THE SPECTRA OF LAMPLIGHTER GROUPS AND CAYLEY MACHINES

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Abstract. We calculate the spectra and spectral measures associated to random walks on restricted wreath products $G \wr \mathbb{Z}$, with $G$ a finite group by calculating the Kesten-von Neumann-Serre spectral measures for the random walks on Schreier graphs of certain groups generated by automata. This generalises the work of Grigorchuk and Żuk on the lamplighter group. In the process we characterise when the usual spectral measure for a group generated by automata coincides with the Kesten-von Neumann-Serre spectral measure.

1. Introduction

The systematic study of random walks on discrete non-abelian groups began with the seminal work of Kesten [20] and has since become an important area of mathematics; it links such diverse fields as probability theory, group theory, geometry and analysis. Many of the properties of a random walk are encapsulated in the spectrum and spectral measure of the associated Markov operators. However, the computation of these is in general quite hard; there are as yet very few examples of complete computations of the spectrum [20, 24, 11, 13] and still fewer of the spectral measure [24, 13].

At the same time, there has been increasing interest in the class of automata groups, or groups generated by transformations defined by finite automata [9, 13]. The study of such groups has given many insights into group theory, and led to the solution of a number of long-standing open problems. The study of random walks on automata groups was initiated by Bartholdi and Grigorchuk [1]; they introduced methods to calculate the spectra and spectral measures associated to random walks of automata groups on the Schreier graphs with respect to their parabolic subgroups.

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These techniques were employed by Grigorchuk and Žuk [13], who computed the spectrum and spectral measure of the Markov operator associated to a random walk on the lamplighter group

\[ \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}, \]

by realizing it as the group generated by a two-state automaton. They showed that the spectral measure was discrete – a previously unseen phenomenon – and subsequently, in collaboration with Linnell and Schick, used their computation to obtain a negative answer to a strong form of the Atiyah conjecture concerning \( L^2 \)-Betti numbers [10].

Subsequently, Dicks and Schick [7] computed the spectral measures for random walks on groups of the form \( G \wr \mathbb{Z} \) with \( G \) a non-trivial finite group using entirely different methods. Their methods were then generalised by Bartholdi and Woess to random walks on a more general class of graphs [3].

The main purpose of this paper is to compute, using finite state automata, the spectral measure for random walks on wreath product groups of the form \( G \wr \mathbb{Z} \) with \( G \) a finite group, thus obtaining a new proof of the results of Dicks and Schick [7] along the lines of the case of \( G = \mathbb{Z}/2\mathbb{Z} \) considered by Grigorchuk and Žuk. This involves showing that the spectral measure for these groups coincides with the so-called Kesten-von Neumann-Serre spectral measure, introduced by Grigorchuk and Žuk [15], for random walks of certain groups generated by automata on their Schreier graphs with respect to parabolic subgroups. In fact, we obtain the spectrum for a more general class of groups as well as some new computations of Kesten-von Neumann-Serre spectral measures.

We also aim to give a coherent, and as self-contained as possible, treatment of the subject of computing spectral measures of random walks on automata groups via calculation of their Kesten-von Neumann-Serre spectral measures. We give necessary and sufficient conditions for these measures to coincide, for groups acting spherically transitively on rooted trees. So far as we know, these criteria have not appeared elsewhere.

In [28] the second and third authors considered a possible connection between automata groups and Krohn-Rhodes theory [21, 22], by studying the automata group \( \Gamma \) generated by the Cayley machines of a finite group \( G \). In the case that \( G \) is abelian, it transpires that \( \Gamma \) is actually the wreath product \( G \wr \mathbb{Z} \). Indeed, when \( G = \mathbb{Z}/2\mathbb{Z} \) the automaton obtained is exactly that used by Grigorchuk and Žuk to generate the lamplighter group. In the non-abelian case, \( \Gamma \) is still a semidirect product of a locally finite group with \( \mathbb{Z} \) [28]; however, we shall see below that \( \Gamma \) is not in general isomorphic to \( G \wr \mathbb{Z} \). The walks that Dicks and Schick considered on \( G \wr \mathbb{Z} \) for \( G \) non-abelian can be obtained from the abelian case.

In this paper, we compute the spectrum of the Markov operator associated to a simple random walk on any automata group generated by a Cayley
machine, proving that it is always the entire interval $[-1, 1]$. We compute also the so-called Kesten-von Neumann-Serre spectral measure \[15\] for the random walks on the Schreier graphs with respect to parabolic subgroups. Interestingly, this measure turns out always to be discrete, with support at the points $\lambda_{p,q} = \cos \frac{p}{q} \pi$ where $1 \leq p < q$, $(p,q) = 1$; the weight at $\lambda_{p,q}$ is\[\frac{(n-1)^2}{n^q-1}\]. In the case that $G$ is abelian, we show that this coincides with the spectral measure of the corresponding simple random walk. We also calculate the Ihara zeta functions \[15\] of the Schreier graphs with respect to parabolic subgroups and hence for the Cayley graph of $G$ wr $\mathbb{Z}$, in the case $G$ is abelian.

2. Markov and Hecke operators, spectral measures, random walks and Ihara zeta functions

In this section, we introduce the fundamental notions from analysis which motivate the work in this area. These include Markov and Hecke operators, spectral measures and Ihara zeta functions. We also see how these concepts relate to the study of random walks on graphs.

2.1. Markov and Hecke-type operators. We consider only real Hilbert spaces. Let $X = (V, E)$ be a $k$-regular (undirected) graph with vertex set $V$ and edge set $E$. We allow multiple edges and loops. The primary example for us is where $\Gamma$ is a finitely generated group with symmetric generating set $S$ of size $k$, $P \leq \Gamma$ is a subgroup and $X = \text{Sch}(\Gamma, P, S)$ is the associated (left) Schreier (or coset) graph. The vertices are the left cosets $\Gamma / P$. The edge set is $S \times \Gamma / P$. The edge $(s, gP)$ goes from $gP$ to $sgP$. In particular, the (left) Cayley graph of $\Gamma$ is $\text{Sch}(\Gamma, 1, S)$.

For $v \in V$, let $E(v)$ be the set of edges incident with $v$. For each edge $e \in E(v)$, let $o_v(e)$ denote the vertex at the other end of $e$ from $v$; for loops $o_v(e)$ is taken to be $v$. The random walk or Markov operator \[20, 13\] on $X$ is the operator $M : \ell^2(V) \to \ell^2(V)$ given by

$$Mf(v) = \frac{1}{k} \sum_{e \in E(v)} f(o_v(e))$$

(2.1)

Here $\ell^2(V)$ is the space of square summable functions from $V$ to $\mathbb{R}$. If $\delta_v : V \to \mathbb{R}$ is the characteristic function of $\{v\}$ for each $v \in V$, and we write the “matrix” for $M$ with respect to the basis $\{\delta_v\}_{v \in V}$, then we obtain the normalized incidence matrix for $X$. That is, the matrix coefficient $\langle M \delta_{v_1}, \delta_{v_2} \rangle$ is the probability that an edge incident on $v_1$ is also incident on $v_2$. For the case of a random walk on a Schreier graph $\text{Sch}(\Gamma, P, S)$, one has that $M : \ell^2(\Gamma / P) \to \ell^2(\Gamma / P)$ is given by

$$Mf(gP) = \frac{1}{|S|} \sum_{s \in S} f(sgP).$$

This is a special case of a Hecke-type operator \[1, 13\].
Let \( \Gamma \) be a (discrete) group with finite symmetric generating set \( S \) and \( \pi : \Gamma \to \mathcal{B}(\mathcal{H}) \) be a unitary representation of \( \Gamma \) on a Hilbert space \( \mathcal{H} \); here \( \mathcal{B}(\mathcal{H}) \) denotes the algebra of bounded linear operators on \( \mathcal{H} \). The associated Hecke-type operator is then \( H_{\pi} : \mathcal{H} \to \mathcal{H} \) given by
\[
H_{\pi} = \frac{1}{|S|} \sum_{s \in S} \pi(s).
\]
Then \( H_{\pi} \) is a self-adjoint operator and \( \|H_{\pi}\| \leq 1 \) by the triangle inequality.

Suppose now that \( P \leq \Gamma \) is a subgroup and let \( \lambda_{\Gamma/P} : \Gamma \to \mathcal{B}(\ell^2(\Gamma/P)) \) be the (left) quasi-regular representation; so \( \lambda_{\Gamma/P}(g)f(hP) = f(g^{-1}hP) \). Then \( H_{\lambda_{\Gamma/P}} \) is precisely the Markov operator associated to the random walk on \( \text{Sch}(\Gamma, P, S) \). The case \( P = 1 \) is called the left regular representation, denoted \( \lambda_\Gamma \).

Another important example is the following. Suppose that \( \Gamma \) acts on a measure space \( (X, \mu) \) by measure-preserving transformations. Then there is an associated unitary representation \( \pi : \Gamma \to \mathcal{B}(L^2(X, \mu)) \) (where \( L^2(X, \mu) \) is the space of square-integrable functions on \( X \)) given by \( \pi(g)f(x) = f(g^{-1}x) \). The case of interest to us arises from the action of an automata group on the boundary of a rooted tree, viewed as a measure space with the product (Bernoulli) measure.

### 2.2. Kesten spectral measures.

The spectrum \( \text{Sp}(T) \) of a bounded operator \( T \) on a Hilbert space consists of all real numbers \( \lambda \) such that \( T - \lambda I \) is not invertible. Then \( \text{Sp}(T) \) is a closed subset of the interval \([-\|T\|, \|T\|]\).

We return now to the Markov operator \( M \) for a random walk on a \( k \)-regular graph \( X \). As in the case of Schreier graphs, one can verify that \( \|M\| \leq 1 \). Also \( M \) is self-adjoint, so it has a spectral decomposition
\[
M = \int_{-1}^{1} \lambda dE(\lambda) \tag{2.2}
\]
where \( E \) is the spectral measure \[25\]. That is, \( E \) is a projection-valued measure defined on the Borel subsets of \([-1, 1] \), taking values in the projections of \( \mathcal{B}(\mathcal{H}) \).

For those unfamiliar with these notions, if \( X \) is a finite graph, then
\[
E(B) = \sum_{\lambda \in \text{Sp}(M)} E_\lambda \tag{2.3}
\]
where \( E_\lambda \) is the projection to the eigenspace associated with \( \lambda \) and \( \text{Sp}(M) \) is the usual orthogonal decomposition of a symmetric matrix into its eigenspaces.

The matrix \( \mu^X \) of measures associated with \( E \) is given by
\[
\mu^X_{v_1, v_2}(B) = \langle E(B)\delta_{v_1}, \delta_{v_2} \rangle. \tag{2.4}
\]

Of particular interest are the diagonal entries \( \mu^X_v := \mu^X_{vv} \). These are called the Kesten spectral measures associated to the random walk \[15\]. To explain the significance of these measures, we remind the reader about the
moments of a measure. If \( \mu \) is a (Borel) probability measure on \( \mathbb{R} \) (we do not necessarily assume the support of the measure is the whole real line) and \( f : \mathbb{R} \to \mathbb{R} \) is a measurable function, then the **expected value** of \( f \), denoted \( E[f] \), (or \( E_\mu[f] \) if we want to emphasize the measure) is given by

\[
E[f] = \int_{\mathbb{R}} f(x) \, d\mu(x).
\]

The \( m \)th **moment** of \( \mu \), denoted \( \mu^{(m)} \), is \( E[x^m] \). The (formal) moment generating function is then the power series

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \mu^{(m)} t^m \in \mathbb{R}[t].
\]

This is the Taylor expansion about 0 of the function \( M(t) = E[e^{tx}] \) (where the integral is taken over \( x \)). Returning to the situation of a simple random walk on a graph \( X \) and the spectral measure \( \mu^X_v \), denote by \( p_m(v) \) the probability of return to \( v \) on the \( m \)th step of the random walk; then \( p_m(v) \) is the \( m \)th moment of \( \mu^X_v \) [20] [13]. In the case of a Schreier graph \( \text{Sch}(\Gamma, P, S) \) we shall use the notation \( \mu^{\Gamma/P} \) to denote \( \mu^\Gamma_P \).

If the automorphism group of \( X \) acts transitively on the vertices (for example this occurs for \( \text{Sch}(\Gamma, P, S) \) when \( P \) is normal) then the Kesten measures are independent of the chosen vertex; moreover the spectral decomposition \( \mu^X \) is determined by the Kesten spectral measure at a single vertex. In particular, this happens for Cayley graphs. In the case of the Cayley graph of \( \Gamma \), one can alternatively use von Neumann traces. The von Neumann trace on the von Neumann algebra generated by the left regular representation of \( \Gamma \) is given by

\[
\text{tr}(T) = \langle T\delta_1, \delta_1 \rangle;
\]

that is, it is the “coefficient” in \( T \) of the identity element (this is literally true for elements of finite support, i.e. elements of \( \mathbb{T} \)). Then \( \mu^\Gamma = \text{tr}(E(B)) \) coincides with the Kesten spectral measure at, for example, the identity, as one easily sees by comparing (2.4) and (2.5).

2.3. **Ihara zeta functions and Kesten-von Neumann-Serre spectral measures.** The following is from [12] [15]. If \( X = (V,E) \) is a graph, one can define the path metric on \( V \) by setting \( d(v_1, v_2) \) to be the number of edges in the shortest path connecting \( v_1 \) and \( v_2 \). Denote by \( B_X(v, r) \) the open ball of radius \( r \) in \( V \) around \( v \).

