The quadric ansatz for the \( mn \)-dispersionless KP equation, and supersymmetric Einstein–Weyl spaces

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Abstract

We consider two multi-dimensional generalisations of the dispersionless Kadomtsev–Petviashvili (dKP) equation, both allowing for arbitrary dimensionality, and non-linearity. For one of these generalisations, we characterise all solutions which are constant on a central quadric. The quadric ansatz leads to a second order ODE which is equivalent to Painlevé I or II for the dKP equation, but fails to pass the Painlevé test in higher dimensions. The second generalisation of the dKP equation leads to a class of Einstein–Weyl (EW) structures in an arbitrary dimension, which is characterised by the existence of a weighted parallel vector field, together with further holonomy reduction. We construct and characterise an explicit new family of EW spaces belonging to this class, and depending on one arbitrary function of one variable.

Keywords: dispersionless KP, Einstein Weyl geometry, Painleve property

1. Introduction

Let \( u : U \rightarrow \mathbb{R} \), where \( U \) is an open set in \( \mathbb{R}^N \) with coordinates \( x^a = (x, y_1, \ldots, y_n, t) \). The \( mn \)-dispersionless Kadomtsev–Petviashvili (\( mn \)-dKP) equation [17, 20] is given by

\[
\Delta u_{tt} - (u^n u_t)_x = \Delta u,
\]

where \( \Delta = \partial^2/\partial y_1^2 + \cdots + \partial^2/\partial y_n^2 \) and \( u_t = \partial_t u \), etc. We refer the readers to [20] for a list of references and applications of equation (1.1) ranging from non-linear optics and acoustics to geometry for different values of integers \((m, n)\).
The aim of this paper is to construct solutions to (1.1) which are constant on central quadrics \( Q \subset \mathbb{R}^N \), i.e. there exists a symmetric \( N \times N \) matrix \( M = M(u) \) such that

\[
\sum_{a,b=1}^{N} M_{ab}(u) x^a x^b = C, \quad \text{where } C = \text{const}, \quad \text{and } a, b = 1, \ldots, N = n + 2. \tag{1.2}
\]

If \( m = n = 1 \), then (1.1) becomes the standard dispersionless Kadomtsev–Petviashvili (dKP) equation, and the ODE for the matrix \( M(u) \) resulting from the quadric ansatz (1.2) is either solvable by quadratures, or is equivalent to Painlevé I or Painlevé II (this last case being generic) [10]. In section 2 we shall perform the analysis of the quadric ansatz for (1.1). The non-generic linearisable case leads to some explicit forms of \( M(u) \), which we shall present in section 3.1. The other two cases lead to 2nd order ODEs generalising Painlevé I and II, which however fail to pass the Painlevé test (theorem 3.3). Thus, in line with the integrability dogma [1], we conclude that (1.1) is only integrable if \( m = n = 1 \) (the dKP case), or \( n = 0 \) (the Riemann equation solvable by the method of characteristics), or \( m = 0 \) (the linear wave equation). It is known [9] that the dKP equation characterises a class of Einstein–Weyl (EW) metrics in \( 2+1 \) dimensions which admit a parallel weighted vector field. While this correspondence does not extend to (1.1) for general \((m, n)\), in section 4 we shall demonstrate that a closely related equation

\[
u_{xt} - (u u_x)_x + \frac{2(n-1)}{n} u_x^2 = \Delta u, \tag{1.3}
\]

classifies a class of EW structures admitting a parallel null weighted vector field, and with a further assumption on the holonomy of the Weyl connection. Our main result in section 4 is theorem 4.1 characterising an explicit new family of EW spaces in arbitrary dimension.

2. The quadric ansatz

The quadric ansatz (1.2) is applicable to equations of the form

\[
\sum_{a,b=1}^{N} \frac{\partial}{\partial x^a} \left( b^{ab}(u) \frac{\partial u}{\partial x^b} \right) = 0, \tag{2.1}
\]

where \( b(u) \) is a symmetric \( N \times N \) matrix with components \( b^{ab}(u) \). While the ansatz can be traced back to some works of Darboux [4], in the ‘modern’ times it has been applied to a class of dispersionless, as well as linear PDEs: the \( SU(\infty) \)-Toda equation [21], the dKP equation [10], the Laplace equation [8], as well as a general class of equations integrable by the method of hydrodynamic reductions [13]. The idea is to find and solve an ODE for \( M(u) \) (and keep in mind that this ODE is not a symmetry reduction of the underlying PDE, and the corresponding solutions in general do not admit any Lie-point, or generalised symmetries). The ODE arises as follows: differentiating (1.2) implicitly w.r.t. \( x^a \) and substituting the resulting expression for \( \partial u / \partial x^a \) into (2.1) yields a matrix ODE

\[
g M' = M b M, \quad \text{where } ' = \frac{d}{du}
\]
and the function \( g = g(u) \) is defined by \( 2g' = \text{Tr}(bM) \). Setting \( N = -M^{-1} \), this matrix ODE simplifies to

\[
gN' = b, \tag{2.2}
\]

and one finds that

\[
g^2 \det N = \zeta, \tag{2.3}
\]

where \( \zeta \) is an arbitrary constant.

