INTRODUCTION

There are several ways to justify the concept of black hole entropy \[1\,\text{,}2\]. One of them is that a black hole is usually formed from the collapse of a quantity of matter or radiation, both of which carry entropy. However, the black hole’s interior and contents are hidden to an exterior observer and it is widely believed that a thermodynamic description of the collapse from that observer’s viewpoint cannot be based on the entropy of that matter or radiation because these are unobservable.

It is clear at the outset that black hole entropy should only depend on the observable properties of the black hole: mass, electric charge and angular momentum. In the following, we will restrict our discussion to the case of a spherical symmetric and stationary (Schwarzschild) black hole. The relevant measurable quantity in this case is given by the Schwarzschild radius which is known as

\[ r_s = \frac{2G}{c^2} M, \]

while \( M \) is the mass inside the sphere corresponding to the radius \( r_s \). When all the mass \( M \) is expressed in terms of the energy \( E/c^2 \) of the black hole, the differential \( dE \), when multiplied by a suitable factor, takes the form \(1\,\text{,}2\)

\[ dE = \frac{c^4}{16\pi G r_s} dA \]

while \( dA \) is the infinitesimal increment of the horizon area

\[ A = 4\pi r_s^2. \]

On the other hand, when near to equilibrium a thermodynamic system at temperature \( T \) changes its state, then the consequent increments of its energy \( E \) and entropy \( S \) are related by the first law of thermodynamics:

\[ dE = T dS. \]

Since \( E \) is the black hole energy, it has been proposed to consider equation (2) as a law for an ordinary thermodynamic system. Actually, it has been considered to be the first law if the black hole entropy is required to be a function of \( A \) and of nothing else. The corresponding expression proportional to the area of the black hole is called Bekenstein-Hawking (BH) entropy \(1\,\text{,}2\,\text{,}3\), defined by

\[ S_{BH} = k_B \frac{A}{4l_p^2}, \]

with Boltzmann constant \( k_B \) and \( l_p \) standing for the Planck length. With the choice in equation (5) the black hole temperature \( T_H \) must be

\[ T_H = \frac{1}{k_B} \frac{\hbar c}{4\pi G r_s}. \]

The reality of this choice was pointed out when Hawking proposed that a non-eternal black hole must spontaneously emit thermal radiation (Hawking radiation) with precisely this temperature \(4\,\text{,}5\). This discovery provided the final calibration of the numerical factor in equation (5).

However, a rigorous interpretation of \( S_{BH} \) as a thermodynamic entropy or a measure of information in terms of Gibbs or Shannon is ambiguous. This emphasizes the fact that contemporary black hole thermodynamics is by no means like ordinary thermodynamics. If the horizon in (5) is expressed in terms of \( A \), then it turns out that the surface area is proportional to the square of the inverse temperature. The entropy according to (5) can in turn be expressed in the following form

\[ S_{BH} = \frac{1}{k_B} \frac{\hbar c^2}{4\pi G r_s} \frac{1}{T_H}. \]

Actually, this is hard to accept from the traditional view of Boltzmann or Gibbs because higher tempera-
tures should be rather representative of a greater disorder and a third law of thermodynamics seems hard to be obtained.

The question we would like to address in the following is: under what circumstances is it possible to define a (or the) proper measure of entropy which is based on the matter or radiation inside the black hole and can nevertheless be expressed in terms of the horizon area \( A \)? Therefore, we introduce an alternative approach in which not only \( r_s \) depends on the energy of the black hole, but the energy itself is in turn also explicitly dependent on \( r_s \) caused by the quantum nature of the particles inside the black hole. Our intention will be to consider a physical situation which is as simple as possible, but still reveals the conceptual mechanism of the interplay between the microscopic and macroscopic structure of the black hole.

In the following section, we uncover the horizon area as an observable in terms of a many-body operator on Fock space. In section 3, the corresponding Gibbs ensemble will briefly be introduced. In Sections 4 and 5, this concept is applied to the case of a canonical ensemble. The question we would like to address in the following is: under what circumstances is it possible to define a quantum statistical degrees of freedom inside the horizon is uniquely determined by its mass and occupation of the energy levels by the many-particle system should be rather representative of a greater dis- order and a third law of thermodynamics seems hard to be obtained.

