On fast bounded locality sensitive hashing

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Abstract. In this paper, we examine the hash functions expressed as scalar products, i.e., \( f(x) = \langle v, x \rangle \), for some bounded random vector \( v \). Such hash functions have numerous applications, but often there is a need to optimize the choice of the distribution of \( v \). In the present work, we focus on so-called anti-concentration bounds, i.e., the upper bounds of \( \mathbb{P}[|\langle v, x \rangle| < \alpha] \). In many applications, \( v \) is a vector of independent random variables with standard normal distribution. In such case, the distribution of \( \langle v, x \rangle \) is also normal and it is easy to approximate \( \mathbb{P}[|\langle v, x \rangle| < \alpha] \). Here, we consider two bounded distributions in the context of the anti-concentration bounds. Particularly, we analyze \( v \) being a random vector from the unit ball in \( l_\infty \) and \( v \) being a random vector from the unit sphere in \( l_2 \). We show optimal up to a constant anti-concentration measures for functions \( f(x) = \langle v, x \rangle \).

As a consequence of our research, we obtain new best results for \( c \)-approximate nearest neighbors without false negatives for \( l_p \) in high dimensional space for all \( p \in [1, \infty] \), for \( c = \Omega(\max\{\sqrt{d}, d^{1/p}\}) \). These results improve over those presented in [16]. Finally, our paper reports progress on answering the open problem by Pagh [17], who considered the nearest neighbor search without false negatives for the Hamming distance.

1 Introduction

Locality sensitive hashing (LSH) functions are hash functions which roughly preserve distance. Namely, for two points 'close' to each other in a given metric, the hashes of these points are also 'close' with large probability. Analogically, two 'distant' points have 'distant' hashes. The concept of LSH is well known and widely used, especially in the high dimension nearest neighbor search [15,17,11,10]. Normally, one uses LSH to reduce the dimension of a given metric space, usually \( l^d \) or a Hamming space. Common choices of the hash functions are \( f(x) = \langle x, v \rangle \) or \( f(x) = \lfloor \langle x, v \rangle \rfloor \), where \( v \) is a vector of numbers drawn independently from some probability distribution. For instance, the famous Johnson-Lindenstrauss Lemma [13] can be seen as LSH where \( v_i \) are independently drawn from the standard normal distribution, for \( i \in \{1, \ldots, d\} \). In fact, any distribution with

\[\footnote{In the introduction, we use imprecise terms such as 'close', 'distant', 'small', 'large', etc. in order to avoid introducing complex notation. These terms are going to be clarified in further sections.} \]
bounded variance produces an LSH function, as \(< x, v >\) is a good approximation of \(\|x\|_2^2\) up to scaling by a constant. Such a choice of hash functions has fine theoretical properties. Moreover, they are very cheap to evaluate, which makes them very useful for practical purposes. The evaluation of a scalar product is proportional to the size of vector representation. We say that hash functions with such property are fast. In this paper, we restrict ourselves to such hash functions.

For the sake of convenience, instead of considering two points \(x, y\) 'close' or 'distant', we can consider one point \(z = x - y\) and call it 'small' or 'large' respectively. Given an LSH function, a false positive is a point which is 'large' but its hash is 'small'. Similarly, a false negative is a point which is 'small' but its hash is 'large'. Naturally, we would like to avoid both false negatives and false positives. Many choices of distributions for LSH functions (e.g. normal distribution) give only probabilistic guarantees for both false negatives and false positives. Pacuk et al. [16] considered hash functions where \(v\) is a vector of independent Rademacher variables. Since Rademacher variable is bounded, the hash of a 'small' vector cannot be too 'large'. Consequently, for such a choice of \(v\), it is possible to eliminate false negatives. The hash functions induced by bounded distributions will also be called bounded.

In this paper, we study the concentration properties of fast bounded LSH functions. The crucial concept of this paper is a so-called anti-concentration measure. For a given random vector \(v\), we are interested in finding the upper bounds of \(\Pr[|\langle v, x \rangle| \leq \alpha]\), where \(x \in S_{d-1}^p\). If variable \(X\) is concentrated in \((-A, A)\), say \(\Pr(|X| < A) = 1 - \epsilon\), and density of \(X\) is symmetric and quasi concave, then \(\Pr[|X| \leq \alpha] \geq \frac{\alpha}{A}(1 - \epsilon)\). We show that the quasi-concaveness is a crucial property of our functions. Actually, the lack of this property was the reason for the inefficiency of the hash functions considered by [16]. With the quasi-concaveness assumption on the density function, we show an optimal, up to a constant, fast bounded hash function.

