Ricci Flow Approach to The Cosmological Constant Problem

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In order to resolve the cosmological constant problem, the fundamental notion, reference frame is re-examined quantum mechanically. By using a quantum non-linear sigma (Q-NLSM) model, a theory of quantum spacetime reference frame (QSRF) is proposed. The underlying mathematical structure is a new geometry endowed with intrinsic 2nd central moment (variance) or even higher moments of its coordinates, which generalizes the classical Riemannian geometry based on only 1st moment (mean) of its coordinates. The 2nd central moment of the coordinates directly modifies the quadratic form distance which is the foundation of the Riemannian geometry. At semi-classical level, the 2nd central moment introduces a flow which continuously deforms the Riemannian geometry driven by its classical Ricci curvature, which is known as the Ricci flow. A generalized equivalence principle of quantum version is also proposed to interpret the new geometry endowed with at least 2nd moment. As a consequence, the spacetime is stabilized against quantum fluctuation, and the cosmological constant problem is resolved within the framework. With an isotropic positive curvature initial condition, the long-time solution of the Ricci flow exists, the accelerating expansion universe at cosmic scale is an observable effect of the spacetime deformation of the normalized Ricci flow. A deceleration parameter -0.67 consistent with measurement is obtained by using the reduced volume method introduced by Perelman. Effective theory of gravity within the framework is also discussed.

I. INTRODUCTION

The incompatibility between Quantum Mechanics (QM) and General Relativity (GR) is not only reflected at the technical level that GR is not renormalizable in ordinary sense, but also at the phenomenology level that GR and the spacetime is fundamentally unstable at the quantum level. If an effective quantum field theory arises infinity in its calculation, it does not necessarily imply a complete disaster. It may call one’s attention to treat the parameters of the theory more seriously that one should discriminate which part of the parameters are physical and which part are not. However, the incompatibility between QM and GR is not just simply the case: if QM is seriously taken into account, the classical spacetime depicted by GR will rapidly collapse and even impossible to exist. More precisely, this severe difficulty concerning the instability of spacetime under quantum fluctuation names the Cosmological Constant (CC) problem.

If someone may think that the CC problem is an non-essential side issue of physics, we consider it is a crisis of fundamental physics [1]. The CC arises as a severe problem is not at the classical level but at the quantum level, because anything including the spacetime is inescapable quantum fluctuating, and the well-tested Equivalence Principle (EP) claims that any quantum fluctuation that contributes to the energy density of the vacuum couples to gravity and behaves like a CC. A standard calculation only concerning the zero-point vacuum oscillating modes up to the Planck scale $\Lambda_{Pl}$, below which the calculations are trustable, gives the energy density $\sim \Lambda_{Pl}^4$ which shows that the vacuum energy densities of quantum fluctuations and CC (if we trust the well-tested EP) should be too large to make the spacetime stable and permanently exist. However, the observation from the accelerating expansion of the universe [2–4] shows that the CC is relatively smaller compared with the prediction. Why so large amount of vacuum energy densities do not gravitate? And if they could be canceled by certain unknown mechanism (e.g. supersymmetry), why they just leave a small remnant to the gravitational effect (i.e. accelerating expansion)? The problem leads to a severe fine-tuning to make the spacetime the way it is under the quantum fluctuation, just like to fine-tune a sharpened pencil standing on a table against perturbation.

There are so many attempts to solve the CC problem, one can find numbers of good review articles [5–9], and references therein. The attempts cover from the phenomenological models, tuning mechanisms, modified gravity, to even anthropic principle. The fundamental incompatibility is sometimes ignored and evaded.

At the fundamental level, one may puzzle that if the quantum fluctuation is real whether the EP is wrong at the quantum level? The fact is that it is well-known that the electron vacuum energy coming from the vacuum polarization measured by the Lamb’s shift does gravitate normally as the EP claims [10, 11]. There is no any evidence that the energy coming from classical and quantum are physically different, the EP is well-tested at very precise level. Essentially speaking, physicists are caught in a dilemma that both quantum fluctuations and the EP are so real and precisely tested in each field, why they give rise to an obvious wrong prediction.

In the paper we start with the assumptions that the validity of the EP is kept and generalized even to the quantum level, and the realness of the quantum fluctuations are also admitted. The approach to resolve the dilemma proposed in the paper is twofold, on the one hand the spacetime geometry is treated in a more quantum manner (Chapter 2) via the
Ricci flow approach (Chapter 3), and on the other hand the principle of QM is treated in a more relational manner [12] (Chapter 4) via a framework of entanglement that a to-be-studied quantum system relative to a Quantum Spacetime Reference Frame (QSRF) system [13]. The Ricci flow describes a continuous deformation of a Riemannian geometry from short distance to long distance scale induced by the quantum fluctuation of the spacetime, which makes the geometry of the universe more and more like an observed accelerating expansion universe (Chapter 5), or equivalently, develops an effective CC consistent with observation in the gravity theory.

The Ricci flow was introduced in 1980s in mathematics by Hamilton [10]. Hamilton used it as a tool to gradually deform a manifolds into a more and more “nice” manifolds whose topology is easily recognized, in order to prove the Poincare’s conjecture. The program was fully realized owing to Perelman’s breakthrough around 2003 [18] by introducing some monotonic functional to successfully deal with the singularities developed in 3-manifolds under the Ricci flow. The Ricci flow approach (see reviews e.g. [22–27]) as a useful tool in mathematics may have important physical applications, e.g. see [28–40], including early attempt applying it to the cosmology as an averaging approach to the spatial inhomogeneous [41–45]. But to the best of our knowledge, its physical meaning is not very clear in the literature, and its connection to the CC problem has yet to be discussed, the goal of the paper is to show their deep relation.

The resolution to the CC problem and the dilemma can be briefly stated as follows. The quantum zero-point fluctuation energies of vacuum are completely unobservable and unphysical, including the Casimir effect [10], when it is relative to a QSRF system which is also zero-point fluctuating quantum mechanically. The leading vacuum energy densities universally coupled to gravity (as EP claims) comes from the two-point vacuum quantum fluctuation given by the Ricci flow, and the accelerating expansion universe at cosmic scale is an observable effect of the spacetime deformation of the normalized Ricci flow.

II. SPACETIME WITH INTRINSIC 2ND MOMENTS: NON-LINEAR SIGMA MODEL

In order to reconcile the incompatibility between QM and GR and hence resolve the cosmological constant problem, a question at least is how to consistently apply the principle of QM to the spacetime geometry, avoiding the instability of spacetime against quantum fluctuations. It is generally believed that we need a new framework of spacetime geometry based on quantum rods and clocks. In classical geometry, the Riemannian geometry, the central concept is to measure the length between two point coordinates. In the language of QM, we measure the mean value or the 1st moment of a coordinate. However, we could imagine that under quantum fluctuation, coordinates of the geometry smear and hence higher moments of a coordinate in measurement naturally appear, for instance, the 2nd central moment, the variance of the coordinate. A crucial question is how to introduce the higher moments to a geometry, making them well-behaved under both the principle of QM and geometry. We find that the classical Riemannian geometry does not explicitly contain the notion of (higher) moments in it. To our knowledge, a new geometry with well-behaved higher moments has not been developed yet. However, the classical Riemannian geometry is not far from the new geometry, because it is a good approximation at the level of 1st moment, the mean value of coordinates, if the variance is not large enough. In the section, we suggest a generalization of Riemannian geometry by considering higher moments on it.

To define the geometry of D dimension with at least 2nd moment, we start with a conventional differentiable map $X$ from a local coordinate patch $x \in \mathbb{R}^d$ to the non-linear manifolds $M^D$. The map in physics can be realized by a kind of fields theory, the non-linear sigma model (NLSM) [28–29, 47–49].

$$S_{X} = \frac{1}{2} \lambda \int d^d x g_{\mu\nu} \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^a}. \quad (II.1)$$

The local coordinate patch $x^a, a = 0, 1, ...d−1$ can be interpreted as a locally flat Laboratory Wall Frame (LWF) which is used as a standard reference to align the coordinate frame of (physical or geometric) measurements in ordinary sense, and the map $X^\mu, \mu = 0, 1, ...D−1$ as the coordinates of the non-linear manifolds be interpreted as the physical spacetime. As quantum fluctuations are inescapable in the fields $X^\mu$, the coordinates of the spacetime geometry must subject to some level of fuzziness. We denote the quantized NLSM as Q-NLSM. Beside the mean value $\langle X^\mu \rangle$ of the coordinates, the 2nd (central) moment $\langle (\delta X^\mu)^2 \rangle = \langle (X^\mu)^2 \rangle - \langle X^\mu \rangle^2$ measures the variance or the fuzziness of the coordinates, or more general, $\langle \delta X^\mu \delta X^\nu \rangle$ measures the covariance and the correlation between two coordinates. In the paper, we ignore the phrase “central”, when mentioning “moment”, we always mean the “central moment” for short, and the bracket $\langle \rangle$ always mean quantum expectation value. Similarly, the 3rd moment “skewness” $\langle (\delta X^\mu)^3 \rangle$ or more general $\langle \delta X^\mu \delta X^\nu \delta X^\rho \rangle$ describes the asymmetry of the fuzziness, the 4th moment “kurtosis” and other higher moments may also exist. If the fluctuation only has 2nd moment, it is Gaussian, and the moments higher than 2nd order we call them non-Gaussian fluctuations.
In the sense of classical Riemannian geometry, the map $X^\mu$ is just a classical coordinate transformation of $x^\mu$, but because of the quantum fluctuations of the coordinates $X^\mu$ and the existence of higher moments of coordinates, the Q-NLSM in fact defines a new geometry with intrinsic higher moments beyond the classical Riemannian geometry, which is not clearly realized before. The NLSM describing a classical Riemannian geometry is nothing but a 1st moment approximation to the new geometry. The quantum fluctuation of the Q-NLSM in fact introduces some extra (quantum) structures to the classical Riemannian geometry, which leads to new phenomenon of spacetime and gravity such as coarse graining process of the spacetime geometry.

