Optical implementation of one-dimensional quantum random walks using orbital angular momentum of a single photon

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Abstract. Photons can carry orbital angular momentum (OAM), which offers a practical realization of a high-dimensional quantum information carrier. In this paper, by employing OAM of a single photon, we propose an experimental scheme to implement one-dimensional two-state quantum random walks on an infinite line. Furthermore, we show that the scheme can be used to implement one-dimensional three-state quantum random walks on an infinite line.

The existence of quantum algorithms for specific problems shows that a quantum computer can in principle provide a tremendous speedup compared to classical computers [1, 2]. However finding new quantum algorithms is a difficult task. Since classical random walks play an essential role in many areas of science [3], it is interesting to consider the quantum version of classical random walks in order to explore whether they might extend the set of quantum algorithms. Indeed, a quantum algorithm based on quantum random walks with exponential speedup has been reported [4]. At present, there are two different types of quantum random walks, namely discrete-time and continuous-time quantum walks. The discrete-time quantum walks were first introduced by Aharonov et al [5] and required a quantum coin as additional degree of freedom in order to allow for discrete time unitary evolution. In [6], Farhi and Gutmann introduced continuous-time quantum random walks, which can be thought of as a quantum generalization of the Markov
Quantum random walks have been a topic of research in the context of quantum information and computation [7]. Recently, possible implementations of one-dimensional two-state quantum random walks have been suggested by considering different physical systems, such as ions in linear traps [8], neutral atoms in an optical lattice [9], cavity QED [5, 10], single photons with linear optical elements [11, 12] and classical optics [13]. Recently, a one-dimensional three-state quantum random walk has been studied, which shows different properties from the two-state quantum random walk [14]. But to the best of our knowledge no scheme has been proposed for implementation of a one-dimensional three-state quantum random walk.

There has been considerable interest in optical implementation of quantum information processing due to the photon’s intrinsic robustness against decoherence and relative ease of manipulation with high precision [15]. Photons can carry both polarization and orbital angular momentum (OAM), both of which offer practical realization of quantum digits. Most of the optical experiments demonstrating quantum information processing are based on polarization of photons [15]–[21]. Recently, increasing experimental efforts have been devoted to using OAM of single photon to encode high-dimensional quantum information, since it allows realization of new types of quantum communication protocols [22]–[26]. In [27], Zeilinger and co-workers demonstrated that an individual photon can be prepared in superpositions of the OAM. In [28, 29], these authors also reported generation of the two-photon OAM entanglement and observed the violation of a generalized Bell inequality of three-dimensional quantum systems. In [30], an interferometric sorter was reported for measuring the OAM of a single photon. In [31], Langford et al performed the first full characterization of an entangled qutrit encoded in OAM of single photons, and outlined a scheme for such systems to implement a quantum communication protocol [26]. Further, a scheme was proposed for realizing an OAM beam splitter which can be used to implement high-dimensional quantum key distribution [32]. In [33], an experiment was reported for generation of hyperentangled photon pairs by combining OAM, time energy and polarization degrees of freedom of photons. These works open up a new possibility for generation and manipulation of higher-dimensional quantum states and implementation of higher-dimensional quantum information processing protocols by using OAM as a quantum information carrier. The aim of the present paper is to propose experimental schemes for the optical implementation of one-dimensional quantum random two- and three-state walks by employing OAM of a single photon.

We briefly review one-dimensional two-state quantum random walks on an infinite line for a single particle which can hop between discrete sites labelled by \(i\). The corresponding Hilbert space spanned by the positions of the particle is given by \(H_p = \{|i\rangle, i \in \mathbb{Z}\}\). Besides the position degree of freedom, the particle has an additional degree of freedom, namely the coin states which span the two-dimensional Hilbert space \(H_c = \{|U\rangle, |D\rangle\}\). Each step of the one-dimensional quantum walk is given by two subsequent operations: (i) the coin state first undergoes the Hadamard transformation \(H\):

