GRUSS-TYPE INEQUALITY BY MEAN OF A FRACTIONAL INTEGRAL

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Abstract. In this paper, using a fractional integral as proposed by Katugampola we establish a generalization of integral inequalities of Gruss-type. We prove two theorems associated with these inequalities and then immediately we enunciate and prove others inequalities associated with these fractional operator.

Keywords: Katugampola fractional integral; generalization inequalities of Gruss-type
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1. Introduction

In 1935, Gruss proved the following integral inequality [1]

\[ \left| \frac{1}{b-a} \int_{a}^{b} f(x) g(x) \, dx - \left( \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right) \left( \frac{1}{b-a} \int_{a}^{b} g(x) \, dx \right) \right| \leq \frac{(M-m)(P-p)}{4}, \]

(1.1)

where \( f \) and \( g \) are two integrable functions on \( [a,b] \) satisfying the conditions

\[ m \leq f(x) \leq M, \quad p \leq g(x) \leq P; \quad m, M, p, P \in \mathbb{R}, \quad x \in [a,b]. \]

(1.2)

The Gruss type inequality has some important applications. We mention: difference equations, integral arithmetic mean and \( h \)-integral arithmetic mean [2, 3]. On the other hand, we also study the Gruss type inequality in spaces with inner product, consequently some applications for the Mellin transform of sequences and polynomials in Hilbert spaces [4].

In this sense, there are another important inequalities using integer order integrals among than we mention: Hölder’s inequality, Jensen’s inequality, Minkowski’s inequality and reverse Minkowskian inequality, [5, 6, 7, 8, 9, 10, 11]. For such inequalities, as well as for the study of functions, integrals and norms, the space of the \( p \)-integrable functions, \( L^p(a,b) \), have a particular importance. However, in this work, we use the space of Lebesgue measurable functions, which admits, as a particular case, the space \( L^p(a,b) \).

The appearance of the fractional calculus allow several consequences, results and important theories in mathematics, physics, engineering, among others areas. Due to this fact, it was possible to define several fractional integrals, for example: Riemann-Liouville, Katugampola,
Hadamard, Erdélyi-Kober, Liouville and Weyl types. There are other fractional integrals that can be found in [12]. Thus, with these fractional integrals some inequalities involving such formulations have been developed over the years, for example: Reverse Minkowiski, Hermite-Hadamard, Ostrowski e Fejér inequalities, [8, 13, 14, 15, 16, 17, 18, 19, 20]. We also mention that, in the literature, there are generalizations for the Eq.(1.1) using, for example, fractional integrals of Riemann-Liouville, Hadamard and also the $q$-fractional integral, [21, 22, 23].

In this paper, using a fractional integral recently introduced [24], we propose a new generalization of Eq.(1.1), i.e., new Gruss-type inequalities that generalize inequalities obtained by means of Riemann-Liouville fractional integral [21].

The paper is organized as follow: In section 2, we present the Katugampola’s fractional integral as well as the convenient space for such definition and the convenient parameters to recover the six fractional integrals as particular cases. In section 3, our main result, we present inequalities of Gruss-type. In section 4, we discuss other inequalities involving the Katugampola’s fractional integral. Conclusions and future perspectives close the paper.

2. Preliminaries

In this section, we define the space $X^p_c(a,b)$, where the Katugampola’s fractional integrals are defined. With a convenient choice of parameters, we recover six well-know fractional integrals, previously mentioned [12, 24].

Definition 1. The space $X^p_c(a,b)$ ($c \in \mathbb{R}, 1 \leq p \leq \infty$) consists of those complex-valued Lebesgue measurable functions $\varphi$ on $(a,b)$ for which $\|\varphi\|_{X^p_c} < \infty$, with

$$
\|\varphi\|_{X^p_c} = \left( \int_a^b |x^c \varphi(x)|^p \frac{dx}{x} \right)^{1/p} \quad (1 \leq p < \infty)
$$

and

$$
\|\varphi\|_{X^\infty_c} = \sup_{x \in (a,b)} [x^c |\varphi(x)|].
$$

In particular, when $c = 1/p$, the space $X^p_c(a,b)$ coincides with the space $L^p(a,b)$.

Definition 2. Let $\varphi \in X^p_c(a,b)$, $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then, the fractional integrals of a function $\varphi$, left- and right-sided, are defined by

\begin{align*}
\rho I^{\alpha,\beta}_{a+,\eta,\kappa} \varphi(x) &:= \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \varphi(\tau) d\tau, \quad 0 \leq a < x < b \leq \infty \\
\rho I^{\alpha,\beta}_{b-,\eta,\kappa} \varphi(x) &:= \frac{\rho^{1-\beta} x^\eta}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} \varphi(\tau) d\tau, \quad 0 \leq a < x < b \leq \infty
\end{align*}

respectively, if the integrals exist.
As previously mentioned, with a convenient choice of parameters, the fractional integral given by Eq. (2.1) admits, as particular cases, six well-known fractional integrals, namely:

1. If \( \kappa = 0, \eta = 0 \) and taking the limit \( \rho \to 1 \), in Eq. (2.1), we obtain the Riemann-Liouville fractional integral, [12, p. 69].

2. If \( \beta = \alpha, \kappa = 0, \eta = 0 \), taking the limit \( \rho \to 0^+ \) and using the \( \ell \)’Hospital’s rule, in Eq. (2.1), we obtain the Hadamard fractional integral, [12, p. 110].

3. When \( \beta = 0 \) and \( \kappa = -\rho(\alpha + \eta) \), in Eq. (2.1), we obtain the Erdélyi-Kober fractional integral, [12, p. 105].

4. Also, for \( \beta = \alpha, \kappa = 0 \) and \( \eta = 0 \), in Eq. (2.1), we recover the Katugampola fractional integral, [25].

5. With the choice \( \kappa = 0, \eta = 0, a = -\infty \) and taking the limit \( \rho \to 1 \), in Eq. (2.1), we have the Weyl fractional integral, [26, p. 50].

6. If \( \kappa = 0, \eta = 0, a = 0 \) and taking the limit \( \rho \to 1 \), in Eq. (2.1), we obtain the Liouville fractional integrals, [12, p. 79].

