EDC3: Ensemble of Deep-Classifiers using Class-specific Copula functions to Improve Semantic Image Segmentation

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Abstract—In the literature, many fusion techniques are registered for the segmentation of images, but they primarily focus on observed output or belief score or probability score of the output classes. In the present work, we have utilized inter source statistical dependency among different classifiers for ensembling of different deep learning techniques for semantic segmentation of images. For this purpose, in the present work, a class-wise Copula-based ensembling method is newly proposed for solving the multi-class segmentation problem. Experimentally, it is observed that the performance has improved more for semantic image segmentation using the proposed class-specific Copula function than the traditionally used single Copula function for the problem. The performance is also compared with three state-of-the-art ensembling methods.

Index Terms—Copula, image segmentation, multiclass classification, copula ensembling, belief score, semantic segmentation

I. INTRODUCTION

IMAGE segmentation techniques partition an image into multiple meaningful regions based on inherent similarities. Segmentation helps to locate objects and boundaries in an image by labeling each pixel so that pixels with the same characteristics have the same label (see Fig. 1). Among different types of segmentations[1][2][3], semantic segmentation [4][5] where each pixel is associated with a class is more challenging. But semantic segmentation is very much essential in different domains like autonomous driving [6], medical image analysis [7][8], object detection and recognition [9], traffic control [10], video surveillance [11] etc. Improving the accuracy of segmentation methods is always a challenge to the research community, and it has been explored heavily during the past few decades. Ensembling of multiple segmentation models is one of the popular techniques which is generally exercised by the researchers to improve the results. Various techniques are found in the literature for fusing data obtained from multiple classifiers [12]. For example, majority voting [13][14][15][16] is the most popular fusion technique. Other ensembling techniques like maximum, minimum, median, weighted [17], average [18], and product [19], etc. of the different classification scores are also popularly used in the literature. Rank-based method [20], the Bayesian approach [15][16], the Dempster-Shafer theory [16], fuzzy-based approaches [21][22][23], probability-based schemes like Linear Opinion Pool [24], Beta-transformed linear opinion pool [25], fusion based on simple logit model [26] etc. are some commonly used techniques in recent times. However, none of them addresses the inter source-statistical dependence, which plays an essential role during the fusion of data, as mentioned in [27].

A Copula based modeling technique is proposed in the present work to use these dependencies. “Copula,” which is a Latin word meaning “a link or a bond,” is a useful statistical tool for determining dependence between multivariate random variables. Copulas are functions that represent the relationship between Multivariate random variables and their marginals. This approach is more flexible than using a single multivariate distribution for multivariate data. In our proposed model, we have obtained the joint probability density of belief scores obtained from different classifiers for each pixel of an image and estimate the fused probability score using Bayes’ theorem.

Copula modeling is used in various fields of research domains like hydrology [28], medical diagnosis [29][30][31], climate and weather research [32], data mining [33] etc. [34][35]. Though Copula is heavily used for identifying risk factors in the finance sector, its applicability is increasing day by day in other kinds of tasks such as classification [36] and evolutionary computation [37]. Even classifier fusion approaches using Copula are also present in the literature, but to the best of our knowledge, they are mainly used for binary classification problems.

Our main contributions in this paper are (1) Use of Copula functions for the first time for ensembling of decisions of different deep learning models for semantic image segmentation, i.e., multi-class classification at pixel level; (2) Development of flexible class-specific Copula based ensemble model for a multi-class problem. A graphical overview of our proposed model is shown in Fig. 2 and Fig. 3.

The remainder of this paper is arranged as follows. In section [1], we formulate our problem. In section [II], we briefly describe all mathematical prerequisites needed for our model. In section [IV], we discuss various methods for fitting a copula to a data. We have presented our experimental setup, results obtained, and analysis of results has been given in section [V] and the conclusion about the paper is given in section [VI].
II. PROBLEM FORMULATION

Image segmentation can be treated as a pixel-level classification problem. Let, the height and width of an image $I_{x,y}$ is represented by $H$ and $W$. After segmentation, a model or a classifier gives an output of size $H \times W \times M$, where $M$ is the total number of associated classes for each pixel. So for each pixel, the model returns an array of $M$ values, which represents the model’s belief scores or confidence scores about that pixel to be in those $M$ classes.

Suppose, there are $L$ number of segmentation models or classifiers. So for each pixel of an image, these models generate belief scores $p_{i}^{(j)}$, $i = 1,2 \ldots ,L; j = 1,2 \ldots M$, which represents their beliefs on a single class $j$. Our objective is to find a function $g : [0,1]^{L} \rightarrow [0,1]$ to compute the fused belief score $p_{i}^{(j)} := g(p_{1}^{(j)}, p_{2}^{(j)}, \ldots ,p_{L}^{(j)})$, which will be the final belief score for the class $j$. To evaluate the mapping $g$, we use $T$ number of training examples and their ground truth labels for each pixel. Most of the fusion techniques assume that participating classifiers are independent of each other. But in real scenarios, there might have some statistical dependence among those classifiers when they are trained on same dataset [27].

The fundamental concept of our strategy is to treat each confidence score $p_{i}^{(j)}$ $i = 1,2 \ldots L; j = 1,2 \ldots M$ as an observation and to construct the joint likelihood of belief scores under class $j$, $f \left( p_{1}^{(j)}, p_{2}^{(j)}, \ldots ,p_{L}^{(j)} \mid \text{Class } j \right)$ for each pixel using training samples. After that, Bayes’ theorem is used to obtain the fused belief score $p_{i}^{(j)}$ as

$$p_{i}^{(j)} := g(p_{1}^{(j)}, p_{2}^{(j)}, \ldots ,p_{L}^{(j)})$$

$$:= P \left( \text{Class } j \mid p_{1}^{(j)}, p_{2}^{(j)}, \ldots ,p_{L}^{(j)} \right)$$

$$\propto f \left( p_{1}^{(j)}, p_{2}^{(j)}, \ldots ,p_{L}^{(j)} \mid \text{Class } j \right) \times P \left( \text{Class } j \right)$$

The challenge is to evaluate the joint likelihood $f \left( p_{1}^{(j)}, p_{2}^{(j)}, \ldots ,p_{L}^{(j)} \mid \text{Class } j \right)$ under unknown statistical dependence. To solve that, Copula function, a popular statistical tool, which is capable of modeling these kinds of dependencies is used here. In the next section, the mathematical background of Copula is revisited.

