Permutations from an arithmetic setting

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Abstract
Let $m, n$ be positive integers such that $\gcd(n, m^2) = m > 1$. In this paper, we introduce a special class of piecewise-affine permutations of the finite set $[1, n] := \{1, \ldots, n\}$ with the property that the reduction $\pmod m$ of $m$ consecutive elements in any of its cycles is, up to a cyclic shift, a fixed permutation of $[1, m]$. Our main result provides the cycle decomposition of such permutations. We further show that such permutations give rise to permutations of finite fields $\mathbb{F}_q$ of order $q$, where $q$ is a prime power. In particular, we explicitly obtain new classes of permutation polynomials of finite fields whose cycle decomposition is explicitly given.

Keywords: permutations; cycle decomposition; $m$-th residues; finite fields

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1. Introduction

Let $m, n$ be positive integers such that $\gcd(n, m^2) = m > 1$. For an integer $k > 1$, set $[1, k] = \{1, 2, \ldots, k\}$. In this paper, we introduce a special class of piecewise-affine permutations of the set $[1, n]$. This class of permutations has the property that the reduction modulo $m$ of $m$ consecutive elements in any of its cycles is, up to a cyclic shift, a fixed permutation of $[1, m]$. In particular, it implies that every cycle of this kind of permutation has length divisible by $m$. One of our main results, Theorem 2.13, provides the explicit cycle decomposition of such permutations.

For integers $a$ and $k > 1$, let $\Psi_k(a)$ denote the unique positive integer $i$ in the set $[1, k]$ such that $a \equiv i \pmod k$. We apply the case that our piecewise-affine permutation $\pi : [1, n] \to [1, n]$ is defined by two affine rules
\[
\pi(x) = \begin{cases} 
\Psi_n(ax + b) & \text{if } x \equiv 0 \pmod m \\
\Psi_n(ax + b) & \text{if } x \not\equiv 0 \pmod m.
\end{cases}
\] (1)
The latter is mainly motivated by the further application of our piecewise-affine permutations in the construction of permutation polynomials over finite fields. Namely, let $q$ be a prime power and $\mathbb{F}_q$ be a finite field with $q$ elements. A polynomial $f \in \mathbb{F}_q[x]$ is called a permutation polynomial if the evaluation map $c \mapsto f(c)$ is a permutation of $\mathbb{F}_q$. It is well known that $\mathbb{F}_q = \mathbb{F}_q^* \cup \{0\}$, where $\mathbb{F}_q^*$ is a multiplicative cyclic group of order $q - 1$. We fix $\theta_q \in \mathbb{F}_q$ a generator of $\mathbb{F}_q^*$.

It turns out that if $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ is the function given by

$$f(0) = 0 \quad \text{and} \quad f(\theta_q^i) = \theta_q^{\pi(i)} \quad \text{for all } 1 \leq i \leq q - 1,$$

where $\pi$ is a piecewise-affine permutation of $[1, q - 1]$ defined by two rules, then it is easy to exhibit the polynomial representation of the permutation $f$ as well as its cycle decomposition. It turns out that permutations like the previous one are piecewise defined as monomials, according to cyclotomic cosets. This kind of permutation was previously explored in full generality by Wang [12]. However, there is no study on their cycle decomposition. It is worth mentioning that, for only few families of permutation polynomials, we know the cycle decomposition without needing to describe the whole permutation; for instance, we have monomials [1], Mobius maps [5], Dickson polynomials [9] and certain linearized polynomials [10].

The idea of bringing piecewise permutations to obtain permutation polynomials was earlier used by Fernando & Hou [6] and by Cao, Hu & Zha [4], who obtained families of permutation polynomials via certain powers of linearized polynomials and using a matrix approach, respectively. Some other algebraic-combinatorial methods to produce large classes of permutation polynomials include linear translators [8] and, most notably, the AGW criterion [2]. See [11, §8] for more details on permutation polynomials over finite fields and [7] for a survey on recent advances.

The structure of the paper is given as follows. In Section 2 we introduce our class of piecewise-affine permutations of $[1, n]$ and present some of its basic properties. In particular, we obtain an explicit description on the cycle decomposition of such permutations. In Section 3 we use the permutations defined by two rules like in Eq. (1) in the construction of permutation polynomials over finite fields and discuss further properties of these polynomials.

2. On piecewise-affine permutations of the set $[1, n]$

We start fixing some notation. The letters $n, m$ always denote positive integers such that $\gcd(n, m^2) = m > 1$. In general, $\vec{a}$ denotes an $m$-tuple of integers in a fixed range (usually $[1, m]$ or $[1, n]$). Also, $a_i$ denotes the $i$-th coordinate of $\vec{a}$. In addition, for a positive integer $N$, let $\text{rad}(N)$ be denote the product of the distinct prime divisors of $N$.

Definition 2.1. Let $\mathcal{C}(m)$ be denote the subset of $[1, m]^m$ of the vectors $\vec{c}$ whose entries comprise a permutation of the set $[1, m]$. 

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Definition 2.2. An \((n,m)\)-piecewise affine permutation (or \((n,m)\)-p.a.p) is a permutation \(\pi\) of the set \([1, n]\) such that there exist \(\vec{a}, \vec{b} \in [1, n]^m\) and \(\vec{c} \in \mathcal{C}(m)\) with the property that
\[
\pi(x) = \Psi_n(a_i x + b_i) \quad \text{and} \quad \Psi_m(\pi(x)) = c_{i+1},
\]
for any \(x \in [1, n]\) with \(\Psi_m(x) = c_i\), where the indexes are taken modulo \(m\). In this case, we say that the triple \((\vec{a}, \vec{b}, \vec{c})\) is \((n,m)\)-admissible and \(\pi\) is the \((n,m)\)-p.a.p with parameters \((\vec{a}, \vec{b}, \vec{c})\).

Example 2.3. Let \(n = 12, m = 3\) and let \(\pi\) be the \((12,3)\)-p.a.p with parameters \((\vec{a}, \vec{b}, \vec{c})\), where \(\vec{a} = (1, 3, 5), \vec{b} = (4, 6, 1)\) and \(\vec{c} = (1, 2, 3)\). In other words, for each \(x \in [1, 12]\),
\[
\pi(x) = \begin{cases} 
\Psi_{12}(x + 4) & \text{if } x \equiv 1 \pmod{3}, \\
\Psi_{12}(3x + 6) & \text{if } x \equiv 2 \pmod{3}, \\
\Psi_{12}(5x + 1) & \text{if } x \equiv 0 \pmod{3}.
\end{cases}
\]
The cycle decomposition of \(\pi\) is given as follows:
\[(1\ 5\ 9\ 10\ 2\ 12)\quad (3\ 4\ 8\ 6\ 7\ 11).\]

In the following theorem, we characterize the \((n,m)\)-admissible triples.

