Isomorphisms of spectral lattices

Martin Bohata

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Abstract
The paper deals with spectral order isomorphisms between certain spectral sublattices of direct sums of AW*-factors. We prove that these maps consist of spectral order isomorphisms between spectral sublattices of individual direct summands. Consequently, we obtain a complete description of spectral order isomorphisms in the case of atomic AW*-algebras. This includes the setting of matrix algebras. Moreover, we also exhibit the general form of spectral order orthoisomorphisms between various spectral sublattices of direct sums of AW*-factors.

Keywords
AW*-algebras · atomic AW*-algebras · spectral order · spectral order isomorphisms · Jordan *-isomorphisms

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1 Introduction
Let \((E^x_\lambda)_{\lambda \in \mathbb{R}}\) and \((E^y_\lambda)_{\lambda \in \mathbb{R}}\) be spectral families of self-adjoint elements \(x\) and \(y\), respectively, in an AW*-algebra \(\mathcal{M}\). We write \(x \leq y\) if \(E^y_\lambda \leq E^x_\lambda\) for all \(\lambda \in \mathbb{R}\). The binary relation \(\leq\) on the self-adjoint part of \(\mathcal{M}\) is a partial order called spectral order. It was first introduced by Olson [17] in the setting of von Neumann algebras. The self-adjoint part, \(\mathcal{M}_{sa}\), of \(\mathcal{M}\) endowed with the spectral order, forms a conditionally complete lattice, the statement which can be proved in the same way as for von Neumann algebras [17]. This result is in a strong contrast to the behavior of the standard order on self-adjoint elements [11, 20]. In the sequel, following the terminology introduced in [4], we shall call \((\mathcal{M}_{sa}, \leq)\) the spectral lattice of \(\mathcal{M}\). By a spectral sublattice of \(\mathcal{M}\) we shall mean a sublattice of the
s spectral lattice of $\mathcal{M}$. It follows from [3, Proposition 3.4] that examples of proper spectral sublattices are $(\mathcal{M}_+, \preceq)$, $(\mathcal{E}(\mathcal{M}), \preceq)$, and $(\mathcal{P}(\mathcal{M}), \preceq)$, where $\mathcal{M}_+$ is the positive part of $\mathcal{M}$, $\mathcal{E}(\mathcal{M})$ is the set of all effects (i.e. the set of all positive elements in the unit ball of $\mathcal{M}$), and $\mathcal{P}(\mathcal{M})$ is the set of all projections in $\mathcal{M}$. It is easy to see that the spectral order coincides with the standard order on projections. Consequently, the spectral lattice can be regarded as a natural extension of the projection lattice of $\mathcal{M}$.

It is worth to note that the spectral order has the following physical interpretation. Let $w((−\infty, λ], x, φ)$ be the probability that a measurement of an observable $x$ gives a value in an interval $(−∞, λ]$ in a state $φ$ of a physical system. Then $x \leq y$ says that $w((−∞, λ], y, φ) \leq w((−∞, λ], x, φ)$ for every $λ \in \mathbb{R}$ and every state $φ$. Thus $x \leq y$ means that, in every state of the physical system, the corresponding distribution functions are pointwise ordered. The interested reader can find physical applications of the spectral order, for example, in the papers [5, 8, 23].

Let $M$ and $N$ be subsets of the self-adjoint parts of two $\mathcal{AW}^\ast$-algebras. A bijection $φ : M \rightarrow N$ is called a spectral order isomorphism if, for all $x, y \in M$, $x \leq y$ if and only if $φ(x) \leq φ(y)$. This paper is devoted to the study of spectral order isomorphisms between certain spectral sublattices of $\mathcal{AW}^\ast$-algebras. We continue the line of research initiated by Molnár and Šemrl [15]. Among other things they described the general form of spectral order automorphisms of the self-adjoint part of the von Neumann algebra $B(H)$ of all bounded operators on a finite-dimensional complex Hilbert space $H$ provided that $\dim H \geq 3$. The two-dimensional setting was later examined by Molnár and Nagy [13]. The open problem of infinite-dimensional case has recently been solved by the author in [3]. More concretely, it has been shown that if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{AW}^\ast$-factors of Type I (i.e. $\mathcal{M}$ and $\mathcal{N}$ are $\ast$-isomorphic to $B(H)$ and $B(K)$, respectively, for some complex Hilbert spaces $H$ and $K$), then every spectral order isomorphism $φ : \mathcal{M}_{sa} \rightarrow \mathcal{N}_{sa}$ has the form $φ(x) = \Theta_φ(f(x))$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing bijection, $τ$ is an isomorphism between projection lattices, and $\Theta_τ$ is defined by $E^\Theta_{\tau}(\lambda) = τ(E^\lambda)$ for all $λ \in \mathbb{R}$. Similar statements are also known for spectral order isomorphisms between various proper spectral sublattices of $\mathcal{AW}^\ast$-factors of Type I [3, 13–15]. However, there are no results on the structure of general spectral order isomorphisms between spectral lattices (or proper spectral sublattices) going beyond $\mathcal{AW}^\ast$-factors of Type I. The aim of this paper is to contribute to fill that gap by investigating spectral order isomorphisms in the context of direct sums of $\mathcal{AW}^\ast$-factors.

A simple observation shows that the mapping $(x, y) \mapsto (x, y^3)$ is a spectral automorphism of the spectral lattice of the $\mathcal{AW}^\ast$-algebra $B(H) \oplus B(H)$ which has not the above canonical form $\Theta_τ(f(x))$. On the other hand, we see that components are transformed in the canonical way. It turns out that the componentwise action of spectral isomorphisms $\mathcal{E}(\mathcal{M}) \rightarrow \mathcal{E}(\mathcal{N})$, $\mathcal{M}_+ \rightarrow \mathcal{N}_+$, and $\mathcal{M}_{sa} \rightarrow \mathcal{N}_{sa}$, where $\mathcal{M}$ and $\mathcal{N}$ are direct sums of $\mathcal{AW}^\ast$-factors, is not exceptional. Indeed, we establish that such isomorphisms are determined by spectral order isomorphisms between the corresponding spectral sublattices of direct summands. This allows us to describe their general form in the case of atomic $\mathcal{AW}^\ast$-algebras using results from [3, 13, 15].
particular, we obtain a complete description of spectral order isomorphisms between spectral lattices of matrix algebras.

A spectral order isomorphism \( \varphi : M \to N \) is called a spectral order orthoisomorphism if, for all \( x, y \in M \), \( xy = 0 \) if and only if \( \varphi(x)\varphi(y) = 0 \). The structure of spectral order orthoisomorphisms is well known in the case of \( AW^* \)-factors. Let \( M \) and \( N \) be \( AW^* \)-factors not of Type I_2. In [9, 10], Hamhalter and Turilova obtained a description of spectral order orthoautomorphism of \( E(M) \) provided that an \( AW^* \)-factor \( M \) is not of Type III. However, the exclusion of \( AW^* \)-factors of Type III is not needed. In fact, it has been proved in [3] that every spectral order orthoisomorphism \( \varphi : E(M) \to E(N) \) has the form \( \varphi(x) = \psi(f(x)) \) for some Jordan \(*\)-isomorphism \( \psi : M \to N \) and some strictly increasing bijection \( f : [0, 1] \to [0, 1] \). It has also been established in [3] that analogous theorems hold for spectral order orthoisomorphisms \( \varphi : M_+ \to N_+ \) and \( \varphi : M_{sa} \to N_{sa} \). In this paper, we use these results to get a general form of spectral order orthoisomorphisms of spectral sublattices of direct sums of \( AW^* \)-factors which are not of Type I_2.

