A general approach for finding exact cosmological solutions in $f(R)$ gravity is discussed. Instead of taking into account phenomenological models, we assume, as a physical criterion, the existence of Noether symmetries in the cosmological $f(R)$ Lagrangian. As a result, the presence of such symmetries leads to the selection of viable models and allows us to solve the equations of motion. We discuss also the case in which no Noether charge is present but general criteria can be used to achieve solutions.

**Keywords:** classical tests of cosmology, gravity

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1. Introduction

The recent interest in investigating alternative theories of gravity has arisen from cosmology, quantum field theory and Mach’s principle. The initial singularity, flatness and horizon problems [1] indicate that the standard cosmological model [2], based on general relativity (GR) and the particle standard model, fails in describing the Universe at extreme regimes. Besides, GR does not work as a fundamental theory capable of giving a quantum description of spacetime. For these reasons and due to the lack of a definitive quantum gravity theory, alternative theories of gravitation have been pursued in order to attempt an at least semi-classical approach to quantization. In particular, extended...
Theories of gravity (ETGs) face the problem of gravitational interaction correcting and enlarging the Einstein theory.

The general paradigm consists in adding, into the effective action, physically motivated higher order curvature invariants and non-minimally coupled scalar fields [3, 4].

The interest of such an approach in early epoch cosmology is due to the fact that ETGs can ‘naturally’ reproduce inflationary behaviors able to overcome the shortcomings of the standard cosmological model and seem also capable of yielding a match with several observations.

From another viewpoint, the Mach principle gives further motivations for modifying GR, stating that the local inertial frame is determined by the average motion of distant astronomical objects [5]. As a consequence, the gravitational coupling can be scale dependent. This means that the concept of inertia and the equivalence principle have to be revised since there is no a priori reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar $R$, minimally coupled with matter [6]–[11].

Very recently, ETGs have been playing an interesting role in describing today’s observed Universe. In fact, an impressive amount of good quality data from the last decade seems to shed new light on the effective picture of the Universe. Type Ia supernovae (SNeIa) [12], anisotropies in the CMBR [13], and the matter–power spectrum derived from wide and deep galaxy surveys [14] represent the strongest evidences for a radical revision of the cosmological standard model also in recent epochs.

Specifically, the concordance $\Lambda$CDM model is showing that baryons contribute only $\sim 4\%$ to the total matter–energy budget, while the cold dark matter (CDM) represents the bulk of the clustered large scale structures ($\sim 25\%$) and the cosmological constant $\Lambda$ plays the role of the so called ‘dark energy’ ($\sim 70\%$) [15].

Although providing the best fit to a wide range of data [16], the $\Lambda$CDM model is afflicted with strong theoretical shortcomings [17] that have motivated the search for alternative models [18, 19].

Dark energy models mainly rely on the implicit assumption that Einstein’s GR is indeed the correct theory of gravity. Nevertheless, its validity on large astrophysical and cosmological scales has never been tested but only assumed [20], and it is therefore conceivable that both cosmic speed up and missing matter are nothing but signals of a breakdown of GR. In this sense, GR could fail in giving self-consistent pictures both at ultraviolet scales (early Universe) and at infrared scales (late Universe).

Following this line of thinking, the ‘minimal’ choice could be to take into account generic functions $f(R)$ of the Ricci scalar $R$. However, such an approach can be encompassed in the ETGs, as the minimal extension of GR. The task for these extended theories should be to match the data under the ‘economic’ requirement that no exotic dark ingredients have to be added, unless these are going to be found with fundamental experiments [21]. This is the underlying philosophy of what is referred to as $f(R)$ gravity (see [19, 22, 23] and references therein).

Although higher order gravity theories have received much attention in cosmology, since they are naturally able to give rise to the accelerating expansion (both in the late and in the early Universe [24]), it is possible to demonstrate that $f(R)$ theories can also play a major role at astrophysical scales. In fact, modifying the gravity Lagrangian affects the gravitational potential in the low energy limit. Provided that the modified potential reduces to the Newtonian one on the Solar System scale, this implication could represent
an intriguing opportunity rather than a shortcoming for $f(R)$ theories. In fact, a corrected gravitational potential could offer the possibility of fitting galaxy rotation curves without the need for huge amounts of dark matter [25–31]. In addition, it is possible to work out a formal analogy between the corrections to the Newtonian potential and the usually adopted galaxy halo models which allow one to reproduce dynamics and observations without dark matter [27].

However, extending the gravitational Lagrangian could give rise to several problems. These theories could have instabilities [32, 33] and ghost-like behaviors [34–36], and they have to be matched with the low energy limit experiments which fairly test GR. Besides, these theories should also be compatible with early Universe tests such as the formation of CMBR anisotropies, big bang nucleosynthesis [37], and baryogenesis [38, 39].

Actually, the debate concerning the weak field limit of $f(R)$ gravity is far from being definitive. In the last few years, several authors have dealt with this matter with contrasting conclusions, in particular with respect to the parameterized post-Newtonian (PPN) limit [40, 42].

In summary, it seems that the paradigm of adopting $f(R)$ gravity leads to interesting results at cosmological, galactic and Solar System scales but, up to now, no definite physical criterion has been found for selecting the final $f(R)$ theory (or class of theories), capable of matching the data at all scales. Interesting results have been achieved along these lines of thinking [21], [43–46] but the approaches are all phenomenological and are not based on some fundamental principle such as the conservation or the invariance of some quantity or some intrinsic symmetry of the theory. Furthermore, as shown in [32], in alternative theories of gravity, it is important to understand the background before exploring other bounds, such as anisotropies in the CMBR. For this goal it is essential to try to find exact analytical solutions for the $f(R)$ theories, and, only after this, to study in more detail the possible evolutions compatible with our data (e.g. Solar System and CMBR bounds).

In some sense, the situation is similar to that of dark matter: we know very well its effect at large astrophysical scales but no final evidence of its existence has been found, up to now, at a fundamental level. In the case of $f(R)$ gravity, we know that the paradigm is working: in principle, the missing matter and accelerated cosmic behavior can be addressed taking into account gravity (in some extended version), baryons and radiation but we do not know of a specific criterion for selecting the final, comprehensive theory.

In this paper, we want to address the following issues. (i) Is there some general principle capable of selecting physically motivated $f(R)$ models? (ii) Can conserved quantities or symmetries be found in relation to specific $f(R)$ theories? (iii) Can such quantities, if they exist, give rise to viable cosmological models?

In this paper, following the so called Noether symmetry approach (see [7, 47, 48], we want to seek for viable $f(R)$ cosmological models. As we will see, the method is twofold: on one hand, the existence of symmetries allows us to solve the dynamics exactly; on the other hand, the Noether charge can always be related to some observable quantity.

The layout of the paper is the following. In section 2, we sketch the dynamics of $f(R)$ gravity in the metric approach and derive the Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmological equations. Section 3 is devoted to the general discussion of the Noether symmetry approach through which it is possible to find conserved quantities and
then symmetries which allow us to exactly solve a dynamical system. In section 4, we apply the method to the \( f(R) \) cosmology. In section 5, we give a detailed summary of the exact solutions, discussing them for the presence and the absence of the Noether charge. Section 6 is devoted to a discussion and conclusions.

2. \( f(R) \) gravity and cosmology

The action

\[
S = \int d^4x \sqrt{-g} f(R) + S_m
\]

describes a theory of gravity where \( f(R) \) is a generic function of the Ricci scalar \( R \). GR is recovered in the particular case \( f(R) = -R/16\pi G \), and \( S_m \) is the action for a perfect fluid minimally coupled with gravity.

This action, in general, leads to fourth-order differential equations for the metric since the field equations are

\[
f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f_{R\mu\nu} + g_{\mu\nu} \Box fR = -\frac{1}{2} T^m_{\mu\nu},
\]

where a subscript \( R \) denotes differentiation with respect to \( R \) and \( T^m_{\mu\nu} \) is the fluid matter stress–energy tensor.

Defining a curvature stress–energy tensor as

\[
T^\text{curv}_{\mu\nu} = \frac{1}{f_R(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[ f(R) - R f(R) \right] + f_R(R) \alpha^\beta (g_{\alpha\mu} g_{\beta\nu} - g_{\mu\nu} g_{\alpha\beta}) \right\},
\]

equations (2) can be recast in the Einstein-like form

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^\text{curv}_{\mu\nu} + T^m_{\mu\nu}/f_R(R),
\]

where matter non-minimally couples to geometry through the term \( 1/f_R(R) \). It is known that these theories can be mapped to a scalar–tensor theory. However, there are two points which should be noticed. First, the two theories might have different quantum descriptions, as they only coincide on the classical solutions. Furthermore, the two theories are classically equivalent if the Brans–Dicke parameter \( \omega_{BD} \) exactly vanishes and if the scalar field possesses a suitable potential. This fact is related to the second point: in the literature, the Brans–Dicke field is commonly taken as a light scalar field for which the local gravity constraint fixes the Brans–Dicke parameter to be greater than 40000. This bound is usually considered when studying Brans–Dicke theories. However, for the \( f(R) \) theories, since \( \omega_{BD} = 0 \), this is not the case, and the presence of a non-negligible potential is essential in order to give an explicit mass to the gravitational scalar degree of freedom. Once one has the solution \( H(t) \) (and consequently \( R(t) \)) for a given \( f(R) \), the scalar field is defined as \( \Phi(t) = -f_R(t) \), and its potential is \( U(\Phi(t)) = R(t) f_R(t) \). An example showing this link between scalar–tensor theories and \( f(R) \) gravity is given in the appendix for one solution which will be found explicitly later on.

In order to derive the cosmological equations in a FLRW metric, one can define a canonical Lagrangian \( \mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R}) \), where \( \mathcal{Q} = \{a, R\} \) is the configuration space and \( T\mathcal{Q} = \{a, \dot{a}, R, \dot{R}\} \) is the related tangent bundle on which \( \mathcal{L} \) is defined. The variables

---

5 We are using the following conventions: \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), and \( R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}, c = h = 1 \).
and then, integrating by parts, the point-like FLRW Lagrangian is

\[ S = 2\pi^2 \int dt a^3 \left\{ f(R) - \lambda \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right) - \frac{\rho_m}{a^3} - \frac{\rho_r}{a^4} \right] \right\}, \quad (5) \]

where \( a \) is the scale factor scaled with respect to today’s value (so that \( a = \tilde{a}/\tilde{a}_0 \) and \( a(t_0) = 1 \)); \( \rho_m \) and \( \rho_r \) represent the standard amounts of dust and radiation fluids as, for example, measured today; finally \( \kappa \) is a canonical function of two coupled fields, \( f \).

It is straightforward to show that, for \( f(R) \equiv -R/16\pi G - \rho_m - \rho_r \), one obtains the usual Friedmann equations.

The variation with respect to \( R \) of the action gives \( \lambda = f_R \). Therefore the previous action can be rewritten as

\[ S = 2\pi^2 \int dt a^3 \left\{ f - f_R \left[ R + 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right) - \frac{\rho_m}{a^3} - \frac{\rho_r}{a^4} \right] \right\}, \quad (6) \]

and then, integrating by parts, the point-like FLRW Lagrangian is

\[ \mathcal{L} = a^3 \left( f - f_R R \right) + 6a^2 f_{RR} \ddot{R} a - 6f_R a \dot{a}^2 - 6\kappa f_R a - \rho_m - \rho_r/a, \quad (7) \]

which is a canonical function of two coupled fields, \( R \) and \( a \), both depending on time \( t \).

