Eventually Positive Semigroups of Linear Operators

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August 8, 2015

Abstract

We develop a systematic theory of eventually positive semigroups of linear operators mainly on spaces of continuous functions. By eventually positive we mean that for every positive initial condition the solution to the corresponding Cauchy problem is positive for large enough time. Characterisations of such semigroups are given by means of resolvent properties of the generator and Perron–Frobenius type spectral conditions. We apply these characterisations to prove eventual positivity of several examples of semigroups including some generated by fourth order elliptic operators and a delay differential equation. We also consider eventually positive semigroups on arbitrary Banach lattices and establish several results for their spectral bound which were previously only known for positive semigroups.

1 Introduction

One of the distinguishing features of many second-order parabolic boundary value problems is their positivity preserving property: if the initial condition is positive, so is the solution at all positive times. Such equations are frequently expressed as an abstract Cauchy problem of the form

\[
\dot{u}(t) = Au(t) \quad \text{if } t \geq 0, \quad u(0) = u_0,
\]

on a (complex) Banach lattice \(E\) such as \(L^p(\Omega)\) or \(C(\overline{\Omega})\), where \(A\) is the generator of a strongly continuous semigroup. If we represent the solution of (1.1) in terms of the

\*=Supported by a scholarship within the scope of the LGFG Baden-Württemberg, Germany.
\†Partly supported by a fellowship of the Alexander von Humboldt Foundation, Germany.
Dedicated with great pleasure to Wolfgang Arendt on the occasion of his 65th birthday

Mathematics Subject Classification (2010): 47D06 47B65 34G10
Published in J. Math. Anal. Appl. 433 (2016), 1561–1593. DOI: 10.1016/j.jmaa.2015.08.050
corresponding semigroup \((e^{tA})_{t\geq 0}\), then positivity means that \(u_0 \geq 0\) implies \(e^{tA}u_0 \geq 0\) for all \(t \geq 0\). There is a sophisticated general theory of positive semigroups, which has found a large number of applications; see for instance \([3]\).

However, if \(A\) is the realisation of a differential operator, such positivity—which is usually obtained as a consequence of the maximum principle—is surprisingly rare. Indeed, under mild auxiliary assumptions on the operator \(A\), \textit{a priori} of arbitrary order, positivity of the semigroup \((e^{tA})_{t\geq 0}\) already implies that \(A\) is second-order elliptic if \(E = L^p(\mathbb{R}^d)\) \([24]\) or \(E = C(\overline{\Omega})\) \([6]\).

In such a case, in an attempt to bypass this restriction, we could weaken the requirement on the semigroup and stipulate that \(e^{tA}u_0\) merely be positive for \(t \geq 0\) “large enough” whenever \(u_0 \geq 0\). Indeed, in recent times various disparate examples of such “eventually positive semigroups” have emerged, all seemingly completely independent of each other. Here are some examples, many of which we will consider in more detail below, in Section 6.

A matrix exponential \(e^{tA}\) can be positive for large \(t\) even if \(A\) has some negative off-diagonal entries, a phenomenon which seems to have been observed only quite recently; see \([26]\) and the references therein.

For elliptic operators of order \(2m\), \(m \geq 2\), there is no maximum principle in general. The resolvent of the bi-Laplacian exhibits positivity properties on very few domains such as balls and perturbations of balls; see \([10, 19]\). The question as to whether the corresponding parabolic problem becomes “essentially” positive for large \(t > 0\) was investigated in \([15, 17]\).

Another example is the \textit{Dirichlet-to-Neumann operator} \(D_\lambda\) associated with \(\Delta u + \lambda u = 0\) on a domain \(\Omega\). For \(\lambda\) on one side of the first Dirichlet eigenvalue, the semigroup generated by \(-D_\lambda\) on \(L^2(\partial\Omega)\) is positive as shown in \([7]\). For other values of \(\lambda\) the semigroup may be positive, eventually positive or neither as the example of the disc shows; see \([12]\). The present paper had its genesis in an attempt to understand this phenomenon better at a theoretical level. We provide a detailed discussion in Section 6.

A further example is provided by certain delay differential equations. Under special assumptions they generate positive semigroups; see \([13, \text{Theorem VI.6.11}]\). We will show in Section 6.5 that there are also situations where the semigroup is eventually positive without being positive.

The variety of examples suggests that eventually positive semigroups could prove more ubiquitous than their positive counterparts, and no doubt more examples will emerge. Quite surprisingly, to date there seems to have been no unified treatment of such objects, in marked contrast to the positive case. Here, and in an envisaged sequel \([11]\), we intend to address this. Our abstract theory will allow us to recover several known results and to prove some new ones in the above-mentioned areas.

Our main focus in this article is the investigation of strongly continuous semigroups with eventual positivity properties on \(C(K)\), the space of complex-valued continuous functions on a compact non-empty Hausdorff space \(K\). In order to give an idea of our results, we first need to introduce some notation. We call \(f\) \textit{positive} if \(f(x) \geq 0\) for all \(x \in K\) and write \(f \geq 0\). If \(f \geq 0\) but \(f \neq 0\) we write \(f > 0\); we call \(f\) \textit{strongly positive} and write \(f \gg 0\) if there exists \(\beta > 0\) such that \(f \geq \beta 1\), where \(1\) is the constant function on \(K\) with value one. A bounded linear operator \(T\) on \(C(K)\) is called \textit{strongly positive}, denoted by \(T \gg 0\), if \(Tf \gg 0\) whenever \(f > 0\), and similarly, a linear functional \(\varphi : C(K) \to \mathbb{C}\) is called \textit{strongly positive}, again denoted by \(\varphi \gg 0\), if \(\varphi(f) > 0\) for each \(f > 0\).
If \( A : D(A) \to E \) is a closed operator with domain \( D(A) \) on the Banach space \( E \) we denote by \( \sigma(A) \) and \( g(A) := \mathbb{C} \setminus \sigma(A) \) the spectrum and resolvent set of \( A \), respectively. Any point in \( \sigma(A) \) is called a spectral value. We call

\[
s(A) := \sup \{ \Re \lambda : \lambda \in \sigma(A) \} \in [-\infty, \infty]
\]

(1.2)

the spectral bound of \( A \). For some classes of positive semigroups the spectral bound \( s(A) \) is necessarily a dominant spectral value, by which we mean that \( s(A) \in \sigma(A) \) and that the peripheral spectrum

\[
\sigma_{\text{per}}(A) := \sigma(A) \cap (s(A) + i\mathbb{R})
\]

(1.3)

consists of \( s(A) \) only. As in the theory of positive semigroups there is a close relationship between positivity properties of the resolvent

\[
g(A) \to \mathcal{L}(E), \quad \lambda \mapsto R(\lambda, A) := (\lambda I - A)^{-1}
\]

(1.4)

and the semigroup \( (e^{tA})_{t \geq 0} \); we shall also see that the spectral projection \( P \) associated with \( s(A) \) plays an important role here in the case that \( s(A) \) is an isolated spectral value. The main part of this paper is, roughly speaking, devoted to characterising the relationship between these three objects. The following theorem provides a rather incomplete but indicative snapshot of our results.

**Theorem 1.1.** Let \( A \) be the generator of a strongly continuous real semigroup \( (e^{tA})_{t \geq 0} \) on \( C(K) \) with spectral bound \( s(A) > -\infty \). Assume that \( \sigma_{\text{per}}(A) \) is finite and consists of poles of the resolvent. Then the following assertions are equivalent:

(i) For every \( f > 0 \) there exists \( t_0 > 0 \) such that \( e^{tA}f \gg 0 \) for all \( t \geq t_0 \);

(ii) The semigroup \( (e^{(A-s(A)I)}t \geq 0 \) is bounded, \( s(A) \) is a dominant spectral value and for every \( f > 0 \) there exists \( \lambda_0 > s(A) \) such that \( R(\lambda, A)f \gg 0 \) for all \( \lambda \in (s(A), \lambda_0] \);

(iii) The semigroup \( (e^{(A-s(A)I)}t \geq 0 \) is bounded, \( s(A) \) is a dominant spectral value and a geometrically simple eigenvalue; moreover, \( \ker(s(A)I-A) \) and \( \ker(s(A)I-A') \) each contain a strongly positive vector.

(iv) The semigroup \( (e^{(A-s(A)I)}t \geq 0 \) is bounded, \( s(A) \) is a dominant spectral value and the associated spectral projection \( P \) fulfils \( P \gg 0 \).

Let us briefly comment on the role of the different assertions in the theorem: The implication “(i) \( \Rightarrow \) (iii)” is a typical Perron–Frobenius type result, since it infers the existence of a dominant spectral value and corresponding positive eigenvectors from positivity properties of the semigroup. The converse implication “(iii) \( \Rightarrow \) (i)” has no analogue for positive semigroups and is characteristic for eventual positivity. The close relationship between properties of the semigroup in (i) and the resolvent in (ii) is reminiscent of the theory of positive semigroups, where the semigroup is positive if and only if the resolvent is positive for all sufficiently large \( \lambda \); see [13, Theorem VI.1.8]. Here, however, it turns out that we need to consider small \( \lambda \) and an additional spectral condition on the generator \( A \). To consider the spectral projection as in (iv) does not seem to be common in classical Perron–Frobenius theory; however, we shall see that it is essential for relating the other conditions to each other.

It is our intention not just to prove a blanket result like Theorem 1.1, but to give a more detailed analysis of each of the different objects considered in the theorem, namely the spectral projection, the resolvent and the semigroup. This is done in Sections 3–5.
Section 3 is concerned with the characterisation of eigenvalues $\lambda_0$ for which $A$ and its dual $A'$ have a strongly positive eigenvector and for which the corresponding eigenspace of $A$ is one-dimensional. We show in Proposition 3.1 that this is equivalent to the spectral projection $P$ associated with $\lambda_0$ being strongly positive.

In Section 4 we characterise strongly positive projections by means of eventual positivity properties of the resolvent. Our main result on resolvents is given in Theorem 4.4. Under some additional assumptions on the peripheral spectrum of $A$ we characterise individually eventually strongly positive semigroups in terms of resolvent and spectral projection in Section 5; see in particular Theorem 5.4 and Corollary 5.6.

The assertions of Theorem 1.1 follow from combining Proposition 3.1, Theorem 4.4 and Theorem 5.4. Note that in general $t_0$ and $\lambda_0$ in (i) and (ii) of Theorem 1.1 depend on the choice of $f > 0$; these are individual rather than uniform conditions with respect to $f$. We will also illustrate the distinction between individual and uniform eventual positivity in Section 5; see in particular Examples 5.7 and 5.8.

In Section 6 we apply our theory to a diverse range of examples: matrix exponentials, a Dirichlet-to-Neumann operator, the square of the Laplacian with Robin boundary conditions, and a delay differential equation. Here, the advantages of our approach become apparent: in practice it is usually easy to check the required condition on the spectral projection of the dominant eigenvalue, and this characterises eventual positivity.

In Section 7 we prove various spectral properties of operators generating eventually positive semigroups on arbitrary Banach lattices, in particular regarding the spectral bound. These generalise corresponding results about positive semigroups, and will be needed in several other places throughout the article. Section 8 is concerned with some remarks on resolvents. The abstract results in Sections 7 and 8 stand apart from the main thrust of the paper, which is concerned with characterisation theorems suitable for concrete applications. Therefore, we defer them until the end.

In [11], we will prove many similar results both for the technically more complicated case of arbitrary Banach lattices and for weaker forms of eventual positivity, which will allow us to establish characterisations which are uniform in the function $f > 0$. This can be used to study various types of higher-order elliptic operators.

2 Preliminaries

We briefly introduce some further notation and basic facts we use throughout the paper. Whenever $E$ is a complex Banach space, we denote by $\mathcal{L}(E)$ the space of bounded linear operators on $E$. If $M \subseteq E$, then $\langle M \rangle$ denotes the span of $M$ in $E$ (over $\mathbb{C}$). If $A \in \mathcal{L}(E)$, then

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

(2.1)

denotes the spectral radius of $A$. We will make extensive use of properties of spectral projections, in particular in connection with poles of the resolvent $R(\cdot, A)$. As an essential tool we make use of the Laurent expansion of $R(\cdot, A)$ about isolated points of the spectrum of $A$, which may be summarised as follows.

Remark 2.1. Let $A$ be a closed operator on a Banach space $E$. The resolvent $\lambda \mapsto R(\lambda, A)$ is an analytic map on $\varrho(A) \subseteq \mathbb{C} \to \mathcal{L}(E)$. If $\lambda_0$ is an isolated point of $\sigma(A)$, then there
exist operators $U, P, B \in \mathcal{L}(E)$ such that the Laurent expansion
\begin{equation}
R(\lambda, A) = \frac{P}{\lambda - \lambda_0} + \sum_{k=1}^{\infty} \frac{U^k}{(\lambda - \lambda_0)^{k+1}} + \sum_{k=0}^{\infty} (-1)^k (\lambda - \lambda_0)^k B^{k+1}
\end{equation}
is valid for $\lambda$ is some neighbourhood of $\lambda_0$. Moreover, $P$ is the spectral projection associated with $\lambda_0$, $U = -(\lambda_0 I - A)P$, $BP = PB = 0$ and $UB = BU = 0$; see [22, Section III-6.5], [31, Section VIII.8], [13, p. 246–248] or [9]. The operator $U$ is called the (quasi-) nilpotent part of $A$ associated with $\lambda_0$. For convenience we set $U^0 := P$. Then $U^n = (-1)^n (\lambda_0 I - A)^nP$ for all $n \in \mathbb{N}_0$.

Assume for the rest of this remark that $\lambda_0$ is a pole of the resolvent, that is, there exists a minimal number $m \geq 1$, the order of the pole, such that $U^{m-1} \neq 0$ and $U^m = 0$. If $\lambda_0$ is a pole of order $m$, then $\lambda_0$ is an eigenvalue since $(\lambda_0 I - A)U^{m-1} = -U^m = 0$ and hence $\{0\} \neq \text{im}(U^{m-1}) \subseteq \ker(\lambda_0 I - A)$. If $\lambda_0$ is a pole of order $m \geq 2$, then there always exists a generalised eigenvector $x \in \ker((\lambda_0 I - A)^2) \setminus \ker(\lambda_0 I - A)$. Indeed, if we choose $y \in E$ such that $U^{m-1} y \neq 0$, then $x = U^{m-2} y$ has the desired properties.

The dimension of the eigenspace $\ker(\lambda_0 I - A)$ is called the geometric multiplicity of $\lambda_0$ as an eigenvalue of $A$. The dimension of the generalised eigenspace $\bigcup_{m \in \mathbb{N}} \ker((\lambda_0 I - A)^m)$ is called the algebraic multiplicity. The generalised eigenspace coincides with $\text{im} P$. The eigenvalue $\lambda_0$ is called geometrically (algebraically) simple if its geometric (algebraic) multiplicity equals $1$. The pole $\lambda_0$ of the resolvent is a simple pole if and only if $\ker(\lambda_0 I - A) = \ker((\lambda_0 I - A)^2)$ if and only if $\ker(\lambda_0 I - A) = \text{im} P$. Assume that the geometric multiplicity is finite. Then it follows that $\lambda_0$ is a simple pole if and only if the algebraic and geometric multiplicities coincide. See also [9, Section 4] for some details.

We assume the reader to be familiar with the basic theory of Banach lattices. As a standard reference for this topic we refer to [27]. If $E$ denotes a complex Banach lattice, then $E$ is by definition the complexification of a real Banach lattice $E_\mathbb{R}$; see [27, Section II.11]. We denote by $E_+ := (E_\mathbb{R})_+ := \{f \in E_\mathbb{R} : f \geq 0\}$ the positive cone in $E$. To avoid any ambiguities we shall adopt the following conventions, which mirror the notation introduced above when $E = C(K)$: we call $f \in E$ positive and write $f \geq 0$ if $f \in E_+$, we write $f > 0$ if $f \geq 0$ and $f \neq 0$. If $E_+ \subseteq E_\mathbb{R}$ has non-empty interior, then we write $f \gg 0$ if $f$ is in the interior of $E_+$ and say $f$ is strongly positive.

A linear operator $T$ between two complex Banach lattices $E$ and $F$ is called positive if $TE_+ \subseteq F_+$. If $F_+ \subseteq F_\mathbb{R}$ has non-empty interior we call $T$ strongly positive and we write $T \gg 0$ if $Tf \gg 0$ whenever $f > 0$. We also apply this notation to elements from the dual space $E'$ and thus say the functional $\varphi \in E'$ is strongly positive if $\varphi \gg 0$ as a linear map from $E$ into $\mathbb{C}$, that is, $\langle \varphi, f \rangle = \varphi(f) > 0$ whenever $f > 0$.

A linear operator $A : E \supset D(A) \rightarrow E$ is called real if $x + iy \in D(A)$ implies that $x, y \in D(A)$ whenever $x, y \in E_\mathbb{R}$, and if $A$ maps $D(A) \cap E_\mathbb{R}$ into $E_\mathbb{R}$. Note that a positive operator $T \in \mathcal{L}(E)$ is automatically real.

We assume the reader to be familiar with the basic theory of $C_0$-semigroups. We will always denote a $C_0$-semigroup on a complex Banach space $E$ by $(e^{tA})_{t \geq 0}$, where $A$ is the generator of the semigroup. The growth bound of the semigroup $(e^{tA})_{t \geq 0}$ will be denoted by $\omega_0(A)$. A $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on a complex Banach lattice is called real if the operator $e^{tA}$ is real for every $t \geq 0$. It is easy to see that $(e^{tA})_{t \geq 0}$ is real if and only if $A$ is real.

The following result will be important to reduce some results for eventually positive semigroups to results for positive semigroups. For real Banach lattices it is stated in [27, Proposition III.11.5], but it easily generalises to complex Banach lattices.
 Proposition 2.2. Let $E$ be a complex Banach lattice and let $P \in \mathcal{L}(E)$ be a positive projection. Then there is an equivalent norm on $PE$ such that $PE_\mathbb{R} \subseteq PE$ is a real Banach lattice for the order induced by $E_\mathbb{R}$ and such that $PE$ becomes the complexification of the real Banach lattice $PE_\mathbb{R}$.

