NORM-INFLATION RESULTS FOR THE BBM EQUATION

JERRY BONA AND MIMI DAI

Abstract. Considered here is the periodic initial-value problem for the regularized long-wave (BBM) equation

\[ u_t + u_x + uu_x - u_{xtt} = 0. \]

Adding to previous work in the literature, it is shown here that for any \( s < 0 \), there is smooth initial data that is small in the \( L^2 \)-based Sobolev spaces \( H^s \), but the solution emanating from it becomes arbitrarily large in arbitrarily small time. This so called norm inflation result has as a consequence the previously determined conclusion that this problem is ill-posed in these negative-norm spaces.

1. Introduction

This note derives from the paper [7] where it was shown that the initial-value problem

\[ u_t + u_x + uu_x - u_{xtt} = 0, \quad u(0, x) = u_0(x), \tag{1.1} \]

for the regularized long-wave or BBM equation is globally well posed in the \( L^2 \)-based Sobolev spaces \( H^r(\mathbb{R}) \) provided \( r \geq 0 \). In the same paper, it was shown that the map that takes initial data to solutions cannot be locally \( C^2 \) if \( r < 0 \). This latter result suggests, but does not prove, that the problem (1.1) is not well posed in \( H^r \) for negative values of \( r \). Later, Panthee [15] showed that this solution map, were it to exist on all of \( H^r(\mathbb{R}) \), could not even be continuous, thus proving that the problem is ill posed in the \( L^2 \)-based Sobolev spaces with negative index. Indeed, he showed that there is a sequence of smooth initial data \( \{ \phi_n \}_{n=1}^{\infty} \) such that \( \phi_n \to 0 \) in \( H^r(\mathbb{R}) \) but the associated solutions, \( \{ u_n \}_{n=1}^{\infty} \) have the property that \( \| u(\cdot, t) \|_{H^r} \) is bounded away from zero for all small values of \( t > 0 \) and all \( n \geq 1 \).

The BBM equation itself was initially put forward in [16] and [3] as an approximate description of long-crested, surface water waves. It is an alternative to the classical Korteweg-de Vries equation and has been shown to be equivalent in that, for physically relevant initial data, the solutions of the two models differ by higher order terms on a long time scale (see [3].) It predicts the propagation of surface water waves pretty well in its range of validity [5]. Finally, it is known rigorously to be a good approximation to solutions of the full, inviscid, water-wave problem by combining results in [11], [4] and [13] (see also [14]).

It is our purpose here to show that in fact, for \( r < 0 \), the problem (1.1) is not only not well posed, but features blow-up in the \( H^r \)-norm in arbitrarily short time. This will be done in the context of the periodic initial-value problem wherein \( u_0 \) is

The author M. Dai was partially supported by NSF grant DMS–1517583.
a periodic distribution lying in $H^r_{\text{per}}$ for some $r < 0$. Similar results hold for $H^r(\mathbb{R})$, but are not explicated here.

More precisely, it will be shown that, for any given $r < 0$, there is a sequence $\{u_n^0\}_{n=1}^\infty$ of smooth initial data such that $u_n^0 \to 0$ in $H^r_{\text{per}}$ and a sequence $\{T_n\}_{n=1}^\infty$ of positive times tending to 0 as $n \to \infty$ such that the corresponding solutions $\{u_n\}_{n=1}^\infty$ emanating from this initial data, whose existence is guaranteed by the periodic version [9] of the theory for the initial-value problem, are such that for $n = 1, 2, 3, \cdots$,

$$\|u(\cdot, T_n)\|_{H^r_{\text{per}}} \geq n.$$  

This insures in particular that the solution map $S$ that associates solutions to initial data, which exists on $L^2$, cannot be extended continuously to all of $H^s_{\text{per}}$, thus reproducing Panthee’s conclusion. Results of this sort go by the appellation norm inflation for obvious reasons. The idea originated in the work of Bourgain and Pavlović [8] for the three-dimensional Navier-Stokes equation. The method of construction there was applied to some other dissipative fluid equations by the second author and her collaborators, see [12, 11, 10]. It suggests that the method is generic as well as sophisticated.

**Notation**

The notation used throughout is standard. For $r \in \mathbb{R}$, the collection $\dot{H}^r_{\text{per}}$ is the homogeneous space of $2\pi$-periodic distributions whose norm

$$\|f\|_{\dot{H}^r_{\text{per}}}^2 = \sum_{k=1}^\infty k^{2r} (|f_k|^2 + |g_k|^2)$$

is finite. Elements in $\dot{H}^r_{\text{per}}$ all have mean zero over the period domain $[0, 2\pi]$. Here, the $\{f_k\}$ are the Fourier sine coefficients and the $\{g_k\}$ are the Fourier cosine coefficients of $f$. Notice that $\dot{H}^r_{\text{per}}$ may be viewed simply as the $L^2$-functions on the period domain $[0, 2\pi]$ with mean zero. If $X$ is any Banach space, the set $C([0, T]; X)$ consists of the continuous functions from the real interval $[0, T]$ into $X$ with its usual norm.