There are evidences that the Q-NLSM \cite{1} with $d = D = 4$ behaves as a good candidate theory of quantum gravity: (1) there are deep analogies between Einstein’s theory of gravity and the NLSM \cite{2}; (2) it has a non-trivial UV fixed point at non-perturbative level so that it is asymptotically safe \cite{1,2}. In the following paper, the asymptotic safety of the Q-NLSM is also confirmed by studying the convergence of its renormalization flow, i.e. the Ricci flow.

III. GEOMETRIC FLOW DRIVEN BY 2ND MOMENT: THE RICCI FLOW

Here we consider the effects of the lowest 2nd (central) moment, the variance, in the geometry. The classical Riemannian geometry is a kind of manifolds based on a metric of quadratic form, i.e. the length measured between two points coordinates. The mean value of the coordinates is not affected by the 2nd moment. However, because in classical Riemannian geometry, the length is the quadratic form of the coordinates, the 2nd moment now gives an extra positive contribution to the length. The length as a quadratic form is generalized to

$$\langle \Delta X^\mu \Delta X_\mu \rangle = \langle \Delta X^\mu \rangle \langle \Delta X_\mu \rangle + \langle \delta X^\mu \delta X_\mu \rangle. \quad (III.1)$$

It is equivalent to deform the metric tensor induced by the 2nd moment at the point $X$

$$g_{\mu\nu}(X) = \left\langle \frac{\partial X_\mu}{\partial x_a} \frac{\partial X_\nu}{\partial x_a} \right\rangle = \frac{\partial \langle X_\mu \rangle}{\partial x_a} \frac{\partial \langle X_\nu \rangle}{\partial x_a} + \frac{1}{(\Delta x)^2} \left\langle \delta X_\mu \delta X_\nu \right\rangle = g_{\mu\nu}^{(1)}(X) + \delta g_{\mu\nu}^{(2)}(X). \quad (III.2)$$

The first term $g_{\mu\nu}^{(1)}(X)$ is the 1st moment contribution gives rise to the standard metric tensor in classical Riemannian geometry, and the second term $\delta g_{\mu\nu}^{(2)}$ related to a cutoff length scale $(\Delta x)^2$ of the base space and proportional to the 2nd moment is a deformation to the classical metric.

The 2nd moment of the geometry as an extra quantum structure introduced by the Q-NLSM can be determined by the 2-point correlation function in the Q-NLSM. In the situation $R_1(X)\delta k^2 \ll \lambda$, where $R_1$ is the scalar curvature given by $g_{\mu\nu}^{(1)}$ at the point, $\delta k^2$ is the cutoff energy scale playing the role of the inverse of cutoff length $(\Delta x)^2$, and $\lambda$ is the prefactor of the NLSM, a lowest order perturbative calculation gives \cite{1,2}

$$\delta g_{\mu\nu}^{(2)} = \frac{R_{\mu\nu}^{(1)}}{32\pi^2\lambda} \delta k^2, \quad (III.3)$$

in which $R_{\mu\nu}^{(1)}$ is the Ricci curvature tensor given by $g_{\mu\nu}^{(1)}$ at the point. The validity of the perturbative calculation gives a definition of the semi-classical approximation: we assume the 2nd moment contribution $\delta g_{\mu\nu}^{(2)}$ is smaller compared with the 1st moment $g_{\mu\nu}^{(1)}$. We will see that the physical interpretation of $\lambda$ is the critical density of the universe $\lambda = \frac{3H_0^2}{8\pi G}$, where $H_0$ is the Hubble’s constant at current epoch and $G$ the Newton’s constant. When we discuss the problem of cosmology, the scalar curvature is approximately $R^{(1)} \sim O(H_0^2)$ and $\delta k^2 \ll 1/G$, so the semi-classical approximation is safe. It worth mentioning that $\lambda$ as the unique parameter of the theory is a combination of the Hubble’s constant and Newton’s constant, which differs from the traditional gravity theory which only has Newton’s constant. The Hubble’s constant here comes into the fundamental theory, and plays an important role in giving a characteristic scale of the universe.

We conclude that at semi-classical approximation, the metric tensor with 2nd moment seem like a classical metric with a deformation driven by its classical Ricci curvature. The deformation introduces a flow of metric tensor in Riemannian geometry, the \cite{1,2} is nothing but the Ricci flow equation

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu}^{(1)}, \quad (III.4)$$

with

$$t = \frac{1}{64\pi^2\lambda^2}. \quad (III.5)$$
Because the 2nd moment only modifies the local quadratic form of length which is not important for the topology of the geometry, so the deformation by the Ricci flow does not change the topology, the flow preserves it.

Averaging procedures in a non-linear gravity system are highly non-trivial and fundamentally an unclear issue yet. The Ricci flow is in essential a process of averaging or coarse graining of the non-linear gravity system. The Ricci flow equation is a non-linear generalization of the heat flow. The $t$ parameter of the flow corresponds to the energy scale $k$, when the flow starting from $t = -\infty$ flows forwardly to $t = 0$, it is equivalent to that the geometry starting from high energy scale $k \to \infty$ (short distance UV scale) flows forwardly to low energy $k \to 0$ (long distance IR scale). At a cut off distance scale $1/k$, the 2nd moment being shorter than $1/k$ are averaged out and gives correction to the quadratic form of length longer than the scale. As the geometry deforms and flow to long distance, it loses its information of shorter distance, so the flow is non-reversible. Because here the manifolds so defined is highly non-linear, the average process is much complex than a conventional average process, quantitatively it describes by the non-linear Ricci flow equation.

The Ricci flow is a diffusion-reaction-like equation, in which the diffusion and reaction compete each other. For some isotropic initial conditions, as it flows to long distance scale, the diffusion wins out, then the local quantum fluctuations and related moments of the geometry are gradually averaged out as the heat equation uniformizes the temperature distribution, the geometry with 2nd moments gradually deforms to a uniform classical Riemannian geometry with constant curvature. The phenomenon indicates that the 2nd moment and Gaussian fluctuation are irrelevant, the spacetime stabilizes against the quantum fluctuation and becomes more and more classical in the IR. The latter of the paper will show that it corresponds to the observation of the accelerating expansion universe at cosmic distance. For some anisotropic initial conditions, when the reaction term wins at the place where local curvature is larger enough than other places, the flow equation in general develops local singularities, near which the Riemannian geometry no longer can be used to model the geometry, the Ricci flow as a semi-classical approximation of the geometry with 2nd moment fails.

Note that the Ricci flow does not explicitly depend on the base-space-dimension $d$ of NLSM, which is hidden in the $t$ parameter. However, we know that the low energy phase structure of the NLSM is sensitive to $d$, so the Ricci flow is obviously an approximation which only describes the phase away from the low energy phase transition in the phase diagram of NLSM. Fortunately, the quantity we need to calculate by the method in the paper is a high energy limit of Perelman’s reduced volume, and we only concern its convergence at UV, so this shortcoming of the Ricci flow does not really bother us. When the NLSM is near a low energy phase transition point, the moments or fluctuation higher than 2nd order can not be ignored, the quantum fluctuation of coordinates are non-Gaussian, then terms being composed of higher power of $R_{\mu\nu}/\lambda$ also come into the flow. The Ricci flow equation as a semi-classical approximation fails, such type of flow is beyond the scope of the paper, some literature show that such type of equations qualitatively behaves similar with the Ricci flow. For instance, for some rescaled Ricci flow equation, the diffusion part determines the qualitative behavior and the non-Gaussian fluctuations are irrelevant, then as an initial geometry with non-Gaussian fluctuations flows, the fluctuation becomes more and more Gaussian, the Ricci flow becomes a good approximation.

