The isotropic s-wave superconductors with non-magnetic scattering is given by

\[ H_{c2}(0) = \mu_h T_c \left( \frac{dH_{c2}}{dT} \right)_{T_c}, \]

(1)

with \( \mu_h \approx 0.4 \) in a range of scattering parameters covering nearly all practical transport scattering rates.

At zero temperature, one replaces the sum with an integral according to \( 2\pi T \sum_{\omega} \to \int_0^\infty d(\hbar \omega) \) to obtain:

\[ \lambda^{-2}(0) = 4\pi^2 e^2 N(0) v^2 \left( 1 + \frac{4 \tan^{-1} \frac{\eta}{\sqrt{1-\eta^2}}}{\pi \sqrt{1-\eta^2}} \right), \]

(4)

where the scattering parameter

\[ \eta = \frac{\hbar}{2\pi \Delta_0} = \frac{\pi}{\sqrt{2}} \frac{\xi_0}{T}. \]

(5)

\( \xi_0 = h/\pi \Delta_0 \) is the BCS zero-T coherence length and \( \ell \) is the transport mean-free path. Eq. (4) works for any \( \eta > 0 \). For \( \eta > 1 \), it can be written in explicitly real form by replacing \( \tan^{-1} \to -\tanh^{-1} \) and \( \sqrt{1-\eta^2} \to \sqrt{\eta^2-1} \).

When the scattering parameter

\[ \rho = \frac{\hbar v}{2\pi T \ell}, \]

(6)

\( \gamma \approx 0.577 \) is the Euler constant. The sum in Eq. (3) is expressed in terms of digamma functions \( \psi \). Doing the algebra one obtains the slope at \( T_c \):

\[ \frac{d\lambda^{-2}}{dT} = \frac{64\pi^2 e^2 N(0) v^2}{21 \zeta(3) v^2 T_c} \left[ \psi \left( \frac{1 + \rho}{2} \right) - \psi \left( \frac{1}{2} \right) - \frac{\pi^2 \rho}{4} \right]. \]

(7)

Following Ref. 1, one defines the quantity

\[ \mu^{-1}_\lambda = -\frac{T_c}{\lambda^{-2}(0)} \left( \frac{d\lambda^{-2}}{dT} \right)_{T_c} = -\left( \frac{d\rho_s}{dt} \right)_{t=1}, \]

(8)

where \( t = T/T_c \), \( \rho_s(t) = \lambda^2(0)/\lambda^2(t) \) is commonly called the superfluid density. We then obtain from Eqs. (7), (4):

\[ \mu^{-1}_\lambda = \frac{16 e^7}{7 \zeta(3) \rho^2} \left[ \psi \left( \frac{1 + \rho}{2} \right) - \psi \left( \frac{1}{2} \right) - \frac{\pi^2 \rho}{4} \right] \]

\[ \left/ \left( 1 + \frac{4 \tan^{-1} \frac{\eta}{\sqrt{1-\eta^2}}}{\pi \sqrt{1-\eta^2}} \right) \right., \quad \eta = \rho e^\gamma. \]

(9)
Remarkably, the scattering parameter does not enter this relation at all. The only material parameter on the RHS is the density of states $N(0)$. The physical reason for this result can be traced to the Rutgers thermodynamic relation, in which the product of slopes in Eq. (12) is proportional to the transport-scattering-independent specific heat jump at $T_c$ (Anderson’s theorem).

In particular, this relation can be checked in the clean limit where at $T_c$

$$\frac{d\lambda^{-2}}{dT} = -\frac{16\pi e^2 N(0)v^2}{3c^2T_c}, \quad \frac{dH_{c2}}{dT} = -\frac{24\pi \phi_0 T_c}{7\zeta(3)\hbar^2 v^2}.$$  \quad (13)

In the dirty limit, we have:

$$\frac{d\lambda^{-2}}{dT} = -\frac{16\pi^3 c^2 N(0)v^2}{21\zeta(3)c^2T_c\rho}, \quad \frac{dH_{c2}}{dT} = -\frac{24\pi \phi_0 T_c}{\pi \nu^2 \hbar^2}.$$  \quad (14)

The slopes of $\lambda^{-2}$ at $T_c$ follow from Eq. (7); one can find slopes of $H_{c2}$ in Ref. 1.

