STRUCTURE OF SOLUTIONS OF THE SKYRME MODEL ON THREE-SPHERE
NUMERICAL RESULTS

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Abstract. The hedgehog Skyrme model on three-sphere admits very rich spectrum of solitonic solutions which can be encompassed by a strikingly simple scheme. The main result of this paper is the statement of the tripartite structure of solutions of the model and the discovery in what configurations these solutions appear. The model contains features of more complicated models in General Relativity and as such can give insight into them.

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1. Introduction to the Skyrme model on S³ and motivation

In this paper I discuss static solutions of the Skyrme model on the three-sphere of radius R as a physical space. In a sense, to be explained later this model includes the Skyrme model on flat space \( \mathbb{R}^3 \) in the limit \( R \to \infty \). The model contains two scales of length which constitute a free dimensionless parameter whose value is crucial to the existence of solutions of a given type. There are known several models in which similar behaviour is observed (e.g. [1]) and because of this it deserves investigation. The Skyrme model on \( S^3 \) is the simplest nonlinear realization of such models which admits very rich structure of solitonic solutions.

The Skyrme model on flat space was originally introduced as minimal nonlinear realization of an effective meson field theory where low energy baryons emerged as solitonic fields. Then nucleons were introduced as topological excitations of these fields. The basic chiral field in that model is \( SU(2) \) valued scalar field \( U(x) \) which is related to the singlet meson field \( \sigma \) and the triplet pion field \( \pi \) by

\[
U(x) = \frac{1}{f_\pi} (\sigma(x) + i\bar{\pi}(x)), \quad (\sigma)^2 + (\bar{\pi})^2 = f_\pi^2.
\]

The model is defined by the Lagrange density

\[
\mathcal{L} = \frac{1}{4} f_\pi^2 \text{Tr} (L_\mu L^\mu) + \frac{1}{32 e^2} \text{Tr} ([L_\mu, L_\nu] [L^\mu, L^\nu]),
\]

where

\[
L_\mu = U^+ \partial_\mu U
\]

is \( su(2) \) valued topological current. The model is characterized by two constants \( f_\pi \) - pion decay constant and \( e \) determining the strength of the fourth order term which was introduced by Skyrme in [2] in order to ensure existence of solitonic solutions, but it can be introduced in quite different way - on geometrical level which I shall sketch now.

The Lagrange density (1.1) has a nice geometrical origin. An \( m \)-parameter Lie group \( G \) valued scalar field \( U \) is a map \( \mathcal{M} \ni p \to U(p) \in G \). If \( x = x(p) \) is a local coordinate system on \( \mathcal{M} \) and \( \xi \) are co-ordinates of the Lie group about the identity of the group, then the field induces a map \( \mathbb{R}^m \ni x \to \xi(x) \in \mathbb{R}^m \) and respective mapping of the tangent bundle \( T\mathcal{M} \) to the Lie algebra \( T_e G \) - the tangent space to the Lie group \( G \) at the identity. Each one-parameter subgroup \( U(t) \) of \( G \) such that \( U(t_1 + t_2) = U(t_1)U(t_2) \) and \( U(0) = e \) defines a smooth curve \( U(t) \) with tangent vector \( \dot{U}(t) = U(t)u \in T_{U(t)}G \) where \( u \in T_e G \) is tangent to \( e \), so that \( U^{-1}\dot{U}(t) \) is an element of \( T_e G \). This fact may be used to define the repere mobile of the Lie algebra at the identity by

\[
a_t(\xi) := U^{-1}\partial_\xi U(\xi).
\]

By that, with each \( v \in T_u \mathcal{M} \) we can associate an element of the Lie algebra

\[
T_p \mathcal{M} \ni v \to a_t(\xi(x)) \frac{\partial^t}{\partial x^\mu} v^\mu(x) \in T_e G,
\]

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which defines a Lie algebra valued one-form $\Omega = -U^{-1}\partial_{\mu}U dx^\mu$ on $\mathcal{M}$ which simultaneously is a matrix of connection one-forms whose curvature identically vanishes $d\Omega - \Omega \wedge \Omega \equiv 0$. From the connection we construct secondary quantities $\Theta := d\Omega$ and, since $\mathcal{M}$ is a Riemannian manifold, $*_\Theta$ and $*\Theta$ (where $*$ is the Hodge operator). Now we can build invariants being full scalar products ($*_\Theta \Omega$) and ($*_\Theta \Theta$), taking the first as the sigma term and the second as the self-interaction term for our field such that, imposing on $\mu$, $\lambda$, and $U$ physical dimensions, we arrive at (1.1) starting from

$$\int_{\mathcal{M}} \mathcal{L}(U, \partial U) \sqrt{-g} d^4x := \int_{\mathcal{M}} \mu \text{Tr} (*\Omega \wedge \Omega) + \lambda \text{Tr} (*\Theta \wedge \Theta).$$

This equation defines, in generally covariant manner, the lagrange density $\mathcal{L}$ and generalizes the flat Skyrme model to an arbitrary curved space-time. Another point of view, though only for static field configurations, is that one-forms whose curvature identically vanishes $d\omega = \omega \wedge \omega \equiv 0$, which in particular reproduces the sigma and self-interaction terms in (1.3), and additionally $c_0 = \det(D)$ which is proportional to the density of topological charge. This approach generalizes the energy density for static solutions to an arbitrary curved three-geometry. (Aside remark: such invariants can be easily derived if one notices that $\det(A) = \exp(\text{Tr} \log(A))$ which is obvious for diagonalizable matrices and, by virtue of continuity, must hold in general. Putting $A = E + tD$, expanding in the variable $t$, and remembering that for matrices $D$ of rank $n$ the Taylor series expansion in $t$ must terminate at $t^n$, we get all invariants for $D$).

The lagrange density (1.1) leads to the action for a field $U$:

$$\mathcal{S}[U] = \int_{\mathcal{M}} \mathcal{L}(U, \partial U) \sqrt{-g} d^4x.$$

Here we immerse the Skyrme model within the simplest possible curved space-time which, from now on, we assume to have the structure of the Cartesian product $\mathcal{M} = \mathbb{R} \times S^3_R$ ($S^3_R$ is the three-sphere of radius $R$) which is mapped into a subset of $\mathbb{R}^4$ by the map $\mathbb{R} \times S^3_R \ni p \rightarrow \Psi(p) = (t, \psi, \vartheta, \varphi)$. In this map the line element takes the form

$$ds^2 = dt^2 - R^2 d\psi^2 - R^2 \sin^2 \psi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$$

and, disregarding singularities of the map, we assume that $t \in (0, +\infty)$, $\psi \in [0, \pi]$, $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. For that space-time is invariant with respect to the group of translations in time, the first theorem of Noether assures for solutions the existence of a constant of motion - the energy integral. Moreover solutions which are static, and we limit ourself in this paper considering only such one, are critical points of the integral

$$E = R^3 \int_{S^3} T^{00} \sqrt{g_{S^3}} d^3\xi,$$

where $T^{00}$ is 00 component of the metrical-energy-momentum tensor, which for static solutions takes the form (then $T^{00} = -\mathcal{L}$)

$$T^{00} = \frac{f^2}{4R^2} g^{ik}_{S^3} \text{Tr} (L_i L_k) + \frac{1}{16e^2 R^4} g^{ik}_{S^3} g^{il}_{S^3} \text{Tr} (L_i L_j [L_k, L_l]).$$

The integral (1.2) measures the energy of a map from the configuration space $S^3$ to the $SU(2)$ group which, as a Riemannian manifold, is equivalent to $S^3$. So that we examine energies for mappings from $S^3$ into $S^3$.

The Euler-Lagrange equations for the Skyrme model on $S^3$, defined above by the energy integral (1.2), lead to a system of three nonlinear partial differential equations of second order for three unknown functions. To bypass this outstanding example of human endeavour, exploiting the invariance of (1.2) under the action of $SO(3)$ group of rotations, we restrict ourselves by considering only those fields which are $SO(3)$ invariant. According to the theorem given by Faddeev in (1) an extremum of a functional in the domain of invariant fields under the group of invariance of the functional is the extremum of the functional in the domain of all fields. The hedgehog Ansatz provides us with a subgroup of fields which are $SO(3)$ invariant. The hedgehog field, which in original Skyrme model describes baryons, is defined by

$$U(\psi, \vartheta, \varphi) = e^{i\sigma \Theta(\varphi, \vartheta) F(\psi)},$$

and in language of Skyrme the pion field $\vec{\pi}$ is identified with the direction $\vec{n}(\varphi, \vartheta)$ in isospin space and $F(\psi)$ measures its amplitude. The form of this Ansatz, of course, comes from the canonical parameterization of $SU(2)$ group with addition of implicitly assumed equivariance of the map $S^3 \to S^3$ i.e. $(\psi, \vartheta, \varphi) \to (\Psi, \Theta, \Phi)$.
such that $\Psi = F(\psi)$, $\Theta = \partial, \Phi \equiv \phi$. This is in accordance with assumptions made in \([1]\) and by that we get a natural field-theoretical generalization of the model of harmonic maps between three-spheres of Bizoń. Our model contains a free parameter $R$ which determines the curvature of physical space and, as we shall see, has crucial effect on the number of solutions. This is a toy model which mimics critical phenomena in formation of solutions in the Einstein-Skyrme model \([2]\) (but not only) and in principle it was the main motivation of the present work.

The hedgehog Ansatz leads to solutions which in the original Skyrme model were called skyrmions. In general we call skyrmions all classical static solutions of the model which minimize energy within a given sector of the baryon number $B$. Equations of motion for the Skyrme fields are the Euler-Lagrange equations for the energy functional and, as such, the solutions may be not minima but saddle points.

