Delta shock solution for a 2 × 2 hyperbolic system with linear damping

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Abstract

In this paper, we propose a time-dependent viscous system and by using the vanishing viscosity method we show the existence of delta shock solution for a particular 2 × 2 system of conservation laws with linear damping.

Keywords: Non strictly hyperbolic system, linear damping, Riemann problem, time-dependent viscous system, delta shock wave solution.

1 Introduction

In this paper, we study the Riemann problem to the following hyperbolic system of conservation laws with linear damping

\[
\begin{align*}
{v_t} + (vu^k)_x &= 0, \\
(vu)_t + (vu^{k+1})_x &= -\alpha vu,
\end{align*}
\]

where \(k\) is an odd natural number, \(\alpha > 0\) is a constant and initial data given by

\[
(v(x,0), u(x,0)) = \begin{cases} 
(v_-, u_-), & \text{if } x < 0, \\
(v_+, u_+), & \text{if } x > 0,
\end{cases}
\]

for arbitrary constant states \((v_\pm, u_\pm)\) with \(v_\pm > 0\). The principal reason to choose the condition on \(k\) is due to physical motivation. In fact, for \(k = 1\) the system (1) becomes the one-dimensional zero-pressure gas dynamics system with linear damping. It is well known that the system (1) is not strictly hyperbolic with eigenvalue \(\lambda = u^k\) and right eigenvector \(r = (1,0)\). Moreover,
∇\lambda \cdot \mathbf{r} = 0 \text{ and therefore the system is linearly degenerate. The system (1) without source term (namely } \alpha = 0 \text{) is a particular case of the following system of conservation laws}

\begin{align*}
  v_t + (vf(u))_x &= 0, \\
  (vu)_t + (vuf(u))_x &= 0.
\end{align*}

The system (3) is called the one-dimensional zero-pressure gas dynamics when } f(u) = u, \text{ where } v \geq 0 \text{ denotes the density of mass and } u \text{ the velocity. The one-dimensional zero-pressure gas dynamics system can be used to describe the motion process of free particles sticking under collision in the low temperature and the information of large-scale structures in the universe [3, 22]. Really, the one-dimensional zero-pressure gas dynamics system arise in a wide variety of models in physics, see for example [2, 10, 18, 15]. For this reason, the system (1) has been studied by many authors and several rigorous results have been obtained for this. So, more details on the studies of the one-dimensional zero-pressure gas dynamics system can be found in [2, 3, 10, 13, 15, 16]. The Riemann problem for system (3) was solved completely in [30] with characteristic analysis and the vanishing viscosity method. In 2016, Shen [23] studied the Riemann problem for the one-dimensional zero-pressure gas dynamics with Coulomb-like friction term and the solutions involve delta shock wave and vacuum state. Shen’s paper is the first work for the one-dimensional zero-pressure gas dynamics system with a source term. Recently, Keita and Bourgault [14] solved the Riemann problem to the one-dimensional zero-pressure gas dynamics system with linear damping (i.e. the system (1) when } k = 1 \text{) and their results include delta shock wave solution.}

In this paper, we are interested in finding delta shock solutions to the Riemann problem for the system (1) with initial data (2). Therefore, we propose the following time-dependent viscous system

\begin{align*}
  v_t^\varepsilon + (vu^k)_x &= 0, \\
  (vu)_t + (vu^{k+1})_x &= \varepsilon \frac{1}{\alpha k} e^{-\alpha kt}(1 - e^{-\alpha kt})u_{xx} - \alpha vu,
\end{align*}

where } k \text{ is an odd natural number and } \alpha > 0 \text{ is a constant. A similar viscous system to (4) was consider in [7] to solve the Riemann problem to the system (1) with } k = 1. \text{ Observe that when } \alpha \to 0^+, \text{ we have that } \lim_{\alpha \to 0^+} \frac{1}{\alpha k} e^{-\alpha kt}(1 - e^{-\alpha kt}) = t \text{ and the system (4) coincides with the viscous system (4.1) in [30]. The viscous system (4) is well motivated by scalar conservation law with time-dependent viscosity}

\begin{equation}
  u_t + F(u)_x = G(t)u_{xx},
\end{equation}

where } G(t) > 0 \text{ for } t > 0. \text{ When } F(u) = u^2 \text{ the scalar equation is called the Burgers equation with time-dependent viscosity. The Burgers equation with time-dependent viscosity was studied as a mathematical model of the propagation of the finite-amplitude sound waves in variable-area ducts, where } u \text{ is an acoustic variable, with the linear effects of changes in the duct area taken out, and the time-dependent viscosity } G(t) \text{ is the duct area [4, 9, 29]. The reader can find results concerning the existence, uniqueness and explicit solutions to the Burgers equation with time-dependent viscosity with suitable conditions for } G(t) \text{ in [4, 5, 9, 24, 28, 29, 32, 33].}
and references cited therein. The Burgers equation with time-dependent viscosity with linear damping was studied in [19] and their results include explicit solutions for different $G(t)$. When $G(t) = \varepsilon t$ and $\varepsilon > 0$, for systems of hyperbolic conservation laws with time-dependent viscosity we refer the works developed by Tupciev in [27] and Dafermos in [6]. The results obtained in [6] and [27] not including the delta shock waves solutions. For systems of hyperbolic conservation laws with delta shock solutions the reader may consult [8, 11, 26, 30, 31].

Note that our proposal of the time-dependent viscous system (4) is a special case of the general systems of conservation laws with time-dependent viscous system. Therefore, we consider the viscous system (4) with initial data (2). Observe that if $(\tilde{v}, \tilde{u})$ solves

\[
\begin{aligned}
\hat{v}_t + e^{-\alpha kt}(\hat{v}\hat{u})_x &= 0, \\
(\hat{v}\hat{u})_t + e^{-\alpha kt}(\hat{v}\hat{u}^{k+1}) &= \varepsilon \frac{1}{\alpha k} e^{-\alpha kt}(1 - e^{-\alpha kt})\hat{u}_{xx},
\end{aligned}
\]

with initial condition

\[
(\hat{v}(x, 0), \hat{u}(x, 0)) = \begin{cases} (v_-, u_-), & \text{if } x < 0, \\ (v_+, u_+), & \text{if } x > 0, \end{cases}
\]

then $(v, u)$ defined by $(v, u) = (\tilde{v}, \tilde{u} e^{-\alpha t})$ solves the problem (4)–(2). In order to solve the problem (5)–(6), we introduce the similarity variable $\xi$ and solutions to (5) should approach for large times a similarity (or self-similar) solution $(\hat{v}, \hat{u})$ to (5) of the form $\hat{v}(x, t) = \hat{v}(\xi)$, $\hat{u}(x, t) = \hat{u}(\xi)$ and $\xi = a(t)x$ for some suitable smooth function $a(t) \geq 0$ for $t > 0$ (more details on the similarity methods can be found in [1, 12, 17, 20, 21, 25] and references therein). Therefore, we introduce the similarity variable $\xi = \frac{akx}{1 - e^{-\alpha x}}$ and the system (5) can be written as

\[
\begin{aligned}
-\xi\hat{v}_\xi + (\hat{v}\hat{u})_\xi &= 0, \\
-\xi(\hat{v}\hat{u})_\xi + (\hat{v}\hat{u}^{k+1})_\xi &= \varepsilon \hat{u}_{\xi\xi},
\end{aligned}
\]

and the initial data becomes the boundary condition

\[
(\hat{v}(\pm\infty), \hat{u}(\pm\infty)) = (v_\pm, u_\pm).
\]

Note that when $\alpha \to 0^+$, the similarity variable $\xi$ converges to $x/t$ which is well used in many methods to study the behavior and structure of solutions of nonlinear hyperbolic systems of conservation laws. If we introduce the similarity transformation $\xi = a(t)x$ with $v(x, t) = \hat{v}(\xi)$ and $u(x, t) = b(t)\hat{u}(\xi)$ for suitable smooth functions $a(t) \geq 0$ and $b(t) \geq 0$ for $t > 0$, then, $a(t) = e^{-\alpha t}$, $b(t) = \frac{ak}{1-e^{-\alpha t}}$ and therefore we can write the system (4) directly as (7). However, in this paper we are not interested in this transformation since we are going to show existence of delta shock solution for the homogeneous system of conservation laws with time-dependent coefficients (5) without viscosity, i.e. for the following system

\[
\begin{aligned}
\hat{v}_t + e^{-\alpha kt}(\hat{v}\hat{u})_x &= 0, \\
(\hat{v}\hat{u})_t + e^{-\alpha kt}(\hat{v}\hat{u}^{k+1})_x &= 0.
\end{aligned}
\]

Using the vanishing viscosity method, and following works by Tan, Zhang and Zheng [26], Li and Yang [16] and Yang [30] with some appropriate modifications, we show the existence of
a delta shock wave solution for the system (5). The main difficulty in applying the vanishing viscosity method developed in [16, 26, 30] is to choose a suitable Banach space and a bounded convex closed subset to use the Schauder fixed point theorem. Therefore, we specifically follow the vanishing viscosity method developed in [8].