Theorem 2.4. Fix \(\vec{a} \in [1, n]^m\) and \(\vec{c} \in \mathcal{C}(m)\). Then there exists an element \(\vec{b} \in [1, n]^m\) such that the triple \((\vec{a}, \vec{b}, \vec{c})\) is \((n,m)\)-admissible if and only if the entries of \(\vec{a}\) are relatively prime with \(n/m\). In this case, the number of distinct such \(\vec{b}\)'s equals \((\frac{n}{m})^m\).

Proof. We observe that \(\pi\) is an \((n,m)\)-p.a.p with parameters \((\vec{a}, \vec{b}, \vec{c})\) if and only if \(\pi\) is of the form given by Eq. (2) and \(\pi\) is one to one. Suppose that \(x, y \in [1, n]\) are such that \(\pi(x) = \pi(y)\). Then \(\pi(x) \equiv \pi(y) \equiv c_{i+1} \pmod{m}\) for some \(i \in [1, m]\) and, by definition, \(x \equiv y \equiv c_i \pmod{m}\), where \(i\) is taken modulo \(m\). Therefore, \(\pi(x) = \Psi_n(a_i x + b_i) = \Psi_n(a_i y + b_i) = \pi(y)\), which is equivalent to \(a_i(x - y) \equiv 0 \pmod{n}\). Since \(\gcd(m, n/m) = 1\) and \(x \equiv y \pmod{m}\), the latter is equivalent to
\[a_i(x - y) \equiv 0 \pmod{n/m},\]
which has the unique solution \(x \equiv y \pmod{n/m}\) if and only if \(\gcd(a_i, n/m) = 1\). In particular, \(\pi\) is one to one if and only if \(\gcd(a_i, n/m) = 1\) for any \(i \in [1, m]\). In this case, each \(b_i\) is uniquely determined modulo \(m\) by
\[b_i \equiv c_{i+1} - a_i c_i \pmod{m}\]
Since the entries of \(\vec{b}\) lie in \([1, n]\), there exist \((\frac{n}{m})^m\) possibilities for \(\vec{b}\). 

From the previous theorem, we obtain the exact number of \((n,m)\)-p.a.p's.
Corollary 2.5. The number of \((n, m)\)-p.a.p’s equals
\[(m - 1)! \cdot \left(n \cdot \varphi \left(\frac{n}{m}\right)\right)^m.\]

Proof. There are \((m - 1)!\) possibilities for \(\vec{c}\), \(m \cdot \varphi \left(\frac{n}{m}\right)\) possibilities for each \(a_i\), hence \([m \cdot \varphi \left(\frac{n}{m}\right)]^m\) possibilities for \(\vec{a}\). In addition, for fixed \(\vec{a}\) and \(\vec{c}\), there are \((\frac{n}{m})^m\) possibilities for \(\vec{b}\). \(\square\)

2.1. Cycle decomposition

We fix \((\vec{a}, \vec{b}, \vec{c})\) an \((n, m)\)-admissible triple and \(\pi = \pi(\vec{a}, \vec{b}, \vec{c})\) the \((n, m)\)-p.a.p with parameters \((\vec{a}, \vec{b}, \vec{c})\). The following proposition provides basic properties of the cycle decomposition of \(\pi\).

Proposition 2.6. For any \(y \in [1, n]\), the following hold:

(i) the cycle of \(y\) determined by \(\pi\) has length divisible by \(m\);
(ii) there exists an element \(z \in [1, n]\) such that \(z\) is divisible by \(m\) and lie in the same cycle of \(\pi\) containing \(y\).

In addition, if a cycle of \(\pi\) has length \(mt\), then for each \(i \in [1, m]\), such a cycle contains exactly \(t\) elements congruent to \(i\) modulo \(m\).

Proof. (i) From Definition 2.2, \(\Psi_m(\pi(x)) = c_{i+1}\) whenever \(\Psi_m(x) = c_i\). This guarantees that the sequence

\[\Psi_m(y), \Psi_m(\pi(y)), \Psi_m(\pi^2(y)), \ldots\]

can only return to \(\Psi_m(y)\) after cyclically running through the entries of \(\vec{c} \in C(m)\). In particular, the sequence

\[y, \pi(y), \pi^2(y), \ldots\]

has minimal period divisible by \(m\).

(ii) In fact, there is an entry of \(\vec{c}\) equals to \(m\), and its correspondent in the above sequence is divisible by \(m\).

We observe that, in a cycle of length \(mt\) of \(\pi\), \(\vec{c}\) is traversed \(t\) times if we consider the reduction modulo \(m\) of its elements. Therefore, each \(i \in [1, m]\) appears exactly \(t\) times. \(\square\)

In particular, in order to compute the cycle decomposition of \(\pi\), Proposition 2.6 entails that it suffices to compute the minimal period of the multiples of \(m\) in the set \([1, n]\). In this context, the following definition is useful.

Definition 2.7. 1. The principal product of \(\pi = \pi(\vec{a}, \vec{b}, \vec{c})\) is

\[P_\pi = \prod_{i=1}^{m} a_i.\]
2. The principal sum of \( \pi = \pi(\vec{a}, \vec{b}, \vec{c}) \) is the unique positive integer \( S_\pi \in [1, n] \) with the property that

\[
\pi^{(m)}(x) = \Psi_n(P_\pi \cdot x + S_\pi),
\]

for any \( x \in [1, n] \) such that \( x \equiv 0 \pmod{m} \).

Example 2.8. If \( \pi \) is the \((12, 3)\)-p.a.p as in Example 2.3, then \( \pi \) has principal product equals \( P_\pi = 15 \) and principal sum equals \( S_\pi = 9 \).

The following lemma provides a way of obtaining the \( mk \)-th iterates of \( \pi \) at elements \( x \in [1, n] \) that are divisible by \( m \).

