The Model of Plane-parallel Ship Movement Based on a Semi-linear System of Differential Equations using the Perturbation Method

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Abstract. The solution of a semi-linear dynamic system, which is the mathematical model of plane-parallel ship motion, is obtained using the perturbation method. It is shown that solution of linear dynamical systems depends on both harmonic functions and integral sine and cosine functions. The perturbation method is used to solve the plane-parallel motion of a ship from nonlinear equations that take into account the influence of static moment. The effect of this value on each of the velocity components is shown. The influence of the linear velocity component on the angular velocity is shown. The resulting solutions can be used to automate the movement of a vessel along a constrained fairway using an automatic control system.

1. Introduction

In recent years, it has become increasingly important to solve the problems of water transport automation. In the short term, it will be necessary to solve the problem of full automation of ship management – the so-called "unmanned navigation". Today, within the framework of state orders, work is already being carried out on the design of such automatic control systems for unmanned boats for various purposes, in which both private organizations and state developers participate. Starting in 2019, the goal of introducing unmanned technologies in water transport has been set as part of the Russian Transport strategy until 2030. Due to the complexity of the task when a ship is moving in a constrained fairway (river vessel, river–sea level vessel), the influence of many factors, and the need for quick decision-making by the control system, the question arises of including formulas and expressions that are convenient for quick calculation in the control program. Such formulas can be obtained (under certain restrictions) by analytical methods or reliable numerical methods. At the same time, such solutions are criteria for the correctness of obtaining solutions by direct numerical methods of integrating dynamical systems with a wide variation of the problem parameters. In this paper, we solve a semi-linear dynamic system of plane-parallel ship motion using the perturbation method.

2. Mathematical model and solution of the problem by the perturbation method

Basic systems of analytical equations of ship motion and their solutions have been formed since the 50s of the last century, and are now widely known [1]. These methods are given, in particular, in [2-5],
their modern refinements are contained, for example, in [6-9]. Laws effects of wind and turbulence on the movement of ships developed and refined by many experts, for example, [10-15]. Simulation of river vessel dynamics by numerical methods can be performed on the basis of the Matlab/Simulink system [16], in the "FlowVision" software package [17], the Unity modeling environment [18], specialized and universal CAD systems, for example, SolidWorks [19], and other software environments. However, a limitation of numerical models is their critical binding to boundary conditions, to specific parameters of ship designs and their operating conditions, which must be set with high accuracy to ensure the adequacy of the results.

In this manuscript, we consider the classical analytical model of ship movement, taking into account its modern refinements [4]:

\[ \begin{align*}
-(m + \lambda_{11}) \frac{dV_x}{dt} + mV_y \omega + X &= 0, \\
-(m + \lambda_{22}) \frac{dV_y}{dt} - mV_x \omega - \lambda_{26} \frac{d\omega}{dt} + Y &= 0, \\
-(I_z + \lambda_{66}) \frac{d\omega}{dt} - \lambda_{26} \frac{dV_y}{dt} + M &= 0
\end{align*} \]

where \( V_x, V_y \) are the velocity components, \( \omega \) - the angular velocity of rotation of the ship's hull, \( X, Y \) - the total forces on the x and y axes, respectively, \( M \) - the total moment on the z axis, \( m \) - the mass of the ship's displacement, \( I_z \) - the moment of inertia of the ship relative to the z axis, \( \lambda_{11}, \lambda_{22}, \lambda_{66}, \lambda_{26} \) - the attached masses when moving along the X, Y axes, when rotating relative to the Z axis and from the static moment, respectively, \( t \) is time. The initial conditions have the form:

\[ \begin{align*}
\omega(t = 0) &= \omega_0, V_x(t = 0) = V_{x0}, V_y(t = 0) = V_{y0}
\end{align*} \]

When obtaining an analytical solution of system (1) in the first approximation, it is possible to neglect the terms of the equations containing a smaller value in comparison with other parameters \( \lambda_{26} \).