The outline of the remaining of the paper is as follows. In Section 2, we show the existence of solutions to the viscous system (7) with boundary condition (8). In Section 3, we study the behavior of the solutions \((\hat{v}^\varepsilon, \hat{u}^\varepsilon)\) as \(\varepsilon \to 0^+\) and we show delta shock solution for the system (5) without viscosity. In Section 4, we show the delta shock solution for the nonhomogeneous system (1). Final remarks are given in Section 5.

2 Existence of solutions to the viscous system \((7)-(8)\)

Let \(R\) be a positive number such that \(R^{1/k} > \max\{|u_-|, |u_+|\}\). We consider the Banach space \(C([-R, R])\), endowed with the supremum norm, and we take the set \(K\) given by

\[
K = \{ U \in C([-R, R]) \mid U \text{ is monotone increasing with } U(-R) = u_- \text{ and } U(R) = u_+ \}
\]

which is bounded and a convex closed set in \(C([-R, R])\).

Lemma 2.1. Suppose \(U \in K \cap C^1([-R, R])\). Let

\[
\hat{v}(\xi) = \begin{cases} 
\hat{v}_1(\xi), & \text{if } -R \leq \xi < \xi^\varepsilon_s, \\
\hat{v}_2(\xi), & \text{if } \xi^\varepsilon_s < \xi \leq R,
\end{cases}
\]

where \(\xi^\varepsilon_s\) is the unique solution of the equation \(\frac{1}{\alpha k}(U(\xi))^k = \xi\) (which solution exists because \(u_- > u_+\) and \(R\) is big enough),

\[
\hat{v}_1(\xi) := v_- \frac{u^k_+ + R}{(U(\xi))^k - \xi} \exp \left( - \int_{-R}^\xi \frac{ds}{(U(s))^k - s} \right)
\]

and

\[
\hat{v}_2(\xi) := v_+ \frac{R - u^k_-}{\xi - (U(\xi))^k} \exp \left( \int_{\xi}^R \frac{ds}{(U(s))^k - s} \right).
\]

Then \(\hat{v} \in L^1([-R, R])\), \(v\) is continuous in \([-R, \xi^\varepsilon_s] \cup (\xi^\varepsilon_s, R]\) and it is a weak solution for

\[
-\xi \hat{v}_\xi + (\hat{v}_U^k)_\xi = 0,
\]

and \(\hat{v}(\pm R) = v_\pm\).

Proof. The equation (13) can be rewritten as

\[
((U(\xi))^k - \xi)\hat{v}' + \hat{v}((U(\xi))^k)' = 0.
\]
Integrating (14) on $[-R, \xi]$ for $-R < \xi < \xi^e$, we get

$$(U(\xi))^k - \xi \hat{v}_1(\xi) - (u_-^k + R)v^- + \int_{-R}^{\xi} \hat{v}_1(s)ds = 0. \quad (15)$$

Let

$$p(\xi) = \int_{-R}^{\xi} \hat{v}_1(s)ds, \quad A_1 = (u_-^k + R)v^- \quad \text{and} \quad a(\xi) = ((U(\xi))^k - \xi).$$

Then (15) can be written as

$$\begin{cases}
a(\xi)p'(\xi) + p(\xi) = A_1, \\
p(-R) = 0.
\end{cases}$$

It follows that

$$p(\xi) = A_1 \left\{ 1 - \exp\left( - \int_{-R}^{\xi} \frac{ds}{a(s)} \right) \right\}.$$ 

Noting that $a(\xi) > 0$ and $a(\xi) = O(|\xi - \xi_\sigma|)$ as $\xi \to \xi^e_\sigma^-$, we obtain

$$\lim_{\xi \to \xi^e_\sigma^-} \int_{-R}^{\xi} \hat{v}_1(s)ds = \lim_{\xi \to \xi^e_\sigma^-} p(\xi) = A_1. \quad (16)$$

Hence

$$\lim_{\xi \to \xi^e_\sigma^-} ((U(\xi))^k - \xi) \hat{v}_1(\xi) = 0. \quad (17)$$

Similarly, one can get

$$\lim_{\xi \to \xi^e_\sigma^+} \int_{-R}^{\xi} \hat{v}_2(s)ds = A_2, \quad \lim_{\xi \to \xi^e_\sigma^+} ((U(\xi))^k - \xi) \hat{v}_2(\xi) = 0, \quad (18)$$

where $A_2 = (u_+^k - R)v_+$. The equalities (16) and (18) imply that $\hat{v}(\xi) \in L^1([-R, R])$.

Now, for arbitrary $\phi \in C_0^\infty([-R, R])$, we verify that

$$I \equiv - \int_{-R}^{R} \left( ((U(\xi))^k - \xi)\hat{v}(\xi)\phi'(\xi) + \hat{v}(\xi)\phi(\xi) \right)d\xi = 0.$$ 

For any $\xi_1, \xi_2$, such that $-R < \xi_1 < \xi^e_\sigma < \xi_2 < R$ we can write $I = I_1 + I_2 + I_3$, where

$$I_1 = \int_{-R}^{\xi_1} (-(U(\xi))^k - \xi)\hat{v}(\xi)\phi'(\xi) + \hat{v}(\xi)\phi(\xi))d\xi,$$ 

$$I_2 = \int_{\xi_1}^{\xi_2} (-(U(\xi))^k - \xi)\hat{v}(\xi)\phi'(\xi) + \hat{v}(\xi)\phi(\xi))d\xi \quad \text{and}$$

$$I_3 = \int_{\xi_2}^{R} (-(U(\xi))^k - \xi)\hat{v}(\xi)\phi'(\xi) + \hat{v}(\xi)\phi(\xi))d\xi.$$
Define an operator \( T \) and observe that
\[
|I_1| = \left| -((U(\xi)) - \xi_1)\tilde{v}(\xi)\phi(\xi) + \int_{-R}^{\xi_1}(((U(\xi)^k - \xi)\tilde{v}(\xi))\phi(\xi) + \tilde{v}(\xi)\phi(\xi))d\xi \right|
= \left| ((U(\xi)) - \xi_1)\tilde{v}(\xi)\phi(\xi) \right|.
\]

By (17), we have that
\[
\lim_{\xi_1 \to \xi_1^+} |I_1| = \lim_{\xi_1 \to \xi_1^+} \left| ((U(\xi)) - \xi_1)\tilde{v}(\xi)\phi(\xi) \right| = 0.
\]

In similar way, we show that
\[
\lim_{\xi_2 \to \xi_2^+} |I_3| = \lim_{\xi_2 \to \xi_2^+} \left| ((U(\xi)) - \xi_2)\tilde{v}(\xi)\phi(\xi) \right| = 0.
\]

Since \( \tilde{v} \in L^1([-R, R]) \),
\[
|I_2| \leq \int_{\xi_1}^{\xi_2} |-(U(\xi)) - \xi_1)\phi'(\xi) + \phi(\xi)|d\xi \to 0, \quad \text{as } \xi_1 \to \xi_1^+, \xi_2 \to \xi_2^+.
\]

But \( I \) is independent of \( \xi_1 \) and \( \xi_2 \), so \( I = 0 \). Therefore, \( \tilde{v} \) defined in (10) is a weak solution.