Based on the hash function, we build an algorithm for the \(c\)-approximate nearest neighbor without false negatives. In the classical nearest neighbor search, given an input set and a query point, we would like to find a point from input set which is the closest to the query point. Another variant involves returning any input point within the distance \(r\) from the query point, for a given parameter \(r\), or reporting that such point does not exist. Unfortunately, these problems do not have efficient solutions for high dimensional spaces. The existence of such algorithms, with the query and preprocessing complexities not depending exponentially on the dimension, would disprove the strong exponential time hypothesis [18]. In order to overcome this obstacle, we consider the \(c\)-approximate nearest neighbor, which allows false positives closer than \(cr\) to the query point.

\(^2\) In practice, we often consider a version of the algorithm which returns all input points within a given radius from the query point. Here we consider a one-point query outputs to keep the calculations plain for the reader’s convenience. However, all presented results easily transfer to multi-point query outputs.
As mentioned, known algorithms for the $c$-approximate nearest neighbor give Monte Carlo guarantees. In this paper, we guaranty no false negatives. Some known derandomizations result only in theoretical gain since it is easy to tune a probabilistic algorithm to have the exponentially small chance of error (e.g., probabilistic prime number testing). This is not true in our case. Consider a situation where there are many possible result points within the radius $r$ from the query point. In such a case, standard LSH algorithms [12] need an exponentially large number of hash functions to be able to exponentially decrease the chance of a false negative. In this paper, we improve complexities of the algorithms for the $c$-approximate nearest neighbor without false negatives in $l_p$ for all $p \in [1, \infty)$.

The presented algorithms have two stages. In the preprocessing stage, we prepare data structures for further queries. In this phase, we use only the input set and the complexity is expected to be polynomial, possibly close to $O(n)$. In the second stage, we perform the queries. Each query should have the complexity $o(n)$, in order to outrun the trivial full scan algorithm. In designing the algorithm, we usually need to choose between different configurations of complexities. Larger processing time can help reduce the query time and vice versa. In this work, we consider different trade-offs between the query and preprocessing times. Improving the hash functions helps us reduce both the query time and the preprocessing time of the $c$-approximate nearest neighbor without false negatives for $c = \Theta(\max(\sqrt{d}, d^{1/p}))$ in comparison with the results of [16]. Under natural assumptions, we show the hash functions with optimal, up to the multiplicative constant, anti-concentration bounds.

2 Related Work

2.1 The anti-concentration measures

In this paper we focus on the anti-concentration measures for $\langle v, x \rangle$, for $x \in S_p^{(d-1)}$. Let us start with a general bound for functions on a sphere. Particularly, in the small ball probability theorem for some function $f$ on the unit sphere $S^{(d-1)}$, we bound $P[|f(x)| \leq \alpha]$. The theorem conjectured in [14] and proved in [6] implies that for any Lipschitz function $f$, with Lipschitz constant $L$, whose average over the sphere is 1, we have $P[|f(x)| \leq \alpha] \leq \alpha^{c/L^2}$, for some constant $c$ and $x \in S^{(d-1)}$.

Carbery and Wright [4] show the following bound for polynomial functions. There exists an absolute constant $c > 0$ such that, if $Q : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree at most $k$ and $\mu$ is a log-concave probability measure on $\mathbb{R}^m$, then for all $\alpha > 0$: 

$\left( \int Q^2 d\mu \right)^{\frac{1}{2}} \mu \{ x \in \mathbb{R}^m : |Q(x)| \leq \alpha \} \leq c k a^{\frac{k}{2}}$.

Since log-concave probability measures are strongly connected with the surface measure (see Lemma 2 in [14]), the above result gives an alternative way of proving the bounds presented in Section 6. The anti-concentration bound achievable using [4], gives worse constants than the alternative proof provided in this
article. This is important since this constant is in the exponent of the complexities of the \textit{c-approximate nearest neighbor without false negatives} algorithm.

The anti-concentration measures are strongly connected with the Littlewood-Offord theory. Consider Lévy concentration function:

$$Q(X, \lambda) = \sup_x P\left[ X \leq x \leq X + \lambda \right].$$

We have $P\left[ |X| \leq \alpha \right] \leq Q(X, 2\alpha)$. So any bound on the Lévy concentration function is also a bound for our problem. Bobkov et al. [3] considered bounds on the Lévy concentration function for $X$ being the sum of independent random variables with log-concave density function. Particularly (Theorem 1.1 in [3]):

**Theorem 1.** If $X_1, \ldots, X_k$ are independent random variables with log-concave distribution, set $S = \sum_i X_k$. Then for all $\lambda \geq 0$

$$Q(S, \lambda) \leq \lambda \sqrt{\text{Var}(S) + \frac{\lambda^2}{12}}.$$

### 2.2 The nearest neighbors

There exist an efficient $c$-nearest neighbor algorithm for $l_1$ [12] with the query and preprocessing complexity equal to $O(n^{1/c})$ and $O(n^{1+1/c})$ respectively and a near to optimal algorithm for $l_2$ [1] with query and preprocessing complexity equal to $O(n^{1/c^2+o(1)})$ and $O(n^{1+1/c^2+o(1)})$ respectively. Moreover, the algorithms presented in [12] work for $l_p$ for any $p \in [1, 2]$. There are also data dependent algorithms which take into account the actual distribution of the input set [2].

