The search for new integrable (3+1)-dimensional partial differential systems is among the most important challenges in the modern integrability theory. It turns out that such a system can be associated to any pair of rational functions of one variable in general position, as established below using contact Lax pairs introduced in [7].

Keywords: multidimensional integrable systems; dispersionless systems; contact Lax pairs

1 Introduction

Nonlinear systems, in particular integrable ones, undoubtedly play an important role in modern mathematics, mechanics and physics, cf. e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein. Integrable systems are particularly interesting as they provide an attractive mix of nonlinearity and tractability. As far as integrable partial differential systems are concerned, (3+1)-dimensional ones are particularly important, cf. e.g. [1, 7] and references therein, as we live in four-dimensional spacetime.

Recently, a novel systematic construction of (3+1)-dimensional integrable systems was discovered in [7] using a new class of Lax pairs, the contact Lax pairs.

Namely, let $x, y, z, t$ be independent and $u_1, \ldots, u_N$ dependent variables. In [7] it was shown that there exist plethora of (3+1)-dimensional integrable systems associated with nonlinear contact Lax pairs of the form

$$\psi_y = \psi_x F(\psi_x/\psi_z, u), \quad \psi_t = \psi_z G(\psi_x/\psi_z, u),$$

where $\psi(x, y, z, t)$ is a scalar function and $u = (u_1, \ldots, u_N)^T$. The superscript $T$ stands here and below for the transposed matrix while the subscripts $x, y, z, t$ refer to partial derivatives. Throughout the present paper it is tacitly assumed that all functions are sufficiently smooth for all computations to make sense. This approach can be made rigorous using the language of differential algebra, cf. [7].

To any nonlinear contact Lax pair (1) there corresponds [7] a linear nonisospectral contact Lax pair, and vice versa; for this reason only nonlinear contact Lax pairs are considered below.

In [7] two large classes of nonlinear contact Lax pairs leading to integrable (3+1)-dimensional systems were found, with the functions $F$ and $G$ being

1) polynomials in $p \equiv \psi_x/\psi_z$ of the form

$$F = p^{m+1} + \sum_{i=0}^{m} v_i p^i, \quad G = p^{n+1} + \frac{n}{m} v_m p^n + \sum_{j=0}^{n-1} w_j p^j,$$

where $u = (v_0, \ldots, v_m, w_0, \ldots, w_{n-1})^T$; $m$ and $n$ are arbitrary natural numbers (here and below natural numbers refer strictly to positive integers not including zero), so $N = m + n + 1$;
2) rational functions of $p$ of the form

\[
F = \sum_{i=1}^{m} \frac{a_i}{p - v_i}, \quad G = \sum_{j=1}^{n} \frac{b_j}{p - w_j},
\]

(3)

where $m$ and $n$ again are arbitrary natural numbers, so now $N = 2(m + n)$ and $\mathbf{u} = (a_1, \ldots, a_m, v_1, \ldots, v_n, b_1, \ldots, b_n, w_1, \ldots, w_n)^T$.

In connection with (3) one is immediately led to wonder what happens in a just slightly more general case, when $F$ and $G$ are rational functions of $p$ in general position, that is,

\[
F = a_0 + \sum_{i=1}^{m} \frac{a_i}{p - v_i}, \quad G = b_0 + \sum_{j=1}^{n} \frac{b_j}{p - w_j},
\]

(4)

so now $N = 2(m + n + 1)$ and $\mathbf{u} = (a_0, a_1, \ldots, a_m, v_1, \ldots, v_n, b_0, b_1, \ldots, b_n, w_1, \ldots, w_n)^T$.

The goal of the present paper is to answer the natural question posed above. Namely, Theorem 1 below introduces a change of variables which turns the nonlinear contact Lax pair associated with $F$ and $G$ from (4) into the one associated with $F$ and $G$ from (3) under essentially a single assumption of compatibility of the former Lax pair. It turns out that $a_0$ and $b_0$ are basically the artifacts of the gauge freedom, and the said change of variables just removes this freedom.

Note that a similar phenomenon occurs [10] in (2+1) dimensions (this corresponds to putting $u_z = 0$ and $\psi_z = 1$ in the notation of present paper), so Theorem 1 can be seen as a generalization of the relevant result of Zakharov [10] to (3+1) dimensions.

2 Rational Lax Pairs

**Theorem 1** Let $F$ and $G$ in (4) be rational functions of $\psi_x/\psi_z$ in general position (4), i.e., (4) has the form

\[
\psi_y = a_0 \psi_z + \psi_z^2 \sum_{i=1}^{m} \frac{a_i}{\psi_x - v_i \psi_z}, \quad \psi_t = b_0 \psi_z + \psi_z^2 \sum_{j=1}^{n} \frac{b_j}{\psi_x - w_j \psi_z},
\]

(5)

where $m$ and $n$ are any natural numbers, so we have $\mathbf{u} = (a_0, a_1, \ldots, a_m, v_1, \ldots, v_m, b_0, b_1, \ldots, b_n, w_1, \ldots, w_n)^T$.

