THEORETICAL ANALYSIS OF A DISCRETE POPULATION BALANCE MODEL FOR SUM KERNEL

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Abstract. The Oort-Hulst-Safronov equation, shortened as OHS is a relevant population balance model. Its discrete form, developed by Dubovski is the main focus of our analysis. The existence and density conservation are established for the coagulation rate $V_{i,j} \leq (i + j)$, $\forall i, j \in \mathbb{N}$. Differentiability of the solutions is investigated for the kernel $V_{i,j} \leq i^\alpha + j^\alpha$ where $0 \leq \alpha \leq 1$. The article finally deals with the uniqueness result that requires the boundedness of the second moment.

1. Introduction

The coagulation is defined as the process when clusters of mass $i$ and $j$ ($i$ and $j$ mers) merge together to generate a $(i + j)$-mer. Coagulation processes have a plethora of real world applications, including collision of asteroids [1], red blood cell aggregation [5], helium bubble formation in nuclear materials [6], colloidal chemistry [7], formation of Saturn’s rings [2], among many others.

This paper discusses a discrete model, i.e., a model for which the properties of the particles namely size, are described by a discrete variable $i \in \mathbb{N}$, known as the Safronov-Dubovski (S-D) coagulation equation [8]. The equation seems to have been first proposed by Dubovski [3] in 1999, long after the introduction of the continuous version, called the Oort-Hult-Safronov (OHS) equation [9, 10], in the context of coagulation of particles in the celestial phenomena.

The S-D model is defined for $t \in [0, \infty)$ as

$$\frac{d\psi_i(t)}{dt} = \delta_{i\geq 2}\psi_{i-1}(t) \sum_{j=1}^{i-1} jV_{i-1,j}\psi_j(t) - \psi_i(t) \sum_{j=1}^{i} jV_{i,j}\psi_j(t) - \sum_{j=i}^{\infty} V_{i,j}\psi_i(t)\psi_j(t), \quad i \in \mathbb{N}, \quad (1)$$

where $\delta_P = 1$ if $P$ is true, and zero otherwise.

The Eqn. (1) deals with the change in concentration of $i$-clusters, and, consequently, the focus of our study is the qualitative behavior of $\psi_i(t)$ to the initial value problems defined by this system with initial conditions

$$\psi_i(0) = \psi_{0i} \geq 0. \quad (2)$$

From a physical point of view, the equations in system (1) are the rate equations describing the time-dependent behaviour of a system of particles whose dynamics can be described as follows: (a) particles of size $i$ are produced when a $i - 1$ cluster is struck by a particle of size $j \leq i - 1$: the result of this collision is that the smaller $j$-cluster is pulverized into $j$ particles of size 1, each of which attaches itself to different particles of size $i - 1$ to form particles of size $i$; this is described mathematically by the first expression in the right part of (1):

$$\delta_{i\geq 2}\psi_{i-1}(t) \sum_{j=1}^{i-1} jV_{i-1,j}\psi_j(t);$$

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(b) particles of size $i$ are destroyed either by being impacted by smaller clusters and thus growing to clusters of size $i+1$, by the mechanism just described, resulting in the second term in the right hand side of (1):

$$-\psi_i(t) \sum_{j=1}^{i} j V_{i,j} \psi_j(t),$$

or by being themselves the smaller clusters in a collision with a larger $j$-cluster, in which case it is the $i$-cluster who is pulverized into a number $i$ of 1-clusters who will then attach to $j$-clusters to produce $(j+1)$-clusters, which results in the last term in the right hand side of (1):

$$- \sum_{j=1}^{\infty} V_{i,j} \psi_i(t) \psi_j(t).$$

The parameters $V_{i,j}$ for $i \neq j$, called the coagulation kernel, are the rate constants for the reaction between clusters of sizes $i$ and $j$, and are assumed to be time independent, non-negative, and symmetric, i.e., $V_{i,j} = V_{j,i}$. As discussed in [3], the rate $V_{i,i}$ is equal to half of the collisions’ rate for the particles of size $i$.

There are various physical properties of the solution that can be investigated with the help of its moments, which is also an extremely useful tool to handle the related mathematical problems. The $r$th moment of a solution $\psi = (\psi_i)$ of (1) is defined by

$$\mu_r(\psi(\cdot)) = \mu_r(\cdot) := \sum_{i=1}^{\infty} i^r \psi_i(\cdot).$$

(3)

Putting $r = 0$ gives the zeroth moment, denoted as $\mu_0(\cdot)$, which is the total number of particles per unit volume. Taking $r = 1$ in [3] we get the first moment, $\mu_1(\cdot)$, which can be physically interpreted as (proportional to) the mass of the system per unit volume. We expect the mass to be a conserved quantity, i.e. $\mu_1(t) = \mu_1(0)$, for kernels with slowly increasing rate of coagulation. Though the physical relevance of the second moment has not been much discussed in the literature, it can be interpreted as the energy dissipated in the process [11]. Bagland [12] established that the solution for S-D model when

$$\lim_{j \to \infty} \frac{V_{i,j}}{j} = 0, \quad i, j \geq 1, \quad \psi_0 \in L_1$$

exists for $t \in [0, \infty)$. Davidson [8] in 2014, presented a global existence theorem, mass conservation result and uniqueness theorem for three types of kernels, namely:

$$j V_{i,j} \leq M \quad \text{for} \quad j \leq i; \quad V_{i,j} \leq C_V h_i h_j, \quad \text{with} \quad \frac{h_j}{i} \to 0 \quad \text{as} \quad i \to \infty; \quad \text{and} \quad V_{i,j} \leq C_V \forall i, j, C_V \in \mathbb{R}^+. $$

Mass is proven to be conserved for $V_{i,j} \leq C_V i^{1/2} j^{1/2}$ and the solution is shown to be unique in the third case, i.e., the bounded kernel is considered. In general, for large classes of kernels such as product kernel, mass is not a conserved quantity, see [3] for the continuous OHS equation. This phenomenon is a consequence of part of the mass of the cluster distribution ($\psi_i$) being transported into larger and larger values of $i$, and part of it being lost to the limit $i \to \infty$, physically interpreted as an infinite size cluster, or gel, in a process called gelation. One can find results on gelation for coagulation type models in, for instance, [13], [14] Chapter 9.

In this paper, we study the existence of mass conserving solutions to the initial value problem [11], [12] with rate kernels satisfying $V_{i,j} \leq (i+j)$, $\forall i, j \in \mathbb{N}$, and initial condition with finite mass. To establish regularity of the solutions, we need to consider a balance between a more restrictive class of kernels $V_{i,j} \leq i^\alpha + j^\alpha$, for $\alpha \in [0, 1]$ and all $i, j \in \mathbb{N}$, and initial condition with some finite higher moment. The uniqueness result is also established for a restrictive classes of kernels, i.e., $V_{i,j} \leq C_V (i+j)$ and $V_{i,j} \leq C_V \min \{i^\eta, j^\eta\}$, $0 \leq \eta \leq 2$, $\forall i, j \in \mathbb{N}$, $C_V \in \mathbb{R}^+$. The boundedness of a higher moment in finite time played a significant role in proving uniqueness. Let us now define some basic notation and notions that are needed throughout.
The set of finite mass sequences is defined by
\[ X = \{ z = (z_k) : ||z|| < \infty \}, \] (4)
with
\[ ||z|| := \sum_{k=1}^{\infty} k|z_k|, \] (5)
where \((X, || \cdot ||)\) is a Banach space. For analysis, we often consider the non-negative cone
\[ X^+ = \{ \psi = (\psi_i) \in X : \psi_i \geq 0 \}. \] (6)

**Definition 1.1.** The solution \( \psi = (\psi_i) \) of the initial value problem (1)–(2) on \( [0, T) \) where \( 0 < T < \infty \), is a function \( \psi : [0, T) \to X^+ \) with following properties

\[ (a) \quad \forall \ i, \ \psi_i \text{ is continuous.} \]
\[ (b) \quad \int_0^t \sum_{j=1}^{\infty} V_i,j \psi_j(s) ds < \infty \text{ for every } i \text{ and } \forall \ 0 \leq t < T \]
\[ (c) \quad \forall \ i \text{ and } \forall \ 0 \leq t < T \]
\[ \psi_i(t) = \psi_{0i} + \int_0^t \left( \sum_{j=1}^{i-1} jV_{i-1,j} \psi_j - \sum_{j=1}^{i} jV_{i,j} \psi_j - \sum_{j=i}^{\infty} V_{i,j} \psi_j \right)(s) ds, \]
(7)
where \( \delta_P = 1 \) if \( P \) is true, and is zero otherwise.