To do so, let \( (\mathcal{M}, g) \) be a 3-dimensional simply connected (pseudo) Riemannian manifold \( \mathcal{M} \) with metric \((g_{ij})\). We consider a particle prepared in an interior ball \( B_r \subset \mathcal{M} \) by a quantum mechanical measurement process in terms of a standard von Neumann-Lüders projection. For a quantum mechanical system of this type the wave function of the particle has to be zero at the boundary \( \partial B_r \). A Hilbert basis of \( L^2(B_r) \), the space of square-integrable functions on \( B_r \), is then defined by the Laplace-Beltrami operator on \( B_r \) with Dirichlet boundary conditions \( \delta \):

\[
\Delta \psi + \lambda \psi = 0 \quad \text{in} \quad B_r, \quad (9)
\]

\[
\psi = 0 \quad \text{on} \quad \partial B_r. \quad (10)
\]

Let \( \{\lambda_i\} \) be the set of eigenvalues and \( \{u_i\} \) the orthonormal basis of eigenfunctions, \( i = 1, 2, \ldots \) It is known that there are an infinite number of eigenvalues with no accumulation point: \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) and \( \lambda_i \to \infty \) as \( i \to \infty \). The scalar product in \( L^2(B_r) \) will be denoted by angular brackets, that is to write \( \langle \phi | \psi \rangle \) for two state vectors \( \phi, \psi \in L^2(B_r) \).

For every wave function \( \psi \in L^2(B_r) \) the eigenvalue problem \( \delta \) is the same for its real part and its imaginary part. Both are co-linear and thus we only have to solve the real valued problem. According to \( \mathcal{H} \), the mean momentum of the particle in \( B_r \) is zero and the square of the momentum standard deviation is given by

\[
\sigma_p^2 = -\hbar^2 \langle \psi | \Delta \psi \rangle. \quad (11)
\]

For the computation of \( \lambda_i \), we consider the spacial part of the Schwarzschild metric inside the black hole which is given by

\[
g(ij) = \text{diag} \left[ \left( \frac{r_s}{r} - 1 \right)^{-1}, -r^2, -r^2 \sin^2 \theta \right] \quad (12)
\]

with the corresponding volume measure

\[
\sqrt{g} = \frac{r^2 \sin \theta}{\left( \frac{r_s}{r} - 1 \right)^{\frac{1}{2}}}. \quad (13)
\]

Actually, the latter is unbounded for \( r \to r_s \), but that will not have any consequences in the following considerations because the relevant terms cancel out so that no divergence will remain at the end. The Laplace-Beltrami operator associated with this coordinate frame is given by

\[
\Delta \psi = \frac{1}{g^{ij}(g^{ij})_{\partial r} \partial_r \psi}
\]

\[
\left( \frac{r_s}{r} - 1 \right)^{1/2} \frac{1}{r^2} \partial_r \left( \left( \frac{r_s}{r} - 1 \right)^{1/2} r^2 \partial_r \psi \right)
\]

