Functional Limit Theorems for the Fractional Ornstein–Uhlenbeck Process

Johann Gehringer · Xue-Mei Li

Received: 21 March 2020 / Revised: 24 August 2020 / Published online: 22 October 2020
© The Author(s) 2020

Abstract
We prove a functional limit theorem for vector-valued functionals of the fractional Ornstein–Uhlenbeck process, providing the foundation for the fluctuation theory of slow/fast systems driven by both long- and short-range-dependent noise. The limit process has both Gaussian and non-Gaussian components. The theorem holds for any $L^2$ functions, whereas for functions with stronger integrability properties the convergence is shown to hold in the Hölder topology, the rough topology for processes in $C^{1/2}$. This leads to a ‘rough creation’ / ‘rough homogenization’ theorem, by which we mean the weak convergence of a family of random smooth curves to a non-Markovian random process with non-differentiable sample paths. In particular, we obtain effective dynamics for the second-order problem and for the kinetic fractional Brownian motion model.

Keywords Passive tracer · Fractional noise · Multi-scale · Mixed functional central and non-central limit theorems · Rough creation · Rough homogenization · Rough topology

Mathematics Subject Classification (2010) 34F05 · 60F05 · 60F17 · 60G18 · 60G22 · 60H05 · 60H07 · 60H10

1 Introduction

The functional limit theorem we study here lays the foundation for the fluctuation problem for a slow/fast system with the fast variable given by a family of non-strong mixing stochastic processes, this will be discussed in [19], see [18] for the preliminary
version. A pivot theorem for obtaining effective dynamics for the slow moving particles in a fast turbulent environment are scaling limit theorems for the convergence of the following functionals

\[ X^\varepsilon := \left(X_1^\varepsilon, \ldots, X_N^\varepsilon\right), \quad X_k^\varepsilon = \alpha_k(\varepsilon) \int_0^t G_k(y_s^\varepsilon) ds, \quad (1.1) \]

with weak convergence in \( C^\alpha([0, T], \mathbb{R}) \), where \( T \) is some finite fixed time horizon, \( \alpha(\varepsilon) \) a suitable scaling and \( G_k : \mathbb{R} \rightarrow \mathbb{R} \). If \( y_t^\varepsilon = y_{t/\varepsilon} \) and \( y_t \) is a strong mixing process, \( \alpha(\varepsilon) = 1/\sqrt{\varepsilon} \) and the limit is a Markov process, for details see, e.g. the book [23] and the references therein. For stochastic processes whose auto-correlation function does not decay sufficiently fast at infinity there is no reason to have the \( \sqrt{\varepsilon} \) scaling or to obtain a diffusive limit. Furthermore, the scaling limit and the limit function may depend on the individual functions \( G_k \).

In this article, we take \( y_t^\varepsilon \) to be the stationary and rescaled fractional Ornstein–Uhlenbeck process with Hurst parameter \( H \), which, for \( H > \frac{1}{2} \), exhibits long-range dependence (LRD) and is not strong-mixing. Our interest for long range dependent / non-strong mixing noise comes the time series data of the river Nile. In a study of water flows of the Nile river, Hurst and his colleagues [20] observed long-range time dependence and found that the time dependence varies proportionally to \( t^H \) where \( H \sim 0.73 \), by contrast, Brownian motions and stable processes have independent increments. Fractional Brownian motions (fBM) were then proposed by Benoit Mandelbrot and John Van Ness [28] for modelling the Hurst phenomenon. A fBM is a continuous mean zero Gaussian process with stationary increments and covariance \( \mathbb{E}(B_t - B_s)^2 = |t - s|^{2H} \), it is self-similar with similarity exponent \( H \), and distinguished by the Gaussian property and stationary, but dependent increments. See, e.g. [26] and the reference therein for stochastic calculus for fBM’s.

Self-similar processes appeared also in mathematically rigorous descriptions of critical phenomena and in renormalisation theory. In [38], Sinai constructed non-Gaussian self-similar fields; while Dobrushin [12] studied self-similar fields subordinated to self-similar Gaussian fields (multiple Wiener integrals). Those self-similar stochastic processes with stationary increments are a particular interesting class. When normalized to begin at 0, to have mean 0 and to have variance 1 at \( t = 1 \), they necessarily have the covariance \( \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) \). Those of Gaussian variety are fBMs. Hermite processes are non-Gaussian self-similar processes with stationary increments and the above-mentioned covariance. They appeared as scaling limits of functionals of long-range dependent Gaussian processes, see [36]. Jona-Lasinio was also concerned with the construction of a systematic theory of limit distributions for sums of ‘strongly dependent’ random variables for which the classical central limit theorems do not hold, [22], see also the book [13].

Let us first consider the convergence of one single component, the scalar case. The scaling constant \( \alpha_k(\varepsilon) \) depends on the function \( G_k \), and is a reflection of the self-similarity exponents of the limiting process. If \( \alpha_k(\varepsilon) = 1/\sqrt{\varepsilon} \), the limit of \( X_k^\varepsilon \) is a Wiener process and the functional central limit theorem is expected to hold. Let \( \mu \) denote the centred and normalized Gaussian measure on \( \mathbb{R} \) and \( m_k \) denote the Hermite
rank of a centred function $G_k \in L^2(\mu)$, which is the smallest nonzero term in its chaos expansion. Let $(H, m) \mapsto H^*(m)$ denote the function given by (1.3), which decreases with $m$. Then, the relevant scaling constants are given as below:

$$\alpha(\varepsilon, H^*(m_k)) = \begin{cases} \frac{1}{\sqrt{\varepsilon}}, & \text{if } H^*(m_k) < \frac{1}{2}, \\ \sqrt{\frac{\varepsilon}{\|\ln(\varepsilon)\|}}, & \text{if } H^*(m_k) = \frac{1}{2}, \\ \varepsilon H^*(m_k)^{-1}, & \text{if } H^*(m_k) > \frac{1}{2}. \end{cases}$$

(1.2)

See Lemma 2.4 for a preliminary computation indicating the scales. In the past, the limit theorems for the non-Gaussian limits had been called non-central limit theorems, we use the terminology ‘functional central limit theorems’ for all cases.

The intuition for this comes from its counterpart for sequences. If $Y_n$ is a mean zero, stationary, and strong mixing sequence, such that $\sigma_n^2 = \mathbb{E}(\sum_{i=1}^n Y_i)^2 \rightarrow \infty$, $\mathbb{E}(\sum_{i=1}^n Y_i)^4 = O(\sigma_n^4)$, then $\frac{1}{\sigma_n} \sum_{i=1}^n Y_i \rightarrow N(0, 1)$. If $Y_n$ is not strong mixing, this CLT may fail. Indeed, if $X_n$ is a stationary mean zero variance 1 Gaussian sequence with auto-correlation $r(n) \sim cn^{-\gamma}$ for some $\gamma \in (0, 1)$ and $c \in \mathbb{R}$ (allowing for negative correlations), $G$ a centred function with Hermite rank $m \geq 1$, and $A(n)$ a sequence such that

$$\lim_{n \to \infty} \text{var} \left( \frac{1}{A(n)} \sum_{k=1}^n G(X_k) \right) = 1.$$

Then, $z_n = \frac{1}{A(n)} \sum_{k=1}^{[nt]} G(X_n)$ is expected to converge in finite dimensional distributions. The scaling constant $A(n)$ is of the order $n^{1-\frac{1}{2m}}$ in the long-range-dependent case, of order $\sqrt{n}$ in the short-range-dependent case, and of order $\sqrt{n \ln n}$ for the borderline case, see [6,8]. By long-range dependence, we mean $\sum_{n=1}^{\infty} |r(n)| = +\infty$.

The limit process, $\lim_{n \to \infty} z_n$, is a Wiener process for fast decaying correlations, i.e. in case $\gamma \in (\frac{1}{m}, 1)$, [6]. In the borderline case, $\gamma = \frac{1}{m}$, the scaling limit is also a Wiener process. However, if $\gamma \in (0, \frac{1}{m})$ the correlations fail to decay sufficiently fast, the scaling limit is a Hermite process in the $m$-th chaos, [8,12]. The first convergence to a non-Gaussian similar process was shown in [36] where the aim was to construct a not strong mixing sequence of random variables, he achieved this by showing the sequence of random variables has a non-Gaussian scaling limit which is now known as the Rosenblatt process. In [8], vector valued combinations of short- and long-range-dependent sums were studied, however, the limit of each component is assumed to be moment determinate (which can only happen when they are in the $L^2$ chaos expansion of order less or equal to 2). This is due to a restriction in the asymptotic independence result in [33], which was extended in [29].

We return to the continuous functional limit theorems. For the scalar case, the continuous version CLT for $\gamma \in (\frac{1}{m}, 1)$ was obtained in [4], the borderline case $\gamma = \frac{1}{m}$ in [2]. These are shown for the convergence in finite dimensional distribution and for $G$ to be a centred $L^2$ function. They also obtained uniform convergences in the continuous topology for a restrictive class of functions $G$ (assuming sufficiently fast decay of the coefficients in the Wiener chaos expansion). This was extended in
[30] to vector valued $X^\varepsilon$, when each component of $X^\varepsilon$ falls in the Brownian case, with convergence understood in the sense of finite dimensional distributions. The result in [30] was improved in [11], where the fast chaos decay restrictions on $G_k$, for $G_k \in L^p$ for $p > 2$, are removed with techniques from Malliavin calculus. In the continuous long-range-dependent case Taqqu, [40], obtained convergence in the continuous topology. These results, although fragmented (in some regimes these are only known for scalar valued processes or only at the level of sequences), provide a fairly good picture of what is going on.

There exists however no vector valued functional limit theorem with joint convergence, when the scaling limit of the components are mixed, in this article we provide a complete description for the joint convergence of $\{X^\varepsilon_k, \varepsilon\}$ for $G_k \in L^2(\mu)$. We have a functional limit theorem for vector valued processes whose components may exhibit both short and long-range dependence. For $G_k$ satisfying a stronger integrability condition, we can also show weak convergence in the $C^0([0, T], \mathbb{R}^d)$-topology and for each fixed time in $L^2$ for the low Hermite rank case, which already have interesting applications. Furthermore, they are the basis for the convergence in a suitable rough topology, which due to the change of the nature of the problem will appear in [18] where rough path theory is used to study slow/fast systems, leading to ‘rough creation’ / ‘rough homogenization’ in which the effective limit is not necessarily a Markov process.

**Application.** Consider the second-order equation on $\mathbb{R}$:

$$\dot{x}^\varepsilon_t = \varepsilon^{H-1} f(x^\varepsilon_t) y^\varepsilon_t, \quad x^\varepsilon_0 = x_0$$

$$\dot{y}^\varepsilon_t = -\frac{1}{\varepsilon} y^\varepsilon_t dt + \frac{\sigma}{\varepsilon^H} dB^H_t, \quad y_0 \sim \varepsilon^{-H} \int_{-\infty}^0 e^{-s/\varepsilon} dB^H_s.$$

Taking $\varepsilon \to 0$, does $x^\varepsilon_t$ converge? In case $H = \frac{1}{2}$ and $f = 1$, this is essentially the Kramer-Smolouchowski limit (this is also called the kinetic Brownian motion model). For $H \neq \frac{1}{2}$ and for $f = 1$ this was shown in [1,7,43] to converge to a fBM, see also [14] for the case with a magnetic field. Given $H > \frac{1}{3}$ and $f \in C_0^3$ (for any $H$ if $f = 1$), we can show $x^\varepsilon_t$ converges to the solution of the equation $\dot{x}_t = f(x_t) \, dB^H_t$ with initial value $x_0$ where the integral in the differential equation is interpreted as a Riemann-Stieltjes integral. Furthermore, we obtain the following bound in $C^{\gamma'}$ where $0 < \gamma' < \gamma < H$:

$$\|x^\varepsilon - x\|_{C^{\gamma'}([0,T])} \lesssim T^{\gamma} \varepsilon^{H-\gamma}.$$

This computation is straightforward, see Propositions 3.1 and 3.2 for detail.

