A 3D-SCHRÖDINGER OPERATOR UNDER MAGNETIC STEPS WITH SEMICLASSICAL APPLICATIONS

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Abstract. We define a Schrödinger operator on the half-space with a discontinuous magnetic field having a piecewise-constant strength and a uniform direction. Motivated by applications in the theory of superconductivity, we study the infimum of the spectrum of the operator. We give sufficient conditions on the strength and the direction of the magnetic field such that the aforementioned infimum is an eigenvalue of a reduced model operator on the half-plane. We use the Schrödinger operator on the half-space to study a new semiclassical problem in bounded domains of the space, considering a magnetic Neumann Laplacian with a piecewise-constant magnetic field. We then make precise the localization of the semiclassical ground state near specific points at the discontinuity jump of the magnetic field.

1. Introduction

We consider a Schrödinger operator defined on the half-space and having a magnetic field with a piecewise-constant strength and a uniform direction. Such operator is interesting to be considered in new situations in the theory of superconductivity as we will describe later. We set the half-space to be \( \mathbb{R}^3_+ := \{ x \in \mathbb{R}^3 \mid x = (x_1, x_2, x_3), \ x_2 > 0 \} \) and we split it in two regions in which the strength of the magnetic field is different as follows. Let \( \alpha \in (0, \pi) \), using spherical coordinates, we define the domains \( D^1_\alpha \) and \( D^2_\alpha \) of \( \mathbb{R}^3_+ \):\[
D^1_\alpha = \left\{ x \in \mathbb{R}^3 \mid x = \rho (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \ \rho \in (0, \infty), \ 0 < \theta < \alpha, \ \phi \in (0, \pi) \right\},
\]
\[
D^2_\alpha = \left\{ x \in \mathbb{R}^3 \mid x = \rho (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \ \rho \in (0, \infty), \ \alpha < \theta < \pi, \ \phi \in (0, \pi) \right\}.
\]

Let \( a \in [-1, 1) \setminus \{0\} \) and \( \gamma \in [0, \pi/2] \), we introduce the following magnetic field\( B_{\alpha, \gamma, a} \) in \( \mathbb{R}^3_+ \):
\[
B_{\alpha, \gamma, a} = (\cos \alpha \sin \gamma, \sin \alpha \sin \gamma, \cos \gamma) \left( s^1_D \alpha + a s^2_D \alpha \right).
\]
Here (and in the sequel) \( s_Z \) denotes the characteristic function corresponding to the set \( Z \) (in this case \( s^1_D \alpha = D^1_\alpha \setminus D^2_\alpha \)). The function \( s_a \alpha \) represents the strength of the magnetic field (see Figure 1). The choice of the values \( a \) in \([-1, 1) \setminus \{0\} \) will be discussed later (see Remark 1.2).

We consider the magnetic Neumann realization of the following self-adjoint operator on \( \mathbb{R}^3_+ \):
\[
L_{\alpha, \gamma, a} = -(\nabla - i A_{\alpha, \gamma, a})^2,
\]
where \( A_{\alpha, \gamma, a} \in H^1_{loc}(\mathbb{R}^3_+, \mathbb{R}^3) \) is a magnetic potential such that \( \text{curl} A_{\alpha, \gamma, a} = B_{\alpha, \gamma, a} \). A choice of the magnetic potential \( A_{\alpha, \gamma, a} \) is fixed in (3.5). The domain of the operator \( L_{\alpha, \gamma, a} \) is

1By symmetry considerations, we restrict the study to the case where \( \gamma \in [0, \pi/2] \).

2The choice of the discontinuous magnetic field \( B_{\alpha, \gamma, a} \) as in (1.3) is motivated by getting the operator \( L_{\alpha, \gamma, a} \) as the right tangent operator in localizations problems on bounded domains, considered later in the paper (see the applications in Section 1.1).
1. Motivation. In the theory of superconductivity and in generic situations, a superconductor submitted to a sufficiently strong magnetic field loses permanently its superconducting properties when the intensity of the magnetic field exceeds a certain (unique) critical value—the so-called third critical field denoted by $H_{C_3}$. We say that the material passes to the normal state (see [20,45]). The Ginzburg–Landau (GL) model is used to study this phase transition from superconducting to normal states. This is naturally a three-dimensional (3D) model, but it is usually reduced to a two-dimensional (2D) one supposing that the superconductor is a long-cylindrical wire and that the direction of the magnetic field is perpendicular to the cross section of the wire (see e.g. [46]). The 2D GL model was extensively used for both constant or smooth variable external magnetic fields in the case of domains with smooth boundary (see e.g. [20,27,32,41,42] or domains with corners (see [10,11]). Recently, [2,3] (see also [4]) examined this phase transition for 2D GL models with piecewise-constant magnetic fields. Also, we refer to [13–17,26] for the study of superconductivity right before the normal state.

Within this context, 3D models were studied in the mathematical literature for more general (bounded or unbounded) domains, not necessarily cylinders, subjected to constant or smooth variable magnetic fields (see e.g. [28,33,38,39,43]). Such studies involved a linear Schrödinger operator, $-(h \nabla - iA)^2$, defined on an open and bounded set $\Omega \subset \mathbb{R}^3$, with smooth boundary or having edges, where $\Lambda \in H^{1 \text{loc}}(\mathbb{R}^3)$ is a magnetic vector potential and $\text{curl}A = B$ is the external magnetic field having a constant or a smooth variable strength. As the semiclassical parameter $h$ goes to 0, the third critical field $H_{C_3}$ is estimated using the asymptotics of the first eigenvalue, $\lambda(B;\Omega,h)$, of this operator (see e.g. [19, Proposition 1.9], [2,20,24,33]). Such asymptotics of $\lambda(B;\Omega,h)$ are usually obtained by using a variational argument where local energies are studied in different zones of the superconductor (like the

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$^{3}$Due to gauge invariance [20, Section 1.1], it is standard that the magnetic potential $\Lambda$ contributes to the spectrum of $-(h \nabla - iA)^2$ only through its associated magnetic field $B$, which justifies the notation $\lambda(B;\Omega,h)$. 

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Figure 1. Let $\alpha \in (0, \pi)$, $\gamma \in [0, \pi/2]$ and $a \in [-1, 1) \setminus \{0\}$. The magnetic field $B_{\alpha,\gamma,a}$ in $\mathbb{R}^2_a$ can have different directions in the two regions $D^1_a$ and $D^2_a$, according to the sign of $a$. The strength of the magnetic field is $s_{\alpha,a} = 1$ in $D^1_a$ and $s_{\alpha,a} = a$ in $D^2_a$. The transition of the strength occurs at the plane $P_\alpha$, of equation $x_1 \sin \alpha - x_2 \cos \alpha = 0$, referred to as the discontinuity plane. The angle $\gamma$ (modulo $-\pi$) represents the angle that $B_{\alpha,\gamma,a}$ makes with the $x_3$-axis.

$D(L_{\alpha,\gamma,a}) = \{ u \in L^2(\mathbb{R}^3_a) : (\nabla - iA_{\alpha,\gamma,a})^n u \in L^2(\mathbb{R}^3_a),$ 

for $n \in \{1, 2\}, (\nabla - iA_{\alpha,\gamma,a})u \cdot (0, 1, 0)|_{\partial \mathbb{R}^3_a} = 0 \}$. (1.5)
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interior, the boundary, or near the edges). The local study involves effective Schrödinger operators of the form $-(\nabla - iA)^2$, with magnetic fields having a constant strength, defined on unbounded domains like $\mathbb{R}^3$, $\mathbb{R}^2_+$ or infinite wedges (see [10,33]). While the operator on $\mathbb{R}^3$ is related to the study in the interior of $\Omega$, that on $\mathbb{R}^2_+$ is related to the study at the smooth boundary of $\Omega$ and depends on the angle between the magnetic field and the boundary $\mathbb{R}^2_+$. Moreover, the operators on infinite wedges are considered ( [35,38,39]) for the study near the edges of $\Omega$ (when exist), and depend on both the direction of the magnetic field and the opening angle of the wedge. Studying the effective models permit to determine the eventual localization of superconductivity in $\Omega$, before its breakdown. We refer the reader to the introduction in [39] for a brief explanation about the link between the original model on $\Omega$ and the various effective models (see also [10] for a more detailed explanation).

Back to the operator $\mathcal{L}_{\alpha,a,\gamma}$ defined in the present contribution, such a 3D operator with a discontinuous magnetic field was not considered yet in the literature. We show that $\mathcal{L}_{\alpha,a,\gamma}$ is a leading operator that plays an essential role in the study of semiclassical problems similar to the aforementioned ones in $\Omega$ (see [21, Section 8] and [23]), but in new situations where the magnetic field is piecewise-constant. As seen later in the paper, we consider the semiclassical problem associated to a piecewise constant magnetic field in $\Omega \subset \mathbb{R}^3$, provide asymptotics of its ground state energy, and establish the concentration of its ground state in specific regions near the discontinuity surface, i.e., the surface at which the jump of the magnetic field occurs (see Section 1.2.2).

1.2. Main results. We present the main results of our paper in Section 1.2.1, and applications to these results in Section 1.2.2.

1.2.1. Bottom of the spectrum of $\mathcal{L}_{\alpha,a,\gamma}$. We recall the operator $\mathcal{L}_{\alpha,a,\gamma}$ introduced in (1.4)

$$\mathcal{L}_{\alpha,a,\gamma} = -(\nabla - iA_{\alpha,a,\gamma})^2, \quad \text{in } \mathbb{R}^3_+,$$

(1.6)

with the domain $\mathcal{D}(\mathcal{L}_{\alpha,a,\gamma})$ defined in (1.5). We consider the bottom of the spectrum of this operator

$$\lambda_{\alpha,a,\gamma} := \inf \text{sp}(\mathcal{L}_{\alpha,a,\gamma}).$$

(1.7)

Using a Fourier transform, the operator $\mathcal{L}_{\alpha,a,\gamma}$ can be decomposed into a family of 2D operators on $\mathbb{R}^2_+$, $\mathcal{L}_{\mathbf{A}_{\alpha,a,\gamma},+} + \mathbf{B}_{\alpha,a,\gamma,+}, \tau$, parametrized by $\tau \in \mathbb{R}$, and defined on $\mathbb{R}^2_+$ as follows

$$\mathcal{L}_{\mathbf{A}_{\alpha,a,\gamma},+} + \mathbf{B}_{\alpha,a,\gamma,+},\tau = -(\nabla - i\mathbf{A}_{\alpha,a,\gamma})^2 + \mathbf{B}_{\alpha,a,\gamma,+},\tau,$$

(1.8)

where $\mathbf{A}_{\alpha,a,\gamma}$ is a vector potential, defined in (3.8), representing a projection of the vector potential $\mathbf{A}_{\alpha,a,\gamma}$ in (1.4) on $\mathbb{R}^2_+$, $\mathbf{B}_{\alpha,a,\gamma} = (b_1, b_2)$ is a magnetic field, defined in (3.13), projecting the field $\mathbf{B}_{\alpha,a,\gamma}$ in (1.3) on $\mathbb{R}^2_+$, and $\mathbf{B}_{\alpha,a,\gamma,+},\tau = (x_1 b_2 - x_2 b_1 - \tau)^2$ is an electric potential defined in (3.14). The bottom of the spectrum of $\mathcal{L}_{\mathbf{A}_{\alpha,a,\gamma},+} + \mathbf{B}_{\alpha,a,\gamma,+},\tau$ is denoted by $\sigma(\alpha, \gamma, a, \tau)$, which highlights its dependence on the parameters $\alpha, \gamma, a$, and $\tau$. Having (see (3.17))

$$\lambda_{\alpha,a,\gamma} = \inf_{\tau \in \mathbb{R}} \sigma(\alpha, \gamma, a, \tau),$$

the examination of $\lambda_{\alpha,a,\gamma}$ reduces to that of the function $\tau \mapsto \sigma(\alpha, \gamma, a, \tau)$. This examination leads to an important comparison between $\lambda_{\alpha,a,\gamma}$ and other well-known spectral values, $\beta_a$ and $\zeta_{\nu_0}$, where $a \in [-1,1) \setminus \{0\}$ is the parameter appearing in the definition of $\mathcal{L}_{\alpha,a,\gamma}$ and $\nu_0 := \arcsin(\sin \alpha \sin \gamma)$. The value $\beta_a$ is the bottom of the spectrum of a Schrödinger operator defined on $\mathbb{R}^3$ in (2.5), with a piecewise-constant magnetic field (splitting $\mathbb{R}^3$ in two half-spaces, the strength of the field takes the values 1 and $a$ in these half-spaces, respectively). The value $\zeta_{\nu_0}$ is the bottom of the spectrum of a magnetic Neumann
Schrödinger operator defined on $\mathbb{R}^3_+$ in (2.8), with a constant magnetic field making an angle $\theta_0$ with the $(x_1x_2)$ plane.

**Theorem 1.1.** Let $a \in [-1, 1) \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in [0, \pi/2]$, and $\nu_0 = \arcsin(\sin \alpha \sin \gamma)$. Let $\lambda_{\alpha,\gamma,a}$ be the bottom of the spectrum of the operator $\mathcal{L}_{\alpha,\gamma,a}$ defined in (1.1). It holds

$$\lambda_{\alpha,\gamma,a} \leq \min (\beta_a, |a| \zeta_{\nu_0}),$$

where $\beta_a$ and $\zeta_{\nu_0}$ are respectively the bottom of the spectrum of the operators defined in (2.5) and (2.8).

Furthermore, if

$$\lambda_{\alpha,\gamma,a} < \min (\beta_a, |a| \zeta_{\nu_0}),$$

then there exists $\tau_* \in \mathbb{R}$ such that

$$\lambda_{\alpha,\gamma,a} = \sigma(\alpha, \gamma, a, \tau_*),$$

and $\sigma(\alpha, \gamma, a, \tau_*)$ is an eigenvalue of the operator $\mathcal{L}_{\mathcal{A}_{\alpha,\gamma,a} + V_{\mathcal{B}_{\alpha,\gamma,a}}, \tau_*}$ defined in (1.8).

**Remark 1.2** (The choice of $a \in [-1, 1) \setminus \{0\}$). One can choose any two distinct real values $b_1$ and $b_2$ for the strength of the magnetic field $B_{\alpha,\gamma,a}$ respectively in $D_\alpha$ and $D_\beta$. However, by a simple scaling argument, one can reduce the study to the case $b_1 = 1$ and $b_2 = a$, where $a$ is a value in $[-1, 1]$.

In the case $a = 0$, the energy $\beta_a$, appearing in Theorem 1.1, is equal to zero (see [29]).

Hence, the comparison between the three energies $\lambda_{\alpha,\gamma,a}, \beta_a$, and $|a| \zeta_{\nu_0}$ is trivial:

$$\lambda_{\alpha,\gamma,a} \geq \min (\beta_a, |a| \zeta_{\nu_0}) = 0.$$

Moreover, our proof technically relies on the assumption $a \neq 0$ in many places, for instance when using translations to link our problem to the toy models in Section 2, which have well-explored spectra. We exclude the case $a = 0$ from our study.

**Remark 1.3** (On the semiclassical problem). As mentioned earlier, the operator $\mathcal{L}_{\alpha,\gamma,a}$ will be used in studying a semiclassical problem on a smooth and bounded domain $\Omega$ of $\mathbb{R}^3$, subjected to a piecewise-constant magnetic field. This semiclassical problem is introduced in Section 1.2. When the bottom of the spectrum, $\lambda_{\alpha,\gamma,a}$, of $\mathcal{L}_{\alpha,\gamma,a}$ is an eigenvalue of a certain $\mathcal{L}_{\mathcal{A}_{\alpha,\gamma,a} + V_{\mathcal{B}_{\alpha,\gamma,a}}, \tau_*}$, one can use its corresponding eigenfunction to construct a trial function in $\Omega$, supported near some point(s) of the discontinuity region of $\partial \Omega$ corresponding to $(\alpha, \gamma, a)$, which yields a desired upper bound in the asymptotic estimates of the semiclassical ground state in $\Omega$ (see the proof of Proposition 5.5). This motivates our interest in setting the condition (1.10) in Theorem 1.1.

In Theorem 1.1, we gave sufficient conditions for $\lambda_{\alpha,\gamma,a}$ to be an eigenvalue of the operator $\mathcal{L}_{\mathcal{A}_{\alpha,\gamma,a} + V_{\mathcal{B}_{\alpha,\gamma,a}}, \tau_*}$ in (3.7), for a certain $\tau_* \in \mathbb{R}$. Our next result provides a condition on $(\alpha, \gamma, a)$ such that (1.10) is realized.

**Proposition 1.4.** Let $a \in [-1, 1) \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in [0, \pi/2]$, and $\nu_0 = \arcsin(\sin \alpha \sin \gamma)$. Consider the function $P[\alpha, \gamma, a]: (0, +\infty) \to \mathbb{R}$ defined by

$$P[\alpha, \gamma, a](x) = A[\alpha, \gamma, a]x^2 - \frac{\pi}{2} \Lambda[\alpha, \gamma, a]x + \frac{\pi}{2},$$

with

$$A[\alpha, \gamma, a] := \frac{1}{128} (-1 + \coth \pi \alpha) \left\{ \pi \cos^2 \gamma \left[ 4(a - 1)((a - e^{\pi})e^{\pi - \alpha} + (ae^{\pi} - 1)e^\alpha) \right. \right.$$}

$$\left. - (a - 1)^2(e^{2\pi - 2\alpha} + e^{2\alpha}) - 2e^{\pi}( - 4a + (3 - 2a + 3a^2) \cos \pi) \right\}$$

$$+ 4(e^{2\pi} - 1) \left[ -(a^2(\pi - a) + a)(-3 + \cos(2\gamma)) + 2(a^2 - 1)(\sin^2 \gamma \sin(2\alpha)) \right] \}$$

and

$$\Lambda[\alpha, \gamma, a] := \min (\beta_a, |a| \zeta_{\nu_0}),$$
where $\beta_\alpha$ and $\zeta_\nu$ are respectively the bottom of the spectrum of the operators defined in (2.5) and (2.8). If there exists $x = x(\alpha, \gamma, a) > 0$ such that $P[\alpha, \gamma, a](x) < 0$, then $\inf_\tau g(\alpha, \gamma, a, \tau)$ is attained in $\mathbb{R}$, i.e. there exists $\tau_\ast \in \mathbb{R}$ satisfying

$$
\inf_\tau g(\alpha, \gamma, a, \tau) = g(\alpha, \gamma, a, \tau_\ast).
$$

Moreover, $g(\alpha, \gamma, a, \tau_\ast)$ is an eigenvalue of the operator $L_{\mathbf{A}, \gamma, a} + V_{\mathbf{B}, \gamma, a, \tau_\ast}$ defined in (3.7).

Remark 1.5 (Admissible triplets $(\alpha, \gamma, a)$). In Section 2, we provide the following lower bound for the value $\Lambda[\alpha, \gamma, a]$ in (1.13)

$$
\Lambda[\alpha, \gamma, a] \geq |a|\Theta_0,
$$

where $\Theta_0$ is the de Gennes constant defined in (2.4). Moreover, [8] gives an explicit lower bound, $\Theta_{\text{low}}$, of $\Theta_0$ equal to $0.590106125 - 10^{-9}$. Hence, if one defines

$$
P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x) := A[\alpha, \gamma, a]x^2 - \frac{\pi}{2}|a|\Theta_{\text{low}}x + \frac{\pi}{2},
$$

for $x > 0$ and $A[\alpha, \gamma, a]$ as in (1.12), one observes that $P[\alpha, \gamma, a](x) \leq P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x)$. By computation, we get that for all $a \in [-1, 1]\{0\}$, $\alpha \in (0, \pi)$ and $\gamma \in [0, \pi/2]$, $P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x)$ admits a minimum $x > 0$. Using Mathematica, we plot the region of triplets $(\alpha, \gamma, a)$ satisfying

$$
\min_{x > 0} P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x) = P_{\Theta_{\text{low}}}[\alpha, \gamma, a](x) < 0.
$$

These triplets are represented by the colored region in Figure 2. Consequently, the corresponding $\lambda_{\alpha, \gamma, a} = \inf_\tau g(\alpha, \gamma, a, \tau)$ is equal to $g(\alpha, \gamma, a, \tau_\ast)$, for a certain $\tau_\ast = \tau_\ast(\alpha, \gamma, a) \in \mathbb{R}$. Furthermore, $g(\alpha, \gamma, a, \tau_\ast)$ is an eigenvalue of the corresponding operator $L_{\mathbf{A}, \gamma, a} + V_{\mathbf{B}, \gamma, a, \tau_\ast}$ defined in (3.7).