Suppose that (5) is compatible; then there exists a ‘potential’ $q$ such that $q_z \neq 0$ and

\[
q_y = a_0 q_z, \quad q_t = b_0 q_z.
\]

(6)

Under the above assumptions, upon the change of variables

\[
\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{z} = q, \quad \tilde{t} = t, \quad \tilde{\psi} = \psi, \quad \tilde{q} = z,
\]

\[
\tilde{a}_i = a_i q_z^2, \quad \tilde{v}_i = v_i - \frac{q_x}{q_z}, \quad i = 1, \ldots, m,
\]

\[
\tilde{b}_j = b_j q_z^2, \quad \tilde{w}_j = w_j - \frac{q_x}{q_z}, \quad j = 1, \ldots, n,
\]

(7)

the system (5) after omitting the tildas takes the form

\[
\psi_y = \psi_z^2 \sum_{i=1}^{m} \frac{a_i}{\psi_x - v_i \psi_z}, \quad \psi_t = \psi_z^2 \sum_{j=1}^{n} \frac{b_j}{\psi_x - w_j \psi_z},
\]

(8)

i.e., it is nothing but the nonlinear contact Lax pair associated with $F$ and $G$ from (3).
Before proceeding to the proof note that the compatibility condition for \( (8) \) is the following (3+1)-dimensional integrable system of 2\((m+n)\) equations for 2\((m+n)\) unknown functions,

\[
\begin{align*}
(v_i)_t - \sum_{l=1}^{n} \left\{ \left( \frac{b_lv_i}{w_l - v_i} \right)_z - \left( \frac{b_l}{w_l - v_i} \right)_x - 2b_l(v_i)_x \right\} &= 0,
\end{align*}
\]

\[
\begin{align*}
(w_j)_y - \sum_{k=1}^{m} \left\{ \left( \frac{a_k}{w_j - v_k} \right)_x - \left( \frac{a_kv_j}{w_j - v_k} \right)_z + 2a_k(w_j)_z \right\} &= 0,
\end{align*}
\]

\[
\begin{align*}
(a_i)_t - \sum_{l=1}^{n} \left\{ \frac{3a_i(b_l)_z}{w_l - v_i} - \frac{3a_ib_l(w_i)_z}{(w_l - v_i)^2} + \left( \frac{a_ib_l(2v_i - w_l)}{(w_l - v_i)^2} \right)_z - \left( \frac{a_ib_l}{(w_l - v_i)^2} \right)_x \right\} &= 0,
\end{align*}
\]

\[
\begin{align*}
(b_j)_y - \sum_{k=1}^{m} \left\{ \frac{3a_k(b_j)_z}{w_j - v_k} - \frac{3a_kb_j(w_j)_z}{(w_j - v_k)^2} + \left( \frac{a_kb_j(2v_k - w_j)}{(w_j - v_k)^2} \right)_z - \left( \frac{a_kb_j}{(w_j - v_k)^2} \right)_x \right\} &= 0,
\end{align*}
\]

where \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \).

The above system is obviously determined; in particular, it has as many equations as dependent variables. Moreover, system (9) can be turned into a system of Cauchy–Kowalewski type by a simple change of independent variables, \( X = x, Y = y - t, Z = z, T = y + t \). Integrability of (9) in the sense of existence of linear Lax pair immediately follows from the general results of [7].

**Proof of Theorem 1**. The left-hand side of the compatibility condition for (11), that is,

\[
(\psi_y)_t - (\psi_t)_y = 0,
\]

where the derivatives are evaluated by virtue of (11), obviously is a rational function of \( p \equiv \psi_x/\psi_z \).

Bringing this rational function to a common denominator and then equating to zero the coefficients at the powers of \( p \) in the numerator yields a system of 2\((n+2m+1)\) PDEs for the 2\((m+2n+2)\) unknown functions \( u_A \), i.e., this system is underdetermined.

Equating to zero the coefficient at the highest power of \( p \) of the said numerator yields the equation

\[
(a_0)_t - (b_0)_y - b_0(a_0)_z + a_0(b_0)_z = 0.
\]

One can easily verify that a general solution of (11) can be written as

\[
a_0 = q_y/q_z, \quad b_0 = q_t/q_z,
\]

where \( q \) is an arbitrary function of \( x, y, z, t \) such that \( q \neq 0 \). The above system is nothing but (6), so \( q \) introduced in Theorem 1 is indeed well-defined if (5) is compatible.

The presence of an arbitrary function \( q \) is a manifestation of the gauge freedom in the system under study, just as in the (2+1)-dimensional case studied in [10].

Straightforward but cumbersome computations show that passing to new variables given by (7) turns, modulo omitting the tildas, (5) into (8). Notice that (8) does not involve \( a_0 \) and \( b_0 \), i.e., using the transformation (7) has removed the gauge freedom associated with \( q \). This remark completes the proof. \( \square \)

## 3 Outlook

It is immediate from the above that there exist non-overdetermined (3+1)-dimensional integrable systems with Lax pairs (5) associated with arbitrary pairs of rational functions of a single variable in general position.
These systems possess hidden gauge freedom just like their (2+1)-dimensional counterparts with Lax pairs of the form

$$\psi_y = a_0 + \sum_{i=1}^{m} \frac{a_i}{\psi_x - v_i}, \quad \psi_t = b_0 + \sum_{j=1}^{n} \frac{b_j}{\psi_x - w_j},$$

studied in [10]. Moreover, upon the removal of the said gauge freedom through the change of variables presented in Theorem 1 the systems in question together with their Lax pairs (5) become equivalent to somewhat simpler systems (9) with Lax pairs (8) studied in [7]. While the systems associated with (5) are underdetermined, this is not the case for systems (9) which are equivalent to systems of Cauchy–Kowalevski type as discussed in Section 2.

The significance of these results consists inter alia in revealing the breadth of the class of (3+1)-dimensional integrable systems in general, as well as of the subclass thereof associated with contact Lax pairs (11).

Moreover, the results of the present paper immediately lead to a natural open problem of going beyond the rational case and finding examples of (3+1)-dimensional systems with Lax pairs (1) whose Lax functions $F$ and $G$ have more involved dependence on $\psi_x/\psi_z$. The author intends to address this in future work.

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