On the convergence of formal Dulac series satisfying an algebraic ODE

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Abstract

We propose a sufficient condition of the convergence of a Dulac series formally satisfying an algebraic ordinary differential equation (ODE). Such formal solutions of algebraic ODEs appear rather often, in particular, the third, fifth, and sixth Painlevé equations possess formal Dulac series solutions, whose convergence follows from the proposed sufficient condition.

1 Introduction

We consider an \(n\)-th order ODE

\[ F(x, y, \delta y, \ldots, \delta^n y) = 0, \]

where \(F = F(x, y_0, y_1, \ldots, y_n)\) is a polynomial of \(n+2\) complex variables and \(\delta\) is the derivation \(x(d/dx)\). Assume (1) has a formal Dulac series solution \(\varphi\) of the form

\[ \varphi = \sum_{k=0}^{\infty} p_k(\ln x) x^k, \quad p_k \in \mathbb{C}[t]. \]

Such series appeared in the 1920’s in the works of Henry Dulac [2] on limit cycles of a planar vector field as asymptotic expansions for the monodromy map (first return map) in a neighbourhood of a polycycle. (More precisely, these were the series with more general powers \(x^{\lambda_k}\) rather than \(x^k\), where \(\lambda_k\) formed a sequence of real numbers increasing to infinity.) Further, in the 1980’s, Dulac series played an important role in finishing proofs of the finiteness theorems for limit cycles (see [5] or [6]).

Dulac series also appear as formal solutions of algebraic ODEs (of the Abel equations, Emden-Fowler type equations, Painlevé equations, \textit{etc.}) and this is a subject of our interest in the present work. First question that one may naturally pose in this context, is that of the convergence of such formal solutions. We propose the following sufficient condition of convergence.

**Theorem 1.** Let the series \(\varphi\) formally satisfy the equation under consideration:

\[ F(x, \Phi) := F(x, \varphi, \delta \varphi, \ldots, \delta^n \varphi) = 0, \]

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and let for each \( j = 0, \ldots, n \) one have

\[
\frac{\partial F}{\partial y_j}(x, \Phi) = a_j x^m + b_j (\ln x) x^{m+1} + \ldots,
\]

where \( a_j \in \mathbb{C}, m \in \mathbb{Z}_+ \) is the same for all \( j \), and \( b_j \in \mathbb{C}[t] \). If \( a_n \neq 0 \) then for any open sector \( S \) of sufficiently small radius, with the vertex at the origin and of the opening less than \( 2\pi \), the series \( \varphi \) converges uniformly in \( S \).

For example, the third, fifth and sixth Painlevé equations have formal solutions in Dulac series. S. Shimomura [8], [9] has proved their convergence for the Painlevé V and VI using the connection of these equations with isomonodromic deformations of linear differential systems. One can also apply Theorem 1 to prove the convergence of formal Dulac series solutions of all the Painlevé equations (see examples at the last section of the paper).

Note that in the case of all \( p_k = \text{const} \), one has a formal power series solution \( \varphi \) of (1) and Theorem 1 becomes a well known sufficient condition of the convergence of such a solution obtained by B. Malgrange [7]. We also note that combining the technique of the present paper with that of [3] on the convergence of generalized power series solutions of (1), one can obtain a theorem similar to Theorem 1 for formal Dulac series of a more general form (with \( x^{\lambda_k}, \lambda_k \in \mathbb{C} \), instead of \( x^k \)).

The paper is organized as follows. Theorem 1 is finally proved in Section 5, which is preceded by some auxiliary statements: in Section 2 we pass from the initial ODE to a reduced one, in Section 3 there is proposed some auxiliary linear algebra, and in Section 4 we construct an ODE which is majorant for the reduced ODE. In the last Section 6 we give two examples of the application of Theorem 1.

2 An equation in the reduced form

**Lemma 1.** Under the conditions of Theorem 1, there is \( \ell' \in \mathbb{Z}_+ \) such that for any \( \ell \geq \ell' \) the transformation

\[
y = \sum_{k=0}^\ell p_k(\ln x)x^k + x^\ell u
\]

reduces the initial equation (1) to the equation

\[
L(\delta)u = x M(x, \ln x, u, \delta u, \ldots, \delta^n u),
\]

where

\[
L(\delta) = \sum_{j=0}^n a_j (\delta + \ell)^j, \quad a_n \neq 0,
\]

and \( M \) is a polynomial of \( n + 3 \) variables. Moreover, the polynomial \( L \) does not vanish in the open right-half plane.

