SPHERICAL TRANSFORM AND JACOBI POLYNOMIALS ON ROOT SYSTEMS OF TYPE BC

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Abstract. Let \( R \) be a root system of type \( BC \) in \( a = \mathbb{R}^r \) of general positive multiplicity. We introduce certain canonical weight function on \( \mathbb{R}^r \) which in the case of symmetric domains corresponds to the integral kernel of the Berezin transform. We compute its spherical transform and prove certain Bernstein-Sato type formula. This generalizes earlier work of Unterberger-Upmeier, van Dijk-Pevsner, Neretin and the author. Associated to the weight functions there are Heckman-Opdam orthogonal polynomials of Jacobi type on the compact torus, after a change of variables they form an orthogonal system on the non-compact space \( a \). We consider their spherical transform and prove that they are the Macdonald-Koornwinder polynomials multiplied by the spherical transform of the canonical weight function. For rank one case this was proved earlier by Koornwinder.

Introduction

The Gaussian functions and the Hermite polynomials play an important role in Fourier transform on Euclidean spaces; the Hermite polynomials diagonalize the harmonic oscillator and the corresponding Hermite type function diagonalize the Fourier transform, which make the Plancherel theory more transparent. The generalization of the Plancherel theory to any non-compact symmetric symmetric spaces has been studied intensively and there are still no general theory generalizing corresponding the results of Hermite polynomials and their Fourier transform, and above all, no concrete orthogonal systems of functions with explicit spherical transforms are constructed. In the present paper we will establish such a theory for root system of type BC.

Now associated with any root system in \( \mathbb{R}^r \) there are two kinds of remarkable orthogonal polynomials, namely the Heckman-Opdam orthogonal polynomials giving the spectral decomposition of the algebra of the Weyl group invariant polynomials of Cherednik operators acting on certain \( L^2 \)-space on a compact torus \( \mathbb{T} \), and the Macdonald-Koornwinder polynomials orthogonal with respect to certain weight functions on \( \mathbb{R}^r \) defined as a product of Gamma functions. Part of the product is in fact the Harish-Chandra Plancherel measure for the spectral decomposition of the algebra of Cherednik operators on the non-compact space \( \mathbb{R}^r \). In the present paper we will introduce certain canonical weight function \( f_{-2\nu}(t) \) for root system of type BC, and we compute its spherical transform \( \widetilde{f}_{-2\nu}(\lambda) \). The product of \( \widetilde{f}_{-2\nu}(\lambda)^2 \) with the Harish-Chandra measure \( |c(\lambda)|^{-2} \) gives precisely weight functions in the Macdonald-Koornwinder orthogonality.

Research supported by the Swedish Science Research Council (VR).
relation; we prove further, roughly speaking, that the Jacobi polynomials on the compact space, after some change of variables, and multiplied by the function $f_{-2\nu}$ give orthogonal system of the $L^2$-space on the non-compact space, and that their spherical transforms are of the form $\tilde{f}_{-2\nu}(\lambda)p(\lambda)$ where $p(\lambda)$ are the Macdonald-Koornwinder polynomials. Thus the function $f_{-2\nu}$ plays the role of the Gaussian functions in the Fourier analysis whereas the Jacobi and Macdonald-Koornwinder polynomials play the role of the Hermite polynomials and their Fourier transforms (up to a multiple of the Gaussian). In rank one case this has been proved earlier by Koornwinder [6].

We remark that only the spherical transform of Weyl group invariant functions are studied in this paper, yet we use the tools of the Cherednik operators acting on general functions with no invariance, which makes the computation much easier. Some of our results can also be easily be generated to that setup.

Our results are motivated by the study of the Berezin transform and branching rule of holomorphic representations on weighted Bergman spaces on bounded symmetric domains $G/K$ ([15] and [16]). To illustrate our result and explain some of the motivations we consider the simplest case of a symmetric domain, namely the unit disk $G/K = \{z \in \mathbb{C}; |z| < 1\}$ with $G = SU(1,1)$ and $K = U(1)$. The root system is $\{\pm 4\varepsilon\}$ on $\mathfrak{a} = \mathbb{R}$ in our notation in Section 1. The Berezin transform in question is a group convolution operator in the space $L^2(G/K) = L^2(G/K, \frac{dz \wedge d\bar{z}}{(1-|z|^2)^{2\nu}})$ with kernel $(1-|z|^2)^{\nu}$. The canonical weight function $f_{-2\nu}(t)$ is just the kernel $(1-|z|^2)^{\nu}$ with $z = \tanh t$ in term of the geodesic coordinate $t$. The spherical transform $\tilde{f}_{-2\nu}(\lambda)$ of $f_{-2\nu}(t)$ gives the spectral symbol of the Berezin transform, and it has been computed for general bounded symmetric domain by Unterberger and Upmeier [10] (see also [12], [8] and [15]). Now there is an orthogonal basis of the space $L^2(G/K)^K$ of radial functions of the form $(1-|z|^2)^{\nu}P(z)$ where $P(z)$ are the Jacobi polynomials, $z = \tanh t \in (-1,-1)$. The Jacobi polynomials (depending on $\nu$) are given by the corresponding spherical polynomials of the same root system with different multiplicity on the compact torus $\mathbb{T} = i\mathbb{R}/i\pi\mathbb{Z}$, with the change of variable $z = \sin s$ for $s \in \mathbb{T} = i\mathbb{R}/i\pi\mathbb{Z}$ (see Section 5). It has been proved by Koornwinder [6] that the spherical transform of $(1-|z|^2)^{\nu}P(z)$ is of the form $\tilde{f}_{-2\nu}(\lambda)Q(\lambda)$ where $Q(\lambda)$ is the Wilson $4F3$ hypergeometric orthogonal polynomials. Thus $Q(\lambda)$ are orthogonal with respect to $\tilde{f}_{-2\nu}(\lambda)^2|c(\lambda)|^{-2}d\lambda$.

The Macdonald-Koornwinder polynomials are the multi-variable generalization of the Wilson’s $4F3$ hypergeometric orthogonal polynomials, and can be viewed as the most general case of a hierarchy of classical polynomials [11]. We introduce the canonical weight function $f_{-2\nu}$ on a general root system of type BC, compute the spherical transform for functions of the form $f_{-2\nu}P$ with $P$ being the Jacobi polynomials studied by Heckman-Opdam.

