Scaling of many-particle correlations in a dissipative sandpile

N. M. Bogoliubov\textsuperscript{1)}, A. G. Pronko\textsuperscript{1,2)}, and J. Timonen\textsuperscript{3)}

Abstract. The two dimensional directed sandpile with dissipation is transformed into a \((1+1)\) dimensional problem with discrete space and continuous ‘time’. The master equation for the conditional probability that \(K\) grains preserve their initial order during an avalanche can thereby be solved exactly, and an explicit expression is given for the asymptotic form of the solution for an infinite as well as for a semi-infinite lattice in the horizontal direction. Non-trivial scaling is found in both cases. This conditional probability of the sandpile model is shown to be equal to a \(K\)-spin correlation function of the Heisenberg XX spin chain, and the sandpile problem is also shown to be equivalent to the ‘random-turns’ version of vicious walkers.

1. Introduction

Non-equilibrium dynamic systems have been for some time of considerable interest as they can exhibit critical behaviour in close analogy with systems at thermal equilibrium. A certain class of such dynamic systems, various sandpile models \([1–7]\), have become a standard framework when analyzing self-organized criticality \([8, 9]\), \textit{i.e.} when the dynamics of the system inevitably drives it to a critical state independent of the initial state. Despite the extensive work on these systems, it is only fairly recently that a more detailed understanding of problems like when exactly sandpile models exhibit self-organized criticality, or what are the possible universality classes of their critical behaviours, have begun to emerge.

Most of the work so far on sandpiles has thus concentrated on properties such as \textit{e.g.} the average duration of avalanches and their size distribution, which both exhibit scaling in a critical state. However, there may well be for example interesting many-particle correlations in sandpiles, which likewise exhibit scaling. If one considers particles with non-intersecting trajectories, interesting connections with problems like vicious walkers \([10,11]\) would probably arise. Non-intersecting Brownian walkers have also been of

\textsuperscript{1)}Saint Petersburg Department of V.A. Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023 Saint Petersburg, Russia
\textsuperscript{2)}Department of Physics, University of Wuppertal, 42097 Wuppertal, Germany
\textsuperscript{3)}Department of Physics, University of Jyväskylä, P.O. Box 35 (YFL), 40014 Jyväskylä, Finland
E-mails: bogoliub@pdmi.ras.ru, agp@pdmi.ras.ru, jussi.t.timonen@jyu.fi.
very recent interest, and they as well seem to display corresponding scaling properties [12].

It is the ‘fermionic’ nature of vicious (and non-intersecting Brownian) walkers, which gives rise to scaling of their (asymptotic) survival probability, and the related scaling exponent depends in a nontrivial way on the number of walkers, as well as on the boundary conditions imposed [13–18]. One would thus expect that suitably defined conditional (many-particle) probabilities of particles with non-intersecting paths in sandpiles should exhibit rather similar properties. If the trajectories of vicious walkers and non-intersecting Brownian particles are equivalent to ‘worldlines’ of free fermions, one would expect in addition that these sandpile probabilities can be transformed into a problem of free fermions, or, equivalently, into one of spin ($S = 1/2$) chains.

In order to address these questions, we consider in this paper the probability that $K$ particles of a two dimensional (2D) directed sandpile of ref. [1] preserve their initial order during an avalanche. First we reformulate the (Abelian) sandpile model such that it becomes one in 1+1 dimensions, and assume for the sake of generality that it is dissipative, i.e. that the number of ‘grains’ is not conserved in the topplings of unstable sites. We also consider two different boundary conditions, an infinite system (in the horizontal direction) and a system with an absorbing boundary at the origin (a ‘semi-infinite’ system). As it is well known by now, the model is critical only at vanishing dissipation [6, 19]. We derive an exact analytic form for the probability, and show that non-zero dissipation introduces an exponential cutoff in its asymptotic form that also includes a power law with a scaling exponent that depends nonlinearly on $K$, and is different for the two boundary conditions.

We also show that this probability is equal to the partition function of $K$ vicious walkers, more precisely the ‘random-turns’ version of such walkers [10,14]. The former probability is thus the generating function for the survival probabilities of the walkers. The scaling exponents of the sandpile probability are not those of the survival probability of the ‘lock-step’ vicious walkers, although the two problems are intimately connected. Finally we show that the sandpile probability is equal to a correlation function of the Heisenberg XX spin chain. This establishes the relation of both the sandpile problem and the walker problem to a problem of free fermions, as the Heisenberg XX chain is equivalent to free fermions via a Jordan-Wigner transformation.

2. Abelian sandpile model

2.1. Discrete model. A 2D directed (Abelian) sandpile model on a lattice (see, e.g., [2]) is constructed such that to each site $(j, n)$ an integer height variable (number of grains) $z(J, n)$ is assigned. The site has a threshold height $z^c_{(j, n)}$ below which it is stable. The dynamics of the model consists
of two steps. First, we choose a site \((j, n)\) at random and add one grain to it, \(i.e., z(j, n) \mapsto z(j, n) + 1\). For \(z(j, n) \geq z_c(j, n)\), site \((j, n)\) becomes unstable and its grains are distributed among the 'downhill' neighbouring sites. In the following we will use the notation by which the locations of lattice sites in the horizontal direction are labelled by \(j, k\) or \(l\), and by \(n\) in the downhill direction. By \(n\) we can equivalently denote the number of steps in a cascade of toppling processes. In a toppling at site \((j, n)\) grains are thus distributed to sites \((j + 1, n + 1)\) and \((j - 1, n + 1)\). By supressing the \(n\) labels (understanding that two adjacent columns in the lattice are connected in a toppling and that there is no \(n\) dependence) we can express a toppling in the form

\[
z_j \mapsto z_j - \Delta_{lj}, \tag{2.1}
\]

in which the elements of the toppling matrix \(\Delta\) satisfy \(\Delta_{jj} > 0\), and \(\Delta_{lj} < 0\) for \(l \neq j\). The condition \(\sum_j \Delta_{lj} \geq 0\) for every \(l\) guarantees that no grains are created in the toppling process. Without loss of generality we can put \(\Delta_{jj} = z_c(j, n)\). The allowed number of grains in a stable site \((j, n)\) is now \(1, 2, \ldots, \Delta_{jj} - 1\). The sites \((j, n)\) such that \(\sum_j \Delta_{lj} > 0\) are called dissipative. Boundary sites are always dissipative so that grains can leave the system through the boundaries. After an initial toppling at a site, neighbouring sites can also become unstable, and sites are kept on relaxing with parallel updating until all sites are stable. In this way an avalanche of topplings is generated. Existence of dissipative sites ensures that all avalanches terminate in a finite time.

