A fiducial approach to nonparametric deconvolution problem: discrete case

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Abstract

Fiducial inference, as generalized by [Hannig et al. (2016)], is applied to nonparametric $g$-modeling (Efron, 2016) in the discrete case. We propose a computationally efficient algorithm to sample from the fiducial distribution, and use the generated samples to construct point estimates and confidence intervals. We study the theoretical properties of the fiducial distribution and perform extensive simulations in various scenarios. The proposed approach yields good statistical performance in terms of the mean squared error of point estimators and the coverage of confidence intervals. Furthermore, we apply the proposed fiducial method to estimate the probability of each satellite site being malignant using gastric adenocarcinoma data with 844 patients (Efron, 2016).

keywords Fiducial inference, Empirical Bayes, Confidence intervals, Nonparametric deconvolution

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1 Introduction

Efron (2014, 2016); Narasimhan and Efron (2016) studied the following important problem: An unknown distribution function $F(\theta)$ yields unobservable realizations $\Theta_1, \Theta_2, \ldots, \Theta_n$, and each $\Theta_i$ produces an observable value $X_i$ according to a known probability mechanism. The goal is to estimate the unknown distribution function from the observed data. In this paper, we aim to provide a generalized fiducial solution to the same problem in the case where $X_i$ given $\Theta_i$ follows a discrete distribution. Following Efron (2016), we use the terminology \textit{deconvolution} here because the marginal distribution of $X_i$ admits the following form (see Equation (6) of Efron (2016) and Equation (7) of Narasimhan and Efron (2016)):

$$
\int g_i(x_i|\theta_i)dF_i(\theta_i),
$$

where $g_i$ is the probability density for $X_i$ given $\Theta_i$.

Efron (2016) proposed an empirical Bayes deconvolution approach to estimating the distribution of $\Theta$ from an observed sample $\{X_i, i = 1, \ldots, n\}$, where the only requirement is a known specification of the distribution for $X_i$ given $\Theta_i$. The empirical Bayes deconvolution since developed has seen tremendous success in many scientific applications including causal inference (Lee and Small, 2019), single-cell analysis (Wang et al., 2018), cancer studies (Gholami et al., 2015; Shen and Xu, 2019), clinical trials (Shen and Li, 2018, 2019) and many other fields (Dulek, 2018). Moreover, for a classic Bayesian data analysis, as noted in (Ross and Markwick, 2018, p19) and Gelman et al. (2013), a single distribution prior may sometimes be unsuitable; hence,
the prior choice is dubious. Efron’s empirical Bayes deconvolution could be one of the alternatives because the obtained estimator of the distribution of \( \Theta \) can be used as a prior distribution to produce posterior approximations (Narasimhan and Efron, 2016).

Fiducial inference can be traced back to R. A. Fisher (Fisher, 1930, 1933) who introduced the concept as a potential replacement of the Bayesian posterior distribution. Hannig (2009); Hannig et al. (2016) showed that fiducial distributions can be related to empirical Bayes methods, which are widely used in large-scale parallel inference problems (Efron, 2012, 2019a,b). Efron (1998) pointed out that objective Bayes theories also have connections with fiducial inference. Other fiducial related approaches include the Dempster-Shafer theory (Dempster, 2008; Edlefsen et al., 2009; Martin et al., 2010; Hannig and Xie, 2012), inferential models (Martin and Liu, 2013, 2015a,b,c), confidence distributions (Schweder and Hjort, 2002; Xie et al., 2011; Xie and Singh, 2013; Xie et al., 2013; Schweder and Hjort, 2016; Hjort and Schweder, 2018; Shen et al., 2019), and higher-order likelihood expansions and implied data-dependent priors (Fraser, 2004, 2011).

We propose a novel fiducial approach to modeling the distribution function \( F \) of \( \Theta \) nonparametrically. In particular, we propose a computationally efficient algorithm to sample from the generalized fiducial distribution (GFD) (i.e., a distribution on a set of distribution functions), and use the generated samples to construct statistical procedures. The pointwise median of the GFD is used as a point estimate, and appropriate quantiles of the GFD evaluated at a given point provide pointwise confidence intervals. We also study the theoretical properties of the fiducial distribution. Extensive sim-
ulations in various scenarios show that the proposed fiducial approach is a good alternative to existing methods, including Efron’s $g$-modeling. We apply the proposed fiducial approach to intestinal surgery data to estimate the probability of each satellite site being malignant for a patient. The resulting fiducial estimate of distribution function reflects the observed patterns of raw data.

The remainder of this paper is organized as follows. In Section 2, we present the mathematical framework for the fiducial approach to the non-parametric deconvolution problem. In Section 3, we establish an asymptotic theory that verifies the frequentist validity of the proposed fiducial approach. Extensive simulation studies are presented in Section 4. We illustrate our method using intestinal surgery data in Section 5. The article concludes with a discussion of future work in Section 6. Some technical results and additional simulations are provided in the Appendix and Supplementary Material.

2 Methodology

2.1 Data generating equation

In this section, we first explain the definition of a generalized fiducial distribution and then demonstrate how to apply it to the deconvolution problem. We start by expressing the relationship between the data $X_i$ and the parameter $\Theta_i$ using

$$X_i = G_i^{-1}(U_i, \Theta_i), \quad \Theta_i = F^{-1}(W_i), \quad i = 1, \ldots, n,$$  

(2)
where $U_i, W_i$ are independent and identically distributed (i.i.d.) Unif(0, 1), $G_i(\cdot, \theta_i)$ are known distribution functions of discrete random variables supported on integers, $G_i$ are defined and non-increasing in $\theta_i \in S \subset \mathbb{R}$ for all $i$, and $F$ is the unknown distribution function with support in the interval $S$.

We are interested in estimating the unknown distribution function $F(\theta)$.

Recall that $F^{-1}(w) = \inf\{\theta : F(\theta) \geq w\}$ (Casella and Berger 2002, p54), and $F^{-1}(w) = \theta$ if and only if $F(\theta) \geq w > F(\theta - \epsilon)$ for all $\epsilon > 0$. We denote $G^*_i(x_i, u_i) = \sup\{\theta : G_i(x_i, \theta) \geq u_i\}$ with the usual understanding that $\sup \emptyset$ is smaller than all elements of $S$. If $G_i(\cdot, \theta_i)$ is continuous in $\theta_i$, $G^*_i(x_i, u_i)$ is the solution (in $\theta_i$) to the equation $G_i(x_i, \theta_i) = u_i$. By Lemma 1 in Section A of the Appendix, $x_i = G_i^{-1}(u_i, \theta_i)$ if and only if $\theta_i \in (G^*_i(x_i - 1, u_i), G^*_i(x_i, u_i))$.

We combine $G^*_i(x_i - 1, u_i) < \theta_i \leq G^*_i(x_i, u_i)$ and $F(\theta_i - \epsilon) < w_i \leq F(\theta_i)$ for all $\epsilon$. Consequently, the inverse of the data generating equation (2) is

$$Q_x(u, w) = \{F : F(G^*_i(x_i - 1, u_i)) < w_i \leq F(G^*_i(x_i, u_i)), i = 1, \ldots, n\}.$$ (3)

Note that $Q_x(u, w)$ is a set of cumulative distribution functions (CDFs). By Lemma 2 in Section A of the Appendix, $Q_x(u, w) \neq \emptyset$ if and only if $u, w$ satisfy:

whenever $G^*_i(x_i, u_i) \leq G^*_j(x_j - 1, u_j)$ then $w_i < w_j$. (4)

A generalized fiducial distribution is obtained by inverting the data generating equation. Hannig et al. (2016) proposed a general definition of GFD. However, to simplify the presentation, we use an earlier, less general version in Hannig (2009). These two definitions are equivalent for the models considered here. Suppose $(U^*, W^*)$ are uniformly distributed on the set
\{(\mathbf{u}^*, \mathbf{w}^*): Q_x(\mathbf{u}^*, \mathbf{w}^*) \neq \emptyset\}. A GFD is then the distribution of any element in the random set \(\overline{Q_x(\mathbf{U}^*, \mathbf{W}^*)}\), where the closure is in the weak topology on the space of probability measures on \(S\), and the element is selected so that it is measurable. Hence, the selected element is a random distribution function on \(S\). Given the observed data \((x_1, \ldots, x_n)\), we define random functions \(F^U\) and \(F^L\) as follows: for each \(\theta \in S\) and \((\mathbf{U}^*, \mathbf{W}^*)\),

\[
F^U(\theta) \equiv \min\{W_i^*: \text{for all } i \text{ such that } \theta < G_i^*(x_i - 1, U_i^*)\},
\]

and

\[
F^L(\theta) \equiv \max\{W_i^*: \text{for all } i \text{ such that } \theta \geq G_i^*(x_i, U_i^*)\},
\]

where \(\min \emptyset = 1\) and \(\max \emptyset = 0\). These functions are clearly non-decreasing and right continuous. Note that if \(Q_x(\mathbf{U}^*, \mathbf{W}^*) \neq \emptyset\), Portmanteau’s theorem (Billingsley, 1999) and (3) imply that a distribution function \(F \in \overline{Q_x(\mathbf{U}^*, \mathbf{W}^*)}\) if and only if \(F^L(\theta) \leq F(\theta) \leq F^U(\theta)\) for all \(\theta \in S\). Thus the functions \(F^U\) and \(F^L\) will be called the upper and lower fiducial bounds, respectively. We generate realizations of \(F^U\) and \(F^L\) using a novel Gibbs sampler in the next section.

