White noise distribution theory for the Fermion system

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In this paper, we give the white noise calculus for the Fermion system and prove the Fock expansion. Each continuous linear operator on Fermionic white noise functionals is uniquely represented by the series of integral kernel operators. This series is called the Fock expansion.

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1 Introduction

In this paper, we give the white noise calculus for the Fermion system and prove the Fock expansion for any continuous linear operators from the space of test functionals to the space of generalized functionals.

To the beginning, we mention the motivation of this study. Our white noise calculus introduced by T. Hida in 1975 is the theory for the space $E$ of test functionals and the space $E^*$ of generalized functionals on the infinite dimensional space, for (continuous) linear operators from $E$ to $E^*$. In quantum mechanical physics, the white noise calculus provides us with a framework of an analysis for the Boson system. (See [5].) In the Boson system, we can represent a continuous linear operator $\Xi$ from $E$ to $E^*$ as a series of integral kernel operators. This representation of $\Xi$ is called the Fock expansion. The Fock expansion is formulated by T. Hida, N. Obata and K. Saitô in [1]. The Fock expansion is applied for determining the commutant. For example, N. Obata [6] used the Fock expansion to obtain the characterization of rotation invariant operators on white noise functionals. Moreover, in [7], the author showed irreducibility of the energy representation of a group of $C^\infty$-mappings from a compact Riemann manifold to a semi-simple compact Lie group.

As mentioned above, we hope the existence of the Fock expansion for the Fermion system since the Fock expansion is useful for determining the commutant. (In fact, we apply the Fock expansion for the Fermion system to the implementability of Bogoliubov automorphisms of canonical anti-commutation relations algebra, and we have a partial solution now. In another paper, we will be able to see an application of the Fock expansion.) As for the white noise calculus for the Fermion system, Y. Liao and K. Liu [4] introduced it and showed Itô’s product formulas for creation, annihilation, and number processes when the one particle space is $L^2(\mathbb{R})$.

In this paper, we define the white noise calculus for the Fermion system when the one particle space is an abstract Hilbert space. Moreover, we prove the Fock expansion of continuous linear operators for the Fermion system.

Next, we describe an outline of the proof of the Fock expansion for continuous linear operators on the Fermion Fock space. Since a pair of Fermions behaves like a Boson, we can show the Fock expansion for the even part of the Fermion system and extend the result of the even part to the whole of the Fermion system with the help of the canonical anti-commutation relations for creation and annihilation operators.

This paper is organized as follows. In section 2, we make the Gelfand triples for the even part, odd part, and the whole of the Fermion system and define $S$-transform of generalized white noise functionals for the even part of the Fermion system. In section 3 and 4 we give the Fock expansion for the even part of the Fermion system. In section 5, we extend the Fock expansion obtained in section 3 to the whole of the Fermion system.

2 $S$-transform

In this section, we make Gelfand triples for the Fermion system and define the $S$-transform of generalized white noise functionals for even part of the Fermion system.

**Definition 2.1.** Let $H$ be a complex Hilbert space with an inner product $(\cdot, \cdot)_0$. Let $A$ be a self-adjoint operator defined on a dense domain $D(A)$. Let $\{\lambda_j\} \in \mathbb{N}$ be eigenvalues of $A$ and $\{e_j\} \in \mathbb{N}$ be normalized eigenvectors for $\{\lambda_j\} \in \mathbb{N}$, i.e., $Ae_j = \lambda_j e_j$, $j \in \mathbb{N}$. Moreover, we also assume the following two conditions:

(i) $\{e_j\} \in \mathbb{N}$ is a C.O.N.S. of $H$, 

(ii) $\sum_j |\lambda_j|^n |e_j|^2 < \infty$ for all $n \in \mathbb{N}$.
(ii) Multiplicity of \( \{ \lambda_j \}_{j \in \mathbb{N}} \) is finite and \( 1 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty \).

Then we have the following properties.

(1) For \( p \in \mathbb{R}_{\geq 0} \) and \( x, y \in D(A^p) \), let \( (x, y)_p := (A^px, A^py)_0 \). Then \( (\cdot, \cdot)_p \) is an inner product on \( D(A^p) \). Moreover, \( D(A^p) \) is complete with respect to the norm \( | \cdot |_p \), that is, the pair \( E_p := (D(A^p), | \cdot |_p) \) is a Hilbert space.

(2) For \( q \geq p \geq 0 \), let \( j_{p,q} : E_q \hookrightarrow E_p \) be the inclusion map. Then every inclusion map is continuous and has a dense image. Then \( \{ E_p, j_{p,q} \} \) is a reduced projective system.

(3) A standard countable Hilbert space

\[
E := \lim_{\leftarrow} E_p = \bigcap_{p \geq 0} E_p
\]

constructed from the pair \((H, A)\) is a reflexive Fréchet space. We call \( E \) a CH-space simply.

(4) From (3), we have \( E^* = \lim_{\rightarrow} E_p^* \) as a topological vector space, i.e. the strong topology on \( E^* \) and the inductive topology on \( \lim_{\rightarrow} E_p^* \) coincide.

(5) Let \( p \in \mathbb{R}_{\geq 0} \) and \((x, y)_- := (A^{-p}x, A^{-p}y)_0 \). Then \((\cdot, \cdot)_- \) is an inner product on \( H \).

(6) Let \( E_{-p} \) be the completion of \( H \) with respect to the norm \( | \cdot |_- \). For \( q \geq p \geq 0 \), we can consider the inclusion map \( i_{-q,-p} : E_{-p} \hookrightarrow E_{-q} \). Then \( \{ E_{-p}, i_{-q,-p} \} \) is an inductive system. Moreover, \( E_{-p} \) and \( E^*_p \) are anti-linear isomorphic and isometric. Thus, from (4), we have

\[
E^* = \lim_{\rightarrow} E_{-p} = \bigcup_{p \geq 0} E_{-p}.
\]

Furthermore, we require for the operator \( A \) that there exists \( \alpha > 0 \) such that \( A^{-\alpha} \) is a Hilbert-Schmidt class operator, namely

\[
\delta^2 := \sum_{j=1}^{\infty} \lambda_j^{-2\alpha} < \infty. \tag{2.1}
\]

From this condition, \( E \) (resp. \( E^* \)) is a nuclear space. Thus we can define the \( \pi \)-tensor topology \( E \otimes_\pi E \) (resp. \( E^* \otimes_\pi E^* \)) of \( E \) (resp. \( E^* \)). If there is no danger of confusion, we will use the notation \( E \otimes E \) (resp. \( E^* \otimes E^* \)) simply.

We denote the canonical bilinear form on \( E^* \times E \) by \( \langle \cdot, \cdot \rangle \). We have the following natural relation between the canonical bilinear form on \( E^* \times E \) and the inner product on \( H \):

\[
\langle f, g \rangle = (Jf, g)_0
\]

for all \( f \in H \) and \( g \in E \). \( Jf \in H \) is the complex conjugate of \( f \in H \).

**Definition 2.2.** Let \( X \) be a Hilbert space, or a CH-space.

(1) Let \( g_1, \ldots, g_n \in X \). We define the anti-symmetrization \( A_n(g_1 \otimes \ldots \otimes g_n) \) of \( g_1 \otimes \ldots \otimes g_n \in X^{\otimes n} \) as follows.

\[
A_n(g_1 \otimes \ldots \otimes g_n) := g_1 \wedge \ldots \wedge g_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) g_{\sigma(1)} \otimes \ldots \otimes g_{\sigma(n)},
\]

where \( \mathfrak{S}_n \) is the set of all permutations of \( \{1, 2, \ldots, n\} \).
(2) If \( f \in X^{\otimes n} \) satisfies \( A_n(f) = f \), then we call \( f \) anti-symmetric. We denote the set of all anti-symmetric elements of \( X^{\otimes n} \) by \( X^{\wedge n} \) and we call \( X^{\wedge n} \) the \( n \)-th anti-symmetric tensor of \( X \). If \( X \) is a Hilbert space, then \( A_n \) is a projection from \( X^{\otimes n} \) to \( X^{\wedge n} \).

(3) Let \( X \) be a CH-space. For \( F \in (X^{\otimes n})^* \) and \( \sigma \in \mathfrak{S}_n \), let \( F^\sigma \) be an element of \( (X^{\otimes n})^* \) satisfying
\[
\langle F^\sigma, g_1 \otimes \ldots \otimes g_n \rangle := \langle F, g_{\sigma^{-1}(1)} \otimes \ldots \otimes g_{\sigma^{-1}(n)} \rangle, \quad g_i \in X.
\]
Then we define the anti-symmetrization \( A_n(F) \) as follows.
\[
A_n(F) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \text{sign}(\sigma) F^\sigma.
\]

(4) If \( F \in (X^{\otimes n})^* \) satisfies \( A_n(F) = F \), we call \( F \) anti-symmetric. We denote the set of all anti-symmetric elements of \( (X^{\otimes n})^* \) by \( (X^{\wedge n})^* \).

From the above discussion, we obtain a Gelfand triple:
\[
E \subset H \subset E^*.
\]

**Lemma 2.3.** Let \( H \) be a Hilbert space.

1. Let
\[
(f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_n)_0 := (f_1, g_1)_0 \cdots (f_n, g_n)_0
\]
for \( f_i, g_j \in H, \ i, j = 1, 2, \ldots, n \). Then
\[
(f_1 \wedge \ldots \wedge f_n, g_1 \wedge \ldots \wedge g_n)_0 = \frac{1}{n!} \det ((f_i, g_j)_0)_{1 \leq i, j \leq n}.
\]
Moreover \( A_n \) is a projection with respect to \( (\cdot, \cdot)_0 \).

2. For \( f \in H^{\wedge n} \) and \( g \in H^{\wedge m} \), we have
\[
|f \wedge g|_0 \leq |f|_0 |g|_0.
\]

Next, we define the Fermion Fock space and the second quantization of a linear operator.

