Gluon Condensate in Pion Superfluid beyond Mean Field Approximation

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We study gluon condensate in a pion superfluid, through calculating the equation of state of the system in the Nambu–Jona-Lasinio model. While in mean field approximation the growing pion condensate leads to an increasing gluon condensate, meson fluctuations reduce the gluon condensate and the broken scalar symmetry can be smoothly restored at finite isospin density.

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Quarks and gluons condense in the vacuum of Quantum Chromodynamics (QCD). From lattice QCD calculations and effective QCD models in hot medium, it is widely accepted that the gluon condensate \( \langle G^a_{\mu\nu} G^{a\mu\nu} \rangle \) \[1, 2\] which describes the degree of the scale symmetry breaking is, however, not so optimistic.

The gluon condensate at finite temperature is investigated in instanton model\[3\], renormalization group\[4\], QCD sum rule\[5\], and effective QCD models at low energy\[6, 12\]. While the results in these calculations are quantitatively different, they show the same temperature trend of the gluon condensate: It is almost invariable at low temperature and starts to decrease rapidly around the critical temperature of QCD phase transitions.

At finite isospin density, both the Lee-Huang-Yang model\[13\] for a dilute Boson gas and the Nambu–Jona-Lasinio (NJL) model\[14\] show a surprising mean field result\[13\]: In the pion superfluid the gluon condensate drops down slightly only at very low isospin density but goes up and even exceeds its vacuum value when the density is high enough. This result is qualitatively in agreement with the calculations for 2-color baryon matter and conductivity and pion superfluidity at moderate temperature. When we neglect the current quark mass \( m \), the gluon condensate decouples from the quark condensate and is purely controlled by the thermodynamics of the system. While the gluon condensate for an ideal gas with \( \epsilon - 3p = 0 \) is medium independent, it will be significantly changed for a strongly coupled system. From the lattice simulation at finite temperature\[19\], the QCD system is a strongly coupled matter around the phase transition temperature \( T_c \) with \( \epsilon - 3p \gg 0 \). This is the reason why the gluon condensate drops down dramatically around \( T_c \).

The NJL model at quark level\[20\] has been successfully used to study chiral symmetry restoration, color superconductivity and pion superfluidity at moderate temperature and density. The flavor SU(2) NJL model is defined through the Lagrangian density

\[
\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m + \mu \gamma_5) + G \left[ \overline{\psi} \psi \right]^2 + \overline{\psi} i \sigma^{\mu\nu} T^a \tau_\mu \psi \right]^3, \tag{2}
\]

where the quark chemical potential potential \( \mu = \text{diag}(\mu_a, \mu_d) = \text{diag}(\mu_B/3 + \mu_I/2, \mu_B/3 - \mu_I/2) \) and the Pauli matrices \( \tau_1, \tau_2, \tau_3 \) are defined in flavor space, \( \mu_B \) and \( \mu_I \) are baryon and isospin chemical potentials, and \( G \) is the four-fermion coupling constant. The NJL thermodynamic potential can be separated into a mean field part and a fluctuation part,

\[
\Omega = \Omega_{MF} + \Omega_{FL}. \tag{3}
\]

The mean field part \( \Omega_{MF} \) contains the mean field potential and the contribution from the quasi-quarks\[21\].

\[
\Omega_{MF} = G \left( \sigma^2 + \pi^2 \right) - 3 \int \frac{d^3k}{(2\pi)^3} \left[ E^+ + E^- - E^+_+ - E^-_+ \right] + 2T \ln \left[ 1 + e^{-E^+_+/T} \right] \left[ 1 + e^{-E^-_/T} \right] + 2T \ln \left[ 1 + e^{E^+_/T} \right] \left[ 1 + e^{E^-_/T} \right], \tag{4}
\]

where the chiral condensate \( \sigma = \langle \overline{\psi} \psi \rangle \) and pion condensate \( \pi = \sqrt{2\langle \overline{\psi} \gamma_5 \tau_+ \psi \rangle} \) with \( \tau_+ = (\tau_1 + i\tau_2)/\sqrt{2} \) are
determined by minimizing the potential,
\[ \frac{\partial \Omega_{MF}}{\partial \sigma} = 0, \quad \frac{\partial \Omega_{MF}}{\partial \pi} = 0, \quad \frac{\partial^2 \Omega_{MF}}{\partial \sigma^2} > 0, \quad \frac{\partial^2 \Omega_{MF}}{\partial \pi^2} > 0, \]
and \( E^\pm_\pi = E^\pm_\pi + \mu_B/3 \) are the quasi-quark energies with
\[ E^\pm_\pi = \sqrt{(E_k \pm \mu/2)^2 + 4G^2q^2}, \quad E_k = \sqrt{k^2 + M_q^2} \]
dynamical quark mass \( M_q = m - 2G\sigma \).

