Landau–Ginzburg/Calabi–Yau correspondence for quintic three-folds via symplectic transformations

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Abstract

We compute the recently introduced Fan–Jarvis–Ruan–Witten theory of \( W \)-curves in genus zero for quintic polynomials in five variables and we show that it matches the Gromov–Witten genus-zero theory of the quintic three-fold via a symplectic transformation. More specifically, we show that the \( J \)-function encoding the Fan–Jarvis–Ruan–Witten theory on the A-side equals via a mirror map the \( I \)-function embodying the period integrals at the Gepner point on the B-side. This identification inscribes the physical Landau–Ginzburg/Calabi–Yau correspondence within the enumerative geometry of moduli of curves, matches the genus-zero invariants computed by the physicists Huang, Klemm, and Quackenbush at the Gepner point, and yields via Givental’s quantization a prediction on the relation between the full higher genus potential of the quintic three-fold and that of Fan–Jarvis–Ruan–Witten theory.

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1 Introduction

During the last twenty years, mirror symmetry has been one of the most inspirational problems arising from physics. There are various formulations of mirror symmetry and each one is important in its own way. The most classical version proposes a conjectural duality in the context of Calabi–Yau (CY) complete intersections of toric varieties, which interchanges quantum cohomology with the Yukawa coupling of the variation of the Hodge structure. In particular, it yields a striking prediction of the genus-zero Gromov–Witten invariants encoding the enumerative geometry of stable maps from curves to these CY manifolds. The most famous example is the quintic three-fold defined by a single degree-five homogeneous polynomial

\[ W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5, \]

for which the genus zero predictions have been completely proven [Gi96] [LLY97].

In the early days of mirror symmetry, physicists noticed that the defining equations of CY hypersurfaces or complete intersections—such as the above quintic polynomial—appear naturally in another context; namely, the Landau–Ginzburg (LG) singularity model. The argument has been made on physical grounds [VW89] [Wi93b] that there should be a Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence connecting CY geometry to the LG singularity model. In this context, CY manifolds are considered from the point of view of Gromov–Witten theory; this correspondence would therefore inevitably yield new predictions on Gromov–Witten invariants and is likely to greatly simplify their calculation (it is generally believed that the LG singularity model is relatively easy to compute). Here, we should mention that in Gromov–Witten theory several general methods have been recently found [LR01] [MP06] and can in principle determine Gromov–Witten invariants of all known examples. These methods, however, are hard to put into practice both when calculating a single invariant and when one needs to effectively compute the full higher genus Gromov–Witten theory. The genus-one theory has been computed only recently by A. Zinger after a great deal of hard work [Zi08]. Computations in higher genera are out of mathematicians’ reach for the moment. There is a physical method put forward by Huang–Klemm–Quackenbush [HKQ]. They worked in the B-model, i.e. on the complex moduli space of the mirror quintic three-fold, and provided the A-model predictions from mirror symmetry: their computations interestingly combines the potential at the “large complex structure point” (mirror of the Calabi–Yau quintic) with the “Gepner point” potential. The regularity of the latter yields predictions for the quintic three-fold up to \( g = 51 \). Unlike the “large complex structure point” the invariants encoded by this “Gepner point” potential lack a mirror geometrical interpretation in terms of enumerative geometry of curves.

In this current unsatisfactory state of affairs, a natural idea is to push through the LG/CY correspondence in enumerative geometry of curves and use the computational power of the LG singularity model as an effective method for determining the higher genus Gromov–Witten invariants of the quintic three-fold. At first sight, it seems surprising that this idea has escaped attention for twenty years; however, a brief investigation reveals that the problem is far more subtle then one might expect. To begin with, the LG/CY correspondence is a physical statement concerning conformal field theory and lower energy effective theories: it does not directly imply an explicit geometric prediction. At a more fundamental level, Gromov–Witten theory embodies all the relevant information on the CY-side, whereas it is not clear which theory plays the same role on the LG-side. Identifying such a counterpart to Gromov–Witten theory is the first step towards establishing a geometric LG/CY correspondence and is likely to be interesting in its own right. For instance, in a different context, the LG/CY correspondence led to identify matrix factorization as the LG counterpart of the derived category of complexes of coherent sheaves [HW04], [Ko].

In [FJR1, FJR2, FJR3], a candidate quantum theory of singularities has recently been constructed. The formulation of this theory is very different and considerably more interesting than Gromov–Witten theory. Naively, Gromov–Witten theory can be thought of as solving the Cauchy–Riemann equation \( \overline{\partial} f = 0 \) for the map \( f: \Sigma \to X_W \), where \( \Sigma \) is a compact Riemann surface and \( X_W \) is the weighted projective hypersurface \( \{ W = 0 \} \). In many ways, the difficulty and the interest of the computation of Gromov–Witten invariants comes from the fact that \( X_W \) is a nonlinear space. On the other hand, the polynomial \( W \) in \( N \) variables in the LG singularity model is treated as a holomorphic function on \( \mathbb{C}^N \). Since there are no nontrivial holomorphic maps from a compact Riemann surface to \( \mathbb{C}^N \), we run
into difficulty at a much more fundamental level. The solution comes from regarding singularities from a rather different point of view. In the early 90’s, Witten conjectured [Wi91] and Kontsevich proved [Ko92] that the intersection theory of Deligne and Mumford’s moduli of curves is governed by the KdV integrable hierarchy—i.e. the integrable system corresponding to the $A_1$-singularity. Witten also pursued the generalization of Deligne–Mumford spaces to new moduli spaces governed by integrable hierarchies attached to other singularities. To this end, he proposed, among other approaches, a remarkable partial differential equation of the form

$$\partial_s \eta_j + \partial_j W(s_1, \cdots, s_N) = 0,$$

where $W$ is a quasihomogeneous polynomial and $\partial_j W$ stands for the partial derivative with respect to the $j$th variable. A comprehensive moduli theory of the above equation has been established by Fan, Jarvis, and Ruan [FJR1, FJR2, FJR3]. Besides extending Witten’s 90’s conjecture (see [FSZ] for $A_n$-singularities and [FJR1] for all simple singularities), one outcome of Fan–Jarvis–Ruan–Witten theory is that it plays the role of Gromov–Witten theory on the LG-side for any quasihomogeneous singularity.

In this perspective, the Witten equation should be viewed as the counterpart in the LG-model to the Cauchy–Riemann equation: we replace a linear equation on a nonlinear target with a nonlinear equation on a linear target.

With the LG-counterpart of Gromov–Witten theory understood, there are two remaining issues: (i) Is this LG-side really easier to compute? (ii) What is the precise mathematical statement for the LG/CY correspondence in terms of Gromov–Witten theory? For (i), there is ample evidence that this is indeed the case. For example, Fan, Jarvis, and Ruan computed their theory for $A_n$-singularities, thereby establishing Witten’s original conjecture claiming that the theory is governed by the $ADE$-hierarchies. For (ii), motivated by a similar conjecture for crepant resolution of orbifolds [CR], a natural conjecture [Ru] can be formulated in terms of Givental’s Lagrangian cones and quantization. In this paper we state for the quintic three-fold and we establish it in genus zero via a symplectic transformation. We achieve this, by computing Fan–Jarvis–Ruan–Witten theory for the quintic polynomial singularity $W = 0$; the result supplies a geometrical interpretation in terms of enumerative geometry of curves of Huang, Klemm, and Quackenbush’s genus-zero potential at the “Gepner point” (Remark 4.2.2). Furthermore, the formula for the symplectic transformation $U$, makes the higher genus statement explicit, for $g \geq 0$ the correspondence is expected to be carried out via Givental’s quantization of $U$.

### 1.1 The main result

We work with quasihomogeneous, or weighted homogeneous, polynomials $W$ in $N$ variables:

$$\exists c_1, \ldots, c_N, d > 0 \; | \; W(\lambda^{c_1} x_1, \ldots, \lambda^{c_N} x_N) = \lambda^d W(x_1, \ldots, x_N) \; \forall \lambda \in \mathbb{C}. \tag{1}$$

The geometric object considered on the CY side is a CY hypersurface inside the weighted projective space $\mathbb{P}(c_1, \ldots, c_N)$. On the LG-side, we regard $W$ as the equation of an isolated singularity in the affine space $\mathbb{C}^N$ (we assume nondegeneracy conditions for the singularity, see Definition 2.1.1).

The Fan–Jarvis–Ruan–Witten genus-zero invariants of the $W$-singularity (LG side)

$$\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_{n-1}}(\phi_{h_{n-1}}), \tau_{a_n}(\phi_{h_n}) \rangle^{FJRW}_{\sigma, n}$$

and the Gromov–Witten genus-zero invariants of the $W$-hypersurface (CY side)

$$\langle \tau_{a_1}(\varphi_{h_1}), \ldots, \tau_{a_{n-1}}(\varphi_{h_{n-1}}), \tau_{a_n}(\varphi_{h_n}) \rangle^{GW}_{\sigma, n, \delta}$$

are intersection numbers defined (see (27) and (28)) for any $(a_1, \ldots, a_n) \in N^n$ and for any entry of the state spaces of the two theories: $H_{FJRW} = \oplus_h \phi_h \mathbb{C}$ and $H_{GW} = \oplus_h \varphi_h \mathbb{C}$. These two sets of numbers arise from two entirely different problems of enumerative geometry of curves. The definition of the Gromov–Witten invariants $\langle \rangle^{GW}$ is well known: the essential ingredient is the moduli space of stable maps to the Calabi–Yau variety. The Fan–Jarvis–Ruan–Witten invariants $\langle \rangle^{FJRW}$ have been recently introduced [FJR1] and are based on a generalization of Witten’s moduli space parametrizing $d$-spin curves: curves
equipped with a line bundle $L$, which is an $d$th root of the canonical bundle: $L^\otimes d = \omega$. We can present this generalization as follows: for $W$ satisfying (1), we endow the curve with $N$ line bundles $L_1, \ldots, L_N$ which are $d$th roots of $\omega^\otimes \epsilon_1, \ldots, \omega^\otimes \epsilon_N$ (the line bundles satisfy further relations in terms of $W$, see Definition 2.3.1).

The two sets of invariants can be incorporated into the Fan–Jarvis–Ruan–Witten partition function and into the Gromov–Witten partition function, which, by standard techniques, can be reconstructed from the generating functions of the one-point descendants: the invariants with not more than one entry $\tau_a(\varphi_n)$ and $\tau_a(\varphi_n)$ having $a \neq 0$: $\langle \tau_0(\phi_{h_1}), \ldots, \tau_0(\phi_{h_{-n-1}}), \tau_a(\phi_{h_n}) \rangle_{GW}^{\text{FJR}}$ and $\langle \tau_0(\varphi_{s_1}), \ldots, \tau_0(\varphi_{s_{-n-1}}), \tau_a(\varphi_{s_n}) \rangle_{GW}^{\text{FJR}}$.

In other words, the two theories are determined by the $J$-functions

$$J_{\text{FJR}}(\sum_k t^k_0 \phi, z) = z \varphi_0 + \sum_k t^k_0 \varphi + \sum_{n \geq 0} \sum_{\epsilon, k} \frac{t^h_0 \cdots t^h_n}{n! z^{k+1}} (\tau_0(\phi_{h_1}), \ldots, \tau_0(\phi_{h_n}), \tau_k(\varphi_{\epsilon}))_{0,n+1} \phi^\epsilon,$$

$$J_{\text{GW}}(\sum_k t^k_0 \varphi, z) = z \varphi_0 + \sum_k t^k_0 \varphi + \sum_{n \geq 0} \sum_{\delta \geq 0} \sum_{\epsilon, k} \frac{t^h_0 \cdots t^h_n}{n! z^{k+1}} (\tau_0(\varphi_{s_1}), \ldots, \tau_0(\varphi_{s_{-n-1}}), \tau_k(\varphi_{\epsilon}))_{0,n+1} \phi^\epsilon,$$

which can be regarded as terms of $H_{\text{FJR}}((z^{-1}))$ and $H_{\text{GW}}((z^{-1}))$, i.e. Laurent series with coefficients in $H_{\text{FJR}}$ and $H_{\text{GW}}$.

The LG/CY correspondence makes the two $J$-functions match. It relies on an isomorphism at the level of state spaces $H_{\text{FJR}}$ and $H_{\text{GW}}$ on which the two $J$-functions are defined. In fact, the state space $H_{\text{GW}}$, the Chen–Ruan cohomology of the hypersurface, and the state space $H_{\text{FJR}}$, an orbifold-type Milnor ring $H_{\text{FJR}}(W)$ (Definition 2.1.12), are isomorphic for all weighted CY hypersurfaces. In this paper we only need the quintic three-fold case, where the isomorphism can be easily shown, see Example 2.1.16. In fact, for $W = x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4$, further simplifications occur: for dimension reasons (see Remarks 3.1.2), the two $J$-functions vanish outside two lines lying in the degree-2 part of the respective state spaces identified via the above isomorphism: the line $t^1_0 \varphi_1$ and the line $t^1_0 \varphi_1$.

On the CY side, Givental’s mirror symmetry theorem [Gi96] for the quintic three-fold sets an equivalence between the above $J$-function and the $H_{\text{GW}}((z^{-1}))$-valued $I$-function

$$I_{\text{GW}}(q, z) = \sum_{d \geq 0} z^d q^H z^{-d} \prod_{i=1}^d (5H + k z) \prod_{i=1}^d (H + k z)^c,$$

where $H$ is the cohomology class corresponding to the hyperplane section and $q = \exp(t^1_0)$ parametrises the above line $\mathbb{C} \varphi_1$ (it is usually regarded as a parameter centred at the “large complex structure point” of the complex moduli space of the quintic three-fold). Expanded in the variable $H$, the $I$-function assembles the period integrals spanning the space of solutions of Picard–Fuchs equation

$$\left[ D_q^4 - 5g \prod_{m=1}^4 (5D_q + m z) \right] I_{\text{GW}} = 0 \quad \text{for } D_q = z^d q^{\partial \over \partial q}.$$

Via an explicit change of variables

$$q' = \frac{g_{\text{GW}}(q)}{f_{\text{GW}}(q)} \quad \text{with } g_{\text{GW}} \text{ and } f_{\text{GW}} \text{ C-valued and } f_{\text{GW}} \text{ invertible}$$

the A-model of the quintic (i.e. $J_{\text{GW}}$), matches the B-model of the quintic (i.e. $I_{\text{GW}}$), via a mirror map

$$\frac{I_{\text{GW}}(q, z)}{f_{\text{GW}}(q)} = J_{\text{GW}}(q', z).$$

We provide the same picture on the LG side (a direct consequence of Thm. 4.1.5, and Rem. 4.2.1-2).
Theorem 1.1.1. Consider the $H_{\text{FJR W}}((z^{-1}))$-valued function (where $[a]_n = a(a+1)\ldots(a+n-1)$)

\[ I_{\text{FJR W}}(t,z) = z \sum_{k=1,2,3,4} \frac{1}{\Gamma(k)} \sum_{l \geq 0} \frac{((k)_5 l^{k+5l})}{[k]_{5l} z^{k-1}} \phi_{k-1}, \]

whose four summands span the solution space of the Picard–Fuchs equation

\[ \left[ D_t^4 - 5^5 t^{-5} \prod_{m=1}^{4} (D_t - mz) \right] I_{\text{FJR W}} = 0 \quad \left( \text{for } D_t = z \frac{\partial}{\partial t} \right) \]

and coincide with the period integrals at the Gepner point computed by Huang, Klemm, and Quackenbush [HKQ]. The above $I$-function and the $J$-function of FJR-w theory are related by an explicit change of variables (mirror map)

\[ t' = \frac{g_{\text{FJR W}}(t)}{f_{\text{FJR W}}(t)} \quad \text{(with } g_{\text{FJR W}} \text{ and } f_{\text{FJR W}} \text{ C-valued and } f_{\text{FJR W}} \text{ invertible)} \]

satisfying

\[ \frac{I_{\text{FJR W}}(t,z)}{f_{\text{FJR W}}(t)} = J_{\text{FJR W}}(t',z). \]

The Picard–Fuchs equation in the above statement coincides with that of the quintic three-fold for $q = t^{-5}$. After the identification $q = t^{-5}$ of the coordinate patch at $t = 0$ with the coordinate patch at $q = \infty$, the two $I$-functions are solutions of the same Picard–Fuchs equation. Since $I_{\text{GW}}$ and $I_{\text{FJR W}}$ take values in two isomorphic state spaces, we can compute the analytic continuation of $I_{\text{GW}}$ and obtain two different bases spanning the space of solutions of the same Picard–Fuchs equation. Therefore, in Section 4, we have the following corollary.

Corollary 4.2.4. There is a $\mathbb{C}[z, z^{-1}]$-valued degree-preserving symplectic transformation $U$ mapping $I_{\text{FJR W}}$ to the analytic continuation of $I_{\text{GW}}$ near $t = 0$. Therefore, the genus-zero LG/CY correspondence holds (Conjecture 3.2.1,(1)).

We have explicitly computed $U$ using the Mellin–Barnes method for analytic continuation, (60). In this way, in the case of the quintic three-fold, the higher genus LG/CY correspondence assumes an explicit form. Indeed, in the terms of the second part of Conjecture 3.2.1, the quantization $\hat{U}$ is a differential operator which we expect to yield the full higher genus Gromov–Witten partition function when applied to the full higher genus Fan–Jarvis–Ruan–Witten partition function.

1.2 Organization of the paper

The rest of the paper is organized in three sections: on FJR theory (Sect. 2), on the LG/CY conjecture (Sect. 3), and on its proof in genus zero (Sect. 4). In the appendix we relate our moduli functor to that of [FJR1] and we show that they yield the same intersection numbers.

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2 Fan–Jarvis–Ruan–Witten theory

We review FJRW theory for quasihomogeneous singularities.

2.1 The state space associated to a singularity

The singularities. We consider singularities \( \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid W(x_1, \ldots, x_N) = 0 \} \)
given by a polynomial \( W \) in \( N \) complex variables
\[
W(x_1, \ldots, x_N) = l_1 \prod_{j=1}^N x_j^{m_{1,j}} + \cdots + l_s \prod_{j=1}^N x_j^{m_{s,j}}.
\]
where \( l_1, \ldots, l_s \) are nonzero complex numbers and \( m_{i,j} \) (for \( 1 \leq i \leq N \) and \( 1 \leq j \leq s \)) are nonnegative integers.

We assume that \( W \) is quasihomogeneous; i.e. there exist positive integers \( c_1, \ldots, c_N \), and \( d \) satisfying
\[
W(\lambda^{c_1}x_1, \ldots, \lambda^{c_N}x_N) = \lambda^d W(x_1, \ldots, x_N) \quad \forall \lambda \in \mathbb{C},
\]
or, equivalently,
\[
W = \sum_{j=1}^N \frac{c_j}{d} x_j \partial_j W.
\]
For \( 1 \leq j \leq N \), we say that the charge of the variable \( x_j \) is \( q_j = c_j/d \). As soon as \( c_1, \ldots, c_N \) and \( d \) are coprime, we say that the degree of \( W \) is \( d \). We assume that the origin is the only critical point of \( W \); i.e. the only solution of
\[
\partial_j W(x_1, \ldots, x_N) = 0 \quad \text{for} \ j = 1, \ldots, N
\]
is \((x_1, \ldots, x_N) = (0, \ldots, 0)\). (By (4), if \((x_1, \ldots, x_N)\) satisfies (5), then \(W(x_1, \ldots, x_N)\) is zero.)

The following definition identifies once and for all the class of singularities considered in this paper.

Definition 2.1.1. We say that \( W \) is a nondegenerate quasihomogeneous polynomial if it is a quasihomogeneous polynomial of degree \( d \) in the variables \( x_1, \ldots, x_N \) of charges \( c_1/d, \ldots, c_N/d > 0 \) satisfying the following conditions:

1. \( W \) has a single critical point at the origin;
2. the charges are uniquely determined by \( W \).

Remark 2.1.2. The second condition above may be regarded as saying that the matrix
\[
M = (m_{i,j})
\]
defined by \( W(x_1, \ldots, x_N) = \sum_{i=1}^s l_i \prod_{j=1}^N x_j^{m_{i,j}} \) has rank \( N \). In particular we have \( N \leq s \) and \( M \) has a left inverse.

We will use the Smith normal form \( D \) of the matrix \( M \). The matrix \( D \) is an \( s \times N \) matrix \( D \) whose entries labeled \((i, j)\) vanish for \( i \neq j \), whose diagonal entries \( d_1, \ldots, d_N \) are positive integers satisfying \( d_i \mid d_{i+1} \ \forall i \). We say that \( D \) is the Smith normal form of \( M \) if there exists an \( N \times N \) matrix \( S \) and an \( s \times s \) matrix \( T \) both invertible over \( \mathbb{Z} \) such that
\[
M = TDS.
\]
The matrix \( D \) is uniquely determined by \( M \) and its \( N \) nonvanishing entries \( d_1, \ldots, d_N \) are the invariant factors of \( M \) and can be computed in the following way: \( d_i = \Delta_i/\Delta_{i-1} \) where \( \Delta_0 = 1 \) and, for \( i > 0 \), \( \Delta_i \) is the gcd of the \( i \times i \) minors of the matrix \( M \).
We point out a consequence of the quasihomogeneity of \( W \): for \( i = 1, \ldots, N \) the sum of the entries of the \( i \)th line of the matrix \( T^{-1} \) is in \( \frac{1}{d} \mathbb{Z} \) and for each of the remaining \( s - N \) lines the sum of the entries is zero:

\[
T^{-1}(d, \ldots, d)^t = T^{-1}M(c_1, \ldots, c_N)^t = T^{-1}TDS(c_1, \ldots, c_N)^t = (d_1s_1, \ldots, d_NS_N, 0, \ldots, 0)^t,
\]

where \((s_1, \ldots, s_N)^t = S(c_1, \ldots, c_N)\).

**Example 2.1.3.** The polynomial \( x^3 + xy^3 \) is nondegenerate of degree 9: the charges of the variables \( x \) and \( y \) are uniquely determined and satisfy \( q_x = 1/3 \) and \( q_y = 2/9 \) and the origin is the only critical point.

**Example 2.1.4.** All homogeneous polynomials are quasihomogeneous. Let us consider the polynomial \( x_1^3 + x_2^3 + x_3^3 + x_4^3 \), the main focus of this paper. This is a nondegenerate quasihomogeneous polynomial of degree 5 for which the charges of all variables equal 1/5.

**Remark 2.1.5.** Not all homogeneous polynomials are nondegenerate. It may happen that the charges are not uniquely determined (e.g. \( xy \)) or that there are critical points outside the origin (e.g \( x^3y^2 + xy^4 \), for which \( d \) equals 5, the charges \( q_x = 1/5 \), \( q_y = 1/5 \) are uniquely determined, but critical points occur all along the \( x \)-axis).

**Notation 2.1.6.** We recall the standard notation for simple ADE-singularities:

- \( A_l = x^{l+1} \) (for \( l \geq 1 \)),
- \( D_l = x^{l-1} + xy^2 \) (for \( l \geq 4 \)),
- \( E_6 = x^3 + y^4 \), \( E_7 = x^3 + xy^3 \), \( E_8 = x^3 + y^5 \).

It is natural to consider the following group of transformations fixing the locus \( \{ W = 0 \} \).