\[
- \frac{1}{r^2 \sin \theta} \left( \partial_{\theta} (\sin \theta \partial_{\theta} \psi) + \frac{1}{\sin \theta} \partial^2_{\varphi} \psi \right). \quad (14)
\]
To treat the eigenvalue problem the first step is to apply the standard product separation
\[ \psi(r, \theta, \varphi) = F(r) Y(\theta, \varphi). \] (15)
After a few algebraic manipulations we find the corresponding radial part of the eigenvalue equation
\[ \left( \frac{r_s}{r} - 1 \right) \frac{d}{dr} \left[ \left( \frac{r_s}{r} - 1 \right) \frac{1}{r} \frac{dF}{dr} \right] + (\lambda r^2 - l(l + 1)) F = 0 \] (16)
and the associated angular equation
\[ \frac{1}{\sin \vartheta} \left( \partial_{\vartheta}(\sin \vartheta \partial_{\vartheta} Y) + \frac{1}{\sin \vartheta} \partial_{\varphi}^2 Y \right) - l(l + 1) Y = 0. \] (17)
The angular quantum numbers are \( l = 0, 1, 2, \ldots \) Because of the rotational symmetry, the angular equation is the same as for the Euclidean case and thus needs not to be discussed in the following. The radial equation can be simplified by applying the transformation
\[ F(r) = \left( \frac{r_s}{r} - 1 \right)^{1/2} G(r), \] (18)
such that the following second order ordinary differential equation is obtained
\[ (r_s - r) r G'' + \left( \frac{r_s}{2} - 2r \right) G' + (\lambda r^2 - l(l + 1)) G = 0, \] (19)
with \( 0 \leq r < r_s \). This equation has regular singularities at \( r = 0 \) and \( r = r_s \). By applying a few simple transformations this equation can be reduced to the standard form of the confluent Heun equation [7]. The corresponding fundamental system can be expressed in terms of the confluent Heun function \( H \), see [8][9][10]:
\[ G_1(r, \lambda) = H(2\sqrt{\lambda} r_s, \frac{1}{2}, \frac{1}{2}, -\lambda r_s^2, l(l + 1) + \frac{1}{8}, \frac{r}{r_s}) \]
\[ \times \frac{e^{\sqrt{\lambda} r}}{(r_s - r)^{1/2} \sqrt{r}} \] and
\[ G_2(r, \lambda) = H(2\sqrt{\lambda} r_s, \frac{1}{2}, -\frac{1}{2}, -\lambda r_s^2, l(l + 1) + \frac{1}{8}, \frac{r}{r_s}) \]
\[ \times \frac{e^{\sqrt{\lambda} r}}{(r_s - r)^{1/2}}. \] (20)
With regard to the transformation [18] the fundamental system of equation [16] can be expressed as
\[ F_1(r, \lambda) = H(2\sqrt{\lambda} r_s, \frac{1}{2}, \frac{1}{2}, -\lambda r_s^2, l(l + 1) + \frac{1}{8}, \frac{r}{r_s}) e^{\sqrt{\lambda} r} \]
\[ \times \frac{1}{\sqrt{r}}. \] (22)
\[ F_2(r, \lambda) = H(2\sqrt{\lambda} r_s, \frac{1}{2}, -\frac{1}{2}, -\lambda r_s^2, l(l + 1) + \frac{1}{8}, \frac{r}{r_s}) \]
\[ \times e^{\sqrt{\lambda} r} \] (23)
According to the properties of Heun functions the first of these solutions is bounded on the closed interval \([0, r_s] \). On the other hand, the second solution is unbounded at \( r = 0 \) because of the square root in the denominator and will not be considered any further. Therefore, the spectrum of all possible eigenvalues is determined by the boundary condition \( F_1(r_s, \lambda) = 0 \). With the abbreviation
\[ x_{nl} := \sqrt{\lambda_{nl} r_s}, \] (24)
for \( n = 1, 2, \ldots \) and \( l = 0, 1, \ldots \), the problem can be summarized by solving the equation
\[ H(2x_{nl}, \frac{1}{2}, \frac{1}{2}, -x_{nl}^2, l(l + 1) + \frac{1}{8}, 1) = 0. \] (25)
For any given \( l \), the value \( x_{nl} \) is the \( n \)-th positive zero of the Heun function and is determined independent of \( r_s \). Since, we are in the position to write down an exact expression for the ultra-relativistic single particle energy spectrum, which has the remarkable property to be inversely proportional to the Schwarzschild radius only
\[ E_{nl} = c \hbar \frac{x_{nl}}{r_s}. \] (26)
Now, this expression is appropriate to be applied in [11], where it can be seen that the Schwarzschild radius is indeed proportional to the energy, but on the other hand, the admissible energy-levels are in turn inversely proportional to the Schwarzschild radius according to expression [20]. As a consequence the Schwarzschild radius cannot take any value but is determined by the countable set of quantum numbers \( n \) and \( l \). By substitution of (24) into (26), and after a few algebraic steps, the spectrum of the possible Schwarzschild radii is given by
\[ r_s^2 = 2l \pi x_{nl}. \] (27)
Thus, we have to accept that the horizon (or the horizon area) cannot further be considered as a classical observable.
At this point it should be mentioned that the necessity that a black hole should have a discrete mass or area spectrum is not new (see cf. [11][12] or more recently [13][14] for an overview). Most (or probably all) of those quantization results can be considered such that the area of the black hole is quantized according to a law like \( \sim n^2 \), where \( n \) is an integer depicting the quantum level and \( \beta \) a certain phenomenological constant. However, all of those schemes are different from the one introduced above because the quantization procedure expressed in (27) follows a law proportional to the Heun zeroes \( x_{nl} \), which is formally obtained from the constraints given by the interior geometric structure of the black hole. Lastly, we have not only treated a single quantum number but instead a set of quantum numbers \( n \) and \( l \) (spin projection does not matter here, see below) corresponding to the radial and the azimuthal space dimensions, which probably
is the main formal difference compared to the concepts of quantization in literature.

