A new approach to the vacuum of inflationary models

Shih-Hung Chen and James B Dent

Department of Physics and School of Earth and Space Exploration, Arizona State University, Tempe, AZ 85287-1404, USA

E-mail: schen102@asu.edu and jbdent@asu.edu

Received 22 May 2011, in final form 3 February 2012
Published 22 March 2012
Online at stacks.iop.org/CQG/29/085002

Abstract

A new approach is given for the implementation of boundary conditions used in solving the Mukhanov–Sasaki equation in the context of inflation. The familiar quantization procedure is reviewed, along with a discussion of where one might expect deviations from the standard approach to arise. The proposed method introduces a (model-dependent) fitting function for the $z''/z$ and $a''/a$ terms in the Mukhanov–Sasaki equation for scalar and tensor modes, as well as imposes the boundary conditions at a finite conformal time. As an example, we employ a fitting function and compute the spectral index, along with its running, for a specific inflationary model, which possesses background equations that are analytically solvable. The observational upper bound on the tensor to scalar ratio is used to constrain the parameters of the boundary conditions in the tensor sector as well. An overview on the generalization of this method is also discussed.

PACS number: 98.80.Cq

(Some figures may appear in colour only in the online journal)

1. Introduction

As is well known, the inflationary paradigm provides solutions to many cosmological problems, such as the flatness problem, the horizon or causality problem, and also dilutes unwanted (and unobserved) relics [1–3]. It also provides a natural mechanism of producing primordial perturbations that seed the inhomogeneities of the universe [4, 5]. The basic idea is that the quantum fluctuations of a classically homogeneous scalar field, the inflaton, source quantum fluctuations of the spacetime metric (the inflaton will create density perturbations that will source the scalar fluctuations of the metric). During the process of inflation, this quantum fluctuation is amplified to become a classical fluctuation, and at the end of inflation, the fluctuation in the metric induces the density fluctuations of matter that were produced during reheating. This primordial perturbation generated during inflation then is what gives rise to the formation of structure in the universe.
In this story, the crucial quantities to be determined are the amplitude of the primordial density and tensor perturbations, as the growth of structure is dependent on their sizes. The subsequent evolution of the primordial perturbations can be inferred from careful observations of the history of the growth of structure. Currently, the spectrum of primordial density perturbations is most stringently constrained by the WMAP 7 year data [6], which measures a nearly scale-independent amplitude of $O(10^{-5})$.

Therefore, it is important to be able to accurately compute these perturbations in order to either preserve or rule out a given inflationary model. For the usual models of inflation, comprised of the Ricci scalar plus a single canonically normalized scalar field, there are two components that will determine this amplitude. As will be described in more detail below, the first is the form of a function $z''/z$, which arises in the Mukhanov–Sasaki equation. This equation is satisfied by a mode function $v_k$, which arises by a redefinition of the co-moving curvature perturbation in momentum space $R_k$ upon having written the original action in terms of $R_k$. Since this equation is crucial to finding the curvature perturbation (the same equation is also obeyed by the tensor modes), it is essential that one accurately specifies its form.

The second component is the input from vacuum selection, which is equivalent to a boundary condition for the Mukhanov–Sasaki equation. The choice of boundary condition depends on the explicit form of $z''/z$. If the background satisfies slow-roll conditions, namely first and second slow-roll parameters and their first derivative are much smaller than 1, the behavior of $z''/z$ is approximately equal to $c/\tau^2$, where $c$ is a positive constant and $\tau$ is the conformal time. The usual boundary condition for such cases is called the Bunch–Davies boundary condition that is imposed at $\tau = -\infty$. Within this class of background solutions, one avenue of study has been to alter the initial state to lie away from the standard Bunch–Davies vacuum [7–15]. Such alternative boundary conditions are typically chosen by conditions set at a given cut-off scale in either momentum or time. These choices will then manifest themselves in physical observables (e.g., as new features in the power spectrum, or enhanced non-Gaussianity), which can allow one to gain knowledge of the initial state from observation. It should be mentioned that there exist arguments [16] that the Bunch–Davies vacuum might be the most probable vacuum to produce the correct power spectrum from the perspective of technical naturalness, although this does not eliminate the possibility of deviations from the Bunch–Davies vacuum.

It is customary to characterize inflation as a period of time where the scale factor grew almost exponentially, a period called de Sitter or quasi-de Sitter inflation (exact exponential growth of the scale factor $a \propto e^{Ht}$, with $H$ being constant, is technically de Sitter inflation, and nearly exponential growth is termed quasi-de Sitter). If the universe inflates in a power-law manner $a \propto \tau^p$, where $p$ is a constant, then the solution of the Mukhanov–Sasaki equation is known. Note that de Sitter inflation corresponds to $p = -1$. The solution for general $p$ is given by a linear combination of Bessel functions. The calculation for the perturbation amplitude has been well established for such a case [17]. However, the generic inflation background solution will not take this simple form. Thus, this commonly used approximation may not apply to all inflationary models (e.g., [18, 19]). If one insists on using the equations derived from the power-law limit, one runs the risk of possibly ruling out phenomenologically viable models, or of preserving models that are ruled out by observational data.

In this work, we would like to address the cases of background solutions that violate slow-roll conditions, where the function $z''/z$ inside the Mukhanov equation is drastically different from $c/\tau^2$. For such cases, the Bunch–Davies boundary condition is not a meaningful choice. One would be forced to find an alternative boundary condition for such cases. The purpose of this paper is to point out such a possibility with a specific example. We also suggest an alternative boundary condition that can be applied to analytic approximated solutions at a
finite conformal time $\tau_p$. It should be stressed that our proposal is not an improvement of the Bunch–Davies boundary condition, rather it is a parallel procedure that applies to different classes of background solutions that were not extensively studied in the past. Although there exist numerical codes for solving the Mukhanov equation [20], an analytic approach may be useful for its transparency and can possibly provide alternative insights than those gleaned from numerical methods.

This paper is organized as follows. In section 2, we review the standard calculation of primordial density and tensor perturbations, where the de Sitter limit is taken. We also explain the physical reasons behind the commonly chosen Bunch–Davies vacuum. In section 3, we introduce a new method of vacuum selection by applying this method to a specific inflation model. We also provides a numerical result that compare our proposal to the numerical solutions. The principles of generalizing this method to other models are also given. In section 4, we give our conclusions.

2. The standard method

We will now outline the key ingredients for the calculation of the primordial perturbations by quantizing the comoving curvature perturbation as well as the tensor perturbation [17][21] (for a recent textbook treatment and lecture notes, see [22]). The theories we consider here contain an Einstein–Hilbert action and a canonically normalized scalar field, minimally coupled to gravity with an arbitrary self-interacting potential

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \delta^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma - V(\sigma) \right\},$$

where $\kappa^2 = 8\pi G = 1/M_{Pl}^2$ and our metric signature is $-+++$.