Note that, we present and discuss our new results associated with the fractional integral using the left-sided operator, only. Moreover, we admit \( a = 0 \) in Eq. (2.1), in order to obtain

\[
\rho x^{\alpha, \beta, \eta, \kappa}(x) = \frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \phi(\tau) \, d\tau.
\]

3. Main results

We enunciate and prove the following lemma in order to use it in the first theorem which generalizes the inequalities of Gruss-type, Eq. (1.1). First of all, let \( \alpha > 0, x > 0 \) and \( \beta, \rho, \eta, \kappa \in \mathbb{R} \). We define the following function

\[
\Lambda^{\rho, \beta, \eta, \kappa}(\alpha, \eta) = \frac{\Gamma(\eta + 1)}{\Gamma(\eta + \alpha + 1)} \rho^{-\beta}x^{\kappa+\rho(\eta+\alpha)},
\]

to simplify the development and notation.

**Lemma 1.** Let \( m, M, \beta, \kappa \in \mathbb{R} \) and \( u \) be an integrable function on \([0, \infty)\). Then for all \( x > 0, \alpha > 0, \rho > 0 \) and \( \eta \geq 0 \), we have

\[
\Lambda^{\rho, \beta, \eta, \kappa}(\alpha, \eta) \rho x^{\alpha, \beta, \eta, \kappa}(x) - (\rho x^{\alpha, \beta, \eta, \kappa}(u(x)))^2
\]

\[
= (M\Lambda^{\rho, \beta, \eta, \kappa}(\alpha, \eta) - \rho x^{\alpha, \beta, \eta, \kappa}(u(x))) \times (\rho x^{\alpha, \beta, \eta, \kappa}(u(x)) - m\Lambda^{\rho, \beta, \eta, \kappa}(\alpha, \eta))
\]

\[
-\Lambda^{\rho, \beta, \eta, \kappa}(\alpha, \eta) \rho x^{\alpha, \beta, \eta, \kappa}(M - u(x))(u(x) - m),
\]

with \( \Lambda^{\rho, \beta, \eta, \kappa}(\alpha, \eta) \) given by Eq. (3.1).
Proof. Let $m, M \in \mathbb{R}$ and $u$ be an integrable function on $[0, \infty)$. For all $\tau, \xi \in [0, \infty)$, we have
\[ (M - u(\xi))(u(\tau) - m) + (M - u(\tau))(u(\xi) - m) - (M - u(\tau))(u(\tau) - m) \]
\[ -(M - u(\xi))(u(\xi) - m) = u(\tau) + u(\xi) - 2u(\tau)u(\xi). \]

Multiplying both sides of Eq. (3.3) by $\frac{\rho^{1-\beta}x^\kappa}{\Gamma(\alpha)} \tau^{\rho(\eta+1)-1}$, where $\tau \in (0, x)$, $x > 0$, and integrating with respect to the variable $\tau$, from 0 to $x$, we obtain
\[ (M - u(\xi)) \left( \rho I_{\eta,\kappa}^{\alpha,\beta} u(x) - m \Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta) \right) + (u(\xi) - m) \]
\[ \times (M \Lambda_{x,\kappa}^{\alpha,\beta}(\alpha, \eta) - \rho I_{\eta,\kappa}^{\alpha,\beta} u(x)) - \left( \rho I_{\eta,\kappa}^{\alpha,\beta} (M - u(x))(u(x) - m) \right) \]
\[ -(M - u(\xi))(u(\xi) - m) \Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta) = \rho I_{\eta,\kappa}^{\alpha,\beta} u^2(x) \]
\[ + u^2(\xi) \Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta) - 2u(\xi)\rho I_{\eta,\kappa}^{\alpha,\beta} u(x). \]

Also, multiplying Eq. (3.4) by $\frac{\rho^{1-\beta}x^\kappa}{\Gamma(\alpha)} \xi^{\rho(\eta+1)-1}$, where $\xi \in (0, x)$ and $x > 0$, and integrating with respect to the variable $\xi$, from 0 to $x$, we get
\[ \left( \rho I_{\eta,\kappa}^{\alpha,\beta} u(x) - m \Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta) \right) \left( M I_{\eta,\kappa}^{\alpha,\beta} u(x) - m \Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta) \right) \]
\[ + (M \Lambda_{x,\kappa}^{\alpha,\beta}(\alpha, \eta) - \rho I_{\eta,\kappa}^{\alpha,\beta} u(x)) - \left( \rho I_{\eta,\kappa}^{\alpha,\beta} (M - u(x))(u(x) - m) \right) \]
\[ - \Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} u^2(x) + \Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} u^2(x) \]
\[ - 2\rho I_{\eta,\kappa}^{\alpha,\beta} u(x) \rho I_{\eta,\kappa}^{\alpha,\beta} u(x), \]
where we have introduced $\Lambda_{\rho,\beta}^{\alpha}(\alpha, \eta)$ in Eq. (3.5).

Rearranging Eq. (3.5), we immediately get, Eq. (3.2).

Note that, when $\eta = 0$, $\kappa = 0$ and $\rho \to 1$ in Eq. (3.2), we have
\[ \left( M x^\alpha \frac{x^\alpha}{\Gamma(\alpha + 1)} - I^\alpha u(x) \right) \left( I^\alpha u(x) - m x^\alpha \frac{x^\alpha}{\Gamma(\alpha + 1)} \right) \]
\[ - \left( \frac{x^\alpha}{\Gamma(\alpha + 1)} - I^\alpha (M - u(x))(u(x) - m) \right) \]
\[ = \frac{x^\alpha}{\Gamma(\alpha + 1)} - I^\alpha u^2(x) - (I^\alpha u(x))^2, \]
where
\[ \lim_{\rho \to 1} \Lambda_{\rho,\beta}^{\alpha}(\alpha, 0) = \Lambda_{x,0}^{\alpha}(\alpha, 0) = \frac{x^\alpha}{\Gamma(\alpha + 1)}. \]

This result, was obtained in [21], considering the Riemann-Liouville fractional integral.
Theorem 1. Let \( f \) and \( g \) be two integrable functions on \([0, \infty)\), satisfying the condition
\[
(3.6) \quad m \leq f(x) \leq M, \quad p \leq g(x) \leq P;
\]
where \( m, M, p, P \in \mathbb{R} \) and \( x \in [0, \infty) \).