The article is organized in six sections. The second section discusses the preliminary results required to establish the main results of the work. Section 3 deals with the existence of solution and its corollary. Further, in Section 4 density conservation is shown for all the solutions of the given equation and the regularity result is proved in Section 5. Finally, the statement and proof of the uniqueness theorem are part of Section 6.

2. A finite dimensional truncation

Our general approach in this paper consists in considering a finite \( n \)-dimensional truncation of (1) and, after obtaining appropriate a priori estimates for its solutions, passing to the limit \( n \to \infty \) and getting corresponding results for (1).

In this section we introduce a truncated system of the S-D model and study some useful results about the moments of its solutions. The finite \( n \)-dimensional truncated system for the equation (1) that we shall consider corresponds to assuming that no particles with size larger than \( n \) can exist initially or be formed by the dynamics. Thus, for the phase variable \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \) the system is
\[ \frac{d\psi_i}{dt} = \Psi^n_i(\psi), \quad \text{for } 1 \leq i \leq n, \] (8)
where
\[ \Psi^n_1(\psi) := -V_{1,1} \psi_1^2 - \sum_{j=1}^{n-1} V_{1,j} \psi_j \]
(9)
\[ \Psi^n_i(\psi) := \psi_{i-1} \sum_{j=1}^{i-1} jV_{i-1,j} \psi_j - \psi_i \sum_{j=1}^{i} jV_{i,j} \psi_j - \psi_i \sum_{j=i}^{\infty} V_{i,j} \psi_j, \quad \text{for } 2 \leq i \leq n - 1, \]
(10)
\[ \Psi^n_n(\psi) := \psi_{n-1} \sum_{j=1}^{n-1} jV_{n-1,j} \psi_j. \]
(11)

From what was stated above the initial conditions of interest are
\[ \psi_i(0) = \psi^0_i \geq 0, \quad \text{for } 1 \leq i \leq n. \] (12)

It can be observed here that we have truncated the last sum up to \( n - 1 \) not \( n \). This was done to make sure that the truncation conserves mass which will be beneficial in proving the existence result. The solutions to (8)–(12) exists and are unique which can be proved using the fact that the right side contains polynomials and the Picard-Lindelöf theorem. The solutions are also
non-negative, established by the addition of positive $\varepsilon$ to the right-part of all equations. Now, if $\psi^f$ satisfies $\psi^f(t_0) > 0$ for some $t_0 \in \mathbb{R}^+ - \{0\}$ and $\psi^f_j(t_0) = 0$ for every $j \in \{1, \ldots, n\}$, then $\frac{d}{dt} \psi^f_j(t_0) > 0$. Finally, taking $\varepsilon \to 0$ gives the non-negativity (see [15 Theorem III-4-5]).

As is usual in the analysis of coagulation type systems, estimates about the time evolution of moments of solutions are of paramount importance. For the truncated system, and in a way analogous to the $r^{th}$ moments defined in [3], we consider the quantities

$$
\mu^n_g(t) := \sum_{i=1}^n g_i \psi_i(t),
$$

where $g = (g_i)$ is a non-negative sequence. The following result on the evolution of $\mu^n_g$ will be relevant:

**Lemma 1.** Let $\psi = (\psi_i)_{i \in \{1, \ldots, n\}}$ be a solution of (8)–(12) defined in an open interval $I$ containing 0. Let $g = (g_i)$ be a real sequence. Then

$$
\frac{d\mu^n_g}{dt} = \sum_{i=1}^n g_i \Psi^n_i(\psi)
$$

$$
= \left(-g_1 V_{i,1} \psi_1^2 - g_1 \psi_1 \sum_{j=1}^{n-1} V_{i,j} \psi_j\right) +
+ \sum_{i=2}^{n-1} g_i \left(\psi_{i-1} \sum_{j=1}^{i-1} j V_{i-1,j} \psi_j - \psi_{i-1} \sum_{j=1}^{i-1} j V_{i-1,j} \psi_j - \psi_{i-1} \sum_{j=i}^{n-1} V_{i,j} \psi_j\right)
+ g_n \psi_{n-1} \sum_{j=1}^{n-1} j V_{n-1,j} \psi_j.
$$

Rewriting the right-hand side by collecting together the first and forth terms, the second and fifth terms, and the third and sixth we obtain

$$
\frac{d\mu^n_g}{dt} = \sum_{i=1}^{n-1} \sum_{j=1}^i (g_{i+1} - g_i) j V_{i,j} \psi_i \psi_j - \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} g_i V_{i,j} \psi_i \psi_j,
$$

and now altering the order of variables in the first equation and using the symmetry of the rate coefficients, $V_{j,i} = V_{i,j}$, we finally conclude (13).\[\square\]

For $g_p := (i^p)$ consider the simplified notation $\mu^n_{g_p} := \mu^n_{g_p}$. It is clear from (14) that, for all $t \in I$,

$$
\frac{d\mu^n_{g_{i^p}}(t)}{dt} \leq 0
$$

and thus $\mu^n_{g_p}(t) \leq \mu^n_{g_p}(0)$, for all $t \in I \cap \{t \geq 0\}$. This $a$ priori bound implies that non negative solutions of the truncated systems (8)–(11) are globally defined forward in time, i.e., $I \supset [0, +\infty)$. Taking $g = g_1$ in (14) we immediately conclude that, for all $t$,

$$
\frac{d\mu^n_{g_1}(t)}{dt} = 0,
$$

which means that solutions to the truncated system conserve mass. For further reference, this is stated in the next lemma.
**Lemma 2.** Solutions to Cauchy problems for the truncated systems (8)-(11) are globally defined forward in time and mass conserving, i.e., satisfy

$$\mu^n_i(t) = \mu^n_i(0), \quad \forall t \geq 0.$$  

(17)

For the existence proof let us consider $\nu^n_m(t)$ defined as in [16] by

$$\nu^n_m(t) := \sum_{i=m}^n i\psi^m_i(t),$$

(18)

where $\psi^n = (\psi^n_1, \ldots, \psi^n_n)$ is a solution of the $n$-dimensional truncated system (8)-(11). From these expressions we immediately obtain

$$\frac{d\nu^n_m(t)}{dt} = \left( \sum_{i=m}^{n-1} \sum_{j=1}^i jV_{i,j}\psi^n_i \psi^n_j + m\psi^n_{m-1} \sum_{j=1}^{m-1} jV_{m-1,j}\psi^n_j - \sum_{i=m}^{n-1} \sum_{j=i}^{n-1} iV_{i,j}\psi^n_i \psi^n_j \right)(t).$$

(19)

Assume now $2m < n$ and consider the function $\kappa^n_m(\cdot)$ defined by

$$\kappa^n_m(t) := \sum_{i=m}^{2m} i\psi^n_i + 2m \sum_{i=2m+1}^n \psi^n_i,$$

(20)

where, again, $\psi^n = (\psi^n_1, \ldots, \psi^n_n)$ is a solution of the $n$-dimensional truncated system. Then, after a few algebraic manipulations, we get

$$\frac{d\kappa^n_m(t)}{dt} = \sum_{i=m}^{2m} i\psi^n_i + 2m \sum_{i=2m+1}^n \psi^n_i$$

$$= m\psi^n_{m-1}(t) \sum_{j=1}^{m-1} jV_{m-1,j}\psi^n_j(t) + \sum_{i=m}^{2m-1} \psi^n_i(t) \sum_{j=1}^i jV_{i,j}\psi^n_j(t) -$$

$$- \sum_{i=m}^{2m-1} \sum_{j=i}^{n-1} i\psi^n_i \psi^n_j V_{i,j}(t) - 2m \sum_{i=2m+1}^n \sum_{j=i}^{n-1} V_{i,j}\psi^n_i \psi^n_j(t).$$

(21)

Finally, so that we can take $n \to \infty$ (or, eventually, only on a subsequence $n_k \to \infty$), we make use of the following lemma.

