THE KOHNEN-ZAGIER FORMULA FOR MAASS FORMS FOR $\Gamma_0(4)$

NICKOLAS ANDERSEN

Abstract. We extend a formula of Duke, Imam\u{g}lu, and T\u{o}th (which itself is a generalization of the Katok-Sarnak formula) to prove the Kohnen-Zagier formula for Maass forms for $\Gamma_0(4)$.

1. Introduction

Let $d$ be a fundamental discriminant. The Kohnen–Zagier formula [12] relates the $|d|$-th coefficient of a holomorphic Hecke eigenform $g$ of half-integral weight on $\Gamma_0(4)$ to $L(1/2, f \times (\frac{d}{\cdot}))$, where $f$ is the Shimura lift of $g$. The formula is an explicit version of the general relation of Waldspurger [16]. Here we show how the ideas of [6] and [1] can be combined to give a short proof of the Kohnen–Zagier formula for Maass cusp forms for $\Gamma_0(4)$. We adopt the notation of [6]; see the next section for details.

Theorem 1.1. Let $\varphi$ be an even Hecke–Maass cusp form of weight $0$ for $\text{SL}_2(\mathbb{Z})$ with Fourier expansion

$$\varphi(z) = 2 \sqrt{y} \sum_{n \neq 0} a(n) K_{ir}(2\pi |n| y) e(nx).$$

Then there exists a unique Maass cusp form $\psi$ of weight $1/2$ for $\Gamma_0(4)$ with Fourier expansion

$$\psi(z) = \sum_{0 \neq n \equiv 0,1(4)} b(n) W_{\frac{1}{4}} \text{sgn}(n), \frac{w}{4} (4\pi |n| y) e(nx),$$

such that for any fundamental discriminant $d \equiv 0, 1 \pmod{4}$ we have

$$12\pi |d| |b(d)|^2 = \langle \varphi, \varphi \rangle |\Gamma(1/2 - \text{sgn}(d)/4 + \frac{1}{4})|^2 L(1/2, \varphi \times \chi_d). \quad (1.1)$$

Here $\chi_d = (\frac{d}{\cdot})$ and $L(s, \varphi \times \chi_d)$ denotes the analytic continuation of the $L$-function

$$L(s, \varphi \times \chi_d) = \sum_{n=1}^{\infty} \frac{a(n) \chi_d(n)}{n^s}.$$

The $d = 1$ case of Theorem 1.1 is a corollary of the main result of Katok and Sarnak [11], which relates the product $b(d)b(1)$ to one of the quantities

$$\sum_{Q \in \mathcal{Q}_d} \varphi(z_Q) \quad \text{or} \quad \sum_{Q \in \mathcal{Q}_d} \int_{C_Q} \varphi(z) y^{-1} |dz|,$$

depending on whether $d$ is negative or positive, respectively. Here $\mathcal{Q}_d$ is a set of integral binary quadratic forms of discriminant $d$, $\Gamma = \text{PSL}_2(\mathbb{Z})$, $z_Q$ is the root of $Q(z, 1)$ in the complex upper half-plane $\mathcal{H}$, and $C_Q$ is a hyperbolic geodesic, finite if $d > 1$ and infinite if $d = 1$ (see Section 2).

Date: March 3, 2022.
for details). In the case $d = 1$ there is one term in the sum, namely $Q = [0, 1, 0]$, and their formula reads
\[
12\sqrt{\pi}|b(1)|^2 = \langle \varphi, \varphi \rangle^{-1} \int_0^\infty \varphi(iy)y^{-1} dy.
\]
The latter integral evaluates to a multiple of $L(\frac{1}{2}, \varphi)$. To prove their formula, Katok and Sarnak modify the theta lift of Shintani \[15\] and Niwa \[13\]. With some extra work, their method can probably produce a formula for $|b(d)|^2$ where $d$ is any positive fundamental discriminant.

Using the Kuznetsov trace formula and some ideas from the proof of the Selberg trace formula, Biró \[3\] extended the Katok–Sarnak formula to general level for a pair of positive discriminants $d$ and $d'$ such that $d$ is fundamental. The main result of \[3\] is a relation between $b(d)b(d')$ and the twisted sums
\[
\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dd'}} \chi_d(Q) \int_{C_Q} \varphi(z)y^{-1}|dz|,
\]
where $\chi_d$ is a character of the finite group $\Gamma \backslash \mathcal{Q}_{dd'}$. When $d = d'$ the sum above evaluates to a multiple of $L(\frac{1}{2}, \varphi \times \chi_d)$. It is not clear whether the methods of \[11\] or \[3\] can be extended to cover the case where $d, d'$ are negative.