1.2.2. Applications: a semiclassical problem in a 3D bounded domain. Let $\Omega \subset \mathbb{R}^3$ be an open bounded and simply connected set, with a smooth boundary. Let $\mathcal{C}$ be a smooth curve in the $(x_1, x_2)$ plane, with an infinite length. We define $S$ as the intersection between $\mathcal{C} \times \mathbb{R}$ and $\Omega$:

$$
S := (\mathcal{C} \times \mathbb{R}) \cap \Omega.
$$

We assume that $S$ cuts $\Omega$ into two disjoint non-empty open sets $\Omega_1$ and $\Omega_2$:

$$
\Omega = \Omega_1 \cup \Omega_2 \cup S.
$$

Let $a \in [-1, 1]\{0\}$, we define a piecewise constant magnetic field $\mathbf{B}$ in $\Omega$ as follows:

$$
\mathbf{B}(x) = s(x)(0, 0, 1), \quad s = 1_{\Omega_1} + a1_{\Omega_2}.
$$

(1.14)
Figure 3. Illustration of the domain $\Omega$, the discontinuity surface $S$ (shaded), and the discontinuity edge $\Gamma := \partial S$. The magnetic field $\mathbf{B}$ is tangent to $S$ and its strength exhibits a jump of discontinuity at $S$.

Note that the magnetic field $\mathbf{B}$ is tangent to $S$. Moreover, its strength $|\mathbf{B}|$ exhibits a discontinuity jump at $S$. We will refer to $S$ as the discontinuity surface. Moreover, we denote by $\Gamma$ the boundary of $S$

$$\Gamma := \partial S = (C \times \mathbb{R}) \cap \partial \Omega.$$  

We refer to $\Gamma$ as the discontinuity curve (see Figure 3).

Let $\mathbf{F} \in H^1(\Omega, \mathbb{R}^3)$ be a vector potential satisfying $\text{curl} \mathbf{F} = \mathbf{B}$ (see [20, Theorem D.3.1]). Let $b > 0$, we consider the Neumann realization of the following self-adjoint operator in $\Omega$:

$$P_{b,\mathbf{F}} = - (\nabla - ib \mathbf{F})^2.$$  

(1.15)

The domain of $P_{b,\mathbf{F}}$ is

$$\mathcal{D}(P_{b,\mathbf{F}}) := \{ u \in L^2(\Omega) : (\nabla - ib \mathbf{F})^j u \in L^2(\Omega) \text{ for } j = 1, 2, \quad (\nabla - ib \mathbf{F}) u \cdot n|_{\partial \Omega} = 0 \},$$

(1.16)

where $n$ is the inward unit normal vector of $\partial \Omega$. We denote by $\lambda(b)$ the bottom of the spectrum (the lowest eigenvalue or the ground state energy) of $P_{b,\mathbf{F}}$. We carry out our analysis in the asymptotic regime\footnote{Taking $b \to +\infty$ in our problem is equivalent to taking $h \to 0$ in the semiclassical problems aforementioned in Section 1.1.} $b \to +\infty$. The result of this section (Theorem 1.8 below) is established under the following assumption:

**Assumption 1.6.** For any point $x \in \Gamma$, we denote by $\alpha_x$ the angle between the tangent plane of $\partial \Omega$ and the discontinuity surface $S$ at $x$ (taken towards $\Omega_1$). We also denote by $\gamma_x$ the angle between the magnetic field $\mathbf{B}$ and the discontinuity edge $\Gamma$ at $x$. We assume that there exists $\varpi \in \Gamma$ such that

$$\lambda_{\alpha_{\varpi}, \gamma_{\varpi}, a} < |a| \Theta_0,$$

where $\lambda_{\alpha_{\varpi}, \gamma_{\varpi}, a}$ is defined in (1.7).

In this case, it obviously holds that

$$\inf_{x \in \Gamma} \lambda_{\alpha_x, \gamma_x, a} < |a| \Theta_0.$$  

**Remark 1.7** (Comments on Assumption 1.6). Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in [0, \pi/2]$.  

(i) We validate the above assumption as follows. Proposition 1.4 (and the computation below it) provide examples of triplets $(\alpha, \gamma, a)$ where
\[\lambda_{\alpha,\gamma,a} < \min(\beta_a, |a|\zeta_0),\]
(1.17)
for $\nu_0 = \arcsin(\sin \alpha \sin \gamma)$. Having at least a point $\bar{\nu} \in \Gamma$ such that $\gamma_{\bar{\nu}} = 0$, we can assume that the corresponding couple $(\alpha_{\bar{\nu}}, a)$ at $\bar{\nu}$ satisfies (1.17), that is
\[\lambda_{\alpha_{\bar{\nu}},0,a} < \min(\beta_a, |a|\zeta_0),\]
(1.18)
($\zeta_0 = \zeta_{\nu_0=0}$). Indeed, one can take $(\alpha_{\bar{\nu}}, a)$ living sufficiently near $(\pi/2, -1)$ to get (1.18) satisfied (see Proposition 1.4 and Figure 2).

Now, by Sections 2.1 and 2.2
\[\zeta_0 = \Theta_0, \text{ and } \beta_a \geq |a|\Theta_0.\]
Hence, (1.18) reads
\[\lambda_{\alpha_{\bar{\nu}},0,a} < |a|\Theta_0.\]
(1.19)

(ii) The conditions in Assumption 1.6 are motivated in what follows.

(a) For the points $\bar{\nu}$ satisfying the foregoing assumption, we have
\[\lambda_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a} < |a|\Theta_0 \leq \min(\beta_a, |a|\zeta_{\bar{\nu}}),\]
(1.20)
where $\bar{\nu} := \arcsin(\sin \bar{\nu} \sin \zeta_{\nu_0})$. The last inequality follows from the fact that $\beta_a \geq |a|\Theta_0$ (see Section 2.1), $\zeta_0 = \Theta_0$, and the function $\nu \mapsto \zeta_\nu$ is strictly increasing on $[0, \pi/2]$ (see Section 2.2). Hence, Assumption 1.6 implies that the energy $\lambda_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a}$ corresponding to each of these points satisfies the condition (1.10) in Theorem 1.1. This theorem ensures that $\lambda_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a}$ is an eigenvalue of the operator $L_{A_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a}} + V_{B_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a},\tau_s}$, for some $\tau_s \in \mathbb{R}$, defined in (3.7). The corresponding eigenfunction will be used in establishing an upper bound of $\lambda(b)$ (see Proposition 5.5).

(b) Furthermore, while the condition
\[\lambda_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a} < |a|\Theta_0\]
in (1.20) is sufficient to get the foregoing upper bound of Proposition 5.5, the more strict condition
\[\lambda_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a} < |a|\Theta_0\]
in Assumption 1.6 is crucial in localizing the ground state in $\Omega$ near the points $\bar{\nu}$ of $\Gamma$ realizing this assumption (see Theorem 1.8 below). Indeed, under this strict condition, the local energies near the points $\bar{\nu}$ will be strictly smaller than those away from these points. The minimum of the latter energies is comparable with $|b|\theta_0$, as $b \to +\infty$ (see Proposition 5.3).

Our next result provides a localization of the eigenfunction (the ground state) corresponding to the eigenvalue $\lambda(b)$ near the points of the discontinuity edge $\Gamma$ satisfying the conditions in Assumption 1.6. Let $D$ be the set of these points
\[D = \{\bar{\nu} \in \Gamma \mid \lambda_{\alpha_{\bar{\nu}},\gamma_{\bar{\nu}},a} < |a|\Theta_0\}.\]
(1.21)

Theorem 1.8. Under Assumption 1.6, there exist positive constants $C, \eta, b_0$ such that for all $b \geq b_0$ and for any ground state $\psi$ of $\mathcal{P}_{b,F}$, it holds
\[\int_{\Omega} e^{2\eta \sqrt{\text{dist}(x,D)}} \left(|\psi|^2 + b^{-1}(|\nabla - ibF\psi|^2)\right) dx \leq C||\psi||^2_{L^2(\Omega)},\]
(1.22)

\[\text{A simple instance of } \Omega \text{ and } \mathcal{B} \text{ satisfying Assumption 1.6 is when having } \Omega \text{ a ball that cuts the plane } \{x_2 = 0\} \text{ by an angle } \alpha = \pi/2 \text{ (or sufficiently near } \pi/2 \text{) and having } a = -1 \text{ (or sufficiently near } -1).\]
Consequently, for any \( N \in \mathbb{N} \), the following localization estimate holds
\[
\int_{\Omega} \text{dist}(x, D)^N |\psi|^2 \, dx = O(b - \frac{N}{2}).
\] (1.23)

1.3. Paper organization. The rest of the paper is organized as follows. In Section 2, we recall some known operators in the plane and the half-space which are useful for our analysis. In Section 3, we decompose our operator into 2D reduced operators. For these reduced operators, we derive some properties of the bottom of essential spectrum and the bottom of the spectrum. The proof of Theorem 1.1 is then established in Section 4. We use Theorem 1.1 in Section 5 to prove Theorem 1.8, establishing localization results in 3D bounded domains under discontinuous magnetic fields.

2. Known effective operators

In this section, we introduce useful linear Schrödinger operators on the plane and the half-space that were explored earlier in the literature.

2.1. An operator with a discontinuous magnetic field on the plane. Let \( a \in [-1, 1) \setminus \{0\} \). We consider a magnetic potential \( A_a \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2) \) with the following associated piecewise-constant magnetic field
\[
\text{curl} A_a(x) = 1_{\{x_2 > 0\}}(x) + a 1_{\{x_2 < 0\}}(x), \quad (x_1, x_2) \in \mathbb{R}^2.
\]
We introduce the self-adjoint operator on \( \mathbb{R}^2 \)
\[
L_a = -\left(\nabla - i A_a\right)^2,
\]
with domain
\[
D(L_a) := \{ u \in L^2(\mathbb{R}^2) : (\nabla - i A_a)^n u \in L^2(\mathbb{R}^2), \text{ for } n \in \{1, 2\} \}.
\]
We denote the bottom of the spectrum by \( \beta_a \). The operator \( L_a \) has been studied in [5, 6, 29, 30]: using a Fourier transform, \( L_a \) was reduced to a family of Schrödinger operators on \( L^2(\mathbb{R}), h_a[\xi] \), parametrized by \( \xi \in \mathbb{R} \). For each fixed \( \xi \in \mathbb{R} \), the operator \( h_a[\xi] \) is defined by
\[
h_a[\xi] = \begin{cases} 
-\frac{d^2}{dt^2} + (at - \xi)^2, & t < 0, \\
-\frac{d^2}{dt^2} + (t - \xi)^2, & t > 0.
\end{cases}
\] (2.1)
We have (see [6])
\[
\beta_a = \inf_{\xi \in \mathbb{R}} \mu_a(\xi),
\] (2.2)
where \( \mu_a(\xi) \) is the bottom of the spectrum of the operator \( h_a[\xi] \), i.e.,
\[
\mu_a(\xi) = \inf \text{ sp}(h_a[\xi]).
\] (2.3)
We collect the following useful properties of \( \beta_a \):
- For \( 0 < a < 1 \), \( \beta_a = a \) and \( \beta_a \) is not attained by \( \mu_a(\xi) \), for all \( \xi \in \mathbb{R} \).
- For \( -1 \leq a < 0 \), \( |a| \Theta_0 \leq \beta_a < |a| \) and \( \beta_a = \mu_a(\xi_a) \), for a certain (unique) \( \xi_a \in \mathbb{R} \). Here, \( \Theta_0 \) is the de Gennes constant defined as the bottom of the spectrum of the magnetic Neumann realization of the Schrödinger operator \( -(\nabla - i A)^2 \), with a unit magnetic field (curl \( A = 1 \)), on the half-plane (see e.g. [20])
\[
\Theta_0 = \inf \text{ sp}[-(\nabla - i A)^2] \cong 0.59.
\] (2.4)
Remark 2.1 (The value $\beta_a$ as the bottom of spectrum of a Schrödinger operator on $\mathbb{R}^3$). Consider the following Schrödinger operator on $\mathbb{R}^3$

$$L_a := -(\nabla - iA_a)^2,$$  
(2.5)

with domain

$$\mathcal{D}(L_a) := \{ u \in L^2(\mathbb{R}^3) | (\nabla - iA_a)^j u \in L^2(\mathbb{R}^3), \quad \text{for} \quad j \in \{1, 2\}\},$$  
(2.6)

where $A_a \in H^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ is a magnetic potential such that the corresponding magnetic field has a piecewise-constant strength equal to $\mathbb{1}_{\{x_2 > 0\}} + a \mathbb{1}_{\{x_2 < 0\}} =: \delta_a$. More precisely, we can fix the gauge and set $A_a = (-\delta_a x_2, 0, 0)$. With this choice of the magnetic potential, performing a partial Fourier transform it is possible to compare the bottom of the spectrum of $L_a$ with the bottom of the spectrum of the operator $\mathcal{L}_a$ introduced above, proving that

$$\inf \text{sp}(L_a) = \beta_a,$$  
(2.7)

where $\beta_a$ is as in (2.2).

2.2. An operator with a constant field on the half-space. Let $\nu \in [0, \pi/2]$. We introduce the following magnetic field with a unit strength on $\mathbb{R}^3_+$

$$B_\nu = (0, \sin \nu, \cos \nu),$$

and an associated magnetic potential $A_\nu \in H^1_{\text{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$, $(\text{curl} A_\nu = B_\nu)$. Note that $B_\nu$ makes an angle $\nu$ with the $(x_1, x_3)$ plane.

Now, we consider the magnetic Neumann realization of the following self-adjoint operator on the half space

$$H_\nu = -(\nabla - iA_\nu)^2 \quad \text{in} \quad L^2(\mathbb{R}^3_+),$$  
(2.8)

We denote by $\zeta_\nu$ the bottom of the spectrum of $H_\nu$,

$$\zeta_\nu = \inf \text{sp}(H_\nu).$$  
(2.9)

We present the following useful properties of $\zeta_\nu$ (see e.g. [32–34]):

$$\zeta_0 = \Theta_0, \quad \zeta_{\pi/2} = 1, \quad \zeta_\nu \in (\Theta_0, 1) \quad \text{for} \quad \nu \in (0, \pi/2),$$  
(2.10)

where $\Theta_0$ is the de Gennes constant defined in (2.4). Moreover, (see e.g. [32–34]), the map $\nu \mapsto \zeta_\nu$ is strictly increasing for $\nu \in [0, \pi/2]$.

3. The operator with magnetic steps on the half space

Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$ and $\gamma \in [0, \pi/2]$. We recall the operator $\mathcal{L}_{a, \gamma, a}$ with a discontinuous field on $\mathbb{R}^3_+$ introduced in (1.4)

$$\mathcal{L}_{a, \gamma, a} = -(\nabla - iA_{a, \gamma, a})^2,$$  
(3.1)

with the domain defined in (1.5) as

$$\mathcal{D}(\mathcal{L}_{a, \gamma, a}) = \{ u \in L^2(\mathbb{R}^3_+) : (\nabla - iA_{a, \gamma, a})^n u \in L^2(\mathbb{R}^3_+), \quad \text{for} \quad n \in \{1, 2\}, (\nabla - iA_{a, \gamma, a}) u \cdot (0, 1, 0)|_{\partial \mathbb{R}^3_+} = 0\}.$$  
(3.2)

Using the min-max principle, we write the bottom of the spectrum of $\mathcal{L}_{a, \gamma, a}$ as

$$\lambda_{a, \gamma, a} = \inf_{u \in \mathcal{D}(Q_{a, \gamma, a})} \frac{Q_{a, \gamma, a}(u)}{\|u\|^2_{L^2(\mathbb{R}^3_+)}},$$  
(3.3)

where $Q_{a, \gamma, a}$ is the quadratic form associated to the operator $\mathcal{L}_{a, \gamma, a}$, defined by

$$Q_{a, \gamma, a}(u) = \| (\nabla - iA_{a, \gamma, a}) u \|^2_{L^2(\mathbb{R}^3_+)}$$
on the domain

$$\mathcal{D}(Q_{a, \gamma, a}) := \{ u \in L^2(\mathbb{R}^3_+) : (\nabla - iA_{a, \gamma, a}) u \in L^2(\mathbb{R}^3_+) \}.$$
We also recall the magnetic field introduced in (1.3), and we denote by \( b_j, j = 1, 2, 3 \) its components:
\[
\mathbf{B}_{\alpha,\gamma,a} = (\cos \alpha \sin \gamma, \sin \alpha \sin \gamma, \cos \gamma)s_{\alpha,a} =: (b_1, b_2, b_3),
\]
where \( s_{\alpha,a} = 1_{D^1_a} + a1_{D^2_a} \) (see Figure 1). Now, we fix the choice of the magnetic potential \( \mathbf{A}_{\alpha,\gamma,a} \). Let
\[
\mathbf{A}_{\alpha,\gamma,a} = (A_1, A_2, A_3)
\]
such that
\[
A_1 = 0
\]
\[
A_2 = \begin{cases} 
\cos \gamma x_1 - (1 - a) \cos \gamma \cot \alpha x_2 & \text{for } x \in D^1_a \\
\alpha \cos \gamma x_1 & \text{for } x \in D^2_a 
\end{cases}
\]
\[
A_3 = \begin{cases} 
x_2 \cos \alpha \sin \gamma - x_1 \sin \alpha \sin \gamma & \text{for } x \in D^1_a \\
\alpha(x_2 \cos \alpha \sin \gamma - x_1 \sin \alpha \sin \gamma) & \text{for } x \in D^2_a 
\end{cases}
\]
This choice of the vector potential guarantees the continuity of \( \mathbf{A}_{\alpha,\gamma,a} \) at the discontinuity plane \( P_a \) (see Figure 1) and that \( \mathbf{A}_{\alpha,\gamma,a} \in H^1_{loc}((\mathbb{R}^+_3, \mathbb{R}^3)) \). Moreover, with this vector potential, the operator \( \mathcal{L}_{\alpha,\gamma,a} \) is translation invariant in the \( x_3 \) variable. Hence, its spectrum is absolutely continuous, and a reduction of the study to a family of 2D operators is allowed as we see below.

### 3.1. A family of reduced 2D operators.
Let \( a \in [-1, 1] \setminus \{0\}, \alpha \in (0, \pi), \gamma \in [0, \pi/2] \). A partial Fourier transform in the \( x_3 \) variable yields the following decomposition of the operator \( \mathcal{L}_{\alpha,\gamma,a} \) (see [44])
\[
\mathcal{L}_{\alpha,\gamma,a} = \int_{\mathbb{R}} \left( \mathcal{L}_{\mathbf{A}_{\alpha,\gamma,a}} + V_{\mathbf{B}_{\alpha,\gamma,a}} \right) d\tau,
\]
where
\[
\mathcal{L}_{\mathbf{A}_{\alpha,\gamma,a}} + V_{\mathbf{B}_{\alpha,\gamma,a}} = -\left( \nabla - i\mathbf{A}_{\alpha,\gamma,a} \right)^2 + V_{\mathbf{B}_{\alpha,\gamma,a}}
\]
is a Schrödinger operator on \( \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} \), parametrized by \( \tau \in \mathbb{R} \) and such that we have the following.