We will need the following result on the range of the spectral projection associated with the peripheral spectrum.

 Proposition 2.3. Let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup on a complex Banach space $E$ such that $\sigma(A) \neq \emptyset$. Suppose that $\sigma_{\text{per}}(A)$ as defined in (1.3) is finite and consists of simple poles of the resolvent. Denote by $P_{\text{per}}$ the spectral projection corresponding to $\sigma_{\text{per}}(A)$. Then there exists a sequence of positive integers $t_n \to \infty$ such that $e^{t_n(A-\sigma(A)I)}f \to f$ for every $f \in \text{im}(P_{\text{per}})$.

Proof. We may assume throughout the proof that $s(A) = 0$ and that $\sigma_{\text{per}}(A) \neq \emptyset$. Let $f \in P_{\text{per}}E$ and let $\sigma_{\text{per}}(A) = \{i\beta_1, \ldots, i\beta_m\}$ for $\beta_1, \ldots, \beta_m \in \mathbb{R}$. If $P_k$ denotes the spectral projection associated with $i\beta_k$, then $P_{\text{per}} = P_1 + \cdots + P_m$. As $i\beta_k$ are simple poles of the resolvent we have $\text{im} P_k = \ker(i\beta_k I - A)$ for every $k = 1, \ldots, m$ and therefore $e^{tA}f = e^{it\beta_1}P_1f + \cdots + e^{it\beta_m}P_m f$ for all $t \geq 0$. Let $\lambda = (e^{i\beta_1}, \ldots, e^{i\beta_m}) \in \mathbb{T}^m$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$ and $\mathbb{T}^m$ the standard $m$-dimensional torus. Define $s_\lambda : \mathbb{T}^m \to \mathbb{T}^m$ to be the group rotation by $\lambda$, that is, $s_\lambda z = \lambda z$ for every $z \in \mathbb{T}^m$; here, $\mathbb{T}^m$ is endowed with pointwise multiplication, with respect to which it is a compact topological group. By a standard result from topological dynamical systems the element $1 = (1, \ldots, 1)$ is recurrent with respect to the group rotation $s_\lambda$, that is, there exists a sequence of positive integers $t_n \to \infty$ such that $s_{t_n} 1 \to 1$; see [16, Definition 1.2 and Theorem 1.2]. We conclude that $\lambda^{t_n} \to 1$ and hence,

$$e^{t_n A}f = e^{it_n\beta_1}P_1f + \cdots + e^{it_n\beta_m}P_m f \to P_1f + \cdots + P_m f = f$$

as $n \to \infty$ as claimed. \hfill \Box

3 Strongly positive projections and Perron–Frobenius properties

One important feature of positive operators is that the spectral radius is itself an element of the spectrum. If the operator $T$ is irreducible and the spectral radius $r(T) > 0$ is a pole of the resolvent of $T$, then the Perron–Frobenius theorem (or Krein–Rutman theorem) asserts that $r(T)$ is an algebraically simple eigenvalue of $T$ and $T'$ with a strongly positive eigenvector; see [27, Theorem V.5.2 and its Corollary and Theorem V.5.4]. We may refer to this as $T$ having a “Perron–Frobenius property”. However, there seems to be no intrinsic reason why this property should only be considered for the spectral radius of a bounded, positive operator. Hence, we start by characterising arbitrary eigenvalues having a Perron–Frobenius type property. We also explain this result geometrically.

 Proposition 3.1. Let $A$ be a closed, densely defined and real operator on $E := C(K)$. Suppose that $\lambda_0 \in \mathbb{R}$ is an eigenvalue of $A$ and a pole of the resolvent $R(\cdot, A)$. Let $P$ be the spectral projection associated with $\lambda_0$. Then the following assertions are equivalent.

(i) $P \gg 0$;

(ii) $\lambda_0$ is algebraically simple.

(iii) $\text{im}(P_{\lambda_0})$ is finite and consists of simple poles of the resolvent.

(iv) $s_{\lambda_0} f \to f$ as $t \to \infty$ for every $f \in \text{im}(P_{\lambda_0})$.

(v) $\lambda_0$ is an algebraically simple pole of $R(\cdot, A)$.

(vi) $\lambda_0$ is an eigenvalue of $A$ with a Perron–Frobenius property.
(ii) The eigenvalue $\lambda_0$ of $A$ is geometrically simple, and $\ker(\lambda_0 I - A)$ and $\ker(\lambda_0 I - A')$ each contain a strongly positive vector;

(iii) The eigenvalue $\lambda_0$ of $A$ is algebraically simple, $\ker(\lambda_0 I - A)$ contains a strongly positive vector and $\text{im}(\lambda_0 I - A) \cap E_+ = \{0\}$.

If assertions (i)–(iii) are fulfilled, then $\lambda_0$ is a simple pole of the resolvents $R(\cdot, A)$ and $R(\cdot, A')$. Moreover, $\dim(\text{im} P) = \dim(\text{im} P') = 1$ and $\lambda_0$ is the only eigenvalue of $A$ having a positive eigenvector.

Proof. By replacing $A$ with $A - \lambda_0 I$, we may assume without loss of generality that $\lambda_0 = 0$.

“(i) $\Rightarrow$ (ii)” As $P$ is a positive operator its image is spanned by positive elements. Let $u, y \in E_+ \cap \text{im} P$ be non-zero. Since $P \gg 0$ we have $u = Pu \gg 0$ and so $\alpha_0 := \inf\{\alpha : \alpha u - y \geq 0\} \in (0, \infty)$. As $E_+$ is closed $\alpha_0 u - y \geq 0$, and moreover $\alpha_0 u - y \in \text{im} P$. Hence, either $\alpha_0 u - y = 0$ or $\alpha_0 u - y \gg 0$. As $E_+$ has non-empty interior in $E_R$, the second possibility cannot occur as it contradicts the definition of $\alpha_0$. Hence $y = \alpha_0 u$ and so $\dim(\text{im} P) = 1$. In particular $\lambda_0$ is algebraically and hence geometrically simple with eigenfunction $u \gg 0$. Note that 0 is also an algebraically simple eigenvalue of $A'$, see [22, Remark III-6.23], and that the dual $P'$ of $P$ is the corresponding spectral projection. As $P \gg 0$ also $P'v \gg 0$ for all $v > 0$ in $E_+^\perp$, that is, $\text{im} P'$ is spanned by an eigenvector $v \gg 0$ of $A'$ corresponding to the eigenvalue 0.

“(ii) $\Rightarrow$ (iii)” By assumption 0 is a geometrically simple eigenvalue. To prove that it is algebraically simple it is sufficient to show that $x \in \ker A^2$ implies $x \in \ker A$, see Remark 2.1. Hence let $x \in \ker A^2$. We know from (ii) that $\ker A$ is spanned by a vector $u \gg 0$ and that $\ker A'$ contains a vector $v \gg 0$. As $Ax \in \ker A$, there exists $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $\alpha Ax \geq 0$. As $\langle v, \alpha Ax \rangle = \langle A'v, \alpha x \rangle = 0$ and $v \gg 0$ we conclude that $Ax = 0$. Therefore $\langle u \rangle = \ker A = \text{im} P$. Now let $y = Ax \in E_+ \cap \text{im} A$. Then $\langle v, y \rangle = \langle v, Ax \rangle = \langle A'v, x \rangle = 0$. As $v \gg 0$ and $y \geq 0$ we conclude that $y = 0$. Hence, $E_+ \cap \text{im} A = \{0\}$.

“(iii) $\Rightarrow$ (i)” As 0 is algebraically simple, (iii) implies that $\text{im} P = \ker A = \langle u \rangle$ for some $u \gg 0$ and $E = \text{im} P \oplus \text{im} A$. Hence, if $x > 0$, then there exist $\alpha \in \mathbb{C}$ and $y \in D(A)$ such that $x = \alpha u + Ay$. Since $A$ is real, so is $P$, and hence $\alpha \in \mathbb{R}$. If $\alpha < 0$, then $x - \alpha u = Ay \in E_+ \cap \text{im} A$, which implies $Ay = 0$ by (iii). Then $x = \alpha u = Px \leq 0$ and $0 < x$, which is a contradiction. Hence, $\alpha > 0$ and thus $Px = \alpha u \gg 0$.

Finally note that the algebraic simplicity of 0 as an eigenvalue of $A$ from (iii) implies that 0 is a simple pole of the resolvent $R(\cdot, A)$; thus, it is also a simple pole of $R(\cdot, A') = R(\cdot, A')^\prime$. Now suppose that $\lambda_1 \neq 0$ is an eigenvalue of $A$ with eigenvector $u_1$. Then $u_1 = \lambda_1^{-1}Az_1 \in \text{im} A$. As $E_+ \cap \text{im} A = \{0\}$ by (iii) we conclude that $u_1$ cannot be positive. Hence, 0 is the only eigenvalue having a positive eigenvector.

To the best of our knowledge, the relationship between strong positivity of the spectral projection and the existence of strongly positive eigenvectors for arbitrary poles of the resolvent as given in Proposition 3.1 has not been investigated before. Our argument which shows that algebraic simplicity follows from geometric simplicity if the corresponding eigenvectors of $A$ and $A'$ are strongly positive is similar to the proof of [18, Theorem 4.12(ii)]. Moreover, a related argument for the eigenspace associated with the spectral bound of a positive semigroup can be found in [3, Remark IV-2.2(c)].

Remark 3.2. There is a simple geometric explanation for the equivalent conditions in Proposition 3.1 if $E = \mathbb{R}^N$ and $\lambda_0 = 0$. Due to the algebraic simplicity of the eigenvalue
the direct sum decomposition $E = \ker(A) \oplus \text{im}(A)$ completely reduces $A$. Recall that $Px$ is the projection of $x$ onto the span of an eigenvector $u$ of $A$. The fact that $P \gg 0$ means that $u \gg 0$ and that $E_+$ is on one side of im $A$. This is also the explicit statement in (iii). To interpret (ii) let $v \gg 0$ be an eigenvector of $A'$ to the eigenvalue 0. Then, $\langle v, Ax \rangle = \langle A'v, x \rangle = 0$ for all $x \in \mathbb{R}^N$, so that im $A = \langle v \rangle^\perp$ is perpendicular to $v$. Hence, if $E_+$ is to be on one side of im $A$, then $v \gg 0$ (or equivalently $v \ll 0$). The configuration is illustrated in Figure 3.1.

![Figure 3.1: Geometric meaning of the Perron–Frobenius property](image)

**Remark 3.3.** In [25, 26], a matrix $A$ is said to have the *strong Perron–Frobenius property* if $r(A)$ is a simple and dominant eigenvalue of $A$ having a strongly positive eigenvector. However, the full conclusion of the Perron–Frobenius theorem is statement (ii) in Proposition 3.1; see for instance [28, Theorem 1.1]. Hence it seems rather more natural to define this latter statement to be the “strong Perron–Frobenius property”. According to Proposition 3.1 one could then summarise the conclusion of the Perron–Frobenius theorem by saying that the spectral projection $P$ associated with $r(A)$ is strongly positive.

### 4 Eventually strongly positive resolvents

It is a feature of Perron–Frobenius theory that the resolvent $R(\lambda, A)$ is positive for all $\lambda > s(A)$. In the context of eventually positive semigroups we cannot expect such a property for all $\lambda > s(A)$, but we show that there is nevertheless a weaker positivity property. We begin with a definition.

**Definition 4.1.** Let $A$ be a closed real operator on $E = C(K)$ and let $\lambda_0$ be either $-\infty$ or a spectral value of $A$ in $\mathbb{R}$.

(a) The resolvent $R(\cdot, A)$ is called *individually eventually (strongly) positive at $\lambda_0$*, if there is a $\lambda_2 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_2] \subseteq \rho(A)$ and for each $f \in E_+ \setminus \{0\}$ there is a $\lambda_1 \in (\lambda_0, \lambda_2]$ such that $R(\lambda, A)f \geq 0$ ($\gg 0$) for all $\lambda \in (\lambda_0, \lambda_1]$.

(b) The resolvent $R(\cdot, A)$ is called *uniformly eventually (strongly) positive at $\lambda_0$*, if there exists $\lambda_1 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_1] \subseteq \rho(A)$ and $R(\lambda, A) \geq 0$ ($\gg 0$) for every $\lambda \in (\lambda_0, \lambda_1]$.

**Example 5.7** below shows that it is necessary to distinguish between individual and uniform properties in Definition 4.1.

Positivity of the resolvent is an important concept within the theory of positive semigroups. For example, it is well known that a $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is positive if and only if $R(\lambda, A) \geq 0$ for all $\lambda > s(A)$. If this is the case, then we even have $R(\lambda, A) \gg 0$ for
some (equivalently all) \( \lambda > s(A) \) if and only if the semigroup is irreducible, see [3, Definition B-III-3.1] (note however that \( R(\lambda, A) \gg 0 \) for some \( \lambda > s(A) \) implies \( R(\lambda, A) \gg 0 \) for all \( \lambda > s(A) \) only if it is already known that \( (e^{tA})_{t \geq 0} \) is a positive semigroup). The assertions in Definition 4.1 are thus generalisations of those well-known resolvent properties to spectral values other than \( s(A) \) and to smaller ranges of \( \lambda \).

We proceed by stating a simple criterion for uniform eventual positivity of the resolvent at some spectral value \( \lambda_0 \).

**Proposition 4.2.** Let \( A \) be a closed real operator on \( E = C(K) \). Let \( \lambda_0 \) be \(-\infty\) or a spectral value of \( A \) in \( \mathbb{R} \). Assume that there exists \( \lambda_1 > \lambda_0 \) such that \( (\lambda_0, \lambda_1] \subseteq \rho(A) \) and \( R(\lambda_1, A) \geq 0 \). Then the following assertions are true.

(i) The resolvent \( R(\cdot, A) \) is uniformly eventually positive at \( \lambda_0 \). More precisely, \( R(\lambda, A) \geq 0 \) for all \( \lambda \in (\lambda_0, \lambda_1] \).

(ii) If \( R(\lambda_1, A)^n \) is strongly positive for some \( n \in \mathbb{N} \), then \( R(\cdot, A) \) is uniformly eventually strongly positive at \( \lambda_0 \). More precisely, \( R(\lambda, A) \gg 0 \) for all \( \lambda \in (\lambda_0, \lambda_1) \).

**Proof.** We prove (i) and (ii) simultaneously. To that end let \( \prec \) denote \( \leq \) in case (i) and \( \ll \) in case (ii). We set \( U := \{ \lambda \in (\lambda_0, \lambda_1) : R(\mu, A) > 0 \text{ for all } \mu \in (\lambda, \lambda_1) \} \) and show that \( U = (\lambda_0, \lambda_1) \). Because \((\lambda_0, \lambda_1)\) is connected it is sufficient to show that \( U \) is non-empty, open and closed in \((\lambda_0, \lambda_1)\).

If \( \lambda_n \in U \) with \( \lambda_n \rightarrow \lambda \in (\lambda_0, \lambda_1) \) and \( \mu > \lambda \), then \( \mu > \lambda_n \) for \( n \) large enough. Hence, by definition of \( U \supset \lambda_n \), we have \( R(\mu, A) > 0 \) for all \( \mu \in (\lambda, \lambda_1) \). Thus \( \lambda \in U \), showing that \( U \) is closed in \((\lambda_0, \lambda_1)\).

Given \( \mu_0 \) in the open set \( g(A) \subseteq \mathbb{C} \), the analytic function \( R(\cdot, A) \) can be expanded as a power series

\[
R(\mu, A) = \sum_{k=0}^{\infty} (\mu_0 - \mu)^k R(\mu_0, A)^{k+1} \tag{4.1}
\]

whenever \( \varepsilon > 0 \) and \( \mu \) are such that \( \mu \in B(\mu_0, \varepsilon) \subseteq \rho(A) \); see [13, Proposition IV.1.3(i)]. If we choose \( \mu_0 = \lambda_1 \), then \( R(\mu_0, A) \geq 0 \) by assumption. In case (ii) at least one of the terms in (4.1) is strongly positive if \( \mu < \mu_0 \). Hence in both cases \((\lambda_1 - \varepsilon, \lambda_1) \subseteq U \) and so \( U \neq \emptyset \).

To show that \( U \) is open let \( \lambda \in U \) and let \( \varepsilon > 0 \) such that \( B(\lambda, \varepsilon) \subseteq \rho(A) \). Then choose \( \mu_0 > \lambda \) and \( \varepsilon_0 > 0 \) such that \( \mu_0 \in U \) and \( \lambda \in B(\mu_0, \varepsilon_0) \subseteq B(\lambda, \varepsilon) \). As \( R(\mu_0, A) > 0 \) it follows from (4.1) that \( R(\mu, A) > 0 \) for all \( \mu \in (\mu_0 - \varepsilon_0, \mu_0) \). By choice of \( \mu_0 \) and \( \varepsilon \) the interval \((\mu_0 - \varepsilon_0, \mu_0)\) is a neighbourhood of \( \lambda \), showing that \( U \) is open in \((\lambda_0, \lambda_1)\). Thus, \( U = (\lambda_0, \lambda_1) \) as claimed.

A consequence of Proposition 4.2 is the following simple but useful criterion for eventual positivity of resolvents, which we will use in Section 6 to study the square of the Robin Laplacian.

**Proposition 4.3.** Let \( A \) be a closed and real operator on \( E = C(K) \). Let \( \lambda_0 < 0 \) be \(-\infty \) or a spectral value of \( A \) and assume that \( (\lambda_0, 0] \subseteq \rho(A) \). Furthermore, suppose that there is a closed operator \( B : C(K) \supseteq D(B) \rightarrow C(K) \) such that \( A = (iB)^2 = -B^2 \). If \( R(0, B) \) is (strongly) positive, then \( R(\cdot, A) \) is uniformly eventually (strongly) positive at \( \lambda_0 \).

**Proof.** Both assertions follow from Proposition 4.2 since \( R(0, A) = R(0, B)^2 \).
We now formulate the main result of this section and characterise eventually positive resolvents by means of positive projections. In conjunction with Proposition 3.1 the following theorem not only contains a Perron–Frobenius (or Krein–Rutman) type theorem but also its converse.

