A.S. Ilyin, K.P. Zybin, A.V. Gurevich.

Dark matter in galaxies and growth of giant black holes.

Abstract

The connection between dark matter and giant black holes in the galactic nuclei is investigated. The joint evolution of dark and luminous matter in averaged self-consistent gravitational potential is considered. It is shown that the distribution of dark matter remains spherically symmetric even in presence of essential asymmetry of luminous matter in the galaxy.

The kinetic equation describing evolution of dark matter particles distribution function which takes into account their dynamics in gravitational potential and scattering on stars is derived. It is shown that significant flux of dark matter on a seed black hole in the center of galaxy should appear. The growth law of seed black hole due to absorption of dark matter is derived. It appears that during the lifetime of galaxy the seed black hole should grow significantly, up to scales $10^7 - 10^8 M_\odot$.

Brief analysis of observational data shows that the presented theory is in reasonable agreement with observations.

1 Introduction

Recently, intensive development of techniques and new methods in astrophysical observations had lead to discovery of more than 80 giant black holes in centers of galaxies [1], [2]. The masses of the black holes are distributed in the range $2 \cdot 10^6 - 3 \cdot 10^9 M_\odot$ ($M_\odot = 2 \cdot 10^{33}$g is solar mass), as is shown at Fig.1. The analysis of observational data revealed a connection between the giant black hole mass in the center of a galaxy and mass of the galaxy’s bulge.

We remind that bulge is the most dense central spherical component of a galaxy, containing mainly old stars. The total rotational velocity of stars in bulge is usually significantly smaller than velocity dispersion $\sigma$, that is why the form of bulge is often close to spherically symmetric. The density of stars in bulge grows rapidly to the center. The size of bulge $r_b \sim 1 - 30$ kpc, it grows with the galaxy mass. The masses of giant black holes $M_{bh}$ are approximately three orders of magnitude smaller than the masses of host galaxy bulges.

According to modern knowledge, the main part of matter in the Universe is made up by nondissipative nonbaryon dark matter. The initial density fluctuations of the dark matter in the early Universe had grown up forming large-scale nonlinear gravitationally bound objects - galactic haloes. The important feature of these haloes is singular distribution of density in their centers [3]. Baryon matter trapped by gravitational field of a halo has settled gradually to its center and formed a galaxy. The presence of singularity in the center of dark matter distribution allows the compressing baryon gas to form a primordial black hole [4] with mass of the order of $10^3 M_\odot$. Note, that in this scenario primordial black hole is situated exactly in the galaxy dynamical center. Later the seed black hole grows intensively due to accretion both baryonic and dark matter from the bulge. The
The present paper is devoted to the analysis of evolution of dark matter and its absorption by the black hole in the center of a galaxy.

The paper is organized as follows.

In the second section, the distribution of dark matter in a galaxy and its evolution under the influence of the averaged self-consistent gravitational field of baryon matter (bulge) is considered. The problem is solved in the adiabatic approximation in assumption of spherical symmetry of baryon matter distribution. The role of deviations from spherical symmetry is analyzed. It is shown that even if the baryon matter distribution is noticeably asymmetric, the distribution of dark matter defined by joint action of gravitational fields of baryon and dark matter remains spherically symmetric.

In the third section, growth of a seed black hole in galactic center on account of direct capture of dark matter particles moving in the averaged self-consistent potential is investigated. It is shown that the mechanism is of small efficiency in real conditions and cannot lead to significant growth of the seed black hole mass.

In the forth and fifth sections, the evolution of dark matter distribution function due to the gravitational scattering of particles by stars is considered. The scattering lead to violation of adiabatic approximation. The important peculiarity of the process is that the mean free path length of particles is much more than the size of the bulge, hence the collisions are rare. Taking this into account, in the section 4 we derive the collision integral averaged over dark matter particles oscillations, and write down the kinetic equation describing the evolution of dark matter distribution function. It is shown that because of special features of initial distribution function, the main role is played by diffusion in angular momentum space.

In the fifth section, the solution of the diffusion equation is obtained, the flux of dark matter on black hole is derived, and the law of the black hole's growth is ascertained.

In conclusion, we make brief analysis of observational data showing that the growth of giant black holes on account of dark matter absorption should be essential.

On the whole, the presented theory is in reasonable agreement with existing observational data concerning giant black holes. Further development both of the observations and the theory and their detailed comparison are undoubtedly of great interest.

2 Adiabatic dynamics of gravitating dark matter and baryonic matter.

Observation show that over distances of the order of the horizon radius our Universe is homogeneous, isotropic and expands uniformly. The expansion leads to a rapid cooling of matter. A cold gravitating gas is unstable because of the action of universal gravitational forces. Growth of the Jeans instability creates regions of strong compression with dimensions much smaller than the horizon radius $R_H$. This is of decisive importance for the formation (in the CDM scenario) of the large-scale structure of matters in the Universe.

The main role in this process is played by dark matter, which manifests itself only in the gravitational interaction. According to the nowadays knowledge the baryonic matter in the Universe contribute $\Omega_B < 0.05$ and thus the gravitational processes are determined.
mostly by nonbaryonic dark matter. It is supposed as usually that the particles of which
dark matter is composed interact very weakly with one another and with baryonic matter.

After recombination initial linear density fluctuations $\delta \rho$ are growing relatively slowly
with time $t$ (due to expansion of Universe) according to power law: $\delta \rho/\rho \propto t^{2/3}$. As soon
as the disturbances amplitude reaches unity $\delta \rho/\rho \propto 1$ the nonlinear processes dominate.
Well known Zeldovich-Harrison spectrum of the initial linear fluctuations is growing with
the wave vector $k$: $\delta \rho(k) \propto k$. Due to this nonlinear stage $\delta \rho(k_n)/\rho \propto 1$ is reached first
for the small $k$. Nowadays the nonlinear boundary $l_n \propto k_n^{-1} \propto 10 - 100 Mps$. Thus the following conditions are fulfilled for the dark matter large-scale structures $l$, which have passed the nonlinear stage of the Jeans instability:

$$\frac{l}{l_f} \to 0, \quad \frac{l}{R_H} \ll 1, \quad \frac{u}{c} \approx \frac{l}{R_H} \ll 1.$$

Here $l_f$ is the free path of dark matter particles which we consider as noninteracting, $u$ - characteristic velocities in the inhomogeneity. We see that the nonlinear stage of Jeans
instability is developing in noninteracting, nonrelativistic, cold gravitating gas.