**Definition 2.1.7.** The group \( G_W \) of diagonal symmetries of a nondegenerate quasihomogeneous polynomial \( W \) in \( N \) variables is the following subgroup of \((\mathbb{C}^\times)^N\)

\[
G_W := \left\{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^\times)^N \mid W(\alpha_1x_1, \ldots, \alpha_Nx_N) = W(x_1, \ldots, x_N) \quad \forall (x_1, \ldots, x_N) \in \mathbb{C}^N \right\}.
\]

**Lemma 2.1.8.** Let \( W \) be a nondegenerate quasihomogeneous polynomial of degree \( d \) in \( N \) variables whose charges equal \( q_1 = c_1/d, \ldots, q_N = c_N/d \). Let us choose coprime indices \( c_1, \ldots, c_N, d \).

Set

\[
J := (\exp(2\pi i c_1/d), \ldots, \exp(2\pi i c_N/d)) \in (\mathbb{C}^\times)^N.
\]

Then, \( J \) belongs to \( G_W \) and generates a cyclic subgroup \( \langle J \rangle \subseteq G_W \) of order \( d \).

Furthermore, let \( d_1 \mid \cdots \mid d_N \) be the invariant factors of the matrix \((m_{i,j})\) introduced in Remark 2.1.2. Then we have

\[
G_W \cong \mathbf{m}_{d_1} \times \cdots \times \mathbf{m}_{d_N} \subset U(1)^N.
\]

**Proof.** The element \( g = (\exp(2\pi i \lambda_j))_{j=1}^N \in (\mathbb{C}^\times)^N \) belongs to \( G_W \) if and only if \( \sum m_{i,j} \lambda_j \in \mathbb{Z} \) for all \( i = 1, \ldots, s \). This shows that \( G_W \) is isomorphic to \( \mathbb{M}^{-1}\mathbb{Z}^s \) modulo \( \mathbb{Z}^N \), where \( M = (m_{i,j}) \) is the injective morphism \( \mathbb{Q}^N \to \mathbb{Q}^s \) (due to the nondegeneracy condition, \( \text{rk}(M) \) equals \( N \)). For suitable bases of \( \mathbb{Z}^s \) and \( \mathbb{Z}^N \), the homomorphism \( M \) can be written in the Smith normal form \((a_1, \ldots, a_N) \mapsto (d_1a_1, \ldots, d_Na_N, 0, \ldots, 0)\); hence, we have \( G_W = \prod_{j=1}^N d_j^{-1}\mathbb{Z}/\mathbb{Z} \).

**Convention 2.1.9.** The index \( d_N \) in the above statement is the exponent of the group \( G_W \); i.e. the smallest integer \( k \) for which \( g^k = 1 \) for all \( g \in G_W \). We usually refer to this index as the exponent of \( W \) or even the exponent of the matrix \( M \) introduced in Remark 2.1.2. We write \( d(W) \) or simply \( d \) when no ambiguity may occur.

**Example 2.1.10.** Note that it may well happen that \( d \neq \bar{d} \). For example for the polynomial \( D_4 = x^3 + xy^2 \) we have

\[
c_x = 1, \quad c_y = 1, \quad d = 3, \quad d_1 = 1, \quad d_2 = 6, \quad \bar{d} = 6.
\]

The same happens for \( x_1^3 + x_1x_2^2 + x_2x_3^2 + \cdots + x_{n-1}x_n^2 \) for which \( d = 3 \) and \( \bar{d} = 2^{n-1} \times 3 \).
Definition 2.1.11. The Milnor ring (or local algebra) of $W$ is given by $\mathbb{C}[x_1, \ldots, x_N]/(\partial_1 W, \ldots, \partial_N W)$.

The dimension of the Milnor ring depends on the charges $q_j = c_j/d$ of $W$ and is given by

$$\mu = \prod_i \left( \frac{1}{q_i} - 1 \right).$$

(7)

We can define a rational grading for the Milnor ring mapping $x_j$ to $q_j = c_j/d$; in this way the highest order term of the Milnor ring is

$$\hat{c}_W = \sum_i (1 - 2q_i).$$

(8)

The index $\hat{c}_W$ is referred as the central charge in physics and will appear in FJR W theory frequently.

FJR W theory is an analogue of Gromov–Witten theory attached to a pair $(W, G)$ satisfying

$$(J) \subset G \subset G_W.$$ The main focus of this paper is the quintic polynomial $W(x_1, \ldots, x_5) = \sum_{j=1}^5 x_j^5$ together with the group $G = \langle J \rangle$. We point out in passing that other choices for $G$ deserve attention: in particular the same polynomial $W$ alongside with the “dual” group $\langle J \rangle'$ = $(\mathbb{Z}/5\mathbb{Z})^4 = \{ (\lambda_1, \lambda_2, \lambda_3, \lambda_5) | \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 1 \}$ are expected to form a pair $(W, \langle J \rangle')$ which is the “mirror” of $(W, \langle J \rangle)$.

The state space. Let $\langle J \rangle \subset G \subset G_W$. For each

$$\gamma = (e^{2\pi i \Theta_1^\gamma}, \ldots, e^{2\pi i \Theta_N^\gamma}) \in G,$$

let $\mathbb{C}_\gamma^N$ be the fixed point set of $\mathbb{C}^N$ with respect to the action of $\gamma$ by multiplication. Write $N_\gamma$ for the complex dimension of $\mathbb{C}_\gamma^N$ and $W_\gamma$ for $W|_{\mathbb{C}^N_\gamma}$. Let $\mathcal{H}_\gamma$ be the invariants of the middle-dimensional relative cohomology of $\mathbb{C}_\gamma^N$

$$\mathcal{H}_\gamma = H^{N_\gamma}(\mathbb{C}_\gamma^N, W_\gamma^{+\infty}, \mathbb{C})^G,$$

where $W^{+\infty} = (\mathbb{R}W_\gamma)^{-1}(\rho, +\infty]$ for $\rho \gg 0$.

The state space of FJRW theory is

$$\mathcal{H}_{\text{FJRW}}(W, G) = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma.$$

(9)

As in Chen–Ruan cohomology, we should shift the degree.

Definition 2.1.12. Consider $\gamma = (e^{2\pi i \Theta_1^\gamma}, \ldots, e^{2\pi i \Theta_N^\gamma}) \in G$. We define the degree-shifting number

$$\iota(\gamma) = \sum_{j=1}^N (\Theta_j^\gamma - q_j).$$

For a class $\alpha \in \mathcal{H}_\gamma$, we define

$$\deg_{W}(\alpha) = \deg(\alpha) + 2\iota(\gamma).$$

Remark 2.1.13. By Proposition 3.2.3 of [FJR1], for any $\gamma \in G_W$ we have

$$\iota(\gamma) + \iota(\gamma^{-1}) = \hat{c}_W - N_\gamma,$$

and for any $\alpha \in \mathcal{H}_\gamma$ and $\beta \in \mathcal{H}_{\gamma^{-1}}$ we have

$$\deg_{W}(\alpha) + \deg_{W}(\beta) = 2\hat{c}_W.$$
The state space $\mathcal{H}_{FJRW}(W, G)$ carries a natural Poincaré pairing as follows. Define

$$I : \mathbb{C}^N \to \mathbb{C}^N$$

by

$$I(x_1, \ldots, x_N) = (\eta^{x_1}, \ldots, \eta^{x_N})$$

with $\eta^d = -1$. Then, $W(I(x)) = -W(x)$. The choice of $I$ is not unique. Note, however, that if $\eta$ is another complex number satisfying $\eta^d = -1$, we have $I \circ I' \in \langle J \rangle \subset G$. It is well known that the natural homological intersection pairing

$$\langle \cdot, \cdot \rangle : H_N(\mathbb{C}^N, \mathbb{C}) \otimes H_N(\mathbb{C}^N, \mathbb{C}) \to \mathbb{C}$$

is perfect (we write $W_{\gamma}^{-\infty}$ for $(\Re W_{\gamma})^{-1} | -\rho |$ for $\rho \gg 0$). This induces a perfect pairing on dual spaces

$$\langle \cdot, \cdot \rangle : H_N^*(\mathbb{C}^N, \mathbb{C}) \otimes H_N^*(\mathbb{C}^N, \mathbb{C}) \to \mathbb{C}.$$  

We define the pairing

$$\langle \cdot, \cdot \rangle : H_N^*(\mathbb{C}^N, \mathbb{C}) \otimes H_N^*(\mathbb{C}^N, \mathbb{C}) \to \mathbb{C}$$

by

$$\langle \alpha, \beta \rangle = \langle \alpha, I^*(\beta) \rangle.$$  

Its restriction on the $G$-invariant subspaces induces a symmetric, nondegenerate pairing

$$\langle \cdot, \cdot \rangle_{FJRW} : H_\gamma \otimes H_{\gamma^{-1}} \to \mathbb{C}$$

independent of the choice of $I$.

**Notation 2.1.14.** In this paper we mainly consider the state space of $W$ with respect to the action of $\langle J \rangle$. Unless otherwise stated, we write $\mathcal{H}_{FJRW}(W)$ for the state space of $W$ with respect to $\langle J \rangle$.

**Remark 2.1.15.** In the geometric LG/CY correspondence, we consider a CY hypersurface $X_W = \{ W = 0 \}$ inside the weighted projective stack $\mathbb{P}(c_1, \ldots, c_N)$. Due to this embedding inside a weighted projective stack, the hypersurface $X_W$ can be naturally regarded as a stack. The state space on the CY-side is the Chen–Ruan cohomology of this stack. In general, there exists a degree-preserving vector space isomorphism between $\mathcal{H}_{FJRW}(W)$ and the Chen–Ruan cohomology of the weighted projective CY hypersurface of equation $W = 0$. In this paper we only need the case of the quintic polynomial, which we analyze in the following example.

**Example 2.1.16.** Consider $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ and the cyclic group $G = \langle J \rangle$ of order 5. For each element $J^m = (e^{2\pi im/5}, \ldots, e^{2\pi im/5}) \in G$ with $m = 0, \ldots, 4$ we compute $\mathcal{H}_{J^m}$ and the degree of its elements. The degree is even if and only if $m$ does not vanish.

Let $m \neq 0$. The degree-shifting number $e(J^m) \in \mathcal{H}_{J^m}$ is a generator, which can be regarded as the constant function 1 on $\mathbb{C}_5^5$. In this way we obtain four elements of degree 0, 2, 4, and 6, which correspond to the generators of $H^0(X_W, \mathbb{C}), H^2(X_W, \mathbb{C}), H^4(X_W, \mathbb{C})$ and $H^6(X_W, \mathbb{C})$.

Let $m = 0$. We have $\mathcal{H}_{J^0} = H^N(\mathbb{C}^N, \mathbb{C})$, which is isomorphic to the degree-3 cohomology group of $X_W$ (this can be checked directly using (7) and the computation of $h^3 = 204$ for the quintic three-fold). In fact this holds in full generality as a consequence of the isomorphism between the Milnor ring and the primitive cohomology. Definition 2.1.12 implies that the degree of the elements of $\mathcal{H}_{J^0}$ is 3.

Therefore, we have a degree-preserving vector space isomorphism

$$\mathcal{H}_{FJRW}(W) \cong H^*(X_W, \mathbb{C}).$$  

The left hand side does not have any natural multiplication in the classical sense. FJR theory builds upon the above isomorphism (11) and provides a quantum multiplication which turns the vector space isomorphism into a quantum ring isomorphism. This generalization requires some preliminaries involving moduli of curves.
2.2 Orbifold curves and roots of line bundles

FJR W theory is a theory of curves equipped with roots of the log-canonical line bundle and its powers. In view of its definition it is crucial to recall the notion of orbifold curve which, with respect to the moduli problem of roots of line bundles, is better suited than the scheme-theoretic notion of algebraic curve.

Definition 2.2.1. An orbifold curve is a 1-dimensional stack of Deligne–Mumford type with nodes as singularities, a finite number of ordered markings, and possibly nontrivial stabilizers only at the markings and at the nodes. The action on a local parameter $z$ at the markings is given by

$$z \mapsto \xi_l z, \quad l \in \mathbb{Z}_{\geq 1}$$

whereas the action on a node $\{xy = 0\}$ is

$$(x, y) \mapsto (\xi_k x, \xi^{-1}_k y), \quad k \in \mathbb{Z}_{\geq 1}.$$

Such a curve $C$ is naturally equipped with a (locally free) sheaf of logarithmic differentials $\omega_{\log}$; i.e. the sheaf of sections of the relative dualizing sheaf $\omega_C$ possibly with poles of order 1 at the markings. We will often refer to $\omega_{\log}$ rather than $\omega_C$, because it corresponds to the pullback of the sheaf of logarithmic differentials from the coarse curve.

Above, $k$ and $l$ denote two orders of stabilizers. It is automatic for a Deligne–Mumford stack-theoretic curve to have cyclic stabilizers; however, in the above expression for the local action at a node, we require that the product of the factors multiplying $x$ and $y$ is 1. This is usually referred as a “balance” condition in [AV02] and insures that the orbifold curves can be smoothed.

An orbifold curve with $n$ markings will be called an $n$-pointed orbifold curve; the above definition naturally extends to the notion of “family of orbifold curves over the base scheme $X$” (or simply “orbifold curve over $X$”): these are flat morphisms $C \to X$ of relative dimension one from a Deligne–Mumford stack $C$ to $X$ for which we extend the local descriptions given above (for example, instead of giving ordered marked points we have to specify ordered cyclic gerbes over the base scheme $X$ embedded in the smooth locus of $C$, we refer the reader to [AV02, Defn. 4.1.2]).

Note that two stabilizers arising in an orbifold curve may have different orders; this feature prevents the moduli stack of orbifold curves from being separated (see e.g. [Ch08a]) and motivates the following definition imposing a stability condition yielding a proper stack.

Definition 2.2.2. For any positive integer $l$, an $l$-stable $n$-pointed genus-$g$ curve is a proper and geometrically connected orbifold curve $C$ of genus $g$ with $n$ distinct smooth markings $\sigma_1, \ldots, \sigma_n$ such that

1. the corresponding coarse $n$-pointed curve is stable;
2. all stabilizers (at the nodes and at the markings) have order $l$.

The moduli stack $\overline{M}_{g,n,l}$ classifying $n$-pointed genus-$g$ $l$-stable curves is proper and smooth and has dimension $3g - 3 + n$. (It differs from the moduli stack of Deligne–Mumford stable curves only because of the stabilizers over the normal crossings boundary divisor, see the discussion in [Ch08a, Thm. 4.1.6].)

The compactifications of the moduli space of smooth curves via these new $l$-stability conditions are well suited to FJRW theory. Indeed, for a fixed singularity $W$ of exponent $\bar{d}$, FJRW theory is a theory of curves coupled with (sections of) roots of order $\bar{d}$ of the log-canonical bundle (and of its powers). In fact the Picard functor of $l$-stable curves is an extension of the Picard functor of ordinary stable curves with a particularly interesting feature: the number of $l$-torsion points in the Picard group is invariant under deformation [Ch08a]. Therefore, choosing to work with $\bar{d}$-stable curves makes the treatment of $\bar{d}$th roots, and consequently that of $W$-curves, straightforward. From now on, we always work with $\bar{d}$-stable curves. We recall a few facts about moduli of roots.

Definition 2.2.3. For any nonnegative integer $c$ and for any divisor $l$ of $\bar{d}$ we define the category $\mathcal{R}_c^l$ of $l$th roots of $\omega_{\log}^c$ over genus-$g$ $n$-pointed $\bar{d}$-stable curves. It is fibred over $\overline{M}_{g,n,\bar{d}}$

$$\mathcal{R}_c^l \to \overline{M}_{g,n,\bar{d}}$$
and it is formed by the following objects: an $\bar{d}$-stable curve, $(C \to X, \sigma_1, \ldots, \sigma_n \in C)$ equipped with a line bundle $L \to C$ and an isomorphism of line bundles over $C$

$$\varphi : L^{\otimes \bar{d}} \to \omega_{log}^\otimes.$$ 

A morphism from $(C' \to X'; \sigma'_1, \ldots, \sigma'_n \in C'; L'; \varphi')$ to $(C'' \to X''; \sigma''_1, \ldots, \sigma''_n \in C''; L'', \varphi'')$ is a morphism $\alpha$ between the $\bar{d}$-stable curves and an isomorphism of line bundles $\rho : L' \to \alpha^* L''$ compatible with $\varphi'$ and $\varphi''$; i.e. we have $\alpha^* \varphi'' \circ \rho^{\otimes \bar{d}} = \varphi'$.

**Remark 2.2.4.** On a $\bar{d}$-stable curve $C$, we often need to twist line bundles by divisors supported on the markings. For a marking $\sigma$ which has stabilizer $G$ of order $\bar{d}$ it is convenient to denote by $(1/\bar{d})D$ the divisor $BG \to C$ of degree $1/\bar{d}$ and to write $D$ for the pullback of the underlying point on the coarse curve. We refer to $D$ as the **integer divisor corresponding to a marking**. This is a good spot to recall the following well known fact, which uses this convention.

**Lemma 2.2.5.** Let $C$ be a $\bar{d}$-stable curve (over $\mathbb{C}$) and let $M$ be a line bundle pulled back from the coarse space (e.g. $\omega_{log}$ and its tensor powers). For $l \mid \bar{d}$, there is an equivalence between two categories of $l$th roots $L$ on $\bar{d}$-stable curves:

$$\left\{ L \mid L^{\otimes \bar{d}} \cong M \right\} \leftrightarrow \bigcup_{0 \leq E < \ell \sum_{i=1}^l \partial_i \left\{ L \mid L^{\otimes \bar{d}} \cong M(-E) \text{ and the stabilizers at the markings act trivially on } L \right\},$$

where the union on the right hand side ranges over all divisors $E$ which are linear combinations of integer divisors $\partial_i$ to the markings with multiplicities in $\{0, \ldots, l-1\}$.

In particular as soon as $n$ is positive or $l$ divides $c(2g-2)$ we have $R^l_i \neq \emptyset$. Furthermore, in these cases $R^l_i \to \overline{M}_{g,n,d}$ is proper and étale and forms a torsor with respect to the group structure of $R^0_i \to \overline{M}_{g,n,d}$.

The fibres of these morphisms to $\overline{M}_{g,n,d}$ are all isomorphic to

$$B\mu_1 \sqcup \cdots \sqcup B\mu_l,$$

$l^{2g-1+n}$ times.

**Proof.** The correspondence is simply the functor $L \mapsto p^* p_* L$ where $p$ is the map to the coarse curve. This implies that each fibre of $R^l_i$ on a smooth curve is nonempty as soon as $n$ is positive or $l$ divides $c(2g-2)$. Indeed, by [Ch08a], the morphism $R^l_i \to \overline{M}_{g,n,d}$ is proper and all fibres are isomorphic. A direct analysis of the fibre over a point representing a smooth curve yields the result.

In fact, consider a smooth curve over $C$ of genus $g$ with $n$ markings and stabilizers $\Gamma_1, \ldots, \Gamma_n$ of order $\bar{d}$. Here, the $l$th roots of $\omega_{log}$ form a torsor under the group of $l$th roots of $O$. The $l$th roots of $O$ on $C$ are represented by as many points as the order of $H^1(C, \mu_l)$; each one of these points is a zero-dimensional stack $B\mu_l$ of degree $1/l$ due to the isomorphism acting by multiplication by $\xi_l$ along the fibres and fixing the curve. We need to compute that the order of $H^1(C, \mu_l)$ equals $l^{2g-1+n}$. This follows from the following exact sequence

$$1 \to H^1(|C|, \mu_l) \to H^1(C, \mu_l) \to \prod_{k=1}^n H^1(\Gamma_k, \mu_l) \to H^2(|C|, \mu_l) \to 1,$$

where $|C|$ is the coarse curve of $C$ and we have $H^1(|C|, \mu_l) \cong (\mu_l)^{2g}$ and $H^2(|C|, \mu_l) \cong \mu_l$. $|C|$ is irreducible because $C$ is smooth and connected). Each group $H^1(\Gamma_k, \mu_l)$ is cyclic of order $l$ and is identified by $\delta$ to $H^2(|C|, \mu_l) = \mu_l$. The claim $h^1(C, \mu_l) = l^{2g-1+n}$ follows.

**Remark 2.2.6.** The stack $R^0_i$ contains natural open and closed substacks

$$R^0_i (L^{\otimes q} \cong O) \subseteq R^0_i$$
where $R^0_q(L^{\otimes q} \cong O)$ is the full subcategory of $R^0$ of objects $(C \to X, L, \varphi)$ such that $L^{\otimes q}$ is trivial. Notice that $R^0_q(L^{\otimes q} \cong O)$ does not coincide with $R^0_q$. In fact, there is a natural morphism

$$R^0_q \to R^0_q(L^{\otimes q} \cong O)$$

which consists in sending $(C \to X; \sigma_1, \ldots, \sigma_n \in C; L; \varphi: L^{\otimes q} \to O)$ to the corresponding $l$th root of $O$ $(C \to X; \sigma_1, \ldots, \sigma_n \in C; L; \varphi^{\otimes l/q}; L^{\otimes l} \to O)$. This functor actually lands in $R^0_q(L^{\otimes q} \cong O)$ because $L^{\otimes q}$ is trivial. Notice, however, that $R^0_q \to R^0_q(L^{\otimes q} \cong O)$ is locally isomorphic to $B\mu_l \to B\mu_l$ and is not an isomorphism unless $q = l$ (its degree is $l/q$). This happens because the points of $R^0_q(L^{\otimes q} \cong O)$ are equipped with the above mentioned $\mu_l$-action acting by multiplication along the fibres, whereas a point of $R^0_q$ is equipped with a $\mu_l$-action.

Finally let us point out that there exist natural maps

$$R^c_q \to R^c_q \circ \, \rho_q$$

$$R^c_q \to R^0_q$$

via the functors $(L, \varphi) \mapsto (L, \varphi^{\otimes q})$ and $(L, \varphi) \mapsto (L^{\otimes l} \otimes \omega^c \to \varphi \otimes \operatorname{id})$. Indeed we have

$$(L^{\otimes l} \otimes \omega^c \to \varphi \otimes \operatorname{id}) \circ \rho_q = (L^{\otimes l} \otimes \omega^c \to \varphi \otimes \operatorname{id})$$

In the rest of the paper we will need two straightforward facts about these morphisms.

The first morphism $R^c_q \to R^0_q$ is an isomorphism when $l = c = 1$:

$$R^0_q \cong R^0_q.$$  \hspace{1cm} (13)

Furthermore, via the second morphism, we can consider the preimage $X$ of $R^0_q(L \cong O)$ inside $R^c_q$; this is an open and closed substack whose fibre over $\overline{M}_{g,n,d}$ consists of $l^{2g-1+n}$ copies of $B\mu_l$. This substack $X \subseteq R^c_q$ is the image of $R^c_q$; we have

$$R^c_q \to X \subseteq R^c_q,$$  \hspace{1cm} (14)

where the morphism $R^c_q \to X$ is locally isomorphic to $B\mu_l \to B\mu_l$ and sends the $l^{2g-1+n}$ points of the fibre of $R^c_q \to \overline{M}_{g,n,d}$ onto the $l^{2g-1+n}$ points of the fibre of $X \to \overline{M}_{g,n,d}$.

### 2.3 Moduli of $W$-curves

In this section we introduce the moduli $W_{g,n}$ of $W$-curves, and we describe the substacks $W(\gamma_1, \ldots, \gamma_n)_{g,n}$ determined by the type of the markings, and the substacks $W_{g,n,G}$ attached to certain subgroups $G$ of $G_W$.

