Topological Gravity in Seven Dimensions

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ABSTRACT

We obtain new topological gravity in seven dimensions by adding two topological terms to the Einstein-Hilbert action. For certain choice of the coupling constants, these terms may have an origin as the $R^4$ correction to the 3-form field equation of eleven-dimensional supergravity. We derive the full set of the equations of motion, and obtain large classes of solutions including static AdS black holes, squashed seven spheres and $Q^{111}$ spaces.
1 Introduction

There has been considerable interest in topological gauge theories [1] because of their wide application in physics. The most studied example is the three-dimensional one. In addition to the Einstein-Hilbert term, the theory has the Chern-Simons term, given by

\[ S = \frac{1}{\mu} \int d^3x \text{Tr} \left( d\omega \wedge \omega - \frac{2}{3} \omega \wedge \omega \wedge \omega \right), \tag{1} \]

where \( \omega \) can be either a Yang-Mills gauge potential or the connection for gravity. Topological Yang-Mills theory can provide a fundamental interpretation for anyons [2]; it can also generate Lorentz violation dynamically [3]. Topological gravity [4] becomes dynamical with a propagating massive particle, with the mass proportional to the coupling constant \( \mu \). Recently, a cosmological constant is added and the corresponding boundary conformal field theory (CFT) is discussed [5]. The three-dimensional massive topological gravity is conjectured to be unitary for certain parameter region even though the theory has higher derivatives in time [6].

The attention on higher dimensional generalizations is considerably less. The five-dimensional Yang-Mills Chern-Simons term was discussed in [7], but there is no gravity counterpart due to the fact that the holonomy group \( SO(1, 4) \) has no invariant rank-3 symmetric tensor. In seven dimensions, Yang-Mills Chern-Simons terms arise naturally from \( \mathcal{N} = 4 \) supergravity [8]. As in the case of three dimensions, we find that such terms in the gravity sector can be obtained directly from those in the Yang-Mills sector by replacing the gauge potential \( A \) to the connection \( \Gamma \). Moreover, as we shall see later, seven-dimensional topological gravity has a direct origin in eleven-dimensional supergravity, while any higher-dimensional origin of the three-dimensional theory remains unknown.

In section 2, we present the two topological terms in seven dimensions, and discuss their properties. Since they are not manifestly invariant under general coordinate transformation, we find it is more convenient to lift the system to eight dimensions in order to derive the equations of motion (EOMs). We obtain the full set. In section 3, we construct large classes of solutions including static Anti-de Sitter (AdS) black holes, squashed \( S^7 \) and \( Q^{111} \). We emphasize that all the previously-known static (AdS) black holes remain to be solutions when the topological terms are added into the action. This is analogous to three dimensions, where the BTZ black hole remains to be a solution in topological massive gravity. We conclude in section 4.
2 The theory

In seven dimensions, there are two topological terms; they are given by

\[ S_1 = \mu \int \Omega_1^{(7)} = \mu \int \text{Tr}(\Gamma \wedge \Theta + \frac{1}{5} \Gamma^3) \wedge \text{Tr}(\Theta^2) = \mu \int \Omega_1^{(3)} \wedge d\Omega_2^{(3)}, \]

\[ S_2 = \nu \int \Omega_2^{(7)} = \nu \int \text{Tr}(\Theta^2 \wedge \Gamma + \frac{2}{5} \Theta^2 \wedge \Gamma^3 + \frac{1}{3} \Theta \wedge \Gamma \wedge \Theta + \frac{1}{5} \Theta \wedge \Gamma^5 + \frac{1}{35} \Gamma^7), \]

with \( \Omega_1^{(3)} = \text{Tr}(d\Gamma \wedge \Gamma - \frac{2}{5} \Gamma^3) \). Here, \( \Theta \) is the curvature 2-form, defined as \( \Theta \equiv d\Gamma - \Gamma \wedge \Gamma \), and \( \mu, \nu \) are two parameters of length dimension 5. (We rescale the total action by the seven-dimensional Newton constant.) The 3-form \( \Omega_1^{(3)} \) has the same structure as the Chern-Simons term in \( D = 3 \), except that now \( \Gamma \) depends on seven coordinates. \( \Omega_1^{(7)} \) and \( \Omega_2^{(7)} \) are topological in the same sense as \( \Omega_1^{(3)} \) being topological in \( D = 3 \). We can lift the system to \( D = 8 \), with the seven-dimensional spacetime as the boundary. Then, we have

\[ d\Omega_1^{(7)} = Y_1^{(8)} \equiv \text{Tr}(\Theta \wedge \Theta) \wedge \text{Tr}(\Theta \wedge \Theta), \quad d\Omega_2^{(7)} = Y_2^{(8)} \equiv \text{Tr}(\Theta \wedge \Theta \wedge \Theta \wedge \Theta). \]

