ELECTRONIC STRUCTURES OF QUANTUM DOTS AND
THE ULTIMATE RESOLUTION OF INTEGERS

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ABSTRACT: The orbital angular momentum $L$ as an integer can be ultimately factorized as a product of prime numbers. We show here a close relation between the resolution of $L$ and the classification of quantum states of an $N$-electron 2-dimensional system. In this scheme, the states are in essence classified into different types according to the $m(k)$-accessibility, namely the ability to get access to symmetric geometric configurations. The $m(k)$-accessibility is an universal concept underlying all kinds of 2-dimensional systems with a center. Numerical calculations have been performed to reveal the electronic structures of the states of different types. This paper supports the Laughlin wave function and the composite fermion model from the aspect of symmetry.

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Introduction, the resolution of integers and the accessibility of the configurations with a $m$-fold axis

The resolution of integers is a basic and important theorem in the primary theory of number. Each integer $I$ can be ultimately factorized as a product of prime numbers as $I=2^{n_2}3^{n_3}5^{n_5}7^{n_7} \cdots$. Thus, like the elementary particles in physics, the prime numbers serve as elementary elements of integers. Let the set of the idempotent indexes be denoted as $\{n_i\}$. Evidently, the character of an integer is determined by this set. It is possible that the speciality of the set $\{n_i\}$ of an integer would affect the role of the integer in nature. In quantum mechanics, integers play a very important role. A number of physical quantities are integers (if specific units are used), e.g., the number of particles in a system, the orbital angular momenta $L$ and their components, the spin $S$, the charge $Z$, etc. Therefore, there might be a connection between the theory of number and quantum mechanics. However, such a connection is not clear until now. In particular, how the idempotent series $\{n_i\}$ play their role directly in physical world is not clear. For three dimensional systems there are magic integers (e.g., the numbers 2, 8, 20, 28, 50, 82, \cdots for the shell structures of nuclei; the numbers 2, 10, 18, 36, 54, \cdots for the periodic table of atoms, the numbers 13, 19, 25, 55, 71 \cdots for the abundance of atomic clusters, etc.). It is not clear whether these magic integers have something special in their $\{n_i\}$. Nonetheless, for two-dimensional quantum dots, we show here a close relation between the set $\{n_i\}$ of $L$ as an integer and the electronic structures.

Let us consider a two-dimensional system of $N$ electrons confined in a quantum dot $^{1-3}$. $N$ may be large or small, but is finite. It is assumed that the potential of confinement is isotropic so that the orbital angular momentum $L$ together with the total spin $S$ are conserved in an eigenstate $\Psi_{LS}$. It is well known that the spin-states are the basis-states of the two-row representation $\tilde{\lambda}$ of the permutation group, $\tilde{\lambda}=\frac{1}{2}+S,\frac{1}{2}-S$. Let the $i$-th spin-state of the $\lambda$ representation be denoted as $\chi_i^\lambda$, $i=1$ to $d$ (the dimension of $\lambda$). $\Psi_{LS}$ can be expanded as

$$
\Psi_{LS} = \sum_i F_i^\lambda \chi_i^\lambda \quad (1)
$$

where $F_i^\lambda$ is a function of spatial coordinates and is a basis-state of the $\lambda$ representation, the conjugate representation of $\tilde{\lambda}$. In (1) the antisymmetrization is assured.

On the other hand, when the number of electrons $N$ or $N-1$ can be factorized as a product of integers $N=N_A N_B$ or $N-1=N_C N_D$, the electrons may surround
the center of confinement and form a geometric configuration with a m-fold axis, 
m=NA, NB, NC, or ND. Such a configuration is called a m(k) configuration, where 
k=N/m or (N−1)/m is the number of homocentric circles, each contains m 
electrons (some circles might have the same radius). When k=(N−1)/m, the m(k) 
configuration would have an electron at the center. Some of the m(k) 
configurations are in the domain of low total potential energy, these m(k) are 
important to the electronic structures as we shall see. Since a rotation of a m(k) about the center by 
\( \frac{2\pi}{m} \) is equivalent to k cyclic permutations of particles, we have a m(k)

\[ e^{i2\pi L/m} F_i^\lambda(12\cdots) = F_i^\lambda(23\cdots) = \sum_j G_{ji}^\lambda(p_c) F_j^\lambda(12\cdots) \] (2)

where \( G_{ji}^\lambda(p_c) \) is the matrix element of the \( \lambda \)-representation associated with the 
k cyclic permutations \( p_c \). There are totally d such equations, they form a set of 
homogeneous linear equations. From the set we define a determinant

\[ D(L, \lambda, m) = |G_{ji}^\lambda(p_c) - \delta_{ij} e^{i2\pi L/m}| \] (3)

