New construction of algebro-geometric solutions to the Camassa–Holm equation and their numerical evaluation

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An independent derivation of solutions to the Camassa–Holm equation in terms of multi-dimensional theta functions is presented using an approach based on Fay’s identities. Reality and smoothness conditions are studied for these solutions from the point of view of the topology of the underlying real hyperelliptic surface. The solutions are studied numerically for concrete examples, also in the limit where the surface degenerates to the Riemann sphere, and where solitons and cuspons appear.

Keywords: shallow water equation; algebro-geometric solutions; numerical evaluation

1. Introduction

The Camassa–Holm (CH) equation,

$$u_t + 3u u_x = u_{xxt} + 2u_x u_{xx} + uu_{xxx} - 2ku_x,$$  \hspace{1cm} (1.1)

was first found by Fokas & Fuchssteiner (1981) with the method of recursion operators and shown to be a bi-Hamiltonian equation with an infinite number of conserved functionals. Camassa & Holm (1993) showed that it appeared as a model for unidirectional propagation of waves in shallow water, $u(x,t)$ representing the height of the free surface about a flat bottom and $k$ being a constant related to the critical shallow water speed. In this context, only real-valued solutions are physically meaningful.

To be able to formulate a Cauchy problem for the CH equation, which implies the solution of (1.1) for a given function $u(x,0)$, it is convenient to write (1.1) in the non-local evolutionary form,

$$u_t = D^{-1}(-3uu_x + 2u_x u_{xx} + uu_{xxx} - 2ku_x),$$  \hspace{1cm} (1.2)

where the operator $D$ is given by $D = 1 - \partial_{xx}$. Since the inverse of the operator $D$ stands for an integral over Green’s function for given boundary conditions, and thus an integral from some base point $x_0$ to $x$, it is not a local operation in $x$. This non-locality has mathematically interesting consequences: the CH equation

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has travelling wave solutions of the form $u(x, t) = c \exp[-|x - vt|]$ ($v$ being the speed, $c = \text{const.}$) called peakons that have a discontinuous first derivative at the wave peak. Camassa & Holm (1993) described the dynamics of the peakons in terms of a finite-dimensional completely integrable Hamiltonian system; namely, each peakon solution is associated with a mechanical system of moving particles. The class of mechanical systems of this type was further extended by Calogero and Françoise in their study (Calogero 1995; Calogero & Françoise 1996). Multi-peakon solutions were studied using different approaches in a series of papers (Beals et al. 1998, 1999, 2000; Camassa 2000). Periodic solutions of the shallow water equation were discussed in McKean & Constantin (1999).

A further consequence of this non-locality is that solutions of the CH equation in terms of multi-dimensional theta functions do not depend explicitly on the physical coordinates. Such solutions were first given by Alber & Fedorov (2000) by solving a generalized Jacobi inversion problem. In contrast to the well-known cases of Korteweg–de Vries (KdV), nonlinear Schrödinger and sine-Gordon equations (see Belokolos et al. (1994) and references therein), complex solutions of the CH equation are not meromorphic functions of $(x, t)$ but have several branches. This is because of the presence of an implicit function $y(x, t)$ of the variables $x$ and $t$ in the argument of the theta function appearing in the solutions. A monodromy effect is thus present in the profile of real-valued solutions such as cusps and peakons. This means that even bounded solutions to the CH equation can have discontinuous or infinite derivatives in contrast to KdV solutions. Algebro-geometric solutions of the CH equation and their properties are studied by Alber et al. (2001), Alber & Fedorov (2000, 2001) and Gesztesy & Holden (2003a, b, 2008).

Our goal in this paper is to give an independent derivation of such solutions based on identities between multi-dimensional theta functions, which naturally arise from Fay’s identity (Fay 1973). This identity states that, for any points $a, b, c, d$ on a compact Riemann surface of genus $g > 0$, and for any $z \in \mathbb{C}^g$, there exist scalars $\gamma_1, \gamma_2, \gamma_3$, depending on the points $a, b, c, d$, such that

$$
\gamma_1 \Theta \left( z + \int_a^c \right) \Theta \left( z + \int_b^d \right) + \gamma_2 \Theta \left( z + \int_b^a \right) \Theta \left( z + \int_c^d \right) = \gamma_3 \Theta(z) \Theta \left( z + \int_a^c + \int_b^d \right),
$$

(1.3)

where $\Theta$ is the multi-dimensional theta function (2.2); here and below we use the notation $\int_a^b$ for the Abel map (2.4) between $a$ and $b$. While Alber and colleagues (Alber & Fedorov 2000, 2001; Alber et al. 2001) used generalized theta functions and generalized Jacobians (going back to investigations of Clebsch & Gordon 1866), we derive the solutions from identity (1.3). This fits into the programme formulated by Mumford (1983, 1984) that all algebro-geometric solutions to integrable equations should be obtained from Fay’s identity and suitable degenerations thereof. Historically, this approach was only able to reproduce solutions already obtained via so-called Baker–Akhiezer functions, generalizations of the exponential function to Riemann surfaces. The first example of new solutions found via the Fay identity was by one of the authors (Kalla 2011) for the multi-component nonlinear Schrödinger equations (see Kalla & Klein 2011). Both methods have specific advantages: for the Baker–Akhiezer approach solutions to the associated linear system, the integrability condition of which

Proc. R. Soc. A (2012)
is the studied equation, have to be constructed on a Riemann surface for a given singularity structure. For the Mumford approach, the non-trivial task is the finding of a suitable degeneration of the Fay identity for the studied equation. Once this is done, the identification of certain constants in the solutions as well as the study of reality and smoothness conditions is then, in general, more straightforward than in the Baker–Akhiezer approach. We provide here the first example of an integrable equation with non-local terms in the evolutionary form (1.2) as explained there. The spectral data for the theta-functional solutions to CH consist of a hyperelliptic curve of the form $\mu^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j)$ with three marked points: two of them are interchanged under the involution $\sigma(\lambda, \mu) = (\lambda, -\mu)$, and the third is a ramification point $(\lambda_0, 0)$.

Our construction of real-valued solutions is based on the description of the real and imaginary parts of the Jacobian associated with a real hyperelliptic curve (i.e. the branch points $\lambda_j$ are real or pairwise conjugate non-real). In this way, one gets purely transcendental conditions on the parameters (i.e. without reference to a divisor defined by the solution of a Jacobi inversion problem), such that the solutions are real-valued and smooth. It turns out that continuous real-valued solutions are either smooth or have an infinite number of cusp-type singularities.

Concrete examples for the resulting solutions are studied numerically by using the code for real hyperelliptic surfaces (Frauendiener & Klein 2004, 2006). This code uses so-called spectral methods to compute periods on the surfaces. It also allows almost degenerate surfaces to be studied numerically where the branch points collapse pairwise. In this limit, the theta functions break down to elementary functions, and the solutions describe solitons or cusps. It is noteworthy that the theta-functional solutions thus contain as limiting cases all known solutions to the CH equation. The quality of the numerics is ensured by testing the identities between theta functions which are used to construct the CH solutions in this paper. In addition, the solutions are computed on a grid and are numerically differentiated. These independent tests ensure that the solutions shown are correct to much better than plotting accuracy.

The paper is organized as follows: in §2, we summarize important facts on Riemann surfaces, especially Fay’s identities for theta functions and results on real surfaces. In §3, we use Fay’s identities to rederive theta-functional solutions to the CH equation, and give reality and smoothness conditions. In §4, we study numerically concrete examples, also in almost degenerate situations. We add some concluding remarks in §5.

2. Theta functions and real Riemann surfaces

In this section, we recall basic facts on Riemann surfaces, in particular real surfaces and multi-dimensional theta functions defined on them.