2.1. Scalar perturbations

We begin with the perturbed Friedmann–Robertson–Walker (FRW) metric including the most general perturbations

$$ds^2 = a^2(\tau) \{- (1 + 2A) d\tau^2 - 2A_{ij} d\tau^i d\tau^j + [(1 + 2R)\delta_{ij} + \partial_i \partial_j H_T] dx^i dx^j\},$$

where $A(\tau, \mathbf{x})$, $B(\tau, \mathbf{x})$, $R(\tau, \mathbf{x})$ and $H_T(\tau, \mathbf{x})$ are small perturbations around the homogeneous FRW metric. We will be concerned with calculating the scalar $R$, which is the gauge-invariant comoving curvature perturbation in the comoving gauge, i.e. $\delta\sigma = 0$.

Variation of the action (1) gives the Einstein equations and the scalar field equation of motion, which at the background level are

$$\frac{(a')^2}{a^2} = \frac{\kappa^2}{3} \left[ \frac{1}{2a^2} (\sigma')^2 + V(\sigma) \right],$$

$$\frac{a''}{a^3} - \frac{(a')^2}{a^2} = -\frac{\kappa^2}{3} \left[ \frac{1}{a^3} (\sigma')^2 - V(\sigma) \right],$$

$$\frac{\sigma''}{a^2} + 2 \frac{a'}{a^2} \sigma' + V'(\sigma) = 0,$$

where a prime denotes a derivative with respect to the conformal time $\tau$, while a dot will indicate a derivative with respect to the coordinate time $t$.

Putting the solutions for the background evolution back into the Einstein–Hilbert action and expanding the action to second order in the perturbations give (setting $\kappa^2 = 1$)

$$S_{(2)} = \frac{1}{2} a^3 \int d^4 x a^2 \frac{\partial^2}{H^2} [\hat{R}^2 - a^{-2} (\partial \hat{R})^2].$$
The above expression can be obtained using the gauge symmetry in the action to choose \( \delta \sigma = 0 \). The derivation of (6) can be found in the appendix of [22]. One may define the Mukhanov variable
\[
v \equiv z \mathcal{R}, \quad \text{where} \quad z^2 = a^2 \frac{\dot{\sigma}^2}{H^2} = -2a^2 \frac{\dot{H}}{H^2} \equiv 2a^2 \epsilon. \tag{7}
\]
We have introduced the slow-roll parameter \( \epsilon \). The second-order action can be rewritten as
\[
S_{(2)} = \frac{1}{2} \int \! dt \, d^3x \left[ (v')^2 - (\partial_v)^2 + \frac{z''}{z} v^2 \right]. \tag{8}
\]
To quantize this action, first define the canonical conjugate momentum of \( v \) and then impose the usual commutation relation
\[
\Pi_v = \frac{\partial L}{\partial \dot{v}} = v', \quad [v(\tau, x), \Pi_v(\tau, x')] = i\hbar \delta^{(3)}(x - x'). \tag{9}
\]
Henceforth, we shall set \( \hbar = 1 \).

Now, one performs a plane-wave expansion of the now quantum operator \( \hat{v}(\tau, x) \) in the Fourier space
\[
\hat{v}(\tau, \vec{x}) = \int \frac{\mathcal{d}^3k}{(2\pi)^3} [v_k(\tau) \hat{a}_k \, e^{i\vec{k} \cdot \vec{x}} + v_k^*(\tau) \hat{a}_k^\dagger \, e^{-i\vec{k} \cdot \vec{x}}]. \tag{10}
\]
Requiring the canonical commutation relation between \( \hat{a}_k(\tau) \) and \( \hat{a}_k^\dagger(\tau) \), i.e. \( [\hat{a}_k(\tau), \hat{a}_k^\dagger(\tau)] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \), we will obtain the Wronskian condition for the mode function \( v_k(\tau) \)
\[
(v_k^* v_{k'} - v_k^* v_{k'}) = -i. \tag{11}
\]
The mode function in momentum space satisfies the Mukhanov–Sasaki equation
\[
v_k''(\tau) + \left( k^2 - \frac{z''}{z} \right) v_k(\tau) = 0. \tag{12}
\]
Upon introducing the second slow-roll parameter
\[
\eta = - \frac{\ddot{\sigma}}{H \dot{a}}, \tag{13}
\]
one can express \( \frac{z''}{z} \) in terms of the first and second slow-roll parameters
\[
\frac{z''}{z} = 2a^2 H^2 \left( 1 - \frac{3}{2} \eta + \epsilon + \frac{1}{2} \eta^2 - \frac{1}{2} \epsilon \eta + \frac{1}{2H} \dot{\epsilon} - \frac{1}{2H} \dot{\eta} \right). \tag{14}
\]
In general, equation (12) with \( \frac{z''}{z} \) given in equation (14) is difficult to solve analytically. For a special subset of general theories, where \( \epsilon \) and \( \eta \) are approximately constants, the equation is analytically solvable. In this special case, \( \frac{z''}{z} \) can be written as
\[
\frac{z''}{z} = \frac{v^2 - \frac{1}{2}}{\tau^2}, \quad \text{where} \quad v = \frac{1 - \eta + \epsilon + \frac{1}{2}}{1 - \epsilon}, \tag{15}
\]
and the analytic solution for \( v_k(\tau) \) is given in terms of Bessel functions:
\[
v_k(\tau) = \alpha \sqrt{\tau} J_\nu(\kappa \tau) + \beta \sqrt{\tau} Y_\nu(\kappa \tau), \tag{16}
\]
where \( \alpha \) and \( \beta \) are the two complex parameters. A well-known example for this special case is that of power-law inflation, i.e. \( a = c t^p \), where \( \epsilon = \eta = \frac{p+1}{p} \). One then obtains \( z = \sqrt{\frac{2^{p+1}}{p}} \cdot c \cdot t^p \), which gives \( z^2 \) or \( v = -p + 1/2 \), and the comoving horizon \( (aH)^{-1} = \tau / p \). The pure de Sitter expansion is the specific case of \( p = -1 \). For genuine de Sitter inflation, \( z \) vanishes, which leads to a divergent perturbation. Such divergence indicates the breakdown of perturbative method and the onset of eternal inflation, which is not observationally accessible.
to us. This implies that inflation must deviate from the pure de Sitter case, at least within our cosmological horizon.

