CONVERGENCE ON GAUSS-SEIDEL ITERATIVE METHODS FOR LINEAR SYSTEMS WITH GENERAL $H$–MATRICES

CHENG-YI ZHANG†, DAN YE‡, CONG-LEI ZHONG§, AND SHUANGHUA LUO¶

Abstract. It is well known that as a famous type of iterative methods in numerical linear algebra, Gauss-Seidel iterative methods are convergent for linear systems with strictly or irreducibly diagonally dominant matrices, invertible $H$–matrices (generalized strictly diagonally dominant matrices) and Hermitian positive definite matrices. But, the same is not necessarily true for linear systems with nonstrictly diagonally dominant matrices and general $H$–matrices. This paper firstly proposes some necessary and sufficient conditions for convergence on Gauss-Seidel iterative methods to establish several new theoretical results on linear systems with nonstrictly diagonally dominant matrices and general $H$–matrices. Then, the convergence results on preconditioned Gauss-Seidel (PGS) iterative methods for general $H$–matrices are presented. Finally, some numerical examples are given to demonstrate the results obtained in this paper.

Key words. Gauss-Seidel iterative methods; Convergence; Nonstrictly diagonally dominant matrices; General $H$–matrices.

AMS subject classifications. 15A15, 15F10.

1. Introduction. In this paper we consider the solution methods for the system of $n$ linear equations

\begin{equation}
Ax = b,
\end{equation}

where $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and is nonsingular, $b, x \in \mathbb{C}^n$ and $x$ unknown. Let us recall the standard decomposition of the coefficient matrix $A \in \mathbb{C}^{n \times n}$,

\begin{equation}
A = D_A - L_A - U_A,
\end{equation}

where $D_A = \text{diag}(a_{11}, a_{22}, \cdots, a_{nn})$ is a diagonal matrix, $L_A$ and $U_A$ are strictly lower and strictly upper triangular matrices, respectively. If $a_{ii} \neq 0$ for all $i \in \{n\}$ =

*Corresponding author: Cheng-yi Zhang, Email: chyzhang08@126.com
†Department of Mathematics and Mechanics of School of Science, Xi’an Polytechnic University, Xi’an, Shaanxi, 710048, P.R. China. Supported by the Science Foundation of the Education Department of Shaanxi Province of China (13JK0593), the Scientific Research Foundation (BS1014) and the Education Reform Foundation (2012JG40) of Xi’an Polytechnic University, and National Natural Science Foundations of China (Grant No. 11201362 and 11271297).
‡School of Science, Xi’an Polytechnic University, Xi’an, Shaanxi, 710048, P.R. China.
§School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, Henan, 417003, P.R. China.
¶Department of Mathematics and Mechanics of School of Science, Xi’an Polytechnic University, Xi’an, Shaanxi, 710048, P.R. China. Supported by the Science Foundation of the Education Department of Shaanxi Province of China (14JK1305).
{1, 2, · · · , n}, the Jacobi iteration matrix associated with the coefficient matrix $A$ is

$$H_J = D_A^{-1}(L_A + U_A);$$

the forward, backward and symmetric Gauss-Seidel (FGS-, BGS- and SGS-) iteration matrices associated with the coefficient matrix $A$ are

$$H_{FGS} = (D_A - L_A)^{-1}U_A, \quad (1.4)$$

$$H_{BGS} = (D_A - U_A)^{-1}L_A, \quad (1.5)$$

and

$$H_{SGS} = H_{BGS}H_{FGS} = (D_A - U_A)^{-1}L_A(D_A - U_A)^{-1}L_A, \quad (1.6)$$

respectively. Then, the Jacobi, FGS, BGS and SGS iterative method can be denoted the following iterative scheme:

$$x^{(i+1)} = Hx^{(i)} + f, \quad i = 0, 1, 2, \cdots \cdots \quad (1.7)$$

where $H$ denotes iteration matrices $H_J$, $H_{FGS}$, $H_{BGS}$ and $H_{SGS}$, respectively, correspondingly, $f$ is equal to $D_A^{-1}b$, $(D_A - L_A)^{-1}b$, $(D_A - U_A)^{-1}b$ and $(D_A - U_A)^{-1}D_A(D_A - L_A)^{-1}b$, respectively. It is well-known that (1.7) converges for any given $x^{(0)}$ if and only if $\rho(H) < 1$ (see [27]), where $\rho(H)$ denotes the spectral radius of the iteration matrix $H$. Thus, to establish the convergence results of iterative scheme (1.7), we mainly study the spectral radius of the iteration matrix in the iterative scheme (1.7).

As is well known in some classical textbooks and monographs, see [27], Jacobi and Gauss-Seidel iterative methods for linear systems with Hermitian positive definite matrices, strictly or irreducibly diagonally dominant matrices and invertible $H$–matrices(generalized strictly diagonally dominant matrices) are convergent. Recently, the class of strictly or irreducibly diagonally dominant matrices and invertible $H$–matrices has been extended to encompass a wider set, known as the set of general $H$–matrices. In a recent paper, Ref. [3, 4, 5], a partition of the $n \times n$ general $H$–matrix set, $H_n$, into three mutually exclusive classes was obtained: the Invertible class, $H_n^I$, where the comparison matrices of all general $H$–matrices are nonsingular, the Singular class, $H_n^S$, formed only by singular $H$–matrices, and the Mixed class, $H_n^M$, in which singular and nonsingular $H$–matrices coexist. Lately, Zhang in [34] proposed some necessary and sufficient conditions for convergence on Jacobi iterative methods for linear systems with general $H$–matrices.
A problem has to be proposed, i.e., whether Gauss-Seidel iterative methods for linear systems with nonstrictly diagonally dominant matrices and general $H$–matrices are convergent or not. Let us investigate the following examples.

**Example 1.1.** Assume that either $A$ or $B$ is the coefficient matrix of linear system (1.1), where $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$. It is verified that both $A$ and $B$ are nonstrictly diagonally dominant and nonsingular. Direct computations yield that $\rho(H_{FGS}^A) = \rho(H_{BGS}^A) = 1$, while $\rho(H_{BGS}^A) = 0.3536 < 1$ and $\rho(H_{SGS}^A) = \rho(H_{SGS}^B) = 0.5797 < 1$. This shows that BGS and SGS iterative methods for the matrix $A$ are convergent, while the same is not FGS iterative method for $A$; However, FGS and SGS iterative methods for the matrix $B$ are convergent, while the same is not BGS iterative method for $B$.

**Example 1.2.** Assume that either $A$ or $B$ is the coefficient matrix of linear system (1.1), where $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$. It is verified that $A$ is nonstrictly diagonally dominant matrix and $B$ is a mixed $H$–matrix. Further, they are nonsingular. By direct computations, it is easy to get that $\rho(H_{FGS}^A) = 0.4215 < 1$, $\rho(H_{BGS}^A) = 0.3536 < 1$ and $\rho(H_{SGS}^A) = 0.3608 < 1$, while $\rho(H_{FGS}^B) = \rho(H_{BGS}^B) = \rho(H_{SGS}^B) = 1$. This shows that FGS, BGS and SGS iterative methods converge for the matrix $A$, while they fail to converge for the matrix $B$.

In fact, the matrices $A$ and $B$ in Example 1.1 and Example 1.2, respectively, are all general $H$–matrices, but are not invertible $H$–matrices. Guass-Seidel iterative methods for these matrices sometime may converge for some given general $H$–matrices, but may fail to converge for other given general $H$–matrices. How do we get the convergence on Guass-Seidel iterative methods for linear systems with this class of matrices without direct computations?

Aim at the problem above, some necessary and sufficient conditions for convergence on Guass-Seidel iterative methods are firstly proposed to establish some new results on nonstrictly diagonally dominant matrices and general $H$–matrices. In particular, the convergence results on preconditioned Gauss-Seidel (PGS) iterative methods for general $H$–matrices are presented. Furthermore, some numerical examples are given to demonstrate the results obtained in this paper.

The paper is organized as follows. Some notations and preliminary results about special matrices are given in Section 2. Some special matrices will be defined, based on which some necessary and sufficient conditions for convergence on Guass-Seidel iterative methods are firstly proposed in Section 3. Some convergence results on
preconditioned Gauss-Seidel iterative methods for general \( H \)-matrices are then presented in Section 4. In Section 5, some numerical examples are given to demonstrate the results obtained in this paper. Conclusions are given in Section 6.

2. Preliminaries. In this section we give some notions and preliminary results about special matrices that are used in this paper.

\( \mathbb{C}^{m \times n} (\mathbb{R}^{m \times n}) \) will be used to denote the set of all \( m \times n \) complex (real) matrices. \( \mathbb{Z} \) denotes the set of all integers. Let \( \alpha \subseteq \langle n \rangle = \{1, 2, \cdots, n\} \subset \mathbb{Z} \). For nonempty index sets \( \alpha, \beta \subseteq \langle n \rangle \), \( A(\alpha, \beta) \) is the submatrix of \( A \in \mathbb{C}^{n \times n} \) with row indices in \( \alpha \) and column indices in \( \beta \). The submatrix \( A(\alpha, \alpha) \) is abbreviated to \( A(\alpha) \). Let \( A \in \mathbb{C}^{n \times n} \), \( \alpha \subseteq \langle n \rangle \) and \( \alpha' = \langle n \rangle - \alpha \). If \( A(\alpha) \) is nonsingular, the matrix

\[
A/\alpha = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha')
\]

is called the Schur complement with respect to \( A(\alpha) \), indices in both \( \alpha \) and \( \alpha' \) are arranged with increasing order. We shall confine ourselves to the nonsingular \( A(\alpha) \) as far as \( A/\alpha \) is concerned.

