On quasi-Poisson Cohomology

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Abstract Let \(A\) be a Poisson algebra and \(Q(A)\) its quasi-Poisson enveloping algebra. In this paper, the Yoneda-Ext algebra \(\text{Ext}^\bullet_{Q(A)}(A, A)\), which we call the quasi-Poisson cohomology algebra of \(A\), is investigated. We construct a projective resolution of \(A\) as \(Q(A)\)-modules, which enables to compute the quasi-Poisson cohomologies in a standard way. To simplify calculation, we also introduce the quasi-Poisson complex and apply to obtain quasi-Poisson cohomologies in some special cases.

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Poisson algebra appears naturally in Hamiltonian mechanics and plays an important role in the study of Poisson geometry. Manifolds with a Poisson algebra structure are known as Poisson manifolds. To meet the development of the deformation quantization of Poisson manifolds, it is helpful to introduce certain deformation theory for Poisson algebras. For this, one needs a more general concept, say the associative product is not restricted to be commutative. Non-commutative Poisson algebra was firstly introduced by P.Xu [13], and has been investigated by many authors from different perspectives since its emergence. Note that there are different versions of non-commutative Poisson algebras other than the one introduced by Xu, see for instance, [2, Definition 1.1] and [12, Definition 2.6.1].

In this paper, we follow the definition as introduced in [3]. A \textbf{Poisson algebra} over a field \(K\) means a triple \((A, \cdot, \{-, -\})\), where \((A, \cdot)\) is an associative \(K\)-algebra and \((A, \{-, -\})\) is a Lie algebra over \(K\), such that the Leibniz rule \(\{a, bc\} = a \{b, c\} + b \{a, c\}\) holds for all \(a, b, c \in A\). This version of Poisson algebras has been widely investigated by many mathematicians recently, for examples, [10,11,13,15]. Notice that as an associative algebra,
A is not required to be commutative.

The (quasi-)Poisson modules over a Poisson algebra were introduced in a natural way in [5], see also [10]. In [14], the (quasi-)Poisson enveloping algebras for a Poisson algebra was introduced, and it was shown that the category of (quasi-)Poisson modules is equivalent to the category of modules over the (quasi-)Poisson enveloping algebra, see section 2 for detail. Consequently, the category of (quasi-)Poisson modules has enough projective and injective objects, which enables the construction of the cohomology theory for a Poisson algebra by using projective or injective resolutions. The present paper aims to develop the quasi-Poisson cohomology theory for a Poisson algebra $A$. The starting point is that $A$ itself is a quasi-Poisson module, and so that one can study its extension group in the category of quasi-Poisson modules.

Throughout $K$ will be a field of characteristic zero. All algebras considered are over $K$ and we write $\otimes = \otimes_K$ for brevity. The paper is organized as follows. Section 1 recalls basic definitions and notions of Poisson algebras and modules. In section 2, we introduce the quasi-Poisson cohomology groups for a Poisson algebra $A$. We construct a projective resolution of $A$ as a module over its quasi-Poisson enveloping algebra, which can be used to compute the quasi-Poisson cohomologies. Moreover, to simplify the calculation we introduce the quasi-Poisson complexes. As an application, we apply it to obtain some special quasi-Poisson cohomology groups in Section 3.

1 Preliminaries

In this paper, we assume that all associative algebras will have a multiplicative identity element.

Let $(A, \cdot, \{-,-\})$ be a Poisson algebra (not necessarily commutative). A quasi-Poisson $A$-module $M$ is an $A$-$A$-bimodule $M$ together with a $K$-bilinear map $\{-,-\}_*: A \times M \to M$, which satisfies

\begin{align}
(1.1) & \quad \{a, bm\}_* = \{a, b\}m + b\{a, m\}_* , \\
(1.2) & \quad \{a, mb\}_* = m\{a, b\} + \{a, m\}_* b , \\
(1.3) & \quad \{\{a, b\}, m\}_* = \{a, \{b, m\}_* \}_* - \{b, \{a, m\}_* \}_* ,
\end{align}

for all $a, b \in A$ and $m \in M$. Clearly, the condition (1.3) just says that $M$ is a Lie module over $A$. If moreover,

\begin{align}
(1.4) & \quad \{ab, m\}_* = a\{b, m\}_* + \{a, m\}_* b ,
\end{align}

then $M$ is a quasi-Poisson $A$-module.
holds for all \(a, b \in A\) and \(m \in M\), then \(M\) is called a **Poisson \(A\)-module**. Let \(M, N\) be quasi-Poisson modules (resp. Poisson modules). A homomorphism of quasi-Poisson \(A\)-module (resp. Poisson modules) is a \(\mathbb{K}\)-linear function \(f: M \rightarrow N\) which is a homomorphism of both \(A\)-\(A\)-bimodules and Lie modules.

Let us recall the definition of **(quasi-)Poisson enveloping algebra** of a Poisson algebra, see [14] for more detail. Before that, we need some convention.

Denote by \(A^{\text{op}}\) the **opposite algebra** of the associative algebra \(A\). Usually we use \(a\) to denote an element in \(A\) and \(a'\) its counterpart in \(A^{\text{op}}\) to show the difference. Let \(\mathcal{U}(A)\) be the **universal enveloping algebra** of the Lie algebra \(A\). Fix a \(\mathbb{K}\)-basis \(\{v_i | i \in S\}\) of \(A\), where \(S\) is an index set with a total ordering \(\leq\). Let \(\alpha = (i(1), \ldots, i(r)) \in S^r\) be a sequence of length \(r\) in \(S\). Usually we call \(r\) the **degree** of \(\alpha\). Denote the element \(v_{i(1)} \otimes \cdots \otimes v_{i(r)}\) by \(\alpha\). If \(i(1) \leq \cdots \leq i(r)\), then we call \(\alpha\) a **homogeneous element** of degree \(r\). The empty sequence, or the sequence of degree 0, is denoted by \(\emptyset\) and we write \(1 = 1_{\mathcal{U}(A)} = \overline{\emptyset}\) for brevity. Then all homogeneous elements of positive degrees together with \(1\) form a PBW-basis of \(\mathcal{U}(A)\). For given \(\alpha = (i(1), \ldots, i(r))\) and \(\beta = (j(1), \ldots, j(s))\), we define \(\alpha \vee \beta := (i(1), \ldots, i(r), j(1), \ldots, j(s))\), hence \(\overline{\alpha \vee \beta} = \overline{\alpha} \overline{\beta}\).

