Two Uniqueness Results on the Unruh Effect and on PCT-symmetry

Bernd Kuckert
Mathematics Institute, Pl. Muidergracht 24
1018 TV Amsterdam, The Netherlands
e-mail:kuckert@science.uva.nl

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Abstract

The Unruh effect and a closely related form of PCT-symmetry have been proved in general for finite-component Wightman fields by Bisognano and Wichmann. While this result incorporates most of the fields occurring in four-dimensional high energy physics, there still are field theories of interest that are not covered (e.g., low-dimensional anyon fields and infinite-component fields). From the spectrum condition, Borchers has derived a couple of commutation relations which “almost, but only almost” imply the Unruh effect and PCT-symmetry. We show that this result does imply Unruh effect and PCT-symmetry provided that the operators involved in Borchers’ commutation relations act geometrically on a local net of observables.

1 Introduction

Unruh’s observation in the seventies that the vacuum state of a free quantum field appears as a temperature state when looked at in a uniformly accelerated frame [49] has been proved for all finite-component Wightman fields by Bisognano and Wichmann, who addressed the problem around the same time and with a more mathematical motivation [7, 8]. Starting from the field operators located in the Rindler wedge, a generic algebraic construction yields a strongly continuous one-parameter group of unitary operators on the one hand and an antiunitary operator on the other. Since these objects arise from the modular theory due to Tomita and Takesaki, they are referred to as the modular unitaries and the modular conjugation of the Rindler wedge,
respectively. The modular unitaries turned out to implement Lorentz boosts, and the modular conjugations give rise to a PCT-operator.

On the other hand, the automorphism group implemented by the modular unitaries, the modular group, exhibits a property that characterizes thermodynamical equilibrium states, the so-called KMS-condition (see, e.g., [32]), so the Bisognano-Wichmann result shows that the vacuum state is a thermodynamical equilibrium state not only in an inertial frame, but also in a uniformly accelerated frame, as Unruh found independently for the free field.

While the result of Bisognano and Wichmann is quite general, there still are field theories of physical interest to which their arguments do not apply, e.g., anyons in 1+2 dimensions or infinite-component Wightman fields in any dimension. What is more, there are examples of theories that do not exhibit the Unruh effect [53, 21], so the long term goal is to find a criterion that tells the theories showing the Unruh effect apart from those without this property. An important application of the Bisognano-Wichmann symmetries in low dimensions, which motivated the subsequent analysis, are a couple of recent proofs of Pauli’s spin-statistics connection and its extension to particles with braid group statistics [30, 31, 36, 41, 43]. As this approach does not depend on the special features of finite-component Wightman fields, it also provides a spin-statistics connection for 1+2-dimensional theories without spinor structure. On the other hand, the original Bisognano-Wichmann analysis is confined to finite-component Wightman fields, whereas the symmetries they established are not confined to these fields; so the further analysis of the Bisognano-Wichmann symmetries is of interest on its own.

During the last decade, several authors have been investigating the Unruh effect and PCT-symmetry in the algebraic approach to relativistic quantum physics [1]. This activity has been initiated by Borchers’ proof of two commutation relations which (essentially) imply the Unruh effect for algebraic theories in 1+1 spacetime dimensions, and in higher dimensions lead to the impression that there is not much room left for theories violating the effect.

This impression has motivated our subsequent analysis. The question is what can “go wrong” if the modular unitaries or conjugation are known to implement a “geometric action” in some way to be made precise.

A first approach to this question specifies this “geometric action” by assuming that the adjoint actions of the corresponding operators map the

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1 See Sect. 4 below for some examples, see [5, 32] for textbooks on the algebraic approach.
algebra of observables associated with a double cone region $\mathcal{O} \subset \mathbb{R}^{1+s}$ onto
the algebra associated with some arbitrary open set $M_\mathcal{O}$. This open set
does, a priori, not need to be the image of the double cone $\mathcal{O}$ under a func-
tion from $\mathbb{R}^{1+s}$ to $\mathbb{R}^{1+s}$. $M_\mathcal{O}$ may be bounded or unbounded, and it may
be connected or disconnected. It has been shown in [37] that the modular
conjugation implements a PCT-symmetry provided that it implements
any geometric action in this sense. If the modular unitaries of the Rindler
wedge act geometrically on the net, they implement the boosts leaving the
wedge invariant plus a possible translation along the edge of the wedge. It is
shown below that this translation degree of freedom can be eliminated by
an application of the Borchers-Vladimirov double cone theorem on analytic
functions in several complex variables. That the double cone theorem can
be used here, was brought to my attention by S. Trebels, whose thesis con-
tains similar results [48]. With this completion, the result provides a first
uniqueness theorem on modular symmetries.

A similar trick allowed the Unruh part of the problem to be treated
in an alternative way: if the modular unitaries of the Rindler wedge map
local observables onto local observables and let their localization regions
change in a continuous way, these regions have to transform as under a boost
which leaves the Rindler wedge invariant. This will be referred to as the
second uniqueness theorem on modular symmetries. In order to investigate
this question, the notion of a localization region of a single local observable
had to be made rigorous. This has been done in [38, 39].

This article is structured as follows: in Sect. 2, the results are stated in
precise terms and compared with each other. The proofs follow in Sect. 3.
Some further discussion also concerning related recent work is given in the
Conclusion.

2 Preliminaries and Results

In what follows, $\mathcal{A}$ denotes a local net of observables that associates a uni-
tal $C^*$-algebra $\mathcal{A}(\mathcal{O})$ in an infinite-dimensional Hilbert space $\mathcal{H}$ with each
bounded open region $\mathcal{O} \subset \mathbb{R}^{1+s}$, where $\mathbb{R}^{1+s}$ denotes a Minkowski spacetime
with at least two spatial dimensions. $\mathcal{A}$ will be assumed to be isotonous,
i.e., $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(P)$ if $\mathcal{O} \subset P$, and to satisfy locality, i.e., if $\mathcal{O}$ and $P$ are
spacelike separated open regions, all elements of $\mathcal{A}(\mathcal{O})$ commute with all
elements of $\mathcal{A}(P)$. For any unbounded open region $R$, one defines $\mathcal{A}(R)$ to
be the $C^*$-algebra generated by the union of all $\mathcal{A}(\mathcal{O})$ for bounded open sets
$\mathcal{O} \subset R$. The elements of the union $\mathcal{A}_{\text{loc}}$ of all $\mathcal{A}(\mathcal{O})$ associated with bounded
open regions $\mathcal{O}$ are called *local observables*.

Throughout this paper, the net $\mathcal{A}$ will be assumed to exhibit the following properties:

**(A) Translation covariance.** There is a strongly continuous unitary representation $U$ of the translation group $(\mathbb{R}^{1+s}, +)$ such that for every bounded open region $\mathcal{O} \subset \mathbb{R}^{1+s}$, one has

$$U(a)\mathcal{A}(\mathcal{O})U(-a) = \mathcal{A}(\mathcal{O} + a) \quad \text{for all } a \in \mathbb{R}^{1+s}.$$

**(B) Spectrum condition.** The (four-dimensional) spectrum of the four-momentum operator generating the representation $U$ is a subset of the closed forward light cone $\mathbf{V}^+$. 

**(C) Existence and uniqueness of the vacuum.** The space of $U$-invariant vectors is one-dimensional. $\Omega$ will denote an arbitrary, but fixed unit vector in this space, the *vacuum vector*. $\Omega$ is cyclic with respect to the algebra $\mathcal{A}(\mathbb{R}^{1+s})$.

**(D) Reeh-Schlieder property.** For every nonempty bounded open region $\mathcal{O}$, the vector $\Omega$ is cyclic with respect to the algebra $\mathcal{A}(\mathcal{O})$.

Conditions (A) and (B) ensure that there is a well-defined four-momentum operator whose energy spectrum is bounded from below, which makes the system energetically stable. Conditions (A) through (C) are characteristic for vacuum states in high energy physics.

Given Conditions (A) and (B), Condition (C) is equivalent to irreducibility of the algebra $\mathcal{A}(\mathbb{R}^{1+s})$. A necessary and sufficient condition for this is that the bicommutant $\mathcal{A}(\mathbb{R}^{1+s})''$ of this algebra is a factor (Theorem III.3.2.6 in [32]), and if $\mathcal{H}$ is separable, this can be assumed without any loss of generality, since there always is a direct-integral decomposition of $\mathcal{A}_{\text{loc}}''$ into factors almost all of which inherit the properties assumed so far (cf. also the remarks in [32], Sect. III.3.2, and references given there).

The Reeh-Schlieder property (Condition (D)) has been established for all Wightman fields [44], and if, in the present setting, Conditions (A) through (C) hold, Condition (D) is well known to hold if and only if one has *weak additivity*, i.e., if for each bounded region $\mathcal{O}$, one has

$$\left( \bigcup_{a \in \mathbb{R}^{1+s}} \mathcal{A}(\mathcal{O} + a) \right)'' = \mathcal{A}_{\text{loc}}''.$$
For a proof that weak additivity is sufficient, see [10], or Thm. 7.3.37 in [6]; a simple proof that it is sufficient as well, can be found, e.g., in [47, 39].

