TOWARDS SHARP BOHNEBLUST–HILLE CONSTANTS

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Abstract. We investigate the optimality problem associated with the best constants in a class of Bohnenblust–Hilde type inequalities for $m$–linear forms. While germinal estimates indicated an exponential growth, in this work we provide strong evidences to the conjecture that the sharp constants in the classical Bohnenblust–Hille inequality are universally bounded, irrespectively of the value of $m$; hereafter referred as the Universality Conjecture. In our approach, we introduce the notions of entropy and complexity, designed to measure, to some extent, the complexity of such optimization problems. We show that the notion of entropy is critically connected to the Universality Conjecture; for instance, that if the entropy grows at most exponentially with respect to $m$, then the optimal constants of the $m$–linear Bohnenblust–Hille inequality for real scalars are indeed bounded universally in $m$. It is likely that indeed the entropy grows as $4^{m-1}$, and in this scenario, we show that the optimal constants are precisely $2^{1-m}$. In the bilinear case, $m = 2$, we show that any extremum of the Littlewood’s $4/3$-inequality has entropy $4$ and complexity $2$, and thus we are able to classify all extrema of the problem. We also prove that, for any mixed $(\ell_1, \ell_2)$-Littlewood inequality, the entropy do grow exponentially and the sharp constants for such a class of inequalities are precisely $(\sqrt{2})^{m-1}$. In addition to the notions of entropy and complexity, the approach we develop in this work makes decisive use of a family of strongly non-symmetric $m$–linear forms, which has further consequences to the theory, as we explain herein.

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1. INTRODUCTION

Let $K$ denote the real or the complex scalar field. Given a positive integer $m$, the Bohnenblust–Hille inequality [7] assures the existence of a constant $B_{K,m} \geq 1$ such that

$$\left( \sum_{j_1, \ldots, j_m = 1}^{\infty} |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{K,m} \|T\|,$$

for all continuous $m$–linear forms $T: c_0 \times \cdots \times c_0 \to K$. Restricting (1.1) to the case $m = 2$ one recovers the famous Littlewood’s $4/3$ inequality [18]. Bohnenblust–Hille inequality is an elegant, far-reaching pearl of classical analysis; for connections with other fields of research, we refer to [5, 12, 20, 24] and references therein.

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The investigation of the sharp constants in the Bohnenblust–Hille inequality, namely the optimization problem

\[ B_{K,m} := \inf \left\{ \left( \sum_{j_1, \ldots, j_m=1}^{\infty} |T(e_{j_1}, \ldots, e_{j_m})|^q \right)^{\frac{1}{q}} \right\}, \text{among all } m\text{-linear forms } T, \text{ with } \|T\| = 1 \]

is fundamental in many aspects of the theory, and a rather challenging mathematical problem. Following classical terminology, a minimum \( T \) for the minimization problem above is called an extremum, or an \( m\)-linear extremum form. It has been known since the seminal work of Bohnenblust–Hille, \([7]\), that

\[ B_{K,m} \leq m^{\frac{1}{2m}} \left( \sqrt{2} \right)^{m-1}, \]

for any \( m \geq 1 \). It was just quite recently that upper bounds for the Bohnenblust–Hille inequality were refined, see for instance \([4]\) and references therein. By means of interpolations, that is, clever usage of Hölder inequality for mixed sums, and with the knowledge of optimal constants in the Khinchin inequality refined, see for instance \([4]\) and references therein, the trivial estimate

\[ B_{K,m} \leq m^{\frac{1}{2m}} \left( \sqrt{2} \right)^{m-1}, \]

where \( \gamma \) is the Euler-Mascheroni constant, and since

\[ m^{\frac{1}{2m}} \approx m^{0.36482}; \quad m^{\frac{1}{m}} \approx m^{0.21139}, \]

the sharp constants grow (at most) sub-linearly with respect to \( m \). The problem of finding good lower bounds for \( B_{K,m} \) turns out to be also delicate. Despite many analytic and numeric attempts, the up-to-now best known lower bounds for \( B_{K,m} \) are still \( 2^{1-\frac{1}{m}} \). In the complex case, nothing is known besides the trivial estimate \( B_{\mathbb{C},m} \geq 1 \).

Recently, the Bohnenblust–Hille inequality has been proved to be part of a much more general class of inequalities, see for instance \([2]\). More than mere generalizations, these broader classes of inequalities reveal importance nuances hidden in the original one. In particular, significant advances in the theory can be acquired by the following extended version of the inequality, see \([2]\):

**Theorem 1.1.** Let \( m \geq 2 \) be a positive integer and \((q_1, \ldots, q_m) \in [1,2]^m\). The following assertions are equivalent:

(A) \((q_1, \ldots, q_m)\) satisfies

\[ \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2}. \]

(B) There exists a constant \( C_{(q_1, \ldots, q_m)}^K \geq 1 \) such that

\[ \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} |T(e_{j_1}, \ldots, e_{j_m})|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_{m-2}}{q_{m-1}}} \right)^{\frac{1}{q_2}} \leq C_{(q_1, \ldots, q_m)}^K \|T\|, \]

for all continuous \( m\)-linear forms \( T: c_0 \times \cdots \times c_0 \to \mathbb{K} \).

When condition (A) is verified, \((q_1, \ldots, q_m) \in [1,2]^m\) is said to be a Bohnenblust–Hille exponent. Hereafter, we will essentially deal with the case \( \mathbb{K} = \mathbb{R} \). It is particularly interesting for our purposes the mixed \((\ell_1, \ell_2)\)-Littlewood inequalities, which refers, in inequality \((1.5)\), to the exponents

\[ \{(1,2,\cdots,2),(2,1,\cdots,2),\cdots(2,2,\cdots,2)\}. \]

Henceforth the mixed \((\ell_1, \ell_2)\)-Littlewood inequalities comprise the existence of positive constants

\[ C_{(1,2,\cdots,2)m}, C_{(2,1,\cdots,2)m}, \ldots, C_{(2,2,\cdots,2,1)m}. \]
such that
\[
\begin{align*}
&\left\{\sum_{j_1=1}^{\infty} \left( \sum_{j_1, j_1+1, \cdots, j_m=1}^{\infty} |U(e_{j_1}, \cdots, e_{j_m})|^2 \right)^{1/2} \right\}^{1/2} \leq C_{(1,2,\cdots,2)m} \|U\|,
&\quad \vdots \nonumber \\
&\quad \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_1, j_1+1, \cdots, j_m=1}^{\infty} |U(e_{j_1}, \cdots, e_{j_m})|^2 \right)^2 \right)^{1/2} \leq C_{(2,2,\cdots,2,1)m} \|U\|,
\end{align*}
\] (1.6)
for all continuous $m$–linear forms $U: c_0 \times \cdots \times c_0 \to \mathbb{R}$, and all $i = 1, \cdots, m$. For the $(\ell_1, \ell_2)$–Littlewood inequalities, it has been proved that
\[
C_{(1,2,\cdots,2)m} = C_{(2,1,2,\cdots,2)m} = \left( \sqrt{2} \right)^{m-1}
\]
gives the sharp constant. While the proof of this fact, see [21], relies on the powerful analysis of Haagerup [15] for finding the best constants of the Khinchin inequality, the strategy cannot be carried out for the other exponents, see [22]. For instance, in the case of the multiple exponent $(2, \cdots, 2, 1)$, previous methods for finding optimal constants simply yield that $\sqrt{2}$ is a lower bound.

In this paper, through a novel approach, we finally show that the optimal constants of all the mixed $(\ell_1, \ell_2)$–Littlewood inequalities are indeed $\left( \sqrt{2} \right)^{m-1}$, that is,
\[
C_{(1,2,\cdots,2)m} = C_{(2,1,2,\cdots,2)m} = \cdots = C_{(2,2,\cdots,2,1)m} = \left( \sqrt{2} \right)^{m-1}.
\] (1.7)

Despite of the infinite-dimensional nature of the Bohnenblust–Hille inequalities, that is, the $m$–linear forms act on $c_0$, it is quite revealing to note that all $m$–linear extrema of the inequalities above mentioned are composed by precisely $4^{m-1} m$ monomials. Such observation raises up the following problem, which seems to play a central role in the theory: how many monomials needed to create an $m$–linear extremum form, i.e., an $m$–linear form which makes the optimal constant of the Bohnenblust–Hille inequality to be attained?

To handle this problem we introduce the notion of entropy (formally defined in the next section) as the minimal number of monomials needed to assemble an extremum of the respective inequality. For instance, the entropy of the exponents of the class of mixed $(\ell_1, \ell_2)$–Littlewood inequalities will be less than or equal to $4^{m-1}$. We expect that the entropy of the classical Bohnenblust–Hille inequality behaves similarly. A stunning implication of such statement is that the optimal constants of the Bohnenblust–Hille inequality would then be uniformly bounded with respect to $m$. Indeed, we will prove in this work that, restricted to $m$–linear forms composed by the combination of up to $4^{m-1} m$ monomials, the optimal constants of the Bohnenblust–Hille inequality for real scalars are precisely $2^{1-\frac{1}{m}}$. This is direct evidence to support the following striking conjecture:

**Universality Conjecture.** The optimal constants in the Bohnenblust–Hille inequality are universally bounded, irrespectively of the value of $m$. In the real case, the best constants should be precisely $2^{1-\frac{1}{m}}$.

Our approach to establish (1.7) makes decisive use of highly non-symmetric $m$–linear forms. As a possible predicament, the most efficient tools known up-to-now to produce upper estimates for the Bohnenblust–Hille constants, namely interpolations, are probably not suited to reach the sharp estimates. We elaborate such considerations in the last section of this article.

2. The notions of entropy and complexity

The optimality problem for the classical Littlewood’s $4/3$ inequality, i.e. the case $m = 2$ in the Bohnenblust–Hille inequality, is relatively well understood in the literature; the optimal constant satisfying (1.1) for the case of real scalars is $\sqrt{2}$. Furthermore, it is attained by the bilinear form
\[
T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2
\] (2.1)
—a simple-looking $2$–linear form comprised of only $4$ monomials. Besides, essentially no other significantly different extremum is known. There is an obvious, empirical relation between the difficulty of establishing the sharp constants in the Bohnenblust–Hille and the algebraic complexity of prospective
extrema. Hence, the following definition is suitable for the purposes of this study and, we deem, itworths further investigation.

