ON THE SHARPNESS OF EMBEDDINGS OF HÖLDER SPACES INTO GAUSSIAN BESOV SPACES

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Abstract. For an interpolation pair \((E_0, E_1)\) of Banach spaces with \(E_1 \hookrightarrow E_0\) we use vectors \(b_1, b_2, \ldots \in E_1\) that satisfy an extremal property with respect to the \(J\)- and \(K\)-functional to construct sub-spaces that are isometric to \(\ell^q(\theta)\). The construction is based on a randomisation using independent Rademacher variables. We verify that systems obtained by re-scaling a function with a certain periodicity property share this extreme property. This implies the sharpness of natural embeddings of Hölder spaces obtained by the real interpolation into the corresponding Gaussian Besov spaces.

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1. Introduction

Let us start with an abstract problem: Assume two interpolation pairs \((F_0, F_1)\) and \((E_0, E_1)\) and a linear operator \(T : F_0 + F_1 \to E_0 + E_1\) such that \(T : F_i \to E_i\) are continuous for \(i = 0, 1\). Find sufficient conditions such that the embedding \(T : (F_0, F_1) \to (E_0, E_1)\) is sharp with respect to the real interpolation method in the sense that

\[ T((F_0, F_1)_{\theta,q}) \not\subseteq (E_0, E_1)_{\theta,q_0} \]

for all \(\theta \in (0, 1)\) and \(1 \leq q_0 < q_1 \leq \infty\). A first typical example that does not share this type of sharpness is obtained by the re-iteration theorem:

**Example 1.1.** Assume that \(F_1 \hookrightarrow F_0\) and let \(E_0 := (F_0, F_1)_{\eta,p}\) and \(E_1 := F_1\), where \((\eta, p) \in (0, 1) \times [1, \infty]\) is fixed. Then one has \(E_0 + E_1 = (F_0, F_1)_{\eta,p}\) and \(F_0 + F_1 = F_0\). Let \(T : E_0 + E_1 \to F_0 + F_1\) be the natural embedding. By the re-iteration theorem we have for \((\theta, q) \in (0, 1) \times [1, \infty]\) that

\[ (E_0, E_1)_{\theta,q} = ((F_0, F_1)_{\eta,p}, F_1)_{\theta,q} = (F_0, F_1)_{(1-\theta)\eta+\theta,q}. \]

This implies that

\[ T((E_0, E_1)_{\theta,\infty}) = (F_0, F_1)_{(1-\theta)\eta+\theta,\infty} \subseteq (F_0, F_1)_{\theta,1} \]

with \((1 - \theta)\eta + \theta > \theta\).

**Example 1.1** relies on the fact that the main interpolation parameter \(\theta\) is already shifted. An example that relies only on the shift of the fine-tuning parameter \(q\) is the following:

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Example 1.2. We let $a = (\alpha_n)_{n=1}^{\infty} \in \ell_r$ for some fixed $r \in [1, \infty)$. Then we get a multiplier on the spaces $\ell_q^{(s)}$, that are recalled in [2.1], by

$$M_a : \ell_q^{(i)} \rightarrow \ell_p^{(i)} \quad \text{defined by} \quad M_a((x_n)_{n=1}^{\infty}) := (\alpha_n x_n)_{n=1}^{\infty}$$

for $i = 0, 1$ and $\frac{1}{p} = \frac{1}{q_0} + \frac{1}{r}, 1 \leq p_i \leq q_i, r \leq \infty$. According to (2.4) real interpolation with parameters $(\theta, q) \in (0, 1) \times [1, \infty]$ gives

$$M_a : \ell_q^{(\theta)} \rightarrow \ell_q^{(\theta)}.$$

Choosing $\alpha_n := n^{-\kappa}$ with $\kappa > 1/r$, one does not have a continuous map

$$M_a : \ell_1^{(\theta)} \rightarrow \ell_\infty^{(\eta)}$$

for any $0 < \theta < \eta < 1$. But still, one has the improvement on the fine-tuning level

$$M_a : \ell_q^{(\theta)} \rightarrow \ell_p^{(\theta)} \quad \text{with} \quad 1/p = 1/q + 1/r,$$

which implies $p < q$ and corresponds to the map

$$M_a : (\ell_0^{(\theta)}, \ell_1^{(1)})_{\theta,q} \rightarrow (\ell_0^{(\theta)}, \ell_1^{(1)})_{\theta,p}.$$

