Nonlinear Integral Equations and high temperature expansion for the $U_q(\hat{sl}(r+1|s+1))$ Perk-Schultz Model

Zengo Tsuboi *
Okayama Institute for Quantum Physics,
Kyoyama 1-9-1, Okayama 700-0015, Japan

Abstract
We propose a system of nonlinear integral equations (NLIE) which gives the free energy of the $U_q(\hat{sl}(r+1|s+1))$ Perk-Schultz model. In contrast with traditional thermodynamic Bethe ansatz equations, our NLIE contain only $r+s+1$ unknown functions. In deriving the NLIE, the quantum (supersymmetric) Jacobi-Trudi and Giambelli formula and a duality for an auxiliary function play important roles. By using our NLIE, we also calculate the high temperature expansion of the free energy. General formulae of the coefficients with respect to arbitrarily rank $r+s+1$, chemical potentials $\{\mu_a\}$ and $q$ have been written down in terms of characters up to the order of 5. In particular for specific values of the parameters, we have calculated the high temperature expansion of the specific heat up to the order of 40.

MSC: 82B23; 45G15; 82B20; 17B80
PACS2003: 02.30.Rz; 02.30.Ik; 02.20.Uw; 05.70.-a
Key words: nonlinear integral equation; Perk-Schultz model; quantum Jacobi-Trudi and Giambelli formula; quantum transfer matrix; thermodynamic Bethe ansatz; $T$-system
OIQP-05-12

1 Introduction
In statistical physics, to calculate the free energy of solvable lattice models for finite temperature is one of the important problems. For this purpose,
thermodynamic Bethe ansatz (TBA) equations have been often used [1]. In general, the TBA equations are an infinite number of coupled nonlinear integral equations (NLIE) with an infinite number unknown functions. Then it is desirable to reduce TBA equations to a finite number of coupled NLIE with a finite number of unknown functions.

Destri, de Vega [2] and Klümper [3, 4] proposed NLIE with two (or one \(^1\)) unknown functions for the \(XXZ\) (or \(XYZ\)) model. To generalize their NLIE to models whose underlying algebras have arbitrary rank seems to be a difficult problem as we need considerable trial and errors to find auxiliary functions which are needed to derive the NLIE. Then there are NLIE of abovementioned type for models whose underlying algebras have at most rank 3 (for example, [5, 6, 7]).

Several years ago, Takahashi discovered [8] an another NLIE for the \(XXZ\) model in simplifying TBA equations. Later, the same NLIE was rederived [9] from fusion relations (\(T\)-system) [10] among quantum transfer matrices (QTM) [11]. In addition, it was also rederived [12] for the \(XXX\) model from a fugacity expansion formula.

In view of these situations, we have derived NLIE of Takahashi type for the \(osp(1|2s)\) model [13], the \(sl(r + 1)\) model [14], the higher spin Heisenberg model [15], the \(U_q(\widehat{sl}(r + 1))\) Perk-Schultz model [16]. In these cases, the number of unknown functions and NLIE coincide with the rank of the underlying algebras. In this paper, we will further derive NLIE with a finite number of unknown functions for the \(U_q(\widehat{sl}(r + 1|s + 1))\) Perk-Schultz model [17, 18], which is a multicomponent generalization of the 6-vertex model and one of the fundamental solvable lattice models in statistical mechanics. For example, a special case of this model is related to the supersymmetric \(t - J\) model, which is important in strongly correlated electron systems.

In section 2, we introduce the \(U_q(\widehat{sl}(r + 1|s + 1))\) Perk-Schultz model, and define the QTM for it. As a summation over tableaux labeled by \(a \times m\) Young (super) diagram, we introduce an auxiliary function (2.13) [19, 20] which includes an eigenvalue formula (2.10) of the QTM as a special case. We also introduce a system of functional relations (\(T\)-system) which is satisfied by this auxiliary function.

In section 3, we derive two kind of NLIE which contain only \(r + s + 1\) unknown functions. The first ones (3.9), (3.10) reduce to the NLIE for the \(U_q(\widehat{sl}(r + 1))\) Perk-Schultz model in [16] if \(s = -1\). However our new NLIE are not straightforward generalization of the ones in our previous paper [16]. In fact for \(r, s \geq 0\) case, a straightforward generation of our previous NLIE becomes a system of an infinite number of coupled NLIE which contains an

\(^1\)if an integral contour with a closed loop is adopted
infinite number of unknown functions (see (3.7)). To overcome this difficulty, we will use the quantum (supersymmetric) Jacobi-Trudi and Giambelli formula (2.25) and a duality (2.23) for the auxiliary function, from which a closed set of NLIE can be derived. We will also propose another NLIE (3.21) and (3.22) in the latter part of the section 3, which have never been considered before even for the $U_q(^{\hat{sl}(2)})$ case. In deriving the NLIE, we assume that $q$ is generic. However we expect that our results can be also analytically continued to the case where $q$ is a root of unity.

In section 4, we calculate the high temperature expansion of the free energy based on our NLIE. In particular, we can derive coefficients (4.2)-(4.6) up to the order of 5 for the arbitrary rank $r+s+1$. The point is that if we fix the degree of the high temperature expansion, we can write down a general formula of the coefficients. On the other hand, if we specialize parameters, we can derive the coefficients for much higher orders. For example for $(r,s) = (2,-1), (-1,2)$, $q = 1$, $\mu_a = 0$ case, coefficients of the high temperature expansion of the specific heat up to the order of 40 are presented in appendix. It will be difficult to derive the coefficients of such a high order by other method.

Section 5 is devoted to concluding remarks.

2 The Perk-Schultz model and the quantum transfer matrix method

In this section, we will introduce the $U_q(^{\hat{sl}(r+1|s+1)})$ Perk-Schultz model \cite{17,18} and the quantum transfer matrix (QTM) method \cite{11,22,23,24,4,3} for it. The QTM method was applied to the Perk-Schultz model in ref. \cite{5} (see also, ref. \cite{25,26,6}).

Let us introduce three sets $B = \{1,2,\ldots, r+s+2\} = B_+ \cup B_-$, where $B_+ \cap B_- = \phi$, $|B_+| = r+1$ and $|B_-| = s+1$ $(r,s \in \mathbb{Z}_{\geq -1})$. We define a grading parameter $p(a)$ ($a \in B$) such that $p(a) = 0$ for $a \in B_+$ and $p(a) = 1$ for $a \in B_-$. The $R$-matrix of the $U_q(^{\hat{sl}(r+1|s+1)})$ Perk-Schultz model \cite{17} is given as

$$R(v) = \sum_{a_1,a_2,b_1,b_2 \in B} R_{a_2,b_2}^{a_1,b_1}(v) E^{a_1,a_2} \otimes E^{b_1,b_2}, \quad (2.1)$$

where $E^{a,b}$ is a $r+s+2$ by $r+s+2$ matrix whose $(i,j)$ element is given as

\footnote{$U_q(^{\hat{sl}(r+1|s+1)})$ is a quantum affine superalgebra, which characterizes the $R$-matrix of this model. See for example, \cite{21}. We assume $\eta \in \mathbb{R}$ \ ($q = e^{2\pi i \eta}$). A rational limit ($q \rightarrow 1$) of the Perk-Schultz model is the Uimin-Sutherland model \cite{27,28}.}
(E^{a,b})_{i,j} = \delta_{ai}\delta_{bj}; R_{a_1,b_1}^{a_2,b_2}(v) is defined as

\begin{align}
R_{a,a}^{a,a}(v) &= (-1)^{(p(a)v + 1)}_q, \\
R_{a,b}^{a,b}(v) &= (-1)^{(p(a)p(b)}_q (a \neq b), \\
R_{a,b}^{b,a}(v) &= q^{(\text{sign}(a-b)v)} (a \neq b),
\end{align}

where \( v \in \mathbb{C} \) is the spectral parameter; \( a, b \in B; [v]_q = (q^v - q^{-v})/(q - q^{-1}); q = e^\eta \). Note that this \( R \)-matrix reduces to the one for the well known 6-vertex model if \((r, s) = (1, -1)\).

Let \( L \) be a positive integer (the number of lattice sites). The row-to-row transfer matrix on \((\mathbb{C}^{r+s+2})^\otimes L\) is defined as \(^3\)

\[
t(v) = \text{tr}_0(R_{0L}(v) \cdots R_{02}(v)R_{01}(v)).
\]

The main part of the Hamiltonian is proportional to the logarithmic derivative of the row-to-row transfer matrix \((\text{QTM})\):

\[
H_{\text{body}} = \left. \frac{J \sinh \eta}{\eta} \frac{d}{dv} \log t(v) \right|_{v=0} = J \sum_{j=1}^L \left\{ \cosh \eta \sum_{a \in B} (-1)^{(p(a)E_j^{a,a}E_{j+1}^{a,a}) + \sum_{a, b \in B} \text{sign}(a-b) \sinh \eta E_j^{a,a}E_{j+1}^{b,b} + (-1)^{(p(a)p(b)}E_j^{b,a}E_{j+1}^{a,b}) \right\},
\]

where we adopt the periodic boundary condition \( E_{L+1}^{a,b} = E_1^{a,b} \). Without breaking the integrability, we can also add the chemical potential term

\[
H_{\text{ch}} = - \sum_{j=1}^L \sum_{a \in B} \mu_a E_j^{a,a}
\]

to \( H_{\text{body}} \). Then the total Hamiltonian is \( H = H_{\text{body}} + H_{\text{ch}} \).

To treat the model at finite temperature \( T \), we introduce the so-called quantum transfer matrix (QTM) \( t_{\text{QTM}}(v) \):

\[
\begin{align}
t_{\text{QTM}}(v) &= \sum_{\{\alpha_k\}, \{\beta_k\}} t_{\text{QTM}}(v)^{\{\beta_1, \ldots, \beta_N\}}_{\{\alpha_1, \ldots, \alpha_N\}} E_1^{\beta_1}\alpha_1 E_2^{\beta_2}\alpha_2 \cdots E_N^{\beta_N}\alpha_N, \\
t_{\text{QTM}}(v)^{\{\beta_1, \ldots, \beta_N\}}_{\{\alpha_1, \ldots, \alpha_N\}} &= \sum_{\{\nu_k\}} \frac{\mu^{\nu_1}}{\nu_1!} \prod_{k=1}^N (\alpha_{2k}^{\nu_{2k}}, \nu_{2k}) (u + iv) R_{\alpha_{2k-1}, \nu_{2k-1}}^{\beta_{2k-1}, \nu_{2k-1}} (u - iv),
\end{align}
\]

\(^3\)The lower index \( i, j \) of \( R_{ij}(v) \) is used as follows: for example, \( E_k^{a,b} \) is defined on \((\mathbb{C}^{r+s+2})^\otimes (L+1); E_k^{a,b} = I^\otimes k \otimes \hat{E}_a^{a,b} \otimes I^\otimes (L-k)\), where \( I \) is \( r+s+2 \) by \( r+s+2 \) identity matrix; \( k = 0, 1, \ldots, L \). Then \( R_{ij}(v) \) is defined as \( R_{ij}(v) = \sum_{a_1, a_2, b_1, b_2} R_{a_2,b_2}^{a_1,b_1}(v) E_i^{a_1,a_2} E_j^{b_1,b_2} \). The trace \( \text{tr}_0 \) is taken over the auxiliary space indexed by 0.
where $N \in 2\mathbb{Z}_{\geq 1}$ is the Trotter number; $\nu_{N+1} = \nu_1$; $\nu_k, \alpha_k, \beta_k \in B$; $u = -\frac{J \sinh \eta}{\eta N T}$; $\tilde{R}_{a_1,b_1}^{a_2,b_2}(v) = R^{b_1,a_2}_{b_2,a_1}(v)$ is the \textquote{90° rotation} of $R(v)$. We can express the free energy per site in terms of only the largest eigenvalue $\Lambda_1$ of the QTM (2.8) at $v = 0$:

$$f = -T \lim_{N \to \infty} \log \Lambda_1,$$

(2.9)

where the Boltzmann constant is set to 1.

