A rotating thin shell in a (2+1)-dimensional asymptotically AdS spacetime is studied. The spacetime exterior to the shell is the rotating BTZ spacetime and the interior is the empty spacetime with a cosmological constant. Through the Einstein equation in (2+1)-dimensions and the corresponding junction conditions we calculate the dynamical relevant quantities, namely, the rest energy-density, the pressure, and the angular momentum flux density. We also analyze the matter in a frame where its energy-momentum tensor has a perfect fluid form. In addition, we show that Machian effects, such as the dragging of inertial frames, also occur in rotating (2+1)-dimensional spacetimes. The weak and the dominant energy condition for these shells are discussed.

Keywords: thin shell; general relativity in 2+1 dimensions; anti-de Sitter spacetime; black hole

1. Introduction

In (2+1)-dimensional general relativity with a cosmological constant, spacetimes with no matter have no propagating degrees of freedom, so that vacuum spacetimes have local constant curvature. Nonetheless, in the presence of a negative cosmological constant, there is a nontrivial black hole solution, the Bañados, Teitelboim, Zanelli (BTZ) black hole. In its most general form the BTZ black hole is a stationary solution representing a rotating black hole, with mass and angular momentum. For zero rotation the BTZ black hole is static. The BTZ black hole,
having a negative cosmological constant, is an asymptotically anti-de Sitter (AdS) solution.

Besides being a pure vacuum solution of a spacetime with a negative cosmological constant, the BTZ solution is also a solution exterior to a (2+1)-dimensional matter configuration. A simple matter configuration is a thin shell. Collapsing nonrotating thin shells with an exterior static BTZ solution have been studied in 3, 4. Static thin shells with an exterior static BTZ solution were worked out in 5. Gravitational collapse and configurations of rotating thin shells with an exterior stationary BTZ solution was analyzed in 6.

Here, we want to study rotating thin shells for which the exterior spacetime is the stationary BTZ solution and the interior spacetime is the empty spacetime with a cosmological constant. We translate to 2+1 dimensions the framework of Poisson to treat rotating shells in (3+1)-dimensional general relativity, and thus, through the Einstein equation in (2+1)-dimensions and the corresponding junction conditions, we calculate the dynamical relevant quantities.

This paper is organized as follows. In Sec. 2 we perform an analysis of the (2+1)-dimensional spacetime with a negative cosmological constant generated by a rotating thin matter shell. The exterior is the BTZ solution, the interior is the empty spacetime, and the energy-density, the pressure, and the angular momentum flux density of the matter in the shell are calculated. In Sec. 3 we analyze the thin shell’s mechanical properties in a frame where one can detect the energy-momentum tensor as a perfect fluid. In Sec. 4 we study rotating properties of the spacetime and find that (2+1)-dimensional spacetimes can display Machian effects, such as the dragging of inertial frames. In Sec. 5 we discuss the weak and dominant energy conditions for the thin shell matter. We conclude in Sec. 6. In an Appendix we rederive the energy-momentum tensor in a perfect fluid form using a different approach.

2. The thin shell spacetime: BTZ outside, AdS inside

2.1. Einstein equation

In 2+1 dimensions, the Einstein equation is

\[ G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \tag{1} \]

where \( G_{\alpha\beta} \) is the Einstein tensor, \( \Lambda \) is the cosmological constant, \( g_{\alpha\beta} \) is the spacetime metric, \( G \) is the gravitational constant in 2+1 dimensions, and \( T_{\alpha\beta} \) is the energy-momentum tensor of the matter fields. We will choose units with the velocity of light equal to one, so that \( G \) has units of inverse of mass. Greek indices run from 0, 1, 2, with 0 being the time index. Since the background spacetime is AdS which has a negative cosmological constant, we define the AdS length \( l \) through the equation

\[ -\Lambda = 1/l^2. \tag{2} \]

We consider a timelike shell with radius \( R \) in a (2+1)-dimensional spacetime. In
two spatial dimensions a shell is a ring. This ring divides spacetime into two regions, an outer region and an inner region.

