Solution of time independent cosmic string for harmonic oscillator plus poschl teller non-central potentials using supersymmetry quantum mechanics methods

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Abstract. Time independent cosmic string equation for Harmonic Oscillator and Poschl Teller non-central potential have been solved using supersymmetry quantum mechanics method. The time independent cosmic string with non-central potential was reduced into three one dimensional Schrodinger equations by using variable separation method. The energy levels and radial wave function were analyzed using supersymmetry quantum mechanics method from radial part of Schrodinger like equation. The angular wave function was obtained from angular part of Schrodinger equation. The effect of the non-central potential caused energy levels increasing. Energy spectra for \( \lambda = 0 \) and \( \kappa = 0 \) have smaller value than energy levels for \( \lambda \neq 0 \) and \( \kappa \neq 0 \). The increased value of parameter \( \lambda \) and \( \kappa \) caused increased in energy levels. However, the decreased of cosmic string parameter \( \alpha \), caused the increased of energy levels.

1. Introduction

String theory is theory of everything which is written in space and time coordinates. This is a theory that combine the theory of quantum mechanics with the theory of relativity, the theory can explain about everything that happen in the universe [1]. Cosmic string theory was studied as a vacuum solution of Kerr general relativity which has been reduced to 1+2 dimensions. Then the three dimensional solution was expanded into four dimensional space time [2]. Cosmic string is linear topology defect which is one of the most important predictions [3,4]. Cosmic string as a stable one dimensional topology defect, whose the length are infinite and straight has an angular parameter \( \alpha \) [5].

A lot of studies on the solution of cosmic string as topological defects on the nonrelativistic and relativistic quantum mechanical systems have been investigated [6,7]. Solution of cosmic string can be obtained by reduce cosmic string equation into Schrodinger equation, Dirac equation and Klein Gordon equation. Solution of cosmic string equation was reduced into Klein Gordon equation have been solved by Mederois and Melo [8]. Cosmic string equation was reduced to Dirac equation also have been solved by Bakke [9]. Solution of a two-dimensional Duffin Petiau Kemmer oscillator in the presence of a coulomb potential in the cosmic string have been solved by Boumali and Mesai [10]. In other research, solution of cosmic string equation for quantum particle by reduce that equation into Schrodinger equation have been solved by Hasanabadi et al [11]. Nonrelativistic system be able to analyzed using quantum mechanics. Behavior of quantum particle of non-central potential in cosmic
string space time have been analyzed by Afshardoost and Hassanabadi [12], they obtained solution of cosmic string for Poschl Teller Double Ring Shaped Coulomb potential. Lots of potentials like Woods-Saxon, Double Ring Shape Oscillator and other non-central potentials [7,13] have been solved in cosmic string. These potential system were solved using some methods like Supersymmetry Quantum Mechanics method, Confluent Hypergeometric method, Ansatz method. That methods can be used to obtain energy spectrum and wave function of the particle [5,11,14]. Ikot et. al, had been solved solution of cosmic string equation coulomb plus improved ring shaped potentials using Nikiforov Uvarov method [15].

In this paper, we find solution of time independent cosmic string equation using Supersymmetry Quantum Mechanics method for Harmonic Oscillator and Poschl Teller non-central potential. The Harmonic Oscillator and Poschl Teller non-central potential can be written:

$$V(r,\theta) = \frac{1}{2}M\omega^2r^2 + \frac{\hbar^2}{2Mr^2}\left(\frac{\kappa(k-1)}{\sin^2\theta} + \frac{\lambda(\lambda-1)}{\cos^2\theta}\right)$$  

(1)

Where $0 \leq r \leq \infty$, $\lambda > \kappa - 1$, and $0 \leq \theta \leq \pi$, $r$ is displacement, $M$ is the reduced mass and $\omega$ is the angular frequency [16,17,18]. This study refers to research is conducted by Hasanabadi et.al and Afshardoost and Hassanabadi [11,12]. This paper is organized as follow. In section 2, we begin with experimental which including time independent cosmic string and supersymmetry quantum mechanics method. In section 3, result and discussion are presented. In section 4, we summarize our result.