III. MATHEMATICAL BACKGROUND

A. Definition:

Let $X_{1}, X_{2}, \ldots , X_{N}$ be $N$ random variables such that their marginal distributions are continuous. We assume that $F_{i}(x_{i})$ is the marginal distribution of $X_{i}$ i.e. $F_{i}(x_{i}) = P[X_{i} \leq x_{i}]$ for $i = 1,2 \ldots N$. The Copula of $(X_{1}, X_{2}, \ldots , X_{n})$ is defined as the joint cumulative distribution over $(U_{1}, U_{2}, \ldots , U_{n})$ as given below :

$$C(u_{1}, u_{2}, \ldots , u_{n}) = P(U_{1} \leq u_{1}, U_{2} \leq u_{2}, \ldots , U_{n} \leq u_{n})$$
where \( U_i = F_i(x_i) \).

**B. Sklar’s Theorem :**

Let \( F(x_1, x_2, ..., x_N) \) be a \( N \) dimensional cumulative distribution function over real valued random variable with margins \( F_1, F_2, ..., F_N \). Then there exists a Copula \( C : [0,1]^N \rightarrow [0,1] \) such that for all \( x_1, x_2, ..., x_N \in [-\infty, \infty] \)

\[
F(x_1, x_2, ..., x_N) = C(F_1(x_1), F_2(x_2), ..., F_N(x_N))
\]

Furthermore if the margins \( F_i(x_i) \) are continuous then \( C \) is unique. Otherwise we can determine \( C \) uniquely on \( \text{Ran}(F_1) \times \text{Ran}(F_2) \times ..., \times \text{Ran}(F_N) \), where \( \text{Ran}(F_i) \) denotes the Range of \( F_i \).

**C. Copula Density :**

Copula function is a multivariate distribution function. Thus we can obtain its density function by partially differentiating with respect to all its variables as :

\[
c(u_1, u_2, ..., u_N) = \frac{\partial^N C(u_1, u_2, ..., u_N)}{\partial u_1 \partial u_2 \cdots \partial u_N}
\]

Here the copula function \( C \) is assumed to be sufficiently differentiable with respect to all its arguments.

To established the relation between copula density function and multivariate density function, we differentiate the above equation as follows:

\[
f(x_1, x_2, ..., x_N) = \frac{\partial^N F(x_1, x_2, ..., x_N)}{\partial x_1 \partial x_2 \cdots \partial x_N} = \frac{\partial^N C(F_1(x_1), F_2(x_2), ..., F_N(x_N))}{\partial x_1 \partial x_2 \cdots \partial x_N} = c(F_1(x_1), F_2(x_2), ..., F_N(x_N)) \prod_{i=1}^{N} f_i(x_i)
\]

(1)

where \( f_i \) being marginal density of \( X_i \) for all \( i \). This copula density is significant for fitting data to a copula and also to determine their likelihood. Some of the important copula functions are discussed below.

**D. Copula Families :**

There are different kinds of copula classes (collection of copula families which have similar properties) present in the literature, viz, fundamental, elliptical, archimedean, and hierarchical archimedean. A brief introduction of different copula families is given in this section.

- **Fundamental Copulas** : There are three fundamental copula families namely comonotonicity (represents perfect positive dependence), countermonotonicity (represents perfect negative dependence) and independent (represents independence) copulas. The simplest is the Independent copula (see Fig. 4), which has the following form

\[
C(u_1, u_2, ..., u_N) = \prod_{i=1}^{N} u_i
\]

- **Elliptical Copulas** : Copulas which describes the dependencies of elliptical multivariate distribution are called Elliptical Copulas. The copula families belonging to this class are gaussian copula and student-t copula.

- **Gaussian copula** : describes the multivariate normal (Gaussian) distribution and has the following form:

\[
C(u_1, u_2, ..., u_N) = \Phi_\Sigma^{-1}(\Phi^{-1}(u_1), \Phi^{-1}(u_2), ..., \Phi^{-1}(u_N))
\]

Where \( \Phi_\Sigma \) is cumulative distribution function of multivariate normal distribution with correlation matrix \( \Sigma \) and \( \Phi^{-1} \) is the quantile function of normal distribution. Gaussian copula can describe a variety of dependences depending on its parameter. For example in bivariate case \( \Sigma \) is is \( 2 \times 2 \) matrix with a single parameter \( \rho = \Sigma_{1,2} = \Sigma_{2,1} \). For \( \rho = -1 \) it represents countermonotonicity copula, for \( \rho = 1 \) it represents Comonotonicity copula, for \( \rho = 0 \) it represents independent copula etc.

- **Student-t Copula** : Similar to Gaussian Copula the Student-t copula can be defined as follows:

\[
C(u_1, u_2, ..., u_N) = t_{\nu,\Sigma}(t_{\nu}^{-1}(u_1), t_{\nu}^{-1}(u_2), ..., t_{\nu}^{-1}(u_N))
\]

Where \( t_{\nu,\Sigma} \) is the cumulative distribution function(CDF) of multivariate Student-t distribution with correlation matrix \( \Sigma \) and degrees of freedom \( \nu \) and \( t_{\nu} \) is the CDF of univariate Student-t distribution with degrees of freedom \( \nu \). As \( \nu \to \infty \) the Student-t copula converge to Gaussian copula(The difference becomes negligible after \( \nu \geq 30 \)). Student-t copula is mostly used in finance studies since it exhibits best fit than other families. And example of
Gaussian Copula and Student-t copula is shown in Fig. 5

- Archimedean Copula: copulas generated by archimedean generator (with one or more parameters) are called Archimedean Copula. Depending on different parameter values, different copulas can be obtained. For a single-parameter family, there exist 22 copulas [39]; among those, we have chosen here three, which are most commonly referred to in literature.

  - Clayton Copula: It can describe the dependence in the lower tail (as shown in Fig. 6), and that is why it is used in finance to detect correlated risk. This type of copulas is represented as:
    \[
    C(u_1, u_2 \cdots u_N) = \left( \sum_{i=1}^{N} u_i^{-\theta} - 1 \right)^{-1/N}, \theta \in [-1, \infty) \setminus \{0\}
    \]

  - Frank Copula: It can describe symmetric dependence (as shown in Fig. 6) also unlike Clayton it can describe positive as well as negative dependence. It has the following form:
    \[
    C(u_1, u_2 \cdots u_N) = -\theta^{-1} \log \left[ 1 + (e^{-\theta} - 1)^{-1} \prod_{i=1}^{N} (e^{-\theta u_i} - 1) \right]
    \]
    Where \(\theta \in \mathbb{R} \setminus \{0\}\)

- Gumbel Copula: It can describe asymmetric dependence (as in Fig. 3). Like clayton it cannot represent negative dependence. It has the following form:
    \[
    C(u_1, u_2 \cdots u_N) = \exp \left( - \left[ \sum_{i=1}^{N} (- \log u_i)^{\theta} \right]^{1/\theta} \right), \theta \in [1, \infty)
    \]

- Hierarchical Archimedean Copula: Archimedean Copula is used in various cases, but they have some limitations like exchangeability of variables, less number of parameters. So, they are not quite useful for higher dimensions. In higher dimensions, a generalized version of Archimedean copulas is usefully called Hierarchical Archimedean Copula(HAC). HACs are constructed by composing two or more simple Archimedean copulas, and they can describe a wide range of dependence structures. The details can be found elsewhere [40-41].

