Determinantal Multivariate Polynomials

Papri Dey*
Department of Electrical Engineering
IIT Bombay, India

Abstract

In this paper, the determinantal multivariate polynomials are the polynomials that can be expressed as the determinant of a definite (monic) symmetric/Hermitian linear matrix polynomial (LMP). These determinantal polynomials can characterize the feasible sets of semidefinite programming problems. We provide a necessary condition for the existence of a symmetric/Hermitian determinantal representation of a multivariate polynomial by establishing a connection between the coefficients of the multivariate polynomial and the eigenvalues of coefficient matrices of the corresponding LMP. We prove that coefficients of a determinantal multivariate polynomial of degree \( d \) can be uniquely determined by the generalized mixed discriminant of coefficient matrices that we define in this paper. We develop an algorithm to determine a monic symmetric/Hermitian determinantal representation of a bivariate polynomial of degree \( d \). Then we propose a heuristic method to obtain a monic symmetric determinantal representation of a multivariate polynomial of degree \( d \).

Keywords. Linear matrix polynomial, determinantal representation, RZ polynomial, semidefinite programming, polynomial optimization.

1 Introduction

One of the main objectives in convex algebraic geometry is to study the geometry of the feasible set of an optimization problem. It is known that if the feasible set of an optimization problem is a definite linear matrix inequality (LMI) representable set, the optimization problem can be transformed into a semidefinite programming (SDP) problem [Ram95], [HV07].

A set \( S \subseteq \mathbb{R}^n \) is said to be LMI representable if

\[
S = \{ \mathbf{x} \in \mathbb{R}^n : L := A_0 + L(\mathbf{x}) := A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \succeq 0 \}    \tag{1}
\]

for some real symmetric matrices \( A_i, i = 0, \ldots, n \) and \( \mathbf{x} = (x_1, \ldots, x_n)^T \). If \( A_0 \succ 0 \) (resp. \( A_0 = I \)), the set \( S \) is called a definite (resp. monic) LMI representable set. By \( A \succ 0(\succeq 0) \) we mean that the matrix \( A \) is positive (semi)-definite. A spectrahedron which is the feasible set of a semidefinite programming (SDP) problem is an another term used for a LMI representable set. One can see [VB96] to study SDP problems in details.

It is proved in [HV07] that if a polynomial \( f(\mathbf{x}) \) is determinantal, then the algebraic interior associated with \( f(\mathbf{x}) \) i.e., the closure of a (arcwise) connected component of \( \{ \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) > 0 \} \) is a spectrahedron. Thus one of the successful techniques to deal with

*Email:papridey@ee.iitb.ac.in, yedirpap@gmail.com
characterizing definite LMI representable sets is to characterize determinantal polynomials. Therefore, in this paper we focus on definite (monic) determinantal representation in order to accomplish the connection between determinantal polynomials and semidefinite programming (SDP) problems.

A polynomial $f(x) \in \mathbb{R}[x]$ is said to have a definite (resp. monic) symmetric /Hermitian determinantal representation if $f(x)$ is the determinant of a definite (resp. monic) symmetric/Hermitian linear matrix polynomial (LMP); i.e.,

$$f(x) = \det(A_0 + x_1 A_1 + x_2 A_2 + \cdots + x_n A_n), \quad (2)$$

where coefficient matrices $A_i$ are symmetric/Hermitian of some order and the constant coefficient matrix $A_0$ is positive definite (resp. identity matrix). The order of the coefficient matrices is called the size of determinantal representation and the size must be greater than or equal to $\deg(f)$.

Determinantal polynomials are special kind of polynomials, known as real zero (RZ) polynomials. A polynomial $f(x) \in \mathbb{R}[x]$ with $f(0) \neq 0$ is said to be real zero (RZ) polynomial [HV07] if its restriction along any line passing through origin has only real roots. The polynomial $f$ is called strictly RZ if all these roots are distinct, for all $x \in \mathbb{R}^n, x \neq 0$. If the polynomial $f(x)$ admits a monic symmetric/Hermitian determinantal representation, then its restriction to any line passing through origin which is univariate polynomial $f(x) = \det(I + t(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n))$ has only real zeros. Thus RZ property of a polynomial is a necessary condition for the existence of monic symmetric/Hermitian determinantal representation.

Helton-Vinnikov have proved that a RZ bivariate polynomial $f(x_1, x_2)$ always admits monic Hermitian as well as symmetric determinantal representations of size $d$ [HV07]. So, RZ property is a necessary and sufficient condition for the existence of a determinantal representation of a bivariate polynomial. The homogenized version of this result is known as Lax conjecture [LPR05]. However, this result is no longer true for a RZ polynomial in more than 2 variables, i.e., it may not be a determinantal polynomial at all. For example, dehomogenized polynomial of Vamos cube $V_8$ is a RZ polynomial without a definite determinantal representation [Bra11]. This leads to the generalized Lax conjecture, for details see [Vin12, KPV15, SP15, NS15]. Hyperbolic polynomials which play an important role in partial differential equations are the homogenized version of RZ polynomials [Brä10].

The authors in [HV07] have raised the attention towards computing such determinantal representations for any bivariate polynomial and then this issue has been widely studied in literature, for example one can see [Dix02, PSV12, Hen10, GKVW14]. To the best of authors’ knowledge nothing is known about multivariate determinantal polynomials except the fact that RZ property is a necessary condition for the existence of determinantal representation of a multivariate polynomial.

In this paper, we establish a connection between the coefficients of a determinantal multivariate polynomial and the eigenvalues of coefficient matrices of corresponding determinantal representation of the given polynomial. We provide analytic expressions of the coefficients of a multivariate determinantal polynomial in terms of the coefficient matrices of the corresponding determinantal representation. We show that the coefficients of a determinantal multivariate polynomial can be uniquely determined by the generalized mixed discriminant of coefficient matrices which is defined in this paper.

In fact, using this result we describe an algebraic method to determine a monic symmetric determinantal representation of size $d$ for a bivariate polynomial of degree $d$ by solving two
Lemma 2.2. For a univariate polynomial of size polynomial $f$ of zeros, i.e., all univariate polynomials. By recalling the definition of RZ polynomial, we know that for any $f$ of the following form

$$f(x) = \det(I + xD_1 + x^2D_2 + \cdots + x^dD_d)$$

where $D_i$ is a diagonal matrix whose diagonal entries are the eigenvalues of $A_i$. So, one can always find a suitable unitary (orthogonal) matrix $U$ such that one of the coefficient matrices becomes diagonal. Without loss of generality, it is enough to consider coefficient matrix $A_1$ associated to $x_1$ as a diagonal matrix $D_1$ and obtain an MHDR (MSDR) of the following form

$$f(x) = \det(I + x_1D_1 + x_2X_1D_2 + \cdots + x_nX_nD_n)$$

where $X_{ij} := U_{ij}(V_{ij}), i \neq j$ is the transition matrix from $D_i$ to $A_{ij} := X_{ij}D_jX_{ij}^*$ and similarly $X_{ij}^* = X_{ji}, i \neq j$ is the transition matrix from $D_j$ to $A_{ji} := X_{ji}^TD_iX_{ij} = X_{ji}D_iX_{ij}$.