Since the vacuum vector is cyclic and, by locality, also separating with respect to the von Neumann algebra \( \mathcal{A}(W_1)'' \) of the Rindler wedge \( W_1 := \{ x \in \mathbb{R}^{1+}\times 0: x_1 > x_0 \} \), the map \( \mathcal{A}\Omega \mapsto \mathcal{A}^*\Omega \), \( A \in \mathcal{A}(W_1)'' \) defines an antilinear operator \( S_{W_1} : \mathcal{A}(W_1)''\Omega \to \mathcal{A}(W_1)''\Omega \) which is closable. Its closure is called the Tomita operator of \( \Omega \) and \( \mathcal{A}(W_1)'' \) and admits a unique polar decomposition \( S_{W_1} = J_{W_1} \Delta_{W_1}^{1/2} \) into an antilinear conjugation \( J_{W_1} \) (the “phase” of \( S_{W_1} \)) which is called the modular conjugation of \((\Omega, \mathcal{A}(W_1)''\))
and a positive operator \( \Delta_{W_1}^{1/2} \) (the “modulus” of \( S_{W_1} \)) whose square \( \Delta_{W_1} \) is referred to as the modular operator of \((\Omega, \mathcal{A}(W_1)''\)).

The main theorem of Tomita-Takesaki theory [46] now implies that the adjoint actions of the operators \( \Delta_{W_1}^{it} \) map the algebras \( \mathcal{A}(W_1)'' \) and \( \mathcal{A}(W_1)' \) onto themselves, whereas the adjoint action of the conjugation \( J_{W_1} \) maps the two algebras onto one another. Bisognano and Wichmann showed that for finite-component Wightman fields, the unitary \( \Delta_{W_1}^{it} \) coincides with the unitary representing the 01-boost by \(-2\pi t\) for all \( t \in \mathbb{R} \), whereas \( J_{W_1} \) implements a charge conjugation together with a time reflection and a spatial reflection in the 1-direction, this combination of discrete transformations will be referred to as a \( P_1CT\)-symmetry.

For the algebraic setting, Borchers proved in [11] that the spectrum condition (without assuming Lorentz covariance) implies the commutation relations

\[
\begin{align*}
(i) \quad & J_{W_1} U(a) J_{W_1} = U(j_1 a), \\
(ii) \quad & \Delta_{W_1}^{it} U(a) \Delta_{W_1}^{-it} = U(\Lambda_1(-2\pi t)a) \quad \text{for all } t \in \mathbb{R},
\end{align*}
\]

where \( \Lambda_1(-2\pi t) \) denotes the Lorentz boost by \(-2\pi t\) in the 1-direction, while \( j_1 \) is the reflection defined by

\[
 j_1 x := (-x_0, -x_1, x_2, \ldots, x_s).
\]

Wiesbrock noted that Borchers’ relations are not only a necessary, but also a sufficient condition for the spectrum condition ([52], cf. also [27]). For 1+1 dimensions, Borchers’ relations immediately imply [11] that the net of observables may be enlarged to a local net which generates the same wedge algebras (and hence the same corresponding modular operator and

\[^2\text{For a considerably simpler proof found recently, see [28].}\]
conjugation) as the original one and which has the property that \( J \) is a
\( P_1 \)CT-operator \((modular \ P_1 \ CT\text{-symmetry})\), whereas \( \Delta \) implements the
Lorentz boost by \(-2\pi t\) for each \( t \in \mathbb{R} \) \((modular \ Lorentz \ symmetry)\).

The first uniqueness theorem for modular symmetries states that even in
higher dimensions, \( J \) or \( \Delta \) can be shown to be a \( P_1 \)CT-operator
or a \( 0-1 \)-Lorentz boost, respectively, provided that \( J \) or \( \Delta \) implement
any geometric action on the net. The first step towards it is the following
lemma. In this lemma and in what follows, \( K \) will denote the class of all
double cones of the form \( \mathcal{O} := (a + V) \cap (b - V), a, b \in \mathbb{R}^{1+s} \).

**Lemma 2.1** Let \( K \) be a unitary or antiunitary operator with the property
that for every double cone \( \mathcal{O} \) there are open sets \( M_\mathcal{O} \) and \( N_\mathcal{O} \) such that
\[
KA(\mathcal{O})K^* = A(M_\mathcal{O}), \quad K^*A(\mathcal{O})K = A(N_\mathcal{O}),
\]
and let \( \kappa \) be a causal automorphism\(^3\) of \( \mathbb{R}^{1+s} \) such that
\[
KU(a)K^* = U(\kappa a) \quad \text{for all } a \in \mathbb{R}^{1+s}.
\]
Then there is a unique \( \xi \in \mathbb{R}^{1+s} \) such that
\[
KA(\mathcal{O})K^* = A(\kappa \mathcal{O} + \xi), \quad \text{for all } \mathcal{O} \in K.
\]

A first proof of Lemma 2.1 was published in [37], but both the statement
and the proof given there were more general, which made the formulation
somewhat technical. For the reader’s convenience a less general, but more
accessible formulation is used here, and a more detailed version of the proof
is given below.

The following theorem is a consequence of Lemma 2.1 and Borchers’
commutation relations.

**Theorem 2.2 (First Uniqueness Theorem)** (i) If for every double cone
\( \mathcal{O} \in K \) there is an open set \( M_\mathcal{O} \) such that
\[
J_{W_1}A(\mathcal{O})J_{W_1} = A(M_\mathcal{O}),
\]

\(^3\)Recall that a causal automorphism of \( \mathbb{R}^{1+s} \) is a bijection \( f : \mathbb{R}^{1+s} \to \mathbb{R}^{1+s} \) which
preserves the causal structure of \( \mathbb{R}^{1+s} \), i.e., \( f(x) \) and \( f(y) \) are timelike with respect to each
other if and only if \( x \) and \( y \) are timelike with respect to each other. Without assuming
linearity or continuity, one can show that the group of all causal automorphisms of \( \mathbb{R}^{1+s} \) is
generated by the elements of the Poincaré group and the dilatations \([1, 3, 2, 54, 15]\). Since
the transformations implemented on the translations by Borchers’ commutation relations
happen to be causal in all applications discussed below, this assumption means no loss of
generality.

6
then

\[ J_{W_1} A(O) J_{W_1} = A(j_1 O) \quad \text{for all } O \in \mathcal{K}. \]

(ii) If for every \( t \in \mathbb{R} \) and for every \( O \in \mathcal{K} \) there is an open set \( M^t_O \) such that

\[ \Delta_{it}^{W_1} A(O) \Delta_{it}^{-W_1} = A(M^t_O), \]

then

\[ \Delta_{it}^{W_1} A(O) \Delta_{it}^{-W_1} = A(\Lambda_1(-2\pi t) O) \quad \text{for all } O \in \mathcal{K}. \]

The statement of part (ii) implies the statement of part (i) \[30\], i.e., the Unruh effect implies modular P\(_1\)CT-symmetry. Further results relating the above statements to each other and to similar conditions can be found in \[26\].

Assuming that \( \Omega \) is separating with respect to the algebra \( A(V_+) \), Borchers also found commutation relations for the corresponding modular conjugation and unitaries: for each \( a \in \mathbb{R}^{1+s} \), he found that

\[
J_+ U(a) J_+ = U(-a); \\
\Delta_+^{it} U(a) \Delta_+^{-it} = U(e^{-2\pi t} a) \quad \text{for all } t \in \mathbb{R}.
\]

These relations, together with Lemma \[7\], imply the following corollary:

**Corollary 2.3 (Uniqueness Theorem “1a”)** Assume \( A \) to be Poincaré covariant, and assume that the vacuum vector \( \Omega \) is separating with respect to the algebra \( A(V_+)^\prime \prime \), and let \( \Delta_{V_+}^{it} \) and \( J_{V_+} \) be the corresponding modular operator and conjugation, respectively.

(i) If for every double cone \( O \) there is an open set \( M_O \) such that

\[ J_{V_+} A(O) J_{V_+} = A(\mathcal{M}_O), \]

then

\[ J_{V_+} A(O) J_{V_+} = A(-O) \quad \text{for all } O \in \mathcal{K}. \]

(ii) If for every \( t \in \mathbb{R} \) and every double cone \( O \) there is an open set \( M^t_O \) such that

\[ \Delta_{V_+}^{it} A(O) \Delta_{V_+}^{-it} = A(M^t_O), \]

then

\[ \Delta_{V_+}^{it} A(O) \Delta_{V_+}^{-it} = A(e^{-2\pi t} O) \quad \text{for all } O \in \mathcal{K}. \]
Since massive theories cannot be dilation invariant unless their mass spectrum is dilation invariant (cf., e.g., \cite{42}), the models concerned by part (ii) of this corollary are massless theories. But it follows from the scattering theory for massless fermions and bosons in 1+3 or 1+1 dimensions (see \cite{17,18,19}) that either of the symmetry properties found in part (i) and part (ii) of the corollary implies a massless theory to be free (i.e., its S-matrix is trivial) (see \cite{18,20,23}).

Note that for the 1+1-dimensional case, all modular symmetries considered in Thm. 2.2 and Cor. 2.3 have been established in \cite{11}.

