Online statistical inference for parameters estimation with linear-equality constraints

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Abstract

Stochastic gradient descent (SGD) and projected stochastic gradient descent (PSGD) are scalable algorithms to compute model parameters in unconstrained and constrained optimization problems. In comparison with SGD, PSGD forces its iterative values into the constrained parameter space via projection. From a statistical point of view, this paper studies the limiting distribution of PSGD-based estimate when the true parameters satisfy some linear-equality constraints. Our theoretical findings reveal the role of projection played in the uncertainty of the PSGD-based estimate. As a byproduct, we propose an online hypothesis testing procedure to test the linear-equality constraints. Simulation studies on synthetic data and an application to a real-world dataset confirm our theory.

Keywords: Online inference, Constrained optimization, Projected stochastic gradient descent algorithm

2020 MSC: Primary 62F12, Secondary 62L20

1. Introduction

With the rapid increase in availability of data in the past two decades or so, many classical optimization methods for statistical problems such as gradient descent, expectation-maximization or Fisher scoring cannot be applied in the presence of large datasets, or when the observations are collected one-by-one in an online fashion\,\cite{5, 20}. To overcome the difficulty in the era of big data, a computationally scalable algorithm called stochastic gradient descent (SGD) proposed in the seminal work\,\cite{17} has been widely applied and achieved great success\,\cite{1, 6, 23}. In comparison with classical optimization methods, one appealing feature of SGD is that the algorithm only requires accessing a single observation during each iteration, which makes it scale well with big data and computationally feasible with streaming data.

Due to the success of SGD, the studies of its theoretical properties have drawn a great deal of attention. The theoretical analysis of SGD can be categorized into two directions based on different research interests. The first direction is about the convergence rate. Existing literature shows that SGD algorithm can achieve a (in terms of regret) $O(1/T)$ convergence rate for strongly convex objective functions (e.g., see\,\cite{2, 9}), and a $O(1/\sqrt{T})$ rate for general convex cases\,\cite{12}, where $T$ is the number of iterations. The second direction focuses on applying SGD to statistical inference. It was proved that the SGD estimate is asymptotic normal (e.g., see\,\cite{13}) under suitable conditions. However, unlike classical parameter estimates, the SGD estimate may not be root-$T$ consistent, and its convergence rate depends on the learning rate. To improve the convergence rate,\,\cite{15} and\,\cite{19} independently proposed...
the averaged stochastic gradient descent (ASGD) estimate, which was obtained by averaging the updated values in all iterations. They showed that the ASGD estimate is root-$T$ consistent, while its asymptotic normality was proved by [16]. Following [16], there is a vast amount of work related to conducting statistical inference based on ASGD estimates. For example, [20] proposed a hierarchical incremental gradient descent (HiGrad) procedure to construct the confidence interval for the unknown parameters. In comparison with ASGD estimate, the flexible structure makes HiGrad easier to parallelize. In [5], the authors developed an online bootstrap algorithm to construct the confidence interval, which is still applicable when there is no explicit formula for the covariance matrix of the ASGD estimate. Recently, [3] proposed a plug-in estimate and a batch-means estimate for the asymptotic covariance matrix. With strong convexity assumption on the objective function, they proved the convergence rate of the estimates.

When there are constraints imposed on the parameters, the SGD algorithm is often combined with projection, which forces the iterated values into the constrained parameter space. The convergence rate of this projected stochastic gradient descent (PSGD) is also well studied (e.g., see [12]), which is proved to be the same as that of SGD. In the view of statistical inference, [10] studied the asymptotic distribution of PSGD estimate when the model parameters are in the interior of the constrained parameter space. It was proved that the projection operation only happens a finite number of times almost surely. As a consequence, the limiting distribution of PSGD estimate is exactly the same as that of SGD estimate. Recently, [7] studied the limiting distribution of averaged projected stochastic gradient descent (APSGD) estimate, which is the averaged version of PSGD. When the model parameters are in the interior of the constrained parameter space, APSGD and ASGD estimates have the same limiting distribution.

This paper aims to quantify the uncertainty in APSGD estimates when the model parameters satisfy some linear-equality constraints. Compared to the existing literature, a significant difference of our model is that the model parameters are not in the interior of the constrained parameter space. Therefore, the projection operation will take place during every iteration, and the limiting distribution of the APSGD estimate turns out to be a degenerate multivariate normal distribution. The contribution of current work is threefold:

(i) We derive the limiting distribution of the APSGD estimate, which is proved to be at least as efficient as ASGD estimate under mild conditions.

(ii) An online specification test for the linear-equality constraints is proposed based on the difference between APSGD and ASGD estimates.

(iii) Our findings reveal that, when the true parameters are not in the interior of the parameter space, the APSGD and ASGD estimates could have different limiting distributions.

This paper is organized as follows. In Section 2, we mathematically formulate the parameters estimation problem with linear-equality constraints. Section 3 proposes the APSGD estimate and studies its asymptotic properties. An online specification test is proposed in Section 4. All the mathematical proofs are deferred to the appendix. A set of Monte Carlo simulations to investigate the finite sample performance of the proposed methods and an application to a real-world dataset are provided in a supplementary material.

2. Problem formulation

We consider the problem to conduct statistical inference about the model parameter

$$\theta^* = \arg \min_{\theta \in \mathbb{R}^p} \{ L(\theta) := E[l(\theta, Z)] \},$$

(2.1)
where $l(\theta, Z)$ is the loss function, and $Z$ is a single copy drawn from an unknown distribution $F_{\theta^*}$. Moreover, we assume that additional information about the truth $\theta^*$ is available:

$$B\theta^* = b, \quad (2.2)$$

where $B$ and $b$ are some prespecified matrix and vector with comfortable dimensions. The loss function specified by (2.1) is quite general and covers many popular statistical models, which are illustrated by the following examples.

Example 1. (Mean Estimation) Suppose $Z \in \mathbb{R}^p$ is random vector with mean $\theta^* = E(Z)$. The loss function becomes $l(\theta, z) = \frac{1}{2}|z - \theta|^2$ with $\theta, z \in \mathbb{R}^p$.

Example 2. (Linear Regression) Let the random vector be $Z = (Y, X^T)\top$ with $Y \in \mathbb{R}$ and $X \in \mathbb{R}^p$ satisfying $Y = X^T\theta^* + \epsilon$. Here $\epsilon \in \mathbb{R}$ is the random noise with zero mean. The loss function can be chosen as $l(\theta, z) = \frac{1}{2}(y - x^T\theta)^2$ with $y \in \mathbb{R}, x, \theta \in \mathbb{R}^p$, and $z = (y, x^T)^\top$.

Example 3. (Logistic Regression) Suppose that the observation $Z = (Y, X^T)\top$ with $Y \in \{-1, 1\}$ and $X \in \mathbb{R}^p$ satisfying $\Pr(Y = y | X = x) = \left[1 + \exp(-yx^T\theta^*)\right]^{-1}$. The loss function is $l(\theta, z) = \log(1 + \exp(-yx^T\theta^*))$ with $y \in \{-1, 1\}, x, \theta \in \mathbb{R}^p$, and $z = (y, x^T)^\top$.

Example 4. (Maximal Likelihood Estimation) Let $F_{\theta^*}$ be the distribution of $Z$, and the function form of $F_{\theta^*}$ is known except the value of $\theta^*$. The loss function is the negative log likelihood: $l(\theta, z) = -\log(F_{\theta^*}(z))$.

In general, the function form of $L(\theta)$ is unknown, as it relies on the distribution $F_{\theta^*}$. Instead, classical statistical methods estimate $\theta^*$ based on the sample counterpart of $L(\theta)$ as follows:

$$\hat{\theta}_T = \arg\min_{\theta \in \mathbb{R}^p} \frac{1}{T} \sum_{t=1}^{T} l(\theta, Z_t), \quad \text{s.t.} \quad B\theta = b, \quad (2.3)$$

where $Z_1, \ldots, Z_T$ are the i.i.d. observations generated from distribution $F_{\theta^*}$. However, the computation of $\hat{\theta}_T$ in (2.3) involves calculating a summation among $T$ terms, which is not efficient when sample size $T$ is large. Moreover, in many real-world scenarios, the observations are collected sequentially in an online fashion. With the growing number of observations, data storage devices cannot store all the collected observations or there is no enough memory to load the whole dataset. In this case, the classical estimation procedures are not computationally feasible.

Before proceeding, we introduction some notation. Let $||v|| = \sqrt{v^\top v}$ denote the Euclidean norm of the vector $v$. For any matrix $A \in \mathbb{R}^{q \times k}$, we define $||A|| = \sup_{x \in \mathbb{R}^k} \sqrt{x^\top A^\top A x}$ as its operator norm, $A^\top$ as its Moore–Penrose inverse, and $\text{rank}(A)$ as its rank. For two symmetric matrices $V_1, V_2 \in \mathbb{R}^{k \times k}$, we say $V_1 \succeq V_2$ if $x^\top V_1 x \geq x^\top V_2 x$ for all $x \in \mathbb{R}^k$. We use the notation $\overset{\text{P}}{\Rightarrow}$ and $\overset{\text{D}}{\Rightarrow}$ to denote convergence in probability and in distribution, respectively. For $t \geq 1$, we denote $\mathcal{F}_t$ as the sigma algebra generated by $\{Z_1, \ldots, Z_t\}$. We denote $\chi^2(k)$ as the chi-square distribution with degree of freedom $k$, and $\chi^2(\delta, k)$ as the non-central chi-squared distribution with noncentrality parameter $\delta$ and degree of freedom $k$, for positive integer $k$ and positive constant $\delta$.

3. Projected Polyak–Ruppert averaging

To overcome the drawbacks of the classical methods, we consider the following PSGD algorithm. Choosing an initial value $\theta_0 \in \mathbb{R}^p$, we recursively update the value as follows:

$$\theta_t = \Pi(\theta_{t-1} - \gamma_t V l(\theta_{t-1}, Z_t)), \quad (3.1)$$
where $\Pi(\cdot)$ is the projection operator onto the affine set $\{\theta \in \mathbb{R}^p : B\theta = b\}$, and $\gamma_t > 0$ is the predetermined learning rate (or step size). The updating equation in (3.1) can be explicitly written in matrix form as

$$\theta_t = c + P[\theta_{t-1} - \gamma_t \nabla l(\theta_{t-1}, Z_t) - c],$$

where $P \in \mathbb{R}^{p \times p}$ is the orthogonal projection matrix onto $\text{Ker}(B)$, and $c \in \mathbb{R}^p$ is any vector satisfying $Bc = b$. Following [16], we define the APSGD estimate as follows:

$$\bar{\theta}_t = \frac{1}{T} \sum_{t=1}^T \theta_t.$$  \hfill (3.2)

By projection operation in (3.1), the estimate $\bar{\theta}_t$ satisfies (3.2). It is worth mentioning that, the average in (3.2) can be updated recursively in an online fashion as

$$\bar{\theta}_t = \frac{t-1}{t} \bar{\theta}_{t-1} + \frac{1}{t} \theta_t,$$

which is also obtainable with a large sample size. To discuss the theoretical properties of $\bar{\theta}_t$, we need the following Assumption.

Assumption A1. There exist constants $K, \epsilon > 0$ such that the following statements hold.

(i) The learning rate satisfies $\gamma_t = \gamma^d_t$, for some constants $\gamma > 0$ and $d \in (1/2, 1)$.

(ii) The objective function $L(\theta)$ is convex and continuously differentiable for all $\theta \in \mathbb{R}^p$. Moreover, it is twice continuously differentiable at $\theta = \theta^*$, where $\theta^*$ is the unique minimizer of $L(\theta)$.

(iii) For all $\theta, \tilde{\theta} \in \mathbb{R}^p$, the inequality $||\nabla L(\theta) - \nabla L(\tilde{\theta})|| \leq K||\theta - \tilde{\theta}||$ holds.

(iv) The Hessian matrix $G := \nabla^2 L(\theta^*) \in \mathbb{R}^{p \times p}$ is positive definite. Furthermore, the inequality $||\nabla^2 L(\theta) - \nabla^2 L(\theta^*)|| \leq K||\theta - \theta^*||$ holds for all $\theta$ with $||\theta - \theta^*|| \leq \epsilon$.

(v) For all $\theta \in \mathbb{R}^p$, it holds that $E(||\nabla l(\theta, Z)||^2) \leq K(1 + ||\theta||^2)$, and the matrix $S := E(\nabla l(\theta^*, Z)\nabla l^\top(\theta^*, Z)) \in \mathbb{R}^{p \times p}$ is positive definite.

(vi) For all $\theta$ with $||\theta - \theta^*|| \leq \epsilon$, it holds that $E(||\nabla l(\theta, Z) - \nabla l(\theta^*, Z)||^2) \leq \delta(||\theta - \theta^*||)$, where $\delta(\cdot)$ is a function such that $\delta(\nu) \rightarrow a$ as $\nu \rightarrow 0$.

(vii) For each $\theta \in \mathbb{R}^p$, there exists a constant $\epsilon_0 > 0$ and a measurable function $M_0(\cdot)$ with $E(M_0(Z)) < \infty$ such that

$$\sup_{\hat{\theta}, \theta - \theta^* \leq \epsilon_0} ||\nabla l(\hat{\theta}, Z)|| \leq M_0(Z) \quad \text{almost surely.}$$

(viii) The projection matrix $P$ satisfies $P^2 = P^\top = P$ and rank($P$) = $d$ for some integer $d \in \{0, \ldots, p\}$.