Observe that the eigenvalues of the coefficient matrices $A_{ij}$ are nothing but the entries of the diagonal matrices $D_i$ for all $i = 2, \ldots, n$. We explain a technique to determine these diagonal matrices. We take restrictions of the given multivariate polynomial $f(x)$ along each $x_i, i = 1, \ldots, n$ that means we restrict the polynomial along one variable at a time by making the rest of the variables zero and generate $n$ univariate polynomials $f_{x_i} = f(0, \ldots, x_i, \ldots, 0)$. It is known that if a multivariate polynomial $f(x)$ admits an MHDR (MSDR), it is a RZ polynomial. By recalling the definition of RZ polynomial, we know that for any $x \in \mathbb{R}^n$, RZ polynomial $f(x)$ when restricted along any line passing through origin, has only real zeros. So when a RZ polynomial $f(x)$ restricted along $x_i, i = 1, \ldots, n$, each of them has only real zeros, i.e., all univariate polynomials $f_{x_i}$ in $x_i$ have only real zeros.

As a consequence of this result we have a necessary condition for the existence of an MHDR (MSDR) of size equal to the degree of the polynomial for a multivariate polynomial of any degree.

**Lemma 2.1.** If a multivariate polynomial $f(x) \in \mathbb{R}[x]$ of degree $d$ has an MHDR (MSDR) of size $d$, then all the roots of $f_{x_i}$ are real for all $i = 1, \ldots, n$.

More interestingly, the entries of the diagonal matrices $D_i$ can be obtained from the roots of $f_{x_i}$ for all $i = 1, \ldots, n$ by the following Lemmas.

**Lemma 2.2.** For a univariate polynomial $f(x) = \det(xI + A)$ of degree $d$, the reversed linear polynomial $\bar{f}(x) := x^d f(1/x) = \det(I + xA)$ has the same coefficients in the reverse order.
Proof: Say, \( f(x) = \sum_{i=0}^{d} a_{d-i}x^i, a_0 = 1 \), then \( \hat{f} := x^df(1/x) = \text{det}(I + xA) = \sum_{i=0}^{d} a_ix^i, a_0 = 1. \) Since there is a one to one correspondence between the roots of \( f \) and \( \hat{f} \) which is given by \( x \to 1/x \), therefore the roots of \( \hat{f} \) are the reciprocals of the nonzero finite eigenvalues of \( A \). If \( A \) is a singular matrix i.e., it has zero eigenvalues, then the degree of \( \hat{f} \) is dropped by the number of multiplicities of the zero eigenvalues of \( A \), but the coefficients of these two polynomials are same in the reverse order. \( \Box \)

Note that

\[
 f(x) = \text{det}(xI + A) = x^d + E_1(A)x^{d-1} + \cdots + E_d(A)
\]

where \( E_k(A) \) is the sum of \( k \times k \) principal minors of \( A, k = 1, \ldots, d \). If \( \lambda_1, \ldots, \lambda_d \) are the eigenvalues of \( A \in M^{d\times d} \), then the sum of \( k \times k \) principal minors of \( A \) is the \( k \)-th elementary symmetric function of the eigenvalues of \( A \) i.e., \( S_k(\lambda_1, \ldots, \lambda_d) = E_k(A) \) [1190]. The \( k \)-th elementary symmetric function of \( d \) numbers \( \lambda_1, \ldots, \lambda_d, k \leq d \) is defined as \( S_k(\lambda_1, \ldots, \lambda_d) = \sum_{1 \leq t_1 < \cdots < t_k = d} \prod_{j=1}^{k} \lambda_{t_j} \). In particular, \( \text{Tr}(A) = \sum_{i=1}^{d} \lambda_i, \text{det}(A) = \prod_{i=1}^{d} \lambda_i. \) Thus

\[
 \hat{f}(x) = \text{det}(I + xA) = E_d(A)x^d + E_{d-1}(A)x^{d-1} + \cdots + 1
\]

Lemma 2.3. The roots of univariate polynomials \( f_{x_i} \) are the negative reciprocal of entries of diagonal matrices \( D_i \) for all \( i = 1, \ldots, n. \)

Proof: By Lemma 2.2 a univariate polynomial \( f(t) \) has only real zeros and \( f(0) \neq 0 \) if and only if the reversed polynomial \( t^d f(1/t) \) has only real zeros. So, a polynomial \( f(x) \) is a RZ polynomial if and only if for any fixed real vector \( x \in \mathbb{R}^n \) the univariate polynomial \( \hat{f}_x(t) := t^d f(x/t) \) in \( t \) has only real zeros. Thus, if a polynomial \( f(x) \in \mathbb{R}[x] \) is a RZ polynomial, then the associated univariate polynomial of the homogenized polynomial \( \hat{f}_x(t) := t^d f_{x_0}(x_i/t), d \) the degree of the polynomial has only real zeros at point \( x = (0, \ldots, x_i, 0) \in \mathbb{R}^n. \) As \( \text{det}(tI + D_i) = \hat{f}_{e_i}(t), \) here map is \( t \rightarrow -t, \) therefore the roots of the reversed polynomials \( \hat{f}_{e_i}(t) := t^d f_{e_i}(e_i/t) \) the negative of the diagonal elements of \( D_i \) for all \( i = 1, \ldots, n. \) Again by Lemma 2.2, we conclude the result of this lemma since polynomials \( f_{x_i} \) can be viewed as the reversed polynomials of \( \hat{f}_{e_i}(t) \) in one parameter \( t \) instead of \( x_i. \) \( \Box \)

Consequently, we have the following corollary.

Corollary 2.4. The non-zero eigenvalues of coefficient matrices of a determinantal polynomial are the negative reciprocal of the roots of univariate polynomials \( f_{x_i}, i = 1, \ldots, n. \)

Remark 2.5. There are many ways to calculate the roots of a univariate polynomial. One of the popular methods is based on using the companion matrix associated to that polynomial. The eigenvalues of the companion matrix \( C_{\hat{f}_{e_i}(t)} \) associated with polynomial \( \hat{f}_{e_i}(t) \) the roots of the polynomial \( \hat{f}_{e_i}(t) \) since \( \text{det}(tI - C_{\hat{f}_{e_i}(t)}) = \hat{f}_{e_i}(t). \)

Remark 2.6. The importance of this result is that it provides a necessary condition (other than RZ property) for the existence of MHDR (MSDR) of size equal to the degree of the polynomial, so one can easily discard a class of a large number of multivariate polynomials which have no MHDR (MSDR) of size equal to the degree of that polynomial.

