HOCHSCHILD COHOMOLOGY OF $U(\mathfrak{sl}_2(k))$

MATTHEW TOWERS

Abstract. We calculate the Hochschild cohomology of $U(\mathfrak{sl}_2(k))$ when $k$ is a field of characteristic $p > 2$.

1. Introduction

Let $k$ be a field, $\mathfrak{g} = \mathfrak{sl}_2(k)$ and $U = U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. In this note we calculate $\text{HH}^*(U)$ when $k$ has characteristic $p > 2$.

$U$ is a Hopf algebra with antipode $\eta(x) = -x$ and comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. Let $U^e = U \otimes_k U^{op}$ be the enveloping algebra of $U$, so there is an algebra homomorphism $(1 \otimes \eta) \circ \Delta : U \to U^e$ making $U^e$ into a free $U$-module (on the generators $\lambda \otimes 1$ for $\lambda$ in a PBW basis of $U$). The induced module $k \otimes_{U^e} U$ is isomorphic to $U$ and the restricted module $U|_{U^e}^U$ is by definition $U^{\text{ad}}$, so Eckmann-Shapiro gives

$$\text{HH}^*(U) = \text{Ext}^*_U(U, U) = \text{Ext}^*_U(k \otimes_{U^e} U, U) \cong \text{Ext}^*_U(k, U^{\text{ad}}).$$

If $k$ has characteristic zero, the structure of $\text{HH}^*(U)$ follows immediately: in that case $\text{Ext}^*_U(k, L) = 0$ whenever $L$ is a nontrivial simple $U$-module so $\text{HH}^*(U) \cong \text{Ext}^*_U(k, k) \otimes_k Z(U)$. The centre $Z(U)$ is generated by the Casimir element of $U$ in the characteristic zero case, and $\text{Ext}^*_U(k, k)$ is an exterior algebra with one generator of degree 3 arising from the Killing form.

Let $S$ be the symmetric algebra on the adjoint $\mathfrak{g}$-module. To calculate the Hochschild cohomology in the case when the characteristic of $k$ is greater than two we use the isomorphisms

$$\text{HH}^*(U) \cong \text{Ext}^*_U(k, U^{\text{ad}}) \cong \text{Ext}^*_U(k, S)$$

where the first follows as above and the second from the isomorphism of $U$-modules $U^{\text{ad}} \cong S$ of [FP87]: this result shows $\text{HH}^*(U)$ agrees with the Poisson cohomology of $S$ equipped with the Poisson bracket induced by the Lie bracket of $\mathfrak{g}$.

Under this isomorphism $\text{HH}^0(U)$ corresponds to $S^2$, and each $\text{HH}^i(U)$ becomes a $S^2$-module. We describe this module structure by generators and relations, and give the Hilbert series (arising from the polynomial grading on $S$) explicitly. The following theorem summarises some of the results to be proved in sections 3, 4 and 5.

**Theorem 1.1.** As $S^2$-modules, $\text{HH}^1(U)$ is generated by three elements of degree $p - 1$ and four of degree $p$, $\text{HH}^2(U)$ is generated by three elements of degree $p - 1$ and one of degree $p$ and $\text{HH}^3(U)$ is generated by one element of degree one and one of degree $p - 2$.

1.1. Notation. Let $k$ be a field of characteristic $p > 2$ and $e, f, h$ denote the usual basis of $\mathfrak{g} = \mathfrak{sl}_2(k)$. Let $S^n$ be the $n$th symmetric power of the adjoint $\mathfrak{g}$-module and $S = \bigoplus_{n \geq 0} S^n$ the symmetric algebra on the adjoint $\mathfrak{g}$-module. Let $c = h^2 + 4ef \in S^2$, let $Z$ be the subalgebra of $S$ generated by $c, e^p, h^p, f^p$ so that $Z = S^2$ and let $Z_0$ be the subalgebra of $Z$ generated by $e^p, f^p$ and $h^p$. Multiplication

\[ Date: Tuesday 28th April, 2015.\]
by an element of $Z$ is a $g$-endomorphism of $S$, so each Hochschild cohomology group is a $Z$-module.

Let $L(r)$ denote the simple highest weight $g$-module with dimension $r + 1$. The natural module $L(1)$ has a basis $x, y$ with $h \cdot x = x, h \cdot y = -y$, and we write $S^n(L(1))$ for the $n$th symmetric power of $L(1)$.

There is a standard resolution of the trivial $U$-module

$$0 \rightarrow U \otimes_k x \otimes_k x \rightarrow U \otimes_k x \otimes_k y \rightarrow U \otimes_k y \rightarrow k \rightarrow 0 \quad (1)$$

in which the differentials are given by

$$\delta_n(1 \otimes x_1 \land \cdots \land x_n) = \sum_i (-1)^{i+1} x_i \otimes x_1 \land \cdots \land \hat{x}_i \land \cdots \land x_n$$

$$+ \sum_{i<j} (-1)^{i+j} [x_i, x_j] \land x_1 \land \cdots \land \hat{x}_i \land \cdots \land \hat{x}_j \land \cdots \land x_n$$

for $n \geq 1$ and $\delta_1(1 \otimes x) = x$ for $x, x_i \in g$. In particular the Hochschild cohomology groups $\text{HH}^i(U)$ are zero for $i \geq 4$.

