itself in a contribution to the electric charge of the monopole. The fermions in effect generate an effective theta angle. The effect is non-trivial because the induced charge depends on the coordinates of the moduli space of vacua. We shall calculate this effect in an $N = 2$ supersymmetric gauge theory with arbitrary gauge group. The contribution to the electric charge has the remarkable feature that for certain magnetic charges it is discontinuous across the moduli space of vacua. This is shown to be consistent because the discontinuities occur precisely on subspaces on which the dyons decay and explains the picture of [5] which concluded that the moduli space of vacua is divided into a number cells separated by walls on which dyons decay. It is important to calculate the allowed dyon electric charges because the behaviour of the theory at strong coupling is governed by the regimes where the dyons become massless [6]. The theory of Seiberg and Witten generalized to arbitrary gauge groups ([7,8,9] for SU($n$), [10] for SO(2$n$), [11] for SO(2$n$ + 1), [12] for Sp($n$), [13] for $G_2$ and [14] for all the other exceptional cases) makes definite predictions for the electric charges of the light dyons at strong coupling, hence for the overall consistency of the theory these charges should match those at weak coupling. (This assumes that there are paths from weak coupling to the dyon singularities on which the dyons do not decay—as in the SU(2) theory [6].)

The mass of a BPS state in an $N = 2$ supersymmetric gauge theory with gauge group $G$ and Lie algebra $g$ is given as [6]

$$M = |g \cdot a_D(u_j) + q \cdot a(u_j)| = |Q \cdot A^T|,$$

where $A = (a_D(u_j), a(u_j))$ are two $r = \text{rank}(g)$ vector functions of the gauge invariant coordinates $u_j$, $j = 1, \ldots, r$, of the moduli space of vacua and $Q = (g, q)$ encodes the magnetic and electric charges of the state with respect to the unbroken U(1)$^r$ symmetry: the exact relationship requires careful explanation due to the presence of a theta term and
quantum corrections. The moduli space of vacua has certain generic features. Firstly there are singularities of co-dimension two on which a certain BPS state is massless defined by

$$Q \cdot A^T = 0.$$ \hspace{1cm} (1.2)

At these points the U(1)^r low energy effective action description breaks down and needs to be supplemented with fields for the massless states. The singularities form the boundaries of submanifolds of co-dimension one on which dyons are kinematically at threshold for decay. If the singularity corresponds to Q becoming massless then this curve, or Curve of Marginal Stability (CMS), allows processes of the form $Q_1 \rightarrow Q_2 \pm Q$ [6,15]. This CMS is defined by the condition:

$$\arg (Q \cdot A^T) = \arg (Q_2 \cdot A^T).$$ \hspace{1cm} (1.3)

The functions $a_D(u_j)$ and $a(u_j)$ are not single-valued functions around a singularity. This implies that associated to each singularity is a cut—a co-dimension one surface whose boundary is also the singularity. The positions of these cuts is otherwise arbitrary. Across the cut $A$ is transformed by a monodromy transformation $A^T \rightarrow MA^T$. This implies that the charges of BPS states are transformed $Q \rightarrow QM^\pm 1$ (where $M$ acts by matrix multiplication to the left) as one crosses the cut in either of the two directions. If a state $Q_1$ is in the spectrum at a particular point then $Q_1M^\pm 1$ will be as well, unless the path around the singularity passes through a CMS on which the dyon decays.

2. The moduli space and BPS states at weak coupling

At weak coupling the moduli space has a very simple description since it is parameterized by the classical Higgs VEV. Up to global gauge transformations we can take the VEV $\Phi_0$ to be in the Cartan subalgebra of $g$. So $\Phi_0 = a \cdot H$ which defines the $r$-dimensional complex vector $a$. However, this does not completely fix the gauge symmetry since it leaves the freedom to perform discrete gauge transformations in the Weyl group of $G$. This then defines the classical moduli space of vacua:

$$\mathcal{M}_{\text{vac}}^\text{cl} \simeq \mathbb{C}^r / W(G),$$ \hspace{1cm} (2.1)

where $W(G)$ is the Weyl group of $G$. More concretely, we can take $\mathcal{M}_{\text{vac}}^\text{cl}$ to be the region for which $\text{Re}(a)$ is in the fundamental Weyl chamber with respect to some choice of simple roots $\alpha_i$:

$$\text{Re}(a) \cdot \alpha_i \geq 0, \hspace{0.5cm} i = 1, 2, \ldots, r.$$ \hspace{1cm} (2.2)

This also defines a notion of a positive or negative root and we write $\Phi = \Phi^+ \cup \Phi^-$, where $\Phi$ is the root system of the Lie algebra $g$ and $\Phi^\pm$ are the set of positive and negative roots.
roots. Notice that the wall $\text{Re}(\mathbf{a}) \cdot \alpha_i = 0$ is identified with itself by $\sigma_i$, the Weyl reflection in $\alpha_i$. As long as $\alpha_i \cdot \mathbf{a} \neq 0$, for some $i$, the unbroken gauge group is abelian $U(1)^r$ generated by the Cartan subalgebra $\mathbf{H}$. Classically, therefore, the submanifolds where $\alpha_i \cdot \mathbf{a} = 0$ correspond to points where additional gauge bosons become massless. In the quantum theory these points correspond to regions where the theory becomes strongly coupled and perturbation theory breaks down. In the neighbourhood of these singularities the classical moduli space is no longer an appropriate description. As long as we avoid the neighbourhoods of these singularities the theory is weakly coupled and semi-classical methods are applicable. As an example, Fig. 1 shows the classical moduli space for $\text{SU}(3)$ where one of the imaginary directions for $\mathbf{a}$ is suppressed. The figure shows the singularities and the cuts as the surfaces identified by the broken arrowed lines.