Set \(\underline{r} = \{1, \ldots, r\}\). By an **ordered bipartition** \(\underline{r} = X \sqcup Y\) of \(\underline{r}\), it is meant that \(X\) and \(Y\) are disjoint subsets of \(\underline{r}\) and \(\underline{r} = X \cup Y\), here "ordered" means that \(X \cup Y\) and \(Y \cup X\) give different bipartitions, which differs from the usual ones. Moreover, \(X\) and \(Y\) are allowed to be empty set. Let \(\alpha = (i(1), \ldots, i(r))\) and \(\underline{r} = X \cup Y\). Suppose \(X = \{X_1, \ldots, X_{|X|}\}\) and \(Y = \{Y_1, \ldots, Y_{|Y|}\}\) with \(X_1 < X_2 < \cdots < X_{|X|}\) and \(Y_1 < Y_2 < \cdots < Y_{|Y|}\). Set \(\alpha_X = (i(X_1), \ldots, i(X_{|X|}))\) and \(\alpha_Y = (i(Y_1), \ldots, i(Y_{|Y|}))\). By definition \(\alpha = \alpha_X \sqcup \alpha_Y\) is called an **ordered bipartition** of \(\alpha\) with respect to the ordered bipartition \(\underline{r} = X \cup Y\). Similarly, one defines **ordered \(n\)-partitions** \(\alpha = \alpha_1 \sqcup \alpha_2 \cdots \sqcup \alpha_n\) for any \(n \geq 2\).

It is well known that the category of Lie modules over \(A\) is equivalent to the category of left \(\mathcal{U}(A)\)-modules. Notice that \(\mathcal{U}(A)\) is a cocommutative Hopf algebra with the coproduct \(\Delta(\alpha) = \sum_{\alpha = \alpha_1 \sqcup \alpha_2} \overline{\alpha_1} \otimes \overline{\alpha_2}\), where the sum is taken over all possible ordered partitions of \(\{1, \ldots, \text{deg}(\alpha)\}\), and the counit given by \(\epsilon(1) = 1\), \(\epsilon(\alpha) = 0\) for any \(\overline{\alpha}\) of degree \(\geq 1\). The Lie bracket makes \(A\) a Lie module and hence a \(\mathcal{U}(A)\)-module with the action given by \(\overline{\alpha}(a) = \{v_{i(1)}, \{v_{i(2)}, \ldots, \{v_{i(r)}, a\}\}\}\) for \(\alpha = (i(1), \ldots, i(r)) \in S^r\) and \(a \in A\). Moreover, by the cocommutativity of \(\mathcal{U}(A)\), the enveloping algebra \(A^e = A \otimes A^{\text{op}}\) of \(A\) in the associative sense is a \(\mathcal{U}(A)\)-module algebra with the action \(\overline{\alpha}(a \otimes b') = \sum_{\alpha = \alpha_1 \sqcup \alpha_2} \overline{\alpha_1}(a) \overline{\alpha_2}(b')\) for all \(\alpha \in S^r\) with \(r \geq 0\) and \(a \otimes b' \in A^e\). Thus we have the following definition.

**Definition 1.1.** ([14]) Let \(A = (A, -,\cdot, -, -)\) be a Poisson algebra. The smash product \(A^e \# \mathcal{U}(A)\) is called the **quasi-Poisson enveloping algebra** of \(A\) and denoted by \(Q(A)\). The **Poisson enveloping algebra** of \(A\), denoted by \(\mathcal{P}(A)\), is the quotient algebra \(Q(A)/J\),
where $J$ is the ideal of $\mathcal{Q}(A)$ generated by \{1_A \otimes 1_A' \#(a \cdot b) - a \otimes 1_A' \#b - 1_A \otimes b' \#a \mid a, b \in A\}.

**Remark 1.2.** By definition, $\mathcal{Q}(A) = A \otimes A^\text{op} \otimes \mathcal{U}(A)$ as a $\mathbb{K}$-vector space. Thus $\mathcal{Q}(A)$ has a PBW-basis given by

\[
\{v_i \otimes v'_j \# \overrightarrow{a} \mid i, j \in S, \overrightarrow{a} = (i(1), \cdots, i(r)) \in S^r, i(1) \leq \cdots \leq i(r), r \geq 0\}.
\]

The multiplication is given by

\[
(v_{i_1} \otimes v'_{j_1} \# \overrightarrow{a})(v_{i_2} \otimes v'_{j_2} \# \overrightarrow{b}) = \sum_{\alpha = \overrightarrow{a}_1 \sqcup \overrightarrow{a}_2 \sqcup \overrightarrow{a}_3} (v_{i_1} \overrightarrow{a}_1(v_{i_2})) \otimes (v'_{j_1}(\overrightarrow{a}_2(v_{j_2}))) \#(\overrightarrow{a}_3 \overrightarrow{b})
\]

for $i_1, j_1, i_2, j_2 \in S$, $\overrightarrow{a} \in S^r$, $\overrightarrow{b} \in S^s$, $r, s \geq 0$, and the identity in $\mathcal{Q}(A)$ is $1_A \otimes 1_A' \#1$.

**Theorem 1.3.** (\cite{14}) The category of quasi-Poisson modules (resp. Poisson modules) is equivalent to the category of $\mathcal{Q}(A)$-modules (resp. $\mathcal{P}(A)$-modules).

Consequently, there are enough projectives and injectives in the category of quasi-Poisson modules, which make it possible to construct the cohomology theory for a Poisson algebra by using projective or injective resolutions in a standard way.

# 2 The quasi-Poisson Cohomology

It is easy to check that under the action $\{-, -\}_\ast = \{-, -\}$, the regular $A$-$A$-bimodule $A$ gives a quasi-Poisson module and hence a left module over $\mathcal{Q}(A)$. Then we may consider the Yoneda-Ext groups $\text{Ext}_{\mathcal{Q}(A)}^n(A, M)$ for any quasi-Poisson module $M$. Note that any quasi-Poisson module is identified with a left $\mathcal{Q}(A)$-module naturally.

**Definition 2.1.** Let $A$ be a Poisson algebra and $\mathcal{Q}(A)$ the quasi-Poisson enveloping algebra of $A$. For any quasi-Poisson module $M$, the extension group $\text{Ext}_{\mathcal{Q}(A)}^n(A, M)$ is called the $n$-th quasi-Poisson cohomology group of $A$ with coefficient in the quasi-Poisson module $M$, and denoted by $\text{HQ}^n(A, M)$.

**Remark 2.2.** The extension group $\text{HQ}^n(A, A)$ is simply denoted by $\text{HQ}^n(A)$. One may consider the Yoneda-Ext algebra $\text{HQ}^*(A) = \bigoplus_{n \geq 0} \text{HQ}^n(A)$ with the multiplication given by the Yoneda product. Clearly, $\text{HQ}^*(A)$ is a positively graded algebra and each $\text{HQ}^*(A, M)$ is a graded right $\text{HQ}^*(A)$-module.