Remark 1. Assumption [A1(i)] specifies the learning rate for $t$-th iteration. The learning rate satisfies $\sum_{t=1}^\infty \gamma_t = \infty$ and $\sum_{t=1}^\infty \gamma_t^2 < \infty$, which is widely used in literature [3, 16, 20]. Assumptions [A1(ii), A1(vii)] are regularity conditions about the objective function $L(\theta)$ and the lose function $l(\theta, z)$, which are standard and also adopted in [3]. Assumption [A1(viii)] is to characterize the linear-equality constraint $B\theta^* = b$. In particular, when $P = I$ and $d = p$, the APSGD estimate $\bar{\theta}_t$ in (3.2) becomes the ASGD estimate without projection in [16].
Theorem 1. Under Assumption [A1] it follows that
\[ \tilde{\theta}_T = \theta^* - \frac{1}{T} \sum_{t=1}^{T} (P GP)\zeta_t + o_p(T^{-1/2}), \]
where \( \zeta_t = \nabla l(\theta_{t-1}, Z_t) - \nabla L(\theta_{t-1}) \). Moreover, the following statement holds:
\[ \sqrt{T}(\tilde{\theta}_T - \theta^*) \xrightarrow{d} N(0, (P GP)\Sigma (P GP)^\top). \]

Theorem 1 provides the asymptotic expansion and limiting distribution of the APSGD estimate \( \tilde{\theta}_T \). Notice that \( \theta_{t-1} \in \mathcal{F}_{t-1} \), and \( Z_t \) is independent from \( \mathcal{F}_{t-1}, \) so \( E(\zeta_t|\mathcal{F}_{t-1}) = 0 \), which implies that \( \zeta_1, \ldots, \zeta_T \) is a martingale-difference process. Under Assumption [A1], we can apply the martingale central limit theorem (e.g., see [14]) to derive the limiting distribution. It is worth mentioning the differences and connections between Theorem 1 and the existing results. First, [16] considered an unconstrained parameter space and showed that the ASGD estimate is asymptotically distributed as \( N(0, G^{-1} SG^{-1}) \). Theorem 1 can be viewed as a generalization of [16] to a general projection matrix \( P \). Second, [7] studied the APSGD estimate when the model parameters are in the interior of the constrained parameter space, and they showed that APSGD have the same limiting distribution as PSGD. However, Theorem 1 reveals the different limiting distributions of APSGD and PSGD in our model. The reason behind this difference is that our model parameter \( \theta^* \) is not in the interior of the constrained parameter space \( \theta \in \mathbb{R}^p : B\theta = b \).

Let us revisit examples in previous section and investigate the limiting distributions of the corresponding APSGD estimates.

Example 1 (Continued). Suppose the covariance of \( Z \) is \( \Sigma \). We can verify \( \nabla l(\theta, z) = -(z - \theta), \nabla^2 l(\theta, z) = I, G = I, \) and \( \Sigma = \Sigma \). So the asymptotic covariance of the APSGD estimate is \( P\Sigma P \).

Example 2 (Continued). Suppose \( \epsilon \) is independent from \( X \) with \( E(\epsilon) = 0, E(\epsilon^2) = \sigma^2 \). It can be verified that \( \nabla l(\theta, z) = -(y - x^\top \theta)x, \nabla^2 l(\theta, z) = xx^\top, G = E(XX^\top), S = \sigma^2 E(XX^\top) = \sigma^2 G \). Hence, the APSGD estimate is asymptotically normal with covariance \( \sigma^2 (P GP)^{-1} \).

Example 3 (Continued). Suppose \( \epsilon \) is independent from \( X \) with \( E(\epsilon) = 0, E(\epsilon^2) = \sigma^2 \), and \( V = E(XX^\top) \). It is not difficult to verify that
\[ \nabla l(\theta, z) = \frac{-yx}{1 + \exp(yx^\top \theta)} \quad \nabla^2 l(\theta, z) = \frac{\exp(yx^\top \theta)}{(1 + \exp(yx^\top \theta))^2} xx^\top, \quad G = S = \frac{\exp(x^\top \theta)}{(1 + \exp(x^\top \theta))^2} XX^\top. \]
As a consequence, the APSGD estimate is asymptotically normal with covariance matrix \( (P GP)^{-1} \).

Example 4 (Continued). Assume almost surely for all \( Z \), the map \( \theta \to F_\theta(Z) \) is twice continuously differentiable. Due to the properties of log likelihood function, the Fisher information matrix satisfies \( I_\theta := E[\nabla^2 l(\theta^*, Z)] = G = S \). Therefore, we show that the covariance matrix is \( (PI_\theta P)^{-1} \).

It is worth discussing the role of the constraint (2.2) played in the estimation. For this purpose, let us denote \( \tilde{\theta}_{T,I} \) and \( \tilde{\theta}_{T,P} \) as the APSGD estimates using projection matrices \( I \) and \( P \), respectively. By Theorem 1 their asymptotic covariance matrices are
\[ V_I := G^{-1}SG^{-1} \quad \text{and} \quad V_P := (P GP)^{-1}S(P GP)^{-1}. \]

For a general loss function \( l(\theta, z) \), the performance \( \tilde{\theta}_{T,P} \) is not necessarily better than \( \tilde{\theta}_{T,I} \). To see this, let us consider a special case of Example 1.
Example 1 (Continued). Suppose \( \theta^* = (\theta_1^*, \theta_2^*)^T \in \mathbb{R}^2 \), \( B = (1, -1) \) and \( b = (0, 0)^T \). The linear-equality constraint in (2.2) becomes \( \theta_1^* = \theta_2^* \). Moreover, we assume \( \Sigma = \text{Diag}(\sigma^2, 3\sigma^2) \). We can verify that

\[
V_p = \begin{pmatrix}
\sigma^2 & \sigma^2 \\
\sigma^2 & 3\sigma^2
\end{pmatrix}
\quad \text{and} \quad
V_I = \begin{pmatrix}
\sigma^2 & 0 \\
0 & 3\sigma^2
\end{pmatrix}.
\]

As a consequence, neither \( V_p \geq V_I \) nor \( V_I \geq V_p \) holds.

However, for a broad class of loss functions, the following Lemma suggests \( \theta_{T,p} \) is at least as efficient as \( \theta_{T,I} \).

**Lemma 1.** Under Assumption A[2] if \( S = cG \) for some constant \( c > 0 \), then \( V_I = cG^{-1} \) and \( V_p = c(PGP)^{-1} \). Moreover, it follows that \( V_I \geq V_p \) and the equality \( V_I = V_p \) holds if and only if \( P = I \).

Lemma 1 indicates that, under an additional condition, the estimation performance of \( \theta_{T,p} \) is improved by utilizing the additional information in (2.2). The additional condition \( S = cG \) holds for many popular models, including Examples 2-4. In particular, for the negative log likelihood loss function in Example 4, the asymptotic covariance matrix \( (P\theta \; P) \) coincides the Cramér–Rao lower bound for constrained maximal likelihood model (e.g., see [811]).

To apply Theorem 1, the unknown covariance matrix needs to be estimated. For this purpose, the following regularity conditions on \( l(\theta, z) \) are imposed.

Assumption A2: There exists a constant \( \epsilon > 0 \) such that, for each \( \theta \) with \( \|\theta - \theta^*\| \leq \epsilon \), the function \( \theta \rightarrow l(\theta, Z) \) has a continuous Hessian matrix \( \nabla^2 l(\theta, Z) \) almost surely. Moreover, there exists a measurable function \( M(Z) \) with \( E(M(Z)) < \infty \) satisfying \( \|\nabla^2 l(\theta, Z)\| \leq M(Z) \) for all \( \theta \) with \( \|\theta - \theta^*\| \leq \epsilon \) almost surely.

The existence of the second-order derivatives of \( \theta \rightarrow l(\theta, z) \) in Assumption A2 is to estimate \( G = \nabla^2 L(\theta^*) \) based on its sample counterpart, while the dominating function \( M(Z) \) is required to allow changing the order of the gradient operator and expectation, namely, \( \nabla^2 E[l(\theta^*, Z)] = E[\nabla^2 l(\theta^*, Z)] \). To estimate the covariance matrix, let us define

\[
\hat{G}_T = \frac{1}{T} \sum_{t=1}^T \nabla^2 l(\hat{\theta}_t, Z_t), \quad \hat{S}_T = \frac{1}{T} \sum_{t=1}^T \nabla l(\hat{\theta}_t, Z_t) \nabla^\top l(\hat{\theta}_t, Z_t),
\]

(3.3)

which both can be recursively calculate by

\[
\hat{G}_t = \frac{t-1}{t} \hat{G}_{t-1} + \frac{1}{t} \nabla^2 l(\hat{\theta}_t, Z_t), \quad \hat{S}_t = \frac{t-1}{t} \hat{S}_{t-1} + \frac{1}{t} \nabla l(\hat{\theta}_t, Z_t) \nabla^\top l(\hat{\theta}_t, Z_t).
\]

The following lemma provides a consistent estimate for the covariance matrix.

**Lemma 2.** Under Assumptions A[1] and A[2] it follows that \( (P\hat{G}_T P)^{-1} \hat{S}_T (P\hat{G}_T P)^{-1} = (PGP)^{-1} S (PGP)^{-1} + o_p(1) \).

Combining Theorem 1 with Lemma 2, we can construct an \( (1 - \alpha) \times 100\% \) confidence interval for the function \( g(\theta) \) as

\[
g(\theta_T) \pm z_{\alpha/2} \sqrt{\frac{\nabla g^\top (\theta_T)(P\hat{G}_T P)^{-1} \hat{S}_T (P\hat{G}_T P)^{-1} \nabla g(\theta_T)}{T}},
\]

(3.4)

where \( z_{\alpha/2} \) is the \( \alpha/2 \times 100\% \) upper quartile of standard normal. Since \( \theta_T, \hat{G}_T \) and \( \hat{S}_T \) can be computed in an online fashion, so is the confidence interval in (3.4).
4. Specification test

As a byproduct of Theorem 1, we propose a specification test for the constraint in (2.2). Specifically, we aim to test the following hypotheses:

\[ H_0 : B\theta^* = b \quad \text{vs.} \quad H_1 : B\theta^* = b + \beta \quad \text{for some } \beta \neq 0. \]

For this purpose, we define the test statistic

\[ \kappa_T = T(\bar{\theta}_{T,P} - \bar{\theta}_{T,J})^T \hat{W}^{-1}(\bar{\theta}_{T,P} - \bar{\theta}_{T,J}). \] (4.1)

Here \( \hat{W} = (I - P)\hat{G}_{T,J}^{-1}\hat{S}_{T,J}\hat{G}_{T,J}^{-1}(I - P) \) is a weight matrix with \( \hat{G}_{T,J} \) and \( \hat{S}_{T,J} \) being the matrices in (3.3) calculated using projection matrix \( I \). Essentially, \( \hat{W} \) estimates the weight matrix \( W = (I - P)G^{-1}S G^{-1}(I - P) \). The idea of the proposed test statistic in (4.1) is simple and straightforward. Under \( H_0 \), both \( \theta_{T,P} \) and \( \theta_{T,I} \) consistently estimate \( \theta^* \). Hence, their difference, as well as \( \kappa_T \), should be around zero. However, under \( H_1 \), due to model misspecification, \( \theta_{T,P} \) is inconsistent, and the difference \( \theta_{T,P} - \theta_{T,I} \) does not vanish. Based on (4.1), we propose the following asymptotic size \( \alpha \) testing procedure:

\[ \text{reject } H_0 \quad \text{if} \quad \kappa_T > \chi^2_{\alpha}(p-d), \] (4.2)

where \( \chi^2_{\alpha}(p-d) \) is the \( \alpha \times 100\% \) upper quartile of \( \chi^2 \) distribution with degree \( p-d \). The following theorem reveals the limiting behavior of the statistic \( \kappa_T \) and the validity of the proposed testing procedure.

**Theorem 2.** Suppose Assumptions (A1) and (A2) are satisfied. Then the following statements are true:

(i) Under \( H_0 : B\theta^* = b \), the convergence \( \kappa_T \xrightarrow{L} \chi^2(p-d) \) holds.

(ii) Under \( H_1 : B\theta^* = b + \beta \) for some \( \beta \neq 0 \), it follows that \( \kappa_T \xrightarrow{P} \infty \) in probability.

(iii) Under \( H_a : B\theta^* = b + \frac{\beta}{\sqrt{T}} \) for some \( \beta \neq 0 \), it holds that \( \kappa_T \xrightarrow{P} \chi^2(\mu^T W^{-1} \mu, p-d) \), where \( \mu \in \mathbb{R}^p \) is any vector satisfying \( B\mu = \beta \).

As a consequence, for any \( \alpha \in (0,1) \), it follows that

\[ \lim_{T \to \infty} \mathbb{P}(\kappa_T > \chi^2_{\alpha}(p-d)|H_0) = \alpha, \quad \lim_{T \to \infty} \mathbb{P}(\kappa_T > \chi^2_{\alpha}(p-d)|H_1) = 1. \]

Theorem 2 provides the asymptotic distributions of \( \kappa_T \) under null hypothesis \( H_0 \) and local alternative hypothesis \( H_a \), which are chi-square and noncentral chi-squared, respectively. Moreover, it shows that \( \kappa_T \) will diverge under alternative hypothesis \( H_1 \). Consequently, it verifies that testing procedure in (4.2) is consistent and has an asymptotic size \( \alpha \).

