Elastic Plates Motions with Transverse Variation of Microrotation.

Lev Steinberg
Department of Mathematical Sciences
University of Puerto Rico
Mayaguez, PR 00681-9018, USA

February 3, 2009

Abstract

The purpose of this paper is to present a new mathematical model for the dynamics of thin Cosserat elastic plates. Our approach, which is based on a generalization of the classical Reissner-Mindlin plate theory, takes into account the transverse variation of microrotation and corresponding microintertia of the elastic plates. The model assumes polynomial approximations over the plate thickness of asymmetric stress, couple stress, displacement, and microrotation, which are consistent with the elastic equilibrium, boundary conditions and the constitutive relationships. Based on the generalized Hellinger-Prange-Reissner variational principle for the dynamics and strain-displacement relation we obtain the complete dynamic theory of Cosserat plate.

AMS Mathematics Subject Classification (2000): 74B99, 74K20, 83C15, 74E20

Key words: Cosserat materials, elastic plates, transverse microrotation, variational principle, elastodynamics

1 Introduction

This paper is straightforward extension of the static theory of Cosserat plates \cite{17} for the dynamic case. In order to describe dynamics of elastic plates with microstructure that possess grains, particles, fibers, and cellular structures A. C. Eringen (1967) was the first to propose a theory of plates in the framework of Cosserat (micropolar) Elasticity \cite{3}. His theory is based on a direct technique of integration of the Cosserat Elasticity and assumes no variation of micropolar rotations in the thickness direction. Eringen’s plate theory in the current form does not produce the Reissner-Mindlin plate equations for zero microrotations. In this paper we propose to use the Reissner-Mindlin’s plate theory as a foundation for the modeling of dynamics Cosserat elastic plates. Our approach, in
addition to the transverse shear deformation, takes into account the second order approximation of couple stresses and the variation of micropolar rotations in the thickness direction and the corresponding inertia characteristics.

2 Micropolar (Cosserat) Linear Elasticity

2.1 Fundamental Equations

Before proceeding some notation convention should be explained. We use the usual summation conventions and all expressions that contain Latin letters as subindices are understood to take values in the set \{1, 2, 3\}. When Greek letters appear as subindices then it will be assumed that they can take the values 1 or 2.

The Cosserat elasticity equilibrium equations without body forces represent the balance of linear and angular momentums of micropolar elasticity and have the following form [3]:

\[
\text{div} \sigma = \rho \ddot{u}, \quad (1)
\]
\[
\varepsilon \cdot \sigma + \text{div} \mu = J \ddot{\varphi}, \quad (2)
\]

where the quantity \(\sigma = \{\sigma_{ji}\}\) is the stress tensor, \(\mu = \{\mu_{ji}\}\) the couple stress tensor, \(u\) and \(\varphi\) the displacement and rotation vectors, \(\rho\) and \(J = \{J_i\}\) the material density and the rotatory inertia characteristics, and \(\varepsilon = \{\varepsilon_{ijk}\}\) is the Levi–Civita tensor, where \(\varepsilon_{ijk}\) equals 1 or -1 according as \((i, j, k)\) is an even or odd permutation of 1, 2, 3 and zero otherwise, and \(\varepsilon \cdot \sigma = \{\varepsilon_{ijk} \sigma_{jk}\}\).

The constitutive equation can be written in Nowacki’s form [12]:

\[
\sigma = (\mu + \alpha)\gamma + (\mu - \alpha)\gamma^T + \lambda (\text{tr} \gamma)1, \quad (3)
\]
\[
\mu = (\gamma + \epsilon)\chi + (\gamma - \epsilon)\chi^T + \beta (\text{tr} \chi)1, \quad (4)
\]

which we consider with the strain-displacement and torsion-rotation relations

\[
\gamma = (\nabla u)^T + \varepsilon \cdot \varphi \quad \text{and} \quad \chi = \nabla \varphi, \quad (5)
\]

where quantities \(\gamma\) and \(\chi\), are the micropolar strain and torsion tensors, \(1\) the identity tensor, \(\mu, \lambda\) are the symmetric and \(\beta, \gamma, \epsilon, \alpha\) the asymmetric Cosserat elasticity constants.

The constitutive equations can be written in the reversible form:

\[
\gamma = (\mu' + \alpha')\sigma + (\mu' - \alpha')\sigma^T + \lambda' (\text{tr} \sigma)1, \quad (6)
\]
\[
\chi = (\gamma' + \epsilon')\mu + (\gamma' - \epsilon')\mu^T + \beta' (\text{tr} \mu)1, \quad (7)
\]

where \(\mu' = \frac{1}{4\mu}, \alpha' = \frac{1}{4\alpha}, \gamma' = \frac{1}{4\gamma}, \epsilon' = \frac{1}{4\epsilon}, \lambda' = \frac{-\lambda}{6\mu(\lambda + 2\mu)}\) and \(\beta' = \frac{-\beta}{6\mu(\beta + 2\mu)}\).

\[^1\]Neff [10] uses the notation \(\mu_c\) for elastic parameter \(\alpha\).
We consider a Cosserat elastic body $B_0$. In this case the equilibrium equations (1) - (2) with constitutive formulas (3) - (4) and kinematics formulas (5) should be accompanied by the following mixed boundary

$$\mathbf{u} = \mathbf{u}_o, \varphi = \varphi_o \text{ on } \partial B \setminus \partial B_\sigma,$$

$$\sigma \cdot \mathbf{n} = \sigma, \mu \cdot \mathbf{n} = \mu_o \text{ on } \partial B_\sigma,$$

and initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{U}_o, \varphi(\mathbf{x},0) = \Phi_o \text{ in } B_0,$$

$$\dot{\mathbf{u}}(\mathbf{x},0) = \dot{\mathbf{U}}_o, \dot{\varphi}(\mathbf{x},0) = \dot{\Phi}_o \text{ on } B_0,$$

where $\mathbf{u}_o, \varphi_o$ are prescribed on $\partial B_1, \sigma_o$ and $\mu$ on $\partial B_2$, and $\mathbf{n}$ denotes the outward unit normal vector to $\partial B_0$.

### 2.2 Cosserat Elastic Energy

The strain stored energy $U_C$ of the body $B_0$ is defined by the integral [12]:

$$U_C = \int_{B_0} W\{\gamma, \chi\} \, dv,$$

where

$$W\{\gamma, \chi\} = \frac{\mu + \alpha}{2} \gamma_{ij} \gamma_{ij} + \frac{\mu - \alpha}{2} \gamma_{ij} \gamma_{ji} + \frac{\lambda}{2} \gamma_{kk} \gamma_{nn}$$

$$+ \gamma + \epsilon \gamma_{ij} \gamma_{ij} + \gamma - \epsilon \gamma_{ij} \gamma_{ji} + \frac{\beta}{2} \chi_{kk} \chi_{nn},$$

then the constitutive relations (3) - (4) can be written in the form:

$$\sigma = C_\sigma \{W\} = \nabla_\gamma W \text{ and } \mu = C_\mu \{W\} = \nabla_\chi W.$$  

The function $W$ is non-negative if and only if [12]

$$\mu > 0, \quad 3\lambda + 2\mu > 0,$$

$$\gamma > 0, \quad 3\beta + 2\gamma > 0,$$

$$\alpha > 0, \quad \mu + \alpha > 0,$$

$$\epsilon > 0, \quad \gamma + \epsilon > 0.$$  

For future convenience, we present the stress energy

$$U_K = \int_{B_0} \Phi\{\sigma, \mu\} \, dv,$$

where

$$\Phi\{\sigma, \mu\} = \frac{\mu' + \alpha'}{2} \sigma_{ij} \sigma_{ij} + \frac{\mu' - \alpha'}{2} \sigma_{ij} \sigma_{ji} + \frac{\lambda'}{2} \sigma_{kk} \sigma_{nn}$$

$$+ \frac{\gamma' + \epsilon'}{2} \mu_{ij} \mu_{ij} + \frac{\gamma' - \epsilon'}{2} \mu_{ij} \mu_{ji} + \frac{\beta'}{2} \mu_{kk} \mu_{nn}.$$  

3
The reversible constitutive relation (6) - (7) can be also written in form:

\[
\gamma = K_\gamma [\sigma] = \frac{\partial \Phi}{\partial \sigma}, \quad \chi = K_\chi [\mu] = \frac{\partial \Phi}{\partial \mu}
\]  

(17)

The total internal work done by the stresses \(\sigma\) and \(\mu\) over the strains \(\gamma\) and \(\chi\) for the body \(B_0\) is

\[
U = \int_{B_0} [\sigma \cdot \gamma + \mu \cdot \chi] \, dv
\]

(18)

and

\[
U = U_K = U_C
\]

provided the constitutive relations (3) - (4) hold.

