EXTENSION OF A CONJECTURAL SUPERCONGRUENCE OF (G.3) OF SWISHER USING ZEILBERGER’S ALGORITHM

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Abstract. Using Zeilberger’s algorithm, we here give a proof of the supercongruence

\[ \sum_{n=0}^{p^r-3} (8n+1) \frac{(1/4)^4}{\binom{1/4}{n}} \equiv -p^3 \sum_{n=0}^{p^{r-2}-3} (8n+1) \frac{(1/4)^4}{\binom{1/4}{n}} \pmod{p^{r-2}}, \]

for any odd integer \( r > 3 \). This extends the third conjectural supercongruence of (G.3) of Swisher to modulo higher prime powers than that expected by Swisher.

1. Introduction and statement of the results

For a complex number \( z \) with \( \text{Re}(z) > 0 \), the gamma function \( \Gamma(z) \) is defined as

\[ \Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx. \]

The functional equation \( z\Gamma(z) = \Gamma(1+z) \) gives the continuation of \( \Gamma(z) \) to a meromorphic function defined for all complex numbers \( z \). Throughout the paper, let \( p \) be an odd prime. Following \([2]\), the \( p \)-adic gamma function is defined as

\[ \Gamma_p(n) = (-1)^n \prod_{0<j<n, p \nmid j} j, \quad n \in \mathbb{N}. \]

Let \( \mathbb{Z}_p \) be the ring of \( p \)-adic integers. We extend \( \Gamma_p \) to all \( x \in \mathbb{Z}_p \) by setting

\[ \Gamma_p(x) = \lim_{n \to x} \Gamma_p(n), \]

where \( n \) runs through any sequence of natural numbers which \( p \)-adically approaches to \( x \) and \( \Gamma_p(0) = 1 \). Suppose \( f_p \left[(x)_n\right] \) denote the product of the \( p \)-factors present in \( (x)_n \), then

\[ (x)_n = (-1)^n f_p \left[(x)_n\right] \frac{\Gamma_p(x+n)}{\Gamma_p(x)}, \]

where the rising factorial (also known as Pochhammer’s symbol) \( (x)_n \) is given by \( (x)_0 := 1 \) and \( (x)_n := x(x+1) \cdots (x+n-1) \) for \( n \geq 1 \).
In [16], Ramanujan discovered 17 infinite series representations of \( \frac{1}{\pi} \), such as
\[
(1.2) \quad \sum_{n=0}^{\infty} (8n + 1) \left( \frac{1}{4} \right)^n_n = \frac{2\sqrt{2}}{\sqrt{\pi} \Gamma \left( \frac{1}{4} \right)^2}.
\]
The identity (1.2), proved by Hardy [6] pp. 495, appeared in the first letter of Ramanujan to Hardy. VanHamme [18] conjectured that the identity (1.2) has a surprising p-adic analogue
\[
(1.3) \quad \sum_{n=0}^{p-1} (8n + 1) \left( \frac{1}{4} \right)^n_n \equiv \frac{p \Gamma_p \left( \frac{1}{2} \right) \Gamma_p \left( \frac{3}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right)^2} \pmod{p^3}
\]
if \( p \equiv 1 \pmod{4} \). Motivated by the work of Long [14], Swisher [17] gave a proof of (G.2) extending to all primes. In particular, she proved that
\[
(1.3) \quad \sum_{n=0}^{p-1} (8n + 1) \left( \frac{1}{4} \right)^n_n \equiv \begin{cases} 
\frac{p \Gamma_p \left( \frac{1}{2} \right) \Gamma_p \left( \frac{3}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right)^2} \pmod{p^4}, & \text{if } p \equiv 1 \pmod{4} \text{ and } t = 1; \\
\frac{-3p^2}{2} (-1)^{\frac{3p-1}{4}} \Gamma_p \left( \frac{1}{2} \right) \Gamma_p \left( \frac{3}{4} \right)^2 \pmod{p^3}, & \text{if } p \equiv -1 \pmod{4} \text{ and } t = 3.
\end{cases}
\]
Swisher further listed a number of general VanHamme-type supercongruence conjectures based on computational evidence computed using Sage, some particular cases of which have been proved by He [7, 8, 9], Chetry and the second author [11], and the authors [10, 13]. The general VanHamme-type supercongruence conjecture of Swisher corresponding to (G.2) states that
\[
(1.3) \quad \left\{ 
\begin{array}{ll}
S \left( \frac{p^r - 1}{4} \right) \equiv (-1)^{\frac{p^r-1}{4}} p \Gamma \left( \frac{1}{2} \right) \Gamma_p \left( \frac{1}{4} \right)^2 \left( \frac{p^{r-1} - 1}{4} \right) \pmod{p^{4r}} & \text{if } p \equiv 1 \pmod{4}; \\
S \left( \frac{p^r - 1}{4} \right) \equiv -p^3 S \left( \frac{p^{r-2} - 1}{4} \right) \pmod{p^{4r-2}} & \text{if } p \equiv 3 \pmod{4}, r \geq 2 \text{ even}; \\
S \left( \frac{p^r - 3}{4} \right) \equiv -p^3 S \left( \frac{p^{r-2} - 3}{4} \right) \pmod{p^{r+1}} & \text{if } p \equiv 3 \pmod{4}, r \geq 3 \text{ odd};
\end{array}
\right.
\]
where \( S(m) := \sum_{n=0}^{m} (8n + 1) \left( \frac{1}{4} \right)^n_n \). Using hypergeometric series identities and evaluations, the authors [13] extended the case \( p \equiv 3 \pmod{4} \) of (1.3) to modulo \( p^4 \), and proved the second supercongruence of (G.3) for the case \( r = 2 \). In [11], the authors have further proved some generalizations of the case \( p \equiv 3 \pmod{4} \) of (1.3).