Pagh [17] considered the $c$-approximate nearest neighbor without false negatives for the Hamming space, obtaining results close to the results of [12]. Pagh [17] showed that the bounds of his algorithm for $c_\tau = \log(\frac{n}{k})$ differ by at most a factor of $\ln 4$ in the exponent in comparison to the bounds of [12]. Indyk [10] provided a deterministic algorithm for $l_\infty$ for $c = \Theta(\log_{1+\rho} \log d)$ with storage $O(n^{1+c^2} \log^{O(1)} n)$ and query time $O(\log^{O(1)} n)$ for some tunable parameter $\rho$. Also, Indyk [11] considered deterministic mappings $l_1 \rightarrow l_m^n$, for $m = n^{1+O(1)}$, which might be useful for constructing efficient algorithms for the $c$-approximate nearest neighbor without false negatives [17].

Eventually the authors of [16] presented algorithms for every $p \in [1, \infty]$ and $c > \tau_p = \sqrt[4]{\max\{d^4 d^{1-p}\}}$. The considered hash function family is of form $h_p(x) = \lfloor <v, x> \rfloor$, with the following properties:

- **Close points transform to close hashes:**
  - If $\|x - y\|_p < 1$ then $|h_p(x) - h_p(y)| \leq 1$.

- **The probability of false positives:**
  - For $x, y \in \mathbb{R}^d$ such that $\|x - y\|_p > c\rho$, it holds:
    $$p_{fp} = P[|h_p(x) - h_p(y)| \leq 1] < 1 - \frac{(1 - \frac{\rho}{d^{1-p}})^2}{2}.$$
For such LSH functions the following holds (Theorems 2. and 3. in [16]):

**Theorem 2.** For $c > \tau_p = \sqrt{8} \text{max}\{d^2d^{1-\frac{1}{p}}\}$ and for a large enough $n$, $|P|$ being the size of the result, $p \in [1, \infty]$ we have the $c$-approximate nearest neighbor without false negatives in $l_p$ with the following complexities:

- **for the 'fast query' version:**
  - Preprocessing time: $\mathcal{O}(n(\gamma d \log n + (\frac{n}{d})^\gamma))$,
  - Memory usage: $\mathcal{O}(n(\frac{n}{d})^\gamma)$,
  - Expected query time: $\mathcal{O}(d(|P| + \gamma \log(n) + \gamma d))$,
  where $\gamma = \frac{1}{\ln \rho_p}$.

- **for the 'fast preprocessing' version:**
  - Preprocessing time: $\mathcal{O}(nd \log n)$,
  - Memory usage: $\mathcal{O}(n \log n)$,
  - Expected query time: $\mathcal{O}(d(|P| + n^{\frac{1}{a}} (\frac{n}{a})^{\frac{1}{b}}))$,
  where $a = -\ln \rho_p$, $b = \ln 3$.

In this paper, we follow the approach of [16]. We provide hash functions that satisfy the property of mapping close points to the same values. Using the enhanced hash functions we decrease the probability of false positives, which leads to the improvement of the algorithms complexities. Theorem 3 in the next Section summarizes the obtained results.

### 3 Our contribution

We introduce two classes of hash functions $\hat{h}_p$ and $\tilde{h}_p$. $\hat{h}_p$ transforms a given point $x$ to $<v, x>$, where $v$ is a random vector from $l_\infty$ ball. In $\tilde{h}_p$, we apply the scalar product with a random vector from sphere $S^{(d-1)}$. We prove the anti-concentration bounds for both function families. We follow the schema described in [16], which gives the following result:

**Theorem 3.** For any $p \in [1, \infty]$ and for any $c > \tau_p$, we show data structures for the $c$-approximate nearest neighbor without false negatives with

- $\mathcal{O}(n^{1+\frac{\ln 3}{\ln c/\tau_p}})$ preprocessing time and $\mathcal{O}(\log n)$ query time for the 'fast query' algorithm,
- $\mathcal{O}(n \log n)$ preprocessing time and $\mathcal{O}(n^{\frac{1}{a} \ln \frac{3}{\ln \rho_p}})$ query time for the 'fast preprocessing' algorithm.

