Effects of impurities on superconductivity in noncentrosymmetric compounds

V. P. Mineev\textsuperscript{1} and K. V. Samokhin\textsuperscript{1,2}
\textsuperscript{1} Commissariat à l’Energie Atomique, DSM/DRFMC/SPSMS, 38054 Grenoble, France
\textsuperscript{2} Department of Physics, Brock University, St.Catharines, Ontario L2S 3A1, Canada
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We microscopically derive the Ginzburg-Landau free energy functional for a noncentrosymmetric superconductor with a large spin-orbit splitting of the electron bands, in the presence of nonmagnetic impurities. The critical temperature is found to be suppressed by disorder, both for conventional and unconventional pairing, in the latter case according to the universal Abrikosov-Gor’kov function. The impurity effect on the upper critical field turns out to be non-universal, determined by the pairing symmetry and the band structure. In a BCS-like model, $T_c$ is not affected, while $H_{c2}$ increases with disorder. For unconventional pairing, both $T_c$ and $H_{c2}$ are suppressed by disorder.

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I. INTRODUCTION

The recent discovery of superconductivity in a heavy-fermion compound CePt$_3$Si, see Ref. \textsuperscript{1}, has renewed interest, both experimental and theoretical, in the properties of superconductors without inversion symmetry. The list of such materials has been steadily growing in recent months and now also includes UIr (Ref. \textsuperscript{2}), CeRhSi$_3$ (Ref. \textsuperscript{3}), CeIrSi$_3$ (Ref. \textsuperscript{3}), Y$_2$C$_3$ (Ref. \textsuperscript{3}), and Li$_2$(Pd$_{1-x}$Pt$_x$)$_3$B (Ref. \textsuperscript{6}).

A peculiar property of noncentrosymmetric crystals is that the spin-orbit (SO) coupling qualitatively changes the nature of single-electron states, namely it leads to the lifting of spin degeneracy and the splitting of the energy bands. This has important consequences for superconductivity. If the typical band splitting $E_{SO}$ is smaller than the superconducting critical temperature, then the effects of the SO coupling can be treated perturbatively. In particular, the pairing interaction can be chosen to be a function of the quasiparticle spins and momenta near the Fermi surface unaffected by the SO coupling. This approach to the theory of noncentrosymmetric superconductivity was introduced by Edelstein in Ref. \textsuperscript{7} and further developed in Refs. \textsuperscript{8,9,10,11,12}. Due to the absence of inversion symmetry the superconducting order parameter does not, in general, have definite parity, becoming a mixture of spin-singlet and spin-triplet components. The triplet component appears, however, only when a spin-triplet channel is explicitly present in the pairing interaction, even for a finite value of the SO band splitting. When the SO coupling increases, only a certain type of the triplet pairing survives, along with the singlet component.

In the limit of strong SO coupling, i.e. when the band splitting exceeds the superconducting energy scales, the Cooper pairing between the electrons with opposite momenta occurs only if they are from the same nondegenerate band. Interband pairing is still possible, but only at the band degeneracy lines or points in the momentum space, and can be neglected. According to Ref. \textsuperscript{12} this is the scenario that is realized in CePt$_3$Si: The band structure calculations show that $E_{SO}$ ranges from 500K to 2000K, which is much larger than the critical temperature $T_c = 0.75$K, even taking into account a possible mass renormalization due to the strong correlation effects. The same is likely to be the case in other materials, e.g. Li$_2$Pd$_3$B and Li$_2$Pt$_3$B, see Ref. \textsuperscript{13}. The pairing interaction in the strong SO coupling case is most naturally introduced using the basis of the exact band states,\textsuperscript{14,15,16} which already incorporate all the effects of the crystal lattice potential and the SO coupling.

In the band representation, the superconducting order parameter is given by a set of complex gap functions, one for each band, which are coupled due to the interband Cooper scattering and other mechanisms, e.g. impurity scattering. The overall structure of the theory resembles that of multi-band superconductors,\textsuperscript{17,18} However, since the bands are nondegenerate, the pairing symmetry is peculiar: While each order parameter is an odd function of momentum, the gap symmetry, in particular the positions of the nodes, is determined by one of the even representations of the point group of the crystal. When expressed in the spin representation, the order parameter becomes a mixture of singlet and triplet components\textsuperscript{15,16} the latter appearing even without any spin-triplet term in the pairing interaction, as an inevitable consequence of the SO band splitting and the difference between the gap magnitudes and the densities of states in different bands.

Noncentrosymmetric superconductors exhibit a variety of unusual features, which are absent in the centrosymmetric case, such as a strongly anisotropic spin susceptibility with a large residual component,\textsuperscript{7,10,14,19,20,21,22,23} magnetoelastic effect,\textsuperscript{7,8,9,20,21,23,24} and helical superconducting states.\textsuperscript{27,28,29,30,31,32,33} Other properties of interest include the nuclear spin relaxation\textsuperscript{34,35} Josephson and quasiparticle tunnelling,\textsuperscript{26,37} electron correlation effects\textsuperscript{25,38} superfluid density and the London penetration depth\textsuperscript{39} and the free energy in the clean case.\textsuperscript{9,31}

In this article we study the effects of nonmagnetic impurities on the superconducting properties. In contrast to the treatment of the same problem in Refs. \textsuperscript{11} and \textsuperscript{24}, we assume that the SO coupling is larger than the superconducting energy scales, which necessitates using
the band representation of the pairing Hamiltonian. Our main goal is to find how the superconducting critical temperature and the upper critical field depend on the impurity concentration, for both conventional and unconventional pairing. The article is organized as follows. In Sec. II we obtain the band representations of the impurity Hamiltonian and of the disorder-averaged Green’s function of electrons in the normal state. In Sec. III we derive the Ginzburg-Landau (GL) free energy functional in the superconducting state. In Sec. IV we calculate the critical temperature and the upper critical field in the presence of impurities and also discuss the application of our results to a model of superconductivity in CePt$_3$Si.

II. IMPURITY SCATTERING IN NORMAL STATE

In a noncentrosymmetric crystal with SO coupling the electron bands are nondegenerate. The formal reason is that without the inversion operation one cannot, in general, have two orthogonal degenerate Bloch states at the same wave vector $k$. In the limit of zero SO coupling there is an additional symmetry in the system – the invariance with respect to arbitrary rotations in spin space – which preserves two-fold degeneracy of the bands. Let us consider one spin-degenerate band with the dispersion given by $\epsilon_0(k)$, and turn on the SO coupling. The Hamiltonian of noninteracting electrons in the presence of scalar impurities can be written in the form $H = H_0 + H_{imp}$, where

$$H_0 = \sum_{k,\alpha,\beta} [\epsilon_0(k)\delta_{\alpha\beta} + \gamma(k)\sigma_{\alpha\beta}]a^\dagger_{k\alpha}a_{k\beta},$$

$$H_{imp} = \int d^3r \sum_{\alpha} U(r)\psi^\dagger_{\alpha}(r)\psi_{\alpha}(r).$$

The impurity potential $U(r)$ is a random function with zero mean and the correlator $\langle U(r_1)U(r_2) \rangle = n_{imp}U_0^2\delta(r_1 - r_2)$, where $n_{imp}$ is the impurity concentration, and $U_0$ is the strength of an individual point-like impurity.

The “bare” band dispersion satisfies $\epsilon_0(-k) = \epsilon_0(k)$, $\epsilon_0(g^{-1}k) = \epsilon_0(k)$, where $g$ is any operation from the point group $G$ of the crystal. The SO coupling of electrons with the crystal lattice is described by the pseudovector $\gamma(k)$, which has the following symmetry properties: $\gamma(-k) = -\gamma(k)$, $(g\gamma)(g^{-1}k) = \gamma(k)$. For example, the point symmetry of CePt$_3$Si, CeRhSi$_3$ and CeIrSi$_3$ is tetragonal and described by $G = C_{4v}$, which is generated by the rotations $C_{4v}$ about the $z$ axis by an angle $\pi/2$ and the reflections $\sigma_x$ in the vertical plane (100). The pseudovector $\gamma(k)$ can be written as

$$\gamma(k) = \gamma_{\perp}[\phi_{E,u}(k) \times \hat{z}] + \gamma_{\parallel}\phi_{A_2,u}(k)\hat{z},$$

where $\gamma_{\perp}$ and $\gamma_{\parallel}$ are constants, and $\phi_{E,u}$ and $\phi_{A_2,u}$ are the odd basis functions of the irreducible representations $E$ (two-dimensional) and $A_2$ (one-dimensional) of $C_{4v}$. The Hamiltonian $H$ with $\gamma(k)$ given by Eq. (3) is a three-dimensional generalization of the Rashba model, which is widely used to describe the effects of SO coupling in two-dimensional semiconductor heterostructures.

The SO coupling strength depends on the quasiparticle moment and might vanish, for symmetry reasons, along some directions or at some isolated points in the Brillouin zone. The former possibility is realized in the tetragonal case: $\gamma(k) = 0$ along the axis $k_x = k_y = 0$, which can be seen from the polynomial expressions for the basis functions: $\phi_{E,u}(k) \sim (k_x, k_y)$ and $\phi_{A_2,u}(k) \sim k_zk_yk_z(k_z^2 - k_y^2)$. However, this is not generic: For example, the point symmetry of Li$_2$(Pd$_{1-x}$Pt$_x$)$_3$B is described by the cubic group $G = O$, which contains only the rotations about the axes of the second, third, and fourth order. In this case the function $\gamma(k)$ transforms according to the vector representation $F_1$:

$$\gamma(k) = \gamma_0\phi_{F_1,u}(k).$$

The representative expression for the basis function is simply $\phi_{F_1,u}(k) \sim (k_x, k_y, k_z)$, so that the SO coupling vanishes at the point $k_x = k_y = k_z = 0$.