IV. SPACETIME WITH INTRINSIC 2ND MOMENT AS A QUANTUM REFERENCE SYSTEM

The next question naturally is how a to-be-studied quantum system is described relative to the spacetime with intrinsic 2nd moment as a reference system. To interpret the above defined spacetime theory, it needs to be reformulated in an effective and semi-classical form as our familiar spacetime theory in physics (as General Relativity does).

A. Quantum Entanglement between a to-be-studied system and a reference system

Ordinary textbook quantum mechanics and quantum field theories are formulated with respect to a classical and absolute parameter background free from any quantum fluctuations, which is not physical and arises severely problem when gravity is taken into account, for example the cosmological constant problem. In a more physical treatment that a to-be-studied system and a spacetime reference system are both quantum, the to-be-studied system are described by a state $|\Psi\rangle$ in Hilbert space $\mathcal{H}_B$, the Quantum Spacetime Reference Frame (QSRF) system are described by a state $|X\rangle$ in Hilbert space $\mathcal{H}_X$, and the states of both systems are given by an entangled state

$$|\Psi[X]\rangle = \sum_{ij} \alpha_{ij} |\Psi\rangle_i \otimes |X\rangle_j$$  \hspace{1cm} (IV.1)
in their direct product Hilbert space $\mathcal{H}_\theta \otimes \mathcal{H}_X$. Here the state denoted by $|\Psi[X]\rangle$ is with respect to the quantum spacetime coordinate $X$, in analogy with a state $|\Psi(x)\rangle$ of the quantum system being with respect to a classical spacetime coordinate $x$.

The reason why the state of both systems are in an entangled state but a direct product state is as follows. Before a quantum measurement is performed, an important step is implicitly carried out. At the step, a one-to-one correlation between a state $|\Psi_i\rangle$ of the to-be-studied system and a state $|X_j\rangle$ of the quantum measuring instrument must be established, which called calibration. This step introduces an entangled state $\sum_{ij} \alpha_{ij} |\Psi_i\rangle \otimes |X_j\rangle$ which describes the state $|\Psi_i\rangle$ with respect to a quantum measuring instrument $|X_j\rangle$. The state is a generalization of a textbook quantum state $|\Psi\rangle$ which is with respect to a classical measuring instrument. In this sense, $|\Psi[X]\rangle$ being a functional is a generalization of $\Psi(x)$ being a function. The functional $\Psi[X]$ can be seen as a function $\Psi$ with a smeared variable $X$. And after that calibration step, every time we read the state $|X_j\rangle$ of the measuring instrument, in the usual sense, to infer a state $|\Psi_i\rangle$ of the to-be-studied system according to the entangled correlation. Now the measuring instrument, for instance rods and clocks measuring the space and time coordinates $X$ as a reference [57, 58], and the to-be-studied system are both quantum.

In the standard interpretation of quantum state, a measuring of $|X\rangle$ can spookily collapse $|\Psi\rangle$, even though one has not touch it at distance. So measuring $|X\rangle$ tells you some information about $|\Psi\rangle$. More precisely, the expanding coefficient $\alpha_{ij}$ gives the amplitude of the to-be-studied system being in state $|\Psi_i\rangle$ and the measuring instrument being in state $|X_j\rangle$. $|\alpha_{ij}|^2$ is the joint probability related to the amplitude. The amplitude $\alpha_{ij}$ of the entangled state in general can not be factorized into a product of each individual absolute amplitude of $|\Psi_i\rangle$ and $|X_j\rangle$, which called “non-separability of entangled state”. In the sense, the interpretation of the state must be relational [12]. The relational nature of entangled state is very important. The state of the system $|\Psi_i\rangle$ makes sense only with respect to the state of the instrument $|X_j\rangle$, that is the essential of relativity and what a reference system $|X_j\rangle$ be used for. Only when the reference system becomes classical, in other words, the 2nd or higher moment of the spacetime coordinates vanish, the coordinates of the spacetime can be seen as a Dirac’s delta distribution for the state $|X_j\rangle$ without any quantum fluctuation, the relational amplitude $\alpha_{ij}$ recovers the standard absolute amplitude $|\Psi_i\rangle$, in textbook quantum mechanics. In general, the reference system at least has non-trivial 2nd moment, it is equivalent to a wavefunction $\Psi$ with blurry coordinates $X$ [59].

B. Action of the to-be-studied system and the reference system (spacetime)

After the step of calibration between the to-be-studied system and the reference system is established and before their next interaction (i.e. measurement), the two systems evolve independently without any interaction. The action of the entangled systems is a direct sum of each individual actions without interaction. Without loss of generality, we consider a conventional scalar field $\Phi(x)$ as the to-be-studied system. The reference system is the spacetime coordinate system given by rods $X^i(x)$ and clocks $X^0(x)$ from the action of NLSM, which shares the base space $x$ as a common background with the scalar field $\Psi(x)$. The action is

$$S[\Psi, X^\mu] = \int d^4x \left[ \frac{1}{2} \frac{\partial^2 \Psi}{\partial x^\alpha \partial x^\alpha} - V_p(\Psi) \right] + \frac{1}{2} g_{\mu\nu} \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\alpha},$$

where $V_p(\Psi)$ is a potential term of the scalar field. The action is formulated by the to-be-studied scalar field $\Phi(x)$ and spacetime fields $X^\mu(x)$ with respect to the parameter background $x^\alpha$. The quantum mechanics and quantum fields theory are formulated requiring certain parameter background, for example, quantum mechanics has only one parameter: Newton’s absolute time, quantum fields theory has four parameters: Minkovski spacetime background $x^\mu$. However, the parameter background is not necessarily interpreted as the physical spacetime, because it is absolute, external, classical and free from any quantum fluctuation. Now the scalar field $\Psi$ must be described with respect to the physical spacetime $X^\mu$ instead of parameter background $x^\alpha$. Since the action describes the entanglement between the field $\Phi(x)$ and spacetime $X^\mu(x)$, it concerns the topology of the quantum states, and hence implies a topological quantum field theory relating to certain “gravity” theory with a proper interpretation.

C. Mean field approximation: recover the classical action

By the semi-classical or Mean Field (M.F.) approximation in which only the mean field value $\langle X \rangle$ is considered and its 2nd moment $\langle \delta X^2 \rangle$ is ignored, the action is simply given by a variable change $x^\alpha \rightarrow X^\mu$

$$S[\Psi, X^\mu] \stackrel{M.F.}{=} \int d^D X \sqrt{\det g} \left[ \frac{1}{4} g_{\mu\nu} \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\alpha} \right] \left( \frac{1}{2} g_{\mu\nu} \frac{\delta \Psi}{\delta X^\mu} \frac{\delta \Psi}{\delta X^\nu} + 2\lambda \right) - V_p(\Psi),$$

where $\psi(x)$ is a smeared variable.
\[ \frac{1}{4} \left( g_{\mu \nu} \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^a} \right) = \frac{1}{4} (g_{\mu \nu} g^{\mu \nu}) = \Omega. \] It is required \( d = D \equiv 4 \) because of the Jacobian determinant \( \sqrt{\det g} = \left\| \frac{\partial x}{\partial X} \right\| \) which is not distinguished \( \sqrt{\det g} \) or \( \sqrt{-\det g} \) for compact or noncompact case throughout the paper. We have

\[ S[\Psi[X^\mu]] \overset{M.F.}{=} \int d^D X \sqrt{\det g} \left[ \frac{1}{2} g^{\mu \nu} \frac{\partial \Psi}{\partial X^\mu} \frac{\partial \Psi}{\partial X^\nu} - V_p(\Psi) + 2\lambda \right]. \] (IV.4)

So when the quantum fluctuation or 2nd moment of spacetime reference system is ignored, the action is degenerated to a textbook scalar field action in a classical Riemannian spacetime up to a constant \( 2\lambda \), the entangled state (IV.1) is degenerated to a textbook quantum state \(|\Psi[X^\mu]\rangle\), and only the original derivative \( \partial \Psi(x)/\partial x^\alpha \) is replaced by a functional derivative \( \delta \Psi[X]/\delta X^\mu \), since now \( \Psi \) is a functional of \( X \).

V. RICCI FLOW FROM AN ISOTROPIC POSITIVE CURVATURE INITIAL SPACETIME

Here we consider the action beyond the mean field approximation, that is to consider the geometry of \( g_{\mu \nu} \) flows governed by the Ricci flow induced by the 2nd moment of the spacetime. For an isotropic initial condition when the diffusion part of the equation wins, it tends to smooth out local fluctuations of the geometry. Although the Ricci flow for a general 4-manifolds remains open, for an isotropic and positive curvature initial condition, the situation is better understood. In such case, when the Ricci flow equation is properly normalized (actually normalized by a CC see later), the flow solution of 4-manifolds exists for all \( t \in (-\infty, 0) \) and be singularity free (singularity located at infinity), the curvature gradually becomes positive, isotropic and uniform.