According to (3) and (26), the total energy of the many-body system inside the black hole is given by the expectation value

\[ E = \langle \hat{H} \rangle = \frac{\hbar}{r_s} \sum_{ilm} x_{ilm} \langle \hat{n}_{ilm} \rangle, \quad (28) \]

with spin projection quantum number \( m = -l, \ldots, l \). That is, the energy of the many-particle system is also proportional to \( 1/r_s \) and so is the total mass \( M \) of the black hole. Therefore, from (11) and (28) it follows that the area \( A = 4\pi r_s^2 \) of the horizon must be written as

\[ A = 8\pi l_p^2 \sum_{ilm} x_{ilm} \langle \hat{n}_{ilm} \rangle \equiv \langle \hat{A} \rangle, \quad (29) \]

with the many-body operator

\[ \hat{A} = \sum_{ilm} a x_{ilm} \hat{n}_{ilm} \quad (30) \]

and the abbreviation

\[ a = 8\pi l_p^2. \quad (31) \]

The explicit \( r_s \)-dependence of the area has been transformed into the internal degrees of freedom corresponding to the occupation number representation of \( A \). By re-substitution of \( r_s = (A/4\pi)^{1/2} \) into (28) and a few algebraic manipulations, we can verify the common relation between surface area and energy according to

\[ A = \frac{\kappa^2}{4\pi} E^2, \quad (32) \]

and \( \kappa = 8\pi G/c^4 \) is Einstein’s constant of gravitation. Since \( E \) is positive, this identity implies a one-to-one correspondence between area and energy and it is obvious to introduce the operator representation \( \hat{A} \) into the standard concept of quantum statistics instead of \( \hat{H} \). In this approach, the degrees of freedom responsible for the entropy will be precisely the ones characterizing the area of the horizon itself. This will be shown in the following section.

**STATISTICAL THERMODYNAMICS OF THE HORIZON AREA**

At first, let us consider the von Neumann entropy

\[ S = -k_B \text{Tr}(\hat{\rho} \log \hat{\rho}) \quad (33) \]

with probability density operator \( \hat{\rho} \) and the notation ‘Tr’ is the trace performed in ordinary Fock space representation. The state function \( A \) is the expectation value of \( \hat{A} \) given by

\[ A = \text{Tr}(\hat{\rho} \hat{A}). \quad (34) \]

The maximum attainable \( S \) that we can obtain by holding this average fixed depends, of course, on the average value we specified, that is

\[ S = k_B (\log Z + \gamma A) \quad (35) \]

with the partition function

\[ Z = \text{Tr}(e^{-\gamma \hat{A}}), \quad (36) \]

and specific density operator

\[ \hat{\rho} = \frac{e^{-\gamma \hat{A}}}{Z}, \quad (37) \]

with the Lagrange multiplier \( \gamma \), which is given in units of inverse area. It is easily verified that the state function (34) can also be expressed by

\[ A = -\frac{\partial \log Z}{\partial \gamma}. \quad (38) \]

At this point it should be mentioned that the von Neumann entropy (33) can be considered as a measure of the ‘amount of uncertainty’ in any probability distribution, which naturally satisfies the axioms of Khinchin [14] and is therefore a proper measure of information. It follows the composition rule by which the entropy of a joint system can be represented as a sum of the entropy of one system and the conditional entropy of another. After that expression has been maximized with respect to the possible probability density operators, it becomes a function (34) of the problem’s given data \( A \). Then it is still a measure of ‘uncertainty’, but this is uncertainty when all the information we have consists of just this data [12].