The diagonalization of $H_0$ yields the following eigenvalues and eigenstates:

$$\xi_\lambda(k) = \epsilon_0(k) + \lambda|\gamma(k)|,$$

and

$$\chi_{\lambda\lambda}(r,\alpha) = \frac{1}{\sqrt{V}} u_{\alpha\lambda}(k)e^{ikr},$$

where $\lambda = \pm$ is the band index, $V$ is the system volume,

$$u_{\lambda\lambda}(k) = e^{i\theta_{\lambda}} \frac{\sqrt{|\gamma| + \lambda\gamma_z}}{2|\gamma|},$$

and $\theta_{\lambda}$ are arbitrary (in general $k$-dependent) phases. The Bloch spinor components $u_{\alpha\lambda}$ form a unitary matrix $\phi(k)$. It follows from Eq. (5) that in the presence of SO coupling the degeneracy of the electron bands is lifted everywhere in the Brillouin zone, except maybe for some high-symmetry lines or points, where $\gamma(k) = 0$ (the band structure might be such that the zeros of $\gamma$ are not located on the Fermi surface). For the typical band splitting $E_{SO}$ one can use, for instance, the Fermi-surface average of $2|\gamma(k)|$.

The band representation of the free-electron Hamiltonian has the following form:

$$H_0 = \sum_{k} \sum_{\lambda = \pm} \xi_\lambda(k)c^\dagger_{k\lambda}c_{k\lambda}.$$
The band dispersion functions \( \tilde{w}(k) \) are even in \( k \) due to time reversal symmetry: the states \( \chi_{k\lambda} \) and \( K\chi_{k\lambda} \) belong to \( k \) and \( -k \), respectively, and have the same energy. Here \( K = i\sigma_2K_0 \) is the time reversal operation, and \( K_0 \) is the complex conjugation.

Writing the field operators in Eq. (2) in the form
\[
\psi_n(r) = \sum_{k\lambda}^\dagger \chi_{k\lambda}(r) c_{k\lambda},
\]
we obtain the band representation of the impurity Hamiltonian:
\[
H_{\text{imp}} = \frac{1}{2} \sum_{kk'} \sum_{\lambda\lambda'} \tilde{U}_{\lambda\lambda'}(k, k') c_{k\lambda}^\dagger c_{k'\lambda'},
\]
where
\[
\tilde{U}_{\lambda\lambda'}(k, k') = U(k - k') w_{\lambda\lambda'}(k, k'),
\]
\( U(q) \) is the Fourier transform of the impurity potential, 
\[
\langle U(q)U(q') \rangle = n_{\text{imp}}U_0^2 \delta_{q,-q'},
\]
and
\[
\tilde{w}(k, k') = \tilde{u}(k)\tilde{u}^\dagger(k').
\]

We see that the matrix elements of the anisotropy factor \( \tilde{w} \) satisfy the following useful identity:
\[
|w_{\lambda\lambda'}(k, k')|^2 = \frac{1 + \lambda\lambda'\tilde{\gamma}(k)\tilde{\gamma}(k')}{2},
\]
which can be easily verified using the explicit expressions
\[\text{(9)}\]
for the Bloch spinors.

The electron Green’s function in the band representation is introduced in the standard fashion: \[\text{(4)}\]
\[
G_{\lambda\lambda'}(k, \tau; k', \tau') = -\langle T_\tau c_{k\lambda}(\tau)c_{k'\lambda'}(\tau') \rangle.
\]
In the absence of impurities it has the following form:
\[
G_{0,\lambda\lambda'}(k, \omega_n) = \frac{\delta_{\lambda\lambda'}}{i\omega_n - \xi_{\lambda}(k)},
\]
where \( \omega_n = (2n + 1)\pi T \) is the fermionic Matsubara frequency (we use the units in which \( k_B = 1 \)). Now we will show that, even when the disorder is included, the average Green’s function remains band-diagonal.

The disorder averaging with the Hamiltonian \[\text{(9)}\] can be performed using the standard methods, see Ref. \[\text{(11)}\]. The result is the matrix Dyson equation of the form
\[
\tilde{G}^{-1} = G_0^{-1} - \tilde{\Sigma},
\]
where \( G \) is the average Green’s function and \( \tilde{\Sigma} \) is the impurity self-energy. In the Born approximation
\[
\tilde{\Sigma}(k, \omega_n) = n_{\text{imp}}U_0^2 \times \int \frac{d^3k'}{(2\pi)^3} \tilde{w}(k, k') \tilde{G}(k', \omega_n) \tilde{w}(k', k)
\]
(here we have taken the thermodynamic limit \( \mathcal{V} \to \infty \)). The diagrammatic representation of the self-energy is given in Fig. 1. We seek solution of the Dyson equation in a band-diagonal form: \( G_{\lambda\lambda'} = G_\lambda \delta_{\lambda\lambda'} \), then the integrand on the right-hand side of Eq. \[\text{(10)}\] becomes
\[
\tilde{w}(k, k') \tilde{G}(k', \omega_n) \tilde{w}(k') = \frac{G_+(k', \omega_n) + G_-(k', \omega_n)}{2} \tilde{\zeta}(k')
\]
\[\text{(11)}\]

where \( \tilde{\zeta} \) is the Pauli matrices, and \( \tilde{\zeta}(k) = \tilde{u}(k)\tilde{\gamma}(k)\tilde{u}(k) \). Using the expressions \[\text{(7)}\], one obtains \( \tilde{\zeta}(k) = \tilde{\gamma}(k)\tilde{\gamma}(k) \), which is an odd function of \( k \) \( (\tilde{\gamma} = \gamma/\gamma) \). Therefore the last line in Eq. \[\text{(10)}\] vanishes after the momentum integration, and \( \tilde{\Sigma}(k, \omega_n) = \Sigma(\omega_n)\tilde{\tau}_0 \). The real part of the self-energy renormalizes the chemical potential, and for the imaginary part we obtain:
\[
\text{Im} \Sigma(\omega_n) = -\pi n_{\text{imp}}U_0^2 N_+ + N_- \frac{\text{sign} \omega_n}{2},
\]
where \( N_\lambda = \mathcal{V}^{-1} \sum_k \delta|\chi_{\lambda}(k)| \) is the Fermi-level density of states in the \( \lambda \)th band. In order to preserve the usual form of the impurity self-energy in the normal state, we introduce the notation \( N_F = (N_+ + N_-)/2 \) and also define the elastic mean free time as \( \tau = \frac{2\pi n_{\text{imp}}U_0^2 N_F^{-1}}{\mathcal{V}} \). Then the average Green’s function takes the form
\[
G_{\lambda\lambda'}(k, \omega_n) = \frac{\delta_{\lambda\lambda'}}{i\omega_n - \xi_{\lambda}(k) + (i/2\tau) \text{sign} \omega_n},
\]
\[\text{(12)}\]

The derivation above is valid under the assumption that the elastic scattering rate is small compared with the Fermi energy \( \epsilon_F \), which justifies neglecting the diagrams with crossed impurity lines in the self-energy in Fig. \[\text{(1)}\].

### A. Effects of magnetic field

In the presence of a nonzero uniform magnetic field \( B \) Eq. \[\text{(3)}\] is replaced by
\[
H_0 = \sum_{k\lambda} \tilde{c}_{k\lambda}^\dagger \mathcal{E}_\lambda(k) \tilde{c}_{k\lambda},
\]
\[\text{(13)}\]
where $\mathcal{E}_\lambda$ is the effective band Hamiltonian in the $k$-space. It was first pointed out by Peierls\textsuperscript{42} based on the requirement of gauge invariance, that the magnetic field can be included in the band electron theory by simply replacing the wave vector $k$ in the zero-field band dispersion $\mathcal{E}_\lambda(k)$ by the operator $\mathbf{K} = k + (e/\hbar c)\mathbf{A}(\mathbf{r})$, where $\mathbf{r} = i\nabla k$ is the position operator in the $k$-representation and $\epsilon$ is the absolute value of the electron charge. We use the symmetric gauge, in which

$$\mathbf{K} = k + \frac{ie}{2\hbar c}(\mathbf{B} \times \nabla k).$$

The Peierls substitution gives only the zero-order term in the expansion of the effective single-band Hamiltonian in powers of $\mathbf{B}$, which in our case can be written as

$$\mathcal{E}_\lambda(k) = \xi_\lambda(k) - \mathbf{Bm}_\lambda(k) + \ldots$$  \hspace{1cm} (19)

The second term on the right-hand side is the analog of the Zeeman interaction for nondegenerate bands.$^{44}$ In the generalized Rashba model it has the following form:

$$m_\lambda(k) = \lambda \mu_B \gamma(k)$$  \hspace{1cm} (20)

($\mu_B$ is the Bohr magneton), which is valid everywhere except for the vicinity of the band crossing points, where the approximation of independent nondegenerate bands fails and the effective band Hamiltonian approach might not work.

It is convenient to introduce the Fourier transform of the Green’s function$^{13}$:

$$G_{\lambda\lambda'}(k, k'; \omega_n) = \frac{1}{V} \sum_{kk'} e^{ikr - ik'r'} G_{\lambda\lambda'}(k, k'; \omega_n).$$  \hspace{1cm} (21)

We would like to stress that this is not the same as the electron Green’s function in the coordinate-spin representation. The latter is defined as $(\langle r| e^{i\mathbf{K}\cdot\mathbf{r}} e^{-i\mathbf{K}'\cdot\mathbf{r}'} |\lambda\rangle) = \frac{1}{\sqrt{V}} \sum_{kk'} e^{ikr - ik'r'} \langle r| G_{\lambda\lambda'}(k, k'; \omega_n) |\lambda\rangle$, where $\langle r| G(k, \omega_n) |\rho\rangle$ is the spinor wave function,$^{9}$ with $\alpha, \beta = \uparrow, \downarrow$.