**Definition 2.4.** Let \( H \) be a Hilbert space and \( A \) be a linear operator on \( H \).

1. Let
\[
\Gamma(H) := \left\{ \sum_{n=0}^{\infty} \phi_n | \phi_n \in H^{\wedge n}, \left\| \sum_{n=0}^{\infty} \phi_n \right\|_0^2 := \sum_{n=0}^{\infty} n! |\phi_n|_0^2 < +\infty \right\},
\]
\[
\left( \sum_{n=0}^{\infty} \phi_n, \sum_{n=0}^{\infty} \psi_n \right)_0 = \sum_{n \in \mathbb{Z}_2} n! \langle \phi_n, \psi_n \rangle_0.
\]

Then we call \( \Gamma(H) \) the Fermion Fock space. The Fermion Fock space \( \Gamma(H) \) is a Hilbert space with respect to the inner product \( (\cdot, \cdot)_0 \). Moreover let
\[
\Gamma^+(H) := \left\{ \sum_{n=0}^{\infty} \phi_{2n} | \phi_{2n} \in H^{\wedge (2n)}, \sum_{n=0}^{\infty} (2n)! |\phi_{2n}|_0^2 < +\infty \right\},
\]
\[
\Gamma^-(H) := \left\{ \sum_{n=0}^{\infty} \phi_{2n+1} | \phi_{2n+1} \in H^{\wedge (2n+1)}, \sum_{n=0}^{\infty} (2n+1)! |\phi_{2n+1}|_0^2 < +\infty \right\}.
\]

Then we call \( \Gamma^+(H) \) (resp. \( \Gamma^-(H) \)) the even part of the Fermion Fock space (resp. the odd part of the Fermion Fock space).
(2) We call
\[ \Gamma(A) := \sum_{n=0}^{\infty} A^\otimes n \]
the second quantization of \( A \). Let
\[ \Gamma^+(A) := \Gamma(A)|\Gamma^+(H), \quad \Gamma^-(A) := \Gamma(A)|\Gamma^-(H). \]

**Definition 2.5.** Let \( H \) be a complex Hilbert space and \( A \) be a self-adjoint operator on \( H \) satisfying the conditions (i) and (ii) in Lemma 2.1 and (2.1). Then we can define a CH-space \( E \) constructed from \( (\Gamma(H), \Gamma(A)) \) and we obtain a Gelfand triple:
\[ E \subset \Gamma(H) \subset E^*. \]
Moreover, let \( E_+ \) (resp. \( E_- \)) be a CH-space constructed from \( (\Gamma^+(H), \Gamma^+(A)) \) (resp. \( (\Gamma^-(H), \Gamma^-(A)) \)) and we obtain Gelfand triples:
\[ E_+ \subset \Gamma^+(H) \subset E_+^*, \quad E_- \subset \Gamma^-(H) \subset E_-^*. \]
Then an element of \( E \) (or \( E_+, E_- \)) is called a test (white noise) functional and an element of \( E^* \) (or \( E_+^*, E_-^* \)) is called a generalized (white noise) functional.

**Corollary 2.6.** Let \( \phi := \sum_{n=0}^{\infty} \phi_n \in \Gamma(H) \), \( \phi_n \in H^{\wedge n} \). Then \( \phi \in E \) if and only if \( \phi_n \in E^{\wedge n} \) for all \( n \geq 0 \). Moreover, it holds that
\[ \|\phi\|^2_p := \|\Gamma(A)^p \phi\|^2_0 = \sum_{n=0}^{\infty} n! |\phi_n|^2_p < +\infty \]
for all \( p \geq 0 \). We can also show this statement in case of \( \phi \in \Gamma^+(H) \) and \( \phi \in \Gamma^-(H) \).

**Remark 2.7.** Let \( H \) be a Hilbert space. Then \( \zeta \wedge \eta = \eta \wedge \zeta \) for \( \zeta, \eta \in H^{\wedge 2} \). Thus we can define \( \zeta^{\wedge n}, n \geq 0 \) for all \( \zeta \in H^{\wedge 2} \). This shows that a pair of Fermions behaves like a Boson.

**Definition 2.8.** For any \( \zeta \in H^{\wedge 2} \), we define an element \( e^+(\zeta) \in \Gamma^+(H) \) as follows:
\[ e^+(\zeta) := \sum_{n=0}^{\infty} \frac{1}{(2n)!} \zeta^{\wedge n}. \]
We can check the well-definedness of \( e^+(\zeta) \) easily.

**Corollary 2.9.** If \( \zeta \in E^{\wedge 2} \), then \( e^+(\zeta) \in E_+ \).

**Proof.** Since
\[ |\phi \otimes \psi|^p \leq |\phi|^p |\psi|^p \]
for all \( \phi \in E^{\otimes l}, \psi \in E^{\otimes m} \), and \( p \geq 0 \) (see Lemma 3.2), we have
\[ |\zeta^{\wedge n}|^p \leq |\zeta|^n_p \]
for all \( \zeta \in E^{\wedge 2}, p \geq 0 \). Thus
\[ \|e^+(\zeta)|^2_p = \sum_{n=0}^{\infty} (2n)! \left| \frac{1}{(2n)!} \zeta^{\wedge n} \right|^2_p \leq \sum_{n=0}^{\infty} \frac{1}{(2n)!} |\zeta|^{2n}_p \leq \sum_{n=0}^{\infty} \frac{1}{n!} |\zeta|^{2n}_p = \exp(|\zeta|^2_p). \]
This implies \( e^+(\zeta) \in E_+ \).
Proposition 2.10. Let $H$ be a Hilbert space and $S := \{e^+(\zeta)|\zeta \in H^{\wedge 2}\}$. Then $\text{span}_C[S]$ is a dense subspace of $\Gamma^+(H)$ with respect to the norm $\|\|_0$. Moreover, for $S' := \{e^+(\zeta)|\zeta \in E^{\wedge 2}\}$, $\text{span}_C[S']$ is a dense subspace of $E_+$ with respect to the topology of $E_+$.

Proof. We prove that $\text{span}_C[S']$ is a dense subspace of $E_+$ first. Let $\{e_i\}_{i=1}^\infty$ be a C.O.N.S. of $H$ satisfying the conditions (i) and (ii) in Lemma 2.1 and (2.1). For

$$\zeta := e_{i_1} \wedge e_{i_2} + \ldots + e_{i_{2n-1}} \wedge e_{i_{2n}} \in E^{\wedge 2},$$

we have

$$\zeta^{\wedge n} = n! e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{2n}}.$$  

We note that $\{(\lambda_{i_1} \ldots \lambda_{i_{2n}})^{-p} e_{i_1} \wedge \ldots \wedge e_{i_{2n}} | i_1 < \ldots < i_{2n}\}$ is a C.O.N.S. of a Hilbert space $E_p^{\wedge n}$. This implies that $\text{span}_C[\{\zeta^{\wedge n}|\zeta \in E^{\wedge 2}\}]$ is a dense subspace of $E_p^{\wedge (2n)}$ with respect to the norm $\|\|_p$ for any $p \geq 0$, i.e.,

$$E_p^{\wedge (2n)} = (E_p^{\wedge 2})^{\wedge n} = \text{span}_C[\{\zeta^{\wedge n}|\zeta \in E^{\wedge 2}\}]. \tag{2.2}$$

On the other hand, we can check that $\zeta^{\wedge n}$, $\zeta \in E^{\wedge 2}$ are elements of the closure of $\text{span}_C[S']$ with respect to the norm $\|\|_p$ for any $p \geq 0$, i.e.,

$$\zeta^{\wedge n} \in \text{span}_C[S']. \tag{2.3}$$

We show (2.3) by induction on $n \geq 0$. For $n = 0$, (2.3) follows from

$$\zeta^{\wedge 0} = 1 = e^+(0) \in S'.$$

Now we assume (2.3) for $n = 1, 2, \ldots, r$. Then $\zeta^{\wedge (r+1)} \in \text{span}_C[S']$ follows from

$$\zeta^{\wedge (r+1)} = (2r + 2)! \|\|_p \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{r+1}} \left(e^+(\varepsilon \zeta) - \sum_{n=0}^r \frac{1}{(2n)!} (\varepsilon \zeta)^{\wedge n}\right).$$

Thus we have (2.3) for all $n \geq 0$. (2.2) and (2.3) imply

$$\text{span}_C[S'] = \text{span}_C[\{\zeta^{\wedge n}|\zeta \in E_+, n \in \mathbb{Z}_{\geq 0}\}] = (D(\Gamma^+(A)^p), \|\|_p)$$

for all $p \geq 0$. Thus $\text{span}_C[S']$ is a dense subspace of $E_+$ with respect to the topology of $E_+$.

In the same manner, we can show that $\text{span}_C[S]$ is a dense subspace of $\Gamma^+(H)$ with respect to the norm $\|\|_0$. \hfill \qed

Definition 2.11. For $\Phi \in E_+$, we define a function $S\Phi$ on $E^{\wedge 2}$ as follows:

$$(S\Phi)(\zeta) := \langle \langle \Phi, e^+(\zeta) \rangle \rangle, \quad \zeta \in E^{\wedge 2}.$$  

Then we call $S\Phi$ the $S$-transform of $\Phi$.

Corollary 2.12. Let $\Phi = (\Phi_{2n})_{n=0}^\infty \in E^*_+$. Then

$$(S\Phi)(\zeta) = \sum_{n=0}^\infty \langle \Phi_{2n}, \zeta^{\wedge n} \rangle, \quad \zeta \in E^{\wedge 2}$$

and the right hand side converges absolutely.
Proof. If \( p \geq 0 \) satisfies \( \|\Phi\|_p < +\infty \), then
\[
\sum_{n=0}^{\infty} |\langle \Phi_{2n}, \zeta^\wedge n \rangle| \leq \sum_{n=0}^{\infty} \sqrt{(2n)!} |\Phi_{2n}|_p \frac{1}{\sqrt{(2n)!}} |\zeta^\wedge n|_p
\]
\[
\leq \left( \sum_{n=0}^{\infty} (2n)! |\Phi_{2n}|^2_p \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{1}{(2n)!} |\zeta^\wedge 2n|_p \right)^{\frac{1}{2}}
\]
\[
\leq \|\Phi\|_p \exp \left( \frac{1}{2} |\zeta^\wedge 2|_p \right) < +\infty.
\]

Proposition 2.13. For any \( \zeta, \eta \in E^\wedge 2 \) and \( \Phi \in E^*_+ \), a function
\[
C \ni z \mapsto S\Phi(z\zeta + \eta) \in C
\]
is holomorphic in \( C \).

