STABILITY OF NON-MONOTONE NON-CRITICAL TRAVELING WAVES IN DISCRETE REACTION-DIFFUSION EQUATIONS WITH TIME DELAY

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ABSTRACT. This paper is concerned with traveling waves for temporally delayed, spatially discrete reaction-diffusion equations without quasi-monotonicity. We first establish the existence of non-critical traveling waves (waves with speeds $c > c^*$, where $c^*$ is minimal speed). Then by using the weighted energy method with a suitably selected weight function, we prove that all noncritical traveling waves $\phi(x + ct)$ (monotone or nonmonotone) are time-asymptotically stable, when the initial perturbations around the wavefronts in a certain weighted Sobolev space are small.

1. Introduction. In this paper, we consider the following spatially discrete reaction-diffusion equation with time delay

$$\frac{\partial u(t, x)}{\partial t} = \Delta_1 u(t, x) - u(t, x) + g(u(t - \tau, x)), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},$$

with the initial data

$$u(s, x) = u_0(s, x), \quad s \in [-\tau, 0], x \in \mathbb{R},$$

where $\tau > 0$, $\Delta_1 u(t, x) = u(t, x + 1) - 2u(t, x) + u(t, x - 1)$. This model is used to describe the spatio-temporal evolution of a single-species population [12, 18]. Here $u(t, x)$ represents the mature population at the time $t$ and the location $x$, and $\tau$ is the maturation delay. The function $g : [0, \infty) \to (0, \infty)$ is called the birth rate function and satisfies

(H1): two constant equilibria of $\{1\}$: 0 is unstable and $K$ is stable, namely $g(0) = 0$, $g(K) = K$, $1 - g'(0) < 0$ and $1 - g'(K) > 0$;

(H2): the unimodality condition: $g(u) > 0$ has only one positive local maximum at the point $u_* \in (0, K)$, and $g(u)$ is increasing on $[0, u_*]$ and decreasing on $[u_*, +\infty)$, which also implies $g'(0) > 0$ and $g'(K) < 0$;

(H3): $g \in C^2[0, \infty)$ and $|g'(u)| \leq g'(0)$ for $u \in [0, \infty)$.

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The equation (1) is a spatially discrete version of the time-delayed reaction-diffusion equation
\[
\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) + g(u(t-\tau,x)).
\] (3)
Model (3) and its non-local versions have been extensively studied, see [3, 14, 15, 19–21, 23] and references therein.

It is well known that traveling wave solution (in short, traveling wave) plays an important role in biology, since it can explain spatial spread or invasion of the species. A traveling wave of (1) is a special translation invariant solution in the form of
\[
u(t,x) = \phi(\xi), \quad \xi = x + ct,
\] where \(c\) is the wave speed. If \(\phi(\xi)\) is monotone in \(\xi \in \mathbb{R}\), then it is called a traveling wavefront. There have been many results on the traveling waves for spatially discrete reaction-diffusion equations [1, 2, 5–8, 18]. For example, Ma and Zou [18] established the existence, uniqueness and asymptotic stability of traveling wavefronts of (1) under the assumption that \(g(u)\) is increasing on the interval \([0, K]\). Chen and Guo [1] investigated the more general equation without delay
\[
u_t(t,x) = [g(u(t,x+1)) + g(u(t,x-1)) - 2g(u(t,x))] + f(u(t,x)).
\] They used a sub-super solution method of Wu and Zuo [24] to show the existence of traveling wavefronts, and a squeezing method to get the asymptotic stability of traveling wavefronts. Recently, Guo and Zimmer [7, 8] studied the traveling wavefronts of the following equation with nonlocal delay effect
\[
\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + f(u(t,x), (h \ast u)(t-\tau,x)),
\] (4)
where \((h \ast u)(t,x) = \int_R h(x-y)u(t,y)dy\). Under the monotonicity assumption \(\partial_2 f(u,v) \geq 0\) for \((u,v) \in [0, K]^2\), Guo and Zimmer [7] established the existence, monotonicity, uniqueness and asymptotic behavior of traveling wavefronts of (4). In [8], by using a combination of the weighted energy method and the Green function technique, Guo and Zimmer further proved that all noncritical traveling wavefronts are globally exponentially stable, and critical traveling wavefronts are globally algebraically stable, when the initial perturbations around the wavefront decay to zero exponentially near minus infinity regardless of the magnitude of time delay. It is easy to see that when \(f(u,v) = -u + g(v)\) and \(h(x) = \delta(x)\) (Dirac delta function), (4) reduces to our model (1).

Note that all above results in [1, 7, 8, 18] are established under the assumption that the whole interaction term is quasi-monotone. When the whole interaction term is non-quasi-monotone, there is no results on the traveling waves. In Remark 1.4 of [18], Ma and Zou pointed out that when \(g(u)\) is not increasing on \([0, K]\), the traveling wave problem of (1) becomes much harder due to the lack of quasi-monotonicity. Inspired by the study on reaction-diffusion equation by Ma [17], in the first part of this paper, we establish the existence of traveling waves of (1) by constructing two auxiliary discrete reaction-diffusion equations with quasi-monotonicity. This method can also be seen in [4, 10] for establishing the spreading speeds. We should point out that, by this method, we can not get any information on the monotonicity of traveling waves. We leave this problem for the future study.

Our main goal of this paper is to prove the stability of traveling waves of (1). We remark that the methods in [1, 8, 18] can not be directly applied to our equation.
due to the lack of the comparison principle. As we know, the stability of traveling waves for non-monotone delayed monostable reaction-diffusion equations and nonlocal dispersal equations have been studied, see [3,11,14,23,25]. Wu, Zhao and Liu [23] used the technical weighted energy method to get the stability of traveling waves of (3) with large speed. In [14], Lin et al. still took the technical weighted energy method, and obtained the stability of traveling waves of (3) with noncritical speeds. Very recently, Chern et al. [3] further proved the stability of traveling waves of (3) with critical speed. Motivated by [14], in this paper, we take the technical weighted energy method to prove the stability of traveling waves of (1) with non-critical speeds. We leave the study on the stability of traveling waves of (1) with critical speed in another paper. We need to point out that the application of the method in [14] to our equation (1) is nontrivial and challenging. For example, (i) comparing with the random diffusion operator \( \partial^2 u(t,x) / \partial x^2 \), the discrete diffusion operator \( \Delta_1 u(t,x) = u(t,x+1) - 2u(t,x) + u(t,x-1) \) produces more complex calculation for obtaining the energy estimates in weighted space; (ii) in order to obtain that the solution of the perturbed equation of (3) belongs to \( X(-\tau,t_0) \), the fundamental solution (like the heat kernel to the linear heat equation) of the perturbed equation plays an important role, see [14]. However, (1) and its perturbed equation (19) do not have fundamental solution as in [14]. More recently, Hu and Li [12] developed the fundamental solution theory to the linear lattice differential equations by using the classical modified Bessel functions. Unfortunately, it also cannot be directly applied to our equation (19) due to the presence of the term of first order derivative with respect to the space variable. Hence, we need to look for a new strategy to treat this problem. Inspired by classical transport equation, we can come over the difficulty caused by no fundamental solution by giving some new forms of the solutions of (19). We should remark that these new forms of the solutions of (19) can help us to get the estimate of solutions of (19) in \( C^\cdot \)-norm and the uniform limit of solutions of (19).

Before stating our main result, let us make the following notation. Throughout the paper, \( C > 0 \) always denotes a generic constant, while \( C_i > 0 (i = 0, 1, 2, \ldots) \) represents a specific constant. Let \( I \) be an interval, typically \( I = \mathbb{R} \). \( L^2(I) \) is the space of the square integrable defined on \( I \), and \( H^k(I)(k \geq 0) \) is the Sobolev space of the \( L^2 \)-functions \( f(x) \) defined on the interval \( I \) whose derivatives \( \frac{d^k}{dx^k} f(i = 1, \ldots k) \) also belong to \( L^2(I) \). \( L^2_w(I) \) denotes the weighted \( L^2 \)-space with a weight function \( w(x) > 0 \), and its norm is defined by

\[
\| f \|_{L^2_w} = \left( \int_I w(x)|f(x)|^2 \, dx \right)^{1/2}.
\]

\( H^k_w(I) \) is the weighted Sobolev space with the norm given by

\[
\| f \|_{H^k_w} = \left( \sum_{i=0}^k \int_I w(x) \left| \frac{d^i}{dx^i} f(x) \right|^2 \, dx \right)^{1/2}.
\]

Let \( T > 0 \) be a number and \( B \) a Banach space. We denote by \( C([0,T];B) \) the space of the \( B \)-valued continuous functions on \([0,T]\) and by \( L^2([0,T];B) \) the space of the \( B \)-valued \( L^2 \)-functions on \([0,T]\).

Define a weight function

\[
w(x) = e^{-2\lambda(x-x_0)}, \quad \lambda \in (\lambda_1, \lambda_2),
\]

where \( \lambda_1, \lambda_2 \) are constants greater than zero.
with a sufficient large number \(x_0 \gg 1\).

Now we state the stability result of traveling waves for (1) with a general non-monotone function \(g(u)\).