2 Preliminaries

We start this section by recalling some basic facts about \( AW^* \)-algebras. For a more detailed exposition of the theory of \( AW^* \)-algebras, we refer the reader to the monographs [2, 19, 21]. An \( AW^* \)-algebra is a \( C^* \)-algebra \( M \) such that the following conditions hold:

(i) Every maximal commutative \( C^* \)-subalgebra of \( M \) is a closed linear span of its projections.

(ii) The set \( P(M) \) of all projections equipped with the standard order is a complete lattice.

Note that every \( AW^* \)-algebra is unital. By the symbol \( 1_M \), we shall denote the unit of \( M \). When no confusion can arise, we shall write \( 1 \) in place of \( 1_M \). A \( C^* \)-subalgebra \( N \) of an \( AW^* \)-algebra \( M \) is called \( AW^* \)-subalgebra of \( M \) if \( N \) is an \( AW^* \)-algebra and the supremum of each family of orthogonal projections in \( N \) computed in the projection lattice of \( M \) is also an element of \( N \). The center of an \( AW^* \)-algebra \( M \) is the set

\[ Z(M) = \{ x \in M \mid xy = yx \text{ for all } y \in M \}. \]

The center \( Z(M) \) forms an \( AW^* \)-subalgebra of \( M \). An important class of \( AW^* \)-algebras consists of von Neumann algebras. It was proved by Kaplansky [12] that an \( AW^* \)-algebra of Type I is a von Neumann algebra if and only if its center is a von Neumann algebra. Consequently, \( AW^* \)-factors of Type I are nothing but von Neumann factors of Type I. On the other hand, it is well known that there are \( AW^* \)-factors which are not von Neumann factors [6, 22, 24].

Let \( (M_\lambda)_{\lambda \in \Lambda} \) be a family of \( AW^* \)-algebras. Suppose that the set
\[ \bigoplus_{j \in \Lambda} M_j := \left\{ (x_j)_{j \in \Lambda} \mid \sup_{j \in \Lambda} \|x_j\| < \infty \right\} \]

is equipped with the pointwise *-algebra operations and the norm \( \| (x_j)_{j \in \Lambda} \| = \sup_{j \in \Lambda} \|x_j\| \). Then it can be shown (see [2]) that \( \bigoplus_{j \in \Lambda} M_j \) forms an AW *-algebra called the direct sum of \( (M_j)_{j \in \Lambda} \). A projection \( p \) in an AW*-algebra is called atomic, if it has no nonzero proper subprojection. An AW*-algebra is said to be atomic if every nonzero projection majorizes an atomic projection. Note that each atomic AW *-algebra is (*-isomorphic to) a direct sum of AW*-factors of Type I. Thus atomic AW *-algebras are precisely atomic von Neumann algebras.

An extremally disconnected compact Hausdorff space is called a Stonean space. It is well known that every abelian AW*-algebra is *-isomorphic to \( C(X) \), where \( X \) is a Stonean space. Thus, there is one-to-one correspondence between projection lattices of abelian AW*-algebras and complete Boolean algebras. Accordingly, the projection lattice, \((P(M), \leq)\), of an abelian AW*-algebra \( M \) is meet-infinitely distributive (see [18, Theorem 5.13]) which means that

\[ \bigwedge_{\lambda \in \Lambda} (p \lor q_\lambda) = p \lor \bigwedge_{\lambda \in \Lambda} q_\lambda \]

for each \( p \in P(M) \) and each family \((q_\lambda)_{\lambda \in \Lambda}\) in \( P(M) \).

Recall that a family \((E_\lambda)_{\lambda \in \mathbb{R}}\) of projections in an AW*-algebra \( M \) is called a (bounded) spectral family if the following conditions hold:

(i) \( E_\lambda \leq E_\mu \) whenever \( \lambda \leq \mu \).

(ii) \( E_\lambda = \bigwedge_{\mu > \lambda} E_\mu \) for every \( \lambda \in \mathbb{R} \).

(iii) There is a positive real number \( \alpha \) such that \( E_\lambda = 0 \) when \( \lambda < -\alpha \) and \( E_\lambda = 1 \) when \( \lambda > \alpha \).

It is part of the folklore of operator theory that there is a bijection between set of all spectral families in \( M \) and the self-adjoint part of \( M \). The spectral family \((E_\lambda)_{\lambda \in \mathbb{R}}\) corresponds to \( x \in M_{sa} \) if and only if \( x E_\lambda \leq \lambda E_\lambda \) and \( \lambda (1 - E_\lambda) \leq x (1 - E_\lambda) \) for each \( \lambda \in \mathbb{R} \). In the sequel, we shall denote by \((E_\lambda^x)_{\lambda \in \mathbb{R}}\) the spectral family corresponding to \( x \in M_{sa} \). It turns out that \( E_\lambda^x \) belongs to the abelian AW*-subalgebra of \( M \) generated by \( \{1, x\} \).

The following two lemmas are well known. We present their proofs for the convenience of the reader.

**Lemma 2.1** Let \((M_j)_{j \in \Lambda}\) be a family of AW*-algebras and \( M = \bigoplus_{j \in \Lambda} M_j \). If \((x_j)_{j \in \Lambda} \in M_{sa} \), then \( E_\lambda^{(x_j)_{j \in \Lambda}} = (E_\lambda^{x_j})_{j \in \Lambda} \) for every \( \lambda \in \mathbb{R} \).

**Proof** Is easy to see that \((E_\lambda^{x_j})_{j \in \Lambda}\) is a spectral family. Since

\[ (x_j)_{j \in \Lambda} (E_\lambda^{x_j})_{j \in \Lambda} = (x_j E_\lambda^{x_j})_{j \in \Lambda} \leq \lambda (E_\lambda^{x_j})_{j \in \Lambda} \]

and
Furthermore, there is a clopen set $M \in \mathcal{M}$. By Lemma 2.1, for every $\lambda \in \mathbb{R}$, $E_\lambda^{(x_j)} = (E_\lambda^{x_j})_{j \in \Lambda}$ for every $\lambda \in \mathbb{R}$. \hfill \square

**Lemma 2.2** Let $(\mathcal{M}_j)_{j \in \Lambda}$ be a family of $AW^*$-algebras and $\mathcal{M} = \bigoplus_{j \in \Lambda} \mathcal{M}_j$. If $(x_j)_{j \in \Lambda}, (y_j)_{j \in \Lambda} \in \mathcal{M}_{sa}$, then $(x_j)_{j \in \Lambda} \leq (y_j)_{j \in \Lambda}$ if and only if $x_j \leq y_j$ for every $j \in \Lambda$.

