Light deflection by squashed Kaluza-Klein black holes in a plasma medium

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Abstract

We study motions of photons in an unmagnetized cold homogeneous plasma medium in the five-dimensional charged static squashed Kaluza-Klein black hole spacetime. In this case, a photon behaves as a massive particle in a four-dimensional spherically symmetric spacetime. We consider the light deflection by the squashed Kaluza-Klein black hole surrounded by the plasma in a weak-field limit. We derive corrections of the deflection angle to general relativity, which are related to the size of the extra dimension, the charge of the black hole and the ratio between the plasma and the photon frequencies.

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I. INTRODUCTION

The direct detection of gravitational waves generated by the coalescence of black hole binaries [1] and the successful imaging of immediate vicinity of a supermassive black hole candidate in the center of the galaxy M87 [2] mean that researches of black holes have entered a new stage. Motivated by these astrophysical observations, we are interested in performing in-depth studies of the optical features of higher-dimensional black hole solutions. Recently, verifications of extra dimensions and braneworld black holes by observations of black hole shadow have been studied [3-6]. In this paper, we focus on the light deflection by higher-dimensional black holes surrounded by a plasma in the spacetime with compact extra dimensions.

Higher-dimensional black hole spacetimes are actively discussed in the context of string theories and braneworld models. If higher-dimensional black hole solutions have compactified extra dimensions, we can regard such black hole solutions as candidates of realistic models, since our observable world is effectively four-dimensional. We call these Kaluza-Klein black holes. In four-dimensional general relativity, the gravitational field in vacuum with spherical symmetry is uniquely described by the Schwarzschild metric. However, in a higher-dimensional spacetime with Kaluza-Klein structure, even if we impose asymptotic flatness in a four-dimensional section, the metric is not determined uniquely. A family of five-dimensional squashed Kaluza-Klein black hole solutions [7-12] asymptote to effective four-dimensional spacetimes with a twisted $S^1$ as an extra dimension at infinity and represent fully five-dimensional black holes near the squashed $S^3$ horizons. Then squashed Kaluza-Klein black hole solutions with a twisted compactified extra dimension would describe the geometry around the compact objects. Several aspects of squashed Kaluza-Klein black holes are discussed, for example, multi-black holes [13-15], stabilities [16, 17], quasi-normal modes [18-20], thin accretion disk [21], X-ray reflection spectroscopy [22], gyroscope precession [23, 24], strong gravitational lensing [25-29] and black hole shadow [30, 31].

The gravitational lensing combines a wide range of phenomena connected with the deflection of light rays by a gravitational field. Most gravitational lensing deal with geometrical optics in vacuum and use a notion of the deflection angle. A basic assumption is the approximation of a small deflection angle of a photon which is well satisfied in some astrophysical situations related to the gravitational lensing. Since the photon trajectories and deflection
angles of photons in vacuum do not depend on the photon frequency or its energy, the gravitational lensing in vacuum is achromatic. Then it is interesting to consider how the light trajectory and its deflection angle change in the presence of a plasma since light rays propagate through plasmas around compact objects, galaxies and galaxy clusters in the Universe. Self-consistent approaches for the geometrical optics in an arbitrary medium in a curved spacetime are discussed in the references [32, 33]. One of the most interesting effects of this kind is a chromatic gravitational deflection of light. In an unmagnetized cold homogeneous plasma medium, the refractive index of a plasma and the propagation of light rays depend on the photon frequency. Then the gravitational deflection of light is different from the vacuum case and its deflection angle depends on the ratio between the plasma and the photon frequencies [34, 35]. Since the effect of difference in gravitational deflection angles is significant for photons of longer wavelengths, the modifications of gravitational lensing due to the presence of the plasma is negligible in the visible spectrum. Then there are some astrophysical observations in the radio spectrum which would detect such plasma effects in low-frequency bands [36–43]. Motivated by these observations, the influence of plasma media on the trajectory of light rays and the deflection angles of photons around compact objects have been studied in a variety of spacetimes including vacuum, electrovacuum and with a vast array of scalar fields or effective fluids at both finite and infinite distances in both weak and strong-field limits [44–75].