#### Moduli of $W$-curves

First, we write $W$ as a sum of distinct monomials in the variables $x_1, \ldots, x_N$

$$W = W_1 + \cdots + W_s$$

(here and in the rest of the paper by distinct monomials we mean monomials with different exponents). Set

$$c_i = c_i \frac{\tilde{d}}{d},$$

where $\tilde{d}$ is the exponent of $W$. 

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Definition 2.3.1. On a $\tilde d$-stable curve $C$, a $W$-structure, is the datum of $N$ $\tilde d$th roots

$$(L_j, \varphi_j : L_j^\otimes \tilde d \to \omega_{\log}^\otimes \tilde d)_{j=1}^N$$

(as many as the variables of $W$) satisfying the following $s$ conditions (as many as the monomials $W_1, \ldots, W_s$): for each $i = 1, \ldots, s$ the line bundle $W_i(L_1, \ldots, L_N) = \bigotimes_{j=1}^N L_j^\otimes m_{i,j}$ satisfies

$$W_i(L_1, \ldots, L_N) \cong \omega_{\log}^\otimes.$$

A $\tilde d$-stable curve equipped with a $W$-structure is called an $n$-pointed genus-$g$ $W$-curve. We denote by $W_{g,n}$ the stack of $n$-pointed genus-$g$ $W$-curves

$$W_{g,n} := \left\{ (L_1, \varphi_1), \ldots, (L_N, \varphi_N) \stackrel{\text{roots}}{\rightarrow} C \ni \sigma_1, \ldots, \sigma_n \stackrel{\text{curve}}{\rightarrow} X \quad W_i(L_1, \ldots, L_N) \cong \omega_{\log}^\otimes \right\} / \cong.$$

Notation 2.3.2. Given an $m$-tuple of line bundles $\vec{E} = (E_1, \ldots, E_m)$ and an $n \times m$ matrix $A = (a_{i,j})$ we denote by $A\vec{E}$ the $n$-tuple of line bundles $(\otimes_j E_j^{a_{i,j}})_{i=1}^n$. Given an $m$-tuple of isomorphisms of line bundles $\vec{\psi} = (\psi_1, \ldots, \psi_m)$ we denote by $A\vec{\psi}$ the $m$-tuple of isomorphisms of line bundles $(\otimes_j \psi_j^{a_{i,j}})_{i=1}^m$.

Remark 2.3.3. We point out that $W_{g,n}$ is, by definition, an open and closed substack of $R_{\tilde d}^\otimes \times_\tilde d \cdots \times_\tilde d R_{\tilde d}^\otimes$, where $\times_\tilde d$ denotes the fibred product over $\mathcal{M}_{g,d}$. Indeed, we can naturally define the morphism

$$p : R_{\tilde d}^\otimes \times_\tilde d \cdots \times_\tilde d R_{\tilde d}^\otimes \to (R_{\tilde d}^\otimes)^s \cong (R_{\tilde d}^0)^s,$$

where $\vec{L}$ is the vector attached to $W$ and the canonical identification (13) has been used. Note that the $i$th entry of $\vec{\psi}$; $M\vec{L}^\otimes \tilde d = M(\omega_{\log}^{\otimes e_1} \cdots, \omega_{\log}^{\otimes e_N})$ identifies the $\tilde d$th tensor powers $W_i(L_1, \ldots, L_N)^{\otimes \tilde d}$ to $\omega_{\log}^{\otimes \tilde d}$. In this way the stack $W_{g,n}$ can be alternatively defined as the open substack where $W_i(L_1, \ldots, L_N)$ is isomorphic to $\omega_{\log}^\otimes$; in other words, we have

$$W_{g,n} = p^{-1} \left( \bigotimes_{i=1}^s R_{\tilde d}^0(L \cong \mathcal{O}) \times_\tilde d \cdots \times_\tilde d R_{\tilde d}^0(L \cong \mathcal{O}) \right).$$

We also point out that $W_{g,n}$ can be naturally regarded as a torsor. Consider two $W$-structures $(L_1', \ldots, L_N')$ (with isomorphisms $\varphi_1', \ldots, \varphi_N'$) and $(L_1'', \ldots, L_N'')$ (with isomorphisms $\varphi_1'', \ldots, \varphi_N''$) on the same $\tilde d$-stable curve. Their “difference”

$$(L_1', \ldots, L_N') \otimes (L_1'', \ldots, L_N'') := (L_1' \otimes (L_1'')^\vee, \ldots, L_N' \otimes (L_N'')^\vee)$$

satisfies

$$\left( L_i' \otimes (L_i'')^\vee \right)^{\otimes \tilde d} \otimes \varphi_i' \otimes ((\varphi_i'')^\vee)^{-1} \otimes \omega_{\log}^{\otimes e_i} \otimes \omega_{\log}^{\otimes e_i^\vee} \cong \mathcal{O} \quad \forall i \in \{1, \ldots, N\}$$

and is therefore an $N$-tuple of $\tilde d$th roots of $\mathcal{O}$. Furthermore $\vec{E} = (L_1', \ldots, L_N') \otimes (L_1'', \ldots, L_N'')^\vee$ satisfies $\vec{M}\vec{E} \cong (\mathcal{O}, \ldots, \mathcal{O})^\vee$. It follows that the $W$-structures on a $\tilde d$-stable curve form a torsor under the group of $W^\vee$-structures in the sense of the following definition.

Definition 2.3.4. Let $c \in \mathbb{N}$. Let $\tilde d$ be the exponent of $W$. On a $\tilde d$-stable curve $C$, a $W^c$-structure, is the datum of $N$ $\tilde d$th roots

$$(L_j, \varphi_j : L_j^\otimes \tilde d \to \omega_{\log}^{\otimes \tilde d})_{j=1}^N$$

satisfying

$$W_i(L_1, \ldots, L_N) \cong \omega_{\log}^{\otimes c}.$$

for each $i = 1, \ldots, s$. We denote by $W_{g,n}^c$ the stack of $n$-pointed genus-$g$ $\tilde d(W)$-stable curves equipped with a $W^c$-structure (or simply $W^c$-curves).
Proposition 2.3.5. Let $W$ be a nondegenerate quasihomogeneous polynomial of degree $d$ in $N$ variables whose charges equal $c_1/d, \ldots, c_N/d$. Let us choose coprime indices $c_1, \ldots, c_N, d$. Let us denote by $d$ the exponent $d(W)$ and set $\bar{c}_j = c_j/d$ for $j = 1, \ldots, N$. For $c \geq 0$ we consider the stack $W_{g,n}^c$.

1. The stack $W_{g,n}^c$ is nonempty if and only if $n > 0$ or $c(2g-2) \in d\mathbb{Z}$. It is a proper, smooth, $3g-3+n$-dimensional Deligne–Mumford stack; more precisely, it is étale over $\overline{\mathcal{M}}_{g,n,d}$ which is a proper and smooth stack of dimension $3g-3+n$.

2. The stack $W_{g,n}^0$ carries a structure of a group over the stack of genus-$g$ $n$-pointed $d$-stable curves $\overline{\mathcal{M}}_{g,n,d}$ with composition law

$$W_{g,n}^0 \times_d W_{g,n}^0 \to W_{g,n}^0,$$

where $\times_d$ denotes the fibred product over $\overline{\mathcal{M}}_{g,n,d}$. The degree of $W_{g,n}^0$ over $\overline{\mathcal{M}}_{g,n,d}$ is equal to $|G_W|^{2g-1+n}/d^n$.

3. The stack $W_{g,n}$ (and more generally $W_{g,n}^c$) is a torsor under the group stack $W_{g,n}^0$ over $\overline{\mathcal{M}}_{g,n,d}$. In particular, its degree over $\overline{\mathcal{M}}_{g,n,d}$ equals that of $W_{g,n}^0$ over $\overline{\mathcal{M}}_{g,n,d}$. We have a surjective étale morphism and an action

$$W_{g,n}^c \to W_{g,n}^0 \times_d W_{g,n}^c \to W_{g,n}^c.$$

Proof. We prove claims (1) and (2) for $W^0$-curves. Let us chose $S$ and $T$ as in Remark 2.1.2 so that $M = TDS$ and $D$ is the Smith normal form of $M$. Recall that $S$ is an $N \times N$ matrix, invertible over $\mathbb{Z}$. Consider the following Cartesian diagram

$$W_{g,n}^0 \xrightarrow{T} R_0^0 \times_d \cdots \times_d R_0^0$$

$$\xrightarrow{\square} R_0^0(L^\otimes d_1 \cong \mathcal{O}) \times_d \cdots \times_d R_0^0(L^\otimes d_N \cong \mathcal{O}) \xrightarrow{\square} R_0^0 \times_d \cdots \times_d R_0^0,$$

where the vertical arrow on the right hand side denotes the automorphism of $R_0^0 \times_d \cdots \times_d R_0^0$

$$(\bar{L}, \bar{\varphi}) \mapsto (S\bar{L}, S\bar{\varphi}).$$

It is clear that the automorphism transforms $W_{g,n}^c$ as indicated by the diagram: the defining condition of the full subcategory $W_{g,n}$ satisfying $M\bar{L} \equiv (\mathcal{O}, \ldots, \mathcal{O})^t$ becomes $D\bar{L} \equiv (\mathcal{O}, \ldots, \mathcal{O})^t$. The group structure, the finiteness, the étaleness and the degree formula for the morphism $W_{g,n}^0 \to \overline{\mathcal{M}}_{g,n,d}$ easily follow from [Ch08a] and Lemma 2.2.5, where these claims are proven for each factor.

Claims (1) and (3) hold for $W^c$-curves. By Lemma 2.2.5, as soon as $n > 0$ or $d \mid 2g-2$, all fibres of $W_{g,n}^c \to \overline{\mathcal{M}}_{g,n,d}$ are nonempty and any two $W$-structures differ by a $W^0$-structure (the same claims extend immediately to $W^c$-structures). Such a $W^0$-structure is unique up to isomorphism, and the isomorphism is unique up to a natural transformation; therefore $W_{g,n}$ (and $W_{g,n}^c$) is a torsor under the action of the group stack $W_{g,n}^0$ over $\overline{\mathcal{M}}_{g,n,d}$ (claim (3)). Notice that the finiteness, the étaleness and the degree formula for $W_{g,n}$ can be proven locally over $\overline{\mathcal{M}}_{g,n,d}$, and are therefore immediate consequences of the torsor structure. ☐
Decomposition of $W_{g,n}$: the type $(\gamma_1, \ldots, \gamma_n)$ of the markings. The natural embedding of $W_{g,n}$ into $R^c_{g,n}$ allows us to generalize the notion of (topological) type already defined in the literature for stacks of spin curves. We recall it for moduli of $l$-spin curves; i.e. for the stack $(A_{l-1})_{g,n}$ and its variants $(A_{l-1})_{g,n}$. In these cases the exponent $d$ is $l$ and we are simply considering $R^c_l$. The stack $R^c_l$ is the disjoint union

$$R^c_l = \bigcup_{0 \leq \Theta_1, \ldots, \Theta_n < 1} R^c_{l}(e^{2\pi i \Theta_1}, \ldots, e^{2\pi i \Theta_n}), \quad (15)$$

where

$$R^c_{l}(e^{2\pi i \Theta_1}, \ldots, e^{2\pi i \Theta_n}) = \left\{ (C; \sigma_1, \ldots, \sigma_n; L; \varphi) \mid \varphi : L^\otimes l \sim \to (\omega_{\log})^{\otimes c} \quad \text{and} \quad \Theta_i = \text{mult}_{\sigma_i} L \in [0, 1[ \quad \forall i \right\}.$$  

The index mult$_{\sigma_i} L$ is determined by the local indices of the universal $l$th root $L$ at the $i$th marking $\sigma_i$. More explicitly, the local picture of $R^c_l$ over $C$ at the $i$th marking $\sigma_i$ is parametrized by the pairs $(x, \lambda) \in C^2$, where $x$ varies along the curve and $\lambda$ varies along the fibres of the line bundle. The stabilizer $\mu_d$ at the marking acts as $(x, \lambda) \mapsto (\exp(2\pi i/d)x, \exp(2\pi i \Theta i)\lambda)$. In this way, the local picture provides an explicit definition of $\Theta_1, \ldots, \Theta_n$.

As a straightforward consequence, the embedding $W_{g,n} \hookrightarrow R^c_{g,n}$, induces a decomposition of the stack $W_{g,n}$ into several connected components obtained via pullback of the fibred products of the components of type (15) for the moduli stacks $R^c_d$ with $i = 1, \ldots, N$. We restate this decomposition in the following definition.

**Definition 2.3.6.** Let us fix $n$ multiindices with $N$ entries $\gamma_i = (e^{2\pi i \Theta_1}, \ldots, e^{2\pi i \Theta_N}) \in U(1)^N$ for $i = 1, \ldots, n$ and $\Theta^j_i \in [0, 1[$. Then $W(\gamma_1, \ldots, \gamma_n)_{g,n}$ is the stack of $n$-pointed genus-$g$ W-curves satisfying the relation $\Theta^i_j$ is the $j$th entry of $\gamma_i$.

**Proposition 2.3.7.** The stack $W^c_{g,n}$ is the disjoint union

$$W^c_{g,n} = \bigcup_{\gamma_1, \ldots, \gamma_n \in U(1)^N} W(\gamma_1, \ldots, \gamma_n)_{g,n}.$$  

For $c = 0$ the connected component $W(1, \ldots, 1)^0_{g,n}$ is a group stack of degree $|G_W|^2g/d^N$, étale, and proper over $\overline{M}_{g,n,d}$. The stack $W(\gamma_1, \ldots, \gamma_n)^0_{g,n}$ is nonempty if and only if

$$\begin{cases} \gamma_i = (e^{2\pi i \Theta_1}, \ldots, e^{2\pi i \Theta_N}) \in G_W \quad i = 1, \ldots, n; \\ q_j c(2g - 2 + n) - \sum_{i=1}^n \Theta^j_i \in \mathbb{Z} \quad j = 1, \ldots, N. \end{cases} \quad (16)$$

In that case it is a torsor under $W(1, \ldots, 1)^0_{g,n}$ étale and proper over $\overline{M}_{g,n,d}$. (In particular the degree $[W(1, \ldots, 1)^0_{g,n} : \overline{M}_{g,n,d}]$ equals $|G_W|^2g/d^N$.)

**Proof.** The decomposition into connected components follows immediately from (15). It is immediate to see that $W(1, \ldots, 1)^0_{g,n}$ is a group stack. The same argument used in the proof of Proposition 2.3.5, shows that the geometric points of the generic fibre of $W(1, \ldots, 1)^0_{g,n}$ over $\overline{M}_{g,n,d}$ are $|G_W|^2g$ and have stabilizers of order $d^N$; this yields the claim (in (12), we can identify these geometric points to the kernel of $m$).

It is clear that $W(\gamma_1, \ldots, \gamma_n)^c_{g,n} \to \overline{M}_{g,n,d}$ is either empty or it is a torsor under the action of $W(1, \ldots, 1)^0_{g,n} \to \overline{M}_{g,n,d}$. It remains to show that $W(\gamma_1, \ldots, \gamma_n)^c_{g,n} \neq \emptyset$ is equivalent to (16). The fact that (16) is necessary follows from degree considerations on the line bundles $L_1, \ldots, L_N$ on the coarse curve and is detailed in [FJR1, Rem. 2.2.14]. The fact that (16) is sufficient follows immediately from the formula $[W^c_{g,n} : \overline{M}_{g,n,d}] = |G_W|^2g - 1 + n/d^N$. Indeed, the number of choices $(\gamma_1, \ldots, \gamma_n)$ compatible with (16) is precisely $|G_W|^{n-1}$ as a consequence of the second condition, which can be regarded as expressing $\gamma_n$
in terms of $\gamma_1, \ldots, \gamma_{n-1}$. In this way, since each component $W(\gamma_1, \ldots, \gamma_n)_{g,n}$ has degree 0 or $|G_W|^{2g}/d^N$, we conclude that, in order to match $|W_{g,n} : \overline{M}_{g,n,d}| = |G_W|^{2g-1+n}/d^N$, every component whose indices satisfy (16) is nonempty. □

**Notation 2.3.8** (markings of Ramond and Neveu–Schwartz type). All the $W$-curves represented in the moduli stack $W(\gamma_1, \ldots, \gamma_n)_{g,n}$ have $n$ markings whose local indices are $\gamma_1, \ldots, \gamma_n$. This allows us to introduce the notion of marking of Neveu–Schwarz (NS) type and Ramond (R) type.

A marking with local index $\gamma = (e^{2\pi i \Theta_1}, \ldots, e^{2\pi i \Theta_N})$ is

$$\begin{cases} 
\text{NS} & \text{if } e^{2\pi i \Theta_j} \neq 1 \forall j \\
R & \text{otherwise.} 
\end{cases}$$

**Example 2.3.9.** Consider $W = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$. The corresponding moduli stack $W_{g,n}$ classifies $n$-pointed genus-$g$ curves equipped with five 5th roots of $\omega_{\log}$. For a marking to be of R type it suffices that one of the local indices equals 1.

In the next sections, the special case where all five roots are isomorphic to each other plays a crucial role. In particular, for $\xi = e^{2\pi i/5}$, we consider the stacks of 3-pointed genus-0 curves equipped with five isomorphic 5th roots of $\omega_{\log} \cong \mathcal{O}(-2 + 3$ markings) $\cong \mathcal{O}(1)$ of local index $\xi^3$ at each marking; note that this stack is nonempty by Proposition 2.3.7 and that each marking is of NS type. Similarly, the stack of 5-pointed genus-0 curves equipped with five isomorphic 5th roots of $\omega_{\log} \cong \mathcal{O}(-2 + 5$ markings) $= \mathcal{O}(3)$ of type $\xi^0$ at the first marking and $\xi^2$ at the four remaining markings is nonempty by equation (16) of Proposition 2.3.7; here, the first marking is of R type whereas the remaining markings are NS as before.

**The subgroups $G \subset G_W$ and the substacks $W_{g,n,G} \subseteq W_{g,n}$.** We have just seen that $G_W$ is the group where the local indices $\gamma_1, \ldots, \gamma_n$ take values. We identify open and closed substacks of $W_{g,n,G}$ where the local indices only belong to a given subgroup $G$ of $G_W$. This happens because $G$ can be regarded as the group of diagonal symmetries (in the sense of Definition 2.1.7) of a polynomial

$$W(x_1, \ldots, x_N) + \text{extra quasihomogeneous terms in the variables } x_1, \ldots, x_N.$$ 

We may allow negative exponents in the extra terms; we only require that the extra monomials are distinct from those of $W$ but involve the same variables $x_1, \ldots, x_N$ with charges $q_1, \ldots, q_N$.

**Definition 2.3.10.** A subgroup $G$ of $G_W$ is admissible if there exists a Laurent power series $Z$ in the same variables $x_1, \ldots, x_N$ as $W$ such that $W(x_1, \ldots, x_N) + Z(x_1, \ldots, x_N)$ is quasihomogeneous in the variables $x_1, \ldots, x_N$ with charges $q_1, \ldots, q_N$ and we have

$$G = G_{W+Z}.$$ 

We require that the monomials of $Z$ are distinct from those of $W$.

To each admissible subgroup $G$ of $G_W$ we can associate a substack $W_{g,n,G}$ of $W_{g,n}$ whose object will be referred to as $(W,G)$-curves. Set

$$W_{g,n,G} := (W + Z)_{g,n},$$

where $Z$ satisfies $G = G_{W+Z}$.

**Remark 2.3.11.** The above definition of $W_{g,n,G}$ makes sense. It is immediate that the definition of $W_{g,n}$ extends when $W$ is a quasihomogeneous power series. It is also straightforward that the definition of $W_{g,n,G}$ does not depend on the choice of $Z$. Assume that there are two polynomials $Z'$ and $Z''$ satisfying $G = G_{W+Z'} = G_{W+Z''}$. We can define a third polynomial $\tilde{Z}$ by summing all distinct monomials of $Z'$ and $Z''$. Then we immediately have $G_{W+\tilde{Z}} = G$ and $(W + Z')_{g,n} \supseteq (W + \tilde{Z})_{g,n} \supseteq (W + Z'')_{g,n}$. Notice that these inclusions cannot be strict: the fibres over $\overline{M}_{g,n,d}$ of all the three moduli stacks involved are zero-dimensional stacks all isomorphic to the disjoint union of $|G|^{2g-1+n}$ copies of $B(\mu_d)^N$.

**Remark 2.3.12.** As in Proposition 2.3.7, we have

$$W_{g,n,G} = \bigcup_{\gamma_1, \ldots, \gamma_n \in G} W(\gamma_1, \ldots, \gamma_n)_{g,n,G},$$

where $\gamma_i \in G$ is the local index at the $i$th marked point.
By construction, an admissible group $H$ in $G_W$ contains $\langle J \rangle$. By [FJR1, 2.3.3], $\langle J \rangle \subset G_W$ is admissible; therefore each stack $W_{g,n}$ contains a distinguished open and closed substack $W_{g,n,J}$. In the following proposition, we provide a map to $W_{g,n}$ which identifies the substack $W_{g,n,J}$.

**Proposition 2.3.13.** The stack $W_{g,n}$ fits in the following diagram

$$\mathcal{R}_d^1 \to W_{g,n,J} \leftarrow W_{g,n} \leftarrow \mathcal{R}_d^{\bar{e}_1} \times \cdots \times \mathcal{R}_d^{\bar{e}_N}.$$  

The surjective morphism $\mathcal{R}_d^1 \to W_{g,n,J}$ is simply

$$(L, \varphi) \mapsto ((L^{\otimes_{\bar{e}_1}}, \varphi^{\otimes_{\bar{e}_1}}), \ldots, (L^{\otimes_{\bar{e}_N}}, \varphi^{\otimes_{\bar{e}_1}}))$$

and is locally isomorphic to $B \mu_d \to B(\mu_d)^N$.

**Proof.** By construction $\mathcal{R}_d^1 \to W_{g,n,J}$ maps to $\mathcal{R}_d^{\bar{e}_1} \times \cdots \times \mathcal{R}_d^{\bar{e}_N}$ in order to prove that it factors through $W_{g,n,J}$, we need to check that $\mathcal{L} = (L^{\otimes_{\bar{e}_1}}, \ldots, L^{\otimes_{\bar{e}_N}})_1$ satisfies $(m_1, \ldots, m_N)\mathcal{L} \cong (\omega_{\log}, \ldots, \omega_{\log})$ whenever $\sum j=1^{N} m_j c_j = d$. In other words, we show that $(L^{\otimes_{\bar{e}_1}}, \ldots, L^{\otimes_{\bar{e}_N}})$ satisfies the condition $W_i(L^{\otimes_{\bar{e}_1}}, \ldots, L^{\otimes_{\bar{e}_N}}) = \omega_{\log}$ imposed by any monomial of the generic Laurent polynomial $W + Z$. Indeed $(m_1, \ldots, m_N)\mathcal{L} = (L^{\otimes d}, \ldots, L^{\otimes d})$ and the isomorphism with $(\omega_{\log}, \ldots, \omega_{\log})^l$ exists because $L$ is a $d$th root of $\omega_{\log}$.