IV. FITTING A COPULA TO A DATA

To fit a Copula to a \(d\)-dimensional data, we need to estimate the parameters of that Copula, which will describe the multivariate distribution of those samples. For example, in the case of elliptical copulas, we need to estimate the correlation matrix of those samples. Similarly, we need to estimate the degree of freedom of the samples for student-t Copula. For Archimedean Copulas, we have to estimate only the generator. In the next section, we will discuss various fitting methods [42-43-44-45].

A. THE MAXIMUM LIKELIHOOD(ML) METHOD [22]:

In this method we estimate both copula parameters and marginal parameters. Given a sample \(\{X_i^1, X_i^2 \cdots X_i^N\}_{t=1}^{T}\) for \(T\) training examples we get the likelihood function \(L(c, \tau)\) from copula density function as:

\[
L(c, \tau) = \prod_{t=1}^{T} f^t(c, \tau)
\]

\[
= \prod_{i=1}^{N} \left( c( F_1(x_1), F_2(x_2), \ldots, F_N(x_N) ) \right) \prod_{i=1}^{T} f^i(x_i)\)
\]

Taking log on both side of (2) we get the log-likelihood function as,

\[
\hat{L}(\hat{c}, \hat{\tau}) = \log(L(c, \tau))
\]

\[
= \log \left( \prod_{t=1}^{T} \left( c( F_1(x_1), F_2(x_2), \cdots, F_N(x_N), \tau) \right)^{N} f^i(x_i) \right)
\]

\[
= \sum_{t=1}^{T} \log \left( c( F_1(x_1), F_2(x_2), \cdots, F_N(x_N), \tau) \prod_{i=1}^{N} f^i(x_i) \right)
\]

\[
= \sum_{t=1}^{T} \log \left( c( F_1(x_1), F_2(x_2), \cdots, F_N(x_N), \tau) \right)
\]

\[+ \left( \sum_{t=1}^{T} \sum_{i=1}^{N} \log f^i(x_i) \right) \]

Hence the ML estimator is

\[
\hat{L}(\hat{c}, \hat{\tau}) = \arg \max L(c, \tau)
\]
If marginals are known then we can rewrite \(4\) as
\[
\hat{L}(\hat{\xi}, \hat{\tau}) = \arg \max_{\in C} \mathcal{L}(c, \tau)
\]
\[
= \arg \max_{\in \mathcal{C} \in C} \sum_{t=1}^{T} \log \left(c(F_1(x_1), F_2(x_2) \ldots F_N(x_N), \tau)\right)
\]
(5)

If the marginals are not known then the ML estimator becomes
\[
\hat{L}(\hat{\xi}, \hat{\tau}, \hat{f}_1(x_1), \hat{f}_2(x_2), \ldots \hat{f}_N(x_N))
\]
\[
= \arg \max_{\in \mathcal{C}, f_i(x_i) \in F} \left[ \sum_{t=1}^{T} \log \left( c(F_1(x_1), F_2(x_2) \ldots F_N(x_N), \tau) \right) \right.
\]
\[
\left. + \left( \sum_{t=1}^{T} \sum_{i=1}^{N} \log f_i^1(x_i) \right) \right]
\]
(6)

Due to the complex structure of equation (6) and presence of huge amount of data ML method is not suitable. Thus Inference Function for Margins Method(IFM) is adopted to solve such problems.

B. The Inference Function for Margins Method (IFM) \(44\) :

The above equation (6) can be simplified by estimating the marginals first then using those marginals to estimate the copula parameters. Equation (6) can be rewritten as :
\[
\mathcal{L}(c, \tau, \xi)
\]
\[
= \sum_{i=1}^{T} \log \left( c(F_1(x_1, \xi_1), F_2(x_2, \xi_2) \ldots F_N(x_N, \xi_N), \tau) \right)
\]
\[
+ \left( \sum_{i=1}^{T} \sum_{i=1}^{N} \log f_i^1(x_i, \xi_i) \right)
\]
(7)

Here \(\xi = (\xi_1, \xi_2 \ldots \xi_N)\) represents the parameters of marginals and \(\tau\) represents the parameter of copula. Now in this method we first estimate \(\xi\) from the second part of (7) i.e.
\[
\hat{\xi}_i = \arg \max_{\xi_i} \left( \sum_{i=1}^{T} \log f_i^1(x_i, \xi_i) \right)
\]
(8)

After that using (8) we estimate the first part of equation (7) as :
\[
\hat{\tau} = \arg \max_{\in \mathcal{C}} \sum_{i=1}^{T} \log \left( c(F_1(x_1, \hat{\xi}_1), F_2(x_2, \hat{\xi}_2) \ldots \right.
\]
\[
\left. \cdots F_N(x_N, \hat{\xi}_N), \tau) \right)
\]
(9)

Hence the IFM estimator will be \((\hat{\xi}, \hat{\tau})\). This method is asymptotically equivalent to the ML method.

C. Estimating Marginals :

In most of the cases the marginals \(f_i(x_i)\) are unknown and have to be estimated for IFM method. To do this we have adapted Kernel Density Estimation(KDE) to fit our data. KDE is a nonparametric technique for estimation of probability density of any random variable. Let \(x_1, x_2 \ldots x_k\) be \(k\) samples drawn from an unknown distribution. Then for any value \(x\) the fromula for KDE is
\[
\hat{f}_i(x) = \frac{1}{kh} \sum_{i=1}^{k} K \left( \frac{x - x_i}{h} \right)
\]
Where \(K(.)\) is the kernel smoothing function and \(h\) is the bandwidth which controls the smoothness of the resulting density curve. For our data we have used Normal kernel or Gaussian kernel in KDE. The formula for gaussian kernel is given by
\[
K(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right)
\]

D. Measuring the fit :

There are various Measuring statistics available in the literature. Some of them are Log-likelihood(LL), Akaike Information Criteria (AIC) \(46\) \(47\), Bayesian Information Criterion (BIC) \(46\) \(47\), Average Kolmogorov-Smirnov distance(AKS), Cramer-von-Mises distance etc. Out of these, we have used Log-likelihood, AIC, and BIC for our experiment.