As an application of our general result [Theorem 3.1] we discuss the sharpness of the embedding

$$(C_0^d(\mathbb{R}^d), \operatorname{Lip}^0(\mathbb{R}^d))_{\theta,q} \hookrightarrow (L_2(\mathbb{R}^d, \gamma_d), D_{1,2}(\mathbb{R}^d, \gamma_d))_{\theta,q},$$

where $\gamma_d$ is the standard Gaussian measure on $\mathbb{R}^d$. Starting with $d = 1$ a typical example of a Hölder continuous function, $f(x) := (x^+)^{\theta}$, one realizes that

$$f \in D_{1,2}(\mathbb{R}, \gamma_1) \quad \text{if} \quad \theta \in \left(\frac{1}{2}, 1\right],$$

so that this type of example does not work for us. The reason is that the Hölder space is much more sensitive to the singularity at $x = 0$ than the Gaussian Besov space is. At least there are two constructions in the literature to overcome this obstacle. Firstly, a problem discussed in [3] within the context of Lévy processes, which is not directly related to ours, gives a hint how to overcome this obstacle: Instead of taking $x \mapsto (x^+)^{\theta}$ one should use appropriate combinations of Schauder functions that have more singularities which can be seen by the Gaussian Besov spaces. Secondly, there exist characterisations for Besov- and Triebel-Lizorkin-spaces, $B_{p,q}^s(\mathbb{R}^d, w)$ and $F_{p,q}^s(\mathbb{R}^d, w)$, where $w$ is a polynomial weight, that are based on Wavelet constructions [4]. On the one hand, our weight is Gaussian and therefore does not fall into the setting of polynomial weights. Secondly, although our construction in [Section 4] is based on the principal idea of wavelets, we do not need the property that we have a basis so that only very mild assumptions on our construction are required.

Our main contribution is that we lift our arguments to a construction for interpolation spaces between general Banach spaces, where we choose elements sharing an extremal relation regarding the $J$- and $K$-functional and a randomisation. This construction might be useful for other problems too.
2. Preliminaries

2.1. Spaces. Given a function \( f : \mathbb{R}^d \to \mathbb{R} \), we denote by \(|f|_{\text{Lip}} := \sup_{x \neq y} |f(x) - f(y)|/|x - y|\) its Lipschitz constant, where \(|x|\) is the euclidean norm on \( \mathbb{R}^d \). We denote by \( \text{Lip}^0(\mathbb{R}^d) \) the space of all Lipschitz functions \( f : \mathbb{R}^d \to \mathbb{R} \) such that \( f(0) = 0 \) which becomes a normed space under \(| \cdot |_{\text{Lip}}\). The space of continuous bounded functions \( f \) equipped with the norm \( \| f \|_{c^0(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d} |f(x)| \). For \((s, q) \in \mathbb{R} \times [1, \infty] \) we use the Banach space

\[
(\theta, q) := \{(2^n x_n)_{n=1}^\infty : \|x\|_{\ell_q} := \|(2^n x_n)_{n=1}^\infty\|_{\ell_q} < \infty\}.
\]

In this note we only consider real Banach spaces. Given Banach spaces \( E_0, E_1 \), the notation \( E_0 \hookrightarrow E_1 \) stands for an injective continuous embedding, so that there is a \( c > 0 \) such that \( \|x\|_{E_0} \leq c \|x\|_{E_1} \) for all \( x \in E_1 \). Moreover, we use the notation \( A \sim_c B \), where \( A, B \geq 0 \) and \( c \geq 1 \), if \((1/c)A \leq B \leq cA\).