The paper is organized as follows. In Section 1 we recall the Plancherel formula for Opdam-Cherednik transform. In Section 2 we find certain Bernstein-Sato type formula
for the so-called canonical weight function $f_\delta$. Its spherical transform is computed in Section 3. Finally we compute the spherical transform of certain Jacobi-type polynomials in Sections 4 and 5.

It is our belief that most results in this paper can be generalized to general root systems, that will provide a theory of Hermite-type functions on non-compact space and thus bring together the orthogonal polynomials on compact torus and spherical transform on non-compact space.

I would like to thank Professors Jacques Faraut, Toshiyuki Kobayashi, Erik Opdam, Jesper Stokman and Harald Upmeier for some helpful discussions.

1. Spherical transform and Plancherel formula

Let $a = \mathbb{R}^r$ be an Euclidean space with inner product $(\cdot, \cdot)$ and let $R \subset a^*$ be a root system of type BC. A prototype of such root systems is the restricted root system of a bounded symmetric domain. We use some familiar notation, in order to be consistent with the notation in the context of bounded symmetric domain and also in the context of Plancherel formula for general root system [9]. We fix an orthonormal basis $\{\xi_j\}_{j=1}^r$ of $a$ and a dual basis $\{\varepsilon_j\}_{j=1}^r$ of $a^*$, i.e., $\varepsilon_j(\xi_k) = \delta_{jk}$, so that the positive roots are $R_+ = \{2\varepsilon_j; j = 1, \cdots, r\} \cup \{4\varepsilon_j; j = 1, \cdots, r\} \cup \{2(\varepsilon_j \pm \varepsilon_k); 1 \leq j < k \leq r\}$, with respective multiplicities $(k_1, k_2, k_3)$ satisfying $2(k_1, k_2, k_3) = (2b, \iota, a)$. We assume that $\iota, a, b > 0$. We order the roots so that $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_r > 0$. The Weyl group is then $W = S_r \times \mathbb{Z}^2$ consisting of signed permutation of $\{\xi_j\}$. Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha$ be the half sum of positive roots. Then

$$\rho = \sum_{j=1}^r \rho_j \varepsilon_j = \sum_{j=1}^r (t + b + (r - j)\alpha)\varepsilon_j.$$  

We recall briefly in this section the Plancherel formula for the spherical transform associated to the root system $R$ developed by Heckman and Opdam ([5], [4] and [9]).

Let $d\mu$ be the measure

$$d\mu(t) = d\mu_k(t) = \prod_{\alpha \in R^+} |2 \sinh(\frac{1}{2} \alpha(t))|^{2k_\alpha} dt$$

on $a$, and let $L^2(a) = L^2(a, d\mu)$ and $L^2(a)^W$ be the corresponding $L^2$ space and its subspace of $W$-invariant functions.

Let

$$D_j = \partial_j - a \sum_{i < j} \frac{1}{1 - e^{-2(t_i - t_j)}(1 - \sigma_{ij})} + a \sum_{j < k} \frac{1}{1 - e^{-2(t_j - t_k)}(1 - \sigma_{jk})} +$$

$$\frac{a}{1 - e^{-2(t_j + t_k)}}(1 - \sigma_{jk}) + 2t \frac{1}{1 - e^{-4t_j}}(1 - \sigma_j) + 2b \frac{1}{1 - e^{-2t_j}}(1 - \sigma_j) - \rho(\xi_j)$$
be the Cherednik operators acting on functions $f(t)$ on $\mathfrak{a}^C$, where we identify a function $f(t)$ on $\mathfrak{a}^C$ with $f(t_1, \cdots, t_r)$, for $t = t_1 \xi_1 + \cdots + t_r \xi_r$. Here $s_{ij}, \sigma_{ij}, \sigma_i$ are the elements in the Weyl group, $s_{ij} = (ij)$ being the permutation of $\xi_i$ and $\xi_j$, $\sigma_{ij}$ the signed permutation, $\sigma_{ij}(\xi_i) = -\xi_j, \sigma_{ij}(\xi_j) = -\xi_i$, $\sigma_i$ the reflection $\sigma_i(\xi_i) = -\xi_i$, and all mapping $\xi_k \to \xi_k$ for $k \neq i, j$. For later purpose we will also rewrite it as

$$D_j = \partial_j - a \sum_{i < j} e^{t_i-t_j} (1 - s_{ij}) + a \sum_{j < k} e^{t_j-t_k} (1 - s_{jk}) + a \sum_{i \neq j} e^{t_j+t_k} (1 - \sigma_{jk}) + 2t \frac{e^{2t_j}}{e^{2t_j} - e^{-2t_j}} (1 - \sigma_j) + 2b \frac{e^{t_j}}{e^{t_j} - e^{-t_j}} (1 - \sigma_j) - \rho_j.$$

The operators $\{D_j\}$ are then commuting, and the decomposition of $L^2(\mathfrak{a})$ with respect to the eigenfunctions of $\{D_j\}$ is given by the Cherednik-Opdam transform [9], formulated in terms of decomposing representations of a Hecke algebra. We will mostly be concerned with the decomposition of $L^2(\mathfrak{a})^W$ under $W$-invariant polynomials of $D_j, j = 1, \cdots, r$, which is given by spherical transform in terms of the Heckman-Opdam theory of hypergeometric functions ([5] and [6]); we shall however use the operators $D_j$ (in place of their symmetric polynomials) to compute the spherical transform of symmetric functions.

For $\lambda \in (\mathfrak{a}^+)^C$ let $\phi_\lambda$ be as in [9] the spherical function. In particular

$$p(D_1, \cdots, D_r) \phi_\lambda = p(\lambda(\xi_1), \cdots, \lambda(\xi_r)) \phi_\lambda,$$

for any $W$-invariant polynomial $p$. The spherical transform of a function $f \in L^2(\mathfrak{a})^W$ is

$$\tilde{f}(\lambda) = \int_\mathfrak{a} f(t) \phi_\lambda(t) d\mu(t).$$

The corresponding Plancherel measure is given by

$$d\tilde{\mu}(\lambda) = \frac{(2\pi)^{-r} c_0^2}{c(\lambda)c(-\lambda)} d\lambda,$$

where

$$c(\lambda) = \prod_{j=1}^r \frac{\Gamma(\lambda_j + b)\Gamma(2\lambda_j)}{\Gamma(\lambda_j + b + \frac{r}{2})\Gamma(2\lambda_j + 2b)} \prod_{1 \leq j < k \leq r, \epsilon = \pm} \frac{\Gamma(\lambda_j + \epsilon\lambda_k)}{\Gamma(\lambda_j + \epsilon\lambda_k + \frac{r}{2})}$$

and

$$c_0 = c(\rho) = \prod_{j=1}^r \frac{\Gamma(\rho_j + b + 1)\Gamma(2\rho_j + 1)}{\Gamma(\rho_j + b + \frac{r}{2} + 1)\Gamma(2\rho_j + b + 1)} \prod_{1 \leq j < k \leq r, \epsilon = \pm} \frac{\Gamma(\rho_j + \epsilon\rho_k + 1)}{\Gamma(\rho_j + \epsilon\rho_k + \frac{r}{2} + 1)}.$$