- The magnetic potential \( \mathbf{A}_{\alpha,\gamma,a} := (A_1, A_2) \) represents the projection of the vector potential \( \mathbf{A}_{\alpha,\gamma,a} \) defined in (3.5) on \( \mathbb{R}^2_+ \), i.e.,
\[
A_1 := 0
\]
\[
A_2 := \begin{cases} 
\cos \gamma x_1 - (1 - a) \cos \gamma \cot \alpha x_2 & \text{for } (x_1, x_2) \in D^1_a \\
\alpha \cos \gamma x_1 & \text{for } (x_1, x_2) \in D^2_a
\end{cases}
\]
where \( D^1_a \) and \( D^2_a \) represent respectively the orthogonal projection of the regions \( D^1_a \) and \( D^2_a \) over the plane \( (x_1 x_2) \):
\[
D^1_a = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2) = \rho(\cos \theta, \sin \theta), \rho \in (0, \infty), 0 < \theta < \alpha \right\},
\]
\[
D^2_a = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2) = \rho(\cos \theta, \sin \theta), \rho \in (0, \infty), \alpha < \theta < \pi \right\}.
\]

Note that \( \mathbf{A}_{\alpha,\gamma,a} \) satisfies
\[
b_3 := \text{curl} \mathbf{A}_{\alpha,\gamma,a} = s_{\alpha,a} \cos \gamma,
\]
where \( s_{\alpha,a} \) is the step function defined in \( \mathbb{R}^2_+ \) by
\[
s_{\alpha,a} = 1_{D^1_a} + a1_{D^2_a}.
\]
The field $\mathbf{B}_{\alpha,\gamma,a}$ is a magnetic field that projects $\mathbf{B}_{\alpha,\gamma,a}$ on $\mathbb{R}^2_+$ and it is defined as follows
\begin{equation}
\mathbf{B}_{\alpha,\gamma,a} = (b_1, b_2) = (\cos \alpha \sin \gamma, \sin \alpha \sin \gamma)\mathbf{s}_{\alpha,a}.
\end{equation}

We have
\begin{align*}
\mathbf{B}_{\alpha,\gamma,a} & = (x_1 b_2 - x_2 b_1 - \tau)^2, \\
& = \left\{(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^1, \\
& \quad |a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma)|^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^2.
\right.
\end{align*}

As a consequence, if $\lambda_{\alpha,a}$ is defined as follows
\begin{align*}
\mathbf{B}_{\alpha,\gamma,a} & = (x_1 b_2 - x_2 b_1 - \tau)^2, \\
& = \left\{(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^1, \\
& \quad |a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma)|^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^2.
\right.
\end{align*}

Theorem 3.1, we know that
\begin{align*}
\lambda_{\alpha,a} & = \inf \{x \in \mathbb{R} : (x_1 b_2 - x_2 b_1 - \tau)^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^1, \\
& \quad |a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma)|^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^2.
\end{align*}

We introduce the quadratic form associated to $\mathcal{L}_{\mathbf{A}_{\alpha,\gamma,a}} + \mathbf{B}_{\alpha,\gamma,a}\tau$:
\begin{equation}
\mathcal{Q}_{\alpha,\gamma,a}(u) = \int_{\mathbb{R}^2_*} \left|(\nabla - i\mathbf{A}_{\alpha,\gamma,a})u\right|^2 + \mathbf{B}_{\alpha,\gamma,a}\tau|u|^2 \, dx_1 dx_2.
\end{equation}

The form domain is
\begin{equation}
\mathcal{D}(\mathcal{Q}_{\alpha,\gamma,a}) = \{ u \in L^2(\mathbb{R}^2_*), \quad (\nabla - i\mathbf{A}_{\alpha,\gamma,a})u \in L^2(\mathbb{R}^2_*), \quad x_1 b_2 - x_2 b_1 |u| \in L^2(\mathbb{R}^2_*). \}
\end{equation}

We denote by $\sigma(\alpha, \gamma, a, \tau)$ the bottom of the spectrum of the operator $\mathcal{L}_{\mathbf{A}_{\alpha,\gamma,a}} + \mathbf{B}_{\alpha,\gamma,a}\tau$. We have
\begin{equation}
\sigma(\alpha, \gamma, a, \tau) = \inf \{x \in \mathbb{R} : (x_1 b_2 - x_2 b_1 - \tau)^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^1, \\
& \quad |a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma)|^2 \quad \text{for} \quad (x_1, x_2) \in D_\alpha^2.
\right.
\end{align*}

By (3.6), we have
\begin{equation}
\lambda_{\alpha,\gamma,a} = \inf \sigma(\alpha, \gamma, a, \tau).
\end{equation}

Hence, the study of $\lambda_{\alpha,\gamma,a}$ transforms to that of the associated band function $\tau \mapsto \sigma(\alpha, \gamma, a, \tau)$. This study will be the subject of the next subsections.

3.2. Case of a magnetic field parallel to the $x_3$–axis. We first treat the simple case when the magnetic field $\mathbf{B}_{\alpha,\gamma,a} = (0, 0, 1)\mathbf{s}_{\alpha,a}$ (i.e. when $\gamma = 0$). In this case, the field is parallel to the $x_3$–axis, thus $\mathbf{B}_{\alpha,0,a} = 0$. The operator $\mathcal{L}_{\mathbf{A}_{\alpha,0,a}} + \mathbf{B}_{\alpha,0,a}\tau$ reduces to a simpler operator
\begin{equation}
\mathcal{L}_{\mathbf{A}_{\alpha,0,a}} + \mathbf{B}_{\alpha,0,a}\tau = -(\nabla - i\mathbf{A}_{\alpha,0,a})^2 + \tau^2.
\end{equation}

For each $\tau \in \mathbb{R}$, the bottom of the spectrum of $\mathcal{L}_{\mathbf{A}_{\alpha,0,a}} + \mathbf{B}_{\alpha,0,a}\tau$ equals
\begin{equation}
\sigma(\alpha, 0, a, \tau) = \mu(\alpha, a) + \tau^2,
\end{equation}

where $\mu(\alpha, a)$ is the bottom of the spectrum of the operator $\mathcal{L}_{\mathbf{A}_{\alpha,0,a}} = -(\nabla + i\mathbf{A}_{\alpha,0,a})^2$. It immediately follows that
\begin{equation}
\lambda_{\alpha,0,a,\mathbb{R}^3_+} = \inf \sigma(\alpha, 0, a, \tau) = \sigma(\alpha, 0, a, 0) + \mu(\alpha, a).
\end{equation}

We present some properties of the operator $\mathcal{L}_{\mathbf{A}_{\alpha,0,a}}$, that is of $\mathcal{L}_{\mathbf{A}_{\alpha,0,a}} + \mathbf{B}_{\alpha,0,a}\tau$, obtained in [2, Section 3]. We denote by $\inf \text{sp}_{\text{ess}}$ the infimum of the essential spectrum. From [2, Theorem 3.1], we know that
\begin{equation}
\inf \text{sp}_{\text{ess}} \mathcal{L}_{\mathbf{A}_{\alpha,0,a}} = \inf \text{sp}_{\text{ess}}(\mathcal{L}_{\mathbf{A}_{\alpha,0,a}} + \mathbf{B}_{\alpha,0,a,\tau}) = |\mu(\Theta_\alpha)|.
\end{equation}

As a consequence, if $\mu(\alpha, a) < |\mu(\Theta_\alpha)|$ then $\mu(\alpha, a)$ is an eigenvalue of $\mathcal{L}_{\mathbf{A}_{\alpha,0,a}} + \mathbf{B}_{\alpha,0,a}\tau$. The foregoing properties will be used in the proof of Theorem 1.1, in the case $\gamma = 0$ (see Section 4).
3.3. Case of a magnetic field non-parallel to the $x_3$-axis. Now, we treat the case where the magnetic field $B_{\alpha,\gamma,a}$ is not parallel to the $x_3$-axis, that is the case when $\gamma \neq 0$ (see Figure 1). In this case, two auxiliary operators will be involved in the analysis. These operators are denoted by $H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]$ and $H_{\alpha,\gamma,a}^{\text{stp}}[\tau]$ and are respectively defined on $\mathbb{R}^2_+$ and $\mathbb{R}^2$ with a constant (resp. piecewise constant) magnetic field. We refer to $H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]$ as the ‘boundary operator’ since it will be used in the proof of Proposition 3.7 while studying it will be used in the study near the discontinuity line away from the boundary (see the proof of Proposition 3.3). We introduce these operators in what follows.

3.3.1. The boundary operator. Let $\tau \in \mathbb{R}$. We define $H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]$ as the magnetic Neumann realization of the following self-adjoint operator on $\mathbb{R}^2_+$

$$H_{\alpha,\gamma,a}^{\text{bnd}}[\tau] = -(\nabla - iA_{\gamma}^{\text{bnd}})^2 + [a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma) - \tau]^2, \quad (3.20)$$

where $A_{\gamma}^{\text{bnd}} \in H^1_{\text{loc}}(\mathbb{R}^2_+)$ is a magnetic potential with an associated constant magnetic field $\text{curl} A_{\gamma}^{\text{bnd}} = \cos \gamma$. This operator was studied in [38] in the case $a = 1$. Using translation, it was proven that the infimum of the spectrum of $H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]$ is independent of $\tau$. More precisely, in [38, Lemma 2.3] it is shown that

$$\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]) = \zeta_{\alpha_0}, \quad \forall \tau \in \mathbb{R}, \quad (3.21)$$

where $\zeta_{\alpha_0}$ is the value defined in (2.9) for $r_0 = \arcsin(\sin \alpha \sin \gamma)$.

Lemma 3.1 (Bottom of the spectrum of the boundary operator). Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$ and let $\gamma \in (0, \pi/2]$. Let $\tau \in \mathbb{R}$. It holds

$$\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]) = |a|\zeta_{\alpha_0}. \quad \text{Proof. By a simple scaling argument, one can prove that}$$

$$\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]) = |a|\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{bnd}}[\tau/a]).$$

Combining this with (3.21) completes the proof. \qed

3.3.2. The step operator. Let $\tau \in \mathbb{R}$. We define $H_{\alpha,\gamma,a}^{\text{stp}}[\tau]$ as the following self-adjoint operator on $\mathbb{R}^2$

$$H_{\alpha,\gamma,a}^{\text{stp}}[\tau] = -(\nabla - iA_{\alpha,\gamma,a}^{\text{stp}})^2 + [(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma)s_{\alpha,a}^{\text{stp}} - \tau]^2, \quad (3.22)$$

where $A_{\alpha,\gamma,a}^{\text{stp}} \in H^1_{\text{loc}}(\mathbb{R}^2)$ is such that $\text{curl} A_{\alpha,\gamma,a}^{\text{stp}} = s_{\alpha,a}^{\text{stp}} \cos \gamma$, and $s_{\alpha,a}^{\text{stp}}$ is the following step function on $\mathbb{R}^2$

$$s_{\alpha,a}^{\text{stp}} := 1_{P_+^a} + a1_{P_-^a},$$

with

$$P_+^a := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \sin \alpha - x_2 \cos \alpha > 0\},$$

$$P_-^a := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \sin \alpha - x_2 \cos \alpha < 0\}.$$

On can see the magnetic field $\text{curl} A_{\alpha,\gamma,a}^{\text{stp}}$ in $\mathbb{R}^2$ as the analogous of the magnetic field $\text{curl} A_{\alpha,\gamma,a}$ in $\mathbb{R}^2_+$, defined in (3.11), with the sets $P_+^a$ and $P_-^a$ as the analogous of the sets $D_+^a$ (in (3.9)) and $D_-^a$ (in (3.10)) respectively.

The next lemma determines the infimum of the spectrum of $H_{\alpha,\gamma,a}^{\text{stp}}[\tau]$.

Lemma 3.2 (Bottom of the spectrum of the step operator). Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$ and let $\gamma \in (0, \pi/2]$. Let $\tau \in \mathbb{R}$. It holds

$$\inf \text{sp}(H_{\alpha,\gamma,a}^{\text{stp}}[\tau]) = \inf_{\xi \in \mathbb{R}} [\mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2],$$

where $\mu_a(\cdot)$ is the value defined in (2.3).
Proof. For simplicity, we denote $H_{a,\gamma,a}^{\mathrm{step}}[\tau]$, $A_{a,\gamma,a}^{\mathrm{step}}$ and $s_{a,\gamma,a}^{\mathrm{step}}$ by $H^{\mathrm{step}}$, $A^{\mathrm{step}}$ and $s^{\mathrm{step}}$ respectively. To estimate the bottom of the spectrum of $H^{\mathrm{step}}$, we perform a rotation of angle $\alpha$ and get that$^6$ the operator $H^{\mathrm{step}}$ is unitarily equivalent to the following operator

$$
\tilde{H}^{\mathrm{step}} := -(\nabla - i\tilde{A}^{\mathrm{step}})^2 + (x_2 \sin \gamma s^{\mathrm{step}} + \tau)^2
$$

defined on $\mathbb{R}^2$, with curl $\tilde{A}^{\mathrm{step}} = \tilde{s}^{\mathrm{step}} \cos \gamma$ and $\tilde{s}^{\mathrm{step}} := 1_{\{x_2 < 0\}} + a 1_{\{x_2 > 0\}}$. Thus, we get

$$
\inf \mathrm{sp}(H^{\mathrm{step}}) = \inf \mathrm{sp}(\tilde{H}^{\mathrm{step}}). \tag{3.23}
$$

Performing a suitable change of gauge, we choose $\tilde{A}^{\mathrm{step}} = -(x_2 \cos \gamma s^{\mathrm{step}}, 0)$. Then, we write the expression of $\tilde{H}^{\mathrm{step}}$ explicitly as

$$
\tilde{H}^{\mathrm{step}} = -(\partial_{x_1} + ix_2 \cos \gamma s^{\mathrm{step}})^2 - \partial_{x_2}^2 + (x_2 \sin \gamma s^{\mathrm{step}} + \tau)^2.
$$

By a Fourier transform in the $x_1$ variable, we get

$$
\tilde{H}^{\mathrm{step}} = \int_{\xi \in \mathbb{R}} \left( - \partial_{x_2}^2 + (\xi + x_2 \cos \gamma s^{\mathrm{step}})^2 + (x_2 \sin \gamma s^{\mathrm{step}} + \tau)^2 \right) d\xi, \tag{3.24}
$$

where $-\partial_{x_2}^2 + (\xi + x_2 \cos \gamma s^{\mathrm{step}})^2 + (x_2 \sin \gamma s^{\mathrm{step}} + \tau)^2$ is a self-adjoint fiber operator on $\mathbb{R}$. Hence

$$
\inf \mathrm{sp}(\tilde{H}^{\mathrm{step}}) = \inf_{\xi} \left[ \inf \mathrm{sp}\left( - \partial_{x_2}^2 + (\xi + x_2 \cos \gamma s^{\mathrm{step}})^2 + (x_2 \sin \gamma s^{\mathrm{step}} + \tau)^2 \right) \right].
$$

We can now rewrite

$$(\xi + s_{a,\gamma} \cos \gamma)^2 + (s_{a,\gamma} \cos \gamma + \tau)^2 = (s_{a,\gamma} \cos \gamma + \xi \cos \gamma)^2 + (\xi \sin \gamma - \tau \cos \gamma)^2.
$$

Then using that $s^{\mathrm{step}} = 1_{\{x_2 < 0\}} + a 1_{\{x_2 > 0\}}$, the fiber operator in (3.24) is unitary equivalent to the operator given by

$$
\mathfrak{h}_a[\tau \sin \gamma + \xi \cos \gamma] + (\xi \sin \gamma - \tau \cos \gamma)^2,
$$

where $\mathfrak{h}_a[\cdot]$ is the operator defined in (2.1). Thus,

$$
\inf \mathrm{sp}(\tilde{H}^{\mathrm{step}}) = \inf \mathrm{sp}(\mathfrak{h}_a[\tau \sin \gamma + \xi \cos \gamma] + (\xi \sin \gamma - \tau \cos \gamma)^2). \tag{3.25}
$$

Moreover, we have

$$
\inf \mathrm{sp}(\mathfrak{h}_a[\tau \sin \gamma + \xi \cos \gamma] + (\xi \sin \gamma - \tau \cos \gamma)^2)
= \inf \mathrm{sp}(\mathfrak{h}_a[\tau \sin \gamma + \xi \cos \gamma] + (\xi \sin \gamma - \tau \cos \gamma)^2)
= \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2, \tag{3.26}
$$

where $\mu_a(\cdot)$ is the bottom of the spectrum of $\mathfrak{h}_a[\tau \sin \gamma + \xi \cos \gamma]$ (see (2.3)). Gathering (3.23), (3.25) and (3.26) completes the proof. $\square$

3.3.3. Bottom of the essential spectrum of the 2D reduced operator. In this section, we determine the infimum of the essential spectrum of the 2D operators $\mathcal{L}_{A_{a,\gamma,a}} + V_{B_{a,\gamma,a}}$ introduced in Section 3.1. For each $a \in [-1, 1] \setminus \{0\}$, $\gamma \in (0, \pi/2]$, $\alpha \in (0, \pi)$ and $\tau \in \mathbb{R}$, let

$$
\sigma_{\mathrm{ess}}(\alpha, \gamma, a, \tau) := \inf \mathrm{sp}_{\mathrm{ess}}(\mathcal{L}_{A_{a,\gamma,a}} + V_{B_{a,\gamma,a}}). \tag{3.27}
$$

Knowing this infimum will be useful in determining values of $(\alpha, \gamma, a, \tau)$ where the bottom of the spectrum $\sigma(\alpha, \gamma, a, \tau)$ of these operators is an eigenvalue. This will be used in establishing Theorem 1.1 later. The next proposition is the main result of this section.

**Proposition 3.3** (Characterization of $\sigma_{\mathrm{ess}}(\alpha, \gamma, a, \tau)$). Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in (0, \pi/2]$ and $\tau \in \mathbb{R}$. Let $\sigma_{\mathrm{ess}}(\alpha, \gamma, a, \tau)$ be as in (3.27), we have

$$
\sigma_{\mathrm{ess}}(\alpha, \gamma, a, \tau) = \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2 \right),
$$

where $\mu_a(\cdot)$ is the value defined in (2.3).

$^6$We refer to [38, Sec.1] for rotation invariance principles.
For the proof of Proposition 3.3 we need the following lemma.

**Lemma 3.4.** Let \( a \in [-1, 1] \setminus \{0\} \), \( \alpha \in (0, \pi) \), \( \gamma \in (0, \pi/2) \) and let \( \tau \in \mathbb{R} \). Let \( \mathcal{L}_{\alpha, \gamma, a} \) be as in (3.27). It holds

\[
\mathcal{L}_{\alpha, \gamma, a} = \lim_{R \to +\infty} \sum \left( \mathcal{L}_{\alpha, \gamma, a} + \nabla \mathcal{B}_{\alpha, \gamma, a, \tau}, R \right),
\]

with

\[
\sum \left( \mathcal{L}_{\alpha, \gamma, a} + \nabla \mathcal{B}_{\alpha, \gamma, a, \tau}, R \right) := \inf_{u \in C_0^\infty \left( \mathbb{R}^2 \setminus \mathcal{C} \right)} \| u \|_2^2 \left( \mathcal{B}_R \right),
\]

where \( \mathcal{B}_R \) is a ball of radius \( R \) centered at the origin, \( \mathcal{C} \) is its complement in \( \mathbb{R}^2 \), and \( \nabla \mathcal{B}_{\alpha, \gamma, a, \tau} \) is the gradient form defined in (3.15).