## 2.1 The projective resolution of $\mathcal{Q}(A)A$

In the sequel, we will construct a projective resolution of $A$ as a $\mathcal{Q}(A)$-module, so that we can compute the cohomology groups $\text{Ext}_{\mathcal{Q}(A)}^n(A, M)$ in a standard way.
To simplify notation, for each $i, j \geq 0$, we denote by
\[ A^i = \underbrace{A \otimes A \otimes \cdots \otimes A}_{i \text{ folds}} , \quad \wedge^j = \underbrace{A \wedge A \wedge \cdots \wedge A}_{j \text{ folds}} \]
the $i$-th tensor product and $j$-th exterior power of the $\mathbb{K}$-space $A$ respectively.

Our construction is based on the following two well-known resolutions. One is
\[ S: \cdots \to A^{i+2} \xrightarrow{\delta_i} A^{i+1} \to \cdots \to A \otimes A \otimes A \xrightarrow{\delta_1} A \otimes A \xrightarrow{\delta_0} A \to 0, \]
the standard resolution of $A$ as an $A^e$-module($A$-$A$-bimodule), where
\[ \delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{i+1}) = \sum_{k=0}^{i} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{i+1}. \]
The other one is the Koszul resolution of $\mathbb{K}$ as a trivial $U(\mathbb{A})$-module, say
\[ C: \cdots \to U(\mathbb{A}) \otimes \wedge^j \xrightarrow{d_i} U(\mathbb{A}) \otimes \wedge^{j-1} \to \cdots \to U(\mathbb{A}) \otimes \wedge^1 \xrightarrow{d_1} U(\mathbb{A}) \to \mathbb{K} \to 0, \]
where $\epsilon$ is the counit map, i.e., $\epsilon(1) = 1$, and $\epsilon(\alpha) = 0$ for all $r > 0$ and $\alpha \in S^r$. The differential is given by
\[ d_j(\alpha \otimes v_1 \wedge \cdots \wedge v_j) = \sum_{l=1}^{j} (-1)^{l+1} (\alpha \otimes v_l) \otimes (v_1 \wedge \cdots \hat{v}_l \cdots \wedge v_j) + \sum_{1 \leq p < q \leq j} (-1)^{p+q} \alpha \otimes (\{v_p, v_q\} \wedge v_1 \wedge \cdots \hat{v}_p \cdots \hat{v}_q \wedge \cdots \wedge v_j), \]
where the symbol $\hat{v}_l$ indicates that $v_l$ is to be omitted.

Denote by $S'$ and $C'$ the deleted complexes of $S$ and $C$ respectively. Consider the double complex $S' \otimes C'$,
and obtain its total complex $Q' = \text{Tot}(S' \otimes C')$,
\begin{equation}
Q' : \cdots \rightarrow Q_n \xrightarrow{\varphi_n} Q_{n-1} \rightarrow \cdots \rightarrow Q_1 \xrightarrow{\varphi_1} Q_0 \rightarrow 0.
\end{equation}

To be precise, $Q_0 = A^2 \otimes U(A)$, and for $n \geq 1$,
\begin{align*}
Q_n &= \bigoplus_{i+j=n} A^{i+2} \otimes U(A) \otimes \wedge^j, \\
\varphi_n &= \bigoplus_{i+j=n} (\delta_i \otimes \text{id} + (-1)^i \text{id} \otimes d_j).
\end{align*}

The following lemmas will be handy for later use. Some of them seem to be well known to experts. For the convenience of the reader, we also include a proof.

**Lemma 2.3.** For any $n \geq 0$, $Q_n$ is a free module over $Q(A)$ with the action given by
\begin{align*}
(v_{i_1} \otimes v_{j_1}^{\#}) (v_1 \otimes \cdots \otimes v_i \otimes \cdot \cdot \cdot \otimes v_j \otimes \wedge^j \otimes \omega^j) := \\
\sum_{\alpha = a_{i_1}^{\cdots} \cdot \cdot \cdot a_{i_{n+1}}^{\alpha}} v_{i_1} \alpha_1^{a_{i_1}}(v_1) \otimes \cdots \otimes \alpha_{i-1}(v_{i-1}) \otimes v_{j_1}(v_j) \otimes \omega^j
\end{align*}
for all $v_{i_1} \otimes v_{j_1}^{\#} \in Q(A)$, $v_1 \otimes \cdots \otimes v_i \otimes \wedge^j \otimes \omega^j \in Q_n$.

**Proof.** Firstly, we show that $Q_n$ is a left $Q(A)$-module. It suffices to check that the equality
\begin{align*}
(a \otimes b')^{\#}((c \otimes d')^{\#})(v_{k(1)} \otimes \cdots \otimes v_{k(i)} \otimes \wedge^i \otimes \omega^j)) \\
= ((a \otimes b')^{\#})(c \otimes d')^{\#})(v_{k(1)} \otimes \cdots \otimes v_{k(i)} \otimes \wedge^i \otimes \omega^j)
\end{align*}
holds. In fact,
\begin{align*}
\text{LHS} &= (a \otimes b')^{\#}((c \otimes d')^{\#})(v_{k(1)} \otimes \cdots \otimes v_{k(i)} \otimes \wedge^i \otimes \omega^j)) \\
&= (a \otimes b')^{\#}((c \otimes d')^{\#})(v_{k(1)}) \otimes \cdots \otimes v_{k(i)}(v_{k(i)}) \otimes \wedge^i \otimes \omega^j) \\
&= \sum_{\alpha = a_{k(1)}^{\cdots} \cdot \cdot \cdot a_{k(i+1)}^{\alpha}} \alpha_1^{a_{k(1)}}(v_{k(1)}) \otimes \cdots \otimes \alpha_i^{a_{k(i)}}(v_{k(i)}) \otimes \wedge^i \otimes \omega^j
\end{align*}
By the Leibniz rule $\{a, bc\} = \{a, b\}c + b\{a, c\}$, we have
\begin{align*}
\alpha_1^{a_{k(1)}}(v_{k(1)}) &= \sum_{\alpha = \alpha_1^{\xi_1} \alpha_2} \xi_1^c((v_{k(1)}) \otimes \wedge^i \otimes \omega^j) \\
\alpha_i^d(v_{k(i)}) &= \sum_{\alpha = \alpha_1^{\xi_1} \alpha_2} ((v_{k(i)}) \otimes \wedge^i \otimes \omega^j) \xi_1^d
\end{align*}
Hence,

$$LHS = \sum_{\alpha=\xi_1\xi_2\cdots\xi_{i-1}\xi_i=1\xi_{i+1}1} a_1'(c)((\xi_2 \beta_2)(v_{k(1)}) \otimes \alpha_2(\beta_2(v_{k(2)})) \otimes \cdots \otimes \alpha_{i-1}(\beta_{i-1}(v_{k(i-1)})) \otimes ((\xi_2 \beta_2)(v_{k(i)})) \xi_1(d) b \otimes \alpha_i+1 \beta_i+1 \gamma \otimes \omega^j$$