Namely, we have

$$\int_\mathfrak{a} |f(t)|^2 d\mu(t) = \int_{\mathfrak{a}^r} |\tilde{f}(\lambda)|^2 d\tilde{\mu}(\lambda).$$
2. Bernstein-Sato type formula for the function \( f_\delta \)

We define the weight function \( f_\delta \) on \( a \) by

\[
f_\delta(t) = \prod_{j=1}^{r} \cosh^{\delta} t_j.
\]

Motivated by the Berezin transform ([10], [12], [15]) we call \( f_\delta \) the canonical function. In the case of bounded symmetric domains \( f_\delta \) is the integral kernel of the Berezin transform considered as a convolution operator defining the so-called canonical representations, and is the analogue of the Gaussian functions in the Euclidean space, see loc. cit..

In this section we prove the following Bernstein-Sato type formula:

**Theorem 2.1.** There is a \( W \)-invariant polynomials of the Cherednik operators \( D_j \) mapping the canonical function \( f_\delta \) to \( f_\delta^{-2} \); more precisely, we have

\[
\prod_{j=1}^{r} (D_j^2 - (\delta + \rho(\xi_1))^2) f_\delta = \prod_{j=1}^{r} (\delta + a(j - 1)) (1 - \delta - \iota - a(r - j)) f_\delta^{-2}
\]

The proof of it will be divided into two technical lemmas.

**Lemma 2.2.** The following formula holds

\[
\prod_{l=1}^{j} (D_l + \delta + \rho(\xi_1)) f_\delta(t) = \prod_{l=1}^{j} (\delta + a(l - 1)) f_\delta(t) \prod_{l=1}^{j} (1 + \tanh t_l)
\]

**Proof.** We prove the lemma by induction on \( j \). First we have, since \( f_\delta \) is Weyl group invariant,

\[
D_1 f_\delta = \delta \tanh t_1 f_\delta - \rho(\xi_1) f_\delta,
\]

where we use \( \frac{d}{dt} \cosh t = \tanh t \cosh t \). Rewriting,

\[
(D_1 + \delta + \rho(\xi_1)) f_\delta = \delta (1 + \tanh t_1) f_\delta,
\]

which is the claim for \( j = 1 \). Assume that the equality is true for \( \prod_{l=1}^{j-1} (D_l + (\delta + \rho(\xi_1))) \). We consider it for \( j \) in place of \( j - 1 \). We need to compute the operator \( D_j + \delta + \rho(\xi_1) \) on the function \( f_\delta \prod_{l=1}^{j-1} (1 + \tanh t_l) \). This function is invariant under the permutations of the first \( j - 1 \) coordinates and the signed permutations of the last \( r - j \) coordinates.
Thus
\[ D_j f_\delta \prod_{l \leq j-1} (1 + \tanh t_l) \]
\[ = \delta \tanh t_j f_\delta \prod_{l \leq j-1} (1 + \tanh t_l) \]
\[- af_\delta \sum_{i < j} \frac{e^{t_i-t_j}}{e^{t_i-t_j} - e^{-(t_i-t_j)}} (\tanh t_i - \tanh t_j) \prod_{l \leq j-1, l \neq i} (1 + \tanh t_l) \]
\[ + af_\delta \sum_{i < j} \frac{e^{t_i+t_j}}{e^{t_i+t_j} - e^{-(t_i+t_j)}} (\tanh t_i + \tanh t_j) \prod_{l \leq j-1, l \neq i} (1 + \tanh t_l) \]
\[- \rho(\xi_j) f_\delta \prod_{l \leq j-1} (1 + \tanh t_l). \]

Using the formulas \( \tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y} \) and \( e^x = (1 + \tanh x) \cosh x \) we see that the sum of the ith terms in the two summations is, apart from the factor \( af_\delta \prod_{l \leq j-1, l \neq i} (1 + \tanh t_l), \)
\[ \frac{e^{t_i+t_j}}{e^{t_i+t_j} - e^{-(t_i+t_j)}} (\tanh t_i + \tanh t_j) - \frac{e^{t_i-t_j}}{e^{t_i-t_j} - e^{-(t_i-t_j)}} (\tanh t_i - \tanh t_j) \]
\[ = 2(1 + \tanh t_i) \tanh t_j, \]
and thus
\[ D_j f_\delta \prod_{l \leq j-1} (1 + \tanh t_l) \]
\[ = ((\delta + a(j - 1)) \tanh t_j - \rho(\xi_j)) f_\delta \prod_{l \leq j-1} (1 + \tanh t_l). \]

It follows then that
\[ (D_j + \delta + \rho(\xi_1)) f_\delta \prod_{l \leq j-1} (1 + \tanh t_l) \]
\[ = (\delta + a(j - 1)) f_\delta \prod_{l \leq j} (1 + \tanh t_l) + (\rho(\xi_1) - \rho(\xi_j) - a(j - 1)) f_\delta \prod_{l \leq j-1} (1 + \tanh t_l). \]

But \( \rho(\xi_j) = \nu + b + a(r - j) \) so the second term vanishes and this completes the proof. \( \square \)

**Lemma 2.3.** The following formula holds
\[ \prod_{l \geq j} (D_l - (\delta + \rho(\xi_1))) f_{\delta-1} e^{t_1+\cdots+t_r} \]
\[ = \prod_{l \geq j} (1 - \delta - \nu - a(r - l)) f_{\delta-1} e^{t_1+\cdots+t_r} \prod_{l \geq j} (1 - \tanh t_l) \]

Accepting temporarily the Lemma, we prove Theorem 2.1. We write the LHS as
\[ \prod_{l=1}^r (D_l - (\delta + \rho(\xi_1))) \prod_{l=1}^r (D_l + (\delta + \rho(\xi_1))) f_\delta \]
Taking \( j = r \) in Lemma 2.2 we see that \( \prod_{l=1}^{r}(D_l + (\delta + r(\xi_1)))f_\delta \) is, disregarding the constant, given by,

\[
 f_\delta \prod_{l=1}^{r}(1 + \tanh t_j) = f_{\delta-1}e^{t_1 + \cdots + t_r}.
\]

Theorem 2.1 then follows from Lemma 2.3 for \( j = 1 \), by using the identity

\[
 f_{\delta-1}e^{t_1 + \cdots + t_r} \prod_{j=1}^{r}(1 - \tanh t_j) = f_{\delta-2}.
\]

It remains to prove the Lemma.

