A MOPERADIC APPROACH TO CYCLOTOMIC ASSOCIATORS

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Abstract. This is a companion paper to Elliptomic associators [4]. We provide a (m)operadic description of Enriquez’s torsor of cyclotomic associators, as well as of its associated cyclotomic Grothendieck–Teichmüller groups.

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Introduction

Since the introduction of associators and Grothendieck–Teichmüller groups by Drinfeld [5], several variations of these have been considered; for instance

- following Grothendieck’s esquisse [10], Lochak–Nakamura–Schneps defined a new version of the Grothendieck–Teichmüller group [14], which acts on more general surface mapping class groups than Drinfeld’s original one;
- cyclotomic [6] and elliptic [7] variants of associators and Grothendieck–Teichmüller groups were discovered by Enriquez;
- ellipsitomic associators, which share both the features of cyclotomic and elliptic associators, have recently been introduced by the authors [4].

It is known, after the original insight of Drinfeld [5] and Bar-Natan [2], and thanks to the recent detailed proof of Fresse [9], that the torsor of associators can be understood as the torsor of isomorphisms between two operads in groupoids. A similar result holds for Enriquez’s torsor of elliptic associators as well, as was recently proven in [4], where one has to consider operadic modules instead of just operads. This need comes from the fact that, while compactified configuration spaces of points in the plane form an operad, compactified configuration spaces of points in a torus form an operadic module on the latter. Still in [4],
ellipsitomic associators are defined as operadic module isomorphisms, and the description à la Drinfeld is derived from it afterwards.

In this companion paper to [4], we prove that Enriquez’s cyclotomic associators torsor (resp. Grothendieck–Teichmüller groups) can also be indentified with isomorphisms (resp. automorphisms) of operadic gadgets. The appropriate notion here is the one of a moperad; it was introduced by Willwacher [18], and it typically encodes the structure of compactified configuration spaces of points in the punctured plane (or, equivalently, the annulus).

After two reminders on moperads (Section 1) on the one hand, and associators (Section 2) on the other hand, we introduce in Section 3 the moperad $\text{PaB}^1$ of parenthesized braids with a frozen strand (obtained as the fundamental groupoid of the configuration moperad of points in the punctured plane) and provide a generators and relations presentation for $\text{PaB}^1$:

**Theorem** (Theorem 3.4). The moperad in groupoid $\text{PaB}^1$ is generated by an arity 1 arrow $E$ and an arity 2 arrow $\Psi$, with relations $(cU)$, $(MP)$, $(RP)$, and $(O)$.

Unsurprisingly, these relations are completely analogous to the axioms for braided module categories from [3]; indeed, one can verify that a braided module category is nothing but a representation of $\text{PaB}^1$ in categories. In Section 4, we decorate the unfrozen strands of our parenthesized braids with elements from a finite quotient $\Gamma = \mathbb{Z}/n\mathbb{Z}$ of the fundamental group of the punctured plane, giving rise to a moperad in groupoids $\text{PaB}^\Gamma$. We show that $\text{PaB}^\Gamma$ admits a presentation by generators and relations similar to the one of $\text{PaB}^1$ (Theorem 4.6), and thus identify the group of $\Gamma$-equivariant automorphisms of $\text{PaB}^\Gamma$ that are the identity on objects with Enriquez’s cyclotomic Grothendieck–Teichmüller group (Proposition 4.10). Finally, in Section 5, we put a moperad structure on the (parenthesized) horizontal $N$-chord diagrams of [3], and prove the following

**Theorem** (Theorem 5.5). The set of $\Gamma$-equivariant moperad isomorphisms that are the identity on objects between $\text{PaB}^\Gamma$ and parenthesized $N$-chord diagrams is in bijection with Enriquez’s cyclotomic associators.

We moreover show that this identification respects the (bi)torsor structures (Theorem 5.13).

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**Convention.** All along the paper, $k$ is a field of characteristic zero.
1. Moperads

In this section we fix a symmetric monoidal category $(C, \otimes, 1)$ having small colimits and such that $\otimes$ commutes with these. We borrow the notation and conventions for $\mathfrak{S}$-modules and operads from [4].

1.1. Moperads over an operad. Let $O$ be an operad. A moperad over an operad $O$ is an $\mathfrak{S}$-module $P$ carrying

• a unital monoid structure for the monoidal product $\otimes$,
• and a left $O$-module structure for the monoidal product $\circ$, that are compatible in the following sense:
  - One first observes that there is a natural map $(O \circ P) \otimes Q \rightarrow O \circ (P \otimes Q)$.
  - Then the compatibility means that the following diagram commutes:

$$
\begin{array}{c}
(O \circ P) \otimes P \\
\downarrow \\
O \circ (P \otimes P) \\
\downarrow \\
P
\end{array}
$$

The map $(O \circ P) \otimes P \rightarrow P$ one obtains decomposes into maps

$$
P(k) \otimes P(m_0) \otimes O(m_1) \otimes \cdots \otimes O(m_k) \rightarrow P(m_0 + \cdots + m_k)
$$

satisfying certain associativity, unit and $\mathfrak{S}$-equivariance relations. We let the reader spell out these conditions explicitly.

We leave it as an exercise to check that, within the symmetric monoidal category of differential graded vector spaces, this definition coincides with Willwacher’s one from [18] (from which we borrowed the name “moperad”). Note that the monoid structure for the monoidal product $\otimes$ encodes precisely the partial composition with respect to the second colour. We will denote this partial composition by $\circ_0$.

1.2. Example of a moperad over an operad: coloured Stasheff polytopes. To any finite set $I$ we associate the configuration space

$$
\text{Conf}(\mathbb{R}_{>0}, I) = \{ x = (x_i)_{i \in I} \in (\mathbb{R}_{>0})^I | x_i \neq x_j \text{ if } i \neq j \}
$$

and its reduced version

$$
\text{C}(\mathbb{R}_{>0}, I) := \text{Conf}(\mathbb{R}_{>0}, I) / \mathbb{R}_{>0}.
$$

The Axelrod–Singer–Fulton–MacPherson compactification $\overline{\text{C}}(\mathbb{R}_{>0}, I)$ of $\text{C}(\mathbb{R}_{>0}, I)$ is a disjoint union of $|I|$-th Stasheff polytopes with two kinds of colours, indexed by $\mathfrak{S}_I$. The boundary

$$
\partial \overline{\text{C}}(\mathbb{R}_{>0}, I) := \overline{\text{C}}(\mathbb{R}_{>0}, I) - \text{C}(\mathbb{R}_{>0}, I)
$$

is the union, over all partitions $I = J_0 \sqcup J_1 \sqcup \cdots \sqcup J_k$, of

$$
\partial_{J_0, \ldots, J_k} \overline{\text{C}}(\mathbb{R}_{>0}, I) := \overline{\text{C}}(\mathbb{R}_{>0}, k) \times \overline{\text{C}}(\mathbb{R}_{>0}, J_0) \times \prod_{i=1}^k \overline{\text{C}}(\mathbb{R}, J_i).
$$
The inclusion of boundary components provides $\overline{C}(\mathbb{R}_{>0}, -)$ with the structure of a $\overline{C}(\mathbb{R}, -)$-moperad in topological spaces.

One can see that $\overline{C}(\mathbb{R}_{>0}, I)$ is a manifold with corners, and that considering only zero-dimensional strata of our configuration spaces we get a sub-moperad $Pa_0 \subset \overline{C}(\mathbb{R}_{>0}, -)$ that can be shortly described as follows:

- $Pa_0(I)$ is the set of pairs $(\sigma, p)$ with $\sigma$ is a linear order on $I$ and $p$ a maximal parenthesization of $\underbrace{0 \cdot \cdot \cdot }_{|I| \text{ times}}$ such that there is no action of $\mathfrak{S}_n$ on 0, but this element can be inside a parenthesis. This means that we allow points to be near the origin.
- The $\overline{C}(\mathbb{R}, -)$-moperad structure is given by substitution as above.

Forgetting the $\overline{C}(\mathbb{R}, -)$-moperad structure on $\overline{C}(\mathbb{R}_{>0}, -)$ and considering a $\overline{C}(\mathbb{R}, -)$-module structure on it amounts to forbidding points to be close to the origin (i.e. the 0-strand cannot be inside a parenthesis in this case).

1.3. Pointing. Recall the operad $\text{Unit}$ defined by

$$\text{Unit}(n) := \begin{cases} 1 & \text{if } n = 0, 1 \\ \emptyset & \text{else} \end{cases}$$

By convention, all our operads $\mathcal{O}$ will be pointed in the sense that they will come equipped with a specific operad morphism $\text{Unit} \to \mathcal{O}$. Morphisms of operads are required to be compatible with this pointing. Actually, all operads appearing in this paper are such that $\mathcal{O}(n) \cong 1$ if $n = 0, 1$.

Similarly, we introduce the moperad $M\text{Unit}$ over $\text{Unit}$, which is such that $M\text{Unit}(n) = 1$ for all $n \geq 0$. By convention, all our moperads will be pointed, in the sense that they will come equipped with a specific $\text{Unit}$-moperad morphism $M\text{Unit} \to \mathcal{Q}$. Morphisms of moperads are required to be compatible with the pointing.

Remar 1.1. In the category of sets, $M\text{Unit}$ is the sub-$\text{Unit}$-moperad of $Pa_0$ that consists only of the left-most maximal parenthesization.

The main reason for these rather strange conventions is that we need the following features, that we have in the case of compactified configuration spaces:

- For operads and moperads, we want to have “deleting operations” $\mathcal{O}(n) \to \mathcal{O}(n - 1)$ that decrease arity.
- For a moperad, we want to be able to “see the operad inside” it, i.e. we want to have a distinguished morphism $\mathcal{O} \to \mathcal{P}$ of $\mathfrak{S}$-modules.

Example 1.2. For instance, being a $Pa$-moperad, $Pa_0$ comes together with a morphism of $\mathfrak{S}$-modules $Pa \to Pa_0$. We let the reader check that it sends a parenthesized permutation $p$ to $0(p)$. 
1.4. Group actions. Let $G$ be a group and $\mathcal{O}$ be an operad. We say that an $\mathcal{O}$-module $P$ carries a $G$-action if

- for every $n \geq 0$, $G^n$ acts $S_n$-equivariantly on $P(n)$, from the left.
- for every $m \geq 0$, $n \geq 0$, and $1 \leq i \leq n$, the partial composition
  \[ \circ_i : P(n) \otimes \mathcal{O}(m) \longrightarrow P(n + m - 1) \]
  is equivariant along the group morphism
  \[ G^m \longrightarrow G^{n+m-1} \]
  \[ (g_1, \ldots, g_n) \longmapsto (g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n) \text{ } \text{-times} \]

If $P$ is a moperad, we additionally require that the partial composition

\[ \circ_0 : P(n) \otimes P(m) \longrightarrow P(n + m) \]

is $G^{n+m}$-equivariant.

A morphism $P \to Q$ of $\mathcal{O}$-moperads with $G$-action is said $G$-equivariant if, for every $n \geq 0$, the map $P(n) \to Q(n)$ is $G^n$-equivariant.

2. Reminders on associators and $G(R)T$

In this Section we recollect some results from [5, 2, 9], following essentially the presentation of [4, Section2].

2.1. Compactified configuration space of the plane. To any finite set $I$ we associate the (reduced) configuration space

\[ C(\mathbb{C}, I) := \{ z = (z_i)_{i \in I} \in \mathbb{C}^I | z_i \neq z_j \text{ if } i \neq j \}/\mathbb{C} \times \mathbb{R}_{>0} \]

of points in the plane. We then consider its Axelrod–Singer–Fulton–MacPherson compactification $\overline{C}(\mathbb{C}, I)$, whose boundary

\[ \partial \overline{C}(\mathbb{C}, I) = \overline{C}(\mathbb{C}, I) - C(\mathbb{C}, I) \]

is made of irreducible components $\partial_{J_1, \ldots, J_k} \overline{C}(\mathbb{C}, I)$ indexed by partitions $I = J_1 \bigsqcup \cdots \bigsqcup J_k$ of $I$:

\[ \partial_{J_1, \ldots, J_k} \overline{C}(\mathbb{C}, I) \cong \overline{C}(\mathbb{C}, k) \times \prod_{i=1}^k \overline{C}(\mathbb{C}, J_i) . \]

The inclusion of boundary components provides $\overline{C}(\mathbb{C}, -)$ with the structure of an operad in topological spaces.
2.2. The operad of parenthesized braids. The inclusions of topological operads
\[ \text{Pa} \subset \mathcal{C}(\mathbb{R}, -) \subset \mathcal{C}(\mathcal{C}, -) \]
allows us to define
\[ \text{PaB} := \pi_1(\mathcal{C}(\mathcal{C}, -), \text{Pa}) , \]
which is an operad in groupoids.

**Example 2.1** (of arrows in small arity). Recall from [4, Examples 2.1] that, in arity two, there is an arrow from \((12)\) to \((12)\), we have an arrow \(R_{1, 2}\) going from \((12)\) to \((21)\), that can be depicted in the following ways:

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{1} \\
\text{2}
\end{array}
\quad
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{2} \\
\text{1}
\end{array}
\]

There is another arrow \(\tilde{R}_{1, 2}\) := \((R_{2, 1})^{-1}\), having the same source and target, that can be depicted as an undercrossing braid.

