Geometrical Properties of Loops and Cluster Boundaries

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We discuss how the statistical properties of the area and radius of gyration of single self-avoiding loops, and of Ising and percolation cluster boundaries, may be calculated using ideas of two-dimensional field theory. For cluster boundaries, we show that almost all loops have area $C \ln L + O(1)$, where $L$ is the size of the system, and $C$ is a calculable constant. We also compute the universal ratios $\langle A \rangle_\ell / \langle R^2 \rangle_\ell$ of the area to the squared radius of gyration of loops of a given large perimeter $\ell$.

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Introduction.

If we look at a typical configuration in a Monte Carlo simulation of the 2D Ising model at its critical point, we see a scale-invariant distribution of structures. If we focus on the boundaries of the clusters, there are some small ones, but others seem to stretch across the system. What is the statistics of the size and shape of these objects? This is a geometric question, not simply related to the conventional correlation functions of the Ising model. In fact, if we study typical configurations of the same model at temperatures $T > T_c$, although it exhibits a finite correlation length in the conventional sense, the geometrical properties of the cluster boundaries appear to be very little different. As we shall show in detail, this is indeed true all the way up to infinite temperature.

We shall discuss two useful measures of the size and shape of these objects, which, it turns out, may be calculated exactly for large clusters. The first is the area contained within a cluster boundary, which in general forms a closed loop. The second is its radius of gyration, defined by $R^2 = (1/2\ell^2) \sum_{r_1,r_2} (r_1 - r_2)^2$, where the sum is over pairs of points on the perimeter, and $\ell$ is the number of such points. As well as the Ising model for $T = T_c$ and $T > T_c$, we shall also discuss the boundaries of percolation clusters at and near the percolation threshold $p = p_c$, and the problem of single self-avoiding loops in the plane. These all turn out to be special cases of an $O(n)$ model, defined as follows.

Consider a honeycomb lattice (this is for convenience in making a clean definition of cluster boundaries, but similar results may be obtained for the square lattice.) At each site $r$ place an $O(n)$ spin with components $s_a(r)$, where $a = (1, 2, \ldots, n)$, normalised so that $\text{Tr} s_a(r)s_b(r') = \delta_{ab}\delta_{rr'}$. The partition function is

$$Z_{O(n)} = \text{Tr} \prod_{\text{links}} \left( 1 + x \sum_a s_a(r)s_a(r') \right)$$

For small $x$ we may expand this as a sum over configurations of bonds on the lattice, with a weight $x$ for each bond which is present. After taking the trace, the only configurations which survive (see Fig. 1) consist of self-avoiding closed loops, with a factor $n$ that arises on taking the trace over the last spin in each loop. Thus

$$Z_{O(n)} = \sum_{\text{loop configurations}} x^{\text{total length}} n^{\text{number of loops}}$$

(1)
Figure 1. Loop gas on the honeycomb lattice, and Ising clusters on the dual triangular lattice.

The advantage of the honeycomb lattice is that it has coordination number three, so that loops never touch at the vertices. This is not of course the usual partition function for the lattice $O(n)$ nonlinear sigma model, which is $Z = \text{Tr} \exp(\beta \sum s(r) \cdot s(r'))$. Nevertheless, they agree at high temperatures, when $x \sim \beta$, and, for the Ising case $n = 1$, they agree exactly with $x = \tanh \beta$. Unlike the non-linear model, (1) makes sense for all $n$, and it has a continuous transition in two dimensions for $n \leq 2$. The phase diagram is best illustrated by the schematic renormalisation group flows in Fig. 2. There is a conventional unstable fixed point at $x = x_{c1}(n)$, but the whole low-temperature phase is also critical in general, being attracted into a stable non-trivial fixed point at $x = x_{c2}(n)$. For the case $n = 1$, $x_{c2} = 1$, corresponding to a conventional $T = 0$ fixed point.

The relation of this model for $n = 1$ to cluster boundaries is through duality. The loops in (1) may be thought of as the domain walls of an Ising model on the dual, triangular lattice. When the original model is at $x = x_{c1}$, the dual spins are also critical, while $x = x_{c2} = 1$ corresponds to the dual spins being at infinite temperature. Since the whole
Figure 2. Schematic RG flows and phase diagram of the $O(n)$ model as a function of $x$.

phase $x > x_{c1}$ is attracted into this fixed point, so is the whole high-temperature phase of the dual model. Thus the universal large distance properties of its cluster boundaries should be the same as those at infinite $T$.