Both $L(1)$ and $g$ admit weight gradings, with $e, h, f$ in degrees 2, 0, $-2$ and $x, y$ in degrees 1, $-1$ respectively, hence so do their symmetric and exterior powers. The differentials in this resolution respect the weight gradings, so $\text{Ext}_U^i(k, S)$ has two gradings, one from the weight grading on $S$ and one from the polynomial grading $S = \bigoplus_{n \geq 0} S^n$.

2. Symmetric powers of the natural module

**Lemma 2.1.** For each $n \geq 2$ there is a short exact sequence of $g$-modules

$$0 \rightarrow S^{n-2} \xrightarrow{\phi} S^n \rightarrow S^{2n}(L(1)) \rightarrow 0 \quad (2)$$

**Proof.** The images of the elements $e^i h^{n-i}, h^n, f^i h^{n-i}$ for $1 \leq i \leq n$ in $S^n/cS^{n-2}$ form a basis, and defining $\phi : S^n \rightarrow S^{2n}(L(1))$ by $\phi(cS^{n-2}) = 0$ and

$$\phi(e^i h^{n-i}) = (-2)^{-i} x^{n+i} y^{n-i}$$

$$\phi(h^n) = x^n y^n$$

$$\phi(f^i h^{n-i}) = 2^{-i} x^{n-i} y^{n+i}$$

induces an isomorphism $S^n/cS^{n-2} \rightarrow S^{2n}(L(1))$. \hfill $\square$

If $2n = qp + r$ with $0 \leq r < p - 1$ then the submodule structure of $S^{2n}(L(1))$ is

$$\begin{array}{c}
L(r') \\
L(r) \\
\cdots \\
L(r') \\
L(r)
\end{array}$$

where $r' = p - 2 - r$ and there are $q + 1$ copies of $L(r)$ and $q$ of $L(r')$. If $2n = qp + (p - 1)$ then $S^{2n}(L(1))$ is a direct sum of copies of $L(p - 1)$.

**Proposition 2.2.** Let $2n = qp + r$ with $0 \leq r \leq p - 1$. Then

$$\dim \text{Ext}_U^1(k, S^{2n}(L(1))) = \begin{cases} 
q + 2 & r = p - 2 \\
2q & r = 0 \\
0 & \text{otherwise.}
\end{cases}$$
Proof. We identify 1-cocycles for the resolution (1) with linear maps \( \alpha : g \to S^{2n}(L(1)) \) such that
\[
\begin{align*}
\epsilon \cdot \alpha(f) - f \cdot \alpha(e) &= \alpha(h) \\
(h - 2) \cdot \alpha(e) &= \epsilon \cdot \alpha(h) \\
(h + 2) \cdot \alpha(f) &= f \cdot \alpha(h)
\end{align*}
\]
so that the coboundaries are the maps \( \delta_{z} : g \to S^{2n}(L(1)) \) given by \( \delta_{z}(r) = r \cdot z \) for \( z \in S^{2n}(L(1)) \). We may assume \( \alpha \) is homogeneous with respect to the weight grading, and if its weight degree \( j \) is not zero mod \( p \) then \( \alpha \) equals the coboundary \( \delta_{\epsilon(h)/j} \). Weight homogeneous cocycles \( \alpha \) whose weight degree is divisible by \( p \) take the form
\[
\alpha(h) = \lambda_{h}x^{i}y^{2n-i} \quad \alpha(e) = \lambda_{e}x^{i+1}y^{2n-i-1} \quad \alpha(f) = \lambda_{f}x^{i-1}y^{2n-i+1}
\]
where \( n \equiv i \mod p, -1 \leq i \leq 2n + 1 \) and \( \lambda_{e}, \lambda_{f}, \lambda_{h} \in k \), with the convention that these coefficients are zero if the corresponding monomial would have a negative exponent. The cocycle condition becomes
\[
(n + 1)(\lambda_{f} - \lambda_{e}) = \lambda_{h} \quad n\lambda_{h} = 0
\]
All coboundaries with the same weight degree are scalar multiples of the map \( \delta_{2, \gamma^{2n-1}} \) which sends \( h \) to 0, \( e \) to \( ix^{i+1}y^{2n-i-1} \) and \( f \) to \( ix^{-i}y^{2n-i+1} \).

Suppose first \( n \neq 0 \), so that \( \lambda_{h} = 0 \). If \( n \neq -1 \) then we must have \( \lambda_{f} = \lambda_{e} \), so \( \alpha \) is a coboundary. This shows that the dimension of the Ext group is as claimed for \( r \neq 0, p - 2 \). If \( n \equiv -1 \), so \( r = p - 2 \), we may choose \( \lambda_{f} \) and \( \lambda_{e} \) freely but \( \alpha \) is a coboundary if \( \lambda_{e} = \lambda_{f} \). Each choice for \( i \) contributes one to the dimension of the Ext group, which is \( q + 2 \).