**Proof.** The method of the Lemma 1 proof is standard and similar to that of the proof of the corresponding reduction lemma for power series solutions (see [3]).
For each non-negative integer $\ell$ the formal Dulac series $\varphi$ can be represented in the form
\[
\varphi = \sum_{k=0}^{\ell} p_k(\ln x)x^k + x^{\ell} \sum_{k=1}^{\infty} p_{k+\ell}(\ln x)x^k =: \varphi_\ell + x^{\ell}\psi,
\]
then
\[
\Phi = (\varphi, \varphi, \delta\varphi, \ldots, \delta^n\varphi) = \Phi_\ell + x^{\ell}\Psi,
\]
where $\Phi_\ell = (\varphi_\ell, \delta\varphi_\ell, \ldots, \delta^n\varphi_\ell)$ and $\Psi = (\psi, (\delta + \ell)\psi, \ldots, (\delta + \ell)^n\psi)$. Applying Taylor’s formula one obtains
\[
0 = F(x, \Phi_\ell + x^{\ell}\Psi) = F(x, \Phi_\ell) + x^{\ell} \sum_{j=0}^{n} \frac{\partial F}{\partial y_j}(x, \Phi_\ell)\psi_j + \frac{x^{2\ell}}{2} \sum_{i,j=0}^{n} \frac{\partial^2 F}{\partial y_i \partial y_j}(x, \Phi_\ell)\psi_i \psi_j + \ldots, \tag{3}
\]
where $\psi_j = (\delta + \ell)^j\psi$.

Let us choose the number $\ell$ in such a way that the following two conditions hold:

1) $\ell > m$,
2) $L(\xi) = \sum_{j=0}^{n} a_j(\xi + \ell)^j \neq 0 \quad \forall \xi \in \{\text{Re} \xi > 0\}$

(recall that the non-negative integer $m$ comes from the condition of Theorem 1).

**Definition.** Define the valuation of an arbitrary Dulac series $\varphi = \sum_{k=0}^{\infty} p_k(\ln x)x^k$ as
\[
\text{val}(\varphi) := \min \{k \mid p_k \neq 0\}.
\]

Since Taylor’s formula gives
\[
\frac{\partial F}{\partial y_j}(x, \Phi) - \frac{\partial F}{\partial y_j}(x, \Phi_\ell) = x^{\ell} \sum_{i=0}^{n} \frac{\partial^2 F}{\partial y_i \partial y_j}(x, \Phi_\ell)\psi_i + \ldots,
\]
and $\text{val}(\psi_i) \geq 1$ for any $i$, one has
\[
\frac{\partial F}{\partial y_j}(x, \Phi_\ell) = a_j x^m + b_j(\ln x)x^{m+1} + \ldots, \quad b_j \in \mathbb{C}[t],
\]
for each $j = 0, 1, \ldots, n$, that is, the leading coefficient $a_j$ is preserved when one substitutes in $\frac{\partial F}{\partial y_j}$ the finite sum $\Phi_\ell$ instead of $\Phi$. Now, the relation (3) implies that
\[
\text{val}(F(x, \Phi_\ell)) \geq m + \ell + 1.
\]
Dividing the relation (3) by $x^{m+\ell}$ one obtains the equation of the prescribed form (2) with the formal Dulac series solution
\[
\psi = \sum_{k=1}^{\infty} p_{k+\ell}(\ln x)x^k = \sum_{k=1}^{\infty} P_k(\ln x)x^k.
\]
The lemma is proved. □

**Lemma 2.** The formal Dulac series solution \( \psi \) of (2) is uniquely determined and the degree \( \nu_k \) of each polynomial \( P_k \) satisfies the estimate \( \nu_k \leq kC \), where \( C \) is the degree of the polynomial \( M \) with respect to \( \ln x \).

**Proof.** First, note that one has the following differentiation rule:

\[
\delta : P_k(\ln x) x^k \mapsto x^k \left( k + \frac{d}{dt} \right) P_k(t)_{t=\ln x},
\]

hence

\[
(\delta + \ell)^j : P_k(\ln x) x^k \mapsto x^k \left( k + \ell + \frac{d}{dt} \right)^j P_k(t)_{t=\ln x},
\]

\[
L(\delta) : P_k(\ln x) x^k \mapsto x^k L \left( k + \frac{d}{dt} \right) P_k(t)_{t=\ln x}.
\]

Thus, substituting \( \psi = \sum_{k=1}^{\infty} P_k(\ln x) x^k \) into the equation

\[
L(\delta)u = x M(x, \ln x, u, \delta u, \ldots, \delta^n u)
\]

we obtain the relation whose both sides are Dulac series. Comparing the polynomials in \( t = \ln x \) at each power of \( x \) on the both sides, first we have

\[
L \left( 1 + \frac{d}{dt} \right) P_1(t) = M(0, t, 0, \ldots, 0).
\]