**Acknowledgments**

The authors gratefully acknowledge the constructive comments and suggestions from the Editor-in-Chief Dr. Dietrich von Rosen, an associate editor, and two anonymous referees. Zuofeng Shang acknowledges supports by NSF DMS-1764280 and DMS-1821157.
Appendix

In the appendix, we collect all the mathematical proofs of the main theorems and related lemmas. Section A.1 provides some preliminary lemmas, while the Sections A.2-A.5 proves Theorem 1, Lemma 1, Lemma 2, and Theorem 3.

A.1. Preliminary lemmas

**Lemma A.1.** Let $H \in \mathbb{R}^{d \times p}$ be a positive definite matrix and suppose that $\gamma_i = \gamma_i t^p$ for some constants $\gamma > 0, \rho \in (1/2, 1)$. Let us define squared matrices

$$ W_j^t = I, \quad W_j^t = (I - \gamma_{t-1}H)W_j^{t-1} = \cdots = \prod_{k=j}^{t-1}(I - \gamma_kH) \quad \text{for } t \geq j, $$

$$ \overline{W}_j^t = \gamma_j \sum_{i=j}^{t-1} W_j^i = \gamma_j \sum_{i=j}^{t-1} \prod_{k=j}^{i-1}(I - \gamma_kH). $$

Then the following statements hold:

1. There are constants $K > 0$ such that $\|\overline{W}_j^t\| \leq K$ for all $j$ and all $t \geq j$.
2. $\frac{1}{t} \sum_{j=0}^{t-1} \|\overline{W}_j^t - H^{-1}\| \to 0$ as $t \to \infty$.

**Proof:** This is Lemma 1 of [10].

**Lemma A.2.** Let $A \in \mathbb{R}^{d \times p}$ be a positive definite matrix and $P$ be a projection matrix such that $P = P^T$ and $\text{rank}(P) = d$. Then there exists an orthonormal matrix $U \in \mathbb{R}^{d \times p}$ such that

$$ U^T P U = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}, \quad U^T P A P U = \begin{pmatrix} \Omega_d & 0 \\ 0 & 0 \end{pmatrix}, \quad U^T (P A P)^T U = \begin{pmatrix} \Omega_d^{-1} & 0 \\ 0 & 0 \end{pmatrix}, $$

where $I_d \in \mathbb{R}^{d \times d}$ is the identity matrix, and $\Omega_d$ is a diagonal matrix with diagonal elements $\rho_1, \ldots, \rho_d > 0$. Moreover, it follows that $(P A P)^T P = (P A P)^T$, $P (P A P)^T = (P A P)^T$ and $(P A P)^T (P A P)x = x$ for all $x$ satisfying $P x = x$.

**Proof:** For any $x \in \mathbb{R}^p$ with $P A P x = 0$, it holds that $x^T P A P x = 0$ and $P x = 0$ by the positive definiteness of $A$. Clearly, $P x = 0$ implies $P A P x = 0$. Therefore, we conclude that $\text{Ker}(P A P) = \text{Ker}(P)$ and $\text{rank}(P A P) = \text{rank}(P) = d$.

For simplicity, we denote $S = P A P$. By direct examination, $S$ and $P$ are diagonalisable, and they commute. By simple linear algebra, there exist eigenvectors $u_1, u_2, \ldots, u_p$ that simultaneously diagonalize $P$ and $S$. W.L.O.G, we assume $P u_i = u_i$ for $i \in \{1, \ldots, d\}$ and $P u_i = 0$ for $i \in \{d+1, \ldots, p\}$. We further assume $\rho_1, \rho_2, \ldots, \rho_p$ to be the eigenvalues of $S$ corresponding to the eigenvectors $u_1, u_2, \ldots, u_p$. By the above notation, it shows that

$$ \rho_i u_i = S u_i = P A P u_i = 0 \quad \text{for } i \in \{d+1, \ldots, p\}. $$

Since $\text{rank}(S) = d$, we conclude that $\rho_i > 0$ for $i \in \{1, \ldots, d\}$. As a consequence, $U = (u_1, \ldots, u_p)$ and $\Omega_d = \text{Diag}(\rho_1, \ldots, \rho_p)$ will be the desired choices. Moreover, it is not difficult to verify that

$$ (P A P)^T P = U \begin{pmatrix} \Omega_d^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} \Omega_d^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T = (P A P)^T. $$

Similarly, we can prove that $P (P A P)^T = (P A P)^T$. Suppose that $x$ satisfies $P x = x$, then $x = \sum_{i=1}^{d} c_i u_i$ for some $c_1, \ldots, c_d \in \mathbb{R}$. As a consequence, it follows that $P A P x = P \sum_{i=1}^{d} c_i u_i = \sum_{i=1}^{d} c_i \rho_i u_i$. Notice that $P A P = \sum_{i=1}^{d} \rho_i u_i u_i^T$ and $(P A P)^T = \sum_{i=1}^{d} \rho_i^{-1} u_i u_i^T$, we have $(P A P)^T (P A P)x = \sum_{i=1}^{d} \rho_i^{-1} u_i u_i^T \sum_{i=1}^{d} c_i \rho_i u_i = \sum_{i=1}^{d} c_i u_i = x$. \qed
Lemma A.3. Under Assumption [11] it follows that
\[ \lim_{t \to \infty} \frac{1}{t} \sum_{j=0}^{t-1} \left\| y_j \sum_{k=j}^{t-1} \prod_{i=j+1}^{k} P(I - \gamma_i G)P - (PGP)^k \right\| = 0. \]
Moreover, there is a constant \( K > 0 \) such that \( \| y_j \sum_{k=j}^{t-1} \prod_{i=j+1}^{k} P(I - \gamma_i G)P \| \leq K \) for all \( j \) and all \( t \geq j \).

Proof: Since \( G \) is positive definite by Assumption [11][4] it follows from Lemma A.2 that
\[ U^\top P(I - \gamma_i G)PU = U^\top P \gamma_i U^\top PGP = \begin{pmatrix} I_d - \gamma_i \Omega_d & 0 \\ 0 & 0 \end{pmatrix}, \]
where \( U \) is an orthonormal matrix, \( I_d \in \mathbb{R}^{d \times d} \) is the identity matrix, and \( \Omega_d \in \mathbb{R}^{d \times d} \) is a diagonal and positive definite matrix. As a consequence, we have
\[ \prod_{i=j+1}^{k} P(I - \gamma_i G)P = U \begin{pmatrix} I_d - \gamma_i \Omega_d \\ 0 \end{pmatrix} U^\top. \]
By Lemma [A.1] we have
\[ \lim_{t \to \infty} \frac{1}{t} \sum_{j=0}^{t-1} \left\| y_j \sum_{k=j}^{t-1} \prod_{i=j+1}^{k} (I_d - \gamma_i \Omega_d) - \Omega_d^{-1} \right\| = 0, \]
which further leads to the first statement according to Lemma A.2. Applying Lemma A.2 again, we obtain the second conclusion.

Lemma A.4. Let \( c_1 \) and \( c_2 \) be arbitrary positive constants. Support that \( \gamma_i = \gamma t^\rho \) for some constants \( \gamma > 0 \) and \( \rho \in (1/2, 1) \). Moreover, assume a sequence \( \{B_t\}_{t=1}^\infty \) satisfies
\[ B_t \leq \frac{\gamma_{t-1}(1 - c_1 \gamma_t)}{\gamma_t} B_{t-1} + c_2 \gamma_t. \]
Then \( \sup_{1 \leq t < \infty} B_t < \infty. \)

Proof: This Lemma A.10 in [20].

Lemma A.5. Let \( F(x) \) be a differentiable convex function defined on \( \mathbb{R}^p \) with an unique minimizer \( x^* \). Suppose there exist constants \( \rho, r > 0 \) such that \( x \to F(x) - \frac{1}{2} \| x \|^2 \) is convex for all \( x \) with \( \| x - x^* \| \leq r \). Then for all \( x \in \mathbb{R}^p \), it holds that \( (x - x^*)^\top \nabla F(x) \geq \rho \| x - x^* \| \min(\| x - x^* \|, r) \).

Proof: This is Lemma B.1 in [20].

A.2. Proof of Theorem [P]

Before stating technical lemmas, we sketch the proof of Theorem [P]. By iteration formula in (3.1), we have
\[ \theta_t = c + P(\theta_{t-1} - \gamma \eta_t) - c, \quad \eta_t = \nabla l(\theta_{t-1}, Z_t) = \nabla L(\theta_{t-1}) + [\nabla l(\theta_{t-1}, Z_t) - \nabla L(\theta_{t-1})] := R(\theta_{t-1}) + \zeta_t, \quad (A.2.1) \]
where \( c \in \mathbb{R}^p \) is any vector satisfying \( Bc = b \). Let \( \Delta_t = \theta_t - \theta^* \). Since \( P(\theta^* - c) = \theta^* - c \), it follows that

\[
\Delta_t = \theta_t - \theta^* = c + P(\theta_{t-1} - \gamma_t \theta_t - c) - \theta^* = c + P(\Delta_{t-1} - \gamma_t \theta_t + \theta^* - c) - \theta^* = P\Delta_{t-1} - \gamma_t P\theta_t
\]

\[
= P(\Delta_t - \gamma_t \theta_t) - \gamma_t P\theta_t = P\Delta_{t-1} - \gamma_t P\theta_t = \gamma_t P R(\theta_{t-1}) - \gamma_t P \Delta_{t-1}
\]

\[
= P(I - \gamma_t G)\Delta_{t-1} - \gamma_t P\theta_t = P(I - \gamma_t G)\Delta_{t-1} + \sum_{j=1}^t \prod_{i=j+1}^t P(I - \gamma_i G)P \gamma_j P R(x_{j-1}) - G\Delta_{j-1})
\]

Taking average, we show that

\[
\frac{1}{T} \sum_{t=1}^T \Delta_t = \frac{1}{T} \sum_{t=1}^T \left[ \prod_{j=1}^t P(I - \gamma_j G)\Delta_0 + \sum_{j=1}^t \prod_{i=j+1}^t P(I - \gamma_i G)P \gamma_j P\theta_t \right] + \frac{1}{T} \sum_{t=1}^T \sum_{j=1}^t \prod_{i=j+1}^t P(I - \gamma_i G)P \gamma_j P R(x_{j-1}) - G\Delta_{j-1}) := S_1 + S_2 + S_3. \tag{A.2.2}
\]

In Lemmas \( A.10 \) and \( A.11 \), we will show that

\[
S_1 + S_2 = \frac{1}{T}(PGP)^T \sum_{t=1}^T \xi_t + o_p(T^{-1/2}), \quad S_3 = o_p(T^{-1/2}).
\]

Finally, we prove the asymptotic normality based on martingale C.L.T. in Lemma \( A.12 \).

**Lemma A.6.** Under Assumption [A] the following statements hold for some constants \( \epsilon, K > 0 \).

(i) \( (\theta - \theta^*)^T R(\theta) \geq \epsilon \|\theta - \theta^*\| \min(||\theta - \theta^*||, \epsilon) \) for all \( \theta \in \mathbb{R}^p \).

(ii) \( E(\xi_t | \mathcal{F}_{t-1}) = 0 \).

(iii) \( E(||\xi_t||^2 | \mathcal{F}_{t-1}) \leq K(1 + ||\theta_{t-1}||^2) \) almost surely.

(iv) \( ||R(\theta)||^2 \leq K(1 + ||\theta||^2) \).

(v) \( ||R(\theta) - G(\theta - \theta^*)|| \leq K||\theta - \theta^*||^2 \) for all \( \theta \) with \( ||\theta - \theta^*|| \leq \epsilon \).

**Proof:** For statement (i) by Assumption [A][iv], we know \( L(\theta) \) satisfies the conditions in Lemma \( A.3 \) with some \( \rho, r > 0 \). Therefore, it follows that

\[
(\theta - \theta^*)^T R(\theta) = (\theta - \theta^*)^T \nabla L(\theta) \geq \epsilon \|\theta - \theta^*\| \min(||\theta - \theta^*||, \epsilon),
\]

where \( \epsilon = \min(\rho, r) \).

For statement (ii), since \( \theta_{t-1} \in \mathcal{F}_{t-1} \) and \( Z_t \) is independent from \( \mathcal{F}_{t-1} \), we have \( E(\nabla l(\theta_{t-1}, Z)|\mathcal{F}_{t-1}) = \nabla L(\theta_{t-1}) \).

Similarly, by Assumption [A][v], the statement (iii) follows from the inequality below:

\[
E(||\xi_t||^2 | \mathcal{F}_{t-1}) = E(||\nabla l(\theta_{t-1}, Z)|\mathcal{F}_{t-1}) = E(||\nabla l(\theta_{t-1}, Z)||^2 | \mathcal{F}_{t-1}) \leq K(1 + ||\theta_{t-1}||^2) .
\]

Statement (iv) follows, as \( ||R(\theta)||^2 = ||\nabla L(\theta)||^2 = ||\nabla L(\theta) - \nabla L(\theta^*)||^2 \leq K^2||\theta - \theta^*||^2 \leq 2K^2(||\theta||^2 + ||\theta^*||^2) \) by Assumption [A][i].