As we have mentioned earlier, these terms can be derived from the Yang-Mills Chern-Simons terms in \([8]\) by changing the gauge potential to the connection. Note that the Pontryagin term is proportional to \( Y_1^{(8)} - 2Y_2^{(8)} \), corresponding to \( \nu = -2\mu \). In eleven-dimensional supergravity, there is an \( R^4 \) correction to the field equation, namely \( d^* F^{(4)} = \frac{1}{2} F^{(4)} \wedge F^{(4)} + X^{(8)} \), where \( X^{(8)} \) is given by

\[ X^{(8)} \propto Y_1^{(8)} - 4Y_2^{(8)}. \]

Thus for \( \nu = -4\mu \), the topological terms can be obtained from the \( S^4 \) reduction of supergravity in \( D = 11 \), and the coupling constant is proportional to the 4-form M5-brane fluxes. For large fluxes, this topological term dominates the higher-order corrections.

To derive the contribution to the EOMs from the Chern-Simons terms, it is necessary to perform their variation with respect to the metric. These topological terms are not manifestly invariant under the general coordinate transformation, but \( Y_1^{(8)} \) and \( Y_2^{(8)} \) are. We find that a convenient way to derive the variation is to lift the system to eight dimensions. Let us first consider the variation of \( S_1 \). In terms of coordinate components, we have

\[ \int d\Omega_1^{(7)} = \frac{1}{16} \int d^8 x \epsilon^{\mu_1 \mu_2 \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6} R_{\mu_1 \mu_2 \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8} R_{\mu_3 \mu_4 \nu_3 \nu_4 \nu_5 \nu_6} R_{\mu_4 \mu_5 \nu_4 \nu_5 \nu_6 \nu_7} R_{\mu_5 \mu_6 \nu_5 \nu_6 \nu_7 \nu_8}. \]
Here we use Greek letters to denote the eight-dimensional coordinates and Latin letters to represent the seven-dimensional ones hereafter. We adopt the convention $\epsilon_1^{2345678} = 1$.

For an infinitesimal variation of the metric $\delta g$, using the Bianchi identity and the following relation

$$\delta R_{\nu \alpha \beta}^\mu = \delta \Gamma_{\nu \beta ; \alpha}^\mu - \delta \Gamma_{\nu \alpha ; \beta}^\mu,$$

we find that

$$\int d\delta \Omega_1^{(7)} = -\frac{1}{2} \int d^8 x \sqrt{g} \left( \frac{1}{\sqrt{g}^{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8}} R_{\mu_1 \mu_2 \nu_1 \nu_2}^\mu R_{\mu_1 \nu_3 \nu_4}^\mu R_{\mu_4 \nu_5 \nu_6}^\mu \delta \Gamma_{\mu_3 \nu_7}^\mu \delta \Gamma_{\mu_5 \nu_8}^\mu \right),$$

where $\cdot \nu$ denotes a covariant derivative and $\ast$ is the Hodge dual. For simplicity, we have introduced a 1-form current $J = J_\alpha dx^\alpha$. Its components are given by

$$J_\alpha = \frac{1}{\sqrt{g}} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8} R_{\mu_1 \nu_1 \nu_2}^\mu R_{\mu_2 \nu_3 \nu_4}^\mu R_{\mu_4 \nu_5 \nu_6}^\mu \delta \Gamma_{\mu_3 \nu_7}^\mu \delta \Gamma_{\mu_5 \nu_8}^\mu.$$ 

Clearly, we have $d\ast J = -\sqrt{g} J_\alpha \delta \Omega_1^{(7)}$, thus we obtain

$$\delta \Omega_1^{(7)} = \frac{1}{2} \ast J,$$

up to a total derivative term. Now restricting the coordinate indices to seven dimensions only, we have

$$\delta S_1 = 4 \mu \int \text{Tr}(\Theta \wedge \Theta) \wedge \text{Tr}(\Theta \wedge \delta \Gamma).$$

The variation of $S_2$ can be obtained in the same manner, given by

$$\delta S_2 = 4 \nu \int \text{Tr}(\Theta \wedge \Theta \wedge \Theta \wedge \delta \Gamma).$$

