Quantum Merlin-Arthur Proof Systems: Are Multiple Merlins More Helpful to Arthur?∗

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Abstract

This paper introduces quantum “multiple-Merlin”-Arthur proof systems in which Arthur receives multiple quantum proofs that are unentangled with each other. Although classical multi-proof systems are obviously equivalent to classical single-proof systems (i.e., usual Merlin-Arthur proof systems), it is unclear whether or not quantum multi-proof systems collapse to quantum single-proof systems (i.e., usual quantum Merlin-Arthur proof systems). This paper presents a necessary and sufficient condition under which the number of quantum proofs is reducible to two. It is also proved that, in the case of perfect soundness, using multiple quantum proofs does not increase the power of quantum Merlin-Arthur proof systems.

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1 Introduction

1.1 Background

Merlin-Arthur proof systems, or Merlin-Arthur games as originally called, were introduced by Babai [9]. In a Merlin-Arthur proof system, powerful Merlin, a prover, presents a proof and Arthur, a verifier, probabilistically verifies its correctness with high success probability. The class of problems having Merlin-Arthur proof systems is denoted by $\text{MA}$, and has played important roles in computational complexity theory [9, 13, 11, 12, 10, 47, 44, 8, 23, 19, 18, 25, 45].

A quantum analogue of $\text{MA}$ was first discussed by Knill [33] and has been studied intensively [30, 48, 29, 26, 49, 7, 39, 28, 1, 86, 38, 3, 5, 37]. In the most commonly-used version of quantum Merlin-Arthur proof systems, a proof presented by Merlin is a pure quantum state called a quantum proof and Arthur’s verification process is a polynomial-time quantum computation. However, all the previous studies only consider the model in which Arthur receives a single quantum proof, and no discussions are done so far on the model in which Arthur receives multiple quantum proofs unentangled with each other.

Classically, multiple proofs can be concatenated into a long single proof, and thus, there is no advantage to use multiple proofs. Quantumly, however, using multiple quantum proofs may not be computationally equivalent to using a single quantum proof, because knowing that a given proof is a tensor product of some quantum states might be advantageous to Arthur. For example, in the case of two quantum proofs versus one, consider the following most straightforward Arthur’s simulation of two quantum proofs by a single quantum proof: given a single quantum proof that is expected to be a tensor product of two pure quantum states, Arthur first runs some preprocessing to rule out any quantum proof far from states of a tensor product of two pure quantum states, and then performs the verification procedure of the original two-proof system. It turns out that this most straightforward method does not work well, since there is no physical method that determines whether a given unknown state is in a tensor product form or even maximally entangled, as will be shown in Section 6. Another fact is that the unpublished proof by Kitaev and Watrous for the upper bound $\text{PP}$ of the class $\text{QMA}$ of problems having single-proof quantum Merlin-Arthur proof systems no longer works well for the multi-proof cases with the most straightforward modification. The simplified proof by Marriott and Watrous [39] for the same statement and even the proof of $\text{QMA} \subseteq \text{PSPACE}$ [30, 31] are also the cases. Furthermore, the existing proofs for the property that parallel repetition of a single-proof system reduces the error probability to be arbitrarily small [32, 48, 21, 59] cannot be applied to the multi-proof cases. Of course, these arguments do not imply that using multiple quantum proofs is more powerful than using only a single quantum proof from the complexity theoretical viewpoint. The authors believe, however, that these at least justify that it is meaningful to consider the multi-proof model of quantum Merlin-Arthur proof systems. It is interesting to note that here the nonexistence of entanglement among proofs may have the possibility of enhancing the verification power, unlike the usual situations of quantum information processing where we make use of the existence of entanglement. Moreover, the multi-proof model has importance even in quantum information theory, because the model is inherently related to entanglement theory. Indeed, after the completion of this work, Aaronson, Beigi, Drucker, Fefferman, and Shor [2] succeeded in proving a strong connection between our model and the famous “Additivity Conjecture” in entanglement theory, which is one of the most important conjectures in quantum information theory.

1.2 Contribution of This Paper

Motivated by the observations listed in the previous subsection, this paper extends the usual single-proof model of quantum Merlin-Arthur proof systems to the multi-proof model by allowing Arthur to use multiple quantum proofs, which are given in a tensor product form of multiple quantum states. One may think of this model as a special case of quantum multi-prover interactive proof systems [34] in which a verifier cannot ask questions to provers, and provers do not share entanglement a priori. Formally, we say that a problem $A = \{A_{\text{yes}}, A_{\text{no}}\}$ has a $(k, c, s)$-quantum Merlin-Arthur proof system if there exists a polynomial-time quantum verifier $V$ such that, for
every input $x$, (i) if $x \in A_{\text{yes}}$, there exists a set of $k(|x|)$ quantum proofs that makes $V$ accept $x$ with probability at least $c(|x|)$, and (ii) if $x \in A_{\text{no}}$, for any set of $k(|x|)$ quantum proofs given, $V$ accepts $x$ with probability at most $s(|x|)$. The resulting complexity class is denoted by $\text{QMA}(k, c, s)$. We often abbreviate $\text{QMA}(k, \frac{2}{3}, \frac{1}{3})$ as $\text{QMA}(k)$ throughout this paper.

Besides our central question whether or not quantum multi-proof Merlin-Arthur proof systems collapse to quantum single-proof systems, it is also unclear if there are $k_1$ and $k_2$ with $k_1 \neq k_2$ such that $\text{QMA}(k_1) = \text{QMA}(k_2)$. Towards settling these questions, this paper presents a necessary and sufficient condition under which the number of quantum proofs is reducible to two. Our condition is related to the possibility of amplifying success probability of quantum two-proof Merlin-Arthur proof systems without increasing the number of quantum proofs. More formally, it is proved that $\text{QMA}(k, c, s) = \text{QMA}(2, \frac{2}{3}, \frac{1}{3})$ for any polynomially-bounded function $k \geq 2$ and any two-sided bounded error probability $(c, s)$ if and only if $\text{QMA}(2, c, s) = \text{QMA}(2, \frac{2}{3}, \frac{1}{3})$ for any two-sided bounded error probability $(c, s)$. Alternatively, it is proved that quantum multi-proof Merlin-Arthur proof systems are equivalent to usual single-proof ones if and only if quantum two-proof Merlin-Arthur proof systems are equivalent to usual single-proof ones. That is, $\text{QMA}(k, c, s) = \text{QMA}(k, c, s)$ for any polynomially-bounded function $k \geq 2$ and any two-sided bounded error probability $(c, s)$ if and only if $\text{QMA}(2, c, s) = \text{QMA}(2, c, s)$ for any two-sided bounded error probability $(c, s)$. The key ingredient to show these properties is the claim that, for any quantum multi-proof Merlin-Arthur proof system with some appropriate condition on completeness and soundness, we can reduce the number of proofs by (almost) two-thirds (where the gap between completeness and soundness becomes worse, but is still bounded by an inverse-polynomial). This is done by using the controlled-swap test, which often plays a key role in quantum computation (e.g., in Refs. [32, 17]).

It is also proved for the case of perfect soundness that, for any polynomially-bounded function $k \geq 2$ and any completeness $c$, $\text{QMA}(k, c, 0) = \text{QMA}(1, c, 0)$. With further analyses, the class $\text{NQP}$, which derives from another concept of “quantum nondeterminism” introduced by Adleman, DeMarrais, and Huang [4] and discussed by a number of studies [22, [21], 51, 50], is characterized by the union of $\text{QMA}(1, c, 0)$ for all error probability functions $c$. This bridges between two existing concepts of “quantum nondeterminism”.

### 1.3 Recent Progresses

After the completion of this work, a number of studies showed very intriguing properties on our model.

Liu, Christandl, and Verstraete [38] showed that the pure state $\mathcal{N}$-representability problem, which naturally arises in quantum chemistry, can be verified by a quantum two-proof Merlin-Arthur proof system with two-sided bounded error. Interestingly, the problem is not known to be in usual $\text{QMA}$.