In fact, the morphism $\mathcal{R}_d^1 \to W_{g,n,J}$ over $\mathcal{M}_{g,n,d}$ can be described fibre by fibre as $d^{2g-1+n}$ disjoint copies of $B \mu_d \to B(\mu_d)^N$. Indeed, a point of $\mathcal{R}_d^1$ lying inside a fibre over $\mathcal{M}_{g,n,d}$ is isomorphic to $B \mu_d$. The image via $\mathcal{R}_d^1 \to W_{g,n,J}$ is a point in the fibre of $W_{g,n,J} \to \mathcal{M}_{g,n,d}$ and is isomorphic to $B(\mu_d)^N$ (as every point of the fibres of $W_{g,n} \to \mathcal{M}_{g,n,d}$). To show that the morphism yields an injection (hence a bijection) at the level of points. For $h_1, \ldots, h_N \in Z$ satisfying $\sum j=1^{N} h_j c_j = 1$, define $\mathcal{R}_d^{\bar{e}_1} \times \cdots \times \mathcal{R}_d^{\bar{e}_N} \to \mathcal{R}_d^{d/d}$ mapping $(\mathcal{L}_i, \varphi_1, \ldots, (\mathcal{L}_N, \varphi_N))$ to $(\otimes_j L_j^{h_j}, \otimes_j \varphi_j^{\varphi_n})$. We point out that the composite morphism

$$\mathcal{R}_d^1 \to \mathcal{R}_d^{\bar{e}_1} \times \cdots \times \mathcal{R}_d^{\bar{e}_N} \to \mathcal{R}_d^{d/d}$$

coincides with (14). The yields the desired description of $\mathcal{R}_d^1 \to W_{g,n,J}$ on $\mathcal{M}_{g,n,d}$ and in particular the bijection at the level of point. 

**Example 2.3.14.** Let $W = \sum_{i=1}^5 x_i^{5}$. The group $G = \langle J \rangle$ can be realized as $G_{W+Z}$ for a generic degree-five homogeneous polynomial $W+Z$. It is easy to show that all coordinates of $(L_1, \ldots, L_5)$ are isomorphic and satisfy $L_i^{\otimes 5} \cong \omega_{\log}$. In other words, $W_{g,n,G}$ is the open and closed substack identified as the image of $\mathcal{R}_d^{1} \to W_{g,n}$ in the statement of Proposition 2.3.13.

For $(W,G) = (\sum_{i=1}^5 x_i^{5}, \langle J \rangle)$ the local indices are determined by the exponent $h = 0, 1, 2, 3, 4$ of $J^h \in \langle J \rangle$ with $J = (e^{2\pi i/5}, e^{2\pi i/5}, e^{2\pi i/5}, e^{2\pi i/5}, e^{2\pi i/5})$. Indeed since the five line bundles are isomorphic the five local indices coincide. As illustrated in Example 2.1.16 the degrees of $J^0, J^1, J^2, J^3, J^4$ in the state space are $3, 0, 2, 4, and 6$.

### 2.4 The enumerative geometry: GRR and the virtual class

In this section we develop the enumerative geometry of $W_{g,n}$: we consider tautological classes and we focus on the FJRW invariants of the degree-five polynomial.

**Tautological classes.** Proposition 2.3.5 allows us to completely clarify the problem of the relations between tautological classes addressed in [FJR1].

We recall that the moduli stacks $W_{g,n}$ are equipped with the classes $\psi_i$:

$$\psi_i \in H^2(W_{g,n}) \quad \text{for } i = 1, \ldots, n.$$  

There are many ways to define these classes, the most straightforward (following [AGV08, §8.3]) is to consider the first Chern class of the bundle whose fiber at a point is the cotangent line to the corresponding curve at the $i$th marked point. We can also define the kappa classes; simply set

$$\kappa_h = \pi_*(c_1(\omega_{\log})^{b+1}) \in H^{2b}(W_{g,n}) \quad \text{for } h \geq 0,$$
where $\pi$ is the universal curve $C_{g,n} \to W_{g,n}$. If we identify each stack $W_{g,n+1}(\gamma_1, \ldots, \gamma_n, 1)$ to the universal curve $\pi: C \to W_{g,n}(\gamma_1, \ldots, \gamma_n)$, then we can express $\kappa_h$ as $\pi_*(\psi_{n+1})$.

A natural question posed in [FJR1, v1, §2.5.4] (and a crucial one for the computation of FJR\- W potentials) is to express in terms of psi classes the Chern character of the higher direct images of the universal W-structure $(L_1, \ldots, L_N)$ on the universal $d$-stable curve $\pi: C_{g,n} \to W_{g,n}$. Let us write the degree-2$h$ term of the Chern character as

$$\text{ch}_h(R\pi_*L_i) \in H^{2h}(W_{g,n}).$$

Indeed, we can express this cohomological class in terms of psi classes and kappa classes.

The statement requires further analysis of the boundary locus $\partial W_{g,n}$, the substack of $W_{g,n}$ representing W-structures on singular curves. The normalization $\overline{\partial W_{g,n}}$ of this locus can be regarded as the stack parametrizing pairs (W-curves, nodes) in the universal curve. Finally, we consider the double étale cover $D$ of this normalization; namely, the moduli space of triples (W-curves, nodes, a branch of the node). The stack $D$ is naturally equipped with two line bundles whose fibres are the cotangent lines to the chosen branch of the coarse curve. We write

$$\psi, \psi' \in H^2(D)$$

for the respective first Chern classes. Note that in this notation we privilege the coarse curve because in this way the classes $\psi, \psi'$ are more easily related to the classes $\psi_i$ introduced above.

Recall, see for example [CZ, §2.2], that for any $c \in \mathbb{Z}$ a $d$th root of $\omega_{\log}^c$ at a node of a $d$-stable curve determines local indices $a, b \in [0, 1]$ such that $a + b \in \mathbb{Z}$ in one-to-one correspondence with the branches of the node. In this way on $D$, naturally maps to $W_{g,n}$, the local index attached to the chosen branch determines a natural decomposition into open and closed substacks and natural restriction morphisms

$$D = \bigsqcup_{\Theta=0,1/d, \ldots, (d-1)/d} D_{\Theta}, \quad j_{\Theta}: D_{\Theta} \to W_{g,n}.$$ 

**Proposition 2.4.1.** Let $W$ be a nondegenerate quasihomogeneous polynomial in $N$ variables whose charges equal $q_1, \ldots, q_N$. For any $j = 1, \ldots, N$, consider the higher direct image $R\pi_*L_j$ of the $j$th component of the universal W-structure. Let $\text{ch}_h$ be the degree-2$h$ term of the restriction of the Chern character to the stack $W(\gamma_1, \ldots, \gamma_n)_{g,n}$, where $\gamma_i = (e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_N})$ for $\theta_j \in [0, 1]^N$. We have

$$\text{ch}_h(R\pi_*L_j) = \frac{B_{h+1}(q_j)}{(h+1)!} \kappa_h - \sum_{i=1}^n \frac{B_{h+1}(\Theta^i_j)}{(h+1)!} \psi^h_i + \frac{d}{2} \sum_{0 \leq \Theta < 1} \frac{B_{h+1}(\Theta)}{(h+1)!} (j_{\Theta})_* \left( \sum_{a + \alpha = h-1} \psi^a (-\psi')^{\alpha'} \right).$$

**Proof.** By Proposition 2.3.5, this is an immediate consequence of the main result of [Ch08b].

The enumerative geometry of $(x_1^5 + x_2^3 + x_3^2 + x_4^5 + x_5^2)$-curves. The crucial ingredient for the enumerative geometry of W-curves is the virtual cycle, which we denote, on each moduli stack $W(\gamma_1, \ldots, \gamma_n)_{g,n,G}$ by

$$[W(\gamma_1, \ldots, \gamma_n)]_{g,n,G}^\text{vir}$$

(we simply write $[W]_{g,n,G}^\text{vir}$ when we refer to the entire stack $W_{g,n,G}$).

The FJR\- W invariants are defined in [FJR1] for any choice of nonnegative integers $a_1, \ldots, a_n$ and any choice of elements $\alpha_1, \ldots, \alpha_n \in \mathcal{H}_{\text{FJR\-W}}(W,G)$.

$$\langle \tau_{a_1}(\phi_{\alpha_1}), \ldots, \tau_{a_n}(\phi_{\alpha_n}) \rangle^\text{FJR\-W}_{g,n}.$$ 

As in GW theory their definition is given by intersecting cohomology classes on the virtual cycle. In FJR\- W theory, the virtual cycle is $[W(\gamma_1, \ldots, \gamma_n)]_{g,n,G}^\text{vir}$ with $\gamma_i$ determined by $\alpha_i \in \mathcal{H}_{\gamma_i}$. The cohomology classes are powers of psi classes $\psi^{a_1}, \ldots, \psi^{a_n}$ and the classes $\alpha_1, \ldots, \alpha_n \in \mathcal{H}_{\text{FJR\-W}}(W,G)$. The precise definition can be found in [FJR1, Def. 4.2.6]; here we will give a self-contained treatment of the case $(W,G) = (\sum_i x_i^5, \langle J \rangle)$. There, three important simplifications occur:

1. the virtual cycle is a fifth power of the $A_4$ virtual cycle (Remark 2.4.2);
2. whenever \( \alpha_i \in \mathcal{H}_{f^0} \) for some \( i \) the invariants vanish (Remark 2.4.3);

3. in genus 0 the virtual cycle is Poincaré dual to a top Chern class of a vector bundle (Remark 2.4.4).

We detail the three points above and we reformulate the definition of the invariants by means of a geometrical definition in Proposition 2.4.5.

**Remark 2.4.2** (the virtual cycle as the fifth power of the \( A_4 \) virtual cycle). By [FJR1, Thm. 4.1.3] the virtual fundamental cycle is compatible with the natural isomorphism

\[
(W_1 + W_2)_{g,n} \cong (W_1)_{g,n} \times_d (W_2)_{g,n},
\]

whenever \( W_1 \) and \( W_2 \) are two nondegenerate quasihomogeneous polynomials in two different sets of variables. In this way, for \( W = x_1^2 + x_2^3 + x_3^5 + x_4^5 \), the virtual cycle \( [W]_{g,n}^{\text{vir}} \) is the tensor product of the five virtual cycles \( [A_4]_{g,n}^{\text{vir}} \) on each factor. Finally, by definition, the virtual cycle of \( [W]_{g,n,G}^{\text{vir}} \) is obtained via pullback of \( [W]_{g,n}^{\text{vir}} \) via \( i: W_{g,n,G} \hookrightarrow W_{g,n} \):

\[
[W]_{g,n,G}^{\text{vir}} = \left[ \frac{G_W}{|G|} \right] i^*[W]_{g,n}^{\text{vir}}
\]

(see Appendix A for more details). By Proposition 2.3.13, the morphism \( s: \mathcal{R}_1^4 \to W_{g,n,(J)} \) is surjective and has degree \( |G_W|/|\langle J \rangle| = 5^4 \); therefore pulling back further to \( \mathcal{R}_1^4 \) simplifies the extra factor appearing on the right hand side above. Since \( (A_4)_{g,n} \) equals \( \mathcal{R}_1^4 \), this means that instead of integrating over the virtual cycle \( [W]_{g,n,G}^{\text{vir}} \) in \( W_{g,n,G} \), we may more directly integrate over the self-intersection of five copies of \( [A_4]_{g,n}^{\text{vir}} \) in \( (A_4)_{g,n} \). In other words, for \( G = (J) \) on each connected component \( W(J^{m_1}, \ldots, J^{m_n})_{g,n,G} \), we have

\[
[W(J^{m_1}, \ldots, J^{m_n})]_{g,n,G}^{\text{vir}} = s_* \left( [A_4(J^{m_1}, \ldots, J^{m_n})]_{g,n}^{\text{vir}} \right)^5.
\]

**Remark 2.4.3** (Ramond vanishing). The state space of \( A_4 \)-singularity is very simple: the group of diagonal symmetries is generated by \( J \), we have \( \mathcal{H}_{f^0} = 0 \) and \( \mathcal{H}_{f^m} \) has a single degree-2\((m-1)\) generator for all \( m \in \{1, 2, 3, 4\} \). This implies, as a consequence of the Picard–Lefschetz Theorem 4.1.2 of [FJR1], the following vanishing condition

\[
[A_4(J^{m_1}, \ldots, J^{m_n})]_{g,n}^{\text{vir}} = 0 \quad \text{if } \exists i \in \{1, \ldots, n\} \mid J^{m_i} = 1.
\]

In other words, we can ignore the cases where there is a marking of Ramond (R) type. In the remaining cases, which are indeed the cases where all markings are of Neveu–Schwartz (NS) type, the definition of the virtual cycle \( [A_4(J^{m_1}, \ldots, J^{m_n})]_{0,n,G}^{\text{vir}} \) is compatible with the definition of Witten’s top Chern class \( c_W(m_1 - 1, \ldots, m_n - 1) \) constructed in [PV01] and [Ch06] (in those papers the indices \( m_1, \ldots, m_n \) are shifted by \( m \mapsto m - 1 \)). Summarizing, we have the following identity

\[
[A_4(J^{m_1}, \ldots, J^{m_n})]_{g,n,G}^{\text{vir}} = c_W(m_1 - 1, \ldots, m_n - 1) \quad \text{for } m_1, \ldots, m_n \in \{1, \ldots, 4\}.
\]

**Remark 2.4.4** (the virtual cycle in genus zero). When we specialize to the genus-zero case and the indices \( m_1, \ldots, m_n \) vary in \( \{1, \ldots, 4\} \) the virtual cycle can be obtained from the vector bundle \( R^1 \pi_* \mathcal{L} \) where \( \pi \) is the universal curve \( \mathcal{C} \to A_4(J^{m_1}, \ldots, J^{m_n})_{0,n} \). The fact that the above higher direct image is a vector bundle is well known (see [Wi93a]), but we recall it for clarity. It is enough to show that \( \mathcal{L} \) has no global sections on every geometric fiber of \( \pi \). If the fibre is smooth (hence reducible) this follows from the fact that \( \mathcal{L} \) has negative degree on the fibre. For a reducible fibre \( \mathcal{C} \) this happens because on each irreducible component \( Z \) the degree \( d_Z \) of \( \mathcal{C} \) is less than the number of points meeting the rest of the fibre minus 1:

\[
d_Z < \#(Z \cap \mathcal{C} \setminus Z) - 1.
\]

Notice that this only holds because \( m_1, \ldots, m_n > 0 \) and \( g = 0 \). In this way, the genus zero virtual cycle \( [W(J^{m_1}, \ldots, J^{m_n})]_{0,n,G}^{\text{vir}} \) can be expressed via the identification \( s \) as an intersection number of ordinary Chern classes on the standard fundamental cycle (see [FJR1, Thm. 4.1.5,(5)])

\[
[W(J^{m_1}, \ldots, J^{m_n})]_{0,n,G}^{\text{vir}} = c_{\text{top}}((R^1 \pi_* \mathcal{L})^*)^5 \cap [A_4(J^{m_1}, \ldots, J^{m_n})]_{0,n}.
\]
The previous remarks yield a self-contained formula for the relevant intersection numbers. The reader who is not familiar with FJRW theory may use it as a definition. We have

\[
\langle \tau_{a_1}(\phi_{a_1}), \ldots, \tau_{a_n}(\phi_{a_n}) \rangle_{g,n}^{\text{FJRW}} = \begin{cases} 
\frac{1}{5^n} \prod_{i=1}^n \psi^{\alpha_i} \cap \left( [A_4(J^{m_1}, \ldots, J^{m_n})]_{g,n} \right)^5 & \text{if } \alpha_i \in \mathcal{H}_{J^{m_i}} \\
0 & \text{for all } i, \text{ with } J^{m_i} \neq 1;
\end{cases}
\]

and, for \( g = 0 \), the first case can be written as

\[
\langle \tau_{a_1}(\phi_{a_1}), \ldots, \tau_{a_n}(\phi_{a_n}) \rangle_{0,n}^{\text{FJRW}} = 5 \prod_{i=1}^n \psi^{\alpha_i} c_{\text{top}}((R^1\pi_* L)^{\vee})^5 \cap [A_4(J^{m_1}, \ldots, J^{m_n})]_{0,n}
\]

In the genus-zero case we can avoid the above dichotomy and assemble the R and NS case into a single geometric definition, which will be useful later. We need a slightly different moduli functor.

**Proposition 2.4.5.** For \( m_1, \ldots, m_n \in \{1, \ldots, 5\} \), consider the stack \( \tilde{A}_4(\frac{m_1}{5}, \ldots, \frac{m_n}{5})_{0,n} \) classifying genus-zero \( n \)-pointed 5-stable curves equipped with 5th roots

\[
\tilde{A}_4(\frac{m_1}{5}, \ldots, \frac{m_n}{5})_{0,n} := \left\{ (C; (\sigma_i)_{i=1}^n; T; \varphi) : T^{\otimes 5} \overset{\sim}{\rightarrow} \omega_\log(-D_0) \text{ and } \text{mult}_{\sigma_i}(T) = 0 \text{ for all } i \right\},
\]

where \( D_0 \) is the linear combination \( m_1D_1 + \cdots + m_nD_n \) of the integer divisors \( D_i \) corresponding to the markings \( \sigma_i \). We point out that \( R^1\pi_* T \) is locally free and \( \pi_* T \) vanishes. Now, let \( h_1, \ldots, h_n \in \{0, \ldots, 4\} \) satisfy \( \alpha_i \in \mathcal{H}_{J^{h_i+1}} \) for all \( i \); then, we have

\[
\langle \tau_{a_1}(\phi_{a_1}), \ldots, \tau_{a_n}(\phi_{a_n}) \rangle_{0,n}^{\text{FJRW}} = 5 \prod_{i=1}^n \psi^{\alpha_i} c_{\text{top}}((R^1\pi_* T)^{\vee})^5 \cap \left[ \tilde{A}_4(\frac{h_1+1}{5}, \ldots, \frac{h_n+1}{5}) \right]_{0,n},
\]

where \( \pi \) is the universal family and \( T \) is the universal root of the moduli functor \( \tilde{A}_4(\frac{h_1+1}{5}, \ldots, \frac{h_n+1}{5})_{0,n} \).

**Proof.** Let us compare the definition of \( \tilde{A}_4 \) and that of \( A_4 \) first. For \( m_1, \ldots, m_n \) ranging in \([1, 5]\) the moduli stacks are canonically isomorphic to the moduli stacks \( A_4(J^{m_1}, \ldots, J^{m_n})_{0,n} \) (an easy consequence of Lemma 2.2.5 and the fact that there is a natural equivalence between 5th roots of two line bundles whenever they differ by \( \mathcal{O}(-5D_i) \) for some \( i \)). It is crucial however, to observe that the universal objects differ: under the identification between the moduli stacks we may relate the two universal 5th roots \( \tilde{T} \) and \( \mathcal{L} \) on the same orbifold curve \( (C; \sigma_1, \ldots, \sigma_n) \) as follows

\[
\tilde{T} = \mathcal{L} \otimes \mathcal{O}(-\sum_{i|m_i=5} D_i).
\]

With this comparison understood, we already know that the claim is true if all entries \( \alpha_1, \ldots, \alpha_n \) belong to \( \mathcal{H}_{J^{m_i}}(W, J) \) with \( J^{m_i} \neq 1 \). In order to prove the rest of the claim it is enough to show that \( c_{\text{top}}((R^1\pi_* T)^{\vee}) \) vanishes as soon as \( J^{m_{i_0}} = 1 \) for some \( i_0 \in \{1, \ldots, n\} \). Consider the integer divisor corresponding to the \( i_0 \)th marking \( D_{i_0} \) and the exact sequence

\[
0 \rightarrow T \rightarrow T(D_{i_0}) \rightarrow T(D_{i_0})|_{D_{i_0}} \rightarrow 0.
\]

We may regard the restriction \( T(D_{i_0})|_{D_{i_0}} \cong \mathcal{L}|_{D_{i_0}} \) as a 5th root of \( \omega_\log|_{D_{i_0}} \cong \mathcal{O}_{D_{i_0}} \); i.e. a line bundle with trivial first Chern class in rational cohomology

\[
c_1(\pi_*(T(D_{i_0})|_{D_{i_0}})) = 0.
\]

Notice that the restriction of \( T \) to every fibre of the universal curve has no global sections. Indeed, the same argument of Remark 2.4.4 applies: on a smooth fibre the degree is negative, whereas on a reducible
fibre $C$ we observe that the degree of $T$ on an irreducible component $Z$ is less than $\#(Z \cap C \setminus Z) - 1$. By a simple induction argument on the number of components, this implies that there are no nonzero global sections on the fibres of $\pi$; hence, $R^1 \pi_\ast T$ is a vector bundle and $\pi_\ast T$ vanishes.

We further notice that the same holds for $T(D_{i_0})$. In fact, for $T$, the above induction applies by iteratively removing tails (rational irreducible components attached to the rest of the fibre at a single point). For this argument it is crucial to observe that $d_Z < \#(Z \cap C \setminus Z) - 1$ simply reads $d_Z < 0$ on a tail. On the other hand, for $(T(D_{i_0}))$, it may well happen that the component carrying the $i_0$th point does not satisfy $d_Z < \#(Z \cap C \setminus Z) - 1$ because of the twisting at $D_{i_0}$; however, we can still apply the induction argument by starting from a different tail (reducible curves of genus zero have at least two tails).

By the vanishing of $\pi_\ast T$ and $\pi_\ast T(D_{i_0})$, we now have the following exact sequence of vector bundles

$$0 \to \pi_\ast T(D_{i_0})|_{D_{i_0}} \to R^1 \pi_\ast T \to R^1 \pi_\ast T(D_{i_0}) \to 0,$$

which immediately yields $c_{\top}(R^1 \pi_\ast T) = 0$ via (25) as required.

We finish this section with two remarks which provide a more insightful explanation from [FJR1] of the above properties. The reader who is satisfied with formula (22) as a definition for the intersection numbers may skip to Section 3.

