1. Artin’s neighborhood

In this section we prove Theorem 3.2, Proposition 3.3 and Proposition 3.4 of [PaSV].

The following Bertini type theorem is an extension of Artin’s result [A, Exp.XI, Thm.2.1].

**Theorem 1.** Let $k$ be an infinite field, and let $V \subset \mathbf{P}_k^n$ be a locally closed sub-scheme of pure dimension $r$. Further, let $V' \subset V$ be an open sub-scheme such that for each point $x \in V'$ the scheme $V$ is $k$-smooth at $x$. Finally, let $p_1, p_2, \ldots, p_m \in \mathbf{P}_k^n$ be a family of pair-wise distinct closed points. For a family $H_1(d), H_2(d), \ldots, H_s(d)$, with $s \leq r$, of hypersurfaces of degree $d$ containing all points $p_i$, $1 \leq i \leq m$, set $Y = H_1(d) \cap H_2(d) \cap \cdots \cap H_s(d)$.

Then there exists an integer $d$ depending on the family $p_1, p_2, \ldots, p_m$ such that if the family $H_1(d), H_2(d), \ldots, H_s(d)$ with $s \leq r$ is sufficiently general, then $Y$ intersects $V$ transversally at each point of $Y \cap V'$.

If, moreover, $V$ is irreducible (respectively, geometrically irreducible) and $s < r$ then for the same integer $d$ and for a sufficiently general family $H_1(d), H_2(d), \ldots, H_s(d)$ the intersection $Y \cap V'$ is irreducible (respectively, geometrically irreducible).

**Proof.** Let $\bar{k}$ be an algebraic closure of $k$. Given a $k$-scheme $X$, we will write $\bar{X}$ for $X \otimes_k \bar{k}$. Let $p = \prod p_i$, let $\bar{p}$ be the corresponding scheme over $\bar{k}$, and let $q_j$ ($j = 1, 2, \ldots, s$) be all its closed points.

Firstly we prove the Theorem for the case of the field $\bar{k}$, the $\bar{k}$-schemes $\bar{V}$ and $\bar{V}'$ and the family of closed points $q_j$ ($j = 1, 2, \ldots, s$). Given that we will be able to choose a sufficiently general hyperplanes $H_1(d), H_2(d), \ldots, H_s(d)$ of the same degree $d$ which are defined over the field $k$ such that the family $\bar{H}_1(d), \bar{H}_2(d), \ldots, \bar{H}_s(d)$ solves our problem for the $\bar{k}$-schemes $\bar{V}$ and $\bar{V}'$ and the family of closed points $q_j$ ($j = 1, 2, \ldots, s$). Then, clearly, the family $H_1(d), H_2(d), \ldots, H_s(d)$ solves our problem for the $k$-schemes $V$ and $V'$ and the family of points $p_1, p_2, \ldots, p_m$.

Now set $q := \prod q_j$ and let $H \subset \mathbf{P}_k^n$ be a hyperplane. Let $\pi : \mathbf{P}_k^n \rightarrow \mathbf{P}_k^n$ be the blow up of $\mathbf{P}_k^n$ at $q$ and let $E = \pi^{-1}(q)$ be the exceptional divisor on $\mathbf{P}_k^n$. By [Ha] Ch.II, Sect.7, Prop.7.10 there exists an integer $N >> 0$ such that for each $c > N$ the invertible sheaf $\mathcal{O}(cH - E)$ on $\pi : \mathbf{P}_k^n \rightarrow \mathbf{P}_k^n$ is very ample relatively to $\mathbf{P}_k^n$. Take such an integer $c$ and set $d = c + 1$. It is easy to see that the sheaf $\mathcal{O}(dH - E)$ is very ample on $\mathbf{P}_k^n$. It follows that its global sections define a closed embedding $\mathbf{P}_k^n \hookrightarrow \mathbf{P}_k^n$. One has

$$H^0(\mathbf{P}_k^n, \mathcal{O}(1)) = H^0(\mathbf{P}_k^n, \mathcal{O}(dH - E)) = H^0(\mathbf{P}_k^n, \pi_*(\mathcal{O}(dH - E))) = H^0(\mathbf{P}_k^n, \pi_*(\mathcal{O}(dH - E) \otimes \mathcal{O}(dH))) = H^0(\mathbf{P}_k^n, I(d)),$$

where $I$ is the ideal sheaf defining $q$ in $\mathbf{P}_k^n$. These identifications of $H^0(\mathbf{P}_k^n, \mathcal{O}(1))$, $H^0(\mathbf{P}_k^n, \mathcal{O}(dH - E))$ and $H^0(\mathbf{P}_k^n, I(d))$ show that the set of homogeneous polynomials of degree $d$ vanishing at all points of $q$ defines a locally closed embedding $f : \mathbf{P}_k^n - q \hookrightarrow \mathbf{P}_k^n$. Let $L_i$ be a hyperplane in $\mathbf{P}_k^n$ corresponding to a degree $d$ hypersurface $H_i$ ($i = 1, 2, \ldots, s$) via the identification of $H^0(\mathbf{P}_k^n, \mathcal{O}(1))$ and $H^0(\mathbf{P}_k^n, I(d))$.

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Set $\tilde{V}'' := \tilde{V}' - (\tilde{V}' \cap q)$, $\tilde{V}'' := f(\tilde{V}'')$. Then $\tilde{V}''$ is isomorphic to $\tilde{V}''$ and the construction of $f$ implies the following: if $L_1 \cap L_2 \cap \cdots \cap L_s$ intersects $\tilde{V}''$ transversally, then $Y = H_1 \cap H_2 \cap \cdots \cap H_s$ intersects $\tilde{V}''$ transversally. Let $\tilde{V}$ be the closure of $\tilde{V}''$ in $P^n_k$. Now the item (i) of Theorem [A Exp.XI, Thm. 2.1] applied to the pair $\tilde{V}'' \subset \tilde{V}$ implies our theorem with $V'$ replaced by $V''$. It remains to consider points from $V' \cap q$. At these points $\tilde{V}$ is $k$-smooth.

