A STUDY ON SPLAY TREES∗

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Abstract. We study the dynamic optimality conjecture, which predicts that splay trees are a form of universally efficient binary search tree, for any access sequence. We reduce this claim to a regular access bound, which seems plausible and might be easier to prove. This approach may be useful to establish dynamic optimality†.

Key words. Data Structures, Binary Search Trees, Splay Trees, Dynamic Optimality, Competitive Analysis, Amortized Analysis

AMS subject classifications. 68P05, 68P10, 05C05, 94A17, 68Q25, 68P20, 68W27, 68W40, 68Q25

1. Introduction.

Binary search trees (BSTs) are ubiquitous in computer science. Their relevance is well established, both in theory and in practise. Figure 1 illustrates this data structure. The numbers in the nodes represent the keys. If read left to right, the keys form an ordered sequence, shown bellow the tree. This property can be used to efficiently determine if a given number exists in the tree. If we want to check if the tree contains the number 0.55, we can start by comparing this value with the one at the root, 0.56. Since the number we are searching for is larger than 0.56, the search continues in the left sub-tree, i.e., it proceeds to the node containing 0.40, thus discarding all the sub-tree to the right. It is the process of eliminating a large portion of its search space that makes BSTs efficient.

Fig. 1. Binary search tree containing keys from [0, 1].

As the search goes through the tree it will eventually reach the node containing 0.52, at which point it becomes apparent that 0.55 is not present in the structure. In fact this search concludes that no number in the open interval ]0.52, 0.56[ exists in the

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tree. This is the only valid conclusion. Even though the search passed through nodes 0.40 and 0.51 it would not be valid to conclude that no number in [0.40, 0.51] exists, because 0.41 is on the tree. Hence whenever a search fails we still obtain information from the data structure. In particular we obtained the numbers in the tree that are the successor (0.56) and the predecessor (0.52) of 0.55.

Consider how the shape of the tree affects the performance of the queries. If we are computing a single, isolated, query, we may want to guarantee that, at each node, the size of the left and right sub-trees is roughly the same. Note that we never know which sub-tree a given search will choose. If both sub-trees have the same size at least half the search space is discarded at each step. Alternatively we could impose that the tree does not contain long branches, since a search might have to traverse all of it.

Either way, some policy must be used to main a tree shape that reduces the search time. Determining “good” tree shapes is a complex task, specially when we consider a sequence of queries, instead of a single query. Moreover, must of the time, this sequence is not known a priori. If a sequence of a couple million queries searches for 0.21 in half of them and only once for 0.85, then it might be better to keep 0.21 at the root and leave 0.85 on the longest existing branch. We would expect such a shape to reduce the overall time. Hence a balanced tree is important if we need to answer several essentially distinct queries. On the other hand if the queries are strongly biased towards a specific region of the keys maintaining certain branches shorter than the rest might be necessary.

Several strategies are known to maintain efficient BST shapes, see Knuth [18] for an introduction to the subject. In this paper we focus on the approach used by splay trees [24]. Our goal is to show that this approach is optimal, in the sense that no other strategy can be asymptotically faster than splay trees. Notice that splay trees dynamically alter their structure as they process queries. Hence we assume that any other BST can do the same. Figure 2 illustrates the tree that results from accessing 0.21 on a splay tree.

Our main contribution is the following:

• We, almost, show that splay trees are dynamically optimal. This means that no other BST is asymptotically faster than splay trees, no matter what is the structure of the query sequence or the re-shaping policy it uses. This property was conjectured to be true, over 30 years ago [24]. Our proof depends on a regular access property, which we deem plausible and conjecture to be true, see Section 3. At this time we do not have a proof of this property.

2. The Problem.

The introduction exemplified how a BST can be used to maintain a finite set $K$ of, keys, elements of a universe set $X$. In our example $X$ is $\mathbb{Q}$. Our goal is to maintain a data structure such that given a fraction $x$ from $\mathbb{Q}$ we can determine if $x$ is in $K$, or not. We refer to these operations as queries. A query is successful it $x$ does belong to $K$ and unsuccessful otherwise.

Several efficient data structures exist for this problem, depending on which resources are critical and on which extra operations are necessary. For our example we also want map behaviour. This means that the elements of $K$ contain information. Whenever $x$ does exist in $K$ we also want to access the associated information. The two major classes of data structures that can be used in this scenario are trees and hashes. The set $\mathbb{Q}$ is, in general, referred to as the key universe and the associated
information is referred to as the value set. In this paper we omit this latter set.

Contrary to hashes, BSTs rely on the order relation among the elements of the universe. In $\mathbb{Q}$ we have for example that $(1/4) < (1/2)$. Hence BSTs can efficiently determine successors and predecessors. The successor of $x$ in $K$ is $\min\{y \in C \mid x < y\}$. To simplify the analysis we do not consider unsuccessful queries, i.e., we assume that all the queries find elements in $K$. An unsuccessful query for $x$ can be modelled by two successful ones, one for the successor and one for the predecessor. In general BSTs support inserting and removing elements from $K$, but we also do not study those operations.

The re-structuring policy of splay trees consists in moving the accessed nodes, and the nodes in the respective path, upwards towards the root. The precise operations are shown in Figures 3, 4 and 5, where the node containing $x$ is the one being accessed. The configuration before the access is represented on the left and the structure after the access on the right. Accesses when $x$ is on the right sub-tree are obtained by symmetry.

To compare splay trees with any other BST we assume that the other BST works in the following way. Besides the tree itself there is a cursor that moves between nodes. An algorithm on a BST may perform any of the following operations:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{splay_tree}
\caption{After splaying 0.21.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{zig_operation}
\caption{Zig operation.}
\end{figure}
**Compare** the key at the cursor with the current search value.