(a) Theta functions

Let $\mathcal{R}_g$ be a compact Riemann surface of genus $g > 0$. Denote by $(\mathcal{A}, \mathcal{B}) := (A_1, \ldots, A_g, B_1, \ldots, B_g)$ a canonical homology basis, and by $(\omega_1, \ldots, \omega_g)$ the basis
of holomorphic differentials normalized via
\[ \int_{A_k} \omega_j = 2i\pi \delta_{kj}, \quad k, j = 1, \ldots, g. \] (2.1)

The matrix \( \mathbb{B} = (\int_{B_k} \omega_j) \) of \( B \)-periods of the normalized holomorphic differentials \( \omega_j, \ j = 1, \ldots, g \), is symmetric and has a negative definite real part. The theta function with (half-integer) characteristics \( \delta = (\delta_1, \delta_2) \) is defined by
\[ \Theta[\delta](z) = \sum_{m \in \mathbb{Z}^g} \exp \left\{ \frac{1}{2} \langle \mathbb{B}(m + \delta_1), m + \delta_1 \rangle + \langle m + \delta_1, z + 2i\pi \delta_2 \rangle \right\}, \] (2.2)
for any \( z \in \mathbb{C}^g \); here \( \delta_1, \delta_2 \in \{0, 1/2\}^g \) are the vectors of the characteristics \( \delta \); \( \langle \ldots \rangle \) denotes the scalar product \( \langle u, v \rangle = \sum_i u_i v_i \) for any \( u, v \in \mathbb{C}^g \). The theta function \( \Theta[\delta](z) \) is even if the characteristics \( \delta \) are even, i.e. \( 4\langle \delta_1, \delta_2 \rangle \) is even, and odd if the characteristics \( \delta \) are odd, i.e. \( 4\langle \delta_1, \delta_2 \rangle \) is odd. An even characteristic is called non-singular if \( \Theta[\delta](0) \neq 0 \), and an odd characteristic is called non-singular if the gradient \( \nabla \Theta[\delta](0) \) is non-zero. The theta function with zero characteristics is called the Riemann theta function and is denoted by \( \Theta \).

Denote by \( \Lambda \) the lattice \( \Lambda = \{2i\pi N + \mathbb{B} M, N, M \in \mathbb{Z}^g\} \) generated by the \( \mathcal{A} \) and \( \mathcal{B} \)-periods of the normalized holomorphic differentials \( \omega_j, \ j = 1, \ldots, g \). The complex torus \( J := J(\mathcal{R}_g) = \mathbb{C}^g/\Lambda \) is called the Jacobian of the Riemann surface \( \mathcal{R}_g \). The theta function (2.2) has the following quasi-periodicity property with respect to the lattice \( \Lambda \):
\[ \Theta[\delta](z + 2i\pi N + \mathbb{B} M) = \Theta[\delta](z) \exp \left\{ -\frac{1}{2} \langle \mathbb{B} M, M \rangle - \langle z, M \rangle + 2i\pi \langle (\delta_1, N) - (\delta_2, M) \rangle \right\}. \] (2.3)

Denote by \( \Pi \) the Abel map \( \Pi : \mathcal{R}_g \mapsto J \) defined by
\[ \Pi(p) = \int_{p_0}^p \omega, \] (2.4)
for any \( p \in \mathcal{R}_g \), where \( p_0 \in \mathcal{R}_g \) is the base point of the application, and where \( \omega = (\omega_1, \ldots, \omega_g)^t \) is the vector of the normalized holomorphic differentials. In the whole paper, we use the notation \( \int_a^b = \Pi(b) - \Pi(a) \).

Now let \( k_a \) denote a local parameter near \( a \in \mathcal{R}_g \) and consider the following expansion of the normalized holomorphic differentials \( \omega_j, \ j = 1, \ldots, g \),
\[ \omega_j(p) = (V_{a,j} + W_{a,j} k_a(p) + \cdots) \, dk_a(p), \] (2.5)
for any point \( p \in \mathcal{R}_g \) in a neighbourhood of \( a \), where \( V_{a,j}, W_{a,j} \in \mathbb{C} \). Let us denote by \( D_a \) the operator of the directional derivative along the vector \( V_a = (V_{a,1}, \ldots, V_{a,g})^t \),
\[ D_a F(z) = \sum_{j=1}^g \partial_j F(z) V_{a,j}, \] (2.6)
where \( F : \mathbb{C}^g \mapsto \mathbb{C} \) is an arbitrary function. According to Mumford (1983, 1984), the theta function satisfies the following identities derived from Fay’s
identity (1.3):

\[ D_b \ln \frac{\Theta(z + \int_a^c)}{\Theta(z)} = p_1 + p_2 \frac{\Theta(z + \int_a^b)\Theta(z + \int_c^b)}{\Theta(z)} \]  

(2.7)

and

\[ D_a D_b \ln \Theta(z) = q_1 + q_2 \frac{\Theta(z + \int_b^a)\Theta(z - \int_a^b)}{\Theta(z)^2} \],  

(2.8)

for any \( z \in \mathbb{C}^g \) and any distinct points \( a, b, c \in \mathcal{R}_g \); here the scalars \( p_i, q_i, i = 1, 2 \) depend on the points \( a, b, c \) and are given by

\[
p_1(a, b, c) = -D_b \ln \frac{\Theta[\delta](\int_c^b)}{\Theta[\delta](\int_c^b)} ,
\]

(2.9)

\[
p_2(a, b, c) = \frac{\Theta[\delta](\int_b^a)}{\Theta[\delta](\int_b^a)\Theta[\delta](\int_b^a)} D_b \Theta[\delta](0),
\]

(2.10)

\[
q_1(a, b) = D_a D_b \ln \Theta[\delta] \left( \int_a^b \right) ,
\]

(2.11)

and

\[
q_2(a, b) = \frac{D_a \Theta[\delta](0) D_b \Theta[\delta](0)}{\Theta[\delta](\int_a^b)^2} ,
\]

(2.12)

where \( \delta \) is a non-singular odd characteristic.

(b) Real Riemann surfaces

A Riemann surface \( \mathcal{R}_g \) is called real if it admits an anti-holomorphic involution, denoted by \( \tau \). An anti-holomorphic involution \( \tau : \mathcal{R}_g \rightarrow \mathcal{R}_g \) satisfies \( \tau^2 = id \) and acts on the local parameter as the complex conjugation. The connected components of the set of fixed points of the anti-involution \( \tau \) are called real ovals of \( \tau \). We denote by \( \mathcal{R}_g(\mathbb{R}) \) the set of fixed points. According to Harnack’s inequality (Harnack 1876), the number \( \chi \) of real ovals of a real Riemann surface of genus \( g \) cannot exceed \( g + 1 \): \( 0 \leq \chi \leq g + 1 \). Curves with the maximal number \( \chi = g + 1 \) of real ovals are called M-curves.

The complement \( \mathcal{R}_g \setminus \mathcal{R}_g(\mathbb{R}) \) has either one or two connected components. The curve \( \mathcal{R}_g \) is called a dividing curve if \( \mathcal{R}_g \setminus \mathcal{R}_g(\mathbb{R}) \) has two components, and \( \mathcal{R}_g \) is called non-dividing if \( \mathcal{R}_g \setminus \mathcal{R}_g(\mathbb{R}) \) is connected (notice that an M-curve is always a dividing curve). In this paper, we only consider hyperelliptic curves, which are discussed in detail §4.

Let \( (\mathcal{A}, \mathcal{B}) \) be a basis of the homology group \( H_1(\mathcal{R}_g) \). According to proposition 2.2 in Vinnikov’s paper (Vinnikov 1993) (see also Gross & Harris 1981), there exists a canonical homology basis such that

\[
\begin{pmatrix} \tau \mathcal{A} \\ \tau \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_g & 0 \\ \mathbb{H} & -\mathbb{I}_g \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} ,
\]

(2.13)

where \( \mathbb{I}_g \) is the \( g \times g \) unit matrix, and \( \mathbb{H} \) is a block diagonal \( g \times g \) matrix which depends on the topological type of \( \mathcal{R}_g \).