Now let \( n \equiv 0 \), so all coboundaries with this weight degree are zero. We have \( r = 0 \) and \( 2n = qp \), and each possibility for \( i \) contributes two to the dimension of the Ext group since \( \lambda_{e} \) and \( \lambda_{f} \) may be chosen freely, except for the extreme values which contribute one: one of \( \lambda_{e} \) or \( \lambda_{f} \) is forced to be zero as the corresponding monomial has a negative exponent. This gives a total of \( 2q \).

\[ \square \]

Proposition 2.3. Let \( 2n = qp + r \) with \( 0 \leq r \leq p - 1 \). Then
\[
\dim \operatorname{Ext}^{2}_{k}(S^{2n}(L(1))) = \begin{cases} 
2q + 2 & r = p - 2 \\
q - 1 & r = 0 \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. We interpret 2-cocycles as maps \( \alpha : \wedge^{2}g \to S^{2n}(L(1)) \) such that
\[
e \cdot \alpha(h \wedge f) + f \cdot \alpha(e \wedge h) = h \cdot \alpha(e \wedge f).
\]
Coboundaries are spanned by maps of the form
\[
\begin{align*}
\rho_{z}(h \wedge f) &= 0 \\
\rho_{z}(e \wedge f) &= -f \cdot z \\
\rho_{z}(e \wedge h) &= (2 - h) \cdot z \\
\tau_{z}(h \wedge f) &= -f \cdot z \\
\tau_{z}(e \wedge f) &= -z \\
\tau_{z}(e \wedge h) &= e \cdot z
\end{align*}
\]
for \( z \in S^{2n}(L(1)) \). If \( \alpha \) is weight homogeneous of degree \( j \neq 0 \) then it equals \((1/j)(\alpha_{h \wedge f} - \tau_{\alpha(e \wedge h)})\) so is a coboundary. Therefore we assume
\[
\alpha(h \wedge f) = \lambda_{f}x^{i-1}y^{2n-i+1}, \alpha(e \wedge f) = \lambda_{e}x^{i+1}y^{2n-i-1}, \alpha(e \wedge h) = \lambda_{e}x^{i}y^{2n-i-1}
\]
where \( n \equiv i \mod p, -1 \leq i \leq 2n + 1 \) and \( \lambda_{e}, \lambda_{f}, \lambda_{h} \in k \), with the convention that these are zero if the corresponding monomial would have a negative exponent. The cocycle condition becomes
\[
(n + 1)(\lambda_{e} + \lambda_{f}) = 0.
\]
Suppose first that \( n + 1 \equiv 0 \mod p \) so this condition is empty. By subtracting a coboundary of the form \( t_2 \) we can assume \( \lambda_h = 0 \), and such a cocycle is not a boundary unless it is zero since if \( r_2 \) or \( s_2 \) have weight degree congruent to zero then they kill \( e \wedge h \) and \( h \wedge f \). It follows that each possible choice of \( i \) contributes two to the dimension of the Ext group, except the extreme values \( i = -1, 2n + 1 \) where one of \( \lambda_e, \lambda_f \) is forced to be zero because the corresponding monomial would have a negative exponent, so the dimension is \( 2q \). This takes care of the case \( r = p - 2 \).

Now suppose \( n + 1 \not\equiv 0 \), so \( \lambda_e = -\lambda_f \) and by subtracting a suitable \( t_2 \) we may assume \( \lambda_h = 0 \). If \( n \not\equiv 0 \) then \( \alpha = -(\lambda_f/n)t_{x_{i-1}}y_{2n+i} - (1/n(n+1))s_{x_{i-1}}y_{2n+i+1} \) is a coboundary. Otherwise as before so each possible choice for \( i \) contributes one to the dimension of the Ext group so the dimension is as claimed.

**Proposition 2.4.** Let \( 2n = qp + r \) with \( 0 \leq r \leq p - 1 \). Then

\[
\dim \Ext^3_U(k, S^{2n}(L(1))) = \begin{cases} 1 & n = 0 \\ q & r = p - 2 \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** The resolution \( \mathbf{1} \) shows \( \Ext^3_U(k, S^{2n}(L(1))) \cong S^{2n}(L(1))/g \cdot S^{2n}(L(1)) \). We have

\[
h \cdot x^a y^b = (a-b)x^a y^b \\
e \cdot x^{a-1} y^{b+1} = (b+1)x^a y^b \\
f \cdot x^{a+1} y^{b-1} = (a+1)x^a y^b
\]

so \( x^a y^b \in S^{2n}(L(1)) \) is in the image of the action of \( g \) if and only if \( a \equiv b \equiv -1 \mod p \) or \( a = b = 0 \). Thus the images of the monomials \( 1 \) and \( x^{p(p+1)}y^{p+1} \) for \( q_x, q_y \geq 0 \) in \( S(L(1))/g \cdot S(L(1)) \) are a basis.

**Proposition 2.5.** The first connecting homomorphism in the long exact sequence obtained by applying \( \hom_U(k, -) \) to \( \mathbf{2} \) is zero.