2.2. Real interpolation. Let \((E_0, E_1)\) be a couple of Banach spaces such that \( E_0 \) and \( E_1 \) are continuously embedded into some topological Hausdorff space \( X \). We equip \( E_0 + E_1 := \{x = x_0 + x_1 : x_1 \in E_1\} \) with the norm \( \|x\|_{E_0 + E_1} := \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1} : x_1 \in E_1, x = x_0 + x_1\} \) and \( E_0 \cap E_1 \) with the norm \( \|x\|_{E_0 \cap E_1} := \max\{\|x\|_{E_0}, \|x\|_{E_1}\} \) to get Banach spaces \( E_0 \cap E_1 \subseteq E_0 + E_1 \). For \( x \in E_0 \cap E_1 \) and \( x \in E_0 + E_1 \), respectively, and \( \lambda \in (0, \infty) \) we define the \( J \) and \( K \)-functional

\[
J(\lambda, x; E_0, E_1) := \max\{\|x\|_{E_0}, \lambda \|x\|_{E_1}\},
\]

\[
K(\lambda, x; E_0, E_1) := \inf\{\|x_0\|_{E_0} + \lambda \|x_1\|_{E_1} : x_1 \in E_1, x = x_0 + x_1\}.
\]

Given \((\theta, q) \in (0, 1) \times [1, \infty] \) we obtain real interpolation spaces by

\[
(\theta, q)_{\theta, q, J} := \{x \in E_0 + E_1 : \|x\|_{(\theta, q)_{\theta, q, J}} < \infty\},
\]

\[
(\theta, q)_{\theta, q, K} := \{x \in E_0 + E_1 : \|x\|_{(\theta, q)_{\theta, q, K}} < \infty\},
\]

with

\[
\|x\|_{(\theta, q)_{\theta, q, J}} := \inf\left\{ \left\| (2^{-k\theta} J(2^k, x_k; E_0, E_1)) \right\|_{k=\infty} : x = \sum_{k=\infty}^{\infty} x_k \text{ in } E_0 + E_1, x_k \in E_0 \cap E_1 \right\},
\]

\[
\|x\|_{(\theta, q)_{\theta, q, K}} := \|\lambda \mapsto \lambda^{-\theta} K(\lambda, x; E_0, E_1)\|_{L_\theta((0, \infty), \ell_q)}.
\]

By \cite{1} Lemma 3.2.3, Theorem 3.3.1 for all \((\theta, q) \in (0, 1) \times [1, \infty] \) there is a constant \( c_{\theta, q} = c(\theta, q) \geq 1 \) such that

\[
\frac{1}{c_{\theta, q}} \|x\|_{(\theta, q)_{\theta, q, J}} \leq \|x\|_{(\theta, q)_{\theta, q, J}} \leq c_{\theta, q} \|x\|_{(\theta, q)_{\theta, q, K}}.
\]

Therefore one defines the two-parameter scale of real interpolation spaces

\[
(\theta, q)_{\theta, q} := (\theta, q)_{\theta, q, J} = (\theta, q)_{\theta, q, K}
\]

and equip \((\theta, q)_{\theta, q}\) with the norm \( \| \cdot \|_{(\theta, q)_{\theta, q}} := \| \cdot \|_{(\theta, q)_{\theta, q, K}} \). We obtain a family of Banach spaces \(((\theta, q)_{\theta, q}, \| \cdot \|_{(\theta, q)_{\theta, q}})\) with the lexicographical ordering

\[
(\theta, q)_{\theta, q_0} \subseteq (\theta, q_1)_{\theta, q_0} \quad \text{for all } \theta \in (0, 1) \text{ and } 1 \leq q_0 < q_1 \leq \infty,
\]
and, under the additional assumption that $E_1 \hookrightarrow E_0$,
\[(E_0, E_1)_{\theta_0, q_0} \subseteq (E_0, E_1)_{\theta_1, q_1} \quad \text{for all } 0 < \theta_1 < \theta_0 < 1 \text{ and } q_0, q_1 \in [1, \infty].\]

For more information about interpolation theory the reader is referred to [127x468][1][2][7].