In this paper, we investigate motions of photons and its deflection angles in a weak-field limit in the five-dimensional charged static squashed Kaluza-Klein black hole spacetime in the presence of a plasma. To the best our knowledge, photon motions around compact objects in plasma media have not been discussed in asymptotically Kaluza-Klein spacetimes. In the present work, we extend the derivations of weak deflection angles of photons in an unmagnetized cold homogeneous plasma medium in four-dimensional black hole spacetimes to the case of the five-dimensional squashed Kaluza-Klein black hole surrounded by such plasma.

This paper is organized as follows. In the section II, we review the properties of five-dimensional charged static Kaluza-Klein black hole solutions with squashed horizons. In the section III, we consider photon motions in a homogeneous plasma medium in squashed Kaluza-Klein geometry and show that there is a stable circular orbit of a photon with no momentum in the direction of the extra dimension. In the section IV, we study the
light deflection by the squashed Kaluza-Klein black hole surrounded by the homogeneous plasma in a weak-field limit. It is shown that the asymptotically Kaluza-Klein structure, the Maxwell field and the plasma modify the deflection angle of photon in the black hole geometry. The section \( V \) is devoted to summary and discussion.

II. SQUASHED KALUZA-KLEIN BLACK HOLES

We consider the charged static Kaluza-Klein black holes with squashed \( S^3 \) horizons, which are exact solutions of the five-dimensional Einstein-Maxwell theory \([9]\). The metric and the Maxwell field are given by

\[
\begin{align*}
    ds^2 &= -F dt^2 + K^2 F d\rho^2 + \rho^2 K^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \frac{r^2}{4K^2} \left( d\psi + \cos \theta d\phi \right)^2, \\
    A_\mu dx^\mu &= \frac{\sqrt{3}Q}{2\rho} dt,
\end{align*}
\]

with

\[
F = 1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}, \quad K^2 = 1 + \frac{\rho_0}{\rho},
\]

where the parameters \( M, \ Q, \ r_\infty \) and \( \rho_0 \) are related as \( r^2_\infty = 4(\rho_0^2 + 2M\rho_0 + Q^2) \). The coordinates run the ranges of \(-\infty < t < \infty, \ 0 < \rho < \infty, \ 0 \leq \theta \leq \pi, \ -\pi \leq \phi \leq \pi \) and \( 0 \leq \psi \leq 4\pi \). The squashed Kaluza-Klein black hole solution is asymptotically locally flat, i.e., the metric asymptotes to a twisted constant \( S^1 \) fiber bundle over the four-dimensional Minkowski spacetime. The parameters \( M, \ Q \) and \( r_\infty \) denote the Komar mass, the charge of the black hole and the size of the compactified extra dimension at infinity, respectively.

In this paper, to avoid the existence of naked singularities on and outside the horizon, we restrict ourselves to the ranges of parameters such that

\[
M \geq Q > 0, \quad r_\infty > 0,
\]

with the relation

\[
\rho_0 = \frac{\sqrt{r^2_\infty + 4(M^2 - Q^2)} - 2M}{2}.
\]

In these parameters, the outer and the inner horizons are located at \( \rho = M + \sqrt{M^2 - Q^2} \) and \( \rho = M - \sqrt{M^2 - Q^2} \), respectively. The parameter \( \rho_0 \) gives the typical scale of transition from five dimensions to effective four dimensions \([76]\). In the limit \( \rho_0 \to 0 \), equivalently
$r_\infty \to 2Q$, we obtain the metric (1) with $K = 1$ which represents the four-dimensional Reissner-Nordström black hole with a twisted constant $S^1$ fiber. We expect the appearance of the higher-dimensional corrections, which are related to the parameter $\rho_0$, to the photon motions and the deflection angle of photon of four-dimensional relativity.