At first, the Lagrange multiplier \( \gamma \) is just unspecified, but eventually we want to know what it is. Therefore, one has to consider the nonlinear equation (35) which must be solved for \( \gamma \) in terms of \( A \). Generally, in a non-trivial problem, it is impractical to solve the \( \gamma \) explicitly (although there is a simple formal solution, (39), below) and we leave the \( \gamma \) as such, expressing things in parametric form. Anyway, the partial derivative of (35) with respect to \( A \) is given by

\[ \frac{\partial S}{\partial A} = k_B \gamma \quad (39) \]

and the associated ”temperature” \( T_A \) of such a system is naturally defined by

\[ T_A \equiv \frac{1}{k_B \gamma}. \quad (40) \]

Of course, the value \( T_A \) is (and has to be) different from the temperature in standard thermodynamics. However, as already mentioned above, at the end all relevant thermodynamic information can be (in principle) expressed in terms of the observable \( A \).
All these relations are just the standard equations of statistical mechanics already given by Gibbs, but now in a slightly modified form with respect to the specific physical situation of the black hole. Moreover, a generalized law of thermodynamics concerning the horizon area and entropy follows from the formalism and is given by
\[
dA = T_A dS. \tag{41}
\]

From the concept introduced so far it becomes clear that the dependence of the entropy on \(A\) can be obtained only after solving the state equation and subsequent elimination of the Lagrange parameter. Actually, as already known from ordinary statistic mechanics, a closed form solution is in general hard to obtain such that one has to consider temperatures not too close to absolute zero. This will be the intention in the following sections.

**THE CANONICAL ENSEMBLE OF \(N\) BOSONS**

According to the operator introduced in \(\text{[30]}\), the ground state of the system of \(N\) bosons is non-degenerated such that all particles are accumulated in the state with quantum numbers \((n,l) = (1,0)\). Thus, for \(T_A \to 0\), the remaining area of the black hole corresponding to the ground state is given by
\[
A = a x_{10} N, \tag{42}
\]
with \(x_{10} = 1.444474883\ldots\). The entropy and the standard deviation of \(A\) are both equal to zero. By a similar argumentation, the energy of the black hole at absolute zero is given by
\[
E = \frac{c h}{l_p} \left( \frac{x_{10} N}{2} \right)^{\frac{1}{2}}. \tag{43}
\]

For the canonical ensemble of \(N\) particles, both of the latter expressions can be respectively considered as a largest possible lower bound.

On the other hand, for high temperatures and low particle densities the standard approximation of the partition function for the canonical ensemble is given by
\[
Z = \frac{1}{N!} z^N \tag{44}
\]
with
\[
z = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (2l+1) e^{-\gamma a x_{nl}}. \tag{45}
\]
Actually, a closed form solution of this expression is hard to obtain because the non-trivial zeros in the exponential function. Numerically, the only software package currently able to work with Heun functions is *Maple* \([10]\). As far as the author knows, alternative ways for evaluations of Heun functions do not exist such that the use of those functions is limited to the routines hidden in the kernel of *Maple*. Actually, there are still some peculiarities with this software because there are values of the parameters where the routines break down leading to infinities or numerical errors. Nevertheless, this mainly concerns complex values outside the unit disc which is not relevant to our purpose. The first 400 zeros \(x_{nl}\) of the condition \(25\) corresponding to the quantum numbers \(n = 1, 2, 3, \ldots, 20\) and \(l = 0, 1, \ldots, 19\) are shown in Fig. 1. As can be seen, there is some evidence for linear asymptotic behaviour in \(n\) and \(l\). For a deeper numerical analysis more zeros have to be computed, but those are hard obtain because of numerical instabilities. Therefore, we do not fix a specific linear law but consider the asymptotic approximation
\[
x_{nl} = a_0 + a_1 n + a_2 l, \tag{46}
\]
with constants \(a_0\) real and \(a_1, a_2 > 0\). In this approach the single particle partition function \(15\) can be computed in a closed form expression and is given by
\[
z(\gamma a) = \frac{e^{-\gamma a x_{n0}} e^{\gamma a x_{n1}}}{e^{\gamma a x_{n1}} - 1} \frac{e^{\gamma a x_{n2}} (e^{\gamma a x_{n2} + 1})}{(e^{\gamma a x_{n2} - 1})^2}. \tag{47}
\]
For \(\gamma a \ll 1\), the leading term of the asymptotic expansion of this expression is \(z(\gamma a) \sim 2/a_1 a_2^2 (\gamma a)^3\) and the corresponding many-body partition function is
\[
Z = \frac{1}{N!} \left( \frac{2}{a_1 a_2^2 (\gamma a)^3} \right)^N. \tag{48}
\]