2.2. **The outside region**

The outside region is described by an exterior metric given by the BTZ line element\(^1, 2\)

\[
\begin{align*}
\text{ds}_o^2 &= g_{\alpha \beta} dx_\alpha^o dx_\beta^o = -\left( \frac{r^2}{l^2} - 8Gm + \frac{16G^2J^2}{r^2} \right) dt_o^2 \\
& \quad + \frac{dr^2}{\left( \frac{r^2}{l^2} - 8Gm + \frac{16G^2J^2}{r^2} \right)} + r^2 \left( d\phi - \frac{4GJ}{r^2} dt_o \right)^2, \quad r > R,
\end{align*}
\]

written in a circularly symmetric outer coordinate system \(x_\alpha^o = (t_o, r, \phi)\), and where \(m\) is the Arnowitt-Deser-Misner (ADM) mass and \(J\) is the spacetime angular momentum.

2.3. **The inside region**

The interior region is the vacuum state empty spacetime, a variant of pure AdS spacetime, with metric given by\(^1, 2\)

\[
\begin{align*}
\text{ds}_i^2 &= g_{i \alpha \beta} dx_\alpha^i dx_\beta^i = -\frac{\rho^2}{l^2} dt_i^2 + \frac{l^2}{\rho^2} d\rho^2 + \rho^2 d\psi^2, \quad \rho < R,
\end{align*}
\]
written in a circularly symmetric inner coordinate system \(x_i^\alpha = (t_i, \rho, \psi)\).

2.4. **The shell-ring region: the matching and junction conditions**

Here we analyze the matching region, i.e., the ring. From the outside, the metric at the shell \(\Sigma, r = R\), is from Eq. \(3\)

\[
\begin{align*}
\text{ds}_\Sigma^2 &= -\left( \frac{R^2}{l^2} - 8Gm + \frac{16G^2J^2}{R^2} \right) dt_o^2 + R^2 \left( d\phi - \frac{4GJ}{R^2} dt_o \right)^2, \quad r = R.
\end{align*}
\]

Now, we want to remove the off-diagonal term in the induced metric \(5\), as viewed from the outer region. For that we go to a corotating frame by defining a new polar coordinate

\[
\psi = \phi - \Omega t_o.
\]

This makes the induced metric diagonal if the angular velocity \(\Omega\) is chosen to be

\[
\Omega = \frac{4GJ}{R^2}.
\]

Then, in coordinates \((t_o, \psi)\), the induced metric at \(\Sigma\) is written as

\[
\begin{align*}
\text{ds}_\Sigma^2 &= -\left( \frac{R^2}{l^2} - 8Gm + \frac{16G^2J^2}{R^2} \right) dt_o^2 + R^2 d\psi^2, \quad r = R.
\end{align*}
\]
From the inside, the induced metric is obtained by setting $\rho = R$ in Eq. (4), i.e.,

$$ds^2 = -\frac{R^2}{l^2}dt^2 + R^2 d\psi^2, \quad \rho = R. \quad (9)$$

On the other hand, at the shell, the metric can be written in terms of its own proper coordinates $x^\alpha = (\tau, \psi)$, with $\tau$ being the proper time of the shell. So, the line element $ds^2 = h_{ab} dx^a dx^b$, with $h_{ab}$ being the intrinsic metric at the shell, can be written as

$$ds^2 = -d\tau^2 + R^2 d\psi^2. \quad (10)$$

We have still to apply the first and second junction conditions, see, e.g., [3]. The first junction condition states that the induced metric on the shell must be the same on both sides of the shell and at the shell. From Eqs. (8)-(10) this gives

$$d\tau^2 = \left(\frac{R^2}{l^2} - 8Gm + \frac{16G^2 J^2}{R^2}\right) dt_o^2 = R^2 dt_i^2. \quad (11)$$