**Move** the cursor to an adjacent node, left child, right child or parent.

**Rotate** the node upwards. The Zig operation is a rotation of node $x$, (Figure 3).

To perform a search in a BST the cursor starts at the root and performs a sequence of the previous operations. The result of the search is the node for which a comparison with $x$ was equal. This node may, or may not, be the last one in the sequence. At the end of a search the cursor must return to the root, those moves are accounted for.

In this model we allow for constant extra information at each node, such as colour for red-black trees or height for AVL trees. The intention is to forbid dynamic memory blocks, which could be used to implement hashing. Hence only a fixed constant amount of information can be added to a node. Other BST models exist, that can be shown to be within a constant factor of the one we consider [29].

Among all BSTs we focus on the one that achieves the best performance for a query sequence. This means choosing the tree after knowing the complete query sequence, in other words, offline. We refer to this optimal tree as $T$.

We do not count compare operations and therefore assume that they cost 0. For the optimal tree we count the number of moves and rotations, but for the splay tree we only count the number of moves. This avoids having to carry around a factor of 2 or 3, which would result from counting rotations and or comparisons.

**3. Analysis Overview.**

In this section we overview the main techniques in the analysis and describe our new potential function.
Amortized analysis. The analysis is amortized, meaning that the time a splay operation takes is not accounted per se, but considering preceding and succeeding operations [28]. Therefore the amount of time an operation requires can be partially shifted to some other operation in the sequence. The resulting time is known as the amortized time. The most common way to transform the total sum into an equivalent telescopic sum is to associate a “potential” value to each state of the data structure $D$, the value $\Phi(D)$. The amortized cost of an operation is then defined as $\hat{c} = c + \Phi(D') - \Phi(D)$, were $c$ is the actual cost and $\Phi(D')$ the potential after the operation is performed. Summing the previous equations over all $m$ queries, and using the fact that it is a telescopic sum, we obtain a global relation $\sum_{1 \leq i \leq m} c_i = (\sum_{1 \leq i \leq m} \hat{c}_i) - \Phi(D_m)$. In this equation $\Phi(D_m)$ represents the potential after the sequence, which might be a negative number. The value $\Phi(D_0)$ represents the potential at the beginning and it was omitted because it is assumed to be 0. This is a classical tool in the analysis of splay trees. We present a new potential function. This function is chosen so that the structure of the optimal tree gets transferred into the splay tree.

Restricted Accesses. After bounding the amortized cost of accesses in $S$ we proceed to amortize the cost of rotations in $T$. Such rotations alter $\Phi(D)$ and therefore must be accounted for. To obtain a constant bound on this variation we introduce extra, organizing splays, in $S$. The amortized time to compute these extra splays will depend on the depth of the node we are rotating in $T$. Therefore if $T$ performs two rotations in a row we need to count the node depth twice. To obtain a valid bound we force $T$ to move its cursor back to the root after performing a rotation. We also further impose that $T$ may not access nodes of depth 3 or more. Although these restrictions seem harsh we show that, with a linear slowdown factor, it is possible to force the optimal sequence to respect them, Lemma 4.3.

Organizing Splays. Splay trees do not know the full sequence of queries in advance, fortunately the analysis does. This means that in the analysis it is possible to verify if the structure of the splay tree is adequate for the upcoming queries. We optimize the tree structure by adding extra organizing splays. These splays introduce the logic dependency on a regular access.

The regular access statement is the following:

Conjecture 3.1 (Regular Access). Let $c_1 + \ldots + c_m$ be the total cost of a sequence of $m$ splay operations and $c'_1 + \ldots + c'_{m+r}$ the cost of performing the same sequence of splay operations, augmented with $r$ extra splays anywhere. Then $c_1 + \ldots + c_m = O(c'_1 + \ldots + c'_{m+r})$.

This conjecture basically states that computing more splays takes longer. We assumed this property in our reasoning, until it became evident that it was not trivial to establish. Our proof relies precisely on this property because the extra splays are used to optimize the tree for future accesses.

We will now define the potential function $\Phi$ and give an overview of the results that we prove in Section 4. $T$ denotes the optimal tree. To every node $v$ of $T$ we assign a weight $w_T(v) = 1/4^{d_T(v)}$, where $d_T(v)$ is the depth of $v$, i.e., the distance to the root. The root itself has depth 0, its children have depth 1 and so on. For each node we add up all the weights of its sub-tree, including $w_T(v)$ itself, the resulting sum is denoted $s_T(v)$. The rank $r_T(v)$ is computed as $\log(s_T(v))$. The potential of
tree \( T \) is given by the following sum:

\[
P(T) = \sum_{v \in T} r_T(v)
\]

Let us consider the trees in Figure 6. We have that \( w_T(a) = w_T(c) = 1/16 \), \( w_T(b) = w_T(e) = 1/4 \) and \( w_T(d) = 1 \). The resulting \( r \) values are \( r_T(a) = r_T(c) = \log(1/4^2) = -4 \), \( r_T(e) = -2 \), \( r_T(b) = \log(1/4 + 1/8) = \log(3/8) \) and \( r_T(d) = \log(1 + 1/2 + 1/8) = \log(13/8) \). Hence the total potential of \( T \) is \( P(T) = r_T(b) + r_T(d) - 10 \).