Finally, we make use of the variation of the connection

$$\delta \Gamma^i_{mj} = \frac{1}{2} g^{in} (\delta g_{nm;j} + \delta g_{nj;m} - \delta g_{n;m})$$

and after integrating by parts, we obtain the contributions to EOMs from the Chern-Simons terms, given by

$$C_1^{ij} = \frac{\delta S_1}{\sqrt{g} \delta \rho_{ij}} = \frac{\mu}{4 \sqrt{g}} \epsilon_{ijkl} (R_{i_1 j_1 k_1}^i R_{i_2 j_2 l_2}^i R_{i_3 j_3 l_3}^i)_{k \leftrightarrow l},$$

$$C_2^{ij} = \frac{\delta S_2}{\sqrt{g} \delta \rho_{ij}} = \frac{\nu}{4 \sqrt{g}} \epsilon_{ijkl} (R_{i_1 j_1 k_1}^i R_{i_2 j_2 l_2}^i R_{i_3 j_3 l_3}^i)_{k \leftrightarrow l}.$$ 

For the total action $S$, which is the sum of the Einstein-Hilbert action, cosmological constant $\Lambda$ and $S_1 + S_2$, the corresponding full set of EOMs is given by

$$R^{ij} - \frac{1}{2} g^{ij} R + \Lambda g^{ij} + C_1^{ij} + C_2^{ij} = 0.$$
It should be remarked that under a large gauge transformation $\Gamma \to O^{-1}\Gamma O - O^{-1}dO$, the action transforms as $S \to S + \mu v(O) + \nu w(O)$, where

$$v(O) = \int -\frac{1}{6} d\left(\text{Tr}(O^{-1}dO)^3 \wedge \Omega^{(3)}\right); \quad w(O) = -\frac{1}{35} \int \text{Tr}(O^{-1}dO)^7.$$  \hspace{1cm} (15)

The $v$ term is trivial and gives no restriction to the parameter $\mu$, while the $w$ term should be classified by the seventh homotopy group of $SO(1,6)$

$$\pi_7[SO(1,6)] \simeq \pi_7[SO(6)] \simeq \mathbb{Z}. \hspace{1cm} (16)$$

The invariance of $e^{iS}$ requires that

$$\nu = 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \hspace{1cm} (17)$$

This quantization condition is clearly consistent with the M5-brane quantization, since it has a direct origin in $D = 11$. This result is completely different from that in three dimensions, where the $SO(1,2)$ is homotopically trivial and the mass parameter is not quantized. Moreover, since $\nu$ is quantized, $S_2$ will not be renormalized in the quantum theory. This suggests some intriguing properties in the corresponding CFT dual.

3 Solutions

Spherically-symmetric solutions:

Having obtained the full set of EOMs for topological gravity in seven dimensions, we are in the position to construct solutions. It is clear that the maximally-symmetric space(time) is unmodified by the inclusion of the topological terms. The next simplest case is to consider the spherically-symmetric ansatz, given by

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{G(r)} + r^2 d\Omega_5^2. \hspace{1cm} (18)$$

We find that for this ansatz, the contributions from the topological terms $C_1^{ij}$ and $C_2^{ij}$ vanish identically. This implies that the previously-known static (AdS) black holes, charged or neutral, are still solutions when the topological terms are added to the action. This is analogous to three dimensions, where the BTZ black hole is still a solution in massive topological gravity. However the thermodynamic quantities such as the mass and entropy will acquire modifications [9, 10].

$S^3$ bundle over $S^4$: 

We now turn our attention to the Euclidean theory. In three dimensions, there exists a large class of squashed $S^3$ or AdS$_3$ \[11\]. We expect the same in seven dimensions. Without loss of generality, we set $\Lambda = 30$ so that it can give rise to a unit round $S^7$. We first consider the squashed $S^7$ that can be viewed as an $S^3$ bundle over $S^4$. The metric ansatz is given by
\[
d s^2 = \alpha \sum_{i=1}^{3} (\sigma_i - \cos^2(\frac{1}{\alpha} \theta) \tilde{\sigma}_i)^2 + \beta \left( d\theta^2 + \frac{1}{4} \sin^2 \theta \sum_{i=1}^{3} \tilde{\sigma}_i^2 \right). \tag{19}\]
where $\sigma_i$ and $\tilde{\sigma}_i$ are the $SU(2)$ left-invariant 1-forms, satisfying $d\sigma_i = \frac{1}{2} \epsilon^{ijk} \sigma_j \wedge \sigma_k$ and $d\tilde{\sigma}_i = \frac{1}{2} \epsilon^{ijk} \tilde{\sigma}_j \wedge \tilde{\sigma}_k$. The metric is Einstein provided that either $\alpha = \beta = \frac{1}{4}$ or $\alpha = \frac{1}{5} \beta = \frac{9}{100}$. The first case corresponds to the round $S^7$ and the second is a squashed $S^7$ that is also Einstein. Now with the contribution from the topological terms, the EOMs can be reduced to
\[
2\alpha^2 + 4\alpha \beta (7\beta^2 - 2) = 0, \tag{20}\]

together with
\[
\sqrt{\alpha}(\alpha - \beta)^3 (4(10\alpha + \beta)\mu - (55\alpha + 7\beta)\nu) + 2\beta^6 (20\alpha\beta - 4\alpha - \beta) = 0. \tag{21}\]
It is clear from \[20\] that there exists one and only one positive $\alpha$ for any positive $\beta$. The squashing parameter $\gamma \equiv \alpha/\beta$ lies in the range $0 < \gamma < 2 + \frac{3}{\sqrt{2}}$. Note that when $2\mu = 3\nu$, the squashed $S^7$ that is Einstein remains Einstein.