We distinguish two cases of the theorem for hash functions $\hat{h}_p$ and $\tilde{h}_p$ respectively:

1. $\tau_p = \hat{\tau}_p = 4\sqrt{3d}\text{max}\{1-1/p, 1/2\}$,
2. $\tau_p = \tilde{\tau}_p = 2d^{1/2 + |1/2 - 1/p|}$.

The $\hat{h}_p$ functions give better results for all $p \in [1, 2)$, while the $\tilde{h}_p$ functions work better for $p \in [2, \infty]$. Let us now proceed to proving the Theorem 3. We prove case 1. and case 2. in Sections 5 and 6 respectively.

3 For simplicity, we omitted the factors dependent on $d$, see [16] for more details.
4 Definitions

The input set will always be assumed to contain \( n \) points. In nearest neighbor algorithms, we would like to find points within given distance \( r \) from a given query point. W.l.o.g., throughout this work we will assume, that \( r - a \) given radius equals 1 (otherwise all vectors might be rescaled by \( 1/r \)). For \( x, y \in \mathbb{R}^d \), \( < x, y > \) denotes the standard scalar product, i.e. \( < x, y > = \sum_{i=1}^{d} x_i y_i \). \( \| x \|_p \) denotes the standard norm in \( l_p \), i.e., \( \| x \|_p = (\sum_{i} |x_i|^p)^{1/p} \). \( S_{d-1}^{(d-1)} \) denotes a sphere in \( l_p \), i.e., \( S_{d-1}^{(d-1)} = \{ x \in \mathbb{R}^d, \| x \|_p = 1 \} \). We will write \( S_{d-1}^{(d-1)} \) instead of \( S_2^{(d-1)} \).

5 The algorithm

The authors of [16] introduced a general framework for solving the \( c \)-approximate nearest neighbor without false negatives in \( l_p \) for any \( p \in [1, \infty] \). The framework was based on the hash functions \( h_p \). Let us recall that \( h_p(x) = \lfloor d^{1/p - 1} \langle x, v \rangle \rfloor \), where \( v \in \{-1, 1\}^d \) is a random vector satisfying: \( \mathbb{P}[v_i = 1] = 1/2 \). In this section, we will introduce new hash functions \( \hat{h}_p \), which improves over the \( h_p \) for \( p \in [1, \infty] \). Particularly, the probability of false positives is decreased, which leads to better complexities of the \( c \)-approximate nearest neighbor without false negatives algorithm for \( c = \Theta(d^{\max \{1/2, 1-1/p\}}) \).

Given a vector \( x \in \mathbb{R}^d \) such that \( \| x \|_p > c \), the probability of a false positive can be bounded as follows [16]:

\[
p_{fp} = \mathbb{P} [ \| h_p(x) - h_p(y) \|_1 \leq 1 ] < 1 - \frac{(1 - \frac{\sqrt{d}}{2})^2}{2}.
\]

Even for very large \( c \), \( p_{fp} \) is always greater than 1/2. This must be the case, since for an arbitrarily large vector \( x = (C, C, 0, 0, \ldots, 0) \), the probability that this vector will be mapped to 0 equals 1/2. To overcome this obstacle, we introduce a new hash function:

\[
\hat{h}_p(x) = \lfloor d^{1/p - 1} \langle w, x \rangle \rfloor,
\]

where \( w \) is a vector of independent random variables: \( w \sim U(-1, 1) \).

To bound the probability of independent random variables, we need to be able to bound the probability of \( \mathbb{P}[ \| \langle w, x \rangle \| < \alpha ] \):

Observation 1 (Anti-concentration bound for a uniform distribution)

Let \( x \in \mathbb{R}^d \) be a fixed vector and \( w \in \mathbb{R}^d \) be a vector of independent random variables with \( U(-1, 1) \) distribution, then

\[
\mathbb{P}[ \| \langle w, x \rangle \| < \alpha ] \leq \frac{2\sqrt{3}\alpha}{\| x \|_2},
\]
Proof. To proof this observation, we apply the general bounds for the Lévy concentration function for log-concave distributions presented in [3]. Let $X_i = w_i x_i$ and $S = \sum X_k$. We have

$$\mathbb{P} [ |\langle w, x \rangle | < \alpha ] = \mathbb{P} [ |S| < \alpha ] \leq Q(S, 2\alpha).$$