2.3 The Generalized Hellinger-Prange-Reissner (HPR) Principle

The HPR principle [5] in the case of Cosserat elasticity states, that for any set \(A\) of all admissible states \(s = [u, \varphi, \gamma, \chi, \sigma, \mu]\) that satisfy the strain-displacement, torsion-rotation relations (5) and the initial condition, the zero variation

\[
\delta \Theta(s) = 0
\]

of the functional

\[
\Theta(s) = U_K - \int_{B_0} [\sigma \cdot \gamma - \rho \ddot{u} \cdot u + \mu \cdot \chi - J \ddot{\varphi} \cdot \varphi] \, dv
\]

\[
+ \int_{\Gamma_1} [\sigma_n \cdot (u - u_o) + \mu_n (\varphi - \varphi_o)] \, da + \int_{\Gamma_2} [\sigma_o \cdot u + \mu_o \cdot \varphi] \, da
\]

at \(s \in A\) is equivalent of \(s\) to be a solution of the system of equilibrium equations (1) - (2), constitutive relations (6) - (7), which satisfies the mixed boundary conditions (8) - (9). The proof is similar to the proof for HPR principle for classic linear elasticity [3].

3 The Cosserat Plate Assumptions

In this section we formulate our stress, couple stress and kinematic assumptions of the Cosserat plate. The set of points \(P = \{\Gamma \times [-h/2, h/2]\} \cup T \cup B\) forms the entire surface of the plate and \(\{\Gamma_u \times [-h/2, h/2]\}\) is the lateral part of the boundary where displacements and microrotations are prescribed. The notation \(\Gamma_o = \Gamma \setminus \Gamma_u\) of the remainder we use to describe the lateral part of the boundary edge \(\{\Gamma_o \times [-h/2, h/2]\}\) where stress and couple stress are prescribed. We also use notation \(P_0\) for the middle plane internal domain of the plate.
In our case we consider the vertical load and pure twisting momentum boundary conditions at the top and bottom of the plate, which can be written in the form:

\[
\begin{align*}
\sigma_{33}|_{x_3 = h/2} &= \sigma^t(x_1, x_2, t), \quad \sigma_{33}|_{x_3 = -h/2} = \sigma^b(x_1, x_2, t), \quad (20) \\
\sigma_{33}|_{x_3 = h/2} &= 0, \quad \sigma_{33}|_{x_3 = -h/2} = 0 \quad (21) \\
\mu_{33}|_{x_3 = h/2} &= \mu^t(x_1, x_2, t), \quad \mu_{33}|_{x_3 = -h/2} = \mu^b(x_1, x_2, t) \quad (22) \\
\mu_{33}|_{x_3 = h/2} &= 0 = 0, \quad \mu_{33}|_{x_3 = -h/2} = 0, \quad (23)
\end{align*}
\]

where \((x_1, x_2) \in P_0\).

### 3.1 Stress, Couple Stress and Kinematics Assumptions

Our approach, which is in the spirit of the Reissner’s theory of plates [13], assumes that the variation of stress \(\sigma_{kl}\) and couple stress \(\mu_{kl}\) components across the thickness can be represented by means of polynomials of \(x_3\). We adapt the expressions for the stress and couple-stress components in the following form [17]:

\[
\sigma_{\alpha\beta} = n_{\alpha\beta}(x_1, x_2, t) + \frac{h}{2} \zeta_3 m_{\alpha\beta}(x_1, x_2, t), \quad (24)
\]

\[
\sigma_{3\beta} = q_\beta(x_1, x_2, t) (1 - \zeta_3^2), \quad (25)
\]

\[
\sigma_{33} = -\frac{3}{4} \left( \frac{1}{3} \zeta_3^3 - \zeta_3 \right) p + \sigma_0, \quad (27)
\]

\[
\mu_{\alpha\beta} = (1 - \zeta_3^2) r_{\alpha\beta}(x_1, x_2, t). \quad (28)
\]

\[
\mu_{33} = \zeta_3 s_3(x_1, x_2, t) + m_3(x_1, x_2, t). \quad (29)
\]

where \(p = \sigma^t(x_1, x_2, t) - \sigma^b(x_1, x_2, t), \quad \sigma_0 = \frac{1}{2} \left( \sigma^t(x_1, x_2, t) + \sigma^b(x_1, x_2, t) \right), \quad v(x_1, x_2) = \frac{1}{2} \left( \mu^t(x_1, x_2) - \mu^b(x_1, x_2) \right), \quad t(x_1, x_2) = \frac{1}{2} \left( \mu^t(x_1, x_2) + \mu^b(x_1, x_2) \right). \)

satisfy the boundary condition requirements. We note that expression (27) is identical to the expression of \(\sigma_{33}\) given in [13] in the case of \(\sigma^b = 0\).

We also assume displacements \(u_\alpha\) are also distributed linearly over the thickness of the plate [3] and that \(u_3\) does not vary over the thickness of the plate, i.e.

\[
\begin{align*}
\quad u_\alpha &= U_\alpha(x_1, x_2, t) - \frac{h}{2} \zeta_3 V_\alpha(x_1, x_2, t), \quad (32) \\
\quad u_3 &= w(x_1, x_2, t), \quad (33)
\end{align*}
\]

5
where the terms $V_\alpha(x_1, x_2, t)$ represent the rotations in middle plane. The variation of microrotation with respect to $x_3$ be represented by means of the second and third order polynomials [17]:

\[
\begin{align*}
\varphi_\alpha &= \Theta_\alpha^0(x_1, x_2, t) \left(1 - \zeta_3^2\right), \\
\varphi_3 &= \Theta_3^0(x_1, x_2, t) + \zeta_3 \left(1 - \frac{1}{3} \zeta_3^2\right) \Theta_3(x_1, x_2, t).
\end{align*}
\]

where $\zeta_3 = \frac{2}{h} x_3$, $\alpha, \beta \in \{1, 2\}$.

We also assume that initial condition can be presented in the similar form, so they can be reduced to the form

\[
\begin{align*}
U_\alpha(x_1, x_2, 0) &= U_\alpha^0(x_1, x_2), \quad V_\alpha(x_1, x_2, 0) = V_\alpha^0(x_1, x_2), \\
w(x_1, x_2, 0) &= w^0(x_1, x_2),
\end{align*}
\]

\[
\begin{align*}
\dot{U}_\alpha(x_1, x_2, 0) &= \dot{U}_\alpha^0(x_1, x_2), \quad \dot{V}_\alpha(x_1, x_2, 0) = \dot{V}_\alpha^0(x_1, x_2), \\
\dot{w}(x_1, x_2, 0) &= \dot{w}^0(x_1, x_2)
\end{align*}
\]

and

\[
\begin{align*}
\Theta_\alpha^0(x_1, x_2, 0) &= \Theta_{\alpha 0}^0(x_1, x_2), \\
\Theta_3^0(x_1, x_2, 0) &= \Theta_{0 3}^0(x_1, x_2) \text{ and } \Theta_3(x_1, x_2, t) = \Theta_{0 3}(x_1, x_2).
\end{align*}
\]

\[
\begin{align*}
\dot{\Theta}_\alpha^0(x_1, x_2, 0) &= \dot{\Theta}_{\alpha 0}^0(x_1, x_2), \\
\dot{\Theta}_3^0(x_1, x_2, 0) &= \dot{\Theta}_{0 3}^0(x_1, x_2) \text{ and } \dot{\Theta}_3(x_1, x_2, t) = \dot{\Theta}_{0 3}(x_1, x_2).
\end{align*}
\]

4 Specification of HPR Variational Principle for the Dynamics of Cosserat Plates

The HPR variational principle for a Cosserat plate is most appropriately expressed in terms of corresponding integrands calculated across the whole thickness. We also introduce the weighted characteristics of displacements, microrotations, strains and stresses of the plate, which will be used to produce the explicit forms of these integrands.