2.2 Gibbs Sampling and GFD based inference

We need to generate \((\mathbf{U}^*, \mathbf{W}^*)\) from the standard uniform distribution on a set described by Equation (4), which is achieved using a Gibbs sampler. For each fixed \(i\), we denote random vectors with the \(i\)-th observation removed by \((\mathbf{U}^*_{[-i]}, \mathbf{W}^*_{[-i]})\). If \((\mathbf{U}^*, \mathbf{W}^*)\) satisfy the constraint \([4]\), so do \((\mathbf{U}^*_{[-i]}, \mathbf{W}^*_{[-i]})\).
The proposed Gibbs sampler is based on the conditional distribution of

$$(U^*_i, W^*_i) \mid U^*_{[-i]}, W^*_{[-i]},$$

which is a bivariate uniform distribution on a set $A$, where $A$ is a disjoint union of small rectangles. The beginnings and ends of the rectangles’ bases are the neighboring points in the set $\bigcup_{j \neq i} \{G^*_i(x_i, U^*_j), G^*_i(x_i - 1, U^*_j)\}$, while the corresponding location and height on the vertical axis are determined by (4). The details are described in Algorithm 1 and a visualization of the rectangles is shown in Figure 4 in the Supplementary Material. Each marginal conditional distribution is supported on the entire $S$; therefore, we expect the proposed Gibbs sampler to mix well.

The proposed Gibbs sampler requires starting points, and we consider two potential initializations. The first one starts with randomly generated $(U^*, W^*)$ from independent $\text{Unif}(0, 1)$ and reorders $W^*$ so that the constraint (4) is satisfied. The second starting value is consistent with having a deterministic $\Theta$, e.g., $p = \sum_{i=1}^n x_i / \sum_{i=1}^n m_i$ for binomial data $X_i \sim \text{Bin}(m_i, p)$ and $\lambda = \sum_{i=1}^n x_i / n$ for Poisson data $X_i \sim \text{Poi}(\lambda)$. As these two starting points are very different, they can be used to monitor convergence. To streamline our presentation, in Sections 4 and 5, we present the numerical results using the first initialization.

From Algorithm 1, we output two distribution functions required for the proposed mixture and conservative confidence intervals. In the remainder of this paper, we denote the Monte Carlo realizations of the lower and upper fiducial bounds by $F^L_l$ and $F^U_l$, respectively, where $l = 1, \ldots, n_{\text{mcmc}},$ and
Algorithm 1: Pseudo algorithm for the fiducial Gibbs sampler

Input: Dataset, e.g., (\(m_i, x_i\)), \(i = 1, \ldots, n\) for binomial data, starting vectors \(u, w\) of length \(n\), \(n_{\text{mcmc}}\), \(n_{\text{burn}}\), and vector \(\theta_{\text{grid}}\) of length \(n_{\text{grid}}\).

1. for \(i = 1\) to \(n\) do
   2. \(\theta_L[i] = G^*_i(x_i - 1, u[i]), \theta_U[i] = G^*_i(x_i, u[i])\);
3. end
4. Run Gibbs Sampler using the initial values \(u, w, \theta_L, \theta_U\);
5. for \(j = 1\) to \(n_{\text{burn}} + n_{\text{mcmc}}\) do
   6. for \(i = 1\) to \(n\) do
   7. \(u^0 = u[-i], w^0 = w[-i], \theta^0_L = \theta_L[-i], \theta^0_U = \theta_U[-i]\);
   8. \(u^0_L = G_i(x_i, \theta^0_L), u^0_U = G_i(x_i - 1, \theta^0_U)\);
   9. Sort \(u^0\) as \(w^0\_	ext{pre} = \theta^0_L, \theta^0_U\), denoted as \(u^\text{sort}\);
   10. Sort \(w^0, 1(n - 1), 1, 0\) according to the order of \(w^0\_	ext{pre} = w^0_U\), where \(1(n - 1)\) is a vector with elements \(1\) of length \(n - 1\);
   11. Sort \(0(n - 1), w^0, 1, 0\) according to the order of \(w^0\_	ext{pre} = w^0_L\), where \(0(n - 1)\) is a vector with elements \(0\) of length \(n - 1\);
   12. \(w^\text{pre}_L = \text{cummin}(w^*_U), w^\text{pre}_U = \text{rev(cummax}(\text{rev}(w^*_L)))\);
   13. Take the component-wise difference of \(w^\text{sort}\), denoted as \(w^\text{diff}\);
   14. for \(k = 1\) to \(2n - 1\) do
   15. \(w^\text{diff}[k] = w^\text{pre}_U[k] - w^\text{pre}_L[k + 1]\);
   16. end
   17. Sample \(i^* \in \{1, \ldots, 2n - 1\}\) with probability \(\propto w^\text{diff} \cdot w^\text{diff}\);
   18. Sample \(a\) and \(b\) from independent \(\text{Unif}(0,1)\), and set
   19. \(u = u^\text{sort}[i^*] + w^\text{diff}[i^*] \cdot a, w = w^\text{pre}_U[i^*] - w^\text{diff}[i^*] \cdot b\),
   20. \(\theta_L = G^*_i(x_i - 1, u), \theta_U = G^*_i(x_i, u)\),
   21. \(u[i] = u, w[i] = w, \theta_L[i] = \theta_L, \theta_U[i] = \theta_U\);
   22. end
   23. Generate \(n\) i.i.d. \(\text{Unif}(0,1)\) and sort them according to the order of \(w\), denoted by \(w^*\). Replace \(w\) by \(w^*\), i.e., \(w = w^*\);
24. end
25. Evaluate the upper and lower bounds on a grid of values \(\theta_{\text{grid}}\) for each MCMC sample after burn-in, indexed by \(l\);
26. for \(j = 1\) to \(n_{\text{grid}}\) do
27. \(F^L_l(\theta_{\text{grid}}[j]) = \max\{w[\theta_U \leq \theta_{\text{grid}}[j]]\}\);
28. \(F^U_l(\theta_{\text{grid}}[j]) = \min\{w[\theta_L \geq \theta_{\text{grid}}[j]]\}\);
29. end
30. return The fiducial samples \(F^U_l, F^L_l\) evaluated on \(\theta_{\text{grid}}\).
\( n_{\text{mcmc}} \) is the number of fiducial samples.

We propose using the median of the \( 2n_{\text{mcmc}} \) samples \( \{F^L_l(\theta), F^U_l(\theta), l = 1, \ldots, n_{\text{mcmc}}\} \) as a point estimator of the distribution function \( F(\theta) \). We construct two types of pointwise confidence intervals, conservative and mixture, using appropriate quantiles of fiducial samples (Hannig, 2009). In particular, the 95\% conservative confidence interval is formed by taking the empirical 0.025 quantile of \( \{F^L_l(\theta), l = 1, \ldots, n_{\text{mcmc}}\} \) as the lower limit and the empirical 0.975 quantile of \( \{F^U_l(\theta), l = 1, \ldots, n_{\text{mcmc}}\} \) as the upper limit. The lower and upper limits of the 95\% mixture confidence interval are formed by taking the empirical 0.025 and 0.975 quantiles of \( \{F^L_l(\theta), F^U_l(\theta), l = 1, \ldots, n_{\text{mcmc}}\} \), respectively.

Remark 1. Algorithm 1 is general to any discrete distribution of \( X_i \). For example, if \( X_i \) follows a binomial distribution, \( G_i \) is the CDF of the binomial distribution with number of trials \( m_i \) and \( G^*_i(x_i, u_i) \) is the \((1 - u_i)\) quantile of Beta\((x_i + 1, m_i - x_i)\); if \( X_i \) follows a Poisson distribution, \( G_i \) is its CDF, and \( G^*_i(x_i, u_i) \) is the \((1 - u_i)\) quantile of Gamma\((x_i + 1, 1)\).

2.3 Further illustration with a simulated dataset

To streamline our presentation, we consider the binomial case as our running example hereinafter, i.e., the observed data are \((m_i, x_i)\), and \( X_i \sim \text{Bin}(m_i, P_i) \), \( i = 1, \ldots, n \), where \( P_i \in S = [0, 1] \) plays the role of \( \Theta_i \). We also provide the details of the proposed approach and some examples for the Poisson data in the Supplementary Material.

We present a toy example to demonstrate the proposed fiducial approach.
Suppose that $F$ follows the beta distribution Beta$(5, 5)$. The number of trials $m_i = 20$, $i = 1, \ldots, n$. The sample size of the simulated binomial data is $n = 20$. The fiducial estimates are based on 10000 iterations after 1000 burn-in times.