Fix a positive integer \( k \) as before. The space of (isomorphism classes of) pointed \( k \)-regular graphs \( (X,v) \) becomes an ultrametric compact totally disconnected space [12] [15] by taking

\[
d(\langle X_1, v_1 \rangle, \langle X_2, v_2 \rangle) = \frac{1}{n+1}
\]

where \( n \) is the largest integer for which \( B_{X_1}(v_1, n) \) is pointedly isometric to \( B_{X_2}(v_2, n) \).
The case of interest to us is the following: \( \Gamma \) is a discrete group with finite symmetric generating set \( S \) and \( P \leq \Gamma \). Suppose \( P_n \leq \Gamma, n \in \mathbb{N} \) are finite index subgroups with \( \bigcap P_n = P \) (so \( P \) is a closed subgroup in the profinite topology on \( \Gamma \)). Then one easily sees that the graphs \( X_n = \text{Sch}(\Gamma, P_n, S) \) converge to the graph \( X = \text{Sch}(\Gamma, P, S) \) (see [12, 15]) where the coset of the identity is taken as the base point.

A sequence of probability measures \( \mu_n \) on a measure space is said to converge weakly to a measure \( \mu \) if, for all measurable subsets \( B \),
\[
\mu_n(B) \rightarrow \mu(B) \quad \text{[5]}
\]
Suppose \( (X_n, v_n) \) is a sequence of pointed finite graphs converging to \( (X, v) \), and let \( N > 0 \) be given. Then \( B_{X_n}(v, N) \) is pointedly isometric to \( B_{X_n}(v_n, N) \) for \( n \) sufficiently large. From this it can be deduced using the method of moments that \( \mu_{X_n}^{v_n} \rightarrow \mu_v^X \) weakly \([12, 15]\).

To motivate the so-called Kesten-von Neumann-Serre spectral measures \([15]\) we recall the definition of the Ihara zeta function \( \zeta_X \) of a finite \( k \)-regular graph \( X \) \([19]\). It is the power series
\[
\zeta_X(t) = \prod_{[C]} (1 - t^{|C|})^{-1}
\]
where \([C]\) runs over the equivalence classes of primitive, cyclically reduced closed paths in \( X \) and \(|C|\) denotes the length of \( C \). It is known \([19, 15]\) that
\[
\ln \zeta_X(t) = \sum_{r=1}^{\infty} \frac{c_r}{r} t^r
\]
where \( c_r \) is the number of cyclically reduced closed paths of length \( r \) in \( X \).

We remark that \( \zeta_X \) can be viewed as a discrete analogue of the Riemann zeta function, and satisfies the Riemann hypothesis if and only if \( X \) is a Ramanujan graph \([15, 6]\).

Let \( M \) be the Markov operator associated to \( X \). Recalling that \( kM \) is the incidence matrix of the graph \( X \), the results of \([19]\) show that
\[
\zeta_X(t) = (1 - t^2)^{-\frac{1}{2}k(k-2)|E(X)|} \det(1 - tkM + (k-1)t^2)^{-1}
\]
where \( E(X) \) denotes the edge set of \( X \).

Grigorchuk and Žuk extended this to infinite graphs that are limits of sequences of finite graphs. Let \( \{X_n\} \) be a sequence of finite \( k \)-regular graphs with associated Markov operators \( M_n \). Denote by \( V_n \) the vertex set of \( X_n \). We use \( \text{tr}(A) \) to denote the trace of a matrix \( A \). Define
\[
\mu_n = \frac{1}{|V_n|} \sum_{v \in V_n} \mu_v^{X_n} = \sum_{\lambda \in \text{Sp}(M_n)} \frac{\text{tr}(E_{\lambda})}{|V_n|} \delta_{\lambda} = \sum_{\lambda \in \text{Sp}(M_n)} \frac{\#_n(\lambda)}{|V_n|} \delta_{\lambda} \quad (2.6)
\]
where \( \#_n(\lambda) \) denotes the multiplicity of \( \lambda \) as an eigenvalue of \( M_n \), where \( E_{\lambda} \) is the projection to the \( \lambda \)-eigenspace of \( M_n \) and where \( \delta_{\lambda} \) is the Dirac measure at \( \lambda \). The second equality of \((2.6)\) follows from \((2.3)\). The probability measure \( \mu_n \) counts the frequency of the eigenvalues weighted by their multiplicities.
Following Serre [27] (see also [15]) the eigenvalues of the Markov operators $M_n$ are said to be equidistributed with respect to a measure $\mu$ having support on $[-1,1]$ if $\mu_n \rightarrow \mu$ weakly.

Suppose now that we are in the situation above with $X = \text{Sch}(\Gamma, P, S)$ and $X_n = \text{Sch}(\Gamma, P_n, S)$. It is shown in [15] that the eigenvalues of the $M_n$ are equidistributed with respect to some measure $\mu$, which Grigorchuk and Žuk call the Kesten-von Neumann-Serre (KNS) spectral measure of $X$ with respect to the approximating sequence $X_n$.

Serre [27] proved that the eigenvalues of $M_n$ are equidistributed with respect to some measure if and only if the sequence $\zeta_{|E(X_n)|} = \frac{1}{|E(X_n)|} \ln \zeta_{X_n}(T)$ converges in $\mathbb{R}[\{t\}]$ with the topology of pointwise convergence of the coefficients. Grigorchuk and Žuk then defined the zeta function of $X$ (in the above context) with respect to the approximating sequence $X_n$ to be given by

$$\ln \zeta_X(t) = \lim_{n \rightarrow \infty} \frac{1}{|E(X_n)|} \ln \zeta_{X_n}(T).$$

They showed [15] that $\ln \zeta_X(t)$ has radius of convergence at least $\frac{1}{1-k}$ and that, for $|t| < \frac{1}{1-k}$,

$$\ln \zeta_X(t) = -\frac{k-2}{2} \ln(1-t^2) - \int_{-1}^{1} \ln(1-tk\lambda + (k-1)t^2)d\mu$$

where $\mu$ is the KNS spectral measure. This serves as a motivation for computing the measure $\mu$. Conversely, it is shown in [15] that $\zeta_X$ determines the moments of $\mu$ and hence determines $\mu$ itself.

They also define in [15] the zeta function of $\Gamma$ (with respect to the generators $S$) by

$$\ln \zeta_{\Gamma}(t) = -\frac{k-2}{2} \ln(1-t^2) - \text{tr} \ln(1-tkM + (k-1)t^2)$$

where $\text{tr}$ is, as above, the von Neumann trace. Notice that this is the same formula as (2.7), but with $\mu$ replaced by the Kesten spectral measure for the random walk on the Cayley graph of $\Gamma$.

### 3. Automata groups and Cayley machines

In this section, we introduce finite automata and the groups they generate.

A finite (Mealy) automaton [22, 8] $A$ is a 4-tuple $(Q, A, \delta, \lambda)$ where $Q$ is a finite set of states, $A$ is a finite alphabet, $\delta : Q \times A \rightarrow Q$ is the transition function and $\lambda : Q \times A \rightarrow A$ is the output function. One writes $qa$ for $\delta(q, a)$ and $q \circ a$ for $\lambda(q, a)$. These functions extend to the free monoid $A^*$ by

$$q(au) = (qa)u \quad (3.1)$$
$$q \circ (au) = (q \circ a)(qa) \circ u. \quad (3.2)$$

We use $A_q$ to denote the initial automaton $A$ with designated start state $q$. There is a function $A_q : A^* \rightarrow A^*$ given by $w \mapsto q \circ w$. This function is
length preserving and extends continuously \cite{11} to the set of right infinite
words \(A^\omega\) via the formula
\[
A_q(a_0a_1\cdots) = \lim_{n \to \infty} A_q(a_0\cdots a_n)
\tag{3.3}
\]
where \(A^\omega\) is given the product topology, making it homeomorphic to a Can-
tor set. If, for each \(q\), the state function \(\lambda_q : A \to A\) given by \(\lambda_q(a) = q \circ a\) is a
permutation, then \(A_q\) is an isometry of \(A^\omega\) for the metric \(d(u, v) = 1/(n+1)\) where \(n\) is the length of the longest common prefix of \(u\) and \(v\) \cite{11}. In this
case the automaton is called invertible. We shall assume here (unless oth-
erwise stated) that all automata are invertible. Let \(\Gamma = G(A)\) be the group
generated by the \(A_q\) with \(q \in Q\).

If we let \(T\) be the Cayley tree of \(A^*\), then \(\Gamma\) acts on the left of \(T\) by rooted
tree automorphisms of \(T\) \cite{11} via the action (3.2). The induced action on
the boundary \(\partial T\) (the space of infinite directed paths from the root) is just
the action (3.3) of \(\Gamma\) on \(A^\omega\).

The automorphism group \(\text{Aut}(T)\) is the iterated (permutational) wreath
product of countably many copies of the left permutation group \((S_A, A)\)
\cite{11}, where \(S_A\) denotes the symmetric group on \(A\). In this paper, our
notation will be such that the wreath product of left permutation groups
has a natural projection to its leftmost factor; this is in contrast to the case
of restricted wreath products of abstract groups where our notation is such
that there is a projection to the rightmost factor. For a group \(\Gamma = G(A)\)
generated by an automaton over \(A\), one has an embedding
\[
(\Gamma, A^\omega) \hookrightarrow (S_{|A|}, A) \wr (\Gamma, A^\omega).
\tag{3.4}
\]
The maps sends \(A_q\) to the element with wreath product coordinates:
\[
A_q = \lambda_q(A_{qa_1}, \ldots, A_{qa_n})
\tag{3.5}
\]
where \(A = \{a_1, \ldots, a_n\}\). See \cite{11, 2, 28} for more details.

The action of \(\Gamma\) on \(T\) is called spherically transitive if \(\Gamma\) acts transitively
on each level of the tree. Here the \(k\)th level of \(T\) is the set of all vertices
corresponding to words of length \(k\). It will be convenient to denote by \(T_k\)
the finite rooted tree obtained by pruning the levels after level \(k\). Denote by
\(\text{St}_\Gamma(k)\) the set of all elements of \(\Gamma\) that fix each vertex of level \(k\); it is a finite
index normal subgroup, being the kernel of the projection \(\Gamma \to \text{Aut}(T_k)\).
Notice that \(\{1\} = \bigcap_{k=0}^\infty \text{St}_\Gamma(k)\) (and so \(\Gamma\) is residually finite).

In this paper, we shall be particularly concerned with one class of exam-
ple. Let \(G\) be a non-trivial finite group. By the Cayley machine \(C(G)\) of \(G\)
we mean the automaton with state set and alphabet \(G\). Both the transition
and the output functions are the multiplication of the group. So in state \(g_0\)
on input \(g\) the machine goes to state \(g_0g\) and outputs \(g_0g\). You can view
\(C(G)\) as the Cayley graph of \(G\) with respect to the generators \(G\) where out-
put is the next state. The state function \(\lambda_g\) is just left translation by \(g\) and
hence a permutation, so \(C(G)\) is invertible. Cayley machines for semigroups
are an important part of Krohn-Rhodes theory \cite{22, 21}. The study of the
An automaton is called a reset automaton if, for each \( a \in A \), \(|Qa| = 1\); that is, each input resets the automaton to a single state. The second and third authors showed that the inverse of a state \( C(G)_g \) is computed by the corresponding state of the reset automaton \( A(G) \) with states \( G \) and input alphabet \( G \). Therefore \( G(C(G)) = G(A(G)) \). In wreath product coordinates \( A(G) = g^{-1}(A_{g_1}, \ldots, A_{g_n}) \) (3.6) where \( G = \{g_1, \ldots, g_n\} \). Hence, in our situation, there is an embedding \( G(A(G)) \hookrightarrow (G, G) \wr (G(A(G)), G^\omega) \). (3.7)

Moreover, the action of \( G(A(G)) \) on the Cayley tree of \( G^* \) is spherically transitive [28].

Let \( x = A(G)_1 \). Notice that \( xA(G)g^{-1} = xC(G)_g = g(1, \ldots, 1) \), so we can identify \( G \) with a subgroup of \( G(A(G)) \) via \( g \leftrightarrow A(G)_g^{-1} \). Let

\[
N = \langle x^nGx^{-n} \mid n \in \mathbb{Z} \rangle. \tag{3.8}
\]

It is shown in [28] that \( x \) has infinite order, \( N \) is a locally finite group and \( G(A(G)) = N \rtimes \langle x \rangle \). If \( G \) is abelian, it is shown [28] that

\[
G(A(G)) = G \wr \mathbb{Z} = \left( \bigoplus_{\mathbb{Z}} G \right) \rtimes \mathbb{Z}
\]

where in the latter semidirect product, \( \mathbb{Z} \) acts by the shift. In particular, \( G(A(\mathbb{Z}/n\mathbb{Z})) \) is the lamplighter-type group \( \mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z} \).

The depth of an element \( \gamma \in \Gamma \) is the least integer \( n \) (if such exists) so that \( \gamma \) only changes the first \( n \) letters of a word. An automorphism of finite depth is often called finitary. An important role is also played by the subgroup

\[
N_0 = \langle x^nGx^{-n} \mid n \geq 0 \rangle. \tag{3.9}
\]

It is shown in [28] that \( x^nGx^{-n} \) has depth \( n + 1 \) and so \( N_0 \) consists of finitary automorphisms. We recall also from [28] that the elements of the form \( A(G)_g \in \Gamma \) with \( g \in G \) generate a free subsemigroup of \( \Gamma \) (this holds for a large class of semigroups generated by invertible reset automata [28]).