**Definition 2.1.** Let m be a positive integer and \((q_1, \ldots, q_m)\) be a Bohnenblust–Hille exponent. The entropy of \((q_1, \ldots, q_m)\) is defined as

\[
\text{ent}^X_m(q_1, \ldots, q_m) = \inf \{ \text{card}(i_1, \ldots, i_m) : \alpha_{i_1, \ldots, i_m} \neq 0 \},
\]

where this infimum is taken over all continuous m–linear forms \(T: c_0 \times \cdots \times c_0 \to \mathbb{K}\) defined by

\[
T(x^{(1)}, \ldots, x^{(m)}) = \sum_{i_1, \ldots, i_m=1}^\infty \alpha_{i_1, \ldots, i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}
\]

such that the optimal constants of the Bohnenblust–Hille inequality with multiple exponents \((q_1, \ldots, q_m)\) is attained by \(T\).

The entropy, \(\text{ent}^X_m(q_1, \ldots, q_m)\), measures, henceforth, the minimal number of monomials necessary to assemble an extremal m–linear form, for which the optimal constant satisfying (1) is reached. Since we will focus our attention in the case of real scalars, hereafter we denote \(\text{ent}^X_m(q_1, \ldots, q_m)\) just by \(\text{ent}_m(q_1, \ldots, q_m)\). We remark that for \(m > 2\), it is not known in general the existence of extrema. In this untoward case, we define \(\text{ent}_m^X(q_1, \ldots, q_m) = \infty\).

Our earlier discussion regarding the explicit 2-form extremum for the Littlewood’s 4/3 inequality, (2.1), reads in the formalism of entropy as

\[
\text{ent}_2 \left( \frac{4}{3}, \frac{4}{3} \right) \leq 4,
\]

and it is in fact simple to verify that equality holds, i.e., \(\text{ent}_2 \left( \frac{4}{3}, \frac{4}{3} \right) = 4\). By similar reasoning, one deduces that

\[
\text{ent}_2(1, 2) = \text{ent}_2(2, 1) = 4.
\]

By obvious reason, \(\text{ent}_1(1) = 1\). We will prove that if the entropies grow like \(4^{m-1}\) (or at least exponentially) with respect to \(m\), then the optimal \(m\)-linear Bohnenblust–Hille constants, for the case of real scalars, should then be bounded (precisely \(2^{1-\frac{4}{m}}\) if the entropies of the Bohnenblust–Hille exponents are \(4^{m-1}\)), confirming henceforth the Conjecture regarding sharp estimates for such class of inequalities.

We conclude this section with a comment on the class of \(m\)-linear forms composed by the sum of monomials whose coefficients are \(\pm 1\) (unimodular monomials), as it plays a relevant – probably central – role in the theory. By way of example, we recall that the proof of the optimality of Theorem [14] or, more precisely, the proof of (B)⇒(A) uses only \(m\)-linear forms composed by the sum of unimodular monomials via the Kahane–Salem–Zygmund inequality; in other words we have (B)⇒(A) even if restrict the inequalities just to \(m\)-linear forms composed by unimodular monomials. As further evidence of the importance of \(m\)-linear forms composed by unimodular monomials, it will be shown later in this work that the optimal constants of any mixed \((\ell_1, \ell_2)\)-Littlewood inequalities are attained by \(m\)-linear forms with coefficients \(\pm 1\) (see also [21]).

**Example 2.2.** In the case of bilinear forms \(T: c_0 \times c_0 \to \mathbb{K}\) of the type \(T(x, y) = ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_1\), using the geometry of the closed unit ball of the space of bilinear forms on \(\ell_\infty^2 \times \ell_\infty^2\), it was proved in [10] that the extrema are essentially the bilinear forms of the type (2.1). If we restrict our attention to \(m\)-linear forms with unimodular monomials, the conjecture that \(2^{1-\frac{4}{m}}\) is sharp constant for the Bohnenblust–Hille inequality is equivalent (using the Krein–Milman Theorem) to the following question in number theory: For all positive integers \(m\) and \(n, i_1, \ldots, i_m \in \{1, \ldots, n\}\) and \(\delta_{i_1, \ldots, i_m} \in \{0, 1, -1\}\), does the following estimate

\[
(2.2) \quad \frac{\text{card} \{ (i_1 \ldots i_m) : \delta_{i_1, \ldots, i_m} \neq 0 \}}{2^{1-\frac{4}{m}}} \leq \max \left\{ \sum_{i_1, \ldots, i_m=1}^n \delta_{i_1, \ldots, i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} : x_{ik} \in \{-1, 1\}, (i) \right\}
\]

hold true? Treating the conjecture in this format brings computational advantages. For instance, equality in (2.2) holds when \(\delta_{i_1, \ldots, i_m}\) follows the pattern of the strategic “classical” \(m\)-linear forms. For \(m = 2\), we know that (2.2) is true.

We conclude this present section by introducing the notion of complexity, which will also play an important role in the analyses to be carried out in the next sections.
Let $T: c_0 \times \cdots \times c_0 \to \mathbb{K}$ be given by

$$T\left(x^{(1)}_1, \ldots, x^{(m)}_m\right) = \sum_{i_1, \ldots, i_m=1}^{\infty} a_{i_1, \ldots, i_m} x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}$$

and define, for all fixed $k=1, \ldots, m$, and fixed $i_k \in \mathbb{N}$, the set

$$(2.3)\quad S_k^{(i_k)}(T) = \{(i_1, \ldots, i_m) : \alpha_{i_1, \ldots, i_m} \neq 0\}.$$

We define the complexity of $T$, and denote by $\text{Comp}(T)$, as

$$(2.4)\quad \text{Comp}(T) = \sup_{k=1, \ldots, m} \sup_{i_k \in \mathbb{N}} \text{card}\left(S_k^{(i_k)}(T)\right),$$

For instance, if for $j = 1, 2$,

$$T_j: c_0 \times c_0 \to \mathbb{K}$$

are given by

$$T_1(x, y) = x_1 \sum_{j=1}^{\infty} \frac{y_j}{j^2},$$

and

$$T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_3,$$

then

$$S_1^{(1)}(T_1) = \{(1, j) : j \in \mathbb{N}\},$$

$$S_1^{(k)}(T_1) = \emptyset \text{ for } k > 1,$$

$$S_2^{(k)}(T_1) = \{(1, k)\} \text{ for all } k,$$

and

$$S_1^{(1)}(T_2) = \{(1, 1), (1, 2)\},$$

$$S_1^{(2)}(T_2) = \{(2, 3)\},$$

$$S_2^{(1)}(T_2) = \{(1, 1)\},$$

$$S_2^{(2)}(T_2) = \{(1, 2)\},$$

$$S_2^{(3)}(T_2) = \{(2, 3)\},$$

and hence

$$\text{Comp}(T_1) = \infty \text{ and } \text{Comp}(T_2) = 2.$$

In essence, the complexity of a $m$-linear form measures the biggest possible degree of combination between the variables.

In the next section we classify all the extrema of the Littlewood’s $4/3$-inequality, that is the $2$-linear Bohnenblust–Hille inequality.

3. Classification of all extrema for Littlewood’s $4/3$ inequality

As it has been previously mentioned, all known extrema for Littlewood’s $4/3$ inequality up to date are bilinear forms like $\mathbf{2.1}$. In this section we show that this is in fact essentially the unique extremum for Littlewood’s $4/3$ inequality. Besides its own interest, this result hints out the possible pattern for the extrema in the $m$-linear case (see Section $\mathbf{4}$ for details). As we will show, this would ultimately prove that, in fact, the optimal constants of the $m$-linear Bohnenblust–Hille inequality are $2^{1-\frac{m}{3}}$, and thus bounded with respect to $m$.

Before we continue, for the sake of the reader’s convenience, let us recall the Khinchin inequality. If $r_n(t)$ denote the Rademacher functions,

$$r_n(t) := \text{sign} (\sin 2^n \pi t),$$

the Khinchin inequality asserts that for any $p > 0$ there are constants $A_p, B_p > 0$ such that

$$(3.1)\quad A_p \left( \sum_{j=1}^{\infty} |a_j|^2 \right)^{\frac{1}{p}} \leq \left( \int_0^1 \left( \sum_{j=1}^{\infty} |r_j(t)a_j| \right)^p dt \right)^{\frac{1}{p}} \leq B_p \left( \sum_{j=1}^{\infty} |a_j|^2 \right)^{\frac{1}{p}}.$$
for all sequence of scalars \((a_i)_{i=1}^{\infty}\). The best constants \(A_p\) are known (see [15, 25]):

- \(A_p = \sqrt[2]{\frac{\Gamma \left( \frac{1+p}{p} \right)}{\sqrt{\pi}}} \) if \(p \geq p_0 \approx 1.8474\);
- \(A_p = 2^{\frac{2}{p} - \frac{2}{q}}\) if \(p < p_0\).

