Gravity–capillary flows over obstacles for the fifth-order forced Korteweg–de Vries equation

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Abstract The aim of this work is to investigate gravity–capillary waves resonantly excited by two topographic obstacles in a shallow water channel. By considering the weakly non-linear regime the forced fifth-order Korteweg–de Vries equation arises as a model for the free-surface displacement. The water surface is initially taken at rest over a uniform flow and the initial value problem for this equation is computed numerically using a pseudospectral method. We study nearly-resonant flows with intermediate capillary effects. Details of the wave interactions are analysed for obstacles with different sizes. Our numerical results indicate that the flow is not necessarily governed by the larger obstacle.

Keywords Bond number · Gravity–capillary waves · KdV equation · Solitary waves

1 Introduction

Waves excited by an external force are of great current interest due to its large number of physical applications. For instance, non-linear electrical lines, superconductive electronics, elementary-particle physics [1,2] and hydrodynamics. Regarding the last one, we mention, flow of water over rocks, ship waves [3] and waves generated by storms [4]. In water waves, the external force usually models a pressure distribution or a topographic obstacle.

In the absence of surface tension, the fundamental parameter used for describing the pattern of waves generated due to a current–topography interaction is the Froude number

$$F = \frac{U_0}{\sqrt{gh_0}}.$$ 

Here, $U_0$ is the velocity of the uniform stream, $g$ is the acceleration of gravity and $h_0$ is the undisturbed depth of a shallow water channel. The Froude number is critical when $F = 1$, i.e. when the linear long-wave phase speed is equal to the mean flow speed. In the weakly non-linear regime the forced Korteweg–de Vries (fKdV) model is
valid to study nearly-resonant flows \((F \approx 1)\) over obstacles with small amplitudes. A detailed study considering the fKdV equation was first done by Wu and Wu [5], later in [6–10], and more recently by Flamarion et al. [11] for a vertically sheared current. All these authors have considered only one obstacle.

Regarding flow over multiple obstacles, Chardard et al. [12] investigated numerically the stability of solitary waves. Lee and Whang [13] studied trapped waves between two obstacles. They considered a bottom topography with two bumps and found numerical solutions for the fKdV which remained bouncing back and forth between the obstacles for a certain period of time. Grimshaw and Malewoong [14] considered a nearly-resonant flow over two obstacles. They defined the development of the flow in three stages. The first stage is characterized by the formation of an undular bore over each obstacle. The second is the interaction of the waves generated between the obstacles, and the third is the evolution at large times when the larger obstacle controls the flow. More recently, these authors studied the interaction of waves generated over two obstacles (bumps and holes) in the nearly-resonant regime describing the dynamic of the wave interactions in great details [15].

When the surface tension is included in the problem, an additional parameter becomes fundamental in the study of waves generated, namely the Bond number

\[B = \frac{\sigma}{\rho gh_0^2},\]

where \(\sigma\) is the coefficient of the surface tension and \(\rho\) is the constant density of the fluid. Gravity–capillary waves can also be described by the Korteweg–de Vries (KdV) nearly-critical equation. However, the dispersion term in the equation vanishes when \(B\) is critical, i.e. \(B = 1/3\). Thus, solitary wave solutions (sech\(^2\)-like) are no longer appropriate since the length scale of these waves becomes zero [16]. Studying flows over obstacles under gravity–capillary effects, Milewski and Vanden-Broeck [17] derived a fifth-order fKdV for \(F \approx 1\) and \(B \approx 1/3\). They showed that this equation has unsteady solitary wave solutions with small oscillating tails. A numerical investigation of solitary waves and collisions for the fifth-order KdV was done by Malomed and Vanden-Broeck [18]. They found solitary waves with oscillatory tails, and when two of these waves interact some of them regain their shape while others split into several solitary waves of different types.

Recently, Hanazaki et al. [19] used the body-fitted curvilinear coordinates to solve Euler’s equations in the presence of an obstacle with a uniform flow and compared the results with the fKdV and the fifth-order fKdV for the resonant flow \((F = 1)\) and intermediate capillary effects \((B \approx 1/3)\). They observed short wave radiation when the effects of surface tension are lesser \((B < 1/3)\). Besides, a train of solitary waves propagating upstream radiates short linear waves whose phase speed is equal to the upstream-advancing speeds of the solitary wave. The fifth-order fKdV captured the wave train propagating upstream, however, it predicted waves of longer length, which is natural since the KdV models are based on a long-wave approach.

In this paper, we investigate numerically the interaction of excited gravity–capillary waves in the nearly-resonant flow over two obstacles for the fifth-order fKdV. More precisely, we focus on the case in which the Froude number is nearly-critical and the capillary effects are intermediate. The problem is studied for obstacles with different sizes. To the best of our knowledge, there are no articles regarding the fifth-order fKdV in the presence of two obstacles. From our experiments, we identify that there are regimes in which the flow is not necessarily driven by the larger obstacle, what is different from the case in which surface tension is neglected [14,15]. Besides, we present in detail the main features of the nearly-resonant flow.

