Abstract

Semidefinite programs (SDPs) – some of the most useful and versatile optimization problems of the last few decades – are often pathological: the optimal values of the primal and dual problems may differ and may not be attained. Such SDPs are both theoretically interesting and often impossible to solve; yet, the pathological SDPs in the literature look strikingly similar.

Based on our recent work [28] we characterize pathological semidefinite systems by certain excluded matrices, which are easy to spot in all published examples. Our main tool is a normal (canonical) form of semidefinite systems, which makes their pathological behavior easy to verify. The normal form is constructed in a surprisingly simple fashion, using mostly elementary row operations inherited from Gaussian elimination. The proofs are elementary and can be followed by a reader at the advanced undergraduate level.

As a byproduct, we show how to transform any linear map acting on symmetric matrices into a normal form, which allows us to quickly check whether the image of the semidefinite cone under the map is closed. We can thus introduce readers to a fundamental issue in convex analysis: the linear image of a closed convex set may not be closed, and often simple conditions are available to verify the closedness, or lack of it.

Key words: semidefinite programming; duality; duality gap; pathological semidefinite programs; closedness of the linear image of the semidefinite cone

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1 Introduction. Main results

Semidefinite programs (SDPs) – optimization problems with semidefinite matrix variables, a linear objective, and linear constraints – are some of the most practical, widespread, and interesting optimization problems of the last three decades. They naturally generalize linear programs, and appear in diverse areas such as combinatorial optimization, polynomial optimization, engineering, and economics. They are covered in many surveys, see e.g. [33] and textbooks, see e.g. [10, 3, 31, 9, 14, 5, 18, 34].

They are also a subject of intensive research: in the last 30 years several thousand papers have been published on SDPs.

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To ground our discussion, let us write an SDP in the form

\[
\sup \sum_{i=1}^{m} c_i x_i \\
\text{s.t.} \quad \sum_{i=1}^{m} x_i A_i \preceq B,
\]

where \(A_1, \ldots, A_m, \) and \(B\) are \(n \times n\) symmetric matrices, \(c_1, \ldots, c_m\) are scalars, and for symmetric matrices \(S\) and \(T\), we write \(S \preceq T\) to say that \(T - S\) is positive semidefinite (psd).

To solve (\(SDP-P\)) we rely on a natural dual, namely

\[
\inf B \bullet Y \\
\text{s.t.} \quad A_i \bullet Y = c_i (i = 1, \ldots, m) \\
Y \succeq 0
\]

(\(SDP-D\)) where the inner product of symmetric matrices \(S\) and \(T\) is \(S \bullet T := \text{trace}(ST)\). Since the weak duality inequality

\[
\sum_{i=1}^{m} c_i x_i \leq B \bullet Y
\]

(1.1)

always holds between feasible solutions \(x\) and \(Y\), if a pair \((x^*, Y^*)\) satisfies (1.1) with equality, then they are both optimal. Indeed, SDP solvers seek to find such an \(x^*\) and \(Y^*\).

However, SDPs often behave pathologically: the optimal values of (\(SDP-P\)) and (\(SDP-D\)) may differ and may not be attained.

The duality theory of SDPs – together with their pathological behaviors – is covered in several references on optimization theory and in textbooks written for broader audiences. For example, [10] gives an extensive, yet concise account of Fenchel duality; [33] and [31] provide very succinct treatments; [3] treats SDP duality as special case of duality theory in infinite dimensional spaces; [9] covers stability and sensitivity analysis; [5] and [14] contain many engineering applications; [18] and [34] are accessible to an audience with combinatorics background; and [8] explores connections to algebraic geometry.

Why are the pathological behaviors interesting? First, they do not appear in linear programs, which makes it apparent that SDPs are a much less innocent generalization of linear programs, than one may think at first. Note that the pathologies can come in “batches”: in extreme cases (\(SDP-P\)) and (\(SDP-D\)) both can have unattained, and different, optimal values! The variety of thought-provoking pathological SDPs makes teaching SDP duality (to students mostly used to clean and pathology-free linear programming) a truly rewarding experience.

Second, these pathologies also appear in other convex optimization problems, thus SDPs make excellent “model problems” to study.

Last but not least: pathological SDPs are often difficult or impossible to solve.

Our recent paper [28] was motivated by the curious similarity of pathological SDPs in the literature. To build intuition, we recall two examples; they or their variants appear in a number of papers and surveys.

Example 1. In the SDP

\[
\sup 2x_1 \\
\text{s.t.} \quad x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

(1.2)

any feasible solution must satisfy \((-x_1 - \frac{1}{x_1}) \geq 0, \text{ i.e., } -x_1^2 \geq 0\), so the only feasible solution is \(x_1 = 0\).
The dual, with a variable matrix $Y = (y_{ij})$, is equivalent to

$$\inf_{y_{11}} \begin{pmatrix} y_{11} & 1 \\ 1 & y_{22} \end{pmatrix} \succeq 0,$$

so it has an unattained 0 infimum.

Example 1 has an interesting connection to conic sections. The primal SDP (1.2) seeks $x_1$ such that $-x_1^2 \geq 0$, meaning a point with nonnegative $y$-coordinate on a downward parabola. This point is unique, so our parabola is “degenerate.” The dual (1.3) seeks the smallest nonnegative $y_{11}$ such that $y_{11}y_{22} \geq 1$, i.e., the leftmost point on a hyperbola. This point, of course, does not exist: see Figure 1.

![Figure 1: Parabola for the primal SDP, vs. hyperbola for the dual SDP in Example 1](image)

**Example 2.** We claim that the SDP

$$\sup_{x_2} x_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has an optimal value that differs from that of its dual. Indeed, in (1.4) we have $x_2 = 0$ in any feasible solution: this follows by a reasoning analogous to the one we used in Example 1. Thus (1.4) has an attained 0 supremum.

On the other hand, letting $Y = (y_{ij})$ be the dual variable matrix, the first dual constraint implies $y_{11} = 0$. By $Y \succeq 0$ the first row and column of $Y$ is zero. By the second dual constraint $y_{22} = 1$ so the optimal value of the dual is 1, hence indeed there is a finite, positive duality gap.

Curiously, while their pathologies differ, Examples 1 and 2 still look similar. First, in both examples a matrix on the left hand side has a certain “antidiagonal” structure. Second, if we delete the second row and second column in all matrices in Example 2, and remove the first matrix, we get back Example 1! This raises the following questions: Do all pathological semidefinite systems “look the same”? Does the system of Example 1 appear in all of them as a “minor”?

The paper [28] made these questions precise and gave a “yes” answer to both.
To proceed, we state our main assumptions and recap needed terminology from [28]. We assume throughout that \( (P_{SD}) \) is feasible, and we say that the semidefinite system

\[
\sum_{i=1}^{m} x_i A_i \preceq B \quad (P_{SD})
\]

is \textit{badly behaved} if there is \( c \in \mathbb{R}^m \) for which the optimal value of \((SDP-P)\) is finite but the dual \((SDP-D)\) has no solution with the same value. We say that \((P_{SD})\) is \textit{well behaved}, if not badly behaved.

A \textit{slack matrix} or \textit{slack} in \((P_{SD})\) is a psd matrix of the form

\[
Z = B - \sum_{i=1}^{m} x_i A_i.
\]

Of course, \((P_{SD})\) has a maximum rank slack matrix, and our characterizations will rely on such a matrix.