4. Dynamics, Free Actions and Spectral Measures

Let \( A \) be a finite alphabet and \( T \) be the Cayley tree of \( A^* \). Let \( \nu \) be the product measure on \( \partial T = A^\omega \); so for each \( u \in A^* \), the cylinder set \( uA^\omega \) is given the measure \( 1/|A|^{|u|} \). Let \( \varphi : T \to T \) be a morphism of trees preserving the distance from the root (and hence the levels); equivalently \( \varphi : A^* \to A^* \) is a function computed by a possibly infinite automaton [11]; such a map is...
called in [26] an elliptic contraction. We also use \( \varphi : \partial T \to \partial T \) to denote the induced morphism on the boundary (which is a contraction [11]). Define

\[
\text{Fix}(\varphi) = \{ w \in \partial T \mid \varphi(w) = w \}.
\]

It is a closed subset of \( \partial T \). Also define, for \( k \geq 0 \),

\[
\text{Fix}_k(\varphi) = \{ w \in A^k \mid \varphi(w) = w \}.
\]

**Proposition 4.1.** Let \( T \) be the Cayley tree of \( A^* \) and \( \varphi : T \to T \) be an elliptic contraction. Then

\[
\nu(\text{Fix}(\varphi)) = \lim_{k \to \infty} \frac{1}{|A|^k} |\text{Fix}_k(\varphi)|.
\]

(4.1)

In particular, (4.1) holds for rooted automorphisms of \( T \). \( \square \)

**Proof.** Set \( F_k = \bigcup \{ wA^\omega \mid w \in \text{Fix}_k(\varphi) \} \). Then \( \{ F_k \}_{k \geq 0} \) is a decreasing sequence of measurable sets and

\[
\text{Fix}(\gamma) = \bigcap_{k=0}^{\infty} F_k.
\]

Hence, since \( \nu \) is a finite measure,

\[
\nu(\text{Fix}(\varphi)) = \lim_{k \to \infty} \nu(F_k) = \lim_{k \to \infty} \frac{1}{|A|^k} |\text{Fix}_k(\varphi)|,
\]

thereby establishing the proposition. \( \square \)

Recall that a subset of a topological space is said to be nowhere dense if it does not contain a non-empty open subset. In general, measure zero is very far from nowhere dense – for instance the irrational numbers in the interval \([0, 1]\) are nowhere dense but have full measure. However, for the case of a contraction of the Cantor set computed by a finite state automata we can prove the following (somewhat surprising) result, which is inspired by a much more complicated argument from [13] for the case of the lamplighter group.

**Theorem 4.2.** Let \( \varphi : A^\omega \to A^\omega \) be a function computed by a finite state automaton. Then \( \text{Fix}(\varphi) \) has measure zero if and only if it is nowhere dense.

**Proof.** Since each non-empty open subset of \( A^\omega \) has positive measure, it suffices to show that if \( \text{Fix}(\varphi) \) is nowhere dense, then it has measure zero. Let \( A = (Q, A, \delta, \lambda) \) be a finite state automaton such that \( \varphi = A_{q_0}, q_0 \in Q \). Let \( Q' \) be the set of \( q \in Q \) such that \( A_q \) is not the identity. The hypothesis on \( \varphi \) then implies that if \( u \in A^* \) is fixed by \( \varphi \), then \( q_0 \cdot u \in Q' \) – if this were not the case \( \varphi \) would fix \( uA^\omega \). Since \( Q \) is finite, we can find an integer \( p > 0 \) such that, for each \( q \in Q' \), \( A_q \) does not fix some element of \( A^p \). Let \( n = |A| \).

We claim:

\[
|\text{Fix}_{pk}(\varphi)| \leq (n^p - 1)^k \quad (4.2)
\]

for all \( k \geq 0 \). We proceed by induction. The result is clear for \( k = 0 \). Suppose that (4.2) holds for \( k - 1 \), with \( k \geq 1 \), and consider the \( n^pk \perp \) words...
of length $pk$. By the inductive hypothesis, at most $(n^p - 1)^{k-1}$ of the possible prefixes of length $p(k - 1)$ of such words are fixed by $\varphi$. Now let $u \in A^{p(k-1)}$ be such a prefix that is fixed by $\varphi$. Then $q_0 \cdot u \in Q'$ by the hypothesis on $\varphi$. By choice of $p$, $A_{q_0 \cdot u}$ does not fix all words of length $p$, so there are at most $(n^p - 1)^{k-1}$ such words that are fixed by $A_{q_0 \cdot u}$. Hence, there are at most $(n^p - 1)^{k-1} \cdot (n^p - 1) = (n^p - 1)^k$ words from $A^{pk}$ that are fixed by $\varphi$, establishing (4.2). Thus

$$\frac{1}{n^{pk}} |\text{Fix}_{pk}(\varphi)| \leq \left(1 - \frac{1}{n^p}\right)^k.$$  

Since the right hand side of the above equation tends to 0 as $k \to \infty$, the theorem follows by an application of Proposition 4.1. □

Let $\Gamma$ be a group acting spherically transitively by rooted automorphisms of $T$. Then $\Gamma$ acts ergodically on the measure space $(\partial T, \nu)$ by measure preserving transformations and topologically transitively on $\partial T$ by isometries [13] [11]. Let $\pi : \Gamma \to B(L^2(\partial T, \nu))$ be the associated representation, $S$ be a finite symmetric generating set for $\Gamma$ and $H_\pi$ be the associated Hecke-type operator. Let $\pi_k : \Gamma \to B(\ell^2(A^k))$ be the permutation representation of $\Gamma$ on the $k^{th}$ level of the tree. Identifying the elements of the $A^k$ with the characteristic functions of the associated cylinder sets, $\pi_k$ can be viewed as a subrepresentation of $\pi$ and $\pi_{k-1}$ as subrepresentation of $\pi_k$. If, for $k \geq 1$, $\pi'_k$ is the orthogonal complement of $\pi_{k-1}$ in $\pi_k$ and $\pi'_0 = \pi_0$, then [11] $\pi = \bigoplus_{k=0}^\infty \pi'_k$ from which one obtains [11]

$$\text{Sp}(H_\pi) = \bigcup_{n \geq 0} \text{Sp}(H_{\pi_k})$$  

(4.3)

Recall [16] that if $\alpha$ is a finite dimensional representation of $\Gamma$, then the associated character $\chi_\alpha$ is given by $\chi_\alpha(\gamma) = \text{tr}(\alpha(\gamma))$. If $\alpha$ is a permutation representation, then $\chi_\alpha$ is the fixed-point character; it simply counts the number of fixed-points of each element of $\Gamma$. It then seems natural to define the fixed-point character of $\pi$ by

$$\chi_\pi(\gamma) = \nu(\text{Fix}(\gamma)).$$

An immediate consequence of Proposition 4.1 is

$$\chi_\pi(\gamma) = \lim_{k \to \infty} \frac{1}{n^k} \chi_{\pi_k}(\gamma).$$  

(4.4)

An action of a group $\Gamma$ on a measure space is said to be free in the sense of ergodic theory if, for all $1 \neq \gamma \in \Gamma$, $\text{Fix}(\gamma)$ has measure zero. We say that the action is free in the sense of Baire category if, for all $1 \neq \gamma \in \Gamma$, $\text{Fix}(\gamma)$ is nowhere dense. Notice that in either of these two cases there is an infinite path $w \in \partial T$ with trivial stabilizer. Indeed, $\bigcup_{1 \neq \gamma \in \Gamma} \text{Fix}(\gamma)$ cannot be all of $\partial T$: in the first case this set has measure zero; in the latter it is a countable union of closed nowhere dense sets and hence cannot be all of $\partial T$ by the Baire category theorem. Theorem 4.2 then has the following interpretation.
Theorem 4.3. Let $\Gamma$ be a group acting on the Cayley tree $T$ of $A^*$ (with $A$ finite) by rooted automorphisms computed by finite state automata. Then the following are equivalent:

1. The action of $\Gamma$ on $(\partial T, \nu)$ is free in the sense of ergodic theory;
2. The action of $\Gamma$ on $\partial T$ is free in the sense of Baire category;

In light of the above result, it seems natural to say that the action of a group $\Gamma$ generated by a finite state automaton is free if the equivalent conditions of Theorem 4.3 hold. The only non-trivial examples of free actions of automata groups in the literature, so far as we know, are the Cayley machines of finite abelian groups [28], including the automaton of the lamp-lighter group considered in [13]. For Cayley machines of non-abelian groups, fixed-point sets can have non-empty interior [28]. It would be interesting to find more examples.

Returning to the spectra of Hecke operators, let us fix $w \in \partial T$ and set $w_k$ to be the prefix of $w$ of length $k$. Let $P = \text{St}_\Gamma(w)$ and $P_k = \text{St}_\Gamma(w_k)$. The subgroup $P$ is called a parabolic subgroup [11]. Then, for all $k \geq 0$, $P_k$ is of finite index in $\Gamma$ and $P = \bigcap_k P_k$. Moreover, since $\Gamma$ acts spherically transitively, $\pi_k$ is equivalent to the quasi-regular representation $\lambda_{\Gamma/P_k}$ and $H_{\pi_k} = M_k$ the Markov operator on $\text{Sch}(\Gamma, P_k, S)$; if $M$ is the Markov operator on $\text{Sch}(\Gamma, P, S)$, then $H_{\lambda_{\Gamma/P}} = M$. If, in addition, $\Gamma$ is amenable, then [1][Theorem 3.6] shows that

$$\text{Sp}(H_{\pi}) = \bigcup_{n \geq 0} \text{Sp}(H_{\pi_k}) = \text{Sp}(H_{\lambda_{\Gamma/P}}) \subseteq \text{Sp}(H_{\lambda_{\Gamma}}) \subseteq [-1, 1]$$

(4.5)

We now relate the fixed-point character to the moments of the KNS spectral measure $\mu$ associated to $\text{Sch}(\Gamma, P, S)$ with respect to the approximating sequence $\text{Sch}(\Gamma, P_k, S)$. If $w \in S^*$, we use $[w]$ to denote the image of $w$ in $G$. Notice that, for $m \geq 0$,

$$M^m = \frac{1}{|S|^m} \sum_{w \in S^m} [w].$$

(4.6)

The moments of $\mu_k$ (see definition (2.6)) are then given by:

$$\mu_k^{(m)} = E_{\mu_k}[\lambda^m] = \sum_{\lambda \in \text{Sp}(M_k)} \lambda^m \frac{\#(\lambda)}{|n_k^{\lambda_k}[w]} = \frac{1}{n_k} \text{tr}(M_k^m) = \frac{1}{|S|^m} \sum_{w \in S^m} \frac{1}{n_k} \chi_{\pi_k}([w])$$

where the last equality holds from (4.6) and the fact that $M_k = \pi_k(M)$. Notice $0 \leq \mu_k^{(m)} \leq 1$, indeed it is the average over all $w \in S^m$ of the probability of $\pi_k([w])$ fixing a vertex on the $k^{th}$ level. Thus the moment generating function of $\mu_k$ is analytic on $\mathbb{R}$ for all $k \geq 0$. Since the $\mu_k$ converge weakly to $\mu$, it follows that, for each $m \geq 0$, $\mu_k^{(m)} \to \mu^{(m)}$ [5]. We
are then led by (4.4) to the following formula for the moments of $\mu$:

$$\mu^{(m)} = \frac{1}{|S|^m} \sum_{w \in S^m} \chi_\pi([w]).$$

(4.7)

Thus the $m$th moment of $\mu$ is the average over all $w \in S^m$ of the probability of $\pi([w])$ fixing an infinite path, whence $0 \leq \mu^{(m)} \leq 1$. This implies that the moment generating function of $\mu$ is analytic on $\mathbb{R}$.

Let us contrast this with the situation for the Kesten spectral measure $\mu^\Gamma$ for the random walk on the Cayley graph of $\Gamma$. As mentioned earlier, the moments correspond to return probabilities. More precisely, if the Markov operator $M$ on $\ell^2(\Gamma)$ has spectral decomposition

$$M = \int_{-1}^1 \lambda dE(\lambda)$$

with (projection-valued) spectral measure $E$, then

$$\mu^\Gamma(B) = \langle \int_B dE(\lambda)\delta_1, \delta_1 \rangle$$

for $B$ a Borel subset of $[-1, 1]$. So the $m$th moment is given by

$$\langle \mu^\Gamma \rangle^{(m)} = \int_{-1}^1 \lambda^m d\mu^\Gamma = \langle \int_{-1}^1 \lambda^m dE(\lambda)\delta_1, \delta_1 \rangle$$

$$= \langle M^m \delta_1, \delta_1 \rangle = \text{tr}(M^m)$$

where the last trace is the von Neumann trace, c.f. (2.5). Since $M^m \in \mathbb{R} \Gamma$, this is just the coefficient of 1. Notice that the return probability $p_m(1)$ is just the fraction of words in $S^m$ representing the identity 1 of $\Gamma$. It follows from (4.6) that $\langle \mu^\Gamma \rangle^{(m)} = p_m(1)$. From this, we may easily deduce that the moment generating function of $\mu^\Gamma$ is analytic on all of $\mathbb{R}$.

Now we are in a position to compare $\mu^{(m)}$ with $\langle \mu^\Gamma \rangle^{(m)}$. Recalling that $0 \leq \chi_\pi(\gamma) \leq 1$ and $\chi_\pi(1) = 1$, it is clear that the right hand side of (4.7) is $p_m(1)$ precisely if, for each $w \in S^m$ such that $[w] \neq 1$, $\chi_\pi([w]) = 0$. In other words, the average probability that a word in $w \in S^n$ of length $m$ fixes an infinite path will be the same as the probability that $w$ represents the identity if and only if for all $w \in A^*$, either $w$ represents the identity, or $w$ almost surely does not fix any infinite path. Recall [5, Theorem 30.1] that if two probability measures on $[-1, 1]$ have the same moment generating function, and this function is analytic on a neighbourhood of 0, then the measures are the same. Recalling that $\chi_\pi(\gamma) = \nu(\text{Fix}(\gamma))$, we may summarize the previous discussion in the following theorem.