Define an operator \( T : K \to C^2([-R, R]) \) as follows: for any \( U \in K \), \( \tilde{u} = TU \) is the unique solution of the boundary value problem
\[
\begin{cases}
\varepsilon\tilde{u}'' = (\tilde{v}(U, \xi) \cdot ((U(\xi))^k - \xi))\tilde{v}'
\tilde{u}(\pm R) = u_{\pm}
\end{cases}
\]
where \( \tilde{v}(U, \xi) \equiv \tilde{v}(\xi) \) is defined in (11) or (12). In fact, the solution to this problem can be found explicitly and it is given by
\[
\tilde{u}(\xi) = u_+ + \frac{(u_+ - u_-)}{\int_{-R}^{\xi} \exp\left(\int_{-R}^{r} \varepsilon^{-1}(U(s))^{k-1}ds \right)dr} \int_{-R}^{\xi} \exp\left(\int_{-R}^{r} \varepsilon^{-1}(U(s))^{k-1}ds \right)dr.
\]

**Lemma 2.2.** \( T : K \to K \) is a continuous operator.

**Proof.** Choose \( \{U_n\} \) in \( K \) such that \( U_n \to U \). As \( U \) belongs to \( K \), then each \( \tilde{u}_n = TU_n \) and \( \tilde{u} = TU \) satisfies the problem (19). Now, we have the following problem
\[
\begin{cases}
\varepsilon(\tilde{u}_n - \tilde{u})'' = (\tilde{v}(U_n, \xi) \cdot ((U_n(\xi))^k - \xi))(\tilde{u}_n - \tilde{u})' + (\tilde{v}(U_n, \xi) \cdot ((U_n(\xi))^k - \xi))\tilde{u}'
(\tilde{u}_n - \tilde{u})(\pm R) = 0.
\end{cases}
\]
Setting \( p_n(\xi) = \hat{v}(U_n, \xi)((U_n(\xi))^k - \xi) \) and \( q_n(\xi) = (\hat{v}(U_n, \xi)((U_n(\xi))^k - \xi) - \hat{v}(U, \xi)((U(\xi))^k - \xi))u^i \), from problem (21) we have

\[
(\hat{u}_n - \hat{u})'(\xi) = \frac{\int_{-R}^{R} \int_{-R}^{y} \frac{q_n(r)}{\varepsilon} \exp \left( \int_{-R}^{y} \frac{p_n(s)}{\varepsilon} ds \right) dy dr}{\int_{-R}^{R} \exp \left( \int_{-R}^{r} \frac{p_n(s)}{\varepsilon} ds \right) dr} \exp \left( \int_{-R}^{\xi} \frac{p_n(s)}{\varepsilon} ds \right) + \int_{-R}^{\xi} \frac{q_n(r)}{\varepsilon} \exp \left( \int_{-R}^{\xi} \frac{p_n(s)}{\varepsilon} ds \right) dr,
\]

(22)

\[
(\hat{u}_n - \hat{u})(\xi) = -\frac{\int_{-R}^{R} \int_{-R}^{y} \frac{q_n(r)}{\varepsilon} \exp \left( \int_{-R}^{y} \frac{p_n(s)}{\varepsilon} ds \right) dy dr \int_{-R}^{\xi} \exp \left( \int_{-R}^{r} \frac{p_n(s)}{\varepsilon} ds \right) dr}{\int_{-R}^{R} \exp \left( \int_{-R}^{r} \frac{p_n(s)}{\varepsilon} ds \right) dr} \int_{-R}^{\xi} \frac{q_n(r)}{\varepsilon} \exp \left( \int_{-R}^{\xi} \frac{p_n(s)}{\varepsilon} ds \right) dr dy.
\]

(23)

From (13), we have

\[
(((U(\xi))^k - \xi)\hat{v}(\xi))' = -\hat{v}(\xi) < 0,
\]

\[
(((U_n(\xi))^k - \xi)\hat{v}_n(\xi))' = -\hat{v}_n(\xi) < 0 \quad \text{for } n = 1, 2, \ldots.
\]

Then, \( \hat{v}(U^k - \xi) \) and \( \hat{v}_n(U^k_n - \xi) \), \( n = 1, 2, \ldots \), are monotone decreasing and continuous functions. Because the sequence of monotone functions which converges to a continuous function must converge uniformly, we get that \( q_n(\xi) \) converges to zero uniformly. Then, from (21), (22) and (23) it follows that

\[
\hat{u}_n \to \hat{u} \text{ in } C^2([-R, R]), \text{ as } n \to \infty.
\]

Therefore \( T : K \to C^2([-R, R]) \) is continuous. In addition, from (20), we have

\[
\hat{u}'(\xi) = \frac{(u^+_n - u^-_n) \exp \left( \int_{-R}^{\xi} \frac{\hat{v}(U(s))((U(s))^k - s)}{\varepsilon} ds \right)}{\int_{-R}^{R} \exp \left( \int_{-R}^{r} \frac{\hat{v}(U(s))((U(s))^k - s)}{\varepsilon} ds \right) dr}
\]

which implies that \( \hat{u} = TU \) is monotone. So we get \( TK \subset K \).

**Lemma 2.3.** \( TK \) is a bounded set in \( C^2([-R, R]). \)

**Proof.** For any \( U \in K \), if \( s < \xi^\varepsilon \), we have

\[
0 < \hat{v}(U, s)((U(s))^k - s) = \hat{v}_-(u^+_n + R) - \int_{-R}^{s} \hat{v}(r) dr < \hat{v}_-(u^+_n + R) \quad (24)
\]

and if \( s > \xi^\varepsilon \),

\[
0 > \hat{v}(U, s)((U(s))^k - s) = \hat{v}_+(u^+_n - R) + \int_{s}^{R} \hat{v}(r) dr > \hat{v}_+(u^+_n - R) \quad (25)
\]
From (19), we can deduce that

\[ \hat{u}''(\xi) < 0, \quad \xi \in [-R, \xi^*_R). \]

Then, \( \hat{u}'(\xi) \leq \hat{u}(-R) < 0, \xi \in [-R, \xi^*_R), \) and

\[ u_- - u_+ > \hat{u}(-R) - \hat{u}(\xi_\sigma) = \hat{u}'(\xi)(-R - \xi_\sigma) > \hat{u}'(\xi)(-R - u^*_R), \quad \xi \in (-R, \xi_\sigma). \]

Thus,

\[ 0 > \hat{u}'(-R) > \hat{u}'(\xi) > -\frac{u_- - u_+}{R + u^*_R}. \]

Also, from (19) we have

\[ \hat{u}'(\xi) = \hat{u}'(-R) \exp \left( \int_{-R}^{\xi} \frac{\hat{u}'(U^k - s)}{\varepsilon} ds \right) \]

and by (24) and (25), we conclude that \( \hat{u}' \) is uniformly bounded. Consequently, \( \hat{u}'' \) is also uniformly bounded. So, \( TK \) is a bounded set in \( C^2([-R, R]). \)

**Lemma 2.4.** \( TK \) is precompact in \( C([-R, R]). \)

**Proof.** This is a consequence of the compact embedding \( C^2([-R, R]) \hookrightarrow C([-R, R]). \)

From the above lemmas, by virtue of Schauder fixed point theorem, we get the following result.

**Theorem 2.1.** For each \( R > \max\{||u_-|^k, |u_+|^k\}, \) there exists a weak solution

\[ (\hat{v}_R, \hat{u}_R) \in L^1([-R, R]) \times C^2([-R, R]) \]

for the system (7) with boundary value \( (\hat{v}_R(\pm R), \hat{u}_R(\pm R)) = (v_\pm, u_\pm), \) and, in addition, being \( \hat{u}_R \) a decreasing function.

The next step is to obtain from this family of solutions a sequence \( R_k \to \infty \) such that \((\hat{v}_{R_k}, \hat{u}_{R_k})\) converges to a weak solution of (7)–(8). To this end, we need the following lemma.

**Lemma 2.5.**

1. \( \hat{u}_R(\xi), \hat{u}'_R(\xi) \) and \( \hat{u}''_R(\xi) \) are uniformly bounded, with respect to \( R \) and \( \xi \in [-R, R]. \)

2. There exist a sequence \( R_k \to \infty \) and a decreasing function \( \hat{u} \in C^1(\mathbb{R}) \) such that \( \hat{u}_{R_k} \) converges to \( \hat{u} \) in \( C^1([-M, M]) \), for each positive number \( M \) (i.e. \( \hat{u}_{R_k}, \hat{u}''_{R_k} \) converge uniformly in compact sets of \( \mathbb{R} \) to \( \hat{u}, \hat{u}' \), respectively).

3. \( \hat{v}_{R_k}(\hat{u}_{R_k}, \xi) \) converges to \( \hat{v}(\hat{u}, \xi) \), as \( R_k \to \infty \), for each \( \xi \in \mathbb{R} \setminus \{\xi^*_R\} \), where \( \hat{v}_{R_k}(\hat{u}_{R_k}, \xi), \) \( \hat{v}(\hat{u}, \xi) \) are defined accordingly with (10), (11) and (12), being \( R = \infty \) for \( \rho(u, \xi) \), and \( \xi^*_R \) satisfies \( (\hat{u}(\xi^*_R))^k = \xi^*_R \).
Proof. 1. To simplify the notation in this proof, we shall use \( \hat{u}, \hat{u}' \) and \( \hat{u}'' \) instead of \( \hat{u}_R, \hat{u}'_R \) and \( \hat{u}''_R \).