Due to the Yang-Baxter equation, the QTM (2.8) forms commuting family for any $v$. Thus it can be diagonalized by the Bethe ansatz. The eigenvalue formula of the QTM (2.8) will be (cf. [5, 6])

$$T^{(1)}_1(v) = \sum_{a \in B} z(a; v),$$

(2.10)

where

$$z(a; v) = \psi_a(v) \xi_a\left(\frac{Q_{a-1}(v - \frac{i \sum_{j=1}^{a-1}(-1)^{p(j)}}{2} - i(-1)^{p(a)} Q_a(v - \frac{i \sum_{j=1}^{a-1}(-1)^{p(j)}}{2} + i(-1)^{p(a)})}{Q_{a-1}(v - \frac{i \sum_{j=1}^{a-1}(-1)^{p(j)}}{2}) Q_a(v - \frac{i \sum_{j=1}^{a-1}(-1)^{p(j)}}{2}}\right),$$

(2.11)

$$\psi_a(v) = e^{\frac{M_a}{T} \phi_-(v - i(-1)^{p(1)} \delta_{a,1}) \phi_+(v + i(-1)^{p(r+s+2)} \delta_{a,r+s+2}),}$$

$$\phi_\pm(v) = \left(\frac{\sin \eta (v \pm i u)}{\sinh \eta}\right)^{\frac{1}{2}},$$

where $M_a \in \mathbb{Z}_{\geq 0}$; $Q_0(v) = Q_{r+s+2}(v) = 1$. $\xi_a \in \{-1, 1\}$ is a parameter which depends on the grading parameter $\{p(b)\}_{b \in B}$; $\{v_k^{(a)}\}$ is a root of the Bethe

---

4To be precise, this formula is a conjecture for general parameters $r, s, q, \mu, N$. In [4], the algebraic Bethe ansatz for a one particle state was executed for the QTM of the $U_q(sl(r+1|s+1))$ Perk-Schultz model. As for the $U_q(sl(2))$ case, a proof of this formula by the algebraic Bethe ansatz is similar to the row-to-row transfer matrix case (cf. [29]). This formula has a quite natural form (dressed vacuum form) from a point of view of the analytic Bethe ansatz [30, 31]. An eigenvalue formula of the row to row transfer matrix (2.5) was derived in [32, 18]. It has essentially same form as (2.10) except for a part which is related to the vacuum eigenvalue. There is also support by numerical calculations for small $r, s$. 

5
ansatz equation (BAE)

\[
\psi_a(v_k^{(a)}) + \frac{i}{2} \sum_{j=1}^{a} (-1)^p(j) = \psi_{a+1}(v_k^{(a)}) + \frac{i}{2} \sum_{j=1}^{a} (-1)^p(j) \tag{2.12}
\]

where \(\varepsilon_a = \frac{\xi_{a+1}}{\xi_a} \in \{-1, 1\}\). From now on, we assume the relation \(p(1) = p(r + s + 2)\) on the grading parameter. In this case, the eigenvalue formula (2.10) of the QTM has good analyticity to derive the NLIE. We expect that this assumption does not spoil generality as the free energy will be independent of the order of the grading parameters.

Let us define an auxiliary function (see also [33]):

\[
T_m(a)(v) = \sum \prod_{j=1}^{a} \prod_{k=1}^{m} z(d_{j,k}; v - \frac{i}{2}(a - m - 2j + 2k)), \tag{2.13}
\]

where \(m, a \in \mathbb{Z}_{\geq 1}\), and the sumation is taken over \(d_{j,k} \in B \ (1 < 2 < \cdots < r + s + 2)\) such that

\[
\begin{align*}
d_{j,k} &\leq d_{j+1,k} \quad \text{and} \quad d_{j,k} \leq d_{j,k+1} \tag{2.14} \\
d_{j,k} &< d_{j,k+1} \quad \text{if} \quad d_{j,k} \in B_- \quad \text{or} \quad d_{j,k+1} \in B_- \tag{2.15} \\
d_{j,k} &< d_{j+1,k} \quad \text{if} \quad d_{j,k} \in B_+ \quad \text{or} \quad d_{j+1,k} \in B_+. \tag{2.16}
\end{align*}
\]

This function contains \(T_{1}^{(1)}(v)\) (2.10) as a special case \((a, m) = (1, 1)\). It can be interpreted as a summation over a Young (super) tableau labeled by \(a \times m\) Young (super) diagram. It is related to a system of eigenvalue formulae of the QTM for fusion models [34]. Note that the condition (2.15) is void if \(s = -1\), then (2.10) reduces to the Bazhanov-Reshetikhin formula [35].

For \(a, m \in \mathbb{Z}_{\geq 1}\), we will normalize (2.13) as \(\widetilde{T}_m(a)(v) = T_m(a)(v)/\mathcal{N}_m(a)(v)\), where

\[
\mathcal{N}_m(a)(v) = \prod_{j=1}^{a} \prod_{k=1}^{m} \phi_-(v - a - m - 2j + 2k) \phi_+(v - a - m - 2j + 2k) \tag{2.17}
\]
Here we introduce a parameter \( \xi \in \{-1, 1\} \). \( T_m^{(a)}(v) \) has no pole on \( v \) due to the BAE (2.12). In contrast, \( \widehat{T}_m^{(a)}(v) \) has poles at \( v = \pm \left( \frac{n+m+\xi}{2} i + i u \right) + \frac{\pi n}{\eta} \) \( (n \in \mathbb{Z}) \) for \( (a, m) \in \mathbb{Z}_{\geq 1} \times \{ 1, 2, \ldots, s + 1 \} \cup \{ 1, 2, \ldots, s + 1 \} \times \mathbb{Z}_{\geq 1} \).

One can show that \( \widehat{T}_m^{(a)}(v) \) satisfies the so called \( T \)-system for \( U_q(\widehat{sl}(r + 1|s + 1)) \) \cite{19} \cite{20} (see also \cite{20} for a derivation of TBA equations from the \( T \)-system). For \( m, a \in \mathbb{Z}_{\geq 1} \),

\[
\widehat{T}_m^{(a)}(v) - \frac{\xi}{2} \widehat{T}_m^{(a)}(v + \frac{\xi}{2}) = \widehat{T}_{m-1}^{(a)}(v) \widehat{T}_{m+1}^{(a)}(v) + \widehat{T}_{m-1}^{(a-1)}(v) \widehat{T}_{m+1}^{(a+1)}(v) \quad (2.18)
\]

for \( a \in \{ 1, 2, \ldots, r \} \) or \( m \in \{ 1, 2, \ldots, s \} \) or \( (a, m) = (r + 1, s + 1) \),

\[
\widehat{T}_m^{(r+1)}(v) - \frac{\xi}{2} \widehat{T}_m^{(r+1)}(v + \frac{\xi}{2}) = \widehat{T}_{m-1}^{(r+1)}(v) \widehat{T}_{m+1}^{(r+1)}(v) \quad (2.19)
\]

\[
\widehat{T}_m^{(a)}(v) - \frac{\xi}{2} \widehat{T}_m^{(a)}(v + \frac{\xi}{2}) = \widehat{T}_{m-1}^{(a-1)}(v) \widehat{T}_{m+1}^{(a+1)}(v) \quad (2.20)
\]

where

\[
\widehat{T}_0^{(a)}(v) = \frac{\phi_-(v - a \frac{\xi}{2} i) \phi_+(v + a \frac{\xi}{2} i)}{\phi_-(v - a \frac{\xi}{2} i) \phi_+(v + a \frac{\xi}{2} i)} \quad (2.21)
\]

\[
\widehat{T}_m^{(0)}(v) = \frac{\phi_-(v + \frac{m \xi}{2} i) \phi_+(v - \frac{m \xi}{2} i)}{\phi_-(v - \frac{m \xi}{2} i) \phi_+(v + \frac{m \xi}{2} i)} \quad (2.22)
\]

There is a duality relation for the auxiliary function.

\[
\widehat{T}_{a+s}^{(r+1)}(v) = \zeta^{a-1} \widehat{T}_{s+1}^{(r+a)}(v) \quad (2.23)
\]

where \( \zeta = \prod_{a \in B_+} \xi_m^{\frac{a}{2}} \prod_{b \in B_-} \xi_m^{-\frac{a}{2}} \). (2.21) (resp. (2.22)) becomes 1 if \( \xi = 1 \) (resp. \( \xi = -1 \)). Note that there is no upper bound for the index \( a \) of \( \widehat{T}_m^{(a)}(v) \) for \( m \in \{ 1, 2, \ldots, s + 1 \} \) if \( s \in \mathbb{Z}_{\geq 2} \). For \( s = -1 \), this \( T \)-system reduces to the one for \( U_q(\widehat{sl}(r + 1)) \) \cite{36} (see also \cite{10}). In this case, (2.23) reduces to \( \widehat{T}_{a-1}^{(r+1)}(v) = \zeta^{a-1} e^{(\xi_1 \xi_1 + \xi_2 + \cdots + \xi_{r+1})} \) if \( \xi = 1 \) (see eq. (2.21) in \cite{16}). From the relations (2.19), (2.20), (2.23) and (2.18) for \( (a, m) = (r + 1, s + 1) \), one can derive the following relation for \( a \in \mathbb{Z}_{\geq 2} \):

\[
\widehat{T}_{a+s}^{(r+1)}(v) = \zeta^{a-1} \widehat{T}_{s+1}^{(r+a)}(v)
\]

\[
= \frac{\zeta^{a-1} \prod_{j=1}^{a} \widehat{T}_{s+1}^{(r+1)}(v + \frac{a-2j+1}{2} i)}{\prod_{j=2}^{a} (\xi \widehat{T}_{s+1}^{(r+1)}(v + \frac{a-2j+2}{2} i) + \widehat{T}_{s+1}^{(r)}(v + \frac{a-2j+2}{2} i))}.
\]