4.1 The Cosserat plate stress energy density

We define the plate stress energy density by the formula:

\[
\Phi(S) = \frac{h}{2} \int_{-1}^{1} \Phi\{\sigma, \mu\} \, d\zeta_3.
\]
Taking into account the stress and couple stress assumptions \([21] - [31]\) and by the integrating \(\Phi \{\sigma, \mu\}\) with respect \(\zeta_3\) in \([-1, 1]\) we obtain the explicit plate stress energy density expression in the form \([17]\):

\[
\Phi(S) = \frac{\lambda + \mu}{2h\mu(3\lambda + 2\mu)} \left[ N_{\alpha\alpha}^2 + \frac{12}{h^2} M_{\alpha\alpha}^2 \right] - \frac{\lambda}{2h\mu(3\lambda + 2\mu)} \left[ N_{11} N_{22} + \frac{12}{h^2} M_{11} M_{22} \right] + \frac{\alpha + \mu}{8h\alpha\mu} \left[ (1 - \delta_{\alpha\beta}) \left( N_{\alpha\beta}^2 + \frac{12}{h^2} M_{\alpha\beta}^2 \right) + \frac{6}{5} (Q_{\alpha} Q_{\alpha} + Q_{\beta} Q_{\beta}) \right] + \frac{3(\alpha - \mu)}{10h\alpha\mu} \left[ Q_{\alpha} Q_{\alpha}^* + \frac{5}{6} N_{12} N_{21} + \frac{10}{h^2} M_{12} M_{21} \right] + \frac{3\lambda}{5h\mu(3\lambda + 2\mu)} Q_{\alpha\alpha}^* M_{\beta\beta}^* + \frac{3}{5h\gamma(3\beta + 2\gamma)} \left[ (\beta + \gamma) R_{\alpha\alpha}^2 - \beta R_{11} R_{22} \right] + \frac{3}{10h} \left( \frac{1}{\gamma} - \frac{1}{\epsilon} \right) R_{12} R_{21} + \frac{17h(\lambda + \mu)}{280h(3\lambda + 2\mu)} \left( Q_{\alpha\alpha}^* \right)^2 + \frac{\lambda}{2\mu(3\lambda + 2\mu)} (N_{\alpha\alpha}) \sigma_0^2 + \frac{h(\lambda + \mu)}{2\mu(3\lambda + 2\mu)} \sigma_0^2 + \frac{\gamma}{h\epsilon} \left[ \frac{1}{8} M_{\alpha}^* M_{\alpha}^* + \frac{3}{2h^2} S_{\alpha}^* S_{\alpha}^* + \frac{3}{20} (1 - \delta_{\beta\gamma}) R_{\beta\gamma}^2 \right] - \frac{\beta}{2(3\beta + 2\gamma)} R_{\alpha\alpha} t + \frac{h(\beta + \gamma)}{2(3\beta + 2\gamma)} t^2 + \frac{h(\beta + \gamma)}{6\gamma(3\beta + 2\gamma)} \sigma^2, \tag{42}\]

where the Cosserat stress set

\[
S = [M_{\alpha\beta}, Q_{\alpha}, Q_{3\alpha}, R_{\alpha\beta}, S_{\alpha}^*, N_{\alpha\beta}, M_{\alpha}^*], \tag{43}\]

where

\[
M_{\alpha\beta} = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \zeta_3 \sigma_{\alpha\beta} d\zeta_3 = \frac{h^3}{12} m_{\alpha\beta}, \tag{44}\]

\[
Q_{\alpha} = \frac{h}{2} \int_{-1}^{1} \sigma_{3\alpha} d\zeta_3 = \frac{2h}{3} q_{\alpha}, \quad Q_{\alpha}^* = \frac{h}{2} \int_{-1}^{1} \sigma_{3\alpha} d\zeta_3 = \frac{2h}{3} q_{\alpha}^* \]

\[
R_{\alpha\beta} = \frac{h}{2} \int_{-1}^{1} \mu_{\alpha\beta} d\zeta_3 = \frac{2h}{3} r_{\alpha\beta}, \]

\[
S_{\alpha}^* = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \zeta_3 \mu_{\alpha3} d\zeta_3 = \frac{h^2}{6} s_{\alpha}^*, \]

\[
N_{\alpha\beta} = \frac{h}{2} \int_{-1}^{1} \sigma_{\alpha3} d\zeta_3 = h n_{\alpha\beta}, \quad M_{\alpha}^* = \frac{h}{2} \int_{-1}^{1} \mu_{\alpha3} d\zeta_3 = h m_{\alpha}^*, \]

Here \(M_{11}\) and \(M_{22}\) are the bending moments, \(M_{12}\) and \(M_{21}\) the twisting moments, \(Q_{\alpha}\) the shear forces, \(Q_{\alpha}^*\) the transverse shear forces, \(R_{11}\) and \(R_{22}\) the micropolar bending moments, \(R_{12}\) and \(R_{21}\) the micropolar twisting moments,
\( S_\alpha^* \) the micropolar couple moments, all defined per unit length, \( N_{11} \) and \( N_{22} \) are the bending forces, \( N_{12} \) and \( N_{21} \) the twisting forces, \( M_\alpha^* \) the micropolar shear couple-stress resultants.

Then the stress energy of the plate \( P \)

\[
U^K_S = \int_{P_0} \Phi(S) \, da, \tag{45}
\]

where \( P_0 \) is the internal domain of the middle plane of the plate \( P \).

4.2 The density of the work done over the Cosserat plate boundary

In the following consideration we also assume that the proposed stress, couple stress, and kinematic assumptions are valid for the lateral boundary of the plate \( P \) as well.

We evaluate the density of the work over the boundary \( \Gamma_u \times [-h/2, h/2] \)

\[
\mathcal{W}_1 = \frac{h}{2} \int_{-1}^{1} [\mathbf{\sigma}_n \cdot \mathbf{u} + \mu_n \Psi] \, d\zeta. \tag{46}
\]

Taking into account the stress and couple stress assumptions (24) - (31) and kinematic assumptions (32) - (35) we are able to represent \( \mathcal{W}_1 \) by the following expression [17]:

\[
\mathcal{W}_1 = \mathcal{S}_n \mathcal{U} = \tilde{M}_\alpha \Psi_{\alpha} + \tilde{Q}^* W + \tilde{R}_\alpha \Omega^0_{\alpha} + \tilde{S}^* \Omega_{3} + \tilde{N}_\alpha U_\alpha + \tilde{M}^* \Omega^0_{3}, \tag{47}
\]

where the sets \( \mathcal{S}_n \) and \( \mathcal{U} \) are defined as

\[
\mathcal{S}_n = [\tilde{M}_\alpha, \tilde{Q}^*, \tilde{R}_\alpha, \tilde{S}^*, \tilde{N}_\alpha, \tilde{M}^*],
\]

\[
\mathcal{U} = [\Psi_{\alpha}, W, \Omega^0_{\alpha}, \Omega_{3}, U_\alpha, \Omega^0_{3}]
\]

and

\[
\tilde{M}_\alpha = M_{\alpha\beta} n_\beta, \quad \tilde{Q}^* = Q^*_{\beta} n_\beta, \quad \tilde{R}_\alpha = R_{\alpha\beta} n_\beta,
\]

\[
\tilde{S}^* = S^*_{\beta} n_\beta, \quad \tilde{N}_\alpha = N_{\alpha\beta} n_\beta, \quad \tilde{M}^* = M^*_{\beta} n_\beta.
\]

In the above \( n_\beta \) is the outward unit normal vector to \( \Gamma_u \), and
Here $\Psi_\alpha$ are the rotations of the middle plane around $x_\alpha$ axis, $W$ the vertical deflection of the middle plate, $\Omega_0^\alpha$ the microrotations in the middle plate around $x_k$ axis, $U_\alpha$ is the in-plane displacements of the middle plane along $x_a$ axis, $\Omega_3$ the rate of change of the microrotation $\varphi_3$ along $x_3$.