With \(48\), we immediately obtain the linear relation between area and temperature
\[
A = 3 N k_B T_A, \tag{49}
\]

\[\text{FIG. 1: Exact zeros } x_{nl} \text{ of the Heun function } 26 \text{ over } n = 1, 2, 3, \ldots, 20, \text{ for } l = 0, 1, \ldots, 19 \text{ (from bottom to top). This gives some evidence for a linear asymptotic behaviour.}\]
which is independent of the coefficients \(a_0, a_1\) and \(a_2\). This state equation relates the generalized temperature \(T_A\) to the measurable quantity \(A\).

On the other hand, the entropy of the internal degrees of freedom is computed by \(63\) and after substitution of \(40\), we obtain

\[
S = k_B N \log \left( \frac{2 e^4}{a_1 a_2^2 N} \left( \frac{k_B T_A}{a} \right)^3 \right). \tag{50}
\]

The temperature in this expression can be eliminated by applying \(66\). Thus we find the following logarithmic \(A\)-dependence of the entropy

\[
S = k_B N \log \left( \frac{2 e^4}{27 a_1 a_2^2 N^4} \left( \frac{A}{a} \right)^3 \right). \tag{51}
\]

Moreover, it is straightforward to compute the 'specific heat', which is

\[
C = \frac{\partial A}{\partial T_A} = 3 N k_B. \tag{52}
\]

While another representation of the specific heat is given by

\[
C = \frac{1}{k_B T_A^2} \left( \langle A^2 \rangle - \langle A \rangle^2 \right) \tag{53}
\]

and by combination of both expressions, the standard deviation \(\sigma_A\) of the surface area can be obtained by

\[
\sigma_A = (3N)^{\frac{1}{2}} k_B T_A. \tag{54}
\]

This expression can also be written in terms of \(A\) as

\[
\sigma_A = \frac{A}{(3N)^{\frac{1}{2}}}. \tag{55}
\]

So far we have considered particles with non-vanishing rest mass. This gave us expressions in terms of the area and the particle number was held fixed. Obviously, this concept can also be applied for bosons and fermions in the grand canonical ensemble with chemical potential \(\mu\) and the standard particle number operator. As a result, the state equation and the entropy stay the same in the relevant limits. Since the derivation is straightforward, we will not go any further in this direction for now.

Instead, in the following section, we consider the statistical mechanics for particles of rest mass zero.

### MASSLESS PARTICLES - THE CASE OF RADIATION

By analogy with the concept of a photon gas, we leave the restriction of a fixed particle number such that every state can take the values \(n_{ilm} = 0, 1, \ldots\). With \(30\) and \(31\), the exact partition function is given by

\[
Z = \prod_{i=1}^{\infty} \prod_{l=0}^{\infty} \left( \frac{1}{1 - e^{-\gamma a x_{il}}} \right)^{2l+1} \tag{56}
\]

and the mean occupation number per state is

\[
\langle n_{ilm} \rangle = \frac{1}{e^{\gamma a x_{il}} - 1}. \tag{57}
\]

