Comments on the little string partition functions of $K3 \times T^2$ via the refined topological vertex

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Abstract

We compute partition functions of the deformed multiple M5-branes theory on $K3 \times T^2$ using the refined topological vertex formalism and the Borcherds lift. The deformation is related to the mass deformation in the corresponding four dimensional $N = 4$ $SU(N)$ gauge theory on $K3$. The seed of the Borcherds-lift is calculated by taking the universal part of the type IIb little string free energy of the CY3-fold $X_{N,1}$. We provide explicit modular covariant expressions, as expansions in the mass parameter $m$, of the genus two Siegel modular forms produced by the Borcherds lift of the first few seed functions. We also discuss the relation between genus-one free energy and Ray-Singer Torsion, and the automorphic properties of the latter.

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1 Introduction and summary

Six dimensional SCFTs do not admit any obvious Lagrangian description in terms of UV degrees of freedom. Embedding these QFTs in M-theory/F-theory has proved to be very useful to study their dynamics. Following this line of inquiry a classification of 6d SCFTs has been proposed by considering F-theory compactification on elliptically fibered Calabi-Yau three folds (CY3-folds) An exciting class of such theories are the so-called Little string theories [14, 18, 17, 15, 1], which are the descendent of usual strings in the limit of vanishing string coupling constant $g_s \to 0$, planck scale approaching infinity and finite string length $l_s$. In type II string theory these strings are stuck to five-branes and do not give rise to spin 2 degrees of freedom. Type IIA little strings of $A_{N-1}$ type possess $N = (1, 1)$ supersymmetry, whereas type IIB little strings of type $A_{N-1}$ possess $N = (2, 0)$ supersymmetry. The former little string can also arise through the decoupling limit of a stack of $N$ NS5 branes in type IIB string theory and the latter as the decoupling limit of a stack of $N$ NS5 branes in type IIA string theory. This dual description is a result of T-dualty that $S^1$-compactified little strings enjoy.

In 11-dimensional lift to M-theory one considers a configuration of M5-branes placed along a compactified transverse direction with M2-branes stretching between consecutive M5 branes. The BPS excitation of little strings are created by the stretched open M2-branes. For $N$ separated M5 branes along $S^1$ there are $N$ intervals and that grades the BPS excitations with $U(1)^N$ quantum numbers. The compactness of the transverse direction makes it evident that little strings are the non-critical counterpart of closed fundamental string just like M-strings are the non-critical counterpart of open fundamental strings. Taking this analogy further the open M2 branes ending on M5-branes can combine to form closed M2-branes and probe the bulk. This analogy breaks down in the decoupling limit where little string are confined to the M5-brane world volume and cannot move in the bulk of 11d space-time.

M-theory compactified on a torus $T^2$ is equivalent to Type IIB compactified on an $S^1$. In this picture the S-duality symmetry $SL(2, Z)$ of type IIB is interpreted as the modular group of $T^2$. The S-duality taken together with T-duality combines into a non-perturbative symmetry of M-theory called U-duality. This duality map can be used to give two equivalent gauge theory interpretations in type IIB strings of the M5-M2-branes configurations. One gauge theory describes the Coulomb branch of a $U(N)$ gauge theory with massive bifundamental engineered by a configuration of one NS5-brane and N D5-branes. The second gauge theory is a circular quiver with N nodes of $U(1)$ gauge theories engineered by the S-dual configuration of one D5-brane and N NS5 branes. On a further lift of the little strings description to F-theory, the type IIB D5-NS5 branes web is translated to a toric web diagram of a particular CY3-fold $X_N$ which is an elliptic fibration over the affine $A_{N-1}$ space, which itself is an elliptic fibration over the complex plane. The presence of two compactified directions in the toric web is translated as double fibration structure. The corresponding topological A-string partition function have been computed in [29, 30] using topological vertex, M-string and Instanton calculus techniques.

We are interested in studying little strings on $K^3 \times T^2$. Given $N$ parallel M5-branes with world volume given by $K^3 \times T^2$ it is a standard result [31] that the compactification of this theory on $T^2$ will result in $N = 4$ $U(N)$ Super Yang Mills on $K^3$. This relation indicates that the partition function of $N$ parallel M5-branes on $K^3 \times T^2$ is related to the partition function of $N = 4$ Super Yang Mills on $K^3$. In this work we want to compute the M5-branes partition function of $K^3 \times T^2$ which corresponds to the partition function of mass deformed $N = 2^*$
SU(N) gauge theory on K3.
The expression for the partition function $Z_{K3 \times T^2}^{U(1)}(\rho, \sigma, \nu)$ of $K3 \times T^2$ corresponding to the $U(1)$ theory was proposed in [15, 17, 18] as the the generating function of elliptic genera of symmetric product $Sym^n(K3)$ of $K3$s. The manifold $Sym^n(K3)$ describes the configuration of instanton moduli space of a $K3$ surface. We know from these works a general result that expresses the orbifold elliptic genera of $Sym^n(M)$ in terms of that of $M$ with $M$ being a 4d (hyper)Kähler manifold,

$$Z(M \times T^2; p, q, y) = \sum_{N=0}^{\infty} p^N \chi(S^N M; q, y) = \prod_{n>0, m \geq 0, l} (1 - p^n q^m y^l)^{-c(mn,l)}$$

where $p = e^{2\pi i \rho}, q = e^{2\pi i \tau}, y = e^{2\pi i z}$ and

$$\chi(M; q, y) = \sum_{m \geq 0, l} c(m, n) q^m y^l.$$ (2)

For $M = K3$ we have for the elliptic genus

$$\chi(K3, \tau, z) = 24 \chi(TN_1, \tau, z) = \sum_{k \geq 0, m \in \mathbb{Z}} 24c(4kl - m^2)e^{2\pi i (hr + mz)}$$ (3)

where the Taub-NUT elliptic genus is defined by

$$\chi(TN_1, \tau, z) = \sum_{k \geq 0, m \in \mathbb{Z}} c(4kl - m^2)e^{2\pi i (hr + mz)}$$ (4)

Roughly speaking the equation (3) indicates that the $K3$ surface under consideration is composed of 24 Taub-NUT spaces $TN_1$. The generating function of $\chi(TN_1, \tau, z)$ can be expressed in terms of the weight 10 automorphic form $\Phi_{10}(\rho, \sigma, \nu)$ of $Sp(2, \mathbb{Z})$ as [4]

$$Z(\rho, \sigma, m) = \sum_N e^{2\pi i N\sigma} \chi_{\rho\sigma}(TN_1^N / S_N)$$

$$= e^{-\pi i (\rho + \sigma + \nu)} \prod_{(k,l,m) > 0} (1 - e^{2\pi i (kp + l\sigma + m\nu)})^{-c(4kl-m^2)}$$

$$= \frac{1}{\Phi_{10}(\rho, \sigma, \nu)^{\frac{1}{24}}}$$ (5)

The genus expansion of the free energy is given by

$$\mathcal{F} = \ln Z = \frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_0 + \mathcal{F}_1 + ...$$ (6)

The exponential of the genus-one part $\mathcal{F}_1$ has the following relationship to $Z_{K3 \times T^2}^{U(1)}(\rho, \sigma, \nu)$

$$e^{24\mathcal{F}_1} = \frac{1}{\Phi_{10}(\rho, \sigma, \nu)} = Z_{K3 \times T^2}^{U(1)}(\rho, \sigma, \nu)$$

$$= \sum_N e^{2\pi i N\sigma} \chi_{\rho\sigma}(K3^N / S_N)$$ (7)
The proof of (1) involves the results about orbifold conformal field theory and is a natural generalisation of orbifold Euler characteristics $\chi_0$ and orbifold $\chi_y$ genus [15],

$$Z_{K3 \times T^2}^G = \sum_{k=0}^{\infty} Q^r_{\tau} \chi_{\text{ell}}(\mathcal{M}_G(K3, r); Q, m)$$  \hspace{1cm} (8)

For a compact manifold $M$ the elliptic genus $\chi_{\text{ell}}(M; Q, m)$ is a holomorphic function on $\mathbb{H} \times \mathbb{C}$ and is a jacobi form of weight zero and index $\frac{d}{2}$ when $c_1(M) = 0$, where $\mathbb{H}$ denotes the complex upper half plane and $\text{dim}_\mathbb{C}(M) = d$. But if $c_1(M)$ is turned on the elliptic genus remains no more a modular form. However, as shown in [24], by introducing an automorphic correction to the elliptic genus it becomes a weak jacobi form of weight 0, index $d_2$. After taking into account $c_1(M) \neq 0$ the elliptic genus can be written as

$$\chi_{\text{ell}}(M; Q, m) = (\frac{\theta_1(\tau, m)}{\eta(\tau)^3})^d \int_M P(M, q, y)W(M, q)$$  \hspace{1cm} (9)

where $P(M, q, y)$ is given by

$$P(M, q, y) = \exp\left( - \sum_{n \geq 2} \frac{\wp^{(n-2)}(\tau, m)}{(2\pi i)^n n!} \sum_{i=1}^{d} x^n_i \right)$$  \hspace{1cm} (10)

and $W(M, q)$, also called the Witten factor, is given by

$$W(M, q) = \exp\left( - \sum_{k \geq 2} \frac{B_{2k} E_{2k}(\tau)}{2k(2k)!} \sum_{i=1}^{d} x^{(2k)}_i \right)$$  \hspace{1cm} (11)

for $x_i$ defined as the roots of the tangent bundle $TM, q = e^{2\pi i \tau}, y = e^{2\pi i m}, \wp(\tau, m)$ is the Weierstrass $\wp$-function and is a jacobi form of weight 2, index 0, and $\wp^{(n-2)}(\tau, m)$ is the $(n-2)$-th derivative of $\wp(\tau, m)$.

If the manifold $M$ is non-compact [27] and admits a $U(1)$ action with isolated fixed points $(m_1, \ldots, m_N)$, then $\chi_{\text{ell}}(M; Q, m)$ can be calculated using equivariant fixed point theorem as

$$\chi_{\text{ell}}(M; Q, m) = \sum_{n=1}^{N} \left( \frac{i\theta_1(\tau, m)}{\eta(\tau)^3} \right)^d (q_{1n} \cdots q_{dn}, a)^{-1} P(M, q, y, q_{a,b})W(M, q, q_{a,b})$$  \hspace{1cm} (12)

where the weights at the fixed points of the $U(1)$ action are denoted here by $q_{a,b}$ for $a = 1, \ldots, d$ and $b = 1, \ldots, N$.

Upon taking the limit $Q \rightarrow 0$ the elliptic genus reduces to the $\chi_y$ genus

$$Z_{K3 \times S^1}^G = \sum_{k=0}^{\infty} Q^r_{\tau} \chi_y(\mathcal{M}_G(K3, r), m)$$  \hspace{1cm} (13)

A further limit $y \rightarrow 0$ results in the generating function of the Euler characteristic of $\mathcal{M}_G$.

$$Z_{K3}^G = \sum_{k=0}^{\infty} Q^r_{\tau} \chi_0(\mathcal{M}_G(K3, r))$$  \hspace{1cm} (14)
where $\mathcal{M}_G(M, r)$ is the moduli space of rank $r$ and gauge group $G$-instantons on K3.

The building block of our computation is the (type IIb little strings) partition function $Z_{X_{1,1}}^{U(1)}$ of the manifold $X_{1,1}$ [29, 30]. It can also be interpreted as the partition function of single wound M-strings configuration. The corresponding free energy $F = \log Z_{X_{1,1}}$ can be expanded in the spacetime equivariant parameters $\epsilon_1, \epsilon_2$. In the unrefined limit $\epsilon_1 = -\epsilon_2 = \epsilon$ we can consider its universal ($\epsilon$ independent) part, which turns out to be the jacobi form of weight zero and index 1, $\phi_{0,1}(\tau, z)$. The Exponential-lift [8, 9, 13] of $\phi_{0,1}$ gives the partition function of a single M5-brane on $K3 \times T^2$, $Z_{K3 \times T^2}^{SU(1)}$.

