A NEW $\frac{1}{2}$-RICCI TYPE FORMULA ON THE SPINOR BUNDLE AND APPLICATIONS

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Abstract. Consider a Riemannian spin manifold $(M^n, g)$ $(n \geq 3)$ endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$, such that $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{2}T$ is the metric connection with skew-torsion $T$. In this note we introduce a generalized $\frac{1}{2}$-Ricci type formula for the spinorial action of the Ricci endomorphism $\text{Ric}^g(X)$, induced by the one-parameter family of metric connections $\nabla^s := \nabla^g + 2sT$. This new identity extends a result described by Th. Friedrich and E. C. Kim, about the action of the Riemannian Ricci endomorphism on spinor fields, and allows us to present a series of applications. For example, we describe a new alternative proof of the generalized Schrödinger-Lichnerowicz formula related to the square of the Dirac operator, i.e. parallel spinors, $\nabla^s$-parallel spinors and twistor spinors with torsion. We illustrate our conclusions for some non-integrable structures satisfying our assumptions, e.g. Sasakian manifolds, nearly Kähler manifolds and nearly parallel $G_2$-manifolds, in dimensions 5, 6 and 7, respectively.

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1. Introduction

Let $(M^n, g)$ $(n \geq 3)$ be a connected Riemannian spin manifold endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$. Consider the one-parameter family of connections $\{\nabla^s : s \in \mathbb{R}\}$, given by

$$\nabla^s = \nabla^g + 2sT.$$ 

This is a line of metric connections with totally skew-symmetric torsion $T^s = 4sT$, which joins the connection $\nabla^{1/4} = \nabla^c$ with torsion $T$, with the Levi-Civita connection $\nabla^0 = \nabla^g$. By an abuse of notation next we shall refer to $\nabla^s$ by the term “characteristic connection”. Let us denote by $\text{Ric}^s$ the Ricci tensor induced by $\nabla^s$. In this note we focus on the action of the associated Ricci endomorphism $\text{Ric}^g(X)$ ($X \in \Gamma(TM)$), on the corresponding spinor bundle $\Sigma^g M$. Under the condition $\nabla^c T = 0$, and for any arbitrary spinor field $\varphi \in F^g := \Gamma(\Sigma^g M)$, we show that this action can be described in terms of the Dirac operator $D^s$ ($s \in \mathbb{R}$) induced by $\nabla^s$. This takes place in Section 3 where we provide the following (generalized) $\frac{1}{2}$-Ricci type formula (see Lemma 3.1)

$$\frac{1}{2} \text{Ric}^g(X) \cdot \varphi = D^g(\nabla^g_X \varphi) - \nabla_X^g(D^g \varphi) - \sum_{j=1}^n e_j \cdot \left[ \nabla^g_{\nabla^g_{e_j} X} \varphi + 4s \nabla^g_{D^g(X,e_j)} \varphi \right] + s(3 - 4s)(X, \sigma_T) \cdot \varphi,$$

(1.1)

for any arbitrary vector field $X \in \Gamma(TM)$, spinor field $\varphi \in F^g$ and $s \in \mathbb{R}$, where $\sigma_T$ is the 4-form

$$\sigma_T := \frac{1}{2} \sum_{i=1}^n (e_i, T) \wedge (e_i, T).$$

From now on we shall mainly refer to (1.1) by the term $\frac{1}{2}$-Ric$^g$-formula, or $\frac{1}{2}$-Ric$^g$-identity. This can be viewed as the analogue of the Riemannian $\frac{1}{2}$-Ricci formula, or in short $\frac{1}{2}$-Ric$^g$-formula, introduced by Friedrich and Kim in [17, Lem. 1.2]. The latter relates the Ricci endomorphism of the Levi-Civita connection with the Riemannian Dirac operator, i.e.

$$\frac{1}{2} \text{Ric}^g(X) \cdot \varphi = D^g(\nabla^g_X \varphi) - \nabla_X^g(D^g \varphi) - \sum_{j=1}^n e_j \cdot \nabla^g_{\nabla^g_{e_j} X} \varphi,$$

(1.2)
In [17] it was shown that the \( \frac{1}{2} \text{Ric}^\varphi \)-identity is stronger than the Schrödinger-Lichnerowicz formula associated to the Riemann Dirac operator \( D^\varphi = D^\varphi \), in the sense that the first formula induces the second one, after a contraction. Here, we extend this result by proving that the new \( \frac{1}{2} \text{Ric}^\varphi \)-formula induces the corresponding generalized formula of Schrödinger-Lichnerowicz type, associated to the Dirac operator \( D^\varphi \) (see for example [13] Thm. 3.1, [2] Thm. 6.1 or [11] Thm. 3.2), under the condition \( \nabla^c T = 0 \). Therefore, when the torsion form \( T \) is \( \nabla^c \)-parallel we provide a new proof for this fundamental formula which is different than the traditional proofs, compare for instance with [13] [11].

The new \( \frac{1}{2} \text{Ric} \)-type identity, being stronger than the generalized SL-formula for \( D^\varphi \), has several nice applications. In fact, it is a spinorial identity which reproduces all \( \frac{1}{2} \text{Ric} \)-type formulas associated to \( \nabla^s \) (in the sense of [17]), even for \( s = 0 \), but also other known results. For example, in [9] we have recently introduced a \textit{twistorial} \( \frac{1}{2} \text{Ric}^s \)-formula for \textit{twistor spinors with torsion} with respect to the family \( \nabla^s \). Such spinors are elements in the kernel of the Penrose operator \( \mathcal{P}^s \), induced by \( \nabla^s \). When \( T \) is \( \nabla^s \)-parallel and \( M^n \) is compact, in [2] Corol. 3.2 it was shown that twistor spinors with torsion realize the equality case of an estimate for the first eigenvalue of the square of the cubic Dirac operator \( D^{1/2} = D^9 + \frac{1}{2}T \), under some additional geometric assumptions (e.g. constant scalar curvature). The twistorial \( \frac{1}{2} \text{Ric}^s \)-formula ([9] Lem. 2.2]) appears also under the condition \( \nabla^c T = 0 \) and in the context of spin geometry with \( \text{(parallel) skew-torsion} \), it establishes the analogue of a basic result of Lichnerowicz [22] (see also [11] p. 123 or [10] Prop. A.2.1.(3a)]). In Section 3 we obtain the twistorial \( \frac{1}{2} \text{Ric}^s \)-formula via a new and easier method, in particular we prove that it coincides with the restriction of the \( \frac{1}{2} \text{Ric}^s \)-identity to the kernel of the twistor operator \( \mathcal{P}^s \) (see Theorem 3.5).

Next we proceed with an examination of \( \nabla^c \)-parallel spinors and more general \( \nabla^s \)-parallel spinors. Recall that when \( \nabla^c \) is the characteristic connection of a non-integrable G-structure on \((M^n,g)\) (in terms for example of [13]), then the condition \( \nabla^c \varphi = 0 \) for some non-trivial spinor field \( \varphi \), imposes restrictions to the holonomy group \( \text{Hol}(\nabla^c) \subset G \). Here, when \( T \) is \( \nabla^c \)-parallel, we deduce that a non-trivial spinor field \( \varphi_0 \in \mathcal{F}^g \) which is parallel with respect to \( \nabla^s \) for some parameter \( s \in \mathbb{R} \), must satisfy the following equations (for any \( X \in \Gamma(TM) \) and for the same \( s ) \)

\[
\text{Ric}^c(X) \cdot \varphi_0 = 2s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi_0, \quad \text{Scal}^c \cdot \varphi_0 = -8s(3 - 4s)\sigma_T \cdot \varphi_0.
\]

For \( s = 0 \) this yields the well-known Ric\(^c\)-flatness of \((M^n,g)\), while for \( \nabla^c \)-parallel spinors we obtain the conditions given by Friedrich and Ivanon [13], i.e. \( \text{Ric}^c(X) \cdot \varphi_0 = (X \cdot \sigma_T) \cdot \varphi_0 \) and \( \sigma_T \cdot \varphi_0 = -\frac{1}{4} \text{Scal}^c \cdot \varphi_0 \). Our most interesting result is related with \( \nabla^c \)-parallel spinors. Such spinors have applications in theoretical physics, especially in type II string theory, where basic models are described in terms of a metric connection with skew-torsion and the corresponding parallel spinors represent the preserved supersymmetries (for more background we refer to [21] [11] [15]). In Section 4 we present the explicit action of the endomorphism \( \text{Ric}^\varphi(X) \) and more general \( \text{Ric}^s(X) \) on \( \ker(\nabla^c) \), where \( \nabla^c \) is any metric connection with skew-torsion \( T \) such that \( \nabla^c T = 0 \) (this means, without assuming that \( \nabla^c \) is the characteristic connection of some underlying special structure). In particular, we provide the following remarkable formula (see Theorem 4.17 and Corollaries 4.10, 4.13)

\[
\text{Ric}^c(X) \cdot \varphi_0 = \frac{(16s^2 - 1)}{4} \sum_{j=1}^{n} T(X, e_j) \cdot (e_j \cdot \sigma_T) \cdot \varphi_0 + \frac{(16s^2 - 3)}{4} (X \cdot \sigma_T) \cdot \varphi_0
\]

\[
= \text{Ric}^c(X) \cdot \varphi_0 - \frac{(16s^2 - 1)}{4} S(X) \cdot \varphi_0,
\]

for any \( \nabla^c \)-parallel spinor \( \varphi_0 \) and \( X \in \Gamma(TM) \), where the endomorphism \( S(X) \) is given by

\[
S(X) := -X \cdot \sigma_T + \sum_{j=1}^{n} e_j \cdot (T(X, e_j) \cdot \sigma_T).
\]

Then, we specialize on some types of non-integrable geometric structures satisfying our assumptions, e.g. 5-dimensional Sasakian manifolds, 6-dimensional nearly Kähler manifolds and 7-dimensional nearly parallel G\(_2\)-manifolds. We illustrate our integrability conditions and describe the action of \( \text{Ric}^c(X) \) on the corresponding \( \nabla^c \)-parallel spinors (adapted to the particular special structure). For the Sasakian case, our result (see Theorem 4.10) nicely extends the integrability conditions given in [15] Thm. 7.3, 7.6], for any \( s \in \mathbb{R} \). For
nearly Kähler structures and nearly parallel G₂-structures we recover some of our conclusions in [9], however by a new method (see Corollary 5.4).

A final contribution of this note is related with the following first-order differential operator acting on spinors, ∇₇(φ) := ∑ᵢ∈{1,...,n} (eᵢ.J) ∇₇ₑᵢ φ. This operator is included in the expression of (D₆)² and in the compact case can be viewed as the main obstruction to a universal estimate of the lowest eigenvalue of (D₆)², see [11 14 4]. In Section 5 we examine ∇₇ and describe some special kinds of ∇₇- eigenspinors (see Proposition 5.2, 5.6). We also provide examples. Different classes of ∇₇- eigenspinors will be presented in a forthcoming work.

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2. Preliminaries

With the aim to set up our conventions relevant to subsequent computations, we begin by recalling basic facts from spin geometry. Since all this material is well-known, we provide an exposition only of the most useful notions (without proofs) and for any further and detailed information, we refer to [11 19 14 4 2 1].

2.1. Spin geometry with torsion. Consider an oriented connected Riemannian manifold (Mⁿ, g) (n ≥ 3) endowed with a spin structure, i.e. a Spin(n)-principal bundle Pᵍ := SO(M, g) → M together with a 2-fold covering Lⁿ : Pⁿ → M, such that Lⁿ(pq) = Lⁿ(p) Ad(q) for any p ∈ Pⁿ and g ∈ Spinₙ. Here, and for the following of this article we denote by Pₙ := SO(M, g) the SOₙ-principal bundle of positively oriented orthonormal frames of M. We also remark that for n ≥ 3, the spin group Spinₙ is the universal covering of SOₙ and Ad : Spinₙ → SOₙ denotes the double covering map. Via the spin representation (which we agree to denote by κₙ), we associate to Pₙ a complex vector bundle Σ := Pₙ ×ₙn Δₙ = Pₙ ×ₙn Δₙ, where Δₙ is the spin module. Notice that the spinor bundle cannot be defined independently of a (semi)-Riemannian metric, in particular the definition of spinor fields, i.e. sections of the spinor bundle, depends in general on g, in contrast to tensors. For the following we set Fⁿ := Γ(Σ). The Clifford multiplication is the bundle morphism μ : TM ⊗ Σ → Σ, defined by μ(X ⊗ φ) := κₙ(X)(φ) = X · φ and it naturally extends to differential forms μ : Λ(M) ⊗ Σ → Σ, for any φ, ψ ∈ Δₙ satisfy the following very useful properties (and similar for sections)

\[
\begin{align*}
-2g_{XY}(X, Y)φ &= X · Y · φ + Y · X · φ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quarter
The (spinorial) curvature operator $R_{X,Y}^s := \{\nabla_X, \nabla_Y\} - \nabla_{[X,Y]} : F^0 \to F^0$ associated to the covariant derivative $\nabla^s$ on the spinor bundle, satisfies the relation $R_{X,Y}^s \varphi = \frac{1}{2} R^s(X \wedge Y) \cdot \varphi$, where $R^s$ is the curvature operator on 2-forms, induced by $\nabla^s$ at the tangent bundle level. Locally, one has the relations
\[
R^s(e_i \wedge e_j) := \sum_{k<l} R^s_{ijkl} e_k \wedge e_l, \quad R_{X,Y}^s \varphi = \frac{1}{2} \sum_{i<j} g(R^s(X,Y)e_i, e_j) e_i \cdot e_j \cdot \varphi.
\]
For $s = 0$, one obtains the (spinorial) Riemannian curvature operator $R_{X,Y}^0$ and then, the first Bianchi identity associated to $\nabla^0$ yields the well-known relation
\[
\frac{1}{2} \text{Ric}^0(X) \cdot \varphi = -\sum_{i=1}^n e_i \cdot R^0_{X,e_i} \varphi,
\]
where $\text{Ric}^0(X)$ is the Riemannian Ricci endomorphism, i.e. $\text{Ric}^0(X,Y) = g(\text{Ric}^0(X), Y)$, for any $X, Y \in \Gamma(TM)$. For our family $\nabla^s$, and under the assumption $\nabla^c T = 0$, the associated Ricci tensor $R^s$ remains symmetric (see also Remark 2.2 below) and the Ricci endomorphism $\text{Ric}^s(X) (X \in \Gamma(TM))$ acts on spinors by the rule
\[
\text{Ric}^s(X) \cdot \varphi := \sum_{i} g(\text{Ric}^s(X), e_i) e_i \cdot \varphi = \sum_{i} \text{Ric}^s(X, e_i) e_i \cdot \varphi.
\]
In the same case, Becker-Bender proved that the first Bianchi identity associated to $\nabla^s$ (cf. [2 Thm. B.1]) induces an analogue of (2.2), namely:

**Lemma 2.1.** ([7 Lem. 1.13]) Under the assumption $\nabla^c T = 0$, the following relation holds
\[
\frac{1}{2} \text{Ric}^s(X) \cdot \varphi = -\sum_{i} e_i \cdot R^s_{X,e_i} \varphi + s(3 - 4s)(X, \sigma_T) \cdot \varphi,
\]
for any arbitrary vector field $X \in \Gamma(TM)$, spinor field $\varphi \in F^0$ and $s \in \mathbb{R}$.