The Green’s function (21) satisfies the equation

$$(i\omega_n \mathbf{\hat{h}}_0 - \mathbf{\hat{h}}_{\text{imp}}) G(r, r'; \omega_n) = \hat{\tau}_0 \delta(r - r').$$  \hspace{1cm} (22)

Here $h_{0,\lambda\lambda'} = \delta_{\lambda\lambda'} \mathcal{E}_\lambda(\mathbf{K}_r)$ is obtained by replacing $\mathbf{K}$ in Eq. (19) by

$$\mathbf{K}_r = -i\nabla_r + \frac{e}{\hbar c} \mathbf{A}(r) = -i\nabla_r + \frac{e}{2\hbar c}(\mathbf{B} \times \mathbf{r}),$$  \hspace{1cm} (23)

and $\mathbf{\hat{h}}_{\text{imp}}$ is the Fourier transform of the impurity Hamiltonian$^{9}$. The latter is nonlocal in real space and can be represented as a differential operator of infinite order:

$$(\mathbf{\hat{h}}_{\text{imp}} G)_{\lambda\lambda'}(r, r'; \omega_n) = \sum_{\lambda k} [U_{0,\lambda\lambda'}(r) + \mathbf{U}_{1,\lambda\lambda'}(r) \mathbf{K}_r + \ldots] G_{\lambda\lambda'}(r, r'; \omega_n),$$  \hspace{1cm} (24)

where

$$U_{0,\lambda\lambda'}(r) = \frac{1}{V} \sum_p e^{ipr} \tilde{U}_{\lambda\lambda'} \left( \frac{p}{2}, -\frac{p}{2} \right),$$

$$U_{1,\lambda\lambda'}(r) = \frac{1}{V} \sum_p e^{ipr} \times 2(\nabla_p - \nabla_{p'}) \tilde{U}_{\lambda\lambda'} \left( \frac{p}{2}, -\frac{p'}{2} \right) \big|_{p'=p},$$

etc. The magnetic field affects the electron Green’s function in Eq. (22) through the vector potential in the operators $\mathbf{K}_r$ (the “orbital effect”) as well as through the $m$-term in Eq. (19) (the “paramagnetic effect”).

In the absence of impurities the solution of Eq. (22) is band-diagonal and can be represented in a factorized form$^{44}$

$$G_{\lambda\lambda'}(r, r'; \omega_n) = \tilde{G}_{\lambda\lambda'}(r, r'; \omega_n) \times \exp \left[ i\frac{e}{\hbar c} \int_r^{r'} \mathbf{A}(r) dr \right],$$  \hspace{1cm} (25)

where the integration in the phase factor is performed along a straight line connecting $r$ and $r'$, and $\tilde{G}_{\lambda\lambda'}(r, r'; \omega_n) = \delta_{\lambda\lambda'} \tilde{G}_{0,\lambda}(r - r'; \omega_n)$. In superconductors the orbital effect of magnetic field on $\tilde{G}_0$ can usually be neglected$^{45}$ and one obtains:

$$\tilde{G}_{0,\lambda}(r, \omega_n) = \frac{1}{i\omega_n - \xi_\lambda(k) + m_\lambda(k)B}.$$  \hspace{1cm} (26)

see also Ref. 31.

The same argument can be used also in the disordered case, to show that the Green’s function before disorder averaging has the form$^{25}$, where $\tilde{G}_{\lambda\lambda'}$ satisfies Eq. (24), in which the vector potential is formally set to zero, but the paramagnetic term is still present. In order to perform disorder averaging of $\tilde{G}_{\lambda\lambda'}$, we go back into the band-momentum representation$^{14}$ and repeat the steps leading to Eq. (15). The only difference is that now the Green’s function is no longer even in $\mathbf{k}$ due to the presence of the paramagnetic term with $m_\lambda(k) = -m_\lambda(-k)$, see Eq. (26). Therefore the last line in Eq. (19) does not vanish identically after the momentum integration, and the impurity self-energy is no longer band-diagonal. One can show however that the off-diagonal contributions to $\Sigma_{\lambda\lambda'}$ are of the order of $\mu_B B N_p$, which can be neglected. Therefore we arrive at the following expression for the impurity-averaged $\tilde{G}_{\lambda\lambda'}$:

$$\langle \tilde{G}_{\lambda\lambda'}(k, k'; \omega_n) \rangle_{\text{imp}} = \delta_{\lambda\lambda'} \delta_{k, k'} \tilde{G}_\lambda(k, \omega_n),$$  \hspace{1cm} (27)

where

$$\tilde{G}_\lambda(k, \omega_n) = \frac{1}{i\omega_n - \xi_\lambda(k) + m_\lambda(k)B + (i/2\gamma) \text{sign} \omega_n}.$$  \hspace{1cm} (28)

While the orbital effect of the field is reduced to the phase factor in the Green’s function, see Eq. (25), the Zeeman
interaction survives in $\tilde{G}_{\lambda\lambda'}$ and affects the quasiparticle energies. The Zeeman term will be eventually dropped, because its effect on the GL free energy turns out to be negligible, see the next section.

III. SUPERCONDUCTING STATE

Now let us take into account an attractive interaction between electrons in the Cooper channel, using the basis of the exact eigenstates of the noninteracting problem. The total Hamiltonian is given by $H = H_0 + H_{\text{imp}} + H_{\text{int}}$, where the first two terms are given by Eqs. (25) and (26), and the last term has the following form:

$$H_{\text{int}} = \frac{1}{2V} \sum_{kk'q} \sum_{\lambda\lambda'} V_{\lambda\lambda'}(k,k') c_{k+q/2,\lambda}^\dagger c_{-k+q/2,\lambda'}^\dagger \times e^{-i\phi_{\lambda}(k)-i\phi_{\lambda'}(k')} (29)$$

We assume, in the spirit of the Bardeen-Cooper-Schrieffer (BCS) theory, that the pairing interaction is nonzero only inside the thin shells of width $\varepsilon_c$ in the vicinity of the Fermi surfaces, i.e. when $|\phi_{\lambda}(k)|,|\phi_{\lambda'}(k')| \leq \varepsilon_c$. The pairing potential satisfies $V_{\lambda\lambda'}(-k,k') = V_{\lambda\lambda'}(k,-k') = -V_{\lambda\lambda'}(k,k')$, which follows from the anti-commutation of the fermionic operators. The diagonal elements of the matrix $V$ describe the intraband Cooper pairing, while the off-diagonal ones correspond to the pair scattering from one band to the other. The SO splitting of the bands is assumed to be large compared with all the energy scales associated with superconductivity. In this case the formation of the pairs of electrons belonging to different bands is strongly suppressed. Although the bands may touch at some isolated points at the Fermi surface, the interband pairing in the vicinity of those points is still suppressed due to the phase space limitations.

The pairing potential can be represented in a factorized form:

$$V_{\lambda\lambda'}(k,k') = -V_{\lambda\lambda'} t_\lambda(k) t_{\lambda'}(k') \phi_{\lambda}(k) \phi_{\lambda'}(k'), (30)$$

where the coupling constants $V_{\lambda\lambda'}$ form a symmetric positive-definite $2 \times 2$ matrix, $t_\lambda(k) = -t_\lambda(-k)$ are non-trivial phase factors, see Refs. 14,15, and $\phi_{\lambda}(k) \equiv \phi_{\lambda}^{(\Lambda)}(k)$ are even basis functions of an irreducible representation $\Gamma$ of the point group $G$ of the crystal (we keep only the representation which corresponds to the pairing channel with the maximum critical temperature).20 While $\phi_{\lambda}(k)$ and $\phi_{\lambda'}(k)$ have the same symmetry, their momentum dependence does not have to be exactly the same. We consider only one-dimensional representations of $G$, because, to the best of our knowledge, there is no experimental evidence of multi-dimensional superconductivity in real noncentrosymmetric materials.

The basis functions are nonzero only inside the BCS shells and are normalized: $\langle \phi_{\lambda}(k) | \phi_{\lambda}(k') \rangle = 1$, where the angular brackets denote the averaging over the Fermi surface in the $\lambda$th band:

$$\langle \phi_{\lambda}(k) | \phi_{\lambda}(k') \rangle = \frac{1}{N_{\lambda}} \sum_k \langle \phi_{\lambda}(k) | \phi_{\lambda}(k') \rangle \delta[\xi_{\lambda}(k)].$$

The phase factors $t_\lambda(k)$ originate in the expression for the time reversal operation for nondegenerate band electrons:

$$K(k\lambda) = t_\lambda(k)|-k,\lambda\rangle. (31)$$

For instance, for the eigenstates (27) we obtain:

$$t_\lambda(k) = \lambda e^{-i[\theta_{\lambda}(k)+\theta_{\lambda'}(k')]} \frac{\gamma_{\lambda}(k) - i\gamma_{\lambda'}(k')}{\sqrt{\gamma_{\lambda}^2(k) + \gamma_{\lambda'}^2(k')}}. (32)$$

Since the factors $t_\lambda$ explicitly depend on the arbitrary phases $\theta_{\lambda}(k)$, they must drop out of the final expressions for all observable quantities.