The general system of equation, describing the nonlinear gravitational compression of
the dark matter can be presented in a form [3]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} (\rho u) = 0$$

$$\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \right) u + \frac{\partial \psi_d}{\partial r} = 0$$

$$\Delta \psi_d = 4\pi G \rho$$

Here $\rho$ is the density, $u$ - velocity of the gas, and $\psi_d$ - gravitational potential. The nonlinear
process is most strongly developing near the density maximums. Let us consider the
vicinity of the one maximum, presenting the initial density distribution $\rho$ in the general
form:

$$\rho_{t=0}(r) = \rho_0 \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \right), \quad u_{t=0} = 0$$

The solution of hydrodynamical system of equations (2), lead to singularity at the moment of time $t = t_k$. After that moment in a dark matter gas the multifluid motion
is developed. It is accompanied by density oscillation. With time $t \to \infty$ the number of
fluids is infinitely growing. The density oscillation amplitudes $a_i(t)$ at the same time is
going to zero: $a_i(t) \propto O(1/t)$. As a result of this kinetic mixing of fluids the stationary
spherically symmetric cluster of dark matter is established. The main peculiarity of dark
matter distribution in this cluster is the existence of a singularity in its core

$$\rho = K r^{-12/7}$$

This result was obtained in analytic theory by Gurevich and Zybin [5].

It should be noted that numerical calculation could not receive this result for a long
time and showed only the plane structures of the Zeldovich pancake type. Only the
usage of special numerical methods allowed to make a significant step demonstrating the existence of the stationary spherical symmetric structures \[6\]. But the singularity of density distribution was different from (4). Further development of numerical methods allowed to improve this result and move on very close to the law (4) \[7\]. So nowadays one can say that the numerical calculation fully confirmed the existence of a stable spherical clusters with the singularity law (4). This cluster was called nondissipative gravitational singularity (NGS).

The main method of analytical solution \[5\] was based on the solution of kinetic equation for dark matter particles using canonical action variables \(I = (I_R, m)\), where

\[
I_R(E, m) = \frac{\sqrt{2}}{\pi} \int_{r_+}^{r_-} \sqrt{E - \psi_d(r) - \frac{m^2}{2r^2}} dr,
\]

- radial action, \(m\) is the angular momentum, \(E\) - energy, \(r_\pm\)-turning points (roots of the equation \(E - \psi_d(r) - m^2/2r^2 = 0\)). According to \[3\] the established distribution function is

\[
f(I_R, m) = f_0 I_R^{1/8} \delta \left( m^2 - l_0^2 I_R^2 \right),
\]

where \(f_0\) and \(l_0\) are constants, determined by the initial distribution (3). In particular the case of spherical symmetric initial density distribution (3): \(a = b = c\), is degenerate. The spherical symmetry is conserved at all stage of evolution, including initial compression, hydrodynamical singularity, kinetic mixing. In this case the angular momentum are always equal to zero, consequently in (6) \(l_0 = 0\). In the case of weak initial asymmetry:

\[
(a - b)^2 + (a - c)^2 << a^2
\]

parameter \(l_0\) is small, proportional to \(\epsilon\):

\[
l_0 = 0.16\epsilon, \quad \epsilon = \frac{1}{a} \sqrt{(a - b)^2 + (a - c)^2 - (a - b)(a - c)}
\]

Accordingly as follows from (6) angular momentum in this case is also small and proportional to parameter \(\epsilon\) and to radial action \(I_R\).

We will consider now the effect of baryonic matter on the distribution of dark matter. Let us take into account, that the baryonic matter particles lose their energy due to inelastic collisions accompanied by optic emission. That is why the baryonic matter falls down to the bottom of the potential wells, formed by the cold dark matter. The luminous baryonic matter then acts as indicator of the dark matter structure identifying in particular the position of the centre of NGS. This produces a unique object: baryonic matter with a halo of dark matter. Examples of such objects are galaxies: in this case the evidence for the presence of a dark matter halo is provided by flat rotation curves \[8\]. Other examples of such objects are clusters of galaxies, which contain a trapped hot gas, again directly confirming the existence of a dark matter halo \[9\].

It should be pointed out that as the baryonic matter drops to the bottom of potential wells, its density rises at the central regions of the wells and can become equal to the density of nonbaryonic matter. That will lead to the considerable distortion of the canonical distribution of the dark matter (4). The main feature of the problem is that the baryonic matter loses its energy and falls to the wells bottom slowly - at the timescales comparable with the lifetime of the Universe. The dark matter particles trapped in the well oscillate
many times. It means that the radial action $I_R$ is integral of motion (adiabatic invariant) and distribution function $f_d$ is conserved and described as previously by formula (6). But the dark matter potential in (5) should be changed to the full potential $\Psi$, determined both by baryonic and dark matter $\Psi = \psi_d + \psi_b$. Consequently the density $\rho_d$ takes the form:

$$\rho_d(r, t) = \frac{\sqrt{2 \pi}}{r^2} \int_0^\infty dm^2 \int_{\Omega E} dE \frac{f(I_R, m)}{\sqrt{E - \Psi(r, t) - m^2/2r^2}}$$

(8)

$\Omega_E$ is the region, where $E - \Psi(r, t) - m^2/2r^2 > 0$.

$$I_R = \frac{\sqrt{2}}{\pi} \int_{r_-}^{r_+} \sqrt{E - \Psi(r, t) - m^2/2r^2} dr.$$  

(9)