From now, we shall regard the system (2.2) and (2.3) as the reduction of (2.1) under the quadric ansatz (1.2). Solving this system for the components of \( N \) gives the matrix \( M \), and thus leads to an implicit solution to the nonlinear PDE (2.1) of the form (1.2), which is therefore constant on a central quadric.

The \( mn \)-dKP equation (1.1) is of the form (2.1), with

\[
b(u) = \begin{pmatrix} -u^n & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & & & & \\ \vdots & & -I_n & & \\ 0 & & & & 0 \\ \frac{1}{2} & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.4}
\]

where \( I_n \) denotes the \( n \times n \) identity matrix. Equation (2.2) then implies that

\[
N = \begin{pmatrix} Y' & \beta_1 & \cdots & \beta_n & Z \\ \beta_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \beta_n & \vdots & \ddots & \ddots & \varepsilon_1 \\ Z & \varepsilon_1 & \cdots & \varepsilon_n & \phi \end{pmatrix}, \tag{2.5}
\]

where \( \beta_i, \varepsilon_i, i = 1, \ldots, n \), and \( \phi \) are constants, \( Y \) and \( Z \) are functions of \( u \), and \( X \) is an \( n \times n \) symmetric matrix, whose diagonal components \( X_1, X_2, \ldots, X_n \) are functions of \( u \) and off-diagonal components \( \alpha_{ij}, i > j \) are all constants. Equation (2.2) also implies that

\[
Y' = -g^{-1}u^n, \quad X_i' = -g^{-1}u^n, \quad Z' = (2g)^{-1}.
\]

The last equation gives \( g \) in terms of \( Z' \), so that

\[
Y' + 2u^nZ' = 0, \tag{2.6}
\]

and

\[
X(u) = -2Z(u)1_n + X_0, \tag{2.7}
\]
where \(X_0\) is a constant matrix with diagonal components \(\gamma_i\) and off-diagonal components \(\alpha_{ij}\). The equation (2.3) becomes
\[
\det N = 4\zeta (Z')^2, \tag{2.8}
\]
and, together with the form (2.5) gives an algebraic expression for \(Y\) in terms of \(Z, Z'\) and constants. Substituting this expression into (2.6) yields a second order scalar ODE for \(Z\) of the form
\[
Z'' = A(Z)(Z')^2 + B(Z) + C(Z)u^m, \tag{2.9}
\]
where \(A, B\) and \(C\) are rational functions of \(Z\), of degrees respectively \(n, 2n + 1\) and \(n\), which will be determined in section 3.

If this equation can be solved, then reversing the steps above we can reconstruct the matrix \(M(u)\), and thus find an explicit solution to (1.1) constant on a central quadric. All solutions in the class (1.2) arise from this construction.

2.1. Eliminating the constants

To make further progress we exploit some symmetries of the \(mn\)-dKP equation (1.1) to eliminate some of the constants in (2.5). If \(m = 1\), then the transformation
\[
x^a \rightarrow \hat{x}^a = A^a_{\ b} x^b, \quad \text{where} \quad A = \begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_n & k \\ 1 & 0 & \cdots & 0 & 2c_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & \ddots & \vdots \\ 0 & & 1 & \ddots & 2c_n \\ 0 & & 0 & \cdots & 1 \end{pmatrix} \tag{2.10}
\]
together with
\[
u \rightarrow \hat{u} = u + c_1^2 + c_2^2 + \cdots + c_n^2 - k, \tag{2.11}
\]
where \(c_i, i = 1, \ldots, n\), and \(k\) are constants is a symmetry of (1.1), which also preserves the quadric ansatz (1.2) with a replacement
\[
M(u) \rightarrow \hat{M}(\hat{u}) = (A^{-1})^T M(u) A^{-1}, \quad \text{or} \quad \hat{N}(\hat{u}) = A N(u) A^T. \tag{2.12}
\]
Using (2.12) we can set some of the constants \((\epsilon, \beta, \alpha, \gamma)\) to 0. There are three cases to consider depending on whether \(\phi\) vanishes, or not.

3. Solutions

3.1. Case I

Assuming \(\phi = \epsilon_i = 0\) for \(i = 1, \ldots, n\) leads to the quadric ansatz equations which are solvable by quadrature,
\[
u = 2 \int \left(\frac{-\zeta}{Z^2 \det X}\right)^{\frac{1}{2}} dZ, \quad Y = -2 \int u^m \frac{dZ}{du} du. \tag{3.1}
\]
To derive these formulae note that in this case \( \det(N) \) is independent on \( Y \). Therefore (3.1) arises from

\[
4\zeta (Z')^2 = -Z^2 \, \det(X) 
\] (3.2)

which is (2.8), where \( \det(X) \) is a polynomial of degree \( n \) in \( Z \). If \( n = 1 \) or \( 2 \), then the corresponding expression for \( u \) is given by elementary functions. Two examples of such solutions are given below:

### 3.1.1. \((m, n) = (2, 1)\)

\[
\frac{1}{\tan^2 u} \left[ \frac{y^2}{4} + (2 \sin^2 u \cos u (\cos u \ln(\cos u) + u \sin u) - u^2 \sin^2 u \\
+ \delta^2 \cos^4 u )^2 - \delta(\cos^2 u)t - (\sin^2 u)xt \right] = C, \quad \text{where } \delta = \text{const.} \tag{3.3}
\]

### 3.1.2. \((m, n) = (1, 2)\)

\[
(4a e^{au} + 1) \left[ 4a^3 (y_1^2 + y_2^2) + 4 e^{-au} (a^2 xt - a^2 \ln(e^{-au} + 4a)) \\
- e^{-2au} (au + \ln(e^{-au} + 4a)) \right] = C, \tag{3.4}
\]

where \( a > 0 \) is a constant.