III. PHOTON MOTIONS IN A PLASMA MEDIUM

We consider motions of photons in the five-dimensional squashed Kaluza-Klein black hole spacetime in the presence of an unmagnetized cold plasma medium. The Hamiltonian for the photon in the metric (1) is

$$H = \frac{1}{2} \left( g^{\mu\nu} p_\mu p_\nu + \omega_e^2 \right)$$

$$= \frac{1}{2} \left( -\frac{p_t^2}{F} + \frac{F}{K^2} p_r^2 + \frac{p_\theta^2}{\rho^2 K^2} + \frac{(p_\phi - p_\psi \cos \theta)^2}{\rho^2 K^2 \sin^2 \theta} + \frac{4K^2}{r_\infty^2} p_\psi^2 + \omega_e^2 \right)$$,  

(6)

with $H = 0$, where $p_\mu$ are the canonical momenta conjugate to the coordinates $x^\mu$ and $\omega_e$ is the electron plasma frequency defined by $\omega_e^2 = 4\pi e^2 N_e/m_e$, where $e$, $N_e$ and $m_e$ are the charge, the number density and the mass of the electron in the plasma, respectively 33, 41, 48, 57. Note that we ignore the self-gravitation of the plasma. In this paper, we consider a homogeneous plasma with the electron number density $N_e = \text{const.}$ and the positive refractive index $n = \sqrt{1 - \omega_e^2 F/\omega_\infty^2}$. From the Hamilton’s equations, we can obtain three constants of motion as

$$\omega_\infty := -p_t, \quad L := p_\phi, \quad \text{and} \quad p_\psi,$$

(7)

where $\omega_\infty$ is the photon frequency measured by an observer at infinity, $L$ and $p_\psi$ are angular momenta of the photon in the $\phi$ and the $\psi$ direction, respectively.

Here, we assume that the photon has no momentum in the direction of the extra dimension, i.e., $p_\psi = 0$. Then the effective Hamiltonian for the photon is

$$H_{\text{eff}} = \frac{1}{2} \left( -\frac{p_t^2}{F} + \frac{F}{K^2} p_r^2 + \frac{p_\theta^2}{\rho^2 K^2} + \frac{p_\phi^2}{\rho^2 K^2 \sin^2 \theta} + \omega_e^2 \right),$$

(8)

where $\omega_e = \text{const.}$ and $H_{\text{eff}} = 0$. We see that this effective Hamiltonian has the same form in the case of the four-dimensional spherically symmetric spacetime filled with a homogeneous plasma where the plasma frequency $\omega_e$ acts like an effective mass for a photon 34, 35. Then
we can concentrate on orbits with $\theta = \pi/2$ and $p_\theta = 0$ on the assumption of $p_\psi = 0$. The Hamilton’s equations in these conditions are given by

$$\frac{dt}{d\lambda} = \frac{\omega_\infty}{F},$$

$$\frac{d\rho}{d\lambda} = \frac{F}{K^2 \rho r},$$

$$\frac{d\phi}{d\lambda} = \frac{L}{\rho^2 K^2},$$

where $\lambda$ is the curve parameter along the photon trajectory. The photon frequency satisfies the condition $\omega_\infty > \omega_e$ for the propagation of the photon through the plasma in the squashed Kaluza-Klein spacetime [1].

Substituting the equation (10) into the Hamiltonian (8), we obtain the energy conservation equation under above conditions

$$\left(1 + \frac{\rho_0}{\rho}\right) \left(\frac{d\rho}{d\lambda}\right)^2 + V_{\text{eff}} = \omega_\infty^2,$$

where the effective potential is given by

$$V_{\text{eff}} = \left(1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}\right) \left(\frac{\omega_e^2}{\rho} + \frac{L^2}{\rho (\rho + \rho_0)}\right).$$

Typical shapes of the potential $V_{\text{eff}}$ are shown in the figure [1]. We see that there is a stable circular orbit of a photon at the local minimum of the effective potential. Then the squashed Kaluza-Klein black holes in a homogeneous plasma medium, where a photon can be stably bounded around the black hole, make a remarkable contrast with the higher-dimensional asymptotically flat black holes in such medium, which have no stable bound state of photon [75].