Proc. R. Soc. A (2012)
In what follows, we choose a canonical homology basis in $H_1(R_g)$ satisfying (2.13) and take $a, b \in R_g$, such that $\tau a = b$. Denote by $\ell$ a contour connecting the points $a$ and $b$ which does not intersect the canonical homology basis. Then the action of $\tau$ on the generators $(A, B, \ell)$ of the relative homology group $H_1(R_g, \{a, b\})$ is given by (see Kalla (2011) for more details)

$$
\begin{pmatrix}
\tau \ell
\tau B
\tau A
\end{pmatrix}
= \begin{pmatrix}
I_g & 0 & 0
\mathbb{H} & -I_g & 0
N^t & 0 & -1
\end{pmatrix}
\begin{pmatrix}
A
B
\ell
\end{pmatrix},
\tag{2.14}
$$

for some $N \in \mathbb{Z}^g$. In the case where $\tau a = a$ and $\tau b = b$, the action of $\tau$ on the generators $(A, B, \ell)$ of the relative homology group $H_1(R_g, \{a, b\})$ reads (see Kalla (2011) for more details)

$$
\begin{pmatrix}
\tau \ell
\tau B
\tau A
\end{pmatrix}
= \begin{pmatrix}
I_g & 0 & 0
\mathbb{H} & -I_g & 0
N^t & M^t & 1
\end{pmatrix}
\begin{pmatrix}
A
B
\ell
\end{pmatrix},
\tag{2.15}
$$

where the vectors $N, M \in \mathbb{Z}^g$ are related by

$$
2N + \mathbb{H}M = 0. \tag{2.16}
$$

Now let us study the action of $\tau$ on Abelian differentials and the action of the complex conjugation on the theta function with zero characteristics. Denote by $\tau^*$ the action of $\tau$ lifted to the space of differentials: $\tau^* \omega(p) = \omega(\tau p)$ for any $p \in R_g$. By (2.13), the $A$-cycles in the homology basis are invariant under $\tau$. Owing to the normalization conditions (2.1), this leads to the following action of $\tau$ on the normalized holomorphic differentials:

$$
\tau^* \omega_j = -\omega_j. \tag{2.17}
$$

Let $a, b \in R_g$ and denote by $\{A, B, S_b\}$ the generators of the homology group $H_1(R_g \setminus \{a, b\})$ of the punctured Riemann surface $R_g \setminus \{a, b\}$, where $S_b$ is a positively oriented small contour around $b$, such that $S_b \circ \ell = 1$. It was proved by Kalla (2011) that, in the case where $\tau a = b$, the $A$-cycles in the homology group $H_1((R_g \setminus \{a, b\})$ are stable under $\tau$. Therefore, by the uniqueness of the normalized differential of the third kind $\Omega_{b-a}$, which has residue 1 at $b$ and residue $-1$ at $a$, we get

$$
\tau^* \Omega_{b-a} = -\Omega_{b-a}. \tag{2.18}
$$

In the case where $\tau a = a$ and $\tau b = b$, proposition A.2 in Kalla (2011) shows that the action of $\tau$ on the $A$-cycles in the homology group $H_1(R_g \setminus \{a, b\})$ is given by

$$
\tau A = A - MS_b, \tag{2.19}
$$

where $M$ is defined in (2.15). Therefore, by the uniqueness of the differential $\Omega_{b-a}$, we deduce that

$$
\tau^* \Omega_{b-a} = \Omega_{b-a} + M^t \omega, \tag{2.20}
$$

where $\omega$ denotes the vector of normalized holomorphic differentials. From (2.13) and (2.17), we obtain the following reality property for the Riemann matrix $\mathbb{B}$:

$$
\mathbb{B} = \mathbb{B} - 2i\pi \mathbb{H}. \tag{2.21}
$$
Moreover, according to proposition 2.3 in Vinnikov (1993), for any \( z \in \mathbb{C}^g \), relation (2.21) implies

\[
\Theta(z) = \kappa \Theta(z - i\pi \text{diag}(\mathbb{H})),
\]

where \( \text{diag}(\mathbb{H}) \) denotes the vector of diagonal elements of the matrix \( \mathbb{H} \), and \( \kappa \) is a root of unity which depends on the matrix \( \mathbb{H} \) (knowledge of the exact value of \( \kappa \) is not needed for our purpose).

(c) Action of \( \tau \) on the Jacobian and the theta divisor of real Riemann surfaces

In this part, we review known results about the theta divisor of real Riemann surfaces (see Dubrovin & Natanzon 1989; Vinnikov 1993). Let us choose a canonical homology basis satisfying (2.13) and consider the Jacobian \( J := J(\mathcal{R}_g) \) of a real Riemann surface \( \mathcal{R}_g \).

The anti-holomorphic involution \( \tau \) on \( \mathcal{R}_g \) gives rise to an anti-holomorphic involution on the Jacobian: if \( D := D_2 - D_1 \) with \( D_1 \) and \( D_2 \) positive divisors on \( \mathcal{R}_g \) (recall that a positive divisor is defined by a finite formal sum of points \( \sum_i n_i a_i \) with \( a_i \in \mathcal{R}_g \) and \( n_i \in \mathbb{N} \)), then \( \tau D \) is the class of the point \( \left( \int_{\tau D_2} \omega \right) = \left( \int_{D_2} \tau^* \omega \right) \) in the Jacobian. Therefore, by (2.17), \( \tau \) lifts to the anti-holomorphic involution on \( J \), denoted also by \( \tau \), given by \( \tau \zeta = -\zeta \) for any \( \zeta \in J \).

Now consider the following two subsets of the Jacobian:

\[
S_1 = \{ \zeta \in J; \zeta + \tau \zeta = i\pi \text{diag}(\mathbb{H}) \}
\]

and

\[
S_2 = \{ \zeta \in J; \zeta - \tau \zeta = i\pi \text{diag}(\mathbb{H}) \},
\]

where the matrix \( \mathbb{H} \) was introduced in (2.13). Below, we study their intersections \( S_1 \cap (\Theta) \) and \( S_2 \cap (\Theta) \) with the theta divisor \( (\Theta) \), the set of zeros of the theta function. Let us introduce the notation: the vectors \( e_i, \ i = 1, \ldots, g \) with components \( e_{ik} = \delta_{jk}, \mathbb{B}_i = \mathbb{B} e_i \).

It is a straightforward computation to prove that the set \( S_1 \) is the disjoint union of the tori \( T_v \) defined by

\[
T_v = \left\{ \zeta \in J; \zeta = i\pi \left( \frac{\text{diag}(\mathbb{H})}{2} + v_1 e_{r+1} + \cdots + v_{g-r} e_g \right) + \sum_{j=1}^g \beta_j \text{Re}(\mathbb{B}_j), \beta_1, \ldots, \beta_r \in \mathbb{R}, \beta_{r+1}, \ldots, \beta_g \in \mathbb{R}/Z \right\},
\]

where \( v = (v_1, \ldots, v_{g-r}) \in (\mathbb{Z}/2\mathbb{Z})^{g-r} \) and \( r \) is the rank of the matrix \( \mathbb{H} \). Therefore, the description of the set \( S_1 \cap (\Theta) \) reduces to the study of the sets \( T_v \cap (\Theta) \). In the case where \( \mathcal{R}_g(\mathbb{R}) \neq \emptyset \) and \( \mathcal{R}_g \) is non-dividing, one can see that for all \( v \) the torus \( T_v \) contains a half-period corresponding to odd half-integer characteristics, which yield \( T_v \cap (\Theta) \neq \emptyset \). The same holds for all \( v \neq 0 \) in the case where the curve is dividing or does not have real ovals. The following result, proved in Vinnikov (1993), provides a complete description of the sets \( T_v \cap (\Theta) \) in the case where the curve admits real ovals:

**Proposition 2.1.** If \( \mathcal{R}_g(\mathbb{R}) \neq \emptyset \), then \( T_v \cap (\Theta) = \emptyset \) if and only if the curve is dividing and \( v = 0 \).

Proc. R. Soc. A (2012)
In other words, among all curves that admit real ovals, the only torus \( T_v \) which does not intersect the theta divisor is the torus \( T_0 \) corresponding to dividing curves.

