The low volatility fluctuations regime of the exponential Ornstein - Uhlenbeck model

G Bormetti\textsuperscript{1,2}, V Cazzola\textsuperscript{1,3,2}, D Delpini\textsuperscript{3,2}, G Montagna\textsuperscript{3,2,1} and O Nicrosini\textsuperscript{2,1}

\textsuperscript{1}Istituto Universitario di Studi Superiori, Centro Studi Rischio e Sicurezza, Viale Lungo Ticino Sforza, 56 27100 Pavia, Italy
\textsuperscript{2}INFN, Sezione di Pavia, Via A.Bassi, 6 27100 Pavia, Italy
\textsuperscript{3}Università degli Studi di Pavia, Dipartimento di Fisica Nucleare e Teorica, Via A.Bassi, 6 27100 Pavia, Italy

E-mail: giacomo.bormetti@pv.infn.it; valentina.cazzola@pv.infn.it; danilo.delpini@pv.infn.it; guido.montagna@pv.infn.it; oreste.nicrosini@pv.infn.it

Abstract. In this work the analytical characterization of the probability density of financial returns in the exponential Ornstein-Uhlenbeck model is addressed. Since prices are driven by a Geometric Brownian motion, whose diffusion coefficient is expressed through an exponential function of the mean-reverting process \( Y \), an exact expression for the density is hard to be drawn. However, we derive the closed-form formula for the characteristic function and for the cumulants of the approximating linear model under the low volatility fluctuations regime. Theoretical results are confirmed numerically by use of Monte Carlo simulations. The effectiveness of the analytical predictions is tested on the German DAX30 Index, finding a good agreement between the empirical data and the theoretical description.

1. Introduction

Since the pioneering work of Bachelier \cite{1} diffusion processes have been natural candidates for the modeling of the stochastic evolution of financial quantities. The assumption of a Geometric Brownian motion as the dynamics driving prices fluctuations led to the famous Black, Scholes and Merton theory of rational option pricing. In their framework the volatility is assumed to be constant, however it has been demonstrated empirically that it fluctuates along time and its stochastic behavior is responsible for the excess of kurtosis observed in the empirical returns distributions. In financial literature the idea of modeling the volatility as a random variable as been widely exploited \cite{2–7}. The research in this field has also been strongly enhanced by the works of Carr and Madan \cite{7}, Lewis \cite{8} and Lipton \cite{9}, through the introduction of generalized Fourier transform. Indeed, they demonstrated how to compute efficiently the price of derivatives contracts once the characteristic function (CF) associated with the process is known analytically.

The econophysicists’ community has analyzed and developed independently models to capture the stochastic nature of volatility, with particular emphasis on the comparison of the models predictions with real market data \cite{10–18}. The present work follows the guidelines of \cite{11} and \cite{16}, the former focusing on the Heston’s model while the latter on the Scott’s model \cite{2}, also known as the exponential Ornstein-Uhlenbeck (ExpOU) model. Both these articles present the analytical characterizations of several stylized facts and in particular the tail behavior of
the probability density function (PDF) of financial returns. Here we derive the closed-form formula for the CF of the linear model to which the ExpOU reduces when the level of volatility fluctuations is low. We check the goodness of the results by fast Fourier transforming the CF and comparing the numerical PDF with the Monte Carlo (MC) simulation of the stochastic models. The effectiveness in financial applications is assessed through the analysis of the German DAX30 Index.

2. The Exponential Ornstein-Uhlenbeck Model

The process we are investigating is a generalization of the stochastic volatility model originally introduced by Scott [2]. The major difference with Scott’s model is the existence of the parameter $\rho$ taking values in $[-1, 1]$. It represents the correlation between the two sources of noise in the diffusive processes and its value is crucial to obtain a non trivial behavior for the skewness of the returns PDF. The following coupled SDEs completely define the model

\begin{align}
\frac{dS(t)}{S(t_0)} &= \mu S(t) dt + m e^{Y(t)} S(t) dW_1(t) , \\
S(t_0) &= S_0 , \\
\frac{dY(t)}{Y(t_0)} &= \alpha (\gamma - Y(t)) dt + k \rho dW_1(t) + k \sqrt{1 - \rho^2} dW_2(t) , \\
Y(t_0) &= Y_0 ,
\end{align}