**Lemma 3.** Take $\psi_0 = (\psi_{0i}) \in X^+$ and, for each $n \in \mathbb{N}$, consider the point $\psi^n_0 \in X^+$ defined by $\psi^n_0 = (\psi_{01}, \psi_{02}, \ldots, \psi_{0m}, 0, 0, \ldots)$ and let it be identified with the point of $\mathbb{R}^n$ obtained by discarding the $j$th components, for $j > n$. Let $\psi^n$ be the solution of the $n$-dimensional truncated system (8)-(11) when $V_{i,j} \leq (i+j)$ with initial condition $\psi^n(0) = \psi^n_0$ such that (17) holds, then $\psi^n$ is relatively compact in $C([0,T])$.

**Proof.** Using the truncated system (8)-(11) and mass conservation of this system (see (17)), it can be shown that $\exists$ a constant $C > 0$ such that $\forall \, n \geq i \geq 1$,

$$\sup_{t \geq 0} \left( \psi^n_i(t) + \left| \frac{d\psi^n_i(t)}{dt} \right| \right) \leq C \mu_1(0)^2.$$

Thus, Ascoli theorem gives the intended result.

Now, we have gathered all the required information to proceed with the existence results.

### 3. Existence Result for the Cauchy Problem

We can now prove the first main result of the paper: the existence of global solutions of the Cauchy problem (1)-(2).

**Theorem 4.** Let, $V_{i,j}$ be nonnegative, symmetric for the exchange of $i$ with $j$, and satisfy $V_{i,j} \leq (i+j), \, \forall \, i,j,$ and let $\psi_0 = (\psi_{0i}) \in X^+, \mu_1(0) < \infty$. Then, $\exists$ a non-negative solution of (1)-(2) defined globally.
Lemma 3. As we stated above, following (12), the initial value problem (8)–(12) has a unique solution, \( \psi^n = (\psi^n_i)_{1 \leq i \leq n} \), which is globally defined, non-negative and, by Lemma 2, density conserving. By defining \( \psi^n_i(t) = 0 \) when \( i > n \) we can consider \( \psi^n_i(t) \) as an element of \( X^+ \), for all \( t \), and thus

\[
\|\psi^n_i(t)\| = \sum_{i=1}^{\infty} i \psi^n_i(t) = \sum_{i=1}^{n} i \psi^n_i(t) = \sum_{i=1}^{n} i \psi^n_{0i} \leq \sum_{i=1}^{\infty} i \psi^n_{0i} = \|\psi^n_i\|. \quad (22)
\]

By Lemma 3 and (17) for each \( n \), \( \exists \) a subsequence of \( \psi^n \) (not relabelled) & a function \( \psi_i : [0, \infty) \to \mathbb{R} \), are of bounded variation on each subset of \( [0, \infty) \), such that \( \psi^n_i(t) \) converges to \( \psi_i(t) \) as \( n \) approaches \( \infty \), for every \( t \in \mathbb{R}^+ \). Thus \( \forall t \geq 0 \),

\[
\psi_i(t) \geq 0 \quad \text{and} \quad ||\psi_i(t)|| \leq ||\psi^n_i||. \quad (23)
\]

Our goal is to prove that this limit function \( \psi \) is a mild solution of the initial value problem (1)–(2), i.e., fulfills the conditions in Definition 1.1. This will be done by passing to the limit \( n \to \infty \) in the integrated version of the truncated problem (8)–(12), namely

\[
\psi^n_i(t) = \psi_{0i} + \int_0^t \left( \psi^n_{i-1}(s) \sum_{j=1}^{i-1} j V_{i-j, \psi^n_j(s)} - \psi^n_i(s) \sum_{j=1}^i j V_{i-j, \psi^n_j(s)} - \sum_{j=1}^{n-1} j V_{i-j, \psi^n_j(s)} \right) ds. \quad (24)
\]

To do this, and also to satisfy condition 2 in Definition 1.1 we need to prove that, for every fixed \( i \in \mathbb{N}, T \geq 0, \) and \( \varepsilon > 0 \), there exists \( m \) and \( N_0 \), with \( N_0 > m \geq i \), such that, for all \( n > N_0 \),

\[
\int_0^T \nu^n_m(t) dt \leq \varepsilon, \quad (25)
\]

where \( \nu^n_m \) was defined in (18). This can be achieved by integrating (19) in \([0, t]\) and using (21) to yield

\[
\nu^n_m(t) = \nu^n_m(0) + \int_0^t \left( \sum_{i=m+1}^{n-1} \sum_{j=1}^i j V_{i-j, \psi^n_j(s)} \psi^n_i(s) + m \psi^n_{m-1} \sum_{j=1}^{m-1} j V_{m-j, \psi^n_j} - \sum_{i=m+1}^{n-1} \sum_{j=i}^{n-1} j V_{i-j, \psi^n_j(s)} \psi^n_i(s) \right) ds
\]

\[
= \nu^n_m(0) + \kappa^n_m(t) - \kappa^n_m(0) + \int_0^t \left( \psi^n_{2m}(s) \sum_{j=1}^{2m} j V_{2m-j, \psi^n_j(s)} + \sum_{i=m+1}^{n-1} \sum_{j=1}^i j V_{i-j, \psi^n_j(s)} \psi^n_i(s) - \sum_{i=m+1}^{n-1} \sum_{j=i}^{n-1} j V_{i-j, \psi^n_j(s)} \psi^n_i(s) \right) ds.
\]

Some algebraic manipulations of the second double sum above provides

\[
\sum_{i=m+1}^{n-1} \sum_{j=1}^i j V_{i-j, \psi^n_j(s)} = \sum_{i=m+1}^{n-1} \sum_{j=1}^{2m} j V_{i-j, \psi^n_j(s)} + \sum_{i=m+1}^{n-1} \sum_{j=m+1}^{n-1} j V_{i-j, \psi^n_j(s)} \psi^n_i(s).
\]

Substituting this into the above expression for \( \nu^n_m(t) \) gives

\[
\nu^n_m(t) = \nu^n_m(0) + \kappa^n_m(t) - \kappa^n_m(0) + \int_0^t \left( \psi^n_{2m}(s) \sum_{j=1}^{2m} j V_{2m-j, \psi^n_j(s)} + \sum_{i=m+1}^{n-1} \sum_{j=1}^{n-1} j V_{i-j, \psi^n_j(s)} \psi^n_i(s) \right) ds. \quad (26)
\]
By (22), (23), and the pointwise convergence of \( \psi^n_i \) to \( \psi_i \) we conclude that, for all \( t \in [0,T] \), \( \forall \varepsilon > 0 \), \( \forall p > \frac{4|V_0|}{\varepsilon} \), \( \exists N_0 : \forall n > N_0 \),

\[
\sum_{i=1}^{p-1} |\psi^n_i(t) - \psi_i(t)| = \sum_{i=1}^{p} |\psi^n_i(t) - \psi_i(t)| + \sum_{i=p}^{\infty} |\psi^n_i(t) - \psi_i(t)| < \frac{\varepsilon}{2} + \frac{2}{p} \|\psi_0\| < \varepsilon,
\]

that enables us to take \( n \to \infty \) in the definition of \( \kappa^n_m(t) \) in (20) and yields

\[
\kappa^n_m(t) = \lim_{n \to \infty} \sum_{i=m}^{2m} i\psi(t) + \sum_{i=2m+1}^{\infty} \psi_i(t) =: \kappa_m(t) \leq \sum_{i=m}^{\infty} i\psi(t),
\]

and so \( \lim_{m \to \infty} \kappa_m(t) = 0 \) and \( |\kappa_m(t)| \leq \|\psi_0\| \), for all \( t \in [0,T] \). Therefore, \( \forall \varepsilon > 0 \), \( \exists M, N_0 \) with \( N_0 > M \), such that, \( \forall m > M, n \geq N_0 \) and \( n \geq 2m + 1 \),

\[
\kappa^n_m(t) \leq \frac{1}{2} \varepsilon,
\]

and

\[
\tau^n_m(0) \leq \frac{1}{2} \varepsilon.
\]