Moving on to the Calabi-Yau 3-fold $X_{N,1}$ which is the orbifold of $X_{1,1}$ by $Z_N \times Z$. The orbifold action is lifted to the K3 instanton moduli space and only the $Z_N \times Z$ invariant configurations will contribute [10]. The description of instanton moduli space in terms of the Hilb$^n[K3]$ will imply projecting out the contribution of those subschemes which are not invariant under $Z_N$. We can again extract the $\epsilon$ independent part of the free energy corresponding to singly wound M-strings to find Jacobi forms of weight zero and index $N$, and take its Exponential-lift to get the partition function of $N$ M5-branes on $K3 \times T^2$, $Z_{K3 \times T^2}^{SU(N)}$. The partition functions $Z_{K3 \times T^2}^{SU(N)}$ turn out to be genus 2 Siegel modular forms. Similar modular forms have been constructed [22] using additive-lift in the context of counting $1/4$-BPS states which are $Z_M$ twisted in the CHL $Z_N$-orbifold. The Siegel(para)modular forms that are constructed by Exponential-lift of certain modular forms, also called the seed modular forms, have a natural interpretation as the generating function of symmetric product of certain CFTs.

We give plan of the paper now. After reviewing the effective 2d/4d description of the M5-branes partition functions of $K3 \times T^2$ in section 2, we compute in section 3 the seed partition functions. In section 4 we compute the little string partition function of $K3 \times T^2$ using Borcherds lift of the seed partition function for $N = 1, 2, 3$. In section 5 we discuss the corresponding Gromov-Witten potentials. In the last section 6 we discuss the relevance of Ray-Singer torsion to the (para)modular forms. Some useful identities and review material is given in the appendices.

2 The effective 2d/4d à la Vafa-Witten limit of the M5-branes Partition function

Consider the setup of $N$ parallel M5 branes wrapped on a six dimensional submanifold $M_4 \times T^2$ of the 11d spacetime. If parallel M5 branes are compactified on $T^2$, it gives rise to effective $U(N)N = 4$ supersymmetric Yang-Mills theory on $M_4$. It means that the partition function of $N$ M5 branes on $M_4 \times T^2$, in the limit of small $T^2$ volume, is the same as the partition function of $U(N), N = 4$ Super Yang Mills on $M_4$ [34]. There is another dual description in which $N$ parallel M5 branes are wrapped on $M_4$ and in the small volume limit it will give rise to string degrees on freedom on $T^2$. The gauge coupling constant is given by the complex structure $\tau$ of $T^2$, i.e $\tau = \frac{4\pi i}{\nu^2} + \frac{\theta}{2\pi}$.

A single M5-brane gives rise to the $U(1)$ theory. The world volume self-dual field strength

\textsuperscript{1}multiplicative-lift, exponential-lift and Borcherds lift are all synonymous
$B_{mn}$ gives rise to 19 left moving and 3 right moving periodic bosonic fields. There are 3 non-compact bosonic fields both for left- and right-movers. Two more scalars both left and right movers transform as sections of trivial canonical line bundle of K3. In total we have 24 left moving and 8 right moving bosonic fields. The fermionic field content comprise of 8 right-moving fields. In the heterotic dual frame this is the field content of heterotic string. In the presence of fermionic zero modes the worldsheet degrees of freedom can be quantified by the regularised elliptic genus

$$Z = \frac{\tau^{5/2}}{\sqrt{3}} \text{Tr}[-1]^F \prod q^{L_0} \bar{q}^{\bar{L}_0} \prod q^J \prod \bar{q}^{\bar{J}}]$$

\[15\]

where the prefactor $\frac{\tau^{5/2}}{\sqrt{3}}$ denotes the contribution of five uncompactified transverse bosons.

A K3 surface has $SU(2)$ holonomy and vanishing first Chern class. Due to the absence of an explicit expression for a metric on K3, orbifold models were extensively used to study certain types of topological invariants. Elliptic genus, for example, is independent of the complex structure moduli of the K3 surface and can equivalently be computed for the orbifold limit of the K3 surface. In the orbifold limit a K3 surface is modelled by moding out $T^2 \times T^2$ by the symmetry groups $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, with orbifold singularities appearing at the fixed points of

$$z_1 \rightarrow e^{\frac{2\pi i}{n}} z_1, \quad z_2 \rightarrow e^{-\frac{2\pi i}{n}} z_2$$ \[16\]

for each $n = 1, 2, 3, 4$. There exists orbifold invariant 2-forms $dz_1 \wedge d\bar{z}_1, dz_2 \wedge d\bar{z}_2$, and $dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2$ which are acted on by an $SU(2)$ symmetry. This allows the $\mathcal{N} = 4$ algebra to enter this picture.

On the K3 surface the $SU(2)$ instanton moduli space is $8k - 12$ dimensional \[35\] for instanton number $k$. The description of instanton moduli space can be given in terms of the space of configuration of $2k - 3$ distinct unordered points on K3. This configuration depends on $8k - 12$ parameters and is hyper-Kähler. The corresponding moduli space has orbifold singularities and a resolution is necessary to give a compactified instanton moduli space. An equivalent description can be given in terms of $\text{Sym}^{2k-3}(K3)$ which upon resolution of orbifold singularities is called Hilb$^{2k-3}(K3)$, the Hilbert scheme of $2k-3$ points on K3. The analysis for $\text{Sym}^{2k-3}(K3)$ and Hilb$^{2k-3}(K3)$ agree because K3 is a hyperkähler manifold. Unlike a general four manifold on which the holonomy is $SU(2)_L \times SU(2)_R$, K3 has $SU(2)_R$ holonomy and hence making a twist using $SU(2)_L$ does not differentiate between twisted and physical theories. In the process of resolving the singularities the hyper-Kähler structure must be preserved and Betti numbers must also remain invariant. The above description of $SU(2)$ instanton moduli space is only valid for $k$ odd. However the partition function for general $k$ can be predicted using modular constraints from S-duality. For $SO(3) \simeq SU(2)/\mathbb{Z}_2$ there is no constraint on the allowed value of the instanton number $k$. We will restrict ourselves to the $SU(N)$. The partition function of K3 corresponding to a single M5-brane is given by \[34\]

$$Z_{U(1)} = G(q) = \frac{1}{\eta(\tau)^{24}}$$ \[17\]

with $q = e^{2\pi i\tau}$ and $\eta(\tau)$ being the Dedekind eta function. The $N = 4$ $SU(2)$ gauge theory partition function of K3 corresponding to two M5-branes is given by

$$Z_{SU(2)} = \frac{1}{4} G(q^2) + \frac{1}{2} G(q^{1/2}) + \frac{1}{2} G(-q^{1/2})$$ \[18\]
This result has a nice interpretation in terms of M5-branes. In this interpretation $Z_{SU(2)}$ can alternatively be viewed as corresponding to single M5-brane wrapped two times over $K3 \times T^2$. This can be done by considering the world volume of two M5-branes to be $K3 \times T^2$ but now with $T^2$ double covering the original spacetime $T^2$. There are then three inequivalent choices of complex structures that we can have

$$\tilde{\tau} = 2\tau, \frac{\tau + 1}{2}$$

(19)

The three terms in the partition function (18) correspond to these three complex structures. We can generalize this picture to the case of $N$ parallel M5-branes wrapping $K3 \times T^2$ with the corresponding $U(N)$ gauge theory on $K3$. In this case $T^2$ covers the original spacetime $T^2$ $N$ times and the inequivalent choices of complex structures are enumerated as follows: given a basis of 1-cycles on $T^2$, the following set of $GL(2)$ transformations enumerate the inequivalent choices of n-fold covers

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

(20)

with $ad = N, b < d$ and $a, b, d \geq 0$.

The $N = 4$ $SU(N)$ gauge theory partition function of K3, $Z_{SU(N)}$, corresponding to $N$ parallel M5-branes can be written as an Hecke transform of $Z_{U(1)}$ in terms of the above matrices as

$$Z_{SU(N)} = \frac{1}{N^2} \sum_{0 \leq a, b, d \in \mathbb{Z}} dG\left(\frac{a\tau + b}{d}\right)$$

(21)

It is a Hecke transformation of order $N$.

## 3 M2-M5 branes and the seed functions

We begin by writing down the relevant M-string partition function for M-theory vacuum described by the 11d space-time $T^2 \times \mathbb{R}^3 \times S^1 \times S^1 \times \mathbb{R}_4^1$. The following table gives the coordinate labels and specifies the world volume directions of the BPS M5-M2-M-string configuration.

| 11D M-theory space-time | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ | $x^9$ | $x^{10}$ |
|-------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| M5                      | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| M2                      | $\times$ | $\times$ |       |       |       |       |       |       |       |       | $\times$ |
| M-string                | $\times$ | $\times$ |       |       |       |       |       |       |       |       |          |

where $T^2$ spans the $\{x^1, x^2\}$ directions and $S^1 \times S^1$ spans $x^5, x^6$ directions respectively,
of the Calabi-Yau 3fold with toric web given in Fig. (1) can be expressed as:

\[ x^1 \sim x^1 + 2\pi R_1; \quad 2\pi i R_1 := \tau; \quad Q_\tau = e^{2\pi i \tau}, \]
\[ x^6 \sim x^6 + 2\pi R_6; \quad 2\pi i R_6 := \rho; \quad Q_\rho = e^{2\pi i \rho}, \]
\[ q = e^{2\pi i e_1} \quad t = e^{-2\pi i e_2} \]

The M5-branes are placed at the following positions

\[ 0 \leq a_1 \leq a_2 \leq \ldots \leq a_N \leq 2\pi R_6 \]

resulting in the corresponding gauge theory Coulomb branch parameters

\[ t_{f_1} = a_2 - a_1, \quad t_{f_2} = a_3 - a_2, \ldots, \quad t_{f_{N-1}} = a_N - a_{N-1}, \]
\[ t_{f_N} = 2\pi R_6 - \sum_{i=1}^{N-1} t_{f_i} = -i\rho - (a_N - a_1). \]

The N-tuple of integers \((k_1, k_2, \ldots, k_N)\) specifies the distribution of \(K = k_1 + k_2 + \ldots + k_N\) M2-branes amongst the N intervals. To render the computation on the non-compact part of the manifold regularised one introduces the so-called \(\Omega\)-deformation as

\[ (z_1, z_2) \rightarrow (e^{2\pi i e_1} z_1, e^{2\pi i e_2} z_2) \]
\[ (\omega_1, \omega_2) \rightarrow (e^{2\pi i m - \pi i (e_1 + e_2)} \omega_1, e^{-2\pi i m - \pi i (e_1 + e_2)} \omega_2) \]

where \((\omega_1, \omega_2) = (x_7 + ix_8, x_9 + ix_{10})\) and \((z_1, z_2) = (x_2 + ix_3, x_4 + ix_5)\). From transverse space-time viewpoint \(m\) is the mass deformation parameter. The partition function can be computed using topological vertex formalism \[32\]. Since it is simpler to study the modular properties of the free energy, below we give its expression for the choice of vertical direction in the toric diagram as the preferred one. This will correspond to the type IIB little string partition function.