Then, for all \( \beta, \kappa \in \mathbb{R} \), \( x > 0 \), \( \alpha > 0 \), \( \rho > 0 \) and \( \eta \geq 0 \), we have
\[
(3.7) \quad \left| \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha, \beta} f(x) - \rho I_{\eta,\kappa}^{\alpha, \beta} f(x) \rho I_{\eta,\kappa}^{\alpha, \beta} g(x) \right| \leq \left( \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \right)^2 (M - m)(P - p).
\]

Proof. Let \( f \) and \( g \) be two integrable functions satisfying the condition Eq.(3.6). Consider
\[
H(\tau, \xi) = (f(\tau) - f(\xi))(g(\tau) - g(\xi));// \tau, \xi \in (0, x) \quad \text{and} \quad x > 0,
\]
or, in the following form,
\[
(3.8) \quad H(\tau, \xi) = f(\tau)g(\tau) - f(\tau)g(\xi) - f(\xi)g(\tau) + f(\xi)g(\xi).
\]
Multiplying both sides of Eq.(3.8) by \( \frac{\rho^{1-\beta} x^\kappa \, \tau^{\rho(\eta+1)-1}}{\Gamma(\alpha)} \frac{(x^\rho - \tau^\rho)^{1-\alpha}}{\Gamma(\alpha)} \), where \( \tau \in (0, x) \), \( x > 0 \), and integrating with respect to the variable \( \tau \), from 0 to \( x \), we obtain
\[
\rho^{1-\beta} x^\kappa \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} H(\tau, \xi) d\tau = \rho I_{\eta,\kappa}^{\alpha, \beta} f(x) - g(\xi) \rho I_{\eta,\kappa}^{\alpha, \beta} f(x)
\]
\[
-f(\xi) \rho I_{\eta,\kappa}^{\alpha, \beta} g(x) + \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) f(\xi)g(\xi).
\]

Multiplying Eq.(3.9) by \( \frac{\rho^{2(1-\beta)} x^{2\kappa} \, \xi^{\rho(\eta+1)-1} \frac{x^\rho - \xi^\rho)^{1-\alpha}}{\Gamma(\alpha)^2}} \), where \( \xi \in (0, x) \), \( x > 0 \), and integrating with respect to the variable \( \xi \), from 0 to \( x \), we have
\[
\rho^{2(1-\beta)} x^{2\kappa} \int_0^x \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\alpha}} H(\tau, \xi) d\tau d\xi
\]
\[
= 2 \left( \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha, \beta} f(x) - \rho I_{\eta,\kappa}^{\alpha, \beta} f(x) \rho I_{\eta,\kappa}^{\alpha, \beta} g(x) \right).
\]

Applying the Cauchy-Schwarz inequality associated with double integrals, [27], we can write
\[
(3.10) \quad \left| \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha, \beta} f(x) - \rho I_{\eta,\kappa}^{\alpha, \beta} f(x) \rho I_{\eta,\kappa}^{\alpha, \beta} g(x) \right| \leq \left( \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \right)^2 \left( \rho I_{\eta,\kappa}^{\alpha, \beta} f^2(x) - (\rho I_{\eta,\kappa}^{\alpha, \beta} f(x))^2 \right)
\]

\[
\times \left( \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha, \beta} g^2(x) - (\rho I_{\eta,\kappa}^{\alpha, \beta} g(x))^2 \right).
\]

Since \((M - f(x))(f(x) - m) \geq 0\) and \((P - g(x))(g(x) - P) \geq 0\), we have
\[
(3.11) \quad \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha, \beta} (M - f(x))(f(x) - m) \geq 0
\]
and
\[
(3.12) \quad \Lambda_{x,\kappa}^{\beta, \alpha} (\alpha, \eta) \rho I_{\eta,\kappa}^{\alpha, \beta} (P - g(x))(g(x) - P) \geq 0.
\]
Thus,
\[
(A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho^2 I_{\eta,k}^{\alpha,\beta} f^2 (x) - (\rho^2 I_{\eta,k}^{\alpha,\beta} f(x))^2) \leq (MA_{x,k}^{\rho,\beta} (\alpha, \eta) - \rho^2 I_{\eta,k}^{\alpha,\beta} f(x)) \times (\rho^2 I_{\eta,k}^{\alpha,\beta} (\alpha, \eta))
\]
(3.13)
and
\[
(A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho^2 I_{\eta,k}^{\alpha,\beta} g^2 (x) - (\rho^2 I_{\eta,k}^{\alpha,\beta} g(x))^2) \leq (PA_{x,k}^{\rho,\beta} (\alpha, \eta) - \rho^2 I_{\eta,k}^{\alpha,\beta} g(x)) \times (\rho^2 I_{\eta,k}^{\alpha,\beta} (\alpha, \eta))
\]
(3.14)
Combining Eq.(3.10), Eq.(3.13) and Eq.(3.14), using Lemma 1, we conclude that
\[
(A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho^2 I_{\eta,k}^{\alpha,\beta} f g(x) - \rho^2 I_{\eta,k}^{\alpha,\beta} f(x) \rho^2 I_{\eta,k}^{\alpha,\beta} g(x))^2 \leq (MA_{x,k}^{\rho,\beta} (\alpha, \eta) - \rho^2 I_{\eta,k}^{\alpha,\beta} f(x)) \times (\rho^2 I_{\eta,k}^{\alpha,\beta} (\alpha, \eta))
\]
(3.15)
Further, using the inequality \(4ab \leq (a + b)^2, a, b \in \mathbb{R}\), we have
\[
4 (MA_{x,k}^{\rho,\beta} (\alpha, \eta) - \rho^2 I_{\eta,k}^{\alpha,\beta} f(x)) (\rho^2 I_{\eta,k}^{\alpha,\beta} f(x) - mA_{x,k}^{\rho,\beta} (\alpha, \eta)) \leq (A_{x,k}^{\rho,\beta} (\alpha, \eta) (M - m)^2)
\]
(3.16)
and
\[
4 (PA_{x,k}^{\rho,\beta} (\alpha, \eta) - \rho^2 I_{\eta,k}^{\alpha,\beta} g(x)) (\rho^2 I_{\eta,k}^{\alpha,\beta} g(x) - pA_{x,k}^{\rho,\beta} (\alpha, \eta)) \leq (A_{x,k}^{\rho,\beta} (\alpha, \eta) (P - P)^2)
\]
(3.17)
Finally, from Eq.(3.15), Eq.(3.16) and Eq.(3.17), we obtain Eq.(3.7). 

Applying Theorem 1 for \(\alpha = 1, \rho \to 1, \eta = 0\) and \(\kappa = 0\), we obtain the classical inequality of Gruss-type, Eq.(1.1). On the other hand, for \(\rho \to 1, \eta = 0\) and \(\kappa = 0\), we recover the Theorem 3.1 in [21].

To prove the next theorem, which generalizes the inequalities of Gruss-type, we need the following lemmas, in which we have introduced the notation as in Eq.(3.1) with \(\alpha = \gamma\), i.e., \(A_{x,k}^{\rho,\beta} (\gamma, \eta)\).