It is assumed above that the adjoint actions of $J_{W_1}$ and $\Delta_{W_1}^t$, $t \in \mathbb{R}$, map each local algebra $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, onto the algebra $\mathcal{A}(M_\mathcal{O})$ associated with some open region $M_\mathcal{O}$ in Minkowski space. This means that, essentially, the net structure has to be preserved. This is the restrictive aspect of the assumption. On the other hand, the shape of the region $M_\mathcal{O}$ is left completely arbitrary, the map $\mathcal{K} \ni \mathcal{O} \mapsto M_\mathcal{O}$ is not even assumed to be induced by a point transformation. In this aspect, the above assumptions are rather weak.

But there are, of course, other ways to specify what a “geometric action” is. Denote by $\mathcal{W}$ the class of all wedges, i.e., all images of the Rindler wedge $W_1$ under Poincaré transformations. For $M \subset \mathbb{R}^{1+s}$, define the causal complement $M^c$ to be the set of all points that are spacelike to $M$, and let $M'$ denote the interior of $M^c$. It has been shown in \cite{38,39} that one can define a nonempty localization region for each local observable $A \notin \mathcal{C}_{\text{id}}$ by

$$L(A) := \bigcap \{\mathcal{W} : \mathcal{W} \in \mathcal{W}, A \in \mathcal{A}(\mathcal{W}')\}.$$  

This localization prescription will be said to satisfy \textit{locality} if any two local observables $A$ and $B$ with the property that $L(A)$ and $L(B)$ are spacelike separated commute. This property does not follow from the locality property of the net alone, but with the following additional assumptions one can derive it for the present setting \cite{39}:

\begin{itemize}
  \item[(E)] \textbf{Wedge duality.} $\mathcal{A}(\mathcal{W}')' = \mathcal{A}(\mathcal{W})''$ for each wedge $W \in \mathcal{W}$.
  \item[(F)] \textbf{Wedge additivity.} For each wedge $W \in \mathcal{W}$ and each double cone $\mathcal{O} \in \mathcal{K}$ with $W \subset W + \mathcal{O}$ one has
    $$\mathcal{A}(W)'' \subset \left( \bigcup_{a \in W} \mathcal{A}(a + \mathcal{O}) \right)'' .$$
\end{itemize}

Wedge duality is a property of all finite-component Wightman fields by the Bisognano-Wichmann theorem, and wedge additivity is a standard property
of Wightman fields as well. Condition (F) is slightly stronger than the definition of wedge additivity used in [17, 39], where the algebras $A(a + O)$ in Condition (F) are replaced by the larger algebras $A(a + O')$, but as this difference is not expected to be substantial for physics, we use the same term for convenience, which is in harmony with the other existing notions of additivity used in algebraic quantum field theory.

Assume now that the localization region of the observable $A_t := \Delta_{W_1}^t A \Delta_{W_1}^{-t}$ depends continuously on $t$, i.e., that for every sequence $(t_\nu)_{\nu \in \mathbb{N}}$ which converges to some $t_\infty \in \mathbb{R}$, the localization region $L(A_{t_\infty})$ consists precisely of all accumulation points of sequences $(x_\nu)_{\nu \in \mathbb{N}}$ with $x_\nu \in L(A_{t_\nu})$.

Then the following lemma establishes a first restriction on how the localization region can depend on $t$.

**Lemma 2.4** With Assumptions (A) – (E), suppose the localization prescription $L$ defined above satisfies locality. Let $A$ be a local observable in $A(W_1)$, and assume that there exists an $\varepsilon > 0$ such that all $A_t$, $t \in [0, \varepsilon]$, are local observables and such that the function $[0, \varepsilon] \ni t \mapsto L(A_t)$ is continuous in the above sense. Then

(i) $L(A_\varepsilon) \subset \Lambda_1(-2\pi \varepsilon) \left( (L(A) + W_1)^{cc} \cap (L(A) - W_1)^{cc} \right)$;

(ii) $L(A_\varepsilon) \subset L(A) - V$;

(iii) $L(A) \subset L(A_\varepsilon) + V$.

It is shown in the Appendix that the continuity assumption made on $t \mapsto L(A_t)$ is equivalent to continuity with respect to a metric first considered by Hausdorff, and that $\bigcup_{t \in [0, \varepsilon]} L(A_t)$ is compact.

Next suppose that $t \mapsto L(A_t)$ is continuous not only for sufficiently small $t$, but for all $t \in \mathbb{R}$, and assume wedge additivity in addition. With these slightly strengthened assumptions one can now prove the following:

**Theorem 2.5 (Second Uniqueness Theorem)** With Assumptions (A) – (F), assume that $\Delta_{W_1}^t A_{loc} \Delta_{W_1}^{-t} = A_{loc}$, and suppose that $L(A_t)$ depends continuously from $t$ for all $t \in \mathbb{R}$ and for all $A \in A_{loc}$. Then

$L(\Delta_{W_1}^t A \Delta_{W_1}^{-t}) = \Lambda_1(-2\pi t)L(A)$ for all $A \in A_{loc}$.

By the result of Guido and Longo, the conclusion of this proposition also implies modular $P_1\text{CT}$-symmetry, but Proposition 2.5 does not provide a proper parallel to the $P_1\text{CT}$-part of the first uniqueness theorem, which may also apply if the modular group does not act in any geometric way.
The assumption that every local observable \( A \) is mapped onto some other local observable under the adjoint action of the modular group prevents \( A \) to be mapped onto an observable localized in an unbounded region. For every bounded open region \( \mathcal{O} \) there are conformal transformations which map \( \mathcal{O} \) onto an unbounded region; these transformations are excluded a priori. In contrast, the assumptions of the first uniqueness theorem do not exclude these symmetries explicitly, while it is evident from this theorem that the modular objects under consideration cannot implement these symmetries.

Another restrictive assumption of the second uniqueness theorem is that wedge duality is assumed there, whereas the first one can be used to derive wedge duality. On the other hand the assumptions made in the second uniqueness theorem admit the situation that the net structure of \( A \) is destroyed completely under the action of the modular group.

3 Proofs

For every algebra \( \mathcal{M} \subset \mathcal{B}(\mathcal{H}) \), define its localization region \( L(\mathcal{M}) \) with respect to the net \( \mathcal{A} \) by

\[
L(\mathcal{M}) := \{ \mathcal{O} \in \mathcal{K} : \mathcal{A}(\mathcal{O}) \subset \mathcal{M} \}.
\]

The only reason to use the class \( \mathcal{K} \) of double cones in this definition is convenience; one could replace \( \mathcal{K} \) by the larger class \( \mathcal{T} \) of all open sets in \( \mathbb{R}^{1+s} \) without affecting the definition. To see this, denote the localization region obtained this way by \( L_T(\mathcal{M}) \); it is trivial that \( L(\mathcal{M}) \subset L_T(\mathcal{M}) \) as \( \mathcal{K} \subset \mathcal{T} \), while from isotony of the net and the fact that each open region \( M \) is the union of all double cones \( \mathcal{O} \subset M \), one finds

\[
L_T(\mathcal{M}) = \bigcup \{ M \in \mathcal{T} : \mathcal{A}(M) \subset \mathcal{M} \}
= \bigcup \{ \mathcal{O} \in \mathcal{K} : \exists M \in \mathcal{T} : \mathcal{O} \subset M, \mathcal{A}(M) \subset \mathcal{M} \}
\subset \bigcup \{ \mathcal{O} \in \mathcal{K} : \mathcal{A}(\mathcal{O}) \subset \mathcal{M} \} = L(\mathcal{M}),
\]

which is the converse inclusion.

It is obvious from the definitions that \( L(\mathcal{A}(M)) \supset M \). For causally complete and convex regions one can prove the converse inclusion, which we recall without proof from [39] (Cor. 5.4) for later use. Here a causally complete region is a region \( \mathcal{R} \) such that \( (\mathcal{R}^c)^c = \mathcal{R} \).

**Lemma 3.1** Let \( \mathcal{R} \subset \mathbb{R}^{1+s} \) be a causally complete convex open region.
For every open region $M \subset \mathbb{R}^{1+s}$, one has $\mathcal{A}(M) \subset \mathcal{A}(R')'$ if and only if $M \subset R$.

(ii) $L(\mathcal{A}(R)) = R$.

One also checks that for any such $R$, one has $L(\mathcal{A}(R)) = L(\mathcal{A}(R)''') = L(\mathcal{A}(R)'')$. We emphasize that the above assumption $s \geq 2$ is crucial for this lemma; in 1+1 dimensions, there are chiral theories which do not obey the statement of the lemma. The repeated use of this lemma in the proofs is the main reason why $s \geq 2$ is assumed throughout this paper.

Proof of Lemma 2.1

In what follows, $K$ and $\kappa$ are defined as in Lemma 2.1. As before, $K$ will denote the class of double cones. For any open region $M \subset \mathbb{R}^{1+s}$, we denote by $\mathcal{K}^M$ the class of all double cones $O \in \mathcal{K}$ with $O \subset M$, and for each subalgebra $M$ of $\mathcal{B}(H)$, we denote by $\mathcal{K}^M$ the class of all double cones $O$ such that $A(O) \subset M$.

The proof will be subdivided into five lemmas. The first implies that for every $O \in \mathcal{K}$, the regions $M_O$ and $N_O$ are bounded. It uses the fact that a region $M$ is bounded if and only if its difference region $M - M$ is bounded, and that difference sets can be expressed in terms of translations. Since the behaviour of translations under the action of the symmetry $K$ is known by assumption, one can prove the following lemma.