The critical exponent \(p_0\) is the unique real number in \(p_0 \in (1, 2)\) verifying

\[
\Gamma \left( \frac{p_0 + 1}{2} \right) = \frac{\sqrt{\pi}}{2}.
\]

**Lemma 3.1.** If \(T(x, y) = \sum_{i,j=1}^{\infty} a_{ij} x_i y_j\) is an extremum for Littlewood’s 4/3 inequality, then

\[
\text{Comp} (T) = 2. \tag{3.2}
\]

Moreover, for all \((i_1,0,j_1),(i_1,0,j_2) \in S_1^{(i_1,0)}\) and \((i_1,2,0),(i_2,2,0) \in S_2^{(i_2,2)}\), we have

\[
|a_{i_1,0,j_1}| = |a_{i_1,0,j_2}| \tag{3.3}
\]

and

\[
|a_{i_1,2,0}| = |a_{i_2,2,0}|. \tag{3.4}
\]

**Proof.** By the Khinchin Inequality we know that

\[
\left( \sum_{j=1}^{\infty} |a_j|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \left( \int_0^1 \left| \sum_{j=1}^{\infty} r_j(t) a_j \right| \, dt \right),
\]

and the equality holds if and only if \((a_j)_{j=1}^{\infty} = \alpha (\pm e_1 \pm e_j)\) for some \(\alpha \neq 0\) and \(i \neq j\) (see [25]). In particular, if (3.2) or (3.3) or (3.4) fails, then

\[
\left( \sum_{j=1}^{\infty} |a_j|^2 \right)^{\frac{1}{2}} < \sqrt{2} \left( \int_0^1 \left| \sum_{j=1}^{\infty} r_j(t) a_j \right| \, dt \right).
\]

In fact, if \(T : c_0 \times c_0 \to \mathbb{R}\) is given by

\[
T(x, y) = \sum_{i,j=1}^{\infty} a_{ij} x_i y_j
\]

with

\[
0 < \text{card} \left( S_2^{(j_0)} \right) \neq 2
\]

for some \(j_0\), we have

\[
\sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} |T(e_i, e_j)|^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{\infty} |T(e_i, e_{j_0})|^2 \right)^{\frac{1}{2}} \sum_{j \neq j_0} \left( \sum_{i=1}^{\infty} |T(e_i, e_j)|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \left( \int_0^1 \left| \sum_{i=1}^{\infty} r_i(t) T(e_i, e_{j_0}) \right| \, dt \right) + \sum_{j \neq j_0} \sqrt{2} \left( \int_0^1 \left| \sum_{i=1}^{\infty} r_i(t) T(e_i, e_j) \right| \, dt \right)
\]

\[
= \sqrt{2} \left( \int_0^1 \left| \sum_{i=1}^{\infty} r_i(t) T(e_i, e_j) \right| \, dt \right)
\]

and thus

\[
\sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} |T(e_i, e_j)|^2 \right)^{\frac{1}{2}} < \sqrt{2} \|T\|. \tag{3.5}
\]
The other cases are similar. On the other hand, it is well known that

\[(3.6) \quad \left( \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} |T(e_i, e_j)| \right)^{2} \right)^{\frac{1}{2}} \leq \sqrt{2} \|T\|.\]

Applying (3.5), (3.6) and the Hölder inequality for mixed sums we conclude that

\[
\left( \sum_{i,j=1}^{\infty} |T(e_i, e_j)|^2 \right)^{3/4} < \sqrt{2} \|T\|.
\]

Therefore, if \(T\) is an extremum we have (3.2), (3.3) and (3.4). \(\square\)

There are three types of bilinear forms on \(c_0 \times c_0\) that verify (3.2), (3.3) and (3.4):

**Type I:** there are \((N_j)_{j=1}^{s}\) and \((M_j)_{j=1}^{s}\), each of one of them a family of pairwise disjoint positive integers with \(\text{card}(N_k) = \text{card}(M_k) = 2\) for all \(k\) and \(s \in [1, \infty]\), such that

\[T(x, y) = \sum_{k=1}^{s} \left( \sum_{(i,j) \in N_k \times M_k} a_{ij}^{(k)} x_i y_j \right),\]

with \(a_{ij}^{(k)} \neq 0\) and, for all \(k\), we have \(|a_{ij}^{(k)}| = |a_{ij}^{(k)}|\) whenever \((i, j) \in N_k \times M_k\). Note that the elementary terms are composed by 4 monomials with the same coefficient. An illustration of bilinear form of type I is:

\[T_1(x, y) = \left( \pm ax_1 y_1 \pm ax_1 y_2 \pm ax_2 y_1 \pm ax_2 y_2 \right) + \left( \pm bx_3 y_3 \pm bx_3 y_4 \pm bx_4 y_3 \pm bx_4 y_4 \right),\]

with \(a \neq 0\) and \(b \neq 0\).

**Type II:** In this case we consider sums of bigger monomials (recall that the complexity is always 2). The simplest case of bilinear forms of type II is a bilinear form like

\[(3.7) \quad T_2(x, y) = (\pm ax_1 y_1 \pm ax_1 y_2) + (\pm ax_3 y_1 \pm ax_4 y_2) + (\pm ax_3 y_5 \pm ax_4 y_6),\]

with \(a \neq 0\). In this case it is simple to prove that

\[\|T_2\| \geq 4 |a|\]

and since

\[\frac{\left(6 |a|^{4/3}\right)^{3/4}}{4} \leq \frac{6^{3/4}}{4} < 1 < \sqrt{2}\]

we do not have an extremum. Another illustration is

\[(3.8) \quad T_2'(x, y) = (\pm ax_1 y_1 \pm ax_1 y_2) + (\pm ax_3 y_1 \pm ax_4 y_2) + (\pm ax_3 y_5 \pm ax_4 y_6) + (\pm ax_5 y_5 \pm ax_5 y_6),\]

with \(a \neq 0\). In this case

\[\|T_2'\| \geq 6 |a|\]

and since

\[\frac{\left(8 |a|^{4/3}\right)^{3/4}}{6} \leq \frac{8^{3/4}}{6} < 0.8 < \sqrt{2}\]
we, again, do not have an extremum and so on. We may also have sums of elementary terms. For instance:

\[ T_{2^r}(x, y) = \begin{cases} \text{elementary term of type II} \\ \sum_{i,j=1}^{\infty} |T_3(e_i, e_j)|^{3/4} \end{cases} \]

with \( a \neq 0 \) and \( b \neq 0 \). This case is even farther from being an extremum, since

\[ \|T_{2^r}\| \geq 4|a| + 6|b| \]

and thus

\[ \left( \frac{6|a|^{3/4} + 8|b|^{3/4}}{4|a| + 6|b|} \right)^{3/4} \]

A simple, though tedious, argument shows that any such combination of elementary terms does not provide extrema.

**Type III:** combinations of elements of the first and second types. Let \( T_3 = R_1 + R_2 \) with \( R_J \neq 0 \) being of type \( J = 1, 2 \). Note that since

\[ \text{Comp}(T_3) = \text{Comp}(R_1) = \text{Comp}(R_2) = 2, \]

there is no overlapping between \( R_1 \) and \( R_2 \) and we have

\[ \|T_3\| = \|R_1\| + \|R_2\|. \]

Thus, since \( R_2 \) is not an extremum,

\[ \left( \sum_{i,j=1}^{\infty} |T_3(e_i, e_j)|^{3/4} \right)^{3/4} \]

and thus \( T_3 \) is not an extremum.

From the previous considerations we conclude that extrema of Littlewood’s 4/3 inequality must satisfy \( (3.2), (3.3) \) and \( (3.4) \) and be of type I. The following theorem gives a final and complete characterization:

**Theorem 3.2.** A bilinear form \( T \) is an extremum of Littlewood’s 4/3 inequality if and only if \( T \) is written as

\[ T(x, y) = 2^{-1/2} \left( x_{i_1} y_{i_2} + x_{i_1} y_{i_3} + x_{i_2} y_{i_4} - x_{i_2} y_{i_3} \right), \]

\[ T(x, y) = 2^{-1/2} \left( x_{i_1} y_{i_2} + x_{i_1} y_{i_3} - x_{i_2} y_{i_4} + x_{i_2} y_{i_3} \right), \]

\[ T(x, y) = 2^{-1/2} \left( x_{i_1} y_{i_2} - x_{i_1} y_{i_3} + x_{i_2} y_{i_4} + x_{i_2} y_{i_3} \right), \]

\[ T(x, y) = 2^{-1/2} \left( -x_{i_1} y_{i_2} + x_{i_1} y_{i_3} + x_{i_2} y_{i_4} + x_{i_2} y_{i_3} \right), \]

for \( i_1 \neq i_2 \) and \( i_3 \neq i_4 \).

**Proof.** We just need to consider bilinear forms of the type I. Denoting

\[ T_k(x, y) = \sum_{(i,j) \in N_k \times M_k} a_{ij}^{(k)} x_i y_j, \]
since \((N_j)_{j=1}^s\) and \((M_j)_{j=1}^s\) are, each of one of them, a family of pairwise disjoint positive integers, we have

\[
\|T\| = \sum_{k=1}^s \|T_k\|
\]

and

\[
\left( \sum_{i,j=1}^\infty |T(e_i, e_j)|^{\frac{3}{2}} \right)^{3/4} = \left( \sum_{i,j=1}^\infty \left| \sum_{k=1}^s T_k(e_i, e_j) \right|^{\frac{3}{2}} \right)^{3/4} \leq \sum_{k=1}^s \left( \sum_{i,j=1}^\infty |T_k(e_i, e_j)|^{\frac{3}{2}} \right)^{3/4}.
\]

If \(T_k\) is not extremum for some \(k\), we have

\[
\left( \sum_{i,j=1}^\infty |T(e_i, e_j)|^{\frac{3}{2}} \right)^{3/4} \leq \sum_{k=1}^s \left( \sum_{i,j=1}^\infty |T_k(e_i, e_j)|^{\frac{3}{2}} \right)^{3/4} < \sqrt{2} \sum_{k=1}^s \|T_k\| = \sqrt{2} \|T\|
\]

and thus \(T\) is not extremum. Thus all \(T_k\) need to be extrema. Recall that for a given \(k\), the coefficients of the monomials of \(T_k\) are all the same in absolute value. So, it is simple to check that

\[
T_k(x, y) = \alpha_k \left( x_{i_k,1} y_{i_k,2} + x_{i_k,2} y_{i_k,3} + x_{i_k,4} y_{i_k,2} - x_{i_k,4} y_{i_k,3} \right),
\]

or

\[
T_k(x, y) = \alpha_k \left( x_{i_k,1} y_{i_k,2} + x_{i_k,1} y_{i_k,3} - x_{i_k,4} y_{i_k,2} + x_{i_k,4} y_{i_k,3} \right),
\]

or

\[
T_k(x, y) = \alpha_k \left( x_{i_k,1} y_{i_k,2} - x_{i_k,1} y_{i_k,3} + x_{i_k,4} y_{i_k,2} + x_{i_k,4} y_{i_k,3} \right),
\]

or

\[
T_k(x, y) = \alpha_k \left( -x_{i_k,1} y_{i_k,2} + x_{i_k,1} y_{i_k,3} + x_{i_k,4} y_{i_k,2} + x_{i_k,4} y_{i_k,3} \right)
\]

for some \(\alpha_k \neq 0\), and there is no overlapping between \(T_k, T_{k_2}\) for \(k_1 \neq k_2\). We have

\[
\left( \sum_{i,j=1}^\infty |T_k(e_i, e_j)|^{\frac{3}{2}} \right)^{3/4} = \left( \sum_{k=1}^s \|T_k\|^{4/3} \right)^{3/4}
\]

and

\[
\|T\| = \|T_1 + \cdots + T_s\| = 2 \sum_{k=1}^s |\alpha_k|.
\]

Since

\[
\frac{\left( \sum_{k=1}^s |\alpha_k|^{4/3} \right)^{3/4}}{2 \sum_{k=1}^s |\alpha_k|} = \sqrt{2},
\]

we conclude that \(s = 1\). In fact, if \(s > 1\), since \(\alpha_k \neq 0\) for all \(k\), we have

\[
\left( \sum_{k=1}^s |\alpha_k|^{4/3} \right)^{3/4} < \sum_{k=1}^s |\alpha_k|,
\]

and thus

\[
\frac{\left( \sum_{i,j=1}^\infty |T_k(e_i, e_j)|^{\frac{3}{2}} \right)^{3/4}}{\|T_1 + \cdots + T_s\|} = \frac{\left( \sum_{k=1}^s |\alpha_k|^{4/3} \right)^{3/4}}{2 \sum_{k=1}^s |\alpha_k|} < \sqrt{2}
\]

and \(T\) is not extremum. \(\square\)
Remark 3.3. We have defined entropy as the minimal number of monomials needed to be combined to generate an extremum. As we have seen, for \( m = 2 \) this number is 4, but this is not only a minimum; this is the unique number of monomials that can be combined to generate an extremum. This is a quite curious property that may be inherited when \( m > 2 \).