**Theorem 1.1 (Stability of traveling waves).** Assume that (H1) — (H3) hold. For any given traveling wave \(\phi(x + ct)\) with \(c > c_*\) to (1), whatever it is monotone or non-monotone, suppose that
\[
 u_0(s, x) - \phi(x + cs) \in C([−τ, 0]; C(\mathbb{R}) \cap H^1_w(\mathbb{R})) \cap L^2([−τ, 0]; H^1_w(\mathbb{R})),
\]
and \(\lim_{x \to +\infty} |u_0(s, x) - \phi(x + cs)| =: u_{0, \infty}(s) \in C[−τ, 0]\) exists uniformly with respect to \(s \in [−τ, 0]\). Then there exist some constants \(\delta_0 > 0\), \(0 < \mu_2 = \mu_2(\tau, g'(K)) < 1\), and \(0 < \mu = \mu(\tau, c, \lambda, g'(K)) < \mu_2\), all independent of \(x, t, u(t, x)\) and \(\phi(x + ct)\), such that, when the initial perturbation is small:
\[
 \max_{s \in [−τ, 0]} \|(u_0 - \phi)(s)\|_{L^2}^2 + \|(u_0 - \phi)(0)\|_{H^1_w}^2 + \int_{−τ}^0 \|(u_0 - \phi)(s)\|_{H^1_w}^2 \, ds \leq \delta_0^2,
\]
the unique solution \(u(t, x)\) of (1) and (2) exists globally and satisfies
\[
 u(t, x) - \phi(x + ct) \in C([−τ, \infty); C(\mathbb{R}) \cap H^1_w(\mathbb{R}))
\]
\[
 \cap L^2([−τ, \infty); H^1_w(\mathbb{R})) \cap C_{unif}[-τ, \infty)
\]
and
\[
 \sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + ct)| \leq C e^{-\mu t}, \quad t > 0,
\]
where \(C_{unif}[-τ, T]\) for \(0 < T \leq \infty\), is defined by
\[
 C_{unif}[-τ, T] = \{ v(t, x) \in C([-τ, T] \times \mathbb{R}) \text{ such that } \lim_{x \to +\infty} v(t, x) \text{ exists uniformly in } t \in [−τ, T] \}.
\]

**Corollary 1 (Uniqueness of traveling waves).** Assume that (H1) — (H3) hold. Then, for any traveling waves \(\phi(x + ct)\) of (1), whatever they are monotone or non-monotone, with the same speed \(c > c_*\) and the same exponential decay at \(\xi \to -\infty\):
\[
 \phi(\xi) = O(1) e^{-\lambda|\xi|} \quad \text{as } \xi \to -\infty,
\]
they are unique up to translation.

The rest of this paper is organized as follows. In section 2, we study the existence of non-critical traveling waves of (1) with a general nonmonotone function \(g(u)\). In section 3, we reformulate the original equation to the perturbed equation around the given non-critical traveling wave, and then give the corresponding stability theorem for the new equation. In section 4, we take the weighted energy method to establish the desired a priori estimates, which play an important role in the proof of stability. Based on the stability theorem, in section 5, we prove the uniqueness of those monotone or nonmonotone traveling waves. In section 6, we give some applications.

2. **Traveling waves.** In this section, we study the existence of non-critical traveling waves of (1) without monotonicity on \(g(u)\). A traveling wave for (1) connecting with two steady states 0 and \(K\) at far fields is a special solution in the form of \(u(t, x) = \phi(x + ct) \geq 0\). Substituting \(\phi(x + ct)\) into (1) and letting \(\xi = x + ct\), we obtain the following wave profile equation with the boundary conditions
\[
 \begin{align*}
 c\phi'(\xi) - \Delta_1 \phi(\xi) + \phi(\xi) &= g(\phi(\xi - c\tau)), \\
 \phi(-\infty) &= 0, \quad \phi(+\infty) = K,
\end{align*}
\]
where \( \frac{d}{d\xi} \), \( \Delta_1 \phi(\xi) = \phi(\xi + 1) - 2\phi(\xi) + \phi(\xi - 1) \) and \( c \) is the wave speed.

It is clear that the characteristic function for \( \phi \) with respect to the trivial equilibrium 0 can be represented by
\[
P(c, \lambda) = c\lambda - (e^\lambda + e^{-\lambda} - 2) + 1 - g'(0)e^{-\lambda c\tau}.
\]

One can easily show that the following result holds.

**Lemma 2.1.** Assume that \( g'(0) > 1 \). Then there exist \( \lambda_* > 0 \) and \( c_* > 0 \) such that
\[
P(c_*, \lambda_*) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda} P(c_*, \lambda)\bigg|_{\lambda=\lambda_*} = 0.
\]
Furthermore, if \( c > c_* \), then \( P(c, \lambda) = 0 \) has two distinct positive real roots \( \lambda_1(c) \) and \( \lambda_2(c) \) with \( \lambda_1(c) < \lambda_* < \lambda_2(c) \), and \( P(c, \lambda) > 0 \) for \( \lambda \in (\lambda_1(c), \lambda_2(c)) \).

When \( g(u) \) is increasing on \([0, K]\), the existence of traveling wavefront has been established in [18] by using sub-super solutions and monotone iteration technique.

**Lemma 2.2.** Assume that (H1) holds and \( g'(0)u-g(u) \leq Mu^{1+\nu} \) for all \( u \in (0, K) \), some \( M > 0 \) and some \( \nu \in (0, 1) \). Let \( c_* > 0 \) be defined as in Lemma 2.1. Then for each \( c \geq c_* \), \([\xi]\) admits a strictly increasing traveling wavefront \( u(x,t) = \phi(x + ct) \) satisfying \( \phi(-\infty) = 0 \) and \( \phi(+\infty) = K \), while for any \( 0 < c < c_* \), \([\xi]\) has no traveling wave \( \phi(x + ct) \) connecting 0 and \( K \). Moreover, when \( c > c_* \),
\[
\lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = 1, \quad \lim_{\xi \to +\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c),
\]
where \( \lambda_1(c) > 0 \) is the smallest solution to the equation \( P(c, \lambda) = 0 \).

Now we devote to studying the case of \( g \) without monotonicity on \([0, K]\). We assume that there exists \( K^* \geq K \) such that \( K^* \geq \max\{g(u)|0 \leq u \leq K^*\} \). Let
\[
K_* := \inf \left\{ u \mid u = \inf_{s \in [0, K^*]} \{g(s)|g(s) \leq s\} \right\} > 0.
\]
Clearly, it is well defined and \( g(u) > u \) for all \( u \in (0, K_*) \). Hence, there exists a small \( \varepsilon_0 \in (0, K_*) \) such that \( K_* - \varepsilon > 0 \) for every \( \varepsilon \in [0, \varepsilon_0] \).

For any \( \varepsilon \in (0, \varepsilon_0) \), define two continuous functions as follows:
\[
g^*(u) = \begin{cases} 
\min\{g'(0)u, K^*\}, & \text{for } u \in [0, K^*], \\
\max\{K^*, g(u)\}, & \text{for } u > K^*, 
\end{cases}
\]
and
\[
g_*(u) = \begin{cases} 
\inf_{\eta \in [u, K^*]} \{g(\eta), K_* - \varepsilon\}, & \text{for } u \in [0, K_*], \\
\min\{g(u), K_* - \varepsilon\}, & \text{for } u > K^*.
\end{cases}
\]

Then we have the following observations:

**Lemma 2.3.** The following statements hold true:

(i): \( g^*(u) \) and \( g_*(u) \) are continuous on \([0, +\infty)\) and nondecreasing on \([0, K^*]\);
(ii): \( g^*(u) \geq g(u) \geq g_*(u) \) for all \( u \geq 0 \);
(iii): \( g'(0)u \geq g^*(u) > 0 \) and \( g'(0)u \geq g_*(u) > 0 \) for all \( u \in (0, K^*) \);
(iv): \( g^*(0) = K^* - g^*(K^*) = 0 \), and \( g^*(u) > u \) for all \( u \in (0, K^*) \);
(v): \( g_*(0) = K_* - \varepsilon - g_*(K_* - \varepsilon) = 0 \), and \( g_*(u) > u \) for all \( u \in (0, K_* - \varepsilon) \);
(vi): \( g_*(0) = g'(0) \) and \( \lim \sup_{u \to 0^+} [g'_*(0) - g_*(u)/u] u^{-\nu} < +\infty \).
Consider the following two auxiliary equations
\[ u_t(t, x) = \Delta_1 u(t, x) - u(t, x) + g^*(u(t - \tau, x)) \]  \hspace{1cm} (9)

and
\[ u_t(t, x) = \Delta_1 u(t, x) - u(t, x) + g_*(u(t - \tau, x)). \]  \hspace{1cm} (10)

Clearly, the wave profile equations corresponding to (9) and (10) read as
\[ c \phi'(\xi) - \Delta_1 \phi(\xi) + \phi(\xi) = g^*(\phi(\xi - c\tau)) \]
\[ c \phi'(\xi) - \Delta_1 \phi(\xi) + \phi(\xi) = g_*(\phi(\xi - c\tau)) \]
\hspace{1cm} (11) and (12) respectively.

From Lemma 2.2, we can obtain the following existence result on traveling wavefronts for (9) and (10).

**Lemma 2.4.** Let \( c_* > 0 \) be defined as in Lemma 2.1. Then for each \( c \geq c_* \), (9) and (10) have strictly increasing traveling wavefronts \( \phi^*(x + ct) \) and \( \phi_*(x + ct) \), respectively, satisfying
\[ \phi^*(-\infty) = \phi_*(-\infty) = 0, \quad \phi^*(+\infty) = K^*, \quad \phi_*(+\infty) = K_* - \epsilon \]
and
\[ \lim_{\xi \to -\infty} \phi^*(\xi) e^{-\lambda_1(c)\xi} = \lim_{\xi \to -\infty} \phi_*(\xi) e^{-\lambda_1(c)\xi} = 1, \]
where \( \lambda_1(c) > 0 \) is the smallest solution to the equation \( P(c, \lambda) = 0 \). 

**Theorem 2.5.** Assume that (H1) – (H3) hold. Then for each \( c > c_* \), (1) admits a traveling wave \( u(t, x) = \phi(x + ct) \) satisfying \( \phi(-\infty) = 0 \) and \( 0 < K_* \leq \liminf_{\xi \to +\infty} \phi(\xi) \leq \limsup_{\xi \to +\infty} \phi(\xi) \leq K^* \).