where $dW_1$ and $dW_2$ are standard Brownian increments, while $S_0$, $\mu$, $m$, $Y_0$, $\alpha$, $\gamma$, $k$ and $\rho$ are constant parameters to be calibrated on real data. If we define $\sigma(t) = m e^{Y(t)}$, in equation (1) we recognize the same dynamics of a Geometric Brownian motion, with a constant drift coefficient $\mu$, while the volatility $\sigma(t)$ characterizes the amplitude of the fluctuations of $S$. By tuning the value of $k \geq 0$ we can strongly modify the behavior of $Y$. The case $k = 0$ switches off the stochastic nature of $Y$ and it evolves deterministically towards its stationary value $\gamma$, with a characteristic time $1/\alpha$ ($\alpha > 0$). If $k$ is strictly greater than zero, it is known that $Y$ follows a Gaussian process, whose mean and variance read

\begin{align}
\mathbb{E}[Y] &= (Y_0 - \gamma)e^{-\alpha(t-t_0)} + \gamma \int_{t_0}^{t+\infty} e^{-\alpha(t-t')} \beta \, dW_2(t) , \\
\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 &= \frac{k^2}{2\alpha} \left[ e^{-2\alpha(t-t_0)} + 1 \right] \int_{t_0}^{t+\infty} \beta \, dW_2(t) ,
\end{align}

where $\mathbb{E}[\cdot]$ denotes expectation with respect to the probability density of the $Y$ process. By means of Itô Lemma applied to the centered logarithmic return $X(t) = \ln S(t) - \ln S_0 - \mu(t-t_0)$, equation (1) reduces to

\begin{align}
\frac{dX(t)}{X(t_0)} &= \frac{1}{2} m^2 e^{2Y(t)} dt + me^{Y(t)} dW_1(t) ,
\end{align}

with boundary condition $X(t_0) = 0$.

We are interested in the characterization of $p_X(X|X_0, Y_0)$, the returns transition probability. The returns PDF $p_X(X)$ is readily obtained taking the expectation over $X_0$ and $Y_0$. In the following section we will consider the limit linear dynamics to which the ExpOU model reduces for small values of $\beta$ and in the final section we will provide empirical evidences supporting the effectiveness of our approach, see also [19] for more details.

3. The Linear Model

Since $\beta$ governs the level of the fluctuations of $Y$ around its stationary value, when $\beta \ll 1$, it is possible to approximate linearly the exponential form of $\sigma(t)$; we will show how to exactly solve the associated Fokker-Planck equation. We limit our investigation to $0 < \beta \leq 0.1$, even
though higher values can be explored and the effectiveness can be tested numerically via MC simulation.

When the $Y$ process thermalizes, the random variable fluctuates around $\gamma$ and we expand the exponential around this value. By defining $\tilde{m} = me^{\gamma}$ and introducing the random variable $Z = Y - \gamma + 1$, equation (3) and equation (2) can be rewritten as

$$dX = -\frac{\tilde{m}^2}{2}(2Z - 1)dt + \tilde{m}ZdW_1,$$

$$X(t_0) = 0;$$

$$dZ = \alpha(1 - Z)dt + k\rho dW_1(t) + k\sqrt{1 - \rho^2}dW_2(t),$$

$$Z(t_0) = Y_0 - \gamma + 1.$$

For the above model there is a non zero probability for the diffusion coefficient $\tilde{m}Z$ to become negative, which can be computed and for $t - t_0 \gg 1/\alpha$ reduces to $\text{Erfc}\left(\frac{1 + \gamma}{\sqrt{\alpha}}\right)/2$, where $\text{Erfc}$ is the complementary error function. Practitioners usually require to keep stochastic volatility positive, however there are no financial or mathematical problems preventing it to become negative, for a discussion on this point see [20].

