Zero-Sum Two Person Perfect Information Semi-Markov Games: A Reduction

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Abstract

Look at the play of a Perfect Information Semi-Markov game (PISMG). As the game has perfect information, at each time point in any play, all but one player is a dummy. Hence on any particular time instant, at most one player has more than one action available to himself. Thus such games lack real conflict throughout its play and no player directly antagonises another ever (in each state, the reward matrix is a row or column vector). Above intuition helps us to show that any zero-sum two person PISMG can be reduced to an one-player game, i.e., to a semi-Markov decision process (SMDP), which has a value (Sinha et al.,(2017) \textsuperscript{14}). In this paper, we use limiting ratio average pay-off (but any standard pay-off function will do) and prove that any PISMG under such an undiscounted pay-off has a value and both the maximiser (player-I) and minimiser (player-II) have pure semi-stationary optimal strategies. To solve such an undiscounted PISMG, we apply Mondal’s algorithm ((2017) \textsuperscript{11}) on the reduced SMDP obtained from the PISMG.

**Keywords:** Semi-Markov games, Semi-Markov Decision Processes, Perfect Information, Semi-Stationary Strategies, Linear Programming.

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1 Introduction

A semi-Markov game (SMG) is a generalisation of a Stochastic (Markov) game Shapley(1953) \textsuperscript{13}. Such games have already been studied in the literature (e.g. Lal-Sinha(1992) \textsuperscript{6}, Luque-Vasquez(1999) \textsuperscript{9}, Sinha-Mondal(2017) \textsuperscript{14}, Mondal(2015)
Single player SMGs are called semi-Markov decision processes (SMDPs) which were introduced by Jewell(1963) and Howard(1971). A perfect information semi-Markov game (PISMG) is a natural extension of perfect information stochastic games (PISGs) Shapley(1953), Raghavan et al.(1997), where at each state all but one player is a dummy (i.e., he has only one available action in that state). Note that for such a game, perfect information is a state property. In this paper, we prove that such games (PISMGs) have a value and both players have pure semi-stationary optimal strategies. We show this by reducing any PISMG to an SMDP and then use the existing result from the theory of SMDP to establish the existence of value and optimal pure semi-stationary strategies for the PISMG. To conclude, we use an algorithm given in Mondal (2017) to solve such PISMGs. The paper is organized as follows. Section 2 contains definitions and properties of an undiscounted two person zero-sum semi-Markov games and semi-Markov decision processes considered under limiting ratio average pay-off. Section 3 contains main result of this paper. In section 4 we propose a linear programming algorithm to compute an optimal semi-stationary strategy pair for the players of such perfect information undiscounted semi-Markov games. Section 5 contains some numerical examples illustrating our theorem and proposed algorithm. Section 6 is reserved for the conclusion.

2 Preliminaries

2.1 Finite two person zero-sum semi-Markov games

A zero-sum two person finite SMG is described by a collection of seven objects $\Gamma = \langle S, \{A(s) : s \in S\}, \{B(s) : s \in S\}, q, P, r, \rangle$, where $S = \{1, 2, \ldots, N\}$ is the finite non-empty state space and $A(s) = \{1, 2, \ldots, m_s\}, B(s) = \{1, 2, \ldots, n_s\}$ are respectively the non-empty sets of admissible actions of the players I and II respectively in the state $s$. Let us denote $K = \{(s, i, j) : s \in S, i \in A(s), j \in B(s)\}$ to be the set of admissible triplets. For each $(s, i, j) \in K$, we denote $q(. \mid s, i, j)$ to be the transition law of the game whereas $P_{ss'}(\cdot \mid i, j)$ is a distribution function on $[0, \infty)$ given $K \times S$, which is called the conditional transition (sojourn) time distribution. Finally $r_1$ and $r_2$ are real valued functions on $K$, which represent the immediate (expected) rewards for the players I and II respectively. Let us consider player I as the maximiser and player II as the minimiser in the zero-sum two person SMG.

The semi-Markov game over infinite time is played as follows. At the 0th decision epoch, the game strats at $s_0 \in S$ and the players I and II simultaneously and independently choose actions $i_0 \in A(s_0)$ and $j_0 \in B(s_0)$ respectively. Consequently player I and II get immediate rewards $r_1(s_0, i_0, j_0)$ and $r_2(s_0, i_0, j_0)$ respectively and the game moves to the state $s_1$ with probability $q(s_1 \mid s_0, i_0, j_0)$. The sojourn time to move from state $s_0$ to the state $s_1$ is determined by the distribution function $P_{s_0s_1}(\cdot \mid i_0, j_0)$. After reaching the state $s_1$ on the next decision epoch, the game is repeated over infinite time with the state $s_0$ replaced by $s_1$.

By a strategy (behavioural) $\pi_1$ of the player I, we mean a sequence $\{(\pi_1)_n(\cdot \mid
\[ hist_n \}^\infty_{n=1}, \] where \((\pi_1)_n\) specifies which action is to be chosen on the \(n\)-th decision epoch by associating with each history \(hist_n\) of the system up to \(n\)th decision epoch (where \(hist_n=(s_1,a_1,b_1,s_2,a_2,b_2 \cdots, s_{n-1},a_{n-1},b_{n-1},s_n)\) for \(n \geq 2\), \(hist_1=(s_1)\) and \((s_k,a_k,j_k) ∈ K\) are respectively the state and actions of the players at the \(k\)-th decision epoch) a probability distribution \((\pi_1)_n(\cdot | hist_n)\) on \(A(s_n)\). Behavioural strategy \(\pi_2\) for player II can be defined analogously. Generally by any unspecified strategy, we mean behavioural strategy here. We denote \(\Pi_1\) and \(\Pi_2\) to be the sets of strategy (behavioural) spaces of the players I and II respectively.

A strategy \(f = \{f_n\}_{n=1}^\infty\) for the player I is called semi-Markov if for each \(n\), \(f_n\) depends on \(s_1, s_n\) and the decision epoch number \(n\). Similarly we can define a semi-Markov strategy \(g = \{g_n\}_{n=1}^\infty\) for the player II.