For the vertical preferred direction, the topological string partition function \(Z_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2)\) of the Calabi-Yau 3fold with toric web given in Fig. (1) can be expressed as:

\[ Z_{X_N}(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2) = Z_2(N, \tau, m, \epsilon_1, \epsilon_2) \tilde{Z}_N(\tau, m, t_{f_1}, \ldots, t_{f_N}, \epsilon_1, \epsilon_2) \]

where

\[ Z_2(N, \tau, m, \epsilon_1, \epsilon_2) = W_{00}(\tau, m, \epsilon_1, \epsilon_2)^N \]

and the open topological string wavefunction \(W_{\nu_{\nu+1}}(Q_\tau, Q_m, t, q)\) is defined by

\[ W_{\nu_{\nu+1}}(Q_\tau, Q_m, t, q) = t^{-\frac{||m+1||^2}{2}} q^{-\frac{||m||^2}{2}} Z_{\nu_{\nu}}(q^{-1}, t^{-1}) \tilde{Z}_{\nu m+1}(t^{-1}, q^{-1}) Q_m^{\frac{||m||+||m+1||}{2}} \prod_{k=1}^{N-1} (1 - Q_{\tau}^k)^{-1} \]
\[ \times \prod_{i,j=1} \frac{1 - Q_{\tau}^i q_{m+1,i-j}^j t_{m+1,i-j}}{1 - Q_{\tau}^i q_{m+1,i-j}^j t_{m+1,i-j}^j - Q_{\tau}^i q_{m+1,i-j}^j t_{m+1,i-j}^j} \]

(29)
where \( \nu_i \) are partitions whose definitions are given in the appendix \[3\]  

The non-perturbative part \( \tilde{Z}_N(\tau, m, t_{f_1}, ..., t_{f_N}, \epsilon_1, \epsilon_2) \) of the topological string partition function can be readily expressed by

\[
\tilde{Z}_N(\tau, m, t_{f_1}, ..., t_{f_N}, \epsilon_1, \epsilon_2) = \sum_{\nu_1, \nu_2, ..., \nu_N} Q_1^{[\nu_1]} Q_2^{[\nu_2]} ... Q_N^{[\nu_N]} Z_{\nu_1 \nu_2 ... \nu_N}(\tau, m, \epsilon_1, \epsilon_2)
\]

where \( Q_a = e^{-t_a} \) and \( Z_{\nu_1 \nu_2 ... \nu_N}(\tau, m, \epsilon_1, \epsilon_2) \) is defined as

\[
Z_{\nu_1 \nu_2 ... \nu_N}(\tau, m, \epsilon_1, \epsilon_2) = \prod_{a=1}^{N} \prod_{(i,j) \in \nu_a} \frac{\theta_1(\tau, z_{ij}^a) \theta_1(\tau, u_{ij}^a)}{\theta_1(\tau, w_{ij}^a) \theta_1(\tau, v_{ij}^a)}
\]

and

\[
\begin{align*}
z_{ij}^a &= m - \epsilon_1(\nu_{a,i} - j + \frac{1}{2}) + \epsilon_2(\nu_{a+1,j} - i + \frac{1}{2}), \\
u_{ij}^a &= -\epsilon_1(\nu_{a,i} - j + 1) - \epsilon_2(\nu_{a+1,j} - i), \\
u_{ij}^a &= m - \epsilon_1(\nu_{a,i} - j + \frac{1}{2}) + \epsilon_2(\nu_{a-1,j} - i + \frac{1}{2}), \\
u_{ij}^a &= \epsilon_1(\nu_{a,i} - j) - \epsilon_2(\nu_{a,j} - i + 1).
\end{align*}
\]

A dual description can be given as the five-dimensional \( U(1)^N \) affine quiver gauge theory. The instanton moduli space corresponding to this gauge theory is given by \( \text{Hilb}^k[\mathbb{C}^2] \times \text{Hilb}^{k_2}[\mathbb{C}^2] \times \cdots \times \text{Hilb}^{k_N}[\mathbb{C}^2] \), where \( \text{Hilb}^n[\mathbb{C}^2] \) is the Hilbert scheme of \( n \) points on \( \mathbb{C}^2 \). The free energy admits the following expansion \[30, 29, 28\]

\[
\Omega_N(\tau, m, t_{f_1}, t_{f_1}, ..., t_{f_N}, \epsilon_1, \epsilon_2) = \text{PLog} \tilde{Z}_N(\tau, m, t_{f_1}, t_{f_1}, ..., t_{f_N}, \epsilon_1, \epsilon_2) = \sum_{k_1, ..., k_N} Q_{f_1}^{k_1} ... Q_{f_N}^{k_N} G_{k_1, ..., k_N}(\tau, m, \epsilon_1, \epsilon_2)
\]

The BPS configurations of the type IIb little string with charges \((k_1, k_2, ..., k_N)\) are encoded by the functions \( G_{k_1, ..., k_N}(\tau, m, \epsilon_1, \epsilon_2) \).
3.1 Universal ($\epsilon_1, \epsilon_2$ independent) part of the free energy $G^{(1, 1, \ldots, 1)}(\tau, m, \epsilon_1, \epsilon_2)$

For our purpose we are interested in the singly wound configuration of M-strings characterized by $G^{(1, 1, \ldots, 1)}(\tau, m, \epsilon_1, \epsilon_2)$ \[29\]

$$G^{(1, 1, \ldots, 1)}(\tau, m, \epsilon_1, \epsilon_2) = W(\tau, m, \epsilon_1, \epsilon_2)^{N-1} \left[ G^{(1)}(\tau, m, \epsilon_1, \epsilon_2) + (N-1)F^{(1)}(\tau, m, \epsilon_1, \epsilon_2) \right]$$  \hspace{1cm} (34)

where

$$W(\tau, m, \epsilon_1, \epsilon_2) = \frac{\theta_1(\tau; m + \epsilon_-)\theta_1(\tau; m - \epsilon_-)}{\theta_1(\tau; \epsilon_1)\theta_1(\tau; \epsilon_2)} - \frac{\theta_1(\tau; m + \epsilon_+)\theta_1(\tau; m - \epsilon_+)}{\theta_1(\tau; \epsilon_1)\theta_1(\tau; \epsilon_2)}$$

$$G^{(1)}(\tau, m, \epsilon_1, -\epsilon_2) = F^{(1)}(\tau, m, \epsilon_1, \epsilon_2) = \frac{\theta_1(\tau; m + \epsilon_+)\theta_1(\tau; m - \epsilon_+)}{\theta_1(\tau; \epsilon_1)\theta_1(\tau; \epsilon_2)}$$  \hspace{1cm} (35)

The importance of the function $G^{(1, 1, \ldots, 1)}$ stems from the fact that its universal part provides the seed function for the Exponential-lifting \[8, 9, 13\] of these seed functions. The seed function is obtained by taking the $\epsilon_1, \epsilon_2$ independent part of $G^{(1, 1, \ldots, 1)}(\tau, m, \epsilon_1, \epsilon_2)$ and turns out to be

$$\chi_N(\tau, m) = 2^{-2N-13}N^{-N} \left( \varphi_{0, 1}(\tau, m) + E_2(\tau)\varphi_{-2, 1}(\tau, m) \right)^{N-2}$$

$$- 2(N-2)E_2(\tau)\varphi_{0, 1}(\tau, m)\varphi_{-2, 1}(\tau, m) -$$

$$(N-1)(E_2(\tau)N + 2 - E_4(\tau)N)\varphi_{-2, 1}(\tau, m)^2 + 2\varphi_{0, 1}(\tau, m)^2$$  \hspace{1cm} (36)

It can be checked that $\chi_N(\tau, m)$ is a Jacobi form of weight 0 and index $N$. To relate it to the computation of $K3 \times T^2$ partition function we have to consider its following multiple

$$\psi_N(\tau, m) = 24\chi_N(\tau, m)$$  \hspace{1cm} (37)

The expression for first few Jacobi forms are as follows

$$\psi_1(\tau, m) = 2\varphi_{0, 1}(\tau, m)$$

$$\psi_2(\tau, m) = \frac{1}{6}\varphi_{0, 1}(\tau, m)^2 - \frac{1}{3}E_2(\tau)^2\varphi_{-2, 1}(\tau, m)^2 + \frac{1}{6}E_4(\tau)\varphi_{-2, 1}(\tau, m)^2$$

$$\psi_3(\tau, m) = -\frac{\varphi_{0, 1}(\tau, m)^3}{72} - \frac{1}{12}E_2(\tau)^2\varphi_{0, 1}(\tau, m)\varphi_{-2, 1}(\tau, m)^2 + \frac{1}{24}E_4(\tau)\varphi_{0, 1}(\tau, m)\varphi_{-2, 1}(\tau, m)^2$$

$$+ \frac{5}{72}E_2(\tau)^3\varphi_{-2, 1}(\tau, m)^3 - \frac{1}{24}E_2(\tau)E_4(\tau)\varphi_{-2, 1}(\tau, m)^3$$  \hspace{1cm} (38)

The functions $\psi_N(\tau, m)$ are what we call the seed functions. The Borcherds-lift of these functions produce the $K3 \times T^2$ partition functions. This choice \[37\] is motivated by the fact that the partition function of $U(1)$ theory is related to the Borcherds-lift of $24\chi_1(\tau, m) = 2\varphi_{0, 1}(\tau, m)$ \[15, 17, 18\]. So for $N = 1$ this choice is correct.

Based on this intuition our main proposal is that the Borcherds-lift of $24\chi_N(\tau, m)$ is related to $N$ parallel M5-branes partition functions $Z_{K3 \times T^2}^{SU(N)}$, with mass deformation, for other values of $N$. It is important to mention that the space of jacobi forms of wight 0 and index $N$ is finite dimensional and moreover it forms a ring structure \[24\].
4 Borcherds Lift and the Little strings partition function

The crucial relationship between (1) and (2) is provided by the Borcherds-lift, that is a way to obtain genus two modular forms by applying Hecke transformation on certain genus one Jacobi forms. More concretely, consider a Jacobi form $J_{w,l}(\tau, m)$ of weight $w$ and index $l$ with Fourier expansion given by

$$J_{w,l}(\tau, m) = \sum_{e,s} C_{J_{w,l}}(e, s) Q_\tau^e Q_m^s$$  \hspace{1cm} (39)

where $Q_\tau = e^{2\pi i \tau}, Q_m = e^{2\pi i m}$. The Borcherds lift of the Jacobi form $J_{0,l}(\tau, m)$ of weight zero and index $l$ is defined as

$$B[J_{0,l}](\rho, \tau, m) = Q_\tau^a Q_m^b \prod_{(d,e,s) > 0} (1 - Q^d_\rho Q_\tau^e Q_m^s)^{C_{J_{0,l}}(d,e,s)}$$  \hspace{1cm} (40)

where $Q_\rho = e^{2\pi i \rho}, a = \sum_{s \in \mathbb{Z}} \frac{C_{J_{0,l}}(0, s)}{24}, b = \sum_{s > 0} \frac{C_{J_{0,l}}(0, s)}{2}, c = \sum_{s > 0} \frac{C_{J_{0,l}}(0, s)^2}{4}$. It is a genus two modular form of weight $\frac{C(0,0)}{2}$ where $C(0,0)$ is the constant term in the Fourier expansion of the corresponding Jacobi form. The Borcherds product is multiplicative by definition. Given two weakly holomorphic Jacobi forms $J_1, J_2$ we have

$$B[J_1 + J_2] = B[J_1]B[J_2]$$  \hspace{1cm} (41)

If we expand the Borcherds product in $Q_\rho$, each coefficient of $Q_\rho$ is a Jacobi form. Although the above can be defined for Jacobi form of any weight, however, for Jacobi forms of weight 0 the above expression can be related to Hecke transforms. The above product can be related to Hecke transforms as we now describe. Let $J_{0,l}(\tau, m)$ be a Jacobi form of weight zero and index $l$. Then the $n$-th Hecke transform of $J_{0,l}(\tau, m)$ is defined as

$$(T_n J_{0,l})(\tau, m) = n^{k-1} \sum_{ad=na, b=0}^{d-1} J_{0,l}(\frac{a \tau + b}{d}; am)$$  \hspace{1cm} (42)

After doing some calculations as described in the appendix [C] we have

$$\sum_{n \geq 1} Q^l_\rho (T_n J_{0,l})(\tau, m) = - \sum_{(d,e,s)} C_{J_{0,l}}(d,e,s) \log(1 - Q^d_\rho Q_\tau^e Q_m^s)$$  \hspace{1cm} (43)

for $s \in \mathbb{Z}, d > 0, e \geq 0$.