We also obtain the correspondence between the weighted displacement and the microrotations (48) and the kinematic variables by applying (??) and (35) in integration of expressions (48):

$$
\Psi_\alpha = V_\alpha(x_1, x_2, t), W = w(x_1, x_2, t),
$$

$$
\Omega_0^\alpha = k_1^* \Theta_0^\alpha(x_1, x_2, t), \quad \Omega_3 = k_2^* \Theta_3(x_1, x_2, t),
$$

$$
U_\alpha = U_\alpha(x_1, x_2, t), \quad \Omega_3^0 = \Theta_3^0(x_1, x_2, t),
$$

where coefficients $k_1^*$ and $k_2^*$ depend on the variation of microrotations. Under the conditions (35) we have that $k_1^* = \frac{1}{4}$ and $k_2^* = \frac{8}{5}$.

The density of the work over the boundary $\Gamma_\sigma \times [-h/2, h/2]$

$$
\mathcal{W}_2 = \frac{h}{2} \int_{-1}^{1} (\sigma_{\alpha\alpha} u_\alpha + m_{\alpha\alpha} \varphi_\alpha) n_\alpha d\zeta_3
$$

can be presented in the form [17]

$$
\mathcal{W}_2 = \mathcal{S}_0 \mathcal{U} = \Pi_{\alpha\alpha} \Psi_\alpha + \Pi_{\alpha3} W + M_{\alpha\alpha} \Omega_0^\alpha + M_{\alpha3}^* \Omega_3 + \Sigma_{\alpha\alpha} U_\alpha + \Upsilon_{\alpha3} \Omega_3^0,
$$

where

$$
\mathcal{S}_0 = [\Pi_{\alpha\alpha}, \Pi_{\alpha3}, M_{\alpha\alpha}, M_{\alpha3}^*, \Sigma_{\alpha\alpha}, \Upsilon_{\alpha3}]
$$

$$
\mathcal{U} = [\Psi_\alpha, W, \Omega_0^\alpha, \Omega_3, U_\alpha, \Omega_3^0],
$$

$$
M_{\alpha\beta} n_\beta = \Pi_{\alpha\alpha}, \quad R_{\alpha\beta} n_\beta = M_{\alpha\alpha},
$$

$$
Q_\alpha^* n_\alpha = \Pi_{\alpha3}, \quad S_\alpha^* n_\alpha = M_{\alpha3}^*.
$$

(50)
\[ N_{\alpha\beta}n_\beta = \Sigma_\alpha, \]
\[ M^*_\alpha n_\alpha = \Upsilon_{3\alpha}. \quad (51) \]

Now \( n_\beta \) is the outward unit normal vector to \( \Gamma_\sigma \), and

\[ \Pi_{o\alpha} = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \zeta_3 \sigma_{o\alpha} d\zeta_3, \quad M_{o\alpha} = \frac{h}{2} \int_{-1}^{1} \mu_{o\alpha} d\zeta_3, \]
\[ \Pi_{3\alpha} = \frac{h}{2} \int_{-1}^{1} (\sigma_{3\alpha} - \sigma_0) d\zeta_3, \quad M^*_{3\alpha} = \frac{h}{2} \int_{-1}^{1} (\mu_{3\alpha} - tn_3) d\zeta_3, \]
\[ \Sigma_{o,\alpha} = \frac{h}{2} \int_{-1}^{1} \sigma_{o\alpha} d\zeta_3, \quad \Upsilon_{3\alpha} = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \zeta_3 (\mu_{3\alpha} - \zeta_3 v) d\zeta_3. \quad (52) \]

We are able to evaluate the work done at the top and bottom of the Cosserat plate by using boundary conditions (20) and (22)

\[ \int_{T \cup B} (\sigma_{o3} u_3 + m_{o3} \varphi_{o3}) n_3 da = \int_{P_0} (pW + vQ_3) da. \]

4.3 The Cosserat plate internal work density

Here we define the density of the work done by the stress and couple stress over the Cosserat strain field:

\[ W_3 = \frac{h}{2} \int_{-1}^{1} (\sigma \cdot \gamma + \mu \cdot \chi) d\zeta_3. \quad (53) \]

Substituting stress and couple stress assumptions (24) - (31) and integrating expression (54) we obtain the following expression (55):

\[ W_3 = \mathcal{E} \cdot \mathcal{S} = M_{\alpha\beta} e_{\alpha\beta} + Q_{\alpha} \omega_{\alpha} + Q_{3\alpha} \omega_{\alpha}^* + R_{\alpha\beta} \tau_{\alpha\beta} + S_{\alpha} \tau_{3\alpha} + N_{\alpha\beta} v_{\alpha\beta} + M^*_{\alpha} r_{3,\alpha}, \]

where \( \mathcal{E} \) is the Cosserat plate strain set of the the weighted averages of strain and torsion tensors

\[ \mathcal{E} = [e_{\alpha\beta}, \omega_{\beta}, \omega_{\alpha}^*, \tau_{3\alpha}, \tau_{\alpha\beta}, v_{\alpha\beta}, r^0_{3,\alpha}]. \]
Here the components of $E$ are

\begin{align*}
e_{\alpha\beta} &= \frac{3}{h} \int_{-1}^{1} \zeta_3 \gamma_{\alpha\beta} d\zeta_3, \quad (56) \\
\omega_\alpha &= \frac{3}{4} \int_{-1}^{1} \gamma_{\alpha3} (1 - \zeta^2) d\zeta_3, \quad (57) \\
\omega^*_\alpha &= \frac{3}{4} \int_{-1}^{1} \gamma_{3\alpha} (1 - \zeta^2) d\zeta_3, \quad (58) \\
\tau_{3\alpha} &= \frac{3}{h} \int_{-1}^{1} \zeta_3 \chi_{3\alpha} d\zeta_3, \quad (59) \\
\tau^0_{\alpha\beta} &= \frac{3}{4} \int_{-1}^{1} \chi_{\alpha\beta} (1 - \zeta^2) d\zeta_3, \quad (60) \\
v_{\alpha\beta} &= \frac{1}{2} \int_{-1}^{1} \gamma_{\alpha\beta} d\zeta_3, \quad (61) \\
\tau^0_{3\alpha} &= \frac{1}{2} \int_{-1}^{1} \chi_{3\alpha} d\zeta_3. \quad (62)
\end{align*}

The components of Cosserat plate strain (56)-(62) can also be represented in terms of the components of set $U$ by the following formulas [17]:

\begin{align*}
e_{\alpha\beta} &= \Psi_{\beta,\alpha} + \varepsilon_{3\alpha\beta} \Omega^0_3, \\
\omega_\alpha &= \Psi_{\alpha} + \varepsilon_{3\alpha\beta} \Omega^0_{\beta}, \\
\omega^*_\alpha &= W_{\alpha} + \varepsilon_{3\alpha\beta} \Omega^0_{\beta}, \\
\tau_{3\alpha} &= \Omega^0_{3,\alpha}, \\
\tau^0_{\alpha\beta} &= \Omega^0_{\beta,\alpha}, \\
v_{\alpha\beta} &= U_{\beta,\alpha} + \varepsilon_{3\alpha\beta} \Omega^0_3, \\
\tau^0_{3\alpha} &= \Omega^0_{3,\alpha}.
\end{align*}

(63)

We call the relation (63) the Cosserat plate strain-displacement relation.