**Proof** Let $\lambda \in \mathbb{R}$. It is easy to see that $(E_\lambda^{x_j})_{j \in \Lambda} \leq (E_\lambda^{y_j})_{j \in \Lambda}$ if and only if $E_\lambda^{x_j} \leq E_\lambda^{y_j}$ for every $j \in \Lambda$. By Lemma 2.1, $(x_j)_{j \in \Lambda} \leq (y_j)_{j \in \Lambda}$ if and only if $x_j \leq y_j$ for every $j \in \Lambda$. \hfill \square

We have pointed out in the introduction that the spectral lattice of an $AW^*$-algebra $\mathcal{M}$ is a conditionally complete lattice. It was established in [3, Proposition 3.4] that suprema and infima of subsets of $\mathcal{E}(\mathcal{M})$ considered in the spectral lattice of $\mathcal{M}$ are the same as those computed in spectral sublattice $(\mathcal{E}(\mathcal{M}), \preceq)$. In addition, it was shown that similar results hold for the sublattices $(P(\mathcal{M}), \preceq) = (\mathcal{P}(\mathcal{M}), \preceq)$ and $(\mathcal{M}_+, \preceq)$ as well. Let us note that suprema and infima can be described in terms of spectral families as follows. Let $M$ be a nonempty set of $\mathcal{M}_{sa}$. If $M$ is bounded above, then its supremum is a self-adjoint element with

$$(E_\lambda^{V_{x \in M} x})_{\lambda \in \mathbb{R}} = \left( \bigwedge_{\lambda \in \mathbb{R}} E_\lambda^{x} \right)_{\lambda \in \mathbb{R}}.$$

If $M$ is bounded below, then its infimum $\bigwedge_{x \in M} x$ is a self-adjoint element with the spectral family

$$(E_\lambda^{\bigwedge_{x \in M} x})_{\lambda \in \mathbb{R}} = \left( \bigvee_{\lambda > \lambda \in \mathbb{R}} E_\lambda^{x} \right)_{\lambda \in \mathbb{R}}.$$

**Lemma 2.3** If $z$ is a central projection and $x \in \mathcal{E}(\mathcal{M})$, then $zx = z \wedge x$.

**Proof** Consider an abelian $AW^*$-subalgebra $\mathcal{N}$ of $\mathcal{M}$ generated by \{z, x, 1\}. Then $\mathcal{N} \cong C(X)$, where $X$ is a Stonean space. Let $f \in C(X)$ correspond to $x$. Since $x \in \mathcal{E}(\mathcal{M})$, $0 \leq f(t) \leq 1$ for all $t \in X$. Furthermore, there is a clopen set $O \subset X$ such that $\chi_O$ corresponds to $z$ because $z$ is a projection. The spectral order $\preceq$ coincides with $\leq$ on abelian algebras. It is easy to see that $\chi_O \wedge f = \chi_O f$. Thus, $z \wedge x = zx$ in the spectral lattice of $\mathcal{N}$ because a *-isomorphism is a spectral order isomorphism. According to [3, Proposition 3.3], the infimum in the spectral lattice of $\mathcal{N}$ coincides with the infimum in the spectral lattice of $\mathcal{M}$ and so $z \wedge x = zx$ in the spectral lattice of $\mathcal{M}$. \hfill \square

**Lemma 2.4** Let $(z_j)_{j \in \Lambda}$ be a family of mutually orthogonal central projections in an $AW^*$-algebra $\mathcal{M}$. If $x \in \mathcal{M}_+$, then
\[ \bigvee_{j \in \Lambda} z_j x = \left( \bigvee_{j \in \Lambda} z_j \right) x. \]

**Proof** First assume that \( x \in \mathcal{E}(\mathcal{M}) \). By Lemma 2.3,
\[
E_{\Lambda}^{\bigvee_{j \in \Lambda} z_j x} = \bigwedge_{j \in \Lambda} E_{\Lambda}^{z_j x} = \bigwedge_{j \in \Lambda} \bigwedge_{\mu > \lambda} (E_{\mu}^{z_j} \lor E_{\mu}^{x}) = \bigwedge_{\mu > \lambda} \bigwedge_{j \in \Lambda} (E_{\mu}^{z_j} \lor E_{\mu}^{x}).
\]

Since the \( AW^* \)-subalgebra of \( \mathcal{M} \) generated by
\[ \{ E_{\Lambda}^{z_j} | j \in \Lambda, \lambda \in \mathbb{R} \} \cup \{ E_{\Lambda}^{x} | \lambda \in \mathbb{R} \} \]
is abelian,
\[
E_{\Lambda}^{\bigvee_{j \in \Lambda} z_j x} = \bigwedge_{\mu > \lambda} \left[ \left( \bigwedge_{j \in \Lambda} E_{\mu}^{z_j} \right) \lor E_{\mu}^{x} \right] = \bigwedge_{\mu > \lambda} \left[ E_{\Lambda}^{\bigvee_{j \in \Lambda} z_j} \lor E_{\Lambda}^{x} \right] = E_{\Lambda}^{\bigvee_{j \in \Lambda} z_j} \lor E_{\Lambda}^{x}.
\]

Now let \( x \in \mathcal{M}_+ \). Then \( x = \alpha e \) for some \( \alpha \in (0, \infty) \) and \( e \in \mathcal{E}(\mathcal{M}) \). Hence,
\[
\bigvee_{j \in \Lambda} z_j x = \alpha \bigvee_{j \in \Lambda} z_j e = \alpha \left( \bigvee_{j \in \Lambda} z_j \right) e = \left( \bigvee_{j \in \Lambda} z_j \right) x.
\]

The goal of the following two propositions is to characterize scalar multiples of atomic projections by means of the spectral order.

**Proposition 2.5** ([3, Proposition 3.7]) Let \( \mathcal{M} \) be an \( AW^* \)-algebra and let \( x \in \mathcal{E}(\mathcal{M}) \) be nonzero. Then the following statements are equivalent:

(i) There is \( \lambda \in (0, 1] \) and an atomic projection \( e \in \mathcal{M} \) such that \( x = \lambda e \).
(ii) If \( y, z \in \mathcal{E}(\mathcal{M}) \) satisfy \( y, z \leq x \), then \( y \leq z \) or \( z \leq y \).

**Proposition 2.6** Let \( \mathcal{M} \) be an \( AW^* \)-algebra and let \( x \in \mathcal{M}_+ \) be nonzero. Then the following statements are equivalent:

(i) There is \( \alpha > 0 \) and an atomic projection \( e \in \mathcal{M} \) such that \( x = \alpha e \).
(ii) If \( y, z \in \mathcal{M}_+ \) satisfy \( y, z \leq x \), then \( y \leq z \) or \( z \leq y \).

**Proof** It follows directly from the previous proposition and [3, Lemma 3.1].
An element \( z \) in a lattice \((P, \leq)\) is said to be **distributive** if

\[
z \lor (x \land y) = (z \lor x) \land (z \lor y).
\]

The set of all distributive elements in \((P, \leq)\) is denoted by \( \mathcal{D}(P, \leq) \). The next statement plays a fundamental role in our discussion of spectral order isomorphisms.

**Proposition 2.7** ([3, Proposition 3.8]) Let \( \mathcal{M} \) be an \( \text{AW}^\ast \)-algebra. Then

(i) \( Z(M)_{sa} = D(M_{sa}, \leq) \);
(ii) \( Z(M)_+ = D(M_+, \leq) \);
(iii) \( E(Z(M)) = D(E(M), \leq) \).

