On testing substitutability

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Abstract

The papers \cite{1} and \cite{2} propose algorithms for testing whether the choice function induced by a (strict) preference list of length $N$ over a universe $U$ is substitutable. The running time of these algorithms is $O(|U|^3 \cdot N^3)$, respectively $O(|U|^2 \cdot N^3)$. In this note we present an algorithm with running time $O(|U|^2 \cdot N^2)$. Note that $N$ may be exponential in the size $|U|$ of the universe.

Keywords: Choice functions, Substitutability, Algorithm complexity

A choice function on a finite set $U$ of alternatives is any function $f$ from subsets of $U$ to subsets of $U$ that maps any set $A$ to a subset of itself, i.e., $f(A) \subseteq A$ for all $A \subseteq U$. A choice function $f$ is substitutable if

$$A \subseteq B \implies f(B) \cap A \subseteq f(A) \text{ for all } A, B \subseteq U,$$

i.e. the additional alternatives provided by $B$ do not promote any $x \in A - f(A)$ to the set of selected elements.

We are interested in choice functions induced by preference lists $Y$ on subsets of $U$. A preference list $Y$ is simply an ordered list of subsets of $U$ and the associated choice function $f_Y$ maps any subset $A$ of $U$ to the first element on the list that is contained in $A$. If $Y$ is understood from the context, we write $f$ instead of $f_Y$. We use $N$ to denote the number of elements on $Y$, and, in order to make $f$ defined for all $A \subseteq U$, we assume that the empty set is the last element of $Y$. For elements $X$ and $Y$ in $Y$, we write $X \succ Y$ if $X$ properly precedes $Y$ on $Y$ and we write $X \succ Y$ for $X \succ Y$ or $X = Y$.

For example, let $U = \{a, b, c, d\}$ and $Y = (\{a, b\}, \{a, c, d\}, \{a, c\}, \{a\}, \{c\}, \emptyset)$. Then $f_Y(\{a, b, c\}) = \{a, b\}$. The function $f_Y$ is not substitutable since $d \in (f_Y(\{a, c, d\}) \cap \{d\}) - f_Y(\{d\})$. We refer to \cite{1} for a discussion of the role of substitutable choice functions in economics.

$Y$ is coherent if $X \succ Y$ implies $X \not\subseteq Y$ for any two elements on $Y$. Assume $X \succ Y$ and $X \subseteq Y$. Then $Y$ does not lie in the range of $f_Y$ and removing $Y$ from $Y$ does not change the function $f$. Thus we may assume that $Y$ is coherent.

From now on, $Y$ denotes a coherent preference list and $f$ stands for $f_Y$. $Y$ is substitutable if $f$ is a substitutable choice function.

Lemma 1. Let $Y$ be a coherent preference list on $U$. Then for any $A \subseteq U$, $f(A) = A$ if and only if $A \in Y$. 

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Proof. Since \( f \) maps the powerset of \( U \) to \( \mathcal{Y} \), \( f(A) = A \) implies \( A \in \mathcal{Y} \). Conversely, assume \( A \in \mathcal{Y} \) and \( f(A) \npreceq A \). Then \( f(A) \) and \( A \) are members of \( \mathcal{Y} \) with \( f(A) \npreceq A \) and \( f(A) \subseteq A \), a contradiction to the coherence of \( \mathcal{Y} \).

An established condition of choice functions known as Aizerman’s *outcast*, or Chernoff’s *postulate* \(^*\), or \( \hat{\alpha} \) (see Brandt and Harrenstein \([3]\)) is

\[(\text{outcast}) : \quad \text{if } f(A) \subseteq B \subseteq A \text{ then } f(B) = f(A).\]

**Lemma 2.** If \( Y \) is a coherent preference list on \( U \), then \( f \) satisfies outcast.

**Proof.** \( B \subseteq A \) implies \( f(A) \supseteq f(B) \) and \( f(A) \subseteq B \) implies \( f(B) \supseteq f(A) = f(A) \), where the last equality uses coherence. Thus \( f(A) = f(B) \).

**Lemma 3.** Let \( \mathcal{Y} \) be a coherent and substitutable preference list on \( U \). If \( X \) is a member of \( \mathcal{Y} \) then also every subset of \( X \) is a member of \( \mathcal{Y} \).

**Proof.** Assume \( X = f(X) \) and \( A \subseteq X \). By substitutability, \( f(X) \cap A \subseteq f(A) \) and hence \( A = X \cap A = f(X) \cap A \subseteq f(A) \). Thus \( f(A) = A \).

A preference list \( \mathcal{Y} \) is *complete* if it contains for each \( x \in Y \) also all of its subsets. Note that complete preference lists are exponentially long in the size of their largest member.

