The study of micro-inextensible piezoelectric cantilever plate

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Abstract. In this paper, a micro-inextensible piezoelectric cantilever plate is analyzed and its nonlinear dynamic behaviour is studied. The nonlinear oscillation differential equation is established by using Hamilton’s principle with the application of strain gradient theory to consider the size effect, and inextensible theory to consider the large deformation and rotation effect of cantilever plate. Based on MATLAB software, using the Runge-Kutta method, we can obtain the response of the nonlinear oscillation differential equation. The influences of the strain gradient length scale parameter and voltage on the dynamic response of micro piezoelectric cantilever plate are investigated separately. The results confirmed an increase of the stiffness of the system by using the strain gradient theory and the amplitude of the vibration is reduced. The vibration of the system can be controlled by applying an active voltage. The effect of external excitation frequency on nonlinear dynamic behaviour is considered by using Poincare surface of section and diagrams of waveforms, phase and bifurcation.

1. Introduction

Piezoelectric materials have both positive and negative piezoelectric effects, thus it can be applied to vibration control, piezoelectric harvester and other engineering fields. With the development of micro-electromechanical technology (MEMS), the piezoelectric element itself is bound to be miniaturized. The deformation of materials in the micron dimension has been shown to be size dependent [1-4]. Therefore, the strain gradient theory is introduced to explain the size effect. Cantilever beams and plates can be assumed to be inextensible under lateral loads. Cantilever plates are important structures and extensive studied in structural engineering, such as bridges, wings and so on. Therefore, the analysis of mechanical properties of cantilever plates can explain many phenomena in practical engineering. There are plenty of studies on inextensible cantilever beams, but few on the plate.

The Kirchhoff hypothesis assumes that the deflection of the midplane is smaller than the plate thickness and the midplane remains unstrained during bending. When the transverse deflection of a plate is large, the midplane is strained and assumptions of Kirchhoff are no longer applicable. Von Kármán strains were derived to systematically account for moderate nonlinearities of thin plates, by assuming large deflection and small rotation. However, even the cantilever plates have a larger displacement when they are under lateral load, the stresses and strains of the midplane are still very small. The nonlinearities of the inextensible plate are mainly derived from the large rotation angles. Consequently, it is inappropriate to follow Von Kármán’s approach where the nonlinearities of rotations are neglected to describe the bending of a cantilever plate. Dowell [5-7] derived the nonlinear modal equations of motion using Hamilton’s principal for inextensible plates based on the full formulation of Novozhilov [8]. It is suitable to describe the deformation of cantilever plates with large deflection and rotations.
According to the experimental results, it is necessary to account for size effects to study microstructures. The strain gradient theory introduces the characteristic length dimension of the material in the constitutive relation, and considers the influence of strain gradient higher order tensor on the variable energy density function. It can describe and explain the size effect phenomenon of the mechanical properties of micro components. Mindlin [9] developed a general higher-order stress theory, which includes higher-order strain gradients. In 1992, Aifantis and Altan [10] introduced the second order strain tensor in the constitutive equation, and propose a theory of gradient elasticity. In 2015, Lei J [11] presented a size-dependent functionally graded (FG) micro-plate model based on a simplified strain gradient theory. The results prove that the inclusion of the size effects results in an increase in the micro-plate's stiffness. In 2016, Ansari and Gholami [12] considered the influences of the magneto electro thermos mechanical coupling to investigate the free vibration with various edge conditions using the Mindlin plate theory and von Karman hypothesis.

Inextensible plates have been studied in some literatures; no research on micro inextensible plates with strain gradient has been reported yet. The piezoelectric effect is also considered on this basis. In this paper, the inextensible theory and strain gradient theory are utilized to establish a size-dependent nonlinear cantilever piezoelectric plate model. The effects of the strain gradient length scale parameter and voltage on the dynamic response of micro piezoelectric cantilever plate are analyzed. The complex nonlinear behavior of the structure is revealed under different external excitation frequencies.

2. Potential energy of micro-inextensible piezoelectric cantilever plate

Figure 1 illustrates the cantilever plate which is studied herein:

![Figure 1. Geometry and coordinates of a micro inextensible piezoelectric cantilever plate.

Note: a, a₁, hₕ, hₗ are the length and thickness of the base plate and piezoelectric plate, respectively. b is the width of the plate.