In the next section we show that the entropy of any mixed \((\ell_1, \ell_2)\)-Littlewood inequalities does grow as predicted. Indeed, we will prove the following lower and upper bounds

\[
2^{m-1} \leq \text{ent}_m(1, 2, \ldots, 2), \ldots, \text{ent}_m(2, 2, \ldots, 2, 1) \leq 4^{m-1},
\]

which is definitive step in the proof of (3.9).

4. Entropies with exponential growth and sharp constants

In this section we deliver a proof of (3.9). Next theorem is the main step in this endeavor, which actually paves the way to all the other optimal estimates we will obtain in this work.

**Theorem 4.1.** Let \( m \geq 2 \) and \( i \geq 1 \) be integers, then

\[
\left( \sum^\infty_{j_1, \ldots, j_m = 1} \left( \sum^\infty_{j_1 = 1} |U(e_{j_1}, \ldots, e_{j_m})| \right)^2 \right)^{1/2} \leq \left( \sqrt{2} \right)^{m-1} \| U \|,
\]

holds for all continuous real \( m \)-linear forms \( U : c_0 \times \cdots \times c_0 \rightarrow \mathbb{R} \). Furthermore,

\[
\left( \sqrt{2} \right)^{m-1} = C(2,2,\ldots,2)_m
\]

is the sharp constant.

A consequence of Theorem 4.1 is that all sharp constants in the mixed \((\ell_1, \ell_2)\)-Littlewood inequality is in fact equal to \( (\sqrt{2})^{m-1} \).

**Corollary 4.2.** For all \( m \geq 2 \), we have

\[
C(1,2,\ldots,2)_m = \cdots = C(2,2,\ldots,2)_m = \left( \sqrt{2} \right)^{m-1}
\]

and

\[
2^{m-1} \leq \text{ent}_m(1, 2, \ldots, 2), \ldots, \text{ent}_m(2, 2, \ldots, 2, 1) \leq 4^{m-1}.
\]

The rest of the section is devoted to the proofs of Theorem 4.1 and Corollary 4.2, which are based on a radical change in the “usual” strategic \( m \)-linear forms.

We start off by delivering a proof of the following simple estimate:

\[
\left( \sum^\infty_{i_1, \ldots, i_m = 1} |U(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{1/2} \leq \| U \|,
\]

holds for all continuous \( m \)-linear forms \( U : c_0 \times \cdots \times c_0 \rightarrow \mathbb{K} \). This is a consequence of Khinchin inequality, or cotype if one prefers. Indeed, applying the Khinchin inequality together with an induction argument, we reach

\[
\left( \sum^\infty_{i_1, \ldots, i_m = 1} |U(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{1/2} \leq \left( \frac{1}{0} \left\| \sum_{i_1, \ldots, i_m = 1} r_{i_1}(t_1) \cdots r_{i_m}(t_m)U(e_{i_1}, \ldots, e_{i_m}) \right\| \right)^{1/2}.
\]

From the multilinearity of \( U \), we can further estimate,

\[
\left( \sum^\infty_{i_1, \ldots, i_m = 1} |U(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{1/2} \leq \left( \frac{1}{0} \left\| U \right\| \left\| \sum_{i_1 = 1}^{\infty} \sum_{i_m = 1}^{\infty} r_{i_1}(t_1)e_{i_1} \cdots r_{i_m}(t_m)e_{i_m} \right\| \right)^{1/2}
\]

\[
\leq \sup_{t_1, \ldots, t_m \in [0,1]} \left\| U \right\| \left\| \sum_{i_1 = 1}^{\infty} r_{i_1}(t_1)e_{i_1} \cdots \sum_{i_m = 1}^{\infty} r_{i_m}(t_m)e_{i_m} \right\|
\]

\[
= \| U \|.
\]
for all continuous $m$–linear forms $U : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

We begin with $m = 2$ and consider the standard bilinear form $S_2 : c_0 \times c_0 \to \mathbb{R}$,

$$S_2(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2.$$ 

As $\|S_2\| = 2$, we conclude that

$$C_{(2,1)2} \geq \sqrt{2}.$$ 

From $m = 3$ and on, we start to deform the standard $m$–linear forms in a non-symmetric fashion. For that define $S_3 : c_0 \times c_0 \times c_0 \to \mathbb{R}$ by

$$S_3(x, y, z) = (z_1 + z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) + (z_1 - z_2)(x_3y_1 + x_3y_2 + x_4y_1 - x_4y_2).$$

Note that $\|S_3\| = 4$ and, since

$$\left( \sum_{i_2, i_3=1}^{\infty} \left| S_3(e_{i_1}, e_{i_2}, e_{i_3}) \right| \right)^2 = 4\sqrt{4},$$

we conclude

$$C_{(2,2,1)3} \geq \left( \sqrt{2} \right)^2.$$ 

For $m = 4$, we consider $S_4 : c_0 \times c_0 \times c_0 \times c_0 \to \mathbb{R}$ given by

$$S_4(x, y, z, w) =$$

$$= (w_1 + w_2) \left( (z_1 + z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) + (z_1 - z_2)(x_3y_1 + x_3y_2 + x_4y_1 - x_4y_2) \right) + (w_1 - w_2) \left( (z_1 + z_2)(x_5y_1 + x_5y_2 + x_6y_1 - x_6y_2) + (z_1 - z_2)(x_7y_1 + x_7y_2 + x_8y_1 - x_8y_2) \right).$$

Similarly, we compute $\|S_4\| = 8$ and

$$\left( \sum_{i_2, i_3, i_4=1}^{\infty} \left| S_4(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) \right| \right)^2 = 8\sqrt{8},$$

Therefore, we obtain

$$C_{(2,2,2,1)4} \geq \frac{8\sqrt{8}}{8} = \left( \sqrt{2} \right)^3.$$ 

The construction can be carried out by induction for all $m$–linear form, through the general non-symmetric procedure:

$$S_m \left( x^{(1)}, \cdots, x^{(m)} \right) = \left( x_1^{(m)} + x_2^{(m)} \right) S_{m-1} \left( x^{(1)}, \cdots, x^{(m-1)} \right) + \left( x_1^{(m)} - x_2^{(m)} \right) S_{m-1} \left( B^{2m-1} \left( x^{(1)} \right), x^{(2)}, \cdots, x^{(m-1)} \right),$$

where

$$B^{2m-1} \left( x^{(1)} \right) = \left( x^{(1)}_{2m-1+1}, x^{(1)}_{2m-1+2}, \cdots \right)$$

for all natural number $m$. As before, such a construction yields

$$C_{(2,2,\ldots,2,1)m} \geq \left( \sqrt{2} \right)^{m-1}.$$ 

It now follows by Hölder inequality that if

$$U \left( x^{(1)}, \cdots, x^{(m)} \right) = \sum_{j_1, \cdots, j_m=1}^{\infty} \alpha_{j_1, \cdots, j_m} x^{(1)}_{j_1} \cdots x^{(m)}_{j_m}$$

for all continuous $m$–linear forms $U : c_0 \times \cdots \times c_0 \to \mathbb{K}$.
is the sum of exactly $k$ monomials, then
\[
\left( \sum_{j_2, \ldots, j_m = 1}^{\infty} \left( \sum_{j_1 = 1}^{\infty} |U(e_{j_1}, \ldots, e_{j_m})| \right)^2 \right)^{1/2} \leq \left( \sum_{j_2, \ldots, j_m = 1}^{\infty} \left( \sum_{j_1 \in A_{j_2 \ldots j_m}} |U(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{1/2} \times \left( \sum_{j_1 \in A_{j_2 \ldots j_m}} 1 \right)^{1/2} \right)^{2/1} \leq k^{1/2} \|U\|,
\]
where $A_{j_2 \ldots j_m} = \{j_1 : \alpha_{j_1 \ldots j_m} \neq 0\}$. In the last inequality we have used (4.3). Thus, if an extremum $m$–linear form is composed by the sum of exactly $k$ monomials, we conclude that
\[
k^{1/2} \geq \left( \sqrt{2}^{m-1} \right),
\]
which means, in terms of entropy, that
\[
\text{ent}_m(2, 2, \ldots, 2, 1) \geq 2^{m-1}.
\]
Next we recall that, if $1 \leq p \leq q$, then
\[
\left( \sum_i \left( \sum_j |a_{ij}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \left( \sum_j \left( \sum_i |a_{ij}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}.
\]
Thus,
\[
C_{(2, 2, \ldots, 2)m} \leq \cdots \leq C_{(1, 2, \ldots, 2)m}.
\]
Easily one shows that
\[
C_{(1, 2, \ldots, 2)m} \leq \left( \sqrt{2} \right)^{m-1},
\]
and hence, combining (4.3), (4.7) and (4.8) we reach
\[
\left( \sqrt{2} \right)^{m-1} \leq C_{(2, 2, \ldots, 2, 1)m} \leq \cdots \leq C_{(1, 2, \ldots, 2)m} \leq \left( \sqrt{2} \right)^{m-1}.
\]
Finally we note that, since each $S_m$ is the sum of exactly $4^{m-1}$ monomials, we can estimate
\[
2^{m-1} \leq \text{ent}_m(2, 2, \ldots, 2, 1) \leq 4^{m-1}.
\]
Hence, the chain of inequalities (4.7) and standard symmetry arguments yield
\[
2^{m-1} \leq \text{ent}_m(1, 2, \ldots, 2), \ldots, \text{ent}_m(2, 2, \ldots, 2, 1) \leq 4^{m-1},
\]
and Corollary 4.2 is finally proved.