Evidently, if \( D(L, \lambda, m) \) is nonzero, the set of linear equations will have only zero 
solutions, and thereby the \( F_i^\lambda \) must all be zero at the m(k). In this case, an inherent 
nodal surface is imposed by symmetry and the m(k) is therefore inaccessible. If a 
wave function is distributed in a domain containing an inaccessible configuration, 
the inherent nodal surface would cause an excited oscillation resulting in a great 
increase in energy. Hence, for low-lying states, the wave function would be far away 
from the inaccessible configuration. Anyway, whether \( D(L, \lambda, m) \) is nonzero or zero 
would affect strongly the electronic structure of the state.

Since \( D(L, \lambda, m) \) depends on L and S, evidently the electronic structures depend 
strongly on L and S. The calculation of \( D(L, \lambda, m) \) is not difficult if N is small. 
However, a general discussion is not easy due to the complexity in the general 
representation of \( S_N \) group. Nonetheless, for polarized systems, the discussion 
becomes much simpler as follows.

Let the group of states having the same L be called a L − series. For polarized 
systems we have \( S=\frac{N}{2} \) and \( \lambda \) is totally-antisymmetric. In this case the discriminant reads

\[ D(L, \lambda, m) = (-1)^{(m-1)k} - e^{i2\pi L/m} = 0 \] (4)

If a couple of m and k fulfill (4), then the corresponding m(k) is accessible to the 
L − series. Evidently, all the m(k) are accessible to the L=0 series except the case 
of m even and k odd. Furthermore, all the m(k) are inaccessible to the L=1 series 
except the case of m=2 and k odd. When \( L \geq 2 \), from the resolution of integers, we have

\[ L = 2^{n_2} 3^{n_3} 5^{n_5} 7^{n_7} (11)^{n_{11}} \cdots \] (5.1)
\[ m = 2^{m_2} 3^{m_3} 5^{m_5} 7^{m_7} (11)^{m_{11}} \cdots \] (5.2)

Inserting (5) into (4), we have

\[ \exp[i\pi (2^{n_2+1-m_2} 3^{n_3-m_3} 5^{n_5-m_5} \cdots)] = (-1)^{(m-1)k} \] (6)

From (6) and by using a little primary knowledge of the theory of number we arrive at the following rules:

RULE 1, If \( m \) is odd, or if \( m \) and \( k \) are both even, then the m(k) is accessible to the L − series with \( n_i \geq m_i \) (here \( i = 2, 3, 5, 7, \cdots \)).

RULE 2, If \( m \) is even and \( k \) is odd, then the m(k) is accessible to the L − series with \( n_2 = m_2 - 1 \) and \( n_i \geq m_i \).

RULE 3, Let \( mk = m'k' = I \). If the integers \( m \) and \( m' \) do not have a common factor, and if both the \( m(k) \) and \( m'(k') \) are accessible to a L − series, then the product-configuration \( mn'(\frac{k}{m'}) \) is also accessible to the L − series.

RULE 4, Let \( mk = m'k' = I \). If the configuration \( mn'(\frac{k}{m'}) \) is accessible to a L − series, then both the \( m(k) \) and \( m'(k') \) are accessible to the L − series. Alternatively, if \( m(k) \) (or \( m'(k') \) ) is inaccessible to a L − series, the \( mn'(\frac{k}{m'}) \) is also inaccessible to the L − series.
RULE 5. If \( L = mk(mk \pm j_o)/2 \geq 0 \), where \( j_o \) is an odd integer, then the \( m(k) \) is accessible to the \( L - series \).