We use \( S \) to denote the splay tree. There is a one to one relation between the nodes of these trees, because both trees share the same key values. For any node \( v \), of \( T \), its correspondent in \( S \) is \( f(v) \), by correspondent we mean that \( v \) and \( f(v) \) store the same key value. This correspondence is used to transfer the weight values from \( T \) to \( S \), more precisely set \( w(f(v)) = w_T(v) \). We omit the \( S \) subscript to simplify notation. The values of \( w \), \( s \) and \( r \) that do not have a subscript refer to \( S \). Moreover we also omit the function \( f \) and assume instead that \( v \) in \( S \) means \( f(v) \). Hence the previous relation will be written as \( w(v) = w_T(v) \). The values \( s(v) \) and \( P(S) \) are computed as before. In general \( s(v) \) and \( s_T(v) \) are not, necessarily, equal. See Figure 6. Likewise \( P(T) \) and \( P(S) \) are also not, necessarily, equal. In fact we use \( \Phi(D) = P(S) - P(T) \) as our potential function.

In our example we have that \( r(a) = r_T(a) \), \( r(c) = r_T(c) \), \( r(e) = r_T(e) \), \( r(b) = r_T(d) \) and \( r(d) = \log(1 + 1/4 + 1/16) = \log(21/16) \). Therefore \( \Phi(D) = \log(21/16) - \log(3/8) = \log(7/2) \).

In the analysis we assume that \( S \) and \( T \) take turns. First \( S \) searches for \( x \) and \( T \) remains idle. Then \( T \) searches for the same \( x \), while \( S \) remains idle. Hence whenever the structure of \( S \) changes \( P(S) \) changes, but \( P(T) \) remains constant. However when the structure of \( T \) changes, both \( P(T) \) and \( P(S) \) change.

The structure of our argument is as follows:

- We show that \( -n < \Phi(D_m) \), Lemma 4.2, where \( n \) is the number of keys.
- We show that the amortized cost to splay a node \( v \), in \( S \), is at most \( O(1 + d_T(v)) \), Lemma 4.3.
- We show that whenever there is a rotation at node \( v \) in \( T \) we have \( \Delta \Phi = O(1) \), by splaying at most 3 nodes, whose depth in \( T \) is at most \( d_T(v) \), Lemma 4.5.
- We combine these results with a restricted optimal sequence, Lemma 4.3, and obtain an optimality result, Theorem 4.6.

Thus we establish that if the optimal tree, with \( n \) nodes, needs \( R \) rotations and \( M \) moves to process a given query sequence then splay trees require \( O(n + R + M) \) time, provided they have the regular access property.

4. The Details.

In this section we fill in the details that are summarized in the previous section. This section is divided into three parts. We start by studying the properties of our potential function \( \Phi \). We then shift the focus to \( T \) and explain how to restrict the general access sequence to ensure that it respects certain desirable conditions. The last part presents bounds for the amortized costs of accessing elements in \( S \) and rotating nodes in \( T \). We then combine these results to obtain an optimality result.

4.1. Properties of \( \Phi \).

Let us start by analyzing the potential function \( \Phi(D) \). We assume that the initial configuration of \( S \) and \( T \) is the same. Therefore \( P(S) = P(T) \) and \( \Phi(D_0) = 0 \).
Fig. 6. Optimal tree $T$ on top, splay tree $T'$ in the bottom, upside down. Example of computing $\Phi$. The $f$ function is illustrated by the dashed arrows. The weight values $w$ are show in between the two trees. Inside the black rectangles we show the $s$ values associated with the nodes.

- $T$: The optimal binary search tree.
- $T'$: A restricted version of $T$.
- $S$: The splay tree.
- $n$: Number of nodes in $T$ and $S$.
- $m$: Number of queries in a sequence.
- $e$: Number of added organizing splays.
- $R$: Number of rotations in $T$.
- $M$: Number of cursor moves in $T$.
- $R'$, $M'$: Same as $R$ and $M$, but in $T'$.
- $v, u$: Nodes.
- $t$: The root node.
- $d_T(v)$: The depth of node $v$ in $T$.
- $w(v)$: Weight of node $v$.
- $s(v)$: Sum of the weights, in the sub-tree below $v$.
- $r(v) = \log(s(v))$: Rank for node $v$.
- $w_T(v), s_T(v), r_T(v)$: Weight, sum and rank in $T$.
- $\Phi(D)$: Potential value for a given state of $S$ and $T$.

Fig. 7. Symbol table.
Given that we are using base 2 logarithms it would be natural to use base (1/2) powers for the weights \( w \) and \( w_T \). Using (1/4) instead causes the amortized access cost to be twice as big. On the other hand we obtain some handy properties.

**Lemma 4.1.** Let \( v \) be a node. The following bounds hold:

\[(4.1) \quad 0 \leq w(v) \quad \text{for any } v \in T \text{ or } S.\]
\[(4.2) \quad w(v) \leq 1 \quad \text{for any } v \in T \text{ or } S.\]
\[(4.3) \quad w(v) \leq s(v) \quad \text{for any } v \in T \text{ or } S.\]
\[(4.4) \quad s_T(v) < 2 \times w_T(v) \quad \text{only for } v \in T.\]
\[(4.5) \quad s(v) < 2 \quad \text{for any } v \in T \text{ or } S.\]
\[(4.6) \quad s(t') = s_T(t) \quad \text{for the root } t' \text{ of } S \text{ and } t \text{ of } T.\]

**Proof.** Inequalities (4.1) and (4.2), follow directly from our definition of weights. Inequality (4.3), follows from the definition of \( s \) and Inequality (4.1).

Inequality (4.4) use a geometric series. For the nodes \( v \) in \( T \) the value \( s_T(v) \) is a sum of powers of 1/4, which depends on the topology of tree. If \( T \) is a single branch then \( s_T(v) < w_T(v)(1 + (1/4) + (1/4)^2 + \ldots + (1/4)^d + \ldots) < (4w_T(v)/3) \). If \( T \) is perfectly balanced, this bound becomes even larger. In that case \( s_T(v) < w_T(v)(1 + 2 \times (1/4) + 2^2 \times (1/4^2) + \ldots + 2^d \times (1/4^d) + \ldots) < 2 \times w_T(v) \). In this case the bound comes from the geometric series of (1/2). Moreover the remaining cases will also be bounded by the geometric series of (1/2).