V. EXPERIMENTS :

To validate the effectiveness of the proposed class-specific Copula based ensemble method i.e EDC3(Ensemble of Deep classifiers using class-specific Copula function), we have evaluated the technique on two publicly available datasets CamVid \(48\), a road scene video sequence consisting of 601 frames and ICCV09 \(49\), a collection of urban and rural scene images consisting of 715 frames. The observed results of the ensembling method are compared with the individual classifiers along with the ensemble of those classifiers using different single copula functions. The proposed technique is also compared with three other state-of-the-art ensembling methods to prove the superiority of the proposed method. The details of the experimental results are shown in Table I,III,IV and V.

A. Data-sets :

1) CamVid: CamVid \(48\) is a road scene video sequence consisting of 367 frames of train set, 101 frames of validation set and 233 frames of test set. In our experiment, we set the dimension of each frame to 360 × 480, which is half of the original dimension. The ground truth of Camvld has 11 semantic segmentation classes, namely Sky, Building, Column-pole, Road, Sidewalk, Tree, Sign-symbol, Fence, Car, Pedestrian, Bicyclist, and one void class. We have trained our base deep models, i.e., SegNet, PSPNet, and Tiramisu, with these 12 semantic classes.

2) ICCV09: We have also used another dataset ICCV09 \(49\), which consists of 715 images of urban and rural scenes assembled from a collection of public image datasets. There are 572 images for training and 143 images for validation purposes. It has a total of 9 semantic segmentation classes, namely Sky, Tree, Grass, Ground, Building, Mountain, Water, Object, and one unknown class. The resolution of each image here is 240 × 340, but in our experiment, we have resized each image to 360 × 480.
parametric marginal estimation of our given data. The data on the CamVid dataset are presented in the Appendix. The copula family fitting statistics for each class of training

\[ \text{BIC}, LL \] after fitting with some well-known copula families.

are determined empirically using their fitting statistics(AIC, the best-fitted Copula family for given input data, which

Family given copula family. The term

Algorithm 1, which returns an estimate of parameters of the input size is

be

H = M \times W, where \( M \) is the total number of classes.

The next step is to determine which Copula will be the best fit for the data (for each class) during the ensemble procedure. To do that, we fit our data for each class to five popular elliptic and Archimedean copulas, namely 1)Gaussian, 2)Students-t, 3)Clayton, 4)Gumbel, and 5) Frank and determine the LL, AIC and BIC statistics \([39] \,[46] \,[47]\). After evaluating those statistics, the best-fitted Copula will be chosen for each class. At the end of this procedure, all the selected class-specific best copulas are used to estimate their parameters. The final class-specific fused distribution will be determined using these parameters and validation data (Classifier outputs on validation samples for each class as a probability distribution across all classes). The whole approach is presented briefly in

Algorithm 1. To fit data to a copula, we have used the function \text{Fit} based on the IFM method (see section IV-B) in Algorithm 1, which returns an estimate of parameters of the given copula family. The term \text{Family} is used here to denote the best-fitted Copula family for given input data, which are determined empirically using their fitting statistics(AIC, BIC, LL) after fitting with some well-known copula families. The copula family fitting statistics for each class of training data on the CamVid dataset are presented in the Appendix. The function \text{KernelDensityEstimation} is the non-parametric marginal estimation of our given data. The \text{Pdf} function returns the probability density function of the selected copula family for the estimated copula parameters at the test data samples. The function \text{KDEpdf} calculates the marginal density of the given samples.

C. Performance Measure :

In this section, we discuss three performance metrics which are used here, namely, the Overall pixel Accuracy(OA), which is the percentage of total number of correctly classified pixels; Mean class accuracy(CA), which is the mean of all the class accuracies and Mean Intersection over Union (mIOU). the formula for OA is formulated as

\[ \text{OA} = \frac{\sum_{j=1}^{N} \hat{y}_j = y_j}{N} \times 100 \]

For a given class \( l \), the class accuracy \( CA_l \) is defined as

\[ \text{CA}_l = \frac{\sum_{j=1}^{N} \hat{y}_j == l \land y_j == l}{\sum_{j=1}^{N} y_j == l} \]

Hence the mean Class Accuracy or CA will be

\[ \text{CA} = \frac{\sum_l CA_l}{l} \times 100 \]

For a given class \( l \), IOU is defined as

\[ \text{IOU}(l) = \frac{\sum_{j=1}^{N} \hat{y}_j == l \land y_j == l}{\sum_{j=1}^{N} \hat{y}_j == l \lor y_j == l} \]

Hence the mean IOU is

\[ \text{mIOU} = \frac{\sum_l \text{IOU}(l)}{l} \]

Where \( \land \) is ‘and’, \( \lor \) is ‘or’ operation, \( \hat{y}_j \) is the prediction and \( y_j \) is the ground truth of the pixel \( j \). Where \( N \) is the total number of pixels in the entire dataset.

D. Other State-of-the-art ensemble methods for comparison

The performance of the proposed ECD3 ensemble model is compared with three other popular classifier ensemble methods namely LOP \([24]\], majority voting \([13], [14], [15], [16]\), and fusion based on simple logit model \([26]\).

LOP or linear opinion pool is just weighted average of all confidence score obtained from different classifiers i.e. if we have \( M \) classifiers and \( p_l \) is a confidence score obtained from classifier \( j \) then the resultant confidence score \( p \) will be

\[ p = \sum_{i=1}^{M} w_i p_i \]

where \( \sum_{i=1}^{M} w_i = 1 \).

In majority voting, we determine the optimal decision according to the decision made by each base architectures. If the majority of classifiers decides that a certain pixel belongs to a particular class then that pixel will belong to that particular class. But if each classifier gives different results on the same sample then the ultimate decision will be made by comparing their confidence score of that pixel.