The second junction condition states that

$$S_{ab} = -\frac{1}{8\pi G} ([K_{ab}] - [K]h_{ab}), \quad (12)$$

where $S_{ab}$ is the energy-momentum tensor of the matter in the shell, $K_{ab} = n_\alpha ; \beta e^\alpha_a e^\beta_b$ is the second fundamental form of the shell, with $n_\alpha$ being the normal vector to the shell, $e^\alpha_a$ are the tangent vectors to the shell, the semicolon denotes covariant derivative, and a quantity in square brackets denotes the jump from the outside to the inside. The tangent vectors $e^\alpha_a$ are such that $h_{ab}$ given through Eq. (10) can be written as the metric induced either from the outside or the inside metric $g_{\alpha\beta}$ at $R$, i.e., $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$, where $g_{\alpha\beta}$ stands to either $g_{\alpha\beta}$ or $g_{i\alpha\beta}$. The three independent components of the surface energy-momentum tensor can then be calculated using Eq. (12) to give

$$S^\tau_\tau = -\frac{1}{8\pi G l} \left(1 - \frac{l}{R} \sqrt{\frac{R^2}{l^2} - 8Gm + \frac{16G^2 J^2}{R^2}}\right), \quad (13)$$

$$S^\psi_\psi = \frac{1}{8\pi G l} \left(\frac{R^4 - 16G^2 l^2 J^2}{l^2 R^2} - 1\right), \quad (14)$$

$$S^\tau_\psi = \frac{J}{2\pi R}. \quad (15)$$

### 2.5. Properties of the matter of the ring shell

These calculations were performed in the shell’s rest frame coordinate $(\tau, \psi)$. In this frame we define the proper frame quantities, namely, the proper rest energy
density $\lambda$, the proper pressure $p$, and the proper angular momentum flux density $j$ as $\lambda = -S_{\tau \tau}^\varphi$, $p = S_{\psi \psi}^\varphi$ and $j = S_{\tau \psi}^\varphi$. So,

$$\lambda = \frac{1}{8\pi G l} \left( 1 - \frac{1}{R} \sqrt{\frac{R^2}{l^2} - 8Gm + \frac{16G^2J^2}{R^2}} \right),$$

(16)

$$p = \frac{1}{8\pi G l} \left( -1 + \frac{R^4 - 16G^2J^2}{lR^3 \sqrt{R^2/l^2 - 8Gm + \frac{16G^2J^2}{R^2}}} \right),$$

(17)

$$j = \frac{J}{2\pi R}.$$  

(18)

If one considers slowly rotation the terms in $J$ in Eqs. (16) and (17) should be discarded, keeping the linear term in Eq. (18).

An interesting quantity is the total rest mass $M$ of the shell, given by

$$M = 2\pi R \lambda.$$  

(19)

Then using Eqs. (16) and (19) one gets an expression for the ADM mass-energy $m$ in terms of the other quantities,

$$m = \frac{R}{l} M - 2GM^2 + 2G\frac{J^2}{R^2}.$$  

(20)

This expression connects shell quantities, namely, $M$ and $R$ with spacetime quantities, namely, $m$, $J$ and $l$.

### 2.6. Properties of the spacetime

Inside the shell the spacetime is empty spacetime. The matching region is the ring-shell region. Outside the shell the spacetime is a rotating BTZ spacetime. The BTZ spacetime has two intrinsic radii, given by the zeros of the metric component $g^{rr}$ in Eq. (3). They are the gravitational radius $r_+$ and the Cauchy radius $r_-$, which have the expressions

$$r_+ = 2l \sqrt{Gm + \sqrt{G^2m^2 - \frac{G^2J^2}{l^2}}},$$

(21)

$$r_- = 2l \sqrt{Gm - \sqrt{G^2m^2 - \frac{G^2J^2}{l^2}}}.$$  

(22)