If we replace the integer $d$ by a larger one, our result is still valid with $V'$ replaced by $V''$. At the same time, enlarging $d$ we may achieve the following: among those hypersurfaces of degree $d$ in $P^n_k$ that contain $q$, each one that is sufficiently general, intersects $\tilde{V}$ transversally at every point of $\tilde{V} \cap q$. This can be easily deduced from the fact that, for a sufficiently large $d$, the sheaf sequence

$$0 \to I^2(d) \to O(d) \to (O/I^2)(d) = O/I^2 \to 0$$

induces an exact sequence of global sections

$$0 \to H^0(P^n_k, I^2(2)) \to H^0(P^n_k, O(d)) \to H^0(P^n_k, O/I^2) \to 0.$$ 

Next we extend a result of Artin from [A] concerning existence of nice neighborhoods. The following notion is due to Artin [A Exp. XI, Déf. 3.1].

**Definition 1.** An elementary fibration is a morphism of schemes $p : X \to S$ that can be included in a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{j} & \overline{X} \\
\downarrow{p} & & \downarrow{q} \\
S & \xrightarrow{i} & Y
\end{array}
$$

of morphisms satisfying the following conditions:

(i) $j$ is an open immersion dense at each fibre of $\overline{p}$, and $X = \overline{X} - Y$;

(ii) $\overline{p}$ is smooth projective all of whose fibres are geometrically irreducible of dimension one;

(iii) $q$ is finite étale all of whose fibres are non-empty.

Using Theorem [A] one can prove the following result which is a slight extension of Artin’s result [A Exp.XI,Prop.3.3].

**Proposition 1.** Let $k$ be an infinite field, $X/k$ be a smooth geometrically irreducible variety, $x_1, x_2, \ldots, x_n \in X$ be closed points. Then there exists a Zariski open neighborhood $X^0$ of the family $\{x_1, x_2, \ldots, x_n\}$ and an elementary fibration $p : X^0 \to S$, where $S$ is an open sub-scheme of the projective space $P^{\dim X - 1}$.

If, moreover, $Z$ is a closed co-dimension one subvariety in $X$, then one can choose $X^0$ and $p$ in such a way that $p|_{Z \cap X^0} : Z \cap X^0 \to S$ is finite surjective.

**Proof.** The proof literally follows the proof of the original Artin’s result. Shrinking $X$ one may assume that $X \subset \mathbb{A}^n$ is affine and still contains the points $x_1, x_2, \ldots, x_n$. Set $x := \prod_{i=1}^{n} x_i$. Let $X_0$ be the closure of $X$ in $P^n$. Let $\overline{X}$ is the normalization of $X_0$ and set $Y = \overline{X} - X$ with the induced reduced structure. Let $S \subset \overline{X}$ be the closed subset of $\overline{X}$ consisting of all singular points. Then one has

(i) $S \subset Y$;

(ii) $\dim \overline{X} = \dim X = n$;

(iii) $\dim Y = n - 1$;

(iv) $\dim S \leq n - 2$.

Embed $\overline{X}$ in a projective space $P^t$. Let $M$ be the restriction of the invertible sheaf $O_{P^t}(t)$ to $\overline{X}$. Take the sheaf $M_{\overline{X}}$ with the integer $d$ from Theorem [A]. Consider the embedding $P^t \hookrightarrow P^N$ via the full linear system of hypersurfaces of degree $d$ in $P^t$. By Theorem [A] there exist hyperplanes $H_1, H_2, \ldots, H_{n-1}$ on $P^N$ such that $x \subset H_1$, $L := H_1 \cap H_2 \cap \cdots \cap H_{n-1}$ has dimension $N - n + 1$ and intersects $\overline{X}$ and $Y$ transversally. The intersection $\overline{X} \cap L$ is a smooth geometrically irreducible
Let \( f \) be a linear form in \( N + 1 \) variables \( t_v, \nu = 1, \ldots, N + 1 \), defining the hyperplane \( H_i \) in \( \mathbb{P}^N \). Consider a rational map \( \mathbb{P}^N \to \mathbb{P}^{n-1} \) sending a point \([t_0 : t_1 : \cdots : t_N]\) to the point \([u_0 : u_1 : \cdots : u_{n-1}]\), where \( u_i = h_i(t_0, t_1, \ldots, t_N) \). That morphism is a rational projection with the projector center \( C = H_0 \cap \cdots \cap H_{n-1} \). Let \( \epsilon : \mathbb{P}' \to \mathbb{P}^N \) be the blow up of \( \mathbb{P}^N \) with the center \( C \). Then the diagram

\[
\begin{array}{ccc}
\mathbb{P}^N & \xrightarrow{\epsilon} & \mathbb{P}' \\
\downarrow{\pi} & & \downarrow{q} \\
\mathbb{P}^{n-1} & & \\
\end{array}
\]

commutes as a diagram of rational morphisms and the arrows \( \epsilon \) and \( \pi \) are regular maps. Let \( \tilde{X}' \subset \mathbb{P}' \) be the proper preimage of \( X \), that is the closure of \( \epsilon^{-1}(\tilde{X} - (\tilde{X} \cap C)) \). By the hypotheses made \( C \) intersects \( \tilde{X} \) transversally, and the morphism \( \tilde{X}' \to \tilde{X} \) is the blow up at the \( k \)-smooth finite center \( \tilde{X} \cap C \). The \( k \)-smoothness means that the scheme \( \tilde{X} \cap C \) consists of finitely many points, and their residue fields are finite separable field extensions of \( k \).

Identify \( X' := X - (\tilde{X} \cap C) \) with an open subscheme of \( \tilde{X}' \). Let \( Y' := \tilde{X}' - X' \) be the closed subscheme of \( \tilde{X}' \) with the reduced subscheme structure. We claim that on the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{j} & \tilde{X}' \\
\downarrow{j'} & & \downarrow{i} \\
Y' & & \\
\end{array}
\]

there exists an open neighborhood \( V \) of the point \( v = f'(x) \) such that the restriction of the diagram (3) to \( V \) satisfies the conditions (i), (ii), (iii) of Definition (1). If this claim is true, it completes the proof of the Proposition. We proceed to verify it.