A. The density \( u \) function and renormalization \( Z, \tilde{Z} \) functions

To handle the difficult Ricci flow equation, there is a trick in Ricci flow literature that introduces a auxiliary scalar function

\[ u(X, t) = \frac{1}{(4\pi t)^{D/2}} e^{-f(X, t)} \] (V.1)

canceling the flow of the measure \( d^D X \) around point \( X \), i.e.

\[ \frac{\partial}{\partial t} \left[ u \sqrt{\det g_{\mu \nu}(X, t)} \right] = 0, \quad \text{or} \quad \frac{\partial}{\partial t} \left[ ud^D X(t) \right] = 0. \] (V.2)

In mathematics the \( u(X, t) \) function is often called the density of a manifolds or a weighted manifolds.

The density function \( u(X, t) \) is useful because, firstly, it provides a fixed measure along the flow, secondly, it just modifies the Ricci flow equation by a family of diffeomorphism equivalent to the standard Ricci flow, and thirdly, the modified Ricci flow equation (called Ricci-DeTurck flow \[ \text{[60]} \])

\[ \frac{\partial g_{\mu \nu}}{\partial t} = -2R_{\mu \nu} - 2\nabla_{\mu} \nabla_{\nu} f \] (V.3)

turns out to be a gradient flow of some functionals. The variational structure of Ricci flow was discovered by Perelman.

Here we discuss its meaning in the language of NLSM, if we consider the flow of the metric is as

\[ g_{\mu \nu}(X, t) = Z(X, t) g_{\mu \nu}(X, t_0) \] (V.4)

and hence the measure flows as

\[ \sqrt{\det g_{\mu \nu}(X, t)} = Z^{D/2}(X, t) \sqrt{\det g_{\mu \nu}(X, t_0)}, \] (V.5)

in which \( t_0 \) is certain starting short distance scale that ordinary physical and volume measurements are relative to, for instance, a Laboratory Wall Frame (LWF) that can be used very precisely as a reference to align the coordinate frame. Since \( u \) cancels the volume flow, we have the relation

\[ u(X, t) = Z^{-D/2}(X, t). \] (V.6)
Thus in fact the invariant measure \( u d^D X \) is nothing but the measure of the base space of the NLSM,
\[
ud^D X(t) = u \sqrt{\det g_{\mu\nu}(X,t)} d^D X(t_0) = u Z^{D/2} \sqrt{\det g_{\mu\nu}(X,t_0)} d^D X(t_0) = \sqrt{\det g_{\mu\nu}(X,t_0)} d^D X(t_0) = d^4 x,
\] (V.7)
i.e. the volume element of LWF as a standard measure is considered not flow. The function \( u(X,t) \) can be seen rescale the geometry \( g_{\mu\nu}(X,t) \) isotropically at each point \( X \) at scale \( t \) and keeping the volume of the base space fixed. Now since \( Z(X,t) = u(X,t)^{-2/D} \) can be interpreted as an isotropic flow of the NLSM action
\[
S_X(t) = \frac{1}{2} \lambda \int d^4 x Z g_{\mu\nu} \partial_\alpha X^\mu \partial_\alpha X^\nu = \frac{1}{2} \lambda \int d^4 x u^{-2/D} g_{\mu\nu} \partial_\alpha X^\mu \partial_\alpha X^\nu,
\] (V.8)
the flow can also be interpreted as a uniform flow of the coupling parameter \( \lambda \), i.e. \( \lambda(t) = Z(t)^{\lambda} \), if \( Z(X,t) \approx Z(t) \) is weakly depends on the coordinate \( X \) and thus be able to take out of the integral, while other quantities are considered fixed.

The flow of the volume in Ricci flow is a forwards heat-like equation, and since the flow of \( u \) cancels the volume flow, it is given by a backwards heat-like equation on the manifolds, from (V.2) we have
\[
\square^+ u = \left( -\frac{\partial}{\partial t} - \Delta + R \right) u = 0,
\] (V.9)
where \( \Delta \) is the Laplace–Beltrami operator and \( \square^+ \) is conjugate to the heat operator \( \square = \frac{\partial^2}{\partial t^2} - \Delta \).

Since the flow of \( u \) completely describes the flow of volume induced by the isotropic flow of metric, so for the isotropic case, we study the flow of \( u \) on the flowing manifolds instead of directly studying the difficult multi-component Ricci flow, it simplifies the problem we deal with. Because the choice of \( u \) leads to the same Ricci flow up to a diffeomorphism, so the the choice of it is analogous to the choice of gauge. We can see obviously that multiplying \( u \) by a constant just gives the action (V.8) an unimportant redefinition and leaves the Ricci-DeTurck flow unchanged. This means that one can choose \( u \) by convenience and it is not unique, it only provides a convenient tool instead of introducing an extra scalar field to the theory. However, in practice, if we impose a starting condition to its flow equation, it can be determined uniquely by solving the flow equation. The partial goal of the following chapter is to determine its IR value \( u_0 \) by imposing a UV renormalization condition.

Naively, the backwards heat flow (V.9) will not exist for general \( u \). However, one of the basic points of view is to let the Ricci flow flow for a IR \( t_\ast > 0 \). At \( t_\ast \) one may then choose an appropriate \( u(t_\ast) = u_0 \) arbitrarily and flow it backwards in \( t \) (\( \tau = t_\ast - t \)) to obtain a solution \( u(t) \) of the backwards equation. Since for the isotropic case, the flow is free from singularity, it simply gives \( t_\ast = 0 \), so we define
\[
\tau = 0 - t = \frac{1}{64\pi^2 \lambda} k^2.
\] (V.10)
It reverses the direction of \( t \) and making the flow of \( u \) a more familiar forwards heat-like equation
\[
\frac{\partial u}{\partial \tau} = \Delta u - Ru,
\] (V.11)
which does admit a solution in \( \tau \) in the sense discussed above, and equivalently, a backwards solution in \( t \).

In a fixed metric with a small scalar curvature, it can be considered degenerate to a heat equation. Thus we expect the fundamental solution forms almost like a standard heat kernel
\[
H(X;Y,g,\tau) = \frac{1}{(4\pi\tau)^{D/2}} e^{-\frac{|X-Y|^2}{4\tau}}.
\] (V.12)
The information of the curvature can be reflected in the Fourier transformation of \( H(X;Y,g,\tau) \) from \( X \) space to its momentum space \( K \)
\[
\hat{u}(K,\tau) = \int_{M^D} H(X;Y,g,\tau) e^{iK_\mu(X^\mu - Y^\mu)} d^D X = e^{-|K|^2 \tau},
\] (V.13)
and the Fourier transformation of \( Z(X,\tau) \) is given by
\[
\hat{Z}(K,\tau) = e^{\frac{\hat{K}^2}{2} |K|^2 \tau} \approx 1 + \frac{2}{D} R\tau.
\] (V.14)
It renormalizes the Fourier components of the metric

$$g_{\mu\nu}(K, \tau) = \hat{Z}(K, \tau)g_{\mu\nu}(K),$$

where $|K|^2$ is the eigenvalue of the operator $-\Delta + R \approx R$ when $\hat{Z}$ is uniform enough.

The definition domain of $u$ can be even $\tau \in (0, \infty)$, called ancient solution in mathematics literature, if the existence of the solution can be traced back to $\tau \to \infty$, in physics it means that the system exists a UV fixed point $k \to \infty$. It is indeed the case for the isotropic and positive curvature spacetime because it is singularity free. So we assume

$$u_{\infty}^{-2/D} = Z_{\infty} = 1$$

(V.16)
to be the final condition of (V.11), which is actually a renormalization condition at UV. The existence of $u_{\infty}$ is called renormalizability in physics. $u_{\infty} = 1$ means that the action (II.1) is considered as a bare action with respect to (V.8).

Our goal is to estimate the initial $u$ and $\hat{Z}$ at IR $\tau \to 0$ by the renormalization condition at UV, then we could finally obtain the IR limit of the action.