**Proof.** For \( c > c_* \), let \( \phi^*(x + ct) \) and \( \phi_*(x + ct) \) be the nondecreasing traveling wavefronts of (9) and (10), respectively, which are given in Lemma 2.4. Let \( a_1 > 0 \) be such that \( e^{\lambda_1(c)a_1} \geq 3 \). Then
\[ \lim_{\xi \to -\infty} \phi^*(\xi + a_1) e^{-\lambda_1(c)\xi} = e^{\lambda_1(c)a_1} \geq 3. \]

Therefore, there exists \( M_1 > 0 \) such that
\[ \phi^*(\xi + a_1) e^{-\lambda_1(c)\xi} > 2 > \phi_*(\xi) e^{-\lambda_1(c)\xi} \]
for all \( \xi \leq -M_1 \). (13)

Since \( \phi^*(+\infty) = K^* > K_* - \epsilon = \phi_*(+\infty) \), we choose \( a_2 > 0 \) sufficiently large such that
\[ \phi^*(\xi + a_2) > \phi_*(\xi) \]
for all \( \xi \geq -M_1 \). (14)

Let \( a_0 = \max\{a_1, a_2\} \). Since \( \phi^*(\cdot) \) is nondecreasing, it follows from (13) and (14) that
\[ \phi^*(\xi + a_0) > \phi_*(\xi) \]
for all \( \xi \in \mathbb{R} \).

Define
\[ H^*[\phi](\xi) := \Delta_1 \phi(\xi) - \phi(\xi) + g^*(\phi(\xi - c\tau)) + c\gamma \phi(\xi), \quad \xi \in \mathbb{R} \]
and
\[ H_*[\phi](\xi) := \Delta_1 \phi(\xi) - \phi(\xi) + g_*(\phi(\xi - c\tau)) + c\gamma \phi(\xi), \quad \xi \in \mathbb{R}. \]

Choose \( \gamma > \frac{3 + g^{(\infty)}}{c} \). Then for any \( \phi, \psi \in C(\mathbb{R}, [0, K^*]) \) with \( \phi(\xi) \geq \psi(\xi) \), \( \xi \in \mathbb{R} \), we have
\[ H^*[\phi](\xi) \geq H^*[\psi](\xi) \quad \text{and} \quad H_*[\phi](\xi) \geq H_*[\psi](\xi) \]
for all \( \xi \in \mathbb{R}. \) (15)
For any $\lambda \in (0, \min\{\lambda_1(c), \lambda_2(c)\})$, let

$$X_\lambda = \left\{ \phi \in C(\mathbb{R}, \mathbb{R}) \left| \sup_{\xi \in \mathbb{R}} |\phi(\xi)|e^{-\lambda \xi} < +\infty \right\} \right., \quad \|\phi\|_\lambda = \sup_{\xi \in \mathbb{R}} |\phi(\xi)|e^{-\lambda \xi}.$$  

Then $(X_\lambda, \| \cdot \|_\lambda)$ is a Banach space. Since $\phi_*(\xi) \leq \phi^*(\xi + a_0)$ for all $\xi \in \mathbb{R}$, it is easy to see that the set

$$\Gamma := \left\{ \phi \in C([0, K^*]) \left| \begin{array}{l}
(i) \phi_*(\xi) \leq \phi^*(\xi + a_0) \quad \text{for all } \xi \in \mathbb{R}; \\
(ii) |\phi(\xi_1) - \phi(\xi_2)| \leq 2\gamma K^*|\xi_1 - \xi_2| \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R}.
\end{array} \right\} \right.$$  

is nonempty, convex and compact in $X_\lambda$.

Define $F : \Gamma \to C([0, K^*])$ by

$$F(\phi)(\xi) = \frac{1}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} H[\phi](y) dy,$$

where $H[\phi](\xi) = \Delta_1 \phi_*(\xi) - \phi^*(\xi + a_0) + g(\phi(\xi) - c\tau) + c\gamma \phi(\xi)$, $\xi \in \mathbb{R}$. Clearly, for any $\phi \in \Gamma \subset C([0, K^*])$, it follows from (15) that

$$0 \leq H_\phi[\phi(\xi) - H^*|\phi(\xi)| \leq -K^* + g^*(K^*) + c\gamma K^* = c\gamma K^*$$

for all $\xi \in \mathbb{R}$. Then we further obtain

$$0 \leq F(\phi)(\xi) \leq \frac{c\gamma K^*}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} dy = K^*,$$

and hence, $F : \Gamma \to C([0, K^*])$ is well defined. Furthermore, it is easily seen that a fixed point of $F$ is a solution of (8).

For any $\phi, \psi \in \Gamma$, we have

$$e^{-\lambda \xi}|H[\phi] - H[\psi]|$$

$$= e^{-\lambda \xi} |\Delta_1 (\phi - \psi) - (\phi - \psi) + (g(\phi(\xi) - c\tau) - g(\psi(\xi) - c\tau) + c\gamma (\phi - \psi)|$$

$$\leq e^{-\lambda \xi} \{ |\Delta_1 (\phi - \psi) + |\phi - \psi| + g'(0)|\phi(\xi) - c\tau) - \psi(\xi - c\tau) + c\gamma |\phi - \psi| \}$$

$$\leq e^{-\lambda \xi} [e^\lambda + e^{-\lambda} - 2 + 1 + g'(0)e^{-\lambda c}\tau + c\gamma]|\phi - \psi|$$

$$\leq L\|\phi - \psi\|_\lambda,$$

where $L := c(\lambda + c) + 2$. Therefore, we have

$$|F(\phi)(\xi) - F(\psi)(\xi)|e^{-\lambda \xi}$$

$$= \left| \frac{1}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} (H[\phi](y) - H[\psi](y)) dy \right| e^{-\lambda \xi},$$

$$\leq \frac{e^{-\lambda \xi}}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} |H[\phi](y) - H[\psi](y)| dy$$

$$\leq \frac{L}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} dy \|\phi - \psi\|_\lambda$$

$$= \frac{L}{c\gamma}\|\phi - \psi\|_\lambda,$$

which imply that $F : \Gamma \to C([0, K^*])$ is continuous.

Next, we shall show that $F(\Gamma) \subseteq \Gamma$. Since $\phi_*(\xi)$ is the solution of (12), we have

$$\phi_*(\xi) = \frac{1}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} H[\phi_*(\xi)](y) dy.$$  

(17)
For any $\phi \in \Gamma$, we have $0 \leq \phi_*(\xi) \leq \phi(\xi) \leq \phi^*(\xi + a_0) \leq K^*$ for all $\xi \in \mathbb{R}$. Therefore, it follows from (15)-(17) that

\[
F(\phi)(\xi) = \frac{1}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} H[\phi](y) \, dy \\
\geq \frac{1}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} H_*[\phi](y) \, dy \\
\geq \frac{1}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} H_*[\phi_*](y) \, dy \\
= \phi_*(\xi).
\]

Since $\phi^*(\xi + a_0)$ is a solution of (11), by using a similar argument, we can show that $F(\phi)(\xi) \leq \phi^*(\xi + a_0)$ for all $\xi \in \mathbb{R}$. For any $\phi \in \Gamma$ and $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_1 < \xi_2$, it follows from (16) that

\[
|F(\phi)(\xi_1) - F(\phi)(\xi_2)| \\
= \frac{1}{c} e^{\gamma \xi_1} \int_{\xi_1}^{\xi_2} e^{-\gamma(y-\xi)} H[\phi](y) \, dy - \int_{\xi_2}^{+\infty} e^{-\gamma(y-\xi_2)} H[\phi](y) \, dy \\
\leq \frac{1}{c} e^{\gamma \xi_1} \int_{\xi_1}^{\xi_2} e^{-\gamma(y-\xi)} H[\phi](y) \, dy + (e^{\gamma \xi_2} - e^{\gamma \xi_1}) \int_{\xi_2}^{+\infty} e^{-\gamma(y-\xi_2)} H[\phi](y) \, dy \\
\leq \frac{1}{c} \sup_{y \in \mathbb{R}} H[\phi](\xi) \left\{ e^{\gamma \xi_1} \int_{\xi_1}^{\xi_2} e^{-\gamma(y-\xi)} \, dy + e^{\gamma \xi_2} (1 - e^{\gamma (\xi_2 - \xi_1)}) \int_{\xi_2}^{+\infty} e^{-\gamma(y-\xi)} \, dy \right\} \\
\leq 2\gamma K^* |\xi_1 - \xi_2|.
\]

Therefore, we conclude that $F(\phi) \in \Gamma$ for all $\phi \in \Gamma$. By virtue of the Schauder’s fixed point theorem, $F$ has a fixed point $\phi$ in $\Gamma \subset X_\lambda$, which satisfies

\[
\phi(\xi) = \frac{1}{c} \int_{\xi}^{+\infty} e^{-\gamma(y-\xi)} H[\phi](y) \, dy
\]

and

\[
\phi_*(\xi) \leq \phi(\xi) \leq \phi^*(\xi + a_0) \quad \text{for all} \ \xi \in \mathbb{R}. \tag{18}
\]

Taking the limit $\xi \to -\infty$ and $\xi \to +\infty$ in (18), respectively, we get $\phi(-\infty) = 0$ and

\[
K_* - \varepsilon \leq \liminf_{\xi \to +\infty} \phi(\xi) \leq \limsup_{\xi \to +\infty} \phi(\xi) \leq K^*.
\]

Since $\phi(\xi)$ is independent of $\varepsilon$, taking the limit as $\varepsilon \to 0^+$ in the last inequality, we get

\[
K_* \leq \liminf_{\xi \to +\infty} \phi(\xi) \leq \limsup_{\xi \to +\infty} \phi(\xi) \leq K^*.
\]

The proof is complete. $\Box$

3. Reformulation of the problem. This section is devoted to the proof of stability of those monotone or non-monotone non-critical traveling waves of (1).

Let $\phi(x + ct) = \phi(\xi)$ be a given traveling wave with speed $c > c_*$, and

\[
v(t, \xi) := u(t, x) - \phi(x + ct) = u(t, \xi - ct) - \phi(\xi), \\
v_0(s, \xi) := u_0(s, x) - \phi(x + cs).
\]

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Then, from (1) and (8), we can see that $v(t,\xi)$ satisfies
\begin{align*}
\begin{cases}
\frac{\partial v}{\partial t} + c \frac{\partial v}{\partial \xi} - \Delta_1 v + v - g'(\phi(\xi - c\tau))v(t - \tau, \xi - c\tau) \\
v(s, \xi) = v_0(s, \xi), \quad s \in [-\tau,0], \quad \xi \in \mathbb{R},
\end{cases}
\end{align*}
(19)
where
\[ Q(v) := g(\phi + v) - g(\phi) - g'(\phi)v, \]
with $\phi = \phi(\xi - c\tau)$ and $v = v(t - \tau, \xi - c\tau)$.