In this case, the system (1) becomes linear. In this case, the time dependence for the angular velocity is determined from the third equation of the system:

\[ \omega = \omega_0 + \frac{Mt}{I_z + \lambda_{66}} \]

We differentiate the second equation of the system, substitute the derivative \( \frac{dV_y}{dt} \) from the first equation of the system and \( \omega \) from (3) in the resulting expression, and obtain a second-order differential equation for \( V_y \):

\[ p \frac{d^2V_y}{dp^2} + \frac{mMD}{(I_z + \lambda_{66})} p^2 V_y - \frac{dV_y}{dp} = -\frac{Y}{(m + \lambda_{22})} - DXp^2 \]

The notation is introduced here:

\[ D = \frac{mM}{(m + \lambda_{11})(m + \lambda_{22})(I_z + \lambda_{66})}, \gamma = \omega_0(I_z + \lambda_{66})/M, p = \gamma + t. \]

When obtaining equation (4), the expression for velocity found from the second equation (1) was used:

\[ V_y = \frac{Y(I_z + \lambda_{66}) - (m + \lambda_{22})(I_z + \lambda_{66}) \frac{dV_y}{dt}}{m(\omega_0(I_z + \lambda_{66}) + Mt) \frac{dV_y}{dt}} \]

The homogeneous equation corresponding to (4) has the form:
\[ \frac{d^2 V_y}{dp^2} + \frac{mMD}{I_z + \lambda_{66}} \frac{dV_y}{dp} - \frac{dV_y}{d\gamma} = 0 \]  
(6)

\[ a = -\frac{mMD}{I_z + \lambda_{66}} \]  
in this case. The last equation is an equation of the form 2.98 from [20]. Solution (7) has the form:

\[ V_{y0} = C_1 \cos(0.5kp^2) + C_2 \sin(0.5kp^2), k = \left( \frac{mMD}{I_z + \lambda_{66}} \right)^{0.5} \]  
(8)

Then, using the Vronsky determinant found from the fundamental solutions (8), we can find the general solution of the equation (4):

\[ V_y = \frac{Y}{2k(m + \lambda_{22})}(\sin(0.5kp^2)(Ci(0.5kp^2) - Ci(0.5k\gamma^2)) - \cos(0.5kp^2)(Si(0.5kp^2) - Si(0.5k\gamma^2))) \]  
(9)

Accordingly, expressions (5) and (9) are used to define an expression for another velocity component:

\[ V_x = \frac{Y(I_z + \lambda_{66})}{mM_0} - C_1 kp \sin(0.5kp^2) + C_2 kp \cos(0.5kp^2) - Y \sqrt{m} / (m + \lambda_{22}) \times \]

\[ (\cos(0.5kp^2)(Ci(0.5kp^2) - Ci(0.5k\gamma^2)) + \sin(0.5kp^2)(Si(0.5kp^2) - Si(0.5k\gamma^2))) \]  
(10)

Here \( Si(x) = \frac{\sin q}{q} dq, Ci(x) = -\frac{\cos q}{q} dq \) are integral sine and cosine. Constants are determined from the initial conditions (2). They are equal:

\[ C_1 = (V_{y0} + \frac{DX}{k^2}) \cos(0.5k\gamma^2) - (V_{y0} - \frac{Y(I_z + \lambda_{66})}{mM_0}) \sin(0.5k\gamma^2) / k\gamma \]

\[ C_2 = (V_{y0} - \frac{Y(I_z + \lambda_{66})}{mM_0}) \cos(0.5k\gamma^2) / k\gamma + (V_{y0} + \frac{DX}{k^2}) \sin(0.5k\gamma^2) \]  
(11)

Thus, in a linear approximation, the solution is determined by the formulas (3), (9)-(11).

Let's denote this solution as follows: \( V_x^{(0)}, V_y^{(0)}, \omega^{(0)} \).

The influence of the terms of the equations of system (1) containing the parameter is taken into account by the perturbation method.