To prove statement (v) by Assumption [A][iv] and Taylor expansion, we have

\[
||R(\theta) - G(\theta - \theta^*)|| = ||R(\theta) - R(\theta^* - G(\theta - \theta^*))|| = ||R(\theta) - R(\theta^*) - \nabla^2 L(\theta^*)(\theta - \theta^*)||
\]

\[
= ||\nabla^2 L(\theta^*)(\theta - \theta^*) - \nabla^2 L(\theta^*)(\theta - \theta^*)|| \leq K||\theta - \theta^*||^2 , \text{ for all } \theta \text{ with } ||\theta - \theta^*|| \leq \epsilon,
\]

where \( \theta^* \) is a vector between \( \theta \) and \( \theta^* \). \( \square \)
Lemma A.7. Suppose Assumption \([\mathbf{A1}]\) holds. Then there exists a constant \(K > 0\) such that
\[
||R(\theta) - G(\theta - \theta^*)|| \leq K||\theta - \theta^*||^2 \quad \text{for all } \theta \in \mathbb{R}^p.
\]

Proof: By Assumptions \([\mathbf{A3}]\) and \([\mathbf{A4}]\) we have
\[
||R(\theta) - G(\theta - \theta^*)|| = ||R(\theta) - R(\theta^*) - G(\theta - \theta^*)|| \leq ||R(\theta) - R(\theta^*)|| + ||G(\theta - \theta^*)|| \leq (K + ||G||)||\theta - \theta^*||
\]
\[
\leq (K + ||G||)||\theta - \theta^*||^2 / \epsilon, \quad \text{for all } \theta \text{ with } ||\theta - \theta^*|| \geq \epsilon.
\]

Combining with statement \([\mathbf{V}]\) in Lemma \([\mathbf{A5}]\), we complete the proof. \(\square\)

Lemma A.8. Under Assumption \([\mathbf{A7}]\) it holds that \(\lim_{t \to \infty} \theta_t = \theta^*\) almost surely.

Proof: Notice that \(\theta_t - c \in \text{Ker}(B)\) for all \(t \geq 1\), so it follows that
\[
\theta_t - \theta^* = c + P[(\theta_{t-1} - \gamma_t V(\theta_{t-1}, Z_t)) - c] - \theta^* = \theta_{t-1} - \theta^* - \gamma_t P[R(\theta_{t-1}) + \zeta].
\]

Moreover \(PD_t = D_t\) for \(t \geq 1\), we have
\[
||\Delta_t||^2 = ||\Delta_{t-1} - \gamma_t P[R(\theta_{t-1}) - \gamma_t P\zeta]||^2 = ||\Delta_{t-1} - \gamma_t P[R(\theta_{t-1})||^2 - 2\gamma_t \Delta_{t-1} R \theta_{t-1} P\zeta + 2\gamma_t^2 R^T \theta_{t-1} P\zeta + \gamma_t^2 \theta_{t-1}||^2
\]
\[
= ||\Delta_{t-1}||^2 + \gamma_t^2 ||P[R(\theta_{t-1})||^2 - 2\gamma_t \Delta_{t-1} R \theta_{t-1} + 2\gamma_t^2 R^T \theta_{t-1} P\zeta + \gamma_t^2 \theta_{t-1}||^2
\]
\[
\leq ||\Delta_{t-1}||^2 + \gamma_t^2 ||P[R(\theta_{t-1})||^2 - 2\gamma_t \Delta_{t-1} R \theta_{t-1} + 2\gamma_t^2 R^T \theta_{t-1} P\zeta + \gamma_t^2 \theta_{t-1}||^2
\]
\[
(\text{A.2.3})
\]

for all \(t \geq 2\). Taking conditional expectation on both sides of (A.2.3) and by Lemma \([\mathbf{A6}]\), we show that there exist constants \(K, \epsilon > 0\) such that
\[
\text{E}(||\Delta_t||^2 | F_{t-1}) \leq ||\Delta_{t-1}||^2 - 2\gamma_t \Delta_{t-1} R \theta_{t-1} + 2\gamma_t^2 K(1 + ||\theta_{t-1}||^2) + \gamma_t^2 \text{E}(||\zeta||^2 | F_{t-1})
\]
\[
\leq ||\Delta_{t-1}||^2 - 2\gamma_t \Delta_{t-1} R \theta_{t-1} + 2\gamma_t^2 K(1 + ||\theta_{t-1}||^2)
\]
\[
(\text{A.2.4})
\]

Since \(\sum_{i=1}^{\infty} \gamma_i = \infty\) and \(\sum_{i=1}^{\infty} \gamma_i^2 < \infty\), applying Robbins-Siegmund Theorem (e.g., see [18]), we have \(||\Delta_t||^2 \to V\) almost surely for some random variable \(V\), and
\[
\sum_{i=1}^{\infty} 2\gamma_i E(||\Delta_{t-1}|| \min(||\Delta_{t-1}||, \epsilon) < \infty \text{ almost surely}
\]

As a consequence, it follows that \(\lim_{t \to \infty} ||\Delta_{t-1}|| \to 0\) almost surely. \(\square\)

Lemma A.9. Suppose Assumption \([\mathbf{A1}]\) holds. Then for any \(M > 0\), there exists a constant \(K_M > 0\) such that
\[
E[||\theta_t - \theta^*||^2 I(\tau M > t)] \leq K_M \gamma_t^2 \quad \text{for all } t \geq 0,
\]

where \(\tau_M = \inf\{i \geq 1 : ||\theta_i - \theta^*|| > M\}\) is a stopping time.
follows that

\[ \|\Delta\|^2 I(\tau_M > t) \leq \|\Delta\|^2 I(\tau_M > t - 1) \]

\[ \leq \left( \|\Delta_{t-1}\|^2 + \gamma_i^2 \|R(\theta_{t-1})\|^2 - 2\gamma_i \Delta_{t-1}^\top R(\theta_{t-1}) - 2\gamma_i \Delta_{t-1}^\top \xi_t + 2\gamma_i^2 R^2(\theta_{t-1}) P_{\xi_t} + \gamma_i^2 \|\xi_t\|^2 \right) I(\tau_M > t - 1). \]

By similar calculation in (A.2.4), we show that there exist constants \(K, \epsilon > 0\) such that

\[ E[\|\Delta\|^2 I(\tau_M > t)/F_{t-1}] \leq (1 + 4\gamma_i^2 K)\|\Delta_{t-1}\|^2 I(\tau_M > t - 1) + 2\gamma_i^2 K(1 + 2\|\theta\|^2) - 2\gamma_i \epsilon \|\Delta_{t-1}\| \epsilon I(\tau_M > t - 1). \]

Notice that \(\|\Delta_{t-1}\| \min(\|\Delta_{t-1}\|, \epsilon) = \|\Delta_{t-1}\|^2\) if \(\|\Delta_{t-1}\| \leq \epsilon\), and \(\|\Delta_{t-1}\| \min(\|\Delta_{t-1}\|, \epsilon) = \|\Delta_{t-1}\| \epsilon \geq \|\Delta_{t-1}\| \epsilon / M\) if \(\epsilon < \|\Delta_{t-1}\| \leq M\), we conclude that

\[ E[\|\Delta\|^2 I(\tau_M > t)/F_{t-1}] \leq (1 - 2\gamma_i \epsilon^2 M^{-1} + 4\gamma_i^2 K)\|\Delta_{t-1}\|^2 I(\tau_M > t - 1) + 2\gamma_i^2 K(1 + 2\|\theta\|^2). \]

where we use the fact that \(\|\Delta_{t-1}\| \leq M\) on event \(\{\tau_M > t - 1\}\). Taking expectation again, if \(\gamma_i \leq \epsilon^2/(4MK)\), then it follows that

\[ E[\|\Delta\|^2 I(\tau_M > t)] \leq (1 - 2\gamma_i \epsilon^2 M^{-1} + 4\gamma_i^2 K)E[\|\Delta_{t-1}\|^2 I(\tau_M > t - 1)] + 2\gamma_i^2 K(1 + 2\|\theta\|^2). \]

Applying Lemma [A.4], we conclude that, there exists a constant \(K_M > 0\) such that \(E[\|\Delta\|^2 I(\tau_M > t)] \leq K_M \gamma_i\) for all \(t \geq 0\).

\[ S_1 + S_2 = \frac{1}{T}(PGP)^{-1} \sum_{t=2}^T \xi_t + o_p(T^{-1/2}), \]

where \(S_1\) and \(S_2\) are defined in (A.2.2).

**Proof:** Let \(\hat{\theta}_0 = \theta_0 \in \mathbb{R}^p\) be the initial value for iteration. We define sequence

\[ \hat{\theta}_t = c + P(\hat{\theta}_{t-1} - \gamma_j h_t - c) \]

with \(h_t = G\hat{\theta}_{t-1} - G\theta' + \xi_t\), for \(t \geq 1\),

where \(G \in \mathbb{R}^{p \times p}\) is the positive definite matrix defined in Assumption [A.4.4], \(\xi_t\) is the process defined in (A.2.1), and \(c \in \mathbb{R}^p\) satisfies \(Bc = b\). The proof is divided into four steps.

Step 1: This step is to show that \(\lim_{n \to \infty} \hat{\theta}_t \hat{\theta}_t = \theta'\) almost surely. Let us define \(\hat{\Delta}_t = \hat{\theta}_t - \theta'\), which is different from \(\Delta_t = \theta_t - \theta'\).

By the fact that \(P(\theta' - c) = \theta' - c\), we have

\[ \hat{\Delta}_t = c + P(\hat{\theta}_{t-1} - \gamma_j h_t - c) - \theta' = c + P(\hat{\Delta}_{t-1} - \gamma_j h_t + \theta' - c) - \theta' = P\hat{\Delta}_{t-1} - \gamma_j P h_t. \]

As a consequence, it follows from (A.2.5) that

\[ \|\hat{\Delta}\|^2 = \|P(I - \gamma_j G)\hat{\Delta}_{t-1}\|^2 - 2\gamma_i \xi_t^\top P(I - \gamma_j G)\hat{\Delta}_{t-1} + \gamma_i^2 \|P_{\xi_t}\|^2 \]

\[ \leq (1 - \gamma_i)\|\hat{\Delta}_{t-1}\|^2 - 2\gamma_i \xi_t^\top P(I - \gamma_j G)\hat{\Delta}_{t-1} + \gamma_i^2 \|\xi_t\|^2. \]
where \( \lambda > 0 \) is the smallest eigenvalue of \( G \). Taking conditional expectation, it follows that

\[
E(||\hat{\Delta}||^2|\mathcal{F}_{t-1}) \leq (1 - 2\gamma_t \lambda + \gamma_t^2 \lambda^2)E(||\hat{\Delta}_{t-1}||^2 + \gamma_t^2 E(||\hat{\zeta}||^2|\mathcal{F}_{t-1})) = (1 + \gamma_t^2 \lambda^2)E(||\hat{\Delta}_{t-1}||^2 + \gamma_t^2 E(||\hat{\zeta}||^2|\mathcal{F}_{t-1}) - 2\gamma_t \lambda E||\hat{\Delta}_{t-1}||^2 \\
\leq (1 + \gamma_t^2 \lambda^2)E(||\hat{\Delta}_{t-1}||^2 + \gamma_t^2 K(1 + ||\theta||^2) - 2\gamma_t \lambda E||\hat{\Delta}_{t-1}||^2 \\
\leq (1 + \gamma_t^2 \lambda^2)E(||\hat{\Delta}_{t-1}||^2 + \gamma_t^2 K(1 + 2||\Delta||^2 + 2||\theta||^2) - 2\gamma_t \lambda E||\hat{\Delta}_{t-1}||^2 ,
\]

where Lemma [A.3 (iii)] is used. Since \( \lim_{t\to\infty} E||\hat{\Delta}|| = 0 \) almost surely by Lemma [A.8] and \( \sum_{i=1}^{\infty} \gamma_i^2 < \infty \) by Assumption [A.1], it follows that \( \sum_{i=1}^{\infty} \gamma_i^2 K(1 + 2||\Delta||^2 + 2||\theta||^2) < \infty \) almost surely. Hence, Robbins-Siegmund Theorem (e.g., see [18]) implies that

\[
\lim_{t\to\infty} ||\hat{\Delta}||^2 \to \tilde{V}, \quad \sum_{i=1}^{\infty} \gamma_i ||\hat{\Delta}||^2 < \infty \text{ almost surely,}
\]

for some random variable \( \tilde{V} \). Since \( \sum_{i=1}^{\infty} \gamma_i = \infty \), we conclude that \( \lim_{t\to\infty} ||\hat{\Delta}||^2 = 0 \) almost surely.