In \[6\], Duke, Imamoğlu, and Tóth generalized the formulas of Katok–Sarnak and Biró in the case of Maass forms for $\Gamma_0(4)$ to allow for two discriminants $d, d'$ of any sign, as long as $d$ is fundamental. In the new case, when $d$ and $d'$ are both negative (and $dd'$ is not a square), Theorem 4 of \[6\] gives a relation between $b(d)b(d')$ and
\[
\sum_{Q \in \Gamma \backslash \mathcal{Q}_{dd'}} \chi_d(Q) \int_{\mathcal{F}_Q} \varphi(z) \frac{dxdy}{y^2},
\]
where $\mathcal{F}_Q$ is a finite area hyperbolic surface with boundary $C_Q$. The case $d = d'$ is not covered in that theorem because the proof relies on being able to compute the integral over $C_Q$ of a certain Poincaré series, and the corresponding integral when $dd'$ is a square does not converge. Here we use the main idea of \[11\] to modify the Poincaré series in the case $d = d'$ and give a short proof of Theorem \[11\].

The generalization of Theorem \[11\] to Maass forms for $\Gamma_0(4N)$, with $N$ odd and squarefree, was proved by Baruch and Mao in \[2\]. Their proof utilizes the powerful tools of automorphic representation theory.

### 2. Background

Throughout this paper we make use of several special functions, especially the Bessel functions $I_\nu(x)$, $J_\nu(x)$, and $K_\nu(x)$, and the Whittaker functions $M_{\mu,\nu}(x)$ and $W_{\mu,\nu}(x)$. Definitions and properties of these functions can be found in Sections 10 and 13 of \[4\]. In the rest of this section, we give some background information on the objects in the introduction, including some standard facts we will need for the proof of the main theorem. We are mostly following the notation and setup of \[6\]. Other standard references are \[7, 9, 14\].

**Maass cusp forms of weight 0.** Let $\Gamma = \text{PSL}_2(\mathbb{Z})$ and let
\[
\Delta_k = y^2(\partial_x^2 + \partial_y^2) - iky\partial_x
\]
denote the weight $k$ hyperbolic Laplacian. A function $\varphi : \mathcal{H} \to \mathbb{C}$ is a Maass form of weight 0 for $\Gamma$ if it is $\Gamma$-invariant and is an eigenfunction of $\Delta_0$ with eigenvalue normalized by $(\Delta_0 +$
\( \lambda \varphi = 0 \) and
\[
\lambda = \frac{1}{4} + r^2 \quad \text{with } r \geq 0.
\]
The quantity \( r \) is called the spectral parameter of \( \varphi \). We say that \( \varphi \) is a Maass cusp form if the constant term in its Fourier expansion is zero, i.e.
\[
\varphi(z) = 2\sqrt{y} \sum_{n \neq 0} a_\varphi(n) K_{ir}(2\pi |n| y) e(nx)
\]
for some coefficients \( a_\varphi(n) \in \mathbb{C} \). For each \( r \geq 0 \) let \( \mathcal{U}_r \) denote the vector space of Maass cusp forms of weight 0 with spectral parameter \( r \).

For each prime \( p \), the Hecke operator \( T_p \) acts on \( \mathcal{U}_r \) via Fourier expansions as
\[
(T_p \varphi)(z) = 2\sqrt{y} \sum_{n \neq 0} (a_\varphi(pm) + p^{-1} a_\varphi(n/p)) K_{ir}(2\pi |n| y) e(nx).
\]
The Hecke operators commute with each other and with \( \Delta \), so we can find an orthogonal (with respect to the Petersson inner product \( \langle \cdot, \cdot \rangle \)) basis \( \mathcal{B}_r \) of \( \mathcal{U}_r \) consisting of Hecke eigenforms. We will normalize the elements of \( \mathcal{B}_r \), which are called Hecke–Maass cusp forms, so that \( a(1) = 1 \). We can also assume that each \( \varphi \) is even or odd, meaning that \( a(-n) = \pm a(n) \) respectively.