2. Experimental
The metric of cosmic string space time for spherical coordinates [15] is defined as:

$$ds^2 = -dt^2 + dr^2 + r^2d\theta^2 + \alpha^2r^2\sin^2\theta\,d\phi^2$$  

(2)

Where $r, \theta$ and $\phi$ were in the range $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. With parameter $\alpha = 1 - 4G\mu$ in $[0,1]$ interval and $\mu$ is density of cosmic string. When $\alpha$ approaches 1, cosmic string space time is reduced to minkowski space time. To solve space component, we can ignore the time component and then the metric become time independent cosmic string. From equation (2) we use separation variable. Matric of the time independent cosmic string is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & \alpha^2r^2\sin^2\theta \end{bmatrix}$$  

(3)

From this matric, we obtain Laplacian equation:

$$\nabla_{LB}^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)$$  

(4)

$$\nabla_{LB}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{\alpha^2r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$  

(5)

where $\nabla_{LB}^2$ is Laplace-Beltrami operator, $\mu$ and $\nu = 1,2,3$ and $g = \det g^{\mu\nu}$.

Supersymmetry quantum mechanics system as a system that consists of a super charge operator $Q_i$ which is commute with supersymmetry Hamiltonian $H_{SS}$ [19] is:

$$[Q_i, H_{SS}] = 0$$  

(6)

where $N$ is the number of super power that meets the anti-commutation relationship

$$[Q_i, Q_j] = \delta_{ij}H_{SS}$$  

(7)

For the simplest supersymmetry system, $N = 2$ with super charge operators $Q_1$ and $Q_2$. Super charge operators $Q_1$ and $Q_2$ are defined as:

$$Q_1 = \frac{1}{\sqrt{2}}\left(-\sigma_1\frac{p}{(2M)^{1/2}} + \sigma_2\phi(x)\right)$$  

and

$$Q_2 = \frac{1}{\sqrt{2}}\left(-\sigma_1\phi(x) + \sigma_2\frac{p}{(2M)^{1/2}}\right)$$  

(8)
where $\sigma_i$ and $\sigma_z$ are Pauli spin matrices, $p = -i\hbar \frac{d}{dx}$ is a linear momentum operator. By using equation (7) and equation (8), components of supersymmetry Hamiltonian are showed as:

$$H_{ss} = \begin{pmatrix}
\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + \frac{\hbar}{(2M)^{\frac{1}{2}}} \frac{d\phi}{dx} + \phi^2 & 0 \\
0 & -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} - \frac{\hbar}{(2M)^{\frac{1}{2}}} \frac{d\phi}{dx} + \phi^2
\end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

We shall write $H_{ss} = H_+$. Supersymmetry Hamiltonian is factorized as

$$H_- = A^+ A^- \quad \text{and} \quad H_+ = A^- A^+$$

where $A^+$ is raising operator and $A^-$ is lowering operator. Supertpartner potential is said to be shape invariant if these potential are following this conditions:

$$V_s(x; a_j) = V(x; a_{j+1}) + R(a_{j+1})$$

$$V_s(x; a_j) = \phi^2(x; a_j) + \frac{\hbar}{(2M)^{\frac{1}{2}}} \phi(x; a_j)$$

$$V_s(x; a_j) = \phi^2(x; a_j) - \frac{\hbar}{(2M)^{\frac{1}{2}}} \phi(x; a_j)$$

where $j = 0, 1, 2, \ldots$ and parameter a determined sequentially and $R$ is constant which don't depend on $x$. Energy level for shape invariant can be find by using:

$$H^{(s)} = -\frac{\hbar}{(2M)^{\frac{1}{2}}} \frac{d}{dx} + V_s(x; a_k) + \sum_{k=0}^{n} R(a_k)$$

if that equation is operated with ground state wave function, than ground state energy level will be obtained from $H$ which is written as:

$$E_n = \sum_{a=0}^{\infty} R(a)$$

Finally, total energy level is obtained:

$$E_n = \varepsilon_0 + E_n^-$$

where $\varepsilon_0$ is a ground state energy of the system. Based on characteristic of lowering operator, ground state operator wave function is obtained as:

$$A^- \cdot \psi_{a_0}^{(-)} = 0$$

The $n$ state wave function can be obtained by multiplying the raising operator with ground state wave function. The $n$ state wave function, following form:

$$\psi_{a_n}^{(+)}(x; a_0) = A^+(x; a_0) \cdot \psi_{a_{n+1}}^{(-)}(x; a_1)$$

That explanation is a simple procedure to construct the hierarchy of Hamiltonian. In the next section, the solution of time independent cosmic string for harmonic oscillator and posch teller non-central potential will be investigated.
3. Results and discussions