Let \( A = (a_{ij}) \in \mathbb{C}^{m \times n} \) and \( B = (b_{ij}) \in \mathbb{C}^{m \times n} \), \( A \otimes B = (a_{ij}b_{ij}) \in \mathbb{C}^{m \times n} \) denotes the Hadamard product of the matrices \( A \) and \( B \). A matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) is called nonnegative if \( a_{ij} \geq 0 \) for all \( i, j \in \langle n \rangle \). A matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) is called a \( Z \)-matrix if \( a_{ij} \leq 0 \) for all \( i \neq j \). We will use \( \mathbb{Z}_n \) to denote the set of all \( n \times n \) \( Z \)-matrices. A matrix \( A = (a_{ij}) \in \mathbb{Z}_n \) is called a \( M \)-matrix if \( A \) can be expressed in the form \( A = sI - B \), where \( B \geq 0 \), and \( s \geq \rho(B) \), the spectral radius of \( B \). If \( s > \rho(B) \), \( A \) is called a nonsingular \( M \)-matrix; if \( s = \rho(B) \), \( A \) is called a singular \( M \)-matrix. \( M_n, M_n^* \) and \( M_n^0 \) will be used to denote the set of all \( n \times n \) \( M \)-matrices, the set of all \( n \times n \) nonsingular \( M \)-matrices and the set of all \( n \times n \) singular \( M \)-matrices, respectively. It is easy to see that

\[
M = M_n^* \cup M_n^0 \quad \text{and} \quad M_n^* \cap M_n^0 = \emptyset.
\]

The comparison matrix of a given matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \), denoted by \( \mu(A) = (\mu_{ij}) \), is defined by

\[
\mu_{ij} = \begin{cases} |a_{ij}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}
\]

It is clear that \( \mu(A) \in \mathbb{Z}_n \) for a matrix \( A \in \mathbb{C}^{n \times n} \). The set of equimodular matrices associated with \( A \), denoted by \( \omega(A) = \{ B \in \mathbb{C}^{n \times n} : \mu(B) = \mu(A) \} \). Note that both \( A \) and \( \mu(A) \) are in \( \omega(A) \). A matrix \( A = a_{ij} \in \mathbb{C}^{n \times n} \) is called a general \( H \)-matrix if \( \mu(A) \in M_n \) (see [2]). If \( \mu(A) \in M_n^* \), \( A \) is called an invertible \( H \)-matrix; if \( \mu(A) \in M_n^0 \) with \( a_{ii} = 0 \) for at least one \( i \in \langle n \rangle \), \( A \) is called a singular \( H \)-matrix; if \( \mu(A) \in M_n^0 \) with \( a_{ii} \neq 0 \) for all \( i \in \langle n \rangle \), \( A \) is called a mixed \( H \)-matrix. \( H_n, H_n^I, H_n^N \) and \( H_n^M \).
will denote the set of all $n \times n$ general $H$–matrices, the set of all $n \times n$ invertible $H$–matrices, the set of all $n \times n$ singular $H$–matrices and the set of all $n \times n$ mixed $H$–matrices, respectively (See [3]). Similar to equalities (2.2), we have

$$H_n = H_n^I \cup H_n^S \cup H_n^M \quad \text{and} \quad H_n^I \cap H_n^S \cap H_n^M = \emptyset.$$  

(2.3)

For $n \geq 2$, an $n \times n$ complex matrix $A$ is reducible if there exists an $n \times n$ permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

(2.4)

where $A_{11}$ is an $r \times r$ submatrix and $A_{22}$ is an $(n - r) \times (n - r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then $A$ is called irreducible. If $A$ is a $1 \times 1$ complex matrix, then $A$ is irreducible if its single entry is nonzero, and reducible otherwise.

**Definition 2.1.** A matrix $A \in \mathbb{C}^{n \times n}$ is called diagonally dominant by row if

$$|a_{ii}| \geq \sum_{j=1, j \neq i}^{n} |a_{ij}|$$

(2.5)

holds for all $i \in \langle n \rangle$. If inequality in (2.5) holds strictly for all $i \in \langle n \rangle$, $A$ is called strictly diagonally dominant by row. If $A$ is irreducible and the inequality in (2.5) holds strictly for at least one $i \in \langle n \rangle$, $A$ is called irreducibly diagonally dominant by row. If (2.5) holds with equality for all $i \in \langle n \rangle$, $A$ is called diagonally equipotent by row.

$D_n(SD_n, ID_n)$ and $DE_n$ will be used to denote the sets of all $n \times n$ (strictly, irreducibly) diagonally dominant matrices and the set of all $n \times n$ diagonally equipotent matrices, respectively.

**Definition 2.2.** A matrix $A \in \mathbb{C}^{n \times n}$ is called generalized diagonally dominant if there exist positive constants $\alpha_i$, $i \in \langle n \rangle$, such that

$$\alpha_i |a_{ii}| \geq \sum_{j=1, j \neq i}^{n} \alpha_j |a_{ij}|$$

(2.6)

holds for all $i \in \langle n \rangle$. If inequality in (2.6) holds strictly for all $i \in \langle n \rangle$, $A$ is called generalized strictly diagonally dominant. If (2.6) holds with equality for all $i \in \langle n \rangle$, $A$ is called generalized diagonally equipotent.

We will denote the sets of all $n \times n$ generalized (strictly) diagonally dominant matrices and the set of all $n \times n$ generalized diagonally equipotent matrices by
Definition 2.3. A matrix $A$ is called nonstrictly diagonally dominant, if either (2.5) or (2.6) holds with equality for at least one $i \in \langle n \rangle$.

Remark 2.4. Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ be nonstrictly diagonally dominant and $\alpha = \langle n \rangle - \alpha' \subset \langle n \rangle$. If $A(\alpha)$ is a (generalized) diagonally equipotent principal submatrix of $A$, then the following hold:

- $A(\alpha, \alpha') = 0$, which shows that $A$ is reducible;
- $A(i_1) = (a_{i_1i_1})$ being (generalized) diagonally equipotent implies $a_{i_1i_1} = 0$.

Remark 2.5. Definition 2.2 and Definition 2.3 show that $D_n \subset GD_n$ and $GSD_n \subset GD_n$.

The following will introduce the relationship of (generalized) diagonally dominant matrices and general $H$–matrices and some properties of general $H$–matrices that will be used in the rest of the paper.

Lemma 2.6. (see [28, 30, 32, 31]) Let $A \in D_n(GD_n)$. Then $A \in H_n^I$ if and only if $A$ has no (generalized) diagonally equipotent principal submatrices. Furthermore, if $A \in D_n \cap Z_n(GD_n \cap Z_n)$, then $A \in M_n^I$ if and only if $A$ has no (generalized) diagonally equipotent principal submatrices.

Lemma 2.7. (see [2]) $SD_n \cup ID_n \subset H_n^I = GSD_n$

Lemma 2.8. (see [3]) $GD_n \subset H_n$.

It is interested in wether $H_n \subseteq GD_n$ is true or not. The answer is ”NOT”. Some counterexamples are given in [3] to show that $H_n \subseteq GD_n$ is not true. But, under the condition of ”irreducibility”, the following conclusion holds.

Lemma 2.9. (see [3]) Let $A \in \mathbb{C}^{n \times n}$ be irreducible. Then $A \in H_n$ if and only if $A \in GD_n$.

More importantly, under the condition of ”reducibility”, we have the following conclusion.

Lemma 2.10. Let $A \in \mathbb{C}^{n \times n}$ be reducible. Then $A \in H_n$ if and only if in the Frobenius normal form of $A$

$$PAP^T = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1s} \\ 0 & R_{22} & \cdots & R_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{ss} \end{bmatrix},$$

(2.7)

each irreducible diagonal square block $R_{ii}$ is generalized diagonally dominant, where $P$
is a permutation matrix, \( R_{ii} = A(\alpha_i) \) is either \( 1 \times 1 \) zero matrices or irreducible square matrices, \( R_{ij} = A(\alpha_i, \alpha_j) \), \( i \neq j \), \( i, j = 1, 2, \cdots, s \), further, \( \alpha_i \cap \alpha_j = \emptyset \) for \( i \neq j \), and \( \cup_{i=1}^s \alpha_i = \{n\} \).

The proof of this theorem follows from Lemma 2.3 and Theorem 5 in [3].

**Lemma 2.11.** A matrix \( A \in H^M_n \cup H^S_n \) if and only if in the Frobenius normal from \([2.7]\) of \( A \), each irreducible diagonal square block \( R_{ii} \) is generalized diagonally dominant and has at least one generalized diagonally equipotent principal submatrix.

**Proof.** It follows from \([2.8]\), Lemma 2.6 and Lemma 2.10 that the conclusion of this lemma is obtained immediately. \( \Box \)

### 3. Some special matrices and their properties.

In order to investigate convergence on Gauss-Seidel iterative methods, some definitions of special matrices will be defined and their properties will be proposed to be used in this paper.

**Definition 3.1.** (see [31]) Let \( E^\theta = (e^{i\theta_{rs}}) \in \mathbb{C}^{n \times n} \), where \( e^{i\theta_{rs}} = \cos \theta_{rs} + i \sin \theta_{rs} \), \( i = \sqrt{-1} \) and \( \theta_{rs} \in \mathbb{R} \) for all \( r, s \in \{n\} \). The matrix \( E^\theta = (e^{i\theta_{rs}}) \in \mathbb{C}^{n \times n} \) is called \( \theta \)-ray pattern matrix if

1. \( \theta_{rs} + \theta_{sr} = 2k\pi \) holds for all \( r, s \in \{n\} \), \( r \neq s \), where \( k \in \mathbb{Z} \);
2. both \( \theta_{rs} - \theta_{rt} = \theta_{ts} + (2k + 1)\pi \) and \( \theta_{sr} - \theta_{st} = \theta_{tr} + (2k + 1)\pi \) hold for all \( r, s, t \in \{n\} \) and \( r \neq s \), \( r \neq t \), \( t \neq s \), where \( k \in \mathbb{Z} \);
3. \( \theta_{rr} = \theta \) for all \( r \in \{n\} \), \( \theta \in [0, 2\pi) \).

**Definition 3.2.** Let \( E^\psi = (e^{i\psi_{rs}}) \in \mathbb{C}^{n \times n} \), where \( e^{i\psi_{rs}} = \cos \psi_{rs} + i \sin \psi_{rs} \), \( i = \sqrt{-1} \) and \( \psi_{rs} \in \mathbb{R} \) for all \( r, s \in \{n\} \). The matrix \( E^\psi = (e^{i\psi_{rs}}) \in \mathbb{C}^{n \times n} \) is called \( \psi \)-ray pattern matrix if

1. \( \psi_{rs} + \psi_{sr} = 2k\pi + \psi \) holds for all \( r, s \in \{n\} \), \( r \neq s \), where \( k \in \mathbb{Z} \);
2. \( \psi_{rs} - \psi_{rt} = \psi_{ts} + (2k + 1)\pi \) if \( r < s < t \) or \( s < t < r \) or \( t < r < s \) for all \( r, s, t \in \{n\} \), where \( k \in \mathbb{Z} \);
3. \( \psi_{rs} - \psi_{rt} = \psi_{ts} - \psi + (2k + 1)\pi \) if \( r < t < s \) or \( t < s < r \) or \( s < r < t \) for all \( r, s, t \in \{n\} \), where \( k \in \mathbb{Z} \);
4. \( \psi_{rr} = 0 \) for all \( r \in \{n\} \).

**Definition 3.3.** Let \( E^\phi = (e^{i\phi_{rs}}) \in \mathbb{C}^{n \times n} \), where \( e^{i\phi_{rs}} = \cos \phi_{rs} + i \sin \phi_{rs} \), \( i = \sqrt{-1} \) and \( \phi_{rs} \in \mathbb{R} \) for all \( r, s \in \{n\} \). The matrix \( E^\phi = (e^{i\phi_{rs}}) \in \mathbb{C}^{n \times n} \) is called \( \phi \)-ray pattern matrix if

1. \( \phi_{rs} + \phi_{sr} = 2k\pi + \phi \) holds for all \( r, s \in \{n\} \), \( r \neq s \), where \( k \in \mathbb{Z} \);
2. \( \phi_{rs} - \phi_{rt} = \phi_{ts} - \phi + (2k + 1)\pi \) if \( r < s < t \) or \( s < t < r \) or \( t < r < s \) for all \( r, s, t \in \{n\} \), where \( k \in \mathbb{Z} \);
3. \( \phi_{rs} - \phi_{rt} = \phi_{ts} + (2k + 1)\pi \) if \( r < t < s \) or \( t < s < r \) or \( s < r < t \) for all \( r, s, t \in \{n\} \), where \( k \in \mathbb{Z} \).
r, s, t ∈ ⟨n⟩, where k ∈ Z.