We close this section with some additional considerations. The mixed $(\ell_1, \ell_2)$–Littlewood inequalities should be regarded, in a natural way, as extremal Bohnenblust–Hille inequalities, in the sense that they correspond to multiple exponents with maximum “diameter”. More precisely, let us define
\[
diam (q_1, \ldots, q_m) = \max |q_i - q_j|.
\]
An exponent $(q_1, \ldots, q_m)$ is said to be extremal if $\diam (q_1, \ldots, q_m) = 1$, and this is the case of the all mixed $(\ell_1, \ell_2)$–Littlewood inequalities.

The classical Bohnenblust–Hille exponents, on the other hand, verify $\diam (q_1, \ldots, q_m) = 0$. Proceeding as in [21] it seems plausible that, when $\diam (q_1, \ldots, q_m)$ is close to 1, optimal constants should grow exponentially. In this regard, Corollary 4.2 suggests an interesting parallel between diameter of the exponent and growth of sharp constants in the Bohnenblust–Hille inequalities:
The relationship between the growth of the sharp constants and the diameter of the exponent seems a natural line of investigation. It suggests that the smaller the diameter, the slower the growth. A quantification of such implication would shed lights on a number of other issues pertaining to the theory of Bohnenblust–Hille inequalities.

5. Estimating the entropy of the classical Bohnenblust–Hille inequality

The Hardy–Littlewood inequalities for $m$–linear forms (see [2] [13] [16] [23]) are a sharp generalization of the Bohnenblust–Hille inequality when we replace $c_0$ by $\ell_p$. It asserts that for any integer $m \geq 2$ and $2m \leq p \leq \infty$, there exists a constant $C_{m,p} \geq 1$ such that,

$$
(5.1) \quad \left( \sum_{i_1, \ldots, i_m=1}^{\infty} |T(e_{i_1}, \ldots, e_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p} \|T\|,
$$

for all continuous $m$–linear forms $T: \ell_p \times \cdots \times \ell_p \to \mathbb{K}$. The exponent $\frac{2mp}{mp+p-2m}$ is optimal. Following usual convention in the field, $c_0$ is understood as the proper substitute of $\ell_\infty$ when the exponent $p \to \infty$. Under such an agreement, one easily checks that taking $p = \infty$ in (5.1) recovers the Bohnenblust–Hille inequality.

Investigations pertaining to the Hardy–Littlewood inequalities are closely related to phenomena observed in the classical Bohnenblust–Hille inequality, and the question regarding optimal constants is not different (see [3]). For this reason, and to support future investigations in the theme, in this section we broaden the definition of entropy to the domain of Hardy–Littlewood inequalities. The results of this section will be proved for both settings.

We shall use the notation $\text{ent}^{HL}_{m}\left(\frac{2mp}{mp+p-2m}, \ldots, \frac{2mp}{mp+p-2m}\right)$ for the entropy of the Hardy–Littlewood inequality. Our first lemma, which is of independent interest, is crucial in this section. It can be understood as a generalization of [13] Lemma 18.14 to $\ell_p$ spaces, which further sharpens the constants to their optimal values.

**Lemma 5.1.** Let $m$ be a positive integer. For all $p > 2m$, we have

$$
\left( \sum_{i_1, \ldots, i_m=1}^{\infty} |U(e_{i_1}, \ldots, e_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq \|U\|,
$$

for all continuous $m$–linear forms $U: \ell_p \times \cdots \times \ell_p \to \mathbb{K}$.

**Proof.** We start off by recalling that

$$
\left( \sum_{i_1, \ldots, i_m=1}^{\infty} |U(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \leq \|U\|,
$$

for all continuous $m$–linear forms $U: \ell_2 \times c_0 \times \cdots \times c_0 \to \mathbb{K}$ (see [8] Proposition 3.5)). So, since $p > 2m$, it is not difficult to see that

$$
\left( \sum_{i_1, \ldots, i_m=1}^{\infty} |U(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{p}} \leq \|U\|,
$$

for all continuous $m$–linear forms $U: \ell_p \times c_0 \times \cdots \times c_0 \to \mathbb{K}$.

The proof we shall deliver of Lemma 5.1 is based on a technique that goes back to the paper of Hardy and Littlewood, [10], see also and [23]. It consists of analyzing the effect on each one of the $m$ exponents 2 when we replace $c_0$ by $\ell_p$. 

| Diameter of $(q_1, \ldots, q_m)$ | Growth of the constants |
|-------------------------------|-------------------------|
| 0 (classical case)             | sublinear ($< m^{\frac{3}{4}}$) |
| 1 (mixed $(\ell_1, \ell_2)$-Littlewood) | Exponential ($= \sqrt{2})^{m-1}$. |
Let us set $s = \frac{2p}{p-2m+2}$ and $\lambda_0 = 2$. From the inclusion of the $\ell_p$ spaces and (4.3) we have

\[
\left( \sum_{j=1}^{\infty} \left( \sum_{j_i=1}^{\infty} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{1}{\lambda_0}} \leq ||T||,
\]

for all $m$–linear forms $T: \ell_p \times c_0 \times \cdots \times c_0 \to K$ and all $i = 1, \ldots, m$. Hereafter, $\sum_{j_i=1}^{n}$ means the sum over all $j_k$ for all $k \neq i$. Note that $\lambda_0 < s$. For all $j = 1, \ldots, m$ let us set

\[
\lambda_j := \frac{\lambda_0 p}{p - \lambda_0 j}.
\]

It is plain to observe that

\[
\lambda_{k-1} < \lambda_k < s
\]

for all $k = 1, \ldots, m - 2$ and $\lambda_{m-1} = s$. In addition, for all $j = 0, \ldots, m - 2$, we have

\[
\left( \frac{p}{\lambda_j} \right)^* = \frac{\lambda_{j+1}}{\lambda_j}.
\]

Now now argue by induction, i.e., assuming for $2 \leq k \leq m - 1$ that

\[
\left( \sum_{j=1}^{n} \left( \sum_{j_i=1}^{n} |T_{k-1}(e_{j_1}, \ldots, e_{j_{m}})|^s \right)^{\frac{1}{s}} \right)^{\frac{1}{\lambda_{k-1}}} \leq ||T_{k-1}||
\]

for all $m$–linear forms $T_{k-1}: \ell_p \times \cdots \times \ell_p \times c_0 \times \cdots \times c_0 \to K$, all $i = 1, \ldots, m$ and all positive integers $n$, we aim to prove that

\[
\left( \sum_{j=1}^{n} \left( \sum_{j_i=1}^{n} |T_k(e_{j_1}, \ldots, e_{j_{m}})|^s \right)^{\frac{1}{s}} \right)^{\frac{1}{\lambda_k}} \leq ||T_k||
\]

for all $m$–linear forms $T_k: \ell_p \times \cdots \times \ell_p \times c_0 \times \cdots \times c_0 \to K$, all $i = 1, \ldots, m$ and all positive integers $n$.

The initial case $k = 2$ in (5.3) is precisely (5.2). Let us then assume (5.3) and consider, for $1 \leq k \leq m$, an $m$–linear form $T_k \in \mathcal{L}(\ell_p, \cdots, \ell_p, c_0; K)$ and, for each $x \in B_{\ell_p}$, define

\[
T_k^{(x)} : \ell_p \times \cdots \times \ell_p \times c_0 \times \cdots \times c_0 \to K
\]

for $k-1$ times
with \( xz^{(k)} = (x_j z_j^{(k)})_{j=1}^{\infty} \). We note that \( \|T_k\| \leq \sup \{ \|T_k^{(x)}\| : x \in B_{\ell_p} \} \) and by the induction hypothesis applied to \( T_k^{(x)} \), we obtain

\[
\left( \sum_{j=1}^{n} \left( \sum_{j_k=1}^{n} |T_k (e_{j_1}, \ldots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s \lambda_{k-1}}} \right)^{\frac{1}{s \lambda_{k-1}}}
\]

\[
= \left( \sum_{j=1}^{n} \left( \sum_{j_k=1}^{n} |T_k (e_{j_1}, \ldots, e_{j_k-1}, x e_{j_k}, e_{j_{k+1}}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s \lambda_{k-1}}} \right)^{\frac{1}{s \lambda_{k-1}}}
\]

\[
= \left( \sum_{j=1}^{n} \left( \sum_{j_k=1}^{n} |T_k^{(x)} (e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s \lambda_{k-1}}} \right)^{\frac{1}{s \lambda_{k-1}}}
\]

\[
\leq \|T_k^{(x)}\|
\]

\[
\leq \|T_k\|
\]

for all \( i = 1, \ldots, m \) and all \( n \).