With the functional limit theorem below, Theorem A, we can conclude also the convergence of solutions of the equations, for $h \in C_b^2(\mathbb{R}^d, \mathbb{R}^d)$ and $g \in C_b(\mathbb{R}, \mathbb{R})$,

$$\dot{x}^\varepsilon_t = \alpha(\varepsilon) f(x^\varepsilon_t) \, G(y^\varepsilon_t) + h(x^\varepsilon_t) \, g(y^\varepsilon_t).$$
We show that \( x_t^{\varepsilon} \) converges in \( C^\gamma([0, T], \mathbb{R}) \) for \( \gamma \in (0, H^*(m)) \vee \frac{1}{2} - \frac{1}{p} \) either to the solution of the equation

\[
d\tilde{x}_t = cf(\tilde{x}_t) dZ_t^{H^*(m),m} + \tilde{g} h(\tilde{x}_t),
\]

where \( Z_t^{H^*(m),m} \) is a Hermite process, or to the solution to the Stratonovich stochastic differential equation

\[
d\tilde{x}_t = cf(\tilde{x}_t) \circ dW_t + \tilde{g} h(\tilde{x}_t)
\]

where \( W_t \) is a standard Wiener process (given enough integrability on \( G \)). Here, \( c \) is a specific constant (c.f. Eq. (3.6) depending on \( G \) arising from the homogenization procedure. For the above, we follow [11] and use Malliavin calculus to obtain suitable moment bounds on \( \int_0^t G(y_{\varepsilon}^s) ds \). These results appeared in the previous version of the current paper [18]. Equations driven by fractional Brownian motions are also studied for the averaging regime, see [21] and [16]. A fluctuation theorem around the effective average was obtained [3].

Main Results We denote our underlying probability space by \( (\Omega, \mathcal{F}, P) \). Let \( \mu \) denote the standard Gaussian distribution and we choose \( \sigma \) such that the stationary scaled fOU process, to be defined below, satisfies \( y_t^{\varepsilon} \sim \mu \). Let \( \{H_m, m \geq 0\} \) be the orthogonal Hermite polynomials on \( L^2(\mu) \), such that they have leading coefficient 1 and \( L^2(\mu) \) norm \( \sqrt{m!} \). Given \( G \in L^2(\mu) \), then it posses an expansion of the form \( G(x) = \sum_{k=0}^{\infty} c_k H_k(x) \), where \( c_k = \frac{1}{k!} \langle G, H_k \rangle_{L^2(\mu)} \). A function \( G \) is centred if and only if \( c_0 = 0 \). The smallest \( m \) with \( c_m \neq 0 \) is called the Hermite rank of \( G \). In case the correlations of \( y_t^{\varepsilon} \) do not decay sufficiently fast the path integral \( \alpha(\varepsilon) \int_0^t G(y_{\varepsilon}^s) ds \) ought to be approximated by that of the first term of its Wiener chaos expansion. By orthogonality of the \( H_m \)'s, it is sufficient to study the asymptotics of \( \alpha(\varepsilon) \int_0^t H_m(y_{\varepsilon}^s) ds \) to deduce \( \alpha(\varepsilon) \).

Although the solutions to the fOU equation converge exponentially fast to each other, their autocorrelation function decays only algebraically. The indicator for the behaviour of \( \alpha(\varepsilon) \int_0^t H_m(y_{\varepsilon}^s) ds \) turns out to be

\[
H^*(m) = m(H - 1) + 1,
\]

and the self-similarity exponent of the limiting process is determined by \( \alpha(\varepsilon, H^*(m)) \). For large \( m \), the limit will be a Wiener process, and otherwise the limit \( Z_t \) should have the scaling property: \( \varepsilon^{H^*(m)} Z^\varepsilon_t \sim Z_t \). Indeed, \( Z_t \) are the self-similar Hermite processes. To state the functional limit theorem concisely, we make the following convention,

**Convention 1.1** Given a collection of functions \( \{G_k \in L^2(\mu), k \leq N\} \), we will label the high rank ones first, so the first \( n \) functions satisfy \( H^*(m_k) \leq \frac{1}{2} \), where \( n \geq 0 \), and the remaining satisfy \( H^*(m_k) > \frac{1}{2} \).

**Theorem A** Let \( y^{\varepsilon} \) be the stationary solution to the scaled fractional Ornstein–Uhlenbeck equation (2.4) with standard Gaussian distribution \( \mu \). Let \( G_k : \mathbb{R} \to \mathbb{R} \) be
centred functions in $L^2(\mu)$ with Hermite ranks $m_k$. Write

$$G_k = \sum_{l=m_k}^{\infty} c_{k,l} H_l, \quad \alpha_k(\varepsilon) = \alpha(H^*(m_k), m_k), \quad X^{k,\varepsilon} = \alpha_k(\varepsilon) \int_0^t G_k(y^\varepsilon_s) ds.$$  

Set

$$X^{W,\varepsilon} = (X^{1,\varepsilon}, \ldots, X^{n,\varepsilon}), \quad X^{Z,\varepsilon} = (X^{n+1,\varepsilon}, \ldots, X^{N,\varepsilon}).$$

Then, the following holds:

1. (a) There exist stochastic processes $X^W = (X^1, \ldots, X^n)$ and $X^Z = (X^{n+1}, \ldots, X^N)$ such that on every finite interval $[0, T]$,

$$(X^{W,\varepsilon}, X^{Z,\varepsilon}) \longrightarrow (X^W, X^Z),$$

in the sense of finite dimensional distributions. Furthermore, for any $t > 0$

$$\lim_{\varepsilon \to 0} \|X^{Z,\varepsilon}_t\|_{L^2(\Omega)} = 0.$$

(b) If furthermore each $G_k$ satisfies Assumption 1.2 below, the convergence is weakly in $C^\gamma([0, T], \mathbb{R}^N)$ for every $\gamma < \frac{1}{2} - \frac{1}{\min_{k \leq n} p_k}$, if there is at least one component converging to a Wiener process. Otherwise they converge in $C^\gamma([0, T], \mathbb{R}^N)$ for every $\gamma < \min_{k > n} H^*(m_k) - \frac{1}{p_k}$.

2. We now describe the limit $X = (X^W, X^Z)$.

(1) $X^W \in \mathbb{R}^n$ and $X^Z \in \mathbb{R}^{N-n}$ are independent.

(2) $X^W = U^\varepsilon W_t$ where $U_t$ is a standard Wiener process and $U$ is a square root of the matrix $(A^{i,j})$,

$$A^{i,j} = \int_0^\infty E(G_i(y_s)G_j(y_0)) ds = \sum_{q=m_i \wedge m_j} c_{i,q} c_{j,q} (q!) \int_0^\infty \varrho(r)^q dr,$$

and $\varrho(r) = E(y_{r} y_0)$. In other words, $E(X_i^s X_j^t) = 2(t \wedge s)A^{i,j}$ for $i, j \leq n$.

(3) Let $Z^H_t^{m_k}$ be the Hermite processes, represented by (2.1). Then,

$$X^Z = c_{n+1,m_{n+1}} Z^1_{n+1} \ldots, c_{N,m_N} Z^N_t,$$

where,

$$Z^k_t = \frac{m_k!}{K(H^*(m_k), m_k)} Z^H_t^{m_k}.$$  \hspace{1cm} (1.4)

We emphasize that the Wiener process $W_t$ defining the Hermite processes is the same for every $k$ (c.f. Eq. (2.1)), which is in addition independent of $W_t$.  

 Springer
Assumption 1.2 (Functional Limit $C'$ assumptions) Let $G_k \in L^2(\mu)$ with Hermite rank $m_k \geq 1$.

- **High rank case.** If $H^*(m_k) \leq \frac{1}{2}$, assume $G_k \in L^{p_k}(\mu)$ where $\frac{1}{2} - \frac{1}{p_k} > \frac{1}{2}$ (i.e. $p_k > 6$).
- **Low rank case.** If $H^*(m_k) > \frac{1}{2}$, assume $G_k \in L^{p_k}(\mu)$ where $H^*(m_k) - \frac{1}{p_k} > \frac{1}{2}$.

**Remark 1.3** The case $H = \frac{1}{2}$ is classical, and is not of interest here. In this case the result is independent of the Hermite rank and the scaling is given by $\alpha(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$, due to the exponential decay of correlations.

An immediate application is the following rough homogenisation theorem for a toy model:

**Theorem B** Let $H \in (\frac{1}{3}, 1) \setminus \{\frac{1}{2}\}$, $f \in C^3_b(\mathbb{R}^d, \mathbb{R}^d)$, $h \in C^2_b(\mathbb{R}^d; \mathbb{R}^d)$, $G \in C(\mathbb{R}, \mathbb{R})$ satisfying Assumption 1.2 and $g \in C_b(\mathbb{R}; \mathbb{R})$. Let $\alpha(\varepsilon) = \alpha(\varepsilon, H^*(m))$. Fix a finite time $T$ and consider

\[ x_t^\varepsilon = \alpha(\varepsilon) f(x_t^\varepsilon) G(y_t^\varepsilon) + h(x_t^\varepsilon) g(y_t^\varepsilon), \quad x_0^\varepsilon = x_0. \tag{1.5} \]

1. If $H^*(m) > \frac{1}{2}$, $x_t^\varepsilon$ converges weakly in $C^r([0, T], \mathbb{R}^d)$ to the solution to the Young differential equation $d\tilde{x}_t = cf(\tilde{x}_t) dZ_t^{H^*(m)} + gh(\tilde{x}_t) dt$ with initial value $x_0$ for $\gamma \in (0, H^*(m) - \frac{1}{p})$.

2. If $H^*(m) \leq \frac{1}{2}$, $x_t^\varepsilon$ converges weakly in $C^r([0, T], \mathbb{R})$ to the solution of the Stratonovich stochastic differential equation $d\tilde{x}_t = cf(\tilde{x}_t) \circ dW_t + gh(\tilde{x}_t) dt$ with $\tilde{x}_0 = x_0$, where $\gamma \in (0, \frac{1}{2} - \frac{1}{p})$.

We also take the liberty to point out an intermediate result on the joint convergence of stochastic processes in finite $L^2$ chaos, for it maybe of service. The proof is a slight modification of results in [29,41].

**Proposition 5.1** Let $q_1 \leq q_2 \leq \cdots \leq q_n \leq p_1 \leq p_2 \leq \cdots \leq p_m$. Let $f_i^\varepsilon \in L^2(\mathbb{R}^{p_i})$, $g_i^\varepsilon \in L^2(\mathbb{R}^{q_i})$, $F^{\varepsilon} = (I_{p_1}(f_1^\varepsilon), \ldots, I_{p_m}(f_m^\varepsilon))$ and $G^{\varepsilon} = (I_{q_1}(g_1^\varepsilon), \ldots, I_{q_\varepsilon}(g_\varepsilon))$, where $I_q$ denotes the Wiener integral of order $q$. Suppose that for every $i$, $j$, and any $1 \leq r \leq q_i$:

\[ \| f_j^\varepsilon \otimes_r g_i^\varepsilon \| \to 0. \]

Then, $F^{\varepsilon} \to U$ and $G^{\varepsilon} \to V$ weakly imply that $(F^{\varepsilon}, G^{\varepsilon}) \to (U, V)$ jointly, where $U$ and $V$ are taken to be independent random variables.

**2 Preliminaries**

We take the Hermite polynomials of degree $m$ to be

\[ H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{\frac{x^2}{2}}. \]
Thus, $H_0(x) = 1$, $H_1(x) = x$. Let $\hat{H}$ be the inverse of $H^*(m) = m(H - 1) + 1$:

$$\hat{H}(m) = \frac{1}{m} (H - 1) + 1.$$ 

### 2.1 Hermite Processes

The rank 1 Hermite processes $Z^{H,1}$ are fractional BMs, the formulation (2.1) below is exactly the Mandelbrot-Van Ness representation for a fBM.

**Definition 2.1** Let $m \in \mathbb{N}$ with $\hat{H}(m) > \frac{1}{2}$. The class of Hermite processes of rank $m$ is given by the following mean-zero processes,

$$Z_t^{H,m} = K(H, m) \frac{m!}{m!} \int_{\mathbb{R}^m} \int_0^t \prod_{j=1}^m (s - \xi_j)^{-\frac{1}{2} \frac{H - 1}{m}} ds dW(\xi_1) \ldots dW(\xi_m), \quad (2.1)$$

where the constant $K(H, m)$ is chosen so their variances are 1 at $t = 1$.

The integration in (2.1) is understood as a multiple Wiener-Itô integral (over the region $\mathbb{R}^n$ without the diagonal). Note $\hat{H}(1) = H$.

By the properties of Wiener integrals, two Hermite processes $Z_t^{H,m}$ and $Z_t^{H',m'}$, defined by the same Wiener process, are uncorrelated if $m \neq m'$. The Hermite processes have stationary increments and finite moments of all orders with covariance

$$E(Z_t^{H,m} Z_s^{H,m}) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (2.2)$$

Therefore, using Kolmogorov’s theorem 2.6, one can show that the Hermite processes $Z_t^{H,m}$ have sample paths of Hölder regularity up to $H$. They are also self-similar with exponent $H$ which means $\lambda^H Z_{\lambda t}^{H,m} \sim Z_t^{H,m}$.