$$RHS = ((a \otimes b' \# \overrightarrow{\alpha}')(c \otimes d' \# \beta})(v_{k(1)} \otimes \cdots \otimes v_{k(i)} \otimes \gamma \otimes \omega^j)$$

\[= \sum_{\alpha=\alpha_1\alpha_2\alpha_3} \alpha \alpha_1'(c) \otimes ((\alpha_2(d)b') \# \alpha_3 \beta))(v_{k(1)} \otimes \cdots \otimes v_{k(i)} \otimes \gamma \otimes \omega^j) \]

\[= \sum_{\alpha=\alpha_1\alpha_2\alpha_3} \alpha \alpha_1'(c) \beta_1(v_{k(1)}) \otimes \beta_2(v_{k(2)}) \otimes \cdots \otimes \beta_{i-1}(v_{k(i-1)}) \otimes \beta_i(v_{k(i)}) \alpha_2(d) b \otimes \beta_{i+1} \gamma \otimes \omega^j.\]

From the significance of the notation \(\sqcup\), we get

\[\sum_{\alpha_3 \beta = \beta_1 \sqcup \cdots \sqcup \beta_{i+1}} = \sum_{\alpha_3 = \xi_1 \sqcup \cdots \sqcup \xi_{i+1}}.\]

Then we have

$$RHS = \sum_{\alpha=\alpha_1\alpha_2\alpha_3} \alpha \alpha_1'(c)(\xi_1 \beta_1(v_{k(1)})) \otimes \xi_2 \beta_2(v_{k(2)}) \otimes \cdots \otimes \xi_{i-1} \beta_{i-1}(v_{k(i-1)}) \otimes \xi_i \beta_i(v_{k(i)}) \alpha_2(d) \alpha_3 \beta \# \xi_i+1 \beta_{i+1} \gamma \otimes \omega^j).$$

Comparing \(LHS\) with \(RHS\), we obtain the equality needed.

Next, we show that \(Q_n\) is free over \(Q(A)\) for each \(n\). Set \(Q_{ij} = A^{i+2} \otimes U(A) \otimes \gamma^j\). We claim that \(Q_{ij}\) is a free \(Q(A)\)-module with a basis

\[\left\{ 1_A \otimes v_{k(1)} \otimes \cdots \otimes v_{k(i)} \otimes 1_A \otimes 1 \otimes 1 \otimes v_{l(1)} \wedge \cdots \wedge v_{l(j)} \mid k(1), \ldots, k(i), l(1), \ldots, l(j) \in S, \quad l(1) \prec \cdots \prec l(j) \right\}.\]

Notice that there exists a PBW-like basis of \(Q(A)\) given by \(v_s \otimes v'_t \# \overrightarrow{\alpha}\), where \(s, t \in S\) and \(\overrightarrow{\alpha}\) is a homogeneous element of degree \(l\) in \(U(A)\). Following the notations in [14], we write \(\overrightarrow{\theta} = v_{k(1)} \otimes \cdots \otimes v_{k(i)}\) if \(\theta = (k(1), \cdots, k(i)) \in S^i\), and \(\overrightarrow{\omega} = v_{l(1)} \wedge \cdots \wedge v_{l(j)}\) if \(\omega = (l(1), \cdots, l(j))\) with \(l(1) \prec \cdots \prec l(j)\).

Assume that some \(Q(A)\)-linear combination equals to zero, that is,

\[\sum \lambda_{s,t,\alpha,\theta,\omega}(v_s \otimes v'_t \# \overrightarrow{\alpha})(1_A \otimes \overrightarrow{\theta} \otimes 1_A \otimes 1 \otimes \overrightarrow{\omega}) = 0,\]

where each \(v_s \otimes v'_t \# \overrightarrow{\alpha}\) is chosen to be in the PBW-basis. Let \(\alpha\) be with highest degree which appears in the sum. Moreover, each term in the left hand side is written as

\[(v_s \otimes v'_t \# \overrightarrow{\alpha})(1_A \otimes \overrightarrow{\theta} \otimes 1_A \otimes 1 \otimes \overrightarrow{\omega}) = \sum_{\alpha_1 \alpha_2} v_s \otimes \alpha_1(\overrightarrow{\theta}) \otimes v'_t \otimes \alpha_2 \otimes \overrightarrow{\omega} \]

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Lemma 2.5. The morphisms 

\[ \alpha_i = \beta_1 \cdots \beta_i \delta_1(v_{k(1)}) \otimes \cdots \otimes \delta_i(v_{k(i)}) . \]

Combining those terms containing \( \alpha \) in the resulting sum, we have

\[ \sum \lambda_{s,t,\alpha,\theta,\omega}(v_s \otimes \delta \otimes v'_t \otimes \alpha \otimes \omega) = 0. \]

Thus \( \lambda_{s,t,\alpha,\theta,\omega} = 0 \) for any \( s, t, \theta \) and \( \omega \), and our claim follows. \( \square \)

Remark 2.4. The corresponding quasi-Poisson action of \( A \) on \( A' \otimes U(A) \otimes \Lambda^j \) is given by

\[ \{a, v_1 \cdots \otimes v_i \otimes \delta \otimes \omega^j\}, \]

\[ := \sum_{k=1}^i v_1 \cdots \otimes \{a, v_k\} \cdots \otimes v_i \otimes \delta \otimes \omega^j + v_1 \cdots \otimes v_i \otimes (\alpha \otimes \delta) \otimes \omega^j. \]

Lemma 2.5. The morphisms \( \varphi_n \) in the total complex (2.1) are \( Q'(A) \)-homomorphisms.

Proof. Clearly, each \( \varphi_n \) in \( Q' \) is a direct sum of \( \left(\begin{array}{c}
\delta_i \otimes \text{id} \\
(-1)^i \text{id} \otimes d_j
\end{array}\right) \) by definition. It suffices to show that \( \delta_i \otimes \text{id} \) and \( \text{id} \otimes d_j \) are homomorphisms of \( Q(A) \) modules. Firstly, \( \delta_i \otimes \text{id} \) and \( \text{id} \otimes d_j \) are \( A^n \)-homomorphisms and hence

\[ (\delta_i \otimes \text{id})(a \otimes b' \# 1)x = (a \otimes b' \# 1)(\delta_i \otimes \text{id})(x), \]

\[ (\text{id} \otimes d_j)(a \otimes b' \# 1)x = (a \otimes b' \# 1)(\text{id} \otimes d_j)(x), \]

for all \( x \in A^{i+2} \otimes U(A) \otimes \Lambda^j \).