Our main aim in this paper is to prove the third supercongruence of the general VanHamme-type supercongruence conjecture (G.3) posed by Swisher. Because of the truncation of the sum at \( \frac{p^{r-1} - 1}{4d} \) for \( p \equiv -1 \pmod{d} \), proof of such supercongruences using hypergeometric series identities fails. Thus the authors have employed the powerful WZ-method in [12] to prove the third supercongruence conjecture of (F.3) where the series truncates at \( \frac{p^{r-1}}{4d} \) for \( p \equiv 3 \pmod{4} \). Following [15, 19], we here use the powerful Zeilberger’s algorithm instead of WZ-method to prove the third supercongruence conjecture of (G.3).
Theorem 1.1. Let $p$ be a prime such that $p \equiv 3 \pmod{4}$. If $r > 1$ is an odd integer, then

$$\sum_{n=0}^{\frac{p^r-3}{2}} (8n+1) \left(\frac{4}{1}\right)_n^4 \equiv 64(-1)^{\frac{p^r-3}{2}+1} p^{\frac{3(r-1)}{2}} \frac{\Gamma_p \left(\frac{3}{4}\right)^2}{\Gamma_p \left(\frac{1}{2}\right) \Gamma_p \left(\frac{1}{4}\right)} \pmod{p^{\frac{3r-1}{2}}}.$$ 

We further prove the following supercongruence confirming the third supercongruence of (G.3) for $r > 3$. In fact, we extend the third conjectural supercongruence of (G.3) to modulo higher prime powers than that expected by Swisher.

Theorem 1.2. Let $p$ be a prime such that $p \equiv 3 \pmod{4}$. If $r > 3$ is any odd integer, then

$$\sum_{n=0}^{\frac{p^r-3}{2}} (8n+1) \left(\frac{4}{1}\right)_n^4 \equiv -p^3 \sum_{n=0}^{\frac{p^r-2-3}{2}} (8n+1) \left(\frac{4}{1}\right)_n^4 \pmod{p^{\frac{3r-1}{2}}}.$$ 

2. Preliminaries

In this section, we state and prove some results concerning $p$-adic gamma function and rising factorials. We first recall some basic properties of $p$-adic gamma function in the following lemma. Let $\mathbb{Q}_p$ and $\nu_p(.)$ denote the field of $p$-adic numbers and the $p$-adic valuation on $\mathbb{Q}_p$, respectively.