From the left panel of the figure [1] we find that the stable circular orbit radius decreases and the unstable one increases with increasing $\omega_e$ for fixed $L/M$, $r_\infty/M$ and $Q/M$. From the right panel of the figure [1] we observe that the stable circular orbit radius increases and the unstable one decreases with increasing $Q/M$ for fixed $L/M$, $r_\infty/M$ and $\omega_e$. Then the presence of a homogeneous plasma increases the radius of critical photon orbits around the Kaluza-Klein black hole similar to the case of the four-dimensional black hole surrounded by such plasma [35].

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1 There exist stable bound orbits of massless particles with nonvanishing angular momenta in two independent angular directions around an asymptotically flat black ring [77].
FIG. 1: Effective potentials \( V_{\text{eff}} \) in various \( \omega_e \) for \( L/M = 4, \ r_\infty/M = 0.1, \ Q/M = 0.03 \) and \( \rho_0/M \approx 0.0008 \) (left panel). \( \omega_e = 0.94 \) (dotted curve), \( \omega_e = 0.97 \) (dashed curve) and \( \omega_e = 1 \) (solid curve). The same ones in various \( Q/M \) for \( L/M = 4, \ r_\infty/M = 1 \) and \( \omega_e = 1 \) (right panel). \( Q = 0 \) \((\rho_0/M \approx 0.1, \) dotted curve), \( Q/M = 0.35 \) \((\rho_0/M \approx 0.06, \) dashed curve) and \( Q/M = 0.5 \) \((\rho_0 = 0, \) solid curve).

From \( dV_{\text{eff}}/d\rho = 0 \) and \( V_{\text{eff}} = \omega_\infty^2, \) we have \( \omega_\infty \) and \( L \) in circular motions with \( \rho = R = \text{const.}, \ p_\psi = 0 \) and \( \theta = \pi/2 \) as

\[
\omega_\infty^2 = \frac{\omega_e^2 (\rho_0 + 2R)(R^2 - 2MR + Q^2)^2}{R^2 [R(\rho_0(R - 4M) + 2R(R - 3M)) + Q^2 (3\rho_0 + 4R)]}, \tag{14}
\]

\[
L^2 = \frac{2\omega_e^2 R(\rho_0 + R)^2 (MR - Q^2)}{R(\rho_0(R - 4M) + 2R(R - 3M)) + Q^2 (3\rho_0 + 4R)}. \tag{15}
\]

Using the equations (9), (11), (14) and (15), the period of a circular orbit is given by

\[
T = 2\pi \frac{d\tau}{d\phi} = \sqrt{2\pi} \sqrt{\frac{R^3(2R + \rho_0)}{MR - Q^2}}. \tag{16}
\]

We see that the orbital period is modified by the extra dimension and the Maxwell field. In the large \( R \) limit, we have

\[
T \to 2\pi \sqrt{\frac{R^3}{M}}. \tag{17}
\]

This means Kepler’s third law.

In the absence of the plasma \( \omega_e = 0, \) the equation (12) with the effective potential (13) describes null geodesics in the five-dimensional charged static squashed Kaluza-Klein spacetime [27]. In this case, an unstable circular orbit of the photon exists at a photonsphere radius. From the denominator of the angular momentum (15), we obtain the photonsphere radius...
radius in the metric (11) as

\[
R = M - \frac{\rho_0}{6} + \frac{\sqrt{(6M + \rho_0)^2 - 24Q^2}}{3} \times \cos \left( \frac{1}{3} \cos^{-1} \left( \frac{216M (M^2 - Q^2) + 18\rho_0 (6M^2 - 7Q^2 - M\rho_0) - \rho_0^3}{((6M + \rho_0)^2 - 24Q^2)^{3/2}} \right) \right). \tag{18}
\]

In a weak-field limit, the photon sphere radius (18) becomes as

\[
R \simeq 3M \left( 1 + \frac{M\rho_0 - 4Q^2}{18M^2} \right). \tag{19}
\]

The second term in the right-hand side is the corrections by the extra dimension and the Maxwell field.