Let us first consider the case \( i = k \). Since, as previously mentioned, \( \left( \frac{p}{\lambda_{j-1}} \right)^* = \frac{\lambda_k}{\lambda_{j-1}} \), for all \( j = 1, \ldots, m \), we have

\[
\left( \sum_{j_k=1}^{n} \left( \sum_{j=1}^{n} |T_k (e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s \lambda_k}} \right)^{\frac{1}{s \lambda_k}}
\]

\[
= \left( \sum_{j_k=1}^{n} \left( \sum_{j=1}^{n} |T_k (e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s \lambda_{k-1}} \left( \frac{p}{\lambda_{j-1}} \right)^*} \right)^{\frac{1}{s \lambda_{k-1}} \left( \frac{p}{\lambda_{j-1}} \right)^*}
\]

\[
= \left( \left\| \left( \sum_{j=1}^{n} |T_k (e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s \lambda_{k-1}}} \right\|_{j_k=1}^{n} \right)^{\frac{1}{s \lambda_{k-1}} \left( \frac{p}{\lambda_{j-1}} \right)^*}
\]

\[
= \left( \sup_{y \in B_{\ell_p}} \sum_{j_k=1}^{n} |y_{j_k}| \left( \sum_{j=1}^{n} |T_k (e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s \lambda_{k-1}}} \right)^{\frac{1}{s \lambda_{k-1}} \left( \frac{p}{\lambda_{j-1}} \right)^*}
\]

\[
= \left( \sup_{x \in B_{\ell_p}} \sum_{j_k=1}^{n} |x_{j_k}|^{\frac{\lambda_{k-1}}{s}} \left( \sum_{j=1}^{n} |T_k (e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s \lambda_{k-1}}} \right)^{\frac{1}{s \lambda_{k-1}} \left( \frac{p}{\lambda_{j-1}} \right)^*}
\]

\[
\leq \|T_k\|,
\]
where in the last inequality we have used by (5.4). Let us now focus in the remaining cases, namely, when \( i \neq k \). Let \( k \in \{ 1, \cdots , m - 1 \} \) and, for \( i = 1, \cdots , m \) and \( n \), denote

\[
S_i = \left( \sum_{j_i=1}^{n} |T_k(e_{j_1}, \cdots , e_{j_m})|^s \right)^{\frac{1}{s}}.
\]

We then have

\[
\sum_{j_i=1}^{n} \left( \sum_{j_i=1}^{n} |T_k(e_{j_1}, \cdots , e_{j_m})|^s \right)^{\frac{\lambda_i}{\lambda_k^s}} = \sum_{j_i=1}^{n} \sum_{j_i=1}^{n} \frac{|T_k(e_{j_1}, \cdots , e_{j_m})|^s}{S_i^{\alpha - \lambda_k}} = \sum_{j_i=1}^{n} \sum_{j_i=1}^{n} \frac{|T_k(e_{j_1}, \cdots , e_{j_m})|^s}{S_i^{\alpha - \lambda_k}}
\]

\[
= \sum_{j_i=1}^{n} \left( \sum_{j_i=1}^{n} \frac{|T_k(e_{j_1}, \cdots , e_{j_m})|^s}{S_i^{\alpha - \lambda_k}} \right) \frac{s - \lambda_k}{s - \lambda_k}.
\]

Hence, applying Hölder’s inequality with exponents \( \left( \frac{s - \lambda_k}{s - \lambda_k - 1}, \frac{s - \lambda_k}{s - \lambda_k} \right) \) and \( \left( \frac{\lambda_k - \lambda_k - 1}{\lambda_k - 1}, \frac{\lambda_k - \lambda_k - 1}{\lambda_k - 1} \right) \) gives

\[
\sum_{j_i=1}^{n} \left( \sum_{j_i=1}^{n} \frac{|T_k(e_{j_1}, \cdots , e_{j_m})|^s}{S_i^{\alpha - \lambda_k}} \right) \frac{s - \lambda_k}{s - \lambda_k - 1} \frac{\lambda_k - \lambda_k - 1}{\lambda_k - 1} \leq \left( \sum_{j_i=1}^{n} \frac{|T_k(e_{j_1}, \cdots , e_{j_m})|^s}{S_i^{\alpha - \lambda_k}} \right)^{\frac{\lambda_k - \lambda_k - 1}{\lambda_k - 1}} \left( \sum_{j_i=1}^{n} |T_k(e_{j_1}, \cdots , e_{j_m})|^s \right)^{\frac{\lambda_k - \lambda_k - 1}{\lambda_k - 1}} \frac{\lambda_k - \lambda_k - 1}{\lambda_k - 1}.
\]

It follows from the estimate already proved in the case \( i = k \) that:

\[
\left( \sum_{j_i=1}^{n} \left( \sum_{j_i=1}^{n} |T(e_{j_1}, \cdots , e_{j_m})|^s \right) \right)^{\frac{1}{s}} \leq \|T\|_{\frac{(\lambda_k - \lambda_k - 1)^s}{\lambda_k - 1}},
\]

(5.5)
Now, applying Hölder’s inequality for \( \left( \frac{s}{s-\lambda_{k-1}}, \frac{s}{\lambda_{m-1}} \right) \) together with \( 5.4 \) yields

\[
\left( \sum_{j_k=1}^n \left( \sum_{j_k=1}^n \frac{|T_k(e_{j_1}, \ldots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_{m-1}}{s-\lambda_{k-1}}} \right)^{\frac{s}{\lambda_{m-1}}} = \left\| \left( \sum_{j_k=1}^n \frac{|T_k(e_{j_1}, \ldots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_{m-1}}{s-\lambda_{k-1}}} \right\|_{S_i}^{\frac{s}{\lambda_{m-1}}}
\]

(5.6)

\[
\sup_{y \in B_{p^{-1}}} \left( \sum_{j_k=1}^n \sum_{j_k=1}^n \frac{|T_k(e_{j_1}, \ldots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_{m-1}}{s-\lambda_{k-1}}} = \sup_{x \in B_{p^{-1}}} \left( \sum_{j_k=1}^n \sum_{j_k=1}^n \frac{|T_k(e_{j_1}, \ldots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_{m-1}}{s-\lambda_{k-1}}}
\]

Therefore, combining (5.5) and (5.6) gives

\[
\left( \sum_{j_k=1}^n \left( \sum_{j_k=1}^n \frac{|T_k(e_{j_1}, \ldots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_{m-1}}{s-\lambda_{k-1}}} \right)^{\frac{s}{\lambda_{m-1}}} \leq \left\| T_k \right\|^{\lambda_{m-1}}.
\]

It finally remains to verify the case \( k = m - 1 \). Since \( \lambda_{m-1} = s \), applying the case \( i = k \) we can estimate

\[
\left( \sum_{j_k=1}^n \left( \sum_{j_k=1}^n |T_k(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{\lambda_{m-1}}{s}} \right)^{\frac{s}{\lambda_{m-1}}} \leq \left\| T_k \right\|,
\]

and the proof of the Lemma is complete. \( \square \)

We can now establish the following lower bound estimate for the entropy in the Hardy-Littlewood inequality. First, we recall a result from \( [9] \), which will be used herein as technical lemma and we sketch its proof for the sake of completeness:

**Lemma 5.2.** Let \( m \geq 2 \) and \( p \geq 2m \). The optimal constants of the Hardy-Littlewood inequalities satisfies

\[
C_{R, m, p} \geq 2^{2m+2m^{-p}-2m^2} \sup_{x \in [0, 1]} \frac{(1+x)^p (1-x)^{p^*+p^*}}{(1+x^p)^{1/p}}.
\]

**Proof.** (Sketch) Consider the natural isometric isomorphism \( \Psi : \mathcal{L} \left( \ell_p^2, \ell_p^2 \right) \to \mathcal{L} \left( \ell_p^2, \ell_p^2 \right) \). For \( T_{2,p} : \ell_p^2 \times \ell_p^2 \to \mathbb{R} \)

\[
((x_1^{(1)}, x_1^{(2)}), (x_2^{(1)}, x_2^{(2)})) \mapsto x_1^{(1)} x_2^{(1)} + x_1^{(1)} x_2^{(2)} + x_1^{(2)} x_2^{(1)} - x_1^{(2)} x_2^{(2)},
\]

we have

\[
\Psi(T_{2,p}) : \ell_p^2 \to \ell_p^2, \quad (x_1) \mapsto (x_1 + x_2, x_1 - x_2).
\]

Therefore

\[
\left\| T_{2,p} \right\| = \left\| \Psi(T_{2,p}) \right\| = \sup_{x \in [0, 1]} \frac{(1+x)^p + (1-x)^{p^*} \lambda_{m-1}}{(1+x^p)^{1/p}}.
\]
where the norm of the linear operator $\Psi(T_{2,p})$ is performed by using the best constants from the Clarkson’s inequality in the real case (see [19 Theorem 2.1]). Defining inductively

$$T_{m,p}: \ell_p^{m-1} \times \cdots \times \ell_p^{m-1} \rightarrow \mathbb{R},$$

$$(x^{(1)}, \ldots, x^{(m)}) \mapsto (x_1^{(m)} + x_2^{(m)})T_{m-1,p}(x^{(1)}, \ldots, x^{(m)}) + (x_1^{(m)} - x_2^{(m)})T_{m-1,p}(B^{p-2}(x^{(1)}), \ldots, B^{p-2}(x^{(m)})),$$

where $x^{(k)} = (x^{(k)}_j)_{j=1}^{m-1} \in \ell_p^{m-1}, 1 \leq k \leq m,$ and $B$ is the backward shift operator in $\ell_p^{m-1},$ we have

$$|T_{m,p}(x^{(1)}, \ldots, x^{(m)})| \leq |x_1^{(m)} + x_2^{(m)}||T_{m-1,p}(x^{(1)}, \ldots, x^{(m)})|$$

$$+ |x_1^{(m)} - x_2^{(m)}||T_{m-1,p}(B^{p-2}(x^{(1)}), B^{p-2}(x^{(2)}), \ldots, B^{p-2}(x^{(m)}))|$$

$$\leq \|T_{m-1,p}\|(|x_1^{(m)} + x_2^{(m)}| + |x_1^{(m)} - x_2^{(m)}|)$$

$$\leq 2\|T_{m-1,p}\||x^{(m)}|_p,$$

i.e.,

$$\|T_{m,p}\| \leq 2^{m-2} \sup_{x \in [0,1]} \frac{(1 + x)^{p^*} + (1 - x)^{p^*}}{(1 + x^p)^{1/p}},$$

and hence

$$C_{R,m,p} \geq \frac{(4^{m-1})^{\frac{mp + 2m}{mp^2}}}{2^{m-2}\|T_{2,p}\|} = \frac{2^{2m + 2m - p - 2m^3}}{2^{m-2}\|T_{2,p}\|} \geq \frac{2^{2(m-2)(m-1)}}{2^{2m + 2m^3 + 2m^2}}.$$  

\[\text{Theorem 5.3.} \quad \text{For } m \geq 2 \text{ and } p > 2m, \text{ there holds}\]

$$\text{ent}_{HL} \left(\frac{2mp}{mp + p - 2m}, \ldots, \frac{2mp}{mp + p - 2m}\right) \geq 2 \frac{2^{2(m-2)(m-1)}}{2^{2m + 2m^3 + 2m^2}}.$$  

\[\text{Proof.} \quad \text{Our starting point is the thesis of previous Lemma which assures that}\]

$$\left(\sum_{i_1, \ldots, i_m=1}^{\infty} |U(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{mp+2m}}\right)^{\frac{mp+2m}{2m}} \leq \|U\|,$$

for all continuous $m$–linear forms $U: \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{R}.$ By Hölder inequality, if an extremum $U$ is composed by the sum of exactly $k$ monomials, we have

$$\left(\sum_{i_1, \ldots, i_m=1}^{\infty} |U(e_{i_1}, \ldots, e_{i_m})|^{\frac{2mp}{mp+2m}}\right)^{\frac{mp+2m}{2m}} \leq \left(\sum_{i_1, \ldots, i_m=1}^{\infty} \frac{1}{|U(e_{i_1}, \ldots, e_{i_m})|^{\frac{2mp}{mp+2m}}}\right)^{\frac{mp+2m}{2m}} \leq \|U\| \frac{2^{m^2 - 4m + 4p}}{2mp}.$$  