Lemma 2.9. The principal sum \( S_\pi \) of \( \pi(\vec{a}, \vec{b}, \vec{c}) \) is well defined and, for any positive integers \( k, x \) such that \( x \in [1, n] \) is divisible by \( m \), we have that

\[
\pi^{(mk)}(x) = \Psi_n \left( P_\pi^k \cdot x + \frac{P_\pi^k - 1}{P_\pi - 1} \cdot S_\pi \right),
\]

whenever \( P_\pi \neq 1 \). For \( P_\pi \equiv 1 \pmod{n} \), we have

\[
\pi^{(mk)}(x) = \Psi_n(x + k \cdot S_\pi),
\]

and for \( P_\pi = 1 \),

\[
\pi^{(mk)}(x) = \Psi_n \left( x + k \cdot \sum_{1 \leq i \leq m} b_i \right).
\]

Proof. In fact, the composition of affine functions is also affine. Since \( \pi^{(m)}(x) \) is the reduction by \( \Psi_n \) of the composition of \( m \) affine functions given by Definition 2.2 where each function has an angular coefficient equal to \( a_i \) (this yields the angular coefficient \( P_\pi \)), \( \pi^{(m)}(x) \) is affine as well. Therefore, the linear coefficient \( S_\pi \) is well-defined and, reindexing \( \vec{c} \) by a cyclic shift if needed, \( S_\pi \) is given by

\[
S_\pi = \Psi_n \left( \sum_{i=1}^{m} a_m a_{m-1} \cdots a_{i+2} a_{i+1} b_i \right). \tag{3}
\]

For the remainder, we proceed by induction on \( k \). The case \( k = 1 \) follows from the definition of principal sum. Suppose that

\[
\pi^{(mk)}(x) = \Psi_n \left( P_\pi^k \cdot x + (P_\pi^{k-1} + \cdots + P_\pi + 1) \cdot S_\pi \right)
\]

for some \( k \geq 1 \). Then

\[
\pi^{(mk+1)}(x) = \pi^{(mk)}(\pi^{(m)}(x)) = \pi^{(mk)}(\Psi_n(P_\pi \cdot x + S_\pi))
\]

\[
= \Psi_n \left( P_\pi^k \cdot (P_\pi \cdot x + S_\pi) + (P_\pi^{k-1} + \cdots + P_\pi + 1) \cdot S_\pi \right)
\]

\[
= \Psi_n \left( P_\pi^{k+1} \cdot x + (P_\pi + P_\pi^{k-1} + \cdots + P_\pi + 1) \cdot S_\pi \right),
\]

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from where we obtain directly the case $P_{\pi} \neq 1$ and $P_{\pi} \equiv 1 \pmod{n}$. If $P_{\pi} = 1$ then $a_i = 1$ for all $1 \leq i \leq m$, which implies $S_{\pi} = \Psi_n \left( \sum_{1 \leq i \leq m} b_i \right)$.

Since $\pi^{(m)}(x) \equiv x \pmod{m}$, it holds $S_{\pi} \equiv 0 \pmod{m}$. The previous lemma implies the following result.

**Proposition 2.10.** Let $\pi$ be an $(n, m)$-p.a.p with principal product $P_{\pi}$ and principal sum $S_{\pi}$. For any positive integer $x \in [1, n]$ divisible by $m$ with $x = mx_0$, the length of the cycle of $\pi$ containing $x$ is given as follows:

(i) $m \cdot \frac{n}{\gcd(n, S_{\pi})}$ if $P_{\pi} \equiv 1 \pmod{n}$;
(ii) if $P_{\pi} \neq 1$, this length is given by $m \cdot \text{ord}_{\kappa(x)} P_{\pi}$, where

$$\kappa(x) = \frac{n \cdot (P_{\pi} - 1)}{\gcd(n \cdot (P_{\pi} - 1), x \cdot (P_{\pi} - 1) + S_{\pi})} = \frac{n \cdot (P_{\pi} - 1)}{g_{\pi} \cdot \gcd \left( \frac{n}{m} \cdot \frac{P_{\pi} - 1}{g_{\pi}}, \frac{x}{m} \cdot \frac{P_{\pi} - 1}{g_{\pi}} + \frac{S_{\pi}}{m \cdot g_{\pi}} \right)},$$

$$g_{\pi} = \gcd \left( \frac{S_{\pi}}{m} \cdot P_{\pi} - 1 \right) = \gcd \left( \frac{S_{\pi}}{m}, P_{\pi} - 1, \frac{n}{m} (P_{\pi} - 1) \right), \quad (5)$$

and $\text{ord}_b a$ denotes the order of $a$ modulo $b$.

**Proof.** (i) In this case, the cycle has length $mk$ if and only if $k$ is minimal such that $\pi^{(mk)}(x) = \Psi_n(x + k \cdot S_{\pi}) = x$, i.e., $kS_{\pi} \equiv 0 \pmod{n}$. It is clear that the minimal $k$ satisfying the latter equals $\frac{n}{\gcd(n, S_{\pi})}$.

(ii) In this case, $\pi^{(mk)}(x) = \Psi_n \left( \frac{P_{\pi}^k \cdot x + P_{\pi}^k - 1}{P_{\pi}^k - 1} \cdot S_{\pi} \right) = x$, and so we have the following equivalent conditions

$$\left( P_{\pi}^k - 1 \right) \cdot x + \frac{P_{\pi}^k - 1}{P_{\pi}^k - 1} \cdot S_{\pi} \equiv 0 \pmod{n},$$

$$\left( P_{\pi}^k - 1 \right) \cdot \left[ x \cdot (P_{\pi} - 1) + S_{\pi} \right] \equiv 0 \pmod{n (P_{\pi} - 1)},$$

$$P_{\pi}^k - 1 \equiv 0 \pmod{k(x)}.$$

Therefore, the smallest possible $k > 0$ is $k = \text{ord}_{\kappa(x)} P_{\pi}$ and the length of the cycle containing $x$ is $m \cdot \text{ord}_{\kappa(x)} P_{\pi}$.

The next lemma displays all the possible values of

$$N_0 = N_0(x_0) := \gcd \left( \frac{n}{m} \cdot \frac{P_{\pi} - 1}{g_{\pi}}, \frac{P_{\pi} - 1}{g_{\pi}} \cdot x_0 + \frac{S_{\pi}}{m \cdot g_{\pi}} \right), \quad (6)$$

and the number of solutions in each case. By Eq. (4) and (6), we observe that

$$\kappa(x) = \frac{\frac{n}{m} \cdot (P_{\pi} - 1)}{g_{\pi} \cdot N_0}. \quad (7)$$
Lemma 2.11. Let $\alpha, \beta, \gamma$ be positive integers such that $\gcd(\alpha, \beta) = 1$, $\alpha | \gamma$ and write $\gamma = \gamma_1 \gamma_2$, where $\text{rad}(\gamma_1) | \alpha$ and $\gcd(\gamma_2, \alpha) = 1$. Then the following hold:

(i) as $y$ runs over $[1, \gamma/\alpha]$, $\gcd(\alpha y + \beta, \gamma)$ varies over all the divisors of $\gamma_2$;

(ii) for each divisor $d$ of $\gamma_2$, the number of solutions $y \in [1, \gamma/\alpha]$ of

$$\gcd(\alpha y + \beta, \gamma) = \gamma_2/d,$$

is $\varphi(d) \cdot \gamma_1$, where $\varphi$ denotes the Euler’s totient function.