**Maass cusp forms of weight \( \frac{1}{2} \).** A function \( \psi : \mathcal{H} \to \mathbb{C} \) is a Maass form of weight 1/2 for \( \Gamma_0(4) \) if it satisfies \( \psi(\gamma z) = J(\gamma, z) \psi(z) \) for all \( \gamma \in \Gamma_0(4) \), where
\[
J(\gamma, z) = \frac{\theta^*(\gamma z)}{\theta^*(z)}, \quad \theta^*(z) = y^{1/4} \sum_{n \in \mathbb{Z}} e(n^2 z),
\]
and if \( (\Delta_k + \lambda) \psi = 0 \) for some \( \lambda \). If \( \psi \) is not a constant multiple of \( \theta^* \) then \( \lambda \geq \frac{1}{4} \) and we define the spectral parameter \( r \) as before. Such a \( \psi \) is a cusp form if the constant term in its Fourier expansion at each of the cusps of \( \Gamma_0(4) \backslash \mathcal{H} \) is zero. In this case the Fourier expansion is written
\[
\psi(z) = \sum_{n \neq 0} b_\psi(n) W_1 \frac{\text{sgn}(n)}{\sqrt{n}} e^n(4\pi |n| y) e(nx).
\]

Let \( \mathcal{V}_r \) denote the vector space of Maass cusp forms of weight 1/2 on \( \Gamma_0(4) \) with spectral parameter \( r/2 \). The Kohnen plus space is the subspace \( \mathcal{V}_r^+ \) of \( \mathcal{V}_r \) comprising forms whose Fourier coefficients are supported on indices \( n \equiv 1, 3 \pmod{4} \). For each prime \( p \geq 3 \), the Hecke operator \( T_p^2 \) acts on \( \mathcal{V}_r^+ \) via Fourier expansions as
\[
(T_p^2 \psi)(z) = \sum_{0 \neq n \equiv 1, 3 \pmod{4}} \left( b_\psi(p^2 n) + \frac{n}{p} b_\psi(p n) + \frac{n}{p^2} b_\psi(n) \right) W_1 \frac{\text{sgn}(n)}{\sqrt{n}} e^n(4\pi |n| y) e(nx).
\]

**The Shimura lift.** In Theorem 1.2 of [2], Baruch and Mao show that for each \( \varphi \in \mathcal{U}_r \) there is a unique \( \psi \in \mathcal{V}_r^+ \), spectrally normalized so that \( \langle \psi, \psi \rangle = 1 \), such that for each prime \( p \geq 3 \) we have
\[
T_p^2 \psi = a_\varphi(p) \psi. \quad (2.1)
\]
The form \( \varphi \) is called the Shimura lift of \( \psi \). A computation involving \( (2.1) \) and \( T_p \varphi = a_\varphi(p) \varphi \) shows that
\[
a_\varphi(m) b_\psi(d) = m \sum_{n|m} n^{-3} \left( \frac{d}{n} \right) b_\psi(m^2 d/n^2)
\]
\(^1\)Having \( \mathcal{U}_r \) and \( \mathcal{V}_r \) correspond to spectral parameters \( r \) and \( r/2 \), respectively, follows [6] and is convenient when working with the Shimura lift, which sends an element of \( \mathcal{V}_r \) into \( \mathcal{U}_r \).
for all fundamental discriminants $d$.

**Quadratic forms and cycles.** For each positive discriminant $D$, let $Q_D$ denote the set of (indefinite) integral binary quadratic forms $Q = [a,b,c]$ with $b^2 - 4ac = D$. The group $\Gamma$ acts on $Q_D$ in the usual way, and the set $\Gamma \setminus Q_D$ is finite. For $Q = [a,b,c] \in Q_D$, the equation $ax^2 + bxy + cy^2 = 0$ has two solutions $(x : y)$ in $\mathbb{P}^1(\mathbb{R})$. When $D$ is not a square, each $x/y$ is a real quadratic irrationality, and when $D$ is a square we have either $(x : y) = (1 : 0)$, corresponding to the point at $i\infty$, or $x/y \in \mathbb{Q}$. Let $S_Q$ denote the geodesic in $\mathcal{H}$ connecting the two solutions, and let $\Gamma_Q \subseteq \Gamma$ denote the isotropy subgroup $\{ \gamma \in \Gamma : \gamma Q = Q \}$. We follow $[\text{6}]$ in orienting $S_Q$ clockwise if $a > 0$, counterclockwise if $a < 0$, and downward if $a = 0$ (if $a = 0$ the geodesic is a vertical line)$^2$ When $D$ is not a square, $\Gamma_Q$ is infinite cyclic, and when $D$ is a square $\Gamma_Q$ is trivial. Let $C_Q = \Gamma_Q \setminus S_Q$ be the cycle corresponding to $Q$; it has finite length when $D$ is not a square, and infinite length otherwise.

Let $D = dd'$ be a factorization of $D$ into a fundamental discriminant $d$ and a discriminant $d'$. The generalized genus character $\chi_d$ associated to the factorization $D = dd'$ is

$$\chi_d(Q) = \begin{cases} \left( \frac{d}{n} \right) & \text{if } \gcd(a,b,c,d) = 1 \text{ and } Q \text{ represents } n, \\ 0 & \text{if } \gcd(a,b,c,d) > 1. \end{cases}$$

In $[\text{5}]$ it is shown that $\chi_d(Q)$ is well-defined on equivalence classes $Q \in \Gamma \setminus Q_D$.