### B. Estimate initial density $u_0$ by the $v$ function and the reduced volume

In order to study the solution $u$ in the non-linear equations (V.11), Perelman introduced a subsolution [19, 23, 26]

$$v = [\tau(2\Delta f - |\nabla f|^2 + R) + f - D]H,$$

(V.17)
to the conjugate heat equation (V.9)

$$\Box^* v = -2\tau \left[R_{\mu\nu} + \nabla_\mu \nabla_\nu f - \frac{1}{2\tau}g_{\mu\nu}\right] H \leq 0$$

(V.18)
as a useful tool to estimate the fundamental solution $H(X; Y, g, \tau)$ of (V.11). The inequality holds as equality when the manifolds is a gradient shrinking soliton solution satisfying

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu f - \frac{1}{2\tau}g_{\mu\nu} = 0.$$

(V.19)

The fundamental subsolution $v$ is the lower bound of the fundamental solution $H$

$$H(X; Y, g, \tau) \geq v(X; Y, g, \tau) = \frac{1}{(4\pi \tau)^{D/2}}e^{-l(X-Y; g, \tau)},$$

(V.20)

where $l(X; Y, g, \tau)$ is the reduced length [19] measuring the minimum distance of a path between the base point $\gamma(0) = X$ and endpoint $\gamma(\tau) = Y$,

$$l(X; Y, g, \tau) = \inf_\gamma \frac{1}{2\sqrt{\tau}} \int_0^\tau \sqrt{\tau'(R + |\gamma'|^2)}d\tau'.$$

(V.21)

The volume integral,

$$\hat{V}(X, g, \tau) \equiv \int_{M^D} d^D Y v(X; Y, g, \tau) \equiv \int_{M^D} d^D Y \frac{1}{(4\pi \tau)^{D/2}}e^{-l(X-Y; g, \tau)}$$

(V.22)
is called the reduced volume [19] of the spacetime $M^D$ with basepoint $X$. The reduced volume is a dimensionless geometric quantity and monotone non-increasing in $\tau$ which even holds on noncompact manifolds. Perelman used the quantity to prove the no-local-collapsing theorem of a manifolds via the monotonic of it.

The reduced volme is a generalization of a volume integral of the heat-kernel (V.12), for example, the integral of (V.12) in a D-dimensional flat manifolds gives

$$\int_{\mathbb{R}^D} d^D Y \frac{1}{(4\pi \tau)^{D/2}}e^{-\frac{|X-Y|^2}{4\tau}} = 1.$$  

(V.23)

And the reduced length $l(X; Y, g, \tau)$ is a variant of the Gaussian distance $f(X; Y, g, \tau) = |X - Y|^2 / 4\tau$ up to a constant. We see that (V.23) leads to a heuristic

$$H(X; Y, g, \tau) \propto \frac{1}{(4\pi \tau)^{D/2}}e^{-\frac{|X-Y|^2}{4\tau}} \approx \frac{1}{\text{Vol}(\mathbb{R}^D)}.$$  

(V.24)
Thus the main difference between the reduced volume in \( M^D \) and in \( \mathbb{R}^D \) comes from the difference in the integrals of volumes, by taking \( H \geq v \), the (V.22) is then

\[
\tilde{V}(g, \tau) \leq \frac{\operatorname{Vol}(M^D, g(\tau))}{\operatorname{Vol}(\mathbb{R}^D)}.
\]

Therefore, the reduced volume is bounded above by the volume ratio, which is well-defined also due to the Bishop-Gromov comparison theorem, and for general \( M^D \) we always have

\[
\tilde{V}(\tau) \leq 1.
\]

Here we discuss a useful observation for \( H \) and \( v \) on an isotropic and positive curvature manifolds having a backwards UV limit. On the one hand, since for positive curvature manifolds, its IR limit is a gradient shrinking soliton satisfying (V.19) and hence (V.20) holds as equality at \( \tau \to 0 \). On the other hand, since the backwards existence of the solution is able to extend to \( \tau \to \infty \), i.e. an ancient solution. There are subtle interplays between ancient solution and soliton, a rescaled ancient solution resembles a soliton, more precisely, its backwards limit is an asymptotic non-flat gradient shrinking soliton \([19, 61]\) also satisfying (V.19), in this situation the equality (V.18) also holds at UV. As a consequence, \( H \) equals \( v \) up to a constant multiple when the solution of Ricci flow is an ancient solution. The constant multiple for the fundamental solution is not important since it can be finally fixed by imposing an initial condition to the general solution \( u \). In fact the ratio

\[
W(\tau) = \frac{v}{H} = \tau (2\Delta f - |\nabla f|^2 + R) + f - D,
\]

whose IR and UV limits are the constant multiples, defines a local entropy \([19]\) which is monotone non-decreasing in \( t = -\tau \). If we start the Ricci flow from an isotropic gradient shrinking soliton at UV the local entropy \( W(\infty) \) has already taken its maximum constant value equaling to the maximum local entropy \( W(0) \) for the IR gradient shrinking soliton, which can be set to 1 by a normalized \( f \) (adding an appropriate constant). So \( H \) must be equal to \( v \) at both IR and UV limits for the ancient solution we concern, and because of their monotonicity, this immediately implies that \( H(X; Y, g, \tau) = v(X; Y, g, \tau) \) for all \( \tau \in [0, \infty) \) (instead of being a subsolution).

As the forward flow makes the density \( u \) more and more homogeneous and isotropic at IR, we set the IR initial condition a constant density \( u_0 = \text{const} \), and using the fundamental solution \( H(X; Y, g, \tau) \) of (V.11) we write its general solution as

\[
u(X, g, \tau) = \int_{M^D} d^D y u_0 H(X; Y, g, \tau).
\]

In the IR limit the fundamental solution \( H(X; Y, g, \tau) \) tends to a Dirac’s delta function, so \( \lim_{\tau \to 0} u(X, g, \tau) = u_0 \), and using the equality between \( H \) and \( v \) at \( \tau \to 0 \)

\[
u_0 = \lim_{\tau \to 0} \int d^D y u_0 H(X; Y, g, \tau) = \lim_{\tau \to 0} \int d^D y u_0 v(X; Y, g, \tau) = u_0 \tilde{V}_0(X),
\]

so we have \( \tilde{V}_0(X) = 1 \). And in the UV limit, by using the renormalization condition (V.16) and the equality between \( H \) and \( v \) also at \( \tau \to \infty \), we have

\[
1 = u_\infty (X) = \lim_{\tau \to \infty} \int d^D y u_0 H(X; Y, g, \tau) = \lim_{\tau \to \infty} \int d^D y u_0 v(X; Y, g, \tau) = u_0 \tilde{V}_\infty (X),
\]

then we obtain

\[
u_0 = \tilde{V}_\infty^{-1},
\]

independent to the basepoint.

This is a basic result of the paper, it shows that the homogeneous IR initial density \( u_0 \) is determined by the UV limit of the reduced volume \( \tilde{V}_\infty = \lim_{\tau \to \infty} \tilde{V}(g, \tau) < 1 \). It is because a subtle relation between IR and UV limit of the Ricci flow that a rescaled IR limit solution with positive curvature resembles an ancient solution whose backwards UV limit converges to a non-flat gradient shrinking soliton, and for the equality in (V.18) holds at UV, the UV reduced volume given by \( v \) fixes the initial density \( u_0 \).
For general $\tau \geq 0$ we have
\[ u(X, g, \tau) = \int_{M^D} d^D Y u_0 H(X; Y, g, \tau) = \int_{M^D} d^D Y u_0 v(X; Y, g, \tau) = u_0 \bar{V}(X, g, \tau), \] (V.32)
so the reduced volume for all $\tau$ is given by
\[ \bar{V}(X, g, \tau) = \frac{u(X, g, \tau)}{u_0} = u(X, g, \tau) \bar{V}_\infty, \] (V.33)
which shows that the reduced volume is in fact monotonic decreasing in $\tau$ as $u(X, g, \tau)$ behaves.

In the language of NLSM, remind that the standard volume of the base space is just the invariant volume constraint of the target space,
\[ \int_{\text{base}} d^4 x = \int_{\text{target}} u(\tau) d^D X = \lim_{\tau \to 0} \int_{\text{target}} d^D X \int_{\text{target}} d^D Y u_0 H(X; Y, g, \tau) = \bar{V}_\infty^{-1} \int_{\text{target}} d^D X(0). \] (V.34)
Thus in the sense of comparison geometry, the UV reduced volume is just a volume ratio between IR volume of the target space and standard volume of the base space
\[ \bar{V}_\infty = \frac{\int_{\text{target}} d^D X(0)}{\int_{\text{base}} d^4 x} < 1. \] (V.35)
It means that if the flow of spacetime starts from a small distance scale (e.g. LWF) forwardly to the long distance scale, the Ricci flow shrinks the spacetime volume and finally converges to a constant volume at IR limit.

The convergence of $\bar{V}_\infty$ and $u_0$ are crucial facts which relate to the renormalizability and asymptotic safety of the Q-NLSM, leading to a convergent value of CC. Actually, $\bar{V}_\infty(g)$ is the Gaussian density $\Theta(g)$ of a UV manifold, its $\ln$ can be given by a limit of the $W$-functional introduced also by Perelman [15, 64, 66],
\[ \ln \bar{V}_\infty(g) = \lim_{\tau \to -\infty} \mathcal{W}(g, f, \tau) = \mathcal{W}_\infty(g) < 0, \] (V.36)
where
\[ \mathcal{W}(g, f, \tau) = \int_{M^D} \left[ \tau (|\nabla f|^2 + R) + f - D \right] u d^D X \] (V.37)
is monotone non-increasing in $\tau$, in other words, it is monotone non-decreasing along the Ricci flow like an entropy.

