Localization of Electrical Flows

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Abstract

We show that in any graph, the average length of a flow path in an electrical flow between the endpoints of a random edge is $O(\log^2 n)$. This is a consequence of a more general result which shows that the spectral norm of the entrywise absolute value of the transfer impedance matrix of a graph is $O(\log^2 n)$. This result implies a simple oblivious routing scheme based on electrical flows in the case of transitive graphs.

1 Introduction

Electrical Flows have have played an important role in several recent advances in graph algorithms — for instance, in the context of exact and approximate maximum flow/minimum cut [ CKM+11, LRS13, Mad13], multicommodity flow [KMP12], oblivious routing [HHN+08, LN09, KM11], graph sparsification [SS11], and random spanning tree generation [KM09, MST15]. This is due to the emergence of nearly linear time Laplacian solvers for computing them, beginning with the work of Spielman and Teng [ST04], and also to their well-known connections with random walks. Using them to solve combinatorial problems is not typically immediate, and may be likened to putting a square peg into a round hole: at a high level, many of the traditional problems of computer science are concerned with finding flows in graphs with controlled $\ell_1$ and $\ell_\infty$ norms (corresponding to distance and congestion, respectively), whereas electrical flows minimize the $\ell_2$ norm (energy). Reducing one to the other often requires some sort of iterative method for combining many electrical flows with varying demands and graphs.

In this work, we ask the following basic structural question about electrical flows in arbitrary unweighted graphs:

What is the typical $\ell_1$ norm of the unit current electrical flow between two neighboring vertices $u, v$ in a graph?

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Recall that the $\ell_1$ norm of a unit circulation-free flow is the average distance traveled by the flow, since any such flow $f_{uv}: E \to \mathbb{R}$ may be decomposed as a convex combination of paths which all have the same direction of flow on every edge:

$$f_{uv} = \sum_{i \in \mathcal{P}_{uv}} \alpha_i f_i,$$

where $\mathcal{P}_{uv}$ is the set of simple paths from $u$ to $v$, and we have

$$\|f_{uv}\|_1 = \sum_{i \in \mathcal{P}_{uv}} \alpha_i \|f_i\|_1 = \sum_{i \in \mathcal{P}_{uv}} \alpha_i \text{length}(f_i).$$

Thus, this question asks when/whether electrical flows in a graph travel a greater distance than shortest paths, and by how much.

### 1.1 Three Examples

To get a feel for this problem, and to set the context for our result and its proof, we begin by presenting three instructive examples. We will use the notation $b_v$ for the indicator vector of a vertex, $b_{uv} = b_u - b_v$ for the signed incidence vector of an edge $uv$, $B$ for the $m \times n$ signed edge-vertex incidence matrix of a graph (where the edges are oriented arbitrarily), and $L = B^T B$ for the Laplacian matrix. For any pair of vertices $u, v$, we will use the notation $\Delta(u, v) := \|f_{uv}\|_1 / \text{dist}(u, v)$, where $f_{uv}$ is the unit electrical flow between $u$ and $v$ and dist is the shortest path distance in the graph.

The first example shows that in general $\Delta(u, v)$ can be quite large for the worst-case edge in a graph.

**Example 1.1 (Parallel Paths).** Consider the graph consisting of a single edge between vertices $u$ and $v$ and $\sqrt{m}$ disjoint parallel paths of length $\sqrt{m}$ with endpoints $u$ and $v$. Since the effective resistance of the parallel paths taken together is 1, half of the unit flow between $u$ and $v$ will use the paths, assigning a flow of $1/2\sqrt{m}$ to each path, and the other half will traverse the edge $uv$. Thus, we have $\Delta(u, v) = (\sqrt{m} + 1)/2$. However, notice that for most of the other edges in the graph, $\Delta$ is tiny. For instance, for any edge $ab$ near the middle of one of the parallel paths, a $1 - O(1/\sqrt{m})$ fraction of the flow will traverse the single edge, so we will have $\Delta(a, b) = O(1)$.

On the other hand, $\Delta(u, v)$ is uniformly bounded for every edge in an expander.