The Fokker-Planck backward equation satisfied by $p_X(X|X_0, Z_0)$, shortly $p_X$, is readily written as

$$\frac{\partial}{\partial t_0}p_X = \frac{\tilde{m}^2}{2}(2z_0 - 1)\frac{\partial}{\partial x_0}p_X - \alpha(1 - z_0)\frac{\partial}{\partial z_0}p_X,$$

$$-\frac{\tilde{m}^2}{2}\frac{\partial^2}{\partial x_0^2}p_X - \rho k\tilde{m}z_0\frac{\partial^2}{\partial x_0\partial z_0}p_X - \frac{k^2}{2}\frac{\partial^2}{\partial z_0^2}p_X. \tag{4}$$

Following Heston [5], we notice that, if we assume as a final time condition the expression $e^{i\omega}$, equation (4) is precisely the partial differential equation governing the evolution of the CF $f(\omega; X_0, Z_0)$. We try a solution of the form

$$f(\omega; X_0, Z_0) = e^{A(t - t_0, \omega) + B(t - t_0, \omega)Z_0 + C(t - t_0, \omega)Z_0^2 + i\omega X_0}. \tag{5}$$

We substitute it in equation (4) and we obtain

$$\dot{A} + BZ_0 + \dot{C}Z_0^2 = \frac{\tilde{m}^2}{2}(2Z_0 - 1)i\omega + \frac{\tilde{m}^2}{2}Z_0^2\omega^2 - \alpha(1 - Z_0)(B + 2CZ_0)$$

$$-\frac{k^2}{2}[(B + 2CZ_0)^2 + 2C] - \rho k\tilde{m}Z_0i\omega(B + 2CZ_0),$$

where the dot stays for a derivative w.r.t. $t_0$. If we collect the coefficients of the different powers of $Z_0$, we have the ODEs satisfied by $C$, $B$ and $A$

$$\dot{C} = \frac{\tilde{m}^2}{2}\omega^2 + 2\alpha C - 2k^2C^2 - \rho k\tilde{m}\omega 2C, \tag{6}$$

$$\dot{B} = \tilde{m}^2i\omega - 2\alpha C + \alpha B - 2k^2BC - \rho k\tilde{m}\omega B, \tag{7}$$

$$\dot{A} = -\frac{\tilde{m}^2}{2}i\omega - \alpha B - \frac{k^2}{2}(B^2 + 2C), \tag{8}$$

with final time conditions $C(0, \omega) = 0$, $B(0, \omega) = 0$, and $A(0, \omega) = 0$. Equation (6) is a Riccati type ODE and once it has been solved, we insert the solution in equations (7) and (8) and we integrate them out in the usual way. The explicit expression for $C$, $B$ and $A$ are quite involved. To improve the readability we introduce the auxiliary variables $b = 2\alpha(1 - i\rho\Omega)$,
\[d = \sqrt{4\alpha^2\Omega^2 + b^2}, \quad g = (b - d)/(b + d), \quad h = i\alpha\bar{\omega}/k, \quad \text{and} \quad n = \alpha(b - d)/(2k^2), \quad \text{where} \quad \Omega = k\bar{\omega}/\alpha.\]

Now the three functions read

\[
C(t - t_0, \omega) = \frac{b - d}{4k^2} \left( 1 - \frac{1 - e^{-d(t-t_0)}}{1 - ge^{-d(t-t_0)}} \right),
\]

\[
B(t - t_0, \omega) = 2 \frac{e^{-\frac{1}{2}d(t-t_0)}((g + 1)h - 2n) + n + e^{-d(t-t_0)}(n - gh) - h}{d(1 - ge^{-d(t-t_0)})},
\]

and

\[
A(t - t_0, \omega) = \left[ \frac{h}{2} + 2\alpha \frac{n - h}{d} + 2k^2 \left( \frac{n - h}{d} \right)^2 + \frac{b - d}{4} \right] (t - t_0),
\]

\[
- \frac{1}{2} \left[ \ln \left( 1 - ge^{-d(t-t_0)} \right) - \ln(1 - g) \right]
\]

\[- 2k^2 \left( \frac{e^{-d(t-t_0)} - 1}{(1 - g)(1 - ge^{-d(t-t_0)})} \right) \left\{ \frac{1}{d^3} \left[ \frac{\alpha}{2k^2} (b + d) - h \right]^2 + \frac{1}{d^3} \left( (g + 1)h - 2n \right)^2 + 2(n - gh)(n - h) + \frac{g(n - h)^2}{d^3} \right\},
\]

\[- 4k^2 \left( \frac{g + 1}{} \right) \left( \frac{\alpha}{k^2} b - 2h \right) \left( 1 + ge^{-\frac{d}{2}(t-t_0)} \right) \left( e^{-\frac{d}{2}(t-t_0)} - 1 \right).
\]