This infinite temperature Ising model on the triangular lattice may also be interpreted as a site percolation problem, where sites are occupied ($\equiv$ spin up) with probability $\frac{1}{2}$. This is precisely the percolation threshold for the site percolation problem on this lattice, so these cluster boundaries may also be thought of as those of critical percolation clusters, and their universal properties should apply to critical site and bond percolation clusters on any lattice.

Area of loops.

In two dimensions, any gauge field is confining, and as a result, Wilson loops obey an exact area law. One way, therefore, to express the area inside a simple closed curve is to regard it as a Wilson loop. To be explicit, consider a unit current $J_\mu$ flowing around the loop in its direction of orientation. Then the area is

$$A_{\text{loop}} = \int \int G_{\mu\nu}(r-r') J_\mu(r) J_\nu(r') d^2r d^2r'$$

where $G_{\mu\nu} = \langle A_\mu(r) A_\nu(r') \rangle$ is the gauge field propagator. It is simplest to take the gauge group to be $U(1)$, so that, in the gauge $A_x = 0$, the area is

$$A_{\text{loop}} = -\frac{1}{2} \int \int |x-x'| \delta(y-y') J_y(x,y) J_y(x',y') d^2r d^2r'$$

which simply corresponds to a decomposition of the interior of the loop into strips of length $|x-x'|$ and width $\delta y = \delta y'$. Although this is written in a continuum notation, in fact
this is valid even on the lattice if we imagine each link slightly broadened so as to form a narrow strip, with a current density $J_\mu$ such that there is unit flux along each link. This makes the transition to the continuum limit particularly clean and unambiguous.

What is this current $J_\mu$? This is most easily understood by thinking of the $O(n)$ model as a complex $O(n/2)$ model, with complex spins $S_a = s_a + is_{a+1}$, and partition function $Z = \text{Tr} \prod (1 + x S^* \cdot S + c.c.)$. $J_\mu$ is then just the $U(1)$ current, proportional to $(1/2i)(S^* \cdot \partial S - c.c.)$, whose integral generates the symmetry $S_a(r) \rightarrow e^{i\theta} S_a(r)$. If we now sum over all loops and take the expectation value, we get

$$n\langle A \rangle = -\frac{1}{2} \int \int |x - x'| \langle J_y(x, y) J_y(x', y') \rangle d^2r d^2r'$$

where the $\langle JJ \rangle$ correlation function is evaluated in the $O(n)$ model, and $A$ is the total area of all loops. Note that contributions to $\langle J_\mu(r) J_\nu(r') \rangle$ cancel in the sum over orientations when $r$ and $r'$ are on different loops. If we use translational invariance, we finally arrive at

$$\langle A \rangle = -\frac{1}{2n} A \int_{-\infty}^{\infty} |x| \langle J_y(x, 0) J_y(0, 0) \rangle dx \tag{3}$$

where $A$ is the total area of the lattice.

At its critical points, the $O(n)$ model, on distance scales much larger than the lattice spacing $a$ is supposed to be described by a conformal field theory. $J_\mu$ is conserved current in this theory with scaling dimension 1, and, by scale invariance, rotational invariance, and conservation, its two-point function must have the form

$$\langle J_\mu(r) J_\nu(0) \rangle = k(n) \left( \frac{r_\mu r_\nu - \frac{1}{2} r^2 g_{\mu\nu}}{r^4} \right)$$

which is unique apart from the universal number $k$ which characterises the theory (and the choice of $U(1)$ current.) [Note that $k$ is sometimes called the chiral anomaly, because if we define the left- and right-moving components of the current to be $J_{L,R} = J_1 \pm i J_2$, strict current conservation requires the existence of a contact term $J_L(r) J_R(r') \propto k \delta^{(2)}(r-r')$.]

In our case we see that the correlation function in (3) behaves like $1/x^2$ so the integral is logarithmically divergent at both short and large distances. Short distances are presumably cut off at the order of the lattice spacing, since the continuum description breaks down there, but the infrared divergence is real, and indicates that the average area has no thermodynamic limit. Instead, if we evaluate this in a finite system of linear size $\sim L$, we should find

$$\langle A \rangle / A \sim (k(n)/2n) \ln(L/a)$$
The right hand side is, of course, much greater than unity for large $L$. This can happen because there are loops with loops, and thus some portions of the total area are counted many times in $\langle A \rangle$.