**Example 1.2 (Expander Graphs).** Let $G$ be a constant degree $d$–regular expander graph with transition matrix $P$ satisfying $\|P - J/n\| \leq \lambda$ for some constant $\lambda$, where $J$ is the all ones matrix. Letting $Q := P - J/n$ and $E = I - J/n$, we have the power series expansion orthogonal to the all ones vector:

$$(L/d)^+ = (E - Q)^+ = E + \sum_{k \geq 1} Q^k.$$

Now for every edge $uv$ we calculate the electrical flow across its endpoints:

$$\|BL^+ b_{uv}\|_1 \leq \|B\|_{1 \to 1} \|L^+\|_{1 \to 1} \|b_{uv}\|_1,$$
where \( \| \cdot \|_{1 \to 1} \) is the \( 1 \to 1 \) operator norm, i.e., maximum column sum of the matrix. Let \( T = O(\log n) \) be the mixing time of \( G \), after which \( \| P^T - J/n \|_{1 \to 1} = \| Q^T \|_{1 \to 1} \leq 1/n \). Noting that \( \| B \|_{1 \to 1} \leq d \) and \( Q^k = P^k - J/n \) and applying the triangle inequality, we obtain:

\[
\| BL^+ b_{uv} \|_1 \leq \frac{2d}{d} \sum_{k=0}^{T} (\| P^k \|_{1 \to 1} + \| J/n \|_{1 \to 1} ) + \| L^+ \|_{1 \to 1} \cdot \| Q^T \|_{1 \to 1}.
\]

Since \( P^k \) is a doubly stochastic matrix we have \( \| P^k \|_{1 \to 1} = 1 \) for all \( k \). Moreover, \( \| L^+ \|_{1 \to 1} \leq \sqrt{n} \| L^+ \| \leq \sqrt{n}/\lambda \). Combining these facts, we get a final bound of \( \Delta(u, v) = O(\log n) \), for every edge \( uv \in G \).

We remark that bounds similar to (and somewhat more general than) the above were shown in the papers [LN09] [KMI11] using different techniques.

Finally, we note that there are highly non-expanding graphs for which \( \Delta(u, v) \) is also uniformly bounded, which means that expansion does not tell the whole story.

**Example 1.3 (The Grid).** Let \( G \) be the \( n \times n \) two dimensional torus (i.e., grid with sides identified). Then it is shown in [LN09] that for every edge \( uv \in G \) we have \( \Delta(u, v) = O(\log n) \), even though \( G \) is clearly not an expander. We briefly sketch an argument explaining where this bound comes from. Let \( uv \) be any horizontal edge in \( G \), and let \( w \) be a vertex in \( G \) at vertical distance \( k \) from \( u \) and \( v \). We will show that the potential at \( w \) in the unit current \( uv \)-electrical flow is small, in particular that

\[
\phi(w) := b_{uw}^T L^+ b_{uv} = O(1/k^2).
\]

First we recall (see, e.g., [Bol13] Chapter IX) that the potential at a vertex \( w \) when \( u, v \) are fixed to potentials \(-1, 1\) is: \( 2(\mathbb{P}_w(t_v < t_u) - \mathbb{P}_w(t_u < t_v)) \), where \( \mathbb{P}_w \) is the law of the random walk started at \( w \) and \( t_u \) is the first time at which the walk hits \( u \). By Ohm’s law, this means that:

\[
\phi(w) \leq \left| \frac{2}{\text{Reff}(u, v)} (\mathbb{P}_w(t_v < t_u) - \mathbb{P}_w(t_u < t_v)) \right|,
\]

where \( \text{Reff}(u, v) := b_{uw}^T L^+ b_{uv} \) is the effective resistance of the edge \( uv \). Since the resistance of every edge in the grid is equal to \( 1/2 \), we find that \( |\phi(w)| = O(\| \mathbb{P}_w(t_v < t_w) - \mathbb{P}_w(t_w < t_v) \|) \).

We now analyze these probabilities. Roughly speaking, the random walk from \( w \) will take time \( O(k^2) \) to reach the horizontal line \( H \) containing \( uv \), at which point its horizontal distance (along \( H \)) from \( w \) will be distributed as a \( k^2 \)-step random horizontal random walk centered at \( w \) (since about half of the steps of the random walk up to that point will be horizontal). The difference in probabilities between any two neighboring points in \( H \) will therefore be at most \( O(1/k^2) \), which implies the bound on \( |\phi(x)| \). Consequently, the potential difference across any edge \( wx \) at distance \( k \) is at most \( O(1/k^2) \); since there are \( O(k) \) edges at distance \( k \), the total contribution from such edges is \( O(1/k) \), and summing over all distances \( k \) (and repeating the argument for vertical edges) yields a bound of \( O(\log n) \).