Lemma 2.1. [2] Section 11.6 Let $p$ be an odd prime and $x, y \in \mathbb{Z}_p$. Then

(i) $\Gamma_p(1) = -1$.

(ii) $\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } \nu_p(x) = 0; \\ 1, & \text{if } \nu_p(x) > 0. \end{cases}$

(iii) $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}$, where $a_0(x) \in \{1, 2, \ldots, p\}$ satisfies $a_0(x) \equiv x \pmod{p}$.

(iv) if $x \equiv y \pmod{p}$, then $\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p}$.

We now state some recurrence relations of certain rising factorials that shall be used in the proofs of our results.

Lemma 2.2. [12] Lemma 2.3] Let $p$ be a prime with $p \equiv 3 \pmod{4}$. If $r > 1$ is an odd positive integer, then

(a) $\left(\frac{4}{3}\right)_{\frac{p^r-3}{2}} = \frac{p^{\frac{r-1}{4}+\frac{r-2}{4}}}{(-1)^{\frac{r-1}{4}+\frac{r-2}{4}}} \Gamma_p \left(\frac{p^r}{4}-\frac{5}{4}\right) \Gamma_p \left(\frac{p^r-1}{4}+\frac{1}{4}\right) \left(\frac{p^{r-2}}{2} - \frac{5}{4}\right) \left(\frac{p^{r-2}}{2} - \frac{1}{4}\right).$

(b) $\left(\frac{1}{3}\right)_{\frac{p^r-3}{2}} = \frac{p^{\frac{r-1}{4}+\frac{r-2}{4}}}{(-1)^{\frac{r-1}{4}+\frac{r-2}{4}}} \Gamma_p \left(\frac{p^r}{4}+\frac{1}{4}\right) \Gamma_p \left(\frac{p^r-1}{4}+\frac{3}{4}\right).$

(c) $\left(\frac{1}{3}\right)_{\frac{p^r-3}{2}} = \frac{p^{\frac{r-1}{4}+\frac{r-2}{4}}}{(-1)^{\frac{r-1}{4}+\frac{r-2}{4}}} \Gamma_p \left(\frac{p^r}{4}-\frac{1}{2}\right) \Gamma_p \left(\frac{p^r-1}{4}+\frac{1}{2}\right) \left(\frac{p^{r-2}}{4} - \frac{1}{2}\right).$

Lemma 2.3. Let $p$ be a prime with $p \equiv 3 \pmod{4}$. If $r > 1$ is an odd integer, then

$$\left(\frac{1}{3}\right)_{\frac{p^r-3}{2}} = \frac{p^{\frac{r-1}{4}+\frac{r-2}{4}}}{(-1)^{\frac{r-1}{4}+\frac{r-2}{4}}} \Gamma_p \left(\frac{p^r}{4}-\frac{1}{2}\right) \Gamma_p \left(\frac{p^r-1}{4}+\frac{1}{2}\right) \left(\frac{p^{r-2}}{4} - \frac{1}{2}\right).$$
Lemma 3.1. Let \( n \geq k \geq 0 \) be integers. Suppose

\[
F(n, k) = (-1)^k \frac{(8n + 1)(\frac{k}{2})^3 (\frac{1}{2})_{n+k}}{(1)_{n} (1)_{n-k} (\frac{3}{2})_{k}^2}
\]

Our motivation of taking the above hypergeometric functions is based on WZ-pairs of \([3, 4, 5, 20]\).