For examples, (i) The \( 8(3) \) configuration of the \( N=24 \) or \( 25 \) system has \( m_2 = 3 \) and \( m_1 = 0 \) \((i \geq 3)\). According to RULE 2, this configuration is accessible to the \( L=4j_o \) series, here \( j_o \) is an arbitrary positive odd integer. (ii) According to RULE 1 the \( 3(6) \) is accessible to the \( L=3j \) series, here \( j \) is an arbitrary positive integer. According to RULE 2 the \( 2(9) \) is accessible to the \( L=j_o \) series. Therefore both the \( 3(6) \) and \( 2(9) \) are accessible to the \( L=3j_o \) series. Since 3 and 2 do not have a common factor, according to RULE 3, the \( 6(3) \) is also accessible to the \( L=3j_o \) series. (iii) Since the \( 8(1) \) is accessible to the \( L=4j_o \) series, according to RULE 4 both the \( 4(2) \) and \( 2(4) \) are also accessible to the series. Alternatively, since the \( 2(4) \) is inaccessible to the \( L=j_o \) series, according to RULE 4 both the \( 4(2) \) and \( 8(1) \) are also inaccessible to the series.

It is straightforward to know from the RULE 5 that all the \( m(k) \) with \( mk = N \) are accessible to the \( L = N(N-1)/2 \) states, and all the \( m(k) \) with \( mk = N \) are accessible to the \( L = N(N-1)/2 \pm j \) states. Therefore, all the \( m(k) \) disregarding \( mk = N \) or \( N-1 \) are accessible to the \( L = j_o N(N-1)/2 \) states, i.e., these special states do not contain inherent nodal surfaces at any \( m(k) \), therefore they are specially stable. When a magnetic field is applied, they are the strongest candidates of ground states. Here the reciprocal of \( j_o \) is associated with the filling factor \( \nu \) of the Hall effect\(^6\), and the \( L = j_o N(N-1)/2 \) states are associated with \( \nu = 1, 1/3, 1/5, \ldots \).

It is recalled that the famous Laughlin wave function\(^8\),

\[
\psi_\nu = |\Pi_{i<j}(z_i - z_j)^j_o\rangle \exp(-\sum_j z_i^* z_j) \tag{7}
\]

has \( L = j_o N(N-1)/2 \), here \( j_o \) is also an odd integer. Thus, we have proved that this state does not contain inherent nodal surfaces at any \( m(k) \) configuration, therefore the wave function can be smoothly distributed without nodes in the domain of low potential energy. This might be a reason that they are close to the exact solutions.

Classification of quantum states in a 9-electron dot

We shall see that the accessibility of the \( m(k) \)-configurations affects the electronic structures greatly. Let us investigate in detail a 9-electron dot. Although such a system has already been more or less concerned in the literatures\(^10\)–\(^13\), a precise calculation beyond the lowest Landau level approximation and a detailed analysis of the wave functions have not yet been performed.

In the view of geometry there are the \( 9(1), 3(3), 8(1), 4(2), \) and \( 2(4) \) configurations. However, due to the RULE 1 and 2, only some of them are accessible to a specific \( L - series \). Furthermore, the \( 9(1) \) can be neglected due to having a much higher potential energy (In \( 9(1) \) only nine bonds can be optimized, while in \( 8(1) \) sixteen bonds can be. Incidentally, the \( 9(1) \) would become more important in a 10-electron system, in that case eighteen bonds can be optimized). Thus the important \( m(k) \) are the other four, they lie in the domain of lower potential energy. Based on their accessibility, the \( L - series \) can be classified into eight types as shown in TABLE 1. Due to the RULE 1 and 2, the scheme depends straight on \( \{n_i\} \). For an example, the \( L - series \) having \( n_2 = 2 \) and \( n_3 \geq 1 \) are both \( 8(1) \)- and \( 3(3) \)-accessible. According to the RULE 4, the \( 4(2) \), and \( 2(4) \) are also accessible to this series. Thus they are inherently nodeless in all the important \( m(k) \), and are grouped to type 1. Consequently, they are superior in stability and therefore particularly important. They have \( L = j_o N(N-1)/6 \), thus the above mentioned Laughlin states \((L = j_o N(N-1)/2)\) are members of this type. The other members of this type are also candidates of the ground state and are associated with the filling factor \( \nu = 3/j_o \).

For another example, the type 4 is \( 3(3) \)-accessible but \( 2(4) \)-, \( 4(2) \)- and \( 8(1) \)-inaccessible. Incidentally, the \( n_i \) with \( i \geq 5 \) are irrelevant to the classification of the
9-electron system and therefore can be arbitrary.