The condition (i) is trivial. To obtain (ii), note that \( \tilde{X} \cap L \) is a \( k \)-smooth geometrically irreducible curve by the hypotheses. It follows that the morphism \((f')^{-1}(v)) \to \tilde{X} \cap L \) induced by \( \epsilon \) is biunivoque. Whence \((f')^{-1}(v))_{\text{red}} \to \tilde{X} \cap L \) is bijective. To check that \( \tilde{f}' \) is smooth over a neighborhood of \( v \), it suffices by a Lemma of Hironaka [SGAI Exp.II, 2.6] to verify that \( \tilde{f}' \) is smooth at a generic point of \((f')^{-1}(v))_{\text{red}}. \) At that point \( \tilde{X}' \) is isomorphic to \( \tilde{X} \) and the morphism is smooth, since \( L \) intersects \( X \) transversally.

It remains to prove that \( g' \) is étale over a neighborhood of \( v \). It is clear that each fibre of \( g' \) is non-empty since \( \dim Y = n - 1 \). One has \( \tilde{X} \cap C = \coprod_p p_i \) scheme-theoretically, where \( p_i \) are points on \( \tilde{X} \) such that their residue fields are separable finite extensions of \( k \). One has \( Y' = e^{-1}(Y) \coprod D_1 \coprod \ldots \coprod D_r \), where \( D_i = e^{-1}(p_i) \cong P^{n-1}_{k(p_i)} \). For each index \( i \), the restriction of \( g' \) to \( P^{n-1}_{k(p_i)} \) coincides with the morphism \( P^{n-1}_{k(p_i)} \to P^{n-1}_k \) induced by the field extension \( k(p_i)/k \). All these field extensions are finite separable. Whence the morphism \( P^{n-1}_{k(p_i)} \to P^{n-1}_k \) is étale. Clearly, one has

\[
g'|_Y \circ e|_{e^{-1}(Y)} = g'|_{e^{-1}(Y)} : e^{-1}(Y) \to P^{n-1}
\]

and \( e|_{e^{-1}(Y)} : e^{-1}(Y) \to Y \) is an isomorphism. The morphism \( g'|_Y : Y \to P^{n-1} \) is étale over a neighborhood of \( v \), since \( L \) intersects \( Y \) transversally. Whence \( g' \) is étale over a neighborhood of \( v \).

In addition, we may choose \( H_0, H_1, \ldots, H_{n-1} \) such that \( H_0 \cap H_1 \cap \cdots \cap H_{n-1} \cap Z = \emptyset \). In that case \( Z \subset X' \) and the morphism \( f'|_{Z} : Z \to P^{n-1} \) is finite surjective. As a consequence the morphism \( f'|_{(f')^{-1}(V) \cap Z} : (f')^{-1}(V) \cap Z \to V \) is finite surjective too.

\( \square \)
Proposition 2. Let \( p : X \to S \) be an elementary fibration. If \( S \) is a regular semi-local scheme, then there exists a commutative diagram of \( S \)-schemes

\[
\begin{array}{ccc}
X & \xrightarrow{j} & \overline{X} \\
\pi & \downarrow & \ \ \ \\
A^1 \times S & \xrightarrow{\text{in}} & P^1 \times S \\
\end{array}
\]

such that \( \bar{\pi} \) is finite surjective, \( Y = \pi^{-1}(\{\infty\} \times S) \) set-theoretically, and the left hand side square is Cartesian. Here \( j \) and \( i \) are the same as in Definition \( \mathbb{I} \) while \( \text{pr}_S \circ \pi = p \), where \( \text{pr}_S \) is the projection \( A^1 \times S \to S \).

In particular, \( \pi : X \to A^1 \times S \) is a finite surjective morphism of \( S \)-schemes, where \( X \) and \( A^1 \times S \) are regarded as \( S \)-schemes via the morphism \( p \) and the projection \( \text{pr}_S \), respectively.

Proof. To prove this Proposition it suffices to construct a finite surjective \( S \)-morphism \( \bar{\pi} : \overline{X} \to P^1 \times S \) such that \( Y = \bar{\pi}^{-1}(\{\infty\} \times S) \) set-theoretically. To do that, we first note that, under the hypotheses of the Proposition, \( Y \subset \overline{X} \) is an effective Cartier divisor. We will construct a desired \( \bar{\pi} \) using two sections \( t_0 \) and \( t_1 \) of the sheaf \( O(Y) \) for a sufficiently large \( n \). Assume that \( t_0 \) and \( t_1 \) are such that the vanishing locus of \( t_0 \) is \( nY \) and the vanishing locus of \( t_1 \) does not intersect \( Y \). Then the pair \( t_0, t_1 \) defines a regular map \( \varphi := [t_0 : t_1] : \overline{X} \to P^1 \). Set \( \bar{\pi} = (\varphi, \bar{p}) : \overline{X} \to P^1 \times S \).

Clearly, \( \bar{\pi} \) is an \( S \)-morphism of the \( S \)-schemes. It is a projective morphism since both \( S \)-schemes are projective \( S \)-schemes. It is a quasi-finite surjective morphism. In fact, for each point \( s \in S \) the morphism \( \bar{\pi} \) induces a non-constant morphism \( \overline{X}_s \to P^1_s \) of two \( k(s) \)-smooth irreducible projective \( k(s) \)-curves. Thus \( \bar{\pi} \) is finite surjective as a quasi-finite projective morphism. It remains to find an appropriate integer \( n \) and two sections \( t_0 \) and \( t_1 \) with the above properties.

Firstly, for each point \( s \) of the scheme \( S \) set \( \overline{X}_s := (\bar{\pi})^{-1}(s) \) scheme-theoretically and note that \( \overline{X}_s \) is a \( k(s) \)-smooth irreducible projective \( k(s) \)-curve. The morphism \( \bar{\pi} \) is smooth. In particular, it is flat. Whence the function \( s \mapsto \chi(\overline{X}_s, O_{\overline{X}_s}) \) is constant by Corollary \( \mathbb{M} \) Ch.II, Sect.5, Cor.1].

The latter means that the genus \( g(\overline{X}_s) \) is the same for all points \( s \in S \). Set \( g = g(\overline{X}_s) \).