Blier and Tapp [16] proved that the NP-complete problem Graph 3-Coloring has a quantum two-proof Merlin-Arthur proof system with one-sided bounded error of perfect completeness, where both of the two unentangled quantum proofs consist of only logarithmically many qubits. The soundness is bounded away from one only by an inverse-polynomial in their proof system.

Aaronson, Beigi, Drucker, Fefferman, and Shor [2] proved that the NP-complete problem 3-SAT has a quantum multi-proof Merlin-Arthur proof system of perfect completeness with constant soundness error, where the number of proofs is almost square root of the instance size, and each quantum proof consists of only logarithmically many qubits. They further showed that the “Additivity Conjecture” would imply that any quantum two-proof Merlin-Arthur proof systems, for every reasonable choices. For instance, $\text{QMA}(k)$ could be defined as the union of $\text{QMA}(k, 1 - \varepsilon, \varepsilon)$ for all negligible functions $\varepsilon$. It is possible that other reasonable definitions of $\text{QMA}(k)$ form different classes from the one defined in this paper, since it is not known how to amplify the success probability of $\text{QMA}(k)$. The authors believe, however, that the choice of $\frac{2}{3}$ and $\frac{1}{3}$ would best highlight the essence of the results in this paper.

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1. Here we choose completeness and soundness accepting probabilities $\frac{2}{3}$ and $\frac{1}{3}$ to define the class $\text{QMA}(k)$, but there may be other reasonable choices. For instance, $\text{QMA}(k)$ could be defined as the union of $\text{QMA}(k, 1 - \varepsilon, \varepsilon)$ for all negligible functions $\varepsilon$. It is possible that other reasonable definitions of $\text{QMA}(k)$ form different classes from the one defined in this paper, since it is not known how to amplify the success probability of $\text{QMA}(k)$. The authors believe, however, that the choice of $\frac{2}{3}$ and $\frac{1}{3}$ would best highlight the essence of the results in this paper.

2. This improves the result proved in our preliminary conference version [35], where we required the amplifiability of the success probability not only for two-proof systems but also for $k$-proof systems, for every $k$. Also, the statement was originally proved only for every constant $k$, whereas the improved statement in the current version holds even for every polynomially-bounded function $k$. The same improvements were done independently by Aaronson, Beigi, Drucker, Fefferman, and Shor [2].
Arthur proof system can be made to have arbitrarily small two-sided bounded error, and thus, $\text{QMA}(k) = \text{QMA}(2)$ for any polynomially bounded function $k \geq 2$.

### 1.4 Organization of This Paper

The remainder of this paper is organized as follows. In Section 2 we give a brief review for several basic notions of quantum computation and information theory used in this paper. In Section 3 we formally define the multi-proof model of quantum Merlin-Arthur proof systems. In Section 4 we show a condition under which $\text{QMA}(k) = \text{QMA}(2)$. In Section 5 we focus on the systems of perfect soundness. In Section 6 we show that there is no physical method that determines whether a given unknown state is in a tensor product form or maximally entangled. Finally, we conclude with Section 7 which summarizes this paper. The conference version of this paper [35] also included the result that there exists an oracle relative to which $\text{QMA}(k)$ does not contain co-UP. The present version omits this result, since it turned out that the statement is easily proved by using the result by Raz and Shpilka [43].

### 2 Preliminaries

We start with reviewing several fundamental notions used in this paper. Throughout this paper we assume that all input strings are over the alphabet $\Sigma = \{0, 1\}$, and $\mathbb{N}$ and $\mathbb{Z}^+$ denote the sets of positive and nonnegative integers, respectively. For any Hilbert space $\mathcal{H}$, let $I_{\mathcal{H}}$ denote the identity operator over $\mathcal{H}$. In this paper, all Hilbert spaces are of dimension power of two.

#### 2.1 Quantum Fundamentals

First we briefly review basic notations and definitions in quantum computation and quantum information theory. Detailed descriptions are found in Refs. [41, 31], for instance.

A pure quantum state, or a pure state in short, is a unit vector $|\psi\rangle$ in some Hilbert space $\mathcal{H}$. For any Hilbert space $\mathcal{H}$, let $|0_{\mathcal{H}}\rangle$ denote the pure quantum state in $\mathcal{H}$ of which all the qubits are in state $|0\rangle$. A mixed quantum state, or a mixed state in short, is a classical probability distribution $(p_i, |\psi_i\rangle)$, $0 \leq p_i \leq 1$, $\sum_i p_i = 1$ over pure states $|\psi_i\rangle \in \mathcal{H}$. This can be interpreted as being in the pure state $|\psi_i\rangle$ with probability $p_i$. A mixed state $(p_i, |\psi_i\rangle)$ is often described in the form of a density operator $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$. Any density operator is positive semidefinite and has trace one. It should be noted that different probabilistic mixtures of pure states can yield mixed states with the identical density operator. It is also noted that there is no physical method (i.e., no measurement) to distinguish mixed states with the identical density operator. Therefore, density operators give complete descriptions of quantum states, and we may use the term “density operator” to indicate the corresponding mixed state. For any Hilbert space $\mathcal{H}$, let $\mathbf{D}(\mathcal{H})$ denote the set of density operators over $\mathcal{H}$.

One of the important operations to density operators is the trace-out operation. Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a quantum state with its density operator $\rho$ in $\mathbf{D}(\mathcal{H} \otimes \mathcal{K})$, the quantum state after tracing out $\mathcal{K}$ has its density operator in $\mathbf{D}(\mathcal{H})$ defined by $\text{tr}_\mathcal{K}\rho = \sum_{i=1}^d (I_{\mathcal{H}} \otimes \langle e_i |) \rho (I_{\mathcal{H}} \otimes | e_i \rangle)$ for any orthonormal basis $\{| e_i \rangle\}$ of $\mathcal{K}$, where $d$ is the dimension of $\mathcal{K}$. To perform this operation on some part of a quantum system gives a partial view of the quantum system with respect to the remaining part.

A positive operator-valued measure (POVM) on a Hilbert space $\mathcal{H}$ is defined to be a set $\mathcal{M} = \{M_1, \ldots, M_k\}$ of nonnegative Hermitian operators over $\mathcal{H}$ such that $\sum_{i=1}^k M_i = I_{\mathcal{H}}$. For any POVM $\mathcal{M}$ on $\mathcal{H}$, there is a quantum mechanical measurement that results in $i$ with probability exactly $\text{tr}(M_i \rho)$ for any $\rho$ in $\mathbf{D}(\mathcal{H})$. See Refs. [24, 42] for more rigorous descriptions on quantum measurements.

The fidelity $F(\rho, \sigma)$ between two density operators $\rho$ and $\sigma$ in $\mathbf{D}(\mathcal{H})$ is defined by $F(\rho, \sigma) = \text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$. This paper uses the following two properties on fidelity.
Lemma 1 ([27]). For any Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and any density operators $\rho_1, \sigma_1 \in \mathcal{D}(\mathcal{H})$ and $\rho_2, \sigma_2 \in \mathcal{D}(\mathcal{K})$,
\[ F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = F(\rho_1, \sigma_1)F(\rho_2, \sigma_2). \]

Lemma 2 ([46, 40]). For any Hilbert space $\mathcal{H}$ and any density operators $\rho, \sigma, \xi \in \mathcal{D}(\mathcal{H})$,
\[ F(\rho, \sigma)^2 + F(\sigma, \xi)^2 \leq 1 + F(\rho, \xi). \]

2.2 Quantum Circuits

Next we review the model of quantum circuits. We use the following notion of polynomial-time uniformly generated families of quantum circuits.

A quantum circuit consists of a finite number of quantum gates that are applied in sequence to a finite number of qubits. A family $\{Q_x\}$ of quantum circuits is polynomial-time uniformly generated if there exists a deterministic procedure that, on every input $x$, outputs a description of $Q_x$ and runs in time polynomial in $|x|$. It is assumed that the circuits in such a family are composed of gates in some reasonable, universal, finite set of quantum gates. Furthermore, it is assumed that the number of gates in any circuit is not more than the length of the description of that circuit. Therefore $Q_x$ must have size polynomial in $|x|$. For convenience, we may identify a circuit $Q_x$ with the unitary operator it induces.