Figure 1 presents the last MCMC sample of the lower fiducial bound $F^L_i(p)$ (blue line) and upper fiducial bound $F^U_i(p)$ (red line) for the two starting points, respectively. As the fiducial distribution reflects the uncertainty, we do not expect every single fiducial curve to be close to the true CDF (black line). Furthermore, Figure 2 presents the mixture (blue line for lower limit; red line for upper limit) and conservative (cyan line for lower limit; magenta line for upper limit) confidence intervals (CIs) with two starting points, respectively, computed from the MCMC sample. We also plot the point estimates of the proposed approach alongside Efron’s $g$-modeling. The brown curve represents the fiducial point estimate $\hat{F}(p)$. The dashed curve is the point estimate of $F(p)$ for Efron’s $g$-modeling without bias correction. Efron’s confidence intervals with bias correction look almost the same as those without correction; thus, we omit them in the figures. As can be seen, the proposed fiducial point estimator and confidence intervals capture the shape of the true CDF fairly well.
Figure 1: The last MCMC sample from generalized fiducial distribution. The blue curve is a realization of the lower fiducial bound $F_L(p)$ and the red curve is a realization of the upper fiducial bound $F_U(p)$. The black curve is the true $F(p)$. Each panel represents a different starting value.

Figure 2: Point estimates and 95% CIs for $F(p)$ given a fixed simulated dataset. Each panel represents an interval computed from a realization of the MCMC chain initiated with different starting values. The orange curve is the fiducial point estimate $\hat{F}(p)$. The dashed curve is the point estimate of $F(p)$ for Efron’s $g$-modeling. The black curve is the true $F(p)$. The blue and red curves are the lower and upper limits of the mixture CIs, respectively. The cyan and magenta curves are the lower and upper limits of the conservative CIs, respectively.
3 Theoretical results

Recall that the GFD is a data-dependent distribution defined for every fixed dataset $x$. It can be made into a random measure in the same way as one defines the usual conditional distribution, i.e., by plugging random variables $X$ into the observed dataset. In this section, we study the asymptotic behavior of this random measure for a binomial distribution $X_i \sim \text{Bin}(m_i, P_i)$ when the rate of $m_i$ is much faster than $n$, i.e., the following assumption holds.

**Assumption 1.** $\lim_{n \to \infty} n^4 (\log n)^{1+\varepsilon}/(\min_{i=1}^{n} m_i) = 0$ for any $\varepsilon > 0$.

We provide a central limit theorem for $F_L(p)$. The proof of Theorem 3.1 is deferred to Section B of the Appendix. A similar result holds for $F_U(p)$.

**Theorem 3.1.** Suppose the true CDF is absolutely continuous with a bounded density. Based on Assumption 1,

$$n^{1/2} \{ F_L(\cdot) - \hat{F}_n(\cdot) \} \to B_F(\cdot), \tag{6}$$

in distribution on the Skorokhod space $\mathcal{D}[0,1]$ in probability, where

$$\hat{F}_n(s) \equiv \frac{1}{n} \sum_{i=1}^{n} I[P_i \leq s], \tag{7}$$

is the oracle empirical CDF constructed based on unobserved $P_i$ used to generate the observed $X_i$, $i = 1, \ldots, n$, and $B_F(\cdot)$ is a mean zero Gaussian process with covariance $\text{cov}(B_F(s), B_F(t)) = F(t \wedge s) - F(t)F(s)$.

Note that the stochastic process on the left-hand side of (6) is naturally in $\mathcal{D}[0,1]$, i.e., the space of functions on $[0,1]$ that are right continuous and
have left limits. Distances on $D[0, 1]$ are measured using Skorokhod’s metric, which makes it a Polish space (Billingsley, 1999). To understand the mode of convergence used here, note that there are two sources of randomness. One is from the fiducial distribution derived based on each fixed dataset. The other is the usual randomness of the data. Thus, (6) can be interpreted as

$$
\rho \left( n^{1/2} \{ F^L(\cdot) - \hat{F}_n(\cdot) \}, B_F(\cdot) \right) \overset{pr}{\to} 0,
$$

(8)

where $\rho$ is any metric metrizing weak convergence of probability measures on the Polish space $D[0, 1]$, e.g., Lévy-Prokhorov or Dudley metric (Shorack, 2017). The distribution of $n^{1/2} \{ F^L(\cdot) - \hat{F}_n(\cdot) \}$ in the argument of $\rho$ is a fiducial distribution, i.e., induced by the randomness of $(U^*, W^*)$ with the data $X_i$ and $P_i$ being fixed. Consequently, the left-hand side of (8) is a function of $X_i$ and $P_i$, and the convergence in probability is based on the distribution of the data.

Theorem 3.1 establishes a Bernstein-von Mises theorem for the fiducial distribution under Assumption 1, which implies that the confidence intervals described in Section 2 have asymptotically correct coverage. Moreover, Theorem 3.1 provides a sufficient condition for the $n^{1/2}$-estimability of the binomial probability parameter’s distribution function. Although Assumption 1 is pretty stringent, it does not seem likely that one can establish a unified asymptotic property of the proposed fiducial approach under a general scheme. This is best seen by the fact that in the binomial case, $m_i$ are uniformly bounded, and there is not enough information in the data to consistently estimate the underlying distribution function of $P$. 

13
Interestingly, looking at the fiducial solution reveals an interesting connection to a different statistical problem (Cui et al., 2021). Recall that quantities $P_i$, $i = 1, \ldots, n$ are only known to be inside random intervals $(G_i^*(x_i - 1, U_i^*), G_i^*(x_i, U_i^*))$. Therefore, the statistical problem we study here is similar to non-parametric estimation under Turnbull’s general censoring scheme (Efron, 1967; Turnbull, 1976), in which case there is no unified theory of the nonparametric maximum likelihood estimator, but properties are investigated under various special and challenging cases, such as the $n^{1/2}$ convergence for right-censored data (Breslow and Crowley, 1974) and the $n^{1/3}$ convergence for current status data (Groeneboom and Wellner, 1992).

Remark 2. Note that

$$\Pr(F \in \mathcal{Q}_n(U^*, W^*)) \propto \prod_{i=1}^{n} \int_0^1 \binom{m}{x_i} p^{x_i} (1-p)^{m-x_i} dF(p)$$

is proportional to the nonparametric likelihood function. Some implications of this observation are provided in Section F of the Supplementary Material.

4 Numerical experiments

We performed simulation studies to compare the frequentist properties of the proposed fiducial confidence intervals with Efron’s $g$-modeling (Efron, 2016; Efron and Narasimhan, 2016), the nonparametric bootstrap, and a nonparametric Bayesian approach (Ross and Markwick, 2019). For each scenario, we first generated $p_i$, $i = 1, \ldots, n$ from the distribution function $F$. Then we drew $X_i$ from the binomial distribution $\text{Bin}(m_i, p_i)$, where $m_i$ is
described in Section 4.1. The simulations were replicated 500 times for each scenario.

For the proposed method and other existing methods, we chose the grid \( p = 0.01, 0.02, \ldots, 0.99 \) following Narasimhan and Efron (2016). The fiducial estimates were based on 2000 iterations of the Gibbs sampler after 500 burn-in times. Efron’s g-modeling was implemented using the R package \texttt{deconvolveR} (Efron and Narasimhan, 2016). We used default values for the degree of the splines, i.e., 5. We considered the regularization strategy with the default value \( c_0 = 1 \). For the nonparametric bootstrap, we first obtained the maximum likelihood estimates \( \hat{p}_i = x_i / m_i \). We then constructed the empirical CDF as the point estimator, and used \( B = 1000 \) bootstrap samples of \( \hat{p}_i \) to construct confidence intervals. We also considered a fully Bayesian approach, i.e., the Dirichlet process mixture of beta binomial distributions that provides more flexibility than a beta binomial model (Ross and Markwick, 2018, p19). The default values for the prior parameters in the R package \texttt{dirichletprocess} (Ross and Markwick, 2019) were used. The Bayesian estimates were based on 2000 MCMC samples after 500 burn-in times.

### 4.1 Simulation settings

We start with the following two scenarios with fixed \( m_i \)'s.

**Scenario 1.** We consider the same setting as in Section 2.3. Let \( F \) follow \( \text{Beta}(5, 5) \), and the number of trials \( m_i = 20, i = 1, \ldots, n \). The sample size \( n \) of the simulated binomial data is set to 50.

**Scenario 2.** Let \( F \) follow a mixture of beta distributions \( 0.5\text{Beta}(10, 30) + \)
0.5Beta(30, 10), and the number of trials be \( m_i = 20, \ i = 1, \ldots, n \). The sample size \( n = 50 \).

Next, we consider three complex settings from Zhang and Liu (2012).

**Scenario 3.** (Beta density) Let \( F \) follow Beta(8, 8), and \( m_i \)'s be integers sampled uniformly between 100 and 200. The sample size \( n = 100 \).

**Scenario 4.** (Multimodal distribution) We consider a mixture of beta distributions 0.5Beta(60, 10) + 0.5Beta(10, 60) for \( F \). The sample size \( n = 100 \), and \( m_i = 100, \ i = 1, \ldots, n \).

**Scenario 5.** (Truncated exponential) Let \( F \) be Exp(8) truncated at 1. The sample size \( n = 200 \) and \( m_i = 100 \) for \( i = 1, \ldots, n \).

We also consider the above five settings for \( n = 1000 \). The results are reported in the Supplementary Material.

### 4.2 Numerical results

In this section, we first compare the mean squared error (MSE) of different methods of \( F(p) \) for the five scenarios. The numerical results for \( p = 0.15, 0.25, 0.5, 0.75, 0.85 \) for each scenario are presented in Table 1. We see that the MSEs of the proposed fiducial point estimates are as good as and sometimes smaller than those of the competing methods.