The total energy \( E_\mathcal{L} \), corresponding to the 0, 0 Einstein equation, is

\[ E_\mathcal{L} = 6f_{RR} a^2 \dddot{a} + 6f_R a \dot{a}^2 - a^3 (f - f_R R) + 6\kappa f_R a + \rho_m + \frac{\rho_r}{a} = 0. \quad (8) \]

As we shall see later, it is convenient to look for parametric solutions in the form \([H(a), f(R(a))]\), so that \( f_R = f'/R' \), where a prime denotes differentiation with respect to the time parameter \( a \). We also have that, if \( R \neq \) constant, \( f_{RR} \dot{R} = df_R/dt = aH f_R = aH [f'' / R' - f' R'' / R'^2] \), so the Friedmann equation can be rewritten as

\[ f - 6a \left( \frac{f''}{R'} - \frac{f' R''}{R'^2} \right) H^2 - \frac{6\kappa}{a^2} + \frac{R'}{R'^2} \right] \frac{f'}{R'} = \frac{\rho_m}{a^3} + \frac{\rho_r}{a^4}. \quad (9) \]

The equations of motion for \( a \) and \( R \) are respectively

\[ f_{RR} \left[ R + 6H^2 + \frac{6}{a} + 6\frac{\kappa}{a^2} \right] = 0, \quad (10) \]

\[ 6f_{RR} \ddot{R} + 6f_R \dot{R} + 12f_R \dot{a} - 3(f - f_R R) - 12f_{RR} H \dot{R} - 6f_R \frac{\kappa}{a^2} + \frac{\rho_r}{a^4}, \quad (11) \]

where \( H \equiv \dot{a}/a \) is the Hubble parameter. Considering \( R \) and \( a \) as the variables, we have, for consistency (excluding the case \( f_{RR} = 0 \)), that \( R \) coincides with the definition of the Ricci scalar in the FLRW metric. Geometrically, this is the Euler constraint of the dynamics. Using (10), only one of the equations (8) and (11) is independent because of the Bianchi identities, as these equations correspond to the first and second modified Einstein equations, and matter is conserved. Equivalently, after multiplying equation (11) by \( a^2 \dot{a} \), and using (10), one can integrate (11) to find (8). Furthermore, as we will show...

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below, constraints on the form of the function \( f(R) \) and, consequently, solutions of the system (8), (10) can be achieved by asking for the existence of Noether symmetries. Such solutions will also solve equation (11) automatically. On the other hand, the existence of the Noether symmetries guarantees the reduction of dynamics and the eventual solvability of the system.

3. The Noether symmetry approach

Solutions for the dynamics given by (7) can be achieved by selecting cyclic variables related to some Noether symmetry. In principle, this approach allows us to select \( f(R) \) gravity models compatible with the symmetry so it can be seen as a physical criterion since the conserved quantities are Noether charges of a sort. Therefore such a criterion might be to look for those \( f(R) \) which have *cosmological* Noether charge. Although this criterion somehow ‘breaks’ Lorentz invariance because we need the FLRW background to formulate it, Lorentz invariance is evidently broken in our Universe by the presence of the CBMR radiation which, by itself, fixes a preferred reference frame.

In general, the Noether theorem states that conserved quantities are related to the existence of cyclic variables in dynamics [49]–[51].

Let \( \mathcal{L}(q^i, \dot{q}^i) \) be a canonical, non-degenerate point-like Lagrangian where

\[
\frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0; \quad \text{det} \ H_{ij} \overset{\text{def}}{=} \text{det} \left| \frac{\partial^2 \mathcal{L}}{\partial q^i \partial q^j} \right| \neq 0,
\]

with \( H_{ij} \) the Hessian matrix related to \( \mathcal{L} \) and a dot denotes differentiation with respect to the affine parameter \( \lambda \). The dot indicates derivatives with respect to the affine parameter \( \lambda \) which, in our case, corresponds to the cosmic time \( t \). In analytical mechanics, \( \mathcal{L} \) is of the form

\[
\mathcal{L} = T(q, \dot{q}) - V(q),
\]

where \( T \) and \( V \) are the ‘kinetic energy’ and ‘potential energy’ respectively. \( T \) is a positive definite quadratic form in \( \dot{q} \). The energy function associated with \( \mathcal{L} \) is

\[
E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L},
\]

which is the total energy \( T + V \). In any case, \( E_{\mathcal{L}} \) is a constant of motion. Since our cosmological problem has a finite number of degrees of freedom, we are going to consider only point transformations. Any invertible transformation of the ‘generalized positions’ \( Q^i = Q^i(q) \) induces a transformation of the ‘generalized velocities’ such that

\[
\dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j;
\]

the matrix \( \mathcal{J} = ||\partial Q^i/\partial q^j|| \) is the Jacobian of the transformation on the positions, and it is assumed to be non-zero. The Jacobian \( \tilde{\mathcal{J}} \) of the induced transformation is easily derived and \( \mathcal{J} \neq 0 \rightarrow \tilde{\mathcal{J}} \neq 0 \). In general, this condition is not satisfied over the whole space but only in the neighborhood of a point. It is a local transformation.

A point transformation \( Q^i = Q^i(q) \) can depend on a (or more than one) parameter. As starting point, we can assume that a point transformation depends on a parameter \( \epsilon \),
$f(R)$ cosmology from Noether’s symmetry

i.e. $Q^i = Q^i(q, \epsilon)$, and that it gives rise to a one-parameter Lie group. For infinitesimal values of $\epsilon$, the transformation is then generated by a vector field: for instance, $\partial/\partial x$ is a translation along the $x$ axis, $x(\partial/\partial y) - y(\partial/\partial x)$ is a rotation around the $z$ axis and so on. The induced transformation (15) is then represented by

$$X = \alpha^i(q) \frac{\partial}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial}{\partial \dot{q}^i}. \quad (16)$$

$X$ is called the ‘complete lift’ of $X$ [51]. A function $F(q, \dot{q})$ is invariant under the transformation $X$ if

$$L_X F \overset{\text{def}}{=} \alpha^i(q) \frac{\partial F}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial F}{\partial \dot{q}^i} = 0, \quad (17)$$

where $L_X F$ is the Lie derivative of $F$. Specifically, if $L_X L = 0$, $X$ is a symmetry for the dynamics derived by $L$. As we shall see later on, we will look for a sufficient condition on the form of $f(R)$ in our Lagrangian which allows $L_X L = 0$ to vanish.

Let us consider now a Lagrangian $L$ and its Euler–Lagrange equations

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} = 0. \quad (18)$$

Let us consider also the vector field (16). Contracting (18) with the $\alpha^i$'s gives

$$\alpha^j \left( \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} \right) = 0. \quad (19)$$

As

$$\alpha^j \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^j} = \frac{d}{d\lambda} \left( \alpha^j \frac{\partial L}{\partial \dot{q}^j} \right) - \left( \frac{d\alpha^j}{d\lambda} \right) \frac{\partial L}{\partial \dot{q}^j}, \quad (20)$$

from (19), we obtain

$$\frac{d}{d\lambda} \left( \alpha^j \frac{\partial L}{\partial \dot{q}^j} \right) = L_X L. \quad (21)$$

The immediate consequence is the Noether theorem which states:

If $L_X L = 0$, then the function

$$\Sigma_0 = \alpha^i \frac{\partial L}{\partial \dot{q}^i} \quad (22)$$

is a constant of motion.

Some comments are necessary at this point. Equation (22) can be expressed independently of coordinates as a contraction of $X$ by a Cartan 1-form

$$\theta_L \overset{\text{def}}{=} \frac{\partial L}{\partial \dot{q}^i} dq^i. \quad (23)$$

For a generic vector field $Y = y^i \partial/\partial x^i$, and the 1-form $\beta = \beta_i dx^i$, we have, by definition, $i_Y \beta = y^i \beta_i$. Thus equation (22) can be written as

$$i_X \theta_L = \Sigma_0. \quad (24)$$
Through a point transformation, the vector field $X$ becomes

$$\tilde{X} = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left( \frac{d}{d\lambda} (i_x dQ^k) \right) \frac{\partial}{\partial Q^k}. \quad (25)$$

We see that $\tilde{X}'$ is still the lift of a vector field defined on the ‘space of positions’. If $X$ is a symmetry and we choose a point transformation such that

$$i_X dQ^1 = 1; \quad i_X dQ^i = 0 \quad i \neq 1, \quad (26)$$

we get

$$\tilde{X} = \frac{\partial}{\partial Q^1}; \quad \frac{\partial L}{\partial Q^1} = 0. \quad (27)$$

Thus $Q^1$ is a cyclic coordinate and the dynamics results reduced \[49,50\].

Furthermore, the change of coordinates given by (26) is not unique and then a clever choice could be very important. In general, the solution of equation (26) is not defined on the whole space. It is local in the sense explained above. Besides, it is possible that more than one $X$ is found, e.g. $X_1$, $X_2$. If they commute, i.e. $[X_1, X_2] = 0$, then it is possible to obtain two cyclic coordinates by solving the system

$$i_{X_1} dQ^1 = 1; \quad i_{X_2} dQ^2 = 1; \quad i_{X_1} dQ^i = 0; \quad i \neq 1; \quad i_{X_2} dQ^i = 0; \quad i \neq 2. \quad (28)$$

The transformed fields will be $\partial/\partial Q^1$, $\partial/\partial Q^2$. If they do not commute, this procedure is clearly not applicable, since commutation relations are preserved by diffeomorphisms. If the relation $X_3 = [X_1, X_2]$ holds, also $X_3$ is a symmetry, as $L_{X_3} = L_{X_1} L_{X_2} L - L_{X_2} L_{X_1} L = 0$. If $X_3$ is independent of $X_1$, $X_2$, we can go on until the vector fields close the Lie algebra.

The usual approach to this situation is to make a Legendre transformation, going to the Hamiltonian formalism, and then derive the Lie algebra of Poisson brackets.

If we seek for a reduction of dynamics via cyclic coordinates, the procedure is possible in the following way: (i) we arbitrarily choose one of the symmetries, or a linear combination of them, searching for new coordinates where, as sketched above, the cyclic variables appear, and after the reduction, we get a new Lagrangian $\tilde{\lambda}(Q)$; (ii) we search again for symmetries in this new configuration space, make a new reduction and so on as far as possible; (iii) if the search fails, we try again with another of the existing symmetries.

Let us now assume that $L$ is of the form (13). As $X$ is of the form (16), $L_X L$ will be a homogeneous polynomial of second degree in the velocities plus a inhomogeneous term in the $q^i$. Since such a polynomial has to be identically zero, each coefficient must be independently zero. If $n$ is the dimension of the configuration space, we get $\{1 + n(n + 1)/2\}$ partial differential equations. The system is overdetermined; therefore, if any solution exists, it will be expressed in terms of integration constants instead of boundary conditions. It is also obvious that an overall constant factor in the Lie vector $X$ is irrelevant. In other words, the Noether symmetry approach can be used to select functions which assign the models and such functions (and then the models) can be physically relevant.

Considering the specific case which we are going to discuss, the $f(R)$ cosmology, the situation is the following. The configuration space is $Q = \{a, R\}$ while the tangent space
for the related tangent bundle is \( TQ = \{ a, \dot{a}, R, \dot{R} \} \). The Lagrangian is an application

\[
\mathcal{L} : TQ \rightarrow \mathbb{R},
\]

where \( \mathbb{R} \) is the set of real numbers. The generator of symmetry is

\[
X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}}.
\]

As discussed above, a symmetry exists if the equation \( L_X \mathcal{L} = 0 \) has solutions. Then there will be a constant of motion on shell, i.e. for the solutions of the Euler equations, as stated above equation (22). In other words, a symmetry exists if at least one of the functions \( \alpha \) or \( \beta \) in equation (30) is different from zero. As a by-product, the form of \( f(R) \), not specified in the point-like Lagrangian (7), is determined in correspondence to such a symmetry.

4. Noether symmetries in \( f(R) \) cosmology

For the existence of a symmetry, we can write the following system of equations (linear in \( \alpha \) and \( \beta \)):

\[
\begin{align*}
f_R(\alpha + 2a\dot{a}\alpha) + a f_{RR}(\beta + a\dot{a}\beta) &= 0, \\
a^2 f_{RR}\dot{R} \alpha &= 0, \\
2 f_R\ddot{R} \alpha + f_{RR}(2\alpha + a\dot{a}\alpha + a\dot{R}\beta) + a\beta f_{RRR} &= 0,
\end{align*}
\]

obtained by setting to zero the coefficients of the terms \( \dot{a}^2 \), \( \dot{R}^2 \) and \( \dot{a}\dot{R} \) in \( L_X \mathcal{L} = 0 \). In order to make \( L_X \mathcal{L} = 0 \) vanish we will also look for those particular \( f \)s which, given the Euler dynamics, also satisfy the constraint

\[
3\alpha(f - Rf_R) - a\beta Rf_{RR} - \frac{6\kappa}{a^2}(\alpha f_R + a\beta f_{RR}) + \frac{\rho\alpha\alpha}{a^4} = 0.
\]

This procedure is different from the usual Noether symmetry approach, in the sense that now \( L_X \mathcal{L} = 0 \) will be solved not for all dynamics (which solve the Euler–Lagrange equations), but only for those \( f \) which allow Euler solutions to solve also the constraint (34). Imposing such a constraint on the form of \( f \) will turn out to be, as we will show, a sufficient condition for finding solutions of the Euler–Lagrange equation which also possess a constant of motion, i.e. a Noether symmetry. As we shall see later on, the system (31)–(33) can be solved exactly. Having a non-trivial solution for \( \alpha \) and \( \beta \) for this system, one finds a constant of motion if the constraint (34) is also satisfied. In fact, with these \( \alpha \) and \( \beta \), only those Euler–Lagrange solutions which also satisfy equation (34) will have a constant of motion. However, this will not happen for all \( f(R) \)s. The task will be finding such forms of \( f \).