$S^1$ bundle over $\mathbb{C}P^3$:

There is another way of squashing an $S^7$, which can be viewed as an $S^1$ bundle over $\mathbb{C}P^3$. The metric ansatz is given by
\[
ds^2 = \alpha (d\tau + \sin^2 \theta (d\psi + B))^2 + \beta ds^2_{\mathbb{C}P^3},
\]
\[
ds^2_{\mathbb{C}P^3} = d\theta^2 + \sin^2 \theta \cos^2 \theta (d\psi + B)^2 + \sin^2 \theta \left( d\tilde{\theta}^2 + \frac{1}{4} \sin^2 \tilde{\theta} \cos^2 \tilde{\theta} (\sigma_1^2 + \sigma_2^2) \right),
\]
\[
B = \frac{1}{2} \sin^2 \tilde{\theta} \sigma_3. \tag{22}\]
It is of a round $S^7$ when $\alpha = \beta = 1$. In general, the EOMs imply that
\[
\alpha = \beta (8 - 7\beta), \quad 8\mu + \nu + \frac{\beta^3}{10976(\beta - 1)^2 \sqrt{\alpha}} = 0. \tag{23}\]
The squashing parameter $\gamma \equiv \alpha/\beta$ lies in the range $(0, 8)$.

Squashed $Q^{111}$ spaces:
The $Q^{111}$ space is an Einstein-Sasaki space of $U(1)$ bundle over $S^2 \times S^2 \times S^2$. We consider the following ansatz

$$ds^2 = \alpha \left( d\psi + \sum_{i=1}^{3} \cos \theta_i \, d\phi_i \right)^2 + \beta \sum_{i=1}^{3} (d\theta_i^2 + \sin \theta_i^2 \, d\phi_i^2). \tag{24}$$

It is of $Q^{111}$ provided that $\alpha = \frac{1}{2} \beta = 1/16$, and it remains so for $\nu = 0$. In general, we have

$$\alpha = 4\beta(1 - 7\beta), \quad 8(\alpha - \beta)(2\alpha - \beta)\mu + \alpha(2\alpha - 3\beta)\nu + \frac{\beta^5(\alpha - 8\beta + 60\beta^2)}{4\alpha^{3/2}} = 0. \tag{25}$$

Thus the squashing parameter $\gamma \equiv \alpha/\beta$ lies in the range $(0, 4)$. We expect that many of the squashed homogeneous spaces in seven dimensions are now solutions in this new gravity theory, and we shall not enumerate them further.

4 Conclusions

This work is motivated by studying the classical solutions of Einstein-Chern-Simons gravity with asymptotic AdS structure. In seven dimensions, there are two topological Chern-Simons terms, and we obtain the full set of equations of motion. We find that spherically-symmetric solutions are unmodified by the inclusion of these topological terms. We also obtain squashed $S^7$ and $Q^{111}$ spaces, where the squashing parameter is related to the coupling constants of the topological terms. It is intriguing to see that these known squashed homogeneous spaces which appear to have no connection can now be unified under our new gravity theory.

As in three dimensions, our topological gravity should play an important role in exploring the AdS$_7$/CFT$_6$ correspondence. The CFT$_6$ that describes the world-volume theory of multiple M5-branes is yet to be known, and our solutions provide many new gravity dual backgrounds. The quantization condition for one of the coupling constant suggests an unusual property of the CFT$_6$ that is absent in lower dimensions. Additional future directions include a classification of all topological gravities in $(4k + 3)$ dimensions, investigating the linearization of $D = 7$ topological gravity and obtaining the propagating degrees of freedom.