In the case of the Skyrme model on flat space, if $U$ tends to a certain limiting matrix at spatial infinity independent of any direction, we can define $U(\infty) = \lim_{|x| \to \infty} U(\vec{x})$. If so, we can attach to $\mathbb{R}^3$ the point at infinity and by that complete the space to a compact one. Next we can identify points from $\mathbb{R}^3 \cup \{\infty\}$ with points from $S^3$ by a suitable mapping - for instance the stereographic projection. A mapping from $\mathbb{R}^3 \cup \{\infty\}$ to $S^3$ and further from $\mathcal{T} \times (\mathbb{R}^3 \cup \{\infty\})$ to $\mathcal{T} \times S^3$ is not an isometry and produces space-time with metric tensor whose curvature is not vanishing. It means that the Skyrme model on flat space is not metrically equivalent to the transformed one but topologically is. By means of that topological equivalence we can examine certain topological aspects of the flat Skyrme model in compact space such as $S^3$ what partly explains my work on the Skyrme model on $S^3$. But we can look at this from quite another point of view. Physically a space-time endowed with non flat metrical tensor stands for gravity but the space-time from this point of view is then only a scene for skyrmions; space-time influences their motion but they do not influence its geometry. It may of course be so only under certain conditions e.g. if the presence of skyrmions only slightly disturbs metric tensor (like gravitational waves far away from their source). Then equations of motion of skyrmions decouple from Einstein’s equations and may be solved separately.

My aim here is not investigation of the full set of equations of the Einstein-Skyrme model (see \([2]\)). I am looking for some aspects of nonlinear field theory coupled to gravity independent on the specific items of the space-time. The Skyrme model on $S^3$ is a toy model and as such aspires only to give insight into these unexplored areas of nonlinearity under gravitation. Here we treat the model only as an example of nonlinear field theory.

2. THE HEDGEHOG SKYRME MODEL ON $S^3$

In this paper I consider only static solutions of the Skyrme model on three-sphere which are critical points of the energy functional \([1,2]\). Using the hedgehog Ansatz \([1,2]\) we get

$$E[F] = \int_0^\pi 4\pi \sin^2 \psi d\psi \left\{ \frac{Rf^2}{2} \left[ F'' + \frac{2\sin^2 F}{\sin^2 \psi} \right] + \frac{1}{Re^2 \sin^2 \psi} \left[ F'^2 + \frac{1}{2} \sin^2 F \right] \right\}.$$  

In the limit $e \to \infty$ we regain the energy functional (with the unit of energy $Rf^2/2$) for harmonic maps between three-spheres which are equivariant with respect to the action of $SO(3)$ group and which were examined by Bizoń in \([1]\). Thus the Skyrme term may be considered as a perturbation which deforms harmonic maps of Bizoń.

Two parameters $e$ and $f_\pi$ (in units in which $c = 1$) provide us with the natural unit of length $L_0 := (ef_\pi)^{-1}$, determining characteristic size of a soliton, and with the unit of energy $E_0 = f_\pi e^{-1}/2$. The second scale of length $L_\pi$ is provided by the radius $R$ of the physical space - the three sphere. Roughly speaking, $R$ artificially introduces gravity. (In the Einstein-Skyrme model, the second scale is introduced by the combination $\sqrt{G}/e$ originating from the full Einstein-Skyrme action \([2]\)). These two scales of length constitute a dimensionless parameter $L := L_\pi / L_0 = ef_\pi R$ which is a free parameter of the Skyrme model on three-sphere. When $e$ and $f_\pi$ are nonzero and fixed, then $L$ has the interpretation of a dimensionless radius of the three-sphere. In these units the energy functional takes the form

\begin{equation}
E[F] = 4\pi L \int_0^\pi \sin^2 \psi d\psi \left\{ \left[ F'' + \frac{2\sin^2 F}{\sin^2 \psi} \right] + \frac{2}{L^2} \left[ F'^2 + \frac{1}{2} \sin^2 F \sin^2 \psi \right] \right\}.
\end{equation}

Critical points of the functional \((2.1)\) are solutions of the Euler-Lagrange equation

\begin{equation}
(\sin^2 \psi + \kappa^2 \sin^2 F)F'' + \sin 2\psi F' + \frac{1}{2} \kappa^2 \sin 2F(F')^2 - \sin 2F - \frac{1}{2} \kappa^2 \frac{\sin^2 F}{\sin^2 \psi} \sin 2F = 0, \quad \kappa^2 := \frac{2}{L^2}.
\end{equation}
This is the quasilinear second order equation for the shape function $F(\psi)$. In what follows we shall refer to the parameter $\kappa^2 = 2/L^2$ as the coupling constant. The only nontrivial solution of this equation, which is known analytically, is the identity map $F(\psi) = \psi$ which has energy

$$E = 6\pi^2(L + L^{-1}) \geq 12\pi^2;$$

where, depending on whether $L > 1$ or $L < 1$, the harmonic or the Skyrme term prevails.

We are interested only in solutions for which both $F(0)$ and $F(\pi)$ are integer multiples of $\pi$. Another possibility are the solutions for which $F(0)$ or $F(\pi)$ (or both) are some odd multiples of $\pi/2$ (see the appendix for appropriate proofs). These solutions behave like the function $\sqrt{\lambda} \sin \ln(\lambda)$ about $\lambda = 0$, oscillating infinitely many times in the vicinity of poles. From the physical reasons we give up considering them, since otherwise the energy integral would be divergent.

For solutions, for which both $F(0)$ and $F(\pi)$ are integer multiples of $\pi$, the Bogomolnyi bound holds, and we get it from (2.1) by completion into the squares

$$(2.4) \quad E[F] = 4\pi L \int_0^\pi \sin^2 \psi d\psi \left\{ \frac{F^2}{L - \sin^2 \psi} + \frac{2}{L^2} \frac{\sin^2 F}{\sin^2 \psi} \left( (F^2 - L)^2 + 3L F^2 \right) \right\} \geq 24\pi \int_{F(0)}^{F(\pi)} \sin^2 F dF = 12\pi^2 Q.$$  

This bound is quite general and holds for all $SU(2)$ fields of the model (1.2). It is clear that the Bogomolnyi bound is saturated by the identity map at $L = 1$. By the way the solution which is the counterpart of the flat one-skyrmion, converges uniformly to the identity when $L \sim \sqrt{2}$. The bound $12\pi^2$ is not attained in the case of flat skyrmions.

By equatorial reflection symmetry of the energy functional (2.1), which does not change energy, we can assume, without loss of generality, that for solutions $F(\pi) - F(0) \geq 0$. For solutions the integer

$$Q = \frac{1}{\pi} (F(\pi) - F(0))$$

is the topological charge which is topological constant of motion and cannot be changed by continuous deformation of a solution.

Because of symmetries of the functional (2.1) we can narrow down the group of solutions. It suffices to take into account only solutions for which $F(0) = 0$ and $F'(0) \geq 0$, for if $F(\psi)$ with $F(0) = k\pi$, $k \in \mathbb{Z}$ is a solution then $F(\psi) - k\pi$ and $-F(\psi)$ are solutions too. Moreover if $F(\psi)$ is a solution with topological charge $Q$ and with $F(0) = 0$, then $G(\psi) := Q\pi - F(\pi - \psi)$ is a solution with $G(0) = 0$ and with the same topological charge $Q$ but with $G'(\psi) = F'(\pi - \psi)$. Hence it suffices to consider only solutions for which $F'(0) \geq F'(\pi)$.

So, to sum up, we can limit ourself, not losing generality, taking into account only solutions for which $F(0) = 0$, $F'(0) \geq 0$ and $F''(0) \geq F''(\pi)$; of course in special cases $F''(\pi)$ can be negative. Numerics shows that for such solutions $Q \geq 0$.

3. Tripartite structure of solutions

3.1. Numerics. The aim of this work is to find the whole spectrum of solutions of the nonlinear equation (2.2). This can be achieved only by finding numerical solutions. For this purpose it is essential to know the Taylor series expansion of solutions in the vicinity of singular points of the equation, namely about $\psi = 0$ and $\psi = \pi$. The formal Taylor series expansion for solutions with $F(0) = 0$ is

$$(3.1) \quad F(\psi) = a\psi - \frac{a(a^2 - 1)(4 + a^2\kappa^2)}{30(1 + a^2\kappa^2)} \psi^3 + O(\psi^5), \quad a = F'(0).$$

This series expansion can be used as a launcher from the singular point to the domain of regular points of equation (2.2) where numerical integration may be successfully carried on. In the left neighbourhood of $\psi = \pi$ the series is similar. By the method of 'shooting to a fitting point' we are able to get effectively numerical solutions of the equation. The series quite accurately approximates solutions at the boundaries and the $n+1$ term can be used to assess an error one does by terminating the series at $n$’th term. This allows us to move away as far as possible from the singular points, within the accuracy desired, and then carry on the integration numerically.

We denote by $F_o$ the function which we get by carrying on an integration forward step by step via equation (2.2), propagating the initial data $F(0) = 0$ and $F'(0) = a_o$ from the north pole of the base three-sphere. Similarly by $F_\pi$ we denote the function derived by propagating the initial data $F(\pi) = Q\pi$ and $F'(\pi) = a_\pi$ backward from the south pole. By assumption $a_o$ is positive. The numbers $a_o$ and $a_\pi$ we call the shooting parameters.

Within a given topological sector with topological charge $Q \geq 0$ and for fixed nonzero radius $L$, there exist some finite number of pairs $\{a_o, a_\pi\}$ of critical shooting parameters, such that the functions $F_o$ and $F_\pi$ and
their first derivatives, launched from the opposite poles, will meet at some fitting point \( \psi_f \) which, from the symmetry, is taken to be the equator \( \psi_f = \frac{1}{2} \pi \). In this way, for these exceptional pairs \( \{ a_o, a_\pi \} \), we get two complementary parts of the same integral curve which can be matched together to form the global solution of equation (2.2) with the initial data \( (0, a_o) \) at \( \psi = 0 \) or equivalently \( (Q \pi, a_\pi) \) at \( \psi = \pi \). In special cases \( a_o = a_\pi \) or \( a_o = -a_\pi < 0 \).