TABLE 1. Classification of states of a polarized 9-electron dot according to the \( \{n_i\} \) of L. The \( m(k) \) configurations accessible to a specific type are listed.

| Type | accessible \( m(k) \) | \( \{n_i\} \) |
|------|----------------------|----------------|
| 1    | 3(3),8(1)            | \( n_2 = 2, n_3 \geq 1 \) |
| 2    | 3(3),4(2)            | \( n_2 \geq 3, n_3 \geq 1 \) |
| 3    | 3(3),2(4)            | \( n_2 = 1, n_3 \geq 1 \) |
| 4    | 3(3)                 | \( n_2 = 0, n_3 \geq 1 \) |
| 5    | 8(1)                 | \( n_2 = 2, n_3 = 0 \) |
| 6    | 4(2)                 | \( n_2 \geq 3, n_3 = 0 \) |
| 7    | 2(4)                 | \( n_2 = 1, n_3 = 0 \) |
| 8    |                      | \( n_2 = 0, n_3 = 0 \) |

The classification according to \( \{n_i\} \) is in essence a classification according to the accessibility of the \( m(k) \), or in other words according to the inherent nodal surfaces. Thus, the classification is model-independent and based simply on the fundamental principle of symmetry. To show the reasonableness of the classification, numerical results are given in the following.

The Hamiltonian reads

\[
H = \sum_{i=1}^{N} \left( \frac{\vec{p}_i^2}{2m_e} + \frac{e^2}{4\pi\varepsilon} \sum_{j>i} \frac{1}{\vec{r}_{ij}} \right) + \frac{\hbar}{2\varepsilon} \sum_{i=1}^{N} \omega_i \frac{\rho_i}{\omega_i} \tag{8}
\]

where \( m^* = \frac{\hbar^2}{2m_e} \), \( \varepsilon \) the dielectric constant. It is assumed that \( m^* = 0.067m_e \), \( \varepsilon = 12.4 \) (for GaAs dots), and \( \hbar\omega_o = 3 \text{ meV} \). In what follows meV and \( \sqrt{\hbar/(m^*\omega_o)} \) will be used as units for energy and length, respectively.

Let the single-electron harmonic oscillator states be denoted as \( |ll',\omega\rangle \), they have energies \( (l + l' + 1)\hbar\omega \) and angular momenta \( l - l' \). With them antisymmetrized harmonic oscillator product states \( \Phi_J = |l_1l'_1,l_2l'_2,\cdots,l_Nl'_N,\omega\rangle \) with \( \sum_j (l_i - l'_i) = L \) are constituted and are used as basis functions of eigenstates. Here \( \omega \) is in general not equal to \( \omega_o \), but is adjustable to minimize the eigenenergies. For all the following calculations we have \( l_i \leq 25 \), \( \sum_i l_i' \leq 3 \), i.e., higher Landau levels are included. In order to depress the number of basis functions, \( \Phi_J \) are arranged in such a way so that \( \langle \Phi_J | H | \Phi_J \rangle \leq \langle \Phi_{J+1} | H | \Phi_{J+1} \rangle \). In such a sequence the one with a very large index \( J \) is not important to the low-lying states. Then, \( H \) is diagonalized first in a space with \( J \) starting from 1 to a given smaller number, and again to a larger number, and repeatedly, until a satisfied convergency is achieved, i.e., the eigenenergies have at least four effective figures and the correlated densities extracted from the related wave functions are nearly unchanged. It was found that, even in the case of \( N=19 \), \( J \leq 8000 \) is enough for our purpose if the variational parameter \( \omega \) has been properly adopted and if \( L \) is not much larger than \( N(N-1)/2 \) (e.g., \( L \leq 100 \) if \( N=9 \)).

Once an eigenstate \( \Psi_{LS} \) is obtained, the associated 1-body, 2-body, and 3-body density functions \( \rho_1, \rho_2 \) and \( \rho_3 \) would be extracted (the results of \( \rho_1 \) will not be given here). For example, the 3-body density function is defined as

\[
\rho_3(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \int d\vec{r}_4 \cdots d\vec{r}_9 |\Psi_{LS}|^2 \tag{9}
\]