We also make the following assumption:

\textbf{Assumption 1.} \textit{The maximum rank slack in \((P_{SD})\) is}

\[
Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \text{ for some } 0 \leq r \leq n. \quad (1.5)
\]

For the rest of the paper we fix this \( r \).

Assumption 1 is easy to satisfy (at least in theory): if \( Z \) is a maximum rank slack in \((P_{SD})\), and \( Q \) is a matrix of suitably scaled eigenvectors of \( Z \), then replacing all \( A_i \) by \( Q^T A_i Q \) and \( B \) by \( Q^T B Q \) puts \( Z \) into the required form.

A slightly strengthened version of the main result of [28] follows.

\textbf{Theorem 1.} \textit{The system \((P_{SD})\) is badly behaved if and only if the \textit{Bad condition} below holds:}

\textbf{Bad condition:} \textit{There is a \( V \) matrix, which is a linear combination of the \( A_i \), and of the form}

\[
V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \text{ where } V_{11} \text{ is } r \times r, V_{22} \succeq 0, \mathcal{R}(V_{12}^T) \not\subseteq \mathcal{R}(V_{22}), \quad (1.6)
\]

\textit{where } \mathcal{R}(\cdot) \text{ stands for rangespace.} \quad \Box

The \( Z \) and \( V \) matrices are \textit{certificates} of the bad behavior. They can be chosen as

\[
Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ in Example 1, and}
\]

\[
Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ in Example 2.}
\]

Theorem 1 is appealing: it is simple, and the excluded matrices \( Z \) and \( V \) are easy to spot in essentially all badly behaved semidefinite systems in the literature. For instance, we invite the reader to spot \( Z \) and \( V \) (after ensuring Assumption 1) in the SDP

\[
\sup x_2 \text{ s.t. } \begin{pmatrix} x_2 - \alpha & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & x_2 & 0 \end{pmatrix} \preceq 0,
\]

\[
\begin{pmatrix} x_2 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{pmatrix} \preceq 0,
\]

\[
\end{pmatrix} \preceq 0,
\]
which is Example 5.79 in [9]. Here \( \alpha > 0 \) is a parameter, and the gap between this SDP and its dual is \( \alpha \).

More examples are in [30, 17, 36, 35, 22, 34]; e.g., in an example [34, page 43] any matrix on the left hand side can serve as a \( V \) certificate matrix! Theorem 1 also easily certifies the bad behavior of some SDPs coming from polynomial optimization, e.g., of the SDPs in [39].

Theorem 1 has an interesting geometric interpretation. Let \( \text{dir}(Z, \mathcal{S}_n^+) \) be the set of feasible directions at \( Z \) in \( \mathcal{S}_n^+ \), i.e.,

\[
\text{dir}(Z, \mathcal{S}_n^+) = \{ Y \mid Z + \epsilon Y \succeq 0 \text{ for some } \epsilon > 0 \}.
\]

Then \( V \) is in the closure of \( \text{dir}(Z, \mathcal{S}_n^+) \), but it is not a feasible direction (see [28, Lemma 3]). That is, for small \( \epsilon > 0 \) the matrix \( Z + \epsilon V \) is “almost” psd, but not quite.

We illustrate this point with the \( Z \) and \( V \) of Example 1. The shaded region of Figure 2 is the set of \( 2 \times 2 \) psd matrices with trace equal to 1. This set is an ellipse, so conic sections make a third appearance! The figure shows \( Z \) and \( Z + \epsilon V \) for a small \( \epsilon > 0 \).

![Figure 2: The matrix \( Z + \epsilon V \) is “almost” psd, but not quite](image)

How do we characterize the good behavior of \( (P_{SD}) \)? We could, of course, say that \( (P_{SD}) \) is well behaved if the \( V \) matrix of Theorem 1 does not exist. However, there is a much more convenient, and easier to check characterization, which we give below:

**Theorem 2.** The system \( (P_{SD}) \) is well behaved if and only if both “Good conditions” below hold.

**Good condition 1:** There is \( U \succ 0 \) such that

\[
A_i \cdot \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix} = 0 \text{ for all } i.
\]
**Good condition 2:** If $V$ is a linear combination of the $A_i$ of the form

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & 0 \end{pmatrix},$$

then $V_{12} = 0$.

In Theorem 2 and the rest of the paper, $U \succ 0$ means that $U$ is symmetric and positive definite, and we use the following convention:

**Convention 1.** If a matrix is partitioned as in Theorems 1 or 2, then we understand that the upper left block is $r \times r$.

**Example 3.** At first glance, the system

$${x_1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.8)$$

looks very similar to the system in Example 1. However, (1.8) is well behaved, and Theorem 2 verifies this by choosing $U = I_2$ in “Good condition 1” (of course “Good condition 2” trivially holds).

In [28] we proved Theorems 1 and 2 from a much more general result (Theorem 1 therein), which characterizes badly (and well) behaved conic linear systems. In this paper we give short proofs of Theorems 1 and 2 using building blocks from [28]. Our proofs mostly use elementary linear algebra: we reformulate $(P_{SD})$ into normal forms that make its bad or good behavior trivial to recognize. The normal forms are inspired by the row echelon form of a linear system of equations, and most of the operations that we use to construct them indeed come from Gaussian elimination.

As a byproduct, we show how to construct normal forms of linear maps $\mathcal{M} : n \times n$ symmetric matrices $\rightarrow \mathbb{R}^m$, to easily verify whether the image of the cone of semidefinite matrices under $\mathcal{M}$ is closed. We can thus introduce students to a fundamental issue in convex analysis: the linear image of a closed convex set is not always closed, and we can often verify its (non)closedness via simple conditions. For recent literature on closedness criteria see e.g., [4, 1, 6, 11, 12, 26]; for connections to duality theory, see e.g. [3, Theorem 7.2], [15, Theorem 2], [28, Lemma 2]. For us the most relevant closedness criteria are in [26, Theorem 1]: these criteria led to the results of [28].

We next describe how to reformulate $(P_{SD})$.

**Definition 1.** A semidefinite system is an elementary reformulation, or reformulation of $(P_{SD})$ if it is obtained from $(P_{SD})$ by a sequence of the following operations:

1. Choose an invertible matrix of the form

$$T = \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix},$$

and replace $A_i$ by $T^T A_i T$ for all $i$ and $B$ by $T^T B T$.

2. Choose $\mu \in \mathbb{R}^m$ and replace $B$ by $B + \sum_{j=1}^m \mu_j A_j$.
(3) Choose indices $i \neq j$ and exchange $A_i$ and $A_j$.

(4) Choose $\lambda \in \mathbb{R}^m$ and an index $i$ such that $\lambda_i \neq 0$, and replace $A_i$ by $\sum_{j=1}^{m} \lambda_j A_j$.

(Of course, we can use just some of these operations and we can use them in any order).

Where do these operations come from? As we mentioned above, mostly from Gaussian elimination: the last three can be viewed as elementary row operations done on $(SDP-D)$ with some $c \in \mathbb{R}^m$. For example, operation (3) exchanges the constraints

$$A_i \bullet Y = c_i \text{ and } A_j \bullet Y = c_j.$$  

Reformulating $(P_{SD})$ keeps the maximum rank slack the same (cf. Assumption 1). Of course, $(P_{SD})$ is badly behaved if and only if its reformulations are.