In order to demonstrate non-substitutability of a preference list, we need to exhibit the search to special subsets of \( U \). A *witness (to non-substitutability)* is a pair \( (X,Y) \) of members of \( \mathcal{Y} \) such that \( X \npreceq Y \), \( f(X \cup Y) = X \) and there is an \( x \in X \setminus Y \) such that \( f(Y \cup \{x\}) = Y \). Note that \( x \) is selected when the set of alternatives is \( X \cup Y \) (this is the set \( B \)) but is not selected when the set of alternatives is \( Y \cup \{x\} \) (this is the set \( A \)).

**Theorem 4.** \( \mathcal{Y} \) is not substitutable if and only if there is a witness to non-substitutability.

**Proof.** Assume first that \( (X,Y) \) is a witness. Then \( X \npreceq Y \), \( f(X \cup Y) = X \) and there is an \( x \in X \setminus Y \) such that \( f(Y \cup \{x\}) = Y \). Let \( A = Y \cup \{x\} \) and \( B = X \cup Y \). Then \( A \subseteq B \) and \( x \in f(B) \cap (A - f(A)) \). Thus \( f \) is not substitutable.

Conversely, assume that \( f \) is not substitutable. Then there are subsets \( A \) and \( B \) of \( U \) with \( A \subseteq B \) and \( f(B) \cap A \not\subseteq f(A) \). Since \( A \subseteq B \), we have \( f(B) \supseteq f(A) \). In fact, \( f(B) \supseteq f(A) \) since \( f(B) = f(A) \) and \( f(A) \subseteq A \) implies \( f(B) \cap A = f(A) \). Since \( f(A) \subseteq A \subseteq B \), we have \( f(A) \cup f(B) \subseteq B \) and hence \( f(B) \subseteq f(A) \cup f(B) \subseteq B \). Thus \( f(f(A) \cup f(B)) = f(B) \) by property (outcast). Let \( x \in (f(B) \cap A) - f(A) \). Then \( f(B) \cup \{x\} \subseteq A \) and \( f(A) \subseteq f(A) \cup \{x\} \subseteq A \) and hence \( f(A) \cup \{x\} = f(A) \) by (outcast). Thus \( (f(B), f(A)) \) is a witness.

Theorem 4 directly translates into an algorithm of running time \( O(N^3|U| + N^2|U|^2) \).

Note first that one can determine \( f(A) \) in time \( O(N|U|) \) by simply scanning the list \( \mathcal{Y} \) and checking each set for containment. The algorithm has two phases. In the first phase, one determines for each \( Y \in \mathcal{Y} \) the set of \( x \) for which \( f(Y \cup \{x\}) = Y \). This requires \( N|U| \) function evaluations and \( O(N^2|U|^2) \) time. Then one checks for every pair \( (X,Y) \) of elements of \( \mathcal{Y} \), whether it is a witness. This requires \( N^2 \) function evaluations and \( N^2|U| \) look-ups of precomputed values and hence takes time \( O(N^3|U|) \).
1. Preprocessing
   
   \[
   \text{for all } X \in \mathcal{Y} \text{ do } \{ \ d_X := 1; \ \text{for all } x \in U \text{ do } \text{sens}(x, X) := false \} \\
   \text{for all } X \in \mathcal{Y} \text{ do } \{
   \text{for all } Y \in \mathcal{Y} \text{ with } X \succ Y \text{ do } \{
   \text{if } X \subseteq Y \text{ then return } \mathcal{Y} \text{ is NOT COHERENT} ; \\
   \text{if } Y \subseteq X \text{ then increment } d_X ; \\
   \text{for all } x \in U - Y \text{ do } \text{if } X \subseteq Y \cup \{x\} \text{ then } \text{sens}(x, Y) := true \}
   \}\}
   \text{for all } X \in \mathcal{Y} \text{ do } \text{if } d_X \neq 2^{\|X\|} \text{ then return } \mathcal{Y} \text{ is NOT COMPLETE} ;
   \]

2. Looking for the first witness to non-substitutability
   
   \[
   \text{for all } X \in \mathcal{Y} \text{ do } \{
   \text{for all } Y \in \mathcal{Y} \text{ with } X \succ Y \text{ do } \\
   \text{if } (\exists x \in X - Y \text{ s.t. } \text{sens}(x, Y) = true) \land (\forall y \in Y - X \text{ sens}(y, X) = false) \text{ then return } \mathcal{Y} \text{ is NOT SUBSTITUTABLE: witness } (X, Y) ;
   \}\}
   \]

3. Success
   
   return \mathcal{Y} \text{ is SUBSTITUTABLE}

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**Figure 1:** Testing if the list \( \mathcal{Y} \) is substitutable

We improve the running time to \( O(N^2|U|^2) \). The crucial insight is as follows. We search for a witness pair \((X, Y)\) in increasing order of \( X \). Of course, we stop the search as soon as we have found a witness. So when we consider a pair \((X, Y)\) we know that there is no witness \((Z, \cdot)\) with \( Z \succ X \). We then have \( f(X \cup Y) = X \) if and only if \( f(X \cup \{x\}) = X \) for all elements \( x \in Y - X \). We stress that this equivalence does not hold in general, it only holds under the assumption that there is no earlier witness. So we can replace the function evaluation \( f(X \cup Y) \) of cost \( O(|N|U|) \) by \(|U|\) look-ups of precomputed values. We next give the details.