4. \(\phi_{rr} = 0\) for all \(r ∈ ⟨n⟩\).

**Definition 3.4.** Any matrix \(A = (a_{rs}) ∈ \mathbb{C}^{n×n}\) has the following form:

\[
A = e^{i\eta} · |A| \otimes E^{i\chi} = (e^{i\eta} · |a_{rs}|e^{i\chi_{rs}}) ∈ \mathbb{C}^{n×n},
\]

where \(\eta ∈ \mathbb{R}\), \(|A| = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})\) and \(E^{i\chi} = (e^{i\chi_{rs}}) ∈ \mathbb{C}^{n×n}, \chi_{rs} ∈ \mathbb{R}\) for \(r, s ∈ ⟨n⟩\). The matrix \(E^{i\chi}\) is called ray pattern matrix of the matrix \(A\). If the ray pattern matrix \(E^{i\chi}\) of the matrix \(A\) is a \(θ\)-ray pattern matrix, then \(A\) is called a \(θ\)-ray matrix; if the ray pattern matrix \(E^{i\chi}\) of the matrix \(A\) is a \(ψ\)-ray pattern matrix, then \(A\) is called a \(ψ\)-ray matrix; and if the ray pattern matrix \(E^{i\chi}\) of the matrix \(A\) is a \(φ\)-ray pattern matrix, then \(A\) is called a \(φ\)-ray matrix.

\(\mathcal{R}_n^θ, \mathcal{U}_n^ψ\) and \(\mathcal{L}_n^φ\) denote the set of all \(n×n \θ\)-ray matrices, the set of all \(n×n \ψ\)-ray matrices and the set of all \(n×n \phi\)-ray matrices, respectively. Obviously, if a matrix \(A ∈ \mathcal{R}_n^θ\), then \(ξ · A ∈ \mathcal{R}_n^θ\) for all \(ξ ∈ \mathbb{C}\), the same is the matrices in \(\mathcal{U}_n^ψ\) and \(\mathcal{L}_n^φ\), respectively.

**Theorem 3.5.** Let a matrix \(A = D_A - L_A - U_A = (a_{rs}) ∈ \mathbb{C}^{n×n}\) with \(D_A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})\). Then \(A ∈ \mathcal{R}_n^θ\) if and only if there exists an \(n×n\) unitary diagonal matrix \(D\) such that \(D^{-1}AD = e^{i\eta} · [|D_A|e^{iθ} - (|L_A| + |U_A|)]\) for \(η ∈ \mathbb{R}\).

**Proof.** According to Definition 3.4, \(A = e^{i\eta} · |A| \otimes E^{iθ} = (e^{i\eta} · |a_{rs}|e^{iθ_{rs}})\). Define a diagonal matrix \(D_θ = \text{diag}(e^{iθ_1}, e^{iθ_2}, \ldots, e^{iθ_n})\) with \(θ_r = θ_{1r} + φ_1 + (2k + 1)π\) for \(φ_1 ∈ \mathbb{R}, r = 2, 3, \ldots, n, k ∈ \mathbb{Z}\). By Definition 3.1, \(D^{-1}AD = e^{i\eta} · [|D_A|e^{iθ} - (|L_A| + |U_A|)]\), which shows that the necessity is true.

The following will prove the sufficiency. Assume that there exists an \(n×n\) unitary diagonal matrix \(D_θ = \text{diag}(e^{iθ_1}, \ldots, e^{iθ_n})\) such that \(D_θ^{-1}AD_θ = e^{i\eta} · [|D_A|e^{iθ} - (|L_A| + |U_A|)]\). Then the following equalities hold:

\[
\begin{align*}
θ_{rs} &= φ_s - φ_r + (2k_1 + 1)π, \\
θ_{sr} &= φ_r - φ_s + (2k_2 + 1)π, \\
θ_{rt} &= θ_t - φ_r + (2k_3 + 1)π, \\
θ_{tr} &= φ_r - φ_t + (2k_4 + 1)π,
\end{align*}
\]

where \(k_1, k_2, k_3, k_4 ∈ \mathbb{Z}\). In (3.2), \(θ_{rs} + θ_{sr} = 2(k_1 + k_2 + 1)π = 2kπ\) with \(k = k_1 + k_2 + 1 ∈ \mathbb{Z}\) and for all \(r, s ∈ ⟨n⟩, r ≠ s\). Following (3.2), \(θ_{ts} = φ_s - φ_t + (2k_5 + 1)π\). Hence, \(φ_s - φ_t = θ_{ts} - (2k_5 + 1)π\). Consequently, \(θ_{sr} - θ_{rt} = φ_s - φ_t + 2(k_1 - k_3)π = θ_{ts} + (2(k_1 - k_3 - k_5 - 1) + 1)θ_{ts} + (2k_4 + 1)π\) for all \(r, s, t ∈ ⟨n⟩\) and \(r ≠ s, r ≠ t, t ≠ s\). Where \(k = k_1 - k_3 - k_5 - 1 ∈ \mathbb{Z}\). In the same method, we can prove that \(θ_{sr} - θ_{tr} = θ_{ts} + (2k_4 + 1)π\) hold for all \(r, s, t ∈ ⟨n⟩\) and \(r ≠ s, r ≠ t, t ≠ s\), where \(k ∈ \mathbb{Z}\). Furthermore, it is obvious that \(θ_{rr} = θ\) for all \(r ∈ ⟨n⟩\). This completes the sufficiency.
In the same method of proof as Theorem 3.5 the following conclusions will be established.

**Theorem 3.6.** Let a matrix \( A = D_A - L_A - U_A = (a_{rs}) \in \mathbb{C}^{n \times n} \) with \( D_A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \). Then \( \mathcal{H}_n^\psi \) if and only if there exists an \( n \times n \) matrix \( D \) such that \( D^{-1}AD = e^{\eta} \cdot [(|D_A| - |L_A|) - e^{\psi}|U_A|] \) for \( \eta \in \mathbb{R} \).

**Theorem 3.7.** Let a matrix \( A = D_A - L_A - U_A = (a_{rs}) \in \mathbb{C}^{n \times n} \) with \( D_A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \). Then \( \mathcal{L}_n^\psi \) if and only if there exists an \( n \times n \) unitary matrix \( D \) such that \( D^{-1}AD = e^{\eta} \cdot [(|D_A| - |U_A|) - e^{\phi}|L_A|] \) for \( \eta \in \mathbb{R} \).

**Corollary 3.8.** \( \mathcal{R}_n^0 = \mathcal{H}_n^0 = \mathcal{L}_n^0 = \mathcal{H}_n^\psi \cap \mathcal{L}_n^\phi \).

**Proof.** By Theorem 3.5, Theorem 3.6 and Theorem 3.7, the proof is obtained immediately.

### 4. Convergence on Gauss-Seidel iterative methods.

In numerical linear algebra, the Gauss-Seidel iterative method, also known as the Liebmann method or the method of successive displacement, is an iterative method used to solve a linear system of equations. It is named after the German mathematician Carl Friedrich Gauss (1777-1855) and Philipp Ludwig von Seidel (1821-1896), and is similar to the Jacobi method. Later, this iterative method was developed as three iterative methods, i.e., the forward, backward and symmetric Gauss-Seidel (FGS-, BGS- and SGS-) iterative methods. Though these iterative methods can be applied to any matrix with non-zero elements on the diagonals, convergence is only guaranteed if the matrix is strictly or irreducibly diagonally dominant matrix, Hermitian positive definite matrix and invertible \( H \)-matrix. Some classic results on convergence on Gauss-Seidel iterative methods as follows:

**Theorem 4.1.** (see [17]) Let \( A \in S_D \cup I_D \). Then \( \rho(H_{FGS}) < 1 \), \( \rho(H_{BGS}) < 1 \) and \( \rho(H_{SGS}) < 1 \), where \( H_{FGS} \), \( H_{BGS} \) and \( H_{SGS} \) are defined in (1.4), (1.5) and (1.6), respectively, and therefore the sequence \( \{x^{(i)}\} \) generated by FGS-, BGS- and SGS-scheme (1.7), respectively, converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \).

**Theorem 4.2.** (see [30, 31]) Let \( A \in H_1^I \). Then \( \rho(H_{FGS}) < 1 \), \( \rho(H_{BGS}) < 1 \) and \( \rho(H_{SGS}) < 1 \), where \( H_{FGS} \), \( H_{BGS} \) and \( H_{SGS} \) are defined in (1.4), (1.5) and (1.6), respectively, and therefore the sequence \( \{x^{(i)}\} \) generated by FGS-, BGS- and SGS-scheme (1.7), respectively, converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \).

**Theorem 4.3.** (see [31, 32]) Let \( A \in \mathbb{C}^{n \times n} \) be a Hermitian positive definite matrix. Then \( \rho(H_{FGS}) < 1 \), \( \rho(H_{BGS}) < 1 \) and \( \rho(H_{SGS}) < 1 \), where \( H_{FGS} \), \( H_{BGS} \) and \( H_{SGS} \) are defined in (1.4), (1.5) and (1.6), respectively, and therefore the sequence
\{x^{(i)}\} generated by FGS-, BGS- and SGS-scheme (1.7), respectively, converges to the unique solution of (1.1) for any choice of the initial guess \(x^{(0)}\).

Following, we consider convergence on Gauss-Seidel iterative methods for general \(H\)-matrices. Let us investigate the case of nonstrictly diagonally dominant matrices. By Lemma 2.3 and Theorem 1.2, the following conclusion is obtained.

**Theorem 4.4.** Let \(A \in D_n(GD_n)\). Then \(\rho(H_{FGS}) < 1\), \(\rho(H_{BGS}) < 1\) and \(\rho(H_{SGS}) < 1\), where \(H_{FGS}\), \(H_{BGS}\) and \(H_{SGS}\) are defined in (1.4), (1.5) and (1.6), respectively, i.e., the sequence \(\{x^{(i)}\}\) generated by FGS-, BGS- and SGS-scheme (1.7), respectively, converges to the unique solution of (1.1) for any choice of the initial guess \(x^{(0)}\) if and only if \(A\) has no (generalized) diagonally equipotent principal submatrices.

Theorem 4.3 indicates that studying convergence on Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices only investigates the case of (generalized) diagonally equipotent matrices. Continuing in this direction, a lemma will be introduced firstly to be used in this section.

**Lemma 4.5.** (see [17, 30]) Let an irreducible matrix \(A \in D_n(GD_n)\). Then \(A\) is singular if and only if \(D_A^{-1}A \in D_n(GD_n) \cap \mathbb{R}_n^0\), where \(D_A = \text{diag}(a_{11}, \ldots, a_{nn})\).

**Theorem 4.6.** Let an irreducible matrix \(A = (a_{ij}) \in GDE_2\). Then \(\rho(H_{FGS}) = \rho(H_{BGS}) = \rho(H_{SGS}) = 1\), where \(H_{FGS}\), \(H_{BGS}\) and \(H_{SGS}\) are defined in (1.4), (1.5) and (1.6), respectively, and therefore the sequence \(\{x^{(i)}\}\) generated by FGS-, BGS- and SGS-scheme (1.7), respectively, doesn’t converge to the unique solution of (1.1) for any choice of the initial guess \(x^{(0)}\).