Similarly to the Heston case, non trivial problems emerge due to the multi-valued nature of complex square roots and logarithms. Due to branching effects, the CF can become discontinuous and to check the smoothness of \( f \) we follow the same arguments discussed in [21–23]. The complete characterization of \( f \) allows us to derive the analytical expression of the cumulants. Here we report the first four cumulants, under the assumptions \( X_0 = 0 \) and \( Z_0 = 1 \) (see [24] for more details about the role played by \( k_1 - k_4 \) when calibrating the model).

\[
k_1 = - \frac{m^2}{2\alpha} \tau,
\]

\[
k_2 = - \frac{m^2}{4\alpha} \left\{ \frac{k^2 m^2}{\alpha^2} (e^{-2\tau} - 4e^{-\tau} - 2\tau + 3) - \frac{k^2}{\alpha} (e^{-2\tau} + 2\tau - 1) - 4\tau \right\}
\]

\[- 2\rho \frac{km^3}{\alpha^2} (e^{-\tau} + \tau - 1),
\]

\[
k_3 = 3 \frac{k^2 m^3}{\alpha^3} (e^{-2\tau} - 4e^{-\tau} - 2\tau + 3)
\]

\[+ \frac{3}{2} \frac{km^3}{\alpha^2} \rho \left\{ \frac{k^2 m^2}{\alpha^2} [-e^{-2\tau}(3 + 2\tau) + 4e^{-\tau}(3 + \tau) + 4\tau - 9] + \frac{k^2}{\alpha} [e^{-2\tau}(1 + \tau) + \tau - 1] \right\}
\]

\[+ 4 [e^{-\tau} + \tau - 1] \right\} - 6 \frac{k^2 m^4}{\alpha^3} \rho^2 [e^{-\tau}(2 + \tau) + \tau - 2],
\]

\[
k_4 = 3 \frac{k^2 m^4}{\alpha^3} \left\{ \frac{1}{2} \frac{k^2 m^2}{\alpha^2} [-e^{-4\tau} + 4e^{-3\tau} - 4e^{-2\tau}(3 + \tau) + 4e^{-\tau}(7 + 2\tau) + 8\tau - 19] \right.
\]

\[+ \frac{k^2}{8\alpha} [e^{-4\tau} + 4e^{-2\tau}(1 + 2\tau) + 4\tau - 5] - 2(e^{-2\tau} - 4e^{-\tau} - 2\tau + 3) \right\}
\]
\[-6 \frac{k^3 m^5}{\alpha^4} \rho [e^{-3\tau} - 2e^{-2\tau}(5 + 2\tau) + e^{-\tau}(35 + 10\tau) + 12\tau - 26] + 3 \frac{k^2 m^4}{\alpha^3} \rho \left\{ 4 \frac{k^2 m^2}{\alpha^2} [-e^{-2\tau}(3 + 3\tau + \tau^2) + e^{-\tau}(12 + 6\tau + \tau^2) + 3\tau - 9] + \frac{k^2}{\alpha} [e^{-2\tau}(3 + 4\tau + 2\tau^2) + 2\tau - 3] - 2 [e^{-2\tau} - 4e^{-\tau}(3 + \tau) - 6\tau + 11] \right\} - 12 \frac{k^3 m^5}{\alpha^4} \rho^3 [e^{-\tau}(6 + 4\tau + \tau^2) + 2\tau - 6], \]

with \( \tau = \alpha(t-t_0) \). The first cumulant does depend only on the parameter \( m \) and this turns to be extremely useful when estimating it. If we introduce the skewness \( \zeta = k_3/k_2^{3/2} \) and the kurtosis \( \kappa = k_4/k_2^2 \), we can study their behavior both in the short and large time limit. Indeed, when \( \alpha(t-t_0) \ll 1 \) we can Taylor expand their expressions finding

\[
\zeta \sim 3k\rho \sqrt{t-t_0}, \quad \text{and} \quad \kappa \sim 4k^2(1+2\rho^2)(t-t_0).
\]