It is instructive to see how the phase factors $t_\lambda(k)$ appear in a simple BCS-like model in which the pairing interaction is local in real space:

$$H_{\text{int}} = -\frac{V}{2} \sum_{\alpha\beta=\uparrow,\downarrow} \int d^3r \psi_\alpha^\dagger(r) \psi_\beta^\dagger(r) \psi_\beta(r) \psi_\alpha(r), (33)$$

where $V > 0$ is the coupling constant. Using the band representation of the field operators, we obtain:

$$H_{\text{int}} = \frac{1}{2V} \sum_{kk'q} \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} V_{\lambda_1\lambda_2\lambda_3\lambda_4}(k,k') \times c_{k+q/2,\lambda_1}^\dagger c_{-k+q/2,\lambda_2}^\dagger c_{-k'+q/2,\lambda_3} c_{k'+q/2,\lambda_4}, (34)$$

where

$$V_{\lambda_1\lambda_2\lambda_3\lambda_4}(k,k') = -V \sum_{\alpha\beta} u_{\alpha\lambda_1}(k) u_{\beta\lambda_2}^*(k') \times e^{-i\phi_{\beta}(k') - i\phi_{\alpha}(k)} u_{\alpha\lambda_4}(k'),$$

Here we replaced $u_{\alpha\lambda}(\pm k + q/2)$ by $u_{\alpha\lambda}(\pm k)$, neglecting the corrections of the order of $O(q/kr)$. The expression (34) contains both intra- and interband pairing terms. For the reasons explained above, we neglect the latter and set $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \lambda_4 = \lambda'$, which reduces the Hamiltonian to the form (29). Using the identities

$$u_{\alpha\lambda}(-k) = t_\lambda^*(k) \sum_{\beta} (i\sigma_2)_{\alpha\beta} u_{\beta\lambda}^*(k'), (35)$$

which follow from the definition (31) of the phase factor $t_\lambda(k)$, we obtain:

$$V_{\lambda\lambda'}(k,k') = -V t_\lambda(k) t_{\lambda'}(k') |w_{\lambda\lambda'}(k,k')|^2 = -V t_\lambda(k) t_{\lambda'}(k') \frac{1 + \lambda' \gamma(k) \gamma(k')}{2},$$

see Eq. (12). The terms in $V$ containing the $\gamma$’s are even in both $k$ and $k'$. Using the anti-commutation of the
fermionic operators one can easily show that these terms drop out of the Hamiltonian. Finally,
\[
H_{\text{int}} = -\frac{V}{4} \sum_{kk'} \sum_{\lambda \lambda'} t_{\lambda}(k) t_{\lambda'}(k') \times c_{k+q/2, \lambda}^\dagger c_{k, \lambda'}^\dagger,
\]

Thus the band representation of the pairing potential in the model, necessarily contains the phase factors \(t_{\lambda}(k)\). The gap symmetry corresponds to the unity representation with \(\phi_\lambda(k) = 1\), and all coupling constants take the same value: \(V_{\lambda \lambda'} = V/2\).

**A. Derivation of the free energy functional**

The superconducting order parameter in the static case can be represented as
\[
\Delta_\lambda(k, q) = \eta_\lambda(q) t_{\lambda}(k) \phi_\lambda(k),
\]

where the bosonic fields \(\eta_\lambda\) play the role of the order parameter components. The free energy \(\mathcal{F}\) (more precisely, the difference between the free energies of the superconducting and the normal states at the same temperature and field) is a functional of \(\eta_\lambda\). In the vicinity of the transition temperature at arbitrary field the order parameter is small, and one can keep in the free energy expansion only the terms quadratic in \(\eta_\lambda\). Following the procedure outlined, e.g. in Ref. 31, we obtain the impurity-averaged free energy in the form \(\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2\), where
\[
\mathcal{F}_1 = \frac{1}{\mathcal{V}} \sum_{\lambda \lambda'} \sum_q \eta_{\lambda'}(q) V_{\lambda \lambda'}^{-1} \eta_{\lambda}(q),
\]

and
\[
\mathcal{F}_2 = \frac{1}{2} \sum_{\lambda \lambda'} \sum_q \eta_{\lambda'}(q) V_{\lambda \lambda'}^{-1} \eta_{\lambda}(q) \times G_{\lambda \lambda'}(-k + \frac{q}{2}, -k' + \frac{q}{2}, \omega_n) \times G_{\lambda \lambda'}(-k + \frac{q}{2}, -k' + \frac{q}{2}, \omega_n). \tag{39}
\]

This expression is applicable for any pairing symmetry described by a one-dimensional representation of \(\mathcal{G}\), and for any band structure. There are corrections to \(\mathcal{F}_2\) of the order of \((T_c/\epsilon_F)^2\), related to the orbital magnetism of the Cooper pairs [38], which we neglect. The GL gradient expansion in all orders can be obtained from the Taylor expansion of Eqs. (38) and (39) in powers of \(q\), by making the replacement
\[
q \rightarrow D = -i \nabla + \frac{2e}{\hbar c} A(r)
\]
in the final expressions.

The disorder averaging in Eq. (39) involves summing the impurity ladder diagrams, in which the average Green’s function is given by Eqs. (27), (28). In the thermodynamic limit \(\mathcal{V} \rightarrow \infty\), the momentum sums are replaced by integrals, and we obtain:
\[
\mathcal{F}_2 = \frac{1}{\mathcal{V}} \sum_{\lambda \lambda'} \int \frac{d^d q}{(2\pi)^3} T \sum_n \int \frac{d^d k}{(2\pi)^3} \Delta_\lambda(k, q) \times G_{\lambda}(k + \frac{q}{2}, \omega_n) G_{\lambda'}(-k + \frac{q}{2}, -\omega_n) \times D_{\lambda}(k, q, \omega_n). \tag{41}
\]

The triangle denotes the gap function \(\Delta_\lambda\), the filled triangle – the impurity-renormalized gap function \(D_{\lambda}\), the dashed line corresponds to \(W_{\lambda \lambda'}(k, k')\), and the solid lines are the average Green’s functions of electrons.

**FIG. 2:** The diagrammatic representation of the gap equation. The triangle denotes the gap function \(\Delta_\lambda\), the filled triangle – the impurity-renormalized gap function \(D_\lambda\), the dashed line corresponds to \(W_{\lambda \lambda'}(k, k')\), and the solid lines are the average Green’s functions of electrons.
We introduce forming them into a system of linear algebraic equations. \( \varepsilon \)
glected. The equations for \( \bar{\Delta} \) as a function of the direction of \( \mathbf{k} \), at given \( q \) and \( \omega_n \):

\[
\Delta_\lambda(\mathbf{k}, q, \omega_n) = \eta_\lambda(q) \phi_\lambda(k) + \frac{1}{2T} \sum_{n} \left( \frac{1 + \lambda \gamma(\mathbf{k}) \gamma'(\mathbf{k}')}{2} L_X(k', q, \omega_n) \right. \\
\times \Delta_\lambda(k', q, \omega_n) \left. \right)_{\lambda}, \tag{48}
\]

The phase factors \( t_\lambda(k) \) drop out of the expression \( \text{[41]} \) for \( F_2 \), which takes the following form:

\[
F_2 = -\frac{\pi N_F}{2} \int \frac{d^3q}{(2\pi)^3} \sum_\lambda \eta_\lambda(q) \\
\times T \sum_n \left\langle \phi_\lambda^*(k)L_X(\mathbf{k}, q, \omega_n)\Delta_\lambda(\mathbf{k}, q, \omega_n) \right\rangle_{\lambda}, \tag{49}
\]

We note that only the diagrams containing \( \bar{G}_+ \) and \( \bar{G}_- \) have been included in the impurity ladder in Eq. \( \text{[42]} \). The contribution from the diagrams with \( \bar{G}_+ \bar{G}_- \) is smaller by a factor of \( \max(\epsilon_c, \tau^{-1})/E_{SO} \ll 1 \) and neglected.

The equations for \( \bar{\Delta}_\lambda(\mathbf{k}, q, \omega_n) \) can be solved by transforming them into a system of linear algebraic equations. We introduce

\[
X_a(\mathbf{k}, q, \omega_n) = \sum_\lambda \left\langle A_{\lambda,a}(\mathbf{k})L_X(\mathbf{k}, q, \omega_n)\Delta_\lambda(\mathbf{k}, q, \omega_n) \right\rangle_{\lambda}, \tag{50}
\]

where \( a = 0, 1, 2, 3, \) and

\[
A_{\lambda,a}(\mathbf{k}) = \begin{cases} 
1, & a = 0 \\
\lambda \gamma_a(\mathbf{k}), & a = 1, 2, 3 
\end{cases}, \tag{51}
\]

Then Eqs. \( \text{[48]} \) can be written as

\[
\bar{\Delta}_\lambda = \eta_\lambda \phi_\lambda + \frac{1}{4T} \sum_{a=0}^3 A_{\lambda,a}X_a, \tag{51}
\]

and the contribution \( \text{[49]} \) to the free energy takes the following form:

\[
F_2 = -\frac{N_F}{2} \int \frac{d^3q}{(2\pi)^3} \\
\times \left\{ \sum_\lambda |\eta_\lambda(q)|^2T \sum_n \left\langle |\phi_\lambda(\mathbf{k})|^2 L_X(\mathbf{k}, q, \omega_n) \right\rangle_{\lambda} \right. \\
+ \frac{1}{4T} \sum_\lambda \eta_\lambda(q) \pi T \sum_n \sum_{a=0}^3 X_a(q, \omega_n) \\
\times \left\langle \phi_\lambda^*(\mathbf{k})A_{\lambda,a}(\mathbf{k})L_X(\mathbf{k}, q, \omega_n) \right\rangle_{\lambda} \tag{52}
\]