### 3.1.3. \(m = 1\)

There is also a simple class of solutions with \( m = 1 \), and arbitrary \( n \). To find it suppose that all constants in the matrices \( N \) and \( X_0 \) are zero. Then equation (3.2) with \( \zeta = (-1)^{n+1} 2^{n-2} \) becomes \( (Z')^2 = Z^{n+2} \), which after integrating (3.1) gives

\[
u \left( 4u(\ln u)t^2 - 4xt + y_1^2 + y_2^2 \right) = C, \quad n = 2, \\
u^{2/n} \left( \frac{8}{n-2} u^2 - 4xt + \sum_{i=1}^{n} y_i^2 \right) = C, \quad n \neq 2.
\]

### 3.2. Case II

We shall now assume that the constant \( \phi \) in (2.5) is zero, but at least one of the \( \varepsilon \)'s is non-zero. If \( m = 1 \), the symmetry (2.10) can be used to eliminate all \( \beta \)s and \( \gamma_1 \). Then (2.8) with \( R_d(Z) \equiv \det(X) \) gives

\[
4 \zeta (Z')^2 = Y Q_{n-1}(Z) - Z^2 R_n(Z), \quad \text{where } \quad Q_{n-1}(Z) \equiv \det \begin{pmatrix} X \\ \varepsilon_1 \\ \vdots \\ \varepsilon_n \\ 0 \end{pmatrix}. \tag{3.5}
\]
Differentiating (3.5) with respect to \( u \) and substituting \( Y' = -2uZ' \) gives a second order ODE for \( Z \) of the form (2.9) where \( m = 1 \), and

\[
A(Z) = \frac{1}{2} \frac{d}{dZ} \ln(Q_{n-1}), \quad B(Z) = \frac{1}{8\zeta} \left( Z^2 R_n \frac{d}{dZ} \ln(Q_{n-1}) - \frac{d}{dZ}(Z^2 R_n) \right), \quad C(Z) = -\frac{Q_{n-1}}{4\zeta}.
\]

**Proposition 3.1.** Let \( \phi = 0 \), and at least one of the \( \varepsilon, \delta \) is non-zero in (2.5). If \((m, n) = (1, 1)\), the equation (2.9) is equivalent to Painlevé I. For \( m = 1 \), and \( n > 1 \) equation (2.9) does not posses the Painlevé property.

**Proof.** For an ODE to have the Painlevé property, its movable singularities can only be poles. Thus we follow the algorithm in [1] to determine whether the general solution of (2.9) admits a movable branch point. For convenience we first conduct the Painlevé test in the special case, where the constant matrix \( X_0 \) is zero, so that \( X = -2ZI_n \), and (with the definition \( \varepsilon^2 \equiv \varepsilon_1^2 + \cdots + \varepsilon_n^2 \))

\[
2\zeta Z'' = \left( \frac{n-1}{Z} \right) (Z')^2 - 3(-2)^{n-2}Z^{n+1} - \varepsilon^2(-2)^{n-2}Z^{n-1} u.
\]

Assume that the dominant behaviour of a solution near a movable singularity \( u_0 \) is of the form

\[
Z \approx a(u - u_0)^p,
\]

where \( a \) and \( p \) are constants. Then substitute (3.7) into (3.6) and balance the power of \( u - u_0 \) of two or more terms. If the balancing terms are dominant, i.e. their power of \( u - u_0 \) is most negative, then other terms can be ignored, and one can solve for \( a \). For equation (3.6), it turns out that the only possible value of \( p \) is \( p = -\frac{2}{\varepsilon} \), which is not an integer for \( n > 2 \) and suggests a movable algebraic branch point. Moreover, this result extends to the general case (2.9). This is because the assumption that leads to the special case (3.6) keeps only the highest degree terms in \( Q_{n-1} \) and \( R_n \). The presence of the lower degree terms in the rational functions \( A(Z), B(Z) \) and \( C(Z) \) will not change the possible dominant behaviour in the first step of the Painlevé test. Therefore we conclude that equation (2.9) does not posses the Painlevé property for \( n > 2 \).

If \( n = 1 \), then (3.6) and (2.9) are equivalent, after constant rescalings of dependent and independent variables, to the Painlevé I equation.

