An inflation model with large variations in spectral index

Bo Feng\textsuperscript{a}, Mingzhe Li\textsuperscript{a}, Ren-Jie Zhang\textsuperscript{b,c}, and Xinmin Zhang\textsuperscript{a}
\textsuperscript{a}Institute of High Energy Physics, Chinese Academy of Science, P.O. Box 918-4, Beijing 100039, P. R. China
\textsuperscript{b}School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA and
\textsuperscript{c}Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, P. R. China.

Recent fits of cosmological parameters by the Wilkinson Microwave Anisotropy Probe (WMAP) measurement favor a primordial scalar spectrum with varying index. This result, if stands, could severely constrain inflation model buildings. Most extant slow-roll inflation models allow for only a tiny amount of scale variations. In this paper we propose a single-field inflation model which is natural theoretically and can generate the required variations of the spectral index as implied by the WMAP for suitable choices of parameters.

PACS number(s): 98.80.Cq, 11.10.Kk

The recently released Wilkinson Microwave Anisotropy Probe (WMAP) data\textsuperscript{1} have been used to fit the cosmological parameters and confront the predictions of inflationary scenarios respectively in Refs.\textsuperscript{2} and\textsuperscript{3}. It is found that the data (with no other significant priors) can be best fitted by a standard ΛCDM model seeded by an almost scale-invariant, adiabatic, Gaussian primordial fluctuation; predictions inferred from the model also agree with other cosmological measurements with high accuracies\textsuperscript{4}. It is noted that there might be possible discrepancies between predictions and observations on the largest and smallest scales. The problem at the smallest scales is improved when combined with data from finer scale CMB experiments (ACBAR and CBI) and structure formation measurements (2dFGRS and Lyman α forest), for which a varying scalar primordial spectrum is favored by the best fit. At the pivot scale $k_0 = 0.05$ Mpc\textsuperscript{−1}, the best-fit values for the scalar power spectrum are $n_s = 0.93 \pm 0.03$ and $d n_s / d \ln k = -0.031^{+0.016}_{-0.018}$\textsuperscript{2}, this means at a 2σ level the spectrum runs from blue to red as the co-moving wave number $k$ increases. As noted, the suppression of the power at the smallest scale might offer an interesting solution to the problem of the standard ΛCDM model at small scales\textsuperscript{5}.

The intriguing result of a varying spectral index needs to be closely evaluated with improved statistics and with future data. If stands, it could severely constrain inflation model buildings. Most extant inflationary models allow for only a tiny amount of scale variations. In this paper we propose a single-field inflation model which is natural theoretically and can generate the required variations of the spectral index as implied by the WMAP.

For the single-field slow-roll inflationary models, one usually defines the following slow-roll parameters\textsuperscript{3},
\begin{align}
\epsilon & \equiv \frac{M_P^2}{2} \left( \frac{V''}{V} \right)^2, \quad \eta \equiv M_P^2 \frac{V'''}{V}, \quad \xi \equiv M_P^2 \frac{V'''}{V^2},
\end{align}
where primes represent derivatives with respect to the inflaton field, $\phi$, and $M_P = 2.4 \times 10^{18}$ GeV is the reduced Planck mass. Slow-roll approximation requires $\epsilon, |\eta|, |\xi| \ll 1$ in the inflationary epoch.

The primordial curvature (scalar) and tensor perturbation power spectra are given by\textsuperscript{8}
\begin{align}
P_R & \approx \frac{V}{24\pi^2 M_P^2 \epsilon} \bigg|_{k=aH}, \quad P_h \approx \frac{2V}{3\pi^2 M_P^4 \xi} \bigg|_{k=aH},
\end{align}
evaluated when a particular mode crosses out the horizon. The tensor to scalar ratio $r \equiv P_h / P_R \approx 16\epsilon$ is generally small in the slow-roll inflation. The spectral indices and their running are the slopes and curvatures of the power spectra. In terms of the slow-roll parameters, for scalar perturbations, they are
\begin{align}
n_s - 1 & \approx -6\epsilon + 2\eta, \quad \frac{d n_s}{d \ln k} \approx 16\epsilon - 24\epsilon^2 - 2\xi.
\end{align}
In most inflationary models, $\epsilon, |\eta| \sim M_P^2 / (\Delta \phi)^2$, $|\xi| \sim M_P^4 / (\Delta \phi)^4$, and the number of $e$-folding $N \sim (\Delta \phi)^2 / M_P^2$, where $\Delta \phi$ is the displacement of the homogeneous field $\phi$, so there is a hierarchy in the slow-roll parameters, $\epsilon, |\eta| \sim N^{-1}$, $|\xi| \sim N^{-2}$, and the variations of the spectral indices, $dn_s / d \ln k$, are negligible.

