\textbf{\textsc{\textit{K}}-HOMOGENEOUS TUPLE OF OPERATORS ON BOUNDED SYMMETRIC DOMAINS}

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\begin{abstract}
Let $\Omega$ be an irreducible bounded symmetric domain of rank $r$ in $\mathbb{C}^d$. Let $\mathbb{K}$ be the maximal compact subgroup of the identity component $G$ of the biholomorphic automorphism group of the domain $\Omega$. The group $\mathbb{K}$ consisting of linear transformations acts naturally on any $d$-tuple $T = (T_1, \ldots, T_d)$ of commuting bounded linear operators. If the orbit of this action modulo unitary equivalence is a singleton, then we say that $T$ is $\mathbb{K}$-homogeneous. In this paper, we obtain a model for all $\mathbb{K}$-homogeneous $d$-tuple $T$ as the operators of multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space of holomorphic functions defined on $\Omega$. Using this model we obtain a criterion for (i) boundedness, (ii) membership in the Cowen-Douglas class (iii) unitary equivalence and similarity of these $d$-tuples. In particular, we show that the adjoint of the $d$-tuple of multiplication by the coordinate functions on the weighted Bergman spaces are in the Cowen-Douglas class $B_1(\Omega)$. For a bounded symmetric domain $\Omega$ of rank 2, an explicit description of the operator $\sum_{i=1}^d T_i T_i^*$ is given. In general, based on this formula, we make a conjecture giving the form of this operator.
\end{abstract}

1. INTRODUCTION

It was noted in \cite{27} Corollary 2 that a weighted shift operator $T$ is circular in the sense that $T$ is unitarily equivalent to $cT$ whenever $|c| = 1$. This class of operators was studied further by several authors \cite{15, 4}. In \cite{27}, Chavan and Yakubovich generalized this notion to spherical tuple of operators. A $d$-tuple $T = (T_1, \ldots, T_d)$ of commuting operators is said to be spherical if $U \cdot T$ is unitarily equivalent to $T$ for all unitary matrix $U$ in the group $\mathcal{U}(d)$ of $d \times d$ unitary matrices. Here $U \cdot T$ is the natural action of $\mathcal{U}(d)$ on the $d$-tuple $T$. Chavan and Yakubovich proved that under some mild hypothesis, every spherical $d$-tuple is unitarily equivalent to the $d$-tuple $M = (M_1, \ldots, M_d)$ of multiplication operators by the coordinate function $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space determined by a $\mathcal{U}(d)$-invariant kernel function $\sum_{j=0}^{\infty} a_j \langle z, w \rangle^j$ defined on the open Euclidean unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. One of our main objectives in this paper is to explore a notion analogous to that of spherical operator tuples in the context of a bounded symmetric domain.

Bounded symmetric domains are the natural generalization of open unit disc in one complex variable and open Euclidean unit ball in several complex variables. A bounded domain $\Omega \subset \mathbb{C}^d$ is said to be \textit{symmetric} if for every $z \in \Omega$, there exists a biholomorphic automorphism on $\Omega$ of period two, having $z$ as an isolated fixed point. The domain $\Omega$ is said to be \textit{irreducible} if it is not biholomorphically isomorphic to a product of two non-trivial domains. We refer to \cite{20}, \cite{1} for the definition and basic properties of bounded symmetric domains.

Let $\Omega$ be an irreducible bounded symmetric domain in $\mathbb{C}^d$ and let $\text{Aut}(\Omega)$ denote the group of biholomorphic automorphisms of $\Omega$, equipped with the topology of uniform convergence on compact subsets of $\Omega$. Let $G$ denote the connected component of identity in $\text{Aut}(\Omega)$. It is known that $G$ acts transitively on $\Omega$. Let $\mathbb{K}$ be the subgroup of linear automorphisms in $G$. By Cartan’s theorem \cite{25} Proposition 2, pp. 67], $\mathbb{K} = \{ \phi \in G : \phi(0) = 0 \}$. $\mathbb{K}$ is known to be a maximal compact subgroup of $G$.

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and $\Omega$ is isomorphic to $G/K$. Note that $\mathcal{U}(d)$ is the subgroup of linear biholomorphic automorphisms of $\text{Aut}(\mathbb{B}^d)$. Therefore, it is natural to replace $\mathcal{U}(d)$ with the subgroup $K$ of linear biholomorphic automorphisms of an irreducible bounded symmetric domain $\Omega$ and study all operator $d$-tuples $T$ such that $k \cdot T$ is unitarily equivalent to $T$ for all $k \in K$. The action of the group $K$ on the $d$-tuples is defined below. The group $K$ acts on $\Omega$ by the rule

$$k \cdot z := (k_1(z), \ldots, k_d(z)), \quad k \in K \text{ and } z \in \Omega.$$ 

Note that $k_1(z), \ldots, k_d(z)$ are linear polynomials. Thus $k \in K$ acts on any commuting $d$-tuple of bounded linear operators $T = (T_1, \ldots, T_d)$, defined on complex separable Hilbert space $\mathcal{H}$, naturally, via the map

$$k \cdot T := (T_1, \ldots, T_d), \ldots, k_d(T_1, \ldots, T_d)).$$

**Definition 1.1.** A $d$-tuple of commuting bounded linear operators $T = (T_1, T_2, \ldots, T_d)$ on $\mathcal{H}$ is said to be $K$-homogeneous if for all $k \in K$ the operators $T$ and $k \cdot T$ are unitarily equivalent, that is, for all $k$ in $K$ there exists a unitary operator $\Gamma(k)$ on $\mathcal{H}$ such that

$$T_j \Gamma(k) = \Gamma(k)T_j(1, \ldots, T_d), \quad j = 1, 2, \ldots, d.$$ 

For brevity, we will write

$$TT' = \Gamma'(k)(k \cdot T).$$

We point out that the commuting operator tuples $T = (T_1, \ldots, T_d)$ such that $T$ and $g(T)$ are unitarily equivalent for all $g$ in $G$, called homogeneous tuples, have been also studied over the past few years, see \cite{22, 23, 18}. Cowen and Douglas introduced a class of operators $B_n(\Omega)$ in the very influential paper \cite{9}, where $\Omega \subset \mathbb{C}$ is a bounded domain and $n$ is a positive integer. In the case of open unit disc $\mathbb{D}$, all homogeneous operators in $B_1(\mathbb{D})$ were classified by Misra in \cite{21}. As a corollary of his abstract classification theorem, Wilkins provided an explicit model for all homogeneous operators in $B_2(\mathbb{D})$, see \cite{31}. Later in 2011, using techniques from complex geometry and representation theory, a complete classification of homogeneous operators in the Cowen-Douglas class $B_n(\mathbb{D})$ was obtained by Misra and Korányi in \cite{17}. Homogeneous operators on an irreducible bounded symmetric domain of type I, see below, were studied by Misra and Bagchi in \cite{6}. Later in \cite{2}, their results were generalized for an arbitrary irreducible bounded symmetric domain by Arazy and Zhang.

First, we fix some notations, which will be used throughout this paper. It will also help us describe our results. For a set $X$ and positive integer $m$, $X^m$ stands for the $m$-fold Cartesian product of $X$. The symbols $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ stand for the set of non-negative integers, field of the real numbers and the field of complex numbers, respectively. Let $\mathcal{K}$ be a complex Hilbert space. Let $\mathcal{B}(\mathcal{K})$ denote the unital Banach algebra of bounded linear operators on $\mathcal{K}$. If $T \in \mathcal{B}(\mathcal{K})$, then $\ker T$ denotes the kernel of $T$, the range of $T$ is denoted by $\text{ran} T$.

Every irreducible bounded symmetric domain $\Omega$ of rank $r$ can be realized as an open unit ball of a Cartan factor $Z = \mathbb{C}^d$. For a fixed frame $e_1, \ldots, e_r$ of pairwise orthogonal minimal tripotents, let

$$Z = \sum_{0 \leq i \leq j \leq r} Z_{ij}$$

be the joint Peirce decomposition of $Z$ (see \cite{29} pp. 57]). Note that $Z_{00} = \{0\}$ and $Z_{ii} = C e_i$ for all $i = 1, \ldots, r$. Moreover,

$$a := \text{dim } Z_{ij}, \quad 1 \leq i < j \leq r$$

is independent of $i, j$ and

$$b := \text{dim } Z_{0j}, \quad 1 \leq j \leq r$$

is independent of $j$. The parameters $a, b$ are known to be the characteristic multiplicities of $Z$ and the numerical invariants $(r, a, b)$ determine the domain $\Omega$ uniquely up to biholomorphic equivalence (see \cite{1}). The dimension $d$ is related to the numerical invariants $(r, a, b)$ as follows:

$$d = r + \frac{a}{2^r}(r - 1) + rb.$$
According to the classification due to E. Cartan [17], there are six types of irreducible bounded symmetric domains up to biholomorphic equivalence (see also [20]). The first four types of these domains are called the classical Cartan domains, while other two types are known as the exceptional domains. In what follows, we consider only the classical domains, that is, an irreducible bounded symmetric domain of one of the following four types:

(i) **Type I**: \( n \times m \ (m \geq n) \) complex matrices \( z \) with \( \|z\| < 1 \). These domains are determined by the triple \((n, 2, m - n)\).