The inequality $r_- \leq r_+$ always holds, with the equality $r_- = r_+$ being equivalent to $J = ml$. Inverting Eqs. (21) and (22) gives $m$ and $J$ in terms of $r_+$ and $r_-,

$$m = \frac{r_+^2 + r_-^2}{8Gl^2},$$

(23)

$$J = \frac{r_+r_-}{4Gl}.$$  

(24)
Up to now we have put no constraints on the angular momentum $J$ of the outside spacetime. There are three cases we should discuss. First, $J < ml$, which is equivalent to $r_+ > r_-$, in which case the shell is underspinning. Second, $J = ml$, which is equivalent to $r_+ = r_-$, and the outside spacetime is extremal BTZ and the shell is called extremal. Third, $J > ml$, in this case both $r_+$ and $r_-$ are imaginary, and the spacetime and the shell are overspinning. For the first two cases $r_+ \geq r_-$, we impose the mechanical constraint that the shell must be outside its own gravitational radius, i.e., there are no trapped surfaces or horizons in the spacetime. This means that the condition

$$R \geq r_+$$

must hold. For the other case, $J > ml$, the spacetime is overspinning and the shell can be taken to $R = 0$ in which case is a naked singularity. Thus, for this case one has to impose simply $R \geq 0$. We will not discuss further this latter case as it is too simple and the analysis done so far is enough. So in the rest of the paper we assume

$$r_+ \geq r_-.$$  (26)

### 2.7. The metric and matter quantities in terms of $r_+$ and $r_-$

We have found all the relevant expressions for the ring-shell spacetime and have displayed the two intrinsic important radii of the spacetime, $r_+$ and $r_-$. It is thus convenient to write the metric and the matter quantities in terms of $r_+$ and $r_-$, instead of $m$ and $J$. We thus use Eqs. (21)-(24). From Eq. (3), the exterior BTZ metric can be written as

$$ds^2_o = -\frac{1}{l^2r^2} \left[ (r^2 - r_+^2) \left( r^2 - r_-^2 \right) \right] dt_o^2 + \frac{l^2r^2 \, dr^2}{(r^2 - r_+^2) \left( r^2 - r_-^2 \right)} + r^2 \left( d\phi - \frac{r_+ r_-}{l^2} dt_o \right)^2, \quad r > R.$$  (27)

The angular velocity $\Omega$ that makes the exterior metric diagonal at $\Sigma$, i.e., Eq. (6), can be written as

$$\Omega = \frac{r_+ r_-}{l R^2}.$$  (28)

In coordinates $(t_o, \psi)$, the induced metric at $\Sigma$ is then

$$ds^2_\Sigma = -\frac{1}{l^2R^2} \left[ (R^2 - r_+^2) \left( R^2 - r_-^2 \right) \right] dt_o^2 + R^2 d\psi^2, \quad r = R.$$  (29)

In the rest frame of the shell, the components of the energy-momentum tensor $S^a_b$, Eqs. (10)-18 are given by

$$\lambda = \frac{1}{8\pi G l} \left[ 1 - \frac{1}{R^2} \sqrt{(R^2 - r_+^2)(R^2 - r_-^2)} \right],$$  (30)

$$p = \frac{1}{8\pi G l} \left[ \frac{R^4 - r_+^2 r_-^2}{R^2 \sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}} - 1 \right],$$  (31)

$$j = \frac{r_+ r_-}{8\pi G l R}.$$  (32)
3. The fluid seen as a perfect fluid

3.1. The perfect fluid energy density and pressure, and the angular velocity of the reference frame that detects a perfect fluid

We now want to pass to a reference frame where the energy-momentum tensor of the matter has the form of a perfect fluid energy-momentum tensor. We will see that this is possible. Thus, we want that the energy-momentum tensor takes the form

\[ S_{ab} = \bar{\lambda} u^a u^b + \bar{p} \left( u^a u^b + h^{ab} \right), \]  

(33)

where \( h_{ab} \) is the metric on the shell given in Eq. (10), \( u^a \) is some velocity field on the shell, and perfect fluid quantities are written as barred quantities, \( \bar{\lambda} \) is the perfect fluid energy density and \( \bar{p} \) its pressure.