### 1.2 Our Results

Our first theorem is the following.
Theorem 1.4. If $G = (V, E)$ is an unweighted graph with $m$ edges, then
\[ \sum_{uv \in E} \Delta(u, v) \leq O(m \log^2 n). \]

This theorem formalizes the intuition in the parallel paths example that there cannot be too many edges in a graph for which the electrical flow uses very long paths. A corollary for edge-transitive graphs is that the above bound holds for every edge, by symmetry. This generalizes our analysis on the grid (which used very specific properties) to a much broader category which includes all Cayley graphs.

Theorem 1.4 is a consequence of a more general result concerning the weighted transfer impedance matrix of a graph. Given a weighted graph $G = (V, E, c)$ with edge weights $c_e \geq 0$, let $C$ be an $m \times m$ diagonal matrix containing the edge weights. Then $L = B^T CB$ is the Laplacian matrix of $G$ and the weighted transfer impedance matrix is the $m \times m$ matrix defined as:
\[ \Pi = C^{1/2} B L^+ B^T C^{1/2}. \]

It is well-known and easy to see that the entry $(BL^+ B)(e, f)$ is the potential difference across the ends of edge $e$ when a unit current is injected across the ends of edge $f$, and vice versa, and that $\Pi$ is a projection matrix with trace $n - 1$. In particular, the latter fact implies that $\|\Pi\|_2 = 1$, where $\| \cdot \|$ is the spectral norm.

Let $\overline{\Pi}$ be the entrywise absolute value matrix of $\Pi$. Our main theorem is:

Theorem 1.5. For an arbitrary weighted graph $G$,
\[ \|\overline{\Pi}\| = O(\log^2 n) \]

Theorem 1.4 follows from Theorem 1.5 by plugging in the the all ones vector $u = (1, \ldots, 1)^T$:
\[ u^T \overline{\Pi} u = \sum_{e \in E} \|\Pi_e\|_1 = \sum_{e \in E} \Delta(e), \]

where $\Pi_e = BL^+ b_{uv}$ is the row of $\Pi$ corresponding to $e = uv$, i.e., the electrical flow across the endpoints of $e$. Since $\|u\|^2 = m$, the spectral norm bound in Theorem 1.5 implies that $u^T \overline{\Pi} u \leq O(m \log^2 n)$.

1.3 Applications to Oblivious Routing

Oblivious routing refers to the following problem: given a graph $G$, specify a set of flows $\{f_{uv}\}$ between pairs of vertices $u, v$ so that for any set of demand pairs $(s_1, t_1), \ldots, (s_k, t_k)$, the congestion of the flow obtained by taking the union of $\{f_{s_it_i}\}_{i \leq k}$ is at most a small factor (called the competitive ratio) greater than the congestion of the optimal multicommodity flow for the given pairs. This is a well-studied problem with a vast literature which we will not attempt to recount; a landmark result is the optimal theorem of Räcke [Rac02] which shows that there is an oblivious routing scheme with competitive ratio $O(\log n)$ for every graph.

In spite of this optimal result, there has been interest in studying whether simpler schemes achieve good competitive ratios. A particularly simple scheme, studied in [HHN+08, LN09, KMT1], is to simply route $f_{uv}$ using the electrical flow. The paper [HHN+08] shows that this scheme has a
good competitive ratio on any graph when restricted to demands which all share a single source. It was shown in [LN09] that the competitive ratio of electrical routing on an unweighted graph is exactly equal to $\|\Pi\|_{1 \rightarrow 1}$, i.e., the maximum of $\Delta(u, v)$ over all edges in a graph. In these papers, it was shown that for grids, hypercubes, and expanders the competitive ratio is $O(\log n)$. Our theorem immediately extends this to all transitive graphs, albeit with a guarantee of $O(\text{polylog}(n))$ rather than $O(\log(n))$.

Corollary 1.6. Electrical Flow Routing achieves a competitive ratio of $O(\log^2 n)$ on every edge-transitive graph.

Proof. By Theorem 1.4 and symmetry, we have that every column sum of $\Pi$ is at most $O(\log^2 n)$. By Proposition 1 and Lemma 4 of [LN09] (or by Theorem 3.1 of [KM11]), this implies that routing each pair by the electrical flow has a competitive ratio of $O(\log^2 n)$ as an oblivious routing scheme. □

1.4 Techniques

Given the expander example above, it may be tempting to attempt to prove Theorem 1.4 by decomposing an arbitrary graph into disjoint expanding clusters. However, using such a decomposition would likely require proving that edge electrical flows do not cross between the clusters, which is what we are trying to show in the first place.