In the limit \(\rho_0 \to 0\), the metric (11) locally has the geometry of the Reissner-Nordström black string and the equation (12) reduces to the energy conservation equation of the photon moving around the four-dimensional Reissner-Nordström black hole surrounded by a homogeneous plasma [74]. In such limit, the energy conservation equation (12) with the replacements \(\omega_e \to 1\), \(\omega_\infty \to E\) describes the geodesic motions of massive test particles with the energy \(E\) measured by an observer at infinity in the four-dimensional Reissner-Nordström spacetime [78]. This is consistent with a correspondence between motions of the photon with the frequency \(\omega_\infty\) in an unmagnetized cold homogeneous plasma characterized by the frequency \(\omega_e\) in a given four-dimensional spacetime and timelike geodesics of the massive test particle on the same background [35, 79]. In the limit \(Q \to 0\), the equation (12) and the effective potential (13) with the replacements \(\omega_e \to 1\), \(\omega_\infty \to E\) describe timelike geodesics of the massive particles with the energy \(E\) in the five-dimensional static squashed Kaluza-Klein spacetime [23]. Then some results for photons in a homogeneous plasma shown in the present paper would be applied to the geodesic motions and the deflection angles of neutral massive particles with \(p_\psi = 0\) and \(\theta = \pi/2\) in the five-dimensional charged static squashed Kaluza-Klein spacetime (1).

IV. DEFLECTION ANGLE OF PHOTON IN A PLASMA MEDIUM

We consider light propagation around the squashed Kaluza-Klein black hole (1) in a homogeneous plasma medium by direct integrations of the photon orbit equation with the conditions \(p_\psi = 0\) and \(\theta = \pi/2\). Substituting the equation (11) into the equation (12) and
introducing a new coordinate $u = \rho^{-1}$, the energy conservation equation becomes as

$$
\left( \frac{du}{d\phi} \right)^2 = \frac{1}{b^2} + \frac{2\sigma^2 M + \rho_0}{b^2} u - \left( 1 + \frac{\sigma^2 (Q^2 - 2M\rho_0)}{b^2} \right) u^2 + \left( 2M - \frac{\sigma^2 Q^2 \rho_0}{b^2} \right) u^3 - Q^2 u^4,
$$

(20)

with

$$
b^2 := \frac{L^2}{\omega_\infty^2 - \omega_e^2},
$$

(21)

$$
\sigma^2 := \frac{\omega_e^2}{\omega_\infty^2 - \omega_e^2}.
$$

(22)

Taking the derivative of the equation (20) with respect to the coordinate $\phi$, we obtain the photon orbit equation as

$$
\frac{d^2 u}{d\phi^2} = 2\sigma^2 M + \rho_0 - \left( 1 + \frac{\sigma^2 (Q^2 - 2M\rho_0)}{b^2} \right) u + 3 \left( M - \frac{\sigma^2 Q^2 \rho_0}{2b^2} \right) u^2 - 2Q^2 u^3.
$$

(23)

To solve the equation (23), we assume that the solution $u$ can be expressed in powers of parameters as

$$
u = u_0 + \frac{M}{b} u_1 + \frac{M^2}{b^2} u_2 + \frac{Q^2}{b^2} u_3 + \frac{\rho_0}{b} u_4 + \frac{\rho_0^2}{b^2} u_5 + \frac{M \rho_0}{b^2} u_6
$$

$$+ O \left( M^3, \rho_0^3, M^2 \rho_0, M Q^2, M \rho_0^2, Q^2 \rho_0 \right),
$$