Due to the circular symmetry and the fact that there are only two components for the velocity \( u^a \), one can write it as

\[ u^a = \gamma (\tau^a + \bar{\omega} \psi^a), \]

(34)

for some angular velocity \( \bar{\omega} \), and \( \tau^a = \partial x^a / \partial \tau \), \( \psi^a = \partial x^a / \partial \psi \). Then the normalization condition gives \( \gamma = 1 / \sqrt{1 - \bar{\omega}^2 R^2} \) and so

\[ u^a = \frac{1}{\sqrt{1 - \bar{\omega}^2 R^2}} (\tau^a + \bar{\omega} \psi^a). \]

(34)

Since \( u^a u_a = -1 \) we have from Eq. (33) that

\[ S^a_b u^b = -\bar{\lambda} u^a, \]

(35)

meaning that the velocity field \( u^a \) is an eigenvector of \( S^a_b \) with eigenvalue \( -\bar{\lambda} \). This yields two equations which enable to calculate \( \bar{\lambda} \) and \( \bar{\omega} \). Indeed, Eq. (35) yields

\[ \bar{\lambda} = -\frac{S^a_{\tau} + R^2 \bar{\omega}^2 \psi^a}{1 + R^2 \bar{\omega}^2}, \]

\[ \bar{\omega} = \frac{S^a_{\psi}}{1 + R^2 \bar{\omega}^2}. \]

Having \( \bar{\lambda} \) and \( u^a \) one finds the pressure \( \bar{p} \) by projecting \( S^{ab} \) into the direction orthogonal to \( u^a \), i.e.,

\[ \bar{p} = (h_{ab} + u_a u_b) S^{ab}. \]

(36)

Then using Eqs. (35)-(36) together with previous equations one gets

\[ \bar{\lambda} = \frac{1}{8\pi G l} \left( 1 - \sqrt{\frac{R^2 - r_+^2}{R^2 - r_-^2}} \right), \]

(37)

\[ \bar{\omega} = \frac{R^2 - r_-^2}{r_+ R} \sqrt{\frac{R^2 - r_+^2}{R^2 - r_-^2} - 1}, \]

(38)

Note that \( \bar{\omega} \) is the angular velocity of the reference frame which detects a perfect fluid relative to the shell’s proper frame, i.e., \( \frac{d\psi}{d\tau} = \bar{\omega} \). The derivation presented so far follows [7]. For an alternative derivation see Appendix.
4. Effects due to rotation: Machian effects and dragging of the inertial frames

4.1. The several rotations and angular velocities

We can now explore some effects due to rotation. The BTZ metric given by Eq. (3), appropriate for the vacuum region outside of the shell, tends to the AdS metric at infinity. Pure AdS metric is a nonrotating metric. Thus, infinity is the standard of a nonrotating frame, the AdS frame, or the fixed stars frame in Machian language. The empty metric given by Eq. (4), appropriate for the vacuum region inside of the shell, is in a coordinate system that is rotating with respect to infinity. Indeed, from Eq. (3), one deduces that the interior region, including the interior neighborhood of the ring, has the property that lines with constant $\psi$ move with respect to $t_o$, the global AdS time at infinity, with angular velocity $d\phi/dt_o = \Omega$, where $\Omega$ is given in Eq. (28) (see also Eq. (7)). Now, since the inside is the empty space metric, lines with constant $\psi$ represent inertial frame lines for the interior spacetime that rotate with $\Omega$ with respect to AdS infinity.

Moreover, the angular velocity $\bar{\omega}$, found previously as the angular velocity of the frame that detects perfect fluid quantities, is measured in the frame that rotates with $\Omega$ relative to infinity, the frame that has proper time $\tau$. Then, the corresponding angular velocity $\omega$ measured in the BTZ global time $t_o$ is

$$\omega = \frac{d\psi}{dt_o} = \frac{d\phi}{dt_o} \frac{dr}{d\phi} = \frac{\sqrt{(R^2 - r^2_+) [(R^2 - r^2_-)]}}{lR} \bar{\omega} = \frac{r_-}{lr_+} - \frac{r_r+}{lR^2},$$

i.e.,

$$\omega = \frac{1}{lr_+} - \frac{1}{l} \frac{r_+ - r_-^2}{R^2}.$$  

This $\omega$ is the angular velocity of the reference frame which detects a perfect fluid relative to the shell’s proper frame redshifted to the global time $t_o$. It is always positive since $R \geq r_+$.