**Proof.** Assume \(A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GDE_2\). By Definition 2.2, \(a_{11}|a_{11}| = a_{22}|a_{22}| \) and \(a_{21}|a_{21}| = a_{12}|a_{12}|\) with \(a_{ij} \neq 0\) and \(\alpha_i > 0\) for all \(i, j = 1, 2\). Consequently, \(A \in GDE_2\) if and only if \(a_{12}/|a_{11}| = 1\). Direct computations give that \(\rho(H_{FGS}) = \rho(H_{BGS}) = \rho(H_{SGS}) = 1\) and consequently, the sequence \(\{x^{(i)}\}\) generated by FGS-, BGS- and SGS-scheme (1.7), respectively, doesn’t converge to the unique solution of (1.1) for any choice of the initial guess \(x^{(0)}\).

**Lemma 4.7.** Let \(A = (a_{ij}) \in D_n \ (n \geq 3)\) be irreducible. Then \(e^{i\psi}\) is an eigenvalue of \(H_{FGS}\) if and only if \(D_A^{-1}A \in \mathbb{H}_n^\psi\), where \(D_A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})\) and \(\psi \in \mathbb{R}\).

**Proof.** We prove the sufficiency firstly. Since \(A = (a_{ij}) \in D_n\) is irreducible, \(a_{ii} \neq 0\) for all \(i \in \langle n \rangle\). Thus, \((D_A + L_A)^{-1}\) exists, and consequently, \(H_{FGS}\) also exists, where \(D_A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})\). Assume \(D_A^{-1}A \in \mathbb{H}_n^\psi\). Theorem 3.6 shows that there exists an unitary diagonal matrix \(D\) such that \(D^{-1}(D_A^{-1}A)D = [(I - |D_A^{-1}L_A|)] - e^{i\psi}|D_A^{-1}U_A|\) for \(\psi \in \mathbb{R}\). Hence, \(D_A^{-1}A = D(I - |D_A^{-1}L_A|)D^{-1} - e^{i\psi}D|D_A^{-1}U_A|D^{-1}\)
and

\[ H_{FGS} = (D_A - L_A)^{-1} U_A = (I - D_A^{-1} L_A)^{-1} D_A^{-1} U_A \]
\[ = [D(I - |D_A^{-1} L_A|)D^{-1}]^{-1}(e^{i\psi} D|D_A^{-1} U_A|D^{-1}) \]
\[ = e^{i\psi} D[(I - |D_A^{-1} L_A|)^{-1} D_A^{-1} U_A]|D^{-1}. \]

Using (4.1),

\[ \det(e^{i\psi} I - H_{FGS}) = \det[e^{i\psi} I - e^{i\psi} D((I - |D_A^{-1} L_A|)^{-1}|D_A^{-1} U_A|)D^{-1}] \]
\[ = e^{i\psi} \cdot \det(I - (I - |D_A^{-1} L_A|)^{-1} D_A^{-1} U_A) \]
\[ = \frac{e^{i\psi} \cdot \det(I - |D_A^{-1} L_A|)}{\det(I - |D_A^{-1} L_A|)}. \]

Since \( A \in DE_n \) is irreducible, so is \( \mu(D_A^{-1} A) \in DE_n \). Then it follows from lemma \( 4.3 \) that \( \mu(D_A^{-1} A) \) is singular. As a result, (4.2) gives \( \det(e^{i\psi} I - H_{FGS}) = 0 \) to reveal that \( e^{i\psi} \) is an eigenvalue of \( H_{FGS} \). This completes the sufficiency.

The following prove the necessity. Let \( e^{i\psi} \) is an eigenvalue of \( H_{FGS} \). Then

\[ \det(e^{i\psi} I - H_{FGS}) = \det(e^{i\psi} I - (D_A - L_A)^{-1} U_A) \]
\[ = \frac{\det[e^{i\psi} (D_A - L_A) - U_A]}{\det(D_A - L_A)} \]
\[ = 0. \]

Thus, \( \det(e^{i\psi} (D_A - L_A) - U_A) = 0 \) which shows that \( e^{i\psi} (D_A - L_A) - U_A \) is singular.

Since \( e^{i\psi} (D_A - L_A) - U_A \in DE_n \) is irreducible for \( A = D_A - L_A - U_A \in DE_n \), it follows from lemma \( 4.3 \) shows that \( I - D_A^{-1} L_A - e^{-i\psi} D_A^{-1} U_A \in \mathbb{R}_n^0 \). Theorem 3.3 shows that there exists an unitary diagonal matrix \( D \) such that

\[ D^{-1}(I - D_A^{-1} L_A - e^{-i\psi} D_A^{-1} U_A)D \]
\[ = I - D^{-1}(D_A^{-1} L_A)D - e^{-i\psi} D^{-1}(D_A^{-1} U_A)D \]
\[ = I - |D_A^{-1} L_A| - |D_A^{-1} U_A|. \]

Equality \( 4.4 \) shows \( D^{-1}(D_A^{-1} L_A)D = |D_A^{-1} L_A| \) and \( D^{-1}(D_A^{-1} U_A)D = e^{i\psi}|D_A^{-1} U_A| \). Therefore,

\[ D^{-1}(D_A^{-1} A)D = I - D^{-1}(D_A^{-1} L_A)D - D^{-1}(D_A^{-1} U_A)D \]
\[ = I - |D_A^{-1} L_A| - e^{i\psi}|D_A^{-1} U_A|, \]

that is, there exists an unitary diagonal matrix \( D \) such that

\[ D^{-1}(D_A^{-1} A)D^{-1} = I - |D_A^{-1} L_A| - e^{i\psi}|D_A^{-1} U_A|. \]
Theorem 3.6 shows that $D_A^{-1}A \in \mathcal{U}_n^\psi$. Here, we finish the necessity.

**Lemma 4.8.** Let $A = (a_{ij}) \in DE_n$ ($n \geq 3$) be irreducible. Then $e^{i\phi}$ is an eigenvalue of $H_{BGS}$ if and only if $D_A^{-1}A \in \mathcal{L}_n^\phi$, where $D_A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ and $\phi \in \mathbb{R}$.

**Proof.** According to Theorem 3.7, Theorem 3.5 and Lemma 4.5, the proof is obtained immediately similar to the proof of Lemma 4.7.

**Theorem 4.9.** Let $A \in DE_n$ ($n \geq 3$) be irreducible. Then $\rho(H_{FGS}) < 1$, where $H_{FGS}$ is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{U}_n^\psi$. 

**Proof.** The sufficiency can be proved by contradiction. We assume that there exists an eigenvalue $\lambda$ of $H_{FGS}$ such that $|\lambda| \geq 1$. Then

$$\det(\lambda I - H_{FGS}) = 0.$$ 

If $|\lambda| > 1$, then $\lambda I - H_{FGS} = (D_A - L_A)^{-1}(\lambda D_A - \lambda L_A - U_A)$. Obviously, $\lambda I - \lambda L - U \in ID_n$ and is nonsingular (see Theorem 1.21 in [27]). As a result, $\det(\lambda I - H_{FGS}) \neq 0$, which contradicts (4.5). Thus, $|\lambda| = 1$. Set $\lambda = e^{i\psi}$, where $\psi \in \mathbb{R}$. Then Lemma 4.7 shows that $D_A^{-1}A \in \mathcal{U}_n^\psi$, which contradicts the assumption $A \notin \mathcal{U}_n^\psi$. Therefore, $\rho(H_{FGS}) < 1$. The sufficiency is finished.

Let us prove the necessity by contradiction. Assume that $D_A^{-1}A \in \mathcal{U}_n^\psi$. It then follows from Lemma 4.7 that $\rho(H_{FGS}) = 1$ which contradicts $\rho(H_{FGS}) < 1$. A contradiction arise to demonstrate that the necessity is true. Thus, we complete the proof.

**Theorem 4.10.** Let $A \in DE_n$ ($n \geq 3$) be irreducible. Then $\rho(H_{BGS}) < 1$, where $H_{BGS}$ is defined in (1.5), i.e., the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{L}_n^\phi$.

**Proof.** Similar to the proof of Theorem 4.9, the proof is obtained immediately by Lemma 4.8.

Following, the conclusions of Theorem 4.9 and Theorem 4.10 will be extended to irreducible matrices that belong to the class of generalized diagonally equipotent matrices and the class of irreducible mixed $H-$matrices.

**Theorem 4.11.** Let $A = (a_{ij}) \in GDE_n$ ($n \geq 3$) be irreducible. Then $\rho(H_{FGS}) < 1$, where $H_{FGS}$ is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{U}_n^\psi$. 

Proof. According to Definition 2.2, the exists a diagonal matrix $E = \text{diag}(e_1, e_2, \ldots, e_n)$ with $e_k > 0$ for all $k \in \langle n \rangle$, such that $AE = (a_{ij}e_j) \in DE_n$. Let $AE = F = (f_{ij})$ with $f_{ij} = a_{ij}e_j$ for all $i, j \in \langle n \rangle$. Then $H_{FGS}^F = E^{-1}H_{FGS}E$ and $D_F = D_{AE}$. Theorem 4.11 yields that $\rho(H_{FGS}^F) < 1$ if and only if $D_F^{-1}F \notin \mathcal{V}^\psi_n$.

Theorem 4.12. Let $A = (a_{ij}) \in GDE_n (n \geq 3)$ be irreducible. Then $\rho(H_{BGS}) < 1$, where $H_{BGS}$ is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{L}^\psi_n$.

Proof. Similar to the proof of Theorem 4.11 we can obtain the proof by Definition 2.2 and Theorem 4.10.

According to Lemma 2.10 and Lemma 2.11 if a matrix is an irreducible mixed $H-$matrix, then it is an irreducible generalized diagonally equipotent matrix. As a consequence, we have the following conclusions.

Theorem 4.13. Let $A = (a_{ij}) \in H_n^M (n \geq 3)$ be irreducible. Then $\rho(H_{FGS}) < 1$, where $H_{FGS}$ is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by FGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{V}^\psi_n$.

Theorem 4.14. Let $A = (a_{ij}) \in H_n^M (n \geq 3)$ be irreducible. Then $\rho(H_{BGS}) < 1$, where $H_{BGS}$ is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by BGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{L}^\psi_n$.

Now, we consider convergence of SGS-iterative method. The following lemma will be used in this section.

Lemma 4.15. (see Lemma 3.13 in [32]) Let $A = \begin{bmatrix} E & U \\ L & F \end{bmatrix} \in \mathbb{C}^{2n \times 2n}$, where $E, F, L, U \in \mathbb{C}^{n \times n}$ and $E$ is nonsingular. Then the Schur complement of $A$ with respect to $E$, i.e., $A/E = F - LE^{-1}U$ is nonsingular if and only if $A$ is nonsingular.