The strain at each point in the plate is:

\[ e_{xx} = \hat{e}_{xx} + z\kappa_{xx}, \quad e_{yy} = \hat{e}_{yy} + z\kappa_{yy}, \quad e_{xy} = \hat{e}_{xy} + z\kappa_{xy} \]

(1)

\[ \hat{e}_{xx}, \hat{e}_{yy}, \hat{e}_{xy} \] are the strain of midplane, \[ \kappa_{xx}, \kappa_{yy}, \kappa_{xy} \] are the curvature of midplane.

Midplane strains can be considered to:

\[ \hat{e}_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \quad \hat{e}_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \quad \hat{e}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \]

(2)

where w is the plate transverse deflection in the z-direction and u, v are the in-plane deformations in x.
and \(y\)-directions.

For the inextensible plate, the following hypotheses are made:

\[
\hat{e}_{xx} = \hat{e}_{yy} = \hat{e}_{xy} = 0
\]  
\[\text{(3)}\]

According to (2.2-2.3), we can obtain expressions:

\[
\frac{\partial u}{\partial x} = -\frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \quad \frac{\partial v}{\partial y} = -\frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right)
\]  
\[\text{(4)}\]

Following Dowell [6], the curvature is considered as:

\[
\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2} \left[ 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right]
\]

\[
\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2} \left[ 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right]
\]

\[
\kappa_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} \left[ 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right]
\]

The expression of strain gradient is proposed by Mindlin [9]:

\[
\eta_{ijk} = \varepsilon_{i,jk} \quad (i, j, k = x, y, z)
\]

\[\text{(9)}\]

For the inextensible plate, from (1), (3) and (5), the expression \(\eta_{ijk}\) are:

\[
\eta_{xx} = -z \frac{\partial \kappa_{xx}}{\partial x}, \quad \eta_{xy} = -z \frac{\partial \kappa_{xx}}{\partial y}, \quad \eta_{yy} = \eta_{yx} = -z \frac{\partial \kappa_{xy}}{\partial x}, \quad \eta_{sy} = \eta_{ys} = -z \frac{\partial \kappa_{yy}}{\partial y}, \quad \eta_{xy} = -z \frac{\partial \kappa_{yx}}{\partial x}, \quad \eta_{yx} = -z \frac{\partial \kappa_{xy}}{\partial y},
\]

\[\text{(10)}\]
\[ \eta_{yy} = -z \frac{\partial \kappa_{yt}}{\partial y}, \quad \eta_{xt} = \kappa_{xt}, \quad \eta_{xy} = \kappa_{xy}, \quad \eta_{yx} = \kappa_{yx} \]

others \( \eta_{jk} = 0, (j \neq k = z) \).

For microstructures, \( \hat{U} \) is expressed by the strain gradient obtained above:
\[ \hat{W}_i = L_i^0 \eta_{kikj} + L_i^1 \eta_{jk} \eta_{lkl} + L_i^2 \eta_{klikj} + L_i^3 \eta_{jk} \eta_{ljk} + L_i^3 \eta_{jk} \eta_{lji}, \quad (\zeta = S, P) \] (11)

from \( L_i^0 \) to \( L_i^5 \) are the material parameters, the reference values for each material parameter is obtained by Ramezani [13]:
\[ L_i^0 = \frac{1}{2} l_{\zeta} \lambda_{\zeta}, \quad L_i^1 = l_{\zeta} \mu_{\zeta}, \quad L_i^2 = L_i^3 = 0, \quad (\zeta = S, P) \] (12)

\( l_{\zeta} \) is the strain gradient length scale parameter, \( \lambda_{\zeta} \) and \( \mu_{\zeta} \) are lame constants:
\[ \lambda_{\zeta} = E_{\zeta} \nu_{\zeta} / (1 - 2 \nu_{\zeta})(1 + \nu_{\zeta}), \quad \mu_{\zeta} = E_{\zeta} / 2(1 + \nu_{\zeta}), \quad (\zeta = S, P) \] (13)