(ii) **Type II**: symmetric complex matrices \( z \) of order \( n \) with \( \|z\| < 1 \). In this case, the triple \((n, 1, 0)\) is complete biholomorphic invariant.

(iii) **Type III**: \( n \times n \) anti-symmetric complex matrices \( z \) of order \( n \) with \( \|z\| < 1 \). Here \( r = \lfloor \frac{n}{2} \rfloor \), \( a = 4 \) and \( b = 0 \) if \( n \) is even and \( b = 2 \) if \( n \) is odd.

(iv) **Type IV** (The Lie ball): all \( z \in \mathbb{C}^d \ (d \geq 5) \) such that \( 1 + \frac{1}{2} |z'|^2 > \Re z \) and \( \Re z' < 2 \), where \( \Re \) is the complex conjugate of the transpose \( z' \). The triple \((2, d - 2, 0)\) is complete biholomorphic invariant for these domains.

Let \( \mathcal{P} \) be the space of all analytic polynomials on \( \Omega \). For all \( n \in \mathbb{N}_0 \), let \( \mathcal{P}_n \) denote the subspace of \( \mathcal{P} \) consisting of all homogeneous polynomials of degree \( n \). Clearly, as a vector space, \( \mathcal{P} \) can be written as the direct sum \( \sum_{n=0}^{\infty} \mathcal{P}_n \). The group \( K \) acts on the space \( \mathcal{P} \) by composition, that is, \((k.p)(z) = p(k^{-1}z)\). Below we describe the irreducible components of this action. An \( r \)-tuple \( s = (s_1, s_2, \ldots, s_r) \) is called a signature if \( s_1 \geq s_2 \geq \ldots \geq s_r \geq 0 \). Let \( \mathfrak{N}_0^{(r)} \) denote the set of all signatures. We associate the conical polynomial, see [29, pp. 128] for the definition, 

\[
\Delta_\mathfrak{s}(z) = \Delta_1^{s_1-s_2}(z) \ldots \Delta_{r-1}^{s_{r-1}-s_r}(z) \Delta_r^{s_r}(z)
\]

for all \( \mathfrak{s} \in \mathfrak{N}_0^{(r)} \) and the polynomial space \( \mathcal{P}_\mathfrak{s} = \vee \{ \Delta_\mathfrak{s} k ; k \in K \} \). It is known that the polynomial space \( \{ \mathcal{P}_\mathfrak{s} \} \mathfrak{s} \in \mathfrak{N}_0^{(r)} \) are precisely the \( K \)-invariant, irreducible subspaces of \( \mathcal{P} \) which are mutually \( K \)-inequivalent, and 

\[
\mathcal{P} = \sum_{\mathfrak{s} \in \mathfrak{N}_0^{(r)}} \mathcal{P}_\mathfrak{s}.
\]

The Fischer-Fock inner product on \( \mathcal{P} \), defined by \((p, q)_F := \frac{1}{2\pi} \int_{\mathbb{C}^d} p(z) \bar{q}(z) e^{-|z|^2} \, dm(z)\), is \( K \)-invariant. The reproducing kernel of the space \( \mathcal{P}_\mathfrak{s} \) with respect to the Fischer-Fock inner product is denoted by \( K_\mathfrak{s}(z, w) \). Note that 

\[
\sum_{\mathfrak{s} \in \mathfrak{N}_0^{(r)}} K_\mathfrak{s}(z, w) = e^{\mathfrak{s} \cdot \overline{w}}.
\]

Further any \( K \)-invariant Hilbert space \( \mathcal{H} \) of analytic functions on \( \Omega \) has the decomposition 

\[
\mathcal{H} = \bigoplus_{\mathfrak{s} \in \mathfrak{N}_0^{(r)}} \mathcal{P}_\mathfrak{s}.
\]

This decomposition is called Peter-Weyl decomposition [28].

Let \( T = (T_1, \ldots, T_d) \) be a commuting \( d \)-tuple of bounded linear operators acting on a complex separable Hilbert space \( \mathcal{H} \). Also, let \( D_T : \mathcal{H} \rightarrow \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) be the operator 

\[
D_T h := (T_1 h, \ldots, T_d h), \ h \in \mathcal{H}.
\]

We note that \( \ker D_T = \cap_{i=1}^d \ker T_i \) is the joint kernel and \( \sigma_p(T) = \{ w \in \mathbb{C}^d : ker D_T - wt \neq 0 \} \) is the joint point spectrum of the \( d \)-tuple \( T = (T_1, \ldots, T_d) \). Throughout this paper we will study a class of \( K \)-homogeneous \( d \)-tuples, which is defined below.

**Definition 1.2.** A commuting \( d \)-tuple of \( K \)-homogeneous operators \( T \) possessing the following properties

(i) \( \dim \ker D_{T^*} = 1 \)

(ii) any non-zero vector \( e \) in \( \ker D_{T^*} \) is cyclic for \( T \).

(iii) \( \Omega \subseteq \sigma_p(T^*) \)

is said to be in the class \( \mathcal{A}_K(\Omega) \).
In this paper, we provide a concrete model for all the commuting \(d\)-tuples \(T\) (which are necessarily of \(K\)-homogeneous) in the class \(A\mathcal{K}(\Omega)\) as multiplication by the coordinate functions \(z_1, \ldots, z_d\) on a reproducing kernel Hilbert space of holomorphic functions \((\mathcal{H}, K)\) defined on \(\Omega\). We describe the kernel \(K\) in terms of the invariant kernels \(K_s\) of the spaces \(P_s\).

Having described the model, we obtain a criterion for boundedness of these operators. Using this criterion, we determine which \(d\)-tuple of multiplication operators on the weighted Bergman spaces are bounded. The boundedness criterion for the multiplication operators on the weighted Bergman spaces has appeared before in [6], [2].

We also obtain criterion for the adjoint of the \(d\)-tuple of operators in \(A\mathcal{K}(\Omega)\) to be in the Cowen-Douglas class \(B_1(\Omega_0)\) for some neighbourhood \(\Omega_0 \subset \Omega\) of \(0 \in \Omega\). In case of weighted Bergman spaces \(\mathcal{H}^{(\nu)}\), we prove that the adjoint of the \(d\)-tuple of multiplication operators by the coordinate functions are in the Cowen-Douglas class \(B_1(\Omega)\).

For any \(T\) in the class \(A\mathcal{K}(\Omega)\), we point out that the operators \(\sum_{i=1}^{d} T_i^* T_i\) and \(\sum_{i=1}^{d} T_i T_i^*\) restricted to the subspace \(P_2\) are scalar times the identity. In particular, for the weighted Bergman spaces \(\mathcal{H}^{(\nu)}\), Proposition 4.4 provides an explicit form for the operator \(\sum_{i=1}^{d} T_i T_i^*\). We extend this formula for any \(T\) in the class \(A\mathcal{K}(\Omega)\). Moreover, for the Hardy space of the Shilov boundary \(S\) of \(\Omega\), we show that \(\sum_{i=1}^{d} M_i^* M_i\) is the rank times identity, see also [5]. Also, for any \(T\) in \(A\mathcal{K}(\Omega)\), we have computed the operator \(\sum_{i=1}^{d} T_i^* T_i\) on certain subspaces of \(\mathcal{H}\), and as a consequence, it is shown that the commutators \([M_i^* M_i, M_j]\), \(i = 1, \ldots, d\), on the weighted Bergman spaces are compact if and only if \(r = 1\). For any domain \(\Omega\) of rank 2, we obtained an explicit description of the operator \(\sum_{i=1}^{d} T_i^* T_i\) and conjectured the form of this operator for a domain of any rank \(r > 2\). This conjecture was proved by Upmeier, see [30].

Finally, we study the question of unitary equivalence and similarity of \(d\)-tuples of operators in the class \(A\mathcal{K}(\Omega)\).

2. Model for operators in \(A\mathcal{K}(\Omega)\)

We begin this section by providing a well known family of examples, namely, the \(d\)-tuple of multiplication by the coordinate functions on the weighted Bergman spaces, which belongs to the class \(A\mathcal{K}(\Omega)\).

For \(\nu \in \{0, \ldots, \frac{\alpha}{2}(r-1)\} \cup \left(\frac{\alpha}{2}(r-1), \infty\right)\), so called Wallach set of \(\Omega\) (see [14]), consider the weighted Bergman kernel

\[
K^{(\nu)}(z, w) = \sum_{s}^{(\nu)_s} K_s(z, w), \quad z, w \in \Omega,
\]

where \((\nu)_s\) is the generalized Pochhammer symbol

\[
(\nu)_s := \prod_{j=1}^{r} \left(\nu - \frac{\alpha}{2}(j-1)\right). 
\]

Let \(\mathcal{H}^{(\nu)}\) denote the weighted Bergman space of holomorphic functions on \(\Omega\) determined by the reproducing kernel \(K^{(\nu)}\). If \(\nu = \frac{d}{r}\) and \(\nu = \frac{\alpha}{2}(r-1) + \frac{d}{2}\), then the weighted Bergman spaces \(\mathcal{H}^{(\nu)}\) coincide with the Hardy space \(H^2(S)\) over the Shilov boundary \(S\) of \(\Omega\) and the classical Bergman space \(\mathcal{A}^2(\Omega)\) respectively. For \(\nu > \frac{\alpha}{2}(r-1)\), the multiplication \(d\)-tuple \(M^{(\nu)} = (M_1^{(\nu)}, \ldots, M_d^{(\nu)})\) on \(\mathcal{H}^{(\nu)}\) is bounded and homogeneous (cf. [6], [2]). One can also verify that \(M^{(\nu)}\) is in \(A\mathcal{K}(\Omega)\). Replacing \((\nu)_s\) by any arbitrary positive number \(\delta_s\) with some boundedness condition, we get a large class of operator tuples in \(A\mathcal{K}(\Omega)\) and we prove that they are all.