### 4.4 The density of the kinetic energy

Here we define the density of the kinetic energy:

$$W_4 = \frac{1}{2} \int_{-1}^{1} (\rho \dddot{u} \cdot u + J \dddot{\phi} \cdot \varphi) d\zeta_3,$$

which can be presented in the form

$$W_4 = K \ddot{U} \cdot U = I_0 \ddot{\Psi}_\alpha \Psi_\alpha + \rho_3 \ddot{W} W + I_0 \ddot{\Omega}^0_\alpha \Omega^0_\alpha + J_3 \ddot{\Omega}_3 \Omega_3 + \rho_o \ddot{\bar{U}}_\alpha U_\alpha + I_0 \ddot{\bar{\Omega}}_3 \bar{\Omega}^0_3,$$

where

$$U = [\ddot{\Psi}_\alpha, \ddot{W}, \ddot{\Omega}^0_\alpha, \ddot{\Omega}_3, \ddot{U}_\alpha, \ddot{\bar{\Omega}}^0_3],$$

11
\[ K = [I_0, \rho_0, I_{o\alpha}, J_3^*, \rho_0, I_{o\beta}], \]

and

\[ K\ddot{U} = \left[ I_0\ddot{\Psi}_\alpha, \rho_0 \ddot{W}, I_{o\alpha}\ddot{\Omega}_2, J_3^*\ddot{\Omega}_3, \rho_0 \ddot{U}_\alpha, I_{o\beta}\ddot{\Omega}_3 \right], \]

where

\[ I_o = \frac{\rho h^3}{12}, \quad \rho_0 = \rho h, \quad I_{o\alpha} = k_3^* J_3 h, \quad J_3^* = k_4^* J_3 h^3, \quad I_{o\beta} = J_3 h, \]

\[ k_3^* = \frac{5}{6}, \quad k_4^* = \frac{25}{32}. \]

### 5 Cosserat Plate HPR Principle

It is natural now to reformulate HPR variational principle for the Cosserat plate \( P \). Let \( \mathcal{A} \) denote the set of all admissible states that satisfy the Cosserat plate strain-displacement relation (63) and let \( \Theta \) be a HPR functional on \( \mathcal{A} \) defined by

\[ \Theta(s) = U_K^S - \int_{P_0} (S \cdot E - K\ddot{U} \cdot U - pW - \varepsilon \Omega_3^0) da + \int_{\Gamma_\sigma} S_\sigma \cdot (\mathcal{U} - \mathcal{U}_o) ds + \int_{\Gamma_u} S_u \mathcal{U} ds, \]

for every \( s = [\mathcal{U}, \mathcal{E}, \mathcal{S}] \in \mathcal{A} \), then

\[ \delta \Theta(s) = 0 \]

is equivalent to the following plate bending (A) and twisting (B) mixed problems.

A. The flexural motions system of equations:

\[ M_{\alpha\beta,\alpha} - Q_\beta = I_0\ddot{\Psi}_\beta, \]

\[ Q_{a,\alpha}^* + p = \rho_0 \ddot{W}, \]

\[ R_{\alpha\beta,\alpha} + \varepsilon_{33\gamma} (Q_\gamma^* - Q_\gamma) = I_{o\beta} \ddot{\Omega}_3, \]

\[ S_{a,\alpha}^* + \varepsilon_{33\gamma} M_{\beta\gamma} = J_3^* \ddot{\Omega}_3, \]

with the resultant traction boundary conditions :

\[ M_{\alpha\beta} n_\beta = \Pi_{\alpha\beta}, \quad R_{\alpha\beta} n_\beta = M_{\alpha\beta}, \]

\[ Q_{a}^* n_\alpha = \Pi_{o3}, \quad S_{a}^* n_\alpha = \Upsilon_{o3}. \]

at the part \( \Gamma_\sigma \) and the resultant displacement boundary conditions
\[ \Psi_\alpha = \Psi_\alpha^0, \ W = W_\alpha, \ \Omega^0_{\alpha} = \Omega^0_{\alpha}, \ \Omega_3 = \Omega_{\alpha3}, \quad (71) \]

at the part \( \Gamma_u \).

The constitutive formulas:

\[ e_{\alpha\alpha} = \frac{\partial \Phi}{\partial M_{\alpha\alpha}} = \frac{12(\lambda + \mu)}{h^3\mu(3\lambda + 2\mu)} M_{\alpha\alpha} \]
\[ - \left| \varepsilon_{\alpha\beta} \right| \frac{6\lambda}{h^3\mu(3\lambda + 2\mu)} M_{\beta\beta} + \frac{3\lambda}{5h\mu(3\lambda + 2\mu)} (Q^*_\beta\beta), \quad (72) \]

\[ e_{\alpha\beta} = \frac{\partial \Phi}{\partial M_{\alpha\beta}} = \frac{3(\alpha + \mu)}{h^3\alpha\mu} M_{\alpha\beta} + \frac{3(\alpha - \mu)}{h^3\alpha\mu} M_{\beta\alpha}, \quad \alpha \neq \beta \]

\[ \omega_\alpha = \frac{\partial \Phi}{\partial Q^*_\alpha} = \frac{3(\alpha - \mu)}{10h\alpha\epsilon} Q^*_\alpha + \frac{3(\alpha + \mu)}{10h\alpha\epsilon} Q^*_\alpha, \quad (73) \]

\[ \omega^*_\alpha = \frac{\partial \Phi}{\partial Q^\alpha} = \frac{3(\alpha - \mu)}{10h\alpha\epsilon} Q^\alpha + \frac{3(\alpha + \mu)}{10h\alpha\epsilon} Q^\alpha, \]

\[ \tau^0_{\alpha\alpha} = \frac{\partial \Phi}{\partial R_{\alpha\alpha}} = \frac{6(\beta + \gamma)}{5h\gamma(3\beta + 2\gamma)} R_{\alpha\alpha} - \frac{3\beta}{5h\gamma(3\beta + 2\gamma)} R_{\beta\beta} - \frac{\beta}{2\gamma(3\beta + 2\gamma)} t, \]
\[ |\varepsilon_{\alpha\beta}| \frac{3\beta}{5h\gamma(3\beta + 2\gamma)} R_{\beta\beta} - \frac{3\beta}{5h\gamma(3\beta + 2\gamma)} R_{\beta\beta} - \frac{\beta}{2\gamma(3\beta + 2\gamma)} t, \]
\[ \tau^0_{\alpha\beta} = \frac{\partial \Phi}{\partial R_{\beta\alpha}} = \frac{3(\epsilon - \gamma)}{10h\gamma\epsilon} R_{\alpha\beta} + \frac{3(\gamma + \epsilon)}{10h\gamma\epsilon} R_{\beta\alpha}, \quad \alpha \neq \beta \]
\[ \tau_{3\alpha} = \frac{\partial \Phi}{\partial S^\alpha} = \frac{3(\gamma + \epsilon)}{h^3\gamma\epsilon} S^\alpha. \quad (76) \]

B. The extensional motions system of equations:

\[ N_{\alpha\beta,\alpha} = \rho_\alpha \tilde{U}_\beta, \quad (77) \]
\[ M^*_{\alpha,\alpha} + \epsilon_{3\beta\gamma} N_{3\beta\gamma} + v = I_3 \tilde{\Omega}^0_{3}, \quad (78) \]

with the resultant traction boundary conditions at \( \Gamma_\sigma \):

\[ N_{\alpha\beta n_\beta} = \Sigma_\alpha, \quad (79) \]
\[ M^*_{\alpha n_\alpha} = M^*_{\alpha3}, \quad (80) \]

and the resultant displacement boundary conditions at \( \Gamma_u \):

\[ U_\alpha = U_{\alpha}, \ \Omega^0_{3} = \Omega^0_{\alpha3}, \quad (81) \]

The constitutive formulas:
\[ \omega_{\alpha\alpha} = \frac{\partial \Phi}{\partial N_{\alpha\alpha}} = \frac{\lambda + \mu}{h\mu(3\lambda + 2\mu)} N_{\alpha\alpha} - \frac{\lambda}{2h\mu(3\lambda + 2\mu)} N_{(\alpha+1)(\alpha+1)} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_0, \quad (82) \]

\[ \omega_{\alpha\beta} = \frac{\partial \Phi}{\partial N_{\alpha\beta}} = \frac{\alpha + \mu}{4h\alpha\mu} N_{\alpha\beta} + \frac{\alpha - \mu}{4h\alpha\mu} N_{\beta\alpha}, \quad \alpha \neq \beta \quad (83) \]

\[ \tau^0_{3\alpha} = \frac{\partial \Phi}{\partial M^*_{\alpha}} = \frac{\gamma + \epsilon}{4h\gamma\epsilon} M^*. \quad (84) \]

We also represent the above constitutive relation in the compact form:

\[ \mathcal{E} = \mathcal{K} [\mathcal{S}] = \mathcal{K} \cdot \mathcal{S}, \]

where we call \( \mathcal{K} \) the compliance Cosserat plate tensor.