Proof. Let \( \Phi = (\Phi_{2n})_{n=0}^\infty \). Then
\[
(S\Phi)(z\zeta + \eta) = \sum_{n=0}^{\infty} \langle \Phi_{2n}, (z\zeta + \eta)^\wedge n \rangle
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} z^k \langle \Phi_{2n}, \zeta^\wedge n_k \wedge \eta^\wedge (n-k) \rangle
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} \binom{n+k}{k} \langle \Phi_{2(n+k)}, \zeta^\wedge n \wedge \eta^\wedge n \rangle \right) z^k.
\]
Now let
\[
a_k := \sum_{n=0}^{\infty} \binom{n+k}{k} \langle \Phi_{2(n+k)}, \zeta^\wedge n \wedge \eta^\wedge n \rangle
\]
and we show that the radius of convergence \( R \) is infinite, that is,
\[
\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{\frac{1}{k}} = 0.
\]

\( |a_k| \) satisfies
\[
|a_k| \leq \sum_{n=0}^{\infty} \binom{n+k}{k}! \frac{(n+k)!}{n!k!} |\Phi_{2(n+k)}|_{-p} |\zeta^\wedge p|_p |\eta^\wedge p|_p
\]
\[
= \frac{|\zeta^\wedge p|_p}{k!} \sum_{n=0}^{\infty} \sqrt{(n+k)!} |\Phi_{2(n+k)}|_{-p} \frac{(n+k)!}{n!} |\eta^\wedge p|_p
\]
\[
\leq \frac{|\zeta^\wedge p|_p}{k!} \left( \sum_{n=0}^{\infty} (n+k)! |\Phi_{2(n+k)}|_{-p}^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} |\eta^\wedge 2n|_p \right)^{\frac{1}{2}}
\]
\[
\leq \frac{|\zeta^\wedge p|_p}{k!} \left( \sum_{n=0}^{\infty} (2n+2k)! |\Phi_{2(n+k)}|_{-p}^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} |\eta^\wedge 2n|_p \right)^{\frac{1}{2}}
\]
\[
= \frac{|\zeta^\wedge p|_p}{k!} \|\Phi\|_{-p} \left( \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} |\eta^\wedge 2n|_p \right)^{\frac{1}{2}}.
\]
Since we have
\[ \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} t^n \leq (t+k)^k \exp t \]
for all \( t \geq 0 \) and \( k \in \mathbb{Z}_{\geq 0} \) (see Lemma 3.2.9 of [5]), it holds that
\[ |a_k| \leq \left( \frac{\zeta_p^k}{k!} \right) \|\Phi\|_{-p} \left( |\eta_p|^2 + k \right)^{\frac{k}{2}} \exp \left( \frac{1}{2} |\eta_p|^2 \right). \]
If we note that \( k^k/k! \leq \exp k \), i.e.,
\[ \frac{1}{k!} \leq \left( \frac{1}{k} \right)^k \exp(k), \]
it holds that
\[ |a_k|^k \leq \left( \frac{\zeta_p}{k!} \right)^k \|\Phi\|_{-p} \frac{\sqrt{|\eta_p|^2 + k}}{k} \exp \left( 1 + \frac{1}{2k} |\eta_p| \right) \rightarrow 0 \text{ as } (k \rightarrow \infty). \]

3 Fock expansion for the even part of the fermion system

In order to discuss integral kernel operators, we define a contraction of tensor product.

**Definition 3.1.** Let \( H \) be a complex Hilbert space and \( A \) be a self-adjoint operator on \( H \) satisfying the conditions (i) and (ii) in Lemma 2.1 and (2.1). Let \( e(i) := e_{i_1} \otimes \ldots \otimes e_{i_l} \), \( i := (i_1, \ldots, i_l) \in \mathbb{N}^l \).

1. Let \( l, m \in \mathbb{N} \). For \( F \in \left( E^{\otimes (l+m)} \right)^* \), let
   \[ |F|_{l,m,p,q}^2 := \sum_{i,j} |\langle F, e(i) \otimes e(j) \rangle|^2 |e(i)|_{p}^2 |e(j)|_{q}^2 \]
   where \( i \) and \( j \) run the whole \( \mathbb{N}^l \) and \( \mathbb{N}^m \) respectively.

2. Let \( l, m, n \in \mathbb{N} \) and \( m \in \mathbb{Z}_{\geq 0} \). For \( F \in \left( E^{\otimes (l+m)} \right)^* \) and \( g \in E^{\otimes (m+n)} \), we define a left contraction \( F \otimes^m g \in \left( E^{\otimes (l+n)} \right)^* \) of \( F \) and \( g \) as follows.
   \[ F \otimes^m g := \sum_{j,k} \left( \sum_i \langle F, e(i) \otimes e(j) \rangle \langle g, e(i) \otimes e(k) \rangle \right) e(j) \otimes e(k) \]
   where \( i, j, \) and \( k \) run the whole \( \mathbb{N}^m, \mathbb{N}^l, \) and \( \mathbb{N}^n \) respectively. Similarly, we define a right contraction \( F \otimes_m g \in \left( E^{\otimes (l+n)} \right)^* \) of \( F \) and \( g \) as follows.
   \[ F \otimes_m g := \sum_{j,k} \left( \sum_i \langle F, e(j) \otimes e(i) \rangle \langle g, e(k) \otimes e(i) \rangle \right) e(j) \otimes e(k) \]
   where \( i, j, \) and \( k \) run the whole \( \mathbb{N}^m, \mathbb{N}^l, \) and \( \mathbb{N}^n \) respectively.
We check the well-definedness of the contraction. Let $\rho$ be the operator norm of $A^{-1}$, that is, $\rho$ is the inverse of the first eigenvalue of $A$. We remark that

$$|e(i)|_p |e(i)|_p = 1$$

and

$$|e(i)|_p \leq \rho^{nr} |e(i)|_{p+r}$$

for all $p \in \mathbb{R}$, $r \geq 0$ and $i \in \mathbb{N}^n$. Then we have

$$|F \otimes^m g|_p^2 $$

$$= |F \otimes^m g|_{l,n,p}^2 $$

$$= \sum_{j,k} \sum_i \langle F, e(i) \otimes e(j) \rangle \langle g, e(i) \otimes e(k) \rangle |e(i)|_{-q} |e(i)|_q^2 |e(j)|_p^2 |e(k)|_p^2 $$

$$\leq \sum_{j,k} \left( \sum_i |\langle F, e(i) \otimes e(j) \rangle|^2 |e(i)|_{-q}^2 \right) \times \left( \sum_i |\langle g, e(i) \otimes e(k) \rangle|^2 \rho^{2nr} |e(i)|_{q+r}^2 \right) |e(j)|_p^2 \rho^{2nr} |e(k)|_{p+r}^2 $$

$$= \rho^{2(m+n)r} |F|_{m,l,-q,p}^2 g|_{l,n,q+r,p+r}^2 $$

$$\leq \rho^{2(m+n)r} |F|_{m,l,-q,p}^2 g|_{\max\{p,q\}+r}^2 $$

for $F \in (E^{\otimes(l+m)})^*$ and $g \in E^{\otimes(m+n)}$. Therefore $F \otimes g \in (E^{\otimes(l+n)})^*$ and we obtain

**Lemma 3.2.** Let $F \in (E^{\otimes(l+m)})^*$ and $g \in E^{\otimes(m+n)}$. Then

$$|F \otimes^m g|_p \leq \rho^{(m+n)r} |F|_{m,l,-q,p} g|_{\max\{p,q\}+r}^2 $$

$$|F \otimes^m g|_p \leq \rho^{(m+n)r} |F|_{l,m,-q,p} g|_{\max\{p,q\}+r}^2 $$

for any $p \in \mathbb{R}$, $q \in \mathbb{R}$, and $r \geq 0$.

The following lemma is easily checked.

**Lemma 3.3.** For $F \in (E^{\otimes(l+m)})^*$ and $g \in E^{\otimes(l+n)}$, put

$$F \wedge^m g := A_{l+n}(F \otimes^m g), $$

$$F \wedge_m g := A_{l+n}(F \otimes_m g). $$

Then $F \wedge^m g$ and $F \wedge_m g$ are elements of $(E^{\wedge(l+n)})^*$. If $F \in (E^{\wedge(l+m)})^*$ and $g \in E^{\wedge(l+n)}$, then

$$F \wedge^m g = (-1)^{m(l+n)} F \wedge_m g. $$

Thus the left contraction $F \wedge^m g$ coincides with the right contraction $F \wedge_m g$ if $m$ is an even number.

Since a map $A_n : E^{\otimes n} \to E^{\wedge n}$ is a projection commuting with $A_{\otimes n}$, we can show the following lemma easily.
Lemma 3.4. Let $F \in (E \wedge (l+m))^*$, $g \in E \wedge (m+n)$. Then
\[
|F \wedge^m g|_p \leq \rho^{(m+n)r} |F|_{m, l-q, p} |g|_{\max\{p, q\}+r},
\]
\[
|F \wedge^l g|_p \leq \rho^{(m+n)r} |F|_{l, m-p, q} |g|_{\max\{p, q\}+r}
\]
for any $p \in \mathbb{R}$, $q \in \mathbb{R}$, and $r \geq 0$.

Put
\[
t_{i, m}(\psi) = \sum_{i,j} (\psi, e(i) \otimes e(j)) e(j) \otimes e(i),
\]
where $i, j$ run the whole $\mathbb{N}^l$ and $\mathbb{N}^m$ respectively.

Lemma 3.5. Let $\kappa \in (E \otimes (l+m))^*$, $\psi \in E \otimes (l+n)$ and $\phi \in E \otimes (m+n)$. Then
\[
\langle \kappa \otimes_m \phi, \psi \rangle = \langle \kappa, t_{i, n}(\psi) \otimes^n \phi \rangle.
\]

Proof. It is easily checked that
\[
((e(i) \otimes e(i')) \otimes_m (e(j) \otimes e(j')), (e(k) \otimes e(k'))
\]
\[
= \langle e(i) \otimes e(i'), (e(k) \otimes e(k)) \otimes^n (e(j) \otimes e(j'))\rangle
\]
for $i, k \in \mathbb{N}^l, i', j' \in \mathbb{N}^m, j, k' \in \mathbb{N}^n$. Thus
\[
\langle \kappa \otimes_m \phi, \psi \rangle := \sum \langle \kappa, e(i) \otimes e(i') \rangle \langle \phi, e(j) \otimes e(j') \rangle \langle \psi, e(k) \otimes e(k') \rangle
\]
\[
\times \langle (e(i) \otimes e(i')) \otimes_m (e(j) \otimes e(j')), (e(k) \otimes e(k')) \rangle
\]
\[
= \langle \kappa, t_{i, n}(\psi) \otimes^n \phi \rangle
\]

Corollary 3.6. For $\kappa \in (E \otimes (l+m))^*$, $\psi \in E \wedge (l+n)$ and $\phi \in E \otimes (m+n)$, we have
\[
\langle \kappa \otimes_{2m} \phi, \psi \rangle = \langle \kappa, \psi \otimes^{2n} \phi \rangle.
\]

Moreover, if $\phi \in E \wedge (m+n)$, then
\[
\langle \kappa \wedge_{2m} \phi, \psi \rangle = \langle \kappa, \psi \otimes^{2n} \phi \rangle.
\]

Proof. This corollary is easily checked.

We mention continuity of linear operators on locally convex spaces before discussing integral kernel operators.

Lemma 3.7. Let $X$ and $Y$ be locally convex spaces with seminorms $\{|| \cdot ||_{X, q}\}_{q \in Q}$ and $\{|| \cdot ||_{Y, p}\}_{p \in P}$ respectively. Let $\mathcal{L}(X, Y)$ be the set of all continuous linear operators from $X$ to $Y$. Then a linear operator $V$ from $X$ to $Y$ is in $\mathcal{L}(X, Y)$ if and only if, for any $p \in P$, there exist $q \in Q$ and $C > 0$ such that
\[
|Vx|_{Y, p} \leq C|x|_{X, q}, \quad x \in X.
\]

Now we define an integral kernel operator.
Proposition 3.8 (Integral kernel operator). Let $\kappa \in ((E^{\otimes 2(l+m)})^*)$. For $\phi := \sum_{n=0}^{\infty} \phi_n \in E_+^\otimes$, $\phi_n \in E^\otimes(2n)$, let

$$\Xi_{l,m}(\kappa) \phi := \sum_{n=0}^{\infty} \frac{(2n + 2m)!}{(2n)!} \kappa \wedge_{2m} \phi_{m+n}.$$  

Then

$$\| \Xi_{l,m}(\kappa) \phi \|_p \leq \rho^{-\frac{r}{2}} ((2l)^{2l}(2m)^{2m})^{\frac{1}{2}} \left( \frac{\rho^{-\frac{e}{2}}}{-re \log \rho} \right)^{l+m} |\kappa|^{2l,2m;p,-q} \| \phi \|_{\max\{p,q\}+r}$$

(3.1)

for $\phi \in E_+$, $p$, $q \in \mathbb{R}$, and $r > 0$. That is, $\Xi_{l,m}(\kappa) \in \mathcal{L}(E_+,E_+^*)$. We call $\Xi_{l,m}(\kappa)$ an integral kernel operator with a kernel distribution $\kappa$.