(2.24)
\(\tilde{T}_m^{(a)}(v)\) can also be written in terms of a determinant (the quantum (super-symmetric) Jacobi-Trudi and Giambelli formula \([19, 20]\) (for \(s = -1\) case, \([35]\); for \(U_q(B_r^{(1)})\) case, \([37]\))

\[
\tilde{T}_m^{(a)}(v) = W_m^{(a)}(v) \det_{1 \leq j, k \leq m} \left( \tilde{T}_{1}^{(a+j-k)} \left( v - \frac{j + k - m - 1}{2} i \right) \right) \quad (2.25)
\]

\[
= Z_m^{(a)}(v) \det_{1 \leq j, k \leq a} \left( \tilde{T}_{m+j-k}^{(1)} \left( v - \frac{a - j - k + 1}{2} i \right) \right), \quad (2.26)
\]

where \(\tilde{T}_1^{(a)}(v) = 0 \text{ for } a < 0 \) and \(\tilde{T}_m^{(1)}(v) = 0 \text{ for } m < 0\). \(W_m^{(a)}(v)\) and \(Z_m^{(a)}(v)\) are normalization functions:

\[
W_m^{(a)}(v) = \frac{1}{\prod_{j=1}^{m-1} \tilde{T}_0^{(a)}(v + \frac{m-2j}{2} i)}, \quad (2.27)
\]

\[
Z_m^{(a)}(v) = \frac{1}{\prod_{j=1}^{a-1} \tilde{T}_m^{(0)}(v - \frac{a-2j}{2} i)}, \quad (2.28)
\]

where \(\prod_{j=1}^{0} (\cdots) = 1\). Substituting (2.25) into (2.28), we obtain an equation

\[
W_{a+s}^{(r+1)}(v) \det_{1 \leq j, k \leq a+s} \left( \tilde{T}_1^{(r+1+j-k)} \left( v - \frac{j + k - a - s - 1}{2} i \right) \right)
\]

\[
= \zeta^{a-1} W_{s+1}^{(r+a)}(v) \det_{1 \leq j, k \leq s+1} \left( \tilde{T}_1^{(r+a+j-k)} \left( v - \frac{j + k - s - 2}{2} i \right) \right)
\]

\[
\quad \text{for } a \in \mathbb{Z}_{\geq 1}. \quad (2.29)
\]

Expanding partially (2.29) on both side, we obtain

\[
\tilde{T}_1^{(a+r+s)}(v) = \frac{\tilde{A}_1(v) - \zeta^{a-1} W_{s+1}^{(r+a)}(v) \tilde{A}_2(v)}{(-1)^{a+s} \tilde{A}_3(v) + (-1)^s \zeta^{a-1} W_{a+s}^{(r+a)}(v) \tilde{A}_4(v)}
\]

\[
\quad \text{for } a \in \mathbb{Z}_{\geq 2}, \quad (2.30)
\]
where

\[ \tilde{A}_1(v) = \det_{1 \leq j, k \leq a+s} \left( \tilde{f}_{j,k} \left( v - \frac{j + k - a - s - 1}{2} i \right) \right) \]  
(2.31)

\[ \tilde{f}_{j,k}(v) = T_1^{(r+1+j-k)}(v) \quad \text{for} \quad (j, k) \neq (a+s, 1), \quad \tilde{f}_{a+s,1}(v) = 0, \]

\[ \tilde{A}_2(v) = \det_{1 \leq j, k \leq s+1} \left( \tilde{g}_{j,k} \left( v - \frac{j + k - s - 2}{2} i \right) \right) \]  
(2.32)

\[ \tilde{g}_{j,k}(v) = T_1^{(r+a+j-k)}(v) \quad \text{for} \quad (j, k) \neq (s+1, 1), \quad \tilde{g}_{s+1,1}(v) = 0, \]

\[ \tilde{A}_3(v) = \det_{1 \leq j, k \leq a+s-1} \left( T_1^{(r+j-k)} \left( v - \frac{j + k - a - s}{2} i \right) \right), \]  
(2.33)

\[ \tilde{A}_4(v) = \det_{1 \leq j, k \leq s} \left( T_1^{(r+a+j-k-1)} \left( v - \frac{j + k - s - 1}{2} i \right) \right). \]  
(2.34)

It turns out that \( T_1^{(a+r+s)}(v) \) is written in terms of \( \{ T_1^{(d)}(v) \} \) where \( \max(0, r - s + 2 - a) \leq d \leq a + r + s - 1 \). Then \( \tilde{T}_1^{(a)}(v) \) for \( a \in \mathbb{Z}_{\geq r+s+2} \) can be expressed in terms of \( \{ \tilde{T}_1^{(d)}(v) \} \) where \( 0 \leq d \leq r + s + 1 \). Similarly, we can derive the following relation from (2.23) and (2.26).

\[ \tilde{T}_1^{(1)}(v) = \frac{\zeta^{a-1} Z_{a+1}^{(r+a)(v)}(v) \tilde{A}_5(v) - \tilde{A}_6(v)}{(-1)^{a+r} \zeta^{a-1} Z_{a+s}^{(r+a)}(v) \tilde{A}_7(v) + (-1)^r \tilde{A}_8(v)} \]
for \( a \in \mathbb{Z}_{\geq 2}, \)  
(2.35)

where

\[ \tilde{A}_5(v) = \det_{1 \leq j, k \leq a+r} \left( \tilde{h}_{j,k} \left( v - \frac{a + r + 1 - j - k}{2} i \right) \right) \]  
(2.36)

\[ \tilde{h}_{j,k}(v) = T_1^{(1)}(v) \quad \text{for} \quad (j, k) \neq (a+r, 1), \quad \tilde{h}_{a+r,1}(v) = 0, \]

\[ \tilde{A}_6(v) = \det_{1 \leq j, k \leq r+1} \left( \tilde{b}_{j,k} \left( v - \frac{r + 2 - j - k}{2} i \right) \right) \]  
(2.37)

\[ \tilde{b}_{j,k}(v) = T_1^{(1)}(v) \quad \text{for} \quad (j, k) \neq (r+1, 1), \quad \tilde{b}_{r+1,1}(v) = 0, \]

\[ \tilde{A}_7(v) = \det_{1 \leq j, k \leq a+r-1} \left( \tilde{T}_1^{(1)}(v) \left( v - \frac{a + r - j - k}{2} i \right) \right), \]  
(2.38)

\[ \tilde{A}_8(v) = \det_{1 \leq j, k \leq r} \left( \tilde{T}_1^{(1)}(v) \left( v - \frac{r + 1 - j - k}{2} i \right) \right). \]  
(2.39)

Let us consider the limit

\[ Q_m^{(a)} := \lim_{\nu \to i \nu^{-1} \infty} \tilde{T}_m^{(a)}(\nu) = \sum_{\{d_{j,k}\}} \prod_{j=1}^{a} \prod_{k=1}^{a} \xi_{d_{j,k}} \exp \left( \frac{\mu_{d_{j,k}}}{T} \right), \]  
(2.40)
where the summation is taken over \( \{d_{j,k}\} \) \((d_{j,k} \in B)\) which obey the rules \(2.14\)-\(2.16\). For example, for \(U_q(\hat{sl}(2|1))\) \((B_+ = \{1, 3\}, B_- = \{2\})\) case, we have,

\[
Q^{(1)}_1 = \xi_1 e^{\frac{\mu_3}{T}} + \xi_2 e^{\frac{\mu_1}{T}} + \xi_3 e^{\frac{\mu_2}{T}},
\]

\[
Q^{(a)}_1 = \xi_1 \xi_2 e^{\frac{(a+1)(a-1)\mu_2}{T}} + \xi_1 \xi_2 e^{\frac{(a-1)\mu_3}{T}} + \xi_2 \xi_3 e^{\frac{a\mu_2}{T}} + \xi_2 \xi_3 e^{\frac{(a-1)\mu_2 + \mu_3}{T}}
\]

\[
= \xi_2 e^{\frac{(a-2)\mu_2}{T}} Q^{(2)}_1 \quad \text{for} \quad a \in \mathbb{Z}_{\geq 2}. \tag{2.42}
\]

We can also rewrite \(2.42\) as

\[
Q^{(a)}_1 = \frac{Q^{(3)}_1}{Q^{(2)}_1} = \frac{Q^{(2)}_1}{(\zeta + Q^{(1)}_1)^{a-2}}. \tag{2.43}
\]

This quantity \(2.40\) corresponds to the character of \(a\)-th anti-(super)symmetric and \(m\)-th (super)symmetric tensor representation. We will use \(Q^{(a)}_1\) and \(Q^{(1)}_m\) later.

\(Q^{(a)}_m\) also satisfies the so called \(Q\)-system, which is the \(T\)-system \(2.18\)-\(2.23\) without the spectral parameter \(v\); for \(m, a \in \mathbb{Z}_{\geq 1}\), we have

\[
Q^{(a)}_m = Q^{(a)}_{m-1} Q^{(a)}_{m+1} + Q^{(a-1)}_{m} Q^{(a+1)}_{m} \tag{2.44}
\]

for \(a \in \{1, 2, \ldots, r\}\) or \(m \in \{1, 2, \ldots, s\}\)

or \((a, m) = (r + 1, s + 1),

\[
Q^{(r+1)}_m = Q^{(r+1)}_{m-1} Q^{(r+1)}_{m+1} \quad \text{for} \quad m \in \mathbb{Z}_{\geq s+2}, \tag{2.45}
\]

\[
Q^{(a)}_{s+1} = Q^{(a-1)}_{s+1} Q^{(a+1)}_{s+1} \quad \text{for} \quad a \in \mathbb{Z}_{\geq r+2}, \tag{2.46}
\]

where

\[
Q^{(0)}_m = Q^{(0)}_m = 1 \quad \text{for} \quad a, m \in \mathbb{Z}_{\geq 1},
\]

\[
Q^{(r+1)}_{a+s} = \zeta^{a-1} Q^{(r+a)}_{s+1} \quad \text{for} \quad a \in \mathbb{Z}_{\geq 1}. \tag{2.47}
\]

The \(Q\)-system was introduced \[38, \ 39\] as functional relations among characters of finite dimensional representations of Yangians (or quantum affine algebras) associated with simple Lie algebras. The above system of equations is a superalgebra version of them.

In closing this section, let us comment on the analyticity of the auxiliary function \(2.13\). As mentioned before, the free energy \(2.9\) is given only by the largest eigenvalue of the QTM \(2.8\). Then we are only interested in a root of the BAE \(2.12\) which gives the largest eigenvalue of the QTM. Judging from numerical calculations \[25, \ 26, \ 14, \ 16\], such a root will exist in
the sector \( N \) of the BAE, and it will form ’one-string’ on the complex plane. For this root, the zeros of the auxiliary function \( \tilde{T}_m(a) \) will exist near the lines \( \text{Im} v = \pm \frac{a+m}{2} \) at least for \( \{\mu_a\} = \{0\} \) and small \( u \) (see, figures in [26, 14, 16]). In this sector, we have

\[
\begin{align*}
\xi_b &= 1 \quad \text{for} \quad b \in B, \\
\varepsilon_b &= 1 \quad \text{for} \quad b \in B - \{r + s + 2\}, \\
\zeta &= \exp\left(\frac{\sum_{a \in B} \mu_a - \sum_{a \in B - \{r + s + 2\}} \mu_a}{T}\right).
\end{align*}
\]

From now on, we only consider the largest eigenvalue of the QTM, and assume these values \( \{2.48\} \) of the parameters.

### 3 The nonlinear integral equations

In this section, we will derive NLIE by using formulae in the previous section. We will treat two kind of NLIE paying attention to the value of the parameter \( \xi \in \{-1, 1\} \). Although the first step of calculations \( \{3.1\}-\{3.6\} \) is similar to \( s = -1 \) case \([9, 14, 16]\), we will present it for reader’s convenience.