The solutions to equation (12) can be written either as the linear combinations of Bessel functions, $J_n(x)$ and $Y_n(x)$, or Hankel functions, $H^{(1)}_n(x)$ and $H^{(2)}_n(x)$:

$$v_k(\tau) = \alpha \sqrt{\tau} J_{-p+\frac{1}{2}}(k\tau) + \beta \sqrt{\tau} Y_{-p+\frac{1}{2}}(k\tau) = \tilde{\alpha} \sqrt{\tau} H^{(1)}_{-p+\frac{1}{2}}(k\tau) + \tilde{\beta} \sqrt{\tau} H^{(2)}_{-p+\frac{1}{2}}(k\tau)$$

(17)

(18)

The Wronskian condition (11) requires

$$\alpha^* \beta - \alpha \beta^* = -\frac{i\pi}{2} \text{ or } |\tilde{\alpha}|^2 - |\tilde{\beta}|^2 = 1.$$  

(19)

When the solution is expressed in terms of Hankel functions, there is a natural place where the boundary condition may be imposed, i.e. when $|k\tau| \gg 1$, or equivalently when the comoving wavelength is deep inside the comoving Hubble radius. The asymptotic forms of the Hankel functions become positive and negative frequency modes:

$$\lim_{|k\tau| \gg 1} \sqrt{\tau} H^{(2)}_{-p+\frac{1}{2}}(k\tau) = e^{-ik\tau}, \quad \lim_{|k\tau| \gg 1} \sqrt{\tau} H^{(1)}_{-p+\frac{1}{2}}(k\tau) = e^{ik\tau}.$$  

(20)

The fact that in the far past the solution approaches those of the Minkowski space can be seen in the behavior of $z''/z$ displayed in figure 1.

The vanishing behavior of $z''/z$ in this asymptotic region ensures that solutions to equation (12) reduce to the Minkowski type in the far past. Thus, for these classes of inflationary
models, there is a natural boundary condition that the solution should approach the positive frequency outgoing mode with no incoming modes

$$\lim_{\tau \to -\infty} v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}. \quad (21)$$

This form can be seen to match that of $H_{1/2}^{(1)}(k\tau)$ in equation (20). The boundary condition equation (21) is known as the Bunch–Davies vacuum [7]. This has the effect of setting $\tilde{\alpha} = 1$ and $\tilde{\beta} = 0$ in equation (17). This appears as a natural choice, as one may think intuitively that at the beginning of time, all the particles (or positive frequency modes) should move forward in time, thus eliminating the possibility of having a contribution from the $H_{3/2}^{(2)}(k\tau)$ term.

We would like to stress that although $\tau \to -\infty$ is a legitimate limit formally, physically there will exist a time where the physical wavelength will be comparable to the Planck length where quantum gravity effects should take place. This means that, in that region, the background evolution can no longer be treated classically. Due to the lack of a full quantum gravity theory, the boundary condition may be imposed at some later time where the physical wavelength is greater than the Planck length. The effect of setting the boundary condition at a finite time may be that the state does not reside in the ground state, but rather in some squeezed or distorted state [8, 9, 23].

For general energy contents of the universe, the form of the scale factor will no longer be a simple power law (although during times where a single component is the dominant contributor to the stress–energy, such as during matter or radiation domination, the power-law form is a good approximation). As an approximation, a standard analytic approach is to assume that the expansion is approximately de Sitter, $p \approx -1$, and therefore, $\epsilon \approx 0$. Together with the smallness condition of the second slow-roll parameter $\eta = -\frac{\dot{\sigma}}{H\dot{\sigma}} \ll 1$ using equation (15), one finds

$$\frac{z''}{z} \approx \frac{a''}{a} \approx \frac{2}{\tau^2}. \quad (22)$$

Under this assumption, the solution of the Mukhanov–Sasaki equation is

$$v_k(\tau) = \tilde{\alpha} \sqrt{k\tau} H_{1/2}^{(1)}(k\tau) + \tilde{\beta} \sqrt{k\tau} H_{3/2}^{(2)}(k\tau), \quad (23)$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are two complex parameters with four degrees of freedom, one of which is fixed by equation (19), two by equation (21) and leaving one irrelevant phase undetermined. With these conditions, the solution for the mode function is

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right). \quad (24)$$

This leads to the well-known relations for the scalar power spectrum $P_R$, the spectral index $n_s$, and the running of the spectral index $\alpha_s$. The definitions (25)–(27) follows the convention in [22]:

$$P_R \equiv \left| \frac{v_k(\tau)^2}{z^2} \right|_{k=aH} = \left| H^2 \frac{4k^3\epsilon}{4k^3\epsilon} \right|_{k=aH}, \quad (25)$$

$$n_s - 1 \equiv \frac{\frac{d\ln(k^3P_R)}{d\ln k}}{\frac{d\ln k}{k^3P_R}} k_0 = \frac{k}{k^3P_R} \left| \frac{d(H^2/\epsilon)}{d\tau} \right|_{\tau=\tau_*} d\tau, \quad (26)$$

$$\alpha_s \equiv \frac{\frac{d\ln k}{d\ln k}}{k^3P_R} k_0. \quad (27)$$

Note that each relation is to be evaluated at horizon crossing, $k = aH$ (or equivalently $\tau = \tau_*$), due to the fact that the amplitude of the perturbation is frozen when the comoving wavelength
becomes stretched outside the comoving Hubble radius. This is a special property of single-field inflation, which is the case investigated in this paper. In a more general situation with multiple scalar fields, the perturbation is classicalized at horizon crossing, i.e. the phase of \( v_k \) becomes constant. However, in the multi-field scenario, the amplitude of the perturbation may still evolve after horizon crossing\(^1\).

The accuracy of this program strongly depends on the quality of how well the approximation of the curve \( z''/z \) near the time of horizon crossing compares with the true function \( z''/z \) [24–26]. Note that, since the observed perturbation is determined by the value at the horizon crossing, the actual fit of \( z''/z \) far away from the horizon crossing point is not important for the prediction of the observed power spectrum. However, if the asymptotic behavior of \( z''/z \) is non-vanishing, then there is no reason to choose the Bunch–Davies vacuum as the boundary condition. Applying these equations to models, where \( z''/z \) is not well fit by the power law and slow-roll approximations, may result in serious deviations from the observable predictions. This is precisely the problem that we will address in section 3. Before doing so we will first give an overview of quantization in the tensor sector.