**Remark 2.2.** Let us denote by $d^s$ and $\delta^s$ the differential and co-differential induced by $\nabla^s$, i.e.
\[
d^s \omega := \sum_{j} e_j \cdot \nabla_{e_j} \omega, \quad \delta^s \omega := -\sum_{j} e_j \cdot \nabla_{e_j} \omega,
\]
respectively. For $s = 0$, we set $d := d^0 \equiv d^0$ and $\delta := \delta^0 \equiv \delta^0$. By [14, 4, 2] it is known that for some 3-form $T \in \Lambda^3 T^* M$, it is $\delta T = \delta^s T$ for any $s \in \mathbb{R}$. Notice also that the 4-from $\sigma_T$ has a distinctive role in the theory of metric connections with skew-torsion. For example, by using [11 Lem. 2.4] one deduces that $\sigma_T$ measures the difference of the differentials $d^s$ and $d$, i.e. $d^c T = dT - 8s \sigma_T$. When the additional condition $\nabla^c T = 0$ holds, it is known that $dT = 2\sigma_T$ [21, 14] and $\nabla^c \sigma_T = 0 = \delta^s T$, for any $s \in \mathbb{R}$ [4, 2]. In the same case, it is also true that the curvature tensor $R^s$ is symmetric, $R^s(X, Y, Z, W) = R^s(Z, W, X, Y)$ and the same holds for the associated Ricci tensor $\text{Ric}^s(X, Y) := \sum_{i} R^s(X, e_i, e_i, Y)$, i.e. $\text{Ric}^s(X, Y) = \text{Ric}^s(Y, X)$, for any $X, Y \in \Gamma(TM)$ and $s \in \mathbb{R}$, see [2 Thm. B.1].

We pass now on differential operators acting on spinors. The (generalized) Dirac operator is the first-order differential operator on $F^0$, defined by
\[
D^s := \mu \circ \nabla^s : F^0 \xrightarrow{\nabla^s} \Gamma(T^* M \otimes \Sigma^g M) \cong \Gamma(TM \otimes \Sigma^g M) \xrightarrow{\mu} F^0.
\]
The Spin$_n$-representation $\mathbb{R}^n \otimes \Delta_n$ splits as $\mathbb{R}^n \otimes \Delta_n = \ker(\mu) \oplus \Delta_n$ and this induces the decomposition $TM \otimes \Sigma^g M = \ker(\mu) \oplus \Sigma^g M$. We shall write $p : \mathbb{R}^n \otimes \Delta_n \to \ker(\mu)$ for the universal projection, which locally is defined by $X \otimes \varphi \mapsto X \otimes \varphi + \frac{1}{n} \sum_{i=1}^n e_i \otimes e_i \cdot X \cdot \varphi$. This naturally extends to sections and yields the (generalized) Penrose, or twistor operator,
\[
\mathcal{P}^s = \mu \circ \nabla^s : F^0 \xrightarrow{\nabla^s} \Gamma(T^* M \otimes \Sigma^g M) \cong \Gamma(TM \otimes \Sigma^g M) \xrightarrow{\mu} \Gamma(\ker(\mu)).
\]
Locally, the operators $D^s$ and $\mathcal{P}^s$ attain the expressions
\[
D^s(\varphi) = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \varphi, \quad \text{and} \quad \mathcal{P}^s(\varphi) := \sum_{i=1}^n e_i \otimes \{\nabla_{e_i} \varphi + \frac{1}{n} e_i \cdot D^s(\varphi)\},
\]
respectively. Finally we mention that the one-parameter family of generalized Dirac operators $\{D^s \equiv D^g + 3s T : s \in \mathbb{R}\}$ and the Riemannian Dirac operator $D^0 \equiv D^g$ are sharing several common properties.
For example, $D^s$ if formally self-adjoint in $L^2(\Sigma^g M)$ for any $s \in \mathbb{R}$, since the torsion $T^s = 4sT$ is a 3-form [18]. Moreover, and similarly with the Riemannian case ($s = 0$) (cf. [11, 19]), one can show that:

**Proposition 2.3.** For any $s \in \mathbb{R}$, $f \in C^\infty(M; \mathbb{R})$, $X \in \Gamma(TM)$, $\xi \in T^*M$, $\omega \in \Lambda^p T^*M$ and $\varphi \in F^g$, the following hold:

1. $D^s(f \varphi) = \text{grad}(f) \cdot \varphi + f D^s(\varphi)$.
2. The principal symbol of $D^s$ is given by $\sigma(D^s) = \xi^2 \varphi$ and hence $D^s$ is elliptic.
3. The operator $-(D^s)^2$ is strongly elliptic, i.e. $\sigma(-(D^s)^2) = \xi^2 |\varphi|^2$.
4. $D^s(X \cdot \varphi) = \sum e_j \cdot (\nabla_{e_j} X) \cdot \varphi - X \cdot D^s(\varphi) - 2\nabla_X \varphi$.
5. $D^s(w \cdot \varphi) = (-1)^p \omega \cdot D^s(\varphi) + (\omega \cdot \delta^s \varphi) \cdot \varphi - 2 \sum e_j \cdot \nabla_{e_j} \varphi$.

3. The generalized $\frac{1}{2}$-Ricci type formula and basic applications

3.1. The generalized $\frac{1}{2}$-Ricci type formula. In this section we shall introduce the generalized $\frac{1}{2}$-Ricci type formula. For this is useful to fix, once and for all, a Riemannian spin manifold $(M^n, g, T)$ ($n \geq 3$) endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$, such that $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{s}{4} T$. As we have already pointed out in Remark 2.2, the reason of our assumption $\nabla^c T = 0$ is the symmetry of the Ricci tensor $R^c$ and Lemma 2.1.

**Lemma 3.1.** (The generalized $\frac{1}{2}$-Ricci type formula, or $\frac{1}{2}$-Ricci$^c$-formula) Assume that $\nabla^c T = 0$. Then, the Ricci endomorphism $R^c(X)$ satisfies

$$\frac{1}{2} \text{Ric}^c(X) \cdot \varphi = D^c (\nabla_X \varphi) - \nabla^c_X (D^c \varphi) - \sum_{j=1}^n e_j \cdot \left[ \nabla_{e_j} \nabla^c_X \varphi - 4s \nabla^c_{T(X,e_j)} \varphi \right] + s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi,$$

for any arbitrary vector field $X \in \Gamma(TM)$, spinor field $\varphi \in F^g$ and $s \in \mathbb{R}$.

**Proof.** We use Lemma 2.1 and replace $R^c_{X,e_j} \varphi$ by $\nabla_X \nabla^c_{e_j} \varphi - \nabla^c_{e_j} \nabla_X \varphi - \nabla^c_{[X,e_j]} \varphi$. This yields the following

$$\frac{1}{2} \text{Ric}^c(X) \cdot \varphi = - \sum_{j=1}^n e_j \cdot R^c_{X,e_j} \varphi + s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi$$

$$= - \sum_{j=1}^n e_j \cdot \left( \nabla^c_X \nabla^c_{e_j} \varphi - \nabla^c_{e_j} \nabla^c_X \varphi - \nabla^c_{[X,e_j]} \varphi \right) + s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi$$

$$= \sum_{j=1}^n e_j \cdot (\nabla^c_{e_j} \nabla^c_X \varphi) - \sum_{j=1}^n e_j \cdot (\nabla^c_{[X,e_j]} \varphi) + \sum_{j=1}^n e_j \cdot \nabla^c_{[X,e_j]} \varphi + s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi$$

$$= D^c (\nabla^c_X \varphi) - \sum_{j=1}^n \nabla^c_X (e_j \cdot (\nabla^c_{e_j} \varphi) + \sum_{j=1}^n (\nabla^c_X e_j) \cdot (\nabla^c_{e_j} \varphi) + \sum_{j=1}^n e_j \cdot \nabla^c_{X,e_j} \varphi$$

$$+ s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi$$

$$= D^c (\nabla^c_X \varphi) - \nabla^c_X (D^c \varphi) + \sum_{j=1}^n (\nabla^c_X e_j) \cdot (\nabla^c_{e_j} \varphi) + \sum_{j=1}^n e_j \cdot \nabla^c_{X,e_j} \varphi$$

$$+ s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi,$$

where for the fourth equality we applied the Liebniz rule to replace $e_j \cdot (\nabla^c_X \nabla^c_{e_j} \varphi) = \nabla^c_X (e_j \cdot \nabla^c_{e_j} \varphi) - (\nabla^c_X e_j) \cdot (\nabla^c_{e_j} \varphi)$. By the definition of $T^s$, we also have that $[X,e_j] = \nabla^c_X e_j - \nabla^c_{e_j} X - T^s (X,e_j) = \nabla^c_X e_j - \nabla^c_{e_j} X - 4sT(X,e_j)$ and consequently $\sum_{j=1}^n e_j \cdot \nabla^c_{X,e_j} \varphi = \sum_{j=1}^n e_j \cdot \left[ \nabla^c_{X,e_j} \varphi - \nabla^c_{[X,e_j]} \varphi - 4s \nabla^c_{T(X,e_j)} \varphi \right]$. Hence, the formula given above reduces to the following one:

$$\frac{1}{2} \text{Ric}^c(X) \cdot \varphi = D^c (\nabla^c_X \varphi) - \nabla^c_X (D^c \varphi) - \sum_{j=1}^n e_j \cdot \nabla^c_{X,e_j} \varphi + \sum_{j=1}^n \left[ (\nabla^c_X e_j) \cdot (\nabla^c_{e_j} \varphi) + e_j \cdot \nabla^c_{X,e_j} \varphi \right]$$

$$- 4s \sum_{j=1}^n e_j \cdot \nabla^c_{T(X,e_j)} \varphi + s(3 - 4s)(X \cdot \sigma_T) \cdot \varphi.$$
Now, it is easy to see that \( \sum_j \left[ (\nabla^s_X e_j) \cdot (\nabla^s_X \varphi) + e_j \cdot \nabla^s_X \nabla^s_X e_j \varphi \right] = 0 \), for any \( s \in \mathbb{R} \), \( X \in \Gamma(TM) \) and \( \varphi \in \mathcal{F}^g \) (for the Riemannian case \( s = 0 \), see also [17] p. 133). For example, this immediately follows after using (without loss of generality) a local \( \nabla^s \)-parallel frame, i.e. a local orthonormal frame \( \{e_j\} \) satisfying \( (\nabla^s e_j)_x = 0 \) at \( x \in M \), for any \( j = 1, \ldots, n \). This finishes the proof. \( \blacksquare \)

**Remark 3.2.** Notice that the \( \frac{1}{2} \)-Ric\(^s\)-formula (1.1) can be simplified a little further. Indeed, in terms of our \( \nabla^s \)-parallel frame \( \{e_i\} \) it is \( [X, e_i] = -\nabla^s_{e_i} X - 4sT(X, e_i) \) and the third term in (1.1) reduces to

\[
-\sum_{j=1}^n e_j \cdot \left[ \nabla^s_{e_j} X \varphi + 4s \nabla^s_{T(e_j, e_i)} \varphi \right] = \sum_{j=1}^n e_j \cdot \nabla^s_{[X, e_j]} \varphi.
\]

Thus, an equivalent expression of the \( \frac{1}{2} \)-Ric\(^s\)-identity is given by

\[
\frac{1}{2} \text{Ric}^s(X) \cdot \varphi = D^s(\nabla^s_X \varphi) - \nabla^s_X (D^s \varphi) + \sum_{j=1}^n e_j \cdot \nabla^s_{[X, e_j]} \varphi + s(3 - 4s)(X \sigma_T) \cdot \varphi.
\]

(3.1)

In introduction we chose to present (1.1), instead of (3.1), since the reduction to (1.2) for \( \phi \) is the first-order differential operator defined by \( \mathcal{D}^s := \sum_i (e_i, T) \cdot \nabla^s_{e_i} \varphi \) and \( \Delta^s := (\nabla^s)^* \nabla^s := -\sum_i \left[ \nabla^s_{e_i} \nabla^s_{e_i} + \nabla^s_{e_i} \nabla^s_{e_i} \right] \) denotes the spin Laplace operator associated to \( \nabla^s \).