This article is organized as follows. In Sect. 2, we present the mathematical formulation of the non-dimensional fifth-order fKdV equation. The numerical results are presented in Sect. 3 and the conclusion in Sect. 4.

### 2 The fifth-order forced Korteweg-de Vries equation

We consider a two-dimensional incompressible and irrotational flow of an inviscid fluid with constant density \((\rho)\) in a shallow water channel of undisturbed depth \((h_0)\) and in the presence of a uniform flow \((U_0)\). Besides, the fluid is under the gravity force \((g)\) and the surface tension \((\sigma)\).
The third-order forced Korteweg–de Vries is obtained considering the following set of dimensionless variables: the horizontal coordinate $x$, time $t$, the free-surface displacement $\zeta(x, t)$ and the obstacle $h(x)$ scaled as $x \rightarrow \epsilon^{-1/2} x$, $t \rightarrow \epsilon^{-3/2} t$, $\zeta(x, t) \rightarrow \epsilon \zeta(x, t) + \mathcal{O}(\epsilon^2)$, $h(x) \rightarrow \epsilon^2 h(x)$, where $\epsilon > 0$ is a small parameter [17]. The Froude number is scaled as $F = 1 + \epsilon f$, where $f$ is a constant. Under these conditions the third-order fKdV equation becomes

$$\zeta_t + f \zeta_x - \frac{3}{2} \zeta_{xx} + \frac{B - 1/3}{2} \zeta_{xxx} = \frac{1}{2} h_x(x).$$

This equation does not have solitary waves as solutions for $B = 1/3$. When the Bond number is nearly-critical a fifth-order fKdV becomes more appropriate to study flow over obstacles.

In order to derive the fifth-order fKdV, Milewski and Vanden-Broeck [17] used the following scalings $x \rightarrow \epsilon^{-1/4} x$, $t \rightarrow \epsilon^{-5/4} t$, $\zeta(x, t) \rightarrow \epsilon \zeta(x, t) + \mathcal{O}(\epsilon^2)$, $h(x) \rightarrow \epsilon^2 h(x)$. The Froude and Bond numbers are scaled as $F = 1 + \epsilon f$ and $B = 1/3 + \epsilon^{1/2} b$, respectively, where $b$ is a constant. This scaling yields the fifth-order forced Korteweg–de Vries (5th-order fKdV)

$$\zeta_t + f \zeta_x - \frac{3}{2} \zeta_{xx} + \frac{b}{2} \zeta_{xxx} - \frac{1}{90} \zeta_{xxxxx} = \frac{1}{2} h_x(x).$$

The parameter $f$ represents a perturbation of the Froude number, i.e., $F = 1 + \epsilon f$, and $b$ is a perturbation of the Bond number $B = 1/3 + \epsilon^{1/2} b$. The flow is supercritical, subcritical or nearly-resonant depending on whether $f > 0$, $f < 0$ or $f \approx 0$. Analogously, the capillary effect is strong, weak or intermediate whether $b > 0$, $b < 0$ or $b \approx 0$.

The 5th-order fKdV equation (1) is solved numerically using a Fourier pseudospectral method with an integrating factor [17,20]. It avoids numerical instabilities due to the higher-order dispersive term. We consider the computational domain $[-L, L]$ with a uniform grid with $N$ points and step $\Delta x = 2L/N$. All derivatives in $x$ are performed spectrally. In addition, the time evolution is computed through the Runge–Kutta fourth-order method (RK4). The initial wave profile is always taken at rest ($\zeta(x, 0) = 0$). Since the fKdV fails when the Bond number is critical, we focus exactly on this case ($b = 0$).

3 Numerical results

In the same fashion as presented in [15], we consider a bottom topography modelled by two localized obstacles

$$h(x) = \epsilon_1 \exp \left( -(x - x_a)^2/w \right) + \epsilon_2 \exp \left( -(x - x_b)^2/w \right),$$

where $\epsilon_1$ and $\epsilon_2$ are the amplitudes of the obstacles, $w$ is the width of the obstacles, and $x_a$ and $x_b$ are their locations. We focus on the cases in which $\epsilon_1$ and $\epsilon_2$ are positive and let the parameter $f$ vary. There are a long list of parameters to be considered so that we fix $x_a = -100$, $x_b = 100$ and $w = 50$. This choice of parameters let us observe waves being generated over the obstacles independently at early times, and later analyse their interactions. A sketch of the physical problem at $t = 0$ is depicted in Fig. 1. Since the current is turned on at $t = 0^+$ waves are immediately generated.