**Proof.** We need that \( \phi^* : \hom_U(k, S^n) \to \hom_U(k, S^{2n}(L(1))) \) is onto, or equivalently that every trivial submodule of \( S^{2n}(L(1)) \) is the image of a trivial submodule of \( S^n \) under \( \phi : S^n \to S^{2n}(L(1)) \). The trivial submodules of \( S^{2n}(L(1)) \) are spanned by monomials \( x^a y^b \) for \( a, b \geq 0 \). If \( a > b \) then \( x^a y^b = \phi(e^{(a-b)/2}h_{p+1}) \), and \( g \) acts trivially on \( e^{(a-b)/2}h_{p+1} \). The cases where \( a \leq b \) are similar.

3. \( \HH^3(U) \)

**Lemma 3.1.** The following are cocycles with values in \( S \):

| \( \alpha \) | \( \alpha(e) \) | \( \alpha(h) \) | \( \alpha(f) \) |
|---|---|---|---|
| \( \delta_e \) | \( f^{p-1} \) | 0 | 0 |
| \( \delta_f \) | 0 | 0 | \( e^{p-1} \) |
| \( \delta \) | \( h^{p-h_{p-1}/2} \) | \( e^{(p-1)/2} \) | \( h^{p-h_{p-1}/2} \) |
| \( E \) | 0 | \( 2e^p \) | \( -h e^{p-1} \) |
| \( F \) | \( h f^{p-1} \) | \( -2f^p \) | 0 |
| \( H \) | \( -2c^p f^{p-1} \) | 0 | \( 2fh^{p-1} \) |
| \( C \) | \( \frac{(p+1)/2-h_{p-1}+1}{4f} \) | \( h^p \) | \( \frac{c(p+1)/2-h_{p-1}+1}{4e} \) |

**Proof.** Checking the cocycle condition \( \mathbf{4} \) is straightforward using the fact that the action of \( g \) on \( S \) is by derivations.
Lemma 3.2.  
(1) \( e^p F + f^p E = (1/2)h^p H \)
(2) \( 2e^p \delta - c(p-1)/2 E - h^p \delta_f = 0 \).
(3) \( 2f^p \delta + c(p-1)/2 F - h^p \delta_e = 0 \).
(4) \( c(p-1)/2 H + 2e^p \delta_e - 2f^p \delta_f = 0 \).
(5) \( c(p-1)/2 C - h^p \delta - e^p \delta_e - f^p \delta_f = 0 \).
(6) \( c(p+1)/2 \delta - h^p C + e^p F - f^p E = 0 \).
(7) \( 2f^p C + h^p F - c(p+1)/2 \delta_e - f^p \delta_f = 0 \).
(8) \( 2e^p C - h^p E - c(p+1)/2 \delta_f + e^p H = 0 \).

Proof.  
(1) \( s = (1/p)(e^p - 4(e,f)^p - h^{2p}) \) makes sense as an element of \( S \) if we expand \( e^p \) and perform the division in an appropriate \( \mathbb{Z} \)-form. Then \( e^p F + f^p E - (1/2)h^p H \) is equal to \(-1/4 \) times the coboundary of \( 1 \implies s \).
(2) From now on we write \( (a, b, c) \) for the linear map \( g \to S \) sending \( e \) to \( a \), \( h \) to \( b \) and \( f \) to \( c \). Then \( 2e^p \delta - c(p-1)/2 E - h^p \delta_f \) is \( e^p \) times the cocycle
\[
\begin{pmatrix}
 h^p - h e^{(p-1)/2} & 0 \\
 0 & -h^p - h e^{(p-1)/2}
\end{pmatrix}
\]
This is the coboundary of \( 1 \implies - \sum_{r=0}^{p-1/3} h^p 2^{r-1} c^r/(2r + 1) \).
(3) Similar to (2).
(4) \( c(p-1)/2 H - 2e^p \delta_e + 2f^p \delta_f \) is the cocycle
\[
- h^p \begin{pmatrix}
 h^p - h e^{(p-1)/2} & 0 \\
 0 & -h^p - h e^{(p-1)/2}
\end{pmatrix}
+ c(p+1)/2 \begin{pmatrix}
 (c^{(p-1)/2} - h^{p-1}) & 0 \\
 0 & -c^{(p-1)/2} - h^{p-1}
\end{pmatrix}
\]
The first term is a coboundary as in (2), and the second bracketed term is the coboundary of \( 1 \implies \sum_{k=0}^{p-3/2} c^k h^p 2^{-k(b+1)/(2(b + 1))} \).
(5) Follows immediately from the definitions.
(6) Follows immediately from the definitions.
(7) \( 2f^p C + h^p F - c(p+1)/2 \delta_e - f^p H \) is the cocycle
\[
-f^p c \begin{pmatrix}
 (c^{(p-1)/2} - h^{p-1}) & 0 \\
 0 & -c^{(p-1)/2} - h^{p-1}
\end{pmatrix}
\]
which is a coboundary as in (4).
(8) Similar to (7). \( \square \)

Proposition 3.3.  
The second connecting homomorphism in the long exact sequence obtained by applying \( \text{hom}_{U}(k, -) \) to (2) is zero.

Proof.  
We show
\[
\phi^* : \text{Ext}_{U}^{1}(k, S^n) \to \text{Ext}_{U}^{1}(k, S^{2n}(L(1)))
\]
is onto by finding preimages under the map induced by \( \phi \) of representing cocycles for the basis of the right hand side provided by Proposition 2.