Lemma 3.4 is a well-known Persson-type result, useful to characterize the bottom of essential spectra. We refer the reader to [1,37,38] for this type of results, and [2, Appendix A] for a detailed proof in similar situations. Moreover, in the proof of Proposition 3.3, we shall see the importance of determining where the electric potential \( \mathcal{B}_{\alpha, \gamma, a, \tau} \) attains its infimum and where it is big. To that end, we define the set

\[
\mathcal{Y}_{\alpha, \gamma, a, \tau} = \left\{ x \in \mathbb{R}^2_+ : \mathcal{B}_{\alpha, \gamma, a, \tau}(x) = \inf_{y \in \mathbb{R}^2_+} \mathcal{B}_{\alpha, \gamma, a, \tau}(y) \right\}.
\]

(3.28)

We note that \( \mathcal{Y}_{\alpha, \gamma, a, \tau} \) is not necessary \( \mathcal{B}_{\alpha, \gamma, a, \tau}^{-1}(\{0\}) \); determining this set depends on the values of \( a \in [-1, 1] \setminus \{0\} \) and \( \tau \in \mathbb{R} \), as shown in what follows. We recall that \( \mathcal{B}_{\alpha, \gamma, a, \tau} \) is defined for \( x = (x_1, x_2) \in \mathbb{R}^2_+ \) as

\[
\mathcal{B}_{\alpha, \gamma, a, \tau}(x) = \left( x_1 b_\alpha - x_2 b_\alpha - \tau \right)^2,
\]

\[
= \begin{cases} (x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2 & \text{for } (x_1, x_2) \in D_\alpha^1, \\ [a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma) - \tau]^2 & \text{for } (x_1, x_2) \in D_\alpha^2, \end{cases}
\]

where \( D_\alpha^1 \) and \( D_\alpha^2 \) are as in (3.9) and (3.10). We now define, for \( x = (x_1, x_2) \in \mathbb{R}^2 \),

\[
\mathcal{B}_{\alpha, \gamma, a, \tau}^{(1)}(x) = (x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma - \tau)^2,
\]

\[
\mathcal{B}_{\alpha, \gamma, a, \tau}^{(2)}(x) = [a(x_1 \sin \alpha \sin \gamma - x_2 \cos \alpha \sin \gamma) - \tau]^2
\]

and the following subsets of \( \mathbb{R}^2 \)

\[
\mathcal{Y}_{\alpha, \gamma, a, \tau}^{(1)} = \left( \mathcal{B}_{\alpha, \gamma, a, \tau}^{(1)} \right)^{-1}(\{0\}) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha = \frac{\tau}{\sin \gamma} \right\},
\]

\[
\mathcal{Y}_{\alpha, \gamma, a, \tau}^{(2)} = \left( \mathcal{B}_{\alpha, \gamma, a, \tau}^{(2)} \right)^{-1}(\{0\}) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha = \frac{\tau}{a \sin \gamma} \right\}.
\]

(3.29)

Note that \( \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(1)} \) and \( \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(2)} \) are two lines parallel to the discontinuity line \( l_\alpha \) of equation \( x_1 \sin \alpha - x_2 \cos \alpha = 0 \). Moreover, for \( x \in \mathbb{R}^2 \)

\[
\mathcal{B}_{\alpha, \gamma, a, \tau}^{(1)}(x) = \sin^2 \gamma \text{dist}^2(x, \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(1)}), \quad \mathcal{B}_{\alpha, \gamma, a, \tau}^{(2)}(x) = a^2 \sin^2 \gamma \text{dist}^2(x, \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(2)}).
\]

(3.30)

We keep denoting by \( \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(1)} \) (resp. \( \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(2)} \)) the intersection between \( \mathbb{R}^2_+ \) and \( \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(1)} \) (resp. \( \mathcal{Y}_{\alpha, \gamma, a, \tau}^{(2)} \)).
Figure 4. For $\alpha \in (0, \pi)$, $\gamma \in (0, \pi/2)$ and $a \in [-1, 0]$, the set $\Upsilon_{a, \gamma, \alpha, \tau}$ is drawn in blue. For $\tau \geq 0$ (at right), $\Upsilon_{a, \gamma, \alpha, \tau} = \Upsilon_{a, \gamma, \tau}^{(1)} \cup \Upsilon_{a, \gamma, a, \tau}^{(2)}$. For $\tau < 0$ (at left), $\Upsilon_{a, \gamma, \alpha, \tau} = l_{\alpha}$.

Figure 5. For $\alpha \in (0, \pi)$, $\gamma \in (0, \pi/2)$ and $a \in (0, 1)$, the set $\Upsilon_{a, \gamma, \alpha, \tau}$ is drawn in blue. For $\tau \geq 0$ (at right), $\Upsilon_{a, \gamma, \alpha, \tau} = \Upsilon_{a, \gamma, \tau}^{(1)} \cup \Upsilon_{a, \gamma, a, \tau}^{(2)}$. For $\tau < 0$ (at left), $\Upsilon_{a, \gamma, \alpha, \tau} = \Upsilon_{a, \gamma, a, \tau}^{(2)}$.

Lemma 3.5 (The set $\Upsilon_{a, \gamma, \alpha, \tau}$). Let $a \in [-1, 1) \setminus \{0\}$, $\alpha \in (0, \pi)$, and $\gamma \in (0, \pi/2]$. Let $\Upsilon_{a, \gamma, \alpha, \tau} \subset \mathbb{R}^2$ be the set defined in (3.28). It holds

$$\Upsilon_{a, \gamma, \alpha, \tau} = \begin{cases} l_{\alpha} & \text{if } a \in [-1, 0), \tau < 0 \\ \Upsilon_{a, \gamma, \tau}^{(1)} \cup \Upsilon_{a, \gamma, a, \tau}^{(2)} & \text{if } a \in [-1, 0), \tau \geq 0 \\ \Upsilon_{a, \gamma, \alpha, \tau}^{(1)} & \text{if } a \in (0, 1), \tau \geq 0 \\ \Upsilon_{a, \gamma, a, \tau}^{(2)} & \text{if } a \in (0, 1), \tau < 0. \end{cases}$$

Indeed (see Figures (4) and (5)),

- Case $a \in [-1, 0)$ and $\tau < 0$. One observes that $V_{\mathcal{B}_{a, \gamma, a, \tau}}^{-1} (\{0\}) = \emptyset$. In this case,
  $$\Upsilon_{a, \gamma, a, \tau} = l_{\alpha} \quad \text{and} \quad \inf_{\mathcal{B}_{a, \gamma, a, \tau}} V_{\mathcal{B}_{a, \gamma, a, \tau}}^{-1} = \tau^2.$$  
  (3.31)

- Case $a \in [-1, 0)$ and $\tau \geq 0$. Here,
  $$\Upsilon_{a, \gamma, a, \tau} = V_{\mathcal{B}_{a, \gamma, a, \tau}}^{-1} (\{0\}) = \Upsilon_{a, \gamma, \tau}^{(1)} \cup \Upsilon_{a, \gamma, a, \tau}^{(2)},$$

Note that this set is $l_{\alpha}$ for $\tau = 0$. 

\[ \sum_{\tau < 0} \]
• Case $a \in (0, 1)$ and $\tau < 0$. In this case,
  \[ \Upsilon_{\alpha, \gamma, a, \tau} = V_{B_{\alpha, \gamma, a, \tau}}^{-1}(\{0\}) = \Upsilon_{\alpha, \gamma, a, \tau}^{(2)}. \]

• Case $a \in (0, 1)$ and $\tau \geq 0$. We have
  \[ \Upsilon_{\alpha, \gamma, a, \tau} = V_{B_{\alpha, \gamma, a, \tau}}^{-1}(\{0\}) = \Upsilon_{\alpha, \gamma, \tau}^{(1)}. \]

Again, this set is $l_\alpha$ for $\tau = 0$.

Now, we prove Proposition 3.3.

**Proof of Proposition 3.3.** The idea of the proof is similar to that in [38, Proposition 3.2] (see also [2, Lemma 3.7]). However, one has to take into consideration the particular properties of the electric potential discussed above, which are induced by the discontinuity of our magnetic field.

The ‘step operator’ $H_{\alpha, \gamma, a, \tau}^{\text{stp}}[r]$ (in (3.22)) and the quadratic form $Q_{\alpha, \gamma, a}$ (in (3.15)) are used frequently in the proof. Throughout the proof, we simplify their notation and denote them respectively by $H_{\alpha}^{\text{stp}}$ and $Q_{\alpha}$.

In light of Lemma 3.4, it suffices to prove that

\[
\lim_{R \to +\infty} \Sigma(L_{\alpha, \gamma, a} + V_{B_{\alpha, \gamma, a, \tau}}, R) = \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2 \right). \tag{3.32}
\]

We establish separately an upper bound and a lower bound for the limit above.

**Upper bound.** Let $\epsilon > 0$ and $R > 0$. Considering the operator $H_{\alpha}^{\text{stp}}$, the min-max principle ensures the existence of a normalized function $u_\epsilon \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$ such that

\[
\langle H_{\alpha}^{\text{stp}}u_\epsilon, u_\epsilon \rangle < \inf_{\xi \in \mathbb{R}}(\mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2) + \epsilon, \tag{3.33}
\]

where the last equality follows from Lemma 3.2. Let the function $u_{\epsilon, r}$ be the translation of $u_\epsilon$ by a vector $r$, i.e., $u_{\epsilon, r}(x) = u_\epsilon(x - r)$ for $x \in \mathbb{R}^2$, where $r$ is an upward direction vector of the discontinuity line $l_\alpha: x_1 \sin \alpha - x_2 \cos \alpha = 0$. We have $u_{\epsilon, r} \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$. Moreover, there exists $r_0 > 0$ such that the function $u_{\epsilon, r}$ is supported in $\mathbb{R}^2_+ \cap \mathcal{C}B_R$, for $|r| > r_0$. Using that $H_{\alpha}^{\text{stp}}$ is invariant by translation in the $l_\alpha$ direction (see (3.22)), we get

\[
\langle H_{\alpha}^{\text{stp}}u_{\epsilon, r}, u_{\epsilon, r} \rangle = \langle H_{\alpha}^{\text{stp}}u_\epsilon, u_\epsilon \rangle. \tag{3.34}
\]

Combining (3.33) and (3.34) gives

\[
\langle H_{\alpha}^{\text{stp}}u_{\epsilon, r}, u_{\epsilon, r} \rangle < \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2 \right) + \epsilon.
\]

Now, using the support properties of $u_{\epsilon, r}$, a direct calculation shows that $\langle H_{\alpha}^{\text{stp}}(u_{\epsilon, r}), u_{\epsilon, r} \rangle = Q(u_{\epsilon, r})$. Then,

\[
Q(u_{\epsilon, r}) < \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2 \right) + \epsilon.
\]

Having $u_{\epsilon, r}$ a non-zero normalized function in $C_0^\infty(\mathbb{R}^2_+ \cap \mathcal{C}B_R)$, we have

\[
\Sigma(L_{\alpha, \gamma, a} + V_{B_{\alpha, \gamma, a, \tau}}, R) = \inf_{u \in C_0^\infty(\mathbb{R}^2_+ \cap \mathcal{C}B_R)} \frac{Q(u)}{||u||_{L^2(\mathbb{R}^2_+)}^2} < \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2 \right) + \epsilon.
\]

Taking first $\epsilon \to 0$ and then $R \to +\infty$, we get the upper bound in (3.32).
**Lower bound.** Let \((\rho, \theta)\) be the polar coordinates in \(\mathbb{R}^2\). We consider a partition of unity \((\chi_{j}^{\text{pol}})_{j=1,2,3} \subset C^\infty(\mathbb{R}_+ \times [0, \pi])\) such that: for \(j \in \{1, 2, 3\}\), \(0 \leq \chi_j^{\text{pol}} \leq 1\) and \(\forall (\rho, \theta) \in \mathbb{R}_+ \times (0, \pi)\), \(\chi_j^{\text{pol}}(\rho, \theta) = \chi_j^{\text{pol}}(1, \theta)\) and
\[
\chi_1^{\text{pol}}(\rho, \theta) = \begin{cases} 1 & \text{for } \theta \in \left(0, \frac{1}{8\alpha}\right), \\ 0 & \text{otherwise} \end{cases},
\chi_2^{\text{pol}}(\rho, \theta) = \begin{cases} 1 & \text{for } \theta \in \left[\frac{1}{4\alpha}, \frac{1}{4\alpha} + \frac{3\pi}{4}\right], \\ 0 & \text{otherwise} \end{cases},
\chi_3^{\text{pol}}(\rho, \theta) = \begin{cases} 1 & \text{for } \theta \in \left[\frac{1}{8\alpha} + \frac{7\pi}{8}, \pi\right), \\ 0 & \text{otherwise} \end{cases}.
\]
Moreover, \(\sum_{j=1}^3 |\chi_j^{\text{pol}}|^2 = 1\) and \(\sum_{j=1}^3 |(\chi_j^{\text{pol}})|^2 \leq C\), where \(C\) is a constant dependent on \(\alpha\) but independent of \(a\). Let \((\chi_j)_{j=1,\ldots,3}\) be the associated functions in Cartesian coordinates
\[
\chi_j(x_1, x_2) = \chi_j^{\text{pol}}(\rho, \theta), \quad (x_1, x_2) \in \mathbb{R}^2.\]
For \(R > 0\) and \(u \in C_0^\infty(\mathbb{R}_+^2 \cap \mathcal{B}_R)\), we use the IMS formula to write (see [12, Theorem 3.2])
\[
Q(u) = \sum_{j=1}^3 Q(\chi_j u) - \sum_{j=1}^3 \|u|\nabla \chi_j\|_{L^2(\mathbb{R}^2_+)}^2. \tag{3.35}
\]
We start by bounding the error term \(\sum_{j=1}^3 \|u|\nabla \chi_j\|_{L^2(\mathbb{R}^2_+)}^2\). For \(x = (x_1, x_2) \in \mathbb{R}^2_+\), we have
\[
|\nabla_x \chi_j(x_1, x_2)|^2 = |\partial_{x_j} \chi_j^{\text{pol}}(\rho, \theta)|^2 + \frac{1}{r^2} |\partial_\theta \chi_j^{\text{pol}}(\rho, \theta)|^2 = \frac{1}{r^2} |\partial_\theta \chi_j^{\text{pol}}(\rho, \theta)|^2,
\]
where the last equality follows from the fact that \(\chi_j^{\text{pol}}\) is constant in the radial coordinate. Thus, using \(\sum_{j=1}^3 |(\chi_j^{\text{pol}})|^2 \leq C\) and that \(u\) is supported outside \(\mathcal{B}_R\), we get
\[
\sum_{j=1}^3 \|u|\nabla \chi_j\|_{L^2(\mathbb{R}^2_+)}^2 \leq C \frac{R^2}{R^2} \|u\|_{L^2(\mathbb{R}^2_+)}^2.
\]
Next, we consider the main term, \(\sum_{j=1}^3 Q(\chi_j u),\) in (3.35). We start by bounding \(Q(\chi_2 u).\) Extending \(\chi_2 u\) by zero over \(\mathbb{R}^2\), we get that \(\chi_2 u\) is in the domain of the operator \(H^{\text{stp}}.\) By Lemma 3.2, we notice that
\[
Q(\chi_2 u) = \langle H^{\text{stp}}(\chi_2 u), \chi_2 u \rangle \geq \inf_{\xi \in \mathbb{R}} (\mu_\alpha(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2) \|\chi_2 u\|^2. \tag{3.36}
\]
Now, we bound \(Q(\chi_j u)\) for \(j = 1, 3.\) Here, we recall the sets \(\Upsilon_{\alpha, \gamma, \tau}^{(1)}\) and \(\Upsilon_{\alpha, \gamma, \tau}^{(2)}\) defined in (3.29). We choose a large \(R_0 > 0\) and assume w.l.o.g that \(\alpha \in (0, \pi/2)\), then an elementary computation yields for \(R > R_0\) (see Figure 6):
Thus, we can write

\[ Q(\chi_1 u) \geq \left| R \sin \left( \frac{3\alpha}{4} \right) \sin \gamma - \tau \right|^2 \| \chi_1 u \|^2 \quad Q(\chi_3 u) \geq |aR \sin \alpha \sin \gamma + \tau|^2 \| \chi_3 u \|^2. \]

Consequently for all \( R > R_0 \), (3.35), (3.36) and (3.37) imply

\[ \Sigma(\mathcal{L}_{\mathcal{A}_{\alpha,\gamma,a}} + V_{\mathcal{B}_{\alpha,\gamma,a}}, R) \geq \inf_{\xi \in \mathbb{R}} \left( \mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2 \right) - \frac{C}{R^2}. \]

Taking the limit \( R \to +\infty \), we establish the lower bound in (3.32). \( \square \)

3.3.4. Bottom of the spectrum of the 2D reduced operator at infinity. Now, we consider the bottom of the spectrum, \( \sigma(\alpha, \gamma, a, \tau) \), of the operator \( \mathcal{L}_{\mathcal{A}_{\alpha,\gamma,a}} + V_{\mathcal{B}_{\alpha,\gamma,a}} \) as a function of \( \tau \). In Proposition 3.7 below, we study the behavior of \( \sigma(\alpha, \gamma, a, \tau) \) as \( |\tau| \) goes to infinity. We will use this proposition to provide a condition on \( (\alpha, \gamma, a) \) such that \( \inf_{\tau} \sigma(\alpha, \gamma, a, \tau) \) that is \( \lambda_{\alpha,\gamma,a} \) (see (3.17)) is attained by some \( \tau \in \mathbb{R} \). This, together with the upper bound of the essential spectrum in Corollary 3.6, will be used to get the result in Theorem 1.1, when the strict inequality in (1.10) is satisfied.
Proposition 3.7. Let $\alpha \in (0, \pi)$ and $\gamma \in (0, \pi/2]$. For $a \in [-1,0)$, we have
\[
\lim_{\tau \rightarrow -\infty} \sigma(\alpha, \gamma, a, \tau) = +\infty, \quad \lim_{\tau \rightarrow +\infty} \sigma(\alpha, \gamma, a, \tau) = |a|\zeta_{\nu_0}.
\]
For $a \in (0,1)$, we have
\[
\lim_{\tau \rightarrow -\infty} \sigma(\alpha, \gamma, a, \tau) = a\zeta_{\nu_0}, \quad \lim_{\tau \rightarrow +\infty} \sigma(\alpha, \gamma, a, \tau) = \zeta_{\nu_0}.
\]
Here, $\zeta_{\nu_0}$ is defined in (2.9) for $\nu_0 = \arcsin(\sin \alpha \sin \gamma)$.

Proof. In this proof, we simplify the notation and write $H_{\alpha,\gamma,a}^{\text{bnd}}$ for the ‘boundary operator’ $H_{\alpha,\gamma,a}^{\text{bnd}}[\tau]$ in (3.21) and $Q$ for the quadratic form $Q_{\alpha,\gamma,a}^\tau$ in (3.31) associated to the operator $L_{A_{\alpha,\gamma,a}} + V_{B_{\alpha,\gamma,a}}[\tau]$.