2.3. Examples of interpolation spaces. For $s_0 \neq s_1$ and $q_0, q_1, q \in [1, \infty]$ the spaces $\ell_q^{(s)}$ interpolate as
\[(\ell_q^{(s)}), \ell_q^{(s)}_{\theta, q} = \ell_q^{(s)} \quad \text{with } s = (1 - \theta)s_0 + \theta s_1,\]
where the norms are equivalent up to multiplicative constants depending at most on $s_0, s_1, q_0, q_1, q, \theta$ (see [2 Theorem 5.6.1]). Regarding the Hölder spaces, we use
\[\text{Hö}^{0}_{\theta, q} (\mathbb{R}^d) := (C^0_k (\mathbb{R}^d), \text{Lip}^0 (\mathbb{R}^d))_{\theta, q}.\]

3. A general result

Let $(M, \Sigma, \rho)$ be a probability space and $\varepsilon_n : M \to \{-1, 1\}$ be a sequence of i.i.d. random variables with $\rho(\varepsilon_n = \pm 1) = \frac{1}{2}$.

**Theorem 3.1.** Assume Banach spaces $(E_0, E_1)$ with $E_1 \hookrightarrow E_0$, $(b_n)_{n=1}^{\infty} \subseteq E_1$, and $\delta, \kappa > 0$ such that
\[(\ell_0^{(s)}), (\ell_1^{(s)})_{\delta, \kappa} = (\ell_0^{(s)}) \quad \text{with } s = (1 - \theta) b_n + \theta b_1,\]
for $n = 1, 2, \ldots$ Then for all $(\theta, q) \in (0, 1) \times [1, \infty)$, there is a constant $c_{\delta, \kappa} = c_{\delta, \kappa}(\theta, q, \kappa, \delta) > 0$ such that for all $(\alpha_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ such that $\#\{n \geq 1 : \alpha_n \neq 0\} < \infty$ one has
\[\frac{1}{c_{\delta, \kappa}} \| (\alpha_n)_{n=1}^{\infty} \|_{\ell_q^{(s)}} \leq \int_M \left\| \sum_{n=1}^{\infty} \alpha_n \varepsilon_n (\xi) b_n \right\|_{(E_0, E_1)_{\theta, q}} \rho(d\xi) \leq \sup_{\varepsilon_n = \pm 1} \left\| \sum_{n=1}^{\infty} \alpha_n \varepsilon_n b_n \right\|_{(E_0, E_1)_{\theta, q}} \leq c_{\delta, \kappa} \| (\alpha_n)_{n=1}^{\infty} \|_{\ell_q^{(s)}}.\]

**Proof.** Using [22], one has
\[\left\| \sum_{n=1}^{\infty} \alpha_n b_n \right\|_{(E_0, E_1)_{\theta, q}} \leq c_{\theta, q} \| (\alpha_n J(2^{-n}, b_n; E_0, E_1))_{n=1}^{\infty} \|_{\ell_q^{(s)}} \leq c_{\theta, q} \kappa \| (\alpha_n)_{n=1}^{\infty} \|_{\ell_q^{(s)}}\]
which proves the right-hand side inequality. The left-hand side inequality is obtained with the help of the Khintchine-Kahane inequality (with a constant $c_q \geq 1$ depending on $q$ only) by
\[\int_M \left\| \sum_{n=1}^{\infty} \alpha_n \varepsilon_n (\xi) b_n \right\|_{(E_0, E_1)_{\theta, q}} \rho(d\xi) \sim_{c_q} \left( \int_M \left\| \sum_{n=1}^{\infty} \alpha_n \varepsilon_n (\xi) b_n \right\|_{(E_0, E_1)_{\theta, q}}^q \rho(d\xi) \right)^{\frac{1}{q}} \]
\[= \left( \int_0^{\infty} \lambda^{-q-1} \int_M K \left( \lambda, \sum_{n=1}^{\infty} \alpha_n \varepsilon_n (\xi) b_n ; E_0, E_1 \right)^{q} \rho(d\xi) d\lambda \right)^{\frac{1}{q}} \]
4. An application to Gaussian Besov spaces