Then we obtain the high order stress tensor \( \chi \):
\[ \chi_{ijk} = \chi_{ijk} = \frac{\partial \hat{W}_i}{\partial \eta_{jk}} \quad (\zeta = S, P) \] (14)

where:
\[ \chi_{xx} = (L_i^0 + L_i^2)(-2z \frac{\partial \kappa_{xt}}{\partial x}), \quad \chi_{xy} = (L_i^0 + L_i^2)(-2z \frac{\partial \kappa_{yt}}{\partial x}), \quad \chi_{yx} = (L_i^0 + L_i^2)(-2z \frac{\partial \kappa_{xt}}{\partial x}), \quad \chi_{yy} = (L_i^0 + L_i^2)(-2z \frac{\partial \kappa_{yt}}{\partial x}), \quad \chi_{yy} = (L_i^0 + L_i^2)(-2z \frac{\partial \kappa_{yt}}{\partial y}), \quad \chi_{yy} = (L_i^0 + L_i^2)(-2z \frac{\partial \kappa_{yt}}{\partial y}) \] (15)

where:
\[ \chi_{xx} = -2L_i^0 \kappa_{xt}, \quad \chi_{xy} = \chi_{yx} = -2L_i^0 \kappa_{yt}, \quad \chi_{yy} = -2L_i^0 \kappa_{yt}, \quad \chi_{yy} = -2L_i^0 \kappa_{yt}, \quad \chi_{yy} = -2L_i^0 \kappa_{yt}, \quad \chi_{yy} = -2L_i^0 \kappa_{yt} \]

Following the equation above, the energy based on the strain gradient can be obtained:
\[ \hat{U} = \int_{V_S} \frac{1}{2} (\chi_{xx} \cdot \eta_{xx} + \chi_{xy} \cdot \eta_{xy} + \chi_{yx} \cdot \eta_{yx}) dxdydz + \int_{V_P} \frac{1}{2} (\chi_{xx} \cdot \eta_{xx} + \chi_{xy} \cdot \eta_{yx} + \chi_{yx} \cdot \eta_{xy}) dxdydz \] (16)

\( (i = x, y, z) \)

3. Nonlinear dynamic model of micro piezoelectric plate

The kinetic energy of the system can be expressed as the sum of the kinetic energy of the base plate and the piezoelectric layer:
\[ T_s = \int_{V_S} \frac{\rho_s}{2} \dot{w}^2 dV, \quad T_p = \int_{V_P} \frac{\rho_p}{2} \dot{w}^2 dV, \quad T = T_s + T_p \] (17)

\( \rho_s \) and \( \rho_p \) are the density of the material of the base plate and the piezoelectric layer, respectively. \( \dot{w} \) is the derivative of \( w \) to time.

The external work of the system divided into three parts:
\[ W_q = \int q \cdot w \, dx dy, \quad W_s = \int r_s \cdot w \cdot dx dy dz, \quad W_p = \int r_p \cdot \dot{w} \cdot dx dy dz \]  
\[ q = q_0 \sin \omega t \]  

is the distributed load, \( r_s \) and \( r_p \) are the damping coefficient of the material of the base plate and the piezoelectric layer, respectively.

The total external power:
\[ W = W_s + W_p + W_q \]  

Based on the boundary conditions of a clamped boundary \((x=0)\) and three free boundaries (other sides), using the combination of beam functions, the form of the solution is assumed to be:
\[ w(x, y, t) = w_1(t) \cdot W_1(x, y) + w_2(t) \cdot W_2(x, y) + w_3(t) \cdot W_3(x, y) \]  

where:
\[ W_1(x, y) = X_1 \cdot Y_1, \quad W_2(x, y) = X_1 \cdot Y_2, \quad W_3(x, y) = X_2 \cdot Y_1 \]
\[ X_i = \cosh \frac{k_i x}{a} - \cos \frac{k_i x}{a} - \beta_i (\sinh \frac{k_i x}{a} - \sin \frac{k_i x}{a}), \quad X_2 = \cosh \frac{k_2 x}{a} - \cos \frac{k_2 x}{a} - \beta_2 (\sinh \frac{k_2 x}{a} - \sin \frac{k_2 x}{a}), \]
\[ Y_1 = 1, \quad Y_2 = \sqrt{(1 - 2y/b)}, \quad k_i^4 = \omega_i^4 \frac{EA}{EI} \frac{i = 1, 2, 3, 4, \ldots, 20}{i = 1, 2, 3, 4, \ldots, 20} \]
\[ Y_2 = \cosh \frac{k_2 x}{a} - \cos \frac{k_2 x}{a} - \beta_2 (\sinh \frac{k_2 x}{a} - \sin \frac{k_2 x}{a}), \]