Proof. Noting that the \( p \)-factors present in \((\frac{1}{2})^{p-\frac{3}{4}}\) are

\[
\left\{ \left( k + \frac{1}{2} \right) p \mid 0 \leq k \leq \frac{p^{r-1} - 5}{4} \right\},
\]

we use (1.1) to obtain

\[
\left( \frac{1}{2} \right)^{p-\frac{3}{4}} = (-1)^{p-\frac{3}{4}} \frac{\Gamma_p \left( \frac{p}{4} - \frac{1}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right)} f_p \left[ \left( \frac{1}{2} \right)^{p-\frac{3}{4}} \right]
\]

\[
= (-1)^{p-\frac{3}{4}} \frac{\Gamma_p \left( \frac{p}{4} - \frac{1}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right)} \prod_{k=0}^{p-1} \left( kp + \frac{p}{2} \right)
\]

\[
= (-1)^{p-\frac{3}{4}} \frac{\Gamma_p \left( \frac{p}{4} - \frac{1}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right)} p^{p-\frac{3}{4}} \left( \frac{1}{2} \right)^{p-\frac{3}{4}}
\]

\[
= (-1)^{p-\frac{3}{4}} \frac{\Gamma_p \left( \frac{p}{4} - \frac{1}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right)} \frac{\Gamma_p \left( \frac{p}{4} + \frac{1}{4} \right)}{\Gamma_p \left( \frac{3}{4} \right)} f_p \left[ \left( \frac{1}{2} \right)^{p-\frac{3}{4}} \right].
\]

Since \((\frac{1}{2})^{p-\frac{3}{4}}\) contains the \( p \)-factors

\[
\left\{ \left( k + \frac{1}{2} \right) p \mid 0 \leq k \leq \frac{p^{r-2} - 3}{4} \right\},
\]

we have

\[
f_p \left[ \left( \frac{1}{2} \right)^{p-\frac{3}{4}} \right] = \prod_{k=0}^{p-2} \left\{ \left( k + \frac{1}{2} \right) p \right\} = p^{p-\frac{2}{4}} \left( \frac{1}{2} \right)^{p-\frac{2}{4}} \left( \frac{p^{r-2} - 3}{4} \right).
\]

As a result, we complete the proof of the lemma. \( \square \)

3. Zeilberger’s algorithm and proof of the results

For integers \( n \geq k \geq 0 \), suppose \( F(n, k) \) is a hypergeometric function in \( n \) and \( k \). The Zeilberger’s algorithm enables us to find another hypergeometric function \( G(n, k) \), as well as polynomials \( p(k) \) and \( q(k) \) such that

\[
p(k)F(n, k - 1) - q(k)F(n, k) = G(n + 1, k) - G(n, k).
\]

However, the process is not so obvious always. If \( p(k) = q(k - 1) \), then \( q(k)F(n, k) \) and \( G(n, k) \) form a WZ-pair. We here employ the Zeilberger’s algorithm for the function

\[
F(n, k) = (-1)^k \frac{(8n + 1)(\frac{k}{2})^3 (\frac{1}{2})_{n+k}}{(1)_{n} (1)_{n-k} (\frac{3}{2})_{k}^2}.
\]

Our motivation of taking the above hypergeometric functions is based on WZ-pairs of \([3, 4, 5, 20]\).

Lemma 3.1. Let \( n \geq k \geq 0 \) be integers. Suppose

\[
F(n, k) = (-1)^k \frac{(8n + 1)(\frac{k}{2})^3 (\frac{1}{2})_{n+k}}{(1)_{n} (1)_{n-k} (\frac{3}{2})_{k}^2}.
\]
and hence we complete the proof of the lemma.

Proof. It is easy to deduce that

\[ G(n, k) = (-1)^{k-1} \frac{16 \left( \frac{1}{4} \right)_{n}^{3} \left( \frac{1}{4} \right)_{n+k-1}}{(1)_{n-1}^{2} (1)_{n-k}^{2}} , \]

where \( 1/(1)_m = 0 \) for \( m = -1, -2, \ldots \). Then

\[ (4k - 3)F(n, k - 1) - (4k - 2)F(n, k) = G(n + 1, k) - G(n, k) . \]

Proof. It is easy to deduce that

\[ \frac{F(n, k - 1)}{F(n, k)} = -\frac{4k^2}{(4n + 4k - 3)(n - k + 1)} , \]

\[ \frac{G(n + 1, k)}{F(n, k)} = -\frac{16 \cdot (4n + 1)^3}{(8n + 1)(n - k + 1)} , \]

and

\[ \frac{G(n, k)}{F(n, k)} = -\frac{16n^3}{(4n + 4k - 3)(8n + 1)} . \]

As a result, we have

\[ (4k - 3) \frac{F(n, k - 1)}{F(n, k)} - (4k - 2) = \frac{G(n + 1, k)}{F(n, k)} - \frac{G(n, k)}{F(n, k)} , \]

and hence the result follows.