(24)

where $u_i \ (i = 0, 1, \ldots, 6)$ are functions of $\phi$. Substituting the ansatz (24) into the equation (23) and solving the differential equations of $u_i$, we obtain the orbit of the photon as

$$
u(\phi) = \frac{\cos \phi}{b} + \frac{(2\sigma^2 + 3) M - M \cos(2\phi) + \rho_0}{2b^2}
$$

$$+ \frac{1}{16b^3} \left[ \left( (8\sigma^2 (\sigma^2 + 6) + 37) M^2 + 24M \rho_0 (\sigma^2 + 1) + 2\rho_0^2 - (8\sigma^2 + 9) Q^2 \right) \cos \phi
$$

$$+ (3M^2 + Q^2) \cos(3\phi) + 4 \left( 3M^2 \left( 4\sigma^2 + 5 \right) + \left( 2\sigma^2 + 3 \right) (2M \rho_0 - Q^2) \right) \phi \sin \phi \right]\n$$

$$+ O \left( M^3, \rho_0^3, M^2 \rho_0, M Q^2, M \rho_0^2, Q^2 \rho_0 \right),
$$

(25)

where $b$ is the impact parameter which represents the minimum value of $\rho$-coordinate for the undeflected light ray, i.e., $M = Q = \rho_0 = 0$. We note that the integration constants are chosen such that the equation (25) has a symmetry $u(\phi) = u(-\phi)$ and satisfies the energy conservation equation (20) up to the second order in the parameters $M$, $Q$ and $\rho_0$. By taking some limits, we obtain particle trajectories in some four-dimensional spacetimes. When $\rho_0 = 0$, $\omega_e = 0$, the equation (25) represents the orbit of the photon in the braneworld.
black hole spacetime with the tidal charge \( q \) which reduces to the four-dimensional Reissner-Nordström spacetime in the limit \( q \to Q^2 \) \cite{80}. When \( \rho_0 = 0, Q = 0 \), the equation \( (25) \) with the replacements \( \phi \to \phi - \pi/2, ~ \sigma^2 \to (1 - v^2)/v^2 \) represents the orbit of the neutral massive particle with the velocity \( v \) measured by an observer at infinity in the four-dimensional Schwarzschild spacetime \cite{57}.

We consider the photon which comes from far away at the distant past, \( \phi = -\pi/2 - \delta\phi/2 \), and is deflected by the black hole then travels towards far away at the distant future, \( \phi = \pi/2 + \delta\phi/2 \), where \( \delta\phi \) is a deflection angle. Since the equation \( (25) \) has a symmetry \( u(\phi) = u(-\phi) \), we solve \( u(\pi/2 + \delta\phi/2) = 0 \) up to the first order in \( \delta\phi \). Then we obtain the deflection angle of photon in a weak-field limit as

\[
\delta\phi = \frac{2M}{b} \left( \frac{1}{1 - \omega_e^2/\omega_\infty^2} + \frac{1}{2M} \frac{1}{2M} \frac{1}{1 - \omega_e^2/\omega_\infty^2} \right) + O \left( \frac{b^3}{b^2} \right),
\]

(26)

where the parameter \( \rho_0 \) is given by the equation \( (5) \) and we use the equation \( (22) \) to represent the deflection angle in terms of the frequency ratio \( \omega_e/\omega_\infty \). The deflection angle in terms of the distance of closest approach is shown in the appendix A.