In this research, we solve the time independent cosmic string equation use spherical coordinates in the part of space using supersymmetry quantum mechanics method. The time independent cosmic string equation is reduced to Schrodinger Equation. Laplace-Beltrami operator equation (5) and Harmonic Oscillator plus Poschl Teller non-central potential (1) are substituted to the three dimensional Schrodinger like equation, and then we obtain the formulation of Schrodinger equation:

\[
\psi(r, \theta, \phi) = E\psi(r, \theta, \phi) \tag{23}
\]

Separation variable is applied to the Schrodinger equation (23). We obtain three part of equation from separation variable, there are radial, angular and azimuth.

\[
\frac{1}{R(r)} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R(r)}{\partial r} \right) - M^2 \frac{1}{r^2} \omega^2 r^2 + \frac{2M}{\hbar^2} E - l(l+1) = 0 \tag{24}
\]

\[
\frac{1}{\Theta(\theta) \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\partial \Theta(\theta)}{\partial \theta} \right) - \frac{\kappa(k-1)}{\sin^2 \theta} + \frac{\lambda(\lambda-1)}{\cos^2 \theta} + l(l+1) - \frac{m^2}{\alpha^2 \sin^2 \theta} = 0 \tag{25}
\]

\[
\frac{1}{\Phi(\phi)} \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + \frac{m^2}{\alpha^2 \sin^2 \theta} = 0 \tag{26}
\]

Angular part must be solved, to obtain value of constants \( l \). We assume that \( \Theta(\theta) = \frac{\hbar(\phi)}{(\sin \theta)^l} \) for equation (25), and we obtain:

\[
\frac{\hbar^2}{2M} \frac{d^2}{d\theta^2} H(\theta) + \frac{\hbar^2}{2M} \left[ \frac{\kappa(k-1)\theta^2 - \frac{1}{4} \alpha^2 + m^2}{\sin^2 \theta} + \frac{\lambda(\lambda-1)}{\cos^2 \theta} \right] H(\theta) = \frac{\hbar^2}{2M} \left[ l(l+1) + \frac{1}{4} \right] H(\theta) \tag{27}
\]

Equation (27) is the one dimensional Schrodinger equation which have effective potential. Formulation of the effective potential is:

\[
V_{\theta} = \frac{\hbar^2}{2M} \left[ \frac{\kappa(k-1)\theta^2 - \frac{1}{4} \alpha^2 + m^2}{\sin^2 \theta} + \frac{\lambda(\lambda-1)}{\cos^2 \theta} \right] \tag{28}
\]

where:

\[
E = \frac{\hbar^2}{2M} \left[ l(l+1) + \frac{1}{4} \right] \tag{29}
\]

To obtain the energy levels and the wave function of angular part we must set a superpotential. Formulation of superpotential:

\[
\phi = A \tan \theta + B \tan \theta \tag{30}
\]

Superpotential is a formulation which like an effective potential. From equation (30) we obtain formulation of \( \phi^2 \) and \( \phi \). Then, we obtain parameters of superpotential:

\[
A = \frac{\hbar}{(2M)^{1/2}} \kappa' \quad \text{and} \quad B = -\frac{\hbar}{(2M)^{1/2}} \kappa' \tag{31}
\]

where, the value of \( \kappa' \) and \( \epsilon_0 \):

\[
\kappa' = \frac{1}{2} \left[ \kappa(k-1) + \frac{m^2}{\alpha^2} \right]^{1/2} \tag{32}
\]

\[
\epsilon_0 = \frac{\hbar^2}{2M} \lambda^2 + \frac{\hbar^2}{2M} \kappa^2 + \frac{\hbar^2}{2M} 2\lambda \kappa \tag{33}
\]
To obtain angular wave function we need raising and lowering operators. Then, we can obtain raising and lowering operators, by using equation (13) and (14):

\[
A_+ = -\frac{\hbar}{(2M)^{1/2}} \frac{d}{d\theta} + \frac{\hbar}{(2M)^{1/2}} \lambda \cot \theta - \frac{\hbar}{(2M)^{1/2}} \kappa \cot \theta
\]  \hspace{1cm} (34)

\[
A_- = \frac{\hbar}{(2M)^{1/2}} \frac{d}{d\theta} + \frac{\hbar}{(2M)^{1/2}} \lambda \tan \theta - \frac{\hbar}{(2M)^{1/2}} \kappa \cot \theta
\]  \hspace{1cm} (35)