In generic situation if we start numerical integration with \( F(0) = 0 \) and with an arbitrary shooting parameter \( a_o \), then there will exist an integer \( l \) such that the numerical solution \( F_o(\psi) \), continued up to \( \psi = \pi \), will attain the value \( F_o(\pi) = (\frac{1}{2} + l) \pi \). In this way we get the function \( y(a_o) = F_o(\pi) \) which is antisymmetric with respect to \( a_o \) and can be restricted to the set \( a_o \in \mathbb{R}^+ \). The function takes values in the set \( \mathcal{X} := \{ x \in \mathbb{R} : x = (\frac{1}{2} + l) \pi, l \in \mathbb{Z} \} \), apart from some discrete number of critical values of \( a_o \) for which the function attains the values being some integer multiples of \( \pi \). Since \( y \) is a step function, if \( a_o \) is critical then the difference \( y(a_o + \eta) - y(a_o) = \pm \pi / 2 \), for sufficiently small \( \eta \), is positive or negative in dependence on the value of \( a_o \), and for fixed \( a_o \) on the sign of \( \eta \). Having found numerically the function \( y(a_o) \) at some fixed radius \( L \), we are able to find all critical shooting parameters \( a_o \). Of course this can be done only approximately since, numerically, one can generate the function \( y(a_o) \) only for a discrete number of values of \( a_o \). It should be clear that among the (approximately) critical values there may exist pairs which are (approximate) shooting parameters of the same global solution. In particular, if a global solution has a reflection symmetry in the base space, its pair of critical shooting parameters may consist of \( \{ a_o, a_\pi \} \) or \( \{ a_o, -a_\pi \} \). Next the pairs of approximate shooting parameters can be used as trial ones to find exact values for \( \{ a_o, a_\pi \} \) and afterwards to find the global numerical solutions of equation (2.2). It is achieved by zeroing a norm of the vector function \( (F_o - F_o, F_o' - F_o') \) of two variables \( a_o \) and \( a_\pi \), taken at the fitting point \( \psi_f = \pi / 2 \). For this purpose it was used the globally convergent Newton-Raphson method for nonlinear systems of equations (4).

### 3.2. Terminology and notation.
Numerics shows that all solutions of the Skyrme model on \( S^3 \) in general consist of three clearly distinguishable parts: a harmonic map in the middle and two skyrmions, each with certain topological charge, attached at the ends of the map. In other words in the vicinity of poles of the base three-sphere the solutions manifest their skyrmionic nature and in between they contain a harmonic map known from the paper of Bizoń [1]. This should be understood in the following way.

Firstly we consider skyrmonic ‘arms’ that is the integral curves of equation (2.2) in the vicinity of poles. We focus our attention on the north pole. (In the case of the south pole one proceeds in the analogous way).

In order to pass from the Skyrme model on \( S^3 \) to the ordinary Skyrme model on flat space, we extract the skyrmionic part from the equation of motion (2.2) introducing new dimensionless variable \( r := 2L \psi \) and defining the function \( S(r) := F(\psi(r)) \). In a small neighborhood of the north pole \( r < 2L \) and consequently \( \sin \psi \approx r / 2L \). For all finite \( r \) we formally get, in the limit of infinite radius \( L \) (the unit of length is fixed), the equation for the limiting function \( S(r) \)

\[
\left( \frac{1}{r^2} + 2 \sin^2 S \right) S'' + \frac{1}{r} S' + S'^2 - \sin 2S - \frac{1}{4} \sin 2S - \frac{\sin^2 S}{r^2} \sin 2S = 0
\]

which is the equation for shape function of the hedgehog skyrmions on flat three-space [10]. Here \( r \) is the radial coordinate and the unit of length is \( e^{-1} F^{-1}_\pi \) with \( F_* = 2F_\pi \). The difference in units comes from the conventions assumed by different authors. In the model, described by equation (3.2), for the reasons similar to those above, one looks for solutions which interpolate between 0 at \( r = 0 \) and \( n \pi \) at \( r = \infty \). Such solutions have finite energy and asymptotically behave as \( S_n(r) = \alpha_n r + O(r^3) \) in the vicinity of \( r = 0 \) and as \( S_n(r) = n \pi + \beta_n / r^2 + c(r^{-2}) \) at infinity. Comparing the first series expansion with (1.1) it is clear that \( F'(0) \) for skyrmionic arms must behave like \( 1 / L \) in the limit \( L \to \infty \). We will denote by \( s_n \) such limiting arm, shrunk into the north pole, where \( n \) is a topological charge which simultaneously is the topological charge we ascribe to the corresponding skyrmionic arm of a solution of equation (2.2) when \( L \) is finite. The skyrmionic arm of the last solution we will denote by \( S_* \), to indicate that it evolves into \( s_n \) as \( L \to \infty \). For the south pole we proceed in the similar way putting \( r = 2L(\pi - \psi) \). In both cases (the north and the south pole) we have to do with skyrmins if \( F'(0) \) or \( F'(\pi) \) diverge to \( +\infty \) or with anti-skyrmions if the respective derivatives diverge to \( -\infty \) as \( L \to \infty \).

In the second step we consider another case in which for a solution of equation (2.2) in the limit \( L \to \infty \) \( F'(0), F'(\pi) \) or both are finite. If in this limit \( F'(0) \) or \( F'(\pi) \) is finite then the respective skyrmionic arm is absent and we have to do with a harmonic map directly attached to the north pole (the south pole) or with pure harmonic map if both derivatives are finite. Harmonic maps appear in our model since if one drops the Skyrme term, what formally can be done by taking the limit \( L \to \infty \) in equation (2.2), we regain the equation for equivariant harmonic maps between three-spheres. These maps were first found and their existence proved
by Bizoń in [1]. They are solutions of the equation

\[ F''(\psi) \sin^2 \psi + F'(\psi) \sin 2\psi - \sin 2F(\psi) = 0. \]

It was proved in [1] that smooth maps \( F : S^3 \to S^3 \), which are solutions of above equation, contain two countable families of harmonic representatives in the homotopy classes of degree 0 and 1. For a given map the index \( k \) of a map is defined as the number of times the map crosses the line \( F = \pi / 2 \). The topological charge is unit for odd and null for even maps. We will denote further these maps by the symbol \( h_k \). Solutions of equation (2.2) which in the limit \( L \to \infty \) tend uniformly to \( h_k \) will be denoted by \( H_k \) and, for convenience, called harmonic maps, though in fact are not harmonic maps.

To sum up saying harmonic or skyrmionic part of a solution of equation (2.2) we mean the limiting behaviour of the solution as \( L \to \infty \) and use these names for convenience and for the sake of clarity of further considerations even if \( L \) is finite, though for finite \( L \) there is no explicit boundary between harmonic and skyrmionic part. Put differently in different regions of physical space - the three-sphere - solutions manifest their different nature. From now on by saying 'a solution is composed of' a harmonic map or a skyrmion, we will mean that the solution in the limit of large \( L \) solves approximately - between the poles - the equation of harmonic maps (3.3) and that in the vicinity of poles (after appropriate rescaling of variables as above) solves approximately the equation known from the flat Skyrme model (3.2). We shall use above terminology regardless of a value of the radius \( L \).

Because of symmetries of the model we can identify solutions which can be transformed one into another using translations and reflections in the target space and reflections in the base space. From now on we will call such class of equivalence a solution. Each class has its own representative for which \( F'(0) > 0 \) and \( Q \geq 0 \) simultaneously, so that by saying a solution we mean its representative. Each solution of (2.2) we denote by the symbol \( S_n, H_n, S_m \) which informs us that in the limit of infinite radius of the base space, the solution takes the following limiting form \( s_n, h_n, s_m \): skyrmions \( s_n \) and \( s_m \) with topological charge \( Q = n \) or \( Q = m \) localized respectively at the north or the south pole of the base space, and a harmonic map \( h_n \) with some index \( k \) which connects them. We admit of \( k = 0 \) and then \( H_n \) denotes the neutral solution which is the representative of trivial solutions \( F \equiv k\pi \). Depending on whether the harmonic map has an odd or an even index, it contributes or not the unit topological charge \( Q_h = \text{sgn}(F'(0)) \), where \( F \) is a solution of equation (3.3). Thus the limiting solution \( s_n, h_n, s_m \) has the topological charge \( Q = n + Q_h + m \) which of course must be the topological charge of the solution \( S_n, H_n, S_m \), since topological charge cannot be changed by continuous deformation within a class of functions fulfilling given boundary conditions. By definition for all solutions of equation (2.2) \( n \geq 0 \) (since \( F'(0) > 0 \)) and \( Q \geq 0 \). In special cases \( k = m = 0 \) and, in the limit of infinite radius, we have a pure skyrmion (localized at the north pole), or \( n = m = 0 \) and then we have a pure harmonic map \( h_n = \lim_{L \to \infty} H_n \) between three-spheres. A skyrmionic solution with negative topological charge we call anti-skyrmion and denote by \( \overline{S} \) i.e. if \( m < 0 \) then \( S_m \equiv \overline{S}_{-m} \). By \( \overline{S}_k \) we denote the harmonic maps for which \( F'(0) < 0 \).

Profile functions of solutions of equation (2.2) and their phase diagrams are shown in this paper using the conformal variable \( x \) which is related to the angle \( \psi \) by the equation \( x = \ln \tan (\frac{\psi}{2}) \). The profile functions are plotted using coordinates \( (x, F(x)) \) and the phase diagrams using coordinates \( (F(x), F'(x)) \).