We see that the gravitational deflection angle (26) depends upon the photon frequency \( \omega_\infty \) and is modified by the squashed Kaluza-Klein geometry, the Maxwell field and the homogeneous plasma through the extra dimension size \( r_\infty \), the black hole charge \( Q \) and the plasma frequency \( \omega_e \), respectively. We find that the deflection angle decreases with increasing \( b/M \) for fixed \( r_\infty/M, Q/M \) and \( \omega_e/\omega_\infty \), while increases with increasing \( r_\infty/M \) for fixed \( b/M, Q/M \) and \( \omega_e/\omega_\infty \). We show the behaviors of the deflection angle \( \delta\phi \) versus \( b/M \) in the figure 2. From the left panel of the figure 2, we see that \( \delta\phi \) increases with increasing \( \omega_e/\omega_\infty \) for fixed \( b/M, r_\infty/M \) and \( Q/M \). Then the presence of plasma changes the deflection angle with the difference from the case of no plasma, \( \omega_e = 0 \) or \( \omega_e/\omega_\infty \ll 1 \), being strongest for photons of smaller frequency or longer wavelength as \( \omega_\infty \) approaches \( \omega_e \). From the right panel of the figure 2, we observe that \( \delta\phi \) decreases with increasing \( Q/M \) for fixed \( b/M, r_\infty/M \) and \( \omega_e/\omega_\infty \). Then the effect of the difference in gravitational deflection angles is significant for smaller charges of the black hole, larger sizes of the extra dimension and larger ratios between the plasma and the photon frequencies.

By taking some limits in the equations \( (26) \) and \( (A3) \), we obtain second-order deflection angles in some four-dimensional spacetimes. First, when \( \rho_0 = 0 \), equivalently \( r_\infty = 2Q \), the
FIG. 2: Deflection angles (26) in various $\omega_e/\omega_\infty$ for $r_\infty/M = 0.1$, $Q/M = 0.03$ and $\rho_0 \simeq 0.0008$ (left panel). $\omega_e = 0$ (dotted curve), $\omega_e^2/\omega_\infty^2 = 0.5$ (dot-dashed curve), $\omega_e^2/\omega_\infty^2 = 0.7$ (dashed curve) and $\omega_e^2/\omega_\infty^2 = 0.8$ (solid curve). The same ones in various $Q/M$ for $r_\infty/M = 1$ and $\omega_e^2/\omega_\infty^2 = 0.5$ (right panel). $Q = 0$ ($\rho_0/M \simeq 0.1$, dotted curve), $Q/M = 0.3$ ($\rho_0/M \simeq 0.08$, dot-dashed curve), $Q/M = 0.4$ ($\rho_0/M \simeq 0.04$, dashed curve) and $Q/M = 0.5$ ($\rho_0 = 0$, solid curve).

equations (26) and (A3) with the replacement $\omega_e^2/\omega_\infty^2 \rightarrow 1 - v^2$ coincide with the deflection angles of neutral massive particles with the velocities $v$ measured by an observer at infinity in the four-dimensional Reissner-Nordström spacetime [81]. Secondly, when $r_\infty = 2Q$, $\omega_e = 0$ or $\omega_e/\omega_\infty \ll 1$, the equations (26) and (A3) represent the deflection angles of photons in the four-dimensional Reissner-Nordström spacetime [82]. Thirdly, when $r_\infty = 2Q$, then taking the limit $Q \rightarrow 0$, the equations (26) and (A3) represent the deflection angles of photons in a homogeneous plasma medium in the four-dimensional Schwarzschild spacetime [57]. Lastly, when $r_\infty = 2Q$, $\omega_e = 0$ or $\omega_e/\omega_\infty \ll 1$, then taking the limit $Q \rightarrow 0$, the equations (26) and (A3) represent the deflection angles of photons in the four-dimensional Schwarzschild spacetime [83, 84].