If \( n = 2 \), with \( \beta \) and \( \gamma_1 \) eliminated by the symmetry for \( m = 1 \), then (2.9) becomes

\[
4\zeta Z'' = \frac{4\zeta}{2Z+\delta} (Z')^2 + \frac{4Z^4 - 2\gamma Z^3 - \alpha^2 Z^2}{2Z+\delta} - 8\gamma Z^2 + \alpha^2 Z - \varepsilon^2 Z^2(2Z+\delta) u,
\]

where we let \( \varepsilon^2 \equiv \varepsilon_1^2 + \varepsilon_2^2, \gamma \equiv \gamma_2 \) and \( \delta \equiv \frac{2\varepsilon_1 \varepsilon_2 - \gamma_2^2}{\varepsilon_1^2 + \varepsilon_2^2} \). Here \( \alpha \) is the off-diagonal component of the matrix \( X_0 \). Substituting (3.7) in (3.8), the only possibility is \((p, a) = (-1, -\zeta/\sqrt{2})\). Let \(-\zeta = \kappa^2\), and take \( a = \kappa \). The remaining steps in the algorithm will determine whether the general solution of (3.8) can be represented near a movable singular point \( u_0 \) by the Laurent series, with the leading term \( \frac{\kappa}{u-u_0} \). It turns out that to satisfy (3.8) one needs to introduce a logarithmic term, which gives

\[
Z(u) \approx \kappa \frac{\kappa}{(u-u_0)} + \gamma \frac{-32\varepsilon^2 u_0 - 3\gamma^2 - 8\alpha^2}{192\kappa}(u-u_0) + \left( \varepsilon + \frac{\varepsilon^2}{8\kappa} \ln(u-u_0) \right) (u-u_0)^2 + O((u-u_0)^3).
\]
where $c$ is an arbitrary constant. The logarithmic term indicates a logarithmic branch point, and this shows that (3.8) does not posses the Painlevé property. Hence we conclude that equation (2.9) does not posses the Painlevé property for $m = 1, n > 1$.

3.3. Case III

This is the generic case, where we assume that $\phi \neq 0$ in (2.5). If $m = n = 1$, then the ODE resulting from the quadric ansatz reduces to Painlevé II [10]. For general $n$, and $m = 1$ the symmetry (2.10) can be used to eliminate $\gamma_1$ and all $\varepsilon$s. Equation (2.8) takes the form

$$4\zeta \left(\zeta^2\right)^2 = (\phi y - Z^2)R_n(Z) + Q_{n-1}(Z),$$

(3.9)

where $R_n(Z) = \det X$, and here $Q_{n-1}(Z) = -\phi \sum_{i=1}^{n} \beta_i \det B_i$ with $B_i$ denoting the matrices obtained from replacing the $i$th column of $X$ (2.7) by the column vector $(\beta_1, \ldots, \beta_n)^T$. Solving this for $Y$ in terms of $Z$ and $Z'$, and differentiating to eliminate $Y'$ by $Y' = -2uZ'$ gives (2.9) where $m = 1$, and

$$A(Z) = \frac{1}{2} \frac{d}{dZ} (\ln R_n), \quad B(Z) = \frac{1}{8\zeta} \left( \frac{dQ_{n-1}}{dZ} - \frac{d}{dZ} (\ln R_n) - 2ZR_n \right), \quad C(Z) = -\frac{\phi}{4\zeta} R_n.$$

**Proposition 3.2.** Let $\phi \neq 0$ in (2.5). If $(m, n) = (1, 1)$, the equation (2.9) is equivalent to Painlevé II. For $m = 1$ and $n > 1$ equation (2.9) does not posses the Painlevé property.

**Proof.** The result is obtained by first performing the Painlevé test on the special case case where $X = -2ZI_n$ (i.e. assuming $X_0 = 0$ in (2.7)) and (2.9) is

$$2\zeta Z'' = \frac{n\zeta}{Z} \left(\zeta^2\right)^2 + (-2)^{n-1} \left( Z^{n+1} + \frac{\phi \beta^2}{4} Z^{n-2} \right) + (-2)^{n-1} \phi Z'u,$$

(3.10)

where $\beta^2 \equiv \beta_1^2 + \cdots + \beta_n^2$. When $n = 1$, then, after a coordinate transformation [10] this family of ODEs (3.10) gives the Painlevé II equation. For $n > 2$, the ODE (3.10) fails the test at the first step of finding the dominant behaviour of the general solution, where it displays the dominant term of the form $a(u - u_0)^p$ with $p = -\frac{2}{n}$. For $n > 2$, this indicates an algebraic branch point of order $-\frac{2}{n}$, hence (3.10) does not have the Painlevé property. Then we argue that this result extends to the general form (2.9) as (2.9) differs from (3.10) only by the lower degree terms in the polynomials appearing in the rational functions $A(Z), B(Z)$ and $C(Z)$, and these will not affect the dominant behaviour analysis.

For $n = 2$, the form of (2.9) is still quite complicated by the presence of the constants $\alpha$ and $\gamma = \gamma_2$ in the matrix $X$. After a translational change of variable $Z \rightarrow Z = Z - \gamma/4$, and then dropping the hat, (2.9) becomes

$$\zeta Z'' = \frac{\zeta Z}{Z^2 - \rho^2} \left(\zeta^2\right)^2 - \left( Z + \frac{\gamma}{4} \right) (Z^2 - \rho^2) - \frac{\phi \beta^2}{4} (2Z + \delta) \frac{Z}{Z^2 - \rho^2} + \frac{\phi \beta^2}{4} - \phi (Z^2 - \rho^2)u,$$

(3.11)
where $\beta^2 = \beta_1^2 + \beta_2^2$, $\delta = \frac{4\alpha_1 \beta_2 + \beta_1 \beta_2 - \beta_1^2}{\beta_1^2}$ and $\rho^2 = \frac{4\alpha_2 \varepsilon_2}{16}$.