Remark 2.4.6 (the Witten equation and the virtual fundamental cycle). The decoupling phenomenon observed in Remark 2.4.2 allowing us to express the virtual fundamental cycle in terms of other virtual cycles can be justified more explicitly by explaining the role of the Witten equation in Fan, Jarvis, and Ruan’s construction. For simplicity, we focus on the components $W(J^{m_1}, \ldots, J^{m_n})_{g,n,G}$ with $J^{m_1}, \ldots, J^{m_n} \neq 1$ (as mentioned above in other cases the virtual cycle vanishes by construction). In this situation the virtual fundamental cycle is constructed in [FJR1] [FJR3] for any quasihomogeneous nondegenerate polynomial $W$ by defining the moduli stack $W(J^{m_1}, \ldots, J^{m_n})_{g,n,G}^{\text{sol}}$ of solutions to the Witten equation

$$\overline{\partial}s_j + \partial_j W(s_1, \ldots, s_N) = 0, \quad j = 1, \ldots, N$$

where $s_j$ is a $C^\infty$-section of $L_j$. This is the moduli stack of $W$-curves paired with the datum of the sections $s_1, \ldots, s_N$ of $L_1, \ldots, L_N$ and is equipped with a forgetful map

$$W(J^{m_1}, \ldots, J^{m_n})_{g,n,G}^{\text{sol}} \to W(J^{m_1}, \ldots, J^{m_n})_{g,n,G}.$$

The idea of [FJR3] is to consider the virtual fundamental cycle of $W(J^{m_1}, \ldots, J^{m_n})_{g,n,G}^{\text{sol}}$ pushed forward to $W(J^{m_1}, \ldots, J^{m_n})_{g,n,G}$. In this way we get

$$[W(J^{m_1}, \ldots, J^{m_n})]_{g,n,G}^{\text{vir}} \in H_*(W(J^{m_1}, \ldots, J^{m_n})_{g,n,G}, \mathbb{Q}).$$

Without getting into the details of this construction let us point out that, in the case of our interest $(W = x_1^5 + x_2^5 + x_3^4 + x_4^3 + x_5^2$ and $G = (J))$ the Witten equation has the form

$$\overline{\partial}s_1 + 5s_1^4 = 0,$$
$$\overline{\partial}s_2 + 5s_2^4 = 0,$$
$$\overline{\partial}s_3 + 5s_3^4 = 0,$$
$$\overline{\partial}s_4 + 5s_4^4 = 0,$$
$$\overline{\partial}s_5 + 5s_5^4 = 0,$$

where $s_1, \ldots, s_5$ are smooth sections of $L$ with $L^{\otimes 5} \cong \omega_{\log}$. The key observation leading to (18) is that the Witten equation decouples into the five equations corresponding to the $A_4$-singularity.
Remark 2.4.7 (virtual codimension). The genus-zero description of
the Chern class does not generalize to higher genus. However, in higher
genus and even beyond the case \((W, G) = (\sum_{i} x_{i}^{2}, \langle J \rangle)\), the codimension
of the virtual cycle can be computed from the \(K\) theory rank of \(-R_{\pi}^{*}L\). In view of (19), Suppose for
simplicity that for each entry \(\gamma_{i}\) of the multiindex \((\gamma_{1}, \ldots, \gamma_{n}) \in G^{n}\) the fixed locus of \(\mathbb{C}^{N}\) is the origin
(i.e. all markings are of NS-type). By [FJR1, Thm. 4.1.5], for any \(W\), the codimension of the cycle
\([W(\gamma_{1}, \ldots, \gamma_{n})]_{g,n}^{vir}\) in the stack \(W(\gamma_{1}, \ldots, \gamma_{n})_{g,n}\) equals \(-\chi(R_{\pi}^{*}E)\) for \(E = \oplus_{j=1}^{N} L_{j}\). By Riemann–Roch
for orbifold curves [AGV08, Thm. 7.2.1], for \(\gamma_{i} = (e^{2\pi i \Theta_{i}^{1}}, \ldots, e^{2\pi i \Theta_{i}^{N}})\), we can explicitly compute
\[-\chi(R_{\pi}^{*}E) = -\text{rk}(E)(1 - g) - \text{deg}(E) + \sum_{i,j} \Theta_{j}^{i}\]
\[= (g - 1)N - \sum_{j=1}^{N} (2g - 2 + n)q_{j} + \sum_{i,j} \Theta_{j}^{i}\]
\[= (g - 1)\sum_{j=1}^{N} (1 - 2q_{j}) + \sum_{i=1}^{n} \sum_{j=1}^{N} (\Theta_{j}^{i} - q_{j})\]
\[= (g - 1)\gamma_{W} + \sum_{i=1}^{n} \iota(\gamma_{i}),\]
where in the last equality we see the role played in FJRW theory by the central charge (8) and by the
degree-shifting number introduced above. In this way, for \(W = x_{1}^{3} + x_{2}^{3} + x_{3}^{3} + x_{4}^{3} + x_{5}^{3}\) and \(G = \langle J \rangle\),
\[\text{codim } [A_{4}(J^{m_{1}}, \ldots, J^{m_{n}})]_{g,n}^{vir} = 3g - 3 - n + \sum_{i=1}^{n} m_{i}. \tag{26}\]

3 LG/CY-correspondence for the quintic three-fold

The FJRW theory of the quintic singularity is definitely different from the GW theory of the quintic
three-fold but also fits in Givental’s formalism. We recall the construction and we state the LG/CY
conjecture for the quintic three-fold.

3.1 Givental’s formalism for GW and FJRW theory

The genus-zero invariants of both theories are encoded by two Lagrangian cones, \(L_{GW}\) and \(L_{FJRW}\), inside
two symplectic vector spaces, \((\mathcal{V}_{GW}, \Omega_{GW})\) and \((\mathcal{V}_{FJRW}, \Omega_{FJRW})\). The two symplectic vector spaces also
allow us to state the conjectural correspondence in higher genera. On both sides of the correspondence
the odd-degree part \(\mathcal{H}_{fJ}(W, \langle J \rangle)\) and \(H^{3}(X_{W}, \mathbb{C})\) do not interfere with the even-degree part: since the
quantum invariants \(\langle \rangle_{GW}\) and \(\langle \rangle_{FJRW}\) vanish for any odd-degree insertion the odd-degree correspondence
boils down to the state space isomorphism of (11): \(H^{3}(X_{W}, \mathbb{C}) \cong \mathcal{H}_{fJ}(W, \langle J \rangle)\). In order to state and
prove the conjecture and simplify the notation, we focus on the even degree part both in GW theory and
in the FJRW theory.

We recall the two settings simultaneously by using the subscript \(W\), which can be read as GW or
FJRW.

The symplectic vector spaces. We define the vector space \(\mathcal{V}_{W}\) and its symplectic form \(\Omega_{W}\).

The elements of the vector space \(\mathcal{V}_{W}\) are the Laurent series with values in a state space \(H_{W}\)
\[\mathcal{V}_{W} = H_{W} \otimes \mathbb{C}((z^{-1})).\]

In FJRW theory the state space is normally the entire space \(\mathcal{H}_{FJRW}(W)\); however, as already mentioned,
the only nonvanishing invariants of FJRW theory occur for the even-degree entries of \(\mathcal{H}_{FJRW}(W)\) (see
(19)). Therefore, in view of the computation of the FJRW potential we can restrict to the even-degree part

$$H_{\text{FJRW}} = H_{\text{FJRW}}^e(W) = \bigoplus_{h=0}^{3} e(J^{h+1})C,$$

with nondegenerate inner product ( , )_{FJRW} (see (10) and recall from Example 2.3.14 that the only element of odd degree is J^0). Similarly, in GW theory, the state space H_W will be the even degree cohomology ring of the variety X_W, which is generated by the hyperplane section $H$:

$$H_{GW} = H_W^e(X_W, C) = \bigoplus_{h=0}^{3} [H^h]C,$$

with a natural nondegenerate inner product induced by Poincaré duality; we denote it by ( , )_{GW}. We express the basis of H_W as $\Phi_0, \ldots, \Phi_3$ and the dual basis $\Phi^0, \ldots, \Phi^3$ as

$$\Phi_h = \begin{cases} \varphi_h = [H^h] & \text{in GW theory,} \\ \phi_h = e(J^{h+1}) & \text{in FJRW theory,} \end{cases} \quad \Phi^h = \begin{cases} \varphi^h = 5^{-1}[H^{3-h}] & \text{in GW theory,} \\ \phi^h = e(J^{3-h}) & \text{in FJRW theory.} \end{cases}$$

Clearly $(\Phi_h, \Phi^k)_W = \delta_{h,k}$ and $\Phi^h = \Phi_{3-h}$.

Convention 3.1.1. When we write $\Phi_h$ or $\Phi^h$ for an integer k lying outside the above range, we mean $\Phi_h$ for $h = 5\left(\frac{k}{2}\right)$, where $\{ \}$ is the fractional part.

The vector space $\mathcal{V}_W$ is equipped with the symplectic form

$$\Omega_W(f_1, f_2) = \text{Res}_{z=0}(f_1(-z), f_2(z))W.$$

In this way $\mathcal{V}_W$ is polarized as $\mathcal{V}_W = \mathcal{V}_W^+ \oplus \mathcal{V}_W^-$, with $\mathcal{V}_W^+ = H_W \otimes C[z]$ and $\mathcal{V}_W^- = z^{-1}H_W \otimes C[[z^{-1}]]$, and can be regarded as the total cotangent space of $\mathcal{V}_W^+$. The points of $\mathcal{V}_W$ are parametrized by Darboux coordinates $\{q^h_{a}, p_{i,j}\}$ and can be written as

$$\sum_{a \geq 0} \sum_{h=0}^{3} q^h_{a} \Phi_h z^a + \sum_{l \geq 0} \sum_{j} p_{l,j} \Phi^j (-z)^{-1-l}.$$

The potentials. We review the definitions of the potentials encoding the invariants of the two theories.

In FJRW theory, the invariants are the intersection numbers

$$\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle_{\text{FJRW}}^{h_1, \ldots, h_n} = \frac{1}{5g-1} \prod_{i=1}^{n} \psi_i^{a_i} \cap [W(J^{h_1+1}, \ldots, J^{h_n+1})]_{g,n,(i)}^\text{vir},$$

where if $g$ equals zero $[W(J^{h_1+1}, \ldots, J^{h_n+1})]_{g,n,(i)}^\text{vir}$ is the cycle given by (21) and for $g \geq 0$ we refer to Remarks 2.4.2-4. In GW theory, the invariants are the intersection numbers

$$\langle \tau_{a_1}(\varphi_{h_1}), \ldots, \tau_{a_n}(\varphi_{h_n}) \rangle_{g,n,\delta}^{\text{GW}} = \prod_{i=1}^{n} \text{ev}_i^*(\varphi_{h_i}) \psi_i^{a_i} \cap [X_W]_{g,n,\delta}^\text{vir}.$$

The generating functions are respectively

$$F^g_{\text{FJRW}} = \sum_{a_1, \ldots, a_n \atop h_1, \ldots, h_n} \langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle_{g,n,\delta}^{h_1, \ldots, h_n} \frac{t^{h_1} \cdots t^{h_n}}{n!},$$

and

$$F^g_{\text{GW}} = \sum_{a_1, \ldots, a_n \atop h_1, \ldots, h_n} \sum_{\delta \geq 0} \langle \tau_{a_1}(\varphi_{h_1}), \ldots, \tau_{a_n}(\varphi_{h_n}) \rangle_{g,n,\delta}^{h_1, \ldots, h_n} \frac{t^{h_1} \cdots t^{h_n}}{n!}.$$
In this way, both theories yield a power series

$$\mathcal{F}_W^g = \sum_{a_1, \ldots, a_n, \delta \geq 0} \sum_{h_1, \ldots, h_n, \delta \geq 0} \left( \tau_{a_1}(\Phi_{h_1}), \ldots, \tau_{a_n}(\Phi_{h_n}) \right)_{g,n,\delta} W^{a_1} \cdots a_n \frac{t_1^{h_1} \cdots t_n^{h_n}}{n!}$$

in the variables $t_i$ (for FJR W theory the contribution of the terms $\delta > 0$ is set to zero, whereas $\langle \rangle_{g,n,0}$ should be read as $\langle \rangle_{g,n}^F_{\text{FJR}}$).

We can also define the partition function

$$\mathcal{D}_W = \exp \left( \sum_{g \geq 0} h^{g-1} \mathcal{F}_W^g \right).$$

Remark 3.1.2. The fact that $X_W$ is Calabi–Yau simplifies the last generating function a great deal. For dimension reasons, for $\delta > 0$, we have

$$\langle \tau_{a_1}(H^{h_1}), \ldots, \tau_{a_n}(H^{h_n}) \rangle_{g,n,\delta} = 0$$

unless $\tau_{a_i}(H^{h_i}) = \tau_1(H^0)$ or $\tau_0(H^1)$. Therefore, the above function is only a power series of $t_1$ and $t_0^1$.

We can use the string equation and the divisor equation to eliminate $\tau_1(H^0)$, $\tau_0(H^1)$ as well and to reduce it to the invariant $\langle \rangle_{0,n,\delta}$. By the divisor axiom, $Q$ and $t_1^0$ are related. In fact, they appear together in the form $Qe^{t_1^0}$. Now, we assume that $\mathcal{F}_W^g$ converges when $|Qe^{t_1^0}|$ is sufficiently small and set $Q = 1$.

This can be achieved in the region where we have $\Re(t_1^0) \ll 0$ (a similar and more detailed discussion can be found in [Co, “The divisor equation”, p.6]).

Notice that there is no $Q$-variable for FJR W theory. For dimension reasons and Ramond vanishing properties, see (26) and (19), the above function is only a power series of $t_1^0$ and $t_0^1$ (the entries $\tau_0(\phi_0)$ also yield zero contribution because all cycles are pulled back via the functor forgetting one point). We can use the string equation to eliminate $t_0^1$. The difference here is that we can not eliminate $t_1^0$ since there is no divisor equation.

The Lagrangian cones. Let us focus on the genus-zero potential $\mathcal{F}_W^0$. The dilaton shift

$$q_a^b = \begin{cases} t_1^0 - 1 & \text{if } (a, h) = (1, 0) \\ t_1^h & \text{otherwise.} \end{cases}$$

makes $\mathcal{F}_W^0$ into a power series in the Darboux coordinates $q_a^b$. Now we can define $\mathcal{L}_W$ as the cone

$$\mathcal{L}_W := \{ p = dq \mathcal{F}_W^0 \} \subset \mathcal{V}_W.$$
which define a locus which uniquely determines the rest of the points of \( \mathcal{L}_W \) (via multiplication by \( \exp(\alpha/z) \) for any \( \alpha \in \mathbb{C} \)—i.e. via the string equation—and via the divisor equation in GW theory). We define the \( J \)-function

\[
t = \sum_{h=0}^{3} t_h^0 \Phi_h \mapsto J_W(t, z)
\]

from the state space \( H_W \) to the symplectic vector space \( \mathcal{V}_W \) so that \( J_W(t, -z) \) equals the expression (32). In this way for \( t_0^0, t_1^0, t_2^0, t_3^0 \) varying in \( \mathbb{C} \) the family \( J_W(t = \sum_{h=0}^{3} t_h^0 \Phi_h, -z) \) varies in \( \mathcal{L}_W \). In fact, this is the only family of elements of \( \mathcal{L}_W \) of the form \(-z + \sum_{h=0}^{3} t_h^0 + O(z^{-1})\).

3.2 The conjecture

The following conjecture can be regarded as a geometric version of the physical Landau–Ginzburg/Calabi–Yau correspondence [VW89] [Wi93b]. A precise mathematical conjecture is proposed by the second author [Ru] and applies to a much more general category including Calabi–Yau complete intersections inside weighted projective spaces. In order to keep the notation simple, we review the conjecture for the case of quintic three-folds. The formalism is analogous to the conjecture of [CR] on crepant resolutions of orbifolds and uses Givental’s quantization from [Gi04], which is naturally defined in the above symplectic spaces \( \mathcal{V}_{FJR\ W} \) and \( \mathcal{V}_{GW} \).

**Conjecture 3.2.1.** Consider the Lagrangian cones \( \mathcal{L}_{FJR\ W} \) and \( \mathcal{L}_{GW} \).

1. There is a degree-preserving \( \mathbb{C}[z, z^{-1}] \)-valued linear symplectic isomorphism

\[
\mathbb{U} : \mathcal{V}_{FJR\ W} \rightarrow \mathcal{V}_{GW}
\]

and a choice of analytic continuation of \( \mathcal{L}_{FJR\ W} \) and \( \mathcal{L}_{GW} \) such that \( \mathbb{U}(\mathcal{L}_{FJR\ W}) = \mathcal{L}_{GW} \).

2. Up to an overall constant, the total potential functions up to a choice of analytic continuation are related by quantization of \( \mathbb{U} \); i.e.

\[
\mathcal{D}_{GW} = \mathbb{U}(\mathcal{D}_{FJR\ W}).
\]

**Remark 3.2.2.** For the readers familiar with the crepant resolution conjecture [CR], an important difference here is the lack of monodromy condition.

By [CR], a direct consequence of the first part of the above conjecture is the following isomorphism between quantum rings.

**Corollary 3.2.3.** For an explicit specialization of the variable \( q \) determined by \( \mathbb{U} \), the quantum ring of \( X_W \) is isomorphic to the quantum ring of the singularity \( \{ W = 0 \} \).

We refer readers to [CR] for the derivation of the above isomorphism.

4 The correspondence for the quintic three-fold in genus zero

4.1 Determining the Lagrangian cone of FJRW theory

**The twisted Lagrangian cone.** This section introduces a symplectic space \( \mathcal{V}_{tw} \) and a Lagrangian cone \( \mathcal{L}_{tw} \), which approximate the symplectic space \( \mathcal{V}_{FJR\ W} \) and the Lagrangian cone \( \mathcal{L}_{FJR\ W} \). This will ultimately allow us to determine the function \( J_{FJR\ W} \) and the entire Lagrangian cone \( \mathcal{L}_{FJR\ W} \). A similar story, with a different symplectic space \( \mathcal{V}_{tw} \), holds on the Gromov–Witten side, see [Gi96].

The definition of \( \mathcal{V}_{tw} \) is based on two observations. First, we already pointed out that the definition of the intersection number \( \langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle^{FJR\ W}_{0,n} \) extends naturally to the entries \( \phi_h \) with \( h = 4 \). By Proposition 2.4.5 this number can be regarded as a fifth power of the top Chern class of \( (R^1 \pi_* T)^\vee \) intersecting all possible powers of psi classes:

\[
5 \prod_{i=1}^{n} \phi_{i}^{a_i} c_{top}(\mathbb{R}^1 \pi_* T)^\vee \cap \left[ \widetilde{A}_4 \left( \frac{b_1+1}{5}, \ldots, \frac{b_n+1}{5} \right) \right]_{0,n},
\]

(33)
The fact that the indexes $h_1, \ldots, h_n$ vary in the extended range $\{0, \ldots, 4\}$ does not affect the FJRW potential (29): the number vanishes as soon $h_j = 4$ for some $j$.

The second observation allows us to set a relation between the well known intersections $\prod_i \psi_i^{a_i}$ and the numbers (33). Since the two numbers differ because of the class $c_{\text{top}}((R^3 \pi_2 T)^\vee)^5$, we now define a class interpolating the fundamental class $\tilde{A}_4((\frac{h_1+1}{5}, \ldots, \frac{h_n+1}{5}))_{0,n}$ and $c_{\text{top}}((R^3 \pi_2 T)^\vee)^5 \cap [\tilde{A}_4((\frac{h_1+1}{5}, \ldots, \frac{h_n+1}{5}))_{0,n}].$

**Lemma 4.1.1.** For any choice of complex parameters $s_d$ with $d \geq 0$ define on the $K$ theory ring the cohomology class

$$K_0(X) \to H^*(X, \mathbb{C}),$$

$$x \mapsto \exp \left( \sum_d s_d \text{ch}_d(x) \right).$$

The above class is multiplicative for any $s_d$ with $d \geq 0$. For $s_d = 0$ $\forall d$ we get the fundamental class, whereas for

$$s_d = \begin{cases} -5 \ln(\lambda) & d = 0, \\ \frac{5(d-1)!}{\lambda^d} & d > 0. \end{cases}$$

(34)

the multiplicative class is related to the equivariant Euler class as follows

$$\exp \left( \sum_d s_d \text{ch}_d([-V]) \right) = e_{C^x}((V^\vee)^5),$$

where $V$ is a $C^x$-equivariant vector bundle over the natural action of $\lambda \in C^x$ scaling the fibres by multiplication. In particular the nonequivariant limit $\lim_{\lambda \to 0}$ yields $c_{\text{top}}(V^\vee)^5$.

**Proof.** For any vector bundle $V$ equipped with the $C^x$-action scaling the fibre; the $C^x$-equivariant Euler class can be expressed in terms of the nonequivariant Chern character as follows:

$$e_{C^x}(V) = \exp \left( \ln(\lambda) \text{ch}_0(V) + \sum_{d>0} (-1)^{d-1} \frac{(d-1)!}{\lambda^d} \text{ch}_d(V) \right).$$

Finally, we can show that, with the above parameters $s_d$, we have

$$e_{C^x}(V^\vee)^5 = \exp \left( 5 \ln(\lambda) \text{ch}_0(V^\vee) + \sum_{d>0} 5(-1)^{d-1} \frac{(d-1)!}{\lambda^d} \text{ch}_d(V^\vee) \right) = \exp \left( -5 \ln(\lambda) \text{ch}_0([-V]^\vee) + \sum_{d>0} 5\frac{(d-1)!}{\lambda^d} \text{ch}_d([-V]) \right) = \exp \left( \sum_d s_d \text{ch}_d([-V]) \right),$$

where the relations $\text{ch}_d(-V) = -\text{ch}_d(V)$ and $\text{ch}_d(V^\vee) = (-1)^d \text{ch}(V)$ have been employed.

The two previous observations justify the introduction of a fifth generator for the state space, the definition of the intersection numbers over the moduli stacks of $A_1$-curves, and the definition of the state space over an extended ground ring $\mathbb{C}[\lambda] \otimes \mathbb{C}[[s_0, s_1, \ldots]]$. We start from the twisted symplectic space.

The twisted symplectic vector space $V_{tw}$ is formed by Laurent series over a modified state space. We have

$$V_{tw} = H_{tw} \otimes \mathbb{C}((z^{-1})).$$

\footnote{The above relation can be easily checked on a line bundle and extends to any vector bundle by the splitting principle; for any line bundle $L$ with $C^x$ acting by multiplication along the fibres, we can express the $C^x$-equivariant Euler class in terms of the first Chern class $\lambda + c_1(L)$ and ultimately as

$$\lambda + c_1(L) = \lambda \left( 1 + \frac{c_1(L)}{\lambda} \right) = \lambda \exp \left( \ln \left( 1 + \frac{c_1(L)}{\lambda} \right) \right) \right) = \lambda \exp \left( \sum_{d>0} (-1)^{d-1} \frac{c_1(L)^d}{\lambda^d} \right) = \exp \left( \ln(\lambda) \text{ch}_0(L) + \sum_{d>0} (-1)^{d-1} \frac{(d-1)!}{\lambda^d} \text{ch}_d(L) \right).$$}
where the state space is modified by adding a new element $\phi_4$ to the base

$$H_{tw} = \phi_0 R \oplus \phi_1 R \oplus \phi_2 R \oplus \phi_3 R \oplus \phi_4 R$$

and by working on the extended ground ring

$$R = H^*_C(pt, \mathbb{C})[[s_0, s_1, \ldots]]$$

of power series in infinitely many variables $s_0, s_1, \ldots$ with values in the ring of $\mathbb{C}^\times$-equivariant cohomology of a point. We regard $H^*_C(pt, \mathbb{C})$ as $\mathbb{C}[[\lambda]]$; so we can identify $R$ with $\mathbb{C}[[\lambda]] \otimes \mathbb{C}[[s_0, s_1, \ldots]]$. The nondegenerate inner product $(\cdot, \cdot)_{FJR}$ is extended to $H_{tw}$ by setting

$$(\phi_4, \phi_h)_{tw} = \begin{cases} \exp(-s_0) & \text{for } h = 4 \\ 0 & \text{otherwise} \end{cases}$$

(this choice allows us use the invariants defined below at (36) to form a Lagrangian cone $L_{tw}$, see in particular Proposition 4.1.4 and the role played by this pairing in the second part of the proof). In this way we get a dual basis $\phi^0, \ldots, \phi^4$ as soon as we set

$$\phi^4 = \exp(s_0)\phi_4.$$  \hspace{1cm} (35)

We extend Convention 3.1.1 to these extended bases: $\phi_{h+5k} = \phi_h$ for all $k$. The symplectic form making $V_{tw}$ into a symplectic vector space is

$$\Omega_{tw}(f_1, f_2) = \text{Res}_{z=0}(f_1(-z), f_2(z))_{tw}.$$  

Again, $V_{tw}$ equals the total cotangent space $T^\ast V_{tw}$, where $V_{tw}^+$ is defined as $H_{tw} \otimes \mathbb{C}[z]$. The Darboux coordinates are again $\{q^i_h, p_{ij}\}$, but this time $h$ and $j$ vary in $[0, 4]$. In view of an approximation of FJRW invariants

$$\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle_{0,n}^{tw} := 5 \int_{\tilde{A}_4 \left( \frac{k+1}{5}, \ldots, \frac{k+1}{5} \right)_{0,n}} \prod_{i=1}^n \psi^0_i \cup \exp \left( \sum_{d \geq 0} s_d \text{ch}_d (R \tau_4 T) \right),$$  \hspace{1cm} (36)

where we point out that the integral is taken on the standard fundamental cycle of $\tilde{A}_4 \left( \frac{k+1}{5}, \ldots, \frac{k+1}{5} \right)_{0,n}$. The definitions of $F^0_{\mathcal{W}}$, $D_{tw}$, and $L_{W}$ of §3.1 generalize word for word and yield the twisted potential $F^0_{tw}$, the partition function $D_{tw}$ and the twisted cone

$$L_{tw} \subseteq V_{tw}.$$  

In Proposition 4.1.4, we prove that this cone is Lagrangian; let us first remark two special cases of the construction.