In the NJL model, the meson modes are regarded as quantum fluctuations above the mean field. The two quark scattering via meson exchange can be effectively expressed in terms of quark bubble summation in random phase approximation.[20] In normal phase without pion condensation, the bubble summation selects its specific isospin channel by choosing at each stage the same proper polarization, and the meson masses \( M_m = (m = \sigma, \pi_+, \pi_-, \pi_0) \) which are determined by poles of the meson propagators, \( 1 - 2G\Pi_{mm}(M_m, 0) = 0 \), are related only to their own polarization functions \( \Pi_{mm}(\theta, q) \). In pole approximation, the meson contribution to the thermodynamic potential can be expressed as[22]
\[ \Omega_{FL} = \sum_m \Omega_m, \]
\[ \Omega_m = \int \frac{d^3q}{(2\pi)^3} \left[ \frac{E_m - \mu_m}{2} + T \ln \left( 1 - e^{-E_m/m} \right) \right] \]
with meson energies \( E_m = \sqrt{q^2 + M_m^2} \) and meson isospin chemical potential \( \mu_{\pi_\pm} = \pm \mu_1 \) and \( \mu_{\pi_0} = \mu_2 = 0 \).

In the pion superfluid phase, the quark propagator contains off-diagonal elements in flavor space, we must consider all possible isospin channels in the bubble summation. In this case, all the possible polarizations form a matrix \( \Pi \) in the four-dimensional meson isospin space with off-diagonal elements \( \Pi_{nm} \). While there is no mixing between \( \pi_0 \) and other mesons, \( \Pi_{\pi_0\pi_0} = \Pi_{\pi_0\pi_+} = \Pi_{\pi_0\pi_-} = 0 \), the other three mesons are coupled to each other. The explicit \( T, \mu_B \) and \( \mu_I \) dependence of all polarization elements \( \Pi_{mm} \) can be found in Appendix B of [21]. When the system goes through the phase transition line and enters the normal phase, all the off-diagonal elements disappear automatically.

The masses of the eigen modes of the Hamiltonian \( \mathcal{H} \) in the pion superfluid are defined through the poles of the meson propagator, \( \text{det}(1 - 2G\Pi(M_\theta, 0)) = 0 \) which can be separated into \( 1 - 2G\Pi_{\pi_0\pi_0}(M_\pi, 0) = 0 \) for \( \theta = \pi_0 \), and 
\( \text{det}(1 - 2G\Pi(M_\theta, 0)) = 0 \) in the three-dimensional isospin subspace for \( \theta = \pi_+, \pi_- \). Different from the normal phase where the meson modes \( \sigma, \pi_+, \pi_-, \pi_0 \) are eigen states of both the Hamiltonian \( \mathcal{H} \) and the isospin operator \( \hat{I}_3 = 1/2 \int d^3x \bar{\psi}_0 \tau_3 \psi \) of the system, only \( \pi_0 \) is still the eigen state of \( \hat{I}_3 \) (we still label it \( \pi_0 \) in the following), but \( \pi_+, \pi_- \) have no longer definite isospin quantum number. The eigen states of \( \hat{I}_3 \) are only related to the diagonal elements \( \Pi_{mm} \) and their masses are defined by \( 1 - 2G\Pi_{mm}(M_m, 0) = 0 \).