We use an alternate scheme inspired by recent Schur-complement based Laplacian solvers. Recall the Schur complement formula for the pseudoinverse of a symmetric block matrix (see e.g. [DKP+17] Section 5):

Fact 1.7. If

$$L = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$$

for symmetric $P, R$ and $R$ invertible, then:

$$L^+ = Z^T \begin{bmatrix} I & 0 \\ -R^{-1}Q^T & I \end{bmatrix} \begin{bmatrix} \text{Schur}(L, P)^+ & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} I & -QR^{-1} \\ 0 & I \end{bmatrix} Z$$

(1)

where $\text{Schur}(L, P) = P - QR^{-1}Q^T$ denotes the Schur complement of $L$ onto $P$, obtained by eliminating $R$ by partial Gaussian elimination, and $Z$ is the projection orthogonal to the nullspace of $L$.

The idea is to apply this formula to compute the terms $|b_e^T L^+ b_f|$ by eliminating vertices one-by-one, as in [KS10], and bounding the original value of $|b_e^T L b_f|$ in terms of the value on small Schur complements. One cannot eliminate arbitrary vertices and get a good bound, though. We use Proposition 3.2 to show that there always exists a vertex whose elimination results in a good bound. Since Laplacian matrices with self loops are closed under taking Schur complements the remaining matrix is the Laplacian of a weighted graph as well. Mapping the demand vectors $b_e$ and $b_f$ to the vertex set of this graph and recurring yields the sum of interest.

2 Schur Complements, Probabilities, and Energies

In this section we collect some preliminary facts about Schur complements of Laplacians and establish some useful correspondences between electrical potentials and probabilities. We do this so
that after recurring on a Schur complement of the graph $G$ that we care about, we can interpret the recursively generated sums that we generate using Fact 1.7 in terms of $G$. We will make frequent use of the fact that for a Laplacian matrix $L_G$ with block $L_S$, the Schur complement $\text{Schur}(L_G, L_S)$ is also a Laplacian. For a graph $G$ and subset of vertices $S$ we will use the notation $\text{Schur}(G, S)$ to denote the graph corresponding to $\text{Schur}(L_G, L_S)$. Since all vectors we will apply pseudoinverses to will be orthogonal to the corresponding nullspaces (the corresponding constant vectors, since all Schur complements will be Laplacians), we will not write the projection $Z$ in Fact 1.7 in what follows.

**Definition 2.1.** Consider a graph $G$. For any set of vertices $S \subseteq V(G)$ with $|S| \geq 2$, a vertex $v \in S$, and a vertex $x \in V(G)$, let

$$p_v^{G,S}(x) := \mathbb{P}_x[t_v < t_{S \setminus v}]$$

For an edge $e = \{x, y\} \in E(G)$, let

$$q_v^{G,S}(e) := |p_v^S(x) - p_v^S(y)|$$

where $t_{S'}$ denotes the hitting time to the set $S'$. Let

$$r_v^{G,S}(e) := \max(p_v^S(x), p_v^S(y), 1/|S|)$$

When $G$ is clear from the context, we omit $G$ from the superscript.

**Corollary 2.2.** For any set $S$ and any $e = \{x, y\} \in E(G)$, $\sum_{v \in S} r_v^{G,S}(x) \leq 3$.

**Proof.** $\{p_v^{G,S}(z)\}_v$ is a distribution for any fixed vertex $z \in V(G)$. Bounding $r_v^{G,S}(e) \leq p_v^{G,S}(x) + p_v^{G,S}(y) + \frac{1}{|S|}$ yields the desired result. $\square$

It is well-known that the above probabilities can be represented as normalized potentials (see, for instance, [Bol13] Chapter IX).

**Fact 2.3.** Let $H$ be the graph obtained by identifying all vertices of $S \setminus \{v\}$ to one vertex $s$. Then $p_v^S(x) := \frac{|b_x^S L_H^+ b_s^S|}{b_v^S L_H^+ b_s^S}$ for any $x \in V(G)$ and $q_v^S(e) := \frac{|b_x^S L_H^+ b_e^S|}{b_v^S L_H^+ b_e^S}$ for any $e \in E(G)$.