To facilitate the study of \(K\)-homogeneous operators, we recall the following result from [1] describing all the \(K\)-invariant kernels on \(\Omega\).

**Proposition 2.1** (Proposition 3.4, [1]). For any \(K\)-invariant semi-inner product \(\langle \cdot, \cdot \rangle\) on the the space of polynomials \(P\), we have the following.
(i) $\mathcal{P}_s$ is orthogonal to $\mathcal{P}_{s'}$ whenever $s \neq s'$.

(ii) There exists a constant $b_s \geq 0$ associated to each signature $s$ such that

$$\langle p, q \rangle = b_s \langle p, q \rangle_{\mathcal{H}} \quad \text{for all } p, q \in \mathcal{P}_s.$$ 

(iii) $b_s > 0$ for all $s$ if and only if $\langle \cdot, \cdot \rangle$ is an inner product.

(iv) If the evaluation at each point of $\Omega$ is continuous on $(\mathcal{P}, \langle \cdot, \cdot \rangle)$ then the completion $\mathcal{H}$ of

$$(\mathcal{P}, \langle \cdot, \cdot \rangle)$$

is a reproducing kernel Hilbert space. Moreover the kernel $K(z, w)$ is of the form

$$K(z, w) = \sum_s b_s^{-1} K_s(z, w),$$

where convergence is both pointwise and uniformly on compact subsets of $\Omega \times \Omega$ and in norm.

The following result is a generalization of [8, Lemma 2.10] which is necessary for the proof of Theorem 2.3 giving a model for commuting $d$-tuple of operators in the class $\mathcal{A}(\Omega)$.

**Lemma 2.2.** Let $T = (T_1, T_2, \ldots, T_d)$ be a $\mathbb{K}$-homogeneous $d$-tuple of commuting operators on $\mathcal{H}$. Suppose that $\ker D_T$ is one-dimensional and is spanned by a vector $e \in \mathcal{H}$ and that $e$ is cyclic for $T$. Then there exists a sequence of non-negative real numbers $a_\mathbb{K}, s \in \mathbb{N}_0$, such that for any polynomial $p \in \mathcal{P}$,

$$||p(T)e||^2 = \sum_{k=0}^{\deg p} \sum_{|\mathbb{K}| = k} a_{\mathbb{K}} ||p_{\mathbb{K}}||_{\mathcal{H}}^2$$

(2.1)

where $\deg p$ is the degree of $p$ and

$$p = \sum_{k=0}^{\deg p} \sum_{|\mathbb{K}| = k} p_{\mathbb{K}}$$

is the Peter-Weyl decomposition.

**Proof.** Since $T$ is $\mathbb{K}$-homogeneous, for each $k \in \mathbb{K}$ there exists a unitary operator $\Gamma(k)$ on $\mathcal{H}$ such that

$$T_j \Gamma(k) = \Gamma(k)(k \cdot T)_j, \quad j = 1, \ldots, d.$$ 

Hence $T_j \Gamma(k) = \Gamma(k)(k \cdot T)_j^*, \quad j = 1, \ldots, d.$ Since $(k \cdot T)_j$ is a linear combination of $T_1, \ldots, T_d$ and $e \in \ker D_T$, it follows that $\Gamma(k)e$ belongs to $\ker D_T$ for all $k \in \mathbb{K}$. Furthermore, since $\ker D_T$ is one dimensional and spanned by $e$, we see that $\Gamma(k)e = \eta(k)e$ for some $\eta(k)$ such that $|\eta(k)| = 1$. We now define a semidefinite sesquilinear product on $\mathcal{P}_s$ for all $s \in \mathbb{N}_0^d$ by the formula

$$\langle p, q \rangle_{\mathcal{P}_s} = \langle p(T)e, q(T)e \rangle_{\mathcal{H}}$$

Now for any $k \in \mathbb{K}$ we have

$$\langle p(k \cdot z), q(k \cdot z) \rangle_{\mathcal{P}_s} = \langle p(k \cdot T)e, q(k \cdot T)e \rangle_{\mathcal{H}}$$

$$= \langle \Gamma(k)^* p(T) \Gamma(k)e, \Gamma(k)^* q(T) \Gamma(k)e \rangle_{\mathcal{H}}$$

$$= \langle p(T) \Gamma(k)e, q(T) \Gamma(k)e \rangle_{\mathcal{H}}$$

$$= \langle p(T) \eta(k)e, q(T) \eta(k)e \rangle_{\mathcal{H}}$$

$$= |\eta(k)|^2 \langle p(T)e, q(T)e \rangle_{\mathcal{H}}$$

$$= \langle p(T)e, q(T)e \rangle_{\mathcal{H}}$$

$$= \langle p, q \rangle_{\mathcal{P}_s}.$$ 

So $(\cdot, \cdot)_{\mathcal{P}_s}$ is an $\mathbb{K}$-invariant semi-inner product on each $\mathcal{P}_s$. Since $\{\mathcal{P}_s\}_{s \in \mathbb{N}_0^d}$ are mutually orthogonal spaces, it follows that $\sum_s (\cdot, \cdot)_{\mathcal{P}_s}$ defines a $\mathbb{K}$-invariant semi-inner product $(\cdot, \cdot)$ on $\oplus \mathcal{P}_s$. Thus by Proposition 2.1 there exists a non-negative constant $a_{\mathbb{K}}$ such that

$$\langle p, q \rangle_{\mathcal{P}_s} = a_{\mathbb{K}} \langle p, q \rangle_{\mathcal{H}}.$$ 

This completes the proof of lemma. □
For all the classical domains, it can be easily verified that $\Omega = \{ w \in \mathbb{C}^d : \overline{w} \in \Omega \}$. Consequently, in the following theorem, the Hilbert space of holomorphic functions that we construct live on $\Omega$ rather than $\{ w \in \mathbb{C}^d : \overline{w} \in \Omega \}$. The next result provides an analytic model for any $d$-tuple of operators $T$ in $A K(\Omega)$.

**Theorem 2.3.** If $T$ is a $d$-tuple of operators in $A K(\Omega)$, then $T$ is unitarily equivalent to a $d$-tuple $M = (M_1, \ldots, M_d)$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space $H_K$ of holomorphic functions defined on $\Omega$ with $K(z, w) = \sum a_2^{-1} K_2(z, w)$ for all $z, w \in \Omega$, for some choice of positive real numbers $a_2$ with $a_2 = 1$.

**Proof.** Since $\Omega \subseteq \sigma_p(T^*)$, for each $w \in \Omega$ there exists a non-zero vector $x \in H$, such that $T^*_j x = \hat{w}_j x$ for all $j = 1, 2, \ldots, d$. Thus for any polynomial $p \in P$, we have $p(T^*) x = p(\overline{w}) x$. If $e \in \ker D_T$, is a cyclic vector for $T$, then

$$p(w)(e, x)_\overline{\mathbb{C}} = \langle e, \overline{p(w)} x \rangle_\overline{\mathbb{C}} = \langle e, p(T^*) x \rangle_\overline{\mathbb{C}} = \langle p(T)e, x \rangle_\overline{\mathbb{C}}.$$ 

Since $e$ is a cyclic vector for $T$ of norm 1 and $x \neq 0$, we have that $\langle e, x \rangle_\overline{\mathbb{C}} = 0$. Thus

$$|p(w)| = \frac{|p(T)e||x||e|_\overline{\mathbb{C}}}{|\langle e, x \rangle_\overline{\mathbb{C}}|}.$$ 

It follows that evaluation at $w \in \Omega$ is bounded and therefore the semi-inner product defined by the rule $(p, q)_{\overline{\mathbb{C}}}_\mathbb{P} = \langle p(T)e, q(T)e \rangle_\overline{\mathbb{C}}$ is an inner product on each $\mathbb{P}_w$. This gives rise to an inner product $(\cdot, \cdot)$ on the space of polynomials $\mathbb{P}$. The sequence $a_2$ of Lemma 2.3, using Proposition 2.2(c), is now evidently positive. Moreover, since $\|e\| = 1$, it follows from (2.1) that $a_2 = 1$. Thus the completion of $(\mathbb{P}, (\cdot, \cdot))$, say $H_K$, is a reproducing kernel Hilbert space, where

$$K(z, w) = \sum a_2^{-1} K_2(z, w), \quad z, w \in \Omega.$$ 

Clearly, the map $p \to p(T)e$ extends to a unitary from $H_K$ to $H$, which intertwines $T$ with the multiplication $d$-tuple $M = (M_1, \ldots, M_d)$ on $H_K$. \hfill $\square$

**Proposition 2.4.** If $T$ is a $d$-tuple of operators in $A K(\Omega)$, then there exists a unitary representation $\Gamma : K \to \mathbb{U}(\mathbb{H})$ such that

$$T \Gamma(k) = \Gamma(k) (k \cdot T).$$

**Proof.** By Theorem 2.3 $T$ is unitarily equivalent to the $d$-tuple $M = (M_1, \ldots, M_d)$ of multiplication operators on a reproducing kernel Hilbert space $H_K$ of holomorphic functions defined on $\Omega$ with a kernel $K(z, w)$ which is $K$-invariant. Clearly, the map $\Gamma$ on $H_K$ given by $\Gamma(k)(f) = f \circ k^{-1}(\cdot)$ is a unitary representation of $K$ satisfying the intertwining condition. \hfill $\square$

**Remark 2.5.** Since $K$ is a subgroup of the group $\mathbb{U}(d)$ of unitary linear transformations on $\mathbb{C}^d$, every spherical $d$-tuple $T = (T_1, \ldots, T_d)$ is $K$-homogeneous. Conversely, a $K$-homogeneous $d$-tuple of theorems 2.2 is spherical if and only if $a_2 = a_2'$ for all signatures $s, s'$ with $|s| = |s'|$.