\( EJ \) is the bending stiffness of section, \( A \) is the cross-sectional area, \( X_i \) are the shape functions of free-beam and \( Y_i \) are the shape functions of free-free beam. Bring the expression of the assumed solution \((20)\) into \((7), (17)\) and \((19)\), according to Hamilton principle:

\[ 0 = \delta \int_0^t (U - T) dt + \int_0^t \delta W dt, \quad t_i \text{ and } t_2 \text{ represent the initial and final moments of the system movement, respectively.} \]

we obtain the ordinary differential motion equation:
\[ \ddot{w}_1 = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 w_1^2 + a_5 w_2^2 + a_6 w_3^2 + a_7 w_1^3 + a_8 w_2^3 + a_9 w_3^3 + a_{10} w_1 w_2 + a_{11} w_1 w_3 + a_{12} w_1 w_2 + a_{13} w_2 w_3 + a_{14} w_3 w_1 + a_{15} w_1 w_2 + a_{16} w_2 w_1 + a_{17} w_3 w_1 + a_{18} w_2 w_3 + a_{19} w_1 w_3 + a_{20} w_1 w_2 + a_{21} w_2 w_3 + a_{22} w_3 w_1 + a_{23} V_0 + a_{24} w_1 V_0 + a_{25} w_2 V_0 + a_{26} w_3 V_0 + a_{27} w_1 w_2 V_0 + a_{28} w_1 w_3 V_0 + a_{29} w_2 w_3 V_0 + a_{30} \omega_0 \sin \omega t \]
\[ \ddot{w}_2 = b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 w_1^2 + b_5 w_2^2 + b_6 w_3^2 + b_7 w_1^3 + b_8 w_2^3 + b_9 w_3^3 + b_{10} w_1 w_2 + b_{11} w_1 w_3 + b_{12} w_2 w_3 + b_{13} w_3 w_1 + b_{14} w_1 w_2 + b_{15} w_2 w_1 + b_{16} w_3 w_1 + b_{17} w_2 w_3 + b_{18} w_1 w_3 + b_{19} w_2 w_3 + b_{20} w_1 w_2 + b_{21} w_2 w_3 + b_{22} w_3 w_1 + b_{23} V_0 + b_{24} w_1 V_0 + b_{25} w_2 V_0 + b_{26} w_3 V_0 + b_{27} w_1 w_2 V_0 + b_{28} w_1 w_3 V_0 + b_{29} w_2 w_3 V_0 + b_{30} \omega_0 \sin \omega t \]
\[ \ddot{w}_3 = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_1^2 + c_5 w_2^2 + c_6 w_3^2 + c_7 w_1^3 + c_8 w_2^3 + c_9 w_3^3 + c_{10} w_1 w_2 + c_{11} w_1 w_3 + c_{12} w_2 w_3 + c_{13} w_3 w_1 + c_{14} w_1 w_2 + c_{15} w_2 w_1 + c_{16} w_3 w_1 + c_{17} w_2 w_3 + c_{18} w_1 w_3 + c_{19} w_2 w_3 + c_{20} w_1 w_2 + c_{21} w_2 w_3 + c_{22} w_3 w_1 + c_{23} V_0 + c_{24} w_1 V_0 + c_{25} w_2 V_0 + c_{26} w_3 V_0 + c_{27} w_1 w_2 V_0 + c_{28} w_1 w_3 V_0 + c_{29} w_2 w_3 V_0 + c_{30} \omega_0 \sin \omega t \]

The parameter \( a_i, b_i, c_i \) \((i=0, 1, \ldots, 30)\), are related to the geometric size and material parameter, as space is limited, the detailed derivation won't be described here.

4. The effect of voltage on the amplitude

By using Runge–Kutta numerical integration method and MATLAB the influences of the scale effect and voltage on the amplitude of piezoelectric plate are studied. The result shows the necessity of
introducing the scale effect and the control effect of the voltage. First, the length scale parameter $l$ is introduced to reflect the effect of the strain gradient. The response of the plate is calculated at $l=0$ and $l=0.5h_S$, respectively, under the same external excitation. It can be seen from figure 2(a) that considering the length parameter, the stiffness of the plate will be obviously increased, then the deflection-thickness ratio is reduced. This is because the thickness of the plate is close to the length scale parameter, and the effect of the scale is obvious, so the strain gradient needs to be considered.