First we recall that

**Theorem 3.3.** ([14] Thm. 3.1), ([14] Thm. 6.1), ([2] Thm. 2.1]) Under the assumption \( \nabla^s T = 0 \), any spinor field \( \varphi \in \mathcal{F}^g := \Gamma(\Sigma^g M) \) on \( (M^n, g, T) \) satisfies the relation

\[
(D^s)^2(\varphi) = \Delta^s(\varphi) + s(3 - 4s)D^s \cdot \varphi - 4s \mathcal{D}^s(\varphi) + \frac{1}{4} \text{Scal}^s \cdot \varphi,
\]

where \( \mathcal{D}^s \) is the first-order differential operator defined by \( \mathcal{D}^s := \sum_i (e_i, T) \cdot \nabla^s_{e_i} \varphi \) and \( \Delta^s := (\nabla^s)^* \nabla^s := -\sum_i \left[ \nabla^s_{e_i} \nabla^s_{e_i} + \nabla^s_{e_i} \nabla^s_{e_i} \right] \) denotes the spin Laplace operator associated to \( \nabla^s \).

First we recall that

**Lemma 3.4.** Consider a p-form \( \omega \in \Lambda^p TM \) and an orthonormal frame \( \{e_j\} \). Then

\[
\sum_{j=1}^n e_j \cdot (e_j \wedge \omega) = p\omega, \quad \sum_{j=1}^n e_j \cdot (e_j \wedge \omega) = (p - n)\omega.
\]

New proof of Theorem 3.3 Since the Ricci tensor \( \text{Ric}^s \) is symmetric, as in the Riemannian case, one can use the generalized \( \frac{1}{2} \)-Ricci formula and apply a contraction with respect to a \( \nabla^s \)-parallel local orthonormal frame \( \{e_i\} \). This means

\[
\sum_i e_i \cdot \text{Ric}^s(e_i) \cdot \varphi = \sum_{i,k} \text{Ric}^s(e_i, e_k) \cdot e_i \cdot e_k \cdot \varphi = -\sum_i \text{Ric}^s(e_i, e_i) \cdot \varphi = -\text{Scal}^s \cdot \varphi
\]

and hence, for the left-hand side part of (1.1) we obtain \( \frac{1}{2} \sum_i e_i \cdot \text{Ric}^s(e_i) \cdot \varphi = -\frac{1}{2} \text{Scal}^s \cdot \varphi \). Next we focus on the right-hand side and write all together:

\[
-\frac{1}{2} \text{Scal}^s \cdot \varphi = \sum_i e_i \cdot \left[ D^s(\nabla^s_{e_i} \varphi) - \nabla^s_{e_i} (D^s \varphi) \right] - \sum_{i,j} e_i \cdot e_j \cdot \left[ \nabla^s_{e_j, e_i} \varphi + 4s \nabla^s_{T(e_i, e_j)} \varphi \right]
\]

\[
+ s(3 - 4s) \sum_i e_i \cdot (e_i \sigma_T) \cdot \varphi
\]

\[
= \sum_i e_i \cdot D^s(\nabla^s_{e_i} \varphi) - \sum_i e_i \cdot \nabla^s_{e_i} (D^s \varphi) - \sum_{i,j} e_i \cdot e_j \cdot \left[ \nabla^s_{e_j, e_i} \varphi + 4s \nabla^s_{T(e_i, e_j)} \varphi \right]
\]

\[
+ 4s(3 - 4s) \sigma_T \cdot \varphi,
\]

\[
= \sum_i e_i \cdot D^s(\nabla^s_{e_i} \varphi) - (D^s)^2(\varphi) - \sum_{i,j} e_i \cdot e_j \cdot \left[ \nabla^s_{e_j, e_i} \varphi + 4s \nabla^s_{T(e_i, e_j)} \varphi \right]
\]

\[
+ 4s(3 - 4s) \sigma_T \cdot \varphi,
\]
where we used the fact that \( \sum_i e_i \cdot (e_i \cdot \sigma T) = 4\sigma T \) (see Lemma [3.3]) and \( - \sum_i e_i \cdot \nabla^s_{e_i} (D^s \varphi) = -(D^s)^2(\varphi) \).

We proceed with the two sums appearing in the resulting formula. For the first one, we are based on the formula given in Proposition 2.3 (4): we replace \( X \) by \( e_i \) and \( \varphi \) by \( \nabla^s_{e_i} \varphi \) and this yields

\[
D^s(e_i \cdot \nabla^s_{e_i} \varphi) = \sum_i e_i \cdot (\nabla^s_{e_i} e_i) \cdot \nabla^s_{e_i} \varphi - e_i \cdot D^s(\nabla^s_{e_i} \varphi) - 2\nabla^s_{e_i} \nabla^s_{e_i} \varphi = -e_i \cdot D^s(\nabla^s_{e_i} \varphi) - 2\nabla^s_{e_i} \nabla^s_{e_i} \varphi.
\]

Consequently,

\[
\sum_i e_i \cdot D^s(\nabla^s_{e_i} \varphi) = - \sum_i D^s(e_i \cdot \nabla^s_{e_i} \varphi) - 2\sum_i \nabla^s_{e_i} \nabla^s_{e_i} \varphi = -(D^s)^2(\varphi) + 2\Delta^s(\varphi),
\]

where in terms of the \( \nabla^s \)-parallel local orthonormal frame \( \{e_i\} \), the spin Laplace operator reads by \( \Delta^s(\varphi) = (\nabla^s)^* \nabla^s \varphi = -\sum_i \nabla^s_{e_i} \nabla^s_{e_i} \varphi \) (observe that the relation \( (\nabla^s e_i)_x = 0 \), yields \( (\nabla^s_{e_i})_x = 0 \)). Therefore, in order to complete the proof of Theorem 3.3, we just need to show that the second sum induces the operator \( \mathcal{D}^s \) with the desired coefficient. Indeed, because \( (\nabla^s_{e_i})_x = 0 \), we obtain

\[
-\sum_{i,j} e_i \cdot e_j \cdot \left( \nabla^s_{e_i} e_j \varphi + 4s\nabla^s_{T(e_i,e_j)} \varphi \right) = -4s \sum_{i,j} e_i \cdot e_j \cdot \nabla^s_{T(e_i,e_j)} \varphi.
\]

For a few, we forgot the factor \(-4s\) and since \( T(e_i,e_j) = \sum_k T^k_{ij} e_k = \sum_k T(e_i,e_j,e_k)e_k \), it follows that

\[
\sum_{i,j} e_i \cdot e_j \cdot \nabla^s_{T(e_i,e_j)} \varphi = \sum_{i,j} e_i \cdot e_j \cdot \nabla^s_{T^{k}_{ij} e_k} \varphi = \sum_{i,j,k} T^{k}_{ij} e_i \cdot e_j \cdot \nabla^s_{e_k} \varphi,
\]

for some non-zero real numbers \( T^{k}_{ij} := T(e_i,e_j,e_k) = g(T(e_i,e_j),e_k) \), with \( T^{k}_{ij} = -T^{k}_{ji} = -T^{i}_{jk} \), etc. Recall now that \( \sum_j T(X,e_j) \cdot e_j = -2(X \cdot T) \) for any \( X \in \Gamma(TM) \) (see for example [2] p. 328]). Thus,

\[
-2(e_k \cdot T) = \sum_j T(e_k,e_j) \cdot e_j = \sum_{i,j} g(T(e_k,e_j),e_i) e_i \cdot e_j
\]

\[
= \sum_{i,j} T(e_k,e_j,e_i) e_i \cdot e_j = -\sum_{i,j} T(e_i,e_j,e_k) e_i \cdot e_j,
\]

and a combination with (3.3) yields

\[
\sum_{i,j} e_i \cdot e_j \cdot \nabla^s_{T(e_i,e_j)} \varphi = 2\sum_k (e_k \cdot T) \cdot \nabla^s_{e_k} \varphi = 2\mathcal{D}^s(\varphi).
\]

Adding all our results together we obtain

\[
-\frac{1}{2} \text{Scal}^s \cdot \varphi = -(D^s)^2(\varphi) + 2\Delta^s(\varphi) - 8s\mathcal{D}^s(\varphi) + 4s(3 - 4s)\sigma T \cdot \varphi.
\]

Since \( \nabla^s T = 0 \) it is \( dT = 2\sigma T \) and consequently the last identity is nothing than the generalized Schrödinger-Lichnerowicz formula under the condition \( \nabla^s T = 0 \).

### 3.3. The new proof of the twistorial 1/2-Ric^s-formula

Here we shall describe the restriction of the 1/2-Ric^s-formula to twistor spinors, providing a simpler proof of the twistorial 1/2-Ric^s-formula. This has been recently introduced by the author in [9] Lem. 2.2], with different however methods. Recall that a **twistor spinor with torsion** (TsT in short) is a (non-trivial) spinor field \( \varphi \in \mathcal{F}^s \), solving the equation

\[
\nabla_X \varphi + \frac{1}{n} X \cdot D^s(\varphi) = 0, \quad \text{for any } X \in \Gamma(TM),
\]

for some parameter \( s \neq 0 \). Hence, a TsT is a spinor field belonging to the kernel of the generalized twistor operator \( \mathcal{P}^s \), see [2] [9] for more details. Obviously, for \( s = 0 \) one obtains the usual notion of Riemannian twistor spinors, i.e. elements \( \varphi \in \ker \mathcal{P}^g \), where \( \mathcal{P}^g \equiv \mathcal{P}^0 \) (cf. [14] [19]).
Theorem 3.5. (Twistorial $\frac{1}{2}$-Ric$^s$-formula) The action of the $\frac{1}{2}$-Ric$^s$-formula on a non-trivial twistor spinor $\varphi \in \ker(P^s)$ is given by
\[
\frac{1}{2} \text{Ric}^s(X) \cdot \varphi = \frac{1}{n} X \cdot (D^s)^2(\varphi) - \frac{n-2}{n} \nabla_X (D^s(\varphi)) + \frac{8s}{n} (X \cdot T) \cdot D^s(\varphi) + (3 - 4s) (X \cdot \sigma_T) \cdot \varphi,
\]
for any $X \in \Gamma(TM)$. In particular, for $s = 0$ and for a Riemannian twistor spinor, it induces the relation
\[
\nabla_X^n (D^s(\varphi)) = \frac{n}{2(n-2)} \left[ - \text{Ric}^0(X) \cdot \varphi + \frac{\text{Sch}^g}{2(n-1)} X \cdot \varphi \right] = \frac{n}{2} \text{Sch}^s(X) \cdot \varphi,
\]
where $\text{Sch}^s(X) := \frac{1}{n-2} \left[ - \text{Ric}^0(X) \cdot \varphi + \frac{\text{Sch}^g}{2(n-1)} X \right]$ is the Schouten endomorphism associated to $\nabla^g$.

Proof. We apply the $\frac{1}{2}$-Ric$^s$-formula \[11\] to a non-trivial twistor spinor $\varphi \in \ker(P^s)$. Any vector field $X \in \Gamma(TM)$ satisfies $\nabla_X^n \varphi = -\frac{1}{n} X \cdot D^s(\varphi)$, hence for the first term in \[11\], Proposition \[2,3\] (4), yields that
\[
D^n(\nabla_X^n \varphi) = D^n(-\frac{1}{n} X \cdot D^s(\varphi)) = -\frac{1}{n} D^n(X \cdot D^s(\varphi))
\]
\[
= -\frac{1}{n} \left[ \sum_j e_j \cdot (\nabla_{e_j} X) \cdot D^s(\varphi) - X \cdot D^s(D^s(\varphi)) - 2 \nabla_X^n (D^s(\varphi)) \right]
\]
\[
= -\frac{1}{n} \sum_j e_j \cdot (\nabla_{e_j} X) \cdot D^s(\varphi) + \frac{1}{n} X \cdot (D^s)^2(\varphi) + \frac{2}{n} \nabla_X^n (D^s(\varphi)).
\]

Therefore, for the first two terms of \[11\] we deduce that
\[
D^n(\nabla_X^n \varphi) - \nabla_X^n (D^s(\varphi)) = \frac{1}{n} X \cdot (D^s)^2(\varphi) - \frac{n-2}{n} \nabla_X^n (D^s(\varphi)) - \frac{1}{n} \sum_j e_j \cdot (\nabla_{e_j} X) \cdot D^s(\varphi).
\]

So, we have already constructed two desired terms with the right coefficients. Now, let us consider the sum
\[-\sum_{j=1}^n e_j \cdot \nabla_{e_j} X \cdot \varphi + 4s \nabla^s_T(X,e_j) \varphi\]
for some non-trivial $\varphi \in \ker(P^s)$. Since $\nabla_{e_j} X \varphi = -\frac{1}{n} (\nabla_{e_j} X) \cdot D^s(\varphi)$, it follows that
\[-\sum_{j=1}^n e_j \cdot \nabla_{e_j} X \cdot \varphi = \frac{1}{n} \sum_{j=1}^n e_j \cdot (\nabla_{e_j} X) \cdot D^s(\varphi),
\]
and this is canceled with the third term in the right-hand side of \[3.6\]. Moreover, based on the identity
\[2(X \cdot T) = \sum e_j \cdot T(X,e_j)\]
we obtain
\[-4s \sum_{j=1}^n e_j \cdot \nabla^s_T(X,e_j) \varphi = \frac{4s}{n} \sum_{j=1}^n e_j \cdot T(X,e_j) \cdot D^s(\varphi) = \frac{8s}{n} (X \cdot T) \cdot D^s(\varphi)
\]
and our claim follows. For more details related to the case $s = 0$ we refer to \[11,19,9\].

The twistorial $\frac{1}{2}$-Ric$^s$-formula gives rise to
\[
\frac{1}{2} \text{Scal}^s \cdot \varphi = -\frac{24s}{n} T \cdot D^s(\varphi) + \frac{2(n-1)}{n} (D^s)^2(\varphi) - 4s(3 - 4s) \sigma_T \cdot \varphi,
\]
which is not hard to see that is equivalent with the generalized SL-formula associated to the Dirac operator $D^s$, when this operator is restricted to $\ker(P^s)$. The most basic consequences of the identity stated in Theorem \[3.5\] have been described in \[9\]. For example:

Proposition 3.6. \[9\] Let $(M^n,g,T)$ $(n \geq 3)$ be a connected Riemannian spin manifold with $\nabla^c T = 0$. Then,
a) The kernel of the twistor operator $P^s$ is finite dimensional, i.e. $\dim \ker(P^s) \leq 2|\mathbb{R}| + 1$.
b) If $\varphi$ and $D^s(\varphi)$ vanish at some point $p \in M$ and $\varphi \in \ker(P^s)$, then $\varphi \equiv 0$.