Lemma 3.2 For every double cone $O \in \mathcal{K}$, one has

$L(\mathcal{K}A(O)K^*) = L(\mathcal{K}A(O)K^*) = \kappa(O - O)$.

Proof. Using the assumptions of Theorem 2.1, one obtains

$L(\mathcal{K}A(O)K^*) - L(\mathcal{K}A(O)K^*) = L(A(M_O)) - L(A(M_O))$

$= \{ a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^A(M_O) : A(P + a) \subset A(M_O) \}$

$= \{ a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^A(M_O) : KU(\kappa^{-1}a)K^*A(P)KU(-\kappa^{-1}a)K^* \subset A(M_O) \}$

$= \kappa \{ a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^A(M_O) : U(a)K^*A(P)KU(a) \subset K^*A(M_O)K \}$

$\subset \kappa \{ a \in \mathbb{R}^{1+s} : \exists P \in \mathcal{K}^A(M_O) : \exists Q \in \mathcal{K}^A(N_P) : A(Q + a) \subset A(O) \}$.

Since the definitions and isotony imply

$\mathcal{K}^A(N_P) = \mathcal{K}^A(P)K \subset \mathcal{K}^K \mathcal{A}(M_O)K = \mathcal{K}^A(O)$,
and since, as remarked above, $K^{A(O)} = K^O$, one obtains
\[
L(KA(O)K^*) - L(KA(O)K^*) \subset \kappa \{a \in R_1^{1+s} : \exists Q \in K^O : A(Q + a) \subset A(O)\} = \kappa(O - O).
\]
Conversely,
\[
\kappa(O - O) = \kappa \{a \in R_1^{1+s} : \exists P \in K^O : A(P + a) \subset A(O)\} = \{a \in R_1^{1+s} : \exists P \in K^O : A(P + \kappa^{-1}a) \subset A(O)\} = \{a \in R_1^{1+s} : \exists P \in K^O : A(M_P + a) \subset A(M_O)\} \subset \{a \in R_1^{1+s} : \exists P \in K^O : \exists Q \in K^{A(M_P)} : A(Q + a) \subset A(M_O)\},
\]
and since $K^{A(M_P)} = K^{KA(P)K^*} \subset K^{K^O}K^* = K^{A(M_O)}$, one obtains
\[
\kappa(O - O) \subset \{a \in R_1^{1+s} : \exists Q \in K^{A(M_O)} : A(Q + a) \subset A(M_O)\} = L(A(M_O)) - L(A(M_O)).
\]

The next lemma proves that strict inclusions of double cones are preserved under the adjoint action of the operator $K$. Again, this boils down to translating local algebras up and down Minkowski space and using the commutation relations between $K$ and the translation operators. One uses the fact that $O \subset P$ if and only if $O$ can be translated within $P$ into all directions.

**Lemma 3.3** For any two double cones $O, P \in K$ with $O \subset P$, one has
\[
L(KA(O)K^*) \subset L(KA(P)K^*).
\]

**Proof.** $O \subset P$ if and only if the set $\{a \in R_1^{1+s} : O + a \subset P\}$ is a neighbourhood of the origin of $R_1^{1+s}$. After using Lemma 3.1, elementary transformations yield
\[
\{a \in R_1^{1+s} : O + a \subset P\} = \{a \in R_1^{1+s} : A(O + a) \subset A(P)\} = \{a \in R_1^{1+s} : K^*U(\kappa a)K^*U(-\kappa a)K \subset A(P)\} = \{a \in R_1^{1+s} : A(M_O + \kappa a) \subset A(M_P)\} = \kappa^{-1}\{a \in R_1^{1+s} : A(M_O + a) \subset A(M_P)\} \subset \kappa^{-1}\{a \in R_1^{1+s} : L(A(M_O)) + a \subset L(A(M_P))\}.
\]

12
Since $\kappa$ is a linear automorphism of $\mathbb{R}^{1+s}$, it follows that $\mathcal{O}$ can be a subset of $P$ only if
\[
\{ a \in \mathbb{R}^{1+s} : L(A(M_\mathcal{O})) + a \subset L(A(M_P)) \}
\]
is a neighbourhood of the origin. This implies the statement. \(\square\)

The next lemma proves that the maps
\[
\mathcal{K} \ni K \mapsto L(KA(O)K^*)
\]
and
\[
\mathcal{K} \ni O \mapsto L(K^*A(O)K)
\]
are induced by continuous functions $\tilde{\kappa} : \mathbb{R}^{1+s} \rightarrow \mathbb{R}^{1+s}$ and $\hat{\kappa} : \mathbb{R}^{1+s} \rightarrow \mathbb{R}^{1+s}$.

**Lemma 3.4** Let $x \in \mathbb{R}^{1+s}$ be arbitrary, and let $(O_\nu)_{\nu \in \mathbb{N}}$ be a neighbourhood base of $x$ consisting of double cones $O_\nu \in \mathcal{K}$.

Then $(L(KA(O_\nu)K^*))_{\nu \in \mathbb{N}}$ is a neighbourhood base of a (naturally, unique) point $\tilde{\kappa}(x) \in \mathbb{R}^{1+s}$, and $(L(K^*A(O_\nu)K))_{\nu \in \mathbb{N}}$ is a neighbourhood base of a point $\hat{\kappa}(x) \in \mathbb{R}^{1+s}$. The functions $x \mapsto \tilde{\kappa}(x)$ and $x \mapsto \hat{\kappa}(x)$ are continuous.

**Proof.** Without loss of generality, one may assume that $O_{\nu+1} \subset O_\nu$ for all $\nu \in \mathbb{N}$. It follows from $L(A(O)) = O^*$ for all $O \in \mathcal{K}$ and Lemma 3.2 that all $L(KA(O_\nu)K^*)$, $\nu \in \mathbb{N}$, are bounded sets, and it follows from Lemma 3.3 that
\[
L(KA(O_{\nu+1})K^*) \subset L(KA(O_\nu)K^*).
\]
Therefore, the intersection of this family is nonempty, and Lemma 3.2 implies that the diameter of $L(KA(O_\nu)K^*)$ tends to zero as $\nu$ tends to infinity. This implies that the intersection contains precisely one point $\tilde{\kappa}(x)$, as stated. The corresponding statements for $K^*$ are proved analogously.

This proves that $x \mapsto \tilde{\kappa}(x)$ is a bijective point transformation. Let $(x_\nu)_{\nu \in \mathbb{N}}$ be a sequence in $\mathbb{R}^{1+s}$ that converges to a point $x_\infty$. Then there is a neighbourhood base $(O_\nu)_{\nu \in \mathbb{N}}$ of $x_\infty$ with $x_\nu \in O_\nu$ for all $\nu \in \mathbb{N}$. But since $\tilde{\kappa}(x_\nu) \in \tilde{\kappa}(O_\nu)$ for all $\nu \in \mathbb{N}$, and since $\tilde{\kappa}(O_\nu)$ is a neighbourhood base of $\tilde{\kappa}(x_\infty)$, it follows that $\tilde{\kappa}(x_\nu)$ tends to $\tilde{\kappa}(x_\infty)$ as $\nu \rightarrow \infty$. This line of argument applies to $\hat{\kappa}$ as well. \(\square\)

The next lemma determines the functions $\tilde{\kappa}$ and $\hat{\kappa}$ up to a constant translation.
Lemma 3.5 For every $x \in \mathbb{R}^{1+s}$, one has

$$\tilde{\kappa}(x) = \tilde{\kappa}(0) + \kappa x,$$

and

$$\hat{\kappa}(x) = \hat{\kappa}(0) + \kappa^{-1} x.$$

Proof. Let $(O_\nu)_{\nu \in \mathbb{N}}$ be a neighbourhood base of $o$. Then $(O_\nu + x)_{\nu \in \mathbb{N}}$ is a neighbourhood base of $x$, and

$$\bigcap_{\nu \in \mathbb{N}} L(KA(O_\nu + x)K^*) = \bigcap_{\nu \in \mathbb{N}} \tilde{\kappa}(O_\nu + x) = \{\tilde{\kappa}(x)\}.$$

On the other hand,

$$\bigcap_{\nu \in \mathbb{N}} L(KA(O_\nu + x)K^*) = \bigcap_{\nu \in \mathbb{N}} L(U(\kappa x)KA(O_\nu)K^*U(-\kappa x))$$

$$= \kappa x + \bigcap_{\nu \in \mathbb{N}} \tilde{\kappa}(O_\nu)$$

$$= \kappa x + \{\tilde{\kappa}(0)\}.$$ 

The corresponding reasoning also leads to the statement made on $\hat{\kappa}$.

It has been shown now that $L(KA(O)K^*) = \tilde{\kappa}(O)$ for each double cone $O \in \mathcal{K}$, and since $KA(O)K^* = A(M_O)$ by assumption, one concludes from $M_O \subset KA(M_O)$ and isotony that

$$KA(O)K^* \subset A(\tilde{\kappa}(O)) \quad \text{for all } O \in \mathcal{K}$$

and that

$$K^*A(O)K \subset A(\hat{\kappa}(O)) \quad \text{for all } O \in \mathcal{K}.$$

Using this, one can now prove that $\tilde{\kappa}$ and $\hat{\kappa}$ are inverse to each other.

Lemma 3.6 $\hat{\kappa} = \tilde{\kappa}^{-1}$, and in particular, $\tilde{\kappa}$ and $\hat{\kappa}$ are homeomorphisms.