In addition, we can calculate the angular velocity of the reference frame at the shell that detects a perfect fluid relative to infinity $t_o$. Call this angular velocity $\omega_\infty$. Then, the angular velocity $\omega_\infty$ of the reference frame at the shell that detects a perfect fluid relative to infinity $t_o$, is clearly the sum of the angular velocity $\omega$ of the reference frame that detects a perfect fluid relative to the proper rest frame with time $\tau$ redshifted to the global time $t_o$, plus the angular velocity of the shell relative to the global time $t_o$, i.e.,

$$\omega_\infty = \frac{d\phi}{dt} = \frac{d\psi}{dt} + \Omega = \omega + \Omega.$$  

The angular velocity $\omega_\infty$ of the reference frame at the shell that detects a perfect fluid relative to a nonrotating frame at infinity with time $t_o$, is then

$$\omega_\infty = \frac{1}{l} \frac{r_-}{r_+},$$

where we have substituted Eqs. (28) and (40) in Eq. (41). Eq. (42) does not depend on $R$, thus $\omega_\infty$ is independent of the shell, it is an intrinsic property of the spacetime.
This does not occur in the corresponding 3+1 dimensional spacetime. Remarkably, the angular velocity $\omega_\infty$ given in Eq. (42) coincides with the horizon’s angular velocity of the BTZ black hole. Indeed, the pure BTZ spacetime, given by Eq. (3) for $0 \leq r < \infty$, possesses a null Killing vector normal to the horizon given by $n^\alpha = t^\alpha_o + \omega_+ \phi^\alpha$, where $\omega_+$ is the horizon angular velocity. The condition $n^\alpha n_\alpha = 0$ gives $g_{00} + 2\omega_+ g_{0\phi} + \omega_+^2 g_{\phi\phi} = 0$, i.e., using Eq. (3),

$$\frac{16G^2 J^2}{r_+^2} - 8\omega_+ GJ + \omega_+^2 r_+^2 = 0.$$  

The solution is $\omega_+ = \frac{4GJ}{r_+}$, i.e.,

$$\omega_+ = \frac{1}{l} \ln \left(\frac{r_+}{r_-}\right).$$  

(43)

Thus, the angular velocity of the special reference frame at the shell $\omega_\infty$ has the same expression as the angular velocity of the BTZ pure vacuum horizon $\omega_+$ as seen by observers at infinity.

### 4.2. Machian effects and the dragging of the inertial frames

We can continue to explore the effects due to the rotation of the ring. As we have discussed, any inner line or interior observer with $\psi$ constant is moving with respect to $t_o$ at infinity with angular velocity $d\phi/dt_o = \Omega$. Now, observers with constant $\psi$ are inertial observers for the empty interior metric and these observers rotate with respect to AdS infinity with $\Omega$. Let us call the angular velocity of the interior observers as $\Omega_{in}$ with $\Omega_{in} = \Omega$. This $\Omega_{in}$ is caused by the presence of the rotation of the shell, and it is called the dragging of inertial frames. From Eq. (28) we have,

$$\Omega_{in} = \frac{1}{l} \ln \left(\frac{r_+}{r_-}\right) \frac{r_+^2}{R^2}.$$  

(44)

We can compare this angular velocity of interior observers with the angular velocity of the shell $\omega_\infty$ given in Eq. (42). The ratio between the two angular velocities $\Omega_{in}/\omega_\infty$ is then

$$\frac{\Omega_{in}}{\omega_\infty} = \frac{r_+^2}{R^2}.$$  

(45)

For $R$ large the inside observers rotate with a small fraction of the shell’s reference frame in question. For $R = r_+$, i.e., when the shell approaches its own gravitational radius, the inside observers corotate with the shell $\omega_{in} = 1$, indicating a very strong dragging effect and a prominent example of Mach’s principle.