In the same way, higher moments of $A$ are given by integrals over higher order correlations of the current $J$. Fortunately, these are simple to write down, since it is an elementary fact of conformal field theory that the connected pieces of such correlation functions all vanish. The way to prove this is to consider the holomorphic and antiholomorphic components $J = J_1 - iJ_2$ and $\overline{J} = J_1 + iJ_2$. Current conservation implies that a correlation function

$$\langle J(z_1)J(z_2)\ldots \overline{J}(\overline{\zeta}_1)\overline{J}(\overline{\zeta}_2)\ldots \rangle$$

depends only holomorphically on the $\{z_j\}$ and antiholomorphically on the $\{\overline{\zeta}_j\}$, so that it is entirely determined by its singularities as these points approach each other. These are given by the operator product expansion

$$J(z_1)J(z_2) \sim \frac{k/4}{(z_1 - z_2)^2} + \text{regular terms}$$

with similar relations for $\overline{J}$. The important point is that, for a $U(1)$ current, there is no term $O((z_1 - z_2)^{-1})$. The consequence is that, if we subtract off the disconnected part of the correlation function, the remainder has no singularities, and falls off sufficiently fast at infinity that it must vanish identically.

Thus, apart from the value of $k$, the correlations of a $U(1)$ current are the same in any conformal field theory. We are therefore free to use a free field representation, which, for convenience, we take to be $J_\mu = \sqrt{k} \xi_\mu \partial_\nu \phi$, where $\phi(r)$ is a free scalar field with action $S_G = (1/2\pi) \int (\partial \phi)^2 d^2 r$. Since the loops do not cross the boundary, the normal component of $J$ should vanish there, corresponding to choosing Dirichlet conditions $\phi = 0$ on the boundary. It is also more convenient to write (2) in a covariant gauge, for which $\partial_\mu G_{\mu\nu} = 0$ and $\partial^2 G_{\mu\nu} = -\delta^{(2)}(r - r')$. It is then readily checked, using Stokes’ theorem, that (2) correctly gives the area.

To generate the higher cumulants of $A$ it is convenient to consider its generating function, given in the free field representation by

$$\langle e^{-uA} \rangle = \langle e^{-u(k/n)} \int G_{\mu\nu}(r-r')\epsilon_{\mu\lambda}\epsilon_{\nu\sigma} \partial_\lambda \partial_\sigma' \phi(r)\phi(r')d^2r d^2r' \rangle$$
Integrating by parts and using the above properties of $G_{\mu\nu}$ then gives a mass term for the $\phi$ field, proportional to $(k/n)u$, so that the result is simply a ratio of two Gaussian integrals

$$\langle e^{-uA} \rangle = \frac{\det(-\partial^2 + (2\pi ku/n))^{-1/2}}{\det(-\partial^2)^{-1/2}} = e^{-\frac{1}{2} \sum_m \ln \left( \frac{1 + \frac{(2\pi ku/n)L^2}{\lambda_m}}{\lambda_m} \right)}$$

where the $\lambda_m/L^2$ are the eigenvalues of $-(\text{laplacian})$, with Dirichlet boundary conditions.

If we expand (5) to first order in $u$, we recover our previous result

$$\langle A \rangle = (\pi k/n)L^2 \sum_m \frac{1}{\lambda_m}$$

since the sum has a divergence like $(1/2\pi)^2 \int d^2p/p^2 \sim (1/2\pi) \ln(L/a)$. However, the higher cumulants behave as

$$\langle A^p \rangle_c \propto L^{2p} \sum_m \frac{1}{\lambda_m^p}$$

and this sum is convergent (but dependent on the shape of the region) for $p > 1$.

This result implies that $A/A$ has mean value $O(\ln L)$, but with fluctuations $O(1)$. However, we are more interested in the area per loop, given by $A/N_l$, where $N_l$ is the number of loops. Since this is conjugate to the fugacity $n$, its cumulants are given by

$$\langle N_l \rangle = (n\partial/\partial n) \ln Z_{O(n)}$$

$$\langle N_l^2 \rangle - \langle N_l \rangle^2 = (n\partial/\partial n)^2 \ln Z_{O(n)}$$

and so on. Each of these is an extensive quantity, proportional to $A$, even at the critical point. (In fact, they are calculable exactly thanks to a series of mappings to the 6-vertex model.) Hence we see that $N_l$ is of order $L^2 \sim A$, with fluctuations of order $L$. Therefore in the ratio $A/N_l$ the numerator fluctuates much more strongly than the denominator, and it is permissible to replace $\langle A/N_l \rangle$ by $\langle A \rangle/\langle N_l \rangle$.