We organize the rest of the paper as follows. In the rest of this section we review preliminaries. In Section 2 we prove Theorems 1 and 2 and show how to construct the normal forms. We prove the chain of implications

$$(P_{SD}) \text{satisfies the “Bad condition”} \implies \text{it has a “Bad reformulation”} \implies \text{it is badly behaved},$$

and the “good” counterpart

$$(P_{SD}) \text{satisfies the “Good conditions”} \implies \text{it has a “Good reformulation”} \implies \text{it is well behaved}.$$  

(1.9)  

(1.10)

In these proofs we only use elementary linear algebra.

Of course, if $(P_{SD})$ is badly behaved, then it is not well behaved. Thus the implication

Any of the “Good conditions” fail $\implies$ the “Bad condition” holds,  

(1.11)

ties everything together and shows that in (1.9) and (1.10) equivalence holds. Only the proof of (1.11) needs some elementary duality theory (all of which we recap in Subsection 1.1), thus all proofs can be followed by a reader at the advanced undergraduate level.

In Section 3 we look at linear maps that act on symmetric matrices. As promised, we show how to bring them into a normal form, to easily check whether the image of the cone of semidefinite matrices under such a map is closed. We also point out connections to asymptotes of convex sets, and weak infeasibility in SDPs. In Section 4 we close with a discussion.

### 1.1 Notation and preliminaries

As usual, we let $\mathcal{S}_n$ be the set of $n \times n$ symmetric matrices, and $\mathcal{S}_n^+$ the set of $n \times n$ symmetric positive semidefinite matrices.

For completeness, we next prove the weak duality inequality (1.1). Let $x$ be feasible in $(SDP-P)$ and $Y$ be feasible in $(SDP-D)$. Then

$$B \bullet Y - \sum_{i=1}^{m} c_i x_i = B \bullet Y - \sum_{i=1}^{m} (A_i \bullet Y) x_i = (B - \sum_{i=1}^{m} x_i A_i) \bullet Y \geq 0,$$

where the last inequality follows, since the $\bullet$ product of two psd matrices is nonnegative. Accordingly, $x$ and $Y$ are both optimal iff the last inequality holds at equality.
We next discuss two well known regularity conditions, both of which ensure that \((P_{SD})\) is well behaved:

- The first is Slater’s condition: this means that there is a positive definite slack in \((P_{SD})\).
- The second requires the \(A_i\) and \(B\) to be diagonal; in that case \((P_{SD})\) is a polyhedron and \((SDP-P)\) is just a linear program.

The sufficiency of these conditions is immediate from Theorem 1. If Slater’s condition holds, then \(Z\) in Theorem 1 is just \(I_n\), so the \(V\) certificate matrix cannot exist; if the \(A_i\) and \(B\) are diagonal, then so are their linear combinations, so again \(V\) cannot exist.

Thus Theorem 1 unifies these two (seemingly unrelated) conditions, and we invite the reader to check that so does Theorem 2.

We mention here that linear programs are sometimes also “pathological,” meaning both primal and dual may be infeasible. However, linear programs do not exhibit the pathologies that we study here.

2 Proofs and examples

In this section we prove and illustrate the implications (1.9), (1.10), and (1.11).

2.1 The Bad

2.1.1 From “Bad condition” to “Bad reformulation”

We assume the “Bad condition” holds in \((P_{SD})\) and show how to reformulate it as

\[
\sum_{i=1}^{k} x_i \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=k+1}^{m} x_i \begin{pmatrix} F_i & G_i \\ G_i^T & H_i \end{pmatrix} \preceq \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Z, \quad (P_{SD,bad})
\]

where

1. matrix \(Z\) is the maximum rank slack,
2. matrices

\[
\begin{pmatrix} G_i \\ H_i \end{pmatrix} (i = k + 1, \ldots, m)
\]

are linearly independent, and
3. \(H_m \succeq 0\).

Hereafter, we shall – informally – say that \((P_{SD,bad})\) is a “Bad reformulation” of \((P_{SD})\). We denote the constraint matrices on the left hand side by \(A_i\) throughout the reformulation process.

To begin, we replace \(B\) by \(Z\) in \((P_{SD})\). We then choose \(V = \sum_{i=1}^{m} \lambda_i A_i\) to satisfy the “Bad condition,” and note that the block of \(V\) comprising the last \(n - r\) columns must be nonzero. Next, we pick an \(i\) such that \(\lambda_i \neq 0\), and we use operation (4) in Definition 1 to replace \(A_i\) by \(V\). We then switch \(A_i\) and \(A_m\).
Next we choose a maximal subset of the \( A_i \) matrices whose blocks comprising the last \( n - r \) columns are linearly independent. We let \( A_m \) be one of these matrices (we can do this since \( A_m \) is now the \( V \) certificate matrix), and permute the \( A_i \) so this special subset becomes \( A_{k+1}, \ldots, A_m \) for some \( k \geq 0 \).

Finally, we take linear combinations of the \( A_i \) to zero out the last \( n - r \) columns of \( A_1, \ldots, A_k \), and arrive at the required reformulation.

Note that the systems in Examples 1 and 2 are already in the normal form of \((P_{SD}, bad)\). The next example is a counterpoint: it is a more complicated badly behaved system, which at first is very far from being in the normal form.

**Example 4. (Large bad example)** The system

\[
\begin{pmatrix}
9 & 7 & 7 & 1 \\
7 & 12 & 8 & -3 \\
7 & 8 & 2 & 4 \\
1 & -3 & 4 & 0
\end{pmatrix} + \begin{pmatrix}
17 & 7 & 8 & -1 \\
7 & 8 & 7 & -3 \\
8 & 7 & 4 & 2 \\
-1 & -3 & 2 & 0
\end{pmatrix} + \begin{pmatrix}
1 & 2 & 2 & 1 \\
2 & 6 & 3 & -1 \\
2 & 3 & 0 & 2 \\
1 & -1 & 2 & 0
\end{pmatrix} + \begin{pmatrix}
9 & 6 & 7 & 1 \\
6 & 13 & 8 & -3 \\
7 & 8 & 2 & 4 \\
1 & -3 & 4 & 0
\end{pmatrix} + \begin{pmatrix}
45 & 26 & 29 & 2 \\
26 & 47 & 31 & -12 \\
29 & 31 & 10 & 14 \\
2 & -12 & 14 & 0
\end{pmatrix}
\]

(2.1)

is badly behaved, but this would be difficult to verify by any ad hoc method.

Let us, however, verify its bad behavior using Theorem 1. System (2.1) satisfies the “Bad condition” with \( Z \) and \( V \) certificate matrices

\[
Z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
7 & 2 & 3 & -1 \\
2 & 1 & 2 & -1 \\
3 & 2 & 2 & 0 \\
-1 & -1 & 0 & 0
\end{pmatrix}.
\]

(2.2)

Indeed, \( Z = B - A_1 - A_2 - 2A_4, \) \( V = A_4 - 2A_3 \) (where we write \( A_i \) for the matrices on the left hand side, and \( B \) for the right hand side), and we explain shortly why \( Z \) is a maximum rank slack.

Let us next reformulate system (2.1): after the operations

\[
\begin{align*}
B & := B - A_1 - A_2 - 2A_4, \\
A_4 & = A_4 - 2A_3, \\
A_2 & = A_2 - A_3 - 2A_4, \\
A_1 & = A_1 - 2A_3 - A_4
\end{align*}
\]

(2.3)

it becomes

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
1 & 2 & 2 & 1 \\
2 & 6 & 3 & -1 \\
2 & 3 & 0 & 2 \\
1 & -1 & 2 & 0
\end{pmatrix} + \begin{pmatrix}
7 & 2 & 3 & -1 \\
2 & 1 & 2 & -1 \\
3 & 2 & 2 & 0 \\
-1 & -1 & 0 & 0
\end{pmatrix} \leq \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(2.4)
which is in the normal form of \((P_{SD,\text{bad}})\). Besides looking simpler than (2.1), the bad behavior of (2.4) is much easier to verify, as we shall see soon.