Proof. Let $p \in \mathbb{R}$, $q \in \mathbb{R}$, and $r > 0$. We have

$$\| \Xi_{l,m}(\kappa) \phi \|_p^2 = \sum_{n=0}^{\infty} \left( \frac{(2n + 2m)!}{(2n)!} \right)^2 (2n + 2l)! |\kappa \wedge_{2m} \phi_{m+n}|_p^2$$

$$\leq \sum_{n=0}^{\infty} \left( \frac{(2n + 2m)!}{(2n)!} \right)^2 (2n + 2l)! \rho^{4rn} |\kappa|_{2l,2m;p,-q}^2 \| \phi_{m+n} \|_{\max\{p,q\}+r}$$

$$= |\kappa|_{2l,2m;p,-q}^2 \sum_{n=0}^{\infty} \frac{(2n + 2m)!}{(2n)!} (2n + 2l)! \rho^{4rn} (2n + 2m)! |\phi_{m+n}|_{\max\{p,q\}+r}$$

$$\leq M^2 |\kappa|_{2l,2m;p,-q}^2 \| \phi \|_{\max\{p,q\}+r}^2,$$

where

$$M := \sup_{n \geq 0} \left( \frac{(2n + 2l)\ldots(2n + 2m)}{(2n)!} \right)^{\frac{1}{2}} \rho^{2rn}$$

$$\leq \left( \sup_{n \geq 0} (2n + 2l)\ldots(2n + 2m) \right)^{\frac{1}{2}} \left( \sup_{n \geq 0} (2n + 2m) \right) \rho^{2rn} \left( \sup_{n \geq 0} (2n + 2l)\ldots(2n + 2m) \right)^{\frac{1}{2}}.$$

Since

$$\sup_{x \geq 0} (x + m)\ldots(x + 1) \rho^{ex} \leq \rho^{\frac{e}{2}m} \left( \frac{\rho^{-\frac{e}{2}}}{-ce \log \rho} \right)^{m}$$

(3.2)

for any $c > 0$, $m \in \mathbb{N}$ (See Lemma 4.1.6 of [5]), we have

$$M \leq \rho^{-\frac{e}{2}} ((2l)^{2l}(2m)^{2m})^{\frac{1}{2}} \left( \frac{\rho^{-\frac{e}{2}}}{-re \log \rho} \right)^{l+m}.$$  

Therefore (3.1) holds.

Note that the following map

$$(E^{\otimes 2(l+m)})^* \ni \kappa \mapsto \Xi_{l,m}(\kappa) \in \mathcal{L}(E_+,E_+^*)$$

is not injective. We define

$$\mathcal{A}_{l,m}(\kappa) := \frac{1}{l!m!} \sum_{\sigma=(\sigma_1,\sigma_2) \in \Theta_l \times \Theta_m} \text{sign}(\sigma_1)\text{sign}(\sigma_2)\kappa^\sigma,$$
where $\kappa^\sigma$ is defined in definition 2.2 (3). Put

$$(E^{\otimes(l+m)})^*_{\text{alt}(l,m)} := \{ \kappa \in (E^{\otimes(l+m)})^* \mid A_{l,m}(\kappa) = \kappa \}.$$  

(“alt” stands for “alternative”.)

**Lemma 3.9.** The map

$$((E^{\wedge 2})^{\otimes(l+m)})^*_{\text{alt}(2l,2m)} \ni \kappa \mapsto \Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{E}_+, \mathcal{E}_+^*)$$

is injective. Moreover, for $\kappa \in ((E^{\wedge 2})^{\otimes(l+m)})^*$ and $\kappa' \in ((E^{\wedge 2})^{\otimes(l'+m')})$, if $\Xi_{l,m}(\kappa) = \Xi_{l',m'}(\kappa')$, then $l = l'$, $m = m'$, and $A_{2l,2m}(\kappa) = A_{2l',2m'}(\kappa')$.

**Proof.** See proposition 4.3.6 of [5].

We also note the following corollary of Proposition 3.8

**Corollary 3.10.** Let $\kappa \in ((E^{\wedge 2})^{\otimes(l+m)})^*$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{E}_+, \Gamma^+(H))$ if and only if $\kappa$ is in $H^{\wedge(2l)} \otimes (E^{\wedge(2m)})^*$. In other words, $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{E}_+, \mathcal{E}_+^*)$ is extended to an element of $\mathcal{L}(\Gamma^+(H), \mathcal{E}_+^*)$ if and only if $\kappa$ is in $(E^{\wedge(2l)})^* \otimes H^{\wedge(2m)}$.

**Proof.** Let $\kappa$ be in $H^{\wedge(2l)} \otimes (E^{\wedge(2m)})^*$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{E}_+, \Gamma^+(H))$ follows from (3.1). Conversely, let $\Xi_{l,m}(\kappa) \in \mathcal{L}(\mathcal{E}_+, \Gamma^+(H))$. From the definition of an integral kernel operator, we have

$$\langle \Xi_{l,m}(\kappa), \psi_m \rangle = (2l)! (2m)! \langle \kappa, \phi_l \otimes \psi_m \rangle$$  

(3.3)

for all $\phi_l \in E^{\wedge(2l)}$, $\psi_m \in E^{\wedge(2m)}$. Due to continuity of $\Xi_{l,m}(\kappa)$, the left hand side of (3.3) is defined for $\phi_l \in H^{\wedge(2l)}$. Therefore $\kappa$ is in $H^{\wedge(2l)} \otimes (E^{\wedge(2m)})^*$.

We are now able to see the Fock expansion for the even part of the Fermion system.

**Theorem 3.11 (Fock expansion).** For any $\Xi \in \mathcal{L}(\mathcal{E}_+, \mathcal{E}_+^*)$, there exists a unique $\{ \kappa_{l,m} \}_{l,m=0}^\infty$, $\kappa_{l,m} \in ((E^{\wedge 2})^{\otimes(l+m)})^*_{\text{alt}(2l,2m)}$ such that

$$\Xi \phi = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m}) \phi, \quad \phi \in \mathcal{E}_+$$  

(3.4)

where the right hand side of (3.4) converges in $\mathcal{E}_+^*$.

If $\Xi \in \mathcal{L}(\mathcal{E}_+, \mathcal{E}_+)$, then

$$\kappa_{l,m} \in E^{\wedge(2l)} \otimes (E^{\wedge(2m)})^*, \quad l, m \geq 0$$

and the right hand side of (3.4) converges in $\mathcal{E}_+$.

We prove Theorem 3.11 in this section and the following section. First, we prove algebraic part of Theorem 3.11. We define a contraction operator.

**Lemma 3.12.** Let

$$c(l, m; n)(x) := \sum_{j,k} \left( \sum_i (x, (e(i) \otimes e(j)) \otimes (e(i) \otimes e(k))) e(j) \otimes e(k) \right),$$

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where \( x \in E^{\otimes (l+m)} \) and \( i, j, k \) run the whole \( N^n, N^{l-n}, N^{m-n} \) respectively. We call
\[
c(l, m; n) \in \mathcal{L}(E^{\otimes (l+m)}, E^{\otimes (l+m-2n)})
\]
a contraction operator. Contraction operators satisfy the following relation.
\[
c(l - n_1, m - n_1; n_2) c(l, m; n_1) = c(l, m; n_1 + n_2).
\]
Moreover,
\[
c(l, m; n)^*|_{E^{\otimes l} \otimes (E^{\otimes (m-n)})^*} \in \mathcal{L}(E^{\otimes (l-n)} \otimes (E^{\otimes (m-n)})^*, E^{\otimes l} \otimes (E^{\otimes m})^*).
\]

**Proof.** Since
\[
|\langle x, c(i) \otimes e(j) \otimes e(i) \otimes e(k) \rangle| \leq |x|_{p+\alpha} |e(i) \otimes e(j) \otimes e(i) \otimes e(k)|_{(p+\alpha)}
\]
\[
\leq |x|_{p+\alpha} |e(i)|_{-\alpha} |e(j)|_{-\alpha} |e(k)|_{-\alpha}
\]
for \( x \in E^{\otimes (l+m)}, p \geq 0, i \in N^n, j \in N^{l-n}, \) and \( k \in N^{m-n} \), we have
\[
|c(l, m; n)(x)|^2_{l,m;p,q} = \sum_{j,k} \sum_{i} \langle x, e(i) \otimes e(j) \otimes e(i) \otimes e(k) \rangle^2 |e(i)|^2_{p} |e(j)|^2_{q} |e(k)|^2_{q}
\]
\[
\leq |x|_{p+\alpha}^2 \sum_{j,k} \sum_{i} |e(i)|^2_{-\alpha} |e(j)|^2_{-\alpha} |e(k)|^2_{-\alpha}
\]
\[
= \delta^{2(l-n)+2(m-n)+2n} |x|_{p+\alpha}^2.
\]
This implies continuity of the linear operator \( c(l, m; n) \) from \( E^{\otimes (l+m)} \) to \( E^{\otimes (l+m-2n)} \).

Next, we consider \( c(l, m; n)^* \). For each \( p > 0 \), there exists \( q \in \mathbb{R} \) such that \( |x|_{l-n,m-n,p,-\max\{p+\alpha,q\}} - \alpha \) is finite. Then
\[
|c(l, m; n)^*(x)|^2_{l,m;p,-}\max\{p+\alpha,q\} - \alpha
\]
\[
= \sum_{i,j,k,k'} |\langle x, c(l, m; n)(e(i) \otimes e(j) \otimes e(i') \otimes e(k) \rangle|^2
\]
\[
\times |e(i) \otimes e(j) \otimes e(i') \otimes e(k)|^2_{l,m;p,-\max\{p+\alpha,q\} - \alpha}
\]
\[
= |x|_{l-n,m-n,p,-\max\{p+\alpha,q\}} - \alpha \sum_{k,k'} |\langle e(i), e(i') \rangle|^2 |e(i)|^2_{p} |e(i')|^2_{p + \alpha - \max\{p+\alpha,q\}} - \alpha
\]
\[
\leq |x|_{l-n,m-n,p,-\max\{p+\alpha,q\}} - \alpha \sum_{k,k'} |e(i)|^2_{-(p+\alpha)} |e(i')|^2_{p + \alpha - \max\{p+\alpha,q\}} - \alpha
\]
\[
\leq \delta^{4n} |x|_{l-n,m-n,p,-\max\{p+\alpha,q\}} - \alpha.
\]
for all \( x \in E^{\otimes (l-n)} \otimes (E^{\otimes (m-n)})^* \). Here \( i, i', j, k \) run the whole \( N^n, N^n, N^{l-n} \) and \( N^{m-n} \) respectively. Therefore the restriction of \( c^* \otimes (l, m; n) \) to \( E^{\otimes (l-n)} \otimes (E^{\otimes (m-n)})^* \) is continuous linear operator from \( E^{\otimes (l-n)} \otimes (E^{\otimes (m-n)})^* \) to \( E^{\otimes l} \otimes (E^{\otimes m})^* \).