On the other hand, in case of fusion based on a simple logit model, the fusion rule is defined as

\[ p = \left[ \prod_{i=1}^{M} \left( \frac{p_i}{1-p_i} \right)^{1/M} \right]^{a} + \left[ \prod_{i=1}^{M} \left( \frac{1-p_i}{p_i} \right)^{1/M} \right]^{a} \]

E. Result and Analysis :

In this section, we have discussed the detailed result and analysis of our experiments. As mentioned in section IV-B we have used the output results obtained from SegNet \([50]\), PSPNet\([51]\), Tiramisu\([52]\) as belief scores to ensemble further. Note that we did not use any pre-trained model or transfer learning for those deep learning networks. We have trained those base models from scratch on CamVid and ICCV09
Algorithm 1: Copula Ensembling

**Constants:**
- \( L \) = Number of Classifiers
- \( M \) = Number of segmentation classes
- \( P \) = Total Number of Pixel for Training Images
- \( Q \) = Total Number of Pixel for Validation Images
- **Family** = The best fitted copula function corresponding marginals for the given data

**Data:**
- \( [X^l_m]_{P \times 1} \) = Pixel-level Probability Distribution for class \( m \), from classifier \( l \).
- \( [X^L_m]_{Q \times 1} \) = Pixel-level Probability Distribution for class \( m \), from classifier \( l \).

**Result:** \( R^m_m = \) Pixel-Level Probability Distribution after ensembling for class \( m \).

1. for \( m = 1 \) to \( M \) do
2. \( X = [X^1_m, X^2_m, \ldots, X^L_m] \)
3. \( U = \text{KernelDensityEstimation}(X) \)
4. Copula-Parameters = \text{Fit}(\text{Family}, U) \quad \triangleright \text{by IFM method}
5. \( XT = [X^1_T, X^2_T, \ldots, X^L_T] \)
6. \( UT = \text{KernelDensityEstimation}(XT) \)
7. \( A_l = \text{Pdf}(\text{Family}, UT, \text{Copula-Parameters}) \)
8. \( R^m_m = A^m_l \ast \text{KDEpdf}(TX) \ast \text{Prior}(m) \quad \triangleright \text{by equation (1)}
9. end for

### Fig. 7: Comparison of accuracies of single class copula ensembling and our proposed method for each class of CamVid dataset which shows our proposed method has better accuracy for every class.

1) Results on CamVid Dataset: First, we need to get the belief scores of the three deep learning methods after pixel-wise distribution in their associated classes, i.e., 12 classes for CamVid data set, to obtain the results of the proposed EDC3. The softmax function is used at the end of the three deep learning techniques to obtain those probabilities. These probability distributions of the training models are used to improve further using Copula based ensemble method. From the pixel level distribution, the determine the best-fitted Copula function for each class empirically by fitting with the five different Copula functions.

The best fitted copula for each class is selected by comparing the fitting statistics as mentioned in section [IV-D].

### TABLE I: The best fitted copula functions for each class determined by comparing fitting statistics for CamVid dataset

| Best Fitted Copula | Classes                          |
|--------------------|---------------------------------|
| Gaussian           | Sky, Building,                  |
| Student-t          | Road, Side-walk                 |
| Frank              | Tree                            |
| Clayton            | Sign-Symbol, Fence, Car, Pedestrian, void |
| Gumbel             | Column-Pole, Bi-Cyclist         |
TABLE II: Results of Ensembling data from SegNet, PSPNet and Tiramisu on CamVid dataset

| Architectures | Overall Accuracy | Mean Accuracy | Mean IOU |
|---------------|-----------------|---------------|----------|
| SegNet[50]    | 84.700303       | 49.519581     | 0.41825  |
| PSPNet[51]    | 92.818602       | 78.782833     | 0.663291 |
| Tiramisu[52]  | 91.637061       | 77.221778     | 0.637337 |
| LOP[24]       | 92.608401       | 81.615978     | 0.635698 |
| Majority_voting[13] | 92.8761 | 80.979356     | 0.652887 |
| Logit[26]     | 92.608401       | 81.615978     | 0.635698 |
| Gaussian      | 90.913687       | 79.188869     | 0.60419  |
| Student-t     | 88.045649       | 73.047487     | 0.556318 |
| Frank         | 87.969128       | 72.89565      | 0.552939 |
| Clayton       | 91.90342        | 81.650784     | 0.65254  |
| Gumbel        | 91.119866       | 79.080911     | 0.579787 |
| Proposed      | 93.091532       | 82.559746     | 0.672016 |

Fig. 8: Performance of single class copula ensembling technique for each class in CamVid dataset.

Fig. 9: Comparision of overall accuracies for each combination between SegNet, PSPNet and Tiramisu net with single class copula ensembling and our proposed method on CamVid dataset.

functions are shown in Table I. From the table we can observe that Gaussian copula is the best-fitted copula for classes Sky and Building, the Student-t copula is for the classes Road and Sidewalk; Frank copula for class Tree; Gumbel is for classes Column-Pole and Bicyclist; finally Clayton copula is the best fitted one for rest of the classes. After obtaining the class-specific best-fitted copulas, we have ensemble our validation data using Algorithm 1 to obtain the fused belief score. We have compared our proposed approach with the three popular state-of-the-art ensembling models on the same dataset, and obtained results of our experiments along with benchmarks are shown in Table II. We can observe from Table II that the majority voting technique performs better than each base model in terms of overall accuracy. On the other hand, all the benchmark models and single class copula models perform significantly better in terms of mean accuracy. However, our proposed class-specific copula method outperforms all the above models in terms of overall accuracy, mean accuracy, and mean IOUs. The comparisons of ensembling with single copula functions and our proposed technique for each class on the CamVid dataset are shown in Fig. 7. From that figure it is clear that ensembling with class-specific copula function performs better than ensembling with single copula function.

In Fig. 8 we have given the class-wise accuracies of ensembling with a single copula. In that figure, we color-coded the result along each row such that boldfaced values indicate the maximum accuracies. From Fig. 8, we can see that Gaussian Copula performs better for classes Sky and Building. Whereas Clayton copula performs better for classes Sidewalk, Sign-symbol, Fence, Car, pedestrian, Bi-cyclist. For class Column-pole Clayton and Gumbel performs closely, but the later has slightly better accuracy. In the case of class Road, gaussian and student-t have similar performance, but student-t has slightly better accuracy. At last for class, Tree has very similar performance with student-t and frank copula, but frank’s performance slightly better. In conclusion, Fig. 8 gives a very convincing proof that the chosen copula functions for each class (as shown in Table I) are pretty accurate.