Since the uniform distribution is log-concave, by applying Theorem 1 we get:

$$\mathbb{P} [ |\langle w, x \rangle | \leq \alpha ] \leq 2\sqrt{\text{Var}(S) + \frac{\alpha^2}{2}} \leq 2\alpha\sqrt{\text{Var}(S)}.$$

Since $\text{Var}(X_i) = x_i^2 / 3$ and $\text{Var}(S) = \|x\|_2^2 / 3$, we have:

$$\mathbb{P} [ |\langle w, x \rangle | \leq \alpha ] \leq 2\sqrt{3\alpha} \|x\|_2.$$

If we assume that variables in $w$ are i.i.d. and bounded, $\langle w, x \rangle$ satisfy assumptions of the Hoeffding inequality [9]. This implies that $\langle w, x \rangle$ is highly concentrated in the interval $(-|x|_2, |x|_2)S$, where $S$ is the standard deviation of $w_i$. Given that, $\hat{h}_p$ is optimal under the assumption that $w$ are i.i.d. In order to analyze the properties of the hash functions, we need the following technical observations:

Observation 2 For any $z \in \mathbb{R}^d$ where, $\delta_q = d^{\min\{1/2 - 1/q, 0\}}$ and $1/p + 1/q = 1$:

$$\|z\|_p \delta_p \leq \|z\|_2 \leq \|z\|_p \delta_q^{-1}.$$ 

This observation is a direct consequence of the inequality between means. Given this technical observation and the anti-concentration bound we prove the crucial properties of $\hat{h}_p$:

Observation 3 (Close points have close hashes for $\hat{h}_p$) For $x, y \in \mathbb{R}^d$, if $\|x - y\|_p \leq 1$ then $\forall \hat{h}_p \|\hat{h}_p(x) - \hat{h}_p(y)\| \leq 1$.

Proof. We have:

$$\mathbb{P} \left[ |\hat{h}_p(x) - \hat{h}_p(y)| \leq 1 \right] \geq \mathbb{P} \left[ |d^{1/p-1} \langle x - y, w \rangle | \leq 1 \right].$$

Since, $|d^{1/p-1} \langle x - y, w \rangle | \leq d^{1/p-1} \|x - y\|_1 \leq \|x - y\|_p \leq 1$, the probability equals 1.

Lemma 1 (Probability of false positives for $\hat{h}_p$). For every $p \in [1, \infty]$, $x, y \in \mathbb{R}^d$ and $c > \hat{c}_p = 4\sqrt{3d^{\max(1-1/p, 1/2)}}$ such that $\|x - y\|_p > c$, it holds:

$$\mathbb{P}_p \left[ |\hat{h}_p(x) - \hat{h}_p(y)| \leq 1 \right] < \hat{c}_p / c.$$
Proof. Let \( z = x - y \). We have:

\[
\Pr \left[ |\hat{h}_p(x) - \hat{h}_p(y)| \leq 1 \right] \leq \Pr \left[ |\langle z, w \rangle| \leq 2d^{1-1/p} \right] \leq \frac{4\sqrt{3}d^{1-1/p}}{\|z\|_2}.
\]

The second inequality follows from the Observation 1. By Observation 2 \( \|z\|_2 \geq \delta_p \|z\|_p \geq \delta_p c \), which gives:

\[
\Pr \left[ |\hat{h}_p(x) - \hat{h}_p(y)| \leq 1 \right] \leq 4\sqrt{3}d^{1-1/p} \delta_p c.
\]

This ends the proof.

\[\Box\]

Theorem 2 applied to the \( \hat{h}_p \) hash functions results in case 1. of Theorem 3. This improves over the complexities presented in [16]. Particularly, when \( c \) goes to infinity, the preprocessing time in our algorithm tends to \( O(n) \), which was not the case in the preceding algorithm in [16]. Still, the preprocessing complexity is worse than the version which does not give the guarantees for false negatives: \( O(n^{1+1/c}) \). This is the price we pay for the certainty, that all the 'close' points will be found by the algorithm.

6 The improved algorithm for \( p \geq 2 \)

In this section, we introduce new LSH function family: \( \tilde{h}_p \) which is tuned up for \( p \geq 2 \). We define \( \tilde{h}_p \) as follows:

\[
\tilde{h}_p(x) = \lfloor \delta_p \langle w, x \rangle \rfloor, \quad \text{where } w \text{ is a random vector from the unit sphere } S^{(d-1)}.
\]