On the other hand, for any \( 1_A \otimes 1_A' \# \alpha \in Q(A) \),

\[ (\delta_i \otimes \text{id})(1_A \otimes 1_A' \# \alpha)(v_0 \cdots \otimes v_{i+1} \otimes \delta \otimes \omega^j) = (\delta_i \otimes \text{id})(\sum_{a=0}^{\alpha} \alpha_0(v_0) \cdots \otimes \alpha_{i+1}(v_{i+1}) \otimes \delta \otimes \omega^j) \]

\[ = (\sum_{0 \leq k \leq i} (-1)^k \alpha_0(v_0) \cdots \otimes \alpha_k(v_k) \cdots \otimes \alpha_{k+1}(v_{k+1}) \cdots \otimes \delta \otimes \omega^j) \]

\[ = (1_A \otimes 1_A' \# \alpha)(\sum_{0 \leq k \leq i} (-1)^k v_0 \cdots \otimes v_k v_{k+1} \cdots \otimes v_{i+1} \otimes \delta \otimes \omega^j) \]

\[ = (1_A \otimes 1_A' \# \alpha)((\delta_i \otimes \text{id})(v_0 \cdots \otimes v_{i+1} \otimes \delta \otimes \omega^j)) \]

By the definition of \( d_j \), it is easy to check that

\[ (\text{id} \otimes d_j)((1_A \otimes 1_A' \# \alpha)(v_0 \cdots \otimes v_{i+1} \otimes \delta \otimes \omega^j)) = (1_A \otimes 1_A' \# \alpha)((\text{id} \otimes d_j)(v_0 \cdots \otimes v_{i+1} \otimes \delta \otimes \omega^j)) \]
Since $a \otimes b' \# 1, 1_A \otimes 1'_A \# \delta' \# /BD$,
$1_A \otimes 1'_A \# \alpha$ are the generators of $Q(A)$, it follows that $\delta_i \otimes \text{id}$ and $\text{id} \otimes d_j$
are $Q(A)$-homomorphisms.

**Lemma 2.6.** Keeping the above notations, we have

\[ H_0(Q') \cong A, \quad \text{and} \quad H_n(Q') = 0, \ n \geq 1. \]

**Proof.** By Künneth’s theorem (see [9]), it is easy to see that $Q'$ is exact at $Q_n$ for each
$n \geq 1$, since both $S'$ and $C'$ are exact for $i, j > 0$, and $Q'$ is the total complex of $S' \otimes C'$.

For $n = 0$, again by Künneth’s theorem, $H_0(Q') \cong H_0(S') \otimes H_0(C') = A \otimes K \cong A$. □

Combining Lemma 2.3, Lemma 2.5 and Lemma 2.6, we obtain a projective resolution
of $A$ as a $Q(A)$-module.

**Theorem 2.7.** Let $A$ be a Poisson algebra, $Q(A)$ the quasi-Poisson enveloping algebra of
$A$, and $\varphi_0: Q_0 \rightarrow A$ the $Q(A)$-homomorphism given by $\varphi_0(a_0 \otimes a_1 \otimes \overline{\alpha}) = \epsilon(\overline{\alpha}) a_0 a_1$. Then
the sequence $Q'$ together with the map $\varphi_0$, say

\[ Q: \cdots \rightarrow Q_n \xrightarrow{\varphi_n} Q_{n-1} \rightarrow \cdots \rightarrow Q_1 \xrightarrow{\varphi_1} Q_0 \xrightarrow{\varphi_0} A \rightarrow 0, \tag{2.2} \]

is a projective resolution of $A$ as a $Q(A)$-module.

Let $M$ be a left $Q(A)$-module and hence a quasi-Poisson module over $A$. Applying the
functor $\text{Hom}_{Q(A)}(-, M)$ to the deleted complex $Q'$, we obtain a complex $\text{Hom}_{Q(A)}(Q', M)$:

\[ 0 \rightarrow \text{Hom}_{Q(A)}(Q_0, M) \rightarrow \text{Hom}_{Q(A)}(Q_1, M) \rightarrow \text{Hom}_{Q(A)}(Q_2, M) \rightarrow \cdots \]

\[ \rightarrow \text{Hom}_{Q(A)}(Q_n, M) \rightarrow \text{Hom}_{Q(A)}(Q_{n+1}, M) \rightarrow \cdots \]

By Theorem 2.7, the $n$-th quasi-Poisson cohomology group is calculated by

\[ HQ^n(A, M) = \text{Ext}^n_{Q(A)}(A, M) = H^n\text{Hom}_{Q(A)}(Q', M). \]

**2.2 The quasi-Poisson complex**

To compute the quasi-Poisson cohomology groups, one uses a simplified complex, the so-called
quasi-Poisson complex. Let $M$ be a quasi-Poisson module. Applying the functor
Hom$_{\mathcal{Q}(A)}(-, M)$ to the bicomplex $S' \otimes C'$, we obtain

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_{\mathcal{Q}(A)}(A^2 \otimes \mathcal{U}(A) \otimes \Lambda^2, M) & \to & \text{Hom}_{\mathcal{Q}(A)}(A^3 \otimes \mathcal{U}(A) \otimes \Lambda^2, M) & \to & \cdots \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_{\mathcal{Q}(A)}(A^2 \otimes \mathcal{U}(A) \otimes \Lambda, M) & \to & \text{Hom}_{\mathcal{Q}(A)}(A^3 \otimes \mathcal{U}(A) \otimes \Lambda, M) & \to & \cdots \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_{\mathcal{Q}(A)}(A^2 \otimes \mathcal{U}(A), M) & \to & \text{Hom}_{\mathcal{Q}(A)}(A^3 \otimes \mathcal{U}(A), M) & \to & \cdots \\
\end{array}
\]

Following from the natural $\mathbb{K}$-isomorphisms

\[
\Phi_{i,j}: \text{Hom}_{\mathcal{Q}(A)}(A^{i+2} \otimes \mathcal{U}(A) \otimes \Lambda^j, M) \xrightarrow{\sim} \text{Hom}_{\mathbb{K}}(A^i \otimes \Lambda^j, M),
\]

\[
\Phi_{i,j}(f)((a_1 \otimes \cdots \otimes a_i) \otimes (x_1 \wedge \cdots \wedge x_j)) = f(1_{A^i} \otimes (a_1 \otimes \cdots \otimes a_i) \otimes 1_\Lambda \otimes 1 \otimes (x_1 \wedge \cdots \wedge x_j)),
\]

the above bicomplex is isomorphic to the bicomplex $\text{Hom}_{\mathbb{K}}(A^\bullet \otimes \Lambda^\bullet, M)$:

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_\mathbb{K}(\Lambda^2, M) & \to & \text{Hom}_\mathbb{K}(A \otimes \Lambda^2, M) & \xrightarrow{\sigma^2_1} & \text{Hom}_\mathbb{K}(A^2 \otimes \Lambda^2, M) & \to & \cdots \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_\mathbb{K}(\Lambda, M) & \to & \text{Hom}_\mathbb{K}(A \otimes \Lambda, M) & \xrightarrow{\sigma^1_1} & \text{Hom}_\mathbb{K}(A^2 \otimes \Lambda, M) & \to & \cdots \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & \to & M & \to & \text{Hom}_\mathbb{K}(A, M) & \xrightarrow{\sigma^0_1} & \text{Hom}_\mathbb{K}(A^2, M) & \to & \cdots \\
\end{array}
\]

where

\[
(\sigma^j_i(f))(a_1 \otimes \cdots \otimes a_{i+1}) \otimes (x_1 \wedge \cdots \wedge x_j)
\]

\[
= a_1 f((a_2 \otimes \cdots \otimes a_i) \otimes (x_1 \wedge \cdots \wedge x_j))
\]

\[
+ \sum_{k=1}^{i} (-1)^k f((a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{i+1}) \otimes (x_1 \wedge \cdots \wedge x_j))
\]

\[
+ (-1)^{i+1} f((a_1 \otimes \cdots \otimes a_i) \otimes (x_1 \wedge \cdots \wedge x_j)) a_{i+1},
\]

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\begin{align*}
&\sigma_{V}(f)\left((a_1 \otimes \cdots \otimes a_i) \otimes (x_1 \wedge \cdots \wedge x_{j+1})\right) \\
&= \sum_{i=1}^{j+1} (-1)^{i+1} \left\{x_i, f((a_1 \otimes \cdots \otimes a_i) \otimes (x_1 \wedge \cdots \wedge x_{j+1}))\right\}_{*} \\
&\quad - \sum_{i=1}^{i} f((a_1 \otimes \cdots \otimes \{x_i, a_i\} \otimes \cdots \otimes a_i) \otimes (x_1 \wedge \cdots \wedge x_{j+1})) \\
&\quad + \sum_{1 \leq p < q \leq j+1} (-1)^{p+q} f((a_1 \otimes \cdots \otimes a_i) \otimes \{x_p, x_q\} \wedge x_1 \wedge \cdots \wedge x_{j+1})
\end{align*}

for all \( f \in \text{Hom}_{K}(A^i \otimes \wedge^j, A) \), and \((a_1 \otimes \cdots \otimes a_i) \otimes (x_1 \wedge \cdots \wedge x_j) \in A^i \otimes \wedge^j, i, j \geq 0\).

**Remark 2.8.** Write \( \delta^n = \sigma_{H}^{n,0} \) and \( d^n = \sigma_{V}^{0,n} \) for each \( n \geq 0 \). Clearly, the bottom row is the Hochschild complex \((\text{Hom}_{K}(A^i \otimes \wedge^j, M), \delta^n)\), and the leftmost column \((\text{Hom}_{K}(\wedge^i, M), d^n)\) calculates the Lie algebra cohomology \(\text{Ext}^n_{\mathfrak{g}(A)}(K, M)\).

**Definition 2.9.** Let \( A \) be a Poisson algebra and \( M \) a quasi-Poisson module. The total complex of \( \text{Hom}_{K}(A^i \otimes \wedge^j, M) \), say

\[
\begin{array}{c}
0 \rightarrow M \rightarrow \text{Hom}_{K}(A^i \otimes \wedge^j, M) \\
\rightarrow \text{Hom}_{K}(\bigoplus_{i+j=n} A^i \otimes \wedge^j, M) \rightarrow \cdots \\
\end{array}
\]

is called the quasi-Poisson complex of \( A \) with coefficients in \( M \), and denoted by \( QC(A, M) \).

An immediate consequence follows.

**Proposition 2.10.** The quasi-Poisson complex is isomorphic to the complex \( \text{Hom}_{\mathbb{Q}(A)}(\mathbb{Q}^{'}, M) \), and hence \( \text{H}^n(QC(A, M)) = \text{H}^n(A, M) \).

### 3 Applications and Examples

#### 3.1 Lower-dimensional quasi-Poisson cohomologies

First examples are lower dimensional quasi-Poisson cohomology groups of a Poisson algebra \((A, -, \{-, -\})\). We denote by \( Z(A) \) and \( Z\{A\} \) the center of the associative algebra and the one of the Lie algebra, respectively. Then we have the following easy result.

**Proposition 3.1.** Keep the above notation. Then \( \text{H}^0(A) = Z(A) \cap Z\{A\} \).
Denote by $\text{Der}(A)$ and $\text{Der}_L(A)$ the $\mathbb{K}$-space of associative derivations and the space of Lie derivations respectively. Consider the maps $\text{ad}: A \to \text{Der}(A)$ and $\text{ad}_L: A \to \text{Der}_L(A)$ given by $\text{ad}(a) = [-, a]$ and $\text{ad}_L(a) = \{-, a\}$ for all $a \in A$. Under these notations, the differential $\sigma^0 = (\text{ad}, \text{ad}_L)$.

Moreover, for any $f = (f_1, f_0) \in \text{Ker} \sigma^1$, by Proposition 2.10 we know that $f_1 \in \text{Der}(A)$ and $f_0 \in \text{Der}_L(A)$ and the equality
\[
f_1(\{x, a\}) - \{x, f_1(a)\} = f_0(x)a - af_0(x)
\]
holds for any $(a, x) \in A \oplus \wedge^1$. Now set
\[
D(A) = \{(f_1, f_0) \in \text{Der}(A) \oplus \text{Der}_L(A) \mid (3.1) \text{ holds for all } a, x \in A\}.
\]
Thus $\text{HQ}^1(A)$ is computed as follows by definition.

**Proposition 3.2.** Keep the above notations. Then we have $\text{HQ}^1 = D(A)/\text{Im}\sigma^0$ and hence
\[
\dim_\mathbb{K} \text{HQ}^1(A) = \dim_\mathbb{K} D(A) - \dim_\mathbb{K} A + \dim_\mathbb{K} \text{HQ}^0(A).
\]

### 3.2 Standard Poisson algebras

Let $A$ be an associative algebra. For any $a, b \in A$, we denote by $[a, b]$ the commutator $ab - ba$ of $a$ and $b$. Then $(A, \cdot, [-, -])$ is a Poisson algebra, which is called a **standard Poisson algebra**. By Proposition 3.1 we have $\text{HQ}^0(A) = Z(A)$.