**Remark 2.2** In some literature, see, e.g., [27] where further details on Hermite processes can also be found, the Hermite processes are defined with a different exponent as below:

$$Z_t^{H,m} = K(H, m) \frac{m!}{m!} \int_{\mathbb{R}^m} \int_0^t \prod_{j=1}^m (s - \xi_j)^{-\frac{H - 3}{2}} ds dW(\xi_1) \ldots dW(\xi_m).$$

The two notions are related by

$$Z_t^{H^*(m),m} = \tilde{Z}_t^{H,m}, \quad Z_t^{H,m} = \tilde{Z}_t^{\hat{H}(m),m}. \quad (2.3)$$

We refer to [10,35,37] for detailed studies of fBM’s which are used in this article.
2.2 Fractional Ornstein–Uhlenbeck Processes

Let us normalise the fractional Brownian motion, so that $B^H_0 = 0$ and $E(B^H_1)^2 = 1$. Disjoint increments of $B^H_t$ have a covariance of the form:

$$E(B_t - B_s)(B_u - B_v) = \frac{1}{2} \left( |t - v|^{2H} + |s - u|^{2H} - |t - u|^{2H} - |s - v|^{2H} \right).$$

We define the stationary fractional Ornstein–Uhlenbeck processes to be $y_t = \sigma \int_{-\infty}^t e^{-\sigma(t-s)} dB^H_s$, where $B^H_t$ is a two-sided fractional BM, $\sigma$ is chosen such that $y_t$ is distributed as $\mu = N(0, 1)$. It is the solution of the following Langevin equation:

$$dy_t = -y_t \, dt + \sigma \, dB^H_t,$$

$$y_0 = \sigma \int_{-\infty}^0 e^s \, dB^H_s.$$

We take $y^\varepsilon_t$, the fast or rescaled fOU, to be the stationary solution of

$$dy^\varepsilon_t = -\frac{1}{\varepsilon} y^\varepsilon_t \, dt + \sigma \varepsilon \, dB^H_t.$$  (2.4)

Observe that $y^\varepsilon$ and $y^\varepsilon_*$ have the same distributions, $y^\varepsilon_t = \frac{\sigma}{\varepsilon} \int_{-\infty}^t e^{-\frac{1}{\varepsilon}(t-s)} dB^H_s$. In particular, both $y_t$ and $y^\varepsilon_t$ are Hölder continuous with Hölder exponents $\gamma \in (0, H)$. Let us denote their correlation functions by $\varrho$ and $\varrho^\varepsilon$:

$$\varrho(s, t) := E(y_s y_t), \quad \varrho^\varepsilon(s, t) := E(y^\varepsilon_s y^\varepsilon_t).$$

Let $\varrho(s) = E(y_0 y_s)$ for $s \geq 0$ and extended to $\mathbb{R}$ by symmetry, then $\varrho(s, t) = \varrho(t-s)$ and similarly for $\varrho^\varepsilon$. For $H > \frac{1}{2}$, the set of functions for which Wiener integrals are defined include $L^2$ functions and so $\varrho$ posses an analytical expression. Indeed, since

$$E(B^H_t B^H_s) = H(2H - 1) \int_0^t \int_0^s |r_1 - r_2|^{2H-2} \, dr_1 \, dr_2,$$

we have

$$\frac{\partial^2}{\partial t \partial s} E(B^H_t B^H_s) = H(2H - 1)|t - s|^{2H-2},$$

which is integrable, and therefore we may use the Wiener isometry to compute the covariances

$$E(y_t y_s) = \sigma^2 H(2H - 1) \int_{-\infty}^t \int_{-\infty}^s e^{-(s+t-r_1-r_2)} |r_1 - r_2|^{2H-2} \, dr_1 \, dr_2.$$

For $u > 0$, we set

$$\varrho(u) = \sigma^2 H(2H - 1) \int_{-\infty}^u \int_{-\infty}^0 e^{-(u-r_1-r_2)} |r_1 - r_2|^{2H-2} \, dr_1 \, dr_2.$$
Using this, the following correlation decay was shown in [10].

**Lemma 2.3** Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then, $\varrho(s) = 2\sigma^2 H (2H - 1)s^{2H-2} + O(s^{2H-4})$ as $s \to \infty$. In particular, for any $t \neq s$,
\[
|\varrho(s, t)| \lesssim 1 \land |t - s|^{2H-2}.
\] (2.5)

Hence, $\int_0^\infty \varrho^m(s)ds$ is finite if and only if $H^*(m) < \frac{1}{2}$, or $H = \frac{1}{2}$ and $m \in \mathbb{N}$, as in the latter the usual OU process admits exponential decay of correlations.

**Lemma 2.4** Let $H \in (0, 1) \setminus \{\frac{1}{2}\}$, fix a finite time horizon $T$, then for $t \in [0, T]$ the following holds uniformly for $\varepsilon \in (0, \frac{1}{2})$:
\[
\left( \int_0^t \int_0^t |\varrho(u, r)|^m \, dr \, du \right)^{\frac{1}{2}} \lesssim \begin{cases} 
\sqrt{\frac{t}{\varepsilon}} \int_0^\infty |\varrho(s)|^m \, ds, & \text{if } H^*(m) < \frac{1}{2}, \\
\sqrt{\frac{t}{\varepsilon}} |\ln (\frac{1}{\varepsilon})|, & \text{if } H^*(m) = \frac{1}{2}, \\
\left(\frac{t}{\varepsilon}\right)^{H^*(m)-1}, & \text{if } H^*(m) > \frac{1}{2}.
\end{cases}
\]
(2.6)

\[
\left( \int_0^t \int_0^t |\varrho^\varepsilon(u, r)|^m \, dr \, du \right)^{\frac{1}{2}} \lesssim \begin{cases} 
\sqrt{\frac{t}{\varepsilon}} \int_0^\infty |\varrho(s)|^m \, ds, & \text{if } H^*(m) < \frac{1}{2}, \\
\sqrt{\frac{t}{\varepsilon}} |\ln (\frac{1}{\varepsilon})|, & \text{if } H^*(m) = \frac{1}{2}, \\
\left(\frac{t}{\varepsilon}\right)^{H^*(m)-1}, & \text{if } H^*(m) > \frac{1}{2}.
\end{cases}
\]
(2.7)

Note, if $H = \frac{1}{2}$, for and any $m \in \mathbb{N}$, the bound is always $\sqrt{\frac{t}{\varepsilon}} \int_0^\infty \varrho^m(s)ds$. In particular,
\[
t \int_0^t |\varrho^\varepsilon(s)|^m \, ds \lesssim \frac{t^{(2H^*(m)+1)}}{\alpha(\varepsilon, H^*(m))^2}.
\] (2.8)

**Proof** We first observe that
\[
\int_0^\infty \varrho^m(s)ds < \infty \iff H^*(m) < \frac{1}{2} \iff H < 1 - \frac{1}{2m}.
\] (2.9)

By a change of variables and using estimate (2.5) on the decay of the auto-correlation function,
\[
\int_0^t \int_0^t |\varrho(|u - r|)|^m \, dr \, du = 2\varepsilon \int_0^t \varrho(s)^m \, ds 
\lesssim \begin{cases} 
\frac{1}{\varepsilon} \int_0^\infty \varrho^m(s)ds, & \text{if } H^*(m) < \frac{1}{2}, \\
\left(\frac{1}{\varepsilon}\right)^{2H^*(m)}, & \text{otherwise}.
\end{cases}
\]
For the case $H^*(m) = \frac{1}{2}$, we use

$$
\int_0^T |q(s)|^mds \leq \int_0^T |q(s)|^mds \lesssim \int_0^T \left( 1 + \frac{1}{s} \right) ds \lesssim |\ln(T)\varepsilon| \lesssim |\ln(\frac{1}{\varepsilon})|.
$$

To complete the proof, we observe that by a simple change of variables,

$$
\int_0^t \int_0^t |\rho(s,r)|^mdsdr = \varepsilon^2 \int_0^T \int_0^T |\rho(s,r)|^mdsdr.
$$

Next, we recall the Garsia-Rodemich-Romsey-Kolmogorov inequality.

**Lemma 2.5** *(Garsia-Rodemich-Romsey-Kolmogorov inequality)* Let $T > 0$. Let $\theta : [0, T] \rightarrow \mathbb{R}^d$. For any positive numbers $\gamma, p$ there exists a constant $C(\gamma, p)$ such that

$$
\sup_{s \neq t, s, t \in [0, T]} \frac{|\theta(t) - \theta(s)|}{|t - s|^{\gamma}} \leq C(\gamma, p) \left( \int_0^T \int_0^T \frac{|\theta_s - \theta_r|^p}{|s - r|^{\gamma p + 2}} dsdr \right)^\frac{1}{p}.
$$

See [17, section A.2] in Appendix, as well as [39], for a proof. As a consequence of this inequality, one obtains the following theorem.

**Theorem 2.6** *(Kolmogorov’s Theorem)* Let $\theta$ be a stochastic process. Suppose that for $s, t \in [0, T], p > 1$ and $\delta > 0$,

$$
E|\theta(t) - \theta(s)|^p \leq c_p |t - s|^{1 + \delta},
$$

where $c_p$ is a constant. Then, for $\gamma < \frac{\delta}{p}$, $\theta \in C^\gamma([0, T], \mathbb{R})$, and in particular

$$
\|\theta\|_{C^\gamma([0, T])}^p \leq C(\gamma, p)(c_p)^\frac{1}{p} \left( \int_0^T \int_0^T |u - v|^{\delta - \gamma p - 1} dudv \right)^\frac{1}{p},
$$

where right hand side is finite when $\gamma \in (0, \frac{\delta}{p})$.

**Lemma 2.7** For any $\gamma \in (0, H), p > 1$, the following estimates hold:

$$
\sup_{s, t \geq 0} \frac{\|y_s - y_t\|_p^p}{1 + |s - t|^H} \lesssim 1, \quad E \sup_{s \neq t, s, t \in [0, T]} \left( \frac{|y_s - y_t|}{|t - s|^{\gamma}} \right)^p \lesssim T C(\gamma, p).
$$

**Proof** We use the fractional Ornstein–Uhlenbeck equation

$$
y_s - y_r = - \int_r^s y_u du + B_s^H - B_r^H,
$$

\[ \Box \] Springer
to obtain $E|y_s - y_r|^2 \lesssim (s-r)^2 E|y_1|^2 + q|s-r|^{2H}$. Using the stationarity of $y_t$, one also has $E|y_s - y_r|^2 \leq 2E|y_1|^2 = 2$. Since for Gaussian random variables the $L^2$ norm controls the $L^p$ norm we have

$$\|y_s - y_r\|_{L^p} \lesssim \begin{cases} 1, & \text{if } |s-r| \geq 1; \\ |s-r|^H, & \text{if } |s-r| \leq 1. \end{cases}$$

Thus, by symmetry and a change of variables, $\int_0^T \int_0^T E|y_s - y_r|^p dsdr \lesssim T$ and application of Kolmogorov’s theorem 2.6 concludes the proof. \hfill $\square$

3 Applications

3.1 The Second-Order Problem

If $x$ is a stochastic process, we write $x_{s,t} = x_t - x_s$.

Proposition 3.1 Let $H \in (0,1)$, $\gamma \in (0, H)$, $p > 1$ and fix a finite time $T$. Let $X^\varepsilon_t = \varepsilon^{H-1} \int_0^t y^\varepsilon_s ds$, then,

$$\sup_{s,t \in [0,T]} \left\| X^\varepsilon_{s,t} - \sigma B^H_{s,t} \right\|_{L^p} \lesssim \varepsilon^H,$$

$$\left\| X^\varepsilon_t - \sigma B^H \right\|_{L^p} \lesssim t^{\gamma} \varepsilon^{H-\gamma},$$

for any $\gamma' < \gamma < H$ and for $t \in [0, T]$.

Proof Set $v^\varepsilon_t = \varepsilon^{H-1} y^\varepsilon_t$, then, $v^\varepsilon_t$ solves the following equation

$$dv^\varepsilon_t = -\frac{1}{\varepsilon} v^\varepsilon_t dt + \frac{\sigma}{\varepsilon} dB^H_t.$$

Using the equation for $v^\varepsilon_t$, we have

$$X^\varepsilon_{s,t} = \varepsilon^{H-1} \int_s^t y^\varepsilon_r dr = \int_s^t v^\varepsilon_r dr = \varepsilon(v^\varepsilon_t - v^\varepsilon_s) + \sigma B^H_{s,t}.$$

Therefore, for any $p > 1$,

$$\sup_{s,t \in [0,T]} \left\| X^\varepsilon_{s,t} - \sigma B^H_{s,t} \right\|_{L^p} = \sup_{s,t \in [0,T]} \left\| \varepsilon(v^\varepsilon_t - v^\varepsilon_s) \right\|_{L^p} \lesssim \varepsilon^H.$$

In the last step, we used the stationarity of $y^\varepsilon_t$. 