This is an inhomogeneous linear ODE with constant coefficients, with respect to the unknown \( P_1 \). Since \( L(1) \neq 0 \) by Lemma 1, zero is not a root of the characteristic polynomial of this equation. Therefore this ODE has a unique polynomial solution and the degree of this polynomial coincides with that of the right hand side:

\[
\deg P_1(t) = \deg M(0, t, 0, \ldots, 0) \leq C.
\]

Let us denote by \( P^j_k(t) \) the polynomial \( \left( k + \frac{d}{dt} \right)^j P_k(t) \), \( j = 0, 1, \ldots, n \) (in particular, \( P^0_k = P_k \)). Then

\[
\delta^j \psi = \sum_{k=1}^{\infty} P^j_k(\ln x) x^k.
\]

Suppose \( M \) is a linear combination of monomials of the form \( x^\mu (\ln x)^\nu u^0 (\delta u)^q_0 \ldots (\delta^n u)^q_n \). Consequently, for each \( P_k(t) \) with \( k \geq 2 \) we have an inhomogeneous linear ODE with constant coefficients,

\[
L \left( k + \frac{d}{dt} \right) P_k(t) = R_k(t), \quad (4)
\]

where \( R_k(t) \) is a linear combination of polynomials of the form

\[
t^\nu \left( P^0_{k_1} \ldots P^0_{k_{q_0}}(P^1_{l_1} \ldots P^1_{l_{q_1}}) \ldots (P^n_{m_1} \ldots P^n_{m_{q_n}}) \right).
\]
furthermore
\[ \nu \leq C \quad \text{and} \quad \sum_{i=1}^{q_0} k_i + \sum_{i=1}^{q_1} l_i + \ldots + \sum_{i=1}^{q_n} m_i \leq k - 1. \]

In view of the inductive assumption,
\[ \deg P^0_{k_1} \ldots P^0_{k_{q_0}} \leq (k_1 + \ldots + k_{q_0})C, \]
\[ \deg P^1_{l_1} \ldots P^1_{l_{q_1}} \leq (l_1 + \ldots + l_{q_1})C, \]
\[ \ldots \ldots \ldots \]
\[ \deg P^m_{m_1} \ldots P^m_{m_{q_n}} \leq (m_1 + \ldots + m_{q_n})C, \]
hence
\[ \deg R_k(t) \leq C + (k - 1)C = kC. \]

Again, since \( L(k) \neq 0 \) by Lemma 1, zero is not a root of the characteristic polynomial of the linear ODE. Therefore this ODE has a unique polynomial solution \( P_k \) whose degree coincides with that of the right hand side:
\[ \deg P_k(t) = \deg R_k(t) \leq kC. \]

The lemma is proved. \( \square \)

3 From ODEs to Linear Algebra

Let us rewrite \( \psi \) in the form
\[ \psi = \sum_{k=1}^{\infty} P_k(-\epsilon \ln x) x^k, \]
where a small constant \( \epsilon > 0 \) will be chosen later (we denote new polynomials by \( P_k \) again). Then the operators \( \delta \) and \( L(\delta) \) act on a monomial \( P_k(-\epsilon \ln x) x^k \) as follows:
\[ \delta: \quad P_k(-\epsilon \ln x) x^k \mapsto x^k \left( k - \epsilon \frac{d}{dt} \right) P_k(t)|_{t=-\epsilon \ln x}, \]
\[ L(\delta): \quad P_k(-\epsilon \ln x) x^k \mapsto x^k L \left( k - \epsilon \frac{d}{dt} \right) P_k(t)|_{t=-\epsilon \ln x}. \]

This action is naturally represented on the level of vectors and matrices: if \( b_k \) is a column of the coefficients of the polynomial \( P_k \) and \( c_k, d_k \) are columns of the coefficients of the polynomials \( (k - \epsilon \frac{d}{dt})P_k, L(k - \epsilon \frac{d}{dt})P_k \) respectively, then
\[ c_k = (kI - N_k) b_k, \quad d_k = L(kI - N_k) b_k, \]
where \( I \) is an identity matrix, \( N_k \) is a nilpotent matrix of the form
\[
N_k = \begin{pmatrix}
0 & \epsilon & 0 & \ldots & 0 & 0 \\
0 & 0 & 2\epsilon & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 & 0 & \nu_k \epsilon \\
0 & \ldots & \ldots & 0 & 0 & 0
\end{pmatrix}, \quad N_k^{\nu_k+1} = 0.
\]
Let us decompose the polynomial \( L(\xi) = a_0 + \ldots + a_n(\xi + \ell)^n = a_n \prod_{j=1}^{n}(\xi + \lambda_j), \text{ Re} \lambda_j \geq 0. \) Then the matrix \( L(kI - N_k) \) is represented in the form

\[
L(kI - N_k) = a_n \prod_{j=1}^{n}((k + \lambda_j)I - N_k) = a_n \prod_{j=1}^{n}(k + \lambda_j) \prod_{j=1}^{n}(I - \frac{N_k}{k + \lambda_j}) = L(k) \prod_{j=1}^{n}(I - \frac{N_k}{k + \lambda_j}).
\]