Proof. (i) We have that $\alpha$ divides $\gamma_1$. Since $\gcd(\gamma_1, \gamma_2) = \gcd(\alpha, \beta) = 1$, we obtain the following equalities

$$\gcd(\alpha y + \beta, \gamma) = \gcd(\alpha y + \beta, \gamma_1 \gamma_2) = \gcd(\alpha y + \beta, \gamma_2).$$

In particular, $\gcd(\alpha y + \beta, \gamma)$ divides $\gamma_2$. Fix $d$ a divisor of $\gamma_2$. As follows, we show that there exists $y \in [1, \gamma/\alpha]$ such that

$$\begin{cases} \alpha y + \beta \equiv \beta \pmod{\gamma_1} \\ \alpha y + \beta \equiv d \pmod{\gamma_2} \end{cases},$$

and this implies that $\gcd(\alpha y + \beta, \gamma_2) = d$. The first congruence is equivalent to $y = t \gamma_1 / \alpha$ for some $t \in \mathbb{Z}$, and the second one is equivalent to $t \gamma_1 \equiv d - \beta \pmod{\gamma_2}$, which has a solution for $t \in [1, \gamma_2]$, so that $y \in [1, \gamma/\alpha]$.

(ii) Let $\omega \in [1, \gamma/\alpha]$ the smallest solution of $\gcd(\alpha y + \beta, \gamma) = \gamma_2/d$. All the other solutions are of the form $\omega + j \frac{\gamma_2}{d}$ with $0 \leq j < \gamma_1 d$. Since $\gcd(\alpha, \gamma_2) = \gcd(\gamma_1, \gamma_2) = 1$, the number $\omega + j \frac{\gamma_2}{d}$ is a solution as well if and only if

$$\gcd\left(\frac{\alpha \omega + \beta}{\gamma_2/d} + \alpha \cdot j, d\right) = 1 \quad \text{and} \quad 0 \leq j < \gamma_1 d.$$

Therefore, the number of solutions $x_0 + j \cdot \frac{\gamma_2}{d} \in [1, \gamma/\alpha]$ of this equation is $\varphi(d) \cdot \gamma_1$.

\[\square\]

Suppose that $P > 1$ and let $\alpha = \frac{P \cdot 1}{g}, \beta = \frac{S \cdot 1}{m \cdot g}$ and $\gamma = \frac{n}{m}$, $\frac{P \cdot 1}{g} = \frac{m}{\alpha}$ as in the lemma above. Write $\frac{m}{n} = N_1 N_2$, where $\text{rad}(N_1)$ divides $a$ and $\gcd\left(\frac{N_2}{g}, \frac{P \cdot 1}{g}\right) = 1$. Hence the number $N_0$ defined by Eq. (3) can be any divisor of $N_2$, that is, $N_0$ can be any divisor of $n/m$ that is relatively prime with $\frac{P \cdot 1}{g}$ when $x_0$ runs over $[1, n/m]$. This observation and Eq. (7) easily implies the following result.

Corollary 2.12. Fix $x = mx_0 \in [1, n]$. Let $\kappa(x)$ and $N_0$ be defined as in Eq. (4) and (5), respectively. We write $\frac{m}{n} = N_1 N_2$ where $\text{rad}(N_1)$ divides $\frac{P \cdot 1}{g}$ and $\gcd\left(\frac{N_2}{g}, \frac{P \cdot 1}{g}\right) = 1$, as in previous paragraph. For a divisor $d$ of $N_2$ such that $N_0 = N_2/d$, we have that $\kappa(x) = N_1 \cdot \frac{P \cdot 1}{g} \cdot d$. In addition, the equation $N_0 = N_2/d$ has exactly $\varphi(d) \cdot N_1$ solutions $x$ with $x_0 \in [1, n/m]$.
Finally, we exhibit the cycle decomposition of an arbitrary \((n, m)\)-p.a.p \(\pi\) with principal product \(P_\pi \neq 1\) (the case \(P_\pi \equiv 1 \pmod{n}\) follows trivially by item (i) of Proposition 2.10). In what follows, \(\text{Cyc}(r)\) denotes a cycle of length \(r\).

**Theorem 2.13.** Let \(\pi\) be an \((n, m)\)-p.a.p with principal product \(P_\pi > 1\) and principal sum \(S_\pi\). Let \(g_\pi\) be defined as in Eq. (5) and write \(n/m = N_1N_2\), where \(\text{rad}(N_1)\) divides \(P_\pi - 1\) and \(\gcd\left(\frac{P_\pi - 1}{g_\pi}, N_2\right) = 1\). For each divisor \(d\) of \(N_2\), set \(\eta(d) = N_1 \cdot P_\pi - 1 \cdot g_\pi \cdot d\). Then the cycle decomposition of \(\pi\) is given by

\[
\bigoplus_{d|N_2} \frac{\varphi(d) \cdot N_1}{\text{ord}_{\eta(d)} P_\pi} \times \text{Cyc}(m \cdot \text{ord}_{\eta(d)} P_\pi),
\]

**Proof.** For each divisor \(d\) of \(N_2\), let \(n_d\) be the number of cycles of \(\pi\) containing an element \(x = mx_0 \in [1, n]\) such that the number \(N_0 = N_0(x_0)\) defined by Eq. (6) satisfy \(N_0 = N_2/d\).

Since every element of \([1, n]\) belongs to a unique cycle, Proposition 2.10 and Corollary 2.12 yield

\[
\sum_{d|N_2} n_d \cdot m \cdot \text{ord}_{\eta(d)} P_\pi = n.
\]

We claim that

\[
n_d \cdot \text{ord}_{\eta(d)} P_\pi \geq \varphi(d) \cdot N_1.
\]

In fact, by Corollary 2.12 for \(x = mx_0\) the equality \(N_0(x_0) = N_2/d\) implies

\[
\kappa(x) = N_1 \cdot \frac{P_\pi - 1}{g_\pi} \cdot d = \eta(d),
\]

where \(\kappa(x)\) is given by Eq. (4). From the same corollary, the latter has exactly \(\varphi(d) \cdot N_1\) solutions \(x = mx_0 \in [1, n]\). Since there exist at most \(\text{ord}_{\eta(d)} P_\pi\) of such \(x = mx_0\) in a same cycle of length \(m \cdot \text{ord}_{\eta(d)} P_\pi\) of \(\pi\), it follows that \(n_d \geq \frac{\varphi(d)N_1}{\text{ord}_{\eta(d)} P_\pi}\), proving the claim. Therefore, we obtain the following inequalities

\[
\frac{n}{m} = \sum_{d|N_2} n_d \cdot \text{ord}_{\eta(d)} P_\pi \geq \sum_{d|N_2} \varphi(d) \cdot N_1 = N_1 \sum_{d|N_2} \varphi(d) = N_1N_2 = \frac{n}{m},
\]

forcing that

\[
n_d = \frac{\varphi(d)N_1}{\text{ord}_{\eta(d)} P_\pi}.
\]