**Theorem 4.4.** Let $A$ be a closed, densely defined and real operator on $E = C(K)$. Suppose that $\lambda_0 \in \mathbb{R}$ is an eigenvalue of $A$ and a pole of the resolvent $R(\cdot, A)$. Let $P$ be the corresponding spectral projection. Then the following assertions are equivalent.

(i) $P \gg 0$.

(ii) The resolvent $R(\cdot, A)$ is individually eventually strongly positive at $\lambda_0$.

If $\lambda_0 = s(A)$, then (i) and (ii) are also equivalent to the following assertions.

(iii) For every $\lambda > s(A)$ and every $f > 0$ there exists $n_0 \in \mathbb{N}$ such that $R(\lambda, A)^n f \gg 0$ for all $n \geq n_0$.

(iv) There exists $\lambda > s(A)$ such that for every $f > 0$ there exists $n_0 \in \mathbb{N}$ such that $R(\lambda, A)^n f \gg 0$ for all $n \geq n_0$.

**Remark 4.5.** Let the assumptions of Theorem 4.4 be satisfied, and suppose that the spectral projection $P$ associated with $\lambda_0$ is strongly positive. Theorem 4.4 shows that $R(\cdot, A)$ is individually eventually strongly positive at $\lambda_0$. We show that $\lambda_0$ can be anywhere on the real axis, independently of the spectral bound $s(A)$. For $\mu \in \mathbb{R}$ we set $A_\mu := A - \mu P$. Then $\sigma(A_\mu) = (\sigma(A) \setminus \{\lambda_0\}) \cup \{\lambda_0 - \mu\}$. Moreover, if $\lambda_0 - \mu \notin \sigma(A)$, then the spectral projection associated with $\lambda_0 - \mu$ is still $P$ and hence $R(\cdot, A_\mu)$ is individually eventually strongly positive at $\lambda_0 - \mu$.

A similar argument can also be used to show that $s(A)$ does not need to be a spectral value of $A$ even if $R(\lambda, A) \geq 0$ in some right neighbourhood of $s(A)$. We refer to Remark 5.3(b) for details.

Assertions (iii) and (iv) in Theorem 4.4 give conditions on large powers of the resolvent. In fact, it is sufficient to consider a single power, provided that this power is a strongly positive operator.

**Proposition 4.6.** Let $A$ be a closed, densely defined and real operator on $E = C(K)$. Suppose that $s(A) \in \mathbb{R}$ is an eigenvalue of $A$ and a pole of the resolvent $R(\cdot, A)$, and that there exist $n \in \mathbb{N}$ and $\lambda_1 > s(A)$ such that $R(\lambda_1, A)^n \gg 0$. Then the equivalent statements of Theorem 4.4 are fulfilled for $\lambda_0 = s(A)$.

The reader might compare the above proposition with Proposition 4.2. If, in addition to the assumptions of Proposition 4.6, the operator $R(\lambda_1, A)$ is positive, then $R(\cdot, A)$ is uniformly eventually strongly positive.

The remainder of this section is devoted to the proofs of Theorem 4.4 and Proposition 4.6. We can assume without loss of generality that $\lambda_0 = 0$ by replacing $A$ with $A - \lambda_0 I$.

The first step towards the proof of Theorem 4.4 is to express the spectral projection $P$ in terms of the resolvent.

**Lemma 4.7.** Let $A$ be a closed operator on a complex Banach space $E$ and assume that $0$ is an eigenvalue of $A$ and a pole of the resolvent $R(\cdot, A)$. Denote by $P$ the corresponding spectral projection.
(i) If \( \lambda > 0 \) is contained in \( \rho(A) \) and if the operator family \( ([\lambda R(\lambda, A)]^n)_{n \in \mathbb{N}} \) is bounded, then 0 is a simple pole of the resolvent.

(ii) Suppose in addition that \( 0 = s(A) \). If \( \lambda > 0 \) and \( s(A) = 0 \) is a simple pole of the resolvent, then

\[
\lim_{n \to \infty} [\lambda R(\lambda, A)]^n = P
\]

in \( \mathcal{L}(E) \).

**Proof.** (i) We give a proof by contrapositive. Suppose that 0 is a pole of order \( m \geq 2 \) of \( R(\cdot, A) \). If we set \( T := \lambda R(\lambda, A) \), then 1 is a pole of order \( m \) of \( R(\cdot, T) \); see [13, Proposition IV.1.18]. As \( m \geq 2 \) there is a generalised eigenvector \( x \in \ker(I - T)^2 \setminus \ker(I - T) \); see Remark 2.1. A short induction argument now shows that \( T^n x = x - n(I - T)x \) for each \( n \in \mathbb{N} \). As \( (I - T)x \neq 0 \), this implies that \( (T^n)_{n \in \mathbb{N}} \) is unbounded.

(ii) By the spectral mapping theorem for resolvents, we have

\[
\sigma(\lambda R(\lambda, A)) \setminus \{0\} = \frac{\lambda}{\lambda - \sigma(A)}
\]

for all \( \lambda > 0 \); see [22, Theorem III-6.15]. The map \( \mu \to \frac{\lambda}{\lambda - \mu} \) is a Möbius transformation mapping the left half plane onto the disc \( B_{1/2}(1/2) \). As 1 is an isolated point of the spectrum of \( \lambda R(\lambda, A) \), there exists \( c \in (0, 1) \) such that \( |\mu| \leq c \) for all \( \mu \in \sigma(\lambda R(\lambda, A)) \setminus \{1\} \). In particular,

\[
r(\lambda R(\lambda, A)(I - P)) \leq c < 1.
\]

Since \( s(A) = 0 \) is a first order pole of the resolvent we have \( \im P = \ker A \) and thus \( \lambda R(\lambda, A)P = P \). As \( \im P \oplus \ker P \) completely reduces \( \lambda R(\lambda, A) \), we obtain

\[
[\lambda R(\lambda, A)]^n = [\lambda R(\lambda, A)P]^n + [\lambda R(\lambda, A)(I - P)]^n = P + [\lambda R(\lambda, A)(I - P)]^n.
\]

Hence \( [\lambda R(\lambda, A)]^n \to P \) as \( n \to \infty \) due to (4.3).

**Lemma 4.8.** Let \( T \in \mathcal{L}(E) \) be an operator on a complex Banach lattice \( E \) with spectral radius \( r(T) = 1 \). Suppose that for every \( x \geq 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( T^n x \geq 0 \) for all \( n \geq n_0 \). If 1 is a pole of the resolvent \( R(\cdot, T) \), then the eigenspace \( \ker(I - T) \) contains a positive, non-zero element.

**Proof.** Let \( m \) be the order of 1 as a pole of \( R(\cdot, T) \). From the Laurent expansion (2.2) we have that

\[
\lim_{\lambda \uparrow 1}(\lambda - 1)^m R(\lambda, T) = U^{m-1}
\]

in \( \mathcal{L}(E) \). Now, let \( \lambda > 1 \) and \( 0 \leq x \in E \). Then there exists \( n_0 \in \mathbb{N} \) such that \( T^n x \geq 0 \) for all \( n \geq n_0 \). Hence we obtain from the Neumann series representation of the resolvent that

\[
R(\lambda, T)x = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} x \geq \sum_{k=0}^{n_0-1} \frac{T^k}{\lambda^{k+1}} x
\]

whenever \( |\lambda| > 1 \). In particular,

\[
U^{m-1} x = \lim_{\lambda \uparrow 1}(\lambda - 1)^m R(\lambda, T)x \geq \lim_{\lambda \uparrow 1}(\lambda - 1)^m \sum_{k=0}^{n_0-1} \frac{T^k}{\lambda^{k+1}} x = 0.
\]

Hence \( U^{m-1} \) is a positive operator. By Remark 2.1 \( \im(U^{m-1}) \) is non-trivial and consists of eigenvalues of \( T \). As \( U^{m-1} \) is positive there exists a positive eigenvector. \( \square \)
The above lemma suggests that it could be interesting to develop a theory of “eventually positive” or “power-positive” operators similar to [25] in infinite dimensions. However, since we are concerned with semigroups we will only use Lemma 4.8 as a technical tool for the proof of Theorem 4.4.

In order to prove Theorem 4.4, one might be tempted to try to use the classical Perron–Frobenius theory, assuming that $R(\lambda, A) \gg 0$ for some $\lambda > 0$. However, here we only assume that the resolvent is \textit{individually} eventually positive, so we need to consider properties of families of operators having a certain weaker pointwise eventual positivity property.

**Lemma 4.9.** Let $E = C(K)$. Let $(J, \leq)$ be a non-empty totally ordered set and let $\mathcal{T} := (T_j)_{j \in J}$ be a family in $\mathcal{L}(E)$ with fixed space $F := \{x \in E : T_j x = x \text{ for all } j \in J\}$.

Assume that for every $x > 0$ there exists $j_x \in J$ such that $T_j x \gg 0$ for all $j \geq j_x$.

(i) If the family $(T_j)_{j \leq j_0}$ is bounded in $\mathcal{L}(E)$ for every $j_0 \in J$ and $F$ contains an element $x_0 > 0$, then $\mathcal{T}$ is bounded in $\mathcal{L}(E)$.

(ii) Let $P > 0$ be a projection on $E$. If every $T \in \mathcal{T}$ leaves $\ker(P)$ invariant and $\im(P) \subseteq F$, then $P \gg 0$.

**Proof.** (i) It suffices to show that the orbit $(T_j x)_{j \in J}$ is bounded for every $x > 0$, since then it follows that $(T_j x)_{j \in J}$ is bounded for every $x \in E$, which in turn implies (i) due to the uniform boundedness principle.

Fix $x > 0$. By assumption there exists $x_0 \in F$ such that $x_0 > 0$. Hence, $x_0 = T_j x_0 \gg 0$ for some $j \in J$, so in particular $x_0 \gg 0$. As $x_0 \gg 0$ there exists $c > 0$ such that $cx_0 \pm x \geq 0$.

Hence there exists $j_0 \in J$ such that $T_j (cx_0 \pm x) \geq 0$ for all $j \geq j_0$ and thus $T_j x \leq cx_0$ for all $j \geq j_0$. As $(T_j)_{j \leq j_0}$ is bounded, this shows that $(T_j x)_{j \in J}$ is indeed bounded.

(ii) If $x > 0$, then by assumption $P x \in F$ and $P x \geq 0$. If $P x > 0$, then by assumption there exists $j \in J$ so that $0 \leq T_j P x = P x$. Hence, for every $x > 0$ either $P x \gg 0$ or $P x = 0$. Let us now show that $P x \neq 0$ whenever $x > 0$. As $P$ is non-zero, there exists $y > 0$ such that $u := Py > 0$. If $x > 0$, then we find a $j \in J$ such that $T_j x \gg 0$, and thus $T_j x - cu \geq 0$ for some $c > 0$. Since $P$ is positive, we conclude that $PT_j x \geq Pcu = cu > 0$. In particular, $PT_j x \neq 0$. As $T_j$ leaves $\ker P$ invariant, we must also have $P x \neq 0$. \hfill $\square$

**Proof of Theorem 4.4.** We may assume that $\lambda_0 = 0$.

“(i) $\Rightarrow$ (ii)” As $P \gg 0$ it follows from Proposition 3.1 that $\lambda_0 = 0$ is a simple pole of $R(\cdot, A)$. Given $f > 0$ it follows from (i) that $P f \gg 0$. By the Laurent expansion (2.2) we conclude that $\lambda R(\lambda, A) Pf \to Pf$ in $E$ as $\lambda \downarrow 0$. As the interior of $E_+$ in $E_\mathbb{R}$ is non-empty and $A$ is real it follows that there exists $\lambda_0 > 0$ such that $R(\lambda, A) f \gg 0$ for all $\lambda \in (0, \lambda_0]$. Hence, $R(\cdot, A)$ is individually eventually strongly positive at $\lambda_0$, proving (ii).

A similar argument using Lemma 4.7(ii) shows that (i) also implies (iii) if $\lambda_0 = s(A) = 0$.

“(ii) $\Rightarrow$ (i)” We first show that $0$ is a simple pole and an eigenvalue with eigenvector $u > 0$. Assumption (ii) implies that there exists $\lambda_2 > 0$ with the following properties: $(0, \lambda_2] \subseteq \rho(A)$ and for every $f > 0$ there exists $\lambda_1 \in (0, \lambda_2]$ such that $R(\lambda, A) f \gg 0$ for all $\lambda \in (0, \lambda_1]$. Let $m$ be the order of 0 as a pole of $R(\lambda, A)$. Using the Laurent expansion (2.2) we see that

$$U^{m-1} f = \lim_{\lambda \downarrow 0} \lambda^m R(\lambda, A) f \geq 0.$$
In particular \( U^{m-1} > 0 \) and hence \( A \) has an eigenvector \( u > 0 \) corresponding to 0 by Remark 2.1. Also \( \lambda_R(\lambda, A)u = u \) for all \( \lambda > 0 \), so \( u \) is in the fixed space \( F \) of the operator family
\[
\mathcal{T} := (\lambda R(\lambda, A))_{\lambda \in (0, \lambda_2]} \tag{4.4}
\]
where the order \( \preceq \) on \( J := (0, \lambda_2] \) is given by \( \succeq \). Clearly, the conditions of Lemma 4.9(i) are satisfied and so the family (4.4) is bounded. Therefore \( \lambda^k R(\lambda, A) \rightarrow 0 \) as \( \lambda \downarrow 0 \) for every \( k \geq 2 \) and hence 0 is a simple pole of \( R(\cdot, A) \). From the above argument \( U^0 = P > 0 \). Moreover \( \text{im} \, P = F \) and thus Lemma 4.9(ii) implies (i).

For the rest of the proof we assume that \( \lambda_0 = s(A) = 0 \). Obviously (iii) implies (iv). ”(iv) \( \Rightarrow \) (i)” We proceed similarly as in the previous paragraph and first show that 0 is a simple pole of \( R(\cdot, A) \). Let \( \lambda > 0 \) be such that for every \( f > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
\[
[R(\lambda, A)]^n f \gg 0 \tag{4.5}
\]
for all \( n \geq n_0 \). The operator \( \lambda R(\lambda, A) \) has spectral radius 1, and 1 is a pole of the resolvent \( R(\cdot, \lambda R(\lambda, A)) \), see [13, Proposition IV.1.18]. By Lemma 4.8 \( \lambda R(\lambda, A) \) has an eigenvector \( u > 0 \) for the eigenvalue 1. In particular, \( u > 0 \) is in the fixed space \( F \) of the family
\[
\mathcal{T} := ((\lambda R(\lambda, A))^n)_{n \in \mathbb{N}}. \tag{4.6}
\]
where the order \( \preceq \) on \( J := \mathbb{N} \) is given by \( \succeq \). Clearly, the conditions of Lemma 4.9(i) are satisfied, so \( \mathcal{T} \) is bounded. Now Lemma 4.7(i) implies that 0 is a simple pole of \( R(\cdot, A) \). Therefore, \( \ker A = \text{im} \, P = F \). Using that \( s(A) = 0 \) we can apply Lemma 4.7(ii) and together with (4.5) we hence obtain \( Pf = \lim_{n \to \infty} [R(\lambda, A)]^n f \geq 0 \) for every \( f \geq 0 \), so \( \mathcal{P} \geq 0 \). Since \( 0 \neq u \in F = \text{im} \, P \) we even have \( \mathcal{P} > 0 \). Now Lemma 4.9(ii) implies (i). □

**Proof of Proposition 4.6.** We may assume that \( \lambda_0 = s(A) = 0 \). If we set \( T := [\lambda R(\lambda, A)]^n \), then \( r(T) = 1 \in \sigma(T) \) and Lemma 4.8 guarantees the existence of a positive fixed vector of \( T \) (alternatively, we could apply the classical Perron–Frobenius theory). Lemma 4.9(i) applied to the operator family \((T^j)_{j \in \mathbb{N}}\) now implies that \( T \) is power-bounded. Therefore, \( \lambda R(\lambda, A) \) is power-bounded as well, and so 0 is a simple pole of \( R(\cdot, A) \) by Lemma 4.7(i).

If \( P \) denotes the spectral projection of \( A \) corresponding to 0, then by Lemma 4.7(ii) we have \( \mathcal{P} = \lim_{j \to \infty} T^j \). As \( T \gg 0 \) we have \( \mathcal{P} \geq 0 \). Since \( \lambda_0 = s(A) = 0 \) is an eigenvalue of \( A \), we also have \( \mathcal{P} \neq 0 \), so \( \mathcal{P} > 0 \). Moreover, \( \text{im}(\mathcal{P}) = \ker(A) \) because 0 is a simple pole of \( R(\cdot, A) \). Thus, \( \text{im}(\mathcal{P}) \) is contained in the fixed space of the operator family \((T^j)_{j \in \mathbb{N}}\) and we can apply Lemma 4.9(ii) to conclude that \( \mathcal{P} \gg 0 \). Hence the equivalent conditions of Theorem 4.4 are fulfilled. □

## 5 Eventually strongly positive semigroups

In this section we come to the heart of the subject. We now use the results of the previous sections to analyse eventual positivity properties of \( C_0 \)-semigroups. Let us start by defining the central notion of this article.

**Definition 5.1.** Let \((e^{tA})_{t \geq 0}\) be a real \( C_0 \)-semigroup on \( E = C(K) \).

(a) The semigroup \((e^{tA})_{t \geq 0}\) is called **individually eventually (strongly) positive** if for every \( f \in E_+ \setminus \{0\} \) there exists \( t_0 \geq 0 \) such that \( e^{tA}f \geq 0 \) (\( \gg 0 \)) for all \( t \geq t_0 \).
(b) The semigroup \((e^{tA})_{t \geq 0}\) is called uniformly eventually (strongly) positive if there exists \(t_0 \geq 0\) such that \(e^{tA} \geq 0\) for all \(t \geq t_0\).

Again, we point out that individual and uniform eventual positivity are not equivalent, see Examples 5.7 and 5.8 below.