For further analysis, we have shown the numerical results of our experiments performed by taking two classifiers at a time in Table III. The combination of SegNet+PSPNet, our model performs better in terms of overall accuracy and mean IOU, but LOP[24] has better mean accuracy among all. On the other hand, in SegNet+Tiramisu, our model outperforms the indi-
TABLE III: Results of Ensembling data by taking two base model at a time on CamVid dataset

| Combination           | SegNet+PSPNet          | SegNet+Tiramisu          | PSPNet+Tiramisu          |
|-----------------------|------------------------|--------------------------|--------------------------|
|                       | Overall Accuracy       | Mean Accuracy            | Mean IOU                 |
|                       |                        |                          |                          |
| LOP [24]              | 90.44951 78.88017      | 90.20362 77.322218       | 93.57038 82.819487       |
| MV [13]               | 91.759322 77.205833    | 90.790956 74.958566      | 93.342855 81.038866      |
| Logit [26]            | 91.613707 78.509435    | 90.684692 75.856698      | 94.422855 81.80753       |
| Gaussian              | 91.491365 77.310408    | 90.947187 76.684308      | 92.338565 81.417598      |
| Student-t             | 92.179364 70.905434    | 88.871516 73.510087      | 90.442771 77.845975      |
| Frank                 | 90.739635 77.813193    | 90.126209 76.568786      | 93.013075 82.535629      |
| Clayton               | 91.828268 78.703963    | 90.060454 77.022918      | 93.712067 83.200367      |
| Gumbel                | 90.928037 76.683425    | 89.851445 76.859128      | 89.76712 80.05377        |

| Proposed              | 93.659201 77.566497    | 91.967287 77.364829      | 93.900994 81.784975      |

TABLE IV: Results of Ensembling data from SegNet, PSPNet and Tiramisu on ICCV09 dataset

| Architectures       | Overall Accuracy | Mean Accuracy | Mean IOU |
|---------------------|-----------------|---------------|----------|
| SegNet [50]         | 75.336665       | 58.313291     | 0.476233 |
| PSPNet [51]         | 83.22131        | 66.745377     | 0.525069 |
| Tiramisu [52]       | 66.064429       | 55.15124      | 0.361695 |
| LOP [24]            | 82.909654       | 68.870868     | 0.538002 |
| Majority_voting [13]| 82.632492       | 0.540236      | 0.511325 |
| Logit [26]          | 83.354601       | 68.274124     | 0.53233  |
| Gaussian            | 82.862361       | 67.139727     | 0.519109 |
| Student-t           | 82.864234       | 67.119442     | 0.519202 |
| Frank               | 82.830263       | 67.029158     | 0.518913 |
| Clayton             | 83.051362       | 67.12981      | 0.517773 |
| Gumbel              | 82.882825       | 67.140376     | 0.519423 |
| Proposed            | 83.821968       | 68.326652     | 0.548254 |

2) ICCV09 dataset: To validate the robustness of the proposed technique, we have also evaluated our method on another dataset, ICCV09, which consists of 9 different classes. Here "Clayton Copula" is the best-fitted Copula for the majority of the classes, namely Sky, Tree, Grass, Ground, Mountain Object. The other selected Copulas are Student-t for Building and water, Frank for the unknown class. It is worthy of mentioning that Gaussian Copula, which normally used popularly [36] is not performed as the best Copula for this dataset. Even for the dataset CamVid, the Gaussian Copula performs better for only two classes among 12 classes. From Table IV, it is observed that a simple Logit model achieves the highest mean accuracy. On the other hand, our proposed model has achieved the highest overall accuracy and mean IOU. The performance is better over the base models as well as their combinations based on single Copula.

We have also experimented with our techniques with pairwise combinations of three deep learning models. From Table V, it can be observed that for each Copula except Gaussian, Segnet+PSPNet combination with single Copula function performs better than the other combinations using the same Copula function. The performance of the Segnet+PSPNet is also better for the proposed Copula. As PSPNet performs better than the other two networks in their base model, hence the combination with PSPNet gives better performance than the combination with the combination without PSPNet.

As shown in Fig. 7 Our model can perform significantly better than ensembling with single copula functions not only for each class of a dataset but also in terms of overall accuracy, mean IOU, and mean accuracy. The results in table [I] II[13] indicates that our model can perform better if we ensemble classifiers which have similar performances. Also, results in [V] and [IV] show that mean accuracy and mean IOU can get affected by the ensembling of classifiers having significantly different performance. However, Fig. 9 and Fig. 10 indicates that in every combination, our proposed model can boost up the overall pixel accuracy. Since we are dealing with multi-class classification on the pixel level, the estimation of marginal distribution and determining the best-fitted copula family for each class are time-consuming, which can be addressed in

vidual base models as well as benchmarks in all performance matrices. In PSPNet+Tiramisu, all the benchmarks, Clayton and frank copula models perform better than the base models, but our proposed models outperform all of them in terms of overall accuracy. Also, note that the overall accuracy of our proposed approach with PSPNet+Tiramisu is better than the combination of SegNet+PSPNet+Tiramisu. That is because the performance of PSPNet and Tiramisu are very similar, and it is expected that the combination of those two will give a better result. On the other hand, SegNet performs poorly compared to the other two. So the combination of SegNet+Tiramisu and SegNet+PSPNet gave significantly lower accuracy.
TABLE V: Results of Ensembling data by taking two architecture at a time on ICCV09 dataset

| Combination        | SegNet[50]+PSPNet[51] | SegNet[50]+Tiramisu[52] | PSPNet[51]+Tiramisu[52] |
|--------------------|-----------------------|-------------------------|--------------------------|
|                    | Overall Accuracy      | Mean Accuracy           | Mean IOU                 | Overall Accuracy      | Mean Accuracy           | Mean IOU                 |
| LOP[24]            | 84.015697             | 68.435561               | **0.550204**             | 77.23354              | 62.325241               | 0.49395                 | 82.318858              | 67.727353               | 0.511844               |
| Majority_voting[13]| 83.656696             | **68.462354**           | 0.542648                 | 77.020235             | 62.224943               | 0.4921                  | 82.302316             | 68.02633                | 0.511325               |
| Logit[26]          | 83.895423             | 68.163356               | 0.548655                 | 77.153738             | 62.274582               | 0.495565                | 82.823065             | **68.087257**           | 0.51364                |
| Gaussian           | 73.515115             | 67.475968               | 0.479339                 | 73.267483             | 62.97528                | 0.474277                | 82.640709             | 67.815459               | 0.509369               |
| Student-t          | 83.268663             | 66.771422               | 0.530653                 | 77.313422             | 62.715795               | **0.497809**            | 82.64648              | 67.815459               | 0.509369               |
| Frank              | 83.229601             | 66.748022               | 0.531364                 | 77.347632             | 62.632267               | 0.497337                | 77.347632             | 62.632267               | 0.497337               |
| Clayton            | 83.23804              | 66.776492               | 0.531374                 | 68.809506             | 64.051407               | 0.459335                | 73.985753             | 67.861775               | 0.461731               |
| Gumbel             | 83.277428             | 66.740492               | 0.53075                  | 68.62281              | **64.085221**           | 0.453662                | 69.993984             | 67.734381               | 0.445879               |
| Proposed           | **84.270355**         | 66.750306               | 0.550144                 | **77.357209**         | 62.603572               | 0.497515                | **83.633214**         | 68.02635                | **0.53625**            |

Fig. 11: Visual representation of performances of our proposed model with the base models on CamVid and ICCV09 datasets. The images indexed with (a),(b),(c) are CamVid samples and (d),(e),(f) are ICCV09 samples.