Let $T > 0$. We define the solution space for (19) as follows
\[ X(-\tau, T) = \left\{ v|v(t, \xi) \in C([-\tau, T]; C(\mathbb{R}) \cap H^1_w(\mathbb{R})) \cap L^2([-\tau, T]; H^1_w(\mathbb{R})) \right\}, \]
equipped with the norm
\[ M_v(T)^2 = \sup_{t \in [-\tau, T]} \left( \|v(t)\|_C^2 + \|v(t)\|_{H^1_w}^2 \right) + \int_{-\tau}^{T} \|v(s)\|_{H^1_w}^2 \, ds. \]
Particularly, when $T = \infty$, we denote the solution space by $X(-\tau, \infty)$ and the norm of the solution space by $M_v(\infty)$.

Now we state the stability result for the perturbed equation (19) which automatically implies Theorem 1.1.

**Theorem 3.1 (Stability).** For any given traveling wave $\phi(x + ct) = \phi(\xi)$ with $c > c_\ast$, suppose that $v_0(s, \xi) \in C([-\tau, 0]; C(\mathbb{R}) \cap H^1_w(\mathbb{R})) \cap L^2([-\tau, 0]; H^1_w(\mathbb{R}))$, and $\lim_{t \to +\infty} v_0(s, \xi) =: v_{0,\infty}(s) \in C[-\tau, 0]$ exists uniformly in $s$, where $w(\xi)$ is defined in (6). Then there exist some constants $\delta_0 > 0$ and $\mu > 0$ which are independent of $v$, $t$ and $\xi$ when $M_v(0) \leq \delta_0$. The solution $v(t, \xi)$ of (19) then uniquely and globally exists in $X(-\tau, \infty)$ and satisfies
\begin{align*}
\|v(t)\|_C^2 + \|v(t)\|_{H^1_w}^2 &+ \int_{-\tau}^{t} e^{-2\mu(t-s)} \|v(s)\|_{H^1_w}^2 ds \\
\leq & Ce^{-2\mu t} \left( \max_{s \in [-\tau, 0]} \|v_0(s)\|_C^2 + \|v_0(0)\|_{H^1_w}^2 + \int_{-\tau}^{0} \|v_0(s)\|_{H^1_w}^2 ds \right)
\end{align*}
for $t \in [0, \infty)$.

By using the continuity extension method [15][16], the global existence of $v(t, \xi)$ directly follows from the local existence result and the a priori estimate given below.

**Proposition 1** (local existence). Under the assumption in Theorem 3.1, for any given traveling wave $\phi(x + ct) = \phi(\xi)$ with $c > c_\ast$, suppose $v_0(s, \xi) \in X(-\tau, 0)$, and $M_v(0) \leq \delta_1$ for a given positive constant $\delta_1 > 0$. Then there exists a small $t_0 = t_0(\delta_1) > 0$ such that the local solution $v(t, \xi)$ of (19) uniquely exists for $t \in [-\tau, t_0]$, and satisfies $v \in X(-\tau, t_0)$ and $M_v(t_0) \leq \tilde{c}_0 M_v(0)$ for some constant $\tilde{c}_0 > 1$.

**Proof.** Let $v^{(0)}(t, \xi) \in X(-\tau, t_0)$ and choose $v^{(0)}(t, \xi) = v_0(t, \xi)$ for $t \in [-\tau, 0]$. Define the iteration $v^{(n+1)} = \mathcal{F}(v^{(n)})$ for $n \geq 0$ by
\begin{align*}
\begin{cases}
\frac{\partial v^{(n+1)}}{\partial t} + c \frac{\partial v^{(n+1)}}{\partial \xi} + 3v^{(n+1)} \\
v^{(n+1)}(s, \xi) = v_0(s, \xi), \quad s \in [-\tau, 0], \quad \xi \in \mathbb{R},
\end{cases}
\end{align*}
(20)
where
\[ P(v^{(n)}) := g(\phi + v^{(n)}) - g(\phi), \]
with \( \phi = \phi(\xi - c\tau) \) and \( v^{(n)} = v^{(n)}(t - \tau, \xi - c\tau) \). By Taylor’s formula and (H3), we obtain
\[
|P(v^{(n)}(t - \tau, \xi - c\tau))| = |g(\phi + v^{(n)}) - g(\phi)| = |g'(\eta)|v^{(n)}(t - \tau, \xi - c\tau)| \leq g'(0)|v^{(n)}(t - \tau, \xi - c\tau)|,
\]
where \( \eta \) is some function between \( \phi \) and \( \phi + v^{(n)} \). It is easy to see that the solution of (20) can be written in the integral form
\[
v^{(n+1)}(t, \xi) = e^{-3t}v_0(0, \xi - ct) + e^{-3t} \int_0^t e^{3s}[v^{(n+1)}(s, \xi + 1 + c(s - t)) + v^{(n+1)}(s, \xi - 1 + c(s - t)) + P(v^{(n)}(s - \tau, \xi + c(s - t - \tau)))] \, ds.
\]
Combining (21) and (22), one has
\[
\|v^{(n+1)}(t)\|_C \leq e^{-3t}\|v_0(0)\|_C + 2 \int_0^t e^{-3(t-s)}\|v^{(n+1)}(s)\|_C \, ds + C \int_0^t e^{-3(t-s)}\|v^{(n)}(s - \tau)\|_C \, ds \leq \|v_0(0)\|_C + Ct_0 \sup_{t \in [-\tau, t_0]} \|v^{(n)}(t)\|_C + 2 \int_0^t \|v^{(n+1)}(s)\|_C \, ds.
\]
Furthermore, by Gronwall’s inequality, we get
\[
\|v^{(n+1)}(t)\|_C \leq \left( \|v_0(0)\|_C + Ct_0 \sup_{t \in [-\tau, t_0]} \|v^{(n)}(t)\|_C \right) e^{2t_0}, \quad t \in [0, t_0].
\]
Notice that \( v^{(n)}(t, \xi) \in C_{unif}[-\tau, t_0] \), namely, \( \lim_{\xi \to \infty} v^{(n)}(t, \xi) = v^{(n)}(t) = v^{(n)}(t, \xi) \in C[-\tau, t_0] \).

Now we prove \( v^{(n+1)}(t, \xi) \in C_{unif}[-\tau, t_0] \). We rewrite the solution of (20) as
\[
v^{(n+1)}(t, \xi) = e^{-t}v_0(0, \xi - ct) + e^{-t} \int_0^t e^{s} \lim_{\xi \to \infty} P(v^{(n)}(s - \tau, \xi + c(s - t - \tau))) \, ds
\]
\[-2v^{(n+1)}(s, \xi + c(s - t)) + v^{(n+1)}(s, \xi - 1 + c(s - t)) + P(v^{(n)}(s - \tau, \xi + c(s - t - \tau))) \, ds.
\]
It is clear that
\[
\lim_{\xi \to \infty} v^{(n+1)}(t, \xi)
\]
\[= e^{-t} \lim_{\xi \to \infty} v_0(0, \xi - ct) + e^{-t} \int_0^t e^{s} \lim_{\xi \to \infty} P(v^{(n)}(s - \tau, \xi + c(s - t - \tau))) \, ds
\]
\[= v_0(0)e^{-t} + \int_0^t e^{-(t-s)}P(v^{(n)}(s - \tau)) \, ds
\]
\[= : v^{(n+1)}(t), \quad \text{uniformly with respect to } t \in [-\tau, t_0].
\]
We can further prove that $v^{(n+1)}(t, \xi)$ is uniformly convergent as $\xi \to \infty$. In fact,

$$\lim_{\xi \to \infty} \limsup_{0 \leq t \leq t_0} \left| v^{(n+1)}(t, \xi) - v^{(n+1)}_\infty(t) \right|$$

$$\leq \left. \lim_{\xi \to \infty} \limsup_{0 \leq t \leq t_0} \int_0^t e^{-(t-s)} \left[ P(v^{(n)}(s-\tau, \xi + c(s-t-\tau))) - P(v^{(n)}_\infty(s-\tau)) \right] ds \right|$$

$$\leq C \lim_{\xi \to \infty} \limsup_{0 \leq t \leq t_0} \int_0^t \sup_{s \in [0, t_0]} \left( e^{-(t-s)} |v^{(n)}(s-\tau, \xi + c(s-t-\tau)) - v^{(n)}_\infty(s-\tau)| \right) ds$$

$$= C \sup_{0 \leq t \leq t_0} \int_0^t \lim_{\xi \to \infty} \sup_{s \in [0, t_0]} \left( e^{-(t-s)} |v^{(n)}(s-\tau, \xi + c(s-t-\tau)) - v^{(n)}_\infty(s-\tau)| \right) ds$$

$$= 0.$$

Next, we shall show the regular energy estimates for $v^{(n+1)}(t, \xi)$. Multiplying (20) by $w(\xi)v^{(n+1)}(t, \xi)$, we have

$$\left\{ \frac{1}{2} w^{(n+1)^2}(t, \xi) \right\}_t + \left\{ \frac{1}{2} c w^{(n+1)^2}(t, \xi) \right\}_\xi + \left[ 3 - \frac{c}{2} w' \right] w^{(n+1)^2}(t, \xi)$$

$$= w(\xi)v^{(n+1)}(t, \xi)v^{(n+1)}(t, \xi + 1) + w(\xi)v^{(n+1)}(t, \xi)v^{(n+1)}(t, \xi - 1)$$

$$+ w(\xi)v^{(n+1)}(t, \xi)P(v^n)(t - \tau, \xi - ct)).$$

Integrating (24) over $\mathbb{R} \times [0, t]$ with respect to $\xi$ and $t$, and noting the vanishing term at far fields (see (36) below),

$$\left\{ \frac{1}{2} c w^{(n+1)^2}(t, \xi) \right\}_\xi = 0,$$

we obtain

$$\|v^{(n+1)}(t)\|_{L^2_w}^2 + \int_0^t \int_{\mathbb{R}} \left\{ 6 - \frac{c}{w(\xi)} \frac{w'(\xi)}{w(\xi)} \right\} w(\xi)(v^{(n+1)^2}(s, \xi) d\xi ds$$

$$= \|v_0(0)\|_{L^2_w}^2 + 2 \int_0^t \int_{\mathbb{R}} w(\xi)v^{(n+1)}(s, \xi)v^{(n+1)}(s, \xi + 1) d\xi ds$$

$$+ 2 \int_0^t \int_{\mathbb{R}} w(\xi)v^{(n+1)}(s, \xi)v^{(n+1)}(s, \xi - 1) d\xi ds$$

$$+ 2 \int_0^t \int_{\mathbb{R}} w(\xi)v^{(n+1)}(s, \xi)P(v^n)(t - \tau, \xi - ct)) d\xi ds.$$ (25)