Inequality (4.5) holds for \( T \), because of inequalities (4.2) and (4.4). For \( S \) note that in general \( T \) cannot have more than \( 2^d \) nodes with depth \( d \) and consequently \( S \) cannot have more that \( 2^d \) nodes with weight \( (1/4^d) \), hence we obtain the geometric series of (1/2) again.

For equality (4.6) notice that the sums at the root contain all the weight values. Since these values are mapped from \( T \) to \( S \) they are globally the same and therefore so is their sum. \( \square \)

In particular from Inequality (4.5) we deduce that \( r(t) \leq 1 \), for the root \( t \) of \( S \).

Using potential functions that can assume negative values implies that the value of \( \Phi(D_m) \) becomes a term in the total time. Hence we bound this value.

**Lemma 4.2.** The bound \( -n < \Phi(D) \) holds for any configuration of \( S \) and \( T \), where \( n \) is the number of keys.

**Proof.** The following derivation establishes the result.

\[(4.7) \quad P(S) - P(T) = \Phi(D)\]
\[(4.8) \quad \sum_{v \in S} \log(w(v)) - P(T) \leq P(S) - P(T)\]
\[(4.9) \quad \sum_{v \in S} \log(w(v)) - \sum_{v \in T} \log(2w_T(v)) < \sum_{v \in S} \log(w(v)) - P(T)\]
\[(4.10) \quad - \sum_{v \in T} 1 = \sum_{v \in S} \log(w(v)) - \sum_{v \in T} \log(2w_T(v))\]
\[(4.11) \quad -n = - \sum_{v \in T} 1\]

To obtain Inequality (4.8) use Inequality (4.3), for all the terms of the sum in \( P(S) \). For Inequality (4.9) use Inequality (4.4) and apply logarithms and change signals.
Equation (4.10) follows from the same argument as Equation (4.6) and the fact that 
\[ \log(2^{w_T(v)}) = 1 + \log(w_T(v)). \]

4.2. Restricting \( T \).

Competing directly with \( T \) is fairly hard, partially because the sequence of accesses in \( T \) is completely free. For our purposes we need that the accesses in \( T \) respect the following properties:

- When a node \( v \) is visited by the cursor or involved in a rotation its depth is less than 3, i.e., \( d_T(v) < 3 \).
- After a rotation the cursor moves back to the root.

We refer to a sequence of nodes that respects these conditions as a restricted sequence. Let us show that given any access sequence of visited nodes in \( T \) it is possible to simulate it in another tree \( T' \), so that the accesses in \( T' \) are restricted. By simulation we mean that the sequence visited by the cursor of \( T \) is a sub-sequence of the one visited by the cursor of \( T' \). Naturally the simulation will be slower than the original sequence.

**Lemma 4.3.** Any sequence of nodes visited by the cursor of \( T \), consisting of \( M \) moves and \( R \) rotations, can be simulated by a restricted sequence of cursor moves on a tree \( T' \) with \( 4M + 3R \) moves and \( 2M + R \) rotations.

*Proof.* Figure 8 illustrates the operations that we consider in this Lemma. Besides the same keys as \( T \), the tree \( T' \) contains two extra keys, min and max, that are, respectively, smaller and larger than all the other keys in \( T \). The simulation works by keeping the node at the cursor of \( T \) in the root of \( T' \).

We assume that the initial state of \( T \) and \( T' \) is almost alike. The same key is stored at the root. The children of \( T' \) are min and max. The left sub-tree of \( T \) is stored in the right child of min and the right sub-tree of \( T \) is stored in the left child of max. Shown at the top in Figure 8.

Whenever the cursor of \( T \) moves down the corresponding node is moved up to the root of \( T' \), the process is a ZigZag operation on \( T' \), or ZagZig depending on which grand-child. Note that between the rotations of the ZigZag operation the cursor must return to the root to comply with the second condition of restricted sequences. These movements are underlined in the example. Hence every downward move in \( T \) originates 4 moves and 2 rotations in \( T' \). This process transforms the trees in the top of Figure 8, to the trees in the middle. In this example the sequence of operations is: `moveTo(min), moveTo(x), rotate(), moveTo(y), moveTo(x) rotate()`.

The cursor may also move upwards on the tree, hence reversing the previous move. In this case the sequence of operations would be: `moveTo(y).rotate(), moveTo(x), moveTo(min), rotate(), moveTo(y)`. This requires 4 moves and 2 rotations. Therefore a move in \( T \) originates 4 moves and 2 rotations in \( T' \).

In this example the initial move was to \( y \), because \( y \) is the child in \( T' \) that corresponds to the parent in \( T \). To determine this property we could store, in \( T' \) the depth of the nodes in \( T \). Only for the nodes in the leftmost and rightmost branches of \( T' \), otherwise updated values would be hard to maintain. However it is not necessary to do so. Recall that \( T' \) is simulating \( T \) and \( T \) is the optimal tree, which does not need to consult the keys to know which move to perform, it only needs to behave as a BST. Therefore \( T' \) also does not need extra information.

Whenever a rotation is performed in \( T \) the structure of \( T' \) must be adapted accordingly. Recall that a rotation on node \( v \) means that \( v \) is moved upwards. In Figure 8 the transition from the middle to the bottom shows how a rotation alters the structure of \( T' \). In this case the sequence of moves is: `moveTo(y), moveTo(max),`
rotate(), moveTo(x). Hence a rotation originates 3 moves and 1 rotation in $T'$. The general procedure is to move to $y$ because it is the child of $T'$ that corresponds to the parent of $x$ in $T$, and move again in the same direction, in this case to max. This node is rotated upwards and the procedure finishes by returning to the cursor back to the root.