![Diagram showing the classical moduli space for SU(3)](image-url)

Figure 1. Classical moduli space for SU(3)

Consider the classical theory of monopoles and dyons in an $N = 2$ supersymmetric theory. The mass of a BPS state can be expressed as

$$M = \frac{1}{e} |Q_E + iQ_M|, \quad (2.3)$$

where $Q_E$ and $Q_M$ are given by integrals over a large sphere at infinity of the inner product in the Lie algebra $g$ of the electric and magnetic fields with the Higgs field:

$$Q_E = \int d\vec{S} \cdot \text{Tr}(\vec{E}\Phi), \quad Q_M = \int d\vec{S} \cdot \text{Tr}(\vec{B}\Phi). \quad (2.4)$$
In unitary gauge all the fields outside the core of dyon solution are valued in the Cartan subalgebra of $g$. This allows us to define the vector electric and magnetic charges of the solution:

$$q_{\text{phys}} \cdot H = \frac{1}{e} \int d\vec{S} \cdot \vec{E}, \quad g \cdot H = \frac{e}{4\pi} \int d\vec{S} \cdot \vec{B}. \quad (2.5)$$

In the above, $q_{\text{phys}}$ is the physical vector electric charge not to be confused with $q$ introduced below. The constants above are included for convenience. Since on the sphere at spatial infinity the Higgs field is equal to $\Phi_0 = a \cdot H$ we have

$$Q_E = e q_{\text{phys}} \cdot a, \quad Q_M = \frac{4\pi}{e} g \cdot a. \quad (2.6)$$

The allowed electric and magnetic charges are quantized in the following way. The magnetic charge vector $g$ has to be a positive co-root of $g$, i.e. a vector of the form:

$$g = \sum_{i=1}^{r} n_i \alpha_i^*, \quad (2.7)$$

where $n_i \in \mathbb{Z} \geq 0$ and the co-root is defined as $\alpha^* = \alpha / \alpha^2$. The allowed electric charges of the dyon are determined by the generalized Dirac-Zwanziger-Schwinger (DZS) quantization condition that requires

$$q \equiv q_{\text{phys}} - \frac{\theta e}{2\pi} g \in \Lambda_R. \quad (2.8)$$

In the above $q$ is not the physical electric charge, rather it is the vector of eigenvalues of the Noether charges corresponding to global gauge transformations in the unbroken $U(1)^r$ global symmetry group. It is well-known that in the presence of a theta term this is not equivalent to the physical electric charge [1].

Putting this together we see that we can label a BPS configuration with the two vectors

$$Q = (g, q) \in (\Lambda_R^*, \Lambda_R), \quad (2.9)$$

and the mass of the state may be written

$$M = |Q \cdot A^T|, \quad (2.10)$$

where $a_D = \tau a$ with

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \quad (2.11)$$

Single monopole solutions can be found by embeddings of the SU(2) monopole solution associated to a regular embedding of the Lie algebra $su(2)$ in the Lie algebra of the gauge

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1 The quantization condition actually allows $q \in \Lambda_W$, the weight lattice, however since in the pure gauge theory all the fields are adjoint valued only the subset $\Lambda_R \subset \Lambda_W$ is actually realized.
group associated to a positive root \( \alpha \) and defined by the three generators \( E_{\pm \alpha} \) and \( \alpha^\ast \cdot H \). We denote by \( \phi^\alpha \) and \( \vec{A}^\alpha \) the SU(2) monopole solution (in the \( A_0 = 0 \) gauge) embedded in the theory with a larger gauge group using this \( su(2) \) Lie subalgebra. The ansatz for the monopole solution is \cite{16,17}

\[
\Phi = e^{i\theta_{\alpha} (\phi^\alpha + \hat{a})}, \quad \vec{A} = \vec{A}^\alpha,
\]

where the phase angle \( \theta_{\alpha} = \arg(a \cdot \alpha) \) and

\[
\hat{a} = e^{-i\theta_{\alpha}} (a - (a \cdot \alpha^\ast)\alpha) \cdot H.
\]

The VEV of the SU(2) Higgs field \( \phi^\alpha \), is fixed to be \( |a \cdot \alpha| \), so that \( \Phi \) has the correct VEV. The solution has magnetic charge vector \( g = \alpha^\ast \). This ansatz is a direct generalization of the ansatz for theories with a single real Higgs field in \cite{18,19}.

To describe a dyon at leading order in the semi-classical approximation we note that a monopole solution generically admits four zero modes corresponding the position of the centre-of-mass and a periodic U(1) charge angle. In the leading order semi-classical analysis, momentum in the periodic direction corresponds to giving the monopole electric charge along \( \alpha \). The dyon receives a contribution to the electric charge from this momentum \( \eta \) and from the theta term \cite{1}:

\[
q_{\text{phys}} = \frac{\theta e}{2\pi} \alpha^\ast + \eta \alpha, \quad \text{i.e.} \ q = \eta \alpha.
\]

We now demand that the semi-classical wavefunction is invariant under gauge transformations in the centre of the gauge group. This is equivalent to the DZS quantization condition (2.8) which requires that \( \eta \in \mathbb{Z} \).

Notice that at leading order in the semi-classical approximation dyons have \( g \propto q \). This means that at leading order it is straightforward to find the CMS for a particular dyon. A dyon whose magnetic charge \( \alpha^\ast \) is a non-simple co-root can decay into a number of dyons of magnetic charge \( \gamma^\ast \) and \( \delta^\ast \), where \( \gamma \) and \( \delta \) are two positive roots of \( g \). The CMS for this process, which we denote \( C_{\gamma, \delta} \), is the submanifold given by (1.3). Defining \( \theta_{\alpha} = \arg(a \cdot \alpha) \) it is defined by

\[
\theta_{\gamma} = \theta_{\delta}.
\]

This surface, along with the walls of the complex Weyl chamber, divides the classical moduli space into two cells which we denote \( D_{\epsilon, \gamma, \delta} \) with

\[
\epsilon = \text{sign} (\theta_{\gamma} - \theta_{\delta}).
\]

We will find later that quantum effects can give corrections to \( q \) which are not proportional to \( g \). These corrections lead to corrections of the positions of the CMS at higher order, an effect that do not analyse here.
For a dyon of given magnetic charge $\alpha^*$ we have the following general description. First of all we need to introduce a set of rank 2 subspaces of the root system $\Phi$ of $g$. These subspaces of roots form the set of roots of either an $A_2$, $B_2$ or $G_2$ subalgebra.\footnote{The latter only occurs in $G_2$ itself, in which case the subspace is full root space.} Let $\gamma_i$ and $\delta_i$ be the two positive roots which are the simple roots of the $i$th rank 2 subalgebra. Any positive root of the original algebra in the subspace may be expanded in terms of $\gamma_i$ and $\delta_i$ with non-negative coefficients. We choose $\delta_i$ to be the shorter of the two roots: $\delta_i^2 \leq \gamma_i^2$.