Next, we present the coverage and average length of confidence intervals for the various methods in Tables 2, 3. Table 2 summarizes the coverage of 95% confidence intervals of various methods, and Table 3 summarizes the average length of these confidence intervals. “M” denotes mixture GFD.
| Scenario | p   | F  | g  | bc | BP | BA |
|----------|-----|----|----|----|----|----|
| 1        | 0.15| 5  | 40 | 39 | 17 | 1  |
|          | 0.25| 20 | 49 | 48 | 69 | 10 |
|          | 0.50| 51 | 47 | 48 | 71 | 78 |
|          | 0.75| 18 | 22 | 21 | 16 | 10 |
|          | 0.85| 5  | 13 | 12 | 4  | 1  |
| 2        | 0.15| 41 | 149| 149| 145|9   |
|          | 0.25| 39 | 17 | 18 | 66 | 101|
|          | 0.50| 49 | 32 | 33 | 54 | 53 |
|          | 0.75| 37 | 18 | 19 | 50 | 88 |
|          | 0.95| 46 | 86 | 84 | 33 | 12 |
| 3        | 0.15| 0.14|8.45|8.16|0.15|0.04|
|          | 0.25| 2.57|14.89|14.41|2.87|1.39|
|          | 0.50| 22.85|26.94|27.04|24.68|21.92|
|          | 0.75| 2.54|5.03|4.81|2.80|1.34|
|          | 0.85| 0.16|1.46|1.38|0.15|0.05|
| 4        | 0.15| 22 | 49 | 49 | 23 | 27 |
|          | 0.25| 27 | 34 | 33 | 26 | 25 |
|          | 0.50| 25 | 24 | 25 | 26 | 26 |
|          | 0.75| 26 | 36 | 35 | 27 | 25 |
|          | 0.85| 20 | 52 | 52 | 26 | 25 |
| 5        | 0.15| 11 | 10 | 10 | 11 | 10*|
|          | 0.25| 6  | 10 | 10 | 6  | 6* |
|          | 0.50| 1  | 5  | 4  | 1  | 1* |
|          | 0.75| 0.13|1.84|1.66|0.12|0.10*|
|          | 0.85| 0.05|0.76|0.68|0.05|0.04*|

Table 1: MSE \((\times 10^{-4})\) of point estimates for \(F(p)\) of each scenario. “F” denotes the fiducial point estimates; “g” denotes Efron’s \(g\)-modeling without bias correction; “bc” denotes Efron’s \(g\)-modeling with bias correction; “BP” denotes the bootstrap method; “BA” denotes the Bayesian method. *The Bayesian results for Scenario 5 are reported based on 489 replications as 11 runs failed because of an error in the R package \texttt{dirichletprocess}. 
| Scenario | $p$ | M | C | g | bc | BP | BA |
|----------|----|---|---|---|----|----|----|
| 1        |    |   |   |   |    |    |    |
|          | 0.15 | 99 | 100 | 25 | 27 | 74 | 88 |
|          | 0.25 | 99 | 100 | 80 | 82 | 65 | 83 |
|          | 0.50 | 100| 100 | 94 | 94 | 91 | 84 |
|          | 0.75 | 100| 100 | 96 | 96 | 92 | 83 |
|          | 0.85 | 100| 100 | 90 | 91 | 54 | 89 |
| 2        |    |   |   |   |    |    |    |
|          | 0.15 | 98 | 99 | 1  | 1  | 27 | 97 |
|          | 0.25 | 100| 100 | 98 | 98 | 90 | 86 |
|          | 0.50 | 98 | 98 | 96 | 96 | 97 | 45 |
|          | 0.75 | 100| 100 | 96 | 96 | 85 | 87 |
|          | 0.85 | 97 | 97 | 31 | 31 | 79 | 97 |
| 3        |    |   |   |   |    |    |    |
|          | 0.15 | 100| 100 | 4  | 5  | 13 | 28 |
|          | 0.25 | 99 | 99 | 69 | 71 | 86 | 28 |
|          | 0.50 | 99 | 99 | 89 | 89 | 96 | 27 |
|          | 0.75 | 98 | 99 | 97 | 97 | 87 | 32 |
|          | 0.85 | 99 | 100| 97 | 98 | 12 | 31 |
| 4        |    |   |   |   |    |    |    |
|          | 0.15 | 99 | 100 | 48 | 49 | 95 | 59 |
|          | 0.25 | 95 | 97 | 89 | 90 | 93 | 12 |
|          | 0.50 | 95 | 96 | 93 | 93 | 95 | 6  |
|          | 0.75 | 95 | 95 | 89 | 89 | 93 | 9  |
|          | 0.85 | 99 | 99 | 75 | 75 | 91 | 59 |
| 5        |    |   |   |   |    |    |    |
|          | 0.15 | 99 | 99 | 95 | 95 | 93 | 18*|
|          | 0.25 | 97 | 98 | 95 | 95 | 93 | 21*|
|          | 0.50 | 98 | 98 | 98 | 98 | 89 | 20*|
|          | 0.75 | 98 | 99 | 100| 100| 36 | 20*|
|          | 0.85 | 99 | 100| 100| 100| 17 | 11*|

Table 2: Coverage (in percent) of 95% CIs for $F(p)$ of each scenario. “M” denotes mixture GFD confidence intervals; “C” denotes conservative GFD confidence intervals; “g” denotes Efron’s $g$-modeling without bias correction; “bc” denotes Efron’s $g$-modeling with bias correction; “BP” denotes the bootstrap method; “BA” denotes the Bayesian method. *The Bayesian results for Scenario 5 are reported based on 489 replications as 11 runs failed because of an error in the R package dirichletprocess.
| Scenario | p   | M   | C   | g   | bc  | BP  | BA  |
|----------|-----|-----|-----|-----|-----|-----|-----|
| 1        | 0.15| 123 | 139 | 101 | 101 | 84  | 28  |
|          | 0.25| 220 | 241 | 171 | 171 | 172 | 95  |
|          | 0.50| 426 | 465 | 246 | 246 | 272 | 279 |
|          | 0.75| 217 | 239 | 154 | 154 | 128 | 89  |
|          | 0.85| 121 | 137 | 82  | 81  | 42  | 28  |
| 2        | 0.15| 243 | 266 | 66  | 66  | 185 | 101 |
|          | 0.25| 357 | 390 | 162 | 162 | 251 | 311 |
|          | 0.50| 305 | 330 | 206 | 206 | 275 | 90  |
|          | 0.75| 352 | 385 | 166 | 166 | 226 | 301 |
|          | 0.85| 245 | 268 | 143 | 143 | 133 | 107 |
| 3        | 0.15| 35  | 42  | 46  | 46  | 4   | 2   |
|          | 0.25| 77  | 84  | 81  | 81  | 51  | 10  |
|          | 0.50| 243 | 259 | 168 | 168 | 195 | 33  |
|          | 0.75| 78  | 86  | 65  | 65  | 51  | 10  |
|          | 0.85| 35  | 42  | 27  | 26  | 4   | 2   |
| 4        | 0.15| 240 | 259 | 125 | 125 | 179 | 101 |
|          | 0.25| 201 | 213 | 191 | 191 | 195 | 16  |
|          | 0.50| 193 | 202 | 188 | 188 | 196 | 0.02|
|          | 0.75| 201 | 213 | 198 | 198 | 195 | 16  |
|          | 0.85| 240 | 259 | 178 | 178 | 173 | 98  |
| 5        | 0.15| 160 | 171 | 125 | 125 | 126 | 18* |
|          | 0.25| 115 | 124 | 114 | 114 | 94  | 13* |
|          | 0.50| 45  | 50  | 74  | 73  | 35  | 6*  |
|          | 0.75| 20  | 24  | 37  | 36  | 7   | 1*  |
|          | 0.85| 17  | 20  | 23  | 22  | 3   | 1*  |

Table 3: Mean length (×10^{-3}) of 95% CIs for F(p) of each scenario. “M” denotes mixture GFD confidence intervals; “C” denotes conservative GFD confidence intervals; “g” denotes Efron’s g-modeling without bias correction; “bc” denotes Efron’s g-modeling with bias correction; “BP” denotes the bootstrap method; “BA” denotes the Bayesian method. *The Bayesian results for Scenario 5 are reported based on 489 replications as 11 runs failed because of an error in the R package dirichletprocess.
confidence intervals; “C” denotes conservative GFD confidence intervals; “g”
denotes Efron’s \( g \)-modeling without bias correction; “bc” denotes Efron’s \( g \)-
modeling with bias correction; “BP” denotes the bootstrap method; “BA”
denotes the Bayesian method.

We see that the GFD confidence intervals maintain or exceed the nominal
coverage everywhere, whereas other methods often have coverage problems.
In particular, the Efron’s confidence intervals and the nonparametric boot-
strap have substantial coverage problems close to the boundary, while the
Bayesian method consistently underestimates the uncertainty, resulting in
credible intervals that are too narrow.