Assume now that all lattice sites are initially in a stationary state (i.e. are stable): \(z(j, n) = z_c(j, n) - 1\). If we add a grain at a randomly chosen site \((l, 0)\), and make site \((j, n)\) dissipative such that the system returns to a stationary state after the extra grain disappears from this site (i.e. after \(n\) steps). The conditional probability \(G_{jl}(n)\) that an extra grain is at site \((j, n)\) satisfies the equation

\[
G_{jl}(n) = \frac{1}{2} \{G_{j+1,l}(n-1) + G_{j-1,l}(n-1)\}, \tag{2.2}
\]

with the initial condition \(G_{jl}(0) = \delta_{jl}\). Since we consider only symmetric topplings, the conditional probability also satisfies \(G_{jl}(n) = G_{lj}(n)\). It is easy to verify that eq. (2.2) is the same as the equation for the corresponding probability expressed in the conventional 'light cone' coordinates, eq. (5) in ref. [1].

\[\textbf{2.2. Continuous 'time' model.}\] We can also express eq. (2.2) in the form

\[
G_{jl}(n + 1) - G_{jl}(n) = \frac{1}{2} \{G_{j+1,l}(n) + G_{j-1,l}(n) - 2G_{jl}(n)\}. \tag{2.3}
\]

Consider now a process in which the discrete number of steps is replaced by a continuous parameter that will be called 'time' in the following. Let \(P_{jl}(t)\) be the conditional probability that a grain is at a horizontal location
at time $t$ after an arbitrary number of steps since it was dropped at a horizontal location $l$ at $t = 0$. Transforming eq. (2.3) into such a continuous time we obtain that, during a short time interval $dt$, the probability $P_{jl}(t)$ changes such that

$$P_{jl}(t + dt) - P_{jl}(t) = \frac{1}{2} \{ P_{j+1,l}(t) + P_{j-1,l}(t) - 2P_{jl}(t) \} \ dt, \quad (2.4)$$

which leads to the master equation

$$\frac{d}{dt} P_{jl}(t) = -\frac{1}{2} \sum_k \Delta_{jk} P_{kl}(t) \quad (2.5)$$

with the toppling matrix

$$\Delta_{jk} = 2\delta_{jk} - (\delta_{j+1,k} + \delta_{j-1,k}). \quad (2.6)$$

This toppling matrix means that, as above, at each toppling two grains are removed from the site and distributed to its nearest-neighbour downhill sites. We consider here only symmetric topplings, $\Delta_{jk} = \Delta_{kj}$, and thus $P_{jk}(t) = P_{kj}(t)$. The initial conditions are $P_{jk}(0) = \delta_{jk}$. For a model of $N$ sites in the horizontal direction, the lateral boundary elements of the toppling matrix can be defined such that $\Delta_{0,1} = \Delta_{N,N+1} = 0$, and hence the boundary sites $j = 1$ and $j = N$ are always dissipative as required.

Notice that the continuous time is not a simple continuum formed by the discrete variables $n$, but $P_{jl}(t)$ includes processes with all possible numbers of steps. In fact function $P_{jl}(t)$ can be considered as the generating function of the conditional probabilities $G_{jl}(n)$ as we find that

$$e^t P_{jl}(t) = \sum_{n=0}^{\infty} G_{jl}(n) \frac{t^n}{n!}. \quad (2.7)$$

The expected number of topplings at site $j$ in an avalanche resulting from a perturbation (adding a grain) at site $l$ is given by $\Gamma_{jl}(0) = \sum_{n=0}^{\infty} G_{jl}(n)$. A Laplace transform of the conditional probability

$$\Gamma_{jl}(f) = \int_0^{\infty} e^{-tf} P_{jl}(t) \ dt, \quad (2.8)$$

is the Green’s function of the master equation eq. (2.5). It is easy to verify that $\Gamma_{jl}(0)$ satisfies [2] the condition $\sum_k \Delta_{jk} \Gamma_{kl}(0) = \delta_{jl}$.

The master equation eq. (2.5) can easily be solved for the toppling matrix of eq. (2.6) with the initial condition $P_{jl}(0) = \delta_{jl}$. Consider first the case of an infinite lattice in the horizontal direction such that $-\infty < j, l < \infty$. In this case we find that

$$P_{jl}(-\infty, \infty)(t) = e^{-t} I_{l-j}(t), \quad (2.9)$$

where $I_{j}(x)$ is a modified Bessel function. Asymptotically, for large $t$, the conditional probability for a single grain thus behaves as

$$P_{jl}(-\infty, \infty)(t) \propto t^{-1/2}. \quad (2.10)$$
There is a pure power law so that the duration of avalanches scales with an exponent $\xi_{(-\infty, \infty)} = 1/2$. This exponent coincides with the known result for 2D directed sandpiles [1] as it should.

We can analyze the effect of boundary conditions by introducing an absorbing boundary at the origin. To this end we first recall the solution for a finite lattice of $N$ sites (see e.g. [19]) for which the boundary conditions are $P_{jl}(t) = 0$ for $j, l = 0$ and $j, l = N + 1$. In this case one has

$$P_{jl}(t) = \frac{2}{N + 1} \sum_{k=1}^{N} e^{-tE_k} \sin \frac{\pi j k}{N + 1} \sin \frac{\pi l k}{N + 1},$$

where the spectrum is of the Bloch form,

$$E_k = 1 - \cos \frac{\pi k}{N + 1}.$$  

In the limit $N \to \infty$ the sum in eq. (2.11) can be replaced by an integral, with the result

$$P^{(0, \infty)}_{jl}(t) = \frac{2}{\pi} \int_{0}^{\pi} e^{-t(1-\cos x)} \sin(jx) \sin(lx) \, dx  
= e^{-t[I_{l-j}(t) - I_{l+j}(t)]}.$$  

The asymptotic behaviour for large $t$ of the conditional probability is now given by

$$P^{(0, \infty)}_{jl}(t) \propto t^{-3/2}.$$  

The scaling exponent indeed depends on having a boundary at a finite distance: $\xi_{(0, \infty)} = \frac{3}{2} = \xi_{(-\infty, \infty)} + 1$. As expected, the same exponent has been found for the scaling of avalanche sizes with the corresponding boundary conditions [20][21].