It will be helpful to have an explicit description of $\Gamma \setminus Q_D$ when $D = d^2$ and $d$ is a fundamental discriminant. The following is a straightforward generalization of Lemma 3 of $[\text{1}]$.

**Lemma 2.1.** If $D = d^2$ then the sets

$$\{Q = [c,|d|,0] : 0 \leq c < |d| \} \quad \text{and} \quad \{Q = [0,|d|,c] : 0 \leq c < |d| \}$$

are both complete sets of representatives for $\Gamma \setminus Q_D$. In both cases we have

$$\chi_d(Q) = \left( \frac{d}{c} \right).$$

3. **Proof of Theorem 1.1**

We begin by borrowing a few intermediate results from $[\text{6}]$. For $\Re(s) > 1$ let $F_m(z,s)$ denote the Poincaré series

$$F_m(z,s) = \sum_{\gamma \in \Gamma \setminus \Gamma} f_m(\gamma z,s),$$

where $f_0(z,s) = y^s$ and for $m \neq 0$

$$f_m(z,s) = \frac{\Gamma(s)}{2\pi \sqrt{|m|} \Gamma(2s)} M_{0,s-\frac{1}{2}}(4\pi |m|y) e(mx).$$

The function $F_0(z,s)$ is the usual real analytic Eisenstein series (see $[\text{10}]$ Chapter 15]) and has Fourier expansion

$$F_0(z,s) = y^s + \frac{\Lambda(2s-1)}{\Lambda(2s)} y^{1-s} + 2\sqrt{y} \sum_{n \neq 0} \left| n \right|^{s-\frac{1}{2}} \frac{\sigma_1-2s(|n|)}{\Lambda(2s)} K_{s-\frac{1}{2}}(2\pi |n|y) e(nx), \quad (3.1)$$

where $\sigma_a(n)$ is the sum of the $a$-th powers of the divisors of $n$, and $\Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$. The modified Eisenstein series $\Lambda(2s) F_0(z,s)$ is analytic in $\mathbb{C} \setminus \{0,1\}$ and is invariant under

$^2$Note that the papers $[\text{5,1}]$ give $S_Q$ and $C_Q$ the opposite orientation.
s \mapsto 1 - s$. For \( m \neq 0 \) the Fourier expansion of \( F_m(z, s) \) is given in Theorem 3.4 of [7] (see also Section 8 of [4]); for \( m \neq 0 \) and \( \text{Re}(s) > 1 \) we have

\[
F_m(z, s) = f_m(z, s) + \frac{2|m|^{1/2-s}\sigma_{2s-1}(|m|)}{(2s-1)\Lambda(2s)}y^{-1-s} + 2\sqrt{y}\sum_{n\neq 0}\Phi(m, n; s)K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),
\]

where

\[
\Phi(m, n; s) = \sum_{c>0} K(m, n, c) \left\{ \begin{array}{ll}
I_{2s-1}(4\pi\sqrt{|mn|}c^{-1}) & \text{if } mn < 0, \\
J_{2s-1}(4\pi\sqrt{|mn|}c^{-1}) & \text{if } mn > 0
\end{array} \right.
\]

and \( K(m, n, c) \) is the ordinary (weight 0) Kloosterman sum. The following result is Proposition 3 of [6].

**Proposition 3.1.** For any \( m \neq 0 \), the function \( F_m(z, s) \) has a meromorphic continuation to \( \text{Re}(s) > 0 \) with

\[
\text{Res}_{s=\frac{1}{2}+ir}(2s-1)F_m(z, s) = \sum_{\varphi \in \mathbb{R}} \langle \varphi, \varphi \rangle^{-1} 2a_{\varphi}(m)\varphi(z).
\]

In Proposition 5 of [6] the authors show that the cycle integrals of \( F_m(z, s) \) and \( \partial_z F_m(z, s) \) over finite geodesics yield weighted sums of Kloosterman sums. The next proposition is a complementary result that evaluates the cycle integrals over infinite geodesics, provided that we make a small modification to the integrand as in [1]. Suppose that \( Q = [a, b, c] \in \mathbb{Q}_D \) with \( D \) a square and let \( a_1, a_2 \) be the rational projective solutions to \( ax^2 + bxy + cy^2 = 0 \). For each \( j = 1, 2 \) there is a unique \( \gamma_j \in \Gamma_{\infty} \setminus \Gamma \) such that \( \gamma_j a_j = \infty \), and we define

\[
F_m,Q(z, s) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} f_m(\gamma z, s).
\]