By the Young’s inequality,

$$2ab \leq \eta a^2 + \frac{1}{\eta} b^2$$

for any $\eta > 0$,

we have

$$|2v^{(n+1)}(s, \xi)v^{(n+1)}(s, \xi + 1)| \leq \eta(v^{(n+1)})^2(s, \xi) + \frac{1}{\eta}(v^{(n+1)})^2(s, \xi + 1)$$ (26)

and

$$|2v^{(n+1)}(s, \xi)v^{(n+1)}(s, \xi - 1)| \leq \eta(v^{(n+1)})^2(s, \xi) + \frac{1}{\eta}(v^{(n+1)})^2(s, \xi - 1).$$ (27)
By choosing $\eta = e^\lambda$ in (26), we can get
\[
\left| 2 \int_0^t \int_\mathbb{R} w(\xi) v^{(n+1)}(s, \xi) v^{(n+1)}(s, \xi + 1) \, d\xi \, ds \right|
\leq e^\lambda \int_0^t \int_\mathbb{R} w(\xi) v^{(n+1)}(s, \xi) d\xi \, ds + \frac{1}{e^\lambda} \int_0^t \int_\mathbb{R} w(\xi) v^{(n+1)}(s, \xi + 1) d\xi \, ds
\]
\[
= e^\lambda \int_0^t \int_\mathbb{R} w(\xi) (v^{(n+1)})^2(s, \xi) d\xi \, ds + \frac{1}{e^\lambda} \int_0^t \int_\mathbb{R} w(\xi - 1) (v^{(n+1)})^2(s, \xi) d\xi \, ds
\]
\[
= 2 e^\lambda \int_0^t \int_\mathbb{R} w(\xi) (v^{(n+1)})^2(s, \xi) d\xi \, ds.
\]

Similarly, taking $\eta = e^{-\lambda}$ in (27), we obtain
\[
\left| 2 \int_0^t \int_\mathbb{R} w(\xi) v^{(n+1)}(s, \xi) v^{(n+1)}(s, \xi - 1) \, d\xi \, ds \right|
\leq 2 e^{-\lambda} \int_0^t \int_\mathbb{R} w(\xi) (v^{(n+1)})^2(s, \xi) d\xi \, ds.
\]

Thus, (25) becomes
\[
\| v^{(n+1)}(t) \|^2_{L^2} + \int_0^t \int_\mathbb{R} \mathcal{M}(\xi) w(\xi) (v^{(n+1)})^2(s, \xi) d\xi \, ds
\leq \| v_0 \|^2_{L^2} + 2 \int_0^t \int_\mathbb{R} w(\xi) v^{(n+1)}(s, \xi) P(v^{(n)}(s-\tau, \xi - c\tau)) \, d\xi \, ds,
\]

where
\[
\mathcal{M}(\xi) := 2 - e^{w(\xi)} \frac{w'(\xi)}{w(\xi)} - 2(e^\lambda + e^{-\lambda} - 2)
\]
\[
= 2 + 2c\lambda - 2(e^\lambda + e^{-\lambda} - 2)
\]
\[
= 2(1 + c\lambda - (e^\lambda + e^{-\lambda} - 2))
\]
\[
> 2g'(0)e^{-\lambda c\tau} > 0.
\]

We then estimate the nonlinear term on the right-hand side of (28). By (21) and the Young’s inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ for any $\varepsilon > 0$, we obtain
\[
\left| 2 \int_0^t \int_\mathbb{R} w(\xi) v^{(n+1)}(s, \xi) P(v^{(n)}(s-\tau, \xi - c\tau)) \, d\xi \, ds \right|
\leq 2 \int_0^t \int_\mathbb{R} w(\xi) (v^{(n+1)})^2(s, \xi) d\xi \, ds + \frac{1}{\varepsilon} (g'(0))^2 \int_0^t \int_\mathbb{R} w(\xi) (v^{(n)})^2(s-\tau, \xi - c\tau) d\xi \, ds
\]
\[
\leq \varepsilon \int_0^t \int_\mathbb{R} w(\xi) (v^{(n+1)})^2(s, \xi) d\xi \, ds + \frac{1}{\varepsilon} (g'(0))^2 \int_0^t \int_\mathbb{R} w(\xi + c\tau) (v^{(n)})^2(s, \xi) d\xi \, ds + \frac{1}{\varepsilon} (g'(0))^2 \int_0^t \int_\mathbb{R} w(\xi + c\tau) (v^{(n)})^2(s, \xi) d\xi \, ds
\]
\[
+ \frac{1}{\varepsilon} (g'(0))^2 \int_0^t \int_\mathbb{R} w(\xi + c\tau) (v^{(n)})^2(s, \xi) d\xi \, ds
\]
\[
(29)
\]

\[
\]
for some small constant $\varepsilon > 0$. Substituting (29) into (25), we have
\[
\|v^{(n+1)}(t)\|^2_{L^2} + \int_0^t (M(\xi) - \varepsilon)\|v^{(n+1)}(s)\|^2_{L^2} \, ds \\
\leq \|v_0(0)\|^2_{L^2} + \frac{1}{\varepsilon} (g'(0))^2 e^{-2\lambda t} \int_0^t \|v^{(n)}(s)\|^2_{L^2} \, ds \\
+ \frac{1}{\varepsilon} (g'(0))^2 e^{-2\lambda t} \int_0^0 \|v_0(s)\|^2_{L^2} \, ds.
\]
Let $\varepsilon \ll 1$. Then we immediately obtain
\[
\|v^{(n+1)}(t)\|^2_{L^2} + \int_0^t \|v^{(n+1)}(s)\|^2_{L^2} \, ds \\
\leq C \left( \|v_0(0)\|^2_{L^2} + \int_0^t \|v^{(n)}(s)\|^2_{L^2} \, ds + \int_{-\tau}^0 \|v_0(s)\|^2_{L^2} \, ds \right). \tag{30}
\]
Similarly, differentiating (29) with respect to $\xi$ and multiplying it by $w(\xi)v^{(n+1)}_\xi(t, \xi)$, then integrating the resultant equation over $\mathbb{R} \times [0, t]$ with respect to $\xi$ and $t$, we can similarly prove
\[
\|v^{(n+1)}_\xi(t)\|^2_{L^2} + \int_0^t \|v^{(n+1)}_\xi(s)\|^2_{L^2} \, ds \\
\leq C \left( \|v_0(0)\|^2_{H^1} + \int_0^t \|v^{(n)}(s)\|^2_{H^1} \, ds + \int_{-\tau}^0 \|v_0(s)\|^2_{H^1} \, ds \right). \tag{31}
\]
Combining (23), (30) and (31), we obtain
\[
M_{v^{(n+1)}(t_0)} \leq C \left( \max_{s \in [-\tau, 0]} \|v_0(s)\|^2_C + \|v_0(0)\|^2_{H^1} + \int_{-\tau}^0 \|v_0(s)\|^2_{H^1} \, ds \right) \\
+ C t_0 M_{v^{(n)}}(t_0).
\]
Choose $0 < t_0 \ll 1$ and $\max_{s \in [-\tau, 0]} \|v_0(s)\|^2_C + \|v_0(0)\|^2_{H^1} + \int_{-\tau}^0 \|v_0(s)\|^2_{H^1} \, ds \ll 1$. Then $v^{(n+1)} = F(v^{(n)})$ defined in (29) maps from $X(-\tau, t_0)$ to $X(-\tau, t_0)$ and is a contraction mapping in $X(-\tau, t_0)$. By Banach fixed point theorem, $F$ has a unique fixed point in $X(-\tau, t_0)$, which is a solution of (19). Since the convergence $\lim_{n \to \infty} v^{(n)}(t, \xi) = v(t, \xi)$ is uniform for $(t, \xi) \in [0, t_0] \times \mathbb{R}$, and $v^{(n)} \in C_{unif}[0, t_0]$, we can also guarantee $v \in C_{unif}[0, t_0]$. The proof is complete. \hfill \square

**Proposition 2** (a priori estimates). Under the assumption in Theorem 3.1, let $v \in X(-\tau, T)$ be a local solution of (19) for a given constant $T > 0$. Then there exist positive constants $\delta_2 > 0$, $C_0 > 1$, and $\mu > 0$ independent of $T$ and $v(t, \xi)$ such that, when $M_v(T) \leq 2$, 
\[
\|v(t)\|^2_C + \|v(t)\|^2_{H^1} + \int_0^t e^{-2\mu(t-s)}\|v(s)\|^2_{H^1} \, ds \\
\leq C_0 e^{-2\mu t} \left( \max_{s \in [-\tau, 0]} \|v_0(s)\|^2_C + \|v_0(0)\|^2_{H^1} + \int_{-\tau}^0 \|v_0(s)\|^2_{H^1} \, ds \right), \tag{32}
\]
for $t \in [0, T]$.

The proof for the a priori estimates of the solution in the designed solution space $X(-\tau, T)$ is technical and plays a crucial role in this paper. We leave this for the next section.
1. Now let us choose By Proposition 1, there exists We first get the energy estimates for $v$

The adopted approach is the weighted energy method but with a new development. A priori estimates.