For each of the $L$ rank 2 subspace for which $\alpha$ is a non-simple root, i.e.

$$\alpha = n_i \gamma_i + m_i \delta_i, \quad i = 1, \ldots, L,$$

(2.17)

for two integers $n_i, m_i \geq 1$, there is a CMS $C_{\gamma_i, \delta_i}$ and two cells $D_{\gamma_i, \delta_i}^{\epsilon_i}$. So for these dyons $M_{\text{vac}}^\text{cl}$ is divided into $2^L$ cells given by the following intersections:

$$D_{\gamma_1, \delta_1}^{\epsilon_1} \cap D_{\gamma_2, \delta_2}^{\epsilon_2} \cap \cdots \cap D_{\gamma_{\ell}, \delta_{\ell}}^{\epsilon_{\ell}},$$

(2.18)

where $\epsilon_i = \text{sign}(\theta_{\gamma_i} - \theta_{\delta_i})$. For example, for the case of SU(3), dyons with magnetic charge $\alpha^*_3 = \alpha^*_1 + \alpha^*_2$ have a CMS $C_{\alpha^*_2, \alpha^*_1}$ indicated by the hatched surface in Fig. 1. This divides the moduli space into two cells $D^\pm = D_{\alpha^*_2, \alpha^*_1}^{\pm 1}$. Notice that the CMS passes through the two singularities $a \cdot \alpha_1 = 0$ and $a \cdot \alpha_2 = 0$, as expected.

### 3. Quantum corrections

In the quantum theory, we shall have to re-assess the DZS quantization condition due to the presence of fermions. First of all, the electric and magnetic charges in (2.6) receive quantum corrections due to the renormalization of the gauge coupling and the generation of effective theta terms for the unbroken $\text{U}(1)^r$ symmetry. These effects can be read off from the low energy effective action. To one-loop \cite{7,8,9,10,11}:

$$a_D = i \frac{\bar{\tau}}{\pi} \sum_{\beta \in \Phi} \beta (\beta \cdot a) \ln \left( \frac{\beta \cdot a}{\Lambda} \right).$$

(3.1)

Writing $a_D = \bar{\tau} a$, where

$$\bar{\tau} = \frac{\bar{\theta}}{2\pi} + \frac{4\pi i}{e^2},$$

(3.2)

is a matrix quantity, we find

$$\bar{\theta} = -2 \sum_{\beta \in \Phi} (\beta \otimes \beta) \arg (\beta \cdot a).$$

(3.3)
The above calculation was done with zero bare theta parameter which contributes simply a constant term to (3.3). This contribution can always be set to zero by performing a U(1)$_R$ transformation which acts on the Higgs VEV as $a \rightarrow e^{i\epsilon}a$.

The form of (3.3) shows how quantum corrections generate theta-like terms in the effective action which depend non-trivially on the moduli space coordinate. We can now read-off the relation between the physical electric charge and the quantum number $q$:

$$q_{\text{phys}} = q - \frac{1}{\pi} \sum_{\beta \in \Phi} \beta (\beta \cdot g) \arg (\beta \cdot a), \quad (3.4)$$

The important point is that the DZS quantization condition still requires $q \in \Lambda_R$. In order to find the quantum numbers $q$ for the dyons in the theory we must calculate the physical electric charge $q_{\text{phys}}$ of the dyons to one-loop and then extract $q$ from (3.4).

These one-loop corrections to the electric charge of a dyon are due to the fermion fields in the theory [2,3,4]. The point is that in the background of a topologically charged soliton, the Dirac vacuum of the fermion field can have a non-trivial fermion number, a phenomenon known as fermion fractionalization. This was originally discovered by Jackiw and Rebbi [20] who showed that fermion number in the background of topological soliton could be half-integer. A generalization to arbitrary, irrational values was obtained by Goldstone and Wilczek [21]. In the present context, the result of this phenomenon is that the fermion number depends non-trivially on the moduli space coordinate, as recognized in the context of the SU(2) by Ferrari [22]. Since the fermions carry electric charge this means that the vacuum also carries a non-trivial electric charge. In the present situation this means that the fermion fields can contribute to the electric charge of a dyon. The electric charge of a dyon (2.14) is modified to

$$q_{\text{phys}} = e\theta \frac{\alpha^*}{2\pi} \eta \alpha + q_f. \quad (3.5)$$

This changes the quantization rule for $\eta$.

In the $N = 2$ pure gauge theory there is a single Dirac fermion which transforms in the adjoint representation of the gauge group. The interaction of the fermion field with the dyon is described by the Dirac equation

$$[i\gamma^m D_m - \text{Re}(\Phi) + i\gamma^5 \text{Im}(\Phi)] \Psi = 0. \quad (3.6)$$

The Dirac field is adjoint valued and can be expanded in a Cartan-Weyl basis. We show in the Appendix, in the sector associated to the root $\beta$, the vacuum has a fermion number

$$N_\beta = \frac{\beta \cdot \alpha^*}{\pi} \arctan \left( \frac{\text{Re} \left( e^{-i\theta a} a \cdot \beta \right)}{\text{Im} \left( e^{-i\theta a} a \cdot \beta \right)} \right), \quad (3.7)$$
where in the above \( \arctan(x) \) is defined over its principal range between \(-\pi/2\) and \(\pi/2\).

Since the component carries an electric charge \( e\beta \) the total electric charge of the Dirac vacuum in the presence of the monopole is

\[
e \mathbf{q}_f = e \sum_{\beta \in \Phi} \beta \mathbf{N}_\beta = \frac{2e}{\pi} \sum_{\beta \in \Phi^+} \beta (\alpha^* \cdot \beta) \arctan \left( \frac{\Re \left( e^{-i\theta_\alpha} \mathbf{a} \cdot \beta \right)}{\Im \left( e^{-i\theta_\alpha} \mathbf{a} \cdot \beta \right)} \right),
\]

where we have used the fact that \( N_{-\beta} = -N_{\beta} \). Notice that the result (3.8) reproduces precisely the functional dependence on \( \mathbf{a} \) of the effective theta parameter (3.3). By comparing (3.8) with (3.4) we can extract the value of the quantum number \( q \):

\[
q = \eta' \alpha + \sum_{\beta \in \Phi^+} \beta (\alpha^* \cdot \beta) \text{sign} (\theta_\beta - \theta_\alpha),
\]

where we have used the fact that \( \sum_{\beta \in \Phi^+} (\alpha^* \cdot \beta) \beta \propto \alpha \) and have absorbed a contribution along \( \alpha \) into a new constant \( \eta' \). In the above, for a positive root \( \beta \) the phase angle \( \theta_\beta \) is constrained by (2.2) to be \(-\pi/2 \leq \theta_\beta \leq \pi/2\). Notice immediately that \( q \) changes discontinuously whenever the surfaces where \( \theta_\beta = \theta_\alpha \) are crossed. These are precisely the CMS which along with the cuts \( \alpha_i \cdot \mathbf{a} = 0 \) separate \( \mathcal{M}_{\text{vac}}^{cl} \) into cells where the dyons have different allowed charges. In fact the result (3.7) and hence (3.9) is only valid if \( \theta_\beta \neq \theta_\alpha \), for some \( \beta \), otherwise the Dirac equation has additional zero modes signaling the fact that, as described in section 5, the dyon will decay.