Firstly let \( n \equiv -1 \mod p \) and write \( 2n = qp + (p-2) \). For any choice of scalars \( \lambda_{i,e} \neq \lambda_{i,f} \) where \(-1 \leq i \leq 2n + 1 \) and \( i \equiv n \mod p \), the cocycles
\[
A_i = (\lambda_{i,e} x^{i+1} y^{2n-i-1}, \lambda_{i,f} x^{-i} y^{2n-i+1})
\]
represent a basis of \( \text{Ext}_{U}^{1}(k, S^{2n}(L(1))) \).

Suppose \( i = n \), and choose \( \lambda_{i,e} = 1, \lambda_{i,f} = -1 \). Since \( \phi(-2ch^{n-1}) = A_n(e) \) and \( \phi(-2fh^{n-1}) = A_n(f) \), a cocycle represents a preimage of \( A_n \) if it is equal modulo \( c \) to
\[
-(2ch^{n-1}, 0, 2fh^{n-1}).
\]
for some $1 ≤ i ≤ n$ have the form $e(2^{n-i-1}h^{i+1}f^{n-i-1}) = A_i(e)$, a cocycle represents a preimage of $A_i$ if it equals $$(2^{n-i-1}h^{i+1}f^{n-i-1}, 0, 0) \mod c.$$

Since $n+1 \equiv 0 \mod p$ a scalar multiple of a power of $h^p$ and of $f^p$ times $δ_e$ works. The case $i > n$ is similar.

Now let $n \equiv 0 \mod p$, so cocycles representing elements of $\text{Ext}^1_U(k, S^{2n}(L(1)))$ have the form $B_i = (λ_{i,e}, x^{i+1}, y^{2n-i-1}, λ_{i,h}, x^i, y^{i+1}, (λ_{i,e} + λ_{i,h})x^i, y^{i+1})$ for some $λ_{i,e}, λ_{i,h}$ and some $i \equiv n \mod p$.

If $i = n$ and $λ_{i,h} = 0$ then as $φ(h^{i+1}e) = (-2)^{-1}x^iy^{i+1}$ and $φ(h^{i+1}f) = 2^{-1}x^iy^{i+1}$ a suitable power of $h^p$ times $H$ is a preimage of $B_n$.

If $i = n$ and $λ_{i,h} = 1, λ_{i,e} = -1/2$ then a preimage of $B_n$ has the form $(eh^{n-1}, h^n, fh^{n-1}) \mod c$ so a power of $h^p$ times $C$ is a preimage, for $C(e) \equiv eh^{p-1}$ and $C(f) \equiv fh^{p-1} \mod c$.

Suppose $n < i < 2n$. If $λ_{i,h} = 0, λ_{i,e} = 1$ then $φ((-2)^{i+1-n}h^{2n-i-1}e^{i+1-n}) = B_i(e)$ and $φ((-2)^{i+1-n}e^{i+1-n}) = B_i(h)$, so $B_i$ has a preimage which is a scalar multiple of some cocycle $e^{(q/2-1)p}E$ is a preimage for $B_{2n}$ since $φ((-2)^n e^{n}) = x^{2n}$. Then a scalar multiple of $e^{(q/2-1)p}E$ is a preimage for $B_{2n}$ since $φ((-2)^{n-1}e^{n-1}h) = x^{2n-1}y$. The cases when $i < n$ are similar.

\[ □ \]

**Proposition 3.4.** $HH^1(U)$ is generated as a $Z$-module by the cohomology classes of the cocycles listed in Lemma 3.3.

**Proof.** Vanishing of the first two connecting homomorphisms gives an exact sequence
\[ 0 \to \text{Ext}^1_U(k, S^{n-2}) \xrightarrow{c} \text{Ext}^1_U(k, S^n) \xrightarrow{δ_e} \text{Ext}^1_U(k, S^{2n}(L(1))) \to 0 \]

for each $n$. Let $H$ be the $Z$-submodule of $\text{Ext}^1_U(k, S)$ generated by the cohomology classes of the cocycles of Lemma 3.1. The proof of the previous proposition showed that every element of $\text{Ext}^1_U(k, S^{2n}(L(1)))$ has a preimage under $φ^*$ in $H$, so the result follows by induction on $n$. \[ □ \]

**Proposition 3.5.** As a $Z_0$-module $HH^1(U)$ is the free module on $c^iδ, c^iδ_e, c^iδ_f$ for $0 ≤ i ≤ (p-1)/2$ and $c^iE, c^iF, c^iH, c^iC$ for $0 ≤ i ≤ (p-3)/2$ modulo relations $c^iR$ where $R$ is the first relation from Lemma 3.2 and $0 ≤ i ≤ (p-3)/2$.

**Proof.** Lemma 3.2 and Proposition 3.4 imply that these elements generate $HH^1(U)$ over $Z_0$ and the $Z_0$-module with these generators and relations surjects onto $HH^1(U)$.