A stationary strategy is a strategy that depends only on the current state. A stationary strategy for player I is defined as \(N\) tuple \(f = (f(1), f(2), \cdots, f(N))\), where each \(f(s)\) is the probability distribution on \(A(s)\) given by \(f(s) = (f(s,1), f(s,2), \cdots, f(s, m_s))\). \(f(s,i)\) denotes the probability of choosing action \(i\) in the state \(s\). By similar manner, one can define a stationary strategy \(g\) for player II as \(g = (g(1), g(2), \cdots, g(N)\) where each \(g(s)\) is the probability distribution on \(B(s)\). Let us denote \(F_1\) and \(F_2\) to be the set of stationary strategies for player I and II respectively.

A stationary strategy is called pure if any player selects a particular action with probability 1 while visiting a state \(s\). We denote \(F_1^s\) and \(F_2^s\) to be the set of pure stationary strategies of the players I and II respectively.

A semi-stationary strategy is a semi-Markov strategy which is independent of the decision epoch \(n\), i.e., for a initial state \(s_0\) and present state \(s_1\), if a semi-Markov strategy \(g(s_0,s_1,n)\) turns out to be independent of \(n\), then we call it a semi-stationary strategy. Let \(\xi_1\) and \(\xi_2\) denotes the set of semi-stationary strategies for the players I and II respectively.

**Definition 1** A zero-sum two person SMG \(\Gamma = \langle S, \{A(s) : s \in S\}, \{B(s) : s \in S\}, q, P, r >\) is called a perfect information semi-Markov game (PISMG) if the following properties hold:

(i) \(S = S_1 \cup S_2, S_1 \cap S_2 = \phi\).

(ii) \(|B(s)| = 1\), for all \(s \in S_1\), i.e., on \(S_1\) player-II is a dummy.

(iii) \(|A(s)| = 1\), for all \(s \in S_2\), i.e., on \(S_2\) player-I is a dummy.

### 2.2 Undiscounted zero-sum two person semi-markov games

Let \((X_1, A_1, B_1, X_2, A_2, B_2 \cdots)\) be a co-ordinate sequence in \(S \times (A \times B \times S)^\infty\). Given behavioural strategy pair \((\pi_1, \pi_2) \in \Pi_1 \times \Pi_2\), initial state \(s \in S\), there exists a unique probability measure \(P_{\pi_1,\pi_2}(\cdot | X_1 = s)\) (hence an expectation \(E_{\pi_1,\pi_2}(\cdot | X_1 = s)\)) on the product \(\sigma\)-field of \(S \times (A \times B \times S)^\infty\) by Kolmogorov’s extension theorem. For a pair of strategies \((\pi_1, \pi_2) \in \Pi_1 \times \Pi_2\) for the players I and II respectively, the limiting ratio average (undiscounted) pay-off for player I, starting from a state \(s \in S\) is defined by:

\[
\phi(s, \pi_1, \pi_2) = \lim \inf_{n \to \infty} \frac{E_{\pi_1,\pi_2} \sum_{m=1}^{\infty} [\tau(X_m,A_m,B_m) | X_1 = s]}{E_{\pi_1,\pi_2} \sum_{m=1}^{\infty} \tau(X_m,A_m,B_m) | X_1 = s}. 
\]
Here $\bar{\tau}(s, i, j) = \sum_{s' \in S} q(s' | s, i, j) \int_0^\infty tdP_{ss'}(t | i, j)$ is the expected sojourn time in the state $s$ for a pair of actions $(i, j) \in A(s) \times B(s)$.

**Definition 2** For each pair of stationary strategies $(f_1, f_2) \in F_1 \times F_2$ we define the transition probability matrix as $Q(f_1, f_2) = [q(s' | s, f_1, f_2)]_{N \times N}$, where $q(s' | s, f_1, f_2) = \sum_{i \in A(s)} \sum_{j \in B(s)} q(s' | s, i, j) f_1(s, i) f_2(s, j)$ is the probability that starting from the state $s$, next state is $s'$ when the players choose strategies $f_1$ and $f_2$ respectively. (For a stationary strategy $f$, $f(s, i)$ denotes the probability of choosing action $i$ in the state $s$)

For any pair of stationary strategies $(f_1, f_2) \in F_1 \times F_2$ of player I and II, we write the undiscounted pay-off for player I as:

$$\phi(s, f_1, f_2) = \lim \inf_{n \to \infty} \frac{\sum_{m=1}^{\infty} r_m(s, f_1, f_2)}{\sum_{m=1}^{\infty} \tau_m(s, f_1, f_2)}$$

for all $s \in S$.

Where $r_m(s, f_1, f_2)$ and $\bar{\tau}_m(s, f_1, f_2)$ are respectively the expected reward and expected sojourn time for player I at the $m$ th decision epoch, when player I chooses $f_1$ and player II chooses $f_2$ respectively and the initial state is $s$. We define $r(f_1, f_2) = [r(s, f_1, f_2)]_{N \times 1}$, $\bar{\tau}(f_1, f_2) = [\bar{\tau}(s, f_1, f_2)]_{N \times 1}$ and $\phi(f_1, f_2) = [\phi(s, f_1, f_2)]_{N \times 1}$ as expected reward, expected sojourn time and undiscounted pay-off vector for a pair of stationary strategy $(f_1, f_2) \in F_1 \times F_2$. Now

$$r_m(s, f_1, f_2) = \sum_{s' \in S} P_{f_1, f_2}(X_m = s' | X_1 = s) r(s', f_1, f_2) = \sum_{s' \in S} r(s', f_1, f_2) q^{m-1}(s' | s, f_1, f_2) = [Q^{m-1}(f_1, f_2) r(f_1, f_2)](s)$$

and

$$\bar{\tau}_m(s, f_1, f_2) = \sum_{s' \in S} P_{f_1, f_2}(X_m = s' | X_1 = s) \bar{\tau}(s', f_1, f_2) = \sum_{s' \in S} \bar{\tau}(s', f_1, f_2) q^{m-1}(s' | s, f_1, f_2) = [Q^{m-1}(f_1, f_2) \bar{\tau}(f_1, f_2)](s)$$

Since $Q(f_1, f_2)$ is a Markov matrix, we have by Kemeny et al., $[5]$

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{m=0}^{n} Q^m(f_1, f_2)$$

exists and equals to $Q^*(f_1, f_2)$.