The final result is that, not only is average size of a loop large, but, with probability one as $L \to \infty$, all loops have size $C \ln L + O(1)$. The amplitude $C$ is a ratio of the universal number $k(n)/2n$ and the non-universal, lattice-dependent, quantity $\langle N_l \rangle/A$. From (5) one may also recover the whole probability distribution $P(A)$ of $A$, in the limit when $a$ and $L$ are large, by inverse Laplace transform. In particular, for

$$A - \langle A \rangle \gg L^2,$$
we find an exponential decay

\[ P(A) \sim e^{-[\lambda_0/(2\pi k/n)](A/L^2)} \]

where \( \lambda_0/L^2 \) is the lowest eigenvalue of \( -\partial^2 \).

The above results for the average area may be generalised away from the point where the loops are critical. In the percolation problem, this would correspond to \( p \neq p_c \), and, for Ising clusters, to adding an external magnetic field \( H \). In either case the integral in (3) will now behave like \((k/2n)\ln \xi\), where the correlation length \( \xi \) behaves as \( \xi \sim (p_c - p)^{-\frac{3}{4}} \) for percolation, and for the Ising model as \( H^{-\frac{15}{8}} \) \((T = T_c)\) and \( H^{-\frac{1}{2}} \) \((T > T_c)\).

**Finite-length loops.**

Although the above calculation told us about the average statistics of loop areas, it did not tell us about the size or shape of individual loops. If we pick out one loop and measure its length to be \( \ell \), how does its area and radius of gyration depend on \( \ell \)?

To study this question we need a generalisation of the \( O(n) \) model, in which the indices \( a \) run from 1 to \( n + n' \):

\[
Z = \text{Tr} \prod_{\text{bonds}} \left( 1 + x \sum_{1}^{n} s_a(r)s_a(r') + x' \sum_{n+1}^{n+n'} s_a(r)s_a(r') \right)
\]

This gives us an expansion in powers of \( x \) and \( x' \) which is a sum over configurations with two different kinds of mutually non-overlapping loops. As \( n' \to 0 \), the term \( O(n') \) corresponds to those configurations with single primed loop:

\[
Z = Z_{O(n)} \left( 1 + n' A \sum_\ell p_\ell x'^\ell + O(n'^2) \right)
\]

which defines the coefficient \( p_\ell \). (As \( n \to 0 \) this is just the total number of self-avoiding loops of length \( \ell \), weighted equally.) This model has a critical point at \( x = x' = x_c \). Turning on the perturbation \((x' - x_c)\) breaks the symmetry down to \( O(n) \times O(n') \), and we expect the degrees of freedom with \( n < a \leq n' \) to become massive.

To measure the area of the chosen loop we may use the current

\[
J' \sim (1/2i) \sum_{n+1}^{n+n'} (s^*_a \partial s_a - \text{c.c.})
\]
whose 2-point function is given by \( \langle J'_r J'_r \rangle = \langle n' / n \rangle \langle JJ \rangle \). Then, as before, on taking the expectation value in this ensemble,

\[
n' \sum_{\ell} p_{\ell}(A)_{\ell} x'^{\ell} = -\frac{1}{2} \int |x| \langle J'_y(x, 0) J'_y(0, 0) \rangle dx
\]

where \( \langle A \rangle_{\ell} \) is the mean area of loops of length \( \ell \) when embedded in the gas of critical loops of the \( O(n) \) model. The integrand once again behaves as \( 1/x^2 \), but it is now cut off at large distances by the length scale \( \xi' \) which is essentially the inverse mass of the massive modes. This will diverge as \( x' \to x_c \) as \( \xi' \sim (x_c - x')^{-\nu'} \), where \( \nu' \) is related to the scaling dimension \( X' \) of the perturbation by \( \nu' = 2/(2 - X') \). Thus the right hand side of (3) behaves like \( \ln(x_c - x') \) with a calculable coefficient. From this we discover the large \( \ell \) behaviour of the coefficient of \( x'^{\ell} \) on the left hand side:

\[
p_{\ell}(A)_{\ell} \sim (k(n)/2n)v'(n)\ell^{-1}x_c^{-1}
\]

Now \( \sum_{\ell} p_{\ell} x'^{\ell} \) itself is like a free energy, which, by hyperscaling, should have a singular part which behaves like \( \xi'^{-2} \sim (x_c - x')^{2\nu'} \), which gives the large \( \ell \) behaviour of \( p_{\ell} \). Putting these results together, we find that \( p_{\ell} \sim \ell^{-1 - 2\nu'}x_c^{-1} \) and \( \langle A \rangle_{\ell} \sim \ell^{2\nu'} \).