More generally, $\text{HQ}^0(A) = Z(A)$ for any inner Poisson algebra since $Z(A) \subset Z\{A\}$ in this case, see Lemma 1.1 in [15]. Recall that a Poisson algebra $(A, \cdot, \{-, -\})$ is said to be inner if $\{a, -\}$ is an inner derivation of $(A, \cdot)$ for any $a \in A$.

Now we turn to $\text{HQ}^1$. Note that in standard case, the equality (3.1) is equivalent to
\[
\text{Im}(f_0 - f_1) \subseteq Z(A),
\]
which holds if and only if $f_1 = f_0 + g$ for some Lie derivation $g$ satisfying $\text{Im} g \subseteq Z(A)$. Since $g([x, y]) = [g(x), y] + [x, g(y)]$, we have $\text{Ker}(g) \supseteq [A, A]$, thus $g$ is obtained from some $\tilde{g} \in \text{Hom}_\mathbb{K}(A/[A, A], Z(A))$. Conversely, each $\tilde{g} \in \text{Hom}_\mathbb{K}(A/[A, A], Z(A))$ gives to a Lie derivation $g$ with $\text{Im} g \subseteq Z(A)$. Thus we have the following characterization.

**Lemma 3.3.** Let $A$ be a standard Poisson algebra. Then
\[
\text{HQ}^1(A) \cong \text{HH}^1(A) \oplus \text{Hom}_\mathbb{K}(A/[A, A], Z(A)),
\]
here $\text{HH}$ denotes the Hochschild cohomology of the associative algebra $A$. 

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In general, quasi-Poisson cohomology groups of higher degrees are difficult to compute, while some special cases can be calculated explicitly.

Example 3.4. Let $A$ be the $\mathbb{K}$-algebra of upper triangular $2 \times 2$ matrices. This algebra is known to be the path algebra of the quiver of $A_2$ type. More explicitly, $A$ has a basis $\{e, f, \alpha\}$, and the multiplication is given by $e^2 = f^2 = ef = fe = \alpha e = f\alpha = 0$ and $e\alpha = \alpha f = \alpha$. Clearly $1_A = e + f$.

Consider the standard Poisson algebra. By direct computation, one shows that as a graded algebra, $\text{HQ}^*(A) \cong \mathbb{K}\langle x, y \rangle / \langle x^2, y^2, xy + yx \rangle$, the exterior algebra in 2 variables.

Example 3.5. Consider the standard Poisson algebra of $A = M_2(\mathbb{K})$, the $\mathbb{K}$-algebra of $2 \times 2$ matrices. Again direct calculation shows that $\text{HQ}^0(A) = \text{HQ}^1(A) = \text{HQ}^3(A) = \text{HQ}^4(A) = \mathbb{K}$ and $\text{HQ}^i = 0$ for $i \neq 0, 1, 3, 4$.

3.3 Poisson algebras with trivial Lie bracket

Let $(A, \cdot, \{\cdot, \cdot\})$ be a finite-dimensional Poisson algebra with trivial Lie structure, i.e. $\{a, b\} = 0$ for any $a, b \in A$. Clearly, $Q(A) = A \otimes A^{\text{op}} \otimes U(A)$ and $U(A) \cong S(A)$, where $S(A)$ is the polynomial algebra of the vector space $A$.

One shows easily that as a $Q(A)$-module, $A$ is the tensor product of the $A \otimes A^{\text{op}}$-module $A$ and the trivial Lie module $\mathbb{K}$ over $A$. Then by a classical result in homological algebra, $\text{HQ}^*(A) \cong \text{HH}^*(A) \otimes \text{Ext}_{S(A)}^*(\mathbb{K}, \mathbb{K})$; see for instance, Theorem 3.1 in [1], Chapter XI. By Koszul duality, $\text{Ext}_{S(A)}^*(\mathbb{K}, \mathbb{K}) \cong \wedge A$, the exterior algebra of the vector space $A$. Thus we have the following result.

Proposition 3.6. Let $(A, \cdot, \{\cdot, \cdot\})$ be a finite-dimensional Poisson algebra with trivial Lie bracket. Then $\text{HQ}^*(A) \cong \text{HH}^*(A) \otimes \wedge A$.

3.4 Poisson algebras with finite Hochschild cohomology dimension

Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra. Suppose the associative algebra $A$ has finite Hochschild cohomology dimension, that is, the $n$-th Hochschild cohomology group of $(A, \cdot)$ vanishes for sufficiently large $n$.

Proposition 3.7. Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra and $k$ a fixed positive integer. Suppose $\text{HH}^n(A) = 0$ for all $n > k$. Set $\Omega^k_n = \text{Hom}_K(\bigoplus_{i+j=n, i \leq k} A^i \otimes \wedge^j, A)$. Then the $n$-th quasi-Poisson cohomology group

$$\text{HQ}^n(A) = \frac{\text{Ker}\sigma^n \cap \Omega^k_n}{\text{Im}\sigma^{n-1} \cap \Omega^k_n}.$$
Proof. Consider the $\mathbb{K}$-homomorphism $\pi : \text{Ker}\sigma^n \cap \Omega^k_n \to \text{HQ}^n(A)$, $f \mapsto f + \text{Im}\sigma^{n-1}$. Suppose $f = (f_n, \cdots, f_1, f_0) \in \text{Ker}\sigma^n$ for some $n > k$. By definition $f_n$ is an $n$-th cocycle in the Hochschild complex, and hence there exists some $g_{n-1} \in \text{Hom}_\mathbb{K}(A^{n-1}, A)$ such that $f_n = \delta^{n-1}g_{n-1}$ since the $n$-th Hochschild cohomology group vanishes, where $\delta^n$ is the $\mathbb{K}$-linear map in the Hochschild complex. Clearly, $\overline{f} = \overline{f_n} - \sigma^n g \in \text{HQ}^n(A)$. Thus,

$$f - \sigma^n g = (0, \tilde{f}_{n-1}, \tilde{f}_{n-2}, \cdots, \tilde{f}_0),$$

For brevity, we still denote $\tilde{f}_{n-1}$ by $f_{n-1}$. Therefore

$$a_1 f_{n-1}(a_2 \otimes \cdots \otimes a_n \otimes x) + \sum_{k=1}^{n-1} f_{n-1}(a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n \otimes x) + (-1)^{n+1} f_{n-1}(a_1 \otimes \cdots \otimes a_{n-1} \otimes x)a_n = 0$$