From Eqs. \( \text{[50]} \) and \( \text{[51]} \) we obtain a system of four algebraic equations for \( X_a(q, \omega_n) \):

\[
\sum_{b=0}^3 (\delta_{ab} - B_{ab})X_b = Y_a, \tag{53}
\]

where

\[
B_{ab}(q, \omega_n) = \frac{1}{4T} \sum_\lambda \left\langle A_{\lambda,a}(\mathbf{k})A_{\lambda,b}(\mathbf{k})L_X(\mathbf{k}, q, \omega_n) \right\rangle_{\lambda}, \\
Y_a(q, \omega_n) = \sum_\lambda \eta_\lambda(q) \left\langle \phi_\lambda^*(\mathbf{k})A_{\lambda,a}(\mathbf{k})L_X(\mathbf{k}, q, \omega_n) \right\rangle_{\lambda}. \tag{54}
\]

We see that, since \( Y_a \) depend linearly on \( \eta_\lambda \), so do \( X_a \), and the right-hand side of Eq. \( \text{[52]} \) is a bilinear functional of the order parameter components. The total free energy, which also includes the contribution \( \text{[38]} \), can be written in the following form:

\[
F = \int \frac{d^3q}{(2\pi)^3} \sum_{\lambda\lambda'} \eta_\lambda(q) f_{\lambda\lambda'}(q, \mathbf{B}) \eta_{\lambda'}(q). \tag{54}
\]

The explicit expressions for the coefficients \( f_{\lambda\lambda'} \) can be derived in principle at arbitrary \( q \) and \( B \), using the solution of the linear equations \( \text{[53]} \). This procedure would allow one to calculate the upper critical field \( H_{c2} \) at any temperature. We focus on the weak-field limit in the vicinity of the critical temperature, in which case Eqs. \( \text{[39]} \) can be solved using a gradient expansion, see Sec. IIIIB below.

We note that the free energy of our system is formally equivalent to that of a two-band singlet superconductor, in which the impurity scattering amplitude, the pairing symmetry, and the Zeeman coupling are all anisotropic. Significant simplifications are achieved, for example, in the case when the SO band splitting is so large that there is only one nondegenerate band (say, the “+”-band) crossing the Fermi level. This is formally described by setting \( \rho_- = 0 \) and \( V_{\lambda+} = V_{\lambda-} = 0 \). We will refer to this case as the single-band limit. On the other hand, in the model \( \text{[38]} \) the matrix \( V \) is degenerate and the whole formalism should be modified. One can show that in this case the pairing interaction \( \text{[39]} \) can be decoupled using just one bosonic field \( \eta(q) \), so that the gap functions are
given by \( \Delta_{\lambda}(k, q) = \eta(q)t_{\lambda}(k) \), and the GL theory has the same form as in the standard, i.e. one-component isotropic singlet, case. The physical properties of these special cases will be studied below by taking the appropriate limits in the general two-band expressions.

### B. Gradient expansion

The gradient and field expansion of the free energy \((54)\) is obtained by expanding both \( L_\lambda \) and \( X_a \) in Eq. \((52)\) in powers of \( \Omega_\lambda \). In order to solve Eq. \((53)\) it is convenient to represent it in the symbolic form \((1 - \delta B) X = Y \). The \(4 \times 4\) matrix \( \delta B \) can be written as \( \delta B = \delta B_0 + \delta \delta B \), where in the first term \( \Omega_\lambda \) is set to zero, while the second term contains the \( \Omega_\lambda \)-dependent corrections. Similarly, \( Y = Y_0 + \delta Y \) and \( X = X_0 + \delta X \), where \( X_0 = (1 - \delta B_0)^{-1} Y_0 \). Then,

\[
\delta X = (1 - \delta B_0 - \delta \delta B)^{-1}(\delta Y + \delta \delta X_0) \\
= (1 - \delta B_0)^{-1}(\delta Y + \delta B X_0) \\
+ (1 - \delta B_0)^{-1}\delta \delta B(1 - \delta B_0)^{-1}(\delta Y + \delta B X_0) + ...
\]

Inserting this in Eq. \((52)\), we obtain after some lengthy but straightforward algebra:

\[
\hat{f} = \hat{f}_0 + \hat{f}_2 + ..., \quad (55)
\]

where \( \hat{f}_m \) denotes the terms of the order of \( \Omega_m \). The terms with odd powers of \( \Omega_\lambda \) are proportional to \( \sin \omega_n \) and vanish after the summation over the Matsubara frequencies.

The uniform and field-independent contribution is given by

\[
A_{\lambda\lambda'} = f_{0,\lambda\lambda'} = \frac{N_F}{2} \left[ \frac{1}{N_F} V_{\lambda\lambda'}^{-1} - \rho_\lambda \delta_{\lambda\lambda'} \cdot S_{01} \\
- \frac{1}{4T} \rho_\lambda \rho_{\lambda'} \langle \phi_\lambda \rangle \langle \phi_{\lambda'} \rangle S_{11} \right]. \quad (56)
\]

To make the notations compact, here and below we omit the arguments of the basis functions and the subscripts \( \lambda, \) in the Fermi-surface averages, and also assume that the basis functions are real. The Matsubara sums \( S_{kl} \) are defined as follows:

\[
S_{kl} = \pi T \sum_n \frac{1}{\omega_n} \frac{1}{(\omega_n + 1/2T)^4}. \quad (57)
\]

The logarithmically divergent sum in \( S_{01} \) is cut off at \( n_c \simeq \varepsilon_c/2\pi T \gg 1 \):

\[
S_{01} = 2\pi T \sum_{n=0}^{n_c} \frac{1}{\omega_n + 1/2T} = \ln \frac{2\gamma_{\varepsilon_c}}{\pi T} - F(\tau T). \quad (58)
\]

Here \( \ln \gamma \simeq 0.577 \) is Euler’s constant,

\[
F(x) = \Psi \left( \frac{1}{2} + \frac{1}{4\pi x} \right) - \Psi \left( \frac{1}{2} \right), \quad (59)
\]

and \( \Psi(x) \) is the digamma function. One can also check that \( S_{11} = 2\pi F(\tau T) \). Therefore, we have

\[
A_{\lambda\lambda'} = \frac{N_F}{2} \left[ \frac{1}{N_F} V_{\lambda\lambda'}^{-1} - \rho_\lambda \delta_{\lambda\lambda'} \ln \frac{2\gamma_{\varepsilon_c}}{\pi T} \\
+ \left( \rho_\lambda \rho_{\lambda'} \langle \phi_\lambda \rangle \langle \phi_{\lambda'} \rangle \right) F(\tau T) \right]. \quad (60)
\]

The second-order term in the expansion \((55)\) has the following form:

\[
f_{2,\lambda\lambda'}(q, B) = \frac{N_F}{2} \left[ \rho_\lambda \rho_{\lambda'} \langle \phi_\lambda \rangle \langle \phi_{\lambda'} \rangle S_{13} \\
+ \frac{1}{4T} \langle \phi_\lambda \rangle \langle \phi_{\lambda'} \rangle \left( \sum_{\lambda_i} \rho_{\lambda_i} \Omega_{\lambda_i}^2 \right) S_{23} \\
+ \lambda' \sum_{\lambda_i} \frac{1}{4T} \left( \sum_{\lambda_i} \rho_{\lambda_i} \Omega_{\lambda_i} \right) \left( \langle \phi_\lambda \rangle \langle \phi_{\lambda'} \rangle + \langle \phi_{\lambda'} \rangle \langle \phi_\lambda \rangle \right) \langle \Omega_{\lambda_i} \rangle \\
+ \left( \frac{1}{4\pi T} \right)^2 \langle \phi_\lambda \rangle \langle \phi_{\lambda'} \rangle \sum_{\lambda_i} \left( \sum_{\lambda_i} \rho_{\lambda_i} \langle \Omega_{\lambda_i} \rangle \right)^2 \langle \Omega_{\lambda_i} \rangle \right], \quad (61)
\]

with

\[
\tilde{S}_{kl,i} = \pi T \sum_n \frac{1}{\omega_n} \frac{1}{(\omega_n + 1/2T)^4} \frac{1}{\omega_n + s_i/2T} \quad (62)
\]

and \( s_i = 1 - \sum_{\lambda} \rho_\lambda \langle \Omega_{\lambda}^2 \rangle/\lambda_i \).

Along with the usual second-order gradient terms, the free energy expansion also contains the terms linear in both \( q \) and \( B \), which give rise to the magnetoelectric effect, see Refs. \[7,22\], and also lead to the helical superconducting phases considered in Refs. \[29,31,32\]. It can be shown, however, that these effects are small. Indeed, a typical contribution to \( f_2 \) is of the form \( N_F (\Omega^2)/T^2 \). Minimizing it with respect to \( q \), we find \( q \sim \mu_B H/\hbar v_F \), therefore the correction to the energy due to the Zee- man term in Eq. \((47)\) is proportional to \( (\mu_B H/\tau_T)^2 \sim (T_c/\varepsilon_F)^2 (m^* m)/(H_c^2)^2 \), where \( m^* \) is the effective mass of quasiparticles and \( H_c^2 \) is the upper critical field at \( T = 0 \). This ratio is typically very small (unless the smallness of \( T_c/\varepsilon_F \) is compensated by a large value of the effective mass), so it is legitimate to drop the Zee- man terms and use

\[
\Omega_\lambda(k, q) = \frac{\hbar v_F (k) q}{2}. \quad (63)
\]

As a result, \( f_2 \) becomes a second-degree polynomial in \( q \): \( f_{2,\lambda\lambda'}(q) = \sum_{ij} K_{\lambda\lambda'} ij q_i q_j \), where the coefficients are found from Eq. \((61)\).
Finally, returning to real space and making the substitution \( q \to D \), see Eq. (60), we obtain the GL free energy in the second order of the gradient expansion:

\[
F = \int d^3 r \sum_{\lambda\lambda'} \eta_\lambda^\star (r) \left[ A_{\lambda\lambda'} + \sum_{ij} K_{\lambda\lambda',ij} D_i D_j \right] \eta_{\lambda'}(r),
\]

with \( i, j = x, y, z \). The explicit expansions for the expansion coefficients are determined by the pairing symmetry and given below.