3 Generalized Mixed Discriminant of Matrices and Determinantal Polynomials

We introduce the concept of generalized mixed discriminant of \( k(\leq n) \)-tuple of \( n \times n \) matrices based on the notion of mixed discriminant of \( n \)-tuple of \( n \times n \) distinct matrices [BR97].
and \( k(< n) \) -tuple \( n \times n \) distinct matrices in [MISS15]. Note that the generalized mixed discriminant of matrices is defined even if the matrices are not distinct. Then by using the notion of generalized mixed discriminant we prove that the coefficients of a determinantal multivariate polynomial can be uniquely determined in terms of the coefficient matrices of its determinantal representation.

**Definition 3.1.** Consider the \( n \times n \) matrices \( A^{(l)} = (a_{ij}^{(l)}) \) for \( l = 1, \ldots, n \). Pick any \( k \)-tuple matrix \( \underbrace{(A^{(1)}, \ldots, A^{(1)}, A^{(2)}, \ldots, A^{(2)}, \ldots, A^{(n)}, \ldots, A^{(n)})}_{\mu_1, \mu_2, \ldots, \mu_n} \), \( \mu_j \in \{0, 1, \ldots, n\}, 1 \leq k \leq n, \) and \( \mu_1 + \cdots + \mu_n = k \). Then the generalized mixed discriminant (GMD) of the \( k \) tuple of \( n \times n \) matrices is defined as

\[
\hat{D}(A^{(1)}, \ldots, A^{(1)}, A^{(2)}, \ldots, A^{(2)}, \ldots, A^{(n)}, \ldots, A^{(n)}) = \sum_{\sigma \in S[k]} \sum_{\sigma \in S_k(v)} \left| \begin{array}{c} \sigma(1) \\ \vdots \\ \sigma(k) \end{array} \right|_{\alpha_1, \alpha_2, \ldots, \alpha_k}
\]

where \( S_n \) is the set of all permutations on \( \{1, \ldots, n\} \) and \( S[k] \) denotes the set of permutations of order \( k \) which are chosen from the set \( S_n \) such that

\[
\alpha = (\alpha_1, \ldots, \alpha_k) \in S[k] \Rightarrow \alpha_1 < \alpha_2 < \cdots < \alpha_k,
\]

\( v = \{1, \ldots, 1, \ldots, n, \ldots, n\} \) and \( S_k(v) \) is the set of all distinct permutations of \( v \).

In order to derive the analytic expressions for the coefficients of a multivariate determinantal polynomial in terms of the coefficient matrices \( A_i \) s we need to prove the following results.

**Notation:** We follow the notation \( | < \nabla m_{i1}(x_1, \ldots, x_n) \ldots \nabla m_{in}(x_1, \ldots, x_n) > | \) to mean that the determinant of \( M(x_1, \ldots, x_n) \) with \( i \) th row being replaced by \( \nabla M(x_1, \ldots, x_n) \) where \( \nabla \) (the nabla symbol) denotes the vector differential operator.

**Lemma 3.2.** Let \( M(x_1, \ldots, x_n) = (m_{ij}(x_1, \ldots, x_n)) \) be a \( d \times d \) matrix with complex entries. Each entry of this matrix, denoted by \( m_{ij}(x_1, \ldots, x_n) \), \( i, j = 1, \ldots, d \) is a linear polynomial in \( x_1, x_2, \ldots, x_n \) with complex coefficients. Then

\[
| \nabla | M(x_1, x_2, \ldots, x_n) | = \sum_{j=1}^{n} | < \nabla m_{j1}(x_1, x_2, \ldots, x_n) \ldots \nabla m_{jn}(x_1, x_2, \ldots, x_n) > |. \tag{3}
\]

**Proof:** Without loss of generality assume that \( M(x_1, \ldots, x_n) := A_0 + x_1 A_1 + \cdots + x_n A_n \).

If \( d = 2 \), then \( M(x_1, \ldots, x_n) := \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \) and \( | \nabla | M(x_1, \ldots, x_n) | = | < \nabla M_1 | + | < \nabla M_2 |. \)

We prove this lemma by induction on the size \( d \) of matrix \( M(x_1, \ldots, x_n) \).

We assume that the equation (3) is true for all \( M(x_1, \ldots, x_n) \in C^{d \times 1}[x] \). Now we have to show that this is true for \( M(x_1, \ldots, x_n) \in C^{d+1 \times (l+1)}[x] \). Let \( C_{ij} \) be the cofactor of \( m_{1j} \). To derive the determinant of \( M(x_1, \ldots, x_n) \) we consider the Laplace expansion of determinant
along the 1st row of $M(x_1, \ldots, x_n)$. Then we have

$$\nabla |M(x_1, \ldots, x_n)| = \sum_{j=1}^{l+1} \nabla (m_{1j}C_{1j}) = \sum_{j=1}^{l+1} [C_{1j} \nabla m_{1j} + m_{1j} \nabla C_{1j}]$$

$$= |\nabla m_{11}(x_1, \ldots, x_n) \ldots \nabla m_{1(l+1)}(x_1, \ldots, x_n)| + m_{11} \sum_{i=2}^{l+1} |\nabla m_{i2}(x_1, \ldots, x_n) \ldots \nabla m_{i(l+1)}(x_1, \ldots, x_n)| +$$

$$\sum_{j=2}^{l+1} (-1)^{j-1} m_{1j} \sum_{k=2}^{l+1} |\nabla m_{k1}(x_1, \ldots, x_n) \ldots \nabla m_{k(j-1)}(x_1, \ldots, x_n) \nabla m_{k(j+1)}(x_1, \ldots, x_n) \ldots \nabla m_{k(l+1)}(x_1, \ldots, x_n)|$$

$$= |\nabla m_{11}(x_1, \ldots, x_n) \ldots \nabla m_{1(l+1)}(x_1, \ldots, x_n)| + \sum_{j=2}^{l+1} |\nabla m_{j1}(x_1, \ldots, x_n) \ldots \nabla m_{j(l+1)}(x_1, \ldots, x_n)|$$

$$= \sum_{j=1}^{l+1} |< \nabla m_{j1}(x_1, \ldots, x_n) \ldots \nabla m_{jn}(x_1, \ldots, x_n)>|$$

Thus it is true for $(l + 1)$. So, by induction we can conclude that this result is true for any $d$.

\[ \square \]

**Lemma 3.3.** Let $M(x_1, \ldots, x_n) = A_0 + x_1 A_1 + \cdots + x_n A_n$ be a linear matrix polynomial of size $d$. For any $k_j \in \{0, \ldots, d\}$ and $\sum_{j=1}^{n} k_j = l \leq d$

$$\frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1}x_2^{k_2} \ldots x_n^{k_n}} |M(x_1, \ldots, x_n)|_{x_1=0, \ldots, x_n=0} = (k_1! \cdots k_n!) \hat{D}(A_0, \ldots, A_0, A_1, \ldots, A_1 \cdots A_n, \ldots, A_n)$$

**Proof:** By Lemma 3.2 it is clear that L.H.S is equal to sum of determinants of some matrices. These matrices are constructed as follows. In the 1st partial derivative of $M$ with respect to $x_j$, new matrices are constructed such that only one row of $M$ is replaced by the corresponding row of the matrix $A_j$. Using the same logic we claim that in the $k_1 k_2 \ldots k_n$-th derivative of $M$, the $k_1$ rows of $M$ will be replaced by the $k_1$ rows of $A_1$, $k_2$ rows of $M$ will be replaced by $A_2$ and $k_n$ rows will be replaced by $k_n$ rows of $A_n$. If $k_1 + \cdots + k_n = l < d$, then the remaining $d - l$ rows of $M$ will be replaced by the corresponding rows of matrix $A_0$ at $x_1 = \cdots = x_n = 0$. From the Definition 3.1 it is clear that we need to construct such matrices in order to calculate generalized mixed discriminant of matrices. So, the partial derivatives of the determinant of any multivariate linear matrix polynomial can be determined by calculating the generalized mixed discriminant of certain matrices.