In arity three, there is an arrow \(\Phi_{1, 2, 3}\), going from \((12)3\) to \(1(23)\), that can be depicted in the following ways:

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\quad
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\]

A version of some claim of Grothendieck [10] about the (genus 0) (Grothendieck–)Teichmüller tower, later proven by Drinfeld [5], can be understood as a generator and relations presentation for \(\text{PaB}\). This was made more explicit by Bar-Natan [2] in a different language, and written in term of operads by Fresse [9, Theorem 6.2.4]. Following the convention and notation from [4, Section 2], it reads as follows

**Theorem 2.2.** As an operad in groupoids having \(\text{Pa}\) as operad of objects, \(\text{PaB}\) is freely generated by \(R := R_{1, 2}^{1, 2}\) and \(\Phi := \Phi_{1, 2, 3}^{1, 2, 3}\) together with the following relations:

1. (U) \(\Phi^{\emptyset, 2, 3} = \Phi^{1, 2, 3} = \Phi^{1, 2, \emptyset} = \text{Id}_{1, 2} \quad \text{in} \ \text{Hom}_{\text{PaB}(3)}((12, 12))\).
2. (H1) \(R_{1, 2}^{1, 2} \Phi^{2, 1, 3} R_{1, 3}^{1, 3} = \Phi^{1, 2, 3} R_{1, 2}^{1, 2, 3} \quad \text{in} \ \text{Hom}_{\text{PaB}(3)}(((12)3, 2(31)))\).
3. (H2) \(\tilde{R}_{1, 2}^{1, 2} \Phi^{2, 1, 3} \tilde{R}_{1, 3}^{1, 3} = \Phi^{1, 2, 3} \tilde{R}_{1, 2}^{1, 2, 3} \quad \text{in} \ \text{Hom}_{\text{PaB}(3)}(((12)3, 2(31)))\).
4. (P) \(\Phi^{12, 3, 4} = \Phi^{12, 3, \emptyset} = \Phi^{1, 23, 4} \Phi^{2, 3, 4} = \Phi^{1, 23, \emptyset} \quad \text{in} \ \text{Hom}_{\text{PaB}(4)}(((12)34, 1(234)))\).

In order to fix our braid group conventions we recall the following. The automorphism group \(\text{Aut}_{\text{PaB}(n)}(p)\) of any parenthesized permutation \(p\) of length \(n\) is exactly the pure braid group \(\text{PB}_n\) on \(n\) strands, which is generated by elementary pure braids \(x_{ij}\), \(1 \leq i < j \leq n\),
which satisfy a certain list of relations (see [4] for more details). In this article we will depict the generator \( x_{ij} \) in the following two equivalent ways:

The group \( \text{PB}_n \) is the kernel of the map \( B_n \to \mathfrak{S}_n \) sending, for all \( 1 \leq i \leq n-1 \), the generators \( \sigma_i \) of \( B_n \) to the permutation \((i, i+1)\). The elements \( \sigma_i \) are depicted in the same way as the \( R \)'s:

### 2.3. The operad of (parenthesized) chord diagrams.
Recall [16, 9] that the collection of Kohno–Drinfeld Lie \( k \)-algebras \( t_n(k) \) is provided with the structure of an operad in the category \( \text{grLie}_k \) of positively graded finite dimensional Lie algebras over \( k \), with symmetric monoidal structure is given by the direct sum \( \oplus \). This is equivalent to Bar-Natan’s cabling operation [2] on chord diagrams.

Taking the degree completion of the universal enveloping algebra functor, we get an operad \( \text{CD}(k) := \hat{U}(t(k)) \) in complete filtered cocommutative Hopf algebras which we view as categories (with only one object) enriched in complete filtered cocommutative coalgebras, that we call the operad of chord diagrams.

By definition, the operad \( \text{Ob}(\text{CD}(k)) \) of object of \( \text{CD}(k) \) is the terminal operad in sets. We can thus define the \( \text{PaCD}(k) \), that we call the operad of parenthesized chord diagrams as the fake pull-back of \( \text{CD}(k) \) along the terminal morphism \( \text{Pa} \to * = \text{Ob}(\text{CD}(k)) \). We refer to [4] for the definition of fake pull-back; it is enough to know that \( \text{PaCD}(k) \) has \( \text{Pa} \) as operad of objects, and that in arity \( n \) the complete filtered cocommutative coalgebra of morphisms between any pair of objects is always \( \hat{U}(t_n(k)) \).

### 2.4. Drinfeld associators.
Recall that

- \( \text{Grpd}_k \) denote the (symmetric monoidal) category of \( k \)-prounipotent groupoids.
- for \( \mathcal{C} \) being \( \text{Grpd}, \text{Grpd}_k, \) or \( \text{Cat} \left( \text{CoAlg}_k \right) \) (see [4]), the notation

\[
\text{Aut}_{\text{OpC}}^* \quad (\text{resp. Is}_{\text{OpC}}^*)
\]

refers to those automorphisms (resp. isomorphisms) which are the identity on objects within the category \( \text{OpC} \) of operads in \( \mathcal{C} \).
A Drinfeld $k$-associator is an isomorphism between the operads $\overline{\text{PB}}(k)$ and $G\text{PaCD}(k)$ in $\text{Grpd}_k$, which is the identity on objects. We denote by

$$\text{Ass}(k) := \text{Iso}^\ast_{\text{op} \text{Grpd}_k}(\overline{\text{PB}}(k), G\text{PaCD}(k))$$

the set of $k$-associators. Drinfeld already implicitly showed in [5] that there is a one-to-one correspondence between the set of Drinfeld $k$-associators and the set $\text{Ass}(k)$ of couples $(\mu, \varphi) \in k^\times \times \exp(\hat{f}_2(k))$, such that

$$\varphi^{3,2,1} = (\varphi^{1,2,3})^{-1} \text{ in } \exp(\hat{f}_3(k)),$$

$$\varphi^{1,2,3} = \varphi^{2,3,1} = \varphi^{3,1,2} = e^{\mu(t_{13} + t_{21})/2} \text{ in } \exp(\hat{f}_3(k)),$$

$$\varphi^{1,2,3} = \varphi^{2,3,4} = \varphi^{1,2,3} \varphi^{3,4} \text{ in } \exp(\hat{f}_4(k)),$$

where $\varphi^{1,2,3} = \varphi(t_{12}, t_{23})$ is viewed as an element of $\exp(\hat{f}_3(k))$ via the inclusion $\hat{f}_2(k) \subset \hat{f}_3(k)$ sending $x$ to $t_{12}$ and $y$ to $t_{23}$. The proof of this result relies on the universal property of $\overline{\text{PB}}$ from Theorem 2.2. In particular, a morphism $F : \overline{\text{PB}}(k) \rightarrow G\text{PaCD}(k)$ is uniquely determined by a scalar parameter $\mu \in k$ and $\varphi \in \exp(\hat{f}_2(k))$ such that we have the following assignment in the morphism sets of the parenthesized chord diagram operad $\text{PaCD}$:

- $F(R^{1,2}) = e^{\mu t_{12}/2} X^{1,2}$,
- $F(\Phi^{1,2,3}) = \varphi(t_{12}, t_{23}) a^{1,2,3},$

where $R$ and $\Phi$ are the ones from Examples 2.1.

An example of such an associator is the $KZ$ associator $\Phi_{KZ}$. It is defined as the renormalized holonomy from 0 to 1 of $G'(z) = \left(\frac{\log z - \log z^{-1}}{z} + \frac{\log z - \log z^{-1}}{z^{-1}}\right) G(z)$, i.e., $\Phi_{KZ} := G_0 G_1^{-1} \in \exp(\hat{f}_3(\mathbb{C}))$, where $G_0, G_1$ are the solutions such that $G_0(z) \sim z^{t_{12}}$ when $z \rightarrow 0^+$ and $G_1(z) \sim (1 - z)^{t_{23}}$ when $z \rightarrow 1^-$. We have that $(2\pi i, \Phi_{KZ})$ is an element of $\text{Ass}(\mathbb{C})$.

2.5. Grothendieck–Teichmüller group. The Grothendieck–Teichmüller group is defined as the group

$$\text{GT} := \text{Aut}^\ast_{\text{op} \text{Grpd}}(\overline{\text{PB}})$$

of automorphisms of the operad in groupoids $\overline{\text{PB}}$ which are the identity of objects and its $k$-pro-unipotent version is

$$\overline{\text{GT}}(k) := \text{Aut}^\ast_{\text{op} \text{Grpd}_k}(\overline{\text{PB}}(k)).$$

In this article we will focus on the $k$-pro-unipotent version of this group in the cyclotomic situation. The group $\overline{\text{GT}}(k)$ is isomorphic to Drinfeld’s Grothendieck–Teichmüller group $\overline{\text{GT}}(k)$ consisting of pairs

$$(\lambda, f) \in k^\times \times \overline{\mathbb{F}}_2(k)$$

which satisfy the following equations:

- $f(x, y) = f(y, x)^{-1}$ in $\overline{\mathbb{F}}_2(k)$,
- $x_1^3 f(x_1, x_2) x_2^5 f(x_2, x_3) x_3^5 f(x_3, x_1) = 1$ in $\overline{\mathbb{F}}_2(k)$,
- $f(x_1 x_2 x_3, x_3 x_2 x_1) f(x_1, x_2) f(x_1 x_2 x_1, x_3 x_2 x_1) f(x_2 x_1, x_3) f(x_1 x_3, x_2) f(x_1 x_2 x_3, x_2) f(x_1 x_2 x_3, x_3) f(x_2, x_3) f(x_3, x_1) f(x_1, x_3) = 1$, and $x_{ij}$ is the elementary pure braid from Subsection 2.2. The multiplication law is given by

$$(\lambda_1, f_1)(\lambda_2, f_2) = (\lambda_1 \lambda_2, f_1(x^{\lambda_1}, f_2(x, y)^y^{\lambda_2} f_2(x, y)^{-1} f_2(x, y)))$$
One obtains the couple \((\lambda, f)\) from an automorphism \(F \in \hat{GT}(k)\), for \(\lambda = 2\nu + 1\) by the assignement
\[
\begin{align*}
F(R^{1,2}) &= (R^{1,2}R^{2,1}R^{1,2})^{-1}, \\
F(\Phi^{1,2,3}) &= f(x_{12}, x_{23})\Phi^{1,2,3}.
\end{align*}
\]

2.6. Graded Grothendieck–Teichmüller group. The graded Grothendieck–Teichmüller group is the group
\[
\text{GRT}(k) := \text{Aut}_{\text{Grpd}}^*(\text{GrPaCD}(k))
\]
of automorphisms of \(\text{GrPaCD}(k)\) that are the identity on objects.

Again, the operadic definition of \(\text{GRT}(k)\) happens to coincide with the one originally given by Drinfeld. Denote by \(\text{GRT}_1\) the set of elements in \(g \in \exp(\hat{f}_2(k)) \subset \exp(\hat{t}_3(k))\) such that
\[
\begin{align*}
g^{1,2,1} &= g^{-1} \text{ and } g^{1,2,3}g^{2,1,2} = 1, \text{ in } \exp(\hat{t}_3(k)), \\
t_{12} + \text{Ad}(g^{1,2,3})(t_{23}) + \text{Ad}(g^{2,1,3})(t_{13}) &= t_{12} + t_{13} + t_{23}, \text{ in } \hat{t}_3(k), \\
g^{1,2,3}g^{1,2,3}g^{2,3,4} &= g^{12,3,4}g^{1,2,34}, \text{ in } \exp(\hat{t}_4(k)).
\end{align*}
\]

One has the following multiplication law on \(\text{GRT}_1\):
\[
(g_1 \ast g_2)(t_{12}, t_{23}) := g_1(t_{12}, \text{Ad}(g_2(t_{12}, t_{23}))(t_{23}))g_2(t_{12}, t_{23}).
\]

Drinfeld showed in [5] that the above \(\text{GRT}_1\) is stable under \(\ast\), that it defines a group structure on it, and that rescaling transformations \(g(x, y) \mapsto \lambda x \cdot g(x, y) = g(\lambda x, \lambda y)\) define an action of \(k^*\) of \(\text{GRT}_1\) by automorphisms and we denote \(\text{GRT}(k)\) the corresponding semi-direct product.

Then, as was shown in [9], the group \(\text{GRT}(k)\) is isomorphic to \(\text{GRT}(k)\). In particular, we obtain the couple \((\lambda, g)\) from an automorphism \(G \in \text{GRT}(k)\) by the assignement
\[
\begin{align*}
G(X^{1,2}) &= X^{1,2}, \\
G(H^{1,2}) &= e^{t_{12}}H^{1,2}, \\
G(a^{1,2,3}) &= g(t_{12}, t_{23})a^{1,2,3}.
\end{align*}
\]

2.7. Bitorsor structure. Recall first that there is a free and transitive left action of \(\hat{GT}(k)\) on \(\text{Ass}(k)\), defined, for \((\lambda, f) \in \hat{GT}(k)\) and \((\mu, \varphi) \in \text{Ass}(k)\), by
\[
((\lambda, f) \ast (\mu, \varphi))(t_{12}, t_{23}) := (\lambda \mu, f(e^{\mu t_{12}}, \text{Ad}(\varphi(t_{12}, t_{23}))e^{\mu t_{23}}))\varphi(t_{12}, t_{23}),
\]
where \(\text{Ad}(f)(g) := fgf^{-1}\), for any symbols \(f, g\).

Recall also that there is a free and transitive right action of \(\text{GRT}(k)\) on \(\text{Ass}(k)\) defined as follows: for \((\lambda, g) \in \text{GRT}(k)\) and \((\mu, \varphi) \in \text{Ass}(k)\), given by
\[
((\mu, \varphi) \ast (\lambda, g))(t_{12}, t_{23}) := (\lambda \mu, \varphi(\mu t_{12}, \text{Ad}(g)(\mu t_{23}))g(t_{12}, t_{23})).
\]

These two action commute with each other, and turn \((\hat{GT}(k), \text{Ass}(k), \text{GRT}(k))\) into a bitorsor. By its very definition, the triple \((\hat{GT}(k), \text{Ass}(k), \text{GRT}(k))\) is also a bitorsor, and it is proven in [9] that the above identifications from subsections 2.4, 2.5, and 2.6, can be promoted to a bitorsor isomorphism
\[
(\hat{GT}(k), \text{Ass}(k), \text{GRT}(k)) \rightarrow (\hat{GT}(k), \text{Ass}(k), \text{GRT}(k)).
\]
3. Parenthesized braids with a frozen strand

3.1. Compactified configuration space of the annulus. For each finite set $I$, let us consider the (reduced) configuration space of $\mathbb{C}^*$:

$$C(C^*, I) := \{ z = (z_i)_{i \in I} \in (\mathbb{C}^*)^I | z_i \neq z_j, \forall i \neq j \} / \mathbb{R}_{>0}.$$ 

We clearly have an isomorphism between $C(C^*, n)$ and $C(C, n+1)$. We then consider the Axelrod–Singer–Fulton–MacPherson compactification $\overline{C}(C^*, n)$ of $C(C^*, n)$. The boundary

$$\partial \overline{C}(C^*, n) = \overline{C}(C^*, n) - C(C^*, n)$$

is made of the following irreducible components: for any partition $[0, n] = J_0 \sqcup \cdots \sqcup J_k$ such that $0 \in J_m$, for some $0 \leq m \leq k$, there is a component

$$\partial J_1, \ldots, J_k \overline{C}(C^*, n) \cong \overline{C}(C^*, k) \times \overline{C}(C^*, J_m) \times \prod_{i=j}^{k} \overline{C}(C, J_i).$$

3.2. The PaB-moperad of parenthesized braids with a frozen strand. We have inclusions of topological moperads

$$\text{Pa}_0 \subset \overline{C}(\mathbb{R}_{>0}, -) \subset \overline{C}(\mathbb{C}^*, -)$$

over

$$\text{Pa} \subset \overline{C}(\mathbb{R}, -) \subset \overline{C}(\mathbb{C}, -).$$

We then define

$$\text{PaB}^1 := \pi_1(\overline{C}(\mathbb{C}^*, -), \text{Pa}_0),$$

which is a moperad over the operad in groupoids $\text{PaB}$.