**Remark 2.6.** We also point out that, by the spectral mapping theorem, the Taylor joint spectrum $\sigma(T)$ of a $K$-homogeneous operator $T$ is $K$-invariant, that is, if $w$ belongs to $\sigma(T)$, then $k.w$ also belongs to $\sigma(T)$ for all $k \in K$.

3. **Boundedness of the multiplication tuple**

Throughout the rest of the paper, let $K^{(a)} : \Omega \times \Omega \to \mathbb{C}$ denote the kernel function given by the formula $K^{(a)}(z, w) = \sum a_2 K_2(z, w)$, $z, w \in \Omega$, for some choice of positive real numbers $a_2$. The positivity of the sequence $a_2$ ensures that $K^{(a)}$ is a positive definite kernel. Thus it determines a unique Hilbert space $\mathcal{H}^{(a)} \subseteq \text{Hol}(\Omega)$ with the reproducing property: $(f, K^{(a)}(\cdot, w)) = f(w)$, $f \in \mathcal{H}^{(a)}$, $w \in \Omega$. It follows from Proposition 2.1 that the polynomial ring $\mathbb{P}$ is dense in $\mathcal{H}^{(a)}$ and $\mathbb{P}$ is orthogonal to $\mathbb{P}'$ whenever $a \neq a'$, that is, $\mathcal{H}^{(a)} = \bigoplus_{a \in N} \mathbb{P}$. In this section, we discuss the boundedness of the $d$-tuple $M^{(a)} := (M^{(a)}_1, \ldots, M^{(a)}_d)$ of multiplication by the coordinate functions
Lemma 3.2. The operators \( \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)} \) and \( \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)} \), acting on the Hilbert space \( \mathcal{H}^{(a)} \), are block diagonal with respect to the decomposition \( \bigoplus_{z \in \mathbb{N}^d} \mathcal{P}_z \), where each block is a non-negative scalar multiple of the identity operator.

**Proof.** It is enough to give the proof for the operator \( \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)} \) since the proof for the operator \( \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)} \) follows exactly in the same way. First, note that \( \Gamma(k)^* M_i^{(a)} \Gamma(k) = M_i^{(a)} \) for \( k \in \mathbb{K} \). Let \( e_1, \ldots, e_d \) be the standard basis vectors in \( \mathbb{C}^d \). Note that \( M_i^{(a)} \mathcal{P}_{z_{i,ok}} = \sum_{j=1}^{d} \langle k^{-1} e_j, e_i \rangle M_j^{(a)} \).

In consequence, we have

\[
\Gamma(k)^* \left( \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)} \right) \Gamma(k) = \sum_{i=1}^{d} \Gamma(k)^* M_i^{(a)} \Gamma(k) \Gamma(k)^* M_i^{(a)} \Gamma(k) = \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)} \]

\[
= \sum_{i=1}^{d} M_i^{(a)*} \mathcal{P}_{z_{i,ok}} M_i^{(a)} = \sum_{i=1}^{d} \sum_{p,q=1}^{d} \langle e_i, k^{-1} e_p \rangle \langle k^{-1} e_q, e_i \rangle M_p^{(a)*} M_q^{(a)} = \sum_{i=1}^{d} \langle k^{-1} e_i, k^{-1} e_p \rangle M_p^{(a)*} M_q^{(a)} = \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)}.
\]

Here the last equality follows from the fact that the subgroup \( \mathbb{K} \) is contained in the group \( \mathfrak{U}(d) \) of unitary linear transformations on \( \mathbb{C}^d \). Since \( \mathcal{H}^{(a)} = \bigoplus_{z \in \mathbb{N}^d} \mathcal{P}_z \) and these subspaces \( \{ \mathcal{P}_z \} \) are all the \( \mathbb{K} \)-irreducible subspaces of \( \mathcal{H}^{(a)} \) and they are mutually \( \mathbb{K} \)-inequivalent, the conclusion follows from the Schur’s lemma. \( \square \)

The following lemma generalizes a known result \cite{2} Proposition 4.4] for the multiplication \( d \)-tuple on the weighted Bergman spaces.

**Lemma 3.2.** For \( f \in \mathcal{P}_z \), we have \( \sum_{i=1}^{d} M_i^{(a)*} M_i^{(a)} f = \tau_\mathfrak{g} f \), where

\[
\tau_\mathfrak{g} = \begin{cases} 
\sum_{j=1}^{r} \frac{a_j - e_j}{a_j} \left( \frac{a_j}{(2)_{k-j}} \right)^{-\frac{k-j}{2}} \prod_{k \neq j} \frac{s_j - s_k + \frac{q}{2}(k-j-1)}{s_j - s_k + \frac{q}{2}(k-j)} & \text{if } \mathfrak{g} \neq 0, \\
0 & \text{if } \mathfrak{g} = 0.
\end{cases}
\]

The proof of the preceding lemma is very similar to the proof of \cite{2} Proposition 4.4] and therefore it is omitted.

For a signature \( \mathfrak{g} \), in the remaining portion of this paper, we set

\[
c_\mathfrak{g}(j) = \prod_{k \neq j} \frac{s_j - s_k + \frac{q}{2}(k-j)}{s_j - s_k + \frac{q}{2}(k-j-1)} \quad \text{and} \quad c_\mathfrak{g}'(j) = \prod_{k \neq j} \frac{s_j - s_k + \frac{q}{2}(k-j-1)}{s_j - s_k + \frac{q}{2}(k-j-1)}
\]

If \( \mathfrak{g} + \varepsilon_j \) is also a signature, then it is easy to see that \( c_\mathfrak{g}(j) > 0 \). Otherwise, \( c_\mathfrak{g}(j) = 0 \). Similarly, if \( \mathfrak{g} - \varepsilon_j \) is a signature, then \( c_\mathfrak{g}'(j) > 0 \). Otherwise, \( c_\mathfrak{g}'(j) = 0 \).
Lemma 3.3. For any fixed but arbitrary signature \( s \), we have
\[
\sum_{j=1}^{r} c'_s(j) = \sum_{j=1}^{r} c_s(j) = r.
\]

Proof. Evidently, we have
\[
\sum_{j=1}^{r} c'_s(j) = \sum_{j=1}^{r} \prod_{k \neq j} \frac{s_j - s_k + \frac{a}{2}(k - j - 1)}{s_j - s_k + \frac{a}{2}(k - j)} = \sum_{j=1}^{r} \prod_{k \neq j} \left( 1 - \frac{\frac{a}{2}}{s_j - s_k + \frac{a}{2}(k - j)} \right) = \sum_{j=1}^{r} \prod_{k \neq j} \left( 1 - \frac{\frac{a}{2}}{(s_j - \frac{a}{2}k) - (s_k - \frac{a}{2}k)} \right).
\]

Setting \( s'_j = \frac{s_j - \frac{a}{2}j}{\frac{a}{2}} \), we see that \( s'_1 > s'_2 > \cdots > s'_r \), and
\[
\sum_{j=1}^{r} c'_s(j) = \sum_{j=1}^{r} \prod_{k \neq j} \left( 1 - \frac{1}{s'_j - s'_k} \right) = r + \sum_{j=1}^{r} \sum_{A \subseteq \{1, \ldots, j-1, j+1, \ldots, r\}} (-1)^{|A|} \prod_{k \in A} \frac{1}{s'_j - s'_k}.
\]

Now, by Corollary 2.3, it follows that \( \sum_{j \in A} \prod_{k \in A \neq j} \frac{1}{s_j - s_k} = 0 \) for all \( A \subseteq \{1, \ldots, r\} \) with \( |A| \geq 2 \).

Therefore, \( \sum_{j=1}^{r} c'_s(j) = r \). The proof of the other part follows exactly in the same way. \( \square \)

Theorem 3.4. The \( d \)-tuple \( M^{(a)} = (M_1^{(a)}, \ldots, M_d^{(a)}) \) of multiplication operators on the Hilbert space \( \mathcal{H}^{(a)} \) is bounded if and only if
\[
A := \sup \left\{ \frac{a_{\underline{x}} - \varepsilon_j}{a_{\underline{x}} (\frac{d}{2})_{\underline{x}} - \varepsilon_j} : a_{\underline{x}} - \varepsilon_j \in \mathbb{N}_{0}^r, \ j = 1, \ldots, r \right\}
\]
is finite.