VI. CONCLUSION

We have proposed EDC3: Ensemble of Deep-Classifer using Class specific Copula functions to improve semantic im-
age segmentation. Our proposed ensemble model can address inter-source statistical dependencies present among different deep learning-based classifiers for image segmentation. The proposed model explored the effectiveness of class-specific Copula, which provides more flexibility than using a single class model for multiclass problems. Though Copula is well known for binary class image segmentation but its property to use multiclass image segmentation hardly used in literature. So multiclass image segmentation using class specific copula function is one of our major contributions in the present work. The proposed model performs significantly better than ensembling with single Copula functions not only for each class of the datasets but also in terms of overall accuracy, mean IOU, and mean accuracy for semantic image segmentation. It also outperforms some other popular ensemble methods, namely LOP, Majority voting. Fusion based on Logit in most of the performance indices. Since we are dealing with multiclass segmentation at a pixel level, the estimation of marginal distribution and determining the best-fitted copula family for each class are time-consuming, which needs to be addressed. The technique suggested can also be used for the multiclass segmentation and classification task effectively where statistical dependencies exist among different classifiers.

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REFERENCES

[1] S. Ghosh, N. Das, I. Das, and U. Maulik, “Understanding deep learning techniques for image segmentation,” arXiv preprint arXiv:1907.06119, 2019.
[2] J. Shi and J. Malik, “Normalized cuts and image segmentation,” Departmental Papers (CIS), p. 107, 2000.
[3] R. Achanta, A. Shaji, K. Smith, A. Lucchi, P. Fua, and S. Süsstrunk, “Slic superpixels,” Tech. Rep., 2010.
[4] J. Long, E. Shelhamer, and T. Darrell, “Fully convolutional networks for semantic segmentation,” in Proceedings of the IEEE conference on computer vision and pattern recognition, 2015, pp. 3431–3440.
[5] L.-C. Chen, G. Papandreou, I. Kokkinos, K. Murphy, and A. L. Yuille, “Semantic image segmentation with deep convolutional nets and fully connected crfs,” arXiv preprint arXiv:1412.7062, 2014.
[6] Ç. Kaymak and A. Ucar, “A brief survey and an application of semantic image segmentation for autonomous driving,” in Handbook of Deep Learning Applications. Springer, 2019, pp. 161–200.
[7] D. L. Pham, C. Xu, and J. L. Prince, “Current methods in medical image segmentation,” Annual Review of Biomedical Engineering, vol. 2, no. 1, pp. 315–337, 2000, pMID: 11701515. [Online]. Available: https://doi.org/10.1146/annurev.bioeng.2.1.315
[8] M. Forouzanfar, N. Forghani, and M. Teshehkar, “Parameter optimization of improved fuzzy c-means clustering algorithm for brain mr image segmentation,” Engineering Applications of Artificial Intelligence, vol. 23, no. 2, pp. 160–168, 2010.
[9] J. A. Delmerico, P. David, and J. J. Corso, “Building facade detection, segmentation, and parameter estimation for mobile robot localization and guidance,” in 2011 IEEE/RSJ International Conference on Intelligent Robots and Systems. IEEE, 2011, pp. 1632–1639.
[10] Z. Liu, L. Wang, G. Hua, Q. Zhang, Z. Niu, Y. Wu, and N. Zheng, “Joint video object discovery and segmentation by coupled dynamic markov networks,” IEEE Transactions on Image Processing, vol. 27, no. 12, pp. 5840–5853, 2018.
[11] L. Wang, X. Duan, Q. Zhang, Z. Niu, G. Hua, and N. Zheng, “Segment-tube: Spatio-temporal action localization in untrimmed videos with per-frame segmentation,” Sensors, vol. 18, no. 5, p. 1657, 2018.
[12] W. Nayer, “Feature based architecture for decision fusion,” Ph.D. dissertation, phD thesis, 2003.
[13] R. Battiti and A. M. Colla, “Democracy in neural nets: Voting schemes for classification,” Neural Networks, vol. 7, no. 4, pp. 691–707, 1994.
[14] C. Ji and S. Ma, “Combinations of weak classifiers,” in Advances in Neural Information Processing Systems, 1997, pp. 494–500.
[15] L. Lam and C. Y. Suen, “Optimal combinations of pattern classifiers,” Pattern Recognition Letters, vol. 16, no. 9, pp. 945–954, 1995.
[16] L. Xu, A. Krzyzak, and C. Y. Suen, “Methods of combining multiple classifiers and their applications to handwriting recognition,” IEEE transactions on systems, man, and cybernetics, vol. 22, no. 3, pp. 418–435, 1992.
[17] L. I. Kuncheva, “A theoretical study on six classifier fusion strategies,” IEEE Transactions on pattern analysis and machine intelligence, vol. 24, no. 2, pp. 281–286, 2002.
[18] P. W. Munro and B. Parmanto, “Competition among networks improves committee performance,” in Advances in Neural Information Processing Systems, 1997, pp. 592–598.
[19] D. M. Tax, M. Van Breukelen, R. P. Duin, and J. Kittler, “Combining multiple classifiers by averaging or by multiplying?” Pattern recognition, vol. 33, no. 9, pp. 1475–1485, 2000.
[20] T. K. Ho, J. J. Hull, and S. N. Sridhara, “Decision combination in multiple classifier systems,” IEEE Transactions on Pattern Analysis & Machine Intelligence, no. 1, pp. 66–75, 1994.
[21] S.-B. Cho and J. H. Kim, “Combining multiple neural networks by fuzzy integral for robust classification,” IEEE Transactions on Systems, Man, and Cybernetics, vol. 25, no. 2, pp. 380–384, 1995.
[22] L. I. Kuncheva, “An application of owa operators to the aggregation of multiple classification decisions,” in The ordered weighted averaging operators. Springer, 1997, pp. 330–343.
[23] L. Kuncheva, J. C. Bezdek, and M. A. Sutton, “On combining multiple classifiers by fuzzy templates,” in 1998 Conference of the North American Fuzzy Information Processing Society-NAFIPS (Cat. No. 98TH8353). IEEE, 1998, pp. 193–197.
[24] C. Genest and K. J. McConway, “Allocating the weights in the linear opinion pool,” Journal of Forecasting, vol. 9, no. 1, pp. 53–73, 1990.
[25] R. Ranjan and T. Gneiting, “Combining probability forecasts,” Journal of the Royal Statistical Society: Series B (Statistical Methodology), vol. 72, no. 1, pp. 71–91, 2010.
[26] V. A. Satopää, J. Barou, D. P. Foster, B. A. Mellers, P. E. Tetlock, and L. H. Ungar, “Combining multiple probability predictions using a simple logit model,” International Journal of Forecasting, vol. 30, no. 2, pp. 344–356, 2014.
[27] O. Ozdemir, T. Allen, S. Choi, T. Wimalajeewa, and P. Varshney, “Copula Based Classifier Fusion Under Statistical Dependence,” 2017.
[28] P. Laux, W. Wagner, A. Wagner, J. Jacobit, A. Bordossy, and H. Kunstmann, “Modelling daily precipitation features in the volta basin of west africa,” International Journal of Climatology: A Journal of the Royal Meteorological Society, vol. 29, no. 7, pp. 937–954, 2009.
[29] E. Eban, G. Rothschild, A. Mizrahi, I. Nelken, and G. Eldan, “Dynamic Copula Networks for Modeling Real-valued Time Series,” Tech. Rep., 2013.
[30] A. Onken, S. Grünewälder, M. H. J. Munk, and K. Obermayer, “Analyzing Short-Term Noise Dependencies of Spike-Counts in Macaque Prefrontal Cortex Using Copulas and the Flashlight Transformation,” PLOS Computational Biology, vol. 5, no. 11, pp. 1–13, 2009. [Online]. Available: https://doi.org/10.1371/journal.pcbi.1000577
[31] I. Pollanen, B. Braithwaite, K. Haataja, T. Ikonen, and P. Toivanen, “Slic superpixels,” Tech. Rep., 2012.
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S. Jégou, M. Drozdzal, D. Vázquez, A. Romero, and Y. Bengio, “The One Hundred Layers Tiramisu: Fully Convolutional DenseNets for Semantic Segmentation,” Tech. Rep. [Online]. Available: https://github.com/SimJeg/FC-DenseNet