Case $a \in [-1,0)$. Establishing the limit when $\tau \rightarrow -\infty$ is straightforward. Indeed, considering the electric potential in (3.14), by (3.31) we have $\inf V_{\alpha,\gamma,a}^\tau = \tau^2$, for any $\tau < 0$. Then, $\lim_{\tau \rightarrow -\infty} \inf V_{\alpha,\gamma,a}^\tau = +\infty$. Hence,
\[
\lim_{\tau \rightarrow -\infty} \sigma(\alpha, \gamma, a, \tau) = +\infty.
\]
Now, we treat the case $\tau \rightarrow +\infty$. Here, the foregoing operator $H_{\alpha,\gamma,a}^{\text{bnd}}$ in (3.20) will be involved. By the min-max principle and Lemma 3.1, for any $\epsilon > 0$, there exists a normalized function $u_\epsilon \in C_0^\infty((\mathbb{R}^2)) \setminus \{0\}$ such that
\[
\langle H_{\alpha,\gamma,a}^{\text{bnd}} u_\epsilon, u_\epsilon \rangle < |a|\zeta_{\nu_0} + \epsilon. \tag{3.38}
\]
We define the function $u_{\epsilon,\tau}$ as follows
\[
u_{\epsilon,\tau}(x) = u_\epsilon \left( x_1 - \frac{\tau}{a \sin \alpha \sin \gamma}, x_2 \right), \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.
\]
For a sufficiently large $\tau$, we have $\text{supp } u_{\epsilon,\tau} \subset D_\alpha^2$, where $D_\alpha^2$ is the set in (3.10). Performing a suitable change of gauge, in which we associate the function $\tilde{u}_{\epsilon,\tau}$ to the function $u_{\epsilon,\tau}$, we get
\[
\frac{Q(\tilde{u}_{\epsilon,\tau})}{\langle (L_{A_{\alpha,\gamma,a}} + V_{B_{\alpha,\gamma,a}}[\tau])\tilde{u}_{\epsilon,\tau}, \tilde{u}_{\epsilon,\tau} \rangle} = \langle H_{\alpha,\gamma,a}^{\text{bnd}} u_{\epsilon,\tau}, u_{\epsilon,\tau} \rangle = \langle H_{\alpha,\gamma,a}^{\text{bnd}} u_\epsilon, u_\epsilon \rangle < |a|\zeta_{\nu_0} + \epsilon,
\]
where in the last inequality we used (3.38). Taking $\tau$ to $+\infty$, we get
\[
\lim_{\tau \rightarrow +\infty} \sup_{\tau \rightarrow +\infty} \sigma(\alpha, \gamma, a, \tau) \leq |a|\zeta_{\nu_0}.
\]
Next, we establish the lower bound for $\lim_{\tau \rightarrow +\infty} \sigma(\alpha, \gamma, a, \tau)$. We consider a partition of unity $(\tilde{\chi}_j)_{j \in \{1,2,3\}} \subset C^\infty(\mathbb{R})$ satisfying
\[
supp \tilde{\chi}_1 \subset \left( \frac{1}{4\sin \gamma}, +\infty \right), \quad sup \tilde{\chi}_2 \subset \left( \frac{1}{2a \sin \gamma}, \frac{1}{2\sin \gamma} \right), \quad sup \tilde{\chi}_3 \subset (-\infty, \frac{1}{4a \sin \gamma})
\]
\[
\sum_j |\tilde{\chi}_j|^2 = 1, \quad \sum_j |\tilde{\chi}_j'|^2 \leq C,
\]
for a certain $C > 0$ independent of $\tau$. Let $(\chi_j)_{j \in \{1,2,3\}} \subset C^\infty(\mathbb{R}^2)$ be the partition of unity of $\mathbb{R}^2$ induced from $(\tilde{\chi}_j)_{j \in \{1,2,3\}}$ as follows
\[
\chi_j(x_1, x_2) = \tilde{\chi}_j \left( \frac{x_1 \sin \alpha - x_2 \cos \alpha}{\tau} \right).
\]
Consequently, we have for $j \in \{1,2,3\}$
\[
supp \chi_j \subset R_j, \quad \sum_j |\chi_j|^2 = 1, \quad \text{and } \sum_j |\nabla \chi_j|^2 \leq \frac{C}{\tau^2}.
\]
where
\[
\begin{align*}
R_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha > \frac{\tau}{4 \sin \gamma}\} \\
R_2 := \{(x_1, x_2) \in \mathbb{R}^2 : \frac{\tau}{2a \sin \gamma} < x_1 \sin \alpha - x_2 \cos \alpha < \frac{\tau}{2 \sin \gamma}\} \\
R_3 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \sin \alpha - x_2 \cos \alpha < \frac{\tau}{4a \sin \gamma}\}
\end{align*}
\]

Thus, for any \(u \in D(Q)\) (see (3.1)), the IMS formula gives
\[
Q(u) = \sum_{j=1}^{3} Q(\chi_j u) - \sum_{j=1}^{3} \|\nabla \chi_j\|^2_{L^2(\mathbb{R}^2)} \geq \sum_{j=1}^{3} Q(\chi_j u) - \frac{C}{\tau^2} \tag{3.39}
\]

We perform a suitable change of gauge and use (3.21), together with the support properties of \(\chi_1 u\), to get
\[
Q(\chi_1 u) = \int_{\mathbb{R}^2} \left( |(\nabla - iA_{\alpha,\gamma,a})(\chi_1 u)|^2 + V_{B_{\alpha,\gamma,a},\tau} |\chi_1 u|^2 \right) dx_1 dx_2 \tag{3.40}
\]
\[
\geq \zeta_0 \|\chi_1 u\|^2.
\]

Similarly, considering the support of \(\chi_3 u\), doing a change of gauge and using Lemma 3.1, we find
\[
Q(\chi_3 u) \geq |a| \zeta_0 \|\chi_3 u\|^2. \tag{3.41}
\]

Finally, considering the support of \(\chi_2 u\), a simple computation using the definition of the electric potential in (3.14) and Lemma 3.5 (see also Figure 4) gives
\[
V_{B_{\alpha,\gamma,a},\tau} \geq \frac{\tau^2}{4}, \quad \text{for } x \in \text{supp } \chi_2 u. \tag{3.42}
\]

Hence, there exists \(\tau_0 > 0\) and \(M > |a| \zeta_0\) such that for \(\tau > \tau_0\)
\[
Q(\chi_2 u) \geq M \|\chi_2 u\|^2. \tag{3.43}
\]

Implementing (3.40), (3.41) and (3.43) in (3.39), we get for \(a \in [-1,0)\)
\[
\lim_{\tau \to +\infty} \inf_{\alpha, \gamma, \tau} \sigma(\alpha, \gamma, a, \tau) \geq |a| \zeta_0.
\]

**Case** \(a \in (0,1)\). Adopting a similar approach as above, using Lemma 3.5 for positive values of \(a\), one can establish the results of the proposition in this case. We omit further computation details. □

4. Proof of Theorem 1.1 and Proposition 1.4

**Proof of Theorem 1.1.** The proof in the case \(\gamma = 0\), is a direct consequence of the results in Section 3.2. Indeed, from (3.18) and (3.19) it follows that
\[
\lambda_{\alpha,0,a} \leq |a| \Theta_0.
\]

Now, from (2.10), we know that \(\zeta_0 = \Theta_0\) and having \(\beta_a \geq |a| \Theta_0\) (see Section 2.1), we get that
\[
\lambda_{\alpha,0,a} \leq \min(\beta_a, |a| \Theta_0) = \min(\beta_a, |a| \zeta_0).
\]

Moreover, it follows from Section 3.2 that if \(\lambda_{\alpha,0,a} < \min(\beta_a, |a| \zeta_0)\), then \(\lambda_{\alpha,0,a}\) is an eigenvalue of the operator \(L_{A_{\alpha,\gamma,a}} + V_{B_{\alpha,\gamma,a},\tau^*}\), with the particular choice \(\tau^* = 0\).

Next, we treat the case \(\gamma \neq 0\). We first establish the upper bound of \(\lambda_{\alpha,\gamma,a}\) in (1.9). The result is a consequence of Proposition 3.3 and Proposition 3.7, as it is shown below. We have (see (3.17))
\[
\lambda_{\alpha,\gamma,a} = \inf_{\tau} \sigma(\alpha, \gamma, a, \tau), \tag{4.1}
\]
where \( g(\alpha, \gamma, a, \tau) \) is as in (3.16). We consider the following two cases.

**Case** \( a \in [-1, 0) \). From Proposition 3.3, we have

\[
\mathcal{E}_{\text{ess}}(\alpha, \gamma, a, \tau) = \inf_{\xi \in \mathbb{R}} (\mu_a(\tau \sin \gamma + \xi \cos \gamma) + (\xi \sin \gamma - \tau \cos \gamma)^2),
\]

where \( \mu_a(\cdot) \) is introduced in (2.3). Let \( \xi_a \) be the unique minimum of \( \mu_a(\cdot) \) (see Section 2.1). For \( \tilde{\tau} := \xi_a \sin \gamma \), one can see that \( \mathcal{E}_{\text{ess}}(\alpha, \gamma, a, \tilde{\tau}) \) is attained by \( \xi = \xi_a \cos \gamma \) and satisfies

\[
\mathcal{E}_{\text{ess}}(\alpha, \gamma, a, \tilde{\tau}) = \mu_a(\xi_a) = \beta_a.
\]

This implies that

\[
g(\alpha, \gamma, a, \tilde{\tau}) \leq \beta_a. \tag{4.2}
\]

Moreover, by Proposition 3.7, we have

\[
g(\alpha, \gamma, a, \tilde{\tau}) \leq |a| \zeta_{\nu_0}. \tag{4.3}
\]

Combining (4.1)–(4.3) yields (1.9).

**Case** \( a \in (0, 1) \). By Proposition 3.7, we have

\[
g(\alpha, \gamma, a, \tau) \leq a \zeta_0.
\]

Moreover, \( \beta_a = a \) for \( a \in (0, 1) \) (see Section 2.1), and \( \zeta_0 \leq 1 \) (see Section 2.2). This yields

\[
\lambda_{\alpha, \gamma, a} \leq a \zeta_{\nu_0} = \min(\beta_a, a \zeta_0).
\]

Now, we consider the case when the strict inequality in (1.10) is satisfied. From Proposition 3.7, we have

\[
\inf_{\tau} g(\alpha, \gamma, a, \tau) = \lambda_{\alpha, \gamma, a} < |a| \zeta_{\nu_0} = \min\left(\lim_{\tau \to -\infty} g(\alpha, \gamma, a, \tau), \lim_{\tau \to +\infty} g(\alpha, \gamma, a, \tau)\right).
\]

Hence, \( \inf_{\tau} g(\alpha, \gamma, a, \tau) \) is attained by some \( \tau_s \in \mathbb{R} \). Moreover, by Corollary 3.6 we know that

\[
\lambda_{\alpha, \gamma, a} = g(\alpha, \gamma, a, \tau_s) < \beta_a \leq \mathcal{E}_{\text{ess}}(\alpha, \gamma, a, \tau_s).
\]

We then deduce that \( \lambda_{\alpha, \gamma, a} \) is an eigenvalue of \( \mathcal{L}_{A, \gamma, a} + V_{B, \alpha, \gamma, a, \tau_s} \). \( \square \)

**Proof of Proposition 1.4.** The proof is inspired by the construction done in [2, Proof of Proposition 3.9] while studying 2D smooth domains under discontinuous magnetic fields, and by [18, Proof of Theorem 1.1] while studying 2D corner domains under constant magnetic fields.

We fix \( a \in [-1, 1] \setminus \{0\}, \) \( \alpha \in (0, \pi), \) and \( \gamma \in [0, \pi/2] \). Let \( \tau = 0 \). We define the function \( \varphi_{\alpha, \gamma, a} \in H^1_{\text{loc}}(\mathbb{R}^2_+) \) by

\[
\varphi_{\alpha, \gamma, a}(x_1, x_2) = \begin{cases} 
\left(\frac{1}{2} x_1 x_2 + \frac{a-1}{2} x_2^2 \cot \alpha\right) \cos \gamma & \text{if } (x_1, x_2) \in D^1_{\alpha}, \\
\frac{a}{2} x_1 x_2 \cos \gamma & \text{if } (x_1, x_2) \in D^2_{\alpha}.
\end{cases}
\]

This function satisfies \( A_{\alpha, \gamma, a} = A_{\alpha, \gamma, a} + \nabla \varphi_{\alpha, \gamma, a} \), where \( A_{\alpha, \gamma, a} \) is the potential in (3.8), and \( A_{\alpha, \gamma, a} = 1/2(-x_2, x_1) \mathbf{s}_{\alpha, a} \cos \gamma \), for \( \mathbf{s}_{\alpha, a} = 1_{D^1_{\alpha}} + a 1_{D^2_{\alpha}} \) being the step function in (3.12) (see [31, Lemma 1.1] for the existence of such gauge functions in more general situations).

We define the quadratic form \( \tilde{Q}_{\alpha, \gamma, a} \) as follows

\[
\tilde{Q}_{\alpha, \gamma, a}(v) = \int_{\mathbb{R}^2_+} \left( |(\nabla - i \tilde{A}_{\alpha, \gamma, a}) v|^2 + V_{B, \alpha, \gamma, a, 0} |v|^2 \right) dx_1 \, dx_2
\]

in the domain

\[
\mathcal{D}(\tilde{Q}_{\alpha, \gamma, a}) = \left\{ v \in L^2(\mathbb{R}^2_+) : (\nabla - i \tilde{A}_{\alpha, \gamma, a}) v \in L^2(\mathbb{R}^2_+), |x_1 \sin \alpha - x_2 \cos \alpha| v \in L^2(\mathbb{R}^2_+) \right\},
\]
where $V_{A,\gamma,a,0} = s_{\alpha,a}^2 (x_1 \sin \gamma \sin \alpha - x_2 \sin \gamma \cos \alpha)^2$ is the electric potential defined in (3.14) for $\tau = 0$. We explicitly express $\bar{Q}_{\alpha,\gamma,a}(v)$ by
\[
\int_{\mathbb{R}^2_+} \left( |(\partial_x + i\frac{1}{2}s_{\alpha,a}x_2 \cos \gamma v|^2 + |(\partial_x - i\frac{1}{2}s_{\alpha,a}x_1 \cos \gamma v)^2 + s_{\alpha,a}^2 \sin^2 \gamma (x_1 \sin \alpha - x_2 \cos \alpha)^2 |v|^2 \right) \, dx_1 dx_2.
\]
For any $v \in \mathcal{D}(\bar{Q}_{\alpha,\gamma,a})$, we have
\[
\bar{Q}_{\alpha,\gamma,a}(v) = Q_{\alpha,\gamma,a}^{\tau=0}(e^{i\varphi_{\alpha,\gamma,a}} v),
\]
where $Q_{\alpha,\gamma,a}^{\tau}$ is the quadratic form in (3.15). In the rest of the proof, we write $\bar{Q}$ for $\bar{Q}_{\alpha,\gamma,a}$ and $s$ for $s_{\alpha,a}$. We now express $\bar{Q}$ in the polar coordinates $(\rho, \theta) \in (0, +\infty) \times (-\pi, \pi) =: \mathcal{D}_{\rho \theta}$ as follows
\[
\bar{Q}_{\rho \theta}(v) = \int_0^\infty \int_0^{2\pi} \left( |\partial_\rho v|^2 + \frac{1}{\rho^2} |\partial_\theta - i\rho s_{\rho \theta} \rho^2 \cos \gamma v|^2 + s_{\rho \theta}^2 \rho^2 \sin^2 \gamma \sin^2 (\alpha - \theta) |v|^2 \right) \rho \, d\rho \, d\theta,
\]
where $s_{\rho \theta}(\rho, \theta) = s(x_1, x_2)$ and
\[
\mathcal{D}(\bar{Q}_{\rho \theta}) = \left\{ v \in L^2_{\rho}(D_{\rho \theta}) : \partial_{\rho} v \in L^2_{\rho}(D_{\rho \theta}), \rho v \in L^2_{\rho}(D_{\rho \theta}), \frac{1}{\rho} \left( \partial_\theta - i s_{\rho \theta} \rho^2 \cos \gamma \right) v \in L^2_{\rho}(D_{\rho \theta}), \rho v \in L^2_{\rho}(D_{\rho \theta}) \right\}.
\]
For any $D \subset \mathbb{R}^2$, we denote by $L^2_{\rho}(D)$ the weighted space of weight $\rho$. Consider further the quadratic form $\bar{Q}_{\rho \theta}$, defined on $\mathcal{D}_{\rho \theta} := (0, +\infty) \times (-\pi + \alpha, \alpha)$ by
\[
\bar{Q}_{\rho \theta}(u) = \int_{-\pi + \alpha}^\alpha \int_0^{2\pi} \left( |\partial_{\rho} u|^2 + \frac{1}{\rho^2} |\partial_\theta + i\rho s_{\rho \theta} \rho^2 \cos \gamma u|^2 + s_{\rho \theta}^2 \rho^2 \sin^2 \gamma \sin^2 \theta |u|^2 \right) \rho \, d\rho \, d\theta,
\]
where $s_{\rho \theta}(\rho, \theta) = s(x_1, x_2)$ and
\[
\mathcal{D}(\bar{Q}_{\rho \theta}) = \left\{ u \in L^2_{\rho}(D_{\rho \theta}) : \partial_{\rho} u \in L^2_{\rho}(D_{\rho \theta}), \rho u \in L^2_{\rho}(D_{\rho \theta}), \frac{1}{\rho} \left( \partial_\theta + i s_{\rho \theta} \rho^2 \cos \gamma \right) u \in L^2_{\rho}(D_{\rho \theta}), \rho u \in L^2_{\rho}(D_{\rho \theta}) \right\},
\]
and
\[
s_{\rho \theta}(\rho, \theta) = \begin{cases} \alpha & \text{if } (\rho, \theta) \in (0, +\infty) \times (-\pi + \alpha, \alpha), \\ 1 & \text{if } (\rho, \theta) \in (0, +\infty) \times (0, \alpha). \end{cases}
\]
For any $u \in \mathcal{D}(\bar{Q}_{\rho \theta})$, we have $\bar{Q}_{\rho \theta}(u) = \bar{Q}_{\rho \theta}(v)$, where $v(\rho, \theta) = u(\rho, -\theta + \alpha)$.

In light of the computation above and from Theorem 1.1, a sufficient condition for $\min_{\tau} \mathcal{I}_{\tau}(\alpha, \gamma, a, \tau)$ to be attained by some $\tau_* \in \mathbb{R}$ and to be an eigenvalue of the operator $\mathcal{L}_{A,\gamma,a} + V_{B,\gamma,a,\tau_*}$ is to find a trial function $u_0 \in \mathcal{D}(\bar{Q}_{\rho \theta})$ satisfying
\[
\bar{Q}_{\rho \theta}(u_0) < \Lambda \|u_0\|_{L^2(\mathbb{R}^2)}^2,
\]
where $\Lambda = \Lambda[\alpha, \gamma, a]$ is the minimum between $\beta_a$ and $|a|\zeta_{\rho \theta}$. Towards this, we consider the function
\[
u_0(\rho, \theta) = e^{-\omega \rho^2} e^{-i\omega \rho(\theta)},
\]
where $g : (-\pi + \alpha, \alpha) \to \mathbb{R}$ is a piecewise-differentiable function and $\omega > 0$. In what follows, we will suitably choose $g$ and $\omega$. We define the functional $\mathcal{J}$ on $\text{Dom} \bar{Q}_{\rho \theta}$ by
\[
u \mapsto \mathcal{J}(\nu) = \bar{Q}_{\rho \theta}(\nu) - \Lambda \|\nu\|_{L^2(\mathbb{R}^2)}^2.
\]
The condition in (4.4) is now equivalent to
\[
\mathcal{J}(u_0) < 0.
\]
We compute $\mathcal{J}[u_0]$ and get
\[
\mathcal{J}[u_0] = \int_0^{+\infty} \rho e^{-\omega \rho^2} d\rho \int_{-\pi + \alpha}^{\pi - \alpha} \left( g^2(\theta) + g' \right) (\theta - \Lambda) d\theta \\
- \int_0^{+\infty} \rho^2 e^{-\omega \rho^2} d\rho \int_{-\pi + \alpha}^{\pi - \alpha} \tilde{s}_p g'(\theta) \cos \gamma d\theta \\
+ \int_0^{+\infty} \rho^3 e^{-\omega \rho^2} d\rho \int_{-\pi + \alpha}^{\pi - \alpha} \left( \omega^2 + \tilde{s}_p^2 \sin^2 \gamma \sin^2 \theta + \frac{1}{4} \tilde{s}_p^2 \cos^2 \gamma \right) d\theta.
\]

We use the following properties of $\mathcal{E}_n = \int_0^{+\infty} \rho^n e^{-\omega \rho^2} d\rho$, for $n \geq 0$: $\mathcal{E}_1 = 1/(2\omega)$, $\mathcal{E}_2 = \sqrt{\pi}/(4\omega^{3/2})$, and $\mathcal{E}_3 = 1/(2\omega^2)$ (see [25, Equations 3.461]). Hence, (4.6) becomes
\[
\mathcal{J}[u_0] = \frac{1}{2\omega} \int_{-\pi + \alpha}^{\pi - \alpha} \left( g^2(\theta) + g' \right) (\theta - \Lambda) d\theta - \frac{\sqrt{\pi}}{4\omega^{3/2}} \int_{-\pi + \alpha}^{\pi - \alpha} \tilde{s}_p g'(\theta) \cos \gamma d\theta \\
+ \frac{1}{2\omega^3} \int_{-\pi + \alpha}^{\pi - \alpha} \left( \omega^2 + \tilde{s}_p^2 \sin^2 \gamma \sin^2 \theta + \frac{1}{4} \tilde{s}_p^2 \cos^2 \gamma \right) d\theta. \tag{4.6}
\]