Analogously, it can be checked that the set \( S_2 \) is the disjoint union of the tori \( \tilde{T}_v \) defined by

\[
\tilde{T}_v = \left\{ \zeta \in J; \zeta = 2i(\alpha_1 e_1 + \cdots + \alpha_g e_g) + \left( \frac{v_1}{2} \right) \mathcal{B}_{r+1} \right. \\
+ \cdots + \left( \frac{v_{g-r}}{2} \right) \mathcal{B}_g, \quad \alpha_1, \ldots, \alpha_g \in \mathbb{R} \mathbb{Z} \right\},
\]

where \( v = (v_1, \ldots, v_{g-r}) \in (\mathbb{Z}/2\mathbb{Z})^{g-r} \) and \( r \) is the rank of the matrix \( \mathbb{H} \). A description of the sets \( \tilde{T}_v \cap (\Theta) \) in the case where the curve admits real ovals was given by Dubrovin & Natanzon (1989):

**Proposition 2.2.** If \( \mathcal{R}_g(\mathbb{R}) \neq \emptyset \), then \( \tilde{T}_v \cap (\Theta) = \emptyset \) if and only if the curve is an M-curve and \( v = 0 \).

3. Algebro-geometric solutions of the Camassa–Holm equation

In this section, we will use Fay’s identities to construct solutions to the CH equation on hyperelliptic surfaces. For the resulting formulae, we establish conditions under which we obtain real and smooth solutions. In what follows \( \mathcal{R}_g \) denotes a hyperelliptic curve of genus \( g > 0 \) represented as a two-sheeted branched covering of the sphere. We denote by \( \sigma \) the hyperelliptic involution defined on it. Note that the CH equation can be expressed in the following simple form:

\[
m_t + um_x + 2mu_x = 0,
\]

where we put \( m := u - u_{xx} + k \).

(a) **Identities between theta functions**

In our approach to construct algebro-geometric solutions of the CH equation, we use the corollaries (2.7) and (2.8) of Fay’s identity.

**Proposition 3.1.** Let \( a, b \in \mathcal{R}_g \) such that \( \sigma(a) = b \) and let \( e \in \mathcal{R}_g \) be a ramification point; namely, \( e = (\lambda_j, 0) \) for some \( j \in \{1, \ldots, 2g + 2\} \). Denote by \( g_1 \) and \( g_2 \) the following functions of the variable \( z \in \mathbb{C}^g \):

\[
g_1(z) = \frac{\Theta(z + r/2)}{\Theta(z)} \quad \text{and} \quad g_2(z) = \frac{\Theta(z - r/2)}{\Theta(z)},
\]

where \( r = \int_{a}^{b} \omega \) and \( \omega \) is the vector of normalized holomorphic differentials. Then the two following identities hold:

\[
D_b D_c \ln \frac{g_1}{g_2} = -\frac{p_2}{g_1 g_2} D_b \ln g_1 g_2
\]

and

\[
D_b D_c \ln (g_1 g_2) = \frac{q_2}{p_2 g_1 g_2} \left( D_b \ln \frac{g_1}{g_2} - 2\tilde{p}_1 \right) - 2\tilde{q}_2 g_1 g_2.
\]
Here we used the notation:

\[ p_2 = p_2(b, e, a), \quad \tilde{p}_i = p_i(e, b, a) \quad \text{and} \quad \tilde{q}_2 = q_2(b, e), \quad (3.5) \]

where the scalars \( q_2(...) \) and \( p_i(..., ...), i = 1, 2, \) are defined in (2.12) and (2.9), (2.10); \( D_b \) (respectively, \( D_e \)) denotes the directional derivative along the vector \( \mathbf{V}_b \) (respectively, \( \mathbf{V}_e \)) defined in (2.5).

**Proof.** Under the changes of variables \((a, b, c) \rightarrow (b, e, a)\) and \( z \rightarrow z - r/2\), identity (2.7) becomes

\[ D_e \ln \frac{g_1}{g_2} = p_1 + \frac{p_2}{g_1 g_2}, \quad (3.6) \]

where we used the notation \( p_i = p_i(b, e, a) \) for \( i = 1, 2 \). Here, we used the fact that \( \int_e^b \omega = \int_e^b \omega = r/2 \), according to the action of \( \sigma^* \) (the action of \( \sigma \) lifted to the space of one-forms) on the normalized holomorphic differentials \( u_j \),

\[ \sigma^* \omega_j = -\omega_j, \quad j = 1, \ldots, g. \quad (3.7) \]

Applying the differential operator \( D_b \) to equation (3.6), one obtains

\[ D_b D_e \ln \frac{g_1}{g_2} = -p_2 D_b \left( \frac{g_1 g_2}{(g_1 g_2)^2} \right) = -\frac{p_2}{g_1 g_2} D_b \ln g_1 g_2, \]

which proves (3.3). To prove (3.4), consider the change of variables \((a, b, c) \rightarrow (e, b, a)\) in (2.7), which leads to

\[ D_b \ln g_1 = \tilde{p}_1 + \tilde{p}_2 g_2 \frac{\Theta(z + r)}{\Theta(z + r/2)}. \]

Changing \( z \) to \(-z\) in the last equality, one obtains

\[ D_b \ln g_2 = -\tilde{p}_1 - \tilde{p}_2 g_1 \frac{\Theta(z - r)}{\Theta(z - r/2)}. \]

From these two identities, it can be deduced that

\[ \frac{\Theta(z + r)}{\Theta(z + r/2)} = (\tilde{p}_2 g_2)^{-1} (D_b \ln g_1 - \tilde{p}_1) \quad (3.8) \]

and

\[ \frac{\Theta(z - r)}{\Theta(z - r/2)} = -(\tilde{p}_2 g_1)^{-1} (D_b \ln g_2 + \tilde{p}_1). \quad (3.9) \]

Moreover, as

\[ D_b D_e \ln(g_1 g_2) = D_b D_e \ln \Theta \left( z + \frac{r}{2} \right) + D_b D_e \ln \Theta \left( z - \frac{r}{2} \right) - 2 D_b D_e \ln \Theta(z), \]

using (2.8) one obtains

\[ D_b D_e \ln(g_1 g_2) = \frac{\tilde{q}_2}{g_1} \frac{\Theta(z + r)}{\Theta(z + r/2)} + \frac{\tilde{q}_2}{g_2} \frac{\Theta(z - r)}{\Theta(z - r/2)} - 2 \tilde{q}_2 g_1 g_2, \quad (3.10) \]

which by (3.8) and (3.9) leads to (3.4).
With identities (3.3) and (3.4), we are now able to construct theta-functional solutions of the CH equation:

**Theorem 3.2.** Let \( a, b \in \mathbb{R}_g \) such that \( \sigma(a) = b \), and let \( e \in \mathbb{R}_g \) be a ramification point. Denote by \( \ell \) an oriented contour between \( a \) and \( b \) which contains the point \( e \). Assume that \( \ell \) does not cross cycles of the canonical homology basis. Choose arbitrary constants \( d \in \mathbb{C}_g \) and \( k, \zeta \in \mathbb{C} \), and put

\[
\alpha_1 = p_1(b, e, a) \quad \text{and} \quad \alpha_2 = 2p_1(e, b, a) + k,
\]

where the function \( p_1 \) is defined in (2.9). Let \( y(x, t) \) be an implicit function of the variables \( x, t \in \mathbb{R} \) defined by

\[
x + \alpha_1 y + \alpha_2 t + \zeta = \ln \frac{\Theta(Z - d + r/2)}{\Theta(Z - d - r/2)},
\]

where \( r = \int_\ell \omega \). Here the vector \( Z \) is given by

\[
Z(x, t) = V_e y(x, t) + V_b t,
\]

where the vectors \( V_e \) and \( V_b \) are defined in (2.5). Then the following function of the variables \( x \) and \( t \) is the solution of the CH equation:

\[
u(x, t) = D_b \ln \frac{\Theta(Z - d + r/2)}{\Theta(Z - d - r/2)} - \alpha_2.
\]

Here \( D_b \) denotes the directional derivative along the vector \( V_b \).