It has been previously suggested\(^{10-12} \) that some of the electrons ( \( N_{\text{out}} \) ) might be located in a ring outside, and form a \( N_{\text{out}} \)-ring-structure. For a 9-electron dot, it turns out that the total potential energy of a ring-structure with \( N_{\text{out}} \leq 4 \) or \( N_{\text{out}}=9 \) is much higher. The domain in coordinate space containing the 5- to 8-ring-structures are broad, where the total potential energy is low and flat. If symmetry is not taken into account, these ring-structures might be equally preferred by low-lying states. However, if the domain of a ring-structure contains an inaccessible \( m(k) \) (e.g., the domain of a 6-ring structure contains the 3(3) which is inaccessible to the type 5 to 8), the ring-structure would be unfavorable because the existing inherent nodal surface would cause a great increase in energy. Thus, which ring-structures would be the better choice depends on the type of states.
Let the \( n \)-th state and its energy of a \( L - \)series be denoted as \( (L)_n \) and \( E_n(L) \). The \( n=1 \) state, namely the lowest of the series, is called the first-state. Let us define

\[
E_{\text{cusp}}(L) = E_1(L) - (E_1(L-1) + E_1(L+1))/2 \quad (10)
\]

Evidently, if \( E_{\text{cusp}}(L) \) is negative, the \( L - \)series would have a downward cusp.

Now, let us first inspect the \( L=60 \) series with \( \nu = N(N-1)/2L = 3/5 \) as an example of type 1. We have \( E_1(60) = 272.86 \) and \( E_{\text{cusp}}(60) = -0.43 \). Thus, the \( L=60 \) series has a downward cusp, a common feature of the type 1. The 2-body densities \( \rho_2 \) of the \( (60)_1 \) and \( (60)_2 \) are plotted in Fig.1a and 1b, where the 8-ring and 6-ring structures originate from the 8(1)- and 3(3)- accessibility, respectively. Since the 8-ring has more electrons in the ring, its moment of inertia is larger resulting in having a smaller rotation energy. Therefore it is lower than the 6-ring in the case of \( L=60 \).

The \( L=36 \) series belongs to the type 1 with \( \nu = 1 \). The \( \nu = 1 \) states are special because they have only one basis function in the lowest Landau level, namely the Laughlin wave function (eq.(7)) with \( j'_o = 1 \). Therefore the \( (36)_1 \) will be dominated by this function. In fact, in our calculation, this state has the weights of the lowest to the fourth lowest Landau levels to be 80.0\%, 16.8\%, 2.5\%, and 0.7\%, respectively. It is noted that the clear geometric features shown in Fig.1a and 1b arise from a coherent mixing of the basis functions. Although the \( (36)_1 \) is allowed by symmetry to get access to symmetric geometric configurations, this state is not able to possess a clear geometric feature as shown in Fig.1c due to the lack of coherent mixing. In fact, the feature of Fig.1c arises simply from the Laughlin wave function. Incidentally, the \( (36)_1 \) has a rather low energy \( E_1(36) = 209.18 \) and a very large gap 4.98 lying between \( E_1(36) \) and \( E_2(36) \), thus this state is superior in stability.

Fig.1d to 1f are examples of type 4. They do not have the 8-ring structure because this type is 8(1)-inaccessible. The \( (63)_1 \) and \( (81)_1 \) have a clear 6-ring originating from the 3(3) accessibility. To see clearer the structure of the core, the \( \rho_3 \) of the \( (81)_1 \) is plotted in Fig.2a, where a clear regular triangle is inside. However, instead of having a 6-ring, the \( (99)_1 \) has a 7-ring structure. It is noted that a 9-particle system does not contain the 7(k)-configuration. Hence, the 7-ring is not constrained by symmetry. Thus, it is not surprising that both the 6- and 7-ring emerge in the type 4. The 6-ring would be better than the 7-ring if \( L \) is smaller (e.g., the \( (63)_2 \) and \( (81)_2 \) are found to have a 7-ring). However, the 7-ring would be better if \( L \) is larger due to having a larger moment of inertia.

It is clear that, although the inherent nodal surfaces have imposed serious constraints on wave functions, the electronic structures are not uniquely determined by them. In addition to the pairwise interaction, the centrifugal barrier and the parabolic potential also play their role. The barrier leads to the preference for the configurations with a larger moment of inertia. Therefore, a critical value(s) of \( L \) denoted as \( L_{\text{crit}} \) might exist for each type so that the first-states with \( L \) smaller than \( L_{\text{crit}} \) and those with \( L \geq L_{\text{crit}} \) are distinct in structure (e.g., the \( (81)_1 \) and \( (99)_1 \)). The parabolic potential confines the number of effective basis functions taking part in coherent mixing. Thus the lower states with \( \nu \) equal or close to 1 are insufficient in coherent mixing and therefore ambiguous in geometric feature (e.g., the \( (36)_1 \)).