Note that in Figure 8 the cursor of $T$ is drawn close to the root. In general the structure of the upward path from the cursor of $T$ to the root gets splitted into the leftmost and rightmost branches of $T'$, but the update procedure is essentially as explained. □
In the following results we continue to use $T$, instead of $T'$, which simplifies the analysis. The tree $T'$ is used only in Theorem 4.6.

4.3. Amortized Costs of $S$ and $T$.

Let us now bound the amortized time of splaying a node of $S$.

Lemma 4.4. Splaying a node $v$ takes at most $4 + 6d_T(v)$, amortized time, where $d_T(v)$ is the depth of the corresponding node in $T$.

Proof. The following derivation establishes this bound:

\begin{align}
    \hat{c}_i & \leq 1 + 3[r(t) - r(v)] \\
    & < 1 + 3[1 - r(v)] \\
    & = 4 - 3r(v) \\
    & \leq 4 - 3\log(w(v)) \\
    & = 4 - 3\log(1/4^{d_T(v)}) \\
    & = 4 + 6d_T(v)
\end{align}

Inequality (4.12) is the classic amortized access Lemma 5.1, see Section 5. Inequality (4.13) follows from (4.5) and recalling that $r(t) = \log(s(t))$. Inequality (4.15) results from applying logarithms and switching signs to (4.3). The remaining equations use the definition of weight $w$ and simplify the result. $lacksquare$

This Lemma shows that the amortized time to splay a node is slightly more than 6 times the time that it is necessary to access the corresponding node in $T$. Let us focus on the amortized cost of accessing nodes in $T$. Whenever $T$ accesses a node there is no real cost for $S$. However if those accesses involve rotations in $T$ then the value of $\Phi$ changes, which will constitute an amortized cost that $S$ must account for.

Lemma 4.5. Whenever a node $v$ of $T$, with $d_T(v) < 3$, gets rotated the bound $\Delta \Phi \leq 11 + \log(11)$ holds, after, at most, 3 nodes of $S$ are splayed. Each splayed node $v'$ has $d_T(v') \leq d_T(v)$.

Proof. Recall that in a rotation the node $v$ moves upwards on the tree. We consider the cases when $d_T(v) = 1$ and $d_T(v) = 2$.

The proof is illustrated in Figure 9, which shows the situation when $d_T(v) = 1$, but from which we can infer properties that apply to all cases.

The optimal tree $T$ is represented on top and the splay tree $S$ in the bottom, upside down. To simplify let us assume that these trees are only slightly different. Meaning that they share some structure. The nodes in sub-trees $A$, $B$ and $C$ are the same, but their shape is not necessarily equal. For example the descendants of node $u$ are not, necessarily, the same.

A fundamental observation for this proof is that almost all the nodes, in the trees, contribute 0 to the value $\Delta \Phi$. The only nodes that effectively contribute to $\Delta \Phi$ belong to, at least, one the following categories:

1. Nodes that contain descendants from more than one of the sets $A$, $B$, $C$, either in $S$ or $T$ or both. Examples of these nodes are the ones containing $x$ and $y$. Moreover some nodes $a \in A$, $b \in B$ and $c \in C$ may appear in $S$ with this property.

2. Nodes for which the set of descendants is altered by the rotation in $T$. Only nodes $x$ and $y$.

Hence we are claiming that the nodes in $A$, $B$ and $C$ do not contribute to $\Delta \Phi$. Consider a node $u$, contained in the sub-tree $A$. Let $w_T(u)$ be the weight of $u$ before the rotation and $w_T'(u)$ be the weight after the rotation. Likewise we consider the $s_T$, $s_{T'}$, $r_T$, $r_{T'}$ values and the values $s$, $s'$, $r$, $r'$, over $S$. 
Let us account for the contribution of \( u \) to \( \Delta \Phi \), i.e.,
\[
    r'(u) - r(u) + r_T(u) - r'_T(u) = \log([s'(u)/s(u)] \times [s_T(u)/s'_T(u)]).
\]
Notice that \( w'(u)/w(u) = 4 \), because the depth of \( u \) decreases in \( T \). Likewise \( s'(u)/s(u) = 4 \), because the same happens for all the elements in \( A \). On the other hand \( s_T(u)/s'_T(u) = 1/4 \) because the order of the factors is reversed. Therefore the previous value is \( \log(4/4) = 0 \). A similar reasoning holds for the elements in \( C \). The nodes in \( B \) do not suffer potential variations and therefore contribute \( 0 \) to the overall variation.

In this scenario the only nodes that contribute to \( \Delta \Phi \) are the ones that contain descendants from more than one of the \( A \), \( B \), \( C \) sets. In our simplification only \( x \) and \( y \). However in the general case the nodes in sub-trees \( A \), \( B \) and \( C \) have different relations in \( S \) and in \( T \). Hence we need to be more precise as to what these nodes should be. The nodes in \( A \) are the descendants of \( x \) that are smaller than \( x \). The nodes in \( B \) are the descendants of both \( x \) and \( y \). The nodes in \( y \) are the descendants of \( y \) that are larger than \( y \).

To force these sets of nodes to be consistent between \( S \) and \( T \) we splay node \( x \) and node \( y \) in \( S \), in this order. Now \( A \) and \( C \) contain the same nodes in \( S \) and \( T \), but it may happen that some ZigZig operation pulls a node \( b \) upwards from \( B \), so that it splits \( B \) and is in between the nodes \( x \) and \( y \). In this case we must also count the contribution from \( b \).

Using an argument similar to the one above we can guarantee that \( s'(b)/s(b) \leq 4 \) and \( s_T(b)/s'_T(b) = 1 \) and therefore \( r'(b) - r(b) + r_T(b) - r'_T(b) \leq 2 + 0 = 2 \).