How do we convince a “user” that \(Z\) in equation (2.2) is indeed a maximum rank slack in system (2.1)? Matrices

\[
Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}
\]

(2.5)

have zero product with all constraint matrices, and hence also with any slack. Thus, if \(S\) is any slack, then \(S \cdot Y_1 = 0\), so the \((4,4)\) element of \(S\) is zero, hence the entire 4th row and column of \(S\) is zero (since \(S \succeq 0\)). Similarly, \(S \cdot Y_2 = 0\) shows the 3rd row and column of \(S\) is zero, thus the rank of \(S\) is at most two. Hence \(Z\) indeed has maximum rank.

In fact, Lemma 5 in [28] proves that \((P_{SD})\) can always be reformulated, so that a similar sequence of matrices certifies that \(Z\) has maximal rank. To do so, we need to use operation (1) in Definition 1.

### 2.1.2 If \((P_{SD})\) has a “Bad reformulation,” then it is badly behaved

For this implication we show that a system in the normal form of \((P_{SD,\text{bad}})\) is badly behaved; and for that, we devise a simple objective function which has a finite optimal value over \((P_{SD,\text{bad}})\), while the dual SDP has no solution with the same value.

To start, let \(x\) be feasible in \((P_{SD,\text{bad}})\) with a corresponding slack \(S\). Observe that the last \(n - r\) rows and columns of \(S\) must be zero, otherwise \(\frac{1}{2}(S + Z)\) would be a slack with larger rank than \(Z\). Hence, by condition (2) (after the statement of \((P_{SD,\text{bad}}))\), we deduce \(x_{k+1} = \ldots = x_m = 0\), so the optimal value of the SDP

\[
\sup \{ -x_m \mid x \text{ is feasible in } (P_{SD,\text{bad}}) \}
\]

(2.6)

is 0. We prove that its dual cannot have a feasible solution with value 0, so suppose that

\[
Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix} \succeq 0
\]

is such a solution. By \(Y \cdot Z = 0\) we get \(Y_{11} = 0\), and since \(Y \succeq 0\) we deduce \(Y_{12} = 0\). Thus

\[
\begin{pmatrix} F_m & G_m \\ G_m^T & H_m \end{pmatrix} \cdot Y = H_m \cdot Y_{22} \geq 0,
\]

so \(Y\) cannot be feasible in the dual of (2.6), a contradiction. \(\square\)

**Example 5.** (Example 4 continued) Revisiting this example, the bad behavior of (2.1) is nontrivial to prove, whereas that of (2.4) is easy: the objective function \(\sup -x_4\) gives a 0 optimal value over it, while there is no dual solution with the same value.
2.2 The Good

2.2.1 From “Good conditions” to “Good reformulation”

Let us assume that both “Good conditions” hold. We show how to reformulate \((P_{SD})\) as

\[
\sum_{i=1}^{k} x_i \left( \begin{array}{cc} F_i & 0 \\ 0 & 0 \end{array} \right) + \sum_{i=k+1}^{m} x_i \left( \begin{array}{cc} F_i & G_i \\ G_i^T & H_i \end{array} \right) \preceq \left( \begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right) = Z, \]

\((P_{SD, \text{good}})\)

with the following attributes:

1. matrix \(Z\) is the maximum rank slack.
2. matrices \(H_i (i = k + 1, \ldots, m)\) are linearly independent.
3. \(H_{k+1} \cdot U = \cdots = H_m \cdot U = 0\) for some \(U \succ 0\).

We shall – again informally – say that \((P_{SD, \text{good}})\) is a “Good reformulation” of \((P_{SD})\). We construct the system \((P_{SD, \text{good}})\) quite similarly to how we constructed \((P_{SD, \text{bad}})\), and, as usual, we denote the matrices on the left hand side by \(A_i\) throughout the process.

We first replace \(B\) by \(Z\) in \((P_{SD})\). We then choose a maximal subset of the \(A_i\) whose lower principal \((n-r) \times (n-r)\) blocks are linearly independent, and permute the \(A_i\), if needed, to make this subset \(A_{k+1}, \ldots, A_m\) for some \(k \geq 0\).

Finally we take linear combinations to zero out the lower principal \((n-r) \times (n-r)\) block of \(A_1, \ldots, A_k\). By “Good condition 2” the upper right \(r \times (n-r)\) block of \(A_1, \ldots, A_k\) (and the symmetric counterpart) also become zero. Thus items (1) and (2) hold.

As to item (3), suppose \(U \succ 0\) satisfies “Good condition 1.” Then \(U\) has zero \(\cdot\) product with the lower principal \((n-r) \times (n-r)\) blocks of the \(A_i\), hence \(H_i \cdot U = 0\) for \(i = k + 1, \ldots, m\). Hence item (3) holds, and the proof is complete.

Example 6. (Large good example) The system

\[
\begin{pmatrix}
9 & 7 & 7 & 1 \\
7 & 12 & 8 & -3 \\
7 & 8 & 2 & 4 \\
1 & -3 & 4 & -2
\end{pmatrix}
+ \begin{pmatrix}
17 & 7 & 8 & -1 \\
7 & 8 & 7 & -3 \\
8 & 7 & 4 & 2 \\
-1 & -3 & 2 & -4
\end{pmatrix} + \begin{pmatrix}
1 & 2 & 2 & 1 \\
2 & 6 & 3 & -1 \\
2 & 3 & 0 & 2 \\
1 & -1 & 2 & 0
\end{pmatrix}
+ \begin{pmatrix}
9 & 6 & 7 & 1 \\
6 & 13 & 8 & -3 \\
7 & 8 & 2 & 4 \\
1 & -3 & 4 & -2
\end{pmatrix} \preceq \begin{pmatrix}
45 & 26 & 29 & 2 \\
26 & 47 & 31 & -12 \\
29 & 31 & 10 & 14 \\
2 & -12 & 14 & -10
\end{pmatrix} \quad (2.7)
\]

is well behaved, but it would be difficult to improvise a method to verify this.

Instead, let us check that the “Good conditions” hold: to do so, we write \(A_i\) for the matrices on the left, and \(B\) for the right hand side.
First, we can see that "Good condition 1" holds with $U = I_2$, since

$$Y := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has zero • product with all $A_i$ (and also with $B$). Luckily, $Y$ also certifies that $Z$ in equation (2.2) is a maximum rank slack in (2.7): as $Y$ has zero • product with any slack, the rank of any slack is at most two. Of course, $Z$ is a rank two slack itself, since $Z = B - A_1 - A_2 - 2A_4$.

Next, let us verify "Good condition 2." Suppose the lower right $2 \times 2$ block of $V := \sum_{i=1}^{4} \lambda_i A_i$ is zero. Then by a direct calculation $\lambda \in \mathbb{R}^4$ is a linear combination of vectors

$$(−2, 1, 3, 0)^T \text{ and } (1, 0, 0, −1)^T,$$

so the upper right $2 \times 2$ block of $V$ (and its symmetric counterpart) is also zero, so "Good condition 2" holds.