In addition to the case of \( \nu = 1 \), an example of \( \nu = 18/19 \) is shown in Fig.1g to 1i belonging to type 7. The \( L=38 \) states have only two basis functions in the lowest Landau level. Consequently, both the \( (38)_1 \) and \( (38)_2 \) are ambiguous in geometric feature. However, the \( (38)_3 \) dominated by the second lowest levels has a clear 7-ring due to having a sufficient coherent mixing. There is a very large gap 3.15 lying between the \( E_2(38) \) and \( E_3(38) \). The two lower states have \( E_1(38) = 220.61 \) and \( E_2(38) = 221.25 \), much higher than the \( E_1(36) \). Noting that the type 7 is both 8(1)- and 3(3)-inaccessible. Therefore, the 7-ring is preferred. Another example of type 7 is given in Fig. 1j. Since a number of the first Landau levels are contained in the
L=50 series, instead of the third-state, the 7-ring appear first in the first-state.

Fig.1k and 1l show the similarity of the two first-states of type 2, both have a 6-ring. Fig.1m and 1n show the similarity of the two second-states, both have a square-structure. Evidently, these structures originates from the 3(3)- and 4(2)-accessibility. To see clearer the square, the $\rho_3$ of the (48)$_2$ is plotted in Fig.2b. Since these states are 8(1)-inaccessible, the inherent nodal surfaces at the 8(1) would spoil the stability of the square. Thus the square-structure is higher.

Fig.1o and 1p are examples of type 5. Since this type is 8(1)-accessible but 3(3)-inaccessible, it is easy to understand why the octagon shape emerges.

**Classification of fermion states of $N \neq 9$ dots**

For the classification of states of a general $N$-electron dot, we have first to figure out which $m(k)$ configurations with $mk=N$ or $N-1$ will be contained in the domain of lower potential energy. Secondly, we have to make sure their accessibility to the $L-\text{series}$.

If $N=6$, the 5(1), 3(2), 2(3) configurations should be considered (the 6(1) is automatically taken into account due to the RULE 3). Then the type 1 has $L=j_015$, which is the intersection of $\{L\equiv0 \text{ mod } 5\}$and $\{L\equiv3 \text{ mod } 6\}$. This is a well known result$^{14}$.

When $N$ is larger, the effect of the $m(k)$-accessibility might reduce, because in the coordinate space the domain of low energy is so broad that the wave function is easy to avoid the inaccessible configuration. However, even if $N$ is as large as 19, the effect of the $m(k)$-accessibility is still explicit. When $N=19$, the 9(2), 3(6) and 2(9) have to be considered (the 6(3) is automatically taken into account due to the RULE 3). Then the classification is shown in TABLE 2.

| Type | accessible $m(k)$ | $\{n_i\}$ |
|------|------------------|------------|
| 1    | 9(2),3(6),2(9)   | $n_2=0$, $n_3 \geq 2$ |
| 2    | 3(6),2(9)       | $n_2=0$, $n_3 = 1$ |
| 3    | 9(2)            | $n_2 \geq 1$, $n_3 \geq 2$ |
| 4    | 3(6)            | $n_2 \geq 1$, $n_3 = 1$ |
| 5    | 2(9)            | $n_2 = 0$, $n_3 = 0$ |
| 6    | $n_2 \geq 1$, $n_3 = 0$ |

It was found that both the type 1 and type 2 are better in stability, they have downward cusps. For examples, we have $E_{\text{cusp}}(177)=-0.032$ and $E_{\text{cusp}}(183)=-0.014$, both belong to type 2. Fig.2c show a 12-ring structure originating from the 6(3)-accessibility (similar to the fact that the 6-ring structure originates from the 3(3)-accessibility, cf. Fig.2a). this 12-ring structure is common to the type 1 and 2. On the other hand, the geometric feature of the (184)$_1$ of type 6 is not very clear, but the ring-structure is explicit (13 electrons are found in the ring). These figures support the ring structure proposed by other authors$^{10-12}$.

In general, not matter how large $N$ is, only the states with a superior stability are interesting. The stability is quite often associated with the geometric symmetry. Once the geometric symmetry is concerned, the effect of the $m(k)$-accessibility should be considered.