In order to determine \( q \), consider the contribution from a pair of roots \( \beta \) and \( \sigma_\alpha(\beta) = \beta - 2(\alpha^* \cdot \beta) \alpha \), the Weyl reflection of \( \beta \) in \( \alpha \). If \( \sigma_\alpha(\beta) \in \Phi^+ \) then since \( \text{sign} (\theta_{\sigma_\alpha(\beta)} - \theta_\alpha) = \text{sign} (\theta_\beta - \theta_\alpha) \) and \( \beta \cdot \alpha = -\sigma_\alpha(\beta) \cdot \alpha \), the contribution from the pair of roots \( \beta \) and \( \sigma_\alpha(\beta) \) is proportional to \( \alpha \). Therefore the only contributions to \( q \) which are not proportional to \( \alpha \) arise when \( \sigma_\alpha(\beta) \in \Phi^- \). Accordingly we have

\[
q = \eta'' \alpha + \sum_{\beta \in \Delta} \beta (\alpha^* \cdot \beta) \text{sign} (\theta_\beta - \theta_\alpha),
\]

where we have absorbed an additional contribution along \( \alpha \) into a new constant \( \eta'' \). The sum is over the set

\[
\Delta = \{ \beta \in \Phi^+ \mid \beta \neq \alpha, \sigma_\alpha(\beta) \in \Phi^- \}.
\]

The generalized Dirac quantization condition (2.8) requires that \( q \) be a root of the Lie algebra and hence determines \( \eta'' \) up to an integer.

At this point, for purposes of illustration, consider the case of gauge group SU(3). For dyons with a magnetic charge vector that is a simple root, \( \alpha_1^* \) or \( \alpha_2^* \), there are no vectors in the set \( \Delta \); for instance \( \sigma_1(\alpha_2) = \alpha_3 = \alpha_1 + \alpha_2 \in \Phi^+ \). Hence in these cases \( q = n\alpha_1 \) and \( n\alpha_2, n \in \mathbb{Z} \), respectively. For a magnetic charge \( \alpha_3^* = \alpha_1^* + \alpha_2^* \) there are two roots in \( \Delta \).
Firstly $\alpha_1$, with $\sigma_3(\alpha_1) = -\alpha_2 \in \Phi^-$, and secondly $\alpha_2$, with $\sigma_3(\alpha_2) = -\alpha_1 \in \Phi^-$. In this case $\mathcal{M}^{\text{vac}}_{\text{cl}}$ is divided into two regions by the CMS $C_{\alpha_2, \alpha_1}$. In the first cell $D^+$, $\theta_{\alpha_2} > \theta_{\alpha_1}$, and so by (3.10) in this region the $\alpha^*_3$ dyon has $q = \eta'_{\alpha_2 - \alpha_3 - \alpha_1}$, where $\eta' = n - 1/2$ and the Dirac quantization condition (2.8) implies that $n \in \mathbb{Z}$. In the other cell $D^-$, $\theta_{\alpha_2} < \theta_{\alpha_1}$, $q = \eta''_{\alpha_2 - \alpha_3 + \alpha_1}$, where $\eta'' = n + 1/2$ with $n \in \mathbb{Z}$. So the discontinuity either side of the CMS is $\mp \alpha_1$. This is precisely the spectrum of dyon derived in [5] using the semi-classical monodromies.

Now consider the general case. It is useful to analyse the set $\Delta$ in terms of the set of $L$ rank 2 root subspaces introduced at the end of section 3. For each such subspace $\alpha$ is a non-simple root, i.e.

$$\alpha = n_i \gamma_i + m_i \delta_i, \quad n_i, m_i \in \mathbb{Z} \geq 1,$$

(3.12)

and there are contributions to the set $\Delta$. (Weyl reflections of positive roots in simple roots are always positive roots again, so there are no contributions to $\Delta$ when $\alpha = \gamma_i$ or $\delta_i$.) Let $\beta$ be some positive root in the subspace so that

$$\beta = c_i \gamma_i + d_i \delta_i, \quad c_i, d_i \in \mathbb{Z} \geq 0.$$

(3.13)

From (3.12) and (3.13) one can show that

$$\text{sign} (\theta_{\beta} - \theta_{\alpha}) = \text{sign} (m_i c_i - n_i d_i) \text{sign} (\theta_{\gamma_i} - \theta_{\delta_i}),$$

(3.14)

and therefore the discontinuity in the charge at $\theta_{\beta} = \theta_{\alpha}$ occurs precisely on the CMS $C_{\gamma_i, \delta_i}$. So for each rank 2 root subspace for which (3.12) is true, there exist contributions $\Delta_i \subset \Delta$ giving a discontinuity in $q$ across $C_{\gamma_i, \delta_i}$ which follows from (3.10). The actual discontinuity can be found explicitly on a case-by-case basis for the three rank 2 algebras. The are 7 different cases (1 in $A_2$, 2 in $B_2$ and 4 in $G_2$) to consider as indicated in Fig. 2. The contributions to $q$ from these subspaces are calculated below using (3.10), up to a contribution along $\alpha$ which will be fixed by the DZS quantization condition.

(a) $A_2$ with $\alpha = \gamma_i + \delta_i$ and $\alpha^* = \gamma^*_i + \delta^*_i$. The set $\Delta_i = \{\gamma_i, \delta_i\}$. In the cells $D^\pm_{\gamma_i, \delta_i}$ we find that the contribution to $q$ is $\mp \delta_i$.

(b) $B_2$ with $\alpha = \gamma_i + 2\delta_i$ and $\alpha^* = \gamma^*_i + \delta^*_i$. The set $\Delta_i = \{\delta_i, \gamma_i + \delta_i\}$. In the cells $D^\pm_{\gamma_i, \delta_i}$ we find that the contribution to $q$ is $\mp \delta_i$.

(c) $B_2$ with $\alpha = \gamma_i + \delta_i$ and $\alpha^* = 2\gamma^*_i + \delta^*_i$.

The set $\Delta_i = \{\gamma_i, \gamma_i + 2\delta_i\}$. In the cells $D^\pm_{\gamma_i, \delta_i}$ we find that the contribution to $q$ is $\mp 2\delta_i$.

(d) $G_2$ with $\alpha = \gamma_i + 3\delta_i$ and $\alpha^* = \gamma^*_i + \delta^*_i$. The set $\Delta_i = \{\gamma_i + 2\delta_i, \delta_i\}$. In the cells $D^\pm_{\gamma_i, \delta_i}$ we find that the contribution to $q$ is $\mp \delta_i$. 