**Example 2.14.** Let \(\pi\) be the \((12, 3)\)-p.a.p given in Example 2.3. In the notation of Theorem 2.13 we have that \(P_\pi = 15, S_\pi = 9, g_\pi = 1\) and \(N_1 = 4, N_2 = 1\). From Theorem 2.13 the cycle decomposition of \(\pi\) is given by

\[
\frac{4}{\text{ord}_{56} 15} \times \text{Cyc}(3 \cdot \text{ord}_{56} 15) = 2 \times \text{Cyc}(6),
\]

as confirmed by Example 2.3.
2.2. The class of p.a.p’s defined by two rules

Here we introduce the special class of \((n, m)\)-p.a.p’s that can be defined by two affine-like rules, one for the multiples of \(m\) and one for the remaining integers in \([1, n]\). More specifically, we have the following definition.

**Definition 2.15.** An \((n, m)\)-p.a.p \(\pi\) is said to be 2-reducible if there exist integers \(a_0, a, b_0, b \in [1, m]\) such that, for any \(x \in [1, n]\), we have that

\[
\pi(x) = \begin{cases} 
\Psi_n(a_0 \cdot x + b_0) & \text{if } x \equiv 0 \pmod{m}, \\
\Psi_n(a \cdot x + b) & \text{if } x \not\equiv 0 \pmod{m}.
\end{cases}
\]

In this case, the quadruple \((a_0, a, b_0, b)\) are the 2-reduced parameters of \(\pi\).

From definition, any \((n, m)\)-p.a.p is 2-reducible if \(m = 2\). Our aim is to provide a complete characterization of the 2-reducible \((n, m)\)-p.a.p’s, where \(m > 2\). We start with the following auxiliary lemmas.

**Lemma 2.16** (Lifting the Exponent Lemma). Let \(p\) be a prime and \(\nu_p\) be the \(p\)-valuation. The following hold:

1. if \(p\) is an odd prime divisor of \(a - 1\),
   \(\nu_p(a^k - 1) = \nu_p(a - 1) + \nu_p(k)\);
2. if \(p = 2\) and \(a > 1\) is odd,
   \(\nu_2(a^k - 1) = \begin{cases} 
\nu_2(a - 1) & \text{if } k \text{ is odd}, \\
\nu_2(a^2 - 1) + \nu_2(k) - 1 & \text{if } k \text{ is even}.
\end{cases}
\)

**Lemma 2.17.** Let \(a, b, m\) be positive integers and set

\[
\text{rad}_2(m) = \begin{cases} 
\text{rad}(m) & \text{if } m \equiv 1 \pmod{2}, \\
2 \cdot \text{rad}(m) & \text{if } m \equiv 0 \pmod{2}.
\end{cases}
\]

Then the reduction modulo \(m\) of the numbers

\[b, b(a + 1), \ldots, b(a^{m-1} + \cdots + a + 1),\]

are all distinct if and only if \(\gcd(b, m) = 1\) and \(a \equiv 1 \pmod{\text{rad}_2(m)}\).

**Proof.** Set \(f_0 = b\) and, for \(1 \leq i \leq m - 1\), set \(f_i = b(a^i + \cdots + a + 1)\). It is clear that \(b\) (resp. \(a\)) must be relatively prime with \(m\), since otherwise the reduction modulo \(m\) of the elements \(f_i\) would not contain the class 1 (resp. the class 0). In particular, for \(0 \leq i < j \leq m - 1\), \(f_i \equiv f_j \pmod{m}\) if and only if \(f_{j-i} \equiv 0 \pmod{m}\). Therefore, it suffices to prove that \(i = m - 1\) is the smallest index such that \(f_i \equiv 0 \pmod{m}\) if and only if \(a \equiv 1 \pmod{\text{rad}_2(m)}\). Of course, this holds for \(a = 1\). Suppose that \(a > 1\) and write \(m = m_0m_1\), where \(\text{rad}(m_0)\) divides \(a - 1\) and \(m_1\) is relatively prime with \(a - 1\). In other words, we want to prove that \(\text{ord}_{m(a-1)} a = m\) if and only if \(m_0 = m\) and \(a \equiv 1 \pmod{4}\) if \(m\) is even. Since \(m_1\) and \(a - 1\) are relatively prime, we have that

\[
\text{ord}_{m(a-1)} a = \text{lcm}(\text{ord}_{m_0(a-1)} a, \text{ord}_{m_1} a) \leq \text{ord}_{m_0(a-1)} a \cdot \text{ord}_{m_1} a,
\]

(8)
with equality if and only if \( \text{ord}_{m_0(a-1)} a \) and \( \text{ord}_m a \) are relatively prime. However, from Lemma \( 2.16 \) \( \text{ord}_{m_0(a-1)} a \leq m_0 \) with equality if and only if \( m_0 \) is odd or \( m_0 \) is even and \( a \equiv 1 \pmod{4} \). In addition, \( \text{ord}_m a \leq \phi(m_1) < m_1 \) whenever \( m_1 > 1 \). Therefore, from Eq. \( 8 \), we have that \( \text{ord}_{m(a-1)} a = m \) if and only if \( m_1 = 1 \) (i.e., \( m_0 = m \)) and \( a \equiv 1 \pmod{4} \) if \( m \) is even.

\[ \square \]

**Proposition 2.18.** Fix \( m > 2 \) a positive integer. For integers \( a_0, a, b_0, b \in [1, n] \), the quadruple \((a_0, a, b_0, b)\) are the 2-reduced parameters of some 2-reducible \((n, m)\)-p.a.p \( \pi \) if and only if the following hold:

(i) \( b \) and \( b_0 \) are relatively prime with \( m \) with \( b \equiv b_0 \pmod{m} \);
(ii) \( a \equiv 1 \pmod{\text{rad}_2(m)} \);
(iii) \( a \) and \( a_0 \) are relatively prime with \( n/m \).