It is well known that a \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\) on a Banach lattice is positive if and only if the resolvent \(R(\lambda, A)\) is positive for all \(\lambda > s(A)\). However, the situation is more complicated for eventual positivity. The point is that the long time behaviour of the semigroup may be influenced by non-real elements of the peripheral spectrum, while those spectral values have only a minor influence on the behaviour of the resolvent \(R(\lambda, A)\) as \(\lambda \downarrow s(A)\). We illustrate the problem with an example in three-dimensional space.

**Example 5.2.** Let \(A\) be the \(3 \times 3\) matrix generating the rotation semigroup \((e^{tA})_{t \geq 0}\) rotating vectors about the line in the direction of the unit vector \(u_1 = 3^{-1/2}(1, 1, 1)\) in \(\mathbb{R}^3\) (more precisely, we consider the extension of this semigroup to \(\mathbb{C}^3\)). Then \(\sigma(A) = \{0, i, -i\}\). Clearly the spectral projection \(P\) associated with \(s(A) = 0\) is given by \(Px = \langle u_1, x \rangle u_1\). Hence \(P \gg 0\) and Theorem 4.4 implies that \(R(\lambda, A)\) is individually eventually strongly positive at \(s(A) = 0\). However, there exist arbitrarily large \(t > 0\) such that \(e^{tA}e_k \not\gg 0\), where \((e_k)\) is the standard basis. Hence, \(e^{tA}\) cannot be eventually positive.

**Remark 5.3.** (a) If we modify the above example in such a way that the semigroup becomes exponentially stable on the orthogonal complement of \(u_1\), then it becomes eventually strongly positive. More precisely, for \(\mu > 0\) we consider the generator \(\tilde{A}_\mu := A - \mu(I - P)\). Then, \(\sigma(\tilde{A}_\mu) = \{0, -\mu + i, -\mu - i\}\), that is, \(0\) is a dominant eigenvalue of \(A_\mu\). We then have

\[
e^{t\tilde{A}_\mu} = P + e^{-\mu t}e^{tA}(I - P) \to P
\]

as \(t \to \infty\). As \(P \gg 0\) it is obvious that \(e^{t\tilde{A}_\mu} \gg 0\) for \(t\) sufficiently large.

(b) Alternatively, we could modify the above example in the following way: as in Remark 4.5 we let \(A_\mu := A - \mu P\). We then have \(\sigma(A_\mu) = \{-\mu, i, -i\}\). If \(\mu > 0\), then \(s(A_\mu) = 0\) and we can make the following observation: We know that \(R(\lambda, A)\) is uniformly eventually strongly positive at \(0\), that is, there exists \(\lambda_1 > 0\) such that \(R(\lambda_1, A) \gg 0\). As \(R(\lambda_1, A_\mu)\) is a continuous function of \(\mu \in \mathbb{R}\) there exists \(\mu > 0\) such that \(R(\lambda_1, A_\mu) \gg 0\). By Proposition 4.2 we have \(R(\lambda, A_\mu) \gg 0\) for all \(\lambda \in (-\mu, \lambda_1]\). In particular \(R(\lambda, A_\mu) \gg 0\) for all \(\lambda \in (0, \lambda_1]\), but \(0 = s(A_\mu) \not\in \sigma(A_\mu)\). This shows that we cannot conclude that \(s(A_\mu)\) is a spectral value of \(A_\mu\) if \(R(\lambda, A_\mu) \geq 0\) for all \(\lambda\) in a right neighbourhood of \(s(A_\mu)\).

Part (a) of the above remark suggests that if \(s(A)\) is a dominant eigenvalue, then eventual strong positivity of the semigroup is equivalent to eventual strong positivity of the resolvent at \(s(A)\). Recall from Theorem 4.4 that individual eventual strong positivity of the resolvent at a pole \(\lambda_0 \in \sigma(A) \cap \mathbb{R}\) of the resolvent has several equivalent manifestations. The most convenient is that the spectral projection \(P\) associated with \(\lambda_0\) is strongly positive; this property can also be characterised by the conditions in Proposition 3.1. Thus, Theorem 4.4 and Proposition 3.1 yield various possibilities to check the second part of condition (ii) in the following theorem.

**Theorem 5.4.** Let \((e^{tA})_{t \geq 0}\) be a real \(C_0\)-semigroup on \(E = C(K)\) with \(\sigma(A) \neq \emptyset\). Suppose that the peripheral spectrum given by \((1.3)\) is finite and consists of poles of the resolvent. Then the following assertions are equivalent:

(i) The semigroup \((e^{tA})_{t \geq 0}\) is individually eventually strongly positive.
Moreover, by (i), for every \( e \) applying Lemma 4.9(i) to the operator family (5.1) with \( J \) with a suitable equivalent norm as stated in Proposition 2.2. Thus, (\( T \) the fixed space of the operator family \( P \) bounded, positive \( \lambda > 0 \leq \)).

Proof. We may assume that \( s(A) = 0 \).

\("(i) \Rightarrow (ii)\)" It follows from Theorem 7.6 and Theorem 7.7(i) below that \( s(A) = 0 \) is an eigenvalue of \( A \) admitting an eigenvector \( x > 0 \). As \( e^{tA}x = x \) for all \( t > 0 \), the vector \( x > 0 \) belongs to the fixed space \( F \) of the operator family

\[ T = (e^{tA})_{t \in [0, \infty)}. \]

Moreover, by (i), for every \( f > 0 \) there exists \( t_0 > 0 \) such that \( e^{tA}f \gg 0 \) for all \( t \geq t_0 \). Finally, by strong continuity the sub-family \( (e^{tA})_{t \in [0, T]} \) is bounded for every \( T > 0 \). Applying Lemma 4.9(i) to the operator family (5.1) with \( J = [0, \infty) \) and the order \( \leq \) given by \( \leq \), we conclude that the semigroup \( (e^{tA})_{t \in [0, \infty)} \) is bounded.

We next show that \( s(A) \) is a simple pole of \( R(\cdot, A) \). Let \( C := \sup_{e \geq 0} \| e^{tA} \| \). By the Laplace transform representation of \( R(\lambda, A) \)

\[ \| \lambda R(\lambda, A) \| = \left\| \int_0^\infty \lambda e^{-\lambda t} e^{tA} dt \right\| \leq C \int_0^\infty \lambda e^{-\lambda t} dt = C \]

for all \( \lambda > 0 \). In particular, \( \lambda^n R(\lambda, A) \to 0 \) as \( \lambda \to 0^+ \) for all \( m \geq 2 \). As \( 0 \) is a pole of the resolvent it must therefore be a simple pole. By Theorem 7.7(ii) below this in turn implies that all poles of \( A \) on the imaginary axis are simple poles of \( R(\cdot, A) \).

Next we show that \( s(A) \) is a dominant spectral value. Denote by \( P_{per} \) the spectral projection corresponding to the peripheral spectrum \( \sigma_{per}(A) = \sigma(A) \cap i\mathbb{R} \); we have \( P_{per} \neq 0 \) since the peripheral spectrum \( \sigma_{per}(A) \) contains \( s(A) \) and is thus non-empty. Let \( f \geq 0 \) and \( t \geq 0 \). By Proposition 2.3, there exists a sequence \( (t_n) \subseteq [0, \infty) \) with \( \lim_{n \to \infty} t_n = \infty \) and \( e^{t_n A}P_{per}f \to P_{per}f \) for every \( f \in E \). Also, \( \sigma(A|_{ker P_{per}}) \subseteq \{ z \in \mathbb{C} : \text{Re } z < 0 \} \), and as shown before, \( (e^{tA})_{t \geq 0} \) is bounded. Now [29, Definition 1.1.3 and Corollary 5.2.6] or [2, Theorem 2.4] implies that \( e^{tA} \) converges strongly to 0 on \( ker(P_{per}) \). As \( e^{tA} \) is individually eventually positive we conclude that

\[ e^{tA}P_{per}f = P_{per}e^{tA}f = \lim_{n \to \infty} e^{t_n A}P_{per}e^{tA}f = \lim_{n \to \infty} e^{t_n A}e^{tA}f \geq 0 \]

for all \( t \geq 0 \) and \( f \geq 0 \). In particular, \( (e^{tA})_{t \geq 0} \) restricted to \( im(P_{per}) \) is positive. Setting \( t = 0 \) we also see that \( P_{per} \geq 0 \) and thus \( im(P_{per}) \) is again a Banach lattice when equipped with a suitable equivalent norm as stated in Proposition 2.2. Thus, \( (e^{tA}|_{im P_{per}})_{t \geq 0} \) is a bounded, positive \( C_0 \) semigroup on the Banach lattice \( im(P_{per}) \) and the spectral bound of its generator is \( s(A|_{im P_{per}}) = 0 \). Therefore, the set \( \sigma_{per}(A|_{im P_{per}}) = \sigma_{per}(A) \) is imaginary additively cyclic, see [3, Definition B-III.2.5, Theorem C-III.2.10 and Proposition C-III-2.9]. By assumption \( \sigma_{per}(A) \) is finite and non-empty, so we conclude that \( \sigma_{per}(A) = \{ 0 \} \); in particular \( P = P_{per} \).

Let us finally show that \( P \gg 0 \). We have already shown that \( P = P_{per} > 0 \) and that 0 is a simple pole of the resolvent. Therefore \( im(P) = ker(A) \) and thus \( im(P) \) coincides with the fixed space of the operator family \( T := (e^{tA})_{t \in [0, \infty)} \). Hence, Lemma 4.9(ii) implies that \( P \gg 0 \).
“(ii) ⇒ (iii)” Since $P \gg 0$ Proposition 3.1 shows that $0$ is a simple pole of the resolvent and therefore $\text{im}(P) = \ker(A)$. Moreover, as $s(A) = 0$ is a dominant spectral value and $(e^{tA})_{t \geq 0}$ is bounded, we conclude from [29, Corollary 5.2.6] or [2, Theorem 2.4] that $e^{tA} \to 0$ strongly on $\ker(P)$. Hence, if $f > 0$ we have $e^{tA}f = Pf + e^{tA}(I - P)f \to Pf \gg 0$ as $t \to \infty$. Therefore, $e^{tA}$ converges strongly to the operator $Q := P \gg 0$ as $t \to \infty$.

“(iii) ⇒ (i)” Suppose that $\lim_{t \to \infty} e^{tA}f = Qf \gg 0$ for all $f > 0$. As the positive cone has non-empty interior in $E_{\geq 0}$ and as the semigroup is real, we conclude that there exists $t_0 > 0$ such that $e^{tA}f \gg 0$ for all $t > t_0$.

Hence we have shown the equivalence of (i)–(iii). The proof of the implication “(ii) ⇒ (iii)” shows that $Q = P$ in (iii). \qed

Remark 5.5. Some assertions of Theorem 5.4 have counterparts in the theory of positive semigroups. For example, if $(e^{tA})_{t \geq 0}$ is a positive semigroup and the spectral assumptions of Theorem 5.4 are fulfilled, then it follows from [3, Theorem C-III-1.1(a) and Corollary C-III-2.12] that $s(A)$ is a dominant spectral value. If the positive semigroup $(e^{tA})_{t \geq 0}$ is irreducible and the assumptions of Theorem 5.4 are fulfilled, then it is also known (see [3, Proposition C-III-3.5]) that the spectral projection corresponding to $s(A)$ is strongly positive and that the corresponding eigenspaces of $A$ and $A'$ have the properties that were proved in a more general situation in Proposition 3.1.

It is also a classical idea in the theory of positive semigroups that, under appropriate assumptions, positivity implies convergence of the semigroup, see e.g. [3, Section C-IV-2].

The converse implications “(ii), (iii) ⇒ (i)” however have no counterparts for positive semigroups; they show that eventual positivity provides the right setting to give characterisations of Perron–Frobenius type properties and of convergence to positive limit operators. In finite dimensions, this has already been demonstrated by similar results; see for example [26, Theorem 3.3]. We also refer to our discussion of the finite-dimensional case in Section 6.1.

Under an additional regularity assumption on the semigroup the boundedness condition in Theorem 5.4(ii) can be removed, as the following corollary shows. In particular such a regularity condition is satisfied for analytic semigroups. The corollary will be useful to check eventual positivity in several applications in Section 6.

Corollary 5.6. Suppose that $(e^{tA})_{t \geq 0}$ is a real $C_0$-semigroup on $E = C(K)$ with $\sigma(A) \neq \emptyset$, and that the peripheral spectrum given by (1.3) is finite and consists of poles of the resolvent. If the semigroup $(e^{tA})_{t \geq 0}$ is eventually norm continuous, then the following assertions are equivalent:

(i) The semigroup $(e^{tA})_{t \geq 0}$ is individually eventually strongly positive.

(ii) $s(A)$ is a dominant spectral value of $A$ and the associated spectral projections $P$ fulfils $P \gg 0$.

Proof. We may assume that $s(A) = 0$ and we note that all assumptions of Theorem 5.4 are fulfilled.

Clearly, (i) implies (ii) by Theorem 5.4. If (ii) holds, then $s(A) = 0$ is a simple pole of $R(\cdot, A)$ by Proposition 3.1, so the semigroup $(e^{tA})_{t \geq 0}$ is bounded on $\text{im} P$. As the semigroup is eventually norm continuous, the set $\{\lambda \in \sigma(A) : \alpha \leq \text{Re } \lambda\}$ is bounded for every $\alpha \in \mathbb{R}$ (see [13, Theorem II.4.18]) and we conclude that $s(A|_{\ker P}) < 0$. From the eventual norm continuity it now follows that the growth bound of $(e^{tA}|_{\ker P})_{t \geq 0}$ is negative. Hence $(e^{tA})_{t \geq 0}$ is also bounded on $\ker P$. Therefore, condition (ii) of Theorem 5.4 is fulfilled, and hence (i) follows. \qed
Since we have now several criteria at hand to check whether a semigroup is individually eventually strongly positive, it is time to give an example which shows that it is necessary to distinguish between the individual and the uniform eventual behaviour of a semigroup.

**Example 5.7.** Consider the Banach lattice $E = C([-1, 1])$. Let $\varphi: E \to \mathbb{C}$ be the continuous linear functional given by $\varphi(f) = \int_{-1}^{1} f(x) \, dx$. We thus have the decomposition

$$ E = \langle 1 \rangle \oplus F \quad \text{with} \quad F := \ker \varphi. $$

By $S$ we denote the reflection operator on $F$, given by $Sf(x) = f(-x)$ for all $f \in F$ and all $x \in [-1, 1]$. Using $S^2 = I_F$ we see that $\sigma(S) = \{1, -1\}$ with corresponding eigenspaces given by even and odd functions, respectively.

Now define a bounded linear operator $A$ on $E$ by

$$ A = 0_{\langle 1 \rangle} \oplus (-2I_F - S). $$

We have $\sigma(A) = \{0, -1, -3\}$ and, using $S^2 = I_F$, we can immediately check that

$$ e^{tA} = I_{\langle 1 \rangle} \oplus e^{-2t}(\cosh(t)I_F - \sinh(t)S) \quad \text{and} \quad R(\lambda, A) = \frac{1}{\lambda} I_{\langle 1 \rangle} \oplus \frac{1}{(\lambda + 2)^2 - 1}((\lambda + 2)I_F - S). $$ (5.3)

for all $t \geq 0$ and all $\lambda \in \rho(A) = \mathbb{C} \setminus \{0, -1, -3\}$. The spectral bound $s(A) = 0$ is a dominant spectral value and the associated spectral projection $P$ is given by $Pf = \frac{1}{2} \varphi(f) \, 1$ and thus strongly positive. Hence, our semigroup is individually eventually strongly positive due to Corollary 5.6, and so is the resolvent at $s(A)$ due to Theorem 4.4.

Now for each $\varepsilon > 0$ choose a function $f_{\varepsilon} \in E_+$ with $\|f_{\varepsilon}\|_{\infty} = 1$, $\varphi(f_{\varepsilon}) = \varepsilon$, $f_{\varepsilon}(1) = 1$ and $f_{\varepsilon}(-1) = 0$. Then

$$ Pf_{\varepsilon} = \frac{\varepsilon}{2} \, 1 \quad \text{and} \quad (I_E - P)f_{\varepsilon} = f_{\varepsilon} - \frac{\varepsilon}{2} \, 1. $$

By (5.2) we obtain for $t \geq 0$ that

$$ e^{tA}f_{\varepsilon}(-1) = \frac{\varepsilon}{2}(1 - e^{-2t} \cosh t + e^{-2t} \sinh t) - e^{-2t} \sinh t. $$

Thus, for each $t \geq 0$, we can choose $\varepsilon > 0$ small enough such that $e^{-tA}f_{\varepsilon} \not\geq 0$. Therefore $(e^{tA})_{t \geq 0}$ is not uniformly eventually positive. In particular, it is not uniformly eventually strongly positive. In a similar way one can check that the resolvent $R(\cdot, A)$ is not uniformly eventually positive at $s(A)$.

Noting that the generator of the semigroup in the previous example is merely bounded, it is natural to ask whether the situation changes if we impose additional compactness conditions on our semigroup. We proceed with a further example which is rather disillusioning. We construct an analytic semigroup with compact resolvent such that the semigroup is individually eventually strongly positive, but again not even uniformly eventually positive. The basic idea of the construction is rather similar to Example 5.7, but it is somewhat more technical.

**Example 5.8.** Let $c(\mathbb{Z})$ be the subspace of $\ell^{\infty}(\mathbb{Z})$ given by

$$ c(\mathbb{Z}) := \langle 1 \rangle \oplus c_0(\mathbb{Z}), $$

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where \( c_0(\mathbb{Z}) \) is as usual the set of sequences \((x_n) \in \ell^\infty(\mathbb{Z}) \) with \( x_n \to 0 \) as \( n \to \pm\infty \). It is easy to see that \( c(\mathbb{Z}) \simeq C(\mathcal{K}) \) for some compact Hausdorff space \( \mathcal{K} \). We can write \( c_0(\mathbb{Z}) \) as a direct sum of the subspaces of symmetric and anti-symmetric sequences, that is, the spaces
\[
\begin{align*}
  c_0^s &:= \{(x_n) \in c_0(\mathbb{Z}) : x_n = x_{-n} \text{ for all } n \in \mathbb{N}\} \\
  c_0^a &:= \{(x_n) \in c_0(\mathbb{Z}) : x_n = -x_{-n} \text{ for all } n \in \mathbb{N}\}
\end{align*}
\]
If \( x = (x_n) \in c_0(\mathbb{Z}) \), we define the reflection operator \( S \) by \( S(x_n) := (x_{-n}) \). Then
\[
x = \frac{1}{2}(x + Sx) + \frac{1}{2}(x - Sx) \in c_0^s \oplus c_0^a
\]
is the unique decomposition into symmetric and anti-symmetric parts, showing that \( c(\mathbb{Z}) = \langle 1 \rangle \oplus c_0^s \oplus c_0^a \).