Example 3.1 (Description of $\text{PaB}^1(1)$). First, observe that $\overline{C}(\mathbb{C}^*, 1) \cong \overline{C}(\mathbb{C}, 2) \cong S^1$. Moreover, $\text{Pa}_0 = \{(01)\}$. Hence $\text{PaB}^1(1) \cong \mathbb{Z}$: it has only one object (01) and is freely generated by an automorphism $E^{0,1}$ of (01), which can be depicted as an elementary pure braid:

Two incarnations of $E^{0,1}$

Example 3.2 (Notable arrow in $\text{PaB}^1(2)$). Let us first recall that $\text{Pa}_0(2) = \mathbb{S}_2 \times \{(\bullet\bullet), (\bullet)(\bullet)\}$ and that $\overline{C}(\mathbb{R}_{>0}, 2) \cong \mathbb{S}_2 \times [0, 1]$. Hence we have an arrow $\Psi^{0,1,2}$ (the identity path in $[0, 1]$) from (01)2 to 0(12) in $\text{PaB}^1(2)$, which can be depicted as follows:

Two incarnations of $\Psi^{0,1,2}$
**Remark 3.3.** Recall from §1.3 that, being a PaB-moperad, PaB\(^1\) comes together with a morphism of \(\mathcal{S}\)-modules \(\text{PaB} \to \text{PaB}^1\). In pictorial terms, this morphism sends a parenthesized braid with \(n\) strands to a parenthesized braid with \(n+1\) strands by adding a frozen strand labelled by 0 on the left. For instance, the images of \(R^{1,2}\) (a morphism in \(\text{PaB}^1(2)\)) and of \(\Phi^{1,2,3}\) (a morphism in \(\text{PaB}(3)\)) can be respectively depicted as follows:

![Pictorial representation of morphisms](image)

We will still denote these images by \(R^{1,2}\) and \(\Phi^{1,2,3}\).

### 3.3. The universal property of PaB\(^1\).

Our main goal in this § is to prove the following generator and relation presentation of \(\text{PaB}^1\).

**Theorem 3.4.** As a PaB-moperad having Pa\(_0\) as Pa-moperad of objects, \(\text{PaB}^1\) is freely generated by \(E \colon E^{0,1} \in \text{PaB}^1(1)\) and \(\Psi \colon \Psi^{0,1,2} \in \text{PaB}^1(2)\) together with the following relations:

- \((cU)\) \(\psi^{0,0,2} = \psi^{0,1,0} = \text{Id}_{01}\) \((\text{in } \text{Hom}_{\text{PaB}^1(1)}(01,01))\),
- \((MP)\) \(\psi^{0,1,2,3} = \psi^{0,1,2} \psi^{0,1,2,3} \psi^{1,2,3}\) \((\text{in } \text{Hom}_{\text{PaB}^1(3)}(\{(01)23,0(123))\})\),
- \((RP)\) \(E^{0,1,2} E^{0,1,2} (\psi^{0,1,2})^{-1} = E^{0,1,2} E^{0,1,2} (\psi^{0,1,2,3})^{-1}\) \((\text{in } \text{Hom}_{\text{PaB}^1(2)}((01)2,(01)2))\),
- \((O)\) \(E^{0,1,2} = \psi^{0,1,2} R^{2,1} (\psi^{0,2,1})^{-1} \psi^{0,1,2} R^{2,1} (\psi^{0,1,2})^{-1}\) \((\text{in } \text{Hom}_{\text{PaB}^1(2)}((01)2,(01)2))\).

**Proof.** Let \(Q^1\) be the PaB-moperad with the above presentation. From Examples 3.1 and 3.2 we deduce that, as a PaB-moperad in groupoid, \(\text{PaB}^1\) contains two morphisms \(E^{0,1}\) (in \(\text{PaB}^1(1)\)) and \(\Psi^{0,1,2}\) (in \(\text{PaB}^1(2)\)). One easily shows, using the following pictures, that they satisfy \textit{mixed pentagon} and \textit{octogon} relations, \((MP)\) and \((O)\), and relation \((RP)\):

![Pictorial representation of relations](image)
Therefore, by the universal property of $Q^1$, there is a morphism of $\text{PaB}$-moperads $Q^1 \to \text{PaB}^1$, which is the identity on objects. In order to show that this is an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of an object arity-wise because all groupoids involved are connected. Let $n \geq 0$, and let $p$ be the object $(\cdots(01)\cdots)$ of $Q^1(n)$ and $\text{PaB}^1(n)$. We want to show that the induced group morphism

$$\text{Aut}_{Q^1(n)}(p) \to \text{Aut}_{\text{PaB}^1(n)}(p) = \pi_1(\bar{C}(n^*, p)),$$

is an isomorphism.

On the one hand, we can replace the base-point $p$ with $p_{\text{reg}} = (1, 2, \ldots, n) \in C(n^*, n)$, as they are in the same path-connected component. Moreover, since the Axelrod–Singer–Fulton–MacPherson compactification does not change the homotopy type of our configuration spaces, we get an isomorphism

$$\pi_1(\bar{C}(n^*, p)) \cong \pi_1(C(n^*, p_{\text{reg}})).$$

On the other hand, in [6, §4.4], Enriquez proves several useful facts:

- Given a braided module category $\mathcal{M}$ over a braided monoidal category $\mathcal{C}$, an object $X$ of $\mathcal{C}$, and an object $M$ of $\mathcal{M}$, there is a group morphism

$$B_n^1 \to \text{Aut}_\mathcal{M}(M \otimes X^{\otimes n}),$$
where, by convention, \( M \otimes X^{\otimes n} \) comes equipped with the left-most parenthesisization \(((M \otimes X) \otimes \ldots) \otimes X\), and \( B_n = B_{n+1} \times_{\mathfrak{S}_{n+1}} \mathfrak{S}_n \) is generated by elements \( \sigma_i \), for \( 1 \leq i \leq n-1 \) and \( \tau \). Seen in \( B_{n+1} \) with generators \( \sigma_0, \ldots, \sigma_{n-1} \), we have \( \tau = \sigma_0^2 = x_0 \).

- There is a universal braided module category \( \text{PaB}^{1, \text{Enr}} \) generated by a single object \( 0 \) over the universal braided monoidal category \( \text{PaB}^{\text{Enr}} \) generated by a single object \( n \).
- Hence objects of \( \text{PaB}^{1, \text{Enr}} \) are parenthesizations of \( 0 \cdots n \), and thus \( p \) determines an object (which we abusively still denote \( p \)).
- the morphism \( \text{Enr}_n \to \text{Aut}_{\text{PaB}^{1, \text{Enr}}}(p) \) is an isomorphism.

One can moreover see that, by construction, \( \text{Aut}_{\text{PaB}^{1, \text{Enr}}}(p) \) is exactly the kernel subgroup
\[
\ker \left( \text{Aut}_{\text{PaB}^{1, \text{Enr}}}(n) \to \mathfrak{S}_n \right) = \text{PB}_{n+1}.
\]

Hence we have a commuting diagram
\[
\begin{array}{ccc}
\text{PB}_n & \xrightarrow{\cong} & \text{Aut}_{\text{PaB}^{1, \text{Enr}}}(p) \\
\downarrow & & \downarrow \\
\text{B}_n & \xrightarrow{\cong} & \text{Aut}_{\text{PaB}^{1, \text{Enr}}}(p) \\
\downarrow & & \downarrow \\
\mathfrak{S}_n & \xrightarrow{\cong} & \mathfrak{S}_n
\end{array}
\]

where all vertical sequences are short exact sequences. Thus, in order to get that the map
\[
\text{Aut}_{\text{PaB}^{1, \text{Enr}}}(p) \to \pi_1(\mathcal{C}(\mathbb{C}, n), p)
\]

is indeed an isomorphism. But this map is, by its very construction, an isomorphism, we are left to prove that the composite map
\[
\text{Enr}_n \to \pi_1(\mathcal{C}(\mathbb{C}, n), [p])
\]

is an isomorphism. This allows us to define the compactification \( \mathcal{C}(\mathbb{C}, I, \Gamma) \) of \( C(\mathbb{C}, I, \Gamma) \), as the closure of \( C(\mathbb{C}, I, \Gamma) \) inside \( \mathcal{C}(\mathbb{C}, I, \Gamma) \). The irreducible components

4. The moperad of twisted parenthesized braids, and cyclotomic GT

4.1. Compactified twisted configuration space of the annulus. Consider, for \( N \geq 1 \), the additive group \( \Gamma = \mathbb{Z}/N\mathbb{Z} \). To every finite set \( I \) let us associate the so-called \( \Gamma \)-twisted configuration space
\[
\text{Conf}(\mathbb{C}^\times, I, \Gamma) = \{ z = (z_i)_{i \in I} \in (\mathbb{C}^\times)^I | |z_i| = \zeta_j, \forall i \neq j, \forall \zeta \in \mu_N \}
\]

(\( \mu_N \) is the set of complex \( N \)th roots of unity) and its reduced version
\[
C(\mathbb{C}^\times, I, \Gamma) := \text{Conf}(\mathbb{C}^\times, I, \Gamma)/\mathbb{R}_{>0}.
\]

Remark 4.1. Observe that \( \text{Conf}(\mathbb{C}^\times, I, \Gamma) \), resp. \( C(\mathbb{C}^\times, I, \Gamma) \), is an \( \Gamma \)-covering space of \( \text{Conf}(\mathbb{C}^\times, I) \), resp. \( C(\mathbb{C}^\times, I) \), the covering mapping being given by \( (z_i)_{i \in I} \mapsto (\zeta_i)_{i \in I} \).

There are also inclusions
\[
\text{Conf}(\mathbb{C}^\times, I, \Gamma) \to \text{Conf}(\mathbb{C}^\times, I \times \mu_N) \quad \text{and} \quad C(\mathbb{C}^\times, I, \Gamma) \to C(\mathbb{C}^\times, I \times \mu_N)
\]
given by \( (z_i)_{i \in I} \mapsto (\zeta z_i)_{(\zeta i)_{i \in I} \in \mu_N} \). This allows us to define the compactification \( \overline{\text{Conf}}(\mathbb{C}^\times, I, \Gamma) \) of \( C(\mathbb{C}^\times, I, \Gamma) \), as the closure of \( C(\mathbb{C}^\times, I, \Gamma) \) inside \( \overline{\text{Conf}}(\mathbb{C}^\times, I, \Gamma) \). The irreducible components
of its boundary \( \partial \overline{C}(\mathbb{C}^x, I, \Gamma) = \overline{C}(\mathbb{C}^x, I, \Gamma) - C(\mathbb{C}^x, I, \Gamma) \) can be described as follows. For an arbitrary partition \( J_0 \sqcup \cdots \sqcup J_k \) of \( \{0\} \cup I \) there is a component

\[
\partial_{J_1,\ldots,J_k} \overline{C}(\mathbb{C}^x, I, \Gamma) \cong \overline{C}(\mathbb{C}^x, k, \Gamma) \times \overline{C}(\mathbb{C}^x, J_m, \Gamma) \times \prod_{i=1}^k \overline{C}(\mathbb{C}, J_i),
\]

where \( m \in \{0, \ldots, k\} \) is the index such that \( 0 \in J_m \). The inclusion of boundary components such that \( m = 0 \) provides \( \overline{C}(\mathbb{C}^x, -, \Gamma) \) with the structure of a moperad over the operad \( \overline{C}(\mathbb{C}, -) \) in topological spaces.

We let the reader check that the covering map \( C(\mathbb{C}^x, I, \Gamma) \rightarrow C(\mathbb{C}^x, I) \) from Remark 4.1 extends to a continuous map \( \phi_n : \overline{C}(\mathbb{C}^x, I, \Gamma) \rightarrow \overline{C}(\mathbb{C}^x, I) \) between their compactifications, and thus leads to a morphism of moperads.

Finally, one observes that the natural action of \( \Gamma^n \) on each \( C(\mathbb{C}^x, I \times \mu N) \), given by

\[
(\alpha \cdot z)(\beta, \gamma) := z\left(\beta, e^{-2\pi \alpha_0} \gamma\right)
\]

induces an action of \( \Gamma \) on the moperad \( \overline{C}(\mathbb{C}^x, -, \Gamma) \), in the sense of §1.4.

4.2. The Pa-moperad of labelled parenthesized permutations. Borrowing the notation from the previous subsection, we define \( \text{Pa}_0^\Gamma(n) := \phi_n^{-1}(\text{Pa}_0(n)) \). Explicitly, \( \text{Pa}_0^\Gamma(n) \) is the set of parenthesized permutations of \( \{0,1,\ldots,n\} \) that fix 0 and that are equipped with a label \( \{1,\ldots,n\} \rightarrow \Gamma \).

**Notation.** As a matter of notation, we will write the label as an index attached to each \( 1,\ldots,n \). For instance, \((02_\alpha)1_\beta \) belongs to \( \text{Pa}_0^\Gamma(2) \) for every \( \alpha, \beta \in \Gamma \).