Note that the function \( y(x, t) \) is the same function as introduced in Alber & Fedorov (2001).

**Proof.** Let \( \beta, \delta \in \mathbb{C} \) and \( \alpha_1, \alpha_2 \in \mathbb{C} \) be arbitrary constants. Let us look for solutions \( u \) of CH having the form

\[
u(x, t) = \beta D_b \ln \frac{\Theta(Z - d + r/2)}{\Theta(Z - d - r/2)} + \delta = \beta D_b \ln \frac{g_1}{g_2} + \delta,
\]

where \( Z(x, t) \) is defined in (3.13), and the functions \( g_1, g_2 \) were introduced in (3.2) with \( z = Z(x, t) - d \). Putting \( \alpha_1 = p_1(b, e, a) \), by (3.12) and (3.6), the derivative with respect to the variable \( x \) of the implicit function \( y(x, t) \) is given by

\[
y_x = \frac{g_1 g_2}{p_2}.
\]

Analogously, it can be checked that

\[
y_t = -y_x \left( \frac{u}{\beta} - \frac{\delta}{\beta} - \alpha_2 \right).
\]

Now, let us express the function \( m(x, t) = u - u_{xx} + k \) introduced in (3.1) in terms of the functions \( g_1 \) and \( g_2 \) of (3.2). By (3.3) and (3.16), the first derivative of the
Algebro-geometric solutions to CH

function $u$ (3.15) with respect to the variable $x$ is given by

$$u_x = -\beta D_b \ln(g_1 g_2).$$

(3.18)

By (3.4) and (3.16), we obtain for the second derivative of $u$ with respect to $x$,

$$u_{xx} = \beta \left( D_b \ln \frac{g_1}{g_2} - 2\tilde{p}_1 \right) - 2\beta \tilde{p}_2 (g_1 g_2)^2;$$

(3.19)

here, we used the identity $\tilde{q}_2 = -\tilde{p}_2 p_2$ relating the scalars $\tilde{q}_2, \tilde{p}_2$ and $p_2$ defined in (3.5). Therefore, with (3.15) and (3.19), the function $m$ reads

$$m(x, t) = \delta + k + 2\beta \tilde{p}_1 + 2\beta \tilde{p}_2 (g_1 g_2)^2.$$

(3.20)

Taking the derivative of $m$ with respect to $x$, and the derivative of $m$ with respect to $t$, one obtains, respectively,

$$m_x(x, t) = 4\beta \tilde{p}_2 (g_1 g_2)^2 y_x D_e \ln(g_1 g_2)$$

(3.21)

and

$$m_t(x, t) = 4\beta \tilde{p}_2 (g_1 g_2)^2 (y_t D_e \ln(g_1 g_2) + D_b \ln(g_1 g_2)).$$

(3.22)

Therefore, the proof is complete substituting the functions (3.15), (3.18), (3.21) and (3.22) in the left-hand side of the CH equation (3.1) with $\beta = 1$, $\delta = -2\tilde{p}_1 - k$ and $\alpha_2 = -\delta$.

(b) Real-valued solutions and smoothness conditions

In this subsection, we identify real-valued and smooth solutions of the CH equation among the solutions given in theorem 3.2. Let us first recall that hyperelliptic M-curves of genus $g$ can be given by the equation

$$\mu^2 = \prod_{i=1}^{2g+2} (\lambda - \lambda_i),$$

(3.23)

where the branch points $\lambda_i$ are real and satisfy $\lambda_i \neq \lambda_j$ if $i \neq j$. On such a curve, we can define two anti-holomorphic involutions $\tau_1$ and $\tau_2$, given, respectively, by $\tau_1(\lambda, \mu) = (\bar{\lambda}, \bar{\mu})$ and $\tau_2(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu})$. Let us show that the curve (3.23) is an M-curve with respect to both anti-involutions $\tau_1$ and $\tau_2$. In the case where $\lambda_i \in \mathbb{R}$ satisfy $\lambda_1 < \cdots < \lambda_{2g+2}$, it can be seen that projections of real ovals of $\tau_1$ on the $\lambda$-plane coincide with the intervals $[\lambda_{2g+2}, \lambda_1], [\lambda_2, \lambda_3], \ldots, [\lambda_{2g}, \lambda_{2g+1}]$, whereas projections of real ovals of $\tau_2$ on the $\lambda$-plane coincide with the intervals $[\lambda_1, \lambda_2], \ldots, [\lambda_{2g+1}, \lambda_{2g+2}]$. Hence, the curve (3.23) has the maximal number $g + 1$ of real ovals with respect to both anti-involutions $\tau_1$ and $\tau_2$.

Now assume that $R_g$ is a real hyperelliptic curve which admits real ovals with respect to an anti-holomorphic involution $\tau$. Let us choose a homology basis satisfying (2.13). Recall that $R_g(\mathbb{R})$ denotes the set of fixed points of the anti-holomorphic involution $\tau$.

The following propositions provide reality and smoothness conditions for the solutions $u(x, t)$ (3.14) in the case where the points $a$ and $b$ are stable under $\tau$, and in the case where $\tau a = b$. It is proved that, for fixed $t_0 \in \mathbb{R}$, the function
u(x, t₀) is smooth with respect to the real variable x when a and b are stable under τ. In the case where τa = b, the function u(x, t₀) is either smooth or it has cusp-like singularities.

**Proposition 3.3.** Assume that \( R_g \) is a hyperelliptic \( M \)-curve of genus \( g \), and denote by \( e \in R_g(\mathbb{R}) \) one of its ramification points. Let \( a, b \in R_g(\mathbb{R}) \) such that \( \sigma(a) = b \). For any \( c \in \{a, b, e\} \), choose a local parameter \( k_c \) such that \( k_c(\tau p) = k_c(p) \) for any point \( p \) in a neighbourhood of \( c \). Denote by \( \ell \) an oriented contour between \( a \) and \( b \) containing point \( e \) which does not intersect cycles of the canonical homology basis. Choose \( \ell \) such that the closed path \( \tau \ell - \ell \) is homologous to zero in \( H_1(\mathcal{R}_g) \). Take \( d \in i\mathbb{R}^g \) and \( k \in \mathbb{R} \). Choose \( \zeta \in \mathbb{C} \) in (3.12) such that \( \text{Im}(\zeta) = \text{arg}[\ln(\Theta(d + \mathbf{r}/2)/\Theta(d - \mathbf{r}/2))] \). Then solutions \( u(x, t) \) of the CH equation given in (3.14) are real-valued, and, for fixed \( t_0 \in \mathbb{R} \), the function \( u(x, t_0) \) is smooth with respect to the real variable \( x \).