The zero-th Hecke operator acting on the Jacobi form $J_{0,l}(\tau, m)$ is defined by

$$T_0(J_{0,l})(\tau, m) = \frac{C_{J_{0,l}}(0,0)}{2} \zeta(1) - \sum_{(0,e,s) > 0} C_{J_{0,l}}(0, s) \log(1 - Q^2_\rho Q_m^s)$$  \hspace{1cm} (44)

where $\zeta(1) = \sum_{h > 0} \frac{1}{h}$. Combining the last two equations we have

$$\sum_{n \geq 0} Q^l_\rho (T_n J_{0,l})(\tau, m) = \frac{C_{J_{0,l}}(0,0)}{2} \zeta(1) - \prod_{(d,e,s) > 0} C_{J_{0,l}}(d,e,s) \log(1 - Q^d_\rho Q_\tau^e Q_m^s)$$  \hspace{1cm} (45)

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By taking the exponential of both sides we obtain
\[
e^{-\sum_{n \geq 0} Q_{\rho}^n(T_n J_{0,l})(\tau,m)} = e^{\frac{C_{J_{0,l}}(0,0)}{2} \zeta(1)} \prod_{(d,e,s)>0} (1 - Q_{\rho}^d Q_{\tau}^e Q_{m}^s)^{C_{J_{0,l}}(d,e,s)}
\]
(46)

Next we consider the following expansion of the product term
\[
\prod_{(d,e,s)>0} (1 - Q_{\rho}^d Q_{\tau}^e Q_{m}^s)^{C_{J_{0,l}}(d,e,s)} = \Psi(\tau,m)(1 + Q_{\rho}^l Z_{J_{0,l}}^{(1)}(\tau,m) + Q_{\rho}^{2l} Z_{J_{0,l}}^{(2)}(\tau,m) + Q_{\rho}^{3l} Z_{J_{0,l}}^{(3)}(\tau,m) + \ldots)(47)
\]

where \(Z_{J_{0,l}}^{(i)}\) are jacobi forms. By substituting the eq.(47) in (46) and comparing the coefficients of \(Q_{\rho}\) expansion we get the functions \(\Psi_{\psi}(\tau,m), Z_{J_{0,l}}^{(1)} Z_{J_{0,l}}^{(2)}, Z_{J_{0,l}}^{(3)}\) in the following form
\[
\Psi_{\psi}(\tau,m) = \prod_{(0,e,s)>0} (1 - Q_{\tau}^e Q_{m}^s)^{C_{J_{0,s}}}
\]
\[
Z_{J_{0,l}}^{(1)}(\tau,m) = -J_{0,l}(\tau,m)
\]
\[
Z_{J_{0,l}}^{(2)}(\tau,m) = \frac{1}{2}(J_{0,l}(\tau,m))^2 - \frac{1}{2}[J_{0,l}(2\tau,2m) + J_{0,l}(\frac{\tau}{2},m) + J_{0,l}(\frac{\tau+1}{2},m)]
\]
\[
Z_{J_{0,l}}^{(3)}(\tau,m) = \frac{1}{6} \left(-T_{1}(J_{0,l}(\tau,m))^3 + 6T_{2}(J_{0,l}(\tau,m))T_{1}(J_{0,l}(\tau,m)) + 6T_{3}(J_{0,l}(\tau,m))\right)
\]
(48)

Next in this procedure we construct \(Z_{J_{0,l}}^{(i)}\) as an expansion in the mass parameter \(m\) by evaluating the Hecke transformations. In the last step we resum the series \(\{47\}\) in the variables \(\rho\) and \(\tau\) to get a mass expansion that is modular covariant in \(\rho\) and \(\tau\). We illustrate this procedure by constructing Borcherds-lift in three examples involving the jacobi forms \(\psi_{1}(\tau,m), \psi_{2}(\tau,m)\) and \(\psi_{3}(\tau,m)\).

4.1 Borcherds lift of \(\psi_{1}(\tau,m)\): Little string partition function corresponding to \(U(1)\) theory

So far we have been discussing the case of an arbitrary jacobi form of weight zero and index \(l\). Let us now specialize to the case \(l = 1\) and consider jacobi form of weight 0 and index 1, \(\psi_{1}(\tau,m)\). From the following Fourier expansion
\[
\psi_{1}(\tau,m) = \sum_{e,s \in \mathbb{Z}} C_{\psi_{1}}(e,s) Q_{\tau}^e Q_{m}^s = 2Q_{m} + 2Q_{m}^{-1} + 20 + O(Q_{\tau})
\]
(49)

we notice that \(C_{\psi_{1}}(0,1) = C_{\psi_{1}}(0,-1) = 2\) and \(C_{\psi_{1}}(0,0) = 20\) with all other \(C_{\psi_{1}}(0,s)\) vanishing. This leads to \(a = b = c = 1\). The function \(\Psi(\tau,m)\) as defined in \(\{47\}\) is given for \(\psi_{1}(\tau,m)\) by
\[
\Psi_{\psi_{1}}(\tau,m) = \prod_{e>0,s \in \mathbb{Z}} (1 - Q_{\tau}^e Q_{m}^s)^{C_{\psi_{1}}(0,s)} \prod_{s<0}(1 - Q_{m}^s)^{C_{\psi_{1}}(0,s)}
\]
\[
= -Q_{\tau}^{-1} Q_{m}^{-1} C_{\psi_{1}}(\tau,m)^2 \eta(\tau)^{18}
\]
\[
= Q_{\tau}^{-1} Q_{m}^{-1} C_{\psi_{1}}(\tau,m) \eta(\tau)^{24}
\]
(50)
where we have used the definition of Jacobi form \( \varphi_{-2,1}(\tau, m) \) of weight 2, index 1 defined in \((113)\). Since \( \varphi_{-2,1}(\tau, m) \) has weight \(-2\) and \( \eta(\tau) \) has weight \( \frac{1}{2} \), \( \Psi_{\psi_1}(\tau, m) \) has weight 10. Using the mass-expansion of \( \varphi_{0,1}(\tau, m) \) calculated in \((114)\) we obtain the following expansion of \( \psi_1(\tau, m) \)

\[
\psi_1(\tau, m) = 24 - 2E_2(\tau)m^2 + \left( \frac{1}{12}E_2(\tau)^2 + \frac{1}{12}E_4(\tau) \right)m^4 + \ldots
\]

The first Hecke transformation \( T_1 \) of \( \psi_1(\tau, m) \) is proportional to \( \psi_1(\tau, m) \). The second and third Hecke transformations of \( \psi_1(\tau, m) \) give us the Jacobi form of weight 0 and index 2, 3 respectively. These Hecke operators expressions are given by

\[
T_2\psi_1(\tau, m) = 36 - 6E_2(\tau)m^2 + \frac{1}{2}(E_2(\tau)^2 + 2E_4(\tau))m^4 + \ldots
\]

\[
T_3\psi_1(\tau, m) = 32 - 8E_2(\tau)m^2 + (E_2(\tau)^2 + \frac{11}{3}E_4(\tau))m^4 + \ldots
\]

By the definition of the Borcherds product

\[
B[\psi_1](\rho, \tau, m) = Q_{\tau}Q_{m}Q_{\rho} \prod_{(d,e,s)>0} \left( 1 - Q_{\rho}^{2d}Q_{e}^{s}Q_{m}^{d} \right)^{C_{\psi_1}(d,e,s)}
\]

We have already seen in \((47)\) the Borcherds product can be written in the following way as well

\[
B[\psi_1](\rho, \tau, m) = Q_{\tau}Q_{m}Q_{\rho} \psi_1(\tau, m) \left( 1 + Q_{\rho}Z^{(1)}_{\psi_1}(\tau, m) + Q_{\rho}^2Z^{(2)}_{\psi_1}(\tau, m) + Q_{\rho}^3Z^{(3)}_{\psi_1}(\tau, m) + \ldots \right)
\]

where \( Z^{(i)}_{\psi_1}(\tau, m) \) are Jacobi forms of weight 0 and index \( i \). \( Z^{(2)}_{\psi_1}(\tau, m) \) and \( Z^{(3)}_{\psi_1}(\tau, m) \) involve second and third Hecke series of \( \psi_1(\tau, m) \). We know the Hecke expansion given in \((52)\) and so we have the following expansions of \( Z^{(1)}_{\psi_1}(\tau, m) \), \( Z^{(2)}_{\psi_1}(\tau, m) \) and \( Z^{(3)}_{\psi_1}(\tau, m) \)

\[
Z^{(1)}_{\psi_1}(\tau, m) = -\psi_1(\tau, m)
\]

\[
Z^{(2)}_{\psi_1}(\tau, m) = 252 - 42E_2(\tau)m^2 + \frac{7}{2}E_2(\tau)^2 + E_4(\tau) + \ldots
\]

\[
Z^{(3)}_{\psi_1}(\tau, m) = -1472 + 368E_2(\tau)m^2 + (-46E_2(\tau)^2 - \frac{2}{3}E_4(\tau))m^4 + \ldots
\]

Using the above expressions of \( Z^{(i)}_{\psi_1}(\tau, m) \) and the expansion of \( \varphi_{-2,1}(\tau, m) \) given in \((113)\) we give first few terms of \( B[\psi_1](\rho, \tau, m) \)

\[
B[\psi_1](\rho, \tau, m) = m^2\eta(\tau)^{24}\eta(\rho)^{24} + \frac{m^4}{12}E_2(\tau)E_2(\rho)\eta(\rho)^{24}\eta(\tau)^{24} + \ldots
\]

\[
= m^8\eta(\tau)^{24}\eta(\rho)^{24} \left( \frac{1}{24}E_2(\tau)^2E_4(\rho) - \frac{13}{24}E_2(\tau)^2E_2(\rho)^2 + \frac{1}{24}E_4(\tau)E_2(\rho)^2 \right) + \ldots
\]

\[
= m^6\eta(\tau)^{24}\eta(\rho)^{24} \left( \frac{1}{373248}E_6(\rho)E_2(\tau)^3 + \frac{91}{746496}E_2(\rho)^3E_2(\tau)^3 \right) + \ldots
\]

\[
= \frac{7}{622080}E_6(\rho)E_2(\tau)E_4(\tau) + \ldots
\]

\[
(56)
\]
By observing this series expansion we can guess the following generic form for \( B[\psi_1](\rho, \tau, m) \)

\[
B[\psi_1](\rho, \tau, m) = \sum_{k \geq 1} \eta(\tau)^{2k} \eta(\rho)^{2k} M_{2k-2}(\tau) M_{2k-2}(\rho) m^{2k}
\]  

(57)

where \( M_{2k-2}(\tau) \) is a quasi modular form of weight \( 2k - 2 \)

\[
M_{2k-2}(\tau) = \sum_{a, b, s, t > 0 \atop 2a + 4b + s + t = 2k - 2} \chi_{a b c} E_2(\tau)^a E_4(\tau)^b E_6(\tau)^c
\]

(58)

where \( \chi_{a b c} \) are numerical coefficients.

The \( B[\psi_1](\rho, \tau, m) \) is a Siegel modular form of weight 10 and is nothing other than the Igusa cusp form \( \Phi_{10}(\rho, \sigma, m) \). So the mass deformed single M5-brane partition function of \( K3 \times T^2 \) is given by

\[
Z^{U(1)}_{K3 \times T^2}(\rho, \tau, m) = \frac{1}{B[\psi_1](\rho, \sigma, m)}
\]

(59)

4.2 Borcherds lift of \( \psi_2(\tau, m) \): Little string partition function corresponding to \( SU(2) \) theory

In section we will discuss the Borcherds lift for Jacobi form of weight 0 having index 2, in particular \( \psi_2(\tau, m) \) which is defined in (38). By substituting \( l = 2 \) in (40) we have the following infinite product for the Borcherd-lift of \( \psi_2(\tau, m) \)

\[
B[\psi_2](\rho, \tau, m) = Q_\rho^a Q_m^b Q_s^c \prod_{(d, e, s) > 0} (1 - Q_\rho^{2d} Q_m^{2e} Q_s^{2s})^{C_{\psi_2}(de, s)}
\]

(60)

where

\[
a = \sum_{s \in \mathbb{Z}} \frac{C_{\psi_2}(0, s)}{24}, b = \sum_{s > 0} \frac{C_{\psi_2}(0, s)}{2}, c = \sum_{s \in \mathbb{Z}} \frac{C_{\psi_2}(0, s) s^2}{4}.
\]

In order to find the values of \( a, b \) and \( c \) we need to know the Fourier coefficient \( C_{\psi_2}(de, s) \). The \( \psi_2(\tau, m) \) has the following Fourier expansion

\[
\psi_2(\tau, m) = \sum_{e, s \in \mathbb{Z}} C_{\psi_2}(e, s) Q_\tau^e Q_m^s = 4Q_m + 4Q_m^{-1} + \ldots
\]

(61)

We see that \( C_{\psi_2}(0, 1) = C_{\psi_2}(0, -1) = 4 \) and \( C_{\psi_2}(0, 0) = 16 \) with all other \( C_{\psi_2}(0, s) \) identically zero. Hence \( a = 1, b = c = 2 \). Again by using the same method of decomposing the product (47) we obtain the following expansion of Borcherds-lift of \( \psi_2(\tau, m) \)

\[
B[\psi_2](\rho, \tau, m) = Q_\rho^2 Q_m^2 Q_s^2 \Psi_{\psi_2}(\tau, m)(1 + Q_\rho^2 Z^{(1)}_{\psi_2}(\tau, m) + Q_\rho^4 Z^{(2)}_{\psi_2}(\tau, m) + Q_\rho^6 Z^{(3)}_{\psi_2}(\tau, m) + \ldots)
\]

(62)

Here \( \Psi_{\psi_2}(\tau, m) \) is given by

\[
\Psi_{\psi_2}(\tau, m) = \prod_{(0, c, s) > 0} (1 - Q_\rho^c Q_m^s)^{C_{\psi_2}(0, s)}
\]

\[
= \prod_{c > 0} (1 - Q_\rho^c Q_m^c)^4 (1 - Q_\rho^c Q_m^{-1})^4 (1 - Q_\rho^c)^16 (1 - Q_m^{-1})^4
\]

(63)
The product form on the right hand side of the above equation can be simplified to

\[
\Psi_{\psi_2}(\tau, m) = Q^{-1}Q^{-2}\theta_1(\tau, m)^4\eta(\tau)^{12} = Q^{-1}Q^{-2}\varphi_{-2,1}(\tau, m)^2\eta(\tau)^{24}
\]  

(64)

The modular form \(\Psi_{\psi_2}\) has weight 8.