Choose strictly positive symmetric sequences \((\alpha_n)_{n \in \mathbb{Z}}\) and \((\beta_n)_{n \in \mathbb{Z}}\) with \( e^{-n\beta_n} - e^{-n\alpha_n} < 0 \) for all \( n \in \mathbb{N} \), and such that \( \alpha_n, \beta_n \to \infty \) as \( |n| \to \infty \). Now let \( g = (g_n) \in \ell^1(\mathbb{Z}) \cap c_0^s \) be such that \( g_n > 0 \) for all \( n \in \mathbb{Z} \). We define a Banach space isomorphism \( B \in \mathcal{L}(c(\mathbb{Z})) \) and its inverse by
\[
\begin{align*}
  B(c \, 1 + x) &:= (c + \langle g, x \rangle) \, 1 + x \quad \text{and} \quad B^{-1}(c \, 1 + x) := (c - \langle g, x \rangle) \, 1 + x.
\end{align*}
\]  
\[
\text{for all } c \in \mathbb{R} \text{ and } x \in c_0(\mathbb{Z}). \quad \text{Define the multiplication operators } M_\alpha \text{ and } M_\beta \text{ on } c_0^s \text{ and } c_0^a \text{ by } M_\alpha x := (\alpha_n x_n) \text{ and } M_\beta x := (\beta_n x_n) \text{ with domains }
\]
\[
D(M_\beta) := \{x \in c_0^s : \beta x \in c_0^s\} \quad \text{and} \quad D(M_\alpha) := \{x \in c_0^a : \alpha x \in c_0^a\}
\]
respectively. Then \(-M_\beta\) and \(-M_\alpha\) generate bounded strongly continuous analytic semigroups on \( c_0^s \) and \( c_0^a \) respectively, see [3, Section A-1.2.3]. We define a semigroup \( (e^{tA})_{t \geq 0} \) on \( c(\mathbb{Z}) \) by using the commutative diagram in (5.6).
\[
\begin{array}{ccc}
c(\mathbb{Z}) & \xrightarrow{e^{tA}} & c(\mathbb{Z}) \\
\downarrow B & & \uparrow B^{-1} \\
\langle 1 \rangle \oplus c_0^s \oplus c_0^a & \xrightarrow{I \oplus e^{-tM_\beta} \oplus e^{-tM_\alpha}} & \langle 1 \rangle \oplus c_0^s \oplus c_0^a
\end{array}
\]  
The generator of that semigroup is given by \( A = -B^{-1}(0_1 \oplus M_\beta \oplus M_\alpha)B \). Clearly, the operator \( A \) has compact resolvent and the semigroup \( (e^{tA})_{t \geq 0} \) on \( c(\mathbb{Z}) \) is real, analytic and bounded. Moreover, 0 is an algebraically simple, isolated and dominant eigenvalue of \( A \) with eigenvector \( 1 \gg 0 \) and 0 is an isolated, dominant eigenvalue of \( A \) with eigenvector \( 1 \gg 0 \). It is an isolated eigenvalue of \( A \) with eigenvector \( 1 \gg 0 \). A short computation shows that the associated spectral projection \( P \) is given by \( P(c \, 1 + x) = (c + \langle g, x \rangle) \, 1 \) for \( c \in \mathbb{C}, \, x \in c_0(\mathbb{Z}) \). Now, if \( c \, 1 + x \gg 0 \), then we can find an element \( 0 < y \in c_0(\mathbb{Z}) \) such that \( e^{tA}1 + x - y \gg 0 \). Hence,
\[
0 < \langle g, y \rangle \leq \langle g, c \, 1 + x \rangle = c + \langle g, x \rangle,
\]
which shows that \( P \) is strongly positive. Hence, we can apply Theorem 5.4 to conclude that \( (e^{tA})_{t \geq 0} \) is individually eventually strongly positive.

We now show that it is not uniformly eventually strongly positive. In fact it is not even uniformly eventually positive. Indeed, if \( t_0 \geq 0 \), then we may choose \( n \in \mathbb{N}, \, n \geq t_0 \) for all \( n \in \mathbb{N} \).
such that (5.4) is fulfilled for this $n$. We now compute $e^{tA} x$ for $x \in c_0(\mathbb{Z})$. From the
definitions (5.5) of $B$ and $B^{-1}$ we conclude that
\[ e^{tA} x = B^{-1} \left( I_{(1)} \oplus e^{-tM_\beta} \oplus e^{-tM_\alpha} \right) B x 
\]
\[ = B^{-1} \left( \langle g, x \rangle \mathbf{1} + \frac{1}{2} e^{-tM_\beta}(x + Sx) + \frac{1}{2} e^{-tM_\alpha}(x - Sx) \right) \]  \hspace{1cm} (5.7)
In particular, if $x > 0$, then
\[ e^{tA} x < \frac{1}{2} \left( 2 \langle g, x \rangle \mathbf{1} + e^{-tM_\beta}(x + Sx) + e^{-tM_\alpha}(x - Sx) \right) \]
for all $t > 0$. Taking $x = 1_{\{n\}}$ we obtain for the $(−n)$-th component of $e^{tA} 1_{\{n\}}$ that
\[ \left( e^{tA} 1_{\{n\}} \right)_{−n} \leq \frac{1}{2} \left( 2g_n + e^{-tM_\beta} - e^{-tM_\alpha} \right), \]
and the last term is negative for $t = n$ due to (5.4). Thus, $(e^{tA})_{t \geq 0}$ is not uniformly
eventually positive.

**Remark 5.9.** Let $(e^{tA})_{t \geq 0}$ be a real $C_0$-semigroup on $C(K)$ and suppose that $s(A) = 0$ is
a dominant spectral value and a first order pole of the resolvent with associated spectral
projection $P$. It does not seem to be easy to find a simple criterion that guarantees the
uniform (strong) eventual positivity of $(e^{tA})_{t \geq 0}$. To provide the reader with a feeling for
the situation we point out that a number of candidate criteria which appear natural at
first glance do not work:

(a) For example, it seems intuitive to require that $P \gg 0$ and that $e^{tA}$ be uniformly
exponentially stable on ker $P$. However, this does not imply uniform eventual pos-
tivity as Example 5.7 shows.

(b) Suppose that $(e^{tA})_{t \geq 0}$ is uniformly exponentially stable on ker $P$. If the eigenvalue
$s(A)$ is algebraically simple and if the subspace $\text{im } A = \ker P$ has strictly positive
distance to the positive normalised functions, then it is indeed possible to show that
$(e^{tA})_{t \geq 0}$ is uniformly eventually positive. However, the reader should be warned that
this seemingly nice criterion can in fact never be applied in infinite dimensions, for
the following reason: if there exists a closed subspace $F \subseteq C(K)$ of co-dimension 1
such that
\[ \inf \{ \| v - u \| : u \in F, v \in E_+, \| v \| = 1 \} > 0, \]
then one can show that $K$ must actually be finite. We omit the elementary proof.

### 6 Applications on $C(K)$

We proceed with several applications of the results presented in Sections 3, 4 and 5.
We begin with a short treatment of the finite-dimensional case, where we obtain several
results, including a slight strengthening of known results, as corollaries of the general
theory on $C(K)$. Then we give an application to the Dirichlet-to-Neumann operator,
which was a major motivation for the development of the theory presented so far. After-
wards, we show that squares of certain generators on $C(\overline{\Omega})$ generate eventually positive
semigroups, and finally, we present an example of a delay differential equation whose
solution semigroup is eventually positive but not positive.
6.1 The finite-dimensional case.

The space \( \mathbb{C}^n \) with the supremum norm \( \| \cdot \|_\infty \) is a complex Banach lattice when its real part \( \mathbb{R}^n \) is endowed with the canonical order. Then \( (\mathbb{C}^n, \| \cdot \|_\infty) = (C(K), \| \cdot \|_\infty) \), where \( K := \{1, \ldots, n\} \) is equipped with the discrete topology, so we can apply our theory.

As noted in the introduction, a sophisticated finite-dimensional theory of eventually positive operators and semigroups has been developed during the last twenty years; see for instance [21, 25] for results about eventually positive matrices and [26] for eventually positive matrix semigroups. Also note that somewhat earlier several results for matrices which possess some positive powers were obtained, see e.g. [8] or [28, p. 48–54].

In this subsection we illustrate how the results from Sections 3–5 imply results from [26, Theorem 3.3] as a special case. The reader should however be aware that our terminology differs in some points: for example, matrices and vectors we call “strongly positive” are simply called “positive” in [26], and what we call “positive” is “non-negative” in [26]. Note also that since it is easy to see that uniform and individual eventual (strong) positivity coincide in the finite-dimensional setting, we will omit the adjectives “uniform” and “individual” in this subsection.

**Theorem 6.1.** For \( A \in \mathbb{R}^{n \times n} \), the following assertions are equivalent:

1. The semigroup \( (e^t A)_{t \geq 0} \) is eventually strongly positive.
2. The spectral bound \( s(A) \) is a dominant and geometrically simple eigenvalue of \( A \) and the eigenspaces \( \ker(s(A)I - A) \) and \( \ker(s(A)I - A^T) \) contain a strongly positive vector.
3. There exists \( c \in \mathbb{R} \) such that \( A + cI \) is eventually strongly positive, that is, there exists \( k_0 > 0 \) such that \( (A + cI)^k \gg 0 \) for all \( k \geq k_0 \).

If assertions (i)–(iii) are fulfilled, then \( s(A) \) is an algebraically simple eigenvalue of \( A \).

**Proof.** “(i) ⇒ (ii)” If (i) holds, then Theorem 5.4 implies that \( s(A) \) is a dominant eigenvalue and that the corresponding spectral projection \( P \) is strongly positive. Hence (ii) follows from Proposition 3.1.

“(ii) ⇒ (iii)” According to Proposition 3.1, \( s(A) \) is an algebraically simple eigenvalue with spectral projection \( P \gg 0 \). The matrix \( A \) has finitely many eigenvalues \( \lambda_k, k = 1, \ldots, m \), other than \( s(A) \). As \( s(A) \) is dominant, there exists \( c \geq 0 \) such that \( \{\lambda_1, \ldots, \lambda_m\} \) is contained in the open ball of radius \( s(A) + c > 0 \) around \( -c \). Hence, \( s(A) + c \) is the only eigenvalue of \( A + cI \) of modulus \( r := r(A + cI) = s(A) + c \) and so \( r((A + cI)(I - P)) < r \).

It follows that

\[
\lim_{k \to \infty} [r^{-1}(A + cI)]^k = P + \lim_{k \to \infty} [r^{-1}(A + cI)(I - P)]^k = P \gg 0.
\]

This is the well-known power method for computing the dominant eigenvalue, see for instance [20, Theorem 8.2.8]. As \( P \gg 0 \) we conclude that \( (A + cI)^k \gg 0 \) for \( k \) large enough.

“(iii) ⇒ (i)” We essentially follow the proof from [26, Theorem 3.3]. Set \( B := A + cI \) and assume that \( B^k \gg 0 \) for all \( k \geq k_0 \). Then

\[
e^t B = \sum_{k=0}^{k_0-1} \frac{t^k}{k!} B^k + \sum_{k=k_0}^{\infty} \frac{t^k}{k!} B^k \gg t^{k_0} \left( \frac{1}{k_0!} B^{k_0} + \sum_{k=0}^{k_0-1} \frac{t^{k-k_0}}{k!} B^k \right)
\]

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As $B^{k_0} \gg 0$ and
\[
\lim_{t \to \infty} \sum_{k=0}^{k_0-1} \frac{t^{k-k_0}}{k!} B^k = 0
\]
there exists $t_0 > 0$ such that $e^{tB} \gg 0$ for all $t > t_0$. Now (i) follows since $e^{tA} = e^{-ct}e^{tB}$.

Finally, (ii) and Proposition 3.1 imply that $s(A)$ is an algebraically simple eigenvalue of $A$.

Note that the other conditions in Proposition 3.1 and Theorem 4.4 together with Corollary 5.6 yield further characterisations of the eventual positivity of a matrix semigroup. However, since those assertions are not simplified in the matrix case, we see no reason to restate them explicitly here.

Let us briefly compare Theorem 6.1 with [26, Theorem 3.3]. Conditions (i) and (iii) in our Theorem above appear also in [26, Theorem 3.3] as conditions (iv) and (ii); our condition (ii) is very similar to condition (i) there. The latter condition is formulated in terms of the spectral radius and can easily be rewritten into our condition on the spectral bound, except for one small difference: the condition in [26] assumes the spectral radius to be an algebraically simple eigenvalue whereas we only assume the spectral bound to be geometrically simple and then deduce the algebraic simplicity. Besides this difference in the assertion of the theorems, we note that many arguments in [26] are based on the fact that $A^k \gg 0$ for all $k$ large enough. Our proof of the implication “(i) $\Rightarrow$ (ii)” in Theorem 6.1 is new, being based on the characterisations of the spectral projection developed in Sections 3–5, which have the advantage of being applicable in the case of unbounded operators in infinite dimensions.

We have seen in Remark 5.3(a) that there are examples of non-positive, eventually positive semigroups in three (and hence all higher) dimensions. On the other hand, a one-dimensional real semigroup is clearly always positive. We now show that in two dimensions, eventual positivity implies positivity.

**Proposition 6.2.** Let $A \in \mathbb{R}^{2 \times 2}$. If $(e^{tA})_{t \geq 0}$ is eventually (strongly) positive, then $e^{tA}$ is (strongly) positive for each $t > 0$.

**Proof.** As usual we assume that $s(A) = 0$. First suppose that $(e^{tA})_{t \geq 0}$ is eventually positive. Then $\lambda_1 := s(A) \in \mathbb{R}$ is an eigenvalue of $A$ as shown in Theorem 7.6 below. Hence $A$ has two real eigenvalues. If $\lambda_1$ has multiplicity two, then either $A = 0$ and $e^{tA} = I \geq 0$, or $A$ is nilpotent and $e^{tA} = I + tA$ is eventually positive if and only if $A \geq 0$. In either case eventual positivity implies positivity.

Now let $-\lambda_2 < 0$ be the second eigenvalue of $A$ with corresponding eigenvector $u_2$. The general solution of $\dot{u} = Au$ is given by $u(t) = au_1 + bu_2e^{-\lambda_2 t}$ for constants $a, b \in \mathbb{R}$, where $u_1$ is an eigenvector for the eigenvalue $s(A) = 0$. If $u(0) = au_1 + bu_2 \geq 0$, then eventual positivity implies that
\[
\lim_{t \to \infty} u(t) = \lim_{t \to \infty} (au_1 + bu_2e^{-\lambda_2 t}) = au_1 \geq 0.
\]

The trajectory for $t \geq 0$ is a line segment connecting $u(0)$ and $au_1$ and thus lies in the positive cone. Hence, $(e^{tA})_{t \geq 0}$ is positive.

Now assume the semigroup is eventually strongly positive. Then the spectral projection $P$ associated with $s(A) = 0$ is strongly positive and $A$ has two distinct eigenvalues; see Theorem 5.4 and Proposition 3.1. If $u(0) > 0$, then $au_1 = Pu(0) \gg 0$. Thus, either $u(0) = u(t) = au_1 \gg 0$ for all $t \geq 0$ or the open line segment between $u(0)$ and $au_1$ is in the interior of the cone. Hence $(e^{tA})_{t \geq 0}$ is strongly positive. \qed
6.2 The Dirichlet-to-Neumann operator

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \) and \( \lambda \not\in \sigma(-\Delta) \), where \( \Delta \) is the Dirichlet Laplacian on \( \Omega \). Given \( \varphi \in L^2(\partial\Omega) \), let \( u \) denote the unique solution of \( \Delta u + \lambda u = 0 \) in \( \Omega \) and \( u = \varphi \) on \( \partial\Omega \). The Dirichlet-to-Neumann operator is defined by

\[
D_\lambda \varphi := \frac{\partial u}{\partial \nu},
\]

where \( \nu \) is the outer unit normal to \( \partial\Omega \). A proper construction of \( D_\lambda \) as the generator of a strongly continuous analytic semigroup on \( L^2(\partial\Omega) \) can be found in [4, 7]. It is shown in [12] that the semigroup \( e^{-tD_\lambda} \) is not positive, but only eventually positive for certain ranges of \( \lambda > \lambda_1 \), where \( \lambda_1 \) is the first eigenvalue of the Dirichlet Laplacian \( -\Delta \) on the unit ball \( \Omega = B_1(0) \) in \( \mathbb{R}^2 \).

Our goal here is to show that this observation continues to hold in \( C(\Gamma) \), where \( \Gamma := \partial B_1(0) \), and can be obtained using Theorem 5.4. However, to do so, we first need to know that \( e^{-tD_\lambda} \) is in fact a \( C_0 \)-semigroup on \( C(\Gamma) \). This is the subject of the main theorem in [14]. However, it appears that the proof given in [14] is not valid without restrictions on the zeroth order term \( a_0 \) of the operator \( \mathcal{A} \) (the general second order elliptic operator considered there). The reason is that in the proofs provided in [14], the Dirichlet-to-Neumann operator is the one associated with the operator \( \mathcal{A} + \alpha^2 I \) for some real \( \alpha \) large enough, not the one associated with \( \mathcal{A} \) as claimed in [14, page 236]. Hence it actually seems to be an open problem to establish that \( D_\lambda \) generates a \( C_0 \)-semigroup on \( C(\partial\Omega) \) whenever \( \lambda > \lambda_1 \) is not in the spectrum of the corresponding Dirichlet Laplacian. Note also that the conclusion on positivity in the main theorem of [14] is not true for the whole range of \( \lambda \in \mathbb{R} \), as was pointed out in [12, page 237].