**Proof.** Let us check that, under the conditions of the proposition, the function \( u(x, t) \) (3.14) is real-valued. Let us fix \( y, t \in \mathbb{R} \). First of all, invariance with respect to the anti-involution \( \tau \) of the points \( e \) and \( b \) implies

\[
\overline{Z} = -Z, \tag{3.24}
\]

where the vector \( Z \) is defined in (3.13). In fact, using the expansion (2.5) of the normalized holomorphic differentials \( \omega_j \) near \( c \in \{e, b\} \), one obtains

\[
\tau^k \omega_j(c)(p) = (\overline{V_{c,j}} + \overline{W_{c,j}} k_c(p) + o(k_c(p)^2)) dk_c(p),
\]

for any point \( p \) in a neighbourhood of \( c \). Then by (2.17), the vectors \( V_e \) and \( V_b \) appearing in the vector \( Z \) are purely imaginary, which leads to (3.24). Moreover, as the closed contour \( \tau \ell - \ell \) is homologous to zero in \( H_1(\mathcal{R}_g) \), from (2.17) one gets

\[
\overline{r} = -r. \tag{3.25}
\]

For arbitrary points \( a_1, a_2, a_3 \in R_g \), using the representation of the differential \( Q_{a_3-a_1} \) in terms of multi-dimensional theta functions (see Belokolos et al. 1994), one obtains

\[
\frac{Q_{a_3-a_1}(p)}{dk_{a_2}(p)} \bigg|_{p=a_2} = p_1(a_1, a_2, a_3). \tag{3.26}
\]

We deduce that the scalars \( p_1(b, e, a) \) and \( p_1(e, b, a) \) appearing, respectively, in \( \alpha_1 \) and \( \alpha_2 \) (see (3.11)) satisfy

\[
\frac{Q_{a-b}(p)}{dk_{e}(p)} \bigg|_{p=e} = p_1(b, e, a) \quad \text{and} \quad \frac{Q_{a-e}(p)}{dk_{b}(p)} \bigg|_{p=b} = p_1(e, b, a). \tag{3.27}
\]

Therefore, as the points \( a, b, e \) are stable under \( \tau \), from (2.20) it can be deduced that \( p_1(b, e, a) \) and \( p_1(e, b, a) \) are real, which involves

\[
\overline{\alpha_1} = \alpha_1 \quad \text{and} \quad \overline{\alpha_2} = \alpha_2. \tag{3.28}
\]

Let \( y, t \in \mathbb{R} \) and denote by \( h \) the function

\[
h(y, t) = \frac{\Theta(Z - d + \mathbf{r}/2)}{\Theta(Z - d - \mathbf{r}/2)}. \tag{3.29}
\]
By (3.12), \( x \) is a real-valued function of the real variables \( y \) and \( t \) if the function \( h \) is real and has a constant sign, and if we choose \( \text{Im}(\zeta) = \arg[\ln(h(0,0))] \). From (2.22), (3.24) and (3.25), we deduce that

\[
\overline{h(y, t)} = \frac{\Theta(Z + \overline{d} + r/2 + i\pi \text{ diag}(H))}{\Theta(Z + \overline{d} - r/2 + i\pi \text{ diag}(H))}. \tag{3.30}
\]

Let us choose a vector \( \mathbf{d} \in \mathbb{C}^g \), such that \( \overline{d} = -d - i\pi \text{ diag}(H) + 2i\pi T + \mathbb{H}L \) for some vectors \( T, L \in \mathbb{Z}^g \). Reality of the vector \( \overline{d} + d \) together with (2.21) implies

\[
d = \frac{1}{2} \text{Re}(\mathbb{H})L + i\mathbf{d}_f \tag{3.31}
\]

for some \( \mathbf{d}_f \in \mathbb{R}^g \) and the relation \( 2T + \mathbb{H}L = \text{diag}(\mathbb{H}) \) for \( T \) and \( L \). For this choice of the vector \( d \), (3.30) becomes

\[
\overline{h(y, t)} = \frac{\Theta(Z - d + r/2)}{\Theta(Z - d - r/2)} \exp\{-\langle r, L \rangle\},
\]

where we used the quasi-periodicity property (2.3) of the theta function. Therefore, the function \( h \) is real if \( L = 0 \), that is, \( d \in i\mathbb{R}^g \). Now let us check that \( h \) has a constant sign with respect to \( y, t \in \mathbb{R} \). Since \( Z - d \pm r/2 \in i\mathbb{R}^g \), by proposition 2.2, the functions \( \Theta(Z - d \pm r/2) \) of the real variables \( y \) and \( t \) do not vanish if the hyperelliptic curve is an M-curve. Hence, \( h \) is a real continuous non-vanishing function with respect to the real variables \( y \) and \( t \), which means it has a constant sign. Therefore, \( x \) is a real-valued function of \( y \) and \( t \) if the constant \( \zeta \) in (3.12) is chosen such that \( \text{Im}(\zeta) = \arg[\ln(h(0,0))] \). It is straightforward to see that the solution \( u \) (3.14) is a real-valued function of the real variables \( y \) and \( t \), and then is real-valued with respect to the real variables \( x \) and \( t \).

Now fix \( t_0 \in \mathbb{R} \) and let us study smoothness conditions for the function \( u(x, t_0) \) with respect to the variable \( x \). First, let us check that the solution \( u \) (3.14) is a smooth function of the real variable \( y \) if it does not have singularities. Since the theta function is entire, singularities of the solution \( u \) are located at the zeros of its denominator. As we saw in the previous paragraph, if the curve is an M-curve and \( d \in i\mathbb{R}^g \), the functions \( \Theta(Z - d \pm r/2) \) and \( \Theta(Z - d) \) do not vanish. In this case, the function \( u \) is smooth with respect to the real variable \( y \). Now let us prove that \( u \) is smooth with respect to the real variable \( x \). By (3.12), the function \( x(y) \) is smooth. Moreover, it can be seen from (3.12) and (3.6) that

\[
x_0(y) = \frac{p_2}{g_1(Z - d)g_2(Z - d)} = \frac{\Theta(Z - d)^2}{\Theta(Z - d + r/2)\Theta(Z - d - r/2)}. \tag{3.32}
\]

Since the functions \( \Theta(Z - d \pm r/2) \) and \( \Theta(Z - d) \) do not vanish, we deduce that \( x(y) \) is a strictly monotonic real function, and thus the inverse function \( y(x) \) has the same property. Therefore, the function \( u(x, t_0) = u(y(x)) \) is a smooth real-valued function with respect to the real variable \( x \).

Now let us study real-valuedness and smoothness of the solutions in the case where \( \tau a = b \).
Proposition 3.4. Assume that $\mathcal{R}_g$ is a hyperelliptic $M$-curve of genus $g$, and denote by $e \in \mathcal{R}_g(\mathbb{R})$ one of its ramification points. Let $a, b \in \mathcal{R}_g$, such that $\sigma(a) = b$ and assume that $\tau a = b$. Choose local parameters such that $\bar{k}_b(\tau p) = k_a(p)$ for any point $p$ in a neighbourhood of $a$, and $k_e(\tau p) = -k_e(p)$ for any $p$ lying in a neighbourhood of $e$. Denote by $\ell$ an oriented contour between $a$ and $b$ containing point $e$, which does not intersect cycles of the canonical homology basis. Assume that $N = 2\mathbf{L}$ for some $\mathbf{L} \in \mathbb{Z}^g$, where $N \in \mathbb{Z}^g$ is defined in (2.14). Take $k \in \mathbb{R}$ and define $d = d_R + (i\pi/2)\mathbf{N}$ for some $d_R \in \mathbb{R}^g$. Choose $\zeta \in \mathbb{C}$ in (3.12) such that $\text{Im}(\zeta) = \arg[\ln(\Theta(d + r/2)/\Theta(d - r/2))]$. Then solutions $u$ (3.14) of the CH equation are real-valued. Moreover, for fixed $t_0 \in \mathbb{R}$, the function $u(x, t_0)$ is smooth with respect to the real variable $x$ in the case where $\mathbf{N} = 0$; otherwise, it has an infinite number of singularities of the type $O((x - x_0)^{2n/(2n+1)})$ for some $n \in \mathbb{N} \setminus \{0\}$ and $x_0 \in \mathbb{R}$, i.e. cusps.