It is obvious that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} r_m(f_1, f_2) = [Q^*(f_1, f_2) r(f_1, f_2)](s)$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \bar{\tau}_m(f_1, f_2) = [Q^*(f_1, f_2) \bar{\tau}(f_1, f_2)](s).$$

Thus we have for any pair of stationary strategies $(f_1, f_2) \in F_1 \times F_2$,

$$\phi(s, f_1, f_2) = \frac{[Q^*(f_1, f_2) r(f_1, f_2)](s)}{[Q^*(f_1, f_2) \bar{\tau}(f_1, f_2)](s)}$$

for all $s \in S$.
where $Q^*(f_1, f_2)$ is the Cesaro limiting matrix of $Q(f_1, f_2)$.

**Definition 3** A zero-sum two person undiscounted semi-Markov game is said to have a value vector $\phi = [\phi(s)]_{N \times 1}$ if $\sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} \phi(s, \pi_1, \pi_2) = \phi(s) = \inf_{\pi_2 \in \Pi_2} \sup_{\pi_1 \in \Pi_1} \phi(s, \pi_1, \pi_2)$ for all $s \in S$. A pair of strategies $(\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2$ is said to be an optimal strategy pair for the players if $\phi(s, \pi_1, \pi_2) \geq \phi(s) \geq \phi(s, \pi_1, \pi_2^*)$ for all $s \in S$ and all $(\pi_1, \pi_2) \in \Pi_1 \times \Pi_2$. Throughout this paper, we use the notion of undiscounted pay-off as limiting ratio average pay-off.

### 2.3 Finite semi-Markov decision processes

A finite (state and action spaces) semi-Markov decision process is defined by a collection of five objects $\Gamma = < S, A = \{A(s) : s \in S\}, \hat{q}, \hat{r}, \hat{\tau} >$, where $S = \{1, 2, \ldots, N\}$ is the finite state space, $A(s)$ is the finite set of admissible actions in the state $s$, $\hat{q}(s', s, a)$ is the transition probability (i.e., $\hat{q}(s' | s, a) \geq 0$ and $\sum_{s' \in S} \hat{q}(s' | s, a) = 1$) that the next state is $s'$, where $s$ is the initial state and the decision maker chooses action $a$ in the state $s$. $\hat{P}_{ss'}(\cdot | a)$ is the distribution function on $[0, \infty)$ called the transition (sojourn) time distribution function and $\hat{r}$ is the immediate reward function. The decision process proceeds over infinite time just as semi-Markov game, where instead of two players we consider a single decision maker. The definition of strategy spaces for the decision maker is same as in the case of semi-Markov games. Let us denote $\Pi$, $F$, $F_s$, $\xi$ as the set of behavioural, stationary, pure-stationary and semi-stationary strategies respectively of the decision maker. Let $(X_1, A_1, X_2, A_2, \ldots)$ be a coordinate sequence in $S \times (A \times S)^\infty$. Given a behavioural strategy $\pi \in \Pi$, initial state $s \in S$, there exists a unique probability measure $P_\pi(\cdot | X_1 = s)$ (hence an expectation $E_\pi(\cdot | X_1 = s)$) on the product $\sigma$- field of $S \times (A \times S)^\infty$ by Kolmogorov’s extension theorem.

For a behavioural strategy $\pi \in \Pi$, the expected limiting ratio average pay-off is defined by

$$\hat{\phi}(s, \pi) = \liminf_{n \to \infty} \frac{E_\pi \sum_{m=1}^n [\hat{r}(X_m, A_m) | X_1 = s]}{E_\pi \sum_{m=1}^n [\hat{P}(X_m, A_m) | X_1 = s]}$$

for all $s \in S$.

Where $\hat{r}(s, a) = \sum_{s' \in S} \hat{q}(s' | s, a) \int_0^\infty t d\hat{P}_{ss'}(t | a)$ is the expected sojourn time in the state $s$ when decision maker chooses the action $a \in \hat{A}(s)$.

**Definition 4** A strategy $\pi^*$ is called optimal under limiting ratio average pay-off criterion if $\hat{\phi}(s, \pi^*) \geq \hat{\phi}(s, \pi)$ for all $\pi \in \Pi$ and all $s \in S$. Let $\hat{\phi}(s) = \sup_{\pi \in \Pi} \hat{\phi}(s, \pi)$ for all $s \in S$. Then $\hat{\phi}(s)$ is the limiting ratio average value of the SMDP for the initial state $s \in S$ and $\hat{\phi} = [\hat{\phi}(s)]_{s \in S}$ is called the value vector.

**Theorem 1** [13] For a finite semi-Markov decision process considered under limiting ratio average pay-offs, there exists a pure semi-stationary strategy $f_s^* \in \xi$ which is optimal for the decision maker.

### 3 Results

**Theorem 2** Any zero-sum two person undiscounted perfect information semi-Markov game has a solution in pure semi-stationary strategies under limiting ratio average pay-off.
pay-offs.

Proof. We prove the above theorem by a strategic equivalence between an undiscounted perfect information semi-Markov game and an undiscounted semi-Markov decision process. Let $\Gamma =< S = S_1 \cup S_2, A = \{A(s) : s \in S_1\}, B = \{B(s) : s \in S_2\}, q, P, r >$ be a zero-sum two person perfect information semi-Markov game under limiting ratio average pay-off, where in $|S_1|$ number of states, player-II is a dummy and from states $\{|S_1|+1, \ldots, |S_1|+|S_2|\}$ player-I is a dummy. We construct an undiscounted SMDP $\hat{\Gamma}$ from the above mentioned perfect information semi-Markov game, which is defined as $\hat{\Gamma} =< S = S_1 \cup S_2, \hat{A} = \{\hat{A}(s) = A(s) : s \in S_1\} \cup \{\hat{A}(s) = B(s) : s \in S_2\}, \hat{q} = q, \hat{P} = P, \hat{r} >$, where $\hat{r}(s,.)$ is defined as

$$\hat{r}(s,.) = \begin{cases} r(s,i) & s \in S_1, i \in \hat{A}(s) = A(s) \\ -r(s,j) & s \in S_2, j \in \hat{A}(s) = B(s) \end{cases}$$