Now we need to prove that the number of determinants in both sides of the equality is same. As there are $d$ rows in the linear matrix polynomial $M$, so there are $d$ determinants in $\frac{\partial}{\partial x_j} |M(x_1, \ldots, x_n)|$. If we differentiate it one more time, each determinant provides $(d - 1)$ nontrivial determinants. So, the number of nontrivial determinants in L.H.S is $d(d-1) \ldots (d-k_1+1) \ldots (d-k_1-\cdots-k_n+1)$. Due to Leibnitz product rule for differentiation there is a functional equality or parity among determinants in the expansion of the L.H.S term. Thus there are repeated determinants in the expansion and each distinct determinant must be repeated the same number of times; i.e.; $k_1! \cdots k_n!$. On the other hand, in order to calculate the generalized mixed discriminant of matrices we follow the lexicographic order; i.e., $(i_j < i_k$ if $j < k)$ to choose $k_1, \ldots, k_n$ rows out of $d$ rows. So, there are$$\binom{d}{k_1} \binom{d-k_1}{k_2} \cdots \binom{d-k_1-k_2-\cdots-k_n-1}{k_n}$$
determinants in $\hat{D}(A_0, \ldots, A_0, A_1, \ldots, A_1, \ldots A_n, \ldots, A_n)$. It satisfies the following identity

$$d(d-1) \cdots (d-k_1-k_2-\cdots-k_n+1) = k_1! \cdots k_n! \left(\frac{d}{k_1}\right) \cdots \left(\frac{d-k_1-k_2-\cdots-k_n}{k_n}\right).$$

Hence the desired result follows. \hfill \square

**Remark 3.4.** Similar kinds of results exist in literature [WL10], [BJ09], [FMS16].

**Theorem 3.5.** *(Generalized Mixed Discriminant Theorem)* The coefficients of a multivariate nonhomogeneous polynomial $f(x_1, \ldots, x_n)$ of degree $d$ that admits a monic symmetric or Hermitian determinantal representation are determined by the generalized mixed discriminants of the coefficient matrices $A_i$ as follows. If the degree of a monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} = k(1 + k_2 + \cdots + k_n) = k \leq d$, then the coefficient $a_{k_1 \cdots k_n}$ of $(x_1^{k_1} \cdots x_n^{k_n})$ is given by

$$\hat{D}(A_1, \ldots, A_1, A_2, \ldots, A_2, \ldots A_n, \ldots, A_n).$$

In particular,

1. $\text{Tr} A_i = \text{the coefficient of } x_i \text{ for all } i = 1, \ldots, n$. 
2. $\det A_i = \text{the coefficient of } x_i^d \text{ for all } i = 1, \ldots, n$ where $d$, the degree of the polynomial is equal to the size $d$ of the coefficient matrix.

**Proof:** Say $f(x_1, \ldots, x_n) = \det(I + x_1 A_1 + \cdots + x_n A_n)$. Using multivariate Taylor series coefficient formula the coefficients of $\det(I + x_1 A_1 + \cdots + x_n A_n)$ can be determined by the given formula [Apo74].

$$a_{k_1 k_2 \cdots k_n} = \frac{1}{k_1! k_2! \cdots k_n!} \frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}} |I + x_1 A_1 + \cdots + x_n A_n|_{x_1 = \cdots = x_n = 0}, k_j \in \{0, \ldots, d\}$$

Therefore by Lemma 3.3 we conclude that

$$a_{k_1 k_2 \cdots k_n} = \hat{D}(A_1, \ldots, A_1, A_2, \ldots, A_2, \ldots A_n, \ldots, A_n)$$

\hfill \square

### 4 Computation of Determinantal Representation of Bivariate Polynomials

In this section, we propose a method to determine an MSDR (MHDR) of size $d$ for a bivariate polynomial of degree $d$ by solving a system of polynomial equations. We would like to mention that though the authors in [PSV12] have proposed a method to determine an MSDR by solving polynomial equations, but note that they have not explained any method to obtain the analytic expressions of the coefficients of a determinantal bivariate polynomial in terms of corresponding coefficient matrices. In this paper, we propose an explicit method to express the coefficients of a determinantal multivariate polynomial in terms of coefficient matrices of the corresponding determinantal representation of that polynomial.
We show that from a given multivariate polynomial \( f(x) \in \mathbb{R}[x] \) of degree \( d \) we can always determine the eigenvalues of coefficient matrices (which are symmetric or Hermitian in our context) of monic determinantal representations of size \( d \) if they exist. Moreover, by our method the diagonal entries of the coefficient matrices can be found by solving systems of linear equations without assuming that the eigenvalues of coefficient matrices are distinct.

4.1 Determining MSDR (MHDR) for Bivariate Polynomials

In this subsection we generalize the idea mentioned above for any bivariate polynomial. In order to determine MSDR (MHDR) we need to find roots of two univariate polynomials. If all of them are real, then solve a system of linear equations followed by finding the real variety of an ideal. We show that the concerned ideal is generically a zero dimensional ideal. Then we propose an algorithm to determine MSDR (MHDR) of size \( d \) for any bivariate polynomial of degree \( d \).

- Construct univariate polynomials \( f_{x_i}, i = 1, 2 \) where \( f_{x_i} := f(0, \ldots, x_i, 0) \), which are constructed by taking restriction of given polynomial \( f(x) \) along each of the coordinates \( x_i \). By Lemma 2.3 the roots of \( f_{x_i} \) are negative reciprocal of entries of diagonal matrices \( D_i \). Find the diagonal entries of \( D_1 \) and \( D_2 \). Say \( D_1 = \text{Diag}(r_1, r_2, \ldots, r_d) \) and \( D_2 = \text{Diag}(s_1, s_2, \ldots, s_d) \). Diagonal entries of \( D_2 \) are actually the eigenvalues of \( A_{12} \).

- Using the generalized mixed discriminant of coefficient matrices, by the Theorem 3.5 we derive the analytic expressions for each coefficient of \( f(x) \) in terms of the entries of \( D_1 \) and \( A_{12} \).