### 2.2. Tensor perturbations

The calculation for tensor perturbations mirrors that of the scalar perturbation. The starting point is once again the perturbed FRW metric with a transverse, traceless tensor perturbation, \( h_{ij} \) (we have omitted the scalar and vector perturbations seen previously which has no effect on the tensor mode evolution due to decoupling of the scalar, vector, and tensor sectors)

\[
\text{d}x^2 = a^2 (-\text{d}\tau^2 + (\delta_{ij} + h_{ij}(\mathbf{x}, \tau)) \text{d}x^i \text{d}x^j).
\]

Next, one expands the Einstein–Hilbert action to second order in the perturbation

\[
S^{(2)} = \frac{M_{\text{pl}}^2}{8} \int \text{d}\tau \text{d}x^3 a^2 \left[ (h_{ij}''(x, \tau))^2 - (\partial_i h_{ij}(x, \tau))^2 \right].
\]

As in the scalar case, one expands \( h_{ij} \) in the Fourier space in terms of plane waves with modes \( h_k^s \):

\[
h_{ij}(\mathbf{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon_{ij}^s h_k^s(\tau) e^{ik \cdot x},
\]

where \( \epsilon_{ij}^s \) are the spin-two polarization tensors. One can then make the definition

\[
\mu_k^s(\tau) = \frac{1}{2} ah_k^s(\tau),
\]

which leads to the action (here, we have set \( M_{\text{pl}} = 1 \))

\[
S^{(2)} = \sum s \frac{1}{2} \int \text{d}\tau \frac{d^3k}{(2\pi)^3} \left[ (\mu_k^s(\tau))^2 - \left( k^2 - \frac{\mu_k^s}{a} \right) (\mu_k^s(\tau))^2 \right].
\]

This action gives similar equations of motion as equation (12):

\[
\mu_k''(\tau) + \left( k^2 - \frac{\mu_k'}{a} \right) \mu_k'(\tau) = 0.
\]

For a scale factor of the power-law form, the calculation follows exactly as in the scalar case, while the fit function in terms of slow-roll parameter is now

\[
\frac{\mu'}{a} = \frac{\mu^2 - \frac{1}{3}}{\tau^2}, \quad \text{where} \quad \mu = \frac{1}{\epsilon - \frac{1}{2}} + \frac{1}{2}.
\]

\(^1\) We thank the referee for pointing out this subtle difference between single-field and multi-field inflation.
In the case of quasi-de Sitter inflation, $\epsilon \approx 0$, the power spectrum for a single polarization of the tensor modes toward the end of inflation is

$$P_t = \frac{4 |\mu_k|^2}{a^2} \bigg|_{k = aH} = \frac{2H^2}{k^3} \bigg|_{k = aH},$$

(34)

which differs from the form of the scalar result in that the slow-roll parameter $\epsilon$ is absent in the denominator. The full power spectrum is then twice of this (due to two polarization states) $P_h = 2P_t = \frac{4H^2}{k^3}$, which leads to a small tensor to scalar ratio $r = P_h/P_R = 16\epsilon$ when the slow-roll parameter is small.

These results for the power spectra are obtained under the assumption that the expansion is de Sitter or very nearly de Sitter in the sense that equation (22) is true. To obtain a more accurate predication, one must solve the Mukhanov equation on a model-by-model basis using the exact $z''/z$ (or $a''/a$) numerically, along with choosing a proper boundary condition accordingly. In the next section, we will institute such a procedure in order to quantize models where equation (22) is not a good approximation.

### 3. Vacuum selection

The mode equation that one needs to solve is

$$v_k'' + (k^2 - f(\tau))v_k = 0.$$  

(35)

One can obtain the analytic expression for $f(\tau)$ from solving the background equations. For the scalar case, $f(\tau)$ is given by $z''/z$, while for the tensor case, it is $a''/a$. In the de Sitter limit, its value is shown in equation (22). In general, the background evolution is not tractable analytically, and for the few cases where the background solution is analytically solvable [19, 27–33], the $\tau$ dependence in $f(\tau)$ may be so complicated that finding an analytic solution for equation (35) becomes impossible. It is worth mentioning that the equation is equivalent to a time-independent Schrödinger equation in one dimension when $\tau$ is regarded as a space coordinate and $f(\tau)$ is the potential for the wavefunction $v_k(\tau)$. This analogy will become apparent when we introduce our method of solving equation (35), which is nothing but the usual WKB approximation in quantum mechanics.

Under the condition that the background evolution of the metric is known for a particular inflationary model, one can determine whether the approximation equation (22) will be applicable for the model under consideration. If so, then the system may be solved using the standard approach outlined in the preceding section. However, this is not the case for a great number of inflationary models. One can understand that this is so because the mechanism used in stopping inflation may falsify the standard approximation. Additionally, the initial conformal time cannot always be pushed back to negative infinity where one would impose the Bunch–Davies boundary condition as is the case when the expansion is truly power law.

The essential idea of our method is that, since the information one needs in order to calculate observables to compare with experiment is the value of the mode function at horizon crossing (which is much later than the asymptotic time $\tau \to -\infty$), it may therefore be advantageous to place the physical boundary condition closer to the time where we require accurate information. Imposing the boundary condition at negative infinity may lead to a situation where the approximate form for $f(\tau)$ in equation (22) has deviated greatly from the actual solution due to the lengthy intervening period of evolution. Although we are placing the boundary condition nearer the era of observable inflation, we will continue to mimic the idea of BD vacuum selection in that we look at the time $\tau = p$, where the wavelength of the mode is deep inside the horizon, and the effect of cosmic expansion is relatively small. The situation can then be approximated as physics in the Minkowski space.
In this section, we will first demonstrate the method with an explicit example before going on to comment on considerations on applying the method in general.

3.1. A specific example

The model we use as an example has background evolution that is analytically solvable [19]. This particular model provides an interesting picture where the big bang is connected to inflation with a specific time delay. The behavior of \( z''/z \) is very different from the slow roll plus de Sitter limit, while the term \( a''/a \) is asymptotically equal to the de Sitter limit. We will apply our method to obtain the scalar power spectrum and constrain the tensor power spectrum from observation.

Our point here is to show that there exist examples, such that \( z''/z \) cannot be approximated by the standard fit function; thus, there is a need of introducing new method of solving the Mukhanov–Sasaki equation. We would like to point out that this model does not provide a mechanism of stopping inflation; therefore, even though our method predicts the correct power spectrum in a certain parameter space, the phenomenologically viable parameter space is expected to change when the stopping mechanism is introduced. Therefore, the model in question may be viewed as a demonstration tool (due to the attractive property that it is analytically solvable at the background level) rather than a fully complete model.

We begin with an action of the form equation (1) with the scalar potential

\[
V(\sigma) = \left( \frac{2}{\kappa^2} \right)^2 c \sinh^4 \left( \sqrt{\frac{\kappa^2}{6}} \sigma \right) + b \cosh^4 \left( \sqrt{\frac{\kappa^2}{6}} \sigma \right).
\]

The potential contains the dimensionless free parameters \( b \) and \( c \), and \( \kappa^{-1} \) is the reduced Planck mass as before. The background solution for all possible combinations of \( c \) and \( b \) has been classified in [19]. We will use the case \( c = 64b > 0 \) as an example to illustrate our method.