Taking note on the limit \( \{2.40\} \) and the fact that \( \tilde{T}_m(a) \) has poles at \( v = \pm \left(\frac{a+m}{2}\xi + \frac{i\pi}{n}\right) \) \((n \in \mathbb{Z})\) for \((a, m) \in \{1, 2, \ldots, r + 1\} \times \mathbb{Z}_{\geq 1} \cup \mathbb{Z}_{\geq 1} \times \{1, 2, \ldots, s + 1\}\), we can expand \( \tilde{T}_m(a) \) as follows.

\[
\tilde{T}_m(a)(v) = Q_m(a)
+ \sum_{n \in \mathbb{Z}} \sum_{j=1}^{\frac{n}{\pi}} \left\{ \frac{A_{m,j}}{(v - \frac{a+m}{2}\xi + i \left(\frac{\pi}{n}\right))^j} + \frac{\bar{A}_{m,j}}{(v + \frac{a+m}{2}\xi + i \left(\frac{\pi}{n}\right))^j} \right\},
\]

where the coefficients \( A_{m,j}, \bar{A}_{m,j} \in \mathbb{C} \) can be expressed as contour integrals:

\[
\begin{align*}
A_{m,j} &= \oint_{\tilde{C}_{m,j}(a)} \frac{dv}{2\pi i} \tilde{T}_m(a)(v) (v - \frac{a+m}{2}\xi - i \left(\frac{\pi}{n}\right))^j, \\
\bar{A}_{m,j} &= \oint_{\bar{C}_{m,j}(a)} \frac{dv}{2\pi i} \tilde{T}_m(a)(v) (v + \frac{a+m}{2}\xi - i \left(\frac{\pi}{n}\right))^j.
\end{align*}
\]

Here the contour \( \tilde{C}_{m,j}(a) \) (resp. \( \bar{C}_{m,j}(a) \)) is a counterclockwise closed loop which surrounds \( v = \frac{a+m}{2}\xi + i \left(\frac{\pi}{n}\right) \) (resp. \( v = -\frac{a+m}{2}\xi - i \left(\frac{\pi}{n}\right) \)) and does not surround \( v = -\frac{a+m}{2}\xi - i \left(\frac{\pi}{n}\right), \frac{a+m}{2}\xi + i \left(\frac{\pi}{n}\right), \frac{a+m}{2}\xi + i \left(\frac{\pi}{n}\right) \), \((n \in \mathbb{Z}, k \in \mathbb{Z} - \{0\}\)), using the \( T \)-system \( \{2.18\}-\{2.20\} \), we can rewrite
\[ (3.2) \] as

\[
A_{m,j}^{(a)} = \oint_{C_{m}^{(a)}} \frac{dv}{2\pi i} \left\{ \frac{\tilde{T}_{m-1}^{(a)}(v - \frac{\xi_i}{2}) \tilde{T}_{m+1}^{(a)}(v - \frac{\xi_i}{2})}{\tilde{T}_{m}^{(a)}(v - \xi i)} + \frac{\tilde{T}_{m}^{(a-1)}(v - \frac{\xi_i}{2}) \tilde{T}_{m+1}^{(a)}(v - \frac{\xi_i}{2})}{\tilde{T}_{m}^{(a)}(v - \xi i)} \right\}(v - \frac{a + m}{2}\xi i - iu)^{j-1},
\]

where we admit \( \tilde{T}_{m}^{(b)}(v) = 0 \) if \((b, n) \in \mathbb{Z}_{\geq r+2} \times \mathbb{Z}_{\geq s+2} \) (cf. [10, 11]). Substituting \((3.3)\) into \((3.1)\) and taking the summation over \(j\), we obtain

\[
\tilde{T}_{m}^{(a)}(v) = Q_{m}^{(a)}
\]

\[ (3.3) \]

\[ (3.4) \]

Here the contours are shifted as follows: the contour \( C_{m}^{(a)} \) (resp. \( \tilde{C}_{m}^{(a)} \)) is a counterclockwise closed loop which surrounds \( y = 0 \) (resp. \( y = 0 \)) and does not surround \( y = -(a+m)\xi i - 2iu - \frac{\pi k}{\eta}, \frac{\pi k}{\eta} \) (resp. \( y = (a+m)\xi i + 2iu + \frac{\pi m}{\eta}, \frac{\pi k}{\eta} \)),
where \( n \in \mathbb{Z}, k \in \mathbb{Z} - \{0\} \). We can neglect the terms \( \left( \frac{y + \frac{a+m}{2} \xi i + iu}{v - \frac{m}{\eta}} \right)^N \) in (3.4) since the poles at \( y = 0 \) in the two brackets \{\ldots\} are canceled by the zeros from these terms. By using the following relation

\[
\lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{v - \frac{m}{\eta}} = \frac{\eta}{\tan \eta v},
\]

we can take the summation over \( n \in \mathbb{Z} \).

\[
\tilde{T}^{(a)}_m(v) = Q^{(a)}_m
\]

\[
+ \oint_{C^{(a)}_m} \frac{dy}{2\pi i \tan \eta (v - y - \frac{a+m}{2} \xi i - iu)}
\times \left\{ \frac{\tilde{T}^{(a)}_{m-1}(v)}{\tilde{T}^{(a)}_m(v)} \left( y + \frac{a+m-1}{2} \xi i + iu \right) \frac{\tilde{T}^{(a)}_{m+1}(v)}{\tilde{T}^{(a)}_m(v)} \left( y + \frac{a+m}{2} \xi i + iu \right) \right. \\
+ \left. \frac{\tilde{T}^{(a)}_{m-1}(v)}{\tilde{T}^{(a)}_m(v)} \left( y - \frac{a+m-1}{2} \xi i - iu \right) \frac{\tilde{T}^{(a)}_{m+1}(v)}{\tilde{T}^{(a)}_m(v)} \left( y - \frac{a+m}{2} \xi i - iu \right) \right\}
\]

\[
(3.5)
\]

for \( (a, m) \in \{1, 2, \ldots, r + 1\} \times \mathbb{Z}_{\geq 1} \cup \mathbb{Z}_{\geq 1} \times \{1, 2, \ldots, s + 1\} \).

In the next subsection, we will consider specializations of this system of NLIE (3.6).

### 3.1 The nonlinear integral equations for \( \xi = 1 \)

Let us consider the NLIE (3.6) for \( \xi = 1 \) and \( m = 1 \). Taking note on the fact \( \tilde{T}_0^{(a)}(v) = 1 \) (cf.(2.21)), we can drop the first terms in the two brackets \{\ldots\} in (3.6) since they have no poles at \( y = 0 \). Then the NLIE (3.6) reduce to the following NLIE on \( \tilde{T}_1^{(a)}(v) = \lim_{N \to \infty} \tilde{T}_1^{(a)}(v) \) after the Trotter limit.
$N \to \infty$ with $u = -\frac{J \sinh \eta}{\eta NT}$.

$$
\mathcal{T}_1^{(a)}(v) = Q_1^{(a)} + \oint_{C_1^{(a)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(a-1)}(y + i\frac{a}{2}) \mathcal{T}_1^{(a+1)}(y + i\frac{a}{2})}{\tan \eta(v - y - i\frac{a+1}{2}) \mathcal{T}_1^{(a)}(y + i\frac{a-1}{2})} \\
+ \oint_{C_1^{(a)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(a-1)}(y - i\frac{a}{2}) \mathcal{T}_1^{(a+1)}(y - i\frac{a}{2})}{\tan \eta(v - y + i\frac{a+1}{2}) \mathcal{T}_1^{(a)}(y - i\frac{a-1}{2})}
$$

for $a \in \mathbb{Z}_{\geq 1}$, (3.7)

where the contour $C_1^{(a)}$ (resp. $\overline{C}_1^{(a)}$) is a counterclockwise closed loop around $y = 0$ (resp. $y = 0$) which satisfies the condition $y \neq -\frac{a+1}{2} + \frac{\pi n}{\eta}$ (resp. $y \neq -\frac{a+1}{2} + \frac{\pi n}{\eta}$) and does not surround $z_1^{(a)} = -\frac{a-1}{2} + \frac{\pi n}{\eta}$, $-(a+1)i + \frac{\pi n}{\eta}$, $\frac{\pi k}{\eta}$ (resp. $z_1^{(a)} = -\frac{a-1}{2} + \frac{\pi n}{\eta}$, $-(a+1)i + \frac{\pi n}{\eta}$, $\frac{\pi k}{\eta}$); ($n \in \mathbb{Z}$, $k \in \mathbb{Z} - \{0\}$). Here we put the zeros of $\mathcal{T}_1^{(a)}(v)$ as $\{z_1^{(a)}\}$: $\mathcal{T}_1^{(a)}(z_1^{(a)}) = 0$. $\mathcal{T}_1^{(0)}(v)$ is a known function:

$$
\mathcal{T}_1^{(0)}(v) = \lim_{N \to \infty} \tilde{T}_1^{(0)}(v) = \exp \left( \frac{2J \sinh \eta}{T(\cosh \eta - \cos(2\eta v))} \right).
$$

Note that (3.7) are an infinite number of couple NLIE if $s \in \mathbb{Z}_{\geq 0}$. This situation is quite different from the $U_q(sl(r + 1))$ case [16, 14, 9]. However these NLIE are not independent, then we will take the first $r + s + 1$ of them ((3.7) for $a \in \{1, 2, \ldots r + s + 1\}$). The NLIE for $a = r + s + 1$ contains $\mathcal{T}_1^{(r+s+2)}(v)$, then we will eliminate this by using the relation (2.30), where $W_m^{(a)}(v) = 1$ for $\xi = 1$.

$$
\mathcal{T}_1^{(a)}(v) = Q_1^{(a)} + \oint_{C_1^{(a)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(a-1)}(y + i\frac{a}{2}) \mathcal{T}_1^{(a+1)}(y + i\frac{a}{2})}{\tan \eta(v - y - i\frac{a+1}{2}) \mathcal{T}_1^{(a)}(y + i\frac{a-1}{2})} \\
+ \oint_{C_1^{(a)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(a-1)}(y - i\frac{a}{2}) \mathcal{T}_1^{(a+1)}(y - i\frac{a}{2})}{\tan \eta(v - y + i\frac{a+1}{2}) \mathcal{T}_1^{(a)}(y - i\frac{a-1}{2})}
$$

for $a \in \{1, 2, \ldots r + s\}$, (3.9)

$$
\mathcal{T}_1^{(r+s+1)}(v) = Q_1^{(r+s+1)} + \oint_{C_1^{(r+s+1)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(r+s)}(y + i\frac{r+s+1}{2}) \mathcal{F}(y + i\frac{r+s+1}{2})}{\tan \eta(v - y - i\frac{r+s+2}{2}) \mathcal{T}_1^{(r+s+1)}(y + i\frac{r+s+1}{2})} \\
+ \oint_{C_1^{(r+s+1)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(r+s)}(y - i\frac{r+s}{2}) \mathcal{F}(y - i\frac{r+s+1}{2})}{\tan \eta(v - y + i\frac{r+s+2}{2}) \mathcal{T}_1^{(r+s+1)}(y - i\frac{r+s+1}{2})},
$$

(3.10)

$$
\mathcal{F}(v) = \lim_{N \to \infty} \tilde{T}_1^{(r+s+2)}(v) = \frac{A_1(v) - \zeta A_2(v)}{(-1)^s A_3(v) + (-1)^s \zeta A_4(v)},
$$