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Figure 2. Dynkin diagrams of the rank 2 root subspaces

\( G_2 \) with \( \alpha = 2\gamma_i + 3\delta_i \) and \( \alpha^* = 2\gamma_i^* + 3\delta_i^* \). The set \( \Delta_i = \{ \gamma_i, \gamma_i + \delta_i, \gamma_i + 2\delta_i, \gamma_i + 3\delta_i \} \). In the cells \( D_{\gamma_i, \delta_i}^{\pm 1} \) we find that the contribution to \( q \) is \( \mp 2\delta_i \).

\( G_2 \) with \( \alpha = \gamma_i + \delta_i \) and \( \alpha^* = 3\gamma_i^* + \delta_i^* \). The set \( \Delta_i = \{ \gamma_i, 2\gamma_i + 3\delta_i \} \). In the cells \( D_{\gamma_i, \delta_i}^{\pm 1} \) we find that the contribution to \( q \) is \( \mp 3\delta_i \).

\( G_2 \) with \( \alpha = \gamma_i + 2\delta_i \) and \( \alpha^* = 3\gamma_i^* + 2\delta_i^* \). The set \( \Delta_i = \{ \gamma_i + \delta_i, \gamma_i + 3\delta_i, 2\gamma_i + 3\delta_i, \delta_i \} \). In the cells \( D_{\gamma_i, \delta_i}^{\pm 1} \) we find that the contribution to \( q \) is \( \mp 4\delta_i \).

These results can be summarized by a single formula for all cases. In the cells \( D_{\gamma_i, \delta_i}^{\pm 1} \) on either side of the CMS \( C_{\gamma_i, \delta_i} \) where

\[
\alpha = n_i \gamma_i + m_i \delta_i, \quad \alpha^* = N_i \gamma_i^* + M_i \delta_i^*,
\]

for \( n_i, m_i, N_i, M_i \in \mathbb{Z} \geq 1 \) and \( N_i = n_i \gamma_i^2 / \alpha_i^2 \) and \( M_i = m_i \delta_i^2 / \alpha_i^2 \), where \( \delta_i \) is the shorter of the two roots \( \gamma_i \) and \( \delta_i \), the contribution to \( q \) is \( \mp (N_i + M_i - 1)\delta_i \). From this we are led to the following general description of the spectrum of dyons with magnetic charge \( \alpha^* \). In each of the cells (2.18) the dyons have

\[
q = n \alpha - \sum_{i=1}^{L} (N_i + M_i - 1) \text{sign} (\theta_{\gamma_i} - \theta_{\delta_i}) \delta_i, \quad n \in \mathbb{Z}.
\]

It follows immediately that the dyons whose magnetic charge is a simple co-root \( \alpha_i^* \) have \( q = n \alpha_i \).
4. Comparison with the semi-classical monodromies

We have calculated the electric charge quantum by considering the fermion fractionalization in the background of a monopole. These quantum numbers can be calculated independently by following the quantum numbers of BPS states as one moves around the classical singularities $\alpha_i \cdot a = 0$ in moduli space [5]. The point is that the function $a_D$ is not single-valued as one moves around a singularity due to the branch-cut of the logarithms from the one-loop contribution in (3.1). A point with coordinate $a$ on the wall containing the singularity $\alpha_i \cdot a = 0$, i.e. $\text{Re}(\alpha_i \cdot a) = 0$, is identified with a point $\sigma_i(a)$. A closed path around the singularity requires an identification of two such points and one finds that under this

$$
\left( \begin{array}{c} a_D \\ a \end{array} \right) \rightarrow M_i^{\pm 1} \left( \begin{array}{c} a_D \\ a \end{array} \right) = \left( \begin{array}{cc} \sigma_i & \mp 2\alpha_i \otimes \alpha_i \\ 0 & \sigma_i \end{array} \right) \left( \begin{array}{c} a_D \\ a \end{array} \right),
$$

(4.1)

where $\sigma_i$ is the Weyl reflection in the simple root $\alpha_i$. The signs in (4.1) are determined by whether the path circles the singularity in a positive or negative sense. This is determined by whether the path goes from a point on the wall with $\text{Im}(\alpha_i \cdot a) > 0$ to one with $\text{Im}(\alpha_i \cdot a) < 0$, or vice-versa. Hence in going around the singularity in a positive or negative sense the quantum numbers $Q = (g, q)$ of a BPS state are transformed as

$$
Q \rightarrow Q M_i^{\pm 1},
$$

(4.2)

respectively, where $M_i$ acts by matrix multiplication to the left. To build up a picture of the spectrum one starts with the states associated to the simple roots with $Q_i = (\pm \alpha_i^*, n\alpha_i)$ which exist throughout the classical moduli space and then considers taking these states along various paths around the singularities.

For example for the SU(3) case, described in [5], Fig. 2 shows four paths $P_i$, $i = 1, 2, 3, 4$, which encircle singularities and involve the monodromy transformations $M_1, M_1^{-1}, M_2$ and $M_2^{-1}$, respectively. Starting with the states $Q_1 = (\pm \alpha_1^*, n\alpha_1)$ and $Q_2 = (\pm \alpha_2^*, n\alpha_2)$ one finds that the two cells $D^\pm$ contains dyons with charges

$$
Q_2 M_1^{\mp 1} \equiv Q_1 M_2^{\pm 1},
$$

(4.3)

respectively. So in $D^\pm$ there are states with quantum numbers

$$
D^\pm: \quad (\pm \alpha_3^*, n\alpha_3 \mp \alpha_1) \equiv (\pm \alpha_3^*, n\alpha_3 \pm \alpha_2).
$$

(4.4)

This matches precisely the result from fermion fractionalization in (3.16).

This equivalence between the two methods of determining the spectrum can be extended to any Lie group. The strategy for proving this is to show that it is true for all the
rank 2 algebras and then the result for general groups follows by an inductive argument. The Dynkin diagrams of all the rank two algebras are illustrated in Fig. 2, where $\alpha_1 \equiv \delta_i$ and $\alpha_2 \equiv \gamma_i$. The moduli spaces of all the rank 2 cases are identical in form to the SU(3) case illustrated in Fig. 1. There is a single decay curve defined by $\theta_{\alpha_1} = \theta_{\alpha_2}$. There are 6 additional cases to consider, with $\alpha$ equal to the roots labelled $b, c, \ldots, g$ in Fig. 2.