Remark 4.1.2 (the untwisted cone). If we set $s_d = 0$ for all $d \geq 0$, then we get the untwisted symplectic vector space $V_{un}$ of Laurent series over the state space $H_{un} := \bigoplus_{h=0}^4 \phi_h R$ with $\phi^4 = \phi_4$ by (35). Correspondingly, we get the cone $L_{un}$ encoding the intersection numbers

$$\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle_{0,n}^{un} = 5 \int_{\tilde{A}_4 \left( \frac{k+1}{5}, \ldots, \frac{k+1}{5} \right)_{0,n}} \prod_{i=1}^n \psi^0_i.$$  

It is easy to prove by carrying out the calculation directly on $\tilde{M}_{0,n}$ that these numbers equal

$$\left( \begin{array}{c} \sum_i a_i \\ a_1, \ldots, a_n \end{array} \right)$$  \hspace{1cm} (37)

as soon as $n - 3 = \sum_i a_i$ and the following selection rule guaranteeing $\tilde{A}_4 \left( \frac{k+1}{5}, \ldots, \frac{k+1}{5} \right)_{0,n} \neq \emptyset$ is satisfied

$$2 + \sum_i h_i \in 5\mathbb{Z}$$  \hspace{1cm} (38)
(the factor 5 in the definition of \( \langle \quad \rangle^{tw} \) is canceled by the degree computation \([A_{A} : \overline{M}_{0,n}] = 1/5\)). The untwisted cone is a Lagrangian cone whose tangent spaces satisfy the geometric condition \( zT = L_{W} \cap T \) at any point (this happens because the equations SE, DE and TRR of [Gi04] can be easily deduced from the analogue equations on \( \overline{M}_{0,n} \)).

**Remark 4.1.3** (the Lagrangian cone \( L_{FJRW} \)). Let us set \( s_{d} \) as in (34). In this way we have \( \phi_{A} = \lambda^{5} \phi_{A} \). By Proposition 2.4.5 the higher direct image \(-R\pi_{*}T\) is represented by the locally free sheaf \( R^{1}\pi_{*}T\). By Lemma 4.1.1 we have

\[
\exp \left( \sum_{d \geq 0} s_{d} ch_{d}(R\pi_{*}T) \right) = \exp \left( \sum_{d \geq 0} s_{d} ch_{d}(-R^{1}\pi_{*}T) \right) = e_{C_{D}}((R^{1}\pi_{*}T)^{\vee})^{5}.
\]

Therefore the nonequivariant limit \( \lim_{\lambda \rightarrow 0} \) of the twisted invariants \( \langle \tau_{\alpha_{1}}(\phi_{h_{1}}), \ldots, \tau_{\alpha_{n}}(\phi_{h_{n}}) \rangle^{tw}_{0,n} \) is the intersection number

\[
5 \int_{A_{A}(b_{1}+1, \ldots, b_{n}+1)}_{0,n} \prod_{i=1}^{n} \psi_{i}^{\alpha_{i}} \cup (c_{top}(R^{1}\pi_{*}T))^{5}.
\]

As a consequence of Proposition 2.4.5 we have

\[
\lim_{\lambda \rightarrow 0} \left( F_{tw}^{0} |_{s_{d} = s_{d}} \right) = F_{FJR}^{0}
\]

(39)

when \( s_{d} \) is the parameters \( s_{d} \) fixed as in (34). Indeed, if all entries \( h_{i} \) are different from 4 the identification with the numbers \( \langle \tau_{\alpha_{1}}(\phi_{h_{1}}), \ldots, \tau_{\alpha_{n}}(\phi_{h_{n}}) \rangle_{FJR}^{0,n,0} \) is immediate, whereas if \( h_{i} = 4 \) for some \( i \) the limit of the twisted invariant \( \langle \tau_{\alpha_{1}}(\phi_{h_{1}}), \ldots, \tau_{\alpha_{n}}(\phi_{h_{n}}) \rangle_{tw}^{0,n} \) vanishes.

In this way we conclude that the Lagrangian cone \( L_{FJRW} \) can be realized as the nonequivariant limit (for \( \lambda \rightarrow 0 \)) of \( L_{tw} \) with the parameters \( s_{d} \) set to the above values.

The following proposition shows that the entire Lagrangian cone \( L_{tw} \) can be reconstructed from \( L_{un} \).

**Proposition 4.1.4.** Consider the symplectic transformation \( \Delta : H_{un} \rightarrow H_{tw} \) given by the direct sum

\[
\Delta = \bigoplus_{i=0}^{4} \exp \left( \sum_{d \geq 0} s_{d} B_{d+1} \frac{(i+1)}{(d+1)!} z_{d} \right).
\]

We have

\[
L_{tw} = \Delta(L_{un}).
\]

(41)

**Proof.** The claim is proven in [CZ], but in a slightly different setting: in [CZ] both the twisted and the untwisted cone correspond to intersection numbers over Witten’s top Chern class \( c_{top}(h) \). With a diagram, we can summarize the main theorem of [CZ] as

\[
\prod_{i} \psi_{i}^{\alpha_{i}} \cap c_{W}^{vir}(h) \quad \rightarrow \quad \prod_{i} \psi_{i}^{\alpha_{i}} \exp \left( \sum_{d \geq 0} s_{d} ch_{d}(R\pi_{*}T) \right) \cap c_{W}^{vir}(h),
\]

where the numbers appearing on the left hand side are those encoded in the untwisted Lagrangian cone of [CZ], whereas the number on the right hand side are those encoded in the twisted Lagrangian cone of [CZ]. In genus zero, the cycle \( c_{W}(h) \) is simply the Poincaré dual of the top Chern class \( c_{top}((R^{1}\pi_{*}T)^{\vee}) \) on the moduli stack \( A_{A}(b_{1}+1, \ldots, b_{n}+1)_{0,n} \), see (20). As we illustrated in Proposition 2.4.5 this class vanishes as soon as one of the entries of \( h \) equal 4.

The present statement claims that the operator \( \Delta \) sets the relation between \( L_{un} \) and \( L_{tw} \). The analogue to the above diagram is

\[
\prod_{i} \psi_{i}^{\alpha_{i}} \quad \rightarrow \quad \prod_{i} \psi_{i}^{\alpha_{i}} \exp \left( \sum_{d \geq 0} s_{d} ch_{d}(R\pi_{*}T) \right).
\]

In one respect the passage from the untwisted to the twisted cone in our setting is simpler: it involves the factorization properties of the fundamental cycle, which are much easier to prove than those of Witten’s
top Chern class. We state the crucial factorization property for the fundamental cycle. Fix a subset $I \subseteq [n] := \{1, \ldots, n\}$. Let $q \in \{0, \ldots, 4\}$ be equivalent to $-2 - \sum_{i \in I} h_i \mod 5$. Consider the stack $D_I$ classifying the following objects: pairs given by an $\tilde{A}_4$-curve and a node dividing the curve into two subcurves with marking set $I$, containing the first marking, and $I' = [n] \setminus I$. Write $q'$ for the index in $\{0, \ldots, 4\}$ satisfying $q + q' = 3 \mod 5$, and let $h_H$ be the multiindex $(h_i \mid i \in H)$ for any set $H \subseteq [n]$. There exist natural morphisms

$$\tilde{A}_4 \left( h_{i, \frac{1}{5}}, q_{\frac{1}{5}} \right)_{0, \# I + 1} \times \tilde{A}_4 \left( h_{i, \frac{1}{5}}, q'_{\frac{1}{5}} \right)_{0, \# I' + 1} \to \mu_I D_I \overset{j_I}{\to} \tilde{A}_4 \left( h_{i, \frac{1}{5}} \right)_{0, n},$$

where $j_I$ is simply the morphism forgetting the node and $\mu_I$ is obtained by normalizing the curve at the node and pulling back. Here, we should notice that the case when $q = q' = 4$ is special: after pulling back we should apply to the line bundles obtained on each component the functor $E \mapsto E \otimes \mathcal{O}(-D)$ where $D$ is the integer divisor associated to the point lifting the node; in this way we obtain an object of the product

$$\tilde{A}_4 \left( h_{i, \frac{1}{5}}, q_{\frac{1}{5}} \right)_{0, \# I + 1} \times \tilde{A}_4 \left( h_{i, \frac{1}{5}}, q'_{\frac{1}{5}} \right)_{0, \# I' + 1} \text{ with } q = q' = 4.$$ 

We can finally state the required factorization property for the fundamental classes. We have

$$\mu_I j_I^* \left[ \tilde{A}_4 \left( h_{i, \frac{1}{5}} \right) \right]_{0, n} = \mu_I [D_I] = \left[ \tilde{A}_4 \left( h_{i, \frac{1}{5}}, q_{\frac{1}{5}} \right) \right]_{0, \# I + 1, 5} \times \left[ \tilde{A}_4 \left( h_{i, \frac{1}{5}}, q'_{\frac{1}{5}} \right) \right]_{0, \# I' + 1, 5}.$$ 

The crucial fact is that the degree of $\mu_I$ equals one\(^3\).

There is one point, however, where the presence of Witten’s top Chern class simplifies things in [CZ]: since the cycle $c_W(h)$ vanishes as soon as one of the entries of $h$ equals 4, in [CZ] we did not need to carry out the calculation outside the range $[0,3]$. We go through the proof focusing on this situation. The claim can be phrased in terms of differential operators on the partitions functions: it says that $\partial D_{\text{tw}}^I / \partial s_I$ equals $P_l D_{\text{tw}}^I$, where $P_l$ is the operator

$$P_l = \frac{B_{l+1} \left( \frac{1}{5} \right)}{(l + 1)!} \frac{\partial}{\partial t^{l + 1}_I} - \sum_{a \geq 0} \frac{B_{l+1} \left( \frac{h_a + 1}{5} \right)}{(l + 1)!} \frac{\partial h_a}{\partial t^{l+1}_I} + \frac{h}{2} \sum_{a + a' = l - 1, 0 \leq h, h' \leq 4} (-1)^a g^{h, h'} \frac{B_{l+1} \left( \frac{h + 1}{5} \right)}{(l + 1)!} \frac{\partial^2}{\partial t^a \partial t^{h'}}, \quad (42)$$

notice that the upper indices $h$ and $h'$ range in $[0,4]$ and we used the convention $(g^{h, h'}) = (g_{h, h'})^{-1}$ for $(g_{h, h'}) = (\cdot, \cdot)_{\text{tw}}$.

For $l = 0$ the above statement simply says that the insertion of $\text{ch}_0 (R_\pi L)$ is equivalent to the multiplication by

$$\left( -\frac{1}{2} + \frac{1}{5} \right) (2g - 2 + n) - \sum_{i=1}^n \left( \frac{1}{2} + \frac{h_i + 1}{5} \right) = -g + 1 + \frac{1}{5} (2g - 2 + n) - \sum_{i=1}^n \frac{h_i + 1}{5}$$

(the third summand is not involved). This is indeed the multiplicty of $\text{ch}_0$ by Riemann–Roch.

For $l > 0$, the third summand is involved. Notice, however, that the coefficient $g^{h, h'}$ appearing there is different from zero only if $(h, h')$ equals $(0,3), (1,2), (2,1), (3,0)$, or $(4,4)$. Therefore, the only new contribution with respect to [CZ] occurs for $(h, h') = (4,4)$. The desired formula follows from the following geometric property. The higher direct image in $K$-theory of a line bundle $L$ on a family of curves $C \to X$ having an node $n$; $X \to C$ differs from the higher direct image of the pullback of $L$ on the family $C' \to Y$ obtained by normalization at the node $n$. The difference is precisely the $K$-class $-n^* L$; in our situation $L$ is the 5th root of $\omega$ and $\text{ch}(-n^* L) = -1$. Summarizing, whenever $I \subset [n]$ satisfies $-2 - \sum_{i \in I} h_i \equiv 4 \mod 5$, we have

$$\mu_I j_I^* (\text{ch}_d) = \begin{cases} \text{ch}_d \times 1 + (1 \times \text{ch}_d) & \text{if } d > 0, \\ \text{ch}_d \times 1 + (1 \times \text{ch}_0) + (1 \times 1) & \text{if } d = 0, \end{cases}$$

\(^3\)The stabilizers on the generic points of both sides are isomorphic to $\mu_5 \times \mu_5$: on the right-hand side the two generators act by rotating the fibres of each $\tilde{A}_4$-structure, whereas on the left-hand side one generator rotates the fibres of the $\tilde{A}_4$-structure and the other operates on the curve by means of an automorphism acting locally as $(x, y) \mapsto (\xi x, y)$ and commonly named the "ghost" automorphism. See discussion of "$\mu_{l, I}$" in [CZ, p.10] for a proof.
(where we write \(ch_d\) for \(ch_d(R\pi,T)\)) and, therefore,
\[
\mu_{1,j}^j(\exp(\sum_d s_d ch_d) = \exp(s_0) \exp(\sum_d s_d [ch_d \times 1]) \exp(\sum_d s_d [1 \times ch_d]).
\] (43)

The appearance of the factor \(\exp(s_0)\) matches the new boundary contribution of (42): indeed, \(g^{4,4}\) equals \(\exp(s_0)\).

**A family on the FJRW Lagrangian cone.** In this section, we exhibit a family of the form

\[
I_{\text{FJRW}}(t,z) \in f(t)z + H_{\text{FJRW}}[[z^{-1}]] \quad \text{with} \quad f(t) \in H_{\text{FJRW}}
\]

such that \(I_{\text{FJRW}}(t,-z)\) belongs to the Lagrangian cone \(L_{\text{FJRW}}\). We will use it to determine the entire Lagrangian cone \(L_{\text{FJRW}}\). In the next statement the Pochhammer symbols \([a]_n = a(a+1)\ldots(a+n-1)\) have been used.

**Theorem 4.1.5.** Let

\[
I_{\text{FJRW}}(t,z) = z \sum_{k=1,2,3,4} \omega_k^{\text{FJRW}}(t) \frac{1}{z^{k-1}} \phi^{4-k}
\]

with

\[
\omega_k^{\text{FJRW}}(t) = \frac{1}{\Gamma(k)} \sum_{l \geq 0} \left( \frac{[k]^5}{[l]!^5} \right) t^{k+5l}.
\]

The family \(C \ni t \mapsto I_{\text{FJRW}}(t,-z)\) lies on the Lagrangian cone \(L_{\text{FJRW}}\).

**Proof.** We proceed as follows:

1. Using the above explicit presentation (32) of \(J_{un}\), we get a function taking values in \(L_{un}\).
2. We operate on \(J_{un}\) with a transformation which sends it to a new function taking values in \(L_{un}\).
3. We apply \(\Delta\) to this function; we get a function \(I_{tw}(t,z)\) taking values on \(L_{tw}\).
4. By setting \(s_0,s_1,\ldots\) as in Lemma 4.1.1 and by taking the limit (\(\lambda \to 0\)), we get \(I_{\text{FJRW}}\).

Step 1. We recall the definition of the \(J\)-function for the untwisted cone

\[
J_{un}(\sum_{0 \leq h \leq 4} t^h_0, -z) = -z \phi_0 + \sum_{0 \leq h \leq 4} t^h_0 \phi_h + \sum_{0 \leq h_1, \ldots, h_n \leq 4} \frac{t^{h_1} \ldots t^{h_n}}{n!^{(1)}(z)^{k+1}} \left( \tau_0(\phi_{h_1}), \ldots, \tau_0(\phi_{h_n}), \tau_k(\phi_{e(h)n+1}) \right)_{0,n+1} \phi_{e(h)n+1},
\]

where the following notation has been used.

**Notation 4.1.6.** Any multiindex with \(n\) integer entries \(h \in [0,4]^n\) can be completed

\[
e(h) = \left( h_1, \ldots, h_n, 5 \left\{ \frac{-2 - |h|}{5} \right\} \right);
\]

so that \(e(h)\) is a multiindex satisfying \(|e(h)| \equiv -2 \mod 5\) (the same selection rule of (38)).

In the expression above, two multiindices \(h\) that coincide after a permutation yield the same contribution. It makes sense to sum with multiplicities over all nonnegative multiindices \((k_0,\ldots,k_4)\) choosing a representant for all permutations of the multiindex

\[
h(k) = (0,\ldots,0,1,\ldots,1,\ldots,4,\ldots,4).
\]

The multiplicities are provided by the multinomial coefficients \((k_0 + \cdots + k_4)!/(k_0! \cdots k_4!)\). Now we can rewrite \(J_{un}(t,z)\) as

\[
J_{un}(t,z) = \sum_{k_0,\ldots,k_4 \geq 0} J_{un}^k(t,z),
\]

\[
J_{un}^k(t,z) = \sum_{l \geq 0} \frac{t^0_{k_0} \ldots t^4_{k_4}}{z^{l+1} k_0! \cdots k_4!} \left( \tau_0(\phi_0), \ldots, \tau_0(\phi_0), \ldots, \tau_0(\phi_4), \ldots, \tau_0(\phi_4), \tau_k(\phi_{e(k)n+1}) \right)_{0,n+1} \phi_{e(k)n+1},
\]

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where, by a slight abuse of notation, we wrote $e(k)$ rather than $c(h(k))$. Using (37) and the fact that $\phi_h = \phi^{h-\delta}$ for all $h$ (we are in the untwisted state space $H_{un}$) we get

$$J_{un}(t, -z) = z \sum_{k_0, \ldots, k_4 \geq 0} \frac{1}{z^{k_1}} \frac{((t_0^h)_{k_0} \ldots (t_4^h)_{k_4})}{k_0! \ldots k_4!} \phi_{\sum i k_i}.$$  (44)

Indeed, we have $\int \psi^d = \delta_l \text{dim}/5$, where dim is the dimension of the moduli stack $\tilde{A}_4$, i.e. $|k| - 2$.

Step 2. We derive from $J_{un}$ a class of functions taking values in $E_{un}$.

**Notation 4.1.7** (the functions $s(x)$ and $G_y(x, z)$). In $\mathbb{C}[y, x, z, z^{-1}][[s_0, s_1, \ldots]]$, we define

$$G_y(x, z) = \sum_{m, l \geq 0} s_{l+m-1} \frac{B_m(y)}{m!} \frac{x^l}{l!} z^{m-1},$$  (45)

with $s_{-1} = 0$. By the definition of Bernoulli polynomials one gets

$$G_y(x, z) = G_0(x + yz, z),$$

$$G_0(x + z, z) = G_0(x, z) + s(x),$$

where $s(x)$ is given by

$$s(x) = \sum_{d \geq 0} s_d \frac{x^d}{d!}.$$  (46, 47)

**Notation 4.1.8** (the dilation vector fields $D_i$). For $i = 0, \ldots, 4$ we write

$$D_i = t_0^i \frac{\partial}{\partial t_0},$$  (48)

for the vector field on the state space $H_{un} = \{t_0^0 \phi_0 + t_0^1 \phi_1 + t_0^2 \phi_2 + t_0^3 \phi_3 + t_0^4 \phi_4\}$ attached to $t_0^i$.

The vector field $D_i$ naturally operates on $J_{un}$. It is easy to check (using (44)) that

$$D_i J_{un} = k_i J_{un}.$$  

In the following lemma we operate on $J_{un}$ by means of the vector field

$$\nabla = \sum_{i=0}^{4} \frac{i}{5} D_i.$$  

**Lemma 4.1.9.** The family $t \mapsto \exp(-G_y(z \nabla, z))J_{un}(t, -z)$ lies on $E_{un}$.

**Proof.** The proof of [CCIT, p.11] extends word for word. The only difference is that here $G_y(z \nabla, z)$ replaces $G_0(z \nabla, z)$; therefore, the coefficient $B_m(\frac{1}{5})$ should replace $B_m(0)$ in the definition of the operator $P_i(z \nabla, z) = \sum_{m=0}^{i+1} \frac{1}{m!(i+1-m)!} z^m B_m(0)(z \nabla)^{i+1-m}$ used there.

For clarity, we go through the generalization step by step. Note that $\exp(-G_y(z \nabla, z))J_{un}(\tau, -z)$ depends on the variables $\{s_0, s_1, \ldots\}$; therefore we write

$$J_s(\tau, -z) = \exp(-G_y(z \nabla, z))J_{un}(\tau, -z).$$

Remark that we can write the point $J_s(\tau, -z)$ as

$$h = -z + \sum_{k \geq 0} t_k z^k + \sum_{j \geq 0} \frac{p_j}{(-z)^{j+1}},$$

31
where \( t_k \) is just a parameter varying in the state space \( H_{un} \), which we can write in terms of parameters \( t_k \) in the ground ring \( R \) as \( t_k = \sum_{k \geq 0} t_k^k \). The claim of the lemma is \( h \in L_{un} \). More explicitly we want to show that the variables \( p_k \) are expressed in terms of the previous variables as follows

\[
p_j = \sum_{n \geq 0} \sum_{a_1, \ldots, a_n \geq 0} \frac{t_1^{a_1} \cdots t_n^{a_n}}{n!} \langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}), \tau_j(\phi_{e(h)_{n+1}}) \rangle_{0,n+1} \phi(e(h)_{n+1}).
\]

(49)

The idea of [CCIT] is to define

\[
E_j(h) = p_j - \sum_{n \geq 0} \sum_{a_1, \ldots, a_n \geq 0} \frac{t_1^{a_1} \cdots t_n^{a_n}}{n!} \langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}), \tau_j(\phi_{e(h)_{n+1}}) \rangle_{0,n+1} \phi(e(h)_{n+1}).
\]

and to prove that for any \( j \) the composition \( E_j \circ J_x \) vanishes.

The claim is evident when we set \( s_0 = s_1 = \cdots = 0 \), so we can proceed by induction: we define the degree of the variable \( s_k \) to be \( k + 1 \) and we proceed by induction on the degree. Namely we show that

\[
E_j(J_x(\tau, z)) \text{ vanishes up to degree } n \implies \frac{\partial}{\partial s_i} E_j(J_x(\tau, z)) \text{ vanishes up to degree } n.
\]

The condition on the left hand side above simply means that we can adjust \( J_x(\tau, z) \) by adding terms as follows

\[
\tilde{J}_x(\tau, z) = J_x(\tau, z) + Y
\]

(50)

so that the degree of \( Y \) in the variables \( s_1, s_2, \ldots \) is greater than \( n \) and \( E_j(\tilde{J}_x(\tau, z)) \) vanishes. (More precisely let us project \( J_x(\tau, z) \) to \( J_x(\tau, z) \) \( \in V^*_m \) and decompose \( J_x(\tau, z) \) as \( J_x(\tau, z) \downarrow + [J_x(\tau, z)]_+ \). We remark that in order to match the desired vanishing condition we only need to modify the term \([J_x(\tau, z)]_+\).)