For the node containing \( x \) we also have that \( s'(x)/s(x) \leq 4 \), but \( s_T(x)/s'_T(x) \) is trickier, because \( s'_T(x) \) now includes the extra terms related to \( w'_T(y) \) and \( s'_T(C) \), these values are non-negative and therefore decrease the fraction \( s_T(x)/s'_T(x) \). Therefore the bound of \( 4 \) holds true. Hence for the node containing \( x \) we count another \( 4 \) units, as \( r'(x) - r(x) + r_T(x) - r'_T(x) \leq 2 + 2 = 4 \).
Therefore the total for \( y \) an ancestor of \( y \) the node is to the left of case we splay \( v \).

Inequality (4.23) follows from Inequality (4.4) that yields

\[
\frac{w_T(y) + s_T(C) + s_T(B) + w_T(x) + s_T(A)}{s'_T(y)} = \frac{w_T(y) + s_T(C) + s_T(B)}{s'_T(y)} + \frac{w_T(x) + s_T(A)}{s'_T(y)} \\
\leq 4 + \frac{w_T(x) + s_T(A)}{s'_T(y)} \\
\leq 4 + \frac{1/4 + s_T(A)}{1/4} \\
= 4 + \frac{(1/4) + (1/8)}{1/4} \\
\leq 4 + 1.5
\]

Equations (4.18) and (4.19) are simple manipulations. Inequality (4.20) is our general bound of 4, for the fraction that does not change the descendants. Inequality (4.3) yields \( 1/4 = w_T'(y) \leq s'_T(y) \), which we use to establish Inequality (4.21). Inequality (4.23) follows from Inequality (4.4) that yields \( s_T(A) \leq 2 \times 1/16 = 1/8 \). Therefore the total for \( y \) is \( r'(y) - r(y) + r_T(y) - r'_T(y) \leq 2 + \log(5.5) = 1 + \log(11) \).

Summing up, in this case we splayed two nodes and have to account for \( b \), \( x \) and \( y \), therefore \( \Phi = 2 + 4 + (1 + \log(11)) = 7 + \log(11) \).

Let us consider the case \( d_T(v) = 2 \). This means that there is a node \( v_r \) which is an ancestor of \( y \), in \( T \). Let us assume that \( v_r \) is to the right of \( C \). The case where the node is to the left of \( A \) is easier. It may happen that \( v_r \) is inside \( C \) in \( S \), in that case we splay \( v_r \). This splay operation may in turn split \( C \) by pulling up a node \( c \). In total we splayed 3 nodes and have 5 nodes that may contribute to \( \Phi \), the nodes \( b \) and \( c \), that got pulled up, and the nodes \( x \), \( y \), \( v_r \).

For node \( c \) we have a bound of 4, using the argument above, which holds because the descendants of \( c \) in \( T \) do not change. Hence \( r'(c) - r(c) + r_T(c) - r'_T(c) \leq 2 + 2 = 4 \).

Note that \( v_t \) is the root of both \( S \) and \( T \). Therefore we can can bound its variation by 0, instead of our pessimistic 4 value. For this node we have \( r'(t) = r'_T(t) \) and \( r(t) = r_T(t) \), by Equation (4.6). Hence in this case the total bound is \( \Phi \leq 2 + 4 + 4 + (1 + \log(11)) + 0 = 11 + \log(11) \).

Notice that all this splaying may change the ancestor relation between \( x \) and \( y \) in \( S \). This may happen when there is no node \( b \) and we are pulling a node \( v_{\ell} \), to the left of \( A \). This relation in \( S \) does not change the argument above, as both \( x \) and \( y \) are being pessimistically bounded in \( S \). The crucial consequence of the extra splays is that the nodes in \( A \), \( B \) and \( C \) are properly encapsulated, and the number of nodes whose descendants belong to more than one of these sets is limited.

We can now prove our optimality result.

**Theorem 4.6.** Consider a sequence of \( m \) queries, to a splay tree \( S \) with \( n + 2 \) keys, for which, a similar, optimal tree \( T \) uses \( R \) rotations and \( M \) cursor movements. Provided that splay trees have regular access, then the total number of operations performed by \( S \) is \( O(n + R + M) \).
Proof. The proof is given by the following deduction:

\[ \sum_{1 \leq i \leq m} c_i - k(n + 2) \leq k \left( \sum_{1 \leq i \leq m+e} \ell + c'_i \right) - k(n + 2) \]

(4.25) \[ \leq k \left( (m + e)\ell - (n + 2) + \sum_{1 \leq i \leq m+e} c'_i \right) \]

(4.26) \[ \leq k \left( (m + e)\ell + \Phi(D_m) + \sum_{1 \leq i \leq m+e} c'_i \right) \]

(4.27) \[ = k \left( (m + e)\ell + \Phi(D_m) - \Phi(D_0) + \sum_{1 \leq i \leq m+e} c'_i \right) \]

(4.28) \[ = k \left( (m + e)\ell + \sum_{1 \leq i \leq m+e} c'_i \right) \]

(4.29) \[ \leq k [(m + e)\ell + (11 + \log 11) R' + (1 + 3)(4 + 6M')] \]

(4.30) \[ \leq k [(m + 3R')\ell + (11 + \log 11) R' + (1 + 3)(4 + 6M')] \]

(4.31) \[ = k [m\ell + 16 + (11 + 3\ell + \log 11) R' + 24M'] \]

(4.32) \[ = k [m\ell + 16 + (11 + 3\ell + \log 11)(2M + R) + 24(4M + 3R)] \]

(4.33) \[ = k [m\ell + 16 + (118 + 6\ell + 2\log 11)M + (83 + 3\ell + \log 11)R] \]

The above deduction uses the restricted sequence over \( T' \), instead of the original optimal sequence over \( T \). We point out which of the results in the previous lemmas apply to \( T' \). The values \( R' \) and \( M' \) refer to the number of rotations and cursor movements in \( T' \).