Obviously, Proposition \[3.6\] generalizes classical properties of Riemannian twistor spinors (cf. \[11,19\]), to the whole family $\{\nabla^s : s \in \mathbb{R}\}$. Notice however that in the Riemannian case the space $\ker(P^0)$ is in addition a conformal invariant of $(M^n,g)$. A similar result for connections with skew-torsion is known to hold only in dimension 4 (cf. \[10\]). For more details on TsT and interesting examples of special geometric structures admitting this kind of spinor fields, we refer to \[4,2,9\] (see also Section \[1.2\] below).
4. $\nabla^s$-parallel spinors and $\nabla^c$-parallel spinors

4.1. Parallel spinors. In this section we present applications of the $\frac{1}{2}$-Ric$^c$-formula, related with $\nabla^s$-parallel spinors. Since Lemma 3 holds only under the assumption $\nabla^T T = 0$, it should be pointed out (even if this is not repeated throughout), that these results are meant for spin manifolds $(M^n, g, T)$ with $\nabla^c T = 0$. Hence, fix a Riemannian spin manifold $(M^n, g)$ endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$ such that $\nabla^c T = 0$, and consider the lift of the 1-parameter family $\{\nabla^s = \nabla^g + 2sT : s \in \mathbb{R}\}$ to the spinor bundle $\Sigma^g M$. By a $\nabla^s$-parallel spinor we mean a non-trivial spinor field $\varphi_0 \in F^g$ satisfying the equation $\nabla_X^s \varphi_0 = 0$, for some $s \in \mathbb{R}$ and any $X \in \Gamma(TM)$. This notion includes the following two well-known kinds of parallel spinors:

- $s = 0$: then we speak about $\nabla^g$-parallel spinors and their existence yields the Ric$^g$-flatness of $(M^n, g)$, i.e. $\text{Ric}^g(X) \cdot \varphi_0 = 0$ for any $X \in \Gamma(TM)$, see for example [11] [19] (notice that an easy way to prove the Ricci flatness is via the $\frac{1}{2}$-Ric$^g$-formula [12]).
- $s = 1/4$: then we speak about $\nabla^c$-parallel spinors and is a simple consequence of [12] Col. 3.2 that when the condition $\nabla^c T = 0$ holds, then a solution of the relation $\nabla^c \varphi_0 = 0$ must satisfies the relations $\text{Ric}^c(X) \cdot \varphi_0 = (X, \sigma_T) \cdot \varphi_0$, for any $X \in \Gamma(TM)$ and $\text{Scal}^c \cdot \varphi_0 = -4\sigma_T \cdot \varphi_0$.

The $\frac{1}{2}$-Ric$^c$-formula immediately yields integrability conditions for any member of the family $\{\nabla^s : s \in \mathbb{R}\}$. Moreover, when a $\nabla^c$-parallel spinor exists it allows us to describe the action of the endomorphism $\text{Ric}^c(X)$ : $F^g \to F^g$ for any other $s$. We begin with the following:

**Corollary 4.1.** Assume that $\nabla^c T = 0$ and let $\varphi_0 \in F^g$ be a non-trivial spinor field which is parallel with respect to $\nabla^s$, for some $s \in \mathbb{R}$. Then, for the same $s$ and for any $X \in \Gamma(TM)$ the spinor $\varphi_0$ must satisfy the following:

\[
\begin{align*}
\text{Ric}^s(X) \cdot \varphi_0 &= 2s(3 - 4s)(X, \sigma_T) \cdot \varphi_0, \\
\text{Scal}^c \cdot \varphi_0 &= -8s(3 - 4s)\sigma_T \cdot \varphi_0.
\end{align*}
\]

**Remark 4.2.** Notice that the connection $\nabla^{3/4}$ has torsion 3T and [14] shows that the existence of a $\nabla^{3/4}$-parallel spinor $\varphi_0$ implies the Ric$^{3/4}$-flatness of $(M^n, g, T)$.

For the record, we also mention that

**Corollary 4.3.** Assume that $\nabla^c T = 0$ and let $\varphi_0$ be a non-trivial $\nabla^s$-parallel spinor for some $s \in \mathbb{R}$ \{0, 3/4\}. Then, $(M, g, T)$ is Ric$^c$-flat for the same parameter $s$, if and only if $(X, \sigma_T) \cdot \varphi_0 = 0$ for any $X \in \Gamma(TM)$.

Finally, since the 4-form $\sigma_T$ vanishes in any dimension $n \leq 4$ [14] [4], we have that

**Corollary 4.4.** A 3-dimensional or 4-dimensional Riemannian spin manifold $(M^n, g, T)$ with $\nabla^c T = 0$, which admits a non-trivial $\nabla^s$-parallel spinor $\varphi_0 \in \ker(\nabla^s)$ for some $s \in \mathbb{R}$, is Ric$^c$-flat for the same parameter $s$.

**Example 4.5.** Consider the round 3-sphere $(S^3, g_{can})$ endowed with the volume form $T = \text{Vol}_{S^3}$. The characteristic connection $\nabla^c = \nabla^{\pm 1/4}$ (which is not unique because $S^3 \cong \text{Spin}_3 \cong SU_2$ is a Lie group), is induced by the Killing spinor equation. The real Killing spinors of $S^3$ trivialize its spinor bundle and they are $\nabla^c$-parallel, see [3] p. 729]. Hence, Corollary 4.1 or Corollary 4.4 apply, and show that any such spinor $\{\varphi_j : 1 \leq j \leq 2[\frac{3}{2}]\}$ must satisfy the equation $\text{Ric}^c(X) \cdot \varphi_j = 0$ for any $X \in \Gamma(T S^3)$, for another approach see for example [3] Prop. 5.1 and p. 133]. More general, any simply connected compact Lie group $G$ with a bi-invariant metric $g$ is flat with respect to the Cartan-Schouten connections $\nabla^{\pm 1/4}$ and there are $\nabla^{\pm 1/4}$-parallel spinors which satisfy $\text{Ric}^{\pm 1/4}(X) \cdot \varphi = 0$ for any $X \in g$. Further Ric$^c$-flat structures carrying $\nabla^c$-parallel spinors can be found in [3] [14] [13], for instance.

**Remark 4.6.** (4) In the compact case, Agricola and Friedrich [4] Thm. 7.1 proved that there are at most three parameters with $\nabla^c$-parallel spinors. Indeed, assume that $(M^n, g, T)$ is a compact Riemannian spin manifold endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$, such that $\nabla^c T = 0$. Then, any $\nabla^c$-parallel spinor $\varphi_0$ of unit length satisfies

\[8s(3 - 4s) \int_M \langle \sigma_T \cdot \varphi_0, \varphi_0 \rangle v^g + \int_M \text{Scal}^c v^g = 0.\]
This follows immediately by integrating the condition (4.2). Based now on (4.2) one can show that if the mean value of \( \langle \sigma_T \cdot \varphi_0, \varphi_0 \rangle \) does not vanish, then the parameter \( s \) equals to \( s = 1/4 \), i.e. \( \varphi_0 \) is necessarily parallel under the characteristic connection. If the mean value of \( \langle \sigma_T \cdot \varphi_0, \varphi_0 \rangle \) vanishes, then the parameter \( s \) depends on Scal and \( \| T \|^2 \). We refer to [4] for further details and examples.

In the following we shall use the \( \frac{1}{2} \)-Ric\(^s\)-formula to describe the spinorial action of the Ricci endomorphism Ric\(^s\)(X) for any \( s \in \mathbb{R} \), when there exists some \( \nabla^s\)-parallel spinor \( \varphi_0 \), without assuming however that \( \nabla^s \) is the characteristic connection of some underlying special structure. An important fact for our approach is that the torsion \( T \) can be viewed as a \( (\nabla^s\)-parallel symmetric endomorphism on \( \Sigma^s M \) in the sense that

\[
\langle T \cdot \varphi, \psi \rangle = \langle \varphi, T \cdot \psi \rangle, \quad \forall \varphi, \psi \in \mathcal{F}^s.
\]

Hence it is diagonalizable with real eigenvalues. Then, one may decompose the spinor bundle \( \Sigma^s M \) into a direct sum of \( T \)-eigenbundles preserved by \( \nabla^s \), i.e. \( \Sigma^s M = \bigoplus_{\gamma \in \text{Spec}(T)} \Sigma^s_{\gamma} M \) with \( \nabla^s \Sigma^s_{\gamma} M \subset \Sigma^s_{\gamma} M \). This induces a splitting also to the space of sections, \( \mathcal{F}^s = \bigoplus_{\gamma \in \text{Spec}(T)} \mathcal{F}^s_{\gamma} \) with \( \mathcal{F}^s_{\gamma} := \Gamma(\Sigma^s_{\gamma} M) \). We finally remind that when the torsion \( T \) is \( \nabla^s\)-parallel, then any non-trivial \( \nabla^s\)-parallel spinor field has constant \( T \)-eigenvalues, i.e. the equations \( \nabla^s T = 0 \), \( \nabla^s \varphi_0 = 0 \) and \( T \cdot \varphi_0 = \gamma \cdot \varphi_0 \) for some \( \gamma \in \text{Spec}(T) \) imply that \( \gamma = \text{constant} \in \mathbb{R} \), see [2] Thm. 1.1.

**Theorem 4.7.** Consider a Riemannian spin manifold \((M^n, g, T)\) \((n \geq 3)\) endowed with a non-trivial 3-form \( T \in \Lambda^3 T^* M \), such that \( \nabla^s T = 0 \), where \( \nabla^s := \nabla^s + \frac{1}{4} T \). Assume that \( \varphi_0 \) is a non-trivial \( \nabla^s\)-parallel spinor field lying in \( \mathcal{F}^s_{\gamma}\), for some (constant) \( \gamma \in \mathbb{R} \). Then, for any \( s \in \mathbb{R} \) and \( X \in \Gamma(TM) \) the following holds

\[
\text{Ric}^s(X) \cdot \varphi_0 = -\frac{(16s^2 - 1)}{4} \sum_j e_j \cdot (T(X, e_j)_s T) \cdot \varphi_0 + \frac{(16s^2 + 3)}{4} (X_{\sigma T}) \cdot \varphi_0
\]

(4.4)

\[
= \left(\frac{16s^2 - 1}{4}\right) \sum_j T(X, e_j) \cdot (e_j)_s T \cdot \varphi_0 + \frac{(16s^2 + 3)}{4} (X_{\sigma T}) \cdot \varphi_0.
\]

**Proof.** The proof is rather long and relies on the \( \frac{1}{2} \)-Ric\(^s\)-formula (1.1) and the \( \nabla^s \)-parallelism of \( T \). To begin with, notice that a \( \nabla^s\)-parallel spinor \( \varphi_0 \) satisfies the following two equations (cf. [2])

\[
\nabla_X^s \varphi_0 = \frac{4s - 1}{4} (X_s T) \cdot \varphi_0, \quad \text{and} \quad D^s(\varphi_0) = \frac{3(4s - 1)}{4} T \cdot \varphi_0,
\]

(4.5)

for any \( X \in \Gamma(TM) \). In particular \( \varphi_0 \) is a \( D^s\)-eigenspinor, \( D^s(\varphi_0) = \frac{3(4s - 1)}{4} \varphi_0 \), for any \( s \in \mathbb{R} \), with \( D^s(\varphi) = 0 \) of course. Let us apply the \( \frac{1}{2} \)-Ric\(^s\)-formula to \( \varphi_0 \). Due to (1.3) and Proposition 2.28 (5), for the first term of (1.1) we deduce that

\[
D^s(\nabla_X^s \varphi_0) = \frac{(4s - 1)}{4} D^s((X_s T) \cdot \varphi_0)
\]

\[
= \frac{(4s - 1)}{4} \left[ (X_s T) \cdot D^s(\varphi_0) + (d^s + \delta^s)(X_s T) \cdot \varphi_0 - 2 \sum_j (e_j)_s X_s T \cdot \nabla^s e_j \varphi_0 \right]
\]

\[
= \frac{3}{4} \left( \frac{4s - 1}{4} \right)^2 (X_s T) \cdot \varphi_0 + \frac{(4s - 1)}{4} \left[ d^s(X_s T) + \delta^s(X_s T) \right] \cdot \varphi_0
\]

\[
- 2 \frac{3}{4} \left( \frac{4s - 1}{4} \right)^2 \sum_j T(X, e_j) \cdot (e_j)_s T \cdot \varphi_0,
\]

where one identifies the 1-form \( T(X, e_j)^s := g(T(X, e_j), -) = T(X, e_j, -) = e_j X_s T \in \Lambda^1 T^* M \) with its dual vector field \( T(X, e_j) \in \Gamma(TM) \), via the metric tensor \( g \). From (1.6) and since \( \gamma \) is a constant, we also obtain

\[
\nabla_X^s(D^s(\varphi_0)) = \frac{3}{4} \left( \frac{4s - 1}{4} \right)^2 \gamma \cdot (X_s T) \cdot \varphi_0 = \frac{3}{4} \left( \frac{4s - 1}{4} \right)^2 (X_s T) \cdot T \cdot \varphi_0
\]

and a combination with the stated expression for \( D^s(\nabla^s \varphi_0) \), yields the difference

\[
D^s(\nabla_X^s \varphi_0) - \nabla_X^s(D^s(\varphi_0)) = -\frac{(4s - 1)^2}{8} \sum_j T(X, e_j) \cdot (e_j)_s T \cdot \varphi_0 + \frac{(4s - 1)}{4} \left[ d^s(X_s T) + \delta^s(X_s T) \right] \cdot \varphi_0.
\]

