Excitations of isolated static charges in the charge $q = 2$ abelian Higgs model

Kazue Matsuyama

Physics and Astronomy Department
San Francisco State University
San Francisco, CA 94132, USA
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We present lattice Monte Carlo evidence of stable excitations of isolated static charges in the Higgs phase of the charge $q = 2$ abelian Higgs model. These localized excitations are excited states of the interacting fields surrounding the static charges. Since the $q = 2$ abelian Higgs model is a relativistic version of the Landau-Ginzburg effective action of a superconductor, we conjecture that excited states of this kind might be relevant in a condensed matter context.

Physical states in gauge field theories are gauge invariant and this property implies that a static charge is necessarily accompanied by a surrounding field. This could be a Coulomb field extending to infinity, as in free field electrodynamics, or the charge of the state could be neutralized in some way by other charged dynamical fields. In an interacting theory in which the surrounding field interacts with itself, there could in principle be a spectrum of localized quantum excitations of the surrounding field. This is certainly true for a static quark-antiquark pair in the confining phase of a pure gauge theory. In that case the color electric field associated with the pair of color charges is collimated into a flux tube, and that flux tube can exist in a number of vibrational modes, as has been shown in various lattice Monte Carlo simulations. By contrast, in free electrodynamics, any disturbance of the field surrounding a static charge can be viewed as the creation of some set of photons superimposed on a Coulombic background. In that case there are no stable (or metastable) localized excitations.

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Recently Greensite has shown that there is indeed a spectrum of localized excitations around an isolated fermion in SU(3) gauge Higgs theory, in the Higgs phase of the theory in four spacetime dimensions. However, we do not know of any system in nature described by such a theory, and an extension of those results to the SU(2)×U(1) electroweak theory is problematic, due to the chiral nature of the gauge symmetry. Our interest here to see if such localized excitations of isolated charges are in any way relevant to condensed matter systems, most especially to superconductors, whose effective low energy theory, i.e. the Landau-Ginzburg model, is an abelian example of a gauge Higgs theory. In this letter we will show that stable localized excitations of the massive photon and Higgs fields surrounding a static charge can in fact exist in a relativistic version of the Landau-Ginzburg model, namely the charge $q = 2$ abelian Higgs model, in at least some regions of the phase diagram. Our letter is limited to this result in the relativistic model; the application to a more realistic model of superconductivity, and to superconductors themselves, is a topic which we defer to later work.

Our strategy is to compute, via lattice Monte Carlo simulations, the energy (above the vacuum) of the ground state containing two static sources of opposite charge, and the energy of a certain excited state of this charge pair whose construction we describe. If the difference in energies is less than the photon mass, then the excited state is stable. This is what we will show below.

Our starting point is the lattice action of the abelian Higgs model

$$S = -\beta \sum_{\mu \text{ plaq}} \Re (U_{\mu}(x)U_{\nu}(x + \hat{\mu})U_{\mu}^{\ast}(x + \hat{\nu})U_{\nu}^{\ast}(x))$$

$$-\gamma \sum_{x, \mu} \Re [\phi^{\ast}(x)U_{\mu}(x)\phi(x + \hat{\mu})].$$

(1)

Here the scalar field has charge $q = 2$ (as do Cooper pairs), and for simplicity we impose a unimodular constraint, $\phi^{\ast}(x)\phi(x) = 1$, corresponding to the $\lambda \to \infty$ limit of a Mexican hat potential $\lambda (\phi \phi^{\ast} - \gamma)^2$, followed by a rescaling to $|\phi| = 1$. We then consider physical states containing a static fermion and anti-fermion at sites $x, y$, each of $\pm 2$ units of electric charge, of the form

$$|\Phi_{\alpha}(R)\rangle = Q_{\alpha}(R)|\Psi_{0}\rangle,$$

(2)

where $\Psi_{0}$ is the vacuum state and

$$Q_{\alpha}(R) = [\nabla(x)\zeta_{\alpha}(x)] \times [\zeta_{\alpha}^{\ast}(y)\psi(y)].$$

(3)