Step 2: Let us define stopping times \( \hat{\tau}_M = \inf\{j \geq 1 : ||\hat{\Delta}|| > M\} \) and \( \tau_M = \inf\{j \geq 1 : ||\Delta|| > M\} \) for \( M > 0 \). This step is to prove that for any \( M > 0 \), there exists a constant \( K_M > 0 \) such that

\[
E[||\hat{\Delta}||^2 I(\hat{\tau}_M > t, \tau_M > t)] \leq K_M \gamma_t \quad \text{for all } t \geq 1. \quad (A.2.7)
\]

Using [A.2.6] again, we have

\[
||\hat{\Delta}||^2 I(\hat{\tau}_M > t, \tau_M > t) \leq ||\hat{\Delta}||^2 I(\hat{\tau}_M > t - 1, \tau_M > t - 1) \\
\leq (1 - \gamma_t \lambda^2)E(||\hat{\Delta}_{t-1}||^2 I(\hat{\tau}_M > t - 1, \tau_M > t - 1) + \gamma_t^2 E(||\hat{\zeta}||^2|\mathcal{F}_{t-1})I(\hat{\tau}_M > t - 1, \tau_M > t - 1) \\
- 2\gamma_t \lambda E||\hat{\Delta}_{t-1}||^2 I(\hat{\tau}_M > t - 1, \tau_M > t - 1).
\]

Taking conditional expectation and noticing that \( \{\hat{\tau}_M > t - 1, \tau_M > t - 1\} \in \mathcal{F}_{t-1} \), Lemma [A.3 (iii)] further leads to

\[
E[||\hat{\Delta}||^2 I(\hat{\tau}_M > t, \tau_M > t)|\mathcal{F}_{t-1}] \leq (1 - \gamma_t \lambda^2)E(||\hat{\Delta}_{t-1}||^2 I(\hat{\tau}_M > t - 1, \tau_M > t - 1) + \gamma_t^2 E(||\hat{\zeta}||^2|\mathcal{F}_{t-1})I(\hat{\tau}_M > t - 1, \tau_M > t - 1) \\
\leq \left(1 - \gamma_t \lambda^2\right)E(||\hat{\Delta}_{t-1}||^2 + \gamma_t^2 K(1 + ||\theta||^2))I(\tau_M > t - 1, \tau_M > t - 1) \\
\leq \left(1 - \gamma_t \lambda^2\right)E(||\hat{\Delta}_{t-1}||^2 + \gamma_t^2 K(1 + 2||\Delta||^2 + 2||\theta||^2))I(\tau_M > t - 1, \tau_M > t - 1) \\
\leq \left(1 - 2\gamma_t \lambda + \lambda^2 \gamma_t^2\right)E(||\hat{\Delta}_{t-1}||^2 I(\tau_M > t - 1, \tau_M > t - 1) + 2\gamma_t^2 K(1 + M^2 + ||\theta||^2),
\]

where we use the fact that \( ||\Delta_{t-1}|| \leq M \) when \( \tau_M > t - 1 \). Taking expectation again, we have

\[
E[||\hat{\Delta}||^2 I(\tau_M > t, \tau_M > t)] \leq (1 - 2\gamma_t \lambda + \lambda^2 \gamma_t^2)E[||\hat{\Delta}_{t-1}||^2 I(\tau_M > t - 1, \tau_M > t - 1)] + 2\gamma_t^2 K(1 + M^2 + ||\theta||^2) \\
\leq (1 - 2\gamma_t \lambda)E[||\hat{\Delta}_{t-1}||^2 I(\tau_M > t - 1, \tau_M > t - 1)] + 2\gamma_t^2 K(1 + M^2 + ||\theta||^2) \text{ when } \gamma_t \leq 1/\lambda,
\]

which further implies that

\[
E[||\hat{\Delta}||^2 I(\tau_M > t, \tau_M > t)] \leq \frac{\gamma_t(1 - 2\gamma_t \lambda)}{\gamma_{t-1}} E[||\hat{\Delta}_{t-1}||^2 I(\tau_M > t - 1, \tau_M > t - 1)] + 2\gamma_t K(1 + M^2 + ||\theta||^2).
\]

Now applying Lemma [A.2.3], we conclude that \( \sup_{t \leq \infty} E[||\hat{\Delta}||^2 I(\tau_M > t, \tau_M > t)]/\gamma_t < \infty \), which further implies [A.2.7].

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Step 3: This step is to show
\[
\frac{1}{\sqrt{T}} \sum_{t=2}^{T} \frac{\hat{\theta}_{t-1} - \hat{\theta}_{t}}{\gamma_{t}} = o_p(1). \tag{A.2.8}
\]

Since both \( \hat{\theta}_{t} \) and \( \theta_{t} \) are strongly consistent by Step 1 and Lemma A.8, for any \( \epsilon > 0 \), there exists a constant \( M > 0 \) such that
\[
\Pr(\sup_{1 \leq t < \infty} \|\hat{\Delta}_{t}\| \leq M) \geq 1 - \epsilon, \quad \Pr(\sup_{1 \leq t < \infty} \|\Delta_{t}\| \leq M) \geq 1 - \epsilon. \tag{A.2.9}
\]

By direction examination, it follows that
\[
\frac{1}{\sqrt{T}} \sum_{t=2}^{T} \frac{\hat{\theta}_{t-1} - \hat{\theta}_{t}}{\gamma_{t}} = \frac{1}{\sqrt{T}}\sum_{t=2}^{T} \frac{\hat{\theta}_{t-1} - \theta_{t} + \theta_{t} - \hat{\theta}_{t}}{\gamma_{t}} = \frac{1}{\sqrt{T}}\sum_{t=2}^{T} \frac{\hat{\theta}_{t-1} - \theta_{t}}{\gamma_{t}} - \frac{1}{\sqrt{T}}\sum_{t=2}^{T} \frac{\hat{\theta}_{t} - \theta_{t}}{\gamma_{t}} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T-1} \frac{\hat{\theta}_{t} - \theta_{t}}{\gamma_{t+1}} - \frac{1}{\sqrt{T}}\sum_{t=2}^{T} \frac{\hat{\theta}_{t} - \theta_{t}}{\gamma_{t}}
\]
\[
= \frac{1}{\sqrt{T}}\hat{\theta}_{1} - \theta_{1} - \frac{1}{\sqrt{T}}\hat{\theta}_{T} + \frac{1}{\sqrt{T}}\sum_{t=2}^{T-1} (\hat{\theta}_{t} - \theta_{t})(\gamma_{t+1}^{-1} - \gamma_{t}^{-1}) := D_{1} - D_{2} + D_{3}.
\]

It suffices to bound the three terms in right side of the last equation. Clearly \( D_{1} = o_p(1) \). For \( D_{2} \), we have the following bound
\[
\|D_{2}\| = \frac{1}{\sqrt{TY_{T}}\|\hat{\Delta}_{T}\|} \|\hat{\theta}_{T} - \theta_{T}\| = \frac{1}{\sqrt{TY_{T}}\|\hat{\Delta}_{T}\|} \|\hat{\Delta}_{T}\| \|I(\hat{\tau}_{T} > T, \tau_{M} > T) + \frac{1}{\sqrt{TY_{T}}}\|\hat{\Delta}_{T}\| \|I(\hat{\tau}_{M} \leq T) + \frac{1}{\sqrt{TY_{T}}}\|\hat{\Delta}_{T}\| \|I(\tau_{M} \leq T) := D_{21} + D_{22} + D_{23}.
\]

By (A.2.7) and Assumption (A.11) we have
\[
E(D_{21}) \leq \frac{1}{\sqrt{TY_{T}}} \sqrt{E[\|\hat{\Delta}_{T}\|^{2} I(\tau_{M} > T, \tau_{M} > T)]} \leq \sqrt{\frac{K_{M}}{TY_{T}}} = \sqrt{\frac{K_{M}}{\gamma T^{1-p}}} \rightarrow 0.
\]

The definitions of \( \hat{\tau}_{M} \) and \( \tau_{M} \) indicate that \( \{\sup_{1 \leq t < \infty} \|\hat{\Delta}_{t}\| \leq M\} \subset \{\hat{\tau}_{M} > T\} \) and \( \{\sup_{1 \leq t < \infty} \|\Delta_{t}\| \leq M\} \subset \{\tau_{M} > T\} \). By (A.2.9), we see that
\[
\Pr(\hat{\tau}_{M} > T) \geq \Pr(\sup_{1 \leq t < \infty} \|\hat{\Delta}_{t}\| \leq M) \geq 1 - \epsilon, \quad \Pr(\tau_{M} > T) \geq \Pr(\sup_{1 \leq t < \infty} \|\Delta_{t}\| \leq M) \geq 1 - \epsilon. \tag{A.2.10}
\]

Since for any \( \delta > 0 \), it follow that
\[
D_{22} = \begin{cases} \frac{1}{\sqrt{TY_{T}}}\|\hat{\Delta}_{T}\| & \text{if } \hat{\tau}_{M} \leq T; \\ 0 & \text{if } \hat{\tau}_{M} > T; \end{cases} \quad D_{23} = \begin{cases} \frac{1}{\sqrt{TY_{T}}}\|\hat{\Delta}_{T}\| & \text{if } \tau_{M} \leq T; \\ 0 & \text{if } \tau_{M} > T, \end{cases}
\]
we see that
\[
\{D_{22} > \delta/3\} \subset \{\hat{\tau}_{M} \leq T\}, \quad \{D_{23} > \delta/3\} \subset \{\tau_{M} \leq T\} \tag{A.2.11}
\]
Combining the above inequalities, for any \( \delta > 0 \), we deduce that
\[
\Pr(\|D_{2}\| > \delta) \leq \Pr(D_{21} > \delta/3) + \Pr(D_{22} > \delta/3) + \Pr(D_{23} > \delta/3)
\]
\[
\leq \frac{3}{\delta} \sqrt{\frac{K_{M}}{\gamma T^{1-p}}} + \Pr(\hat{\tau}_{M} \leq T) + \Pr(\tau_{M} \leq T) \leq \frac{3}{\delta} \sqrt{\frac{K_{M}}{\gamma T^{1-p}}} + 2\epsilon,
\]
which further implies that \( \lim_{T \to \infty} \Pr(||D_2|| > \delta) \leq 2\epsilon \). Since \( \epsilon > 0 \) can be arbitrarily chosen, we show that \( D_2 = o_p(1) \).

To handle \( D_3 \), we use the following decomposition

\[
\|D_3\| \leq \frac{1}{\sqrt{T}} \sum_{i=2}^{T-1} |\hat{\theta}_i - \theta^*||y_{i+1}^{-1} - y_i^{-1}|I(\hat{\tau}_M > t, \tau_M > t) + \frac{1}{\sqrt{T}} \sum_{i=2}^{T-1} |\hat{\theta}_i - \theta^*||y_{i+1}^{-1} - y_i^{-1}|I(\hat{\tau}_M \leq t) \\
+ \frac{1}{\sqrt{T}} \sum_{i=2}^{T-1} |\hat{\theta}_i - \theta^*||y_{i+1}^{-1} - y_i^{-1}|I(\tau_M \leq t) \\
\leq \frac{1}{\sqrt{T}} \sum_{i=2}^{T-1} |\hat{\Delta}_i||y_{i+1}^{-1} - y_i^{-1}|I(\hat{\tau}_M > t, \tau_M > t) + \frac{1}{\sqrt{T}} \sum_{i=2}^{T-1} |\hat{\Delta}_i||y_{i+1}^{-1} - y_i^{-1}|I(\hat{\tau}_M \leq T) \\
+ \frac{1}{\sqrt{T}} \sum_{i=2}^{T-1} |\hat{\Delta}_i||y_{i+1}^{-1} - y_i^{-1}|I(\tau_M \leq T) := D_{31} + D_{32} + D_{33}.
\]

We obtain from (A.2.10) that

\[
E\left( \frac{1}{\sqrt{T}} \sum_{i=2}^{T-1} |\hat{\Delta}_i||y_{i+1}^{-1} - y_i^{-1}|I(\hat{\tau}_M > t, \tau_M > t) \right) \leq \sum_{i=2}^{\infty} \frac{|y_{i+1}^{-1} - y_i^{-1}|}{\sqrt{t}} \left( \frac{1}{\sqrt{t}} \sqrt{\E[|\hat{\Delta}_i||I(\hat{\tau}_M > t, \tau_M > t)]} \right) \leq \sum_{i=2}^{\infty} \frac{|y_{i+1}^{-1} - y_i^{-1}|}{\sqrt{t}} \sqrt{K_Mt} \\
= \sqrt{K_M} \sum_{i=2}^{\infty} \frac{1}{\sqrt{t}} \left( t + 1 \right)^{\rho} t^{-\rho} \sqrt{t} \gamma t^\rho = \sqrt{\gamma K_M} \sum_{i=2}^{\infty} \frac{1}{\sqrt{t}} \left( \left( t + 1 \right)^{\rho} t^{-\rho} \right) < \infty,
\]

where we use Assumption (A.1.1) that \( y_t = y^\rho \) for some \( \rho \in (1/2, 1) \). The above inequality also implies that

\[
\sum_{i=2}^{\infty} \frac{1}{\sqrt{t}} \left| \frac{\hat{\Delta}_i}{y_{i+1}^{-1}} - \frac{\hat{\Delta}_i}{y_i^{-1}} \right|I(\hat{\tau}_M > t, \tau_M > t) < \infty \quad \text{almost surely.}
\]

As a consequence of Kronecker’s lemma, we show that \( D_{31} = o_p(1) \). Using (A.2.10) and similar arguments as (A.2.11), for any \( \delta > 0 \), we have

\[
\Pr(||D_3|| > \delta) \leq \Pr(D_{31} > \delta/3) + \Pr(D_{32} > \delta/3) + \Pr(D_{33} > \delta/3) \\
\leq \Pr(D_{31} > \delta/3) + \Pr(\hat{\tau}_M \leq T) + \Pr(\tau_M \leq T) \leq \Pr(D_{31} > \delta/3) + 2\epsilon.
\]

Taking limit, it holds that \( \lim_{T \to \infty} \Pr(||D_3|| > \delta) \leq 2\epsilon \). Since \( \epsilon > 0 \) can be arbitrarily chosen, we show that \( D_3 = o_p(1) \).