**Radius of gyration.**

The radius of gyration of a single loop is given by \( R^2 = (1/2\ell^2) \sum_{r_1, r_2} (r_1 - r_2)^2 \). We need to find a way of counting all loops going through two chosen bonds at \( r_1 \) and \( r_2 \). Let us define the energy density of the perturbation of the \( O(n + n') \) model on the bond at \( r \) as \( E'(r) = \sum_{n+1} s_a \cdot s_a \). Then \( E'(r_1)E'(r_2) \) has only contributions from the selected loops. In the limit \( n' \to 0 \), \( r_1 \) and \( r_2 \) must be on the same loop, since there is only one. Hence \( \sum_{r_1, r_2} (r_1 - r_2)^2 E'(r_1)E'(r_2) \), when inserted into a correlation function, gives the radius of gyration (squared) of the loop. On averaging, we thus find that

\[
2n' \mathcal{A} \sum_{\ell} p_{\ell} \ell^2 \langle R^2 \rangle_{\ell} x'^{\ell-2} = \sum_{r_1, r_2} \langle (r_1 - r_1)^2 E'(r_1)E'(r_2) \rangle \sim \mathcal{A} \int r^2 \langle E'(r)E'(0) \rangle d^2 r
\]

where this last correlation function is to be evaluated in the perturbed \( O(n + n') \) model. This last integral may be evaluated by a corollary of Zamolodchikov’s \( c \)-theorem.[1]

The addition of the term \( \sum_{r} (x' - x_c)E'(r) \) to the action takes the theory away from the conformal point, and gives rise to a non-zero trace \( \Theta = T_\mu^\mu \) of the stress tensor. In fact, \( \Theta = 2\pi\nu'^{-1}(x' - x_c)E'(r) \), which may be demonstrated by considering the response of the
theory to a dilatation.\[1\] Thus we need the second moment of 2-point function $\langle \Theta \Theta \rangle$. A simple way of deriving this is as follows: in two dimensions, conservation and rotational symmetry force two-point function of the stress tensor to have the form

$$\langle T_{\mu\nu}(r)T_{\lambda\sigma}(r') \rangle = (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2)(\partial'_\lambda \partial'_\sigma - g_{\lambda\sigma} \partial'^2)G(r-r')$$

where $G$ is a scalar function. At short distances, the theory is equivalent to some conformal field theory with central charge $c_{UV}$, so that

$$\langle T_{zz}(z)T_{zz}(z') \rangle \sim \frac{c_{UV}/2}{(z-z')^4} \sim \partial^2 \partial'^2 G$$

so $G(r) \sim (c_{UV}/6) \ln r$ as $r \to 0$. Similarly, as $r \to \infty$, $G(r) \sim (c_{IR}/6) \ln r$. Now $\int r^2 \langle \Theta(r)\Theta(0) \rangle d^2r = \int r^2 \partial^4 G(r) d^2r$. Integrating by parts twice kills the factor $r^2$, and the result is the integral of a total derivative, which is not zero because of the surface terms at $r = 0$ and infinity. The final result is that

$$c_{UV} - c_{IR} = \frac{3}{4\pi} \int r^2 \langle \Theta(r)\Theta(0) \rangle d^2r \quad (7)$$

In our case, the UV theory is the $O(n + n')$ model. In the IR, we are left with the massless modes of the $O(n)$ model. Hence the left hand side of (7) is $c(n + n') - c(n) \sim n'(dc/dn)$, so that

$$2 \sum_\ell p_\ell \ell^2 \langle R^2 \rangle_\ell x'^{\ell-2} \sim \left(2\pi \nu'^{-1} (x' - x_c) \right)^{-2} (4\pi/3)(dc/dn)$$

This tells us about the large $\ell$ behaviour of $p_\ell \langle R^2 \rangle_\ell$, and, in particular that $\langle R^2 \rangle_\ell \sim \ell^{2\nu'}$. This implies that a loop of finite (but large) length, immersed in the gas of other critical loops, has a fractal dimension of $\nu'^{-1}$. The amplitudes in the mean area and $R^2$ are not universal, but the ratio, in which all dimensionful and cut-off dependent quantities cancel, is, and we find, eliminating $p_\ell$, $\langle A \rangle_\ell / \langle R^2 \rangle_\ell \sim 3\pi (k(n)/n) / (dc/dn)\nu'(n)$

The remarkable thing about this equation is that it relates expectation values at finite $\ell$, that is, in the non-critical $O(n) \times O(n')$ model, to quantities defined at the conformal points.
Calculation of $k$, $c$ and $\nu'$.