Observe that \( u^k_+ < \xi_\varepsilon < u^k_- \). We choose \( \xi_1 \) such that \( -R < \xi_1 < u^k_+ \). From (7) it follows that

\[
\hat{u}'(\xi) = \hat{u}'(\xi_1) \exp\left( \int_{\xi_1}^{\xi} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds \right).
\]

As \( \hat{u}''(\xi) < 0 \) for \( \xi \in (-R, \xi_\varepsilon) \), then we have that \( \hat{u}'(\xi) < \hat{u}'(\xi_1) < 0 \), \( \xi \in (\xi_1, \xi_\varepsilon) \). Since \( u_- - u_+ > \hat{u}(\xi_1) - \hat{\xi}_\varepsilon = \hat{u}'(\xi_1)(\xi_1 - \xi_\varepsilon) > \hat{u}'(\xi_1)(u^k_1 - u^k_+) \),

where \( \xi \in (\xi_1, \xi_\varepsilon) \), we get

\[
\hat{u}'(\xi) > \frac{u_- - u_+}{\xi_1 - u^k_+}, \quad \xi \in (\xi_1, \xi_\varepsilon).
\]

It follows that

\[
0 > \hat{u}'(\xi_1) > \frac{u_- - u_+}{\xi_1 - u^k_+}.
\]

When \( \xi < \xi_1 \),

\[
\exp\left( \int_{\xi_1}^{\xi} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds \right) < 1.
\]

When \( \xi_1 < \xi < \xi_\varepsilon \), observe that

\[
\hat{v}_1(\xi_1) = \hat{v}_- \frac{u^k_+ + R}{(\hat{u}(\xi_1))^k - \xi_1} \exp\left( - \int_{-R}^{\xi_1} \frac{ds}{(\hat{u}(s))^k - s}\right)
\]

\[
\leq \hat{v}_- \frac{u^k_+ + R}{(\hat{u}(\xi_1))^k - \xi_1} \exp\left( - \int_{-R}^{\xi_1} \frac{ds}{\hat{u}_- - s}\right) = \hat{v}_- \frac{u^k_+ - \xi_1}{(\hat{u}(\xi_1))^k - \xi_1}
\]

and

\[
\hat{v}(\xi)((\hat{u}(\xi))^k) - \xi_1 = \hat{v}(\xi_1)((\hat{u}(\xi_1))^k) - \xi_1 - \int_{\xi_1}^{\xi} \hat{v}(s) ds
\]

\[
\leq \hat{v}(\xi_1)((\hat{u}(\xi_1))^k) - \xi_1 \leq v_-(u^k_1 - \xi_1),
\]

and we obtain

\[
\exp\left( \int_{\xi_1}^{\xi} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds \right) \leq \exp\left( \frac{(v_-(u^k_1 - \xi_1))^2}{\varepsilon} \right).
\]

When \( \xi > \xi_\varepsilon \), we have

\[
\int_{\xi_1}^{\xi} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds = \int_{\xi_1}^{\xi_\varepsilon} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds + \int_{\xi_\varepsilon}^{\xi} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds
\]

\[
< \int_{\xi_1}^{\xi_\varepsilon} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds
\]
or
\[
\exp\left(\int_{\xi_1}^{\xi} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds\right) < \exp\left(\int_{\xi_1}^{\xi} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds\right).
\]
Therefore, \(\hat{u}'(\xi)\) and \(\hat{u}(\xi)\) are uniformly bounded. From (19) it follows that \(\hat{u}'(\xi)\) is also uniformly bounded, with respect to \(R\) and \(\xi \in [-R, R]\).

2. Fixing \(M > 0\), we consider \(R >> M\) and apply the Arzelà-Ascoli theorem to obtain a sequence \((\hat{u}_{R_k})\) converging in \(C^1([-M, M])\) to a decreasing function \(u\). Then, by a diagonalization process, we obtain a sequence \((\hat{u}_{R_k})\), which we do not relabel, such that \((\hat{u}_{R_k})\) converges to a decreasing function \(\hat{u} \in C^1(\mathbb{R})\), uniformly in compact sets in \(\mathbb{R}\), \((\hat{u}_{R_k})\) also converges uniformly in compact sets in \(\mathbb{R}\) to \(\hat{u}'\), and \(\hat{u}(-\infty) = u_-\), \(\hat{u}(\infty) = u_+\).

3. Claim 3 is obtained from (11), (12) by passing to the limit as \(R_k \to \infty\), for each fixed \(\xi \neq \xi^e\), noting that, up to a subsequence, we can assume that \(\xi^e_{\sigma^k}\), defined by \((u_{R_k}(\xi^e_{\sigma^k})) = \xi^e_{\sigma^k}\), converges to \(\xi^e\) (where \(\xi^e\) is defined by \((u(\xi^e))k = \xi^e\)).

\[\square\]

**Theorem 2.2.** Let \(\hat{u}\) be the function obtained in Lemma 2.5. Then, for each \(\varepsilon > 0\), \(\hat{u}\) satisfies

\[
\begin{aligned}
\varepsilon u'' &= (\hat{v}(\hat{u}, \xi)(\hat{u}^k - \xi))\hat{u}', \\
\hat{u}(\pm\infty) &= u_{\pm},
\end{aligned}
\]

and

\[
\hat{v}(\xi) = \begin{cases} 
\hat{v}_1(\xi), & \text{if } -\infty < \xi < \xi^e, \\
\hat{v}_2(\xi), & \text{if } \xi^e < \xi < \infty,
\end{cases}
\]

where \(\xi^e\) satisfies \((u(\xi^e))k = \xi^e\),

\[
\hat{v}_1(\xi) = v_+ \exp\left(-\int_{-\infty}^{\xi} \frac{((u(s))^k)'}{u(s)^k - s} ds\right) \quad \text{and} \quad \hat{v}_2(\xi) = v_+ \exp\left(\int_{\xi}^{+\infty} \frac{((u(s))^k)'}{u(s)^k - s} ds\right).
\]

**Proof.** Denote by \((\hat{v}_R(\xi), \hat{u}_R(\xi))\) the solution of the problem (7) with boundary value \((\hat{v}(\pm R), \hat{u}(\pm R)) = (v_+, u_+)\). Fixing \(\xi_2\) and integrating (19) from \(\xi_2\) to \(\xi\), we obtain

\[
\varepsilon(\hat{u}_R(\xi) - \hat{u}_R(\xi_2)) = (\hat{v}_R(\xi)((\hat{u}_R(\xi))^k - \xi))\hat{u}_R(\xi) - (\hat{v}_R(\xi)((\hat{u}_R(\xi_2))^k - \xi_2))\hat{u}_R(\xi_2) + \int_{\xi_2}^{\xi} \hat{v}_R(s)\hat{u}_R(s) ds
\]

(independently of whether \(\xi^e_{\sigma^k}\) is between \(\xi_2\) and \(\xi\)). Letting \(R \to +\infty\), by the Lebesgue Convergence Theorem it follows that

\[
\varepsilon(\hat{u}'(\xi) - \hat{u}'(\xi_2)) = (\hat{v}(\xi)((\hat{u}(\xi))^k - \xi))\hat{u}(\xi) - (\hat{v}(\xi)((\hat{u}(\xi_2))^k - \xi_2))\hat{u}(\xi_2) + \int_{\xi_2}^{\xi} \hat{v}(s)\hat{u}(s) ds.
\]

Differentiating (26) with respect to \(\xi\), we obtain

\[
\varepsilon \hat{u}'' = (\hat{v}(\hat{u}^k - \xi))\hat{u}',
\]

and from (20) we have \(\hat{u}(\pm\infty) = u_{\pm}\).
**Theorem 2.3.** There exists a weak solution $(\hat{v}, \hat{u}) \in L^1_{\text{loc}}((-\infty, +\infty)) \times C^2((-\infty, +\infty))$ for the boundary value problem (7)–(8).

**Proof.** Let $(\hat{v}, \hat{u})$ be defined in Theorem 2.2. By Lemma 2.5 we know that $\hat{u}$ is decreasing and of class $C^1$ in $\mathbb{R}$. Then $\hat{v}$ is of class $C^1$ in $(-\infty, \xi^\ast) \cup (\xi^\ast, \infty)$. In addition, it is also bounded, hence, locally integrable. From (26) it follows that $\hat{u}$ is of class $C^2$. The first equation in (7) comes by differentiating $\hat{v}_1$ and $\hat{v}_2$, and the second is equivalent to the first and the equation stated in Theorem 2.2.

### 3 The limit solutions of (5)–(6) as viscosity vanishes

We continue this section studying the case when $u_- > u_+$ and we are interested in analyzing the behavior of the solutions $(\hat{v}^\varepsilon, \hat{u}^\varepsilon)$ of (7)–(8) as $\varepsilon \to 0+$.