A solution of (31)–(33) exists if explicit forms of \( \alpha \), \( \beta \) are found. If at least one of them is different from zero, a Noether symmetry exists.

If \( f_{RR} \neq 0 \), equation (32) can be immediately solved:

\[
\alpha = \alpha(a).
\]
The case \( f_{RR} = 0 \) is trivial since it corresponds to the standard GR. We can rewrite equations (31) and (33) as follows:

\[
f_R \left( \alpha + 2a \frac{d\alpha}{da} \right) + af_{RR}(\beta + a\partial_a\beta) = 0, \tag{36}
\]

\[
f_{RR} \left( 2\alpha + a \frac{d\alpha}{da} + a\partial_R\beta \right) + a\beta f_{RRR} = 0. \tag{37}
\]

Since \( f = f(R) \), then \( \partial f / \partial a = 0 \); even if we consider \( R = R(a) \), it is possible to solve equation (37), by writing it as

\[
\partial_R(\beta f_{RR}) = -f_{RR} \left( \frac{2\alpha}{a} + \frac{d\alpha}{da} \right), \tag{38}
\]

whose general solution can be written as

\[
\beta = - \left[ \frac{2\alpha}{a} + \frac{d\alpha}{da} \right] \frac{f_R}{f_{RR}} + \frac{h(a)}{f_{RR}}. \tag{39}
\]

Therefore one finds that equation (36) gives

\[
f_R \left[ \alpha - a^2 \frac{d^2\alpha}{da^2} - a \frac{d\alpha}{da} \right] + a \left[ h - a \frac{dh}{da} \right] = 0, \tag{40}
\]

which has the solution

\[
\alpha = c_1 a + \frac{c_2}{a} \quad \text{and} \quad h = \frac{\bar{c}}{a}, \tag{41}
\]

where, \( a \) being dimensionless, \( c_1 \) and \( c_2 \) have the same dimensions. We can further fix \( \alpha \) to be dimensionless; this fixes the dimensions of \( \beta \) to be \([\beta] = M^2\). Then also \([\bar{c}] = M^2\), so we have

\[
\beta = - \left[ 3c_1 + \frac{c_2}{a^2} \right] \frac{f_R}{f_{RR}} + \frac{\bar{c}}{a f_{RR}}. \tag{42}
\]

We can now use the expressions for \( \alpha \) and \( \beta \) in equation (34) as follows:

\[
f_R = \frac{3a(c_1 a^2 + c_2) \rho - \bar{c}(a^2 R + 6\kappa)}{2a(c_2 R - 6c_1\kappa)} + \frac{(c_1 a^2 + c_2) \rho_0}{2a^4(c_2 R - 6c_1\kappa)}, \tag{43}
\]

if \( c_2 R - 6\kappa c_1 \neq 0 \). It is clear that, for a general \( f \), it will not be possible to solve at the same time the Euler–Lagrange equation and this constraint. Therefore we have to use the Noether constraint in order to find the subset of those \( f \) which make this possible. As we shall see later, it is convenient to look for a parametric solution in the form \([H(a), f(R(a))]\). In this case, since \( f_R = f'/R' \), the Noether condition corresponds to the following ODE:

\[
\frac{f'(a)}{R'(a)} = \frac{3a(c_1 a^2 + c_2) f(a) - \bar{c}(a^2 R(a) + 6\kappa)}{2a(c_2 R(a) - 6c_1\kappa)} + \frac{(c_1 a^2 + c_2) \rho_0}{2a^4(c_2 R(a) - 6c_1\kappa)}. \tag{44}
\]

It should be noted that this change of variable is defined only if \( R' \neq 0 \), that is if \( R \) is not constant during the evolution. When this happens equation (34) or (45) sets \( a = a_0 = \) constant, which corresponds to an uninteresting solution.
Any Euler–Lagrange solution, by definition, satisfies the Einstein equations. However we will show that there are forms of $f(R)$ for which a subset of those solution will also be a Noether solution. In fact, equation (43) can also be rewritten as
\begin{equation}
c_{1} a^{2}(\rho_{r0} + 3a^{4}f + 12\kappa a^{2}f_{R}) + c_{2}[\rho_{m0} + a^{4}(3f - 2Rf_{R})] = \bar{c}a^{3}(a^{2}R + 6\kappa).
\end{equation}
Therefore we look for a family of solutions that, as a Noether symmetry, gives a class of $f(R)$ models.

This symmetry implies the existence of the following constant of motion:
\begin{equation}
\alpha(6f_{RR}a^{2}\dot{R} + 12f_{R}a\ddot{a}) + \beta(6f_{RR}a^{2}\dot{a}) = 6\mu_{0}^{3} = \text{constant},
\end{equation}
where $\mu_{0}$ has the dimensions of a mass. Equation (46) can be recast in the form
\begin{equation}
\frac{d(f_{R})}{dt} = f_{RR}\dot{R} = \frac{\mu_{0}^{3}}{a(c_{1}a^{2} + c_{2})} + \frac{c_{1}a^{2} - c_{2}}{c_{1}a^{2} + c_{2}} f_{R} H - \frac{\bar{c}a}{c_{1}a^{2} + c_{2}} H,
\end{equation}
or, using the time parameter $a$,
\begin{equation}
aH(a) \left( \frac{f^{\prime}(a)}{R^\prime(a)} - \frac{f(a)R^{\prime\prime}(a)}{R^\prime(a)^{2}} \right) - \frac{(a^{2}c_{1} - c_{2}) H(a)f^{\prime}(a)}{(c_{1}a^{2} + c_{2}) R^\prime(a)} = \frac{\mu_{0}^{3}}{a(c_{1}a^{2} + c_{2})} - \frac{\bar{c}a}{c_{1}a^{2} + c_{2}} H(a).
\end{equation}
Once equation (44) is solved, because the Noether constraint is satisfied, the solution $[H(a), f(R(a))]$ will automatically solve also (48) for a particular $\mu_{0}$. Equation (46) can be used to reduce the order of the Friedmann equation. In fact, writing equation (8) as
\begin{equation}
f - 6f_{RR}\dot{R}H - 6f_{R}H^{2} - f_{R} \left( R + \frac{6\kappa}{a^{2}} \right) - \frac{\rho_{m0}}{a^{3}} - \frac{\rho_{r0}}{a^{4}} = 0,
\end{equation}
we have
\begin{equation}
f - \frac{12c_{1}a^{2}}{c_{1}a^{2} + c_{2}} f_{R} H^{2} - f_{R} \left( R + \frac{6\kappa}{a^{2}} \right) + \frac{6\bar{c}a}{c_{1}a^{2} + c_{2}} H^{2} - \frac{6\mu_{0}^{3} H}{a(c_{1}a^{2} + c_{2})} + \frac{\rho_{m0}}{a^{3}} + \frac{\rho_{r0}}{a^{4}},
\end{equation}
where $f_{R}$ is given by (43). We will use this relation in order to find exact cosmological solutions. Namely, we will search for solutions depending on the constant of motion $\mu_{0}$ determined by the Noether symmetry.

5. Exact cosmological solutions

In order to find exact cosmological solutions, let us discuss the Noether condition equation (45) and the dynamical system (8), (10) with respect to the values of the integration constants $c_{1,2}$, the structural parameters $k, \rho_{r0}, \rho_{m0}$ and the Noether charge $\mu_{0}$. Beside cosmological solutions, the explicit form of $f(R)$ will also be fixed in the various cases. As we shall see later on, analytical solutions can be easily found for the case where both $\bar{c}$ and $\mu_{0}$ vanish at the same time. Therefore throughout this section, except one subsection, we will set $\bar{c} = 0$.

5.1. Case $c_{1} = 0$

In this case, the Noether condition (45) reduces to
\begin{equation}
2Rf_{R} - 3f = \frac{\rho_{r0}}{a^{4}}.
\end{equation}
This relation gives

\[ f = f_0 \left( \frac{R}{R_0} \right)^{3/2}. \]  

(52)

This solution, for the vacuum case \( \rho_{r0} = \rho_{m0} = 0 \), has already been found [48]. The absence of a ghost imposes that \( f_R < 0 \), i.e. \( f_0 > 0 \) since \( R_0 < 0 \). In the case of dust and no radiation (\( \rho_{m0} \neq 0, \rho_{r0} = 0 \)), one can substitute equation (52) into (50) and find

\[ \left( \frac{R}{R_0} \right)^{3/2} + \frac{18\kappa}{a^2 R_0} \left( \frac{R}{R_0} \right)^{1/2} = -\frac{12\mu_0^2 H}{c_2 a f_0} - \frac{2\rho_{m0}}{a^3 f_0}. \]  

(53)

(1) \( k = 0 \). In this case, for consistency, we need the right-hand side of (53) to be positive. If \( \mu_0 = 0 \) (the case for which analytical solutions could be given), this is impossible as \( f_0 > 0 \); therefore there is no ghost-free solution. For the more general case \( \mu_0^2 / c_2 < 0 \), there could be a physical solution: the non-linearity of the equations does not allow us to find analytical solutions for this case. Nevertheless, solutions (to be found numerically) may still exist.

(2) \( k \neq 0 \). The Ricci scalar can be found as the solution of equation (53). For \( \mu_0 = 0 \), we have a cubic equation in \( (R/R_0)^{1/2} \), for which a real solution always exists (which may not be positive though). Looking at equation (53), the case \( \mu_0 = 0, k = -1 \) has no ghost-free solutions \( (f_0 < 0) \). Also the case \( \mu_0 = 0, k = 1 \) has no solution, because we have

\[ \sqrt{\frac{R}{R_0}} = \left[ \frac{\tilde{B}_0^{1/3}}{f_0 R_0} - \frac{6\kappa f_0}{\tilde{B}_0^{1/3}} \right] \frac{1}{a}, \]  

(54)

where we have defined the constant

\[ \tilde{B}_0 = \sqrt{f_0^4 \rho_{m0}^2 R_0^6 + 216 f_0^6 \kappa^3 R_0^3} - f_0^2 \rho_{m0} R_0^3, \]  

(55)

which implies that \( (f_0 / \rho_{m0})^2 (\kappa/R_0)^3 > -1/216 \). If this is the case, then, since \( R_0 < 0 \), \( \tilde{B}_0 > 0 \). However, this would lead to a negative value for \( (R/R_0)^{1/2} \).