Proof. We first compute \( D_r f_\delta e^{t_1 + \cdots + t_r} \)

\[
 D_r f_\delta e^{t_1 + \cdots + t_r} = f_\delta \prod_{i \leq r} e^{t_i} + (\delta - 1) f_\delta \prod_{i \leq r} e^{t_i} + a f_\delta \sum_{i < r} \frac{e^{t_i + t_r}}{e^{t_i + t_r} - e^{-(t_i + t_r)}} (e^{t_i + t_r} - e^{-(t_i + t_r)}) \prod_{l \leq r, l \neq i, r} e^{t_l} + 2 t f_\delta \frac{e^{2t_r}}{e^{2t_r} - e^{-2t_r}} (e^{t_r} - e^{-t_r}) \prod_{l \leq r-1} e^{t_l} + 2 b f_\delta \frac{e^{t_r}}{e^{t_r} - e^{-t_r}} (e^{t_r} - e^{-t_r}) \prod_{l \leq r-1} e^{t_l} - \rho(\xi_r) f_{\delta-1} \prod_{l \leq r} e^{t_l}.
\]

The third term can be simplified as

\[
 a(r - 1) f_{\delta-1} \prod_{l \leq r} e^{t_l},
\]

and the next two terms are

\[
 \iota \tanh t_r f_{\delta-1} \prod_{l \leq r} e^{t_l} + (\iota + 2b) f_{\delta-1} \prod_{l \leq r} e^{t_l}.
\]

So that the previous formula is then

\[
 (\delta - 1 + \iota) \tanh t_r f_{\delta-1} \prod_{l \leq r} e^{t_l} + (1 + \iota + 2b + a(r - 1) - \rho(\xi_r)) f_{\delta-1} \prod_{l \leq r} e^{t_l} + (\delta - 1 + \iota) \tanh t_r f_{\delta-1} \prod_{l \leq r} e^{t_l} + (1 + b + \rho(\xi_1) - \rho(\xi_r)) f_{\delta-1} \prod_{l \leq r} e^{t_l}
\]

and consequently

\[
 (D_r - (\rho_1 + \delta(\xi_1))) f_{\delta-1} e^{t_1 + \cdots + t_r} = (\delta - 1 + \iota) \tanh t_r f_{\delta-1} \prod_{l \leq r} e^{t_l} + (1 + b - \delta - \rho(\xi_r)) f_{\delta-1} \prod_{l \leq r} e^{t_l} = (\delta - 1 + \iota)(\tanh t_r - 1) f_{\delta-1} \prod_{l \leq r} e^{t_l},
\]

since \( 1 + b - \delta - \rho(\xi_r) = -(\delta - 1 + \iota) \). This is the Lemma for \( j = r \).
Assume the lemma is true for the action of \( \prod_{l \geq j+1} (D_l - (\delta + \rho(\xi_1))) \). We compute

\[
(D_j - (\delta + \rho(\xi_1))) f_{\delta-1} \prod_{l \geq j+1} (1 - \tanh t_l) \prod_{l \leq r} e^{t_l}
\]

and find it is

\[
-(1 - \tanh t_j)(a(r - j) + \iota + \delta - 1) f_{\delta-1} \prod_{l \geq j} (1 - \tanh t_l) \prod_{l \leq r} e^{t_l}
\]

by a straightforward yet tedious computation. \( \square \)

**Remark 2.4.** Consider the double Hecke algebra generated by the Cherednik operators, the group ring \( \mathbb{C}[W] \), and \( \mathbb{C}[P] \) of polynomials \( e^{\varepsilon_j} \) as multiplication operators on \( \mathbb{C}[P] \). Some refinements of the above computation then gives certain commutation relations of the Cherednik operators with \( e^{\varepsilon_1 + \cdots + \varepsilon_r} \), which might be of independent interests. This Theorem (or its equivalent version under the spherical transform, see the proof of Theorem 3.2) can possibly be obtained also by using intertwining relations of the Cherednik-Opdam transform with the double Hecke algebras proved by Cherednik [2], it would involves nevertheless many intriguing computations.

### 3. Spherical transform of the function \( f_\delta \)

In this section we shall use Bernstein-Sato type formula for the function \( f_{-2\nu} \) to derive a recursive formula for its spherical transform \( \widehat{f_{-2\nu}}(\lambda) \) (for sufficiently large \( \nu \)). By using a limit formula we then derive a product and, consequently, a Gamma function formula for \( \widehat{f_{-2\nu}}(\lambda) \).

Denote

\[
\Gamma_a(\sigma) = \prod_{j=1}^r \Gamma(\sigma - \frac{a}{2}(j - 1)),
\]

which in the symmetric domain case is the Gindikin Gamma function; it will be used to simplify certain product formulas.

We compute first some normalization constant, which corresponds to the spherical transform of \( f_{-2\nu} \) at \( \lambda = \rho \).