4.1. Gaussian Besov spaces. We let $d \geq 1$ and let $\gamma_d$ be the standard Gaussian measure on $\mathbb{R}^d$. The space $L^2(\mathbb{R}^d, \gamma_d)$ is equipped with the orthonormal basis of generalised Hermite polynomials $(h_k)_{k=0}^\infty$ given by

$$h_{k_1,\ldots,k_d}(x_1,\ldots,x_d) := h_{k_1}(x_1)\cdots h_{k_d}(x_d),$$

where $(h_k)_{k=0}^\infty \subset L^2(\mathbb{R}, \gamma_1)$ is the orthonormal basis of Hermite polynomials. The Gaussian Sobolev space $D_{1,2}(\mathbb{R}^d, \gamma_d)$ consists of all $f \in L^2(\mathbb{R}^d, \gamma_d)$ such that

$$\sum_{k_1,\ldots,k_d=0}^\infty \langle f, h_{k_1,\ldots,k_d} \rangle_{L^2(\mathbb{R}^d, \gamma_d)}^2 \|\nabla h_{k_1,\ldots,k_d}\|_{L^2(\mathbb{R}^d, \gamma_d)}^2 < \infty.$$ 

We equip $D_{1,2}(\mathbb{R}^d, \gamma_d)$ with the norm

$$\|f\|_{D_{1,2}(\mathbb{R}^d, \gamma_d)} := \sqrt{\|f\|_{L^2(\mathbb{R}^d, \gamma_d)}^2 + \|Df\|_{L^2(\mathbb{R}^d, \gamma_d)}^2}$$

and obtain a Banach space. For $(\theta, q) \in (0, 1) \times [1, \infty)$ we let

$$\mathbb{B}^\theta_{2,q}(\mathbb{R}^d, \gamma_d) := (L^2(\mathbb{R}^d, \gamma_d), D_{1,2}(\mathbb{R}^d, \gamma_d))_{\theta,q}$$

be the Gaussian Besov space of smoothness $\theta$ and with fine-index $q$. 

This concludes the proof. \(\square\)
4.2. Description of the $K$-functional. Assume a basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $(\mathcal{F}_t)_{t \geq 0}$ is the augmented natural filtration of a standard $d$-dimensional standard Brownian motion $(W_t)_{t \geq 0}$, where for convenience $W_0 \equiv 0$ and all paths of $W$ are assumed to be continuous. Assume a corresponding copy $(\Omega', \mathcal{F}', \mathbb{P}', (\mathcal{F}_t')_{t \geq 0}, (W_t')_{t \geq 0})$. From [5, Proposition 3.4] we know that, for $t \in (0, 1)$,

\begin{align}
(4.1) \quad & K(f, \sqrt{1-t}; L_2(\mathbb{R}^d, \gamma_d), D_{1,2}(\mathbb{R}^d, \gamma_d)) \\
& \sim \epsilon_0 \|f(W_t) - f(W_t + W_{1-t})\|_{L_2} + \sqrt{1-t} \|f\|_{L_2(\mathbb{R}^d, \gamma_d)}
\end{align}

where $\epsilon_0 > 0$ is an absolute constant.

4.3. A construction for condition (3.1). Assuming Borel functions $b_n : \mathbb{R}^d \to \mathbb{R}$, we replace condition (3.1) as follows: there exist $\delta, \kappa > 0$ such that, for all $n = 1, 2, \ldots$,

\begin{align}
(4.2) \quad & \max\{\|b_n\|_{L_2(\mathbb{R}^d, \gamma_d)}, 2^{-n}\|b_n\|_{D_{1,2}(\mathbb{R}^d, \gamma_d)}\} \leq \kappa, \\
(4.3) \quad & \|b_n(W_1) - b_n(W_{1-\frac{1}{2^n}} + W_{\frac{1}{2^n}}')\|_{L_2} \geq \delta.
\end{align}

We use the following assumptions to construct such $b_n$:

**Assumption 4.1.** We assume a Borel measurable $b : \mathbb{R}^d \to \mathbb{R}$ such that

1. $\sup_{x \in \mathbb{R}^d} |b(x)| \leq \kappa,$
2. $|b|_{\text{Lip}} \leq \kappa,$
3. there is an $M > 0$ such that

$$\inf_{x \in \mathbb{R}^d} \int_{|y| \leq M} \int_{|\tilde{y}| \leq M} |b(x + y) - b(x + \tilde{y})|^2 dyd\tilde{y} > 0.$$  

**Lemma 4.2.** Assume the conditions in Assumption 4.1 and define

$$b_n(x) := b(2^{n-1}x) \quad \text{for} \quad n \geq 1.$$

Then (4.2) and (4.3) are satisfied.

**Proof.** Condition (4.2) follows from assumptions (1) and (2). To verify condition (4.3), we use the scaling property, set $\sigma := \frac{1}{2^n}$, and get that

$$\|b_n(W_1) - b_n(W_{1-\frac{1}{2^n}} + W_{\frac{1}{2^n}}')\|_{L_2}^2$$

$$= \|b(W_{4^n-1}) - b \left(W_{4^n-1} - \frac{x}{4^n} + W_{\frac{x}{4^n}}'\right)\|_{L_2}^2$$

$$\geq \inf_{x \in \mathbb{R}^d} \|b(x + W_{\sigma}) - b(x + W_{\sigma}')\|_{L_2}^2$$

$$\geq \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x|^2}{2\sigma^2}}\right)^n \inf_{x \in \mathbb{R}^d} \int_{|y| \leq M} \int_{|\tilde{y}| \leq M} |b_1(x + y) - b_1(x + \tilde{y})|^2 dyd\tilde{y} > 0,$$

which verifies (4.3).

**Lemma 4.3.** Assume $b : \mathbb{R}^d \to \mathbb{R}$ to be continuous and bounded and an $R > 0$ such that the following is satisfied:

1. The function $b$ is not constant on $\{x \in \mathbb{R}^d : |x| \leq R\}$.
2. For all $x \in \mathbb{R}^d$ with $|x| > R$ there is $z \in \mathbb{R}^d$ with $|z| \leq R$ such that

$$b(x + y) = b(z + y) \quad \text{for all} \quad y \in \mathbb{R}^d.$$
Then condition (3) of Assumption 4.1 is satisfied.

Proof. We choose $M := 2R$ and get
\[
\inf_{x \in \mathbb{R}^d} \int_{|y| \leq M} \int_{|\bar{y}| \leq M} |b(x + y) - b(x + \bar{y})|^2 \, dy \, d\bar{y}
\]
\[
= \inf_{|z| \leq R} \int_{|y| \leq 2R} \int_{|\bar{y}| \leq 2R} |b(z + y) - b(z + \bar{y})|^2 \, dy \, d\bar{y}.
\]
Assuming the right-hand side to be zero implies by the continuity of $b$ that there is an $z_0 \in \mathbb{R}^d$ with $|z_0| \leq R$ such that
\[
\int_{|y| \leq 2R} \int_{|\bar{y}| \leq 2R} |b(z_0 + y) - b(z_0 + \bar{y})|^2 \, dy \, d\bar{y} = 0.
\]
Therefore, $b(z_0 + y) = b(z_0)$ for all $|y| \leq 2R$ and $b$ would be constant on $\{x \in \mathbb{R}^d : |x| \leq R\}$ which is a contradiction. \qed

4.4. An application to Hölder spaces. Now we apply Theorem 3.1 to the relation between Hölder functions and Gaussian Besov spaces.