Proof of the theorem. The variation of \( \Theta(s) \)

\[ \delta\Theta(s) = \int_{P_0} \left\{ (\mathcal{K} [\mathcal{S}] - \mathcal{E}) \cdot \delta \mathcal{S} - K\mathcal{U} \cdot \delta \mathcal{U} - S\delta \mathcal{E} + p\delta W + v\delta \Omega^0_{3}\right\} da \]

\[ + \int_{\Gamma_s} \{ \delta \mathcal{S}_o \cdot (\mathcal{U} - \mathcal{U}_o) + \mathcal{S}_o \cdot \delta \mathcal{U} \} ds + \int_{\Gamma_u} \mathcal{S}_n \cdot \delta \mathcal{U} ds. \]

We apply Green’s theorem and integration by parts for \( \mathcal{S} \) and \( \delta \mathcal{U} \) to the expression:

\[ \int_{P_0} \mathcal{S} \cdot \delta \mathcal{E} da = \int_{P_0} \mathcal{S}_o \delta \mathcal{U} \ ds - \int_{P_0} \left\{ (M_{\alpha\beta,\alpha} - Q\beta) \delta \Psi + Q^{*\alpha\alpha} \delta W \right\} da \]

\[ + (R_{\alpha\beta,\alpha} + \epsilon_{3\beta\gamma} (Q^*_\gamma - Q_\gamma) R_{\alpha\beta,\alpha}) \delta \Omega^0_{\beta} \]

\[ + (S^{*\alpha\alpha} + \epsilon_{3\beta\gamma} M_{\beta\gamma}) \delta \Omega^0_3 + N_{\alpha\beta,\alpha} \delta U_\beta \]

\[ + (M^{*\alpha\alpha} + \epsilon_{3\beta\gamma} N_{\beta\gamma}) \delta \Omega^0_3 \} da. \]

Then based on the fact that \( \delta \mathcal{U} \) and \( \delta \mathcal{E} \) satisfy the Cosserat plate strain-displacement relation \([53]\), we obtain
\[
\delta \Theta (s) = \int_{P_B} \{(K[S] - E) \cdot \delta S - S \delta E \} \, da \\
+ \int_{P_B} \{ (M_{\alpha \beta, \alpha} - Q_\beta - I_\alpha \ddot{\Psi}_\beta) \, \delta \Psi_\beta + (Q^*_{\alpha, \alpha} + p - \rho_o \ddot{W}) \, \delta W \\
+ \left( R_{\alpha \beta, \alpha} + \varepsilon_{33 \gamma} (Q^*_\gamma - Q_\gamma) R_{\alpha \beta, \alpha} - I_{\alpha \beta} \ddot{\Omega}_3^0 \right) \, \delta \Omega_3^0 \\
+ \left( S^*_{\alpha, \alpha} + \varepsilon_{33 \gamma} M_{\beta \gamma} - J_3 \ddot{\Omega}_3 \right) \delta \Omega_3 + \left( N_{\alpha \beta, \alpha} - \rho_o \ddot{U}_\beta \right) \delta U_\beta \\
+ \left( M^*_{\alpha, \alpha} + \varepsilon_{33 \gamma} N_{\beta \gamma} + v - I_{\alpha \beta} \ddot{\Omega}_3^0 \right) \delta \Omega_3^0 \, da \\
+ \int_{\Gamma_s} \delta S_o \cdot (U - U_o) \, ds + \int_{\Gamma_u} (S_o - S_n) \delta U \, ds.
\]

If \( s \) is a solution of the mixed problem, then

\[ \delta \Theta (s) = 0. \]

On the other hand, some extensions of the fundamental lemma of calculus of variations \[5\] together with the fact that \( U \) and \( E \) satisfy the Cosserat plate strain-displacement relation \[63\] imply that \( S \) is a solution of the A and B mixed problems.

### 6 Field Equations Governing Flexural and Extensional Motions in terms of Kinematics Variables

For the future consideration we represent the constitutive relations in the following form\[2\]

\[
M_{\alpha \alpha} = D (\Psi_{\alpha, \alpha} + \nu \Psi_{\alpha', \alpha'}) + \frac{\nu h^2}{10(1 - \nu)} p, \quad (85)
\]

\[
M_{\alpha' \alpha} = \frac{D}{2} \frac{(1 + \nu)}{(1 - N^2)} \left( \Psi_{\alpha', \alpha} + \Psi_{\alpha, \alpha'} + 2N^2(-1)^{\alpha+1} (\Omega_3 - \Psi_{\alpha', \alpha}) \right), \quad (86)
\]

\[
R_{\alpha' \alpha} = \frac{\kappa^2 G h}{2} \left( (l^2 - 2 l^2_b) \Omega^0_{\alpha', \alpha} + 2l^2 \Omega^0_{\alpha, \alpha'} \right), \quad (87)
\]

\[
R_{\alpha \alpha} = \frac{\kappa^2 G h \bar{t}^2}{2} \left( \Omega^0_{\alpha, \alpha} + (1 - \Psi) \left( \Omega^0_{\alpha, \alpha'} + \Omega^0_{\alpha', \alpha'} \right) \right) + \frac{2G l_t^2 (1 - \Psi) \bar{t}}{\Psi},
\]

\[2\] In the following formulas a subindex \( \alpha' = 1 \) if \( \alpha = 2 \) and \( \alpha' = 2 \) if \( \alpha = 1 \)
\[ Q_\alpha = \kappa_1^2 \frac{Gh}{(1 - N^2)} \left( W_{\alpha} + \Psi_{\alpha} - 2N^2 \left( W_{\alpha} + (-1)^\alpha \Omega_0^0 \right) \right), \]

\[ Q^*_\alpha = \kappa_1^2 \frac{Gh}{(1 - N^2)} \left( W_{\alpha} + \Psi_{\alpha} - 2N^2 \left( \Psi_{\alpha} + (-1)^\alpha \Omega_0^0 \right) \right), \quad (88) \]

\[ S^*_\alpha = \frac{Gl^2 (4l^2_b - l^2_t) h^3}{12l^4_b} \Omega_{3,\alpha}, \quad (89) \]

\[ N_{\alpha\alpha} = \frac{Eh}{(1 - \nu^2)} (U_{\alpha,\alpha} + \nu U_{\alpha',\alpha'}) + \frac{h\nu}{1 - \nu} \sigma_0, \quad (90) \]

\[ N_{\alpha'\alpha} = \frac{Gh}{(1 - N^2)} \left( U_{\alpha',\alpha} + U_{\alpha,\alpha'} - 2N^2 \left( U_{\alpha',\alpha} + (-1)^\alpha \Omega_3^0 \right) \right), \quad (91) \]

\[ M^*_{\alpha} = \frac{Gl^2 (4l^2_b - l^2_t) h^3}{l^4_b} \Omega_{3,\alpha}, \quad (92) \]

where we use the following technical constants [4], [6]: the Young’s modulus \( E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \), the Poisson’s ratio \( \nu = \frac{\lambda}{2(\lambda + \mu)} \), the shear modulus \( G = \frac{E}{2(1 + \nu)} \), the flexural rigidity of the plate \( D = \frac{Eh^3}{12(1 - \nu^2)} \), the characteristic length for torsion \( l_t = \sqrt{\frac{\gamma}{\mu + \nu}} \), the characteristic length for bending \( l_b = \frac{1}{2} \sqrt{\frac{\gamma + \pi}{\mu}} \), the coupling number \( N = \sqrt{\frac{\alpha}{\mu + \nu}} \), the polar ratio \( \Psi = \frac{2\gamma}{\beta + \gamma} \), \( \kappa_1^2 = \frac{5}{6} \) and \( \kappa_2^2 = \frac{5}{3} \).

Remark: The values of \( \kappa_1 \) and \( \kappa_2 \) depend on the form of approximation. For instance, in the Mindlin’s case of dynamics [11] the value of \( \kappa_1^2 = \frac{\pi}{12} \), i.e. is slightly different.