(3.11)
where

\[
A_1(v) = \det_{1 \leq j,k \leq s+2} \left( f_{j,k} \left( v - \frac{j + k - s - 3}{2} \right) \right) \quad (3.12)
\]

\[
f_{j,k}(v) = T_1^{(r+1+j-k)}(v) \quad \text{for} \quad (j,k) \neq (s+2,1), \quad f_{s+2,1}(v) = 0,
\]

\[
A_2(v) = \det_{1 \leq j,k \leq s+1} \left( g_{j,k} \left( v - \frac{j + k - s - 2}{2} \right) \right) \quad (3.13)
\]

\[
g_{j,k}(v) = T_1^{(r+2+j-k)}(v) \quad \text{for} \quad (j,k) \neq (s+1,1), \quad g_{s+1,1}(v) = 0,
\]

\[
A_3(v) = \det_{1 \leq j,k \leq s} \left( T_1^{(r+j-k)}(v) \left( v - \frac{j + k - s}{2} \right) \right) \quad (3.14)
\]

\[
A_4(v) = \det_{1 \leq j,k \leq s} \left( T_1^{(r+j-k+1)}(v) \left( v - \frac{j + k - s - 1}{2} \right) \right) \quad (3.15)
\]

If \( s = -1 \), then \( A_1(v) = A_4(v) = 0 \) and \( A_2(v) = A_3(v) = 1 \), and consequently \( (3.10) \) reduces to \( \mathcal{F}(v) = T_1^{(r+1)}(v) = Q_1^{(r+1)} = \zeta = e^{\frac{\mu_1 + \cdots + \mu_{r+1}}{r}} \), where the determinants should be interpreted as \( \det_{1 \leq j,k \leq 0(\cdots)} = 1, \det_{1 \leq j,k \leq -1(\cdots)} = 0 \). Thus \( (3.9) \) and \( (3.10) \) reduce to the NLIE for \( U_q(sl(r+1)) \) in [16]. In particular for \( s = 0 \) \( (U_q(sl(r+1))) \) case, we can use \( (2.24) \):

\[
\mathcal{T}_1^{(a)}(v) = Q_1^{(a)} + \int_{C_1^{(a)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(a-1)}(y + \frac{ia}{2}) \mathcal{T}_1^{(a+1)}(y + \frac{ia}{2})}{\tan \eta(v - y - \frac{(a+1)(a-1)}{2})} = \int_{C_1^{(a)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_1^{(a-1)}(y - \frac{ia}{2}) \mathcal{T}_1^{(a+1)}(y - \frac{ia}{2})}{\tan \eta(v - y + \frac{a(a+1)}{2})}
\]

\[
\mathcal{T}_1^{(r+1)}(v) = Q_1^{(r+1)} \quad \text{for} \quad a \in \{1, 2, \ldots, r\}, \quad (3.16)
\]

\[
\mathcal{T}_1^{(r)}(v) = \mathcal{T}_1^{(r)}(v) \quad \text{for} \quad a \in \{1, 2, \ldots, r\}, \quad (3.17)
\]

The free energy per site is given by a solution of these NLIE \( (3.9)-(3.17) \)

\[
f = J \cosh \eta - T \log \mathcal{T}_1^{(1)}(0). \quad (3.18)
\]

In these NLIE \( (3.9)-(3.17) \), the number of unknown functions and equations is \( r + s + 1 \), which contrasts with TBA equations \[42, 43, 44, 26, 45\].
3.2 The nonlinear integral equations for $\xi = -1$

Next, let us consider the NLIE (3.6) for $\xi = -1$ and $a = 1$. Taking note on the fact $\tilde{T}_m^{(0)}(v) = 1$ (cf. (2.22)), we can drop the second terms in the two brackets $\{ \cdots \}$ in (3.6) since they have no poles at $y = 0$. Then the NLIE (3.6) reduce to the following NLIE on $T_m^{(1)}(v)$ after the Trotter limit $N \to \infty$ with $u = -\frac{J\sinh \eta}{\eta N T}$.

$$T_m^{(1)}(v) = Q_m^{(1)} + \int_{C_m^{(1)}} dy \frac{\eta T_{m-1}^{(1)}(y - \frac{im}{2}) T_{m+1}^{(1)}(y - \frac{im}{2})}{2\pi i \tan \eta (v - y + \frac{i(m+1)}{2}) T_m^{(1)}(y - \frac{i(m-1)}{2})}$$

$$+ \int_{C_m^{(1)}} dy \frac{\eta T_{m-1}^{(1)}(y + \frac{im}{2}) T_{m+1}^{(1)}(y + \frac{im}{2})}{2\pi i \tan \eta (v - y - \frac{i(m+1)}{2}) T_m^{(1)}(y + \frac{i(m-1)}{2})}$$

for $m \in \mathbb{Z}_{\geq 1}$,

(3.19)

where

$$T_0^{(1)}(v) = \lim_{N \to \infty} \tilde{T}_0^{(1)}(v) = \exp \left( -\frac{2J(\sinh \eta)^2}{T(\cosh \eta - \cos(2\eta v))} \right),$$

(3.20)

and the contour $C_m^{(1)}$ (resp. $C_m^{(1)}$) is a counterclockwise closed loop around $y = 0$ (resp. $y = 0$) which satisfies the condition $y \neq v + \frac{m+1}{2}i + \frac{\pi n}{\eta}$ (resp. $y \neq v - \frac{m+1}{2}i + \frac{\pi n}{\eta}$) and does not surround $z_m^{(1)} + \frac{m-1}{2}i + \frac{\pi n}{\eta}, (1 + m)i + \frac{\pi n}{\eta}, \frac{\pi k}{\eta}$ (resp. $z_m^{(1)} - \frac{m-1}{2}i + \frac{\pi n}{\eta}, -(1 + m)i + \frac{\pi n}{\eta}, \frac{\pi k}{\eta}$) ($n \in \mathbb{Z}, k \in \mathbb{Z} - \{0\}$). Here $\{z_m^{(1)}\}$ are zeros of $T_m^{(1)}(v)$: $T_m^{(1)}(z_m^{(1)}) = 0$. These are an infinite number of coupled NLIE. We can reduce them as $\xi = 1$ case. By using (2.23) in the limit $N \to \infty$, we can reduce (3.19) as follows, where $Z_m^{(a)}(v) = 1$ for $\xi = -1$.

$$T_m^{(1)}(v) = Q_m^{(1)} + \int_{C_m^{(1)}} dy \frac{\eta T_{m-1}^{(1)}(y - \frac{im}{2}) T_{m+1}^{(1)}(y - \frac{im}{2})}{2\pi i \tan \eta (v - y + \frac{i(m+1)}{2}) T_m^{(1)}(y - \frac{i(m-1)}{2})}$$

$$+ \int_{C_m^{(1)}} dy \frac{\eta T_{m-1}^{(1)}(y + \frac{im}{2}) T_{m+1}^{(1)}(y + \frac{im}{2})}{2\pi i \tan \eta (v - y - \frac{i(m+1)}{2}) T_m^{(1)}(y + \frac{i(m-1)}{2})}$$

for $m \in \{1, 2, \ldots r + s\}$,

(3.21)

$$T_{r+s+1}^{(1)}(v) = Q_{r+s+1}^{(1)} + \int_{C_{r+s+1}^{(1)}} dy \frac{\eta T_{r+s}^{(1)}(y - \frac{i(r+s+1)}{2}) G(y - \frac{i(r+s+1)}{2})}{2\pi i \tan \eta (v - y + \frac{i(r+s+2)}{2}) T_{r+s+1}^{(1)}(y - \frac{i(r+s)}{2})}$$

$$+ \int_{C_{r+s+1}^{(1)}} dy \frac{\eta T_{r+s}^{(1)}(y + \frac{i(r+s+1)}{2}) G(y + \frac{i(r+s+1)}{2})}{2\pi i \tan \eta (v - y - \frac{i(r+s+2)}{2}) T_{r+s+1}^{(1)}(y + \frac{i(r+s)}{2})},$$

(3.22)
where
\[
A_5(v) = \det_{1 \leq j,k \leq r+2} \left( h_{j,k} \left( \frac{v - r + 3 - j - k}{2} \right) \right)
\] (3.24)

\[
h_{j,k}(v) = \mathcal{T}_{s+1+j-k}^{(1)}(v) \quad \text{for} \quad (j,k) \neq (2+r,1), \quad h_{r+2,1}(v) = 0,
\]

\[
A_6(v) = \det_{1 \leq j,k \leq r+1} \left( b_{j,k} \left( \frac{v - r + 2 - j - k}{2} \right) \right)
\] (3.25)

\[
b_{j,k}(v) = \mathcal{T}_{s+2+j-k}^{(1)}(v) \quad \text{for} \quad (j,k) \neq (r+1,1), \quad b_{r+1,1}(v) = 0,
\]

\[
A_7(v) = \det_{1 \leq j,k \leq r} \left( \mathcal{T}_{s+j-k}^{(1)} \left( \frac{v - r + 2 - j - k}{2} \right) \right),
\]

\[
A_8(v) = \det_{1 \leq j,k \leq r} \left( \mathcal{T}_{s+1+j-k}^{(1)} \left( \frac{v - r + 1 - j - k}{2} \right) \right),
\]

(3.26)

(3.27)

where \( \mathcal{T}_m^{(1)}(v) = 0 \) for \( m < 0 \).

In particular for \( r = 0 \) (\( U_q(\hat{s}l(1|s+1)) \) case), we can use (2.24):

\[
\mathcal{T}_m^{(1)}(v) = Q_m^{(1)} + \int_{C_m^{(s+1)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_{m-1}^{(1)}(y - \frac{im}{2}) \mathcal{T}_{m+1}^{(1)}(y - \frac{im}{2})}{\tan \eta(v - y - \frac{im}{2}) \mathcal{T}_{m}^{(1)}(y - \frac{im}{2})}
\] (3.28)

\[
+ \int_{C_1^{(1)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_{m-1}^{(1)}(y + \frac{im}{2}) \mathcal{T}_{m+1}^{(1)}(y + \frac{im}{2})}{\tan \eta(v - y - \frac{im}{2}) \mathcal{T}_{m}^{(1)}(y + \frac{im}{2})}
\]

for \( m \in \{1,2,\ldots,s\} \).

\[
\mathcal{T}_{s+1}^{(1)}(v) = Q_{s+1}^{(1)} + \int_{C_{s+1}^{(s+1)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_{s}^{(1)}(y - \frac{i(s+1)}{2}) \mathcal{T}_{s+1}^{(1)}(y - \frac{i(s+2)}{2})}{\tan \eta(v - y - \frac{i(s+2)}{2})(\zeta^{-1} + \mathcal{T}_{s}^{(1)}(y - \frac{i(s+1)}{2}))}
\] (3.29)

\[
+ \int_{C_{s+1}^{(s+1)}} \frac{dy}{2\pi i} \frac{\eta \mathcal{T}_{s}^{(1)}(y + \frac{i(s+1)}{2}) \mathcal{T}_{s+1}^{(1)}(y + \frac{i(s+2)}{2})}{\tan \eta(v - y - \frac{i(s+2)}{2})(\zeta^{-1} + \mathcal{T}_{s}^{(1)}(y + \frac{i(s+1)}{2}))}
\]