### 3. Multiple-grain correlations

#### 3.1. Master equation

Having established that our master equation method indeed reproduces previously known results, we turn now to a more interesting problem of correlations between multiple grains during ‘avalanche dynamics’. To this end, let us address the following problem. Consider the same lattice as above with all its sites in a stationary state: $z_{j,n} = z_{c(j,n)} - 1$, and add $K$ grains at randomly chosen $K$ horizontal locations: $l_1 > l_2 > \cdots > l_K$. The toppling rules are the same as above, at each toppling two grains are removed from the toppling site $j$, and a grain can jump to each of the two nearest-neighbour sites in the downhill direction. However, if $z_{j,n} - z_{j+1,n+1} = 0$, site $(j,n)$ cannot topple. The probability that the additional grains will be at dissipative sites $j_1 > j_2 > \cdots > j_K$ at time $t$ (after an arbitrary number of topplings) satisfies a generalized version of eq.
Supplemented by the condition $P_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = 0$, if $j_r = j_{r+1}$, for all $r = 1, \ldots, K - 1$. The solution to this equation is given by

$$P_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = \det_{1\leq r,s\leq K} \{P_{j_r,l_s}(t)\}, \quad (3.2)$$

where $P_{jl}(t)$ is the one-grain conditional probability which satisfies eq. (2.5) with the same boundary conditions as the solution of eq. (3.1).

As for the single grain, in the multi-grain case the continuous conditional probabilities $P_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t)$ are generating functions of the discrete ones, $G_{j_1,\ldots,j_K;l_1,\ldots,l_K}(n)$, and we find that

$$e^{Kt}P_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = \sum_{n=0}^{\infty} G_{j_1,\ldots,j_K;l_1,\ldots,l_K}(n) \frac{K^n n^n}{n!}. \quad (3.3)$$

The discrete probabilities satisfy the equation

$$G_{j_1,\ldots,j_K;l_1,\ldots,l_K}(n) = \frac{1}{2K} \sum_{r=1}^{K} \{G_{j_1,\ldots,j_r-1,j_r+1,j_r+1,\ldots,j_K;l_1,\ldots,l_K}(n-1) + G_{j_1,\ldots,j_r-1,j_r-1,j_r+1,\ldots,j_K;l_1,\ldots,l_K}(n-1)\}, \quad (3.4)$$

supplemented by the condition $G_{j_1,\ldots,j_K;l_1,\ldots,l_K}(n) = 0$, if $j_r = j_{r+1}$, for all $r = 1, \ldots, K - 1$.

### 3.2. Infinite lattice

Let us now consider the asymptotic behaviour for $t \to \infty$ of the above multi-grain conditional probability. We first consider the case of an infinite lattice in the horizontal direction when the one-grain probability is given by eq. (2.9). Using the integral representation for the modified Bessel function, we arrive at the expression

$$P_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = \frac{1}{(2\pi)^K} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K \ e^{-t \sum_{m=1}^{K} (1-\cos x_m)} \times \det_{1\leq r,s\leq K} \{e^{i(l_s-j_r)x_r}\}. \quad (3.5)$$

Making use of the symmetry of the integrand with respect to permutations of integration variables $x_1, \ldots, x_K$, the determinant in this expression can
be transformed such that
\[
\begin{align*}
\det_{1 \leq r, s \leq K} \left\{ e^{i(l_s - j_r)x_r} \right\} & \longrightarrow \det_{1 \leq r, s \leq K} \left\{ e^{i x_r} \right\} \prod_{r=1}^{K} e^{-i j_r x_r} \\
& \longrightarrow \frac{1}{K!} \det_{1 \leq r, s \leq K} \left\{ e^{-i j_r x_r} \right\} \det_{1 \leq r, s \leq K} \left\{ e^{i x_r} \right\}.
\end{align*}
\]
(3.6)

The two determinants above can be represented in terms of Schur functions (for a survey on Schur functions see e.g. [22]):
\[
s_\lambda(x_1, x_2, \ldots, x_K) := \frac{\det_{1 \leq s, k \leq K} (x_1^{s_k + k - k})}{\det_{1 \leq s, k \leq K} (x_1^{K - k})} = \det_{1 \leq s, k \leq K} (x_1^{s_k + K - k}) \prod_{1 \leq s < k \leq K} (x_s - x_k)^{-1},
\]
(3.7)

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K) \) is a partition of a non-increasing series of the non-negative integers \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_K \geq 0 \). If we consider the case \( j_r \geq -K \) and \( l_r \geq -K \), we find that
\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) = \frac{1}{(2\pi)^K K!} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-t \sum_{m=1}^{K} (1 - \cos x_m)}
\times s_\lambda(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_K}) s_\mu(e^{-ix_1}, e^{-ix_2}, \ldots, e^{-ix_K})
\times \prod_{1 \leq r < s \leq K} |e^{ix_r} - e^{ix_s}|^2,
\]
(3.8)

where \( \lambda_r = j_r - K + r \) and \( \mu_r = l_r - K + r \).