The rotating effect on the inside region relative to infinity caused by the presence of a rotating shell as an example of the dragging of inertial frames effect is well known in 3+1 general relativity, see e.g.[7]. We see that in 2+1 dimensions this phenomenon also occurs.
5. Energy Conditions

5.1. The weak energy condition

In Eq. (25) we have imposed that the shell’s radius is always larger than the gravitational radius, \( R \geq r_+ \). We discuss here the weak and the dominant energy conditions. The weak energy condition is automatically satisfied since we have \( \lambda \) and \( p \) non-negative.

5.2. The dominant energy condition

The dominant energy condition can be discussed in the the frame that detects the matter as a perfect fluid, where the quantities \( \bar{\lambda} \) and \( \bar{p} \) are the relevant ones. In this frame the dominant energy condition states that

\[
\bar{p} \leq \bar{\lambda}.
\]  

(46)

From Eqs. (37) and (38) this means

\[
R^2 - r_+^2 \leq R^2 - r_+^2.
\]

(47)

So, first, Eq. (17) can be obeyed when \( R \to \infty \). In this case one has from Eqs. (37) and (38) that \( \bar{\lambda} = \frac{1}{16 \pi G l} \frac{r_+^2 - r_2^2}{R^2} \) and \( \bar{p} = \frac{1}{16 \pi G l} \frac{r_+^2 - r_2^2}{R^2} \). Now, from Eq. (20) one has that for large \( R \) that \( m = \frac{2}{l} M \). So in this limit \( \bar{\lambda} = \frac{M}{2 \pi R}, \bar{p} = \frac{M}{2 \pi R} \), and \( \bar{p} = \bar{\lambda} \). For \( M \) constant the matter disappears in the \( R \to \infty \) limit. However, it can be the case that \( M \) grows proportional to \( R \), so that \( \bar{\lambda} \) is constant. In this case the limit \( R \to \infty \) is also well defined and the shell obeys the dominant energy condition since \( \bar{p} = \bar{\lambda} \).

Second, Eq. (17) is satisfied when \( r_+ = r_- \) (or \( J = ml \)), i.e., the outer BTZ spacetime is extremal and the observers that see a perfect fluid are rotating at the extremality limit, i.e., at the speed of light, \( R \bar{\omega} = 1 \), see Eq. (39). In this case, for these observers the matter in the shell becomes massless as expected and so \( \bar{\lambda} = 0 \) as one can check from Eq. (37). Now, the pressure for these observer also vanish, see Eq. (38). This can be interpreted considering that the AdS attraction due to the negative cosmological constant is now purely balanced by some kind of centrifugal force due to this maximum rotating speed, making the pressure going to zero. In the limit one has \( \bar{p} = \bar{\lambda} \) and so the dominant energy condition is satisfied for any \( R \). For all other settings the dominant energy condition is violated. This is due to the fact that the spacetime is AdS, with a negative cosmological constant, which as seen as a perfect fluid does not obey the dominant energy condition.

6. Conclusions

We have studied the dynamics of a rotating thin matter ring shell in (2+1)-dimensional spacetime with a negative cosmological constant. The outside metric is the BTZ metric, asymptotically AdS, and the inside metric is the empty spacetime metric with a negative cosmological constant. We obtained the shell’s rest energy
density, the pressure and angular momentum flux density by using the junction conditions. Due to rotation, the shell’s energy-momentum tensor was not in the form of a perfect fluid. We then attempted to write the energy-momentum tensor in a perfect fluid form and obtained the thin shell’s energy density, pressure and angular velocity for the frame where one sees the thin shell as a perfect fluid with no angular momentum flux. We have found that Machian effects occur in 2+1 dimensions and that frame dragging is present in these spacetimes. We analyzed the weak and dominant energy conditions of the system. The dominant energy condition implies that there are only two valid configurations: the $R \to \infty$ case with its subcases, the trivial $M = 0$ shell and the nontrivial $M = 2\pi\lambda R$ shell, and the extremal case $r_+ \to r_-$ (or $J = ml$).