Since non-unitary and unitary quantum circuits are equivalent in computational power [6], it is sufficient to treat only unitary quantum circuits, which justifies the above definition. For avoiding unnecessary complication, however, the descriptions of procedures may include non-unitary operations in the subsequent sections. Even in such cases, it is always possible to construct unitary quantum circuits that essentially achieve the same procedures described.

3 Definitions

Here we formally define quantum multi-proof Merlin-Arthur proof systems. Although all the statements in this paper can be proved only in terms of languages without using promise problems [20], in what follows we define models and prove statements in terms of promise problems, for generality and for the compatibility with some subsequent studies on our model [38, 37].

A quantum proof of size $q$ is a pure quantum state of $q$ qubits.

A quantum verifier $V$ for quantum $k$-proof Merlin-Arthur proof systems is a polynomial-time computable mapping of the form $V: \Sigma^* \to \Sigma^*$. For every input $x \in \Sigma^*$, the string $V(x)$ is interpreted as a description of a polynomial-size quantum circuit. In other words, $\{V(x)\}$ forms a polynomial-time uniformly generated family of quantum circuits. The qubits upon which each $V(x)$ acts are divided into $k + 1$ sets: one set, consisting of $q_V(|x|)$ qubits, serves as work space of $V$, and each of the rest $k$ sets serves as “witness space” of $V$ that is used for storing a quantum proof of size $q_M(|x|)$, for some polynomially bounded functions $q_V, q_M: \mathbb{Z}^+ \to \mathbb{N}$. One of the qubits in the work space of $V$ is designated as the output qubit.

A set of $k$ quantum proofs is compatible with a quantum verifier $V$ if the size of every quantum proof coincides with the size of witness space of $V$.

Suppose that $V$ receives $k$ quantum proofs $|\phi_1\rangle, \ldots, |\phi_k\rangle$. The probability that $V$ accepts $x$ is defined to be the probability that an observation of the output qubit in the $\{|0\rangle, |1\rangle\}$ basis yields $|1\rangle$, after the circuit $V(x)$ is applied to the state $|0_V\rangle \otimes |\phi_1\rangle \otimes \cdots \otimes |\phi_k\rangle$, where $V$ is the Hilbert space corresponding to the work space of $V$.

More generally, the number of quantum proofs may not necessarily be a constant, and may be a polynomially bounded function $k: \mathbb{Z}^+ \to \mathbb{N}$ of the input length.
Definition 3. Given a polynomially bounded function \( k: \mathbb{Z}^+ \rightarrow \mathbb{N} \) and functions \( c, s: \mathbb{Z}^+ \rightarrow [0, 1] \), a problem \( A = \{ A_{\text{yes}}, A_{\text{no}} \} \) is in QMA\((k, c, s)\) if there exists a quantum verifier \( V \) for \( k \)-proof quantum Merlin-Arthur proof systems such that, for every \( x \),

- Completeness if \( x \in A_{\text{yes}} \), there exists a set of quantum proofs \( |\phi_1\rangle, \ldots, |\phi_k(|x|)\rangle \) compatible with \( V \) that makes \( V \) accept \( x \) with probability at least \( c(|x|) \),

- Soundness if \( x \in A_{\text{no}} \), for any set of quantum proofs \( |\phi_1\rangle, \ldots, |\phi_k(|x|)\rangle \) compatible with \( V \), \( V \) accepts \( x \) with probability at most \( s(|x|) \).

We say that a problem \( A = \{ A_{\text{yes}}, A_{\text{no}} \} \) has a \((k, c, s)\)-quantum Merlin-Arthur proof system, or a QMA\((k, c, s)\) proof system in short, if and only if \( A \) is in QMA\((k, c, s)\). For simplicity, we abbreviate QMA\((k, \frac{2}{3}, \frac{1}{3})\) as QMA\((k)\) for every \( k \).

Note that allowing quantum proofs of mixed states does not increase the maximal accepting probability of proof systems, which justifies the model defined above. For readability, in what follows, the arguments \( x \) and \(|x|\) may be dropped in various functions, if it is not confusing.

4 Condition under which QMA\((k) = \text{QMA}(2)\)

Classically, it is trivial to show that classical multi-proof Merlin-Arthur proof systems are essentially equivalent to single-proof ones. However, it is unclear whether quantum multi-proof Merlin-Arthur proof systems collapse to quantum single-proof systems. Moreover, it is also unclear whether there are \( k_1 \) and \( k_2 \) of \( k_1 \neq k_2 \) such that QMA\((k_1) = \text{QMA}(k_2)\). Towards settling these questions, here we give a condition under which QMA\((k) = \text{QMA}(2)\) for every polynomially-bounded function \( k \geq 2 \).

Formally, we consider the following condition on the possibility of amplifying the success probability of quantum two-proof Merlin-Arthur proof systems without increasing the number of quantum proofs:

\[
(*) \text{ For any two-sided bounded error probability } (c, s), \text{ QMA}(2, c, s) \text{ coincides with QMA}(2, \frac{2}{3}, \frac{1}{3}).
\]

Our main result is the following theorem, which will be shown in this section.

Theorem 4. QMA\((k, c, s) = \text{QMA}(2, \frac{2}{3}, \frac{1}{3})\) for any polynomially-bounded function \( k: \mathbb{Z}^+ \rightarrow \mathbb{N} \) satisfying \( k \geq 2 \) and any two-sided bounded error probability \( (c, s) \) if and only if the condition \((*)\) is satisfied.

4.1 Achieving Exponentially Small Completeness Error

We first show a simple way of achieving exponentially small completeness error while keeping soundness error bounded away from one, which works well for any proof systems. The same result was independently proved by Aaronson, Beigi, Drucker, Fefferman, and Shor [2, Lemma 6].

Lemma 5. Let \( c, s: \mathbb{Z}^+ \rightarrow [0, 1] \) be any functions that satisfy \( c - s \geq \frac{1}{q} \) for some polynomially bounded function \( q: \mathbb{Z}^+ \rightarrow \mathbb{N} \), and let \( \Pi \) be any proof system with completeness at least \( c \) and soundness at most \( s \). Consider another proof system \( \Pi' \) such that, for every input of length \( n \), \( \Pi' \) carries out \( N = 2p(n)(q(n))^2 \) attempts of \( \Pi \) in parallel for a polynomially bounded function \( p: \mathbb{Z}^+ \rightarrow \mathbb{N} \), and accepts iff at least \( \frac{c(n) + s(n)}{2} \) fraction of these \( N \) attempts results in acceptance in \( \Pi \). Then \( \Pi' \) has completeness at least \( 1 - 2^{-p} \) and soundness at most \( \frac{2k}{c + s} \leq 1 - \frac{c - s}{2} \leq 1 - \frac{1}{2q} \).

Proof. Let \( X_i \) be the random variable that takes \( 1 \) iff the \( i \)th attempt of \( \Pi \) in \( \Pi' \) results in acceptance and otherwise takes \( 0 \), for each \( 1 \leq i \leq N \), and let \( Y \) be the random variable defined by \( Y = \sum_{i=1}^N X_i \).

Noticing that the accepting probability in \( \Pi' \) is given by \( \Pr[Y \geq \frac{(c(n) + s(n))}{2}] \), and that \( \mathbb{E}[Y] = \sum_{i=1}^N \mathbb{E}[X_i] \), the completeness bound of \( \Pi' \) directly follows from the Hoeffding bound while the soundness bound of \( \Pi' \) directly follows from Markov’s inequality. \(\square\)
Controlled-Swap Test

1. Apply the Hadamard transformation $H$ to $B$.

2. Apply the controlled-swap operator to $R_1$ and $R_2$ using $B$ as a control qubit. That is, swap the contents of $R_1$ and $R_2$ if $B$ contains 1, and do nothing if $B$ contains 0.