Assume that \( n \geq 2g - 1 \), then \( H^0(\overline{X}_s, O_{\overline{X}_s}(nY_s)) = \chi(\overline{X}_s, O_{\overline{X}_s}(nY_s)) = n - g + 1 \). Let \( \mathcal{E}_n := \bar{p}_*(O_{\overline{X}}(nY_s)) \). By Corollary \( \mathbb{M} \) II, Sect.5, Cor.1] and Lemma \( \mathbb{M} \) II, Sect.5, Lem.1] the sheaf \( \mathcal{E}_n \) on \( S \) is locally free of rank \( n - g + 1 \), and for each point \( s \in S \) one has \( \mathcal{E}_n \otimes O_S k(s) \cong H^0(\overline{X}_s, O_{\overline{X}_s}(nY_s)) \).

Let \( s = \prod s_i \), where \( s_i \) are all closed points of the semi-local scheme \( S \). Let \( k(s) = \prod k(s_i) \), where \( k(s_i) \) is the residue field of the point \( s_i \). Consider a commutative diagram

\[
\begin{array}{ccc}
H^0(S, \mathcal{E}_n) & \xrightarrow{\text{id}} & H^0(\overline{X}, O_{\overline{X}}(nY)) \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\mathcal{E}_n \otimes O_S k(s) & \xrightarrow{\text{can}} & H^0(\overline{X}_s, O_{\overline{X}_s}(nY_s))
\end{array}
\]

where \( \alpha \), \( \beta \) and \( \text{can} \) are the canonical homomorphisms. As it was mentioned in the previous paragraph, the map \( \text{can} \) is an isomorphism. The map \( \alpha \) is surjective, since \( s \) is a closed subscheme of an affine scheme \( S \). Whence the map \( \beta \) is surjective.

For each \( s_i \in s \) the curve \( X_{s_i} \) is a smooth geometrically irreducible curve of genus \( g \). Whence there exists an integer \( n_0 \) such that for each \( n \geq n_0 \) the sheaf \( O_{\overline{X}_s}(nY_s) \) is very ample. By the Bertini type theorem \( \mathbb{A} \) Exp.XI,Thm.2.1 there exists for any \( i \) a section \( t_{1,i} \) of \( O_{\overline{X}_s}(nY_s) \) that does not vanish on \( Y_{s_i} \). By the surjectivity of \( \beta \) we may choose a section \( t_1 \) of \( O_{\overline{X}}(nY) \) such that \( \beta(t_1)|_{\overline{X}_s} = t_{1,i} \) for any \( i \). The vanishing locus of \( t_1 \) does not intersect \( Y_s \), whence it does not intersect \( Y \). Clearly, \( t_1 \) is the desired section of \( O_{\overline{X}}(nY) \). It remains to take for \( t_0 \) a section of \( O_{\overline{X}}(nY) \) with the vanishing locus \( nY \).

\( \square \)
2. Equating group schemes

The following result is Proposition 9.1 of [PaSV].

**Proposition 3.** Let $S$ be a regular semi-local irreducible scheme and let $G_1, G_2$ be two semi-simple simply connected $S$-group schemes which are twisted forms of each other. Further, let $T \subset S$ be a closed subscheme of $S$ and $\phi : G_1|_T \to G_2|_T$ be an $S$-group scheme isomorphism. Then there exist a finite étale morphism $\tilde{S} \to S$ together with a morphism $\delta : T \to \tilde{S}$ of schemes over $S$, and a $\tilde{S}$-group scheme isomorphism $\tilde{\Phi} : \pi^* G_1 \to \pi^* G_2$ such that $\delta^*(\tilde{\Phi}) = \phi$.

Since the proof of the Proposition is rather long we first give an outline. Clearly, $G_1$ and $G_2$ are of the same type. By [SGA3] Vol.153, Exp.XXIV, Cor.1.8 there exists an $S$-scheme $\text{Isom}_S(G_1, G_2)$ representing the functor that sends an $S$-scheme $W$ to the set of all $W$-group scheme isomorphisms from $W \times_S G_1$ to $W \times_S G_2$. The isomorphism $\phi$ from the hypothesis of Proposition 3 determines a section $\delta : T \to \text{Isom}_S(G_1, G_2)$ of the structure map $\text{Isom}_S(G_1, G_2) \to S$. By Lemmas 4 and 1 below there exists a closed subscheme $\tilde{S}$ of $\text{Isom}_S(G_1, G_2)$ which is finite étale over $S$ and contains $\delta(T)$. So, we have a commutative diagram of $S$-schemes

$$
\begin{array}{ccc}
T & \xrightarrow{\delta} & \tilde{S} \\
\downarrow & & \downarrow \pi \\
S & & \text{Isom}_S(G_1, G_2)
\end{array}
$$

such that the horizontal arrows are closed embeddings. Thus we get an isomorphism $\Phi : \pi^* (G_1) \to \pi^* (G_2)$ such that $\delta^*(\Phi) = \phi$.

The precise proof of the Proposition requires some auxiliary results and will be postponed till the very end of the Section. Clearly, $G_1$ and $G_2$ are of the same type. Let $G_0$ be a split semi-simple simply connected algebraic group over the ground field $k$ such that $G_1$ and $G_2$ are twisted forms of the $S$-group scheme $S \times_{\text{Spec}(k)} G_0$. Let $\text{Aut}_k(G_0)$ be the automorphism scheme of the algebraic $k$-group $G_0$. It is known that $\text{Aut}_k(G_0)$ is a semi-direct product of the algebraic $k$-group $G_0^{ad}$ and a finite group, where $G_0^{ad}$ is a group adjoint to $G_0$. Also, $\text{Aut}_k(G_0)$ is a smooth affine algebraic $k$-group (for example, by [SGA3] Vol.153, Exp.XXIV, Cor.1.8]). Set for short $\text{Aut} := \text{Aut}_k(G_0)$ and $\text{Aut}_S$ for the $S$-group scheme $S \times_{\text{Spec}(k)} \text{Aut}$.

Consider an $S$-scheme $\text{Isom}_S(G_{0,S}, G_2)$ constructed in [SGA3] Vol.153, Exp.XXIV, Cor.1.8] and representing a functor that sends an $S$-scheme $W$ to the set of all $W$-group scheme isomorphisms $\varphi_2 : W \times_S G_{0,S} \to W \times_S G_2$. Similarly, consider an $S$-scheme $\text{Aut}_S(G_2)$ constructed in [SGA3] Vol.153, Exp.XXIV, Cor.1.8] and representing a functor that sends an $S$-scheme $W$ to the set of all $W$-group scheme automorphisms $\alpha : W \times_S G_2 \to W \times_S G_2$.