Since \( F_{m,\sigma Q}(z, s) = F_m,Q(\sigma z, s) \) for all \( \sigma \in \Gamma \), the integrals

\[
\int_{C_Q} F_m,Q(z, s)y^{-1}|dz| \quad \text{and} \quad \int_{C_Q} \partial_z F_m,Q(z, s)\,dz
\]

are well-defined, assuming they converge. To show convergence, using Lemma 2.1 we may assume that \( Q = [0, |d|, c] \) and \( 0 < c < |d| \) for \( D = d^2 \). Then we can take \( a_1 = \infty \) and \( \gamma_1 = I \). The Fourier expansions (3.1) and (3.2) show that for \( \text{Re}(s) > 1 \) the integrals in (3.3) converge at \( \infty \), and the observation

\[
F_m,Q(\gamma_2^{-1}z, s) = \sum_{\gamma \neq \gamma_1, \gamma_2^{-1}} f_m(\gamma z, s)
\]

shows that the integrals converge at \( a_2 \). For \( \text{Re}(s) > 1 \) and \( d \) a fundamental discriminant, define

\[
T_m(d) = \sum_{Q \in \Gamma \setminus \mathbb{Q}_d} \chi_d(Q) \left\{ \begin{array}{ll}
\int_{C_Q} F_m,Q(z, s)y^{-1}|dz| & \text{if } d > 0, \\
\int_{C_Q} i\partial_z F_m,Q(z, s)\,dz & \text{if } d < 0.
\end{array} \right.
\]

Then we have the following analogue of Proposition 5 of [6].
Proposition 3.2. Let \( m \geq 0 \) and \( \text{Re}(s) > 1 \). Suppose that \( d \) is a fundamental discriminant. Then

\[
T_m(d) = \begin{cases} 
6\pi^{1/2}|d|^{3/2}m \sum_{n|m} n^{-3/2} \left( \frac{d}{n} \right) \Phi^+(d, \frac{m^2}{n^2}d; \frac{2s+1}{4}) & \text{if } m > 0, \\
\frac{\Gamma\left(\frac{s}{2} + \frac{1 - \text{sgn}d}{2}\right) |d|^s L(s, \chi_d)^2}{\Gamma(s)\zeta(2s)} & \text{if } m = 0,
\end{cases}
\]

where, for \( p, q \equiv 0, 1 \pmod{4} \) and \( pq > 0 \) we have

\[
\Phi^+(p, q, s) = \frac{\Gamma(s - \frac{\text{sgn}p}{4})\Gamma(s - \frac{\text{sgn}q}{4})}{3\sqrt{\pi} 2^{2-2s}\Gamma(2s - \frac{1}{2})} (pq)^{-\frac{1}{2}} \sum_{4c>0} K^+(p, q, c) J_{2s-1} \left( \frac{4\pi \sqrt{pq}}{c} \right).
\]

Here \( K^+(p, q, c) \) is the half-integral weight Kloosterman sum

\[
K^+(p, q, c) = (1 - i) \sum_{d \mod c} \left( \frac{c}{d} \right) \varepsilon_d e \left( \frac{pd + qd}{c} \right) \times \begin{cases} 
1 & \text{if } c/4 \text{ is even}, \\
2 & \text{if } c/4 \text{ is odd},
\end{cases}
\]

with \( \varepsilon_d = 1 \) if \( d \equiv 1 \pmod{4} \) and \( \varepsilon_d = i \) if \( d \equiv 3 \pmod{4} \).

Proof. When \( d, m > 0 \) this is (4.4) of [1] (see also Proposition 4 of that paper). For the case \( d > 0, m = 0 \), (4.4) of [1] reads

\[
T_0(d) = \frac{\Gamma\left(\frac{s}{2}\right)^2}{4\Gamma(s)} d^s L(s, \chi_d) \sum_{c=1}^{\infty} K^+(d, 0; 4c) c^{s+1/2}.
\]

By Lemma 4 of [5] we have

\[
T_0(d) = \frac{\Gamma\left(\frac{s}{2}\right)^2 d^s L(s, \chi_d)^2}{\Gamma(s)\zeta(2s)}.
\]