**Lemma 3.1.** Let $\xi^\ast$ be the unique point satisfying $(\hat{u}^\varepsilon(\xi^\ast))^k = \xi^\ast$, and let $\xi_\sigma$ be the limit

$$\xi_\sigma = \lim_{\varepsilon \to 0+} \xi^\varepsilon_{\sigma}$$

(passing to a subsequence if necessary). Then for any $\eta > 0$,

$$\lim_{\varepsilon \to 0+} \hat{u}^\varepsilon(\xi) = 0, \quad \text{for } |\xi - \xi_\sigma| \geq \eta,$$

and

$$\lim_{\varepsilon \to 0+} \hat{u}^\varepsilon(\xi) = \begin{cases} u_-, & \text{if } \xi \leq \xi_\sigma - \eta, \\ u_+, & \text{if } \xi \geq \xi_\sigma + \eta, \end{cases}$$

uniformly in the above intervals.

**Proof.** To simplify the notation in this proof, we shall use $\hat{v}, \hat{u}$ instead of $\hat{v}^\varepsilon, \hat{u}^\varepsilon$.

Take $\xi_3 = \xi_\sigma + \eta/2$, and let $\varepsilon$ be so small such that $\xi^\ast < \xi_3 - \eta/4$. For $\xi > \xi_\sigma$,

$$\hat{v}(\xi) = v_+ \exp \left( \int_{\xi}^{+\infty} \frac{((\hat{u}(s))^k)'}{(\hat{u}(s))^k - s} \, ds \right) = \lim_{R \to +\infty} \frac{R - u_+^k}{\xi - (\hat{u}(\xi))^k} \exp \left( \int_{\xi}^{R} \frac{ds}{\hat{u}(s)} \right)$$

$$\leq \lim_{R \to +\infty} \frac{R - u_+^k}{\xi - (\hat{u}(\xi))^k} \exp \left( \int_{\xi}^{R} \frac{ds}{u_+^k - s} \right) = \lim_{R \to +\infty} \frac{R - u_+^k}{\xi - (\hat{u}(\xi))^k} \frac{\xi - u_+^k}{R - u_+^k}$$

$$= v_+ \frac{\xi - u_+^k}{\xi - (\hat{u}(\xi))^k},$$

and we have

$$\hat{v}(\xi)((\hat{u}(\xi))^k - \xi) \geq v_+(u_+^k - \xi), \quad \xi \in (\xi_\sigma, +\infty).$$
Now, integrating the second equation of (7) twice on [ξ₃, ξ], we get

\[ \hat{u}(ξ₃) - \hat{u}(ξ) = -\hat{u}'(ξ₃) \int_{ξ₃}^{ξ} \exp \left( \int_{ξ₃}^{r} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{ε} \, ds \right) \, dr \]

\[ \geq -\hat{u}'(ξ₃) \int_{ξ₃}^{ξ} \exp \left( \int_{ξ₃}^{r} \frac{v_+(u_+^k - s)}{ε} \, ds \right) \, dr \]

\[ = -\hat{u}'(ξ₃) \int_{ξ₃}^{ξ} \exp \left( \frac{v_+}{ε} \left( (u_+^k - ξ₃) (r - ξ₃) - \frac{1}{2}(r - ξ₃)^2 \right) \right) \, dr \]

\[ = -\hat{u}'(ξ₃) \int_{0}^{ξ - ξ₃} \exp \left( \frac{v_+}{ε} \left( (u_+^k - ξ₃) r - \frac{1}{2}r^2 \right) \right) \, dr. \]

Letting ξ → +∞, we get

\[ u_- - u_+ \geq -\hat{u}'(ξ₃) \int_{0}^{r + ∞} \exp \left( \frac{v_+}{ε} \left( (u_+^k - ξ₃) r - \frac{1}{2}r^2 \right) \right) \, dr \]

\[ \geq -\hat{u}'(ξ₃) \int_{0}^{2ε} \exp \left( \frac{v_+}{ε} \left( (u_+^k - ξ₃) r - \frac{1}{2}r^2 \right) \right) \, dr \]

\[ \geq -\hat{u}'(ξ₃) √ε A₃ \]

for 0 ≤ ε ≤ 1, where A₃ is a constant independent of ε. Thus

\[ |\hat{u}'(ξ₃)| \leq \frac{u_- - u_+}{√ε A₃}. \]

So

\[ |\hat{u}'(ξ)| \leq \frac{u_- - u_+}{√ε A₃} \exp \left( \int_{ξ₃}^{ξ} \frac{\hat{v}(s)((\hat{u}(s))^k - s)}{ε} \, ds \right). \]

(27)

For ξ > ξ₃,

\[ \hat{v}(ξ) = \lim_{R → ∞} v_+ \frac{R - u_+^k}{ξ - (\hat{u}(ξ))^k} \exp \left( \int_{ξ}^{R} \frac{ds}{(\hat{u}(s))^k - s} \right) \]

\[ \geq \lim_{R → ∞} v_+ \frac{R - u_+^k}{ξ - (\hat{u}(ξ))^k} \exp \left( \int_{ξ}^{R} \frac{ds}{(\hat{u}(ξ))^k - s} \right) \]

\[ = v_+ \frac{ξ - (\hat{u}(ξ))^k}{ξ - (\hat{u}(ξ))^k} \lim_{R → ∞} \frac{R - u_+^k}{R - (\hat{u}(ξ))^k} = v_+ \frac{ξ - (\hat{u}(ξ))^k}{ξ - (\hat{u}(ξ))^k} \]

and we have

\[ \hat{v}(ξ)((\hat{u}(ξ))^k - ξ) \leq v_+((\hat{u}(ξ))^k - ξ), \]

(28)

ξ > ξ₃.

From (27) and (28) we have

\[ |\hat{u}'(ξ)| \leq \frac{u_- - u_+}{√ε A₃} \exp \left( -\frac{v_+}{ε} \int_{ξ₃}^{ξ} (s - (\hat{u}(ξ))^k) \, ds \right) \]
which implies that
\[
\lim_{\varepsilon \to 0^+} \hat{u}_\varepsilon(\xi) = 0, \quad \text{uniformly for } \xi \geq \xi_\sigma + \eta.
\]
Now, we choose \(\xi\) and \(\xi_4\) such that \(\xi > \xi_4 \geq \xi_\sigma + \eta\). From
\[
\hat{u}(\xi_4) - \hat{u}(\xi) = -\hat{u}'(\xi_4) \int_{\xi_4}^{\xi} \exp \left( \int_{\xi_4}^{r} \frac{\hat{u}(s)((\hat{u}(s))^k - s)}{\varepsilon} ds \right) dr,
\]
we get
\[
|\hat{u}(\xi_4) - \hat{u}(\xi)| \leq |\hat{u}'(\xi_4)| \int_{\xi_4}^{\xi} \exp \left( -\frac{A_4}{\varepsilon} (r - \xi_4) \right) dr \leq \frac{\varepsilon}{A_4} |\hat{u}'(\xi_4)| \left( 1 - \exp \left( \frac{A_4}{\varepsilon} (\xi_4 - \xi) \right) \right),
\]
where \(A_4 = v_+ (\xi_4 - (\hat{u}(\xi_4))^k)\). When \(\xi \to +\infty\), we obtain
\[
|\hat{u}(\xi_4) - u_+| \leq \frac{\varepsilon}{A_4} |\hat{u}'(\xi_4)|,
\]
which implies that
\[
\lim_{\varepsilon \to 0^+} \hat{u}_\varepsilon(\xi) = u_+, \quad \text{uniformly for } \xi \geq \xi_\sigma + \eta.
\]
The results for \(\xi < \xi_\sigma - \eta\) can be obtained analogously. \(\Box\)

**Lemma 3.2.** For any \(\eta > 0\),
\[
\lim_{\varepsilon \to 0^+} \hat{v}_\varepsilon(\xi) = \begin{cases} v_-, & \text{if } \xi < \xi_\sigma - \eta, \\ v_+, & \text{if } \xi > \xi_\sigma + \eta, \end{cases}
\]
uniformly, with respect to \(\xi\).