If the lowering operator is multiplied by the ground state wave function, the result will be zero like equation (21). Then we obtain ground state wave function:

\[
H_0^{-1}(\theta) = C \cos \theta^d \sin \theta
\]  \hspace{1cm} (36)

Where \( C \) was constant. Equation (36) is non-normalized angular ground state wave function. The first state wave function for angular part is obtained by multiplying the raising operator with ground state wave function like equation (22). The first state wave function for angular part is obtained (37).

\[
H_0^{+1}(\theta) = \left\{ \frac{\hbar}{(2M)^{1/2}} C \left( (\lambda + 1) \sin \theta^2 - \frac{1}{2} + \left[ \kappa (\kappa - 1) + \frac{m^2}{\alpha^2} \right]^{1/2} \right) \cos \theta^2 \right\}
\]

To obtain solution of energy levels, we use of superpartner potential equation (16) and equation (17). Superpartner potential for angular part are:

\[
V_-(\theta; \kappa' : \lambda_0) = \left\lfloor \frac{\hbar^2}{2M} \lambda (\lambda - 1) \sec^2 \theta + \frac{\hbar^2}{2M} \kappa (\kappa - 1) \csc^2 \theta - \frac{\hbar^2}{2M} \left( \lambda + \kappa \right)^2 \right\rfloor
\]  \hspace{1cm} (38)

\[
V_+(\theta; \kappa' : \lambda_0) = \left\lfloor \frac{\hbar^2}{2M} \lambda (\lambda + 1) \sec^2 \theta + \frac{\hbar^2}{2M} \kappa (\kappa + 1) \csc^2 \theta - \frac{\hbar^2}{2M} \left( \lambda + \kappa \right)^2 \right\rfloor
\]  \hspace{1cm} (39)

Energy levels for angular part based on equation (20), following form:

\[
E_{n,\theta} = \frac{\hbar^2}{2M} \left[ (\lambda + n_\theta) + (\kappa' + n_\theta) \right]
\]  \hspace{1cm} (40)

By using equation (30) energy levels following form:

\[
E_{n,\theta} = \frac{\hbar^2}{2M} \left[ \left( \lambda + n_\theta \right) + \left\lfloor \frac{1}{2} + \left[ \kappa (\kappa - 1) + \frac{m^2}{\alpha^2} \right]^{1/2} + n_\theta \right\rfloor \right]^2
\]  \hspace{1cm} (41)

By substitution equation (41) to equation (29), we obtain value of constant \( I \):

\[
I = (\lambda + n_\theta) + \left\lfloor \kappa (\kappa - 1) + \frac{m^2}{\alpha^2} \right\rfloor + n_\theta
\]  \hspace{1cm} (42)

To solve radial part, we assume that \( R(x) = \frac{x^{(l+1)}}{r} \) and we obtain:

\[
-\frac{\hbar^2}{2M} \frac{d^2X(r)}{dr^2} + \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2M} \frac{l(l+1)}{r^2} X(r) = EX(r)
\]  \hspace{1cm} (43)

Equation (43) is one dimension Schrodinger equation which have an effective potential, following formulation:

\[
V_{\text{eff}} = \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2M} \frac{l(l+1)}{r^2}
\]  \hspace{1cm} (44)
We must write a superpotential. From superpotential formulation we also obtain \( \phi^2 \) and \( \phi \). which that formulation will be the same formulation with effective potential,

\[
\phi = Cr + D
\]

and then, we obtain parameters of superpotential:

\[
C = \left(\frac{1}{2} M \right)^{1/2} \omega \quad \text{and} \quad D = -\frac{\hbar}{(2M)^{1/2}} (l+1)
\]

\[
e_0 = \hbar \omega \left(l + \frac{3}{2}\right)
\]

Then, to obtained radial wave function we also need raising and lowering operators. We can write raising and lowering operators for radial part base on equation (13) and (14), following form:

\[
A^+ = -\frac{\hbar}{(2M)^{1/2}} \frac{d}{dx} + \left(\frac{1}{2} M \right)^{1/2} \omega r - \frac{\hbar}{(2M)^{1/2}} \frac{1}{r}
\]

\[
A^- = \frac{\hbar}{(2M)^{1/2}} \frac{d}{dx} + \left(\frac{1}{2} M \right)^{1/2} \omega r - \frac{\hbar}{(2M)^{1/2}} \frac{1}{r}
\]