It is not surprising that the GFD confidence intervals are often longer
than those of the other methods, as the GFD approach aims to provide a
conservative way to quantify uncertainty. As expected, the mean length of
the mixture GFD confidence intervals is slightly shorter than the conservative
GFD confidence intervals. A potential reason for the fiducial approach out-
performing Efron’s \( g \)-modeling in terms of coverage is that Efron’s \( g \)-modeling
relies on an exponential family parametric model, and the proposed fiducial
approach is nonparametric. Therefore, the proposed method is expected to
perform better when the true model does not belong to an exponential family.

## 5 Intestinal surgery data

In this section, we consider an intestinal surgery study on gastric adenocar-
cinoma involving \( n = 844 \) cancer patients (Gholami et al., 2015). Resection
of the primary tumor with appropriate dissection of the surrounding lymph
nodes is the foundation of curative care. In addition to the primary tumor, surgeons also removed the satellite nodes for later testing. Efron’s deconvolution was used to estimate the prior distribution of the probability of one satellite being malignant in this study (Gholami et al., 2015).

The dataset consists of pairs \((m_i, X_i), i = 1, \ldots, n\), where \(m_i\) is the number of satellites removed, and \(X_i\) is the number of satellites found to be malignant. The \(m_i\) varies from 1 to 69. Among all cases, 322 have \(X_i = 0\). For the rest, \(X_i/m_i\) has an approximate \(\text{Unif}(0, 1)\) (Efron, 2016). We are interested in estimating the distribution function of the probability of one satellite being malignant. Following the model proposed in Efron (2016), we assume a binomial model, i.e., \(X_i \sim \text{Bin}(m_i, P_i)\), where \(P_i\) is the \(i\)-th patient’s probability of any one satellite being malignant.

We compared the proposed mixture and conservative GFD confidence intervals to Efron’s with and without bias correction, the bootstrap method, and the fully Bayesian approach. For all methods, we used the grid \([0.01, 0.02, \ldots, 0.99]\) for the discretization of \(p\). The fiducial and Bayesian estimates were based on 10000 iterations after 1000 burn-in times. Other tuning parameters for each method were chosen in the same manner as in Section 4.

The overall shapes of the bootstrap and Bayesian confidence intervals are similar to the fiducial ones but much narrower. We provide bootstrap and Bayesian point estimates and 95% confidence intervals in the Supplementary Material. Figure 3 shows the point estimates and 95% confidence intervals of the distribution function \(F\) for the proposed GFD approach and Efron’s \(g\)-modeling. Overall, the GFD confidence intervals are more conservative. The GFD confidence intervals cover Efron’s almost everywhere.
Figure 3: GFD vs. g-modeling. Estimated CDF (dashed line) and 95% CIs for \( F(p) \) of GFD and Efron’s g-modeling. The red and orange curves are mixture and conservative confidence intervals, respectively, and the blue curve is Efron’s confidence intervals without bias correction. Efron’s confidence intervals with bias correction look almost the same as those without correction; thus, we omit the bias-corrected ones in the figure.
For the proposed fiducial approach, there is a large mode for the upper fiducial confidence interval near \( p = 0 \), which reflects the fact that approximately 38\% of the \( X_i \)'s are 0 in the surgery data. However, the Bayesian method and Efron’s \( g \)-modeling seem to quantify the uncertainty of this proportion to be lower. One exception is the nonparametric bootstrap, which gives the estimation of point mass at zero 0.38 with 95\% confidence intervals (0.35, 0.41). We note that this might be overestimated, as \( X_i = 0 \) may correspond to a non-zero probability \( p \) especially when \( m_i \) is small.

Moreover, the generalized fiducial confidence intervals provide a unimodal density, whereas Efron’s gives a bimodal density. Thus, we believe that the fiducial, bootstrap, and nonparametric Bayesian answers are more in line with Efron’s observation that \( X_i / m_i \) has an approximate Unif(0, 1) for those \( X_i \neq 0 \) in the surgery data (Efron, 2016).

6 Discussion

In this paper, we proposed a prior-free approach to the nonparametric deconvolution problem, and obtained valid point estimates and confidence intervals. This was accomplished through a novel algorithm to sample from the GFD. The median of the GFD was used as the point estimate, and appropriate quantiles of the GFD evaluated at a given \( p \) provided pointwise confidence intervals. We also studied the theoretical properties of the fiducial distribution. Extensive simulations show that the proposed fiducial approach is a good alternative to existing methods, including Efron’s \( g \)-modeling. We applied the proposed fiducial approach to intestinal surgery data to estimate
the probability that each satellite site is malignant for patients.

We conclude by listing some open research problems:

1. The proposed fiducial method seems to be a powerful nonparametric approach. It would be interesting to implement it inside other statistical procedures, such as tree or random forest models, to include covariates (Wu et al., 2019).

2. This paper focuses on discrete data. The proposed approach can be extended to continuous data, such as Normal(Θ, 1), where Θ follows a distribution function $F$. This part is currently under investigation.

3. As we can see from the simulations, the GFD approach is sometimes over-conservative. It might be possible to consider a different choice of fiducial samples, such as log-interpolation (Cui and Hannig, 2019a) or monotonic spline interpolation (Taraldsen and Lindqvist, 2019; Cui and Hannig, 2019b).

4. The asymptotic distribution results in Section 3 only hold under Assumption 1. To investigate a non-$n^{1/2}$ rate of convergence should provide a fruitful avenue for future research.

5. It should be possible to use the GFD in conjunction with various functional norms to construct simultaneous confidence bands (Cui and Hannig, 2019a; Nair, 1984; Martin, 2019).
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Appendix

A Lemmas

Recall that the observed data points $x_i$ are integers.

**Lemma 1.** $x_i = G_i^{-1}(u_i, \theta_i)$ if and only if $G_i^*(x_i - 1, u_i) < \theta_i \leq G_i^*(x_i, u_i)$.

**Proof.** Recall the definition of $G_i^{-1}(u_i, \theta_i) = \inf\{x_i : G_i(x_i, \theta_i) \geq u_i\}$. So $x_i = G_i^{-1}(u_i, \theta_i)$ if and only if $G_i(x_i, \theta_i) \geq u_i > G_i(x_i - \epsilon, \theta_i)$ for all $\epsilon > 0$. Then by the definition of $G_i^*$, it is further equivalent to $G_i^*(x_i - 1, u_i) < \theta_i \leq G_i^*(x_i, u_i)$. □

**Lemma 2.** $Q_x(u, w) \neq \emptyset$ if and only if $u, w$ satisfy: whenever $G_i^*(x_i, u_i) \leq G_j^*(x_j - 1, u_j)$ then $w_i < w_j$.

**Proof.** Sufficiency: If $Q_x(u, w) \neq \emptyset$ holds, and $G_i^*(x_i, u_i) \leq G_j^*(x_j - 1, u_j)$, then we know that $w_i \leq F(G_i^*(x_i, u_i)) \leq F(G_j^*(x_j - 1, u_j)) < w_j$.

Necessity: We prove it by contradiction. If $Q_x(u, w)$ is empty, then there must exist indices $i$ and $j$ such that, $(G_j^*(x_j - 1, u_j), G_j^*(x_j, u_j))$ is strictly larger than $(G_i^*(x_i - 1, u_i), G_i^*(x_i, u_i))$ but $w_i \geq w_j$. This contradicts with whenever $G_i^*(x_i, u_i) \leq G_j^*(x_j - 1, u_j)$ then $w_i < w_j$. □

B Proof of Theorem 3.1

**Proof.** Recall that the data generating equation is

$$X_i = G_i^{-1}(U_i, P_i), \quad P_i = F^{-1}(W_i),$$  \hspace{1cm} (9)
where $G_i$ is the CDF of binomial distribution. Define the oracle fiducial distribution based on unobserved $P_1, \ldots, P_n$ as

$$
\tilde{F}(s) = \sum_{i=0}^{n} I[P(i) \leq s < P(i+1)] W^*_i, \quad (10)
$$

where $P(0) \equiv 0$, $P(n+1) \equiv 1$, $W^*_i$ are uniform order statistics, and $W^*_0 \equiv 0$. Note that (10) is the lower fiducial distribution $F^L$ in Cui and Hannig (2019a) when there is no censoring. By Corollary 1 of the Supplementary Material, we have

$$
n^{1/2} \{ \tilde{F}(\cdot) - \hat{F}_n(\cdot) \} \to \{ 1 - F(\cdot) \} B(\gamma(\cdot)),
$$
in distribution on the Skorokhod space $D[0,1]$ in probability, where $B$ is a Brownian motion, $\gamma(t) = \int_0^t \frac{f(s)}{[1-F(s)]^2} ds = \frac{F(t)}{1-F(t)}$, and $\hat{F}_n$ is the oracle empirical distribution function defined in (7). We also define

$$
\tilde{F}^L(s) = \sum_{i=0}^{n} I[G^U_i \leq s < G^U_{i+1}] W^*_i,
$$

and

$$
\tilde{F}^U(s) = \sum_{i=0}^{n} I[G^L_i \leq s < G^L_{i+1}] W^*_i,
$$

where $G^U(1), \ldots, G^U(n)$ are order statistics of $\{G^*_i(x_i, U^*_i), i = 1, \ldots, n\}$, $G^L(1), \ldots, G^L(n)$ are order statistics of $\{G^*_i(x_i - 1, U^*_i), i = 1, \ldots, n\}$, $G^U(0) \equiv 0$, $G^U(n+1) \equiv 1$, $G^L(0) \equiv 0$, $G^L(n+1) \equiv 1$, and $W^*_0 \equiv 1$. Note that $\tilde{F}^L$ and $\tilde{F}^U$ can be regarded as the lower and upper bounds of $F^L$ and $F^U$, respectively.
In order to obtain

\[ n^{1/2}\{F^L(\cdot) - \hat{F}_n(\cdot)\} \to \{1 - F(\cdot)\}B(\gamma(\cdot)), \]

in distribution in probability, it is enough to show that

\[ \sup n^{1/2}\left| \tilde{F}(s) - F^L(s) \right| \to 0, \]

in probability, which would be implied by

\[ \sup n^{1/2}\{\tilde{F}^U(s) - \tilde{F}^L(s)\} \to 0, \tag{11} \]

in probability. In order to show Equation (11), one essentially needs to show that for any \( \epsilon > 0 \),

\[ pr\left( \sup n^{1/2}\{\tilde{F}^U(s) - \tilde{F}^L(s)\} > \epsilon \right) \to 0. \]

By Lemma 4, given in the Supplementary Material, there is no intersection between \{\((G_i^*(x_i - 1, U^*_i), G_i^*(x_i, U^*_i)), i = 1, \ldots, n\}\) with a large probability converging to 1.