\hfill Springer
Thus, applying Kolmogorov’s theorem 2.6 to \( X^\varepsilon - \sigma B^H \), we see that the following holds for any \( t \in [0, T] \), \( p > 1 \) and any \( \gamma' < \gamma \)
\[
\| X^\varepsilon - \sigma B^H \|_{C^{\gamma'}([0,t], \mathbb{R})} \lesssim \varepsilon^H \left( \frac{t}{\varepsilon} \right)^{\gamma},
\]
hence, the claim follows. \( \square \)

As an application, we consider the following slow/fast system,
\[
\begin{aligned}
\dot{x}^\varepsilon_t &= \varepsilon^{H-1} f(x^\varepsilon_t) y_t^\varepsilon, \\
x^\varepsilon_0 &= x_0.
\end{aligned}
\tag{3.1}
\]
To describe its limit, we review the concept of Young integrals. If \( f, g : [0, T] \to \mathbb{R} \) with \( f \in C^\alpha \) and \( g \in C^\beta \) such that \( \alpha + \beta > 1 \), then the Riemann-Stieljes integral makes sense, and
\[
\int_0^t f_s d g_s = \lim_{|P| \to 0} \sum_{[u,v] \subset P} u^{\varepsilon} f_u (g_u - g_v) \in C^\beta.
\]
For details see [42]; this integral is called a Young integral. Since Young integrals have the regularity of its integrand, for \( H > \frac{1}{2} \) the equation \( \dot{x} = f(x_t) d B^H_t \) makes sense. In [25], it was shown that if \( f \in C^3_b \), then the equation has a unique global solution from each initial value. This type of equations are Young equation, the simplest type of rough differential equation. The notation \( C^3_b \) denotes the space of bounded functions such that their first three derivatives are bounded.

**Proposition 3.2** Let \( H \in (\frac{1}{2}, 1) \) and \( f \in C^3_b(\mathbb{R}^d, \mathbb{R}^d) \). Then for any \( \gamma \in (0, H), \gamma' < \gamma, x^\varepsilon_t \) converges in \( L^p \) in \( C^{\gamma'}([0, T], \mathbb{R}^d) \) to the solution of the rough differential equation:
\[
\dot{x}_t = \sigma f(x_t) d B^H_t, \quad x_0 = x_0.
\tag{3.2}
\]
Furthermore, for \( t \in [0, T] \),
\[
\| x^\varepsilon - x \|_{C^{\gamma'}([0,t], \mathbb{R})} \lesssim t^{\gamma} \varepsilon^{H-\gamma}.
\]

**Proof** The idea is to consider Eq. (3.1) as a Young/rough differential equation. In case \( H \in (\frac{1}{2}, 1) \), we can rewrite our equation as
\[
\dot{x}^\varepsilon_t = f(x^\varepsilon_t) d X^\varepsilon_t,
\]
where \( X^\varepsilon = \varepsilon^{H-1} \int_0^t y^\varepsilon_s d s \) is as in the previous lemma. Young’s continuity theorem, see Theorem 3.6, states that \( x^\varepsilon \) converges weakly provided \( X^\varepsilon \) converges weakly in a Hölder space of regularity greater than \( \frac{1}{2} \).
For $H \in \left( \frac{1}{3}, \frac{1}{2} \right)$, we need to rewrite our equation into a rough differential equation,
\[ \dot{x}_t^\epsilon = f(x_t^\epsilon) \, dX^\epsilon_t, \]
where $X^\epsilon$ is given by $X^\epsilon$ enhanced with its canonical lift
\[ X^\epsilon_{s,t} = \int_s^t (X^\epsilon_r - X^\epsilon_s) \, dX^\epsilon_r. \]
As we restrict ourselves to one dimension we obtain, by symmetry,
\[ X^\epsilon_{s,t} = \frac{1}{2} \left( X^\epsilon_{s,t} \right)^2, \]
and hence, $X^\epsilon$ converges to a fBm $\sigma B^H$ enhanced with $B^H = \frac{1}{2} (\sigma B^H_{s,t})^2$. As the solution map, $\Phi$, to a RDE satisfies, see [15] or Theorem 3.6 below,
\[ |\Phi(X^\epsilon) - \Phi(B^H)|_{C^{\gamma'}} \lesssim \varrho_{\gamma}(X^\epsilon, B^H), \]
where $\rho^{\gamma}$ denotes the inhomogeneous rough path norm of regularity $\gamma$ and $B^H = \frac{1}{2} (\sigma B^H_{s,t})^2$. Thus, the $L^p$ convergence of the solutions follows from the $L^p$ convergence of the drivers, hence, we can conclude the proof by Proposition 3.1. \[ \square \]

**Remark 3.3** Krammer-Smoluchowski limits/Kinetic fBM’s are studied in [1,7,43]. See also [14–16].

**Remark 3.4**
1. For $H < \frac{1}{2}$ and $m = 1$, Theorem A appears to contradict with Proposition 3.1; in the first we claim the limit is a Brownian motion, whereas in the second we claim that it is a fractional Brownian motion. Both results are correct and can be easily explained. It lies in the fact that $\int_R \varrho(s) \, ds$ vanishes if $H < \frac{1}{2}$, and so the Brownian motion limit is degenerate. Since according to [10],
\begin{equation}
\varrho(s) = \sigma^2 \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi} \int_R e^{isx} \frac{|x|^{1-2H}}{1 + x^2} \, dx,
\end{equation}
and by the decay estimate from (2.5), $\varrho$ is integrable, $s(\lambda)$ is the value at zero of the inverse Fourier transform of $\varrho(s)$, which is up to a multiplicative constant $|\lambda|^{1-2H}$. This is also the spectral density of $y_t$ and has value 0 at 0. This means we have scaled too much and the correct scaling is to multiply the integral $\int_0^t \varrho^\epsilon \, ds$ by $e^{H-1}$ in which case we obtain a fBm as limit.
2. For $m > 1$ and $H < \frac{1}{2}$ the Wiener limit is not trivial. Indeed,
\[ \int_R \varrho(s)^m \, ds = C \int_R \ldots \int_R \prod_{k=1}^m e^{isx_k} \frac{|x_k|^{1-2H}}{1 + |x_k|^2} \, dx_1 \ldots dx_m \, ds \]
\[ = C \int_R \ldots \int_R |x_2 + \ldots + x_m|^{1-2H} \prod_{k=2}^m |x_k|^{1-2H} \neq 0. \]
3.2 The 1d Fluctuation Problem

In this section, we give an application of Theorem A. Given a function \( g \in L^1(\mu) \) we denote \( \bar{g} = \int_{\mathbb{R}} g(y) \mu(dy) \).

**Lemma 3.5** The stationary Ornstein–Uhlenbeck process is ergodic. Thus, \( \int_0^t g(y_s^\varepsilon)ds \to t \bar{g} \) in probability for every \( g \in L^1(\mu) \).

**Proof** A stationary Gaussian process is ergodic if its spectral measure has no atom, see [9,37]. The spectral measure \( F \) of a stationary Gaussian process is obtained from Fourier transforming its correlation function and \( \varrho(\lambda) = \int_{\mathbb{R}} e^{ix\lambda} dF(x) \). According to [10]:

\[
\varrho(s) = \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi} \int_{\mathbb{R}} e^{ixs} |x|^{-2H} \frac{dx}{1 + x^2},
\]

so the spectral measure is absolutely continuous with respect to the Lebesgue measure with spectral density, up to a nonzero constant, given by \( s(x) = \frac{|x|^{-2H}}{1 + x^2} \). Since \( \int_0^t g(y_s^\varepsilon)dt \) equals \( \int_0^t g(y_s)ds \) in law, the former converges in law to \( t \bar{g} \) by Birkhoff’s ergodic theorem. The claim now follows as weak convergence to a constant implies convergence in probability. \( \square \)

In the following proof we will need the following theorem from Young/rough path theory for details we refer to [15,17,24,25]. We denote the space of rough paths of regularity \( \beta \) by \( \mathcal{C}^\beta \).

**Theorem 3.6** Let \( Y_0 \in \mathbb{R}^d, \beta \in (\frac{1}{2}, 1), \gamma \in (\frac{1}{2}, 1) \), \( f \in \mathcal{C}^3_b(\mathbb{R}^d, \mathbb{R}^d), h \in \mathcal{C}^2_b(\mathbb{R}^d, \mathbb{R}^d), X \in \mathcal{C}^\beta([0, T], \mathbb{R}) \) and \( Z \in \mathcal{C}^\gamma([0, T], \mathbb{R}) \) such that \( \beta + \gamma > 1 \). Then, the differential equation

\[
Y_t = Y_0 + \int_0^t f(Y_s)dX_s + \int_0^t h(Y_s)dZ_s
\]

has a unique solution which belongs to \( \mathcal{C}^{\beta + \gamma} \). Furthermore, the solution map \( \Phi_{f,h} : \mathbb{R} \times \mathcal{C}^\beta([0, T], \mathbb{R}) \times \mathcal{C}^\gamma([0, T], \mathbb{R}) \to \mathcal{C}^{\beta + \gamma}([0, T], \mathbb{R}) \), where the first component is the initial condition and the second and third components the drivers, is continuous.

Given a centred function \( G \in L^2(\mu) \), with chaos expansion \( G = \sum_{k=m}^{\infty} c_k H_k \) and let \( c \geq 0 \) be given by

\[
c^2 = \begin{cases} \frac{(c_m m!)^2}{K(H,m)}, & H^*(m) > \frac{1}{2} \\ 2 \sum_{k=m}^{\infty} (c_k)^2 k! \int_0^\infty q^k(s)ds, & H^*(m) < \frac{1}{2} \\ 2m!(c_m)^2, & H^*(m) = \frac{1}{2}. \end{cases}
\]

**Theorem 3.7** Let \( H \in (\frac{1}{2}, 1) \setminus \{\frac{1}{2}\} \), \( f \in \mathcal{C}^3_b(\mathbb{R}^d, \mathbb{R}^d), h \in \mathcal{C}^2_b(\mathbb{R}^d; \mathbb{R}^d), G \in C(\mathbb{R}, \mathbb{R}) \) satisfying Assumption 1.2 and \( g \in \mathcal{C}_b(\mathbb{R}; \mathbb{R}) \). Let \( \alpha(\varepsilon) = \alpha(\varepsilon, H^*(m)) \). Fix a finite time \( T \) and consider

\[
\dot{x}_t^\varepsilon = \alpha(\varepsilon) f(x_t^\varepsilon) G(y_t^\varepsilon) + h(x_t^\varepsilon) g(y_t^\varepsilon), \quad x_0^\varepsilon = x_0.
\]
1. If $H^*(m) > \frac{1}{2}$, $x^\varepsilon_t$ converges weakly in $C^\gamma([0, T], \mathbb{R}^d)$ to the solution to the Young differential equation $d\tilde{x}_t = cf(\tilde{x}_t) dZ_{H^*(m)}^m + \tilde{g} h(\tilde{x}_t) dt$ with initial value $x_0$ for $\gamma \in (0, H^*(m) - \frac{1}{p})$.

2. If $H^*(m) \leq \frac{1}{2}$, $x^\varepsilon_t$ converges weakly in $C^\gamma([0, T], \mathbb{R}^d)$ to the solution of the Stratonovich stochastic differential equation $d\tilde{x}_t = cf(\tilde{x}_t) \circ dW_t + \tilde{g} h(\tilde{x}_t) dt$ with $\tilde{x}_0 = x_0$, where $\gamma \in (0, \frac{1}{2} - \frac{1}{p})$.