Observe that the \( \mathcal{S} \)-module (in sets) \( \text{Pa}_0^\Gamma \) carries the structure of a \( \text{Pa} \)-moperad. Indeed, it is a fiber product

\[
\text{Pa}_0^\Gamma = \text{Pa}_0 \times_{\overline{C}(\mathbb{C}^x, -, \Gamma)} \overline{C}(\mathbb{C}^x, -, \Gamma)
\]

in the category of \( \text{Pa} \)-moperads (in topological spaces). Here are two self-explanatory examples of partial compositions:

\[
(02_\alpha)1_\beta \circ_2 (12)3 = (0((2_\alpha 3_\beta)4_\alpha))1_\beta \quad \text{and} \quad (02_\alpha)1_\beta \circ_0 (02_\gamma)1_\delta = (((02_\gamma)1_\delta)4_\alpha)3_\beta.
\]

**Remark 4.2.** As we have seen in Subsection 1.3, our conventions are such that the Pa-moperad structure on \( \text{Pa}_0^\Gamma \) gives in particular a morphism of Pa-modules \( \text{Pa} \rightarrow \text{Pa}_0^\Gamma \). One can see that it is the map that sends a parentheised permutation \( p \) to \( 0(p) \) together with the trivial label function \( i \mapsto 0 \).

Finally, \( \text{Pa}_0^\Gamma \) is acted on by \( \Gamma \) in the following way: for \( n \geq 0, \Gamma^n \) only acts on the labellings, via the group law of \( \Gamma \). For instance, if \( f : \{1,\ldots,n\} \rightarrow \Gamma \) and \( \alpha \in \Gamma^n \), then \( (\alpha \cdot f)(i) = f(i) + \alpha_i \).

4.3. The PaB-moperad of parenthesized cyclotomic braids. We define

\[
\text{PaB}^\Gamma := \pi_1 \left( \overline{C}(\mathbb{C}^x, -, \Gamma), \text{Pa}_0^\Gamma \right).
\]

It is a \( \text{PaB} \)-moperad (in groupoids), that carries an action of the group \( \Gamma \). The maps \( \phi_n : \overline{C}(\mathbb{C}^x, n, \Gamma) \rightarrow \overline{C}(\mathbb{C}^x, n) \) induce a \( \text{PaB} \)-moperad morphism \( \text{PaB}^\Gamma \rightarrow \text{PaB}^1 \).
Example 4.3 (Description of $\mathcal{P}aB^\Gamma(1)$). First observe that $\mathcal{P}a\mathcal{B}^\Gamma_0(1) \rightarrow \mathcal{P}a\mathcal{B}_0(1)$ is the terminal map $\mu_N \simeq \{01_\alpha | \alpha \in \Gamma\} \rightarrow \{01\} = \ast$. Then observe that the map $\overline{\mathbb{C}}(\mathbb{C}^\times, 1, \Gamma) \rightarrow \overline{\mathbb{C}}(\mathbb{C}^\times, 1)$ is nothing but the path-connected $\Gamma$-cover $S^1 \rightarrow S^1$. Hence we in particular have morphisms $E^{0,1_\alpha}, \alpha \in \Gamma$ from $01_\alpha$ to $01_\alpha + \bar{1}$ in $\mathcal{P}a\mathcal{B}^\Gamma(1)$, being the unique lift of $E^{0,1}$ that starts at $01_\alpha \in \mathcal{P}a\mathcal{B}^\Gamma_0(1)$. Pictorially:

In the above picture, on the right we have pictured a path in the twisted configuration space, together with its image under the covering map, which is a loop. Diagrammatically (see the left of the above picture), we depict it as a pure braid (a loop in the base configuration space) together with appropriate base points (which uniquely determines the lift in the covering twisted configuration space).

Example 4.4 (Notable arrow in $\mathcal{P}a\mathcal{B}^\Gamma(2)$). Let $\Psi^{0,1_0,2_0}$ be the unique lift of $\Psi^{0,1,2}$ (a morphism in $\mathcal{P}a\mathcal{B}^\Gamma(1)$) starting at $(01_0)2_0$. It can be depicted as follows:

Remark 4.5. As in Remark 3.3, one can see from §1.3 that there is a morphism of $\mathfrak{S}$-modules $\mathcal{P}a\mathcal{B} \rightarrow \mathcal{P}a\mathcal{B}^\Gamma$. In pictorial terms, it sends a parenthesized braid with $n$ strands to a labelled parenthesized braid with $n + 1$ strands by adding a frozen strand labelled by 0 on the left and choosing the trivial label. For instance, the images $R^{1_0,2_0}$ of $R^{1,2}$ and $\Phi^{1_0,2_0,3_0}$ of $\Phi^{1,2,3}$ can be respectively depicted as follows:

Notation. (i) First of all, for any arrow $X = X^{0,1_0,\ldots,n_0}$ in $\mathcal{P}a\mathcal{B}^\Gamma(n)$ starting at a parenthesized permutation $x$ equipped with the constant labelling equal to $0$, and for any $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \Gamma^n$, we write $X^{0,1_{\alpha_1},\ldots,n_\alpha_n} := \underline{\alpha} \cdot X$, which starts now at the same parenthesized permutation $x$ equipped with the labelling $\underline{\alpha}$. 

(ii) Second of all, for \( p \geq 0 \), if \( X \) ends at the same parenthesized permutation \( x \), but equipped with a possibly non-trivial labelling \( \alpha \), then we write

\[
X^{(p)} := \prod_{k=0, \ldots, p-1} (k\alpha) \cdot X = X^{0, 1_0, \ldots, n_0} X^{0, 1_{n_1}, \ldots, n_{n_1}} \cdots X^{0, 1_{(p-1)n_p}, \ldots, n_{(p-1)n_P}}
\]

which starts at \((x, 0)\) and ends at \((x, p\alpha)\).

(iii) Finally, if \( \gamma \in \Gamma \) and \( 1 \leq i \leq n \), then we write \( \gamma_i := (\bar{0}, \ldots, \bar{0}, \gamma, \bar{0}, \ldots, \bar{0}) \). In particular,

\[
(E^{0,1_0}(p)) := \prod_{k=0, \ldots, p-1} E^{0, 1_k} = E^{0, 1_0} E^{0, 1_1} \cdots E^{0, 1_{p-1}},
\]

which is an element in \( \text{Hom}_{\text{PaB}^\Gamma}(1)((0, 1_0), (0, 1_P)) \).

4.4. The universal property of \( \text{PaB}^\Gamma \). We are now ready to provide an explicit presentation for the \( \text{PaB} \)-moperad \( \text{PaB}^\Gamma \):

**Theorem 4.6.** As a \( \text{PaB} \)-moperad in groupoids with a \( \Gamma \)-action having \( \text{PaB}^\Gamma \) as \( \text{PaB}^\Gamma \)-moperad of objects, \( \text{PaB}^\Gamma \) is freely generated by \( E^{0,1_0} \) and \( \Psi^{0,1_0,2_0} \) together with the following relations:

\[ (tU) \quad \Psi^{0,\emptyset,1_0} = \Psi^{0,1_0,\emptyset} = \text{Id}_{0,1_0} \quad \left( \text{in } \text{Hom}_{\text{PaB}^\Gamma}(1)(01_0, 01_0) \right), \]

\[ (MP) \quad \Psi^{01_0,2_0;3_0} \Psi^{0,1_0,2_0;3_0} = \Psi^{0,1_0,2_0} \Psi^{0,1_0,2_0;3_0} \Psi^{1_0,2_0;3_0} \quad \left( \text{in } \text{Hom}_{\text{PaB}^\Gamma}(3)(((01_0)2_0)3_0, 01_0(2_03_0)) \right), \]

\[ (t\text{RP}) \quad \Psi^{1_0,2_0} E^{0,1_0} \Psi^{0,1_0,2_0} (\Psi^{0,1_0,2_0})^{-1} = E^{0,1_0} \Psi^{1_0,2_0} \left( \text{in } \text{Hom}_{\text{PaB}^\Gamma}(2)(((01_0)2_0), (01_0)2_0) \right), \]

\[ (tO) \quad E^{0,1_0} \Psi^{0,1_0,2_0} R^{1_0,2_0}(\Psi^{0,2_0,1_0})^{-1} E^{0,2_0} \Psi^{0,2_0,1_0} R^{2_0,1_0}(\Psi^{0,1_0,2_0})^{-1}, \]

\[ \left( \text{in } \text{Hom}_{\text{PaB}^\Gamma}(2)(((01_0)2_0), (01_0)2_0) \right). \]

**Proof.** Let \( Q^\Gamma \) be the \( \text{PaB} \)-moperad with the above presentation, and recall that \( Q^\Gamma \) is the \( \text{PaB} \)-moperad with the presentation of Theorem 3.4. Our first goal is to show that there is a morphism \( Q^\Gamma \to \text{PaB}^\Gamma \) of \( \text{PaB} \)-moperads, commuting with the \( \Gamma \)-action. We have already seen in the Examples above that there are morphisms \( E^{0,1_0} \) and \( \Psi^{0,1_0,2_0} \), in \( \text{PaB}^\Gamma(1) \) and \( \text{PaB}^\Gamma(2) \), respectively. We have to prove that they satisfy the mixed pentagon and twisted octagon relation, (MP) and (tO) and (tRP).

These relations are the unique lifts of the similar relations (MP), (RP) and (O) in \( \text{PaB}^3 \) from Theorem 3.4, starting at \(((01_0)2_0)3_0 \) and \((01_0)2_0 \) respectively. They can be depicted as follows:
By universal property of $Q^\Gamma$ there is a $\Gamma$-equivariant morphism of $PaB$-moperads $Q^\Gamma \longrightarrow PaB^\Gamma$, which is the identity on objects. As before, in order to show that this is an isomorphism, it suffices to show that it is an isomorphism at the level of automorphism groups of an object arity-wise (because all groupoids involved are connected). Let $n \geq 0$, and let $\tilde{p}$ be the object $(\cdots(01_0)2_0\cdots)n_0$ of $Q^\Gamma(n)$ and $PaB^\Gamma(n)$, which lifts the object $p = (\cdots(01)2\cdots)n$ of $Q^1(n) \simeq PaB^1(n)$. We want to show that the induced group morphism

$$\text{Aut}_{Q^\Gamma(n)}(\tilde{p}) \longrightarrow \text{Aut}_{PaB^\Gamma(n)}(\tilde{p}) = \pi_1(\tilde{C}(\mathbb{C}^x, n, \Gamma), \tilde{p})$$

is an isomorphism.
We claim that it fits into a commuting diagram

\[
\begin{array}{c}
\text{Aut}_{\mathcal{Q}_1^g(n)}(\tilde{p}) \ar[r]^{\pi_1(\mathcal{C}^*, n, \Gamma)} & \pi_1(\mathcal{C}^*, n, \Gamma, \tilde{p}_{reg}) \\
\end{array}
\]

where only the left-most vertical arrows remain to be described.

The morphism \(\text{Aut}_{\mathcal{Q}_1^g(n)}(p) \to \Gamma^n\). Let \(*\) be the terminal operad in groupoids. We have a \(*\)-moperad structure on the following \(\mathcal{S}\)-module in groupoids: \(\Gamma = \{\Gamma^n\}_{n \geq 0}\), where we view a group as a groupoid with only one object, and where the action of the symmetric group is by permutation. The moperad structure is described as follows:

- \(\circ_0 : \Gamma^n \times \Gamma^m \to \Gamma^{n+m}\) is the concatenation of sequences.
- for every \(i \neq 0\), \(\circ_i : \Gamma^n \to \Gamma^{n+m-1}\) is the partial diagonal
  \[(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n).\]

We let the reader check that sending \(E\) to \(\bar{1} \in \Gamma\) and \(\Psi\) to \((\bar{0}, \bar{0}) \in \Gamma^2\) defines a moperad morphism \(\text{PaB}_1^g \to \Gamma\) along the terminal operad morphism \(\text{PaB} \to *\). This in particular induces a group morphism

\[\text{Aut}_{\mathcal{Q}_1^g(n)}(p) \to \Gamma^n,\]

for every \(n \geq 0\). Heuristically, this morphism counts, for every \(i\), and modulo \(N\), the number of times that \(E^{0,i}\) appears in an element of \(\text{Aut}_{\mathcal{Q}_1^g(n)}(p)\). It is obviously surjective, and we let the reader check that the following triangle commutes:

\[
\begin{array}{c}
\text{Aut}_{\mathcal{Q}_1^g(n)}(p) \ar[r]^{\pi_1(\mathcal{C}^*, n, p)} & \Gamma^n \\
\end{array}
\]

The morphism \(\text{Aut}_{\mathcal{Q}_1^g(n)}(\tilde{p}) \to \text{Aut}_{\mathcal{Q}_1^g(n)}(p)\). We have a \(\Gamma\)-equivariant morphism of \(\text{PaB}\)-moperads \(\mathcal{Q}_1^\Gamma \to \mathcal{Q}_1^1\), where \(\Gamma\) acts trivially on \(\mathcal{Q}_1^1\), which forgets the label on objects, and sends the generators \(E^{0,1}_0\) and \(\Psi^{0,1,2}_0\) to \(E\) and \(\Psi\), respectively. It obviously fits into a commuting square

\[
\begin{array}{c}
\mathcal{Q}_1^\Gamma \ar[r] & \text{PaB}_1^\Gamma \\
\mathcal{Q}_1^1 \ar[u] & \text{PaB}_1^1 \\
\end{array}
\]

of \(\text{PaB}\)-moperads. This induces in particular a group morphism

\[\text{Aut}_{\mathcal{Q}_1^g(n)}(\tilde{p}) \to \text{Aut}_{\mathcal{Q}_1^g(n)}(p),\]
Indeed, an automorphism of \( \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(\bar{p}) \) is an isomorphism. We already know that the morphism \( \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(p) \rightarrow \Gamma^n \) is surjective. The morphism \( \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(\bar{p}) \rightarrow \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(p) \) is injective. Indeed, an automorphism of \( \bar{p} \) in \( Q^r(n) \) can be represented by a finite sequence \( S \) of \( R's, \Phi's, E's, \Psi's, \) and their images under the action of \( \Gamma^n \). The image of such an automorphism under \( Q^r \rightarrow Q^l \) is represented by the corresponding finite sequence \( S \) of \( R's, \Phi's, E's \) and \( \Psi's \). Every modification of \( S \) using the relations (MP), (RP) and (O) can be lifted (uniquely) to a modification of \( \bar{S} \) using (MP), (tRP) and (tO) or their images under the action of \( \Gamma^n \). Hence, if an automorphism has trivial image, then it must be trivial.