Thus, we have that

\[ pr\left( \sup n^{1/2}\{\tilde{F}^U(s) - \tilde{F}^L(s)\} > \epsilon \right) \leq \sum_{i=0}^{n} pr(W^*_{i+1} - W^*_i > \frac{\epsilon}{n^{1/2}}) \]

\[ = (n + 1) \times pr(Beta(1, n) > \frac{\epsilon}{n^{1/2}}) \]

\[ = (n + 1) \times (1 - \frac{\epsilon}{n^{1/2}})^n \to 0. \]
Therefore, we have

\[ n^{1/2} \{ F^L(\cdot) - \hat{F}_n(\cdot) \} \to \{ 1 - F(\cdot) \} B(\gamma(\cdot)), \]

in distribution in probability. Note that for any \( t < s \),

\[ \text{cov}[\{ 1 - F(s) \} B(\gamma(s)), \{ 1 - F(t) \} B(\gamma(t))] = \gamma(t)\{ 1 - F(s) \}\{ 1 - F(t) \} = F(t)\{ 1 - F(s) \}, \]

which completes the proof.
Lemma 3 and its proof

Lemma 3. Assume the conditions of Theorem 3.1. Suppose that $X_i \sim \text{Bin}(m_i, P_i)$, where $P_i$ are unobserved i.i.d. random variables with a distribution function $F$. We have that

$$\min_{i,j \in \{0,\ldots,n\}} \left\{ \frac{X_i}{m_i} - \frac{X_j}{m_j} \right\} \geq O\left( \frac{1}{n^2 \log n \sqrt{\epsilon/2}} \right),$$

for any $\epsilon > 0$, with a large probability converging to 1.

Proof. We first show that the unobserved $P_i = F^{-1}(W_i)$ are well-separated. Straightforward calculation with uniform order statistics shows that

$$\Pr\left( \min_{i \in \{0,\ldots,n\}} \left\{ W_{(i+1)} - W_{(i)} \right\} > \frac{t}{n(n+1)} \right) \geq \left( 1 - \frac{t}{n} \right)^n,$$

for any $t > 0$, where $W_{(0)} \equiv 0$ and $W_{(n+1)} \equiv 1$. Therefore,

$$\min_{i \in \{0,\ldots,n\}} \left\{ P_{(i+1)} - P_{(i)} \right\} \geq O\left( \frac{1}{n^2 \log n \sqrt{\epsilon/2}} \right),$$

with a large probability converging to 1, where $P_{(0)} \equiv 0$ and $P_{(n+1)} \equiv 1$. Furthermore, by the following Bernstein inequality for binomial $X \sim \text{Bin}(m, P)$,

$$\Pr(|X - mP| \geq t) \leq 2 \exp \left\{ - \frac{t^2}{2[mP(1-P) + t/3]} \right\},$$
where \( t > 0 \). Taking \( t \sim O\left(\frac{m}{n^2(\log n)^{\sqrt{\epsilon}/2}}\right) \), we have that

\[
\min_{i,j} \left\{ \frac{X_i}{m_i} - \frac{X_j}{m_j} \right\} \geq O\left(\frac{1}{n^2(\log n)^{\sqrt{\epsilon}/2}}\right),
\]

with a large probability converging to 1 because \( \lim_{n \to \infty} n^4(\log n)^{1+\epsilon}/m_i = 0 \) for any \( m = m_i \) given by Assumption 1.

\[\square\]

D Lemma 4 and its proof

**Lemma 4.** Given data that satisfy Equation (12) of Lemma 3, if \((U^*, W^*)\) are uniformly distributed on the set \(\{(u^*, w^*) : Q_x(u^*, w^*) \neq \emptyset\}\), we have that

\[
\Pr\{(G^*_i(x_i-1, U^*_i), G^*_i(x_i, U^*_i)) \cap (G^*_j(x_j-1, U^*_j), G^*_j(x_j, U^*_j)) \neq \emptyset \text{ for some } i \neq j\} \to 0.
\]

(13)

**Proof.** Let \( \tilde{U}_i \) be i.i.d. \( U(0, 1) \) and denote \( L_i = G^*_i(x_i-1, \tilde{U}_i) \sim \text{Beta}(x_i, m_i - x_i + 1) \), and \( R_i = G^*_i(x_i, \tilde{U}_i) \sim \text{Beta}(x_i + 1, m_i - x_i) \). Recall that the proposed Algorithm 1 can be regarded as an importance sampling in the following way: if there is one \( k \)-intersections (i.e., \((k + 1)\) intervals share one common area) in \(\{(L_i, R_i), i = 1, \ldots, n\}\), the corresponding \( W^* \) has \((k + 1)!\) possible permutations. So we need to show \( \sum_{k=1}^{n-1}(k+1)!q_k \to 0 \) as \( n \to 0 \), where \( q_k \) is defined as the probability of \(\{(L_i, R_i), i = 1, \ldots, n\}\) having one or more \( k \)-intersections, and \( W^*_{i_1} < \ldots < W^*_{i_k} \) (say \( i_1, \ldots, i_k \) are indices of intervals that intersect).

We start with one-intersection between the \( i \)-th and \( j \)-th intervals, i.e.,
(L_i, R_i) \cap (L_j, R_j) \neq \emptyset$, which is equivalent to $L_1 \leq R_2$ and $L_2 \leq R_1$. Without loss of generality, we focus on $L_1 \leq R_2$ as

$$\text{pr}(A_1 \cap A_2) \leq \min\{\text{pr}(A_1), \text{pr}(A_2)\}.$$  

For a random variable $Y \sim \text{Beta}(\alpha, \beta)$, let $\mu = E[Y] = \frac{\alpha}{\alpha + \beta}$. By Theorem 2.1 of [Marchal and Arbel (2017)], $Y$ is sub-Gaussian with the variance proxy parameter

$$\Sigma \equiv \frac{1}{4(\alpha + \beta + 1)}.$$  

Hence, by the definition of a sub-Gaussian random variable, for any $c, t \in \mathbb{R}$,

$$\text{pr} (Y - \mu \geq t) \leq \exp \left\{ -\frac{t^2}{2\Sigma} \right\}.$$  

Note that

$$\text{pr}(L_1 \leq R_2) = \text{pr}(R_2 - L_1 \geq 0), \quad (14)$$  

where $L_1$ is sub-Gaussian with mean $x_1/(m_1 + 1)$ and $\Sigma_1 = 1/(m_1 + 2)$, and $R_2$ is sub-Gaussian with mean $(x_2 + 1)/(m_2 + 1)$ and $\Sigma_2 = 1/(m_2 + 2)$. Therefore, Equation (14) equals $\text{pr}(Z \geq x_1/(m_1 + 1) - (x_2 + 1)/(m_2 + 1))$, where $Z$ is a sub-Gaussian with mean 0 and $\Sigma_Z = 1/(m_1 + 2) + 1/(m_2 + 2)$.
Then we further have that Equation (14) equals

\[ \Pr\{Z \geq x_1/(m_1 + 1) - (x_2 + 1)/(m_2 + 1)\} \]
\[ \leq \exp\left\{ -\frac{T^2}{2\Sigma Z} \right\} \]
\[ \leq \exp\{-\tilde{m}T^2\}, \]

where \( \tilde{m} = \min_{i=1,\ldots,n} m_i \), and \( T = x_1/(m_1 + 1) - (x_2 + 1)/(m_2 + 1) \).

Thus, the probability of a single intersection \( 2! \times q_1 \) is bounded by

\[ 2! \times \binom{n}{2} \times \exp \left[-\tilde{m}T^2\right] \leq \exp \left\{ c_1 \log n - c_2 (\log n)^{1+\epsilon/2} \right\}, \]

for any \( \epsilon > 0 \), and some constants \( c_1 \) and \( c_2 \). We note that the coefficient \( \binom{n}{2} \) refers to the number of possible pairs.