Now, the same operations that are listed in equation (2.3) turn system (2.7) into

$$x_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 6 & 3 & -1 \\ 2 & 3 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix}$$

$$+ x_4 \begin{pmatrix} 7 & 2 & 3 & -1 \\ 2 & 1 & 2 & -1 \\ 3 & 2 & 2 & 0 \\ -1 & -1 & 0 & -2 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is in the normal form of $(P_{SD,good})$. As we shall see soon, the good behavior of (2.8) is much easier to verify.

2.2.2 If $(P_{SD})$ has a “Good reformulation,” then it is well behaved

For this implication we show that the system $(P_{SD,good})$ is well behaved; and for that, we let $c$ be such that

$$v := \sup \left\{ \sum_{i=1}^{m} c_i x_i \mid x \text{ is feasible in } (P_{SD,good}) \right\} \quad \text{(2.9)}$$

is finite. An argument like the one in Subsubsection 2.1.2 proves that $x_{k+1} = \cdots = x_m = 0$ holds for any $x$ feasible in (2.9), so

$$v = \sup \left\{ \sum_{i=1}^{k} c_i x_i \mid \sum_{i=1}^{k} x_i F_i \preceq I_r \right\}. \quad \text{(2.10)}$$

Since (2.10) satisfies Slater’s condition, there is $Y_{11}$ feasible in its dual with $Y_{11} \bullet I_r = v$.

We next choose a $Y_{22}$ symmetric matrix (which may not be not positive semidefinite), such that

$$Y := \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}$$
satisfies the equality constraints of the dual of (2.9) (this can be done, by condition (2)). We then replace \( Y_{22} \) by \( Y_{22} + \lambda U \) for some \( \lambda > 0 \) to make it psd: we can do this by a simple linesearch. After this, \( Y \) is feasible in the dual of (2.9) (by condition (3)), and clearly \( Y \cdot Z = v \) holds. The proof is now complete.

The above proof is illustrated in Figure 3 by a commutative diagram. The horizontal arrows represent “elementary” constructions, i.e., we find the object at the head of the arrow from the object at the tail of the arrow by a basic argument or computation.

\[
\begin{align*}
\text{(2.9)} & \quad \text{prove } x_{k+1} = \cdots = x_m = 0 \\
\downarrow \text{dual solution} & \quad \downarrow \text{dual solution} \\
\begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} + \lambda U \end{pmatrix} & \quad \text{solve for } Y_{22} \text{ and do a linesearch} \\
& \quad \text{Y}_{11}
\end{align*}
\]

Figure 3: How to construct an optimal dual solution of (2.9)

**Example 7.** (Example 6 (Large good example) continued.) We now illustrate how to verify the good behavior of system (2.8): we pick an objective function with a finite optimal value over this system, and show how to construct an optimal dual solution.

We thus consider the SDP

\[
\begin{align*}
\sup & \quad 2x_2 + 5x_3 + 7x_4 \\
\text{s.t.} & \quad (x_1, x_2, x_3, x_4) \text{ is feasible in (2.8)},
\end{align*}
\]

in which \( x_3 = x_4 = 0 \) holds whenever \( x \) is feasible, since in (2.8) the right hand side is the maximum rank slack, and the lower right \( 2 \times 2 \) blocks of \( A_3 \) and \( A_4 \) are linearly independent.

So the optimal value of (2.11) is the same as that of

\[
\begin{align*}
\sup & \quad 2x_2 \\
\text{s.t.} & \quad x_1 \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

Next, let

\[
Y_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_{22} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}.
\]

Here \( Y_{11} \) is an optimal solution of the dual of (2.12): this follows since it has the same value as the primal optimal solution \((x_1, x_2) = (-\frac{1}{2}, \frac{1}{2})\). Further, \( Y_{22} \) is chosen so that \( Y \) satisfies the equality constraints of the dual of (2.11).

Of course, \( Y_{22} \) is not psd, hence neither is \( Y \). As a remedy, we replace \( Y_{22} \) by \( Y_{22} + \lambda I_2 \) for some \( \lambda \geq 1 \). This operation makes \( Y \) feasible, because \( U := I_2 \) verifies item (3) (after the statement of (\( P_{SD, \text{good}} \))). Now \( Y \) is optimal in the dual of (2.11) and the process is complete.

We remark that the procedure of constructing \( Y \) from \( Y_{11} \) was recently generalized in [29] to the case when \( (P_{SD}) \) satisfies only “Good condition 2.”
2.3 Tying everything together

Now we tie everything together: we show that if any of the “Good conditions” fail, then the “Bad condition” holds.

Clearly, if “Good condition 2” fails, then the “Bad condition” holds, so assume that “Good condition 1” fails.

First, we shall produce a matrix $V$ which is a linear combination of the $A_i$ such that

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix} \text{ with } V_{22} \succeq 0, \; V_{22} \neq 0.$$  \hfill (2.13)

To achieve that goal, we let $B_i$ be the lower right order $n - r$ principal block of $A_i$ for $i = 1, \ldots, m$ and for some $\ell \geq 1$ choose matrices $C_1, \ldots, C_\ell$ such that the set of their linear combinations is

$$\{ U \in S^{n-r} : B_1 \bullet U = \cdots = B_\ell \bullet U = 0 \}.$$

Consider next the primal-dual pair of SDPs

\begin{align*}
\sup_t & \quad \text{s.t. } tI + \sum_{i=1}^\ell x_i C_i \preceq 0 & \text{inf } 0 \\
& \quad \text{s.t. } I \bullet W = 1 & \text{s.t. } C_i \bullet W = 0 \; (i = 1, \ldots, \ell) \\
& \quad & W \succeq 0.
\end{align*} \hfill (2.14) \hfill (2.15)

Since “Good condition 1” fails, the primal (2.14) has optimal value zero. The primal (2.14) also satisfies Slater’s condition (with $x = 0$ and $t = -1$) so the dual (2.15) has a feasible solution $W$. This $W$ is of course nonzero, and a linear combination of the $B_i$, say

$$W = \sum_{i=1}^m \lambda_i B_i \text{ for some } \lambda \in \mathbb{R}^m.$$  

Thus, $V := \sum_{i=1}^m \lambda_i A_i$ passes requirement (2.13).

We are done if we show $\mathbb{R}(V_{12}^T) \nsubseteq \mathbb{R}(V_{22})$, so assume otherwise, i.e., assume $V_{12}^T = V_{22}D$ for some $D \in \mathbb{R}^{(n-r) \times r}$. Define

$$M = \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix},$$

and replace $A_i$ by $M^T A_i M$ for all $i$ and $B$ by $M^T B M$. After this, the maximum rank slack $Z$ in $(P_{SD})$ remains the same (see equation (1.5)) and $V$ is transformed into

$$M^T V M = \begin{pmatrix} V_{11} - D^T V_{12}^T & 0 \\ 0 & V_{22} \end{pmatrix}.$$  

Since $V_{22} \neq 0$, we deduce $Z + \epsilon V$ has larger rank than $Z$ for a small $\epsilon > 0$, which is a contradiction. The proof is complete. \hfill \Box

We thus proved the following corollary:

**Corollary 1.** The system $(P_{SD})$ is badly behaved if and only if it has a bad reformulation of the form $(P_{SD,\text{bad}})$.

It is well behaved if and only if it has a good reformulation of the form $(P_{SD,\text{good}})$.  