A particular solution of this model is to initiate expansion at \( \tau = \tau_{BB} \approx 0.92 \), where the scale factor is exactly zero. Beginning at a later time, \( \tau_l \approx 2.87 \), there is an inflationary period, and finally, when \( \tau \) approaches \( \tau_\infty \approx 7.4 \), the scale factor diverges. At this point, the physical time, \( t = \int a(\tau) \, d\tau \), will also diverge. Of course, the finite value \( \tau_{BB} \approx 0.92 \) is not physically significant since \( \tau \) can be translated by an arbitrary amount. The solutions for the background evolution of the scale factor \( a(\tau) \) and inflaton \( \sigma(\tau) \) are as follows:

\[
a(\tau) = \sqrt{\frac{1}{12\kappa}} \left( \frac{E}{b} \right)^{\frac{1}{2}} \left\{ 2 \left[ 1 - \frac{c}{1 + cn \left( \frac{1}{\sqrt{2}} \tilde{\tau} \right)} - \frac{1}{4} \left[ cn \left( \frac{1}{\sqrt{2}} \tilde{\tau} \right) \right]^2 \right\}^{\frac{1}{2}} \tag{37}
\]

\[
\sigma(\tau) = \frac{1}{\kappa} \sqrt{\frac{3}{2}} \ln \left( 1 + \frac{1}{\sqrt{2\kappa}} cn(\tilde{\tau}) \left[ \frac{1}{1 - cn \left( \frac{1}{\sqrt{2}} \tilde{\tau} \right)^2} \right]^\frac{1}{2} \right) \left[ 1 - \frac{1}{\sqrt{2\kappa}} cn(\tilde{\tau}) \left[ \frac{1}{1 - cn \left( \frac{1}{\sqrt{2}} \tilde{\tau} \right)^2} \right]^\frac{1}{2} \right], \tag{38}
\]

where \( \tilde{\tau} \equiv 2[cE]^\frac{1}{2} \tau \). \( E \) is a free parameter of this model, which determines the scale of \( a \). In the present application, we will choose \( E \) so that

\[
2|64bE|^{\frac{1}{2}} = 1 \text{ or } \tilde{\tau} \equiv \tau. \tag{39}
\]

The term \( cn \left( \frac{1}{\sqrt{2}} \tilde{\tau} \right) \equiv cn \left( \frac{1}{\sqrt{2}} \tau \right) \) is the Jacobi elliptic function[34]. Following the standard method, one needs to determine whether \( \epsilon \) and \( \eta \) are approximately constants. If so, following equations (15) and (33), one can determine the fit function of \( z''/z \) and \( a''/a \). Since the analytic solution of this model is known, we can plot the exact curves for \( \epsilon \) and \( \eta \) as in figures 2 and 3.
One can easily see that the asymptotic values of $\epsilon$ and $\eta$ approach the constant values 0 and 1, respectively. These values will result in the breakdown of equation (15) for $z''/z$ and $2/(\tau - \tau_\infty)^2$ for $a'/a$. From the above expression, we can also compute $z''/z$ and $a'/a$. The result is shown in the plots figures 4 and 5 along with the fitting function.

The fit function plotted in figure 5 is the predicted fit function for $a'/a$, which is $2/(\tau - 7.4)^2$, while the fit function for figure 4 is a quadratic function

$$f_{fit}(\tau) = m(\tau - p)^2 + h,$$

with parameters

$$m = -1.2, \quad p = 5.66, \quad h = -0.23,$$

In the case of the scalar perturbation, since the formula for the standard fit function breaks down, we now introduce a new quadratic fit function according to the following rules. The parameter $p$ is designed to match the point where $f(\tau)$ has slope zero, while $h$ is designed to match the value of $f(\tau)$ at $\tau = p$. The value of $m$ is the only parameter in the fit function, but it has to be adjusted such that the curve of $f_{fit}(\tau)$ approaches the actual $f(\tau)$ in the region of physical interest, namely the region where the mode exits the horizon. That is to say, we want a better fit for $\tau > p$. The region $\tau < p$ corresponds to the time before the beginning of observable inflation, or before about 60 e-folds before the end of inflation. The perturbation generated during the preceding periods have yet to re-enter our horizon; thus, they have produced no observable effect. (Since the universe is currently accelerating again, it is not clear to what extent regions corresponding to time intervals before the last 60 e-folds will contribute to observable effects in the future.)

Once the fit function is chosen, one can then proceed to solve the Mukhanov–Sasaki equation with the new fit function. In the current example, the two independent solutions to
equation (35) with \( f(\tau) = f_{\text{fit}}(\tau) \) are the hypergeometric functions \( S_{k1} \) and \( S_{k2} \). The general mode function is thus

\[
v_k(\tau) = \alpha S_{k1}(\tau) + \beta S_{k2}(\tau),
\]

where \( \alpha \) and \( \beta \) are the two complex parameters that can be parametrized by four real parameters

\[
\alpha = r_1 e^{i\theta_1}, \quad \beta = r_2 e^{i\theta_2}
\]

written in the real parameters

\[
v_k(\tau) = e^{i\theta_1} (r_1 S_1 + r_2 e^{i(\theta_2 - \theta_1)} S_2).
\]

Here, \( r_1, r_2, \theta_1, \theta_2 \) are to be determined by the boundary conditions. It can be easily seen that when the quantity of interest is the expectation value of \( v_k(\tau) \), the parameter \( \theta_1 \) is an irrelevant phase.

The exact solutions \( S_{k1} \) and \( S_{k2} \) are

\[
S_{k1}(\tau) = (e^{-\frac{3}{2} + p})^{\frac{\sqrt{m}}{2}} F_1 \left( \frac{\sqrt{m}}{4} \tau^2 + \frac{1}{2}, \frac{3}{2}, \sqrt{m}(\tau - p)^2 \right),
\]

\[
S_{k2}(\tau) = (\tau - p) e^{\frac{3}{2} + p} \frac{3}{2} F_1 \left( \frac{3\sqrt{m}}{4} \tau^2 + \frac{3}{2}, \frac{3}{2}, \sqrt{m}(\tau - p)^2 \right),
\]

where the hypergeometric function is defined as [34]

\[
_1 F_1 (a, b, z) = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \cdots.
\]
The Wronskian condition (11) will require $\alpha$ and $\beta$ to satisfy

$$1 = i (v_k^* v_k' - v_k v_k'^*) = \begin{cases} \frac{i(\alpha^* \beta - \alpha \beta^*)}{i(\alpha^* \beta - \alpha \beta^*)} & \text{if } m < 0 \\ i(\alpha^* \beta - \alpha \beta^*) e^{\sqrt{mp^2}} & \text{if } m > 0 \end{cases}$$

or

$$\alpha^* \beta - \alpha \beta^* = -iC(m, p), \quad (47)$$

where $C$ is a number that depends on the values of $m$ and $p$. When $m < 0$, $C = 1$, and when $m > 0$, $C = e^{-\sqrt{mp^2}}$. In the specific example discussed in this paper, $m = -1.2 < 0$. For completeness purposes, we have also included the expression for $C$ for positive $m$.