At the point the initial condition is given by $u_0 = e^{-W_\infty}$, and $\bar{Z}(K, \tau)$ satisfying renormalization condition (V.16) becomes
\[ \bar{Z}(K, \tau) = e^{\frac{\tau}{D} + |K|^2} \approx 1 + \delta_2 + \frac{2}{D} R \tau, \] (V.38)
where $\delta_2 = \bar{Z}(K, 0) - 1 \approx \frac{2}{D} W_\infty$ is a counter term completely canceling $\frac{2}{D} R \tau$ at $\tau \to -\infty$. By substituting $\tau = \frac{1}{64\pi^2} k^2$, it coincides with the eq.(23) in ref.[15] leaving the constant $\delta_2$ determined latter.

Let us summarize this section, the result (V.33) shows that the density $u$ starting from $u_0 = \bar{V}_\infty^{-1} > 1$ at IR flows backwardly to $u_\infty = 1$ at UV, due to the fact that the reduced volume starting from $\bar{V}_0 = 1$ flows backwardly to $\bar{V}_\infty < 1$, both processes are monotone non-increasing in $\tau$. As a consequence, the function $\bar{Z}$ starting from $\bar{Z}_\infty = 1$ at UV flows forwardly to an IR value $\bar{Z}_0$ related to $\bar{V}_\infty$ as follows
\[ \bar{Z}_0 = Z_0 = u_0^{-\frac{\tau}{D}} = \bar{V}_\infty^{-\frac{\tau}{D}} = \Theta^{\frac{\tau}{D}} = e^{\frac{\tau}{D} W_\infty} < 1. \] (V.39)
If $\bar{Z}$ is interpreted as a renormalization of the UV bare parameter $\lambda$ while other quantities fixed, it uniformly flows from $\lambda$ to $\bar{Z}_0\lambda < \lambda$ at IR.

### C. Asymptotic UV reduced volume of a maximally symmetric spacetime

We have seen that $\delta_2$ is just a counter term for a global and isotropic Ricci flow $\frac{2}{D} R \tau$ in (V.38). To calculate $\delta_2$ and related $\bar{V}_\infty$, we need to flow the reduced volume $\bar{V}(g, \tau)$ from a initial spacetime to the UV limit. A physical choice of initial condition for such flow at current epoch is a maximally symmetric spacetime, for instance, a comoving
Friedman-Robertson-Walker (FRW) or de Sitter metric, which is an isotropic and homogeneous spacetime geometry with a positive curvature, \( M^D = S^3 \times \mathbb{R} \),
\[
    ds^2 = a^2(T_0, \tau) \left[-(dT)^2 + (d\Sigma_3)^2\right].
\] (V.40)

Since the CC corresponds to the late (physical) time universe, we concern the isotropic and homogeneous scale factor \( a(T_0, \tau) \) at late (physical) time or current epoch \( T_0 \) at scale \( \tau \) (compared with the very early universe \( T \rightarrow 0 \) when the spatial part approaches to a singularity differing from the temporal part can not be isotropically flowed), \((dT)^2\) is the unit temporal distance metric, and \((d\Sigma_3)^2\) is the rotational symmetric round metric of the unit 3-sphere \( S^3 \).

The metric (V.40) is a noncompact gradient shrinking soliton and resembles a 4-dimensional round cylinder at the asymptotic infinity or late (physical) time. Such spacetime is in fact an Einstein manifolds. Einstein manifolds is a special soliton solution as a flow limit, in the sense that the Ricci flow only uniformly rescales its backwards flow limit at UV when the local inhomogeneity and anisotropy are not large enough so that they will gradually be smoothed out and is to calculate the UV limit of the reduced volume by the soliton metric or its Gaussian density.

\( c \) is the unit temporal distance metric, and \((a, \tau)\) is the unit temporal distance metric, and \((d\Sigma_3)^2\) is the rotational symmetric round metric of the unit 3-sphere \( S^3 \).

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If the chosen initial metric is slightly not such form due to inhomogeneity and anisotropy, it is naturally considered when the local inhomogeneity and anisotropy are not large enough so that they will gradually be smoothed out and the metric tends to form like that at IR, so we can always find an initial metric sufficiently close to such form so that its backwards flow limit at UV \( \tau \rightarrow \infty \) must also be such form because of the scaling nature of soliton. The goal here is to calculate the UV limit of the reduced volume by the soliton metric or its Gaussian density.

By regarding the UV asymptotic noncompact maximally symmetric manifolds as a \( D \)-sphere punctured at the north and south poles: \( S^{D-1} \times \mathbb{R} \approx S^D - \{ \text{poles} \} \) with radius \( a(T_0, \infty) \), using a normalized Einstein manifolds of positive curvature
\[
    R_{\mu\nu} = \frac{D - 1}{a^2} g_{\mu\nu} = \frac{1}{2\tau} g_{\mu\nu},
\] (V.41)
so taking \( \tau = \frac{a^2}{2(D-1)} \rightarrow \infty \), and according to (V.22), we obtain
\[
\begin{align*}
    \tilde{V}_\infty(S^D - \{\text{poles}\}) = & \int_{S^D - \{\text{poles}\}} d^D X \left(\frac{1}{4\pi \tau} \right)^{D/2} e^{-l_\infty} \\
    & \approx \left[ \frac{D - 1}{2\pi a^2(T_0, \infty)} \right]^{D/2} \int_0^{a(T_0, \infty)} e^{-\frac{\pi}{4\tau}(D\omega_D)r^{D-1}} dr \\
    & \approx \frac{D=4}{0.442}.
\end{align*}
\] (V.42)

in which \( l_\infty \sim -\frac{\tau^2}{4\tau} = -\frac{1|X|^2}{4\tau} \) has been used, and \( D\omega_D = D\pi^{D/2}/\Gamma(\frac{D}{2}+1) \) is the surface area of the unit \( D \)-sphere. The final results are given by
\[
    \tilde{Z}_0 = \frac{\tilde{V}_\infty^{2/D}}{D} \approx 0.663, \quad \delta_\tilde{Z} = \tilde{Z}_0 - 1 = -0.337.
\] (V.43)

These are central numerical results of the paper. The counter term \( \delta_\tilde{Z} \) relating to CC (see next section) is calculated by the UV reduced volume from a maximally symmetric spacetime. The limit quantity \( \tilde{Z}_0 \) in essentially normalizes the Ricci flow of spacetime so that it converges globally to a spacetime with constant positive curvature instead of shrinking to a singularity. The value of \( \delta_\tilde{Z} \) arises approximately as the counter term to the scaling exponent to satisfy the renormalization condition.

**D. Effective Action**

To get a precise physical interpretation to the above results, especially the relation between \( \delta_\tilde{Z} \) and CC, we apply them to the effective action (V.4) which now needs correction from (V.35). At the cutoff scale \( k \) the effective action is
\[
    S_k = \int d^4 X \sqrt{|g|} \left[ \mathcal{L}_M + 2(1 + \delta_\tilde{Z})\lambda + \frac{R_k}{16\pi^2 D} k^2 \right],
\] (V.44)
where we have replaced \( \mathcal{L}_M = \frac{1}{2} g_{\mu\nu} \frac{\partial \phi}{\partial X^\mu} \frac{\partial \phi}{\partial X^\nu} - V_\phi(\Psi) \) for short. In the action, \( 2(1 + \delta_\tilde{Z})\lambda \) can be interpreted as an IR constant vacuum energy density and \( \lambda \) is a unique input constant of the NLSM and the theory as well. \( \frac{R_k}{16\pi^2 D} k^2 \) is
a flow term coming from the Ricci flow reflecting the quantum correction at scale \( k \) on the background metric, the dynamic of spacetime or gravity comes from this term.

We consider the action \( \Psi \) as an effective action of matter field coupling with gravity, so it must recover the action of matter field coupling with the standard Einstein-Hilbert (EH) action. A major difference between this effective action and the standard EH action is that the action has gradient flow but the standard EH action does not. The standard EH action without CC is an action at certain fixed scale much shorter than the cosmic scale where it is successfully tested, for instance the scale from LWF to the scale of the solar system. However, at the cosmic scale, the standard EH action deviates from observations, where CC becomes important. The correspondence between the effective action and the standard EH action is that the action has gradient flow but the standard EH action does not. The standard EH action without CC is an action at certain fixed scale much shorter than the cosmic scale where it is successfully tested, for instance the scale from LWF to the scale of the solar system. However, at the cosmic scale, the standard EH action deviates from observations, where CC becomes important. The correspondence between the effective action and the standard EH action is that the action has gradient flow but the standard EH action does not.