After substitution (85) - (92) into (65) - (68) and (77) - (78) we obtain bending and twisting governing systems. We write system for the flexural motions in the form:

\[ \mathbf{L} (\partial_\mathbf{x}) \mathbf{H} - \mathbf{F} = \partial_0 \mathbf{P}, \quad \mathbf{x} \in \mathbf{P}_0, \quad (93) \]

where \( \mathbf{L} (\partial_\mathbf{x}) = \mathbf{L} \left( \frac{\partial}{\partial \mathbf{x}_\alpha} \right) \),

\[ \mathbf{L} (\mathbf{\xi}) = \mathbf{L} (\mathbf{\xi}_0) = \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 & L_{16} \\ L_{12} & L_{22} & L_{23} & L_{24} & -L_{16} & 0 \\ L_{13} & L_{23} & L_{33} & 0 & L_{35} & L_{36} \\ -L_{14} & L_{24} & 0 & L_{44} & 0 & 0 \\ 0 & L_{16} & -L_{35} & 0 & L_{55} & L_{56} \\ L_{16} & 0 & L_{36} & 0 & -L_{56} & L_{66} \end{bmatrix}, \]

\[ \mathbf{H}^T = \begin{bmatrix} \Psi_1 & \Psi_2 & W & \Omega_3 & \Omega_0^0 & \Omega_0^0 \end{bmatrix}, \]

and
\[ F^T = [ F_1 \ F_2 \ F_3 \ F_4 \ F_5 \ F_6 ] . \]

In the above
\[ p^T = \left[ \frac{\partial h^3}{\partial t} \frac{\partial \psi}{\partial t} \frac{\partial h}{\partial t} \frac{\partial \Psi}{\partial t} k_4 J_3 h^3 \frac{\partial \psi}{\partial t} k_5 J_2 h^3 \frac{\partial \Psi}{\partial t} \right] , \]

\[ L_{11} = L_{11}(\xi_1, \xi_2) = k_1 \xi_1^2 + k_2 \xi_2^2 - k_4, \quad L_{22} = L_{11}(\xi_2, \xi_1), \quad L_{33} = k_4 \Delta, \]
\[ L_{44} = k_5 \Delta - k_6, \quad L_{55} = L_{55}(\xi_1, \xi_2) = k_7 \xi_1^2 + k_8 \xi_2^2 - k_9, \quad L_{66} = L_{55}(\xi_2, \xi_1), \]
\[ L_{12} = k_{10} \xi_1 \xi_2, \quad L_{13} = k_{11} \xi_1, \quad L_{14} = k_{12} \xi_2, \quad L_{16} = k_{13}, \quad L_{23} = k_{11} \xi_2, \]
\[ L_{24} = k_{12} \xi_1, \quad L_{35} = -k_{13} \xi_2, \quad L_{36} = k_{13} \xi_1, \quad L_{56} = k_{14} \xi_1 \xi_2, \quad \Delta = \xi_1^2 + \xi_2^2, \]
\[ F_1 = -\frac{h \nu (1 - N^2) \partial p}{10(1 - \nu)} \frac{\partial x_1}{\partial t}, \quad F_2 = -\frac{h \nu (1 - N^2) \partial p}{10(1 - \nu)} \frac{\partial x_2}{\partial t}, \]
\[ F_3 = -(1 - N^2) p, \quad F_4 = 0, \quad F_5 = -\frac{5h(1 - N^2)}{6} (1 - \Psi) \frac{\partial t}{\partial x_2}, \]
\[ F_6 = \frac{5h(1 - N^2)}{6} (1 - \Psi) \frac{\partial t}{\partial x_2} \]

Here
\[ k_1 = D(1 - N^2), \quad k_2 = \frac{D(1 - \nu)}{2}, \quad k_3 = -\frac{5Gh}{6}, \quad k_4 = \frac{5Gh}{6}, \]
\[ k_5 = \frac{D(1 - \nu)}{2} \frac{(4l_2^2 - l_1^2)(1 - N^2)}{2l_1^2}, \quad k_6 = 2N^2 D(1 - \nu), \]
\[ k_7 = \frac{5h(1 - N^2) G l_1^2 (2 - \Psi)}{3}, \quad k_8 = \frac{10h(1 - N^2) G l_1^2}{3}, \quad k_9 = \frac{10h GN^2}{3}, \]
\[ k_{10} = \frac{D(1 + \nu - 2N^2)}{2}, \quad k_{11} = \frac{5Gh(2N^2 - 1)}{6}, \quad k_{12} = DN^2(1 - \nu), \]
\[ k_{13} = \frac{5Gh N^2}{3}, \quad k_{14} = \frac{5h(1 - N^2) G (l_2^2 (2 - \Psi) - 2l_1^2)}{3}. \]

The correspondent boundary and initial conditions are
\[ T(\partial_x) H - F^* = 0, \quad x \in \Gamma_{\sigma}, \quad (94) \]
\[ H - H_0 = 0, \quad x \in \Gamma_{u}, \quad (95) \]

and
\[ H(x, 0) = H^0 \]
\[ \partial_t H(x, 0) = \dot{H}^0 \]

where differential operator \( T(\partial_x) = T \left( \frac{\partial}{\partial x_n} \right) \).
\[ T(\xi) = T(\xi_o) = \begin{bmatrix} T_{11} & T_{12} & 0 & T_{14} & 0 & 0 \\ T_{21} & T_{22} & 0 & T_{24} & 0 & 0 \\ T_{31} & T_{32} & T_{33} & 0 & 0 & T_{36} \\ 0 & 0 & 0 & T_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & T_{55} & T_{56} \\ 0 & 0 & 0 & 0 & T_{65} & T_{66} \end{bmatrix}, \]

and

\[ (F^*)^T = \begin{bmatrix} F_1^* & F_2^* & F_3^* & F_4^* & F_5^* & F_6^* \end{bmatrix}, \]

\[ H^T_o = \begin{bmatrix} \Psi_{o1} & \Psi_{o2} & W_o & \Omega_o & \Omega^0_{o1} & \Omega^0_{o2} \end{bmatrix}. \]

In the above

\[ T_{11} = T_1(\xi_1, \xi_2), T_{22} = T_1(\xi_2, \xi_1), T_1(\xi_1, \xi_2) = Dn_1\xi_1 + \frac{D(1 + \nu)}{2(1 - N^2)}n_2\xi_2, \]

\[ T_{33} = \frac{5Gh}{6(1 - N^2)}(n_1\xi_1 + n_2\xi_2), T_{44} = \frac{GL^2(4l_2^2 - l_1^2)h^3}{12l_6^2}(n_1\xi_1 + n_2\xi_2), \]

\[ T_{55} = \frac{5Gh}{3}(l_1^2n_1(2 - \Psi)\xi_1 + 2l_2^2n_2\xi_2), T_{66} = \frac{5Gh}{3}(2l_2^2n_1\xi_1 + l_1^2(2 - \Psi)n_2\xi_2), \]

\[ T_{12} = Dn_1\xi_2 + \frac{D(1 + \nu)(1 - 2N^2)}{2(1 - N^2)}n_2\xi_1, T_{14} = \frac{D(1 + \nu)N^2}{1 - N^2}n_2, \]

\[ T_{21} = Dn_2\xi_1 + \frac{D(1 + \nu)(1 - 2N^2)}{2(1 - N^2)}n_1\xi_2, T_{24} = -\frac{D(1 + \nu)N^2}{1 - N^2}n_1, \]

\[ T_{31} = \frac{5Gh(1 - N^2)}{6(1 - N^2)}n_1, T_{32} = \frac{5Gh(1 - 2N^2)}{6(1 - N^2)}n_2, T_{36} = \frac{5GhN^2}{3(1 - N^2)}(n_1 - n_2), \]

The governing system for the extensional motions is

\[ \tilde{L}(\partial_x)\tilde{\mathbf{H}} - \tilde{\mathbf{F}} = \frac{\partial\tilde{\mathbf{P}}}{\partial t}, \mathbf{x} \in P_0, \quad (96) \]

where

\[ \tilde{L}(\xi) = \tilde{L}(\xi_o) = \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} \\ \tilde{L}_{21} & \tilde{L}_{22} & \tilde{L}_{23} \\ \tilde{L}_{31} & \tilde{L}_{32} & \tilde{L}_{33} \end{bmatrix} \]
\[ \begin{aligned}
\tilde{F}^T &= \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 & \tilde{F}_3 \end{bmatrix}, \\
\tilde{H}^T &= \begin{bmatrix} U_1 & U_2 & \Omega_0^3 \end{bmatrix}, \\
\tilde{p}^T &= \partial_t \left[ \begin{array}{ccc}
\rho_0 \frac{\partial u_1}{\partial t} & \rho_0 \frac{\partial u_2}{\partial t} & I_0 \frac{\partial \Omega_0^3}{\partial t}
\end{array} \right].
\end{aligned} \]