Corollary 4.4. Let $\theta \in (0, 1)$ and $1 \leq q < \bar{q} \leq \infty$. Then
\[
\text{Hö}^0_{\bar{q}}(\mathbb{R}^d) \not\subseteq B_{2, \bar{q}}(\mathbb{R}^d, \gamma_d).
\]

Proof. (a) Without loss of generality we can assume that $\bar{q} < \infty$. We define the norms
\[
\|f\|_0 := \max \left\{ \|f\|_{\text{Hö}^0_{\bar{q}}(\mathbb{R}^d)}, \|f\|_{B_{2, \bar{q}}(\mathbb{R}^d, \gamma_d)} \right\},
\]
\[
\|f\|_1 := \max \left\{ \|f\|_{\text{Hö}^0_{\bar{q}}(\mathbb{R}^d)}, \|f\|_{B_{2, \bar{q}}(\mathbb{R}^d, \gamma_d)} \right\}.
\]
Then one has that $\|\cdot\|_0 \leq c \|\cdot\|_1$ for some $c > 0$ because of (2.3) (which is a continuous embedding). If we find $f_N$ such that
\[
\sup_{N=1,2,\ldots} \|f_N\|_0 < \infty \quad \text{and} \quad \sup_{N=1,2,\ldots} \|f_N\|_1 = \infty,
\]
then, by the open mapping theorem, there is an $f$ such that $\|f\|_1 = \infty$ and $\|f\|_0 < \infty$. This would imply $f \in \text{Hö}^0_{\bar{q}}(\mathbb{R}^d)$ but $f \notin B_{2, \bar{q}}(\mathbb{R}^d, \gamma_d)$ and therefore the theorem would be verified.

(b) To construct such $f_N$ we first choose a non-constant $B : \mathbb{R}^d \to [0, \kappa] \subset C^\infty$ with $|B|_{\text{Lip}} \leq \kappa$ for some $\kappa > 0$ and with $\text{supp}(B) \subseteq [-1, 1]^d$. From $B$ we construct
\[
b(\mathbf{x}) := \sum_{k_1, \ldots, k_d = -\infty}^{\infty} 1_{(2k_1, \ldots, 2k_d) + [-1, 1]^d}(\mathbf{x}) B(\mathbf{x} - (2k_1, \ldots, 2k_d)).
\]
Using Lemma 4.3 (with $R = \sqrt{d}$) and Lemma 4.2 we construct $(b_n)_{n=1}^\infty$ satisfying (4.2) and (4.3). Then we take an $a = (\alpha_n)_{n \geq 1} \in \ell^q_\mathcal{P} \setminus \ell^q_\mathcal{Q}$ and let $a_N := (a_1, \ldots, a_N, 0, \ldots)$. From Theorem 3.1 (applied to $(\theta, q)$) it follows that for each $N$ there are signs $\varepsilon_1^N, \ldots, \varepsilon_N^N \in \{-1, 1\}$ such that for $f_N := \sum_{n=1}^N \varepsilon_n^N \alpha_n b_n$ one has
\[
\frac{1}{(\bar{q})^{1/\bar{q}}} \|a_N\|_0 \|\cdot\|_{\bar{q}} \leq \int_{\mathbb{R}^d} \left\| \sum_{n=1}^N \alpha_n \varepsilon_n^N (\xi) b_n \right\|_{B_{2, \bar{q}}(\mathbb{R}^d, \gamma_d)} \rho(d\xi) \leq \|f_N\|_{B_{2, \bar{q}}(\mathbb{R}^d, \gamma_d)}.
with \(\|a^N\|_{(e)} \to \infty\) as \(N \to \infty\). Therefore, \(\sup_{N=1,2,\ldots} \|f_N\|_1 = \infty\).

(c) Using (Theorem 3.1) (this time applied to \((\theta, \tau))\) gives
\[
\|f_N\|_{B^0_\infty(\mathbb{R}^d)} \leq c \|a^N\|_{(e)} \leq c \|a\|_{(e)} < \infty.
\]

(d) Finally, we show that there is a constant \(c > 0\) such that
\[
\|f_N\|_{\text{Hö}^0_{\theta,\bar{\theta},\tau}(\mathbb{R}^d)} \leq c \|a^N\|_{(e)} \leq c \|a\|_{(e)} < \infty.
\]
For this purpose we define a linear operator bounded as operator between
\[
T : \ell_1^{(0)}(\mathbb{R}^d) \to C^0_b(\mathbb{R}^d) \quad \text{and} \quad T : \ell_1^{(1)}(\mathbb{R}^d)
\]
by \(T((\beta_n)_{n=1}^\infty) := \sum_{n=1}^\infty \beta_n b_n\). By interpolation we get that
\[
T : \ell_1^{(\theta)}(\mathbb{R}^d), \text{Lip}^0(\mathbb{R}^d))_{\theta,\bar{\theta}}
\]
is bounded as well.