In proving the desired result, it will help to recursively compute Schur complements with respect to certain sets of vertices $S$. We now relate the Schur complement to the above probabilities, which will be central to our proof; the following proposition is likely to be known but we include it for completeness.

**Proposition 2.4.** For a set of vertices $S \subseteq V(G)$ and a vertex $x \in V(G)$ possibly not in $S$, let $b_x \in \mathbb{R}^{V(G)}$ denote the indicator vector of $x$. Let $b_x^S \in \mathbb{R}^S$ denote the vector with coordinates $b_x^S(v) = p_v^S(x)$ for all $v \in S$. Write $L_G$ as a two-by-two block matrix:

$$L = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$$

where $P$, $Q$, and $R$ have index pairs $S \times S$, $S \times (V(G) \setminus S)$, and $V(G) \setminus S \times V(G) \setminus S$ respectively. Then

$$b_x^S = M_S b_x$$

where

$$M_S = \begin{bmatrix} I & -QR^{-1} \end{bmatrix}.$$
Proof. If \( x \in S \), then \( b^S_x \) is the indicator vector of \( x \) and \( x \) is in the identity block of \( M_S \). Therefore, \( b^S_x = M_S b_x \).

If \( x \notin S \), then let \( b^C_x \) denote the coordinate restriction of \( b_x \) to \( V(G) \setminus S \). We want to show that \( b^S_x = -QR^{-1}b^C_x \). Consider the linear system

\[
b^C_x = Rp
\]

Let \( H \) be the graph obtained by identifying all vertices in \( S \) within \( G \) to a single vertex \( s \). Then the vector \( p' \) with \( p'_v = 0 \) and \( p'_s = p_v \) for all \( v \in V(H) \setminus \{s\} \) is a solution to a boundary value problem with \( p'_v = 0 \) and \( p'_s \) having the maximum potential of any vertex. The block matrix \( Q \) can be viewed as mapping the potentials \( p' \) to a flow proportional to the \( xs \)-flows on edges incident with \( s \). By Proposition 2.2 of [LP16], for example, the incoming flow on edges to \( s \) is equal to the probability that an \( x \to s \) random walk first visits \( s \) by crossing that edge. Grouping edges according to their common endpoints shows that \( -QR^{-1}b^C_x \) is a scalar multiple of \( b^S_x \).

However, notice that

\[
1^T (-QR^{-1})b^C_x = (-1^T Q)R^{-1}b^C_x = 1^T RR^{-1}b^C_x = 1 = 1^T b^S_x
\]

so \( b^S_x = -QR^{-1}b^C_x \), as desired. \( \square \)

Once one views the the \( q^S_v(e) \)'s in the above way, it makes sense to discuss the energy of the \( q^S_v(e) \)'s in relation to the probabilities \( p^S_v(x) \). It turns out that the total energy contributed by edges with both endpoints having potential at most \( p \) is at most a \( p \) fraction of the total energy.

**Proposition 2.5.** For any \( p \in (0,1) \), let \( F \) be the set of edges \( \{x,y\} \) with \( \max_{z \in \{x,y\}} p^S_v(z) \leq p \). Then the total energy of those edges is at most a \( p \) fraction of the overall energy. More formally,

\[
\sum_{e \in F} c_e (q^S_v(e))^2 \leq p \sum_{e \in E(G)} c_e (q^S_v(e))^2
\]

Proof. Let \( H \) be the graph obtained by identifying \( S' \setminus \{v\} \) to a vertex \( s \) in \( G \). By Fact 2.3, we can show the desired proposition by proving the following:

\[
\sum_{e \in F} c_e (b^T_{vs} L^+_L b_v)^2 \leq pb^T_{vs} L^+_L b_{vs}
\]

for an arbitrary graph \( H \) and the subset of edges \( F \subseteq E(H) \) with \( \max_{z \in \{x,y\}} |b^T_{vs} L^+_L b_{zs}| \leq pb^T_{vs} L^+_L b_{vs} \).