By (28) and (29) and using the assumption \( V_{ij} \leq (i + j) \) we can estimate the right-hand side of (26) as follows (redefining)

\[
x^n_m(t) \leq \varepsilon + \int_0^t \left( \sum_{i=m}^{2m} \sum_{j=1}^{\infty} j(i + j) \psi_i^n(s) \psi_j^n(s) + \sum_{i=2m+1}^{\infty} \sum_{j=i}^{\infty} (i + j) \psi_i^n(s) \psi_j^n(s) \right) ds
\]

\[
\leq \varepsilon + \int_0^t \left( \sum_{i=m}^{2m} \sum_{j=1}^{\infty} i j \psi_i^n(s) \psi_j^n(s) + \sum_{i=2m+1}^{\infty} \sum_{j=i}^{\infty} i \psi_i^n(s) \psi_j^n(s) \right) ds
\]

\[
\leq \varepsilon + \int_0^t \left( \sum_{i=m}^{\infty} i \psi_i^n(s) \sum_{j=1}^{n} j \psi_j^n(s) + \sum_{i=m}^{n} i \psi_i^n(s) \sum_{j=1}^{n} j \psi_j^n(s) \right) ds
\]

\[
\leq \varepsilon + 6 \|\psi_0\| \int_0^t \nu^n_m(s) ds.
\]

Hence, thanks to Gronwall’s lemma, we get, for all \( t \in [0,T] \),

\[
\nu^n_m(t) \leq k_1 \varepsilon
\]

where \( k_1 = e^{\delta T} \), which implies that \( \forall \varepsilon > 0 \), \( \exists M, N_0 \) with \( N_0 > M \), such that, \( \forall m > M, n \geq N_0 \) and \( n \geq 2m + 1 \),

\[
\int_0^t \nu^n_m(s) ds \leq \varepsilon k_1 T, \quad \text{for all } t \in [0,T].
\]

Since, \( \psi^n_i(t) \) is point-wise convergent to \( \psi_i(t) \), the above expression entails that, for all \( \varepsilon > 0 \), there exists \( M \) such that, for all \( m > M \), we have

\[
\int_0^T \sum_{i=m}^{\infty} i \psi_i(t) dt \leq \varepsilon.
\]

Hence, when \( V_{ij} \leq (i + j) \), for all \( i \geq 1 \),

\[
\int_0^T \sum_{j=1}^{\infty} V_{ij} \psi_j(t) dt < \infty,
\]

thus establishing (b) in Definition 1.1.
Now, for every fixed $i$, take $n > i$ sufficiently large and, for any $\ell$ such that $i < \ell < n - 1$, write (24) as

$$\left| \psi^n_i(t) - \psi_i(0) - \int_0^t \left( \psi^n_{i-1}(s) \sum_{j=1}^{i-1} jV_{i-1,j} \psi^n_j - \psi^n_i(s) \sum_{j=1}^i jV_{i,j} \psi^n_j(s) - \psi^n_i(s) \sum_{j=i}^\ell V_{i,j} \psi^n_j(s) \right) ds \right|$$

$$= \psi^n_i(s) \int_0^t \sum_{j=i+1}^{n-1} V_{i,j} \psi^n_j(s) ds \leq 2\|\psi_0\| \int_0^t \nu_{\ell+1}(s) ds.$$  

Thus, from (31), for all $\varepsilon > 0$, there exists $M$ such that, for all $\ell + 1 > M$ and all $n$ sufficiently large, the right hand side can be bounded above by $2\|\psi_0\|k_1T$. Considering that each sum in the left hand side has a fixed and finite number of terms, that $\psi^n_j(s) \to \psi_j(s)$ pointwise as $n \to \infty$, and each of the three terms inside the integral is bounded by $2\|\psi_0\|^2$, we can use the dominated convergence theorem and take $n \to \infty$ to conclude that, for every $\varepsilon > 0$, there exists $M$ such that, for all $\ell \geq M$, we have

$$\left| \psi_i(t) - \psi_i(0) - \int_0^t \left( \psi_{i-1}(s) \sum_{j=1}^{i-1} jV_{i-1,j} \psi_j - \psi_i(s) \sum_{j=1}^i jV_{i,j} \psi_j(s) - \psi_i(s) \sum_{j=i}^\ell V_{i,j} \psi_j(s) \right) ds \right|$$

$$\leq 2\varepsilon\|\psi_0\|k_1T.$$  

Hence, by the arbitrariness of $\varepsilon$, we can let $\ell \to \infty$ and conclude that $\psi = (\psi_i)$ satisfy (17), which completes the proof. \(\square\)

Next we establish that the subsequence $\psi^{n_k}$ of solutions to the truncated system which converges to the solution $\psi$ of (10)–(12) actually does so in the strong topology of $X$, uniformly for $t$ in compact subsets of $[0, \infty)$.

**Corollary 4.1.** Let $\psi^{n_k}$ be the pointwise convergent subsequence of solutions to (10)–(12). Then, $\psi^{n_k} \to \psi$ in $X$ uniformly on compact subsets of $[0, \infty)$.

**Proof.** To prove this, we prove that $\psi^{n_k}_i(t) \to \psi_i(t)$ for each $i$ uniformly on the compact subsets of $[0, \infty)$. For this, let $n_k$ be $n$ and for each $I < m$

$$\xi^m_n(t) := e^{-t} \left[ \psi^m_1(t) - \sum_{i=1}^{m-1} i\psi^m_i(t) + (2m + 2)\mu^n_1(0)^2 \right].$$  

(33)

Now, differentiating (33) gives

$$\frac{d\xi^m_n(t)}{dt} = e^{-t} \left[ \frac{d\mu^n_1(t)}{dt} - \sum_{i=1}^{m-1} i \frac{d\psi^m_i(t)}{dt} \right] - e^{-t} \left[ \mu^n_1(t) - \sum_{i=1}^{m-1} i\psi^m_i(t) + (2m + 2)\mu^n_1(0)^2 \right]$$

(34)

where

$$\frac{d}{dt} \sum_{i=1}^{m-1} i\psi^m_i(t) = \sum_{i=1}^{m-2} jV_{i,j} \psi^m_i \psi^m_j - \sum_{i=1}^{m-1} n_{i,j} \psi^m_i \psi^m_j - (m - 1)\psi^m_{m-1} \sum_{j=1}^{m-1} jV_{m-1,j} \psi^m_j.$$  

(35)

Using the above expression, Lemma 2 and $V_{i,j} \leq (i + j) \forall i, j$ in (34), one can obtain

$$\frac{d\xi^m_n(t)}{dt} \leq e^{-t} \left[ \sum_{i=1}^{m-1} \sum_{j=i}^{m-1} j(i + j)\psi^m_i \psi^m_j + (m - 1)\psi^m_{m-1} \sum_{j=1}^{m-1} j(m - 1 + j)\psi^m_j - 2\mu^n_1(0)^2 - 2m\mu^n_1(0)^2 \right].$$

Some simplifications guarantee that

$$\frac{d\xi^m_n(t)}{dt} \leq 0, \quad n \geq m, \quad t \in [0, T].$$

Hence, $\xi^m_n(t) \to \xi_m(t)$ uniformly on compact subsets of $[0, T]$ where

$$\xi_m(t) := e^{-t} \left[ \psi_1(t) - \sum_{i=1}^{m-1} i\psi_i(t) + (2m + 2)\mu_1(0)^2 \right].$$
Let, $K \subset [0, \infty)$ be compact and $t_n \to t$ in $K$, then
\[
\lim_{n \to \infty} \|\psi^n(t_n)\| = \lim_{n \to \infty} \sum_{i=1}^{\infty} i\psi_i^n(t_n) = \sum_{i=1}^{\infty} i\psi_i(t) = \|\psi(t)\|
\]
which ensures that $\|\psi^n\| \to \|\psi\|$, in $C(K, X)$.