Figure 1 illustrates above terminology using as an example the solution \( S_n, H_n, S_m \) together with its phase diagram for different sizes of the base space. This solution, whose representant is \( S_n, H_n, S_m \), bifurcates from the solution \( S_n, H_n, S_m \) at the critical value \( L \approx 126.56 \) of the radius of the base three-sphere. The greater the radius the more skyrmionic branches of the solution flatten and become tangent to the lines \( F = \pi \) and \( F = 2\pi \) respectively for 1-skyrmion in the vicinity of the north pole and for anti 2-skyrmion in the vicinity of the south pole. The inner part of the solution in the limit of infinite radius overlaps the harmonic map \( h_n \) for which \( F(0) = \pi \) and \( F(\pi) = 2\pi \).

In the following we shall show that the extremely rich structure of solutions of the nonlinear equation (2.2), undergoes a strikingly simple scheme which enables us doing predictions of the properties of the solutions and how do they appear.
with topological charge $Q$ which for integer $Q$ is seen on phase diagrams (fig.2) for small asymptotic behaviour of energies of solutions with odd and even topological charge in the two limits (fig.3). $S$ and $S$ are quite different - charge $Q$ which in the limit $L = f_x e R$ has the interpretation of a dimensionless radius of the base three-sphere. I emphasize the trivial fact the 'coupling constant' $\kappa^2 = 2 / L^2$ of the model can be changed both by changing physical radius $R$ or by changing $e$ or $f_x$. In particular the limit $\kappa^2 = 0$ can be attained either by taking $e \to \infty$, then we recover the sigma model with the unit of energy $R f^2 / 2$ (energies of harmonic maps are then finite and the same as in (1)); or by flattening the physical space putting $R \to \infty$, then we recover the flat Skyrme model with the unit of energy $f_x e^{-1 / 2}$ (and then the energies of skyrmionic solutions coincide with those from the flat Skyrme model). As we shall see this fact partly explains why the Skyrme model on three-sphere admits existence of branches of solutions of harmonic, skyrmionic or mixed type, which meet at some critical value of $\kappa^2$ and disappear.

4.1. Solutions which exist for all values of the coupling constant. For all $\kappa^2$ there exist solutions which in the limit $\kappa^2 \to 0$ tend to configurations $H_0$, $S, H_0 S_1$, $S, H_0 S_2$, ..., $S, H_0 S_n$, ... , with topological charge $Q = 2 n$ and solutions which in this limit tend to configurations $H_1$, $S, H_1 S_1$, $S, H_1 S_2$, ..., $S, H_1 S_n$, ..., with topological charge $Q = 2 n + 1$. They all are invariant under the transformation $F(x) \to Q \pi - F(-x)$ which for integer $Q$ is a symmetry of the equation of motion (and in general is broken by generic solutions). As is seen on phase diagrams (fig.2) for small $L$ there is no qualitative difference between solutions $S, H_0 S_n$ and $S, H_1 S_n$ except for the number of maxima for the function $F'$. However, in the limit of infinite $L$, they are quite different - $S, H_0 S_n$ contain additionally the harmonic map $H_1$. This is reflected especially in the asymptotic behaviour of energies of solutions with odd and even topological charge in the two limits (fig.3). In the limit $f_x \to 0$ (no sigma term) $F(\psi)$ solves the equation

$$F'' + F^2 \cot F - \frac{1}{2} \frac{\sin 2F}{\sin^2 \psi} = 0$$

which can be formally obtained by taking the limit $\kappa^2 \to \infty$ in equation (2.2). As is seen on phase diagrams (fig.2) in the limit $L \to 0$ the maxima of $F'(\psi)$, localized at such $\psi_k$ for which $F(\psi_k) = k \pi$ where $k$ is some positive integer, tend to infinity. This follows from the formal Taylor series expansion of solutions of equation (1.1) whose singular part for $F'$ in the neighbourhood of $\psi_k$ is

$$F'(\psi) \sim \frac{(k \pi)^3}{48} |\psi - \psi_k|^{-1 / 2}.$$

In the limit $L \to 0$ the shooting parameters for the solutions tend to finite values e.g. 2.5635, 4.1047, 5.641, 7.1755, 8.7089 respectively for $S, H_0 S_1$, $S, H_1 S_2$, $S, H_0 S_3$, $S, H_1 S_4$, $S, H_0 S_n$, $S, H_1 S_n$. These values form a sequence which asymptotically is arithmetic. The same (but with different values) holds for solutions $S, H_0 S_n$. For $L$ small enough the energies $E_Q$ of solutions $S, H_0 S_n$ and $S, H_0 S_n$ behave like $e_Q / L$ where constants $e_Q$ form an ascending sequence in raising topological charge $Q$ and asymptotically scale with $Q^2$ that is

$$\lim_{L \to 0} L E_Q(L) = e_Q, \quad \lim_{Q \to \infty} \frac{e_Q}{Q^2} = \text{const.} > 0.$$

These energies would become finite again if, instead of $f_x e^{-1 / 2}$ as the unit of energy, we chose $R^{-1} e^{-2}$ for $R$ and $e$ fixed (but then we would get quite different model). In the limit of large $L$ the asymptotic behaviour
Figure 2. Solutions $S_1 H S_1$ ($\kappa^2 = 10^{-5}, \ldots, 10^3$) and $S_2 H S_2$ ($\kappa^2 = 10^{-3}, \ldots, 10^2$) and their phase diagrams for different sizes of the base space. The smaller the radius $L$ the higher the maxima for $F'$. In the limit $L \to 0$ the maxima become infinite and then the solutions tend to the solutions of the limiting equation (4.1).

Figure 3. Energies (in units $f\pi e^{-1}/2$) for solutions $S_n H_0 S_n$ or $S_n H_1 S_n$ for $n \geq 1$. Counting from the bottom the curves are $H_1, S_1 H_0 S_1, S_1 H_1 S_1, S_2 S_2$ and so on with growing topological charge $Q$ up to $S_4 H_1 S_4$. It should be evident how this curves will behave for $Q > 9$. The lowest line - the identity $H_1$ - can be explicitly calculated from equation (2.3) hence we get the asymptotic behaviour of energies for other solutions.

The difference between the energies of solutions $S_n H_0 S_n$ is very different from the behaviour of $S_n H_1 S_n$. This difference comes from the fact (which is the numerical observation) that energies of n-skyrmions are bounded while the energies of harmonic maps are unbounded in the limit $L \to \infty$ (with $f\pi e^{-1}/2$ as the unit of energy). Due to the tripartite structure of solutions and the fact the integral of energy is additive, this behaviour is clear. This fact (clarified on fig.3) is quite general, that is if we take a solution $S_n H_1 S_m$ then

$$\lim_{L \to \infty} (E_{S_n H_1 S_m}(L) - E_{H_1}(L)) = \lim_{L \to \infty} E_{S_n}(L) + \lim_{L \to \infty} E_{S_m}(L)$$
then these energies (with energies of harmonic maps scale with minimum of energies as follows from the Bogomolnyi bound (2.4). Numerics shows that in this limit the explicitly from (2.1) coincide with those from [1]. If the limit $\kappa \to \infty$ understood in the sense to those found by Bizoń in [1]. These limiting maps are shown in figure 5. In the case when this limit is containing the same harmonic map inside, have the same limiting ‘energies’ This energy is infinite in the limit $L \to \infty$ tend pointwise but not uniformly to those of harmonic maps they contain. This observation qualitatively solutions tend to zero so in the limit of large $L$ different solutions, with different topological charges, but containing the same harmonic map inside, have the same limiting ‘energies’ $\tilde{E}$. Moreover their profile functions tend pointwise but not uniformly to those of harmonic maps they contain. This observation qualitatively

From the tripartite structure it is also clear, that in the limit $L \to \infty$, the shooting parameters for solutions $S_n H S_n$ and $S_n H S_n$ divided by $L$ are the same as for n-skyrmions (see pt. 4.3).

We can assume that the arms of solutions $S_n H S_n$ and $S_n H S_n$ pass from the regime described by equation (4.1) to the regime when they are skyrmonic arms (the n-skyrmions), if their phase diagrams flatten at $x = 0$.

4.2. Harmonic maps and accompanying solutions. There exist a descending sequence of critical values of coupling constants $\{\kappa_n^2\}_{k \in \mathbb{N}}$, $k \geq 2$ at which there appear harmonic maps with index $k$. The only exception is the identity map $H_1$ which exists for all values of $\kappa^2$. In the limit $\kappa^2 \to 0$ all these maps tend uniformly to those found by Bizoń in [1]. These limiting maps are shown in figure 3. In the case when this limit is understood in the sense $\varepsilon \to \infty$ (with $Rf^2/2$ as the unit of energy) the energies of limiting harmonic maps coincide with those from [1]. If the limit $\kappa^2 \to 0$ was understood in the sense $R \to \infty$ (with $f_\pi$ and $\varepsilon$ fixed), then these energies (with $f_\pi \varepsilon^{-1}/2$ as the unit of energy) would be infinite. The energies of harmonic maps and the accompanying solutions are shown in fig.4. The energy of the identity solution $H_1$ can be computed explicitly from (4.1)

$$E_{H_1}(L) = 6\pi^2(L + L^{-1}) \geq 12\pi^2.$$  

This energy is infinite in the limit $L \to \infty$ and attains its minimum $12\pi^2$ at $L = 1$ which is the global minimum of energies as follows from the Bogomolnyi bound (2.4). Numerics shows that in this limit the energies of harmonic maps scale with $L$ and that the functions $E_{H_k}(L)/(12\pi^2L)$ have finite limits

$$\lim_{L \to \infty} \frac{E_{H_k}(L)}{12\pi^2L} = \tilde{E}_k < \frac{2}{3},$$

where the sequence $\{\tilde{E}_k\}_{k \in \mathbb{N}}$ (in units in which $Rf^2/2 = 1$) is ascending and converges to $2/3$. This limiting value is the energy of the singular map $F \equiv \pi/2$ to which the sequence of functions $H_k(\psi)$ converges pointwise (but not uniformly because the singular map does not fulfill the boundary conditions). In the limit $\varepsilon \to \infty$ the respective sequence of energies $\tilde{E}_k$ is precisely the same as in the sigma model on $S^3$ examined in [1]. Furthermore, as we shall see below, after the above rescaling $E \to \tilde{E} = E/L$, the energies of skyrmionic solutions tend to zero so in the limit of large $L$ different solutions, with different topological charges, but containing the same harmonic map inside, have the same limiting ‘energies’ $\tilde{E}$. Moreover their profile functions tend pointwise but not uniformly to those of harmonic maps they contain. This observation qualitatively
Figure 5. Even and odd harmonic maps $H_k$ for $k = 2, 4, 6, 8, 10, 12$ and for $k = 1, 3, 5, 7, 9, 11$ respectively and phase diagrams for $k = 12$ and $k = 11$ in the limit of infinite $e$. In the limit $k \to \infty$ phase diagrams have infinite self-similar structure.