We call \( X \in \mathcal{Y} \) insensitive to \( x \in U \) if \( f(X \cup \{x\}) = X \) and sensitive otherwise.

**Lemma 5.** Let \( X, Y \in \mathcal{Y} \) with \( X \succ Y \). If \( f(X \cup Y) = X \), then \( X \) is insensitive to all \( x \in Y - X \). If \( X \) is insensitive to all \( x \in Y - X \) and there is no witness \((Z, \cdot)\) with \( Z \succ X \), then \( f(X \cup Y) = X \).

**Proof.** Let \( x \in Y - X \) be arbitrary. Then \( X \subseteq X \cup \{x\} \subseteq X \cup Y \) and hence \( X = f(X \cup Y) \supseteq f(X \cup \{x\}) \supseteq f(X) = X \). Thus \( f(X \cup \{x\}) = X \) and \( X \) is insensitive to \( x \).

For the second part, assume \( f(X \cup Y) = Z \) with \( Z \succ X \). Then \( Z \subseteq X \cup Y \) and hence \( Z \cup X \subseteq X \cup Y \). Thus \( Z \supseteq f(X \cup Y) \supseteq f(X \cup Z) \supseteq Z \), where the last inequality follows from \( Z \subseteq X \cup Z \). Thus \( f(X \cup Z) = Z \). Since \((Z, X)\) is not a witness, we must have \( f(X \cup \{x\}) \neq X \) for every \( x \in Z - X \). On the other hand, \( Z - X \subseteq Y - X \) (since \( Z \subseteq X \cup Y \)) and \( f(X \cup \{x\}) = X \) since \( X \) is insensitive to all \( x \in Y - X \), a contradiction. \( \square \)

Lemma 5 suggests a way to find the non-substitutability witness \((X, \cdot)\) with minimal first component.
Theorem 6. Let $X, Y \in \mathcal{Y}$ with $X \succ Y$ and assume that there is no witness $(Z, \cdot)$ with $Z \succ X$. Then $(X, Y)$ is a witness if and only if $X$ is insensitive to all $x \in Y - X$ and $Y$ is sensitive to some $x \in X - Y$.

Proof. Assume first that $(X, Y)$ is a witness pair. Then $Y$ is sensitive to some $x \in X - Y$ and $f(X \cup Y) = X$. The latter implies that $X$ is insensitive to all elements of $Y - X$.

Conversely, assume that $X$ is insensitive to all $x \in Y - X$ and $Y$ is sensitive to some $x \in X - Y$. Then, $f(X \cup Y) = X$ by Lemma 5 and hence $(X, Y)$ is a witness pair.

We are now ready for the algorithm. The algorithm has two phases. In a preprocessing phase, we determine whether $\mathcal{Y}$ is coherent, complete, and, most importantly, compute the Boolean flags $sens(x, X)$ which is true if $X \in \mathcal{Y}$ is sensitive to $x$.

In the main computation, we search for the first witness to non-substitutability. We iterate over the elements of $X$ of $\mathcal{Y}$ in increasing order. Assume that there is no witness $(Z, \cdot)$ with $Z \succ X$. We then iterate over the $Y \in \mathcal{Y}$ with $X \succ Y$ and use Theorem 6 to determine whether $(X, Y)$ is a witness pair.

The most expensive task of the first phase is the construction of the Boolean matrix $\text{sens}$ of size $|\mathcal{U}| \times N$. Since an inclusion test needs $O(|\mathcal{U}|)$ time, the overall time is therefore $O(|\mathcal{U}|^2 \cdot N^2)$. The time complexity of the second phase is $O(|\mathcal{U}| \cdot N^2)$ (the $|\mathcal{U}|$ factor is given by the inspection of the Boolean matrix $\text{sens}$ in order to apply Theorem 6).

By Theorems 4 and 6 and the above discussion, the following corollary holds.

Corollary 1. The algorithm in Figure 1 tests in $O(|\mathcal{U}|^2 \cdot N^2)$ time if a given preference list of size $N$ over an universe $\mathcal{U}$ is substitutable.

Remarks. The $O(N)$ speed-up over the existing algorithms is significant since (as we noted after the definition of complete lists) $N$ is exponential in the size of the largest member of $\mathcal{Y}$. The algorithm in [2] also applies to weak preferences. We leave it as an open problem whether this also holds for our algorithm.

References

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