Step 4: Using (A.2.5), we have

\[
\hat{\Delta}_t = P(I - y_tG)\hat{\Delta}_{t-1} - y_tP\zeta = P\hat{\Delta}_{t-1} - y_tPG\hat{\Delta}_{t-1} - y_tP\zeta.
\]

Since \( \hat{P}\hat{\Delta}_t = \hat{\Delta}_t \) for \( t \geq 1 \), it further leads to

\[
y_tPGP\hat{\Delta}_{t-1} = y_tPG\hat{\Delta}_{t-1} = -P(\hat{\Delta}_t - \hat{\Delta}_{t-1}) - y_tP\zeta = -P(\hat{\theta}_t - \hat{\theta}_{t-1}) - y_tP\zeta \quad \text{for all } t \geq 2.
\]

Taking summation, we show that

\[
\sum_{t=2}^{T} PGP\hat{\Delta}_{t-1} = PGP\hat{\Delta}_T + \sum_{t=2}^{T} PGP\hat{\Delta}_{t-1} = PGP\hat{\Delta}_T - P \sum_{t=2}^{T} \frac{\hat{\theta}_t - \hat{\theta}_{t-1}}{y_t} - P \sum_{t=2}^{T} \zeta_t,
\]
which further implies that
\[
\frac{1}{\sqrt{T}} PGP \sum_{t=1}^{T} \hat{\Delta}_t = \frac{1}{\sqrt{T}} PGP \hat{\Delta}_T - \frac{1}{\sqrt{T}} P \sum_{t=2}^{T} \frac{\theta_t - \theta_{t-1}}{\gamma_t} - \frac{1}{\sqrt{T}} P T \sum_{t=2}^{T} \zeta_t := E_1 - E_2 - E_3.
\]

Using the strong consistency of \( \hat{\theta} \) in Step 1 and (A.2.3) and (A.2.5) in Step 3, we show that \( E_1 = o_p(1) \) and \( E_2 = o_p(1) \). Moreover, by iterative substitution and the fact that \( \hat{\theta}_0 = \theta_0 \), (A.2.5) leads to
\[
\hat{\Delta}_t = P(I - \gamma_t^* \hat{G}) \Delta_{t-1} - \gamma_t P \zeta_t = \left[ \prod_{j=1}^{t} \left[ P(I - \gamma_j G) \right] \right] \Delta_0 + \sum_{j=1}^{t} \left[ \prod_{i=j+1}^{t} P(I - \gamma_i G) \right] \gamma_j P \zeta_j,
\]
which, by averaging, further implies that
\[
\frac{1}{T} \sum_{t=1}^{T} \hat{\Delta}_t = \frac{1}{T} \sum_{t=1}^{T} \left[ \prod_{j=1}^{t} \left[ P(I - \gamma_j A) \right] \right] \Delta_0 + \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{t} \left[ \prod_{i=j+1}^{t} P(I - \gamma_i G) \right] \gamma_j P \zeta_j = S_1 + S_2.
\]
Notice that \((PGP)^{-1} P = (PGP)^{-1}\) by Lemma A.2, we complete the proof.

**Lemma A.11.** Under Assumption (A), it follows that \( S_3 = o_p(T^{-1/2}) \), where \( S_3 \) is defined in (A.2.2).

**Proof:** Changing the order of summation leads to
\[
S_3 = \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{t} \left[ \prod_{i=j+1}^{t} P(I - \gamma_i G) \right] \gamma_j P(R(\theta_{j-1}) - G \Delta_{j-1}) = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=j+1}^{t} \sum_{j=1}^{t} \left[ \prod_{i=j+1}^{t} P(I - \gamma_i G) \right] \gamma_j P(R(\theta_{j-1}) - G \Delta_{j-1}).
\]
By Lemma A.8 for any \( \epsilon > 0 \), there exists a constant \( M > 0 \) such that
\[
\Pr(\tau_M > T) \geq \Pr(\sup_{1 \leq j < \infty} ||\Delta_j|| \leq M) \geq 1 - \epsilon,
\]
where \( \tau_M = \inf\{j \geq 1 : ||\Delta_j|| > M\} \) is the stopping time defined in Lemma A.5. Setting \( \alpha^*_j = \gamma_j \sum_{i=j+1}^{T} P(I - \gamma_i G) \), Lemmas A.3 and A.7 lead to
\[
\|\sqrt{T} S_3\| \leq \left\| \frac{1}{\sqrt{T}} \sum_{j=1}^{T} \alpha^*_j P(R(\theta_{j-1}) - G \Delta_{j-1}) \right\| \leq \frac{1}{\sqrt{T}} \sum_{j=1}^{T} ||\alpha^*_j|| ||P(R(\theta_{j-1}) - G \Delta_{j-1})|| \leq \frac{K}{\sqrt{T}} \sum_{j=1}^{T} ||\Delta_{j-1}|| \leq \frac{K^2}{\sqrt{T}} \sum_{j=1}^{T} ||\Delta_{j-1}||^2 I(\tau_M \leq j - 1) + \frac{K^2}{\sqrt{T}} \sum_{j=1}^{T} ||\Delta_{j-1}||^2 I(\tau_M > j - 1)
\]
\[
\leq \frac{K^2}{\sqrt{T}} \sum_{j=1}^{T} ||\Delta_{j-1}||^2 I(\sup_{1 \leq j < \infty} ||\Delta_j|| > M) + \frac{K^2}{\sqrt{T}} \sum_{j=1}^{T} ||\Delta_{j-1}||^2 I(\tau_M > j - 1) := S_{31} + S_{32}.
\]
For the first term, using (A.2.12) and similar arguments as (A.2.11), for any \( \delta > 0 \), we have
\[
\Pr(S_{31} > \delta/2) \leq \Pr(\sup_{1 \leq j < \infty} ||\Delta_j|| > M) \leq \epsilon.
\]
For the second term, Lemma A.9 implies that
\[
E \left( \sum_{j=1}^{\infty} ||\Delta_j||^2 I(\tau_M > j) \right) \leq K_M \sum_{j=1}^{\infty} \frac{\gamma_j}{j^{1/2}} = K_M \sum_{j=1}^{\infty} \frac{\gamma_j}{j^{1/2}} < \infty,
\]
where we use Assumption \([\text{A}1(1)]\) that \(\gamma_j = \gamma j^\rho\) for some \(\rho \in (1/2, 1)\). The above inequality also implies that 
\[
\Pr\left(\sum_{j=1}^{\infty} j^{1/2} |\Delta_j|^2 I(T_M > j) < \epsilon \right) = 1.
\]
Applying Kronecker’s lemma, we show that \(S_{32} \to 0\) almost surely as \(T \to \infty\). Combining the bounds of \(S_{31}\) and \(S_{32}\), we conclude that 
\[
\lim_{T \to \infty} \Pr(|TS_3| > \delta) \leq \lim_{T \to \infty} \Pr(S_{31} > \delta/2) + \lim_{T \to \infty} \Pr(S_{32} > \delta/2) \leq \epsilon.
\]
Since \(\epsilon > 0\) can be arbitrarily chosen, we show that \(\sqrt{T}S_3 = o_p(1)\). 

**Lemma A.12.** Under Assumption \([\text{A}1]\) it follows that \(T^{-1/2} \sum_{t=1}^{T} \zeta_t \overset{L}{\to} N(0, S)\).

**Proof:** We decompose the process \(\zeta_t\) as follows:
\[
\zeta_t = \nabla(l(\theta_{t-1}, Z_t) - \nabla l(\theta^*, Z_t)) + [\nabla l(\theta_{t-1}, Z_t) - \nabla l(\theta^*, Z_t) - \nabla L(\theta_{t-1}) + \nabla L(\theta^*)] := \eta_t + \xi_t.
\]
Assumption \([\text{A}1(v)]\) and Lemma \([\text{A.3}]\) imply that
\[
\text{E}([\eta_t \xi_t^\top] | F_{t-1}) \leq \text{E}(\|\nabla l(\theta_{t-1}, Z_t) - \nabla l(\theta^*, Z_t)\|_2^2 | F_{t-1}) \leq \delta(\|\theta_{t-1} - \theta^*\|) \to 0 \quad \text{almost surely.}
\]
Moreover, by Cauchy–Schwarz inequality, it follows that
\[
\text{E}(\|\eta_t \xi_t^\top\|_2 | F_{t-1}) \leq \text{E}(\|\eta_t\| \|\xi_t\| | F_{t-1}) \leq \sqrt{\text{E}(\|\eta_t\|^2)} \sqrt{\text{E}(\|\xi_t\|^2 | F_{t-1})} \to 0 \quad \text{almost surely.}
\]
As a consequence of the above two inequalities, we show that
\[
\text{E}(\xi_t^\top F_{t-1}) = \text{E}(\eta_t \xi_t^\top) + 2\text{E}(\eta_t^2 | F_{t-1}) + \text{E}(\xi_t \xi_t^\top | F_{t-1}) \to 0 \quad \text{almost surely,}
\]
where \(S = \text{E}[\nabla l(\theta^*, Z)\nabla l^\top(\theta^*, Z)] \in \mathbb{R}^{p \times p}\) is a positive definite matrix defined in Assumption \([\text{A}3]\). For any \(\epsilon > 0\), direct calculation leads to
\[
\text{E}(\|\xi_t\|^2) \leq \text{E}(\|\xi_t\|^2 | F_{t-1}) \leq \text{E}(\|\eta_t\|^2 + \|\xi_t\|^2) \leq \text{E}(\|\eta_t\|^2 | F_{t-1}) + \text{E}(\|\xi_t\|^2 | F_{t-1}) \leq 4\text{E}(\|\eta_t\|^2 | F_{t-1}) + 4\text{E}(\|\xi_t\|^2 | F_{t-1}) \leq 4\text{E}(\|\eta_t\|^2 | F_{t-1}) + 4\text{E}(\|\xi_t\|^2 | F_{t-1}) \leq 4\text{E}(\|\eta_t\|^2 | F_{t-1}) + 4\text{E}(\|\xi_t\|^2 | F_{t-1}) \leq 4\text{E}(\|\eta_t\|^2 | F_{t-1}) + 4\text{E}(\|\xi_t\|^2 | F_{t-1}) \leq 4\delta(\|\theta_{t-1} - \theta^*\|).
\]
Since \(\theta_t = \nabla(l(\theta^*, Z_t))\) are i.i.d., and \(\lim_{T \to \infty} \delta(\|\theta_{t-1} - \theta^*\|) = 0\) almost surely, we conclude that
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \text{E}(\|\xi_t\|^2 | F_{t-1}) \leq \epsilon \sqrt{T} | F_{t-1} | \to 0 \quad \text{almost surely.}
\]
By the C.L.T. for martingale-difference arrays (e.g., see \([14]\)), we prove the asymptotic normality. 

**A.3. Proof of Lemma [1]**

It suffices to show that \(G^{-1} - (PGP)^{-1}\) is positive semidefinite and has rank \(p - d\). Since \(\text{rank}(P) = d\) by Assumption \([\text{A}1(vii)]\) and \(P\) is diagonalisable, there exists an orthogonal matrix \(U \in \mathbb{R}^{p \times p}\) such that
\[
P = U \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} U^\top, \quad U^\top GU = \begin{pmatrix} X & Y \\ Y^\top & Z \end{pmatrix},
\]
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for some matrices $X,Y,Z$ with comfortable dimensions. As a consequence, it follows that
\[
P_{GP} = U \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} U^T G U \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U^T.
\]

Let $S = Z - Y^T X^{-1} Y \in \mathbb{R}^{(p-d) \times (p-d)}$ be the Schur complement of $X$. Since $G$ is positive definite by Assumption [A.1iv], so is $S$. The matrix block inversion formula implies that
\[
G^{-1} = U \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix}^{-1} U^T = U \begin{pmatrix} X^{-1} + X^{-1} Y S^{-1} Y^T X^{-1} & -X^{-1} Y S^{-1} \\ -S^{-1} Y^T X^{-1} & S^{-1} \end{pmatrix} U^T
\]
\[
= U \begin{pmatrix} X^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T + U \begin{pmatrix} X^{-1} Y S^{-1/2} & -X^{-1} Y S^{-1/2} \\ -S^{-1/2} Y^T X^{-1} & S^{-1/2} \end{pmatrix} U^T = (P_{GP})^* + U \begin{pmatrix} X^{-1} Y S^{-1/2} & -X^{-1} Y S^{-1/2} \\ -S^{-1/2} Y^T X^{-1} & S^{-1/2} \end{pmatrix} U^T,
\]
which proves the positive semidefiniteness. Because $\text{rank}(S^{-1/2}) = p - d$ and $(S^{-1/2} Y^T X^{-1}, -S^{-1/2}) \in \mathbb{R}^{(p-d) \times p}$, we verify that $G^{-1} - (P_{GP})^*$ has rank $p - d$.