**Proof.** Take \(\varepsilon_0 > 0\) so small such that \(|\xi_\sigma^\varepsilon - \xi_\sigma| < \frac{\eta}{2}\) whenever \(0 < \varepsilon < \varepsilon_0\). For any \(\xi > \xi_\sigma + \eta\) and \(\varepsilon < \varepsilon_0\), we have
\[
\xi > \xi_\sigma + \frac{\eta}{2}
\]
and
\[
\hat{v}_\varepsilon(\xi) = v_+ \exp \left( \int_{\xi}^{\infty} \frac{((\hat{u}_\varepsilon(s))^k)' ((\hat{u}_\varepsilon(s))^k)' - s}{((\hat{u}_\varepsilon(s))^k)' - s} ds \right).
\]
For any \(s \in [\xi, +\infty)\), we have
\[
((\hat{u}_\varepsilon(s))^k)' - s < ((\hat{u}_\varepsilon(s))^k)' - \xi = (1 - ((\hat{u}_\varepsilon(s))^k)')(\xi_\sigma^\varepsilon - \xi) \leq -\frac{\eta}{2}.
\]
As \(\hat{u}\) is decreasing, we have that \(((\hat{u}(s))^k)' = k(\hat{u}(s))^k - 1\hat{u}'(s) < 0\), and
\[
\frac{((\hat{u}(s))^k)'}{(\hat{u}(s))^k - s} < -\frac{2}{\eta}((\hat{u}(s))^k)', \quad \text{for any } s \in [\xi, +\infty).
\]
Now, in the last inequality, integrating on \([\xi, +\infty)\) we have
\[
0 \leq \int_\xi^\infty \frac{((\hat{v}^\varepsilon)(s))'k'}{(\hat{u}^\varepsilon(s))k' - s} ds \leq - \frac{2}{\eta} \int_\xi^\infty \frac{((u^\varepsilon(s))'k'}{(u^\varepsilon(s))k' - s} ds = - \frac{2}{\eta}(u^k_+ - (\hat{v}^\varepsilon(\xi))^k),
\]
so
\[
1 \leq \exp \left( \int_\xi^\infty \frac{((\hat{u}^\varepsilon(s))'k')}{(\hat{u}^\varepsilon(s))k' - s} ds \right) \leq \exp \left( - \frac{2}{\eta}(u^k_+ - (\hat{v}^\varepsilon(\xi))^k) \right) .
\]
By Lemma 3.1 we have that \(\lim_{\varepsilon \to 0^+} \hat{u}^\varepsilon(\xi) = u_+\), and from (29) we have
\[
\lim_{\varepsilon \to 0^+} \exp \left( \int_\xi^\infty \frac{((\hat{u}^\varepsilon(s))'k')}{(\hat{u}^\varepsilon(s))k' - s} ds \right) = 1
\]
and
\[
\lim_{\varepsilon \to 0^+} \hat{v}^\varepsilon(\xi) = \lim_{\varepsilon \to 0^+} v_+ \exp \left( \int_\xi^\infty \frac{((\hat{u}^\varepsilon(s))'k')}{(\hat{u}^\varepsilon(s))k' - s} ds \right) = v_+, \quad \text{uniformly for } \xi > \xi_{\sigma} + \eta.
\]
Similarly, we obtain also \(\lim_{\varepsilon \to 0^+} \hat{v}^\varepsilon(\xi) = v_-, \) uniformly for \(\xi < \xi_{\sigma} - \eta\). \(\Box\)

Now, we study the limit behavior of \((\hat{v}^\varepsilon, \hat{u}^\varepsilon)\) in the neighborhood of \(\xi_{\sigma}\) as \(\varepsilon \to 0^+\).

**Theorem 3.1.** Denote
\[
\sigma = \xi_{\sigma} = \lim_{\varepsilon \to 0^+} \xi^\varepsilon_{\sigma} = \lim_{\varepsilon \to 0^+} \left( \hat{u}^\varepsilon(\xi^\varepsilon_{\sigma}) \right)^k = \left( \hat{u}(\sigma) \right)^k.
\]
Then
\[
\lim_{\varepsilon \to 0^+} (\hat{v}^\varepsilon(\xi), \hat{u}^\varepsilon(\xi)) = \begin{cases} (v_-, u_-), & \text{if } \xi < \sigma, \\
(w_0 \cdot \delta, u_3), & \text{if } \xi = \sigma, \\
(v_+, u_+), & \text{if } \xi > \sigma,
\end{cases}
\]
where \(\hat{v}(\xi)\) converges in the sense of the distributions to the sum of a step function and a Dirac measure \(\delta\) with weight \(w_0 = -\sigma(v_- - v_+) + (v_-u_- - v_+u_+)\).

**Proof.** As \(\sigma = \xi_{\sigma} = \lim_{\varepsilon \to 0^+} \left( \hat{u}^\varepsilon(\xi^\varepsilon_{\sigma}) \right)^k = \left( \hat{u}(\sigma) \right)^k\), then we have
\[
u_+^k < \sigma < u_-^k.
\]

Let \(\xi_1\) and \(\xi_2\) be real numbers such that \(\xi_1 < \sigma < \xi_2\) and \(\phi \in C_0^\infty([\xi_1, \xi_2])\) such that \(\phi(\xi) \equiv \phi(\sigma)\) for \(\xi\) in a neighborhood \(\Omega\) of \(\sigma\), \(\Omega \subset (\xi_1, \xi_2)\). Then \(\xi^\varepsilon_{\sigma} \in \Omega\) whenever \(0 < \varepsilon < \varepsilon_0\). From (7) we have
\[
- \int_{\xi_1}^{\xi_2} \hat{v}(\xi^\varepsilon)(\hat{u}(\xi^\varepsilon) - \xi) \phi' d\xi + \int_{\xi_1}^{\xi_2} \hat{v}(\xi^\varepsilon) \phi d\xi = 0.
\]
\[\text{[26]}\]
For \( \alpha_1, \alpha_2 \in \Omega, \alpha_1, \alpha_2 \) near \( \sigma \) such that \( \alpha_1 < \sigma < \alpha_2 \), we write

\[
\int_{\xi_1}^{\xi_2} \tilde{v}^* ((\tilde{w}^*)^k - \xi) \phi' d\xi = \int_{\xi_1}^{\alpha_1} \tilde{v}^* ((\tilde{w}^*)^k - \xi) \phi' d\xi + \int_{\alpha_2}^{\xi_2} \tilde{v}^* ((\tilde{w}^*)^k - \xi) \phi' d\xi,
\]

and from Lemmas 3.1 and 3.2, we obtain

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} \tilde{v}^* ((\tilde{w}^*)^k - \xi) \phi' d\xi = \int_{\xi_1}^{\alpha_1} v_- (u_-^k - \xi) \phi' d\xi + \int_{\alpha_2}^{\xi_2} \tilde{v}_+ (u_+^k - \xi) \phi' d\xi
\]

\[
= (v_- u_-^k - v_+ u_+^k - v_1 - v_2) \phi(\sigma)
\]

\[
+ \int_{\xi_1}^{\alpha_1} v_- \phi(\xi) d\xi + \int_{\alpha_2}^{\xi_2} v_+ \phi(\xi) d\xi
\]

Then taking \( \alpha_1 \to \sigma^- \), \( \alpha_2 \to \sigma^+ \), we arrive at

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} \tilde{v}^* ((\tilde{w}^*)^k - \xi) \phi' d\xi = (-[\tilde{v}]\sigma + [\tilde{v} u^k]) \phi(\sigma) + \int_{\xi_1}^{\xi_2} J(\xi - \sigma) \phi(\xi) d\xi
\] (33)

where \([q] = q_- - q_+\) and

\[
J(x) = \begin{cases} v_- & \text{if } x < 0, \\ v_+ & \text{if } x > 0. \end{cases}
\]

From (32) and (33), we get

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\tilde{v}^* - J(\xi - \sigma)) \phi(\xi) d\xi = (-[\tilde{v}]\sigma + [\tilde{v} u^k]) \phi(\sigma).
\]

for all sloping test functions \( \phi \in C_0^\infty([\xi_1, \xi_2]) \).