Let $T$

Proof of Theorem 3.1.

In order to establish the energy estimate (33), sufficient regularity of the solution to (19) is required. We thus mollify the initial data as follows

Proof. In order to establish the energy estimate (33), sufficient regularity of the solution to (19) is required. We thus mollify the initial data as follows $v_{0\varepsilon}(s, \xi) = (J_{\varepsilon} * v_0)(s, \xi)$

where $J_{\varepsilon}(\xi)$ is the mollifier. Let $v_{\varepsilon}(t, \xi)$ be the solution to (19) with this mollified initial data. By Theorem 3.1, one has

$$v_{\varepsilon}(t, \xi) \in C([0, \infty); C(\mathbb{R}) \cap H^2_w(\mathbb{R}^2)) \cap L^2([0, \infty); H^2_w(\mathbb{R}^2)).$$

4. A priori estimates. In this section, we shall establish the a priori estimates. The adopted approach is the weighted energy method but with a new development. We first get the energy estimates for $v(t, \xi)$ in the weighted Sobolev space $H^1_w(\mathbb{R})$.

Lemma 4.1. Let $v(t, \xi) \in X(-\tau, T)$. Then

$$\|v(t)\|_{L^2_w}^2 + \int_0^T e^{-2\mu(t-s)} \int_{\mathbb{R}} |B_{\eta, w}(\xi) - CM_w(T)|v^2(s, \xi) d\xi ds$$

$$\leq C e^{-2\mu t} \left( \|v_0(0)\|_{L^2_w}^2 + \int_{-\tau}^0 \|v_0(s)\|_{L^2_w}^2 ds \right),$$

(33)

where

$$B_{\eta, w}(\xi) := A_{\eta, w}(\xi) - 2\mu - \frac{1}{\eta} \left( e^{2\mu \tau} - 1 \right) |g'(\phi(\xi))| \frac{w(\xi + c\tau)}{w(\xi)}$$

(34)

and

$$A_{\eta, w}(\xi) := -\epsilon \frac{w'(\xi)}{w(\xi)} + 2 - 2(e^\lambda + e^{-\lambda} - 2)$$

$$- \eta |g'(\phi(\xi - c\tau))| - \frac{1}{\eta} |g'(\phi(\xi))| \frac{w(\xi + c\tau)}{w(\xi)},$$

(35)

and $\mu$ and $\eta$ both are arbitrarily give positive constants at this moment, but will be specified later.

Proof. In order to establish the energy estimate (33), sufficient regularity of the solution to (19) is required. We thus mollify the initial data as follows $v_{0\varepsilon}(s, \xi) = (J_{\varepsilon} * v_0)(s, \xi)$
Then we take the limit $\varepsilon \to 0$ to obtain the corresponding energy estimate for the original solution $v(t, \xi)$. For the sake of simplicity, below we give the formal calculation using $v(t, \xi)$ directly to establish the desired energy estimates.

Multiplying (19) by $e^{2\mu t}w(t, \xi)$, where $\mu > 0$ is a constant and will be specified later in Lemma 4.3 we have

\[
\left\{ \frac{1}{2} e^{2\mu t} w^2 \right\}_t + e^{2\mu t} \left\{ \frac{1}{2} c w^2 \right\}_t + \left[ 3 - \mu - \frac{c w'}{w} \right] e^{2\mu t} w^2 (t, \xi)
\]

\[
- \frac{2}{\mu} \int_0^t \int_R e^{2\mu s} w(t, \xi) v(t, \xi + 1) d\xi ds
\]

\[
- \frac{2}{\mu} \int_0^t \int_R e^{2\mu s} w(t, \xi) v_t(t, \xi - 1) d\xi ds
\]

\[
- \frac{2}{\mu} \int_0^t \int_R e^{2\mu s} g' (\phi (\xi - ct)) v(t, \xi) v(s - \tau, \xi - cr) d\xi ds
\]

\[
= e^{2\mu t} w(t, \xi) v(t, \xi) Q(v(t - \tau, \xi - cr)).
\]

By (30), we get

\[
\left\{ \frac{1}{2} c w^2 \right\}_{\xi = -\infty}^{\xi = \infty} = 0.
\]

Integrating (37) over $\mathbb{R} \times [0, t]$ with respect to $\xi$ and $t$ yields

\[
e^{2\mu t} \|v(t)\|^2_{L^2_w} + \int_0^t \int_R e^{2\mu s} \left[ 6 - 2\mu - \frac{c w'}{w} \right] w(t, \xi) v^2 (s, \xi) d\xi ds
\]

\[
- \frac{2}{\mu} \int_0^t \int_R e^{2\mu s} w(t, \xi) v(t, \xi + 1) d\xi ds
\]

\[
- \frac{2}{\mu} \int_0^t \int_R e^{2\mu s} w(t, \xi) v_t(t, \xi - 1) d\xi ds
\]

\[
- \frac{2}{\mu} \int_0^t \int_R e^{2\mu s} g' (\phi (\xi - ct)) v(t, \xi) v(s - \tau, \xi - cr) d\xi ds
\]

\[
= \|v_0(0)\|^2_{L^2_w} + 2 \int_0^t \int_R e^{2\mu s} w(t, \xi) v(s, \xi) Q(v(s - \tau, \xi - cr)) d\xi ds.
\]

By a similar argument as in Proposition 4, we further have

\[
e^{2\mu t} \|v(t)\|^2_{L^2_w} + \int_0^t \int_R e^{2\mu s} \left[ 2 - 2\mu - \frac{c w'}{w} \right] w(t, \xi) v^2 (s, \xi) d\xi ds
\]

\[
- \frac{2}{\mu} \int_0^t \int_R e^{2\mu s} g' (\phi (\xi - ct)) w(t, \xi) v(s, \xi) v(s - \tau, \xi - cr) d\xi ds
\]

\[
\leq \|v_0(0)\|^2_{L^2_w} + 2 \int_0^t \int_R e^{2\mu s} w(t, \xi) v(s, \xi) Q(v(s - \tau, \xi - cr)) d\xi ds.
\]

By the Cauchy-Schwarz inequality, one has

\[
|e^{2\mu s} g' (\phi (\xi - ct)) w(t, \xi) v(s, \xi) v(s - \tau, \xi - cr)|
\]

\[
\leq e^{2\mu s} |g' (\phi (\xi - ct))| w(t, \xi) \left[ v^2 (s, \xi) + \frac{1}{\eta} v^2 (s - \tau, \xi - cr) \right],
\]
for any \( \eta > 0 \). Furthermore, by change of variables, we obtain

\[
\left| 2 \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} g'(\phi(\xi - ct)) w(\xi) v(s, \xi) v(s - \tau, \xi - ct) \, d\xi \, ds \right|
\]

\[
\leq \eta \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi - ct))| w(\xi) v^2(s, \xi) \, d\xi \, ds
\]

\[
+ \frac{1}{\eta} \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi - ct))| w(\xi) v^2(s, \xi) \, d\xi \, ds
\]

\[
= \eta \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi - ct))| w(\xi) v^2(s, \xi) \, d\xi \, ds
\]

\[
+ \frac{1}{\eta} e^{2\mu \tau} \int_{-\tau}^{0} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi))| w(\xi + \tau) v^2(s, \xi) \, d\xi \, ds
\]

[by setting \( \xi - ct \to \xi, s - \tau \to s \)]

\[
\leq \eta \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi - ct))| w(\xi) v^2(s, \xi) \, d\xi \, ds
\]

\[
+ \frac{1}{\eta} e^{2\mu \tau} \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi))| w(\xi + \tau) v^2(s, \xi) \, d\xi \, ds
\]

\[
+ \frac{1}{\eta} e^{2\mu \tau} \int_{-\tau}^{0} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi))| w(\xi + \tau) v^2(s, \xi) \, d\xi \, ds.
\]

Substituting (39) into (38) yields

\[
e^{2\mu t} \| v(t) \|_{L_2^w}^2
\]

\[
+ \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} \left[ 2 - 2\mu - e^{w'(\xi)} - 2(e^\lambda + e^{-\lambda} - 2) \right] w(\xi) v^2(s, \xi) \, d\xi \, ds
\]

\[
- \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} \left[ \eta |g'(\phi(\xi - ct))| + \frac{1}{\eta} e^{2\mu \tau} |g'(\phi(\xi))| \frac{w(\xi + \tau)}{w(\xi)} \right] w(\xi) v^2(s, \xi) \, d\xi \, ds
\]

\[
\leq \| v_0(0) \|_{L_2^w}^2 + 2 \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} w(\xi) v(s, \xi) Q(v(s - \tau, \xi - ct)) \, d\xi \, ds
\]

\[
+ \frac{1}{\eta} e^{2\mu \tau} \int_{-\tau}^{0} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi))| w(\xi + \tau) v^2(s, \xi) \, d\xi \, ds.
\]

Namely,

\[
e^{2\mu t} \| v(t) \|_{L_2^w}^2 + \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} B_{\eta, \mu, w}(\xi) w(\xi) v^2(s, \xi) \, d\xi \, ds
\]

\[
\leq \| v_0(0) \|_{L_2^w}^2 + 2 \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} w(\xi) v(s, \xi) Q(v(s - \tau, \xi - ct)) \, d\xi \, ds
\]

\[
+ \frac{1}{\eta} e^{2\mu \tau} \int_{-\tau}^{0} \int_{\mathbb{R}} e^{2\mu s} |g'(\phi(\xi))| w(\xi + \tau) v^2(s, \xi) \, d\xi \, ds,
\]

where \( B_{\eta, \mu, w}(\xi) \) is given in (34).