A.4. Proof of Lemma 2

**Lemma A.13.** Let $\Sigma, \tilde{\Sigma} \in \mathbb{R}^{p \times p}$ be symmetric with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_p$ and $\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_p$. Fixing $1 \leq r \leq s \leq p$, let us define $d = s - r + 1$, and let $V = (v_1, v_{r+1}, \ldots, v_s) \in \mathbb{R}^{d \times d}$, $\tilde{V} = (\tilde{v}_1, \tilde{v}_{r+1}, \ldots, \tilde{v}_s) \in \mathbb{R}^{d \times d}$ have orthonormal columns satisfying $\Sigma v_j = \lambda_j v_j$ and $\tilde{\Sigma} v_j = \tilde{\lambda}_j v_j$ for $j \in \{r, r + 1, \ldots, s\}$. If $e := \inf|\tilde{\lambda} - \lambda| : \lambda \in \{\lambda_r, \lambda_s\}, \tilde{\lambda} \in (-\infty, \tilde{\lambda}_{r-1}] \cup [\tilde{\lambda}_{s+1}, \infty) > 0$, where $\tilde{\lambda}_0 := -\infty$ and $\tilde{\lambda}_{p+1} := \infty$, then it follows that $||V^T \tilde{V}^T - \tilde{V}^T V|| \leq 2||\tilde{\Sigma} - \Sigma||e$. Moreover, the eigenvalues satisfy $|\tilde{\lambda}_i - \lambda_i| \leq ||\tilde{\Sigma} - \Sigma||e$.

**Proof:** It follows from Davis-Kahan Theorem (e.g., see [22]) and Weyl’s inequality [21].

**Lemma A.14.** Let $\Sigma, \tilde{\Sigma}_n \in \mathbb{R}^{p \times p}$ be positive semidefinite matrices such that $\text{rank}(\tilde{\Sigma}_n) = \text{rank}(\Sigma)$ and $\tilde{\Sigma}_n \to \Sigma$ as $n \to \infty$. Then $\lim_{n \to \infty} \tilde{\Sigma}_n = \Sigma$.

**Proof:** Let distinct eigenvalues of $\Sigma$ be $\rho_1 > \rho_2 > \cdots > \rho_d = 0$, and suppose that there are $k_j \geq 1$ eigenvalues $\lambda_{j,1} = \lambda_{j,2} = \cdots = \lambda_{j,k_j} = \rho_j$ for $j \in \{1, \ldots, d\}$. We denote $v_{j,s}$ as the eigenvector corresponding to eigenvalue $\lambda_{j,s}$. Similarly, we define $(\tilde{\lambda}_{j,1}, \tilde{\lambda}_{j,2}, \ldots, \tilde{\lambda}_{j,k_j})$ as the eigenpairs of $\Sigma_n$ for $j \in \{1, \ldots, d\}$ and $n \in \{1, \ldots, k_j\}$. However, in general, we do not have $\tilde{\lambda}_{j,s} = \lambda_{j,s} = \cdots = \lambda_{j,k_j}$. Moreover, the eigenvalues can be chosen to be in an increasing order such that
\[
\tilde{\lambda}_{j,1} \geq \tilde{\lambda}_{j,1} \geq \cdots \geq \tilde{\lambda}_{j,k_j} \quad \text{for all } j \in \{1, \ldots, d\},
\]
\[
\lambda_{1,n_1} \geq \lambda_{2,n_2} \geq \cdots \geq \lambda_{d,n_d} \quad \text{for all } s_j \in \{1, \ldots, k_j\} \text{ and } j \in \{1, \ldots, d\}.
\]

By Lemma A.13, we see that $\tilde{\lambda}_{j,s} \to \lambda_{j,s} = \rho_j$ for all $j \in \{1, \ldots, d\}$. As a consequence, when $n$ is sufficiently large, there exists a constant $\epsilon > 0$ such that
\[
\rho_{j+1} < \rho_j - \epsilon \leq \tilde{\lambda}_{j,s} \leq \rho_j + \epsilon < \rho_{j-1} \quad \text{for all } s_j \in \{1, \ldots, k_j\} \text{ and } j \in \{1, \ldots, d - 1\}.
\]

Since $\text{rank}(\Sigma_n) = \text{rank}(\Sigma)$, it holds that $\lambda_{d,s} = \rho_d = 0$. For each $j \in \{1, \ldots, d - 1\}$, applying Lemma A.13 to eigenpairs $(\lambda_{j,s}, v_{j,s})$ and $(\tilde{\lambda}_{j,s}, \tilde{v}_{j,s})$ with $s \in \{1, \ldots, k_j\}$, we have $e \geq \epsilon$ and
\[
\left\| \sum_{s=1}^{k_j} \tilde{v}_{j,s}^T v_{j,s} \right\|_2 \leq 2||\tilde{\Sigma}_n - \Sigma||/\epsilon = o_p(1),
\]

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By Assumption A2, Lebesgue's Dominated Convergence Theorem, and L.L.N., we can see

\[ \lim_{n \to \infty} \sum_{j=1}^{n} A_{j,1} v_{j,1} v_{j,1}^T = \sum_{j=1}^{d-1} A_{j,1} v_{j,1} v_{j,1}^T. \]

Similarly, we have \( \lim_{n \to \infty} \sum_{j=1}^{n} A_{j,d} v_{j,d} v_{j,d}^T = \sum_{j=1}^{d-1} A_{j,d} v_{j,d} v_{j,d}^T. \) Combining the above, we show that \( \hat{\theta} = \theta. \) In a similar way, we can show that \( \hat{\mathbf{v}} = \mathbf{v}. \) The desired result follows from Lemma A.14.

**Proof:** Since A is positive definite, so is \( A_n \) when \( n \) is sufficiently large. Hence \( PA_n P \) and \( PAP \) both have the same rank as \( P. \) The desired result follows from Lemma A.14.

**Lemma A.15.** Suppose a sequence of matrices \( \{A_n\}_{n=1}^{\infty} \in \mathbb{R}^{p \times p} \) satisfies \( \lim_{n \to \infty} A_n = A \) where \( A \in \mathbb{R}^{p \times p} \) is positive definite. Let \( P \in \mathbb{R}^{p \times p} \) be a projection matrix such that \( P^2 = P \) and \( P^T = P. \) Then \( \lim_{n \to \infty} (PA_n P)^- = (PAP)^-. \)

**Proof:** Since \( A \) is positive definite, so is \( A_n \) when \( n \) is sufficiently large. Hence \( PA_n P \) and \( PAP \) both have the same rank as \( P. \) The desired result follows from Lemma A.14.

**Lemma A.16.** Under Assumptions [A1] and [A2] it follows that \( \hat{G}_T = G + o_p(1), \) \( (P\hat{G}_T P)^- = (PGP)^- + o_p(1), \) and \( \hat{\mathbf{S}}_T = S + o_p(1). \)

**Proof:** Since \( \mathbf{\hat{\theta}}_T \to \mathbf{\theta}' \) almost surely as \( T \to \infty \) by Lemma A.8 it follows from the continuity of \( \mathbf{\theta} \to \nabla^2 l(\mathbf{\theta}, \mathbf{Z}) \) at \( \mathbf{\theta}' \) in Assumption A2 that \( \lim_{T \to \infty} \|\nabla^2 l(\mathbf{\hat{\theta}}_T, \mathbf{Z}_T) - \nabla^2 l(\mathbf{\theta}', \mathbf{Z}_T)\| = 0 \) almost surely. As a consequence, when \( T \to \infty, \) we have

\[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left\| \nabla^2 l(\mathbf{\theta}', \mathbf{Z}_T) \right\| = \frac{1}{T} \sum_{t=1}^{T} \left\| \nabla^2 l(\mathbf{\theta}', \mathbf{Z}_T) \right\| \to 0 \text{ almost surely.} \]

By Assumption A2 Lebesgue's Dominated Convergence Theorem, and L.L.N., we can see

\[ \frac{1}{T} \sum_{t=1}^{T} \nabla^2 l(\mathbf{\theta}', \mathbf{Z}_T) = E[\nabla^2 l(\mathbf{\theta}', \mathbf{Z}_T)] + o_p(1) = \nabla^2 L(\mathbf{\theta}') + o_p(1). \]

Combining the above, we show that \( \hat{G}_T = G + o_p(1). \)

Similarly, we have \( \lim_{T \to \infty} \|\nabla l(\mathbf{\hat{\theta}}_T, \mathbf{Z}_T) - \nabla l(\mathbf{\theta}', \mathbf{Z}_T)\| = 0 \) almost surely by the differentiability of \( \mathbf{\theta} \to \nabla l(\mathbf{\theta}, \mathbf{Z}) \) in Assumption A2. Moreover, by L.L.N., we can derive \( \hat{\mathbf{S}}_T = S + o_p(1). \) Finally, applying Lemma A.15 we complete the proof. 

**Lemma A.15** is a direct consequence of Lemma A.16.
A.5. Proof of Theorem 2

Under $H_0$, by Theorem 1 it follows that

$$
\bar{\theta}_{T,p} - \theta^* = -\frac{1}{T} \sum_{t=1}^{T} (PGP)^{\top} \xi_t + o_p(T^{-1/2}),
$$

$$
\bar{\theta}_{T,J} - \theta^* = -\frac{1}{T} \sum_{t=1}^{T} G^{-1} \xi_t + o_p(T^{-1/2}).
$$

Since $P(PGP)^{\top} = (PGP)^{\top}$ by Lemma A.2, we have

$$(I - P)(\bar{\theta}_{T,p} - \bar{\theta}_{T,J}) = \frac{1}{T} \sum_{t=1}^{T} (I - P)[G^{-1} - (PGP)^{\top}] \xi_t + o_p(T^{-1/2}) = \frac{1}{T} \sum_{t=1}^{T} (I - P)G^{-1} \xi_t + o_p(T^{-1/2}).$$

By Lemma A.12 we show that $\sqrt{T}(\bar{\theta}_{T,p} - \bar{\theta}_{T,J}) \xrightarrow{d} N(0, W)$, where $W = (I - P)G^{-1} S G^{-1}(I - P)$. By delta method, we have $\sqrt{T}[W^{-\frac{1}{2}}(\bar{\theta}_{T,p} - \bar{\theta}_{T,J})] \xrightarrow{d} N(0, V)$, where $V = \text{Diag}(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{p \times p}$ is a squared matrix with rank $p - d$. The above convergence further leads to $T(\bar{\theta}_{T,p} - \bar{\theta}_{T,J})^\top W^{-\frac{1}{2}} (\bar{\theta}_{T,p} - \bar{\theta}_{T,J}) \xrightarrow{d} \chi^2(p - d)$. By Lemma A.16 it follows that $\hat{G}_{T,J} = G + o_p(1)$ and $\hat{S}_{T,J} = S + o_p(1)$. Moreover, both $W$ and $\hat{W}$ are of rank $p - d$. As a consequence of Lemma A.14 it follows $\hat{W} = W + o_p(1)$. Applying Slutsky’s Theorem, we complete the proof of the result under $H_0$.

Under $H_1$, since $B\theta^* = b + \beta$, for some $\beta \neq 0$. Consider the following decomposition $\theta^* = \theta^* + \mu$ with $B\theta^* = b$ and $B\mu = \beta$. Clearly, $(I - P)\mu = 0$, as $(I - P)\mu = 0$ implies $P\mu = \mu$ and $\mu \in \text{Ker}(B)$, which is impossible. Since $B\bar{\theta}_{T,p} = B\theta^* = b$, we have

$$(I - P)(\bar{\theta}_{T,p} - \bar{\theta}_{T,J}) = (I - P)(\bar{\theta}_{T,p} - \theta^* + \theta^* - \bar{\theta}_{T,J}) = (I - P)(\bar{\theta}_{T,p} - \theta^* - \theta^* - \bar{\theta}_{T,J}) = -(I - P)\mu - (I - P)(\bar{\theta}_{T,J} - \theta^*).$$

(A.5.1)

Moreover, by Lemma A.2, we have $\hat{W}^- (I - P) = (I - P)\hat{W}^- = W^-$. Following (A.5.1), we have

$$T(\bar{\theta}_{T,p} - \bar{\theta}_{T,J})^\top W^- (\bar{\theta}_{T,p} - \bar{\theta}_{T,J}) = T\mu^\top W^- \mu + T(\bar{\theta}_{T,J} - \theta^*)^\top W^- (\bar{\theta}_{T,J} - \theta^*) + 2T(\bar{\theta}_{T,J} - \theta^*)^\top W^- \mu := J_1 + J_2 + J_3.$$

For $S_1$, let $\hat{A}_1, \hat{A}_{p-d}$ and $A_1, A_{p-d}$ be the largest and smallest non-zero eigenvalues of $\hat{W}$ and $W$ respectively. By Lemma A.14 we know $\hat{A}_1 \leq 2A_1$ and $\hat{A}_{p-d} \geq A_{p-d}/2$ with probability approaching 1. Then by Lemma A.2, we conclude that

$$J_1 \geq \frac{T}{A} ||(I - P)\mu||^2 \geq \frac{T}{2A} ||(I - P)\mu||^2,$$

with probability approaching 1.