In this case, \( \pi \) is the \((n, m)\)-p.a.p with parameters \((\bar{a}, \bar{b}, \bar{c})\), where \( \bar{a} = (a_0, a, \ldots, a) \), \( \bar{b} = (b_0, b, \ldots, b) \) and \( \bar{c} \) is a cyclic permutation of the vector \((c_1, \ldots, c_m)\) with

\[
c_i = \begin{cases} 
m & \text{if } i = 1 \\
\Psi_m(b) & \text{if } i = 2 \\
\Psi_m(b \cdot (a^{i-1} + a^{i-2} + \ldots + a + 1)) & \text{if } 3 \leq i \leq m.
\end{cases}
\]

In particular, the principal product of \( \pi \) equals \( P_\pi = a_0 a^{m-1} \) and the principal sum of \( \pi \) equals \( S_\pi = \Psi_n(b_0 a^{m-1} + b(a^{m-2} + \ldots + a + 1)) \).

**Proof.** We just need to show that, if \((a_0, a, b_0, b)\) are the 2-reduced parameters of the 2-reducible \((n, m)\)-p.a.p \( \pi \), then we necessarily have that \( b \equiv b_0 \pmod{m} \). The remainder ‘if and only if’ part follows from Theorem \( 2.11 \) and Lemma \( 2.17 \) and the further identities follow directly by calculations. We observe that, since \( \pi \) is an \((n, m)\)-p.a.p, for any \( t \in [1, m] \) such that \( t \neq \Psi_m(b_0) \), there exists \( y = y(t) \in [1, m-1] \) such that \( ay + b \equiv t \pmod{m} \). In particular, if \( b \not\equiv b_0 \pmod{m} \), there exists \( y \in [1, m-1] \) such that \( ay + b \equiv b \pmod{m} \) and so \( ay \equiv 0 \pmod{m} \). This implies that \( d := \gcd(a, m) > 1 \). However, in this case, the set \( \{ \Psi_m(ay + b) \mid y \in [1, m-1] \} \) has at most \( \frac{m}{d} \) elements. Since \( m > 2 \), we have that \( \frac{m}{d} < m - 1 \) and so we get a contradiction with the property of \( y(t) \).

\[ \square \]

From the previous proposition, a formula for the number of 2-reducible \((n, m)\)-p.a.p’s is derived.

**Corollary 2.19.** The number of 2-reducible \((n, m)\)-p.a.p’s equals

\[ \phi(n) \cdot \phi \left( \frac{n}{m} \right) \cdot \frac{n^2}{m} \cdot \left[ \frac{m}{\text{rad}_2(m)} \right]. \]

**Proof.** There are \( \phi(m) \cdot \frac{m}{d} \) ways to choose \( b_0 \) and, from these, there are \( \frac{n}{m} \) ways to choose \( b \), since \( b \equiv b_0 \pmod{m} \). Also, there are \( \phi(n/m) \cdot m \) ways to choose \( a_0 \) and \( \phi(n/m) \cdot \left[ \frac{m}{\text{rad}_2(m)} \right] \) ways to choose \( a \). Since \( \gcd \left( m, \frac{n}{m} \right) = 1 \), the result follows.

\[ \square \]
3. Application: permutation polynomials over finite fields

Throughout this section, we fix \( q \) a prime power and let \( \mathbb{F}_q \) denote the finite field with \( q \) elements. We consider the set
\[
C_q = \{ m > 1 \mid \gcd(q - 1, m^2) = m \}.
\]
Let \( q - 1 = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \) be the prime factorization of \( q - 1 \). We claim that \( C_q \) has cardinality \( 2^s - 1 \). In fact, let \( m = p_1^{\beta_1} \cdots p_s^{\beta_s} \) for some \( 0 \leq \beta_i \leq \alpha_i \). If \( 0 < \beta_i < \alpha_i \) for some \( 1 \leq i \leq s \) then \( p_i^{\beta_i + 1} \) divides \( \gcd(q - 1, m^2) \), therefore \( m \notin C_q \). Thus, \( \beta_i \in \{0, \alpha_i\} \) for every \( 1 \leq i \leq s \) and, if not all \( \beta_i \in [1, s] \) are such that \( \beta_i = 0 \), we indeed have that \( \gcd(q - 1, m^2) = m > 1 \).

We observe that, for each \( m \in C_q \), we may construct many \((q - 1, m)\)-p.a.p's. It turns out that such p.a.p’s extend to permutations of the finite field \( \mathbb{F}_q \). Let \( \theta_q \in \mathbb{F}_q \) be a primitive element, i.e. a generator of the multiplicative group \( \mathbb{F}_q^* \). If \( m \in C_q \) and \( \pi \) is any \((q - 1, m)\)-p.a.p, we define its \( \theta_q \)-lift as the permutation \( F_{\pi, \theta_q} : \mathbb{F}_q \to \mathbb{F}_q \) given by
\[
\begin{align*}
F_{\pi, \theta_q}(0) &= 0 \\
F_{\pi, \theta_q}(\theta_q^i) &= \theta_q^{\pi(i)} \quad \text{for any } 1 \leq i \leq q - 1.
\end{align*}
\]
Of course, \( F_{\pi, \theta_q} \) defines a permutation of the finite field \( \mathbb{F}_q \). We observe that, from construction, such permutation defines a piecewise monomial function on \( m \)-cyclotomic cosets of \( \mathbb{F}_q^* \). In other words, if \( D_m \) denotes the set of perfect \( m \)-th powers of \( \mathbb{F}_q^* \), restricted to each coset \( \mathbb{F}_q^*/D_m \), \( F_{\pi, \theta_q}(x) \) assumes the form \( \alpha x^\beta \).

Remark 3.1. We emphasize that functions defined by different monomials on cyclotomic cosets of \( \mathbb{F}_q^* \) were previously studied in full generality: see Theorem 2 of [12]. Our aim here is to apply our \((q - 1, m)\)-p.a.p’s in the construction of permutation polynomials where the explicit polynomial representation (over the whole field \( \mathbb{F}_q \)) can be easily given.

Definition 3.2. For each divisor \( m \) of \( q - 1 \), set \( E_m(x) = \sum_{j=0}^{m-1} x^{\frac{(q-1)j}{m}} \in \mathbb{F}_q[x] \).