(4.6)
We proceed with the action of the term $\mathcal{E}(\varphi_0) := -\sum_{j=1}^{n} e_{j} \cdot \left[ \nabla_{\tilde{\nabla}_{e_j}} X \varphi_0 + \frac{4s}{4} \nabla_{T(X,e_j)} \varphi_0 \right]$ on $\varphi_0$. Because
\[
\nabla_{\tilde{\nabla}_{e_j}} X \varphi_0 = \frac{4s - 1}{4} \left( (\nabla_{e_j} X) \cdot T \right) \cdot \varphi_0, \quad \nabla_{T(X,e_j)} \varphi_0 = \frac{4s - 1}{4} \left( T(X,e_j) \cdot T \right) \cdot \varphi_0,
\]
and $\sum_{j} e_{j} \cdot (T(X,e_j) \cdot T) = -\sum_{j} T(X,e_j) \cdot (e_{j} \cdot T)$ (see [2, p. 325]), we conclude that
\[
\mathcal{E}(\varphi_0) = -\frac{(4s - 1)}{4} \sum_{j} e_{j} \cdot \left( (\nabla_{e_j} X) \cdot T \right) \cdot \varphi_0 - s(4s - 1) \sum_{j} e_{j} \cdot (T(X,e_j) \cdot T) \cdot \varphi_0 - \frac{(4s - 1)}{4} \sum_{j} e_{j} \cdot (e_{j} \cdot T) \cdot \varphi_0.
\]

Now, by the definition of $d^s$ and since $\nabla_X^s(Y \cdot T) = (\nabla_X^s Y) \cdot T + Y \cdot (\nabla_X^s T)$ for any $X, Y \in \Gamma(TM)$ and $s \in \mathbb{R}$, it also follows that
\[
d^s(X \cdot T) = \sum_{j} e_{j} \cdot (\nabla_{e_j}^s (X \cdot T)) = \sum_{j} e_{j} \cdot ((\nabla_{e_j}^s X) \cdot T) + \sum_{j} e_{j} \cdot (X \cdot (\nabla_{e_j}^s T)).
\]

Having in mind the isomorphism $X \cdot \simeq X - X_{\omega}$, we combine the last relation with a part of (4.7), i.e.
\[
\mathcal{A}(\varphi_0) := -\frac{(4s - 1)}{4} \sum_{j=1}^{n} e_{j} \cdot \nabla_{\tilde{\nabla}_{e_j}} X \varphi_0
\]

On the other hand, for any $X \in \Gamma(TM)$, $T \in \Lambda^{3}T^*M$ and $s \in \mathbb{R}$ it holds that
\[
\delta^s(X \cdot T) = -\sum_{j} e_{j} \cdot (\nabla_{e_j}^s (X \cdot T)) = -\sum_{j} e_{j} \cdot ((\nabla_{e_j}^s X) \cdot T) - \sum_{j} e_{j} \cdot (X \cdot (\nabla_{e_j}^s T)).
\]

Therefore, adding appropriately with (4.8), the first sums cancel each other and we obtain
\[
\mathcal{A}(\varphi_0) + \frac{(4s - 1)}{4} \delta^s(X \cdot T) \cdot \varphi_0 := -\frac{(4s - 1)}{4} \left[ d^s(X \cdot T) + \delta^s(X \cdot T) \right] \cdot \varphi_0 - \sum_{j=1}^{n} e_{j} \cdot \nabla_{\tilde{\nabla}_{e_j}} X \varphi_0
\]

where for the last equality we apply again the isomorphism $X \cdot \simeq X - X_{\omega}$. In this way and by combining the relations (4.9), (4.7), (4.5), (4.10) and (4.11), we conclude that
\[
\frac{1}{2} \text{Ric}^s(X) \cdot \varphi_0 = -\frac{(4s - 1)}{4} \sum_{j} e_{j} \cdot (X \cdot (\nabla_{e_j}^s T)) \cdot \varphi_0 + \frac{(16s^2 - 1)}{8} \sum_{j} T(X,e_j) \cdot (e_{j} \cdot T) \cdot \varphi_0
\]

The final step is based on the fact that under the condition $\nabla^c T = 0$, it holds that (see [2, Thm. B.1])
\[
(\nabla_X^s T)(X,Y,Z,W) = \frac{4s - 1}{2} \sigma_T(Y,Z,W,X) = -\frac{4s - 1}{2} \sigma_T(X,Y,Z,W).
\]

Thus, for any $X \in \Gamma(TM)$ the 3-form $(\nabla_X^s T)$ equals to
\[
(\nabla_X^s T) = -\frac{4s - 1}{2} (X_{\omega} \sigma_T).
\]
Corollary 4.10. Consider a triple operator acts on $\nabla^c \gamma$ can appear in the corresponding expression, as well).

Lemma (observe that a similar reformulation as (4.11) applies also to Theorem 4.7 and hence the eigenvalue $\sigma_n^X := X_\sigma T$ is a 3-form, see also Lemma 3.4. In combination with (4.10), this observation completes the proof. ■

Remark 4.8. For the parameter $s = 1/4$, Theorem 4.7 reduces to $\text{Ric}^c(X) \cdot \varphi_0 = (X \cdot \sigma T) \cdot \varphi_0$, for any $X \in \Gamma(TM)$, as it should be according to [14], or our Corollary 4.1.

For completeness, let us use (4.4) to verify the relation between the scalar curvatures $\text{Scal}^c$ and $\text{Scal}^g$, namely $\text{Scal}^c = \text{Scal}^g - 24s^2 \|T\|^2$ (see [14]). Indeed, as in the proof of Theorem 3.3 we consider a local orthonormal frame $\{e_i\}$ and we write $\sum_i e_i \cdot \text{Ric}^c(e_i) \cdot \varphi_0 = - \text{Scal}^c \cdot \varphi$. On the other hand, it is $T(e_i, e_j) = \sum_k T^k_{ij} e_k = \sum_k T(e_i, e_j, e_k) e_k$ and by Theorem 4.7 for a non-trivial $\nabla^c$-parallel spinor $\varphi_0$, we see that

$$\sum_i e_i \cdot \text{Ric}^c(e_i) \cdot \varphi_0 = - \frac{(16s^2 - 1)2}{4} \sum_{i,j} e_i \cdot (T(e_i, e_j) \cdot \sigma T) \cdot \varphi_0 + \frac{(16s^2 + 3)2}{4} \sum_i e_i \cdot (e_i \cdot \sigma T) \cdot \varphi_0$$

(4.4)

$$= - \frac{(16s^2 - 1)2}{4} \sum_{i,j,k} T^k_{ij} e_i \cdot e_j \cdot (e_k \cdot T) \cdot \varphi_0 + (16s^2 + 3) \sigma T \cdot \varphi_0$$

$$= - \frac{(16s^2 - 1)2}{2} \sum_k (e_k \cdot T) \cdot (e_k \cdot T) \cdot \varphi_0 + (16s^2 + 3) \sigma T \cdot \varphi_0$$

(4.5)

$$= - \frac{(16s^2 - 1)2}{2} [2\sigma T - 3 \|T\|^2] \cdot \varphi_0 + (16s^2 + 3) \sigma T \cdot \varphi_0$$

$$= 4\sigma T \cdot \varphi_0 + \frac{3(16s^2 - 1)2}{2} \|T\|^2 \cdot \varphi_0 = - \text{Scal}^c \cdot \varphi_0 + \frac{3(16s^2 - 1)2}{2} \|T\|^2 \cdot \varphi_0,$$

where for (*) we used the fact $\sum_k (e_k \cdot T) \cdot (e_k \cdot T) = 2\sigma T - 3 \|T\|^2$ (see [2] p. 328). We deduce that

$$\text{Scal}^c \cdot \varphi_0 = \text{Scal}^g \cdot \varphi_0 - \frac{3(16s^2 - 1)2}{2} \|T\|^2 \cdot \varphi_0 = \text{Scal}^g \cdot \varphi_0 - 24s^2 \|T\|^2 \cdot \varphi_0,$$

and the assertion follows since $\varphi_0$ does not have zeros.

Remark 4.9. For these computations one could even proceed as follows: Based on (16.5), in (*) we replace $(e_k \cdot T) \cdot \varphi_0$ by $\frac{1}{\|T\|} \nabla^c_{e_k} \varphi_0$ for any $s \neq 1/4$. Then, we use (3.3) to obtain (a multiple of) the operator $\mathcal{D}^s$, appearing in Theorem 6.3. For the final step one needs a description of the $\mathcal{D}^s$-eigenvalues when this operator acts on the $\nabla^c$-parallel spinors, which we present in Section 5, see Proposition 5.2.

Corollary 4.10. Consider a triple $(M^n, g, T)$ as in Theorem 4.7 admitting a non-trivial $\nabla^c$-parallel spinor $\varphi_0 \in \mathcal{F}^g(\gamma)$ ($\gamma \in \mathbb{R}$). Then, the Riemannian Ricci endormorphism acts on $\varphi_0$ as

$$\text{Ric}^g(X) \cdot \varphi_0 = \frac{1}{4} \sum_j e_j \cdot (T(X, e_j) \cdot \varphi_0 + \frac{3}{4} (X \cdot \sigma T) \cdot \varphi_0$$

(4.11)

$$= \frac{1}{8} \sum_j e_j \cdot (X \cdot T) \cdot (e_j \cdot T) \cdot \varphi_0 - \frac{3\gamma}{8} (X \cdot T) \cdot \varphi_0 + \frac{3}{4} (X \cdot \sigma T) \cdot \varphi_0.$$

Proof. The first expression occurs by Theorem 4.7 for $s = 0$. The second one is based on the following lemma (observe that a similar reformulation as (4.11) applies also to Theorem 4.7 and hence the eigenvalue $\gamma$ can appear in the corresponding expression, as well). ■
Lemma 4.11. For any vector field $X \in \Gamma(TM)$, $3$-form $T \in \Lambda^3T^*M$ and orthonormal frame $\{e_j\}$, the following holds:

$$\sum_{j=1}^{n} T(X, e_j) \cdot (e_j, T) = \frac{1}{2} \sum_{j=1}^{n} e_j \cdot (X, T) \cdot (e_j, T) + \frac{3}{2} (X, T) \cdot T.$$ 

Proof. By [2,11] we see that $e_j \cdot \omega - (-1)^p \omega \cdot e_j = -2(e_j, \omega)$ for any $p$-form $\omega \in \Lambda^pT^*M$. Since for any $X \in \Gamma(TM)$ the quantity $\omega := X, T$ is a 2-form, it follows that $-\frac{1}{2} \left[ e_j \cdot (X, T) - (X, T) \cdot e_j \right] = e_j \cdot X, T.$ Consequently, recalling that $\sum_{j=1}^{n} e_j \cdot (e_j, T) = 3T$ (cf. [2] p. 328) or Lemma [3,2], one gets the result:

$$\sum_{j=1}^{n} (e_j, X, T) \cdot (e_j, T) = -\frac{1}{2} \sum_{j=1}^{n} e_j \cdot (X, T) - (X, T) \cdot e_j \cdot (e_j, T)$$

$$= -\frac{1}{2} \sum_{j=1}^{n} e_j \cdot (X, T) \cdot (e_j, T) + \frac{1}{2} \sum_{j=1}^{n} (X, T) \cdot e_j \cdot (e_j, T)$$

$$= -\frac{1}{2} \sum_{j=1}^{n} e_j \cdot (X, T) \cdot (e_j, T) + \frac{3}{2} (X, T) \cdot T. \quad \blacksquare$$

Remark 4.12. As we explained in Remark [2,2] for $\nabla^c$-parallel torsion $T$, the Ricci tensor satisfies the relation $\text{Ric}^c(X,Y) = \text{Ric}^c(X,Y) - 4s^2S(X,Y)$, where $S$ is a symmetric covariant 2-tensor defined by $S(X,Y) := \sum_{j} g(T(X, e_j), T(Y, e_j))$, see for example [3, p. 110]. Hence, the Ricci endomorphism $\text{Ric}^c(X)$ is given by $\text{Ric}^c(X) = \text{Ric}^c(X) - 4s^2S(X)$, where $S(X)$ is the symmetric endomorphism associated to $S$, i.e. $g(S(X), Y) = S(X,Y)$, for any $X, Y \in \Gamma(TM)$. In particular, $\text{Ric}^c(X) = \text{Ric}^c(X) + \frac{1}{2} S(X)$ (cf. [13]). Therefore, a direct combination of Corollaries 4.10 and 13, allows us to describe the explicit action of $S(X)$ on $\nabla^c$-parallel spinors, for any metric connection $\nabla^c$ with $\nabla^c T = 0$.

Corollary 4.13. Consider a triple $(M^n, g, T)$ as in Theorem 4.7, admitting a non-trivial $\nabla^c$-parallel spinor $\varphi_0 \in \mathcal{F}(\gamma)$ ($\gamma \in \mathbb{R}$). Then, for any $X \in \Gamma(TM)$, the action of the symmetric endomorphism $S(X)$ on $\varphi_0$ is given by

$$S(X) \cdot \varphi_0 = \sum_{j=1}^{n} e_j \cdot (T(X, e_j), T) \cdot \varphi_0 - (X, \sigma T) \cdot \varphi_0$$

$$= \frac{1}{2} \sum_{j=1}^{n} e_j \cdot (X, T) \cdot (e_j, T) \cdot \varphi_0 - \frac{3\gamma}{2} (X, T) \cdot \varphi_0 - (X, \sigma T) \cdot \varphi_0.$$ 

Theorem 4.7 and Corollaries 4.10 and 4.13 can be applied on any triple $(M^n, g, T)$ endowed with a non-integrable $G$-structure $R \subset \mathcal{P}^3 (G \subset SO_n)$ and a $\nabla^c$-parallel spinor $\varphi_0$ with respect to the adapted (unique) characteristic connection $\nabla^c = \nabla^b + \frac{1}{2} T$, under the assumption $\nabla^c T = 0$. Special structures fitting in this setting are plentiful, e.g. Sasakian manifolds in any odd dimension [14][15], almost hermitian structures in even dimensions [6][23], co-calibrated $G_2$-structures in dimension 7 [14][12], (non-parallel) Spin$_7$-structures in dimension 8 [23][20], to name some of them. In the following, we are going to illustrate our integrability conditions on nearly parallel $G_2$-structures, nearly Kähler structures and Sasakian structures.