Let us consider central part of the object where $M_b > M_d$ and $\Psi \approx \psi_b$. Potential $\psi_b(r)$ is determined by the mass $M_b(r)$ of a baryonic matter. $M_b(r)$ is slowly changing with time $t$ due to slowly falling down into the well of a baryonic component. Let us suppose, for example, that $M_b$ has a power-like dependence on $r$:

$$M_b(r) \propto r^n, \quad \psi_b \propto r^{n-1}, \quad n \geq 0.$$  

(10)

From (6), (8), (9) and (10) it follows:

$$\rho_d(r, t) = \rho_0 r^{(9n-39)/16}$$

(11)

Thus we see that in conditions (10) the power scaling for $\rho_d$ exists also and just as in the case of purely dark matter (4), the scaling law (11) does not depend on parameters of dark matter initial distribution $l_0$ and $f_0$. For example, if $M_b(r) \propto r^{9/7}$ both $\rho_d \propto \rho_b \propto r^{-12/7}$. If $\rho_b \propto r^{-2}$, $M_b \propto r$ (isothermal sphere) and $\rho_d \propto r^{-15/8}$. We see, that dark matter is compressed by the gravitational action of baryonic gas. For point-like baryonic mass distribution could be reached maximal compression of a dark matter

$$\rho_d \propto r^{-39/16}.$$  

(12)

We note that the mass distribution $\rho \propto r^{-m}$ with $m > 2$ must lead to the black hole creation and the problem should be considered in more details taking this fact into account.

We discussed above the spherically symmetric distribution of baryonic matter. In real conditions due to rotational motion baryonic component even in the central parts both of the spiral and elliptic galaxies can have a noticeable nonspherical components. Can they lead to a significant discrepancy of dark matter distribution in comparison with the case of purely spherical symmetric baryonic matter? There are some general theorems concerning the behaviour of systems close to integrable ones. According to these theorems, the average action variables $I_R, m, m_z$ remain constant during particles motion in weakly asymmetric, slowly varying potential [10]. That is why the distribution function (6) does not change significantly during evolution of the slowly rotating baryonic matter [11].
3 Black hole growth in the center of the dark matter halo (self-consistent approximation).

The aim of the present section is the investigation of possibility of seed black holes growth within the bounds of self-consistent, collisionless approximation.

The growth of central black hole leads to decreasing of dark matter particles number in the loss cone. On the other hand, it leads to the growth of the loss cone. As the result of the two competing processes the flux of a dark matter to black hole can decrease. Let us assume that the growth of black hole stops at total mass

$$M_{bh} = M_b + M_d,$$  

(13)

where $M_b$ is an initial baryonic mass of a black hole and $M_d$ is absorbed dark mass. The flux of dark matter on a black hole with mass $M_{bh}$ is composed of particles with angular momentum

$$m < m_g = 2c r_g,$$  

(14)

where $r_g = \frac{2M_{bh}}{c^2}$ is the gravitational radius of a black hole.

In spherically symmetric potential $\Psi (r, t)$ (with arbitrary time dependence) the angular momentum of particles does not change. That is why total mass of particles captured by a black hole is described by initial distribution function $f_i$:

$$N \left( f_i; M_{bh} \right) = \int d^3r d^3v f_i(r, v) \theta (m_g - m),$$

where

$$\theta(x) = 1, \text{ if } x > 0, \text{ and } \theta(x) = 0, \text{ if } x < 0.$$ 

The input of a dark matter particles in total mass of black hole is

$$M_d = N \left( f_i; M_b + M_d \right)$$  

(15)

Thus the scenario of growth of black hole mass caused by dark matter particles flux depends on initial distribution function $f_i$ and black holes baryonic mass $M_b$ only. If the equation (15) has the solution $M_d = M_d (f_i; M_b) > 0$ the mass of a black hole will not exceed $M_b + M_d (f_i; M_b)$. If the equation (15) has no solution, the seed black hole will grow beyond all bounds.

As an example let us consider initial isothermal distribution of a dark matter. In this case the distribution function, density and potential have the form:

$$f_i(E, m) = \frac{\rho_0}{(2\pi \sigma_d^2)^{3/2}} e^{-\frac{E}{2\sigma_d^2}}, \quad \rho_0 = \frac{\sigma_d^2}{2\pi G},$$  

(16)

$$\rho(r) = \frac{\rho_0}{r^2},$$  

(17)

$$\psi_d(r) = 4\pi G \rho_0 \ln(r),$$  

(18)
where $\sigma_d$ is velocity dispersion of dark matter particles. Density of particles with angular momentum smaller than $m_g$ is

$$\rho_g = \frac{\sqrt{2\pi}}{r^2} \int dE \int dm^2 \frac{f(E, m^2)}{\sqrt{E - \psi_d - \frac{m^2}{2r^2}}} \theta \left(E - \psi_d - \frac{m^2}{2r^2}\right) \theta(m_g - m)$$

or, after calculations

$$\rho_g(r) = \frac{\rho_0}{r^2} \left(1 - e^{-\frac{m_g^2}{2\sigma_d^2}}\right),$$

hence total mass of particles with the angular momentum smaller than $m_g$ is

$$N_g = 2\pi \left(\frac{\rho_0}{G}\right)^{1/2} m_g.$$  \hfill (19)

Taking into account (19) one can rewrite the equation (15) in the form:

$$\frac{M_d}{M_b} = Q \left(1 + \frac{M_d}{M_b}\right),$$  \hfill (20)

where

$$Q = \frac{8\pi}{c} (G\rho_0)^{1/2}.$$

The positive solution of the equation (20) exists only at $Q < 1$. Thus, scenario of growth of a black hole in the case of isothermal distribution of a dark matter does not depend on black hole baryonic mass but depends strongly on the dark matter density determined by factor $\rho_0$. If $\rho_0$ is anomaly big: $\rho_0 > G^{-1}(c/8\pi)^2$ the initial isothermal distribution is unstable: any small seed black hole will result in absorption of all dark matter halo. At $\rho_0 < G^{-1}(c/8\pi)^2$ distribution is steady. The absorbed dark matter mass appears proportional to baryonic mass and is determined by expression (20).