We aim to bound the value on the left. The first inequality is our regular access conjecture. In here we are using explicit constants, \( k \) and \( \ell \), instead of the notation \( O \), to determine the hidden factors. Equation (4.25) rearranges the sums. Equation (4.26) follows from Lemma 4.2. In Equation (4.27) we add the value \( \Phi(D_0) = 0 \), because we are assuming that the initial structure of \( S \) and \( T' \) is the same. Equation (4.28) follows from the telescopic nature of the definition of amortized costs.

Inequality (4.29) follows from Lemmas 4.4 and 4.5. From Lemma 4.4 we conclude that the sum of the \( c'_i \) that corresponds to splay operations that answer queries is at most \( (4 + 6M') \), because \( T' \) must also move its cursor to those nodes and between any such nodes the cursor returns to the root. Therefore for \( T' \) to answer a query it must perform at least \( d_T(v) \) cursor movements, where \( v \) is the node containing the key that corresponds to the query.

To bound the sum of the \( c'_i \) that corresponds to the costs to update \( S \), due to rotations in \( T' \), we use Lemma 4.5. Note that we can apply this Lemma because of the first property of restricted sequences. The fixed cost yields the term \((11 + \log 11)R' \). The amortized cost of splaying, at most, 3 nodes of \( S \) is bounded by the term \( 3(4 + 6M') \). This bound holds because the second property of restricted sequences implies that \( T' \) must perform at least \( d_T(v) \) cursor movements before rotating node \( v \). This property states that after each rotation the cursor of \( T' \) returns to the root, see Section 4.2.
In Equation (4.30) we use the fact that $e \leq 3R'$, i.e., we use at most 3 extra splays for each rotation in $T'$. Equation (4.31) results by rearranging terms. In Equation (4.32) we use the bounds from Lemma 4.3. Equation (4.33) obtains the final bound, by rearranging terms. □

Let us now survey related work, so that we can discuss our result in context.

5. Related Work.

This section describes splay trees and some previous results. These trees were proposed by Sleator and Tarjan [24] in 1985. An extensive up to date survey on dynamic optimally was given by Iacono [16].

There are several restricted optimally results for BSTs. Assuming that the optimal BST is static than the best performance is the entropy $O(\sum_{i=1}^{m} f_i \log(m/f_i))$, where $m$ is the total number of queries and $f_i$ is the number of queries for the $i$-th key. Knuth presented the first algorithm to determine such an optimal tree [18]. This dynamic programming algorithm requires $O(n^2)$ time to determine the optimal tree. Kurt Mehlhorn obtained a faster, $O(n)$ time algorithm, which approximates the optimal tree [21]. In the context of this paper this result would be referred to as statically optimal, because we are hiding factors into the $O$ notation. Splay trees also obtain this kind of optimality and moreover do not need to know the $f_i$ values. Static optimality becomes the best upper bound when the keys in the query sequence are independent of each other.

We are only interested in approximating dynamic optimality, i.e., using $O$ notation and being a factor away from the optimal value, because it is expected that the exact problem is NP-Complete [7]. Even in this case the only BST that achieves the dynamic optimality bound, assumes free rotations and takes exponential time to select operations [3].

The good performance properties of splay trees are due to the access Lemma, which was establish in the paper on splay trees [24]. Here we repeat the original argument and highlight the fact that it follows from Jenssen’s Inequality [17].

**Lemma 5.1 (Amortized Access).** Splaying a node $v$ to the root $t$ takes at most $1 + 3[r'(t) - r(v)]$, amortized time.

**Proof.** Recall Figures 3, 4 and 5. When $v$ is at the root, the bound is trivial. Hence let us focus on the last operation that is used to access $v$, which contains $x$ in our figures.

**Zig** In this case $x$ is a child of $y$, which is at the root. The amortized cost is computed as follows:

\[
\hat{c}_{Zig} = 1 + r'(x) + r'(y) - r(x) - r(y) \\
\leq 1 + r'(x) - r(x) \\
\leq 1 + 3[r'(x) - r(x)]
\]

The only nodes that change rank are $x$ and $y$. The Inequality (5.1) follows from the fact that $r'(y) \leq r(y)$. On the other hand Inequality (5.2) is true because $r(x) \leq r'(x)$.

**ZigZig** In this case $x$, $y$ and $z$ change rank

\[
\hat{c}_{ZigZig} = 2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \\
= 2 + r'(y) + r'(z) - r(x) - r(y) \\
\leq 2 + r'(x) + r'(z) - 2r(x) \\
\leq 3[r'(x) - r(x)]
\]
Equation (5.5) follows from the fact that \( r'(x) = r(z) \). Inequality (5.6) is true because \( r'(y) \leq r'(x) \) and \(-r(y) \leq -r(x)\). Inequality (5.7) reduces to proving that \((r(x) + r'(z))/2 \leq r'(x) - 1\), this relation is known as Jensen’s inequality and it holds because the log function is concave. More explicitly the relation is the following:

\[
\frac{\log(s(x)) + \log(s'(z))}{2} \leq \log \left( \frac{s(x) + s'(z)}{2} \right)
\]

**ZigZag** In this case \( x, y \) and \( z \) change rank.

\[
\begin{align*}
\hat{c}_{Zigzag} & = 2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \\
& = 2 + r'(y) + r'(z) - r(x) - r(y) \\
& \leq 2 + r'(y) + r'(z) - 2r(x) \\
& \leq 2[r'(x) - r(x)] \\
& \leq 3[r'(x) - r(x)]
\end{align*}
\]

Equation (5.9) follows from the fact that \( r'(x) = r(z) \). Inequality (5.10) is true because \(-r(y) \leq -r(x)\). Inequality (5.11) is also Jensen’s inequality, in this case it reduces to \((r'(y) + r'(z))/2 \leq r'(x) - 1\). The Inequality (5.12) is true because \( r(x) \leq r'(x) \).