Figure 6. Energies (in units where $Rf^2/2 = 1$) for the first five harmonic maps and the associated solutions. Solid lines shows harmonic branches $H_k$ (only five for clarity), dotted lines - branches of associated solutions $S_1 H_{k-2} S_1$ and $S_1 H_{k-2} S_1$ for odd and even $k \geq 2$, respectively. These branches coalesce in a cusp for descending $L$. The short dotted lines are branches of solutions $S_1 H_{k-1}$ bifurcating from harmonic maps $H_k$. The identity map $H_1$ is exceptional and exists for all $L$. The solution $S_1$ bifurcates from $H_1$ at $L = \sqrt{2}$.

Explains the conjecture made by Y. Brihaye and C. Gabriel in [5], who found several solutions, that (using my terminology)

$$\lim_{\kappa \to \kappa_{2k}} \tilde{E}_{H_{2k}}(\kappa) = \lim_{\kappa \to \kappa_{2k}} \tilde{E}_{S_1 \overline{H}_{2k-2} S_1}(\kappa) = \tilde{E}_{H_{2k}}(\kappa_{2k}), \quad \lim_{\kappa \to 0} \tilde{E}_{S_1 \overline{H}_{2k-1} S_1}(\kappa) = \tilde{E}_{H_{2k-1}}(\kappa) = \tilde{E}_{H_{2k-1}}(0),$$

The numerical evidence for these limits, which are made here far more plausible, is shown in fig.6 and 7.

It should be clear, that in the limit of large $L$ (understood as large $e$), the Skyrme term can be treated
The mechanisms in the case of families of harmonic maps are described below. Further, is generic and will be repeated during appearing of other solutions with higher topological charges. Of a pair and next the bifurcation of a second solution from the just created harmonic branch, as we shall see with higher index appears (see fig. 7). These two mechanisms of appearing of solutions, namely the creation of a solution as a function of $L$ they evolve on separate branches. (By a branch we mean the plot of the energy or of the shooting parameters of a solution as a function of $L$ or $\kappa^2$). A solution of the second type forms another branch which bifurcates from the harmonic map’s branch elsewhere at a larger critical radius $L$, but before the next harmonic map with higher index appears (see fig. [3]). These two mechanisms of appearing of solutions, namely the creation of a pair and next the bifurcation of a second solution from the just created harmonic branch, as we shall see further, is generic and will be repeated during appearing of other solutions with higher topological charges. The mechanisms in the case of families of harmonic maps are described below.

**Mechanism I:** Each harmonic map $H_k$, $k = 2, 3 \ldots$ appears together with an accompanying solution. The pattern of appearance is as follows. The even harmonic map $H_{2k}$ appears together with the solution $S_1 \overline{\mathcal{P}}_{2k-2} S_1$ e.g. $H_2$ together with $S_1 H_0 \overline{S}_1$, while the odd harmonic map $H_{2k+1}$ appears together with $S_1 \overline{\mathcal{P}}_{2k-1} S_1$ e.g. $H_3$ together with $S_1 \overline{\mathcal{P}}_1 S_1$. See tab. [1] and fig. [3].

The higher the index of a map, the larger $L$ at which it appears. Here again, in connection with harmonic maps, as in [1] there appears the characteristic number $e^{2\pi/\sqrt{7}}$ but this time it is the limit of a sequence defined in the following numerical hypothesis:

**Hypothesis 4.1.** The sequence $\left\{ \frac{\kappa_k}{\kappa_{k+2}} \right\}_{k \in \mathbb{N}}$ of quotients of critical coupling constants at which there appear odd or even harmonic maps is decreasing and attains the limiting value ($\sim 10.749087$)

$$\lim_{n \to \infty} \frac{\kappa_k}{\kappa_{k+2}} = e^{2\pi/\sqrt{7}}$$
Table 1. Solutions accompanying harmonic maps appear in two mechanisms. In the first mechanism there appear even harmonic maps $H_{2k}$ together with $S, \overline{H}_{2k-1}, S$, or odd harmonic maps $H_{2k+1}$ together with $S, \overline{H}_{2k+1}, S$. In the second mechanism new solutions bifurcate from already existing harmonic maps i.e. $S, \overline{H}_{2k-1}$, simultaneously with $H_{2k-1}, \overline{S}$, bifurcate from $H_{2k}$ while $S, \overline{H}_{2k}$, simultaneously with $H_{2k}, S$, bifurcate from $H_{2k+1}$.

Figure 8. This figure illustrates the evolution of even and odd harmonic maps together with accompanying solutions. The profile functions on the left and respective phase diagrams on the right are plotted for different values of coupling constant. The evolution is exemplified by pairs of solutions: $H_2$ together with $S, \overline{H}, S$, at the top and $H_3$ together with $S, \overline{H}, S$, at the bottom. Dotted lines shows the limiting solutions in the vicinity of critical values of coupling constant $\kappa_{H_2}^2 = 1.593848 \cdot 10^{-3}$ and $\kappa_{H_3}^2 = 1.391083 \cdot 10^{-4}$.

Using this hypothesis it is easy to predict when and together with what solution other harmonic maps will appear. This reveals the whole structure of appearance of harmonic maps together with their companions. But that is not the whole story. Numerics shows that there have to exist some (conformal) instabilities in solutions containing harmonic maps which lead to new solutions which bifurcate from the previous. This leads to the existence of the second mechanism of how solutions are born.
Mechanism II: from each even harmonic map $H_{2k}$ there bifurcates the solution $S_1 \overline{H}_{2k-1}$ and simultaneously its reflection in the base space $H_{2k-2} \overline{S}_1$. Similarly from each odd harmonic map $H_{2k+1}$ there bifurcates the solution $S_1 \overline{H}_{2k}$ together with $\overline{H}_{2k} S_1$, which is derived from the first by reflection both in the base and in the target space. See tab.3 and fig.5.

Figure 9. The figure shows profiles and phase diagrams for solutions appearing in the second mechanism in which new solutions bifurcate from already existing harmonic maps. Upper figures shows $S_1 \overline{H}_k$ for several $\kappa^2$ up to $10^{-6}$ (solid lines) and $H_1 \overline{S}_1$ (dashed line); both bifurcate from $H_2$ (dotted line) at $\kappa^2 = 5.245019 \times 10^{-2}$. The figures in the bottom show $S_1 \overline{H}_k$ for several $\kappa^2$ up to $10^{-7}$ (solid lines) and $H_2 S_1$ (dashed line) - both bifurcate from $H_3$ at $\kappa^2 = 5.70371 \times 10^{-3}$. It is easy to distinguish skyrmionic and harmonic components of solutions (e.g. see phase diagrams). In general from harmonic map $H_k$ $k \geq 1$ there bifurcates $S_1 \overline{H}_{k-1}$ and simultaneously $H_{k-1} \overline{S}_1$ for $k$ even or $H_{k-2} S_1$ for $k$ odd. In special case when $k = 1$ from harmonic map $H_1$ there bifurcates 1-skyrmion $S_1$ at the north and at the south pole.

Here again, as in the case of the first mechanism, one can form the sequence $\left\{ \frac{a_k}{a_{k+1}} \right\}_{k \in \mathbb{N}}$ of quotients of critical coupling constants at which new pair of solutions bifurcates from odd or even harmonic maps. Again this limit attains the magic value $e^{2\pi/\sqrt{7}}$.

It is worth noting that the index and topological charge of solutions appearing in pairs is the same. Similarly the same is in the case of an already existing solution and the one which bifurcates from it. I state this as an empirical fact which is true for all solutions of the Skyrme model on $S^3$ irrespective of a value of the topological charge, but by an index we mean then the index of a solution within the $\pi$-wide strip which contains the harmonic map. Summing up, the formation of solutions in the second mechanism is governed by the laws of conservation of the index and of the topological charge of appearing solution.

