A cosmological dust model with extended $f(\chi)$ gravity.

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Introducing a fundamental constant of nature with dimensions of acceleration into the theory of gravity makes it possible to extend gravity in a very consistent manner. At the non-relativistic level a MOND-like theory with a modification in the force sector is obtained, which is the limit of a very general metric relativistic theory of gravity. Since the mass and length scales involved in the dynamics of the whole universe require small accelerations of the order of Milgrom’s acceleration constant $a_0$, it turns out that the relativistic theory of gravity can be used to explain the expansion of the universe. In this work it is explained how to use that relativistic theory of gravity in such a way that the overall large-scale dynamics of the universe can be treated in a pure metric approach without the need to introduce dark matter and/or dark energy components.

In this article we explore the consequences of the relativistic extended metric $f(\chi)$ theory of gravity [3] applied to the present expansion of the universe. The article is organised as follows. Section II gives a brief summary of the extended theory of gravity by Bernal et al. [2], generalising the Newnotian description by Mendoza et al. [16]. Section III interconnects this extended relativistic description of gravity with a metric description of gravity for which the energy-momentum tensor appears in the gravitational field’s action. On section IV we use the developed theory of gravity for cosmological applications in a dust universe and see how it is a coherent representation of gravity at cosmological scales. Finally on section V we discuss the consequences of the developed approach of gravity and some of the future developments of the theory.

I. INTRODUCTION

Current cosmological data is generally explained introducing two unknown mysterious dark components, namely dark matter and dark energy. These ad hoc hypothesis represent a big cosmological paradigm, since they arise due to the fact that Einstein’s field equations are forced to remain unchanged under certain observed astrophysical and cosmological anomalies.

A natural alternative scenario would be to see whether viable cosmological solutions can be found if dark unknown entities are assumed non-existent. The price to pay with this assumption is that the field equations of the theory of gravity need to be extended, and so new Friedmann-like equations will arise. The most natural approach to extend gravity arises when a metric extension $f(R)$ is introduced into the theory [see e.g. 2 and references therein].

In a series of recent articles, Bernal et al. [2, 3], Hernandez, Jiménez, and Allen [11], Hernandez et al. [12], Mendoza et al. [16] have shown how relevant the introduction of a new fundamental physical constant $a_0 \approx 10^{-10}$ m/s$^2$ with dimensions of acceleration is in excellent agreement with different phenomenology at many different astrophysical and length dimensions, from solar-system to extragalactic scales. The introduction of the so called Milgrom’s acceleration constant $a_0$ in a description of gravity means that any gravitational field produced by a certain distribution of mass (and hence energy) needs to incorporate the acceleration $a_0$ together with Newton’s gravitational constant $G$ and the speed of light $c$ in the description of gravity.

II. RELATIVISTIC METRIC EXTENSION

Finding a relativistic theory of gravity for which one of its non-relativistic limits converges to MOND yields usually strange assumptions and/or complicated ideas [see e.g. 1, 4, 18]. A good first approach was provided by a slight modification of Einstein’s field equations by Sobouti [21], but the attempt is not complete.

In order to find an elegant and simple theory of gravity with a non-relativistic weak-field limit MONDian solution, Bernal et al. [3] used a correct metric interpretation of Hilbert’s action $S_f$ in such a way that:

$$S_f = -\frac{c^3}{16\pi G L^2_M} \int f(\chi) \sqrt{-g} d^4x,$$

which slightly differs from its traditional form (see e.g. [3, 7, 22]) since the dimensionless Ricci scalar:
Einstein-Hilbert action is obtained. The matter action \( R \) and \( \nabla \) Laplace-Beltrami operator has been written as \( \Delta := \nabla^\mu \nabla_\mu \) its argument. The energy-momentum tensor \( T_{\mu\nu} \), a (\( + \)stein’s summation convention over repeated indices.

\[
\delta S = \int \frac{1}{2c^2} \int L_m \sqrt{-g} \, d^4 x, \tag{3}
\]
with \( L_m \) the matter Lagrangian density of the system.

The null variations of the complete action, i.e. \( \delta S = \int \delta L \, \sqrt{-g} \, d^4 x \), yield the following field equations:

\[
f'(\chi) \chi_{\mu\nu} \frac{1}{2} f(\chi) g_{\mu\nu} - L_M^2 \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \Delta \right) f'(\chi) = \frac{8 \pi G L_M^2}{c^4} T_{\mu\nu}, \tag{4}
\]
where the dimensionless Ricci tensor \( \chi_{\mu\nu} \) is given by:

\[
\chi_{\mu\nu} := L_M^2 R_{\mu\nu}, \tag{5}
\]
and \( R_{\mu\nu} \) represents the standard Ricci tensor. The Laplace-Beltrami operator has been written as \( \Delta := \nabla^\alpha \nabla_\alpha \) and the prime denotes derivative with respect to its argument. The energy-momentum tensor \( T_{\mu\nu} \) is defined through the following standard relation: \( \delta S_m = -\frac{1}{2c^2} \int \delta g \, d^4 x \), \( \delta \) being the variation.

In other words, general relativity is recovered when \( \chi \gg 1 \) in the strong field regime and the relativistic version of MOND with \( \chi^{3/2} \) is recovered for the weak field regime of gravity when \( \chi \ll 1 \).