## 2. Norm inflation

The principal result of our study is the following theorem.

**Theorem 2.1.** Let $r < 0$ by given. Then there is a sequence $\{u_j^0\}_{j=1}^\infty$ of $C^\infty$, periodic initial data such that

$$u_j^0 \to 0 \quad \text{as} \quad j \to \infty$$

in $\dot{H}^r_{\text{per}}$ and a sequence $\{T_j\}_{j=1}^\infty$ of positive times tending to zero as $j \to \infty$ such that if $u_j(x, t)$ is the solution emanating from $u_j^0$, then

$$\|u(\cdot, T_j)\|_{\dot{H}^r_{\text{per}}} \geq j$$

for all $j = 1, 2, \cdots$.

**Proof:** Fix $s > 0$, let $r = -s$ and consider a wavenumber $k_1 \in \mathbb{N}$ which, in due course, will be taken to be large. Let $k_2 = k_1 + 1$, define $\bar{u}$ by $\bar{u} = \sin(k_1 x) + \sin(k_2 x)$ and consider the $2\pi$-periodic, men zero initial data $u_0 = k_1^2 \bar{u}$ for (1.1) where $\gamma > 0$ will be restricted presently. Of course, $u_0$ is smooth, so the theory developed in [9] implies that a unique, global, smooth solution emanates from this initial data.
Notice also that the solution preserves the property of having zero mean, so it lies in $C([0, T]; \dot{H}^s_{\text{per}})$ for all $\rho \in \mathbb{R}$.

Let $\varphi(D_x)$ be the Fourier multiplier operator given in terms of its Fourier transform by $\hat{\varphi}(\xi) u(\xi) = \frac{k}{1 + k^2} \hat{u}(\xi)$. The equation (1.1) can be rewritten as

$$
\begin{align*}
  iu_t &= \varphi(D_x) u + \frac{1}{2} \varphi(D_x) (u^2), \\
  u(0, x) &= u_0(x).
\end{align*}
$$

Let $S(t) = e^{-it\varphi(D_x)}$ be the unitary group defining the evolution of the linear BBM equation. Then, Duhamel’s principle allows the solution of (1.1)-(2.2) to be written in the form

$$
(2.3) \quad u(x, t) = S(t)u_0(x) + u_1(s, t) + y(x, t)
$$

where

$$
(2.4) \quad y(x, t) = \int_0^t S(t - \tau) \varphi(D_x) [G_0(\tau) + G_1(\tau) + G_2(\tau)] d\tau
$$

with

$$
\begin{align*}
  G_0(\tau) &= \frac{1}{2} u_1^2(\tau) + u_1(\tau) S(\tau) u_0, \\
  G_1(\tau) &= u_1(\tau)y(\tau) + y(\tau)S(\tau)u_0, \\
  G_2(\tau) &= \frac{1}{2} y^2(\tau),
\end{align*}
$$

where the spatial dependence has been suppressed for ease of reading. The strategy is to show that by choosing $k_1$ sufficiently large, $u_1$ becomes large in a short time in the space $\dot{H}^s_{\text{per}} = \dot{H}^{s - \rho}_{\text{per}}$, while the error term $y$ remains under control in the same space.

In contrast to dissipative equations, the linear dispersion operator $S(t)$ only translates the wave, but does not change its magnitude; more precisely, for $k = 1, 2, \cdots$,

$$
(2.5) \quad S(t) \sin(kx) = \sin \left( kx - \frac{k}{1 + k^2} t \right), \quad S(t) \cos(kx) = \cos \left( kx - \frac{k}{1 + k^2} t \right).
$$

On the other hand, the operator $\varphi(D_x)$ both decreases the amplitude of its argument and adds rotation viz.

$$
(2.6) \quad \varphi(D_x) \sin kx = -i \frac{k}{1 + k^2} \cos kx, \quad \varphi(D_x) \cos kx = i \frac{k}{1 + k^2} \sin kx.
$$

It follows from this that $\varphi(D_x)$ vanishes on constant functions.