The inverse matrix has the form

\[
 L(kI - N_k)^{-1} = \frac{1}{L(k)} \prod_{j=1}^{n} \left( I - \frac{N_k}{k + \lambda_j} \right)^{-1},
\]

where

\[
 \left( I - \frac{N_k}{k + \lambda_j} \right)^{-1} = I + \frac{N_k}{k + \lambda_j} + \left( \frac{N_k}{k + \lambda_j} \right)^2 + \ldots + \left( \frac{N_k}{k + \lambda_j} \right)^{\nu_k}.
\]

Further for matrices we will use the matrix 1-norm \( \| A \|_1 = \| A \| = \max \sum_i |a_{ij}| \) corresponding to the vector 1-norm \( \| x \|_1 = \sum_i |x_i| \).

**Lemma 3.** If \( \epsilon \) is small enough then there is \( 0 < \varepsilon < 1 \) such that

\[
\| L(kI - N_k) \| \leq (1 + \varepsilon)^n |L(k)|,
\]

\[
\| L(kI - N_k)^{-1} \| \leq \frac{1}{(1 - \varepsilon)^n |L(k)|},
\]

for any polynomial \( L \) of degree \( n \) not vanishing in the open right-half plane. In particular, \( \| kI - N_k \| \leq (1 + \varepsilon)k \) and \( \| (kI - N_k)^{-1} \| \leq \frac{1}{(1 - \varepsilon)k} \).

**Proof.** If \( \epsilon \) is small enough then \( \| N_k \| = \nu_k \varepsilon \leq \varepsilon k \), with \( \varepsilon \) small enough. Hence, for any integer \( k > 0 \) one has

\[
\left| I - \frac{N_k}{k + \lambda_j} \right| \leq 1 + \varepsilon, \quad j = 1, \ldots, n.
\]

Therefore,

\[
\| L(kI - N_k) \| = \| L(k) \prod_{j=1}^{n} \left( I - \frac{N_k}{k + \lambda_j} \right) \| \leq |L(k)| \prod_{j=1}^{n} \left| \left( I - \frac{N_k}{k + \lambda_j} \right) \right| \leq |L(k)| (1 + \varepsilon)^n.
\]

To estimate the norm of the inverse matrix, we note that

\[
\left| \left( I - \frac{N_k}{k + \lambda_j} \right)^{-1} \right| \leq \| I \| + \left| \frac{N_k}{k + \lambda_j} \right| + \ldots + \left| \frac{N_k}{k + \lambda_j} \right|^{\nu_k} \leq 1 + \varepsilon + \ldots + \varepsilon^{\nu_k} \leq \frac{1}{1 - \varepsilon},
\]

whence

\[
\| L(kI - N_k)^{-1} \| = \left| \frac{1}{L(k)} \prod_{j=1}^{n} \left( I - \frac{N_k}{k + \lambda_j} \right)^{-1} \right| \leq \frac{1}{|L(k)| (1 - \varepsilon)^n}.
\]

The lemma is proved. \( \square \)
4 A majorant equation

Let us rewrite the reduced equation (2) in the form

\[ L(\delta)u = xM(x, -\epsilon \ln x, u, \delta u, \ldots, \delta^n u), \]

(5)
denoting the new polynomial on the right hand side by the same letter \( M \). Consider another equation which, as we will see later, is a majorant one for (5) in some sense:

\[ \sigma \delta^n u = x\tilde{M}(x, -\epsilon \ln x, \delta u), \]

(6)
where

\[ \frac{1}{\sigma} := \sup_{k \geq 1} \|L(kI - N_k)^{-1}\| \cdot \|(kI - N_k)^n\| < +\infty \]

by Lemma 3, and a polynomial \( \tilde{M} \) is constructed as follows. Suppose \( M \) is a sum of monomials of the form

\[ \alpha x^\mu (-\epsilon \ln x)^\nu u^{q_0}(\delta u)^{q_1} \cdots (\delta^n u)^{q_n}, \quad \alpha \in \mathbb{C}. \]

(7)
To construct the polynomial \( \tilde{M} \), one changes each such a summand to

\[ |\alpha| x^\mu (-\epsilon \ln x)^\nu (c_1 \delta u)^{q_0} (c_2 \delta^n u)^{q_1} \cdots (c_n \delta^n u)^{q_n}, \quad c = \left(1 + \frac{\epsilon}{1 - \epsilon}\right)^n. \]

(8)

**Lemma 4.** The equation (6) possesses a uniquely determined formal Dulac series solution

\[ \tilde{\psi} = \sum_{k=1}^{\infty} \tilde{P}_k(-\epsilon \ln x) x^k, \]

where \( \tilde{P}_k \in \mathbb{R}_+[t] \) are polynomials with non-negative real coefficients of degree \( \deg \tilde{P}_k \leq kC \).