Now assume that \( d < 0 \). We will closely follow the proof of Proposition 4 of [1]. From the proof of Lemma 5 of [6] (see (9.2) especially) we have

\[
2i\partial_z F_{m, Q}(z, s) = \sum_{\gamma \in \Gamma \backslash \Gamma \setminus \Gamma_{\gamma \neq \gamma_1, \gamma_2}} f_{2,m}(\gamma z, s) \frac{d(\gamma z)}{dz},
\]

where \( f_{2,m}(z, s) = \phi_{2,m}(y, s)e(mx) \) and

\[
\phi_{2,m}(y, s) = \begin{cases} 
sy^{s-1} & \text{if } m = 0, \\
sm^{-1/2}(2\pi y)^{-1} \Gamma(s) \Gamma(2s) M_{1,s-rac{1}{2}}(4\pi my) & \text{if } m > 0.
\end{cases}
\]

We choose representatives \([c, |d|, 0]\) for \( \Gamma \backslash Q_{d^2} \) as in Lemma 2.1 so that \( \chi_d(Q) = \left( \frac{d}{c} \right) \). Since \( \left( \frac{d}{c} \right) = 0 \) when \( \gcd(c, d) > 1 \) we can restrict the sum to those \( c \) which are coprime to \( d \). For \( Q = [c, |d|, 0] \) with \( \gcd(c, d) = 1 \) we have

\[
a_1 = (0 : 1), \gamma_1 = \left( 0 \begin{array}{c} 1 \\ -1 \end{array} \right), \quad a_2 = (d : c), \gamma_2 = \left( \begin{array}{c} a \\ b \\ c \end{array} \right),
\]

for some \( a, b \in \mathbb{Z} \). Thus

\[
T_m(d) = \frac{1}{2} \sum_{c \mod |d| \atop \gcd(c, d) = 1 \atop Q = [c, |d|, 0]} \left( \frac{d}{c} \right) \sum_{\gamma \in \Gamma \backslash \Gamma_{\gamma \neq \gamma_1, \gamma_2} Q} e(mx)\phi_{2,m}(y, s) dz.
\]
The map \((\gamma, Q) \mapsto \gamma Q\) is a bijection from \(\Gamma_\infty \Gamma \times \Gamma \backslash Q_{d^2}\) to \(\Gamma_\infty \backslash Q_{d^2}\) which sends \((\Gamma_\infty \gamma_1, [c, |d|, 0])\) to \([0, d, c + d\mathbb{Z}]\) and \((\Gamma_\infty \gamma_2, [c, |d|, 0])\) to \([0, |d|, -b + d\mathbb{Z}]\). It follows that

\[
T_m(d) = \frac{1}{2} \sum_{Q \in \Gamma_\infty \backslash Q_{d^2}} \chi_d(Q) \int_{C_Q} e(mx) \phi_{2,m}(y, s) \, dz.
\]

Since \(\chi_d(-Q) = -\chi_d(Q)\) and the geodesic \(C_{-Q}\) is the same set as \(C_Q\) but with opposite orientation, we have

\[
T_m(d) = \sum_{Q \in \Gamma_\infty \backslash Q_{d^2}, Q = [a, b, c] \neq 0} \chi_d(Q) \int_{C_Q} e(mx) \phi_{2,m}(y, s) \, dz.
\]

Each cycle \(C_Q\) with \(Q = [a, b, c]\) and \(a > 0\) can be parametrized by

\[
z = \text{Re} \, z_Q - e^{-i\theta} \text{Im} \, z_Q, \quad 0 \leq \theta \leq \pi,
\]

where

\[
z_Q = -\frac{b}{2a} + i\frac{|d|}{2a}
\]

is the apex of the geodesic. Thus

\[
\int_{C_Q} e(mx) \phi_{2,m}(y, s) \, dz = e\left(\frac{-mb}{2a}\right) H_m\left(\frac{|d|}{2a}\right),
\]

where

\[
H_m(t) = it \int_0^\pi e(-mt \cos \theta) \phi_{2,m}(t \sin \theta, s) e^{-i\theta} \, d\theta.
\]

It follows that

\[
T_m(d) = \sum_{a=1}^{\infty} H_m\left(\frac{|d|}{2a}\right) \sum_{b(2a)} \chi_d\left([a, b, \frac{b^2-d^2}{4a}]\right) e\left(\frac{-mb}{2a}\right).
\]

By Lemma 7 of [6] we have

\[
H_m(t) = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{s+1}{2}\right)} J_{s-\frac{1}{2}}(2\pi|m|t)
\]

when \(m \neq 0\), while when \(m = 0\) we have by [4] \((5.12.2)\) that

\[
H_0(t) = 2\sqrt{\pi} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s}{2}\right).
\]

The result follows after using Lemma 8 of [6].