The free energy per site is given by a solution of these NLIE (3.24)-(3.29)

\[
f = -J \cosh \eta - T \log \mathcal{T}_1^{(1)}(0).
\] (3.30)

In some sense, these NLIE are ‘dual’ to the ones in the previous section. The NLIE (3.24)-(3.29) have only \( r + s + 1 \) unknown functions. These NLIE have never been considered before even for \( U_q(\hat{s}l(2)) \) case.
4 High temperature expansions

In this section, we will calculate the high temperature expansion of the free energy from our new NLIE. For large $T/|J|$, we assume the following expansion:

\[
T^{(a)}_1(v) = \exp\left(\sum_{n=0}^{\text{deg}} b_n^{(a)}(v)\left(\frac{J}{T}\right)^n + O\left(\left(\frac{J}{T}\right)^{\text{deg}+1}\right)\right)
\]

\[
= Q_1^{(a)}\left\{1 + b_1^{(a)}(v)\frac{J}{T} + \left(b_2^{(a)}(v) + \frac{(b_1^{(a)}(v))^2}{2}\right)\left(\frac{J}{T}\right)^2 + \left(b_3^{(a)}(v) + b_2^{(a)}(v)b_1^{(a)}(v) + \frac{(b_1^{(a)}(v))^3}{6}\right)\left(\frac{J}{T}\right)^3 + \cdots \right\} + O\left(\left(\frac{J}{T}\right)^{\text{deg}+1}\right),
\]

where $b_0^{(a)}(v) = \log Q_1^{(a)}$. Here we do not expand $\{Q_1^{(b)}\}_{b \geq 1}$ with respect to $\frac{J}{T}$. Thus the coefficients $\{b_n^{(a)}(v)\}$ themselves depend on $\frac{1}{T}$. In this sense, our high temperature expansion formula is different from ordinary one. Substituting this (4.1) into some of the NLIE (3.7)-(3.17), we can calculate the coefficients $\{b_n^{(a)}(v)\}$ up to the order of $n = \text{deg}$. Note that we only need $\{b_n^{(1)}(0)\}$ to calculate the free energy (3.18). Taking note on this fact, firstly we use \(^5\) a subset (NLIE for $a \in \{1, 2, \ldots, \text{deg}\}$) of the non-reduced NLIE (3.7) rather than the reduced NLIE (3.9)-(3.17). We have observed that $b_n^{(1)}(0)$ can be expressed in terms of \(^6\) $Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(n+1)}$. We have calculated the coefficients by using Mathematica. As examples, we shall enumerate the

\(^5\)As for numerical calculations of the free energy, we expect that the reduced NLIE (3.9)-(3.17) are easier to use than the non-reduced NLIE (3.7).

\(^6\)For $s = -1$ case, they are $Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(d)}$: $d = \min(n + 1, r + 1)$ since $Q_1^{(a)} = 0$ if $a \geq r + 2$. 

18
coefficients \( \{ b_0^{(1)}(0) \} \) up to the order of 5, where we put \( \Delta = \cosh \eta \).

\[
b_1^{(1)}(0) = \frac{2\Delta Q_1^{(2)}}{Q_1^{(1)^2}} ,
\]

\[
b_2^{(1)}(0) = -\frac{6\Delta^2 Q_1^{(2)^2}}{Q_1^{(1)^4}} + \frac{(2\Delta^2 + 1) Q_1^{(2)} Q_1^{(1)^2}}{Q_1^{(1)^2}} + \frac{(4\Delta^2 - 1) Q_1^{(3)}}{Q_1^{(1)^4}} ,
\]

\[
b_3^{(1)}(0) = \frac{80Q_1^{(2)^3} \Delta^3}{3Q_1^{(1)^6}} + \frac{8Q_1^{(3)} \Delta^3}{Q_1^{(1)^3}} + \frac{(\frac{4\Delta^3}{3} + 2\Delta) Q_1^{(2)}}{Q_1^{(1)^2}}
\]

\[
+ \frac{(8\Delta - 32\Delta^3) Q_1^{(2)} Q_1^{(3)}}{Q_1^{(1)^3}} + \frac{(-12\Delta^3 - 6\Delta) Q_1^{(2)^2} + (8\Delta^3 - 4\Delta) Q_1^{(4)}}{Q_1^{(1)^4}} ,
\]

\[
b_4^{(1)}(0) = -\frac{140\Delta^4 Q_1^{(2)^4}}{Q_1^{(1)^8}} + \frac{(240\Delta^4 - 60\Delta^2) Q_1^{(3)} Q_1^{(2)^2}}{Q_1^{(1)^7}}
\]

\[
+ \frac{(\frac{2\Delta^4}{3} + 2\Delta^2 + \frac{4}{3}) Q_1^{(2)^2}}{Q_1^{(1)^2}} + \frac{(\frac{28\Delta^4}{3} + 14\Delta^2 - \frac{1}{3}) Q_1^{(3)}}{Q_1^{(1)^3}}
\]

\[
+ \frac{(-14\Delta^4 - \frac{56\Delta^2}{3} - \frac{3}{2}) Q_1^{(2)^2} + (24\Delta^4 - 8\Delta^2 - 1) Q_1^{(4)}}{Q_1^{(1)^4}}
\]

\[
+ \frac{(80\Delta^4 + 40\Delta^2) Q_1^{(2)^3} + (40\Delta^2 - 80\Delta^4) Q_1^{(4)} Q_1^{(2)}}{Q_1^{(1)^6}}
\]

\[
+ \frac{(-40\Delta^4 + 20\Delta^2 - \frac{5}{2}) Q_1^{(3)^2}}{Q_1^{(1)^6}}
\]

\[
+ \frac{(-96\Delta^4 - 8\Delta^2 + 4) Q_1^{(2)} Q_1^{(3)} + (16\Delta^4 - 12\Delta^2 + 1) Q_1^{(5)}}{Q_1^{(1)^5}} ,
\]
\[ b_5^{(1)}(0) = \frac{4032\Delta^5 Q_1^{(2)^5}}{5Q_1^{(1)^6}} + \frac{(448\Delta^3 - 1792\Delta^5) Q_1^{(3)^3}}{Q_1^{(1)^9}} \]
\[ + \left( \frac{44\Delta^3 + 44\Delta^3 + \frac{\Delta}{2}}{Q_1^{(1)^2}} \right) Q_1^{(2)} + \left( \frac{8\Delta^5 + 10\Delta^3 + \frac{\Delta}{2}}{Q_1^{(1)^3}} \right) Q_1^{(3)} \]
\[ + \left( \frac{-12\Delta^5 - 30\Delta^3 - 8\Delta}{Q_1^{(1)^4}} \right) Q_1^{(2)^2} + \left( \frac{40\Delta^5 - 6\Delta}{Q_1^{(1)^4}} \right) Q_1^{(4)} \]
\[ + \left( \frac{-560\Delta^5 - 280\Delta^3}{Q_1^{(1)^8}} \right) Q_1^{(2)^4} + \left( \frac{672\Delta^5 - 336\Delta^3}{Q_1^{(1)^8}} \right) Q_1^{(4)^2} \]
\[ + \left( \frac{672\Delta^5 - 336\Delta^3 + 42\Delta}{Q_1^{(1)^8}} \right) Q_1^{(3)^2} Q_1^{(2)} \]
\[ + \left( \frac{-160\Delta^5 - 100\Delta^3 + 11\Delta}{Q_1^{(1)^8}} \right) Q_1^{(2)^2} Q_1^{(3)} + \left( \frac{64\Delta^5 - 40\Delta^3}{Q_1^{(4)^2}} \right) Q_1^{(1)^5} \]
\[ + \left( \frac{960\Delta^5 + 120\Delta^3 - 60\Delta}{Q_1^{(1)^8}} \right) Q_1^{(2)^2} Q_1^{(3)} + \left( \frac{-240\Delta^5 + 12\Delta}{Q_1^{(1)^8}} \right) Q_1^{(2)^2} Q_1^{(4)} \]
\[ + \left( \frac{-192\Delta^5 + 144\Delta^3 - 24\Delta}{Q_1^{(1)^7}} \right) Q_1^{(3)^2} Q_1^{(4)} \]
\[ + \left( \frac{400\Delta^5 + 500\Delta^3 + 20\Delta}{Q_1^{(1)^6}} \right) Q_1^{(2)^3} Q_1^{(4)} + \left( \frac{-320\Delta^5 + 80\Delta^3 + 30\Delta}{Q_1^{(1)^6}} \right) Q_1^{(4)^2} Q_1^{(2)} \]
\[ + \left( \frac{40\Delta^3 - 160\Delta^5}{Q_1^{(1)^6}} \right) Q_1^{(3)^2} + \left( \frac{32\Delta^5 - 32\Delta^3 + 6\Delta}{Q_1^{(1)^6}} \right) Q_1^{(6)}. \]

In deriving these coefficients (4.2)-(4.6), we did not assume (2.40). Of course, when one calculate the free energy of the model, one must assume (2.40) and (2.48). We can also rewrite the coefficient \( b_n^{(1)}(0) \) in terms of \( Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(d)} \) and \( \zeta (d = \min(n + 1, r + s + 2)) \) since \( Q_1^{(a)} \) for \( a \in \mathbb{Z}_{\geq r+s+2} \) can be written in terms of \( Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(r+s+1)} \) and \( \zeta \) due to the relation (2.30) in the limit \( v \to i\eta^{-1} \infty \) (see also an example: (2.41)-(2.43)). If \( b_n^{(a)}(0) \) is written in terms of \( Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_1^{(d)} \) and \( \zeta (d = \min(n + 1, r + s + 1)) \), it should be the coefficient of the high temperature expansion directly derived from the reduced NLIE (3.9)-(3.17). Of course these two expressions of the coefficient

\[ Q_1^{(r+1)} = \zeta \text{ if } s = -1. \]
Figure 1: Temperature dependence of the high temperature expansion of the specific heat $C$ for the rank 2 case ($r + s = 1$, $J = 1$, $q = 1$, $\mu_a = 0$ ($a \in B$)). We have plotted plan series (dotted lines) of $C$ in Appendix and their Pade approximations of order $[n,d]$ (numerator: a degree $n$ polynomial of $1/T$, denominator: a degree $d$ polynomial of $1/T$) by using Mathematica: each line denotes $C$ for $sl(3|0)$ with [20,20] (thin), $sl(2|1)$ with [17,17] (medium), $sl(1|2)$ with [17,17] (thick), $sl(0|3)$ [20,20] (dashed thick) respectively. We have also plotted (thick dots) a result of numerical calculation from another NLIE by Jüttner and Klümper [53] for the $sl(2|1)$ case. $C$ for the $sl(3|0)$ case was also considered in [57, 6].

$b_n^{(1)}(0)$ are equivalent under the relations (2.40) and (2.48).

For fixed values of parameters, we have calculated the high temperature expansion for much higher order (see, appendix). We have plotted the high temperature expansion of the specific heat (Figure 1). Here we have adopted the Pade approximation method. There is a duality among the specific heats with respect to interchange of $r$ and $s$. In particular, $r = s$ case is self-dual, then the specific heat becomes an even function of $T$ (see (A.1.6)). In Figure 1 we have also plotted a result of a numerical calculation by another NLIE [53]. We find a good agreement between our result and their result except for very low temperature region.