**Theorem 4.4.** Let $\Gamma$ be a group acting spherically transitively on the Cayley tree of $A^*$ with finite symmetric generating set $S$. Let $w \in A^\omega$ be an infinite path and, for all $k \geq 0$, let $w_k$ be the prefix of $w$ of length $k$. Set $P = \text{St}_\Gamma(w)$ and $P_k = \text{St}_\Gamma(w_k)$ and let $X$ and $X_k$ be the respective Schreier graphs with respect to a finite generating set $S$. Then the following are equivalent:
The KNS spectral measure for $X$, with respect to the approximating sequence $\{X_k\}$, coincides with the Kesten spectral measure for the simple random walk on the Cayley graph of $\Gamma$.

(2) The action of $\Gamma$ is free in the sense of ergodic theory.

An immediate consequence of Theorems 4.3 and 4.4 is:

**Corollary 4.5.** Let $\Gamma$ be an automata group acting freely on the boundary of $T$. Choose $w \in \partial T$ with trivial stabilizer and define $P_k$, $k \geq 0$, as per Theorem 4.4. Then $\zeta_\Gamma$ coincides with the limit zeta function from the approximating sequence $\text{Sch}(\Gamma, P_k, S)$.

5. Calculation of the spectral measures for Cayley machines

In this section, we present a computation of the spectra and spectral measures associated to random walks on automata groups generated by the Cayley machines of finite groups. This not only provides an interesting new class of examples of spectra and spectral measures of random walks, but also serves to illustrate the material introduced in the previous sections.

Fix for this section a non-trivial finite group $G = \{g_1, \ldots, g_n\}$. As standing notation we set $\Gamma = G(\mathcal{A}(G))$. It is locally finite-by-infinite cyclic [28] and hence amenable so (4.5) applies to computing the spectrum. In the case that $G$ is abelian, the results of [28] show the action of $\Gamma$ is free.

Let us establish some notation. We use $\overline{g}$ as a shorthand notation for the element $A_{g} \in \Gamma$. Let

$$S = \{\overline{g_1}, \ldots, \overline{g_n}, \overline{g_1^{-1}}, \ldots, \overline{g_n^{-1}}\};$$

(5.1)

so $|S| = 2n$. We fix a parabolic subgroup $P$. If $G$ is abelian, we choose $P$ to be trivial. If $G$ is non-abelian, we can choose $P$ to be a locally finite group [28]. Let $w$ be an infinite path with stabilizer $P$ and for $k \geq 0$ let $w_k$ denote the prefix of $w$ of length $k$. Let $P_k \leq \Gamma$ be the stabilizer of $w_k$. Now $P_k$ has index $n^k$ in $\Gamma$ (by spherical transitivity) and $P = \bigcap P_k$. We shall show that all the spectra considered in equation (4.5) are the entire interval $[-1, 1]$. We shall also calculate the KNS spectral measure associated to the Schreier graph $X = \text{Sch}(\Gamma, P, S)$ with respect to the approximating sequence of graphs $X_k = \text{Sch}(\Gamma, P_k, S)$, in particular obtaining the Kesten spectral measure for $\Gamma$ in the case $G$ is abelian.

5.1. Operator recursion, wreath products and the monomial representation.

We begin by recalling some standard facts about the matrix representation of wreath products of permutations groups [16, 22]. Let $(H, X)$ and $(K, Y)$ be left permutation groups and let $(W, X \times Y) = (H, X) \wr (K, Y)$. The associated monomial representation [16, 22] is described as follows. Let $(h, f) \in W = H \times K^X$. The monomial matrix for $(h, f)$ is obtained from the $|X| \times |X|$ permutation matrix for $h$ by replacing the 1 in column $i \in X$ by $f(i)$. The action on $X \times Y$ is recovered by considering column vectors of size
Let \( T \) given in block matrix form:

\[
\begin{pmatrix}
  \pi & 0 \\
  0 & \pi
\end{pmatrix}
\]

Since all these matrices are permutation matrices, the inverse of a matrix \( T \) is obtained by taking the permutation matrix from the left regular representation of \( G \) corresponding to \( g_i^{-1} \) and replacing the 1 in column \( j \) by \( \pi_k(\gamma_j) \) and the zeroes by the \( n^k \times n^k \) zero matrix. Thus the matrices for \( (W, X \times Y) \) are block monomial.

Observe that the wreath product coordinates (3.6), restricted to \( T_{k+1} \), show

\[
(\pi_k(\Gamma), G^{k+1}) \leq (G, G) \ast (\pi_k(\Gamma), G^k).
\]

On the automata generators, the embedding is given by

\[
\pi_{k+1}(\gamma_i) \mapsto g_i^{-1}(\pi_k(\gamma_1), \ldots, \pi_k(\gamma_n))
\]

This lets us construct inductively the matrices for the representations \( \pi_k \) using the monomial representation; this procedure is called operator recursion [13]. For \( i = 1, \ldots, n \), set \( \pi_0(\gamma_i) = 1 \). Then, for \( k \geq 0 \), the matrix for \( \pi_{k+1}(\gamma_i) \) is obtained by taking the permutation matrix from the left regular representation of \( G \) corresponding to \( g_i^{-1} \) and replacing the 1 in column \( j \) by \( \pi_k(\gamma_j) \) and the zeroes by the \( n^k \times n^k \) zero matrix. So \( \pi_{k+1}(\gamma_i) \) is an \( n \times n \) block monomial matrix with blocks of size \( n^k \). There is exactly one non-zero block in column \( j \), namely in row \( l \), where \( g_i^{-1} g_j = g_l \); this block is \( \pi_k(\gamma_j) \).

Since all these matrices are permutation matrices, the inverse of a matrix \( \pi_{k+1}(\gamma_i) \) is simply the transpose \( \pi_{k+1}(\gamma_i)^T \).

For example, if \( G = \mathbb{Z}/2\mathbb{Z} = \{a, b\} \) where \( a \) is the identity and \( b \) the non-trivial element, then (c.f. [13])

\[
\pi_{k+1}(\alpha) = \begin{pmatrix} \pi_k(\alpha) & 0 \\ 0 & \pi_k(\beta) \end{pmatrix}, \quad \pi_{k+1}(\beta) = \begin{pmatrix} 0 & \pi_k(\beta) \\ \pi_k(\alpha) & 0 \end{pmatrix}.
\]

If \( G = \mathbb{Z}/3\mathbb{Z} = \{a, b, c\} \) where \( a \) is the identity, then

\[
\pi_{k+1}(\alpha) = \begin{pmatrix} \pi_k(\alpha) & 0 & 0 \\ 0 & \pi_k(\beta) & 0 \\ 0 & 0 & \pi_k(\gamma) \end{pmatrix}, \quad \pi_{k+1}(\beta) = \begin{pmatrix} 0 & \pi_k(\beta) & 0 \\ 0 & 0 & \pi_k(\gamma) \\ \pi_k(\alpha) & 0 & 0 \end{pmatrix}.
\]

\[
\pi_{k+1}(\gamma) = \begin{pmatrix} 0 & 0 & \pi_k(\gamma) \\ \pi_k(\alpha) & 0 & 0 \\ 0 & \pi_k(\beta) & 0 \end{pmatrix}.
\]

Given square matrices \( A \) and \( B \) we write \( A \otimes B \) for their tensor product, given in block matrix form:

\[
A \otimes B = [A_{ij}B_{k,l}]_{i,j,k,l=1}^n.
\]

Let \( T \) be the \( n \times n \) matrix defined by

\[
T_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}
\]
and set
\[ S_0 = n - 1, \quad S_k = T \otimes I_n^k, \quad k > 0. \]

Notice that \( T \) is the sum of the permutation matrices corresponding to the non-identity elements of \( G \) under the left regular representation; indeed for each \( i \neq j \) there is a unique non-identity permutation of \( G \) taking \( i \) to \( j \).

We shall need the following lemma later.

**Lemma 5.1.** For all \( k \in \mathbb{N} \), we have
\[
\sum_{i,j=1, i \neq j}^{n} \pi_k(\overline{g}_i) \pi_k(\overline{g}_j)^T = nS_k
\]

*Proof.* For \( k = 0 \) this is clear, since, for each \( i = 1, \ldots, n \), we add up 1 exactly \( n - 1 \) times. For \( k > 0 \), observe that (5.2) easily implies that in wreath product coordinates
\[
\pi_k(\overline{g}_i) \pi_k(\overline{g}_j)^{-1} = g_i^{-1} g_j(1, \ldots, 1).
\]
So in matrix form, we obtain \( A \otimes I_n^k \) where \( A \) is the matrix for \( g_i^{-1} g_j \) in the regular representation. If we fix \( i \) and sum over \( j \neq i \) we get the sum of all permutation matrices from the regular representation of \( G \) except the identity, that is, \( T \), tensored with \( I_n^k \). But this is precisely \( S_k \). Since we have \( n \) choices for \( i \), we obtain the formula of the lemma. \( \square \)

### 5.2. Calculation of the characteristic polynomial.

It will be convenient to not have to always divide by \( 2n \) so set \( A_k = 2n M_k \); \( A_k \) is the incidence matrix of the Schreier graph \( \text{Sch}(\Gamma, P_k, S) \). More explicitly,
\[
A_k = \sum_{i=1}^{n} (\pi_k(\overline{g}_i) + \pi_k(\overline{g}_i)^T).
\]

Our first objective is to calculate the spectrum of the matrix \( A_k \). To this end, define a function of two variables by
\[
\Phi_k(\lambda, \mu) = |A_k - \lambda I_n^k - \mu S_k|
\]
so that \( \Phi_k(\lambda, 0) \) is the characteristic polynomial of \( A_k \). Our objective, then, is to find the roots of \( \Phi_k(\lambda, 0) \). To facilitate this, we shall obtain a recursive formula for \( \Phi_{k+1} \) in terms of \( \Phi_k \). Of course
\[
\Phi_0(\lambda, \mu) = 2n - \lambda - (n - 1)\mu \quad (5.3)
\]
The additional term \( \mu S_k \) serves as a garbage collecting term; it arises when one tries to express the characteristic polynomial of \( A_{k+1} \) in terms of the characteristic polynomial of \( A_k \).

The matrices \( \pi_k(\overline{g}_i) \) and \( \pi_k(\overline{g}_i)^T, \ i = 1, \ldots, n \), are block monomial matrices coming from the image of \( g_i^{-1} \), respectively, \( g_i \), under the left regular representation of \( G \). Since the sum of the permutation matrices from the left regular representation of \( G \) is the matrix of all ones (see the discussion above concerning \( T \), but now add in the identity matrix), we see that
and so \( A_{k+1} - \lambda I_n + \mu S_{k+1} \) is given by
\[
\begin{pmatrix}
\pi_k(\mathcal{G}_1) + \pi_k(\mathcal{G}_1)^T - \lambda I & \pi_k(\mathcal{G}_2) + \pi_k(\mathcal{G}_1)^T - \mu I & \ldots & \pi_k(\mathcal{G}_n) + \pi_k(\mathcal{G}_1)^T - \mu I \\
\pi_k(\mathcal{G}_1) + \pi_k(\mathcal{G}_2)^T - \mu I & \pi_k(\mathcal{G}_2) + \pi_k(\mathcal{G}_2)^T - \lambda I & \ldots & \pi_k(\mathcal{G}_n) + \pi_k(\mathcal{G}_2)^T - \lambda I \\
\vdots & \vdots & \ddots & \vdots \\
\pi_k(\mathcal{G}_1) + \pi_k(\mathcal{G}_n)^T - \mu I & \pi_k(\mathcal{G}_2) + \pi_k(\mathcal{G}_n)^T - \mu I & \ldots & \pi_k(\mathcal{G}_n) + \pi_k(\mathcal{G}_n)^T - \lambda I \\
\end{pmatrix}
\]
where \( I \) denotes the \( n \times n \) identity matrix.

We now apply some row and column operations at the block level, designed to simplify the computation of the determinant. Applying the operation \( C_i \mapsto C_i - C_n, \) for \( i = 1, \ldots, n - 1, \) yields the matrix
\[
\begin{pmatrix}
\pi_k(\mathcal{G}_1) - \pi_k(\mathcal{G}_1) - (\lambda - \mu)I & \pi_k(\mathcal{G}_2) - \pi_k(\mathcal{G}_1) - (\lambda - \mu)I & \ldots & \pi_k(\mathcal{G}_n) - \pi_k(\mathcal{G}_1) - (\lambda - \mu)I \\
\pi_k(\mathcal{G}_1) - \pi_k(\mathcal{G}_2) + (\lambda - \mu)I & \pi_k(\mathcal{G}_2) - \pi_k(\mathcal{G}_2) + (\lambda - \mu)I & \ldots & \pi_k(\mathcal{G}_n) - \pi_k(\mathcal{G}_2) + (\lambda - \mu)I \\
\vdots & \vdots & \ddots & \vdots \\
\pi_k(\mathcal{G}_1) - \pi_k(\mathcal{G}_n) + (\lambda - \mu)I & \pi_k(\mathcal{G}_2) - \pi_k(\mathcal{G}_n) + (\lambda - \mu)I & \ldots & \pi_k(\mathcal{G}_n) - \pi_k(\mathcal{G}_n) + (\lambda - \mu)I \\
\end{pmatrix}
\]

Applying the operation \( R_i \mapsto R_i - R_n, \) for \( i = 1, \ldots, n - 1, \) we obtain the matrix (*):
\[
\begin{pmatrix}
-2(\lambda - \mu)I & -2(\lambda - \mu)I & \ldots & -2(\lambda - \mu)I \\
-(\lambda - \mu)I & -2(\lambda - \mu)I & \ldots & -2(\lambda - \mu)I \\
\vdots & \vdots & \ddots & \vdots \\
-(\lambda - \mu)I & -(\lambda - \mu)I & \ldots & -(\lambda - \mu)I \\
\end{pmatrix}
\]

To calculate the determinant of this matrix, we need some technical results.