Proof. For every double cone $O$, it follows from the preceding results that

$$A(O) = K^*KA(O)K^*K \subset K^*A(\tilde{\kappa}(O))K \subset A(\hat{\kappa}(\tilde{\kappa}(O)))$$

and since $\hat{\kappa}(\tilde{\kappa}(O))$ is a double cone by Lemma 3.5, one can use Lemma 3.1 to conclude that $O \subset \hat{\kappa}(\tilde{\kappa}(O))$. On the other hand, it follows from Lemma
that the radii of the double cones $\mathcal{O}$ and $\tilde{\kappa}(\tilde{\kappa}(\mathcal{O}))$ are equal, so these double cones coincide, and as this applies for any double cone $\mathcal{O}$, it follows that $\tilde{\kappa} = \tilde{\kappa}^{-1}$, as stated. 

The proof of Lemma 2.1 is now almost complete. For each $\mathcal{O} \in \mathcal{K}$, one has

$$KA(\mathcal{O})K^* \subset A(\tilde{\kappa}(\mathcal{O})),$$

and conversely,

$$A(\tilde{\kappa}(\mathcal{O})) = KK^* A(\tilde{\kappa}(\mathcal{O})) KK^* \subset KA(\tilde{\kappa}^{-1}(\tilde{\kappa}(\mathcal{O}))) K^* = KA(\mathcal{O})K^*,$$

so

$$KA(\mathcal{O})K^* = A(\tilde{\kappa}(\mathcal{O})),$$

and with $\xi := \tilde{\kappa}(0)$ it follows from Lemma 3.5 that

$$KA(\mathcal{O})K^* = A(\kappa\mathcal{O} + \xi) \quad \text{for all } \mathcal{O} \in \mathcal{K}.$$ 

That $\xi$ is unique, immediately follows from Lemma 3.1, so the proof of Lemma 2.1 is complete. 

Proof of Theorem 2.2 (i)

It follows from Lemma 2.1 that there is a unique $\iota \in \mathbb{R}^{1+s}$ such that

$$J_{W_1}A(\mathcal{O})J_{W_1} = A(j_1\mathcal{O} + \iota) \quad \text{for all } \mathcal{O} \in \mathcal{K}.$$ 

It remains to be shown that $\iota = 0$. Since $J$ is an involution, one has

$$x = j_1(j_1x + \iota) + \iota = x + j_1\iota + \iota \quad \text{for all } x \in \mathbb{R}^{1+s},$$

which gives $\iota = -j_1\iota$, hence $\iota_2 = \cdots = \iota_s = 0$. Furthermore, one has

$$A(W_1' + \iota)''' = J_{W_1}A(W_1)''J_{W_1} = A(W_1'),$$

from Lemma 2.1 and the Tomita-Takesaki theorem, so on the one hand, it follows from Lemma 3.1 that

$$W_1' + \iota \subset W_1',$$

and on the other hand, locality implies

$$A(W_1)'' \subset A(W_1)' = A(W_1' + \iota)''' \subset A(W_1 + \iota),$$

15
so using Lemma 3.1 once more one finds
\[ W_1' \subset W_1' + \iota, \]
arriving at \( W_1' + \iota = W_1' \) and \( \iota_0 = \iota_1 = 0 \), as stated. \( \square \)

In what follows, a well-known generalization of Asgeirsson’s Lemma will be used repeatedly. It is called the double cone theorem of Borchers and Vladimirov \[50, 4, 71, 12\]. Below, it will be applied together with the edge of the wedge theorem due to Bogoliubov (cf., e.g., \[45, 41, 12\]). For the reader’s convenience, both theorems are recalled here. For \( \varepsilon > 0 \), \( B_\varepsilon \) will denote the open \( \varepsilon \)-ball centered at the origin of \( \mathbb{R}^2 \), and \( n \) will denote some natural number.

**Theorem 3.7** (Edge of the Wedge Theorem) *Let \( C \) be a nonempty, open and convex cone in \( \mathbb{R}^n \). For some \( \varepsilon > 0 \), assume that \( g_+ \) is a function analytic in the tube \( \mathbb{R}^n + i(C \cap B_\varepsilon) \), and that \( g_- \) is a function analytic in the tube \( \mathbb{R}^n - i(C \cap B_\varepsilon) \). If there is an open region \( \gamma \subset \mathbb{R}^n \) where \( g_+ \) and \( g_- \) have a common boundary value in the sense of distributions, then \( g_+ \) and \( g_- \) are branches of a function \( g \) which is analytic in a complex neighbourhood \( \Gamma \) of \( \gamma \).*

**Theorem 3.8** *Given the assumptions and notation of Theorem 3.7, let \( c \) be any smooth curve in \( \gamma \) which has all its tangent vectors in \( C \). Then \( g \) is analytic in a complex neighbourhood of the double cone \( (c + C) \cap (c - C) \).*

Another well known lemma that will be used repeatedly is the following (cf. e.g., part (i) of Lemma 2.4.1 in \[39\]).

**Lemma 3.9** *Let \( R \subset \mathbb{R}^{1+s} \) be a region that contains an open cone, and let \( A \in A_{\text{loc}} \) be a local observable such that \( \langle \Omega, AB \Omega \rangle = \langle \Omega, BA \Omega \rangle \) for all \( B \in A(R) \). Then \( A \in A(R)' \).*

**Proof of Theorem 2.2** (ii)

In what follows, \( e_0 \) and \( e_1 \) denote the unit vectors pointing into the 0- and the 1-direction, respectively.
For every $t \in \mathbb{R}$, Theorem 2.1 implies the existence of a unique $\xi(t) \in \mathbb{R}^{1+s}$ with

$$\Delta_{W_1}^{it} A(O) \Delta_{W_1}^{-it} = A(\xi(t) + \Lambda_1(-2\pi t)O) \quad \text{for all } O \in \mathcal{K}.$$ 

By Corollary 3.1 it is clear that $\xi(t) + W_1 = W_1$, so for all $s \in \mathbb{R}$, one has $\Lambda_1(-2\pi s)\xi(t) = \xi(t)$ and

$$A(\xi(s + t) + \Lambda_1(-2\pi(t + s))O) = \Delta_{W_1}^{is} \Delta_{W_1}^{it} A(O) \Delta_{W_1}^{-is} \Delta_{W_1}^{-it}$$

$$= A(\xi(s) + \Lambda_1(-2\pi s)(\xi(t) + \Lambda_1(-2\pi t)O))$$

$$= A(\xi(s) + \Lambda_1(-2\pi s)\xi(t) + \Lambda_1(-2\pi(t + s))O)$$

$$= A(\xi(s) + \xi(t) + \Lambda_1(-2\pi(t + s))O),$$

so $\xi(s + t) = \xi(s) + \xi(t)$ follows from Lemma 3.1. One now concludes that $\xi(\lambda t) = \lambda \xi(t)$ for $\lambda \in \mathbb{Q}$, so $t \mapsto \xi(t)$ is $\mathbb{Q}$-linear.

Next we prove that the function $\mathbb{R} \ni t \mapsto \xi(t)$ is continuous and, hence, $\mathbb{R}$-linear. As $\xi$ is additive, it is sufficient to prove continuity at $t = 0$. Assume $\xi$ were not continuous there, then there would exist a sequence $(t_\nu)_{\nu \in \mathbb{N}}$ in $\mathbb{R}$ that tends to zero, while $|\xi(t_\nu)| > \varepsilon$ for some $\varepsilon > 0$. Define the double cone

$$O := \left( -\frac{\varepsilon}{3}e_0 + V_+ \right) \cap \left( \frac{\varepsilon}{3}e_0 - V_+ \right).$$

By the above results and locality, there is an $N_\varepsilon \in \mathbb{N}$ such that for any $A, B \in A(O)$, one has

$$[\Delta_{W_1}^{it}, A \Delta_{W_1}^{-it}, B] = 0 \quad \text{for all } \nu > N_\varepsilon.$$ 

But as $\Delta_{W_1}^{it}$ depends strongly continuously on $t$, one concludes that $A$ and $B$ commute, and since $A$ and $B$ are arbitrary elements of $A(O)$, it follows that $A(O)$ is abelian. Additivity implies that $A''$ is abelian as well, so $\mathcal{H} = \mathbb{C}$ by irreducibility, which contradicts the assumption that $\mathcal{H}$ is infinite-dimensional. It follows that $\xi$ is continuous and, hence, $\mathbb{R}$-linear, so there is a $\xi \in \mathbb{R}^{1+s}$ with $\xi(t) = \xi t$ for all $t \in \mathbb{R}$.

It remains to be shown that $\xi = 0$. To this end, define the double cone $O := (\rho e_1 + V_+) \cap (\rho e_1 + \rho e_0 - V_+) \subset W_1$ for some $\rho > 0$. If one chooses $\rho$ sufficiently small, there are $a \in \mathbb{R}^{1+s}$ and $\varepsilon, \delta > 0$ such that

1. $\Lambda_1(-2\pi t)O + t\xi - \delta te_0 \subset a + V_+$ for all $t \in [0, \varepsilon]$;
2. $\overline{O} \nsubseteq a + V_+$. 