**Lemma 2.** Let \(f\) and \(g\) be two integrable functions on \([0, \infty)\). Then for all \(\beta, \kappa \in \mathbb{R}, x > 0, \alpha > 0, \rho > 0, \eta \geq 0\) and \(\gamma > 0\), we have
\[
(A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,k}^{\alpha,\beta} f g(x) + A_{x,k}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,k}^{\alpha,\beta} f g(x))
\]
\[
\rho I_{\eta,k}^{\alpha,\beta} f(x) \rho I_{\eta,k}^{\alpha,\beta} g(x) - \rho I_{\eta,k}^{\alpha,\beta} f(x) \rho I_{\eta,k}^{\alpha,\beta} g(x))^2 \leq (A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,k}^{\alpha,\beta} f^2 (x) + A_{x,k}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,k}^{\alpha,\beta} f^2 (x))
\]
\[
-2 \rho I_{\eta,k}^{\alpha,\beta} f(x) \rho I_{\eta,k}^{\alpha,\beta} g(x)) \times (A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,k}^{\alpha,\beta} g^2 (x))
\]
(3.18)
Proof. Multiplying both sides of Eq. (3.9) by \(\frac{\rho^{1-\beta} x^\kappa}{\Gamma(\gamma)} \frac{\xi^{\rho(\gamma+1)-1}}{(x^\rho - \xi^\rho)^{1-\gamma}}\), where \(\xi \in (0, x), x > 0\), and integrating with respect to the variable \(\xi\), from 0 to \(x\), we obtain

\[
\frac{\rho^{2(1-\beta)} x^{2\kappa}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^x \int_0^x \frac{\tau^{\rho(\gamma+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \frac{\xi^{\rho(\gamma+1)-1}}{(x^\rho - \xi^\rho)^{1-\gamma}} H(\tau, \xi) d\tau d\xi
= \Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} f g(x) + \Lambda_{x,\kappa}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,\kappa}^{\gamma,\beta} f g(x)
- \rho I_{\eta,\kappa}^{\alpha,\beta} f(x) \rho I_{\eta,\kappa}^{\gamma,\beta} f(x) - \rho I_{\eta,\kappa}^{\alpha,\beta} g(x) \rho I_{\eta,\kappa}^{\gamma,\beta} g(x).
\]

Using the Cauchy-Schwarz inequality associated with double integrals, we have

\[
\left| \Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} f g(x) + \Lambda_{x,\kappa}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,\kappa}^{\gamma,\beta} f g(x)
- \rho I_{\eta,\kappa}^{\alpha,\beta} f(x) \rho I_{\eta,\kappa}^{\gamma,\beta} f(x) - \rho I_{\eta,\kappa}^{\alpha,\beta} g(x) \rho I_{\eta,\kappa}^{\gamma,\beta} g(x) \right|^2
\leq \left( \Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} f^2(x) + \Lambda_{x,\kappa}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,\kappa}^{\gamma,\beta} f^2(x)
- 2 \rho I_{\eta,\kappa}^{\alpha,\beta} f(x) \rho I_{\eta,\kappa}^{\gamma,\beta} f(x) \right) \left( \Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} g^2(x)
- 2 \rho I_{\eta,\kappa}^{\alpha,\beta} g(x) \rho I_{\eta,\kappa}^{\gamma,\beta} g(x) \right),
\]

which is Eq. (3.18). \(\square\)

Lemma 3. Let \(u\) be an integrable function on \([0, \infty)\) satisfying Eq. (3.6) on \([0, \infty)\). Then for all \(\beta, \kappa \in \mathbb{R}, x > 0, \alpha > 0, \rho > 0, \eta \geq 0\) and \(\gamma > 0\), we have

\[
\Lambda_{x,\kappa}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,\kappa}^{\gamma,\beta} u^2(x) + \Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} u^2(x)
- 2 \rho I_{\eta,\kappa}^{\alpha,\beta} u(x) \rho I_{\eta,\kappa}^{\gamma,\beta} u(x) = \left( \Lambda_{x,\kappa}^{\rho,\beta} (\alpha, \eta) - \rho I_{\eta,\kappa}^{\alpha,\beta} u(x) \right)
\times \left( \rho I_{\eta,\kappa}^{\gamma,\beta} u(x) - m\Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) \right) + \left( \Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) - \rho I_{\eta,\kappa}^{\gamma,\beta} u(x) \right)
\times \left( \rho I_{\eta,\kappa}^{\alpha,\beta} u(x) - m\Lambda_{x,\kappa}^{\rho,\beta} (\alpha, \eta) \right) - \Lambda_{x,\kappa}^{\rho,\beta} (\alpha, \eta) \rho I_{\eta,\kappa}^{\gamma,\beta} (M - u(x))(u(x) - m),
\]

(3.19)

\[-\Lambda_{x,\kappa}^{\rho,\beta} (\gamma, \eta) \rho I_{\eta,\kappa}^{\alpha,\beta} (M - u(x))(u(x) - m).\]
Proof. Multiplying both sides of Eq. (3.4) by \( \rho^{1-\beta} x^{\kappa} \frac{\xi^{\rho(\eta+1)-1}}{\Gamma(\gamma)} \), where \( \xi \in (0, x), \ x > 0 \), and integrating with respect to variable \( \xi \), from 0 to \( x \) we obtain

\[
\left( \rho \gamma_{\eta, \kappa} u(x) - m \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) \right) \left( \frac{\rho^{1-\beta} x^{\kappa}}{\Gamma(\gamma)} \int_0^x \frac{\xi^{\rho(\eta+1)-1}}{\Gamma(\gamma)} (M - u(\xi)) d\xi \right) \\
+ \left( M \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) - \rho \gamma_{\eta, \kappa} u(x) \right) \left( \frac{\rho^{1-\beta} x^{\kappa}}{\Gamma(\gamma)} \int_0^x \frac{\xi^{\rho(\eta+1)-1}}{\Gamma(\gamma)} (u(\xi) - m) d\xi \right) \\
- \left( \rho \gamma_{\eta, \kappa} (M - u(x)) u(x) - m \right) \rho^{1-\beta} x^{\kappa} \int_0^x \frac{\xi^{\rho(\eta+1)-1}}{\Gamma(\gamma)} (u(\xi) - m) d\xi \\
\Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) \rho^{1-\beta} x^{\kappa} \int_0^x \frac{\xi^{\rho(\eta+1)-1}}{\Gamma(\gamma)} (M - u(\xi)) (u(\xi) - m) d\xi \\
+ A x_{,k}^{\rho,\beta} (\alpha, \eta) \rho^{1-\beta} x^{\kappa} \int_0^x \frac{\xi^{\rho(\eta+1)-1}}{\Gamma(\gamma)} u^2(\xi) d\xi \\
- 2 \rho \gamma_{\eta, \kappa} u(x) \rho^{1-\beta} x^{\kappa} \int_0^x \frac{\xi^{\rho(\eta+1)-1}}{\Gamma(\gamma)} u(\xi) d\xi.
\]

From this last expression, follows immediately, Eq. (3.19).