The Painlevé test then shows that the general solution of (3.11) has a logarithmic branch point

$$Z(u) \approx \frac{\kappa}{(u - u_0)} - \frac{\phi u_0}{2} + \frac{\gamma}{8} + \left( e + \frac{\phi}{3} \ln(u - u_0) \right) (u - u_0) + O(u - u_0),$$

where $\kappa = \sqrt{-c}$ and $c$ and $u_0$ are arbitrary constants. Hence we conclude that (2.9) does not have the Painlevé property for $m = 1$, $n = 2$.

If $m > 1$, there is no obvious symmetry to eliminate the constants in $N$. Nevertheless, the Painlevé analysis shows that the case $m = n = 1$ is the only case that the quadric ansatz reduction possesses the Painlevé property.

**Theorem 3.3.** The quadric ansatz reduction of the mn-dKP equation does not possess the Painlevé property unless $m = n = 1$.

**Proof.** The quadric ansatz reduction is of the form (2.9), where $A(Z)$ and $B(Z)$ are rational functions of degrees respectively $n$ and $2n + 1$, and $C(Z)$ is a polynomial of degree $n$. The non-zero constants in $N$ for $m \geq 1$ only contribute to the lower degree terms in the polynomials appearing in $A(Z)$, $B(Z)$, and $C(Z)$ and thus will not change the dominant behaviour of a solution near a movable singularity $u_0$. Also, the term $C(Z)u^m$ is not leading for any $m$. Therefore, from the proofs of propositions 3.1 and 3.2 we conclude that for $n > 2$, and any $m \geq 1$ the general solution has a movable algebraic branch point of order $-\frac{2}{n}$.

It remains to settle the case where $n = 1$ or 2, and $m > 1$. Performing the Painlevé test in these cases we find that the general solution exhibits a logarithmic branch point. In particular, we have the following form of the general solution:

**Case I.** $\phi = 0$ and at least one of the $\varepsilon_i$,s is non-zero in (2.5).

$n = 1$:

$$Z(u) \approx \frac{8\zeta}{(u - u_0)^2} + \frac{\gamma}{6} - \frac{12\varepsilon_1(a_0^m \varepsilon_1 + \beta) - \gamma^2}{480\zeta} (u - u_0)^2$$

$$- \frac{m_0^{m-1} \varepsilon_2^2}{24\zeta} (u - u_0)^3 + \left( e + \frac{m(m-1)u_0^{-2} \varepsilon_2^2}{56\zeta} \ln(u - u_0) \right)$$

$$\times (u - u_0)^4 + O((u - u_0)^4)$$

$n = 2$:

$$Z(u) \approx \frac{\kappa}{(u - u_0)} + \frac{\gamma_1 + \gamma_2}{8}(u - u_0)$$

$$+ \frac{32(a_0^m (\varepsilon_1^2 + \varepsilon_2^2) + \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2) - 8\alpha^2 - 3(\gamma_1^2 + \gamma_2^2) + 2\gamma_1 \gamma_2}{192\kappa} (u - u_0)$$

$$+ \left( e + \frac{m_0^{m-1} (\varepsilon_1^2 + \varepsilon_2^2)}{8\kappa} \ln(u - u_0) \right) (u - u_0)^2 + O((u - u_0)^2).$$
Case III. $\phi \neq 0$ in (2.5).

$n = 1$:

$$Z(u) \approx \frac{8\zeta}{(u - u_0)^2} - \frac{2}{3} u_0^m \phi + \frac{\gamma}{6} - mu_0^{m-1} \phi (u - u_0)$$

$$+ \left( c + \frac{2m(m-1)u_0^{m-2} \phi}{5} \ln(u - u_0) \right) (u - u_0)^2 + O((u - u_0)^3)$$

$n = 2$:

$$Z(u) \approx \frac{\kappa}{(u - u_0)^2} - \frac{u_0^m \phi}{2} + \frac{\gamma_1 + \gamma_2}{8}$$

$$+ \left( c + \frac{mu_0^{m-1} \phi}{3} \ln(u - u_0) \right) (u - u_0) + O(u - u_0).$$

Here, $\kappa = \sqrt{-\zeta}$, $c$ is an arbitrary constant, and $\gamma \equiv \gamma_1$ for $n = 1$.

4. Einstein–Weyl geometry

There is a, by now well-established, link between 2 + 1 dimensional EW geometry, and dispersionless integrable systems [2, 7, 9, 11, 14, 22]. In particular, the Manakov–Santini equation [16] is known to be the general local normal form of the EW equations [12].

In this section we construct a Weyl structure in an arbitrary dimension $N = n + 2$, such that the EW condition reduces to a single dispersionless PDE. In the case when $n = 1$ this PDE is the dKP equation, and the EW structure is that of [9], and for $n > 1$ the PDE is (1.3). The main result (theorem 4.1) is an explicit class of EW spaces depending on one arbitrary function of one variable.