\section{4. All Solutions Conserve Density}

In this section we prove that, under the assumption on the rate coefficients we have been using, all solutions of \eqref{(1)}-\eqref{(2)} conserve density.

Let $\psi = (\psi_i) \in X^+$ be a solution of \eqref{(7)} in $[0, T]$. Multiplying each equation in \eqref{(7)} by $g_i$ and adding from $i = 1$ to $n$, we have, after some algebraic manipulations, for all $t \in [0, T]$,
\[
\sum_{i=1}^{n} g_i \psi_i(t) - \sum_{i=1}^{n} g_i \psi_0 i = \int_{0}^{t} \sum_{i=1}^{n} \sum_{j=i}^{\infty} (jg_{i+1} - jg_i - g_j)V_{i,j} \psi_i(s) \psi_j(s) ds \\
- \int_{0}^{t} \sum_{j=1}^{n} \sum_{i=n+1}^{\infty} g_j V_{i,j} \psi_i(s) \psi_j(s) ds - \int_{0}^{t} g_{n+1} \psi_n(s) \sum_{j=1}^{n} jV_{n,j} \psi_j(s) ds. \tag{36}
\]

We start by observing that, taking $g_i \equiv i$ in \eqref{(36)} we conclude that
\[
\sum_{i=1}^{n} i\psi_i(t) \leq \sum_{i=1}^{n} i\psi_0 i \leq \|\psi_0\|,
\]
and, as this inequality is valid for all $n$, we can take the limit as $n \to \infty$ and conclude the \textit{a priori} bound $\|\psi(t)\| \leq \|\psi_0\|$.

We now use \eqref{(36)} to prove that, under the assumed conditions on $V_{i,j}$ all solutions conserve density:

\textbf{Theorem 5.} Let $V_{i,j} \leq (i + j)$ for all $i$ and $j$. Let $\psi = (\psi_i) \in X^+$ be a solution of the Safronov-Dubovski equation \eqref{(7)}. Then the total density of $\psi$ is constant.

\textbf{Proof.} Let $A \in \mathbb{N}$ be fixed, and consider the sequence $(g^A_i) \in \ell^\infty$ defined by
\[
g^A_i = i \wedge A, \tag{37}
\]
where $a \wedge b = \min\{a, b\}$. Then
\[
jg^A_{i+1} - jg^A_i - g^A_j = \begin{cases} -A & \text{on } \{(i, j) : A \leq i \leq j \leq n\}, \\
0 & \text{on } \{(i, j) : 1 \leq i \leq A - 1 \text{ and } i \leq j \leq n\},
\end{cases}
\]
and \eqref{(36)} becomes, for $n > A$,
\[
\sum_{i=1}^{n} g^A_i \psi_i(t) - \sum_{i=1}^{n} g^A_i \psi_0 i = \tag{38}
\]
\[
= -\int_{0}^{t} A \sum_{j=A}^{n} \sum_{i=A}^{j} V_{i,j} \psi_i(s) \psi_j(s) ds \tag{39}
\]
\[
- \int_{0}^{t} \left( \sum_{j=1}^{n} \sum_{i=n+1}^{\infty} jV_{i,j} \psi_i(s) \psi_j(s) + A \sum_{j=A+1}^{n} \sum_{i=n+1}^{\infty} V_{i,j} \psi_i(s) \psi_j(s) \right) ds \tag{40}
\]
\[
- \int_{0}^{t} A \psi_n(s) \sum_{j=1}^{n} jV_{n,j} \psi_j(s) ds. \tag{41}
\]
Thus, by (43) and (44) we conclude that the integrand function in (40) is bounded by 4:

\[ A \sum_{j=A}^{n} \sum_{i=A}^{j} V_{i,j} \psi_{i} \psi_{j} \leq A \sum_{j=A}^{n} \psi_{j} \sum_{i=A}^{j} i \psi_{i} + A \sum_{j=A}^{n} j \psi_{j} \sum_{i=A}^{j} \psi_{i} \]

\[ = A \sum_{j=A}^{n} \frac{1}{j} j \psi_{j} \sum_{i=A}^{j} i \psi_{i} + A \sum_{j=A}^{n} j \psi_{j} \sum_{i=A}^{j} \frac{1}{j} i \psi_{i} \]

\[ \leq 2 \sum_{j=A}^{n} j \psi_{j} \sum_{i=A}^{j} i \psi_{i} \]

\[ \leq 2 \sum_{j=A}^{\infty} \sum_{i=A}^{\infty} i \psi_{i}. \]  

Thus, \( \psi \in X^{+} \) implies that (42) converges to zero as \( A \to \infty \). Furthermore, since (42) is bounded above by \( 2\|\psi_{0}\|^{2} \), the dominated convergence theorem implies that, for all \( \varepsilon > 0 \) there exists \( A_{0} \) such that, for all \( n > A \geq A_{0} \) the absolute value of (39) is smaller than \( \varepsilon \).

Consider now (40). For the first double sum, observe that for \( n > A \),

\[ \sum_{j=1}^{A} \sum_{i=n+1}^{\infty} j V_{i,j} \psi_{i} \psi_{j} \leq \sum_{j=1}^{A} j \psi_{j} \sum_{i=n+1}^{\infty} i \psi_{i} + \sum_{j=1}^{A} j^{2} \psi_{j} \sum_{i=n+1}^{\infty} \psi_{i} \]

\[ \leq \|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i} + \sum_{j=1}^{A} j^{2} \psi_{j} \sum_{i=n+1}^{\infty} i \psi_{i} \]

\[ \leq \|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i} + \sum_{j=1}^{A} j \psi_{j} \sum_{i=n+1}^{\infty} i \psi_{i} \]

\[ \leq \|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i} + \|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i} \]

\[ \leq 2\|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i}. \]  

(43)

For the second double sum in (40) we have a similar estimate:

\[ A \sum_{j=A+1}^{n} \sum_{i=n+1}^{\infty} V_{i,j} \psi_{i} \psi_{j} \leq A \sum_{j=A+1}^{n} \psi_{j} \sum_{i=n+1}^{\infty} i \psi_{i} + A \sum_{j=A+1}^{n} j \psi_{j} \sum_{i=n+1}^{\infty} \psi_{i} \]

\[ \leq \frac{A}{A+1} \sum_{j=A+1}^{n} j \psi_{j} \sum_{i=n+1}^{\infty} i \psi_{i} + A \sum_{j=A+1}^{n} j \psi_{j} \sum_{i=n+1}^{\infty} \frac{1}{n+1} i \psi_{i} \]

\[ \leq \|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i} + \frac{A}{n+1} \|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i} \]

\[ \leq 2\|\psi_{0}\| \sum_{i=n+1}^{\infty} i \psi_{i}. \]  

(44)

Thus, by (43) and (44) we conclude that the integrand function in (40) is bounded by \( 4\|\psi_{0}\|^{2} \) and converges pointwise to zero as \( n \to \infty \), for each fixed \( A \). Hence, again by the dominated convergence theorem we conclude that, as previously, for all \( \varepsilon > 0 \), there exists \( A_{0} \) such that, for all \( n > A_{0} \), the absolute value of the integral (40) is smaller than \( \varepsilon \).
Finally, let us consider (41)

\[ A\psi_n \sum_{j=1}^{n} jV_{n,j} \psi_j \leq A\psi_n \sum_{j=1}^{n} j(n+j) \psi_j \]

\[ \leq 2A\psi_n \sum_{j=1}^{n} j \psi_j \]

\[ \leq 2\|\psi_0\| A\psi_n. \]  

(45)

Clearly, for each fixed \( A \) (45) converges to zero as \( n \to \infty \) and it is bounded above by \( A\|\psi_0\|^2 \), and so the dominated convergence theorem implies that, for every \( \varepsilon > 0 \), there exists \( A_0 = A_0(\varepsilon) \) such that, for any fixed \( A > A_0 \), there exists \( n_0 = n_0(\varepsilon, A) \) such that, for all \( n > n_0 \vee A \), the absolute value of (41) is smaller than \( \varepsilon \).