The functor transformation $(\varphi_2, \alpha_2) \mapsto \varphi_2 \circ \alpha_2^{-1}$ defines an $S$-scheme morphism

$$
\text{Isom}_S(G_{0,S}, G_2) \times_S \text{Aut}_S \to \text{Isom}_S(G_{0,S}, G_2)
$$

which makes the $S$-scheme $\text{Isom}_S(G_{0,S}, G_2)$ a principal right $\text{Aut}_S$-bundle. The functor transformation $(\beta_2, \varphi_2) \mapsto \beta_2 \circ \varphi_2$ defines an $S$-scheme morphism

$$
\text{Aut}_S(G_2) \times_S \text{Isom}_S(G_{0,S}, G_2) \to \text{Isom}_S(G_{0,S}, G_2)
$$

which makes the $S$-scheme $\text{Isom}_S(G_{0,S}, G_2)$ a principal left $G_2$-bundle.

Similarly, the functor transformation $(\alpha_1, \varphi_1) \mapsto \alpha_1 \circ \varphi_1$ makes the $S$-scheme $\text{Isom}_S(G_1, G_{0,S})$ a principal left $\text{Aut}_S$-bundle and the functor transformation $(\varphi_1, \beta_1) \mapsto \varphi_1 \circ \beta_1$ makes the $S$-scheme $\text{Isom}_S(G_1, G_{0,S})$ a principal right $G_1$-bundle.

Let $2P_r$ be a left principal $G_2$-bundle and at the same time a right principal $\text{Aut}_S$-bundle such that the two actions commute. Let $1P_1$ be a left principal $\text{Aut}_S$-bundle and at the same time a right principal $G_1$-bundle such that the two actions commute. Let $Y$ be a $k$-variety equipped with a left and a right $\text{Aut}_k$-actions which commute. Then the $k$-scheme

$$
(2P_r) \times_S (Y_S) \times_S (1P_1)
$$
is equipped with a left $Aut_k \times Aut_k$-action given by $(\alpha_2, \alpha_1)(p_2, y, p_1) = (p_2\alpha_2^{-1}, \alpha_2y\alpha_1^{-1}, \alpha_1p_1)$. The orbit space does exist (it can be constructed by descent). Denote it by $2Y_1$. We now show that it is an $S$-scheme. Indeed, the structure morphism $Y \to Spec(k)$ defines a morphism

$$(2P_r) \times_S (Y_S) \times_S (iP_1) \to (2P_r) \times_S (iP_1)$$

respecting the $Aut_k \times Aut_k$-actions on both sides. Thus it defines a morphism of the orbit spaces $2Y_1 \to (2Spec(k)_1) = S$.

The latter equality holds since $(2P_r) \times_S (iP_1)$ is a principal left $Aut \times Aut$-bundle with respect to the left action given by $(\alpha_2, \alpha_1)(p_2, p_1) = (p_2\alpha_2^{-1}, \alpha_1p_1)$.

The construction $Y \mapsto 2Y_1$ has several nice properties. Namely,

(i) it is natural with respect to $k$-morphisms of $k$-varieties $Y \to Y'$ commuting with the given two-sided $Aut \times Aut$-actions on $Y$ and $Y'$,

(ii) it takes closed embeddings to closed embeddings,

(iii) it takes open embeddings to open embeddings,

(iv) it takes $k$-products to $S$-products,

(v) locally in the étale topology on $S$, the $S$-schemes $Y_S$ and $2Y_1$ are isomorphic.

Set $2P_r = Isoms_S(G_0, S_2)$ and $iP_1 = Isoms_S(G_1, G_0, S)$. The functor transformation $(\varphi_2, \alpha, \varphi_1) \mapsto \varphi_2 \circ \alpha \circ \varphi_1$ gives a morphism of representable $S$-functors

$$Isoms_S(G_0, S_2) \times_S (Aut_S) \times_S Isoms_S(G_1, G_0, S) \to Isoms_S(G_1, G_2).$$

The equality

$$\varphi_2 \circ \alpha \circ \varphi_1 = (\varphi_2 \circ \alpha_2^{-1}) \circ (\alpha_2 \circ \alpha \circ \alpha_1^{-1}) \circ (\alpha_1 \circ \varphi_1)$$

shows that the morphism $\Phi$ induces a morphism $\Phi' : 2(Aut_S)_1 \to Isoms_S(G_1, G_2)$.

**Lemma 1.** The $S$-morphism

$$\Phi' : 2(Aut_S)_1 \to Isoms_S(G_1, G_2)$$

is an isomorphism.

**Proof.** It suffices to prove that $\Phi'$ is an isomorphism locally in the étale topology on $S$. The latter follows from the property (v). \qed

Now let $G_0$ and $Aut$ be as above. There is a closed embedding of algebraic groups $\rho : Aut \hookrightarrow GL_{V,k}$ for an $n$-dimensional $k$-vector space $V$. Replacing $\rho$ with $\rho \oplus det^{-1} \circ \rho$ we get a closed embedding of algebraic $k$-groups $p_1 : Aut \hookrightarrow SL_{W,k}$, where $W = V \oplus k$. Let $End := End_k(W)$. Clearly, the composition $in : Aut \overset{p_1}{\to} SL_{W,k} \hookrightarrow End$ is a closed embedding. We will identify $Aut$ with its image in $End$. Let $\overline{Aut}$ be the closure of $Aut$ in the projective space $P(k \oplus End)$. Set $Aut_\infty := \overline{Aut} - Aut$ regarded as a reduced scheme. So, we get a commutative diagram of $k$-varieties

\begin{equation}
\begin{array}{ccc}
Aut & \overset{j}{\to} & \overline{Aut} \leftarrow i & \overset{i}{\to} & Aut_\infty \\
\downarrow{in} & & \downarrow{in} & & \downarrow{in_\infty} \\
End & \overset{j}{\to} & P(k \oplus End) \leftarrow i & \overset{i}{\to} & P(End)
\end{array}
\end{equation}