By theorem 1, $\hat{\Gamma}$ admits an optimal pure semi-stationary strategy. Suppose $\hat{f}^*$ is a pure semi-stationary strategy which is optimal in the SMDP $\hat{\Gamma}$. Let $s_0$ be an initial state, thus $f^* = \{\hat{f}^*(s_0, s) : \hat{f}^*(s_0, s) \in \mathbb{P}_{\hat{A}(s)} \forall s_0, s \in S\}$, where $\mathbb{P}_{\hat{A}(s)}$ is the set of probability distributions over the action space $\hat{A}(s)$. We extract a pure semi-stationary strategy pair $(f^*, g^*)$ in $\Gamma$ from $\hat{f}^*$ as follows:

$$f^*(s_0, s) = \begin{cases} \hat{f}^*(s_0, s) & s \in S_1 \\ 1 & s \in S_2 \end{cases}$$

$$g^*(s_0, s) = \begin{cases} 1 & s \in S_1 \\ \hat{f}^*(s_0, s) & s \in S_2 \end{cases}$$

We denote $\phi$ and $\hat{\phi}$ to be the undiscounted pay-off functions for the PISMG $\Gamma$ and the SMDP $\hat{\Gamma}$ respectively. Let $\xi_1$ and $\xi_2$ be the set of semi-stationary strategies for player-I and player-II respectively in the PISMG $\Gamma$. By theorem 9 of [10], we know that if there exists a pure semi-stationary strategy pair $(f^*, g^*)$ such that for all $s \in S$, $\phi(s, f, g^*) \leq \phi(s, f^*, g^*) \leq \phi(s, f^*, g)$ for all $(f, g) \in \xi_1 \times \xi_2$. Then $(f^*, g^*)$ is a pair of optimal strategies of the game. So, we can concentrate only on the set of semi-stationary strategies instead of behavioural strategies.

Now, if we fix the strategy $g^*$ for the states $\{|S_1|+1, \ldots, |S_1|+|S_2|\}$ in the SMDP $\hat{\Gamma}$, then it reduces to another SMDP model $\hat{\Gamma}_1$, where for the states $\{|S_1|+1, \ldots, |S_1|+|S_2|\}$, the decision maker has a fixed strategy $g^*(s_0, |S_1|+j)$, where $j \in \{1, 2, \ldots, |S_2|\}$. Similarly, we can derive another SMDP model $\hat{\Gamma}_2$ by fixing the strategy $f^*$ for the states $\{1, 2, \ldots, |S_1|\}$ in the SMDP $\hat{\Gamma}$. If we prove that $f^*$ is an optimal pure semi-stationary strategy in the reduced SMDP $\hat{\Gamma}_1$ for fixed $g^*$ and $g^*$ is an optimal pure semi-stationary strategy in the reduced SMDP $\hat{\Gamma}_2$ for fixed $f^*$, then eventually $(f^*, g^*)$ becomes an optimal pure semi-stationary strategy pair for both the players in the PISMG $\Gamma$. We now prove the following lemma.

Lemma 1. $f^*$ is an optimal pure semi-stationary strategy for the player I in the reduced SMDP $\hat{\Gamma}_1$. 
Proof. We assume by contradiction, that \( f^* \) is not an optimal pure semi-stationary strategy in \( \hat{\Gamma}_1 \). Let \( \hat{\phi}_1 \) be the undiscounted pay-off function of the SMDP \( \hat{\Gamma}_1 \). Suppose, \( f_1^* \) is an optimal pure semi-stationary strategy in \( \hat{\Gamma}_1 \). Thus we have \( \hat{\phi}_1(s, f_1^*) \geq \hat{\phi}_1(s, f^*) \) \( \forall s \in S_1 \cup S_2 \), with strict inequality for at least one state. Suppose for \( s_1 \in S_1 \), \( \hat{\phi}_1(s_1, f_1^*) > \hat{\phi}_1(s_1, f^*) \) holds. This implies, \( \phi(s_1, f_1^*, g^*) > \phi(s_1, f^*, g^*) \).

Let us construct the pure semi-stationary strategy \( \hat{f}_1^* \), which consists of the strategy pair \( (f_1^*, g^*) \), such that it coincides with \( f_1^* \) in \( S_1 \) and in \( S_2 \) it coincides with \( g^* \). i.e.,

\[
\hat{f}_1^*(s_0, s) = \begin{cases} 
  f_1^*(s_0, s) & s \in S_1 \\
  g^*(s_0, s) & s \in S_2
\end{cases}
\]

Thus we get the inequality \( \hat{\phi}(s_1, \hat{f}_1^*) > \hat{\phi}(s_1, f^*) \), which contradicts that \( f^* \) is an optimal pure semi-stationary strategy in the SMDP \( \hat{\Gamma} \).

Similarly we can prove the following lemma.

**Lemma 2** \( g^* \) is an optimal pure semi-stationary strategy in the reduced SMDP \( \hat{\Gamma}_2 \).

**Proof.** Proof is similar to the proof of lemma 1. \( \square \)

From lemma 1 and lemma 2, we conclude that \( (f^*, g^*) \) is an optimal pure semi-stationary strategy pair for both players in the PISMG \( \Gamma \). \( \square \)

### 4 Algorithm For Solving A Zero-Sum Two Person Perfect Information Semi-Markov Game

Mondal (2017) [11] has described an algorithm which solves a limiting ratio average SMDP. To solve a PISMG, we first reduce it to an SMDP and use Mondal’s algorithm to solve the SMDP. Lastly, we get the game value and optimal pure semi-stationary strategies for the players following the proof of theorem 2. Below, we describe the algorithm in detail:

#### 4.1 The Algorithm

Suppose \( s_0 \) be a fixed but arbitrary initial state. We consider the following linear programming problem in the variables \( v(s_0), g = (g_s : s \in S) \) and \( h = (h_s : s \in S) \) as:

\[
LP : \min v(s_0)
\]

subject to

\[
g_s \geq \sum_{s' \in S} \hat{q}(s' | s, a)g_{s'} \forall s \in S, a \in \hat{A}(s).
\]
\[ g_s + h_s \geq r(s, a) - v(s_0) + \sum_{s' \in S} \hat{q}(s' \mid s, a) h_{s'} \quad \forall s \in S, a \in \hat{A}(s). \quad (4.2) \]

\[ g_{s_0} \leq 0. \quad (4.3) \]