**Lemma 5.2.** Suppose that we have a block matrix
\[
A = \begin{pmatrix}
A_{11} & 0 & \ldots & 0 & A_{1n} \\
0 & A_{22} & 0 & \ldots & A_{2n} \\
0 & 0 & \ddots & 0 & \vdots \\
0 & 0 & \cdots & A_{n-1,n-1} & A_{n-1,n} \\
A_{n1} & A_{n2} & \ldots & A_{n,n-1} & A_{nn}
\end{pmatrix}
\]
where the \( A_{ij} \) are square matrices of the same size. Moreover, suppose that \( A_{11}, A_{22}, \ldots, A_{n-1,n-1} \) commute with all the other matrices. Then
\[
|A| = \left| A_{11} \cdots A_{nn} - \sum_{i=1}^{n-1} A_{1i} \cdots \widehat{A}_{ii} \cdots A_{n-1,n-1} A_{ni} A_{in} \right| \quad (5.4)
\]
where \( \widehat{A}_{ii} \) means omit \( A_{ii} \).
Proof. Since the invertible matrices are dense in the space of matrices, we may assume without loss of generality that $A_{ii}$ is invertible, $i = 1, \ldots, n - 1$. Let $I$ be the identity matrix of the same size as the $A_{ij}$. Then one verifies directly that

$$A = \begin{pmatrix} A_{11} & 0 & \ldots & 0 & 0 \\ 0 & A_{22} & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & A_{n-1,n-1} & 0 \\ A_{n1} & A_{n2} & \ldots & A_{n,n-1} & I \end{pmatrix} \times \begin{pmatrix} I & 0 & \ldots & 0 & A_{11^{-1}}A_{1n} \\ 0 & I & 0 & \vdots & A_{22^{-1}}A_{2n} \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & I & A_{n-1,n-1^{-1}}A_{n-1,n} \\ 0 & 0 & \ldots & 0 & A_{nn} - \sum_{i=1}^{n-1} A_{ni}A_{ii^{-1}}A_{in} \end{pmatrix} \times \begin{pmatrix} A_{11} & 0 & \ldots & 0 & 0 \\ 0 & A_{22} & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & A_{n-1,n-1} & 0 \\ A_{n1} & A_{n2} & \ldots & A_{n,n-1} & I \end{pmatrix}$$

Using that the determinant of a block upper (lower) triangular matrix is the product of the determinant of the diagonal blocks and that $A_{ii}$, $i = 1, \ldots, n - 1$, commutes with the remaining matrices gives (5.4). □

Corollary 5.3. Consider a block matrix

$$M = \begin{pmatrix} 2A & A & \ldots & A & B_{1n} \\ A & 2A & \ldots & A & B_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & \ldots & 2A & B_{n-1,n} \\ B_{n1} & B_{n2} & \ldots & B_{n,n-1} & B_{nn} \end{pmatrix}$$

where $A$ is a square matrix and $B_{1n}, \ldots, B_{nn}, B_{n1}, \ldots, B_{nn}$ are square matrices of the same size as $A$, commuting with $A$. Then

$$|M| = \left| A^{n-2} \left( nAB_{nn} - (n - 1) \sum_{i=1}^{n-1} B_{ni}B_{in} + \sum_{i \neq j}^{n-1} B_{ni}B_{jn} \right) \right| .$$

Proof. We proceed by applying the following elementary row and column operations to the rows and columns of blocks in $M$.

(i) $C_i \mapsto C_i - C_{i+1}$ for $i = 1, \ldots, n - 2$;
(ii) $R_i \mapsto \sum_{j=1}^{i} R_j$ for $i = n - 1, \ldots, 2$;
(iii) $C_{n-1} \mapsto C_{n-1} - \sum_{j=1}^{n-2} jC_j$. 

These operations leave the determinant unchanged and it is easy to verify that they result in the matrix:

$$
\begin{pmatrix}
A & 0 & \ldots & 0 & 0 & B_{1n} \\
0 & A & \ldots & 0 & 0 & B_{1n} + B_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & A & 0 & \sum_{j=1}^{n-2} B_{jn} \\
0 & 0 & \ldots & 0 & 0 & \sum_{j=1}^{n-2} B_{jn} \\
B_{n1} - B_{n2} & B_{n2} - B_{n3} & \ldots & B_{n,n-2} - B_{n,n-1} & (n-1)B_{n,n-1} - \sum_{j=1}^{n-2} B_{nj} & B_{nn}
\end{pmatrix}
$$

Since this matrix has the same determinant as $M$, applying Lemma 5.2 gives us that

$$
|M| = nA^{n-1}B_{nn} - nA^{n-2} \sum_{i=1}^{n-2} \left( [B_{ni} - B_{n,i+1}] \sum_{j=1}^{i} B_{jn} \right)
$$

$$
- A^{n-2} \left( (n-1)B_{n,n-1} - \sum_{i=1}^{n-2} B_{ni} \right) \left( \sum_{j=1}^{n-1} B_{jn} \right).
$$

(5.5)

By telescoping we obtain

$$
\sum_{i=1}^{n-2} \left( [B_{ni} - B_{n,i+1}] \sum_{j=1}^{i} B_{jn} \right) = \sum_{i=1}^{n-2} B_{ni}B_{in} - B_{n,n-1} \sum_{i=1}^{n-2} B_{in}.
$$

(5.6)

Substituting (5.6) into (5.5) gives

$$
|M| = A^{n-2} \left( nAB_{nn} - n \sum_{i=1}^{n-2} B_{ni}B_{in} + nB_{n,n-1} \sum_{i=1}^{n-2} B_{in} 
\right)
$$

$$
- (n-1)B_{n,n-1} \sum_{j=1}^{n-1} B_{jn} + \sum_{i=1}^{n-2} B_{ni}B_{jn} 
$$

$$
= A^{n-2} \left( nAB_{nn} - n \sum_{i=1}^{n-2} B_{ni}B_{in} + nB_{n,n-1} \sum_{i=1}^{n-2} B_{in} 
\right)
$$

$$
- (n-1)B_{n,n-1} \sum_{j=1}^{n-1} B_{jn} + \sum_{i=1}^{n-2} B_{ni}B_{jn} 
$$

$$
= A^{n-2} \left( nAB_{nn} - n \sum_{i=1}^{n-1} B_{ni}B_{in} + \sum_{i=1,j=1}^{n-1} B_{ni}B_{jn} 
\right)
$$

$$
= A^{n-2} \left( nAB_{nn} - (n-1) \sum_{i=1}^{n-1} B_{ni}B_{in} + \sum_{i\neq j}^{n-1} B_{ni}B_{jn} 
\right)
$$

as desired. □
Corollary 5.3 gives us a method of calculating the determinant of \((*)\). To apply it we set
\[
A = (\mu - \lambda)I
\]
\[
B_{ni} = \pi_k(\mathbf{g}_i) - \pi_k(\mathbf{g}_n) - (\mu - \lambda)I, \text{ } i = 1, \ldots, n-1
\]
\[
B_{jn} = \pi_k(\mathbf{g}_j)^T - \pi_k(\mathbf{g}_n)^T - (\mu - \lambda)I, \text{ } j = 1, \ldots, n-1
\]
\[
B_{nn} = \pi_k(\mathbf{g}_n) + \pi_k(\mathbf{g}_n)^T - \lambda I
\]
From this we obtain for \(i, j = 1, \ldots, n - 1\)
\[
B_{ni}B_{jn} = \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_j)^T - \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_n)^T - \pi_k(\mathbf{g}_n)\pi_k(\mathbf{g}_j)^T + I
\]
\[
+ (\mu - \lambda)^2 I - (\mu - \lambda) \left( \pi_k(\mathbf{g}_i) + \pi_k(\mathbf{g}_j) \right) - \pi_k(\mathbf{g}_n) - \pi_k(\mathbf{g}_n)^T)
\]
Notice that if \(i = j\), then the first term becomes \(I\).
Substituting the above values into the formula from Corollary 5.3 gives
\[
\Phi_{k+1}(\lambda, \mu) = (\mu - \lambda)^{(n-2)n^k} |n(\mu - \lambda)(\pi_k(\mathbf{g}_n) + \pi_k(\mathbf{g}_n)^T) - \lambda I|
\]
\[
-(n - 1) \sum_{i=1}^{n-1} \left[ 2I - \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_n)^T - \pi_k(\mathbf{g}_n)\pi_k(\mathbf{g}_i)^T + (\lambda - \mu)^2I
\]
\[
- (\mu - \lambda) \left( \pi_k(\mathbf{g}_i) + \pi_k(\mathbf{g}_i) \right) T - \pi_k(\mathbf{g}_n) - \pi_k(\mathbf{g}_n)^T])
\]
\[
+ \sum_{i \neq j} \left[ \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_j)^T - \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_n)^T - \pi_k(\mathbf{g}_n)\pi_k(\mathbf{g}_j)^T + I
\]
\[
+ (\mu - \lambda)^2 I - (\mu - \lambda) \left( \pi_k(\mathbf{g}_i) + \pi_k(\mathbf{g}_j) \right) T - \pi_k(\mathbf{g}_n) - \pi_k(\mathbf{g}_n)^T]]
\]
Judicious rearranging of the various terms shows that the above summation equals:
\[
(\mu - \lambda)^{(n-2)n^k} |n(\mu - \lambda)(\pi_k(\mathbf{g}_n) + \pi_k(\mathbf{g}_n)^T) - \lambda I|
\]
\[
- 2(n - 1)^2 I + (n - 1) \sum_{i=1}^{n-1} \left( \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_n)^T + \pi_k(\mathbf{g}_n)\pi_k(\mathbf{g}_i)^T + (\mu - \lambda)^2I
\]
\[
- (n - 1)^2(\mu - \lambda)^2 I + (n - 1)(\mu - \lambda) \sum_{i=1}^{n-1} \left( \pi_k(\mathbf{g}_i) \right) + \pi_k(\mathbf{g}_n) \right)
\]
\[
- (n - 1)^2(\mu - \lambda)(\pi_k(\mathbf{g}_n) + \pi_k(\mathbf{g}_n)^T) + \sum_{i \neq j} \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_j)^T
\]
\[
- (n - 2) \sum_{i=1}^{n-1} \left( \pi_k(\mathbf{g}_i)\pi_k(\mathbf{g}_n)^T + \pi_k(\mathbf{g}_n)\pi_k(\mathbf{g}_i)^T + (\mu - \lambda)^2I
\]
\[
+ (n - 1)(n - 2) I + (n - 1)(n - 2)(\mu - \lambda)^2 I
\]
Further rearrangement of the terms in attempt to obtain an expression involving $A_k$ shows that the above summation equals:

$$(\mu - \lambda)(n-2)n^k \times$$

$$|\mu - \lambda)(A_k - \lambda I - (n - 1)\mu I) - n(n-1)I + \sum_{i \neq j}^{n} \pi_k(g_i)\pi_k(g_j)^T|$$

Applying Lemma 5.1 we obtain

$$\Phi_{k+1}(\lambda, \mu)$$

$$= (\mu - \lambda)(n-2)n^k |(\mu - \lambda)(A_k - \lambda I - (n - 1)\mu I)$$

$$- n(n-1)I + nS_k|$$

$$= (\mu - \lambda)(n-1)n^k |A_k - \left(\lambda + (n - 1)\mu + \frac{n(n-1)}{\mu - \lambda}\right) I$$

$$+ \frac{n}{\mu - \lambda} S_k|$$

$$= (\mu - \lambda)(n-1)n^k$$

$$\Phi_k \left(\lambda + (n - 1)\mu + \frac{n(n-1)}{\mu - \lambda}, -\frac{n}{\mu - \lambda}\right)$$

$$= (\mu - \lambda)(n-1)n^k$$

$$\times$$

$$\Phi_k \left(-\frac{\lambda^2 + (n - 1)\mu^2 + (2 - n)\lambda\mu + n(n-1)}{\mu - \lambda}, -\frac{n}{\mu - \lambda}\right)$$

Next, we seek to solve the recursion and obtain an explicit formula.