**Definition 3.13.** Let \( \Xi \in \mathcal{L}(E^+, E^*) \). We define a continuous linear functional \( \kappa_{l,m} \) on \( E^{\otimes (2l)} \otimes E^{\otimes (2m)} \), i.e., \( \kappa_{l,m} \in (E^{\otimes (2l)} \otimes E^{\otimes (2m)})^* \), \( l, m \in \mathbb{Z}_{\geq 0} \) inductively as follows:
\[
\langle \kappa_{l,m}, x \rangle := \frac{1}{(2l)!(2m)!} \langle \Xi \otimes (1)^* \alpha, x \rangle - \sum_{n=1}^{\min(l,m)} \frac{1}{(2n)!} \langle \kappa_{l-n,m-n}, c(2l, 2m; 2n)(x) \rangle \quad (3.5)
\]
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for \( x \in E^{\langle 2l \rangle} \otimes E^{\langle 2m \rangle} \). Here \( \tau \in \mathcal{E}_+ \otimes \mathcal{E}_+^* \) is defined by

\[
\tau(\Psi \otimes \phi) = \langle \langle \Psi, \phi \rangle \rangle, \quad \Psi \in \mathcal{E}_+^*, \ \phi \in \mathcal{E}_+.
\]

(Continuity of \( \kappa_{l,m} \) is considered in the following section.) When \( x = \psi_l \otimes \phi_m \in E^{\langle 2l \rangle} \otimes E^{\langle 2m \rangle} \), (3.5) implies

\[
\langle \langle \Xi, \psi_l \rangle \rangle = \sum_{l,m=0}^{\infty} \langle \langle \Xi, \psi_l \rangle \rangle
\]

This shows

\[
\langle \langle \Xi, \psi \rangle \rangle = \sum_{l,m=0}^{\infty} \langle \langle \Xi, \psi_l \rangle \rangle
\]

for \( \phi = \sum_{m} \phi_{m}, \ \psi = \sum_{l} \psi_l \in \mathcal{E}, \ \phi_{m} \in E^{\langle 2m \rangle}, \ \psi_l \in E^{\langle 2l \rangle} \). Thus we obtain the formal or algebraic part of Theorem 3.11. It is helpful to give another algebraic expression of \( \kappa_{l,m} \) before proving the analytic part of Theorem 3.11. This expression is used in Lemma 4.9.

Let

\[
K_{l,m} := \sum_{n=0}^{\min(l,m)} \frac{(2l)! (2m)!}{(2n)!} c(2l, 2m; 2n) (\kappa_{l-n,m-n} \wedge 2m-2n \phi_m, \psi_l)
\]

for all \( l, m \in \mathbb{Z}_{\geq 0} \). (\( K_{l,m} \) is related to the symbol of \( \Xi \). The “symbol” of \( \Xi \) is defined in Definition 4.1.)

**Lemma 3.14.** Let

\[
\Pi_k := \{(k_1, \ldots, k_t) \in \mathbb{N}^t \mid t \in \{1, 2, \ldots, \min\{l, m\}\}, k_1 + \ldots + k_t = k\}
\]

for \( k \in \mathbb{N} \). For \( (k_1, \ldots, k_t) \in \Pi_k \), put

\[
A(k_1, \ldots, k_t) := \frac{1}{(2k_1)! \ldots (2k_t)!} c(2l, 2m; 2k) (K_{l-k,m-k})^*, \quad a(k_1, \ldots, k_t) := \frac{1}{(2k_1)! \ldots (2k_t)!} c(2l, 2m; 2k) (\kappa_{l-k,m-k})^*.
\]
(1) It holds that 

\[ a(k_1, \ldots, k_t) = \frac{1}{(2l - 2k)! (2m - 2k)!} A(k_1, \ldots, k_t) - \sum_{n=1}^{\min\{l - k, m - k\}} a(k_1, \ldots, k_t, n). \]

(2) From (1), we have

\[ \kappa_{l,m} = \frac{1}{(2l)! (2m)!} K_{l,m} \]

\[ + \sum_{k=1}^{\min\{l,m\}} \frac{1}{(2l - 2k)! (2m - 2k)!} \left( \sum_{(k_1, \ldots, k_t) \in \Pi_k} (-1)^t \right) \]

\[ \times c(2l, 2m; 2k)^* (K_{l-k,m-k}) \]

for all \( l, m \in \mathbb{Z}_{\geq 0} \) with \( \min\{l, m\} \geq 1 \).

\[ \kappa_{l,m} = \frac{1}{(2l)! (2m)!} K_{l,m} \]

holds if \( \min\{l, m\} = 0 \).

Proof. (1) Since

\[ \kappa_{l,m} = \frac{1}{(2l)! (2m)!} K_{l,m} - \sum_{n=1}^{\min\{l,m\}} \frac{1}{(2n)!} c(2l, 2m; 2n)^* (\kappa_{l-n,m-n}) \quad (3.7) \]

from the definition of \( K_{l,m} \), we have

\[ a(k_1, \ldots, k_t) \]

\[ = \frac{1}{(2k_1)! \ldots (2k_t)!} c(2l, 2m; 2k)^* (\kappa_{l-k,m-k}) \]

\[ = (2k_1)! \ldots (2k_t)! c(2l, 2m; 2k)^* \left\{ \frac{1}{(2l - 2k)! (2m - 2k)!} (K_{l-k,m-k}) \right. \]

\[ - \sum_{n=1}^{\min\{l-k,m-k\}} \frac{1}{(2n)!} c(2l - 2k, 2m - 2k; 2n)^* (\kappa_{l-k,n,m-n}) \} \right\} \]

\[ = \frac{1}{(2l - 2k)! (2m - 2k)!} A(k_1, \ldots, k_t) - \sum_{n=1}^{\min\{l-k,m-k\}} a(k_1, \ldots, k_t, n). \]
(2) is proved immediately since we have (3.7) and can show
\[
c(2l,2m;2k_1)^{*}(\kappa_{l-k_1,m-k_1})
= a(k_1)
= \frac{1}{(2l - 2k_1)!(2m - 2k_1)!} A(k_1)^
\min\{l-k_1,m-k_1\}
\times \sum_{k_2=1}^{\min\{l-k_1,m-k_1\}} A(k_1,k_2)
+ \ldots
(-1)^{l-1} \frac{1}{(2l - 2k_1 - \ldots - 2k_t)!(2m - 2k_1 - \ldots - 2k_t)!}
\times \sum_{k_2=1}^{\min\{l-k_1,m-k_1\}} \ldots \sum_{k_t=1}^{\min\{l-k_1-\ldots-k_{t-1},m-k_1-\ldots-k_{t-1}\}} \frac{1}{(2k_2)! \ldots (2k_t)!} A(k_1,\ldots,k_t)
\]
by using (1) inductively.

4 The proof of the analytic part of Theorem 3.11

In this section, we prove the analytic part of Theorem 3.11.

Definition 4.1. For \( \Xi \in \mathcal{L}(E_+, E_+^*) \), let
\[
\hat{\Xi}(\zeta, \eta) := \langle \langle \Xi e^+(\zeta), e^+(\eta) \rangle \rangle, \quad \zeta, \eta \in E^\wedge 2.
\]
Then we call \( \hat{\Xi} \) the symbol of \( \Xi \).

Proposition 4.2. For any \( \zeta_1, \zeta_2, \eta_1, \eta_2 \in E^\wedge 2 \), a function
\[
C^2 \ni (z,w) \mapsto \hat{\Xi}(z\zeta_1 + \zeta_2, w\eta_1 + \eta_2) \in C
\]
is holomorphic in \( C^2 \).

Proof. This statement follows from proposition 2.13 and
\[
\hat{\Xi}(z\zeta_1 + \zeta_2, w\eta_1 + \eta_2) = S(\Xi e^+(z\zeta_1 + \zeta_2))(w\eta_1 + \eta_2)
= S(\Xi^* e^+(w\eta_1 + \eta_2))(z\zeta_1 + \zeta_2).
\]
Moreover, symbol \( \hat{\Xi} \) satisfies the two following lemmas.

Lemma 4.3.

1. Let \( \Xi \in \mathcal{L}(E_+, E_+^*) \) and \( r \geq 0 \). Then there exist \( p \in \mathbb{R}, q \in \mathbb{R} \) and \( C_0 > 0 \) such that
\[
|\hat{\Xi}(\zeta, \eta)| \leq C_0 \exp \left[ \frac{\rho^{dr}}{8} (|\zeta|^2_{\max\{p,q\}+r} + |\eta|^2_{-p}) \right]. \tag{4.1}
\]
(2) Let $Ξ ∈ L(E^+, E^+)$ and $r ≥ 0$. Then, for any $p ≥ 0$, there exist $q > 0$, and $C_0 > 0$ satisfying (4.1).

Proof. We remark that

$$\|e^+(ζ)\|_α^2 = \sum_{n=0}^{∞} (2n)! \left| \frac{ζ^n}{(2n)!} \right|_α^2 ≤ \sum_{n=0}^{∞} \frac{1}{n!} \left( \frac{1}{4} |ζ|_α^2 \right)^n = \exp \left( \frac{1}{4} |ζ|_α^2 \right)$$

for all $α ∈ \mathbb{R}$ and $ζ ∈ E^{∧^2}$.

(1) For $r ≥ 0$, there exists $p ≤ −r$ and $C_0 > 0$ such that

$$\|Ξe^+(ζ)\|_{p+r} ≤ C_0\|e^+(ζ)\|_{−(p+r)} ≤ C_0 \exp \left( \frac{1}{8} |ζ|_{−(p+r)}^2 \right) ≤ C_0 \exp \left( \frac{ρ_{4r}}{8} |ζ|_{−p}^2 \right)$$

for all $ζ ∈ E^{∧^2}$. Thus

$$|ˆΞ(ζ, η)| ≤ \|Ξe^+(ζ)\|_{p+r}\|e^+(η)\|_{−(p+r)} ≤ C_0 \exp \left[ \frac{ρ_{4r}}{8} (|ζ|_{−p}^2 + |η|_{−p}^2) \right] ≤ C_0 \exp \left[ \frac{ρ_{4r}}{8} (|ζ|_{\max(p, −p)}^2 + |η|_{−p}^2) \right].$$

(2) For any $r ≥ 0$ and $p ≥ 0$, there exist $q > 0$ and $C_0 > 0$ such that

$$\|Ξe^+(ζ)\|_{p+r} ≤ C_0\|e^+(ζ)\|_q ≤ C_0 \exp \left( \frac{1}{8} |ζ|_q^2 \right) ≤ C_0 \exp \left( \frac{ρ_{4r}}{8} |ζ|_{q+r}^2 \right).$$

Hence

$$|ˆΞ(ζ, η)| ≤ \|Ξe^+(ζ)\|_{p+r}\|e^+(η)\|_{−(p+r)} ≤ C_0 \exp \left[ \frac{ρ_{4r}}{8} (|ζ|_{q+r}^2 + |η|_{−p}^2) \right] ≤ C_0 \exp \left[ \frac{ρ_{4r}}{8} (|ζ|_{\max(q, p)}^2 + |η|_{−p}^2) \right].$$

Now we have to remark the following lemma.