**Proof** As in Proposition 3.2 we can rewrite our equations as Young/rough differential equations and therefore reduce our analysis to the drivers $(\alpha(\varepsilon) \int_0^t G(\tilde{y}^\varepsilon_s) ds, \int_0^t g(\tilde{y}^\varepsilon_s) ds)$. By Theorem A, $\alpha(\varepsilon) \int_0^t G(\tilde{y}^\varepsilon_s) ds$ converges in finite dimensional distributions either to a Wiener or a Hermite process. By Lemma 3.5, $\int_0^t g(\tilde{y}^\varepsilon_s) ds$ converges in probability to the deterministic path $\tilde{t} \tilde{g}$. Hence, $(\alpha(\varepsilon) \int_0^t G(\tilde{y}^\varepsilon_s) ds, \int_0^t g(\tilde{y}^\varepsilon_s) ds)$ converges jointly in finite dimensional distributions. Furthermore, $\|\int_0^t g(\tilde{y}^\varepsilon_s) ds\|_\infty \leq t \|g\|_\infty$, this combined with the moment bounds obtained in Theorem A enables us to apply Theorem 3.6 to conclude the proof.

**Remark 3.8** The constant $c$ could be 0, for further details see Remark 3.4.

### 4 Proof of Theorem A

We first establish the $L^2(\Omega)$ convergence of $X^\varepsilon_t = \alpha(\varepsilon) \int_0^t G(\tilde{y}^\varepsilon_s) ds$, where $G = \sum_{k=0}^\infty c_k H_k$ has low Hermite rank, followed by a reduction theorem. We then prove moment bounds and conclude the proof of Theorem A.

#### 4.1 Preliminary Lemmas

The basic scalar valued functional limit theorem, for low rank Hermite functions, was proved in [40] for $e^{H^*(m)} \int_0^t G(X_s) ds$ with $X_t = \int_\mathbb{R} p(t - \xi) dW_\xi$ a moving average, where in order to prove convergence one uses the self-similarity of a Wiener process leading to weak convergence as this equivalence of course is only in law. Nevertheless, in our case we can choose a properly scaled fast variable and write, $y^\varepsilon_t = \int_\mathbb{R} \hat{g}(\frac{t - \xi}{\varepsilon}) dW_\xi$ for a function $\hat{g}$, and thus avoid using self-similarity. The key idea is to write a Wiener integral representation beginning with

$$y^\varepsilon_t = e^{-H} \int_{-\infty}^t e^{-\frac{r-s}{\varepsilon}} dB_r^H = \int_\mathbb{R} h_\varepsilon(t, s) dW_s,$$

where,

$$h_\varepsilon(t, s) = e^{-\frac{1}{2} \sigma \int_{c_1(H)} e^{-\frac{r-u}{\sigma}} \int_0^{\frac{r-s}{\sigma}} e^v v^H H \frac{1}{2} dv},$$

(4.1)
and \( c_1 (H) = \sqrt{\int_{-\infty}^{0} \left( (1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H}} \). This can be obtained by applying the integral representation for fBM’s:

\[
B^H_t = \int_{-\infty}^{\infty} g(t, s) dW_s, \quad \text{where} \quad g(t, s) = \frac{1}{c_1 (H)} \int_{0}^{t} (r-s)^{H-\frac{3}{2}} + \frac{1}{2} H ds, \quad (4.2)
\]

and by repeated applications of integration by parts (to the Young integrals):

\[
\sigma \int_{-\infty}^{t} e^{-\frac{s-t}{\varepsilon}} dB^H_s = \sigma \int_{-\infty}^{t} e^{-\frac{s-t}{\varepsilon}} B^H_s ds - \sigma \int_{-\infty}^{t} \int_{-\infty}^{r} e^{-\frac{s-t}{\varepsilon}} \partial_s g(s, r) ds dW_r
\]

\[
= \frac{\sigma}{c_1 (H)} \int_{R} \int_{-\infty}^{t} e^{-\frac{s-t}{\varepsilon}} (s-r)^{H-\frac{3}{2}} ds dW_r.
\]

One may also use the following, see [34], taking \( f \in L^1 \cap L^2 \):

\[
\int_{R} f(u) dB^H_u = \frac{1}{c_1 (H)} \int_{R} \int_{R} f(u)(u-s)^{H-\frac{3}{2}} du dW_s.
\]

**Lemma 4.1** Let \( \lambda \) denote the Lebesgue measure, then as \( \varepsilon \to 0 \), \( \varepsilon H^*(m)^{-1} \int_{0}^{t} H_m (\gamma_{s}^\varepsilon) ds \) converges to \( \frac{m!}{K(H, m)} Z_{H^*(m), m} \) in \( L^2 (\Omega) \). Equivalently,

\[
\left\| \int_{0}^{t} \prod_{i=1}^{m} h_{\varepsilon}(s, u_i) ds - \int_{0}^{t} \prod_{i=1}^{m} (s - u_i)^{H-\frac{3}{2}} ds \right\|_{L^2 (R^m, \lambda)} \to 0. \quad (4.3)
\]

**Proof** This can be shown by applying [40, Theorem 4.7], where weak convergence is obtained. With a small modification and using [40, Lemma 4.5, Lemma 4.6] directly we obtain the \( L^2 (\Omega) \) convergence:

\[
E \left( \int_{R^m} \int_{0}^{t} \prod_{i=1}^{m} p \left( \frac{s - \xi_i}{\varepsilon} \right) (s - \xi_i)^{H-\frac{3}{2}} ds dW_{\xi} - \frac{Z_{H^*(m), m}}{K(H, m)} \right)^2 \to 0,
\]

using the Wiener integral representation of the Hermite processes this is equivalent, by a multiple Wiener-Itô isometry, to

\[
\int_{R^m} \left( \int_{0}^{t} \prod_{i=1}^{m} p \left( \frac{s - \xi_i}{\varepsilon} \right) (s - \xi_i)^{H-\frac{3}{2}} ds - \int_{0}^{t} \prod_{i=1}^{m} (s - \xi_i)^{H-\frac{3}{2}} ds \right)^2 d\xi_1 \ldots d\xi_m \to 0. \quad (4.4)
\]

Examining Taqqu’s proof, we note that in fact the \( L^2 \) convergence of (4.4) is obtained under the following conditions.
\[ \int_{\mathbb{R}} p(s)^2 ds < \infty. \]
\[ |p(s)| \leq C s^{H-\frac{3}{2}} L(u) \text{ for almost all } s > 0. \]
\[ p(s) \sim s^{H-\frac{3}{2}} L(s) \text{ as } s \to \infty. \]
4 There exists a constant \( \gamma \) such that \( 0 < \gamma < (1 - H) \wedge (H - (1 - \frac{1}{2m})) \) such that \( \int_{-\infty}^{0} |p(s) p(xy+s)| ds = o(x^{2H-2} L^2(x)) y^{2H-2-2\gamma} \) as \( x \to \infty \) uniformly in \( y \in (0, t] \).

where \( L \) denotes a slowly varying function (for every \( \lambda > 0 \) \( \lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1 \)). Set

\[ \hat{g}(s) = \frac{\sigma}{c_1(H)} e^{-s} \int_{0}^{s} e^u u^{H-\frac{3}{2}} du, \tag{4.5} \]

then,

\[ y_t^e = e^{-\frac{1}{2}} \int_{-\infty}^{t} \hat{g}\left( \frac{t-s}{\varepsilon} \right) dW_s. \]

We are now in Taqqu’s framework and it is only left the check \( \hat{g} \) defined by (4.5) satisfies these conditions. To increase readability, we suppress the constant \( \frac{\sigma}{c_1(H)} \) in the following computations. For \( s < 1 \),

\[ e^{-s} \int_{0}^{s} e^u u^{H-\frac{3}{2}} du \leq \int_{0}^{s} u^{H-\frac{3}{2}} du \lesssim s^{H-\frac{3}{2}}. \]

We calculate for \( s > 1 \) via integration by parts

\[ e^{-s} \int_{0}^{s} e^u u^{H-\frac{3}{2}} du \leq e^{-s} \int_{0}^{1} e^u u^{H-\frac{3}{2}} du + e^{-s} \int_{1}^{s} e^u u^{H-\frac{3}{2}} du \]
\[ \lesssim e^{-s} + s^{H-\frac{3}{2}} + e^{-s} \int_{1}^{s} e^u u^{H-\frac{5}{2}} du \lesssim s^{H-\frac{3}{2}}. \]

This of course implies that \( \hat{g} \) is \( L^2(\lambda) \) integrable. Finally, observe that \( \int_{-\infty}^{0} |\hat{g}(s) \hat{g}(xy+s)| ds = 0 \) as \( \hat{g}(s) = 0 \) for \( s < 0 \). With these we apply [40, Theorem 4.7] to conclude the \( L^2 \) convergence of the kernels. \( \square \)

**Lemma 4.2** Let \( G \in L^2(\mu) \) be a centred function with Hermite rank \( m \) satisfying \( H^*(m) > \frac{1}{2} \). Let \( H \in \left( \frac{1}{2}, 1 \right) \). Then, the following statements hold for the stationary scaled fOU process \( y_t^e \). Fix \( t > 0 \), then,

\[ \left\| e^{H^*(m)-1} \int_{0}^{t} G(y_s^e) ds - \frac{c_m m!}{K(H^*(m), m)} Z_{t}^{H^*(m), m} \right\|_{L^2(\Omega)} \to 0. \]
For $G = H_m$, the claim has already been shown in Lemma 4.1. To conclude the claim in the case of a general $G$, we compute,

$$\left\| \epsilon^{H_m(m) - 1} \int_0^t (G - c_m H_m)(y_s^\epsilon) ds \right\|_{L^2(\Omega)}^2 = \epsilon^{2H_m(m) - 2} \sum_{k=m+1}^{\infty} c_k^2 k! o(\epsilon^{2H_m(m) - 2}) \to 0$$

as $\sum_{k=m+1}^{\infty} c_k^2 \sqrt{k!} < \infty$ as $G \in L^2(\mu)$. This finishes the proof.

The fact that only the first term in the chaos expansion gives a contributions is in the literature often called a reduction lemma. In the high Hermite rank case however it is not possible to restrict one’s analysis to a pure Hermite polynomial, but as the next lemma shows finite linear combinations are indeed sufficient. To make the application later on easier, we directly prove it in the multi-dimensional case.

**Lemma 4.3** *(Reduction Lemma)* Fix $H \in (0, 1) \setminus \{\frac{1}{2}\}$. For $M \in \mathbb{N}$, define the truncated functions:

$$G_{k,M} = \sum_{j=m_k}^{M} c_{k,j} H_j, \quad X_{M,M}^{k,\epsilon}(t) = \alpha_k(\epsilon) \int_0^t G_{k,M}(y_s^\epsilon) ds.$$  

If for every $M \in \mathbb{N}$, $(X_{M,1}^{1,\epsilon}, \ldots, X_{M,N}^{N,\epsilon}) \overset{(\epsilon \to 0)}{\to} (X_{M,1}^{1}, \ldots, X_{M,N}^{N})$ in finite dimensional distributions, then,

$$(X_{1,1}^{1,\epsilon}, \ldots, X_{N,N}^{N,\epsilon}) \overset{(\epsilon \to 0)}{\to} (X_{1}, \ldots, X_{N})$$

in finite dimensional distributions.

**Proof** Firstly,

$$X_{M}^{k,\epsilon}(t) - X_{M,M}^{k,\epsilon}(t) = \alpha_k(\epsilon) \int_0^t \left( G_k(y_s^\epsilon) - G_{k,M}(y_s^\epsilon) \right) ds$$

$$= \alpha_k(\epsilon) \int_0^t \sum_{j=M+1}^{\infty} c_{k,j} H_j(y_s^\epsilon) ds.$$
Using properties of the Hermite polynomials, we obtain

\[ E \left( \alpha_k(\varepsilon) \int_0^t \sum_{j=M+1}^{\infty} c_{k,j} H_j(y^\varepsilon_s) ds \right)^2 \]

\[ = \alpha_k(\varepsilon)^2 \int_0^t \int_0^t \sum_{j=M+1}^{\infty} (c_{k,j})^2 E(H_j(y^\varepsilon_s)H_j(y^\varepsilon_r)) dr ds \]

\[ = \alpha_k(\varepsilon)^2 \sum_{j=M+1}^{\infty} (c_{k,j})^2 j! \int_0^t \int_0^t \varepsilon(|s-r|) dr ds \lesssim \sum_{j=M+1}^{\infty} (c_{k,j})^2 j! \]

As \( \sum_{j=m}^{\infty} (c_{k,j})^2 j! < \infty \) we obtain \( \sum_{j=M+1}^{\infty} (c_{k,j})^2 j! \to 0 \) as \( M \to \infty \). Thus,

\[ \lim_{M \to \infty} \lim_{\varepsilon \to 0} E \left( \alpha_k(\varepsilon) \int_0^t G_k(y^\varepsilon_s) ds - \alpha_k(\varepsilon) \int_0^t G_k,M(y^\varepsilon_s) ds \right)^2 \to 0, \quad (4.7) \]

Let \( \{t_{y,k,l}, k \leq N, l \leq A\} \) be a sequence of positive numbers. Now, by the triangle inequality,

\[ \lim_{M \to \infty} \lim_{\varepsilon \to 0} \left\| \sum_{k,l} y_{k,l} \left( X^{k,\varepsilon}(t_l) - X^{k,\varepsilon}_M(t_l) \right) \right\|_{L^2(\Omega)} \to 0. \]

With Theorem 3.2 in [5], this proves the claim. \( \square \)

### 4.2 Moment Bounds

We will use some results from Malliavin Calculus. Let \( x_s \) be a stationary Gaussian process with \( \beta(s) = E(x_s x_0) \), such that \( \beta(0) = 1 \). As a real separable Hilbert space we use \( \mathcal{H} = L^2(\mathbb{R}_+, \nu) \) where for a Borel-set \( A \) we have \( \nu(A) = \int_{\mathbb{R}_+} \mathbf{1}_A(s) d\beta_s \).