A bijection \( \tau : P(\mathcal{M}) \to P(\mathcal{N}) \) between projection lattices of \( \text{AW}^\ast \)-algebras \( \mathcal{M} \) and \( \mathcal{N} \) is called a **projection isomorphism** if it preserves the order in both directions (i.e., for all \( p, q \in P(\mathcal{M}) \), \( p \leq q \) if and only if \( \tau(p) \leq \tau(q) \)). Let \( H \) and \( K \) be Hilbert spaces of dimension at least 3. By the fundamental theorem of projective geometry [1, p. 203] and the result of Fillmore and Longstaff [7], the form of projective isomorphisms is well known when \( \mathcal{M} = B(H) \) and \( \mathcal{N} = B(K) \). An interesting result on projection isomorphisms was recently proved by Mori [16]. He described projection isomorphisms between projection lattices of von Neumann algebras by means of ring isomorphisms between algebras of locally measurable operators.

Let \( \tau : P(\mathcal{M}) \to P(\mathcal{N}) \) be a projection isomorphism. In the sequel, we shall denote by \( \Theta_\tau \) the bijection from \( M_{sa} \) onto \( N_{sa} \) defined by

\[
E^{\Theta_\tau(x)}_{\lambda} = \tau(E^x_{\lambda}).
\]

This map is indeed a spectral order isomorphism. Since \( \Theta_\tau(E(\mathcal{M})) = E(\mathcal{N}) \) and \( \Theta_\tau(M_+) = N_+ \), the corresponding restrictions of \( \Theta_\tau \) are spectral order isomorphisms \( \varphi : E(\mathcal{M}) \to E(\mathcal{N}) \) and \( \psi : M_+ \to N_+ \).

An important example of a projection isomorphism is given by a restriction of Jordan \(*\)-isomorphism. By a Jordan \(*\)-isomorphism we mean a linear bijection \( \psi : \mathcal{M} \to \mathcal{N} \) such that, for all \( x \in \mathcal{M} \), \( \psi(x^2) = \psi(x)^2 \) and \( \psi(x^*) = \psi(x)^* \). If a projection isomorphism \( \tau \) is a restriction of a Jordan \(*\)-isomorphism \( \psi : \mathcal{M} \to \mathcal{N} \), then \( \Theta_\tau(x) = \psi(x) \) for all \( x \in M_{sa} \). As \( \psi \) preserves orthogonality relation in both directions, a restriction of \( \psi \) is a spectral order orthoisomorphism.

### 3 Isomorphisms between lattices of effects

In the sequel, we shall denote by \( P_{at}(\mathcal{M}) \) the set of all atomic projections in an \( \text{AW}^\ast \)-algebra \( \mathcal{M} \). The proof of the following lemma is based on arguments used in [15].
Lemma 3.1 Let $\mathcal{M}$ and $\mathcal{N}$ be AW*-algebras. If $\varphi : \mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{N})$ is a spectral order isomorphism, then $\varphi(P_{at}(\mathcal{M})) = P_{at}(\mathcal{N})$.

Proof By Proposition 2.5, $\varphi(M) = N$, where

$$ M = \{ \lambda p \mid p \in P_{at}(\mathcal{M}), \lambda \in (0, 1] \}, $$

$$ N = \{ \lambda p \mid p \in P_{at}(\mathcal{N}), \lambda \in (0, 1] \}. $$

The set of all maximal elements of $(M, \leq)$ (resp. $(N, \leq)$) is $P_{at}(\mathcal{M})$ (resp. $P_{at}(\mathcal{N})$). Thus, $\varphi(P_{at}(\mathcal{M})) = P_{at}(\mathcal{N})$. \hfill $\square$

Throughout the rest of this paper, $\mathcal{M}_j \in \Lambda$ and $\mathcal{N}_k \in \Gamma$ will be (nonempty) families of AW*-factors. Let $\delta_{jk}$ be the Kronecker delta. For each $j \in \Lambda$ and $k \in \Gamma$, we set

$$ z_j = (\delta_{jk} \mathbf{1}_{\mathcal{M}_j})_{j \in \Lambda}, $$

$$ w_k = (\delta_{jk} \mathbf{1}_{\mathcal{N}_k})_{j \in \Gamma}. $$

Note that elements $z_j$ and $w_k$ belong to the center of $\bigoplus_{j \in \Lambda} \mathcal{M}_j$ and $\bigoplus_{k \in \Gamma} \mathcal{N}_k$, respectively.

Theorem 3.2 Let $\mathcal{M} = \bigoplus_{j \in \Lambda} \mathcal{M}_j$ and $\mathcal{N} = \bigoplus_{k \in \Gamma} \mathcal{N}_k$, where $\mathcal{M}_j$ and $\mathcal{N}_k$ are AW*-factors. If $\Phi : \mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{N})$ is a spectral order isomorphism, then there are a bijection $\pi : \Gamma \to \Lambda$ and a family $\varphi_j \in \Lambda$ of spectral order isomorphisms $\varphi_j : \mathcal{E}(\mathcal{M}_j) \to \mathcal{E}(\mathcal{N}_{\pi^{-1}(j)})$ such that

$$ \Phi((x_j)_{j \in \Lambda}) = (\varphi_{\pi(k)}(x_{\pi(k)}))_{k \in \Gamma}. $$

Proof Using Proposition 2.7, we see that $\Phi(\mathcal{E}(\mathcal{Z}(\mathcal{M}))) = \mathcal{E}(\mathcal{Z}(\mathcal{N}))$. Since $\mathcal{Z}(\mathcal{M}) = \bigoplus_{j \in \Lambda} \mathcal{Z}(\mathcal{M}_j)$ and each $\mathcal{Z}(\mathcal{M}_j)$ is *-isomorphic to $\mathbb{C}$, $\mathcal{Z}(\mathcal{M})$ is an atomic AW*-algebra. Similarly, $\mathcal{Z}(\mathcal{N})$ is an atomic AW*-algebra. Furthermore, Lemma 3.1 establishes that

$$ \Phi(P_{at}(\mathcal{Z}(\mathcal{M}))) = P_{at}(\mathcal{Z}(\mathcal{N})). $$

Let $z_j$ and $w_k$ be elements defined in (3.1) and (3.2), respectively. Clearly,

$$ P_{at}(\mathcal{Z}(\mathcal{M})) = \{ z_j \mid j \in \Lambda \} \quad \text{and} \quad P_{at}(\mathcal{Z}(\mathcal{N})) = \{ w_k \mid k \in \Gamma \}. $$

It follows from $\Phi(P_{at}(\mathcal{Z}(\mathcal{M}))) = P_{at}(\mathcal{Z}(\mathcal{N}))$ that there is a bijection $\pi : \Gamma \to \Lambda$ such that $\Phi(z_j) = w_{\pi^{-1}(j)}$ for all $j \in \Lambda$. If $x \in z_j \mathcal{E}(\mathcal{M})$, then

$$ \Phi(x) = \Phi(z_j x) = \Phi(z_j) \wedge x = w_{\pi^{-1}(j)} \wedge \Phi(x) = \Phi(x) \in w_{\pi^{-1}(j)} \mathcal{E}(\mathcal{N}) $$

by Lemma 2.3. On the other hand, if $y \in w_{\pi^{-1}(j)} \mathcal{E}(\mathcal{N})$, then
\[ \Phi(z_j \Phi^{-1}(y)) = \Phi(z_j \land \Phi^{-1}(y)) = w_{x^{-1}(y)} \land y = y. \]