- Find the diagonal entries of coefficient matrix \( A_{12} \): They can be determined by solving a system of linear equations of the form \( Gz_2 = y_2 \) where \( y_2 \) denotes the vector consisting of the diagonal entries of \( A_{12} \), \( z_2 = \begin{bmatrix} \text{coeff of } x_1x_2 \\ \text{coeff of } x_1^{d-1}x_2 \\ \vdots \\ \text{coeff of } x_1x_2 \\ \vdots \\ \text{coeff of } x_1^{d-1}x_2 \end{bmatrix} \) and \( G = (g_{ij}) = \begin{bmatrix} 1 \\ \sum_{i=2}^d r_i \\ \sum_{i=1, i \neq 2}^d r_i \\ \vdots \\ \sum_{i_k, i_l 
eq 1, i_k < i_l}^d r_k r_l \\ \vdots \\ r_2r_3 \ldots r_d \end{bmatrix} \end{bmatrix} \). Note that the number of mixed monomials which are of the form \( x_1^{\alpha_1}x_2 \) with \( 0 \leq \alpha_1 \leq d - 1 \) is \( d \) for a bivariate polynomial and the diagonal entries of coefficient matrix \( A_{12} \) is also \( d \). So, if monic symmetric/Hermitian determinantal representation exists for a bivariate polynomial, one can always determine the diagonal entries of coefficient matrix \( A_{12} \) by solving a system of \( d \) linear equations in \( d \) unknowns. However, they may not be unique.
Lemma 4.1. A matrix $G = \begin{bmatrix} \sum_{i=2}^{d} r_i & 1 & \cdots & 1 \\ \sum_{i,k,i \neq k} r_i r_{ik} & \sum_{i=1,i \neq 2}^{d} r_i & \cdots & \sum_{i,k,i \neq d} r_i r_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ r_2 r_3 \ldots r_d & r_1 r_3 \ldots r_d & \cdots & r_1 \ldots r_{d-1} \end{bmatrix}$ is invertible if and only if $r_i$’s, $i = 1, \ldots, d$ are all distinct.

Proof: As $\det(G) = (r_1 - r_2)(r_1 - r_3) \ldots (r_1 - r_d) \ldots (r_2 - r_3) \ldots (r_2 - r_d) \ldots (r_d - r_{d-1})$, so the result follows. \hfill \square

Using the Lemma 4.1, we prove that if all the eigenvalues of one coefficient matrix are distinct, the diagonal entries of other coefficient matrix can be uniquely determined. Indeed, the diagonal entries of $A_{12}$ are uniquely determined up to ordering if all the diagonal entries of coefficient matrix $D_1$ are distinct. As the diagonal entries of $A_{12}$ are evaluated, so the number of unknown variables which are the off-diagonal entries of the symmetric coefficient matrix $A_{12}$ is $(\binom{d+1}{2}) - d = \binom{d}{2}$.

- Consider the expressions associated with the coefficients of monomials $x_1^{\alpha_1} x_2^{\alpha_2}, 0 \leq \alpha_1 \leq d - 2, 2 \leq \alpha_2 \leq d$ by the Theorem [35]. The number of monomials of the the form $x_1^{\alpha_1} x_2^{\alpha_2}$, where $0 \leq \alpha_1 \leq d - 2$ and $2 \leq \alpha_2 \leq d$ is $(d - 1 + d - 2 + d - 3 + \cdots + 2 + 1) = \binom{d}{2}$.

Then by comparing the expressions associated with the coefficients of the remaining monomials $x_1^{\alpha_1} x_2^{\alpha_2}$, where $0 \leq \alpha_1 \leq d - 2$ and $2 \leq \alpha_2 \leq d$, we obtain generically a zero dimensional ideal $\mathcal{I}$ generated by $\binom{d}{2}$ polynomial equations in $\binom{d}{2}$ parameters.

- Find an element of the real variety $V_\mathbb{R}(\mathcal{I})$. This can be obtained by using available softwares like Singular, Bertini, Maple and Sage. For instance, using Sage find Groebner basis of the ideal $\mathcal{I}$ and then find real roots of that Groebner basis. Check whether there exists at least one real root, otherwise exit-no MSDR (MHD) of size $d$ possible. Construct off diagonal entries of coefficient matrix $A_{12}$.

Computation of MHD

As it is known that for a bivariate polynomial, MHD exists if and only if MSDR exists, so we focus on constructing MSDR first and then we would like to explain the method to determine MHD of size $d$ from a given MSDR. We explain the idea for a quartic bivariate polynomial and the method can be generalized by choosing suitable phases for bivariate polynomials of degree $d$.

For a quartic bivariate polynomial by choosing the cosine of the following phases equal to one; i.e., by choosing

$$\cos(\theta_1 + \theta_4 - \theta_2) = 1, \cos(\theta_1 + \theta_5 - \theta_3) = 1, \cos(\theta_2 + \theta_6 - \theta_3) = 1, \cos(\theta_4 + \theta_6 - \theta_5) = 1$$

$$\cos(\theta_1 + \theta_4 + \theta_6 - \theta_3) = 1, \cos(\theta_1 + \theta_5 - \theta_2 - \theta_6) = 1, \cos(\theta_2 + \theta_5 - \theta_3 - \theta_4) = 1$$

we obtain infinitely many MHDs such that the magnitudes of the off diagonal entries of Hermitian coefficient matrices associated with $x_2$ variable is same as the off diagonal entries of the corresponding symmetric coefficient matrix of the given MSDR. As it is shown in [Chap-4, [Dev17]] that two unitarily equivalent MHDs are same up to the phases of the off-diagonal entries, so all of these MHDs are unitarily equivalent. Therefore for the different choice of phases such that cosine values of the mentioned phases equal to one each equivalence class of
an MHDR contains equivalence class of corresponding MSDR. By choosing the cosine values of the mentioned phases lying between \((-1, 1)\) we can generate a different ideal and each point of the real variety of this ideal provides an MHDR of size 4. Similarly we can deal with bivariate polynomials of any degree. Therefore, there is a continuum of unitarily non-equivalent MHDRs for a bivariate polynomial of degree \(d\).