(b) $B_2$ with $\alpha = 2\alpha_1 + \alpha_2$. A dyon with magnetic charge $(2\alpha_1 + \alpha_2)^*$ can decay on the CMS to $(\alpha_1^*) + (\alpha_2^*)$. In the cells $D^\pm$ the result (3.16) gives

$$q = n(\alpha_1 + \alpha_2) \mp \alpha_1,$$

(4.5)

respectively. Since $\sigma_1(\alpha_2) = 2\alpha_1 + \alpha_2$, these states will be generated by taking dyons with magnetic charge $\alpha_2^*$ around the singularity $a \cdot \alpha_1 = 0$. The monodromy gives charges

$$Q_2 M_1^{\pm 1} = ((\alpha_1 + \alpha_2)^*, n(\alpha_1 + \alpha_2) \pm \alpha_1),$$

(4.6)

in $D^\mp$, respectively, which matches (4.5) exactly.

(c) $B_2$ with $\alpha = \alpha_1 + \alpha_2$. A dyon with magnetic charge $(\alpha_1 + \alpha_2)^*$ can decay on the CMS to $2(\alpha_1^*) + (\alpha_2^*)$. In the cells $D^\pm$ the result (3.16) gives

$$q = n(\alpha_1 + \alpha_2) \mp 2\alpha_1,$$

(4.7)
respectively. Since \( \sigma_2(\alpha_1) = \alpha_1 + \alpha_2 \), these states will be generated by taking a dyon with magnetic charge \( \alpha_1^* \) around the singularity \( a \cdot \alpha_2 = 0 \). The monodromy gives charges

\[
Q_1 M_2^{\pm 1} = ((\alpha_1 + \alpha_2)^*, n(\alpha_1 + \alpha_2) \mp 2\alpha_2) = ((\alpha_1 + \alpha_2)^*, n'(\alpha_1 + \alpha_2) \mp 2\alpha_1)
\] (4.8)

in \( D^\pm \), respectively, which matches (4.7) exactly.

\((d)\) \( G_2 \) with \( \alpha = 3\alpha_1 + \alpha_2 \). A dyon of magnetic charge \( (3\alpha_1 + \alpha_2)^* \) can decay on the CMS to \((\alpha_1^*) + (\alpha_2^*)\). In the cells \( D^\pm \) the result (3.16) gives

\[
q = n(3\alpha_1 + \alpha_2) \mp \alpha_1,
\] (4.9)

respectively. Since \( \sigma_1(\alpha_2) = 3\alpha_1 + \alpha_2 \), dyons with magnetic charge \((\alpha_1 + \alpha_2)^*\) are generated by taking dyons with magnetic charge \( \alpha_2 \) around the singularity \( a \cdot \alpha_1 \). The monodromies lead to dyons with charges

\[
Q_2 M_1^{\pm 1} = (3\alpha_1 + \alpha_2)^*, n(3\alpha_1 + \alpha_2) \pm \alpha_1),
\] (4.10)

in cells \( D^\mp \), respectively, which matches (4.9) precisely.

\((e)\) \( G_2 \) with \( \alpha = 3\alpha_1 + 2\alpha_2 \). A dyon of magnetic charge \((3\alpha_1 + 2\alpha_2)^*\) can decay to \((\alpha_1^*) + 2(\alpha_2^*)\). In the cells \( D^\pm \) the result (3.16) gives

\[
q = n(3\alpha_1 + 2\alpha_2) \mp 2\alpha_1,
\] (4.11)

respectively. Since \( \sigma_2 \sigma_1(\alpha_2) = 3\alpha_1 + 2\alpha_2 \), dyons with magnetic charge \((3\alpha_1 + 2\alpha_2)^*\) are generated by taking dyons with magnetic charge \( \alpha_2 \) around the singularity \( a \cdot \alpha_1 \) in a positive (negative) sense and then around the singularity \( \alpha \cdot \alpha_2 = 0 \) in a positive (negative) sense. The monodromies lead to dyons with charges

\[
Q_2 M_1^{\pm 1} M_2^{\pm 1} = ((3\alpha_1 + 2\alpha_2)^*, n(3\alpha_1 + 2\alpha_2) \mp \alpha_1) M_2^{\pm 1} = ((3\alpha_1 + 2\alpha_2)^*, n(3\alpha_1 + 2\alpha_2) \mp (\alpha_1 + \alpha_2) \mp \alpha_2)
\] (4.12)

\[
\equiv ((3\alpha_1 + 2\alpha_2)^*, n'(3\alpha_1 + 2\alpha_2) \mp 2\alpha_1),
\]

in cells \( D^\pm \), respectively, which matches (4.11) precisely. It is important that the monodromy is equal to \( M_1^{\pm 1} M_2^{\pm 1} \) and not \( M_1 M_2^{-1} \) or \( M_1^{-1} M_2 \). These latter monodromy transformations cannot be realized on the states \( Q_2 \) because they involve paths that cross the CMS where the dyons \( Q_2 M_1 \) and \( Q_2 M_1^{-1} \), respectively, decay.

\((f)\) \( G_2 \) with \( \alpha = \alpha_1 + \alpha_2 \). A dyon of magnetic charge \((\alpha_1 + \alpha_2)^*\) can decay to \((\alpha_1^*) + 3(\alpha_2^*)\). In the cells \( D^\pm \) the result (3.16) gives

\[
q = n(\alpha_1 + \alpha_2) \mp 3\alpha_1,
\] (4.13)
respectively. Since \( \sigma_2(\alpha_1) = \alpha_1 + \alpha_2 \), dyons with magnetic charge \((\alpha_1 + \alpha_2)^*\) are generated by taking dyons with magnetic charge \(\alpha_1\) around the singularity \(\mathbf{a} \cdot \alpha_2\). The monodromies lead to dyons with charges

\[
Q_1 M^{\pm 1}_2 = ((\alpha_1 + \alpha_2)^*, n(\alpha_1 + \alpha_2) \pm 3\alpha_2)
\equiv ((\alpha_1 + \alpha_2)^*, n'(\alpha_1 + \alpha_2) \mp 3\alpha_1),
\]

in cells \(D^\pm\), respectively, which matches (4.13) precisely.