Recall [19] that a Weyl structure on a manifold $U$ (which is really just an open set in $\mathbb{R}^N$, as our considerations are local) consists of a conformal structure $[h]$ represented by a metric $h$, and a torsion-free connection $D$ which is compatible with $[h]$ in the sense that $Dh = \nu \otimes h$ for some one-form $\nu$. This compatibility is invariant under the conformal change of metric:

$$h \rightarrow \Omega^2 h, \quad \nu \rightarrow \nu + 2d(\ln \Omega), \quad (4.1)$$

where $\Omega : U \rightarrow \mathbb{R}^+$. A Weyl structure is said to be non-closed if $d\nu \neq 0$, or equivalently if $D$ is not a Levi-Civita connection of any metric in the class $[h]$. The EW equations hold if the symmetrised Ricci tensor of $D$ is proportional to some metric $h$ in $[h]$. The EW equations can be regarded as a system of PDEs for the representative metric $h$ and the associated one-form $\nu$:

$$\chi_{ab} \equiv R_{ab} + \frac{N - 2}{2} \nabla_d (\nu^b) + \frac{N - 2}{4} \nu_d \nu_b - \frac{1}{N} h_{ab} \left( R + \frac{N - 2}{4} \nabla_c \nu^c + \frac{N - 4}{4} \nu^c \nu^c \right) = 0, \quad (4.2)$$

where $\nabla, R_{ab}$ and $R$ are respectively the Levi-Civita connection, the Ricci tensor, and the Ricci scalar of $h$. A tensor $V$ is said to be of weight $k$, if $V \rightarrow \Omega^k V$ under the conformal rescaling.
(4.1). If \( V \) is a vector field of weight \( k \), then the weighted covariant derivative of \( V \) with respect to the Weyl connection is given by
\[
\widetilde{D}_a V^b = \nabla_a V^b - \frac{1}{2} \nu_c V^c \delta_a^b - \frac{1}{2} (k + 1) \nu_a V^b + \frac{1}{2} V_a \nu^b.
\]

The \( mn \)-dKP equation (1.1) can be written in the form \( d \star du = 0 \), where the Hodge endomorphism \( \star : \Lambda^1 \to \Lambda^{n+1} \) corresponds to the metric (note that the inverse metric corresponds to the matrix \( b(u) \) in (2.1))
\[
h = dy_1^2 + \cdots + dy_n^2 - 4dx^2 dt - 4u^2 dr^2.
\]

In the case \( m = n = 1 \) there exists a one-form \( \nu = -4u \, dt \) such that the EW condition reduces to the dKP equation [9]. It turns out that there is no one-form which, together with the metric (4.4) gives the \( mn \)-dKP equation if \( n > 1 \). We shall instead take the metric (4.4) with \( m = 1 \) (which can always be achieved by re-defining the function \( u \)) as a starting point. It can then be verified by an explicit computation of (4.2) that for the Weyl structure represented by
\[
h = dy_1^2 + \cdots + dy_n^2 - 4dx^2 dt - 4u^2 dr^2, \quad \nu = -\frac{4}{n} u \, dr
\]
with \( \chi^0 = t, \chi^i = y_i, i = 1, \ldots, n, \) and \( \chi^{n+1} = x \), all components of \( \chi_{ab} \) except \( \chi_{00} \) vanish identically. The resulting EW equation \( \chi_{00} = 0 \) is a scalar PDE (1.3)
\[
u_{x_\ell} - (uu_x)_x + \frac{2(n - 1)}{n} u_x^2 = \Delta u.
\]
Moreover, the vector field \( V = \partial/\partial x \) is null, and covariantly constant with weight \( -\frac{n}{2} \) with respect to \( D \). equation (1.3) is the dKP equation if \( n = 1 \), or its generalisation [5, 18] if \( n > 1 \).

This class of EW structures falls into a larger class of solutions which admit a parallel weighted spinor [5]. The particular case (4.5) corresponds to example 4 in this reference. To understand a coordinate invariant characterisation of (4.5), assume that an \( (n + 2) \)-dimensional Weyl space represented by a pair \((h, \nu)\) admits a covariantly constant null vector field with weight \(-n/2\). Then the one-form \( V = h(V, \cdot) \) dual to \( V \) satisfies
\[
dV = \frac{4 - n}{4} \nu \wedge V.
\]
The Frobenius theorem implies that there exist functions \((\nu, t)\) on \( U \) so that \( V = \nu \, dt \). We shall use \( t \) as one of the local coordinates on \( U \). The existence of canonical (up to some freedom) remaining \((n + 1)\) coordinates \((y_1, \ldots, y_n, x)\) is also guaranteed by the Frobenius theorem: the distribution \( V \) of null curves is spanned by \( V = \partial/\partial x \), and its integrable orthogonal complement \( V^\perp \) is spanned by \( \{\partial/\partial y_1, \partial/\partial y_2, \ldots, \partial/\partial y_n\} \).