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Following this transformation, (ii) we apply a shift operation \(S_1\), which moves the particle left or right depending on the coin states:

\[
S_1|U\rangle|i\rangle = |U\rangle|i + 1\rangle, \quad S_1|D\rangle|i\rangle = |D\rangle|i - 1\rangle.
\]
Thus, transformation for one step of the particle from an arbitrary position state $|i\rangle$ is given by

\[ |U\rangle|i\rangle \rightarrow \frac{1}{\sqrt{2}}(|U\rangle|i+1\rangle + |D\rangle|i-1\rangle), \]
\[ |D\rangle|i\rangle \rightarrow \frac{1}{\sqrt{2}}(|U\rangle|i+1\rangle - |D\rangle|i-1\rangle). \]  

(3)

By iterating the above steps, we could implement a one-dimensional two-state random walk on an infinite line with unlimited steps in principle.

To generalize the two-state quantum walk to the three-state case, the coin states span the three-dimensional Hilbert space $H_c = \{|U\rangle, |D\rangle, |O\rangle\}$. There exists a wide variety of unitary transformations on the coin states that could be used as a generalization of the Hadamard transformation for the one-dimensional three-state quantum walk. Here we assume that the coin states are transformed at each time step by the following transformation

\[ U_3 = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \]

(4)

which is called Grover’s diffusion operator and has been used in three-state quantum walk studies [14, 34]. In contrast to a one-dimensional Hadamard walk with two states, the quantum particle is trapped with high probability near the initial position if the particle in the three-state walk based on equation (4) exists at only one site initially. A similar localization was first studied in the two-dimensional Grover walk [34, 35].

We define a shift operation $S_2$, which, depending on the coin states, moves the particle as follows

\[ S_2|U\rangle|m\rangle = |U\rangle|m+1\rangle, \quad S_2|D\rangle|m\rangle = |D\rangle|m-1\rangle, \quad S_2|O\rangle|m\rangle = |O\rangle|m\rangle. \]

(5)

By combining equations (4) and (5), the transformation for one step of particle from the arbitrary position state $|m\rangle$ is given by

\[ |m\rangle|U\rangle \rightarrow \frac{1}{3}(-|m+1\rangle|U\rangle + 2|m\rangle|O\rangle + 2|m-1\rangle|D\rangle), \]
\[ |m\rangle|O\rangle \rightarrow \frac{1}{3}(2|m+1\rangle|U\rangle - |m\rangle|O\rangle + 2|m-1\rangle|D\rangle), \]
\[ |m\rangle|D\rangle \rightarrow \frac{1}{3}(2|m+1\rangle|U\rangle + 2|m\rangle|O\rangle - |m-1\rangle|D\rangle). \]

(6)

By iterating the above steps, we could implement a one-dimensional three-state random walk on an infinite line with unlimited steps in principle.

In figure 1, we propose a simple experimental setup for implementing one step of the one-dimensional two-state quantum random walk, which consists of one symmetric beam splitter, holograms with charge $m = \pm 1$. In the scheme, the two coin states correspond to the two different spatial paths of a single photon pulse and the positions of the particle are represented by the OAM states. As shown in figure 1, the logic state $|U\rangle$ in equation (2) is the presence of a photon in optical mode $u$, and logic state $|D\rangle$ is the presence of single photon in photon mode $d$. The Hadamard transformation (1) acting on the coin states can be implemented by mixing the optical modes $u$ and $d$ with the symmetric beam splitter. The action of the hologram with the charge $m$ transforms OAM state of a single photon $|l\rangle$ into state $|l+m\rangle$ [28]-[30], [36], where $|l\rangle$ is the
Figure 1. (a) The schematic shows the implementation of one step of a one-dimensional two-state quantum walk. BS is a symmetric beam splitter. Holo$_i$ denotes the hologram with charge $i$ ($i = \pm 1$). (b) Optical network with $N$ steps, where $G$ is the experimental setup shown in (a).