For our Galaxy $Q \sim 0.01$ and $M_d$ is less that 1%, what means, that the accretion of dark matter on the initial black hole $M_b$ lead to a very small growth of its mass only.

Actually, distribution function of dark matter (6) differs essentially from isothermal one. In this case total mass of particles with angular momentum smaller than $m_g$ is determined by expression

$$N_g = (2\pi)^3 \int_0^\infty dI_0 \int_0^\infty dm \int_{-m}^m dm' f_0 I_0^{1/8} \delta \left(m^2 - l_0^2 I_0^2\right) \theta \left(m_g - m\right),$$

hence

$$N_g = (2\pi)^3 \frac{8}{9} f_0 \left(\frac{m_g}{f_0}\right)^{9/8}. \hfill (21)$$

The equation (15) takes the form:

$$\frac{M_d}{M_b} = Q \left(1 + \frac{M_d}{M_b}\right)^{9/8},$$  \hfill (22)

where $Q = Q'M_b^{1/8}$, $Q' = (2\pi)^{3/8} f_0 \left(\frac{4G}{f_0c}\right)^{9/8}$ and $f_0$ can be calculated by expression

7
\[ M_H = (2\pi)^3 \int_0^{I_{\text{max}}} dI_R \int_0^\infty dm^2 f_0 I_R^{1/8} \delta \left( m^2 - l_R^2 \right), \]

where the radial action of particles near the halo's boundary is related to the halo's full mass \( M_H \) and radius \( R_H \) by expression \( I_{\text{max}} \approx \frac{1}{\pi} G^{1/2} M_H^{1/2} R_H^{1/2} \), hence

\[ f_0 = \frac{9}{8} \frac{\pi^{9/8}}{(2\pi)^3} \frac{M_H^{7/16}}{(G R_H)^{9/16}}, \quad (23) \]

Using this relation, one can show that in our Galaxy the value of parameter \( Q \) is again small (of the order of \( 10^{-2} \)), and mass of dark matter captured by black hole is not significant.

We would like to pay attention that investigation of this problem in previous papers [12] results to opposite statement: the initial black hole was growing without any limits. The reason of this discrepancy is in the fact that authors did not take into account the changing of dark matter particles distribution function caused by black hole absorption. That is why the loss cone was filled constantly and the flux on black hole was not exhausted. Considering the transformation of distribution function one comes to our result.

Thus, dark matter particles cannot cause a noticeable growth of the black hole because of conservation of their angular momentum in the spherically symmetric potential. As it follows from the remark in the end of Section 2, weak asymmetry of the potential does not change significantly this conclusion. The simplest process that can fill the loss cone is collisions of dark matter particles with stars. We shall discuss this point in next sections.

### 4 Kinetic equation

Gravitational interaction of dark matter particles and their collisions with stars is described by distribution function \( f(\mathbf{r}, \mathbf{v}, t) \), which satisfies the kinetic equation [13]:

\[ \frac{\partial f}{\partial t} + \{H_0, f\} = St[f], \quad (24) \]

where \( H_0 \) is Hamilton function corresponding to the motion of particles in the averaged self-consistent potential, \( St[f] \) is the collision term. Gravitational interaction of dark matter particles with individual stars is described by Coulomb law, hence the collision term may be written in Landau form [14]

\[ St[f] = \frac{\partial}{\partial v_k} W_{kp} \frac{\partial}{\partial v_p} f. \quad (25) \]

Here \( W_{kp} = 2\pi G^2 M_\odot \Lambda^2 \int d^3 v w_{kp} F(v', r) \), \( w_{kp} = (u^2 \delta_{kp} - u_k u_p) / u^3 \), \( u_k \) are the dark matter velocity \( \mathbf{v} \) components, \( F(v', r) \) is the stars distribution function, \( \mathbf{u} = \mathbf{v}' - \mathbf{v} \) is the relative velocity of a star and dark matter particle, \( \Lambda \sim 10 \) is the gravitational Coulomb logarithm.

In fact, as we have already noticed, the frequency of collisions of particles with stars is much less than the frequency of their orbital motion. Thus the kinetic equation (24) may
be significantly simplified. For this purpose, let us rewrite it in the action-angle variables $I, \phi$:

$$\frac{\partial f}{\partial t} + \omega_k \frac{\partial f}{\partial \phi_k} = St_{I,\phi}[f]$$

Taking into account that the initial distribution function (6) does not depend on angle variables $\phi$, we note that its time variation is determined by the collision term (25) only. However, the collisions are rare and $St[f] \propto \nu f$, where $\nu$ is the collision frequency. Hence we search a solution of the equation (26) in the form $f = f_0 + \nu f_1$, where $\nu f_1$ is a small correction to $f_0$, and both the terms $f_0, f_1$ depend on "fast" $t_0 = t$ and "slow" $t_1 = \nu t$ times. In zero approximation by the parameter $\nu$ we obtain

$$\frac{\partial f_0}{\partial t_0} + \omega_k \frac{\partial f_0}{\partial \phi_k} = 0$$

From this equation follows that the main part of distribution function $f_0$ does not depend on angle variables and "fast" time. The first approximation gives

$$\nu \left( \frac{\partial f_0}{\partial t_1} + \frac{\partial f_1}{\partial t_0} + \omega_k \frac{\partial f_1}{\partial \phi_k} \right) = St_{I,\phi}[f_0].$$

Averaging this equation over the angle variables and "fast" time, taking into account that the second and third terms become zero, we obtain equation

$$\frac{\partial f(I,t)}{\partial t} = St[f(I,t)]$$

(Here we omit the index "0" in distribution function and returned to usual time $t$.)