In the following subsection, we present our results in the non-resonant and nearly-resonant regimes. In order to avoid the effects of the spatial periodicity, for instance the return of small-amplitude radiation, we consider a large computational domain. In addition, we point out that the results present in the next section do not depend on the grid. The resolution was tested with different numbers of Fourier modes.

3.1 Non-resonant regime

In this regime, we let $|f|$ be sufficiently large and investigate how changes of the amplitude of the obstacles affect the flow.
Fig. 1 Sketch of the physical problem at $t = 0$

![Sketch of the physical problem at $t = 0$]

Fig. 2 Supercritical non-resonant regime free-surface evolution with $f = 0.2$, $\epsilon_1 = 0.01$ and $\epsilon_2 = 0.01$

![Supercritical non-resonant regime free-surface evolution with $f = 0.2$, $\epsilon_1 = 0.01$ and $\epsilon_2 = 0.01$]

Supercritical regime

In a presence of one single obstacle, the supercritical non-resonant flow is characterized by the formation of an elevation wave over the obstacle and depression solitary waves propagating downstream. Besides, when $t \to \infty$, the steady state is reached after a train of downstream waves move away from the obstacle. Radiation of short waves is not observed [17].

We first consider two obstacles with same amplitudes $\epsilon_1 = \epsilon_2 = 0.01$. In this case, we observe depression solitary waves propagating downstream and the formation of two elevation waves over the obstacles. These elevation waves no longer reach the steady state because radiation of short waves is emitted from the right obstacle to the left one. Part of the radiation is reflected back remaining between the bumps and the other moves away from both obstacles. Since capillary effects are under consideration, short waves travel faster, therefore, the downstream solitary waves are constantly disturbed by the radiation. Figure 2 illustrates this flow motion.

Now, we increase the amplitude of the second obstacle by considering $\epsilon_2 = 0.03$. In this regime, the flow is governed by the larger obstacle as can be seen in Fig. 3. At early times, an elevation wave forms above both obstacles, as time goes on a wave train emitted from the larger obstacle swallows the wave over the smaller one destroying the expected steady elevation wave. Differently from the previous case, depression solitary waves propagate downstream without radiation being observed. Waves are trapped between the obstacles with part of it being radiated upstream.

Lastly, we choose $\epsilon_1 = 0.03$ and $\epsilon_2 = 0.01$. The dynamic is just a simple reflection from the previous case at early times. A elevation wave is generated over the two obstacles with depression solitary waves being emanated from both obstacles, being larger the ones that come from the larger obstacle. These waves pass the smaller obstacle.
which radiates short waves upstream. Once the radiation reaches the larger obstacle, it reflects part of it and the elevation wave is no longer steady. This situation is displayed in detail in Fig. 4. We see that radiation moves back and forth between the bumps with some portion moving away from the obstacles. Although the elevation wave over the smaller obstacle is not steady, it is not destroyed as in the previous case.

**Subcritical regime**

In the one-obstacle problem, the subcritical non-resonant flow is mainly described for the formation of a depression solitary wave over the obstacle and a wave train propagating upstream. Moreover, when $t \to \infty$, the solution of (1) reaches a steady state, which in this regime is a free-surface depression wave above the obstacle [17].

In order to study the two-obstacle problem, we initially fix $\epsilon_1 = \epsilon_2 = 0.01$. In this scenario, the dynamic of the free surface behaves as the following: at early times, an elevation wave is generated above both obstacles and wave...
trains are emanated upstream. Then the wave train generated from the right obstacle passes over the left one, gains kinetic energy and interacts with the other wave train. The depression waves over the obstacles remain intact. The steady state is obtained when the upstream train waves are shed. Figure 5 illustrates this dynamic. We observe that the short wave train generated from the right obstacles overtakes the one from the left one.

For $\epsilon_1 = 0.01$ and $\epsilon_2 = 0.03$, the flow is quite different as depicted in Fig. 6. Depression solitary waves are now emitted periodically downstream from the larger obstacle. Steady waves no longer occur. An undular bore forms between the bumps with a wave train ahead. This wave train interacts with the elevation wave formed above the smaller obstacle destroying its shape. As time elapses, an elevation wave rises again, but it is no longer a steady wave because an undular bore remains between the obstacles.

Now, we reverse the role of the obstacles by taking $\epsilon_1 = 0.03$ and $\epsilon_2 = 0.01$. As in the previous case the larger obstacle generates depression solitary waves periodically which propagates downstream. This generation radiates small short waves upstream. The depression solitary waves have enough energy to overcome the small obstacle. Although their shape is changed during the passage over the second obstacle, after the interaction it regains its form. This dynamic is displayed in Fig. 7. The initial steady elevation waves above the obstacles no longer persist.
3.2 Nearly-resonant regime

In this regime, we let \( f \) be closer to 0 and consider the same types of obstacles as before. As we will show the wave interaction is somehow more non-linear, so differently from the non-resonant regime it is difficult to capture a pattern in the waves generated.