Since Theorem 1 implies that $\bar{\theta}_{T,J} - \theta^* = O_p(T^{-1/2})$, it follows that

$$J_2 \leq T||W^-|| ||\bar{\theta}_{T,J} - \theta^*||^2 \leq \frac{T}{A_{p-d}} ||\bar{\theta}_{T,J} - \theta^*||^2 \leq \frac{2T}{A_{p-d}} ||\bar{\theta}_{T,J} - \theta^*||^2 = O_p(1).$$

Similarly, by Cauchy–Schwarz inequality, we can show

$$J_3 \leq 2T||\hat{W}^-|| ||\bar{\theta}_{T,J} - \theta^*|| ||\mu|| \leq \frac{2T}{A_{p-d}} ||\bar{\theta}_{T,J} - \theta^*|| ||\mu|| \leq \frac{4T}{A_{p-d}} ||\bar{\theta}_{T,J} - \theta^*|| ||\mu|| = O_p(T^{1/2}).$$

Combining the three bounds, we prove that $T(\bar{\theta}_{T,p} - \bar{\theta}_{T,J})^\top W^- (\bar{\theta}_{T,p} - \bar{\theta}_{T,J}) \xrightarrow{d} \infty$ with probability approaching 1.

Suppose the local alternative $H_a : B\theta^* = b + \beta/\sqrt{T}$ holds. Consider the following decomposition $\theta^* = \theta^* + \mu/\sqrt{T}$ with $B\theta^* = b$ and $B\mu = \beta$. By Lemma A.2 we have $(\hat{W}^-)^{1/2}(I - P) = (I - P)(\hat{W}^-)^{1/2} = (W^-)^{1/2}$. By similar proof to (A.5.1), we have

$$(I - P)(\bar{\theta}_{T,p} - \bar{\theta}_{T,J}) = -(I - P)\mu/\sqrt{T} - (I - P)(\bar{\theta}_{T,J} - \theta^*),$$

with probability approaching 1.
which further leads to 

\[(\hat{W}^{-1}2(\hat{\theta}_{T,P} - \hat{\theta}_{T,J}) = (\hat{W}^{-1}2(I - P)(\hat{\theta}_{T,P} - \hat{\theta}_{T,J}) = -(\hat{W}^{-1}2(I - P)\mu)\sqrt{T} - (\hat{W}^{-1}2(\hat{\theta}_{T,J} - \theta^*) := R_1 - R_2.\]

Since \(\hat{W} = W + o_p(1)\), it follows that \(\sqrt{T}R_1 = -(W^{-1}2(I - P)\mu + o_p(1) = -(W^{-1}2\mu + o_p(1)\)

Moreover, Theorem \[\] implies that 

\[\sqrt{T}R_2 = (W^{-1}2(\hat{\theta}_{T,J} - \theta^*) + o_p(1) \overset{L}{\rightarrow} N(0, (W^{-1}2G^{-1}SG^{-1}(W^{-1}2)).\]

By direct calculation, it can be verified that 

\[(W^{-1}2G^{-1}SG^{-1}(W^{-1}2 = (W^{-1}2(I - P)G^{-1}SG^{-1}(I - P)(W^{-1}2 = (W^{-1}2W(W^{-1}2 = \text{Diag}(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{p \times p}.\]

As a consequence, we show that \(T(\hat{\theta}_{T,P} - \hat{\theta}_{T,J})^\top \hat{W}^{-1}(\hat{\theta}_{T,P} - \hat{\theta}_{T,J}) \overset{L}{\rightarrow} \chi^2(\mu^\top W^{-1}\mu, p - d).\]

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Supplementary Material for “Online Statistical Inference for Parameters Estimation with Linear-Equality Constraints”

This supplementary material summarizes several simulation results and an application to a real-world dataset.

S.1. Estimation error and coverage probability

DGP 1 (Linear Regression): Consider the model $Y = \sum_{j=1}^{4} \beta_j X_j + \epsilon$, with the true parameters $\theta^* = (\beta_1, \beta_2, \beta_3, \beta_4)^T = (1.5, -3, 2, 1)^T$. The covariates $X_1, X_2, X_3, X_4 \sim N(0, 1)$ and the error term $\epsilon \sim N(0, 9)$ are independent. The linear-equality constraint $\beta_2 + \beta_3 + \beta_4 = 0$ is used.

DGP 2 (Logistic Regression): We generate the model $P(Y = y|X_1, X_2, X_3, X_4) = [1 + e^{X_j^T \beta_j}]^{-1}$ for $y \in \{-1, 1\}$, with the true parameters $\theta^* = (\beta_1, \beta_2, \beta_3, \beta_4)^T = (1, -2, -2, 1.5)^T$. The covariates $X_1, X_2, X_3, X_4$ follow the same distributions as DGP 1. The linear-equality constraint $\beta_2 - \beta_3 = 0$ is applied.

We consider the APSGD estimate $\hat{\theta}_{T,P} = (\hat{\beta}_{1,P}, \hat{\beta}_{2,P}, \hat{\beta}_{3,P}, \hat{\beta}_{4,P})^T$ using the proper projection matrix $P$ and the ASGD estimate $\hat{\theta}_{T,J} = (\hat{\beta}_{1,J}, \hat{\beta}_{2,J}, \hat{\beta}_{3,J}, \hat{\beta}_{4,J})^T$ using identity projection matrix. The estimation error is evaluated by $|\hat{\beta}_j - \beta_j|$ and $|\hat{\beta}_{ij} - \beta_{ij}|$ over 500 runs. Moreover, during each run, we construct a 95% level confidence interval for $\beta_j$, and examine whether the true $\beta_j$ is in the confidence interval or not. The learning rate is selected as $\tau^{-0.505}$. The estimation error and coverage probability are reported in Tables 1-2. First, from Table 1, we see that the estimation errors of $\hat{\theta}_{T,P}$ and $\hat{\theta}_{T,J}$ decrease when the sample size $T$ increases. Second, for both linear and logistic models, the estimation error of $\hat{\theta}_{T,P}$ is uniformly smaller than $\hat{\theta}_{T,J}$ regardless of the sample size. Third, Table 2 reveals that the coverage probabilities of the 95% confidence intervals for $\beta_1, \ldots, \beta_4$ are around 95%, which confirms the validity of our theoretical results.

S.2. Size and power

To examine the empirical performance of the specification test in (4.2), we modify the settings of DGP 1 and DGP 2 as follows.

DGP 1 (Linear Regression): The coefficients are chosen to be $\theta^* = (\beta_1, \beta_2, \beta_3, \beta_4)^T = (1.5, -3, 2, 1 + r)^T$, with $r = 0, 0.005, 0.01, 0.015, 0.02, 0.025$. The hypothesis to be tested is $H_0 : \beta_2 + \beta_3 + \beta_4 = 0$.

DGP 2 (Logistic Regression): The coefficients are chosen to be $\theta^* = (\beta_1, \beta_2, \beta_3, \beta_4)^T = (3, -2, -2 + r, 1)^T$, with $r = 0, 0.005, 0.01, 0.015, 0.02, 0.025$. The null hypothesis to be tested is $H_0 : \beta_2 - \beta_3 = 0$.

The scalar $r$ measures the level of model misspecification. When $r = 0$, the model is correctly specified by the linear-equality constraint. We repeat the experiment 500 times with significance level $\alpha = 0.05$ for different choices of $r$ and $T$, and the average rejection probabilities are reported in Figure 1. First, Figure 1 indicates that the probabilities of rejecting the null hypothesis are around 0.95 when $r = 0$, which suggests the proposed specification test has an correct asymptotic size ($\alpha = 0.05$). Second, for different sample sizes, the rejection probability is monotonically increasing with respect to $r$. In particular, the specification test almost 100% rejects $H_0$ when $r = 0.02, 0.025$ for both linear and logistic models, which confirms the consistency of the specification test.
Table 1: Estimation error of $\hat{\theta}_{T,p}$ and $\hat{\theta}_{T,i}$.

|          | Linear |          | Logistic |
|----------|--------|----------|----------|
| $T$      |        |          |          |
| $10^4$   | 0.0080 | 0.0080   | 0.0099   |
| $2 \times 10^4$ | 0.0060 | 0.0077   | 0.0119   |
| $5 \times 10^4$ | 0.0058 | 0.0072   | 0.0119   |
| $10^5$   | 0.0062 | 0.0077   | 0.0113   |
| $\beta_1$ |        |          |          |
| $10^4$   | 0.0055 | 0.0055   | 0.0065   |
| $2 \times 10^4$ | 0.0043 | 0.0057   | 0.0076   |
| $5 \times 10^4$ | 0.0043 | 0.0048   | 0.0076   |
| $10^5$   | 0.0045 | 0.0054   | 0.0073   |
| $\beta_2$ |        |          |          |
| $10^4$   | 0.0035 | 0.0036   | 0.0038   |
| $2 \times 10^4$ | 0.0028 | 0.0035   | 0.0049   |
| $5 \times 10^4$ | 0.0025 | 0.0031   | 0.0049   |
| $10^5$   | 0.0029 | 0.0034   | 0.0048   |
| $\beta_3$ |        |          |          |
| $10^4$   | 0.0023 | 0.0023   | 0.0028   |
| $2 \times 10^4$ | 0.0020 | 0.0023   | 0.0035   |
| $5 \times 10^4$ | 0.0021 | 0.0026   | 0.0035   |
| $10^5$   | 0.0020 | 0.0024   | 0.0032   |
| $\beta_4$ |        |          |          |
| $10^4$   | 0.0023 | 0.0023   | 0.0028   |
| $2 \times 10^4$ | 0.0020 | 0.0023   | 0.0035   |
| $5 \times 10^4$ | 0.0021 | 0.0026   | 0.0035   |
| $10^5$   | 0.0020 | 0.0024   | 0.0032   |

Table 2: Coverage probability of $\hat{\theta}_{T,p}$.

|          | Linear Model |          | Logistic Model |
|----------|--------------|----------|----------------|
| $T$      | $10^5$   | $2 \times 10^5$ | $5 \times 10^5$ | $10^6$ |
| $\beta_1$ | 0.942    | 0.934    | 0.944    | 0.966 |
| $\beta_2$ | 0.946    | 0.938    | 0.942    | 0.944 |
| $\beta_3$ | 0.960    | 0.968    | 0.944    | 0.966 |
| $\beta_4$ | 0.938    | 0.960    | 0.962    | 0.962 |
|          | $10^5$   | $2 \times 10^5$ | $5 \times 10^5$ | $10^6$ |
|          | 0.918    | 0.938    | 0.954    | 0.956 |
|          | 0.924    | 0.924    | 0.932    | 0.948 |
|          | 0.924    | 0.924    | 0.932    | 0.945 |
|          | 0.934    | 0.936    | 0.938    | 0.962 |

Fig. 1: Rejection probability of the specification test.
Table 3: ASGD estimate. The symbols $\ast$ and $\cdot$ stand for p-value $<0.05$ and p-value $<0.1$.

|       | V1   | V2    | V3    | V4    | V5    | V6    | V7    | V8    | V9    |
|-------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| PSGD  | 1.187| 3.094 | 0.901 | -4.713| 1.019 | -2.303| 1.159 | 0.884 | -0.231|
| SE    | 0.896| 0.343 | 0.146 | 0.312 | 0.759 | 0.333 | 0.637 | 0.041 | 0.180 |
| P-value| 0.185| 0.000\ast| 0.000\ast| 0.000\ast| 0.179 | 0.000\ast| 0.069\ast| 0.000\ast| 0.198 |

Table 4: APSGD estimate with constraint V1=V9=0. The symbols $\ast$ and $\cdot$ stand for p-value $<0.05$ and p-value $<0.1$.

|       | V2    | V3    | V4    | V5    | V6    | V7    |
|-------|-------|-------|-------|-------|-------|-------|
| Estimates | 3.468 | 0.671 | -4.971| 2.086 | -1.681| 0.466 |
| SE    | 0.208 | 0.085 | 0.149 | 0.216 | 0.188 | 0.243 |
| P-value | 0.000\ast| 0.000\ast| 0.000\ast| 0.000\ast| 0.055\ast| 0.000\ast|

S.3. Empirical application

In this section, we apply our method to the protein tertiary structure dataset from UCI machine learning repository ([4]). The dataset contains response variable *size of the residue* and nine explanatory variables (denoted by V1-V9) measuring the physicochemical properties of the protein tertiary structure. There are 45730 observations in the dataset. After standardizing the explanatory variables, we first obtain the ASGD estimate (using $P=I$) and its standard error. We next calculate the corresponding p-values to examine whether the coefficients are significant, and the results are summarized in Table 3. Based on the ASGD estimate, the p-values of the explanatory variables V1, V5, V9 are 0.185, 0.179 and 0.198, which are not significant under significant level $\alpha=0.05$. Meanwhile, the variable V7 has a p-value 0.069, which is close to 0.05. To determine whether removing these insignificant variables or not, we sequentially apply the testing procedure to test the following null hypotheses based on the p-value of the insignificant variables: V1=V5=V7=V9=0, V1=V5=V9=0, V1=V9=0, V9=0. The corresponding p-values of the specification test are 0, 0, 0.207 and 0.198. Based on the results, we fail to reject the null hypotheses V1=V9=0 and V9=0. Therefore, we calculate the APSGD estimate using the linear-equality constraint V1=V9=0, and the results are reported in Table 4. In comparison with the ASGD estimate, the APSGD estimate gives a smaller standard error for the estimated coefficients. Moreover, all the variables, except V7, are highly significant with p-value being almost zero. The p-values of V7 in APSGD and ASGD estimates are 0.055 and 0.069, respectively, which suggests its significance is slightly improved.