For \( \alpha(t-t_0) \gg 1 \), we have

\[
\zeta \sim -6k^2 \frac{\left[ \frac{1}{\alpha} - k\rho \left( \frac{1}{4} + \frac{m^2}{\alpha} + \frac{\rho}{k} \right) + m\rho^2 \right]}{\left[ \left( \frac{km}{\alpha} \right)^2 + \frac{k^2}{\alpha} - 1 - 2\rho \frac{km}{\alpha} \right]^{3/2}} \frac{1}{\sqrt{t-t_0}},
\]

\[
\kappa \sim 3 \frac{k^2}{\alpha^2} \left\{ 4 \frac{(km)}{\alpha^2} + \frac{k^2}{\alpha} + 4 - 24 \frac{km}{\alpha} \rho + 2\rho^2 \left( 6 \frac{(km)}{\alpha^2} + \frac{k^2}{\alpha} + 6 \right) - 8 \frac{km}{\alpha} \rho^3 \right\} \frac{1}{\left[ \left( \frac{km}{\alpha} \right)^2 + \frac{k^2}{\alpha} - 1 - 2\rho \frac{km}{\alpha} \right]} \frac{1}{t-t_0}.
\]

The peculiar time scaling of the previous expressions predicts no excess of skewness and kurtosis for long horizons. It would be interesting to explore the tail properties of the \( p_X \) density to verify whether the long time behavior of this model is consistent with a Gaussian dynamics.

4. Numerical Results

The Fourier anti-transform of \( f \) cannot be computed analytically, but exploiting the symmetries of the CF we can evaluate it efficiently with a Fast Fourier Transform algorithm. All the results presented in the figures have been obtained with \( 2^{22} \) points on the fixed interval of integration \([0,10^3]\). To simulate the dynamics of the ExpOU and linear models, we set \( \Delta t = 10^{-4} \) and the number of MC paths equal to \( 5 \cdot 10^6 \). In figure 1 we report the returns histograms and the \( p_X \) curve given by the numerical Fourier anti-transform; \( t-t_0 = 1 \), \( m = 0.1 \), \( \alpha = 10 \), \( \gamma = 0 \) and \( Y_0 = 0 \). Each group of three curves corresponds to \( \beta \) values 0.5%, 1%, 2%, 5% and 10%, from bottom to top respectively. Curves has been shifted for the sake of readability. The top frame has been obtained with a positive \( \rho \) value (0.5), which is evident from the rightward asymmetry of the curves, while, in the bottom panel, curves have large leftward asymmetry corresponding to \( \rho = -0.9 \). The agreement between theory prediction and MC simulation is very good for \( \beta = 0.5\% \), 1\% and 2\%, in both cases of \( \rho \) values. The returns PDFs for the exponential model are well reproduced by the ones for the linear model and, as a consequence, by the semi-analytical approximation \( p_X \). The agreement worsen when \( \beta \) increases to 5\% and 10\%, as it can be seen by observing the fatter tail in each panel.

In figure 2 the scaling with time of mean, variance, skewness and kurtosis is shown for \( \beta = 0.5\% \) and 5\%, \( \rho = -0.9 \). Then in table 1 we detail the numerical values of normalized cumulants for \( \beta = 2\% \) and \( t-t_0 = 0.01, 0.1, 0.2, 0.5, 1 \) (MC for the ExpOU and analytical for the linear model). Since the linear limit is an approximating dynamics of the ExpOU model, we
do not expect necessarily a statistical compatibility between the MC estimates and the analytical values. However, we are able to quantify the relative differences among cumulants values and we can evaluate the degree of accuracy of the approximation. For example, for $\beta = 2\%$ the relative disagreement between $k_{2}^{\text{Exp}}$ and $k_{2}^{\text{Lin}}$ increases with time from 0.1% to 2%. As expected, this effect is enhanced by higher values of $\beta$.