It is clear that if $s > 0$, then

$$
(2.7) \quad \| S(t)u_0 \|_{-s} = \| u_0 \|_{-s} \sim k_1^{-s}, \quad \text{while} \quad \| S(t)u_0 \|_0 = \| u_0 \|_0 \sim k_1^s.
$$
As we want the initial data to be small in $H^{-s}_{per}$, $\gamma$ is restricted to the range $(0, s)$. The formulas in (2.5) imply

$$S(\tau)\bar{u} = \sin \left( k_1 x - \frac{k_1}{1 + k_1^2} \tau \right) + \sin \left( k_2 x - \frac{k_2}{1 + k_2^2} \tau \right),$$

so that

$$[S(\tau)\bar{u}]^2 = \frac{1}{2} \left[ 1 - \cos \left( 2k_1 x - \frac{2k_1}{1 + k_1^2} \tau \right) \right] + \frac{1}{2} \left[ 1 - \cos \left( 2k_2 x - \frac{2k_2}{1 + k_2^2} \tau \right) \right]$$

$$+ \cos \left( (k_1 - k_2)x - \left( \frac{k_1}{1 + k_1^2} - \frac{k_2}{1 + k_2^2} \right) \tau \right)$$

$$- \cos \left( (k_1 + k_2)x - \left( \frac{k_1}{1 + k_1^2} + \frac{k_2}{1 + k_2^2} \right) \tau \right).$$

It then follows from (2.6) that

$$\frac{1}{2} \varphi(D_x)[S(\tau)\bar{u}]^2 = -i \frac{2k_1}{4 + 4k_1^2} \sin \left( 2k_1 x - \frac{2k_1}{1 + k_1^2} \tau \right)$$

$$- i \frac{2k_2}{4 + 4k_2^2} \sin \left( 2k_2 x - \frac{2k_2}{1 + k_2^2} \tau \right)$$

$$+ i \frac{k_1 - k_2}{2 + (k_1 - k_2)^2} \sin \left( (k_1 - k_2)x - \left( \frac{k_1}{1 + k_1^2} - \frac{k_2}{1 + k_2^2} \right) \tau \right)$$

$$- i \frac{k_1 + k_2}{2 + (k_1 + k_2)^2} \sin \left( (k_1 + k_2)x - \left( \frac{k_1}{1 + k_1^2} + \frac{k_2}{1 + k_2^2} \right) \tau \right)$$

$$\equiv I_1 + I_2 + I_3 + I_4.$$

Consider now the function $\sin (kx - \omega t)$ and calculate as follows:

$$\int_0^t S(t - \tau) \sin(kx - \omega \tau) d\tau = \int_0^t \sin \left( kx - \frac{k}{1 + k^2} (t - \tau) - \omega \tau \right) d\tau$$

$$= \left( \frac{k}{1 + k^2} - \omega \right)^{-1} \left( \cos \left( kx - \frac{k}{1 + k^2} t \right) - \cos \left( kx - \omega t \right) \right)$$

where use has been made of (2.5).
The latter formula, applied four times, allows us to calculate \( u_1 \) explicitly, to wit:

\[
\begin{align*}
u_1 &= k_1^{2\gamma} \int_0^t S(t - \tau) \left[ I_1 + I_2 + I_3 + I_4 \right] d\tau \\
&= -\frac{ik_1^{2\gamma}}{12} \left[ \frac{1 + k_1^2}{k_1^2} \right] \left[ \cos \left( 2k_1 x - \frac{2k_1 t}{1 + k_1^2} \right) - \cos \left( 2k_1 x - \frac{2k_1}{1 + 4k_1^2} t \right) \right] \\
&\quad - \frac{ik_1^{2\gamma}}{12} \left[ \frac{1 + k_2^2}{k_2^2} \right] \left[ \cos \left( 2k_2 x - \frac{2k_2 t}{1 + k_2^2} \right) - \cos \left( 2k_2 x - \frac{2k_2}{1 + 4k_2^2} t \right) \right] \\
&\quad + \frac{ik_1^{2\gamma}}{2} \left[ \frac{1 + (k_1 - k_2)^2}{1 + (k_1 - k_2)^2} \right] \left[ \cos \left( (k_1 - k_2) x - \left( \frac{k_1}{1 + k_1^2} - \frac{k_2}{1 + k_2^2} \right) t \right) - \cos \left( (k_1 - k_2) x - \frac{k_1 - k_2}{1 + (k_1 - k_2)^2} t \right) \right] \\
&\quad - \frac{ik_1^{2\gamma}}{2} \left[ \frac{1 + (k_1 + k_2)^2}{1 + (k_1 + k_2)^2} \right] \left[ \cos \left( (k_1 + k_2) x - \left( \frac{k_1}{1 + k_1^2} + \frac{k_2}{1 + k_2^2} \right) t \right) - \cos \left( (k_1 + k_2) x - \frac{k_1 + k_2}{1 + (k_1 + k_2)^2} t \right) \right].
\end{align*}
\]