Theorem 4.16. Let $A \in DE_n (n \geq 3)$ be irreducible. Then $\rho(H_{SGS}) < 1$, where $H_{SGS}$ is defined in (1.7), i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{V}^\psi_n$.
Proof. The sufficiency can be proved by contradiction. We assume that there exists an eigenvalue \( \lambda \) of \( H_{SGS} \) such that \( |\lambda| \geq 1 \). According to equality (1.6),

\[
\det(\lambda I - H_{SGS}) = \det(\lambda I - (D_A - U_A)^{-1}L_A(D_A - L_A)^{-1}U_A)
\]

\[
= \det((D_A - U_A)^{-1}) \det(\lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A)
\]

\[
= \frac{\det(\lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A)}{\det(D_A - U_A)}
\]

(4.6)

Equality (4.6) gives

\[
\det B = \det(\lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A) = 0,
\]

i.e., \( B := \lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A \) is singular. Let \( E = D_A - L_A \), \( F = \lambda(D_A - U_A) \) and

\[
C = \begin{bmatrix} E & -U_A \\ -L_A & F \end{bmatrix} = \begin{bmatrix} D_A - L_A & -U_A \\ -L_A & \lambda(D_A - U_A) \end{bmatrix}
\]

(4.8)

Then \( B = F - L_A E^{-1} U_A \) is the Schur complement of \( C \) with respect to the principal submatrix \( E \). Now, we investigate the matrix \( C \). Since \( A \) is irreducible, both \( L_A \neq 0 \) and \( U_A \neq 0 \). As a result, \( C \) is also irreducible. If \( |\lambda| > 1 \), then \( (4.8) \) indicates \( C \in ID_{2n} \). Consequently, \( C \) is nonsingular, so is \( B = \lambda(D_A - U_A) - L_A(D_A - L_A)^{-1}U_A \) coming from Lemma 4.15, i.e., \( \det B \neq 0 \), which contradicts (4.7). Therefore, \( |\lambda| = 1 \). Let \( \lambda = e^{i\theta} \) with \( \theta \in \mathbb{R} \). (4.7) and Lemma 4.15 yield that

\[
C = \begin{bmatrix} D_A - L_A & -U_A \\ -L_A & \lambda(D_A - U_A) \end{bmatrix}
\]

and hence \( C_1 = \begin{bmatrix} D_A - L_A & -U_A \\ -e^{-i\theta}L_A & \lambda(D_A - U_A) \end{bmatrix} \) are singular. Since \( A = I - L - U \in DE_n \) and is irreducible, both \( C \) and \( C_1 \) are irreducible diagonal equicontemporaneous. The singularity of \( C_1 \) and Lemma 4.16 yield that \( D_{C_1}^{-1}C_1 \in R_{2n} \), where \( D_{C_1} = \text{diag}(D_A, D_A) \), i.e., there exists an \( n \times n \) unitary diagonal matrix \( D \) such that \( \tilde{D} = \text{diag}(D, D) \) and

\[
\tilde{D}^{-1}(D_{C_1}^{-1}C_1){\tilde{D}} = \begin{bmatrix} I - D^{-1}(D_A^{-1}L_A)D & -D^{-1}(D_A^{-1}U_A)D \\ -e^{-i\theta}D^{-1}(D_A^{-1}L_A)D & I - D^{-1}(D_A^{-1}U_A)D \end{bmatrix}
\]

(4.9)

\[
= \begin{bmatrix} I - |D_A^{-1}L_A| & -|D_A^{-1}U_A| \\ -|D_A^{-1}L_A| & I - |D_A^{-1}U_A| \end{bmatrix}.
\]

(1.9) indicates that \( \theta = 2k\pi \), where \( k \) is an integer and thus \( \lambda = e^{i2k\pi} = 1 \), and there exists an \( n \times n \) unitary diagonal matrix \( D \) such that \( D^{-1}(D_A^{-1}A)D = I - |D_A^{-1}L_A|\) \( - |D_A^{-1}U_A| \rangle \), i.e., \( D_A^{-1}A \in R_{n}^{0} \). However, this contradicts \( D_A^{-1}A \notin \mathbb{R}_{n}^{0} \). Thus, \( |\lambda| \neq 1 \). According to the proof above, we have that \( |\lambda| \geq 1 \) is not true. Therefore, \( \rho(H_{SGS}) < 1 \), i.e., the sequence \( \{x^{(i)}\} \) generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \).
Convergence on Gauss-Seidel iterative methods for linear systems with general $H$–matrices

The following will prove the necessity by contradiction. Assume that $D_A^{-1}A \in \mathcal{S}_n^0$. Then there exists an $n \times n$ unitary diagonal matrix $D$ such that $D_A^{-1}A = I - D_A^{-1}L_A - D_A^{-1}U_A = I - D[D_A^{-1}L_A]D^{-1} = D[D_A^{-1}U_A]D^{-1}$ and

$$H_{SGS} = (D_A - U_A)^{-1}L_A(D_A - L_A)^{-1}U_A$$

(4.10)

$$= [I - (D_A^{-1}U_A)^{-1}(D_A^{-1}L_A)[I - (D_A^{-1}L_A)]^{-1}(D_A^{-1}U_A)]$$

$$= D[(I - |D_A^{-1}U|)^{-1}|D_A^{-1}L|(|I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|]D^{-1}.$$ 

Hence,

$$\det(I - H_{SGS})$$

$$= \det(I - D[(I - |D_A^{-1}U|)^{-1}|D_A^{-1}L|(|I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|]D^{-1})$$

$$= \det(I - [(I - |D_A^{-1}U|)^{-1}|D_A^{-1}L|(|I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|]D^{-1}[(I - |D_A^{-1}U|)^{-1}|D_A^{-1}L|(|I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|])$$

$$= \det(I - |D_A^{-1}U_A|)$$

(4.11)

Let $V = \begin{bmatrix}
I - |D_A^{-1}L_A| & -|D_A^{-1}U_A| \\
-|D_A^{-1}L_A| & I - |D_A^{-1}U_A|
\end{bmatrix}$ and $W = (I - |D_A^{-1}U_A|) - |D_A^{-1}L|(|I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|].$ Then $W$ is the Schur complement of $V$ with respect to $I - |D_A^{-1}L|$. Since $A = I - L - U \in DE_n$ is irreducible, $D_A^{-1}A = I - D_A^{-1}L_A - D_A^{-1}U_A \in DE_n$ is irreducible. Therefore, $V \in DE_{2n} \cap \mathcal{S}_{2n}^0$ and is irreducible. Lemma 4.13 shows that $V$ is singular and hence

$$\det V = \det((I - |D_A^{-1}U_A|) - |D_A^{-1}L|(|I - |D_A^{-1}L|)^{-1}|D_A^{-1}U|] = 0.$$ 

Therefore, (4.11) yields $\det(I - H_{SGS}) = 0$, which shows that $1$ is an eigenvalue of $H_{SGS}$. Thus, $\rho(H_{SGS}) \geq 1$, i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.7) does not converge to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. This is a contradiction which shows that the assumption is incorrect. Therefore, $A \notin \mathcal{S}_n^0$. This completes the proof. \]

**Corollary 4.17.** Let $A \in DE_n (n \geq 3)$ be irreducible and nonsingular. Then $\rho(H_{SGS}) < 1$, where $H_{SGS}$ is defined in (1.7), i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.

**Theorem 4.18.** Let $A \in H_n^M$ ($GDE_n$) be irreducible for $n \geq 3$. Then $\rho(H_{SGS}) < 1$, where $H_{SGS}$ is defined in (1.7), i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $D_A^{-1}A \notin \mathcal{S}_n^0$.

**Proof.** According to Lemma 2.9 and Lemma 2.11 under the condition of irreducibility, $H_n^M = GDE_n$. Then, similar to the proof of Theorem 1.11, we can obtain
Corollary 4.19. Let \( A \in H_n^M \) (\( n \geq 3 \)) be irreducible and nonsingular. Then \( \rho(H_{SGS}) < 1 \), where \( H_{SGS} \) is defined in (1.4), i.e., the sequence \( \{ x^{(i)} \} \) generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \).

Proof. It follows from Lemma 4.8 that the proof of this corollary is obtained immediately.

In what follows we establish some convergence results on Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices.

Theorem 4.20. Let \( A = (a_{ij}) \in D_n(GD_n) \) with \( a_{ii} \neq 0 \) for all \( i \in \langle n \rangle \). Then \( \rho(H_{FGS}) < 1 \), where \( H_{FGS} \) is defined in (1.4), i.e., the sequence \( \{ x^{(i)} \} \) generated by FGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \) if and only if \( A \) has neither \( 2 \times 2 \) irreducibly (generalized) diagonally equipotent principal submatrix nor irreducibly principal submatrix \( A_k = A(i_1, i_2, \cdots, i_k), 3 \leq k \leq n \), such that \( D_{A_k}^{-1} A_k \notin \mathcal{P}_k \cap DE_k(\mathcal{P}_k \cap GD_k) \), where \( D_{A_k} = \text{diag}(a_{i_1i_1}, a_{i_2i_2}, \cdots, a_{i_ki_k}) \).

Proof. The proof is obtained immediately by Theorem 4.4, Theorem 4.6, Theorem 4.9 and Theorem 4.11.

Theorem 4.21. Let \( A = (a_{ij}) \in D_n(GD_n) \) with \( a_{ii} \neq 0 \) for all \( i \in \langle n \rangle \). Then \( \rho(H_{BGS}) < 1 \), where \( H_{BGS} \) is defined in (1.4), i.e., the sequence \( \{ x^{(i)} \} \) generated by BGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \) if and only if \( A \) has neither \( 2 \times 2 \) irreducibly (generalized) diagonally equipotent principal submatrix nor irreducibly principal submatrix \( A_k = A(i_1, i_2, \cdots, i_k), 3 \leq k \leq n \), such that \( D_{A_k}^{-1} A_k \notin \mathcal{P}_k \cap DE_k(\mathcal{P}_k \cap GD_k) \), where \( D_{A_k} = \text{diag}(a_{i_1i_1}, a_{i_2i_2}, \cdots, a_{i_ki_k}) \).

Proof. According to Theorem 4.4, Theorem 4.6, Theorem 4.10 and Theorem 4.12 we can obtain the proof of this theorem immediately.

Theorem 4.22. Let \( A = (a_{ij}) \in D_n(GD_n) \) with \( a_{ii} \neq 0 \) for all \( i \in \langle n \rangle \). Then \( \rho(H_{SGS}) < 1 \), where \( H_{SGS} \) is defined in (1.4), i.e., the sequence \( \{ x^{(i)} \} \) generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \) if and only if \( A \) has neither \( 2 \times 2 \) irreducibly (generalized) diagonally equipotent principal submatrix nor irreducibly principal submatrix \( A_k = A(i_1, i_2, \cdots, i_k), 3 \leq k \leq n \), such that \( D_{A_k}^{-1} A_k \notin \mathcal{P}_k \cap DE_k(\mathcal{P}_k \cap GD_k) \).

Proof. It follows from Theorem 4.4, Theorem 4.6, Theorem 4.10 and Theorem 4.12.
that the proof of this theorem is obtained immediately. □

**Theorem 4.23.** Let \( A \in GD_n \) be nonsingular. Then \( \rho(H_{SGS}) < 1 \), where \( H_{SGS} \) is defined in (1.4), i.e., the sequence \( \{x^{(i)}\} \) generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \) if and only if \( A \) has no \( 2 \times 2 \) irreducibly generalized diagonally equipotent principal submatrices.