5. Real data analysis
Analytical results of section 3 are tested on the German DAX30 time series, consisting of 12173 daily close prices, from 4th January 1960 until 30th June 2008. Time is measured on a yearly base, for DAX30 we have a 48.5 years time window. The model described by equations (1) and (2) depends on 8 parameters. However, $S_{0}$ corresponds to the asset spot price so the true free parameters are $\mu$, $m$, $Y_{0}$, $\alpha$, $\gamma$, $\beta$ and $\rho$. In order to estimate their values we adopt the following strategy:

$\mu$: from the discretized version of equation (1)

$$\frac{\Delta S_{i}}{S_{i}} = \mu \Delta t + m \sqrt{\Delta t} e^{Y_{i} \epsilon_{i}},$$

where $\Delta S_{i} = S_{i+1} - S_{i}$ and $\epsilon_{i} \sim \mathcal{N}(0, 1)$, we can conclude that the expectation $\langle \Delta S_{i}/S_{i} \rangle/\Delta t$ over the real data sample provides an estimate of $\mu$; for DAX30 $\Delta t = 3.98 \times 10^{-3}$.
Figure 2. From top left clockwise: scaling with time of mean, variance, kurtosis and skewness.

Table 1. Scaling with time of normalized cumulants (between parenthesis the error on the last significant digit, 95% confidence level).

| $t - t_0$ (yr) | $\beta = 2\%$ |
|----------------|----------------|
|                | 0.01 | 0.1 | 0.2 | 0.5 | 1 |
| $k_1^{\text{Exp}} (10^{-4})$ | -0.50(8) | -5.0(2) | -10.1(4) | -25.8(6) | -51.8(9) |
| $k_1^{\text{Lin}} (10^{-4})$ | -0.50 | -5.0 | -10.0 | -25.0 | -49.9 |
| $k_2^{\text{Exp}} (10^{-4})$ | 1.004(1) | 10.28(1) | 20.77(2) | 52.35(7) | 104.8(1) |
| $k_2^{\text{Lin}} (10^{-4})$ | 1.003 | 10.16 | 20.43 | 51.36 | 102.9 |
| $\kappa^{\text{Exp}}$ | -0.151(4) | -0.402(4) | -0.443(4) | -0.393(4) | -0.311(4) |
| $\kappa^{\text{Lin}}$ | -0.165 | -0.393 | -0.427 | -0.379 | -0.301 |
| $\gamma^{\text{Exp}}$ | 0.04(2) | 0.30(2) | 0.36(2) | 0.28(2) | 0.17(2) |
| $\gamma^{\text{Lin}}$ | 0.04 | 0.22 | 0.26 | 0.21 | 0.13 |

$\gamma, y_0$: Remembering the definition $\sigma(t) = me^{Y(t)}$ and under the assumption that the volatility process has already reached the stationary state for the observed time series, it is readily proved that $E[\sigma(t)^n] = (me^\gamma)^n e^{\frac{n^2}{2}\beta}$. Since all the moments do not depend on $Y_0$ and $e^\gamma$ is always coupled with $m$, we set $Y_0$ equal to zero and estimate $\bar{m}$.

$m, \beta$: These two parameters completely specify the stationary log-normal distribution of the stochastic variable $\sigma$. To extract the distribution of the hidden variable $\sigma$ from the series of financial returns, we implement the methodology described in [25].
Table 2. Estimated values for the model parameters for DAX30 Index.

| Parameter | Value       |
|-----------|-------------|
| $\mu$ (yr$^{-1}$) | 7.39 × 10$^{-2}$ |
| $y_0$ | 0 |
| $\gamma$ | 14.52 × 10$^{-2}$ |
| $\bar{m}$ (yr$^{-2}$) | 11.16 × 10$^{-2}$ |
| $\beta$ | 30.76 |
| $\alpha$ (yr$^{-1}$) | -0.54 |
| $\rho$ | -0.52 |