To estimate (38) observe that, for every \( n > A \), we can write

\[ \sum_{i=1}^{n} g_i^A \psi_i(t) - \sum_{i=1}^{n} g_i^A \psi_0 i \geq \left| \sum_{i=1}^{A} i \psi_i(t) - \sum_{i=1}^{A} i \psi_0 i \right| - A \left| \sum_{i=A+1}^{n} \psi_i(t) - \sum_{i=A+1}^{n} \psi_0 i \right|, \]

and thus,

\[ \sum_{i=1}^{A} i \psi_i(t) - \sum_{i=1}^{A} i \psi_0 i \leq \sum_{i=A+1}^{\infty} i \psi_i(t) + \sum_{i=A+1}^{\infty} i \psi_0 i + \sum_{i=1}^{n} g_i^A \psi_i(t) - \sum_{i=1}^{n} g_i^A \psi_0 i. \]  

(46)

Now, for every \( \varepsilon > 0 \) there exists \( A_0 \) such that, for all \( A > A_0 \), each of the first two sums in the right hand side of (46) can be made smaller than \( \frac{3}{5} \varepsilon \), and since the estimates of (39)-(41) obtained previously allow us to have the last term in the right hand side of (46) is smaller than \( \frac{3}{5} \varepsilon \), we conclude that

\[ \forall \varepsilon > 0, \exists A_0 : \forall A > A_0, \sum_{i=1}^{A} i \psi_i(t) - \sum_{i=1}^{A} i \psi_0 i < \varepsilon, \]

which proves the result. \( \square \)

5. Differentiability

This section is devoted in proving that the solution of the S-D model is first-order differentiable, if the rate coefficients satisfy \( V_{i,j} \leq i^\alpha + j^\alpha \) for \( \alpha \in [0, 1] \). This requires the boundedness of \( (\alpha + 1)\)-moments of the solutions and an invariance result, which are proved below in Lemma 6 and Theorem 7, respectively.

**Lemma 6.** Let the non-negative kernel \( V_{i,j} \) satisfy \( V_{i,j} \leq i^\alpha + j^\alpha \), for all \( i, j \geq 1 \), and for some fixed \( 0 \leq \alpha \leq 1 \). For any \( T \in (0, \infty) \), let \( \psi \) be a solution to (1)-(2) in \([0, T]\) with initial condition \( \psi_0 \in X^+ \). If the \( (\alpha + 1)\)-moment of \( \psi_0, \mu_{1+\alpha}(\psi_0) \), is bounded, then the \( (\alpha + 1)\)-moment of \( \psi(t) \) is also bounded for all \( t \in [0, T] \).

**Proof.** Let \( \psi = (\psi_i) \) be a solution of (1)-(2) in \([0, T]\) with initial condition \( \psi_0 \). By (7), adding the components from \( i = 1 \) to \( n \), we have, for all \( t \in [0, T] \),

\[ \sum_{i=1}^{n} i^{1+\alpha} \psi_i(t) + \int_0^t \sum_{i=1}^{\infty} \sum_{j=1}^{i} i^{1+\alpha} V_{i,j} \psi_i(s) \psi_j(s) ds + \int_0^t (n+1)^{1+\alpha} \psi_n(s) \sum_{j=1}^{n} j V_{n,j} \psi_j(s) ds \]

\[ = \sum_{i=1}^{n} i^{1+\alpha} \psi_0 i + \int_0^t \sum_{i=1}^{n} \sum_{j=1}^{i} (j+1)^{1+\alpha} - j^{1+\alpha} V_{i,j} \psi_i(s) \psi_j(s) ds. \]  

(47)

Due to the non-negativity of solutions, (47) implies that

\[ \sum_{i=1}^{n} i^{1+\alpha} \psi_i(t) \leq \sum_{i=1}^{n} i^{1+\alpha} \psi_0 i + \int_0^t \sum_{i=1}^{n} \sum_{j=1}^{i} (j+1)^{1+\alpha} - j^{1+\alpha} V_{i,j} \psi_i(s) \psi_j(s) ds. \]  

(48)
Since $\alpha \in [0,1]$ and $i \geq 1$ we have $(i+1)^{1+\alpha} - i^{1+\alpha} \leq (1 + \alpha)i^\alpha + \frac{(1+\alpha)^\alpha}{2i^\alpha}$, and so,

$$j((i+1)^{1+\alpha} - i^{1+\alpha})V_{i,j} \leq (1 + \alpha)j^2\alpha + (1 + \alpha)j^{1+\alpha}\alpha + \frac{(1 + \alpha)^\alpha}{2i}j^\alpha + \frac{(1 + \alpha)^\alpha}{2i}j^{1+\alpha},$$

from where, using $\sum_{i=1}^n i\psi_i(s) \leq \|\psi_0\|$, we obtain

$$\sum_{i=1}^n \sum_{j=1}^i (j(i+1)^{1+\alpha} - j^{1+\alpha})V_{i,j}\psi_i(s)\psi_j(s) \leq \frac{1}{2}(1 + \alpha)\|\psi_0\|^2 + \frac{1}{2}(1 + \alpha)(4 + \alpha)\|\psi_0\| \sum_{i=1}^n j^{1+\alpha}\psi_i(s) \leq \|\psi_0\|^2 + 5\|\psi_0\| \sum_{i=1}^n j^{1+\alpha}\psi_i(s)$$

which, upon substitution in (48), gives

$$\sum_{i=1}^n j^{1+\alpha}\psi_i(t) \leq \sum_{i=1}^n i^{1+\alpha}\psi_0 i + \|\psi_0\|^2T + \int_0^t 5\|\psi_0\| \sum_{i=1}^n i^{1+\alpha}\psi_i(s) ds.$$

Hence, by Gronwall’s lemma, we conclude that, for all $t \in [0,T]$ and $n \geq 1$,

$$\sum_{i=1}^n j^{1+\alpha}\psi_i(t) \leq \left( \sum_{i=1}^n i^{1+\alpha}\psi_0 i + T\|\psi_0\|^2 \right) e^{5\|\psi_0\|t} \leq (\mu_{1+\alpha}(\psi_0) + T\|\psi_0\|^2) e^{5\|\psi_0\|t},$$

where the inequality (49) is due to the assumption about the boundedness of the $(1+\alpha)$-moment of the initial condition $\psi_0$. Since the right-hand side (49) does not depend on $n$ we conclude that the same is valid in the limit $n \to \infty$, which proves the result. 

An important result regarding the evaluation of the higher moments of the solution is analyzed here.