Here the $\nabla, \psi$ are operators creating double-charged static fermions of opposite charge, transforming as $\psi(x) \to e^{2i\theta(x)}\psi(x)$, and the $\{\zeta_{\alpha}(x)\}$ are a set of operators, which may depend on some (possibly non-local) combination of the Higgs and gauge fields, also transforming as $\zeta(x) \to e^{2i\theta(x)}\zeta(x)$, under a gauge transformation $U_{\mu}(x) \to \exp(i\theta(x))U_{\mu}(x)\exp(-i\theta(x + \hat{\mu}))$. One possible choice for $\zeta$ is the Higgs field $\phi(x)$. Another set is provided by eigenstates $\zeta = \zeta_{\alpha}$ of the covariant Laplacian, where

$$(-D_{i}D_{i})_{xy} \zeta_{\alpha}(y, U) = \lambda_{\alpha} \zeta_{\alpha}(x; U)$$

(4)
and
\[
(-D_{ij}D_{jk}) = \sum_{k=1}^{3} [2\delta_{kj} - U_{k}^{2}(x)\delta_{j,k} - U_{k}^{2}(x)\delta_{j,k}].
\]

Because the covariant Laplacian depends only on the squared link variable, the \(\xi_{\alpha}(x; U)\), which we have elsewhere referred to as “pseudomatter” fields \[2\], transform like \(q = 2\) charged matter fields. Pseudomatter fields depend nonlocally on the gauge fields, and the low-lying eigenstates and eigenvalues of the covariant Laplacian, which is a sparse matrix, can be computed numerically via the Arnoldi algorithm \[8\]. In our calculation we make use of the two lowest-lying eigenstates,\(\xi_{3,2} = \phi\) as the scalar (or “Landau-Ginzburg,” or “Higgs”) field. In general the three states \(\Phi_{\alpha}(R)\) are non-orthogonal at finite \(R\).

We express the operator \(Q_{\alpha}\) in eq. \(3\) in terms of a non-local operator \(V_{\alpha}(x, y; U)\)
\[
Q_{\alpha}(R) = \Psi(x)V_{\alpha}(x, y; U)\Psi(y)
\]
\[
V_{\alpha}(x, y; U) = \xi_{\alpha}(x; U)\xi_{\alpha}^{*}(y; U),
\]
and also define \(T = e^{-(H - \delta_{0})T}\) as the Euclidean time evolution operator of the lattice abelian Higgs model. This is the operator corresponding to the transfer matrix, multiplied by a constant \(e^{\delta_{0}}\) where \(\delta_{0}\) is the vacuum energy, evolving states for one unit of discretized time. Let
\[
[Q]_{\alpha\beta} = \langle \Phi_{\alpha}|e^{-(H - \delta_{0})T}|\Phi_{\beta}\rangle = \langle Q_{\alpha}^{\dagger}(R, 1)Q_{\beta}(R, 0)\rangle
\]
\[
[O]_{\alpha\beta} = \langle \Phi_{\alpha}|\Phi_{\beta}\rangle = \langle Q_{\alpha}^{\dagger}(R, 0)Q_{\beta}(R, 0)\rangle
\]

(7)
denote matrix elements of \(T\), in the three non-orthogonal states \(\Phi_{\alpha}\), with \([O]\) the matrix of overlaps of such states. We obtain the three orthogonal eigenstates of \(T\) in the subspace spanned by the \(\Phi_{\alpha}\) by solving the generalized eigenvalue problem
\[
T_{\alpha\beta}v_{\beta}^{(n)} = \lambda_{n}O_{\alpha\beta}v_{\beta}^{(n)},
\]
and we have energy expectation values above the vacuum energy given by
\[
E_{n}(R, 1) = -\log(\epsilon_{n}),
\]
ordered such that \(E_{n}\) increases with \(n\), corresponding to eigenstates in the subspace
\[
\Psi_{n}(R) = \sum_{\alpha=1}^{3} v_{\alpha}^{(n)}\Phi_{\alpha}(R).
\]