The single photon state of the LG$_{0l}$ mode, which can be written as

$$|l\rangle = \int dr \text{LG}_0^l(r)a^\dagger(r)|0\rangle,$$

where $r$ is the radial coordinate in the transverse $X-Y$ plane and the normalized Laguerre–Gaussian modes in polar coordinates are given by

$$\text{LG}_p^l(\rho, \varphi) = \frac{2p!}{\pi([l] + p)!} \frac{1}{w} \left( \frac{\sqrt{2}\rho}{w} \right)^{|l|} L_p^{|l|} \left( \frac{2\rho^2}{w^2} \right) e^{-\rho^2/w^2} e^{-il\varphi},$$

where the $Z$-dependent phase was omitted and $L_p^{|l|}$ denotes the associated Laguerre polynomial. The index $l$ is referred to as the winding number and $p$ is the number of nonaxial radial nodes. In this paper, we consider only the case $p = 0$. The customary Gaussian mode can be viewed as a LG mode with $l = 0$. LG beams with an index $l$ carry an OAM of $l\hbar$ per photon.

A detailed analysis of the proposed scheme shown in figure 1 is described as follows. We assume that the light beam, before entering input ports $u$ and $d$, is in a state of the form

$$|\Psi\rangle_{\text{in}} = \sum_l (U_l|l\rangle|U\rangle + D_l|l\rangle|D\rangle),$$

where $|l\rangle_u|U\rangle$ and $|l\rangle_d|D\rangle$ denote OAM states $|l\rangle$ in spatial modes $u$ and $d$, respectively. As shown in figure 1, we emit modes $u$ and $d$ into two input ports of the symmetric beam splitter BS, whose action can be described as follows

$$|U\rangle \rightarrow \frac{1}{\sqrt{2}}(|U\rangle + |D\rangle), \quad |D\rangle \rightarrow \frac{1}{\sqrt{2}}(|U\rangle - |D\rangle),$$

where $|U\rangle$ and $|D\rangle$ denote the input states for the symmetric beam splitter.
which exactly implement the Hadamard transformation (1) acting on the coin states. After passing the beam splitter BS, the state of the system becomes

$$|\Psi_1\rangle = \sum_l \frac{1}{\sqrt{2}} [(U_l + D_l)|l\rangle_u|U\rangle + (U_l - D_l)|l\rangle_d|D\rangle].$$

(11)

In order to implement conditional operation (2), we send the modes $u$ and $d$ to pass through the hologram with charge $m = 1$ and $m = -1$, respectively. Thus the OAM in spatial mode $u$ ($d$) is increased (decreased) by an amount $1$

$$|\rangle_u|U\rangle \rightarrow |\rangle_u|U\rangle, \quad |\rangle_d|D\rangle \rightarrow |\rangle_d|D\rangle,$$

(12)

which is exactly equivalent to the transformation (2). The state of the system is evolved into

$$|\Psi_2\rangle = \sum_l \frac{1}{\sqrt{2}} [(U_l + D_l)|l+\rangle_u|U\rangle + (U_l - D_l)|l-\rangle_d|D\rangle].$$

(13)

If we choose $U_l = \delta_{l,i}$ or $D_l = \delta_{l,i}$, equation (13), i.e. exactly equivalent to equation (3), i.e. the joint actions of symmetric beam splitter and holograms on a single photon exactly correspond to one step of a one-dimensional quantum random walk. By iterating the above steps, we could implement a random walk on an infinite line with unlimited steps in principle. The optical network shown in figure 1(b) shows the optical implementation of the one-dimensional quantum random walk for arbitrary $N$ steps.

A simple experimental setup for implementing one step of the one-dimensional three-state quantum random walk is shown in figure 2(a), which consists of one six-port device $D_3$ and holograms with charge $m = \pm 1$. In the scheme, the coin state $|M\rangle$ in equation (5) is the presence of a photon in optical mode $m$ ($m = u, d, o$). The transformation $U_3$ acting on the coin states can be implemented by mixing the optical modes $u$, $d$ and $o$ with the six-port device. In [37], Reck et al. provide an algorithm to use a triangular array of standard beam splitters, phase shifters and mirrors to realize any unitary matrix. The optical implementation of the six-port device is given in figure 2(b).

