Estimating the boundary of the asymptotic stability region of Lotka–Volterra system by using the trajectory reversing method

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Abstract. This paper proposes a topological approach for plotting the boundary of the region of asymptotic stability (RAS) of Lotka–Volterra predator-prey system. First, stability analysis was used to determine the specific saddle point that has eigenvalues with one positive and two negative real parts in a linearized Jacobian matrix. A set of initial states located around the saddle point on the specific eigenplane spanned by the two stable eigenvectors was then selected. Finally, the trajectory reversing method was used and the trajectories that had initial states on the eigenplane delineated the boundary of the asymptotic stability region. The trajectories of the initial states that started from the opposite sides of the RAS exhibited different dynamic behaviour. The numerical simulation are presented to demonstrate the effectiveness of the proposed approach.

1. Introduction
Determining the boundary of an asymptotic stability region around an equilibrium point is a conventional technique in the study of nonlinear systems. The region of asymptotic stability (RAS) is defined as a set of initial conditions in which the system approaches a specific equilibrium point. The analysis of the RAS can clarify many nonlinear characteristics of a system. Studies have proposed numerous methods for finding the boundary of the RAS. These methods can be classified into two categories: Lyapunov and non-Lyapunov methods [1].
The Lyapunov method consists of two steps. First, Lyapunov’s direct method is used to derive a Lyapunov function and prove the local asymptotical stability of an equilibrium point. Second, optimization techniques are used to enlarge the region defined by the Lyapunov function [2-3]. Because Lyapunov functions are not exclusive, an RAS estimated using a Lyapunov function is usually conservative. No conclusion can be reached regarding the actual boundary of the RAS [4]. Most non-Lyapunov methods are based on topological approaches. The stability characteristics of a nonlinear system can be determined by analysing the type of equilibrium point of the linearized system. There are three types of equilibrium points: sink, source, and saddle points. Saddle points have specific dynamic behaviours that are distinct from those of sink and source points. For example, if some of the initial points are determined to be in the neighborhood of a saddle point, parts of the trajectories converge to the saddle point when the time is increasing or reversing, but others do not. Therefore, these converging trajectories can be used to form the surface of a stability boundary. The estimation of the boundary of an RAS in the second-order system using the trajectory reversing method has been demonstrated previously [5-6]. For presenting a clear boundary for an RAS in a three-dimensional space, Lee and Han proposed the phase portrait method, in which some special sets

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of initial points are selected on the surface spanned by the stable eigenvector of the saddle point; the trajectory reversing method is then used to make the trajectories form a manifold of the RAS in a nonlinear system [7].

The Lotka–Volterra systems are crucial mathematical models. They are the proper starting point for numerous classes of models in ecology, biology, physics, and chemistry. Numerous interesting dynamic behaviours (e.g., global asymptotic behaviour, bifurcation, and phase space analysis) of Lotka–Volterra systems have been identified [8-10]. In addition, the conditions of chaotic dynamics were evaluated by using equilibria analysis [11]. However, few studies have discussed the boundary of RAS or the edge of chaotic attractors in Lotka–Volterra systems. The technique for determining the exact basins of chaotic attractors when a chaotic attractor and a stable equilibrium point coexist in phase space has seldom been investigated. Clarify the clear boundary of RAS can be used to predict the dynamic characteristics of initial states.

This study analysed the stability characteristics of equilibrium points and determined the actual boundaries of the RAS in the three-dimensional Lotka–Volterra predator-prey system. First, the necessary conditions for the saddle point and the RAS are introduced. Furthermore, the three-dimensional Lotka–Volterra predator-prey system is presented and the stability properties of the equilibrium points are investigated. Last, the numerical results for plotting the boundary of an RAS are given.

2. Method

Consider a third-order autonomous system expressed by

$$\dot{X} = F(X),$$

(1)

where $X = [x_1 \ x_2 \ x_3]^T$ and $F(X) = [f_1(X) \ f_2(X) \ f_3(X)]^T$.

Linearizing the third-order system at an equilibrium point $X_{ep}$, the Jacobian matrix can be written as

$$A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}
\end{bmatrix}. $$

(2)

Assume that the column vector $V_{Ri}$ and the row vector $V_{Li}$ satisfy the following equations

$$A \cdot V_{Ri} = \sum_{i=1}^{3} \lambda_i \cdot V_{Ri},$$

(3)

$$V_{Li}^T \cdot A = \sum_{i=1}^{3} \lambda_i \cdot V_{Li}^T,$$

(4)

where $\lambda_i$ is the eigenvalues of $A$; $V_{Ri}$ is named the right eigenvector of $A$. Likewise, $V_{Li}^T$ is named the left eigenvector of $A$.

Rewrite Eq. (3) and (4) as

$$A \cdot V_{R} = V_{R} \cdot D \Rightarrow V_{R}^{-1} \cdot A = D \cdot V_{R}^{-1},$$

(5)

$$V_{Li}^T \cdot A = D \cdot V_{Li}^T,$$

(6)

where $D$ is a diagonal matrix if all the eigenvalue of $A$ are different.