Now, we choose
\[
g(\theta) = \begin{cases} 
    c_1 e^\theta + c_2 e^{-\theta} & \text{if } -\pi + \alpha < \theta \leq 0, \\
    c_3 e^\theta + c_4 e^{-\theta} & \text{if } 0 < \theta < \alpha,
\end{cases}
\]
where $c_i$, $i = 1, \ldots, 4$, are real coefficients satisfying the condition $c_1 + c_2 = c_3 + c_4$ which makes the function $g$ continuous on $(-\pi + \alpha, \alpha)$. Implementing this choice in (4.6) yields
\[
\mathcal{J}[u_0] = \frac{2 - e^{-2\alpha} - e^{-2\pi + 2\alpha}}{2\omega} c_1^2 + \frac{(-e^{-2\alpha} + e^{2\pi - 2\alpha})}{2\omega} c_2^2 + \frac{(-e^{-2\alpha} + e^{2\alpha})}{2\omega} c_3^2 + \frac{(1 - e^{-2\alpha})}{\omega} c_1 c_2 + \frac{(-1 + e^{-2\alpha})}{\omega} c_1 c_3 + \frac{(-1 + e^{-2\alpha})}{\omega} c_2 c_3 + \\
\frac{(1 - e^{-\alpha} + a e^{-\pi + \alpha})}{4\omega^2} \sqrt{\pi} \cos \gamma c_1 + \frac{(1 - a - e^{-\alpha} + a e^{-\pi + \alpha})}{4\omega^2} \sqrt{\pi} \cos \gamma c_2 + \\
\frac{(e^{-\alpha} - e^\alpha)}{4\omega^2} \sqrt{\pi} \cos \gamma c_3 + \frac{4\pi \omega^2 - 4\pi \omega \Lambda + (a^2(\pi - \alpha) + \alpha) \cos^2 \gamma}{8\omega^2} + \\
\frac{2(a^2(\pi - \alpha) + \alpha + (a^2 - 1) \cos \alpha \sin \alpha) \sin^2 \gamma}{8\omega^2}
\]
Notice that $\mathcal{J}[u_0]$ is quadratic in $c_1$, $c_2$ and $c_3$. Minimizing $\mathcal{J}[u_0]$ with respect to these coefficients gives a unique solution $(c_1, c_2, c_3)$, which is
\[
c_1 = \frac{e^{-2\alpha} ((-1 + a) e^\pi + (-1 + a) e^{\pi + 2\alpha} + 2 e^\alpha (-a + e^\pi)) \sqrt{\pi} \cos \gamma (-1 + \coth \pi)}{16 \sqrt{\omega}}
\]
\[
c_2 = \frac{(-1 + a + (-1 + a) e^{2\pi} - 2(-1 + a e^\pi) e^\alpha) \sqrt{\pi} \cos \gamma (-1 + \coth \pi)}{16 \sqrt{\omega}}
\]
\[
c_3 = \frac{e^{-\alpha} (-a + e^\pi + (-1 + a) \cosh(\pi - \alpha)) \sqrt{\pi} \cos \gamma \cosh \pi}{8 \sqrt{\omega}}
\]
We compute $\mathcal{J}[u_0]$ corresponding to the coefficients above, and get $\mathcal{J}[u_0] = P[\alpha, \gamma, a](x)$ with $x = \frac{1}{\omega} > 0$, where $P[\alpha, \gamma, a]$ is as in (1.11). This, together with the condition in (4.5), complete the proof. \qed

We consider cases of $(\alpha, \gamma, a, \tau)$ where the infimum of the spectrum of $\mathcal{L}_{A_{\alpha, \gamma, a}} + V_{B_{\alpha, \gamma, a, \tau}}$ is an eigenvalue below the essential spectrum. The following theorem reveals an exponential decay of the corresponding eigenfunction, for large values of $|x|$. This is a standard Agmon-estimate result. For the proof details, we refer the reader to similar results in [7, Theorem 9.1] and [9].
Theorem 4.1. Let $a \in [-1, 1] \setminus \{0\}$, $\alpha \in (0, \pi)$, $\gamma \in (0, \pi/2]$ and $\tau \in \mathbb{R}$. Consider the case where $\sigma(\alpha, \gamma, a, \tau) < \sigma_{\text{ess}}(\alpha, \gamma, a, \tau)$. Let $v_{\alpha, \gamma, a, \tau}$ be the normalized eigenfunction corresponding to $\sigma(\alpha, \gamma, a, \tau)$. For all $\eta \in (0, \sqrt{\sigma_{\text{ess}}(\alpha, \gamma, a, \tau) - \sigma(\alpha, \gamma, a, \tau)})$, there exists a constant $C$ which depends on $\alpha$ and $\eta$ such that

$$Q^\tau_{\alpha, \gamma, a}(\exp(\eta) v_{\alpha, \gamma, a, \tau}) \leq C,$$

where $\phi(x) = |x|$, for $x \in \mathbb{R}^2$.

5. Proof of Theorem 1.8

We consider the open and bounded set $\Omega \subset \mathbb{R}^2$ defined in the settings of Section 1.2.2. Let $a \in [-1, 1] \setminus \{0\}$ and $b > 0$, we recall the piecewise-constant magnetic field $\mathbf{B}$ in (1.14)

$$\mathbf{B}(x) = s(x)(0, 0, 1), \quad s = 1_{\Omega_1} + a1_{\Omega_2}, \quad (5.1)$$

and the linear operator $P_{b, \mathbf{F}}$ introduced in (1.15). Recall also the discontinuity surface $S$, at which the strength $|\mathbf{B}|$ exhibits a discontinuity jump, and the discontinuity edge $\Gamma$, which is the boundary of $S$.

We denote by $Q_{b, \mathbf{F}}$ the quadratic form associated to $P_{b, \mathbf{F}}$, defined by

$$Q_{b, \mathbf{F}}(u) = \int_{\Omega} |(\nabla - i b \mathbf{F})u|^2 \, dx, \quad D(Q_{b, \mathbf{F}}) = H^1(\Omega). \quad (5.2)$$

The bottom of the spectrum $\lambda(b)$ is equal to

$$\lambda(b) = \inf_{u \in D(Q_{b, \mathbf{F}}) \setminus \{0\}} \frac{Q_{b, \mathbf{F}}(u)}{\|u\|^2_{L^2(\Omega)}}. \quad (5.3)$$

We consider large values of $b$. The main goal of this section is to prove Theorem 1.8, that is to establish the localization of the eigenfunction corresponding to the eigenvalue $\lambda(b)$ near the set $D$ of points of the discontinuity edge $\Gamma$, given by

$$D = \{ \pi \in \Gamma \mid \lambda_{\alpha\pi, \gamma\pi, a} < |a|\Theta_0 \}. \quad (5.4)$$

But first, as seen below, the discussion leading to the proof of Theorem 1.8 establishes as a by-product the following rough asymptotics of $\lambda(b)$, as $b \to +\infty$.

Theorem 5.1 (Asymptotics for $\lambda(b)$). Under Assumption 1.6, it holds

$$\lambda(b) = b \inf_{x \in \Gamma} \lambda_{\alpha_x, \gamma_x, a} + o(b), \quad \text{as} \quad b \to +\infty.$$

The proof of the theorem above is split in two parts, in Proposition 5.4 we prove the lower bound and in Proposition 5.5 we establish the corresponding upper bound. Proposition 5.4 is a particular result induced from Proposition 5.3. The latter proposition is essential in establishing the Agmon estimates in Theorem 1.8.

5.1. Change of variables. We will localise the study of the energy in different regions of $\overline{\Omega}$, which we classify into four categories: regions away from the discontinuity surface $S$ and the boundary $\partial \Omega$, regions meeting $S$ and away from $\partial \Omega$, regions meeting $\partial \Omega$ and away from $S$ and its boundary $\Gamma$, and regions meeting $\Gamma$. A rigorous definition is given later (see Section 5.2). In each of these regions, we will use suitable local coordinates. When working away from the discontinuity surface $S$ and its boundary $\Gamma$, the situation is well-known and already analysed in previous papers (see e.g. [20, Chapter 9]). We focus then on new situations when the foregoing regions meet $\overline{S}$. Below, we describe the appropriate local coordinates to use in two cases: the first one is when the regions meet $\Gamma$ and the second one is when these regions meet $S$ away from $\Gamma$. 


5.1.1. Boundary coordinates near the discontinuity edge $\Gamma$. In this section, we will define a local change of coordinates near the discontinuity edge $\Gamma$.

Let $x_0 \in \Gamma$. After performing a translation, we may assume that the Cartesian coordinates of the point $x_0$ are all 0 ($x_0 = 0$). In what follows, we work near the point $x_0$.

As seen below, our problem will have $L_{\alpha_0,\gamma_0,a}$ as a leading operator, where $L_{\alpha_0,\gamma_0,a}$ is defined in (1.4), for $\alpha_0$ being the angle between the tangent plane of $\partial \Omega$ and the discontinuity surface $S$ at $x_0$, and $\gamma_0$ being the angle between the magnetic field $B$ and the discontinuity edge $\Gamma$ at this point. To show this link with the leading operator, we define a coordinates-transformation $\Phi = \Phi_{x_0} : (x_1, x_2, x_3) \mapsto (y_1, y_2, y_3)$, in a neighborhood $\mathcal{N}_{x_0}$ of $x_0$, s.t. $\Phi(x_0) = (0, 0, 0)$ and there exists a neighborhood $\mathcal{U}_0$ of $(0, 0, 0)$ where

$$\Phi(\mathcal{N}_{x_0} \cap \Gamma) = \mathcal{U}_0 \cap (y_3\text{-axis}) \quad (5.5)$$

$$\Phi(\mathcal{N}_{x_0} \cap S) = \mathcal{U}_0 \cap \mathcal{P}_{\alpha_0} \quad (5.6)$$

$$\Phi(\mathcal{N}_{x_0} \cap (\partial \Omega_1 \setminus \tilde{S})) = \mathcal{U}_0 \cap \{(y_1, 0, y_3) : y_1 > 0\} \quad (5.7)$$

$$\Phi(\mathcal{N}_{x_0} \cap (\partial \Omega_2 \setminus \Gamma)) = \mathcal{U}_0 \cap \{(y_1, 0, y_3) : y_1 < 0\} \quad (5.8)$$

$$\Phi(\mathcal{N}_{x_0} \cap \Omega_1) = \mathcal{U}_0 \cap \mathcal{D}_{\alpha_0}^1$$

and

$$\Phi(\mathcal{N}_{x_0} \cap \Omega_2) = \mathcal{U}_0 \cap \mathcal{D}_{\alpha_0}^2 \quad (5.9)$$

Here, $\mathcal{D}_{\alpha_0}^1$ and $\mathcal{D}_{\alpha_0}^2$ are the sets in (1.1) and (1.2), and $\mathcal{P}_{\alpha_0} = \mathcal{D}_{\alpha_0}^1 \cap \mathcal{D}_{\alpha_0}^2$ is a semi plane making an angle of $\alpha_0$ with $(y_1y_3)$ (see Figure 7 for illustration):

$$P_{\alpha_0} : \begin{cases} y_2 = y_1 \tan \alpha_0, & y_1 > 0, \text{ if } \alpha_0 \in (0, \frac{\pi}{2}) \\ y_2 = y_1 \tan \alpha_0, & y_1 < 0, \text{ if } \alpha_0 \in (\frac{\pi}{2}, \pi) \\ y_2 = 0, & \text{if } \alpha_0 = \frac{\pi}{2}. \end{cases}$$

To that end, we use the same ‘magnetic normal coordinates’ transformation in [40], which we denote by $\Phi$ in our paper. In [40], $\Phi$ is the composition of two local diffeomorphisms $\Phi_1$ and $\Phi_2$ respectively defined near $x_0$ and $\Phi_1(x_0)$ (see the precise definitions below in this section).

Roughly speaking, $\Phi_1 : (x_1, x_2, x_3) \mapsto (r, t, s)$ is a standard tubular coordinates-transformation that straightens $\Gamma$, and sends the boundary $\partial \Omega$ and the surface $S$ (near $x_0$) respectively to surfaces ($\partial \tilde{\Omega}$) and $\tilde{S}$ which make at the point $\Phi_1(x)$ the same (opening) angle $\alpha_x$ made between $\partial \tilde{\Omega}$ and $S$ at the point $x$, for $x \in \Gamma$. However, these surfaces are not necessarily planar surfaces (see Figure 7).

To straighten ($\partial \tilde{\Omega}$) and $\tilde{S}$, and to transform the variable opening angle $\alpha_0$ to the forgoing constant angle $\alpha_0$, we perform a second transformation, $\Phi_2$, near $\Phi_1(x_0) = (0, 0, 0)$. In other words, the local diffeomorphism $\Phi_2 : (r, t, s) \mapsto (y_1, y_2, y_3)$ is defined such that ($\partial \tilde{\Omega}$) is sent to a patch of the $(y_1y_3)$-plane and $\tilde{S}$ to a patch of the aforementioned semi plane $P_{\alpha_0}$ (again see Figure 7).

In what follows, we will make more precise the rough discussion above. We will borrow from [40] the following terminology: we refer to $\Phi_1$ (resp. $\Phi_2$) as the first (resp. second) normalization transformation.

The first normalization. We first consider the tubular coordinates-transformation $\Phi$ defined in a neighborhood of $(0, 0, 0)$ by (see [40], also e.g. [36, Section 3] or [22])

$$\Phi^{-1}_1(r, t, s) = rV(s) + \xi(s) + tn(s),$$

where $s \mapsto \xi(s)$ is a parametrization by arc length of the edge $\Gamma$, $n(s)$ is the inward unit normal vector at the point $\xi(s)$, and $V(s)$ is the unit vector normal to $\Gamma$ at $\xi(s)$ in the tangent plane of $\partial \Omega$, pointing toward $\Omega_1$. The orientation of the parametrization of $\Gamma$ is fixed such that $\det(V(s), n(s), \xi'(s)) > 0$.

Let now $g_0$ be the Riemann metric on $\mathbb{R}^3$. Under $\Phi_1$, the matrix $G_1$ of the metric $g_0$ satisfies

$$G_1^{-1} = Id + O(|r|).$$
The Jacobian $J_{\Phi_1}$ satisfies

$$|J_{\Phi_1}| = 1 + O(|r|).$$

Note that $\Phi_1$ transforms the discontinuity surface $S$ near $x_0$ to a surface-neighborhood $\tilde{S}$ of $(0,0,0)$ making an angle $\alpha(s) := \alpha_{\Phi_1^{-1}(0,0,s)}$ with the boundary $(\tilde{\partial}\Omega)$ at the point $(0,0,s)$ (see Figure 7). Clearly, $s \mapsto \alpha(s)$ is a smooth function.

**The second normalization.** We now introduce the second change of coordinates, $\Phi_2$, in a neighborhood of $(0,0,0) = \Phi_1(x_0)$. First we underline that we can take the neighborhood of $(0,0,0)$ such that $(\tilde{\partial}\Omega) \cup \tilde{S}$ is included in

$$\{(r,t,s) | \phi_s(r) = t \text{ or } r = \psi_s(t)\},$$

where $\phi_s$ and $\psi_s$ are smooth functions depending smoothly on the parameter $s$ and satisfying

$$\phi_s(0) = 0, \quad \phi'_s(0) = 0, \quad \psi_s(0) = 0, \quad \psi'_s(0) = \cot(\alpha(s)).$$

Following [40, Section 2.2.1], we first introduce the change of variables

$$(u,v) = C_s(r,t),$$

where $C_s$ is a local diffeomorphism near $(0,0)$ defined by

$$\begin{cases}
  u = r - \psi_s(t) + \cot(\alpha(t - \phi_s(r)), \\
  v = t - \phi_s(r).
\end{cases}$$

We then define a local diffeomorphism near $(0,0,0)$ by

$$\tilde{\Phi}(r,t,s) = (u,v,s) := (C_s(r,t),s),$$

The matrix $\tilde{G}$ associated to $\tilde{\Phi}$ satisfies (see [40])

$$\tilde{G}^{-1} = Id + O(|u|),$$

where $|u|$ denotes the norm of $(u,v,s)$. The jacobian associated to $\tilde{\Phi}$ satisfies

$$|J_{\tilde{\Phi}}| = 1 + O(|u|).$$

Consequently, in $(\tilde{\partial}\Omega) \cup \tilde{S}$ there exists a neighborhood of $(0,0,0)$ which is sent by $\tilde{\Phi}$ to the following region

$$\{(u,v,s) | v = 0 \text{ or } v = \tan(\alpha(s))u\}.$$  

(5.10)
Finally, we want to replace the variable angle \( \alpha(s) \) in (5.10) with the constant opening angle \( \alpha_0 = \alpha(0) \). To do that, we first perform a rotation \( R_{-\alpha(s)/2} \) of angle \( -\alpha(s)/2 \) to get

\[
(\tilde{u}, \tilde{v}) = R_{-\alpha(s)/2}(u, v).
\]

Let \( \tau(s) = \tan(\alpha(s)/2) \) (notice that \( \tau(0) = \tan(\alpha_0/2) \)). We do the following rescaling

\[
\tilde{u} = \tilde{u}, \quad \tilde{v} = \tau(s)^{-1}\tau(0)v, \quad s = s.
\]

We then perform an inverse rotation \( R_{\alpha(s)/2} \) and define

\[
(y_1, y_2) := R_{\alpha(s)/2}(\tilde{u}, \tilde{v}), \quad y_3 = s.
\]

Now, we introduce the diffeomorphism \( \Phi_2 \) near \( (0, 0, 0) \) by setting

\[
\Phi_2(r, t, s) = (y_1, y_2, y_3).
\]

By \( \Phi_2 \), one can map \( (\partial \Omega) \cup \hat{S} \) near \( (0, 0, 0) \) into a subset of

\[
\{(y_1, y_2, y_3) | y_2 = 0 \text{ or } y_2 = y_1 \tan \alpha_0 \}.
\]

The composition of the two normalization. We define \( \Phi \) in a neighborhood of \( N_{x_0} \) as the composition of \( \Phi_2 \) and \( \Phi_1 \)

\[
\Phi = \Phi_2 \circ \Phi_1 : (x_1, x_2, x_3) \mapsto y = (y_1, y_2, y_3).
\]

One can see now that the properties of \( \Phi \) in (5.5)–(5.9) hold true for a suitable \( N_{x_0} \) (see Figure 7). Moreover, let \( G \) be the matrix of the metric \( g_0 \) corresponding to \( \Phi \), and \( G^{-1} \) be its inverse. By the above discussion, we have

\[
G^{-1} = (g^{lm}) = Id + O(|y|).
\]

The Jacobian \( J_{\Phi} \) satisfies

\[
|J_{\Phi}| = 1 + O(|y|).
\]

The quadratic form in the boundary coordinates. We consider the diffeomorphism \( \Phi \) defined above near \( x_0 \). Under \( \Phi \), the Lebesgue measure \( dx \) transforms into \( dy = |J_{\Phi}|dy \).