**Proof.** Since \( A \in GD_n \) is nonsingular, it follows from Theorem 3.11 in [34] that \( A \) hasn’t any irreducibly principal submatrix \( A_k = A(i_1, i_2, \ldots, i_k), 3 \leq k \leq n \), such that \( D_k^{-1}A_k \in \mathcal{R}_k^0 \), and hence \( D_k^{-1}A_k \notin \mathcal{R}_k^0 \cap GDE_k \). Then the conclusion of this theorem follows Theorem 4.22. □

In the rest of this section, the convergence results on Gauss-Seidel iterative method for nonstrictly diagonally dominant matrices will be extended to general \( H \)-matrices.

**Theorem 4.24.** Let \( A = (a_{ij}) \in H_n \) with \( a_{ii} \neq 0 \) for all \( i \in \langle n \rangle \). Then \( \rho(H_{FGS}) < 1 \), where \( H_{FGS} \) is defined in (1.4), i.e., the sequence \( \{x^{(i)}\} \) generated by FGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \) if and only if \( A \) has neither \( 2 \times 2 \) irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix \( A_k = A(i_1, i_2, \ldots, i_k), 3 \leq k \leq n \), such that \( D_k^{-1}A_k \notin \mathcal{R}_k^0 \cap GDE_k \).

**Proof.** If \( A \in H_n \) is irreducible, it follows from Theorem 4.6 and Theorem 4.13 that the conclusion of this theorem is true. If \( A \in H_n \) is reducible, since \( A \in H_n \) with \( a_{ii} \neq 0 \) for all \( i \in \langle n \rangle \), Theorem 2.10 shows that each diagonal square block \( R_{ii} \) in the Frobenius normal form of (2.7) of \( A \) is irreducible and generalized diagonally dominant for \( i = 1, 2, \ldots, s \). Let \( H_{FGS} \) denote the Gauss-Seidel iteration matrix associated with diagonal square block \( R_{ii} \). Direct computations give

\[
\rho(H_{FGS}) = \max_{1 \leq i \leq s} \rho(H_{FGS}^{R_{ii}}).
\]

Since \( R_{ii} \) is irreducible and generalized diagonally dominant, Theorem 4.4 Theorem 4.9 Theorem 4.11 and Theorem 4.29 show that \( \rho(H_{FGS}) = \max_{1 \leq i \leq s} \rho(H_{FGS}^{R_{ii}}) < 1 \), i.e., the sequence \( \{x^{(i)}\} \) generated by FGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \) if and only if \( A \) has neither \( 2 \times 2 \) irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix \( A_k = A(i_1, i_2, \ldots, i_k), 3 \leq k \leq n \), such that \( D_k^{-1}A_k \notin \mathcal{R}_k^0 \cap GDE_k \). □

**Theorem 4.25.** Let \( A = (a_{ij}) \in H_n \) with \( a_{ii} \neq 0 \) for all \( i \in \langle n \rangle \). Then \( \rho(H_{BGS}) < 1 \), where \( H_{BGS} \) is defined in (1.4), i.e., the sequence \( \{x^{(i)}\} \) generated by BGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \) if and only if \( A \) has neither \( 2 \times 2 \) irreducibly generalized diagonally equipotent...
principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \cdots, i_k), 3 \leq k \leq n$, such that $D_{A_k}^{-1}A_k \notin \mathcal{L}_k^0 \cap GDE_k$.

**Proof.** Similar to the proof of Theorem 4.24, we can obtain the proof immediately by Theorem 2.10 and Theorem 4.21.

**Theorem 4.26.** Let $A = (a_{ij}) \in H_n$ with $a_{ii} \neq 0$ for all $i \in \langle n \rangle$. Then $\rho(H_{SGS}) < 1$, where $H_{SGS}$ is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $A$ has neither $2 \times 2$ irreducibly generalized diagonally equipotent principal submatrix nor irreducibly principal submatrix $A_k = A(i_1, i_2, \cdots, i_k), 3 \leq k \leq n$, such that $D_{A_k}^{-1}A_k \notin \mathcal{L}_k^0 \cap GDE_k$.

**Proof.** Similar to the proof of Theorem 4.24, we can obtain the proof immediately by Theorem 2.10 and Theorem 4.22.

**Theorem 4.27.** Let $A \in H_n$ be nonsingular. Then $\rho(H_{SGS}) < 1$, where $H_{SGS}$ is defined in (1.4), i.e., the sequence $\{x^{(i)}\}$ generated by SGS-scheme (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$ if and only if $A$ has no $2 \times 2$ irreducibly generalized diagonally equipotent principal submatrices.

**Proof.** The proof is similar to the proof of Theorem 4.23.

The research in this section shows that the FGS iterative method associated with the irreducible matrix $A \in H_n^M \cap B_n^0$ fails to converge, the same does for the BGS iterative method associated with the irreducible matrix $A \in H_n^M \cap \mathcal{L}_n^0$ and the SGS iterative method associated with the irreducible matrix $A \in H_n^M \cap \mathcal{R}_n^0$. It is natural to consider convergence on preconditioned Gauss-Seidel iterative methods for nonsingular general $H$–matrices.

**5. Convergence on preconditioned Gauss-Seidel iterative methods.** In this section, Gauss-type preconditioning techniques for linear systems with nonsingular general $H$–matrices are chosen such that the coefficient matrices are invertible $H$–matrices. Then based on structure heredity of the Schur complements for general $H$–matrices in [34], convergence on preconditioned Gauss-Seidel iterative methods will be studied and some results will be established.

Many researchers have considered the left Gauss-type preconditioner applied to linear system (1.1) such that the associated Jacobi and Gauss-Seidel methods converge
faster than the original ones. Milaszewicz [24] considered the preconditioner

\[ P_1 = \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  -a_{21} & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  -a_{n1} & 0 & \cdots & 1
\end{bmatrix}. \]

Later, Hadjidimos et al [13] generalized Milaszewicz’s preconditioning technique and presented the preconditioner

\[ P_1(\alpha) = \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  -\alpha a_{21} & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  -\alpha a_{n1} & 0 & \cdots & 1
\end{bmatrix}. \]

Recently, Zhang et al. [36] proposed the left Gauss type preconditioning techniques which utilizes the Gauss transformation [11] matrices as the base of the Gauss type preconditioner based on Hadjidimos et al. [13], Milaszewicz [24] and LU factorization method [11]. The construction of Gauss transformation matrices is as follows:

\[ M_k = \begin{bmatrix}
  1 & \cdots & 0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & \cdots & 1 & 0 & \cdots & 0 \\
  0 & \cdots & -\tau_{k+1} & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & \cdots & -\tau_n & 0 & \cdots & 1
\end{bmatrix}, \]

where \( \tau_i = a_{ik}/a_{kk}, \ i = k+1, \ldots, n \) and \( k = 1, 2, \ldots, n-1 \). Zhang et al. [36] consider the following left preconditioners:

\[ \mathcal{P}_1 = M_1, \ \mathcal{P}_2 = M_2 M_1, \ \cdots, \ \mathcal{P}_{n-1} = M_{n-1} M_{n-2} \cdots M_2 M_1. \]

Let \( \mathcal{H}_n = \{ A \in H_n : A \text{ is nonsingular} \} \). Then \( H^1_n \subset \mathcal{H}_n \) while \( H^1_n \neq \mathcal{H}_n \). Again, let \( \check{H}^M_n = \{ A \in H^M_n : A \text{ is nonsingular} \} \). In fact, \( \mathcal{H}_n = H^1_n \cup \check{H}^M_n \). Thus, nonsingular general \( H \)-matrices that the matrices in \( \mathcal{H}_n \) differ from invertible \( H \)-matrices. In this section we will propose some Gauss-type preconditioning techniques for linear systems with the coefficient matrices belong to \( \mathcal{H}_n \) and establish some convergence results on preconditioned Gauss-Seidel iterative methods.

Firstly, we consider the case that the coefficient matrix \( A \in \mathcal{H}_n \) is irreducible.
Then let us generalize the preconditioner of (5.1), (5.2) and (5.3) as follows:

\[
\mathcal{P}_k = \begin{bmatrix}
1 & \cdots & 0 & -\tau_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & -\tau_{k-1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -\tau_{k+1} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -\tau_n & 0 & \cdots & 1 
\end{bmatrix},
\]

(5.5)

where \(\tau_i = a_{ik}/a_{kk}, \ i = 1, \ldots, n; i \neq k\) and \(k \in \langle n \rangle\). Assume that \(\hat{A}_k = \mathcal{P}_k A\) for \(k \in \langle n \rangle\), \(H^A_1, H^A_{FGS}, H^A_{BGS}\) and \(H^A_{SGS}\) denote the Jacobi and the forward, backward and symmetric Gauss-Seidel (FGS-, BGS- and SGS-) iteration matrices associated with the coefficient matrix \(A\), respectively.

**Theorem 5.1.** Let \(A \in \mathcal{H}_n\) be irreducible. Then \(\hat{A}_k = \mathcal{P}_k A \in \mathcal{H}_1^k\) for all \(k \in \langle n \rangle\), where \(\mathcal{P}_k\) is defined in (5.5). Furthermore, the following conclusions hold:

1. \(\rho(H^A_1) \leq \rho(H^A_1^{(A/k)}) < 1\) for all \(k \in \langle n \rangle\), where \(A/k = A/\alpha\) with \(\alpha = \{k\}\);
2. \(\rho(H^A_{FGS}) \leq \rho(H^A_{FGS}^{(A/k)}) < 1\) for all \(k \in \langle n \rangle\);
3. \(\rho(H^A_{BGS}) \leq \rho(H^A_{BGS}^{(A/k)}) < 1\) for all \(k \in \langle n \rangle\);
4. \(\rho(H^A_{SGS}) \leq \rho(H^A_{SGS}^{(A/k)}) < 1\) for all \(k \in \langle n \rangle\),

i.e., the sequence \(\{x^{(i)}\}\) generated by the preconditioned Jacobi, FGS, BGS and SGS iterative schemes (1.7) converge to the unique solution of (1.1) for any choice of the initial guess \(x^{(0)}\).

**Proof.** Since \(A \in \mathcal{H}_n^M\) is irreducible and nonsingular for \(A \in \mathcal{H}_n\) is irreducible, it follows from Theorem 5.9 in [34] that \(A/\alpha\) is an invertible \(H\)-matrix, where \(\alpha = \{k\}\). For the preconditioner \(\mathcal{P}_k\), there exists a permutation matrix \(P_k\) such that \(P_k \mathcal{P}_k P_k^T = \begin{bmatrix} 1 & 0 \\ -\tau & I_{n-1} \end{bmatrix}\), where \(\tau = (\tau_1, \cdots, \tau_{k-1}, \tau_{k+1}, \cdots, \tau_n)^T\). As a consequence,

\[
P_k(\mathcal{P}_k A) P_k^T = P_k \mathcal{P}_k P_k^T P_k A P_k^T = \begin{bmatrix} a_{kk} & \alpha_k \\ 0 & A/\alpha \end{bmatrix}
\]

is an invertible \(H\)-matrix, so is \(\mathcal{P}_k A\). Following, Theorem 4.1 in [34] and Theorem 4.2 show that the four conclusions hold. This completes the proof.