We can also calculate the high temperature expansion from the NLIE for
Figure 2: Temperature dependence of the high temperature expansion of the specific heat $C$ for the rank 3 case ($r + s = 2, J = 1, q = 1, \mu_a = 0$ ($a \in B$)). We have plotted plan series (dotted lines) of $C$ in Appendix and their Pade approximations of order $[n,d]$ (numerator: a degree $n$ polynomial of $1/T$, denominator: a degree $d$ polynomial of $1/T$): each line denotes $C$ for $sl(4|0)$ with [19,20] (thin), $sl(3|1)$ with [17,17] (medium), $sl(2|2)$ with [16,16] (thick), $sl(1|3)$ with [17,17] (dashed medium), $sl(0|4)$ with [18,21] (dashed thick) respectively. $C$ for the $sl(4|0)$ case was also considered in [57].
Figure 3: Temperature dependence of the high temperature expansion of the specific heat $C$ for the rank 4 case ($r + s = 3$, $J = 1$, $q = 1$, $\mu_a = 0$ ($a \in B$)). We have plotted plan series (dotted lines) of $C$ in Appendix and their Pade approximations of order $[n,d]$ (numerator: a degree $n$ polynomial of $1/T$, denominator: a degree $d$ polynomial of $1/T$): each line denotes $C$ for $sl(5|0)$ with $[17,21]$ (thin), $sl(4|1)$ with $[16,18]$ (medium), $sl(3|2)$ with $[17,17]$ (thick), $sl(2|3)$ with $[16,17]$ (dashed thin), $sl(1|4)$ with $[16,18]$ (dashed medium), $sl(0|5)$ with $[17,21]$ (dashed thick) respectively.
\(\xi = -1\) in subsection 3.2. Similar to \(\xi = 1\) case, we assume

\[
\tau_{m}^{(1)}(v) = \exp \left( \sum_{n=0}^{\text{deg}} \tilde{b}_{m,n}(v) \left( \frac{J}{T} \right)^{n} + O(\left( \frac{J}{T} \right)^{\text{deg}+1}) \right),
\]

(4.7)

where \(\tilde{b}_{m,0}(v) = \log Q_{m}^{(1)}\). Here we do not expand \(\{Q_{k}^{(1)}\}_{k \geq 1}\) with respect to \(\frac{J}{T}\). (4.1) for \(a = 1\) should coincide with (4.7) for \(m = 1\) up to a factor from the normalization function (2.17). Thus we have

\[
b_{n}^{(1)}(0) = \tilde{b}_{1,n}(0) + 2\Delta \delta_{n,1}
\]

(4.8)

Due to symmetry between the NLIE for \(\xi = 1\) and the one for \(\xi = -1\), the following relation follows:

\[
\tilde{b}_{1,n}(0) = (-1)^{n}b_{n}^{(1)}(0)|_{Q_{1}^{(a)} \to Q_{a}^{(1)}} \text{ for } a \geq 1.
\]

(4.9)

For example, (4.8) and (4.9) for \(n = 1\) and (4.2) reproduce the \(Q\)-system (2.44) for \((a,m) = (1,1)\). From the relations (4.8) and (4.9) for \(n = 2\) and (4.3), we obtain identities among characters

\[
-3Q_{1}^{(2)}Q_{1}^{(1)} + 2Q_{1}^{(3)}Q_{1}^{(1)} = -3Q_{2}^{(1)}Q_{1}^{(1)} + Q_{2}^{(1)}Q_{1}^{(1)} + 2Q_{3}^{(1)}Q_{1}^{(1)},
\]

(4.10)

\[
Q_{1}^{(2)}Q_{1}^{(1)} - Q_{1}^{(3)} = Q_{2}^{(1)}Q_{1}^{(1)} - Q_{3}^{(1)},
\]

(4.11)

where we have used the fact that \(Q_{m}^{(a)}\) does not depend on \(\Delta\). These relations can be proved from the relations (2.25), (2.26) and (2.40).

Some comments on references on the high temperature expansion are in order. The high temperature expansion of the free energy was calculated from the Takahashi’s NLIE for the XXX-model up to the order of 100 [46]; the XXZ-model up to the order of 99 [16]. As for the higher rank or higher spin case, we have some results [13, 14, 15, 16] from NLIE. In particular, our result on the \(sl(r+1)\) Uimin-Sutherland model in [14] was applied [17, 48, 49, 50, 51, 52] to spin ladder models and good agreement was seen between theoretical results and experimental data. We note that the coefficients (4.2)-(4.4) coincide with eqs. (4.14)-(4.16) in [16]. Note however that the coefficients in our paper are more general than the ones in [16] since the value of \(Q_{1}^{(a)}\) (2.40) was restricted to \(s = -1\) case in [16]. There are also several works on high temperature expansions by different methods (see for example, [53, 54, 55, 56, 57, 58]).

5 Concluding remarks

In this paper, we have derived NLIE which contain only \(r + s + 1\) unknown functions for the \(U(\hat{sl}(r+1|s+1))\) Perk-Schultz model. The key is a duality
for the auxiliary function (2.23) and the quantum (supersymmetric) Jacobi- 
Trudi and Giambelli formula (2.25) and (2.26). Although we assumed that 
$q$ is generic, we expect that our NLIE (at least reduced ones (3.9)-(3.17), 
(3.21)-(3.29)) will also be valid even for the case where $q$ is root of unity 
as we will not need to take into account truncation of the $T$-system. The 
high temperature expansion of the free energy in terms of characters was 
calculated from our NLIE.

There are NLIE with a finite number of unknown functions for algebras 
of arbitrary rank in different context [59, 60]. These NLIE are different from 
Takahashi-type. Whether one can generalize (or modify) their NLIE for finite 
temperature case is still not clear. A deeper understanding of this subject is 
desirable.

There is another kind of formulation of transfer matrices which is based 
on the graded formulation of the quantum inverse scattering method. In this 
formulation, the row-to-row transfer matrix is defined as a supertrace: 
$\hat{t}(v) = \text{str}_0(\hat{R}_{0L}(v) \cdots \hat{R}_{02}(v)\hat{R}_{01}(v))$, where the $R$-matrix is defined as 
$\hat{R}^{a_1,b_1}_{a_2,b_2}(v) = (-1)^{p(a_1)p(b_1)}R^{a_1,b_1}_{a_2,b_2}(v)$ and the graded tensor product is adopted. As far as 
the free energy (in the thermodynamic limit) is concerned, we think that 
there is no difference between this graded formulation and the one we have 
adopted.

Acknowledgments

The author would like to thank A. Klümper and K. Sakai for comments 
on a figure of specific heats. He also thank Y. Nagatani for a remark on 
programming of Mathematica.
Appendix: The high temperature expansion of the specific heat

We will list the high temperature expansion of the specific heat \( C_{s(r+1|s+1)} \) for the \( U_q(\hat{sl}(r + 1|s + 1)) \) Perk-Schultz model at \( q = 1 \), \( \mu_a = 0 \) (\( a \in B \)). Here we put \( t = \frac{j}{f} \). In this case, \( Q_{1}^{(a)} \) (cf. (2.30)) becomes

\[
Q_{1}^{(a)} = \sum_{j=0}^{\alpha} \binom{r+1}{j} \binom{a+s-j}{a-j},
\]

which is the dimension of \( a \)-th anti-(super)symmetric tensor representation of \( sl(r + 1|s + 1) \). If one substitute \( \Delta = 1 \) and the values of \( (r, s) \) into (4.2)- (4.6), one can recover (A.1.2)-(A.1.9) up to the order of 5 through \( C = -T \frac{\partial^2 E}{\partial T^2} \). A formula for \( r < s \) can be obtained from the relation \( C_{s(r+1|s+1)} = C_{s(r+1|s+1)}|_{t\rightarrow -t} \).

\[
\begin{align*}
C_{J(3)}(3;0) &= \frac{81}{9} + \frac{161}{27}t^3 - \frac{405}{27}t^4 - \frac{4045}{9}t^5 + \frac{12466}{3}t^6 + 12287t^7 + 43343t^8 + 49908t^9 + 120769t^{10} + \frac{555937}{3}t^{11} + \frac{3695379}{12}t^{12} + 433458857t^{13} + \cdots \nonumber \\
&+ \frac{2982227709}{14}t^{14}
\end{align*}
\]

\[
\begin{align*}
C_{J(3)}(3;1) &= \frac{81}{9} + \frac{161}{27}t^3 - \frac{405}{27}t^4 - \frac{4045}{9}t^5 + \frac{12466}{3}t^6 + 12287t^7 + 43343t^8 + 49908t^9 + 120769t^{10} + \frac{555937}{3}t^{11} + \frac{3695379}{12}t^{12} + 433458857t^{13} + \cdots \nonumber \\
&+ \frac{584353}{45}t^{14}
\end{align*}
\]

\[
\begin{align*}
C_{J(3)}(3;2) &= \frac{81}{9} + \frac{161}{27}t^3 - \frac{405}{27}t^4 - \frac{4045}{9}t^5 + \frac{12466}{3}t^6 + 12287t^7 + 43343t^8 + 49908t^9 + 120769t^{10} + \frac{555937}{3}t^{11} + \frac{3695379}{12}t^{12} + 433458857t^{13} + \cdots \nonumber \\
&+ \frac{584353}{45}t^{14}
\end{align*}
\]

\[
\begin{align*}
C_{J(3)}(3;3) &= \frac{81}{9} + \frac{161}{27}t^3 - \frac{405}{27}t^4 - \frac{4045}{9}t^5 + \frac{12466}{3}t^6 + 12287t^7 + 43343t^8 + 49908t^9 + 120769t^{10} + \frac{555937}{3}t^{11} + \frac{3695379}{12}t^{12} + 433458857t^{13} + \cdots \nonumber \\
&+ \frac{584353}{45}t^{14}
\end{align*}
\]

\[
\begin{align*}
C_{J(3)}(3;4) &= \frac{81}{9} + \frac{161}{27}t^3 - \frac{405}{27}t^4 - \frac{4045}{9}t^5 + \frac{12466}{3}t^6 + 12287t^7 + 43343t^8 + 49908t^9 + 120769t^{10} + \frac{555937}{3}t^{11} + \frac{3695379}{12}t^{12} + 433458857t^{13} + \cdots \nonumber \\
&+ \frac{584353}{45}t^{14}
\end{align*}
\]

\[
\begin{align*}
C_{J(3)}(3;5) &= \frac{81}{9} + \frac{161}{27}t^3 - \frac{405}{27}t^4 - \frac{4045}{9}t^5 + \frac{12466}{3}t^6 + 12287t^7 + 43343t^8 + 49908t^9 + 120769t^{10} + \frac{555937}{3}t^{11} + \frac{3695379}{12}t^{12} + 433458857t^{13} + \cdots \nonumber \\
&+ \frac{584353}{45}t^{14}
\end{align*}
\]
\[ C_{1}(21) = \frac{321}{2} + \frac{304}{27} + \frac{584}{217} + \frac{8320}{561} + \frac{736708}{17174} + \frac{1470644}{53144} + \frac{1468348918}{2391445} + \frac{1419515400}{27119432} + \frac{22823126005}{27119432} + \frac{33006989909}{3661236205} + \frac{860462113532}{790820721668} \]