3. Apply the Hadamard transformation $H$ to $B$. Accept if $B$ contains 0, and reject otherwise.

Figure 1: The controlled-swap test.

The following is an immediate corollary of Lemma 5.

Corollary 6. For any polynomially bounded functions $k, p: \mathbb{Z}^+ \to \mathbb{N}$ and any two-sided bounded error probability $(c, s)$,

$$QMA(k, c, s) \subseteq QMA\left(k, 1 - 2^{-p}, \frac{2s}{c + s}\right) \subseteq QMA\left(k, 1 - 2^{-p}, 1 - \frac{c - s}{2}\right).$$

4.2 Controlled-Swap Test with Mixed States

Next we show a fundamental property of the controlled-swap test when applied to a pair of mixed states. This property is easy to prove. To the best knowledge of the authors, however, it has not appeared previously, and the authors believe that this property will be useful in many cases.

The controlled-swap operator exchanges the contents of two registers $R_1$ and $R_2$ if the control register $B$ contains 1, and does nothing if $B$ contains 0.

Given a pair of mixed states $\rho$ and $\sigma$ of $n$ qubits of the form $\rho \otimes \sigma$, prepare quantum registers $B$, $R_1$, and $R_2$. The register $B$ consists of only one qubit that is initially set to state $|0\rangle$, while the registers $R_1$ and $R_2$ consist of $n$ qubits and $\rho$ and $\sigma$ are initially set in $R_1$ and $R_2$, respectively.

The controlled-swap test is performed by running the algorithm described in Figure 1.

Proposition 7. The probability that the input pair of mixed states $\rho$ and $\sigma$ is accepted in the controlled-swap test is exactly $\frac{1}{2} + \frac{1}{2} |\langle e_i | f_j \rangle|^2$.

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the Hilbert spaces corresponding to $R_1$ and $R_2$, respectively.

Let $\rho = \sum_i p_i |e_i\rangle\langle e_i|$ and $\sigma = \sum_i q_i |f_i\rangle\langle f_i|$ be the decompositions of $\rho$ and $\sigma$ with respect to some orthonormal bases $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Then the state in $(\mathcal{R}_1, \mathcal{R}_2)$ is $|e_i\rangle \otimes |f_j\rangle$ with probability $p_i q_j$, and in such a case, the test results in acceptance with probability $\frac{1}{2} + \frac{|\langle e_i | f_j \rangle|^2}{2}$.

Therefore, the states $\rho$ and $\sigma$ are accepted with probability

$$\sum_i \sum_j p_i q_j \left(\frac{1}{2} + \frac{|\langle e_i | f_j \rangle|^2}{2}\right) = \frac{1}{2} + \frac{1}{2} \sum_i \sum_j p_i q_j |\langle e_i | f_j \rangle|^2$$

$$= \frac{1}{2} + \frac{1}{2} \sum_i p_i \sum_j q_j |\langle e_i | f_j \rangle|^2$$

$$= \frac{1}{2} + \frac{1}{2} \text{tr} \left(\left(\sum_i p_i |e_i\rangle\langle e_i|\right) \left(\sum_j q_j |f_j\rangle\langle f_j|\right)\right)$$

$$= \frac{1}{2} + \frac{1}{2} \text{tr}(\rho \sigma),$$
as desired.

\[ \square \]

### 4.3 Reducing the Number of Proofs

Using Proposition 7, we can show the following lemma, which is the key to proving Theorem 4.

**Lemma 8.** For any polynomially bounded function \( k: \mathbb{Z}^+ \rightarrow \mathbb{N} \), any \( r \in \{0, 1, 2\} \), and any functions \( \varepsilon, \delta: \mathbb{Z}^+ \rightarrow [0, 1] \) satisfying \( \delta > 10\varepsilon \),

\[ \text{QMA}(3k + r, 1 - \varepsilon, 1 - \delta) \subseteq \text{QMA}\left(2k + r, 1 - \frac{\varepsilon}{2}, 1 - \frac{\delta}{20}\right). \]

The essence of the proof of Lemma 8 is the basis case where \( k = 1 \) and \( r = 0 \). We first give a proof for this particular case, which will be helpful to see the idea.

**Proposition 9.** For any functions \( \varepsilon, \delta: \mathbb{Z}^+ \rightarrow [0, 1] \) satisfying \( \delta > 10\varepsilon \),

\[ \text{QMA}(3, 1 - \varepsilon, 1 - \delta) \subseteq \text{QMA}\left(2, 1 - \frac{\varepsilon}{2}, 1 - \frac{\delta}{20}\right). \]

**Proof.** Let \( A = \{A_{\text{yes}}, A_{\text{no}}\} \) be a problem in QMA(3, 1 - \( \varepsilon \), 1 - \( \delta \)). Given a QMA(3, 1 - \( \varepsilon \), 1 - \( \delta \)) proof system for \( A \), we construct a QMA(2, 1 - \( \frac{\varepsilon}{2} \), 1 - \( \frac{\delta}{20} \)) proof system for \( A \) in the following way.

Let \( V \) be the quantum verifier of the original QMA(3, 1 - \( \varepsilon \), 1 - \( \delta \)) proof system. For every input \( x \), suppose that \( V \) uses \( q_V(|x|) \) private qubits, and each of the quantum proofs \( V \) receives consists of \( q_M(|x|) \) qubits, for some polynomially bounded functions \( q_V, q_M: \mathbb{Z}^+ \rightarrow \mathbb{N} \). Let \( V(x) \) be the unitary transformation \( V \) applies.

Our new quantum verifier \( W \) in the QMA(2, 1 - \( \frac{\varepsilon}{2} \), 1 - \( \frac{\delta}{20} \)) proof system prepares quantum registers \( R_1, R_2, S_1 \), and \( S_2 \) for quantum proofs and quantum registers \( V \) and \( B \) for his private computation. Each \( R_i \) and \( S_i \) consists of \( q_M(|x|) \) qubits, \( V \) consists of \( q_V(|x|) \) qubits, and \( B \) consists of a single qubit. All the qubits in \( (V, B) \) are initialized to state \( |0\rangle \). \( W \) receives two quantum proofs \( |\psi_1\rangle \) and \( |\psi_2\rangle \) of \( 2q_M(|x|) \) qubits in \( (R_1, S_1) \) and \( (R_2, S_2) \), respectively, which are expected to be of the form

\[ |\psi_1\rangle = |\phi_1\rangle \otimes |\phi_3\rangle, \quad |\psi_2\rangle = |\phi_2\rangle \otimes |\phi_3\rangle, \]

where each \( |\phi_i\rangle \) is the \( i \)th quantum proof the original quantum verifier \( V \) would receive. Of course, each \( |\psi_i\rangle \) may not be of the form above and the first and the second \( q_M(|x|) \) qubits of \( |\psi_i\rangle \) may be entangled. Let \( V, B, \) each \( R_i, \) and each \( S_i \) be the Hilbert spaces corresponding to the quantum registers \( V, B, R_i, \) and \( S_i \), respectively.

The protocol of \( W \) is described in Figure 2.

For the completeness, assume that the input \( x \) is in \( A_{\text{yes}} \). In the original proof system, there exist quantum proofs \( |\phi_1\rangle, |\phi_2\rangle, \) and \( |\phi_3\rangle \) that cause the original quantum verifier \( V \) to accept \( x \) with probability at least \( 1 - \varepsilon(|x|) \). In the constructed protocol, let the quantum proofs \( |\psi_1\rangle \) and \( |\psi_2\rangle \) be of the form \( |\psi_1\rangle = |\phi_1\rangle \otimes |\phi_3\rangle \) and \( |\psi_2\rangle = |\phi_2\rangle \otimes |\phi_3\rangle \). Then it is obvious that the constructed quantum verifier \( W \) accepts \( x \) with certainty in the SEPARABILITY TEST and with probability at least \( 1 - \varepsilon(|x|) \) in the CONSISTENCY TEST, and thus, the completeness follows.

Now for the soundness, assume that the input \( x \) is in \( A_{\text{no}} \).