17
As an example, choose \( a := \rho e_1 + \xi - |\xi|e_0 \), where \( |\xi| := \sqrt{|\xi|^2} \). Defining

\[
f(t) := (\Lambda_1(-2\pi t)\rho e_1 + t\xi - \delta te_0 - a)^2,
\]

one computes

\[
f'(0) = 2|\xi|(-2\pi\rho + |\xi| - \delta).
\]

If one chooses \( \rho < \frac{|\xi|}{2\pi} \), one can choose \( \delta \) such that \( 0 < \delta < -2\pi\rho + |\xi| \). With this choice one has \( f'(0) > 0 \), and as \( f \) is smooth and satisfies \( f(0) = 0 \), there is an \( \varepsilon > 0 \) such that \( f(t) \geq 0 \) for all \( t \in [0, \varepsilon] \), which immediately implies Condition (1), whereas Condition (2) follows from \( f(0) = 0 \).

As the set \( \bigcup_{0 \leq t \leq \varepsilon} (\Lambda_1(-2\pi t)O + t\xi) \) is bounded, there is a \( b \in \mathbb{R}^{1+s} \) such that \( (3) \Lambda_1(-2\pi t)O + t\xi \subset b - V_+ \) for all \( t \in [0, \varepsilon] \).

Now denote \( P := (a + V_+) \cap (b - V_+) \) (Fig. 1), choose \( A \in \mathcal{A}(O) \) and \( B \in \mathcal{A}(P') \), denote by \( e_0 \) the unit vector in the time direction, and consider the function \( g_{A,B} \) defined by

\[
\mathbb{R}^2 \ni (t, s) \mapsto g_{A,B}(t, s) := \left\langle \Omega, [B, U(se_0)\Delta_{W_1}^H A\Delta_{W_1}^{-H}U(-se_0)]\Omega \right\rangle.
\]

By Conditions (1) and (3), this function vanishes in the closure of the open triangle \( \gamma \) with corners \((0, 0), (\varepsilon, 0)\) and \((\varepsilon, -\delta\varepsilon)\) (Fig. 2). Clearly, \( \gamma \) contains a smooth curve that joins \((0, 0)\) to \((\varepsilon, -\delta\varepsilon)\) and that has tangent vectors in the cone \( C := \{(t, s) \in \mathbb{R}^2 : t > 0, s < 0\} \). It will be shown that by the double cone theorem, \( g_{A,B} \) vanishes in the whole open rectangle \([0, \varepsilon] \times [-\delta\varepsilon, 0]\). Since \( g_{A,B} \) is continuous, it follows that it even vanishes in the closed rectangle \([0, \varepsilon] \times [-\delta\varepsilon, 0] \). Since \( B \in \mathcal{A}(P') \) and \( A \in \mathcal{A}(O) \) are arbitrary, Lemma 3.9 implies that \( \mathcal{A}(O - \delta\varepsilon e_0) \subset \mathcal{A}(P')' \). But since by Condition (2), the double cone \( O - \delta\varepsilon e_0 \) cannot be contained in \( P \) no matter how small \( \delta\varepsilon \) is, this is in conflict with Lemma 3.1, so it follows that \( \xi = 0 \), which completes the proof.

It remains to be shown that the function \( g_{A,B} \) fulfills the assumptions of the double cone theorem. To this end, first note that

\[
g_{A,B} = \left\langle \Omega, BU(se_0)\Delta^H A\Omega \right\rangle - \left\langle \Omega, A\Delta^{-H}U(-se_0)B\Omega \right\rangle
= \left\langle \Omega, BU(se_0)\Delta^H A\Omega \right\rangle - \left\langle \Omega, B^*U(se_0)\Delta^H A^*\Omega \right\rangle
=: g_+(t, s) - g_-(t, s).
\]

Using elementary arguments from spectral theory it can be shown that given any \( \rho > 0 \), any vector \( \phi \) in the domain of \( \Delta^\rho \) and any \( \psi \in \mathcal{H} \), the function
Figure 1: The double cone \( P \) in the proof of Thm. 2.2 (ii)

\[ b \]

\[ P \]

\[ V_l(-2\pi t)O + \varepsilon \xi \]

\[ a \]

\[ \mathbb{R} \ni t \mapsto \langle \psi, \Delta^t \phi \rangle \] has an extension to a function that is continuous on the strip \( \{ t \in \mathbb{C} : -\rho \leq \text{Im} t \leq 0 \} \) and analytic on the interior of this strip (cf. [40], Lemma 8.1.10 (p. 351)).

As \( \mathcal{O} \subset W_1 \), the vectors \( A\Omega \) and \( A^*\Omega \) are in the domain of \( \Delta^\frac{1}{2} \), and it follows that for every \( \psi \in \mathcal{H} \), the functions \( \mathbb{R} \ni t \mapsto \langle \psi, \Delta^t A\Omega \rangle \) and \( \mathbb{R} \ni t \mapsto \langle \psi, \Delta^t A^*\Omega \rangle \) have extensions that are continuous in the strips \( \{ t \in \mathbb{C} : \frac{1}{2} \leq \text{Im} t \leq 0 \} \) and \( \{ t \in \mathbb{C} : 0 \leq \text{Im} t \leq \frac{1}{2} \} \), respectively, and that are analytic in the interior of these strips.

On the other hand, it follows from the spectrum condition that for any two vectors \( \phi, \psi \in \mathcal{H} \), the functions \( \mathbb{R} \ni s \mapsto \langle \psi, U(se_0)\phi \rangle \) and \( \mathbb{R} \ni s \mapsto \langle \psi, \tilde{U}(se_0)\phi \rangle \) have extensions that are continuous in the (complex) closed
upper and lower half plane, respectively, and analytic in the interior of these half planes.

This proves that the function \( g_+ \) has a continuous extension to the tube \( T_+ := \{(t, s) \in \mathbb{C}^2 : \frac{1}{2} \leq \text{Im } t \leq 0, \text{Im } s \geq 0\} \) and that at every interior point of this strip, this extension is analytic separately in \( t \) and in \( s \). Using Hartogs’ fundamental theorem stating that a function of several complex variables is holomorphic if and only if it is holomorphic separately in each of these variables \([33, 51]\), it follows that \( g_+ \), as a function in two complex variables, is analytic in the interior of \( T_+ \). It follows in the same way that \( g_- \) has the corresponding properties for the tube \(-T_+ =: T_-\). The tubes \( T_+ \) and \( T_- \) contain the smaller tubes \( \mathbb{R}^2 - iC \cap \overline{B_{\frac{1}{2}}} \) and \( \mathbb{R}^2 + iC \cap \overline{B_{\frac{1}{2}}} \).

Since \( g_+ \) and \( g_- \) coincide as continuous functions in the closure of \( \gamma \), they coincide as distributions in the open region \( \gamma \), and it follows from the edge of the wedge theorem that they are branches of a function \( g \) that is analytic in a complex neighbourhood \( \Gamma \) of \( \gamma \). But since \( \gamma \) contains a smooth curve joining the points \((0, 0)\) and \((\varepsilon, -\delta\varepsilon)\) with tangent vectors in \( C \), it follows from the double cone theorem that the function \( g \) is analytic in the region

\[
((0, 0) + C) \cap ((\varepsilon, -\delta\varepsilon) - C) = \{0, \varepsilon[\times] - \delta\varepsilon, 0\}.
\]

This implies that \( g_{A,B} \) vanishes in this region, which is all that remained to be shown, so the proof is complete.
Proof of Corollary 2.3

If $J_+$ or $\Delta^t_+$ behave the way assumed in (i) or (ii), respectively, the commutation relations recalled in the remark preceding the corollary, together with Lemma 2.1, imply that its geometrical action can differ from the stated symmetry at most by a translation. Since $V_+$ is Lorentz-invariant, $J_+$ and $\Delta^t_+, t \in \mathbb{R}$, commute with all $U(g), g \in L^+_\uparrow$. However, there are no nontrivial translations that commute with all $g \in L^+_\uparrow$. □

Proof of Lemma 2.4

It follows from the Tomita-Takesaki Theorem that the modular group under consideration leaves the algebras $\mathcal{A}(W_1)'$ and $\mathcal{A}(W_1)$ invariant. By wedge duality, it also leaves the algebra $\mathcal{A}(W_1)' = \mathcal{A}(-W_1)$ invariant. Borchers' commutation relations now imply

$$\Delta^{i\varepsilon} W_1 \mathcal{A}(a \pm W_1)' \Delta^{-i\varepsilon} W_1 = \mathcal{A}(\Lambda_1(-2\pi\varepsilon)a \pm W_1)' .$$

$L(A) + W_1$ is a union of translates of $W_1$, so $(L(A) + W_1)^\text{cc}$ is a translate of $W_1$. In particular,

$$(L(A) + W_1)^\text{cc} = \bigcap \{a + W_1 : a \in \mathbb{R}^{1+s}, (L(A) + W_1)^\text{cc} \subset a + W_1\} .$$