After taking bubble summation and Matsubara frequency summation, the fluctuation part of the thermodynamic potential can be generally written as[22]
\[ \Omega_{FL} = -\int \frac{d^3q}{(2\pi)^3} \sum_{\omega} \frac{2}{\pi} \frac{1}{2} \left[ \omega + T \ln \left( 1 - e^{-\omega/T} \right) \right] \]
\[ \times \frac{\text{det}(1 - 2GM_{\omega + i\epsilon, \pi})}{\text{det}(1 - 2GM_{\omega - i\epsilon, \pi})}, \]
where the two polarization matrices are respectively defined in the top and bottom complex meson energy plane. An often used simplification to calculate \( \Omega_{FL} \) is the pole approximation, namely neglecting the scattering phase shifts and considering only the contribution from the quasi particles, like (6) for the normal phase. In this case, we have \( \Omega_{FL} = \sum_{\theta} \Omega_{\theta} \). To explicitly show the isospin dependence, we further make a transformation from the basis \( (\pi_+, \pi_-) \) to the basis \( (\sigma, \pi_+, \pi_-) \). The element states in the former basis do not carry definite isospin quantum numbers, but the later is constructed by the eigen states of the isospin operator \( \hat{I}_3 \). Since the two spaces are both complete, such a transformation will not lose any information. Taking into account the orthogonal condition for the two spaces, \( \Omega \) can be expanded as a linear combination of \( \Omega_{\theta} \). Finally, we have
\[ \Omega_{FL} = \sum_{\theta} \Omega_{\theta} = \Omega_{\pi_0} + \sum_{m} c_m \Omega_m \]
with the coefficients
\[ c_m = \sum_{\theta} |\langle \theta | m \rangle|^2 = \frac{\mathcal{M}_{nm}(M_\theta)}{\sum_{\theta} \mathcal{M}_{nm}(M_\theta)}. \]
where \( \mathcal{M} \) is a matrix defined in the three dimensional meson isospin subspace,
\[ \mathcal{M}(M_\theta) = \frac{\text{det}(1 - 2G\Pi(M_\theta, 0))}{1 - 2G\Pi(M_\theta, 0)}. \]

It is easy to see the normalization condition for the coefficients, \( \sum_m c_m = \sum_{\theta} = 3 \), it means that only two of the three coefficients are independent. The coefficients \( c_m \) as functions of temperature at fixed chemical potentials are shown in Fig.[1] Their strong deviation from unit indicate a strong mixing of \( \sigma, \pi_+, \pi_- \) in the pion superfluid. For the Goldstone mode \( \pi_+ \), its linear combination is \( |\pi_+\rangle = 1/\sqrt{2}(|\pi_+\rangle - |\pi_-\rangle) \), and the two fractions are equal and medium independent. Therefore, at the critical point the coefficient \( c_{\pi_+} \) jumps up from 0.5 to 1 and \( c_{\pi_-} \) drops down from 1.5 to 1. For \( T > T_c \) in the normal phase, all the three coefficients are unit. For \( \mu_I = 200 \) MeV and \( \mu_B = 600 \) MeV in Fig.[1] \( T_c \) is about 110 MeV. It is necessary to note that the discontinuity of the coefficients \( c_{\pi_+} \) and \( c_{\pi_-} \) happens on the whole phase transition border. However, when we approach to the border from the pion superfluid side, the pion condensate goes to zero continuously, and this can smooth the thermodynamics on the border, see the calculations below.

Now we use the trace anomaly relation [11] to calculate the gluon condensate, under the assumption that the NJL model can describe reasonably well the QCD thermodynamics in the pion superfluid. From the thermodynamic potential relative to the vacuum \( \Pi(T, \mu_B, \mu_I) = 0 \),
\[ \Omega(T, \mu_B, \mu_I) - \Omega(0, 0, 0), \] we obtain the pressure \( p = -\Omega \) and energy density \( \epsilon = -p + Ts + \mu_Bn_B + \mu_In_I \) with the entropy density \( s = -\partial \Omega / \partial T \), baryon number density \( n_B = -\partial \Omega / \partial \mu_B \) and isospin number density \( -\partial \Omega / \partial \mu_I \).

Before we make numerical calculations, we first determine the parameters in the model. Since the NJL model is non-renormalizable, we can employ a hard three momentum cutoff \( \Lambda \) to regularize the gap equations for quarks and pole equations for mesons. In the following numerical calculations, we take the current quark mass \( m_0 = 5 \text{ MeV} \), the coupling constant \( G = 4.93 \text{ GeV}^{-2} \) and the cutoff \( \Lambda = 653 \text{ MeV} \). This group of parameters corresponds to the pion mass \( m_\pi = 134 \text{ MeV} \), the pion decay constant \( f_\pi = 93 \text{ MeV} \) and the effective quark mass \( M_q = 310 \text{ MeV} \) in vacuum.

