Hamiltonian-minimal Lagrangian submanifolds in toric varieties

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Hamiltonian minimality (\textit{H-minimality} for short) for Lagrangian submanifolds is a symplectic analogue of Riemannian minimality. A Lagrangian immersion is said to be \textit{H-minimal} if the variations of its volume along all Hamiltonian vector fields are zero. This notion was introduced in the paper [1] of Oh in connection with the well-known \textit{Arnold conjecture} on the number of fixed points of a Hamiltonian symplectomorphism.

In [2] and [3] the authors defined and studied a family of \textit{H-minimal} Lagrangian submanifolds of $\mathbb{C}^m$ arising from intersections of real quadrics. Here we extend this construction to define \textit{H-minimal} submanifolds in toric varieties.

The initial data of the construction is an intersection of $m-n$ Hermitian quadrics in $\mathbb{C}^m$:

$$Z = \{ z = (z_1, \ldots, z_m) \in \mathbb{C}^m : \sum_{k=1}^{m} \gamma_{jk} |z_k|^2 = \delta_j \text{ for } j = 1, \ldots, m-n \}. \quad (1)$$

We assume that this intersection is non-empty, non-degenerate, and rational. These conditions can be expressed in terms of the coefficient vectors $\gamma_k = (\gamma_{1k}, \ldots, \gamma_{m-n,k})^t \in \mathbb{R}^{m-n}$ as follows:

a) $\delta \in \mathbb{R}^{\langle \gamma_1, \ldots, \gamma_m \rangle}$ (that is, $\delta$ is in the cone generated by $\gamma_1, \ldots, \gamma_m$);

b) if $\delta \in \mathbb{R}^{\langle \gamma_{l1}, \ldots, \gamma_{lp} \rangle}$, then $p \geq m-n$;

c) the vectors $\gamma_1, \ldots, \gamma_m$ span a lattice $L$ of full rank in $\mathbb{R}^{m-n}$.

Under these conditions, $Z$ is a smooth $(m+n)$-dimensional submanifold of $\mathbb{C}^m$, and

$$T_\Gamma = \{ (e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, e^{2\pi i \langle \gamma_m, \varphi \rangle}, \varphi \in \mathbb{R}^{m-n}) \} = \mathbb{R}^{m-n}/L^*$$

is an $(m-n)$-dimensional torus. We also define

$$D_\Gamma = (\frac{1}{2} L^*)/L^* \cong (\mathbb{Z}_2)^{m-n}.$$ 

We note that $D_\Gamma$ embeds canonically as a subgroup of $T_\Gamma$.

Let $R \subset Z$ be the subset of real points, which can be written by the same equations in real coordinates:

$$R = \{ u = (u_1, \ldots, u_m) \in \mathbb{R}^m : \sum_{k=1}^{m} \gamma_{jk} u_k^2 = \delta_j \text{ for } j = 1, \ldots, m-n \}. \quad (2)$$

We ‘spread’ $R$ by the action of $T_\Gamma$, that is, we consider the set of $T_\Gamma$-orbits passing through $R$. More precisely, we consider the map

$$j : R \times T_\Gamma \longrightarrow \mathbb{C}^m, \quad (u, \varphi) \mapsto u \cdot \varphi = (u_1 e^{2\pi i \langle \gamma_1, \varphi \rangle}, \ldots, u_m e^{2\pi i \langle \gamma_m, \varphi \rangle})$$

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and observe that \( j(\mathcal{R} \times T_G) \subset \mathcal{L} \). We let \( D_G \) act on \( \mathcal{R} \times T_G \) diagonally; this action is free since it is free on the second factor. The quotient space

\[
N = \mathcal{R} \times D_G \mathcal{T}_G
\]

is an \( m \)-dimensional manifold.

**Theorem 1** [2]. The map \( j: \mathcal{R} \times T_G \rightarrow \mathbb{C}^m \) induces an H-minimal Lagrangian immersion \( i: N \rightarrow \mathbb{C}^m \).

The intersection of quadrics (1) is invariant with respect to the diagonal action of the standard torus \( \mathbb{T}^m \subset \mathbb{C}^m \). The quotient \( \mathcal{L}/\mathbb{T}^m \) is identified with the set of non-negative solutions of the system of linear equations \( \sum_{k=1}^{m} \gamma_k y_k = \delta \). This set may be described as a convex \( n \)-dimensional polyhedron

\[
P = \left\{ \mathbf{x} \in \mathbb{R}^n : (a_i, \mathbf{x}) + b_i \geq 0 \text{ for } i = 1, \ldots, m \right\},
\]

where \( (b_1, \ldots, b_m) \) is any solution, and the vectors \( a_1, \ldots, a_m \in \mathbb{R}^n \) form the transpose of a basis of solutions of the homogeneous system \( \sum_{k=1}^{m} \gamma_k y_k = 0 \). We refer to \( P \) as the associated polyhedron of the intersection of quadrics (1).