In the following, a detailed analysis of the proposed scheme shown in figure 2(a) is given. We assume that the light beam, before entering input ports $u$, $d$ and $o$ is in a state of the form

$$|\Psi\rangle_{in} = \sum_m (U_m|m\rangle_u|U\rangle + D_m|m\rangle_d|D\rangle + O_m|m\rangle_o|O\rangle),$$

(14)

where $|\rangle_u|U\rangle$ denote OAM states $|l\rangle$ in spatial modes $u$; the others are similar. As shown in figure 2(a), let us emit three modes into input ports of the six-port device $D_3$, whose action can be described as follows

$$|U\rangle \rightarrow \frac{1}{3} (-|U\rangle + 2|O\rangle + 2|D\rangle), \quad |O\rangle \rightarrow \frac{1}{3} (2|U\rangle - |O\rangle + 2|D\rangle),$$

$$|D\rangle \rightarrow \frac{1}{3} (2|U\rangle + 2|O\rangle - |D\rangle),$$

(15)
Figure 2. (a) The schematic shows the implementation of one step of a one-dimensional three-state quantum walk. $D_3$ is a six-port device shown in (b). $\text{Holo}_i$ denotes the hologram with charge $i$ ($i = \pm 1$). (b) The optical implementation of a six-port device $D_3$. The $\text{BS}_1$ and $\text{BS}_3$ beam splitters have a reflectivity $R = 1/2$; for the other $R = 8/9$.

which exactly implement the transformation (4) acting on the coin states. After passing the six-port device $D_3$, the state of the system becomes

$$|\Psi_i\rangle = \frac{1}{3} \sum_m \left[ (-U_m + 2O_m + 2D_m) |m\rangle_u |U\rangle + (2U_{m,n} - O_m + 2D_m) |m\rangle_o |O\rangle + (2U_m - 2O_m + D_m) |m\rangle_d |D\rangle \right]. \quad (16)$$

In order to implement conditional operation (5), we send the modes $u$ and $d$ to pass through the hologram with charge $m = 1$ and $m = -1$, respectively. Thus the OAM in spatial mode $u$ ($d$) is increased (decreased) by an amount 1, and the state (16) of the system is evolved into

$$|\Psi_i\rangle = \frac{1}{3} \sum_m \left[ (-U_m + 2O_m + 2D_m) |m+1\rangle_u |U\rangle + (2U_m - O_m + 2D_m) |m\rangle_o |O\rangle + (2U_m + 2O_m - D_m) |m-1\rangle_d |D\rangle \right]. \quad (17)$$

If we choose $U_l = \delta_{l,i}$ or $O_l = \delta_{l,i}$ or $D_l = \delta_{l,i}$, equation (17) i.e. exactly equivalent to equation (7), i.e. the joint actions of the six-port device $D_3$ and two holograms implement the transformation (7), which demonstrates implementation of one step of a two-dimensional quantum random walk. By iterating the above steps, we could implement a two-dimensional quantum random walk on an infinite plane with unlimited steps in principle.

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In summary, we have proposed experimental schemes for the optical implementation of one-dimensional two- and three-state quantum random walks by using OAM of a single photon. The proposed scheme requires linear optical elements (beam splitters, phase shifters and mirrors) and holograms.

We now give a brief discussion on the experimental feasibility of the proposed scheme within the current experimental technology. The scheme presented here requires: (i) preparation of a single photon source, (ii) manufacture of a hologram with charges $m = \pm 1$, (iii) the measurement of an OAM state. Based on recent experiments performed by Zeilinger and co-workers [36], a single photon source is available with the parametric down-conversion by performing a measurement on one photon. Holograms with charges $m = \pm 1$ have also been successfully manufactured [30, 36]. An experiment was also reported for distinguishing the OAM of a single photon. Therefore, these experiments demonstrated that the experimental requirements of the present scheme can be satisfied; schemes to realize basic building blocks in figures 1(a) and 2(a) are realizable in the near future.

Finally we should mention that the approach is mainly limited by the diffraction efficiency $p$ of the holograms. With increasing the number of steps $N$ of the walk, the efficiency of implementing walks decreases exponentially $p^N$. Therefore, to combine these setups together to implement a multi-step quantum walk is an experimental challenge.

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