But if $a \in \mathbb{R}^{1+s}$ is chosen such that $(L(A) + W_1)^\text{cc} \subset a + W_1$, Lemma 3.1 above and wedge duality imply $A \in \mathcal{A}(a + W_1)' = \mathcal{A}(a + W_1)'$, so one finds

$$\bigcap \{a + W_1 : a \in \mathbb{R}^{1+s}, A \in \mathcal{A}(a + W_1)'\} \subset (L(A) + W_1)^\text{cc} ,$$

and one concludes

$$L(A_\varepsilon) \subset \bigcap \{a + W_1 : a \in \mathbb{R}^{1+s}, \Delta^{i\varepsilon} W_1 A \Delta^{-i\varepsilon} W_1 \in \mathcal{A}(a + W_1)'\}$$

$$= \bigcap \{a + W_1 : a \in \mathbb{R}^{1+s}, A \in \mathcal{A}(a + W_1)'\}$$

$$= \bigcap \{a + W_1 : a \in \mathbb{R}^{1+s}, A \in \mathcal{A}(\Lambda_1(2\pi\varepsilon)a + W_1)'\}$$

$$= \Lambda_1(-2\pi\varepsilon) \bigcap \{a + W_1 : a \in \mathbb{R}^{1+s}, A \in \mathcal{A}(a + W_1)'\}$$

$$\subset \Lambda_1(-2\pi t)(L(A) + W_1)^\text{cc} .$$

The proof that $L(A_\varepsilon) \subset \Lambda_1(-2\pi t)(L(A) - W_1)^\text{cc}$ is completely analogous, so the proof of (i) is complete.
It remains to prove (ii) and (iii). We prove (iii); (ii) can be established along precisely the same line of argument by replacing $\Delta_{W_1}^t$ by $\Delta_{-W_1}^t$ and by exchanging, respectively, $V_+$ and $-V_+$, $A$ and $A_\varepsilon$ with one another. Due to Borchers’ commutation relations it suffices to consider $A \in \mathcal{A}(W_1)^\prime$, which, as in the proof of Theorem 2.2 (ii), will ensure that $A\Omega \in D(\Delta^{1/2})$ in the following argument.

Assume that $L(A) \not\subset L(A_\varepsilon) + V_+$. Then one finds an $a \in \mathbb{R}^1$ such that

1. $L(A_\varepsilon) \subset a + V_+$, while
2. $L(A) \not\subset a + V_+$.

This can be seen as follows. The assumption that $L(A) \not\subset L(A_\varepsilon) + V_+$ and Statement (i) just proved imply that there is a double cone $O \subset L(A)$ such that $O$ and $L(A_\varepsilon)$ are spacelike separated, so there is a double cone $P \supset L(A_\varepsilon)$ such that $O$ and $P$ are spacelike separated (cf., e.g., Prop. 3.8 (b) in [47]); choosing $a$ to be the lower tip of $P$, one arrives at both Conditions (1) and Condition (2).

By Condition (1), $L(A_\varepsilon)$ is a compact subset of the open set $a + V_+$, and as $L(A_t)$ depends continuously on $t$ by assumption, there exist $\sigma^b > 0$ and $\delta > 0$ such that

1. $L(A_\varepsilon) - \sigma^b e_0 \subset a + V_+$ for all $t \in [\varepsilon - \delta, \varepsilon]$,

and this condition is, of course, equivalent to Condition (1).

Since $L(A_t)$ depends continuously on $t \in [0, \varepsilon]$, the set $\bigcup_{0 \leq t \leq \varepsilon} L(A_t)$ is bounded, so one finds a $\sigma^s \geq 0$ such that

3. $L(A_t) + \sigma^s e_0 \subset a + V_+$ for all $t \in [0, \varepsilon]$,

and for the same reason there is a $b \in \mathbb{R}^1$ such that

4. $L(A_t) + 2\sigma^s e_0 \subset b - V_+$ for all $t \in [0, \varepsilon]$.

Now define $P := (a + V_+) \cap (b - V_+)$, and for any $B \in \mathcal{A}(P')$, consider – as in the proof of Proposition 2.2 – the function $g_{A,B}$ defined by

$$
\mathbb{R}^2 \ni (t, s) \mapsto g_{A,B}(t, s) := \langle \Omega, [B, U(se_0)A_tU(-se_0)]\Omega \rangle.
$$

Locality and Conditions (3) and (4) imply that this function vanishes in the rectangle $[0, \varepsilon] \times [\sigma^s, 2\sigma^s]$, and Condition (1') implies that it also vanishes in the rectangle $[\varepsilon - \delta, \varepsilon] \times [-\sigma^b, \sigma^s]$. By the double cone theorem, $g_{A,B}$ vanishes throughout the whole rectangle $[0, \varepsilon] \times [-\sigma^b, 2\sigma^s]$ (Fig. 3). In particular, one
obtains \( g_{A,B}(0, -\sigma^b) = 0 \) for all \( B \in \mathcal{A}(P') \), so one can use Lemma 3.9 to conclude that \( A \in \mathcal{A}(\sigma^b e_0 + P')' \). By the definition of \( L(A) \), one finds

\[
L(A) - \sigma^b e_0 \subset \mathcal{T} \subset a + \mathcal{V}_+,
\]

and as \( \sigma^b > 0 \), this implies \( L(A) \subset a + \mathcal{V}_+ \), which is in conflict with Condition (2) above and completes the proof. \( \square \)

**Proof of Theorem 2.5**

Fix any \( \rho > 0 \), and define the double cones

\[
\mathcal{O}_1 := (\rho(2e_1 + e_0) + V_+) \cap (\rho(2e_1 + 2e_0) - V_+), \\
\mathcal{O}_2 := (\rho(2e_1 - 2e_0) + V_+) \cap (\rho(2e_1 + 2e_0) - V_+),
\]

and

\[
\mathcal{O}_3 := (\rho(2e_1 - 3e_0) + V_+) \cap (\rho(2e_1 + 3e_0) - V_+),
\]

(Fig. 4) and choose \( A \in \mathcal{A}(\mathcal{O}_1) \). As \( L(A) \subset \overline{\mathcal{O}_1} \), it follows from Lemma 2.4
Figure 4: The double cones $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ in the proof of Thm. 2.5

(i) and (ii) that

$$L(A_t) \subset (\Lambda_1(-2\pi t)\rho (\frac{3}{2}e_1 + \frac{3}{2}e_0) + \overline{W}_1)$$
$$\cap (\Lambda_1(-2\pi t)\rho (\frac{3}{2}e_1 + \frac{3}{2}e_0) - \overline{W}_1)$$
$$\cap (\rho(2e_1 + 2e_0) - \overline{V}_+) =: R_t,$$

and there is an $\varepsilon > 0$ such that

$$R_t \subset \overline{\mathcal{O}_2} \quad \text{for all } t \in [0, \varepsilon].$$

Note that by the linearity of the Lorentz boosts, $\varepsilon$ does not depend on $\rho$. One now has $L(A_t) \subset \overline{\mathcal{O}_2}$ for all $A \in \mathcal{A}(\mathcal{O}_1)$, and with Corollary 5.4 in [39], it follows that

$$\Delta_{W_1}^{it} A(\mathcal{O}_1) \Delta_{W_1}^{-it} \subset \mathcal{A}(\mathcal{O}_3)' \quad \text{for all } t \in [0, \varepsilon].$$

Using Borchers’ commutation relations, one finds

$$\Delta_{W_1}^{it} A(a + \mathcal{O}_1) \Delta_{W_1}^{-it} \subset \mathcal{A}(\Lambda_1(-2\pi t)a + \mathcal{O}_3')'$$
for all $a \in \mathbb{R}^{1+s}$ and all $t \in [0, \varepsilon]$. Defining $x := \rho(2e_1 + e_0)$, $P_1 := \mathcal{O}_1 - x$, and $P_3 := \mathcal{O}_3 - x$, one obtains

$$\Delta_{W_1}^{it} A(a + P_1) \Delta_{W_1}^{-it} \subset A(\Lambda_1 (-2\pi t)a + (x - \Lambda_1 (-2\pi t)x) + P_3')'. $$

Note that the euclidean length of the vector $x - \Lambda_1 (-2\pi t)x$ is $\leq 3\rho$ for all $t \in [0, \varepsilon]$, as $\Lambda_1 (-2\pi t)x \in R_1 \subset \mathcal{O}_2$ by the above choice of $\varepsilon$.

Now choose any wedge $W \in \mathcal{W}$. As $W \subset W + P_1$, it follows from wedge additivity that

$$A(W)'' \subset \left( \bigcup_{a \in W} A(a + P_1) \right)'' = \left( \bigcup_{a \in W} A(a + P_3')' \right)'',$

and as the euclidean length of the vector $(\Lambda_1 (-2\pi t)x - x)$ is $\leq 3\rho$, one arrives at

$$\left( \bigcup_{a \in W} A(a + (x - \Lambda_1 (-2\pi t)x) + P_3')' \right)'' \subset A(W^{(7\rho)})''.

For $t \in [0, \varepsilon]$, one now obtains

$$\Delta_{W_1}^{it} A(\Lambda_1 (2\pi t)W)'' \Delta_{W_1}^{-it} \subset \left( \bigcup_{a \in \Lambda_1 (2\pi t)W} \Delta_{W_1}^{it} A(a + P_1) \Delta_{W_1}^{-it} \right)''

\subset \left( \bigcup_{a \in \Lambda_1 (2\pi t)W} A(\Lambda_1 (-2\pi t)a + (x - \Lambda_1 (-2\pi t)x) + P_3')' \right)''

\subset A(W^{(7\rho)})'',$

and as $W = (W^{(-7\rho)})^{(7\rho)}$, this can be rewritten

$$\Delta_{W_1}^{-it} A(W)'' \Delta_{W_1}^{it} \subset A(\Lambda_1 (2\pi t)W^{(-7\rho)})''''.