Proof. Analogous to the case where $a$ and $b$ are stable under $\tau$, let us prove that solutions $u$ (3.14) are real-valued. Fix $y, t \in \mathbb{R}$. First let us check that the vector $\mathbf{Z}$ (3.13) satisfies

$$\bar{\mathbf{Z}} = \mathbf{Z}.$$  \hfill (3.33)

From (2.5) and (2.17), one gets $V_a = -V_b$ and $V_e = V_e$. Moreover, the vectors $V_a$ and $V_b$ satisfy $V_a + V_b = 0$ because of (3.7); thus, we have $\bar{V}_b = V_b$ and $\bar{V}_e = V_e$, which proves (3.33). By (2.18) and (2.27), it can be deduced that $\alpha_1$ and $\alpha_2$ (3.11) satisfy

$$\bar{\alpha}_1 = \alpha_1 \quad \text{and} \quad \bar{\alpha}_2 = \alpha_2.$$  \hfill (3.34)

Moreover, from (2.17) and (2.14) one obtains

$$\bar{\mathbf{r}} = \mathbf{r} - 2i\pi \mathbf{N},$$  \hfill (3.35)

where $\mathbf{N} \in \mathbb{Z}^g$ is defined in (2.14). Let us check that $x$ (3.12) is a real-valued function of the real variables $y$ and $t$. By (3.34), this holds if the function $h$ (3.29) is real and has a constant sign with respect to the real variables $y$ and $t$, and if we choose $\text{Im}(\zeta) = \arg[\ln(h(0,0))]$. By (2.22), (3.33) and (3.35), it follows that

$$\frac{h(y, t)}{\Theta(\mathbf{Z} - \bar{\mathbf{d}} + r/2 + \mathbf{p})} = \frac{\Theta(\mathbf{Z} - \bar{\mathbf{d}} - r/2 + \mathbf{p})}{\Theta(\mathbf{Z} - \bar{\mathbf{d}} + r/2 + \mathbf{p})},$$  \hfill (3.36)

where $\mathbf{p} = -i\pi \mathbf{N} - i\pi \text{diag}(\mathbb{H})$. Now let us choose the vector $d \in \mathbb{C}^g$, such that $\bar{\mathbf{d}} \equiv \mathbf{d} + \mathbf{p}$ (mod $2i\pi \mathbb{Z}^g + \mathbb{B} \mathbb{Z}^g$), which is equivalent to

$$d = d_R + \frac{i\pi}{2}(\mathbf{N} + \text{diag}(\mathbb{H}) - 2\mathbf{T}),$$  \hfill (3.37)

for some $d_R \in \mathbb{R}^g$ and $\mathbf{T} \in \mathbb{Z}^g$. Here we used the action (2.21) of the complex conjugation on the Riemann matrix $\mathbb{B}$, and the fact that $\mathbb{B}$ has a negative definite real part. For this choice of the vector $d$, by (3.36) the function $h$ is a real-valued function of the real variables $y$ and $t$. Now let us study in which cases the function $h$ has a constant sign. The sign of the function $h$ is constant with respect to $y$ and $t$ if the functions $\Theta(\mathbf{Z} - \mathbf{d} \pm r/2)$ do not vanish. By (3.33), (3.35) and (3.37), the vectors $\mathbf{Z} - \mathbf{d} \pm r/2$ belong to the set $\mathcal{S}_l$ introduced in (2.23). Hence, by proposition 2.1, the functions $\Theta(\mathbf{Z} - \mathbf{d} \pm r/2)$ do not vanish if the hyperelliptic curve is dividing (in this case $\text{diag}(\mathbb{H}) = 0$), and if the arguments $\mathbf{Z} - \mathbf{d} \pm r/2$
in the theta function are real (modulo $2\pi i\mathbb{Z}^g$). The vector \( \mathbf{Z} - \mathbf{d} + \mathbf{r}/2 \) is real if \( \mathbf{T} = 0 \) in (3.37). With this choice of the vector \( \mathbf{T} \), the imaginary part of the vector \( \mathbf{Z} - \mathbf{d} - \mathbf{r}/2 \) equals $-i\pi \mathbf{N}$. Therefore, the vector \( \mathbf{Z} - \mathbf{d} - \mathbf{r}/2 \) is real modulo $2\pi i\mathbb{Z}^g$ if all components of the vector \( \mathbf{N} \) are even. To summarize, the function \( h(y, t) \) defined in (3.29) is a real-valued function with constant sign if the hyperelliptic curve is dividing (i.e. all ramification points are stable under \( \tau \), as the ramification point \( e \) is stable under \( \tau \) and since dividing curves have either only real branch points or pairwise conjugate ones), if \( \mathbf{T} = 0 \) and \( \mathbf{N} = 2\mathbf{L} \) for some \( \mathbf{L} \in \mathbb{Z}^g \), where vector \( \mathbf{N} \in \mathbb{Z}^g \) is defined in (2.14). Analogous to the proof of proposition 3.3, we conclude that \( x \) is a real-valued continuous function of the real variables \( y \) and \( t \), and thus solutions \( u(x, t) \) (3.14) are real-valued functions of the real variables \( x \) and \( t \).

Now let us study smoothness conditions for fixed \( t_0 \in \mathbb{R} \). Notice that the function \( u(y) \) (3.14) is a smooth function of the real variable \( y \) since the denominator does not vanish, as we have seen before. Put \( \mathbf{z} = \mathbf{Z} - \mathbf{d} \). Let us consider the function \( x_g(y) \) given in (3.32) in both cases: \( \mathbf{N} = 0 \) and \( \mathbf{N} \neq 0 \).

— If \( \mathbf{N} = 0 \), the function \( x_g(y) \) does not vanish, as in this case \( \mathbf{z} \in \mathbb{R}^g \), which implies that the function \( \Theta(\mathbf{z}) \) does not vanish. Hence, analogous to the case where \( a \) and \( b \) are stable under \( \tau \), for fixed \( t_0 \in \mathbb{R} \), the function \( u(x, t_0) \) is smooth with respect to the real variable \( x \).

— If \( \mathbf{N} \neq 0 \), the function \( \Theta(\mathbf{z}) \) vanishes when \( \mathbf{z} \) belongs to the theta divisor. Fix \( x_0, t_0 \in \mathbb{R} \) and denote by \( \mathbf{z}_0 \) and \( y_0 \) the corresponding values of \( \mathbf{z} \) and \( y \). Assume that \( \mathbf{z}_0 \) is a zero of the theta function of order \( n \geq 1 \). Then, by (3.32), the function \( x_g(y) \) has a zero at \( y_0 \) of order \( 2n \). It follows that function \( x(y) - x(y_0) \) has a zero of order \( 2n + 1 \) at \( y_0 \), and then

\[
y(x) - y_0 = O((x - x_0)^{1/(2n+1)}). \tag{3.38}
\]

On the other hand, it can be seen from (3.3) that

\[
u_g(y) = p^2 \frac{\Theta(\mathbf{z})}{\Theta(\mathbf{z} + \mathbf{r}/2)\Theta(\mathbf{z} - \mathbf{r}/2)}[\Theta(\mathbf{z})\psi(\mathbf{z}) - 2D_b\Theta(\mathbf{z})], \tag{3.39}
\]

where \( \psi(\mathbf{z}) = D_b \ln(\Theta(\mathbf{z} + \mathbf{r}/2)\Theta(\mathbf{z} - \mathbf{r}/2)) \). Identity (3.39) implies that function \( u_g(y) \) has a zero at \( y_0 \) of order \( 2n - 1 \); namely,

\[
u(y) - u(y_0) = O((y - y_0)^{2n}). \tag{3.40}
\]

Finally with (3.38) and (3.40), the function \( u(x, t_0) \) has an infinite number of singularities of type \( O((x - x_0)^{2n/(2n+1)}) \), i.e. cusps.

4. Numerical study of algebro-geometric solutions to the Camassa–Holm equation

In this section, we will numerically study concrete examples of the CH solutions (3.14). As shown in the previous sections, real and bounded solutions are obtained on hyperelliptic M-curves, i.e. curves of the form (3.23).

Proc. R. Soc. A (2012)
For the numerical evaluation of the CH solutions (3.14), we use the code presented in Frauendiener & Klein (2004, 2006) for real hyperelliptic Riemann surfaces. The reader is referred to these publications for details. The basic idea is to introduce a convenient homology basis on the related surfaces (figure 1). It is related to the basis used in the previous sections by the simple relation $A \rightarrow -B, B \rightarrow A$. This choice of the homology basis has the advantage that the limit in which branch points encircled by the same $A$-cycle collide can be treated essentially numerically. Below we will consider examples where the distance of such a pair of branch points is of the order of machine precision ($10^{-14}$) and thus numerically zero. This limit is interesting as the $B$-periods diverge, which implies that the corresponding theta functions reduce to elementary functions. The CH solutions (3.14) reduce in this case to solitons or cuspons. Since we also want to study this limit numerically, we use the homology basis of figure 1, and not the one of the previous sections.