where the left square is Cartesian. All varieties are equipped with the left $Aut \times Aut$-action induced by $Aut \times Aut$-action on the affine space $k \oplus End$ given by $(g_1, g_2)(\alpha, c) = (c, g_1ag_2^{-1})$. All the arrows in this diagram respect this action. Applying to this diagram the above construction $Y \mapsto 2Y_1$, we obtain a commutative diagram of $S$-schemes

\begin{equation}
\begin{array}{ccc}
2Aut & \overset{j}{\to} & 2(Aut)_1 \leftarrow i & \overset{i}{\to} & 2(Aut)_\infty \\
\downarrow{in} & & \downarrow{in} & & \downarrow{in_\infty} \\
2End & \overset{j}{\to} & P(O_S \oplus 2End_1) \leftarrow i & \overset{i}{\to} & P(2End_1)
\end{array}
\end{equation}
where the square on the left is Cartesian.

From now on we assume that $S$ is a semi-local irreducible scheme. Then the vector bundle $2\text{End}_1$ is trivial. Since it is trivial, we may choose homogeneous coordinates $Y_i$’s on $\mathbb{P}(\mathcal{O}_S \oplus 2\text{End}_1)$ such that the closed subschemes $\{Y_0 = 0\}$ and $\mathbb{P}(2\text{End}_1)$ of the scheme $\mathbb{P}(\mathcal{O}_S \oplus 2\text{End}_1)$ coincide and the $S$-scheme $\mathbb{P}(\mathcal{O}_S \oplus 2\text{End}_1)$ itself is isomorphic to the projective space $\mathbb{P}^2_S$. Thus the diagram (S) of $S$-schemes and of $S$-scheme morphisms can be rewritten as follows

\[
\begin{array}{ccc}
2\text{Aut}_1 & \xrightarrow{j} & 2(\text{Aut}_1) \quad & \xrightarrow{i} \quad & 2(\text{Aut}_{\infty})_1 \\
\downarrow{\text{in}} & & \downarrow{\text{in}} & & \\
\{Y_0 \neq 0\} & \xrightarrow{j} & \mathbb{P}^2_S & \xrightarrow{i} & \{Y_0 = 0\}
\end{array}
\]

where the square on the left is Cartesian. Since $2(\text{Aut}_{\infty})_1 = 2(\text{Aut}_1) - 2\text{Aut}_1$, the set-theoretic intersection $2(\text{Aut}_1) \cap \{Y_0 = 0\}$ in $\mathbb{P}^2_S$ coincides with $2(\text{Aut}_{\infty})_1$.

The following Lemma is the lemma [OjPa, Lemma 7.2].

**Lemma 2.** Let $S = \text{Spec}(R)$ be a regular semi-local scheme and $T$ a closed subscheme of $S$. Let $\bar{X}$ be a closed subscheme of $\mathbb{P}^2_S = \text{Proj}(S[Y_0, \ldots, Y_N])$ and $X = \bar{X} \cap \mathbb{A}^n_S$, where $\mathbb{A}^n_S$ is the affine space defined by $Y_0 \neq 0$. Let $X_{\infty} = \bar{X} \setminus X$ be the intersection of $\bar{X}$ with the hyperplane at infinity $Y_0 = 0$. Assume further that

1. $X$ is smooth and equidimensional over $S$, of relative dimension $r$.
2. For every closed point $s \in S$ the closed fibres of $X_{\infty}$ and $X$ satisfy
   \[
   \dim(X_{\infty}(s)) < \dim(X(s)) = r.
   \]
3. Over $T$ there exists a section $\delta : T \to X$ of the canonical projection $X \to S$.

Then there exists a closed subscheme $\bar{S}$ of $X$ which is finite étale over $S$ and contains $\delta(T)$.

The diagram (9) shows that the $S$-schemes $X = 2\text{Aut}_1$, $\bar{X} = 2(\text{Aut}_1)$ and $X_{\infty} = 2(\text{Aut}_{\infty})_1$ satisfy all the hypotheses of Lemma 2 except possibly the conditions (2) and (3). To check (2), observe that the diagram of $S$-schemes

\[
\begin{array}{ccc}
2\text{Aut}_1 & \xrightarrow{j} & 2(\text{Aut}_1) \quad & \xrightarrow{i} \quad & 2(\text{Aut}_{\infty})_1 \\
\downarrow{\text{in}} & & \downarrow{\text{in}} & & \\
\{Y_0 \neq 0\} & \xrightarrow{j} & \mathbb{P}^2_S & \xrightarrow{i} & \{Y_0 = 0\}
\end{array}
\]

locally in the étale topology on $S$ is isomorphic to the diagram of $S$-schemes

\[
\begin{array}{ccc}
\text{Aut} \times S & \xrightarrow{j} & (\text{Aut}_1) \times S \quad & \xrightarrow{i} \quad & (\text{Aut}_{\infty})_1 \times S.
\end{array}
\]

This follows from the property (v) of the construction $Z \to 2Z_1$. Since $\text{Aut}$ is equidimensional and $2\text{Aut}_1$ is the closure of $\text{Aut}$ in $\mathbb{P}(\text{End} \oplus k)$, one has

\[
\dim(\text{Aut}_{\infty}) < \dim(2\text{Aut}_1) = \dim \text{Aut}.
\]

Thus the assumption (2) of Lemma 2 is fulfilled. Whence we have proved the following

**Lemma 3.** Assume $S$ is a regular semi-local irreducible scheme and assume we are given with a closed subscheme $T \subset S$ equipped with a section $\delta : T \to 2\text{Aut}_1$ of the structure map $2\text{Aut}_1 \to S$. Then there exists a closed subscheme $\bar{S}$ of $2\text{Aut}_1$ which is finite and étale over $S$ and contains $\delta(T)$.