To achieve a running spectrum in the order favored by the WMAP, one has to consider inflation models with more exotic potentials\textsuperscript{2}, such as the running mass model\textsuperscript{10} and models with oscillating primordial spectrum\textsuperscript{11}. These models can generate significant spectral runnings, which could be as large as $n_s - 1$.

The Achilles heel of these inflationary models (and of the inflation paradigm in general) lies, arguably, in the difficulty of obtaining a sufficiently flat and stable (against radiative corrections) inflaton potential from the perspectives of particle physics. Symmetry principles must be invoked. There are only two known symmetries which can protect the flatness of a scalar potential: supersymmetry and the shift symmetry for a Pseudo Nambu-Goldstone Boson (PNGB). However, as shown in Ref.\textsuperscript{12} and recently re-emphasized in Refs.\textsuperscript{13} and\textsuperscript{14}, supersymmetry alone cannot protect the flatness of the inflaton potential, since it is explicitly broken during inflation, and the gravitational effects generically give a Hubble-scale mass correction to the inflaton.

The shift symmetry was first realized in the natural inflation model\textsuperscript{15}. Still there are some difficulties in this
model. The flatness condition, in the simplest scenario with a single PNGB, requires the scale of spontaneous
symmetry breaking and the values of the inflaton during
the slow roll above the Planck scale, taking the model
outside the regime of validity of an effective field theory
description. Moreover, it is expected that the gravity-
induced higher-dimensional operators are not suppressed.

These issues have been recently re-examined in the
context of extra dimensions (called extra-natural inflation
in Ref. [13]). Consider a five-dimensional Abelian
gauge field model with the fifth dimension compactified
on a circle of radius $R$, we identify the inflaton field $\theta$ with
the gauge-invariant Wilson loop of the extra component
$A_5$ propagating in the bulk,

$$\theta = g_5 \oint dx^5 A_5 ,$$

(4)

where $g_5$ is the five-dimensional gauge coupling constant.
At energies below $1/R$, $\theta$ is a four-dimensional field with
an effective Lagrangian

$$L = \frac{1}{2 g_4^2 (2\pi R)^2} (\partial \theta)^2 - V(\theta) ,$$

(5)

with $g_4^2 = g_5^2 / 2\pi R$ the four-dimensional effective
gauge coupling constant. The non-local potential $V(\theta)$ is
generated in the presence of particles charged under the
Abelian symmetry [16].

For bulk fields with bare masses $M_a$ and charges $q_a$ the
torque potential takes the form [17]

$$V(\theta) = \frac{1}{128\pi^6 R^4} \text{Tr} \left[ V(r_a^F, \theta) - V(r_a^B, \theta) \right] ,$$

(6)

where the trace is over the number of degrees of freedom,
and the superscripts $F$ and $B$ stand for fermions and bosons respectively. Here

$$V(r_a, \theta) = \frac{x_a^2}{2} \text{Li}_2(r_a e^{-x_a}) + 3 x_a \text{Li}_3(r_a e^{-x_a})$$
$$+ 3 \text{Li}_5(r_a e^{-x_a}) + \text{h.c.} ,$$

(7)

with

$$r_a = e^{i q_a \theta} , \quad x_a = 2\pi R M_a ,$$

(8)

and the poly-logarithm function $\text{Li}_k(z)$ are

$$\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} .$$

(9)

For massless particles ($x_a = 0$) considered in Ref. [13] the
potential is

$$V(\theta) = -\frac{3}{64\pi^6 R^4} \sum_I (-F_I) \sum_{n=1}^{\infty} \frac{\cos(nq \theta)}{n^5} ,$$

(10)

where $F_I = 0$ and 1 stand for massless bosonic and
fermionic fields, respectively.