Consider any pair of quantum proofs \( |\psi_1'\rangle \) and \( |\psi_2'\rangle \) of \( 2q_M(|x|) \) qubits, which are set in the pairs of the quantum registers \( (R_1, S_1) \) and \( (R_2, S_2) \), respectively. Let \( \rho = \text{tr}_{R_1}|\psi_1'\rangle \langle \psi_1'| \) and \( \sigma = \text{tr}_{R_2}|\psi_2'\rangle \langle \psi_2'| \).
Verifier’s Protocol in Two-Proof System

1. Receive the first quantum proof $|\psi_1\rangle$ in $(R_1, S_1)$ and the second quantum proof $|\psi_2\rangle$ in $(R_2, S_2)$.

2. Do one of the following two tests uniformly at random.

2.1 (SEPARABILITY TEST)
   Perform the controlled-swap test over $S_1$ and $S_2$ using $B$ as a control qubit. That is, perform the following:
   2.1.1 Apply the Hadamard transformation $H$ to $B$.
   2.1.2 Apply the controlled-swap operator to $S_1$ and $S_2$ using $B$ as a control qubit.
   2.1.3 Apply the Hadamard transformation $H$ to $B$. Accept if $B$ contains $0$, and reject otherwise.

2.2 (CONSISTENCY TEST)
   Apply $V(x)$ to the qubits in $(V, R_1, R_2, S_1)$. Accept if the result corresponds to the accepting computation of the original quantum verifier.

Figure 2: Verifier’s protocol in two-proof system.

(i) In the case $\text{tr}(\rho\sigma) \leq 1 - \frac{\delta}{5}$.
   In this case, by Proposition 7, the probability $p_{\text{sep}}$ that the input $x$ is accepted in the SEPARABILITY TEST is at most
   $$p_{\text{sep}} \leq \frac{1}{2} + \frac{1}{2}(1 - \frac{\delta}{5}) = 1 - \frac{\delta}{10}.$$
   Thus the verifier $W$ accepts the input $x$ with probability at most $\frac{1}{2} + \frac{p_{\text{sep}}}{2} \leq 1 - \frac{\delta}{20}$.

(ii) In the case $\text{tr}(\rho\sigma) > 1 - \frac{\delta}{5}$.
   Let $\tilde{V} = V(x) \otimes I_{S_2}$ and $\Pi_{\text{acc}} = \Pi_{\text{acc}} \otimes I_{S_2}$, where $\Pi_{\text{acc}}$ is the projection onto accepting states of the original proof system. For notational convenience, here it is assumed that $\tilde{V}$ and $\Pi_{\text{acc}}$ are applied to $(V, R_1, S_1, R_2, S_2)$ in this order of registers, although the registers to which $V(x)$ and $\Pi_{\text{acc}}$ are applied are assumed to be in order of $V, R_1, R_2$, and $S_1$. Let
   $$|\alpha\rangle = |0\rangle \otimes |\psi_1\rangle \otimes |\psi_2\rangle,$$
   and
   $$|\beta\rangle = |0\rangle \otimes |\psi_1\rangle \otimes |\psi_2\rangle.$$
   Then the probability $p_{\text{cons}}$ that the input $x$ is accepted in the CONSISTENCY TEST is at most
   $${p_{\text{cons}}} \leq F(\tilde{V}^\dagger|\alpha\rangle\langle\alpha|\tilde{V}, |\beta\rangle\langle\beta|).$$

   The fact $\text{tr}(\rho\sigma) > 1 - \frac{\delta}{5}$ implies that the maximum eigenvalue $\lambda$ of $\rho$ satisfies $\lambda > 1 - \frac{\delta}{5}$. Thus there exists a pure state $|\phi_1\rangle \in R_1 \otimes S_1$ of the form $|\phi_1\rangle = |\xi_1\rangle \otimes |\eta_1\rangle$ for some pure states $|\xi_1\rangle \in R_1$ and $|\eta_1\rangle \in S_1$ such that
   $$F(|\phi_1\rangle\langle\phi_1|, |\psi_1\rangle\langle\psi_1|) > \sqrt{1 - \frac{\delta}{5}},$$
   since $\rho = \text{tr}_{R_1} |\phi_1\rangle\langle\phi_1|$. Similarly, the maximum eigenvalue of $\sigma$ is more than $1 - \frac{\delta}{5}$, and there exists a pure state $|\phi_2\rangle \in R_2 \otimes S_2$ of the form $|\phi_2\rangle = |\xi_2\rangle \otimes |\eta_2\rangle$ for some pure states $|\xi_2\rangle \in R_2$ and $|\eta_2\rangle \in S_2$ such that
   $$F(|\phi_2\rangle\langle\phi_2|, |\psi_2\rangle\langle\psi_2|) > \sqrt{1 - \frac{\delta}{5}}.$$
   Therefore, letting $|\gamma\rangle = |0\rangle \otimes |\psi_1\rangle \otimes |\phi_2\rangle$, we have from Lemma 1 that
   $$F(|\beta\rangle\langle\beta|, |\gamma\rangle\langle\gamma|) > 1 - \frac{\delta}{5}.$$
Furthermore, from the soundness condition of the original proof system, it is easy to see that

\[ F(\bar{V}^\dagger |\alpha\rangle \langle \alpha| \bar{V}, |\gamma\rangle \langle \gamma|) = F(|\alpha\rangle \langle \alpha|, \bar{V} |\gamma\rangle \langle \gamma| \bar{V}^\dagger) \leq \sqrt{1 - \delta}. \]

Using Lemma 2, we have that

\[ F(\bar{V}^\dagger |\alpha\rangle \langle \alpha| \bar{V}, |\beta\rangle \langle \beta|) \leq F(|\beta\rangle \langle \beta|, |\gamma\rangle \langle \gamma|) \leq 1 + F(\bar{V}^\dagger |\alpha\rangle \langle \alpha| \bar{V}, |\gamma\rangle \langle \gamma|). \]

It follows that

\[ p_{\text{cons}} \leq 1 + F(\bar{V}^\dagger |\alpha\rangle \langle \alpha| \bar{V}, |\gamma\rangle \langle \gamma|) - F(|\beta\rangle \langle \beta|, |\gamma\rangle \langle \gamma|)^2 \leq 2 - \frac{\delta}{2} - 1 + \frac{2\delta}{5} - \frac{\delta^2}{25} \leq 1 - \frac{\delta}{10}. \]

Thus the verifier \( W \) accepts the input \( x \) with probability at most \( \frac{1}{2} + \frac{p_{\text{cons}}}{2} \leq 1 - \frac{\delta}{20} \).

Hence the soundness is at most \( 1 - \frac{\delta}{20} \), as required.

Now we prove Lemma 8.

**Proof of Lemma 8** The proof is a simple generalization of the case of Proposition 9.

Let \( A = \{ A_{\text{yes}}, A_{\text{no}} \} \) be a problem in QMA(3k + r, 1 - \( \varepsilon \), 1 - \( \delta \)). Given a QMA(3k + r, 1 - \( \varepsilon \), 1 - \( \delta \)) proof system for \( A \), we construct a QMA(2k + r, 1 - \( \varepsilon \), 1 - \( \delta \)) proof system for \( A \) in the following way.

Let \( V \) be the quantum verifier of the original QMA(3k + r, 1 - \( \varepsilon \), 1 - \( \delta \)) proof system. For every input \( x \), suppose that \( V \) uses \( q_V(|x|) \) private qubits, and each of quantum proofs \( V \) receives consists of \( q_M(|x|) \) qubits, for some polynomially bounded functions \( q_V, q_M : \mathbb{Z}^+ \rightarrow \mathbb{N} \). Let \( V(x) \) be the unitary transformation \( V \) applies.