This shows that \( \Phi(z_j \mathcal{E}(\mathcal{M})) = w_{x^{-1}(y)} \mathcal{E}(\mathcal{N}) \) for all \( j \in \Lambda \). In other words, there is a family \((\varphi_j)_{j \in \Lambda}\) of spectral order isomorphisms \( \varphi_j : \mathcal{E}(\mathcal{M}_j) \to \mathcal{E}(\mathcal{N}_{x^{-1}(y)}) \) such that

\[
\Phi(z_j(x_k))_{k \in \Gamma} = w_{x^{-1}(y)}(\varphi_k(x_k))_{k \in \Gamma}
\]

for all \( l \in \Lambda \). According to Lemma 2.4,

\[
\Phi((x_j)_{j \in \Lambda}) = \Phi(\bigvee_{l \in \Lambda} z_l(x_j))_{j \in \Lambda}) = \bigvee_{l \in \Lambda} w_{x^{-1}(y)}(\varphi_k(x_k))_{k \in \Gamma} = (\varphi_k(x_k))_{k \in \Gamma}.
\]

\[ \square \]

It was shown in [3, Corollary 4.2] (see also [13, 15] for the special case of automorphisms) that if \( \varphi : \mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{N}) \) is a spectral order isomorphism, where \( \mathcal{M} \) and \( \mathcal{N} \) are AW*-factors of Type I, then there are a projection isomorphism \( \tau : P(\mathcal{M}) \to P(\mathcal{N}) \) and a bijection \( f : [0, 1] \to [0, 1] \) such that \( \varphi(x) = \Theta(f(x)) \) for all \( x \in \mathcal{E}(\mathcal{M}) \). With this fact in mind, Theorem 3.2 leads immediately to a complete description of spectral order isomorphisms between spectral sublattices of all effects of direct sums of Type I factors. We formulate an explicit statement in the case of direct sums of full matrix algebras. Of course, the special case of general matrix algebras is covered by this result.

**Corollary 3.3** Let \( \mathcal{M} = \bigoplus_{j \in \Lambda} B(\mathbb{C}^{m_j}) \) and \( \mathcal{N} = \bigoplus_{k \in \Gamma} B(\mathbb{C}^{n_k}) \), where \( m_j \) and \( n_k \) are natural numbers for each \( j \in \Lambda \) and each \( k \in \Gamma \). If \( \Phi : \mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{N}) \) is a spectral order isomorphism, then there are a bijection \( \pi : \Gamma \to \Lambda \) with \( m_j = n_{x^{-1}(y)} \) for all \( j \in \Lambda \), a family \((f_j)_{j \in \Lambda}\) of strictly increasing bijections \( f_j : [0, 1] \to [0, 1] \), and a family \((\tau_j)_{j \in \Lambda}\) of projection automorphisms \( \tau_j : P(B(\mathbb{C}^{m_j})) \to P(B(\mathbb{C}^{n_{x^{-1}(y)}})) \) such that

\[
\Phi((x_j)_{j \in \Lambda}) = (\Theta_{\tau_j}(f_j(x_k)))_{k \in \Gamma}.
\]

**Proof** Theorem 3.2 together with [3, Corollary 4.2] ensure the existence of a bijection \( \pi : \Gamma \to \Lambda \), a family \((f_j)_{j \in \Lambda}\) of strictly increasing bijections \( f_j : [0, 1] \to [0, 1] \), and a family \((\tau_j)_{j \in \Lambda}\) of projection isomorphisms \( \tau_j : \mathcal{E}(B(\mathbb{C}^{m_j})) \to \mathcal{E}(B(\mathbb{C}^{n_{x^{-1}(y)}})) \) such that

\[
\Phi((x_j)_{j \in \Lambda}) = (\Theta_{\tau_j}(f_j(x_k)))_{k \in \Gamma}.
\]

As there is a projection isomorphism \( \tau_r : \mathcal{E}(B(\mathbb{C}^{m_r})) \to \mathcal{E}(B(\mathbb{C}^{n_{x^{-1}(y)}})) \), we have \( m_r = n_{x^{-1}(y)} \). \[ \square \]
The next simple consequence of Theorem 3.2 is the following description of spectral order orthoisomorphisms.

**Corollary 3.4** Suppose that $\mathcal{M} = \bigoplus_{j \in \Lambda} \mathcal{M}_j$ and $\mathcal{N} = \bigoplus_{k \in \Gamma} \mathcal{N}_k$, where $\mathcal{M}_j$ and $\mathcal{N}_k$ are AW*-factors not of Type I$_2$. If $\Phi : \mathcal{E}(\mathcal{M}) \to \mathcal{E}(\mathcal{N})$ is a spectral order orthoisomorphism, then there are a bijection $\pi : \Gamma \to \Lambda$, a family $(\psi_j)_{j \in \Lambda}$ of Jordan *-isomorphisms $\psi_j : \mathcal{M}_j \to \mathcal{N}_{\pi^{-1}(j)}$, and a family $(f_j)_{j \in \Lambda}$ of strictly increasing bijections $f_j : [0,1] \to [0,1]$ such that

$$\Phi((x_j)_{j \in \Lambda}) = (\psi_j(x_{\pi(k)}))_{k \in \Gamma}. $$

**Proof** This follows immediately from [3, Corollary 5.2] and Theorem 3.2. \qed 

### 4 Isomorphisms between lattices of positive elements

**Theorem 4.1** Let $\mathcal{M} = \bigoplus_{j \in \Lambda} \mathcal{M}_j$ and $\mathcal{N} = \bigoplus_{k \in \Gamma} \mathcal{N}_k$, where $\mathcal{M}_j$ and $\mathcal{N}_k$ are AW*-factors. If $\Phi : \mathcal{M}_+ \to \mathcal{N}_+$ is a spectral order isomorphism, then there are a bijection $\pi : \Gamma \to \Lambda$ and a family $(\phi_j)_{j \in \Lambda}$ of spectral order isomorphisms $\phi_j$ from the positive part of $\mathcal{M}_j$ onto positive part of $\mathcal{N}_{\pi^{-1}(j)}$ such that

$$\Phi((x_j)_{j \in \Lambda}) = (\phi_j(x_{\pi(k)}))_{k \in \Gamma}. $$

**Proof** It follows from Proposition 2.7 that $\Phi(\mathcal{Z}(\mathcal{M})_+) = \mathcal{Z}(\mathcal{N})_+$. Thus a restriction of $\Phi$ is a spectral order isomorphism from $\mathcal{Z}(\mathcal{M})_+$ onto $\mathcal{Z}(\mathcal{N})_+$. Since $\mathcal{M}$ and $\mathcal{N}$ are direct sums of factors, $\mathcal{Z}(\mathcal{M})$ and $\mathcal{Z}(\mathcal{N})$ are atomic AW*-algebras. Let $z_j$ and $w_k$ be elements defined in (3.1) and (3.2), respectively. Suppose that $j \in \Lambda$. By Proposition 2.6, for each $\lambda \in (0, \infty)$, there is $k \in \Gamma$ and a positive number $f_j(\lambda)$ such that $\Phi(\lambda z_j) = f_j(\lambda)w_k$. We show that $k$ does not depend on $\lambda$. For this, choose $\lambda, \mu \in (0, \infty)$. Then

$$f_j(\max\{\lambda, \mu\})w_m = \Phi(\max\{\lambda, \mu\}z_j) = \Phi(\lambda z_j \lor \mu z_j) = f_j(\lambda)w_k \lor f_j(\mu)w_l$$

for some $k, l, m \in \Gamma$. If $k \neq l$, then we see from [3, Lemma 3.5] that $f_j(\lambda) = 0$ or $f_j(\mu) = 0$ which is a contradiction. Consequently, there is a permutation $\pi : \Gamma \to \Lambda$ and a family $(f_j)_{j \in \Lambda}$ of strictly increasing bijections $f_j : [0, \infty) \to [0, \infty)$ such that

$$\Phi(\lambda z_j) = f_j(\lambda)w_{\pi^{-1}(j)}$$

for all $\lambda \in (0, \infty)$ and all $j \in \Lambda$.