**Theorem 7.** Assume $(g_i)$ be a real valued non-negative sequence such that $g_i = O(i^{\alpha+1})$. Let $\psi$ be a solution of (11) when $V_{i,j} \leq i^\alpha + j^\alpha$, $\alpha \in [0,1]$ under the assumption that $\mu_{\alpha+1}(\psi_0)$ is bounded on some interval $[0,T)$, for $0 < T < \infty$. Let $0 \leq t_1 < t_2 < T$. If the following hypotheses hold

\[(H1)\]

$$\int_{t_1}^{t_2} \sum_{i=1}^\infty \sum_{j=1}^\infty j(g_{i+1} - g_i)V_{i,j}\psi_j(s)\psi_i(s) ds < \infty$$

\[(H2)\]

$$\int_{t_1}^{t_2} \sum_{i=1}^\infty \sum_{j=1}^\infty g_jV_{i,j}\psi_i(s)\psi_j(s) ds < \infty$$

then, for every $m \in \mathbb{N}$,

$$\sum_{i=m} g_i\psi_i(t_2) - \sum_{i=m} g_i\psi_i(t_1) = \int_{t_1}^{t_2} \sum_{i=m}^\infty \sum_{j=1}^\infty (jg_{i+1} - jg_i)V_{i,j}\psi_i(s)\psi_j(s) ds + \delta_{m \geq 2} \int_{t_1}^{t_2} \sum_{i=m}^\infty \sum_{j=1}^{m-1} g_jV_{i,j}\psi_i(s)\psi_j(s) ds + \delta_{m \geq 2} \int_{t_1}^{t_2} g_m\psi_m(s) \sum_{j=1}^{m-1} jV_{m-1,j}\psi_j(s) ds$$

where $\delta_P = 1$ if $P$ holds, and is equal to zero otherwise.
Proof. Take positive integers \( m < n \). Multiplying each equation in (7) by \( g_i \) and summing over \( i \) from \( m \) to \( n \), we obtain

\[
\sum_{i=m}^{n} g_i \psi_i(t_2) - \sum_{i=m}^{n} g_i \psi_i(t_1) = \int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=1}^{i} (g_{i+1} - g_{j}) V_{i,j} \psi_i(s) \psi_j(s) ds
\]

(53)

\[
+ \delta_{m \geq 2} \int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=1}^{m-1} g_j V_{i,j} \psi_i(s) \psi_j(s) ds
\]

(54)

\[
+ \delta_{m \geq 2} \int_{t_1}^{t_2} g_m \psi_{m-1}(s) \sum_{j=1}^{m-1} j V_{m-1,j} \psi_j(s) ds
\]

(55)

\[
- \int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=n+1}^{\infty} g_i V_{i,j} \psi_i(s) \psi_j(s) ds
\]

(56)

\[
- \int_{t_1}^{t_2} g_{n+1} \psi_n(s) \sum_{j=1}^{n} j V_{n,j} \psi_j(s) ds.
\]

(57)

We need to prove that, as \( n \to \infty \), the integrals in (56) and (57) converge to zero, and the other integrals converge to the corresponding ones in the right-hand side of (52).

Using (H2) by interchanging the order of summation and replacing \( i \) for \( j \) and then following (a) in Definition 1.1, one can obtain

\[
\lim_{n \to \infty} \int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=n+1}^{\infty} g_i V_{i,j} \psi_i(s) \psi_j(s) ds = 0
\]

(58)

which proves the convergence of (56) to zero. With \( g_i = 1 \) we can write (53)-(57) as follows

\[
\sum_{i=m}^{n} \psi_i(t_2) - \sum_{i=m}^{n} \psi_i(t_1) = \int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=1}^{i} (-V_{i,j} \psi_i(s) \psi_j(s)) ds + \delta_{m \geq 2} \int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=1}^{m-1} V_{i,j} \psi_i(s) \psi_j(s) ds
\]

\[
+ \delta_{m \geq 2} \int_{t_1}^{t_2} \psi_{m-1}(s) \sum_{j=1}^{m-1} j V_{m-1,j} \psi_j(s) ds - \int_{t_1}^{t_2} \sum_{i=m}^{n} \sum_{j=n+1}^{\infty} V_{i,j} \psi_i(s) \psi_j(s) ds
\]

\[
- \int_{t_1}^{t_2} \psi_n(s) \sum_{j=1}^{n} j V_{n,j} \psi_j(s) ds.
\]

Further, thanks to relation (H2) and the fact that \( g_j = O(j^{\alpha+1}) \), the last two integrals in the above expression tend to zero as \( n \to \infty \). Therefore,

\[
\sum_{i=m}^{\infty} \psi_i(t_2) - \sum_{i=m}^{\infty} \psi_i(t_1) = \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{i} (-V_{i,j} \psi_i(s) \psi_j(s)) ds + \delta_{m \geq 2} \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{m-1} V_{i,j} \psi_i(s) \psi_j(s) ds
\]

\[
+ \delta_{m \geq 2} \int_{t_1}^{t_2} \psi_{m-1}(s) \sum_{j=1}^{m-1} j V_{m-1,j} \psi_j(s) ds.
\]

(59)

For \( p = 1, 2 \), consider,

\[
|g_{n+1}| \sum_{i=n+1}^{\infty} \psi_i(t_p) \leq C(n+1)^{\alpha+1} \sum_{i=n+1}^{\infty} \psi_i(t_p)
\]

\[
\leq C \sum_{i=n+1}^{\infty} i^{\alpha+1} \psi_i(t_p)
\]
for $C \in \mathbb{R}^+$ and thus Lemma 6 guarantees that
\[
\lim_{n \to \infty} |g_{n+1}| \sum_{i=1}^{\infty} \psi_i(t_p) = 0.
\] (60)

Replacing $m$ by $n + 1$ in (59), multiplying both sides by $g_{n+1}$, letting $n \to \infty$, and using (H2) together with (60) confirms that
\[
\int_{t_1}^{t_2} g_{n+1} \psi_n(s) \sum_{j=1}^{n} jV_{n,j} \psi_j(s) ds \to 0.
\] (61)

By Definition 1.1, the boundedness of $\psi(t)$, (H1) and (H2), we conclude that
\[
\int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{n} (jg_{i+1} - jg_i - jg_j)V_{i,j} \psi_i(s) \psi_j(s) ds \to \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{n} (jg_{i+1} - jg_i - jg_j)V_{i,j} \psi_i(s) \psi_j(s) ds
\]
and
\[
\delta_{m+2} \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{n} g_j V_{i,j} \psi_i(s) \psi_j(s) ds \to \delta_{m+2} \int_{t_1}^{t_2} \sum_{i=m}^{\infty} \sum_{j=1}^{n} g_j V_{i,j} \psi_i(s) \psi_j(s) ds.
\] (62)

Thus, using Definition 1.1 together with equations (58), (61)-(63) and the bounded convergence theorem, the result follows. □

Finally, the following proposition is discussed which is essential in showing that the solution of the S-D model is first-order differentiable.

**Proposition 8.** Let $\{V_{i,j}\}_{i,j \in \mathbb{N}}$ be non-negative and $V_{i,j} \leq i^\alpha + j^\alpha$, $0 \leq \alpha \leq 1$. Let $\psi = (\psi_i)$ be a solution on some interval $[0, T \rangle$, where $0 < T \leq \infty$, of the equation (11) with initial condition $f_0$ and having bounded $\mu_{i+1}(\psi_0)$. Then, the series $\sum_{i=1}^{\infty} jV_{i,j} \psi_j(t) \psi_i(t)$ and $\sum_{i=1}^{\infty} V_{i,j} \psi_j(t) \psi_i(t)$ are absolutely continuous on the compact sub-intervals of $[0, T \rangle$.