4.2 Real Killing spinors which are $\nabla^c$-parallel. For the first two special structures mentioned above, the description can be globalized and this is because on these manifolds the existent $\nabla^c$-parallel spinors coincide with the real Killing spinors, i.e. they satisfy the additional equation $\nabla^c_{X} \varphi_0 = \kappa X \cdot \varphi_0$ for any $X \in \Gamma(TM)$ and some $\kappa \in \mathbb{R}^*$ (with respect to the same metric $g$ that holds $\nabla^c \varphi_0 = 0$). Friedrich and Ivanov [14] Thm. 5.6, 10.8 were the first who provided this identification and moreover proved that any such Einstein manifold is also $\nabla^c$-Einstein. In [3] Prop. 5.1 we generalise these results by showing that the Ricci endomorphism $\text{Ric}^c(X)$ on such Einstein manifolds is a multiple of the identity operator for any $s \in \mathbb{R}$, i.e. $\text{Ric}^c = \frac{1+s}{n} \text{Id}$, and moreover that the existent $\nabla^c$-parallel spinors are Killing spinor with torsion with respect to $\nabla^c$ (or twistor spinors with torsion) for any $s \in \mathbb{R} \setminus \{0, 1/4\}$ (for $s = 5/12$ and 6-dimensional nearly Kähler manifolds this result was known by [2] Thm. 6.1]). A direct and very simplified proof of the first conclusion now in terms of Theorem 4.7 as follows.
Let us denote by $K^c(M, g) := \{ \varphi \in \mathcal{F}^g : \nabla^c_X \varphi = \zeta X \cdot \varphi, \forall X \in \Gamma(TM) \}$ the set of all Killing spinors with torsion (KsT in short), with respect to the family $\nabla^c = \nabla^g + 2sT$ (s $\neq 0$), with Killing number $\zeta \neq 0$ (we refer to [2, 9] for a detailed exposition related to this kind of spinors). Similarly, we shall write $K^0(M^n, g)_\kappa$ for the set of all real Killing spinors with Killing number $\kappa \neq 0$. Assume that $(M^n, g, T)$ is a compact connected Riemannian spin manifold $(M^n, g, T)$, with $\nabla^c T = 0$ and positive scalar curvature given by $\text{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$, for some constant $0 \neq \gamma \in \text{Spec}(T)$. In [1, Thm. 3.7] we extended the identification mentioned above, namely

$$\text{Ker}(\nabla^c) \cong \bigoplus_{\gamma \in \text{Spec}(T)} \left[ \Gamma(\Sigma \gamma) \cap K^0(M^n, g) \right] \frac{2n}{\gamma},$$

by proving that for any non-trivial $\nabla^c$-parallel spinor $\varphi_0$ the following conditions are equivalent:

1. $\varphi_0 \in \mathcal{F}^g(\gamma) \cap \text{Ker}(\mathcal{P}^s) := \text{Ker}(\mathcal{P}^s|_{\Sigma^sM})$ with respect to the family $\{\nabla^s : s \in \mathbb{R}\setminus\{1/4\}\}$,

2. $\varphi_0 \in K^c(M, g)_{\zeta}$ with respect to the family $\{\nabla^s : s \in \mathbb{R}\setminus\{0, 1/4\}\}$ with $\zeta := 3(1 - 4s)\gamma/4n$,

3. $\varphi_0 \in K^0(M, g)_{\kappa}$ with $\kappa := 3\gamma/4n$.

This correspondence allows us now to proceed with the following (see [1, Prop. 5.1] for another method):

**Corollary 4.14.** Let $(M^n, g, T)$ $(n \geq 3)$ be a compact Riemannian spin manifold, endowed with the characteristic connection $\nabla^c = \nabla^g + \frac{2}{3}T$, such that $\nabla^c T = 0$. Assume that $0 \neq \varphi_0 \in \Sigma^2_M$ $(R \ni \gamma \neq 0)$ is a non-trivial $\nabla^c$-parallel spinor, which satisfies one of the conditions (a), (b), or (c). Then, the $\frac{1}{2}$-Ric$^c$-formula gives rise to the equation

$$\text{Ric}^c(X) \cdot \varphi_0 = \frac{\text{Scal}^s}{n} X \cdot \varphi_0 = \frac{3\gamma^2(-3 + 3n - 144s^2 + 16ns^2)}{4n^2} X \cdot \varphi_0, \quad (4.1)$$

for any $X \in \Gamma(TM)$, where $\text{Scal}^s := \frac{4n^2}{n} \left[ \frac{n(n-1)(1-4s)^2+96s(1-4s)+16s(3-4s)(n-3)}{16} \right] = \frac{3\gamma^2(-3 + 3n - 144s^2 + 16ns^2)}{4n^2}$.

Moreover, for $n \neq 9$ the symmetric endomorphism $S(X)$ acts on $\varphi_0$ as a multiple of the identity,

$$S(X) \cdot \varphi_0 = -\frac{3\gamma^2(n-9)}{n^2} X \cdot \varphi_0. \quad (4.12)$$

**Proof.** Assume that $0 \neq \varphi_0 \in \Sigma^2_M$ is a $\nabla^c$-parallel spinor which satisfies any of the conditions (a), (b), or (c). Then, by [9, Prop. 3.2, Thm. 3.7] this is equivalent to say that $\varphi_0$ is a solution of the equation

$$X \cdot T \cdot \varphi_0 + \frac{3\gamma}{n} X \cdot \varphi_0 = 0, \quad (4.13)$$

for any $X \in \Gamma(TM)$. Moreover, the Ricci tensor $\text{Ric}^c$ is computed algebraically (see for example the proof of [9, Prop. 5.1, (b)])

$$\text{Ric}^c(X) \cdot \varphi_0 = (X \cdot \sigma_T) \cdot \varphi_0 = \frac{3\gamma^2(n-3)}{n^2} X \cdot \varphi_0. \quad (4.14)$$

By (4.13) it follows that an arbitrary vector field $X$ satisfies the equation $(T(X, e_j) \cdot T) \cdot \varphi_0 = -\frac{3\gamma}{n} T(X, e_j) \cdot \varphi_0$ and because $\sum_j e_j \cdot T(X, e_j) = 2(X \cdot T)$, an application of Theorem 4.7 gives rise to

$$\text{Ric}^c(X) \cdot \varphi_0 = -\frac{18\gamma^2(16s^2 - 1)}{4n^2} X \cdot \varphi_0 + \frac{3\gamma^2(n-3)(16s^2 + 3)}{4n^2} X \cdot \varphi_0,$$

which equals to the given relation. Finally, Corollary 4.13 yields the expression for the action of $S(X)$. Therefore, as in [9, one concludes that a (complete) Riemannian manifold $(M^n, g, T)$ $(n \geq 3)$ satisfying Corollary 4.13 is a compact Einstein manifold with constant positive scalar curvature $\text{Scal}^g = \frac{9(n-1)\gamma^2}{4n}$, a $\nabla^c$-Einstein manifold with parallel torsion and constant positive scalar curvature $\text{Scal}^c = \frac{3(n-3)(n-9)^2}{4n^2}$ and moreover that satisfies the “harmony equation” (2) for any other $s$. For $n = 3$, $(M^3, g, T)$ is $\text{Ric}^c$-flat and hence isometric to the 3-sphere. Notice also that under our point of view, the study of 6-dimensional nearly Kähler manifolds and 7-dimensional nearly parallel $G_2$-manifolds reduces to be qualitatively the same. Therefore, below we illustrate our conclusions only for one of these two classes, e.g. nearly parallel $G_2$-manifolds and similarly is treated the former class (see also Examples [5,6] and [1, 2, 9] for useful details).
Example 4.15. Consider a nearly parallel $G_2$-manifold $(M^7, g, \omega)$, i.e. a 7-dimensional oriented Riemannian manifold with a $G_2$-structure $\omega \in \Gamma(A^2_+TM)$ satisfying the differential equation $d\omega = -\tau_0 * \omega$, for some real constant $\tau_0 \neq 0$ (we refer to [10, 13, 12] for an introduction to $G_2$-structures and also to $G_2$-structures carrying a characteristic connection). In [13] Cor. 4.9 it was shown that a nearly parallel $G_2$-manifold admits a unique characteristic connection $\nabla^c$ with parallel skew-torsion $T$, given by $T := \frac{1}{6} (d\omega, *\omega) \cdot \omega$, in particular $T = -\frac{1}{6} \omega$ and $\|T\|^2 = \frac{\tau_0^2}{36}$. Moreover, there exists a unique spinor field $\varphi_0$ which is $\nabla^c$-parallel (cf. [12] Prop. 3.2) and satisfies the equation $T \cdot \varphi_0 = -\frac{\tau_0}{6} \varphi_0 = -\sqrt{7} \|T\| \varphi_0$, i.e. $\gamma = -\sqrt{7} \|T\| \|T\|^2$. In fact, $\varphi_0$ is a real Killing spinor and hence $\text{Ric}^c(X) \cdot \varphi_0 = \frac{\tau_0}{6} T \parallel T \parallel^2 X \cdot \varphi_0 = \frac{3\tau_0^2}{2} X \cdot \varphi_0$ ([16, 14]). More general, in [9] Exam. 5.3 we deduced the "harmony equation" (2), i.e.

$$\text{Ric}^c(X) \cdot \varphi_0 = \frac{3(9 - 16s^2)}{14} \|T\|^2 X \cdot \varphi_0,$$

for any $s \in \mathbb{R}$ and $X \in \Gamma(TM)$. Let us provide a new proof of this result, via Theorem 4.17. As in the proof of Corollary 4.14 the key point is that $\varphi_0$ satisfies the equations (4.13) and (4.14) respectively, i.e.

$$(X \cdot T) \cdot \varphi_0 = \frac{\tau_0}{2} X \cdot \varphi_0 = \frac{3\|T\|}{\sqrt{7}} X \cdot \varphi_0, \quad \text{Ric}^c(X) \cdot \varphi_0 = (X \cdot T) \cdot \varphi_0 = \frac{12}{7} \|T\|^2 X \cdot \varphi_0,$$

for any $X \in \Gamma(TM)$. Both of them can be found [9] (see also [3] Lem. 2.3 and [13] p. 318). The first represents the Killing equation, or equivalent the twistor equation [9] Prop. 3.2, Thm. 4.2], while the second one states that $(M^7, g, \omega)$ is a $\nabla^c$-Einstein manifold. Hence, [13] yields that

$$\text{Ric}^c(X) \cdot \varphi_0 = -\frac{3(16s^2 - 1)\|T\|}{4\sqrt{7}} \sum_{j=1}^7 e_j \cdot T(X, e_j) \cdot \varphi_0 + \frac{12(16s^2 + 3)\|T\|^2}{28} X \cdot \varphi_0 = -\frac{6(16s^2 - 1)\|T\|}{4\sqrt{7}} \frac{2(16s^2 + 3)\|T\|^2}{28} X \cdot \varphi_0 \frac{18(16s^2 - 1)\|T\|^2}{28} X \cdot \varphi_0 + \frac{12(16s^2 + 3)\|T\|^2}{28} X \cdot \varphi_0,$$

which gives rises to the result. Finally, by (4.12) we compute $S(X) \cdot \varphi_0 = \frac{6\|T\|^2}{7} X \cdot \varphi_0$ (cf. [13]).

4.3 5-dimensional Sasakian structures. Recall that a Sasakian structure on a Riemannian manifold $(M^{2n+1}, g)$ consists of a Killing vector field $\xi$ of unit length, the so-called Reeb vector field, such that the endomorphism $\phi : TM \to TM$ given by $\phi(X) = -\nabla^c_\xi X$, satisfies $(\nabla^c_\xi \phi)(Y) = g(X, Y) \xi - g(\xi, Y)X$ for any $X, Y \in \Gamma(TM)$. The dual 1-form $\eta$ of $\xi$ solves the equation $d\eta = 2F$, where $F(X, Y) := g(X, \phi(Y))$ is the fundamental 2-form, see for example [17, 18] for equivalent definitions and more details. Let us focus on 5-dimensional Sasakian manifolds $(M^5, g, \xi, \eta, \phi)$. We fix an orthonormal basis $e_1, \ldots, e_5$ of $T_xM \cong \mathbb{R}^5$ and use the abbreviation $e_{i_1, \ldots, i_p}$ for the $p$-form $e_{i_1} \wedge \ldots \wedge e_{i_p}$. It is

$$\xi := e_5, \quad \phi := -(e_{12} + e_{34}), \quad F := e_{12} + e_{34},$$

and in terms of $\phi$, our orthonormal frame reads by $\{e_1, e_2 := -\phi(e_1), e_3, e_4 = -\phi(e_3), e_5 = \xi\}$, with $\phi(\xi) = 0$. By [13] Prop. 7.1 it is known that there exists a unique metric connection $\nabla^c$ with parallel skew-torsion $T = \eta \wedge d\eta = 2\eta \wedge F = 2(e_{125} + e_{345})$, preserving the Sasakian structure, $\nabla^c g = \nabla^c \eta = \nabla^c \phi = 0$. The torsion form $T$ acts on the 5-dimensional spin representation $\Delta_5$ with eigenvalues $(-4, 0, 0, 4, 0)$. Hence, the spinor bundle $\Sigma^g M$ splits into two 1-dimensional subbundles and one 2-dimensional subbundle, i.e. $\Sigma^g M = \Sigma^g_1 M \oplus \Sigma^g_0 M \oplus \Sigma^g_2 M$ with $\Sigma^g_{\pm 4} := \{\varphi \in \Sigma^g M : T \cdot \varphi = \pm 4\varphi\}$ and $\Sigma^g_0 M := \{\varphi \in \Sigma^g M : T \cdot \varphi = 0\}$, respectively.