Write the sum on the left side in terms of an integral over sweep cuts. For \( p \in (0,1) \), let \( C_p \) denote the set of edges cut by the normalized potential \( p \). More precisely, let \( C_p \) be the set of edges \( \{x,y\} \) with \( |b^T_{vs} L^+_L b_{xs}| \geq pb^T_{vs} L^+_L b_{vs} \) and \( |b^T_{vs} L^+_L b_{ys}| \leq pb^T_{vs} L^+_L b_{vs} \). Notice that
\[
\sum_{e \in F} c_e (b_{us}^T L_H^+ b_e)^2 \leq \int_0^{p b_{us}^T L_H^+ b_{us}} \sum_{e \in C_q} c_e |b_{us}^T L_H^+ b_e| dq \\
= \int_0^{p b_{us}^T L_H^+ b_{us}} 1 dq \\
\leq p b_{us}^T L_H^+ b_{us}
\]

where the equality follows from the fact that \( C_q \) is a threshold cut for the \( v-s \) electrical flow and the first inequality follows from splitting the contribution of \( e \) to the sum in terms of threshold cuts. This inequality is the desired result.

Finally, we relate the weighted degrees of vertices in \( \text{Schur}(G, S) \) to energies in \( G \) with respect to \( S \).

**Definition 2.6.** Let \( c_v^H \) denote the sum of the conductances of edges incident with \( v \) in \( H \).

**Proposition 2.7.** Let \( G \) be a graph. Consider a set \( S \) and let \( H = \text{Schur}(G, S) \). Then

\[
c_v^H = \sum_{e \in E(G)} c_e^G (q_{G, S}(e))^2
\]

**Proof.** Let \( I \) be the graph obtained by identifying \( S \setminus \{v\} \) to \( s \) in \( H \). Since the effective conductance of parallel edges is the sum of the conductance of those edges, \( c_v^H = \frac{1}{b_{us}^T L_I^+ b_{us}} \).

By commutativity of Schur complements, \( I \) can also be obtained by identifying \( S \setminus \{v\} \) in \( G \) before eliminating all vertices besides \( s \) and \( v \). Let \( J \) be the graph obtained by just doing the first step (identifying \( S \setminus \{v\} \)). By definition of Schur complements,

\[
b_{us}^T L_I^+ b_{us} = b_{us}^T L_J^+ b_{us}
\]

By Fact 2.3

\[
b_{us}^T L_J^+ b_{us} = \frac{1}{\sum_{e \in E(G)} c_e^G (q_{G, S}(e))^2}
\]

Substitution therefore shows the desired result. \( \Box \)

### 3 Proof of Theorem 1.5

We will deduce Theorem 1.5 from the following seemingly weaker statement regarding positive test vectors.

\footnote{To avoid confusion, we remind the reader that by conductances we always mean electrical conductances, i.e., weights in the graph, and not conductances in the sense of expansion.}
Theorem 3.1. Let $G$ be a graph. Then for any vector $w \in \mathbb{R}^E_{\geq 0}$,

$$
\sum_{e,f \in E} w_e b_f \sqrt{c_e c_f} |b_T e C_G^1| \leq O(\log^2 n) ||w||_2^2
$$

Theorem 1.5 can be deduced from this by a Perron-Frobenius argument.

Proof of Theorem 1.5. Since the matrix $M = |C_G^{1/2} B_G^T L_G^+ B_G C_G^{1/2}|$ has nonnegative entries, there is an eigenvector with maximum eigenvalue with nonnegative coordinates by Perron-Frobenius. Such an eigenvector corresponds to a positive eigenvalue. Theorem 3.1 bounds the value of the quadratic form of this eigenvector. In particular, the quadratic form is at most $O(\log^2 n)$ times the $\ell_2$ norm of the vector, as desired. \QED

The proof hinges on the following key quantity. Define

$$
\text{Degree}_S(u) := \frac{(\sum_{e \in E(G)} w_e \sqrt{c_e q_S^0(e)})^2}{\sum_{e \in E(G)} c_e q_S^0(e)^2}
$$

The quantity $\text{Degree}_S(u)$ may be interpreted as a measure of the sparsity of the vector $(q_S^0(e))_e$, since it is the ratio of the (weighted) $\ell_2^1$ norm of this vector and its $\ell_2^2$ norm. Note that when $S = V(G)$, $w = 1$, and $G$ is unweighted, $\text{Degree}_S(u)$ is simply the degree of $u$.