**Lemma 3.1.** Let \( \nu > \iota + b + a(r - 1) \). The integral

\[
N_\nu = \int_a \! f_{-2\nu}(t) d\mu(t)
\]

is given by

\[
N_\nu = 2^{r(2\iota + 2b + a(r-1)) r!} \Gamma_a \left( \frac{\nu}{2} \right) \Gamma_a \left( \frac{a(r-1) + \iota + b}{2} \right) \prod_{1 \leq i < j \leq r} \Gamma \left( \frac{a}{2}(j - i + 1) \right) \Gamma \left( \frac{a}{2}(j - i) \right).
\]
Proof. By symmetry we need only to integrate over all \( s = (s_1, \ldots, s_r) \) with \( s_j \geq 0 \). Making the change of variables \( z_j = \tanh^2 t_j \), we find first that

\[
N_\nu = 2^{(2 + 2b + a(r - 1))} r \int_{[0,1]^r} \prod_{i<j} (z_i - z_j)^a \prod_{i=1}^r \prod_{i<j} (1 - z_i)^{\nu} (1 + i + b + a(r - 1)) dz_1 \cdots dz_r
\]

which is evaluated by the beta integral (see [7, Ex. 7, Sect. 10, Chapt. VII]) that

\[
\int_{[0,1]^r} \prod_{i<j} (t_i - t_j)^a \prod_{i=1}^r t_i^{\frac{\alpha}{2} - \frac{a}{2} + (r - 1)a - 1} \prod_{i=1}^r (1 - t_i)^{\beta - \frac{\alpha}{2} + (r - 1)a - 1} dt_1 \cdots dt_r
= r! \prod_{1 \leq i < j \leq r} \frac{\Gamma\left(\frac{\alpha}{2} (j - i + 1)\right) \Gamma_k(\alpha) \Gamma_k(\beta)}{\Gamma(\frac{\alpha}{2} (j - i)) \Gamma_k(\alpha + \beta)}.
\]

In particular it follows that

\[
\frac{N_{\nu+1}}{N_\nu} = \prod_{j=1}^r \left[ \nu - (\frac{\alpha}{2} (r - 1) + i + b) - \frac{\alpha}{2} (j - 1) \right] \frac{\nu + \frac{1}{2} - \frac{a}{2} (j - 1)}{\nu + \frac{1}{2} - \frac{a}{2} (j - 1)}.
\]

Changing \( j - 1 \) to \( r - j \) and using \( \rho(\xi) = i + b + a(r - 1) \), we can rewrite it as the following

\[
\frac{N_{\nu+1}}{N_\nu} = \prod_{j=1}^r \left[ \nu - \frac{1}{2} \rho(\xi) - \frac{1}{2} (i + b + a(j - 1)) \right] \frac{\nu + \frac{1}{2} - \frac{a}{2} (j - 1)}{\nu + \frac{1}{2} - \frac{a}{2} (j - 1)}.
\]

Theorem 3.2. Let \( \nu > i + b + a(r - 1) \). The spherical transform of \( f_{-2\nu} \) is given by

\[
\widetilde{f_{-2\nu}}(\lambda) = N_\nu \prod_{j=1}^r \prod_{\epsilon = \pm} \frac{\Gamma(\nu - \frac{1}{2} \rho(\xi) + \epsilon \frac{1}{2} \lambda(\xi_j))}{\Gamma(\nu - \frac{1}{2} \rho(\xi) + \epsilon \frac{1}{2} (i + b + a(j - 1)))}, \quad \lambda \in iA^*.
\]

The following result is elementary and we omit its proof; it is proved in [13] in the case when the root system corresponds to a bounded symmetric domain.

Lemma 3.3. Let \( \phi \) be a bounded and continuous function on \( A \). Then

\[
\lim_{\nu \to \infty} \frac{1}{N_\nu} \int_A f_{-2\nu}(t) \phi(t) d\mu(t) = \phi(0)
\]

We prove now Theorem 3.2.

Proof. We write \( \beta_\nu(\lambda) = \frac{\widetilde{f_{-2\nu}}(\lambda)}{N_\nu} \) and compute it in terms of \( \beta_{\nu+1}(\lambda) \). We note first that it is well-defined for \( \lambda \in iA^* \). Indeed the spherical function \( \phi_\lambda(t) \) is a bounded function \([9]\), and the function \( f_{-2\nu} \) is in \( L^1(A^*, d\mu) \) by Lemma 3.1. We perform the spherical transform on the identity (2.1) with \( \delta = -2\nu \). Using [13] we have,

\[
\prod_{j=1}^r (\lambda(\xi_j)^2 - (-2\nu + \rho(\xi_1)^2)) \beta_\nu(\lambda) = \prod_{j=1}^r (-2\nu + a(j - 1))(1 + 2\nu - i - a(r - j)) \beta_{\nu+1}(\lambda),
\]

and we have
and

\[
\beta_\nu(\lambda) = \prod_{j=1}^{r} \frac{(-2\nu + a(j-1))(1 + 2\nu - \iota - a(r - j))}{(\lambda(\xi_j)^2 - (-2\nu + \rho(\xi_j))^2)} \beta_{\nu+1}(\lambda)
= \prod_{j=1}^{r} \frac{(\nu - \frac{a}{2}(j - 1))(\nu + \frac{1 - \iota}{2} - \frac{a}{2}(r - j))}{((\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}\lambda(\xi_j))((\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}\lambda(\xi_j))} \beta_{\nu+1}(\lambda)
\]

We write further the denominator as

\[
\prod_{j=1}^{r} (\nu - \frac{a}{2}(j - 1))(\nu + \frac{1 - \iota}{2} - \frac{a}{2}(r - j))
= \prod_{j=1}^{r} (\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}(\iota + b + a(j - 1)))(\nu + \frac{1 - \iota}{2} - \frac{a}{2}(j - 1)).
\]

Thus

\[
\beta_\nu(\lambda) = \frac{N_{\nu+1}}{N_\nu} \prod_{j=1}^{r} \frac{(\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}(\iota + b + a(j - 1)))(\nu + \frac{1 - \iota}{2} - \frac{a}{2}(j - 1))}{(\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}\lambda(\xi_j))((\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}\lambda(\xi_j))} \beta_{\nu+1}(\lambda).
\]

Using \(3.1\) this becomes

\[
\beta_\nu(\lambda) = \beta_{\nu+1}(\lambda) \prod_{j=1}^{r} \frac{(\nu + \frac{1}{2}\rho(\xi_1) + \frac{1}{2}(\iota + b + a(j - 1)))(\nu + \frac{1}{2}\rho(\xi_1) - \frac{1}{2}(\iota + b + a(j - 1)))}{(\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}\lambda(\xi_j))((\nu - \frac{1}{2}\rho(\xi_1) + \frac{1}{2}\lambda(\xi_j))}
= \beta_{\nu+1}(\lambda) \prod_{j=1}^{r} \left(1 + \frac{\frac{1}{2}(\iota + b + a(j - 1))}{\nu - \frac{1}{2}\rho(\xi_1)}\right) \left(1 - \frac{\frac{1}{2}(\iota + b + a(j - 1))}{\nu + \frac{1}{2}\rho(\xi_1)}\right)
\times \left(1 + \frac{\frac{1}{2}\lambda(\xi_j)}{\nu - \frac{1}{2}\rho(\xi_1)}\right)^{-1} \left(1 - \frac{\frac{1}{2}\lambda(\xi_j)}{\nu - \frac{1}{2}\rho(\xi_1)}\right)^{-1}.
\]