![Figure 2](image-url)

**Figure 2.** The influence of $l$ and $V_0$, $l=0$ (↔) and $l=0.5h_S$ (↔) in (a), $V_0=-2V$ (↔) , $V_0=0V$ (−) and $V_0=2V$ (↔) in (b).

The geometric parameters and material parameters of the micro piezoelectric cantilever laminates are shown in table 1 below.

| physical quantity | value | physical quantity | value |
|-------------------|-------|-------------------|-------|
| $a$               | 2 mm  | $v_S$             | 0.278 |
| $a_p$             | 1 mm  | $v_p$             | 0.35  |
| $b$               | 1 mm  | $e_{31}$          | $-20C \cdot m^2$ |
| $h_S$             | 4 μm  | $e_{32}$          | $-20C \cdot m^2$ |
| $h_p$             | 2 μm  | $e_{33}$          | $6500 e_i$ |
| $E_S$             | 210 Gpa | $\rho_S$         | 2330 kg/m$^3$ |
| $E_p$             | 1.4 Gpa | $\rho_p$         | 1800 kg/m$^3$ |

Next, the impact of the voltage is studied, where $V=V_0 \sin \omega t$. Both positive and negative voltages are applied to the piezoelectric plate respectively.

From figure 2(b), it can be seen that the application of both positive and negative voltages has a diametrically opposite effect on the amplitude of the plate compared to the plate without loaded voltage. Therefore, the application of voltage can play a controlling effect.

5. Complex nonlinear behavior

This section mainly studies the nonlinear dynamic behavior of micro piezoelectric plate. The nonlinear dynamic equation (21) mentioned above is solved by Runge-Kutta method. With the help of MATLAB software, the nonlinear vibration waveform diagram (ratio of deflection to thickness and time), phase diagram (velocity and deflection), Poincare cross section (velocity and deflection) and bifurcation diagram (deflection and excitation force frequency) of the micro piezoelectric inextensible cantilever
under the influence of external vibration are obtained using numerical simulation. The results of these numerical simulations are observed and analyzed. The length scale parameter \( l=0.5h_S \).

The frequency of the applied voltage is adjusted to \( \omega_v=2\omega_p \), where \( \omega_v \) and \( \omega_p \) are the frequency of the external voltage and the external excitation force, respectively. Changing the frequency of external excitation, we get a series of waveforms of deflection-thickness ratio and phase diagrams of displacement and velocity, and bifurcation diagram with frequency variation of \( \omega_p \). From figure 3(a), we can find the micro piezoelectric plate has been maintaining a single periodic harmonic vibration before \( \omega_p=2200 \) rad/s. When \( \omega_p=2200 \) rad/s, the system becomes bifurcate, the single period vibration becomes two times period, then changes into three and four times period. After \( \omega_p=2250 \) rad/s, the system goes into a stable period. When the external excitation frequency increases to 2550 rad/s, the micro piezoelectric plate gradually appeared \( 2T-3T-4T-3T-2T \) of periodic motion, shown in the figure 3(b). The vibration returns to the single periodic vibration after 2900 rad/s. With the increase of frequency, the micro piezoelectric cantilever plate returns to single periodic harmonic vibration.\( 2T-3T-4T \) of period occurred again between \( \omega_p=3600 \) rad/s and \( \omega_p=3700 \) rad/s. The amplitude of vibration increases obviously, when the system appears bifurcation. The waveform, phase diagram and Poincare cross section of period \( -1 \) to period \( -4 \) are shown in figure 4.

![Figure 3. Bifurcation diagram of \( \omega \) (rad/s) and \( w \) (m), (b) is the partial enlarged drawing of (a).](image-url)
6. Conclusion
In this paper, a nonlinear dynamic model of a micro cantilever piezoelectric plate is established, considering the inextensible midplane, the scale effect and the mechatronic coupling effect. When the geometric size of the microstructure approaches the material scale parameters, the strain gradient theory can fully reflect the influence of the scale effect. Therefore, it is necessary to consider the size effect of the microstructure. The positive and negative voltages are applied to enlarge and restrain the response so that the vibration response is controlled. The complex nonlinear dynamic behavior of the plate is generated by changing the external excitation frequency. The amplitude increases sharply, when the system appears bifurcation. This phenomenon can be used as a theoretical basis for the application of the piezoelectric plate in the field of energy harvester.

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