The sequence is exact. We already know from the commuting diagram that the image of \( \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(\bar{p}) \) in \( \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(p) \) lies in the kernel of \( \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(p) \rightarrow \Gamma^n \). We finally can show that the image is exactly the kernel. Indeed:

- Using (O), every element \( g \) in \( \mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(p) \) can be written as a product of \( \Phi's, R's, \Psi's \) and \( E's \) where the only \( E's \) appearing are of the form \( E^0,i \).
- Such an element admits a unique lift to a morphism \( \tilde{g} \) in \( Q^r(n) \), with source being \( \bar{p} \) (one just replace \( \Phi's, R's, \Psi's \) and \( E's \) in the expression for \( g \) by the same symbols, maybe acted on by \( \Gamma^n \) in order to get the correct starting objects).
- An element \( g \) as above lies in \( \ker (\mathring{\text{Aut}}_{\mathcal{Q}^r(n)}(p) \rightarrow \Gamma^n) \) if and only if for every \( i \), the number of occurence of \( E^0,i \) (counted in an algebraic way) is a multiple of \( N \). This tells us in particular that the target of the lifted morphism shall be the same as its source, so that \( \tilde{g} \) lies in the kernel.

This ends the proof of the Proposition.

4.5. **Cyclotomic Grothendieck-Teichmüller groups.** We let \( \text{MopC} \) be the category of pairs \((\mathcal{O}, \mathcal{M})\), with \( \mathcal{O} \) an operad and \( \mathcal{M} \) a \( \mathcal{O} \)-moperad, in a symmetric monoidal category \( \mathcal{C} \). A morphism \( (\mathcal{O}, \mathcal{M}) \rightarrow (\mathcal{P}, \mathcal{N}) \) is a pair \((F, G)\), with \( F : \mathcal{O} \rightarrow \mathcal{P} \) an operad morphism and \( G : \mathcal{M} \rightarrow \mathcal{N} \) a \( \mathcal{O} \)-moperad morphism, where the \( \mathcal{O} \)-moperad structure on \( \mathcal{N} \) is defined from its \( \mathcal{P} \)-moperad structure by applying \( F \).

In addition to the superscript “\( \ast \)”, wich means, as in §2.4, that we are considering morphisms of groupoids/categories that are the identity on objects, we may also add, as usual, a superscript “\( \Gamma \)” for \( \Gamma \)-equivariant morphisms.
Definition 4.7. The \((k\text{-pro-unipotent version of the})\) cyclotomic Grothendieck-Teichmüller group is defined as the group

\[
\overline{GT^\Gamma}(k) := \text{Aut}^\Gamma_{\text{MopGrpd}_k}(\overline{PaB}(k), \overline{PaB}^\Gamma(k))
\]

of \(\Gamma\)-equivariant automorphisms of the pair \((\overline{PaB}(k), \overline{PaB}^\Gamma(k))\) which are the identity on objects.

Our main goal in this subsection is to relate this cyclotomic Grothendieck-Teichmüller group with one of those introduced by Enriquez in \([6]\).

Let us recall that \(PB_2 \cong PB_1^3\) is identified with the free group \(F_1\) generated by a single generators \(x\) (being \(x_{12}\) in \(PB_2\), and \(x_{01}\) in \(PB_1^3\)). Let us also recall that the quotient of \(PB_3 \cong PB_1^3\) by its center (which is freely generated by a single element) is a free group \(F_2\) generated by two elements \(x, y\). As usual, we will consider the inclusion of \(F_2\) in \(PB_3\) (resp. \(PB_1^3\)) sending \(x\) to \(x_{12}\) (resp. \(x_{01}\)), and \(y\) to \(x_{23}\) (resp. \(x_{12}\)). Recall finally that \(PB_1^n\) is the kernel of the morphism \(PB_1^n \to \Gamma^n\) sending \(x_{0j}\) to \(1_j\), and the other generators to \((0, \ldots, 0)\).

Whenever \(n = 1\), this is nothing but the morphism \(F_1 \to \Gamma\) sending \(x\) to \(\bar{1}\), having kernel freely generated by \(X = x^N\). Finally notice that the morphism \(\phi_N : F_2 \to \Gamma\) sending \(x\) to \(\bar{1}\) and \(y\) to \(0\) fits into the following commuting square:

\[
\begin{array}{ccc}
F_2 & \longrightarrow & PB_1^3 \\
\downarrow & & \downarrow \\
\Gamma & \longrightarrow & \Gamma^2
\end{array}
\]

It induces a morphism between the kernels \(F_{N+1} \cong \ker \phi_N \to PB_2^N\). The generators of \(\ker \phi_N\) are \(X = x^N\) and \(y(a) = x^{-a}y^a\), \(1 \leq a \leq N - 1\).

An element of the cyclotomic Grothendieck-Teichmüller group \(\overline{GT^\Gamma}(k)\) first depends on an automorphism \(F\) of \(\overline{PaB}(k)\), which is determined by a pair \((\lambda, f)\), where \(\lambda \in k^\times\) and \(f \in \hat{F}_2(k)\) satisfying the relations from §2.4:

- \(F(R^{1,2}) = x_{12}^{1,2}\),
- \(F(\Phi^{1,2,3}) = f(x_{12}, x_{23})\Phi^{1,2,3}\).

Then we have an automorphism \(G\) of \(\overline{PaB}^\Gamma(k)\), compatible with \(F\), which is likewise determined by the images of \(E^{0,1_0} \in \text{Hom}_{\overline{PaB}^\Gamma(k)(1)}(1_{01}, 0_{11})\) and \(\Psi^{0,1_0,2_0} \in \text{Hom}_{\overline{PaB}^\Gamma(k)(2)}((01_{02}, 0_{12})\):

- \(G(E^{0,1_0}) = u E^{0,1_0}\), where \(u = X^{\mu_1} = x^{N_{\mu_1}}\) for some \(\mu_1 \in k^\times\), necessarily,
- \(G(\Psi^{0,1_0,2_0}) = v \Psi^{0,1_0,2_0}\),

where \(v \in \overline{PB}_2^\Gamma(k) \subset \overline{PB}_2^1(k)\) can be written as \(C^{\mu_2} g(x_{01}, x_{12})\), with \(C\) a central generator of \(\ker \phi_N\) and \(g \in \ker \phi_N(k) \subset \hat{F}_2(k)\).

Notation. We will also write \(g(X, y(0), \ldots, y(N - 1))\) when we want to view \(g\) in \(\hat{F}_{N+1}(k) \cong \ker \phi_N(k)\).
Relation (tU) tells us that $X^2 = v^{0,1} = 1$, and thus that $\mu = 0$. Indeed, the morphism $\langle - \rangle^{0,1} \colon \mathbb{P}^1 \to \mathbb{P}^1 \simeq \mathbb{F}_1$ sends $\ker N$ to 1, and $x_0^N$ (as well as the central generator) to $X = x^N$. We conclude that $v = g(x_0, x_2) = g(X, y(0), \ldots, y(N - 1))$.

**Proposition 4.8.** The elements $(\lambda, f, \mu, g)$ satisfy

\begin{equation}
(2) \quad g(x_0, x_2)g(x_0, x_2) = g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2),
\end{equation}

and, for $\alpha \in \Gamma$,

\begin{equation}
(3) \quad x^\mu g(x, y) \sim \frac{1}{\alpha} g(z, y)^{-1} z^\mu \alpha \cdot (g(z, y) \sim \frac{1}{\alpha} g(x, y)^{-1}) = 1 \quad (\text{in } \hat{\Gamma}_2(\phi_N, k)) \quad xyz = 1.
\end{equation}

**Proof.** First of all, the fact that relation (2) is satisfied is straightforward. Second of all, suppose $N = 1$ and consider the image of (0) by $G$:

\begin{equation}
(4) \quad G(E^{0,2}) = g(x_0, x_2)g(x_0, x_2)g(x_0, x_2) = g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2).
\end{equation}

Now, by using $x_21 = x_12$, $\sigma_1 x_01 \sigma_1^{-1} = x_02$ and $\sigma_1 x_12 \sigma_1^{-1} = x_12$, we get

\begin{equation}
(5) \quad G(E^{0,2}) = \sigma_1^{-1} g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2).
\end{equation}

Now, since $\Psi^{0,1,2} R^{1,2}(\Psi^{0,1,2})^{-1}$ is nothing but $\sigma_1$, we obtain

\begin{equation}
(6) \quad G(E^{0,2}) = \sigma_1^{-1} g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2).
\end{equation}

Next, $G(E^{0,1}) = x_0$ so, by using relation $\sigma_1^{-1} x_0 = x_0 \sigma_1^{-1}$, we obtain

\begin{equation}
(7) \quad G(E^{0,1}) = \sigma_1^{-1} g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2).
\end{equation}

The above equation is then equivalent to

\begin{equation}
(8) \quad g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2).
\end{equation}

Finally, by writing $x_0 = x_0 x_0 x_0 x_0$, we obtain, by absorbing the central element $z$ and simplifying it from the equation, the following result:

\begin{equation}
1 = x_0^\mu g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2)g(x_0, x_2).
\end{equation}
By denoting \( x = x_{02}, y = x_{12} \) and \( z = y^{-1}x^{-1} \) we obtain
\[
x^\mu g(x,y)g(\tilde{z},y)^{-1}g(z,y)g(\tilde{z},y)^{-1}g(x,y)^{-1} = 1.
\]
Finally, when \( N \geq 1 \), one takes the chosen lifts of each term of the above equation to obtain equation (3).

**Lemma 4.9.** We have \( \lambda = 1 + \mu_1 N \).

**Proof.** It is proven in [6] that, if we have a quadruple \((\lambda, \mu, f, g)\), with \((\lambda, f) \in \widetilde{\text{GT}}(\mathbf{k})\), \( \mu \in (a, \mu_1) \in \Gamma \times \mathbf{k} \), and \( g \in \ker \varphi_N(\mathbf{k}) \), satisfying the above two equations (2) and (3), then \( \lambda = [\mu] := \tilde{\alpha} + \mu_1 N \), where \( 0 \leq \tilde{\alpha} \leq N - 1 \) is a representative of \( a \in \Gamma \). In our case, we are in the situation where \( \mu = (1, \mu_1) \).

As a consequence, we can identify the underlying set of our operadically defined cyclotomic Grothendieck-Teichmüller group \( \widetilde{\text{GT}}(\mathbf{k}) \) with the underlying set of the group GTM\(_1(N, \mathbf{k})\) introduced in [6].

Indeed, GTM\(_1(N, \mathbf{k})\) is defined as the set of triples \((\lambda, f, g)\) with \((\lambda, f) \in \widetilde{\text{GT}}(\mathbf{k})\) and \( g \in \ker \varphi_N(\mathbf{k}) \), and satisfies equations (2) and (3) with \( \mu = \frac{\lambda_1}{N} \). It carries a group structure, which is defined by
\[
(\lambda_1, f_1, g_1) * (\lambda_2, f_2, g_2) = (\lambda, f, g)
\]
with
- \( \lambda = \lambda_1 \lambda_2 \),
- \( f(x, y) = f_1(f_2(x, y)^{-1}x^{\lambda_2}f_2(x, y), y^{\lambda_2}) \cdot f_2(x, y) \),
- \( g(x, y) = g_1(g_2(x, y)^{-1}x^{\mu_2}g_2(x, y), y^{\lambda_2}) \cdot g_2(x, y) \).

Therefore it follows from the above discussion that we get an assignment
\[
\widetilde{\text{GT}}^{\Gamma}(\mathbf{k}) \rightarrow \text{GTM}_1(N, \mathbf{k}); (F, G) \mapsto (\lambda, f, g).
\]
It is obviously injective.

**Proposition 4.10.** The injective map (6) is a group morphism.

We will see later in Theorem 5.13 that (6) is actually an isomorphism.

**Proof.** We have to prove that the assignment \((F, G) \rightarrow (\lambda, f, g)\) constructed above is a group morphism. As we already know, the composition of automorphisms \( F_1 \) and \( F_2 \) in \( \text{Aut}_{\text{Grpd}_{\mathbf{k}}}^{\mathbf{PaB}(\mathbf{k})} \) corresponds to the composition law in \( \widetilde{\text{GT}}(\mathbf{k}) \), that is, the associated couples \((\lambda_1, f_1)\) and \((\lambda_2, f_2)\) in \( k^* \times F_2(\mathbf{k}) \) satisfy
\[
(F_2 \circ F_1)(R_{1,2}^{1,2}) = (R_{1,2}^{1,2})^{\lambda_1 \lambda_2}
\]
and
\[
(F_2 \circ F_1)(\Phi^{1,2,3}) = f_1(x^{\lambda_2}, f_2(x, y)^{\lambda_2} f_2(x, y)^{-1}) f_2(x, y)^{\Phi^{1,2,3}}.
\]
(here \( F_2 \) is generated by \( x := x_{12} \) and \( y := x_{23} \)). We also already showed that any two automorphisms \( G_1 \) and \( G_2 \) in the group \( \widetilde{\text{GT}}^{\Gamma}(\mathbf{k}) \), depending on \( F_1 \) and \( F_2 \) respectively, are associated to couples \((\mu_1, g_1(X|y(0), \ldots, y(N - 1)))\) and \((\mu_2, g_2(X|y(0), \ldots, y(N - 1)))\) where \( g_1 \) and \( g_2 \) are elements of in \( \tilde{F}_{N+1}(\mathbf{k}) \).
In the group $A := \text{Aut}_{\text{FB}_l(k)(2)}((01)_02_0)$, we have

$$X = x_{01}^N = (E^{0,1_0})^{(N)}$$

and $G_1(X) = X^{\lambda_1}$ for $\lambda_1 \in k^\times$. Next, we want to compute $G_1(y(0))$, where $y(0) = x_{12}$ decomposes in $A$ as follows:

$$0(1_0)_2 \xrightarrow{\Phi_{0_1}^{0,1_0,2_0}} (01)_02_0 \xrightarrow{R^{0_1}2_0^{0_1}R^{0_2}1_0} (01)_02_0 \xrightarrow{(\Phi_{0_1}^{0,1_0,2_0})^{-1}} (01)_02_0.$$  