Lemma 4.4. Let $f$ be a holomorphic function on $\mathbb{C}$ with Taylor expansion

$$f(z, w) = \sum_{l,m=0}^{∞} a_{l,m} z^l w^m$$

and let $f$ satisfy

$$|f(z, w)| ≤ C \exp(K_1 |z|^2 + K_2 |w|^2), \quad z, w ∈ \mathbb{C}$$

for some $C ≥ 0$ and $K_1, K_2 ≥ 0$. Then

$$|a_{l,m}| ≤ C \left( \frac{2eK_1}{l} \right)^{\frac{l}{2}} \left( \frac{2eK_2}{m} \right)^{\frac{m}{2}}.$$
Lemma 4.5. Let \( K_{i,m} \) be elements of \((E^{\lambda(2l)} \otimes E^{\lambda(2m)})^*\) given by (3.6). Let \( p, q \in \mathbb{R}, \ r \geq 0, C_0 > 0 \) be numbers given in Lemma 4.3. Then it holds that

\[
|\langle K_{l,m}, \eta^l \otimes \zeta^m \rangle| \leq C_0 \left( \frac{e^4r}{4l} \right)^l \left( \frac{e^4r}{4m} \right)^m |\eta|_{-p}^l |\zeta|_{\max\{p+q\}+r}^m \tag{4.2}
\]

for any \( \zeta, \eta \in E^{\lambda(2)} \) and \( l, m \geq 0 \). Therefore we have

\[
|K_{i,m}|_{2i,2m;p,\max\{p+q\}+r} \leq C_0 (e^2r)^{l+m} \left( \frac{1}{(2l)!(2m)!} \right)^{l+m} \tag{4.3}
\]

Proof. (4.2) follows from Lemma 4.3, Lemma 4.4 and

\[
\hat{E}(z \zeta, w \eta) = \sum_{l,m=0}^{\infty} \left( \frac{\zeta^l \times \eta^m}{(2m)! (2l)!} \right) w^l z^m = \sum_{l,m=0}^{\infty} \langle K_{l,m}, \eta^l \otimes \zeta^m \rangle w^l z^m
\]

for any \( z, w \in \mathbb{C} \). We prove (4.3). Let \( e_{i_1}, \lambda_{i_1} \) be C.O.N.S. and eigenvalues given in Definition 2.1 respectively. Now fix \( i_1 < \ldots < i_{2l}, j_1 < \ldots < j_{2m} \) and we put

\[
\eta = (\lambda_{i_1} \ldots \lambda_{i_{2l}})^{-p+\alpha} \sum_{s=1}^{l} (\lambda_{i_{2s-1}} \lambda_{i_{2s}})^{p+\alpha} e_{i_{2s-1}} \wedge e_{i_{2s}}
\]

\[
\zeta = (\lambda_{j_1} \ldots \lambda_{j_{2m}})^{\max\{p+q\}+r} \sum_{t=1}^{m} (\lambda_{j_{2t-1}} \lambda_{j_{2t}})^{-\max\{p+q\}+r} e_{j_{2t-1}} \wedge e_{j_{2t}}
\]

Then

\[
e_{i_1} \wedge \ldots \wedge e_{i_{2l}} = \frac{1}{l!} (\lambda_{i_1} \ldots \lambda_{i_{2l}})^{(p+\alpha)(l-1)} \eta^l,
\]

\[
e_{j_1} \wedge \ldots \wedge e_{j_{2m}} = \frac{1}{m!} (\lambda_{j_1} \ldots \lambda_{j_{2m}})^{-\max\{p+q\}+r} (m-1) \zeta^m,
\]

\[
|\eta|_{-p}^{l} = l! (\lambda_{i_1} \ldots \lambda_{i_{2l}})^{-2(p+\alpha)l},
\]

\[
|\zeta|_{\max\{p+q\}+r}^{m} = m! (\lambda_{j_1} \ldots \lambda_{j_{2m}})^{2(\max\{p+q\}+r)m}.
\]

Thus, from (4.2), we have

\[
|\langle K_{l,m}, (e_{i_1} \wedge \ldots \wedge e_{i_{2l}}) \otimes (e_{j_1} \wedge \ldots \wedge e_{j_{2m}}) \rangle|^2 \leq C_0^2 \left( \frac{1}{l!m!} \right)^2 \left( \frac{e^4r}{l!} \right)^l \left( \frac{e^4r}{m!} \right)^m l!m!
\]

\[
\times (\lambda_{i_1} \ldots \lambda_{i_{2l}})^{-2(p+\alpha)} (\lambda_{j_1} \ldots \lambda_{j_{2m}})^{2(\max\{p+q\}+r)}
\]

\[
\leq C_0^2 \left( e^{4r} \right)^{l+m-1} \left( \frac{1}{(2l)!(2m)!} \right) (\lambda_{i_1} \ldots \lambda_{i_{2l}})^{-2(p+\alpha)} (\lambda_{j_1} \ldots \lambda_{j_{2m}})^{2(\max\{p+q\}+r)}.
\]

Here we used the following inequality

\[
(2l)! = 2l \cdot (2l-1) \cdot \ldots \cdot 1 \leq 2l \cdot 2l \cdot \ldots \cdot 2 = (2^l l^l)^2.
\]
Lemma 4.6. Therefore we obtain (4.4) by using

\[ |K_{l,m}|_{2l,2m;p,-(\max\{p+\alpha,q\}+r+\alpha)}^2 = \sum_{i_1<\cdots<i_{2l},j_1<\cdots<j_{2m}} |\langle K_{l,m}, (e_{i_1} \& \ldots \& e_{i_{2l}}) \otimes (e_{j_1} \& \ldots \& e_{j_{2m}}) \rangle|^2 \times |e_{i_1} \& \ldots \& e_{i_{2l}}|^2 |e_{j_1} \& \ldots \& e_{j_{2m}}|^2 \leq C_0^2 (\rho r)^{t+m} \frac{1}{(2l)! (2m)!} \sum_{i_1<\cdots<i_{2l},j_1<\cdots<j_{2m}} (\lambda_{i_1} \& \ldots \& \lambda_{i_{2l}})^{-2\alpha} (\lambda_{j_1} \& \ldots \& \lambda_{j_{2m}})^{-2\alpha} \]

\[ \leq C_0^2 (\rho \delta r)^{t+m} \frac{1}{(2l)! (2m)!}. \]

Next, we discuss estimations for \( c(2l,2m;2k)^*(K_{l-k,m-k}) \), \( k = 1, 2, \ldots, \min\{l,m\} \).

**Lemma 4.6.**

1. Let \( \lambda_{l,m} \in (E^{\wedge(2l)} \otimes E^{\wedge(2m)})^* \) and \( r \geq 0 \). Then there exist \( p \in \mathbb{R} \) and \( q \in \mathbb{R} \) with \( |\lambda_{l,m}|_{2l,2m;p+r,-q} < +\infty \) and we have

\[ |\Xi_{l,m}(\lambda_{l,m})(\zeta, \eta)| \leq (l! m!)^{\frac{1}{2}} |\lambda_{l,m}|_{2l,2m;p+r,-q} \exp \left( \frac{\rho r^4}{2} (|\zeta|_{\max\{p,q\}+2r}^2 + |\eta|_{-p}^2) \right). \quad (4.4) \]

2. Let \( \lambda_{l,m} \in E^{\wedge(2l)} \otimes (E^{\wedge(2m)})^* \) and \( r \geq 0 \). Then, for any \( p \geq 0 \), there exists \( q \in \mathbb{R} \) satisfying (4.4).

**Proof.** (1) From definition of an integral kernel operator, we have

\[ |\Xi_{l,m}(\lambda_{l,m})(\zeta, \eta)| = \left| \sum_{n=0}^{\infty} (2n)! \langle \lambda_{l,m} \wedge 2n \zeta^{\wedge(m+n)}, \eta^{\wedge(l+n)} \rangle \right| \]

\[ \leq \sum_{n=0}^{\infty} \frac{(2n)!}{(2n)!} |\lambda_{l,m} \wedge 2n \zeta^{\wedge(m+n)}|_{p+r} \eta^{\wedge(l+n)}|_{-(p+r)} \]

\[ \leq \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left( \rho^{2(m+n)r} |\lambda_{l,m}|_{2l,2m;p+r,-q} \zeta^{\wedge(m+n)}|_{\max\{p+r,q\}+r} \right) \left( \rho^{2(l+n)r} |\eta|_{-(l+n)} \right) \]

\[ \leq |\lambda_{l,m}|_{2l,2m;p+r,-q} \rho^{4r} \zeta_{\max\{p,q\}+2r}^n \eta_{-p}^n \frac{\sqrt{n}}{(2n)!} \left( \sum_{n=0}^{\infty} \frac{1}{(2n)!} \rho^{4nr} |\zeta|_{\max\{p,q\}+2r}^n \eta_{-p}^n \right). \]

Therefore we obtain (4.4) by using

\[ \sum_{n=0}^{\infty} \frac{1}{(2n)!} \rho^{4nr} |\zeta|_{\max\{p,q\}+2r}^n \eta_{-p}^n \leq \sum_{n=0}^{\infty} \left( \frac{\rho^{2r} |\zeta|_{\max\{p,q\}+2r}^n}{(n!)^{\frac{1}{2}}} \cdot \frac{\rho^{2r} |\eta|_{-p}^n}{(n!)^{\frac{1}{2}}} \right) \]

\[ \leq \left\{ \sum_{n=0}^{\infty} \left( \frac{\rho^{2r} |\zeta|_{\max\{p,q\}+2r}^n}{n!} \right) \right\}^{\frac{1}{2}} \left\{ \sum_{n=0}^{\infty} \left( \frac{\rho^{2r} |\eta|_{-p}^n}{n!} \right) \right\}^{\frac{1}{2}} \]

\[ = \exp \left( \frac{\rho^{2r} |\zeta|_{\max\{p,q\}+2r}^2}{2} \right) \exp \left( \frac{\rho^{2r} |\eta|_{-p}^2}{2} \right). \]
and
\[(\rho^{4r}|\zeta|_{\text{max}(p,q)+2}^2)^m = m! \cdot \frac{1}{m!}(\rho^{4r}|\zeta|_{\text{max}(p,q)+2}^2)^m \leq m! \exp(\rho^{4r}|\zeta|_{\text{max}(p,q)+2}^2).
\]

(2) is easily checked in the same manner as (1).