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Seven-Dimensional Gravity with Topological Terms

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ABSTRACT

We construct new seven-dimensional gravity by adding two topological terms to the Einstein-Hilbert action. For certain choice of the coupling constants, these terms may be related to the $R^4$ correction to the 3-form field equation of eleven-dimensional supergravity. We derive the full set of the equations of motion. We find that the static spherically-symmetric black holes are unmodified by the topological terms. We obtain squashed AdS₇, and also squashed seven spheres and $Q^{111}$ spaces in Euclidean signature.
1 Introduction

There has been considerable interest in topological gauge theories [1] because of their wide application in physics. The most studied example is the three-dimensional one. In addition to the Einstein-Hilbert term, the theory has the Chern-Simons term, given by

$$S = \frac{1}{\mu} \int d^3x \text{Tr} (d\omega \wedge \omega + \frac{2}{3} \omega \wedge \omega \wedge \omega),$$

where $\omega$ can be either a Yang-Mills gauge potential or the connection for gravity. Topological Yang-Mills theory can provide a fundamental interpretation for anyons [2]; it can also generate Lorentz violation dynamically [3]. Topologically massive gravity [4] becomes dynamical with a propagating massive particle, with the mass proportional to the coupling constant $\mu$. Recently, a cosmological constant is added and the corresponding boundary conformal field theory (CFT) is discussed [5]. The three-dimensional massive topological gravity is conjectured to be unitary for certain parameter region even though the theory has higher derivatives in time [6].

The attention on higher dimensional generalizations is considerably less. The five-dimensional Yang-Mills Chern-Simons term was discussed in [7], but there is no gravity counterpart due to the fact that the holonomy group $SO(1,4)$ has no invariant rank-3 symmetric tensor. In seven dimensions, Yang-Mills Chern-Simons terms arise naturally from $\mathcal{N} = 4$ supergravity [8]. As in the case of three dimensions, we find that such terms in the gravity sector can be obtained directly from those in the Yang-Mills sector by replacing the gauge potential $A$ to the connection $\Gamma$. As we shall see later, these topological terms in seven dimensions may be related to the anomaly cancelation terms in eleven-dimensional supergravity.

In section 2, we present the two topological terms in seven dimensions, and discuss their properties. Since they are not manifestly invariant under general coordinate transformation, we find it is more convenient to lift the system to eight dimensions in order to derive the equations of motion (EOMs). We obtain the full set. In section 3, we construct large classes of solutions. We find that the static spherically-symmetric black holes are unmodified by the topological terms. This is analogous to three dimensions, where the BTZ black hole remains to be a solution in topologically massive gravity. In Euclidean signature, we obtain squashed $S^7$ and $Q^{111}$ spaces. In particular, one of the squashed seven sphere can be Wick rotated to become squashed AdS7. We conclude in section 4.
2 The theory

In seven dimensions, there are two topological terms; they are given by

\[ S_1 = \tilde{\mu} \int \Omega_1^{(7)} = \tilde{\mu} \int \text{Tr}(\Gamma \land \Theta - \frac{1}{3} \Gamma^3) \land \text{Tr}(\Theta^2) = \tilde{\mu} \int \Omega^{(3)} \land d\Omega^{(3)}, \]

\[ S_2 = \tilde{\nu} \int \Omega_2^{(7)} = \tilde{\nu} \int \text{Tr}(\Theta^3 \land \Gamma - \frac{2}{5} \Theta^2 \land \Gamma^3 - \frac{1}{5} \Theta \land \Theta \land \Gamma + \frac{1}{5} \Theta \land \Gamma^5 - \frac{1}{35} \Gamma^7), \]

with \( \Omega^{(3)} = \text{Tr}(d\Gamma \land \Gamma + \frac{3}{2} \Gamma^3) \). Here, \( \Theta \) is the curvature 2-form, defined as \( \Theta \equiv d\Gamma + \Gamma \land \Gamma \), and \( \tilde{\mu}, \tilde{\nu} \) are two parameters of length dimension 5. (We rescale the total action by the seven-dimensional Newton constant.) The 3-form \( \Omega^{(3)} \) has the same structure as the Chern-Simons term in \( D = 3 \), except that now \( \Gamma \) depends on seven coordinates. \( \Omega_1^{(7)} \) and \( \Omega_2^{(7)} \) are topological in the same sense as \( \Omega^{(3)} \) being topological in \( D = 3 \). We can lift the system to \( D = 8 \), with the seven-dimensional spacetime as the boundary. Then, we have

\[ d\Omega_1^{(7)} = Y_1^{(8)} \equiv \text{Tr}(\Theta \land \Theta) \land \text{Tr}(\Theta \land \Theta), \quad d\Omega_2^{(7)} = Y_2^{(8)} \equiv \text{Tr}(\Theta \land \Theta \land \Theta \land \Theta). \]

As we have mentioned earlier, these terms can be derived from the Yang-Mills Chern-Simons terms in [8] by changing the gauge potential to the connection. Note that the Pontryagin term is proportional to \( Y_1^{(8)} - 2Y_2^{(8)} \), corresponding to \( \tilde{\nu} = -2\tilde{\mu} \). In eleven-dimensional supergravity, there is an \( R^4 \) correction to the field equation, namely \( d{*F}^{(4)} = \frac{1}{2} F^{(4)} \land F^{(4)} + X^{(8)} \), where \( X^{(8)} \) is given by

\[ X^{(8)} \propto Y_1^{(8)} - 4Y_2^{(8)}. \]

Thus for \( \tilde{\nu} = -4\tilde{\mu} \), the topological terms can be obtained from the \( S^4 \) reduction of supergravity in \( D = 11 \), and the coupling constant is proportional to the 4-form M5-brane fluxes. For large fluxes, this topological term dominates the higher-order corrections.