Then, as

$$G_1(\Psi_{0_1}^{0,1_0,2_0}) = g_1(X|y(0), \ldots, y(N - 1))\Psi_{0_1}^{0,1_0,2_0}$$

and

$$G_1(R^{0_12_0}R^{2_01_2}) = (x_{12})^{\lambda_1},$$

we obtain

$$G_1(y(0)) = g_1(X|y(0), \ldots, y(N - 1))y(0)^{\lambda_1}g_1^{-1}(X|y(0), \ldots, y(N - 1)).$$

More generally, $y(a) = x_{01}^a x_{12} x_{01}^a$ decomposes as

$$0(1_0)_2 \xrightarrow{\Phi_{0_1}^{0_1,2_0}} (01)_a2_0 \xrightarrow{R^{0_1}2_0^{0_1}R^{0_2}1_0} (01)_a2_0 \xrightarrow{(\Phi_{0_1}^{0_1,2_0})^{-1}} (01)_a2_0.$$  

Therefore, by $\Gamma$-equivariance we get

$$x_{01}^a g(X|y(0), \ldots, y(N - 1)) = g(X|y(a), \ldots, y(a + N - 1)) x_{01}^a,$$

and thus

\begin{align*}
G_1(y(a)) &= G_1((E^{0_1_0})^{(a)}\Psi_{0_1}^{0_1,2_0}R^{0_1}2_0 R^{2_01_2} \Psi_{0_1}^{0,1_0,2_0})^{-1}(E^{0_1_0})^{(-a)}) \\
&= G_1((E^{0_1_0})^{(a)}G_1(\Psi_{0_1}^{0,1_0,2_0}G_1(R^{0_1}2_0 R^{2_01_2} G_1(\Psi_{0_1}^{0,1_0,2_0})^{-1})G_1((E^{0_1_0})^{(-a)})) \\
&= \text{Ad}((X^{N\lambda_1}E^{0_1_0})^{(a)}g(X|y(0), \ldots, y(N - 1)))(x_{12}^{\lambda_1}) \\
&= \text{Ad}(X^{a N\lambda_1}E^{0_1_0})^{(a)}g(X|y(0), \ldots, y(N - 1)))(x_{12}^{\lambda_1}) \\
&= \text{Ad}(X^{a N\lambda_1}g(X|y(a), \ldots, y(a + N - 1)))(y(a)^{\lambda_1}).
\end{align*}

Finally we obtain

\begin{align*}
(G_2 \circ G_1)(\Psi_{0_1}^{0,1_0,2_0}) &= G_2(g_1(X|y(0), \ldots, y(N - 1))\Psi_{0_1}^{0,1_0,2_0}) \\
&= g_1(G_2(X)g_2(y(0)), \ldots, G_2(y(N - 1)))g_2(X|y(0), \ldots, y(N - 1))\Psi_{0_1}^{0,1_0,2_0} \\
&= g_1(X^{\lambda_2} \text{Ad}(g_2(X|y(0), \ldots, y(N - 1))))(y(0)^{\lambda_2}), \\
&\quad \text{Ad}(X^{\lambda_2} g_2(X|y(1), \ldots, y(N)))(y(1)^{\lambda_2}), \ldots, \\
&\quad \text{Ad}(X^{(N - 1)\lambda_2} g_2(X|y(N - 1), \ldots, y(2N - 2)))(y(N - 1)^{\lambda_2})) \\
&\quad g_2(X|y(0), \ldots, y(N - 1))\Psi_{0_1}^{0,1_0,2_0}.
\end{align*}

which is nothing but the composition law in the group $\text{GTM}_1(N, k)$. This concludes the proof, as the composite of moperad morphisms $G_2 \circ G_1$ is compatible with the composition of operad morphisms $F_2 \circ F_1$. \qed
5. The moperad of $N$-chord diagrams, and cyclotomic associators

5.1. Infinitesimal cyclotomic braids. Let $\Gamma = \mathbb{Z}/N\mathbb{Z}$, $I$ a finite set, and let $t^\Gamma_I(k)$ be the Lie $k$-algebra with generators $t_{ij}$, $(i \in I)$, and $t^\Gamma_{ij}$, $(i \neq j \in I$, $\alpha \in \mathbb{Z}/N\mathbb{Z})$, and relations:

(tS) $t^\alpha_{ij} = t^{-\alpha}_{ji}$,

(tL) $[t_{0i}, t^\alpha_{jk}] = 0$ and $[t^\alpha_{ij}, t^\beta_{kl}] = 0$,

(t4T) $[t^\alpha_{ij}, t^\beta_{jk} + t^\gamma_{jk}] = 0$,

(t4T') $[t_{0i}, t_{0j} + \sum_{\alpha \in \mathbb{Z}/N\mathbb{Z}} t^\alpha_{ij}] = 0$,

(t6T) $[t_{0i} + t_{0j} + \sum_{\beta \in \mathbb{Z}/N\mathbb{Z}} t^\beta_{ij}, t^\alpha_{ij}] = 0$,

where $i, j, k, l \in I$ are pairwise distinct and $\alpha, \beta \in \mathbb{Z}/N\mathbb{Z}$. We will call it the $k$-Lie algebra of infinitesimal cyclotomic braids. This definition is obviously functorial with respect to bijections, exhibiting $t^\Gamma_I(k) := \{t^\Gamma_I(k)\}_I$ as an $\mathcal{S}$-module. It moreover also has the structure of a $t(k)$-moperad, where partial compositions are defined as follows:\footnote{We just re-package Enriquez’s insertion-coproduct morphisms $[6, \S 2.1.1]$ in a moperadic manner.}

$o_i: t^\Gamma_I(k) \otimes t_I(k) \rightarrow t^\Gamma_{I \setminus \{i\}}(k)$

$(0, t_{pq}) \mapsto t^\Gamma_{pq}$

$(t^\alpha_{jk}, 0) \mapsto \begin{cases} t^\alpha_{jk} & \text{if } i \notin \{j, k\} \\ \sum_{\alpha \in \mathbb{Z}/N\mathbb{Z}} t^\alpha_{jk} & \text{if } j = i \\ \sum_{\alpha \in \mathbb{Z}/N\mathbb{Z}} t^\alpha_{jk} & \text{if } k = i \end{cases}$

$(t_{0i}, 0) \mapsto \begin{cases} t_{0i} & \text{if } j \neq i \\ \sum_{\alpha \in \mathbb{Z}/N\mathbb{Z}} \sum_{\gamma \neq i} t^\gamma_{qr} & \text{if } j = i \end{cases}$

and

$o_0: t^\Gamma_I(k) \otimes t^\Gamma_I(k) \rightarrow t^\Gamma_I(k)$

$(0, t_{0p}) \mapsto t_{0p}$

$(0, t^\alpha_{pq}) \mapsto t^\alpha_{pq}$

$(t^\alpha_{jk}, 0) \mapsto t^\alpha_{jk}$

$(t_{0i}, 0) \mapsto t_{0i} + \sum_{\gamma \neq i} \sum_{\gamma \neq j} t^\gamma_{ji}$

We call $t^\Gamma_I(k)$ the moperad of infinitesimal cyclotomic braids. It is acted on by $\Gamma$: for $\gamma \in \Gamma$ and $1 \leq i \leq n$, $\gamma_i \in \Gamma^n$ acts as

$\gamma_i \cdot t_{0p} = t_{0p}$ \hspace{1cm} $(p \in \{1, \ldots, n\})$,

$\gamma_i \cdot t^\alpha_{qr} = t^\alpha_{qr}$ \hspace{1cm} $(\alpha \in \Gamma$ and $q, r \neq i)$,

$\gamma_i \cdot t^\alpha_{qr} = t^{r-\gamma}_{qr}$ \hspace{1cm} $(\alpha \in \Gamma$ and $r \neq i)$,

$\gamma_i \cdot t^\alpha_{qr} = t^{\gamma-\gamma}_{qr}$ \hspace{1cm} $(\alpha \in \Gamma$ and $q \neq i)$.
5.2. **Horizontal \( N \)-chord diagrams.** We now consider the \( \text{CD}(k) \)-moperad \( \text{CD}_0^N(k) := \mathcal{U}(\mathfrak{t}^N(k)) \) in \( \text{Cat} (\text{CoAlg}_k) \). Morphisms in \( \text{CD}_0^N(k)(n) \) can be represented as linear combinations of diagrams of chords on \( n + 1 \) vertical strands, together with a labelling (by elements of \( \Gamma \)) of chords that are not connected to the leftmost strand (i.e. the frozen one). On can equivalently represent them as certain horizontal \( N \)-diagrams according to the terminology of [3, Definition 6.4] (where the relation to Vassiliev invariants has been explored): more precisely, those horizontal \( N \)-diagrams for which the sum of labels on each strand is \( \bar{0} \). Using the representation from [3], i.e. the one with labels on (non frozen) strands rather than on chords, the diagram corresponding to \( t_{0i} \) is

\[
\begin{array}{ccc}
0 & i & 0 \\
\hline
\alpha & & -\alpha \\
\end{array}
\]

while the one corresponding to \( t_{ij}^\alpha = t_{ji}^{-\alpha} \) is

\[
\begin{array}{ccc}
i & j & i \\
\hline
\alpha & & -\alpha \\
\end{array}
\]

and relations can be depicted as follows:

\[(tL)\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\alpha & & -\beta & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\beta & & -\alpha & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\alpha & & -\beta & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\beta & & -\alpha & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\alpha & & -\beta & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\beta & & -\alpha & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\alpha & & -\beta & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
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\beta & & -\alpha & \\
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\begin{array}{c}
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\beta & & -\alpha & \\
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\begin{array}{c}
i & j & k & l \\
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\beta & & -\alpha & \\
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\[
\begin{array}{c}
i & j & k & l \\
\hline
\alpha & & -\beta & \\
\end{array}
\]

\[
\begin{array}{c}
i & j & k & l \\
\hline
\beta & & -\alpha & \\
\end{array}
\]
Let us now introduce another $CD(k)$-moperad $CD^Γ(k)$, which will be made of all horizontal $N$-diagrams. In arity $n$, objects of $CD^Γ(k)$ are just labellings $\{1, \ldots, n\} \to Γ$, $\text{Ob}(CD^Γ(k))(n) = Γ^n$, and the $*$-moperad structure is given as follows:

- $α_0 : Γ^n \times Γ^m \to Γ^{n+m}$ is the concatenation of sequences.
- for every $i \neq 0$, $α_i : Γ^n \to Γ^{n+m-1}$ is the partial diagonal

\[
(α_1, \ldots, α_n) \mapsto (α_1, \ldots, α_{i-1}, α_i, α_{i+1}, \ldots, α_n).
\]

Given two labellings $α = (α_1, \ldots, α_n)$ and $β = (β_1, \ldots, β_n)$, the $k$-vector space of morphisms from $α$ to $β$ in $CD^Γ(k)$ is the vector space of horizontal $N$-chord diagrams such that, on the
$i$-th strand, the sum of labels $i \beta_i - \alpha_i$. The $\text{CD}(k)$-moperad structure on morphisms is the exact same as the one for $\text{CD}_0(k)$.

We call $\text{CD}^\Gamma(k)$ the $\text{CD}(k)$-moperad of $N$-chord diagrams. It carries an obvious action of $\Gamma$, by translation on the labelling of objects.

**Example 5.1** (Notable arrows of $\text{CD}^\Gamma(k)(1)$). We have the following arrows in $\text{CD}^\Gamma(k)(1)$:

\[
K^{0,1}_i := t_{01}.
\]

\[
L^{0,1}_i := 1.
\]

Note that, by definition, we have

\[
t_{ij}^\alpha = L^{0,i_0} t_{ij}^0 (L^{0,j_0})^{-1}.
\]

There is an obvious $\Gamma$-equivariant morphism of moperads $\omega : \mathbb{P}_0 \rightarrow \text{Ob} \text{CD}(k)$, over the terminal operad morphism $\mathbb{P} \rightarrow * = \text{Ob} \text{CD}(k)$, that forgets the underlying parenthesized permutation and just remembers the labelling. Hence we can consider the fake pull-back $\mathbb{P}_0 \text{CD}(k)$-moperad

\[
\mathbb{P}_0 \text{CD}^\Gamma(k) := \omega^* \text{CD}^\Gamma(k)
\]

of parenthesized $N$-chord diagrams, which is still acted on by $\Gamma$.

**Example 5.2** (Notable arrow in $\mathbb{P}_0 \text{CD}^\Gamma(k)(2)$). We also have the following arrow in $\mathbb{P}_0 \text{CD}^\Gamma(k)(2)$:

\[
b^{0,1_0,2_0} := 1.
\]

**Remark 5.3.** As explained in subsection 1.3, there is a map of $\mathcal{S}$-modules $\mathbb{P}_0 \text{CD} \rightarrow \mathbb{P}_0 \text{CD}^\Gamma$ and we denote (borrowing our previous conventions) $X^{1_0,2_0}$, $H^{1_0,2_0}$ and $a^{1_0,2_0,3_0}$ the images in $\mathbb{P}_0 \text{CD}^\Gamma$ of the corresponding arrows in $\mathbb{P}_0 \text{CD}$. The elements $K^{0,1_0}$, $L^{0,1_0}$ and $b^{0,1_0,2_0}$ are generators of the $\mathbb{P}_0 \text{CD}(k)$-moperad with $\Gamma$-action $\mathbb{P}_0 \text{CD}^\Gamma(k)$. They satisfy the following
relations:

(7) \((L_{0,10}^{0,10})(\alpha) \coloneqq L_{0,10}^{0,10} \cdot L_{0,11}^{0,11} \cdot \cdots \cdot L_{0,10}^{0,10} = \text{Id}_{0,10}\),

(8) \(L_{0,10}^{0,10} \cdot K_{0,11}^{0,11} = K_{0,10}^{0,10} \cdot L_{0,11}^{0,11}\),

(9) \(Y_{0,10}^{0,10} \cdot Y_{0,11}^{0,11} = Y_{0,10}^{0,10} \cdot Y_{0,11}^{0,11}\),

(10) \(Y_{0,10}^{0,10} \cdot Y_{0,11}^{0,11} = Y_{0,10}^{0,10} \cdot Y_{0,11}^{0,11}\),

(11) \(Y_{0,10}^{0,10} \cdot Y_{0,11}^{0,11} = Y_{0,10}^{0,10} \cdot Y_{0,11}^{0,11}\),

(12) \(K_{0,10}^{0,10} = \sum_{k=0}^{\infty} (L_{0,10}^{0,10} Y_{0,11}^{0,11}) H^{1,2} (L_{0,10}^{0,11})^{-1}\),

(13) \(K_{0,10}^{0,10} = \sum_{k=0}^{\infty} (L_{0,10}^{0,10} Y_{0,11}^{0,11}) H^{1,2} (L_{0,10}^{0,11})^{-1}\).