Appendix: The energy-momentum tensor in a perfect fluid form, another derivation

Here we derive properties of the matter in a frame that sees it as a perfect fluid. The intrinsic metric on the shell is given by

$$ds^2 = -d\tau^2 + R^2 d\psi^2.$$  \hspace{1cm} (48)

In the rest frame of the shell, the energy-momentum tensor $S^a_b$ is given by

$$\lambda = \frac{1}{8\pi Gl} \left( 1 - \frac{1}{R^2} \sqrt{(R^2 - r_+^2)(R^2 - r_-^2)} \right),$$  \hspace{1cm} (49)

$$p = \frac{1}{8\pi Gl} \left( \frac{R^4 - r_+^2 r_-^2}{R^2 \sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}} - 1 \right),$$  \hspace{1cm} (50)

$$j = \frac{r_+ r_-}{8\pi Gl R}.$$  \hspace{1cm} (51)

Now the metric on the shell, Eq. (48), is invariant under the boost

$$\bar{\tau} = \gamma \left( \tau - \bar{\omega} R^2 \psi \right), \quad \bar{\psi} = \gamma \left( \psi - \bar{\omega} \tau \right),$$  \hspace{1cm} (52)

with

$$\gamma = \frac{1}{\sqrt{1 - \bar{\omega}^2 R^2}}.$$  \hspace{1cm} (53)

Indeed, using Eq. (52) in (48) one gets,

$$ds^2 = -d\bar{\tau}^2 + R^2 d\bar{\psi}^2.$$  \hspace{1cm} (54)

Defining $S^\tau_{\bar{\tau}} = -\bar{\lambda}$, $S^{\bar{\psi}}_{\bar{\psi}} = \bar{p}$, and $S^{\bar{\tau}}_{\bar{\psi}} = \bar{j}$, and transforming $S^a_b$ appropriately, one obtains

$$\bar{\lambda} = \gamma^2 \left( \lambda - 2\bar{\omega} j + \bar{\omega}^2 R^2 p \right),$$  \hspace{1cm} (55)

$$\bar{p} = \gamma^2 \left( p - 2\bar{\omega} j + \bar{\omega}^2 R^2 \lambda \right),$$  \hspace{1cm} (56)

$$\bar{j} = \gamma^2 \left\{ \left( 1 + \bar{\omega}^2 R^2 \right) j - \bar{\omega} R^2 (\lambda + p) \right\}.$$  \hspace{1cm} (57)
Imposing $j = 0$ fixes

$$\dot{\lambda} = \frac{1}{8\pi Gl} \left(1 - \sqrt{\frac{R^2 - r_+^2}{R^2 - r_-^2}}\right), \quad (58)$$

$$\ddot{p} = \frac{1}{8\pi Gl} \left(\frac{R^2 - r_+^2}{R^2 - r_-^2} - 1\right), \quad (59)$$

$$\ddot{\omega} = \frac{r_-}{R^2} \frac{R^2 - r_+^2}{R^2 - r_-^2}, \quad (60)$$

which agreed with the results in Sec. 3.1. Thus,

$$\lambda = \gamma^2 \left(\dot{\lambda} + \omega^2 R^2 \ddot{p}\right), \quad (61)$$

$$p = \gamma^2 \left(\ddot{p} + \omega^2 R^2 \dot{\lambda}\right), \quad (62)$$

$$j = \ddot{\omega} R^2 \gamma^2 \left(\dot{\lambda} + \ddot{p}\right), \quad (63)$$

The set of Eqs. (61)-(62) can be written in a perfect fluid form

$$S^a_b = \dot{\lambda} u^a u_b + \ddot{p} (u^a u_b + h^a_b), \quad (64)$$

with

$$u^\tau = \gamma, \quad u^\psi = \gamma \ddot{\omega}, \quad u_r = -\gamma, \quad u_\psi = \gamma \ddot{\omega} R^2, \quad (65)$$

$$h_{\tau\tau} = -1, \quad h_{\psi\psi} = R^2, \quad h_{\tau\psi} = 0. \quad (66)$$

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