If \( n \neq 4 \) (so that the \( \dim(U) \neq 6 \)), the we can rescale the metric so that \( V = -2 \, dt \) and
\[
h = f^{ij} dy_i dy_j - 4dx^2 dt + 2A^i dy_i dt - 4u^2 dr^2, \quad \nu = b \, dt,
\]
where the functions \( f^{ij}, A^i, b \) and \( u \) at this stage depend on all coordinates. Going back to (4.3) with \( V = 0/\partial x \) and \( k = -n/2 \), and considering its symmetrised part shows that \( f^{ij} \) and \( A^i \) do not depend on \( x \). Moreover a coordinate transformation \( y_i \to \tilde{y}_i(y_i, t) \) together with a redefinition of \( f^{ij}(y, t) \) and \( u(x, y, t) \) can be used to set \( A^i = 0 \). The parallel weighted condition on \( V \) also implies that \( b = -(4/n)u_x \).
The final step reducing the functions \( f^{ij} \) to the identity \( n \times n \) matrix is achieved in [5] by considering the connection induced by \( D \) on the screen bundle (see [15])

\[
S \equiv \mathcal{V}^\perp / \mathcal{V} \subset TU,
\]

and restricting its holonomy to \( \mathbb{R} \otimes \text{Id} \) (equivalently the \( \mathfrak{so}(n) \) projection of the holonomy algebra of this connection is zero). Now the metric and the one-form are given by (4.5), and the EW equations reduce to \(^3\) (1.3).

To this end we shall construct an explicit subclass of examples of (4.5) and (1.3) under the additional assumption that the screen bundle distribution \( \mathcal{V}^\perp / \mathcal{V} \) generates an isometric action of \( \mathbb{R}^n \) or \( T^n \) on the EW space. This will be done by linearising (1.3) by a contact transformation.

**Theorem 4.1.** Let \((h, \nu)\) be an \((n+2)\)-dimensional EW structure \( U \) which admits a parallel weighted null vector field \( V \) with weight \( -\frac{n}{2} \), and such that

- The connection on the screen bundle \( S \) defined by (4.7) induced by \( D \) has holonomy \( \mathbb{R} \otimes \text{Id} \).
- The sections of the screen bundle \( S \) generate the isometric action of the group of translations \( \mathbb{R}^n \) on \( U \).

Then there exist local coordinates \((t, y_i, s)\) such that the one-form \( V \equiv h(V, \cdot) = -2 dt \), the isometric action is generated by \( \{\partial / \partial y_1, \ldots, \partial / \partial y_n\} \) and the EW structure is given by

\[
h = dy_1^2 + \cdots + dy_n^2 + \frac{4G(s)}{(t-s)^{n-2}} ds \, dt, \quad \nu = -\frac{4}{(n-2)(t-s)} dt \quad \text{if } n \neq 2
\]

\[
h = dy_1^2 + dy_2^2 + 4G(s)e^{-st}ds \, dt, \quad \nu = -2s dt \quad \text{if } n = 2,
\]

where \( G = G(s) \) is an arbitrary function of one variable.

**Proof.** The existence of the parallel weighed null vector field, and the holonomy reduction in theorem 4.1 implies—as explained above—that the metric, and the one-form take the local normal form (4.5). The additional symmetry assumption in the theorem then implies that \( u = u(x, t) \), and the EW condition (1.3) becomes

\[
u_{xt} - uu_{sx} + \kappa u_x^2 = 0, \quad \text{where } \kappa = \frac{n-2}{n}.
\]

Rewrite (4.9) as a differential ideal

\[
\omega_1 \equiv du - u_t \, dx - u_x \, dt = 0,
\]

\[
\omega_2 \equiv du_x \wedge dx + u \, du_x \wedge dt + \kappa u_x^2 \, dt \wedge dx = 0
\]

and set

\[
H = u - xu_x, \quad p = u_x.
\]

\(^3\) In the special case \( n = 4 \), we start-off with the metric (4.5), and the general one-form \( \nu \) and impose the weighted parallel condition on \( V = \partial / \partial x \) to reduce the one-form \( \nu \) to (4.5).
Rewriting $\omega_1$ (4.10) as
\[ dH = u_t \, dt - x \, du_t 
= H_t \, dt + H_p \, dp \]
gives
\[ x = -H_p, \quad u_t = H_t, \quad u = H - pH_p, \]  
(4.11)
where now $H = H(p,t)$. Substituting (4.11) in $\omega_2$ in (4.10) gives
\[ H_{pt} + pH_p - H - \kappa p^2 H_{pp} = 0. \]  
(4.12)
The corresponding EW structure (4.5) takes the form
\[ h = dy_1^2 + \cdots + dy_n^2 + 4F(dp \, dt + \kappa p^2 \, dt^2), \]
\[ \nu = -\frac{4}{n} \rho \, dt, \quad \text{where } F \equiv H_{pp}. \]  
(4.13)
This only depends on the second derivatives of the function $H$, so the function $F$ is constrained by one PDE obtained from differentiating (4.12) with respect to $p$:
\[ F_t + (1 - 2k)pF - \kappa p^2 F_p = 0. \]  
(4.14)
This PDE can be solved explicitly, and the form of the general solution depends on $n = 2/(1 - k)$:
\[ F = \begin{cases} 
  p^{-\frac{n}{n-1}} G \left( t - \frac{n}{(n-2)p} \right) & \text{if } n \neq 2, \\
  G(p) e^{-p}, & \text{if } n = 2,
\end{cases} \]  
(4.15)
where in both cases $G$ is an arbitrary function of one variable. Introducing a new variable $s$ by
\[ s = \begin{cases} 
  t - \frac{n}{(n-2)p} & \text{if } n \neq 2, \\
  p & \text{if } n = 2,
\end{cases} \]
absorbing the overall constant into the arbitrary function $G$, and adopting $(y_i, t, s)$ as local coordinates on $\mathcal{U}$ yields (4.8).