Lemma 3.2. Let \( p \equiv 3 \pmod{4} \) and \( r > 1 \) an odd integer. For \( k = 1, 2, \ldots, \frac{p^r - 3}{4} \), we have

\[ G \left( \frac{p^r + 1}{4}, k \right) \equiv 0 \pmod{p^{2r-1}} . \]

Proof. Noting that \((a)_{n+1} = (a + n)(a)_n, (a)_{n+k} = (a)_{n}(a + n)_k, \) and \((a)_{n-k} = (1-a-n)_{n-k} (1-a-n)_k \), we have

\[ G \left( \frac{p^r + 1}{4}, k \right) = (-1)^{k-1} 16 \frac{\left( \frac{1}{4} \right)_{n}^{3} \left( \frac{1}{4} \right)_{n+k}^{3}}{(1)_{n-1}^{2} (1)_{n-k}^{2} (\frac{1}{4})^k} . \]

\[ = -16 \left( \frac{p^r - 2}{4} \right)^3 \left( \frac{1}{4} \right)_{\frac{p^r - 3}{4}} \frac{4 \left( \frac{p^r - 2}{4} \right)_{k} (\frac{1}{4})_{\frac{p^r - 3}{4}}}{(\frac{1}{4})^{2k}} . \]

Using Lemma 2.2, one can easily see that

\[ \nu_p \left\{ \left( \frac{1}{4} \right)_{\frac{p^r - 3}{4}} \right\} = \frac{r-1}{2} . \]

Since \( \nu_p \left( \left( \frac{p^r - 2}{4} \right)_{k} \right) \geq \nu_p \left( \left( \frac{1}{4} \right)_{k} \right) \) and \( \nu_p \left( \left( \frac{1}{4} \right)_{\frac{p^r - 3}{4}} \right) \geq \nu_p \left( \left( \frac{1}{4} \right)_{k} \right) \), we must have

\[ \nu_p \left\{ \left( \frac{p^r - 2}{4} \right)_{k} (\frac{1}{4})_{\frac{p^r - 3}{4}} (\frac{1}{4})^k \right\} \geq 0 , \]