We know the mass-expansion of \(\varphi_{-2,1}(\tau, m)\) and \(\varphi_{0,1}(\tau, m)\) as given in (113,114), therefore by using these expansions we have the mass-expansion of \(\psi_2(\tau, m)\)

\[
\psi_2(\tau, m) = 24 - 4m^2E_2(\tau) + \frac{1}{3}E_4(\tau)m^4 + ...
\]  

(65)

The functions \(Z^{(i)}(\tau, m)\) are Jacobi form of weight 0 index 2. To find the expansion of these functions we have to know the Hecke transformations of \(\psi_2(\tau, m)\). The first Hecke transformation \(T_1\) of \(\psi_2(\tau, m)\) is proportional to \(\psi_2(\tau, m)\). The second and third Hecke expansions of \(\psi_2(\tau, m)\) are

\[
T_2\psi_2(\tau, m) = 36 - 12E_2(\tau)m^2 + 3E_4(\tau)m^4 + ...
\]  

\[
T_3\psi_2(\tau, m) = 32 - 16E_2(\tau)m^2 + \frac{28}{3}E_4(\tau)m^4 + ...
\]  

(66)

The function \(Z^{(1)}(\tau, m)\) is equal to \(-\psi_2(\tau, m)\) and using \(T_2\psi_2(\tau, m)\), \(T_3\psi_2(\tau, m)\) the functions \(Z^{(2)}(\tau, m), Z^{(3)}(\tau, m)\) are given as

\[
Z^{(2)}_{\psi_2}(\tau, m) = 252 - 84E_2(\tau)m^2 + (8E_2(\tau)^2 + 5E_4(\tau))m^4 + ...
\]  

\[
Z^{(3)}_{\psi_2}(\tau, m) = -1472 + 736E_2(\tau)m^2 + (-144E_2(\tau)^2 - \frac{64}{3}E_4(\tau))m^4 + ...
\]  

(67)

Using (62) and the last equations the first few terms of Borcherds product of \(\psi_2(\tau)\) are

\[
B[\psi_2](\rho, \tau, m) = m^4\eta(\tau)^{24}\eta(2\rho)^{24} - m^6\frac{1}{6}\eta(\tau)^{24}\eta(2\rho)^{24}E_2(\tau)E_2(2\rho)
\]  

\[+ m^8\eta(\tau)^{24}\eta(2\rho)^{24}\left(-\frac{1}{720}E_4(\tau)E_4(2\rho) + \frac{1}{72}E_2(\tau)^2E_2(2\rho)^2\right) + ...
\]  

(68)

This is a Siegel modular form of weight 8. It describes the partition function of two parallel M5-branes on \(K3 \times T^2\) with mass deformation

\[
Z^{SU(2)}_{K3\times T^2}(\rho, \tau, m) = \frac{1}{B[\psi_2](\rho, \tau, m)}
\]  

(69)

It is related to the \(N = 2^* SU(2)\) gauge theory partition function of the \(K3\) surface.

### 4.3 Borcherds lift of \(\psi_3(\tau, m)\): Little string partition function corresponding to \(SU(3)\) theory

In this section we will discuss the Borcherds-lift of \(\psi_3(\tau, m)\) as we defined in the previous two sections for \(\psi_1(\tau, m)\) and \(\psi_2(\tau, m)\). Consider \(\psi_3(\tau, m)\) given in (38) and its Fourier expansion given by

\[
\psi_3(\tau, m) = \sum_{e, s \in \mathbb{Z}} C_{\psi_3}(e, s)Q_{\tau}eQ_{m} = 6Q_m + 6Q_m^{-1} + 12 + ...
\]  

(70)
This shows that \( C_{\psi_3}(0,0) = 12, C_{\psi_3}(0,-1) = C_{\psi_3}(0,1) = 6 \) with all other \( C_{\psi_3}(0,s) \) identically zero. This implies that \( a = 1, b = c = 3 \).

Proceeding as in the previous sections for the computation of \( B[\psi_1](\rho, \tau, m) \) and \( B[\psi_2](\rho, \tau, m) \) we first have

\[
B[\psi_3](\rho, \tau, m) = Q_\tau \varphi_{m}^3 Q_\rho^3 \psi_3(\tau,m) (1 + Q_\rho^3 Z^{(1)}_\psi(\tau,m) + Q_\rho^6 Z^{(2)}_\psi(\tau,m) + Q_\rho^9 Z^{(3)}_\psi(\tau,m) + \ldots) \quad (71)
\]

The function \( \Psi_3(\tau,m) \) turns out to be

\[
\Psi_3(\tau,m) = \prod_{(0, e, s) > 0} (1 - Q_e^* Q_m^*)^{C_{\psi_3}(0,s)} = Q_\tau^{-1} Q_m^{-3} \varphi_{-2,1}(\tau, m)^3 \eta(\tau)^{24} \quad (72)
\]

\( \Psi_3(\tau,m) \) has weight 6. The mass-expansion of \( \psi_3(\tau,m) \) is given by

\[
\psi_3(\tau,m) = 24 - 6E_2(\tau)m^2 + \frac{1}{4}(3E_4(\tau) - E_2(\tau)^2)m^4 + \ldots \quad (73)
\]

To compute \( Z^{(i)}_{\psi_3}(\tau,m) \) we first have to compute the Hecke expansions of \( \psi_3(\tau,m) \)

\[
T_2(\psi_3(\tau,m)) = 36 - 18m^2 E_2(\tau) + m^4(6E_4(\tau) - \frac{3}{2}E_2(\tau)^2) + \ldots
\]

\[
T_3(\psi_3(\tau,m)) = 32 - 24E_2(\tau)m^2 + (17E_4 - 3E_2(\tau)^2)m^4 + \ldots \quad (74)
\]

Using these Hecke transformations we can give the expressions for \( Z^{(i)}_{\psi_3} \) as follows

\[
Z^{(1)}_{\psi_3}(\tau,m) = -\psi_3(\tau,m)
\]

\[
Z^{(2)}_{\psi_3}(\tau,m) = 252 - 126E_2(\tau)m^2 + \left( \frac{27}{2}E_2(\tau)^2 + 12E_4(\tau) \right) + \ldots
\]

\[
Z^{(3)}_{\psi_3}(\tau,m) = -1472 + 1104E_2(\tau)m^2 + (-294E_2(\tau)^2 - 62E_4(\tau))m^4 + \ldots \quad (75)
\]

Using the results in equations \[72\] \[74\] \[75\] we can write down the m-expansion of \( B[\psi_3](\rho, \tau, m) \) as follows

\[
B[\psi_3](\rho, \tau, m) = -m^6 \eta(\tau)^{24} \eta(3\rho)^{24} + \frac{m^8}{4} \eta(\tau)^{24} \eta(3\rho)^{24} E_2(\tau) E_2(3\rho) + \ldots
\]

\[
+ \ m^{10} \eta(\tau)^{24} \eta(3\rho)^{24} \left( - \frac{1}{1152} E_2(\tau)^2 E_4(3\rho) - \frac{35}{1152} E_2(\tau)^2 E_2(3\rho)^2
\]

\[
- \frac{1}{1152} E_4(\tau) E_2(3\rho)^2 + \frac{17}{5760} E_4(3\rho) E_4(\tau) \right) + \ldots \quad (76)
\]

This is a Siegel modular form of weight 6. It describes the partition function of three parallel M5-branes on \( K3 \times T^2 \) with mass deformation

\[
Z^{SU(3)}_{K3 \times T^2}(\rho, \tau, m) = \frac{1}{B[\psi_3](\rho, \tau, m)} \quad (77)
\]

and is related to \( N = 2^* \) \( SU(3) \) partition function of the \( K3 \) surface.

By following the same procedure we can workout \( B[\psi_N](\rho, \tau, m) \) and hence \( Z^{SU(N)}_{K3 \times T^2}(\rho, \tau, m) \) for other values of \( N \).


5 Gromov-Witten potentials

The gauge/geometry correspondence [21] allows us to compute gauge theory prepotential from the type IIA topological string amplitude. Compactifying type IIA superstrings on a CY3-fold one finds, among other terms, F-terms

\[ \int d^4 x F_g(t_i) R_+^2 F_{2g-2}^2, \quad g \geq 1 \]  

(78)

where \( R_+ \), \( F_+ \) are self dual Riemann tensor and graviphoton field strength. The constant background value \( F_+ = \lambda \) is used as a parameter for asymptotic expansion and is called topological string coupling constant. \( F_g \) is the A-model topological string amplitude of the 3-fold. These A-twisted topological string amplitudes are also interpreted as the generating functions of the genus \( g \) Gromov-Witten invariants and appear as integrals on the moduli spaces of genus \( g \) Riemann surfaces. An observer on the world sheet will interpret \( F_g \) as roughly the number of maps form genus \( g \) Riemann surface, possibly with boundary, to the CY3-fold. The presence of world sheet boundary translates to the presence of Lagrangian-branes on the target CY.

The genus \( g \) topological A-model amplitudes are given by [33]

\[
\begin{align*}
F_0 &= F_0^0 + F_0^1 + \ldots = \frac{c_{b_4} c_{b_4} c_{b_4}}{6} + \sum_{\beta \in H_2(X, \mathbb{Z})} N^0_\beta e^{-f_\beta \omega}, \\
F_1 &= F_1^0 + F_1^1 + \ldots = \sum_{a=1}^{h^{1,1}} t_a \int_X c_2(X) \wedge \omega_a + \sum_{\beta \in H_2(X, \mathbb{Z})} N^1_\beta e^{-f_\beta \omega}, \\
F_{g \geq 2} &= F_g + \ldots = (-1)^{g} \left( \int_{\mathcal{M}_g} \chi^3 \right) \frac{\chi(X)}{24} + \sum_{\beta \in H_2(X, \mathbb{Z})} N^g_\beta e^{-f_\beta \omega} 
\end{align*}
\]

(79)

where \( H_2(X, \mathbb{Z}) \) is spanned by the classes \( \{ \omega_1, \omega_2, \ldots, \omega_{h^{1,1}} \} \). \( D_a \) are 4-cycles dual to \( \omega_a \), \( N^g_\beta \) are genus \( g \) Gromov-Witten invariants and \( \lambda_{g-1} \) is the \((g-1)th\) Chern class of the Hodge bundle over the moduli space of genus \( g \) curves \( \mathcal{M}_g \) and

\[
\int_{\mathcal{M}_g} \lambda^3_{g-1} = \frac{|B_{2g}||B_{2g-2}|}{(2g)(2g-2)(2g-2)!}.
\]

(80)

with \( B_{2g} \) being the Bernoulli numbers.