Applying the asymptotic approximation \(46\), the total number of particles is obtained by

\[
N = \sum_{ilm} \langle n_{ilm} \rangle \tag{58}
\]

\[
= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{l=0}^{\infty} (2l+1) e^{-\gamma a x_{il}} \tag{59}
\]

\[
= \sum_{n=1}^{\infty} z(\gamma a n). \tag{60}
\]

with \(z(\cdot)\) given by \(17\). For \(\gamma a \ll 1\), we have the asymptotic behaviour

\[
z(\gamma a n) \sim \frac{2}{a_1 a_2^2 (\gamma a)^3} \frac{1}{n^3}. \tag{61}
\]

while for larger \(n\) the contribution becomes less relevant. By this approximation the temperature dependency of the mean particle number is

\[
N = \frac{2 \zeta(3)}{a_1 a_2^2} \left( \frac{k_B T_A}{a} \right)^3. \tag{62}
\]

The mean area of the horizon is given by \(58\) and after some computations, we obtain the following relation between area and temperature

\[
A = \sigma T_A^4. \tag{63}
\]

with the constant

\[
\sigma = \frac{\pi^4 k_B^4}{15 a_1 a_2^2 a^3}. \tag{64}
\]

Sometimes it is useful to express \(63\) in terms of the particle number and temperature

\[
A = \frac{\zeta(4)}{\zeta(3)} 3 N k_B T_A \tag{65}
\]

\[
= 2.7 k_B T_A. \tag{66}
\]

The entropy is computed by integration

\[
S = \int \frac{dA}{T_A} = \frac{4 \sigma}{3} T_A^3 \tag{67}
\]

and by substitution of \(62\), we obtain the entropy of the black hole in terms of the surface area

\[
S = k_B \frac{4 \pi}{3} \left( \frac{1}{15 a_1 a_2^2} \right)^{\frac{3}{2}} \left( \frac{A}{a} \right)^{\frac{3}{2}}. \tag{68}
\]
This result implies that for massless particles the entropy depends only on the area of the black hole and is given by the power law \( S \propto A^{3/4} \). With (72) and (67), it is even possible to eliminate the dependency on \( A \) and to express the entropy in terms of the mean particle number only

\[
S = \frac{\zeta(4)}{\zeta(3)} 4 k_B N = 3.6 k_B N. \tag{69}
\]

The heat capacity is obtained by derivation of (63) with respect to \( T_A \) and after subsequent re-substitution in (67), we obtain

\[
C = 3 S. \tag{70}
\]

That is, the heat capacity is positive definite and (up to a multiplicative constant) the same as the entropy of the black hole. This is interesting because the heat capacity is also related to the standard deviation \( \sigma_A \) of the state variable, which can be computed by

\[
\sigma_A = 2 \left( \frac{15 a_1 a_2 a_3}{\pi^4} \right)^{1/4} A^{5/4}. \tag{71}
\]

This expression tells us the amount of fluctuations of the area in terms of the area itself and is indeed independent of the Boltzmann constant.

For the very low temperature regime in the limes \( \gamma a \to \infty \), the mean area scales according to the exponential law

\[
A \sim (a_0 + a_1) a e^{-\frac{a_0 + a_1}{T_A}} \tag{72}
\]

which tends to zero for \( T_A \to 0 \). Let \( A_0 = (a_0 + a_1) a > 0 \), then by integrating over \( dA/T \), we obtain the low temperature behaviour of the entropy

\[
S = k_B \left( 1 + \frac{A_0}{k_B T_A} \right) e^{-\frac{A_0}{k_B T_A}}, \tag{73}
\]

which satisfies

\[
\lim_{T_A \to 0} S = 0. \tag{74}
\]

Expression (73) can also be expressed in terms of the area and is given by

\[
S = k_B \left( 1 - \log \frac{A}{A_0} \right) \frac{A}{A_0} \tag{75}
\]

which is valid for \( A \ll A_0 \). For the heat capacity near absolute zero we have

\[
C = A_0^2 \frac{e^{-\frac{A_0}{k_B T_A}}}{k_B T_A}, \tag{76}
\]

such that the standard deviation of the area is given by

\[
\sigma_A = A_0 e^{-\frac{A_0}{k_B T_A}}. \tag{77}
\]

This confirms that all fluctuations of the area are suppressed when \( T_A \) tends to absolute zero.