We need only check the dimensions of their graded pieces agree. Since $Z_0$ is polynomial on three generators of degree $p$, its degree $n$ part has dimension $d_n = (n/p+2)/2$ if $p$ divides $n$ and zero otherwise. Thus the degree $n$ part of the $Z_0$-module with these generators and relations has dimension
\[ e_n = 3 \sum_{i=0}^{(p-1)/2} d_{n-(p-1)-2i} + 4 \sum_{i=0}^{(p-1)/2} d_{n-p-2i} - \sum_{i=0}^{(p-3)/2} d_{n-2p-2i}. \]
Let \( e_n = \dim \text{Ext}_1^U(k, S^n) \). Then \([4]\) and Proposition \([2, 2]\) tell us that \( e_0 = e_1 = 0 \) and if \( n = qp + r \) for \( 0 \leq r < p \),

\[
e_n - e_{n-2} = \begin{cases} 
4q & r = 0 \\
2q + 3 & r = p - 1 \\
0 & \text{otherwise}.
\end{cases}
\]

It is straightforward to check that \( e_n \) obeys the same initial conditions and recurrence relation.

**Corollary 3.6.** Let \( n = qp + r \) with \( 0 \leq r \leq p - 1 \). Then

\[
\dim \text{Ext}_1^U(k, S^n) = \begin{cases} 
3\left(\binom{q+2}{2} + 4\binom{q+1}{2}\right) & r = p - 1 \\
4\binom{q+1}{2} - \binom{q}{2} & r \neq p - 1 \text{ even} \\
3\binom{q+1}{2} & r \text{ odd}.
\end{cases}
\]

**Proof.** This follows from \([5]\). \(\square\)

4. \( \text{HH}^2(U) \)

**Lemma 4.1.** The following are cocycles with values in \( S \):

| \( \alpha \) | \( \alpha(h \wedge f) \) | \( \alpha(e \wedge f) \) | \( \alpha(e \wedge h) \) |
|---|---|---|---|
| \( R_e \) | 0 | 0 | \( f \) |
| \( R_h \) | \( f h^{p-2} \) | \( f \) | \( e f h^{p-2} \) |
| \( R_f \) | \( e f^{p-1} \) | 0 | 0 |
| \( T \) | \( f h^{p-1} \) | 0 | \( e f h^{p-1} \) |

**Proof.** Check the cocycle condition \([5]\). \(\square\)

**Lemma 4.2.** \( e^p R_e + f^p R_f - h^p R_h = c^{(p-1)/2} T \)

**Proof.** \( e^p R_e + f^p R_f - c^{(p-1)/2} T \) equals

\[
(f((ef)^{p-1} - c^{(p-1)/2} h^{p-1}), 0, 0, (ef)^{p-1} - c^{(p-1)/2} h^{p-1})
\]

and since \((ef)^{p-1} = (c - h^2)^{p-1} = \sum_{i=0}^{p-1} c^i h^{2p-2-2i}\) this is

\[
\left( \sum_{i=0, i \neq (p-1)/2}^{p-1} f^i c^i h^{2p-2-2i}, 0, \sum_{i=0, i \neq (p-1)/2}^{p-1} e c^i h^{2p-2-2i} \right).
\]

The coboundary \( c^i h^{2p-1-2i} \) equals

\[
(2(2i + 1)) f c^i h^{2p-2-2i} - c^i h^{2p-1-2i} - 2(2i + 1) e c^i h^{2p-2-2i}
\]

so \( e^p R_e + f^p R_f - c^{(p-1)/2} T \) differs by a coboundary from

\[
\left( 0, \sum_{i=0, i \neq (p-1)/2}^{p-1} c^i h^{2p-1-2i} - 2(2i + 1), 0 \right).
\]

Each \((0, c^i h^{2p-1-2i}, 0)\) with \( 0 < i \leq p - 1 \) is the coboundary \( s_{x_i} \), where \( x_i = (1/i) \sum_{j=0}^{i-1} f c^j h^{2p-2-2i} \), so up to coboundaries \( e^p R_e + f^p R_f - c^{(p-1)/2} T \) equals

\[
(0, h^{2p-1}/2, 0).
\]

Since \( t_{h^{p-1}} = (2fh^{p-2} - h^{p-1}, 2eh^{p-2}) \)

\( h^p R_h \) differs by \((0, h^{p-1}/2, 0)\) from a boundary, and the result follows. \(\square\)

**Proposition 4.3.** The third connecting homomorphism in the long exact sequence obtained by applying \( \text{hom}_U(k, -) \) to \([3]\) is zero except in the case \( n = p - 1 \) when it has image of dimension one.
Proof. We must show
\[ \phi^* : \text{Ext}_U^2(k, S^n) \to \text{Ext}_U^2(k, S^{2n}(L(1))) \]
is onto unless \( n \neq p - 1 \) when its image has codimension one. To do this, as before, we find preimages of cocycles representing a basis of the right hand side.

Suppose \( n \equiv 0 \mod p \). The cocycles
\[ A_i = (x^{-1}y^{2n-i+1}, 0, -x^{i+1}y^{2n-i-1}) \]
for \( 0 < i < 2n \) and \( i \equiv 0 \mod p \) represent a basis of \( \text{Ext}_U^2(k, S^{2n}(L(1))) \), where we adopt the notation \((a, b, c)\) for the linear map on \( \Lambda^2 g \) such that \( h \wedge f \mapsto a, e \wedge f \mapsto b, e \wedge h \mapsto c \).