As \( t \to \infty \) (and \( j_s - l_r \ll t \) for all \( r, s = 1, \ldots, K \)), the main contributions to the above integrals come from near the origin of the integration variables, and in leading order we find that
\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) \sim \frac{s_\lambda(1, 1, \ldots, 1) s_\mu(1, 1, \ldots, 1)}{(2\pi)^K K!}
\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2} \sum_{m=1}^{K} x_m^2} \prod_{1 \leq r < s \leq K} (x_r - x_s)^2.
\]
(3.9)

The prefactor of the integral can be computed (see e.g. [22]) using the well known result
\[
s_\lambda(1, 1, \ldots, 1) = \frac{\prod_{1 \leq r < s \leq K} (\lambda_r - r - \lambda_s + s)}{\prod_{m=1}^{K-1} m!},
\]
(3.10)
while the integral is the Mehta integral of the gaussian unitary ensemble of random matrices \[^{23}\], which can be explicitly evaluated:

\[
\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_K e^{-\frac{1}{2} \sum_{m=1}^{K} x_m^2} \prod_{1 \leq r < s \leq K} (x_r - x_s)^2 = \frac{(2\pi)^{K/2}}{t^{K^2/2}} \prod_{m=1}^{K} m!.
\]

(3.11)

We thus find that, as \(t \to \infty\), in leading order the multi-grain conditional probability is given by

\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) \sim A_{j_1, \ldots, j_K; l_1, \ldots, l_K} t^{-\gamma}
\]

(3.12)

with the scaling exponent

\[
\gamma = \frac{K^2}{2}
\]

(3.13)

and the amplitude

\[
A_{j_1, \ldots, j_K; l_1, \ldots, l_K} = \frac{\prod_{1 \leq s < r \leq K} (l_r - l_s)(j_r - j_s)}{(2\pi)^{K} \prod_{m=1}^{K-1} m!}.
\]

(3.14)

3.3. Semi-infinite lattice. Let us consider the conditional probability in the presence of an absorbing boundary at the origin. As in the one-grain case, let us start with a finite lattice of \(N\) sites in the horizontal direction. Substituting eq. (2.11) into eq. (3.2), we find that

\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) = \frac{2^K}{(N+1)^K} \sum_{k_1=1}^{N} \cdots \sum_{k_K=1}^{N} e^{-t \sum_{m=1}^{K} E_{km}}
\]

\[
\times \det_{1 \leq r, s \leq K} \left\{ \frac{\sin \pi j_r}{N+1} \frac{\sin \pi l_s}{N+1} \right\},
\]

(3.15)

where \(E_k\) is given by eq. (2.12). The multi-grain conditional probability for the semi-infinite lattice follows from this result by taking the large \(N\) limit; the resulting expression is similar to eq. (3.5), but with a determinant that now contains sine functions instead of exponential functions:

\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) = \frac{1}{\pi^K} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^{K} (1-\cos x_m)}
\]

\[
\times \det_{1 \leq r, s \leq K} \{ \sin(j_r x_r) \sin(l_s x_r) \}. \quad (3.16)
\]

Again, using the symmetry with respect to permutations of the integration variables \(x_1, \ldots, x_K\), we can transform the determinant in this expression
such that

\[
\begin{align*}
\det_{1 \leq r, s \leq K} \{\sin(j_r x_r) \sin(l_s x_r)\} & \longrightarrow \det_{1 \leq r, s \leq K} \{\sin(l_s x_r)\} \prod_{r=1}^{K} \sin(j_r x_r) \\
& \longrightarrow \frac{1}{K!} \det_{1 \leq r, s \leq K} \{\sin(j_s x_r)\} \det_{1 \leq r, s \leq K} \{\sin(l_s x_r)\}.
\end{align*}
\]

(3.17)

Using the character of the irreducible representation corresponding to a partition \(\lambda\) of the symplectic Lie algebra,

\[
sp_{\lambda}(x_1, x_2, \ldots, x_K) := \frac{\det_{1 \leq j, k \leq K} (x^\lambda_j + K - k + 1 - x^{-\lambda_j + K + k + 1})}{\det_{1 \leq j, k \leq K} (x^K_j + 1 - x^{-K - k + 1})},
\]

(3.18)

we can express eq. (3.16) in the form

\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) = \frac{1}{\pi^K K!} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^{K} (1 - \cos x_m)} \times \left( \det_{1 \leq r, s \leq K} \{\sin s x_r\} \right)^2 sp_{\lambda}(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_K}) \times sp_{\mu}(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_K}),
\]

(3.19)

where \(\lambda_r = j_r - K + r - 1\) and \(\mu_r = l_r - K + r - 1\). The determinant in the above integrand can be evaluated using the identity (for a proof, see [24])

\[
\det_{1 \leq r, s \leq K} \{\sin s x_r\} = 2^{K(K-1)} \prod_{r=1}^{K} \sin x_r \prod_{1 \leq j < k \leq K} \sin \frac{x_j - x_k}{2} \sin \frac{x_j + x_k}{2}.
\]

(3.20)

We finally obtain for the conditional probability the expression

\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) = \frac{2^{2K(K-1)}}{\pi^K K!} \int_{-\pi}^{\pi} dx_1 \cdots \int_{-\pi}^{\pi} dx_K e^{-t \sum_{m=1}^{K} (1 - \cos x_m)} \times \prod_{r=1}^{K} \sin^2 x_r \prod_{1 \leq j < k \leq K} \sin^2 \frac{x_j - x_k}{2} \sin^2 \frac{x_j + x_k}{2} \times sp_{\lambda}(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_K})sp_{\mu}(e^{ix_1}, e^{ix_2}, \ldots, e^{ix_K}).
\]

(3.21)
In the limit \( t \to \infty \) we can approximate the integrals in the above expression with the integrals

\[
\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_K e^{-\frac{t}{2} \sum_{m=1}^{K} x_m^2} \prod_{1 \leq j < k \leq K} (x_j^2 - x_k^2)^2 \prod_{j=1}^{K} x_j^2 = \frac{\prod_{m=1}^{K} (2m)!}{(2\pi)^{K/2} t^{K/2} (2K + 1)/2}.
\] (3.22)

For a proof of eq. (3.22), see [23]. We find thereby for the leading asymptotic term of the generating function

\[
P_{j_1, \ldots, j_K; l_1, \ldots, l_K}(t) \sim A_{j_1, \ldots, j_K; l_1, \ldots, l_K} t^{-\gamma}.
\] (3.23)

Here the scaling exponent is given by

\[
\gamma = \frac{K(2K + 1)}{2}
\] (3.24)

and the amplitude is

\[
A_{j_1, \ldots, j_K; l_1, \ldots, l_K} = \frac{\prod_{m=1}^{K} (2m)!}{2K(K+1)\pi \frac{1}{2} K!} sp_\lambda(1, \ldots, 1) sp_\mu(1, \ldots, 1),
\] (3.25)

in which

\[
sp_\lambda(1, \ldots, 1) = \prod_{1 \leq r < s \leq K} (j_r^2 - j_s^2) \prod_{m=1}^{K-1} \frac{[2(K - m) + 1]!}{m!(K + m)!}.
\] (3.26)

A similar expression can be found for \( sp_\mu(1, \ldots, 1) \) with the \( j_r \)'s replaced by \( l_r \)'s.