The variables \( v(s_0), (g_s : s \in S, s_0 \neq s) \) and \((h_s : s \in S)\) are unrestricted in sign. The dual linear programming problem of this primal for the variables \( x = (x_{sa} : s \in S, a \in \hat{A}(s)) \) and \( y = (y_{sa} : s \in S, a \in \hat{A}(s)) \) and \( t \) is given by

\[
DLP : \max R_s, \quad \text{where } R_s = \sum_{s \in S} \sum_{a \in \hat{A}(s)} \hat{r}(s, a)x_{sa}
\]

subject to

\[
\sum_{s \in S} \sum_{a \in \hat{A}(s)} \{\delta_{ss'} - \hat{q}(s' \mid s, a)\}x_{sa} = 0 \quad \forall s' \in S.
\]

\[
\sum_{a \in \hat{A}(s')} \sum_{s \in S} \sum_{a \in \hat{A}(s)} \{\delta_{ss'} - \hat{q}(s' \mid s, a)\}y_{sa} = 0 \quad \forall s' \in S - \{s_0\}.
\]

\[
\sum_{a \in \hat{A}(s_0)} \sum_{s \in S} \sum_{a \in \hat{A}(s)} \{\delta_{ss_0} - \hat{q}(s_0 \mid s, a)\}y_{sa} - t = 0 \quad \forall s' \in S - \{s_0\}.
\]

\[
\sum_{s \in S} \sum_{a \in \hat{A}(s)} \hat{r}(s, a)x_{sa} = 1.
\]

\[
x_{sa}, y_{sa} \geq 0 \quad \forall s \in S, a \in \hat{A}(s), t \geq 0.
\]

where \( \delta_{ss'} \) is the Kronecker delta function. For a feasible solution \((x, y, t)\) of the \(DLP\), we define the following sets associated with the feasible solution:

\[
S_x = \{s \in S : \sum_{a \in \hat{A}(s)} x_{sa} > 0\}
\]

\[
S_y = \{s \in S : \sum_{a \in \hat{A}(s)} x_{sa} = 0 \quad \text{and} \quad \sum_{a \in \hat{A}(s)} y_{sa} > 0\}
\]

\[
S_{xy} = \{s \in S : \sum_{a \in \hat{A}(s)} x_{sa} = 0 \quad \text{and} \quad \sum_{a \in \hat{A}(s)} y_{sa} = 0\}.
\]

Thus \( S = S_x \cup S_y \cup S_{xy} \), where \( S_x, S_y \) and \( S_{xy} \) are pairwise disjoint sets. A pure stationary strategy corresponding to the feasible solution \((x, y, t)\) of the \(DLP\) is defined by \( f_{pq0}^{ps0} \), where \( s_0 \) is the fixed but arbitrary initial state \( f_{pq0}^{ps0}(s) = a_s, s \in S \) such that:

\[
a_s = \begin{cases} 
a & \text{if } s \in S_x \text{ and } x_{sa} > 0 \\ a' & \text{if } s \in S_y \text{ and } y_{sa} > 0 \\ \text{arbitrary} & \text{if } s \in S_{xy} \end{cases}
\]

From [11], we have the following theorem.

**Theorem 3** Let \((x^*, y^*, t^*)\) be an optimal solution of the \(DLP\). Then \( f_{pq0}^{ps0}^{x^*y^*} \) is a pure stationary optimal strategy of the SMDP for the initial state \( s_0 \).

We conclude from the above theorem that an optimal pure semi-stationary strategy of an SMDP can be found by observing the optimal solution of the dual linear
programming problem. Thus this algorithm is useful to obtain an optimal pure semi-stationary strategy for both the players in the perfect information semi-Markov game following the proof of theorem 2. Now for a fixed initial state \( s_0 \), we have \( f^{ps}_{x^0y^0t^0} \) as an optimal pure stationary strategy, where \( f^{ps}_{x^0y^0t^0}(s) = a^*_s \) and \( a^*_s \) is defined above. So, for different initial states, we have an optimal pure stationary strategy of the SMDP with initial state \( \hat{s} \). By corollary 1 of [11], we obtain an optimal pure semi-stationary strategy \( \hat{f} \) of the decision maker in the undiscounted SMDP. Thus this algorithm is useful to obtain an optimal pure semi-stationary strategy for both the players in the perfect information semi-Markov game.

If \( R_\hat{s} \) is the objective function of the DLP for the initial state \( s \) (from the definition of DLP), then \( \hat{\phi} = (\hat{\phi}(1), \hat{\phi}(2), \cdots, \hat{\phi}(s), \cdots, \hat{\phi}(N)) \) is the undiscounted value vector of the SMDP \( \hat{\Gamma} \), where \( \hat{\phi}(s) = \max R_\hat{s} \). The undiscounted value vector of the PISMG \( \hat{\Gamma} \) is given by \( \hat{\phi} = (\hat{\phi}(1), \hat{\phi}(2), \cdots, \hat{\phi}(|S_1|), -\hat{\phi}(|S_1|+1), \cdots, -\hat{\phi}(|S_1|+|S_2|)) \). The negative sign occurs for the states \( |S_1|+1, \cdots, |S_1|+|S_2| \) as player II is the minimiser for those states.

### 4.2 Complete enumeration method to obtain pure semi-stationary optimal strategy

By complete enumeration method, we calculate the undiscounted value of the SMDP for a pure stationary strategy \( f \) as:

\[
\hat{\phi}(s, f) = \frac{[Q^*(f) r(f)[s] \tau(f)[s]]}{[Q^*(f) r(f)[s]]} \quad \text{for all } s \in S
\]

where \( r(f) \) is the reward vector and \( \tau(f) \) is the expected sojourn time vector for a pure stationary strategy \( f \). We extend the algorithm of Lazari et al., (2020) [7] for an SMDP model to compute Cesaro limiting matrix for a SMDP with \( N \) number of states is given as follows:

**Input:** The transition matrix \( Q \in M_N(\mathbb{R}) \) (where \( M_N(\mathbb{R}) \) is the set of \( N \times N \) matrices over real numbers).

**Output:** The Cesaro limiting matrix \( Q^* \in M_N(\mathbb{R}) \).

**Step 1:** Determine the characteristic polynomial \( Ch_Q(z) = |Q - zI_n| \).

**Step 2:** Divide the polynomial \( Ch_Q(z) \) by \( (z-1)^{m(1)} \) (where \( m(1) \) is the algebraic multiplicity of the eigenvalue \( z_0 = 1 \)) and call it quotient \( D(z) \).