Thus we propose the following algorithm to determine an MSDR of size \(d\) for bivariate polynomial by solving a system of polynomial equations.

\[ \text{Algorithm 1 Algorithm to Determine an MSDR of size } d \text{ for Bivariate Polynomial of degree } d \]

Input: Bivariate polynomial
Output: Coefficient matrices \(D_1, A_{12}\) such that
\[ f(x) = \det(I + x_1D_1 + x_2A_{12}) \]

1. Construct the univariate polynomials \(f_{x_i}, i = 1, 2\).
2. Determine the eigenvalues of coefficient matrices \(D_1, A_{12}\).
3. Check that the eigenvalues of coefficient matrices \(D_1, A_{12}\) are real. If not, exit-no MSDR of size \(d\) possible.
4. Obtain analytic expressions for the vector coefficient of mixed monomials \(x_1^{\alpha_1}x_2^{\alpha_2}\), where \(1 \leq \alpha_1 \leq d - 1\) and \(1 \leq \alpha_2 \leq d - 1\) by the Theorem 3.5.
5. Find the diagonal entries of the matrix \(A_{12}\) by solving a system of linear equations of the form \(Gy = z\).
6. Comparing the expressions associated with the coefficients of monomials \(x_1^{\alpha_1}x_2^{\alpha_2}\), where \(0 \leq \alpha_1 \leq d - 2\) and \(2 \leq \alpha_2 \leq d\), obtain generically a zero dimensional ideal \(I\) generated by \(\binom{d}{2}\) polynomial equations in \(\binom{d}{2}\) parameters.
7. Find an element of real variety \(V_R(I)\).
8. Check whether there exists at least one REAL root, otherwise exit-no MSDR of size \(d\) possible.
9. Construct \(D_1\) and \(A_{12}\).

Quartic Bivariate Polynomials

Consider a determinantal quartic bivariate polynomial
\[ f(x) := f_{40}x_1^4 + f_{31}x_1^3x_2 + f_{22}x_1^2x_2^2 + f_{13}x_1x_2^3 + f_{30}x_1^3 + f_{21}x_1^2x_2 + f_{12}x_1x_2^2 + f_{20}x_2^2 + f_{11}x_1x_2 + f_{02}x_2^2 + f_{10}x_1 + f_{01}x_2 + 1 \]

such that \(f(x) = \det(I + x_1D_1 + x_2A_{12})\), where \(A_{12} = \begin{bmatrix} a & e & f & g \\ e & b & h & k \\ f & h & c & l \\ g & k & l & d \end{bmatrix}\).

At first, determine the eigenvalues of \(D_1\) and \(A_{12}\) by Lemma 2.3. Next, determine the diagonal entries of \(A_{12}\) by solving a system of linear equations of the form \(Gy = z\). Then
using the generalized mixed discriminant of coefficient matrices, by the Theorem 3.5 we have the following relations between the coefficients of \( f(x) \) and the entries of \( D_1 \) and \( A_{12} \).

\[
\text{coeff of } x_2^2 = ab + ac + ad + bc + bd + cd - e^2 - f^2 - g^2 - h^2 - k^2 - l^2
\]
\[
\text{coeff of } x_1 x_2^2 = d_1 (bd + cd + bc) + d_2 (ad + cd + ac) + d_3 (ad + bd + ab) + d_4 (ab + ac + bc) - e^2 (d_3 + d_4) - f^2 (d_2 + d_4) - g^2 (d_2 + d_3) - h^2 (d_1 + d_4) - k^2 (d_1 + d_3) - l^2 (d_1 + d_2)
\]
\[
\text{coeff of } x_1^2 x_2^2 = d_1 d_2 (cd - l^2) + d_1 d_3 (bd - k^2) + d_1 d_4 (bc - h^2) + d_3 d_4 (ab - e^2) + d_2 d_3 (ad - g^2) + d_2 d_4 (ac - f^2)
\]
\[
\text{coeff of } x_3^2 = (abc + acd + abd + bcd) - e^2 (c + d) - f^2 (b + d) - g^2 (b + c) - h^2 (a + d) - k^2 (a + c) - l^2 (a + b) + 2 (hkl + fgl + egk + efh)
\]
\[
\text{coeff of } x_1 x_3^2 = d_1 (bcd + 2hkl) + d_2 (acd + 2fgl) + d_3 (abd + 2egk) + d_4 (abc + 2efh) - e^2 (d_3 d + d_4 c) - f^2 (d_2 d + d_4 b) - g^2 (d_2 c + d_3 b) - h^2 (d_4 a + d_1 c) - k^2 (d_3 a + d_1 c) - l^2 (d_1 b + d_2 a)
\]
\[
\text{coeff of } x_1^2 = abcd - cde^2 - bdf^2 - bge^2 - adh^2 - aeh^2 - abf^2 + 2 (fglb + hkl + egk + efh)
\]
\[
+ e^2 l^2 + f^2 k^2 + g^2 h^2 - 2 (ef kl + eg lh + ef k h) + 2 (hkl + fgl + egk + efh)
\]

By comparing the expressions of the coefficients of monomials \( x_2^2, x_1 x_2^2, x_1^2 x_2^2, x_1^3 x_2^2, x_1 x_3^2, x_2^3 \) we obtain an ideal \( \mathcal{I} \) generated by three quadratic, two cubic and one quartic equations. As the diagonal entries of \( D_1 \) and \( A_{12} \) are known and diagonal entries of coefficient matrix \( D_1 \) are generically distinct, so we obtain the zero dimensional ideal \( \mathcal{I} \) generated by six polynomials in six parameters.

In order to compute \( A_{12} \) one needs to find (magnitude of) off-diagonal entries of \( A_{12} \) that can be obtained by finding an element of the real variety \( V_2(\mathcal{I}) \).

We illustrate our idea through an example.

**Example 4.2.** Consider a Helton-Vinnikov curve

\[
f(x_1, x_2) = 1/2 x_1^4 + 1/2 x_2^4 - 1.5 x_1^2 - 1.5 x_2^2 + 1/2 x_1^3 x_2^2 + 1
\]
 whose homogeneous version is given in the paper [PSV11]. The diagonal entries of \( D_1 \) and \( D_2 \) are \( (1, 1/\sqrt{2}, -1/\sqrt{2}, -1) \) and \( (0, 0, 0, 0) \) respectively. Derive analytic expressions of the coefficients in terms of the coefficient matrices \( D_1 \) and \( A_2 = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix} \) as follows:

\[
\text{coeff of } x_2^2 = -a^2 - b^2 - c^2 - d^2 - e^2 - f^2
\]
\[
\text{coeff of } x_1 x_2^2 = 1.7071 a^2 + .2929 b^2 - .2929 c^2 - 1.7071 f^2
\]
\[
\text{coeff of } x_1^2 x_2^2 = - .7071 a^2 + .7071 b^2 + .5 c^2 + d^2 + .7071 e^2 - .7071 f^2
\]
\[
\text{coeff of } x_3^2 = 2 (def + bcf + ace + abd)
\]
\[
\text{coeff of } x_1 x_3^2 = 2 def + 1.4142 bc f - 1.4142 ace - 2 abd
\]
\[
\text{coeff of } x_4^2 = a^2 f^2 + b^2 e^2 + c^2 d^2 - 2 (abe f + acdf + bcd e)
\]

Solving the system of six equations in six unknowns we derive one of the possible represen-
tations as follows:

\[ f(x_1, x_2) = \det(I + x_1D_1 + x_2D_2 V^T_1 + \cdots + x_nD_n V^T_1) = \det(I + x_1D_1 + x_2A_{12} \cdots + x_nA_{1n}) \]

where \( V_{ij} \) is the transition matrix from \( A_{ij} := V_{ij}D_j V^T_{ij} \) and similarly \( V^T_{ij} = V_{ji}, i \neq j \) is the transition matrix from \( D_j \) to \( A_{ji} := V^T_{ij}D_i V_{ij} = V_{ji}D_i V_{ji} \). In order to determine an MSDR of size \( d \) for a multivariate polynomial of degree \( d \) we do the following steps.