We would like to apply Proposition 3.1 to the integrals appearing in Proposition 3.2, but the integrals in (3.3) do not converge for \(\text{Re}(s) = \frac{1}{2}\). However, the integrals

\[
\int_{C_Q} (F_{m,Q}(z, s) - c(s) F_{0,Q}(z, s)) y^{-1} |dz| \quad \text{and} \quad \int_{C_Q} \partial_z (F_{m,Q}(z, s) - c(s) F_{0,Q}(z, s)) \, dz,
\]

where

\[
c(s) = \frac{2|m|^{1/2-s} \sigma_{2s-1}(|m|)}{(2s-1) \Lambda(2s-1)},
\]
do converge for \( \text{Re}(s) > 0 \), as long as \( s \) is not one of the poles of the integrands. This is because the coefficient of \( y^{1-s} \) equals zero in the Fourier expansion of \( F_{m,Q}(z,s) - c(s)F_{0,Q}(z,s) \) at the cusps corresponding to the endpoints of \( C_Q \). Note that by Proposition 3.1 we have

\[
\text{Res}_{s=\frac{1}{2}+ir}(2s-1)(F_{m,Q}(z,s) - c(s)F_{0,Q}(z,s)) = \sum_{\varphi \in B_r} \langle \varphi, \varphi \rangle^{-1}a_\varphi(m)\varphi(z)
\]

because \( c(s)F_{0,Q}(z,s) \) is analytic at \( s = \frac{1}{2} + ir, \ r \neq 0 \), and \( f_m(\gamma j z, s) \) is analytic for \( s \in \mathbb{C} \). So we get the next result by following the proof of Proposition 6 of [6] with only minor changes.

**Proposition 3.3.** For any even Hecke–Maass cusp form \( \varphi \in \mathcal{U}_r \) there is a unique Hecke–Maass cusp form \( \psi \in \mathcal{V}_r \) such that \( \varphi \) is the Shimura lift of \( \psi \) and such that for any fundamental discriminant \( d \) we have

\[
12\pi^{1/2}|d|^{3/2}|b_\psi(d)|^2 = \frac{1}{\langle \varphi, \varphi \rangle} \sum_{Q \in \Gamma\backslash \mathcal{Q}_{d^2}} \chi_d(Q) \left\{ \begin{array}{ll}
\int_{C_Q} \varphi(z)y^{-1}|dz| & \text{if } d > 0, \\
\int_{C_Q} i\partial_z\varphi(z)dz & \text{if } d < 0.
\end{array} \right.
\]

**Proof of Theorem 1.1.** Suppose that \( d \) is a fundamental discriminant. Then by Lemma 2.1 the quadratic forms \([0, |d|, c] \) with \( 0 \leq c < |d| \) form a complete set of representatives for \( \Gamma\backslash \mathcal{Q}_{d^2} \) and

\[
\chi_d([0, |d|, c]) = \left( \frac{d}{c} \right).
\]

Suppose first that \( d > 0 \). If \( \text{Re}(s) > 1 \) then

\[
\sum_{Q \in \Gamma\backslash \mathcal{Q}_d} \chi_d(Q) \int_{C_Q} \varphi(z)y^{s-1}|dz| = \sum_{c \equiv d \mod |d|} \left( \frac{d}{c} \right) \int_0^\infty \varphi(-\frac{d}{2} + iy)y^{s-1}dy
\]

\[
= 2\sum_{n \neq 0} a_\varphi(n)G(-n,d) \int_0^{\infty} y^{s-\frac{1}{2}}K_{ir}(2\pi|n|y)dy,
\]

where \( G(n,d) \) is the Gauss sum

\[
G(n,d) = \sum_{c \equiv d \mod |d|} \chi_d(c)e\left( \frac{nc}{|d|} \right) = \chi_d(n)\sqrt{|d|} \times \left\{ \begin{array}{l}
1 & \text{if } d > 0, \\
i & \text{if } d < 0.
\end{array} \right.
\]

By [4] (10.43.19) we have

\[
\int_0^{\infty} y^{s-\frac{1}{2}}K_{ir}(2\pi|n|y)dy = \frac{1}{4}(\pi|n|)^{-s-\frac{1}{2}}\Gamma\left( \frac{s}{2} + \frac{i r}{2} + \frac{1}{4} \right)\Gamma\left( \frac{s}{2} - \frac{i r}{2} + \frac{1}{4} \right).
\]

Thus, using that \( a_\varphi(n) = a_\varphi(-n) \) we find that

\[
\sum_{Q \in \Gamma\backslash \mathcal{Q}_d} \chi_d(Q) \int_{C_Q} \varphi(z)y^{s-1}|dz| = \pi^{-s-\frac{1}{2}}\sqrt{d}\Gamma\left( \frac{s}{2} + \frac{i r}{2} + \frac{1}{4} \right)\Gamma\left( \frac{s}{2} - \frac{i r}{2} + \frac{1}{4} \right)L(s + \frac{1}{2}, \psi \times \chi_d).
\]

Setting \( s = 0 \) and using Proposition 3.3 we get (1.1).