Iterating we find

\[
\beta_\nu(\lambda) = \beta_{\nu+k}(\lambda) \prod_{l=1}^{k} \prod_{j=1}^{r} \left(1 + \frac{\frac{1}{2}(\iota + b + a(j - 1))}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right) \left(1 - \frac{\frac{1}{2}(\iota + b + a(j - 1))}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right)
\times \left(1 + \frac{\frac{1}{2}\lambda(\xi_j)}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right)^{-1} \left(1 - \frac{\frac{1}{2}\lambda(\xi_j)}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right)^{-1}.
\]

However \(\frac{1}{N_{\nu+k}}b_{\nu+k}(\lambda) \to 1\) as \(k \to \infty\) by Lemma 3.3 since the function \(\phi_\lambda(t)\) is a continuous bounded function for \(\lambda \in i\mathbb{R}^*\). We get thus

\[
\beta_\nu(\lambda) = \prod_{l=1}^{\infty} \prod_{j=1}^{r} \left(1 + \frac{\frac{1}{2}(\iota + b + a(j - 1))}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right) \left(1 - \frac{\frac{1}{2}(\iota + b + a(j - 1))}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right)
\times \left(1 + \frac{\frac{1}{2}\lambda(\xi_j)}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right)^{-1} \left(1 - \frac{\frac{1}{2}\lambda(\xi_j)}{\nu + l - 1 - \frac{1}{2}\rho(\xi_1)}\right)^{-1}.
\]
Our result follows rewriting the infinite product in terms of the Gamma function, using
\[
\frac{\Gamma(A)\Gamma(B)}{\Gamma(A+C)\Gamma(B-C)} = \prod_{s=0}^{\infty} \left(1 + \frac{C}{A+s}(1 - \frac{C}{B+s})\right);
\]
see [3, p.5].

**Remark 3.4.** The function \(\widetilde{f}_{-2\nu}(\lambda)\) for \(\nu\) as above has exponential decay and is a bounded function for \(\lambda \in i\mathfrak{a}^*\), and holomorphic in a strip around \(i\mathfrak{a}^*\). We can then define the Berezin transform on \(L^2(\mathfrak{a}, d\mu)^W\) by the inverse spherical transform, namely
\[
B_\nu F(t) = \int_{i\mathfrak{a}^*} \widetilde{f}_{-2\nu}(\lambda) \widetilde{F}(\lambda) \phi_\lambda(t) c(\lambda)^{-2} d\lambda.
\]

\(B_\nu\) is then a bounded and positive operator and it has an integral kernel \(B(t, s)\). In particular \(B(t, 0) = f_{-2\nu}(t)\). It follows then from the selfadjoint property of \(B\) that \(B(x, y) = f_{-2\nu}(t)f_{-2\nu}(s)L(t, s)\). It would be interesting to find a series expansion for the kernel \(L(t, s)\) in terms of the Jack symmetric polynomials (in the variables \((\tanh^2 t_1, \ldots, \tanh^2 t_r))\). In the case of bounded symmetric domain, this is indeed possible, and some degenerate cases of the expansion have been studied in [16] (see also [1] for related problems).

**4. Spherical transform of a class of functions**

We shall compute the spherical transform of a class of functions of the form
\[
f_{-2\nu}(t)p(x_1^2, \ldots, x_r^2), \quad x_j = \tanh t_j, \quad j = 1, \ldots, r,
\]
where \(p(x_1^2, \ldots, x_r^2)\) is a symmetric polynomial in \((x_1, \ldots, x_r)\). We will prove that they are of the form \(\widetilde{f}_{-2\nu}(\lambda)q(\lambda)\) where \(q\) is a symmetric polynomial in \(\lambda^2_1, \ldots, \lambda^2_r\). Eventually we will specify \(p\) to be Jacobi-type polynomials in the next section.

We denote \(D_j\) the conjugation of \(D_j\) by the canonical function \(f_\delta\),
\[
D_j = D_j(\delta) = f_{-\delta} D_j f_\delta, \quad j = 1, \ldots, r.
\]
The operators \(\{D_j(\delta)\}\) under the change of variables \(x_j = \tanh t_j\) have the form
\[
D_j = \delta x_j + (1 - x_j^2) \partial_j - \frac{a}{2} \sum_{i<j} \frac{1 + x_i - x_j - x_i x_j}{x_i - x_j}(1 - s_{ij})
\]
\[
+ \frac{a}{2} \sum_{j<k} \frac{1 + x_j - x_k - x_j x_k}{x_j - x_k}(1 - s_{jk}) + \frac{a}{2} \sum_{k \neq j} \frac{1 + x_j + x_k + x_k x_j}{x_j + x_k}(1 - \sigma_{jk})
\]
\[
+ i \left(1 + \frac{1}{2}(x_j + \frac{1}{x_j})\right)(1 - \sigma_j) + b(1 + \frac{1}{x_j})(1 - \sigma_j) - \rho_j
\]

We shall consider the operators \(\{D_j\}\) acting on polynomials in \(x\). For that purpose we recall the natural ordering \(\geq\) on the set of all partitions \(\eta = (\eta_1, \ldots, \eta_r) \in \mathbb{N}^r\), \(\eta_1 \geq \cdots \geq \eta_r \geq 0\). For two partitions \(\zeta\) and \(\eta\), \(\eta \geq \zeta\) if \(\eta_1 \geq \zeta_1, \eta_1 + \eta_2 \geq \zeta_1 + \zeta_2, \ldots\),
Lemma 4.1. The action of the operators $D_j$ on the monomials $x^n = x_1^{n_1} \cdots x_r^{n_r}$, $n \in \mathbb{N}^r$, preserves the order in the sense that

$$D_j x^n = \sum_{\xi, \xi \leq (\eta^j)^*} a_{\xi, \eta} x^\xi,$$

where $\eta^j = \eta + (0, \cdots, 0, 1, 0, \cdots, 0)$ and the coefficient of $x^{\eta^j}$ is

$$a_{\eta^j, \eta} = \delta + a \#\{i < j : \eta_i > \eta_j\} + \frac{t}{2} (1 - (-1)^{\eta_j}).$$

Remark 4.2. Some refinement of the above lemma can be obtained by using the Bruhat ordering on the set $\mathbb{N}^r$, however we will not need it here.