We then consider evolving the states \(\Psi_{n}\) in Euclidean time
\[
\mathcal{T}_{n}(R, T) = \langle \Psi_{n}|e^{-(H - \delta_{0})T}|\Psi_{n}\rangle
\]
\[
= v_{\alpha}^{(n)}\langle \Phi_{\alpha}|e^{-(H - \delta_{0})T}|\Phi_{\beta}\rangle v_{\beta}^{(n)}
\]
\[
= v_{\alpha}^{(n)}\langle Q_{\alpha}^{\dagger}(R, T)Q_{\beta}(R, 0)\rangle v_{\beta}^{(n)},
\]
(11)
where Latin indices indicate matrix elements with respect to the \(\Psi_{n}\) rather than the \(\Phi_{\alpha}\), and there is a sum over repeated Greek indices.

To calculate this expression, we first define timelike Wilson lines of length \(T\)
\[
P(x, t, T) = U_{0}(x, t)U_{0}(x, t + 1)...U_{0}(x, t + T - 1).
\]
(12)

After integrating out the massive fermions, whose worldlines lie along timelike Wilson lines, we have
\[
\langle Q_{\alpha}^{\dagger}(R, T)Q_{\beta}(R, 0)\rangle = \langle \text{Tr}[V_{\alpha}(x, y; U(t + T))P_{\alpha}(x, t, T)V_{\beta}(x, y; U(t))P_{\beta}(y, t, T)]\rangle.
\]
(13)

In principle,
\[
E_{n}(R, T) = -\log\left[\frac{\mathcal{T}_{n}(R, T)}{\mathcal{T}_{n}(R, T - 1)}\right],
\]
(14)
can be regarded (for \(T\) an odd integer) as the energy expectation value of a state \(\Psi_{n}(R)\) which has evolved for \((T - 1)/2\) units of Euclidean time. We are interested in finding this energy in the large \(T\) limit, which in practice we extract by fitting \(\mathcal{T}_{n}(R)\) to an exponential \(e^{-E_{n}(R)T}\).

Of course one might expect that the \(E_{n}(R, T)\) will all rapidly converge, in Euclidean time \(T\), to the ground state energy. This will be true for all \(n\) unless one or more of the \(\Psi_{n}(R)\), constructed as just described, has only a very small overlap with the true ground state. In that case \(E_{n}(R, T)\) will converge to the energy of one of the excited states, at least for some moderate range of \(T\).

We proceed to the numerical results. The phase diagram of the \(q = 2\) abelian Higgs model was first obtained from a lattice Monte Carlo simulation by Ranft et al in \[10\], and more recently and accurately by Greensite and the author in \[11\], with the result shown in Fig. 1. We are interested in determining \(E_{n}(R)\) in the Higgs phase, and, because the calculation involves fitting exponential decay, we would like both the mass of the photon and the energies \(E_{n}(R)\) to be not much larger than unity in lattice units. For this reason we choose to work at the edge of the phase diagram shown in Fig. 1 just above the massless-to-Higgs transition line at \(\beta = 3.0, \gamma = 0.5\).

We compute the photon mass from the gauge invariant on-axis plaquette-plaquette correlator with the same \(\mu, \nu\) orienta-
n example of these fits for data taken every 100 sweeps, computing independent runs, each of 77000 sweeps after thermalization, with is Fig. 3. The data and errors were obtained from ten independent runs, each of 77000 sweeps after thermalization, with

\[ T / a \sim 1 \] ("conf" denotes the confinement phase).

We have checked that if the calculation is done just below the transition, in the massless phase, then

\[ E \beta = \gamma \] \( \beta = 3, \gamma = 0.5 \) parameters we have chosen is shown in Fig. 2. From an exponential fit, disregarding the initial points, we find a photon mass of 5.7058 is shown.

\[ T(R, T) \]

The points shown are the average of the ten sets, with the error taken as the standard error of the mean. The fits shown in Fig. 3 are through the points at \( n = 1, 2 \) at fixed \( R = 4.58 \). Horizontal lines through the \( n = 1, 2 \) data points are to guide the eye. The top line is the threshold energy, which is the sum of the (average) ground state energy plus the photon mass.

\[ E(R) \]