We thus find that the scaling exponent of the multi-grain sandpile problem considered here is not equal to the one found previously for the 'lock-step' version of vicious walkers, for which \( \gamma = K(K - 1)/4 \) [10][11]. Instead, our sandpile problem corresponds to the 'random-turns' version of vicious walkers [10][14]. The connection to the 'random-turns' version of vicious walkers is discussed in more detail below.

### 4. Connection to vicious walkers

#### 4.1. Heisenberg chain.

Before addressing the relation between the above sandpile problem and the 'random-turns' vicious walkers, we first outline its relation to free fermions. It turns out that it has a straightforward connection to the Heisenberg XX spin chain that can be mapped, as is well known, to a free fermion problem by the Jordan-Wigner transformation.

The Heisenberg XX chain has the Hamiltonian

\[
\hat{H} = -\frac{1}{2} \sum_{i,k} \Lambda_{ik} \sigma_i^- \sigma_k^+.
\] (4.1)
where summation is over all lattice sites, and
\[ \Lambda_{ik} = \delta_{i,k+1} + \delta_{i,k-1}. \] (4.2)
We use the standard notations \( \sigma_i^\pm, \sigma_i^z \) for Pauli spin operators that satisfy the commutation relations
\[ [\sigma_i^+, \sigma_k^-] = \sigma_i^z \delta_{ik}, \quad [\sigma_i^z, \sigma_k^\pm] = \pm 2\sigma_i^\pm \delta_{ik}, \] (4.3)
and have in addition the properties
\[ (\sigma_i^\pm)^2 = 0, \quad (\sigma_i^z)^2 = 1. \] (4.4)

In what follows we use the fact that the ferromagnetic state with all spins up, \( |\uparrow\rangle = \otimes_i |\uparrow_i \rangle \), which satisfies \( \sigma_k^+ |\uparrow\rangle = 0 \) for all \( k \) and normalized such that \( \langle \uparrow | \uparrow \rangle = 1 \), is annihilated by the Hamiltonian,
\[ \hat{H} |\uparrow\rangle = 0. \] (4.5)

Our aim is to study the ‘temporal’ evolution of states with a finite number of down spins, which can be constructed by acting with operators \( \sigma_j^- \) on the state \( |\uparrow\rangle \). We thus consider the matrix elements
\[ F_{j_1,\ldots,j_K; l_1,\ldots,l_K}(t) = \langle \uparrow | \sigma_{j_1}^+ \cdots \sigma_{j_K}^+ e^{-t\hat{H}} \sigma_{l_1}^- \cdots \sigma_{l_K}^- |\uparrow\rangle. \] (4.6)

Parameter \( t \) will play the role of ‘time’ in the context of the sandpile model.

Before proceeding with the general case, let us first consider the case \( K = 1 \), i.e. the ‘temporal’ evolution of a single reversed spin. Differentiating the function \( F_{jl}(t) = \langle \uparrow | \sigma_j^+ e^{-t\hat{H}} \sigma_l^- |\uparrow\rangle \) with respect to \( t \) and using the commutation relation
\[ [\sigma_j^+, \hat{H}] = -\frac{1}{2} \sum_k \Lambda_{jk} \sigma_j^z \sigma_k^+ = -\frac{1}{2} \sigma_j^z (\sigma_{j-1}^+ + \sigma_{j+1}^+) \] (4.7)

together with the property \( \langle \uparrow | \sigma_j^z = (\langle \uparrow | \) \), we find that
\[ \frac{d}{dt} F_{jl}(t) = -\langle \uparrow | \sigma_j^+ \hat{H} e^{-t\hat{H}} \sigma_l^- |\uparrow\rangle = \frac{1}{2} \langle \uparrow | (\sigma_{j-1}^+ + \sigma_{j+1}^+) e^{-t\hat{H}} \sigma_l^- |\uparrow\rangle. \] (4.8)

Hence the correlation function eq. (4.6) for \( K = 1 \) satisfies the equation
\[ \frac{d}{dt} F_{jl}(t) = \frac{1}{2} (F_{j+1l}(t) + F_{j-1l}(t)). \] (4.9)

Similarly, by commuting \( \hat{H} \) with \( \sigma_l^- \), a difference equation similar to eq. (4.9) follows, but for subscript \( j \) with fixed subscript \( l \). Both equations are subject to the initial condition \( F_{jl}(0) = \delta_{jl} \), and to boundary conditions that depend on the type of the lattice: for the semi-infinite lattice \( F_{jl} = 0 \) for \( j,l = 0 \), while for a finite lattice \( F_{jl} = 0 \) for \( j,l = 0 \) and \( j,l = N + 1 \).

As a result, comparing eqs. (4.9) and (2.5) together with their initial and boundary conditions, we find that the one-spin correlation function of the Heisenberg XX chain is equal, modulo a trivial factor, to the one-grain conditional probability of the sandpile model, i.e. \( P_{jl}(t) = e^{-t} F_{jl}(t) \).
Let us now consider the case of general $K$. Differentiating eq. (4.6) with respect to $t$, taking into account the differential property of the commutation relations,

$$[\sigma^+_{j_1}\sigma^+_{j_2}\cdots\sigma^+_{j_K}, \hat{H}] = \sum_{k=1}^{K} \sigma^+_{j_1}\cdots\sigma^+_{j_k-1}[\sigma^+_{j_k}, \hat{H}]\sigma^+_{j_k+1}\cdots\sigma^+_{j_K}$$  \hspace{1cm} (4.10)

and applying the commutation relation eq. (4.7), we find that

$$\frac{d}{dt}F_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = \frac{1}{2} \sum_{r=1}^{K} \left( F_{j_1,\ldots,j_{r-1},j_r+1,j_{r+1},\ldots,j_K;l_1,\ldots,l_K}(t) ight.$$  
$$
+ F_{j_1,\ldots,j_{r-1},j_r-1,j_{r+1},\ldots,j_K;l_1,\ldots,l_K}(t) \bigg) \bigg) \hspace{1cm} (4.11)$$

A similar equation can be found with respect to subscripts $l_r$ with the $j_r$’s kept fixed. The initial condition is $F_{j_1,\ldots,j_K;l_1,\ldots,l_K}(0) = \delta_{j_1l_1}\cdots\delta_{j_Kl_K}$. The correlation function also satisfies the conditions $F_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = 0$ if $l_r = l_s$ or $j_r = j_s$ ($r, s = 1, \ldots, K$), which follow from the nilpotency of the Pauli spin operators, eq. (4.4).