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Remark 1. Can we actually compute the $Z$ and $V$ matrices of Theorem 1, or the $U$ of Theorem 2? Regrettably, we don’t know how to do this in polynomial time either in the Turing model, or in the real number model of computing. However, we shall argue below that we can reduce this task to solving SDPs.

To start with the theoretical aspect of the reduction, we can find $Z$ by running a facial reduction algorithm \cite{13, 38, 34, 27}. These algorithms must solve a sequence of SDPs in exact arithmetic. We can then verify whether “Good condition 1” holds by solving the pair of SDPs (2.14)-(2.15). If it does hold, we can extract a $U$ matrix that satisfies it from an optimal solution of (2.14). If it does not, we can extract a $V$ certificate matrix that satisfies the “Bad condition” from an optimal solution of the dual (2.15).

In practice, heuristic and reasonably effective implementations of facial reduction algorithms exist \cite{29, 40}, and we may solve (2.14)-(2.15) approximately, to deduce that $(P_{SD})$ is nearly badly or well behaved.

We mention here that the complexity of checking attainment and the existence of a positive gap in SDPs is unknown.

3 When is the linear image of the semidefinite cone closed?

We now address a question of independent interest in convex analysis/convex geometry:

\begin{center}
\begin{tabular}{|c|}
\hline
\textbf{Given a linear map, is the image of $S^n_+$ under the map closed?}\tabularnewline \hline
\end{tabular}
\end{center}

This question fits in a much broader context. More generally, we can ask: when is the linear image of a closed convex set, say $C$, closed? Such closedness criteria are fundamental in convex analysis, and Chapter 9 in Rockafellar’s classic text \cite{32} is entirely dedicated to them. For more closedness criteria see Chapter 2.3 in \cite{1}, and for more recent work on this subject we refer to \cite{4, 6, 11, 12}. The latter paper shows that the set of linear maps under which the image of a closed convex cone is not closed is small both in measure and in category.

The closedness of the linear image of a closed convex cone ensures that a conic linear system is well-behaved (in the same sense as $(P_{SD})$); see e.g., \cite[Theorem 7.2]{3}, \cite[Theorem 2]{15}, \cite[Lemma 2]{28}. We studied criteria for the closedness of the linear image of a closed convex cone in \cite{26}, and the results therein led to \cite{28}, and to this paper.

The special case $C = S^2_+$ is interesting, since the semidefinite cone is one of the simplest nonpolyhedral sets whose geometry is well understood, see, e.g. \cite{2, 25} for a characterization of its faces. It turns out that the (non)closedness of the image of $S^2_+$ admits simple combinatorial characterizations.

We need some basic notation: for a set $S$ we define its \textbf{frontier} \text{front}(S) as the difference between its closure and the set itself:

\[ \text{front}(S) := \text{closure}(S) \setminus S. \]

\textbf{Example 8.} Define the map

\[ S^2 \ni Y \rightarrow (y_{11}, 2y_{12}). \]

\text{The image of $S^2_+$ -- shown on Figure 4 in blue, and its frontier in red -- is}

\[ \{(0,0)\} \cup \{ (\alpha, \beta) : \alpha > 0 \}, \]

\[ \{(0,0)\} \cup \{ (\alpha, \beta) : \alpha > 0 \}, \]
so it is not closed. For example, \((0, 2)\) is in the frontier since \((\epsilon, 2)\) is the image of the psd matrix
\[
\begin{pmatrix}
\epsilon & 1 \\
1 & 1/\epsilon
\end{pmatrix}
\]
for all \(\epsilon > 0\), but no psd matrix is mapped to \((0, 2)\).

In more involved examples, however, the (non)closedness of the image is much harder to check.

**Example 9.** This example is based on Example 6 in [20]. Define the linear map
\[
S^3 \ni Y \to (5y_{11} + 4y_{22} + 4y_{13}, 3y_{11} + 3y_{22} + 2y_{13}, 2y_{11} + 2y_{22} + 2y_{13}).
\]
As we shall see, the image of \(S^3\) is not closed, but verifying this by any ad hoc method seems very difficult.

For convenience, we shall represent linear maps from \(S^n\) to \(\mathbb{R}^m\) by matrices \(A_1, \ldots, A_m \in S^n\) and write
\[
A(x) = \sum_{i=1}^m x_i A_i, \quad \text{and} \quad A^*(Y) = (A_1 \bullet Y, \ldots, A_m \bullet Y).
\]
That is, we consider a linear map from \(S^n\) to \(\mathbb{R}^m\) as the adjoint of a suitable linear map in the opposite direction, to better fit the framework of [26, 28].

The next proposition connects the closedness of the linear image of \(S^n_+\) and the bad (or good) behavior of a homogeneous semidefinite system. A simple proof follows, e.g., from the classic separation theorem [10, Theorem 1.1.1].

**Proposition 1.** Given a linear map \(A\) and its adjoint \(A^*\) as in (3.4), the set \(A^*(S^n_+)\) is not closed if and only if the system
\[
\sum_{i=1}^m x_i A_i \preceq 0 \quad \text{(P \_SDH)}
\]
is badly behaved. In particular, \(c \in \text{front}(A^*(S^n_+))\) if and only if the SDP
\[
\begin{align*}
\sup_{s.t.} & \quad \sum_{i=1}^m c_i x_i \\
& \quad \sum_{i=1}^m x_i A_i \preceq 0
\end{align*}
\]

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has optimal value zero, but its dual is infeasible.

Thus, if \((P_{SDH})\) satisfies Assumption 1, then the characterizations of Theorems 1 and 2 apply.

More interestingly, Corollary 1 and Proposition 1 together imply the following:

**Corollary 2.** Suppose \(A\) and \(A^*\) are represented as in (3.4). Then \(A^*(\mathcal{S}^n_+)\) is

1. not closed if and only if the homogeneous system \((P_{SDH})\) has a bad reformulation (of the form \((P_{SD,bad})\));
2. closed if and only if the homogeneous system \((P_{SDH})\) has a good reformulation (of the form \((P_{SD,good})\)).

We next illustrate Corollary 2 by continuing the previous examples. On the one hand, reformulating the map of Example 8 does not help either to verify nonclosedness of the image set, or to exhibit a vector in its frontier. Reformulating, however, does help a lot in Example 9.

**Example 10.** (Example 8 continued) We can write the map in (3.1) as

\[
\mathcal{S}^2 \ni Y \mapsto \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \cdot Y, \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \cdot Y .
\]

so the corresponding homogeneous semidefinite system is

\[
x_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + x_2 \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \preceq 0,
\]

whose bad reformulation is essentially the same:

\[
x_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) + x_2 \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \preceq \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)
\]

(we just replaced the the right hand side by the maximum rank slack).