In order to bound the probability false positive, we need to be able to bound the probability of \( \Pr \left[ |\langle w, x \rangle| < \alpha \right] \). We cannot use the techniques introduced in Section 5 because random variables in \( w \) are not independent. Instead, the probability can be elegantly expressed in geometrical terms. \( \langle w, x \rangle \) can be seen as the first coefficient of a random point from \( S^{(d-1)} \). The probability of the complementary event is proportional to the area of two spherical caps of distance \( \alpha \) from the origin of \( S^{(d-1)} \). The fraction between the area of these spherical caps and the area of the unit ball can be expressed as \( I_{\alpha z}(1/2, (d-1)/2) \) for \( |x|_2 = 1 \), where \( I_{x}(a, b) \) is a regularized incomplete beta function [15]. Bounding the incomplete beta function gives the following observation:

**Observation 4 (The anti-concentration bound for \( S^{(d-1)} \))** Let \( x \in S^{(d-1)} \) be a given unit vector and \( w \in S^{(d-1)} \) be a random unit vector, then

\[
\Pr \left[ |\langle w, x \rangle| < \alpha \right] \leq \alpha \sqrt{d}.
\]

**Proof.** As stated before, the complement of the above probability equals the area of two spherical caps of the normalized \( (d-1) \)-dimensional sphere (i.e. the area
of the sphere equals 1). For a spherical cap let $0 \leq \phi \leq \pi/2$ denote a colatitude angle, i.e. the largest angle between $e_1$ and a vector from the spherical cap. As stated in [15], the area of the spherical cap is given by $\frac{1}{2} I \sin^2 \phi((d-1)/2,1/2)$. Substituting $\alpha = \cos \phi$, we have:

$$f(\alpha) = \mathbb{P} \{ \| w, x \| < \alpha \} = I_{\sin^2 \phi}((d-1)/2,1/2)$$

$$= I_{1-\alpha^2}((d-1)/2,1/2) = I_{\alpha^2}(1/2,(d-1)/2),$$

where the last equality follows from the fact that $I_x(a,b) = I_1-a$ $I_1-b$. By the definition of $I_x(a,b)$, we have

$$f'(\alpha) = \frac{2\alpha^{-1}(1-\alpha^2)^{d-1}}{B(1/2,(d-1)/2)} = \frac{2(1-\alpha^2)^{d-1}}{B(1/2,(d-1)/2)}$$

and

$$f''(\alpha) = \frac{-2\alpha(d-3)(1-\alpha^2)^{d-3}}{B(1/2,(d-1)/2)},$$

where $B(a,b)$ is a beta function. For $d=2$ the function $f$ is convex, so

$$f(\alpha) \leq (1-\alpha)f(0) + \alpha f(1) = \alpha.$$

For $d > 2$, the function is concave and

$$f(\alpha) \leq f(0) + \alpha f'(0) = \frac{2\alpha}{B(1/2,(d-1)/2)}.$$

The last step is proving, that $B(1/2,(d-1)/2) \geq \frac{2}{\sqrt{d}}$. Grenié et al. [8] proved that:

$$B(x,y) \geq \frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}}.$$

Applying this inequality gives the following bound:

$$B(1/2,(d-1)/2) \geq \frac{(1/2)^{-1/2}(d-1)^{d-3}}{(d/2)^{d/2}} = \frac{(1/2)^{-1/2}(d-1)^{d-3}}{(d/2)^{d/2}} = \frac{2(d-1)^{d-3}}{\sqrt{d}},$$

which ends the proof, since $g(d) = (d-1)^{d-3}$ is decreasing for $d \geq 3$ and $g(3) = 1$.

For large $d$, $g(d) \approx e^{-1/2}$, what gives a slightly better bound. Given the above anti-concentration bound we prove the crucial properties of $\tilde{h}_p$:

**Observation 5 (Close points have close hashes for $\tilde{h}_p$)** For $x, y \in \mathbb{R}^d$, if $\|x-y\|_p < 1$ then $\forall h_p |\tilde{h}_p(x) - \tilde{h}_p(y)| \leq 1$. 


Proof. We have:

\[ P \left[ \left| \tilde{h}_p(x) - \tilde{h}_p(y) \right| \leq 1 \right] \geq P \left[ \| x - y \| \delta_q \leq 1 \right]. \]

Applying, in turn, the Schwarz inequality and Observation 2, we get:

\[ \delta_q \| x - y, w \| \leq \delta_q \| x - y \| \leq \| x - y \|_p \leq 1. \]

Hence, the points will inevitably hash into the same or adjacent buckets. \qed

Lemma 2 (Probability of false positives for \( \tilde{h}_p \)). For every \( p \in [1, \infty] \), \( x, y \in \mathbb{R}^d \) and \( c > \tilde{\tau}_p = 2d^{1/2 + 1/2 - 1/p} \) such that \( \| x - y \|_p < c \), it holds:

\[ p_{fp} = P \left[ \left| \tilde{h}_p(x) - \tilde{h}_p(y) \right| \leq 1 \right] < \tilde{\tau}_p / c. \]