- Generate \( n \) univariate polynomial \( f_{x_i} := f(0, \ldots, x_i, \ldots, o) \) from the given polynomial by taking its restriction along each of \( x_i, i = 1, \ldots, n \)-th coordinates. By Lemma 2.1 if a multivariate polynomial \( f(x) \in \mathbb{R}[x] \) of degree \( d \) has an MSDR (MHDR) of size \( d \), then all the roots of \( f_{x_i} \) are real for all \( i = 1, \ldots, n \). This provides a necessary condition for the existence of an MSDR of a multivariate polynomial.

- Determine the eigenvalues of coefficient matrices by using Lemma 2.3. The roots of univariate polynomials \( f_{x_i} \) are the negative reciprocal of entries of diagonal matrices \( D_i \) for all \( i = 1, \ldots, n \). The entries of \( D_i \) are the eigenvalues of coefficient matrices \( A_{ji} \) for all \( i = 2, \ldots, n \). Say \( D_1 = \text{Diag}(r_1, r_2, \ldots, r_d) \).

- The diagonal entries of a coefficient matrix \( A_{ji} \) can be determined by solving a system of linear equations of the form \( Gy_i = z_i \) for all \( i = 1, \ldots, d \) where \( y_i \) denotes the vector consisting of the diagonal entries of \( A_{ji} \), \( z_i = \begin{bmatrix} \text{coef of } x_i \\ \text{coef of } x_1x_i \\ \vdots \\ \text{coef of } x_1^{d-1}x_i \end{bmatrix} \) and \( G = (g_{ij}) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i=1}^d r_i & \sum_{i \neq j}^d r_i & \cdots & \sum_{i=1}^{d-1} r_i \\ \sum_{i \neq j, i < j} r_{ik}r_{ij} & \cdots & \sum_{i \neq j, i < j} r_{ik}r_{ij} & \cdots & \sum_{i \neq j, i < j} r_{ik}r_{ij} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_2r_3 \cdots r_d & r_1r_3 \cdots r_d & \cdots & r_1 \cdots r_{d-1} \end{bmatrix} \).

By Lemma 4.1 the diagonal entries of coefficient matrices of a determinantal multivariate polynomial can be uniquely determined if and only if one of the coefficient matrices have all distinct eigenvalues.

5 Computation of Determinantal Representation of Multivariate Polynomials

In this section, we propose a heuristic method to determine an MSDR of size \( d \) for a multivariate polynomial of degree \( d \). Here we assume the eigenvalues of all the coefficient matrices are distinct and the reason is explained later. If a multivariate polynomial of degree \( d \) admits an MSDR of size \( d \), then without loss of generality we can make the coefficient matrix associated with \( x_1 \) variable a diagonal matrix and it is discussed in section 2 that the polynomial can be expressed as

\[ f(x) = \det(I + x_1D_1 + x_2V_{12}D_2 V^T_{12} + \cdots + x_nD_n V^T_{1n}) = \det(I + x_1D_1 + x_2A_{12} \cdots + x_nA_{1n}) \]
Let matrix $D_{\pm}$ denotes a diagonal matrix with diagonal entries being $\pm 1$, i.e.,

$$D_{\pm} = \begin{bmatrix} 
\pm 1 & 0 & \ldots & 0 \\
0 & \pm 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \pm 1 
\end{bmatrix},$$

known as signature matrix.

**Lemma 5.1.** If all the eigenvalues of a diagonal matrix $D$ are distinct, matrix $D$ is invariant under pre and post multiplication by an orthogonal matrix $W$ if and only if $W = D_{\pm}$ where $D_{\pm}$ is a signature matrix.

**Proof:** A diagonal matrix $D$ is invariant under pre and post multiplication by the orthogonal matrix, i.e., $WDW^T = D$ if and only if $D$ and $W$ commute if and only if they have a common eigenvectors. Since all the eigenvalues of $D$ are distinct, so the corresponding non-degenerate eigenvectors are unique up to the sign. Therefore, orthogonal matrix $W$ must be a signature matrix, i.e., $W = D_{\pm}$.

From a given $n$-variate polynomial construct $\binom{n}{2}$ bivariate polynomials $f_{x_i x_j}, i, j = 1, \ldots, n, i < j$ by making $n - 2$ variables zero at a time. Determine an MSDR of size $d$ for those $\binom{n}{2}$ bivariate polynomials and then check the compatibility condition. Note that determinantal representation for a bivariate polynomial may not be unique up to equivalence class.

**Compatibility Conditions:** Therefore, we need to define compatibility conditions among transition matrices to choose a suitable $n$-tuple of $(V_{12}, V_{13}, \ldots, V_{1n})$. Using the method to determine MSDR for a bivariate polynomial we find the orthogonal matrices $V_{1i}, V_{ij}, V_{ij}$ such that the polynomials $f_{x_1 x_i} = \det(I + x_1 D_1 + x_i V_{1i} D_i V_{1i}^T), f_{x_1 x_j} = \det(I + x_1 D_1 + x_j V_{1j} D_j V_{1j}^T)$, and $f_{x_i x_j} = \det(I + x_i D_i + x_j V_{ij} D_j V_{ij}^T)$. Observe that

$$\det(I + x_i D_i + x_j V_{ij} D_j V_{ij}^T) = \det(I + x_i V_{1i} D_i V_{1i}^T + x_j V_{1j} D_j V_{1j}^T V_{ij} V_{ij}^T V_{1i}^T).$$

So, by Lemma 5.1 the triplet $(V_{1i}, V_{ij}, V_{ij})$ is compatible if and only if $V_{ij} = D_{\pm} V_{1i} V_{ij}$.

However, it is shown in [PV13] that if the eigenvalues of all coefficient matrices are distinct, finitely many orthogonally non-equivalent MSDRs of size $d$ are possible for a bivariate polynomial of degree $d$ and the number of non-equivalent real definite representations is precisely $2^{\binom{d-1}{2}}$. On other hand, if a multivariate polynomial $f(x)$ is determinantal, $\binom{n}{2}$ bivariate polynomials $f_{x_i x_j}, i, j = 1, \ldots, n, i < j$ are also determinantal, though the converse is not true. So, it is evident that there are finitely many orthogonally non-equivalent MSDRs for a multivariate polynomial if the eigenvalues of all the coefficient matrices are distinct. So, the assumption of distinct eigenvalues of all coefficient matrices make sure that the process of testing compatibility condition will end at finite steps.