So, under assumptions on collision frequency $\nu$ made above, it is possible to find the distribution depending on action variables only, as a solution of kinetic equation (27) with time-averaged collision term:

$$St[f] = \frac{1}{(2\pi)^3} \int d^3\phi St_{I,\phi}[f]$$

The time-independent kinetic equation in the averaged form (27) describing the distribution of stars in the vicinity of a giant black hole was first studied in [15], [16] and [17]. A purely Coulomb potential field was considered in these works.

Our aim is to examine time-dependent solutions of the kinetic equation (27) describing dynamics of particles in arbitrary central symmetric field. First, it is necessary to obtain the time-averaged collision integral (28). To do this we will follow the technique, developed in [18].

Let us consider tensor differential form (25) formally in six-dimensional phase space $X = (v, x)$, assuming coordinate components of the tensor $W$ being zero:

$$\frac{\partial}{\partial v_k} W_{kp} \frac{\partial}{\partial v_k} = \frac{\partial}{\partial X_\mu} W_{\mu\nu} \frac{\partial}{\partial X_\nu}; \quad k, p = 1, 2, 3; \quad \mu, \nu = 1, \ldots, 6.$$
In this six-dimensional space we make canonical transformation to the action-angle variables

\[ X_\mu \mapsto Y_\mu'; \quad Y = \{ I_R, m, m_z; \phi_1, \phi_2, \phi_3 \} \]

The differential form then takes the form

\[ \frac{\partial}{\partial X_\mu} W_{\mu\nu} \frac{\partial}{\partial X_\nu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial Y_{\mu'}} \sqrt{g} R_{\mu'\nu'} \frac{\partial}{\partial Y_{\nu'}}, \quad R_{\mu'\nu'} = \frac{\partial Y_{\mu'}}{\partial X_\mu} \frac{\partial Y_{\nu'}}{\partial X_\nu} W_{\mu\nu} \]

where \( \sqrt{g} \) is the Jacobian. However, Jacobian of canonical transformation is unity, hence the differential form in new variables preserves in the form:

\[ \frac{\partial}{\partial X_\mu} W_{\mu\nu} \frac{\partial}{\partial X_\nu} = \frac{\partial}{\partial Y_{\mu'}} R_{\mu'\nu'} \frac{\partial}{\partial Y_{\nu'}}. \]

We search for the solution of kinetic equation (27) which does not depend on angle variables \( \phi \), hence \( \frac{\partial}{\partial Y} R \frac{\partial}{\partial \phi} = 0 \). After averaging over \( \phi \) terms \( \frac{\partial}{\partial \phi} R \frac{\partial}{\partial I} \) is vanished, and the expression for averaged collision integral becomes again three-dimensional:

\[ \overline{St}[f] = \frac{\partial}{\partial I_{k'}} \overline{R_{k'p'}} \frac{\partial}{\partial I_{p'}} f, \quad (29) \]

where

\[ \overline{R_{k'p'}} = \frac{1}{(2\pi)^3} \int d^3 \phi \frac{\partial I_k}{\partial v_k} \frac{\partial I_{p'}}{\partial v_{p'}} W_{kp}. \quad (30) \]

To give concrete expression to the differential (29), one should know the distribution function of stars \( F \). The observations of stellar dynamics in the bulge indicate that in the first approximation the distribution function of stars could be assumed isotropic, i.e., depending on energy \( E' = \frac{v'^2}{2} + \Psi(r) \) only. For simplicity we also assume that the dependence of stellar distribution function on energy is power-like:

\[ F(v', r) = F_0 E'^{-\beta} \quad (31) \]

One can show that the potential produced by stars distributed in accordance with (31) is also power function of \( r \):

\[ \Psi(r) = \Psi_0 r^\alpha, \quad \alpha = 4/(2\beta - 1) \quad (32) \]

We note that both distribution function (31) and potential (32) can be determined if the velocity dispersion of stars \( \sigma = \sqrt{\langle v_k^2 \rangle} = \sqrt{\langle v^2 / 3 \rangle} \) is known. The dispersion could be choosen as a function of distance to the galaxy center:

\[ \sigma(r) = \sigma_0 r^{\alpha/2}. \quad (33) \]

The distribution function (31) and the potential (32) are uniquely dependent functions of the velocity dispersion, and hence the parameters \( \sigma_0, \alpha \) only:

\[ \Psi_0 = 3\sigma_0^2 \int_0^\infty x^2 (1 + x^2/2)^{-\beta} dx, \quad F_0 = \frac{\alpha(1 + \alpha)}{(4\pi)^2 G \int_0^\infty x^2 (1 + x^2/2)^{-\beta} dx} \Psi_0^{2/\alpha} \]
Note that in the case $\alpha = 0$ the relations formally lose their mathematical meaning. However, one can show that when $\alpha \to 0$ the stellar distribution function becomes isothermal:

$$F(E') = \frac{\rho_0}{(2\pi \sigma^2)^{3/2}} e^{-\frac{\sigma^2 E'}{2\sigma^2}}, \quad \rho_0 = \frac{\sigma^2}{2\pi G},$$

$$\Psi(r) = 2\sigma^2 \ln(r),$$
and the velocity $\sigma$ dispersion does not depend on distance. As a rule, the parameter $\alpha$ in a bulge is rather small.