Neglecting the higher power terms in Eq. (10), one
obtains the same form of the potential as that of the
natural inflation model. The effective decay constant of
the spontaneously broken Abelian symmetry is

$$f_{\text{eff}} = \frac{1}{2 \pi g_4 R} ,$$

(11)

which can be naturally greater than $M_P$ for a sufficiently
small coupling constant $g_4$ [13]. Moreover, due to the extra dimension nature, gravity-induced higher-
dimensional operators are generally exponentially suppressed.
This solves the aforementioned problems of the four-dimensional natural inflation model[1]. The extranatural model predicts a red-tilted scalar spectrum with
negligible spectral runnings, same as that of the natural
inflation.

The model we propose in this paper includes one massless and one massive fields coupled to $A_5$, i.e.,
$M_1 = 0$, $M_2 \geq 1/R$, the corresponding potential for $\theta$ is

$$V(\theta) = -\frac{3}{64\pi^6 R^4} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ (-F_1) \cos(nq_1 \theta)$$
$$+ (-F_2) e^{-nq_2 x_2} \left( \frac{x_2^2}{3} + \frac{x_2}{n} + \frac{1}{n^2} \right) \cos(nq_2 \theta) \right] ,$$

(12)

Neglecting the higher power terms in the sum, and defining
a canonical field $\phi = f_{\text{eff}} \theta$, the effective Lagrangian of our model becomes

$$L = \frac{1}{2} (\partial \phi)^2 - V_0 \left[ 1 - \cos \left( \frac{q_1 \phi}{f_{\text{eff}}} \right) - \sigma \cos \left( \frac{q_2 \phi}{f_{\text{eff}}} \right) \right] ,$$

(13)

where

$$\sigma = (-F_2 + 1) e^{-nq_2 x_2} \left( \frac{x_2^2}{3} + x_2 + 1 \right) , \quad V_0 = \frac{3}{64\pi^6 R^4} .$$

(14)

Note that in our calculations we have added a $\sigma$-
dependent term to the potential, Eq. (13), to make it
vanish at the minimum. For $\sigma = 0$, this potential coincides
with that of the natural inflation model.

The slow-roll parameters of Eq. (1) are

$$\epsilon = \frac{\mu^2 (\sin \tilde{\theta} + \sigma \kappa \sin \kappa \tilde{\theta})^2}{2 \left[ 1 - \cos \tilde{\theta} - \sigma \cos \kappa \tilde{\theta} \right]^2} ,$$

(15)

1 If natural inflation model includes a large $Z$ in the kinetic
term[19], redefining the field gives rise to an effective decay constant $f_{\text{eff}} = \sqrt{Z} f$, which (as well as the inflaton field itself)
can also be larger than $M_2$, however a question remained is
how to get a large $Z$ naturally[17].
2 Massive particles have also been considered in Ref. [21]
in the context of extra-dimensional quintessence models.
3 For a specific presentation, we set $F_1 = 1$. The $F_1 = 0$ case is
equivalent since it only corresponds to a co-ordinate shift in the
potential.
\[
\begin{align*}
\eta &= \frac{\mu^2(\cos \tilde{\theta} + \sigma \kappa^2 \cos \kappa \tilde{\theta})}{1 - \cos \tilde{\theta} - \sigma \cos \kappa \tilde{\theta}}, \\
\xi &= \frac{-\mu^4(\sin \tilde{\theta} + \sigma \kappa \sin \kappa \tilde{\theta})(\sin \tilde{\theta} + \sigma \kappa^3 \sin \kappa \tilde{\theta})}{[1 - \cos \tilde{\theta} - \sigma \cos \kappa \tilde{\theta}]^2},
\end{align*}
\]
where we have defined \( \tilde{\theta} \equiv q_1 \theta \) and \( \mu \equiv q_1 M_p / f_{\epsilon \eta} \), \( \kappa \equiv q_2 / q_1 \).