V. SUMMARY AND DISCUSSION

We consider motions of photons around a spherical compact object in an unmagnetized cold homogeneous plasma medium. We assume that the five-dimensional charged static squashed Kaluza-Klein black hole metric describes the geometry of the region outside the compact object and a photon has no momentum in the direction of the extra dimension. We show that the five-dimensional squashed Kaluza-Klein spacetime filled with a homogeneous
plasma admits stable circular orbits of photons similar to the four-dimensional spherically symmetric black holes surrounded by such plasma. We solve the photon orbit equation in the plasma medium in the squashed Kaluza-Klein spacetime and derive the deflection angle of photon in a weak-field limit with corrections by the extra dimension, the Maxwell field and the plasma. Some known deflection angles in four-dimensional spacetimes are obtained by taking limits in our deflection angle of photon. We see that, for fixed values of the photon frequency and the black hole mass, the deflection angle of photon increases with increasing the extra dimension size and the plasma frequency, while decreases with increasing the impact parameter and the black hole charge. The variations of these parameters provide specific signatures on the optical features of the squashed Kaluza-Klein black hole solutions in the plasma medium which open the possibility of testing such higher-dimensional models by using astronomical and astrophysical observations.

We note that the exterior spacetimes of standard general relativistic spherical compact objects are described by the Schwarzschild metric. However, in higher-dimensional spacetime models with Kaluza-Klein structures, the Schwarzschild metric is no longer the exterior metric of a static compact object. Even if we impose asymptotic flatness to the four-dimensional part of the spacetime, there are various possibilities of fiber bundle structures of the extra dimensions as the fiber over the four-dimensional base spacetime. For example, the direct product of four-dimensional Schwarzschild spacetime with a small $S^1$ is a possible metric to describe the exterior of the compact object. In this case, no higher-dimensional correction of light deflection appears without a momentum of the photon in the direction of the extra dimension. In contrast, it is interesting that the correction exists even if the photon moves along the four-dimensional spacetime in the squashed Kaluza-Klein geometry.

Since the asymptotically Kaluza-Klein structure, the Maxwell field and the homogeneous plasma modify the unstable circular orbits of photons in the four-dimensional Schwarzschild spacetime, shadows of black holes would be influenced by such corrections. Moreover, generalizations of the present study to light deflections in inhomogeneous plasma media \[35, 47, 59\] in another class of Kaluza-Klein type metrics \[85, 86\] would be interesting. We leave the analysis of these topics for the future.
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Appendix A: Deflection angle of photon in terms of distance of closest approach

While the deflection angle is given in terms of the impact parameter $b$, it would be useful to represent the deflection angle in terms of the distance of closest approach $\rho_{\text{min}}$. Substituting $\phi = 0$ and $u = \rho_{\text{min}}^{-1}$ into the equation \(25\), the relation between the distance of closest approach and the impact parameter is given by

$$\frac{1}{\rho_{\text{min}}} = \frac{1}{b} \left( 1 + \frac{2M(\sigma^2 + 1) + \rho_0}{2b} + \frac{4(\sigma^2 + 1)((\sigma^2 + 5)M^2 - Q^2) + 12M\rho_0(\sigma^2 + 1) + \rho_0^2}{8b^2} \right).$$  \(\text{(A1)}\)

Solving this equation for $b^{-1}$ up to the second order in $\rho_{\text{min}}^{-1}$, we have

$$\frac{1}{b} \simeq \frac{1}{\rho_{\text{min}}} \left( 1 - \frac{2M(\sigma^2 + 1) + \rho_0}{2\rho_{\text{min}}} \right).$$  \(\text{(A2)}\)

Then we obtain the deflection angle in terms of the distance of closest approach as

$$\delta \phi = \frac{4M}{\rho_{\text{min}}} \left( 1 + \frac{\sigma^2}{2} + \frac{\rho_0}{4M} \right) + \frac{15\pi M^2}{4\rho_{\text{min}}^2} \left( 1 + \frac{4\sigma^2}{5} + \frac{(2\sigma^2 + 3)(2M\rho_0 - Q^2)}{15M^2} \right)$$

$$- \frac{4M^2}{\rho_{\text{min}}^2} \left( 1 + \frac{\sigma^2(\sigma^2 + 3)}{2} + \frac{\rho_0(2M(2\sigma^2 + 3) + \rho_0)}{8M^2} \right) + O(\rho_{\text{min}}^{-3}).$$  \(\text{(A3)}\)

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