On the other hand, by the definition of \( M_\nu(T) \), we have

\[ |v(t, \xi)| \leq CM_\nu(T) \quad \text{for} \ t \in [0, T], \ \xi \in \mathbb{R}. \]

By the Taylor’s formula, we get

\[ |Q(v)| = |g(v + \phi) - g(\phi) - g'(\phi)v| \leq C|v|^2, \]
Hence, Lemma 4.2. Let

where $C > 0$ is independent of $\nu$. Then we can estimate the nonlinear term as

\[
\left| 2 \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) \nu(s, \xi) Q(v(s - \tau, \xi - c\tau)) \, d\xi \, ds \right|
\leq C M_v(T) \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) |v(s - \tau, \xi - c\tau)|^2 \, d\xi \, ds
\]
\[= CM_v(T) \int_{-\tau}^t \int_{\mathbb{R}} e^{2\mu(s - \tau)} w(\xi + c\tau) |v(s, \xi)|^2 \, d\xi \, ds
\]
\[\leq CM_v(T) \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) |v(s, \xi)|^2 \, d\xi \, ds
\]
\[+ CM_v(T) \int_{-\tau}^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) |v_0(s, \xi)|^2 \, d\xi \, ds. \tag{41}
\]

Here, $w(\xi + c\tau) = e^{-2\lambda c\tau} w(\xi)$. Substituting (41) into (40), we get

\[
e^{2\mu t} \|v(t)\|^2_{L^2_\nu} + \int_0^t e^{2\mu s} \int_{\mathbb{R}} [B_{\eta, \mu, w}(\xi) - CM_v(T)] w(\xi) v^2(s, \xi) \, d\xi \, ds
\]
\[\leq C \left( \|v_0(0)\|^2_{L^2_\nu} + \int_{-\tau}^t \|v_0(s)\|^2_{L^2_{w, \nu}} \, ds \right).
\]

The proof is complete. \qed

**Lemma 4.2.** Let $\eta = e^{-\lambda c\tau}$. Then there exists a constant $C_1 > 0$ such that

\[A_{\eta, w}(\xi) \geq C_1 > 0 \quad \text{for} \ \xi \in \mathbb{R}, \tag{42}\]

where $A_{\eta, w}(\xi)$ can be seen in [35].

**Proof.** By (H3) and (5), we have

\[|g'(\phi)| \leq g'(0) \quad \text{for all} \ \phi \geq 0\]

and

\[\frac{w'(\xi)}{w(\xi)} = -2\lambda, \quad \frac{w(\xi + c\tau)}{w(\xi)} = e^{-2\lambda c\tau}.
\]

In addition,

\[c\lambda - (e^\lambda + e^{-\lambda} - 2) + 1 - g'(0)e^{-\lambda c\tau} > 0 \quad \text{for} \ c > c_* \text{ and} \ \lambda \in (\lambda_1(c), \lambda_2(c)).
\]

Hence,

\[A_{\eta, w}(\xi) = -\frac{w'(\xi)}{w(\xi)} + 2 - 2(e^\lambda + e^{-\lambda} - 2)
\]
\[\quad - \eta |g'(\phi(\xi - c\tau))| - \frac{1}{\eta} |g'(\phi(\xi))| \frac{w(\xi + c\tau)}{w(\xi)}
\]
\[= 2c\lambda + 2 - 2(e^\lambda + e^{-\lambda} - 2) - e^{-\lambda c\tau} |g'(\phi(\xi - c\tau))| - e^{-\lambda c\tau} |g'(\phi(\xi))| \]
\[\geq 2c\lambda + 1 - (e^\lambda + e^{-\lambda} - 2) - g'(0)e^{-\lambda c\tau}
\]
\[= : C_1 > 0,
\]

for $\xi \in \mathbb{R}$ with $c > c_*$ and $\lambda \in (\lambda_1(c), \lambda_2(c))$. The proof is complete. \qed
Lemma 4.3. Let \( v(t, \xi) \in X(-\tau, T) \). Then there exists a constant \( \mu_1 > 0 \) such that, for \( 0 < \mu < \mu_1 \), it holds that

\[
\| v(t) \|_{L^2_w}^2 + \int_0^t e^{-2\mu(t-s)} \| v(s) \|_{L^2_w}^2 ds \\
\leq C e^{-2\mu t} \left( \| v_0(0) \|_{L^2_w}^2 + \int_{-\tau}^0 \| v_0(s) \|_{L^2_w}^2 ds \right),
\]

provided \( M_v(T) \ll 1 \).

Proof. By (42), one can estimate \( B_{\eta,\mu,w}(\xi) \) as defined in (34) as

\[
B_{\eta,\mu,w}(\xi) := A_{\eta,w}(\xi) - 2\mu - 1 \eta (e^{2\mu \tau} - 1) |g'(\phi(\xi))| \frac{w(\xi + c\tau + \xi)}{w(\xi)} \geq C_1 - 2\mu - g'(0)e^{-\lambda c \tau} (e^{2\mu \tau} - 1) =: C_2 > 0
\]

by selecting \( \mu \) to be small enough such that

\[
0 < \mu < \mu_1,
\]

where \( \mu_1 > 0 \) is the unique positive solution of the equation

\[
C_1 - 2\mu - g'(0)e^{-\lambda c \tau} (e^{2\mu \tau} - 1) = 0.
\]

Substituting (44) into (33) yields

\[
\| v(t) \|_{L^2_w}^2 + \int_0^t e^{-2\mu(t-s)} \int_R \left[ C_2 - CM_v(T) \right] w(\xi + \xi) v^2(s, \xi) d\xi ds \\
\leq C e^{-2\mu t} \left( \| v_0(0) \|_{L^2_w}^2 + \int_{-\tau}^0 \| v_0(s) \|_{L^2_w}^2 ds \right),
\]

which implies (43) by letting \( M_v(T) \ll 1 \). The proof is complete.

Next, we establish the estimate for the one order derivative \( v_\xi(t, \xi) \) of the solution \( v(t, \xi) \).

Lemma 4.4. Let \( v(t, \xi) \in X(-\tau, T) \). Then it holds that

\[
\| v_\xi(t) \|_{L^2_w}^2 + \int_0^t e^{-2\mu(t-s)} \| v_\xi(s) \|_{L^2_w}^2 ds \\
\leq C e^{-2\mu t} \left( \| v_0(0) \|_{H^1_w}^2 + \int_{-\tau}^0 \| v_0(s) \|_{H^1_w}^2 ds \right),
\]

provided \( M_v(T) \ll 1 \).

Proof. Differentiating (19) with respect to \( \xi \) and multiplying it by \( e^{2\mu_t}w(\xi)v_\xi(t, \xi) \), then integrating the resultant equation with respect to \( \xi \) and \( t \) over \( \mathbb{R} \times [0, t] \) and applying Lemma 4.3 we can similarly prove (45). Thus, we omit the details.

Remark 1. As discussed in the proof of Lemma 4.1, in order to establish the energy estimate (45), we need a good enough regularity for the solution \( v(t, \xi) \). Actually, the same mollification procedure is also needed in the proof of Lemma 4.4. Here, we omit the details.

Finally, combining Lemmas 4.3 and 4.4 we obtain the following a priori estimates.
Lemma 4.5. Let \( v(t, \xi) \in X(-\tau, T) \). Then
\[
\|v(t)\|^2_{H^1_w} + \int_0^t e^{-2\mu(t-s)}\|v(s)\|^2_{H^1_w} \, ds
\leq C e^{-2\mu t} \left( \|v_0(0)\|^2_{H^1_w} + \int_{-\tau}^0 \|v_0(s)\|^2_{H^1_w} \, ds \right),
\]
provided \( M_v(T) \ll 1 \).

By the Sobolev inequality, we obtain the following decay for \( v(t, \xi) \).

Lemma 4.6. Let \( v(t, \xi) \in H^1_w(\mathbb{R}) \). Then it is equivalent to \( \sqrt{wv} \in H^1(\mathbb{R}) \) and
\[
\|\sqrt{wv}\| L^\infty \leq C \|v\|_{H^1_w}
\]
and
\[
\sup_{\xi \in (-\infty, x_0]} |v(t, \xi)| \leq C \delta_0 e^{-\mu t}, \quad t > 0.
\]

Proof. Since \( v \in H^1_w(\mathbb{R}) \), i.e., \( \sqrt{wv} \in L^2(\mathbb{R}) \), \( \sqrt{wv_{\xi}} \in L^2(\mathbb{R}) \), it immediately yields
\[
\partial_t (\sqrt{wv}) = \sqrt{wv_{\xi}} + \lambda \sqrt{wv} \in L^2(\mathbb{R}).
\]
So we prove that \( \sqrt{wv} \in H^1(\mathbb{R}) \). Thus, by using the Sobolev inequality \( H^1 \hookrightarrow C^0 \), we get \( \|\sqrt{wv}\|_{C^0} \leq C \|v\|_{H^1_w} \), which implies, from lemma 4.5 and the fact \( w(\xi) = e^{-2\lambda(\xi-x_0)} \geq 1 \) for \( \xi \in (-\infty, x_0] \), that
\[
|v| \leq \|\sqrt{wv}\|_{C^0} \leq C \|v\|_{H^1_w} \leq C \delta_0 e^{-\mu t},
\]
namely,
\[
\sup_{\xi \in (-\infty, x_0]} |v(t, \xi)| \leq C \delta_0 e^{-\mu t}, \quad t > 0.
\]
This completes the proof. \( \square \)

Next, we prove the time-exponential decay of \( v(t, \xi) \) at \( \xi = +\infty \).