In the large volume limit of the base [33] of certain elliptically and the K3 fibered Calabi-Yau 3-folds, the corresponding Gromov-Witten potential at genus \( g \) can be written as a lifting of a Jacobi form \( \varphi_{2g-2,m}(\tau, z) \) of weight \( 2g - 2 \) and index \( m \).

\[
\mathcal{F}_g := \sum_{l \geq 0} p^l \varphi_{2g-2,m}|_{V_l}(\tau, z) = \frac{c_g(0,0)}{2} \zeta(3 - 2g) + \sum_{(l,n,\gamma) > 0} c_g(ln, \gamma) Li_{3-2g}(p^l q^n \zeta^\gamma)
\]

(81)

where \( Li_r(p^l q^n \zeta^\gamma) \) is the usual Polylogarithm for \( r > 0 \) and a rational function for \( r \leq 0 \). The triplet \((l, n, \gamma)\) is said to be positive if (i) \( l > 0 \) (ii) \( l = 0, n > 0 \) (iii) \( l = n = 0, \gamma > 0 \). Notice that in our case the jacobi forms \( \psi_l(\tau, m) \) have zero weights and hence only the genus one contribution will be there. This additive-lift generates genus two modular forms.
GW-potential for $N = 1$

$$\psi_1(\tau, m) = \sum_{e,s \in \mathbb{Z}} c_1(e, s) Q_e^r Q_s^m = 2Q_m + 2Q_m^{-1} + 20 + ...$$  \hfill (82)

$$F_1 = \sum_{l=0}^{\infty} Q_l^1 \psi_1 |_{T_l}(\tau, m) = \frac{c_1(0, 0)}{2} \zeta(1) + \sum_{(l,n,\gamma) > 0} c_1(ln, \gamma) Li_1(Q_l^1 Q_n^p Q_m^q)$$  \hfill (83)

GW-potential for $N = 2$

$$\psi_2(\tau, m) = \sum_{e,s \in \mathbb{Z}} c_2(e, s) Q_e^r Q_s^m = 4Q_m + 4Q_m^{-1} + 16 + ...$$  \hfill (84)

$$F_1 = \sum_{l=0}^{\infty} Q_l^1 \psi_2 |_{T_l}(\tau, m) = \frac{c_2(0, 0)}{2} \zeta(1) + \sum_{(l,n,\gamma) > 0} c_2(ln, \gamma) Li_1(Q_l^1 Q_n^p Q_m^q)$$  \hfill (85)

GW-potential for $N = 3$

$$\psi_3(\tau, m) = \sum_{e,s \in \mathbb{Z}} c_3(e, s) Q_e^r Q_s^m = 6Q_m + 6Q_m^{-1} + 12 + ...$$  \hfill (86)

$$F_1 = \sum_{l=0}^{\infty} Q_l^1 \psi_3 |_{T_l}(\tau, m) = \frac{c_3(0, 0)}{2} \zeta(1) + \sum_{(l,n,\gamma) > 0} c_3(ln, \gamma) Li_1(Q_l^1 Q_n^p Q_m^q)$$  \hfill (87)

6 Modular forms, and Ray-Singer torsion as the discriminant of the mirror curve

The Ray-Singer torsion is the analytic analogue of the Reidemeister torsion. Intuitively the Reidemeister torsion of a smooth manifold counts the number of closed orbits of the vector fields which are associated to the smooth maps $f : X \rightarrow S^1$. The maps $f$ must not have any critical points. Consider the differential operator $\bar{\partial}_V$ coupled with a vector bundle $V$ on a Kähler manifold $M$

$$\bar{\partial}_V : \bigwedge^p \bar{T}^* \otimes V \rightarrow \bigwedge^{p+1} \bar{T}^* \otimes V$$  \hfill (88)

where $p = 0, 1, ..., \dim(M) - 1$. The existence of a suitable norm and a compatible connection on $V$ is assumed. Formally the holomorphic Ray-Singer torsion of $V$ is defined as the regularised product of the determinants over the spectrum of the Laplacian $\Delta_V = \bar{\partial}_V \bar{\partial}_V^* + \bar{\partial}_V^* \bar{\partial}_V$ acting on $\bigwedge^p \bar{T}^* \otimes V$

$$I(V) = \prod (\det' \Delta^{(p)})(-1)^p$$  \hfill (89)
where the prime indicates that the zero modes are projected out. The Ray-Singer torsion is independent of the choice of metric that goes into the definition of the Laplacian $\Delta^p$. A useful, although formal, integral representation of $I(V)$ is given by

$$
\log(I(V)) = \int_{\epsilon}^{\infty} \frac{ds}{s} \text{Tr}'(-1)^p p e^{-s\Delta_V}
$$

Interestingly in analogy with the genus one free energy the holomorphic Ray-Singer torsion satisfies the Quillen anomaly equation given by

$$
\partial \bar{\partial} \log I(V) = \partial \bar{\partial} \sum_p (-1)^p \log \det (g^{(p)}) + 2\pi i \int_M \text{Td}(T) \text{Ch}(V)|_{(1,1)}
$$

where $T$ denotes the tangent bundle of $M$, $\det(g^{(p)})$ is determinant of the matrix $g^{(p)}$ whose entries are the inner products in the space of the kernel of $\bar{\partial}_V$. This anomaly arises in dealing with the determinant of the differential operators that depends on a parameter. The left-hand side and the right-hand side of the equation (91) denote a $(1,1)$-form on the complex structure moduli space.

In terms of the operators of the twisted $N = 2$ conformal theory the relevant quantity is the index defined by

$$
\mathcal{I} = \text{Tr}[(-1)^F F_L F_R q^{H_L} \bar{q}^{H_R}]
$$

where $q = e^{2\pi i \tau}$, $\bar{q} = e^{2\pi i \bar{\tau}}$ for the torus period $\tau, 2\pi H_L = L_0 - c/24, 2\pi H_R = \bar{L}_0 - c/24$. Moreover the right and left Fermi numbers operator $F_R, F_L$ are conserved. We can construct topological string genus one free energy (with $SL(2, \mathbb{Z})$ covariance) by taking an average over the fundamental domain $\mathcal{F}$ of the upper half plane

$$
F_1 = -4 \int_{\mathcal{F}} \frac{\mathrm{d}^2 \tau}{\tau_2} \text{Tr}[(-1)^F F_L F_R q^{H_L} \bar{q}^{H_R}]
$$

where $\tau_2 = \text{Im}(\tau)$. If the $N = 2$ theory is a sigma model the total hamiltonian is the Laplacian $\Delta = \bar{\partial} \partial^* + \partial^* \bar{\partial}$, and the harmonic $(p,q)$-forms describe the susy vacua. The action of the operators $F_L$ and $F_R$ is translated to the holomorphic and anti-holomorphic degrees of the differential forms denoted by $p$ and $q$ respectively, here. Combining the last equation with (90), we get $F_1$ in terms of the holomorphic Ray-Singer torsion as

$$
F_1 = \frac{1}{2} \sum_q (-1)^q q \log I(\wedge^q T^*)
$$

For $K3 \times T^2$

$$
F_1 = \frac{1}{2} \sum_{q=0}^{3} \sum_{p=0}^{3} (-1)^{p+q} pq \log(\det'\Delta^{(p,q)})
= \log \frac{\det'(\Delta^{(0,0)})^2 \det'(\Delta^{(1,1)})^{1/2}}{\det'(\Delta^{(1,0)})^3}
$$

The CY3-fold $K3 \times T^2$ is a self mirror manifold [25] and therefore one can use the complex structure moduli or Kähler moduli. From general considerations of topological strings, $F_1$
satisfies a holomorphic anomaly equation \[^{6}\mathbf{3}\]. If we denote by \(G_{ij}\) the Zamolodchikov metric and by \(C_i\) respectively \(C_j\) the action of operator \(\phi_i\) respectively \(\phi_j\) on the Ramond ground states then the holomorphic anomaly equation satisfied by \(F_1\) is given by \[^{16}\]

\[
\partial_j \partial_i F_1 = \text{Tr}(\mathbf{-1)^FC}_iC_j - \frac{1}{12} G_{ij}\text{Tr}(\mathbf{-1)^F} \\
= \text{Tr}(\mathbf{-1)^FC}_iC_j - \frac{1}{12} G_{ij}\chi(M) \quad (96)
\]

In fact the Quillen anomaly equation \((91)\) is identical to the holomorphic anomaly equation \((96)\) satisfied by the genus-one topological string free energy. The topological strings partition function of \(K3 \times T^2\) is a genus one partition function that is expressible in terms of Ray-Singer analytic torsion. For our purposes the important fact is that the Ray-Singer analytic torsion is the discriminant of the mirror curve. As shown in \[^{31},^{37}\], the mirror curves for the non-compact CY3-fold \(X_{m,1}\) are the theta divisors on the abelian varieties \(A_r = \mathbb{C}^g/\Lambda_r\) and are defined as

\[
\Theta_r := \{ z \in A_r; \theta(z, \tau) = 0 \}, \quad \Theta_m := \{(u, z, \tau) \in \mathbb{P}(V_m) \times A : \sum u_a \theta_{a,0}(z, \tau) = 0 \} \quad (97)
\]

where \(V_n = \mathbb{C}^{g \times n}\) is the space of coordinates \(u_a\) and we can define a projection map \(\pi = id_{\mathbb{P}(V_m)} \times p : \mathbb{P}(V_m) \times A \to \mathbb{P}(V_m) \times \mathbb{H}_g\) and

\[
\theta(z, \tau) = \sum m \in \mathbb{Z}^g \exp \left( \pi i m^i \tau m + 2 \pi i m z \right) \quad (98)
\]

\[
\theta_{a,b}(z, \tau) = \sum n \in \mathbb{Z}_g \exp \left( \pi i (n + a)^i \tau (n + a) + 2 \pi i (n + a)^i (n + b) \right)
\]

for \(a, b \in \mathbb{R}^g\) and \(\mathbb{H}_g\) is the genus \(g\) Siegel upper half plane. For a review see the appendix \[^{14}\]. The fibers \(\Theta_{m,(u,\tau)} = \pi^{-1}(u, \tau)\) are projective spaces and are elements of a complete linear system \[^{23}\] denoted by \([L_m, \tau]\). The singular locus or the discriminant locus of \(\pi : \Theta_m \to \mathbb{P}(V_m) \times \mathbb{H}_g\) is defined by \(D_{g,m} := \{(u, \tau) \in \mathbb{P}(V_m) \times \mathbb{H}_g : \text{Sing}\Theta_{m,(u,\tau)} \neq \emptyset\}\). One of the main results proven in \[^{37}\] is that the Ray-Singer torsion \(I(\Theta_r)\) is related to the Siegel cusp form of weight \(\frac{(g+3)g!}{2}\) as

\[
I(\Theta_r) = \left( \frac{(\det \text{im}\tau)^{(g+3)g!}}{2} |\Delta_g(\tau)|^{\frac{(g-1)^g}{(g+1)}} \right) \quad (99)
\]

Moreover \(\Delta_g\) can be factorized as

\[
\Delta_g(\tau) = \chi_g(\tau) J_g(\tau)^2 \quad (100)
\]

where \(J_g(\tau)\) is a Siegel modular form of weight \(\frac{(g+3)g!}{4} - 2^{g-3}(2^g + 1)\) and \(\chi_g(\tau)\) corresponding to \(\theta_{\text{null},g}\) can be written in terms of the characteristic theta constants as

\[
\chi_g(\tau) := \prod_{(a,b)\text{even}} \theta_{a,b}(0, \tau) \quad (101)
\]

A genus two theta constant is defined as

\[
\theta_{a_1, a_2, b_1, b_2} \left( \begin{array}{cc} \tau & 0 \\ 0 & \tau \end{array} \right) = \theta_{a_1 b_1}(0, \tau) \theta_{a_2 b_2}(0, \tau) \quad (102)
\]
There are ten even theta constants given by

\[ \theta_{0000}, \theta_{1000}, \theta_{0100}, \theta_{0010}, \theta_{0110}, \theta_{0001}, \theta_{0011}, \theta_{1011}, \theta_{1111} \]  

(103)

In many cases the modular forms can be expressed in terms of the theta constants, such as the Siegel theta constant \( \Delta_4(\Omega) \) can be expressed as

\[ \Delta_4(\Omega)^2 = \left( \frac{1}{2} \theta_{1111}(\Omega') \right)^2 \]  

(104)

for \( \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \) and \( \Omega' = \begin{pmatrix} \tau & 2z \\ 2z & 2\sigma \end{pmatrix} \).