Now suppose that \( d < 0 \). A computation involving [4] §10.29, (10.30.2), and (10.40.2) shows that

\[
\partial_z[\sqrt{y}K_{ir}(2\pi|n|y)e(nx)] = \pi in\sqrt{y}K_{ir}(2\pi|n|y)e(nx) + g(n, y)e(nx)
\]
for some function \( g(n, y) \) which satisfies \( g(-n, y) = g(n, y) \) and \( g(n, y) \ll |n|^{1/2} e^{-2\pi |n|y} \) as \( |n|y \to \infty \) and \( g(n, y) \ll n^{-1/2} \) as \( y \to 0 \). So if \( \text{Re}(s) > 1 \) we have

\[
\sum_{Q \in \Gamma \setminus \mathcal{Q}} \chi_d(Q) \int_{C_Q} i \partial_{\bar{z}} \varphi(z) y^s \, dz = -2\pi i \sum_{n \neq 0} na_{\varphi}(n) G(n, d) \int_0^\infty y^{s+\frac{1}{2}} K_{iv}(2\pi |n|y) \, dy
\]

because \( a_{\varphi}(-n) G(-n, d) g(-n, y) = -a_{\varphi}(n) G(n, d) g(n, y) \). Again using (3.4) we find that

\[
\sum_{Q \in \Gamma \setminus \mathcal{Q}} \chi_d(Q) \int_{C_Q} i \partial_{\bar{z}} \varphi(z) y^s \, dz = \pi^{s-\frac{1}{2}} \sqrt{|d|} \Gamma\left(\frac{s}{2} + \frac{ir}{2} + \frac{3}{4}\right) \Gamma\left(\frac{s}{2} - \frac{ir}{2} + \frac{3}{4}\right) L\left(s + \frac{1}{2}, \varphi \times \chi_d\right).
\]

The result follows as in the previous case.

\[\square\]

References

[1] Nickolas Andersen. Periods of the \( j \)-function along infinite geodesics and mock modular forms. Bull. Lond. Math. Soc., 47(3):407–417, 2015.
[2] Ehud Moshe Baruch and Zhengyu Mao. A generalized Kohnen-Zagier formula for Maass forms. J. Lond. Math. Soc. (2), 82(1):1–16, 2010.
[3] A. Biró. Cycle integrals of Maass forms of weight 0 and Fourier coefficients of Maass forms of weight 1/2. Acta Arith., 94(2):103–152, 2000.
[4] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.1.4 of 2022-01-15.
F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
[5] W. Duke, Ö. Imamoğlu, and Á. Tóth. Cycle integrals of the \( j \)-function and mock modular forms. Ann. of Math. (2), 173(2):947–981, 2011.
[6] W. Duke, Ö. Imamoğlu, and Á. Tóth. Geometric invariants for real quadratic fields. Ann. of Math. (2), 184(3):949–990, 2016.
[7] John D. Fay. Fourier coefficients of the resolvent for a Fuchsian group. J. Reine Angew. Math., 293(294):143–203, 1977.
[8] B. Gross, W. Kohnen, and D. Zagier. Heegner points and derivatives of \( L \)-series. II. Math. Ann., 278(1-4):497–562, 1987.
[9] Dennis A. Hejhal. The Selberg trace formula for \( \text{PSL}(2, \mathbb{R}) \). Vol. 2, volume 1001 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1983.
[10] Henryk Iwaniec and Emmanuel Kowalski. Analytic number theory, volume 53 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.
[11] Svetlana Katok and Peter Sarnak. Heegner points, cycles and Maass forms. Israel J. Math., 84(1-2):193–227, 1993.
[12] W. Kohnen and D. Zagier. Values of \( L \)-series of modular forms at the center of the critical strip. Invent. Math., 64(2):175–198, 1981.
[13] Shinji Niwa. Modular forms of half integral weight and the integral of certain theta-functions. Nagoya Math. J., 56:147–161, 1975.
[14] Walter Roelcke. Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II. Math. Ann. 167 (1966), 292–337; ibid., 168:261–324, 1966.
[15] Takuro Shintani. On construction of holomorphic cusp forms of half integral weight. Nagoya Math. J., 58:83–126, 1975.
[16] J.-L. Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. J. Math. Pures Appl. (9), 60(4):375–484, 1981.

Email address: nick@math.byu.edu

Brigham Young University, Provo, UT 84602