Proof. Clearly, the multiplication \( d \)-tuple \( M^{(a)} \) on the Hilbert space \( \mathcal{H}^{(a)} \) is bounded if and only if \( \sum_{i=1}^{d} M_i^{(a)} \) is bounded. Therefore, using Lemma 3.2, it is enough to show that \( \tau(\underline{s}) \) is bounded for all \( \underline{s} \in \mathbb{N}_{0}^r \) if and only if \( A \) is finite. First assume that \( A \) is finite. Then
\[
\tau(\underline{s}) = \sum_{j=1}^{r} \frac{a_{\underline{x}} - \varepsilon_j}{a_{\underline{x}} (\frac{d}{2})_{\underline{x}} - \varepsilon_j} \frac{a}{2}(r - j) + s_j \frac{\alpha}{2}(r - j) + s_j c'_s(j) \leq A \sum_{j=1}^{r} \frac{\alpha}{2}(r - j) + s_j c'_s(j) \leq A \sum_{j=1}^{r} c'_s(j) = Ar,
\]
for any signature \( \underline{s} \). Here, the last equality follows from Lemma 3.3. To prove the other direction, assume that \( \tau(\underline{s}) \) is bounded, that is, \( \tau(\underline{s}) \leq B \) for some positive real number \( B \) and for all \( \underline{s} \in \mathbb{N}_{0}^r \).
The multiplication Corollary 3.5. □
This completes the proof.

Thus
\[
\frac{a_{d-\varepsilon_j}}{a_s} \frac{(\frac{d}{r})_{d-\varepsilon_j}}{(\frac{d}{r})_{d-\varepsilon_j}} \cdot \frac{a_s}{b + \frac{r}{2}(r - j) + s_j} \leq \tau(s) \leq B.
\]

Now, note that if \( s - \varepsilon_j \in \overrightarrow{\mathbb{N}}_0 \), then
\[
\frac{1}{c_j'(j)} = \prod_{k \neq j} \frac{s_j - s_k + \frac{r}{2}(k - j)}{s_j - s_k + \frac{r}{2}(k - j - 1)}
= \prod_{k < j} \frac{s_j - s_k + \frac{r}{2}(k - j)}{s_j - s_k + \frac{r}{2}(k - j - 1)} \prod_{k > j} \frac{s_j - s_k + \frac{r}{2}(k - j)}{s_j - s_k + \frac{r}{2}(k - j - 1)}
\leq \prod_{k > j} \frac{s_j - s_k + \frac{r}{2}(k - j)}{s_j - s_k}
\leq \prod_{k > j} (1 + \frac{\frac{r}{2}(k - j)}{s_j - s_k})
\leq (1 + \frac{\frac{r}{2}(r - 1)}{s_j - s_k})^r.
\]

Here the third inequality holds since \( \frac{s_j - s_k + \frac{r}{2}(k - j)}{s_j - s_k + \frac{r}{2}(k - j - 1)} \leq 1 \) for \( k < j \). Finally, see that
\[
\frac{a_{d-\varepsilon_j}}{a_s} \frac{(\frac{d}{r})_{d-\varepsilon_j}}{(\frac{d}{r})_{d-\varepsilon_j}} \leq \frac{B}{b + \frac{r}{2}(r - j) + s_j} \leq B(1 + \frac{a}{2}(r - 1))^r(1 + b).
\]
This completes the proof. □

Corollary 3.5. The multiplication \( d \)-tuple \( M^{(\nu)} \) on the Hilbert space \( \mathcal{H}^{(\nu)} \) is bounded if \( \nu > \frac{r}{2}(r - 1) \).

Proof. If \( \nu > \frac{r}{2}(r - 1) \), then
\[
\frac{\nu}{\nu} \frac{(\frac{d}{r})_{d-\varepsilon_j}}{(\frac{d}{r})_{d-\varepsilon_j}} = \frac{\nu}{\nu} \frac{\frac{d}{r} - \frac{r}{2}(j - 1) + s_j - 1}{\nu - \frac{r}{2}(j - 1) + s_j - 1} \leq \max \left\{ 1, \frac{1 + b}{\nu - \frac{r}{2}(r - 1)} \right\}.
\]
Therefore, from Theorem 3.4, it follows that \( M^{(\nu)} \) is bounded. □

Having determined (a) the condition for boundedness of the operator \( M^{(a)} \), (b) noting that each \( w \) in \( \Omega \) is a joint eigenvector for the multiplication \( d \)-tuple \( M^{(a)} \) and finally since the constant vector 1 is cyclic for \( M^{(a)} \), it is natural to investigate the question of which of these are in the Cowen-Douglas class \( B_1(\Omega) \), see [9], [10] for the definition of this very important class of operators. As shown in [12, pp. 285], the cyclicity implies that the dimension of the joint eigenspace at each \( w \) in \( \Omega \) is 1. Thus to determine the membership in the Cowen-Douglas class in a neighbourhood of origin contained in \( \Omega \), we only need to find when \( \text{ran}D_{M^{(a)}} \) is closed. The following theorem provides the precise condition for this.

Theorem 3.6. For a multiplication \( d \)-tuple \( M^{(a)} = (M^{(a)}_1, \ldots, M^{(a)}_d) \) on \( \mathcal{H}^{(a)} \), \( \text{ran}D_{M^{(a)}} \) is closed if and only if
\[
B := \inf \left\{ \sum_{j=1}^{r} \frac{a_{d-\varepsilon_j}}{a_s} \frac{(\frac{d}{r})_{d-\varepsilon_j}}{(\frac{d}{r})_{d-\varepsilon_j}} : \begin{array}{l}
\varepsilon_j, \overrightarrow{\mathbb{N}}_0, j = 1, \ldots, r \n\end{array}\right\}
\]
is non-zero positive.
Hence, noting that positive number, say $C$.

**Proof.** It is elementary to see that $\text{ran}D_{M^{(a)}}$ is closed if and only if $\sum_{i=1}^{d} M_i^{(a)} M_i^{(a)*}$ is bounded below on $\ker D_{M^{(a)}}$. Also, for the $d$-tuple $M^{(a)}$ on $\mathcal{H}^{(a)}$, we have $\ker D_{M^{(a)}} = \mathcal{P}_0$, the space of constant functions. Therefore, in view of Lemma 3.2, it suffices to show that $B$ is non-zero positive if and only if $\inf \{ \tau(\mathcal{S}) : \mathcal{S} \neq 0 \}$ is non-zero positive. Suppose that $B$ is a non-zero positive number. Now, for any signature $\mathcal{S} \neq 0$, we have

$$
\tau(\mathcal{S}) = \sum_{j=1}^{r} \frac{a_{j-\epsilon_j}}{a_{j}} \left( \frac{d}{r} \right)_{j-\epsilon_j} \frac{a_{j}}{b + \frac{a_{j}}{2}(r-j) + s_j} c_j'(j) \\
\geq \sum_{j=1}^{r} \frac{a_{j-\epsilon_j}}{a_{j}} \left( \frac{d}{r} \right)_{j-\epsilon_j} \frac{b + \frac{a_{j}}{2}(r-j) + s_j}{1} c_j'(j) \\
\geq \sum_{j=1}^{r} \frac{a_{j-\epsilon_j}}{a_{j}} \left( \frac{d}{r} \right)_{j-\epsilon_j} \frac{1}{(b+1)(1+\frac{a_{j}}{2}(r-j))^{r}}
$$

Here the third inequality follows from (3.2). Conversely, assume that $\inf \{ \tau(\mathcal{S}) : \mathcal{S} \neq 0 \}$ is a non-zero positive number, say $C$. Thus for each signature $\mathcal{S} \neq 0$

$$
\sum_{j=1}^{r} \frac{a_{j-\epsilon_j}}{a_{j}} \left( \frac{d}{r} \right)_{j-\epsilon_j} \frac{b + \frac{a_{j}}{2}(r-j) + s_j}{1} c_j'(j) \geq C.
$$

Hence, noting that $c_j'(j) \leq r$ by Lemma 3.3 and $\frac{2(r-j)+s_j}{b+\frac{a_{j}}{2}(r-j)} \leq 1$, it follows that

$$
\sum_{j=1}^{r} \frac{a_{j-\epsilon_j}}{a_{j}} \left( \frac{d}{r} \right)_{j-\epsilon_j} \geq \frac{C}{r}.
$$

**Corollary 3.7.** The range of $D_{M^{(a)}}$ is closed if $\nu > \frac{a}{2}(r-1)$.

**Proof.** Suppose $\nu = \frac{a}{2}(r-1) + \varepsilon$ for some $\varepsilon > 0$. Then

$$
\sum_{j=1}^{r} \frac{(\nu)_{j-\epsilon_j}}{(\nu)_{j}} \left( \frac{d}{r} \right)_{j-\epsilon_j} \frac{b + \frac{a_{j}}{2}(r-j) + s_j}{1} c_j'(j) \geq \frac{C}{r}
$$

which is always bounded below by 1 if $\varepsilon \leq b + 1$. On the other hand, for $\varepsilon \geq b + 1$, it is bounded below by $\frac{1}{\varepsilon}$. Hence, by Theorem 3.3, $\text{ran}D_{M^{(a)}}$ is closed.