**Proof.** A proof is analogous to that of Lemma 2: each \( \tilde{P}_k \) is obtained as a solution of an inhomogeneous linear ODE with constant coefficients. Starting with

\[ \sigma \left(1 - \epsilon \frac{d}{dt}\right)^n \tilde{P}_1(t) = \tilde{M}(0, t, 0) \in \mathbb{R}_+[t], \]
then one finds the other \( \tilde{P}_k, k \geq 2 \), as unique polynomial solutions of the corresponding ODE

\[ \sigma \left(k - \epsilon \frac{d}{dt}\right)^n \tilde{P}_k(t) = \tilde{Q}_k(t) \in \mathbb{R}_+[t], \quad k \geq 2, \]
where the polynomial \( \tilde{Q}_k \) is expressed via the polynomials \( \tilde{Q}_1 = \tilde{M}(0, t, 0), \tilde{Q}_2, \ldots, \tilde{Q}_{k-1} \) and its degree does not exceed \( kC \) (see the detailed expressions in the proof of the next Lemma 5). Thus, the non-negativity of the coefficients of each polynomial \( \tilde{P}_k \) follows from the non-negativity of the coefficients of the corresponding \( \tilde{Q}_k \) and from the non-negativity of the elements of the matrix \( (kI - N_k)^{-1} \). \( \square \)
For an arbitrary polynomial \( P \in \mathbb{C}[t] \), let us define its norm \( \|P\| \) as the 1-norm of the column of its coefficients. Besides the standard norm properties, this satisfies the following ones which are not difficult to check:

1) for any \( P, Q \in \mathbb{C}[t] \) one has \( \|PQ\| \leq \|P\| \cdot \|Q\| \);
2) if \( P, Q \in \mathbb{R}_+[t] \) then \( \|P + Q\| = \|P\| + \|Q\| \) and \( \|PQ\| = \|P\| \cdot \|Q\| \).

Now we will prove that the constructed equation (6) is a majorant one for the initial equation (3) in the following sense.

**Lemma 5.** The formal Dulac series solution \( \tilde{\psi} \) of the equation (6) is majorant for the formal Dulac series solution \( \psi \) of the equation (5) : \( \|P_k\| \leq \|\tilde{P}_k\| \) for all \( k \).

**Proof.** We have already understood that the polynomials \( P_k \) and \( \tilde{P}_k \) are solutions of the corresponding inhomogeneous linear ODEs with constant coefficients:

\[
L \left( k - \epsilon \frac{d}{dt} \right) P_k(t) = Q_k(t), \\
\sigma \left( k - \epsilon \frac{d}{dt} \right) \tilde{P}_k(t) = \tilde{Q}_k(t),
\]

where \( Q_1(t) = M(0, t, 0, \ldots) \) and \( \tilde{Q}_1(t) = \tilde{M}(0, t, 0) \), whence \( \|Q_1\| = \|\tilde{Q}_1\| \). Therefore

\[
\|P_1\| \leq \|L(I - N_1)^{-1}\| \cdot \|Q_1\| = \|L(I - N_1)^{-1}\| \cdot \|\tilde{Q}_1\| \leq \sigma \|L(I - N_1)^{-1}\| \cdot \|(I - N_1)^n\| \cdot \|\tilde{P}_1\| \leq \|\tilde{P}_1\|.
\]

To obtain analogous estimates for all the other \( k \geq 2 \), let us study expressions for the corresponding \( Q_k \) and \( \tilde{Q}_k \) in more details. We will use the notation \( P^j_k(t) \) for the polynomial \( (k - \epsilon \frac{d}{dt})^j P_k(t) \), \( j = 0, 1, \ldots, n \) (in particular, \( P^0_k = P_k \)). Then

\[
\delta^j \psi = \sum_{k=1}^{\infty} P^j_k(-\epsilon \ln x) x^k, \\
\delta^n \tilde{\psi} = \sum_{k=1}^{\infty} \frac{1}{\sigma} \tilde{Q}_k(-\epsilon \ln x) x^k.
\]