(e) Combining (b), (c), and (d) gives
\[
\sup_{N=1,2,\ldots} \|f_N\|_0 < \infty \quad \text{and} \quad \sup_{N=1,2,\ldots} \|f_N\|_1 = \infty
\]
and the proof is complete. \(\Box\)

5. An application to parabolic PDEs

We use \(d = 1\) and a stochastic basis and Brownian motion \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,1]})\) and \((W_t)_{t \in [0,1]}\) as in Section 4.2 here restricted to \(t \in [0,1]\) and with \(\mathcal{F} = \mathcal{F}_1\). For \(f \in L^2(\mathbb{R}, \gamma_1)\) we let \(F : [0, T] \times \mathbb{R} \to \mathbb{R}\) be given by
\[
F(t, x) := \mathbb{E}f(x + W_{t-}).
\]
Moreover, for \(t \in [0,1]\) and \(\alpha > 0\) we let
\[
\varphi_t := \frac{\partial F}{\partial x}(t, W_t) \quad \text{and} \quad \mathcal{I}^\alpha_x \varphi := \alpha \int_0^1 (1 - u)^{\alpha-1} \varphi_{u \wedge t} \, du.
\]
For \(\theta \in (0, 1)\) we know from [4] Theorem 6.6 that
\[
(5.1) \quad \mathcal{I}^\alpha_x \varphi - \varphi_0 \in \text{BMO}_2([0,1]) \quad \text{for} \quad f \in \text{Hö}^0_{\theta,2}(\mathbb{R}) \quad \text{and} \quad \alpha := \frac{1 - \theta}{2}.
\]
The question is whether we can use the condition \(f \in \text{Hö}^0_{\theta, q}(\mathbb{R})\) for some \(q \in (2, \infty]\). We will disprove this:

**Theorem 5.1.** If \((\theta, q) \in (0, 1) \times (2, \infty]\), then there is an \(f \in \text{Hö}^0_{\theta, q}(\mathbb{R})\) such that
\[
\mathcal{I}^\alpha_x \varphi - \varphi_0 \notin \text{BMO}_2([0,1]) \quad \text{for} \quad \alpha := \frac{1 - \theta}{2}.
\]

**Proof.** Assume \(f \in \text{Hö}^0_{\theta, q}(\mathbb{R})\). From [4] formula (3.3) of Proposition 3.8 one gets
\[
\mathcal{I}^\alpha_x \varphi = \varphi_0 + \int_{(0, t]} (1 - u)^{\alpha} \, d\varphi_u \quad \text{a.s.}
\]
If \(\mathcal{I}^\alpha_x \varphi - \varphi_0 \in \text{BMO}_2([0,1])\) would hold, then Itô’s isometry would give
\[
\sqrt{\mathbb{E} \int_0^1 (1 - u)^{1-\theta} \left| \frac{\partial^2 F}{\partial x^2}(u, W_u) \right|^2 \, du} = \sqrt{\mathbb{E} \int_0^1 (1 - u)^{2\alpha} \left| \frac{\partial^2 F}{\partial x^2}(u, W_u) \right|^2 \, du}
\]
\[
\sqrt{\mathbb{E} \int_0^1 (1-u)^{2\alpha} \, d[\varphi]_u}
\]
\[
= \sup_{t \in [0,1)} \|I^\alpha_t \varphi - \varphi_0\|_{L^2}
\]
\[
\leq \|I^\alpha \varphi - \varphi_0\|_{\text{BMO}_2((0,1))} < \infty.
\]

By \cite{5} Theorem 3.1] this would imply \( f \in \mathcal{B}^{\theta}_{2,2}(\mathbb{R}, \gamma_1) \). Now Corollary 4.4 applied to \((q,\gamma) = (2,q)\) implies our statement. \( \square \)

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