To see analytically the effects of \( \sigma \) term, we consider the following choice of parameters: \( \kappa \gg 1, \sigma \ll 1, |\sigma| \kappa \ll 1 \) and \( |\phi| \kappa^2 \sim O(1) \). When cosmological scales begin to cross out the horizon in the inflationary epoch, \( \tilde{\theta} \sim \pi \), the slow-roll parameters are approximately
\[
\begin{align*}
\epsilon &\sim \mu^2(\pi - \tilde{\theta})^2, \\
\eta &\sim \mu^2(-1 + \sigma \kappa^2 \cos \kappa \tilde{\theta}), \\
\xi &\sim -\mu^4 \sigma \kappa^3(\pi - \tilde{\theta}) \sin \kappa \tilde{\theta}.
\end{align*}
\]

Hence \( \epsilon \ll |\eta| \). The spectral index is determined by the \( \eta \) parameter, \( n_s - 1 \approx 2 \eta \), and the tensor fluctuation is negligible.

From the above equations, one generically would have \( |\xi| \sim \eta^2 \sim 10^{-4} \), corresponding to a negligible spectral variation. However, the large \( \sigma \kappa^3 \) term in Eq. 24 can enhance \( \xi \) substantially (to the size as large as \( \eta \)), and lead to a scale varying spectrum. Furthermore, for appropriate choices of \( \sigma \) and \( \kappa \), the scalar power spectrum can run from blue to red as the co-moving scale \( k \) increases. Such a tilted spectrum is favored by the current data of WMAP 2.

In Fig. 1, for the following choices of parameters: (a) \( \sigma = -3.8 \times 10^{-4}, \mu = 1/2 \) and \( \kappa = 100 \) (solid lines), and (b) \( \sigma = 0.012, \mu = 1/3, \kappa = 15 \) (dashed lines). To ensure the accuracy of our results, we have retained the higher power terms (up to \( n = 6 \) for the massless particle) in the numerical calculations. We have also set the number of e-folding \( N(k_*) = 50 \) at a reference co-moving scale \( k_* \); the WMAP analyses used pivot scales \( k_0 = 0.002 \) and 0.05 Mpc\(^{-1} \) 2 3. (This scale arbitrariness is the inherent theoretical uncertainty in our analysis; it can be resolved if the detail reheating history is known 21.) Normalizing the spectral index to the WMAP central value \( n_s = 1.1 \) at \( k_0 = 0.002 \) Mpc\(^{-1} \) 3, we find \( dn_s / d \ln k \approx -0.041 \) and \( \approx -0.021 \) for case (a) and (b) respectively; they are in good agreements with the current WMAP fits 2 3. Such a running feature of the spectral index does not exist in the usual natural or extra-natural inflation models.

\[
\text{FIG. 1: Spectral indices } n_s \text{ and their running } dn_s / d \ln k \text{ for } \sigma = -3.8 \times 10^{-4}, \mu = 1/2, \kappa = 100 \text{ (solid lines), and for } \sigma = 0.012, \mu = 1/3, \kappa = 15 \text{ (dashed lines). } k_0 = 0.002 \text{ Mpc}^{-1} \text{ is the pivot scale of WMAP.}
\]

In Fig. 2 we delineate the parameter space that gives the required amount of spectral running \((-0.084 < dn_s / d \ln k < -0.027)\) in our model. We have normalized the spectral index \( n_s = 1.13 \) at the same pivot scale \( k_0 \). We show allowed regions in the \( \kappa-\sigma \) plane, for \( \mu = 1/3 \). The correlation between \( \kappa \) and \( \sigma \) can be understood because one needs a certain amount of cancellations between the two terms in Eq. 20 (to achieve a spectral running from blue to red).