The next step is to choose a corresponding boundary condition. Since figure 4 shows $z''/z < 0$ in the whole region of physical conformal time, $0.92 < \tau < 7.4$, the solution to equation (35) will be a combination of oscillating waves. We introduce the WKB-type solution

$$v_k(\tau) = A(\tau) e^{i\phi(\tau)}, \quad (48)$$

where $A(\tau)$ and $\phi(\tau)$ are the two real functions. Inserting this ansatz into equation (35), one obtains

$$A'' + 2iA' \phi' + iA \phi'' - A (\phi')^2 = -\omega^2 A, \quad (49)$$

where $\omega(\tau) = \sqrt{k^2 - f(\tau)}$. This complex equation can be separated into two real equations:

$$A'' - A (\phi')^2 = -\omega^2 A, \quad (50)$$

$$(A^2 \phi')' = 0. \quad (51)$$
The second equation can be solved easily

\[ A = \pm \frac{q}{\sqrt{\phi}} \]  \hspace{1cm} (52)

where \( q \) is an arbitrary constant.

To solve equation (50), we invoke the WKB approximation, which assumes that \( A \) is slowly varying with conformal time, and then, the second derivative term can be neglected. Such an approximation obtains when the potential \( f(\tau) \) is slowly varying with conformal time. This occurs when the function \( f(\tau) \) is at a local extremum, which corresponds to the very moment when we impose the boundary condition. Under this approximation, equation (50) can be solved analytically

\[ \phi(\tau) = \pm \int \omega(\tau) \, d\tau. \]  \hspace{1cm} (53)

The full solution can then be expressed as a linear combination of two modes:

\[ v_k(\tau) = c_1 \frac{e^{i \int \omega(\tau) \, d\tau}}{\sqrt{\omega}} + c_2 \frac{e^{-i \int \omega(\tau) \, d\tau}}{\sqrt{\omega}}. \]  \hspace{1cm} (54)

In the vicinity of \( \tau = p \), \( \omega(\tau) \) is a constant, i.e. \( \omega_s = \sqrt{k^2 - h} \). The boundary condition that we impose is to choose the parameters \( \alpha \) and \( \beta \) so that in the vicinity of \( \tau \approx p \), the solution is a linear combination of incoming and outgoing plane waves satisfying the usual Klein–Gordon normalization:

\[ \text{when } \tau \approx p, \quad v_k(\tau) \approx a \frac{e^{-i\omega_s \tau}}{\sqrt{2\omega_s}} + b \frac{e^{i\omega_s \tau}}{\sqrt{2\omega_s}}, \quad \text{where } |a|^2 - |b|^2 = 1. \]  \hspace{1cm} (55)
The proposed boundary condition does invoke more parameters than the Bunch–Davies boundary condition since we are not setting \(a\) to 1 and \(b\) to 0, but instead keeping them as general parameters. As will be shown later, the phenomenologically viable parameter space requires \(a\) close to 1 and \(b\) close to 0. This means that the boundary condition is close to a purely outgoing wave with small amount of incoming wave and, thus, close to the standard Bunch–Davies vacuum. We want to emphasize that we are treating \(a\) and \(b\) as parameters to fit observations; therefore, we are not predicting the power spectrum, rather finding a parameter space that gives correct phenomenology. How to derive \(a\) and \(b\) from a fundamental theory is undetermined and beyond the scope of the purpose of this method.

Unlike the Bunch–Davies boundary condition, which is imposed at the beginning of time, our method imposes the boundary condition at a finite time, \(\tau = p\), where there is no reason to assume there only exists an outgoing mode. One may criticize that our method introduces too many new parameters in the boundary condition. However, this may also be the case if \(z''/z\) can be approximated by equation (15), where the standard Bunch–Davies condition is meaningful. This is because trans-Planckian physics could significantly alter the initial condition such that the Bunch–Davies condition is not valid at all \([8]\). Thus, although more parameters are introduced in our method, the freedom is not necessarily more than the standard method.

Now, we express \(a\) and \(b\) in terms of real parameters: \(a = a_0 e^{i\theta_a}, b = b_0 e^{i\theta_b}, |a|^2 - |b|^2 = 1\). According to the proposed boundary condition, the mode function and its derivative are given by

\[
|v_k(p)|^2 = \left| a \frac{e^{-i\omega p}}{\sqrt{2\omega}} + b \frac{e^{i\omega p}}{\sqrt{2\omega}} \right|^2 = \frac{1}{2\omega} + \frac{b_0^2 + b_0 \sqrt{1 + b_0^2} \cos(\Delta)}{\omega},
\]

\[
|v'_k(p)|^2 = \left| -i\omega a \frac{e^{-i\omega p}}{\sqrt{2\omega}} + i\omega b \frac{e^{i\omega p}}{\sqrt{2\omega}} \right|^2 = \frac{\omega}{2} + \omega \left( b_0^2 - b_0 \sqrt{1 + b_0^2} \cos(\Delta) \right),
\]

where \(\Delta = 2\omega p - \theta_a + \theta_b\). Here, we have assumed \(\omega = \sqrt{k^2 - \hbar} > 0\). In the example, we are discussing \(h = -0.23\); thus, this assumption is always true for this case. However, with a different fit function, it is conceivable that \(h\) could be positive, giving the condition \(k^2 > h\) in such an instance. The idea behind this choice is that at \(\tau = p\), the equation becomes a harmonic oscillator with the constant frequency \(\omega\). For a small period of time near \(p\), the solution should then approach the usual harmonic oscillator where the vacuum is given by the lowest energy state. From equation (57), we find two more constraints that can be used to solve \(r_1\), \(r_2\) and \((\theta_2 - \theta_1)\):

\[
|v_k(p)|^2 = |\alpha|^2 |e^{i\frac{\pi}{2} m}|^2 = \frac{1}{2\omega} + \frac{b_0^2 + b_0 \sqrt{1 + b_0^2} \cos(\Delta)}{\omega},
\]

\[
|v'_k(p)|^2 = |\beta|^2 |e^{i\frac{\pi}{2} m}|^2 = \frac{\omega}{2} + \omega \left( b_0^2 - b_0 \sqrt{1 + b_0^2} \cos(\Delta) \right).
\]