In the direction of the Reeb vector field $\xi$ the Riemannian Ricci endomorphism must occur with eigenvalue 4, i.e. $\text{Ric}^c(\xi) = 4\xi$ (cf. [17]). Let us explain how Corollary 4.10 fits with this result. Assume that there exists some $\nabla^c$-parallel spinor $\varphi_1$, which for instance belongs to $\Sigma^g_{-4} M$, i.e. $T \cdot \varphi_1 = -4\varphi_1$. Any vector field $X$ satisfies $X \cdot F = -\phi(X)$, hence by (4.1) we get that (see also [17] Lem. 6.3)

$$X \cdot d\eta - d\eta \cdot X = -2(X \cdot d\eta) = -4(X \cdot F) = 4\phi(X), \quad \forall X \in \Gamma(TM).$$

\[\text{Notice that here our 3-form } \omega \text{ is such that } \omega \cdot \varphi_0 = 7\varphi_0, \text{ see [3] Lem. 2.3}.\]
It is $\sigma T = 4e_{1234}$, $\xi, \sigma T = 0$ and $\xi, T = dp$. Thus, by applying for example 4.11 (for a description based on the first expression of Corollary 4.10, see the proof of Theorem 4.16 below), we obtain

$$\text{Ric}^g(\xi) \cdot \varphi_1 = 1 \sum_j e_j \cdot d\eta \cdot (e_j, T) \cdot \varphi_1 + \frac{3}{2} d\eta \cdot \varphi_1$$

One also computes $e_2, T = -2e_{15}$, $e_1, T = 2e_{25}$, $e_4, T = -2e_{35}$, and $e_3, T = 2e_{45}$, which finally yields the desired assertion: $\text{Ric}^g(\xi) \cdot \varphi_1 = ( -e_2 \cdot e_{25} - e_1 \cdot e_{15} - e_4 \cdot e_{45} - e_3 \cdot e_{35} ) \cdot \varphi_1 = 4e_5 \cdot \varphi_1$. In fact, the relation $\text{Ric}^g(\xi) = 4\xi$ can be also obtained by using a $\nabla^c$-parallel spinor in $\Sigma^g_5 \mathcal{M}$, or in $\Sigma^g_6 \mathcal{M}$. More general, in the simply-connected case one can use Corollary 4.10 to verify [14, Thm. 7.3, 7.6], i.e. the characteristic connection

\[ (1) \text{There exists a } \nabla^c \text{-parallel spinor } \varphi_1 \in \Sigma^g_4 \mathcal{M}, \text{ or } \varphi_1 \in \Sigma^g_4 \mathcal{M}, \text{ if and only if for any } s \in \mathbb{R} \text{ the eigenvalues of the Ricci tensor } \text{Ric}^g \text{ are given by } \{ (6-32s^2), (6-32s^2), (6-32s^2), (6-32s^2), -4(16s^2-1) \}. \]

\[ (2) \text{There exists a } \nabla^c \text{-parallel spinor } \varphi_0 \in \Sigma^g_5 \mathcal{M}, \text{ if and only if for any } s \in \mathbb{R} \text{ the eigenvalues of the Ricci tensor } \text{Ric}^g \text{ are given by } \{ -(2+32s^2), -(2+32s^2), -(2+32s^2), -(2+32s^2), -(2+32s^2), -4(16s^2-1) \}. \]

**Proof.** We begin again with the action of the endormorphism $\text{Ric}^g(\xi)$. As before, this is independent of which subbundle $\Sigma^g_6 \mathcal{M}$ ($\gamma \in (-4, 0, 4)$) the $\nabla^c$-parallel spinor is lying in. So, assume that $\psi$ is a $\nabla^c$-parallel spinor such that $\psi \in \Sigma^g_6 \mathcal{M}$ for some (constant) $\gamma \in \mathbb{R}$. For the computation of the first term in Theorem 4.16 it is useful to remind that the Reeb vector field $\xi$ is $\nabla^c$-parallel, in particular $\xi$ is a Killing vector field and consequently $\nabla^c_\xi \psi = \frac{1}{2} \xi \cdot \psi$, see [15, 4]. Thus we compute

\[ \nabla^c_{e_1} \xi = e_2, \quad \nabla^c_{e_2} \xi = -e_1, \quad \nabla^c_{e_3} \xi = e_4, \quad \nabla^c_{e_4} \xi = -e_3, \quad \nabla^c_{e_5} \xi = 0. \]

Let us also set $\mathcal{W} := - \sum_{j=1}^{5} e_j \cdot (T(\xi, e_j), T) = \sum_{j=1}^{5} \cdot j (T(\xi, e_j), (e_j, T))$. Then we deduce that

\[ \mathcal{W} = 2 \left[ \nabla^c_{e_1} \xi \cdot (e_1, T) + \nabla^c_{e_2} \xi \cdot (e_2, T) + \nabla^c_{e_3} \xi \cdot (e_3, T) + \nabla^c_{e_4} \xi \cdot (e_4, T) \right] \]

\[ = 4 \left[ e_2 \cdot e_{25} + e_1 \cdot e_{15} + e_4 \cdot e_{45} + e_3 \cdot e_{35} \right] = -16\xi. \]

One finishes with the first term, after a multiplication with the coefficient $(16s^2-1)/4$,

\[ -\frac{16s^2-1}{4} \sum_{j=1}^{5} e_j \cdot (T(\xi, e_j), T) \cdot \psi = \frac{16s^2-1}{4} \sum_{j=1}^{5} T(X, e_j) \cdot (e_j, T) \cdot \psi = -4(16s^2-1) \xi \cdot \psi. \]

Since $\xi, \sigma T = 0$, our claim follows, $\text{Ric}^g(\xi) \cdot \psi = -4(16s^2-1) \xi \cdot \psi$, for any $s \in \mathbb{R}$. Let us proceed now with the action of $\text{Ric}^g(X)$, for some $X \in \{e_1, \ldots, e_4\}$. We analyse only the case $X = e_1$ and similarly are
treated the other vectors. At this point it is sufficient to assume that \( \psi \in \Sigma^g M \) \((\gamma \in \mathbb{R})\) is a \(\nabla^c\)-parallel spinor (we use the fact that \( \psi := \varphi_1 \in \Sigma^g_{2,4} M \), or \( \psi := \varphi_0 \in \Sigma^g_0 M \), only at the final step). We compute
\[
T(e_1, e_1) = T(e_1, e_3) = T(e_1, e_4) = 0 \quad \text{and} \quad T(e_1, e_2) = 2e_5, \quad T(e_1, \xi) = -T(\xi, e_1) = -2\nabla^g_\xi \xi = -2e_2.
\]
Hence for the first term in Theorem 4.7 we deduce that
\[
- \sum_{j} e_j \cdot (T(e_1, e_j) \cdot T) = \sum_{j} T(e_1, e_j) \cdot (e_j \cdot T) = \left[ T(e_1, e_2) \cdot (e_2 \cdot T) + T(e_1, \xi) \cdot (\xi \cdot T) \right]
\]
\[
= -2 \left[ 2e_5 \cdot e_1 + e_2 \cdot d\eta \right] = -4 \left[ 2e_1 + H \right],
\]
since inside \( \mathcal{C}(5) \) we get \( e_2 \cdot d\eta = 2(e_1 + H) \), where \( H := e_{234} = \frac{1}{4}(e_1, \sigma_T) \). Multiplying with the coefficient \((16s^2 - 1)/4\), this gives rise to
\[
- \frac{16s^2 - 1}{4} \sum_{j=1}^5 e_j \cdot (T(e_1, e_j) \cdot T) \cdot \psi = \sum_{j=1}^5 T(e_1, e_j) \cdot (e_j \cdot T) \cdot \psi = -(16s^2 - 1) \left[ 2e_1 + H \right] \cdot \psi.
\]
Moreover, it is \((16s^2 + 3)H \cdot \psi \) and thus
\[
\text{Ric}^s(e_1) \cdot \psi = 2(1 - 16s^2)e_1 \cdot \psi + 4H \cdot \psi.
\]
The final step includes the action of the 3-form \( H := e_{234} = \frac{1}{4}(e_1, \sigma_T) \) on \( \Sigma^g_{2,4} M \) and \( \Sigma^g_0 M \), respectively, which of course is related to the endomorphism \( \text{Ric}^s(e_1) = (e_1, \sigma_T) \) and is computed algebraically, see [14] pp. 324-325:
\[
H \cdot \psi = \begin{cases} 
e \psi, & \text{if} \quad \psi := \varphi_1 \in \Sigma^g_{2,4} M, \\
-e_1 \cdot \psi, & \text{if} \quad \psi := \varphi_0 \in \Sigma^g_0 M. 
\end{cases}
\]
Consequently
\[
(4.16) \quad \text{Ric}^s(e_1) \cdot \psi = \begin{cases} 2(1 - 16s^2)e_1 \cdot \psi + 4e_1 \cdot \psi = (6 - 32s^2)\psi, & \text{if} \quad \psi := \varphi_1 \in \Sigma^g_{2,4} M, \\
2(1 - 16s^2)e_1 \cdot \psi - 4e_1 \cdot \psi = -(2 + 32s^2)\psi, & \text{if} \quad \psi := \varphi_0 \in \Sigma^g_0 M, 
\end{cases}
\]
for any \( s \in \mathbb{R} \). For the converse, assume that \((M^5, g, \xi, \eta, \phi)\) is a 5-dimensional simply-connected Sasakian spin manifold whose Ricci tensor \( \text{Ric}^s \) \((s \in \mathbb{R})\) satisfies \([14,15]\). Then, for \( s = 0, 1/4 \) we see that \( 4.10 \) induces the desired prescribed conditions, i.e. \( \text{Ric}^g = \text{diag}(6, 6, 6, 6, 4), \text{Ric}^c = \text{diag}(4, 4, 4, 4, 0) \) for \( \varphi_1 \in \Sigma^g_{2,4} M \) and \( \text{Ric}^g = \text{diag}(-2, -2, -2, -2, 4), \text{Ric}^c = \text{diag}(-4, -4, -4, 0) \) for \( \varphi_0 \in \Sigma^g_0 M \), respectively. Hence the assertion follows as in [14]. This finishes the proof. \( \blacksquare \)

**Remark 4.17.** The stated expressions of the Ricci endomorphism \( \text{Ric}^s(X) \) can be also obtained by applying the general type \( \text{Ric}^s = \text{Ric}^g - 4s^2S = \text{Ric}^c - (16s^2 - 12)S \), where for the action of the symmetric endomorphism \( S(X) \) on the related \( \nabla^c \)-parallel spinor \( \psi \in \Sigma^g M \) one can apply Lemma [14,13]. We refer also to [14] for \( S(X) \).

5. **On the differential operator** \( \mathcal{D}^s = \sum_i (e_i \cdot T) \cdot \nabla^s \)

### 5.1. Special \( \mathcal{D}^s \)-eigenspinors.

Next we examine some special eigenspinors of the differential operator
\[
\mathcal{D}^s(\varphi) = \sum_i (e_i \cdot T) \cdot \nabla^s \varphi = \mathcal{D}^0(\varphi) + s \sum_i (e_i \cdot T) \cdot (e_i \cdot T) \cdot \varphi = \mathcal{D}^0(\varphi) + sT \cdot \varphi_0,
\]
appearing in Theorem [3.3] see [14,14]. Here, \( \mathcal{D}^0 \) denotes the part corresponding to the Riemannian connection \( \nabla^0 \equiv \nabla^g \) and \( T := \sum_j (e_j \cdot T) \cdot (e_j \cdot T) = 2sT - 3||T||^2. \) Notice also by [3.5], that
\[
\mathcal{D}^s(\varphi) = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \nabla^s T(e_i, e_j) \varphi.
\]
In fact, this formula holds also in the general case (although in the proof of Theorem [3.3] we use the assumption \( \nabla^c T = 0 \), this does not effect to the computations related to \( \mathcal{D}^s \)). By Proposition [2.2] (5) and Remark [2.2] one also has (see [14,11,2])
\[
(5.1) \quad \mathcal{D}^s(\varphi) = -\frac{1}{2} \left[ D^s(T \cdot \varphi) + T \cdot D^s(\varphi) - (dT + \delta T) \cdot \varphi + 8s\sigma_T \cdot \varphi \right].
\]
Let us focus now on triples \((M^5, g, T)\) with \( \nabla^c \)-parallel skew-torsion, \( \nabla^c T = 0 \). In this case the operator \( \mathcal{D}^s \) has more equivalent expressions.
Lemma 5.1. (4) Consider a Riemannian spin manifold \((M^n, g, T)\) \((n \geq 3)\) endowed with a non-trivial 3-form \(T \in \Lambda^n T^*M\), such that \(\nabla^c T = 0\), where \(\nabla^c := \nabla^g + \frac{1}{2} T\). Then, the operator \(\mathfrak{D}^s\) is given by

\[
\mathfrak{D}^s(\varphi) = -\frac{1}{2} \left[ D^s(T \cdot \varphi) + T \cdot D^s(\varphi) - 2(1 - 4s)\sigma_T \cdot \varphi \right]
\]

(5.2)

\[
\mathfrak{D}^s(\varphi) = -\frac{1}{2} \sum_j \langle e_j \cdot T \cdot \nabla^s_{e_j} \varphi \rangle - \frac{1}{2} T \cdot D^s(\varphi),
\]

where \(D^s\) is the (generalized) Dirac operator induced by \(\nabla^s\).

\[\text{Proof.}\] The first formula is an immediate consequence of (5.1). For the second description, we use (2.1), the definition of \(\mathfrak{D}^s\) and relation (4.3). Then, for some arbitrary spinor fields \(\varphi, \psi\) we conclude that

\[
\langle \mathfrak{D}^s(\varphi), \psi \rangle = -\frac{1}{2} \sum_j \langle e_j \cdot T \cdot \nabla^s_{e_j} \varphi, \psi \rangle - \frac{1}{2} \sum_j \langle T \cdot e_j \cdot \nabla^s_{e_j} \varphi, \psi \rangle
\]

\[
= -\frac{1}{2} \sum_j \langle e_j \cdot T \cdot \nabla^s_{e_j} \varphi, \psi \rangle - \frac{1}{2} \sum_j \langle e_j \cdot \nabla^s_{e_j} \varphi, T \cdot \psi \rangle
\]

\[
= -\frac{1}{2} \sum_j \langle e_j \cdot T \cdot \nabla^s_{e_j} \varphi, \psi \rangle - \frac{1}{2} \langle T \cdot D^s(\varphi), \psi \rangle,
\]

which gives rise to (5.3). \(\square\)

Therefore, when the torsion is \(\nabla^c\)-parallel, it is \(\sum_j e_j \cdot T \cdot \nabla^s_{e_j} \varphi = D^s(T \cdot \varphi) - 2(1 - 4s)\sigma_T \cdot \varphi = \text{grad}(\gamma) \cdot \varphi + \gamma D^c(\varphi) - 2(1 - 4s)\sigma_T \cdot \varphi\), where \(\gamma \in \text{Spec}(T)\) denotes an eigenvalue of \(T\), i.e. we assume (without loss of generality) that \(\varphi \in \Sigma^g\) for some real function \(\gamma\), not necessarily constant. Let us begin our investigation with \(\nabla^c\)-parallel spinors, where \(\gamma\) is a real constant.