**Supercritical regime**

We start choosing \( \epsilon_1 = 0.01 \) and \( \epsilon_2 = 0.01 \). In this case, we observe a formation of a hydraulic jump above the left obstacle. It propagates very slowly upstream led by a series of wave trains. In the region between the obstacles, at early times, wave trains propagate upstream. Part of the wave trains are reflected from the first obstacle and part of it radiates. So, radiation is observed between the bumps for large times. Above the right obstacle, we notice an unstable
Fig. 9  Supercritical nearly-resonant regime free-surface evolution with \( f = 0.1, \epsilon_1 = 0.01 \) and \( \epsilon_2 = 0.03 \)

elevation wave, which is shown in Fig. 8. Although the dynamic between the obstacles is very unpredictable, the formation of the hydraulic jump is very clear.

When the second obstacle is larger, \( \epsilon_1 = 0.01 \) and \( \epsilon_2 = 0.03 \), the flow structure is very distinct. A formation of a hydraulic jump is observed above in both the obstacles. The one over the small obstacle propagates upstream and downstream while the other upstream. As a consequence of that, wave trains accumulate between the bumps. Later, the accumulated wave trains are compressed and then the undular bore collapses. After a while, waves start moving upon the bore, this is depicted in Figure 9. Around \( t = 5800 \) the hydraulic jump starts deteriorating and waves run over it. Such behaviour is very intricate, and may be related to the choice \( b = 0 \), since it eliminates the most dispersive term of the fKdV equation (1). It indicates that a dispersive shock may occur in the full Euler equations model. It is important to notice that it is not clear which obstacle controls the flow for large times. If we look at Fig. 9 for \( t > 8200 \), we cannot point out which obstacle is larger, what is different from the supercritical non-resonant regime (see Fig. 3).

Now, let us consider the case where the first obstacle is larger, namely \( \epsilon_1 = 0.03 \) and \( \epsilon_2 = 0.01 \). This case is much clear than the previous one. We see the formation of an hydraulic jump above the larger obstacle and radiation between the obstacles which is expelled downstream. Figure 10 shows the free-surface evolution for \( f = 0.1 \). The flow is driven by the larger obstacle.

Subcritical regime

Our study starts considering obstacles with the same amplitude \( (\epsilon_1 = 0.01 \) and \( \epsilon_2 = 0.01 \)). Figure 11 shows the free-surface evolution for \( f = -0.1 \). At early times, the right obstacle emits depression solitary waves which travel downstream and the formation of an undular bore above the obstacle so does the left one. It is different from what happens in the non-resonant subcritical case, in which a steady state is reached (see Fig. 5). We point out that from the collision between the depression solitary waves generated by the left obstacle and the bore over the right bump, we have short waves being radiated upstream. Once the depression solitary waves reach the right obstacle, a series of wave collision happens and eventually they escape out.

For bumps with amplitudes \( \epsilon_1 = 0.01 \) and \( \epsilon_2 = 0.03 \), an interesting scenario arises as can be seen in Fig. 12. At early times, the left obstacle emanates depression solitary waves which travel downstream, then this mechanism ceases. Meanwhile, depression solitary waves are being emitted frenetically from the second obstacle producing a series of collisions. It is worth noting that as the time elapses the depression solitary waves generated by the left obstacle remain trapped bouncing between the bumps. When \( \epsilon_1 = 0.03 \) and \( \epsilon_2 = 0.01 \) both obstacles generate
depression solitary waves periodically with the larger one more rapidly. Since the flux of wave generation of the large obstacle is high, wave collisions highlight the dynamic. More details of this case are given in Fig. 13.

4 Conclusion

In this paper, we have investigated capillary–gravity flows over two obstacles. Through a pseudospectral numerical method, we showed that the flow is not necessarily governed by the larger obstacle as it is in the absence of surface tension. In the supercritical nearly-resonant case, the flow is mainly described by the formation of a hydraulic jump. In certain regimes, this structure generated by both obstacles collides and collapses. In addition, we find a regime that for large times, it is not clear which obstacle controls the flow. In the subcritical nearly-resonant case, the flow is mainly characterized by the generation of depression solitary waves periodically that propagate downstream.
Fig. 12 Subcritical resonant regime free-surface evolution with $f = -0.1, \epsilon_1 = 0.01$ and $\epsilon_2 = 0.03$

Fig. 13 Subcritical resonant regime free-surface evolution with $f = -0.1, \epsilon_1 = 0.03$ and $\epsilon_2 = 0.01$

Besides, we also found cases in which depression solitary waves remain trapped between the bumps bouncing back and forth.

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