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A Modular forms and paramodular forms

A.1 Modular forms

To begin with the simplest examples of modular forms \([20, 12]\), the Eisenstein series \( E_k(\tau) \) for \( k > 2 \) are modular forms of weight \( k \)

\[ E_k(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k E_k(\tau) \]  

(105)

under the \( SL(2,\mathbb{Z}) \) transformations \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \) and for \( \tau \) in the upper half plane \( \mathcal{H} \).

The second Eisenstein series on the other hand transforms anomalously

\[ E_2(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{4\pi i} \]  

(106)

The modular forms \( E_k, k > 2 \) form a ring which is generated by \( \{ E_4(\tau), E_6(\tau) \} \).

The Jacobi form \( J_{w,l} \) of weight \( w \) and index \( l \) is a holomorphic function with the following properties

\[ J(\frac{a\tau + b}{c\tau + d}, \frac{m}{c\tau + d}) = (c\tau + d)^w e^{2\pi ilm^2 \sigma/c\tau + d} J(\tau, m) \]

\[ J(\tau, m + u\tau + v) = e^{-2\pi i(lu^2 \tau + 2um)} J(\tau, m) \]  

(107)

for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \) and \( u, v, l \in \mathbb{Z} \).

Two important Jacobi forms of index 1 that are often used in the text are \( \varphi_{-2,1}(\tau, m), \varphi_{0,1}(\tau, m) \).
They can be expressed as

\[
\varphi_{0,1}(\tau, m) = 4 \sum_{a=2}^{4} \frac{\theta_a(\tau, m)^2}{\theta_a(\tau, 0)^2} = \frac{3}{\pi} \varphi(\tau, m) \frac{\theta_1(\tau, m)^2}{\eta(\tau)^6} \\
\varphi_{-2,1}(\tau, m) = -\frac{\theta_1(\tau, m)^2}{\eta(\tau)^6} = -m^2 e^{\pi i \tau} \sum_{k \geq 1} \frac{(-1)^k B_{2k} E_{2k}(\tau)}{(2k)!} m^{2k}
\]

(108)

where the Weierstrass \( \wp(z, w) \) function is defined as

\[
\wp(z, w) = \frac{1}{z^2} + \sum_{w \in \mathbb{Z} + \tau \mathbb{Z}, w \neq 0} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right)
\]

(109)

and the Bernoulli numbers \( B_{2k} \) are defined by

\[
x = \sum_{k \geq 0} B_k \frac{m^k}{k!}
\]

(110)

The Bernoulli number \( B_k \) for \( k \) odd is zero.

The theta function \( \eta(\tau) \) and Jacobi theta functions \( \theta_i, i = 1, 4 \) are given by

\[
\eta(\tau) = e^{\frac{\pi i}{24} \sum_{n \geq 1} \left( 1 - Q_n^\tau \right)}
\]

\[
\theta_1(\tau, m) = -i Q_r(Q_m^{-\frac{1}{2}} - Q_m^\frac{1}{2}) \prod_{r \geq 1} (1 - Q_r^\tau)(1 - Q_r^\tau Q_m)(1 - Q_r^\tau Q_m^{-1})
\]

\[
\theta_2(\tau, m) = Q_r^\frac{1}{2} \prod_{r \geq 1} (1 - Q_r^\tau)(1 + Q_r^\tau Q_m)(1 + Q_r^\tau Q_m^{-1})
\]

\[
\theta_3(\tau, m) = \prod_{r \geq 1} (1 - Q_r^\tau)(1 + Q_r^{-\frac{1}{2}} Q_m)(1 + Q_r^{-\frac{1}{2}} Q_m^{-1})
\]

\[
\theta_4(\tau, m) = \prod_{r \geq 1} (1 - Q_r^\tau)(1 - Q_r^{-\frac{1}{2}} Q_m)(1 - Q_r^{-\frac{1}{2}} Q_m^{-1})
\]

(111)

for \( Q_\tau = e^{2\pi i \tau}, Q_m = e^{2\pi i m} \).

The Jacobi theta function \( \theta_1(\tau, m) \) satisfies the following important properties

\[
\theta_1(\tau, m + \frac{\zeta}{2\pi}) = \theta_1(\tau, m) e^{\frac{1}{24} E_2(\tau) \zeta^2} + \frac{\theta_1' \zeta}{2\pi \theta_1(\tau, m)} - \sum_{n \geq 2} \varphi^{(n-2)}(\tau, m) \frac{\zeta^n}{n!}
\]

(112)

\( \varphi_{-2,1}(\tau, m) \) and \( \varphi_{0,1}(\tau, m) \) are the two most important examples of Jacobi forms of index 1 and they are defined as

\[
\varphi_{-2,1}(\tau, m) = -\frac{\theta_1(\tau, m)}{\eta(\tau)^6}
\]

\[
= -m^2 e^{\pi i \tau} \sum_{k \geq 1} \frac{(-1)^k B_{2k} E_{2k}(\tau)}{k(2k)!}
\]

\[
= -m^2 + \frac{E_2(\tau)}{12} m^4 + \frac{-5 E_2(\tau)^2 + E_4(\tau)}{1440} m^6 + ...
\]

(113)
\[ \varphi_{0,1}(\tau, m) = 4 \sum_{i=2}^{4} \frac{\theta_i(\tau, m)^2}{\theta_i(\tau, 0)^2} = 12 - E_2(\tau)m^2 + \frac{1}{24}(E_2(\tau)^2 + E_4(\tau))m^4 + ... \] (114)

### A.2 Paramodular forms

The paramodular group of level \( t \) \([3,5]\), \( \Gamma_t \), is defined as

\[
\Gamma_t = \begin{bmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap Sp(4, \mathbb{Q}) \] (115)

An extension of \( \Gamma_t \) can be defined as

\[ \Gamma_t^+ = \Gamma_t \cup \Gamma_t V_t \] (116)

where

\[ V_t = \begin{bmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \end{bmatrix} \] (117)

A matrix \( M \) of \( \Gamma_t^+ \) can be decomposed into block form as

\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \] (118)

Recall that the Siegel upper half plane \( \mathbb{H}_2 \) is defined in terms of the matrix \( \Omega = \begin{pmatrix} \tau & z \\ z & \rho \end{pmatrix} \) as

\[ \det(\text{Im}(\Omega)) > 0, \quad \text{Tr}(\text{Im}(\Omega)) > 0 \] (119)

The action of \( M \) on \( \Omega \) is given by

\[ M(\Omega) = (A\Omega + B)(C\Omega + D)^{-1} \] (120)

A weight \( k \) meromorphic paramodular form \( \Phi_k(\Omega) \) satisfies

\[ \Phi_k(M(\Omega)) = \det(C\Omega + D)^k \Phi_k(\Omega) \] (121)

### B Partitions

A sequence \( \nu = (\nu_1, ..., \nu_k) \) with non-increasing order is called the partition of some non-negative integer. The length of the partition \( \nu \) is denoted by \( l(\nu) \) and size of the partition is denoted by \( |\nu| \). A Young diagram gives a pictorial representation of the partition \( \nu \) as

\[ \{(i, j) \in \mathbb{N} | 1 \leq j \leq \nu_i\} \] (122)
The transpose of partition \( \nu' \) is obtained by reflecting \( \nu \) around the diagonal. The arm length \( a(\nu) \), leg length \( l(\nu) \) and the hook length \( h(\nu) \) are defined as

\[
\begin{align*}
a(\nu) &= \nu_i - j, \\
l(\nu) &= \nu_j' - i, \\
h(\nu) &= \nu_i - j + \nu_j' - i + 1
\end{align*}
\]  

(123)

C Borcherds Lift

For weight zero Jacobi forms the Borcherds product can be expressed in terms of the exponential lift of Hecke operator. To elaborate on this procedure [2, 13, 19, 8, 9] first consider the group \( G_0(N) \) consisting of matrices with integer entries of the block-diagonal form

\[
\begin{bmatrix}
A & B \\
NC & D
\end{bmatrix} \in \text{Sp}(2, \mathbb{Z})
\]  

(124)

\( G_0(N) \) admits an action of the Hecke operator \( T_n \) for integer \( n \), such that for a weak Jacobi form \( \phi_{k,m} \) of weight \( k \) and index \( m \), \( T_n(\phi_{k,m}) = \phi_{k,mm} \) is another weak Jacobi form of weight \( k \) and index \( mn \). Since \( G_0(N) \) has multiple cusps in its fundamental domain the Hecke operator is more involved as compared to that of \( SL(2, \mathbb{Z}) \). For a given weak Jacobi form \( \phi^k(\rho, z) \) we define

\[
L_{\phi}^k(\rho, \sigma, z) = \sum_{n=1}^{\infty} \left( \frac{1}{n} \right) \sum_{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)/\Delta_N(n)} e^{\frac{2\pi imcz}{c+d}} e^{2\pi i\sigma n} \phi\left(\frac{a\rho + b}{d}, az\right)
\]  

(125)

For \( L_{\phi}^k(\rho, \sigma, z) \) to be a Siegel modular form of zero weight it must be invariant under modular transformation, elliptic transformation and must have a Fourier expansion. The modular form is defined on the Siegel upper half plane \( \mathcal{H}_2 \) defined as \( \mathcal{H}_2 = \{ \Omega \in \mathcal{M}_2(\mathbb{C})|\Omega = \Omega^T, \text{Im}(\Omega) > 0 \} \). Clearly \( T_n(\phi^k)(\rho, z)e^{2\pi i\sigma n} \) is invariant under the modular and elliptic transformations. To investigate the third condition consider writing \( L_{\phi}^k \) in a product form. Using the explicit form of Hecke transformation the equation (125) can be written as

\[
L_{\phi} = \sum_{n=1}^{\infty} \left( \frac{1}{n} \right) \sum_{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)/\Delta_N(n)} e^{\frac{2\pi imcz^2}{c+d}} e^{2\pi i\sigma n} \phi\left(\frac{a\rho + b}{d}, az\right) e^{2\pi i\sigma n}
\]  

(126)

To write a representative element of \( \Gamma_0(N)/\Delta_N(n) \) choose the complete set of cusps \( \{s\} \) of \( \Gamma_0(N) \) given by a set of matrices \( g_s \),

\[
g_s \in SL(2, \mathbb{Z}) = \begin{bmatrix} x_s & y_s \\ z_s & w_s \end{bmatrix}
\]  

(127)

Next define a natural number \( h_s \) by

\[
g_s^{-1}\Gamma_0(N)g_s \cap P(\mathbb{Z}) = \{ \pm \begin{bmatrix} 1 & h_sn \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \}
\]  