Next, we consider the case of \( k \)-intersections. We only need to consider two intervals corresponding to the two farthest \( P_i \) among \( k \) intervals. Thus, the probability of existing \( k \)-intersections \( (k+1)! \times q_k \) is bounded by

\[ (k+1)! \times \binom{n}{k} \times \exp \left\{ -\tilde{m} \left[ O \left( \frac{k}{n^2(\log n)^{\sqrt{\epsilon/2}}} \right) \right]^2 \right\} \]
\[ \leq \exp \left\{ c_1 (k \log n + k \log k) - c_2 k^2 (\log n)^{1+\epsilon/2} \right\}, \]

for any \( \epsilon > 0 \), and some constants \( c_1 \) and \( c_2 \). Because \( \sum_{k=1}^{n-1} (k+1)! \times q_k \rightarrow 0 \), we conclude (13).
E  Corollary 1 and its proof

We present a Bernstein-von Mises theorem for fiducial distribution associated with the empirical distribution function. This result can be viewed either as a special special case (without censoring) of Cui and Hannig (2019a), or as a particular case of the exchangeably weighted bootstrap in Praestgaard and Wellner (1993).

Corollary 1. Assume the conditions of Theorem 3.1. We have

\[ n^{1/2} \{ \tilde{F}(\cdot) - \hat{F}_n(\cdot) \} \to \{ 1 - F(\cdot) \} B(\gamma(\cdot)), \]

in distribution on the Skorokhod space \( \mathcal{D}[0,1] \) in probability, where \( B \) is a Brownian motion, \( \gamma(t) = \int_0^t \frac{f(s)}{[1-F(s)]^2} ds = \frac{F(t)}{1-F(t)} \), \( \hat{F}_n \) is defined in Equation (7) of Section 3.1, and \( \tilde{F} \) is defined in Equation (10) of Appendix B.

Proof. By Theorem 2 of Cui and Hannig (2019a), we need to check their Assumptions 1-3. Their Assumption 1 satisfies with their \( \pi(p) = 1 - F(p) \); their Assumption 2 satisfies as we assume true CDF is absolutely continuous; their Assumption 3 satisfies as we assume true CDF is absolutely continuous;

\[
\int_0^p \frac{g_n(s)}{\sum_{i=1}^n I(P_i \geq s)} d\sum_{i=1}^n I(P_i \leq s) \to \int_0^p \frac{f(s)}{[1-F(s)]^2} ds,
\]

for any \( p \) such that \( 1 - F(p) > 0 \) and any sequence of functions \( g_n \to \frac{1}{1-F} \) uniformly. \qed
Remark on binomial and Poisson data

In the following two theorems for binomial and Poisson data, respectively, we show implications of the fact that \( pr(F \in Q_x(U^*, W^*)) \) is proportional to the nonparametric likelihood function. In particular, maximizing the scaled fiducial probability in its limit provides the underlying true CDF.

**Theorem F.1.** Suppose \( \Theta_i \equiv P_i \), and \( X_i \mid P_i \) follows Bin\((m, P_i)\), maximizing \( \lim_{n \to \infty} [c_n \times pr(F \in Q_x(U^*, W^*))]^{1/n} \) leads to a CDF matching the first \( m \)-moments of the true \( F(p) \), where \( c_n \) is the normalizing constant.

**Theorem F.2.** Suppose \( \Theta_i \equiv \Lambda_i \), and \( X_i \mid \Lambda_i \) follows Poi\((\Lambda_i)\), maximizing \( \lim_{n \to \infty} [c_n \times pr(F \in Q_x(U^*, W^*))]^{1/n} \) leads to the true \( F(\lambda) \) almost surely, where \( c_n \) is the normalizing constant.

**Proof of Theorem F.1.** Recall the data generating equation in (9), where \( G_i \) is the CDF of binomial distribution. The fiducial probability is

\[
pr(F \in Q_x(U^*, W^*)) \propto \prod_{i=1}^{n} \left[ \int_{0}^{1} \binom{m}{x_i} p^{x_i} (1 - p)^{m-x_i} dF(p) \right]
\]

\[
= \exp \left\{ \sum_{i=1}^{n} \log \left( E_F \left[ \binom{m}{x_i} p^{x_i} (1 - p)^{m-x_i} \right] \right) \right\}
\]

\[
= \exp \left\{ n_m \log(E_F[\binom{m}{P}]) + n_{m-1} \log \left( E_F \left[ \binom{m}{m-1} P^{m-1}(1 - P) \right] \right) + \ldots \right. 
\]

\[
+ n_0 \log(E_F[(1 - P)^m]) \right\},
\]

where \( n_k \) is the number of samples with \( X_i = k \), and \( E_F \) refers to the expec-
tation with respect to $F$ that is evaluated. Thus, as $n$ goes to infinity,

$$[c_n \times pr(F \in Q_x(U^*, W^*))]^{1/n} \to \exp \left\{ E[P^m] \log(E_F[P^m]) + E \left[ \left( \frac{m}{m-1} \right)^{P^m-1}(1 - P) \right] \log \left( E_F \left[ \left( \frac{m}{m-1} \right)^{P^m-1}(1 - P) \right] \right) 
+ \cdots + E[(1 - P)^m] \log(E_F[(1 - P)^m]) \right\},$$ (15)

where $c_n$ is the normalizing constant, and $E$ refers to the expectation with respect to the true distribution function that generates data. By the method of Lagrange multipliers,

$$H(x) = \sum_i y_i \log x_i \text{ subject to } \sum_i x_i = 1, \sum_i y_i = 1,$$

is maximized with respect to $x$ by setting $x_k = y_k$, $k \in \mathbb{R}^+$. Maximizing Equation (15) gives

$$E[P^m] = E_F[P^m],$$
$$E[P^{m-1}(1 - P)] = E_F[P^{m-1}(1 - P)],$$
$$\cdots$$
$$E[(1 - P)^m] = E_F[(1 - P)^m].$$

The above equations are restrictions on the first $m$-moments of $P$, which completes the proof. \qed
Proof of Theorem F.2. Recall the data generating equation is

\[ X_i = G_i^{-1}(U_i, \Lambda_i), \quad \Lambda_i = F^{-1}(W_i), \]

where \( G_i \) is the CDF of Poisson distribution. The fiducial probability is

\[ pr(F \in Q_x(U^*, W^*)) \propto \prod_{i=1}^{n} \int_0^\infty \frac{\lambda^{x_i} \exp\{-\lambda\}}{x_i!} dF(\lambda) \]

\[ = \left\{ \frac{E_F[\exp(-\Lambda)]}{0!} \right\}^{n_0} \times \left\{ \frac{E_F[\Lambda \exp(-\Lambda)]}{1!} \right\}^{n_1} \times \cdots \times \left\{ \frac{E_F[\Lambda^k \exp(-\Lambda)]}{k!} \right\}^{n_k} \times \cdots \]

\[ = \exp \left\{ n \left[ \frac{n_0}{n} \log \frac{E_F[\exp(-\Lambda)]}{0!} + \frac{n_1}{n} \log \frac{E_F[\Lambda \exp(-\Lambda)]}{1!} + \cdots + \frac{n_k}{n} \log \frac{E_F[\Lambda^k \exp(-\Lambda)]}{k!} + \cdots \right] \right\}, \]

where \( n_k \) is the count of \( X_i = k \), and \( E_F \) refers to the expectation with respect to \( F \) that is evaluated. Thus, as \( n \) goes to infinity,

\[ [c_n \times pr(F \in Q_x(U^*, W^*))]^{1/n} \rightarrow \exp \left\{ \left[ \frac{E[\exp(-\Lambda)]}{0!} \log \frac{E_F[\exp(-\Lambda)]}{0!} + \frac{E[\Lambda \exp(-\Lambda)]}{1!} \log \frac{E_F[\Lambda \exp(-\Lambda)]}{1!} + \cdots \right. \]

\[ + \left. \frac{E[\Lambda^k \exp(-\Lambda)]}{k!} \log \frac{E_F[\Lambda^k \exp(-\Lambda)]}{k!} + \cdots \right\}, \tag{16} \]

where \( c_n \) is the normalizing constant, and \( E \) refers to the expectation with respect to the true distribution function that generates data. By the method of Lagrange multipliers, maximizing Equation (16) gives

\[ \frac{E[\Lambda^k \exp(-\Lambda)]}{k!} = \frac{E_F[\Lambda^k \exp(-\Lambda)]}{k!}, \quad k \in \mathbb{R}^+. \]
The above equations are essentially restrictions on all derivatives of the Laplace transform at 1. The property of the Laplace transform being analytic in the region of absolute convergence implies the uniqueness of a distribution, which completes the proof.

\[ \square \]

**G Rectangles of Algorithm 1**

![Figure 4: A visualization of rectangles in Algorithm 1 for the final step of the Gibbs sampler \((j = n_{\text{burn}} + n_{\text{mcmc}}, i = n)\) for the toy example of Section 2.3. The last \((U_n, W_n)\) was generated from the gray area, with the rectangles described in Section 2.2 indicated by black lines. Just like in Figure 1, each panel represents a different starting value.]