**Proof of Proposition 3.** By Lemma 1 the $S$-schemes $\text{Isom}_S(G_1, G_2)$ and $2\text{Aut}_1$ are naturally isomorphic as $S$-schemes. The isomorphism $\varphi$ from the hypotheses of the Proposition 3 determines a section $\delta : T \to \text{Isom}_S(G_1, G_2) = 2\text{Aut}_1$ of the structure map $\text{Isom}_S(G_1, G_2) \to 2\text{Aut}_1 \to S$. By Lemma 3 there exists a closed subscheme $\bar{S}$ of $2\text{Aut}_1 = \text{Isom}_S(G_1, G_2)$ which is finite étale over $S$.
and contains \( \delta(T) \). So, we have morphisms (even closed inclusions) of \( S \)-schemes

\[
T \xrightarrow{\delta} S \xrightarrow{\pi} Isom_S(G_1, G_2)
\]

Thus we get an isomorphism \( \Phi : \pi^*(G_1) \to \pi^*(G_2) \) such that \( \delta^*(\Phi) = \varphi \).

3. Horrocks type theorem

In this section we give a proof of Theorem 8.6 of [PaSV].

In the Theorem, let \( k \) be an infinite field, \( \mathcal{O} \) be the semi-local ring of finitely many points on a \( k \)-smooth affine scheme \( X; x = \{x_1, x_2, \ldots, x_n\} \subset \text{Spec}(\mathcal{O}) \) be the set of all closed points, \( l = \prod_{i=1}^n k(x_i) \). Let \( G \) be a simple simply-connected \( \mathcal{O} \)-group scheme, \( G_1 = G \otimes k \), \( \mathbb{A}^1 := \mathbb{A}^1_k \).

**Theorem 2.** Let \( G \) be an simple simply connected \( \mathcal{O} \)-group scheme. Let \( E \) be a principal \( G \)-bundle over \( \mathbb{P}^1 \), whose restriction to each closed fibre is trivial, that is \( E_{\mathbb{P}^1} \) is trivial over \( \mathbb{P}^1 \). Then \( E \) is of the form: \( E = \text{pr}^*(E_0) \), where \( E_0 \) is a principal \( G \)-bundle over \( \text{Spec}(\mathcal{O}) \) and \( \text{pr} : \mathbb{P}^1 \to \text{Spec}(\mathcal{O}) \) is the canonical projection.

The proof of this Theorem is rather standard, and for the most part follows [RL]. However, our group scheme \( G \) does not come from the ground field \( k \). Therefore, we have to somewhat modify Raghunathan’s arguments. We will use the following lemma.

**Lemma 4.** Let \( W \) be a semi-local irreducible Noetherian scheme over an arbitrary field \( k \). Let \( H \) and \( H' \) be two reductive group schemes over \( W \), such that \( H \) is a closed \( W \)-subgroup scheme of \( H' \), and denote by \( j : H \hookrightarrow H' \) the corresponding embedding. Denote by \( \mathbb{P}^1_W \) the projective line over \( W \).

Let \( F \in H^1(\mathbb{P}^1_W, H) \) be a principal \( H \)-bundle, and let \( M := j_*(F) \in H^1(\mathbb{P}^1_W, H') \) be the corresponding principal \( H' \)-bundle. If \( M \) is a trivial \( H' \)-bundle, then there exists a principal \( H \)-bundle \( F_0 \) over \( W \) such that \( \text{pr}^*(F_0) \cong F \), where \( \text{pr} : \mathbb{P}^1_W \to W \) is the canonical projection.

**Proof.** Set \( X = H'/j(H) \). Locally in the étale topology on \( W \) this scheme is isomorphic to the \( W \)-scheme \( W \times_{\text{Spec}(k)} H_{0,k}/H_{0,k} \), where \( H_{0,k} \) and \( H_{0,k} \) are the split reductive \( k \)-group schemes of the same types as \( H \) and \( H' \) respectively. By results of Haboush [HAB] and Nagata [NA] (see Nisnevich [NI] Corollary]) the \( k \)-scheme \( H_{0,k}/H_{0,k} \) is an affine \( k \)-scheme. Thus \( X \) is an affine \( W \)-scheme.

Consider the long exact sequence of pointed sets

\[
1 \to H(\mathbb{P}^1_W) \xrightarrow{j_*} H'(\mathbb{P}^1_W) \to X(\mathbb{P}^1_W) \xrightarrow{\partial} H^1_{\text{et}}(\mathbb{P}^1_W, H) \xrightarrow{\text{pr}^*_{\mathbb{P}^1_W}} H^1_{\text{et}}(\mathbb{P}^1_W, H').
\]

Since \( j_*(F) \) is trivial, there is \( \varphi \in X(\mathbb{P}^1_W) \) such that \( \text{pr}^*(\varphi) = F \).

The \( W \)-morphism \( \varphi : \mathbb{P}^1_W \to X \) is a \( W \)-morphism of a \( W \)-projective scheme to a \( W \)-affine scheme. Thus \( \varphi \) is "constant", that is, there exists a section \( s : W \to X \) such that \( \varphi = s \circ \text{pr} \). Consider another long exact sequence of pointed sets, this time the one corresponding to the scheme \( W \), and the morphism of the first sequence to the second one induced by the projection \( \text{pr} \). We get a big commutative diagram. In particular, we get the following commutative square

\[
\begin{array}{ccc}
X(W) & \xrightarrow{\partial} & H^1_{\text{et}}(W, H) \\
\text{pr}^*_{\mathbb{P}^1_W} \downarrow & & \downarrow \text{pr}^*_{\mathbb{P}^1_W} \\
X(\mathbb{P}^1_W) & \xrightarrow{\partial} & H^1_{\text{et}}(\mathbb{P}^1_W, H).
\end{array}
\]

We have \( \text{pr}^*_W(s) = \varphi \). Hence

\[
F = \text{pr}(\varphi) = \text{pr}(\text{pr}^*_W(s)) = \text{pr}^*_W(\partial(s)).
\]

Setting \( F_0 = \partial(s) \) we see that \( F = \text{pr}^*_W(F_0) \). The Lemma is proved.
Proof of Theorem 2. Let $U = \text{Spec}(O)$. The $U$-group scheme $G$ is given by a 1-cocycle $\xi \in Z^1(U, \text{Aut})$, where $\text{Aut}$ is the $k$-algebraic group from Section 2 (the automorphism group of the split group $G_0$ from that Section). Recall that $\text{Aut} \cong G_0^{ad} \rtimes N$, where $N$ is the finite group of automorphisms of the Dynkin diagram of $G_0$, and $G_0^{ad}$ is the adjoint group corresponding to $G_0$. Since $\text{Aut} \cong G_0^{ad} \rtimes N$, we have an exact sequence of pointed sets

$$\{1\} \rightarrow H^1(U, G_0^{ad}) \rightarrow H^1(U, G_0^{ad} \rtimes N) \rightarrow H^1(U, N).$$

Thus there is a finite étale morphism $\pi : V \rightarrow U$ such that $G_V := G \times_U V$ is given by a 1-cocycle $\xi_V \in Z^1(U, G_0^{ad})$.