Proof. Let \( z = x - y \) and \( X = \| z \|^{-1} \langle w, z \rangle \), be a random variable. We have:

\[ P \left[ \left| \tilde{h}_p(x) - \tilde{h}_p(y) \right| \leq 1 \right] \leq P \left[ \| X \| \| z \|_2 \leq 2\delta_q^{-1} \right] \leq P \left[ \| X \| \leq 2(\| z \|_p \delta_q \delta_p)^{-1} \right]. \]

The second inequality follows from the Observation 2. Since \( \delta_q \delta_p = d^{-1/2 - 1/p} \), we have:

\[ P \left[ \left| \tilde{h}_p(x) - \tilde{h}_p(y) \right| \leq 1 \right] \leq P \left[ \| X \| \leq 2\| z \|_p^{-1}d^{1/2 - 1/p} \right] \leq P \left[ \| X \| \leq 2c^{-1}d^{1/2 - 1/p} \right]. \]

Applying the anti-concentration bound ends the proof. \qed

Theorem 2 applied to the \( \tilde{h}_p \) hash functions results in case 2. of Theorem 3. For \( p \in [2, \infty] \) we have asymptotically the same constraints on \( c \) (\( c = \mathcal{O}(d^{1 - 1/p}) \)). In addition, for any \( p \in [1, \infty] \) we have \( \tilde{h}_p < \tilde{\rho}_p \). Although the improvement in the bound for \( \tilde{\rho}_p \) is only in constant, this might be important for practical cases, because this constant is present in the exponent of the complexities of the \( c \)-approximate nearest neighbor without false negatives algorithm. For \( p \in [1, 2] \) there are discrepancies between the constraints on \( c \), depending on the hash functions used. Particularly, the hash functions \( h_p \) and \( \tilde{h}_p \) work for any \( c = \Omega(\sqrt{d}) \) for \( p \in [1, 2] \), while the \( h_p \) works for \( c = \Omega(d^{1/p}) \).

A natural approach for optimizing both the probability of false positives and the constraint on \( c \) would be to consider hash functions of form \( h_p = \langle x, w > \rangle \), where \( w \) is a random point from \( S_q^{(d-1)} \) for \( 1/q + 1/p = 1 \). The Hölder inequality implies the property of ‘close’ points being hashed to adjacent buckets. In order to prove the bounds for false positives, we need to bound \( P \left[ \| x, w > | < c \right] \). We conjecture that this probability can be bounded by \( \mathcal{O}(\epsilon \sqrt{d}) \) for any \( p \in [1, 2] \).

\[ ^4 \text{There are many possibilities of choosing a random point from a sphere in } l_p. \text{ We conjecture that the bounds should hold for both geometric surface measure and cone measure.} \]
This is true for $p = 2$, since $h_2 = \hat{h}_2$. Also for large $d$, $h_1 \approx \hat{h}_1$, because these two functions differ only by the factor of $\max_i |u_i|$, where $u_i \sim U(-1, 1)$. This factor will be close to 1 for large $d$. Still, techniques used to prove bounds for $\hat{h}_p$ and $\hat{h}_p$ seem to be insufficient to prove more general bounds for $\hat{h}_p$.

7 Conclusion and Future Work

We introduced hash functions $\hat{h}_p$ and $\tilde{h}_p$. Using these functions, we were able to improve the query and the preprocessing time complexities for the $c$-approximate nearest neighbor without false negatives for any $p \in [1, \infty)$. This is a major improvement over the results presented in [10].

The future work concerns further relaxing of the restrictions on the approximation factor $c$ and reducing the time complexity of the algorithm or proving that these restrictions are essential. We wish to match the time complexities given in [12] or show that the achieved bounds are optimal.

Also, many interesting theoretical problems arise. Consider for instance a random (e.g., random in cone measure) point $v$ from $S^{(d-1)}_q$ and a fixed point $w$ from $S^{(d-1)}_p (1/p + 1/q = 1, p \in [1, 2))$. A problem can be posed, whether the probability $P[|\langle w, x \rangle| < \epsilon]$ can be bounded. We conjecture, that this probability is $O(\epsilon \sqrt{d})$.

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References

1. Alexandr Andoni and Piotr Indyk. Near-optimal hashing algorithms for approximate nearest neighbor in high dimensions. Commun. ACM, 51(1):117–122, 2008. URL: http://doi.acm.org/10.1145/1327452.1327494, doi:10.1145/1327452.1327494.