The solutions can be categorized according to the sign of \(m\). We find that if \(m < 0\)

\[
r_1 = \sqrt{\frac{1}{2\omega} + \frac{b_0^2 + b_0 \sqrt{1 + b_0^2} \cos(\Delta)}{\omega}},
\]

\[
r_2 = \sqrt{\frac{\omega}{2} + \omega \left( b_0^2 - b_0 \sqrt{1 + b_0^2} \cos(\Delta) \right)},
\]

\[
(\theta_2 - \theta_1) = -\sin^{-1} \left( \frac{1}{\sqrt{1 + 4(b_0^2 + b_0^2) \sin^2 \Delta}} \right).
\]
while for $m > 0$ we obtain

$$r_1 = e^{-\frac{\pi^2}{2} \rho^2} \sqrt{\left( \frac{1}{2\alpha} + \frac{b_0^2 + b_0 \sqrt{1 + b_0^2 \cos(\Delta)}}{\omega} \right)}. \quad (63)$$

$$r_2 = e^{-\frac{\pi^2}{2} \rho^2} \sqrt{\left( \frac{\omega}{2} + \omega(b_0^2 - b_0 \sqrt{1 + b_0^2 \cos(\Delta)}) \right)}. \quad (64)$$

$$(\theta_2 - \theta_1) = -\sin^{-1} \left( \frac{1}{\sqrt{1 + 4(b_0^2 + b_0^4) \sin^2 \Delta}} \right). \quad (65)$$

Then, for the mode functions, we have for $m < 0$

$$v_k(\tau) = e^{i\theta_1} \sqrt{\frac{1}{2\omega} + \frac{b_0^2 + b_0 \sqrt{1 + b_0^2 \cos(\Delta)}}{\omega} S_i}$$

$$+ e^{i \sin^{-1} \left(\frac{i}{1+i(\xi_0 + \omega) \sin \theta} \right)} \sqrt{\frac{\omega}{2} + \omega(b_0^2 - b_0 \sqrt{1 + b_0^2 \cos(\Delta)}) S_2} \quad (66)$$

and for $m > 0$

$$v_k(\tau) = e^{i\theta_0} \sqrt{\frac{1}{2\omega} + \frac{b_0^2 + b_0 \sqrt{1 + b_0^2 \cos(\Delta)}}{\omega} S_i}$$

$$+ e^{i \sin^{-1} \left(\frac{i}{1+i(\xi_0 + \omega) \sin \theta} \right)} \sqrt{\frac{\omega}{2} + \omega(b_0^2 - b_0 \sqrt{1 + b_0^2 \cos(\Delta)}) S_2} \quad (67)$$

With the above expressions, the mode function is uniquely determined (up to an irrelevant phase $\theta_1$) by the fit function $f_{\text{fit}}(\tau)$ along with the parameters $b_0$ and $(\theta_0 - \theta_b)$ from the boundary condition. As usual, the mode function will be evaluated at horizon crossing, where $k = aH$.

In order to produce an observationally consistent spectral index and its running, we find that the parameters $\Delta$ and $b_0$ are 0 and 0.075, respectively. The mode function then satisfies the boundary condition

$$v_k(\tau) \approx 1.0028 e^{-i\omega r} + 0.075 e^{i\omega r} \sqrt{2\omega} \quad (68)$$

One can derive the spectral index as well as the running of the spectral index using definitions (26) and (27) as a function of conformal time (see figures 6 and 7).

At $\tau \approx 7.05$, the spectral index is $n_r \approx 0.95$, and the running of the spectral index is $a_r \approx 0$. This is within the allowed value of WMAP constraints [6].

The above results were obtained using the mode functions $v_k(\tau)$ computed from the Mukhanov–Sasaki equation with a quadratic fit for $z'/z$. To check the validity of this approximation, we performed a numerical computation of the mode function $v_{\text{exact}}(\tau)$ using the exact numerical value of $z'/z$ in solving the Mukhanov–Sasaki equation with the same boundary condition (68). Using $k = aH|_{\tau=7.05} = 2.73$ as an example, the plot of $v_k(\tau)$ and $v_{\text{exact}}(\tau)$ is given in figure 8. It can be seen qualitatively that $v_k(\tau)$ agrees with $v_{\text{exact}}(\tau)$.
very well. Quantitatively, the error of the mode function at the horizon crossing $\tau = 7.05$ is only about 1.5%, thereby confirming the validity of our approximation.

Now, consider the tensor perturbation. The mismatch of the standard fit function is shown in figure 5. The fit function $2/(\tau - 7.4)^2$ is a good approximation in the asymptotic region. This is because the first slow-roll parameter is approaching zero when $\tau > 6$. Despite the success in the asymptotic region, the validity of Bunch–Davies boundary condition is questionable. The reason is that the Bunch–Davies vacuum imposes a condition at a fictitious conformal time $\tau \rightarrow -\infty$, while the physical universe actually starts its expansion at a finite conformal time $\tau \approx 0.92$. The limit $\tau \rightarrow -\infty$ is inapplicable in our model; thus, the mode function evaluated from the evolution of such an ill-defined boundary condition is not reliable. Thus, we shall keep the standard fit function but abandon the boundary condition that is imposed at an unphysical time. We will proceed by obtaining a general solution of the mode function and subsequently use the observational constraints to restrict the possible parameters. Since the power spectrum of tensor perturbations has yet to be observed, what we have to constrain experimentally is the ratio of tensor perturbation to the scalar perturbation.
The most general mode function for the tensor modes is
\[ \mu_k(\tau) = a\sqrt{k\tau}H_{3/2}^{(1)}(k\tau) + b\sqrt{k\tau}H_{3/2}^{(2)}(k\tau) \]
\[ = a e^{-ik\tau} \left( 1 - \frac{i}{k\tau} \right) + b e^{ik\tau} \left( 1 + \frac{i}{k\tau} \right). \]

From the definition (34), one obtains
\[ P_h = 4\frac{H^2}{k^3} + 8\frac{H^2}{k^3} \left( \beta_0^2 - b_0 \sqrt{1 + \beta_0^2 \cos(\Delta)} \right). \]

With \( P_R \) computed from equation (68) as well as the observational restriction on \( r \), one can deduce an upper bound for \( (2\beta_0^2 - 2b_0 \sqrt{1 + \beta_0^2 \cos(\Delta)}) \). From the WMAP7 data [6] \( r < 0.2 \), which corresponds to \( (2\beta_0^2 - 2b_0 \sqrt{1 + \beta_0^2 \cos(\Delta)}) < 0.16 \).

3.2. The general case

We would like to comment on how to generalize this program, such that it may be implemented for a given background evolution of an inflationary model. There is no reason, a priori, that either equation (15) in the standard method, or a quadratic function, as introduced in the previous section, should be a good fit for \( z''/z \) in general.

Our proposal in this paper has been to point out that one should not apply the standard fit function to all inflationary models without carefully examining whether it is actually a good fit. If the analytic solution for the Mukhanov–Sasaki equation is not attainable, one may wish to find a fit function that is analytically solvable. There exist a number of analytically solvable fit functions for \( z''/z \); the quadratic function introduced in the previous section is just one simple example.