E. Diday and M. Vrac, “Mixture decomposition of distributions by copulas in the symbolic data analysis framework,” Discrete Applied Mathematics, 2005.

E. Diday and M. Vrac, “Mixture decomposition of distributions by copulas in the symbolic data analysis framework,” Discrete Applied Mathematics, 2005.

R. Salinas-Gutiérrez, A. Hernández-Aguirre, M. J. J. Rivera-Meraz, and E. R. Villa-Diharce, “Using Gaussian Copulas in Supervised Probabilistic Classification,” Tech. Rep.

S. Conant-Pablos, D. J. Magaña-Lozano, and H. Terashima-Marin, “Pipelining memetic algorithms, constraint satisfaction, and local search for course timetabling,” in Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), 2009.

E. Kole, K. Koedijk, and M. Verbeek, “Testing copulas to model financial dependence,” Department of Financial Management, RSM Erasmus University, Rotterdam, The Netherlands, 2005.

C. Savu and M. Trede, “Hierarchies of archimedean copulas,” Quantitative Finance, vol. 10, no. 3, pp. 295–304, 2010.

O. Okhrin and A. Ristig, “Hierarchical archimedean copulas: the hac package,” SFB 649 Discussion Paper, Tech. Rep., 2012.

G. Weiβ, “Copula parameter estimation by maximum-likelihood and minimum-distance estimators: a simulation study,” Computational Statistics, vol. 26, no. 1, pp. 31–54, 2011.

K. Aho, D. Derryberry, and T. Peterson, “Model selection for ecologists: the worldviews of AIC and BIC,” Tech. Rep. 3, 2014.

R. Salinas-Gutiérrez, A. Hernández-Aguirre, M. J. J. Rivera-Meraz, and E. R. Villa-Diharce, “Using Gaussian Copulas in Supervised Probabilistic Classification,” Tech. Rep.

K. P. Burnham and D. R. Anderson, “Multimodel inference: Understanding AIC and BIC in model selection,” 2004.

K. Aho, D. Derryberry, and T. Peterson, “Model selection for ecologists: the worldviews of AIC and BIC,” Tech. Rep. 3, 2014.

G. J. Brostow, J. Fauqueur, and R. Cipolla, “Semantic object classes in video: A high-definition ground truth database,” Pattern Recognition Letters, vol. 30, no. 2, pp. 88–97, 2009.

S. Gould, R. Fulton, and D. Koller, “Decomposing a scene into geometric and semantically consistent regions,” in 2009 IEEE 12th international conference on computer vision. IEEE, 2009, pp. 1–8.

V. Badrinarayanan, A. Kendall, and R. Cipolla, “SegNet: A Deep Convolutional Encoder-Decoder Architecture for Image Segmentation,” Tech. Rep. [Online]. Available: http://mi.eng.cam.ac.uk/projects/segnet/

H. Zhao, J. Shi, X. Qi, X. Wang, and J. Jia, “Pyramid Scene Parsing Network,” Tech. Rep. [Online]. Available: https://github.com/hszhao/PSPNet

S. Jégou, M. Drozdzal, D. Vázquez, A. Romero, and Y. Bengio, “The One Hundred Layers Tiramisu: Fully Convolutional DenseNets for Semantic Segmentation,” Tech. Rep. [Online]. Available: https://github.com/SimJeg/FC-DenseNet

E. Kole, K. Koedijk, and M. Verbeek, “Testing copulas to model financial dependence,” Department of Financial Management, RSM Erasmus University, Rotterdam, The Netherlands, 2005.

C. Savu and M. Trede, “Hierarchies of archimedean copulas,” Quantitative Finance, vol. 10, no. 3, pp. 295–304, 2010.

O. Okhrin and A. Ristig, “Hierarchical archimedean copulas: the hac package,” SFB 649 Discussion Paper, Tech. Rep., 2012.

G. Weiβ, “Copula parameter estimation by maximum-likelihood and minimum-distance estimators: a simulation study,” Computational Statistics, vol. 26, no. 1, pp. 31–54, 2011.

K. Aho, D. Derryberry, and T. Peterson, “Model selection for ecologists: the worldviews of AIC and BIC,” Tech. Rep. 3, 2014.

R. Salinas-Gutiérrez, A. Hernández-Aguirre, M. J. J. Rivera-Meraz, and E. R. Villa-Diharce, “Using Gaussian Copulas in Supervised Probabilistic Classification,” Tech. Rep.