4.3. The n-skyrmions and their companions. The solutions whose shooting parameters in the limit of large $L$ have asymptotic behaviour of the type $a_0 \sim O(L)$ and simultaneously $a_\pi \sim O(L^{-2})$, where $a_0$ and $a_\pi$ are the values of $F'(0)$ and $F'(\pi)$ respectively, we refer to as n-skyrmions and denote them by $S_n$ where $n$ is topological charge. The name skyrmion for these solutions is justified by the observation that in the limit of infinite radius $L$ of the base three-sphere their energies are just the same as in ordinary flat Skyrme model (see tab.5 and fig.10). As we have already seen, after appropriate rescaling of independent variable, in the limit $L \to \infty$ the skyrmionic solutions fulfil the Skyrme equation (3.2) in flat space. These solutions we order with increasing topological charge which simultaneously is the order in which they appear as the radius $L$ increases. A solution $S_n$, for which $n \geq 2$, appears together with its companion $S_{n-1} H_1$, which is composed of the harmonic map $H_1$ and attached to it $(n-1)$-skyrmion (see fig.11). In the special case when $n = 1$ the scheme of appearance is quite different - the solution $S_1$ bifurcates from the already existing $H_1$ at the critical
A fit gives the value $s_c \approx 1.7169$. From the table 3, it is clear that shooting parameters $a_0$ and $a_\pi$ for $n$-skyrmions $S_n$ scale asymptotically with the radius $L$ that is the limits $\lim_{L \to \infty} a_0/L$ and $\lim_{L \to \infty} a_\pi L^2$. Numerics shows that critical values $L_n$ scale with topological charge $Q = n$ and supports the hypothesis that the limit of the sequence $\{L_n/n\}_{n \in \mathbb{N}}$ exists and is finite

$$\lim_{n \to \infty} \frac{L_n}{n} = s_c.$$
are not spherically symmetric). This suggests that n-skyrmions are unstable (for instance in the case of flat space skyrmions stable configurations exist and are finite (see fig 12). In addition this asymptotic equalities, as well limiting energies, scale again with topological charge and this all is summed up by the following limits:

\[
\lim_{n \to \infty} \lim_{L \to \infty} \frac{E_n(L)}{12n^2} = s_c, \quad \lim_{n \to \infty} \lim_{L \to \infty} \frac{a_{0,n}(L)}{Ln} = s_o, \quad \lim_{n \to \infty} \lim_{L \to \infty} L^2 \frac{a_{\pi,n}(L)}{n^2} = s_\pi.
\]

Fits give approximate values to the above limits: \( s_c \approx 0.688, s_o \approx 1.77, s_\pi \approx 0.24 \). The value of the asymptotic energy of the skyrmionic solution \( S \) is the same as the energy of 1-skyrmion on flat space and confirms results given in [9] where the Skyrme model on \( S^3 \) was investigated. From the table 2, it is clear that n-skyrmions with \( Q \geq 2 \) are not energetically stable because \( E_n/E_1 > n \). In particular, \( E_2 > 2E_1 \) suggests that dynamically \( S_2 \) would decay into a system of two 1-skyrmions (maybe \( S, H_1, S_1 \) or). This suggest that n-skyrmions are unstable (for instance in the case of flat space skyrmions stable configurations are not spherically symmetric).
5. Full structure of solutions of the Skyrm model on $S^3$

The main result of this work, besides the statement of the tripartite structure of solutions of the Skyrme model on $S^3$, is the discovery in what configurations do the solutions appear. My analysis of a vast number of numerical solutions of equation (2.2) and their evolution with the radius of the base three-sphere, had led to stinkingly simple picture. If a new solution appears when the radius exceeds its critical value, characteristic for that solution, then either it appears together with the second accompanying solution or it bifurcates from an existing solution. The clue in guesswork was to divide all the solutions into disjoint families $n, ki$ as shown in the table. The nonnegative index $n$ means that within the family $n, ki$ we ask how the solution $S_n \overline{\Pi}_{(i,k)} S_r$ contains an odd or an even harmonic map. To see how this all works I give here several examples:

| $n$ | $n, k \geq 0$ | $Q$ | $n, k \geq 1$ | $Q$ |
|-----|----------------|-----|----------------|-----|
|      | pairs of solutions       |     | pairs of solutions       |     |
| $k > 0$ | $S_n H_{2k-1} S_n$ | 0 | $S_{n-1} H_2 S_{n-1}$ * | 0 |
|      | $S_n \overline{\Pi}_{2k-2} S_{n-m}$ | $m$ | $S_{n-1} H_{2k-1} S_{n-m}$ | $m-1$ |
|      | $S_n \overline{\Pi}_{2k-1} S_{n-1}$ * | $2n-1$ | $S_{n-1} H_{2k+1} S_{n-1}$ | $2n-1$ |
| $k = 0$ | $S_k S_n$ | $2n$ | $S_n H_1 S_n$ | $2n+1$ |

Table 4. Disjoint families of solutions $n, ki$ of the Skyrm model on $S^3$ where $n \in \mathbb{N} \cup \{0\}$, $i = 0, 1$ (we say that $i = 0$ is opposite to $i = 1$). Solutions $S_n, S_n$ and $S_n H_1 S_n$ which exist for all radii $L$ of base three-sphere are included conventionally in the families. The solutions from $n, ki$ marked by the star * belong to families with subscript $n - 1$ and with the opposite $i$ and are unstable at certain critical value of $L$ when they ’emit’ a solution written on its left side. The positive integer $m$ takes the values $1, 2, \ldots, 2n - 2$ if $i = 0$ or $2, 3, \ldots, 2n - 1$ if $i = 1$.

which contains the n-skyrmion at the north pole, harmonic map $\overline{H}$ with index $p$ inside, and r-skyrmion at the south pole; appears under assumption that $r$ takes such values that $r$ does not exceed $n$ and that the topological charge of the solution is not negative. The topological charge depends of course on whether the solution contains an odd or an even harmonic map. To see how this all works I give here several examples:

- **Solutions $S_n H_n S_n$ and $S_n H_1 S_n$:** The solutions $S_n H_n S_n$ from $n_00$ never decay in contrast to the solutions $S_n H_1 S_n$ from $n_01$ from which $S_n, S_n$ bifurcate belonging to the families $n+1, 1, 10$. As we saw before the solutions $S_n S_n$ and $S_n H_1 S_n$ exist for all radii of the base three-sphere.

- **Solutions $H_n$ (‘harmonic maps’):** Families $1, k0$ and $1, k1$ contain even and odd harmonic maps respectively. From the table, we read off that the maps $H_2, H_3, \ldots$ appear together with $S_n \overline{\Pi}_1$, $S_n \overline{\Pi}_2$, $S_n \overline{\Pi}_3$, $\ldots$, respectively, and odd maps $H_1, H_3, \ldots$, appear respectively together with $S_n \overline{\Pi}_1 S_1$, $S_n \overline{\Pi}_3 S_3$, $\ldots$. We saw in (1.2) that the greater the index of a harmonic map, the greater the radius $L$ at which it appears. Nevertheless, before $H_{p+1}$ appears, $S_n \overline{\Pi}_{p-1}$ bifurcates from $H_p$. Of course, the solutions $H_n$ for $p = 2, 3, \ldots$ are not harmonic maps (we have used this name for convenience), but in the limit of infinite radius of the base three-sphere they also fulfill the equation of harmonic maps between three-spheres.

- **Skyrmionic solutions:** Families $n, 10$ for $n = 2, 3, \ldots$ contain n-skyrmions $S_n$, which appear together with $S_{n-1} H_1$. We remember from (3.4) how to interpret the name n-skyrmion of the solution $S_n$, which, for finite radii of the base three-sphere, is not n-skyrmion at all. But the similarities and proper interpretation introduced in previous paragraphs had justified the terminology.

- **Solutions with $Q = 0$:** contain even harmonic maps which can appear only in pairs $S_n H_{2k-1} S_n$ together with $S_{n-1} \overline{\Pi}_{2k} S_{n-1}$, whereas solutions containing odd harmonic maps $S_n \overline{\Pi}_{2k-1} S_{n+1}$ appear from decay of $S_n H_{2k} S_n$, where $k \geq 1$ and $n \geq 0$. Further it turns out that the greater $n$ with fixed $k$ or the greater $k$ at fixed $n$, the greater critical radius at which given solution appears.

Numerics shows that in general there is no rule which could be used to "forecast" the order of appearance of arbitrary solutions (at least if one uses numerical methods of finding solutions). If we take the family of solutions with n-skyrmion at the north and m-skyrmion at the south pole fixed but which are distinguished by the harmonic maps with different index $k$ they contain, then the cusps of the type as in fig. [10] or knee-like shapes as in fig. [9] locate, in the limit of large $k$, along straight lines (if we use logarithmic scales) whose slopes are different in dependence on a pair $(n, m)$. In this limit, along given straight line, these critical values are distributed homogeneously what implies that asymptotically they form a geometrical sequence with base quotient being certain power of $10^{2\pi/\sqrt{7}}$. Now it is clear that the sequences of critical
values from different lines are not commensurate with each other and that is why it is impossible to predict which solution would appear as the next one after the one arbitrarily chosen.

In spite of above we can distinguish certain 'directions' in which such 'putting in order of appearance' is possible. Here are some of them:

1. If we take the set of solutions \( S_n H_p S_m \), with arbitrary but fixed \( n \) and \( m \), with different \( p \geq 0 \), then the larger \( p \), the greater the radius at which the solution containing harmonic map \( H_p \) appears.

2. For fixed topological charge, the solution \( S_{n+s} H_p S_{m+s} \) appears at larger radius than \( S_n H_p S_m \).

3. For fixed nonnegative \( n \), the solution \( S_{n+s} S_m \) appears at larger radius than \( S_n S_{m+s} \) where \( s \geq 1 \) and \( 0 \leq n - m < 2n \).

4. If positive \( n \) is fixed then sequentially there appear \( S_n S_{n-1}, S_n S_{n-2}, \ldots, S_n S_0 \) (see fig.13).