Our new quantum verifier \( W \) in the QMA(2k + r, 1 - \( \varepsilon \), 1 - \( \delta \)) proof system prepares quantum registers \( R_{1,1}, \ldots, R_{1,k}, R_{2,1}, \ldots, R_{2,k}, S_{1,1}, \ldots, S_{1,j}, S_{2,1}, \ldots, S_{2,j}, R_{3,1}, \ldots, R_{3,3}, S_{3,1}, \ldots, S_{3,j} \) for quantum proofs and quantum registers \( V, B \) for his private computation. Each of \( R_{i,j} \) and \( S_{i,j} \) consists of \( q_M(|x|) \) qubits, \( V \) consists of \( q_V(|x|) \) qubits, and \( B \) consists of a single qubit. All the qubits in \( (V, B) \) are initialized to state \( |0\rangle \). Let \( V, B \), each \( R_{i,j} \), and each \( S_{i,j} \) be the Hilbert spaces corresponding to the quantum registers \( V, B, R_{i,j}, \) and \( S_{i,j} \), respectively. \( W \) receives \( 2k + r \) quantum proofs \( |\psi_{1,1}\rangle, |\psi_{1,2}\rangle, \ldots, |\psi_{1,k}\rangle, |\psi_{2,1}\rangle, \ldots, |\psi_{2,k}\rangle, |\psi_{3,1}\rangle, \ldots, |\psi_{3,r}\rangle \) of \( q_M(|x|) \) qubits in \((R_{1,1}, S_{1,1}), \ldots, (R_{1,k}, S_{1,j}), (R_{2,1}, S_{2,1}), \ldots, (R_{2,k}, S_{2,j}), (R_{3,1}, S_{3,1}), \ldots, (R_{3,3}, S_{3,j})\) respectively, which are expected to be of the form

\[
|\psi_{i,j}\rangle = |\phi_{i,j}\rangle \otimes |\phi_{2k+j,j}\rangle,
|\psi'_{i,j}\rangle = |\phi'_{i,j}\rangle \otimes |\phi'_{2k+j,j}\rangle,
|\psi_{3,j}\rangle = |\phi_{3k+j,2}\rangle \otimes |0_{S_{i,j}}\rangle,
\]

for each \( 1 \leq j_1 \leq k \) and \( 1 \leq j_2 \leq r \), where each \( |\phi_{i,j}\rangle \) is the \( i \)th quantum proof the original quantum verifier \( V \) would receive.

The protocol of \( W \) is described in Figure 5.

The rest of the proof is essentially the same as in the case of Proposition 9. When analyzing soundness, consider any set of \( 2k + r \) quantum proofs \( |\psi_{1,1}\rangle, \ldots, |\psi_{1,k}\rangle, |\psi_{2,1}\rangle, \ldots, |\psi_{2,k}\rangle, |\psi_{3,1}\rangle, \ldots, |\psi_{3,r}\rangle \) of \( q_M(|x|) \) qubits, which are set in the quantum registers \( (R_{1,1}, S_{1,1}), \ldots, (R_{1,k}, S_{1,k}), (R_{2,1}, S_{2,1}), \ldots, (R_{2,k}, S_{2,k}), \ldots, (R_{3,3}, S_{3,3}) \), respectively, and let \( |\psi_{i,j}\rangle = |\psi_{1,1}\rangle \otimes \cdots \otimes |\psi_{1,k}\rangle \otimes |\psi_{2,1}\rangle \otimes \cdots \otimes |\psi_{2,k}\rangle \otimes |\psi_{3,1}\rangle \otimes \cdots \otimes |\psi_{3,r}\rangle \) and \( |\sigma\rangle = |\psi_{i,j}\rangle \otimes |\psi_{i,j}\rangle \otimes |\psi_{i,j}\rangle \) for each \( 1 \leq j \leq k \). Let \( \rho = p_1 \otimes \cdots \otimes p_k \) and \( \sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \), where \( \rho_j = \text{tr}_{R_{1,j}} |\psi_{1,j}\rangle \langle \psi_{1,j}| \) and \( \sigma_j = \text{tr}_{R_{2,j}} |\psi_{2,j}\rangle \langle \psi_{2,j}| \) for each \( 1 \leq j \leq k \) such that the
Verifier’s Protocol in $(2k + r)$-Proof System

1. For each $(i, j) \in \{(1, 1), \ldots, (1, k), (2, 1), \ldots, (2, k), (3, 1), \ldots, (3, r)\}$, receive the quantum proof $|\psi_{i,j}\rangle$ in $(R_{i,j}, S_{i,j})$. Reject if any of the qubits in $S_{3,j}$ contains 1, for $1 \leq j \leq r$.

2. Do one of the following two tests uniformly at random.

2.1 (Separability Test)
Perform the controlled-swap test over $(S_{1,1}, \ldots, S_{1,k})$ and $(S_{2,1}, \ldots, S_{2,k})$ using $B$ as a control qubit. That is, perform the following:

2.1.1 Apply the Hadamard transformation $H$ to $B$.
2.1.2 Apply the controlled-swap operator to $(S_{1,1}, \ldots, S_{1,k})$ and $(S_{2,1}, \ldots, S_{2,k})$ using $B$ as a control qubit.
2.1.3 Apply the Hadamard transformation $H$ to $B$. Accept if $B$ contains 0, and reject otherwise.

2.2 (Consistency Test)
Apply $V(x)$ to the qubits in $(V, R_{1,1}, \ldots, R_{1,k}, R_{2,1}, \ldots, R_{2,k}, S_{1,1}, \ldots, S_{1,k}, R_{3,1}, \ldots, R_{3,r})$. Accept if and only if the result corresponds to the accepting computation of the original quantum verifier.

Figure 3: Verifier’s protocol in $(2k + r)$-proof system.

Now Theorem 4 can be proved by using the transformation in Lemma 8 repeatedly.

Proof of Theorem 4 The “only if” part is obvious and we show the “if” part.

From Corollary 6 we have that $QMA(k, c, s) \subseteq QMA(k, 1 - 2^{-p}, 1 - \frac{c - s}{2})$ for any polynomially bounded function $p: \mathbb{Z}^+ \rightarrow \mathbb{N}$. Now we repeatedly apply the transformation in Lemma 8 $O(\log k)$ times, and finally we can show the inclusion $QMA(k, 1 - 2^{-p}, 1 - \frac{c - s}{2}) \subseteq QMA(2, 1 - 2^{-p}, 1 - \frac{1}{q})$ for some polynomially bounded function $q: \mathbb{Z}^+ \rightarrow \mathbb{N}$. Note that the size of the circuit of the verifier after each application of the transformation in Lemma 8 is at most some constant times that of the original verifier plus an amount bounded by a polynomial in the input length. Thus, given a description of the circuit of the verifier in the original $k$-proof system, one can compute in time polynomial in the input length a description of the circuit of the verifier in the resulting two-proof system. From our assumption, $QMA(2, 1 - 2^{-p}, 1 - \frac{1}{q}) = QMA(2, \frac{2}{3}, \frac{1}{3})$, and thus, the inclusion $QMA(k, c, s) \subseteq QMA(2, \frac{2}{3}, \frac{1}{3})$ follows. The other inclusion is trivial since our assumption implies that $QMA(2, \frac{2}{3}, \frac{1}{3}) = QMA(2, c, s)$, and we have the theorem.

The following is an immediate corollary of Theorem 4.

Corollary 10. $QMA(k, c, s) = QMA$ for any polynomially-bounded function $k: \mathbb{Z}^+ \rightarrow \mathbb{N}$ satisfying $k \geq 2$ and any two-sided bounded error probability $(c, s)$ if and only if $QMA(2, c, s) = QMA$ for any two-sided bounded error probability $(c, s)$.