In our models the size of the fifth dimension is determined to be of the order \( R \sim 10/M_p \) from the COBE normalization, \( P_R^{1/2} \sim 10^{-5} \). For our choices of parameters, \( \mu \sim 0.1 \), this implies the four- and five-dimensional gauge coupling constants \( g_4 \lesssim 10^{-3}/q_1, g_5^2 \lesssim 10^{-4}/q_1^2 M_p \).

These parameter choices could be accommodated in models of fundamental theories. For example, string theory predicts a host of exotic particles with fractional
charges; one can obtain large charge ratios of the massive to massless particles ($\kappa \sim 10-100$) as required in our models. The massive particle has a mass of the order $M_2 \sim O(1)R^{-1}$, falls in the range of massive string states. It remains to be seen whether this kind of model can arise naturally from specific string models.

The small gauge coupling constant ($g_4 \lesssim 10^{-3}$ when $q_1 \sim O(1)$) might seem worrisome, since string theory compactification generally requires $g_4 M_P R \gtrsim 1$. This problem can be improved by introducing many Abelian gauge fields in higher dimensions as in the assisted inflation model [22]. For a system with $N$ Wilson loops $\theta_i$, the Lagrangian is given by

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2g_4^2(2\pi R)^2}(\partial \theta_i)^2 - \sum_{i=1}^{N} V(\theta_i). \quad (22)$$

This system has a solution where all the $\theta_i$ fields are equal, consequently it can be described by a single-field model through the redefinitions, $f_{\text{eff}} = \sqrt{N}/2\pi g_4 R$, $V = NV_i$. The COBE normalization gives $RN^{-1/4} \sim 10M_P^{-1}$, which implies $g_4 \lesssim 10^{-3}N^{1/4}$. The above requirement can be naturally satisfied if $N \gtrsim 100$.

Before concluding, we comment on the possible discrepancy between predictions and COBE [23], WMAP observations on the largest scales [2]. As argued in Ref. [2], the need of large running is due to the suppressed multipoles at $l = 2, 3, 4$. However as shown in Ref. [2] the possibility of finding a lower value of quadrupole in the presence of a constant running of the spectral index is no more than 0.9 percent for a spacial flat $\Lambda$CDM cosmology. In our model, the running index $dn_s/d\ln k$ is not a constant, but we have checked that it does not improve the fit significantly. With many Abelian gauge fields in the higher dimensions, we will show now this discrepancy can be alleviated. For simplicity of our discussion, consider two Wilson lines $\theta_1$ and $\theta_2$, the Lagrangian for such a system is given by

$$\mathcal{L} = \frac{1}{2}(\partial \phi_1)^2 + \frac{1}{2}(\partial \phi_2)^2 - V_0 \left[1 - \cos \left(\frac{\theta_1}{f_{\text{eff}}}\right)\right] - V_0 \left[1 - \cos \left(\frac{\theta_2}{f_{\text{eff}}}\right)\right]. \quad (23)$$

Assuming adiabatic process with certain choice of parameters, e.g., $q_1 = 5q_2$, inflation will be firstly driven by the heavy field $\phi_1$ and then by $\phi_2$, with no interruption in between. The "double natural inflation" can generate a primordial spectrum with a feature at the largest scales favored by current data. In Fig. 3 we give an example with $q_1 = 5$ and $q_2 = 1$. The details of our calculations is similar to Ref. [23] we numerically calculate the primordial scalar and tensor spectra and fit the resulting CMB TT and TE spectra using WMAP likelihood code [26] with CMBfast program [27]. We compare our result with WMAP team’s best fit power law $\Lambda$CDM ($\Lambda$CDM) and running spectral index $\Lambda$CDM (R$\Lambda$CDM) model. The CMB quadrupole as shown in Fig. 3 is better suppressed in this model.

In summary, the WMAP result of a varying spectral index, if stands, could be used as a discriminator for inflationary models; most extant models allow for only a tiny amount of scale variations in the spectral index and could face a severe challenge. We have studied in this paper the possibility of building inflation models with large running spectral indices, and specifically proposed a higher-dimensional model which can generate $dn/d\ln k \sim O(10^{-2})$, favored by the WMAP analyses.