Proposition 5.2. Consider a Riemannian spin manifold \((M^n, g, T)\) \((n \geq 3)\) with \(\nabla^c T = 0\), where \(\nabla^c := \nabla^g + \frac{1}{2} T\) is the metric connection with skew-torsion \(0 \neq T \in \Lambda^n T^*M\). Assume that \(\varphi_0 \in \Sigma^g\) is a non-trivial \(\nabla^c\)-parallel spinor and \(\gamma \in \text{Spec}(T)\) is an eigenvalue of \(T\). Then, \(\varphi_0\) is an eigenspinor of the operator \(\mathfrak{D}^s\) for any \(s \in \mathbb{R}\),

\[
\mathfrak{D}^s(\varphi_0) = -\frac{(4s - 1)}{4} \left[ T^2 + 2\|T\|^2 \right] \cdot \varphi_0 = -\frac{(4s - 1)}{4} \left[ \gamma^2 + 2\|T\|^2 \right] \varphi_0.
\]

\[\text{Proof.}\] Based on (4.5) and the definition of \(\mathfrak{D}^s\), we see that any spinor field \(\varphi \in F^g\) satisfies

\[
\mathfrak{D}^s(\varphi) = \mathfrak{D}^c(\varphi) + \frac{(4s - 1)}{4} T \cdot \varphi = \mathfrak{D}^s(\varphi) + \frac{(4s - 1)}{4} \left[ 2\sigma_T - 3\|T\|^2 \right] \cdot \varphi,
\]

where \(\mathfrak{D}^c := \sum_j (e_j \cdot J) \cdot \nabla^c_{e_j}\) is the operator associated to \(\nabla^c\). Thus, if \(\nabla^c \varphi_0 = 0\), then \(\mathfrak{D}^c(\varphi_0) = 0\) and the claim immediately follows in combination with \(\sigma_T \cdot \varphi_0 = \frac{1}{2} (\|T\|^2 - T^2) \cdot \varphi_0\) (cf. (4)). Of course, the same occurs by applying (5.3). Indeed, we rely again on (4.6) and compute that

\[
\mathfrak{D}^s(\varphi_0) = -\frac{1}{2} \sum_j e_j \cdot T \cdot \nabla^s_{e_j} \varphi_0 - \frac{1}{2} T \cdot D^s(\varphi_0)
\]

\[
= -\frac{4s - 1}{8} \sum_j e_j \cdot T \cdot (e_j \cdot J) \cdot \varphi_0 - \frac{3(4s - 1)}{8} \varphi_0.
\]

However, it is \(e_j \cdot T = -T \cdot e_j - 2(e_j \cdot J)\), hence one can write

\[
\mathfrak{D}^s(\varphi_0) = \frac{4s - 1}{8} \sum_j T \cdot e_j \cdot (e_j \cdot J) \cdot \varphi_0 + \frac{2(4s - 1)}{8} \sum_j (e_j \cdot J) \cdot (e_j \cdot J) \cdot \varphi_0 - \frac{3(4s - 1)}{8} \varphi_0
\]

\[
= \frac{3(4s - 1)}{8} \varphi_0 + \frac{(4s - 1)}{4} (2\sigma_T - 3\|T\|^2) \cdot \varphi_0 - \frac{3(4s - 1)}{8} \varphi_0
\]

\[
= \frac{(4s - 1)}{4} (2\sigma_T - 3\|T\|^2) \cdot \varphi_0 = -\frac{(4s - 1)}{4} T \cdot \varphi_0.
\]
Thus the assertion follows by using the relations $T = -(T^2 + 2\|T\|^2)$ and $T^2 \cdot \varphi_0 = \gamma^2 \cdot \varphi_0$. ■

The action of the operator $\mathcal{D}^s$ on Killing spinors and twistor spinors (with torsion or not), with respect to the family $\nabla^s$, is known by [9]. In particular, for a non-trivial element $\varphi_0 \in \ker(P^s)$ for some $s \in \mathbb{R}$ and independently of the assumption $\nabla^s T = 0$, it is not hard to show that

**Proposition 5.3.** (9) Consider a Riemannian spin manifold $(M^n, g, T)$ $(n \geq 3)$ endowed with a non-trivial 3-form $T \in \Lambda^3 T^* M$ and the one-parameter family of metric connections $\nabla^s = \nabla^g + 2sT$. Then, any twistor spinor $\varphi_0 \in \ker(P^s)$ (with torsion or not), with respect to $\nabla^s$ for some $s \in \mathbb{R}$, satisfies

\[
\mathcal{D}^s(\varphi_0) = -\frac{3}{n} T \cdot D^s(\varphi_0).
\]

Moreover, if $\varphi_0 \in K^s(M, g)\zeta$ for some $s \in \mathbb{R} \setminus \{0, 1/4\}$ and $\zeta \neq 0$, then $\mathcal{D}^s(\varphi_0) = 3\zeta T \cdot \varphi_0$ and similarly, if $\varphi_0 \in K^\kappa(M, g)\kappa$ for some $\kappa \neq 0$, then $\mathcal{D}^\kappa(\varphi_0) = 3\kappa T \cdot \varphi_0$.

**Corollary 5.4.** Whenever $\nabla^c T = 0$, a non-trivial $KsT$ (resp. real Killing spinor) $\varphi_0$ induces a non-trivial eigenspinor of $\mathcal{D}^s$ for the same $s$ (resp. for $s = 0$) with eigenvalue $\beta = 3\gamma\zeta$, (resp. $\beta = 3\gamma\kappa$), where $\gamma \in \Spec(T)$ is the corresponding $T$-eigenvalue.

**Example 5.5.** Consider a 6-dimensional (strict) nearly Kähler manifold $(M^6, g, J)$, an almost Hermitian manifold endowed with a non-integrable almost complex structure $J$ such that $(\nabla^J_X) X = 0$. By [13, Thm. 10.1] it is known that $M^6$ admits an (unique) characteristic connection $\nabla^c$ with parallel skew-torsion, given by $T(X, Y) := (\nabla^J_X)JY$. Moreover, there exist two $\nabla^c$-parallel spinors $\varphi^\pm$ such that $\mathcal{F}^g(\pm 2\|T\|)$, i.e. $T \cdot \varphi^\pm = \pm 2\|T\| \cdot \varphi^\pm$. Thus, by Proposition 5.3 we get

\[
\mathcal{D}^s(\varphi^\pm) = -\frac{3(4s - 1)^2\|T\|^2}{2} \varphi^\pm.
\]

On the other hand, $\varphi^\pm$ are real Killing spinors with $\kappa := \mp\|T\|/4$, $\mathrm{Ts}T$ with torsion for any $s \neq 1/4$, i.e. $\varphi^\pm \in \ker(P^s|\Sigma^g_2|\Sigma^c_1)M$ and $\mathrm{Ts}T$ for any $s \neq 0, 1/4$, with Killing number $\zeta := \mp\frac{4s - 1}{4}\|T\|$, see [9, Thm. 4.1]. Therefore, (5.3) is deduced also by applying Corollary 5.4.

Under the condition $\nabla^c T = 0$, a kind of converse of Proposition 5.3 reads as follows:

**Proposition 5.6.** Consider a triple $(M^n, g, T)$ $(n \geq 3)$ with $\nabla^c T = 0$, where $\nabla^c := \nabla^g + \frac{1}{4}T$ is the metric connection with skew-torsion $0 \neq T \in \Lambda^3 T^* M$. Assume that $\varphi_0 \in \Gamma(\Sigma^g_2) \cap \ker(P^s) := \ker(P^s|\Sigma^g_2)$ is a non-trivial restricted twistor spinor (with torsion or not), for some $s \in \mathbb{R}$ and some non-zero constant eigenvalue $0 \neq \gamma \in \Spec(T)$. If $\varphi_0$ is a $\mathcal{D}^s$-eigenspinor, i.e. $\mathcal{D}^s(\varphi_0) = \beta\varphi_0$ for some constant eigenvalue $\beta$, then

\[
D^s(\varphi_0) = \frac{(n - 6)\beta}{3\gamma} \varphi_0 + \frac{2(1 - 4s\beta)}{\gamma} \sigma^T \cdot \varphi_0.
\]

If $n = 6$ or $\beta = 0$, i.e. $\varphi_0 \in \ker(\mathcal{D}^s)$, then $D^s(\varphi_0) = \frac{2(1 - 4s\beta)}{\gamma} \sigma^T \cdot \varphi_0$.

**Proof.** Since $\mathcal{D}^s(\varphi_0) = \beta\varphi_0$ and $\varphi_0 \in \ker(P^s|\Sigma^g_2 M)$, the type (5.4) reduces to $\sigma^T \cdot D^s(\varphi_0) = -\frac{n\beta}{3\gamma} \varphi_0$ and since $\gamma \neq 0$ is a real constant such that $\sigma^T \varphi_0 = \gamma \varphi_0$, our claim follows by relation (5.2). ■

**Corollary 5.7.** If $\varphi_0 \in \ker(P^s|\Sigma^g_2 M)$ is a non-trivial restricted twistor spinor with torsion with respect to $\nabla^c$ and $\varphi_0$ is $\mathcal{D}^s$-harmonic, i.e. $\beta = 0$ and hence $\mathcal{D}^s(\varphi_0) = 0$, then $D^s(\varphi_0) = 0$, in particular $\varphi_0$ is $\nabla^c$-parallel, i.e. $\varphi_0 \in \ker(\nabla^c)$.

**Proof.** This follows by Proposition 5.6 in combination with [9] Lem. 2.2], see also [9, p. 119] for details. ■

We deduce that the relation $\varphi_0 \in \ker(\mathcal{D}) \cap \ker(P^s|\Sigma^g_2 M)$ for some constant $\gamma \neq 0$, is a very strong condition which in fact implies the $\nabla^c$-parallelism of $\varphi_0$, similarly with the condition $\varphi_0 \in \ker(D^c) \cap \ker(P^c|\Sigma^c_2 M)$. Hence, in general we avoid to consider this kind of $\mathrm{Ts}T$, as in [9].

Recall finally by Proposition 5.2 that for a $\nabla^c$-parallel spinor field $\varphi_0$ the relation $\mathcal{D}^s(\varphi_0) = \beta\varphi_0$ is always verified with $\beta = \frac{4s - 1}{4\sigma^T - 3\|T\|^2}$. Adding now the extra condition $\varphi_0 \in \ker(P^s|\Sigma^g_2 M)$ for some constant
γ ≠ 0 and s ≠ 1/4, then for 3 ≤ n ≤ 8 we see that relation (5.3) gives rise to an alternative way to verify that ϕ₀ is actually a KsT with ζ = \frac{3(4s - 1)}{4n} (see [11] Thm. 3.7), i.e. \( D^s(ϕ₀) = \frac{3(4s - 1)}{4n} \cdot ϕ₀ \) as it should be according to (3.20). For such a proof one may use the formulas \( γ^2 = \frac{2n}{3n - 7} ||T||^2 \) and \( σ_T \cdot ϕ₀ = -\frac{3s(n - 3)}{4n} ϕ₀ \), given in [12] Prop. 3.2. For n = 6, relation (5.6) is simplified and we do not need the explicit form of β. This case of course applies on nearly Kähler manifolds. For nearly parallel \( G_2 \)-manifolds \((M^7, g, ω)\) and for the unique \( \nabla^c \)-parallel spinor \( ϕ₀ ∈ \text{Ker}(P^s|_{\nabla^c = 0})_{M} \) one computes \( β = \frac{9(4s - 11)}{4} ||T||^2 \) via Proposition 5.2 hence (5.10) in combination with \( σ_T \cdot ϕ₀ = -3 ||T||^2 \) yield the result:
\[
D^s(ϕ₀) = \frac{-21(4s - 1)}{4\sqrt{7}} ||T|| \cdot ϕ₀ = \frac{3(4s - 1)γ}{4} \cdot ϕ₀,
\]
thus \( ϕ₀ ∈ K^s(M^7, g)_{\zeta} \) with \( ζ = -\frac{3(4s - 1)}{4\sqrt{7}} ||T|| \) (cf. [11] Thm. 4.2).

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