\(G_2\) with \(\alpha = 2\alpha_1 + \alpha_2\). A dyon of magnetic charge \((2\alpha_1 + \alpha_2)^*\) can decay to \(2(\alpha_1^*) + 3(\alpha_2^*)\). In the cells \(D^\pm\) the result (3.16) gives

\[
q = n(2\alpha_1 + \alpha_2) \mp 4\alpha_1,
\]

respectively. Since \(\sigma_1 \sigma_2(\alpha_1)\), dyons with magnetic charge \((2\alpha_1 + \alpha_2)^*\) are generated by taking dyons with magnetic charge \(\alpha_1\) around the singularity \(\mathbf{a} \cdot \alpha_2 = 0\) in a positive (negative) sense and then around the singularity \(\mathbf{a} \cdot \alpha_1 = 0\) in a positive (negative) sense. The monodromies lead to dyons with charges

\[
Q_1 M^{\pm 1}_2 M_1^{\pm 1} = ((\alpha_1 + \alpha_2)^*, n(\alpha_1 + \alpha_2) \pm 3\alpha_2) M_1^{\pm 1}
= ((2\alpha_1 + \alpha_2)^*, n(2\alpha_1 + \alpha_2) \pm 3(3\alpha_1 + \alpha_2) \pm \alpha_1)
\equiv ((2\alpha_1 + \alpha_2)^*, n'(2\alpha_1 + \alpha_2) \mp 4\alpha_1),
\]

present in cells \(D^{\mp}\), respectively, which matches (4.15) precisely.

Fortunately it is only a little bit harder to establish the exact relation between the semi-classical monodromies and the result (3.16) for all the gauge groups of rank > 2. The reason is that it is not necessary to prove the result case-by-case, since the result can be proved by induction assuming that it is true for the rank 2 cases, a fact that has been established above. It is useful at this stage to consider in more detail the relation between the Weyl group and the semi-classical monodromies. The Weyl reflections in the simple roots \(\sigma_j\) are associated to the pair of monodromy transformations \(M_j^{\pm 1}\). In fact the semi-classical monodromies generate a representation of the Brieskorn Braid Group [23]. To each element \(\sigma\) of the Weyl group \(W\), which can be written \(\sigma = \sigma_{a_1} \cdots \sigma_{a_p}\), there exist the following elements of the Braid group: \(M = M^\epsilon_{a_1} \cdots M^\epsilon_{a_p}\), where \(\epsilon_i = \pm 1\). These elements all have the form

\[
M = \begin{pmatrix}
\sigma & A \\
0 & \sigma
\end{pmatrix},
\]

for some matrix \(A\) depending on the \(\epsilon_i\).

Now consider the spectrum of dyons of magnetic charge \(\alpha^* = N_1 \gamma_i^* + M_i \delta_i^*\) either side of the CMS \(C_{\gamma_i, \delta_i}\). There always exists an element \(\sigma \in W\) such that \(\alpha_j = \sigma(\gamma_i)\) and \(\alpha_k = \sigma(\delta_i)\), where \(\alpha_j\) and \(\alpha_k\) are two simple roots (with some choice of labelling) which
form either an $A_2$ or $B_2$ type sub-root space of $\Phi$. The choice of $\sigma$, $\alpha_j$ and $\alpha_k$ are not unique, but this will not affect the result. In some cell

$$D = D^\epsilon_1 \cap \cdots \cap D^\epsilon_{i-1} \cap D^\epsilon_{i+1} \cap \cdots \cap D^\epsilon_L$$

(4.18)

which straddles $C_{\gamma_i,\delta_i}$, the dyons of magnetic charge $\gamma_i^*$ and $\delta_i^*$ have charges

$$Q_{\gamma_i} = Q_j M, \quad Q_{\delta_i} = Q_k M,$$

(4.19)

where $M$ is an element of the Braid group of the form (4.17) for some $A$ which depends on $D$, i.e. on the $\epsilon_l$ for $l \neq i$. To work out the charges of $\alpha^*$ we can map the problem back to the rank 2 subspace defined by $\alpha_j$ and $\alpha_k$. In particular, $\sigma(\alpha^*) = N_i \alpha_j^* + M_i \alpha_k^*$ and $\sigma(\alpha) = n_i \alpha_j + m_i \alpha_k$. We have previously established in this section (by identifying $\alpha_j$ with $\alpha_2$ and $\alpha_k$ with $\alpha_1$) that in the regions $D^\pm_1$ there are dyons of charge

$$Q^\pm = (N_i \alpha_j^* + M_i \alpha_k^*, n(n_i \alpha_j + m_i \alpha_k) \mp (N_i + M_i - 1)\alpha_k), \quad n \in \mathbb{Z}.$$

(4.20)

By following a path with monodromy $M$, which does not pass through a CMS on which the dyons decay, these dyons give dyons of charge $Q^\pm M$ in $D^\pm_1 \cap D$. Explicitly

$$Q^\pm M = (\alpha^*, n\alpha \mp (N_i + M_i - 1)\sigma^T(\alpha_k) + A^T \sigma(\alpha^*)) = (\alpha^*, n\alpha + \beta \mp (N_i + M_i - 1)\delta_i),$$

(4.21)

where $\beta = A^T \sigma(\alpha^*)$ is some root which depends on $D$. In (4.21) we have used the expression for $M$ in (4.17) and the fact that $\sigma^T = \sigma^{-1}$ implying $\gamma_i = \sigma^T(\alpha_j)$ and $\delta_i = \sigma^T(\alpha_k)$. Following this argument for each of the cells in (2.18) for $i = 1, \ldots, L$, one deduces the general result (3.16).

5. Dyon decays

For the consistency of the spectrum that we have deduced in (3.16) it is important that the dyon with magnetic charge $\alpha^*$ decays on the CMS $C_{\gamma_i,\delta_i}, i = 1, \ldots, L$. In certain cases, this fact can also be established in the semi-classical approximation [17].

At a generic point in $\mathcal{M}_{\text{vac}}^{\text{cl}}$, a monopole solution has four bosonic zero modes, corresponding to the four collective coordinates which specify the centre-of-mass and the $U(1)$ charge angle [16,17,24]. The moduli space of the collective coordinates has the form $M = \mathbb{R}^3 \times S^1$. In semi-classical quantization the states of the monopole are described by a supersymmetric quantum mechanics on the moduli space of collective coordinates [25]. This yields the usual tower of dyon states, where the electric charge corresponds to
momentum in the compact direction. These states transform in a hypermultiplet representation of $N = 2$ supersymmetry. On the CMS $C_{\gamma_i, \delta_i}$, where $\alpha^* = N_i \gamma_i^* + M_i \delta_i^*$, there are $4(N_i + M_i - 1)$ additional zero modes. This can be deduced from the index theory calculation of Weinberg [19]. The space of collective coordinates enlarges discontinuously to space which has the form [26]:

$$\mathbb{R}^3 \times S^1 \rightarrow M' = \mathbb{R}^3 \times \frac{\mathbb{R} \times M_0}{D},$$

(5.1)

where $D$ is a certain discrete group, the factor $\mathbb{R}$ is associated with the overall charge angle and $M_0$, the centred moduli space, is a hyper-kahler manifold of dimension $4(N_i + M_i - 1)$. The various different cases described previously give the possible dimensions of $M_0$ as 4,8,12 or 16. In the case when $\alpha^* = \gamma_i^* + \delta_i^*$, $M_0$ has dimension 4 and is a Euclidean Taub-NUT space [26,27].