On the other hand, if an irreducible matrix \(A \in \mathcal{H}_n\) has a principal submatrix \(A(\alpha)\) which is easy to get its inverse matrix or is a (block)triangular matrix, there exists a permutation matrix \(P_\alpha\) such that
(5.6) \[ P_\alpha A P_\alpha^T = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha', \alpha') \end{bmatrix}, \]

where \( \alpha' = \langle n \rangle - \alpha \). Let

(5.7) \[ M = \begin{bmatrix} I_{|\alpha|} & 0 \\ -[A(\alpha)]^{-1}A(\alpha', \alpha) & I \end{bmatrix}. \]

Then

(5.8) \[ M P_\alpha A P_\alpha^T = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ 0 & A/\alpha \end{bmatrix}, \]

where \( A(\alpha) \) and \( A/\alpha \) are both invertible \( H \)-matrices, so is \( M P_\alpha A P_\alpha^T \). As a result, \( P_\alpha^T M P_\alpha A = P^T (M P_\alpha A P_\alpha^T) P \) is an invertible \( H \)-matrix. Therefore, we consider the following preconditioner

(5.9) \[ \mathcal{P}_\alpha = P_\alpha^T M P_\alpha, \]

where \( P_\alpha \) and \( M \) are defined by (5.6) and (5.6), respectively.

**Theorem 5.2.** Let \( A \in \mathcal{H}_n \) be irreducible. Then \( \tilde{A}_\alpha = \mathcal{P}_\alpha A \in H^I_n \) for all \( \alpha \subset \langle n \rangle, \alpha \neq \emptyset \), where \( \mathcal{P}_\alpha \) is defined in (5.9). Furthermore, the following conclusions hold:

1. \( \rho(H^A_\alpha) \leq \max\{\rho(H^\mu_{\text{Jacobi}}(\alpha)), \rho(H^\mu_{\text{FGS}}(\alpha))\} < 1 \) for all \( \alpha \subseteq \langle n \rangle \);
2. \( \rho(H^\mu_{\text{FGS}}) \leq \max\{\rho(H^\mu_{\text{FGS}}(\alpha)), \rho(H^\mu_{\text{FGS}}(\alpha'))\} < 1 \) for all \( \alpha \subseteq \langle n \rangle \);
3. \( \rho(H^\mu_{\text{BGS}}) \leq \max\{\rho(H^\mu_{\text{BGS}}(\alpha)), \rho(H^\mu_{\text{BGS}}(\alpha'))\} < 1 \) for all \( \alpha \subseteq \langle n \rangle \);
4. \( \rho(H^\mu_{\text{SGS}}) \leq \max\{\rho(H^\mu_{\text{SGS}}(\alpha)), \rho(H^\mu_{\text{SGS}}(\alpha'))\} < 1 \) for all \( \alpha \subseteq \langle n \rangle \)

i.e., the sequence \( \{x^{(i)}\} \) generated by the preconditioned Jacobi, FGS, BGS and SGS iterative schemes (1.7) converge to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \).

**Proof.** The proof is similar to the proof of Theorem 5.3.

Following, we consider the case that the coefficient matrix \( A \in \mathcal{H}_n \) is reducible. If there exists a proper \( \alpha = \langle n \rangle - \alpha' \subset \langle n \rangle \) such that \( A(\alpha) \) and \( A(\alpha') \) are both invertible \( H \)-matrices, we consider the preconditioner (5.9) and have the following conclusion.

**Theorem 5.3.** Let \( A \in \mathcal{H}_n \) and a proper \( \alpha = \langle n \rangle - \alpha' \subset \langle n \rangle, \alpha \neq \emptyset \), such that \( A(\alpha) \) and \( A(\alpha') \) are both invertible \( H \)-matrices. Then \( \tilde{A}_\alpha = \mathcal{P}_\alpha A \in H^I_n \), where \( \mathcal{P}_\alpha \) is defined in (5.9). Furthermore, the following conclusions hold:

1. \( \rho(H^A_\alpha) \leq \max\{\rho(H^\mu_{\text{Jacobi}}(\alpha)), \rho(H^\mu_{\text{Jacobi}}(\alpha'))\} < 1 \) for all \( \alpha \subseteq \langle n \rangle \);
2. \( \rho(H_{\text{FGS}}^{\alpha}) \leq \max\{\rho(H_{\text{FGS}}^{\mu(A(\alpha))}), \rho(H_{\text{FGS}}^{\mu(A/\alpha)})\} < 1 \) for all \( \alpha \in \langle n \rangle \); 
3. \( \rho(H_{\text{BGS}}^{\alpha}) \leq \max\{\rho(H_{\text{BGS}}^{\mu(A(\alpha))}), \rho(H_{\text{BGS}}^{\mu(A/\alpha)})\} < 1 \) for all \( \alpha \in \langle n \rangle \); 
4. \( \rho(H_{\text{SGS}}^{\alpha}) \leq \max\{\rho(H_{\text{SGS}}^{\mu(A(\alpha))}), \rho(H_{\text{SGS}}^{\mu(A/\alpha)})\} < 1 \) for all \( \alpha \in \langle n \rangle \), 

i.e., the sequence \( \{x^{(i)}\} \) generated by the preconditioned Jacobi, FGS, BGS and SGS iterative schemes (1.7) converge to the unique solution of (1.1) for any choice of the initial guess \( x^{(0)} \).

Proof. It is obvious that there exists a permutation matrix \( P_{\alpha} \) such that (5.6) holds. Further, 

\[
(5.10) \quad P_{\alpha} A = P_{\alpha}^{T} \mu(P_{\alpha}) A = P_{\alpha}^{T} (\mu(P_{\alpha}) A) P = P_{\alpha}^{T} \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ 0 & A/\alpha \end{bmatrix} P.
\]

Since \( A \in \mathcal{H}, A \in H^{I}_{n} \cap H^{M}_{n} \) is nonsingular. Again, \( A(\alpha) \) and \( A(\alpha') \) are both invertible \( H \)-matrices, it follows from Theorem 5.2 and Theorem 5.11 in [34] that \( A/\alpha \) is an invertible \( H \)-matrix. Therefore, \( A_{\alpha} = P_{\alpha} A \in H^{I}_{n} \) coming from (5.10). Following, Theorem 4.1 in [34] and Theorem 4.2 yield that the four conclusions hold, which completes the proof.

It is noted that the preconditioner \( P_{\alpha} \) has at least two shortcomings when the coefficient matrix \( A \in \mathcal{H} \) is reducible. One is choice of \( \alpha \). For a large scale reducible matrix \( A \in \mathcal{H} \cap H^{M}_{n} \), we are not easy to choose \( \alpha \) such that \( A(\alpha) \) and \( A(\alpha') \) are both invertible \( H \)-matrices. The other is the computation of \( [A(\alpha)]^{-1} \). Although \( A(\alpha) \) is an invertible \( H \)-matrices, it is difficult to obtain its inverse matrix for large \( A(\alpha) \). These shortcomings above are our further research topics.

6. Numerical examples. In this section some examples are given to illustrate the results obtained in Section 4 and Section 5.

Example 6.1. Let the coefficient matrix \( A \) of linear system (1.1) be given by the following \( n \times n \) matrix

\[
(6.1) \quad A_{n} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 
\end{bmatrix}.
\]

It is easy to see that \( A_{n} \in DE_{n} \subset H_{n} \) is irreducible and \( A_{n} \notin H^{I}_{n} \), but Lemma
4.3 in [30] shows that $A_n$ is nonsingular. Thus, $A_n \in \mathcal{R}_n$ is irreducible. Since
\[ D_n^{-1} A_n D_n = |D_{A_n}| - |L_{A_n}| - c^T |U_{A_n}|, \]
where
\[ D_n = \text{diag}[1, -1, \ldots, (-1)^{k-1}, \ldots, (-1)^{n-1}], \]
it follows from Theorem 3.6 that $A_n \in \mathcal{R}_n$. In addition, it is obvious that $A_n \in \mathcal{L}_n$. Therefore, Theorem 4.9 and Theorem 4.10 show that
\[ \rho(H_{FGS}(A_{100})) = \rho(H_{BGS}(A_{100})) = 1. \]
Further, Theorem 4.16 shows that $\rho(H_{FGS}(A_{100})) < 1$. In fact, direct computations also get $\rho(H_{FGS}(A_{100})) = \rho(H_{BGS}(A_{100})) = 1$ and $\rho(H_{SGS}(A_{100})) = 0.3497 < 1$ which demonstrates that the conclusions of Theorem 4.9, Theorem 4.10 and Theorem 4.16 in Section 4 are correct and effective.

The discussion above shows that FGS and BGS iterative schemes fail to converge to the unique solution of linear system (1.1) with the coefficient matrix (6.1) of for any choice of the initial guess $x^{(0)}$, but SGS iterative schemes does. Now we consider preconditioned Gauss-Seidel iterative methods for linear system (1.1) with the coefficient matrix (6.1).

Choose two set $\alpha = \{1\} \in \langle n \rangle$ and $\beta = \{1, n\} \in \langle n \rangle$ and partition $A_n$ into
\[ A_n = \begin{bmatrix} 1 & -a^T \\ a & A_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -b^T & 0 \\ b & A_{n-2} & -c^T \\ 0 & c & 1 \end{bmatrix}, \]
where $a = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n-1}$, $b = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n-2}$, $c^T = (0, \ldots, 0, 1)^T \in \mathbb{R}^{n-2}$ and $A_{n-2} = \text{tr}[1, 2, -1] \in \mathbb{R}^{(n-2) \times (n-2)}$, we get two preconditioners
\[ \mathcal{P}_\alpha = \begin{bmatrix} 1 & 0 \\ -a & I_{n-1} \end{bmatrix} \quad \text{and} \quad \mathcal{P}_\beta = \begin{bmatrix} 1 & 0 & 0 \\ -b & I_{n-2} & c^T \\ 0 & c & 1 \end{bmatrix}, \]
where $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix. Then Theorem 5.9 in [34] shows that $\tilde{A}_1 = \mathcal{P}_\alpha A_n = \begin{bmatrix} 1 & -a^T \\ 0 & A_n/\alpha \end{bmatrix}$ and $\tilde{A}_\beta = \mathcal{P}_\beta A_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_n/\beta & 0 \\ 0 & c & 1 \end{bmatrix}$ are both invertible $H-$matrices. According to Theorem 5.1 and Theorem 5.2, for these two preconditioners, the preconditioned FGS, BGS and SGS iterative schemes converge to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.

In fact, by direct computations, Table 6.1 in the following is obtained to show that $\rho(H_{FGS}^{\tilde{A}_1}) = \rho(H_{FGS}^{\tilde{A}_\beta}) = 0.9970 < 1$, $\rho(H_{BGS}^{\tilde{A}_1}) = \rho(H_{BGS}^{\tilde{A}_\beta}) = 0.9970 < 1$ and
\[ \rho(H_{SGS}^{A_1}) = 0.3333 < \rho(H_{SGS}^{A_1}) = 0.9950 < 1, \rho(H_{SGS}^{A_\beta}) = 0.3158 < \rho(H_{SGS}^{A_\beta}) = 0.9979 < 1, \] which illustrate specifically that Theorem 5.1 and Theorem 5.2 are both valid.