**Step 3:** Compute the quotient matrix \( R = D(Q) \).

**Step 4:** Determine the limiting matrix \( Q^* \) by dividing the matrix \( R \) by the sum of its elements of any arbitrary row.
We can observe that the LP algorithm is much faster than the complete enumeration method, i.e., less number of steps are required to calculate the optimal pure semi-stationary strategy of the decision maker by LP algorithm than the complete enumeration method. We elaborate this fact in the following examples.

5 Numerical examples

Example 1: Consider a PISMG $\Gamma$ with five states $S = \{1, 2, 3, 4, 5\}$, $A(1) = \{1, 2\} = A(2) = A(3)$, $A(4) = \{1\} = A(5)$ and $B(1) = \{1\} = B(2) = B(3) = B(4) = B(5)$. In this example player-II is the dummy player here for all the states. Rewards, transition probabilities and expected sojourn times for the players are given below

Clearly, we can think the above PISMG as an undiscounted (under limiting ratio average pay-off criteria) semi-Markov decision process $\hat{\Gamma}$, where there is only one decision maker (i.e., player-I). Now we implement our LP algorithm to solve this SMDP and obtain pure stationary strategy of the decision maker. For a fixed initial state $s_0$, the DLP in the variables $x = (x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{51}), y = (y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}, y_{41}, y_{51})$ and $t$ can be written as:

$$\max R_{s_0} = 13x_{11} + 9x_{12} + 4x_{21} + 3x_{22} + 7x_{31} + 3x_{32} + 15x_{41} + 5x_{51}$$
subject to

\[
\begin{align*}
3x_{12} - x_{31} - 3x_{41} &= 0 \quad (5.1) \\
x_{31} &= 0 \quad (5.2) \\
x_{31} + x_{32} - x_{51} &= 0 \quad (5.3) \\
-x_{12} + x_{41} &= 0 \quad (5.4) \\
-x_{32} + x_{51} &= 0 \quad (5.5) \\
3x_{11} + 3x_{12} + 3y_{12} - y_{31} - 3y_{41} - 3\delta_{s01} t &= 0 \quad (5.6) \\
3x_{12} + 3x_{22} - 2y_{31} - 3\delta_{s02} t &= 0 \quad (5.7) \\
x_{31} + x_{32} + y_{31} + y_{32} - y_{51} - \delta_{s03} t &= 0 \quad (5.8) \\
-y_{12} + x_{41} + y_{41} - \delta_{s04} t &= 0 \quad (5.9) \\
x_{51} - y_{32} + y_{51} - \delta_{s05} t &= 0 \quad (5.10) \\
4x_{11} + 2x_{12} + 2x_{21} + 1.6x_{22} + 2x_{31} + 1.5x_{32} + 5x_{41} + 3x_{51} &= 1 \quad (5.11)
\end{align*}
\]

Thus we conclude from the above example that the SMDP \(\hat{\Gamma}\) has the value vector \(\hat{\phi} = (3.428, 2.2686, 3.428, 2.685)\) and an optimal pure stationary strategy is given by \(f^*_1 = (f^*_1, f^*_2, f^*_3, f^*_4, f^*_5)\), where \(f^*_1 = f^*_3 = f^*_4 = f^*_5 = (2, 2, 1, 1, 1)\) and \(f^*_2 = (1, 1, 1, 1, 1)\), which we can calculate from the definition of \(f^*_{xy0}\), in section 4.

Now, we calculate the undiscounted value by complete enumeration method. Observe that there are 8 pure stationary strategies available in the SMDP model \(\Gamma\), which are given by \(f_1 = (1, 1, 1, 1, 1), f_2 = (1, 2, 1, 1, 1), f_3 = (1, 1, 2, 1, 1), f_4 = (1, 2, 2, 1, 1), f_5 = (2, 1, 1, 1, 1), f_6 = (2, 2, 1, 1, 1), f_7 = (2, 1, 2, 1, 1)\) and \(f_8 = (2, 2, 2, 1, 1)\). By our proposed algorithm we calculate the Cesaro limiting matrix \(Q^*(f_1)\) for the pure stationary strategy \(f_1\) as follows:
1. The transition probability matrix $Q(f_1)$ is given as $Q(f_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

The characteristic polynomial for the matrix $Q(f_1)$ is $x^5 - 2x^4 + x^3$. The eigenvalues are 0, 0, 0, 1, 1. Hence the algebraic multiplicity of the eigenvalue 1 is $2 = m(1)$. Now dividing the characteristic polynomial by $(x - 1)^2$, we get the polynomial $x^3$.

Thus our quotient matrix is $Q^3(f_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Now dividing each row of this matrix by the sum of elements of any arbitrary row, we get the limiting matrix

$$Q^*(f_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \end{bmatrix}.$$  

Now the reward vector $\hat{r}(f_1) = (13, 4, 7, 15, 5)$ and expected sojourn time vector $\hat{\tau}(f_1) = (4, 2, 2, 5, 3)$. Thus by using the definition of $\hat{\phi}$, we get $\hat{\phi}(f_1) = (3.25, 2, 2.625, 3.25, 2.625)$. By a similar manner we get the Cesaro limiting matrices $Q^*(f_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \end{bmatrix}$, $Q^*(f_3) = Q^*(f_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$.

Thus we get $\hat{\phi}(f_2) = (3.25, 1.875, 2.639, 3.25, 2.639), \hat{\phi}(f_3) = (3.25, 2, 1.778, 3.25, 1.778), \hat{\phi}(f_4) = (3.25, 1.875, 1.778, 3.25, 1.778), \hat{\phi}(f_5) = (3.429, 2, 2.667, 3.429, 2.667), \hat{\phi}(f_6) = (3.429, 1.875, 2.687, 3.429, 2.687), \hat{\phi}(f_7) = (3.429, 2, 1.778, 3.429, 1.778)$ and $\hat{\phi}(f_8) = (3.429, 1.875, 1.778, 3.429, 1.778)$. Thus by our algorithm, we have $(3.429, 2, 2.687, 3.429, 2.687)$ as the value vector and the optimal pure semi-stationary strategy for $\hat{\Gamma}$ is $\hat{f}^* = (f_6, f_1, f_6, f_6, f_6)$. Note that this optimal pure semi-stationary strategy coincides with the pure semi-stationary strategy we got form the LP algorithm. So, we conclude that $\hat{f}_1^* = \hat{f}^*$. The optimal pure stationary strategy pair $(f^*, g^*)$ for the players in the PISMG $\Gamma$ is given as:

$$f^*(s_0, s) = \begin{cases} \hat{f}_1^*(s_0, s) & s \in S_1 \\ 1 & s \in S_2 \end{cases}$$
\[ g^*(s_0, s) = \begin{cases} 1 & s \in S_1 \\ \hat{f}_1^*(s_0, s) & s \in S_2 \end{cases} \]

and the value of the game is given by \( \phi = (3.429, 2, 2.2687, 3.429, 2.687) \). Next, we calculate the total number of iterations to calculate the optimal pure semi-stationary strategy of the SMDP \( \Gamma \) by LP algorithm. The following table gives us number of iterations against each state:

| Initial state | Number of iterations needed |
|---------------|-----------------------------|
| \( s_0 = 1 \) | 3                           |
| \( s_0 = 2 \) | 3                           |
| \( s_0 = 3 \) | 5                           |
| \( s_0 = 4 \) | 3                           |
| \( s_0 = 5 \) | 4                           |

From the above table we see that the total number of iterations to solve the LP in a coding software is 18. In the complete enumeration method, we need 9 operations to calculate the undiscounted value of the SMDP for each pure stationary strategy. So, for 8 pure stationary strategies, the number of steps required to calculate the undiscounted value is 72.

Example 2: Consider a PISMG \( \Gamma \) with four states \( S = \{1, 2, 3, 4\} \), \( A(1) = \{1, 2\} = A(2) \), \( B(1) = B(2) = \{1\}, B(3) = B(4) = \{1, 2\}, A(3) = A(4) = \{1\} \). Player II is the dummy player in the state 1 and 2 and player I is the dummy player for the states 3 and 4. Rewards, transition probabilities and expected sojourn times for the players are given below

| State-1 | 1.1 |
|---------|-----|
| (\( \frac{1}{2}, \frac{1}{2} \), 0, 0) | 1   |
| (\( \frac{1}{2}, 1 \), 0, 0)     | 0.9 |

| State-2 | 3.1 |
|---------|-----|
| (\( \frac{1}{2}, \frac{1}{2} \), 0, 0) | 1   |
| (\( \frac{1}{2}, 1 \), 0, 0)     | 0.9 |

| State-3 | \( \frac{3}{2} \) |
|---------|------------------|
| (0, 0, 1, 0) | 1 |
| (0, 0, 1, 0) | 2 |

| State-4 | 4 |
|---------|-----|
| (\( \frac{1}{2}, 0 \), \( \frac{1}{2} \), 0) | 2   |
| (\( \frac{1}{2}, 0 \), \( \frac{1}{2} \), 0) | 1.1 |

Here player I is the row player and player II is the column player. The undiscounted SMDP \( \hat{\Gamma} \) extracted from this PISMG is given below:
Next we implement our LP algorithm to solve this SMDP and obtain pure semi-stationary strategy of the decision maker. For a fixed initial state $s_0$, the DLP in the variables $x = (x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42})$, $y = (y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}, y_{41}, y_{42})$ and $t$ can be written as

$$\max R_{s_0} = 1.1x_{11} + x_{12} + 3.1x_{21} + 3x_{22} - 3x_{31} - 5.8x_{32} - 4x_{41} - 2x_{42}$$

subject to

$$3x_{12} + 4x_{12} - 3x_{21} - 4x_{22} - 3x_{41} - 3x_{42} = 0 \quad (5.13)$$
$$-3x_{11} - 4x_{12} + 3x_{21} + 4x_{22} = 0 \quad (5.14)$$
$$x_{41} + x_{42} = 0 \quad (5.15)$$
$$-x_{12} + x_{41} = 0 \quad (5.16)$$
$$6x_{11} + 6x_{12} + 3y_{11} + 4y_{12} - 3y_{21} - 4y_{22} - 3y_{41} - 3y_{42} - 6\delta_{s_0}t = 0 \quad (5.17)$$
$$6x_{12} + 6x_{22} - 3y_{11} - 4y_{12} + 3y_{21} + 4y_{22} - 6\delta_{s_0}t = 0 \quad (5.18)$$
$$2x_{31} + 2x_{32} - y_{41} - y_{42} - 2\delta_{s_0}t = 0 \quad (5.19)$$
$$-y_{12} + x_{41} + y_{41} - \delta_{s_0}t = 0 \quad (5.20)$$
$$x_{41} - x_{42} - \delta_{s_0}t = 0 \quad (5.21)$$
$$x_{11} + 0.9x_{12} + x_{21} + 1.1x_{22} + x_{31} + 2x_{32} + 2x_{41} + 1.1x_{42} = 1 \quad (5.22)$$
$$x, y, t \geq 0. \quad (5.23)$$

The solution of the above linear programming problem by dual-simplex method for different initial states are given by:

(i) For $s_0 = 1$: \[\max R_1 = 2.2985, \ x = (0, 0.4478, 0.5970, 0, 0, 0, 0, 0), \ y = (0, 0.8955, 0, 0, 0, 0, 0, 0), \ t = 1.0448.\]

(ii) For $s_0 = 2$: \[\max R_2 = 2.2985, \ x = (0, 0.4478, 0.5970, 0, 0, 0, 0, 0), \ y = (0, 0.8955, 0, 0, 0, 0, 0, 0), \ t = 1.04.\]

(iii) For $s_0 = 3$: \[\max R_3 = -2.9, \ x = (0, 0, 0, 0, 0, 0, 0, 0), \ y = (0, 0, 0, 0, 0, 0, 0, 0), \ t = 0.5.\]

(iv) For $s_0 = 4$: \[\max R_4 = -0.4088, \ x = (0, 0.2190, 0.2920, 0, 0.5109, 0, 0, 0), \ y = (0, 0.4380, 0, 0, 0, 0, 0, 1.0219), \ t = 1.0219.\]

The negative sign occurs for the value of the objective (his loss) function in state 3 and 4 because player-II wants to minimize the objective function in those states and we have considered a maximization problem here. Just like the previous example, we find SMDP $\Gamma$ has the value vector $\hat{\phi} = (2.2985, 2.2985, -2.9, -0.4088)$ and an optimal pure semi-stationary strategy is given by $f^* = (f_1^*, f_2^*, f_3^*, f_4^*)$, where $f_1^* = (2, 1, 1, 1)$
and $f_2^* = (2, 1, 2, 1)$ and $f_3^* = (2, 1, 1, 2)$, which we can calculate from the definition of $f_{390}^{ps}$ in section 4.