Calculations show that for isotropic stellar distribution function $F(E')$ tensor $W_{kp}$ in (25) takes the form

$$W_{kp} = A(E, r)\delta_{kp} - B(E, r)\frac{v_k v_p}{v^2},$$

where

$$A = \frac{16\pi^2}{3} G^2 M_\odot \Lambda \int_{\Psi(r)}^\infty dE' F(E') \left\{ \begin{array}{ll} \frac{3}{2} \frac{\sigma^2}{v^2} (1 - \frac{1}{3\pi}) & , \ E < E' \\ \frac{1}{\rho^4} & , \ E > E' \end{array} \right.,$$

$$A - B = \frac{16\pi^2}{3} G^2 M_\odot \Lambda \int_{\Psi(r)}^\infty dE' F(E') \left\{ \begin{array}{ll} \frac{1}{\rho^4} & , \ E < E' \\ \frac{1}{\rho^4} & , \ E > E' \end{array} \right..$$

We note, that the radial action $I_R$ of particles moving in the potential (32) could be with good accuracy approximated by the expression

$$I_R(E, m) \approx J(E) - b_\alpha m,$$

where $b_\alpha \approx 1$ is some positive constant,

$$J(E) = \frac{\sqrt{2} r^1_0 (1 - x^\alpha)^{1/2} dx}{\pi \Psi_0^{1/\alpha} E^{1/2 + \frac{1}{2}}, \ \alpha > 0,}$$

$$J(E) = \frac{\sigma}{\sqrt{\pi}} e^{E^2/2}, \ \alpha = 0.$$

(We notice that in the case of Coulomb or oscillator potentials the equality (35) becomes exact and the constant $b = 1$. In the case of isothermal stellar distribution $b \approx 0.6$)

The initial function (6) does not depend on the variable $m_z$, hence one can search for solution of the equation (29) as a function of $I_R$ and $m$. Calculating the coefficients of quadratic form (30) and taking into account (34), we rewrite the collision term in a form

$$\overline{S}[f] = \frac{1}{m} \frac{\partial}{\partial m} \left( \overline{R}_{22} \frac{\partial f}{\partial m} + \overline{R}_{12} \frac{\partial f}{\partial I_R} \right) + \frac{\partial}{\partial I_R} \left( \overline{R}_{12} \frac{\partial f}{\partial m} + \overline{R}_{11} \frac{\partial f}{\partial I_R} \right),$$

where

$$\overline{R}_{11} = p - 2b_\alpha q + b_\alpha^2 s,$$

$$\overline{R}_{12} = q - b_\alpha s, \quad \overline{R}_{22} = s.$$
\[ p = \left( \frac{dJ}{dE} \right)^2 \left\langle (A - B)v^2 \right\rangle_{\phi}, \quad q = \left( \frac{dJ}{dE} \right) \left\langle (A - B)m \right\rangle_{\phi}, \]
\[ s = \left\langle Ar^2 - B \frac{m^2}{v^2} \right\rangle_{\phi}, \]

where \( \left\langle \ldots \right\rangle_{\phi} = \frac{2}{T(E,m)} \int_{r_1}^{r_2} \frac{dr}{v_r} \ldots \), and \( T = 2 \int_{r_1}^{r_2} \frac{dr}{v_r} \) is the period of dark matters partial radial oscillations, \( v_r = \sqrt{2 \left( E - \Psi(r) - \frac{m^2}{2r} \right)} \) is radial velocity. Taking into account that angular momentum of dark matter particles are small, in the first approximation we shall calculate the coefficients \( R_{ab}(I_R, m) \) in (36) at \( m = 0 \). Further, from the initial distribution function (6) one can see that for sufficiently large time, as far as the distribution function has sharp maximum at \( m = l_0 I_R, l_0 \ll 1 \), one can neglect the diffusion by \( I_R \) compared to diffusion by \( m \). Summarizing, we rewrite the kinetic equation (27) as the diffusion equation depending on \( I_R \) only parametrically:

\[ \frac{\partial f(I_R, m, t)}{\partial t} = R(I_R) \frac{1}{m} \frac{\partial}{\partial m} m^3 \frac{\partial}{\partial m} f(I_R, m, t) \quad \text{(37)} \]

Here the diffusion coefficient \( R = \overline{\mathcal{F}}_{22} |_{m=0} = \left\langle Ar^2 \right\rangle_{\phi} \) calculated with account of (33), takes the form

\[ R(I_R) = 0.46GM_g \Lambda \sigma_0 \frac{2}{v^2} I_R^{1/2} \]

\[ (38) \]

We note that in the case of isothermal distribution of stars in the bulge \( \alpha = 0 \), and the diffusion coefficient \( R \) does not depend on \( I_R \). We emphasize that the fact that \( I_R \) comes into the final equation only as parameter is conditioned by the particular form of initial distribution function (6) containing small parameter \( l_0 \).

5 Flow of dark matter on a black hole.

As a result of direct capture of particles in the loss cone of seed black hole, the distribution function of dark matter particles takes the form

\[ f(I_R, m, 0) = f_0 I_R^{1/2} \delta \left( m^2 - l_0^2 I_R^2 \right) \theta (m - m_g). \quad \text{(39)} \]

It differs from (6) by factor \( \theta (m - m_g) \).

The diffusion of dark matter particles in angular momentum space, their drift into the loss cone (14) and further capture by black hole can be described by the diffusion equation (37) with initial condition (39) and boundary condition

\[ f |_{m=m_g} = 0. \quad \text{(40)} \]

The solution of the diffusion equation can be presented in a form

\[ f(I_R, m, t) = \int_{m_g}^{\infty} dm_1 G(I_R, m, m_1, t) f(I_R, m, 0) \quad \text{(41)} \]

where \( G \) is the Green function of the boundary problem (37)-(40).
\[ G = \int_0^\infty d\lambda m_1 Z_\lambda(m_1, m_g) Z_\lambda(m, m_g) e^{-\lambda R(I_R)t}, \]

\( Z \) is the orthogonal system of fundamental functions of the boundary problem (37)-(40)

\[ Z_\lambda(m, m_g) = \frac{J_0(\sqrt{\lambda m_g}) N_0(\sqrt{\lambda m}) - N_0(\sqrt{\lambda m_g}) J_0(\sqrt{\lambda m})}{\left(J_0^2(\sqrt{\lambda m_g}) + N_0^2(\sqrt{\lambda m_g})\right)^{1/2}}, \]

\( J_0, N_0 \) are Bessel and Neumann functions of zeroth order.