By Lemma 5.2, we have

$$C_{m,p} \geq \frac{2^{2mp - 2m^3 + 2m^2}}{2^{mp + p - 2m}} \geq \frac{2^{2(m-2)(m-1)}}{2^{2m + 2m^3 + 2m^2}}.$$  

Last estimate finally yields

$$k \frac{2^{m^2 - 4m + 4p}}{2mp} \geq 2 \frac{2^{mp - 2m^3 + 2m^2}}{2^{mp + p - 2m}},$$

and hence

$$k \geq 2 \frac{2^{2mp - 2m^3 + 2m^2}}{2^{mp + p - 2m}}.$$  

which concludes the proof of the current theorem.
Corollary 5.4. For \( m \geq 2 \),
\[
\text{ent}_m \left( \frac{2m}{m+1}, \cdots, \frac{2m}{m+1} \right) \geq 4^{m-1}.
\]

The following two corollaries are immediate consequences of the proof of Theorem 5.3:

Corollary 5.5. Let \( m \geq 2 \) be an integer and \( K \geq 1 \) be a real number. If we are restricted to \( m \)-linear forms with up to \( K^m \) monomials, then the optimal constants of the \( m \)-linear Bohnenblust–Hille inequalities are bounded by \( \sqrt{K} \).

Corollary 5.6. If we are restricted to \( m \)-linear forms with up to \( 4^{m-1} \) monomials, the optimal constants of the \( m \)-linear Bohnenblust–Hille inequalities are precisely \( 2^{1 - \frac{1}{m}} \).

Remark 5.7. Initially, we note that in Lemma 5.4, we have \( \frac{2p}{p-2m+2} = 2 \) if and only if \( p = \infty \). Interestingly enough, it is not possible to attain the exponent \( 2 \) when \( p < \infty \). In fact, if a universal estimate as
\[
(5.7) \quad \left( \sum_{j_1, \ldots, j_m=1}^{\infty} |T(e_{j_1}, \cdots, e_{j_m})|^\eta \right)^{\frac{1}{\eta}} \leq \|T\|,
\]
holds for some \( \eta \geq 1 \), then by plugging the \( m \)-linear from Lemma 5.2 into (5.7), one reaches
\[
1 \geq \left( \sum_{j_1, \ldots, j_m=1}^{\infty} |T(e_{j_1}, \cdots, e_{j_m})|^\eta \right)^{\frac{1}{\eta}} \geq \frac{(4^{m-1})^{\frac{1}{\eta}}}{2^{m-2} \sup_{x \in [0,1]} \frac{(1+x)^p + (1-x)^p}{(1+x)^{p/2} (1+x)^{p/2}}},
\]
and thus
\[
\eta \geq \log_2 \left( \sup_{x \in [0,1]} \frac{(1+x)^p + (1-x)^p}{(1+x)^{p/2} (1+x)^{p/2}} \right) + (m-2) > 2.
\]

6. Why does interpolation seem not to be the optimal approach?

In this final section, we discuss the eventual (and, in our opinion, quite likely) impossibility of finding sharp constants in Bohnenblust–Hille inequalities by means of interpolation techniques, which ultimately suggests that new technologies will be required to fully access such an optimality problem.

The results established so far in this work gather evidences to support the conjecture that the optimal constants of the Bohnenblust–Hille inequality are uniformly bounded in \( m \); they are likely to be precisely \( 2^{1 - \frac{1}{m}} \). We have noted that for \( m = 2 \) (the case \( m = 1 \) is obvious), the entropy of the classical exponent of the Bohnenblust–Hille inequality coincides with the entropy of the exponents of the mixed \( (\ell_1, \ell_2) \)-Littlewood inequalities. The entropy of the exponents of any mixed \( (\ell_1, \ell_2) \)-Littlewood inequalities grow exponentially with \( m \). Restricting to \( m \)-forms composed by the sum of \( 4^{m-1} \) monomials (and this is the number of monomials needed to attain the optimal constants of any mixed \( (\ell_1, \ell_2) \)-Littlewood inequalities) the optimal constants of the Bohnenblust–Hille inequalities are \( 2^{1 - \frac{1}{m}} \). In this final section, we offer a further technical argument which enforces such evidences, at the expenses, however, of the eminence of interpolation techniques.

For the sake of fairness, we open the floor by emphasizing that interpolation methods have promoted significant advances in the theory of Bohnenblust–Hille inequalities and several recent developments indicate that eventually they were a definitive (optimal) approach. By way of example, we mention the radical improved of the original exponential upper bounds to sublinear growth. That is, a celebrated result established by means of interpolation methods, in [4], assures the existence of a constant \( \kappa \) such that
\[
B_{R,m} < \kappa m \sqrt{\log \frac{2^{2m}}{m}},
\]
where \( \gamma \) is the Euler-Mascheroni constant. Hence, it gives a sublinear growth as \( m^{2 - \log \frac{2^{2m}}{m}} \approx m^{0.36482} \). Nonetheless, despite several attempts, the best known lower bounds for \( B_{R,m} \) are still \( 2^{1 - \frac{1}{m}} \). In what follows we will establish an abstract formula for the upper bounds for \( B_{R,m} \), and hereafter in this section
we consider both the real and complex fields, which aim to call the attention on why the interpolation procedure used in the proof of (6.1) is potentially not optimal.

Let us revisit the interpolative procedure used in [2, 4] in the illustrative case $m = 3$.

\[ B_3 \leq \left( \sup_{ijk} \{ C_{(1,2,2);3}C_{(2,1,2);3}C_{(2,2,1);3} \} \right)^{1/3}. \]

If we use the multiple exponents $(\frac{4}{3}, \frac{4}{3}, 2), (\frac{4}{3}, 2, \frac{4}{3})$ and $(2, \frac{4}{3}, \frac{4}{3})$ we further obtain

\[ B_3 \leq \left( \sup_{ijk} \{ C_{(\frac{4}{3}, \frac{4}{3}, 2);3}C_{(2, \frac{4}{3}, \frac{4}{3});3}C_{(\frac{4}{3}, 2, \frac{4}{3});3} \} \right)^{1/3}. \]

where the notation $\sup_{ijk}$ means that the supremum is taken over the product of the constants, considering the same $3$–linear form when estimating the constants and the order of the sums is always kept the same. This is, by all possible means, a rather uniform information, and the estimate (6.3) is, at least, as good as the best known estimate for the constants of the $3$–linear Bohnenblust–Hille inequality, i.e., $2^{3/4}$.

It has been observed in [2] that the interpolation of the mixed $(\ell_1, \ell_2)$–Littlewood inequalities is not an optimal choice. In fact, using interpolation in this setting one simply gets $(\frac{3}{2})^{m-1}$ as the upper bounds for the $m$–linear Bohnenblust–Hille constants, and this is far from optimal. On the other hand, in [4] it is noted that the interpolation of

\[ \frac{2(m-1)}{(m-1)+1}, \frac{2(m-1)}{(m-1)+1}, \frac{2(m-1)}{(m-1)+1}, \cdots, \frac{2(m-1)}{(m-1)+1}, \cdots, \frac{2(m-1)}{(m-1)+1}, \frac{2(m-1)}{(m-1)+1}, \cdots, \frac{2(m-1)}{(m-1)+1}, \cdots \]

is much more effective, which ultimately gives sublinear upper bounds as (6.1). In what follows, however, we will remark that, insofar as optimality of the constants are concerned, this is yet not the most favorable procedure. Rather, the key point seems to rely on keeping the uniformity when carrying interpolation procedures: the same $m$–linear form be considered in the whole process, which obviously yields a potentially sub optimal constant.

In a broad sense, this approach shows that the optimal Bohnenblust–Hille constants for $m$–linear forms is

\[ B_m \leq \inf \left\{ \sup_{i_1, \ldots, i_m} \left\{ (C_{(1,2,\ldots,2);m} \cdots C_{(2,2,\ldots,2);m} )^{1/m} \right\}, \cdots, \sup_{i_1, \ldots, i_m} \left\{ (C_{(\frac{2(m-1)}{(m-1)+1}, \ldots, \frac{2(m-1)}{(m-1)+1},2);m} \cdots C_{(\frac{2(m-1)}{(m-1)+1}, \ldots, \frac{2(m-1)}{(m-1)+1},2);m} )^{1/m} \right\} \right\}, \]

where, again, the supremum is taken over the product of the constants, considering the same $m$–linear form when estimating the constants and, moreover, the order of the sums is always the same. It turns out that this new strategy is rather sensible to the $m$–linear form chosen to estimate the optimal constants.
of the mixed \((\ell_1, \ell_2)\)-Littlewood inequalities, as performed in Section 4. In general we should not expect that the same \(m\)-linear form optimizes all the extrema \(m\)-linear forms used in the interpolation process and this is why the interpolation process seems to be sub-optimal. For reasons of symmetry, we can choose the natural order of the sums \(\sum_{i_1, \ldots, i_m=1}^{\infty} \prod_{s=1}^{m} T_s(e_{i_1}, \ldots, e_{i_m})\) in (6.3).