Considering \( \eta = 0, \ \kappa = 0 \) \( \rho \to 1 \), in Lemma 2 and Lemma 3, we obtain the results of Lemma 3.4 and Lemma 3.5 in [21].

To prove the next theorem, we use Lemma 2 and Lemma 3, previously proved.

**Theorem 2.** Let \( f \) and \( g \) be two integrable functions on \([0, \infty)\) satisfying the condition Eq. (3.6) on \([0, \infty)\). Then for all \( \beta, \kappa \in \mathbb{R}, \ x > 0, \ \alpha > 0, \ \gamma > 0 \) and \( \eta \geq 0 \), we have

\[
\begin{align*}
&\left( \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) \right) \rho \gamma_{\eta, \kappa} f g(x) + \Lambda x_{,k}^{\rho,\beta} (\gamma, \eta) \rho \gamma_{\eta, \kappa} f g(x) \\
&- \rho \gamma_{\eta, \kappa} f (x) \rho \gamma_{\eta, \kappa} g(x) - \rho \gamma_{\eta, \kappa} f (x) \rho \gamma_{\eta, \kappa} g(x) \bigg) \\
&\leq \left[ \left( M \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) - \rho \gamma_{\eta, \kappa} f (x) \right) \left( \rho \gamma_{\eta, \kappa} f (x) - m \Lambda x_{,k}^{\rho,\beta} (\gamma, \eta) \right) \\
&+ \left( \rho \gamma_{\eta, \kappa} f (x) - m \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) \right) \left( M \Lambda x_{,k}^{\rho,\beta} (\gamma, \eta) - \rho \gamma_{\eta, \kappa} f (x) \right) \right] \\
&\times \left[ \left( M \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) - p \Lambda x_{,k}^{\rho,\beta} (\gamma, \eta) \right) \left( \rho \gamma_{\eta, \kappa} g (x) - p \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) \right) \\
&+ \left( p \Lambda x_{,k}^{\rho,\beta} (x) - p \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) \right) \left( \rho \gamma_{\eta, \kappa} g (x) - \Lambda x_{,k}^{\rho,\beta} (\gamma, \eta) \right) \right].
\end{align*}
\]

Proof. Since \((M - f(x))(f(x) - m) \geq 0, (P - g(x))(g(x) - p) \geq 0, x > 0 \) and \( \rho > 0 \), we can write

\[
\begin{align*}
&- \Lambda x_{,k}^{\rho,\beta} (\alpha, \eta) \rho \gamma_{\eta, \kappa} (M - f(x))(f(x) - m) \\
&- \Lambda x_{,k}^{\rho,\beta} (\gamma, \eta) \rho \gamma_{\eta, \kappa} (M - f(x))(f(x) - m) \leq 0
\end{align*}
\]
and

\[-\Lambda^{\rho,\beta}_{x,\kappa}(\alpha, \eta) \rho I^{\gamma,\beta}_{\eta,\kappa}(P - g(x))(g(x) - p)\]
\(-\Lambda^{\rho,\beta}_{x,\kappa}(\gamma, \eta) \rho I^{\alpha,\beta}_{\eta,\kappa}(P - g(x))(g(x) - p) \leq 0.\] (3.22)

Applying Lemma 3 for \(f\) and \(g\), using Eq.(3.21) and Eq.(3.22), we obtain

\[(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha, \eta) \rho I^{\gamma,\beta}_{\eta,\kappa} f(x) + \Lambda^{\rho,\beta}_{x,\kappa}(\gamma, \eta) \rho I^{\alpha,\beta}_{\eta,\kappa} f(x)) \leq (M\Lambda^{\rho,\beta}_{x,\kappa}(\alpha, \eta) - \rho I^{\alpha,\beta}_{\eta,\kappa} f(x)) \times (\Lambda^{\rho,\beta}_{x,\kappa}(\gamma, \eta) - \rho I^{\gamma,\beta}_{\eta,\kappa} f(x))\] (3.23)

and

\[(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha, \eta) \rho I^{\gamma,\beta}_{\eta,\kappa} g(x) + \Lambda^{\rho,\beta}_{x,\kappa}(\gamma, \eta) \rho I^{\alpha,\beta}_{\eta,\kappa} g(x)) \leq (P\Lambda^{\rho,\beta}_{x,\kappa}(\alpha, \eta) - \rho I^{\alpha,\beta}_{\eta,\kappa} g(x)) \times (\Lambda^{\rho,\beta}_{x,\kappa}(\gamma, \eta) - \rho I^{\gamma,\beta}_{\eta,\kappa} g(x))\] (3.24)

Considering the product Eq.(3.23) by Eq.(3.24), using Lemma 2, follows immediately, Eq.(3.20).

Taking \(\alpha = \gamma\) in Theorem 2, we obtain Theorem 1. On the other hand, considering \(\eta = 0, \kappa = 0\) and \(\rho \to 1\), in Theorem 2, we obtain Theorem 3.3 in [21]. Considering Theorem 2 with \(\alpha = \gamma = 1, \eta = 0, \kappa = 0\) and \(\rho \to 1\), we recover Eq.(1.1).

We mention that, the results obtained in Lemmas 1, 2, 3 and Theorems 1 and 2 can be proved, considering the integral in Eq.(2.3), from \(a = 1\) to \(x\), in order to obtain, as a particular cases, the generalized inequalities of Gruss-type, discussed in [22], with the Hadamard fractional integral.

4. OTHER FRACTIONAL INTEGRAL INEQUALITIES

In this section, we present some integral inequalities involving the Katugampola’s fractional operator. The results obtained were adapted from the paper Chinchane & Pachpatte [22], in which makes a brief approach with respect inequalities of Gruss-type, but in Hadamard sense.

A priori, it should be emphasize that, although the following results are adapted for the Katugampola’s operator, the left-sided, Eq.(2.3), and that with convenient condition on the parameters of the fractional operator it is possible to obtain the Hadamard’s operator. In this sense, the results presented here are, in fact, true for the Hadamard’s operator when we admit \(a = 1\) in Eq.(2.1).