Now, we wish to show that the adjoint $M^{(a)*}$ of the $d$-tuple of multiplication operators on $\mathcal{H}^{(\nu)}$ is in the Cowen-Douglas class $B_1(\Omega)$ for $\nu > \frac{a}{2}(r-1)$.

Recall that the left essential spectrum $\pi^{(\nu)}_e(T)$ of a commuting $d$-tuple of operators $T$ is defined to be the complement of the set of all $w \in \mathbb{C}^d$ with the property:

1. $\dim \ker D_{(T-wI)}$ is finite,
2. $\text{ran}D_{(T-wI)}$ is closed.

If $0 \notin \pi^{(\nu)}_e(T)$, then the $d$-tuple $T$ is said to be left semi-Fredholm.

The essential ingredient of the proof of the following theorem is based on the spectral mapping property of the left essential spectrum, which appears in [13] and was pointed out by J. Eschmeier to G. Misra during a conversation at University of Saarbrucken in February 2014.

**Theorem 3.8.** The adjoint $M^{(\nu)*}$ of the multiplication $d$-tuple on $\mathcal{H}^{(\nu)}$ is in the Cowen-Douglas class $B_1(\Omega)$ whenever $\nu > \frac{a}{2}(r-1)$. 
Proof. Since polynomials are dense in the Hilbert space $\mathcal{H}(\nu)$, it follows that $\dim \ker D_{M^{(\nu)^*}}$ is 1. By Corollary 3.7, we also have that $\operatorname{ran} D_{M^{(\nu)^*}}$ is closed. Therefore, $D_{M^{(\nu)^*}}$ is left semi-Fredholm and hence there is a $\varepsilon > 0$ such that for $w \in \Omega$ with $\|w\|_2 < \varepsilon$, the operators $D_{(M^j - wI)^*}$ are left Fedholm. Thus $M^{(\nu)^*}_j$ is in the Cowen-Douglas class $B_1(\Omega_\varepsilon)$, where $\Omega_\varepsilon = \{w \in \Omega : \|w\|_2 < \varepsilon\}$. Now, using the homogeneity of $M^{(\nu)}$ and the spectral mapping property of the left essential spectrum, we show that $M^{(\nu)^*}$ is actually in $B_1(\Omega)$.

To complete the proof, first note that if $w \in \Omega$ is any fixed but arbitrary point, then there exists a bi-holomorphic automorphism $\varphi$ of $\Omega$ with the property: $\varphi(0) = w$. We have seen that $0 \notin \pi_e(\nu(M^{(\nu)^*}))$. An analytic spectral mapping property for the left essential spectrum is ensured by [13 Corollary 2.6.9]. It follows that

$$w = \varphi(0) \notin \varphi(\pi_e(\nu(M^{(\nu)^*}))) = \pi_e(\varphi(M^{(\nu)^*})) = \pi_e(\nu(M^{(\nu)^*})).$$

Here the last equality follows from the homogeneity assumption. \qed

4. Computation of the operator $\sum M_i^* M_i$ on $\mathcal{H}(\alpha)$

In this section, we wish to compute the operator $M^{(\nu)^*} a M^{(\nu)} := \sum_{i=1}^d M_i^{(\nu)^*} M_i^{(\nu)}$ on the Hilbert space $\mathcal{H}(\alpha)$. First, we note that the bounded symmetric domain $\Omega$ sits inside a linear space of dimension $d$ in its Harish-Chandra realization. The type I domains are realized as the open unit ball, with respect to the operator norm, in the linear space of $n \times m$ matrices. The situation becomes somewhat different when we consider domains of type II. Pick one of these domains of dimension $\frac{n(n+1)}{2}$. It is convenient to put $\frac{n(n+1)}{2}$ variables in the form of a symmetric matrix, where the inner product is given by $\operatorname{tr}(AB^*)$. Now, in the space of these symmetric matrices of size $n$, the matrices $E_{ii}^\nu, i = 1, \ldots, n$ together with $E_{ij} + E_{ji}$ for $1 \leq i \neq j \leq n$ form an orthonormal basis. Consequently, the coordinates of this domain is of the form $z_{11}, \sqrt{2} z_{12}, \ldots, \sqrt{2} z_{1n}, z_{22} \ldots \sqrt{2} z_{2n}, \ldots, z_{n-1, n-1}, \sqrt{2} z_{n,n}, z_{nn}$, see [16 pp. 130]. One may pick coordinates similarly for the type III domains consisting of the $n \times n$ antisymmetric matrices of norm at most 1. Finally, the type IV domains, in its Harish-Chandra realization are described in [24 pp. 76]:

$$\left\{ z := (z_1, \ldots, z_d) : \frac{d}{i=1} |z_i|^2 < 2 \text{ and } \sum_{i=1}^d |z_i|^2 < \frac{1}{2} \sum_{i=1}^d z_i^2 \right\}.$$

The following theorem appears in [5] in a slightly different form. The difference arises since we take the multiplication by the coordinate functions to be the ones described in the previous paragraph, while in the paper [5], these are the usual coordinates. Thus it makes no difference in the case of the type I domains, while for the other domains, the answer is different.

Theorem 4.1. Let $M^{(S)} = (M_1^{(S)}, \ldots, M_d^{(S)})$ be the $d$-tuple of multiplication by the coordinate functions $z_1, \ldots, z_d$ on the Hardy space $H^2(S)$, where $S$ is the Shilov boundary of $\Omega$. Then

$$\sum_{i=1}^d M_i^{(S)^*} M_i^{(S)} = r I. \quad (4.4)$$

By Lemma 3.1, note that $M^{(\nu)^*} a M^{(\nu)}$ is a block diagonal operator with respect to the decomposition $\oplus \mathbb{P}_d$, where each block is a non-negative scalar multiple of the identity, that is, $M_i^{(\nu)^*} a M^{(\nu)} p = \delta(s)p$, $p \in \mathbb{P}_d$ for some positive real number $\delta(s)$. Therefore, we need to compute the constant $\delta(s)$ for all $s \in \mathbb{N}_0$. Unfortunately, we are only able to find $\delta(s)$ when $s$ is a signature such that $s + \varepsilon j$ is a signature for at most two $j$, where $1 \leq j \leq r$. In particular, we have the complete answer in case the rank $r = 2$. 

\[\int\]
Proposition 4.2. For any polynomial \( p \in \mathcal{P}_s \),
\[
M_i^{(a)} p = \sum_{j=1}^{r} \frac{a_{s+\varepsilon_j}}{a_s} (\partial_j p)_{s+\varepsilon_j}.
\]

Proof. By theorem [29] 4.11.86, we see that \( z_i \mathcal{P}_s \) is contained in \( \mathcal{P}_{s+\varepsilon_j} \). Thus, for any polynomial \( p \in \mathcal{P}_s \), it follows that \( M_i^* p \) belongs to \( \mathcal{P}_{s+\varepsilon_j} \). Now for \( q \in \mathcal{P}_{s+\varepsilon_j} \), we have
\[
\langle M_i^* p, q \rangle_{3(\alpha)} = \langle p, (z_i q)_{3(\alpha)} \rangle_{3(\alpha)} = \frac{1}{a_s} \langle p, (z_i q) \rangle_{3(\alpha)} = \frac{1}{a_s} \langle p, z_i q \rangle_{s+\varepsilon_j} = \frac{1}{a_s} \langle \partial_j p, q \rangle_{s+\varepsilon_j} = \frac{1}{a_s} \langle \partial_j p - \varepsilon_j, q \rangle_{s+\varepsilon_j}.
\]
Therefore the conclusion follows. \( \square \)

Theorem 4.3. (i) Let \( \bar{s} \) be a signature such that \( \bar{s} + \varepsilon_j \) is a signature if and only if \( j = 1 \). Then \( M^{(a)} M^{(a)} p = \delta(\bar{s}) p, p \in \mathcal{P}_s \), where
\[
\delta(\bar{s}) = \frac{a_s}{a_{s+\varepsilon_1}} r \left( \frac{d}{r} + s_1 \right).
\]

(ii) Let \( \bar{s} \) be a signature such that \( \bar{s} + \varepsilon_j \) is a signature if and only if \( j = 1, k \), where \( 2 \leq k \leq r \). Then \( M^{(a)} M^{(a)} p = \delta(\bar{s}) p, p \in \mathcal{P}_s \), where
\[
\delta(\bar{s}) = \frac{a_s}{a_{s+\varepsilon_1}} \frac{(k-1) \left( \frac{d}{r} + s_1 \right) (s_1 - s_k + \frac{a_r}{r})}{(s_1 - s_k + \frac{a_r}{r}(k-1))} + \frac{a_s}{a_{s+\varepsilon_k}} \frac{(r-k+1) \left( \frac{d}{r} - \frac{a_r}{r}(k-1) + s_k \right) (s_1 - s_k)}{(s_1 - s_k + \frac{a_r}{r}(k-1))}.
\]