It is evident that the differential equation eq. (4.11), which the correlation function eq. (4.6) satisfies, coincides with eq. (3.1) for the multi-grain probability up to a trivial ‘diagonal’ term. It is also easy to verify that the solution of eq. (4.11) can be expressed in a determinant form,

$$F_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = \det_{1 \leq r,s \leq K} \{ F_{j_rl_s}(t) \},$$  \hspace{1cm} (4.12)

where $F_{j_l}(t)$ are the one-particle correlation functions satisfying eq. (4.9). We thus find that

$$P_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t) = e^{-Kt}F_{j_1,\ldots,j_K;l_1,\ldots,l_K}(t).$$  \hspace{1cm} (4.13)

Hence, the multi-grain probabilities in our sandpile problem are also multi-spin correlation functions in the Heisenberg $XX$ spin chain.

4.2. Quantum trajectories. We are now in the position to establish an explicit relation of our sandpile model with random walks. To this end we exploit the known connection of the Heisenberg XX spin chain with the random-turns vicious walk model \cite{25,27}. Our starting point, which follows from eqs. (4.13) and (3.3), is that the discrete multi-grain probability can be expressed as a matrix element:

$$G_{j_1,\ldots,j_K;l_1,\ldots,l_K}(n) = \frac{1}{K^n} \langle \uparrow | \sigma^+_{j_1} \cdots \sigma^+_{j_K} (\hat{H})^n \sigma^-_{l_1} \cdots \sigma^-_{l_K} | \uparrow \rangle.$$  \hspace{1cm} (4.14)

By evaluating this matrix element, it can be shown that the discrete probability of the sandpile model is equal, modulo a simple factor, to the number of paths of random-turns vicious walkers subject to the same boundary conditions.
We start again with the single-grain case in which the discrete probability
\( G_{jl}(n) \) is the one-spin matrix element \( \langle \uparrow | \sigma_j^+ (-\hat{H})^n \sigma_l^- | \uparrow \rangle \). Taking into
account eq. \((4.5)\), we can write
\[
\hat{H} \sigma_l^- | \uparrow \rangle = [\hat{H}, \sigma_l^-] | \uparrow \rangle = -\frac{1}{2} \sum_{k_1} \Lambda_{k_1} \sigma_{k_1}^- | \uparrow \rangle,
\]
and repeating the procedure for the \( n \)th power, we find that
\[
\hat{H}^n \sigma_l^- | \uparrow \rangle = \left( -1 \right)^n \frac{1}{2^n} \sum_{k_1, \ldots, k_n} \Lambda_{k_n, k_{n-1}} \cdots \Lambda_{k_2 k_1} \Lambda_{k_1} \sigma_{k_n}^- | \uparrow \rangle.
\]
(4.15)

Multiplying this expression from the left by \( \langle \uparrow | \sigma_j^+ \rangle \) and using the orthogonality of the spin states, \( \langle \uparrow | \sigma_j^+ \sigma_l^- | \uparrow \rangle = \delta_{jl} \), we find that
\[
\langle \uparrow | \sigma_j^+ (-\hat{H})^n \sigma_l^- | \uparrow \rangle = \frac{1}{2^n} \sum_{k_1, \ldots, k_n} \Lambda_{j k_{n-1}} \cdots \Lambda_{k_2 k_1} \Lambda_{k_1} \sigma_{k_n}^- | \uparrow \rangle.
\]
(4.16)

The sum in eq. \((4.17)\) can be interpreted as one over all possible quantum
trajectories (lattice paths) of \( n \) time steps of a particle (corresponding to
the down spin state) from site \( l \) to site \( j \) subject to the boundary conditions.
In this interpretation \( \Lambda \) appears as the transfer matrix. From eq. \((4.2)\) it
follows that we deal with a random walk on a lattice. Denoting by \( P_n(l \rightarrow j) \)
the number of all admissible paths of \( n \) steps from site \( l \) to site \( j \), we have
\[
G_{jl}(n) = \frac{1}{2^n} P_n(l \rightarrow j).
\]
(4.18)

Let us now consider the general case. Acting with the Hamiltonian \( \hat{H} \)
on the state \( \sigma_{l_1}^- \sigma_{l_2}^- \cdots \sigma_{l_K}^- | \uparrow \rangle \), for which we assume that \( l_1 > l_2 > \cdots > l_K \),
we find that
\[
\hat{H} \sigma_{l_1}^- \sigma_{l_2}^- \cdots \sigma_{l_K}^- | \uparrow \rangle = \sum_{r=1}^K \sigma_{l_r}^- \cdots \sigma_{lr-1}^- \left[ \hat{H}, \sigma_{lr}^- \right] \sigma_{lr+1}^- \cdots \sigma_{l_K}^- | \uparrow \rangle

= -\frac{1}{2} \sum_{r=1}^K \sum_m \Lambda_{ml_r} \sigma_{l_r}^- \cdots \sigma_{lr-1}^- \sigma_m^- \sigma_{lr+1}^- \cdots \sigma_{l_K}^- | \uparrow \rangle,

= -\frac{1}{2} \sum_{m_1, \ldots, m_K} T_{m_1, \ldots, m_K ; l_1, \ldots, l_K} \sigma_{m_1}^- \cdots \sigma_{m_K}^- | \uparrow \rangle,
\]
(4.19)

where
\[
T_{m_1, \ldots, m_K ; l_1, \ldots, l_K} = \sum_{r=1}^K \delta_{m_1 l_1} \cdots \delta_{m_{r-1} l_{r-1}} \Lambda_{ml_r} \delta_{m_{r+1} l_{r+1}} \cdots \delta_{m_K l_K}
\]
(4.20)

for \( m_1 > m_2 > \cdots > m_K \), and \( T_{m_1, \ldots, m_K ; l_1, \ldots, l_K} = 0 \) for \( m_r = m_{r+1} \) \((r = 1, \ldots, K-1)\). Interpreting \( T \) as a transfer matrix, we find for the multi-spin
matrix element an expression in terms of a matrix element of the $n$th power of this transfer matrix,
\[
\langle \uparrow \mid \sigma_{j_1}^+ \ldots \sigma_{j_K}^+ (-\hat{H})^n \sigma_{l_1}^- \ldots \sigma_{l_K}^- \mid \uparrow \rangle = \frac{1}{2^n} (T^n)_{j_1 \ldots j_K; l_1 \ldots l_K},
\]
which generalizes eq. (4.16) to the case of $K$ random walkers.