5.3. Cyclotomic associators. We borrow the notation from §4.5.

**Definition 5.4.** A cyclotomic associator is a couple \((F, G)\) where \(F\) is in \(\text{Assoc}(\mathbf{k})\) and \(G\) is a \(\Gamma\)-equivariant isomorphism of \(\text{PaB}(\mathbf{k})\)-moperads from \(\mathbf{PaB}^F(\mathbf{k})\) to \(\mathbf{GPaCD} ^\Gamma(\mathbf{k})\) which is the identity on objects (the \(\mathbf{PaB}(\mathbf{k})\)-moperad structure on \(\mathbf{GPaCD} ^\Gamma(\mathbf{k})\) is given by \(F\)). We denote by

\[ \text{Assoc}^\Gamma(\mathbf{k}) := \text{Iso}^*_{\text{Mop}} \left( \left( \text{PaB}(\mathbf{k}), \mathbf{PaB}^F(\mathbf{k}) \right), \left( \mathbf{GPaCD}(\mathbf{k}), \mathbf{GPaCD} ^\Gamma(\mathbf{k}) \right) \right) ^\Gamma \]

the set of cyclotomic associators.

**Notation.** The Lie algebra \(t_2^\Gamma(\mathbf{k})\) is the direct sum of its center, that is one dimensional and generated by \(c := t_{01} + t_{02} + \sum_{\alpha \in \Gamma} t_{01}^\alpha\), with the free Lie algebra \(f_{N+1}(\mathbf{k}) = \mathbf{k}^\alpha \times \exp(i_\alpha(\mathbf{k})) \times \exp(i_\alpha^0(\mathbf{k}))\) generated by \(t_0\) and the \(t_0^\alpha\)'s \((\alpha \in \Gamma)\). The quotient of \(t_2^\Gamma(\mathbf{k})\) by its center will be denoted \(\overline{t}_2^\Gamma(\mathbf{k})\), and is thus isomorphic to \(f_{N+1}(\mathbf{k})\). For every \(\psi \in \overline{t}_2^\Gamma(\mathbf{k})\), we write \(\psi^{0,1,2} := \psi(t_{01}, t_{02}, \ldots, t_{N+1}^{N-1})\).

We then have the following theorem:

**Theorem 5.5.** There is a one-to-one correspondence between elements of \(\text{Assoc}^\Gamma(\mathbf{k})\) and those of the set \(\text{Ass}^\Gamma(\mathbf{k})\) consisting of triples \((\mu, \varphi, \psi) \in \mathbf{k}^2 \times \exp(i_\alpha(\mathbf{k})) \times \exp(i_\alpha^0(\mathbf{k}))\), such that \((\mu, \varphi) \in \text{Ass}(\mathbf{k})\) and \(\psi\) satisfies

(14) \(\psi^{0,1,2,3,1,2,3} = \psi^{0,1,3,1,2,3} \cdot \exp(\overline{i}_2^\Gamma(\mathbf{k}))\),

(15) \(e^{\text{exp}(\overline{i}_2^\Gamma(\mathbf{k}))} \cdot \psi^{0,1,2,3,1,2,3} = \psi^{0,1,3,1,2,3} \cdot \exp(\overline{i}_2^\Gamma(\mathbf{k})),\)

where \(\alpha = (0, 1) \in \Gamma^2\).

**Proof.** Let \(\tilde{F}\) be a \(\mathbf{k}\)-associator \(\mathbf{PaB}(\mathbf{k}) \to \mathbf{GPaCD}(\mathbf{k})\), and let \(\tilde{G}\) be a \(\Gamma\)-equivariant isomorphism

\[ \mathbf{PaB}^\Gamma(\mathbf{k}) \to \mathbf{GPaCD}^\Gamma(\mathbf{k})\]
of \( \overline{\text{PaB}}(k) \)-moperads, which is the identity on objects. It corresponds to a unique \( \Gamma \)-equivariant morphism \( G : \overline{\text{PaB}}^\Gamma \rightarrow \text{GPaCD}^\Gamma(k) \). From the presentation of \( \overline{\text{PaB}}^\Gamma \), we know that \( G \) is uniquely determined by the images of the morphisms \( E^{0,1}_0 \in \text{Hom}_{\overline{\text{PaB}}^\Gamma(k)}((01)_0,01_0) \) and \( \Psi^{0,1}_0 \in \text{Hom}_{\overline{\text{PaB}}^\Gamma(k)(2)}((01)_0,01_0(01)_2) \). Thus, there are elements \( u \in \exp(t^\Gamma_1(k)) \) and \( v \in \exp(t^\Gamma_2(k)) \) such that

- \( G(E^{0,1}_0) = uL^{0,1}_0 \), with \( u = e^{\mu_1 t^{01}} \) for some \( \mu_1 \in k \), necessarily;
- \( G(\Psi^{0,1}_0) = v\psi^{0,1}_0. \)

These elements must satisfy the following relations, that are images of (19), (20), and (21), respectively:

\[
\begin{align*}
(16) \quad & v^{0,1}_0 u^{0,12}(\bar{1}, \bar{1}) \cdot (v^{0,12})^{-1} = u^{0,1}(\bar{1}, 0) \cdot u^{0,1} (\text{in } \exp(t^\Gamma_2(k))), \\
(17) \quad & v^{0,1}_0 v^{0,12} = v^{0,1}_0 v^{0,12}_0 \cdot (v^{0,12}_0) (\text{in } \exp(t^\Gamma_3(k))), \\
(18) \quad & u^{0,1}_0 = v^{0,1}_0 e^{\psi^{0,1}_2} (v^{0,12}_1)^{-1} u^{0,2}_0 (\text{in } \exp(t^\Gamma_2(k))).
\end{align*}
\]

**Lemma 5.6.** Equation (16) is satisfied for arbitrary \( u \) and \( v \).

**Proof.** First of all, observe that the diagonal action of \( \Gamma \) on \( \overline{t}^\Gamma_0(k) \) is trivial. Hence \( (\bar{1}, \bar{1}) \cdot u^{0,1} = u^{0,1} \). Second of all, recall the formulae for the moperadic structure on \( \overline{t}^\Gamma_1(k) \):

\[
(t_0)^{0,12} = t_{01} + t_{02} + \sum_{\gamma \in \Gamma} t^{\gamma}_{12} = t_{01} + (t_0)^{0,12}.
\]

Therefore, \( (\bar{1}, 0) \cdot u^{0,1} = u^{0,1} \), and \( u^{0,12} = u^{0,1} u^{0,12} \). Finally, the above element (19) is central in \( t^\Gamma_2(k) \), thus so is \( v^{0,12} = u^{0,1} v^{0,12} \), and equation (16) is satisfied. \( \square \)

Considering that we have a Lie algebra isomorphism

\[
\overline{t}^\Gamma_2(k) = k e \oplus j_{N+1}(k),
\]

where \( c = t_{01} + t_{02} + \sum_{\gamma \in \Gamma} t^{\gamma}_{12} \), then \( v \) is of the form \( e^{\mu_2 e} \psi(t_{01}, t^0_{12}, \ldots, t^{N-1}_{12}) \) for some \( \mu_2 \in k \).

**Lemma 5.7.** We have \( \mu_1 = k \). In particular, \( u = e^{\psi^{0,1}} t^{01}_0 \).

**Proof of the Lemma.** Denote by \( \varepsilon : t^\Gamma_0 \rightarrow t^\Gamma_0 \) the Lie algebra morphism sending \( t_0 \) to 0, and \( t^\gamma_{ij} \) to \( \frac{1}{\varepsilon} \bar{t}^\gamma_{i-1,j-1} \) if \( 0 < i < j \). We have \( \varepsilon(u^{0,12}) = e^{\varepsilon t_{01}} e^{\exp(t^\Gamma_1)} \), for some \( s \in k \). Now, the image of the mixed pentagon relation (17) by \( \varepsilon \) yields:

\[
(20) \quad u^{0,12} e^{s(t_{01} + t_{02})} = e^{s t_{01}} e^{s(t_{02} + t_{12})} \varphi^{0,12}_0.
\]

Moreover, as \( [t_{01}, t_{02} + t_{12}] = 0 \), we further have \( e^{s t_{01}} e^{s(t_{02} + t_{12})} = e^{s(t_{01} + t_{02} + t_{12})} \). Thus equation (20) is equivalent to

\[
\psi^{0,12} = e^{s(t_{01} + t_{02} + t_{12})} \varphi^{0,12} e^{s(t_{01} + t_{02})} = \varphi^{0,12} e^{s t_{12}}.
\]
where we have used that $t_{01} + t_{02} + t_{12}$ is central in $t_{12}^\gamma(k)$. Now, we consider the image of equation (18) in $t_{01} = t_{01}^\gamma((t_{01} + t_{02} + t_{12})$. Using that $\psi^{01,12} = \varphi^{01,12} e^{t_{12}}$ we have
\[
e^{-\mu_1 t_{01}} = \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1} e^{\mu_1 t_{02}} \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1}
\Leftrightarrow 1 = e^{-\mu_1 t_{01}} \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1} e^{\mu_1 t_{02}} \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1}
\Leftrightarrow 1 = e^{-\mu_1 t_{01}} \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1} e^{\mu_1 t_{02}} \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1}
\Leftrightarrow 1 = e^{-\mu_1 t_{01}} \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1} e^{\mu_1 t_{02}} \varphi^{01,12} e^{t_{12}} \varphi(\varphi^{01,12})^{-1}

where for the last equivalence we have used the hexagon relation for the couple $(\mu, \varphi)$ twice. This is now equivalent to
\[
1 = e\varphi^{01,2} e^{(\mu_1 - \varphi)} t_{02} e^{2\mu_1} e(\mu_1 - \varphi) t_{01}
\]
Sending the generator $t_{02}$ to 0, we get that $e^{(\mu_1 - \varphi)} t_{01} = 1$, and thus $\mu_1 = e^{\varphi}$. 

\textit{End of the proof of the Theorem.} Finally, relation (U) is equivalent to
\[
e^{\mu_2 \hat{\partial}(c)} \partial t_{12} \psi((t_{01}, t_{02},\ldots,t_{12})^{N-1}) = 1,
\]
where we consider the restriction operator $\partial : t_{12}^{\gamma}(k) \to t_{12}^{\gamma}(k)$. We have $\partial_1 t_{02} = t_{01}$ and $\partial_1 t_{01} = t_1(t_{01}) = 0$. Therefore this equation reduces to $e^{\mu_2 \hat{\partial}(c)} = 1$ which implies $\mu_2 = 0$. We conclude that $v = \psi(t_{01}, t_{02}^2, \ldots, t_{12}^{N-1})$. This automatically shows that relations (17) and (14) are equivalent. Now, in order to show that relation (18) is equivalent to relation (15), we notice that $[c, -] = 0$ and that $\psi((t_{01}, t_{02}^2, \ldots, t_{12}^{N-1})$ has no component in weight 1 so one can collect the factors $e^{t_{12}}$ in equation (18). The element $\psi^{1,0,2} e^{t_{12}}$ is of the form $\psi^{t_{02} + \sum \tau_1 t_{12}^{t_{12}}}$.

Then, by noticing that $-t_{01} = t_{02} + \sum \tau_1 t_{12}^{t_{12}}$, we obtain the equivalence between (18) and (15). 

\textbf{Remark 5.8.} The set $\text{Ass}^\Gamma(k)$ is denoted $\textbf{Pseudo}_\gamma(N, k)$ in [6]. One more generally has a set $\textbf{Pseudo}_\gamma(N, k)$ for every $\gamma \in (\mathbb{Z}/(N\mathbb{Z}))^\ast$: one just has to replace $\alpha = (0, 1)$ with $(0, \gamma)$ in the definition appearing in the statement of Theorem 5.5. A variation on the proof of Theorem 5.5 shows that $\textbf{Pseudo}_\gamma(N, k)$ can be identified with $\Gamma$-equivariant isomorphisms between the $\textbf{P}_{\alpha}(k)$-moperad $\textbf{P}_{\alpha}^\gamma(k)$ and the $\textbf{G}_{\alpha}(k)$-moperad $\textbf{G}_{\alpha}^\gamma(k)$ which, on objects, is the global relabeling given by the automorphism of $\Gamma$ sending $1$ to $\gamma$.

\textbf{Example 5.9} (Cyclotomic KZ Associator). Consider the differential equation
\[
\frac{d}{dz} H(z) = \left( \frac{t_{01}}{z} + \sum_{\alpha \in \mathbb{Z}/(N\mathbb{Z})} \frac{t_{12}^\alpha}{z - \zeta^\alpha} \right) H(z),
\]
where $\zeta$ is a primitive $N$th root of unity, and let $H_0^\gamma, H_1^\gamma$ be the solutions such that $H_0^\gamma(z) \sim z^{t_{01}}$ when $z \to 0$ and $H_1^\gamma(z) \sim z^{t_{12}}$ when $z \to 1$. Then, in our conventions, the renormalized holonomy $\Psi_{\text{KZ}} = H_0, H_1^\gamma \in \exp(t_{12}^\gamma(k))$ from 0 to 1 of the above differential equation is the cyclotomic KZ associator constructed by Enriquez in [6]. More precisely, Enriquez showed that the triple $(2i\pi, \Phi_{\text{KZ}}, \Psi_{\text{KZ}})$ is in $\textbf{Pseudo}_\gamma(N, C)$. 