Here, to have at least one example (that of the disk in \( \mathbb{R}^2 \)) valid for the complete range of admissible \( \lambda \in \mathbb{R} \), we start by providing a proof of the following theorem.

**Theorem 6.3.** Let \( \Gamma = \partial B_1(0) \) be the unit circle in \( \mathbb{R}^2 \) and let \( \lambda \in \mathbb{R} \setminus \sigma(-\Delta) \). When restricted to \( C(\Gamma) \), the family \( (e^{-tD_\lambda})_{t \geq 0} \) is a \( C_0 \)-semigroup on \( C(\Gamma) \).

**Proof.** The semigroup can be represented by a convolution kernel

\[
e^{-tD_\lambda} \varphi = G_{\lambda,t} \ast \varphi = \int_{-\pi}^{\pi} G_{\lambda,t}(\cdot - s)\varphi(s) \, ds,
\]

where the kernel \( G_{\lambda,t} \) is given by the Fourier series

\[
G_{\lambda,t}(\theta) = \frac{1}{2\pi} \sum_{k=\infty}^{\infty} e^{-t\mu_k(\lambda)}e^{ik\theta}.
\]

Here, \( \mu_k(\lambda) = \mu_{-k}(\lambda) \) are the eigenvalues of \( D_\lambda \) with eigenfunctions \( e^{\pm ik\theta} \). As \( \mu_k(\lambda) \) behaves asymptotically like \( k \) as \( k \to \infty \), the Fourier coefficients of \( G_{\lambda,t} \) decay exponentially; see [12, Lemma 4.2]. Hence, as \( C(\Gamma) \hookrightarrow L^2(\Gamma) \) we conclude that \( e^{-tD_\lambda} \) is analytic as a map from \( (0, \infty) \) into \( \mathcal{L}(C(\Gamma)) \). We only need to prove the strong continuity at \( t = 0 \). As shown in [12, Proposition 4.6] we can represent \( G_{\lambda,t} \) in terms of the Fejér kernels \( K_n \geq 0 \) in the form

\[
G_{\lambda,t}(\theta) = \sum_{n=1}^{\infty} nb_n(\lambda,t)K_{n-1}(\theta)
\]
with \( b_n(\lambda, t) := e^{-t\mu_{n+1}(\lambda)} + e^{-t\mu_{n-1}(\lambda)} - 2e^{-t\mu_n(\lambda)} \). As shown in [12, Proposition 4.6], for fixed \( \lambda \), there exists \( n_0 \geq 1 \) such that \( b_n(\lambda, t) \geq 0 \) for all \( n \geq n_0 \) and all \( t > 0 \). An elementary but not entirely trivial argument now yields

\[
M := \sup_{t \in (0, 1]} \int_{-\pi}^{\pi} |G_{\lambda,t}(s)| \, ds < \infty, \tag{6.2}
\]

\[
\lim_{t \to 0} \int_{\alpha}^{2\pi - \alpha} |G_{\lambda,t}(s)| \, ds = 0 \quad \text{for all } \alpha \in (0, \pi). \tag{6.3}
\]

Using these properties we now show that for every \( \varphi \in C(\Gamma) \) the family \( e^{-tD_\lambda} \varphi = G_{\lambda,t} \ast \varphi \); \( t \in (0, 1] \) is bounded and equicontinuous and therefore relatively compact in \( C(\Gamma) \) by the Arzelà–Ascoli theorem. First, we obtain from (6.2) that

\[
\sup_{t \in (0, 1]} |u(t)| = \left| \int_{-\pi}^{\pi} G_{\lambda,t}(\theta - s)\varphi(s) \, ds \right| \leq M \| \varphi \|_\infty,
\]

so the family is bounded. As \( u(t) := e^{-tD_\lambda} \varphi \to \varphi \) in \( L^2(\Gamma) \) as \( t \to 0 \) this implies that we also have convergence in \( C(\Gamma) \). Indeed, for fixed \( \alpha \in (0, \pi) \) we have

\[
|u(t, \theta + \eta) - u(t, \theta)| \\
= \left| \int_{-\pi}^{\pi} G_{\lambda,t}(\theta + \eta - s)\varphi(s) \, ds - \int_{-\pi}^{\pi} G_{\lambda,t}(\theta - s)\varphi(s) \, ds \right| \\
\leq \int_{-\pi}^{\pi} |G_{\lambda,t}(\theta - s)||\varphi(s - \eta) - \varphi(s)| \, ds \\
\leq 4 \int_{\alpha}^{\pi} |G_{\lambda,t}(s)| \, ds \| \varphi \|_\infty \\
+ \int_{\alpha}^{\pi} |G_{\lambda,t}(s)| \, ds \sup_{s \in [-\pi, \pi]} |\varphi(s - \eta) - \varphi(s)| \\
\leq 4 \int_{\alpha}^{\pi} |G_{\lambda,t}(s)| \, ds \| \varphi \|_\infty + M \sup_{s \in [-\pi, \pi]} |\varphi(s - \eta) - \varphi(s)|
\]

for all \( t \in (0, 1] \), where we used (6.2) in the last inequality. Fix \( \varepsilon > 0 \). Due to the uniform continuity of \( \varphi \) on the compact set \( \Gamma \) we can choose \( \delta > 0 \) such that

\[
M \sup_{s \in [-\pi, \pi]} |\varphi(s - \eta) - \varphi(s)| < \frac{\varepsilon}{2}
\]

whenever \( |\eta| < \delta \). By (6.3) there exists \( t_0 > 0 \) such that

\[
4 \int_{\alpha}^{\pi} |G_{\lambda,t}(s)| \, ds \| \varphi \|_\infty < \frac{\varepsilon}{2}
\]

for all \( t \in (0, t_0] \). Hence,

\[
|u(t, \theta + \eta) - u(t, \theta)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{6.4}
\]

whenever \( |\eta| < \delta \) and \( t \in (0, t_0] \). As \( u \in C([t_0, 1]; C(\Gamma)) \) there exists a possibly smaller \( \delta > 0 \) such that (6.4) holds whenever \( |\eta| < \delta \) and \( t \in (0, 1] \). Hence, \( u(t) \to u(0) \) in \( C(\Gamma) \), showing that \( e^{-tD_\lambda} \) is a strongly continuous semigroup on \( C(\Gamma) \).
We finally consider the positivity properties of the semigroup. Regarding eventual positivity, Theorem 5.4 allows us to prove the following proposition. In fact, if one considers the Fourier series representation of the semigroup, one can actually show that it is uniformly eventually strongly positive.

**Proposition 6.4.** There exists \( \lambda^* \in (\lambda_3, \lambda_4) \) such that \( e^{-tD_{\lambda}} \) is individually eventually strongly positive but not positive on \( C(\Gamma) \) for all \( \lambda \in (\lambda_3, \lambda^*) \).

There are in fact infinitely many small intervals in which this holds. We merely discuss one in detail, as an illustration of the principle.

**Proof.** Recall that \( \lambda \in \sigma(-\Delta) \) if and only if \( J_{k}(\sqrt{\lambda}) = 0 \) for some Bessel function \( J_k \), \( k \in \mathbb{N} \). It is shown in [12] that the eigenvalues of \( D_{\lambda} \) are of the form

\[
\mu_k(\lambda) = \frac{\sqrt{\lambda} J'_{k}(\sqrt{\lambda})}{J_{k}(\sqrt{\lambda})}
\]

(6.5)

with the corresponding eigenspaces spanned by \( 1 \) if \( k = 0 \) and by \( \cos(kt), \sin(kt) \) if \( k \geq 1 \). A plot of the first few eigenvalues as a function of \( \lambda \) is shown in [12, Fig. 3]. The curves have vertical asymptotes at the strictly ordered eigenvalues \( \lambda_1 < \lambda_3 < \lambda_3 < \ldots \) of the negative Dirichlet Laplacian on the unit disc \( B_1(0) \). The only eigenvalue having a strictly positive eigenvector, namely \( 1 \), is \( \mu_0(\lambda) \). The corresponding projection is given by

\[
P \varphi = \int_{-\pi}^{\pi} \varphi(\theta) \, d\theta \, 1
\]

and hence \( P \gg 0 \). However, note that \( \mu_0(\lambda) \) is not always the dominant eigenvalue.

From the explicit values for \( \mu_k(\lambda) \) given in (6.5) we can see that \( \mu_0(\lambda) \) is dominant if \( \lambda \in (\lambda_3, \lambda_4) \). This can also clearly be seen from [12, Fig. 3], where \( \mu_0(\lambda) \) is represented by a solid line. Hence, we conclude from Theorem 5.4 that \( e^{-tD_{\lambda}} \) is individually eventually positive for \( \lambda \in (\lambda_3, \lambda_4) \). It is shown in [12] that \( (e^{-tD_{\lambda}})_{\geq 0} \) is a positive semigroup for \( \lambda \) close enough to \( \lambda_4 \). We now show that it is not positive if \( \lambda \) is in a right neighbourhood of \( \lambda_3 \). To do so we take as an initial condition the Fejér kernel

\[
u_0(\theta) := 2K_3(\theta) = 2 + 3\cos \theta + 2\cos 2\theta + \cos 3\theta = \frac{1}{2} \left( \frac{\sin(2\theta)}{\sin(\theta/2)} \right)^2;
\]

(6.6)

see [23, p. 12]. Let \( u_\lambda(t) := e^{-tD_{\lambda}}u_0 \). We show that \( u_\lambda(t) \) is not positive for \( t \) sufficiently small if \( \lambda \) is in a right neighbourhood of \( \lambda_3 \). We do that by showing that \( u_\lambda \) has a negative time derivative at a point where the initial condition is zero. Using the formula from [12, Proposition 4.3(ii)] we see that

\[
\dot{u}_\lambda(\cdot, 0) = \frac{d}{dt} e^{-tD_{\lambda}}u_0 \bigg|_{t=0} = -D_{\lambda}u_0
\]

\[
= -2\mu_0(\lambda) - 3\mu_1(\lambda) \cos \theta - 2\mu_2(\lambda) \cos 2\theta - \mu_3(\lambda) \cos 3\theta.
\]

Clearly \( u_0 > 0 \), \( u_0(\pi) = 0 \) and

\[
\dot{u}_\lambda(\pi, 0) = -2\mu_0(\lambda) + 3\mu_1(\lambda) - 2\mu_2(\lambda) + \mu_3(\lambda)
\]

(6.7)

for all \( \lambda \in (\lambda_3, \lambda_4) \). Further note that \( J_2(\sqrt{\lambda_3}) = 0 \), so that \( \mu_2(\lambda) \to \infty \) as \( \lambda \downarrow \lambda_3 \); see [12, p. 244]. As the eigenvalues \( \mu_0(\lambda), \mu_1(\lambda) \) and \( \mu_3(\lambda) \) remain bounded in a right neighbourhood of \( \lambda_3 \), (6.7) implies that \( \dot{u}_\lambda(\pi, 0) \to -\infty \) as \( \lambda \downarrow \lambda_3 \). This can be seen in [12, Fig. 3]. In particular, because \( u(\pi, 0) = u_0(0) = 0 \) we conclude that \( \dot{u}_\lambda(\pi, 0) < 0 \) if \( \lambda \) is in a right neighbourhood of \( \lambda_3 \). Hence \( e^{-tD_{\lambda}}u_0 \) is not positive, but only eventually positive.

\[\square\]
6.3 Squares of Generators

We saw in Proposition 4.3 that the resolvent of the operator $A := (iB)^2 = -B^2$ is eventually positive at $\lambda_0 < 0$ if $(\lambda_0, 0) \subseteq \rho(A)$ and if $B$ is resolvent positive in 0. We now show that such a situation gives rise to eventually positive semigroups. However, note that even if $B$ generates a strongly continuous semigroup, that is not automatically the case for $A = -B^2$. There are special conditions when this is the case, namely if $B$ generates a group; see [3, Theorem A-II-1.15] or [13, Corollary II.4.9]. We do not wish to assume this, but instead work with sectorial operators.

Let us therefore recall some important notions: Let $E$ be a complex Banach space and $\theta \in (0, \pi]$. By $\Sigma_\theta := \{re^{i\varphi}: r > 0, \varphi \in (-\theta, \theta)\}$ we denote the open sector of angle $\theta$. Now, let $\theta \in (0, \pi/2]$. A $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is called analytic of angle $\theta$ if it has an extension $(e^{zA})_{z \in \Sigma_\theta \cup \{0\}}$ which is analytic on $\Sigma_\theta$ and which is bounded on $\{z \in \Sigma_\theta: |z| < 1\}$ for each $\theta' \in (0, \theta)$. The semigroup is called analytic if it is analytic of some angle $\theta \in (0, \pi/2]$. The $C_0$-semigroup $(e^{tA})_{t \geq 0}$ is called bounded analytic of angle $\theta \in (0, \pi/2]$ if it is analytic of angle $\theta$ and if its extension is bounded on $\Sigma_{\theta'}$ for each $\theta' \in (0, \theta)$. Finally, an operator $A$ on $E$ is called sectorial of angle $\theta \in (0, \pi/2]$ if $\rho(A) \supset \Sigma_{\pi/2+\theta}$ and if $\sup_{\lambda \in \Sigma_{\pi/2+\theta}} |\lambda R(A, \lambda)| < \infty$ for each $\theta' \in (0, \theta)$. Here we use the definition of sectorial operators in [13, Definition II.4.1], which differs from that in other sources.

Let $\theta \in (0, \pi/2]$. It is well known that a densely defined operator $A$ generates a $C_0$-semigroup which is bounded analytic of angle $\theta$ if and only if $A$ is sectorial of angle $\theta$, see [5, Theorem 3.7.11 and Corollary 3.3.11]. For our subsequent application to the Robin Laplace operator we will need the following observation:

**Lemma 6.5.** Let $(e^{tA})_{t \geq 0}$ be an analytic $C_0$-semigroup of angle $\theta \in (0, \pi/2]$ such that $\rho(A) \supset \Sigma_{\theta+\pi/2}$ and $0 \not\in \sigma(A)$. Then $(e^{tA})_{t \geq 0}$ is bounded analytic of angle $\theta$.

**Proof.** Let $\theta' \in (0, \theta)$. Using [5, Proposition 3.7.2(d)] we see that $(e^{te^{i\varphi}A})_{t \geq 0}$ and $(e^{te^{-i\varphi}A})_{t \geq 0}$ are analytic $C_0$-semigroups. Moreover, $s(e^{i\varphi}A) < 0$ and $s(e^{-i\varphi}A) < 0$. Hence, both semigroups $(e^{te^{i\varphi}A})_{t \geq 0}$ and $(e^{te^{-i\varphi}A})_{t \geq 0}$ converge to 0 with respect to the operator norm and are therefore bounded in norm by some constant $M \geq 1$. This in turn implies that $(e^{zA})_{z \in \Sigma_{\theta'}}$ is bounded by $M^2$, which shows the assertion. \qed

Let us now prove a result on squares of generators and eventual positivity.

**Proposition 6.6.** Let $B$ generate an analytic $C_0$-semigroup of angle $\pi/2$ on $E = C(K)$, and suppose that $\sigma(B) \subseteq (-\infty, 0)$ is non-empty and that $B$ has compact resolvent. If $R(0, B) \gg 0$, then $A := -B^2$ generates an analytic $C_0$-semigroup on $E$ which is individually eventually strongly positive.

**Proof.** By Lemma 6.5, the operator $B$ is sectorial of angle $\pi/2$. As is well known (and easy to check), this implies that $A$ is sectorial of angle $\pi/2$, too. Hence, $A$ generates an analytic $C_0$-semigroup.

Since $\sigma(A) \subseteq (-\infty, 0)$ is non-empty, $s(A) < 0$ is clearly a dominant spectral value of $A$. By assumption $R(0, B) \gg 0$ and hence an application of Proposition 4.3 shows that $R(\cdot, A)$ is uniformly eventually strongly positive at $s(A)$. As $B$ has compact resolvent, the same is true for $A$ and hence $s(A)$ is a pole of $R(\cdot, A)$, see [13, Corollary IV.1.19]. Now Theorem 4.4 implies that the spectral projection of $A$ associated with $s(A)$ is strongly positive. Hence, Corollary 5.6 shows that $(e^{tA})_{t \geq 0}$ is individually eventually strongly positive. \qed
6.4 The square of the Robin Laplacian on $C(\overline{\Omega})$.

We will apply Proposition 6.6 to a particular operator, the Robin Laplacian. To that end, let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain of class $C^2$ and let $\beta \in C^1(\partial\Omega)$ with $\beta \gg 0$. Denote by $\Delta^c_R$ the realisation of the Laplacian on $C(\Omega)$ subject to the Robin boundary condition

$$\frac{\partial}{\partial \nu} u + \beta u = 0 \quad \text{on } \partial\Omega.$$ 

It is shown in [1, Theorems 8.2 and 6.1] that $\Delta^c_R$ generates a compact and strongly positive semigroup on $C(\Omega)$, which is analytic of angle $\pi/2$ by [30, Theorem 3.3]. Moreover, as $\beta \gg 0$, we certainly have $\sigma(\Delta^c_R) \subseteq (-\infty, 0)$ and clearly, $\sigma(\Delta^c_R) \neq \emptyset$.

Proposition 6.7. Under our assumptions on $\beta$ and $\Omega$, the operator $A = -(\Delta^c_R)^2$ generates a $C_0$-semigroup on $C(\Omega)$ which is individually eventually strongly positive but not positive.

Proof. From the above discussion and from Proposition 6.6 it follows that $A$ generates an analytic $C_0$-semigroup $(e^{tA})_{t \geq 0}$ which is individually eventually strongly positive. The semigroup $(e^{tA})_{t \geq 0}$ is not positive because the restriction of $A$ to $C^\infty_c(\Omega)$ is the bi-Laplacian acting on $C^\infty_c(\Omega)$, whose extensions cannot generate a positive semigroup by [6, Proposition 2.2].