Notice that in any access to \( S \) there is at most 1 Zig operation and several ZigZig and ZigZag. This justifies why it is important to omit the term 1 in the bound for the ZigZig and ZigZag operations. The overall bound is obtained by summing the bounds of the respective operations. This sum telescopes to the expression in the Lemma.

A recent detailed study of this Lemma is available [4]. An important consequence of the splay operation is that most of the nodes in the splayed branch are moved upwards, i.e., their depth gets reduced, essentially in half. A detailed study on depth reduction was given by Subramanian [25].

Several good performance theorem follow from this Lemma:

**Static Optimality**, obtaining the entropy bound we mentioned in the beginning of the Section.

**Static Finger**, the performance of splay trees can be made to make dependent on the position of a given element of the key set \( X \). Let \( f \) be the position of a certain element \( x \) in the ordered set \( X \) and \(|x - f|\) the distance between that element and the element \( x_i \), the \( i \)-th element in the query sequence. Then processing a sequence of queries with a splay tree takes \( O(n \log n + m + \sum_{i=1}^{m} \log(|x_i - f| + 1)) \). In fact splay trees have an even better better performance because the chosen element at position \( f \) can be made dynamic [6, 5], but this result requires a longer analysis.

**Working Set**, meaning that recently accessed are quicker to access again. Let \( t(x) \) be the number of different items accessed before accessing \( x \) since the last time \( x \) was accessed, or since the beginning of the sequence if \( x \) was never accessed before. The the total time to process a sequence of queries is \( O(n \log n + m + \sum_{i=1}^{m} \log(t(i) + 1)) \). An important consequence of this bound is that if the keys are assigned arbitrarily (but consistently) to unordered data, splay trees behave as dynamically optimal [15].

**Unified bound**, combining the best performance of the 3 previous bounds. Dedicated structures where designed to achieve an even better bound which uses
the dynamic finger bound [2], instead of the static version. Generalized versions of this bound, with tighter values for structured sequences where also proposed [14].

Another important performance bound is the sequential access, or scanning, bound, which states that accessing the \( n \) keys sequentially takes only \( O(n) \) time. This was initially established by Tarjan [27] with a factor of 9, the current best factor of 4.5 [11] was proven by Elmasry.

Besides dynamic optimality Sleator and Tarjan [24] proposed the traversal conjecture, which remains unproven. It is similar to the sequential access but the key order is taken to be the preorder of another BST.

Other open conjectures on splay trees include the Deque conjecture, which claims that splay trees can be used to implement a deque, with \( O(1) \) amortized time per operation [26, 23]. The split conjecture claims that deleting all the nodes of a splay tree takes \( O(n) \) time, in whatever order [20].

Studying binary search trees from a geometric point of view has yielded several important results [7]. It presented an online algorithm, which may yield dynamically optimal BSTs. This algorithm is referred to as greedy, and it was originally proposed as an offline algorithm [19, 22]. The first \( O(\log n) \) performance bound, of this algorithm, was established by Fox [12].

The same authors proposed Tango trees which are \( O(\log \log n) \) competitive to the optimal BST [8]. This was the first structure to obtain a proven non-trivial competitive ratio. The worst case performance of these trees was further improved [10] to \( O(\log n) \) per operation. The \( O(\log \log n) \) ratio was also proved for the chain-splay variation of splaying [13].

Another alternative to try and obtain, proven, dynamic optimality consists in combining BSTs. In [9] the authors show that given any constant number of online BST algorithms (subject to certain technical restrictions), there is an online BST algorithm that performs asymptotically as well on any sequence as the best input BST algorithm.

Iacono [16] recently proposed another approach, the Weighted Majority Algorithm. This approach is proven to yield a dynamically optimal BST, provided any such data structure exists.

A considerable amount of research as also gone into the study of lower bounds, i.e., formulating expressions that are smaller than \( R + M \), i.e., the amount of operations required by \( T \). The first lower bounds where established by Wilber [29]. These bounds where further improved by the geometric view of BSTs [7]. In this view accesses to nodes in a tree are drawn as points in the plane. One coordinate stores the ordered keys and the other coordinate represents time. We can then consider rectangles using these points as corners. Two rectangles rectangles are independent if their corners are outside their interception or in the boundary of the interception. An access sequence in a BST must respect any independent set of rectangles. Hence the maximum independent set of rectangles yields a lower bound for the number of visited nodes in a BST.

The alternation lower and the funnel bounds can also be explained with the geometric view of BSTs. Moreover this latter bound is also related to the number of turns that it is necessary to move a key to root, with a procedure proposed by Allen and Munro [1].

6. Conclusions and Further Work.

In this paper we studied the dynamic optimality conjecture of splay trees, which
was proposed by the authors of these trees [24] and has stood for more than 30 years. During this time a vast amount of results and approaches have been proposed for this problem and related conjectures. Note that a proof of dynamic optimality would establish other conjectures.

All this research has resulted in new data structures, several of which where proven to be \((\log \log n)\) competitive. The results we present are a step forward. We reduced dynamic optimality to the regular access conjecture. This conjecture seems very plausible. On the other hand, if it is false that would also be a remarkable property.

In the future we plan to research this conjecture. An exhaustive literature review may provide the necessary tools to solve it. We also plan to investigate if our potential function can be applied to the analysis of other dynamically optimal candidates, namely the greedy algorithm, combining BSTs or the weighted majority algorithm.

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