We denote any vector potential \( F \in H^1(\Omega, \mathbb{R}^3) \) satisfying \( \text{curl} F = B \) (as in (5.1)) by \( \tilde{F} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3) \) in the new coordinates near \( x_0 \). We have

\[
F_1dx_1 + F_2dx_2 + F_3dx_3 = \tilde{F}_1dy_1 + \tilde{F}_2dy_2 + \tilde{F}_3dy_3.
\]

Denoting the magnetic field \( B \) by \( \tilde{B} \) in the local coordinates, we have (see [28, Section 5])

\[
\text{curl} \tilde{F} = |J_{\Phi}| \tilde{B}.
\]

In addition, for any \( u \in H^1(\Omega) \) supported in \( N_{x_0} \), the quadratic form \( Q_{B,F}(u) \) becomes

\[
Q_{B,F}(u) = \int_{N_{x_0} \cap \Omega} |(\nabla - i\tilde{B})u|^2 dx = \int_{\overline{N}_{x_0} \cap \mathbb{R}^3_+} |J_{\Phi}|\left[ \sum_{1 \leq \ell, m \leq 3} g^{lm}(\partial_{y_\ell} - i\tilde{B}_\ell)\tilde{u}(\partial_{y_m} - i\tilde{B}_m)\tilde{u} \right] dy
\]

where \( \overline{N}_{x_0} = \Phi(\overline{N}_{x_0}) \) and \( \tilde{u}(\cdot) = u(\Phi^{-1}(\cdot)) \).

Finally, the following lemma presents a gauge transformation of the potential \( \tilde{F} \) that will be useful in comparing the operator \( (\nabla - i\tilde{B})^2 \) (in (1.15)) with the leading operator \( \mathcal{L}_{\alpha, \gamma, \alpha} \) (in (1.4)) near the discontinuity edge \( \Gamma \).

Lemma 5.2. Let \( a \in [-1, 1] \setminus \{0\} \) and \( B(0, \ell) \) be a ball of radius \( \ell \) such that \( B(0, \ell) \cap \mathbb{R}^3_+ \subset \overline{N}_{x_0} \). Consider the vector potential \( \tilde{F} \in H^1(\Omega) \) such that \( \text{curl} \tilde{F} = B \), where \( B \) is as in (5.1). There exists \( \omega \in H^2(B(0, \ell) \cap \mathbb{R}^3_+) \) such that the vector potential \( \tilde{F} \) can be written as

\[
\tilde{F}(y) = A_{\alpha_0, \gamma_0, \alpha}(y) + \nabla \omega(y) + O(|y|^2), \quad \forall y \in B(0, \ell),
\]
where \( A_{\alpha_0,\gamma_0,a} \) is the vector potential introduced in (3.5). Here \( \alpha_0 \) is the angle between the discontinuity surface \( S \) and the tangent plane to \( \partial \Omega \) at \( x_0 \), and \( \gamma_0 \) is the angle between the magnetic field \( B \) and the discontinuity edge \( \Gamma \) at \( x_0 \).

Proof. We consider the magnetic field \( B_{\alpha_0,\gamma_0,a} = \text{curl} A_{\alpha_0,\gamma_0,a} \) introduced in (1.3). Let now \( \tilde{B} \) (resp. \( \tilde{F} \)) be the magnetic field \( B \) (resp. the vector potential \( F \)) expressed in the local coordinates.

Notice that the magnetic field \( B \) is constant in each of \( \Omega_1 \) and \( \Omega_2 \). Indeed, it is equal to \( b \) in \( \Omega_1 \) and \( ab \) in \( \Omega_2 \), where \( b \) is a unit magnetic field making an angle \( \alpha_0 \) with the boundary and \( \gamma_0 \) with \( \Gamma \) at \( x_0 \). Also, notice that \( \Omega_1 \) is sent to \( D_{\alpha_0}^1 \) and \( \Omega_2 \) to \( D_{\alpha_0}^2 \) near \( x_0 \) (see (5.9)). Consequently, using the properties of the local coordinates transformation and a Taylor expansion of the unit field yields (see e.g. [33, Section 5] or [28])

\[
\tilde{B} = B_{\alpha_0,\gamma_0,a} + \mathcal{O}(|y|). \tag{5.16}
\]

In addition, by (5.14) and (5.13), we have

\[
\text{curl} \tilde{F} = \tilde{B}(1 + \mathcal{O}(|y|)). \tag{5.17}
\]

Then, (5.16) and (5.17) give

\[
\text{curl} \tilde{F} = B_{\alpha_0,\gamma_0,a} + \mathcal{O}(|y|). \tag{5.18}
\]

Using the inverse curl formula, we define a vector potential \( A_{\text{diff}} \) such that

\[
\text{curl} A_{\text{diff}} = \text{curl}(\tilde{F} - A_{\alpha_0,\gamma_0,a}) = \text{curl} \tilde{F} - B_{\alpha_0,\gamma_0,a}. \]

We take

\[
A_{\text{diff}}(y) = \int_0^1 (\text{curl} \tilde{F}((\lambda y) - B_{\alpha_0,\gamma_0,a}(\lambda y)) \wedge (\lambda y) \, d\lambda. \tag{5.19}
\]

It follows from (5.18) that \( A_{\text{diff}}(y) = \mathcal{O}(|y|^2) \), for each \( y \in B(0, \ell) \cap \mathbb{R}^3_+ \). Moreover, since the two vector potentials \( \tilde{F} - A_{\alpha_0,\gamma_0,a} \) and \( A_{\text{diff}} \) are generating the same magnetic field in \( B(0, \ell) \cap \mathbb{R}^3_+ \), there exists a function \( \omega \in H^2(B(0, \ell) \cap \mathbb{R}^3_+) \) such that

\[
\tilde{F} - A_{\alpha_0,\gamma_0,a} = A_{\text{diff}} + \nabla \omega. \tag{5.20}
\]

Using that \( A_{\text{diff}}(y) = \mathcal{O}(|y|^2) \) completes the proof. \( \square \)

5.1.2. Local coordinates near the discontinuity surface \( S \) (away from \( \Gamma \)). Let \( S \) be the discontinuity surface introduced earlier. Under our hypothesis, this is a smooth simply connected set with boundary \( \Gamma \). We will consider standard tubular coordinates near \( S \) away from the discontinuity edge \( \Gamma \).

Let \( x_0 \in S \) (away from \( \Gamma \)), we consider a small neighborhood \( V_{x_0} \) of \( x_0 \) in \( S \), in which a local coordinate system is well defined. We denote such coordinates by \( (r,s,t) \). Let \( \phi \) be the local diffeomorphism corresponding to these coordinates

\[
\phi : V_{x_0} \to U \subset \mathbb{R}^2, \text{ such that } \phi(x) = (r,s). \]

Denoting by \( n \) the unit normal to \( S \) at the point \( \phi^{-1}(r,s) \), we define the coordinate transformation \( \Phi \) in a small neighborhood of \( x_0 \) such that

\[
(x_1, x_2, x_3) = \Phi^{-1}(r,s,t) = \phi^{-1}(r,s) + tn, \tag{5.21}
\]

where \( t = \text{dist}(x,S) \) if \( x \in \Omega_1 \) and \( t = -\text{dist}(x,S) \) if \( x \in \Omega_2 \).

We denote by \( y = (y_1,y_2,y_3) = (r,t,s) \). The coordinates \( (y_1,y_2,y_3) \) have similar properties to those of the tubular coordinates defined in the previous section (see also e.g. [20]). Similar estimates to those in (5.12)–(5.15) are valid.
5.2. Lower bound.

**Proposition 5.3** (Lower bound of local energies). Under Assumption 1.6, there exist \( C > 0 \) and \( b_0 > 0 \) such that for \( b \geq b_0 \), for all \( R_0 > 1 \) and \( u \in \mathcal{D}(Q_{b,F}) \), it holds

\[
Q_{b,F}(u) \geq \int_{\Omega} (U_b(x) - C R_0^{-2} b)|u(x)|^2 dx,
\]

where

\[
U_b(x) = \begin{cases} |a| b & \text{if } \text{dist}(x, \partial \Omega \cup S) \geq R_0 b^{-\frac{1}{2}}, \\
|a| \Theta_0 b - C R_0^2 b^2 & \text{if } \text{dist}(x, \partial \Omega) < R_0 b^{-\frac{1}{2}} \leq \text{dist}(x, S), \\
\beta_x b - C R_0^2 b^2 & \text{if } \text{dist}(x, S) < R_0 b^{-\frac{1}{2}} \leq \text{dist}(x, \partial \Omega), \\
\lambda_{\alpha_j, \gamma_j, a} b - C R_0^2 b^2 & \text{if } \text{dist}(x, x_j) < R_0 b^{-\frac{1}{2}}, \ x_j \in \Gamma,
\end{cases}
\]

where \( \Theta_0, \beta_x, \) and \( \lambda_{\alpha_j, \gamma_j, a} \) are respectively the values in (2.4), (2.2) and (1.7), with \( \alpha_j \) being the angle between the tangent plane of \( \partial \Omega \) and the discontinuity surface \( S \) at \( x_j \) (taken towards \( \Omega_j \)) and \( \gamma_j \) being the angle between the magnetic field \( B \) and the discontinuity edge \( \Gamma \) at \( x_j \).

**Proof.** We will work locally in \( \Omega \), proceeding similarly as in [20, Theorem 9.1.1]. Let \( 0 < \rho < 1 \), we can define a partition of unity, \( (\chi_j)_{j \in \mathcal{I}} \) of \( \mathbb{R}^3 \), and assume that it satisfies the following.

For \( (\chi_j) \) with a support intersecting \( \overline{\Omega} \), we suppose that they are supported in balls of centers \( x_j \in \overline{\Omega} \) and radius \( R_0 b^{-\rho} \), and

\[
\chi_j \in C_c^\infty(\mathbb{R}^3; \mathbb{R}), \quad \sum_j \chi_j^2 = 1, \quad \sum_j |\nabla \chi_j|^2 \leq C R_0^{-2} b^{2\rho}.
\]  

(5.24)

Moreover, we assume that each ball is either entirely contained in \( \Omega \) or it is centered at the boundary \( \partial \Omega \). In the first case, we further assume that the ball is either disjoint with \( S \) or centered at \( S \). In the second case, we assume that the ball is either disjoint with \( S \) or centered at \( \Gamma \). We then define the following sets of centers \( x_j \), which we assume to constitute a partition of the set of the entire centers:

\[
\begin{align*}
(1) \quad D_1 & := \{ x_j \in \Omega | \text{ supp}(\chi_j) \cap (\partial \Omega \cup S) = \emptyset \}, \\
(2) \quad D_2 & := \{ x_j \in \partial \Omega | \text{ supp}(\chi_j) \cap (S \cup \Gamma) = \emptyset \}, \\
(3) \quad D_3 & := \{ x_j \in S | \text{ supp}(\chi_j) \cap \partial \Omega = \emptyset \}, \\
(4) \quad D_4 & := \{ x_j \in \Gamma \}.
\end{align*}
\]

We also denote by \( \mathcal{I}_\ell \), for \( \ell = 1, 2, 3, 4 \), the indices subsets of \( \mathcal{I} \) such that \( \mathcal{I}_\ell := \{ j \in \mathcal{I} : x_j \in D_\ell \} \). We now use the IMS formula to write

\[
Q_{b,F}(u) = \sum_{\ell=1}^4 \sum_{j \in \mathcal{I}_\ell} Q_{b,F}(\chi_j u) - \sum_j \|\nabla \chi_j |u|^2\|_{L^2(\Omega)}^2.
\]

(5.25)

First, using the properties in (5.24), we can estimate the (error) term in the RHS of the equality above as follows

\[
\sum_j \|\nabla \chi_j |u|^2\|_{L^2(\Omega)}^2 \leq C R_0^{-2} b^{2\rho} \|u\|_{L^2(\Omega)}^2.
\]

(5.26)

Next, we estimate the (main) term in the foregoing RHS, i.e. \( \sum_{\ell=1}^4 \sum_{j \in \mathcal{I}_\ell} Q_{b,F}(\chi_j u) \).

**Estimates in D_1.** For \( j \in \mathcal{I}_1 \), we work as in Theorem [20, Theorem 9.1.1], and get

\[
Q_{b,F}(\chi_j u) \geq \min(1, |a| b) \sum_{i \in \mathcal{I}_1} \int_{\Omega} dx |\chi_j u|^2 \geq |a| b \sum_{i \in \mathcal{I}_1} \int_{\Omega} dx |\chi_j u|^2,
\]

(5.27)

having \( |a| \leq 1 \).
Estimates in $D_3$. Let $j \in \mathcal{I}_3$. Mainly ([20, Theorem 9.1.1]), local tubular coordinates can be defined near $\partial \Omega$ away from the discontinuity zone $\overline{\Sigma}$, which permits to compare the quadratic form with the one associated to the model operator introduced in Section 2.2. Using the monotonicity properties of the bottom of the spectrum of the aforementioned model operator (see Section 2.2), one can write

$$Q_{b,F}(\chi_j u) \geq |a| \Theta_0 b \int_\Omega |\chi_j u|^2 - Cc_1(R_0, b)\|\chi_j u\|^2, \quad (5.28)$$

where

$$c_1(R_0, b) = R_0 b^{1-\rho} + R_0^3 b^{2-4\rho} + R_0^6 b^{3-2\rho}. \quad (5.29)$$

Estimates in $D_3$. Now, we consider the balls centered at the discontinuity surface $S$ and do not meet the boundary $\partial \Omega$. In this situation, the local coordinates in Section 5.1.2 are involved. For any $j \in \mathcal{J}_3$, we take $b$ sufficiently large so that the ball $B(x_j, R_0 b^{-\rho})$ is included in the domain of the local diffeomorphism $\Phi = \Phi_{x_j}$, introduced in Section 5.1.2 and where $\Phi^{-1}(x_j) = (0, 0, 0)$. Using the properties of $\Phi$, one gets that $\Phi(B(x_j, R_0 b^{-\rho})) \subset B(0, cR_0 b^{-\rho})$ for some positive constant $c > 0$. Moreover, for any $v \in D(Q_{b,F})$ s.t. $\text{supp} v \subset B(x_j, R_0 b^{-\rho})$, we have

$$(1 - CR_0 b^{-\rho}) \int_{\Phi(B(x_j, R_0 b^{-\rho}))} dy |(\nabla - ib \tilde{F}) \tilde{v}|^2 \leq Q_{b,F}(v) \leq (1 + CR_0 b^{-\rho}) \int_{\Phi(B(x_j, R_0 b^{-\rho}))} dy |(\nabla - ib \tilde{F}) \tilde{v}|^2, \quad (5.30)$$

where $\tilde{F}$ and $\tilde{v}$ are respectively the transforms of the vector potential $F$ and the function $v$ by $\Phi$.

Similarly as in [22], we can perform a change of gauge that replaces the vector potential $\tilde{F}$ by a new one $\tilde{F} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3)$ ($\tilde{F} = \tilde{F} + \nabla \varphi$ for a certain $H^2$-function $\varphi$) such that

$$\tilde{F}_1 = \begin{cases} -y_2 + O(|y|^2), & \text{if } y_2 > 0, \\ -a y_2 + O(|y|^2), & \text{if } y_2 < 0. \end{cases}, \quad \tilde{F}_2 = O(|y|^2), \quad \tilde{F}_3 = O(|y|^2).$$

Above, we used that the magnetic field is tangent to the surface $S$. We consider the function $\tilde{v} = e^{ib \tilde{F}} v$. By gauge invariance properties of the energy, we have

$$\int_{\Phi(B(x_j, R_0 b^{-\rho}))} dy |(\nabla - ib \tilde{F}) \tilde{v}|^2 = \int_{\Phi(B(x_j, R_0 b^{-\rho}))} dy |(\nabla - ib \tilde{F}) \tilde{v}|^2. \quad (5.31)$$

Now we use Cauchy’s inequality in the RHS of the equation above to further replace the vector potential $\tilde{F}$ by the vector potential $A_a$ defined (in Section 2.1) by $A_a(y) = (-\delta_a(y) y_2, 0, 0)$ with $\delta_a = \mathbb{1}_{y_2 > 0} + a \mathbb{1}_{y_2 < 0}$. We then get

$$\int_{\Phi(B(x_j, R_0 b^{-\rho}))} dy |(\nabla - ib \tilde{F}) \tilde{v}|^2 \geq (1 - b^{-\delta}) \int_{\mathbb{R}^3} dy |(\nabla - i A_a) \tilde{v}|^2 - C b^2 (R_0^2 b^{-2\rho})^2 b^\delta \int_{\mathbb{R}^3} dy |\tilde{v}|^2, \quad (5.32)$$

for $\delta \in (0, 1)$. Using Remark 2.1, we have

$$\int_{\mathbb{R}^3} dy |(\nabla - i A_a) \tilde{v}|^2 \geq \beta_a \int_{\mathbb{R}^3} dy |\tilde{v}|^2, \quad (5.33)$$

where $\beta_a$ is the bottom of the spectrum of the operator $(\nabla - i A_a)$ in Section 2.1. Then, using (5.30), (5.31), and a scaling argument in (5.33), we get for $v = \chi_j u$

$$Q_{b,F}(\chi_j u) \geq (\beta_a b - Cc_2(R_0, b)) \int_\Omega |\chi_j u|^2, \quad (5.34)$$
where $c_2(R_0, b)$ is a constant depending on $R_0$ and $b$ which explicitly reads as
\begin{equation}
  c_2(R_0, b) = R_0 b^{1-\rho} + b^{1-\delta} + R_0^4 b^{2-4\rho+\delta},
\end{equation}
and where, recalling the Jacobian, $J_{\Phi_{-1}}$, of the change of coordinates function $\Phi^{-1}$, we used that
\begin{equation}
  \int_{\mathbb{R}^3} dy |\hat{\psi}|^2 = \int_{\mathbb{R}^3} dy |\tilde{\psi}|^2 = \int_{\Omega} dx |\chi_j u|^2 (J_{\Phi_{-1}}),
\end{equation}
along with the estimate (5.13).

**Estimates in $D_4$.** Finally, we consider the balls centered at the discontinuity edge $\Gamma$. For $j \in I_4$, we have $\text{supp} \chi_j \subset B(x_j, R_0 b^{-\rho})$ where $x_j \in \Gamma$. Here, the proof outline is quite similar to the one above (for the balls centered in $D_3$), but with using the local coordinates and the notation introduced in Section 5.1.1. Thus, we will omit some computation details. Let $\Phi = \Phi_{x_j}$ be the diffeomorphism introduced in the foregoing section with $\Phi(x_j) = (0, 0, 0)$. We suppose that there exists a positive constant $c > 0$ such that $\Phi(B(x_j, R_0 b^{-\rho}) \cap \Omega) \subset B(0, c R_0 b^{-\rho})$. We consider $v \in \mathcal{D}(Q_{b, F})$ s.t. $\text{supp} v \subset B(x_j, R_0 b^{-\rho})$.

As above, using the estimates (5.12) and (5.13), we can write
\begin{equation}
  (1 - C R_0 b^{-\rho}) \int_{\Phi(B(x_j, R_0 b^{-\rho}) \cap \Omega)} dy |(\nabla - ib \hat{F})\tilde{\psi}|^2 \leq Q_{b, F}(v)
  \leq (1 + C R_0 b^{-\rho}) \int_{\Phi(B(x_j, R_0 b^{-\rho}) \cap \Omega)} dy |(\nabla - ib \hat{F})\tilde{\psi}|^2.
\end{equation}
Using now the gauge transform result in Lemma 5.2 with $B(0, \ell) = B(0, c R_0 b^{-\rho})$ and $x_0 = x_j$ in this lemma, then applying Cauchy’s inequality, we get for $\delta \in (0, 1)$
\begin{equation}
  \int_{\Phi(B(x_j, R_0 b^{-\rho}) \cap \Omega)} dy |(\nabla - ib \hat{F})\tilde{\psi}|^2 \geq (1 - b^{-\delta}) \int_{\Phi(B(x_j, R_0 b^{-\rho}) \cap \Omega)} dy |(\nabla - ib A_{\alpha_j, \gamma_j, a})\hat{\psi}|^2 - C R_0^4 b^{2-4\rho+\delta} \int_{\mathbb{R}^3} dy |\hat{\psi}|^2,
\end{equation}
where $\hat{\psi} = e^{ib\omega \hat{\varphi}} \tilde{\psi}$ and $\omega$ is the gauge function in Lemma 5.2. Using (1.6) and (1.7) and proceeding similarly as in establishing (5.34) above, we get
\begin{equation}
  Q_{b, F}(\chi_j u) \geq (b \lambda_{\alpha_j, \gamma_j, a} - C c_3(R_0, b)) \int_{\mathbb{R}^3} dx |\chi_j u|^2,
\end{equation}
where
\begin{equation}
  c_3(R_0, b) = R_0 b^{1-\rho} + b^{1-\delta} + R_0^4 b^{2-4\rho+\delta}.
\end{equation}

Finally, we choose $\rho = \delta = 1/2$ and $R_0 > 1$ large. This yields
\begin{equation}
  |c_j(R_0, b)| \leq C R_0^4 b^2, \quad \text{for } j = 1, 2, 3.
\end{equation}
Gathering the results in the proof above establishes the proposition statement. $\Box$

Actually, the proof of Proposition 5.3 yields the following particular result on the lower bound of the ground state energy:

**Proposition 5.4 (Lower bound of $\lambda(b)$).** Under Assumption 1.6, there exist $C > 0$ and $b_0 > 0$ such that for all $b \geq b_0$, it holds
\begin{equation}
  \lambda(b) \geq b \inf_{x \in \Gamma} \lambda_{x, \gamma x, a} - C b^2.
\end{equation}

**Proof.** The result is a consequence of the min-max principle in (5.3), the proof in Proposition 5.3 (taking $\rho = 3/8$, $\delta = 1/4$, $R_0 = 1$ in the proof of this proposition), and the fact that
\begin{equation}
  \inf_{x \in \Gamma} \lambda_{x, \gamma x, a} < \min(|a|, \Theta_0 |a|, \beta_0) = \Theta_0 |a|
\end{equation}
under Assumption 1.6. $\Box$
5.3. Upper bound. In this section, we will establish an upper bound of \( \lambda(b) \). Note that this upper bound is not optimal. However, it will be sufficient to get the right order in the localization estimates in Theorem 1.8.