For each fundamental weight $\lambda$ of $G_0$, there is a central (also called center preserving, see \cite{PS}) representation $\rho_\lambda : G_0 \rightarrow GL_{V_\lambda, k}$, where $V_\lambda$ is the Weyl module corresponding to $\lambda$. This gives a commutative diagram of $k$-group morphisms

$$\begin{array}{ccc}
G_0 & \rightarrow & GL_{V_\lambda, k} \\
\rho_\lambda & & \\
\downarrow & & \\
G_0^{ad} & \rightarrow & PGL_{V_\lambda, k}.
\end{array}$$

Considering the product of $\rho_\lambda$'s with $\lambda$ running over the set $\Lambda$ of all fundamental weights, we obtain the following commutative diagram of algebraic $k$-group homomorphisms:

$$\begin{array}{ccc}
G_0 & \rightarrow & \prod_{\lambda \in \Lambda} GL_{V_\lambda, k} \\
\rho & & \\
\downarrow & & \\
G_0^{ad} & \rightarrow & \prod_{\lambda \in \Lambda} PGL_{V_\lambda, k}.
\end{array}$$

Note that $\rho$ is a closed embedding. This fact will be used below.

Twisting the $k$-group morphism $\rho$ with the 1-cocycle $\xi_V$ we get an $V$-group scheme morphism $\rho_V : G_V \rightarrow \prod_{\lambda \in \Lambda} GL_{1, A_\lambda}$, where the product is taken over $V$, and each $A_\lambda$ is an Azumaya algebra over $V$ obtained from $\text{End}(V_\lambda)$ via the 1-cocycle $\theta_\lambda = (\bar{\rho}_\lambda)_*(\xi_V) \in Z^1(V, PGL_{V_\lambda})$. Set $H = \prod_{\lambda \in \Lambda} GL_{1, A_\lambda}$. One has

$$\text{Hom}_V(G_V, H) = \text{Hom}_U(G, R_{V/U}(H)),$$

where $R_{V/U}$ is the Weil restriction functor. Thus $\rho_V$ determines an $U$-morphism $\rho_U : G \hookrightarrow R_{V/U}(H)$.

Well-known properties of the functor $R_{V/U}$ imply that $\rho_U$ is a $U$-group scheme morphism. Note that, since $\rho$ is a closed embedding, $\rho_U$ is a closed embedding as well (by étale descent).

Recall that we are given with a principal $G$-bundle $E \in H^1(P^1, G)$. The map

$$(\rho_U)_* : H^1(P^1, G) \rightarrow H^1(P^1, R_{V/U}(H))$$

produces a principal $R_{V/U}(H)$-bundle $(\rho_U)_*(E)$ over $P^1$.

By the Shapiro–Faddeev Lemma (for example, \cite{SGA3} Exp. XXIV Prop. 8.4), there is an isomorphism of pointed sets

$$H^1(P^1, R_{V/U}(H)) \cong H^1(P^1 \times_U V, H).$$

Let $F$ be a principal $H$-bundle over $P^1 \times_U V$ corresponding to the bundle $(\rho_U)_*(E)$ via the latter isomorphism. Recall that $H = \prod_{\lambda \in \Lambda} GL_{1, A_\lambda}$. Consider the compositions of the closed embeddings

$$j : H \hookrightarrow \prod_{\lambda \in \Lambda} GL_{A_\lambda} \hookrightarrow GL_{\oplus A_\lambda},$$

where $GL_{A_\lambda}$ (resp. $GL_{\oplus A_\lambda}$) is the $V$-group scheme of automorphisms of $A_\lambda$ (resp. $\oplus A_\lambda$) regarded as an $O_V$-module.

Recall that the original $G$-bundle $E$ is trivial on $P^1 = P^1 \times_U x \subset P^1$. It follows that the $R_{V/U}(H)$-bundle $(\rho_U)_*(E)$ is trivial on $P^1 \times_U x \subset P^1$. Hence the $H$-bundle $F$ is trivial on
$\mathbb{P}^1 \times_U \pi^{-1}(x) \subseteq \mathbb{P}^1 \times_U V$. Hence the $GL_{\oplus A_\lambda}$-bundle $j_*(F)$ is trivial on $\mathbb{P}^1 \times_U \pi^{-1}(x)$. The $V$-group scheme $GL_{\oplus A_\lambda}$ is just the ordinary general linear group, since $V$ is semi-local. Thus $j_*(F)$ corresponds to a vector bundle $M$ over $\mathbb{P}^1 \times_U V$. Moreover, this vector bundle is trivial on $\mathbb{P}^1 \times_U \pi^{-1}(x)$. By Horrocks’ theorem the bundle $M$ is trivial on $\mathbb{P}^1 \times_U V$. Thus $j_*(F)$ is a trivial $GL_{\oplus A_\lambda}$-bundle.

Now, applying Lemma 4 to the embedding $j : H \hookrightarrow GL_{\oplus A_\lambda}$, we see that $F = pr^*(F_0)$ for some $F_0 \in H^1(V, H)$. Since $H = \prod_{\lambda \in \Lambda} GL_{1, A_\lambda}$ and $V$ is semi-local, the bundle $F_0$ is trivial by Hilbert 90 for Azumaya algebras. Whence $F$ is trivial as well. The isomorphism (16) shows that $(\rho_U)_*(E)$ is trivial too. Now, applying Lemma 4 to $W = U$, $j = \rho_U : G \hookrightarrow R_{V/U}(H)$ and $F = E$, we see that there exists a principal $G$-bundle $E_0$ over $U$ such that $pr^*_U(E_0) \cong E$. The theorem is proved.

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