5.1.2. Dust and radiation case. In this case we have

\[ f_R = \frac{3}{2} \frac{f}{R} + \frac{\rho_{r0}}{2a^4 R}. \]  

(56)

Once again, in order to have \( f_R < 0 \) and \( R < 0 \) during the evolution of the Universe one requires

\[ f > -\frac{\rho_{r0}}{3a^2}. \]  

(57)

If we substitute the expression for \( f_R \) into the reduced Friedmann equation (50) we find

\[ f = -\frac{12\mu_0^2 a HR}{c_2 (Ra^2 + 18\kappa)} - \frac{6\kappa \rho_{r0}}{a^4 (Ra^2 + 18\kappa)} - \frac{3\rho_{r0} R}{a^2 (Ra^2 + 18\kappa)} - \frac{2\rho_{m0} R}{a (Ra^2 + 18\kappa)}. \]  

(58)

This relation gives \( f \) as a function of \( a \) as \( R = R(a) \). It has to be the case that \( c_2 \neq 0 \); otherwise the Noether condition becomes trivial. This expression can be inserted
back into (56). Assuming $R = R(a)$ as a monotonic function of $a$, one finds that $f_R = (df/da)/(dR/da)$, and equation (51) becomes a differential equation for $R(a)$, which can be written as

$$R' = \frac{6}{a^3(18a^3H\mu_0^3 + 4c_2\rho_{r0} + 3ac_2\rho_{m0}) (Ra^2 + 6\kappa)} \times \{- R^2[2a^3(H - aH')\mu_0^3 + c_2(2\rho_{r0} + a\rho_{m0})]a^4 + 6\kappa R[6a^3\mu_0^3(H + aH') - c_2(4\rho_{r0} + a\rho_{m0})]a^2 - 72c_2\kappa^2\rho_{r0}\},$$

(59)

where the prime denotes differentiation with respect to the scale factor $a$. Equation (59) can be further rewritten as a second-order differential equation in $H(a)$, by using equation (10),

$$R = -12H^2 - 6aH' - 6\frac{\kappa}{a^2}.$$  

(60)

Substituting (60) into (59) one finds

$$H'' = -\frac{1}{a^4H^2 (18a^3H\mu_0^3 + 4c_2\rho_{r0} + 3ac_2\rho_{m0})} \times \{24a\kappa^2\mu_0^3 + H[a^2(6a^3H\mu_0^3 + 4c_2\rho_{r0} + 3ac_2\rho_{m0})H^2 + a[12a\kappa\mu_0^3 + H(78a^3H\mu_0^3 + 32c_2\rho_{r0} + 21ac_2\rho_{m0})]H' + 12H(2a^3H\mu_0^3 + 2c_2\rho_{r0} + ac_2\rho_{m0})] - 8c_2\kappa\rho_{r0}]\}.$$  

(61)

This differential equation selects those $f(R)$ models which satisfy, at the same time, both the Friedmann equation and the Noether condition. It has to be stressed that, having chosen $a$ as the time variable, finding the $H(a)$s which solve (61) uniquely fixes the metric tensor. Hence, $H(a)$ represents a fully solved exact solution for the Einstein equations. Of course, if one wants to know the link between $a$ and the proper time, $a = a(t)$, one needs to find the integral $t = \int da/(aH)$.

The case $\mu_0 = 0$ is interesting as it allows us to find analytical solutions, as the differential equation becomes (second order and) linear for the variable $H^2$. In this case, the solution of the equation will be a family $H = H(a, d_1, d_2, c_2, \mu_0, \kappa, \rho_{r0}, \rho_{m0})$, where $d_{1,2}$ are two constants coming from the integration of equation (61). In turn, by using equation (60), it is possible to define a function $R = R(a, d_1, d_2, c_2, \mu_0, \rho_{r0}, \rho_{m0})$, which can then be substituted into equation (58) in order to find the explicit parametric form of $f(R)$, i.e. $f = f(a, d_1, d_2, c_2, \mu_0, \rho_{r0}, \rho_{m0})$. In other words, we find the explicit parametric form for $f(R)$ where the parameter used to describe the $f(R)$ is the scale factor $a$ (see also [21] for a comparison with observations; however, in that case, the adopted $f(R)$ models were constructed by phenomenological considerations and not derived from some first principle, such as the existence of symmetries discussed here).

We can distinguish some relevant cases.

(1) $k = 0$, $\mu_0 = 0$. In this case, by exactly integrating equation (61), we find

$$H^2 = d_2^2 \frac{d_1 + 8a\rho_{r0} + 3\rho_{m0}a^2}{a^4},$$

(62)

where $d_{1,2}$ are integration constants, with $[d_1] = M^4$ and $[d_2] = M^{-2}$. This expression for $H(a)$ together with (58) and (60) form a solution for the set of ODEs (9), and (44), so equation (48) is satisfied giving $\mu_0 = 0$. Although this solution is analytical
it cannot be accepted because it allows for a negative Newton constant. In fact, equation (57) cannot be satisfied by equation (58) if \( k = 0, \mu_0 = 0 \). However the non-linear case \( \mu_0/c^2 < 0 \) could actually lead to physical solutions (to be discussed elsewhere in a forthcoming paper). For the same reason, the case \( k = -1, \mu_0 = 0 \) should also be rejected.

(2) \( k = 1, \mu_0 = 0 \). As long as \( R < -18\kappa/a^2 \), the second term in the lhs of equation (58) becomes positive, allowing for the possibility of finding a physical solution. The integration of (61) leads to

\[
H^2 = \left( \sqrt{2}d_1 - \frac{32\rho_{r0}^2\kappa}{9\rho_{m0}^2} \right) \frac{1}{a^4} + \left( 8d_2\rho_{r0} - \frac{16\rho_{r0}\kappa}{3\rho_{m0}} \right) \frac{1}{a^3} + \frac{3d_2\rho_{m0}}{a^2}, \tag{63}
\]

with \([d_1] = M^2\) and \([d_2] = M^{-2}\). In order to find \(d_1\) and \(d_2\) one can fit this formula with the standard Friedmann equation of GR with only matter, radiation and curvature. Therefore, one has to consider

\[
\sqrt{2}d_1 - \frac{32\rho_{r0}^2\kappa}{9\rho_{m0}^2} = H_0^2\Omega_{r0}^{eff},
\]

\[
8d_2\rho_{r0} - \frac{16\rho_{r0}\kappa}{3\rho_{m0}} = H_0^2\Omega_{m0}^{eff},
\]

\[
3d_2\rho_{m0} = H_0^2\Omega_{k0}^{eff},
\]

but this system admits no solutions as one finds

\[
\kappa = \frac{1}{2}H_0^2\Omega_{k0}^{eff} - \frac{3}{16}\rho_{r0}H_0^2\Omega_{m0}^{eff} < 0 \tag{67}
\]

using today’s data [52].

5.2. Case \( c_2 = 0 \)

In this case, the Noether condition (45) reduces to

\[
\rho_{r0} + 3a^4f + 12\kappa a^2f_R = 0. \tag{68}
\]

5.2.1. Vacuum and dust only case. In this case we have \( \rho_{r0} = 0 \), and a flat Universe cannot be a solution as one would obtain \( f = 0 \). Considering \( k \neq 0 \) one finds

\[
f_R = -\frac{a^2f}{4\kappa}. \tag{69}
\]

Since \( f_R < 0 \), then \( f \) is positive when \( k < 0 \) and vice versa. Substituting this into the Friedmann equation one finds

\[
\{a^3c_1[(12H^2 + R)a^2 + 10\kappa]\}f = 4\kappa(6H\mu_0^3 + c_1\rho_{m0}). \tag{70}
\]
Restricting ourselves to the study of the simple and linear case of a vanishing $\mu_0$, we can distinguish two cases:

1. $\rho_{m0} = 0, \mu_0 = 0$. In this case one needs to impose

$$R = -12H^2 - 10\frac{\kappa}{a^2},$$

which, together with the definition of $R$, gives

$$H^2 = 2d_1 - \frac{2\kappa}{3a^2},$$

where $d_1$ is a constant of integration with dimensions $M^2$. This behavior describes a Universe with only a cosmological constant and curvature. Equation (68) can now be solved for $f(a)$ giving

$$f = \frac{d_2}{a} = d_2 \left[ -\frac{R + 24d_1}{2\kappa} \right]^{1/2},$$

where $d_2$ is a constant of integration with dimensions $M^4$.

2. $\rho_{m0} \neq 0, \mu_0 = 0$. In this case the Friedmann equation and (69) give

$$f = -\frac{4\kappa\rho_{m0}}{(12H^2 + R) a^5 + 10\kappa a^3}.$$ 

Substituting this expression in (69), and using the definition for $R$ in terms of $H(a)$ one finds a linear second-order differential equation in $H^2(a)$, which has the solution

$$H^2 = \frac{d_1}{2a^4} + 2d_2 - \frac{2\kappa}{3a^2},$$

where $d_{1,2}$ are integration constants, and $[d_1] = [d_2] = M^2$. Therefore one has

$$R = -24d_2 - \frac{2\kappa}{a^2},$$

$$f = -\frac{2\kappa \rho_{m0}}{3ad_1}.$$ 

5.2.2. Radiation and dust case. Also in this case, we have three possibilities, according to the values of $k$.

1. $k = 0$. In this case one finds that

$$f = -\frac{\rho_{r0}}{3a^4}.$$ 

Therefore we have

$$f_R = \frac{f'}{R'} = \frac{4}{3} \frac{\rho_{r0}}{a^5 R'}.$$ 

A well-behaved background evolution requires, with our conventions, $R' > 0$, so $f_R > 0$. This means a negative effective Newton constant, i.e. the solution cannot be accepted.
where $c \neq 0$. In this case, using equation (68) one finds
\[
f_R = -\frac{\rho_\nu}{12\kappa a^2} - \frac{f a^2}{4\kappa},
\]
and then using Friedmann equation (50) one can solve for $f$, as follows:
\[
f = -c_1 (12 R^2) \rho_\nu a^2 + 12\kappa (6 H \mu_0^3 + c_1 \rho_\nu) a + 6c_1 \kappa \rho_\nu.
\]
By plugging this relation into the Noether condition (68), and using the definition of $H$ in terms of $H, H'$, and $a$, one finds the following differential equation for $H(a)$:
\[
H'' = \{aH(-18aH\mu_0^3 + 3ac_1 \rho_\nu + 4c_1 \rho_\nu)H^2 a^4 - 3(aH(30aH\mu_0^3 + 5ac_1 \rho_\nu + 8c_1 \rho_\nu))
- 4\kappa \mu_0^3 H^2 a^2\} \{a^2 H^2 (18aH\mu_0^3 + 3ac_1 \rho_\nu + 4c_1 \rho_\nu)\}^{-1}
+ \frac{4\kappa (6aH\mu_0^3 + ac_1 \rho_\nu + 2c_1 \rho_\nu)}{a^2 H^2 (18aH\mu_0^3 + 3ac_1 \rho_\nu + 4c_1 \rho_\nu)}.
\]
In the case $\mu = 0, \rho_\nu \neq 0$, this differential equation can be exactly integrated to give
\[
H^2 = \frac{256\kappa \rho_\nu^3}{405a^5 \rho_\nu^3} + \frac{16\kappa \rho_\nu}{27a^4 \rho_\nu^3} + \frac{8d_1 \rho_\nu}{5a^5} - \frac{2\kappa}{3a^2} + \frac{3\rho_\nu d_1}{2a^4} + 2d_2,
\]
where $d_1, d_2$ are two constants of integration with dimensions $[d_1] = M^{-2} = [d_2]^{-1}$.
It is interesting to note the presence of a new cosmological term in this Friedmann equation, which goes as $a^{-5}$, which would correspond to a matter term with equation of state parameter $w = 2/3$. If $\mu = 0, \rho_\nu = 0$, i.e. a Universe filled with radiation only, equation (82) has the following solution:
\[
H^2 = 2d_2 + \frac{2d_1}{5a^5} - \frac{2\kappa}{3a^2},
\]
with $[d_1] = [d_2] = M^2$.

5.3. Case $c_1, c_2 \neq 0$.

In this case, one can divide equation (45) by $c_1$, finding
\[
f_R = -\frac{a^2 + c_3 \rho_\nu + 3a^4 f}{c_3 R - 6k a^4},
\]
where $c_3 = c_2 / c_1 \neq 0$. This implies that
\[
f_{RR}\dot{R} = \frac{\dot{\mu}_0^3}{a(a^2 + c_3)} + \frac{a^2 - c_3 f_R}{a^2 + c_3} f_H,
\]
where $\dot{\mu}_0^3 = \mu_0^3 / c_1$.