Comparing Eq. (5) and (6), it can be seen that

$$V_{Li}^T = V_{R}^{-1} \Rightarrow V_{Li}^T \cdot V_{R} = I.$$

(7)

Assume the boundary of the RAS in the phase space is the manifold surface passing through a saddle point. Using linearizing process at a saddle point, this boundary will be a tangent plane of the manifold surface at this saddle point. Suppose that the manifold including the point ($\hat{x}_1, \hat{x}_2, \hat{x}_3$) = 0 has the normal vector $V^T$. Then

$$V^T \cdot \hat{X} = 0.$$ 

(8)
Because all the states on this plane must satisfy the Eq. (1), we have
\[ \frac{d}{dt}(V^T \cdot \hat{x}) = 0 \Rightarrow V^T \cdot \hat{x} = 0 \Rightarrow V^T \cdot A \cdot \hat{x} = 0. \] (9)
Referring to Eq. (6), if we want to satisfy Eq. (8) and (9) simultaneously, we have
\[ V^T \cdot A \cdot \hat{x} = \lambda \cdot V^T \cdot \hat{x} \Rightarrow V^T \cdot A = \lambda \cdot V^T. \] (10)
Eq. (10) indicates that \( V^T \) must be the left eigenvector of Jacobian matrix \( A \). Therefore, the manifold surface with normal vector \( V^T \) can be spanned by the two stable right eigenvectors.

According to the preceding analysis, the procedure for identifying the boundary of the RAS is as follows:

Step 1: Find all the equilibrium points \( (F(X) = 0) \).
Step 2: Find the specific saddle point having one positive and two negative real parts of eigenvalues in the linearized Jacobian matrix. Then calculate the left eigenvectors that are associated with the eigenplane spanned by the two stable eigenvectors of the saddle point.
Step 3: Select a set of initial points around the saddle point on the eigenplane.
Step 4: Utilize the backward integrations of the system equation and plot all the reversing trajectories in phase space.

3. Stability analysis of the three-dimensional Lotka-Volterra system
Consider the three-dimensional Lotka-Volterra predator-prey system [12]:
\[ \frac{dx_1}{dt} = ax_1 - bx_1x_2 + ex_1^2 - sx_1^2 x_3, \] (11)
\[ \frac{dx_2}{dt} = -cx_2 + dx_1x_2, \] (12)
\[ \frac{dx_3}{dt} = -px_3 + sx_1^2 x_3. \] (13)
The Jacobian matrix of the predator-prey system is
\[ A_p = \begin{bmatrix} a + 2ex_1 - bx_2 - 2sx_1x_3 & -bx_1 & -sx_1^2 \\ -c + dx_1 & 0 & 0 \\ 2sx_1x_3 & 0 & -p + sx_1^2 \end{bmatrix}. \] (14)
Consider the following system parameters: \( a=1, b=1, c=1.2, d=1, e=2, p=3 \) and \( s=2.7 \). The predator-prey system has five equilibrium points \( E_i^p, i=1,2,3 \): \( E_1^p = (0,0,0) \), \( E_2^p = (-0.5,0,0) \), \( E_3^p = (-1.054,0.0,3.893) \), \( E_4^p = (1.054,0.1,0.921) \) and \( E_5^p = (1.3,0) \). The corresponding eigenvalues of the Jacobian matrix (14) evaluated at equilibrium points are: \( \lambda_1 = -3, \lambda_2 = -1, \lambda_3 = 1 \) for \( E_1^p \) (a saddle having two negative eigenvalues); \( \lambda_1 = -1, \lambda_2 = -1.5, \lambda_3 = -2.325 \) for \( E_2^p \) (a stable node having three negative eigenvalues); \( \lambda_1 = -3.1266, \lambda_2 = 2.1256, \lambda_3 = -0.054 \) for \( E_3^p \) (a saddle having two negative eigenvalues); \( \lambda_1 = 0.5409, \lambda_2 = 0.5409 \), \( \lambda_3 = -0.5409 \) for \( E_4^p \) (a spiral-saddle having two complex eigenvalues with negative real part); \( \lambda_1 = 0.5409, \lambda_2 = 1+1.414j, \lambda_3 = 1-1.414j \) for \( E_5^p \) (a spiral-saddle having two complex eigenvalues with positive real part).
Both spiral-saddle points \( E_4^p \) and \( E_5^p \) are the chaotic attractor. The interaction between these two chaotic attractors is responsible for the chaotic dynamics of the system. Regarding the saddle point \( E_3^p \), the two stable eigenvectors are related to the stable manifold separating the boundary of RAS of the unstable region and the stable equilibrium point \( E_2^p \).

4. Numerical simulation of an RAS
Regarding the saddle point \( E_3^p \), the left eigenvector related to the positive eigenvalue is \( V_L = [0.572, 0.144, -0.807] \), which can be defined the normal vector of the tangent plane of boundary of RAS at \( E_3^p \).
Regarding the neighborhood of the saddle point $E^p_2$, by employing the backward integration method for selecting a set of initial conditions on the tangent plane, the boundary of the RAS manifold can be found, as shown in Fig. 1. For example, Fig. 2 shows that the trajectories with initial points (-0.8, 0.031, 0.69), (-1.71, 0.058, 0.185) in the unstable region (upper side of RAS) will diverge to infinity. However, Fig. 3 shows that the trajectories with initial points (-0.8, 0.031, 0.59), (-1.71, 0.058, 0.085) in the stable region (opposite side of RAS) will converge to the equilibrium point $E^p_2$. Therefore, the boundary in Figure 1 is really the boundary of RAS of Lotka-Volterra predator-prey system.

**Figure 1.** The boundary of RAS of the Lotka-Volterra predator-prey system.

**Figure 2.** The trajectories with initial states on the unstable region (upper side of RAS).
Figure 3. The trajectories with initial states on the stable region (opposite side of RAS).

5. Conclusion
In this paper, stability analyses of the third-order Lotka–Volterra predator-prey system are conducted. The boundary of the RAS of the system is plotted using the trajectory reversing method. The trajectories of the initial states starting from opposite sides of the RAS exhibit different dynamic behaviour. The simulation results verify the effectiveness of the proposed method.

6. References
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Acknowledgments
This paper was supported by Education Department of Fujian Province (no. FBJG20180093) and Construction Project of Mechanical Engineering Application-oriented Discipline in Fujian Province.