Proof. The proof is almost parallel to the proof of Theorem 4. Again the “only if” part is obvious and we show the “if” part.
Using the same argument as in the proof of Theorem 11, we can show the inclusion $\text{QMA}(k, c, s) \subseteq \text{QMA}(2, 1 - 2^{-p}, 1 - 1/q)$ for some polynomially bounded function $q: \mathbb{Z}^+ \to \mathbb{N}$. From our assumption, $\text{QMA}(2, 1 - 2^{-p}, 1 - 1/q) = \text{QMA}$, and thus, the inclusion $\text{QMA}(k, c, s) \subseteq \text{QMA}$ follows. The other inclusion is trivial since our assumption implies that $\text{QMA} \subseteq \text{QMA}(2, c, s)$, and we have the corollary. \hfill $\Box$

**Remark.** Theorem 11 improves the original statement in our conference version [35] in two ways. First, the condition (*) now only requires the amplifiability of the success probability for two-proof systems, whereas our original condition required it for every $k$-proof system. Second, now $\text{QMA}(k)$ even with every polynomially-bounded function $k$ coincides with $\text{QMA}(2)$ if the condition holds. Previously, we showed it only for $\text{QMA}(k)$ with every constant $k$. The same improvements were independently done by Aaronson, Beigi, Drucker, Fefferman, and Shor [2] but with a different proof. Instead of repeatedly applying the transformation that reduces the number of proofs by two-thirds as above, they showed a direct method of reducing the number of proofs to two [2, Theorem 23].

Although the resulting two-proof system from their transformation also has soundness only polynomially bounded away from one, their soundness is better than ours in most cases (except for the case where the gap between completeness $c$ and soundness $s$ in the original system is so small relative to the number $k$ of proofs that $c - s \in o(k^{-\alpha})$, where $\alpha = \log_{20}/\log_{3-1} - 1 \approx 6.388 \cdots$, in which case our analysis gives better soundness).

## 5 Cases with Perfect Soundness

This section focuses on the quantum multi-proof Merlin-Arthur proof systems of perfect soundness. In the case of perfect soundness, it is proved that multiple quantum proofs do not increase the verification power, which also gives a connection between two existing concepts of “quantum nondeterminism”. Formally, the following is proved.

**Theorem 11.** For any polynomially bounded function $k: \mathbb{Z}^+ \to \mathbb{N}$ and any function $c: \mathbb{Z}^+ \to [0, 1]$, $\text{QMA}(k, c, 0) = \text{QMA}(1, c, 0)$.

**Proof.** Let $A = \{A_{\text{yes}}, A_{\text{no}}\}$ be a problem in $\text{QMA}(k, c, 0)$. Given a $\text{QMA}(k, c, 0)$ proof system for $A$, we construct a $\text{QMA}(1, c, 0)$ proof system for $A$ in the following way.

Let $V$ be a quantum verifier of the $\text{QMA}(k, c, 0)$ proof system. For every input $x$, assume that each quantum proof $V$ receives is of size $q(|x|)$.

Our new quantum verifier $W$ in the $\text{QMA}(1, c, 0)$ proof system receives one quantum proof of size $k(|x|)q(|x|)$ and simulates $V$ with this quantum proof.

The completeness is clearly at least $c$.

For the soundness, assume that the input $x$ is in $A_{\text{no}}$. Let $|\phi\rangle$ be any quantum proof of size $k(|x|)q(|x|)$. Let $e_i$ be the lexicographically $i$th string in $\Sigma^{k(|x|)q(|x|)}$. Note that, for every $i$, the original verifier $V$ never accepts $x$ when the $k(|x|)$ quantum proofs he receives form the state $|e_i\rangle$. Since any $|\phi\rangle$ is expressed as a linear combination of these $|e_i\rangle$, it follows that $W$ rejects $x$ with certainty. \hfill $\Box$

Let $\text{EQMA}(k) = \text{QMA}(k, 1, 0)$ and $\text{RQMA}(k) = \text{QMA}(k, 1/2, 0)$ for every $k$. Theorem 11 implies that $\text{EQMA}(k) = \text{EQMA}(1)$ and $\text{RQMA}(k) = \text{RQMA}(1)$. Furthermore, one can consider the complexity class $\text{NQMA}(k)$ that combines two existing concepts of “quantum nondeterminism”, $\text{QMA}(k)$ and $\text{NQP}$.

**Definition 12.** A problem $A = \{A_{\text{yes}}, A_{\text{no}}\}$ is in $\text{NQMA}(k)$ if there exists a function $c: \mathbb{Z}^+ \to (0, 1]$ such that $A$ is in $\text{QMA}(k, c, 0)$.

Note that $\text{NQMA}(k) = \text{NQMA}(1)$ is also immediate from Theorem 11. The next theorem shows that $\text{NQMA}(1)$ coincides with the class $\text{NQP}$.
NQP Simulation of NQMA Proof System

1. Apply the Hadamard transformation $H$ to every qubit in $S_1$.

2. Copy the contents of $S_1$ to those of $S_2$.

3. Apply $V(x)$ to the pair of quantum registers $(R, S_1)$. Accept if the contents of $(R, S_1)$ make the original verifier accept.

Figure 4: NQP simulation of an NQMA proof system.

Theorem 13. EQMA(1) $\subseteq$ RQMA(1) $\subseteq$ NQMA(1) = NQP.

Proof. It is sufficient to show that NQMA(1) $\subseteq$ NQP, since EQMA(1) $\subseteq$ RQMA(1) $\subseteq$ NQMA(1) and NQMA(1) $\supseteq$ NQP hold obviously.

Let $A = \{A_{\text{yes}}, A_{\text{no}}\}$ be a problem in NQMA(1). Given an NQMA(1) proof system for $A$, we construct an NQP algorithm for $A$.

Let $V$ be the quantum verifier of the NQMA(1) proof system. For every input $x$, suppose that $V$ uses $q_V(|x|)$ private qubits, and each quantum proof $V$ receives consists of $q_M(|x|)$ qubits, for some polynomially bounded functions $q_V, q_M : \mathbb{Z}^+ \rightarrow \mathbb{N}$. Let $V(x)$ be the unitary transformation $V$ applies.

In the NQP algorithm for $A$, we prepare quantum registers $R, S_1,$ and $S_2$, where $R$ consists of $q_V(|x|)$ qubits and each $S_i$ consists of $q_M(|x|)$ qubits. All the qubits in $R, S_1,$ and $S_2$ are initialized to state $|0\rangle$. The precise algorithm is described in Figure 4.

For the completeness, suppose that the input $x$ is in $A_{\text{yes}}$. In the original NQMA(1) proof system for $A$, there exists a quantum proof $|\phi\rangle$ of size $q_M(|x|)$ that causes $V$ to accept $x$ with non-zero probability. Suppose that $V$ never accepts $x$ with any given quantum proof $|e_i\rangle$ for $1 \leq i \leq 2^{q_M(|x|)}$, where $e_i$ is the lexicographically $i$th string in $\Sigma^{q_M(|x|)}$. Then with a similar argument to the proof of Theorem 13, $V$ never accepts $x$ with any given quantum proof $|\phi\rangle$ of size $q_M(|x|)$, which contradicts the assumption. Thus there is at least one $|e_i\rangle$ that causes $V$ to accept $x$ with non-zero probability. Hence, in the algorithm in Figure 4, the probability of acceptance must be non-zero, since it simulates with probability $2^{-q_M(|x|)}$ the case where $V$ is given a proof $|e_i\rangle$ for every $i$.

Now for the soundness, suppose that the input $x$ is in $A_{\text{no}}$. In the original NQMA(1) proof system for $A$, no matter which quantum proof $|\phi\rangle$ of size $q_M(|x|)$ is given, $V$ never accepts $x$. Hence, in the algorithm in Figure 4, the probability of acceptance is zero and the soundness follows.

Now the following characterization of NQP is immediate.

Corollary 14. $\text{NQP} = \bigcup_{c: \mathbb{Z}^+ \rightarrow (0,1]} \text{QMA}(1,c,0)$.

6 Discussions

This section shows that there is no POVM measurement that determines whether a given unknown state is in a tensor product form or even maximally entangled.