$\alpha, \rho$: Finally, to estimate $\alpha$ and $\rho$, we search for values able to reproduce the empirical scaling with time of real data skewness and kurtosis. We consider time horizons from one day to one hundred days and normalized cumulants are evaluated with standard estimators. The empirical skewness, $\varsigma_{Ph}$, and kurtosis, $\kappa_{Ph}$, and corresponding errors, $\epsilon_{\varsigma_{Ph}}$ and $\epsilon_{\kappa_{Ph}}$, are obtained from historical returns, while $\varsigma_{MC}$, $\kappa_{MC}$ and associated errors, $\epsilon_{\varsigma_{MC}}$ and $\epsilon_{\kappa_{MC}}$, are drawn by generating 10000 MC paths. The optimal $\alpha$ and $\rho$ are given by those values minimizing the sum of the normalized squared differences

$$
(\alpha^*, \rho^*) = \arg\min_{\alpha > 0, \rho \in (-1, 1)} \sum_{i=1}^{100} \left[ \frac{(\varsigma_{i,Ph} - \varsigma_{i,MC})^2}{\epsilon_{\varsigma_{i,Ph}}^2 + \epsilon_{\varsigma_{i,MC}}^2} + \frac{(\kappa_{i,Ph} - \kappa_{i,MC})^2}{\epsilon_{\kappa_{i,Ph}}^2 + \epsilon_{\kappa_{i,MC}}^2} \right].
$$

The volatility density $P(\sigma)$ is plotted in figure 3. Tails suffer low statistics effects, but in the central region they are well fitted by a log-normal distribution

$$
p(\sigma) = \frac{1}{\sqrt{2\pi s} \sigma} \exp \left[ -\frac{1}{2} \left( \frac{\log \sigma - \log \sigma_0}{s} \right)^2 \right].
$$

The fit is performed in the range $0.0004 \leq \sigma \leq 0.015$ and gives $\log \sigma_0 = -4.492 \pm 0.001$ and $s = 0.334 \pm 0.001$. Table 2 details the corresponding values for $\bar{m} = \sigma_0/\sqrt{\Delta t}$ and $\beta = s^2$. Finally, we present the probability densities for DAX30, see figure 4, computed according to the models we have discussed in this work. The first row is obtained by setting the time horizon equal to 25 trading days, while the second one corresponds to the 45 trading days horizon. The densities are presented both in log-linear and linear scales, in order to allow the reader to appreciate the behavior on the tails and in the central region, respectively. The boxed-lines represent the empirical histograms, while black points correspond to the histograms from the MC simulation of the ExpOU model. Moreover, we report the Fourier transform of the CF for the linear model (solid black line) and the Normal Maximum Likelihood fit (solid red line). The Normal approximation is scarcely representative of the true empirical density, while the exponential model captures in a quite effective way the leftward asymmetry, the fatter tails and the narrower central region. Figure 4 clearly shows the good performance of the linear model for the DAX30 Index.
Figure 4. DAX30 index returns distributions for two different time horizons, 25 trading days (first row) and 45 trading days (second row).

6. Conclusions

The ExpOU dynamics has been widely studied in the econophysics literature from the point of view of multi-time scale properties, leverage effect and stationary volatility distribution [11–13, 16, 17, 26]. However, since our interest is driven by the possibility of a financial application in the context of option pricing and risk management we focused on the analytical characterization of the CF. Several attempts to obtain an approximated closed-form expression for the PDF can be found in [16, 19, 27]. In this work we explored the possibility to approximate linearly the exponential form of the volatility under a low fluctuations regime of the secondary process $Y$. Following Heston [5], we solved exactly the Fokker-Planck backward equation associated with the linear model and in Section 3 we reported the full expression of the CF. The numerical PDF obtained by anti-transforming our analytical CF was checked against the MC simulation of the linear process, finding a perfect agreement. The analytical predictions were tested over a data set of market returns. We detailed the performances for the German DAX30 Index, confirming the capability of the linear model to capture the statistical properties of the returns distribution for moderate values of $\beta$ and over appropriate time horizons. The results discussed in this paper have been exploited successfully for the characterization of the ExpOU model and its linear limit in a risk-neutral framework, see [24].

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