We observe that if \( z \in \mathbb{F}_q^* \) then
\[
E_m(z) = \begin{cases} 
m & \text{if } z^{\frac{m}{\gcd(m, q-1)}} = 1, \\
0 & \text{otherwise.} \end{cases}
\]
In particular,
\[
E_m(z) = \begin{cases} 
m & \text{if } z = \theta_q^i \text{ with } i \equiv 0 \pmod{m}, \\
0 & \text{otherwise.} \end{cases}
\]
In the following theorem we show that, if \( \pi \) is 2-reducible, we can explicitly give a polynomial expression for \( F_{\pi, \theta_q} \).
Theorem 3.3. Let $q$ be a power prime, $\theta_q \in \mathbb{F}_q$ be a primitive element and $m \in \mathbb{C}_q$. If $\pi$ is the 2-reducible $(q-1,m)$-p.a.p with reduced parameters $(a_0, a, b_0, b)$, then the $\theta_q$-lift $F_{\pi, \theta_q}$ of $\pi$ admits the following polynomial representation

$$F_{\pi, \theta_q}(x) = x^a \theta_q^b + \left(\frac{x^{a_0} \theta_q^{b_0} - x^a \theta_q^b}{m}\right) E_m(x) \in \mathbb{F}_q[x].$$

In particular, $F_{\pi, \theta_q}$ has at most $2m$ nonzero coefficients.

Proof. Since $\pi$ is a 2-reducible $(q-1,m)$-p.a.p with reduced parameters $(a_0, a, b_0, b)$, we have that the function given by

$$F_{\pi, \theta_q}(0) = 0$$
and

$$F_{\pi, \theta_q}(\theta_q^i) = \frac{1}{m} E_m(\theta_q^i) \cdot \theta_q^{a_i+b_0} + \left(1 - \frac{1}{m} E_m(\theta_q^i)\right) \cdot \theta_q^{a_i+b} = \begin{cases} \theta_q^{a_i+b_0} & \text{if } m | i, \\ \theta_q^{a_i+b} & \text{otherwise}, \end{cases}$$

represents a permutation of $\mathbb{F}_q$. The polynomial representation (over $\mathbb{F}_q$) of such permutation is given by

$$F_{\pi, \theta_q}(x) = \frac{1}{m} E_m(x) \cdot x^{a_0} \theta_q^{b_0} + \left(1 - \frac{1}{m} E_m(x)\right) \cdot x^a \theta_q^b,$$
from where the theorem follows. \hfill \square

3.1. On the cycle decomposition

We observe that the cycle decomposition of the permutation polynomials given in Theorem 3.3 can be explicitly computed. In fact, for a given 2-reducible $(q-1,m)$-p.a.p $\pi$ with reduced parameters $(a_0, a, b_0, b)$, we can readily obtain its principal sum and product. In particular, the cycle decomposition of $\pi$ is explicitly obtained from Theorem 2.13. Moreover, for a fixed primitive element $\theta_q \in \mathbb{F}_q$, the cycle decomposition of the $\theta_q$-lift permutation $F_{\pi, \theta_q}$ is obtained by the one of $\pi$, adding a loop corresponding to the fixed point $0 \in \mathbb{F}_q$.

Example 3.4. Let $q = 25$ and let $\pi$ be the 2-reduced $(24,3)$-p.a.p with reduced parameters $(5, 7, 2, 8)$ so that

$$\pi(x) = \begin{cases} \Psi_{24}(5x + 2) & \text{if } x \equiv 0 \pmod{3}, \\ \Psi_{24}(7x + 8) & \text{otherwise}. \end{cases}$$

Its principal product equals $P_\pi = 245$ and its principal sum equals $S_\pi = 18$. In the notation of Theorem 2.13, we have that $g_\pi = 2$ and $N_1 = 8$, $N_2 = 1$. From Theorem 2.13 the cycle decomposition of $\pi$ is given by

$$\frac{8}{\text{ord}_{122,8}245} \times \text{Cyc}(3 \cdot \text{ord}_{122,8}245) = 2 \times \text{Cyc}(12).$$
Let $F_{25} = F_5(\alpha)$, where $\alpha^2 - \alpha + 2 = 0$ so $\alpha$ is a primitive element. In particular, the $\alpha$-lift of $\pi$ yields the permutation polynomial

$$F_{\pi, \alpha}(x) = \frac{1}{3}E_3(x) \cdot x^5 \alpha^2 + \left(1 - \frac{1}{3}E_3(x)\right) \cdot x^7 \alpha^8$$

$$= (2\alpha + 1)(x^5 - x^7 + x^{13} + x^{21}) + (\alpha + 3)(x^{15} + x^{23}),$$

over $F_{25}$, whose cycle decomposition is

$$\text{Cyc}(1) \oplus (2 \times \text{Cyc}(12)).$$

**3.1.1. Involutions**

Involutions are frequently used in cryptographic applications. More specifically, they are used as $S$-boxes, a basic component in key-algorithms used to cover the relation between the key and the encrypted message. We observe that if $P$ is an involution over $F_q$, then any element $a \in F_q$ either belongs to a cycle of length two or is a fixed point, i.e., $P(a) = a$. There are some cryptographic attacks that explore the number of fixed points of a permutation and according to [3], for secure implementations, involutions should have few fixed points. In the following proposition, we determine all the possible involutions arising from the class of permutations given in Theorem 3.3. In particular, we see that they have only one fixed point.

**Proposition 3.5.** Let $q$ be a power prime, $\theta_q \in F_q$ be a primitive element and $m \in C_q$. If $\pi$ is 2-reducible $(q - 1, m)$-p.a.p with reduced parameters $(a_0, a, b_0, b)$, then the $\theta_q$-lift $F_{\pi, \theta_q}$ of $\pi$ is an involution if and only if the following hold:

(i) $m = 2$;
(ii) $q \equiv 3 \pmod{4}$;
(iii) $a_0 a \equiv 1 \pmod{\frac{q - 1}{2}}$;
(iv) $b_0 a + b \equiv 0 \pmod{q - 1}$.

In this case, the cycle decomposition of $F_{\pi, \theta_q}$ over $F_q$ is given by

$$\text{Cyc}(1) \oplus \left(\frac{q - 1}{2} \times \text{Cyc}(2)\right).$$

and so it has only one fixed point. Moreover, in this case, $F_{\pi, \theta_q}$ has the following polynomial representation

$$F_{\pi, \theta_q} = \theta_q^{b_0} \cdot \frac{x^{\frac{q-1}{2} + a_0} + x^{a_0}}{2} + \theta_q^b \cdot \frac{x^a - x^{\frac{q-1}{2} + a}}{2}.$$

**Proof.** We observe that $F_{\pi, \theta_q}$ is an involution if and only if any of its cycles has length one or two. From definition, the latter holds if and only if $\pi$ is an involution itself. From construction, any $(q - 1, m)$-p.a.p decomposes into cycles of length divisible by $m$, forcing $m = 2$ and $\gcd(q - 1, 4) = 2$, i.e., $q \equiv 3 \pmod{4}$. Since $m = 2$ and $\text{ord}_a b$ divides $\text{ord}_a c$ whenever $\gcd(bc, a) = 1$ and $b$ divides $c$,
Theorem 2.13 entails that $\pi$ is an involution if and only if $P_\pi - 1 = a_0a - 1$ is divisible by

$$\frac{(q - 1)(P_\pi - 1)}{2g_\pi},$$

where $g_\pi = \gcd(S_\pi, P_\pi - 1)$ and $S_\pi = b_0a + b$. The latter is equivalent to

$$g_\pi \equiv 0 \pmod{q - 1},$$

i.e., $b_0a + b \equiv 0 \pmod{q - 1}$ and $a_0a \equiv 1 \pmod{q - 1}$.