(128)
where \( P(\mathbb{Z}) \) is the set of all upper-triangular matrices over the field of integers with unit determinant. A more explicit form for \( \Gamma_0(N)/\Delta_N(n) \) is then given by
\[
\Gamma_0(N)/\Delta_N(n) = \bigcup_s \{ g_s = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}; a, b, d \in \mathbb{Z}, \quad ad = n, \quad az_s = 0 \mod N, \quad b = 0, 1, ..., h_s d - 1 \big\}
\]
For each cusp define \( n_s = \begin{align} \frac{N}{g.c.d(z_s, N)} \end{align} \) and the form
\[
\phi_s(\rho, z) = \phi\left( \frac{x_s \rho + y_s}{z_s \rho + w_s}, \frac{z}{z_s \rho + w_s} \right)
\]
that can be Fourier expanded as
\[
\phi_s(\rho, z) = \sum_{n, l} c_s(n, l) e^{2\pi i (n \rho + l z)}
\]
\[c_s(n, l) = c_{s, l}(4n - l^2).\] Next following [13]
\[
L \phi = \sum_s \sum_{n=1}^{N} \sum_{a \equiv 0 \mod N} b_s d^{-1} \phi_s(\frac{a \rho + b}{d}, az) e^{2\pi i n_s \sigma}
\]
\[
= \sum_s \sum_{n=1}^{N} \sum_{a \equiv 0 \mod N} (ad)^{-1} d \sum_{n, l \in \mathbb{Z}} c_{s, l}(4n_1 d - l^2) e^{2\pi i (an_1 \rho + az + n \sigma)}
\]
\[
= \sum_s h_s \sum_{a=1}^{N} \sum_{m=1}^{N} \sum_{n, l \in \mathbb{Z}} c_{s, l}(4n_1 d - l^2) e^{2\pi i (an_1 \rho + az + n \sigma)}
\]
\[
= \sum_s h_s \log \left( \prod_{l, m, n \in \mathbb{Z}, m \geq 1} (1 - e^{n_s (n_1 \rho + l z + m \sigma)}) c_{s, l}(4mn_1 - l^2) \right)
\]
or
\[
e^{L \phi} = \prod_{(n, l, m) > 0} (1 - (q^n y^l p^m)^{n_s} h_s n_s^{-1} c_{s, l}(4mn - l^2))
\]
for \( q = e^{2\pi i p}, y = e^{2\pi i z}, p = e^{2\pi i a} \). In the product representation [133] although the coefficients \( c_{s, l}(4mn - l^2) \) are symmetric under \( m \leftrightarrow n \), the ranges of \( m \) and \( n \) are not. To make the expression symmetric we have to multiply it by an appropriate factor that can be determined by inspection. The result of this is that one obtains a Siegel modular form as the multiplicative-lift of the weak Jacobi form \( \phi(\rho, z) \),
\[
\Phi_k = Q_n^a Q_m^b Q_p^c \prod_{(n, l, m) > 0} (1 - (Q_n^a Q_m^b Q_p^c)^{n_s}) h_s n_s^{-1} c_{s, l}(4mn - l^2)
\]
for \( b \) some integer and \( a \geq 0, c \geq 0 \). The condition \( (l, m, n) > 0 \) means one of the following
\[(i) m > 0, n, l \in \mathbb{Z} \text{ or } (ii) m = 0, n > 0, l \in \mathbb{Z} \]
Next note that

\[ Q^\rho_m \prod_{n=m,l<0} (1 - Q^l_m)^{c(0,l)} = \prod_{l<0} Q_m^{\frac{c(0,l)}{2}} (1 - Q_m^l)^{c(0,l)} = \prod_{l>0} (Q_m^l - Q_m^{-l})^{c(0,l)} \]  

(135)

Then

\[ Q^\rho_m Q^\rho_n \prod_{n=m,l<0} (1 - Q^l_m)^{c(0,l)} \prod_{n=m,l<0} (1 - Q^l_n)^{c(0,l)} \]

\[ = Q^\rho_m Q^\rho_n \prod_{n>0} (1 - Q^n_m)^{c(0,n)} \prod_{l>0} \left( \frac{Q_m^l - Q_m^{-l}}{Q_m^{c(0,l)}} (1 - Q_m^n)^{c(0,l)} \right) \]

\[ = Q^\rho_m \eta(\tau)^{c(0,0)} \prod_{l>0} \frac{\theta_1(\tau; l\tau)}{\eta(\tau)^{c(0,l)}} \]  

(137)

Therefore we can write the Borcherds product as

\[ \Phi_k = Q^\rho_m \eta(\tau)^{c(0,0)} \prod_{l>0} \frac{\theta_1(\tau; l\tau)}{\eta(\tau)^{c(0,l)}} e^{L \phi_{k,1}} \]  

(138)

### D Determinant bundles and Ray-Singer Torsion

Consider a proper smooth morphism (37) \( \pi : X \to S \) of Kähler manifolds X and S and a determinant line bundle \( \lambda(\mathcal{O}_X) \) on X defined as (37):

\[ \lambda(\mathcal{O}_X) := \otimes_{q \geq 0} (\det R^q \pi_* \mathcal{O}_X)^{(-1)^q} \]  

(139)

There exists a Hermitian metric \( g_{X/S} \) on \( TX/S := \text{mer} \pi_* \) whose restriction to any fiber \( X_t = \pi^{-1}(t) \) is Kähler. The line bundle \( \lambda(\mathcal{O}_X)_t \) can thus be expressed as

\[ \lambda(\mathcal{O}_X)_t := \otimes_{q \geq 0} (\wedge^{\max} H^{0,q}(X_t))^{(-1)^q} \]  

(140)

for the space of \((0,q)\)-forms \( H^{0,q} \). This identification of determinant line bundle endows it with an \( L^2 \)-metric relative to \( g_{X/S} \) denoted by \( ||.||_L^2 \). The norm \( ||.||_L^2(t) \) is not smooth but can be made so by multiplying it by holomorphic torsion \( I(X_t) \) of the fiber \( X_t \). The modified smooth metric is called Quillen metric \( ||.||_Q(t) \) of \( \lambda(\mathcal{O}_X)_t \) and expressed as

\[ ||.||_Q(t) := I(X_t)||.||_L^2(t) \]  

(141)
The Quillen anomaly has its origin in the non-triviality of the line bundle \( \lambda(O_X)_t \). The symplectic group \( \Gamma_g = Sp(2g, \mathbb{Z}) \) acts on the theta divisors and is uplifted to its action on the corresponding line bundles \( L \). However only the subgroup \( \Gamma_g(1,2) := \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \} \in \Gamma_g(A^t C)_0 \equiv (B^t D)_0 \equiv 0 \mod 2 \) where \( X_0 = x_{ij} \delta_{ij} \) is the diagonal of \( X = x_{ij} \), preserves the line bundle. The space of sections \( s_c \in H^0(\mathbb{P}(V_m) \times \mathbb{H}_g, \lambda(O_{\Theta_m})^{(-1)^g}) \) is generated by \[ (142) \]

\[
s_c := \wedge_{u \in B_m} \frac{u_c \theta_u}{\sum_{b \in B_m} u_b \theta_b} dz_1 \wedge dz_2 \wedge ... \wedge dz_g
\]

In fact for \( g > 1 \) and \( m \geq 2 \) the space of functions \( \{ \sigma_J \}_{|J|=m^g} \) defined as \[ (143) \]

\[
\sigma_J|_{U \times \mathbb{H}_g} := \frac{u^J}{u^{m^g}} s_c
\]
gives a map to a well defined basis of \( \lambda(O_{\Theta_m})^{(-1)^g} \). For \( m = 1 \) one gets an expression for section of \( \lambda_{\Theta_n} \) as \[ (144) \]

\[
\sigma_{\Theta_n} := 1_h \otimes (dz_1 \wedge ... \wedge dz_g)^{(-1)^g}
\]

The Quillen norm of \( \sigma_J \) is given by \[ (145) \]

\[
||\sigma_J||^2_Q = (\det \text{Im} \tau)^{(g-1)m^g} \left| \frac{u^J}{\Delta_{g,m}(u, \tau)^{(g+1)^g}} \right|^2
\]

where \( \Delta_{g,m}(u, \tau)^{(g+1)^g} \) is a holomorphic function homogenous in variable \( u_i \) and satisfies the following automorphy property \[ (146) \]

\[
\Delta_{g,m}(\gamma, u, \gamma.\tau) = \gamma_{g,m}(\gamma, u, \tau) \gamma(\tau, \gamma) \frac{(g+3)m^g}{2} \Delta_{g,m}(u, \tau)
\]

Moreover \( \Delta_{g,m}(u, \tau) \) defines the projective surface \( D_{g,m,\tau} \) which is the fiber of the map \( p : D_{g,m} \to \mathbb{H}_g \).

We can specialise to \( m = 1 \) corresponding to the smooth theta divisor \( \Theta \) and find the Quillen norm as \[ (147) \]

\[
||\sigma_\Theta||^2_Q = (\det \text{Im} \tau)^{(-1)^g} \left| \frac{1}{\Delta_\Theta(\tau)^{(1)^g}} \right|^2
\]

On the other hand the \( L^2 \)-norm can be calculated to be \[ (148) \]

\[
||\sigma_\Theta||^2_{L^2(\Theta)} = (\det 2 \text{Im} \tau)^{(-1)^g}
\]

The ratio \( \frac{||\sigma_\Theta||^2_Q}{||\sigma_\Theta||^2_{L^2(\Theta)}} \) defines the Ray-Singer torsion \( I \).

The embedding or projective compactification corresponding to the complete linear system \( |L_{m,n,\tau}| \) is defined by the theta variables \( \theta_{a,b} \) with characteristics \( a \in \frac{1}{m} \mathbb{Z}^g, b \in \frac{1}{n} \mathbb{Z}^g \). The theta coordinates \( \theta_{a,b} \) define the sections of line bundle \( L_{m,n} \) and also the embedding in the projective space \( \mathbb{P}^{m^g n^g - 1} \) by the algebraic relations for \( m, n \geq 4 \) \[ (36) \]

\[
\theta_{y_1,\rho} \sum_{z \in \mathbb{Z}_2} \rho(z)X_{y+y_z+2}X_{y-y_z+2} = \theta_{y_2,\rho}(0) \sum_{z \in \mathbb{Z}_2} \rho(z)X_{y+y_1+z}X_{y-y_1+z}
\]

28
where $\rho(z)$ are the characters of $\text{Hom}(X_2, \pm 1)$ and

$$
\theta_{y_1 \cdot \rho}(0) = \sum_{z \in \mathbb{Z}_2} \rho(z) \theta_{y_1 + z}(0)^2,
$$

$$
\theta_y : = \theta \left( \begin{array}{c} a \\ b \end{array} \right) (z)
$$

(150)

for $y, y_1, y_2 \in \mathbb{Z}_m \oplus \mathbb{Z}_n$ and $y \equiv y_1 \equiv y_2 \mod 2\mathbb{Z}_m \oplus 2\mathbb{Z}_n$. The eq. (149) describes the projectification of the linear system $|L_{m, \tau}|$ in terms of the homogeneous coordinates $X_a$ for $a \in B_m$. As indicated before $\text{Divisor}(\Delta_{g,m}) = \mathcal{D}_{g,m}$. For a given $J$ we can expand $\Delta_{g,m}(u, \tau)$ in terms of holomorphic functions $f_J(\tau)$ as follows

$$
\Delta_{g,m}(u, \tau) = \sum_J f_J(\tau) u^J
$$

(151)

and it can be shown that

$$
f_J(\tau) = f_{(m g (g+1), 0, \ldots, 0)}(\tau) = \Delta_g(\tau)^{m^g}
$$

(152)

To elucidate the structure we can consider the normalised modular form $\frac{\Delta_{g,m}(u, \tau)}{m^{\frac{g(g+1)}{2}} \Delta_g(\tau)^{m^g}}$

$$
\tilde{\Delta}_{g,m}(u, \tau) : = \frac{\Delta_{g,m}(u, \tau)}{m^{\frac{g(g+1)}{2}} \Delta_g(\tau)^{m^g}}
$$

$$
= u_0^{m^g} + \sum_{J \neq J_0} f_J(\tau) u^J
$$

(153)

The algebraic equations (149) carve an algebraic variety $\mathcal{A}_m$ in $\mathbb{P}_k^{m^g}$. The projective dual variety $\mathcal{\tilde{A}}_m$ is described by the equation of the modular form $\tilde{\Delta}_{g,m}(u, \tau)$. Notice that this modular form is homogeneous in variables $u_i$ and monic in $u_0$. The modular form $\tilde{\Delta}_{g,m}(u, \tau)$ can be expressed in terms of theta constants i.e. $\tilde{\Delta}_{g,m}(u, \tau) \in \mathbb{Q}[^{\theta_{a,b}(0,\tau)}_{a,b \in B_m}]$, moreover there exists a constant $C_g$ such that the ring $\mathbb{Z}[^{\theta_{a,b}(0,\tau)}_{a,b \in B_m}]$ contains $C_g^{-1} \Delta_g(\tau)$ with the Fourier coefficients of the later taking values in $\mathbb{Q}$.

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