**5.3. Calculation of the eigenvalues.** Given $\lambda, \mu \in \mathbb{C}$, we write

$$\lambda' = \frac{-\lambda^2 + (n - 1)\mu^2 + (2 - n)\lambda\mu + n(n-1)}{\mu - \lambda}$$

and

$$\mu' = -\frac{n}{\mu - \lambda}$$

We define a sequence of functions $F_k(\lambda, \mu)$ inductively by

$$F_1(\lambda, \mu) = \mu - \lambda$$

and for $k \geq 1$

$$F_{k+1}(\lambda, \mu) = F_k(\lambda', \mu').$$

**Lemma 5.4.** Define a sequence by $(\lambda_1, \mu_1) = (\lambda, \mu)$ and $(\lambda_{k+1}, \mu_{k+1}) = (\lambda'_k, \mu'_k)$ for $k \geq 1$. Then for any $k \geq 1$ we have

(i) $\lambda_{k+1} + (n-1)\mu_{k+1} = \lambda_k + (n-1)\mu_k$;
\[\begin{align*}
(ii) \quad \mu_{k+1} - \lambda_{k+1} &= -(\lambda + (n-1)\mu) - \frac{n^2}{\mu_k - \lambda_k}; \\
(iii) \quad F_k(\lambda, \mu) &= \mu_k - \lambda_k; \text{ and} \\
(iv) \quad F_{k+1}(\lambda, \mu) &= -(\lambda + (n-1)\mu) - \frac{n^2}{F_k(\lambda, \mu)}.
\end{align*}\]

**Proof.** It is a straightforward computation to verify claims (i) and (ii) using the definitions of \(\lambda_{k+1}\) and \(\mu_{k+1}\), while claims (iii) and (iv) are immediate consequences of (ii), together with the definitions. \(\square\)

We remark that the rational function \(f : \mathbb{R}^2 \to \mathbb{R}^2\) given by \(f(\lambda, \mu) = (\lambda', \mu')\) is integrable in the sense of [15]. Namely, if \(\psi : \mathbb{R}^2 \to \mathbb{R}\) is given by \(\psi(\lambda, \mu) = \lambda + (n-1)\mu\) and \(\alpha : \mathbb{R} \to \mathbb{R}\) is the identity, then the previous lemma implies \(\alpha \psi = \psi f\).

**Lemma 5.5.** For all \(\mu, \lambda\) and all \(k \geq 0\)

\[\Phi_k(\lambda, \mu) = (2n - \lambda - (n-1)\mu) \prod_{i=1}^{k} (F_i(\lambda, \mu))^{(n-1)n^{k-i}}\]

**Proof.** For \(k = 0\), this is clear from (5.3). Now suppose the lemma holds for \(k \geq 0\). Then by (5.7),

\[\Phi_{k+1}(\lambda, \mu) = (\mu - \lambda)^{(n-1)n^k} \Phi_k(\lambda', \mu')\]

\[= (\mu - \lambda)^{(n-1)n^k} (2n - \lambda' - (n-1)\mu') \prod_{i=1}^{k} (F_i(\lambda', \mu'))^{(n-1)n^{k-i}}\]

\[= (\mu - \lambda)^{(n-1)n^k} (2n - \lambda - (n-1)\mu) \prod_{i=1}^{k} (F_{i+1}(\lambda, \mu))^{(n-1)n^{k-i}}\]

(by Lemma 5.4(i))

\[= (2n - \lambda - (n-1)\mu) (F_1(\lambda, \mu))^{(n-1)n^k} \prod_{i=2}^{k+1} (F_i(\lambda, \mu))^{(n-1)n^{k+1-i}}\]

establishing the lemma. \(\square\)

We now want to express each \(F_k\) as a rational function \(P_k/Q_k\) so that we can compute our determinant. We define inductively polynomials

\[P_1(\lambda, \mu) = \mu - \lambda, \quad P_{k+1}(\lambda, \mu) = -(\lambda + (n-1)\mu) P_k(\lambda, \mu) - n^2 Q_k(\lambda, \mu)\]

\[Q_1(\lambda, \mu) = 1, \quad Q_{k+1}(\lambda, \mu) = P_k(\lambda, \mu).
\]

**Lemma 5.6.** For \(k \geq 1\) we have

\[F_k(\lambda, \mu) = \frac{P_k(\lambda, \mu)}{Q_k(\lambda, \mu)}.
\]

**Proof.** For \(k = 1\) the result is clear. Now let \(k \geq 1\) and assume by induction that

\[F_k(\lambda, \mu) = \frac{P_k(\lambda, \mu)}{Q_k(\lambda, \mu)}.
\]
Applying this decomposition to solve the above recursion we obtain

\[ F_{k+1}(\lambda, \mu) = -(\lambda + (n-1)\mu) - \frac{n^2}{F_k(\lambda, \mu)} \]

so we obtain

\[
F_{k+1}(\lambda, \mu) = -(\lambda + (n-1)\mu) - \frac{n^2Q_k(\lambda, \mu)}{P_k(\lambda, \mu)} \\
= -\frac{(\lambda + (n-1)\mu)P_k(\lambda, \mu) - n^2Q_k(\lambda, \mu)}{P_k(\lambda, \mu)} \\
= \frac{P_{k+1}(\lambda, \mu)}{Q_{k+1}(\lambda, \mu)},
\]

as required.

We are primarily interested in the case \( \mu = 0 \), so set \( P_k(\lambda) = P_k(\lambda, 0) \) and \( Q_k(\lambda) = Q_k(\lambda, 0) \). Now \( P_k(\lambda) \) and \( Q_k(\lambda) \) satisfy:

\[
\begin{pmatrix}
P_k(\lambda) \\
Q_k(\lambda)
\end{pmatrix} = \begin{pmatrix}
-\lambda & -n^2 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
P_{k-1}(\lambda) \\
Q_{k-1}(\lambda)
\end{pmatrix} = \begin{pmatrix}
-\lambda & -n^2 \\
1 & 0
\end{pmatrix}^k \begin{pmatrix}
1 \\
0
\end{pmatrix}.
\]

Calculating the eigenvalues and eigenvectors and then diagonalizing, we obtain:

\[
\begin{pmatrix}
-\lambda & -n^2 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{-2n^2}{\lambda + \sqrt{\lambda^2 - 4n^2}} & \frac{-2n^2}{\lambda - \sqrt{\lambda^2 - 4n^2}} \\
\frac{-1}{\lambda + \sqrt{\lambda^2 - 4n^2}} & \frac{\lambda}{\lambda + \sqrt{\lambda^2 - 4n^2}}
\end{pmatrix} \begin{pmatrix}
\frac{-2n^2}{\lambda - \sqrt{\lambda^2 - 4n^2}} \\
0
\end{pmatrix} \begin{pmatrix}
\frac{-2n^2}{\lambda + \sqrt{\lambda^2 - 4n^2}} \\
0
\end{pmatrix}
\]

where the last matrix on the right hand side is the inverse of the first.

We now make a change of variables by setting \( \lambda = 2n \cos z \) for \( z \in [0, \pi] \). This change of variables gives a bijection between \([0, \pi]\) and \([-2n, 2n]\). Since \( \|A_k\| \leq 2n \), all our eigenvalues belong to \([-2n, 2n]\) and so we can use this change of variables to compute the eigenvalues. Then (5.8) becomes

\[
\begin{pmatrix}
-2n \cos z & n^2 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{n}{\cos z + i \sin z} & \frac{n}{\cos z - i \sin z} \\
\frac{-\cos z + i \sin z}{\cos z - i \sin z} & \frac{\cos z - i \sin z}{\cos z + i \sin z}
\end{pmatrix} \times
\begin{pmatrix}
\frac{1}{\cos z} & 0 \\
\frac{-\cos z + i \sin z}{\cos z - i \sin z} & \frac{\cos z - i \sin z}{\cos z + i \sin z}
\end{pmatrix}.
\]

Applying this decomposition to solve the above recursion we obtain

\[
\begin{pmatrix}
P_k(2n \cos z) \\
Q_k(2n \cos z)
\end{pmatrix} = \begin{pmatrix}
\frac{-n}{e^{iz}} & \frac{n}{e^{iz}} \\
1 & 1
\end{pmatrix} \begin{pmatrix}
\frac{e^{iz}}{n^2} & \frac{e^{-iz}}{n^2} \\
0 & 0
\end{pmatrix} \times
\begin{pmatrix}
\frac{-1}{2ni \sin z} & \frac{n \cos z}{2n i \sin z} \\
\frac{1}{2ni \sin z} & \frac{n \cos z}{2n i \sin z}
\end{pmatrix} \times
\begin{pmatrix}
\frac{1}{2ni \sin z} & \frac{1}{2ni \sin z} \\
\frac{-\cos z}{2n i \sin z} & \frac{\cos z}{2n i \sin z}
\end{pmatrix} \begin{pmatrix}
1 \\
0
\end{pmatrix}.
\]
\[
\begin{align*}
\frac{1}{2n \sin z} \left( (-ne^{iz})^{k+1} - (-ne^{-iz})^{k+1} \right) &
\frac{1}{2n \sin z} \left( (-ne^{iz})^k - (-ne^{-iz})^k \right) \\
\end{align*}
\]

Thus, we have
\[
F_k(\lambda, 0) = \frac{P_k(\lambda)}{Q_k(\lambda)} = -n \left( \frac{e^{iz(k+1)} - e^{-iz(k+1)}}{e^{izk} - e^{-izk}} \right) = -n \left( \frac{\sin(z(k+1))}{\sin(zk)} \right).
\]

It now follows from Lemma 5.5 that
\[
\Phi_k(\lambda, 0) = \Phi_k(2n \cos z, 0) = \left( 2n - 2n \cos z \right) \prod_{j=1}^{k} \left[ -n \left( \frac{\sin(z(j+1))}{\sin(zj)} \right) \right]^{(n-1)n^{k-j}}
\]
\[
= 2n(1 - \cos z)(-n)^{(n-1)n^{k-1}} \prod_{j=2}^{k} \left( \frac{1}{\sin z} \right)^{n^{k-j}} \left( \sin(z(k+1)) \right)^{(n-1)n^{k-1}}
\]

Thus, \( \Phi_k(2n \cos z, 0) = 0 \) if and only if either \( \cos z = 1 \) or \( \sin(zj) = 0 \) for some \( j \in \{2, \ldots, k+1\} \). That is, if and only if either \( z = 0 \) or \( z = l\pi \) for some \( j \in \{2, \ldots, k+1\} \) and integer \( l \). That is, if and only if \( z = 0 \) or \( z = \frac{l}{j}\pi \) with \( j \in \{2, \ldots, k+1\} \) and \( 1 \leq l \leq j \). The corresponding values of \( \lambda \) give the set of eigenvalues:

\[
\{2n\} \cup \left\{ 2n \cos \frac{p}{q} \pi \mid q \in \{2, \ldots, k+1\}, 1 \leq p < q \right\}.
\]

Notice that \(-2n\) is not an eigenvalue because the factor \( \left( \frac{1}{\sin z} \right)^{(c-1)n^{k-1}} \) compensates exactly for the remaining factors.

Our next objective is to determine the multiplicities of the eigenvalues. First we determine the multiplicity of \( 2n \) as an eigenvalue. It is a basic result in Perron-Frobenius theory \cite{22, 6} that, for a connected \( 2n \)-regular graph, the multiplicity of \( 2n \) as an eigenvalue of the incidence matrix is 1. Hence, \( 2n \) has multiplicity 1 as an eigenvalue of \( A_k \).

The multiplicities of the remaining eigenvalues can be computed from the formula \(5.9\). Suppose \( p \) and \( q \) are such that \( q \in \{2, \ldots, k+1\}, 1 \leq p < q \) and \( (p, q) = 1 \); we wish to calculate the multiplicity of \( 2n \cos \frac{p}{q} \pi \). If \( q = k+1 \), then only the last term of \(5.9\) contributes to the multiplicity, so we have multiplicity \( n-1 \). Suppose now \( q \in \{2, \ldots, k\} \). Let \( j \in \{2, \ldots, k+1\} \). Then we have \( \sin \frac{pj}{q} \pi = 0 \) if and only if \( q \mid pj \), that is, if and only if \( q \mid j \). Thus,
setting \([r]\) to be the integer part of a real number \(r\) and \(\chi_{\text{Div}(k+1)}\) to be the characteristic function for the set of divisors of \(k + 1\), we obtain that the eigenvalue \(2n \cos \frac{p}{q} \pi\) has multiplicity:

\[
(n - 1)^2 \sum_{i=1}^{\lfloor \frac{k}{q} \rfloor} n^{k-i} + (n - 1) \chi_{\text{Div}(k+1)}(q)
= n^k(n - 1)^2 \left(\frac{1 - n^{-q(\lfloor \frac{k}{q} \rfloor + 1)}}{1 - n^{-q}} - 1\right) + (n - 1) \chi_{\text{Div}(k+1)}(q).
\]

Summing up, we have proven the following:

**Theorem 5.7.** The spectrum of the Markov operator \(M_k\) from the random walk on \(\text{Sch}(\Gamma, P_k, S)\) is:

\[
\{1\} \cup \left\{ \cos \frac{p}{q} \pi \mid q \in \{2, \ldots, k + 1\}, 1 \leq p < q \right\}.
\]

The eigenvalue 1 has multiplicity 1. For 1 \(\leq p < q\) with \(p\) and \(q\) coprime, the multiplicity is

\[
\# \left( \cos \frac{p}{q} \pi \right) = \begin{cases} 
  n - 1 & \text{if } q = k + 1 \\
  n^k(n - 1)^2 \left(\frac{1 - n^{-q(\lfloor \frac{k}{q} \rfloor + 1)}}{1 - n^{-q}} - 1\right) & \text{else.}
\end{cases}
\]

□

It is interesting to note that this result only depends on the size \(n\) of our finite group \(G\) and not on the structure of \(G\). At this point, it would be easy to compute the Ihara zeta function of \(X_k\). We instead wait to compute the zeta function for \(X\).

Let \(\phi\) be the Euler totient function, so that \(\phi(q)\) denotes the number of positive integers less than or equal to \(q\) and coprime to \(q\). Using Theorem 5.7, we obtain a proof of a classical result from number theory [17, Theorem 309]. A probabilistic interpretation can be given to this result from our computation of the KNS spectral measure; see Proposition 5.9 in the next subsection.