For an arbitrary \( \psi \in C_0^\infty([\xi_1, \xi_2]) \), we take a sloping test function \( \phi \), such that \( \phi(\sigma) = \psi(\sigma) \) and

\[
\max_{[\xi_1, \xi_2]} |\psi - \phi| < \mu,
\]

for a sufficiently small \( \mu > 0 \). As \( \tilde{v}^* \in L^1([\xi_1, \xi_2]) \) uniformly, we obtain

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\tilde{v}^* - J(\xi - \sigma)) \psi(\xi) d\xi = \lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\tilde{v}^* - J(\xi - \sigma)) \phi(\xi) d\xi + O(\mu)
\]

\[
= (-[\tilde{v}]\sigma + [\tilde{v} u^k]) \phi(\sigma) + O(\mu)
\]

\[
= (-[\tilde{v}]\sigma + [\tilde{v} u^k]) \psi(\sigma) + O(\mu).
\]

Then, when \( \mu \to 0^+ \), we find that

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\tilde{v}^* - J(\xi - \sigma)) \psi(\xi) d\xi = (-[\tilde{v}]\sigma + [\tilde{v} u^k]) \psi(\sigma)
\] (34)
Thus, $\tilde{\nu}^\varepsilon$ converges in the sense of the distributions to the sum of a step function and a Dirac delta function with strength $-\left[\tilde{\nu}\right]\sigma + \frac{1}{\alpha k}[\tilde{\nu}\hat{u}^+]$. In a similar way, from

$$-\int_{\xi_1}^{\xi_2} \left( \tilde{\nu}^\varepsilon ((\hat{u}^\varepsilon)^k - \xi) \right) \hat{u}^\varepsilon \phi^\prime d\xi + \int_{\xi_1}^{\xi_2} \tilde{\nu}^\varepsilon \hat{u}^\varepsilon \phi d\xi = \varepsilon \int_{\xi_1}^{\xi_2} (\hat{u}^\varepsilon)^\prime \phi d\xi,$$  

we can obtain

$$\lim_{\varepsilon \to 0+} \int_{\xi_1}^{\xi_2} \left( \tilde{\nu}^\varepsilon \hat{u}^\varepsilon - \tilde{J}(\xi - \sigma) \right) \phi(\xi) d\xi = (-[\tilde{\nu}\hat{u}]\sigma + \tilde{\nu}\hat{u}^{k+1}] ) \phi(\sigma)$$

for all test functions $\phi \in C_0^\infty([\xi_1, \xi_2])$, where

$$\tilde{J}(x) = \begin{cases} v_-, & \text{if } x < 0, \\ v_+u_+, & \text{if } x > 0. \end{cases}$$

Thus, $\tilde{\nu}\hat{u}$ also converges in the sense of the distributions to the sum of a step function and a Dirac delta function with strength $-\left[\tilde{\nu}\hat{u}\right]\sigma + \tilde{\nu}\hat{u}^{k+1}]$. If we take the test function in (35) as $\frac{\psi}{\tilde{\nu}\hat{u}^+}$, $\nu > 0$, where $\tilde{u}^\varepsilon$ is a modified function satisfying $\hat{u}^\varepsilon(\sigma)$ in $\Omega$ and $\tilde{u}^\varepsilon$ outside $\Omega$, and let $\nu \to 0+$, we find

$$\lim_{\varepsilon \to 0+} \int_{\xi_1}^{\xi_2} \left( \tilde{\nu}^\varepsilon - \tilde{J}(\xi - \sigma) \right) \psi d\xi \cdot \hat{u}(\sigma) = (-[\tilde{\nu}\hat{u}]\sigma + \tilde{\nu}\hat{u}^{k+1}] ) \psi(\sigma)$$

for all test functions $\psi \in C_0^\infty([\xi_1, \xi_2])$. Let $w_0$ be the strength of the Dirac delta function in $\tilde{\nu}$, and denote

$$\hat{u}_\delta = \lim_{\varepsilon \to 0+} \hat{u}^\varepsilon(\xi_\delta) = \hat{u}(\sigma).$$

From (30), (34) and (36) it follows that

$$\begin{cases} \sigma = (\hat{u}_\delta)^k, \\ w_0 = -\sigma[\tilde{\nu}] + \tilde{\nu}\hat{u}^k, \\ w_0u_\delta = -\sigma[\tilde{\nu}] + \tilde{\nu}\hat{u}^{k+1}. \end{cases}$$  

(37)

Under the entropy condition (31) the system (37) admits a unique solution $(\sigma, w_0, u_\delta)$. Then we get the following theorem.

**Theorem 3.2.** Suppose $u_- > u_+$. Let $(\tilde{\nu}^\varepsilon(x,t), \tilde{u}^\varepsilon(x,t))$ be the similarity solution of (5)–(6). Then the limit

$$\lim_{\varepsilon \to 0+} (\tilde{\nu}^\varepsilon(x,t), \tilde{u}^\varepsilon(x,t)) = (\tilde{\nu}(x,t), \tilde{u}(x,t))$$

exists in the measure sense and $(\hat{\nu}, \hat{u})$ solves (9)–(6). Moreover,

$$\hat{\nu}(x,t), \hat{u}(x,t) = \begin{cases} (v_-, u_-), & \text{if } x < \sigma \left(1 - e^{-\alpha k}\right), \\ \left(\frac{w_0}{\alpha k}(1 - e^{-\alpha k}) \delta(x - \sigma \left(1 - e^{-\alpha k}\right)), u_\delta\right), & \text{if } x \geq \sigma \left(1 - e^{-\alpha k}\right), \end{cases}$$

for $x < \sigma (1 - e^{-\alpha k})$.
where the constants $\sigma$, $w_0$, and $u_\delta$ are determined uniquely by the entropy condition $u_\uparrow^k < \sigma < u_\downarrow^k$ and
\[
\begin{cases}
\sigma = u_\delta^k, \\
w_0 = -\sigma [\bar{v}] + [\bar{v}^{} u]^k, \\
w_0 u_\delta = -\sigma [\bar{v}^{} u] + [\bar{v}^{} u]^k+1.
\end{cases}
\]

### 4 Delta shock solutions for the system (1)

In this section, we study the Riemann problem to the system (1) with initial data (2) when $u_- > u_+$. We need recall the following definition:

**Definition 4.1.** A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{(x(s), t(s)) : a < s < b\}$, for $w \in L^1((a, b))$, is defined as
\[
\langle w(\cdot) \delta_L, \phi(\cdot, \cdot) \rangle = \int_a^b w(s) \phi(x(s), t(s)) ds, \quad \phi \in C_0^\infty(\mathbb{R} \times [0, \infty)).
\]
Now, we define a delta shock wave solution for the system (1) with initial data (2).

**Definition 4.2.** A distribution pair $(v, u)$ is a delta shock wave solution of (1) and (2) in the sense of distribution if there exist a smooth curve $L$ and a function $w \in C^1(L)$ such that $v$ and $u$ are represented in the following form
\[
v = \bar{v}(x, t) + w \delta_L \quad \text{and} \quad u = \bar{u}(x, t),
\]
$\bar{v}, \bar{u} \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R})$ and
\[
\begin{cases}
\langle v, \varphi_t \rangle + \langle vu^k, \varphi_x \rangle = 0, \\
\langle vu, \varphi_t \rangle + \langle vu^{k+1}, \varphi_x \rangle = \langle \alpha u^{} v, \varphi \rangle,
\end{cases}
\] (38)
for all the test functions $\varphi \in C_0^\infty(\mathbb{R} \times (0, \infty))$, where $u|_L = u_\delta(t)$ and
\[
\langle v, \varphi \rangle = \int_0^\infty \int_\mathbb{R} \bar{v} \varphi \, dxdt + \langle w \delta_L, \varphi \rangle,
\]
\[
\langle vG(u), \varphi \rangle = \int_0^\infty \int_\mathbb{R} \bar{v}G(\bar{u}) \varphi \, dxdt + \langle wG(u_\delta) \delta_L, \varphi \rangle.
\]

With the previous definitions, we are going to find a solution with discontinuity $x = x(t)$ for (1) of the form
\[
(v(x, t), u(x, t)) = \begin{cases}
(v_- (x, t), u_- (x, t)), & \text{if } x < x(t), \\
(w(t) \delta_L, u_\delta(t)), & \text{if } x = x(t), \\
(v_+ (x, t), u_+ (x, t)), & \text{if } x > x(t),
\end{cases}
\] (39)
where $v_\pm(x, t)$, $u_\pm(x, t)$ are piecewise smooth solutions of system (1), $\delta(\cdot)$ is the Dirac measure supported on the curve $x(t) \in C^1$, and $x(t)$, $w(t)$ and $u_\delta(t)$ are to be determined.