When the cutoff scale \( k \) is small, i.e. at short distance scale even at UV, (V.44) recovers the standard EH at the scale, where we know that the flow term \( \frac{R_k}{16\pi G} k^2 \) almost cancels \( 2\delta_2 \lambda \) in order to satisfy the UV renormalization condition \( \tilde{Z} = 1 \) leaving only \( 2\lambda \) term; while when the cutoff scale \( k \) is large, i.e. at cosmic scale, the flow term can be neglected, and leaving \( 2\delta_2 \) together with the short distance \( 2\lambda \) term. In this sense, the \( 2\lambda \) term plays the role of a EH term without CC at short distance scale, and \( 2\delta_2 \lambda \) plays the role of a CC which is important at cosmic scale.

In the interpretation of \( 2\lambda \), and as the flow terms shows, \( 2\lambda \) can be reformulated by a constant scalar curvature \( R_0 \) and a constant UV energy scale \( k_{UV} \), most naturally the Planck energy scale \( k_{UV} \sim G^{-1/2} \). So like the standard EH term without CC, for instance, we have \( 2\lambda = \frac{R_0}{16\pi G} \). To interpret the scalar curvature \( R_0 \) introduced, it is indicated from observations that at the short distance scale the scalar curvature \( R_0 \) is very small and qualitatively given by the Hubble’s constant at current epoch \( H_0 \), i.e. \( R_0 = D(D-1)H_0^2 = 12H_0^2 \). As a consequence, \( \lambda \) is nothing but the Critical Density in cosmology

\[
\lambda = \frac{R_0}{32\pi G} = \frac{3H_0^2}{8\pi G} = \rho_c.
\]

Then the action has a CC term given by,

\[
2\delta_2 \lambda = -0.67 \rho_c = -\rho_c = \frac{-2\lambda}{16\pi G},
\]

with the fraction \( \Omega_L = \rho_L/\rho_c = -2\delta_2 \approx 0.67 \) consistent with the observations. The fraction \( \Omega_L \approx 0.67 \) estimated by the Ricci flow approximation is close to the result \( \Omega_L \approx 2/\pi \approx 0.64 \) by using the effective dimensional reduction method [12].

From another point of view, if we consider the backwards flow of \( u \) is interpreted as a uniform flow of the scalar curvature, in other words, \( 2\lambda + \frac{R_k}{16\pi G} k^2 \) in the action plays the role of an effective EH term without CC

\[
2\lambda + \frac{R_k}{16\pi G} k^2 = \frac{R_k}{16\pi G},
\]

so in this interpretation we obtain the backwards flow of the scalar curvature at scale \( k \), or equivalently at \( \tau \),

\[
R_k = \frac{R_0}{1 - \frac{1}{D\pi Gk^2}}, \quad \text{or} \quad R_\tau = \frac{R_0}{1 - \frac{2}{D}R_0 \tau}.
\]

It is clear that \( R_0 \) can also be interpreted as the IR value of the scalar curvature, being a homogeneous and isotropic positive lowest curvature background of the spacetime at cosmic scale, and \( R_\tau \) satisfies

\[
\frac{\partial R}{\partial \tau} = \frac{2}{D} R^2,
\]

which is a homogeneous backwards flow equation (to UV) of the scalar curvature when it starts from an IR initial uniform geometry so that the diffusion term \( \Delta R \) is small. The flow equation is due to the backwards flow of \( u \) but the forwards Ricci flow of \( g_{\mu\nu} \). The equation means that as the flow goes backwardly to the short distance scale the quantum fluctuations gradually concentrate its curvature, consequently, as the flow goes forwardly to the long distance scale the curvature fluctuations are gradually removed and the spacetime becomes uniform. Naively speaking, if we calculate the gradient flow of the standard EH action, it gives a backwards flow to the scalar curvature whose solution typically may not admit. That is the reason why the standard EH action in general does not have a gradient flow, which is known as a weak point of the standard EH theory to be a possible quantum gravity theory. However, in the situation we concern the backwards flow solution of \( u \) in \( \Psi \) does exist shown previously, so a homogeneous backwards flow solution of the scalar curvature and the effective EH action induced by the backwards flow of \( u \) makes sense. The effective EH action does have a homogeneous gradient flow unlike the standard EH action which does not.
and the resulting classical field equation is by using (V.45) and (V.46) the effective action (V.44) of gravity can be rewritten as our familiar form

\[ S_k = \int d^4 X \sqrt{\det g} \left[ \mathcal{L}_M + \frac{R_k}{16\pi G} + 2\delta^2 \lambda \right] \approx \int d^4 X \sqrt{\det g} \left[ \mathcal{L}_M + \frac{R_k}{16\pi G} - 0.67\lambda \right]. \]  

(V.49)

In fact, if one does not introduce the cut off energy scale, the Newton’s constant \( G \), the theory with the only input constant \( \lambda \) can also be formulated and well-defined in general

\[ S_k = \int d^4 X \sqrt{\det g} \left[ \mathcal{L}_M + 2\lambda (\mathcal{R}_k - 0.34) \right], \]  

(V.50)

where \( \mathcal{R}_k \) is just a dimensionless scalar curvature equivalent to the conventional scalar curvature \( R_k \) rescaled by the IR scalar curvature, i.e. \( \mathcal{R}_k = \frac{R_k}{R_0} = \frac{R_k}{\frac{\lambda}{k^2}} \). In the context of traditional Einstein’s gravity theory with Newton’s constant, the scalar curvature of the theory is seem bound from below by \( R_0 \) in the isotropic case. By using \( \mathcal{R} \) the resulting classical field equation is

\[ (\mathcal{R}_k)_{\mu\nu} - \frac{1}{2} (g_k)_{\mu\nu} \mathcal{R}_k + 0.67 (g_k)_{\mu\nu} = \frac{(T_{\mu\nu})_k}{4\lambda}. \]  

(V.51)

which is a rescaled Einstein’s equation but Newton’s constant plays no role in it, if we note that \( (\mathcal{R}_k)_{\mu\nu} \lambda = \frac{(R_k)_{\mu\nu}}{2\pi G} \).

This equation is in analogy with the Friedman equation \( H^2 / H_0^2 = \rho / \rho_c \) in which \( \rho \) are densities of matter components rescaled by the critical density \( \rho_c \) and the Hubble’s parameter \( H \) is rescaled by its current value \( H_0 \) while the Newton’s constant is absent as well.

It is worth mentioning that in the theory based on \( \mathcal{R} \), the critical density \( \lambda = \rho_c \) is the only characteristic energy scale instead of the Planck scale. The traditional gravity theory with a non-vanishing CC has two fundamental constants, the Newton’s constant and the CC, and hence has two characteristic scales, the Planck scale and Hubble scale. As is shown above, this theory with only one constant \( \lambda \) can also reproduce the traditional gravity theory with CC by choosing a specific cut off scale, the Planck scale. However, this theory allows the cut off goes beyond the Planck scale \( k \to \infty \) while keeping \( \lambda \) finite, in the limit the spacetime is infinitely large \( R_0 \sim \lambda / k^2 \to 0 \). Individually, the Planck scale is not necessarily a characteristic energy scale of this theory, neither the individual Hubble scale, one can go beyond each individual scale and keeps their combination (the only characteristic scale \( \lambda \)) the same. The gravity theory is independent to how you define the Planck scale by an absolute ruler, just like the fact that there is no specific scale such as the absolute Planck scale in the Ricci flow. This is a major difference between this theory and the traditional gravity theory.

E. Observation of the IR Limit: Distance-Redshift Relation

We have described that the reason why the spacetime metric continuously deforms governed by the Ricci flow semi-classically is because the existence of the non-trivial 2nd moment of the spacetime coordinates, or equivalently, the quantum fluctuation of the spacetime. Here we will see what is the physical effects of the 2nd moment of the spacetime coordinates and the resulting flow of the spacetime metric in observations. Reminding III.2 and (V.13), we have the flow of the scale factor

\[ \langle \Delta a(\tau) \rangle^2 = \langle \Delta a(0) \rangle^2 + \langle \delta a^2 \rangle = \hat{Z}(\tau) \langle \Delta a(0) \rangle^2 = \left( 1 + \delta^2 + \frac{2}{D} R \tau \right) \langle \Delta a(0) \rangle^2, \]  

(V.52)

in which \( \langle \Delta a(\tau) \rangle = \langle a(T, \tau) \rangle - \langle a(T_0, \tau) \rangle \) is the classical displacement of the scale factors between different epochs \( T \) and \( T_0 \), and \( \langle \delta a^2 \rangle \) is the 2nd moment contribution on it, so

\[ \frac{\langle \delta a^2 \rangle}{\langle \Delta a(0) \rangle^2} = \delta^2 + \frac{2}{D} R \tau. \]  

(V.53)