Let \( \Lambda \) denote the lattice of rank \( n \) spanned by \( a_1, \ldots, a_m \). The polyhedron (2) is called a Delzant polyhedron if, for any vertex \( \mathbf{x} \in P \), the vectors \( a_{i_1}, \ldots, a_{i_k} \) normal to the facets meeting at \( \mathbf{x} \) form a basis of the lattice \( \Lambda \).

**Theorem 2** [3]. The immersion \( i: N \rightarrow \mathbb{C}^m \) is an embedding if and only if the associated \( P \) is a Delzant polyhedron.

We now consider two sets of quadrics:

\[
\mathcal{L}_\Gamma = \left\{ \mathbf{z} \in \mathbb{C}^m : \sum_{k=1}^{m} \gamma_k |z_k|^2 = c \right\}, \quad \mathcal{L}_\Delta = \left\{ \mathbf{z} \in \mathbb{C}^m : \sum_{k=1}^{m} \delta_k |z_k|^2 = d \right\},
\]

\( \gamma_k, c, \delta_k, d \in \mathbb{R}^{m-n}, \) \( \gamma_k, c, \delta_k, d \in \mathbb{R}^{m-\ell} \) such that \( \mathcal{L}_\Gamma, \mathcal{L}_\Delta, \) and \( \mathcal{L}_\Gamma \cap \mathcal{L}_\Delta \) satisfy conditions a)–c) above. Assume also that the polyhedra associated with \( \mathcal{L}_\Gamma, \mathcal{L}_\Delta, \) and \( \mathcal{L}_\Gamma \cap \mathcal{L}_\Delta \) are Delzant polyhedra.

We define the real intersections of quadrics \( \mathcal{R}_\Gamma, \mathcal{R}_\Delta, \) the tori \( T_\Gamma \cong \mathbb{T}^{m-n}, T_\Delta \cong \mathbb{T}^{m-\ell}, \) and the groups \( D_\Gamma \cong \mathbb{Z}_2^{m-n}, D_\Delta \cong \mathbb{Z}_2^{m-\ell} \) as above.

Let us consider the toric variety \( \hat{V} \) obtained as the symplectic quotient of \( \mathbb{C}^m \) by the torus corresponding to the first set of quadrics: \( V = \mathcal{L}_\Gamma/T_\Gamma \). It is a Kähler manifold of real dimension \( 2n \). The quotient space \( \mathcal{R}_\Gamma/D_\Gamma \) is the set of real points of \( V \); it has dimension \( n \). Consider the subset of \( \mathcal{R}_\Gamma/D_\Gamma \) defined by the second set of quadrics:

\[
\mathcal{J} = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/D_\Gamma;
\]

then \( \dim \mathcal{J} = n + \ell - m \). Finally, we define an \( n \)-dimensional submanifold of \( V \):

\[
N = \mathcal{J} \times D_\Delta T_\Delta.
\]

**Theorem 3.** \( N \) is an H-minimal Lagrangian submanifold in \( V \).

**Proof.** Let \( \hat{V} \) be the symplectic quotient of \( V \) by the torus corresponding to the second set of quadrics, that is, \( \hat{V} = (V \cap \mathcal{L}_\Delta)/T_\Delta = (\mathcal{L}_\Gamma \cap \mathcal{L}_\Delta)/(T_\Gamma \times T_\Delta) \). It is a toric manifold of real dimension \( 2(n + \ell - m) \). The submanifold of real points

\[
\hat{N} = N/T_\Delta = (\mathcal{R}_\Gamma \cap \mathcal{R}_\Delta)/(D_\Gamma \times D_\Delta) \hookrightarrow (\mathcal{L}_\Gamma \cap \mathcal{L}_\Delta)/(T_\Gamma \times T_\Delta) = \hat{V}
\]

is the fixed point set of the complex conjugation, and hence it is a totally geodesic submanifold. In particular, \( \hat{N} \) is a minimal submanifold in \( \hat{V} \). According to Corollary 2.7 in [4], \( N \) is an H-minimal submanifold in \( V \).
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