Since $v(x, t) = \bar{v}(x, t)$ and $u(x, t) = \bar{u}(x, t) e^{-\alpha t}$, from Theorem 3.2, we can establish a solution of the form (39) to the system (1) with initial data (2). Thus, we have the following result.
Theorem 4.1. Assume that $u_+ > u_-$. Then the Riemann problem (1)-(2) admits one and only one measure solution of the form

$$
(v(x, t), u(x, t)) = \begin{cases} 
(v_-, u_+ e^{-\alpha t}), & \text{if } x < x(t), \\
(w(t)\delta(x - x(t)), u_{\delta} e^{-\alpha t}), & \text{if } x = x(t), \\
(v_+, u_+ e^{-\alpha t}), & \text{if } x > x(t),
\end{cases}
$$

(40)

where $w(t) = \frac{w_0}{\alpha k}(1 - e^{-\alpha k t})$, $x(t) = \frac{\sigma}{\alpha k}(1 - e^{-\alpha k t})$ and the constants $\sigma$, $w_0$, and $u_{\delta}$ are determined uniquely by the entropy condition $u_+^k < \sigma < u_-^k$ and

$$
\begin{cases}
\sigma = u_+^k, \\
\sigma = u_-^k, \\
w_0 = -\sigma(v_+ - v_+^k) + (v_- u_+^k - v_+ u_-), \\
w_0 u_{\delta} = -\sigma(v_- u_+ - v_+ u_+^k) + (v_- u_-^{k+1} - v_+ u_+^{k+1}).
\end{cases}
$$

(41)

Proof. We need show that (40) is a solution to the problem (1)-(2) which can be found with $(v, u) = (\hat{v}, \hat{u} e^{-\alpha t})$ and the result obtained in Theorem 3.2. Therefore, for any test function $\varphi \in C_0^\infty(\mathbb{R} \times (0, \infty))$ we have

$$
\langle vu, \varphi \rangle + \langle vu^{k+1}, \varphi \rangle = \int_0^\infty \int_{\mathbb{R}} (\nu \varphi t + vu^{k+1} \varphi x) dx dt + \int_0^\infty w(t)u_{\delta} e^{-\alpha t}(\varphi t + u_{\delta}^k e^{-\alpha k t} \varphi x) dt
$$

$$
= \int_0^\infty \int_{-\infty}^{x(t)} (v_- u_+ e^{-\alpha t} \varphi t + v_- u_+^k e^{-\alpha (k+1) t} \varphi x) dx dt + \int_0^\infty \int_{x(t)}^{\infty} (v_- u_+ e^{-\alpha t} \varphi t + v_- u_+^k e^{-\alpha (k+1) t} \varphi x) dx dt
$$

$$
+ \int_0^\infty w(t)u_{\delta} e^{-\alpha t}(\varphi t + u_{\delta}^k e^{-\alpha k t} \varphi x) dt
$$

$$
= -\int_0^\infty \int_{-\infty}^{x(t)} (v_- u_+^k e^{-\alpha (k+1) t} \varphi t) dx dt + (v_- u_+ e^{-\alpha t} \varphi) dx
$$

$$
+ \int_0^\infty \int_{-\infty}^{x(t)} (v_- u_+^k e^{-\alpha (k+1) t} \varphi t) dx dt + (v_- u_+ e^{-\alpha t} \varphi) dx
$$

$$
+ \int_0^\infty \int_{-\infty}^{x(t)} \alpha v \varphi dx dt + \int_0^\infty w(t)u_{\delta} e^{-\alpha t}(\varphi t + \sigma e^{-\alpha k t} \varphi x) dt
$$

$$
= \int_0^\infty \left( (v_- u_+^{k+1} - v_+ u_+^k) e^{-\alpha (k+1) t} - \frac{dx(t)}{dt}(v_- u_+ - v_+ u_+^k) e^{-\alpha t} \right) \varphi dt
$$

$$
+ \int_0^\infty \int_{\mathbb{R}} \alpha v \varphi dx dt + \int_0^\infty w(t)u_{\delta} e^{-\alpha t}(\varphi t + \frac{dx(t)}{dt} \varphi x) dt
$$

$$
= \int_0^\infty \left( (v_- u_+^{k+1} - v_+ u_+^k) e^{-\alpha (k+1) t} - \frac{dx(t)}{dt}(v_- u_+ - v_+ u_+^k) e^{-\alpha t} \right) \varphi dt
$$

$$
+ \int_0^\infty \int_{\mathbb{R}} \alpha v \varphi dx dt + \int_0^\infty w(t)u_{\delta} e^{-\alpha t}(\varphi t + \frac{dx(t)}{dt} \varphi x) dt
$$

$$
= \int_0^\infty \int_{\mathbb{R}} \alpha v \varphi dx dt + \int_0^\infty \alpha w(t)u_{\delta} e^{-\alpha t} \varphi dt = \langle \alpha v, \varphi \rangle
$$
which implies the second equation of (38). A completely similar argument leads to the first equation of (38).

As an application of Theorem 4.1 we have the following result:

**Corollary 4.1** (Keita and Bourgault [14]). Consider the following Eulerian droplet model

\[
\begin{cases}
  v_t + (vu)_x = 0, \\
  (vu)_t + (vu^2)_x = -\alpha uv,
\end{cases}
\]

with initial data given by

\[
(v(x, 0), u(x, 0)) = \begin{cases}
(v_-, u_-), & \text{if } x < 0, \\
(v_+, u_+), & \text{if } x > 0,
\end{cases}
\]

and assume that \(v_+ > 0\) and \(u_- > u_+\). Then the Riemann solution to the Eulerian droplet model is given by

\[
(v(x, t), u(x, t)) = \begin{cases}
(v_-, u_- e^{-\alpha t}), & \text{if } x < x(t), \\
(w(t)\delta(x - x(t)), u_\delta e^{-\alpha t}), & \text{if } x = x(t), \\
(v_+, u_+ e^{-\alpha t}), & \text{if } x > x(t),
\end{cases}
\]

where \(w(t) = \sqrt{v_- v_+} (u_- - u_+)^{\frac{1-e^{-\alpha t}}{\alpha}}, x(t) = \frac{\sqrt{v_-+\sqrt{v_+}}(1-e^{-\alpha t})}{\sqrt{v_-}+\sqrt{v_+}}, u_\delta = \frac{\sqrt{v_-+\sqrt{v_+}} u_-}{\sqrt{v_-}+\sqrt{v_+}}\), when \(v_- \neq v_+\).

For the case \(v_- = v_+\), \(w(t) = v_- (u_- - u_+)^{\frac{1-e^{-\alpha t}}{\alpha}}, x(t) = \frac{(u_-+u_+)(1-e^{-\alpha t})}{2\alpha}\) and \(u_\delta = \frac{1}{2}(u_- + u_+)\).

**Proof.** Using the Theorem 4.1 with \(k = 1\), we need solve the following system

\[
\begin{cases}
  \sigma = u_\delta, \\
  w_0 = -\sigma(v_- - v_+) + (v_- u_- - v_+ u_+), \\
  w_0 u_\delta = -\sigma(v_- u_- - v_+ u_+) + (v_- u_-^2 - v_+ u_+^2).
\end{cases}
\]

subject to \(u_+ < \sigma < u_-\). Thus, when \(v_- \neq v_+\), from the system (42) we have

\[
(v_- - v_+)^2 u_\delta^{(1)} - 2(v_- u_- - v_+ u_+)u_\delta + (v_- u_-^2 - v_+ u_+^2) = 0
\]

and therefore we can find

\[
u_\delta^{(1)} = \frac{\sqrt{v_- u_-} - \sqrt{v_+ u_+}}{\sqrt{v_-} - \sqrt{v_+}} \quad \text{and} \quad u_\delta^{(2)} = \frac{\sqrt{v_- u_-} + \sqrt{v_+ u_+}}{\sqrt{v_-} + \sqrt{v_+}}.
\]

Observe that only \(u_\delta^{(2)}\) satisfies the condition \(u_+ < \sigma < u_-\). Thus, with \(u_\delta = u_\delta^{(2)}\), from (42) we have that

\[
w_0 = \sqrt{v_- v_+} (u_- - u_+).
\]

When \(v_- = v_+\), from (42) we have

\[
2(u_- - u_+)u_\delta - (u_-^2 - u_+^2) = 0
\]

and therefore

\[
u_\delta = \frac{1}{2}(u_- + u_+) \quad \text{and} \quad w_0 = v_- (u_- - u_+). \tag*{\square}
\]
5 Final remarks

1. From Theorem 4.1, we can observe that when \( \alpha \to 0^+ \),

\[
\lim_{\alpha \to 0^+} x(t) = u_0^k t \quad \text{and} \quad \lim_{\alpha \to 0^+} w(t) = -u_0^k (v_+ - v_+) t + (v_- u_- - v_+ u_+^k) t.
\]

So, the limit behavior of the solution given in Theorem 4.1 as \( \alpha \to 0^+ \) corresponds to the solution given in Theorem 5.6 of [30].

2. It is easy see that (39) is a delta shock solution with discontinuity \( x = x(t) \) for the problem (1)–(2) if and only if the following generalized Rankine-Hugoniot conditions are satisfied

\[
\begin{align*}
\frac{dx(t)}{dt} &= (u_0(t))^k = \sigma(t), \\
\frac{dw(t)}{dt} &= -[\nu] \sigma(t) + [\nu u^k] \\
\frac{d[w(t) u_0(t)]}{dt} &= -[\nu u] \sigma(t) + [\nu u^{k+1}] - \alpha w(t) u_0(t)
\end{align*}
\]

where \([q] = q(x(t)^-, t) - q(x(t)^+, t)\). Observe that the solution given in Theorem 4.1 satisfies the generalized Rankine-Hugoniot conditions. Moreover, in this case the generalized Rankine-Hugoniot conditions (43) are equivalent to the equations given in the system (41).

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