**Proposition 5.5** (Upper bound of \( \lambda(b) \)). Let \( \overline{\sigma} \in D \). Under Assumption 1.6, there exists \( C_{\overline{\sigma}} > 0 \) depending on \( \overline{\sigma} \) and \( b_0 > 0 \) such that for all \( b \geq b_0 \), it holds

\[
\lambda(b) \leq b \lambda_{\alpha_{\overline{\sigma}},\gamma,a} + C_{\overline{\sigma}} b^{\frac{5}{2}}.
\]

**Proof.** Recall the set \( D \) defined in (1.21) by

\[
D = \{ \overline{\sigma} \in \Gamma \mid \lambda_{\alpha_{\overline{\sigma}},\gamma,a} < |\alpha|_{\Theta_0} \}.
\]

Let \( \overline{\sigma} \in D \). Below, we define a suitable trial state supported near \( \overline{\sigma} \). In a convenient neighborhood \( \overline{\mathcal{N}} \) of \( \overline{\sigma} \), we can consider the local coordinates \((y_1, y_2, y_3)\) introduced in Section 5.1.1, with the related diffeomorphism \( \Phi = \Phi_{\overline{\sigma}} \) which satisfies \( \Phi(\overline{\sigma}) = (0, 0, 0) \). For \( b > 0 \) sufficiently large, we can assume that

\[
\mathcal{N}_{\rho,\eta} := (-b^{-\rho}, b^{-\rho}) \times (0, b^{-\rho}) \times (-b^{-\eta}, b^{-\eta}) \subset \Phi(\overline{\mathcal{N}}),
\]

for some \( \rho \in (0, 1/2) \) and \( \eta > 0 \) to be fixed later. We also set \( \mathcal{N}_\rho := (-b^{-\rho}, b^{-\rho}) \times (0, b^{-\rho}) \) for a later use. Let \( \chi \in C^\infty(\mathbb{R}) \) such that \( \|\chi\|_2 = 1 \) and

\[
0 \leq \chi \leq 1, \quad \chi = 1 \text{ in } (-1/2, 1/2), \quad \text{and supp } \chi \subset (-1, 1).
\]

We define the functions \( \chi_\rho(\cdot) = \chi(b^{\rho} \cdot) \) and \( \chi_\eta(\cdot) = \chi(b^{\eta} \cdot) \).

Under Assumption 1.6, we can consider a normalized eigenfunction, \( \pi \), of the operator \( \mathcal{L}_{\alpha_{\overline{\sigma}},\gamma,a} + V_{\overline{\mathcal{N}}_{\rho,\gamma,a,\tau}} \), mentioned in Theorem 1.1, for some \( \tau \in \mathbb{R} \), and satisfying \( \sigma(\alpha_{\overline{\sigma}},\gamma,a,\tau) = \lambda_{\alpha_{\overline{\sigma}},\gamma,a} \). The existence of \( \pi \) is ensured by Theorem 1.1 (see Remark 1.7).

We define the function \( \overline{u} \) by\(^7\) \( \overline{u}(y) = \sqrt{\nu}\overline{v}(\sqrt{\nu} y) \). We then introduce our trial function \( u_{\text{trial}} \) such that

\[
u_{\text{trial}}(x) = \begin{cases} (\tilde{u} \circ \Phi)(x) & \text{if } x \in \Phi^{-1}(\mathcal{N}_{\rho,\eta}) \cap \Omega, \\ 0 & \text{otherwise}. \end{cases}
\]

with

\[
\tilde{u}(y) = \tilde{u}(y_1, y_2, y_3) := \begin{cases} b^{\eta/2} \chi_\eta(y_3) \chi_\rho(y_1) \chi_\rho(y_2) \overline{\pi}(y_1, y_2) e^{i\omega(y)} e^{i\sqrt{\nu}y_3} & \text{if } y \in \mathcal{N}_{\rho,\eta} \cap \mathbb{R}^3_+, \\ 0 & \text{otherwise}. \end{cases}
\]

where \( \omega \) is the gauge function in Lemma 5.2 and the normalization factor \( b^{\eta/2} \) ensures that \( \|b^{\eta/2} \chi_\eta\|_2 = 1 \).

Next, we evaluate \( Q_{b,F}(u_{\text{trial}}) \) and \( \|u_{\text{trial}}\|_2 \). In what follows, the constants on the estimates will depend on the point \( \overline{\sigma} \). But for simplicity, we omit this dependence and write \( C \) instead of \( C_{\overline{\sigma}} \) to denote these constants. From the definition of \( u_{\text{trial}} \) and the property of the diffeomorphism \( \Phi \) (see (5.13)), we can write, doing a change of variables,

\[
\int_{\Omega} dx |u_{\text{trial}}|^2 \geq (1 - C(b^{-\rho} + b^{-\eta})) \int_{\mathcal{N}_{\rho,\eta} \cap \mathbb{R}^3_+} dy |\tilde{u}|^2
\]

\[
= (1 - C(b^{-\rho} + b^{-\eta})) \int_{\mathcal{N}_{\rho} \cap \mathbb{R}^2_+} dy_1 dy_2 |\chi_\rho(y_1) \chi_\rho(y_2) \overline{\pi}(y_1, y_2)|^2
\]

\[
= (1 - C(b^{-\rho} + b^{-\eta})) \int_{\mathcal{N}_{\rho-1/2} \cap \mathbb{R}^2_+} dy_1 dy_2 |\chi_\rho(b^{-\frac{1}{2}} y_1) \chi_\rho(b^{-\frac{1}{2}} y_2) \overline{\pi}(y_1, y_2)|^2
\]

\[
\geq (1 - C(b^{-\rho} + b^{-\eta})) \left( 1 - \int_{\mathcal{N}_{\rho-1/2} \cap \mathbb{R}^2_+} dy_1 dy_2 |\overline{\pi}(y_1, y_2)|^2 \right)
\]

---

\(^7\)This rescaling is useful to get the leading order in the upper bound of \( \lambda(b) \) as \( O(b) \).
where $\mathcal{N}^c_{\rho-1/2}$ denotes the complement of $\mathcal{N}_{\rho-1/2}$. For $\rho \in (0, 1/2)$, using the exponential decay of $v$ stated in Theorem 4.1, one gets that
\[
\int_{\Omega} dx |u_{\text{trial}}|^2 \geq (1 - Cb^{-\rho} - Cb^{-\eta}).
\]
We now estimate $Q_{b,F}(u_{\text{trial}})$. Similarly as in Proposition 5.3, using (5.12) and (5.13), we can write
\[
Q_{b,F}(u_{\text{trial}}) \leq (1 + Cb^{-\rho} + Cb^{-\eta}) \int_{R^3_+} dy \left| (\nabla - ib\bar{F})\bar{u} \right|^2,
\]
where $\bar{F}$ is the transform of the vector potential $F$ by $\Phi$. Using the change of gauge in Lemma 5.2, we have
\[
\int_{R^3_+} dy \left| (\nabla - ib\bar{F})\bar{u} \right|^2
= b^n \int_{R^3_+} dy \left| (\nabla - ib(A_{(\pi,\tau,a} \gamma,a) + O(|y|^2)) (\chi_\gamma(y_3) \chi_\rho(y_1) \chi_\rho(y_2) \bar{\pi}(y_1, y_2) e^{i\sqrt{\gamma} \tau, y_3}) |^2,
\]
where $A_{(\pi,\tau,a} \gamma,a$ is the vector potential in this lemma (introduced in (3.5)), for $\pi = a\pi$ and $\gamma_3$. $A_{(\pi,\tau,a} \gamma,a$ was chosen independent of $y_3$. We now use Cauchy’s inequality to write for some $0 < \epsilon < 1$,
\[
b^n \int_{R^3_+} dy \left| (\nabla - ib(A_{(\pi,\tau,a} \gamma,a) + O(|y|^2)) (\chi_\gamma(y_3) \chi_\rho(y_1) \chi_\rho(y_2) \bar{\pi}(y_1, y_2) e^{i\sqrt{\gamma} \tau, y_3}) |^2
\leq b^n(1 + b^{-\epsilon}) \int_{R^3_+} dy \left| (\nabla - ibA_{(\pi,\tau,a} \gamma,a) \bar{\pi}(y_1, y_2) e^{i\sqrt{\gamma} \tau, y_3}) |^2 |
+ Cb^{\epsilon + \eta} ||\chi_\gamma'||^2 \int_{b^{-\eta} \leq |y_3| \leq b^{n-\eta}} \int_{R^3_+} dy \bar{\pi}(y_1, y_2) dy
+ Cb^{\epsilon + \eta} \int_{R^3_+} dy |\chi_\gamma(y_3)|^2 |\chi_\rho(y_1)|^2 |\chi_\rho(y_2)||^2 \bar{\pi}(y_1, y_2)|^2
+ Cb^{2 + \epsilon + \eta} \int_{R^3_+} dy |y|^4 |\chi_\gamma(y_3)|^2 |\pi(y_1, y_2)|^2. \quad (5.46)
\]
We now take into account separately the integrals in the RHS of the inequality above. Via a change of coordinates,
\[
I := b^n(1 + b^{-\epsilon}) \int_{R^3_+} dy \left| (\nabla - ibA_{(\pi,\tau,a} \gamma,a) \bar{\pi}(y_1, y_2) e^{i\sqrt{\gamma} \tau, y_3}) |^2 |
= b^{1 + \eta}(1 + b^{-\epsilon}) \int_{R^3_+} dy \left| (\nabla - ibA_{(\pi,\tau,a} \gamma,a) \bar{\pi}(y_1, y_2) e^{i\sqrt{\gamma} \tau, y_3}) |^2 |
= b^{1 + \eta}(1 + b^{-\epsilon}) \int_{R^3_+} dy \left| (\nabla - ibA_{(\pi,\tau,a} \gamma,a) \bar{\pi}(y_1, y_2) e^{i\sqrt{\gamma} \tau, y_3}) |^2 |
= b(1 + b^{-\epsilon}) Q_{(\pi,\tau,a} \gamma,a (\pi),
\]
where $Q_{(\pi,\tau,a} \gamma,a$ is the quadratic form defined in (3.15) and where we used that $b^{-\eta} = ||\chi_\gamma||^2$. Using that
\[
Q_{(\pi,\tau,a} \gamma,a (\pi) = c(\pi, \gamma, a, \tau_\gamma) = \lambda_{(\pi,\tau,a} \gamma,a,
\]
we conclude that
\[
I = b(1 + b^{-\epsilon}) \lambda_{(\pi,\tau,a} \gamma,a. \quad (5.47)
\]
Next, let
\[
II := Cb^{\epsilon + \eta} ||\chi_\gamma||^2 \int_{\frac{b^{-\eta}}{2} \leq |y_3| \leq b^{\eta}} \int_{R^3_+} dy \bar{\pi}(y_1, y_2)|^2.
\]
We bound \( \| \chi' \|_\infty \leq Cb^\gamma \), and consequently get
\[
\| \Pi \| \leq Cb^{\epsilon+2\eta} \int_{\mathbb{R}^2_+} dy_1 dy_2 |\pi(y_1, y_2)|^2 \leq Cb^{\epsilon+2\eta}.
\] (5.48)

For the term
\[
\Pi := Cb^{\epsilon+\eta} \int_{\mathbb{R}^2_+} dy |\chi(y_3)|^2 \left( |\chi_{\rho}(y_1)|^2 |\chi_{\rho}(y_2)|^2 + |\chi_{\rho}(y_1)|^2 |\chi'_{\rho}(y_2)|^2 \right) |\pi(y_1, y_2)|^2,
\]
we use the support of \( \chi_{\rho} \) and the exponential decay of \( \pi(y_1, y_2) = \sqrt{b} \pi(\sqrt{b}(y_1, y_2)) \) in Theorem 4.1, and get
\[
\| \Pi \| \leq Cb^{\epsilon}.
\] (5.49)

To bound the term,
\[
\Pi := Cb^{2+\epsilon+\eta} \int_{\mathbb{R}^2_+} dy |y|^4 |\chi(y_3)|^2 |\pi(y_1, y_2)|^2,
\]
we use again the decay of \( \pi \) from Theorem 4.1, which implies that \( \int |y|^4 |\pi|^2 \leq C \). We write
\[
\| \Pi \| \leq Cb^{2+\epsilon+\eta} \int_{\mathbb{R}^2_+} dy (|y_1|^4 + |y_2|^4 + |y_3|^4) |\chi(y_3)|^2 |\pi(y_1, y_2)|^2 \leq Cb^{2+\epsilon-4\eta} + Cb^{\epsilon}.
\] (5.50)

Inserting (5.47)–(5.50) in (5.45), we get
\[
\int_{\mathcal{N}_{\rho,\eta}} dy |(\nabla - ib\tilde{F})\tilde{u}|^2 \leq b(1 + b^{-\epsilon}) \lambda_{\alpha_x, \gamma_x, a} + Cb^{\epsilon+2\eta} + Cb^{2+\epsilon-4\eta}.
\] (5.51)

Then, using the bound (5.51) in (5.44), we have for \( \rho \in (0, 1/2), \epsilon \in (0, 1) \), and \( \eta > 0 \),
\[
Q_{b, F}(u_{\text{trial}}) \leq (1 + Cb^{-\rho} + Cb^{-\eta}) \left[ b(1 + b^{-\epsilon}) \lambda_{\alpha_x, \gamma_x, a} + Cb^{\epsilon+2\eta} + Cb^{2+\epsilon-4\eta} \right].
\]

Taking \( \epsilon = \rho = 1/6 \) and \( \eta = 1/3 \) gives
\[
\frac{Q_{b, F}(u_{\text{trial}})}{\| u_{\text{trial}} \|_2^2} \leq b \lambda_{\alpha_{\mathfrak{p}}, \gamma_{\mathfrak{p}}, a} + Cb^{\frac{1}{2}}.
\] (5.52)

\[\Box\]

5.4. Proof of Theorem 5.1. The lower bound of \( \lambda(b) \) is established in Proposition 5.4. For the upper bound, introducing the \( \lim \sup \) (as \( b \to +\infty \)) in Equation (5.43) of Proposition 5.5 gives
\[
\limsup_{b \to +\infty} \frac{\lambda(b)}{b} \leq \lambda_{\alpha_{\mathfrak{p}}, \gamma_{\mathfrak{p}}, a}.
\]

Taking then the infimum over \( \mathfrak{p} \in D \) and noticing that \( \inf_{x \in \Gamma} \lambda_{\alpha_x, \gamma_x, a} = \inf_{x \in D} \lambda_{\alpha_x, \gamma_x, a} \) yield the desired upper bound.

5.5. Proof of Theorem 1.8. The proof of Theorem 1.8 follows now by standard arguments.

**Proof.** We use the common proof for Agmon-type estimates (see e.g. [20, Theorem 9.4.1]). Let \( R_0 > 1, \eta > 0 \) to be chosen later and let
\[
g(x) := \eta \max(\text{dist}(x, D), R_0 b^{-\frac{1}{2}}), \quad x \in \Omega.
\] (5.53)

Notice that
\[
\text{supp}(\nabla g) \subset \{ \text{dist}(x, D) \geq R_0 b^{-\frac{1}{2}} \} \text{ and } |\nabla g| \leq C\eta.
\] (5.54)

By an integration by parts, the ground state \( \psi \) of the operator \( \mathcal{P}_{b, F} \) satisfies
\[
\lambda(b) \| e^{\sqrt{b} g} \psi \|^2 = \text{Re}(\mathcal{P}_{b, F} \psi, e^{2\sqrt{b} g} \psi) = Q_{b, F}(e^{\sqrt{b} g} \psi) - b \| \nabla g e^{\sqrt{b} g} \psi \|^2.
\] (5.55)
Moreover, under Assumption 1.6 and from Proposition 5.3 we get
\begin{equation}
Q_{b,F}(e^{\sqrt{\theta} \psi}) \geq \int_{\Omega} dx \left( U_b(x) - CR_0^{-2} \right) |e^{\sqrt{\theta} \psi}|^2 \geq \int_{\text{dist}(x,D) \geq R_0b^{-1/2}} dx \left( |a| \Theta_0 b - CR_0^{1/2} - CR_0^{-2} b \right) |e^{\sqrt{\theta} \psi}|^2 + \int_{\text{dist}(x,D) < R_0b^{-1/2}} dx \left( \lambda_D b - CR_0^{1/2} - CR_0^{-2} b \right) |e^{\sqrt{\theta} \psi}|^2,
\end{equation}

where \( \lambda_D := \inf_{x \in \Omega} \lambda_{\alpha, \gamma, \varepsilon, a} = \inf_{x \in \Gamma} \lambda_{\alpha, \gamma, \varepsilon, a} \). By Assumption 1.6, we have that \( \lambda < |a| \Theta_0 \).

Now, by Theorem 5.1 (more precisely see Proposition 5.5), we have
\begin{equation}
\lambda(b) \leq b \lambda_D + o(b).
\end{equation}

Inserting (5.56) and (5.57) in (5.55) implies
\begin{equation}
\int_{\text{dist}(x,D) \geq R_0b^{-1/2}} dx \left( |a| \Theta_0 b - CR_0^{1/2} - CR_0^{-2} b \right) |e^{\sqrt{\theta} \psi}|^2 + \int_{\text{dist}(x,D) < R_0b^{-1/2}} dx \left( \lambda_D b - CR_0^{1/2} - CR_0^{-2} b \right) |e^{\sqrt{\theta} \psi}|^2 \leq (\lambda_D + o(1)) \| e^{\sqrt{\theta} \psi} \|^2 + \| \nabla g e^{\sqrt{\theta} \psi} \|^2.
\end{equation}

Finally, using the properties of \( g \) in (5.54), we write
\[ \| \nabla g e^{\sqrt{\theta} \psi} \|^2 \leq \eta^2 \| e^{\sqrt{\theta} \psi} \|^2, \]
and insert this equation in (5.58). This yields, for \( 0 < \eta < \sqrt{|a| \Theta_0 - \lambda_D} \), the existence of \( R_0 > 1 \) and (a sufficiently large) \( b_0 \) such that
\begin{equation}
\int_{\Omega} dx \ e^{2\eta \sqrt{\text{dist}(x,D)}} |\psi|^2 \leq C(R_0, \eta) \| \psi \|^2.
\end{equation}

The estimate for the gradient term easily follows from the inequality above, using that \( \psi \) is a ground state of \( P_{b,F} \). \( \square \)

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