A study of the various constants appearing above reveals that, up to absolute constants,

\[
\begin{align*}
u_1 &\sim -ik_1^{2\gamma} \left[ \cos \left( 2k_1 x - \frac{2k_1 t}{1 + k_1^2} \right) - \cos \left( 2k_1 x - \frac{2k_1}{1 + 4k_1^2} t \right) \right] \\
&\quad - ik_1^{2\gamma} \left[ \cos \left( 2k_2 x - \frac{2k_2 t}{1 + k_2^2} \right) - \cos \left( 2k_2 x - \frac{2k_2}{1 + 4k_2^2} t \right) \right] \\
&\quad + ik_1^{2\gamma} \left[ \cos \left( x - \left( \frac{k_1}{1 + k_1^2} - \frac{k_2}{1 + k_2^2} \right) t \right) - \cos \left( x - \frac{t}{2} \right) \right] \\
&\quad - ik_1^{2\gamma} \left[ \cos \left( (k_1 + k_2) x - \left( \frac{k_1}{1 + k_1^2} + \frac{k_2}{1 + k_2^2} \right) t \right) \right] \\
&\quad - \cos \left( (k_1 + k_2) x - \frac{k_1 + k_2}{1 + (k_1 + k_2)^2} t \right). \end{align*}
\]

as \( k_1 \) becomes large. Since

\[
| \cos(kx - \omega_1 t) - \cos(kx - \omega_2 t) | \leq |\omega_1 - \omega_2| t,
\]

straightforward calculations show that the first, second and fourth terms above are uniformly small compared to the third term, for large values of \( k_1 \). Indeed, they are all of order \( k_1^{2\gamma -1} t \), whereas the third term is of order \( k_1^{2\gamma} t \).

It follows from this that for all \( t \geq 0 \),

\[
\| u_1(t, \cdot) \|_{-s} \sim k_1^{2\gamma} t \quad \text{and likewise} \quad \| u_1(t, \cdot) \|_0 \sim k_1^{2\gamma} t.
\]

Thus, by taking \( k_1 \) large, the \( \dot{H}^{-s}_{per} \)-norm of \( u_1 \) can be made as big as we like.

As mentioned earlier, an estimate of the error term \( y \) is needed to complete the argument. It will in fact be shown that \( y \) is even bounded in \( L_2 \), let along \( \dot{H}^{-s}_{per} \).
To this end, use is made of one of a periodic version of one the bilinear estimates in [7].

Lemma 2.2. Let $u, v \in H^q_{per}$ with $q \geq 0$. Then

$$\|\varphi(D_x)(uv)\|_q \lesssim \|u\|_q \|v\|_q$$

where the implied constant only depends upon $q$.

The proof of this result is the same as the proof of Lemma 1 in [7], with sums replacing integrals.

Introduce the abbreviation $X_T$ for $C([0, T]; L^2)$ for ease of reading. The value of $T > 0$ will be specified momentarily. It follows from (2.10) and the implicit relationship (2.4) for the remainder $y$ that

$$\|y\|_{X_T} \lesssim T \|u_0\|_{X_T}^2 + T \|S(t)u_0\|_{X_T} \|y\|_{X_T} + T \|u_1\|_{X_T} \|y\|_{X_T}$$

$$+ T \|S(t)u_0\|_{X_T} \|y\|_{X_T} + T \|y\|_{X_T}^2$$

(2.11)

$$\lesssim T^3k_1^{4\gamma} + T^2k_1^{3\gamma} + (k_1^{2\gamma}T^2 + k_1^{4\gamma})|y|_{X_T} + T \|y\|_{X_T}^2$$

$$= A + BY + T^2,$$

where $Y = Y(T) = \|y\|_{X_T}$. As $y \in C([0, M]; L^2)$ for all $M > 0$, it follows that $Y(T)$ is a continuous function of $T$. Moreover, $Y(0) = 0$.

Choose $T_0 = k_1^{-\nu\gamma}$, where $\mu > \frac{\nu}{2}$. With this choice, we see that for $T \leq T_0$,

$$A = O(k_1^{\gamma(4-3\mu)} + k_1^{\gamma(3-2\mu)}) \quad \text{and} \quad B = O(k_1^{2\gamma(1-\mu)} + k_1^{\gamma(1-\mu)}),$$

as $k_1 \to \infty$ and all the exponents are negative.