5. Conclusions

We have demonstrated that solutions to the $mn$-dKP equation (1.1) constant on central quadrics are characterised by solutions to a 2nd order scalar ODE. In the generic case this ODE is of Painlevé type if $m = n = 1$, but does not posses the Painlevé property if $m \cdot n > 1$. This rules out the integrability of (1.1) for these values of $m, n$. There are other approaches to dispersionless integrability of (1.1) discussed in [20], and in particular Kruglikov informed us that the approach taken in references [3, 14] could also be used to rule out non-integrable cases, and perhaps narrow them down to $m = n = 1$. 

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Equation (1.1) with \( n = 1 \) has been studied numerically in [6], where an asymptotic description of a gradient catastrophe in generalised KP equation was conjectured, and related to special solutions of Painlevé I. In [20] the analytical approach to this shock formation has been presented. It would be interesting to understand whether our explicit solutions shed more light on these shock formations.

In section 4 we have related another multi-dimensional generalisation of the dKP equation (1.3) to a class of Lorentzian EW structures. This class admits a parallel weighted null vector, and thus a parallel weighted spinor which makes it interesting in both physics (supersymmetric solutions to EW equations [18]), and geometry, where the existence of such spinor corresponds to a holonomy reduction of the Weyl connection [5]. In theorem 4.1 an explicit subclass of these supersymmetric solutions depending on one arbitrary function has been constructed, and invariantly characterised.

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Data availability statement

No new data were created or analysed in this study.

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References

[1] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P-type. J. Math. Phys. 21 715–21
[2] Calderbank D M J 2014 Integrable background geometries Symmetry, Integrability Geometry Methods Appl. 10 034
[3] Calderbank D M J and Kruglikov B 2021 Integrability via geometry: dispersionless differential equations in three and four dimensions Commun. Math. Phys. 382 1811–41
[4] Darboux G 1910 Lecons sur les systmes orthogonaux et les coordonnes curvilignes (Paris: Gauthiers-Villars)
[5] Dikarev A and Galaev A S 2021 Parallel spinors on Lorentzian Weyl spaces Monatsh. Math. 196 39–58
[6] Dubrovin B, Grava T and Klein C 2016 On critical behaviour in generalized Kadomtsev–Petviashvili equations Nonlinearity 29 1384–416
[7] Dunajski M 2009 Solitons, Instantons and Twistors (Oxford Graduate Texts in Mathematics) vol 19 (Oxford: Oxford University Press)
[8] Dunajski M 2003 Harmonic functions, central quadrics, and twistor theory Class. Quantum Grav. 20 3427–40
[9] Dunajski M, Mason I J and Tod P 2001 Einstein–Weyl geometry, the dKP equation and twistor theory J. Geom. Phys. 37 63–93
[10] Dunajski M and Tod P 2002 Einstein–Weyl spaces and dispersionless Kadomtsev–Petviashvili equation from Painlevé I and II Phys. Lett. A 303 253–64
[11] Dunajski M and Kręski W 2014 Einstein–Weyl geometry, dispersionless Hirota equation and Veronese webs Math. Proc. Camb. Phil. Soc. 157 139–50
[12] Dunajski M, Ferapontov E and Kruglikov B 2015 On the Einstein–Weyl and conformal self-duality equations J. Math. Phys. 56 083501
[13] Ferapontov E V, Huard B and Zhang A 2012 On the central quadric ansatz: integrable models and Painlevé reductions J. Phys. A: Math. Theor. 45 195204
[14] Ferapontov E and Kruglikov B 2014 Dispersionless integrable systems in 3D and Einstein–Weyl geometry J. Differ. Geom. 97 215–54
[15] Leistner T 2006 Screen bundles of Lorentzian manifolds and some generalisations of pp-waves J. Geom. Phys. 56 2117–34
[16] Manakov S V and Santini P M 2006 The Cauchy problem on the plane for the dispersionless Kadomtsev–Petviashvili equation JETP Lett. 83 462–6
[17] Manakov S V and Santini P M 2011 On the dispersionless Kadomtsev–Petviashvili equation in $n + 1$ dimensions: exact solutions, the Cauchy problem for small initial data and wave breaking J. Phys. A: Math. Theor. 44 405203
[18] Meessen P, Ortín T and Palomo-Lozano A 2012 On supersymmetric Einstein–Weyl spaces J. Geom. Phys. 62 301
[19] Pedersen H and Tod K P 1993 Three-dimensional Einstein–Weyl geometry Adv. Math. 97 74–109
[20] Santucci F and Santini P M 2016 On the dispersionless Kadomtsev–Petviashvili equation with arbitrary nonlinearity and dimensionality: exact solutions, longtime asymptotics of the Cauchy problem, wave breaking and shocks J. Phys. A: Math. Theor. 49 405203
[21] Tod K P 1995 Scalar-flat Kähler and hyper-Kähler metrics from Painlevé-III Class. Quantum Grav. 12 1535–47
[22] Ward R S 1990 Einstein–Weyl spaces and $SU(\infty)$ Toda fields Class. Quantum Grav. 7 L95–8