**Unconventional pairing:** The order parameter transforms according to a nontrivial one-dimensional representation of the point group of the system, and \( \langle \phi_\lambda (k) \rangle_\lambda = 0 \). From Eq. (60) we obtain:

\[
A_{\lambda\lambda'} = \frac{N_F}{2} \left[ \frac{1}{N_F} V_{\lambda\lambda'}^{-1} - \rho_\lambda \delta_{\lambda\lambda'} \ln \frac{2 \gamma \varepsilon_c}{\pi T} + \rho_{\lambda} \delta_{\lambda\lambda'} F(\tau T) \right],
\]

with many of the terms in Eq. (61) vanish, giving

\[
K_{\lambda\lambda',ij} = \frac{\hbar^2 N_F}{8} \left[ \rho_\lambda \delta_{\lambda\lambda'} \langle \phi_\lambda^\star v_{\lambda,i} v_{\lambda,j} \rangle S_{03} + \frac{1}{4\tau} \lambda' \rho_\lambda \rho_{\lambda'} \sum_{k=x,y,z} \langle \gamma_k \phi_\lambda v_{\lambda,i} \rangle \langle \gamma_k \phi_\lambda \varepsilon_{\lambda',j} \rangle \tilde{S}_{03,k} \right].
\]

**Conventional pairing:** The order parameter transforms according to the trivial (unity) representation of the point group. For simplicity, we consider only a fully isotropic pairing, for which \( \phi_\lambda (k) = 1 \), then

\[
A_{\lambda\lambda'} = \frac{N_F}{2} \left[ \frac{1}{N_F} V_{\lambda\lambda'}^{-1} - \rho_\lambda \delta_{\lambda\lambda'} \ln \frac{2 \gamma \varepsilon_c}{\pi T} + \frac{\lambda'^2}{2} \rho_{\lambda} + \rho_{\lambda'} F(\tau T) \right],
\]

and

\[
K_{\lambda\lambda',ij} = \frac{\hbar^2 N_F}{8} \left\{ \rho_\lambda \delta_{\lambda\lambda'} \langle \phi_{\lambda,i} v_{\lambda,j} \rangle S_{03} + \frac{1}{4\tau} \rho_\lambda \rho_{\lambda'} \left[ \langle v_{\lambda,i} \rangle \langle v_{\lambda,j} \rangle + \frac{1}{4\tau} \rho_\lambda \rho_{\lambda'} \left[ \langle v_{\lambda,i} \rangle \langle v_{\lambda,j} \rangle \right] S_{13} + \frac{1}{4\tau} \rho_\lambda \rho_{\lambda'} \left[ \langle v_{\lambda,i} \rangle \langle v_{\lambda,j} \rangle \right] S_{23} + \lambda' \sum_{k=x,y,z} \langle \gamma_k \phi_\lambda v_{\lambda,i} \rangle \langle \gamma_k \phi_\lambda v_{\lambda',j} \rangle \tilde{S}_{03,k} + \frac{1}{4\tau} \sum_{k=x,y,z} \left[ \rho_\lambda \langle \gamma_k v_{\lambda,i} \rangle - \rho_\lambda \langle \gamma_k v_{\lambda,j} \rangle \right] \times \left( \lambda' \langle \gamma_k v_{\lambda,i} \rangle + \lambda' \langle \gamma_k v_{\lambda,j} \rangle \right) S_{13,k} + \frac{1}{4\tau^2} \sum_{k=x,y,z} \left( \rho_\lambda \langle \gamma_k v_{\lambda,i} \rangle - \rho_\lambda \langle \gamma_k v_{\lambda,j} \rangle \right) \times \left( \rho_\lambda \langle \gamma_k v_{\lambda,i} \rangle - \rho_\lambda \langle \gamma_k v_{\lambda,j} \rangle \right) \tilde{S}_{23,k} \right\}. \tag{68}
\]

In the next section we use the GL expansion (64) to study the effects of impurities on the critical temperature and the upper critical field.

### IV. Calculation of Observables

#### A. Critical temperature

At high temperatures the matrix \( \hat{A} \) in Eq. (64) is positive definite, therefore the minimum of the free energy is achieved at \( \eta_+ = \eta_- = 0 \), which corresponds to the normal state. The superconducting critical temperature \( T_c \) defined as the temperature below which one of eigenvalues of \( A \) turns negative at zero field, is found from the equation \( det A = 0 \).

It is convenient to introduce the following notation:

\[
g_{\lambda\lambda'} = N_F V_{\lambda\lambda'} \rho_{\lambda} = V_{\lambda\lambda'} N_\lambda. \tag{69}
\]

Since \( \hat{V} \) is positive definite, we have \( g_{++} > 0, g_{--} > 0 \), det \( \hat{g} > 0 \) (note that the matrix \( \hat{g} \) is not symmetric, in general). Using the expression (60), the equation for the critical temperature can be written as \( det(\tau_0 + \hat{g} M) = 0 \), where

\[
M_{\lambda\lambda'} = -\delta_{\lambda\lambda'} \ln \frac{2 \gamma \varepsilon_c}{\pi T} + \left( \delta_{\lambda\lambda'} - \frac{\rho_\lambda}{2} \langle \phi_\lambda \rangle \langle \phi_{\lambda'} \rangle \right) F(\tau T_c). \tag{70}
\]

At \( \tau = \infty \) the second term in \( M_{\lambda\lambda'} \) vanishes, and we obtain the critical temperature of the clean superconductor:

\[
T_{c0} = \frac{2 \gamma \varepsilon_c}{\pi} e^{-1/g}, \tag{71}
\]

where

\[
g = \frac{g_{++} + g_{--}}{2} + \sqrt{\left( \frac{g_{++} - g_{--}}{2} \right)^2 + g_{+-} g_{-+}} \tag{72}
\]

is the effective coupling constant.

We note that in the model (63), in which the pairing is described by a single coupling constant \( V_{\lambda\lambda'} = V/2 \), we have \( g = N_F V \). Although the expression for the critical temperature in this case has the usual BCS form, the superconductivity is non-BCS, because the order parameter resides in two nondegenerate bands (see the discussion in the end of Sec. IIIA), with the critical temperature independent of the band splitting.

In the presence of impurities the cases of conventional and unconventional pairing have to be considered separately.

**Unconventional pairing:** \( \langle \phi_\lambda \rangle = 0 \). In this case, we obtain the following equation for \( T_c \):

\[
\frac{\ln T_{c0}}{T_c} = F(\tau T_c), \tag{73}
\]

where \( F(x) \) is defined by the expression (69). The reduction of the critical temperature is described by a universal function, similar to the suppression of superconductivity by paramagnetic impurities, see Ref. 48. In particular,
at weak disorder, i.e. in the limit $\tau T_{c0} \gg 1$, we have
\[ F(T_c) \approx \pi / 8 \tau T_{c0}, \]
therefore
\[ T_c = T_{c0} - \frac{\pi}{8 \tau}. \]  
(74)

Using the small-$x$ asymptotics
\[ F(x) = \ln \left( \frac{\gamma}{\pi x} \right) + \frac{2 \pi^2}{3} x^2 + O(x^3), \]
we find from Eq. (73) that the superconductivity is completely suppressed at $\tau T_{c0} = \gamma / \pi \simeq 0.567$.

Conventional pairing: $\phi_\lambda = 1$. Instead of Eq. (63) we obtain:
\[ \ln \frac{T_{c0}}{T_c} = \frac{1 + c_1 F(x)}{c_2 + c_3 F(x) + \sqrt{c_4 + c_5 F(x) + c_6 F^2(x)}} - \frac{1}{g}, \]
(76)
where $x = \tau T_c$, and
\[
\begin{align*}
c_1 &= \frac{\rho_+ (g-- - g++) + \rho_- (g++ - g--)}{2}, \\
c_2 &= \frac{g++ + g--}{2}, \\
c_3 &= \frac{\det \hat{g}}{2}, \\
c_4 &= \left( \frac{g++ - g--}{2} \right)^2 + g+-g--, \\
c_5 &= (c_2 - c_1) \det \hat{g}, \\
c_6 &= c_3^2.
\end{align*}
\]

We see that the critical temperature depends on nonmagnetic disorder, but in contrast to the unconventional case, the effect is not described by a universal Abrikosov-Gor’kov function. At weak disorder the suppression is in the linear scattering rate, but with a non-universal slope:
\[ T_c = T_{c0} - \frac{1}{g} \left[ c_1 - \frac{1}{g} \left( c_3 + \frac{c_5}{2 \sqrt{c_4}} \right) \right] \frac{\pi}{8 \tau}. \]  
(77)

In the opposite limit of strong impurity scattering, $\tau T_{c0} \ll 1$ (but still $\tau \varepsilon_c \gg 1$), see Ref. [47], we use $F(x) = \ln(1/x) + O(1) \gg 1$ at $x \to 0$, to find that the critical temperature approaches
\[ T_\ast_c = T_{c0} \exp \left( \frac{1}{g} - \frac{c_1}{2 c_3} \right), \]  
(78)
i.e. superconductivity is not completely destroyed by impurities. The explanation is the same as in the conventional two-gap superconductors, see e.g. Refs. [49],[50],[51].