Table 6.1 The comparison result of spectral radii of PGS iterative matrices

| X  | \(\rho(H_X^{A_1})\) | \(\rho(H_X^{A_\alpha})\) | \(\rho(H_X^{A_\beta})\) | \(\rho(H_X^{A_\gamma})\) |
|----|---------------------|---------------------|---------------------|---------------------|
| FGS| 0.9970              | 0.9970              | 0.9900              | 0.9900              |
| BFS| 0.9970              | 0.9970              | 0.9900              | 0.9900              |
| SGS| 0.3333              | 0.9950              | 0.3158              | 0.9979              |

Example 6.2. Let the coefficient matrix \(A\) of linear system (1.1) be given by the following 6 \(\times\) 6 matrix

\[
A = \begin{bmatrix}
5 & -1 & 1 & 1 & 1 & -1 \\
1 & 5 & -1 & 1 & 1 & 1 \\
1 & 1 & 5 & -1 & 1 & 1 \\
0 & 0 & 0 & 2 & -1 & 1 \\
0 & 0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 1 & 1 & 2
\end{bmatrix}
\] (6.4)

Although \(A \in DE_6\) are reducible but there is not any principal submatrix \(A_k\) \((k < 6)\) in \(A\) such that \(D_k^{-1}A_k \in \mathbb{R}_k^{\mathbb{0}}\). Theorem 3.16 in [34] shows that \(A\) is nonsingular. Thus, \(A_k \in H_6\) is reducible. Furthermore, there is not any principal submatrix \(A_k\) in \(A\) such that \(D_k^{-1}A_k \in \mathbb{R}_k^{\mathbb{0}}\) and \(D_k^{-1}A \in \mathbb{R}_k^{\mathbb{0}}\). It follows from Theorem 4.20, Theorem 4.21 and Theorem 4.22 that FGS, BGS and SGS iterative schemes converge to the unique solution of (1.1) for any choice of the initial guess \(x^{(0)}\).

From the first column in Table 6.2, one has \(\rho(H_{FGS}) = \rho(H_{BGS}) = 0.3536 < 1\) and \(\rho(H_{SGS}) = 0.2500 < 1\). This naturally verifies the results of Theorem 4.20, Theorem 4.21 and Theorem 4.22.

Table 6.2 The comparison result of spectral radii of GS and PGS iterative matrices

| X  | \(\rho(H_X)\) | \(\rho(H_X^{A_\alpha})\) | \(\rho(H_X^{A_\beta})\) | \(\rho(H_X^{A_\gamma})\) |
|----|----------------|---------------------|---------------------|---------------------|
| FGS| 0.3536         | 0.6000              | 0.6000              |
| BFS| 0.3536         | 0.6000              | 0.6000              |
| SGS| 0.2500         | 0.6000              | 0.6000              |

Now, we consider convergence on preconditioned Gauss-Seidel iterative methods. Set \(\alpha = \{3, 4\} \subset \langle 6 \rangle = \{1, 2, 3, 4, 5, 6\}\), and set \(\beta = \{3, 4\} \subset \langle 6 \rangle\) and \(\gamma = \{3, 4\} \subset \langle 6 \rangle\). Since \(A(\beta \cup \gamma) \in H_4^1\), it follows from Theorem 4.3 in [32] that \(A/\alpha \in H_4^1\). Thus, we
choose a preconditioner

\( \mathcal{P}_\alpha = \begin{bmatrix} I_2 & -A(\beta, \alpha)[A(\alpha)]^{-1} & 0 \\ 0 & I_2 & c^T \\ 0 & -A(\gamma, \alpha)[A(\alpha)]^{-1} & I_2 \end{bmatrix} \) \tag{6.5}

such that \( \tilde{A}_\alpha = \mathcal{P}_\alpha A \in H^n \). From Theorem 5.3, it is obvious to see that the preconditioned FGS, BGS and SGS iterative schemes converge to the unique solution for any choice of the initial guess \( x^{(0)} \).

As is shown in Table 6.2, \( \rho(H_{FGS}^{\tilde{A}_\alpha}) = \rho(H_{BGS}^{\tilde{A}_\alpha}) = 0.0000 < 1 \) and \( \rho(H_{SGS}^{\tilde{A}_\alpha}) = 0.0000 < 1 \), which directly verifies the results of Theorem 5.3.

7. Conclusions. This paper studies convergence on Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices and general \( H \)-matrices. The definitions of some special matrices are firstly proposed to establish some new results on convergence of Gauss-Seidel iterative methods for nonstrictly diagonally dominant matrices and general \( H \)-matrices. Following, convergence of Gauss-Seidel iterative methods for preconditioned linear systems with general \( H \)-matrices is established. Finally, some numerical examples are given to demonstrate the results obtained in this paper.

Acknowledgment. The authors would like to thank the anonymous referees for their valuable comments and suggestions, which actually stimulated this work.

REFERENCES

[1] M. Alanelli and A. Hadjidimos. A New Iterative Criterion for \( H \)-Matrices. SIAM J. Matrix Anal. Appl. 29(1):160-176, 2006.
[2] A. Berman, R.J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. Academic, New York, 1979.
[3] R. Bru, C. Corral, I. Gimenez and J. Mas. Classes of general \( H \)-matrices, Linear Algebra Appl. 429: 2358-2366, 2008.
[4] R. Bru, C. Corral, I. Gimenez and J. Mas. Schur complement of general \( H \)-matrices, Numer. Linear Algebra Appl. 16(11-12): 935-974-2366, 2009.
[5] R. Bru, I. Gimenez and A. Hadjidimos. \( A \in C^{n,n} \) a general \( H \)-matrices, Linear Algebra Appl. 436: 364-380, 2012.
[6] D. Carlson, T. Markham. Schur Complements of Diagonally Dominant Matrices, Czech. Math. J. 29(104): 246-251, 1979.
[7] L. Cvetković, V. Kostić, M. Kovačević and T. Szulc. Further results on \( H \)-matrices and their Schur complements. Linear Algebra Appl. 198: 506-510, 2008.
[8] H. Elman, D. Silvester, and A. Wathen, Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, UK, 2005.
[9] M. Fiedler and V. Ptak. On matrices with nonpositive offdiagonal elements and positive principal minors. *Czechoslovak Mathematical Journal*, 12(87): 382-400, 1962.

[10] M. Fiedler and V. Ptak. Diagonally dominant matrices, *Czechoslovak Mathematical Journal*, 17(92): 420-433, 1967.

[11] G. H. Golub and C.F. Van Loan. *Matrix Computations*, third ed., Johns Hopkins University Press, Baltimore, 1996.

[12] A. D. Gunawardena, S. K. Jain and L. Snyder. Modified iterative methods for consistent linear systems, *Linear Algebra Appl.* 154-156: 123-143, 1991.

[13] A. Hadjidimos, D. Noutsos and M. Tzoumas. More on modifications and improvements of classical iterative schemes for M-matrices, *Linear Algebra Appl.* 364: 253-279, 2003.

[14] C. R. Johnson. Inverse M-matrices, *Linear Algebra Appl.* 47:195-216, 1982.

[15] R. Kress. *Numerical Analysis*. Springer, New York, 1998.

[16] T. Kohn, H. Kotakemori, H. Niki and M. Usui. Improving the Gauss-Seidel method for Z-matrices, *Linear Algebra Appl.* 267: 113-132, 1997.

[17] L. Yu. Kolotilina. Nonsingularity/singularity criteria for nonstrictly block diagonally dominant matrices, *Linear Algebra Appl.* 359:133-159, 2003.

[18] TG Lei, CW Woo, JZ Liu and F Zhang. On the Schur Complements of Diagonally Dominant Matrices. *Proceedings of the SIAM Conference on Applied Linear Algebra*, 2003.

[19] X. Liao. *The Stability Theory and Application of Dynamic System*, National Defence industry Press, Beijing, 2000.

[20] Jianzhou Liu and Yungqing Huang. Some Properties on Schur Complements of H-matrix and Diagonally Dominant Matrices. *Linear Algebra Appl.* 389: 365-380, 2004.

[21] Jianzhou Liu, Yungqing Huang and Fuqin Zhang. The Schur complements of generalized doubly diagonally dominant matrices. *Linear Algebra Appl.* 378:231-244, 2004.

[22] Jianzhou Liu and Fuqin Zhang. Disc Separation of the Schur Complement of Diagonally Dominant Matrices and Determinantal Bounds. *SIAM J. Matrix Anal. Appl.* 27(3): 665-674, 2005.

[23] Jianzhou Liu, Jicheng Li, Zhuohong Huang and Xu Kong. Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices. *Linear Algebra Appl.* 428: 1009-1030, 2008.

[24] J. P. Milaszewicz. Improving Jacobi and Gauss-Seidel iterations, *Linear Algebra Appl.* 93: 161-170, 1987.

[25] B. Poland. Incomplete blockwise factorizations of (block) H-matrices, *Linear Algebra Appl.*, 90:119-132, 1987.

[26] A. M. Ostrowski. Über die determinanten mit überwiegender hauptdiagonale, *Commentari Mathematici Helvetici*, 10:69-96, 1937.

[27] R. S. Varga. *Matrix Iterative Analysis*. Prentice Hall, Englewoods Cliffs and New Jersey, 1962(reprinted and updated, Springer, Berlin, 2000).

[28] Chengyi Zhang and Yao-tang Li. Diagonal Dominant Matrices and the Determining of H-matrices and M-matrices. *Guangxi Sciences*. 12(3): 1161-164, 2005.

[29] Chengyi Zhang, Yao-tang Li and Feng Chen. On Schur complements of block diagonally dominant matrices. *Linear Algebra Appl.* 414: 533-546, 2006.

[30] Chengyi Zhang, Chengxian Xu and Yao-tang Li. The Eigenvalue Distribution on Schur Complements of H-matrices. *Linear Algebra Appl.* 422: 250-264, 2007.

[31] Chengyi Zhang, Chengxian Xu and Yao-tang Li. Nonsingularity/singularity criteria for nonstrictly generalized diagonally dominant matrices, *Advances in Matrix Theory and its Applications* (Vo. II), Proceedings of the Eighth International Conference on Matrix Theory and Its Applications, Taiyuan, China, July 2008, Edited by Er-Xiong Jiang, Chuan-long Whang, pp. 425-428.

[32] Chengyi Zhang, Shuanghua Luo, Chengxian Xu and Hongying Jiang. Schur complements of generally diagonally dominant matrices and criterion for irreducibility of matrices, *Elec-
Convergence on Gauss-Seidel iterative methods for linear systems with general $H$–matrices

Electronic Journal of Linear Algebra. 18: 69-87, 2009.

[33] CHENG-YI ZHANG, SHUANGHUA LUO, FENGMIN XU AND CHENGXIAN XU. The eigenvalue distribution on Schur complement of nonstrictly diagonally dominant matrices and general $H$-matrices, Electronic Journal of Linear Algebra, 18: 801-820, 2009.

[34] CHENG-YI ZHANG, FENGMIN XU, ZONGBEN XU AND JICHENG LI. General $H$–matrices and their Schur complements. Frontiers of Mathematics in China, 5(9): 1141-1168, 2014.

[35] FUZHEN ZHANG. The Schur Complement and Its Applications. Springer, New York, 2005.

[36] Y. ZHANG, T. Z. HUANG, AND X. P. LIU. Modified iterative methods for nonnegative matrices and $M$–matrices linear systems, Comput. Math. Appl. 50: 1587-1602, 2005.