Suppose that there is a quantum subroutine that answers which of the following (a) and (b) is true for a given proof $|\Psi\rangle \in \mathcal{H}^\otimes 2$ of $2n$ qubits, where $\mathcal{H}$ is the Hilbert space consisting of $n$ qubits:

(a) $|\Psi\rangle\langle\Psi|$ is in $H_0 = \{|\Psi_0\rangle\langle\Psi_0| : |\Psi_0\rangle \in \mathcal{H}^\otimes 2, \exists|\phi\rangle, |\psi\rangle \in \mathcal{H}, |\Psi_0\rangle = |\phi\rangle \otimes |\psi\rangle\}$,

(b) $|\Psi\rangle\langle\Psi|$ is in $H_1 = \{|\Psi_1\rangle\langle\Psi_1| : |\Psi_1\rangle \in \mathcal{H}^\otimes 2, \max_{|\phi\rangle,|\psi\rangle} F(|\Psi_1\rangle\langle\Psi_1|, |\phi\rangle\langle\phi\rangle \otimes |\psi\rangle\langle\psi|) \leq 1 - \varepsilon\}$. 

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As for the proof \(|\Psi\rangle\) that does not satisfy (a) nor (b), this subroutine may answer (a) or (b) arbitrarily. The rest of this section proves that this kind of subroutines cannot be realized by any physical method. In fact, we prove a stronger statement that the set of states in a tensor product form cannot be distinguished even from the set of maximally entangled states by any physical operation. Here, following Ref. [14], we say that the \(n\)-qubit state \(\rho = |\Psi\rangle \langle \Psi|\) is maximally entangled if \(|\Psi\rangle\) can be written as

\[
|\Psi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle \otimes |f_i\rangle, \quad |\alpha_i|^2 = \frac{1}{d},
\]

where \(d = 2^n\) is the dimension of \(H\) and \(\{|e_i\}\) and \(\{|f_i\}\) are orthonormal bases of \(H\). Among all states, maximally entangled states are farthest away from states in a tensor product form, and

\[
\min_{|\psi\rangle \in H^\otimes 2} \max_{|\phi\rangle \in H} F(|\Psi\rangle \langle \Psi|, |\phi\rangle \otimes |\psi\rangle \langle \psi|) = \frac{1}{\sqrt{d}} = 2^{-\frac{n}{2}}
\]

is achieved by maximally entangled states. Thus Arthur cannot rule out quantum proofs that are far from states of a tensor product of pure states.

**Theorem 15.** Suppose that one of the following two is true for a given proof \(|\Psi\rangle \in H^\otimes 2\) of \(2n\) qubits:

(a) \(|\Psi\rangle \langle \Psi|\) is in \(H_0 = \{|\psi_0\rangle \langle \psi_0|: |\psi_0\rangle \in H^\otimes 2, \exists |\phi\rangle, |\psi\rangle \in H, |\psi_0\rangle = |\phi\rangle \otimes |\psi\rangle\},\)

(b) \(|\Psi\rangle \langle \Psi|\) is in \(H_1 = \{|\psi_1\rangle \langle \psi_1|: |\psi_1\rangle \in H^\otimes 2\) is maximally entangled\}.

Then, in determining which of (a) and (b) is true, no POVM measurement is better than the trivial strategy in which one guesses at random without any operation at all.

**Proof.** Let \(M = \{M_0, M_1\}\) be a POVM on \(H^\otimes 2\). With \(M\) we conclude \(|\Psi\rangle \langle \Psi|\) \(\in H_i\) if \(M\) results in \(i, i \in \{0, 1\}\). Let \(P_{i \rightarrow j}^M(|\Psi\rangle \langle \Psi|)\) denote the probability that \(|\Psi\rangle \langle \Psi|\) \(\in H_j\) is concluded by \(M\) while \(|\Psi\rangle \langle \Psi|\) \(\in H_i\) is true. We want to find the measurement that minimizes \(P_{0 \rightarrow 1}^M(|\Psi\rangle \langle \Psi|)\) keeping the other side of error small enough. More precisely, we consider \(E\) defined and bounded as follows.

\[
E = \min_M \left\{ \max_{\rho \in H_0} P_{0 \rightarrow 1}^M(\rho): \max_{\rho \in H_1} P_{1 \rightarrow 0}^M(\rho) \leq \delta \right\}
\]

\[
\geq \min_M \left\{ \int_{\rho \in H_0} P_{0 \rightarrow 1}^M(\rho) \mu_0(\rho) \, d\rho: \int_{\rho \in H_1} P_{1 \rightarrow 0}^M(\rho) \mu_1(\rho) \, d\rho \leq \delta \right\}
\]

\[
= \min_M \left\{ P_{0 \rightarrow 1}^M \left( \int_{\rho \in H_0} \rho \mu_0(\rho) \, d\rho \right): \int_{\rho \in H_1} \rho \mu_1(\rho) \, d\rho \leq \delta \right\},
\]

where each \(\mu_i\) is an arbitrary probability measure in \(H_i\). It follows that \(E\) is larger than the error probability in distinguishing \(\int_{\rho \in H_0} \rho \mu_0(\rho) \, d\rho\) from \(\int_{\rho \in H_1} \rho \mu_1(\rho) \, d\rho\).

Take \(\mu_0\) as a uniform distribution over the set \(\{|e_i\rangle \langle e_i| \otimes |e_j\rangle \langle e_j| \}_{1 \leq i, j \leq d}\), that is, \(\mu_0(|e_i\rangle \langle e_i| \otimes |e_j\rangle \langle e_j|) = \frac{1}{d}\) for each \(i\) and \(j\), where \(\{|e_i\}\) is an orthonormal basis of \(H\), and take \(\mu_1\) as a uniform distribution over the set \(\{|g_{k,l}\rangle \langle g_{k,l}| \}_{1 \leq k, l \leq d}\), that is, \(\mu_1(|g_{k,l}\rangle \langle g_{k,l}|) = \frac{1}{d}\) for each \(k\) and \(l\), where

\[
|g_{k,l}\rangle = \frac{1}{d} \sum_{j=1}^{d} (e^{2\pi \sqrt{-1} \frac{k}{d}} |e_j\rangle \otimes |e j+l \text{ mod } d\rangle).
\]

This \(\{|g_{k,l}\}\) forms an orthonormal basis of \(H^\otimes 2\) [15], and thus

\[
\int_{\rho \in H_0} \rho \mu_0(\rho) \, d\rho = \int_{\rho \in H_1} \rho \mu_1(\rho) \, d\rho = \frac{1}{d^2} I_{H^\otimes 2}.
\]

Hence we have the assertion. \(\square\)
From Theorem 15, it is easy to show the following corollary.

**Corollary 16.** Suppose one of the following two is true for the proof $|\Psi\rangle \in \mathcal{H}^{\otimes 2}$ of $2n$ qubits:

(a) $|\Psi\rangle \langle \Psi |$ is in $H_0 = \{ |\Psi_0\rangle \langle \Psi_0 | : |\Psi_0\rangle \in \mathcal{H}^{\otimes 2}, \exists |\phi\rangle, |\psi\rangle \in \mathcal{H}, |\Psi_0\rangle = |\phi\rangle \otimes |\psi\rangle \}$,

(b) $|\Psi\rangle \langle \Psi |$ is in $H_1 = \{ |\Psi_1\rangle \langle \Psi_1 | : |\Psi_1\rangle \in \mathcal{H}^{\otimes 2}, \max_{|\phi\rangle, |\psi\rangle \in \mathcal{H}} F(|\Psi_1\rangle \langle \Psi_1 |, |\phi\rangle \langle \phi| \otimes |\psi\rangle \langle \psi |) \leq 1 - \varepsilon \}.$

Then, for any $0 \leq \varepsilon \leq 1 - 2^{-\frac{n}{2}}$, in determining which of (a) and (b) is true, no POVM measurement is better than the trivial strategy in which one guesses at random without any operation at all.

7 Conclusions

This paper introduced the multi-proof version of quantum Merlin-Arthur proof systems. To investigate the possibility that multi-proof quantum Merlin-Arthur proof systems collapse to usual single-proof ones, this paper proved several basic properties such as a necessary and sufficient condition under which the number of quantum proofs is reducible to two. However, the central question whether multiple quantum proofs are indeed more helpful to Arthur still remains open. The authors hope that this paper sheds light on new features on quantum Merlin-Arthur proof systems and entanglement theory, and more widely on quantum computational complexity and quantum information theory.

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