Let $(x_j)_{j \in \Lambda}$ belong to $\mathcal{M}_+$. Then there is $c \in (0, \infty)$ such that $x_j \leq c 1_{\mathcal{M}_j}$ for all $j \in \Lambda$. We observe that, for each $j \in \Lambda,$

$$f_j^{-1}(c)z_j = \Phi^{-1}(cw_{\pi^{-1}(j)}) \leq \Phi^{-1}(c 1_{\mathcal{N}})$$

and so $0 \leq f_j^{-1}(c)z_j \leq \Phi^{-1}(c 1_{\mathcal{N}}).$ Hence,
As \( f_j^{-1} \) defines a spectral order automorphism of the positive part of \( M_j \),
\[
  f_j^{-1}(x_j) \preceq f_j^{-1}(c)1_{M_j} \preceq \|\Phi^{-1}(c1_N)\|1_{M_j}
\]
which implies that
\[
  \sup_{j\in\Lambda} \|f_j^{-1}(x_j)\| \leq \|\Phi^{-1}(c1_N)\| < \infty.
\]
This allows us to conclude that \((f_j^{-1}(x_j))_{j\in\Lambda} \in M_+\). Therefore, a map \( \Psi : M_+ \rightarrow N_+ \)
given by
\[
  \Psi((x_j)_{j\in\Lambda}) = \Phi((f_j^{-1}(x_j))_{j\in\Lambda})
\]
for all \((x_j)_{j\in\Lambda} \in M_+\) is well defined. It is easy to see that \( \Psi \) is a spectral order isomorphism. Moreover, \( \Psi(\lambda z_j) = \lambda w_{x^{-1}(j)} \) for all \( j \in \Lambda \) and all \( \lambda \in [0, \infty) \).

Suppose that \( x \in z_j M_+ \). Then there are \( \alpha \in (0, \infty) \) and \( e \in z_j \mathcal{E}(\mathcal{M}) \) such that \( x = \alpha e \). Using Lemma 2.3 and the fact that \( \Psi \) is a spectral order isomorphism,
\[
  \Psi(x) = \Psi(\alpha(z_j \wedge e)) = \Psi((\alpha z_j) \wedge (\alpha e)) = (\alpha w_{x^{-1}(j)}) \wedge \Psi(x)
\]
\[
  = \alpha \left[ w_{x^{-1}(j)} \wedge \frac{1}{\alpha} \Psi(x) \right].
\]
As
\[
  \Psi(\alpha 1_{M_j}) = \Psi(\bigvee_{j \in \Lambda} \alpha z_j) = \bigvee_{j \in \Lambda} \Psi(\alpha z_j) = \bigvee_{j \in \Lambda} \alpha w_{x^{-1}(j)} = \alpha 1_N,
\]
we have \( \frac{1}{\alpha} \Psi(x) \in \mathcal{E}(N) \). It follows from Lemma 2.3 that \( \Psi(x) = w_{x^{-1}(j)} \Psi(x) \). Similarly, we show that \( \Psi^{-1}(y) \in z_j M_+ \) whenever \( y \in w_{x^{-1}(j)} N_+ \). This proves that \( \Psi(z_j M_+) = w_{x^{-1}(j)} N_+ \) for all \( j \in \Lambda \) and so
\[
  \Psi(z_j(x_k)_{k\in\Lambda}) = w_{x^{-1}(j)}(\psi_{\pi(k)}(x_{\pi(k)}))_{k\in\Gamma}
\]
for some family \((\psi_j)_{j\in\Lambda}\) of spectral order isomorphisms \( \psi_j \) from the positive part of \( M_j \) onto the positive part of \( N_{x^{-1}(j)} \). By Lemma 2.4,
As in the case of effects, one can formulate the following corollaries.

**Corollary 4.2** Let $M = \bigoplus_{j \in \Lambda} B(\mathbb{C}^{m_j})$ and $N = \bigoplus_{k \in \Gamma} B(\mathbb{C}^{n_k})$, where $m_j$ and $n_k$ are natural numbers for each $j \in \Lambda$ and each $k \in \Gamma$. If $\Phi : M_+ \to N_+$ is a spectral order isomorphism, then there are a bijection $\psi : \Lambda \to \Gamma$ with $m_j = n_{\psi(j)}$ for all $j \in \Lambda$, a family $(f_j)_{j \in \Lambda}$ of strictly increasing bijections $f_j : [0, \infty) \to [0, \infty)$, and a family $(\tau_j)_{j \in \Lambda}$ of projection automorphisms $\tau_j : P(B(\mathbb{C}^{m_j})) \to P(B(\mathbb{C}^{n_k}))$ such that

$$\Phi((x_j)_{j \in \Lambda}) = (\Theta_{\tau_j}(f_j(x_{\psi(j)})))_{k \in \Gamma}.$$ 

**Proof** The corollary is a direct consequence of [3, Theorem 4.5] and Theorem 4.1.

**Corollary 4.3** Suppose that $M = \bigoplus_{j \in \Lambda} M_j$ and $N = \bigoplus_{k \in \Gamma} N_k$, where $M_j$ and $N_k$ are AW*-factors not of Type $I_2$. If $\Phi : M_+ \to N_+$ is a spectral order orthoisomorphism, then there are a bijection $\pi : \Gamma \to \Lambda$, a family $(\psi_j)_{j \in \Lambda}$ of Jordan *-isomorphisms $\psi_j : M_j \to N_j$, and a family $(f_j)_{j \in \Lambda}$ of strictly increasing bijections $f_j : [0, \infty) \to [0, \infty)$ such that

$$\Phi((x_j)_{j \in \Lambda}) = (\psi_{\pi(j)}(f_j(x_{\pi(j)})))_{k \in \Gamma}.$$ 

**Proof** This follows immediately from [3, Corollary 5.4] and Theorem 4.1.

## 5 Isomorphisms between spectral lattices

Let us fix a notation. By $x^+$ and $x^-$ we denote, respectively, the positive part and the negative part of a self-adjoint element $x$ in an AW*-algebra. First, we recall a useful lemma proved in [3].

**Lemma 5.1** [3, Lemma 5.5] Let $\varphi : M \to N$ be a spectral order isomorphism between AW*-algebras $M$ and $N$ with $\varphi(0) = 0$. If $x \in M_{sa}$, then $\varphi(x^+) = \varphi(x^+)$ and $\varphi(x^-) = -\varphi(-x^-)$. 