As a suitable dimensionless small parameter, you can choose one of the following:

\[ \epsilon_1 = \frac{\lambda_{26} M}{V_0^2 m L}, \epsilon_2 = \frac{\lambda_{26} V_0^2}{Q L^2}, Q = (X^2 + Y^2)^{0.5} \neq 0 \]

\[ \epsilon_3 = \frac{\lambda_{26} L}{I_z}, \epsilon_4 = \frac{\lambda_{26} m}{m L}, \epsilon_5 = \frac{\lambda_{26} Q}{V_0^2 m^2} \]  
(12)

Here \( L \) is the characteristic size of the vessel, \( \epsilon \equiv \epsilon = \lambda_{26} \sigma, \lambda_{26} = \frac{\epsilon}{\sigma} \), \( \sigma \) is a value that depends on the problem parameters. Then the solution of the system can be represented as follows:
\[ V_x = V_x^{(0)} + \varepsilon V_x^{(1)}; V_y = V_y^{(0)} + \varepsilon V_y^{(1)}, \omega = \omega^{(0)} + \varepsilon\omega^{(1)} \]  

Substituting (13) in the system (1) and the initial conditions (2) and equating the expressions when we get the following system and the initial conditions for \( V_x^{(1)}, V_y^{(1)}, \omega^{(1)} \):

\[
-k \frac{dV_x^{(1)}}{dt} + m\omega_x^{(0)} + V_y^{(0)} \omega^{(1)} = 0,
\]

\[
-k \frac{dV_y^{(1)}}{dt} + m\omega_y^{(0)} + V_x^{(0)} \omega^{(1)} = 0,
\]

\[
(I_z + \lambda_{66}) \frac{d\omega^{(1)}}{dt} + \frac{1}{\sigma} \frac{dV_y^{(0)}}{dt} = 0,
\]

\[ \omega^{(1)}(t = 0) = 0, V_x^{(1)}(t = 0) = V_y^{(1)}(t = 0) = 0 \]

The solution for the third equation of the system (14) is obtained most simply:

\[ \omega(t) = \frac{\varepsilon}{\sigma(I_z + \lambda_{66})} (V_y^{(0)} - V_x^{(0)}(t)) \]

By differentiating the first equation of the system (14) and using the second of the equations (14) and the solution (16), we arrive at the following second-order differential equation:

\[
-k \frac{d^2V_x^{(1)}}{dt^2} = \frac{1}{\sigma} \frac{d^2V_y^{(1)}}{dt^2} + \frac{k^2}{p^2} V_x^{(1)} = F(p)
\]

Accordingly, the homogeneous equation has the form:

\[
-k \frac{d^2V_x^{(1)}}{dp^2} = \frac{1}{\sigma} \frac{d^2V_y^{(1)}}{dp^2} + \frac{k^2}{p^2} V_x^{(1)} = 0
\]

This equation is an equation of the form 2.162 from [20]:

\[ y = x^{(1-\alpha)/2} Z_\alpha \left( \frac{2}{m} \sqrt{b} x^{\alpha/2} \right), \quad \nu = \frac{1}{m} \sqrt{1-a^2} - 4c \]

Here \( Z_\alpha(x) \) are the Bessel functions of the first and second kind. Then, given expressions for Bessel functions of order 1/2, we can write the solution of equation (18) in the following form:

\[ F(p) = F(p) / \sigma \]

\[ \tilde{F}(p) = \frac{(m + \lambda_{22})DV_y^{(0)}(p)\tilde{\omega}^{(1)}(p)}{\omega^{(0)}(p)} - m\kappa\omega^{(0)}(p)\tilde{\omega}^{(1)}(p)V_y^{(0)}(p) - \frac{D}{\sigma} \omega^{(0)}(p) + \]

\[ \frac{m}{m + \lambda_{11}} \tilde{\omega}^{(1)}(p) \frac{dV_y^{(0)}}{dp} - \frac{\kappa}{\sigma} V_y^{(0)}(p) \frac{dV_y^{(0)}}{dp}, \tilde{\omega}^{(1)}(p) = \frac{V_y^{(0)}(p)}{I_z + \lambda_{66}}, \kappa = \frac{m}{(m + \lambda_{11})(m + \lambda_{22})}, \]

\[ \omega^{(0)}(p) = \frac{M_p}{I_z + \lambda_{66}}, \frac{dV_y^{(0)}}{dp} = \left( Y - \frac{mM_pV_y^{(0)}(p)}{I_z + \lambda_{66}} \right) / (m + \lambda_{22}) \]