There are two parts to the proof: (1) recursively reducing the original problem to a number of problems involving sums of simpler inner products and (2) bounding those sums. The difference between the value of a problem and the subproblem after eliminating $u$ is at most $\text{Degree}_S(u)$. We want to show that there always is a choice of $u$ with a small value of $\text{Degree}_S(u)$. The following proposition shows this:

Proposition 3.2. For any $\{c_e\}_e$-weighted graph $G$, set $S \subseteq V(G)$ with $|S| \geq 2$, and nonnegative weights $\{w_e\}_{e \in E(G)}$, the following holds:

$$
\sum_{u \in S} \text{Degree}_S(u) \leq O(\log |S|) \sum_{e \in E(G)} w_e^2
$$

We now reduce Theorem 3.1 to this proposition by picking the vertex $u$ with that minimizes the summand $\text{Degree}_S(u)$ of Proposition 3.2 and recurring on the Schur complement with $u$ eliminated. The summand of Proposition 3.2 is an upper bound on the decrease due to eliminating $u$.

Proof of Theorem 3.1 given Proposition 3.2. Define the following:

- $G_0 \leftarrow G$, $c^{(0)} \leftarrow c$, $x_0 \leftarrow \arg\min_{x \in V(G)} \text{Degree}_{S_0}(x)$, $S_0 \leftarrow V(G)$, $i \leftarrow 0$.

- While $|V(G_i)| > 2$:
  - $i \leftarrow i + 1$
  - $G_i \leftarrow \text{Schur}(G_{i-1}, V(G_{i-1}) \setminus \{x_{i-1}\})$
  - $c^{(i)} \leftarrow \text{conductance vector for } G_i$
  - $S_i \leftarrow V(G_i)$
  - $x_i \leftarrow \arg\min_{x \in V(G_i)} \text{Degree}_{S_i}(x)$
Let $L_i \leftarrow L_{G_i}$ and let $m_i = L_{x_i x_i}$. We start by understanding how to express the left hand side of the desired inequality in $G_i$ for all $i$. For a vertex $x \in V(G)$, let $b_x^{(i)} \in \mathbb{R}^{V(G_i)}$ denote the vector with $b_x^{(i)}(v) = p_v^{G_i, S_i}(x)$ for all $v \in V(G_i)$. For an edge $\{x, y\} \in V(G)$, let $b_{xy}^{(i)} = b_x^{(i)} - b_y^{(i)}$. Let

$$V_i := \sum_{e, f \in E(G)} w_e \sqrt{c_e} |b_e^{(i)T} L_i^+ b_f^{(i)}| \sqrt{c_f w_f}$$

We now bound $V_i$ in terms of $V_{i+1}$ for all nonnegative integers $i < T$. By Fact 1.7 and Proposition 2.4

$$V_i = \sum_{e, f \in E(G)} w_e \sqrt{c_e} |b_e^{(i)T} L_i^+ b_f^{(i)}| \sqrt{c_f w_f}$$

$$= \sum_{e, f \in E(G)} w_e \sqrt{c_e} |b_e^{(i+1)T} L_{i+1}^+ b_f^{(i+1)} + x^{(i)}_e x^{(i)}_f \frac{1}{m_i} \sqrt{c_f w_f}$$

$$\leq V_{i+1} + \sum_{e, f \in E(G)} w_e \sqrt{c_e} |x^{(i)}_e \frac{1}{m_i} x^{(i)}_f| \sqrt{c_f w_f}$$

where $x^{(i)}_e := b^{(i)}_e(x_i)$. Since the $x^{(i)}_e$ are scalars, we can further simplify the above sum:

$$\sum_{e, f \in E(G)} w_e \sqrt{c_e} |x^{(i)}_e \frac{1}{m_i} x^{(i)}_f| \sqrt{c_f w_f} = \frac{1}{m_i} \left( \sum_{e \in E(G)} w_e \sqrt{c_e} |x^{(i)}_e| \right)^2$$

$$= \frac{1}{c_{x_i}} \left( \sum_{e \in E(G)} w_e \sqrt{c_e} x^{(i)}_e \right)^2$$

$$= \frac{\left( \sum_{e \in E(G)} w_e \sqrt{c_e} x^{(i)}_e \right)^2}{\sum_{e \in E(G)} c_e x^{(i)}_e^2}$$

where the second-to-last denominator equality follows from Proposition 2.7. Since $x_i$ minimizes $\text{Degree}_{S_i}(x_i)$, Proposition 3.2 with $S \leftarrow S_i$ and $G \leftarrow G$ implies that

$$\text{Degree}_{S_i}(x_i) \leq O \left( \frac{\log n}{|S_i| ||w||_2^2} \right)$$