Both clean and dirty limits are not quite realistic. According to Fig. 1, within the range $1 < \rho < 10$ one has approximate relations:

$$\left(\frac{d\lambda^{-2}}{dT}\right)_{T_c} \approx -\lambda^{-2}(0) \frac{0.4}{T_c},$$  \quad (15)

while

$$\left(\frac{dH_{c2}}{dT}\right)_{T_c} \approx \frac{H_{c2}(0)}{0.7T_c}.$$  \quad (16)

Using Eq. (12) along with (15) and (16) one obtains:

$$H_{c2}(0)\lambda^{-2}(0) \approx 415 \frac{T_c^2 N(0)}{\phi_0}.$$  \quad (17)

This, however, holds if there is no low-temperature paramagnetic limiting of $H_{c2}$. Utilizing only Eq. (15), one gets:

$$-\lambda^{-2}(0) \left(\frac{dH_{c2}}{dT}\right)_{T_c} \approx 593 \frac{T_cN(0)}{\phi_0},$$  \quad (18)

a potentially useful relation, since the paramagnetism is not involved here and one can express $\lambda^{-2}(0)$ in terms of the slope $(dH_{c2}/dT)_{T_c}$, and the density of states $N(0)$. Noting that $N(0) = 3\gamma/2\pi^2$ ($\gamma$ is the coefficient in linear low-$T$ dependence of the specific heat) and the slope $H_{c2}(T_c)$ are usually accessible, one can estimate a difficult to measure $\lambda(0)$. Note: in Eq. (18), temperatures are given in energy units (erg); with temperature in Kelvins one has:

$$-\lambda^{-2}(0) \left(\frac{dH_{c2}}{dT}\right)_{T_c} \left(\frac{G}{\text{cm}^2\text{K}}\right) \approx 593 \frac{k_B T_c N(0)}{\phi_0} \approx 4.5 \times 10^8 \left(\frac{1}{\text{cm}^2\text{G}}\right) T_c(K) \gamma \left(\frac{\text{erg}}{\text{cm}^3\text{K}^2}\right).$$  \quad (19)

This is perhaps the most useful result of our paper since it relates a difficult to measure $\lambda(0)$ to easily accessible slope $(dH_{c2}/dT)_{T_c}$.
IV. DISCUSSION

The above arguments hold for isotropic s-wave materials with non-magnetic scattering. In the presence of pair-breaking or for other than s-wave order parameter and general Fermi surfaces, the Anderson theorem does not work, and Eq. (12) is not expected to be valid. Eqs. (2) may still hold, however with the factor $\mu_H$ changing significantly with scattering, unlike the situation considered here. On the other hand, if the order parameter is constant at the Fermi surface of any shape (including multiband structures), there is no obvious reason for our results to be inapplicable.

To show how the obtained results can be applied for real materials, we estimate $\lambda(0)$ of V$_3$Si and Nb$_3$Sn using data of Orlando et al.\textsuperscript{8} For a sample of V$_3$Si with $T_c = 16.4$ K, $dH_{c2}/dT = -1.84 \times 10^4$ Oe/K, $\gamma = 2.2 \times 10^4$ erg/cm$^3$K$^2$, Eq. (19) yields $\lambda(0) \approx 106$ nm. Near $T_c$, $\lambda = \lambda_{GL}/\sqrt{T - T_c}$ and Ref. 8 provides $\lambda_{GL} = 62$ nm. In the clean limit we have $\lambda(0) = \lambda_{GL}/\sqrt{2} \approx 88$ nm, a reasonably close to 106 nm given uncertain scattering parameters for these data (as shown below, the dirty limit assumption would give $\lambda(0) = 1.63\lambda_{GL} \approx 102$ nm).

For Nb$_3$Sn with $T_c = 17.9$ K, $dH_{c2}/dT = -1.83 \times 10^4$ Oe/K, $\gamma = 1.1 \times 10^4$ erg/cm$^3$K$^2$, one obtains $\lambda(0) \approx 144$ nm, that corresponds to $\lambda_{GL} = \lambda(0)/\sqrt{2} \approx 102$ nm, whereas Ref. 8 cites $\lambda_{GL} = 64$ nm (the dirty limit assumption would have given $\lambda_{GL} \approx 88$ nm).