In this way, we discuss some theorems involving fractional integrals inequalities.
Theorem 3. Let $\alpha > 0$, $\beta, \rho, \eta, \kappa \in \mathbb{R}$ and $f, g \in X^p(0, x)$ be two positive functions defined on $[0, \infty)$, $x > 0$ and $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequalities hold:

1. $\frac{\rho T_{\eta, \kappa}^{\alpha, \beta} f^p (x)}{p} + \frac{\rho T_{\eta, \kappa}^{\alpha, \beta} g^q (x)}{q} \geq \frac{1}{\Gamma (\eta + 1)} (\rho T_{\eta, \kappa}^{\alpha, \beta} f (x) T_{\eta, \kappa}^{\alpha, \beta} g (x)).$

2. $\frac{\rho T_{\eta, \kappa}^{\alpha, \beta} f^p (x)}{p} \frac{\rho T_{\eta, \kappa}^{\alpha, \beta} f^q (x)}{q} + \frac{\rho T_{\eta, \kappa}^{\alpha, \beta} g^q (x)}{q} \geq (\rho T_{\eta, \kappa}^{\alpha, \beta} f (x) g (x))^2.$

3. $\frac{\rho T_{\eta, \kappa}^{\alpha, \beta} f^p (x)}{p} \frac{\rho T_{\eta, \kappa}^{\alpha, \beta} g^q (x)}{q} + \frac{\rho T_{\eta, \kappa}^{\alpha, \beta} g^q (x)}{q} \geq (\rho T_{\eta, \kappa}^{\alpha, \beta} f^p (x) g^q (x)) (\rho T_{\eta, \kappa}^{\alpha, \beta} f^q (x) g^q (x)).$

4. $\frac{\rho T_{\eta, \kappa}^{\alpha, \beta} f^p (x)}{p} \frac{\rho T_{\eta, \kappa}^{\alpha, \beta} f^q (x)}{q} = (\rho T_{\eta, \kappa}^{\alpha, \beta} f^p (x) g^q (x)).$

Proof. 1. Considering Young inequality [28],

$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \forall a, b \geq 0, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1,$

and putting $a = f(t)$ and $b = f(s)$, $s > 0$, in inequality Eq.(4.1), we have

$\int f^p (t) + \int g^q (s) \geq f (t) g (s), \forall f (t) g (s) \geq 0.$

Multiplying by $\frac{\rho^{1-\beta} x^{\kappa (\eta + 1) - 1}}{p \Gamma (\alpha)}$ both sides of Eq.(4.2), and integrating with respect to the variable $t$ on $(0, x)$, $x > 0$, we have

$\frac{\rho^{1-\beta} x^{\kappa}}{p \Gamma (\alpha)} \int_0^x t^{\rho (\eta + 1) - 1} f^p (t) dt + \frac{\rho^q (s)}{q \Gamma (\alpha)} \int_0^x t^{\rho (\eta + 1) - 1} f (t) dt,$

which can be rewritten as follows,

$\int_0^x t^{\rho (\eta + 1) - 1} f^p (t) dt \geq g (s) \int_0^x t^{\rho (\eta + 1) - 1} f (t) dt,$

Further with the variable change $u = \frac{x^\rho}{t^\rho}$ in the integral $\int_0^1 \frac{t^{\rho (\eta + 1) - 1}}{(x^\rho - t^\rho)} dt$, we obtain

$\int_0^x \frac{t^{\rho (\eta + 1) - 1}}{(x^\rho - t^\rho)} dt = \int_0^1 \frac{u^{\rho (\eta + 1) - 1}}{(1 - u)} du = \frac{x^{\rho (\eta + 1)}}{\rho} B (\eta + 1, \alpha),$

where $B (a, b)$ is the Beta function. Using the following identity $B (a, b) = \frac{\Gamma (a) \Gamma (b)}{\Gamma (a + b)}$ in Eq.(4.5) and replacing the result in Eq.(4.4), we have

$\frac{\rho T_{\eta, \kappa}^{\alpha, \beta} f^p (x)}{p} + \frac{\rho T_{\eta, \kappa}^{\alpha, \beta} g^q (x)}{q \rho^q} \int_0^x \frac{t^{\rho (\eta + 1) - 1}}{(x^\rho - t^\rho)} dt = \frac{x^{\rho (\eta + 1)}}{\rho} B (\eta + 1, \alpha),$
Multiplying by $\frac{\rho^{1-\beta}s^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - s^\rho)^{1-\alpha}}$ both sides of Eq. (4.6) and integrating with respect to the variable $s$ on $(0, x)$, $x > 0$, we have

$$
\frac{\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}f^p(x)}{p\Gamma(\alpha)} \rho^{1-\beta} x^\kappa \int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} ds + \frac{\Gamma(\eta+1)x^{\rho(\eta+\alpha)+2\kappa} \rho^{1-\beta}}{\Gamma(\alpha)(\eta + \alpha + 1)q\rho^3} \int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} g^q(s) ds
$$

which can be rewritten as follows,

$$
\left(\int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} ds\right) \left(\int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} ds\right) + \frac{\Gamma(\eta+1)x^{\rho(\eta+\alpha)+\kappa}}{q\rho^2 \Gamma(\eta + \alpha + 1)} \left(\frac{\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}f^p(x)}{p\Gamma(\alpha)} \rho^{1-\beta} x^\kappa \int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} ds \right)
$$

(4.7) \quad \geq \left(\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}f^p(x) \mathcal{T}_{\eta,k}^{\alpha,\beta}g^q(x)\right).

Further with the variable change $u = \frac{s^\rho}{x^\rho}$ in integral $\int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} ds$, we obtain

$$
\int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} ds = \frac{\Gamma(\eta+1)\Gamma(\alpha)}{\rho\Gamma(\eta + \alpha + 1)}.
$$

(4.8)

Replacing Eq. (4.8) in Eq. (4.7), we conclude that

$$
\frac{\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}f^p(x)}{p} + \frac{\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}g^q(x)}{q} \geq \frac{\Gamma(\eta+1)x^{\rho(\eta+\alpha)+\kappa}}{\rho\Gamma(\eta + \alpha + 1)} \left(\frac{\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}f^p(x)}{p\Gamma(\alpha)} \rho^{1-\beta} x^\kappa \int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} ds \right).
$$

2. For the proof item (2), we take $a = f(t)g(s)$ and $b = f(s)g(t)$ and replacing in Young inequality following the same as in item (1).

3. To prove item (3), we take $a = \frac{f(t)}{g(t)}$ and $b = \frac{g(s)}{f(s)}$, then replacing in Young inequality, in the same way as in item (1).