The Friedmann equation, equation (50), can be rewritten as
\[
f - \frac{12a^2}{a^2 + c_3} f_R H^2 = f_R \left( R + \frac{6k}{a^2} \right) = \frac{6\dot{\mu}_0^3 H}{a(a^2 + c_3)} + \frac{\rho_\nu d_1}{a^4} + \frac{\rho_\nu}{a^4}.
\]
By substituting (85) into (87), and solving for $f$, one finds

$$f = \frac{12\tilde{\mu}^3_0 a^5 H(6k - c_3 R)}{a^4(a^2 + c_3)[3(12H^2 + R)a^4 + (30k + c_3 R)a^2 + 18c_3 k]} - \frac{\rho_{r0}(12H^2 + R)a^4 + 2\rho_{n0}(c_3 R - 6k)a^3 + 3\rho_{r0}(c_3 R - 2k)a^2 + 6c_3 k \rho_{r0}}{a^4[3(12H^2 + R)a^4 + (30k + c_3 R)a^2 + 18c_3 k]},$$

(88)

which means that the Noether symmetry, combined with the dynamics, determines the form of $f$. In this case $f$ is a function of $a$ since both $R$ and $H$ are functions of $a$. We can still go further by using the same trick as was used in the previous section, i.e. considering $f$ as an implicit function of $a$ in the Noether condition (85). Since $f = f(R(a))$ one finds

$$f_R = \frac{df}{dR} = \frac{df}{da} = \frac{f'}{R'}.$$  

(89)

Plugging equations (88) and (89) into (85), one finds a second-order differential equation for $H$, as follows:

$$H'' = -\frac{1}{a^4(a^2 + c_3)(3a^2 + c_3)H^2[18\tilde{\mu}^3_0 H^5 a^3 + (a^2 + c_3)(4\rho_{r0} + 3a\rho_{n0})]} \times \left\{-24c_3(3a^2 + c_3)\tilde{\mu}^3_0 H^4 a^5 - 24(a^2 + c_3)^2 k^2 \tilde{\mu}^3_0 a 
- H^2 \left[6(3a^2 + c_3)^2 \tilde{\mu}^3_0 H^2 a^4 + 24(-3a^4 - 2c_3 a^2 + c_3^2)k^2 \tilde{\mu}^3_0 
+ (a^2 + c_3)^2(45\rho_{n0} a^3 + 72\rho_{r0} a^2 + 21c_3\rho_{n0} a + 32c_3^2 \rho_{r0}) H'] a^3 
- 6H^3 \left[(3a^2 + c_3)(15a^2 + 13c_3)\tilde{\mu}^3_0 H' a^4 + 2c_3(a^2 + c_3)^2(2\rho_{r0} + a\rho_{n0})] a^2 
- (a^2 + c_3)H[a^4 H'[12(c_3 - 3a^2)k^2 \tilde{\mu}^3_0 + (a^2 + c_3)(3a^2 + c_3)(4\rho_{r0} + 3a\rho_{n0}) H'] 
- 4(a^2 + c_3)k(3\rho_{n0} a^3 + 6\rho_{r0} a^2 + 2c_3 \rho_{r0}) \right]\right\}. $$

(90)

This differential equation defines the dynamics of the Noether solutions for a generic $f(R)$ model compatible with the Noether symmetry. This result is relevant since there is a free parameter $c_3$, which, together with the initial conditions for $H_0$ and $H'_0$, uniquely specifies the dynamics. This non-linear ODE is still of second order in $H(a)$ as is the 0,0 Einstein equation for any $f(R)$ theory. However, there is a huge improvement as this equation is independent of the explicit form $f(R)$, having as the only unknown parameters two real numbers, $c_3$ and $\mu_0$, the Noether charge. This also says that for any value of the Noether charge there is a solution, the solution of (90). Therefore all the solutions of (90), as $c_3, \mu_0$ vary, represent the whole set of Noether-charged cosmological solutions of the $f(R)$ theories.

5.3.1. Vacuum and pure dust case. In this case equation (85) reduces to

$$f_R = \frac{3f(a^2 + c_3)}{2(Rc_3 - 6\kappa)},$$

(91)

whereas $f$ can be written as

$$f = \frac{2(6\kappa - Rc_3)((6H\tilde{\mu}^3_0 + \rho_{n0}) a^2 + \rho_{n0} c_3)}{a(a^2 + c_3)(3(12H^2 + R)a^4 + (30\kappa + Rc_3)a^2 + 18\kappa c_3)}.$$  

(92)
The case $\rho_{m0} = 0, \mu_0 = 0$ admits no solutions; therefore, as before, we will only discuss the case $\mu_0 = 0, \rho_{m0} \neq 0$, for which we can recast $f$ in the following form:

$$f = 3 \left( 12H^2 + R \right) a^4 + (30\kappa + Rc_3) a^2 + 18\kappa c_3. \quad (93)$$

Inserting this relation into (91) together with the definition of $\rho$ whose general solution reads

$$H'' = \frac{4c_3H^2 - a \left( 15a^2 + 7c_3 \right) H'H - a^2 \left( 3a^2 + c_3 \right) H'^2 + 4\kappa}{a^2 \left( 3a^2 + c_3 \right) H},$$

whose general solution reads

$$H^2 = -\frac{c_3\kappa}{9a^4} - \frac{2\kappa}{3a^2} + \frac{2d_1}{a^2} + \frac{2c_3d_2}{a^2} + 3d_2. \quad (95)$$

5.3.2. Pure radiation case. Once again, studying equation (90) for the case $\mu_0 = 0$ and $\rho_{m0} = 0$, we find the following equation:

$$(H^2)' = -\frac{18a^2 + 8c_3}{a \left( 3a^2 + c_3 \right)} (H^2)' - \frac{12c_3H^2}{a^2 \left( 3a^2 + c_3 \right)} + \frac{2k(6a^2 + 2c_3)}{a^4 \left( 3a^2 + c_3 \right)}. \quad (96)$$

The general solution, when $c_3 > 0$, for this ODE is

$$H^2 = \frac{3c_3d_1}{a^2} + \frac{27d_1}{c_3} + \frac{18d_1}{a^2} + \frac{5\sqrt{3}c_3d_2}{a^3} + \frac{9\sqrt{3}d_2}{a^3 \sqrt{c_3}} + \frac{4\kappa}{a \sqrt{c_3}} + \frac{2\kappa}{a^2} + \frac{3c_3d_2 \arctan \left( \sqrt{3a}/\sqrt{c_3} \right)}{a^4} + \frac{27d_2 \arctan \left( \sqrt{3a}/\sqrt{c_3} \right)}{c_3} + \frac{18d_2 \arctan \left( \sqrt{3a}/\sqrt{c_3} \right)}{a^2}, \quad (97)$$

whereas, for $c_3 < 0$, one finds

$$H^2 = \frac{3c_3d_1}{a^2} + \frac{27d_1}{c_3} + \frac{18d_1}{a^2} - \frac{5\sqrt{3}c_3d_2}{a^3} + \frac{9\sqrt{3}d_2}{a^3 \sqrt{-c_3}} + \frac{4\kappa}{a \sqrt{-c_3}} + \frac{2\kappa}{a^2} + \frac{3c_3d_2 \arctanh \left( \sqrt{3a}/\sqrt{-c_3} \right)}{a^4} + \frac{27d_2 \arctanh \left( \sqrt{3a}/\sqrt{-c_3} \right)}{c_3} + \frac{18d_2 \arctanh \left( \sqrt{3a}/\sqrt{-c_3} \right)}{a^2}. \quad (98)$$

Either expression for $H(a)$, together with equations (88) and (60), forms a solution for (9), and (44), and possesses $\mu_0 = 0$ Noether charge.

5.3.3. Matter and radiation case. Let us restrict our study to the case $\tilde{\mu} = 0$, for which we can find analytical solutions. Equation (90) reduces to

$$(H^2)'' = -\frac{(45\rho_{m0}a^2 + 72\rho_{r0}a^2 + 21c_3\rho_{m0}a + 32c_3\rho_{r0})}{a \left( 3a^2 + c_3 \right) \left( 4\rho_{r0} + 3a\rho_{m0} \right)} (H^2)' - \frac{24c_3 \left( \rho_{m0}a + 2\rho_{r0} \right) H^2}{a^2 \left( 3a^2 + c_3 \right) \left( 4\rho_{r0} + 3a\rho_{m0} \right)} + \frac{8k(3\rho_{m0}a^3 + 6\rho_{r0}a^2 + 2c_3\rho_{r0})}{a^4 \left( 3a^2 + c_3 \right) \left( 4\rho_{r0} + 3a\rho_{m0} \right)}.$$

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It is remarkable that this differential equation is linear in $H^2$. This makes the problem of solving it much easier. In fact, analytical solutions for $k = 0, \pm 1$ can be achieved. Let us discuss them.

(1) $k = 0$. The solution of equation (99) is

$$H^2 = \frac{4d_1 d_2 c_3^{9/2}}{a^4} + \frac{24d_1 d_2 c_3^{7/2}}{a^2} - \frac{\rho_{m0} d_2 c_3^{5/2}}{a^4} + 36d_1 d_2 c_3^{5/2}$$

$$+ \frac{2\sqrt{3}\rho_{r0} \arctan \left( \sqrt{3a/\sqrt{c_3}} \right) d_2 c_3^2}{a^4} + \frac{10\rho_{r0} d_2 c_3^{3/2}}{a^3}$$

$$+ \frac{12\sqrt{3}\rho_{r0} \arctan \left( \sqrt{3a/\sqrt{c_3}} \right) d_2 c_3}{a^2}$$

$$+ \frac{18\rho_{r0} d_2 \sqrt{c_3}}{a} + 18\sqrt{3}\rho_{r0} \arctan \left( \frac{\sqrt{3a}}{\sqrt{c_3}} \right) d_2, \quad (100)$$

where $d_1$ and $d_2$ are integration constants with dimensions $[d_1] = M^4$ and $[d_2] = M^{-2}$. This is clearly a deviation from standard GR, because there is a $1/a$ term, which leads to an accelerated behavior if it dominates. Furthermore there are terms, all involving $\rho_{r0}$, which include the arctangent of $a$, where $c_3$ is supposed to be positive. These terms have different behavior at low and high redshift. In fact since $\lim_{a \to 0} \arctan(a) \sim a$ at high redshifts, these terms behave as dust, $1/a$ and $a$ respectively, and are subdominant with respect to the radiation. On the other hand, since $\lim_{a \to \infty} \arctan(a) \sim \pi/2$ for large and positive $a$, these terms will behave as radiation, curvature and the cosmological constant respectively. It is also interesting that in order to have a true dust matter component at late times, it has to be the case that

$$10\rho_{r0} d_2 c_3^{3/2} = \frac{8\pi G}{3} \rho_{m0}. \quad (101)$$

This means that $\rho_{r0}$ behaves as the source of matter component in this modified Friedmann equation. A cosmological constant term is also present. It is determined by the integration constants of the Noether condition. As for the case $c_3 < 0$, the solution of equation (99) can be written as follows:

$$H^2 = -\frac{4d_1 d_2 (-c_3)^{9/2}}{a^4} + \frac{24d_1 d_2 (-c_3)^{7/2}}{a^2} + \frac{\rho_{m0} d_2 (-c_3)^{5/2}}{a^4} - 36d_1 d_2 (-c_3)^{5/2}$$

$$+ \frac{2\sqrt{3}\rho_{r0} \arctanh \left( \sqrt{3a/\sqrt{-c_3}} \right) d_2 c_3^2}{a^4} + \frac{10\rho_{r0} d_2 (-c_3)^{3/2}}{a^3}$$

$$+ \frac{12\sqrt{3}\rho_{r0} \arctanh \left( \sqrt{3a/\sqrt{-c_3}} \right) d_2 c_3}{a^2}$$

$$- \frac{18\rho_{r0} d_2 \sqrt{-c_3}}{a} + 18\sqrt{3}\rho_{r0} \arctanh \left( \frac{\sqrt{3a}}{\sqrt{-c_3}} \right) d_2. \quad (102)$$

For this solution, as a pedagogical example, more detailed calculations and a link with scalar–tensor theories are given in the appendix.
\( k \neq 0 \). The general solution is

\[
H^2 = \frac{32\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \rho_{r0}^3}{9\sqrt{3}a^3 \rho_{m0}^3 \sqrt{c_3}} - \frac{160\kappa \rho_{r0}^3}{27a^3 \rho_{m0}^3 c_3} - \frac{64\kappa \arctan\left(\frac{\sqrt{3}a}{\sqrt{c_3}}\right) \rho_{r0}^3}{3\sqrt{3}a^2 \rho_{m0}^3 c_3^{3/2}}
\]

Also in these cases we have interesting behaviors matching the main cosmological eras.