**Lemma 4.7.** Let \(\lambda_{l,m}\) be an element of \((E^{(2l)} \otimes E^{(2m)})^*\) and \(p, q \in \mathbb{R}\), \(r \geq 0\) be numbers given in Lemma 4.6. Then it holds that
\[
|\langle (c(2l + 2n, 2m + 2n; 2n)^*(\lambda_{l,m}), \eta^{(l+n)} \otimes \zeta^{(m+n)}\rangle|
\[
\leq (l!m!)^{\frac{1}{2}}|\lambda_{l,m}|^{|2l,2m;p+r−q|} \left(\frac{e\rho^{4r}}{l+n}\right)^{\frac{|l+n|}{2}} \left(\frac{e\rho^{4r}}{m+n}\right)^{\frac{|m+n|}{2}} |\eta|_{−p} |\zeta|_{\text{max}(p,q)+2r} \tag{4.5}
\]
for all \(\zeta, \eta \in E^{\otimes 2}\).

**Proof.** Since
\[
\Xi(\lambda_{l,m})(z\zeta, w\eta) = \sum_{l,m=0}^{\infty} \langle (c(2l + 2n, 2m + 2n; 2n)^*(\lambda_{l,m}), \eta^{(l+n)} \otimes \zeta^{(m+n)}\rangle w^{l+n} z^{m+n}
\]
for \(z, w \in \mathbb{C}\), Lemma 4.6 and Lemma 4.4 imply (4.5).

**Lemma 4.8.** Let \(\lambda_{l−k,m−k}\) be an element of \((E^{(2l−2k)} \otimes E^{(2m−2k)})^*\) and \(p, q \in \mathbb{R}\), \(r \geq 0\) be numbers given in Lemma 4.6. Then
\[
|\langle (c(2l, 2m; 2k)^*(\lambda_{l−k,m−k}), (e_{i_1} \wedge \ldots \wedge e_{i_{2l}}) \otimes (e_{j_1} \wedge \ldots \wedge e_{j_{2m}})\rangle|^2
\[
\leq (l−k)!|m−k|!|\lambda_{l−k,m−k}|^2_{(l−k),2(m−k);p+r−q} \left(\frac{e\rho^{4r}}{l}\right)^{l} \left(\frac{e\rho^{4r}}{m}\right)^{m} \times \left(\frac{1}{l!m!}\right)^2 l^m m_{(\lambda_{i_1} \ldots \lambda_{i_{2l}})^{−2(p+\alpha)} (\lambda_{j_1} \ldots \lambda_{j_{2m}})^{2(\text{max}(p+q)+2r)}}
\]
\[
\leq \frac{1}{(2l)!(2m)!} (l−k)!|m−k|!|\lambda_{l−k,m−k}|^2_{(l−k),2(m−k);p+r−q} (4e\rho^{4r})^{l+m} \times (\lambda_{i_1} \ldots \lambda_{i_{2l}})^{−2(p+\alpha)} (\lambda_{j_1} \ldots \lambda_{j_{2m}})^{2(\text{max}(p+q)+2r)}
\]
(See Lemma 4.5) Thus we can show (4.6) immediately.

We are now ready to give the estimation for \(\kappa_{l,m}\). Recall that \(\kappa_{l,m}\) is determined by \(\Xi \in \mathcal{L}(\mathcal{E}_+, \mathcal{E}_+^*)\) via (3.5)

**Lemma 4.9.**

(1) Let \(\Xi \in \mathcal{L}(\mathcal{E}_+, \mathcal{E}_+^*)\) and \(r \geq 0\). Then there exist \(p \in \mathbb{R}\), \(q \in \mathbb{R}\), \(C_1 > 0\) and \(C_2 > 0\) such that
\[
|\kappa_{l,m}|_{2l,2m;p−\text{max}(p+r+q)+2r−2\alpha} \leq C_1(C_2\rho^{2r})^{l+m} \left(\frac{1}{(2l)!(2m)!}\right)^{\frac{1}{2}} \tag{4.7}
\]
(2) Let $\Xi \in \mathcal{L}(\mathcal{E}_+)$ and $r > 0$. Then, for any $p \geq 0$, there exist $q > 0$, $C_1 > 0$, and $C_2 > 0$ satisfying (4.7).

Proof. We see only (1). From Lemma 4.5, we can take $p \in \mathbb{R}$, $q \in \mathbb{R}$, and $C_0 > 0$ satisfying

$$|K_{l,k,m-k}|_{2(l-k),2(m-k);p+r,-Q} \leq C_0(e^{\delta^4}p^r)^{\frac{l+m-2k}{2}} \left(\frac{1}{(2l-2k)!(2m-2k)!}\right)^{\frac{1}{2}}$$

for all $l$, $m \geq 0$, $0 \leq k \leq \min\{l, m\}$, where

$$Q := \max\{p + r + \alpha, q\} + r + \alpha.$$

Hence (4.6) implies

$$|c(2l, 2m; 2k)^{(k-l,m-k)}|_{2l,2m,p,-\max\{p+r+\alpha,q\}+2r-\alpha} \leq \left(\frac{1}{(2l)!(2m)!}\right)^{\frac{1}{2}} ((l-k)!(m-k)!)^{\frac{1}{2}} (4e\delta^4 p^r)^{\frac{l+m}{2}} |K_{l,k,m-k}|_{2(l-k),2(m-k);p+r,-Q} \leq C_0 \left(\frac{1}{(2l)!(2m)!}\right)^{\frac{1}{2}} (4e\delta^4 p^r)^{\frac{l+m}{2}} (e\delta^4)^{\frac{l+m-2k}{2}}$$

Here

$$\max\{p + \alpha, Q\} + 2r + \alpha = Q + 2r + \alpha = \max\{p + r + \alpha, q\} + 3r + 2\alpha$$

and we obtain the following estimation for $\kappa_{l,m}$:

$$|\kappa_{l,m}|_{2l,2m,p,-\max\{p+r+\alpha,q\}+3r-2\alpha} \leq C_0 \left(\frac{1}{(2l)!(2m)!}\right)^{\frac{1}{2}} p^{2r(l+m)} \left\{ \frac{C_3^{l+m}}{(2l)!(2m)!} + (2C_3)^{l+m} \sum_{k=1}^{\min\{l,m\}} \frac{C_3^{l-k+(m-k)}}{(2l-2k)!(2m-2k)!} \sum_{(k_1,\ldots,k_t) \in \Pi_k} \frac{1}{(2k_1)!(2k_t)!} \right\}$$

where $C_3 := (e\delta^4)^{\frac{1}{2}}$. Since

$$\sum_{(k_1,\ldots,k_t) \in \Pi_k} \frac{1}{(2k_1)!(2k_t)!} \leq \frac{1}{k!} \sum_{(k_1,\ldots,k_t) \in \Pi_k} \frac{k!}{k_1!\ldots k_t!} = \frac{1}{k!} \sum_{t=1}^{k} \left(\frac{1}{t!}\right)^{k} = 1 \leq e \int_1^{k+1} t^k dt = \frac{(k+1)^{k+1}}{(k+1)!},$$

it holds that

$$\sum_{k=1}^{\min\{l,m\}} \frac{C_3^{l-k+(m-k)}}{(2l-2k)!(2m-2k)!} \sum_{(k_1,\ldots,k_t) \in \Pi_k} \frac{1}{(2k_1)!(2k_t)!} \leq \left( \sum_{k=1}^{\min\{l,m\}} \frac{(\sqrt{C_3})^{2l-2k}}{(2l-2k)!} \right) \left( \sum_{k=1}^{\min\{l,m\}} \frac{(\sqrt{C_3})^{2m-2k}}{(2m-2k)!} \right) e^{\min\{l,m\}+1} \leq e^{2\sqrt{C_3}}e^{\min\{l,m\}+1}.$$
Therefore we obtain

\[
|\kappa_{l,m}|2l,2m;p, - \max\{p + r + \alpha, q\} - 3r - 2\alpha \\
\leq C_0 \left( \frac{1}{(2l)!(2m)!} \right)^{\frac{1}{2}} \rho^{2r(l+m)} e^{1+2\sqrt{C_3}} (1 + e^{\min\{l,m\}} (2C_3)^{l+m}) \\
\leq C_0 \left( \frac{1}{(2l)!(2m)!} \right)^{\frac{1}{2}} \rho^{2r(l+m)} e^{1+2\sqrt{C_3}} \cdot 2 \max\{1, 2eC_3\}^{l+m} \\
= C_1(C_2\rho^{2r})^{l+m} \left( \frac{1}{(2l)!(2m)!} \right)^{\frac{1}{2}},
\]

where

\[ C_1 := 2C_0 e^{1+2\sqrt{C_3}}, \quad C_2 := \max\{1, 2eC_3\}. \]

Lemma 4.10. Fock expansion \((3.4)\) converges in \(E^*_+\) (resp. \(E_+\)) with respect to the topology of \(E^*_+\) (resp. \(E_+\)) if \(\Xi \in \mathcal{L}(E_+, E^*_+)\) (resp. \(\Xi \in \mathcal{L}(E_+, E_+)\)).

Proof. Due to \((3.3)\) and \((4.7)\),

\[
\|\Xi_{l,m}(\kappa_{l,m})\phi\|_p \\
\leq \rho^{-\frac{7}{2}} (2m)^n (2l)^l \left( \frac{\rho^{-\frac{7}{2}}}{-re \log \rho} \right)^{l+m} \\
\times |\kappa_{l,m}|2l,2m;p, - \max\{p + r + \alpha, q\} - 3r - 2\alpha \|\phi\|_{\max\{p, \max\{p + r + \alpha, q\} + 3r + 2\alpha\} + r} \\
\leq C_1 \rho^{-\frac{7}{2}} \left( \frac{(2m)^2 (2l)^2}{(2l)!(2m)!} \right)^{\frac{1}{2}} \left( C_2\rho^{2r} \frac{\rho^{-\frac{7}{2}}}{-re \log \rho} \right)^{l+m} \|\phi\|_{\max\{p + \alpha, q\} + 4r + 2\alpha}.
\]

Put

\[ R := \frac{C_2}{2} \rho^{2r} \frac{\rho^{-\frac{7}{2}}}{\log \rho^{-\frac{7}{2}}} > 0 \]

By using

\[
\frac{(2l)^2}{(2l)!} \leq e^{2l}, \quad \frac{(2m)^2 (2l)^2}{(2m)!} \leq e^{2m},
\]

we have

\[
\|\Xi_{l,m}(\kappa_{l,m})\phi\|_p \leq C_1 \rho^{-\frac{7}{2}} R^{l+m} \|\phi\|_{\max\{p + \alpha, q\} + 4r + 2\alpha}.
\]

Now, if we choose sufficiently large \(r > 0\), then \(R < 1\) holds. Therefore

\[
\sum_{l,m=0}^\infty \|\Xi_{l,m}(\kappa_{l,m})\phi\|_p \leq \frac{C_1 \rho^{-\frac{7}{2}}}{(1 - R)^2} \|\phi\|_{\max\{p + \alpha, q\} + 4r + 2\alpha}
\]

and this implies \(\sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m})\phi\) converges in \(E^*_+\) (resp. \(E_+\)) with respect to the topology of \(E^*_+\) (resp. \(E_+\)). 

\[ \]
5 Fock expansion for the fermion system

In this section, we extend result of section 3 to the whole of the Fermion system. In order
to make the white noise calculus for the Fermion system, we mention properties for creation
and annihilation operators for the Fermion system.