Let us now calculate the flux of dark matter on a black hole. Let \( D \) be the area in the phase space where \( I_R \geq 0, m \geq m_g, -m \leq m_z \leq m \). The whole mass of dark matter is \( N(t) = \int_D d^3 \phi f(I_R, m, t) \). Now, from the condition of mass conservation it follows that the flux through the boundary is \( S = -dN/dt \). Taking into account (25) we obtain

\[ S(t) = -\int d^3 \phi \frac{\partial}{\partial I_k} R_{kp} \frac{\partial}{\partial I_p} f(I_R, m, t). \]

Let us transform now the integral over volume into an integral over surface according to Stokes formula. Taking into account that the only flow of matter is through the surface \( m = m_g \), we find that

\[ S(t) = 2(2\pi)^3 \int_0^\infty dI_R R(I_R) m_g \frac{\partial}{\partial m} f(I_R, m, t) \bigg|_{m=m_g}. \quad (42) \]

Substituting the solution of diffusion equation (41) into (42), we obtain

\[ S(t) = 2 \frac{(2\pi)^3}{\pi} f_0 R_\alpha \int_0^\infty dI_R \gamma I_R^{\frac{\alpha}{2} + \frac{1}{2}} \int_0^\infty d\lambda e^{-\lambda R_\alpha I_R^{\frac{1}{2}}} \frac{Z_\lambda(l_0 I_R; m_g)}{\left(J_0^2(\sqrt{\lambda m_g}) + N_0^2(\sqrt{\lambda m_g})\right)^{1/2}}, \]

where

\[ \gamma = \alpha/(2 + \alpha), R_\alpha = 0.46\Lambda GM_\odot \sigma_0^{\frac{2}{\alpha + 2}}. \quad (43) \]

It is convenient to introduce a normalized time \( T = l_0^{-\gamma} R_\alpha t \) and make a change of variables

\[ \lambda \mapsto \eta = \lambda T^{-\frac{2}{\alpha + 2}}, \quad I_R \mapsto y = l_0 I_R T^{-\frac{1}{\alpha + 2}}, \quad m_g \mapsto x = m_g T^{-\frac{1}{\alpha + 2}}. \]

The expression for the flux takes then a form

\[ S(t) = 2 \frac{(2\pi)^3}{\pi} \frac{f_0 R_\alpha}{l_0^{\frac{4}{\alpha + 2}}} T^{\zeta} \Phi_\gamma(x), \quad (44) \]

\[ \zeta = \frac{7 - 8\gamma}{16 - 8\gamma}, \quad \Phi_\gamma(x) = \int_x^\infty dy H_\gamma(x, y), \]

\[ H_\gamma(x, y) = y^{\frac{1}{2} + \gamma} \int_0^\infty d\eta e^{-\eta y} \frac{J_0(\sqrt{\eta x}) N_0(\sqrt{\eta y}) - N_0(\sqrt{\eta x}) J_0(\sqrt{\eta y})}{J_0^2(\sqrt{\eta x}) + N_0^2(\sqrt{\eta x})}. \]
One can show that \( H_\gamma(x, y) \) is finite positive function, it is equal to zero at \( y = x \) and exponentially small at \( y > x + 4 \). From this follows that at the moment \( t \) the flow of dark matter to the black hole is made up by particles with radial action range in an interval

\[
\frac{m_g}{l_0} < I_R < \frac{m_g}{l_0} + \frac{4}{l_0}T^{1/(2-\gamma)}.
\]  

(45)

Thus, the area from where the dark matter flows into black hole, grows with time as \( t^{1/(2-\gamma)} \). For function \( \Phi \) in expression (45), there is an estimate \( \Phi_\gamma(x) \approx x^{\frac{1}{2}+\frac{\gamma}{2}} \).

Finally, we find the expression for the flux of dark matter onto a black hole through the boundary of the loss cone \( m = m_g \):

\[
S(t) = 2 \left(\frac{2\pi}{3}\right)^{\frac{3}{2}} \frac{f_0}{l_0^{\frac{1}{2}+\gamma}} R^{\frac{1}{2}} \frac{M_g^{\frac{1}{2}+\frac{\gamma}{2}}}{t^{1/2}}.
\]

(46)

The expression (46) has been obtained in assumption that the boundary of the loss cone \( m_g \) does not depend on time. In reality, the boundary moves in accordance with the whole black hole mass. However, from (46) follows that the flux depends only weakly on the value \( m_g \). It means, that the growing of \( m_g \) is a quasistationary process and one can use the expression (46), assuming \( t \) as a parameter: \( m_g(t) = 4GM_{bh}(t)/c \). Hence, the growth of the black hole is described by

\[
\frac{dM_{bh}}{dt} = 2 \left(\frac{2\pi}{3}\right)^{\frac{3}{2}} \frac{f_0}{l_0^{\frac{1}{2}+\gamma}} R^{\frac{1}{2}} \frac{M_{bh}^{\frac{1}{2}+\frac{\gamma}{2}}}{t^{1/2}}.
\]

Assuming the seed black hole mass small, we obtain the solution of this equation

\[
M_{bh} = Ct^{\frac{4}{7-\gamma}},
\]  

(47)

\[
C = \left(\frac{7 - 4\gamma}{2}\right) \frac{(2\pi)^3}{\pi} \left(\frac{4G}{c}\right)^{\frac{1}{2}+\frac{\gamma}{2}} \frac{f_0}{l_0^{\frac{1}{2}+\gamma}} R^{\frac{1}{2}} \left(\frac{M_{bh}^{\frac{1}{2}+\frac{\gamma}{2}}}{t^{1/2}}\right)^{\frac{8}{7-\gamma}}.
\]

To estimate the parameter \( f_0 \) one can use the relation (23). Thus, the growth of black hole at the expense of absorption of dark matter scattered by stars follows a power law \( M_{bh} \propto t^a, \ a \approx 4/7 \).