**Bosonic systems**

Since the above discussion is model-independent and is simply based on symmetry consideration, it can be generalized to bosonic systems as well. In this case the wave functions should be completely symmetric with respect to particle permutation. Thus, instead of eq.(4), we have the criterion

$$1 - e^{i2\pi L/m} = 0 \quad (10)$$
and accordingly the \( m(k) \) configuration is accessible to the \( L - \text{series} \) with \( n_i \geq m_i \) disregarding \( m \) and \( k \) are even or odd. In particular, all the \( m(k) \) are accessible to the \( L=0 \) states.

In recent years the Bose-Einstein condensation has been extensively studied, and the trapped atoms gases have been shown to Bose condense\(^{15,16} \). Cooper and Wilkin have studied the properties of rotating Bose-Einstein condensates in parabolic traps. When the rotation frequency is larger, they found the ground state angular momentum \( L \) for \( N=3 \) to 10- boson systems are 6, 12, 20, 30, 42, 56, 72, and 90, respectively\(^{17} \). This can be explained based on the composite fermion model\(^{17,18} \). Alternatively, we now provide a model-independent explanation simply based on the \( m(k) \)-accessibility. For an example, the important \( m(k) \) for the \( N=8 \) system is 7(1) and 4(2) (here the 8(1) is much less important than the 7(1), the former has only eight bonds to be optimized while the latter has fourteen ), thus the type 1 has \( n_2 \geq 2 \) and \( n_7 \geq 1 \). Therefore the \( L=56 \) state would appear as a ground state when the rotation frequency lies in a specific region. The important \( m(k) \) for the \( N=9 \) system is 8(1) and 3(3), thus the type 1 has \( n_2 \geq 3 \) and \( n_3 \geq 1 \), therefore the \( L=72 \) state would appear as a ground state. The important \( m(k) \) for the \( N=10 \) system is 9(1), 5(2), 3(3), and 2(5), thus the type 1 has \( n_2 \geq 1 \), \( n_3 \geq 2 \), and \( n_5 \geq 1 \), therefore the \( L=90 \) state would appear as a ground state.

In fact, the downward cusps found in the ref.17 are closely related to the \( m(k) \)-accessibility. For examples, from the TABLE I of the ref.17 , we know that the \( N=6 \) system has a cusp at \( L=6 \) and 12 which are associated with the 3(2) and 2(3) accessibility, a cusp at \( L=10 \) which is associated with the 5(1) and 2(3) accessibility, a cusp at \( L=15 \) which is associated with the 5(1) and 3(2) accessibility, etc.

Concluding remarks
An exact diagonalization of the Hamiltonian has been performed. Since higher Landau levels have been considered, the numerical results (even in the case of \( N=19 \)) are very accurate in the qualitative sense. Since the \( \omega \) of the basis functions is a variational parameter, the convergency is thereby greatly improved. Furthermore, in addition to the usually given \( \rho_1 \) and \( \rho_2 \), \( \rho_3 \) have also been calculated to help the analysis.

A scheme of classification according to the idempotent series \( \{n_i\} \) of \( L \) has been proposed. In this scheme each type has its own \( m(k) \)-accessibility, or its own inherent nodal surfaces. The classification is objective and model-independent. Although the electronic structures are not uniquely determined by the \( m(k) \)-accessibility, its great effect has been confirmed by the numerical results. Since the type 1 is inherently nodeless in the domain of low potential energy, the first-states of this type are superior in stability and are the strongest candidates of the ground states. These noticeable states can be easily identified in our scheme.

The analysis of this paper supports the Laughlin wave function and the composite fermion model from the aspect of symmetry.

The \( m(k) \)-accessibility is an universal concept for all kinds of 2-dimensional systems with a center. The introduction of this concept would lead to a better understanding of these systems.

The theory of number and quantum mechanics are two previously unrelated areas of science. Here we show a direct relation between the ultimate resolution of \( L \) as an integer and the accessibility of the \( m(k) \) configuration. This finding might lead to a closer relation between these two important areas of science.

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FIGURE CAPTIONS

Fig.1, Contour plots of $\rho_2$ of a 9-electron system. The given electron is marked by a black spot, its distance from the origin is given (in the unit $\sqrt{\hbar/(m^*\omega_o)} = 194.7\text{Å}$). This marked distance serves as a scale for both the X and Y directions (slightly different scales have been used for distinct states). The inmost contour (associated with the highest peak) is marked by a double-line.

Fig.2, Contour plots of $\rho_3$ of 9- and 19-electron systems. Different scales have been used for different states.