We can include solutions containing harmonic maps into (3) and (4). Using above rules, reflections in the base and/or in the target space, and using the table 4, we can generate much more rules. But they are not all of course. Using above empirical rules we are able to 'predict' for example that the solution \( S_2 S_1 \) appears earlier (at lower \( L \)) than \( S_2 \); it means that although there exists a solution (e.g. \( S_2 S_1 \)) there do not exist all its components yet (i.e. \( S_2 \) does not exist yet).

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**Appendix A.**

A.1. **Local existence theorem and asymptotics for solutions.** In section (4.3) by change of variable \( \psi = r/2L \) we defined the function \( S_L(r) = F(\psi)|_{\psi=r/2L} \), which in the limit \( L \to \infty \) fulfills the Skyrme equation (3.2). Solutions of the equation are critical points of the energy functional

\[
E[S] = 4\pi \int_0^\infty \left[ \left( S''^2 + \frac{2 \sin^2 S}{r^2} \right) + \alpha^2 \left( S'^2 + \frac{1}{2} \frac{\sin^2 S}{r^2} \right) \sin^2 S \right] r^2 dr,
\]

where \( \alpha^2 \) depends on a unit of length one chooses (the second integrand above is not scale invariant) and here \( \alpha^2 = 8 \). For proofs of asymptotic behaviour of solutions of the equation

\[
(\sin^2 \psi + \kappa^2 \sin^2 F)F'' + \sin 2\psi F' + \frac{1}{2} \kappa^2 \sin 2F(F')^2 - \sin 2F - \frac{1}{2} \kappa^2 \frac{\sin^2 F}{\sin^2 \psi} \sin 2F = 0,
\]
the crucial point is the introduction of a function $Q$ which (by analogy with $\tilde{F}$) can be simply guessed by comparison of the functionals $\tilde{F}$ and $F$.

**Definition A.1.**

\[ Q(\psi) := \frac{1}{4} F''^2(\sin^2 \psi + \kappa^2 \sin^2 F) - \frac{1}{2} \sin^2 F - \frac{1}{8} \kappa^2 \sin^4 F / \sin^2 \psi, \quad \psi \in [0, \pi]. \]

In this section I exploit methods elaborated by McLeod and Troy in their work [8] where they gave theorems concerning solutions of the Skyrme model on $\mathbb{R}^3$, and in principle I repeat their ideas extending them on equation (A.2). In this (numerical) work I need only the proof that the series expansion (3.1) for solutions of the Skyrme model on $S^3$ for which $F(0)$ and $F(\pi)$ are integer multiples of $\pi$ is valid.

**Lemma A.2.** As $\psi \searrow 0$ the function $Q(\psi) \sin^2 \psi$ converges to a finite limit.

△. Let $\psi \in (0, \frac{1}{2} \pi]$ then the function $g = \sin^2 \psi(Q + \frac{1}{2})$ is increasing because from the definition of $Q$

\[ g' = \sin^2 \psi(Q' + 2Q \cot \psi + \cot \psi) = 4 \sin^2 F \cot \psi(\frac{1}{2} \kappa^2 F''^2 - \frac{1}{4}) + \cot \psi \geq \cos^2 F \cot \psi \geq 0. \]

So that $g$ decreases as $\psi \searrow 0$ and the function $Q \sin^2 \psi$ has to attain a limit which is finite or not. But it cannot equal $-\infty$ what would imply from $Q \sin^2 \psi$ that $\frac{1}{2} \sin^2 \psi F''^2(\sin^2 \psi + \kappa^2 \sin^2 F)$ was negative what is impossible. Hence there exists a constant $a$ such that

\[ \lim_{\psi \to 0^+} Q(\psi) \sin^2 \psi = a, \quad a \text{ is finite.} \]

What is the consequence of this limiting behaviour of $Q(\psi)$ for convergence of $F(\psi)$ at $\psi = 0$? The answer is the following lemma.

**Lemma A.3.** As $\psi \searrow 0$, the function $F(\psi)$ tends to a limit.

△. For contradiction suppose that $F(\psi)$ does not tend to a limit as $\psi \searrow 0$, hence $\sin F$ does not tend to a limit too. From (A.3) and (A.4) it is seen that

\[ \frac{1}{4} F''^2 \sin^2 \psi(\sin^2 \psi + \kappa^2 \sin^2 F) - \frac{1}{2} \sin^2 \psi \sin^2 F \sim a + \frac{1}{8} \kappa^2 \sin^4 F \]

and next that

\[ \frac{1}{4} \kappa^2 F''^2 \sin^2 \psi \sin^2 F \sim a + \frac{1}{8} \kappa^2 \sin^4 F \]

where we used the fact that since $\sin^2 F$ does not tend to a limit then $(\sin^2 \psi + \kappa^2 \sin^2 F) \sim \kappa^2 \sin^2 F$ and since $\sin F$ is bounded then $-\frac{1}{2} \sin^2 \psi \sin^2 F \to 0$. Moreover we can choose a descending sequence of points $\{\psi_n\}$ in such a way that $\lim_{n \to \infty} \psi_n = 0$ and that simultaneously $\frac{1}{4} \kappa^2 \sin^4 F(\psi_n) \neq -a$ (if $a < 0$) and $\frac{1}{8} \kappa^2 \sin^4 F(\psi_n) \neq 0$ for all $n$. For these $\psi_n$, $\frac{1}{4} \kappa^2 F''^2 \sin^2 \psi \sin^2 F$ is finite and different from 0. Having noticed that

\[ F' \sin \psi = \psi F'(1 + c(\psi)) \quad \text{if} \quad \psi < \varepsilon, \]

we see that $\psi_n F'(\psi_n)$ is finite and different from 0. In particular there exist such $\Delta$ that $0 < \Delta < 1$ and for all $\psi$ $F'(\psi)$ is bounded and different from zero in some interval $\psi_n(1 - \Delta) \leq \psi \leq \psi_n(1 + \Delta)$. On the other hand

\[ (Q(\psi) \sin^2 \psi)' = \sin^2 F \cot \psi(\frac{1}{2} \kappa^2 F''^2 \sin^2 \psi - \sin^2 \psi), \]

and since $F'$ is finite and bounded from zero there must exist such constant $M > 0$ that for sufficiently large $n$

\[ (Q(\psi) \sin^2 \psi)' \geq \frac{M}{\psi_n} \quad \text{if} \quad \psi_n(1 - \Delta) \leq \psi \leq \psi_n(1 + \Delta). \]

Integrating this in the interval $\psi_n^- \leq \psi \leq \psi_n^+$ where $\psi_n^+ = \psi_n(1 + \Delta)$ we get that

\[ s(\psi_n^+) - s(\psi_n^-) \geq 2M\Delta, \quad \text{and} \quad s(\psi) = Q(\psi) \sin^2 \psi. \]

But this says that $Q(\psi) \sin^2 \psi$ as a function does not fulfills the Heine criterion for convergence, what contradicts the fact proved in previous lemma that $Q(\psi) \sin^2 \psi$ tends to a limit if $\psi \searrow 0$. Thus $F(\psi)$ tends to a limit if $\psi \searrow 0$.

△.

**Lemma A.4.** As $\psi \uparrow \pi$, the function $F(\psi)$ tends to a limit.

△. The function $\tilde{F}(\tilde{\psi}) = F(\psi)$, $\tilde{\psi} \in [0, \pi]$ where $\tilde{\psi} = \pi - \psi$ fulfils identical equation like $F(\psi)$ that is why all qualitative properties of the function $F(\psi)$ pass to the function $\tilde{F}(\tilde{\psi})$. In particular $\tilde{F}(\tilde{\psi})$ tends to a limit as $\tilde{\psi} \searrow 0$ what is equivalent to that $F(\psi)$ tends to a limit as $\psi \uparrow \pi$. 

\[ \therefore \]
Theorem A.5. As \( \psi \searrow 0 \) then every solution tends to a limit which may equal either \( k\pi \) or \( (k + \frac{1}{2})\pi \), where \( k \in \mathbb{Z} \). As \( \psi \uparrow \pi \) then every solution tends to a limit which may equal \( l\pi \) or \( (l + \frac{1}{2})\pi \), where \( l \in \mathbb{Z} \).

Proof. Suppose that the limit \( \lim_{\psi \to 0^+} \sin F(\psi) \), which exists by virtue of lemma (A.3), is different from zero what means that \( \sin F_0 \neq 0 \) where \( F_0 = \lim_{\psi \to 0^+} F(\psi) \). From definition (A.1) and from the lemma (A.2) we get then

\[
\kappa^2 F''(\psi) = \frac{1}{\kappa^2} \sin^2 \psi \sim \frac{1}{\kappa^2} \sin^2 F_0 + 4a,
\]

and further that

\[
F'' \sim \frac{8a + \kappa^2 \sin^4 F_0}{2\kappa^2 \sin^2 F_0 \sin^2 \psi} \sim \frac{C_1}{\psi^2}.
\]

We must put \( 8a + \kappa^2 \sin^4 F_0 = 0 \) otherwise \( F(\psi) \) would not be bounded because then \( F(\psi) = O(\ln \psi) \) and it would contradict the lemma (A.3). But then \( F'(\psi) = o(\psi^{-1}) \) and taking it into consideration in equation (A.2) multiplied by \( \sin^2 \psi \) and taking the limit \( \psi \searrow 0 \) we get

\[
\lim_{\psi \to 0^+} \sin^2 \psi F''(\psi) = \sin F_0 \cos F_0.
\]

Now if \( \cos F_0 \neq 0 \) (sin \( F_0 \neq 0 \) from assumption) then \( F'(\psi) = O(\psi^{-1}) \) what cannot be because it contradicts just now derived formula \( F'(\psi) = o(\psi^{-1}) \). Thus if \( \sin F_0 \neq 0 \) then \( \cos F_0 = 0 \). So the conclusion is that the only possible limiting values of \( F(\psi) \) at zero may be looked for among the numbers from the countable set \( \{k\pi : k \in \mathbb{Z} \} \). As in previous proof we state that the behaviour of solutions in the left-sided neighbourhood of \( \psi = \pi \) is similar i.e. the only possible limiting values of \( F(\psi) \) at \( \pi \) may be looked for among the numbers from the countable set \( \{l\pi : l \in \mathbb{Z} \} \). This completes the proof.