Looking at (7) and (8) one concludes that \( Q_k(t) \) is a sum of polynomials of the form

\[
\alpha t^n \left( P^0_{k_1} \ldots P^0_{k_{q_0}} (P^1_{l_1} \ldots P^1_{l_{q_1}}) \ldots (P^n_{m_1} \ldots P^n_{m_{q_n}}) \right),
\]

where \( \sum_{i=1}^{q_0} k_i + \sum_{i=1}^{q_1} l_i + \ldots + \sum_{i=1}^{q_n} m_i \leq k - 1 \), and \( \tilde{Q}_k(t) \) is a sum of the corresponding polynomials

\[
|\alpha| t^n \left( \frac{c}{\sigma} \tilde{Q}_{k_1} \ldots \frac{c}{\sigma} \tilde{Q}_{k_{q_0}} \right) \left( \frac{c}{\sigma} \tilde{Q}_{l_1} \ldots \frac{c}{\sigma} \tilde{Q}_{l_{q_1}} \right) \ldots \left( \frac{c}{\sigma} \tilde{Q}_{m_1} \ldots \frac{c}{\sigma} \tilde{Q}_{m_{q_n}} \right).
\]

The norm of (10) does not exceed

\[
|\alpha| \cdot \|P^0_{k_1} \| \ldots \|P^0_{k_{q_0}} \| \cdot \|P^1_{l_1} \| \ldots \|P^1_{l_{q_1}} \| \ldots \cdot \|P^n_{m_1} \| \ldots \|P^n_{m_{q_n}} \|.
\]
Using the inductive assumption and the relations (9) we can estimate each factor \( \|P^j_s\|, s < k \), in the product above:

\[
\|P^j_s\| \leq \|(sI - N_s)^j \| \cdot \|P_s\| \leq \|(sI - N_s)^j \| \cdot \|\tilde{P}_s\| \\
\leq \|(sI - N_s)^j \| \cdot \|(sI - N_s)^{-n} \| \cdot \|\tilde{Q}_s/\sigma\| \leq \|sI - N_s\| \cdot \|(sI - N_s)^{-1}\| \cdot \|\tilde{Q}_s/\sigma\| \\
\leq \frac{(1 + \varepsilon)^j s^j}{(1 - \varepsilon)^{n_s}} \|\tilde{Q}_s/\sigma\| \leq \frac{c}{\sigma} \|\tilde{Q}_s\|.
\]

Hence, the norm of (10) does not exceed the norm of (11) (recall that the norm of a product equals the product of norms for polynomials with non-negative real coefficients) and, therefore, \( \|Q_k\| \leq \|\tilde{Q}_k\| \) (again, the norm of a sum equals the sum of norms for polynomials with non-negative real coefficients).

Finally, we conclude

\[
\|P_k\| \leq \|L(kI - N_k)^{-1}\| \cdot \|Q_k\| \leq \|L(kI - N_k)^{-1}\| \cdot \|\tilde{Q}_k\| \\
\leq \sigma \|L(kI - N_k)^{-1}\| \cdot \|(kI - N_k)^n\| \cdot \|\tilde{P}_k\| \leq \|\tilde{P}_k\|.
\]

The lemma is proved. \( \square \)

5 Proof of the theorem on the convergence

Since the function \( \ln x \) is transcendental, the majorant equation (6) can be regarded as an algebraic one,

\[
\sigma U = x \tilde{M}(x,t,U)
\]

(with two independent variables \( x,t \) and the unknown \( U = \delta^n u \)) having a formal solution

\[
\hat{U} = \sum_{k=1}^{\infty} \frac{1}{\sigma} \tilde{Q}_k(t) x^k, \quad \tilde{Q}_k \in \mathbb{R}_+ [t].
\]

After opening the brackets in the series \( \hat{U} \) we get the bivariate power series \( \hat{U}_{\text{pow}} = \sum_{k=1}^{\infty} \sum_{l=0}^{\tilde{d}_k} c_{kl} t^lx^k, \) \( c_{kl} \in \mathbb{R}_+ \), which also formally satisfies the equation (12) (to open the brackets in \( \hat{U} \) and then to substitute \( \hat{U}_{\text{pow}} \) into the both sides of (12) is the same as to substitute \( \hat{U} \) into (12) at first and then to open the brackets on its both sides). By the implicit function theorem, this series converges absolutely for small \( t \) and \( x \) but this is not what we finally need, as the variable \( t \) responses for \( \ln x \) in the Dulac series, and the latter is unbounded for small \( x \). Therefore we take an integer \( r > C \) and consider an open sector \( S \) with the vertex at the origin and of the opening less than \( 2\pi \), such that

\[
|\varepsilon \ln x| < |x|^{-1/r} \quad \forall x \in S,
\]

which implies

\[
|\tilde{Q}_k(-\varepsilon \ln x)| < \tilde{Q}_k(|x|^{-1/r}) \quad \forall x \in S.
\]