4. Putting $a = \frac{f(s)}{f(t)}$ and $b = \frac{g(s)}{g(t)}$, $f(t), g(t) \neq 0$, and replacing in the Eq. (4.1), we get

$$
\frac{fp(t)}{p} + \frac{gq(s)}{q} \geq f(t)g(s), \quad \forall f(t)g(s) \geq 0.
$$

(4.9)

Multiplying by $\frac{\rho^{1-\beta}x^\kappa t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^\rho - t^\rho)^{1-\alpha}}$ both sides of Eq. (4.9), and integrating with respect to the variable $t$ on $(0, x)$, $x > 0$, we have

$$
fp(s)\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}f^{p-1}(x)g^{q-1}(x) \geq f(s)g(s)\rho\mathcal{T}_{\eta,k}^{\alpha,\beta}f^{p-1}(x)g^{q-1}(x).
$$

(4.10)
Again multiplying by \( \frac{\rho^{1-\beta} x^\kappa s^{\rho(\eta+1)-1}}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}} \) both sides of Eq. (4.10), and integrating with respect to the variable \( s \) on \((0, x), x > 0\), we get

\[
\frac{\rho \mathcal{T}_{\eta, \kappa}^\alpha g^q(x) \rho^{1-\beta} x^\kappa}{p \Gamma (\alpha)} \int_0^x \frac{s^{\rho(q+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} f^p(s) \, ds + \frac{\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^p(x) \rho^{1-\beta} x^\kappa}{q \Gamma (\alpha)} \int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} g^q(s) \, ds
\]

\[
\geq (4 \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^{p-1}(x) g^{q-1}(x)) \frac{\rho^{1-\beta} x^\kappa}{\Gamma (\alpha)} \int_0^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho - s^\rho)^{1-\alpha}} f(s) g(s) \, ds.
\]

Using the identity \( \frac{1}{p} + \frac{1}{q} = 1 \) in Eq. (4.11), we conclude that

\[
\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^q(x) \rho^{1-\beta} x^\kappa \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^p(x) \geq (\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^p(x) g(x)) (\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^{p-1}(x) g^{q-1}(x)),
\]

which is the item (4).

**Theorem 4.** Let \( \alpha > 0 \), \( \beta, \rho, \eta, \kappa \in \mathbb{R} \) and \( f, g \in X_\rho^p(0, x) \) be two positive functions on \([0, \infty), x > 0\) and \( p, q > 1 \) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, follow the inequalities:

1. \[
\frac{\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^p(x) \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^q(x)}{p} + \frac{\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f(x) \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^q(x)}{q} \geq (\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^p(x) g(x)) \left( \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^{p-1}(x) g^{q-1}(x) \right).
\]
2. \[
\frac{\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^q(x) \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^p(x)}{p} + \frac{\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f(x) \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^p(x)}{q} \geq (\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^q(x) g(x)) \left( \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^{q-1}(x) g^{p-1}(x) \right).
\]
3. \[
\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^2(x) \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} \left( \frac{g^p(x)}{p} \right) + \frac{g^q(x)}{q} \geq (\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^2(x) g(x)) \left( \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^{q-1}(x) g^{p-1}(x) \right).
\]

**Proof.**

1. Taking \( a = f(t) g^\frac{q}{p} (s) \) and \( b = f^\frac{p}{q} (s) g(t) \) and replacing in Eq. (4.11), we have

\[
(4.12) \quad \frac{f^p(t) g^2(s)}{p} + \frac{g^q(s) f(s)}{q} \geq f(t) g(s) f^\frac{p}{q} (t) g^\frac{2}{q} (s).
\]

Multiplying by \( \frac{\rho^{1-\beta} x^\kappa s^{\rho(\eta+1)-1}}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}} \) both sides of Eq. (4.12), and integrating with respect to the variable \( t \) on \((0, x), x > 0\), we have

\[
(4.13) \quad \frac{g^2(s)}{p} \left( \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^{p}(x) \right) + \frac{f^2(s)}{q} \left( \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^{q}(x) \right) \leq g^\frac{2}{p} (s) f^\frac{2}{q} (s) \left( \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f(x) g(x) \right).
\]

Again multiplying by \( \frac{\rho^{1-\beta} x^\kappa s^{\rho(\eta+1)-1}}{\Gamma (\alpha) (x^\rho - s^\rho)^{1-\alpha}} \) both sides of Eq. (4.13), and integrating with respect to the variable \( s \) on \((0, x), x > 0\), we conclude that

\[
\frac{\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^p(x) \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^q(x)}{p} + \frac{\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} g^p(x) \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f^q(x)}{q} \geq (\rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} f(x) g(x)) \left( \rho \mathcal{T}_{\eta, \kappa}^{\alpha, \beta} \frac{g^p}{p} (x) f^\frac{q}{p} (x) \right).\]
2. Taking \( a = \frac{\int_0^x f(t) \, dt}{f(x)} \) and \( b = \frac{\int_0^x g(t) \, dt}{g(x)} \) with \( f(s), g(s) \neq 0 \), and replacing in Eq. (4.11), we have
\[
\frac{f^2(t)}{pf^p(s)} + \frac{g^2(t)}{pg^q(s)} > \frac{f^\frac{2}{p}(t) g^\frac{2}{q}(t)}{f(s) g(s)},
\]
which can be rewritten as follows,
\[
\frac{f^2(t)}{p} g^\frac{2}{q}(s) + \frac{g^2(t)}{q} f^p(s) \geq f^{p-1}(s) g^{q-1}(s) f^\frac{2}{p}(t) g^\frac{2}{q}(t).
\] (4.14)

Again multiplying by \( \frac{\rho^{1-\beta} x^\kappa t^{\rho (\eta + 1)-1}}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}} \) both sides of Eq. (4.14), and integrating with respect to the variable \( t \) on \((0, x), x > 0\), we have
\[
\frac{g^q(s)}{p \Gamma (\alpha)} \rho^{1-\beta} x^\kappa \int_0^x t^{\rho (\eta + 1)-1} (x^\rho - t^\rho)^{1-\alpha} f^2(t) \, dt + \frac{f^p(s)}{q \Gamma (\alpha)} \rho^{1-\beta} x^\kappa \int_0^x t^{\rho (\eta + 1)-1} (x^\rho - t^\rho)^{1-\alpha} g^2(t) \, dt
\geq
\frac{f^{p-1}(s) g^{q-1}(s)}{\Gamma (\alpha)} \rho^{1-\beta} x^\kappa \int_0^x t^{\rho (\eta + 1)-1} (x^\rho - t^\rho)^{1-\alpha} f^\frac{2}{p}(t) g^\frac{2}{q}(t) \, dt,
\]
or in the following form
\[
\frac{g^q(s)}{p} (\rho T_{\eta, \kappa}^\alpha f^2 (x)) + \frac{f^p(s)}{q} (\rho T_{\eta, \kappa}^\alpha g^2 (x)) \geq f^{p-1}(s) g^{q-1}(s) (\rho T_{\eta, \kappa}^\alpha f^\frac{2}{p} (x) g^\frac{2}{q} (x)).
\] (4.15)