The presence of just single factor $\Lambda$ in each term of the transfer matrix eq. (4.20) implies that, at each time step, only a single walker moves out of the total $K$. Thus the above matrix element of the $n$th power of $T$ gives the number of all lattice paths of $n$ steps made by $K$ random-turns vicious walkers. We recall that, in the random-turns vicious walkers model, at each time step only a single randomly chosen walker moves one step to the ‘left’ or one step to the ‘right’ with the constraint that two walkers cannot occupy the same site. These random-turns vicious walkers are different from the more common lock-step vicious walkers all of which, at each time step, must move left or right with the same constraint that two walkers cannot occupy the same site [10].

Denoting by $P_n(l_1, \ldots, l_K \rightarrow j_1, \ldots, j_K)$ the number of all admissible configurations in which the $K$ walkers are initially located on the lattice sites $l_1 > l_2 > \cdots > l_K$, and have after $n$ steps arrived at the positions $j_1 > j_2 > \cdots > j_K$, we find the result
\[
G_{j_1, \ldots, j_K; l_1, \ldots, l_K}(n) = \frac{1}{(2K)^n} P_n(l_1, \ldots, l_K \rightarrow RT \mapsto j_1, \ldots, j_K),
\]
where RT stands for random-turns vicious walks.

4.3. Large $n$ limit. Having established a connection of our sandpile model with the random-turns vicious walkers, it is natural to consider the large $n$ limit of discrete conditional probabilities. This is useful for a direct comparison with the random-turns walkers, for both an infinite and a semi-infinite lattice in the horizontal direction.

For definiteness we consider here an infinite lattice in the horizontal direction; a semi-infinite lattice can be considered similarly, and below we outline the results for both cases. Using the relation between the continuous and discrete conditional probabilities, see eq. (3.3), we find from eq. (3.5) in the case of an infinite lattice the representation
\[
G_{j_1, \ldots, j_K; l_1, \ldots, l_K}(n) = \frac{1}{(2\pi)^K} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left( \sum_{m=1}^{K} \cos x_m \right)^n \times \det_{1 \leq r, s \leq K} \left\{ e^{i(l_r-j_s)x_r} \right\}.
\]
We are interested here in the large $n$ limit with the $l_r$’s and $j_s$’s kept fixed. In order to apply the standard saddle-point approximation, we express the first factor of the integrand in the above equation in the form
\[
\exp \{ n \log (\sum_m \cos x_m) \}, \text{ and obtain thereby the following system of saddle-point equations:}
\]
\[
\frac{\sin x_r}{\sum_{m=1}^{K} \cos x_m} = 0, \quad r = 1, \ldots, K.
\] (4.24)

It is evident that the solutions to this system of equations satisfy \( \sin x_r = 0 (r = 1, \ldots, K) \) with the restriction that \( \sum_m \cos x_m \neq 0 \). Requiring that the matrix of second derivatives
\[
\frac{\partial^2}{\partial x_r \partial x_s} \log \left( \sum_{m=1}^{K} \cos x_m \right) = -\frac{\cos x_r}{\sum_{m=1}^{K} \cos x_m} \delta_{rs} - \frac{\sin x_r \sin x_s}{\left( \sum_{m=1}^{K} \cos x_m \right)^2}
\] (4.25)
is a negatively definite matrix for the solution of eq. (4.24), we find that the steepest descent corresponds to the solution for which \( \cos x_r = 1 \) \( (r = 1, \ldots, K) \), i.e. the main contribution to the integrals in eq. (4.23) comes from near the points \( x_r = 0 \) \( (r = 1, \ldots, K) \), similarly to the case of continuous conditional probability considered in sect. 3.2.

Therefore, replacing the first factor of the integrand in eq. (4.23) by its approximation near the origin of the integration variables, \( \text{i.e.} \)
\[
\left( \sum_{m=1}^{K} \cos x_m \right)^n \propto (2\pi)^K K! \times \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_K e^{-\left(n/2K\right)\sum_{m=1}^{K} x_m^2} \prod_{1 \leq r < s \leq K} (x_r - x_s)^2.
\] (4.27)

Clearly this expression is the same as eq. (3.39) with \( t \) replaced by \( n/K \).

It is easy to check that a similar result is valid for a semi-infinite lattice, now using the procedure of sect. 3.3. Hence, the leading terms of the large \( n \) limits of discrete multi-grain probabilities can be obtained from those of the large \( t \) limits of the continuous probabilities simply by the replacement \( t \to n/K \).

We have thus shown that, as \( n \to \infty \) for fixed \( l_r \)'s and \( j_r \)'s, the discrete multi-grain conditional probabilities scale as
\[
G_{j_1, \ldots, j_K; l_1, \ldots, l_K} (n) \sim B_{j_1, \ldots, j_K; l_1, \ldots, l_K} n^{-\gamma}
\] (4.28)

with
\[
B_{j_1, \ldots, j_K; l_1, \ldots, l_K} = K^\gamma A_{j_1, \ldots, j_K; l_1, \ldots, l_K},
\] (4.29)
where the exponent \( \gamma \) and amplitude \( A_{j_1, \ldots, j_K; l_1, \ldots, l_K} \) are given by eqs. (3.13) and (3.14), respectively, in the case of an infinite lattice, and by eqs. (3.24).
and (3.25), respectively, in the case of a semi-infinite lattice. The discrete forms of the conditional probabilities scale exactly as the continuous ones, as they should. These results, in view of eq. (4.22), are also in agreement with the known scaling properties of the random-turns vicious walkers \[13\].