**Example 11.** (Example 9 continued) The homogeneous semidefinite system corresponding to the map in (3.3) is

\[
x_1 \begin{pmatrix} 5 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq 0. \quad (3.6)
\]

Its bad reformulation is

\[
x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.7)
\]

(How exactly did we obtain (3.7)? To explain, let us call the matrices \(A_1, A_2,\) and \(A_3\) on the left hand side in (3.6). Then (3.7) is obtained by performing the operations \(A_2 = A_2 - A_3; A_1 = A_1 - 2A_3; A_3 = A_3 - A_1 - A_2,\) then replacing the right hand side by \(A_2.)\)
Let \( A(x) \) be the left hand side in (3.6) and \( A'(x) \) the left hand side in (3.7). Then
\[
A^*(Y) = (y_{11}, y_{11} + y_{22}, y_{22} + 2y_{13}),
\]
and a calculation shows (for details, see Example 6 in [20])
\[
\begin{align*}
closure(A^*S_3^+ &= \{ (\alpha, \beta, \gamma) : \beta \geq \alpha \geq 0 \}, \\
\text{front}(A^*S_3^+) &= \{ (0, \beta, \gamma) | \beta \geq 0, \beta \neq \gamma \}. 
\end{align*}
\tag{3.8}
\]

The set \( A^*(S_3^+) \) is shown in Figure 5 in blue, and its frontier in red. Note that the blue diagonal segment on the red facet actually belongs to \( A^*(S_3^+) \).

The exact algebraic description of \( A^*(S_3^+) \) (or of its closure and frontier) is still not trivial to find. However, its nonclosedness readily follows from Proposition 1 and Theorem 1, since (3.7) is badly behaved: we can choose \( Z \) as the right hand side in (3.7) and \( V \) as the coefficient matrix of \( x_3 \).

We can also quickly exhibit an element in \( \text{front}(A^*(S_3^+)) \): the optimal value of the SDP
\[
\sup \{ x_3 | \text{s.t. } A'(x) \preceq 0 \}
\]
is 0, but its dual is infeasible, hence by Proposition 1 we deduce
\[
(0, 0, 1) \in \text{front}(A^*(S_3^+)).
\]

**Remark 2.** We next connect our work to two other areas of convex analysis. The first area, asymptotes of convex sets, is classical; the second area, weak infeasibility in SDPs, is more recent.

Let us define the distance of sets \( S_1 \) and \( S_2 \) as
\[
dist(S_1, S_2) := \inf \{ \| x_1 - x_2 \| | x_1 \in S_1, x_2 \in S_2 \}.
\]
Let \( H := \{ Y | A^*(Y) = c \} \). Then by a standard argument the following three statements are equivalent:
(1) $c \in \text{front}(A^*(S^n_+))$;
(2) $H \cap S^n_+ = \emptyset$, and $\text{dist}(H, S^n_+)=0$;
(3) (SDP-D) is infeasible, and its alternative system

$$\begin{align*}
\sum_{i=1}^{m} c_i x_i &= 1 \\
\sum_{i=1}^{m} x_i A_i &\preceq 0
\end{align*}$$

is also infeasible.

(The interested reader may want to work out the equivalences: for example, one can use Theorem 11.4 in [32] which shows that two convex sets have a positive distance iff they can be separated in a strong sense.)

Note that whenever (3.9) happens to be feasible, it is an easy certificate that (SDP-D) is infeasible, as an argument analogous to proving weak duality shows that both cannot be feasible (hence the jargon “alternative system”).

Two terminologies are used to express the equivalent statements (1)-(3) above.

The first terminology says that $H$ is an (affine) asymptote of $S^n_+$. Asymptotes of convex sets were introduced in the classical paper [16]. For example,

$$H = \{ Y \in S^2 \mid Y = \begin{pmatrix} 0 & 1 \\ 1 & y_{22} \end{pmatrix} \text{ for some } y_{22} \in \mathbb{R} \}$$

is an asymptote of $S^2_+$: evidently $H$ and $S^2_+$ do not intersect, but their distance is zero, since

$$\begin{pmatrix} \epsilon & 1 \\ 1 & 1/\epsilon \end{pmatrix} \succeq 0 \text{ for all } \epsilon > 0.$$

Alternatively, we can intersect $S^2_+$ with the hyperplane $\{ Y \in S^2 : y_{12} = y_{21} = 1 \}$ and check that $\{ (0, y_{22}) : y_{22} \in \mathbb{R} \}$ is an asymptote of the resulting convex set (the area above a hyperbola). See the second part of Figure 1.

For more recent work on asymptotes, see [23], which shows that a convex set $C$ has an asymptote if and only if there is a quadratic function that is convex and lower bounded on $C$, but does not attain its infimum.

The second terminology says that (SDP-D) is weakly infeasible. Observe that when (SDP-P) has finite optimal value and the dual (SDP-D) is infeasible, it must be weakly infeasible. Indeed, suppose not; then the alternative system (3.9) has a feasible solution $x$, and adding a large multiple of $x$ to a feasible solution of (SDP-P) proves the latter is unbounded, which is a contradiction.

In more recent work [21] proved that a weakly infeasible SDP over $S^n_+$ has a “small” weakly infeasible subsystem of dimension at most $n-1$. This result was generalized in Corollary 1 in [20] to conic linear programs, using a fundamental geometric parameter of the underlying cone, namely the length of the longest chain of faces.

4 Discussion and conclusion

We presented an elementary, in fact almost purely linear algebraic, proof of a combinatorial characterization of pathological semidefinite systems. En route, we showed how to transform semidefinite
systems into normal forms to easily verify their pathological (or good) behavior. The normal forms also turned out to be useful for a related problem: they allow one to easily verify whether the linear image of $\mathcal{S}_+^n$ is closed.

We conclude with a discussion.

- As we assumed throughout that $(P_{SD})$ is feasible, we may ask: does studying its bad behavior help us understand all pathologies in SDPs?
  
  It certainly helps us understand many. In particular, it helps understand weak infeasibility, a pathology of infeasible SDPs: Remark 2 and Proposition 1 show that all $c$ that make $(SDP-D)$ weakly infeasible are suitable objective functions associated with badly behaved *homogeneous* (hence feasible) systems.

  However, we cannot yet distinguish among bad objective functions; for example, we cannot tell which $c \in \mathbb{R}^m$ gives a finite positive duality gap, and which gives the more benign pathology of zero duality gap coupled with unattained dual optimal value.

Since the interplay of semidefinite programming and algebraic geometry is a very active recent research area (some recent references are [8, 7, 24, 37]), it would be interesting to connect our results to algebraic geometry.

- Let us look again at the semidefinite systems in their normal forms $(P_{SD, \text{bad}})$ and $(P_{SD, \text{good}})$ and note an interesting feature they share. They are both naturally split into two parts:

  - a “Slater part,” namely the system $\sum_{i=1}^k x_i F_i \preceq I_r$, and
  - a “Redundant part,” which corresponds to always zero variables $x_{k+1}, \ldots, x_m$.

In $(P_{SD, \text{bad}})$ the “Redundant part” is responsible for the bad behavior.

In $(P_{SD, \text{good}})$ the “Redundant part” is essentially linear: we can find the corresponding dual variable $Y_{22}$ by solving a system of equations, then doing a linesearch.

- Here (and in [28]) we showed how normal forms of semidefinite systems help to verify their bad or good behavior. In more recent work, such normal forms turned out to be useful for other purposes:

  - to verify the infeasibility of an SDP (see [19]) and
  - to verify the infeasibility and weak infeasibility of conic linear programs: see [20].

- To construct the normal forms, the bulk of the work is transforming the linear map

$$\mathbb{R}^m \ni x \rightarrow A(x) = \sum_{i=1}^m x_i A_i.$$ 

Indeed, operations (3)-(4) of Definition 1 find an invertible linear map $M : \mathbb{R}^m \rightarrow \mathbb{R}^m$ so that $A M$ is in an easier-to-handle form.

Normal forms of linear maps are ubiquitous in linear algebra: see, for example, the row echelon form, or the eigenvector decomposition of a matrix. This work (as well as [19] and [20]) shows that they are also useful in a somewhat unexpected area, the duality theory of conic linear programs.