9
Now let us consider an equation

$$\sigma U = x \tilde{M}(x, x^{-1/r}, U)$$

(14)

obtained from (12) by putting $t = x^{-1/r}$. This has a formal Puiseux series solution

$$\phi = \sum_{k=1}^{\infty} \sum_{l=0}^{\tilde{\nu}_k} c_{kl} x^{k-l/r}$$

obtained, respectively, from the bivariate power series $\hat{U}_{\text{pow}}$. Indeed, to put $t = x^{-1/r}$ in $\hat{U}_{\text{pow}}$ and then to substitute the obtained Puiseux series into the both sides of (14) is the same as to substitute $\hat{U}_{\text{pow}}$ into the equation (12) at first and then to put $t = x^{-1/r}$ on its both sides. (Note that $\tilde{\nu}_k \leq k C < kr$, and by the same reason $k_1 - l_1/r \neq k_2 - l_2/r$ if $(k_1, l_1) \neq (k_2, l_2)$.) The Puiseux series $\phi$ converges in $S$ absolutely for $x$ small enough (to see this, it is sufficient to make a change $x = z^r$ in (14) and apply the implicit function theorem).

The series

$$\phi^\circ (|x|) = \sum_{k=1}^{\infty} \frac{1}{\sigma} \tilde{Q}_k(|x|^{-1/r}) |x|^k$$

is another representation of the Puiseux series $\phi(|x|) = \sum_{k=1}^{\infty} \sum_{l=0}^{\tilde{\nu}_k} c_{kl} |x|^{k-l/r}$, hence it also converges in $S$ for $x$ small enough (say, for $|x| < \rho$) by the corresponding well known property of convergent positive series (see [4, Ch. VIII]). From the inequality (13) it follows that the series

$$\delta^n \tilde{\psi} = \sum_{k=1}^{\infty} \frac{1}{\sigma} \tilde{Q}_k(-\epsilon \ln x) x^k$$

converges in $S$ absolutely for $|x| < \rho$, therefore $\tilde{\psi} = \sum_{k=1}^{\infty} \tilde{P}_k(-\epsilon \ln x) x^k$ does. Since $-\epsilon \ln |x| \leq |\epsilon \ln x|$, we then have the convergence of the series

$$\sum_{k=1}^{\infty} \tilde{P}_k(-\epsilon \ln |x|) |x|^k, \quad |x| < \rho.$$ 

Now we finish the proof of Theorem 1 by proving the convergence of the series $\psi = \sum_{k=1}^{\infty} P_k(-\epsilon \ln x) x^k$. We have

$$|P_k(-\epsilon \ln x) x^k| \leq \|P_k\| \cdot |\epsilon \ln x|^p_k |x|^k \leq \|P_k\| \cdot |x|^{(1-C/r)k} \leq \|\tilde{P}_k\| \cdot |x|^{(1-C/r)k}.$$ 

Thus, for all $x$ such that $w = |x|^{1-C/r} < \rho$ one has

$$|P_k(-\epsilon \ln x) x^k| \leq \|\tilde{P}_k\| w^k \leq \tilde{P}_k(-\epsilon \ln w) w^k,$$

whence the convergence of $\psi$ follows. The theorem is proved.
Remark. Using the results of the present work and article [3] one can prove a statement similar to Theorem 1 but for formal Dulac series of a more general form (a proof is more technical though). Let the series
\[ \varphi = \sum_{k=0}^{\infty} p_k (\ln x)^{x^k}, \quad \lambda_k \in \mathbb{C}, \quad 0 \leq \text{Re} \lambda_0 \leq \text{Re} \lambda_1 \leq \cdots \rightarrow +\infty, \]
formally satisfy the equation under consideration:
\[ F(x, \Phi) := F(x, \varphi, \delta \varphi, \ldots, \delta^n \varphi) = 0, \]
and let for each \( j = 0, \ldots, n \) one have
\[ \frac{\partial F}{\partial y_j}(x, \Phi) = a_j x^\alpha + b_j (\ln x) x^{\alpha_j} + \ldots, \quad \text{Re} \alpha < \text{Re} \alpha_j, \]
where \( a_j \in \mathbb{C}, \alpha \in \mathbb{C} \) is the same for all \( j \), and \( b_j \in \mathbb{C}[t] \). If \( a_n \neq 0 \) then for any open sector \( S \) of sufficiently small radius, with the vertex at the origin and of the opening less than \( 2\pi \), the series \( \varphi \) converges uniformly in \( S \).