The sheets are identified at the point $a$ by the sign of the root picked by Matlab. We denote a point in the first sheet with projection $\lambda$ into the complex plane by $\lambda^{(1)}$, and a point in the second sheet with the same projection by $\lambda^{(2)}$. The theta functions are in general approximated numerically by a truncated series as explained in Frauendiener & Klein (2004) and Bobenko & Klein (2011). A vector of holomorphic differentials for these surfaces is given by $(1, \lambda, \ldots, \lambda^{g-1})^t d\lambda/\mu$. The periods of the surface are computed as integrals between branch points of these differentials as detailed in Frauendiener & Klein (2006). The Abel map of the point $a$ (and analogously for $b$) is computed in a similar way (Kalla & Klein 2011) to the integral between $a$ and the branch point with minimal distance to $a$. It is well known (see Belokolos et al. 1994) that the Abel map between two branch points is a half-period.

To control the accuracy of the numerical solutions, we use essentially two approaches. First we check the theta identity (2.7), which is the underlying reason for the studied functions being solutions to CH. Since this identity is not built into the code, it provides a strong test. This check for various combinations of the points $a, b$ and $e$ ensures that the theta functions are computed with sufficient precision, and that the quantities $\alpha_1$ and $\alpha_2$ in (3.12) are known with the required precision (we always use machine precision here). In addition, the smooth solutions are computed on Chebyshev collocation points (see Trefethen 2000) for $x$ and $t$. This can be used to approximate the computed solution via Chebyshev polynomials, a so-called spectral method having in practice exponential convergence for smooth functions. Since the
derivatives of the Chebyshev polynomials can be expressed linearly in terms of Chebyshev polynomials, a derivative acts on the space of polynomials via a so-called differentiation matrix. With these standard Chebyshev differentiation matrices (see Trefethen 2000), the solution can be numerically differentiated. The computed derivatives allow us to check with which numerical precision the partial differential equation (PDE) is satisfied by a numerical solution. With these two independent tests, we ensure that the solutions shown are correct to much better than plotting accuracy (the code reports a warning if the above tests are not satisfied to better than $10^{-6}$). We do not use the expansion in terms of Chebyshev polynomials for the cusped solutions as the convergence is slow for functions with cusps.

We first consider smooth solutions $u$ \(3.14\) in the case $\tau a = a$, $\tau b = b$. To obtain non-trivial solutions in the solitonic limit, we use a vector $d$ corresponding to the characteristics $\frac{1}{2} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^t$ in all examples. To plot a solution $u$ in dependence of $x$ and $t$, we compute it on a numerical grid for $y$ and $t$ to obtain $x(y,t)$ defined in (3.12) and $u(y,t)$ given by (3.14). These are then used to obtain a plot of $u(x,t)$ without having to solve the implicit relation (3.12). In all examples, we have $k = 1$. The solutions for $\tau a = b$, e.g. for points $a$ and $b$ on the cuts encircled by the $A$-periods in figure 1, look very similar and are therefore not shown here.

Solutions on elliptic surfaces describe travelling waves and will not be discussed here. In genus 2, we obtain CH solutions of the form shown in figure 2. The typical soliton collision known from the KdV equation is also present here, the unchanged shape of the solitons after the collision, but an asymptotic change of phase.

In genus 6, the CH solutions have the form shown in figure 3. In the solitonic limit, one can recognize a six-soliton event.

The reality properties of the quantities entering the solution (3.14) depend on the choice of the homology basis. For instance, in the homology basis of figure 1 and for $a$ and $b$ stable under $\tau$, the Abel map $r$ up to a vector proportional to $i\pi$ and the vectors $V_b, V_e$ are real, whereas these quantities are purely imaginary in the homology basis used in the previous sections. Thus, the easiest way to obtain cusped solutions is in this case to put $e = \lambda_{\frac{1}{2}}$ and to choose $d$ corresponding to the characteristics $\frac{1}{2} \begin{bmatrix} 1 & \cdots & \frac{1}{2} \end{bmatrix}^t$. It can be easily checked that the theta functions

Figure 2. Solution (3.14) to the CH equation on a hyperelliptic curve of genus 2 with branch points $-3, -2, 0, \epsilon, 2, 2 + \epsilon$ and $a = (-4)^{(1)}$, $b = (-4)^{(2)}$ and $e = (-3, 0)$ for $\epsilon = 1$ on the left and $\epsilon = 10^{-14}$, the almost solitonic limit, on the right. (Online version in colour.)
Figure 3. Solution (3.14) to the CH equation on a hyperelliptic curve of genus 6 with branch points \(-7, -6, -5, -5 + \epsilon, -3, -3 + \epsilon, -1, -1 + \epsilon, 1.1 + \epsilon, 3.3 + \epsilon, 5.5 + \epsilon\) and \(a = (-8)^{(1)}\), \(b = (-8)^{(2)}\) and \(e = (-7, 0)\) for (a) \(\epsilon = 1\) and (b) \(\epsilon = 10^{-14}\), the almost solitonic limit. (Online version in colour.)

Figure 4. Cusped solution (3.14) to the CH equation on a hyperelliptic curve of genus 2 with branch points \(-3, -2, 0, \epsilon, 2, 2 + \epsilon\) and \(a = (-4)^{(1)}\), \(b = (-4)^{(2)}\) and \(e = (-2, 0)\) for (a) \(\epsilon = 1\) and (b) \(\epsilon = 10^{-14}\), the almost solitonic limit. (Online version in colour.)

Figure 5. Solution (3.14) to the CH equation on a hyperelliptic curve of genus 6 with branch points \(-7, -6, -5, -5 + \epsilon, -3, -3 + \epsilon, -1, -1 + \epsilon, 1.1 + \epsilon, 3.3 + \epsilon, 5.5 + \epsilon\) and \(a = (-8)^{(1)}\), \(b = (-8)^{(2)}\) and \(e = (-6, 0)\) for (a) \(\epsilon = 1\) and (b) \(\epsilon = 10^{-14}\), the almost solitonic limit. (Online version in colour.)
\[ \Theta(Z - d \pm r/2) \] cannot vanish as the argument is real, whereas the \( \Theta(Z - d) \) will have zeros as the argument is complex. This implies that the derivative \( u_x \) in (3.18) diverges, which corresponds to cusps for the solution \( u \). Peakons do not appear in such a limit of theta-functional solutions to CH and are thus not discussed here. To obtain them one would have to glue solutions in the solitonic limit on finite intervals to obtain a continuous solution that is piecewise \( C^1 \).

We show the cusped solutions always in a co-moving frame \( x' = x + vt \) to allow a better visualization of the solutions. In genus 2, we obtain cusped CH solutions of the form shown in figure 4, where cusped solitons also can be seen in the degenerate situation. Obviously, the collision between cuspons is analogous to soliton collisions.

In genus 6, the CH solutions have the form shown in figure 5. In the solitonic limit one can recognize a six-cuspon.

5. Conclusion

In this paper, we have shown in the example of the CH equation that Mumford’s programme to construct algebro-geometric solutions to integrable PDEs can also be applied to non-local (here in \( x \)) equations. For the case studied, the solutions in terms of multi-dimensional theta functions do not depend directly on the physical coordinates \( x \) and \( t \), but via an implicit function. One consequence of this non-locality is the existence of non-smooth solitons. A numerical study of smooth and non-smooth solutions was presented.

A further example in this context would be the equation from the Dym-hierarchy for which theta-functional solutions were studied by Alber & Fedorov (2000), which will be treated elsewhere with Mumford’s approach. It is an interesting question whether the \( 2 + 1 \)-dimensional generalization of the CH equation (Chen et al. 2005; Falqui 2006), for which algebro-geometric solutions are so far unknown, can also be treated with these methods.

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