If $n-1 > k$, consider the $\mathbb{K}$-linear isomorphism $\text{Hom}_\mathbb{K}(A^{n-1} \otimes \wedge^1, A) \to \text{Hom}_\mathbb{K}(A^{n-1}, A) \otimes A^*$ such that $f_{n-1} \mapsto f'_{n-1} \otimes f''_1$, where $A^*$ is the dual $\mathbb{K}$-vector space of $\wedge^1 = A$. Clearly, we have $f'_{n-1} \in \text{Ker}\delta^{n-1}$. By the assumption $\text{HH}^{n-1}(A) = 0$, there exists $g'_{n-2} \in \text{Hom}_\mathbb{K}(A^{n-2}, A)$ such that $f'_{n-1} = \delta^{n-2}(g'_{n-2})$. Suppose $g_{n-2} = g'_{n-2} \otimes f''_1 \in \text{Hom}_\mathbb{K}(A^{n-2}, A) \otimes A^* \cong \text{Hom}_\mathbb{K}(A^{n-2} \otimes \wedge^1, A)$ and $g = (0, g_{n-2}, 0, \cdots, 0)$, then we have $f_{n-1} = (\sigma^{n-1}(g))_{n-1}$ and

$$\overline{f} = \overline{f_n} - \sigma^{n-1}(g) \in \text{HQ}^{n-1}(A).$$

Clearly,

$$f - \sigma^{n-1}(g) = (0, 0, \tilde{f}_{n-2}, \tilde{f}_{n-3}, \cdots, \tilde{f}_0)$$

Denote still $\tilde{f}_{n-2} = f_{n-2}$.

Repeat the above argument, we know that each $f \in \text{HQ}^n(A)$ can be written as

$$\overline{f} = (0, \cdots, 0, f_k, \cdots, f_0).$$

Therefore, the $\mathbb{K}$-homomorphism $\pi$ is surjective. Clearly, $\text{Ker}(\pi) = \Omega^k_n \cap \text{Im}\sigma^{n-1}$, and hence

$$\text{HQ}^n(A) = \frac{\text{Ker}\sigma^n \cap \Omega^k_n}{\text{Im}\sigma^{n-1} \cap \Omega^k_n}.$$ 

\[\square\]

Corollary 3.8. Let $(A, \cdot, \{-,-\})$ be a Poisson algebra over $\mathbb{K}$ with $\text{HH}^i(A) = 0$ for all $i > 0$. Then $\text{HQ}^n(A) \cong HL^p(A, Z(A))$. Moreover, we have $\text{HH}^* = Z(A) \otimes \text{HL}^*(A)$ as graded algebras, where $\text{HL}^*(A) = \bigoplus_{i \geq 0} \text{Ext}_A^i(\mathbb{K}, \mathbb{K})$ is the Lie algebra cohomology of $A$ with coefficients in the trivial Lie module $\mathbb{K}$. 

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Proof. Recall that in Remark 2.8 we have defined a complex \((\text{Hom}_K(\wedge^*, M), d^n)\) for any quasi-Poisson module \(M\), which is by definition the leftmost column of the bicomplex \(\text{Hom}_K(A^* \otimes \wedge^*, M)\).

To compute \(HQ^*(A)\), we need only to consider the quasi-Poisson complex \(QC(A, A)\) and the subcomplex \((\text{Hom}_K(\wedge^*, A), d^n)\). We claim that for each \(n\),

\[
HQ^n(A) = \frac{\text{Ker} d^n \cap \text{Hom}_K(\wedge^n, Z(A))}{\text{d}^{n-1}(\text{Hom}_K(\wedge^{n-1}, Z(A))},
\]

notice that we view \(\text{Hom}_K(\wedge^n, Z(A))\) as a subspace of \(\text{Hom}_K(\wedge^n, A)\) here.

Firstly we have an isomorphism of \(K\)-vector spaces,

\[
\Omega^n \cap \text{Ker} \sigma^n \cong \text{Ker} d^n \cap \text{Hom}_K(\wedge^n, Z(A)), \quad (0, \cdots, 0, f_0) \mapsto f_0.
\]

In fact, \(f = (0, \cdots, 0, f_0) \in \text{Ker} \sigma^n\) if and only if \(f_0 \in \text{Ker} d^n\) and \(\text{Im} f_0 \subseteq Z(A)\).

Next, it is easy to show that there is a well-defined \(K\)-linear map

\[
d^{n-1}(\text{Hom}_K(\wedge^{n-1}, Z(A)) \rightarrow \Omega^n \cap \text{Im} \sigma^{n-1}, f_0 \mapsto (0, \cdots, 0, f_0),
\]

which is obviously injective. We show that the map is also surjective as follows. For any \(f = (0, \cdots, 0, f_0) \in \Omega^n \cap \text{Im} \sigma^{n-1}\), there exists some \(g \in \text{Hom}_K(\bigoplus_{i+j=n-1} A^i \otimes \wedge^j, A)\) such that \(f = \sigma^{n-1} g\). From the proof of Proposition 3.7 there exists some

\[
g' = (0, \cdots, 0, g'_0) \in \text{Hom}_K(\bigoplus_{i+j=n-1} A^i \otimes \wedge^j, A),
\]

such that \(g' - g \in \text{Im} \sigma^{n-2}\). Thus \(g'_0 \in \text{Hom}_K(\wedge^{n-1}, Z(A))\) and \(d^{n-1}(g'_0) = d^{n-1}g_0 = f_0\).

Now the claim follows from Proposition 3.7. Thus we have shown that as an abelian group, \(HQ^*(A)\) is computed by the cohomology groups of the complex \((\text{Hom}_K(\wedge^*, Z(A), d^n)\).

Since the Lie algebra \(A\) acts trivially on \(Z(A)\), it is direct to show the isomorphism of complexes \(\text{Hom}_K(\wedge^*, Z(A)) \cong \text{Hom}_K(\wedge^*, K) \otimes Z(A)\). It follows that \(HQ^*(A) \cong Z(A) \otimes \text{HL}^*(A)\) as abelian groups as well as \(Z(A)\)-modules, and \(Z(A)\) acts centrally on \(HQ^*(A)\). The rest part follows by comparing the Yoneda product.

Example 3.9. Let \(Q\) be a finite quiver with underlying graph being a tree. Denote by \(KQ\) the path algebra of \(Q\). Then we have \(\text{HH}^i(KQ) = 0\) for any \(i \geq 1\), see Section 1.6 in [7]. Consider the standard Poisson algebra of \(KQ\), by Proposition 3.7 and it is immediate that \(HQ^n(KQ) = \text{HL}^n(KQ)\), the usual \(n\)-th Lie algebra cohomology group of \((A, [-, -])\) with coefficients in \(K\).

Remark 3.10. We have shown in Proposition 3.6 and Corollary 2.8 that in two extreme cases, say either \(A\) has trivial Lie bracket or trivial Hochschild cohomologies, there is an
isomorphism of graded algebras $HQ^*(A) \cong HH^*(A) \otimes HL^*(A)$. Naively one may expect that for general $A$, the algebra $HQ^*(A)$ is still given by certain twisted tensor product of $HH^*(A)$ and $HL^*(A)$. Up to now, few results in this direction are known.

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