Interband impurity scattering tends to reduce the difference between the gap magnitudes in the two bands, which costs energy and thus suppresses $T_c$, but only until both gaps become equal. We have checked numerically that, for any positive-definite matrix $\hat{g}$ and for any $\rho_\lambda$ satisfying the constraint $\rho_+ + \rho_- = 2$, both the coefficient in front of $\tau^{-1}$ in Eq. (77) and the exponent in Eq. (78) are negative, i.e. $T_\ast_c < T_{c0}$.

In the model [53], in which the pairing is isotropic and the order parameter has only one component, there is an analog of Anderson’s theorem: The nonmagnetic disorder has no effect on the critical temperature, since the right-hand side of Eq. (76) is identically zero. The same is also true in the single-band limit, i.e. when the only nonzero constants are $g_{++} = g$ and $\rho_+ = 2$.

### B. Upper critical field

To illustrate the effect of disorder on the upper critical field, we consider a tetragonal crystal, $G = C_{4v}$, in the field $B = B_\perp \hat{z}$, in which case $H_{c2}(T)$ can be calculated analytically, see also Ref. [52]. The GL free energy (64) takes the form
\[
\mathcal{F} = \int d^3 r \sum_{\lambda \lambda'} \eta_\lambda^*(r) \left( A_{\lambda \lambda'} + [K^+_{\lambda \lambda'} (D_x^2 + D_y^2) + K^\parallel_{\lambda \lambda'} D_z^2] \right) \eta_{\lambda'}(r),
\]  
(79)
where $A, K^\perp, K^\parallel$ are real symmetric matrices, see Eqs. (67) and (68) in the conventional pairing case, and Eqs. (65) and (66) in the unconventional pairing case.

In order to find the spectrum of the matrix differential operator $O_{\lambda \lambda'} = K^+_{\lambda \lambda'} (D_x^2 + D_y^2) + K^\parallel_{\lambda \lambda'} D_z^2$, we introduce the operators
\[
a_\pm = \sqrt{\frac{\hbar c}{eB}} D_x \pm i D_y, \quad a_3 = \sqrt{\frac{\hbar c}{eB}} D_z.
\]  
(80)
It is easy to check that $a_+ = a_+^\dagger$ and $[a_-, a_+] = 1$, and therefore $a_\pm$ have the meaning of the raising and lowering operators, while $a_3 = a_3^\dagger$ commutes with both of them: $[a_3, a_\pm] = 0$. Representing $O_{\lambda \lambda'}$ in terms of the operators (80), we have
\[
O_{\lambda \lambda'} = \frac{eB}{\hbar c} [K^+_{\lambda \lambda'} (4 a_+ a_- + 2) + K^\parallel_{\lambda \lambda'} a_3^2].
\]  
(81)
To calculate the eigenvalues explicitly, it is convenient to use the basis of the Landau levels $|N,p\rangle$, which satisfy
\[
\begin{align*}
a_+ |N,p\rangle &= \sqrt{N+1} |N+1,p\rangle, \\
a_- |N,p\rangle &= \sqrt{N} |N-1,p\rangle, \\
a_3 |N,p\rangle &= |p,N\rangle,
\end{align*}
\]
where $N = 0, 1, \ldots$ and $p$ determines the modulation along the $z$-axis: $p = k_z \sqrt{\hbar c/eB}$. Writing the order parameter as a linear combination of the Landau levels,
\[
\eta_\lambda(r) = \sum_{N,p} \eta_{\lambda,N,p}(r) |N,p\rangle,
\]

\[ \]
we obtain:

\[ \mathcal{F} = \sum_{N,p} \sum_{\lambda \lambda'} \mathcal{L}_{\lambda \lambda'}(N,p) \eta_{\lambda,N,p} \eta_{\lambda',N,p}, \]  

(82)

where

\[ \mathcal{L}_{\lambda \lambda'}(N,p) = \lambda_{\lambda \lambda'} + \left( K_{\lambda \lambda'}^r (4N + 2) + K_{\lambda \lambda'}^l \right) \frac{eB}{\hbar c}. \]  

(83)

The upper critical field \( H_{c2}(T) \) can be found from the equation \( \det \hat{\mathcal{L}} = 0 \), after maximization with respect to \( N \) and \( p \). We assume that the maximum is achieved at \( N = p = 0 \) and find the following expression for the slope of \( H_{c2}(T) \) at \( B \to 0 \):

\[ R = \left| \frac{dH_{c2}}{dT} \right|_{T=T_c} = \frac{\Phi_0}{2\pi} \left| A_{++}a_{--} + A_{--}a_{++} - 2A_{++}a_{--} \right|, \]  

(84)

where \( \Phi_0 = \pi \hbar c/e \) is the magnetic flux quantum, and

\[ a_{\lambda \lambda'} = \frac{N_F}{2T_c} \left[ \rho_\lambda \delta_{\lambda \lambda'} - \left( \rho_\lambda \delta_{\lambda \lambda'} - \frac{1}{2} \rho_\lambda \rho_\lambda \langle \phi_\lambda \rangle \langle \phi_\lambda \rangle \right) \right. \]

\[ \times \left. \left( \frac{1}{\pi} - \frac{1}{\pi T_c} \right) \right], \]

(85)

While the expression (84) is valid for arbitrary disorder strength, we are especially interested in the limit of weak disorder. At \( \tau T_c > 1 \) the critical temperature can be represented as \( T_c = T_{c0} \left[ 1 - b(\pi/8\tau T_{c0}) \right] \), where the dimensionless coefficients \( b \) takes different values for conventional and unconventional pairing, see Eqs. (77) and (78), respectively. The matrices \( \hat{A}, \hat{K}^\perp, \hat{a} \) can also be expanded in powers of \( \tau^{-1} \):

\[ \hat{A} = \hat{A}_0 + \delta \hat{A}, \quad \hat{K}^\perp = \hat{K}_0^\perp + \delta \hat{K}^\perp, \quad \hat{a} = \hat{a}_0 + \delta \hat{a}, \]  

(86)

where

\[ \hat{A}_{0,\lambda \lambda'} = \frac{N_F}{2} \left( \frac{1}{N_F} V_{\lambda \lambda'} - \rho_\lambda \delta_{\lambda \lambda'} \frac{1}{g} \right), \]

\[ K_{0,\alpha \lambda'}^\perp = \delta_{\alpha \lambda} \left( \frac{7\zeta(3) J_k^2 N_F}{48 \pi^2 T_c^2} \rho_\lambda \langle \phi_\lambda^2 \rangle \right), \]

\[ a_{0,\lambda \lambda'} = \frac{N_F}{2T_c} \rho_\lambda \delta_{\lambda \lambda'}, \]

\[ \zeta(z) \] is the Riemann zeta-function, and

\[ \delta A_{\lambda \lambda'} = \frac{N_F}{2} \left[ (1 - b) \rho_\lambda \delta_{\lambda \lambda'} - \frac{1}{2} \rho_\lambda \rho_\lambda \langle \phi_\lambda \rangle \langle \phi_\lambda \rangle \right] \pi \frac{1}{8\tau T_{c0}}, \]

\[ \delta K_{0,\lambda}^\perp = \frac{h^2 N_F}{96T_{c0}^3} \left[ -3 + \frac{42 \zeta(3) b}{\pi^2} \rho_\lambda \delta_{\lambda \lambda'} \langle \phi_\lambda^2 \rangle \right] \]

\[ + \frac{1}{2} \rho_\lambda \rho_\lambda \left( \langle \phi_\lambda \rangle \langle \phi_\lambda \rangle^2 + \langle \phi_\lambda \rangle \langle \phi_\lambda \rangle^2 \right) \]

\[ + \frac{\lambda'}{2} \rho_\lambda \rho_\lambda \sum_{i=x,y,z} \langle \gamma_i \phi_\lambda v_{\lambda,x} \rangle \langle \gamma_i \phi_\lambda v_{\lambda,x} \rangle \pi \frac{1}{8\tau T_{c0}}, \]

(87)

\[ \delta a_{\lambda \lambda'} = -\frac{1}{T_{c0}} \delta A_{\lambda \lambda'}, \]

[to get this we used \( \zeta(4) = \pi^4/90 \)].

In the clean case, \( A_{0,\lambda \lambda'} \) can be expressed in terms of the matrix elements of \( \hat{g} \), see Eq. (69), which yields

\[ R_0 = \frac{8\pi \Phi_0 T_{c0}}{7 \zeta(3) h^2} \left( 1 - b \right) \frac{g_{++} + g_{--} - 2g^{-1} \det \hat{g}}{\sum_{\lambda}(g_{\lambda \lambda} - g^{-1} \det \hat{g}) \langle \phi_\lambda^2 \rangle^2}, \]

(86)

with \( g \) defined by Eq. (72). In the presence of impurities, we substitute the expansions (85) in Eq. (84) and obtain the correction to the upper critical field slope:

\[ \frac{\delta R}{R_0} = \left[ \frac{D_1}{g_{++} + g_{--} - 2g^{-1} \det \hat{g}} \right] \frac{1}{\sum_{\lambda}(g_{\lambda \lambda} - g^{-1} \det \hat{g}) \langle \phi_\lambda^2 \rangle^2} \pi \frac{1}{8\tau T_{c0}}, \]