6 Conclusion.

In conclusion, we compare roughly the results of the theory presented above with the observational data. Fig.2 represents the dependence of stellar velocity dispersion on distance to the galactic center for our Galaxy and galaxies M31, NGC4258 [19]. One can see that the distribution of stars in bulges of M31 and NGC4258 is close to isothermal, i.e. up to the area of action of black hole (\( r < (4 - 7)pc \)) the velocity dispersion is almost constant, it does not depend on \( r \): \( \sigma = \sigma_0 \approx 200km/s \). In the case of isothermal distribution of stars in the bulge we can rewrite the expression (47):
Let us suppose that parameter \( l_0 \) of the dark matter haloes around these galaxies is close to average value \( \langle l_0 \rangle = 0.1 \). Implying, that masses \( M_H \) and sizes \( R_H \) of haloes are roughly equal and constitute \( 10^{12} M_\odot \) and \( 100 \text{kpc} \) respectively we obtain from (48) the mass of the black hole \( M_{bh} \approx 8 \cdot 10^7 M_\odot \) (We assume, that the age \( t \) of black hole is comparable with the age of the Universe \( t \approx 3 \cdot 10^{17} \text{s} \)). Thus, theoretical predictions of masses of black holes in these galaxies are close enough to the observed values: \( M_{bh} = (2.0 - 8.5) \cdot 10^7 M_\odot \) for M31 and \( M_{bh} = (3.8 - 4.0) \cdot 10^7 M_\odot \), for NGC4258.

The central part of our Galaxy’s bulge does not satisfy the assumption of isothermal distribution of stars at \( r \leq 100 \text{pc} \). At the same time, as it is clear from (45), the dark matter accumulates mainly from the central region. The dependence of stellar velocity dispersion on distance is shown at Fig. 3 [20]. One can see that at the scales from 10pc to 100pc the dispersion can be roughly represented as \( \sigma = 60 \text{km/s} \left( \frac{r}{10 \text{pc}} \right)^{0.3} \). From (33) then follows \( \alpha \approx 0.6 \). In the case, the formula (47) gives for the black hole mass a value \( 2 \cdot 10^7 M_\odot \), which is much more than observed value \( M_{bh} \approx 2.6 \cdot 10^6 M_\odot \). To obtain more precise estimation of the black hole mass, more information on the stellar and dark matter distribution functions and their evolution is needed. Also, the contribution of the black hole into the joint gravitational potential may be important. However, we see that even the rough calculation of absorption of dark matter only may give a reasonable estimate of black hole mass. This estimate is consistent with observed masses of most giant black holes (see Fig.1). The fact may be considered as an additional prove of the general theory of the role of CDM in the large-scale structuring of the Universe [5].

Note also that the kinetic theory developed in Section 5 allows to describe the absorption of luminous (baryon) matter as well as dark matter. More precisely, it is possible to take into account the capture of stars by black hole because of their gravitational scattering. This process is especially important for giant black holes having mass of the order \( M_{bh} \sim 10^9 M_\odot \) and active galactic nuclei.

Another important process described by the kinetic equation (27) is the possibility of diminishing of amount of dark matter in bulge. The change of dark matter particles’ energy as a result of gravitational scattering by stars leads to their ”heating” and pushing them out of bulge – the phenomenon analogous to Fermi acceleration. Therefore, the density of dark matter compared to that of luminous matter decreases. Qualitatively this corresponds to the results of recent observations discussed in the literature [21], [22].

Further development of the consistent kinetic theory discussed above and its detailed comparison with observational data would be undoubtedly useful for understanding of main physical processes in the galactic nuclei.

The authors are grateful to V.A. Sirota and M.I. Zelnikov for numerous and useful discussions.
References

[1] Cherepaschyk A.M. Sov. Physics-Uspekhi 173, 4, 2003.

[2] Ho L. C. in Observational Evidence for Black Holes in the Universe. Astrophys. and Space Sci. Library, Vol. 234, 1999.

[3] Gurevich A. V., Zybin K. P. Sov. Phys. JETF (67), 1957, 1988.

[4] Gurevich A. V., Zybin K. P. Sov. Phys. JETF (70), 10, 1990.

[5] Gurevich A.V., Zybin K.P. Sov. Physics-Uspekhi 38 (7) 687-722 (1995)

[6] Navaro F., Frenk C.S., White C.D. Aph. J. 462, 563, 1996.

[7] Fukshige L., Makino J. Aph. J. 477, L9, 1997.

[8] Persic M., Salucci P. Aph. J. Suppl., 99, 501, 1995.

[9] Puche D., Carignan C. Aph. J., 378, 487, 1991.

[10] Arnold V.I., Mathematical Methods of Classical Mechanics, Moscow, Nauka (1989) (in russian)

[11] A.S. Ilyin, K.P. Zybin, A.V. Gurevich., Sov. Phys. JETF (in press).

[12] MacMillan J. D., Henriksen R.N. astro-ph/0201153

[13] V.L. Polyachenko, A.M. Fridman. Equilibrium and stability of gravitating systems, Moscow, Nauka (1976).

[14] Lifshitz E.M., Pitaevskii L.P., Physical Kinetics. Oxford: Pergamon press. 1981.

[15] Gurevich A.V. Geomagnetism and Aeronoma, 4, 247, 1964.

[16] Dokuchaev V.I., Ozernoy L.M., Sov. Phys. JETF. 73, 5, 1587, 1977.

[17] Lightman A.P., Shapiro S.L. ApJ., 211, 244, 1977.

[18] Budker G.I., Belyaev S.T. Plasma Physics and Controlable fusion reaction. 1958. (in russian)

[19] Yoshiaki Sofue, Vera Rubin, astro-ph/0010594

[20] Scott Tremaine, et al., astro-ph/0203468

[21] Napolitano N., Arnaboldi M., Capaccioli M. Astronomy and Astrophys. 383, 791, 2002.

[22] Combes F. astro-ph/0206126
This figure "fig_1.gif" is available in "gif" format from:

http://arxiv.org/ps/astro-ph/0306490v2
This figure "fig_2.gif" is available in "gif" format from:

http://arxiv.org/ps/astro-ph/0306490v2
This figure "fig_3.gif" is available in "gif" format from:

http://arxiv.org/ps/astro-ph/0306490v2