Here
\[ \begin{aligned}
\tilde{T}_{11} &= \kappa_1 \xi_1^2 + \kappa_2 \xi_2^2, \\
\tilde{T}_{12} &= \kappa_3 \xi_2, \\
\tilde{T}_{13} &= 2 \kappa_4 \xi_2,
\end{aligned} \]
\[ \begin{aligned}
\tilde{T}_{21} &= \tilde{T}_{12}, \\
\tilde{T}_{22} &= \tilde{T}_{11} + \tilde{T}_{23} = 2 \kappa_4 \xi_1, \\
\tilde{T}_{31} &= -\kappa_4 \xi_1, \\
\tilde{T}_{32} &= \kappa_4 \xi_1, \\
\tilde{T}_{33} &= \kappa_5 (\xi_1^2 + \xi_2^2) - \kappa_2,
\end{aligned} \]
\[ \begin{aligned}
\tilde{F}^*_{1} &= -\frac{\nu \kappa_1}{2G} \frac{\partial \sigma_0}{\partial x_1}, \\
\tilde{F}^*_{2} &= -\frac{\nu \kappa_1}{2G} \frac{\partial \sigma_0}{\partial x_2}, \\
\tilde{F}^*_{3} &= -(1 - N^2) \frac{G h v}{\tilde{T}_{33}},
\end{aligned} \]
\[ \begin{aligned}
\kappa_1 &= \frac{2(1 - N^2)}{1 - \nu}, \\
\kappa_2 &= 2N^2, \\
\kappa_3 &= 1 - \kappa_1 = \frac{(1 + \nu - 2N^2)}{(1 - \nu)},
\end{aligned} \]
\[ \begin{aligned}
\kappa_4 &= N^2, \\
\kappa_5 &= \frac{l_2^2(4l_0^2 - l_3^2)(1 - N^2)}{2l_0^2}.
\end{aligned} \]

The boundary and initial conditions for the extensional system have the following form:
\[ \begin{aligned}
\tilde{T}(\partial_x)\tilde{H} - \tilde{F}^* &= 0, \quad x \in \Gamma_\sigma \tag{97} \\
\tilde{H} - \tilde{H}_o &= 0, \quad x \in \Gamma_u, \tag{98}
\end{aligned} \]

and
\[ \begin{aligned}
\tilde{H}(x,0) &= \tilde{H}_0^0, \\
\partial_t \tilde{H}(x,0) &= \tilde{H}_0^0,
\end{aligned} \]

where differential operator \( \tilde{T}(\partial_x) = \tilde{T} \left( \frac{\partial}{\partial x_1} \right) \),
\[ \begin{aligned}
\tilde{T}(\xi) &= \tilde{T}(\xi_o) = \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} & \tilde{T}_{13} \\
\tilde{T}_{21} & \tilde{T}_{22} & \tilde{T}_{23} \\
0 & 0 & \tilde{T}_{33} \end{bmatrix}, \\
(\tilde{F}^*)^T &= \begin{bmatrix} \tilde{F}^*_1 & \tilde{F}^*_2 & \tilde{F}^*_3 \end{bmatrix}, \\
(\tilde{H}_o)^T &= \begin{bmatrix} U_{o1} & U_{o2} & \Omega_0^{o3} \end{bmatrix}.
\end{aligned} \]
In the above

\[ \tilde{T}_{11} = \frac{Eh_{n_1}}{1 - \nu^2} \xi_1 + \frac{Ghn_{2}}{1 - N^2} \xi_2, \quad \tilde{T}_{12} = \frac{Eh_{n_1}}{1 - \nu^2} \xi_2 + \frac{Ghn_{2}(1 - 2N^2)}{1 - N^2} \xi_1, \]

\[ \tilde{T}_{13} = \frac{2N^2Ghn_{2}}{1 - N^2}, \quad \tilde{T}_{21} = \frac{Eh_{n_2}}{1 - \nu^2} \xi_1 + \frac{Ghn_{1}(1 - 2N^2)}{1 - N^2} \xi_2, \]

\[ \tilde{T}_{22} = \frac{Eh_{n_2}}{1 - \nu^2} \xi_1 + \frac{Ghn_{1}}{1 - N^2} \xi_2, \quad \tilde{T}_{23} = -\frac{2N^2Ghn_{1}}{1 - N^2}, \]

\[ \tilde{T}_{33} = \frac{Gl^2(4l_b^2 - l_b^2)}{l_b^2} (\xi_1 n_1 + \xi_2 n_2), \]

\[ \tilde{F}_1^* = -\Sigma_{0,1} - \frac{h_{n_1}}{1 - \nu} \sigma_0, \quad \tilde{F}_2^* = -\Sigma_{0,2} - \frac{h_{n_2}}{1 - \nu} \sigma_0, \quad \tilde{F}_3^* = -M_{03}. \]

## 7 Conclusion

We proposed a new mathematical model for dynamics of Cosserat elastic plates based on Reissner-Mindlin’s plate theory. The polynomial approximations of the variation of couple stress and micropolar rotations in the thickness direction allowed us to project Cosserat 3D Elasticity dynamics equations into the dynamics equations in the middle plane of the plate. We generalized for the dynamics Hellinger-Prange-Reissner (HPR) principle to derive the dynamics equations in the middle plane and constitutive relationships for the plate. In terms of the kinematic variables, the total system of dynamic equations describes the flexural (subsystem of 6 equations) and the extensional (subsystem of 3 equations) motions of the plate.

## References

[1] Bathe K. J., Brezzi F.: On the convergence of a four node plate bending element based on Mindlin-Reissner plate theory and a mixed interpolation, The Mathematics of Finite Elements and Applications V (Uxbridge,1984), Academic Press, London, 491-503, (1985)

[2] Donnell L. H.: Beams Plates and Shells, McGraw-Hill, New York (1976)

[3] Eringen A. C.: Theory of micropolar plates, Journal of Applied Mathematics and Physics, Vol. 18, 12-31, (1967)

[4] Gauthier R.D., Jahsman W.E.: A quest for micropolar elastic constants, Journal of Applied Mechanics, 42, 369-374 (1975)

[5] Gurtin M. E.: The Linear Theory of Elasticity in Handbuch der Physik, Vol. VIa/2; C. Truesdell (editor), Springer-Verlag, 1-296 (1972)

[6] Lakes R.: Experimental methods for study of Cosserat elastic solids and other generalized elastic continua. In Mühlhaus H (ed.), Continuum Models for Materials with Microstructures, Wiley J, 1-22, New York (1995)
[7] Love A. E. H.: A Treatise on the Mathematical Theory of Elasticity, Dover, New York (1986)

[8] Naghdi P. M.: The Theory of Shells and Plates, in Handbuch der Physik, Vol. VIa/2; C. Truesdell (editor), Springer-Verlag, 425-640 (1972)

[9] Neff P.: A geometrically exact Cosserat-shell model including size effects, avoiding degeneracy in the thin shell limit. Part I: Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus. Cont. Mech. Thermodynamics, 16: 577-628 (2004)

[10] Neff P.: The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric. Preprint 2409, http://www3.mathematik.tudarmstadt.de/fb/mathe/bibliothek/preprints.html (2005)

[11] Mindlin R.D.: Influence of Rotary Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates, Journal of Applied Mechanics, 73, 31-38 (1951)

[12] Nowacki W.: Theory of Asymmetric Elasticity, Pergamon Press, Oxford, New York (1986)

[13] Reissner E.: The effect of transverse shear deformation on the bending of elastic plates, Journal of Applied Mechanics, June, 69-77 (1945)

[14] Reissner E.: On the theory of Elastic Plates, Journal of Mathematics and Physics, 23, 184-191, (1944)

[15] Reissner E.: Reflections on the theory of elastic plates, Applied Mechanics Reviews, 38, 1453-1464 (1985)

[16] Rössle A., Bischoff M., Wendland W., Ramm E.: On the mathematical foundation of the (1,1,2)-plate model, International Journal of Solids and Structures, 36, 2143-2168 (1999)

[17] Steinberg L.: Elastic Plate Deformation with Transverse Variation of Microrotation, arXiv:0811.1534v1, November (2008)

[18] Timoshenko S. and Woinowsky-Krieger S.: Theory of Plates and Shells, McGraw-Hill (1959)

[19] Wan F.Y.M.: Lectures Notes on Problems in Elasticity: II Linear Plate Theory. Tech.Rep. No. 83-15, Institute of Applied Mathematics, University of British Columbia (1983)