**Corollary 5.8.** Let \(n \geq 2\) be an integer. Then

\[
(n - 1)^2 \sum_{q=2}^{\infty} \frac{\phi(q)}{n^q - 1} = 1.
\]

**Proof.** The operator \(M_k\), being symmetric, has \(n^k\) eigenvalues with multiplicity. Using Theorem 5.7 and observing that the multiplicity of \(\cos \frac{p}{q} \pi\),
where $1 \leq p < q$, $(p,q) = 1$, depends only on $q$, we obtain

$$n^k = 1 + n^k(n - 1)^2 \sum_{q=2}^{k} \phi(q) \left( \frac{1 - n^{-q\left(\frac{k}{q}\right)+1}}{1 - n^{-q}} - 1 \right) + (n - 1) \sum_{q|k+1, q \neq 1} \phi(q).$$

Dividing both sides by $n^k$ gives:

$$1 = \frac{1}{n^k} + (n - 1)^2 \sum_{q=2}^{k} \phi(q) \left( \frac{1 - n^{-q\left(\frac{k}{q}\right)+1}}{1 - n^{-q}} - 1 \right) + \frac{(n - 1)}{n^k} \sum_{q|k+1, q \neq 1} \phi(q).$$

Since $\phi(q) \leq q$, and so

$$\sum_{q|k+1, q \neq 1} \phi(q) \leq \frac{(k + 1)(k + 2)}{2} - 1,$$

we see, by taking the limit as $k \to \infty$, that

$$1 = \lim_{k \to \infty} \frac{1}{n^k} + (n - 1)^2 \sum_{q=2}^{k} \phi(q) \left( \frac{1 - n^{-q\left(\frac{k}{q}\right)+1}}{1 - n^{-q}} - 1 \right) + \frac{(n - 1)}{n^k} \sum_{q|k+1, q \neq 1} \phi(q).$$

Thus, to establish the result, we need only to show that the last term vanishes as $k \to \infty$. But

$$\sum_{q=2}^{k} \phi(q) \left( \frac{n^{-q\left(\frac{k}{q}\right)}}{n^q - 1} \right) \leq k \sum_{q=2}^{k} \left( \frac{n^{-q\left(\frac{k}{q}\right)-1}}{n^q - 1} \right) = k \sum_{q=2}^{k} \frac{1}{n^k - n^{k-q}} \leq k(k - 1) \frac{1}{n^{k-2}(n^2 - 1)} \to 0 \text{ as } k \to \infty.$$

\[\square\]

5.4. Calculating the KNS measure. If $\psi$ is a probability measure defined on the Borel subsets of $\mathbb{R}$, then the associated (cumulative) distribution function $\psi \circ F : \mathbb{R} \to [0,1]$ is given by

$$F_\psi(x) = \psi((-\infty, x]).$$

One has that $F_\psi$ is non-decreasing, right continuous and

$$\lim_{x \to -\infty} F_\psi(x) = 0, \quad \lim_{x \to \infty} F_\psi(x) = 1. \quad (5.10)$$

Conversely, if $F : \mathbb{R} \to [0,1]$ is a non-decreasing, right continuous function satisfying $F_\psi$ then there is a unique probability measure $\psi \circ F$ on the Borel sets of $\mathbb{R}$ such that $\psi \circ F((a,b]) = F(b) - F(a)$. The function $F$ can have at most countabley many discontinuity points. If the sum of the jumps
of these points is 1, then $\psi_F$ will be a discrete measure supported at the discontinuity points of $F$ with the weight at a discontinuity point equal to the amount of the jump.

Suppose $\{\psi_k\}$ is a sequence of probability measures on $\mathbb{R}$. Then $\psi_k \to \psi$ weakly if and only if $F_{\psi_n}(x) \to F_{\psi}(x)$ at each point of continuity of $F_\psi$ [5]. Let $\mu_k$ be the measure associated to $M_k$ as per (2.6). The KNS spectral measure $\mu$ is the weak limit of $\mu_k$. We shall compute it by computing its distribution function. To ease notation, we shall perform a change of variables. Let $f : [0, 1] \to [-1, 1]$ be given by $f(x) = \cos \pi x$. Let us define the measures $\sigma_k$, $k \geq 0$, and $\sigma$ on the Borel sets of $[0, 1]$ by

$$\sigma_k(B) = \mu_k(f(B)), \quad \sigma(B) = \mu(f(B)).$$

Since $f$ is a homeomorphism it follows that $\sigma$ and $\mu$ determine each other and that $\sigma_k \to \sigma$ weakly.

To calculate $F_\sigma$, we define a one-parameter family of Euler $\phi$-functions $\{\phi_x : \mathbb{N} \to \mathbb{N}\}_{x \in [0, 1]}$, by

$$\phi_x(q) = \left| \left\{ p \in \mathbb{N} \mid (p, q) = 1 \text{ and } \frac{p}{q} \leq x \right\} \right|.$$

Then, $\phi_0(q) = 0$, $\phi(q_1) = \phi(q)$ and $\phi_x(q)$ is non-decreasing as a function of $x$ for fixed $q \in \mathbb{N}$.

**Proposition 5.9.** For all $x \in [0, 1]$,

$$\lim_{k \to \infty} F_{\sigma_k}(x) = (n - 1)^2 \sum_{q=2}^{\infty} \frac{\phi_x(q)}{n^q - 1}.$$

**Proof.** This proof is exactly like the proof of Corollary 5.8 only in the right hand side of the various equations, the role of $\phi(q)$ is taken by $\phi_x(q)$, while in the left hand side the role of 1 is taken by $F_{\sigma_k}(x)$ before taking limits. The same estimates apply since $\phi_x(q) \leq \phi(q)$. $\square$

Set $F = \lim F_{\sigma_k}$. We know that $F = F_\sigma$ since $\sigma_k \to \sigma$ weakly, but we prefer to verify directly that $F$ is indeed a distribution function, thereby giving a direct proof, independent of [15], that the sequence of measures $\sigma_k$ has a weak limit.

**Proposition 5.10.** $F$ is a probability distribution function defining a discrete measure $\sigma$ supported on the rational points of the interval $(0, 1)$. More precisely,

$$\sigma = (n - 1)^2 \sum_{q=2}^{\infty} \left( \sum_{1 \leq p < q, (p, q) = 1} \frac{1}{n^q - 1} \delta_{\frac{p}{q}} \right)$$

(5.11)

where $\delta_{\frac{p}{q}}$ is a Dirac measure.

**Proof.** Since $\phi_x(q)$ is non-decreasing as a function of $x$ (for $q$ fixed), $F$ is clearly non-decreasing. By Corollary 5.8 $F(1) = 1$, while clearly $F(0) = 0.$
Now we show right continuity. It is immediate from Proposition 5.9 that if \( \frac{p}{q} \in (0, 1) \) is a rational point, then
\[
\lim_{x \to \frac{p}{q}} \left( F\left( \frac{p}{q} \right) - F(x) \right) = (n - 1)^2 \frac{1}{n^q - 1}
\] (5.12)

Hence, by Corollary 5.8, the sum of jumps at the rational points is 1. It follows that \( F(x) \) is continuous at irrational points and the jump at a rational point \( \frac{p}{q} \) is \( \frac{(n - 1)^2}{n^q - 1} \). From this, (5.11) is immediate. □

Changing variables, observing that the set \( \{ \cos \frac{p}{q} \pi \mid 1 \leq p < q \} \) is dense in \([-1, 1]\) and using the freeness of the action in the case \( G \) is abelian (in conjunction with Theorem 4.4), we obtain our main result.

**Theorem 5.11.** Let \( G \) be a non-trivial finite group of order \( n \). Then the KNS spectral measure \( \mu \) for the Schreier graph of \( \mathcal{G}(C(G)) \) with respect to a parabolic subgroup \( P \) and generators (5.1) is a discrete measure given by
\[
\mu = (n - 1)^2 \sum_{q=2}^{\infty} \left( \sum_{1 \leq p < q, (p, q) = 1} \frac{1}{n^q - 1} \delta_{\cos \frac{p}{q} \pi} \right)
\] (5.13)

The following equalities of spectra of Hecke operators hold:
\[ [-1, 1] = \text{Sp}(H_\pi) = \text{Sp}(H_{\lambda_{\mathcal{G}(C(G))}/P}) = \text{Sp}(H_{\lambda_{\mathcal{G}(C(G))}}), \]
so the Markov operator for the simple random walk on the Cayley graph of \( \mathcal{G}(C(G)) \) has spectrum \([-1, 1]\).

In the case \( G \) is an abelian group, \( \mathcal{G}(C(G)) = G \wr \mathbb{Z} \) and (5.13) gives the Kesten spectral measure of the Markov operator for the simple random walk on the Cayley graph of \( G \wr \mathbb{Z} \) with respect to the automaton generators.

For the case \( G \) is abelian, the results for the Markov operator were obtained in a different way by Dicks and Schick [7]. Their result concerned random walks on wreath products \( G \wr \mathbb{Z} \) with \( G \) a finite non-trivial group. If \( G = \{g_1, \ldots, g_n\} \) and \( \mathbb{Z} = \langle t \rangle \), then Dicks and Schick used the symmetric generating set
\[
S = \{tg_1, \ldots tg_n, g_1t^{-1}, \ldots, g_nt^{-1}\}.
\] (5.14)

It is easy to check that in the case \( G \) is abelian, these are the generators obtained using the Cayley machine for \( G \). It is also straightforward to verify that if \( G \) and \( H \) are finite groups of the same order \( n \), then the walks on \( G \wr \mathbb{Z} \) and \( H \wr \mathbb{Z} \) with the above generators give rise to isomorphic Markov chains and hence have the same spectral measure. One can think of the system as being the Cayley graph of \( \mathbb{Z} \), with at each point a lamp which can either be off or illuminated in any one of \( n - 1 \) different colours. There is a lamplighter who at each move, with equal probability, either moves to the right and changes the lamp at his new position to off or to any of the \( n - 1 \) colours, or he can change the lamp where he currently is to off or to any of the \( n - 1 \) colours and then move to the left. The starting configuration has...
all lamps off and the lamplighter at the origin. Hence Theorem 5.11 gives the following result.

**Theorem 5.12.** Let \( G \) be a non-trivial finite group of order \( n \). Then the spectral measure for the random walk on \( G \wr \mathbb{Z} \) with respect to the generators (5.14) is

\[
\mu = (n - 1)^2 \sum_{q=2}^{\infty} \left( \sum_{1 \leq p < q, (p, q) = 1} \frac{1}{n^q - 1} \delta_{\cos \frac{p}{q} \pi} \right)
\]

(5.15)

We can now calculate the zeta function using Theorem 5.11 and (2.7).

**Corollary 5.13.** Let \( G \) be a non-trivial finite group and \( P \) be a parabolic subgroup of \( G(C(G)) \). Let \( X = \text{Sch}(G(C(G)), P, S) \) as per (5.1). Then

\[
\zeta_X(t) = (1 - t^2)^{-\frac{1}{2}} \prod_{q=2}^{\infty} \prod_{1 \leq p < q, (p, q) = 1} \left( 1 - 2nt^2 \cos \frac{p}{q} \pi + (2n - 1)t^2 \right)^{(n-1)^2 \frac{1}{n-1}}.
\]

This product converges for \(|t| < \frac{1}{2n-1}\). Moreover, if \( G \) is abelian, then \( \zeta_X = \zeta_{G \wr \mathbb{Z}} \).

6. **The structure of Cayley machines of non-abelian groups**

In [28], the second and third authors showed that for finite abelian groups \( G \), the automata group \( G(C(G)) \) is isomorphic to the wreath product \( G \wr \mathbb{Z} \). In this section, we consider the case in which \( G \) is not abelian, showing that in most cases, the group \( G(C(G)) \) cannot be expressed as a wreath product of any finite group with any torsion-free group. The following simple proposition allows us to consider separately two different cases.

**Proposition 6.1.** Let \( G \) be a finite group. Then either:

(i) \( G \) has a non-central element of odd order; or

(ii) \( G \) is the direct product of a 2-group and an abelian group.

**Proof.** Suppose (i) does not hold, that is, that all odd order elements of \( G \) are central. We claim first that \( G/Z(G) \) is a 2-group. Indeed, suppose not. Then some coset \( gZ(G) \in G/Z(G) \) has order an odd prime \( q \). It follows that \( g^q \in Z(G) \) if and only if \( q \) divides \( n \), and in particular that \( q \) divides the order of \( g \). Suppose the order of \( g \) is \( q^i r \) where \( q \nmid r \). Then \( g^q \) has order \( q^i \) but is not contained in \( Z(G) \), which contradicts the supposition that all odd order elements of \( G \) are central. Hence, \( G/Z(G) \) is a 2-group.

In particular, \( G \) is a central extension of a nilpotent group, and so is nilpotent. Hence, \( G \) is a direct product of its Sylow subgroups. But for odd primes \( p \), the \( p \)-Sylow subgroups are central by assumption and so in particular must be abelian. Thus \( G \) is a direct product of a 2-group and an abelian group. \( \square \)
We shall show that in the first case, our group $G(C(G))$ cannot be a wreath product of a finite group with a torsion-free group. The second case is slightly more involved and we can only handle the case where the 2-group component is not nilpotent of class 2. In this case we again show that $G(C(G))$ cannot embed in a wreath product of a finite group with a torsion-free group.

Fix now a finite group $G$. We consider the free monoid $G^*$ over the elements of $G$ and write elements as bracketed, comma-separated sequences, to avoid confusion with the multiplication in $G$. Set $x = C(G)^{-1} \in \Gamma$. Then we recall from [28] that

$$x(g_0, g_1, \ldots, g_n) = (g_0, g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_n^{-1} g_n)$$

$$x^{-1}(g_0, g_1, \ldots, g_n) = (g_0, g_0 g_1, \ldots, g_0 g_1 \cdots g_n)$$

(6.1)

$$xC(G)_g(g_0, g_1, \ldots, g_n) = (g g_0, g_1, \ldots, g_n)$$

for every $g \in G$. It follows that the elements of the form $xC(G)_g$ form a subgroup of $G(C(G))$ isomorphic to $G$. For notational convenience, we identify $g$ with $xC(G)_g$ and view $G$ as embedded in $G(C(G))$. It is shown in [28] that the element $x^n g x^{-n}$ has depth exactly $n + 1$.