Finally, let us consider the transmission of Planck’s radiation law. Since we are working in spherical geometry its derivation is not straightforward. However, the law does not essentially depend on whether there is a spherical or cubic symmetry. Therefore, one can expect an expression similar to the original Planck law. The basic conditions for its derivation are given by the expression for the particle number \( (62) \) and the total area \( (63) \). According to \( (67) \), the particle number per state of area \( A \) can be defined by

\[
n(A) = \frac{1}{e^{\frac{A}{k_B T_A}} - 1} \tag{78}
\]

and furthermore the quadratic density of states in three dimensions is

\[
D(A) = \frac{A^2}{a_1 a_2 a_3}, \tag{79}
\]

Then, the particle number and the mean area mentioned above are verified by integration

\[
N = \int_0^\infty n(A) D(A) dA, \tag{80}
\]

\[
A = \int_0^\infty B(A, T_A) dA, \tag{81}
\]

while the density function is

\[
B(A, T_A) = \frac{1}{a_1 a_2 a_3} \frac{A^3}{e^{\frac{A}{k_B T_A}} - 1}. \tag{82}
\]

The latter might be considered as a generalized Planck law for the horizon area of the black hole. The maxima of this density is given by the generalized Wien displacement law

\[
A_{max} = 2.82 k_B T_A. \tag{83}
\]

At this point, one is compelled to question the spacetime perturbation which is caused by the quantum fluctuations of the horizon. Statistical deviations in \( A \) also imply variations of the Schwarzschild metric which might give reason to think of fluctuating spacetime caused by the black hole’s internal degrees of freedom. According to the Birkhoff theorem such a system might not emit gravity waves at first glance \( [16] \[17] \). However, a crucial assumption for the theorem is that the underlying matter (energy) is fixed. In the situation discussed above, this assumption is only satisfied from the macroscopic point of view (or in mean). In our situation, the non-stationary behaviour of the matter is caused by the microscopic quantum fluctuations of the particles. This mechanism makes it possible that information from the
inside of the black hole can be transformed into gravitational waves outside the horizon. The manifestation of this transformation between entropy and surface area of the black hole is already indicated by the thermodynamic law \( dA = T_A dS \).

**SUMMARY**

There are at least two ways by which the mass of an ordinary black hole can be related to the Schwarzschild radius. On the one hand, there is the standard relation \[ M = \frac{A}{4\pi} \] obtained by the classical Einstein vacuum field equations. On the other hand, because of the fact that no particle can ever escape a black hole, there is an inverse dependency \[ M \sim \frac{1}{A^{3/4}} \] of the mass on the horizon, if the corresponding energy spectrum is expressed by a quantum mechanical (many-body) Hamiltonian. Taking this nonlinearity into account, it is obtained that the area of the horizon becomes a \((a)\) proper thermodynamic state function depending only on the internal degrees of freedom of the black hole. The corresponding thermodynamic formalism provides a consistent framework to obtain a well-defined notion of (Gibbs/von Neumann) entropy, temperature as a Lagrange parameter, three laws of thermodynamics and a reasonable behaviour for the regime of low and high temperatures.

By ordinary elimination of the Lagrange parameter with respect to the state function \( A \), we obtained that the entropy is proportional to the logarithm of \( A \) in the case of an ideal gas of ultra-relativistic particles or otherwise proportional to \( A^{3/4} \) in the case of an ideal gas of massless particles (radiation). These results are in contrast to the contemporary belief that the entropy of a black hole should be proportional to the first power of \( A \) over the whole range of its domain. Actually, a comparison of both concepts does not seem obvious because the entropy definition in the approach of Bekenstein and Hawking does in fact not satisfy the requirements of a quantity suitable to be embedded in the standard formalism of Gibbs.

The intention of the present approach was to introduce a consistent concept as simple as possible but still sufficient to explain the basic thermodynamic structure of a Schwarzschild black hole. Several aspects might be further refined. For the description of the single particle black hole energy spectrum, we only regarded an asymptotic expansion of Heun zeros. This is still an approximation which could be further improved by also regarding higher order corrections. Actually, the research concerning the computation of Heun functions is part of present activities in mathematical research. In addition, we believe that certain properties of the interior geometry of general black holes might be negligible whenever the ultra-relativistic limit is dominant. An extension of the present formalism to the case of charged and rotating black holes is a challenge which is left to a future contribution.

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