Let \( \mathcal{H}^\otimes q \) denote the \( q \)-th tensor product of \( \mathcal{H} \). For \( h \in \mathcal{H} \), we may define the Wiener integrals \( W(h) = \int_0^\infty h_s dx_s \) by \( W([a,b]) = x(b) - x(a) \) (where \( a, b \geq 0 \), linearity and the Wiener isometry \( \langle \mathbf{1}_{[0,1]}, \mathbf{1}_{[0,1]} \rangle = \beta(t - s) \)). Iterated Wiener integrals are defined similarly and by its values on indctor functions: \( I_m(\mathbf{1}_{A_1} \times \cdots \times A_m) = \prod_{i=1}^m W(A_i) \) where \( A_i \) are pairwise disjoint Borel subsets of \( \mathbb{R}_+ \).

If \( F \) denotes the \( \sigma \)-field generated by \( x \), then any \( \mathcal{F} \)-measurable \( L^2(\Omega) \) function \( F \) has the chaos expansion: \( F = \mathbf{E}F + \sum_{m=0}^{\infty} I_m(f_m) \) where \( f_m \in L^2(\mathbb{R}_+^m) \). This is due to the fact that \( L^2(\Omega) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m \) where \( \mathcal{H}_m \) is the closed linear space generated by \( \{ H_m(W(h)) : \|h\|_{L^2} = 1 \} \), \( H_m \) are the \( m \)-th Hermite polynomials, and \( \mathcal{H}_m = I_m(L^2_{\text{sym}}(\mathbb{R}_+^m)) \). The last fact is due to \( H_m(W(h)) = I_m(h^{\otimes m}) \). In the following \( \mathbb{D}^{k,p}(\mathcal{H}^\otimes m) \) denotes the closure of Malliavin smooth random variables under the following norm \( \|u\|_{\mathbb{D}^{k,p}(\mathcal{H}^\otimes m)} = \left( \sum_{j=0}^{k} \mathbf{E}(\|D^j u\|^p_{\mathcal{H}^\otimes m}) \right)^{\frac{1}{p}} \).
Lemma 4.4 (Meyer’s inequality) [32] Let $\delta$ denote the divergence operator. Then for $u \in D^k, p(\mathcal{H} \otimes \mathcal{M})$,
\[
\| \delta^m(u) \|_{L^p(\Omega)} \lesssim \sum_{k=0}^m \| u \|_{D^k, p(\mathcal{H} \otimes \mathcal{M})}. \tag{4.8}
\]

Lemma 4.5 [11] If $G : \mathbb{R} \to \mathbb{R}$ is a function of Hermite rank $m$, then $G$ has the following multiple Wiener-Itô-integral representation:
\[
G(x_s) = \delta^m \left( G_m(x_s) \mathbb{I}_{[0,s]} \right), \tag{4.9}
\]
where $G_m$ has the following properties:

1. $\| G_m \|_{L^p(\mu)} \lesssim \| G \|_{L^p(\mu)}$,
2. $G_m(x_1)$ is $m$ times Malliavin differentiable and its $k^{th}$ derivative, denoted by $G^{(k)}_m(x_1) \mathbb{I}_{[0,1]}$, satisfies $\| G^{(k)}_m \|_{L^p(\mu)} \lesssim \| G \|_{L^p(\mu)}$.

In the lemma below we estimate the moments of $\int_0^t G(x_\varepsilon r) dr$, where we need the multiple Wiener-Itô-integral representation above to transfer the correlation function to $L^2$ norms of indicator functions. We use an idea from [11] for the estimates below.

Lemma 4.6 Let $x_t = W([0,t])$ be a stationary Gaussian process with correlation $\beta(t) = |E(x_t x_0)|$, stationary distribution $\mu$ and $\mathcal{H}$ the $L^2$ space over $\mathbb{R}_+$ with measure $\beta(r) dr$. If $G$ is a function of Hermite rank $m$ and $G \in L^p(\mu)$, for $p > 2$, then,
\[
\left\| \frac{1}{\varepsilon} \int_0^t G(x_\varepsilon r) dr \right\|_{L^p(\Omega)} \lesssim \| G \|_{L^p(\mu)} \left( \int_0^t \int_0^t \beta(|u - r|) m^2 dr du \right)^{\frac{1}{2}}. \tag{4.10}
\]

For the stationary scaled fractional OU process $y^\varepsilon_t$, we have
\[
\left\| \frac{1}{\varepsilon} \int_0^t G(y^\varepsilon_r) dr \right\|_{L^p(\Omega)} \lesssim \begin{cases} \| G \|_{L^p(\mu)} \left( \int_0^\infty q^m(s) ds \right)^{\frac{1}{2}}, & \text{if } H^*(m) < \frac{1}{2}, \\ \| G \|_{L^p(\mu)} \sqrt{\ln \frac{1}{\varepsilon} \frac{1}{\varepsilon}}, & \text{if } H^*(m) = \frac{1}{2}, \\ \| G \|_{L^p(\mu)} \left( \frac{1}{\varepsilon} \right)^{H^*(m)}, & \text{otherwise}. \end{cases} \tag{4.11}
\]
in particular,
\[
\left\| \int_0^t G(y^\varepsilon_r) dr \right\|_{L^p(\Omega)} \lesssim \| G \|_{L^p(\mu)} t^{\frac{H^*(m)\sqrt{\frac{1}{2}}}{\alpha(\varepsilon, H^*(m))}}. \tag{4.12}
\]
Proof We first use Lemma 4.5 and then apply Meyer’s inequality from Lemma 4.4 to obtain

\[
\left\| \frac{1}{\varepsilon} \int_0^t G(x_r) dr \right\|_{L^p(\Omega)} = \left\| \int_0^t G(x_r) dr \right\|_{L^p(\Omega)}
\]

\[
\leq \left\| \int_0^t \delta_m \left( G_m(x_r) 1_{[0,r]}^{\otimes m} \right) dr \right\|_{L^p(\Omega)}
\]

\[
\leq m \sum_{k=0}^m \left\| \int_0^t D^k \left( G_m(x_r) 1_{[0,r]}^{\otimes m} \right) dr \right\|_{L^p(\Omega, \mathcal{H}^{\otimes m+k})}
\]

\[
= m \sum_{k=0}^m \left\| \int_0^t G_m^{(k)}(x_r) 1_{[0,r]}^{\otimes m+k} dr \right\|_{L^p(\Omega, \mathcal{H}^{\otimes m+k})}.
\]

Here for \( G_m \) is as given in Lemma 4.5 and \( G_m^{(k)} \) denotes its k-th order Malliavin derivative. We estimate the individual terms using the linearity of the inner product and the isometry \( \langle 1_{[0,r]} 1_{[0,s]} \rangle_{\mathcal{H}} = E(x_r x_s) = \beta(r - s) \),

\[
\left( \left\| \int_0^t G_m^{(k)}(x_r) 1_{[0,r]}^{\otimes m+k} dr \right\|_{\mathcal{H}^{\otimes m+k}} \right)^2
\]

\[
= \left( \int_0^t G_m^{(k)}(x_r) 1_{[0,r]}^{\otimes m+k} dr , \int_0^t G_m^{(k)}(x_u) 1_{[0,u]}^{\otimes m+k} du \right)_{\mathcal{H}^{\otimes m+k}}
\]

\[
= \int_0^t \int_0^t G_m^{(k)}(x_r) G_m^{(k)}(x_u) (1_{[0,r]}^{\otimes m+k} , 1_{[0,u]}^{\otimes m+k})_{\mathcal{H}^{\otimes m+k}} dr du
\]

\[
= \int_0^t \int_0^t G_m^{(k)}(x_r) G_m^{(k)}(x_u) \beta(r - u)^{m+k} dr du.
\]

Using Minkowski’s inequality, we obtain

\[
\sum_{k=0}^m \left\| \int_0^t G_m^{(k)}(x_r) 1_{[0,r]}^{\otimes m+k} dr \right\|_{L^p(\Omega, \mathcal{H}^{\otimes m+k})}
\]

\[
\leq m \sum_{k=0}^m \left( \left\| \int_0^t \int_0^t G_m^{(k)}(x_r) G_m^{(k)}(x_u) \beta(r - u)^{m+k} dr du \right\|_{L^p(\Omega)} \right)^{1/2}
\]

\[
\leq m \sum_{k=0}^m \left( \int_0^t \int_0^t \left\| G_m^{(k)}(x_r) G_m^{(k)}(x_u) \right\|_{L^p(\Omega)} \beta(r - u)^{m+k} dr du \right)^{1/2}.
\]
We then estimate $E|G_m^{(k)}(x_r)G_m^{(k)}(x_u)|^{\frac{p}{2}}$ by Hölder’s inequality and use the fact that $x_t$ is stationary. Since the right hand side is then controlled by

$$RHS \leq \sum_{k=0}^{m} \|G_m^{(k)}\|_{L^p(\mu)} \left( \int_0^t \int_0^t |\beta(|u - r|)|^{m+k} dr du \right)^{\frac{1}{2}}$$

$$\lesssim \|G\|_{L^p(\mu)} \left( \int_0^t \int_0^t |\beta(|u - r|)|^{m} dr du \right)^{\frac{1}{2}},$$

concluding (4.10). We finally apply Lemma 2.4 to conclude (4.11). \qed

Now we are ready to prove our main theorem.

### 4.3 Concluding the Proof

**Proof** Step 1, CLT in the pure Wiener case

We first deal with the high Hermite rank components. For $k \leq n$ we define the truncated functions $G_{k,M} = \sum_{j=m_k}^{\infty} c_{k,j} H_j$ and set

$$X_{k,M}^{\varepsilon,\delta} = \alpha_k(\varepsilon) \int_0^t G_{k,M}(y_s^\varepsilon) ds.$$

Then, by the reduction Lemma 4.3 above, it is sufficient to show the convergence of $(X_{k,M}^{1,\varepsilon}, \ldots, X_{n,M}^{n,\varepsilon})$ for every $M$. By [4] and [2] each component alone converges to a Wiener process. Hence, as each $X_{k,M}^{\varepsilon,\delta}$ belongs to a finite chaos we can make use of the normal approximation theorem from [32, Theorem 6.2.3]: if each component of a family of mean zero vector valued stochastic processes, with components of the form $I_{q_i}(f_{i,n})$, where $f_{i,n}$ are symmetric $L^2$ functions in $q_i$ variables, converges in law to a Gaussian process, then they converge jointly in law to a vector valued Gaussian process, provided that their covariance functions converge. Furthermore, the covariance functions of the limit distribution are $\lim_{\varepsilon \to 0} E[X_{i,\varepsilon}(t)X_{j,\varepsilon}(s)]$. Let $m = \min(m_i, m_j)$ we use

$$E(H_k(y_s^\varepsilon)H_l(y_s^\varepsilon)) = \delta_{k,l} \left( E(y_s^\varepsilon y_t^\varepsilon) \right)^k.$$
to obtain, for \( s \leq t \),

\[
\mathbb{E} \left[ \alpha_i(\varepsilon)\alpha_j(\varepsilon) \int_0^t G_{i,M}(y_u^\varepsilon) du \int_0^s G_{j,M}(y_r^\varepsilon) dr \right] \\
= \sum_{k=m}^{M} \alpha_i(\varepsilon)\alpha_j(\varepsilon)c_{i,k}c_{j,k}(k!)^2 \int_0^t \int_0^s \left( \mathbb{E}(y_u^\varepsilon y_r^\varepsilon) \right)^k dr du \\
= \sum_{k=m}^{M} \alpha_i(\varepsilon)\alpha_j(\varepsilon)c_{i,k}c_{j,k}(k!)^2 \left( \int_0^s \int_0^s \varphi^\varepsilon(u-r)^k dr du + \int_s^t \int_0^s \varphi^\varepsilon(u-r)^k dr du \right).
\]

By Lemma 2.4, we obtain, for \( \varepsilon \to 0 \),

\[
\alpha_i(\varepsilon)\alpha_j(\varepsilon) \int_s^t \int_0^s \varphi^\varepsilon(u-r)^k dr du \to 0.
\]

Hence,

\[
\lim_{\varepsilon \to 0} RHS = 2 \sum_{k=m}^{M} c_{i,k}c_{j,k}(k!)^2 \int_0^\infty \varphi(u)^k du
\]

proving the finite chaos case. We now prove that the correlations of the limit converge as \( M \to \infty \). Indeed,

\[
\lim_{M \to \infty} 2(s \land t) \sum_{k=m}^{M} c_{i,k}c_{j,k}(k!)^2 \int_0^\infty \varphi(u)^k du = 2(s \land t) \int_0^\infty \mathbb{E}(G_{i,M}(y_s)G_{j,M}(y_0)) ds.
\]

As \( G_{i,M} \to G_i \) in \( L^2(\mu) \), and similarly for \( j \), this proves the joint convergence of the high Hermite rank components.