**Acknowledgments:** This work was supported in part by National Natural Science Foundation of China and by Ministry of Science and Technology of China under Grant No. NKBRSF G19990754, and in part by an NSF grant NSF-0070928.

[1] C. L. Bennett et al., astro-ph/0302207, to appear in ApJ; also see the WMAP homepage, http://map.gsfc.nasa.gov
[2] D. N. Spergel et al., astro-ph/0302209, to appear in ApJ.
[3] H. V. Peiris et al., astro-ph/0302225, to appear in ApJ.
[4] A. Melchiorri and C. J. Ødman, Phys. Rev. D 67, 081302 (2003).
[5] S. L. Bridle, A. M. Lewis, J. Weller, and G. Efstathiou, astro-ph/0302306, to appear in MNRAS.
[6] V. Barger, H.-S. Lee, and D. Marfatia, Phys. Lett. B 565, 33 (2003).
[7] B. Moore, Nature 370, 629 (1994); R. A. Flores and J. A. Primack, ApJ 427, L1 (1994); W. J. G. De Blok

---

**FIG. 3:** Suppression of CMB quadrupole in the double natural inflation model. The CMB data with error bars are taken from WMAP collaborations [2].
and S. S. McCaugh, MNRAS 290, 533 (1997); D. N. Spergel and P. J. Steinhardt, Phys. Rev. Lett. 84, 3760 (2000); M. Kamionkowski and A. R. Liddle, Phys. Rev. Lett. 84, 4525 (2000); A. R. Zentner and J. S. Bullock, Phys. Rev. D 66, 43003 (2002); W. Lin, D. Huang and X. Zhang, Phys. Rev. Lett. 86, 954 (2001).

[8] See, e.g., A. R. Liddle and D. H. Lyth, Cosmological inflation and large-scale structure, Cambridge University Press (2000).

[9] S. Dodelson and E. D. Stewart, Phys. Rev. D 65, 101301 (2002).

[10] E. D. Stewart, Phys. Lett. B 391, 34 (1997); Phys. Rev. D 56, 2019 (1997).

[11] X. Wang et al., astro-ph/0209242

[12] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart, and D. Wands, Phys. Rev. D 49, 6410 (1994).

[13] N. Arkani-Hamed, H.-C. Cheng, P. Creminelli, and L. Randall, Phys. Rev. Lett. 90, 221302 (2003); hep-th/0302034 to appear in JCAP.

[14] D. E. Kaplan and N. Weiner, hep-ph/0302014; M. Fairbairn, L. L. Honorez, and M. H. G. Tytgat, Phys. Rev. D 67, 101302 (2003).

[15] K. Freese, J. A. Frieman, and A. V. Olinto, Phys. Rev. Lett. 65, 3233 (1990); F. Adams, J. R. Bond, K. Freese, J. A. Frieman, and A. V. Olinto, Phys. Rev. D 47, 426 (1993).

[16] Y. Hosotani, Phys. Lett. B 126, 309 (1983); 129, 193 (1983); Ann. Phys. 190, 233 (1989).

[17] A. Delgado, A. Pomarol, and M. Quiros, Phys. Rev. D 60, 095008 (1999).

[18] S. Dimopoulos and S. Thomas, hep-th/0307004

[19] B. Feng, M. Li and X. Zhang, in preparation.

[20] L. Pilo, D. A. J. Rayner, and A. Riotto, hep-ph/0302087

[21] B. Feng, X. Gong, and X. Wang, astro-ph/0301111

[22] A. R. Liddle, A. Mazumdar, and F. E. Schunck, Phys. Rev. D 58, 061301 (1998).

[23] C. L. Bennett et al., Astrophys. J. 464, L1 (1996).

[24] A. Oliveira-Costa, M. Tegmark, M. Zaldarriaga and A. Hamilton, astro-ph/0307282

[25] Bo Feng and Xinmin Zhang, astro-ph/0305020

[26] L. Verde et al., astro-ph/0302218 to appear in ApJ.

[27] U. Seljak and M. Zaldarriaga, Astrophys. J. 469, 437 (1996).