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We can now state the main result of this paper.

**Theorem 5.2** Let $\mathcal{M} = \bigoplus_{j \in \Lambda} \mathcal{M}_j$ and $\mathcal{N} = \bigoplus_{k \in \Gamma} \mathcal{N}_k$, where $\mathcal{M}_j$ and $\mathcal{N}_k$ are AW *-factors. If $\Phi : \mathcal{M}_{sa} \to \mathcal{N}_{sa}$ is a spectral order isomorphism, then there are a bijection $\pi : \Gamma \to \Lambda$ and a family $(\varphi_j)_{j \in \Lambda}$ of spectral order isomorphisms $\varphi_j$ from the self-adjoint part of $\mathcal{M}_j$ onto self-adjoint part of $\mathcal{N}_{\pi^{-1}(j)}$ such that

$$\Phi((x_j)_{j \in \Lambda}) = (\varphi_{\pi(k)}(x_{\pi(k)}))_{k \in \Gamma}.$$

**Proof** By Proposition 2.7, $\Phi$ restricts to a spectral order isomorphism from the atomic AW*-algebra $\mathcal{Z}(\mathcal{M})_{sa}$ onto the atomic AW*-algebra $\mathcal{Z}(\mathcal{N})_{sa}$. Combining Lemma 2.2 with [3, Theorem 4.1], we see that the map

$$x \mapsto \Phi(x) - \Phi(0)$$

is a spectral order isomorphism from $\mathcal{M}_{sa}$ onto $\mathcal{N}_{sa}$ because $\Phi(0)$ is a central element and so $\Phi(0) = (\alpha_j 1_{N_j})$ for some family $(\alpha_j)_{j \in \Lambda}$ of real numbers. Therefore, we can assume without loss of generality that $\Phi(0) = 0$. Using this assumption, we have $\Phi(\mathcal{M}_+) = \mathcal{N}_+$ and $\Phi(\mathcal{M}_-) = \mathcal{N}_-$, where $\mathcal{M}_- = -\mathcal{M}_+$ and $\mathcal{N}_- = -\mathcal{N}_+$. By Theorem 4.1 and [3, Theorem 4.5], there are a family $(g_j)_{j \in \Lambda}$ of strictly increasing bijections $g_j : [0, \infty) \to [0, \infty)$ and a permutation $\pi : \Gamma \to \Lambda$ such that

$$\Phi(\lambda z_j) = g_j(\lambda) w_{\pi^{-1}(j)}$$

for all $\lambda \in [0, \infty)$ and all $j \in \Lambda$, where $z_j$ and $w_k$ are elements defined in (3.1) and (3.2), respectively.

Set $\Psi(x) = -\Phi(-x)$, $x \in \mathcal{M}_{sa}$. Then $\Psi : \mathcal{M}_{sa} \to \mathcal{N}_{sa}$ is a spectral order isomorphism from $\mathcal{M}_{sa}$ onto $\mathcal{N}_{sa}$ because the multiplication by $-1$ is order-reversing. Furthermore, $\Psi(0) = 0$, $\Psi(\mathcal{M}_+) = \mathcal{N}_+$, and $\Psi(\mathcal{M}_-) = \mathcal{N}_-$. If $\lambda \leq 0$, then we conclude from the above discussion that $\Psi(\lambda z_j) = -g_j(-\lambda) w_{\pi^{-1}(j)}$ for each $j \in \Lambda$. If $\lambda \geq 0$, then we obtain from Theorem 4.1, [3, Theorem 4.5], and $\Psi(\mathcal{Z}(\mathcal{M})_{sa}) = \mathcal{Z}(\mathcal{N})_{sa}$ that there are a family $(h_j)_{j \in \Lambda}$ of strictly increasing bijections and a permutation $\sigma : \Gamma \to \Lambda$ such that $\Psi(\lambda z_j) = h_j(\lambda) w_{\sigma^{-1}(j)}$. We are going to show that $\pi = \sigma$. To prove this we suppose that $\pi \neq \sigma$. Then there are two different indices $k, l \in \Lambda$ such that $\pi^{-1}(l) = \sigma^{-1}(k)$. Applying Lemma 5.1,

$$\Psi(z_k - z_l) = \Psi(z_k) + \Psi(-z_l) = h_k(1) w_{\sigma^{-1}(k)} - g_l(1) w_{\pi^{-1}(l)}$$

However, $(h_k(1) - g_l(1)) w_{\sigma^{-1}(k)}$ is in $\mathcal{N}_+$ or $\mathcal{N}_-$ which is a contradiction because $z_k - z_l$ does not belong to $\mathcal{M}_+$ or $\mathcal{M}_-$. Therefore, $\pi = \sigma$. As a consequence, $\Psi(\lambda z_j) = f_j(\lambda) w_{\pi^{-1}(j)}$, where

$$f_j(\lambda) = \begin{cases} h_j(\lambda), & \text{if } \lambda \geq 0; \\ -g_j(-\lambda) & \text{if } \lambda < 0. \end{cases}$$