Denoting $m_\eta$ the symmetric power sum

$$(4.1) \quad m_\eta(y_1, \cdots, y_r) = \sum_{\zeta \in S_r \eta} y_1^{\zeta_1} \cdots y_r^{\zeta_r}.$$

We compute now $m_\eta(D_1^2, \cdots, D_r^2)1$.

Lemma 4.3. The operators $m_\eta(D_1^2, \cdots, D_r^2)$ acting on the constant monomial 1 gives

$$m_\eta(D_1^2, \cdots, D_r^2)1 = d_\eta m_\eta(x_1^2, \cdots, x_r^2) + \text{lower terms},$$

where the “lower terms” here stand for a linear combination of symmetric power sums $m_{\eta'}$ of $x_1^2, \cdots, x_r^2$ with $\eta' \prec \eta$, and the leading coefficient is

$$(4.2) \quad d_\eta = \prod_{j=0}^{\eta_1-1} \prod_{k=0}^{r-1} (\delta + (r - j) - 2k) (\delta + (r - j)a + \eta_1 - 1 - 2k).$$

Proof. This can be obtained by using the fact that $m_\eta(D_1^2, \cdots, D_r^2)1$ is Weyl group invariant and by applying Lemma 4.1 successively. \hfill \Box

Proposition 4.4. Let $\nu = \iota + b + a(r - 1)$ and let $\eta : \eta_1 \geq \eta_2 \geq \cdots \geq \eta_r \geq 0$ be a partition. The spherical transform of the function $f_{-2\nu}(t)m_\eta(tanh^2 t_1, \cdots, tanh^2 t_r)$ is given by

$$\tilde{f}_{-2\nu}(\lambda) l_\eta(\lambda),$$

and $l_\eta(\lambda)$ is $W$-invariant polynomial of $\lambda$ with leading term $d_\eta^{-1} m_\eta(\lambda(\xi_1)^2, \cdots, \lambda(\xi_r)^2)$.

Proof. We apply Lemma 4.2 for $\delta = -2\nu$. Observe that $d_\eta \neq 0$ for any partition $\eta$. It follows that for each partition $\eta$,

$$m_\eta(x_1^2, \cdots, x_r^2) = d_\eta^{-1} m_\eta(D_1^2, \cdots, D_r^2)1 + \sum_{\eta' \prec \eta} c_{\eta', \eta} m_{\eta'}(D_1^2, \cdots, D_r^2)1.$$
In other words
\[ f_{-2\nu}m_\eta(x_1^2, \cdots, x_r^2) = d_{2\eta}^{-1} m_\eta(D_1^2, \cdots, D_r^2) f_{-2\nu} + \sum_{\eta' < \eta} c_{\eta', \eta} m_{\eta'}(D_1^2, \cdots, D_r^2) f_{-2\nu}. \]

For each \( \zeta \), the spherical transform of \( m_\zeta(D_1^2, \cdots, D_r^2) f_{-2\nu} \) is given by, in view of (1.1),
\[ m_\zeta(\lambda(\xi_1)^2, \cdots, \lambda(\xi_r)^2) \widehat{f}_{-2\nu} (\lambda). \]

Our result then follows from the previous formula. \( \square \)

5. Jacobi type functions and Macdonald-Koornwinder polynomials

We recall briefly the Heckman-Opdam theory on Jacobi polynomials. We will then construct certain functions of Jacobi type and prove that their spherical transforms are Macdonald-Koornwinder type polynomials.

Let \( \nu > t + b + a(r - 1) \). Consider now the same root system \( R \) with \( R_+ = \{ 2\varepsilon_j; j = 1, \cdots, r \} \cup \{ 4\varepsilon_j; j = 1, \cdots, r \} \cup \{ 2\varepsilon_j \pm \varepsilon_k; 1 \leq j < k \leq r \} \) and with respective multiplicities \( k^{(\nu)} = (k_1^{(\nu)}, k_2^{(\nu)}, k_3^{(\nu)}) \), such that
\[ 2k_2^{(\nu)} = 2(2\nu - (1 + t + b + a(r - 1))) + 1, \quad 2(k_1^{(\nu)} + k_2^{(\nu)}) = t + 2b, \quad 2k_3^{(\nu)} = a. \]
(Note that one of the multiplicities, \( k_1^{(\nu)} \), is negative, however they still satisfy the condition \( k_\alpha + k_\beta \geq 0, \forall \alpha \in R \), see \[9\].)

The corresponding dual coroot lattice is \( 1/2(\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_r) = (1/2\mathbb{Z})^r \) and weight lattice is \( P = 2(\mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_r) = (2\mathbb{Z})^r \). The set of dominant weights is
\[ P^+ = \{ 2(\eta_1\varepsilon_1 + \cdots + \eta_r\varepsilon_r); \eta = (\eta_1, \cdots, \eta_r) \in \mathbb{N}^r, \eta_1 \geq \cdots \geq \eta_r \geq 0 \}, \]
and for simplicity we will identify the elements \( 2\eta \) in \( P^+ \) with the partitions \( \eta \in \mathbb{N}^r \), and all \( \eta \) appeared below will be assumed to be partitions. The partial ordering on the weight lattice \( P \) defined in terms of positive roots is exactly the natural ordering \( \geq \).

The polynomial algebra \( \mathbb{C}[P] \) of integral weights is then the polynomial algebra generated by \( e^{\pm 2\varepsilon_j}, j = 1, \cdots, r \) as functions defined on the compact torus \( T^r = i\mathbb{R}/(\pi i\mathbb{Z})^r \).

Consider the inner product in the space \( \mathbb{C}[P] \),
\[ (f, g)_{k^{(\nu)}} = \int_{[0, \pi]^r} f(s) \overline{g(s)} |\delta_{k^{(\nu)}}(s)\| ds, \]
where \( \delta_{k^{(\nu)}}(s) = \prod_{\alpha \in R_+} |2 \sin \frac{1}{2} \alpha(s)\|^{2k^{(\nu)}_{\alpha}}. \)

We recall first the Heckman-Opdam theory of Jacobi polynomials, see \[3, 4, 9\].