To derive the contribution to the EOMs from the Chern-Simons terms, it is necessary to perform their variation with respect to the metric. These topological terms are not manifestly invariant under the general coordinate transformation, but \( Y_1^{(8)} \) and \( Y_2^{(8)} \) are. We find that a convenient way to derive the variation is to lift the system to eight dimensions. Let us first consider the variation of \( S_1 \). In terms of coordinate components, we have

\[ \int d\Omega_1^{(7)} = \frac{1}{16} \int d^8 x \epsilon^{\mu_1\mu_2\mu_3\mu_4\nu_5\nu_6\nu_7\nu_8} R_{\mu_1}^{\mu_2} R_{\mu_2}^{\mu_3} R_{\mu_3}^{\mu_4} R_{\mu_4}^{\nu_5} R_{\nu_5}^{\nu_6} R_{\nu_6}^{\nu_7} R_{\nu_7}^{\nu_8}. \]
Here we use Greek letters to denote the eight-dimensional coordinates and Latin letters to represent the seven-dimensional ones hereafter. We adopt the convention $\epsilon^{12345678} = 1$.

For an infinitesimal variation of the metric $\delta g$, using the Bianchi identity and the following relation

$$
\delta R_{\nu\alpha\beta} = \delta\Gamma_{\nu\beta;\alpha} - \delta\Gamma_{\nu\alpha;\beta},
$$

we find that

$$
\int d\delta\Omega^{(7)} = -\frac{1}{2} \int d^8x \sqrt{g} \left( \frac{1}{\sqrt{g}} \epsilon^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8} R_{\mu_1\nu_1\nu_2}^{\mu_2} R_{\mu_3\nu_3\nu_4}^{\mu_4} R_{\mu_5\nu_5\nu_6}^{\mu_6} \delta \Gamma_{\mu_7}^{\mu_8} \right)_{\nu_8}
$$

(6)

where ";" denotes a covariant derivative and $*$ is the Hodge dual. For simplicity, we have introduced a 1-form current $J = J_\alpha dx^\alpha$. Its components are given by

$$
J_\alpha = \frac{1}{\sqrt{g}} \epsilon^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8} R_{\mu_1\nu_1\nu_2}^{\mu_2} R_{\mu_3\nu_3\nu_4}^{\mu_4} R_{\mu_5\nu_5\nu_6}^{\mu_6} \delta \Gamma_{\mu_7}^{\mu_8}. \tag{7}
$$

Clearly, we have $d*J = -\frac{1}{\sqrt{g}} J_\alpha d^8x$, Thus we obtain

$$
\delta\Omega^{(7)} = \frac{1}{2} *J, \tag{8}
$$

(7)

up to a total derivative term. Now restricting the coordinate indices to seven dimensions only, we have

$$
\delta S_1 = 4\tilde{\mu} \int \text{Tr}(\Theta \wedge \Theta) \wedge \text{Tr}(\Theta \wedge \delta\Gamma).
$$

(9)

The variation of $S_2$ can be obtained in the same manner, given by

$$
\delta S_2 = 4\tilde{\nu} \int \text{Tr}(\Theta \wedge \Theta \wedge \Theta \wedge \delta\Gamma).
$$

(10)

Finally, we make use of the variation of the connection

$$
\delta\Gamma^i_{\mu j} = \frac{1}{2} g^{in} (\delta g_{nm;j} + \delta g_{nj;m} - \delta g_{nl;n}) \tag{11}
$$

(11)

and after integrating by parts, we obtain the contributions to EOMs from the Chern-Simons terms, given by

$$
C^{ij}_1 = \frac{\delta S_1}{\sqrt{g} g_{ij}} = \frac{\mu}{4 \sqrt{g}} \left[ \epsilon^{i j_1 j_2 j_3 j_4 j_5 j_6} (R_{i j_1 j_2}^{i j_3 j_4} R_{i j_1 j_2}^{i j_3 j_4} R_{i j_1 j_2}^{i j_3 j_4})_{j_1 j_2 j_3 j_4 j_5 j_6} \right],
$$

$$
C^{ij}_2 = \frac{\delta S_2}{\sqrt{g} g_{ij}} = \frac{\nu}{4 \sqrt{g}} \left[ \epsilon^{i j_1 j_2 j_3 j_4 j_5 j_6} (R_{i j_1 j_2}^{i j_3 j_4} R_{i j_1 j_2}^{i j_3 j_4} R_{i j_1 j_2}^{i j_3 j_4})_{j_1 j_2 j_3 j_4 j_5 j_6} \right]. \tag{12}
$$