5. Conclusion

In conclusion, we showed that the probability of multiple grains in a two dimensional directed sandpile to preserve their order during an avalanche, displays non-trivial scaling properties that are similar to those of the ‘random-turns’ version of vicious walkers. The conditional probability in the case when the downhill direction of the lattice was transformed into a continuous variable (‘time’), which we found as the solution to a master equation, was found to be the generating function of directed lattice paths in the original lattice. This continuous time conditional probability then provided the connection with the Heisenberg XX spin chain as it was found to be the same as the corresponding many-spin correlation function of the Heisenberg chain. Notice that connection with spin-1/2 variables was based on having a directional lattice with the chosen initial condition. After a toppling on ‘row’ \(n\) there were only sites with a critical or subcritical (critical minus one) number of grains on row \(n+1\), i.e. there were effectively only two possible states per site. For other situations connection would possibly be to spin systems of higher spin. Connection with the Heisenberg chain also established a relation between sandpile models and free fermions. Similarly to the sandpile problem, the spin correlation function was shown to be the generating function of ‘vicious’ quantum trajectories of spins.

Acknowledgements

This research was partially supported by the Russian Foundation for Basic Research, grant 10-01-00600, the Russian Academy of Sciences programme Mathematical Methods in Nonlinear Dynamics and the University of Jyväskylä. A.G.P. was supported by the Alexander von Humboldt Foundation research fellowship.

References

[1] D. Dhar and R. Ramaswamy, Exactly solved model of self-organized critical phenomena, Phys. Rev. Lett. 63 (1989), no. 16, 1659–1662.

[2] D. Dhar, Self-organized critical state of sandpile automaton models, Phys. Rev. Lett. 64 (1990), no. 14, 1613–1616.

[3] D. Dhar and P. Pradhan, Probability distribution of residence times of grains in sandpile models, J. Stat. Mech. Theor. Exp. 2004 (2004), no. 05, P05002, arXiv: cond-mat/0404019.

[4] S. S. Manna, Two-state model of self-organized criticality, J. Phys. A 24 (1991), no. 7, L363–L370.

[5] R. Karmakar, S. S. Manna, and A. L. Stella, Precise toppling balance, quenched disorder, and universality for sandpiles, Phys. Rev. Lett. 94 (2005), no. 8, 088002, arXiv: cond-mat/0312127.
[6] T. Tsuchiya and M. Katori, *Proof of breaking of self-organized criticality in a non-conservative Abelian sandpile model*, Phys. Rev. E 61 (2000), no. 2, 1183–1188.

[7] M. Stapleton and K. Christensen, *Universality class of one-dimensional directed sandpile models*, Phys. Rev. E 72 (2005), no. 6, 066103, arXiv:cond-mat/0506746.

[8] P. Bak, Ch. Tang, and K. Wiesenfeld, *Self-organized criticality*, Phys. Rev. A 38 (1988), no. 1, 364–374.

[9] M. Alava, *Self-organized criticality as a phase transition* (2003), arXiv:cond-mat/0307688.

[10] M. E. Fisher, *Walks, walls, wetting, and melting*, J. Stat. Phys. 34 (1984), no. 5, 667–729.

[11] D. A. Huse and M. E. Fisher, *Commensurate melting, domain walls, and dislocations*, Phys. Rev. B 29 (1984), no. 1, 239–270.

[12] P. J. Forrester, S. N. Majumdar, and G. Schehr, *Non-intersecting Brownian walkers and Yang-Mills theory on the sphere*, Nucl. Phys. B 844 (2011), no. 3, 500–526, arXiv:1009.2362.

[13] P. J. Forrester, *Exact results for vicious walker models of domain walls*, J. Phys. A 24 (1991), no. 1, 203–218.

[14] P. J. Forrester, *Random walks and random permutations*, J. Phys. A 34 (2001), no. 31, L417–L424, arXiv:math/9907037.

[15] M. Katori and H. Tanemura, *Scaling limit of vicious walks and two-matrix model*, Phys. Rev. E 66 (2002), no. 1, 011105, arXiv:cond-mat/0203549.

[16] M. Katori, H. Tanemura, T. Nagao, and N. Komatsuda, *Vicious walks with a wall, noncolliding meanders, and chiral and Bogoliubov-de Gennes random matrices*, Phys. Rev. E 68 (2003), no. 2, 021112, arXiv:cond-mat/0303573.

[17] G. Schehr, S. N. Majumdar, A. Comtet, and J. Randon-Furling, *Exact distribution of the maximal height of p vicious walkers*, Phys. Rev. Lett. 101 (2008), no. 15, 150601, arXiv:0807.0522.

[18] Y. Shilo and O. Biham, *Sandpile models and random walkers on finite lattices*, Phys. Rev. E 67 (2003), no. 6, 066102, arXiv:cond-mat/0303495.

[19] C. Vanderzande and F. Daerden, *Dissipative Abelian sandpiles and random walks*, Phys. Rev. E 63 (2001), no. 3, 030301, arXiv:cond-mat/0101024.

[20] B. Tadić, U. Nowak, K. D. Usadel, R. Ramaswamy, and S. Padlewski, *Scaling behavior in disordered sandpile automata*, Phys. Rev. A 45 (1992), no. 12, 8536–8545.

[21] J. Theiler, *Scaling behavior of a directed sandpile automata with random defects*, Phys. Rev. E 47 (1993), no. 1, 733–734.

[22] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995.

[23] M. L. Mehta, *Random matrices*, 3rd ed., Pure and Applied Mathematics, vol. 142, Elsevier and Academic Press, Amsterdam, 2004.

[24] C. Krattenthaler, A. J. Guttmann, and X. G. Viennot, *Vicious walkers, friendly walkers and Young tableaux: II. With a wall*, J. Phys. A 33 (2000), no. 48, 8835–8866, arXiv:cond-mat/0006367.

[25] N. M. Bogoliubov, *XX0 Heisenberg chain and random walks*, J. Math. Sci. 138 (2006), no. 3, 5636–5643.

[26] N. M. Bogoliubov, *Integrable models for vicious and friendly walkers*, J. Math. Sci. 143 (2007), no. 1, 2729–2737.

[27] N. M. Bogoliubov and C. L. Malyshev, *Correlation functions of the XX Heisenberg magnet and random walks of vicious walkers*, Theor. Math. Phys. 159 (2009), no. 2, 563–574.