Lemma 4.7. There exists large number \( x_0 \gg 1 \) (independent of \( t \)) such that
\[
\sup_{\xi \in [0, x_\infty]} |v(t, \xi)| \leq C M_v(0) e^{-\mu_2 t}, \quad t > 0.
\]

Proof. Since \( v(t, \xi) \in X(\tau, T) \), by the definition of \( C_{unif}[0, T] \), we have that
\[
\lim_{\xi \to +\infty} v(t, \xi) \exists \text{ uniformly with respect to } t \in [0, T].
\]
Let us go back to the original equations (1), (2) and (8), and denote
\[
V(t, x) = u(t, x) - \phi(x + ct).
\]
Then \( V(t, x) = v(t, \xi) \) and satisfies
\[
\begin{aligned}
\frac{\partial V}{\partial t} - \Delta_1 V + V - g'(\phi(x + c(t-\tau))V(t-\tau, x + c(t-\tau))
&= Q(V(t-\tau, x + c(t-\tau))), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},
V(s, x) = V_0(s, x), \quad s \in [-\tau, 0], \quad x \in \mathbb{R}.
\end{aligned}
\]
Denote \( z(t) := v(t, \infty) = V(t, \infty) \) and \( z_0(s) := V_0(s, \infty) \) for \( s \in [-\tau, 0] \). Since \( v(t, \xi) \in C_{unif}[0, T] \), namely, \( \lim_{\xi \to +\infty} v(t, \xi) = \lim_{x \to +\infty} V(t, x) = z(t) \) uniformly with respect to \( t \in [0, T] \). Thus, by taking \( x \to +\infty \) to equation (49), we have
\[
\begin{aligned}
z'(t) + z(t) - g'(K)z(t - \tau) &= Q(z(t - \tau)),
z(s) = z_0(s), \quad s \in [-\tau, 0].
\end{aligned}
\]
Applying the nonlinear Halanay’s inequality given in [14], we have that:
\[
|z(t)| \leq C \|z_0\|_{L^\infty(-\tau, 0)} e^{-\mu_2 t}, \quad t > 0,
\]
for some \( \mu_2 > 0 \), provided \( M_{\varepsilon}(T) \ll 1 \). That is
\[
|V(t, \infty)| \leq C\|z_0\|_{L^\infty}(0)e^{-\mu_2 t} \leq CM_V(0)e^{-\mu_2 t}, \quad t > 0.
\]
(50)
By the proof of Proposition 3, we can see that
\[
\lim_{x \to \infty} |e^{\mu_2 t}V(t, x) - e^{\mu_2 t}V(t, \infty)| = 0 \quad \text{uniformly in } t \in [0, T].
\]
Therefore, for any given positive number \( \varepsilon > 0 \), there exists a number \( x_0 = x_0(\varepsilon) \gg 1 \) but independent of \( t \), such that when \( x \geq x_0 \),
\[
|e^{\mu_2 t}V(t, x) - e^{\mu_2 t}V(t, \infty)| < \varepsilon,
\]
which implies
\[
|e^{\mu_2 t}|V(t, x)| - e^{\mu_2 t}|V(t, \infty)|| \leq |e^{\mu_2 t}V(t, x) - e^{\mu_2 t}V(t, \infty)| < \varepsilon.
\]
Notice that, from (50), \( e^{\mu_2 t}|V(t, \infty)| \leq CM_V(0) \) is uniformly bounded with respect to \( t \), we then immediately obtain
\[
e^{\mu_2 t}|V(t, x)| \leq CM_V(0) + \varepsilon \quad \text{for } x \geq x_0, \ t \in [0, T].
\]
Let us take \( \varepsilon = M_V(0) \), we immediately get
\[
\sup_{x \in [x_0, \infty]} |V(t, x)| \leq CM_v(0)e^{-\mu_2 t}, \quad t > 0.
\]
(51)
Again, notice that \( v(t, \xi) = V(t, x) \) and \( \xi = x + ct \geq x_0 \) for \( x \geq x_0 \) and \( t > 0 \). Then (51) immediately implies
\[
\sup_{\xi \in [x_0, \infty]} |v(t, \xi)| \leq CM_v(0)e^{-\mu_2 t}, \quad t > 0.
\]
The proof is complete.

Proof of Proposition 2
Combining (46), (47) and (48), we can immediately prove (32), namely
\[
\|v(t)\|_{H_{w}^2}^2 + \|v(t)\|_{H_{w}^1}^2 + \int_0^t e^{-2\mu(t-s)}\|v(s)\|_{H_{w}^1}^2 \, ds
\]
\[
\leq C_0 e^{-2\mu t} \left( \max_{s \in [-\tau, 0]} \|v_0(s)\|_{H_{w}^2}^2 + \|v_0(0)\|_{H_{w}^1}^2 + \int_{-\tau}^0 \|v_0(s)\|_{H_{w}^1}^2 \, ds \right),
\]
for some positive constant \( C_0 \), where \( \mu \) is taken as \( 0 < \mu \leq \min\{\mu_1, \mu_2\} \). The proof of Proposition 2 is complete.

5. Uniqueness of traveling waves.

Proof of Corollary 2
Let \( \phi_1(x + ct) \) and \( \phi_2(x + ct) \) be two different traveling waves with the same speed \( c > c_* \) and the same exponential decay at \( -\infty \):
\[
\phi_1(\xi) = Ae^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty
\]
and
\[
\phi_2(\xi) = Be^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty
\]
for some positive constant \( A \) and \( B \), where \( \lambda_1 = \lambda_1(c) > 0 \) is defined in Lemma 2.1.
Let us shift \( \phi_2(x + ct) \) to \( \phi_2(x + ct + \xi_0) \) with some constant shift \( \xi_0 \). By taking \( \xi \to -\infty \), obviously \( \xi + \xi_0 < 0 \), we obtain
\[
\phi_2(\xi + \xi_0) = Be^{-\lambda_1|\xi + \xi_0|} = Be^{\lambda_1(\xi + \xi_0)} = Be^{\lambda_1\xi_0}e^{-\lambda_1|\xi|} = Ae^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty
\]
by selecting \( \xi_0 \) as
\[
\xi_0 = \frac{1}{\lambda_1} \ln \frac{A}{B}
\]
Thus, we have
\[
|\phi_2(\xi + \xi_0) - \phi_1(\xi)| = O(1)e^{-\alpha|\xi|} \quad \text{for} \quad \alpha > \lambda_1 \quad \text{as} \quad \xi \to -\infty.
\]
This implies
\[
\phi_2(\xi + \xi_0) - \phi_1(\xi) \in C(\mathbb{R}) \cap H^1_w(\mathbb{R}).
\]
Now we take the initial data for (1) by
\[
v_0(s, x) = \phi_2(x + cs + \xi_0), \quad x \in \mathbb{R}, s \in [-\tau, 0].
\]
Obviously, with such selected initial data, the corresponding solution to (1) is
\[
v(t, x) = \phi_2(x + ct + \xi_0).
\]
Applying the stability theorem (Theorem [1.1]), we have
\[
\lim_{t \to \infty, x \in \mathbb{R}} |\phi_2(x + ct + \xi_0) - \phi_1(x + ct)| = 0,
\]
namely, \( \phi_2(x + ct + \xi_0) = \phi_1(x + ct) \) for all \( x \in \mathbb{R} \) as \( t \gg 1 \). This proves the uniqueness of the traveling waves up to a translation. \( \square \)

6. Applications. Now we are going to state the stability result for Nicholson’s birth rate function \( g \), i.e.,
\[
g(u) = pue^{-au}, \quad a > 0, p > 0,
\]
where \( p > 0 \) is the impact of the birth on the immature population. Equation (11) with the particular birth rate function \( g \) in (52) is called Nicholson’s blowflies equation.
It is easy to see that the so-called Nicholson’s blowflies equation possesses two constant equilibria 0 and \( K = \frac{1}{a} \ln p \). Clearly, \( K > 0 \) for \( p > 1 \). Notice that \( g'(0) - 1 = p - 1 > 0 \) and \( g'(K) - 1 = -\ln p < 0 \). Thus, Nicholson’s blowflies equation (11) possesses one unstable node 0 and one stable node \( K \). When \( p > e \), the birth rate function \( g(u) \) is unimodal on \( u \in [0, K] \), and reaches its unique global maximum at \( u_* = \frac{1}{a} \in (0, K) \). Furthermore, it can be verified that \( |g'(u)| \leq g'(0) \) for \( u \in [0, \infty) \). Therefore, \( g(u) = pue^{-au} \) satisfies the conditions (H1) – (H3).
As a direct application of Theorem 2.3, Theorem 1.1 and Corollary 1 we immediately obtain the following existence, stability and uniqueness of traveling waves for (11) with Nicholson’s birth rate function \( g(u) = pue^{-au} \).

**Theorem 6.1.** Let \( g(u) = pue^{-au} \) with \( e < p \leq e^2 \). Then there exists \( c_* > 0 \) such that for every \( c > c_* \), (11) admits a traveling wave \( u(t, x) = \phi(x + ct) \) satisfying \( \phi(-\infty) = 0 \) and \( 0 < K_* \leq \liminf_{\xi \to +\infty} \phi(\xi) \leq \limsup_{\xi \to +\infty} \phi(\xi) \leq K^* \), where
\[
K^* = \frac{p}{ae}, \quad K_* = \frac{e^2}{ae} e^{-\frac{p}{a}}.
\]

**Theorem 6.2.** Let \( g(u) = pue^{-au} \) with \( e < p \leq e^2 \). For any given traveling wave \( \phi(x + ct) \) with \( c > c_* \), connecting with 0 and \( K = \frac{1}{a} \ln p \), whatever it is monotone or non-monotone, suppose that
\[
u_0(s, x) - \phi(x + cs) \in C([-\tau, 0]; C(\mathbb{R}) \cap H^1_w(\mathbb{R}))) \cap L^2([-\tau, 0]; H^1_w(\mathbb{R})))
\]
and \(\lim_{x \to +\infty} [u_0(s, x) - \phi(x + cs)] =: u_{0, \infty}(s) \in C[-\tau, 0]\) exists uniformly with respect to \(s \in [-\tau, 0]\), and the initial perturbation is small:

\[
\max_{s \in [-\tau, 0]} \|u_0 - \phi(s)\|_C^2 + \|u_0(0)\|_{H^2}^2 + \int_{-\tau}^0 \|u_0(s)\|_{H^1}^2 \, ds \leq \delta_0^2
\]

for some positive number \(\delta_0 > 0\). Then the solution \(u(t, x)\) of (1) and (2) uniquely exists and satisfies (6) and (7).

**Corollary 2.** Let \(g(u) = pue^{-au}\) with \(e < p \leq e^2\). Then, for all traveling waves \(\phi(x + ct)\) of (1) with the same speed \(c > c^*_e\) and the same exponential decay at \(-\infty\), whether they are monotone or non-monotone, these waves are unique up to translation.

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