The photon mass is obtained from the slope of the line shown.

\[ \hat{G}(R) \]

where \( \gamma = x + \hat{R} \hat{k} \), and \( \hat{k} \) is a unit vector orthogonal to the \( \hat{\mu}, \hat{\nu} \) directions. The result for the \( \beta = 3, \gamma = 0.5 \) parameters we have chosen is shown in Fig. 2. From an exponential fit, disregarding the initial points, we find a photon mass of \( m_R = 1.57(1) \) in lattice units. Data was obtained on a \( 16^4 \) lattice with 1,600,000 sweeps and data taken every 100 sweeps. We have checked that if the calculation is done just below the transition, in the massless phase, then \( G(R) \) is fit quite well by a \( 1/R^4 \) falloff, as expected.

The energies \( E_n(R) \) are also obtained by fitting the data for \( T_n(R, T) \) vs. \( T \), at each \( R \), to an exponential falloff. An example of these fits for \( n = 1 \) and \( n = 2 \) at \( R = 4.58 \) is shown is Fig. 3. The data and errors were obtained from ten independent runs, each of 77000 sweeps after thermalization, with data taken every 100 sweeps, computing \( T_n \) from each independent run. The lattice volume was again \( 16^4 \), with couplings \( \beta = 3, \gamma = 0.5 \). The points shown are the average of the ten sets, with the error taken as the standard error of the mean. The fits shown in Fig. 3 are through the points at \( T = 2 - 5 \), with \( E_1 = 0.2911(1) \) and \( E_2(R) = 1.02(1) \) in this case. It would, of course, be desirable to go to larger \( T \) values, but extracting reliable data for \( T > 6 \) at \( n = 2 \) is numerically rather demanding. But there is no evidence whatever of convergence of the \( n = 2 \) data to the slope corresponding to the ground state energy \( (n = 1 \) data), at least up to \( T = 6 \).

The final result of our procedure is shown in Fig. 4. Here we show only the values for \( E_1(R) \) and \( E_2(R) \), as the data for \( E_3(R, T) \) is rather noisy beyond \( T = 3 \), and extrapolation to large \( T \) is unreliable. We note that there is little \( R \) dependence (as one would expect); the interaction between the static charges appears to be negligible beyond \( R = 2 \). But the main observation is that \( E_2(R) \) is well below the threshold \( E_1(R) + m_\gamma \) for production of a massive photon. This means that the \( n = 2 \) state, obtained after a short evolution in Euclidean time, is stable; it obviously cannot decay into the ground state by massive photon emission.

\[ \hat{G}(R) = \left\langle \text{Im}[U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x)] \right\rangle \times \text{Im}[U_\mu(y)U_\nu(y + \hat{\mu})U_\mu^\dagger(y + \hat{\nu})U_\nu^\dagger(y)] \right\rangle, \] (15)

where \( y = x + R \hat{k} \), and \( \hat{k} \) is a unit vector orthogonal to the \( \hat{\mu}, \hat{\nu} \) directions. The result for the \( \beta = 3, \gamma = 0.5 \) parameters we have chosen is shown in Fig. 2. From an exponential fit, disregarding the initial points, we find a photon mass of \( m_R = 1.57(1) \) in lattice units. Data was obtained on a \( 16^4 \) lattice with 1,600,000 sweeps and data taken every 100 sweeps. We have checked that if the calculation is done just below the transition, in the massless phase, then \( G(R) \) is fit quite well by a \( 1/R^4 \) falloff, as expected.

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To summarize, we have presented lattice Monte Carlo evidence for the existence of stable excitations of the quantized fields surrounding isolated static charges, in the Higgs phase of the $q = 2$ abelian Higgs model in $D = 4$ spacetime dimensions. The $q = 2$ abelian Higgs model is a close relative of the non-relativistic Ginzburg-Landau effective action of superconductivity. So the obvious next question is whether excitations of the type seen in the abelian Higgs model would also be found in non-relativistic models of that kind. If such excitations are found to exist in a realistic effective model, then the follow-up question is how they might be observed experimentally. We leave this for later investigation.

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