2. Alexandr Andoni and Ilya Razenshteyn. Optimal data-dependent hashing for approximate near neighbors. In Rocco A. Servedio and Ronitt Rubinfeld, editors, Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, June 14-17, 2015, pages 793–801. ACM, 2015. URL: http://doi.acm.org/10.1145/2746539.2746553, doi:10.1145/2746539.2746553.

3. Sergey G. Bobkov and Gennadiy P. Chistyakov. On concentration functions of random variables. Journal of Theoretical Probability, 28(3):976–988, 2015. URL: http://dx.doi.org/10.1007/s10959-013-0504-1, doi:10.1007/s10959-013-0504-1.

4. A Carbery and J Wright. Distributional and l-q norm inequalities for polynomials over convex bodies in r-n. Mathematical Research Letters, 8(3):233–248, 5 2001.
5. Bernard Chazelle, Ding Liu, and Avner Magen. Approximate range searching in higher dimension. *Computational Geometry*, 39(1):24 – 29, 2008. URL: http://www.sciencedirect.com/science/article/pii/S092577210700065X
   doi:10.1016/j.comgeo.2007.05.008

6. D Cordero-Erausquin, M Fradelizi, and B Maurey. The (b) conjecture for the gaussian measure of dilates of symmetric convex sets and related problems. *Journal of Functional Analysis*, 214(2):410 – 427, 2004. URL: http://www.sciencedirect.com/science/article/pii/S0022123604000205
   doi:10.1016/j.jfa.2003.12.001

7. Mayur Datar and Piotr Indyk. Locality-sensitive hashing scheme based on p-stable distributions. In *In SCG 04: Proceedings of the twentieth annual symposium on Computational geometry*, pages 253–262. ACM Press, 2004.

8. Łośc Grenié and Giuseppe Molteni. Inequalities for the beta function. *Math. Inequal. Appl.*, 18(4):1427–1442, 2015. URL: http://dx.doi.org/10.7153/mia-18-111

9. Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, March 1963. URL: http://www.jstor.org/stable/2282952?

10. Piotr Indyk. On approximate nearest neighbors in non-euclidean spaces. In *39th Annual Symposium on Foundations of Computer Science, FOCS ’98*, November 9-11, 1998, Palo Alto, California, USA, pages 148–155, 1998. URL: http://dx.doi.org/10.1109/SFCS.1998.743438
    doi:10.1109/SFCS.1998.743438

11. Piotr Indyk. Uncertainty principles, extractors, and explicit embeddings of $l_2$ into $l_1$. In *Proceedings of the Thirty-ninth Annual ACM Symposium on Theory of Computing*, STOC ’07, pages 615–620, New York, NY, USA, 2007. ACM. URL: http://doi.acm.org/10.1145/1250790.1250881
    doi:10.1145/1250790.1250881

12. Piotr Indyk and Rajeev Motwani. Approximate nearest neighbors: Towards removing the curse of dimensionality. In *Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing*, STOC ’98, pages 604–613, New York, NY, USA, 1998. ACM. URL: http://doi.acm.org/10.1145/276698.276876
    doi:10.1145/276698.276876

13. William Johnson and Joram Lindenstrauss. Extensions of Lipschitz mappings into a Hilbert space. In *Conference in modern analysis and probability (New Haven, Conn., 1982)*, volume 26 of *Contemporary Mathematics*, pages 189–206. American Mathematical Society, 1984.

14. Rafał Lataa and Krzysztof Oleszkiewicz. Small ball probability estimates in terms of width. *STUDIA MATHEMATICA*, 169(3):305–314, 2005. doi:10.4064/sm169-3-6

15. S. Li. Concise formulas for the area and volume of a hyperspherical cap. *Asian Journal of Mathematics and Statistics*, pages 4(1):66–70, 2011.

16. Andrzej Pacuk, Piotr Sankowski, Karol Wegrzycki, and Piotr Wygocki. Locality-sensitive hashing without false negatives for $l_p$. In *Computing and Combinatorics - 22nd International Conference, COCOON 2016, Ho Chi Minh City, Vietnam, August 2-4, 2016, Proceedings*, pages 105–118, 2016. URL: http://dx.doi.org/10.1007/978-3-319-42634-1_9
    doi:10.1007/978-3-319-42634-1_9

17. Rasmus Pagh. Locality-sensitive hashing without false negatives. In *Proceedings of the Twenty-seventh Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’16, pages 1–9, Philadelphia, PA,
18. Ryan Williams. A new algorithm for optimal 2-constraint satisfaction and its implications. *Theor. Comput. Sci.*, 348(2):357–365, December 2005. URL: [http://dx.doi.org/10.1016/j.tcs.2005.09.023](http://dx.doi.org/10.1016/j.tcs.2005.09.023)