The compatibility condition on transition matrices is not sufficient condition for the existence of an MSDR for a multivariate polynomial as it does not ensure to satisfy the coefficients of monomials in more than two variables. In order to find compatible $n$-tuple $(A_{12}, \ldots, A_{1n})$ of coefficient matrices of a determinantal representation of a multivariate polynomial we use the following iterative process. In order to determine a determinantal representation for the part of $f(x)$ which is a $d$ degree polynomial in three variables we want to find a 3 tuple compatible coefficient matrices. Without loss of generality we choose one possible coefficient matrix $A_{12}$ out of $2^{\binom{d-1}{2}}$ choices and do the
experiment as follows. Suppose we want to find $A_{13}$ such that $(D_1, A_{12}, A_{13})$ provides a determinantal representation for polynomial $f_{x_1x_2x_3} := f(x_1, x_2, x_3, 0, \ldots, 0)$ of $f(x)$.

Note that the diagonal entries of matrix $A_{13}$ are uniquely determined by Lemma 4.1. We determine the set of $\binom{d}{2}$ off-diagonal entries of $A_{13}$ by solving another system of linear equations associated with the coefficients of monomials $x_2x_3, \ldots, x_2^{d-1}x_3, x_1^{\alpha_1}x_2^{\alpha_2}x_3$, $1 \leq \alpha_1 \leq d-2, 1 \leq \alpha_2 \leq d-2, \alpha_1 + \alpha_2 \leq d-1$. The number of such monomials is $\binom{d}{2}$. If the system of $\binom{d}{2}$ linear equations in $\binom{d}{2}$ variables is inconsistent, start with another $A_{12}$. Otherwise, we repeat the same method by including one more variable at each step to get a $n$-tuple compatible coefficient matrices. Note that generically the set of off-diagonal entries of $A_{13}$ is uniquely determined when the eigenvalues of coefficient matrices are distinct for a fixed $A_{12}$. If the method fails for all possible coefficient matrix $A_{12}$, we declare that the multivariate polynomial $f(x)$ has no MSDR of size $d$.

6 Conclusion

We have introduced the notion of generalized mixed discriminant of $k(\leq n)$-tuple $n \times n$ matrices. Using this concept we have established a connection between the coefficients of a multivariate determinantal polynomials and the coefficient matrices of the corresponding determinantal representation for that multivariate polynomial. This connection enables us to obtain analytic expressions for the coefficients of a multivariate polynomial in terms of coefficient matrices. We have developed an algorithm to determine an MSDR (MHDR) of size $d$ for a bivariate polynomial of degree $d$ by solving a system of polynomial equations followed by finding roots of two univariate polynomials and solving two systems of linear equations. Finally we have proposed a heuristic method to compute an MSDR for a multivariate polynomial of degree $d$ if it exists.

Acknowledgements

As this work is a part of the author’s PhD work, so the author would like to thank her supervisor Prof. Harish K. Pillai and the author is grateful to CSIR for providing financial assistance in the form of JRF and SRF for carrying out her doctoral work in IIT Bombay.

References

[Apo74] Tom M Apostol. Mathematical analysis. Addison Wesley Publishing Company, 1974.

[BJ09] Rajendra Bhatia and Tanvi Jain. Higher order derivatives and perturbation bounds for determinants. Linear Algebra and its Applications, 431(11):2102–2108, 2009.

[BR97] R. B. Bapat and T. E. S. Raghavan. Nonnegative Matrices and Applications. Cambridge University Press, 1997.

[Bräi0] Petter Brändén. Notes on hyperbolicity cones, 2010.

[Bra11] Petter Brasden. Obstruction to determinantal representability. Advances in Mathematics, 226:1202–1212, 2011.
References

[Dey17] Papri Dey. *Monic Symmetric/Hermitian Determinantal Representations of Multivariate polynomials*. PhD thesis, 2017.

[Dix02] Alfred Cardew Dixon. Note on the reduction of a ternary quantic to a symmetrical determinant. In *Proc. Cambridge Philos. Soc*, volume 5, pages 350–351, 1902.

[FMS16] D Florentin, V Milman, and R Schneider. A characterization of the mixed discriminant. *Proceedings of the American Mathematical Society*, 144(5):2197–2204, 2016.

[GKVVW14] Anatolii Grinshpan, Dmitry S Kaliuzhnyi-Verbovetskyi, Victor Vinnikov, and Hugo J Woerdeman. Stable and real-zero polynomials in two variables. *Multidimensional Systems and Signal Processing*, 27(1):1–26, 2014.

[Hen10] Didier Henrion. Detecting rigid convexity of bivariate polynomials. *Linear Algebra and its Applications*, 432:1218–1233, 2010.

[HJ90] R A Horn and C R Johnson. *Matrix Analysis*. Cambridge University Press, 1990.

[HV07] J.W. Helton and Vinnikov. Linear matrix inequality representation of sets. *Communications on Pure and Applied Mathematics*, 60:654–674, 2007.

[KPV15] Mario Kummer, Daniel Plaumann, and Cynthia Vinzant. Hyperbolic polynomials, interlacers, and sums of squares. *Mathematical Programming*, 153(1):223–245, 2015.

[LPR05] A. S. Lewis, P. A. Parrilo, and M. V. Ramana. The lax conjecture is true. *Proceedings of The American Mathematical Society*, 133:2495–2499, 2005.

[MSS15] Adam Marcus, Daniel A Spielman, and Nikhil Srivastava. Interlacing families ii: Mixed characteristic polynomials and the kadison-singer problem. *Annals of Mathematics*, 182:327–350, 2015.

[NS15] Tim Netzer and Raman Sanyal. Smooth hyperbolicity cones are spectrahedral shadows. *Mathematical Programming*, 153(1):213–221, 2015.

[PSV11] Daniel Plaumann, Bernd Sturmfels, and Cynthia Vinzant. Quartic curves and their bitangents. *Journal of Symbolic Computation*, 46(6):712–733, 2011.

[PSV12] Daniel Plaumann, Bernd Sturmfels, and Cynthia Vinzant. Computing linear matrix representations of helton-vinnikov curves. In *Mathematical methods in systems, optimization, and control*, pages 259–277. Springer, 2012.

[PV13] Daniel Plaumann and Cynthia Vinzant. Determinantal representations of hyperbolic plane curves:an elementary approach. *Journal of Symbolic Computation*, 2013.

[Ram95] Motakuri Ramana. Some geometric results in semidefinite programming. *Journal of Global Optimization*, 7:33–50, 1995.
[SP15] James Saunderson and Pablo A Parrilo. Polynomial-sized semidefinite representations of derivative relaxations of spectrahedral cones. Mathematical Programming, 153(2):309–331, 2015.

[VB96] Lieven Vandenberghe and Stephen Boyd. Semidefinite programming. SIAM review, 38(1):49–95, 1996.

[Vin12] Victor Vinnikov. Lmi representations of convex semialgebraic sets and determinantal representations of algebraic hypersurfaces: Past, present, and future. Operator Theory: Advances and Applications, 222:325–348, 2012.

[WL10] Yan Wu and Phillip D Lorren. On the characteristic polynomail of regular linear matrix pencil. Alexandria Journal of Mathematics, 2010.