6.5 A delay differential equation

We consider the time evolution of a complex value $y(t)$, where the rate of change of $y(t)$ depends on the values of $y$ on the past time interval $[t-2, t]$, more precisely being given by

$$y'(t) = \int_{t-2}^{t-1} y(s) \, ds - \int_{t-1}^{t} y(s) \, ds. \quad (6.8)$$

This is called a delay differential equation and it can be analysed by means of evolution semigroups as described in [13, Section IV.2.8] (with a different time scale) and in [13, Section VI.6]. Note that the latter section deals with a more general situation; in their notation, we obtain the setting for our example by defining $Y := \mathbb{C}$ and $B := 0$. We can reformulate (6.8) as the abstract Cauchy problem $\dot{u}(t, \cdot) = Au(t, \cdot)$ on the space $C([-2, 0])$, where the operator $A$ is given by

$$D(A) := \left\{ f \in C^1([-2, 0]): f'(0) = \int_{-2}^{-1} f(x) \, dx - \int_{-1}^{0} f(x) \, dx \right\},$$

$$Af := f'.$$  

(6.9)

For a derivation of this reformulation, we refer to the references quoted above. There, it is also shown that the operator $A$ generates a $C_0$-semigroup on $C([-2, 0])$. Our aim here is to prove that this semigroup is individually eventually strongly positive.

Proposition 6.8. The operator $A: C([-2, 0]) \supset D(A) \to C([-2, 0])$ given by (6.9) has the following properties:

(i) The spectral bound $s(A)$ equals 0; moreover, it is a dominant spectral value and a pole of the resolvent.
(ii) The spectral projection \( P \) associated with \( s(A) \) is strongly positive.

(iii) The semigroup \((e^{tA})_{t \geq 0} \) on \( C([-2,0]) \) is individually eventually strongly positive.

(iv) The semigroup \((e^{tA})_{t \geq 0} \) is not positive.

Proof. For the proof we introduce the functional \( \Phi: C([-2,0]) \to \mathbb{C} \) given by

\[
\Phi(f) = \int_{-2}^{-1} f(x) \, dx - \int_{-1}^{0} f(x) \, dx.
\]

(i) Since the embedding \( D(A) \hookrightarrow C([-2,0]) \) is compact due to the Arzelà–Ascoli theorem, \( A \) has compact resolvent (cf. also [13, p. 256]). Hence, all spectral values are poles of the resolvent \( R(\cdot, A) \) (see [13, Corollary IV.1.19]).

Let us now show that \( 0 \) is a dominant spectral value of \( A \). By [13, formula (6.11) on p. 427], the spectral values of \( A \) are exactly the complex numbers \( \lambda \) which fulfil the equation

\[
\lambda - \Phi(e^{\lambda \cdot}) = 0.
\]

Using our definition of \( \Phi \), we obtain after a short computation that

\[
\lambda \in \sigma(A) \iff -\lambda^2 = (1 - e^{-\lambda}^2).
\]

Clearly, \( 0 \) is a solution of this equation, so \( 0 \in \sigma(A) \). One can also see directly that \( 1_{[-2,0]} \) is an eigenfunction for \( 0 \). To show that there are no other spectral values with non-negative real part, note that the above equation is satisfied if and only if one of the following two equations is fulfilled:

\[
i\lambda = 1 - e^{-\lambda}, \quad \text{(6.10)}
\]
\[
-i\lambda = 1 - e^{-\lambda}. \quad \text{(6.11)}
\]

Since \( \lambda \in \mathbb{C} \) fulfils (6.10) if and only if \( \overline{\lambda} \) fulfils (6.11), it is sufficient to consider the first equation. Writing \( \lambda = \alpha + i\beta \), where \( \alpha, \beta \in \mathbb{R} \), we obtain that (6.10) is equivalent to the system

\[
-\beta = 1 - e^{-\alpha} \cos \beta, \quad \text{(6.12)}
\]
\[
\alpha = e^{-\alpha} \sin \beta. \quad \text{(6.13)}
\]

Suppose that \( \alpha \geq 0 \). Then (6.12) yields that \( \beta \in [-2,0] \). Hence, we obtain from (6.13) that \( 0 \leq \alpha = e^{-\alpha} \sin \beta \leq 0 \). Thus we conclude that \( \sin \beta = 0 \), so \( \beta = 0 \) and, finally, \( \alpha = 0 \). So we have shown that the only spectral value of \( A \) with non-negative real part is given by \( \lambda = 0 \).

(ii) Clearly, the eigenvalue \( s(A) = 0 \) of \( A \) is geometrically simple and its eigenspace is spanned by \( 1_{[-2,0]} \). Moreover, consider the functional \( \varphi \in C([-2,0])' \) which is given by

\[
\varphi(f) = f(0) + \int_{-2}^{-1} (2 + x)f(x) \, dx + \int_{-1}^{0} -xf(x) \, dx.
\]

The functional \( \varphi \) is strongly positive and using the definition of the adjoint \( A' \), it is easy to check that \( \varphi \) is an eigenvector of \( A' \) for the eigenvalue 0. Hence, we conclude from Proposition 3.1 that \( P \gg 0 \).

Alternatively, we could use the explicit formula for the resolvent of \( A \) which is given in [13, Proposition VI.6.7] to compute that \( R(\cdot, A) \) is individually eventually strongly positive at \( s(A) = 0 \). Then it follows from Theorem 4.4 that \( P \gg 0 \).

(iii) Since \((e^{tA})_{t \geq 0} \) is eventually norm continuous (see [13, Theorem VI.6.6]), assertion (iii) follows from (i), (ii) and Corollary 5.6.

(iv) This follows from [3, Example B-II.1.22] by rescaling the time scale from \([-1,0] \) to \([-2,0] \).
7 The spectral bound of eventually positive semigroups

In this section we consider eventual positivity not only on \(C(K)\)-spaces but also on arbitrary Banach lattices. By analogy with Definition 5.1, we call a \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\) on a complex Banach lattice \(E\) \textit{individually eventually positive} if for each \(0 \leq f \in E\) there is a \(t_0 \geq 0\) such that \(e^{tA}f \geq 0\) for all \(t \geq t_0\). Our aim is to show that such semigroups have many properties which are already well known for positive semigroups.

We note that some of the results in this section should also hold on more general ordered spaces than Banach lattices. For example, Proposition 7.1 and its corollaries also hold on ordered Banach spaces with normal cones. One could also try to consider eventually positive semigroups on operator algebras, as was done for positive semigroups in [3, Chapter IV]. However, we shall not pursue this here.

Recall that if \(A\) generates a \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\), then \(\omega_0(A)\) denotes the growth bound of this semigroup. We start with the following representation formula for the resolvent: if \((e^{tA})_{t \geq 0}\) is a \(C_0\)-semigroup on a Banach space \(E\), it is well known that for \(\text{Re}\, \lambda > \omega_0(A)\) the resolvent \(R(\lambda, A)\) can by represented as the Laplace transform of the semigroup, that is,

\[
R(\lambda, A)f = \int_0^\infty e^{-t\lambda}e^{tA}f \, dt
\]  

(7.1)

for all \(f \in E\), where the integral is absolutely convergent. If the spectral bound and the growth bound \(\omega_0(A)\) of \(A\) do not coincide, this formula may in general fail for \(s(A) < \text{Re}\, \lambda \leq \omega_0(A)\); see [5, Example 5.1.10, Theorem 5.1.9 and the end of p. 342]. It is a special feature of positive semigroups that (7.1) holds in the strip \(s(A) < \text{Re}\, \lambda \leq \omega_0(A)\), where the integral is to be understood as an improper Riemann integral; see [3, Theorem C-III-1.2] or [5, Theorem 5.3.1 and Proposition 5.1.4]. We now show that this property holds for individually eventually positive semigroups as well.

**Proposition 7.1.** Let \((e^{tA})_{t \geq 0}\) be an individually eventually positive \(C_0\)-semigroup on a complex Banach lattice \(E\). Then the Laplace transform representation (7.1) is valid whenever \(\text{Re}\, \lambda > s(A)\) and \(f \in E\), where the integral converges as an improper Riemann integral.

**Proof.** We may assume that \(f \geq 0\). Let \(t_0 \geq 0\) such that \(e^{tA}f \geq 0\) for all \(t \geq t_0\) and consider the functions \(u, v: [0, \infty) \to E\), \(u(t) = e^{tA}f, v(t) = e^{(t_0+t)A}f\). By abs \(u, abs v\) we denote the abscissas of convergence of the Laplace transforms \(\hat{u}, \hat{v}\), as for instance defined in [5, Section 1.4]. Clearly, \(\text{abs}(u) \leq \omega_0(A)\). From the formula

\[
\int_0^T e^{-t\lambda}u(t) \, dt = \int_0^{t_0} e^{-t\lambda}u(t) \, dt + e^{-t_0\lambda}\int_{t_0}^{T} e^{-t\lambda}v(t) \, dt
\]

for all \(T > t_0\), we conclude that \(\text{abs}(u) = \text{abs}(v)\). Both Laplace transforms \(\hat{u}(\lambda)\) and \(\hat{v}(\lambda)\) exist and are analytic on the half plane \(\text{Re}\, \lambda > \text{abs}(v)\) (see [5, Proposition 1.4.1 and Theorem 1.5.1]). The function \(\hat{u}(\lambda)\) coincides with \(R(\lambda, A)f\) for \(\text{Re}\, \lambda > \omega_0(A)\) and so, due to the identity theorem for analytic functions, also for \(\text{Re}\, \lambda > \text{abs}(v)\). Hence, we only have to show \(\text{abs}(v) \leq s(A)\).

Assume for a contradiction that \(\text{abs}(v) > s(A)\). Then \(\text{abs}(v) > -\infty\) and \(\text{abs}(v) \notin \sigma(A)\). Since \(v(t) \geq 0\) for all \(t \geq 0\), [5, Theorem 1.5.3] implies that \(\hat{v}\) has a singularity at
abs(v), i.e. \( \hat{v} \) cannot be analytically extended to an open neighbourhood of abs(v). As we have

\[
R(\lambda, A)f = \hat{u}(\lambda) = \int_0^{t_0} e^{-t\lambda}u(t)\,dt + e^{-t_0\lambda}\hat{v}(\lambda)
\]

for Re \( \lambda > \) abs(v), we conclude that \( R(\cdot, A)f \) also has a a singularity at abs(v). This contradicts abs(v) \( \not\in \sigma(A) \).

Note that the proof of the above proposition in fact shows that [5, Theorem 1.5.3] holds for eventually positive functions.

Proposition 7.1 yields the following stability result, which is already known for positive semigroups; see [13, Proposition VI.1.14].

**Corollary 7.2.** Let \((e^At)_{t \geq 0}\) be an individually eventually positive \(C_0\)-semigroup on a complex Banach lattice \(E\). Then \(s(A) < 0\) if and only if there exists \(\varepsilon > 0\) such that \(e^{\varepsilon t}\|e^Atf\| \to 0\) as \(t \to \infty\) for every \(f \in D(A)\).

**Proof.** If we use the notation from [5, p. 343], then Proposition 7.1 shows that \(\text{abs}(e^{-A}) = s(A)\) and hence

\[
s(A) = \omega_1(e^{-A}) := \inf\{\omega \in \mathbb{R} : \forall x \in D(A) \exists M \geq 1 \text{ with } \|e^{tA}x\| \leq Me^{\omega t} \forall t \geq 0\}
\]

by [5, Theorem 5.1.9]. This implies the assertion. \(\Box\)

Another corollary of Proposition 7.1 is the following “asymptotic positivity” of the resolvent.

**Corollary 7.3.** Let \((e^At)_{t \geq 0}\) be an individually eventually positive \(C_0\)-semigroup on a Banach lattice \(E\) with \(s(A) > -\infty\). Then for every \(f \geq 0\) we have

\[
\text{dist}\left((\lambda - s(A))R(\lambda, A)f, E_+\right) \to 0 \quad \text{as } \lambda \downarrow s(A).
\]

**Proof.** We may assume \(s(A) = 0\). Let \(f \geq 0\) and choose \(t_0\) such that \(e^{t_0}f \geq 0\) whenever \(t \geq t_0\). By Proposition 7.1, we obtain for \(\lambda > 0\) that

\[
\text{dist}\left(\lambda R(\lambda, A)f, E_+\right) \leq \left\|\lambda \int_0^{t_0} e^{-t\lambda}e^{tA}f\,dt\right\| \leq C(1 - e^{-t_0\lambda}),
\]

where \(C = \sup_{0 \leq t \leq t_0} \|e^Atf\|\). The Corollary follows by letting \(\lambda \downarrow 0\). \(\Box\)

For a positive semigroup \((e^At)_{t \geq 0}\) the estimate \(|R(\lambda, A)f| \leq R(\text{Re } \lambda, A)|f|\) holds for all \(f \in E\) whenever \(\text{Re } \lambda > s(A)\); this is an easy consequence of the validity of formula (7.1) for \(\text{Re } \lambda > s(A)\); see [3, Corollary C-III-1.3]. The following lemma provides us with a slightly weaker result for individually eventually positive semigroups and for real elements \(f \in E_\mathbb{R}\).

**Lemma 7.4.** Let \((e^At)_{t \geq 0}\) be an individually eventually positive \(C_0\)-semigroup on a complex Banach lattice \(E\). For each \(f \in E_\mathbb{R}\) there is a bounded map \(r_f : (s(A), \infty) \to E\) which satisfies the following properties:

(i) We have \(|R(\lambda, A)f| \leq R(\text{Re } \lambda, A)|f| + r_f(\text{Re } \lambda)\) for all \(\text{Re } \lambda > s(A)\);
(ii) If \(s(A) > -\infty\), then \(r_f\) is norm-bounded on \((s(A), \infty)\).
(iii) If \( s(A) = -\infty \), then \( r_f \) is norm-bounded on \( (\alpha, \infty) \) for every \( \alpha \in \mathbb{R} \).

**Proof.** Let \( f \in E_R \) and let \( t_0 \geq 0 \) such that \( e^{tA}f^+ \geq 0 \) and \( e^{tA}f^- \geq 0 \) for all \( t \geq t_0 \). Then \( |e^{tA}f| \leq e^{tA}|f| \) for all \( t \geq t_0 \). For \( \text{Re} \lambda > s(A) \) and \( T \geq t_0 \) we have

\[
\left| \int_0^T e^{-\lambda t} e^{tA} f \, dt \right| \leq \int_0^T e^{-t \text{Re} \lambda} |e^{tA}f| \, dt \\
\leq \int_0^T e^{-t \text{Re} \lambda} e^{tA}|f| \, dt + \int_0^{t_0} e^{-t \text{Re} \lambda} \left( |e^{tA}f| - e^{tA}|f| \right) \, dt.
\]

Letting \( T \to \infty \) we conclude from Proposition 7.1 that

\[
|R(\lambda, A)f| \leq R(\text{Re} \lambda, A)|f| + \int_0^{t_0} e^{-t \text{Re} \lambda} \left( |e^{tA}f| - e^{tA}|f| \right) \, dt.
\]

Defining the last integral as \( r_f(\text{Re} \lambda) \), we obtain (i)–(iii). \( \square \)

Recall that for a \( C_0 \)-semigroup \((e^{tA})_{t \geq 0}\) on a complex Banach space, the quantity

\[ s_0(A) := \inf \{ \omega > s(A) : \sup_{\text{Re} \lambda > \omega} \| R(\lambda, A) \| < \infty \} \]

is called the **abscissa of uniform boundedness of the resolvent** or the **pseudo-spectral bound** of \( A \).

**Corollary 7.5.** Let \((e^{tA})_{t \geq 0}\) be an individually eventually positive semigroups on a complex Banach lattice \( E \). Then \( s(A) = s_0(A) \).

**Proof.** This readily follows from Lemma 7.4 and the uniform boundedness principle. \( \square \)

Using Lemma 7.4, we are able to prove a generalisation of a well-known result for positive semigroups as for instance given in [3, Corollary C-III.1.4] or [5, Theorem 5.3.1]

**Theorem 7.6.** Let \((e^{tA})_{t \geq 0}\) be an individually eventually positive \( C_0 \)-semigroup on a complex Banach lattice \( E \). If \( \sigma(A) \neq \emptyset \), then \( s(A) \in \sigma(A) \).

**Proof.** Let \( \sigma(A) \neq \emptyset \) and choose a sequence \((\lambda_n)\) with \( \text{Re} \lambda_n > s(A) \) such that \( \text{dist}(\lambda_n, \sigma(A)) \to 0 \). Then \( \text{Re} \lambda_n \to s(A) \) and \( \| R(\lambda_n, A) \| \to \infty \). By the uniform boundedness principle, there is an \( f \in E \) and a subsequence \((\lambda_{n_k})\) of \((\lambda_n)\) such that \( \| R(\lambda_{n_k}, A)f\| \to \infty \). We may in fact choose \( f \) to be real. Thus, Lemma 7.4 implies that \( \| R(\text{Re} \lambda_{n_k}, A)f\| \to \infty \). As \( \text{Re} \lambda_{n_k} \to s(A) \), we conclude that \( s(A) \in \sigma(A) \). \( \square \)

If we know that the spectral bound \( s(A) \) is a pole of the resolvent, then we can draw a conclusion on the order of any other pole in the peripheral spectrum \( \sigma_{\text{per}}(A) \), similar to the case of positive semigroups; see [3, Corollary C-III-1.5].

**Theorem 7.7.** Let \((e^{tA})_{t \geq 0}\) be an individually eventually positive \( C_0 \)-semigroup on a complex Banach lattice \( E \). Suppose that \( s(A) > -\infty \) is a pole of \( R(\cdot, A) \) of order \( m \in \mathbb{N} \). Then we have the following assertions.

(i) The number \( s(A) \) is an eigenvalue of \( A \) admitting a positive eigenvector.

(ii) Every pole of \( R(\cdot, A) \) in \( \sigma_{\text{per}}(A) \) has order at most \( m \).
\textit{Proof.} As usual, without loss of generality we assume that $s(A) = 0$. To prove (i) we use the Laurent expansion (2.2) of $R(\cdot, A)$ about $s(A) = 0$. As 0 is a pole of order $m$ it follows that $\lambda^m R(\lambda, A) \to U^{m-1}$ in $\mathcal{L}(E)$. Also recall from Remark 2.1 that $\text{im}(U^{m-1}) \neq \{0\}$ consists of eigenvectors of $A$ to the eigenvalue 0. It follows from Corollary 7.3 that $U^{m-1}$ is positive, and hence $A$ has a positive eigenvector.