Let us focus on \( \frac{\partial}{\partial s_i} E_j(J_x(\tau, z)) \). By the chain rule, we have

\[
\frac{\partial}{\partial s_i} E_j(J_x(\tau, z)) = d_{J_x(\tau, z)} E_j \left( \frac{\partial}{\partial s_i} J_x(\tau, z) \right).
\]

By the definition of \( G^x_z(x, z) \), we have

\[
\frac{\partial}{\partial s_i} E_j(J_x(\tau, z)) = d_{J_x(\tau, z)} E_j \left( z^{-1} P_i J_x(\tau, z) \right),
\]

(51)

where

\[
P_i = \sum_{m=0}^{i+1} \frac{1}{m! (i + 1 - m)!} z^m B_m \left( \frac{1}{z} \right) (z \nabla)^{i+1-m}.
\]

Note that the right hand side of (51) coincides with

\[
d_{J_x(\tau, z)} E_j \left( z^{-1} P_i J_x(\tau, z) \right),
\]

(52)

up to degree \( n \) in the variables \( s_k \).

We need to show that \( \frac{\partial}{\partial s_i} E_j(J_x(\tau, z)) \) vanishes. Recall that the point \( J_x(\tau, z) \) lies on \( L_{un} \). We want to show that \( P_i J_x(\tau, z) \) is a point in \( zT \), where \( T \) is the tangent space of \( L_{un} \) at \( J_x(\tau, z) \). Recall that \( T \) and \( L_{un} \) satisfy Givental’s property

\[
zT = L_{un} \cap T.
\]

Notice that \( P_i \) is a linear combination of \( \lambda_0 z^{i+1} + \lambda_1 z^i (z \nabla) + \cdots + \lambda_{i+1} (z \nabla)^{i+1} \); so we can prove the claim term by term. First, consider the (possibly iterated) multiplication by \( z \). By the string equation, we can
by applying again $\tilde{z}^T$. In fact, by (46), the above expression equals $\Delta$; $H_{un} \to H_{tw}$ from Proposition 4.1.4. Write $J_{un}(t, -z)$ as the sum over the multiindices $k$ of the terms $J_{un}^k(t, -z)$. Then we can write the family of Lemma 4.1.9 as

$$t \mapsto \exp(-G^2_\Delta(z \nabla, z)) J_{un}(t, -z) = \sum_k \exp\left(-G_\Delta \left(\frac{1 + \sum_{i=0}^4 i k_i}{5} z, z\right)\right) J_{un}^k(t, -z).$$

By (46), the above expression equals

$$\sum_k \exp\left(-G_\Delta \left(\frac{1 + \sum_{i=0}^4 i k_i}{5} z, z\right)\right) J_{un}^k(t, -z).$$

Step 3. We apply to the function $t \mapsto \exp(-G_\Delta(z \nabla, z)) J_{un}(t, -z)$ the symplectic transformation $\Delta: H_{un} \to H_{tw}$ from Proposition 4.1.4. Write $J_{un}(t, -z)$ as the sum over the multiindices $k$ of the terms $J_{un}^k(t, -z)$. Then we can write the family of Lemma 4.1.9 as

$$t \mapsto \exp(-G_\Delta(z \nabla, z)) J_{un}(t, -z) = \sum_k \exp\left(-G_\Delta \left(\frac{1 + \sum_{i=0}^4 i k_i}{5} z, z\right)\right) J_{un}^k(t, -z).$$

By applying the operator from Proposition 4.1.4

$$\Delta = \bigoplus_{i=0}^4 \exp\left(\sum_{d \geq 0} \frac{s_d}{d !} \frac{(i + 1)^{d+1}}{5^d} z^d\right) \bigoplus_{i=0}^4 \exp\left(G_0 \left(\frac{i + 1}{5} z, z\right)\right)$$

to (53) we get

$$\sum_k \exp\left(G_0 \left(\left(\frac{1}{5} + \left\{\frac{ik_i}{5}\right\}\right) z, z\right) - G_0 \left(\frac{1 + \sum i k_i}{5} z, z\right)\right) J_{un}^k(t, -z).$$

The relation $s(x) = G_0(x + z, z) - G_0(x, z)$ yields

$$\sum_k \exp\left(- \sum_{0 \leq m < \left\{\sum i k_i / 5\right\}} s \left(\frac{1}{5} z + \left\{\frac{ik_i}{5}\right\} z + mz\right)\right) J_{un}^k(t, -z).$$

In this way, Step 2 and Proposition 4.1.4 imply the following lemma.

**Lemma 4.1.10.** Let $M_k(z)$ be the modification function

$$M_k(z) = \exp\left(- \sum_{0 \leq m < \left\{\sum i k_i / 5\right\}} s \left(-\frac{1}{5} z - \left\{\frac{ik_i}{5}\right\} z - mz\right)\right).$$

Define $I_{tw}(t, z)$ as the sum

$$I_{tw}(t, z) = \sum_k M_k(z) J_{un}^k(t, -z).$$

Then, the family $t \mapsto I_{tw}(t, -z)$ lies on $L_{tw}$. □

Step 4. As we pointed out in Remark 4.1.3 the Lagrangian cone $L_{FIRW}$ can be realized from $L_{tw}$. This can be done by setting the parameters $s_0, s_1, \ldots$ as in (34) and by taking the nonequivariant limit.
First, observe that $M_k(z)$ assumes a simple form under the conditions (34). We have

$$M_k(z) = \exp \left( - \sum_{0 \leq m < |\sum ik_i/5|} s \left( \frac{-1}{5} z - \left( \frac{\sum ik_i}{5} \right) z - mz \right) \right)$$

$$= \prod_{0 \leq m < |\sum ik_i/5|} \exp \left( -s_0 - \sum_{d > 0} s_d \left( \frac{-1}{5} z - \left( \frac{\sum ik_i}{5} \right) z - mz \right)^d \right)$$

$$= \prod_{0 \leq m < |\sum ik_i/5|} \exp \left( 5 \ln(\lambda) + \sum_{d > 0} 5(-1)^{d-1} \frac{1}{d!} \left( \frac{\frac{1}{5} z + \left( \frac{\sum ik_i}{5} \right) z - mz}{\lambda} \right)^d \right)$$

$$= \prod_{0 \leq m < |\sum ik_i/5|} \left( \lambda \exp \left( \ln \left( 1 + \frac{\frac{1}{5} z + \left( \frac{\sum ik_i}{5} \right) z - mz}{\lambda} \right) \right) \right)^5$$

$$= \prod_{0 \leq b < |\sum ik_i/5| \atop \{b\} = |\sum ik_i/5\|} \left( \lambda + \frac{1}{5} z + bz \right)^5.$$

By using the formula for $J^k_{\text{FJR}}(t, z)$ we deduce that

$$\sum_{k_0, \ldots, k_4 \geq 0} \prod_{0 \leq b < k/5, \{b\} = |k/5\|} \left( \lambda - \frac{1}{5} z - bz \right)^5 \frac{1}{(-z)^{k-1} k!} \frac{1}{(t_0^{k_0}) \cdots (t_4^{k_4})} \phi_{\sum ik_i}$$

lies on the cone $L_{tw}$ when the parameters $s_d$ are fixed as above. In particular, if $t$ ranges over $\{t_0^i = 0 \mid i \neq 1\} = \{(0, t, 0, 0, 0)\} \subset H_{tw}$, we have

$$I_{tw}(t, z) = \sum_{k \geq 0} \prod_{0 \leq b < k/5, \{b\} = |k/5\|} \left( \lambda - \frac{1}{5} z - bz \right)^5 \frac{1}{(-z)^{k-1} k!} \phi_k.$$ 

Recall that $\phi_0 = \phi^3$, $\phi_1 = \phi^2$, $\phi_2 = \phi^1$, $\phi_3 = \phi^0$, and $\phi_4 = \exp(-s_0)\phi^4 = \lambda^5\phi^4$; therefore, in order to compute the twisted invariants we write $I_{tw}$ in the form

$$I_{tw}(t, z) = \omega_1 \phi^3 + \omega_2 \phi^2 + \omega_3 \phi^1 + \omega_4 \phi^0 + \lambda^5 \omega_5 \phi^4,$$

and the fifth term vanishes for $\lambda \to 0$. We have

$$\prod_{0 \leq b < k/5, \{b\} = |k/5\|} \left( -\frac{1}{5} z - bz \right)^5 = \frac{\Gamma \left( \frac{k+1}{5} \right)}{\Gamma \left( \frac{k+1}{5} \right)} (-z)^{5(k/5)}.$$

In this way

$$-z \sum_{0 \leq k \neq 4} \frac{\Gamma \left( \frac{k+1}{5} \right)}{\Gamma \left( \frac{k+1}{5} \right)} (-z)^{5(k/5)} \frac{1}{k!} t^k \phi^{3-k}$$

lies on $L_{\text{FJR}}$ for any value of $t$. Note that, since $L_{\text{FJR}}$ is a cone, if we modify the above expression by multiplying by $t$, then we still get a family of elements of $L_{\text{FJR}}$. In this way, shifting by one the variable $k$ and expressing the $\Gamma$-functions in terms of the Pochhammer symbols by $[a]_n = \Gamma(a+n)/\Gamma(a)$, we get the $I$-function of the statement.
4.2 Relating the $I$-functions of FJRW theory and GW theory

Remark 4.2.1 (Proof of Theorem 1.1.1). The functions $\omega^F_{k}\text{JRW}$ in the statement of the theorem form a well known basis of the solution space of the Picard–Fuchs equation in the variable $t$: $[D^4_t - 5q^4 \prod_{m=1}^4 (D_t - mz)]f = 0$. As mentioned in the introduction the entire calculation of $I_{FJRW}$ is parallel to Givental’s calculation [Gi96] of $I_{GW}$ for the quintic three-fold $X_W$. There, we get

\[ I_{GW}(q, z) = \sum_{d \geq 0} z^d q^{H/z + d} \prod_{k=1}^d (H + k z)^5, \]  

which (by $H^4 = 0$) is a solution of the same equation in the variable $q$: $[D^4_q - 5q \prod_{m=1}^4 (5D_q + mz)]f = 0$. By expanding $I_{GW}$ in the variable $H$, we obtain the four period integrals spanning the solution space

\[ I_{GW} = \omega^GW_1 z + \omega^GW_2 H + \omega^GW_3 \frac{1}{z} H^2 + \omega^GW_4 \frac{1}{z^2} H^3. \]

This implies Theorem 1.1.1 stated in the introduction.

Remark 4.2.2. The $I$-functions $I_{GW}$ and $I_{FJRW}$ may be regarded as multivalued functions $^4$ in the variables $q$ and $t^5$ (from this point of view, the parameter $H$ should be considered as a complex variable). In the literature, the function $I_{GW}$ is regarded as a solution of the Picard–Fuchs equation in the sense that it satisfies the equation up to order 4 in the variable $H$ and the first four terms of the Taylor expansion yield a basis of the solution space of the differential equation. The picture at a neighbourhood of $t = 0$ is simpler: the elements of the basis $\omega^F_{k}\text{JRW}$, $k = 1, \ldots, 4$ admit the explicit expression given in Theorem 4.1.5 and assemble into a regular function in the variable $t$. In this respect, Theorem 1.1.1 is the interpretation in terms of enumerative geometry of curves of a well known analytic picture.

String theory provides another geometric interpretation of the same analytic picture. As mentioned in the introduction there is a physical interpretation of the Landau–Ginzburg/Calabi–Yau correspondence in terms of period integrals. In the framework of mirror symmetry, this is usually referred to as the A-model picture and it reflects information on the A-model which is only partly understood and incorporates, in particular, the GW theory of the quintic three-fold. From this B-model interpretation, in [HKQ], Huang, Klemm, and Quackenbush build upon physical grounds new predictions in higher genera. Near $q = 0$, the B-model higher genus potential is determined up to $q = 51$ and is expected to match the (A-model) higher genus GW invariants of the quintic three-fold. Near $t = 0$, the B-model higher genus potential is explicitly determined in low genus (see [HKQ, §3.4]): FJRW theory supplies the A-model geometrical interpretation of these invariants usually referred to as the “Gepner point potential” in the physical literature (in [HKQ] the Gepner point is called “orbifold point” $F_{orb}$ due to the fact that the point $t = 0$ is actually an orbifold point with stabilizer of order 5). The functions $\omega^F_{k}\text{JRW}$ of Theorem 4.1.5 match [HKQ, 3.57] up to a constant factor (due to the fact that the period integrals have been rescaled in [HKQ, (3.64-66)]$^5$). As we now illustrate, the genus-zero formulae of [HKQ, §3.4] are matched by our $I_{FJRW}$ formula. It would be an extremely interesting problem to compute the higher genus FJRW theory to match [HKQ] Gepner point potential.

Remark 4.2.3 (intersection numbers). We derive intersection numbers in Givental’s formalism. The function $I_{FJRW}(t, z)$ of the statement of the theorem above can be modified by multiplication by any function taking values in the ground ring (we already used this property in the last step of the proof). We can expand $I_{FJRW}$ in the variable $z$ and get the four function $\omega^F_{k}\text{JRW}$:

\[ I_{FJRW}(t, z) = \omega^F_{1}\text{JRW}(t)\phi^3 z^3 + \omega^F_{2}\text{JRW}(t)\phi^2 + \omega^F_{3}\text{JRW}(t)\phi z^{-1} + \omega^F_{4}\text{JRW}(t)\phi z^{-2} \]  

with $\omega^F_{3}\text{JRW}$ invertible; therefore, we can multiply $I_{FJRW}$ by $1/\omega^F_{1}\text{JRW}$ and get a function lying on the cone $L_{FJRW}$. In fact, by $\phi_k = \phi^{3-k}$, we have

\[ J_{FJRW} \left( \frac{\omega^F_{2}\text{JRW}}{\omega^F_{1}\text{JRW}}(t)\phi_1, -z \right) = -z\phi_0 + \frac{\omega^F_{2}\text{JRW}}{\omega^F_{1}\text{JRW}}(t)\phi_1 - \frac{\omega^F_{2}\text{JRW}}{\omega^F_{1}\text{JRW}}(t)\phi_2 z^{-1} + \frac{\omega^F_{4}\text{JRW}}{\omega^F_{1}\text{JRW}}(t)\phi_3 z^{-2}; \]  

$^4$The multivaluedness comes from the factor $q^{H/z}$, which should be read as $\exp(H \log(q)/z)$, and from the fact that $I_{FJRW}$ actually depends on the 5th root $t$ of $t^5$.

$^5$We are grateful to Klemm for explaining this to us.
indeed the $J$-function $J_{FJRW}(\sum_n \gamma_n \phi_n, -z)$ is characterized by being on the cone $\mathcal{L}_{FJRW}$ and admitting an expression of the form $-\gamma_0 + \sum_n \gamma_n \phi_n + O(-1)$. If we write $\tau$ for $(\omega_2^{FJRW}/\omega_1^{FJRW})(t)$ we can regard the above expression as the value of $J_{FJRW}(\tau \phi_1, -z)$.

We can invert the relation $\tau = (\omega_2^{FJRW}/\omega_1^{FJRW})(t)$ explicitly in low degree
\[
t = \tau - \frac{13}{1125000} \tau^6 - \frac{31991}{974531250000000} \tau^{11} - \frac{294146129}{997676367187500000000} \tau^{16} + O(\tau^{21}),
\]
plug it into $(\omega_2^{FJRW}/\omega_1^{FJRW})(t)$, and get the coefficient of $z^{-1}$ in $J_{FJRW}(\tau \phi_1, z)$ in low degree in the variable $\tau$:
\[
\frac{1}{2} \tau^2 + \frac{1}{39375} \tau^7 + \frac{239}{1559250000000} \tau^{12} + \frac{6904357}{45227995312500000000} \tau^{17} + O(\tau^{22}).
\]
By the definition (32) of $J_{FJRW}$, the above power series coincides with the power series
\[
\sum_{h \geq 0} \left( \frac{\tau_0(\phi_1) \ldots \tau_0(\phi_1))^{FJRW}_{0h+3}}{5h+2} \right)^{5h+2};
\]
therefore, we get
\[
\begin{array}{c|ccc}
  n & 3 & 8 & 13 \\
\hline
  \langle \tau_0(\phi_1) \ldots \tau_0(\phi_1) \rangle_{FJRW}^{0,0} & 1 & 8 & 5736 \\
\end{array}
\]
\[
\begin{array}{c|ccc}
  n & 13 & 18 \\
\hline
  \langle \tau_0(\phi_1) \ldots \tau_0(\phi_1) \rangle_{FJRW}^{0,0} & 1325636544 & 1220703125 \\
\end{array}
\]
After the identification $s = t/5$ and multiplication by 5 (due to the rescaling of the period integrals in [HKQ]), the generating function
\[
\frac{1}{5} \tau^3 + \frac{1}{3125} \tau^8 + \frac{5736}{390625} \tau^{13} + \frac{1325636544}{1220703125} \tau^{18} + \ldots
\]
matches the genus-zero potential $F_{0b}^0$ near $t = 0$ computed at page 21 of [HKQ].

A similar calculation yields the coefficient of $z^{-2}$ in $J_{FJRW}(\tau \phi_1, z)$ in low degree:
\[
\frac{1}{6} \tau^3 + \frac{1}{525000} \tau^8 + \frac{239}{1842750000000} \tau^{13} + \frac{6904357}{508814947265625000000} \tau^{18} + O(\tau^{23})
\]
which coincides by (32) with the generating function of the invariants $\langle \tau_0(\phi_1) \ldots \tau_0(\phi_1) \tau_1(\phi_0) \rangle_{FJRW}^{0,0,5h+4}$. In this way we get
\[
\begin{array}{c|cccc}
  n & 4 & 9 & 14 & 19 \\
\hline
  \langle \tau_0(\phi_1) \ldots \tau_0(\phi_1) \tau_1(\phi_0) \rangle_{FJRW}^{0,0} & 1 & 48 & 63096 & 21210184704 \\
\end{array}
\]
but these values can be easily deduced from the previous table via the string equation\(^6\).

As expected (see Remark 3.1.2) we only find nonvanishing numbers $\langle \cdots \rangle_{FJRW}$ with insertions of type $\tau_0(\phi_1)$ and $\tau_1(\phi_0)$ (the reader may also refer to Example 2.3.14 which describes how for 5-pointed curves with 4 points with local index $J^2$ we necessarily have a Ramond markings on the fifth marking—hence, a vanishing invariant).

Carel Faber’s computer programme for computation with tautological classes together with the formula of Proposition 2.4.1 also allowed us to compute explicitly the first coefficients of the above lists. Clearly, this algorithm is definitely less efficient than the one illustrated above. Indeed, in this respect, Givental’s quantization may be regarded as a setup embodying the GRR algorithm used by Faber’s computer programme.

---

\(^6\)The value corresponding to $n$ in the first table coincides with that corresponding to $n + 1$ in the second table after multiplication by $2g - 2 + n = n - 2$ (string equation).
Corollary 4.2.4. There is a \( C[z, z^{-1}] \)-valued degree-preserving symplectic transformation \( U \) mapping \( L_{FJR} \) to the analytic continuation of \( L_{GW} \) near \( t = 0 \). The genus-zero LG/CY correspondence holds (Conjecture 3.2.1,(1)).

Proof. The Lagrangian cone \( L_{FJR} \) is uniquely determined by \( I_{FJR} \). Indeed all the intersection numbers defining \( L_{FJR} \) can be computed via the string equation from the invariants appearing in the function \( J_{FJR}(\tau, \phi, z) \). We only need to analytically continue the function \( I_{GW} \) near \( t = 0 \) and derive the change of basis matrix.

By the formula \( z^{-l} \prod_{k=1}^{l} (x + k z) = \Gamma(1 + \frac{z}{x} + l)/\Gamma(1 + \frac{z}{x}) \), we rewrite \( I \) as

\[
I_{GW}(q, z) = z q^{H/z} \sum_{d \geq 0} q^d \frac{\Gamma(1 + 5 \frac{H}{z} + 5d)}{\Gamma(1 + \frac{H}{z} + d)^5}.
\]

(58)

hence, we have

\[
I_{GW}(q, z) = z q^{H/z} \frac{\Gamma(1 + \frac{H}{z})^5}{\Gamma(1 + 5 \frac{H}{z})} \sum_{d \geq 0} q^d \frac{\Gamma(1 + 5 \frac{H}{z} + 5d)}{\Gamma(1 + \frac{H}{z} + d)^5},
\]

\[
I_{GW}(q, z) = z q^{H/z} \frac{\Gamma(1 + \frac{H}{z})^5}{\Gamma(1 + 5 \frac{H}{z})} \sum_{d \geq 0} \frac{1}{2\pi i} \frac{1}{\Gamma(1 + \frac{H}{z} + s)} \frac{\Gamma(1 + 5 \frac{H}{z} + 5s)}{\exp^{2\pi i s} - 1} q^s.
\]

Consider the contour integral

\[
z q^{H/z} \frac{\Gamma(1 + \frac{H}{z})^5}{\Gamma(1 + 5 \frac{H}{z})} \int_C \frac{1}{\exp^{2\pi i s} - 1} \frac{\Gamma(1 + 5 \frac{H}{z} + 5s)}{\Gamma(1 + \frac{H}{z} + s)^5} q^s,
\]

(59)

where the \( C \) is curve shown below. For \( |q| < 1/5^5 \), we can close the contour to the right; then (59) equals

| Figure 1: the contour of integration. |

(58). For \( |q| > 1/5^5 \), we can close the contour to the left; then (59) is the sum of residues at

\[
s = -1 - l, \ l \geq 0 \quad \text{and} \quad s = -\frac{H}{z} - \frac{m}{5}, \ m \geq 1.
\]

The residues at \( s = -1 - l \) vanish as they are multiples of \( H^4 = 0 \). Thus the analytic continuation of \( I_{GW}(q, z) \) is given by evaluating the remaining residues. Since we have

\[
\text{Res}_{s=-\frac{H}{z} - \frac{m}{5}} \left( 1 + 5 \frac{H}{z} + 5s \right) = \frac{1}{5} \frac{(-1)^m}{\Gamma(m)},
\]