It is obvious that the estimates given in (6.3) are at least as good as the best known estimates given in [4]. While, it still seems hard to give a definite step forward as to improve the upper estimates of the \(m\)-linear Bohnenblust–Hille constants, we believe that some computational assisted simulations could accomplish some advances. In fact, we conjecture that the optimal constants for the \(m\)-linear Bohnenblust–Hille for real scalars are precisely \(2^{\frac{1}{2}} \pi\).

The effective computation of the above formulas seem to be somewhat puzzling in view of the uniformity (the same multilinear form has to be chosen, and the same order of the sums), and it is our belief that this is the reason why the previous proofs of the Bohnenblust–Hille inequality could not provide better constants.

The notion of universally extremum \(m\)-linear form is strongly connected to what we have discussed in this section so far. We will say that a continuous \(m\)-linear form \(T_{ext}: c_0 \times \cdots \times c_0 \to \mathbb{K}\) is universally extremum for the mixed \((\ell_1, \ell_2)\)-Littlewood inequalities if

\[
\begin{align*}
\sum_{j_1=1}^{\infty} \left( \sum_{j_2, \ldots, j_m=1}^{\infty} |T_{ext}(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{\frac{1}{2}} &\leq \left( \sqrt{2} \right)^{m-1} \|U\|, \\
\vdots &\\
\left( \sum_{j_1, \ldots, j_{m-1}=1}^{\infty} \left( \sum_{j_m=1}^{\infty} |T_{ext}(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{\frac{1}{2}} \right)^{1/2} &\leq \left( \sqrt{2} \right)^{m-1} \|U\|.
\end{align*}
\]

Note that the order of the sums in all \(m\) inequalities is preserved. We note that our approach to reach the optimal constants of the mixed \((\ell_1, \ell_2)\)-Littlewood inequalities have shown that different inequalities potentially need different extrema multilinear forms and, moreover, the order of the sums is rather important for the effectiveness of the computations. We are led to believe, henceforth, that it is quite unlikely that there exists an universally extremum \(m\)-linear form \(T_{ext}\) for \(m > 2\) as we have defined.

Of course the same definition of universally extremum multilinear forms will be stated for the other interpolation sequences

\[
\left( \left( \frac{2k}{k+1}, \ldots, \frac{2k}{k+1} \right) \prod_{s=1}^{m} \frac{1}{T_s(e_{i_1}, \ldots, e_{i_m})} \right)^{\frac{1}{m+1}} \leq \left[ D_m^c \prod_{s=1}^{m} \|T_s\|^{\frac{1}{m}} \right]^{\frac{1}{m+1}}
\]

After so many attention devoted to discuss the Bohnenblust–Hille inequality we think that it worths to present, with a quite simple proof, a formally stronger version of (1.1); the interesting case is the classical one, so this is just a formally stronger result:

**Theorem 6.1.** Let \(m\) be a positive integer. There is a constant \(D_m^c \geq 1\) such that

\[
(6.5) \quad \left( \sum_{i_1, \ldots, i_m=1}^{\infty} \left| \prod_{s=1}^{m} T_s(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{1}{m+1}} \right)^{m+1} \leq D_m^c \prod_{s=1}^{m} \|T_s\|^{\frac{1}{m}}
\]

for all continuous \(m\)-linear forms \(T_s: c_0 \times \cdots \times c_0 \to \mathbb{K}\) and all \(s = 1, \ldots, m\).

In (6.5), when \(T_1 = \cdots = T_m\), we recover the classical Bohnenblust–Hille inequality; the proof keeps for \(D_m^c\) the best known upper bounds for (1.1), i.e., there are constants \(\kappa_1, \kappa_2\) such that

\[
(6.6) \quad B_{\mathbb{R}, m} < \kappa_1 m^{2 - \log_2 \frac{2}{\gamma}}
\]

\[
(6.7) \quad B_{\mathbb{C}, m} < \kappa_2 m^{\frac{1 - \gamma}{\gamma}}
\]

where \(\gamma\) is the Euler-Mascheroni constant \((m^{2 - \log_2 \frac{2}{\gamma}} \approx m^{0.36482} \quad \text{and} \quad m^{\frac{1 - \gamma}{\gamma}} \approx m^{0.21139})\).

**Proof of Theorem 6.1.** Since

\[
\frac{1}{m+1} = \frac{1}{2(m-1)+1} + \cdots + \frac{1}{2(m-1)+1} + \frac{1}{2},
\]

...
we can apply Hölder inequality for mixed exponents (see [1] page 50) or [6]) as to reach

\[
\left( \sum_{i_1, \ldots, i_m=1}^{\infty} \left| \prod_{s=1}^{m} T_s(e_{i_1}, \ldots, e_{i_m}) \right|^\frac{m+1}{m} \right)^{\frac{m}{m+1}}
\]

\[= \left[ \sum_{i_1, \ldots, i_m=1}^{\infty} \left| \prod_{s=1}^{m} T_s(e_{i_1}, \ldots, e_{i_m}) \right|^\frac{m+1}{m} \right]^{\frac{m}{m+1}}
\]

\[\leq \left( \sum_{i_1=1}^{\infty} \left( \sum_{i_2, \ldots, i_m=1}^{\infty} |T_1(e_{i_1}, \ldots, e_{i_m})|^\frac{2(m-1)}{(m-1)+1} \right)^{\frac{m+1}{2(m-1)+1}} \right)^{\frac{1}{2}} \times \cdots \times \left( \sum_{i_m=1}^{\infty} \left( \sum_{i_1, \ldots, i_{m-1}=1}^{\infty} |T_m(e_{i_1}, \ldots, e_{i_m})|^\frac{2(m-1)}{m+1} \right)^{\frac{m+1}{2(m-1)+1}} \right)^{\frac{1}{2}} \]

In view of the Khinchin inequality and the optimal constants for the case of Rademacher functions for real scalars and Steinhaus functions for complex scalars (see [15], [17]) as in (4) and (10), each one of the \(m\) products above are dominated by

\[\left( \prod_{j=2}^{m} \Gamma \left( 2 - \frac{1}{j} \right) \right)^{\frac{1}{2}} \left( T_s \right) \]

for complex scalars and

\[2^{\frac{4(m-3)}{3(m-1)}} \frac{m}{2} \prod_{j=1}^{m} \left( \frac{\Gamma \left( \frac{3}{2} - \frac{1}{j} \right)}{\sqrt{\pi}} \right)^{\frac{1}{2}} \left( T_s \right) \]

for real scalars. The proof now follows as in (4); the above estimates yield (6.6) and (6.7).

We conclude the current manuscript with a list of guiding questions which we believe are relevant to advance the understanding in the theory of Bohnenblust–Hille inequalities.

**Problem 6.2.** Is the asymptotic growth of \(\text{ent}_m \left( \frac{2m}{m+1}, \ldots, \frac{2m}{m+1} \right)\) in fact exponential?

**Problem 6.3.** Is there some intermediate (between the classical and extremals) exponent \((q_1, \ldots, q_m)\) such that the optimal constants associated have a polynomial growth? In other words, is there \(\delta \in (0, 1)\) and \((q_1, \ldots, q_m)\) with

\[
\text{diam} (q_1, \ldots, q_m) = \delta
\]

such that the optimal growth of the constants associated to \((q_1, \ldots, q_m)\) is polynomial?

**Problem 6.4.** Is it true that for \(m > 2\) there are no universally extremum \(m\)-linear forms for the mixed \((\ell_1, \ell_2)\)-Littlewood inequalities?

**Problem 6.5.** Is it true that for \(m > 2\) there are no universally extremum \(m\)-linear forms for

\[
\left( \frac{2(m-1)}{(m-1)+1}, \ldots, \frac{2(m-1)}{(m-1)+1} \right)\cdots, \left( \frac{2}{m-1} \right)\cdots, \left( \frac{2}{m-1} \right)\cdots, \left( \frac{2(m-1)}{(m-1)+1}, \ldots, \frac{2(m-1)}{(m-1)+1} \right)\?
\]

A positive solution to Problem 6.2 would imply that the sequence of optimal constants of the \(m\)-linear Bohnenblust–Hille inequality is bounded. Positive solutions to Problems 6.3 and/or 6.5 are essentially a confirmation that the interpolation procedure is not optimal.
6.1. **Final comments.** It is evident that the Khinchin inequality is a key result in the proof of the Bohnenblust–Hille inequality. As we have remarked in Section 3, for $1 \leq p < p_0$ the optimal constants are $A_p = 2^{p-\frac{1}{2}}$. It is quite simple to check that these estimates are attained when $(a_j)_{j=1}^\infty = \alpha (\pm e_i, \pm e_j)$ for some $\alpha \neq 0$ and $i \neq j$. When $p = 1$ a decisive fact used in Section 3 is that the optimal estimates of the Khinchin inequality are achieved if and only if $(a_j)_{j=1}^\infty = \alpha (\pm e_i, \pm e_j)$ (Szarek’s result, see also [11] for a robust Khinchin Inequality with a qualitative approach to Szarek’s result). This was one of the main ingredients to prove Proposition 3.1. Although we have not found in the literature, we believe that Szarek’s result (and eventually a qualitative version of it) also holds for $p < p_0 \approx 1.8474$ (we are using the notation of Section 3).

**Conjecture 1.** For $p < p_0$ the optimal constants of the Khinchin inequality are attained if and only if $(a_j)_{j=1}^\infty = \alpha (\pm e_i, \pm e_j)$ for some $\alpha \neq 0$ and $i \neq j$.

Up to now, the best upper bound for $B_{R,4}$ is $2^{3/4}$. Following the lines of the proof of Proposition 3.1 and supposing the above conjecture verified, we can prove that if $2^{3/4}$ is in fact sharp and

$$T(x,y,z) = \sum_{i,j,k=1}^\infty a_{ijk}x_iy_jz_k$$

is an extremum, then

$$\text{Comp} (T) = 2.$$ 

Furthermore, a condition analogous to (3.3) and (3.4) is verified in this context. As we have mentioned before, we believe that $2^{3/4}$ is not sharp and there are not extrema as described above. Similar considerations hold for $3 < m \leq 13$ because $\frac{2(m-1)}{m(m-1)} < p_0$ and the optimal constants of the Khinchin inequality still behave as in the case $m = 2$. We stress that if we show that the in fact $2^{3/4}$ is not the optimal constant for $m = 3$, it is immediate that the other best known values for $m > 3$ are also not optimal.

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