By using equation (21) and (22), Ground state wave function for radial part and first state wave function are obtained, following:

\[
R_0^{(-)} = C_r^{(l+1)} \exp \left[-\frac{1}{2} \frac{M \omega^2}{\hbar} \right]
\]

\[
R_1^{(-)} = \left\{-\frac{\hbar}{(2M)^{1/2}} C \left[2l+3\right] r^{l+3} - \frac{M \omega}{\hbar} R^{(l+3)} \right\} + \left(\frac{1}{2} M \right)^{1/2} \omega \left(\frac{M \omega}{\hbar} + 1\right) R^{(l+1)} \right\} \exp \left[-\frac{1}{2} \frac{M \omega^2}{\hbar} \right]
\]

By using equation (16) and (17), Superpartner potential for radial part can be written:

\[
V_r(r; l) = \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2M} l(l+1) - \frac{\hbar \omega}{\hbar} \left(l + \frac{3}{2}\right)
\]

\[
V_r(r; l) = \frac{1}{2} M \omega^2 r^2 + \frac{\hbar^2}{2M} \left(l(l+1)(l+2) - \frac{(l+1)(l+2)}{2}\right) - \frac{\hbar \omega l}{\hbar} \left(l + \frac{3}{2}\right)
\]

and by using superpartner potential for radial part, energy level is obtained:

\[
E_n = \hbar \omega \left(l + \frac{3}{2} + 2n_r\right)
\]

by using equation (54) and equation (42) energy levels of time independent cosmic string for Harmonic Oscillator and Poschl Teller non-central potential.

\[
E_{n, \nu} = \hbar \omega \left\{\left(\nu + n_\nu\right) + \left[\kappa(\kappa - 1) + \frac{m^2}{\alpha^2}\right]^{1/2} + n_\nu\right\} + \frac{3}{2} + 2n_r
\]

Wave function of angular and radial part can be visualized. Ground state angular wave function correspond to the equation (36). Figure 1, is three dimensional ground state angular wave function in spherical coordinates for several parameter variations. Based on Figure 1, is show that probability of particle distribution based on spherical coordinates which depended by \( \theta \). These wave function are result of visualization by assuming \( 0 \leq \theta \leq \pi \). Based on Figure 1, variation of parameters \( \alpha \), \( \lambda \) and \( \kappa \), cause variation of wave amplitude in the ground state wave function for angular part. In the Figure 1(a) and Figure 1(b), value of parameter \( \alpha \) affect to the wave characteristic. If value of parameter \( \alpha \) decrease, amplitude of wave function also decrease. So, for the smaller value of parameter \( \alpha \), probability of particle distribution decrease.
Figure 1. Three dimensional ground state angular wave function with $H_{(0,\lambda,\kappa,\sigma)}$ (a) $H_{(0,1,1.5,0.8)}$, (b) $H_{(0,1,5,0.5)}$, (c) $H_{(0,2,1.5,0.9)}$, (d) $H_{(0,3,1.5,0.9)}$, (e) $H_{(0,1,2.5,0.9)}$, (f) $H_{(0,1,3.5,0.9)}$

Value of parameter $\lambda$ and $\kappa$ also affect to the amplitude of ground state wave function. In the Figure 1(c) and Figure 1(d), if value of parameter $\lambda$ increase, amplitude of wave function decrease. But there are different waveform, if value of parameter $\lambda$ is even number, and odd number. In the Figure 1(e) and Figure 1(f), if value of parameter $\kappa$ increase then amplitude of wave function decrease. For increasing value of parameter $\kappa$, probability of particle distribution decrease.
Figure 2. Three dimensional first state angular wave function with $H_{(n, \lambda, \kappa, \alpha)}$ (a) $H_{(1,1,5,0.8)}$, (b) $H_{(1,1,5,0.5)}$, (c) $H_{(1,2,1,5,0.9)}$, (d) $H_{(1,3,1,5,0.9)}$, (e) $H_{(1,1,2,5,0.9)}$, (f) $H_{(1,1,3,5,0.9)}$

First state angular wave function correspond to the equation (37). Figure 2, is three dimensional first state angular wave function in spherical coordinates for several parameter variations. Based on Figure 2, show that variation of parameters $\alpha$, $\lambda$ and $\kappa$, obtain variation of wave amplitude in the first state wave function for angular part. In the Figure 2(a) and Figure 2(b), if value of parameter $\alpha$ decrease, amplitude of wave functions also decrease. For the smaller value of parameter $\alpha$, probability of particle distribution decrease. Value of parameter $\lambda$ and $\kappa$ also affect to the amplitude of ground state wave function. In the Figure 2(c) and Figure 2(d), if value of parameter $\lambda$ increase,
amplitude of wave function decrease. For increasing value of parameter $\lambda$, probability of particle distribution decrease. In the Figure 2(e) and Figure 2(f), if value of parameter $\kappa$ increase then amplitude of wave function decrease. For increasing value of parameter $\kappa$, probability of particle distribution decrease.