There is an infinite sequence of words that will play an important technical role in what follows. Let $X = \{t_0, t_1, t_2, \ldots\}$ be a countably infinite set, which we view as a set of variables. In what follows, we take the viewpoint that $X^*$ consists of terms. If $w \in X^*$, We sometimes write $w(t_0, \ldots, t_n)$ if $w \in \{t_0, \ldots, t_n\}^*$.

If $m_0, \ldots, m_n$ are elements of a monoid $M$, we write $w(m_0, \ldots, m_n)$ to denote the element of $M$ obtained by substituting $m_i$ for $t_i$. Our sequence $\{w_n\}$ is defined recursively as follows:

- $w_0(t_0) = t_0$ and, for $n \geq 0$,
- $w_{n+1}(t_0, t_1, \ldots, t_n, t_{n+1}) = w_n(t_0, t_0 t_1, \ldots, t_0 t_1 \cdots t_n) t_0 t_1 \cdots t_{n+1}$

Notice that $w_n$ has content $\{t_0, \ldots, t_n\}$. The first four terms of $\{w_n\}$ are: $t_0$, $t_0 t_1$, $t_0 t_0 t_1 t_0 t_1 t_2$, and $t_0 t_0 t_0 t_0 t_1 t_0 t_1 t_0 t_1 t_2 t_0 t_1 t_2 t_3$. Sometimes it will be convenient to set $w_{-1}$ to be the empty word $\varepsilon$, as the recursion still applies if we follow the usual conventions regarding empty variable sets.

If $w \in X^*$, we denote by $|w|_{t_i}$ the number of occurrences of $t_i$ in $w$.

**Lemma 6.2.** For $0 \leq i \leq n$, $|w_n|_{t_i} = 2^{n-i}$.

**Proof.** The proof proceeds by induction on $n$. For $n = 0$, $w_0 = t_0$ and so the lemma holds for this case. Suppose that the lemma holds for $n \geq 0$. Then

$$w_{n+1}(t_0, \ldots, t_{n+1}) = w_n(t_0, t_0 t_1, \ldots, t_0 t_1 \cdots t_n) t_0 \cdots t_{n+1}.$$

First we consider $1 \leq i \leq n$. In $w_n(t_0, t_0 t_1, \ldots, t_0 \cdots t_n)$, there is one occurrence of $t_i$ for each occurrence of $t_j$, $i \leq j \leq n$, in $w_n(t_0, \ldots, t_n)$. So by induction, we obtain

$$|w_{n+1}|_{t_i} = 1 + \sum_{j=i}^{n} 2^{n-j} = 2^{n+1-i}.$$
Clearly \(|w_{n+1}|_{t_{n+1}} = 1 = 2^0\). This completes the induction, thereby establishing the lemma.

The next lemma connects the sequence \(\{w_n\}\) to our automata groups.

**Lemma 6.3.** Let \(g \in G\). Then the last entry of \(x^n gx^{-n}(g_0, \ldots, g_n)\) is \((g^{(-1)^n})_{w_{n-1}(g_0, \ldots, g_{n-1})} g_n\).

**Proof.** The proof is by induction on \(n\). For \(n = 0\), \(g(g_0) = (gg_0)\), while \((g^{(-1)^0})e g_0 = gg_0\) (recall that \(w_{-1} = e\)). Let us assume, by way of induction, that the lemma holds for \(n \geq 0\). Then

\[
x^{n+1} gx^{-(n+1)}(g_0, \ldots, g_{n+1}) = x^n gx^{-n}(g_0, g_0 g_1, \ldots, g_0 \cdots g_{n+1})
\]

(6.2)

for certain \(u_i \in G\) (since \(x^n gx^{-n}\) has depth \(n + 1\)). Moreover, we know by induction that

\[
u_n = (g^{(-1)^n})w_{n-1}(g_0, g_0 g_1, \ldots, g_0 \cdots g_{n-1}) g_0 \cdots g_n.
\]

Hence, the last entry in (6.3) is

\[
u_n^{-1} g_0 \cdots g_{n+1} = (g_0 \cdots g_n)^{-1}(g^{(-1)^{n+1}})_{w_{n-1}(g_0, g_0 g_1, \ldots, g_0 \cdots g_{n-1})} (g_0 \cdots g_n) g_{n+1}
\]

\[
= (g^{(-1)^{n+1}})_{w_{n-1}(g_0, g_0 g_1, \ldots, g_0 \cdots g_{n-1})} g_0 \cdots g_n g_{n+1}
\]

\[
= (g^{(-1)^{n+1}})w_n(g_0, g_1, \ldots, g_{n+1}) g_{n+1},
\]

as required.

Our key obstruction to embedding in wreath products is presented by the following observation.

**Lemma 6.4.** Let \(A = G \wr H\) with \(G\) a finite group and \(H\) a torsion-free group. Then the set of torsion elements of \(A\) is the subgroup \(N = \oplus_H G\). Hence every conjugacy class of \(N\) is finite.

**Proof.** Since \(A = (\oplus_H G) \times H\) and \(H\) is torsion-free, the torsion elements of \(A\) are exactly the elements of the subgroup \(N = \oplus_H G\). Since in the group \(\oplus_H G\), conjugate elements have the same support and the direct sum only contains elements of finite support, the conjugacy classes of \(N\) are finite.

**Theorem 6.5.** Let \(G\) be a finite group with a non-central element of odd order. Then \(G(\mathcal{C}(G))\) does not embed in the wreath product of a finite group with a torsion-free group.

**Proof.** Let \(g \in G\) be a non-central element of odd order. Let \(h \in G\) be an element of minimal order amongst those elements that do not commute with \(g\). Let \(p\) be a prime factor of the order of \(h\). Then \(h^p\) has order less than that of \(h\), and so commutes with \(g\). Let \(v = gh^{-1}\).

For each \(n \in \mathbb{N}\), we consider the element

\[
\gamma_n = (x^{p^n} h^{-1} x^{-p^n})^{-1} v(x^{p^n} hx^{-p^n}) \in G(\mathcal{C}(G)).
\]
Each such element is a conjugate of the torsion element $v$ by another torsion element; see (3.9). Our objective is to show that the $\gamma_n$ are all distinct. By Lemma 6.4, this cannot happen in a wreath product of a finite group with a torsion-free group, so it will follow that $G(C(G))$ cannot embed in such a wreath product.

To this end, we consider the action of $\gamma_n$ on the word $(1, 1, \ldots, 1) \in G^{p^n+1}$ and in particular compute the last letter. Our goal is to show that the action is non-trivial on the last letter. Since $\gamma_n$ has depth at most $p^n+1$, it will then follow that $\gamma_n$ has depth exactly $p^n+1$ and so the various $\gamma_n$ are all distinct. Using (6.1) and Lemma 6.3,

$$\gamma_n(1, \ldots, 1) = x^{p^n} h^{-1} x^{-p^n} v x^{p^n} (h, 1, \ldots, 1)$$

$$= x^{p^n} h^{-1} x^{-p^n} \left( g, h^{-(p^n)}(1), h^{(p^n)_2}, \ldots, h^{(-1)p^{n-1}}(p^n) \right)$$

$$= \left( \ldots, (h^{-1}(-1)^{p^n}) w \right) h^{(-1)^{p^n}}$$

where

$$w = w_{p^n-1} \left( g, h^{-(p^n)}(1), h^{(p^n)_2}, \ldots, h^{(-1)p^{n-1}}(p^n-1) \right) \in G.$$ 

Now it is well-known that $p$ divides $\binom{p^n}{i}$ for every prime $p$, every $n \geq 1$ and every $1 \leq i \leq p^n - 1$. In view of the fact that $h^p$ commutes with $g$, we see using Lemma 6.2 that

$$w = g^{2^{p^n}-1} h^s,$$

where $s = \sum_{i=1}^{p^n-1} (-1)^i \binom{p^n}{i} 2^{p^n-1-i}$. Now

$$((h^{-1})(-1)^{p^n}) w h^{(-1)^{p^n}} = 1$$

$$\iff h^{-s} g^{2^{p^n}-1} h g^{2^{p^n}-1} h^{s} = h$$

$$\iff g^{2^{p^n}-1} h g^{2^{p^n}-1} = h$$

$$\iff h g^{2^{p^n}-1} = g^{2^{p^n}-1} h$$

(6.4)

But $g$ has odd order, so there is an integer $k$ such that $(g^{2^{p^n}-1})^k = g$. Since $g$ does not commute with $h$, we conclude $g^{2^{p^n}-1}$ does not commute with $h$. Therefore, none of the equalities in (6.4) hold and so $\gamma_n$ acts non-trivially on the last letter of $(1, \ldots, 1) \in G^{p^n+1}$, finishing the proof. \[\square\]

We now consider the case in which $G$ contains a 2-group, which is not nilpotent of class 2.

**Theorem 6.6.** Let $G$ be a finite group containing a 2-subgroup that is not nilpotent of class 2. Then $G(C(G))$ does not embed in a wreath product of a finite group with a torsion-free group.
Proof. As in the previous proof, it will suffice to show that some torsion element in $\mathcal{G}(C(G))$ has infinitely many distinct conjugates by other torsion elements.

Let $K$ be a 2-group in $G$ that is not nilpotent of class 2. Let $g, f, h \in K$ be such that $h^{-1} fh f^{-1}$ does not commute with $g$. Set

$$\gamma_n = x^n g x^{-n} h x^n g^{-1} x^{-n}.$$ 

We claim that it suffices to show that the last entry of

$$x^n g x^{-n} (1, \ldots, 1, f, 1, 1)$$

differs from the last entry of

$$x^n g x^{-n} (h, \ldots, 1, f, 1, 1)$$

for infinitely many positive integers $n$. (Here each of the strings we are acting on has length $n + 1$). Indeed, for each $n$ such that this is true, we see that

$$\gamma_n [x^n g x^{-n} (1, 1, \ldots, 1, f, 1, 1)] = x^n g x^{-n} h (1, 1, \ldots, 1, f, 1, 1)$$

differs from

$$x^n g x^{-n} (1, 1, \ldots, 1, f, 1, 1)$$

in position $n + 1$. Thus, the element $\gamma_n$ acts non-trivially on the $n + 1$ level; we know that $\gamma_n$ can have depth at most $n + 1$, so it must have depth exactly $n + 1$. So if we have infinitely many $n$s for which this is the case, we have conjugates of a given torsion element with arbitrarily large depth, so there must be infinitely many of them.

By Lemma 6.3, we have that the last entry of

$$x^n g x^{-n} (1, \ldots, 1, f, 1, 1)$$

is $(g^{(-1)^n})^{w_{n-1}(1, \ldots, 1, f, 1)}$ and the last entry of

$$x^n g x^{-n} (h, 1, \ldots, 1, f, 1, 1)$$

is $(g^{(-1)^n})^{w_{n-1}(h, \ldots, 1, f, 1)}$.

By Lemma 6.2 $|w_{n-1}|_{t_{n-2}} = 2$. Hence

$$w_{n-1}(1, \ldots, 1, f, 1) = f^2.$$

Let $k = w_{n-1}(h, \ldots, 1, f, 1) \in G$. Then it will suffice to show that for infinitely many $n$,

$$f^{-2} gf^2 \neq k^{-1} gk,$$

that is, that for infinitely many $n$, $g$ does not commute with $kf^{-2}$.

Now from the definition of $w_{n-1}$, we have

$$w_{n-1}(t_0, 1, \ldots, 1, t_{n-2}, 1) = w_{n-2}(t_0, t_0, \ldots, t_0, t_0 t_{n-2}) t_0 t_{n-2}$$

$$= w_{n-3}(t_0, t_0^2, \ldots, t_0^{n-2}) t_0^{-1} t_{n-2} t_0 t_{n-2}.$$
Observing that $w_{n-3}(t_0, t_0^2, \ldots, t_0^{n-2})$ must be a power of $t_0$ and recalling that $|w_{n-1}|_{t_0} = 2^{n-1}$ by Lemma 6.2, we obtain

$$w_{n-1}(t_0, 1, \ldots, t_{n-2}, 1) = t_0^{2^{n-1} - 1} t_{n-2} t_0 t_{n-2}$$

Substituting $h$ for $t_0$ and $f$ for $t_{n-2}$, we obtain $k = h^{2^{n-1} - 1} f h f$, whence

$$k f^{-2} = h^{2^{n-1} - 1} f h f^{-1}.$$ 

But $h$ is in the 2-group $K$, and so for infinitely many $n$, we will have $h^{2^{n-1}} = 1$. But then $k f^{-2} = h^{-1} f h^{-1}$, which by assumption does not commute with $g$, as required. □

The question of whether $G(C(G))$ is a wreath product of a finite group and a torsion-free group remains open for a small class of groups, namely for those groups $G$ that are direct products of an abelian group with a 2-group of nilpotency class 2. Examples include $D_4$ and the 8-element quaternion group.

Another interesting question is that of whether, for non-abelian $G$, the group $G(C(G))$ has bounded torsion. We conjecture that this is never the case.

If, as we suspect, the group $G(C(G))$ is not isomorphic to $G \wr \mathbb{Z}$ for any non-abelian $G$, then one might ask whether this wreath product arises as an automata group at all and, if so, what conditions can be placed upon the automaton. In particular, can a reset automaton be found?

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