If \( i = n \) then \( 2h^{n-p}T \) is a preimage. If \( i > n \) then a scalar multiple of \( e^{i-n}h^{2n-i-p}T \) is a preimage, and if \( i < n \) we get a similar result.

Now suppose \( n \equiv -1 \mod p \) and \( n > p - 1 \), so that cocycles have the form
\[ (\lambda_f x^{-1}y^{2n-i+1}, 0, \lambda_e x^{i+1}y^{2n-i-1}) \]
for \(-1 \leq i \leq 2n + 1 \) and \( i \equiv n \mod p \).

When \( i = 2n + 1 \), we need only find a preimage for
\[ (x^{2n}, 0, 0) \]
and \( e^{n+1-p}R_f \).

When \( n < i < 2n + 1 - p \) we have to find preimages for two linearly independent cocycles. Taking \( \lambda_f = 1, \lambda_e = 0 \) we get the cocycle
\[ (x^{-1}y^{2n-i+1}, 0, 0) \]
which has a scalar multiple of \( h^{2n-i+1}e^{i-n-p}R_f \) as a preimage. Also \( h^{2n-i+1-p}e^{i-n}R_h \)
is congruent modulo \( c \) to
\[ (e^{i-n}h^{2n-i+1}/4, 0, e^{i-n+p}h^{2n-i-1}) \]
so it is a preimage for the cocycle with \( \lambda_e = \lambda_f = (-2)^{i-n-1} \). The cases where \( i < n \) are similar.

When \( i = n \), \( 2fph^{n+1-2p}R_f \) is a preimage of the cocycle with \( \lambda_f = 1, \lambda_e = 0 \) since \((f e)p^{-1} = h^{2p-2} \mod c \). A preimage for the cocycle with \( \lambda_f = 0, \lambda_e = 1 \) can be found similarly.

Finally suppose \( n = p - 1 \). Preimages when \( i = -1 \) or \( 2n + 1 \) work exactly as before, so we look only at cocycles with weight degree zero. By subtracting an appropriate coboundary we may assume these send \( e \wedge f \) to zero, so they take the form
\[ \left( f \sum_{i=0}^{(p-3)/2} \phi_i e^i h^{p-2-i}, 0, e \sum_{i=0}^{(p-3)/2} \epsilon_i e^i h^{p-2-i} \right) \]
for some scalars \( \phi_i, \epsilon_i \). Such a map is a a cocycle if and only if \( \phi_{i-3/2} = \epsilon_{i-3/2} \) and
\[ \phi_{i-1} - \phi_i = \epsilon_{i-1} - \epsilon_i \]
for \( 1 \leq i \leq (p - 3)/2 \). This has the general solution \( \phi_i = \epsilon_i \) for \( 0 \leq i \leq (p - 3)/2 \). On the other hand, it shows that if \( z \in S \) has weight \(-2 \) then
\[ (-f : e \cdot z, 0, e \cdot e \cdot z) \]
is a coboundary. Taking \( z = f e^i h^{p-2-2i} \) we get
\[ -4i(2i + 1)(f e^i h^{p-2(2i+1)}, 0, e e^i h^{p-2(2i+1)}) \]
\[ + 4(i + 1)(2i + 1)(f e^{i+1} h^{p-2(i+2)}, 0, e e^{i+1} h^{p-2(i+2)}) \]
and so each
\[ (f e^i h^{p-2(i+1)}, 0, e e^i h^{p-2(i+1)}) \]
for $0 < i \leq (p - 3)/2$ is a coboundary. Therefore the weight zero ($i = n$) part of $\text{Ext}_U^1(k, S)$ has dimension at most one, and since $(fh^{p-2}, 0, eh^{p-2}) = R_h$ is a preimage of the nonbounding cocycle 
\[(x^{p-2}y^{p}/2, 0, -x^{p}y^{p-2}/2)\]
it is exactly one.

**Proposition 4.4.** $\text{HH}^2(U)$ is generated as a $Z$-module by $\bar{R}_e$, $\bar{R}_f$, $\bar{R}_h$ and $\bar{T}$. As a $Z$-module it is isomorphic to the free module on those generators modulo the relation of Lemma 4.3.

**Proof.** That these elements generate $\text{HH}^2(U)$ as a $Z$-module follows from the preimages computed in the proof of Proposition 4.3 as for $\text{HH}^1(U)$. We only need show that the graded pieces of the free module with these generators and relations have the same dimensions as those of $\text{HH}^2(U)$.