\[ C_{1}(24) = \frac{152}{16} + \frac{153}{25} + \frac{435}{256} + \frac{955}{8192} + \frac{21917}{63816} + \frac{1729678}{65360} + \frac{30524459}{888128} + \frac{536848710}{22020096} + \frac{413815511}{2944146120} + \frac{24190001597}{834872190} + \frac{746297434613}{186625771008} + \frac{59210699497}{1616642000} \]

\[ C_{4}(40) = \frac{152}{16} + \frac{153}{25} + \frac{435}{256} + \frac{955}{8192} + \frac{21917}{63816} + \frac{1729678}{65360} + \frac{30524459}{888128} + \frac{536848710}{22020096} + \frac{413815511}{2944146120} + \frac{24190001597}{834872190} + \frac{746297434613}{186625771008} + \frac{59210699497}{1616642000} \]

\[ C_{4}(41) = \frac{152}{16} + \frac{153}{25} + \frac{435}{256} + \frac{955}{8192} + \frac{21917}{63816} + \frac{1729678}{65360} + \frac{30524459}{888128} + \frac{536848710}{22020096} + \frac{413815511}{2944146120} + \frac{24190001597}{834872190} + \frac{746297434613}{186625771008} + \frac{59210699497}{1616642000} \]

\[ A(1.3) = \sum_{n=2}^{14} b_n \]
\[ C_{s}(3|1) = \left( \frac{69^2}{64} - \frac{45^3}{128} - \frac{8745}{128} + \frac{4065}{128} + \frac{44705}{128} - \frac{273683}{128} + \frac{405927}{128} - \frac{22229515}{128} + \frac{359910583}{128} + \frac{21225460233}{128} + \frac{14333277601113}{128} \right) + \ldots \]

\[ C_{s}(2|2) = \left( \frac{9^2}{8} - \frac{305^1}{128} + \frac{4105^1}{1024} - \frac{1028409^1}{1024} + \frac{7654369^1}{1024} - \frac{77150078171^1}{1024} + \frac{210818365469^1}{1024} - \frac{360863729270302071^1}{1024} + \frac{14501013836240^1}{1024} + \frac{3656522554631240^1}{1024} \right) + \ldots \]
\[ C_{r}(50) = \frac{24r^2}{25} + \frac{48r^3}{125} + \frac{112r^4}{625} + \frac{350r^5}{3125} + \frac{8274r^6}{15625} + \frac{17844r^7}{78125} - \frac{1306457r^8}{62515625} + \frac{160692418r^9}{39062515625} + \frac{3091451869r^{10}}{1953125390625} + \frac{252294795727190428811r^{11}}{46142578125390625} - 4227122266577r^{12} + 483388437500 \]

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]
\[\sum_{t=1}^{4} = \frac{64t^2}{2^5} + \frac{16t^3}{2^3} - \frac{159t^4}{2^5} - \frac{159300t^5}{2^6} + \frac{1061500t^6}{2^6} - \frac{13995060t^7}{2^7} + \frac{501937706t^8}{2^8} + \frac{1451470293t^9}{2^9} + O(t^{36}) \] (A.1.8)
\[
\begin{align*}
\omega_{\ell}(3/2) &= \frac{672^2}{625} + \frac{336^3}{3125} - \frac{172296^4}{78125} + \frac{304554^5}{9765625} + \frac{4080719^6}{6103515625} - \frac{19837763521^7}{30517578125} + \frac{1492774189466571^8}{1620172815625} + \frac{3004704966796875^9}{19073846328125} + \\
&\quad \frac{632031923446610911^{10}}{3004704966796875}
\end{align*}
\]

\[\begin{align*}
&= 450611104051995312500 + 30635898995526019977863317971^{11} + 47197534344910699471973057151154694151190425911^{12} + \\
&\quad 201674923511371499868697141948469250060808000 + 3413036547484807670116424560546875000000000000000
\end{align*}\]
References

[1] M. Takahashi, Thermodynamics of One-Dimensional Solvable models, (Cambridge University Press, Cambridge, 1999).

[2] C. Destri and H. J. de Vega, Phys. Rev. Lett. 69 (1992) 2313-2317.

[3] A. Klümper, Z. Phys. B91 (1993) 507-519.

[4] A. Klümper, Ann. Physik 1 (1992) 540-553.

[5] A. Klümper, T. Wehner and J. Zittartz, J. Phys. A: Math. Gen. 30 (1997) 1897-1912.

[6] A. Fujii and A. Klümper, Nucl. Phys. B546 (1999) 751-764; cond-mat/9811234.

[7] Recently NLIE for the rank 3 case have been proposed. J. Damerau, a seminar in Bergische Universität Wuppertal, July (2005).

[8] M. Takahashi, in Physics and Combinatorics, eds. A. N. Kirillov and N. Liskova, (2001) 299-304 (World Scientific, Singapore); cond-mat/0010486.

[9] M. Takahashi, M. Shiroishi and A. Klümper, J. Phys. A: Math. Gen. 34 (2001) L187–L194; cond-mat/0102027.

[10] A. N. Kirillov and N. Yu. Reshetikhin, J. Phys. A: Math. Gen. 20 (1987) 1565-1585.

[11] M. Suzuki, Phys. Rev. B31 (1985) 2957-2965.

[12] G. Kato and M. Wadati, J. Math. Phys. 43 (2002) 5060-5078; cond-mat/0212325.

[13] Z. Tsuboi, Phys. Lett. B544 (2002) 222-230; math-ph/0209024.

[14] Z. Tsuboi, J. Phys. A: Math. Gen. 36 (2003) 1493-1507; cond-mat/0212280.

[15] Z. Tsuboi, J. Phys. A: Math. Gen. 37 (2004) 1747–1758; cond-mat/0308333.

[16] Z. Tsuboi and M. Takahashi, J. Phys. Soc. Jpn. 74 (2005) 898; cond-mat/0412698.
[17] J. H. H. Perk and C. L. Schultz, Phys. Lett. 84A (1981) 407-410.
[18] C. L. Schultz, Physica A122 (1983) 71-88.
[19] Z. Tsuboi, J. Phys. A: Math. Gen. 30 (1997) 7975-7991.
[20] Z. Tsuboi, Physica A 252 (1998) 565-585.
[21] H. Yamane, Publ. RIMS, Kyoto Univ. 35 (1999) 321-390; errata: RIMS, Kyoto Univ. 37 (2001) 615-619; q-alg/9603015
[22] M. Suzuki and M. Inoue, Prog. Theor. Phys. 78 (1987) 787-799.
[23] T. Koma, Prog. Theor. Phys. 78 (1987) 1213–1218.
[24] J. Suzuki, Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn. 59 (1990) 2667-2680.
[25] G. Jüttner, A. Klümper and J. Suzuki, Nucl. Phys. B487 (1997) 650-674; cond-mat/9611058
[26] G. Jüttner, A. Klümper and J. Suzuki, Nucl. Phys. B512 (1998) 581-600; hep-th/9707074
[27] G.V. Uimin, JETP Lett. 12 (1970) 225-228.
[28] B. Sutherland, Phys. Rev. B12 (1975) 3795-3805.
[29] F. Göhmann, A. Klümper and A. Seel, J. Phys. A: Math. Gen. 37 (2004) 7625-7651; hep-th/0405089
[30] N. Yu. Reshetikhin, Sov. Phys. JETP 57 (1983) 691-696.
[31] A. Kuniba and J. Suzuki, Commun. Math. Phys. 173 (1995) 225-264; hep-th/9406180
[32] O. Babelon, H. J. de Vega and C-M. Viallet, Nucl. Phys. B200 (1982) 266-280.
[33] Z. Tsuboi, J. Phys. A: Math. Gen. 31 (1998) 5485-5498.
[34] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, Lett. Math. Phys. 5 (1981) 393-403.
[35] V. V. Bazhanov and N. Reshetikhin, J. Phys. A Math. Gen. 23 (1990) 1477-1492.
[36] A. Kuniba, T. Nakanishi and J. Suzuki, Int. J. Mod. Phys. A9 (1994) 5215-5266; hep-th/9309137.

[37] A. Kuniba, Y. Ohta and J. Suzuki, J.Phys. A28 (1995) 6211-6226; hep-th/9506167.

[38] A. N. Kirillov, J. Sov. Math. 47 (1989) 2450-2459.

[39] A. N. Kirillov and N. Yu. Reshetikhin, J. Sov. Math. 52 (1990) 3156-3164.

[40] T. Deguchi and P. P. Martin, Int. J. Mod. Phys. A7, Suppl. 1A (1992) 165-196.

[41] P. P. Martin and V. Rittenberg, Int. J. Mod. Phys. A7 Suppl. 1B (1992) 707-730.

[42] P. Schlottmann, Phys. Rev. B36 (1987) 5177-5185.

[43] P. Schlottmann, J. Phys.: Condens. Matter 4 (1992) 7565-7578.

[44] F. H. L. Essler and V. E. Korepin, Int. J. Mod. Phys. B8 (1994) 3243-3279; cond-mat/9307019.

[45] H. Saleur, Nucl.Phys. B578 (2000) 552-576; solv-int/9905007.

[46] M. Shiroishi and M. Takahashi, Phys. Rev. Lett. 89 (2002) 117201; cond-mat/0205180.

[47] M. T. Batchelor, X. W. Guan, N. Oelkers, K. Sakai, Z. Tsuboi and A. Foerster, Phys. Rev. Lett. 91 (2003) 217202; cond-mat/0309244.

[48] Zu-Jian Ying, I. Roditi, A. Foerster and B. Chen, Euro. Phys. J. B41 (2004) 67-74; cond-mat/0403520.

[49] Zu-Jian Ying, I. Roditi and Huan-Qiang Zhou, cond-mat/0405274.

[50] M. T. Batchelor, X. W. Guan and N. Oelkers, Phys. Rev. B70 (2004) 184408; cond-mat/0409310.

[51] M. T. Batchelor, X. W. Guan, N. Oelkers and A. Foerster, JSTAT (2004) P10017; cond-mat/0409311.

[52] M. T. Batchelor, X. W. Guan, N. Oelkers and Z. Tsuboi, Integrable models and quantum spin ladders: comparison between theory and experiment for the strong coupling compounds (review), in preparation.
[53] G. Jüttner and A. Klümper, Europhys. Lett. 37 (1997) 335-340.

[54] C. Destri and H. J. de Vega, Nucl. Phys. B438 [FS] (1995) 413-454; hep-th/9407117

[55] O. Rojas, S. M. de Souza and M. T. Thomaz, J. Math. Phys. 43 (2002) 1390-1407; hep-ph/0012368

[56] A. Bühler, N. Elstner and G. S. Uhrig, Eur. Phys. J B16 (2000) 475-486; cond-mat/0003221

[57] N. Fukushima, Y. Kuramoto, J. Phys. Soc. Jpn. 71 (2002) 1238-1241; cond-mat/0110550

[58] N. Fukushima, J. Stat. Phys. 111 (2003) 1049-1090; cond-mat/0212123

[59] P. Zinn-Justin, J. Phys. A: Math. Gen. 31 (1998) 6747-6770; hep-th/9712222

[60] P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Gen. 33 (2000) 8427-8441; hep-th/0008039