and hence we complete the proof of the lemma.
Proof of Theorem 1.1. From Lemma 3.1 we have
\[(4k - 3) \sum_{n=0}^{r-3} F(n, k - 1) - (4k - 2) \sum_{n=0}^{r-3} F(n, k) = G \left( \frac{p^r + 1}{4}, k \right) - G(0, k).\]
Thus Lemma 3.2 yields
\[\sum_{n=0}^{r-3} F(n, k - 1) \equiv \frac{(4k - 2)}{(4k - 3)} \sum_{n=0}^{r-3} F(n, k) \pmod{p^{2(r-1)}}.\]
Using this repeatedly, we deduce that
\[\sum_{n=0}^{r-3} F(n, 0) = \left( \prod_{k=1}^{n} \frac{4k - 2}{4k - 3} \right) \sum_{n=0}^{r-3} F \left( n, \frac{p^r - 3}{4} \right) \pmod{p^{2(r-1)}}\]
\[\equiv \frac{(\frac{1}{2})^{r-3}}{(\frac{1}{2})^{r-3}} F \left( \frac{p^r - 3}{4}, \frac{p^r - 3}{4} \right) \pmod{p^{2(r-1)}}.\]
Noting that
\[\sum_{n=0}^{r-3} (8n + 1) \frac{4}{(1)^n} = \sum_{n=0}^{r-3} F(n, 0),\]
we have
\[\sum_{n=0}^{r-3} (8n + 1) \frac{4}{(1)^n} \equiv \frac{(\frac{1}{2})^{r-3}}{(\frac{1}{2})^{r-3}} F \left( \frac{p^r - 3}{4}, \frac{p^r - 3}{4} \right) \pmod{p^{2(r-1)}}\]
(3.1)
\[\equiv (-1)^{r-3} (2p^r - 5) \frac{(\frac{1}{2})^{r-3}}{(\frac{1}{2})^{r-3}} \equiv \frac{(\frac{1}{2})^{r-3}}{(\frac{1}{2})^{r-3}} A(p),\]
where
\[A(p) = \prod_{j=1}^{r-3} \left\{ \begin{array}{l}
\left( \frac{x^j - 1}{4} \right) \left( \frac{x^{j+1} - 1}{4} \right) \Gamma_p \left( \frac{x^{j+1} + 1}{4} \right) \\
\Gamma_p \left( \frac{x^{j+1} + 1}{4} + \frac{1}{4} \right) \Gamma_p \left( \frac{x^{j+1} + 1}{4} - \frac{1}{4} \right) \Gamma_p \left( \frac{x^{j+1} + 1}{4} \right)
\end{array} \right\} \pmod{p}
\]
\[= \frac{\Gamma_p \left( \frac{1}{4} \right) \Gamma_p \left( \frac{3}{4} \right) \Gamma_p \left( \frac{1}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right) \Gamma_p \left( \frac{3}{4} \right)} \pmod{p}
\]
\[= (-1)^{r-3} \pmod{p}
\]
because of Lemma 2.1. In view of (1.1) and Lemma 2.1 note that
\[\left( \frac{1}{2} \right)^{r-3} \frac{\Gamma_p \left( \frac{1}{4} \right) \Gamma_p \left( \frac{3}{4} \right)}{\Gamma_p \left( \frac{1}{4} \right)} \equiv 4(-1)^{r-3} \frac{\Gamma_p \left( \frac{1}{2} \right)}{\Gamma_p \left( \frac{1}{2} \right)} \pmod{p},\]
\[
\left(\frac{1}{4}\right)_{r-3} = (-1)^{r-3} \frac{\Gamma_p \left(\frac{r}{2} - \frac{3}{4}\right)}{\Gamma_p \left(\frac{3}{4}\right)} = \frac{16}{5} (-1)^{\frac{r-1}{2}} \frac{\Gamma_p \left(\frac{3}{4}\right)}{\Gamma_p \left(\frac{1}{4}\right)} \pmod{p},
\]
and
\[
(1)_{r-1} = (-1)^{r-1} \frac{\Gamma_p \left(\frac{r}{2} + \frac{1}{4}\right)}{\Gamma_p (1)} = (-1)^{\frac{r+1}{2}} \frac{1}{4} \pmod{p}.
\]

As a result, we have from (3.2) that
\[
\frac{(1/2)_{r-1} \left(\frac{3}{4}\right)_{r-3}}{(1)_{r}^{4}} \equiv \frac{64}{5} (-1)^{\frac{r+1}{2}} p^{\frac{3(r-1)}{2}} \frac{\Gamma_p \left(\frac{4}{1}\right)}{\Gamma_p \left(\frac{1}{4}\right)} \pmod{p}.\]

Since \(\min \{2(r - 1), \frac{3r - 1}{2}\} = \frac{3r - 1}{2}\), (3.1) yields
\[
\sum_{n=0}^{\frac{r-2}{3}} (8n + 1) \frac{\left(\frac{1}{4}\right)_{n}^{4}}{(1)_{n}^{4}} \equiv 64(-1)^{\frac{r-1}{2}} p^{\frac{3(r-1)}{2}} \frac{\Gamma_p \left(\frac{4}{1}\right)}{\Gamma_p \left(\frac{1}{4}\right)} \pmod{p},
\]
completing the proof of the theorem.