We show in Fig[2] the ratios for gluon, chiral and pion condensates, \( R_g = \langle \sigma = \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu}^a \rangle_0 / \langle \pi = \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu}^a \rangle_0, R_\pi = \sigma / \sigma_0 \) and \( R_\sigma = \pi / \pi_0 \), where \( \langle \alpha = \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu}^a \rangle_0 \) and \( \sigma_0 \) are the condensates in vacuum, and \( \langle \pi = \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu}^a \rangle_0 \), \( \sigma \) and \( \pi \) are the total condensates including the vacuum and matter parts. To reduce the model dependence and focus on the medium effect, we take an empirical value for the vacuum part of the gluon condensate, \( \langle \sigma = \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu}^a \rangle_0 = (360 \text{ MeV})^4 \) [23], (the value of the vacuum part will not change the trend of the gluon condensate in the medium). At \( T = \mu_B = 0 \) in the top panel of Fig[2], the ratio \( R_g^{\text{MF+FL}}, \) calculated with the total thermodynamic potential \( \Omega = \Omega_{\text{MF}} + \Omega_{\text{FL}} \), is a constant in the normal phase with \( \mu_I < m_\pi \) and drops down monotonously in the pion superfluid phase with \( \mu_I > m_\pi \). Therefore, the behavior of the gluon condensate at finite isospin density is qualitatively the same as in the case at finite temperature: The broken scale symmetry of the system is gradually restored in hot and dense medium. However, in mean field approximation, the gluon condensate behaves very differently. The ratio \( R_g^{\text{MF}} \) decreases slightly only in the beginning of the pion superfluid and then goes up monotonously and even exceeds the vacuum value when \( \mu_I \) is high enough. For \( T = 50 \text{ MeV} \) and \( \mu_B = 600 \text{ MeV} \) shown in the bottom panel of Fig[2], while the mean field calculation is changed slightly, the finite temperature and baryon chemical potential effect results in stronger meson fluctuations, and the ratio \( R_g^{\text{MF+FL}} \) drops down much faster.
and the corresponding trace of the Noether current for the scalar transformation is $T^{\mu}_{\mu} = m\bar{\psi}\psi - 2G(\sigma^2 + \pi^2)$. Taking the identification of the trace of the energy-momentum tensor in QCD and in the NJL model, the gluon condensate is characterized only by the two condensates,

$$\left(\frac{G^2}{\pi}G_{\mu\nu}G_{\mu\nu}\right)_{T,\mu} = \frac{16}{9} G (\sigma^2 - \sigma^4 + \pi^2)
= \frac{16}{9} G \sigma R^2 (R^2 + R^2 - 1). \quad (12)$$

In the pion superfluid phase, the two ratios $R_\sigma$ and $R_\pi$ behave in an opposite way, $R_\sigma$ drops down but $R_\pi$ goes up, and the trend of the gluon condensate is controlled by the competition between the chiral and pion condensates. When $\mu_\pi$ is above but close to the critical point $\mu_\pi^* = m_\pi$, the chiral and pion condensates are equally important and their competition may result in a possible decreasing gluon condensate. However, when $\mu_\pi$ is large enough, the chiral condensate becomes small and the pion condensate dominates the system. In this case, the gluon condensate increases with increasing pion condensate.

It is necessary to emphasize again that the trace anomaly relation [11] between the gluon condensate and the thermodynamics of the system is valid only at quantum level. At classical or mean field level, the relation is not true, and the scale symmetry of QCD is only explicitly broken by the current quark mass $m$, $\langle T^\mu_{\mu} \rangle = m\bar{\psi}\psi$. In the NJL model, the quantum fluctuations or the meson modes can not be neglected. At mean field level, there are only quarks in the model which control the thermodynamics only at high temperature and density. At moderate temperature and density, both quarks and mesons are important. At low temperature and density, mesons become the dominant contribution to the thermodynamics. Therefore, we need quantum fluctuations to describe the system in the whole temperature and density region.

In summary, we have studied the gluon condensate beyond mean field approximation in a pion superfluid described by the NJL model. Since the trace anomaly relation is valid only at quantum level, the quantum fluctuations in the model must be considered in the calculation of gluon condensate. At classical or mean field level, the growing pion condensate in the superfluid leads to a surprising increase of the gluon condensate. However, when the quantum fluctuations are included, the meson contribution dominates the thermodynamics of the system at low and intermediate temperature and density, and the gluon condensate becomes to decrease gradually in the pion superfluid. Therefore, the scale symmetry can be restored at both finite temperature and density.

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