Now, we calculate the undiscounted value by complete enumeration method. Observe that there are 16 pure stationary strategies available in the SMDP model $\hat{\Gamma}$, which are given by $f_1 = (1, 1, 1, 1)$, $f_2 = (1, 1, 1, 2)$, $f_3 = (1, 1, 2, 1)$, $f_4 = (1, 1, 2, 2)$, $f_5 = (1, 2, 1, 1)$, $f_6 = (1, 2, 2, 1)$, $f_7 = (1, 2, 1, 2)$ and $f_8 = (1, 2, 2, 2)$, $f_9 = (2, 1, 1, 1)$, $f_{10} = (2, 1, 1, 2)$, $f_{11} = (2, 1, 2, 1)$, $f_{12} = (2, 1, 2, 2)$, $f_{13} = (2, 2, 1, 1)$, $f_{14} = (2, 2, 1, 2)$, $f_{15} = (2, 2, 1, 2)$, $f_{16} = (2, 2, 2, 2)$. As before, we get the Cesaro limiting matrices

$$Q^*(f_1) = Q^*(f_2) = Q^*(f_3) = Q^*(f_4) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}, Q^*(f_5) = Q^*(f_6) = Q^*(f_7) = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix},$$

$$Q^*(f_8) = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}, Q^*(f_9) = Q^*(f_{10}) = Q^*(f_{11}) = Q^*(f_{12}) = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}.$$}

Thus we get $\hat{\phi}(f_1) = (2.1, 2.1, -3, -0.9)$, $\hat{\phi}(f_2) = (2.1, 2.1, -3, -0.9)$, $\hat{\phi}(f_3) = (2.1, 2.1, -2.9, -0.9)$, $\hat{\phi}(f_4) = (2.1, 2.1, -2.9, -0.9)$, $\hat{\phi}(f_5) = (1.8353, 1.8362, -3, -0.53)$, $\hat{\phi}(f_6) = (1.8353, 1.8362, -3, 0.53)$, $\hat{\phi}(f_7) = (1.8353, 1.8362, -3, -0.58)$, $\hat{\phi}(f_8) = (1.8353, 1.8362, -2.9, -1.2773)$, $\hat{\phi}(f_9) = (2.2985, 2.2985, -3, -0.4088)$, $\hat{\phi}(f_{10}) = (2.2985, 2.2985, -3, -0.4088)$, $\hat{\phi}(f_{11}) = (2.2979, 2.2985, -2.9, -0.4088)$, $\hat{\phi}(f_{12}) = (2.2979, 2.2985, -2.9, -1.2182)$, $\hat{\phi}(f_{13}) = (2.2979, 2.2985, -3, -0.4277)$, $\hat{\phi}(f_{14}) = (2.112, 2.1129, -2.9, -1.2141)$, $\hat{\phi}(f_{15}) = (2.112, 2.1129, -3, -0.4267)$, $\hat{\phi}(f_{16}) = (2.112, 2.1129, -2.9, -1.2137)$. By Sinha et al.,(2017) [14], we have $(2.2985, 2.2985, -2.9, -0.4088)$ as the value vector and the optimal pure semi-stationary strategy for $\hat{\Gamma}$ is $\hat{f}^* = (f_9, f_9, f_{11}, f_1) = \hat{f}_1^*$. Player-I’s optimal pure semi-stationary strategy is denoted by $f^*$, which is given as: $f^*(1, 1) = f^*(2, 1) = f^*(3, 1) = f^*(4, 1) = 2$, $f^*(1, 2) = f^*(2, 2) = f^*(3, 2) = f^*(4, 2) = 1$. Thus, player-I has an optimal pure stationary strategy in this game, given by, $f^*(1) = 2$, $f^*(2) = 1$. Player-II’s optimal pure semi stationary strategy $g^*$ is given by: $g^*(1, 3) = g^*(1, 4) = 1$, $g^*(2, 3) = 2$, $g^*(2, 4) = 1$, $g^*(3, 3) = 2 = g^*(4, 4)$, $g^*(3, 4) = g^*(4, 3) = 1$. Note that $g^*$ is a pure semi-stationary strategy.

The value of the game is given by $\phi = (2.2985, 2.2985, 2.9, 0.4088)$. Next, we calculate the total number of iterations to calculate the optimal pure semi-stationary strategy of the SMDP $\Gamma$ by LP algorithm. The following table gives us number of iterations against each initial state:
| Initial state | Number of iterations needed |
|---------------|----------------------------|
| $s_0 = 1$     | 2                          |
| $s_0 = 2$     | 2                          |
| $s_0 = 3$     | 1                          |
| $s_0 = 4$     | 3                          |

From the above table we see that the total number of iterations to solve the LP in a coding software is 8. In the complete enumeration method, we need 9 operations to calculate the undiscounted value of the SMDP for each pure stationary strategy. So, for 16 pure stationary strategies, the number of steps required to calculate the undiscounted value is 144. So, from the above examples, we conclude that our algorithm is much more fast than complete enumeration method to calculate value and optimal pure semi-stationary strategies for the players in the PISMG.

6 Conclusion

The purpose of this paper is to show that any Perfect Information Markov/ semi-Markov game can be treated as a (Markov/ semi-Markov) Decision Process and thus proving the existence of the game value and optimal strategies for the players follows from the corresponding result from the literature in Markov/ semi-Markov Decisiion Processes. Thus, the existence of the value and a pair of pure stationary optimals for the players in a zero-sum two person Perfect Information undiscounted stochastic (Markov) game (Gillette (1957) [2], Liggett-Lippman (1969) [8]) can be obtained as a corollary of Derman’s paper (1970) [1] directly. Furthermore, same result holds also for an $N$ person Perfect Information non-cooperative Markov/ semi-Markov game under any standard (discounted/ undiscounted) pay-off criteria. We shall elaborate on this in a forthcoming paper.

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