The integration constants \( d_1, d_2 \) have dimensions respectively \([d_1] = M^2\) and \([d_2] = M^{-2}\). The analysis, for both this and the previous case \((k = 0)\), of the set of parameters \(\{d_1, d_2, c_3\}\) which can be bounded by observations will be carried out in a forthcoming paper.

Equation (99), for the case \(c_3 < 0\), has the solution

\[
H^2 = \frac{32\kappa \arctanh\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \rho_{r0}^3}{9\sqrt{3}a^4 \rho_{m0}^3 \sqrt{-c_3}} - \frac{160\kappa \rho_{r0}^3}{27a^4 \rho_{m0}^3 c_3} - \frac{64\kappa \arctanh\left(\frac{\sqrt{3}a}{\sqrt{-c_3}}\right) \rho_{r0}^3}{3\sqrt{3}a^2 \rho_{m0}^3 (-c_3)^{3/2}}
\]

Also in these cases we have interesting behaviors matching the main cosmological eras.
It is worthy of note that once the free parameters are constrained by the data (the set of allowed parameters might be empty anyway), one can select physically interesting $f(R)$ models as in [21].

5.3.4. Non-linear case, $\bar{\rho}_0 \neq 0$. In this more general case, equation (90) cannot be written as a linear differential equation in $H^2$; therefore it is not possible to achieve an analytical general solution. However, after fixing initial conditions for $H$ and giving suitable values for the parameters, one can solve it numerically. These initial conditions fix, in turn, the $f(R)$ model and the behavior of $H(a)$.

5.3.5. General non-linear case, $\bar{c} \neq 0$ and $\bar{\rho}_0 \neq 0$. By using equation (43) inside equation (50) one finds the following expression for $f$:

\[
f = \frac{c_1 \bar{c} R (12H^2 + R) a^5}{(c_1 a^2 + c_2) \Delta} + \frac{\bar{c} R (12c_2 H^2 + 12c_1 \kappa + c_2 R) a^3}{(c_1 a^2 + c_2) \Delta} + \frac{2 (36c_1 \kappa H^3 \mu_0^3 - 6c_2 H R \mu_0^3 + 18c_1 \kappa c_2^2 + 6c_1^2 \kappa \rho_{m0} + 6c_2 \kappa R - c_1 c_2 \rho_{m0} R) a}{(c_1 a^2 + c_2) \Delta} - \frac{2c_2 (-18c_1 \kappa - 6c_1 \rho_{m0} \kappa + c_2 \rho_{m0} R)}{(c_1 a^2 + c_2) \Delta a} \rho_{r0} (12c_1 H^2 a^4 + c_1 R a^4 - 6c_1 \kappa a^2 + 3c_2 R a^2 + 6c_2 \kappa),
\]

where

\[
\Delta = 36c_1 H^2 a^4 + 3c_1 R a^4 + 30c_1 \kappa a^2 + c_2 R a^2 + 18c_2 \kappa.
\]

The Friedmann equation gives us the expression for $f$ in terms of $R(a)$, $H(a)$ and $a$. Equation (44), which can be rewritten here as

\[
\frac{f'(a)}{R'(a)} = \frac{3a(c_1 a^2 + c_2) f(a) - \bar{c}(a^2 R(a) + 6 \kappa)}{2a(c_2 R(a) - 6c_1 \kappa) + \frac{(c_1 a^2 + c_2) \rho_{r0}}{2a^4(c_2 R(a) - 6c_1 \kappa)},
\]

giving a dynamics for $f$, defines a second-order differential equation for $H$, given by

\[
H'' = \left[ (c_1 a^2 + c_2) H \Gamma \right]^{-1} H' \left( 12c_1^2 \bar{c} \kappa a^7 + 9c_1^3 \rho_{m0} a^7 + 54c_1^2 \mu^3 H a^7 + 12c_1^2 \rho_{r0} a^6 + 24c_1 c_2 \bar{c} \kappa a^5 + 21c_2^2 \rho_{m0} a^5 + 36c_1^2 c_2 \mu^3 H a^5 + 28c_1 c_2 \rho_{r0} a^4 + 12c_2 \bar{c} \kappa a^3 + 15c_1^2 \rho_{m0} a^3 + 6c_2^2 \mu^3 H a^3 + 20c_1 c_2 \rho_{r0} a^2 + 3c_2 \rho_{m0} a + 4c_2^2 \rho_{r0} \right) - \left[ a \left( c_1 a^2 + c_2 \right) H \Gamma \right]^{-1} H' \left( 54c_1^2 \bar{c} \mu^3 H a^9 + 108c_1 c_2 \bar{c} H a^7 \right.
\]

\[
- 270c_1^2 \mu^3 H^2 a^7 - 60c_1^2 \bar{c} \kappa H a^7 - 45c_1^3 \rho_{m0} H a^7 - 72c_1^2 \rho_{r0} H a^6
\]

\[
+ 36c_1^2 \kappa \mu^3 a^5 + 54c_1 c_2 \mu^3 H^2 a^5 - 324c_1 c_2 \mu^3 H^2 a^5 - 120c_1 c_2 \bar{c} \kappa H a^5 - 111c_1^2 c_2 \rho_{m0} H a^5 - 176c_1^2 c_2 \rho_{r0} H a^4 + 24c_1 c_2 \kappa \mu^3 a^3 - 78c_1^2 \mu^3 H^2 a^3.
\]
write the Einstein equation as a second-order differential equation for $R$ can be found using different methods [24]. Assuming, in general, a power law and do not admit cosmological solutions compatible with a Noether charge. Friedmann equations. In doing this, it is easy to show that, for $\kappa f_\mu^\nu \Gamma_{\mu\nu\rho} \propto \rho_m a^3$, whereas all other quantities ($H$ and $R$) are given functions of $a$. The same argument holds for the redshift $z$ [21].

5.4. Non-Noether solutions

In general it is not possible to find a solution of the Friedmann equations which is also a Noether symmetry since, in principle, such symmetries do not exist for any $f(R)$ theory. In general, a solution of the cosmological equations is not a solution compatible with the condition $L X L = 0$. This is a peculiar situation which holds only if conserved quantities (Noether’s charges) are intrinsically present in the structure of the theory (in our case, the form of $f(R)$). For example, imposing a power law solution, $a \propto t^p$, defines a function of $R = R(a)$, which can be put into the Noether symmetry equations in order to find $f = f(R(a))$. Finally one can substitute the expressions for $f(a), R(a)$, and $H$ in the Friedmann equations. In doing this, it is easy to show that, for $k = 0$, there are no simple power law solutions compatible with a Noether charge.

The method discussed above allows one to discriminate between theories which admit and do not admit cosmological solutions compatible with a Noether charge.

It is also clear that power law solutions do exist in general for $f(R)$ models, but they can be found using different methods [24]. Assuming, in general, a power law $H(a)$, one finds $R$ as a function of $a$, and then, in principle, $f = f(R(a))$. It is therefore possible to write the Einstein equation as a second-order differential equation for $f$ as a function of $a$, whereas all other quantities ($H$ and $R$) are given functions of $a$. The same argument holds for the redshift $z$ [21].

For example, let us rewrite the Friedmann equation (8) as

$$f - 6f_R R H - 6f_R H^2 - f_R \left( R + \frac{6k}{a^2} \right) = \frac{\rho_m}{a^3} + \frac{\rho_r}{a^4}, \tag{110}$$

and let us consider $H = \bar{H}(a)$ and $R = \bar{R}(a)$ as given functions of $a$, where, as above,

$$\bar{R} = -12 \bar{H}^2 - 6a \bar{H} \bar{H}' - 6 \frac{k}{a^2}. \tag{111}$$
The Friedmann equation can be written as
\[ f'' + \left[ \frac{1}{a} - \frac{\bar{R}''}{\bar{R}'} + \frac{1}{6aH^2} \left( \bar{R} + \frac{6k}{a^2} \right) \right] f' - \frac{\bar{R}'}{6aH^2} f = -\frac{\rho_{m0}a + \rho_{r0}}{6a^5H^2} \bar{R}' . \] (112)

This is a second-order linear equation in \( f \), whose general solution depends on two parameters, \( f_0 \) and \( f'_0 \). Specifically, the equation being linear, the general solution is the linear combination of two solutions of the homogeneous ODE plus a particular solution. It is then clear that more than one \( f(R) \) model can have the same behavior for \( H(a) \), i.e. more theories share the same cosmological evolution. This situation is due to the fact that one has a fourth-order gravity theory. The singular points of this differential equation are those for which either \( \bar{H} \) or \( d\bar{R}/da \) vanishes.

Starting from these considerations, interesting classes of solutions can be found.

5.4.1. Radiation solutions. Let us seek for all the \( f(R) \) models which have the particular solution \( a = \sqrt{t/t_0} \), which means
\[ \bar{H} = \frac{1}{2t_0a^2} = \frac{H_0}{a^2}, \quad \text{and so} \quad \bar{R} = -\frac{6k}{a^2}, \] (113)
where \( H_0 \equiv (2t_0)^{-1} \). We have three interesting cases.

1. For \( k = 0 \), we have \( R = 0 \), leading to the Friedmann equation
\[ f(0) - 6f_R(0) \bar{H}^2 = \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4}, \] (114)
which, if \( \rho_{m0} \neq 0 \), cannot be solved for \( \bar{H} \sim a^{-2} \) since \( f(0) \) and \( f'(0) \) cannot be functions of \( a \), but only constants. If \( \rho_{m0} = 0 \), standard GR is of course recovered.

2. For the case \( k = -1 \) we have the following differential equation for \( f \):
\[ f'' + \frac{4}{a} f' + \frac{2\kappa}{H_0^2} f = \frac{2\kappa(\rho_{r0} + a\rho_{m0})}{H_0^2a^4}, \] (115)
whose general solution can be written as
\[ R = -\frac{6\kappa}{a^2}, \] (116)
\[ f = \frac{\sqrt{a\sqrt{\kappa}H_0d_2 \cos (a\sqrt{2\kappa}/H_0) H_0^2}}{\sqrt{2a^{7/2}\kappa\sqrt{\pi}}} - \frac{\sqrt{a\sqrt{\kappa}/H_0d_1 \sin (a\sqrt{2\kappa}/H_0) H_0^3}}{\sqrt{2a^{7/2}\kappa\sqrt{\pi}}} - \frac{\sqrt{2\sqrt{a\sqrt{\kappa}/H_0d_1 \cos (a\sqrt{2\kappa}/H_0) H_0)}}{a^{5/2}\sqrt{\kappa\sqrt{\pi}}} - \frac{\sqrt{2\sqrt{a\sqrt{\kappa}/H_0d_2 \sin (a\sqrt{2\kappa}/H_0) H_0)}}{a^{5/2}\sqrt{\kappa\sqrt{\pi}}} + \frac{\rho_{m0}}{a^3} + \rho_{r0}\sqrt{\kappa} \text{Ci} \left( \sqrt{2a\sqrt{\kappa}/H_0} \right) \sin (a\sqrt{2\kappa}/H_0) \frac{\sqrt{2a^3H_0}}{\sqrt{2a^3H_0}} - \rho_{r0}\sqrt{\kappa} \cos (a\sqrt{2\kappa}/H_0) \text{Si} \left( a\sqrt{2\kappa}/H_0 \right) \frac{\sqrt{2a^3H_0}}{\sqrt{2a^3H_0}}. \]
5.4.2. Matter solutions.

In this case, we search for a behavior, that is

\[ f(R) \text{ cosmology from Noether’s symmetry} \]

\[ R = \frac{6\kappa}{a^2}, \]

\[ f = \sqrt{a\sqrt{2\kappa}/H_0}d_1 \cosh \left( \sqrt{2\kappa a}/H_0 \right) H_0^2 \]

\[ + \frac{\rho_{0}\kappa \cos \left( a\sqrt{-2\kappa}/H_0 \right) \text{Ci} \left( a\sqrt{-2\kappa}/H_0 \right)}{a^2 H_0^2} \]

\[ + \frac{\rho_{0}\kappa \sin \left( a\sqrt{-2\kappa}/H_0 \right) \text{Si} \left( a\sqrt{-2\kappa}/H_0 \right)}{a^2 H_0^2}, \quad (117) \]

where the sin-integral and cos-integral functions, Si and Ci respectively, are defined as

\[ \text{Si}(x) = \int_0^x \sin(t) \frac{dt}{t} \quad \text{Ci}(x) = -\int_x^\infty \cos(t) \frac{dt}{t}. \quad (118) \]

The integration constants \( d_{1,2} \) have dimensions \( [d_1] = [d_2] = M^4 \).