(87)

where
\[ D_1 = \frac{1}{2} (\rho_g + g - \rho_g - g) \langle \phi^+ \phi^- \rangle - \sum_{\lambda = \pm} \left[ g_{\lambda\lambda} - \left( 1 + \frac{1}{g} \right) \det \hat{g} \right] \left( 1 - b - \frac{\rho_\alpha}{2} \langle \phi \phi \rangle^2 \right), \]

\[ D_2 = \frac{\pi^2}{21\zeta(3)} \left( \sum_{\lambda = \pm} \left( g_{\lambda\lambda} - \det \hat{g} \right) \left[ \left( \frac{42\zeta(3)b}{\pi^2} - 3 \right) \langle \phi_{\lambda \lambda}^2 \rangle_\text{x,y,z} + \rho_\lambda \langle \phi_{\lambda} \phi_{\lambda}^2 \rangle_\text{x,y,z} + \frac{\rho_\lambda}{2} \sum_{i=x,y,z} \langle \tilde{g}_i \phi_{\lambda} \phi_{\lambda} \rangle_\text{x,y,z} \right] \right. \]

\[ + \frac{1}{2} (\rho_g + g - \rho_g - g) \left( \langle \phi^+ \phi^+ \rangle - \langle \phi^- \phi^- \rangle - \sum_{i=x,y,z} \langle \tilde{g}_i \phi^+ \phi^+ \rangle \langle \tilde{g}_i \phi^- \phi^- \rangle \right) \]

\[ + \det \hat{g} \left[ \left( 1 - b - \frac{\rho_\alpha}{2} \langle \phi^+ \phi^+ \rangle \right) \langle \phi_{x x}^2 \rangle + \left( 1 - b - \frac{\rho_\alpha}{2} \langle \phi^- \phi^- \rangle \right) \langle \phi_{x x}^2 \rangle \right]. \]

It does not seem possible to draw any conclusions from Eq. (87) about the effect of impurities on the slope of the upper critical field enhancement by disorder. For this reason, we just look at two limiting cases.

In the model, the expression takes the form

\[ R_0 = \frac{16\pi^2 \Phi_0 T_{c0}}{7\zeta(3)\hbar^2} \frac{1}{\rho_+(v_{x x}^2) + \rho_-(v_{x x}^2)}, \] (88)

Neglecting the differences between the Fermi velocities and the densities of states in the two bands (these differences are of the order of \( g/\tau_F \) and assuming a spherical Fermi surface, we obtain: \( \rho_+(v_{x x}^2) + \rho_-(v_{x x}^2) \rightarrow 2v_F^2/3 \). In this way we recover Gorkov’s expression for the slope of the upper critical field in a clean isotropic superconductor. In the same approximation, Eq. (87) becomes

\[ \frac{\delta R}{R_0} = \frac{\pi^3}{168\zeta(3)\tau T_{c0}}, \] (89)

which coincides with the one obtained in Ref. 53 for the impurity-induced enhancement of the upper critical field in a conventional isotropic BCS superconductor.

In the single-band limit, setting \( g_{\lambda\lambda} = g_{\lambda\lambda} + \delta_{\lambda+,+} \), \( \rho_\lambda = 2\delta_{\lambda+,+} \), \( v_+^\gamma(k) = v(k) \), \( v_+^\gamma(k) = \phi(k) \), we obtain from Eq. (89):

\[ R_0 = \frac{8\pi^2 \Phi_0 T_{c0}}{7\zeta(3)\hbar^2} \frac{1}{\langle \phi^2 v_{x x}^2 \rangle}, \] (90)

and from Eq. (87):

\[ \frac{\delta R}{R_0} = C \frac{\pi^3}{168\zeta(3)\tau T_{c0}}. \] (91)

The coefficient \( C \) is given by

\[ C = C_{\text{con}} = 1 - \sum_{i=x,y,z} \langle \tilde{g}_i v_x \rangle^2 \] (92)

in the conventional pairing case, and

\[ C = C_{\text{uncon}} = 3 - \frac{42\zeta(3)}{\pi^2} \sum_{i=x,y,z} \langle \tilde{g}_i \phi v_x \rangle^2 \] (93)

in the unconventional pairing case. Since \( 42\zeta(3)/\pi^2 \approx 5.115 \), \( C_{\text{uncon}} \) is always negative, i.e. the upper critical field for unconventional pairing is reduced in the presence of impurities. In the conventional case, the sign and the magnitude of the correction depend on the impurity-induced enhancement of the upper critical field.

C. Application to CePt$_3$Si

Application of our results to real noncentrosymmetric materials is complicated by the lack of a definite information about the superconducting gap symmetry and the distribution of the pairing strength between the bands. One can only make progress by using some simple models. For example, there are experimental indications that most of the Fermi surface in CePt$_3$Si remains normal. Therefore it is plausible that the superconductivity resides in one of the two quasi-two-dimensional $g$ bands. Therefore we take into account only the $g$ bands and assume that, while the “+”-band is superconducting, the “-”-band remains normal, i.e. \( V_+ = V_- = 0 \). In contrast to the pure single-band limit, in the case under consideration \( \rho_\alpha \neq 0 \).

As far as the pairing symmetry is concerned, there is strong experimental evidence that the superconducting order parameter in CePt$_3$Si has lines of gap nodes. The lines of nodes are required by symmetry for all nontrivial one-dimensional representations of $C_{4v}$ (\( A_2, B_1, \) and \( B_2 \)), so that the superconductivity in CePt$_3$Si is most likely unconventional. This can be verified using the measurements of the dependence of $T_c$ on the impurity concentration: For all types of unconventional pairing, the suppression of the critical temperature is described by the universal Abrikosov-Gor’kov function, see Eq. (79). For the upper critical field, one...
obtains from Eq. (87):

$$\frac{\delta R}{R_0} = C \frac{\pi^3}{168 \zeta(3)} \frac{1}{\tau T_c} \rho$$

where

$$C = 3 - \frac{42 \zeta(3)}{\pi^2} - \frac{\rho_+}{2} \sum_i \left( \langle \gamma_i \phi_+ v_{+,x} \rangle \right)^2$$

One can see that for all types of unconventional pairing $C < 0$, i.e. the slope of $H_{c2}$ is reduced by disorder.

It should be mentioned that the lines of gap nodes can exist also for conventional pairing ($A_1$ representation), in which case they are purely accidental. While the accidental nodes would be consistent with the power-law behavior of physical properties observed experimentally, the impurity effect on $T_c$ in this case is qualitatively different from the unconventional case. Indeed, using Eq. (70) we obtain the following equation for the critical temperature:

$$\ln \frac{T_{c0}}{T_{c}} = \left( 1 - \frac{\rho_+}{2} \langle \phi_+ \rangle^2 \right) F(\tau T_c),$$

from which we have

$$T_c = T_{c0} \left( 1 - \frac{b}{8 \pi \tau T_{c0}} \right), \quad b = 1 - \frac{\rho_+}{2} \langle \phi_+ \rangle^2$$

at weak disorder ($\tau T_{c0} \gg 1$), and, using the asymptotical expression (75),

$$T_c = T_{c0} \left( \frac{\pi \tau T_{c0}}{\gamma} \right)^{\alpha}, \quad \alpha = \frac{2}{\rho_+ \langle \phi_+ \rangle^2} - 1$$

at strong disorder ($\tau T_{c0} \ll 1$). From the Schwarz inequality $\langle \phi_+ \rangle^2 \leq \langle \phi_+^2 \rangle = 1$, we find $b \geq \rho_-/2$ and $\alpha \geq \rho_- / \rho_+$. This means that anisotropy of the conventional order parameter increases the rate at which $T_c$ is suppressed by impurities. Unlike the unconventional case, however, the superconductivity is never completely destroyed, even at strong disorder.

V. CONCLUSIONS

We have derived the Ginzburg-Landau free energy of a noncentrosymmetric superconductor with isotropic nonmagnetic impurities, using the microscopic model with a large spin-orbit splitting of the electron bands. If the pairing corresponds to a one-dimensional representation of the crystalline point group, then the order parameter has two components, one for each band, and the GL functional has the same form as for a two-band superconductor (in the spin representation, the order parameter is a mixture of spin-singlet and spin-triplet states, even without a spin-triplet term in the pairing interaction, see Ref. [16]).

We have also studied the impurity effect on the critical temperature $T_c$ and the upper critical field $H_{c2}$. Although $T_c$ is generally suppressed by impurities, this happens differently for conventional and unconventional pairing: Any deviation of $T_c(\tau)$ from the Abrikosov-Gor’kov curve, see Eq. (73), in particular an incomplete suppression of superconductivity by strong disorder, is a signature of conventional pairing symmetry. The impurity effect on the slope of $H_{c2}$ turns out to be sensitive to the pairing symmetry and the band structure, and therefore non-universal.

In the general case, i.e. for arbitrary values of the intraband and interband coupling constants and the densities of states, the microscopic expressions for the coefficients in the GL functional are rather cumbersome and therefore of limited utility. Considerable simplifications occur only in some cases, for instance, in the BCS-like model (33), whose properties resemble a conventional isotropic superconductor: The order parameter has only one component, $T_c$ is not affected by disorder, while $H_{c2}$ increases with disorder. On the other hand, in CePt$_3$Si, where the pairing is likely unconventional, with lines of the gap nodes, we obtain that both $T_c$ and $H_{c2}$ are suppressed by disorder.

As far as experiment is concerned, the upper critical field measurements in CePt$_3$Si, both in polycrystals and in single crystals give the zero-temperature values of $H_{c2}$ exceeding the Clogston-Chandrasekhar expression for the paramagnetic limit, $H_P \simeq 1.24 T_c / \mu_B$. This can be understood as being due to the large residual spin paramagnetic susceptibility, see Refs. [14] [21] [22]. Surprisingly, the values of both the critical temperature and the upper critical field in polycrystals are higher than in single crystals. This is opposite to what has been observed in other unconventional superconductors and also disagrees with our theoretical predictions (assuming that the polycrystalline samples are intrinsically “more disordered” than the single crystals). To resolve this puzzle more work needs to be done on the purity dependence of the properties of noncentrosymmetric superconductors.

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