Using the Vronsky determinant found from the fundamental solutions (20), one can find the General solution of equation (17) with the right-hand side (21), and then from the first equation (14)
find the solution for the velocity. Substituting the general solutions in the initial conditions (15) leads to equalities: \( K_2 = K_3 = 0 \). Thus, the solutions for the velocities due to the perturbation have the following form:

\[
V^{(i)}_x = \sqrt{p} \int (\sigma k)(\sin(0.5k^2\rho) \cos(0.5k^2\rho) \bar{F}(\rho) - \cos(0.5k^2\rho) \bar{F}(\rho) ) p^{-3/2} dp \]
\[
V^{(i)}_y = -\frac{V^{(0)}_y V^{(0)}_x - V^{(0)}_x V^{(0)}_y}{\sigma M\rho} + \frac{(m + \lambda_1\lambda_2)}{m M\rho\sigma} \int (\sin(0.5k^2\rho) i(2k\sqrt{p} - p^{3/2} \cos(0.5k^2))) \times \int \cos(0.5k^2\rho) \bar{F}(\rho) p^{-3/2} dp + (p^{3/2} \sin(0.5k^2\rho) - \cos(0.5k^2\rho) i(2k\sqrt{p})) \int \sin(0.5k^2\rho) \bar{F}(\rho) p^{-3/2} dp)
\]

Since the integrals included in (22) are not taken in elementary functions and can be integrated numerically.

3. Discussion
Thus, as obtained in the previous section, the solution of the original semilinear system by the perturbation method leads to the following expressions:

\[
V_x = V^{(0)}_x + \varepsilon V^{(1)}_x = V^{(0)}_x + \lambda_{26} \tilde{V}^{(i)}_x, V_y = V^{(0)}_y + \varepsilon V^{(1)}_y = V^{(0)}_y + \lambda_{26} \tilde{V}^{(i)}_y, \tilde{V}^{(i)}_x = \frac{V^{(i)}_x}{\sigma}, \tilde{V}^{(i)}_y = \frac{V^{(i)}_y}{\sigma} \quad (23)
\]

The values included here are determined by the formulas: (3), (9)-(11), (16), (21)-(22). The obtained solutions allow us to take into account the influence of all the main parameters on the linear and angular velocities. In this case, it follows from formula (3) that the linear increase in angular velocity over time must be limited, which is produced by restatements of the helm. From formula (16), it follows that the parameter \( \lambda_{26} \) leads to an influence on the angular velocity of the value \( V^{(0)}_y \). The small parameter can be selected in one way or another according to the formulas (12). Note that the value \( \varepsilon \) depends only on the parameters included in the system (1) and does not depend on the characteristic size of the vessel. If necessary, it is possible to decompose the velocities, not limited only to the linear approximation by, but the formulas for the velocities are much more complicated.

The value \( \lambda_{26} \) can be calculated, for example, using the following formula:

\[
\lambda_{26} = 0.5 \rho T^2 (1 - \frac{8}{3} (\frac{T - 0.18}{1 + T - 0.18}))
\]

Here \( T \)-precipitation, \( \rho \)-water density. We evaluated the effect of the perturbation on the velocity in order to find out whether the method can be applied. The data of the project 1743 (Omsk) dry cargo ship were considered: length \( L=108.4 \) m, width \( B=15.2 \) m, \( T=3.3 \) m. The angular velocity was considered to be influenced only by the moment caused by the wave (the wave was assumed to be 4 points on the Beaufort scale) \( M_w \), the time of steering was determined using the formula (3) and the value of the limit angle: \( \omega f + \frac{M(t')^2}{2(I_z + \lambda_{26})} = \psi_{lim} \). The necessary formulas for the calculation \( M_w \) were taken from [4]. The contribution estimate \( \varepsilon \omega^{(1)} \) relative to \( \omega^{(0)} \) due only to the moment in an irregular wave is about 3%, which is a small parameter. In general, when considering the total moment and total forces, the contribution increases, but it allows us to apply the perturbation method.

4. Conclusion
An analytical solution for plane-parallel ship motion in a linear approximation is obtained. It is shown that this solution depends on both harmonic functions and integral sine and cosine functions.
The perturbation method is used to solve the plane-parallel motion of a ship from nonlinear equations that take into account the influence of static moment. The effect of this value on each of the velocity components is shown. The influence of the linear velocity component on the angular velocity is shown. The resulting solutions can be used to automate the movement of a vessel along a constrained fairway using an automatic control system.

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