![Figure 3](image)

**Figure 3.** Three dimensional angular wave function with $H_{(0,1,1.5,0.9)}$ (a) $H_{(0,1,1.5,0.9)}$, (b) $H_{(0,1,1.5,0.9)}$.

Based on Figure 3, the difference between ground state wave function and first state wave function is the largest number of wave, that can be concluded that increasing value of $n_p$ caused the number of wave also increase. Beside that, angular wave function also describe the probability of particles in the spherical coordinate system. The amount of deviation on the x, y axis indicated that probability value of a particle being located. There is a difference between amplitude of ground state wave function and amplitude of first state wave function, so there is a possibility of particles moving.

Ground state wave function in Figure 4 correspond to the equation (50). Based on Figure 4, show that variation of parameters $\alpha$, $\lambda$, and $\kappa$, obtain variation of wave amplitude in the ground state wave function for radial part. Waveform is damped along of the r axis and probability of particle distribution decrease when it is far from the center. In the Figure 4(a), if value of parameter $\alpha$ decrease, amplitude of wave function increase and the amplitude shift smaller in the r axis. For the smaller value of parameter $\alpha$, probability of particle distribution increase. In the Figure 4(b), if value of parameter $\lambda$ increase, amplitude of wave function increase and the amplitude shift further in the r axis. For increasing value of parameter $\lambda$, probability of particle distribution also increase. In the Figure 4(c), if value of parameter $\kappa$ increase then amplitude of wave function increase. For increasing value of
parameter \( \kappa \), the amplitude shift further in the \( r \) axis, and probability of particle distribution increase. The biggest probability of distribution in ground state wave function is \( \lambda \) variation.

![Figure 4](image1)

**Figure 4.** Ground state wave function for radial part with (a) \( \alpha \) variation, (b) \( \lambda \) variation and (c) \( \kappa \) variation

First state wave function in Figure 5 correspond to the equation (51). Based on Figure 5, show that variation of parameters \( \alpha \), \( \lambda \) and \( \kappa \), obtain variation of wave amplitude in the first state wave function for radial part. Waveform is damped along of the \( r \) axis and probability of particle distribution decrease when it is far from the center. In the Figure 5(a), if value of parameter \( \alpha \) decrease, amplitude of wave function increase but value of amplitude is negative and the amplitude shift smaller in the \( r \) axis. For the smaller value of parameter \( \alpha \), probability of particle distribution increase. In the Figure 3(b), if value of parameter \( \lambda \) increase, amplitude of wave function increase but value of amplitude is negative and the amplitude shift further in the \( r \) axis. In the Figure 5(c), if value of parameter \( \kappa \)
increase then amplitude of wave function increase but value of amplitude is negative and the amplitude shift further in the $r$ axis. For increasing value of parameters $\lambda$ and $\kappa$, probability of particle distribution increase. The biggest probability of distribution in first state wave function is $\lambda$ variation. The greater the value of $\eta$, then the greater the wave form is damped. So, the probability of particle distribution is smaller and there is a difference between amplitude of ground state wave function and amplitude of first state wave function. There is a possibility of particles moving.

Based on equation (55), energy levels can be obtained. Based on Table 1, show energy levels time independent for harmonic oscillator plus poschl teller non-central potential. The value of parameter $\lambda$ and $\kappa$ have a significant effect. Energy levels for $\lambda = 0$ and $\kappa = 0$ is smaller than energy levels for $\lambda \neq 0$ and $\kappa \neq 0$. Parameter $\lambda \neq 0$ and $\kappa \neq 0$ caused increasing energy levels.