Another more sophisticated example is the quartic function, which renders equation (12) solvable by linear combinations of the Heun triconfluent function. Unfortunately, in practice, the introduction of a more complicated fitting function, such as the quartic function, may reduce the predictivity of the model, as there will tend to be more parameters in the fit function.
that require matching rules to pin down their values. However, the rule of thumb is simple; the employed fit function should resemble the actual curve at the region of interest, which is the moment of horizon crossing. After that, one should impose the boundary condition at a meaningful time. As we have demonstrated, using the WKB approximation at the moment when $z''/z$ is flat is one sensible choice. Finally, one can use the current observational data for the tensor to scalar ratio to constrain tensor perturbation. However, since the power spectrum of the tensor mode has not been observed, the constraints from the tensor sector are not overly restrictive.

4. Conclusions

In order to make predictions testable by observations, inflation needs not only a model, but suitable boundary conditions. Some models of inflation do not seem to fall within the realm where the standard boundary conditions may be naturally applied. With this in mind, we have discussed the introduction of an alternative method that generalizes the standard approach of computing the scalar and tensor power spectra. It is suggested that for those models whose background is analytically solvable, one should re-examine their power spectra using our method and determine how their spectral indices compare with the results from the standard method.

In general, this procedure will introduce additional parameters into the model, thus allowing more accurate phenomenology, with the usual drawback that the introduction of more parameters decreases predictivity. This new method is implemented on a model-by-model basis; hence, the generic effects of this approach have yet to be determined. For the specific example discussed in this paper, we explored an inflationary model that has analytically solvable background dynamics. We introduced a quadratic fit (other models may require more complicated fitting functions) for the function $z''/z$, which appears in the Mukhanov–Sasaki equation for the scalar modes, and imposed boundary conditions at finite conformal time $\tau$. It was found that near $\tau = 7.05$ the spectral index and its running both fall into a phenomenologically acceptable range. This calculation gives an example for the implementation of our approach, although the model in question is not fully realized in the sense that it lacks a proper accounting for the cessation of inflation in order to produce the requisite amount of e-fold expansion.

The capacity for altering the calculation, and thus the values, of observables predicted by inflation via this new approach is clear. It may therefore be possible that models that were hitherto discarded may need to be re-investigated in the framework of this method.

Acknowledgments

We thank Itzhak Bars, Yi-Fu Cai, Yi-Zen Chu and Tanmay Vachaspati for helpful discussions. The work of SHC was supported in part by the US Department of Energy, grant number DE-FG03-84ER40168. The work of SCH and JBD was supported in part by the Arizona State University Cosmology Initiative.

References

[1] Guth A H 1981 Phys. Rev. D 23 347
[2] Linde A D 1982 Phys. Lett. B 108 389
[3] Albrecht A and Steinhardt P J 1982 Phys. Rev. Lett. 48 1220
[4] Kofman L A and Mukhanov V F 1986 JETP Lett. 44 619
[5] Sasaki M 1986 Prog. Theor. Phys. 76 (5) 1036–46
[6] Komatsu E et al 2010 arXiv:1001.4538
[7] Bunch T S and Davies P C W 1978 Proc. R. Soc. A 360 117
[8] Danielsson U H 2002 Phys. Rev. D 66 23511
[9] Green B, Schalm K, Schaar J Pieter van der and Shiu G 2005 arXiv:astro-ph/0503458
[10] Easther R, Kinney W H and Peiris H 2005 J. Cosmol. Astropart. Phys. JCAP08(2005)001 (arXiv:astro-ph/0505426)
[11] Chen X, Huang M-X, Kachru S and Shiu G 2007 J. Cosmol. Astropart. Phys. JCAP01(2007)002 (arXiv:hep-th/0605045)
[12] Holman R and Tolley A J 2008 J. Cosmol. Astropart. Phys. JCAP05(2008)001 (arXiv:0710.1302)
[13] Meerbart P D, Schaar J P van der and Corasaniti P S 2009 J. Cosmol. Astropart. Phys. JCAP05(2009)018 (arXiv:0901.4044)
[14] Sriramkumar L and Padmanabhan T 2005 Phys. Rev. D 71 103521 (arXiv:gr-qc/0408034)
[15] Jackson M G and Schalm K 2010 arXiv:hep-th:1007.0185
[16] Kaloper N, Kleban M, Lawrence A, Shenker S and Susskind L 2002 J. High Energy Phys. JHEP11(2002)037
[17] Stewart E D and Lyth D H 1993 Phys. Lett. B 302 171 (arXiv:gr-qc/9302019)
[18] Arkani-Hamed N, Creminelli P, Mukohyama S and Zaldarriaga M 2004 J. Cosmol. Astropart. Phys. JCAP04(2004)001 (arXiv:hep-th/0312100)
[19] Bars I and Chen S H 2010 arXiv:1004.0752
[20] Mortonson M J, Peiris H V and Easther R 2011 Phys. Rev. D 83 043505 (arXiv:astro-ph:1007.4205)
[21] Lidsey J E, Liddle A R, Kolb E W, Copeland E J, Barreiro T and Abney M 1997 Rev. Mod. Phys. 69 373 (arXiv:astro-ph/9508078)
[22] Weinberg S 2008 Cosmology (New York: Oxford University Press)
[23] Baumann D 2009 TASI lectures on inflation arXiv:0907.5424
[24] Grishchuk L P 1993 Class. Quantum Grav. 10 2449–78 (arXiv:gr-qc/9302036)
[25] Lyth D H and Riotto A 1999 Phys. Rep. 314 1 (arXiv:hep-ph/9807278)
[26] Nibbelink S G and van Tent B J W 2002 Class. Quantum Grav. 19 613 (arXiv:hep-ph/0107272)
[27] Maldacona J 2003 J. High Energy Phys. JHEP05(2003)013 (arXiv:astro-ph/0210603)
[28] Easther R 1993 Class. Quantum Grav. 10 2203 (arXiv:gr-qc/9308010)
[29] Barrow J D 1993 Phys. Rev. D 48 1585
[30] Barrow J D 1994 Phys. Rev. D 49 3055
[31] Ellis G F R and Madsen M S 1991 Class. Quantum Grav. 8 667
[32] Lidsey J E 1991 Class. Quantum Grav. 8 923
[33] Easther R 1996 Class. Quantum Grav. 13 1775 (arXiv:astro-ph/9511143)
[34] Mimoso J P and Charters T 2010 J. Phys.: Conf. Ser. 229 012051
[35] Abramowitz M and Stegan I A 1965 Handbook of Mathematical Functions (New York: Dover)