(3) Along the same lines, the case \( k = 1 \) has the following solution:

\[ R = \frac{6\kappa}{a^2}, \]

\[ f = \sqrt{a\sqrt{2\kappa}/H_0}d_1 \cosh \left( \sqrt{2\kappa a}/H_0 \right) H_0^2 \]

\[ - \frac{\sqrt{2\kappa a}}{a^{5/2} \sqrt{\pi}} a_{d_1} \sinh \left( \sqrt{2\kappa a}/H_0 \right) H_0^2 \]

\[ - \frac{\sqrt{2\kappa a}}{a^{5/2} \sqrt{\pi}} a_{d_1} \sinh \left( \sqrt{2\kappa a}/H_0 \right) H_0^2 \]

\[ + \frac{\rho_{0}\kappa \text{Chi} \left( \sqrt{2\kappa a}/H_0 \right) \sinh \left( \sqrt{2\kappa a}/H_0 \right)}{a^{5/2} \sqrt{\pi}} \]

\[ + \frac{\rho_{0}\kappa \cos \left( \sqrt{2\kappa a}/H_0 \right) \text{Shi} \left( \sqrt{2\kappa a}/H_0 \right)}{a^{5/2} \sqrt{\pi}} \]

\[ + \frac{\rho_{0}\kappa \cosh \left( \sqrt{2\kappa a}/H_0 \right) \text{Chi} \left( \sqrt{2\kappa a}/H_0 \right)}{a^{5/2} \sqrt{\pi}} \]

\[ + \frac{\rho_{0}\kappa \sinh \left( \sqrt{2\kappa a}/H_0 \right) \text{Shi} \left( \sqrt{2\kappa a}/H_0 \right)}{a^{5/2} \sqrt{\pi}} \], \quad (120) \]

where the hyperbolic sin-integral and cos-integral, Shi and Chi respectively, are defined as

\[ \text{Shi}(x) = \int_0^x \sinh(t) \frac{dt}{t} \quad \text{Chi}(x) = \gamma_{EM} \ln(x) + \int_0^x \cosh(t) - 1 \frac{dt}{t}, \quad (121) \]

and \( \gamma_{EM} \approx 0.577 \) is the Euler–Mascheroni constant. Both \( d_1 \) and \( d_2 \) are integration constants with dimensions \( M^4 \).

5.4.2. Matter solutions. In this case, we search for \( f(R) \) models which have a dust matter behavior, that is \( a = (t/t_0)^{2/3} \),

\[ \bar{H} = \frac{2}{3t_0a^{3/2}} = \frac{H_0}{a^{3/2}}, \quad \text{and} \quad \bar{R} = -\frac{2(2/t_0^2 + 9ka)}{3a^3}, \quad (122) \]
where \( H_0 \equiv 2/(3t_0) \). For the case \( k = 0 \), we find the explicit analytic solution

\[
R = \frac{4}{3t_0^2 a^3},
\]

(123)

\[
f(a) = a^{-\left(7+\sqrt{73}\right)/4} \left( d_1 a^{\sqrt{73}/2} + d_2 \right) + \frac{\rho_m(a) - 6\rho_r}{2a^4}.
\]

(124)

This is a two-parameter family of solutions, depending on the two integration constants \( d_{1,2} \) both with dimensions \( M^4 \). The Einstein–Hilbert case \( f(R) = R \) belongs to this family, when \( d_1, d_2, \) and \( \rho_{m0} \) all vanish.

5.4.3. Exponential solutions. In this case, we look for the behavior

\[
\bar{H} = H_0 = \text{constant}, \quad \text{which is } \bar{R} = -12H_0^2 - \frac{6k}{a^2}.
\]

(125)

As above, we have three cases depending on \( k \).

1. \( k = 0 \). Both \( H \) and \( R \) are constants, and \( R = R_0 \equiv -12H_0^2 \). The Friedmann equation is

\[
f(R_0) - \frac{1}{2}f_R(R_0)R_0 = \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4},
\]

(126)

and it has solutions only for \( \rho_{m0} = \rho_{r0} = 0 \), \( R_0 \) being a constant (see also [53]).

2. \( k = 1 \). In this case, \( H \) is still a constant but \( R \) is not. One finds

\[
R = -12H_0^2 - \frac{6\kappa}{a^2},
\]

(127)

\[
f = d_1 \cosh \left( \frac{\sqrt{2\kappa}}{H_0 a} \right) + d_2 \sinh \left( \frac{\sqrt{2\kappa}}{H_0 a} \right) + 6\rho_{r0}H_0^4 \frac{\kappa}{\kappa^2} + 3\rho_{m0}H_0^2 \frac{\kappa}{a\kappa} + \frac{\rho_{r0}a^2}{a^2} + \frac{\rho_{m0}a^3}{a^3},
\]

(128)

3. \( k = -1 \). The solution is

\[
R = -12H_0^2 - \frac{6\kappa}{a^2},
\]

(129)

\[
f = d_1 \cos \left( \frac{\sqrt{-2\kappa}}{H_0 a} \right) + d_2 \sin \left( \frac{\sqrt{-2\kappa}}{H_0 a} \right) + 6\rho_{r0}H_0^4 \frac{\kappa}{\kappa^2} + 3\rho_{m0}H_0^2 \frac{\kappa}{a\kappa} + \frac{\rho_{r0}a^2}{a^2} + \frac{\rho_{m0}a^3}{a^3}.
\]

(130)
5.4.4. $\Lambda$CDM solutions. Let us now look for $f(R)$ models which are compatible with the $\Lambda$CDM while being solutions of Friedmann equations. This analysis could be extremely important for comparing the $f(R)$ approach with observations (see also [47]). One defines

$$H^2 = \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + 1 - \Omega_{m0} - \Omega_{r0}.$$  

(131)

The differential equation to solve is therefore the following:

$$f'' + \left[ \frac{6\Omega_{m0}H_0^2}{3\Omega_{m0}H_0^2 + 4ak} - \frac{4(\Omega_{m0} + \Omega_{r0} - 1)a^4 - 7\Omega_{m0}a - 8\Omega_{r0}}{-(\Omega_{m0} + \Omega_{r0} - 1)a^4 + \Omega_{m0}a + \Omega_{r0}} \right] f' = \frac{3\Omega_{m0}H_0^2 + 4ak}{2a} - \frac{2a[-(\Omega_{m0} + \Omega_{r0} - 1)a^4 + \Omega_{m0}a + \Omega_{r0}]}{(3\Omega_{m0}H_0^2 + 4ak)(\rho_{r0} + a\rho_{m0})} H_0^2 f.$$  

(132)

The general integration can be numerically achieved by giving suitable initial conditions for $f_0$, $f_0'$. This analysis will be pursued in a forthcoming paper.

6. Discussion and conclusions

In this paper, we have discussed a general method for finding exact/analytical cosmological solutions in $f(R)$ gravity. The approach is based on the search for Noether symmetries which allow one to reduce the dynamics and, in principle, to solve more easily the equations of motion. Besides, due to the fact that such symmetries are always related to conserved quantities, such a method can be seen as based on a physically motivated criterion.

The main point is that the existence of the symmetry allows us to fix the form of $f(R)$ models assumed in a point-like cosmological action where the FLRW metric is imposed. It is worth noticing that, starting from a point-like FLRW Lagrangian and then deriving the Euler–Lagrange equations of motion leads to exactly the same equations as are obtained by imposing the FLRW metric in the Einstein field equations. This circumstance allows us to search ‘directly’ for the Noether symmetries in the point-like Lagrangian and then to plug the related conserved quantities into the equations of motion. As a result (i) the form of the $f(R)$ is fixed directly by the symmetry existence conditions and (ii) the dynamical system is reduced since some of its variables (at least one) are cyclic.

The method is useful not only in a cosmological context but it works, in principle, every time a canonical, point-like Lagrangian is achieved (in [54], it has been used to find spherically symmetric solutions in $f(R)$ gravity).

In this paper, we have considered a generic $f(R)$ theory where standard fluid matter (dust and radiation) is present. The Noether conditions for symmetry select forms of $f(R)$ depending on a set of cosmological parameters such as $\{\rho_{r0}, \rho_{m0}, k, H_0\}$ and the effective gravitational coupling. Such a dependence can be easily translated into the more suitable set of observational parameters $\{\Omega_{r0}, \Omega_{m0}, \Omega_k, H_0\}$ and then matched with data. This situation has a twofold relevance: on one hand, it could contribute to removing the well-known problem of degeneracy (several dark energy models fit the same data and, essentially, reproduce the $\Lambda$CDM model); on the other hand, the search for Noether symmetries being a relevant approach for finding conserved quantities in physics, this
could be an interesting method for selecting models motivated at a fundamental level. It is worth noticing that the Noether constant of motion, which we have found, has the dimension of a mass and is directly related to the various sources present into dynamics. In some sense, the Noether constant ‘determines’ the bulk of the various sources as \( \rho_m \), \( \rho_r \) and the effective \( \rho_\Lambda \) and thus could greatly contribute to solving the dark energy and dark matter puzzles. In a forthcoming paper, we will directly compare the solutions which we have presented here with observational data.

The ‘non-Noether solutions’ deserve a final remark. In this case, we do not ask for a Noether symmetry but finding these solutions can be related to the previous general method. We have shown that the standard cosmological behaviors of the usual Einstein–Friedmann cosmology can be achieved also in generic \( f(R) \) models, assuming that the cosmological quantities \( H \) and \( R \) depend on the scale factor \( a \). As result, we find general \( f(R(a)) \) where the standard solutions of the linear \( f(R) = R \) case are easily recovered.

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**Appendix. Solutions and the link with scalar–tensor theories**

We will explicitly show, as an example, that equation (102) is indeed a Noether solution (with \( k = 0 \), flat space, and \( \mu_0 = 0 \), zero Noether charge). First, from \( H(a) \), given by

\[
H^2 = \frac{-4d_1d_2(-c_3)^{9/2}}{a^4} + \frac{24d_1d_2(-c_3)^{7/2}}{a^2} + \frac{\rho_m d_2(-c_3)^{5/2}}{a^4} - \frac{36d_1d_2(-c_3)^{5/2}}{a^4} + \frac{2\sqrt{3}\rho_r a \text{arctanh} (\sqrt{3a}/\sqrt{-c_3})}{a^2} \frac{d_2^2 c_3^2}{a^4} + \frac{10\rho_r a \text{arctanh} (\sqrt{3a}/\sqrt{-c_3})}{a^2} \frac{d_2^2 c_3}{a^4} - \frac{18\rho_r a \text{arctanh}(\sqrt{3a}/\sqrt{-c_3})}{a^2} d_2^2 c_3 + 18\sqrt{3}\rho_r a \text{arctanh}(\sqrt{3a}/\sqrt{-c_3}) d_2.
\]

we can calculate the expression for \( R(a) \) as follows:

\[
R = -12H^2 - 6aHH' = -\frac{144d_1d_2(-c_3)^{7/2}}{a^2} + \frac{432d_1d_2(-c_3)^{5/2}}{a^2} - \frac{48d_2\rho_r(-c_3)^{3/2}}{a^2} - \frac{72\sqrt{3}\rho_r a \text{arctanh} (\sqrt{3a}/\sqrt{-c_3})}{a^2} c_3 + \frac{216d_2\rho_r a \sqrt{-c_3}}{a} - 216\sqrt{3}\rho_r a \text{arctanh}(\sqrt{3a}/\sqrt{-c_3}).
\]
Since we know both \( H \) and \( R \), now, by using equation (88), we can find \( f(a) \) as follows:

\[
f = -\frac{8d_1c_3^2}{a^3} - \frac{24d_1c_3}{a} - \frac{3\rho_0}{a^4} + \frac{4\sqrt{3}\rho_0\text{arctanh}\left(\sqrt{3}a/\sqrt{-c_3}\right)}{a^4\sqrt{-c_3}} - \frac{12\rho_0}{a^2c_3} - \frac{12\sqrt{3}\rho_0\text{arctanh}\left(\sqrt{3}a/\sqrt{-c_3}\right)}{a(-c_3)^{3/2}}.
\]

(A.3)

These expressions for \( f, R, H \) fulfil equation (9). The system has also a constant of motion \( \mu_0 = 0 \) given by equation (48), as the Lagrangian possesses a Noether symmetry.