**Proof.** Let, $(g_i)$ satisfy the conditions in the statement of Theorem [1]. For (H1) to hold, proving the boundedness of the series $\sum_{i=1}^{\infty} j^\alpha V_{i,j} \psi_j(t) \psi_i(t)$ is enough. Using the fact that $g_{i+1} - g_i = O(i^\alpha)$, we have the following
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{i} j^\alpha V_{i,j} \psi_j(t) \psi_i(t) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} C j^\alpha V_{i,j} \psi_j(t) \psi_i(t)
\]
\[
\leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} C j^\alpha (i^\alpha + j^\alpha) \psi_j(t) \psi_i(t) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} C (j^\alpha)^{i+1} + j^\alpha j^{\alpha+1} \psi_j(t) \psi_i(t)
\]
for $C$ being some positive constant. Hence, by Lemma 6 and Section 4, one can obtain
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{i} j^\alpha V_{i,j} \psi_j(t) \psi_i(t) \leq 2CN\alpha+1\mu_1(0).
\] (64)

Thus, (H1) holds true. Further, to establish the relation (H2), consider the expression,
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{i} g_j V_{i,j} \psi_i(s) \psi_j(s) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} C (j^{2\alpha+1} + j^\alpha j^{\alpha+1}) \psi_i(t) \psi_j(t)
\]
\[
\leq 2CN\alpha+1\mu_1(0)
\]
which is finite by Lemma 6. Therefore, all the hypothesis of Theorem 7 are satisfied for any $t_1, t_2 \in [0, T \rangle$. Hence, considering $m = 1$ for $t \in [0, T \rangle$, equation (52) implies the uniform convergence of the series $\sum_{i=1}^{\infty} g_i \psi_i(t)$. Since, the series $\sum_{j=1}^{i} jV_{i,j} \psi_j(t)$ is bounded by this series, as $jV_{i,j} = O(i^{\alpha+1})$ when $j < i$, we conclude the uniform convergence of $\sum_{j=1}^{i} jV_{i,j} \psi_j(t)$. Now, the boundedness of $\psi_i(t)$ ensures the absolute continuity of $\sum_{j=1}^{i} jV_{i,j} \psi_j(t) \psi_i(t)$. Also,
the series $\sum_{j=i}^{\infty} V_{i,j} \psi_j(t)$ is bounded by $\sum_{j=1}^{\infty} g_j \psi_j(t)$, which yields its uniform convergence. Finally, the boundedness of $\psi_i(t)$ gives the desired result.

The Definition 1.1 (a), hypotheses $(H1) – (H2)$ of Theorem 7 and Proposition 8 ensure that the solution $f$ is differentiable in the classical sense in $[0, T]$.  

6. Uniqueness

In this section, we discuss the uniqueness of solutions for the S-D model for a restrictive class of kernels. It should be mentioned here that it was not easy to deal with $V_{i,j} \leq C V(i + j)$ $\forall i, j$, so the kernel was restricted to establish uniqueness.

**Theorem 9.** Let, the kernel $V_{i,j} \leq C V(i + j)$ and $V_{i,j} \leq C V \min\{i^n, j^n\}$, $0 \leq n \leq 2$ valid $\forall i, j \in \mathbb{N}$, $C V \in \mathbb{R}^+$. If the Lemma 6 holds, then the equations (11)–(2) have a unique solution in $X^+$.  

**Proof.** We shall use an apprach that revolves around defining a function (say $u(t)$) that is difference of two solutions of the equation (11) (let $\psi_i$ and $\rho_i$), both satisfying the initial condition (2). Furthermore, it makes use of the properties of the signum function such as

- (P1) $\sgn(C(t)) \frac{dC(t)}{dt} = \frac{d|C(t)|}{dt}$,
- (P2) $\sgn(a) \sgn(b) = \sgn(ab)$ and $|a| = a \sgn(a)$ for any real numbers $a, b$.

Our goal here is to use Gronwall’s lemma to the function $u(t)$ to reach at the required result. Define

$$u(t) := \sum_{i=1}^{\infty} |\psi_i(t) - \rho_i(t)| = \sum_{i=1}^{\infty} |u_i(t)|.$$  

(65)

Using the expression of the equation (11), we obtain

$$\frac{d u_i(t)}{dt} = \left( \delta_{i,2} \psi_{i-1}(t) \sum_{j=1}^{i-1} j V_{i-1,j} \psi_j(t) - \psi_i(t) \sum_{j=1}^{i} j V_{i,j} \psi_j(t) \sum_{j=1}^{\infty} V_{i,j} \psi_i(t) \psi_j(t) \right)$$

$$- \left( \delta_{i,2} \rho_{i-1}(t) \sum_{j=1}^{i-1} j V_{i-1,j} \rho_j(t) - \rho_i(t) \sum_{j=1}^{i} j V_{i,j} \rho_j(t) - \sum_{j=1}^{\infty} V_{i,j} \rho_i(t) \rho_j(t) \right).$$

Multiplying both sides by $\sgn(u_i(t))$ and then using (P1), the above equation reduces to

$$\frac{d}{dt} |u_i(t)| = \sgn(u_i(t)) \left( \delta_{i,2} \psi_{i-1}(t) \sum_{j=1}^{i-1} j V_{i-1,j} \psi_j(t) - \psi_i(t) \sum_{j=1}^{i} j V_{i,j} \psi_j(t) - \sum_{j=1}^{\infty} V_{i,j} \psi_i(t) \psi_j(t) \right)$$

$$- \sgn(u_i(t)) \left( \delta_{i,2} \rho_{i-1}(t) \sum_{j=1}^{i-1} j V_{i-1,j} \rho_j(t) - \rho_i(t) \sum_{j=1}^{i} j V_{i,j} \rho_j(t) - \sum_{j=1}^{\infty} V_{i,j} \rho_i(t) \rho_j(t) \right).$$

Further, integrating both sides, using $u(0) = 0$ and summing over $i$ from 1 to $\infty$ yield

$$u(t) = \int_{0}^{t} \sum_{i=1}^{\infty} \sgn(u_i(h)) \left( \delta_{i,2} \psi_{i-1}(h) \sum_{j=1}^{i-1} j V_{i-1,j} \psi_j(h) - \psi_i(h) \sum_{j=1}^{i} j V_{i,j} \psi_j(h) - \sum_{j=1}^{\infty} V_{i,j} \psi_i(h) \psi_j(h) \right) dh$$

$$- \int_{0}^{t} \sum_{i=1}^{\infty} \sgn(u_i(h)) \left( \delta_{i,2} \rho_{i-1}(h) \sum_{j=1}^{i-1} j V_{i-1,j} \rho_j(h) - \rho_i(h) \sum_{j=1}^{i} j V_{i,j} \rho_j(h) - \sum_{j=1}^{\infty} V_{i,j} \rho_i(h) \rho_j(h) \right) dh.$$

Replacing $i - 1$ by $i'$, then changing $i'$ by $i$ in the first and fourth sums and finally using

$$(\psi_i \psi_j - \rho_i \rho_j)(t) = (\psi_i u_j + \rho_i u_i)(t),$$
simplify to
\[ u(t) = \int_0^t \left( \sum_{i=1}^\infty \left( \text{sgn}(u_{i+1}) \left( \delta_{i \geq 1} \sum_{j=1}^i j V_{i,j} |\psi_i u_j + |u_i| \rho_j \right) - \text{sgn}(u_i) \left( \sum_{j=1}^\infty j V_{i,j} (\psi_i u_j + \rho_j u_i) \right) \right) \right) (h) dh. \]

Using (P1) and (P2), one can obtain the following expression
\[ u(t) \leq \int_0^t \left( \sum_{i=1}^\infty \sum_{j=i}^\infty j V_{i,j} |u_j| \psi_i + \sum_{i=1}^\infty \sum_{j=i}^\infty V_{i,j} |u_j| \psi_i \right) (h) dh. \]

Finally, inserting the value of \( V_{i,j} \) we write
\[ u(t) \leq \int_0^t \left( 2 \sum_{i=1}^\infty \sum_{j=1}^\infty C_{V,i} (i + j) |u_j| \psi_i + \sum_{i=1}^\infty \sum_{j=i}^\infty C_{V,i} |u_j| \psi_i \right) (h) dh. \]

The application of Lemma 3 for \( \alpha = 1 \) leads to
\[ u(t) \leq \int_0^t (4C_{V} \mu_2 + C_{V} \mu_2) u(h) dh. \]

The application of Gronwall’s lemma enables us to have \( u(t) \equiv 0 \) which implies that \( \psi_i(t) = \rho_i(t) \) \( \forall 0 \leq t \leq T \). Since, \( T \) is arbitrary, we get uniqueness of \( \psi_i(t) \).

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