Choose $k_1$ large enough that $B < \frac{A}{2}$ and $T$ and $A$ are both small. It follows in this circumstance that the quadratic polynomial

$$p(z) = A + (B - 1)z + Tz^2$$

has two positive roots, the smaller of which is denoted $z$ and the larger $\bar{z}$. Of course, $p(z) < 0$ for $z \in (z, \bar{z})$.

The inequality (2.11) may be expressed as

$$p(Y(T)) \geq 0.$$ 

As $Y(T)$ is continuous and $Y(0) = 0$, it follows that $Y(T) \leq \bar{z}$ for all $T \in [0, T_0]$. For $k_1$ large, $T_0 < 1$. When combined with the fact that $B < \frac{A}{2}$, it is readily deduced that

$$\bar{z} \leq 4A, \quad \text{whence} \quad Y(T) \leq 4A,$$

thus assuring that the remainder $y(\cdot, t)$ is indeed uniformly bounded in $\dot{H}_{per}^{-s}$ for $t \leq T_0$ and large choices of $k_1$.

Taking a suitably chosen, increasing sequence $\{k_1^{(j)}\}_{j=1}^{\infty}$ of wavenumbers for which

$$\lim_{j \to \infty} k_1^{(j)} = +\infty,$$

and with the indicated choices of $\gamma$ and $\mu$, (2.7) assures the initial data tends to zero in $\dot{H}_{per}^{-s}$. The decomposition (2.4) together with (2.7), (2.10) and the bound just obtained on $y$ then implies that the solutions $u_j$ blow up at times $T_j = (k_1^{(j)})^{-\nu\gamma}$. The latter tend to zero as $j \to \infty$ since $\mu$ and $\gamma$ are both positive. This completes the proof of the theorem.
References

[1] A.A. Alazman, J.P. Albert, J.L. Bona, M. Chen and J. Wu. *Comparisons between the BBM equation and a Boussinesq system*. Advances Differential. Eq. 11 (2006) 121–166.

[2] D. Ambrose, J.L. Bona, and D. Nicholls. *On ill-posedness of truncated series models for water waves*. Proc. Royal Soc. London, Series A 470 (2014) 1–16.

[3] T.B. Benjamin, J.L. Bona and J.J. Mahony. *Model equations for long waves in nonlinear dispersive media*. Philos. Trans. Royal Soc. London Series A 272 (1972) 47–78.

[4] J.L. Bona, T. Colin and D. Lannes. *Long wave approximations for water waves*, Archive Rat. Mech. Anal. 178 (2005) 373–410.

[5] J.L. Bona, W.G. Pritchard and L.R. Scott. *An evaluation of a model equation for water waves*, Philos. Trans. Royal Soc. London Series A 302 (1981) 457–510.

[6] J.L. Bona, W.G. Pritchard and L.R. Scott. *A comparison of solutions of two model equations for long waves*, In Lectures in Applied Mathematics 20 (ed. N. Lebovitz) American Mathematical Society: Providence (1983) 235–267.

[7] J.L. Bona and N. Tzvetkov. *Sharp well-posedness results for the BBM equation, Discrete & Continuous Dynamical Systems, Series A* 23 (2009) 1241–1252.

[8] Ill-posedness of the Navier-Stokes equations in a critical space in 3D. Journal of Functional Analysis, 255 (2008) 2233–2247.

[9] H. Chen. *Periodic initial-value problem for the BBM-equation*, Computers and Mathematics with Applications, Special Issue on Computational Methods in Analysis 48 (2004) 1305–1318.

[10] A. Cheskidov and M. Dai. *Norm inflation for generalized Magneto-hydrodynamic system*. Nonlinearity, 28 (2015) 129–142.

[11] A. Cheskidov and M. Dai. *Norm inflation for generalized Navier-Stokes equations*. Indiana University Mathematics Journal, 63 (2014), No. 3 : 869–884.

[12] M. Dai, J. Qing, and M. Schonbek. *Norm inflation for incompressible Magneto-hydrodynamic system in B_{1,-1}^{1,\infty}*. Advances in Differential Equations, 16 (2011), No. 7-8, 725–746.

[13] D. Lannes. *Well-posedness of the water-waves equations*, J. American Math. Soc. 18 (2005) 605–654.

[14] D. Lannes. *The water waves problem: mathematical analysis and asymptotics*. Mathematical Surveys and Monographs 188 American Math. Soc.: Providence (2013).

[15] M. Panthee, *On the ill-posedness result for the BBM equation Discrete & Continuous Dynamical Systems* 30 (2011) 253–259.

[16] D.H. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech. 25 (1966) 321–330.