**H Additional simulation results with \(n = 1000\)**

In this section, we present the results for both point estimates and 95% CIs for \(n = 1000\). The simulations were replicated 500 times for each scenario. We again observe a consistent pattern that the proposed methods are comparable
to and sometimes better than the competing methods.

| Scenario | $p$ | $F$ | $g$ | bc | BP | BA |
|----------|-----|-----|-----|----|----|----|
| 1        | 0.15 | 0.50 | 1.79 | 1.74 | 10.65 | 0.04 |
|          | 0.25 | 1.49 | 1.46 | 1.44 | 52.03 | 0.65 |
|          | 0.50 | 3.92 | 9.54 | 9.52 | 27.67 | 2.94 |
|          | 0.75 | 1.32 | 0.61 | 0.62 | 5.48  | 0.67 |
|          | 0.85 | 0.48 | 0.17 | 0.16 | 1.30  | 0.04 |
| 2        | 0.15 | 5    | 10   | 9   | 122   | 2*  |
|          | 0.25 | 4    | 8    | 8   | 19    | 14* |
|          | 0.50 | 3    | 3    | 3   | 3     | 4*  |
|          | 0.75 | 4    | 3    | 3   | 17    | 16* |
|          | 0.95 | 4    | 1    | 1   | 18    | 2*  |
| 3        | 0.15 | 0.02 | 0.33 | 0.30 | 0.02  | 0.003 |
|          | 0.25 | 0.28 | 1.27 | 1.21 | 0.65  | 0.10 |
|          | 0.50 | 3.02 | 6.98 | 6.95 | 2.81  | 2.21 |
|          | 0.75 | 0.29 | 0.11 | 0.11 | 0.54  | 0.12 |
|          | 0.85 | 0.02 | 0.01 | 0.01 | 0.02  | 0.002 |
| 4        | 0.15 | 3    | 5    | 5   | 3     | 2   |
|          | 0.25 | 3    | 2    | 2   | 3     | 2   |
|          | 0.50 | 2    | 2    | 2   | 2     | 2   |
|          | 0.75 | 3    | 2    | 2   | 5     | 2   |
|          | 0.85 | 3    | 34   | 34  | 8     | 2   |
| 5        | 0.15 | 2    | 3    | 3   | 2     | 2*  |
|          | 0.25 | 1    | 1    | 1   | 1     | 1*  |
|          | 0.50 | 0.20 | 0.26 | 0.25 | 0.20  | 0.17* |
|          | 0.75 | 0.02 | 0.11 | 0.10 | 0.02  | 0.02* |
|          | 0.85 | 0.01 | 0.04 | 0.04 | 0.01  | 0.01* |

Table 4: MSE (×10⁻⁴) of point estimates for $F(p)$ of each scenario. “F” denotes the fiducial point estimates; “g” denotes Efron’s $g$-modeling without bias correction; “bc” denotes Efron’s $g$-modeling with bias correction; “BP” denotes the bootstrap method; “BA” denotes the Bayesian method. *The Bayesian results for Scenarios 2, 5 are reported based on 495, 476 replications as 5, 24 runs failed because of an error in the R package dirichletprocess.
Table 5: Coverage (in percent) of 95% CIs for $F(p)$ of each scenario. “M” denotes mixture GFD confidence intervals; “C” denotes conservative GFD confidence intervals; “g” denotes Efron’s $g$-modeling without bias correction; “bc” denotes Efron’s $g$-modeling with bias correction; “BP” denotes the bootstrap method; “BA” denotes the Bayesian method. *The Bayesian results for Scenarios 2, 5 are reported based on 495, 476 replications as 5, 24 runs failed because of an error in the R package `dirichletprocess`.  

| Scenario | $p$ | M | C | g | bc | BP | BA |
|----------|----|---|---|---|----|----|----|
| 1        |    | 0.15 | 98 | 99 | 28 | 31 | 0  | 93 |
|          |    | 0.25 | 100 | 100 | 91 | 92 | 0  | 83 |
|          |    | 0.50 | 100 | 100 | 84 | 84 | 10 | 70 |
|          |    | 0.75 | 100 | 100 | 98 | 98 | 17 | 80 |
|          |    | 0.85 | 98  | 99  | 98 | 98 | 10 | 88 |
| 2        |    | 0.15 | 93  | 98  | 10 | 12 | 0  | 89*|
|          |    | 0.25 | 100 | 100 | 82 | 82 | 17 | 87*|
|          |    | 0.50 | 97  | 98  | 93 | 93 | 91 | 65*|
|          |    | 0.75 | 100 | 100 | 97 | 97 | 15 | 85*|
|          |    | 0.85 | 95  | 97  | 98 | 98 | 0  | 89*|
| 3        |    | 0.15 | 99  | 99  | 74 | 80 | 72 | 31 |
|          |    | 0.25 | 98  | 99  | 68 | 71 | 77 | 29 |
|          |    | 0.50 | 99  | 100 | 71 | 71 | 94 | 20 |
|          |    | 0.75 | 99  | 99  | 100| 100| 82 | 28 |
|          |    | 0.85 | 99  | 99  | 100| 100| 69 | 28 |
| 4        |    | 0.15 | 100 | 100 | 75 | 76 | 89 | 49 |
|          |    | 0.25 | 95  | 95  | 96 | 96 | 91 | 13 |
|          |    | 0.50 | 96  | 96  | 96 | 96 | 96 | 3  |
|          |    | 0.75 | 94  | 96  | 96 | 96 | 85 | 12 |
|          |    | 0.85 | 100 | 100 | 2  | 2  | 55 | 53 |
| 5        |    | 0.15 | 99  | 100 | 87 | 87 | 93 | 22*|
|          |    | 0.25 | 99  | 99  | 96 | 96 | 96 | 21*|
|          |    | 0.50 | 98  | 98  | 98 | 98 | 94 | 29*|
|          |    | 0.75 | 99  | 100| 99 | 99 | 90 | 29*|
|          |    | 0.85 | 99  | 99  | 100| 100| 56 | 26*|
| Scenario | $p$ | M  | C  | g  | bc | BP | BA |
|----------|-----|-----|-----|-----|----|----|----|
| 1        | 0.15| 27  | 29  | 23  | 23 | 23 | 8  |
|          | 0.25| 64  | 69  | 39  | 39 | 40 | 26 |
|          | 0.50| 147 | 156 | 86  | 86 | 62 | 47 |
|          | 0.75| 64  | 69  | 35  | 35 | 32 | 26 |
|          | 0.85| 26  | 29  | 15  | 15 | 16 | 8  |
| 2        | 0.15| 69  | 75  | 43  | 43 | 43 | 45*|
|          | 0.25| 132 | 142 | 74  | 74 | 57 | 120*|
|          | 0.50| 72  | 76  | 65  | 65 | 62 | 35*|
|          | 0.75| 132 | 142 | 72  | 72 | 51 | 121*|
|          | 0.85| 70  | 75  | 35  | 35 | 32 | 45*|
| 3        | 0.15| 7   | 7   | 12  | 12 | 3  | 0.49|
|          | 0.25| 25  | 27  | 24  | 24 | 19 | 3  |
|          | 0.50| 92  | 96  | 60  | 60 | 62 | 8  |
|          | 0.75| 25  | 27  | 17  | 17 | 18 | 3  |
|          | 0.85| 7   | 7   | 4   | 4  | 3  | 0.49|
| 4        | 0.15| 99  | 105 | 56  | 56 | 57 | 24 |
|          | 0.25| 65  | 67  | 62  | 62 | 62 | 6  |
|          | 0.50| 62  | 63  | 62  | 62 | 62 | 0.02|
|          | 0.75| 65  | 67  | 61  | 61 | 62 | 5  |
|          | 0.85| 99  | 105 | 57  | 57 | 55 | 24 |
| 5        | 0.15| 81  | 86  | 52  | 52 | 56 | 8* |
|          | 0.25| 57  | 60  | 44  | 44 | 43 | 6* |
|          | 0.50| 21  | 22  | 20  | 20 | 17 | 3* |
|          | 0.75| 7   | 8   | 11  | 11 | 5  | 1* |
|          | 0.85| 5   | 6   | 6   | 6  | 2  | 1* |

Table 6: Mean length ($\times 10^{-3}$) of 95% CIs for $F(p)$ of each scenario. “M” denotes mixture GFD confidence intervals; “C” denotes conservative GFD confidence intervals; “g” denotes Efron’s $g$-modeling without bias correction; “bc” denotes Efron’s $g$-modeling with bias correction; “BP” denotes the bootstrap method; “BA” denotes the Bayesian method. *The Bayesian results for Scenarios 2, 5 are reported based on 495, 476 replications as 5, 24 runs failed because of an error in the R package dirichletprocess.
I Trace plots of the proposed Gibbs sampler for intestinal surgery data

In this section, we present trace plots of the proposed Gibbs sampler for intestinal surgery data. As can be seen from these figures, the fiducial MCMC samples have good variability and mix well.

Figure 5: Trace plots of mean and variance of $P \sim \frac{[F_L(p) + F_U(p)]}{2}$ for intestinal surgery data, respectively.
Additional plots for intestinal surgery data

In Figure 6, we plot the point estimates and 95% confidence intervals of $F(p)$ for the Bayesian method and nonparametric bootstrap, respectively.

Figure 6: Estimated CDF (dashed line) and 95% CIs for $F(p)$. Left panel: fiducial vs. Bayesian. Right panel: fiducial vs. nonparametric bootstrap. The red and orange curves are the mixture and conservative confidence intervals, respectively. The brown and black curves are the Bayesian and bootstrap confidence intervals, respectively.

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