**Step 2, CLT in the pure Hermite case**

In this step, we focus on the vector component whose entries satisfy \( H^*(m_k) > \frac{1}{2} \). Recall, this implies \( H > \frac{1}{2} \). By Lemma 4.2, evaluations of each component of
\((X_{n+1}^{N}, \ldots, X^N)\) converge in \(L^2(\Omega)\). Hence, they converge as well jointly in \(L^2(\Omega)\).

Now, choose finitely many points \(t_{k,j} \in [0, T]\) and constants \(a_{k,j} \in \mathbb{R}\), then, \(\sum_{k,j} a_{k,j} X_{t_{k,j}}^k\) converges in \(L^2(\Omega)\) to \(\sum_{k,j} a_{k,j} X_{t_{k,j}}^k\) and thus we may conclude joint convergence in finite dimensional distributions by an application of the Cramer-Wold theorem.

**Step 3, Joint convergence**

We have already shown that \(X_{W,\varepsilon}^W \to X^W\) and \(X_{Z,\varepsilon}^Z \to X^Z\) in finite dimensional distributions, it is only left to prove their joint convergence. By Lemma 4.3 and Eq. (4.6), we may again reduce the problem to

\[G_i = \sum_{k=m_i}^M c_{i,k} H_k, \quad G_j = c_{j,m_j} H_{m_j}, \quad 1 \leq i \leq n, \quad j > n.\]

Now, we can rewrite \(H_m(y_{s}^{\varepsilon}) = I_m(f_s^{m,\varepsilon})\), where \(I_m\) denotes a \(m\)-fold Wiener-Itô integral and a function \(f_s^{m,\varepsilon} \in L^2(\mathbb{R}^m, \mu)\). Now, for \(1 \leq i \leq n\) we obtain,

\[
\alpha_i(\varepsilon) \int_0^t G_i(y_{s}^{\varepsilon}) ds = \alpha_i(\varepsilon) \int_0^t \sum_{k=m_i}^M c_{i,k} H_k(y_{s}^{\varepsilon}) ds \\
= \alpha_i(\varepsilon) \int_0^t \sum_{k=m_i}^M c_{i,k} I_k(f_s^{k,\varepsilon}) ds = \sum_{k=m_i}^M c_{i,k} I_k(\hat{f}_s^{k,\varepsilon}),
\]

where

\[\hat{f}_s^{k,\varepsilon} = \int_0^t f_s^{k,\varepsilon} ds.\]

Similarly for \(j > n\),

\[
\int_0^t G_j(y_{s}^{\varepsilon}) ds = \int_0^t c_{j,m_j} H_{m_j}(y_{s}^{\varepsilon}) ds = c_{j,m_j} I_{m_j}(f_s^{m_j,\varepsilon}).
\]

Hence, we only need to show that the collection of stochastic processes of the form \(I_{m_k}(\hat{f}_{t_{l_i}}^{m,\varepsilon})\) converges jointly in finite dimensional distribution. It is thus sufficient to show that for every finite collection of times, \(t_1, \ldots, t_Q \in [0, T]\), the vector \(\{I_k(\hat{f}_{t_{l_i}}^{k,\varepsilon}), k = m, \ldots, M, l = 1, \ldots, Q\}\) converges jointly, where \(m = \min_{k=1,\ldots,N} m_k\). Let \(n_0\) denote the smallest natural number such that \(H^*(n_0) < \frac{1}{2}\).

For \(k > n_0\), the collection \(I_k(\hat{f}_{t_{l_i}}^{k,\varepsilon})\) converges to a normal distribution, hence, by the moment bounds in Lemma 4.6 \(\| (k!) I_k(\hat{f}_{t_{l_i}}^{k,\varepsilon}) \|_{H^k} = \sqrt{E(I_k(\hat{f}_{t_{l_i}}^{k,\varepsilon})^2)}\) converges to a constant. The convergence of such a sequence is equivalent to the convergence of the contractions, as defined in Eq. (5.1), of their kernels. Indeed, by a generalised fourth
moment theorem [31, Theorem 1], we have
\[ \| \hat{f}^{k,\varepsilon}_{t_1} \otimes_r \hat{f}^{k,\varepsilon}_{t_2} \|_{\mathcal{H}^{2k-2r}} \to 0, \quad r = 1, \ldots, k - 1. \]

By Cauchy-Schwarz, we obtain for \( r = 1, \ldots, k_1, \)
\[ \| \hat{f}^{k_1,\varepsilon}_{t_1} \otimes_r \hat{f}^{k_2,\varepsilon}_{t_2} \|_{\mathcal{H}^{k_1+k_2-2r}} \leq \| \hat{f}^{k_1,\varepsilon}_{t_1} \otimes_r \hat{f}^{k_1,\varepsilon}_{t_1} \|_{\mathcal{H}^{2k-r}} \| \hat{f}^{k_2,\varepsilon}_{t_2} \otimes_r \hat{f}^{k_2,\varepsilon}_{t_2} \|_{\mathcal{H}^q-r} \to 0, \]
for all \( t_1, t_2 \in [0, T], 1 \leq k_1 \leq n_0 < k_2 \leq M. \) We can now apply an asymptotic independence result, Proposition 5.1 in Appendix, to conclude the joint convergence in finite dimensional distributions of \( X^\varepsilon \) to \( (X^W, X^Z) \). Furthermore, \( X^W \) is independent of \( X^Z \).

The correlations between \( X_i^j \) and \( X_j^j, \) where \( i, j > n, \) are 0 if \( m_i \neq m_j, \) otherwise given by the \( L^2 \) norm of their integrands, which follows from the Wiener-Itô isometry and are given by
\[ c_i, m_i c_j, m_j \int_0^T \int_0^T \prod_{i=1}^{m_i} \frac{1}{\sqrt{2 \pi (r - \xi_i)^+}} \frac{1}{\sqrt{2 \pi (r - \xi_i)^+}} d\xi_1 \ldots d\xi_{m_i} dr ds. \]

Step 4, Convergence in the Hölder norms.
We first choose \( \gamma' \in \left( \gamma, (H^*(m_k) \wedge \frac{1}{p_k}) \right) \). Then, using the Markov-Chebyshev inequality, we obtain that, as \( M \) tends to \( \infty, \)
\[ \mathbb{P} \left( |X^k,\varepsilon|_{C^{\gamma'}([0,T],\mathbb{R})} > M \right) \leq \frac{\|\theta|_{C^{\gamma'}([0,T],\mathbb{R})}\|_{p_k}}{M^{p_k}} \to 0, \]
since \( \|\theta|_{C^{\gamma'}}\|_{p_k} < \infty \) by Lemma 4.6 and an application of Kolmogorov’s theorem 2.6. Furthermore, since \( C^{\gamma'}([0,T],\mathbb{R}) \) is compactly embedded in \( C^{\gamma'}([0,T],\mathbb{R}) \) for \( 1 > \gamma' \geq \gamma, \) by an type Arzela-Ascoli argument, the sets \( \{X^k,\varepsilon \subseteq [X^k,\varepsilon]_{C^{\gamma'}([0,T],\mathbb{R})} \leq M \} \) are sequentially compact in \( C^{\gamma'}([0,T],\mathbb{R}) \). Hence, \( \{X^k,\varepsilon \} \) is tight in \( C^{\gamma'}([0,T],\mathbb{R}). \) See, e.g. [17]. As tightness in each component implies joint tightness, we obtain that \( X^\varepsilon \) is tight in \( C^{\gamma'}([0,T],\mathbb{R}^N), \) where \( \gamma \in (0, \frac{1}{2} - \frac{1}{\min_{k\leq p_k} p_k}) \) in case \( 0 < n \) and \( \gamma \in (0, \min_{k\geq n} H^*(m_k) - \frac{1}{p_k}) \) otherwise. By the above discussion, any limit of a converging subsequence has the same finite dimensional distributions, hence, every subsequence converges to the same limit. This concludes the proof for the convergence in the Hölder norm.

5 Appendix: Joint Convergence by Asymptotic Independence

For the proof in the previous section, we need the following which modifies results from [33] and [29]. Let \( I^p(f) \) denote the \( p^{th} \) iterated Itô-Wiener integral of a symmetric
function \( f \) of \( p \) variables,

\[
I_p(f) = p! \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(s_1, \ldots, s_p) dW_{s_1} dW_{s_2} \cdots dW_{s_p}.
\]

If \( f \in L^2(\mathbb{R}^p) \) and \( g \in L^2(\mathbb{R}^q) \) are symmetric functions and \( p, q \geq 1 \), their \( r \)th contraction is given by

\[
f \otimes_r g = \int_{\mathbb{R}^r} f(x_1, \ldots, x_{p-r}, s_1, \ldots, s_r) g(y_1, \ldots, y_{q-r}, s_1, \ldots, s_r) ds_1 \cdots ds_r,
\]

where \( r \leq p \wedge q \). If \( f \otimes_1 g = \int_{\mathbb{R}} f(x_1, \ldots, x_{p-1}, s) g(y_1, \ldots, y_{q-1}, s) ds \) vanishes, so do all higher order contractions.

**Proposition 5.1** Let \( q_1 \leq q_2 \leq \cdots \leq q_n \leq p_1 \leq p_2 \leq \cdots \leq p_m \). Let \( f_i^\varepsilon \in L^2(\mathbb{R}^{p_i}) \), \( g_i^\varepsilon \in L^2(\mathbb{R}^{q_i}) \), \( F_\varepsilon = (I_{p_1}(f_1^\varepsilon), \ldots, I_{p_m}(f_m^\varepsilon)) \) and \( G_\varepsilon = (I_{q_1}(g_1^\varepsilon), \ldots, I_{q_n}(g_n^\varepsilon)) \).

Suppose that for every \( i, j, \) and any \( 1 \leq r \leq q_i \):

\[
\| f_j^\varepsilon \otimes_r g_i^\varepsilon \| \to 0.
\]

Then \( F_\varepsilon \to U \) and \( G_\varepsilon \to V \) weakly imply that \((F_\varepsilon, G_\varepsilon) \to (U, V)\) jointly, where \( U \) and \( V \) are taken to be independent random variables.

These results benefit from the insights of Üstünel-Zakai [41] on the independence of two iterated integrals \( I_p(f) \) and \( I_q(g) \). They are independent if and only if the 1-contraction between \( f \) and \( g \) vanishes almost surely with respect to the Lebesgue measure. An asymptotic independence result follows as below,

**Lemma 5.2** \([29, \text{Thm. 3.1}]\) Let \( F_\varepsilon = I_p(f_\varepsilon) \) and \( G_\varepsilon = I_q(g_\varepsilon) \), where \( f_\varepsilon \in L^2(\mathbb{R}^p) \) and \( g_\varepsilon \in L^2(\mathbb{R}^q) \). Then,

\[
\text{Cov}\left(\left(\begin{array}{c} F_\varepsilon \\ G_\varepsilon \end{array}\right)^2\right) \to 0
\]

is equivalent to \( \| f_\varepsilon \otimes_r g_\varepsilon \| \to 0 \), for \( 1 \leq r \leq p \wedge q \).