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we get
\[
I_{GW}(q, z) = \frac{z}{5} \frac{\Gamma^5(1 + \frac{4}{z})}{\Gamma(1 + 5 \frac{4}{z})} \sum_{0 < m \neq 0} (-1)^m \frac{(2\pi i)^m}{\exp -2\pi i \frac{4}{m} - \xi^m \Gamma(m) \Gamma(1 - \frac{m}{5})^5} \text{ where } \xi = \exp \frac{2\pi i}{z}.
\]
Note that in the above equation we have omitted the terms corresponding to \( m \in 5\mathbb{Z} \), because in these cases the residues vanish. In terms of the coordinate \( t \), which satisfies \( t^5 = q^{-1} \), we have (using \( \Gamma(x) \Gamma(1 - x) = \pi / \sin(\pi x) \))
\[
I_{GW}(t, z) = \frac{z}{5} \frac{\Gamma^5(1 + \frac{4}{z})}{\Gamma(1 + 5 \frac{4}{z})} \sum_{0 < m \neq 0} (-1)^m \frac{(2\pi i)^m}{\exp -2\pi i \frac{4}{m} - \xi^m \Gamma(m) \sin^6(\frac{m}{5} \pi)} \cdot t^m
\]
\[
= \frac{z}{5} \frac{\Gamma^5(1 + \frac{4}{z})}{\Gamma(1 + 5 \frac{4}{z})} \sum_{k=1,2,3,4} (-1)^k \frac{(2\pi i)^k}{\exp -2\pi i \frac{4}{k} - \xi^k \Gamma(k) \Gamma(1 - \frac{k}{5})^5} \sum_{l \geq 0} 5 \frac{\Gamma^5(\frac{k+5l}{5})}{\Gamma(k+5l)} t^{k+5l}
\]
\[
= \frac{z}{5} \frac{\Gamma^5(1 + \frac{4}{z})}{\Gamma(1 + 5 \frac{4}{z})} \sum_{k=1,2,3,4} \frac{(2\pi i)^k}{\exp -2\pi i \frac{4}{k} - \xi^k \Gamma(k) \Gamma(1 - \frac{k}{5})^5} \sum_{l \geq 0} \left( \frac{\Gamma(\frac{k+5l}{5})}{\Gamma(k+5l)} \right)^5 t^{k+5l} \Gamma(5(1 - \frac{k}{5})^5)
\]
\[
= \frac{z}{5} \frac{\Gamma^5(1 + \frac{4}{z})}{\Gamma(1 + 5 \frac{4}{z})} \sum_{k=1,2,3,4} \frac{(2\pi i)^k}{\exp -2\pi i \frac{4}{k} - \xi^k \Gamma(k) \Gamma(1 - \frac{k}{5})^5} \sum_{l \geq 0} \left( \frac{\Gamma(\frac{k+5l}{5})}{\Gamma(k+5l)} \right)^5 t^{k+5l} \Gamma(5(1 - \frac{k}{5})^5)
\]
\[
= \frac{z}{5} \frac{\Gamma^5(1 + \frac{4}{z})}{\Gamma(1 + 5 \frac{4}{z})} \sum_{k=1,2,3,4} \frac{(2\pi i)^k}{\exp -2\pi i \frac{4}{k} - \xi^k \Gamma(k) \Gamma(1 - \frac{k}{5})^5} \sum_{l \geq 0} \left( \frac{\Gamma(\frac{k+5l}{5})}{\Gamma(k+5l)} \right)^5 t^{k+5l} \Gamma(5(1 - \frac{k}{5})^5)
\]

The Taylor expansions of the functions
\[
g(\rho) = \frac{\rho^{k}}{\exp -2\pi i \rho - \xi^{k}} \quad \text{and} \quad f(\rho) = \frac{\Gamma^5(1 + \rho)}{\Gamma(1 + 5 \rho)}
\]
read
\[
g(\rho) = \frac{\xi^k}{1 - \xi^k} + \frac{\xi^k}{(1 - \xi^k)^2} (2\pi i) \rho + \frac{\xi^k(1 + \xi^k)}{2(1 - \xi^k)^3} (2\pi i)^2 \rho^2 + \frac{\xi^k(1 + 4\xi^k + \xi^{2k})}{6(1 - \xi^k)^4} (2\pi i)^3 \rho^3 + O(\rho^4).
\]
and
\[
f(\rho) = 1 + C(2\pi i)^3 \rho^2 - E(2\pi i)^3 \rho^3 + O(\rho^4),
\]
where \( C = 5/12 \) and \( E = -\zeta(3)40/(2\pi i)^3 \) (with \( \zeta(3) \) equal to Apéry’s constant) are related to the intersection theory of the quintic three-fold \( (C = c_2(X_W)[H]/\deg(W))24 \) and \( E = \zeta(3)(X_W)/\deg(W)(2\pi i)^3) \).

In this way, by expanding \( I_{GW} \) in the variable \( H \), we get \( \omega_1^{GW}, \omega_2^{GW}, \omega_3^{GW}, \omega_4^{GW} \), the analytic continuations of \( \omega_1^{JRW}, \omega_2^{JRW}, \omega_3^{JRW}, \omega_4^{JRW} \) to \( \omega_1^{GW}, \omega_2^{GW}, \omega_3^{GW}, \omega_4^{GW} \), from (54). After identifying \( \phi^k \) with \( [H]/5 \), the change of basis matrix from \( \omega_1^{JRW}, \omega_2^{JRW}, \omega_3^{JRW}, \omega_4^{JRW} \) to \( \omega_1^{GW}, \omega_2^{GW}, \omega_3^{GW}, \omega_4^{GW} \) is defined column by column as
\[
U = \begin{pmatrix}
\frac{(1)^k}{\Gamma(1+\xi)} z^{k-1} \\
\frac{(1)^k}{\Gamma(1+\xi)} z^{k-2} \\
\frac{(1)^k}{\Gamma(1+\xi)} z^{k-3} \\
\frac{(1)^k}{\Gamma(1+\xi)} z^{k-4}
\end{pmatrix}
\]

Clearly \( U \) is degree-preserving with \( \deg z = 2 \). It can be easily seen via a direct computation that \( U \) is symplectic (i.e. we have \( U^\dagger U = 1 \)).}

\footnote{Choosing a suitable basis on GW side, \cite{HKQ} shows that this transformation can be defined entirely inside \( Sp(4, \mathbb{Q}(\xi)) \). See also Iritani \cite{Iritani} where a formalism based on Grothendieck–Riemann–Roch for orbifolds is proposed (this uses the above interpretation of \( C \) and \( E \) in terms of intersection theory).}
A Appendix. Compatibility with previous definitions

A.1 Another definition of the moduli functor

In [FJR1] a definition of a slightly different moduli stack is given. This is a natural étale cover of \( W_{g,n} \), locally isomorphic to

\[
BG_W \to B(\mu_3)^N,
\]

where \( G_W \to (\mu_3)^N \) is the homomorphism of Lemma 2.1.8, and allows a more direct treatment of the construction of Witten’s cycle. For sake of completeness (and in order to provide the statement needed in [FJR1, Rem. 2.3.6]), we state the definition from [FJR1] and we relate it to our setting.

One needs to proceed as follows: consider a quasihomogeneous polynomial \( W \) in \( N \) variables \( x_1, \ldots, x_N \) whose charges are \( c_1/d, \ldots, c_N/d \). Let \( M \) be the matrix corresponding to \( W \), let \( D \) be the Smith normal form of \( M \) and let us fix two matrices \( T \) and \( S \) invertible over \( \mathbb{Z} \) such that

\[
M = TDS
\]

(see Remark 2.1.2). The chosen matrices \( T \) and \( S \) play a role in the following definition: we stress it by writing them in the notation: \( W^{TDS}_{g,n} \) and we illustrate it with Example A.1.3. In §A.2 we explain why it is equivalent to work with \( W_{g,n} \) and we extend the discussion to \( W_{g,n,G} \).

Definition A.1.1. The moduli stack of \( W^{TDS}_{g,n} \) is formed by \( d \)-stable \( n \)-pointed genus-\( g \) curves equipped with \( N \) line bundles \((L_1, \ldots, L_N)\) alongside with \( N \) isomorphisms \((\tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)\) between the first \( N \) entries of \( DS(L_1, \ldots, L_N)^t \) and the first \( N \) entries of \( T^{-1}(\omega_{\log}, \ldots, \omega_{\log})^t \).

A morphism starting from the object \((C' \to X'; \sigma'_1, \ldots, \sigma'_n; L'_1, \ldots, L'_N; \tilde{\varphi}_1', \ldots, \tilde{\varphi}_N')\) and ending at the object \((C'' \to X''; \sigma''_1, \ldots, \sigma''_n; L''_1, \ldots, L''_N; \tilde{\varphi}_1'', \ldots, \tilde{\varphi}_N'')\) is given by \((\alpha; \rho_1, \ldots, \rho_N)\) where \( \alpha: C' \to C'' \) is a morphism of \( d \)-stable curves and \( \rho_j: L'_j \to \alpha^*L''_j \) for \( j = 1, \ldots, N \) are isomorphisms of line bundles compatible with \( \tilde{\varphi}_1, \ldots, \tilde{\varphi}_N \) in the sense that the composite of \((\alpha^*\tilde{\varphi}_1', \ldots, \alpha^*\tilde{\varphi}_N')\) with the first \( N \) entries of \( DS(\rho_1, \ldots, \rho_N) \) equals \((\tilde{\varphi}_1', \ldots, \tilde{\varphi}_N')\).

Remark A.1.2. In the above definition, we can drop the “first \( N \) entries” whenever it appears, and consider objects involving \( s \) isomorphisms \((C \to X; \sigma_1, \ldots, \sigma_s; L_1, \ldots, L_N; \tilde{\varphi}_1, \ldots, \tilde{\varphi}_s)\), where the last \( s - N \) isomorphisms are the identities \( \mathcal{O} \to \mathcal{O} \). Indeed, recall that \( DS \) is by construction a matrix with vanishing entries apart form the first \( N \) lines and recall that \( T^{-1}(1, \ldots, 1)^t \) is of the form \((u_1, \ldots, u_N, 0, \ldots, 0)\) (see Remark 2.1.2, (6)).

There is a natural functor

\[
\rho_{TDS}: W^{TDS}_{g,n} \to W_{g,n}
\]

over the moduli stack \( \mathfrak{M}_{g,n,d} \):

\[
(\tilde{L} = (L_1, \ldots, L_N)^t; \tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_s)^t) \mapsto (\tilde{L}; S^{-1}\tilde{D}\tilde{\varphi}),
\]

where \( \tilde{D} \) is an \( N \times s \) diagonal matrix whose \( N \) diagonal entries are \( d_1/d, \ldots, d_N/d \) (in this way \( \tilde{D}D = dI_N \)). Indeed the \( N \) entries of \( S^{-1}\tilde{D}\tilde{\varphi} \) map from \( \tilde{d}\tilde{L} \) to \( S^{-1}\tilde{D}^{-1}(\omega_{\log}, \ldots, \omega_{\log})^t \) which equals \((\omega_{\log}^{\tilde{e}_1}, \ldots, \omega_{\log}^{\tilde{e}_N})\) because the \( N \) lines of \( S^{-1}\tilde{D}^{-1} \) add up to \( \tilde{e}_1, \ldots, \tilde{e}_N \):

\[
S^{-1}\tilde{D}T^{-1}(1, \ldots, 1)^t = S^{-1}\tilde{D}^{-1}M(c_1/d, \ldots, c_N/d)^t
\]

\[
= S^{-1}\tilde{D}^{-1}TDS(c_1/d, \ldots, c_N/d)^t = \tilde{d}(c_1/d, \ldots, c_N/d)^t = (\tilde{e}_1, \ldots, \tilde{e}_N)^t.
\]

In order to see that the functor actually lands inside the full subcategory \( W_{g,n} \) of \( \mathcal{R}_d^c \times_\mathbb{Z} \cdots \times_\mathbb{Z} \mathcal{R}_d^c \) satisfying \( M\tilde{L} \cong (\omega_{\log}, \ldots, \omega_{\log})^t \) we simply remark that \( T\tilde{\varphi} \) sets the desired isomorphism.

We may observe that an object of the stack \( W^{TDS}_{g,n} \) embodies a privileged choice of an \( s \)-tuple of isomorphisms \( M\tilde{L} \cong (\omega_{\log}, \ldots, \omega_{\log})^t \) which is not needed in order to specify an object of \( W_{g,n} \). The morphism \( \rho_{TDS} \) is in fact a forgetful functor. If we regard the map \( \rho_{TDS} \), fibre by fibre over \( \mathfrak{M}_{g,n,d} \),
we get $|G_W|^{2g-1+n}$ copies of $BG_W$ inside $W_{g,n}^{\text{vir}}$ mapping to $|G_W|^{2g-1+n}$ copies of $B(\mu_2)^N$ inside $W_{g,n}$.

Indeed $p_{TDS}$ is proper, étale and locally isomorphic to $BG_W \to B(\mu_2)^N$; in particular, we have

$$\deg(p_{TDS}) = \tilde{d}^N / |G_W|.$$

**Example A.1.3.** As we just pointed out the $N$ isomorphisms $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$ embody an $s$-tuple of isomorphisms $M\tilde{L} \cong (\omega_{\log}, \ldots, \omega_{\log})^s$, which we may also write as $W_i(L_1, \ldots, L_N) \to \omega_{\log}$. By specifying these $s$ isomorphisms in this way, we actually get a proper stack. We illustrate with an example how this may fail to be the case if we do not proceed as above via the Smith normal form; this may be regarded as a motivation for the use of the Smith normal form in the definition of $W_{g,n}^{\text{vir}}$.

Consider $(xy + y^2 + x^2)_{g,n}$. This is a full subcategory of $\mathbb{R}_2^1 \times 2 \mathbb{R}_2^1$ of 2-stable curves equipped with two square roots of $\omega_{\log}$ (we have $d = d = 2, c_x = c_y = 1$): $\varphi_x : L_x^{\otimes 2} \to \omega_{\log}$ and $\varphi_y : L_y^{\otimes 2} \to \omega_{\log}$. The full subcategory is characterized by the following condition $L_x \otimes L_y \cong \omega_{\log}$. Notice that this amounts to requiring that $L_x$ and $L_y$ are isomorphic to each other: $L_x \cong \omega_{\log} \otimes L_y' \cong L_y^{\otimes 2} \otimes L_y \cong L_y$.

If we write the matrix $M$ attached to the above polynomial in the form $TDS$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

we may regard $(xy + y^2 + x^2)_{g,n}^{TDS}$ as the stack classifying 2-stable curves equipped with two line bundles $L_x$ and $L_y$ and two isomorphisms $\tilde{\varphi}_1 : L_x \otimes L_y \to \omega_{\log}$ and $\tilde{\varphi}_2 : L_y^{\otimes 2} \to \omega_{\log}$. As pointed out in Remark A.1.2 we may add a third trivial datum $\tilde{\varphi}_3 = \text{id}$. Notice that this amounts to requiring $\tilde{\varphi}_3$ is id equals $(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$. This explains why this definition does not yield a proper stack.

**A.2 Compatibility**

Consider two different choices of matrices such that $T' DS' = M = T'' DS''$. We have an isomorphism

$$j : W_{g,n}^{TDS'} \xrightarrow{\sim} W_{g,n}^{T''DS''}$$

$$(\hat{L}, \tilde{\varphi}) \mapsto (\hat{L}, (T'')^{-1}T'T, \tilde{\varphi}),$$

which commutes with $p_{T' DS'}$ and $p_{T'' DS''}$. First notice that $(T'')^{-1}T'DS' = DS''$ which guarantees that $(T'')^{-1}T'$ yields indeed a functor mapping to $W_{g,n}^{T''DS''}$ (the matrix $(T'')^{-1}T'$ transforms a morphism of the form $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_N, \text{id}, \ldots, \text{id})$ from $DS' \hat{L}$ to $T''^{-1}(\omega_{\log}, \ldots, \omega_{\log})^s$ into a morphism of the form $(\tilde{\varphi}_1', \ldots, \tilde{\varphi}_N', \text{id}, \ldots, \text{id})$ from $DS'' \hat{L}$ to $T''^{-1}(\omega_{\log}, \ldots, \omega_{\log})^s$). Second, we have

$$j \circ p_{T'' DS''} = p_{T' DS'}$$

hence

$$\tilde{d}(S'')^{-1}\hat{D}(T'')^{-1}T' = (S')^{-1}\hat{D}(T')^{-1}T' DS'(S'')^{-1}\hat{D}(T'')^{-1}T' = (S')^{-1}\hat{D}(T')^{-1}T' DS''(S'')^{-1}\hat{D}(T'')^{-1}T' = \tilde{d}(S')^{-1}\hat{D}.$$

In [FJR1][FJR2], the stack $W_{g,n}^{TDS}$ is used in order to define Witten’s virtual cycle $[W_{TDS}^{\text{vir}}]_{g,n}$. Via $p_{TDS}$ we get a the cycle used in this paper

$$[W]_{g,n}^{\text{vir}} := (p_{TDS})_* [W_{TDS}^{\text{vir}}]_{g,n},$$

40
which by (64) does not depend on $T$ and $S$. Note that $W_{g,n}^{T \Delta S} \to \mathcal{M}_{g,n,d}$ can be regarded as $W_{g,n}^{T \Delta S} \to W_{g,n}$; therefore, using the fact that psi classes are pullbacks from $\mathcal{M}_{g,n,d}$ and the projection formula, the FJRW invariants (27) do not differ if we replace $W$ with $W^{T \Delta S}$. Hence, the FJRW invariants of [FJR1, Defn. 4.2.6] equal those introduced in (27).

In this paper we only consider FJRW invariants for $(W, G)$-curves with $G = \langle J \rangle$; in this case Proposition 2.3.13 provides an étale cover of $W_{g,n,\langle J \rangle}$ analogous to $W_{g,n}^{T \Delta S} \to W_{g,n}$ allowing us to define the virtual cycle naturally (see Remark 2.4.2). We point out, that the above procedure allows us, more generally, to define an étale cover of $W_{g,n,G}$ and, via pushforward, a virtual cycle inside $W_{g,n,G}$ compatible with (17).

Remark A.2.1. First consider the case where $W + \mathcal{Z}$ and $W + \mathcal{Z}$ are two nondegenerate quasihomogeneous polynomials such that $M_{W + \mathcal{Z}}$ is an $\bar{s} \times N$ submatrix of the $\bar{s} \times N$ matrix $M_{W + \mathcal{Z}}$:

$$M_{W + \mathcal{Z}} = (I_{\bar{s}} \ 0) \ M_{W + \mathcal{Z}}, \quad M_{W + \mathcal{Z}} = \tilde{T} \tilde{D} \tilde{S}, \quad M_{W + \mathcal{Z}} = \tilde{T} \tilde{D} \tilde{S},$$

(65)

where $\tilde{T} \tilde{D} \tilde{S}$ and $\tilde{T} \tilde{D} \tilde{S}$ relate $M_{W + \mathcal{Z}}$ and $M_{W + \mathcal{Z}}$ to the respective Smith normal forms. We define $\tilde{i}$ fitting in the commutative diagram (since $M_{W + \mathcal{Z}}$ is a submatrix of $M_{W + \mathcal{Z}}$, $i$ is the obvious inclusion)

$$\begin{array}{ccc}
(W + \mathcal{Z})_{g,n} & \xrightarrow{\tilde{i}} & (W + \mathcal{Z})_{g,n} \\
\downarrow p_{\tilde{T} \tilde{D} \tilde{S}} & & \downarrow p_{\tilde{T} \tilde{D} \tilde{S}} \\
(W + \mathcal{Z})_{g,n} & \xrightarrow{i} & (W + \mathcal{Z})_{g,n}
\end{array}$$

locally isomorphic to

$$\begin{array}{ccc}
BG_{W + \mathcal{Z}} & \longrightarrow & BG_{W + \mathcal{Z}} \\
\downarrow B(\mu_d)^{\tilde{N}} & & \downarrow B(\mu_d)^{\tilde{N}} \\
B(\mu_d)^{\tilde{N}} & \longrightarrow & B(\mu_d)^{\tilde{N}}
\end{array}$$

(66)

Indeed, using the usual notations $\hat{D}$ and $\hat{D}$ we get $\tilde{S}^{-1} \hat{D} \tilde{D}^{-1} (I_{\bar{s}} \ 0) \tilde{T} \left( I_{\tilde{N}} \begin{array}{c} 0 \\ 0 \end{array} \right)$ and

$$\tilde{i}: (W + \mathcal{Z})_{g,n} \longrightarrow (W + \mathcal{Z})_{g,n}$$

$$(\tilde{L}, \tilde{\mathcal{Z}}) \longrightarrow \left( \tilde{L}, \tilde{T}^{-1} (I_{\bar{s}} \ 0) \tilde{T} \tilde{\mathcal{Z}} \right).$$

Now, we apply the construction of [FJR1] to $W_{g,n,G} \subseteq W_{g,n}$. Let $G$ be an admissible subgroup of $G_W$ and consider $Z$ such that $G_{W + \mathcal{Z}} = G$. Write the matrix $M_{W + \mathcal{Z}}$ as $T \Delta S$ and consider $(W + \mathcal{Z})_{g,n}^{T \Delta S}$ which is an étale cover of $W_{g,n,G} := (W + \mathcal{Z})_{g,n}$ via $p_{T \Delta S}$. Clearly we may realize $G$ by choosing a different polynomial $Z$ such that $G = G_{W + Z'}$; in this way we obtain different matrices and different Smith normal forms $M_{W + Z'} = T' \Delta S'$ and $M_{W + \mathcal{Z}} = T \Delta S$ and two stacks $(W + Z')_{g,n}^{T' \Delta S'}$ and $(W + \mathcal{Z})_{g,n}^{T \Delta S}$ covering $W_{g,n,G}$. We prove that these two covering stacks are related by an isomorphism which commutes with the maps to $W_{g,n,G}$.

Indeed it is enough to consider the case where $W + \mathcal{Z}$ and $W + \mathcal{Z}$ are two polynomials such that $G_{W + \mathcal{Z}} = G_{W + \mathcal{Z}}$ inside $G_W$ and $M_{W + \mathcal{Z}}$ is an $\bar{s} \times N$ submatrix of the $\bar{s} \times N$ matrix $M_{W + \mathcal{Z}}$ satisfying (65). We are in the conditions of Remark A.2.1. As observed in Remark 2.3.11, $G_{W + \mathcal{Z}} = G_{W + \mathcal{Z}}$ implies that $\tilde{i}$ is an isomorphism identifying $(W + \mathcal{Z})_{g,n}$ to $(W + \mathcal{Z})_{g,n}$ and $\tilde{i}$ commutes with the morphisms $p_{T \Delta S}$ and $p_{T \Delta S}$. Since $\tilde{i}$ is locally isomorphic to $BG \to BG$, and induces an isomorphism on every fibre, the morphism $\tilde{i}$ is an isomorphism between the two covering stacks over $W_{g,n,G}$. This justifies the fact that, in [FJR1] the authors adopt a notation for this covering stack of $W_{g,n,G}$ which only involves $W, g, n,$ and the group $G$.

In particular pushing cycles forward into $W_{g,n,G}$ does not depend on the polynomial $\mathcal{Z}$ chosen to represent $G$ nor on the matrices chosen to relate $M_{W + \mathcal{Z}}$ to the Smith normal form. This justifies the definition of [FJR1] of the virtual cycle via pullback via $\tilde{i}: (W + \mathcal{Z})_{g,n}^{T \Delta S} \to W_{g,n}^{T \Delta S}$ where $\tilde{T} \tilde{D} \tilde{S} = M_{W + \mathcal{Z}}$. In equation (17), we defined the virtual cycle inside $W_{g,n,G}$ directly via pullback via $W_{g,n,G} \to W_{g,n}$. After pushforward to $W_{g,n,G}$ of the virtual cycle of [FJR1], the two definitions are equivalent. Indeed, set

$$[W]_{g,n,G}^{vir} := (p_{T \Delta S})_* \tilde{i}^* [W_{g,n}^{T \Delta S}]_{g,n}^{vir}.$$
We have \((I_0, 0) M_{W+\tilde{Z}} = M_W\); therefore Remark A.2.1 applies and (66) yields
\[
(p_{TD})^\star \tilde{\eta}^* = \frac{|G_W|}{|G|} i^* (p_{\tilde{T} \tilde{D} \tilde{S}})^\star.
\]
The compatibility between the above definition of \([W]_{vir}^{\star, G, G}\) and that of (17) follows.

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