Using the fact that the transformations $\Lambda_1 (2\pi t)$ are linear and, hence, bounded maps in $\mathbb{R}^{1+s}$, which map the euclidean $7\rho$-ball onto some bounded set with radius proportional to $\rho$, and using the facts that this radius continuously
depends on $t \in [0, \varepsilon]$, that the interval $[0, \varepsilon]$ is compact, and that $\varepsilon$ does not depend on the choice of $\rho$, one concludes that there is an $M > 0$ which is independent from $\rho$ and satisfies

$$\Lambda_1(2\pi t)W^{(-7\rho)} \supset (\Lambda_1(2\pi t)W)^{(-M\rho)}$$

for all $t \in [0, \varepsilon]$, so with the above specifications of $\varepsilon$ and $M$, one obtains

$$\Delta^{-it}_{W_1}A(W)^{\prime\prime} \Delta^{-it}_{W_1} \supset A((\Lambda_1(2\pi t)W)^{(-M\rho)})^{\prime\prime}$$

for all wedges $W \in \mathcal{W}$ and all $\rho > 0$. For each $A \in \mathcal{A}_{\text{loc}}$, one now concludes

$$L(A_t) = \bigcap \{W : W \in \mathcal{W}, \Delta^{-it}_{W_1}A\Delta^{-it}_{W_1} \in A(W)^{\prime\prime}\}$$

$$= \bigcap \{W : W \in \mathcal{W}, A \in \Delta^{-it}_{W_1}A(W)^{\prime\prime}\}$$

$$\subset \bigcap_{\rho > 0} \bigcap \{W : W \in \mathcal{W}, A \in \mathcal{A}(\Lambda_1(2\pi t)W)^{(-M\rho)})^{\prime\prime}\}$$

$$= \bigcap_{\rho > 0} \bigcap \{\Lambda_1(-2\pi t)X : X \in \mathcal{W}, A \in \mathcal{A}(X^{(-M\rho)})^{\prime\prime}\}$$

$$= \Lambda_1(-2\pi t) \bigcap_{\rho > 0} \bigcap \{X^{(-M\rho)} : X \in \mathcal{W}, A \in \mathcal{A}(X)^{\prime\prime}\}$$

$$= \Lambda_1(-2\pi t) L(A).$$

To prove the converse inclusion, one proves $L(A_t) \subset \Lambda_1(-2\pi t)$ for $t \in [-\varepsilon,0]$ by mimicking the above argument: one defines the double cone

$$\mathcal{O}_1 := \rho(2e_1 - 2e_0) + V_+ \cap (\rho(2e_1 - e_0) - V_+),$$

keeps $\mathcal{O}_2$ and $\mathcal{O}_3$ as before, defines $x := \rho(2e_1 - e_0)$ and proceeds like above with $t \in [-\varepsilon,0]$, using Lemma 2.4 (iii) instead of Part (ii) of the same lemma. Now having proved $L(A_t) \subset \Lambda_1(-2\pi t)L(A)$ for all $t \in [-\varepsilon,\varepsilon]$ and for all $A \in \mathcal{A}_{\text{loc}}$, one concludes $L(A_t) = \Lambda_1(-2\pi t)L(A)$ for all $t \in [-\varepsilon,\varepsilon]$ and for all $A \in \mathcal{A}_{\text{loc}}$. As this immediately implies the statement for all $t \in \mathbb{R}$ and all $A \in \mathcal{A}_{\text{loc}}$, the proof is complete.

$$\square$$

4 Conclusion

By the above results, the modular group of a theory that does not exhibit the Unruh effect acts in a completely “non-geometric” fashion, in the sense
that it can neither preserve the net structure nor act on the local observables in such a way that localization regions evolve continuously. In particular, it cannot implement any equilibrium dynamics in this case.

The above results imply that the only observer who can possibly experience the vacuum in thermodynamical equilibrium is the uniformly accelerated one (whose acceleration may, of course, be zero). Physically, this result reflects the fact that any non-uniformly accelerated observer would feel non-stationary inertial forces destroying any thermodynamical equilibrium, while the constant acceleration felt by a uniformly accelerated observer does not affect thermodynamical equilibrium provided the theory exhibits the Unruh effect.

The first results similar to the above ones have been obtained by Araki and by Keyl [4, 35]. These authors avoid the spectrum condition and assume stronger a priori restrictions on the possible geometric behaviour instead. Recently, more results in this spirit have been found by Buchholz et al. and by Trebels [21, 27, 29, 48]. One aim of these approaches is to obtain new insight on quantum fields on curved spacetimes by avoiding the spectrum condition. So far, results have been obtained for de Sitter, Anti-de Sitter, and certain Robertson-Walker spacetimes [21, 22, 24].

For the vacuum states in Minkowski space considered above, the spectrum condition is a reasonable physical assumption. The assumptions made above on the possible geometric behaviour of the modular objects (in particular those made in the first uniqueness theorem) are less restrictive than those made in any of the other approaches, since a small class of regions, namely, the double cones, is assumed to be mapped into an extremely large class of regions, namely, the open sets. In this sense the above results are, at present, the most general uniqueness results in Minkowski space that point towards the Unruh effect and modular $P_1$CT-symmetry.

Even more than a uniqueness result can be found if conformal symmetry holds in addition to our above Conditions (A) through (C). In this case, the whole representation of the conformal group arises from the modular objects of the theory, and in particular, the Bisognano-Wichmann symmetries can be established [16].

Appendix. A Remark on the Continuity of $t \mapsto L(A_t)$

In the discussion of the second uniqueness theorem it was assumed that $L(A_t)$ depends continuously on $t$ for $t \in [0, \varepsilon]$ in the sense that for each sequence $(t_\nu)_{\nu \in \mathbb{N}}$ tending to a $t_\infty \in [0, \varepsilon]$, the localization region $L(A_{t_\infty})$
consists precisely of all accumulation points of sequences \((x_\nu)_{\nu \in \mathbb{N}}\) with \(x_\nu \in L(A_t_\nu)\). In this appendix we show that this notion of convergence, which we refer to as pointwise convergence, is equivalent to the convergence according to a metric first considered by Hausdorff, which one can introduce on the set \(\mathcal{C}\) of compact convex subsets of \(\mathbb{R}^{1+s}\) by defining, for any two such sets \(K, L \in \mathcal{C}\),

\[
\delta_H(K, L) := \inf\{\delta > 0 : K \subset B_\delta(L) \text{ and } L \subset B_\delta(K)\}
\]

(cf. Problem 4D (p. 131) in [34]). It is evident that continuity of \([0, \varepsilon] \ni t \mapsto L(A_t)\) with respect to this metric, which we refer to as uniform continuity, implies the pointwise continuity for this map. Conversely, one can also show that pointwise continuity implies uniform continuity for \(t \mapsto L(A_t)\).

To prove this indirectly, assume that \(t \mapsto L(A_t)\) is pointwise continuous for \(t \in [0, \varepsilon]\) and that this map is not continuous with respect to Hausdorff’s metric. Then there exists a \(\rho > 0\) and a sequence \((t_\nu)_{\nu \in \mathbb{N}}\) of points in \([0, \varepsilon]\) which converges to a point \(t_\infty \in [0, \varepsilon]\) and has the property that \(\delta_H(L(A_t_\nu), L(A_t_\infty)) \geq \rho\).

On the other hand, there is a subsequence \((s_\nu)_{\nu \in \mathbb{N}}\) of \((t_\nu)_{\nu \in \mathbb{N}}\) with the property that all \(L(A_s_\nu)\) have nonempty intersection with \(B_\rho(L(A_t_\infty))\), as otherwise \(L(A_t_\infty)\) would be empty by the assumption of pointwise continuity.

As \(\delta_H(L(A_s_\nu), L(A_t_\infty)) \geq \rho\), there exists a sequence \((x_\nu)_{\nu \in \mathbb{N}}\) such that the euclidean distance \(\delta(x_\nu, L(A_t_\infty))\) between \(x_\nu\) and \(L(A_t_\infty)\) is \(\geq \rho/2\) for all \(\nu \in \mathbb{N}\), and as all \(L(A_s_\nu)\) are convex sets with a nonempty intersection with \(B_\rho(L(A_t_\infty))\), this sequence can be chosen such that it is bounded and, hence, has an accumulation point \(\tilde{x}\). As \(\delta(x_\nu, L(A_t_\infty)) \geq \rho/2\) for all \(\nu \in \mathbb{N}\), one finds \(\delta(\tilde{x}, L(A_t_\infty)) \geq \rho/2\), so \(\tilde{x} \notin L(A_t_\infty)\). But this contradicts the assumption that \(t \mapsto L(A_t)\) is pointwise continuous and proves that this map is pointwise continuous if and only if it is uniformly continuous, as stated.

It is now easy to see that \(\bigcup_{t \in [0, \varepsilon]} L(A_t)\) is bounded, as stated in the text. Namely, the function \([0, \varepsilon] \ni t \mapsto \delta_H(L(A), L(A_t))\) is continuous and, hence, has a maximum \(\rho > 0\) in the compact interval \([0, \varepsilon]\). It follows that \(\bigcup_{t \in [0, \varepsilon]} L(A_t) \subset B_\rho(L(A))\), which is a bounded set.

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