The result concerning the polynomial expression and the cycle decomposition of $F_{\pi, \theta}$ follows directly from Theorem 3.3 and the previous observations. □

Example 3.6. Let $q = 27$ and let $\pi$ be the 2-reduced $(26, 2)$-p.a.p with reduced parameters $(5, 8, 3, 2)$ so that $\pi(x) = \begin{cases} \Psi_{24}(5x + 3) & \text{if } x \text{ is even}, \\ \Psi_{24}(8x + 2) & \text{if } x \text{ is odd}. \end{cases}$

Let $F_{27} = F_3(\alpha)$ where $\alpha^3 - \alpha + 1 = 0$, so $\alpha$ is a primitive element. In particular, the $\alpha$-lift of $\pi$ yields the involution

$$F_{\pi, \alpha}(x) = \alpha^3 \left( \frac{x^{18} + x^5}{2} \right) + \alpha^2 \left( \frac{x^8 - x^{21}}{2} \right)$$

$$= \alpha^2x^{21} - (\alpha - 1)x^{18} - \alpha^2x^8 - (\alpha - 1)x^5,$$

over $\mathbb{F}_{27}$, whose cycle decomposition is

$$\text{Cyc}(1) \oplus (13 \times \text{Cyc}(2)).$$

3.2. More explicit results

We observe that Theorem 3.3 provides classes of permutation polynomials over $\mathbb{F}_q$, that depend on a primitive element $\theta_q \in \mathbb{F}_q$. If $q$ is large, it can be hard to find such $\theta_q$. Here we consider special cases where such permutation polynomials can be obtained without going through a primitive element of $\mathbb{F}_q$. Instead, we only need certain primitive roots of unity in the base field $\mathbb{F}_p$ of $\mathbb{F}_q$. This is done in the following proposition.

Proposition 3.7. Let $p$ be a prime and $m, k$ be positive integers such that $\gcd(p - 1, m^2) = m > 1$ and $\gcd(k, m) = 1$. Let $\theta \in \mathbb{F}_p$ be any primitive $m$-th root of unity, write $q = p^k$ and let $a, a_0$ be positive integers such that $\gcd(a_0a, \frac{q - 1}{m}) = 1$ and $a \equiv 1 \pmod{\text{rad}(m)}$. Then, for any positive integer $b < m$ such that $\gcd(b, m) = 1$,

$$F_{a_0, a, b}(x) = \theta^b \left( \frac{1}{m}E_m(x) \cdot x^{a_0} + \left( 1 - \frac{1}{m}E_m(x) \right) \cdot x^a \right),$$

is a permutation polynomial over $\mathbb{F}_q$. Set $g = \gcd \left( \frac{q - 1}{m}, a_0a^{m-1} - 1 \right)$ and write $\frac{q - 1}{m} = N_1N_2$, where $\text{rad}(N_1) | \frac{a_0a^{m-1} - 1}{g}$ and $\gcd \left( \frac{a_0a^{m-1} - 1}{g}, N_2 \right) = 1$. Then the
cycle decomposition of the permutation polynomial $F_{a_0,a,b}$ over $\mathbb{F}_q$ is given by

$$\text{Cyc}(1) \oplus \left( \bigoplus_{d|N_2} \frac{\varphi(d) \cdot N_1}{\text{ord}_q(a_0a^{m-1})} \times \text{Cyc} \left( m \cdot \text{ord}_q(a_0a^{m-1}) \right) \right), \quad (10)$$

where $\eta(d) = N_1 \cdot \frac{a_0a^{m-1} - 1}{g} \cdot d$.

Proof. We observe that, since $\gcd(m, k) = 1$, Lemma 2.16 entails that $\gcd(q - 1, m^2) = \gcd(p - 1, m^2) = m$.

From construction and Proposition 2.18 $(a_0, a, B, B)$ are the reduced parameters of a $(q - 1, m)$-p.a.p, where $B = \frac{b(q-1)}{m}$. Therefore, as $\theta_q$ runs over the primitive elements of $\mathbb{F}_q$, $\theta_q^{m-1}$ runs over the primitive $m$-th roots of unity in $\mathbb{F}_p$.

In particular, the fact that $F_{a_0,a,b}(x)$ permutes $\mathbb{F}_q$ follows from Theorem 3.3.

Let $\pi$ be the $(q - 1, m)$-p.a.p with reduced parameters $(a_0, a, B, B)$ where $B$ is as before. Therefore, $\pi$ has principal product $P_{a_0,a}(x) = a_0a^{m-1}$ and principal sum $S_{x} = q - 1$. In particular, Eq. (10) follows from Theorem 2.13.

Some cases of the previous proposition readily yield explicit results.

**Corollary 3.8.** Let $q$ be a prime power such that $q \equiv 3 \pmod{4}$ and let $a, a_0$ be positive integers such that $\gcd(a_0a^{\frac{q-1}{2}}) = 1$ and $a \equiv 1 \pmod{4}$. Then

$$P_{a_0,a}(x) = x^{\frac{q-1}{2}} + 1 \cdot x^{a_0} + x^{\frac{q-1}{2}} - 1 \cdot x^a,$$

is a permutation polynomial over $\mathbb{F}_q$ with cycle decomposition given by Eq. (10).

**Corollary 3.9.** Let $q = 7^k$ with $\gcd(k, 3) = 1$ and let $a, a_0$ be positive integers such that $\gcd(a_0a^{\frac{q-1}{3}}) = 1$ and $a \equiv 1 \pmod{3}$. Then for $j = 1, 2$,

$$P_{a_0,a,j}(x) = 2^j \left( \frac{1 + x^{a_0} + x^{\frac{2(q-1)}{3}}}{3} \cdot x^{a_0} + \left( 1 - \frac{1 + x^{a_0} + x^{\frac{2(q-1)}{3}}}{3} \right) \cdot x^a \right),$$

is a permutation polynomial over $\mathbb{F}_q$ with cycle decomposition given by Eq. (10).

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