Proposition 4.3 implies 
\[0 \to \text{Ext}_U^2(k, S^{n-2}) \xrightarrow{\phi} \text{Ext}_U^2(k, S^n) \xrightarrow{\phi^*} \text{Ext}_U^2(k, S^{2n}(L(1))) \to 0\]
is exact except when $n = p - 1$ when $\phi^*$ has image of dimension three, which together with Proposition 4.3 gives a recurrence relation for $e_n = \text{dim Ext}_U^2(k, S^{n-2})$:

\[e_n - e_{n-2} = \begin{cases} 
3 & n = p - 1 \\
4q + 4 & n = qp + p - 1 \text{ for } q > 0 \\
2q - 1 & n = qp \\
0 & \text{otherwise.} 
\end{cases}\]

Since $Z$ is free as a $Z_0$-module on $e^i$ for $0 \leq i < p$, the dimension $f_n$ of the polynomial degree $n$ part of $Z$ satisfies 
\[(9) \quad f_n = \begin{cases} 
\binom{q + 2}{2} & n = qp + r, r \text{ even} \\
\binom{q + 1}{2} & n = qp + r, r \text{ odd} 
\end{cases}\]
so the free $Z$-module with our generators and relations has degree $n$ part of dimension $3f_{n-(p-1)} + f_{n-p} - f_{n-(2p-1)}$ which equals 
\[
\begin{align*}
3\binom{q + 2}{2} + \binom{q + 1}{2} - \binom{q - 1}{2} & \quad n = qp + r, r \neq p - 1 \text{ even} \\
3\binom{q + 2}{2} & \quad n = qp + p - 1 \\
3\binom{q + 1}{2} & \quad n = qp + r, r \text{ odd.}
\end{align*}
\]
This satisfies the same recurrence relation and initial conditions as $e_n$. \hfill \Box

5. $\text{HH}^3(U)$

**Proposition 5.1.** Let $n = qp + r$ for $0 \leq r < p$. Then 
\[\text{dim Ext}_U^3(k, S^n) = \begin{cases} 
1 & n \leq p - 3 \text{ even} \\
\binom{q + 2}{2} & r = p - 1 \\
\binom{q + 1}{2} & r \text{ odd} \\
\binom{q}{2} & r < p - 1 \text{ even.}
\end{cases}\]

**Proof.** Proposition 4.3 implies that 
\[(10) \quad 0 \to \text{Ext}_U^3(k, S^{n-2}) \xrightarrow{\phi} \text{Ext}_U^3(k, S^n) \xrightarrow{\phi^*} \text{Ext}_U^3(k, S^{2n}(L(1))) \to 0\]
is exact, except when \( n = p - 1 \) in which case the map \( \text{Ext}^3_U(k, S^{n-2}) \to \text{Ext}^3_U(k, S^n) \) has kernel of dimension one. Using Proposition 2.4 this gives the following recurrence relation for \( e_n = \dim \text{Ext}^3_U(k, S^n) \):

\[
e_n - e_{n-2} = \begin{cases} 
1 & n = 0 \\
2q + 1 & n = qp + p - 1, q > 0 \\
0 & \text{otherwise.}
\end{cases}
\]

The claimed dimensions obey this recurrence relation and agree with \( e_n \) for \( n = -1, 0 \) so the result follows. \( \square \)

Let \( I \) and \( J \) be the cohomology classes of the 3-cocycles sending \( f \wedge h \wedge e \) to 1 and \( h^{p-1} \) respectively.

**Proposition 5.2.** As a \( \mathbb{Z} \)-module, \( \text{Ext}^3_U(k, S) \) is generated by \( I \) and \( J \) subject to the relations \( e^p I = f^p I = h^p I = c^{(p-1)/2} I = 0 \).

**Proof.** The free \( \mathbb{Z} \)-module with these generators and relations has degree \( n \) part of dimension 1 if \( n \leq (p - 3)/2 \) is even and \( f_{n-(p-1)} \) otherwise, where \( f_n \) is as in (9). This agrees with \( \dim \text{Ext}^3_U(k, S^n) \) by the previous proposition, so we only need show that the given relations hold in \( \text{Ext}^3_U(k, S) \) and that \( I \) and \( J \) generate it as a \( \mathbb{Z} \)-module.

That \( e^p I = f^p I = h^p I = 0 \) is because \( \dim \text{Ext}^3_U(k, S^p) = 0 \), and \( c^{(p-1)/2} I = 0 \) because of the \( n = p - 1 \) case of (10). That \( I \) and \( J \) generate \( \text{Ext}^3_U(k, S) \) as a \( \mathbb{Z} \)-module follows, as for the lower degree Hochschild cohomology groups, by induction on \( n \) using (10); the right hand term has a basis consisting of the element represented by the cocycle \( f \wedge h \wedge e \to 1 \), which has \( I \) as a preimage under \( \phi^* \), and elements represented by cocycles

\[
f \wedge h \wedge e \mapsto x^{ap+b-1} y^{bp+p-1}
\]

for \( a, b \geq 0, a + b \) even by Proposition 2.4. If \( a > b \) these have \( e^{(a-b)p/2} h^{bp} J \) as a preimage under \( \phi^* \), if \( a < b \) \( f^{(b-a)p/2} h^{ap} J \) is a preimage and if \( a = b \) then \( h^{ap} J \) is a preimage. \( \square \)

**References**

[FP87] Eric M. Friedlander and Brian J. Parshall, *Rational actions associated to the adjoint representation*, Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 2, 215–226. MR 911755 (88k:14026)

E-mail address: m.towers@imperial.ac.uk