An important observable in cosmology is the redshift, the measurement of its mean value is given by scale factors at different epochs (\( T \) and \( T_0 \))

\[ 1 + \langle z \rangle = \frac{\langle a(T_0, 0) \rangle}{\langle a(T, 0) \rangle}. \]  

(V.54)
Its variance can be defined via Taylor expansion at a fixed epoch $T$ but at different scale $\tau$ of the Ricci flow

$$1 + \frac{1}{2} \langle \delta z^2 \rangle = \frac{\langle a^2(T, 0) \rangle}{\langle a^2(T, \tau) \rangle},$$

so we have

$$\frac{\langle \delta z^2 \rangle}{\langle z \rangle^2} = -2 \frac{\langle a^2(T, \tau) - a^2(T, 0) \rangle}{\langle a(T, 0) - a(T_0, 0) \rangle^2} = -2 \frac{\langle \delta a^2 \rangle}{\langle a \delta(0) \rangle^2} = -2 \delta z - \frac{4}{D} R \tau.$$  \hspace{1cm} (V.56)

From the formula we see that the 2nd moment of the redshift renormalized by the squared 1st moment redshift is monotone along $\tau$, which is finite $-2 \delta z \approx 0.67$ at IR $\tau \to 0$ and approaches to zero at UV $\tau \to \infty$. In other words, at small scale the variance of the redshift can be ignored, but it is significant (order $O(1)$) at cosmic scale observations. The proportional relation between the 2nd and 1st moment of redshift gives a correction to Distance-Redshift Relation at order $O(z^2)$ which can not be ignored at large redshift

$$\langle D(z) \rangle = \frac{1}{H_0} \left[ \langle z \rangle + \frac{1}{2} \langle z^2 \rangle + O(z^3) \right] \overset{\tau \to 0}{=} \frac{1}{H_0} \left[ \langle z \rangle + \frac{1}{2} (1 - 2 \delta z) \langle z \rangle^2 + O(z^3) \right],$$

where we have used $\langle z^2 \rangle = \langle z \rangle^2 + \langle \delta z^2 \rangle$ and $\langle D(z) \rangle$ is the distance between e.g. supernovas and the observer. The 2nd moment of the redshift coming from the 2nd moment of the spacetime coordinates does not modified the relation at order $O(z)$ which describes the expansion rate of the universe, but modified it at order $O(z^2)$ which describes the accelerating or deceleration of the expansion. More precisely, the 2nd moment of the redshift gives an additional deceleration parameter $q_0 = 2 \delta z \approx -0.67$, which is clearly redshift independent and uniform behaving like a dark energy.

The uniformness and universal of the quantum variance of the redshift is also an indication that the Equivalence Principle (EP) could be valid at the quantum level. The gravity is not only universally depicted by the 1st moment of the metric (expansion rate) but also the 2nd moment (acceleration). Phenomenologically speaking, the spectral lines taking different energies universally free-fall: not only they universally redshift (i.e. expansion rate) but also the 2nd moment (acceleration). Phenomenologically speaking, the spectral lines taking different energies universally free-fall: not only they universally redshift (i.e. expansion rate) but also the 2nd moment (acceleration).

We see that the spacetime coordinates become more and more fuzzy as the Ricci flow driving the 2nd moment of the spacetime geometry becomes more and more significant at cosmic scale. As a consequence, the quantum variance of the redshift as physical observable becomes more and more non-ignorable at large redshift regime. It is the reason why the universe seems accelerating expansion. Indeed, we do not directly measure the quantum variance of the redshift, instead of measuring the modified Distance-Redshift Relation. So if this theory is true, it is an important proposal to try to measure the almost linear dependence between the quantum variance of redshift $\langle \delta z^2 \rangle$ and the squared redshift $\langle z \rangle^2$ shown in (V.56). At first glance, the measurement of the variance or width $\langle \delta z^2 \rangle$ may have numbers of dirty non-quantum origins, such as the thermo-widening, so that to single out the clean quantum part of the variance seems difficult. But as the distance scale becomes larger and larger, the ratio $\langle \delta z^2 \rangle/\langle z \rangle^2$ becomes of order one as predicted, so that the quantum part of the variance may become dominant compared with other effects. On the other hands, unlike other non-quantum effects, the quantum part of the variance is universal as the EP claims which differs it from other noises. Therefore we think the measurement of the quantum variance versus squared redshift may be feasible.

**VI. CONCLUSIONS**

We summarize the results as follows. When the quantum fluctuations are inescapable in the quantum measurement of the spacetime coordinates, the Riemannian geometry can not be realized in rigor, so that the 2nd central moment or even higher moment of the spacetime coordinates must not be ignored. We consider the effects and corrections of 2nd central moment to the Riemannian geometry by a quantum non-linear sigma models (Q-NLSM) interpreted as a quantum spacetime reference frame (QSRF) system. The Ricci flow as the semi-classical approximation of the renormalization flow of Q-NLSM, have been studied by powerful tools, such as the reduced volume, which provides us a framework to calculate how the geometry of the isotropic universe at current epoch continuously deforms and finally how it looks like at very long distance or cosmic scale.

We show that as the spacetime isotropically flows to IR by the Ricci flow, under the QSRF interpretation to the Q-NLSM, (1) an effective Einstein-Hilbert-like action and a correct cosmological constant (CC) emerge, (2) the universe becomes more and more homogeneous and isotropic as the cosmological principle asserts, and the metric tends to an Einstein metric at IR, moreover, (3) it gives rise to an accelerating expansion universe with a fraction of “dark energy” $\Omega_\Lambda \approx 0.67$ in the IR limit, which is consistent with current observations. Therefore, in the sense of effective quantum field theory of gravity, the CC problem is so resolved.
In the conceptual sense, the leading energy density coupled to gravity is not anymore the quartic of the Planck scale cutoff, $A^4_{pl}$ coming from the zero-point fluctuation of the vacuum of the spacetime, which is the main puzzle of the CC problem. It is resolved by noticing in the theory that, the parameter background $x, y, z, t$, which is absolute, external, classical and free from any quantum fluctuation, in fact has nothing to do with the physical spacetime. The unphysical nature of the parameter background makes the zero-point energy densities $A^4_{pl}$ completely unobservable, including the Casimir effect $[46]$, and hence disappears in the effective action $(V.44)$. In contrast, the leading energy density coupled to gravity is given by the two-point quantum fluctuation of the physical spacetime $\langle \delta X_\mu \delta X_\nu \rangle \neq 0$ or $\delta g^{(2)}_{\mu \nu} \neq 0$ depicted semi-classically by the Ricci flow. In this sense, the Equivalence Principle (EP) is kept and generalized to the quantum level: energy densities which universally coupled to gravity are those with respect to the physical spacetime $X^\mu$ which is inescapably quantum fluctuating.

A measurement to test the theory is also proposed. In this theory, the phenomenological existence of CC or the “dark energy” is all about the deviation of the Distance-Redshift relation from the standard Hubble’s law at relative large redshift regime. The theory suggests that the deviation is due to the quantum variance of the redshift $\langle \delta z^2 \rangle$ induced by the 2nd central moment of the spacetime coordinates. An almost linear dependence between the quantum variance $\langle \delta z^2 \rangle$ and the squared-redshift $\langle z \rangle^2$ is predicted $(V.50)$, and the proportionality constant at long distance limit is $-2\delta_2 \approx 0.67$ being close to the “acceleration” parameter $-q_0$. And we argue that to measure the clean quantum part of the variance of the redshift seems feasible.

Here we discuss the prospect of other Ricci flow’s applications to the cosmology, for instance, the very early universe. The CC is shown as a special example of the application of the Ricci flow with isotropic and positive curvature initial condition, where the space and time coordinates are renormalized on an equal footing. Such situation is relatively easy to deal with, since a CC-normalized Ricci flow with isotropic positive curvature initial condition is free from singularities and diffusion term dominants. It is not necessarily the case when the universe is in the very early epoch, where the spatial part of the universe approaches to a singularity. The space and time differ from each other in the very early universe, in such anisotropic initial condition, the Ricci flow may develop local singularities. The application of the Ricci flow (or its generalization) to a more general initial condition, such as the very early universe, is still a challenge. Because, firstly in the regime near the singularity, the validity of the Ricci flow approximation is unclear; and secondly the mathematical tools to deal with the singularities developed by the Ricci flow is also highly technical. Some qualitative results from the studies of such case gives us confidence that the application of Ricci flow to the very early universe is also worth pursuing: (1) the local singularity can be well modeled by its soliton solution whose profile is seen like an spatially inflation universe; (2) the linear fluctuations at the vicinity of the singularity is governed by a linearized version of the Ricci flow, the Lichnerowicz equation, which gives rise to an evolution equation for the fluctuation being similar with the primordial fluctuation of the universe. The detail studies are leaving for our future works.

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