Another example is Rh$_3$In$_4$S$_4$ with $T_c = 2.25$ K, $dH_{c2}/dT = -1.69 \times 10^4$ G/K, $\gamma = 3 \times 10^4$ erg/mol K$^2 = 0.21 \times 10^4$ erg/cm$^3$K$^2$, and $\lambda_{GL} = 575$ nm.\textsuperscript{9} The ratio of zero-T BCS coherence length to the mean-free path for the sample studied was $\xi_0/\ell \sim 20 - 200$, i.e., it is the dirty limit. From Eq. (3) with $\eta \gg 1$ one obtains

$$\lambda^2 = \frac{8\pi T_c^2 N(0)D}{c^2\hbar}\frac{\Delta \tanh \Delta}{2T}, \quad (20)$$

($D = \nu\ell/3$ is the diffusivity). It is now readily shown that

$$\frac{\lambda^2(0)}{\lambda_{GL}^2} = \frac{4\pi^2 T_c}{\gamma(3)\Delta_0}. \quad (21)$$

Since, $\Delta_0/T_c \approx 1.76$, we estimate $\lambda(0) = 1.63\lambda_{GL} \approx 939$ nm.

On the other hand, Eq. (19) yields $\lambda(0) \approx 892$ nm. A reasonable agreement between our model and the data of this case might be due to the fact that the strong scattering washes away anisotropies of the order parameter thus making the material “more BCS-like”. Besides, the strong scattering excludes possibility of other than s-wave symmetry since even the transport scattering for a non-s-wave symmetry is pair breaking and superconductivity disappears well before the dirty limit is reached.

The multi-band MgB$_2$ is an example for which our model should not work. Still, taking $T_c \approx 39$ K, $dH_{c2}/dT = -0.41 \times 10^4$ G/K as given in Ref. 10 along with $\gamma = 7.2 \times 10^2$ erg/cm$^3$K$^2$ as provided by Ref. 11, with the help of Eq. (19) we estimate $\lambda(0) \approx 176$ nm. Ref. 11 cites 185 nm obtained from thermodynamic data, and a close value of 180 nm reported from analysis of microwave response.\textsuperscript{12} One may say that proximity of these numbers does not mean much. On the other hand, it shows that formulas based on the penetration depth property represented by Eq. (2), are quite robust and can be used for rough estimates in variety of situations, similar to what is commonly done with the $H_{c2}$ property of Eq. (1).

Still, since the above derivation of $\lambda(0)$, Eq. (4), has been done for s-wave superconductors with only transport scattering, one should not expect consequences of this equation to hold in the presence of pair breaking, be it due to the spin-flip or to other than s-wave order parameter. To check this we turn to well-studied CeCoIn$_5$, a clean superconductor with $T_c = 2.3$ K and $\lambda = 3 \times 10^4$ erg/cm$^3$K$^2$.\textsuperscript{13} The penetration depth in this material turned out\textsuperscript{14} to satisfy with high accuracy the relation $\lambda = \lambda(0)/\sqrt{T - T_c}$ established by Abrikosov-Gor’kov for a strong pair breaking.\textsuperscript{15} The fit to the data gives $\lambda(0) = 358$ nm and $dH_{c2}/dT = -11.5 \times 10^4$ G/K.\textsuperscript{14} We thus have all information needed to calculate $\lambda(0)$ with the help of Eq. (19) that gives $\lambda(0) = 608$ nm. The discrepancy is larger yet, if we compare with $\lambda(0) = 196$ nm, as found in microwave measurements.\textsuperscript{16} Clearly, our model fails in this case indicating the pair breaking as a possible culprit.

Also, our model fails being applied to KFe$_2$As$_2$ with $T_c = 3.5$ K, $\gamma = 1.53 \times 10^4$ erg/cm$^3$K$^2$ and $dH_{c2}/dT = -0.6$ T/K for the field along the $c$ direction.\textsuperscript{17-19} Equation (19) of the model yields $\lambda(0) = 158$ nm while the literature values are close to 200 nm.\textsuperscript{17,20-22} One of the reasons for disagreement is possible d-wave order parameter and a relatively strong anisotropy of superconducting properties.

In conclusion, we consider isotropic s-wave superconductors without pair-breaking scattering, but with arbitrary potential scattering. We establish new relations of the London penetration depth at $T = 0$ with its variation at $T_c$ and with the slope of the upper critical field $H_{c2}$. This gives a relatively simple way to estimate difficult to measure $\lambda(0)$ from measured $H_{c2}$ near $T_c$ and the specific heat (needed to estimate the density of states $N(0)$ at the Fermi level). On the other hand, if the obtained estimate comes out unreasonable or very different from independent measurement, this may signal the unconventional superconductivity or pair breaking in the studied material. We, therefore, believe that our work provides a useful practical tool for researchers dealing with experimental superconductivity.

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