$$\leq O \left( \frac{\log n}{n - i ||w||_2^2} \right)$$
Plugging this in shows that

$$ V_i \leq V_{i+1} + O\left(\frac{\log n}{n - i}\right) ||w||_2^2 $$

for all $i < T$. Therefore, to bound $V_0$, it suffices to bound $V_T$. Let $S_T = \{a, b\}$. Then

$$ V_T = \sum_{e,f \in E(G)} w_e \sqrt{c_e |b_e^{(T)T} L_T^{+} b_f^{(T)}|} \sqrt{c_f w_f} $$

$$ = \sum_{e,f \in E(G)} w_e \sqrt{c_e q_{u_i}^{G,S_i}(e)} \sqrt{b_i^{T} L_T^{+} b_{ab} q_{u_i}^{G,S_i}(f)} \sqrt{c_f w_f} $$

$$ = \sum_{e,f \in E(G)} w_e \sqrt{c_e \frac{|b_i^{T} L_T^{+} b_{ab}|}{b_i^{T} L_T^{+} b_{ab}} \frac{|b_i^{T} L_T^{+} b_f|}{b_i^{T} L_T^{+} b_{ab}} \sqrt{c_f w_f}} $$

$$ \leq ||w||_2^2 $$

where the last line follows from Cauchy-Schwarz. Therefore,

$$ V_T \leq ||w||_2^2 $$

Combining these bounds yields a harmonic sum that proves the desired result. \qedhere

Now, we prove Proposition 3.2.

**Proof.** For each vertex $v \in S$ and each integer $i \in [0, \log |S|]$, let $X_v^{(i)} \subseteq E(G)$ denote the set of edges $e = \{x, y\}$ for which $r_v^S(e) \leq 2^{-i}$. Let $T := \log |S|$. For each $0 \leq i < T$, let $Y_v^{(i)} = X_v^{(i)} \setminus X_v^{(i+1)}$. Let $Y_v^{(T)} = X_v^{(T)}$.

For each $v$ and each $i \geq 0$, $X_v^{(0)} = E(G)$, and $X_v^{(i+1)} \subseteq X_v^{(i)}$. Therefore, $\{Y_v^{(i)}\}_{i=0}^{T}$ is a partition of $E(G)$ for each $v \in S$. By Cauchy-Schwarz,

$$ \sum_{u \in S} \text{Degree}_S(u) \leq \sum_{u \in S} \left( \sum_{e \in E(G)} \frac{w_e \sqrt{c_e q_u^S(e)^2}}{\sum_{e \in E(G)} c_e q_u^S(e)^2} \right)^2 $$

$$ \leq \sum_{u \in S} \left( \sum_{e \in E(G)} r_u^S(e) w_e^2 \right) \left( \sum_{e \in E(G)} \frac{c_e q_u^S(e)^2}{\sum_{e \in E(G)} c_e q_u^S(e)^2} \right) $$

By the definition of $X_v^{(i+1)}$,

$$ \sum_{u \in S} \left( \sum_{e \in E(G)} r_u^S(e) w_e^2 \right) \left( \sum_{e \in E(G)} \frac{c_e q_u^S(e)^2}{\sum_{e \in E(G)} c_e q_u^S(e)^2} \right) \leq \sum_{u \in S} \left( \sum_{e \in E(G)} r_u^S(e) w_e^2 \right) \left( \sum_{i=0}^{T} 2^{i+1} \frac{c_e q_u^S(e)^2}{\sum_{e \in E(G)} c_e q_u^S(e)^2} \right) $$

By the definition of $X_v^{(i)}$ and Proposition 2.5.
By Proposition 2.2,

\[ \sum_{u \in S} \left( \sum_{e \in E(G)} r_u^S(e)w_e^2 \right) \left( \sum_{i=0}^{T} 2^{i+1} \sum_{e \in E^{(i)}} c_e q_u^S(e)^2 \right) \leq \sum_{u \in S} \left( \sum_{e \in E(G)} r_u^S(e)w_e^2 \right) \left( \sum_{i=0}^{T} 2 \right) \leq \sum_{u \in S} \left( \sum_{e \in E(G)} r_u^S(e)w_e^2 \right) (2T + 2) \]

Combining these bounds shows that

\[ \sum_{u \in S} \text{Degree}_S(u) \leq (6T + 6) \sum_{e \in E(G)} w_e^2 \leq O(\log |S|)||w||_2^2 \]

as desired.

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