To prove (ii) assume that $\lambda_0 \in i\mathbb{R}$ is a pole of $R(\cdot, A)$. Applying Lemma 7.4 we see that for $\lambda = \lambda_0 + \alpha$ with $\alpha > 0$ we have

$$| (\lambda - \lambda_0)^k R(\lambda, A) f | \leq \alpha^k R(\alpha, A)|f| + \alpha^k r(\alpha)$$

(7.2)

for all $k \in \mathbb{N}$ and all $f \in E_\mathbb{R}$. If $\lambda_0$ is a pole of order $k_0$, then $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{k_0} R(\lambda, A)$ exists in $\mathcal{L}(E)$ and the limit is non-zero. However, the right hand side of (7.2) converges to 0 as $\alpha \downarrow 0$ if $k > m$. Hence, $k_0 \leq m$. \hfill \Box

It is another remarkable property of positive semigroups that on many important spaces their spectral bound and their growth bound always coincide (see [3, Theorem C-IV-1.1(a)]). In the next theorem, we show that this remains true for individually eventually positive semigroups, for essentially the same reasons.

\textbf{Theorem 7.8.} Let $(e^{tA})_{t \geq 0}$ be an individually eventually positive $C_0$-semigroup on a complex Banach lattice $E$. Then $s(A) = \omega_0(A)$ in any of the following cases:

(i) $E$ is a Hilbert space.

(ii) $E = L^1(\Omega, \Sigma, \mu)$ for an arbitrary measure space $(\Omega, \Sigma, \mu)$ with $\mu \geq 0$.

(iii) $E = C(K)$ for a compact Hausdorff space $K$ and $A$ is real.

\textit{Proof.} Note that it is sufficient to prove in each case that every individually eventually positive (and, in case (iii), real) semigroup on $E$ satisfies the implication

$$s(A) < 0 \implies (e^{tA})_{t \geq 0} \text{ is bounded.}$$

(7.3)

Indeed, (7.3) yields for every individually eventually positive (and, in case (iii), real) semigroup $(e^{tA})_{t \geq 0}$ that the rescaled semigroup $(e^{t(A-\alpha)})_{t \geq 0}$ is bounded whenever $\alpha > s(A)$; this in turn implies $\omega_0(A) \leq \alpha$ and hence $\omega_0(A) \leq s(A)$.

(i) Suppose that $s(A) < 0$. Then we have $s_0(A) < 0$ according to Corollary 7.5. The Gearhart–Prüss theorem, see [13, Theorem V.1.11] or [5, Theorem 5.2.1], now implies that $e^{tA}$ converges to 0 with respect to the operator norm as $t \to \infty$; in particular, $(e^{tA})_{t \geq 0}$ is bounded.

(ii) Let $E = L^1(\Omega, \Sigma, \mu)$ and suppose that $s(A) < 0$. Given $f \in E_+$, choose $t_0 \geq 0$ such that $e^{t_0A} f \geq 0$ for all $t \geq t_0$. Proposition 7.1 yields that $R(0, A) f = \int_0^\infty e^{tA} f \, dt$ exists as an improper Riemann integral. Hence, due to the additivity of $\| \cdot \|_1$ on the positive cone $E_+$, we obtain

$$\int_0^\infty \| e^{tA} f \|_1 \, dt \leq \left\| \int_0^\infty e^{tA} f \, dt \right\|_1 + 2 \int_0^{t_0} \| e^{tA} f \|_1 \, dt.$$ 

Therefore, $\int_0^\infty \| e^{tA} f \|_1 \, dt < \infty$ for every $f \in E_+$ and thus for every $f \in E$. By a theorem due to Datko and Pazy (see [5, Theorem 5.1.2] or [13, Theorem V.1.8]) we conclude that $(e^{tA})_{t \geq 0}$ converges to 0 with respect to the operator norm and is in particular bounded.

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(iii) Let $E = C(K)$ and suppose $A$ is real and $s(A) < 0$. Given $f \in E$, Proposition 7.1 implies that the Laplace transform of the trajectory 

$$[0, \infty) \to E, \quad t \mapsto e^{tA}f$$

has an abscissa of convergence which is no larger than $s(A)$ (see [5, Start of Section 1.4] for a definition of the abscissa of convergence). Hence, [5, Proposition 1.4.5(a)] implies that \( \lim_{t \to \infty} \int_0^t e^{-\lambda s} e^{sA} ds \) exists in the operator norm whenever $\Re \lambda > s(A)$. Setting $\lambda = 0$ we obtain in particular that

$$M := \sup_{t \geq 0} \left\| \int_0^t e^{sA} ds \right\|_{\infty} < \infty.$$ 

Next, we choose an element $u \in D(A)$ with $u \gg 0$. Such an element $u$ exists since $D(A)$ is dense in $C(K)$ and $A$ is real. Hence, for all $t > 0$

$$\left\| e^{tA}u - u \right\|_{\infty} = \left\| \int_0^t e^{sA}Au \ ds \right\|_{\infty} \leq M \left\| Au \right\|_{\infty},$$

so the trajectory $(e^{tA})_{t \geq 0}$ is bounded. Now let $f \in E_+$ with $\|f\|_{\infty} = 1$. As $u \gg 0$ there exists $\beta > 0$ such that $0 \leq f \leq 1 \leq \beta u$. Hence, by the eventual positivity of the semigroup there exists $t_0 > 0$ such that $0 \leq e^{tA}f \leq \beta e^{tA}u$ for all $t \geq t_0$. This shows that the trajectory $(e^{tA}f)_{t \geq 0}$ is bounded for all $f \in E_+$ and hence for all $f \in E$. The uniform boundedness principle finally implies that $(e^{tA})_{t \geq 0}$ is bounded.

For positive semigroups, the assertion of Theorem 7.8 is also known to hold on $L^p(\Omega, \Sigma, \mu)$ for $\sigma$-finite measure spaces $(\Omega, \Sigma, \mu)$ (see [5, Theorem 5.3.6] or [29, Theorem 3.5.3]) and on $C_0(L)$-spaces for locally compact Hausdorff spaces $L$ (see [3, Theorem B-IV-1.4]). It would be interesting to know whether those results remain true for individually or at least for uniformly eventually positive semigroups.

On $C(K)$ spaces, Theorem 7.8 yields the following result on the non-emptiness of the spectrum of the generator.

**Corollary 7.9.** Let $(e^{tA})_{t \geq 0}$ be a real and uniformly eventually strongly positive $C_0$-semigroup on $C(K)$ for some compact Hausdorff space $K$. Then $\sigma(A) \neq \emptyset$.

**Proof.** For sufficiently large $t_0 > 0$, the operator $e^{t_0A}$ is strongly positive. Thus, we have $e^{t_0A} 1 \geq \varepsilon 1$ for some $\varepsilon > 0$. Iterating this inequality, we obtain that $(e^{t_0A})^n 1 \geq \varepsilon^n 1$ for all $n \in \mathbb{N}$. Hence, $\| (e^{t_0A})^n \| \geq \varepsilon^n$ and therefore $r(e^{t_0A}) \geq \varepsilon$. As $r(e^{t_0A}) = e^{t_0} \omega_0(A)$ (see [13, Proposition IV.2.2]), we conclude that $\omega_0(A) > -\infty$. Since the semigroup is real, so is $A$ and Theorem 7.8 thus implies $\omega_0(A) = s(A)$. Hence, $s(A) > -\infty$. \qed

**Remark 7.10.** For the generator $A$ of a positive semigroup on $C(K)$, the fact that $\sigma(A) \neq \emptyset$ is true without any irreducibility or strong positivity assumptions, see [3, Theorem B-III-1.1]. It does not seem clear whether $\sigma(A) \neq \emptyset$ for a uniformly or individually eventually positive semigroup on $C(K)$ in general.
8 Final remarks on eventually positive resolvents

After discussing the spectral bound of individually eventually positive semigroups, let us finish with a few notes on individually eventually positive resolvents on arbitrary Banach lattices. If $E$ is a complex Banach lattice, $A$ is a closed operator on $E$ and $\lambda_0$ is either $-\infty$ or a spectral value of $A$ in $\mathbb{R}$ then, in complete analogy to Section 4, the resolvent of $A$ is called \textit{individually eventually positive at} $\lambda_0$ if there is a number $\lambda_2 > \lambda_0$ with the following properties: $(\lambda_0, \lambda_2] \subseteq \rho(A)$ and for each $f \in E_+$ there exists $\lambda_1 \in (\lambda_0, \lambda_2]$ such that $R(\lambda, A)f \geq 0$ for all $\lambda \in (\lambda_0, \lambda_1]$. Let us first make the following simple observation.

**Proposition 8.1.** Let $A$ be a closed real operator on a complex Banach lattice $E$, let $\lambda_0$ be either $-\infty$ or a spectral value of $A$ in $\mathbb{R}$ and suppose that the resolvent of $A$ is individually eventually positive at $\lambda_0$. Then the cone $D(A)_+ := D(A) \cap E_+$ is generating in $D(A)$, that is, $D(A) = D(A)_+ - D(A)_+$.

**Proof.** Let $f \in D(A)$ and choose $\lambda > \lambda_0$ sufficiently small such that $\lambda \in \rho(A)$ and such that $R(\lambda, A)f^+ \geq 0$, $R(\lambda, A)f^- \geq 0$, $R(\lambda, A)(Af)^+ \geq 0$ and $R(\lambda, A)(Af)^- \geq 0$. We then have

$$f = R(\lambda, A)(\lambda - A)f = \lambda R(\lambda, A)f^+ - \lambda R(\lambda, A)f^- - R(\lambda, A)(Af)^+ + R(\lambda, A)(Af)^-,$$

which is clearly contained in $D(A)_+ - D(A)_+$. \hfill \Box

One might ask whether $D(A) \cap E_+$ is also generating in $D(A)$ if $A$ is the generator of an individually eventually positive semigroup, and one might also wonder whether the resolvent of the generator $A$ of an individually eventually positive semigroup is always individually eventually positive at $s(A)$. The following example shows that the answer to both questions is negative, even if the semigroup is assumed to be uniformly eventually positive.

**Example 8.2.** There is a real $C_0$-semigroup $(e^{tA})_{t \geq 0}$ on a complex Banach lattice $E$ with the following properties:

(a) The semigroup $(e^{tA})_{t \geq 0}$ is nilpotent and therefore uniformly eventually positive.

(b) The resolvent $R(\cdot, A)$ is not individually eventually positive at $s(A) = -\infty$.

(c) $D(A)_+ := D(A) \cap E_+$ is not generating in $D(A)$.

Indeed, let $p \in [1, \infty)$ and let $E = L^p((0,1)) \oplus L^p((0,1))$. We define a “sign-flipping left shift semigroup” $(e^{tA})_{t \geq 0}$ on $E$ in the following way: For $(f_1, f_2) \in E$ we set $e^{tA}(f_1, f_2) = (g_1(t), g_2(t))$, where

$$g_1(x, t) = \begin{cases} f_1(x + t) & \text{if } 0 \leq x + t \leq 1, \\ -f_2(x - 1 + t) & \text{if } 1 < x + t \leq 2, \\ 0 & \text{if } 2 < x + t, \end{cases}$$

and

$$g_2(x, t) = \begin{cases} f_2(x + t) & \text{if } 0 \leq x + t \leq 1, \\ 0 & \text{if } 1 < x + t. \end{cases}$$

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Clearly this semigroup is nilpotent. Similarly as in [3, Section A-I-2.6], it can be proved that its generator \( A \) is given by
\[
D(A) = \{(f_1, f_2) \in W^{1,p}((0,1)) \oplus W^{1,p}((0,1)) : f_1(1) = -f_2(0) \text{ and } f_2(1) = 0\},
\]
\[
A(f_1, f_2) = (f'_1, f'_2).
\]
In particular we have \( f_1(1) = f_2(1) = 0 \) for each tuple \((f_1, f_2) \in D(A)_+\). Hence, \( D(A)_+ \) is not generating in \( D(A) \). By Proposition 8.1 this implies that the resolvent \( R(\cdot, A) \) is not individually eventually positive at \(-\infty\).

Acknowledgements The authors would like to express their warmest thanks to Wolfgang Arendt for many enlightening discussions and for contributing several ideas and proofs. They would also like to thank Anna Dall’Acqua for pointing out several references, and the referee for a careful and thoughtful reading of the manuscript. This paper was initiated during a very pleasant visit of the first author to Ulm University.

References

[1] H. Amann, Dual semigroups and second order linear elliptic boundary value problems, Israel J. Math. 45 (1983), 225–254. DOI: 10.1007/BF02774019

[2] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, Trans. Amer. Math. Soc. 306 (1988), 837–852. DOI: 10.2307/2000826

[3] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck, One-parameter semigroups of positive operators, Lecture Notes in Mathematics, vol. 1184, Springer-Verlag, Berlin, 1986. DOI: 10.1007/BFb0074922

[4] W. Arendt, A. F. M. ter Elst, J. B. Kennedy, and M. Sauter, The Dirichlet-to-Neumann operator via hidden compactness, J. Funct. Anal. 266 (2014), 1757–1786. DOI: 10.1016/j.jfa.2013.09.012

[5] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, second ed., Monographs in Mathematics, vol. 96, Birkhäuser/Springer Basel AG, Basel, 2011. DOI: 10.1007/978-3-0348-0087-7

[6] W. Arendt, C. J. K. Batty, and D. W. Robinson, Positive semigroups generated by elliptic operators on Lie groups, J. Operator Theory 23 (1990), 369–407.

[7] W. Arendt and R. Mazzeo, Friedlander’s eigenvalue inequalities and the Dirichlet-to-Neumann semigroup, Commun. Pure Appl. Anal. 11 (2012), 2201–2212. DOI: 10.3934/cpaa.2012.11.2201

[8] A. Brauer, On the characteristic roots of power-positive matrices, Duke Math. J. 28 (1961), 439–445. DOI: 10.1215/S0012-7094-61-02840-X

[9] A. P. Campbell and D. Daners, Linear Algebra via Complex Analysis, Amer. Math. Monthly 120 (2013), 877–892. DOI: 10.4169/amer.math.monthly.120.10.877

[10] A. Dall’Acqua and G. Sweers, The clamped-plate equation for the limaçon, Ann. Mat. Pura Appl. (4) 184 (2005), 361–374. DOI: 10.1007/s10231-004-0121-9

[11] D. Daners, J. Glöckler, and J. Kennedy, Eventually and asymptotically positive semigroups on Banach lattices, In preparation.

[12] D. Daners, Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator, Positivity 18 (2014), 235–256. DOI: 10.1007/s11117-013-0243-7

[13] K.-J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000, With contributions
by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. DOI: 10.1007/b97696

[14] J. Escher, The Dirichlet-Neumann operator on continuous functions, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 21 (1994), 235–266. Available at http://www.numdam.org/item?id=ASNSP_1994_4_21_2_235_0

[15] A. Ferrero, F. Gazzola, and H.-C. Grunau, Decay and eventual local positivity for biharmonic parabolic equations, Discrete Contin. Dyn. Syst. 21 (2008), 1129–1157. DOI: 10.3934/dcds.2008.21.1129

[16] H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures.

[17] F. Gazzola and H.-C. Grunau, Eventual local positivity for a biharmonic heat equation in \(\mathbb{R}^n\), Discrete Contin. Dyn. Syst. Ser. S 1 (2008), 83–87. DOI: 10.3934/dcdss.2008.1.83

[18] J. J. Grobler, Spectral theory in Banach lattices, Operator theory in function spaces and Banach lattices, Oper. Theory Adv. Appl., vol. 75, Birkhäuser, Basel, 1995, pp. 133–172. DOI: 10.1007/978-3-0348-9076-2_10

[19] H.-C. Grunau and G. Sweers, Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions, Math. Nachr. 179 (1996), 89–102. DOI: 10.1002/mana.19961790106

[20] R. A. Horn and C. R. Johnson, Matrix analysis, Cambridge University Press, Cambridge, 1985. DOI: 10.1017/CBO9780511810817

[21] C. R. Johnson and P. Tarazaga, On matrices with Perron-Frobenius properties and some negative entries, Positivity 8 (2004), 327–338. DOI: 10.1007/s11117-003-3881-3

[22] T. Kato, Perturbation theory for linear operators, second ed., Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, Band 132. DOI: 10.1007/978-3-642-66282-9

[23] Y. Katznelson, An introduction to harmonic analysis, third ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004.

[24] S. Miyajima and N. Okazawa, Generators of positive \(C_0\)-semigroups, Pacific J. Math. 125 (1986), 161–176. Available at http://projecteuclid.org/euclid.pjm/1102700217

[25] D. Noutsos, On Perron-Frobenius property of matrices having some negative entries, Linear Algebra Appl. 412 (2006), 132–153. DOI: 10.1016/j.laa.2005.06.037

[26] D. Noutsos and M. J. Tsatsomeros, Reachability and holdability of nonnegative states, SIAM J. Matrix Anal. Appl. 30 (2008), 700–712. DOI: 10.1137/070693850

[27] H. H. Schaefer, Banach lattices and positive operators, Springer-Verlag, New York, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 215. DOI: 10.1007/978-3-642-65970-6

[28] E. Seneta, Nonnegative matrices and Markov chains, second ed., Springer Series in Statistics, Springer-Verlag, New York, 1981. DOI: 10.1007/0-387-32792-4

[29] J. van Neerven, The asymptotic behaviour of semigroups of linear operators, Operator Theory: Advances and Applications, vol. 88, Birkhäuser Verlag, Basel, 1996. DOI: 10.1007/978-3-0348-9206-3

[30] M. Warma, The Robin and Wentzell-Robin Laplacians on Lipschitz domains, Semigroup Forum 73 (2006), 10–30. DOI: 10.1007/s00233-006-0617-2

[31] K. Yosida, Functional analysis, Classics in Mathematics, Springer-Verlag, Berlin, 1995. DOI: 10.1007/978-3-642-61859-8