We are interested in behaviour of solutions of equation (A.2) in the vicinity of the singular points \( \psi = 0 \) and \( \psi = \pi \) for which \( F(\psi) \to k\pi \) with an integer \( k \). Because of reflection symmetries of the equation we can limit ourself by considering only the solutions for which \( F(0) = 0 \) and \( F'(0) > 0 \). For such solutions \( F(\psi) > 0 \) and \( F''(\psi) > 0 \) for sufficiently small \( \psi \). Irrespective of any behaviour of \( F \) as \( \psi \to 0 \) we can for contradiction take a sequence of points \( \{\psi_n\}_{n \in \mathbb{N}} \) for which \( \psi_n \to 0 \) and \( F(\psi_n) > 0, F'(\psi_n) > 0 \). If for large enough \( n \) there was \( F(\psi) > 0 \) and \( F'(\psi) = 0 \), where \( \psi > \psi_n \), then from (A.2) we see that \( F''(\psi) \) would be positive and so \( F \) would have local minimum at this \( \psi \). Thus for sufficiently small \( \psi \) we have always \( F(\psi) > 0 \) and \( F'(\psi) > 0 \). We shall use this fact further.

The formal expansion (2.1) suggests that \( \psi F'/F \to 1 \) as \( \psi \to 0 \). This is the hint for the proof of asymptotics of solutions of the equation (2.2).

Lemma A.6. As \( \psi \searrow 0 \) then \( f(\psi) < 1 \) holds for the function \( f(\psi) := \frac{F'(\psi) \sin 2\psi}{2F(\psi)} \).

Proof. For contradiction suppose that \( F' \sin 2\psi > 2F \) then

\[
\left( \frac{F' \sin 2\psi}{2F} \right)' = \frac{F'' \sin 2\psi}{2F} + \frac{F'}{2F^2} (2F \cos 2\psi - F' \sin 2\psi) < \frac{\sin 2\psi}{2F} F'',
\]

since from assumption \( 2F \cos 2\psi - F' \sin 2\psi < 2F (\cos 2\psi - 1) < 0 \). Moreover

\[
\sin^2 F - (F')^2 \sin^2 \psi < F^2 - \frac{1}{4} (F')^2 \sin^2 2\psi,
\]

so from (A.2) we get \( F'' < 0 \). But then \( f(\psi) \) increases as \( \psi \searrow 0 \) and \( f(\psi) > c > 1 \) for \( \psi \) small enough. In particular \( F > 2cF'/\sin 2\psi \) and integration in the interval \( (\varepsilon_n, \psi_n) \) (where \( \varepsilon_n = \psi_n/n \)) gives

\[
\frac{F(\psi_n)}{\psi_n} > c \frac{2F(\delta_n)}{\sin 2\delta_n} \left( 1 - \frac{1}{n} \right) + \frac{F(\varepsilon_n)}{\varepsilon_n} \frac{1}{n} > c \frac{2F(\delta_n)}{\sin 2\delta_n} \left( 1 - \frac{1}{n} \right), \quad 0 < \varepsilon_n < \delta_n < \psi_n,
\]

and hence, taking the limit \( n \to \infty \), we get \( g \geq cg \) for \( c > 1 \) where the nonnegative number is the limit \( g := \lim_{\psi \to 0} F(\psi)/\psi \). But this can be true only if \( g = 0 \) and then from (A.2) we get

\[
\psi F'' = O(1) \left( \frac{\sin 2\psi}{\psi} - O(1) F' + \frac{1}{2} \kappa^2 \frac{\sin 2F}{\sin^2 \psi} \left( \frac{2F}{\sin^2 \psi} - (F')^2 \right) \right) < -mF'
\]

fore some positive \( m \). Integration of \( F''/F' \to -m/\psi \) gives

\[
\ln \frac{F(\psi)}{F(\varepsilon)} < -m \ln \frac{\psi}{\varepsilon}, \quad 0 < \varepsilon < \psi.
\]

But this implies that as \( \varepsilon \searrow 0 \) (at fixed \( \psi \)) then \( F'(\varepsilon) \) diverges to \( +\infty \) what in turn contradicts that \( F(\psi) = o(\psi) \). Thus the assumption \( F' \sin 2\psi > 2F \) is false and always, for sufficiently small \( \psi \), \( f(\psi) < 1 \) must hold. This proves the lemma.
Lemma A.7. As \( \psi \searrow 0 \) then \( f(\psi) > 1 \) holds for the function \( f(\psi) := \frac{2F'(\psi) \tan \psi}{\sin 2F(\psi)} \).

\( \square \).

For contradiction suppose that \( 2F' \tan \psi < \sin 2F \) then

\[
\left( \frac{2F' \tan \psi}{\sin 2F} \right)' = \frac{2 \tan \psi}{\sin 2F} F'' + \frac{2F'}{\sin^2 2F} \left( \sin 2F - F' \cos 2F \sin 2\psi \right) > \frac{2 \tan \psi}{\sin 2F} F'' ,
\]

since from assumption \( \sin 2F - F' \cos 2F \sin 2\psi > 0 \). Moreover

\[
\sin 2F - F' \sin 2\psi > F'(2 \tan \psi - \sin 2\psi) > 0,
\]

and

\[
\sin^2 2F - (F')^2 \sin^2 \psi > (F')^2 (4 \tan^2 \psi - \sin^2 \psi) > 0,
\]

so from (A.2) we get \( F'' > 0 \). But then \( f(\psi) \) decreases as \( \psi \searrow 0 \) and in fact \( f(\psi) < c < 1 \) for \( \psi \) small enough. In particular \( F' < c \sin 2F/2 \tan \psi \) and integration in the interval \((\varepsilon_n, \psi_n)\) (where \( \varepsilon_n = \psi_n/n \)) gives

\[
\frac{F(\psi_n)}{\psi_n} < c \frac{\sin 2F(\delta_n)}{2 \tan \delta_n} \frac{\psi_n - \varepsilon_n}{\psi_n} + \frac{F(\varepsilon_n)}{\varepsilon_n} \frac{\varepsilon_n}{\psi_n} < c \frac{\sin 2F(\delta_n)}{2 \tan \delta_n} \frac{F(\varepsilon_n)}{\varepsilon_n} \frac{1}{n \varepsilon_n}, \quad 0 < \varepsilon_n < \delta_n < \psi_n ,
\]

hence, taking the limit \( n \to \infty \), \( g \leq cg \) for \( 0 < c < 1 \) where the nonnegative number \( g \) is the limit \( g := \lim_{\psi \to 0} F(\psi)/\psi \). But this can be true only if \( g = 0 \). The same arguments as in the previous lemma show that \( F'(\psi) \) would be infinite in the limit \( \psi \searrow 0 \) what would contradict that \( F(\psi) = o(\psi) \). Thus the assumption \( 2F' \tan \psi < \sin 2F \) is false and always, for sufficiently small \( \psi \), \( f(\psi) > 1 \) must hold, what proves the lemma.

\( \square \).

The inequalities we have already proved can be recast in to the form

\[
\frac{\psi^2 \sin 2F(\psi)}{2 \tan \psi} < \psi F'(\psi) < \frac{2 \psi F(\psi)}{\sin 2\psi}
\]

and they are valid for sufficiently small \( \psi \). The Taylor series expansion about \( \psi = 0 \) and \( F = 0 \) gives

\[
-\frac{2}{3} F^3 - \frac{1}{3} \psi^2 + o(\psi^3) < \psi F' - F < \frac{2}{3} \psi^2 F + o(\psi^3)
\]

what implies that

\[
F(\psi) = a \psi + \mathcal{O}(\psi^3), \quad \text{where} \quad a = F'(0),
\]

and this proves (3.3). The same of course, must hold at \( \psi = \pi \) (the only difference is that \( F(\pi) = k\pi \) with an integer \( k \), which is irrelevant to the proof).

References

1. P. Bizoń, Harmonic maps between three-spheres, Proc. Roy. Soc. Lond. A (1995), no. 451, 779–793.
2. P. Bizoń & T. Chmaj, Gravitating skyrmions, Phys. Lett. B (1992), no. 297, 55–62.
3. L. D. Faddeev, The search for multidimensional solitons, Proc. of the IV Int. Symp. on Nonlocal Field Theories (1976).
4. W. H. Press & S. A. Teukolsky & W. T. Vetterling & B. P. Flannery, Numerical recipes in fortran 77. The art of scientific computing, Cambridge University Press.
5. Y. Brihaye & C. Gabriel, Skyrme model on \( S^3 \) and harmonic maps, Mod. Phys. Lett. A (1999), no. 14, 893–901.
6. S. Krush, \( S^3 \) skyrmions and the rational map ansatz, Nonlin. (2000), no. 13, 2163–2185.
7. T.H.R. Skyrme, A non-linear field theory, Proc. Roy. Soc. Lond. A (1961), no. 260, 127–138.
8. J. B. McLeod & W. C. Troy, The skyrme model for nucleons under spherical symmetry, Proc. Roy. Soc. Edinburgh A (1991), no. 118, 271–288.
9. A.D. Jackson & N.S. Manton & A. Wirzba, New skyrmeon solutions on a 3-sphere, Nucl. Phys. A (1989), no. 495, 499–522.
10. G. S. Adkins & C. R. Nappi & E. Witten, Static properties of nucleons in the skyrme model, Nucl. Phys. B (1983), no. 228, 552–566.