5.4. Graded cyclotomic Grothendieck–Teichmüller groups.

**Definition 5.10.** The **graded cyclotomic Grothendieck–Teichmüller group** is the group

\[
\text{GRT}^\Gamma(k) := \text{Aut}_\text{Mop}^+ (\text{PaCD}(k), \text{PaCD}^\Gamma(k))^\Gamma
\]

of \(\Gamma\)-equivariant automorphisms of \((\text{PaCD}(k), \text{PaCD}^\Gamma(k))\) which are the identity on objects.

In the rest of this subsection, we again compare our operadic definition with the ones given by Enriquez in [6].

**Definition 5.11.** Define \(\text{GRT}^\Gamma_{(1,1)}(k)\) as the set of pairs \((g, h)\), with \(g \in \text{GRT}_1(k)\) and \(h \in \exp(\hat{U}^\Gamma_2(k))\), such that

\[
(23) \quad h^{0,1,2}(h^{0,2,1})^{-1} h(t_{02}|_1^1, t_{12}^{1,2}, \ldots, t_{12}^{N-1}) h(t_{01}|_1^1, \ldots, t_{12}^{N-1})^{-1} = 1 \quad \text{in} \exp(\hat{U}^\Gamma_2(k)),
\]

\[
(24) \quad t_{01} + \sum_{a=0}^{N-1} \text{Ad}(h(t_{01}|_1^1, \ldots, t_{12}^{a+N-1})) (t_{12}^a) + \text{Ad}(h^{0,1,2}(h^{0,2,1})^{-1}) (t_{02}) = 0 \quad \text{in} \hat{U}^\Gamma_2(k),
\]

\[
(25) \quad h^{0,1,2} h^{0,1,2} = h^{0,1,2} h^{0,1,2}, \quad g^{1,2,3} \quad \text{in} \exp(\hat{U}^\Gamma_2(k)).
\]

One can show that \(\text{GRT}^\Gamma_{(1,1)}(k)\) is a group when equipped with the product

\[
(g_1, h_1) \ast (g_2, h_2) = (g, h),
\]

where

- \(g(t_{12}, t_{23}) = g_1(t_{12}, \text{Ad} g_2(t_{12}, t_{23}) (t_{23})) g_2(t_{12}, t_{23})\),
- \(h^{0,1,2} = h_1(t_{01} \text{Ad}((h_2^{0,1,2}) (t_{12}^0), \ldots, \text{Ad}(h_2(t_{01}|_1^1, \ldots, t_{12}^{N-1}))) (t_{12}^{-1}) \cdot h_2^{0,1,2}\).

The action of \((\mathbb{Z}/N\mathbb{Z})^* \times k^x\) by automorphisms of \(\hat{U}^\Gamma_3\) (resp. \(\hat{U}^\Gamma_4\)) given by \((c, \gamma) \cdot t_{03} = \gamma t_{03}\), \((c, \gamma) \cdot t_{ij} = \gamma t_{ij}\)) induces its action by automorphisms of \(\text{GRT}^\Gamma_{(1,1)}(k)\). We denote by \(\text{GRT}^\Gamma_1(k)\) the corresponding semidirect product, and \(\text{GRT}^\Gamma_{(1,1)}(k) = \text{GRT}^\Gamma_1(k) \times k^x\).

**Proposition 5.12.** There is an injective group morphism \(\text{GRT}^\Gamma(k) \to \text{GRT}^\Gamma_1(k)\).

**Proof.** Let \((G, H)\) be an element in \(\text{GRT}^\Gamma(k)\). We have

- \(G(X^{1,2}) = X^{1,2}\),
- \(G(H^{1,2}) = \lambda H^{1,2}\),
- \(G(a^{1,2,3}) = g(t_{12}, t_{23}) a^{1,2,3}\),
- \(H(h^{0,1,2}) = \mu h^{0,1,2}\),
- \(H(H^{0,1,0}) = \mu K^{0,1,0}\),
- \(H(L^{0,1,0}) = L^{0,1,0}\),

where \((\lambda, g) \in \text{GRT}(k)\) and \((\mu, w) \in k^x \times \exp(\hat{U}_2^\Gamma(k))\). Borrowing the notation from the previous subsection, let us write \(w = e^{\nu h(t_{01}, t_{02}, \ldots, t_{N-1})}\), with \(h \in \hat{F}_{N,1}(k) \simeq \exp(\hat{U}_2^\Gamma(k))\). First of all, observe that one can show, along the same lines as in the previous subsection\(^2\), that \(\mu = \lambda\)

\(^2\)See the proof of Lemma 5.7, and the end of the proof of Theorem 5.5.
and $\nu = 0$. Second of all, (25) follows directly from (9), and (23) follows directly from (11). Finally, (13) implies that

$$t_{02} + \sum_{a=0}^{N-1} t_{a2}^a = \text{Ad} \left( h^{0,1,2}(h^{0,2,1})^{-1} \right) (t_{02}) + \sum_{a=0}^{N-1} \text{Ad} \left( h(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1}) \right) (t_{12}^a),$$

and since $t_{02} + \sum_{a=0}^{N-1} t_{a2}^a = -t_{01}$ in $T^2(k)$, we get (24).

As a consequence of the above discussion, the assignment

$$\begin{align*}
& (G, H) \mapsto (\lambda, g(t_{12}, t_{23}), h(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1}))
& \end{align*}$$

defines a map $\text{GRT}^F(k) \rightarrow \text{GRT}^I(k)$, that is obviously injective. It remains to prove that the composition of automorphisms in $\text{GRT}^F(k)$ corresponds to the composition law of the group $\text{GRT}^I(k)$. We already know that the composition of automorphisms $G_1$ and $G_2$ in $\text{GRT}(k)$ corresponds to the composition law in $\text{GRT}(k)$, that is, the associated couples $(\lambda_1, g_1)$ and $(\lambda_2, g_2)$ in $k^* \times \exp(t_3(k))$ satisfy

1. $(G_1 \circ G_2)(H^{1,2}) = \lambda_1 \lambda_2 H^{1,2}$,
2. $(G_1 \circ G_2)(a^{1,2,3}) = g_2(\lambda_1 t_{12}, g_1(t_{12}, t_{23}) \lambda_1 t_{23} \lambda_1 t_{23}^{-1}) g_1(t_{12}, t_{23}) a^{1,2,3}$.

We also already showed that any two $H_1$ and $H_2$ such that $(G_1, H_1)$ and $(G_2, H_2)$ lie in $\text{GRT}^F(k)$ are determined by elements $h_1(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1})$ and $h_2(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1})$ which represent automorphisms of the parenthesized word $(01_0)2_0$ in the groupoid $\text{GP} \text{aCD}^F(k)(2)$. Note that the group $\text{Aut}_{\text{GP} \text{aCD}^F(k)(3)}((01_0)2_0)$ is canonically identified with $\exp(t_3(k))$. Within this identification, $t_{01} = K^{0,15}$ for instance, but $t_{12}^0 = b^{0,15,2_0} H_{1a}^{1a} (b^{0,15,2_0})^{-1}$. Therefore

$$H_1(t_{01}) = \lambda t_{01} \text{ and } H_1(t_{12}^0) = \text{Ad} \left( h_1(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1}) \right) (\lambda, t_{12}^0).$$

More generally, $t_{12}^0 = (L^{0,16})^{(a)} b^{0,1a,2a} H_{1a}^{1a} (b^{0,1a,2a})^{-1} (L^{0,16})^{(-a)}$, and thus

$$H_1(t_{12}^a) = \text{Ad} \left( h_1(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1}) \right) (\lambda, t_{12}^a).$$

Therefore, we compute

$$\begin{align*}
(H_1 \circ H_2) & (b^{0,1a,2a}) = H_1 \left( h_2(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1}) b^{0,1a,2a} \right) \\
& = h_2 \left( h_1(t_{01}), H_1(t_{12}^0), \ldots, H_1(t_{12}^{N-1}) \right) h_1(t_{01}, t_{12}^0, \ldots, t_{12}^{N-1}) b^{0,1a,2a} \\
& = h_2 \left( \lambda t_{01}, \text{Ad} \left( h_1(t_{01}^0, t_{12}^0, \ldots, t_{12}^{N-1}) \right) (\lambda, t_{12}^0), \ldots, \text{Ad} \left( h_1(t_{01}, t_{12}^{N-1}), \ldots, t_{12}^{N-2} \right) (\lambda, t_{12}^{N-1}) \right) h_1(t_{01}, t_{12}^0, \ldots, t_{12}^{N-1}) b^{0,1a,2a},
\end{align*}$$

which is nothing but the composition law in the group $\text{GRT}^I(k)$. This concludes the proof, as the composite of operad morphisms $H_1 \circ H_2$ is compatible with the composition of operad morphisms $G_1 \circ G_2$.

\[\square\]

In the next § we will show, among other things, that this injective morphism is actually an isomorphism. We could prove surjectivity by proving that the relations from Remark 5.3 completely determine $\text{PaCD}^F(k)$ (which we believe is true), but we use instead the torsor structure.
5.5. **Bitorsors.** Our main goal in this final § is to promote the one-to-one correspondence from Theorem 5.5 to a bitorsor isomorphism. Recall from [6] that the group $\text{GTM}_1(N,k)$ acts freely and transitively on $\text{Ass}^F(k)$ from the left, in the following manner:

$$(\lambda, f, g) \cdot (\lambda', \varphi', \psi') = (\lambda\lambda', \varphi', \psi')$$

where $\varphi'(t_{12}, t_{23}) := \varphi'(t_{12}, t_{23})f(\exp t_{12}, \Ad_0(t_{12}, t_{23}))$, and

$\psi'(t_{12}^0, t_{23}^0, \ldots, t_{23}^{N-1}) := \psi'(t_{12}^0, t_{23}^0, \ldots, t_{23}^{N-1})$.

For the sake of completeness, let us recall how this right action is defined. For every $\mu \in k^*$, the group $\text{GRT}^F(1,1)(k)$ acts on

$$\text{Ass}^F_k(k) := \{ (\varphi, \psi) \in \exp(t_{12}^0(k)) \times \exp(t_{23}^0(k)) ; (\mu, \varphi, \psi) \in \text{Ass}^F(k) \}$$

from the right by $(\varphi, \psi) \cdot (g, h) = (\varphi', \psi')$, where

$$\varphi'(t_{12}, t_{23}) = \varphi(t_{12}, \Ad_0(t_{12}, t_{23}))$$

and

$$\psi'(t_{01}, t_{12}^0, \ldots, t_{12}^{N-1}) = \psi(t_{01}, \Ad_0(t_{01}, t_{12}^0, \ldots, t_{12}^{N-1})).$$

This naturally extends to an action of $\text{GRT}^F(1,1)(k)$ on $\text{Ass}^F(k)$, which is compatible with the scaling action of $k^*$ on $k$.

We already know that, by definition,

$$\left(\text{GTM}_1(N,k), \text{Ass}^F(k), \text{GRT}^F(k)\right)$$

has a natural bitorsor structure.

**Theorem 5.13.** There is a bitorsor isomorphism

$$\left(\text{GTM}_1(N,k), \text{Ass}^F(k), \text{GRT}^F(k)\right) \to \left(\text{GTM}_1(N,k), \text{Ass}^F(k), \text{GRT}^F(k)\right).$$

**Proof.** This is a summary of most of the above results. Indeed, we proved that

- There is an injective group morphism $\text{GTM}^F(1,1)(k) \to \text{GTM}_1(N,k)$ (Proposition 4.10);
- There is an injective group morphism $\text{GRT}^F(k) \to \text{GRT}^F(1,1)(k)$ (Proposition 5.12);
There is a bijection $\text{Assoc}^\Gamma(k) \to \text{Ass}^\Gamma(k)$ (Theorem 5.5).

Hence, in order to conclude it is sufficient to prove that the three above maps take the respective actions $\text{GT}^\Gamma(k)$ and $\text{GRT}^\Gamma(k)$ on $\text{Assoc}^\Gamma(k)$, to the ones of $\text{GTM}_1(N,k)$ and $\text{GRT}_1(k)$ on $\text{Ass}^\Gamma(k)$. The proof is similar to the proofs that $\text{GT}^\Gamma(k) \to \text{GTM}_1(N,k)$ and $\text{GRT}^\Gamma(k) \to \text{GRT}_1(k)$ are group morphisms (see the proofs of Propositions 4.10 and 5.12), and is left to the reader. $\square$
LIST OF NOTATION

Operads.

- **PaB**: Operad of parenthesized braids. 5
- **PaCD(k)**: Operad of parenthesized chord diagrams. 7
- **PaB^1**: PaB-moperad of parenthesized braids with a frozen strand. 10
- **PaB^Γ**: PaB-moperad of parenthesized cyclotomic braids. 14
- **PaCD^Γ(k)**: PaCD(k)-moperad of parenthesized N-chord diagrams. 27

Groups.

- **GT**: Operadic Grothendieck-Teichmüller group. 8
- **GT(k)**: k-pro-unipotent Grothendieck-Teichmüller group. 8
- **GRT(k)**: Operadic graded Grothendieck-Teichmüller group. 9
- **GT^Γ(k)**: Operadic k-pro-unipotent cyclotomic Grothendieck-Teichmüller group. 19
- **GTM^Γ(N,k)**: k-pro-unipotent (1,N)-cyclotomic Grothendieck-Teichmüller group. 22
- **GRT^Γ(k)**: Operadic graded cyclotomic Grothendieck-Teichmüller group. 30
- **GRT^Γ^Γ(k)**: Graded (1,N)-cyclotomic Grothendieck-Teichmüller group. 31

Spaces.

- **C(ℂ, I)**: Reduced configuration space of I-indexed points in ℂ. 5
- **C^Γ(ℂ, I)**: Fulton-MacPherson compactification of C(ℂ, I). 5
- **C(ℂ^x, I)**: Reduced configuration space of I-indexed points in ℂ^x. 9
- **C(ℂ^x, I, Γ)**: Reduced Γ-decorated configuration space of I-indexed points in ℂ^x. 13

Torsor sets.

- **Assoc(k)**: Operadic k-associators. 7
- **Ass(k)**: k-associators. 8
- **Assoc^Γ(k)**: Operadic cyclotomic k-associators. 28
- **Ass^Γ(k)**: Cyclotomic (1,k)-associators. 28

Series.

- **Φ_KZ**: KZ associator. 8
- **Ψ_KZ**: Cyclotomic KZ associator. 30
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