**Lemma 5.1.** For each \( \eta = (\eta_1, \cdots, \eta_r) \in \mathbb{N}^r, \eta_1 \geq \cdots \geq \eta_r \geq 0 \), there exists a unique polynomial \( P_{\nu, \eta} \) on \( T \) such that
\[ P_{\nu, \eta} = p_\eta^W + \sum_{\eta' < \eta} c_{\eta', \eta} p_{\eta'}^W \]
and
\[ (P_{\nu, \eta}, p_{\eta'}^W)_{k^{(\nu)}} = 0, \]
where \( p_\eta^W = \sum_{w \in W} w(e^{2(\eta_1 \xi_1 + \cdots + \eta_r \xi_r)}) \) is the Weyl group orbit sum of the power function. The polynomials \( \{P_{\nu, \eta}\}_\eta \) forms an orthogonal basis for \( L^2(\mathbb{T}, |\delta_{k^{(\nu)}}(s)|ds)^W \).

The inner product \( \langle P_{\nu, \eta}, P_{\nu, \eta}\rangle_{k^{(\nu)}} \) is explicitly computed in \([5]\) and \([9]\).

The Jacobi polynomials on \( T \) are symmetric with respect to the Weyl group \( W \), thus they are symmetric polynomials of the functions \( \frac{e^{2is_j} + e^{-2is_j}}{2} = \cos 2s_j \), which can be further identified as symmetric polynomials of \( (x_1^2, \ldots, x_r^2) \), with

\[
(5.2) \quad x_j := \sin s_j \in [-1, 1], s = s_1 \xi_1 + \cdots + s_r \xi_r \in \mathbb{T} = i\mathbb{R}/(\pi i\mathbb{Z})^r.
\]

With some abuse of notations we denote it also by \( P_{\nu, \eta}(x_1, \ldots, x_r) \). We rewrite the characterization of \( P_{\nu, \eta} \) in terms of the variables \( (x_1, \ldots, x_r) \).

The Jacobi polynomial \( P_{\nu, \eta}(x_1, \ldots, x_r) \) is then characterized as the unique polynomial so that

\[
P_{\nu, \eta}(x_1, \ldots, x_r) = 2^{2m + \cdots + 2r} m_\eta(x_1^2, \ldots, x_r^2) + \sum_{\zeta < \eta} c_{\eta, \zeta} m_\zeta(x_1^2, \ldots, x_r^2)
\]

and that

\[
(P_{\nu, \eta}, m_\zeta)_{k^{(\nu)}} = 0, \quad \zeta < \eta
\]

where \((f, g)_{k^{(\nu)}}\) is the inner product \([5.1]\) after the change of variables \([5.2]\),

\[
(5.3) \quad (f, g)_{k^{(\nu)}} = 2^{2(r+2b+a(r-1))} \int_{[-1, 1]^r} f(x)g(x) \prod_{i < j} |x_i^2 - x_j^2|^a \prod_j |x_i^{2b+1}| \prod_j (1 - x_i^2)^{2\nu - (1+b+a(r-1))} dx_1 \cdots dx_r.
\]

We now define the Jacobi type function on \( a \), with the variable \( x = \tanh t, t \in \mathfrak{a} \), in \( P_{\nu, \eta} \) being \( x = \tanh t, t \in \mathfrak{a} \),

\[
(5.4) \quad H_{\nu, \eta}(t) := f_{-2\nu}(t)P_{\nu, \eta}(\tanh t_1, \cdots, \tanh t_r).
\]

**Lemma 5.2.** The functions \( \{H_{\nu, \eta}(t)\}_\eta \) form an orthogonal basis for the space \( L^2(\mathfrak{a})^W \) and

\[
\langle H_{\nu, \eta}, H_{\nu, \eta}\rangle_{L^2(\mathfrak{a})} = \langle P_{\nu, \eta}, P_{\nu, \eta}\rangle_{k^{(\nu)}}
\]

**Proof.** Changing of variables \( t \in \mathfrak{a} \to x \in (-1, 1)^r, x_j := \tanh t_j \) in the definition of inner product in \( L^2(\mathfrak{a}) \), we get the above formula. The remaining claim follows from (the symmetric version of) Weierstrass approximation theorem.

To state the next result we recall briefly (with some slight reformulation using the Harish-Chandra \( c \)-function and the function \( \widetilde{f}_{-2\nu}(\lambda) \)) the Macdonald-Koornwinder polynomials of type BC (also called multi-variable Wilson polynomials), see \([\Pi]\). There exists a unique system of polynomials \( \{p_{\eta}^{MK}(\lambda)\}_\eta \) on \( \mathbb{R}^r \), identified with \( i\mathfrak{a}^* \), such that they are orthogonal in the space \( L^2(i\mathfrak{a}^*, \widetilde{f}_{-2\nu}(\lambda)^2|c(\lambda)|^{-2}d\lambda)^W \) and such that \( p_{\eta}^{MK}(\lambda) \) has leading term \( m_\eta(\lambda(\xi_1)^2, \ldots, \lambda(\xi_r)^2) \).
Theorem 5.3. The spherical transforms of the Jacobi type function $H_{\nu,\eta}(t)$ on $\mathfrak{a}$ are given by

$$\tilde{f}_{-2\nu}(\lambda)q_{\nu,\eta}(\lambda), \quad \lambda \in i\mathfrak{a}^*$$

where $q_{\nu,\eta}(\lambda)$ is a $W$-invariant polynomial. The polynomials $\{q_{\nu,\eta}(\lambda)\}$ form an orthogonal basis of the space $L^2(i\mathfrak{a}^*, \tilde{f}_{-2\nu}(\lambda)^2|c(\lambda)|^{-2}d\lambda)^W$, their norm in that space is given by

$$\|q_{\nu,\eta}\|_2 = \|H_{\nu,\eta}\|_{L^2(\mathfrak{a})} = \|P_{\nu,\eta}\|_{k(\nu)},$$

and they are up to constant multiples the Macdonald-Koornwinder polynomials $\{p_{MK}(\lambda)\}$.

Proof. That the spherical transform is of the given form follows from Proposition 4.4, the orthogonality relation is by the Plancherel formula (1.2). Now the polynomials $q_{\nu,\eta}(\lambda)$ are orthogonal, and have the leading term $m_{\eta}(\lambda(\xi_1)^2, \cdots, \lambda(\xi_r)^2)$, again by Proposition 4.4, thus they are multiples of the Macdonald-Koornwinder polynomials $p_{MK}(\lambda)$, by the uniqueness of the latter [11].

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