It turns out that these universal quantities may be computed using Coulomb gas techniques. These are explained at length in the review article by Nienhuis[2] (see also [3]), so they will be only summarised here. We first rewrite the partition function (1) of the $O(n)$ model in a form where the Boltzmann weights are local. This may be done by assigning to each vertex of each oriented loop a factor $e^{i\chi}$ or $e^{-i\chi}$ according to whether the oriented walk turns to the left or the right as it passes through the vertex. In this way, each anticlockwise loop will accumulate a factor of $e^{6i\chi}$, and each clockwise loop $e^{-6i\chi}$. On summing over orientations, we may reproduce the factor $n$ for each loop if we choose $\chi$ such that $n = 2 \cos 6\chi$.

The next step is to write the loop gas as a solid-on-solid (SOS) model. Define a height variable $\phi(r)$, conventionally normalised so that $\phi(r)/\pi$ is an integer, at each site of the dual triangular lattice. There is 1-1 mapping between configurations of oriented loops and SOS heights as follows: assign $\phi = 0$ on the boundary, and increase (decrease) $\phi$ by $\pi$ each time a loop is crossed which goes to the left (right). The fact that all loops are closed makes this a consistent procedure. The resulting Boltzmann weight for each configuration is now a product of weights for each elementary triangle, each of which depends only on the differences of the $\phi$s at the vertices of the triangle. In the large distance limit, we now assume that this model renormalises onto one where the discreteness of the $\phi$s is irrelevant. This may be justified $a$ $posteriori$. Since the continuum limit must be a conformal field theory, depending on a single scalar field $\phi$, the simplest guess is the Gaussian model with action $S = (g/4\pi) \int (\partial \phi)^2 d^2r$. (Note that this is different from that introduced to give the free field representation of $J_\mu$.) The only difficult part of this argument is in assigning the value of $g$, which determines the various scaling dimensions. The argument is quite involved[2] and we quote only the result $g = 1 - (6\chi/\pi)$, which corresponds to $n = -2 \cos \pi g$. The correct branches are given by $1 \leq g \leq 2$, corresponding to $x = x_{c1}$, and $0 \leq g \leq 1$, corresponding to $x = x_{c2}$.

The free field theory is often called a Coulomb gas, because operators $e^{iq\phi}$ behave like charges of strength $q$: $\langle e^{iq\phi(r)} e^{-iq\phi(0)} \rangle \sim e^{-(q^2/g) \ln r}$. However, the SOS model is not completely equivalent to a free field theory, since if we calculate $\langle e^{-12i\chi\phi/\pi} \rangle$ in the SOS model we find that it is identically equal to 1. This may be seen to occur to all orders in the expansion in powers of $x$, the first few terms of which are

$$
\langle e^{-12i\chi\phi/\pi} \rangle = \frac{1 + e^{-12\chi} \cdot e^{6\chi} + e^{12\chi} \cdot e^{-6\chi} + \ldots}{1 + e^{6\chi} + e^{-6\chi} + \ldots}
$$
Since the only non-zero expectation values in the free field theory are those with total charge zero (as a consequence of the $U(1)$ symmetry $\phi \to \phi + \text{const.}$), there must be a charge $+12\chi/\pi$ distributed on the boundary. The only non-zero expectation values are those of products of operators with total charge $-12\chi/\pi$ in the interior.