Proof. First note that, for \( p \in \mathcal{P}_s \), we have
\[
\sum_{i=1}^{d} M_i^{(a)} M_i^{(a)} p = \sum_{i=1}^{d} M_i^{(a)} \left( \sum_{j=1}^{r} (z_i p)_{s+\varepsilon_j} \right) = \sum_{i=1}^{d} \left( \sum_{j=1}^{r} \frac{a_{s+\varepsilon_j}}{a_s} \partial_j \left( (z_i p)_{s+\varepsilon_j} \right) \right) = \sum_{j=1}^{r} \frac{a_s}{a_{s+\varepsilon_j}} \sum_{i=1}^{d} \left( \partial_j \left( (z_i p)_{s+\varepsilon_j} \right) \right),
\]
where the third equality follows from Theorem 4.2. Let \( Q_j \) be the linear map on the space of polynomials given by
\[
Q_j(p) = \sum_{i=1}^{d} \left( \partial_j \left( (z_i p)_{s+\varepsilon_j} \right) \right), p \in \mathcal{P}_s.
\]
Then clearly,
\[
\delta(\bar{s}) p = \sum_{j=1}^{r} \frac{a_s}{a_{s+\varepsilon_j}} Q_j(p).
\]
Note that, for $p \in \mathcal{P}_d$, $Q_j$ satisfies the following:

$$
\sum_{j=1}^{r} Q_j(p) = \sum_{i=1}^{d} \sum_{j=1}^{r} \left( \partial_i((z_i p)_{s+\varepsilon_j}) \right) = \sum_{i=1}^{d} \left( \sum_{j=1}^{r} (z_i p)_{s+\varepsilon_j} \right) = \sum_{i=1}^{d} \left( \partial_i(z_i p) \right) + dp + \sum_{i=1}^{d} (z_i \partial_i p).
$$

Therefore, by Euler’s formula

$$
\sum_{j=1}^{r} Q_j(p) = (d + |s|)p. \quad (4.6)
$$

If $s$ is a signature such that $s + \varepsilon_j$ is a signature if and only if $j = 1$, then it follows easily from (4.5) and (4.6) that $\delta(s) = \frac{a_s}{a_{s+1}} r (\frac{d}{2} + s_1)$, proving the first part of the theorem.

To prove the second part, note that by Theorem 4.4 we have

$$
\sum_{j=1}^{r} \left( \frac{d}{2} \right) \frac{Q_j(p)}{s+\varepsilon_j} = rp. \quad (4.7)
$$

If $s$ is a signature such that $s + \varepsilon_j$ is a signature if and only if $j = 1, k$, where $2 \leq k \leq r$, then in the summation in (4.5) and (4.6) only two terms survive, namely, $Q_1(p)$ and $Q_k(p)$. By solving these two equations, it is easily verified that

$$
Q_1(p) = \frac{(k-1)(\frac{d}{2} + s_1)(s_1 - s_k + \frac{a_k}{2})}{(s_1 - s_k + \frac{a_k}{2})(k-1)} p,
$$

and

$$
Q_k(p) = \frac{(r - k + 1)(\frac{d}{2} - \frac{a_k}{2}(k - 1) + s_k)(s_1 - s_k)}{(s_1 - s_k + \frac{a_k}{2})(k-1)} p.
$$

This completes the proof. \qed

**Corollary 4.4.** Let $\nu > \frac{a}{2}(r - 1)$ and $M^{(\nu)} = (M_1^{(\nu)}, \ldots, M_d^{(\nu)})$ be the $d$-tuple of multiplication operators on $\mathcal{H}^{(\nu)}$. Then the operator $M_i^{(\nu)}$ is essentially normal, that is, the commutator $M_i^{(\nu)} M_i^{(\nu)^*} - M_i^{(\nu)^*} M_i^{(\nu)}$ is compact for all $i = 1, \ldots, d$ if and only if $r = 1$.

**Proof.** If $r = 1$, then by a direct computation it is easily verified that each $M_i^{(\nu)}$ is essentially normal. For the converse part, first set $p$ to be the signature $(l, 0, \ldots, 0)$, where $l$ is a positive integer. Then, by Lemma 3.2 and Theorem 4.3 we see that, $\sum_{i=1}^{d} (M_i^{(\nu)*} M_i^{(\nu)} - M_i^{(\nu)*} M_i^{(\nu)}) p = \eta(p)$, $p \in \mathcal{P}_d$, where

$$
\eta(p) = \frac{(l + \frac{a_l}{2})(l + \frac{a_l}{2})}{(\nu + l)(l + \frac{a_l}{2})} - \frac{l}{\nu + l - 1}. \quad (4.8)
$$

Suppose that each $M_i^{(\nu)}$ is essentially normal. Then the operator $\sum_{i=1}^{d} (M_i^{(\nu)*} M_i^{(\nu)} - M_i^{(\nu)*} M_i^{(\nu)})$ is compact. Hence $\eta(p)$ must converge to 0 as $l \to \infty$. Thus, from (4.8), we obtain that $\frac{(r-1)(\frac{d}{2} - \frac{a_k}{2})}{\nu - \frac{a_k}{2}} = 0$. Finally, since $\frac{d}{2} = (r-1)\frac{a}{2} + b + 1$, we get that $r = 1$. \qed

**Conjecture 4.5.** For $p \in \mathcal{P}_d$, $\sum_{i=1}^{d} M_i^* M_i p = \delta(s)p$ on the Hilbert space $\mathcal{H}^{(\nu)}$, where

$$
\delta(s) = \sum_{j=1}^{r} \frac{a_{s_j}}{a_{s+\varepsilon_j}} \frac{\left(\frac{d}{2}\right)_{s+\varepsilon_j}}{\left(\frac{d}{2}\right)_s} \prod_{k \neq j} \frac{s_j - s_k + \frac{a_k}{2}(k - j + 1)}{s_j - s_k + \frac{a_k}{2}(k - j)}. \quad (4.9)
$$
5. UNITARY EQUIVALENCe AND SIMILARITY

In this section, we study the question of unitary equivalence and similarity of two commuting \(d\)-tuple of operators in the class \(\mathcal{KK}(\Omega)\). In the particular case when \(K\) is the unit circle \(T\), these results were obtained by Shields in \([27]\). The higher-dimensional counterpart of similarity result is obtained in \([19]\) Lemma 2.2.

By Theorem 4.3 any \(d\)-tuple of operator \(T\) in \(\mathcal{KK}(\Omega)\) is unitarily equivalent to \(M^{(a)}\) consisting of multiplication by the coordinate functions \(z_1, \ldots, z_d\) on the reproducing kernel \(\mathcal{H}^{(a)}\) with the reproducing kernel \(K^{(a)}(z, w) = \sum_{\underline{a}} a_{\underline{a}} K_{\underline{a}}(z, w)\), where \(a_{\underline{a}} > 0\) with \(a_{\underline{0}} = 1\). Thus we assume, without loss of generality, that \(T \sim_u M^{(a)}\).

**Theorem 5.1.** Let \(T_1\) and \(T_2\) be two \(K\)-homogeneous operator tuples in \(\mathcal{KK}(\Omega)\). Suppose that \(T_1 \sim_u M^{(a)}\) and \(T_2 \sim_u M^{(b)}\). Then the following statements are equivalent.

(i) \(T_1\) and \(T_2\) are unitarily equivalent.
(ii) \(\alpha \sim B\) for all \(\alpha \in N_0^d\).
(iii) \(K^{(a)} = K^{(b)}\).

**Proof.** It is easy to see that (ii) and (iii) are equivalent. It is obvious that (iii) implies (i). Therefore it remains to verify that (i) implies (iii). Assume that the \(d\)-tuples \(T_1\) and \(T_2\) are unitarily equivalent. Then so are the operators \(M^{(a)}\) and \(M^{(b)}\). By \([11]\) Theorem 3.7, there exists a holomorphic function \(g\) on \(\Omega\) such that

\[K^{(a)}(z, w) = g(z)K^{(b)}(z, w)g(w), \quad z, w \in \Omega.\]

In particular, \(K^{(a)}(z, 0) = g(z)K^{(b)}(z, 0)g(0), \ z \in \Omega\). Therefore, \(a_{\underline{a}} = b_{\underline{a}}g(z)g(0)\), and consequently, \(g(z)g(0) = 1\) since \(a_{\underline{a}} = b_{\underline{a}} = 1\). Hence \(K^{(a)} = K^{(b)}\). \(\square\)

Recall that two commuting \(d\)-tuples \(A = (A_1, \ldots, A_d)\) and \(B = (B_1, \ldots, B_d)\), defined on \(\mathcal{H}_1\) and \(\mathcal{H}_2\) respectively, are said to be similar if there exists an invertible operator \(X : \mathcal{H}_1 \rightarrow \mathcal{H}_2\) such that \(XA_i = B_iX\) for all \(i = 1, \ldots, d\). For a non-negative integer \(n\), as before, \(\mathcal{P}_n\) denote the space of homogeneous polynomials in \(d\) variables of degree \(n\).

**Theorem 5.2.** Let \(\Omega \subseteq \mathbb{C}^d\) be a bounded domain, and let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be two reproducing kernel Hilbert spaces determined by two positive definite kernels \(K_1\) and \(K_2\) respectively. Suppose that

(i) the space of polynomials \(\mathcal{P}\) is dense in both \(\mathcal{H}_1\) and \(\mathcal{H}_2\),
(ii) \(\mathcal{P}_n\) is orthogonal to \(\mathcal{P}_m\) if \(m \neq n\),
(iii) for each \(i = 1, 2\), the \(d\)-tuple \(M^{(i)} = (M_1^{(i)}, \ldots, M_d^{(i)})\) of multiplication operators by the coordinate functions \(z_1, \ldots, z_d\) on \(\mathcal{H}_i\) is bounded.