**Figure 5.** First state wave function for radial part with (a) $\alpha$ variation, (b) $\lambda$ variation and (c) $\kappa$ variation
Table 1. Energy levels time independent for harmonic oscillator and poschl teller non-central potential

| $n_r$ | $n_\theta$ | $\lambda$ | $\kappa$ | $\alpha$ | $m$ | $l$ | Energy |
|-------|------------|-----------|---------|--------|-----|-----|--------|
| 0     | 0          | 0         | 0       | 0.9    | 1   | 2.1111 | 4.6111 |
| 1     | 0          | 0         | 0       | 0.9    | 1   | 2.1111 | 6.6111 |
| 2     | 0          | 0         | 0       | 0.9    | 1   | 2.1111 | 8.6111 |
| 3     | 0          | 0         | 0       | 0.9    | 1   | 2.1111 | 10.6111|
| 0     | 1          | 1         | 1.5     | 0.9    | 1   | 2.4087 | 5.9087 |
| 1     | 1          | 1         | 1.5     | 0.9    | 1   | 2.4087 | 7.9087 |
| 2     | 1          | 1         | 1.5     | 0.9    | 1   | 2.4087 | 9.9087 |
| 3     | 1          | 1         | 1.5     | 0.9    | 1   | 2.4087 | 11.9087|

Table 2. Energy levels time independent for harmonic oscillator and poschl teller non-central potential ($\kappa, \lambda, \alpha$ and $n_\theta$ variation)

| $n_r$ | $n_\theta$ | $\lambda$ | $\kappa$ | $\alpha$ | $m$ | $l$ | Energy |
|-------|------------|-----------|---------|--------|-----|-----|--------|
| 1     | 1          | 1         | 1.5     | 0.9    | 1   | 2.4087 | 7.9087 |
| 1     | 1          | 1         | 2.5     | 0.9    | 1   | 3.2326 | 8.7326 |
| 1     | 1          | 1         | 3.5     | 0.9    | 1   | 4.1598 | 9.6598 |
| 1     | 1          | 2         | 1.5     | 0.9    | 1   | 2.4087 | 8.9087 |
| 1     | 1          | 3         | 1.5     | 0.9    | 1   | 2.4087 | 9.9087 |
| 1     | 1          | 1         | 1.5     | 0.8    | 1   | 2.5207 | 8.0207 |
| 1     | 1          | 1         | 1.5     | 0.5    | 1   | 3.1794 | 8.6794 |
| 1     | 2          | 1         | 1.5     | 0.9    | 1   | 2.4087 | 9.9087 |
| 1     | 3          | 1         | 1.5     | 0.9    | 1   | 2.4087 | 11.9087|

Based on Table 2, show that the value of parameter $\kappa$ have a significant effect to the energy levels. The increasing value of parameter $\kappa$ caused increasing in energy levels. The value of parameter $\lambda$ also have a significant effect to the energy levels. The increasing in energy levels also given by $\lambda$ variation. Increasing of the value of parameter $\lambda$ caused increasing in energy levels. But different result is only given by variation of parameter $\alpha$. The smaller value of parameter $\alpha$ caused increasing in energy levels. Energy levels also associate with $n$. From variation of quantum number $n_r$ and $n_\theta$, both give the same effect. These number give the same increasing in energy levels. The biggest increasing in energy levels is caused by increasing number of $n_r$ and $n_\theta$.

4. Conclusion

We investigate energy levels and wave function of time independent cosmic string equation for Harmonic Oscillator plus Poschl Teller non-central potential. Equation have been solved using supersymmetry quantum mechanics method. By using variable separation method, the time independent cosmic string with non-central potential be modified into one dimensional Schrodinger equation. Energy levels and wave function of corresponding the Schrodinger equation have been solved using supersymmetry quantum mechanics method. We use superpartner potential to solve the energy levels and use raising and lowering operators to obtain wave function. The energy levels and radial wave function are from radial part of Schrodinger like equation. The angular wave function are obtained from angular part of Schrodinger equation. Energy levels for $\lambda = 0$ and $\kappa = 0$ have smaller value than energy levels by using parameter $\lambda \neq 0$ and $\kappa \neq 0$. Parameters $\kappa$, $\lambda$, $\alpha$ and $n_\theta$ have the significant effect to the energy levels. The effect of the non-central potential caused energy level
increase for variation of parameter $\kappa$, $\lambda$, $\alpha$ and $n_\theta$. Increasing value of parameters $\lambda$ or $\kappa$ or $n_\theta$ cause increasing in energy levels. But, the decreasing of parameter $\alpha$ cause increase in energy levels.

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