Multiplying by \( \frac{\rho^{1-\beta} x^\kappa t^{\rho (\eta + 1)-1}}{\Gamma (\alpha) (x^\rho - t^\rho)^{1-\alpha}} \) both sides of Eq. (4.15), and integrating with respect to the variable \( s \) on \((0, x), x > 0\), we conclude that
\[
\frac{(\rho T_{\eta, \kappa}^\alpha f^2 (x))}{p} (\rho T_{\eta, \kappa}^\alpha g^2 (x)) + \frac{(\rho T_{\eta, \kappa}^\alpha g^2 (x))}{q} (\rho T_{\eta, \kappa}^\alpha f^\frac{2}{p} (x) g^\frac{2}{q} (x))
\geq
(\rho T_{\eta, \kappa}^\alpha f^\frac{2}{p} (x) g^\frac{2}{q} (x)) (\rho T_{\eta, \kappa}^\alpha f^{p-1} (x) g^{q-1} (x)).
\]

3. To prove item (3), we consider \( a = \frac{\int_0^x f(t) \, dt}{f(x)} \) and \( b = \frac{\int_0^x g(t) \, dt}{g(x)} \) with \( g(s), g(t) \neq 0 \) and replace in the Young inequality, in the same way as in item (2).

\( \square \)

**Theorem 5.** Let \( \alpha > 0, \beta, \rho, \eta, \kappa \in \mathbb{R} \) and \( f, g \in X^p(0, x) \) be two positive functions on \([0, \infty), x > 0 \) and \( p, q > 1 \) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). Let
\[
m = \min_{0 \leq t \leq x} \frac{f(t)}{g(t)} \quad \text{and} \quad M = \max_{0 \leq t \leq x} \frac{f(t)}{g(t)}.
\]
Then, follow the inequalities:

1. \( 0 \leq (\rho T_{\eta, \kappa}^\alpha f^2 (x) \rho T_{\eta, \kappa}^\alpha g^2 (x)) \leq \frac{(M + m)^2}{4Mm} (\rho T_{\eta, \kappa}^\alpha (fg) (x))^2 \)
2. \(0 \leq \sqrt{\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x) \rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x)} - (\rho T_{\eta,\kappa}^{\alpha,\beta} (fg) (x)) \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{Mm}} (\rho T_{\eta,\kappa}^{\alpha,\beta} (fg) (x))\)

3. \(0 \leq \rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x) \rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x) - (\rho T_{\eta,\kappa}^{\alpha,\beta} (fg) (x))^2 \leq \frac{(M - m)^2}{4Mm} (\rho T_{\eta,\kappa}^{\alpha,\beta} (fg) (x))^2\)

Proof.

1. From Eq. (4.16) and the inequality

\[
(f (t) - m) \left( \frac{f (t)}{g (t)} - m \right) \left( \frac{M - f (t)}{g (t)} \right) g^2 (t) \geq 0, 0 \leq t \leq x,
\]

we can write as

\[
(f (t) - mg (t)) (Mg (t) - f (t)) \geq 0,
\]

obtaining the expression,

\[
(M + m) f (t) g (t) \geq f^2 (t) + mM g^2 (t).
\]

Multiplying by \(\frac{\rho^{1-\beta} e^{-\rho \eta}}{\Gamma (\alpha) (\rho^\eta t^\rho)^{1-\alpha}}\) both sides of Eq. (4.18), and integrating with respect to the variable \(t\) on \((0, x), x > 0\), we have

\[
(M + m)^\rho T_{\eta,\kappa}^{\alpha,\beta} f (x) g (x) \geq (\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x)) + mM (\rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x)) .
\]

On the other hand, it follows from \(Mm > 0\) and

\[
\sqrt{\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x)} - \sqrt{mM (\rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x))} \geq 0,
\]

that

\[
\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x) + mM \rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x) \geq 2 \sqrt{\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x)} \sqrt{mM (\rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x))} .
\]

From Eq. (4.19) and Eq. (4.21), we get

\[
2 \sqrt{\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x)} \sqrt{mM (\rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x))} \leq (M + m) (\rho T_{\eta,\kappa}^{\alpha,\beta} f (x) g (x)) .
\]

Thus, we conclude that

\[
(\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x)) (\rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x)) \leq \frac{(M + m)^2}{4} (\rho T_{\eta,\kappa}^{\alpha,\beta} f (x) g (x))^2 .
\]

2. From Eq. (4.22) we have

\[
\sqrt{\rho T_{\eta,\kappa}^{\alpha,\beta} f^2 (x)} (\rho T_{\eta,\kappa}^{\alpha,\beta} g^2 (x)) \leq \frac{(M + m)}{2\sqrt{mM}} (\rho T_{\eta,\kappa}^{\alpha,\beta} f (x) g (x)) .
\]
Subtracting $\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f(x)g(x)$ in both sides of Eq. (4.23), we get
\[
\sqrt{\left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f^2(x)\right) \left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} g^2(x)\right)} - \left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f(x)g(x)\right) \leq \frac{(M + m)}{2\sqrt{mM}} \left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f(x)g(x)\right).
\]
Thus, we conclude that
\[
\sqrt{\left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f^2(x)\right) \left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} g^2(x)\right)} - \left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f(x)g(x)\right) \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f(x)g(x)\right).
\]
(3) Subtracting $(\rho \mathcal{I}^{\alpha,\beta}_{\eta,\kappa} f(x)g(x))^2$ of the Eq. (4.22) and with the same procedure as in item (2), we conclude the proof. \qed

**Concluding remarks**

From the fractional integral unifying six existing fractional integrals, as proposed by Katugampola, it was possible to generalize inequalities of Gruss-type obtaining, as a particular case, the well-known inequality involving Riemann-Liouville fractional integral. In this sense, we also proved other inequalities using the Katugampola’s fractional integral. A natural continuation of this paper, with this formulation, consists into generalize inequalities of Hermite-Hadamard and Hermite-Hadamard-Fejér [17], as well as to propose inequalities using the so-called $M$-fractional integral recently introduced [29].

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