Then the following statements are equivalent.

(i) \(M^{(1)}\) and \(M^{(2)}\) are similar.
(ii) There exist constants \(\alpha, \beta > 0\) such that

\[\alpha \|p\|_{\mathcal{H}_1} \leq \|p\|_{\mathcal{H}_2} \leq \beta \|p\|_{\mathcal{H}_1}, \quad p \in \mathcal{P}.\]  

(iii) \(\mathcal{H}_1 = \mathcal{H}_2\).
(iv) There exists constants \(\alpha, \beta > 0\) such that

\[\alpha K_1 \leq K_2 \leq \beta K_1.\]

**Proof.** The equivalence of (iii) and (iv) follows from the standard theory of reproducing kernel Hilbert spaces (cf. \([3], [26]\)). Also it is clear that (ii) implies (iii), since the polynomials are dense in both \(\mathcal{H}_1\) and \(\mathcal{H}_2\). If \(\mathcal{H}_1 = \mathcal{H}_2\), then the identity operator from \(\mathcal{H}_1\) to \(\mathcal{H}_2\) is a bounded invertible operator which intertwines the multiplication \(d\)-tuples \(M^{(1)}\) and \(M^{(2)}\), and consequently, (iii) implies (i).

Now, to complete the proof, it remains to show that (i) implies (ii).

Suppose that \(M^{(1)}\) and \(M^{(2)}\) are similar. Then there exists an invertible operator \(X : \mathcal{H}_1 \rightarrow \mathcal{H}_2\) such that

\[XM_j^{(1)} = M_j^{(2)}X, \quad j = 1, \ldots, d.\]
Since the subspaces \( P_n, n \geq 0 \), are mutually orthogonal, it suffices to show that (5.10) is satisfied for all \( p \in P_n \) and for some \( \alpha, \beta > 0 \) (which is independent of \( n \)). Fix a polynomial \( p \) in \( P_n \). Clearly, it follows from (5.11) that

\[
X M_p^{(1)} = M_p^{(2)} X.
\]

(5.12)

Let \((X_{r,s})_{r,s=0}^{\infty}\) be the matrix representation of \( X \) with respect to \( \oplus P_n \), that is, \( X_{r,s} = P_r X_{p_s} \).

Similarly, let \( M_p^{(i)} = ((M_p^{(i)})_{r,s})_{r,s=0}^{\infty} \) be the matrix representation of \( M_p^{(i)}, i=1,2 \). Since \( M_p^{(i)} \) maps \( P_s \) into \( P_{s+n}, i=1,2 \), it clear that

\[
(M_p^{(i)})_{r,s} = \begin{cases} 
(M_p^{(i)})_{p_s}, & \text{if } r = s + n \\
0, & \text{otherwise.}
\end{cases}
\]

Therefore it follows from (5.12) that

\[
X_{r,s+n}(M_p^{(i)})_{s+n,s} = \begin{cases} 
(M_p^{(2)})_{r,r-n} X_{r-n,s}, & \text{if } r - n \geq 0 \\
0, & \text{otherwise.}
\end{cases}
\]

(5.14)

Choosing \( r = n \) and \( s = 0 \), we see that

\[
(M_p^{(2)})_{n,0} X_{0,0} = X_{n,n}(M_p^{(1)})_{n,0}.
\]

(5.15)

Therefore

\[
(M_p^{(1)})_{n,0}^* X_n^* n,n (M_p^{(1)})_{n,0} = X_{0,0}^* (M_p^{(2)})_{n,0}^* (M_p^{(2)})_{n,0} X_{0,0}.
\]

(5.16)

Since \( \|X_{n,n}\| \leq \|X\| \), we have

\[
X_n^* n,n \leq \|X\|^2 I.
\]

Hence from (5.16) we obtain

\[
X_{0,0}^* (M_p^{(2)})_{n,0}^* (M_p^{(2)})_{n,0} X_{0,0} \leq \|X\|^2 (M_p^{(1)})_{n,0}^* (M_p^{(1)})_{n,0}.
\]

(5.17)

Note that \( X_{0,0} \) is a linear map from \( P_0 \) to \( P_0 \) and \( \dim P_0 = 1 \). Hence \( X_{0,0} = \eta 1 \) for some \( \eta \in \mathbb{C} \). Also, taking \( p \) to be the polynomial \( z_j, 1 \leq j \leq d, \) and \( r = 0 \) in (5.14) we see that

\[
X_{0,s+1} (M^{(1)}_{s+1})_{s+1,s} = 0, \quad \text{for all } s \geq 0.
\]

(5.18)

Since this is true for all \( j = 1, \ldots, d \), it follows that \( X_{0,s+1} = 0 \) for all \( s \geq 0 \). Moreover, since \( X \) is invertible we must have \( X_{0,0} \neq 0 \). Otherwise, if \( X_{0,s} = 0 \) for all \( s \), then it is easy to see that \( P_0 \) is orthogonal to range of \( X \), which is a contradiction. Hence \( X_{0,0} \neq 0 \), and consequently \( \eta \neq 0 \). Therefore (5.17) gives

\[
\langle (M_p^{(2)})_{n,0} X_{0,0} 1, (M_p^{(2)})_{n,0} X_{0,0} 1 \rangle \leq \|X\|^2 \langle (M_p^{(1)})_{n,0} 1, (M_p^{(1)})_{n,0} 1 \rangle.
\]

Consequently,

\[
|\eta|^2 \|p\|_{\ell_2}^2 \leq \|X\|^2 \|p\|_{\ell_1}^2.
\]

To finish the proof, note that (5.11) implies

\[
X^{-1} M_p^{(2)} = M_p^{(1)} X^{-1}, \quad j = 1, \ldots, d.
\]

(5.18)

Hence following the arguments used in the first part of this proof we obtain that

\[
|\zeta|^2 \|p\|_{\ell_2}^2 \leq \|X^{-1}\|^2 \|p\|_{\ell_1}^2.
\]

where \( (X^{-1})_{0,0} = \zeta, 1, \zeta 
eq 0 \). This completes the proof. \( \square \)

**Remark 5.3.** In the proof given above, we have shown that \( X_{0,s} = 0 \) for all \( s > 0 \). But using (5.14), it can be easily verified that \( X_{r,s} = 0 \) for all \( s > r \), that is, \( X \) is lower triangular with respect to the decomposition \( \oplus P_n \). Consequently, \( \zeta = \frac{1}{\eta} \).

**Theorem 5.4.** Let \( T_1 \) and \( T_2 \) be two operator tuples in \( \mathcal{A}(\Omega) \). Suppose that \( T_1 \sim_u M^{(a)} \) and \( T_2 \sim_u M^{(b)} \). Then the following statements are equivalent.

(i) \( T_1 \) and \( T_2 \) are similar.
(ii) There exist constants $\alpha, \beta > 0$ such that
\[ \alpha \| p \|_{\mathcal{H}(a)} \leq \| p \|_{\mathcal{H}(b)} \leq \beta \| p \|_{\mathcal{H}(a)}, \quad p \in \mathcal{P}. \]  
(5.19)

(iii) $\mathcal{H}(a) = \mathcal{H}(b)$.

(iv) There exist constants $\alpha, \beta > 0$ such that
\[ \alpha K(a) \preceq K(b) \preceq \beta K(a). \]

(v) there exist constants $\alpha, \beta > 0$ such that
\[ \alpha a_s \leq b_s \leq \beta a_s, \quad s \in \mathbb{N}_0. \]

Proof. The equivalence of (i), (ii), (iii) and (iv) follows easily from Theorem 5.2. Assume that (ii) holds. Then (v) is easily verified by choosing any polynomial $p$ in $\mathcal{P}_s$ and using $\| p \|_{\mathcal{H}(a)}^2 = \frac{\| p \|_s^2}{a_s}$ and $\| p \|_{\mathcal{H}(b)}^2 = \frac{\| p \|_s^2}{b_s}$ in (5.10). Also, it is trivial to see that (v) implies (iv). \qed

Corollary 5.5. Let $\nu_1, \nu_2 > \frac{2}{r}(r-1)$. Then the $d$-tuple of multiplication operators $M^{(\nu_1)}$ on $\mathcal{H}(\nu_1)$ and $M^{(\nu_2)}$ on $\mathcal{H}(\nu_2)$ are similar if and only if $\nu_1 = \nu_2$.

Proof. Suppose that $M^{(\nu_1)}$ and $M^{(\nu_2)}$ are similar. Then, by Theorem 5.4, there exist constants $\alpha, \beta > 0$ such that $\alpha \nu_1  \preceq \nu_2 \preceq \beta \nu_1$ for all $s \in \mathbb{N}_0$. Take $s = (t,0,\ldots,0), \ t \in \mathbb{N}_0$. By the properties of the Gamma function we have $\frac{\nu_1}{\nu_2} = \frac{(\nu_1)_{t}}{(\nu_2)_t} \sim t^{\nu_1-\nu_2}$. Hence $\nu_1 = \nu_2$. The other implication is trivial. \qed

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