**Proof of Theorem 1.2** Let \(r \geq 5\) be an odd integer. Replacing \(r\) by \(r - 2\) in Theorem 1.1, we obtain
\[
\sum_{n=0}^{\frac{r-2}{3}} (8n + 1) \frac{\left(\frac{1}{4}\right)_{n}^{4}}{(1)_{n}^{4}} \equiv 64(-1)^{\frac{r-3}{2}} p^{\frac{3(r-3)}{2}} \frac{\Gamma_p \left(\frac{3}{4}\right)}{\Gamma_p \left(\frac{1}{4}\right)} \pmod{p}.
\]

Consequently,
\[
-p^{3} \sum_{n=0}^{\frac{r-2}{3}} (8n + 1) \frac{\left(\frac{1}{4}\right)_{n}^{4}}{(1)_{n}^{4}} \equiv 64(-1)^{\frac{r-1}{2}} p^{\frac{3(r-1)}{2}} \frac{\Gamma_p \left(\frac{3}{4}\right)}{\Gamma_p \left(\frac{1}{4}\right)} \pmod{p}.
\]

Hence we complete the proof of the theorem because of Theorem 1.1.

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**References**

[1] A. S. Chetry and G. Kalita, *On a general VanHamme-type supercongruence*, J. Ramanujan Math. Soc., to appear.
[2] H. Cohen, *Number Theory* Vol. II, Analytic and Modern Tools, Graduate Text in Mathematics 240, Springer, New York (2007).
[3] S. B. Ekhad and D. Zeilberger, *A WZ proof of Ramanujan’s formula for \(\pi\)*, Geometry, Analysis, and Mechanics, J. M. Rassias (ed.), World Scientific, Singapore (1994), 107–108.
[4] J. Guillera, *Some binomial series obtained by the WZ-method*, Adv. Appl. Math. 29 (2002), no. 4, 599-603.
[5] J. Guillera, *Generators of some Ramanujan formulas*, Ramanujan J. 11 (2006), no. 1, 41–48.
[6] G. H. Hardy, *A chapter from Ramanujan’s note-book*, Math. Proc. Cambridge Philos. Soc. 21 (1923), no. 2, 492–503.
[7] B. He, *Some congruences on conjectures of van Hamme*, J. Number Theory 166 (2016), 406–414.
[8] B. He, *On some conjectures of Swisher*, Results Math. 71 (2017), 1223–1234.
[9] B. He, *On extensions of van Hamme’s conjectures* Proc. Roy. Soc. Edinburgh Sect. A 148 (2018), no. 5, 1017–1027.
[10] A. Jana and G. Kalita, *Supercongruences for sums involving rising factorial* \(\left(\frac{1}{k}\right)^3\), Integral Transforms Spec. Funct. **30** (2019), no. 9, 683–692.
[11] A. Jana and G. Kalita, *Supercongruence conjectures involving fourth power of some rising factorials*, Proc. Math. Sci. **130** (2020), Art. 59, 13 pp.
[12] A. Jana and G. Kalita, *Proof of a supercongruence conjecture of (F.3) of Swisher using the WZ-method*, submitted for publication.
[13] G. Kalita and A. Jana, *On some supercongruence conjectures for truncated hypergeometric series*, Indian J. Pure Appl. Math., to appear.
[14] L. Long, *Hypergeometric evaluation identities and supercongruences*, Pacific J. Math. **249** (2011), no. 2, 405–418.
[15] M. Petkovšek, H. S. Wilf, and D. Zeilberger, *A=B*, A K Peters, Ltd., Wellesley, MA, (1996).
[16] S. Ramanujan, *Modular equations and approximations to \(\pi\)*, Quart. J. Math. **45** (1914), 350–372. In Collected papers of Srinivasa Ramanujan, pages 23–39. AMS Chelsea Publ., Providence, RI, 2000.
[17] H. Swisher, *On the supercongruence conjectures of Van Hamme*, Res. Math. Sci. **2** (2015), Art. 2, 18 pp.
[18] L. Van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, \(p\)-adic functional analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math. **192** (1997), 223–230.
[19] S.-D. Wang, *Some supercongruences involving \(\left(\frac{2k}{k}\right)^4\)*, J. Differ. Equ. Appl. **24** (2018), no. 9, 1375–1383.
[20] W. Zudilin, *Ramanujan-type supercongruences*, J. Number Theory **129** (2009), no. 8, 1848–1857.

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