We will discuss how to link this solution (extending this procedure to the other solutions is straightforward) to the scalar–tensor picture, by finding the potential for the scalar non-minimally coupled with gravity. In fact, starting from the action

\[
S = \int d^4x \sqrt{-g} f(R) + S_m,
\]

(A.4)

one can rewrite it (at least at the classical level) in the following form:

\[
S = \int d^4x \sqrt{-g}[f(\varphi)R - V(\varphi)] + S_m,
\]

(A.5)

where \( V = \varphi f_\varphi - f(\varphi) \), and \( f_\varphi = \partial f/\partial \varphi \). The classical equation of motion for \( \varphi \) leads to \( \varphi = R \). One can make a field redefinition to bring the action into the form

\[
S = \int d^4x \sqrt{-g}[-\chi R - V(\chi)] + S_m,
\]

(A.6)

where \( \chi = -f(\varphi) \).

In this case we can use our solutions in order to find \( V(\chi) \), the only unknown in the theory. One can do it as follows:

\[
\chi = -f(\varphi) = -f_R = -\frac{f}{R'},
\]

(A.7)

\[
V = \varphi f_\varphi - f = Rf_R - f = R\frac{f}{R'} - R,
\]

(A.8)

where these relations are correct on shell, i.e. for the solutions of the equations of motion. Using equations (A.1)–(A.3), one can explicitly write down the potential, at least for this case, as follows:

\[
V(\chi) = 3456d_1d_2^3\chi^3(-c_3)^{13/2} - 10368d_2^3\rho_0\chi^4c_3^2
\]

\[
-1728\sqrt{3}d_2^3\rho_0\chi^3\text{arctanh}\left[\frac{\sqrt{3}(6d_2\chi(-c_3)^5/2 + \sqrt{-36d_2^2\chi^2c_3^3 - c_3})}{\sqrt{-c_3}}\right]c_3^4
\]

\[
+1728d_2^3\rho_0\chi^3\sqrt{-36d_2^2\chi^2c_3^3 - c_3}(-c_3)^{7/2} - 288d_1d_2\chi(-c_3)^{5/2}
\]

\[
+432d_2^3\rho_0\chi^2c_3^2 + 288\sqrt{3}d_2^2\rho_0\chi^2\sqrt{-36d_2^2\chi^2c_3^3 - c_3}
\]

\[
x\text{arctanh}\left[\frac{\sqrt{3}(6d_2\chi(-c_3)^5/2 + \sqrt{-36d_2^2\chi^2c_3^3 - c_3})}{\sqrt{-c_3}}\right](-c_3)^{3/2}
\]
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\[
+ 144 \sqrt{3} d_2 \rho_0 \chi \arctanh \left[ \frac{\sqrt{3}(6d_2 \chi(-c_3)^{5/2} + \sqrt{-36d_2^4 \chi^2 c_3^5 - c_3})}{\sqrt{-c_3}} \right]
\]

\[-16d_1 \sqrt{-36d_2^4 \chi^2 c_3^5 - c_3} - \frac{96d_2 \rho_0 \chi \sqrt{-36d_2^4 \chi^2 c_3^5 - c_3}}{\sqrt{-c_3}}
\]

\[-\frac{9 \rho_0}{c_3^2} - 576d_1 d_2 \chi^2 \sqrt{-36d_2^4 \chi^2 c_3^5 - c_3 c_3^4}
\]

\[+ 8\sqrt{3} \rho_0 (-c_3)^{-5/2} \sqrt{-36d_2^4 \chi^2 c_3^5 - c_3}
\]

\[\times \arctanh \left[ \frac{\sqrt{3}(6d_2 \chi(-c_3)^{5/2} + \sqrt{-36d_2^4 \chi^2 c_3^5 - c_3})}{\sqrt{-c_3}} \right]. \quad (A.9)
\]

In order to study the evolution of the background, whether or not it leads to a viable dynamics for the Universe, it is already sufficient to check whether the Hubble parameter given by (A.1) can fit the data, from big bang nucleosynthesis up to dark energy domination.

References

[1] Guth A, 1981 Phys. Rev. D 23 347 [SPIRES]
[2] Weinberg S, 1972 Gravitation and Cosmology (New York: Wiley)
[3] Buchbinder I L, Odintsov S D and Shapiro I L, 1992
[4] Bondi H, 1952 Cosmology (Cambridge: Cambridge University Press)
[5] Bondi H, 1952 Cosmology (Cambridge: Cambridge University Press)
[6] Brans C and Dicke R H, 1961 Phys. Rev. 124 925 [SPIRES]
[7] Capozziello S, de Ritis R, Rubano C and Scudellaro P, 1996 Riv. Nuovo Cimento 4 1
[8] Sciama D W, 1953 Mon. Not. R. Astron. Soc. 113 34
[9] Faraoni V, 2004 Cosmology in Scalar–Tensor Gravity (Dordrecht: Kluwer–Academic)
[10] Capozziello S, Cardone V F and Troisi A, 2005 Phys. Rev. D 71 606513 [SPIRES] [astro-ph/0410031]
[11] Carroll S M, De Felice A and Trodden M, 2005 Phys. Rev. D 71 023525 [SPIRES] [astro-ph/0408081]
[12] Riess A G et al, 2004 Astrophys. J. 607 665 [SPIRES]
[13] Spergel D N et al, 2003 Astrophys. J. Suppl. 148 175
[14] Cole S et al, 2005 Mon. Not. R. Astron. Soc. 362 505
[15] Bahcall N A, Ostriker J P, Perlmutter S and Steinhardt P J, 1999 Science 284 1481
[16] Seljak U et al, 2005 Phys. Rev. D 71 103515 [SPIRES]
[17] Carroll S M, Press W H and Turner E L, 1992 Ann. Rev. Astron. Astrophys. 30 499 [SPIRES]
[18] Peebles P J E and Ratra B, 2003 Rev. Mod. Phys. 75 559 [SPIRES]
[19] Padmanabhan T, 2003 Phys. Rep. 380 235 [SPIRES]
[20] Copeland E J, Sami M and Tsujikawa S, 2006 Int. J. Mod. Phys. D 15 1753 [SPIRES]
[21] Will C M, 2006 Liv. Rev. Relat. 9 3 [gr-qc/0510072]
[22] Capozziello S, Cardone V F and Troisi A, 2005 Phys. Rev. D 71 043503 [SPIRES]
[23] Capozziello S and Francaviglia M, 2008 Gen. Rel. Grav. 40 357 [SPIRES]
[24] Capozziello S, 2003 Int. J. Mod. Phys. D 12 483 [SPIRES]
[25] Capozziello S, Cardone V F and Troisi A, 2005 Rec. Res. Dev. Astron. Astrophys. 1 1 [astro-ph/0303041]
[26] Odintsov S D and Nojiri S, 2003 Phys. Lett. B 576 5 [SPIRES]
[27] Capozziello S, Carloni S and Troisi A, 2003 Int. J. Mod. Phys. D 12 1969 [SPIRES]
[28] Capozziello S, Duvvuri V, Trodden M and Turner M, 2004 Phys. Rev. D 70 043528 [SPIRES]
[29] Allemendi G, Borowiec A and Francaviglia M, 2004 Phys. Rev. D 70 103503 [SPIRES]
[30] Nojiri S and Odintsov S D, 2004 Gen. Rel. Grav. 36 1765 [SPIRES]
[31] Cognola G, Elizalde E, Nojiri S, Odintsov S D and Zerbini S, 2005 J. Cosmol. Astropart. Phys. JCAP02(2005)010 [SPIRES]
$f(R)$ cosmology from Noether's symmetry

[25] Capozziello S, Cardone V F, Carloni S and Troisi A, 2004 Phys. Lett. A 326 292 [SPIRES]

[26] Milgrom M, 1983 Astrophys. J. 270 365 [SPIRES]

Bekenstein J, 2004 Phys. Rev. D 70 083509 [SPIRES]

[27] Capozziello S, Cardone V F and Troisi A, 2006 J. Cosmol. Astropart. Phys. JCAP08(2006)001 [SPIRES]

[28] Capozziello S, Cardone V F and Troisi A, 2007 Mon. Not. R. Astron. Soc. 375 1423

[29] Sobouti Y, 2007 Astron. Astrophys. 464 921 [SPIRES]

[30] Frigerio Martins C and Salucci P, 2007 Mon. Not. R. Astron. Soc. 381 1103 [astro-ph/0703243]

[31] Mendoza S and Rosas-Guevara Y M, 2007 Astron. Astrophys. 472 367 [SPIRES]

[32] De Felice A and Hindmarsh M, 2007 J. Cosmol. Astropart. Phys. JCAP08(2007)028 [SPIRES] [0705.3375] [astro-ph]

[33] Faraoni V, 2005 [astro-ph/0403019]

[34] De Felice A and Trodden M, 2005 Phys. Rev. D 72 043512 [SPIRES] [hep-ph/0412020]

[35] Olmo G J, 2005 Gen. Rel. Grav. 37 1891 [SPIRES]

[36] Capozziello S, Stabile A and Troisi A, 2007 Preprint gr-qc/0701138

[37] De Felice A, Hindmarsh M and Trodden M, 2006 Phys. Rev. D 74 103005 [SPIRES] [astro-ph/0510359]

[38] Davoudiasl H, Kitano R, Kribs G D, Murayama H and Steinhardt P J, 2004 Phys. Rev. Lett. 93 201301 [SPIRES]

[39] De Felice A and Trodden M, 2005 Phys. Rev. D 72 043512 [SPIRES] [hep-ph/0412020]

[40] Chiba T, 2005 Phys. Lett. B 575 1 [SPIRES]

[41] Faraoni V, 2006 Phys. Rev. D 74 023529 [SPIRES]

[42] Allevand G, Francaviglia M, Ruggiero M L and Tartaglia A, 2005 Gen. Rel. Grav. 37 1891 [SPIRES]

[43] Capozziello S and Troisi A, 2005 Phys. Rev. D 72 044022 [SPIRES]

[44] Chiba T, 2005 Phys. Lett. B 575 1 [SPIRES]

[45] De Felice A, Mukherjee P and Wang Y, 2008 Phys. Rev. D 77 024017 [SPIRES] [0706.1197] [astro-ph]

[46] Faraoni V, 2006 Phys. Rev. D 74 023529 [SPIRES]

[47] Erickcek A L, Smith T L and Kamionkowski M, 2006 Phys. Rev. D 74 121501 [SPIRES]

[48] Capozziello S, Stabile A and Troisi A, 2007 Phys. Rev. D 76 104019 [SPIRES]

[49] Faraoni V, 2006 Phys. Rev. D 74 023529 [SPIRES]

[50] Allevand G and Ruggiero M L, 2006 Preprint astro-ph/0610661

[51] Faraoni V, 2006 Phys. Rev. D 74 023529 [SPIRES]

[52] Allevand G and Ruggiero M L, 2006 Preprint astro-ph/0610661

[53] Capozziello S, Nesseris S and Perivolaropoulos L, 2007 J. Cosmol. Astropart. Phys. JCAP12(2007)009 [SPIRES] [0705.3586] [astro-ph]

[54] Capozziello S and Lambiase G, 2000 Gen. Rel. Grav. 32 295 [SPIRES]

[55] Arnold V I, 1978 Mathematical Methods of Classical Mechanics (Berlin: Springer)

[56] Marmo G, Saletan E J, Simoni A and Vitale B, 1985 Dynamical Systems. A Differential Geometric Approach to Symmetry and Reduction (New York: Wiley)

[57] Morandi G, Ferrario C, Lo Vecchio G, Marmo G and Rubano C, 1990 Phys. Rep. 186 149 [SPIRES]

[58] Spergel D N et al (WMAP Collaboration), 2007 Astrophys. J. Suppl. 170 377 [astro-ph/0603449]

[59] Barbosa J and Ottewill A C, 1983 J. Phys. A: Math. Gen. 16 2757 [SPIRES]

[60] Capozziello S, Stabile A and Troisi A, 2007 Class. Quantum Grav. 24 2153 [SPIRES]