It is also known that if two integrals \( I_p(f) \) and \( I_q(g) \) are independent, then their Malliavin derivatives are orthogonal, see [41]. This explains why Malliavin calculus comes into prominent play, which has been developed to its perfection in [29, Lemma 3.2]. Given a smooth test function \( \varphi \) we define,

\[
\| \varphi \|_q = \| \varphi \|_\infty + \sum_{|k|=1}^q \left\| \frac{\partial^k}{\partial^k x} \right\|_\infty,
\]

where the sum runs over multi-indices \( k = (k_1, \ldots, k_m) \). Let \( L = -\delta D \) and throughout this section \( f_i : \mathbb{R}^{p_i} \to \mathbb{R} \) and \( g : \mathbb{R}^{q} \to \mathbb{R} \) denote symmetric functions.
Lemma 5.3  [29] Let \( q \leq p_i, g \in L^2(\mathbb{R}^q), G = I_q(g), f_i \in L^2(\mathbb{R}^{p_i}), \) and \( F_i = I_{p_i}(f_i) \) with \( E(F_i^2) = 1 \). Set \( F = (F_1, \ldots, F_m) \) and let \( \theta \) be a smooth test function. Then,

\[
E \left[ \left| (I - L)^{-1} \theta(F) DF_j, DG \right|_{\mathcal{H}} \right] \leq c \left\| \theta \right\|_q \text{Cov}(F_j^2, G^2),
\]

where \( c \) is a constant depending on \( \|F\|_{L^2}, \|G\|_{L^2}, \) and \( q, m, p_1, \ldots, p_m \).

The final piece of the puzzle is the observation that the defect in being independent is quantitatively controlled by the covariance of the squares of the relative components. The following is from [29], our only modification is to take \( G \) to be vector valued. Let \( g_i : \mathbb{R}^{q_i} \to \mathbb{R} \) be symmetric functions.

Lemma 5.4 Given \( F = (I_{p_1}(f_1), \ldots, I_{p_m}(f_m)) \) and \( G = (I_{q_1}(g_1), \ldots, I_{q_n}(g_n)) \) such that \( p_k \geq q_l \) for every pair of \( k, l \). Then, for all test functions \( \varphi \) and \( \psi \), the following holds for some constant \( c \), depending on \( \|F\|_{L^2}, \|G\|_{L^2}, \) and \( m, n, p_1, \ldots, p_m, q_1, \ldots, q_n \),

\[
E(\varphi(F)\psi(G)) - E(\varphi(F))E(\psi(G)) \leq c\|D\psi\|_\infty \|\varphi\|_{q_n} \sum_{i=1}^{m} \sum_{j=1}^{n} \text{Cov}(F_i^2, G_j^2).
\]

Proof Define \( L^{-1}(\sum_{k=0}^{\infty} I_k(h_m)) = \sum_{k=1}^{\infty} \frac{1}{k} I_k(h_m) \in \mathbb{D}^{2,2} \). The key equality is \( -DL^{-1} = (I - L)^{-1}D \). As in [29],

\[
\varphi(F) - E(\varphi(F)) = LL^{-1}\varphi(F) = \sum_{j=1}^{m} \delta((I - L)^{-1} \partial_j \varphi(F) DF_j).
\]

Multiplying both sides by \( \psi(G) \), taking expectations and using integration by parts we obtain

\[
E(\varphi(F)\psi(G)) - E(\varphi(F))E(\psi(G))
\]

\[
= \sum_{j=1}^{m} \sum_{i=1}^{n} \text{E} \left( \left| \left((I - L)^{-1} \partial_j \varphi(F) DF_j, DG_i \right)_{\mathcal{H}} \right| \partial_i \psi(G) \right)
\]

\[
\leq \|D\psi\|_\infty \sum_{j=1}^{m} \sum_{i=1}^{n} \text{E} \left( \left| \left((I - L)^{-1} \partial_j \varphi(F) DF_j, DG_i \right)_{\mathcal{H}} \right| \right).
\]

To conclude, apply to each summand Lemma 5.3 with \( \theta = \partial_j \varphi \) and \( G = G_i \). \( \square \)

Lemma 5.5 Let \( F^\varepsilon = (I_{p_1}(f_1^\varepsilon), \ldots, I_{p_m}(f_m^\varepsilon)) \) and \( G^\varepsilon = (I_{q_1}(g_1^\varepsilon), \ldots, I_{q_n}(g_n^\varepsilon)) \) with \( q_1 \leq q_2, \ldots, q_n \leq p_1 \leq p_2 \leq \cdots \leq p_m \). Then for every \( i \leq m, j \leq n \),

\[
\|f_j^\varepsilon \otimes_r g_i^\varepsilon\| \to 0, \quad 1 \leq r \leq p_j \wedge q_i.
\]
implies that for any smooth test functions \( \varphi \) and \( \psi \),

\[
E(\psi(F^\varepsilon)\psi(G^\varepsilon)) - E(\psi(F^\varepsilon))E(\psi(G^\varepsilon)) \to 0.
\]

**Proof** Just combine Lemmas 5.4 and 5.2.

Finally, we finish the proof of Proposition 5.1.

**Proof** Since \((F^\varepsilon, G^\varepsilon)\) is bounded in \(L^2(\Omega)\) it is tight. Now choose a weakly converging subsequence \((F^n, G^n)\) with limit denoted by \((X, Y)\). Let \(\varphi\) and \(\psi\) be smooth test functions, then by Lemma 5.5 and the bounds on \(\varphi, \psi\), we pass to the limit under the expectation sign and obtain

\[
E(\varphi(X)\psi(Y)) = E(\varphi(X))E(\psi(Y)).
\]

Thus, every limit measure is the product measure determined by \(U\) and \(V\), hence, \((F^\varepsilon, G^\varepsilon)\) converges as claimed.

**Acknowledgements** This research is partially funded by an EPSRC Roth studentship.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**References**

1. Al-Talibi, H., Hilbert, A.: Differentiable approximation by solutions of Newton equations driven by fractional Brownian motion. Preprint (2012)
2. Buchmann, B., Chan, N.H.: Integrated functionals of normal and fractional processes. Ann. Appl. Probab. 19(1), 49–70 (2009)
3. Bourguin, S., Gailus, S., Spiliopoulos, K.: Typical dynamics and fluctuation analysis of slow-fast systems driven by fractional Brownian motion (2019)
4. Hariz, S.B.: Limit theorems for the non-linear functional of stationary Gaussian processes. J. Multivariate Anal. 80(2), 191–216 (2002)
5. Billingsley, P.: Convergence of probability measures. Wiley Series in Probability and Statistics: Probability and Statistics. Wiley, New York, second edition (1999)
6. Breuer, P., Major, P.: Central limit theorems for nonlinear functionals of Gaussian fields. J. Multivariate Anal. 13(3), 425–441 (1983)
7. Boufoussi, B., Tudor, C.A.: Kramers-Smoluchowski approximation for stochastic evolution equations with FBM. Rev. Roumaine Math. Pures Appl. 50(2), 125–136 (2005)
8. Bai, S., Taqqu, M.S.: Multivariate limit theorems in the context of long-range dependence. J. Time Series Anal. 34(6), 717–743 (2013)
9. Cornfeld, I. P., Fomin, S. V., Sinai, Ya. G.: Ergodic theory volume 245 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, New York. Translated from the Russian by A. B. Sosinskii (1982)
10. Cheridito, P., Kawaguchi, H., Maejima, M.: Fractional Ornstein-Uhlenbeck processes. Electron. J. Probab. 8(3), 14 (2003)
11. Campese, S., Nourdin, I., Nualart, D.: Continuous Breuer-major theorem: tightness and nonstationarity. Ann. Probab. 48(1), 147–177 (2020)
12. Dobrushin, R.L.: Gaussian and their subordinated self-similar random generalized fields. Ann. Probab. 7(1), 1–28 (1979)
13. Embrechts, P., Maejima, M.: Selfsimilar Processes. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ (2002)
14. Friz, P., Gassiat, P., Lyons, T.: Physical Brownian motion in a magnetic field as a rough path. Trans. Am. Math. Soc. 367(11), 7939–7955 (2015)
15. Friz, P. K., Hairer, M.: A Course on Rough Paths. Universitext. Springer, Cham, With an introduction to regularity structures (2014)
16. Fannjiang, A., Komorowski, T.: Fractional Brownian motions in a limit of turbulent transport. Ann. Appl. Probab. 10(4), 1100–1120 (2000)
17. Friz, P. K., Victoir, N. B.: Multidimensional stochastic processes as rough paths, volume 120 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, Theory and applications (2010)
18. Gehring, J., Li, X.-M.: Homogenization with fractional random fields. arXiv:1911.12600. This is now improved and split into ‘Functional limit theorem for fractional OU’ and Diffusive and rough homogenisation in fractional noise field (2019)
19. Gehring, J., Li, X.-M.: Diffusive and rough homogenisation in fractional noise field. This is an improved version of arXiv:1911.12600 (part II) (2020)
20. Hurst, H.E., Black, R.P., Simaika, Y.M.: Long-term storage: an experimental study. Constable, London (1965)
21. Hairer, M., Li, X.-M.: Averaging dynamics driven by fractional Brownian motion. Ann. Probab. 48(4), 1826–1860 (2020)
22. Jona-Lasinio, G.: Probabilistic approach to critical behavior. In: New developments in quantum field theory and statistical mechanics (Proc. Cargèse Summer Inst., Cargèse, 1976), pp. 419–446. NATO Adv. Study Inst. Ser., Ser. B: Physics, 26 (1977)
23. Komorowski, T., Landim, C., Olla, S.: Fluctuations in Markov processes, volume 345 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, Time symmetry and martingale approximation (2012)
24. Lyons, T. J., Caruana, M., Lévy, T.: Differential equations driven by rough paths, volume 1908 of Lecture Notes in Mathematics. Springer, Berlin, Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard (2007)
25. Lyons, T.: Differential equations driven by rough signals. I. An extension of an inequality of LC Young. Math. Res. Lett. 1(4), 451–464 (1994)
26. Mishura, Y.S.: Stochastic Calculus for Fractional Brownian Motion and Related Processes. Lecture Notes in Mathematics, vol. 1929. Springer, Berlin (2008)
27. Maejima, M., Tudor, C.A.: Wiener integrals with respect to the Hermite process and a non-central limit theorem. Stoch. Anal. Appl. 25(5), 1043–1056 (2007)
28. Mandelbrot, B.B., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10, 422–437 (1968)
29. Nourdin, I., Nualart, D., Peccati, G.: Strong asymptotic independence on Wiener chaos. Proc. Am. Math. Soc. 144(2), 875–886 (2016)
30. Nourdin, I., Nualart, D., Zintout, R.: Multivariate central limit theorems for averages of fractional Volterra processes and applications to parameter estimation. Stat. Inference Stoch. Process. 19(2), 219–234 (2016)
31. Nualart, D., Peccati, G.: Central limit theorems for sequences of multiple stochastic integrals. Ann. Probab. 33(1), 177–193 (2005)
32. Nourdin, I., Peccati, G.: Normal approximations with Malliavin calculus, volume 192 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, From Stein’s method to universality (2012)
33. Nourdin, I., Rosinski, J.: Asymptotic independence of multiple wiener-ito integrals and the resulting limit laws. Ann. Probab. 42(2), 497–526 (2014)
34. Pipiras, V., Taqqu, M.S.: Integration questions related to fractional Brownian motion. Probab. Theory Related Fields 118(2), 251–291 (2000)
35. Pipiras, V., Taqqu, M.S.: Long-Range Dependence and Self-Similarity. Cambridge Series in Statistical and Probabilistic Mathematics, vol. 45. Cambridge University Press, Cambridge (2017)
36. Rosenblatt, M.: Independence and dependence. In: Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. II, pp. 431–443. Univ. California Press, Berkeley, Calif (1961)
37. Samorodnitsky, G.: Long range dependence. Found. Trends Stoch. Syst. 1(3), 163–257 (2006)
38. Sinai, Y.G.: Self-similar probability distributions. Theory Probab. Appl 21(1), 63–80 (1976)
39. Stroock, D. W., Srinivasa Varadhan, S. R.: Multidimensional Diffusion Processes. Classics in Mathematics. Springer, Berlin, Reprint of the 1997 edition (2006)
40. Taqqu, M.S.: Convergence of integrated processes of arbitrary Hermite rank. Z. Wahrsch. Verw. Gebiete 50(1), 53–83 (1979)
41. Üstünel, A.S., Zakai, M.: On independence and conditioning on Wiener space. Ann. Probab. 17(4), 1441–1453 (1989)
42. Young, L.C.: An inequality of the Hölder type, connected with Stieltjes integration. Acta Math. 67(1), 251–282 (1936)
43. Zhang, S.: Smoluchowski-Kramers approximations for stochastic equations with Lévy noise. PhD thesis, Purdue University (2008)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.