Note that $f_j : \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijection.
If \((x_j)_{j \in \Lambda} \in \mathcal{M}_{sa}\), then there is \(c \in (0, \infty)\) such that \(-c1_{\mathcal{M}_j} \leq x_j \leq c1_{\mathcal{M}_j}\) for each \(j \in \Lambda\). Arguments used in the proof of Theorem 4.1 establish that 
\[ \sup_{j \in \Lambda} f_j^{-1}(c) \leq \|\Psi^{-1}(c1_{\Lambda^+})\|, \] 
We can prove similarly that 
\[ \sup_{j \in \Lambda} -f_j^{-1}(-c) \leq \|\Psi^{-1}(-c1_{\Lambda^+})\|. \] 
Put 
\[ \alpha = \max \left\{ \|\Psi^{-1}(c1_{\Lambda^+})\|, \|\Psi^{-1}(-c1_{\Lambda^+})\| \right\}. \]
We deduce from \(-c1_{\mathcal{M}_j} \leq x_j \leq c1_{\mathcal{M}_j}\) that 
\[ -\alpha 1_{\mathcal{M}} \leq f_j^{-1}(-c)1_{\mathcal{M}_j} \leq f_j^{-1}(x_j) \leq f_j^{-1}(c)1_{\mathcal{M}_j} \leq \alpha 1_{\mathcal{M}} \]
for all \(j \in \Lambda\). Accordingly, 
\[ \sup_{j \in \Lambda} \|f_j^{-1}(x_j)\| \leq \|\alpha 1_{\mathcal{M}}\| < \infty \]
Thus \((x_j)_{j \in \Lambda} \mapsto \Psi((f_j^{-1}(x_j))_{j \in \Lambda})\) is a well defined spectral order isomorphism from \(\mathcal{M}_{sa}\) onto \(\mathcal{N}_{sa}\) and so we can assume without loss of generality that \(f_j\) is the identity function for each \(j \in \Lambda\). In other words, we shall suppose that 
\[ \Psi(\lambda z_j) = \lambda w_{x^{-1}(j)} \]
for all \(\lambda \in \mathbb{R}\) and all \(j \in \Lambda\).
To complete the proof of our assertion it is sufficient to show that there is a family \((\psi_j)_{j \in \Lambda}\) of spectral order isomorphisms \(\psi_j\) from the self-adjoint part of \(\mathcal{M}_j\) onto self-adjoint part of \(\mathcal{N}_{x^{-1}(j)}\) such that 
\[ \Psi((x_j)_{j \in \Lambda}) = (\psi_{\pi(k)}(x_{\pi(k)}))_{k \in \Gamma}. \]
for every \((x_j)_{j \in \Lambda} \in \mathcal{M}_{sa}\). If \(x \in z_j\mathcal{M}_{sa}\), then \(x^+\) and \(x^-\) belong to \(z_j\mathcal{M}_{sa}\). We can write \(x^+\) and \(x^-\) in the form \(x^+ = \alpha u\) and \(x^- = \beta v\) for some \(\alpha, \beta \in (0, \infty)\) and \(u, v \in z_j\mathcal{E}(\mathcal{M})\). It was pointed out in the proof of Theorem 4.1 that \(\frac{1}{\beta} \Psi(x^+) \in \mathcal{E}(\mathcal{N})\). We can apply a similar reasoning to prove that 
\[ -\frac{1}{\beta} \Psi(-x^-) \in \mathcal{E}(\mathcal{N}) \]
Therefore, we obtain from Lemma 2.3 and Lemma 5.1 that 
\[ \Psi(x) = \Psi(x^+) - \Psi(x^-) = \Psi(x^+) + \Psi(-x^-) \]
\[ = \Psi(\alpha(z_j \wedge u)) + \Psi(-\beta(z_j \wedge v)) \]
\[ = \Psi(\alpha z_j) \wedge \Psi(x^+) + \Psi(-\beta z_j) \vee \Psi(-x^-) \]
\[ = a \left[ w_{x^{-1}(j)} \wedge \frac{1}{\alpha} \Psi(x^+) \right] - \beta \left[ w_{x^{-1}(j)} \wedge \left( -\frac{1}{\beta} \Psi(-x^-) \right) \right] \]
\[ = w_{x^{-1}(j)} \left[ \Psi(x^+) + \Psi(-x^-) \right] = w_{x^{-1}(j)} \Psi(x). \]
We also observe from analogous arguments that \(\Psi^{-1}(y) \in z_j\mathcal{M}_{sa}\) whenever \(y \in w_{x^{-1}(j)}\mathcal{N}_{sa}\). This means that \(\Psi(z_j\mathcal{M}_{sa}) = w_{x^{-1}(j)}\mathcal{N}_{sa}\) for all \(j \in \Lambda\). Thus, there
exists a family \((\psi_j)_{j \in \Lambda}\) of spectral order isomorphisms from the self-adjoint part of \(\mathcal{M}_j\) onto the self-adjoint part of \(\mathcal{N}_{\pi^{-1}(j)}\) such that

\[
\psi(z_j(x_k)) = w_{\pi^{-1}(j)}(\psi_{\pi(k)}(x_{\pi(k)}))_{k \in \Gamma}
\]

whenever \((x_k)_{k \in \Lambda} \in \mathcal{M}_{sa}\). Taking into account Lemma 2.4 together with Lemma 5.1, we get

\[
\psi((x_k)_{k \in \Lambda}) = \psi((x_k^+)_{k \in \Lambda}) + \psi(-(x_k^-)_{k \in \Lambda})
\]

\[
= \psi(\bigvee_{j \in \Lambda} z_j(x_k^+)_{k \in \Lambda}) + \psi(-\bigvee_{j \in \Lambda} z_j(x_k^-)_{k \in \Lambda})
\]

\[
= \bigvee_{j \in \Lambda} \psi(z_j(x_k^+)_{k \in \Lambda}) + \bigwedge_{j \in \Lambda} \psi(-z_j(x_k^-)_{k \in \Lambda})
\]

\[
= \bigvee_{j \in \Lambda} w_{\pi^{-1}(j)}(\psi_{\pi(k)}(x_{\pi(k)}^+))_{k \in \Gamma}
\]

\[
- \bigvee_{j \in \Lambda} w_{\pi^{-1}(j)}(\psi_{\pi(k)}(x_{\pi(k)}^-))_{k \in \Gamma}
\]

\[
= (\psi_{\pi(k)}(x_{\pi(k)}^+) - \psi_{\pi(k)}(x_{\pi(k)}^-))_{k \in \Gamma} = (\psi_{\pi(k)}(x_{\pi(k)}))_{k \in \Gamma}
\]

for all \((x_k)_{k \in \Lambda} \in \mathcal{M}_{sa}\). \(\square\)

For the sake of completeness, we state some consequences of the previous theorem. The first result is concerned with spectral order isomorphisms between spectral lattices of direct sum of full matrix algebras. The second assertion describes spectral order orthoisomorphisms between spectral lattices of direct sum of AW*-factor.

**Corollary 5.3** Let \(\mathcal{M} = \bigoplus_{j \in \Lambda} B(\mathbb{C}^{m_j})\) and \(\mathcal{N} = \bigoplus_{k \in \Gamma} B(\mathbb{C}^{n_k})\), where \(m_j\) and \(n_k\) are natural numbers for each \(j \in \Lambda\) and each \(k \in \Gamma\). If \(\Phi : \mathcal{M}_{sa} \to \mathcal{N}_{sa}\) is a spectral order isomorphism, then there are a bijection \(\pi : \Gamma \to \Lambda\) with \(m_j = n_{\pi^{-1}(j)}\) for all \(j \in \Lambda\), a family \((f_j)_{j \in \Lambda}\) of strictly increasing bijections \(f_j : \mathbb{R} \to \mathbb{R}\), and a family \((\tau_j)_{j \in \Lambda}\) of projection automorphisms \(\tau_j : P(B(\mathbb{C}^{m_j})) \to P(B(\mathbb{C}^{m_j}))\) such that

\[
\Phi((x_j)_{j \in \Lambda}) = (\Theta_{\tau_{\pi(j)}(f_j(\pi(j))))_{k \in \Gamma}}.
\]

**Proof** It follows from [3, Theorem 4.7] and Theorem 5.2. \(\square\)

**Corollary 5.4** Suppose that \(\mathcal{M} = \bigoplus_{j \in \Lambda} \mathcal{M}_j\) and \(\mathcal{N} = \bigoplus_{k \in \Gamma} \mathcal{N}_k\), where \(\mathcal{M}_j\) and \(\mathcal{N}_k\) are AW*-factors not of Type I_2. If \(\Phi : \mathcal{M}_{sa} \to \mathcal{N}_{sa}\) is a spectral order orthoisomorphism, then there are a bijection \(\pi : \Gamma \to \Lambda\), a family \((\psi_j)_{j \in \Lambda}\) of Jordan *-isomorphisms \(\psi_j : \mathcal{M}_j \to \mathcal{N}_j\), and a family \((f_j)_{j \in \Lambda}\) of strictly increasing bijections \(f_j : \mathbb{R} \to \mathbb{R}\) such that

\[
\Phi((x_j)_{j \in \Lambda}) = (\psi_{\pi(k)}(f_{\pi(k)}(x_{\pi(k)})))_{k \in \Gamma}.
\]

**Proof** This follows directly from [3, Corollary 5.7] and Theorem 5.2. \(\square\)
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