6 Examples

In this section we give two examples, of the Abel equation and of the Painlevé VI equation, and apply Theorem 1 for proving the convergence of their formal Dulac series solutions.

Example 1. The Abel equation of the second kind
\[ w \frac{dw}{dx} = -w - \frac{1}{x^3} \]
has a one-parameter family of formal Dulac series solutions
\[ \hat{w} = \frac{1}{x} \left( 1 + (C - \ln x)x^2 + \sum_{k=2}^{\infty} P_k(\ln x)x^{2k} \right), \quad C \in \mathbb{C}. \]
Making the power transformation \( w = y/x \) in the equation under consideration and rewriting the result by means of the operator \( \delta \), we obtain the equation
\[ f(x, y, \delta y) := y \delta y - y^2 + x^2 y + 1 = 0 \]
with a family of formal Dulac series solutions
\[ \varphi = 1 + (C - \ln x)x^2 + \sum_{k=2}^{\infty} P_k(\ln x)x^{2k}. \]
Let us check their convergence by applying Theorem 1. As \( f(x, y_0, y_1) = y_0 y_1 - y_0^2 + x^2 y_0 + 1 \), one has
\[ \frac{\partial f}{\partial y_0} = y_1 - 2y_0 + x^2, \quad \frac{\partial f}{\partial y_1} = y_0. \]
Substituting
\[ y_0 = \varphi = 1 + (C - \ln x)x^2 + \ldots, \]
\[ y_1 = \delta\varphi = (2C - 1 - 2\ln x)x^2 + \ldots, \]
we thus obtain
\[ \frac{\partial f}{\partial y_0} = -2 + \ldots, \quad \frac{\partial f}{\partial y_1} = 1 + (C - \ln x)x^2 + \ldots. \]
Hence, the condition of Theorem 1 is fulfilled and the series \( \varphi \) converges in any open sector \( S \subset \mathbb{C} \) with the vertex at the origin, of sufficiently small radius and of the opening less than \( 2\pi \).

**Example 2.** The Painlevé VI equation
\[ y'' = \frac{(y')^2}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) - y' \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) + \]
\[ + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ a + b\frac{x}{y^2} + c\frac{x-1}{(y-1)^2} + d\frac{x(x-1)}{(y-x)^2} \right], \quad a, b, c, d \in \mathbb{C}, \]
with \( a, c \neq 0 \), \( a \neq c \), and \( \sqrt{2a} \pm \sqrt{2c} \in \mathbb{N} \) has two one-parameter families of formal Dulac series solutions
\[ \varphi = 1 \pm \sqrt{c/a} + \sum_{k=1}^{\infty} P_k(\ln x) x^k, \]
where a free parameter is contained in a polynomial \( P_k \) with \( k = \sqrt{2a} \pm \sqrt{2c} \), see [1]. One rewrites equation (15) in the form
\[ f(x, y_0, y_1, y_2) := 2y_2(x-1)^2y_0(y_0 - 1)(y_0 - x) - y_1^2(x-1)^2[(y_0 - 1)(y_0 - x) + \]
\[ + y_0(y_0 - x) + y_0(y_0 - 1)] + 2y_1x(x-1)y_0(y_0 - 1)^2 - \]
\[ - [2ay_0^2(y_0 - 1)^2(y_0 - x)^2 + 2bx(y_0 - 1)^2(y_0 - x)^2 + \]
\[ + 2c(x - 1)y_0^2(y_0 - x)^2 + 2dx(x - 1)y_0^2(y_0 - 1)^2] = 0. \]
The partial derivative of \( f \) with respect to \( y_2 \) is
\[ \frac{\partial f}{\partial y_2} = 2(x-1)^2y_0(y_0 - 1)(y_0 - x). \]
Substituting \( y_0 = \varphi \) in this expression we obtain
\[ \frac{\partial f}{\partial y_2} = \pm 2 \left( 1 \pm \sqrt{c/a} \right)^2 \sqrt{c/a} + b_2(\ln x)x + \ldots, \]
for some polynomial \( b_2 \). As \( c \neq 0 \) and \( a \neq c \), this Dulac series begins with a non-zero constant, which is sufficient for the condition of Theorem 1 being fulfilled. Thus, the series \( \varphi \) converges in any open sector \( S \subset \mathbb{C} \) with the vertex at the origin, of sufficiently small radius and of the opening less than \( 2\pi \).

As we mentioned in Introduction, in such a way one can prove the convergence of all formal Dulac series solutions of the third, fifth and sixth Painlevé equations (which has been also done by S. Shimomura for the Painlevé V and VI in [3] and [8], respectively).
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