How do we identify the current $J_\mu$ of the $O(n)$ model in the Coulomb gas language? The simplest guess is to take $J_\mu = (1/\pi)\epsilon_{\mu\nu}\Delta_\nu\phi$, where $\Delta_\nu$ is a lattice derivative, since this will ensure a unit current to the left or right according to whether $\phi$ steps up or down by $\pi$. However, this has the wrong charge, and in fact is not correct, as may be seen by computing its expectation value for a given loop, on summing over orientations:

$$\langle J \rangle = 1 \cdot e^{6i\chi} + (-1) \cdot e^{-6i\chi} \neq 0$$

Instead consider $\tilde{J}_\mu \propto \epsilon_{\mu\nu}\Delta_\nu (e^{-12i\chi\phi/\pi})$, which does have the correct total charge. Now

$$\langle \tilde{J} \rangle \propto (e^{-12i\chi} - 1)e^{6i\chi} + (e^{12i\chi} - 1)e^{-6i\chi} = 0$$

as required. However, the 2-point function cannot be $\langle \tilde{J}\tilde{J} \rangle$, since, once again, this has the wrong total charge. In fact, what works is $\langle \tilde{J}(r)J(r') \rangle$. When $r$ and $r'$ are on different loops this vanishes, because we get the same factor as in (8) on summing over the orientations of the loop passing through $r$. When they are on the same loop, we get something proportional to

$$1 \cdot (e^{-12i\chi} - 1) \cdot e^{6i\chi} + (-1) \cdot (e^{12i\chi} - 1) \cdot e^{-6i\chi} = -4i \sin 6\chi$$

which may be absorbed into the normalisation of $\tilde{J}$. This is an example of how the mapping to the Coulomb gas is not at the level of operators, but of correlation functions. It is now a simple matter to evaluate $\langle \tilde{J}(r)J(r') \rangle$ in the Gaussian model. It is of the required form (4), with

$$\frac{k(n)}{n} = \frac{1 - g}{2\pi^2 g \sin \pi g}$$

The central charge $c(n)$ of the $O(n)$ model is known by a variety of methods. A simple way is to note that it is related to the free energy of a system when placed in a box of linear size $L$:

$$F = -\ln Z \sim -(c\chi_E/6) \ln L$$

where $\chi_E$ is the Euler character. For the free field theory $c = 1$, but the charge on the boundary gives an additional contribution to the electrostatic energy. Choosing for convenience a disc with $\chi_E = 1$, we then find

$$\frac{c}{6} = 1 - \frac{(6\chi)^2}{g} \quad \text{or} \quad c = 1 - \frac{6(g - 1)^2}{g}$$
The final ingredient is $\nu'$, related to the scaling dimension $X'$ of the perturbation $\sum_{n+1}^{n+n'} s_a(r)s_a(r')$. In the context of spin systems, this is a symmetry-breaking perturbation inducing a crossover to lower spin dimensionality, and its analysis is somewhat subtle. It is first necessary to decompose the perturbation into irreducible representations of $O(n+n')$, by writing it as

$$\sum_{n+1}^{n+n'} \left( s_a(r)s_a(r') - \frac{1}{n+n'} \delta_{aa} E(r) \right) + \frac{n'}{n+n'} E(r)$$

where $E(r) = \sum_{1}^{n+n'} s_a(r)s_a(r')$ is the energy density of the $O(n+n')$ model. The second term contributes merely to a shift in the critical point $x_c$ when we turn on the perturbation, and is not important. In the first term it is permissible to set $r' = r$ (since the terms have the same symmetry) and we notice that each term in the sum transforms according to the representation of traceless rank 2 tensors. The scaling dimension is therefore independent of $n'$, and we may use Nienhuis’ result\[2\] for the uniaxial case $n' = 1$, obtained by Coulomb gas methods. The result is $\nu' = 2g/(1 + 2g)$. Finally, putting all the pieces together, we find

$$\langle A \rangle_{\ell} \langle R^2 \rangle_{\ell} \sim \frac{1 + 2g}{2(1 + g)} \pi$$

This result tells us about the compactness of the loops. If all loops were circular, the ratio would be $\pi$. Compact loops would have a ratio larger than this. In fact, we see for all $g$ that this ratio lies between $\frac{1}{2} \pi$ and $\pi$. For example, for Ising clusters above $T_c$ ($g = \frac{2}{3}$) we find $\frac{5}{16} \pi$, for Ising clusters at $T_c$ ($g = \frac{4}{3}$) the slightly greater ratio $\frac{11}{14} \pi$, and for single self-avoiding loops\[3\] ($g = \frac{3}{2}$), the value $\frac{4}{5} \pi \approx 2.513$. This latter value has been checked numerically by extrapolating the results of exact enumerations for finite $\ell$. The result is $2.515\ldots \[4\]$, which agrees well with the analytic result.
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