(12)

For the total action $S$, which is the sum of the Einstein-Hilbert action, cosmological constant $\Lambda$ and $S_1 + S_2$, the corresponding full set of EOMs is given by

$$
R^{ij} - \frac{1}{2} g^{ij} R + \Lambda g^{ij} + C^{ij}_1 + C^{ij}_2 = 0. \tag{13}
$$

(13)
It should be remarked that under a large gauge transformation \( \Gamma \rightarrow O \Gamma O^{-1} - dO \), the action transforms as \( S \rightarrow S + \tilde{\mu} v(O) + \tilde{\nu} w(O) \), where

\[
v(O) = \int \frac{1}{8} d(Tr(dO O^{-1})^3 \wedge \Omega(3)) ; \quad w(O) = \frac{1}{36} \int Tr(dO O^{-1})^7. \tag{15}
\]

The \( v \) term is trivial and gives no restriction to the parameter \( \tilde{\mu} \), while the \( w \) term should be classified by the seventh homotopy group of \( SO(1,6) \)

\[
\pi_7[SO(1,6)] \simeq \pi_7[SO(6)] \simeq \mathbb{Z}. \tag{16}
\]

The invariance of \( e^{iS} \) requires that

\[
64\pi^4 \tilde{\nu} = 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \tag{17}
\]

This result is completely different from that in three dimensions, where the \( SO(1,2) \) is homotopically trivial and the mass parameter is not quantized. Moreover, since \( \tilde{\nu} \) is quantized, \( S_2 \) will not be renormalized in the quantum theory. This suggests some intriguing properties in the corresponding CFT dual.

### 3 Solutions

**Spherically-symmetric solutions:**

Having obtained the full set of EOMs for topological gravity in seven dimensions, we are in the position to construct solutions. It is clear that the maximally-symmetric space(time) is unmodified by the inclusion of the topological terms. The next simplest case is to consider the spherically-symmetric ansatz, given by

\[
ds^2 = -F(r) dt^2 + \frac{dr^2}{G(r)} + r^2 d\Omega_5^2. \tag{18}
\]

We find that for this ansatz, the contributions from the topological terms \( C^{ij}_1 \) and \( C^{ij}_2 \) vanish identically. This implies that the previously-known static (AdS) black holes, charged or neutral, are still solutions when the topological terms are added to the action. This is analogous to three dimensions, where the BTZ black hole is still a solution in massive topological gravity. However the thermodynamic quantities such as the mass and entropy will acquire modifications \[9, 10\].

As we shall discuss presently, there also exist squashed AdS\(_7\) solutions.

**S\(^3\) bundle over S\(^4\):**
We now turn our attention to the Euclidean theory. In three dimensions, there exists a large class of squashed \( S^3 \) or AdS\(_3 \) [11]. We expect the same in seven dimensions. Without loss of generality, we set \( \Lambda = 30 \) so that it can give rise to a unit round \( S^7 \). We first consider the squashed \( S^7 \) that can be viewed as an \( S^3 \) bundle over \( S^4 \). The metric ansatz is given by

\[
\begin{align*}
\text{ds}^2 &= \alpha \sum_{i=1}^{3} (\sigma_i - \cos^2(\frac{1}{2}\theta) \tilde{\sigma}_i)^2 + \beta \left( d\theta^2 + \frac{1}{4} \sin^2 \theta \sum_{i=1}^{3} \tilde{\sigma}_i^2 \right),
\end{align*}
\]

where \( \sigma_i \) and \( \tilde{\sigma}_i \) are the \( SU(2) \) left-invariant 1-forms, satisfying \( d\sigma_i = \frac{1}{2} \epsilon^{ijk} \sigma_j \wedge \sigma_k \) and \( d\tilde{\sigma}_i = \frac{1}{2} \epsilon^{ijk} \tilde{\sigma}_j \wedge \tilde{\sigma}_k \). The metric is Einstein provided that either \( \alpha = \beta = \frac{1}{4} \) or \( \alpha = \frac{1}{5} \beta = \frac{9}{100} \). The first case corresponds to the round \( S^7 \) and the second is a squashed \( S^7 \) that is also Einstein. Now with the contribution from the topological terms, the EOMs can be reduced to

\[
\begin{align*}
2\alpha^2 + 4\alpha \beta(7\beta - 2) - \beta^2 &= 0, \\
\sqrt{\alpha(\alpha - \beta)^3} (4(10\alpha + \beta)\tilde{\mu} - (55\alpha + 7\beta)\nu) + 2\beta^6(20\alpha\beta - 4\alpha - \beta) &= 0.
\end{align*}
\]