Definition 5.1.

(1) For \( f \in E^\ast \), we define an annihilation operator \( a(f) \in \mathcal{L}(E, E) \) as follows:
\[
a(f) : E \ni \phi = (\phi_n)_{n \in \mathbb{Z}_0} \mapsto a(f)\phi \in E,
\]
\[
a(f)\phi_n := n \cdot f \wedge_1 \phi_n, \quad n \geq 1,
\]
\[
a(f)\phi_0 = 0.
\]
(Well-definedness and continuity of \( a(f) \) is discussed in the following lemma.)

(2) For \( f \in E^\ast \), we define a creation operator \( a^\dagger(f) \in \mathcal{L}(E^\ast, E^\ast) \) as follows:
\[
a^\dagger(f) : E^\ast \ni \phi = (\phi_n)_{n \in \mathbb{Z}_0} \mapsto a^\dagger(f)\phi \in E^\ast,
\]
\[
a^\dagger(f)\phi_n := f \wedge \phi_n, \quad n \geq 0.
\]

(3) For \( f \in E^\ast \) and \((l,m) \in \{(1,0), (0,1)\} \), put
\[
a_{(l,m)}(f) := \begin{cases} a^\dagger(f), & \text{if } (l,m) = (1,0), \\ a(f), & \text{if } (l,m) = (0,1). \end{cases}
\]

\( a(f) \) is a map from \( E \) to \( E \) and is a continuous map as follows:

Lemma 5.2. Let \( p, q \in \mathbb{R}, r > 0 \), and \( f \in E^\ast \). Then
\[
\|a_{(l,m)}(f)\phi\|_p \leq \left(\frac{\rho^{-2r}}{2re \log \rho}\right)^{\frac{1}{2}} |f|_{m,l;-(q+r),p} \|\phi\|_{\max\{p,q\}+r}, \quad \phi \in E. \tag{5.1}
\]
Thus we have the following properties (1)–(3). Let \( \sigma \) be +, −, or a blank.

(1) \( a(f)|_{E_\sigma} \in \mathcal{L}(E_\sigma, E_{-\sigma}) \) for \( f \in E^\ast \),

(2) \( a^\dagger(f)|_{E_\sigma} \in \mathcal{L}(E_\sigma, E_{-\sigma}) \) for \( f \in E \),

(3) \( a^\dagger(f)|_{E^\ast_\sigma} = (a(f)^*)|_{E^\ast_\sigma} \in \mathcal{L}(E^\ast_\sigma, E_{-\sigma}^\ast) \) for \( f \in E^\ast \).

Proof. (5.1) can be shown by using Lemma 3.4 and (3.2). In fact, we have
\[
\|a(f)\phi\|_p^2 = \sum_{n=1}^\infty (n - 1)! |nf \wedge_1 \phi_n|^2
\]
\[
\leq \sum_{n=1}^\infty n! \cdot n \rho^{2nr} |f|_{-q}^2 |\phi_n|^2_{\max\{p,q\}+r}
\]
\[
\leq \left(\sup_{n \geq 1} (n + 1) \rho^{2nr}\right) |f|_{-q}^2 \|\phi\|_{\max\{p,q\}+r}^2
\]
\[
\leq \left(\frac{\rho^{-2r}}{-2re \log \rho}\right) |f|_{-q}^2 \|\phi\|_{\max\{p,q\}+r}^2
\]

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and also have
\[
\|a^\dagger(f)\phi\|^2_p = \sum_{n=0}^{\infty} (n+1)! |f \wedge \phi_n|^2_p
\]
\[
\leq \sum_{n=0}^{\infty} n! \cdot (n+1) \rho^{2nr} |f_p|^2 |\phi_n|^2_{\max\{p,q\} + r}
\]
\[
\leq \left( \sup_{n \geq 1} (n+1) \rho^{2nr} \right) |f_p|^2 |\phi|_{\max\{p,q\} + r}^2
\]
\[
\leq \left( \frac{\rho^{-2r}}{-2r \log \rho} \right) |f_p|^2 |\phi|_{\max\{p,q\} + r}^2.
\]
(1) We take any \( p > 0 \) and \( r > 0 \). Then we choose \( q > 0 \) with \( |f|_{-q} < +\infty \) and this implies that \( a(f)|_{E_\sigma} \in \mathcal{L}(E_\sigma, E_{-\sigma}) \). (2) is obvious. (3) follows from (1).

Creation and annihilation operators satisfy the following commutation relation, called canonical anti-commutation relations.

**Proposition 5.3.** For \( f \in E \) and \( g \in E^* \), we have
\[
\{a^\dagger(f), a(g)\} \phi = \langle g, f \rangle \phi
\]
for all \( \phi \in E_\sigma \). (\( \sigma \) is +,−, or a blank.). Moreover
\[
(a^\dagger(f) + a(Jf))^2 \phi = \phi
\]
for all \( \phi \in \text{on } E_\sigma \) for \( f \in E \) with \( (f, f)_0 = 1 \). Here \( Jf \in E \) is the complex conjugate of \( f \in E \).

Put
\[
d\Gamma(A)^{(n)} := \sum_{i=1}^{n} 1^{\otimes (i-1)} \otimes A \otimes 1^{\otimes (n-i)}
\]
for a linear operator \( A \in \mathcal{L}(E, E^*) \).

**Lemma 5.4.** (1) For \( f, g \in E^* \), we have
\[
a^\dagger(f)a^\dagger(g)|_{E_+} = \Xi_{1,0}(f \wedge g),
\]
\[
a(f)a(g)|_{E_+} = \Xi_{0,1}(f \wedge g).
\]
(2) Let \( (f \otimes g)h := \langle g, h \rangle f \) for \( f, g \in E^* \), and \( h \in E \). Then we have
\[
a^\dagger(f)a(g)|_{E_+} = \Xi_{1,1}((1_2 \otimes d\Gamma(f \otimes g)^{(2)})^* \tau)
\]
where \( \tau \in (E^{\wedge 2})^* \otimes (E^{\wedge 2})^* \) is defined by \( \tau(\zeta, \eta) := \langle \zeta, \eta \rangle \) for all \( \zeta \in (E^{\wedge 2})^* \) and \( \eta \in E^{\wedge 2} \).

**Proof.** (1) is easily checked. We show only (2). For any \( h_i \in E \) (\( i = 1, 2, \ldots, 2n \)), we have
\[
a^\dagger(f)a(g)h_1 \wedge \ldots \wedge h_{2n} = \sum_{i=1}^{2n} (-1)^{i-1} \langle g, h_i \rangle f \wedge h_1 \wedge \ldots \wedge h_{i-1} \wedge h_{i+1} \wedge \ldots \wedge h_{2n}
\]
\[
= \sum_{i=1}^{2n} (-1)^{i-1} (f \otimes g)h_i \wedge h_1 \wedge \ldots \wedge h_{i-1} \wedge h_{i+1} \wedge \ldots \wedge h_{2n}
\]
\[
= d\Gamma(f \otimes g)^{(2n)}h_1 \wedge \ldots \wedge h_{2n}
\]
\[
= d\Gamma(d\Gamma(f \otimes g)^{(2)})^{(n)}h_1 \wedge \ldots \wedge h_{2n}.
\]
Thus we have (5.4). (See proposition 4.6.13 of [3].)
Let \( \Xi_{l,m}(\kappa) \in \mathcal{L}(E^*, E^*) \) be an integral kernel operator with a kernel distribution \( \kappa \). Let \( f \) be an element of \( E \) with \( (f, f)_0 = 1 \) and \( W(f) := a^f(f) + a(Jf) \). Then we also call all operators

\[
\Xi_{l,m}(\kappa)W(f), \quad W(f)^*\Xi_{l,m}(\kappa), \quad W(f)^*\Xi_{l,m}(\kappa)W(f).
\]

integral kernel operators for the sake of convenience. Now we give the main theorem of this paper.

**Theorem 5.5.** Every \( \Xi \in \mathcal{L}(E, E^*) \) is realized as a series of integral kernel operators.

**Proof.** Note that for \( \Xi \in \mathcal{L}(E, E^*) \) there exist unique \( \Xi_{\alpha\beta} \in \mathcal{L}(E^\beta, E^\alpha) \) \((\alpha, \beta \in \{+, -\})\) such that

\[
\Xi = \Xi_{++} + \Xi_{+-} + \Xi_{-+} + \Xi_{--}.
\]

Thus we have only to show that each \( \Xi_{\alpha\beta} \in \mathcal{L}(E^\beta, E^\alpha) \) is realized as a series of integral kernel operators.

Let \( \Xi_{+-} \) be an element of \( \mathcal{L}(E^-, E^*_+) \) and \( f \in E \) satisfy \( (f, f)_0 = 1 \). Note that \( \Xi_{+-}W(f) \) is an element of \( \mathcal{L}(E^+, E^*_+) \). Then, from theorem 3.11, there exists a unique kernel distribution \( \kappa_{+-}(l, m; f) \) such that

\[
\Xi_{+-}W(f) = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{+-}(l, m; f)).
\]

Thus we have

\[
\Xi_{+-} = \Xi_{+-}W(f)^2 = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{+-}(l, m; f))W(f).
\]

In the same manner, for \( \Xi_{-+} \in \mathcal{L}(E_+, E^*_+) \), we have \( \kappa_{-+}(l, m; f) \) with

\[
\Xi_{-+} = (W(f)^*)^2 \Xi_{-+} = \sum_{l,m=0}^{\infty} W(f)^*\Xi_{l,m}(\kappa_{-+}(l, m; f)).
\]

Since \( W(f)^*\Xi_{-+}W(f) \in \mathcal{L}(E_+, E^*_+) \), there exists a unique kernel distribution \( \kappa_{-+}(l, m; f) \) satisfying

\[
\Xi_{-+} = (W(f)^*)^2 \Xi_{-+}W(f)^2 = \sum_{l,m=0}^{\infty} W(f)^*\Xi_{l,m}(\kappa_{-+}(l, m; f))W(f).
\]

For \( \Xi_{++} \in \mathcal{L}(E_+, E^*_+) \), we also have a unique kernel distribution \( \kappa_{++}(l, m) \) satisfying

\[
\Xi_{++} = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{++}(l, m)).
\]

Therefore we obtain

\[
\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m},
\]

where

\[
\Xi_{l,m} = \Xi_{l,m}(\kappa_{++}(l, m)) + \Xi_{l,m}(\kappa_{-+}(l, m; f))W(f) + W(f)^*\Xi_{l,m}(\kappa_{-+}(l, m; f)) + W(f)^*\Xi_{l,m}(\kappa_{-+}(l, m))W(f).
\]

\[ \square \]
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