The question of whether the dyon with magnetic charge $\alpha^*$ decays on $C_{\gamma_i, \delta_i}$ can be answered by enquiring whether or not there is a threshold bound-state in the supersymmetric quantum mechanics on $M'$ [17]. Such bound-states require a normalizable holomorphic harmonic form (or spinor) on $M_0$. For the case when $M_0$ is the Euclidean Taub-NUT space it is known that there is no such form and so there are no bound-states and the dyons decay. The description of the spectrum that we have established strongly suggests that no such threshold bound-states exists for the higher-dimensional cases as well.

6. Discussion

We have constructed the dyon spectrum of $N = 2$ gauge theories at weak coupling for theories with an arbitrary gauge group. The spectrum of states that arise is consistent with the semi-classical monodromies. The effect that allows us to determine the charges of the dyon states involves fermion fractionalization which is a one-loop effect. It is known that for these BPS states there are no higher perturbative corrections to the electric charge. In other words the vector quantum numbers $q$ that we have derived are not subject to any additional corrections. These states can then be followed into the strong coupling regime as long as one does not cross a CMS on which they decay. These states should then be responsible for the singularities that occur at strong coupling. It would be interesting to investigate whether the dyons present at weak coupling can account for all the strong coupling singularities as checked for the SU($N$) theory in [5].

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Appendix

In this appendix we calculate the fractional fermion number of the vacuum in the background of a monopole. The monopole solution is given via the embedding in (2.12).

The Dirac equation for the fermion fields in the background of the monopole solution is

\[
\begin{align*}
\left[i\gamma^m D_m - \text{Re}(\Phi) + i\gamma^5 \text{Im}(\Phi)\right]\Psi &= 0. \\
(A.1)
\end{align*}
\]

The background is time-independent and so we look for stationary solutions \(\Psi = e^{iEt}\psi\):

\[
\begin{align*}
\left[i\gamma^0\gamma^i D_i - \gamma^0 \text{Re}(\Phi) + i\gamma^0\gamma^5 \text{Im}(\Phi)\right]\psi &= E\psi. \\
(A.2)
\end{align*}
\]

In order to write this equation in a recognizable form we first perform a U(1) transformation \(\Phi \rightarrow e^{-i\theta_\alpha}\Phi\). With a suitable basis for the gamma matrices, the Hamiltonian equation (A.2) now takes the form

\[
H\psi = \begin{pmatrix}
\text{Im}(\hat{a}) & D \\
D^\dagger & -\text{Im}(\hat{a})
\end{pmatrix}\psi = E\psi, \\
(A.3)
\]

where

\[
D = i\sigma_i D_i + i\phi^\alpha + i\text{Re}(\hat{a}), \quad D^\dagger = i\sigma_i D_i - i\phi^\alpha - i\text{Re}(\hat{a}).
(A.4)
\]

The fermion number of the vacuum is related to the spectral asymmetry of the Dirac Hamiltonian \(H\). Writing,

\[
H = H_0 + \text{Im}(\hat{a})\Gamma^c, \quad H_0 = \begin{pmatrix}
0 & D \\
D^\dagger & 0
\end{pmatrix}, \quad \Gamma^c = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
(A.5)
\]

the Hamiltonian \(H_0\) anti-commutes with the charge conjugation matrix \(\Gamma^c\). This means that for \(H_0\) there is a precise mapping between states of positive and negative energy. \(H_0\) is precisely the Dirac operator consider by Weinberg for the theory with a single real Higgs field [19]. \(H\) on the other hand does not admit a conjugation symmetry. Even so there is a mapping between positive and negative energy eigenvectors. If \(\psi\) has positive energy \(E\) then \([H, \Gamma^c]\psi\) is an eigenvector with energy \(-E\). However, since this mapping involves a derivative it does not guarantee that the densities of the positive and negative energy modes of the continuum part of the spectrum are identical. Hence in general the operator \(H\) has a non-trivial spectral asymmetry which is defined formally as

\[
\eta_H = \text{Tr}(\text{sign}(H)) = \frac{2}{\pi} \int_0^\infty d\tau \text{Tr}\left(\frac{H}{H^2 + \tau^2}\right), \\
(A.6)
\]

\[17\]
where the trace is over the whole Hilbert space and the second expression involves an integral representation of the sign function. The fermion number of the vacuum is given by [4]

\[ N = -\frac{1}{2} \eta_H = -\frac{1}{\pi} \int_0^\infty dz \text{Tr} \left( \frac{H}{H^2 + z^2} \right) \]  
(A.7)

This can be written as [4]:

\[ N = \frac{1}{\pi} \int_0^\infty dz \text{Tr} \left( \frac{\text{Im}(\hat{a})}{D^\dagger D + \text{Im}(\hat{a})^2 + z^2} - \frac{\text{Im}(\hat{a})}{DD^\dagger + \text{Im}(\hat{a})^2 + z^2} \right). \]  
(A.8)

A trace of the kind above was calculated by Weinberg [19] by an adaptation of the Callias index theorem [28]. It turns out that the only non-zero contribution comes from a surface term and hence only involves the fields at spatial infinity. The relevant result for the trace is

\[ \sum_{\beta \in \Phi} \text{Re} \left( e^{-i\theta_\alpha a \cdot \beta} \right) \text{Im} \left( e^{-i\theta_\alpha a \cdot \beta} \right) \beta \cdot \alpha^* \left( |a \cdot \beta|^2 + z^2 \right)^{1/2} \left( \text{Im} \left( e^{-i\theta_\alpha a \cdot \beta} \right)^2 + z^2 \right). \]  
(A.9)

The fact that the integrand separates into a sum over all the roots in the Lie algebra allows us to interpret the result as a sum over the fractional fermion numbers \( N_\beta \) associated to the expansion of the fermion field in a Cartan-Weyl basis. By evaluating the integral, we have for each mode

\[ N_\beta = \frac{\beta \cdot \alpha^*}{\pi} \text{arctan} \left( \frac{\text{Re} \left( e^{-i\theta_\alpha a \cdot \beta} \right)}{\text{Im} \left( e^{-i\theta_\alpha a \cdot \beta} \right)} \right). \]  
(A.10)

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