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References

[1] Alfred Auslender and Marc Teboulle. *Asymptotic cones and functions in optimization and variational inequalities*. Springer Science & Business Media, 2006. 6, 15

[2] George Phillip Barker and David Carlson. Cones of diagonally dominant matrices. *Pacific J. Math.*, 57:15–32, 1975. 15

[3] Alexander Barvinok. *A Course in Convexity*. Graduate Studies in Mathematics. AMS, 2002. 1, 2, 6, 15

[4] Heinz Bauschke and Jonathan M. Borwein. Conical open mapping theorems and regularity. In *Proceedings of the Centre for Mathematics and its Applications* 36, pages 1–10. Australian National University, 1999. 6, 15

[5] Aharon Ben-Tal and Arkadii Nemirovskii. *Lectures on modern convex optimization*. MPS/SIAM Series on Optimization. SIAM, Philadelphia, PA, 2001. 1, 2

[6] Dimitri Bertsekas and Paul Tseng. Set intersection theorems and existence of optimal solutions. *Math. Program.*, 110:287–314, 2007. 6, 15

[7] Avinash Bhardwaj, Philipp Rostalski, and Raman Sanyal. Deciding polyhedrality of spectrahedra. *SIAM J. Opt.*, 25(3):1873–1884, 2015. 20

[8] Grigoriy Blekherman, Pablo Parrilo, and Rekha Thomas, editors. *Semidefinite Optimization and Convex Algebraic Geometry*. MOS/SIAM Series in Optimization. SIAM, 2012. 2, 20

[9] Frédéric J. Bonnans and Alexander Shapiro. *Perturbation analysis of optimization problems*. Springer Series in Operations Research. Springer-Verlag, 2000. 1, 2, 5

[10] Jonathan M. Borwein and Adrian S. Lewis. *Convex Analysis and Nonlinear Optimization: Theory and Examples, Second Edition*. CMS Books in Mathematics. Springer, 2005. 1, 2, 16

[11] Jonathan M. Borwein and Warren B. Moors. Stability of closedness of convex cones under linear mappings. *J. Convex Anal.*, 16(3–4):699–705, 2009. 6, 15

[12] Jonathan M. Borwein and Warren B. Moors. Stability of closedness of convex cones under linear mappings ii. *Journal of Nonlinear Analysis and Optimization: Theory & Applications*, 1(1), 2010. 6, 15

[13] Jonathan M. Borwein and Henry Wolkowicz. Regularizing the abstract convex program. *J. Math. Anal. App.*, 83:495–530, 1981. 15

[14] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004. 1, 2

[15] Didier Henrion and Milan Korda. Convex computation of the region of attraction of polynomial control systems. *IEEE Trans. Autom. Control*, 59(2):297–312, 2014. 6, 15

[16] Victor Klee. Asymptotes and projections of convex sets. *Mathematica Scandinavica*, 8(2):356–362, 1961. 19

[17] Igor Klep and Markus Schweighofer. An exact duality theory for semidefinite programming based on sums of squares. *Math. Oper. Res.*, 38(3):569–590, 2013. 5

[18] Monique Laurent and Frank Vallentin. *Semidefinite Optimization*. Available from “http://homepages.cwi.nl/~monique/master_SDP_2016.pdf”. 1, 2
[19] Minghui Liu and Gábor Pataki. Exact duality in semidefinite programming based on elementary reformulations. *SIAM J. Opt.*, 25(3):1441–1454, 2015. 20

[20] Minghui Liu and Gábor Pataki. Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming. *Math. Program. Ser. A, to appear*, 2017. 16, 18, 19, 20

[21] Bruno Lourenco, Masakazu Muramatsu, and Takashi Tsuchiya. A structural geometrical analysis of weakly infeasible SDPs. *Journal of the Operations Research Society of Japan*, 59(3):241–257, 2015. 19

[22] Zhi-Quan Luo, Jos Sturm, and Shuzhong Zhang. Duality results for conic convex programming. Technical Report Report 9719/A, Erasmus University Rotterdam, Econometric Institute, The Netherlands, 1997. 5

[23] Juan-Enrique Martinez-Legaz, Dominikus Noll, and Wilfredo Sosa. Minimization of quadratic functions on convex sets without asymptotes. *Journal of Convex Analysis*, 25(2):623–641, 2018. 19

[24] Jiawang Nie, Kristian Ranestad, and Bernd Sturmfels. The algebraic degree of semidefinite programming. *Mathematical Programming*, 122(2):379–405, 2010. 20

[25] Gábor Pataki. The geometry of semidefinite programming. In Romesh Saigal, Lieven Vandenberghe, and Henry Wolkowicz, editors, *Handbook of semidefinite programming*. Kluwer Academic Publishers, also available from www.unc.edu/~pataki, 2000. 15

[26] Gábor Pataki. On the closedness of the linear image of a closed convex cone. *Math. Oper. Res.*, 32(2):395–412, 2007. 6, 15, 16

[27] Gábor Pataki. Strong duality in conic linear programming: facial reduction and extended duals. In David Bailey, Heinz H. Bauschke, Frank Garvan, Michel Théra, Jon D. Vanderwerff, and Henry Wolkowicz, editors, *Proceedings of Jonfest: a conference in honour of the 60th birthday of Jon Borwein*. Springer, also available from http://arxiv.org/abs/1301.7717, 2013. 15

[28] Gábor Pataki. Bad semidefinite programs: they all look the same. *SIAM J. Opt.*, 27(1):146–172, 2017. 1, 2, 3, 4, 5, 6, 10, 15, 16, 20

[29] Frank Permenter and Pablo Parrilo. Partial facial reduction: simplified, equivalent sdps via approximations of the psd cone. *Mathematical Programming*, pages 1–54, 2014. 13, 15

[30] Motakuri V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Math. Program. Ser. B*, 77:129–162, 1997. 5

[31] James Renegar. *A Mathematical View of Interior-Point Methods in Convex Optimization*. MPS-SIAM Series on Optimization. SIAM, Philadelphia, USA, 2001. 1, 2

[32] Tyrrel R. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, USA, 1970. 15, 19

[33] Michael J. Todd. Semidefinite optimization. *Acta Numer.*, 10:515–560, 2001. 1, 2

[34] Levent Tunçel. *Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization*. Fields Institute Monographs, 2011. 1, 2, 5, 15

[35] Levent Tunçel and Henry Wolkowicz. Strong duality and minimal representations for cone optimization. *Comput. Optim. Appl.*, 53:619–648, 2012. 5

[36] Lieven Vandenberghe and Steven Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996. 5
[37] Cynthia Vinzant. What is ... a spectrahedron? Notices Amer. Math. Soc., 61(5):492–494, 2014.

[38] Hayato Waki and Masakazu Muramatsu. Facial reduction algorithms for conic optimization problems. J. Optim. Theory Appl., 158(1):188–215, 2013.

[39] Hayato Waki, Maho Nakata, and Masakazu Muramatsu. Strange behaviors of interior-point methods for solving semidefinite programming problems in polynomial optimization. Computational Optimization and Applications, 53(3):823–844, 2012.

[40] Yuzixuan Zhu, Gábor Pataki, and Quoc Tran-Dinh. Sieve-sdp: a simple facial reduction algorithm to preprocess semidefinite programs. Mathematical Programming Computation, 11(3):503–586, 2019.