It is clear from (20) that there exists one and only one positive \( \alpha \) for any positive \( \beta \). The squashing parameter \( \gamma \equiv \alpha/\beta \) lies in the range \( 0 < \gamma < 2 + \frac{3}{\sqrt{2}} \). Note that when \( 2\tilde{\mu} = 3\tilde{\nu} \), the squashed \( S^7 \) that is Einstein remains Einstein.

**\( S^1 \) bundle over \( \mathbb{C}P^3 \):**

There is another way of squashing an \( S^7 \), which can be viewed as an \( S^1 \) bundle over \( \mathbb{C}P^3 \). This example can be generalized to Minkowskian signature to give rise to squashed AdS\(_7 \) [12]. The metric ansatz is given by

\[
\begin{align*}
\text{ds}^2 &= \alpha (d\tau + \sin^2 \theta (d\psi + B))^2 + \beta \text{ds}^2_{\mathbb{C}P^3}, \\
\text{ds}^2_{\mathbb{C}P^3} &= d\theta^2 + \sin^2 \theta \cos^2 \theta (d\psi + B)^2 + \sin^2 \theta \left( d\tilde{\theta}^2 + \frac{1}{4} \sin^2 \tilde{\theta} \cos^2 \tilde{\theta} \sigma_3^2 \right), \\
B &= \frac{1}{2} \sin^2 \tilde{\theta} \sigma_3.
\end{align*}
\]

It is of a round \( S^7 \) when \( \alpha = \beta = 1 \). In general, the EOMs imply that

\[
\alpha = \beta(8 - 7\beta), \quad 8\tilde{\mu} + \tilde{\nu} + \frac{\beta^3}{10976(\beta - 1)^2\sqrt{\alpha}} = 0.
\]

The squashing parameter \( \gamma \equiv \alpha/\beta \) lies in the range \( (0, 8) \).

**Squashed \( Q^{111} \) spaces:**
The $Q^{111}$ space is an Einstein-Sasaki space of $U(1)$ bundle over $S^2 \times S^2 \times S^2$. We consider the following ansatz

$$ds^2 = \alpha \left( d\psi + \sum_{i=1}^{3} \cos \theta_i d\phi_i \right)^2 + \beta \sum_{i=1}^{3} (d\theta_i^2 + \sin \theta_i^2 d\phi_i^2).$$

(24)

It is of $Q^{111}$ provided that $\alpha = \frac{1}{2} \beta = 1/16$, and it remains so for $\tilde{\nu} = 0$. In general, we have

$$\alpha = 4\beta(1 - 7\beta), \quad 8(\alpha - \beta)(2\alpha - \beta)\tilde{\mu} + \alpha(2\alpha - 3\beta)\tilde{\nu} + \frac{\beta^5(\alpha - 8\beta + 60\beta^2)}{4\alpha^{3/2}} = 0.$$

(25)

Thus the squashing parameter $\gamma \equiv \alpha/\beta$ lies in the range $(0, 4)$. We expect that many of the squashed homogeneous spaces in seven dimensions are now solutions in this new gravity theory, and we shall not enumerate them further.

4 Conclusions

This work is motivated by studying the classical solutions of Einstein-Chern-Simons gravity with asymptotic AdS structure. In seven dimensions, there are two topological Chern-Simons terms, and we obtain the full set of equations of motion. We find that spherically-symmetric solutions are unmodified by the inclusion of these topological terms. We also obtain squashed AdS$_7$, and squashed $S^7$ and $Q^{111}$ spaces in Euclidean signature, where the squashing parameter is related to the coupling constants of the topological terms. It is intriguing to see that these known squashed homogeneous spaces which appear to have no connection can now be unified under our new gravity theory.

As in three dimensions, our topological gravity should play an important role in exploring the AdS$_7$/CFT$_6$ correspondence. The CFT$_6$ that describes the world-volume theory of multiple M5-branes is yet to be known, and our solutions provide many new gravity dual backgrounds. The quantization condition for one of the coupling constant suggests an unusual property of the CFT$_6$ that is absent in lower dimensions. Additional future directions include a classification of all topological gravities in $(4k + 3)$ dimensions, investigating the linearization of $D = 7$ topological gravity and obtaining the propagating degrees of freedom.

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