The mass dependence of \( \chi \) and \( L_M \) means that Hilbert’s action \( S \) is a function of the mass \( M \). This is usually not assumed, since that action is thought to be purely a function of the geometry of space-time due to the presence of mass and energy sources. However, it was Sobouti \([\text{21}]\) who first encountered this peculiarity in the Hilbert action when dealing with a metric generalisation of MOND. Following the remarks by Sobouti \([\text{21}]\) and Mendoza and Rosas-Guevara \([\text{17}]\) one should not be surprised if some of the commonly accepted notions, even at the fundamental level of the action, require generalisations and re-thinking. An extended metric theory of gravity goes beyond the traditional general relativity ideas and in this way, we need to change our standard view of its fundamental principles.

\section{F(R,T) Connections}

For the description of gravity shown in section \([\text{II}]\) it follows that an adequate way of writing up the gravitational field’s action is given by:

\[
S_t = -\frac{c^3}{16 \pi G} \int \frac{f(\chi)}{L_M^2} \sqrt{-g} \, d^4 x. \tag{11}
\]

The function \( L_M \) is a function of the mass of the system and in general terms it is a function of the space-time coordinates. For the particular case of a spherically symmetric space-time it coincides with the mass of the central object generating the gravitational field as expressed in equations \((5)\) and \((6)\). Generally speaking what the meaning of \( M \) would be for a particular distribution of mass and energy needs further development, beyond the scope of this work. Nevertheless one expects that for a systems with high degree of symmetry (such as as a spherically symmetric space-time or a Friedmann-Lemaître-Robertson-Walker one), the function \( M \) would be given by the standard mass-energy relation [see e.g. \([\text{19}]\)]:

\[
M := \frac{4 \pi}{c^4} \int T \, r^2 \, dr, \tag{12}
\]
where \( r \) is the “radial” coordinate.
The field equations produced by the null variations of the addition of the field’s action $S_I + S_m$ can be constructed in the following form. Harko et al. [10] have built an $F(R, T)$ theory of gravity, so making the natural identification:

$$F(R, T) := \frac{f(\chi)}{L_M^2},$$  \hspace{1cm} (13)

it is possible to use all their results for our particular case expressed in equation [13]. For example, the null variations of the complete action $S_I + S_m$ for the particular case of equation (13) yield the following field equations [10]:

$$(f_R) R_{\mu\nu} - \frac{1}{2L_M^2} f g_{\mu\nu} + \left[ g_{\mu\nu} \Delta - \Delta_{\mu} \Delta_{\nu} \right] \left( \frac{f_R}{L_M^2} \right) = \frac{8\pi G}{c^4} T_{\mu\nu} - \left( \frac{f}{L_M^2} \right)_T \left[ T_{\mu\nu} + \Theta_{\mu\nu} \right],$$  \hspace{1cm} (14)

with a trace:

$$(f_R) R - \frac{2f}{L_M^2} + 3\Delta \left( \frac{f_R}{L_M^2} \right) = \frac{8\pi G}{c^4} T - \left( \frac{f}{L_M^2} \right)_T \left[ T + \Theta \right],$$  \hspace{1cm} (15)

where the subscripts $R$ and $T$ stand for the partial derivatives with respect to those quantities, i.e.

$$\left( \begin{array}{c} \frac{\partial}{\partial R} \\ \frac{\partial}{\partial T} \end{array} \right) := \left( \begin{array}{c} R \\ T \end{array} \right).$$  \hspace{1cm} (16)

The tensor $\Theta_{\mu\nu}$ is such that $\Theta_{\mu\nu} \delta g^{\mu\nu} := g^{\alpha\beta} \delta T_{\alpha\beta}$ and for the case of an ideal fluid it can be written as [10]:

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu},$$  \hspace{1cm} (17)

Note that equation (14) or (15) converges to the field and trace relations as discussed in section III when one considers a point mass generating the gravitational field, i.e. when $L_M = \text{const.}$ and so $\partial/\partial R = L_M^2 \partial/\partial \chi$.

In general terms, the $F(R, T)$ theory described by Harko et al. [10] produces non-geodesic motion of test particles since:

$$\nabla^\mu T_{\mu\nu} = \left( \frac{f}{L_M^2} \right) T \left[ \frac{8\pi G}{c^4} - \left( \frac{f}{L_M^2} \right)_T \left[ T_{\mu\nu} + \Theta_{\mu\nu} \right] \right] \nabla^\mu \ln \left( \frac{f}{L_M^2} \right)_T (R, T) + \nabla^\mu \Theta_{\mu\nu} \neq 0,$$  \hspace{1cm} (18)

and as such the geodesic equation has a force term:

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = \lambda^\mu,$$  \hspace{1cm} (19)

where the four-force

$$\lambda^\mu := \frac{8\pi G}{c^4} \left( \rho c^2 + p \right)^{-1} \left[ \frac{8\pi G}{c^4} + \left( \frac{f}{L_M^2} \right)_T \right] \times \left( g^{\mu\nu} - u^\mu u^\nu \right) \nabla_\nu p,$$  \hspace{1cm} (20)

is perpendicular to the four velocity $dx^\alpha/ds$. As explained by Harko et al. [10], the motion of test particles is geodesic, i.e. $\lambda^\mu = 0$ and/or $\nabla^\alpha T_{\alpha\beta} = 0$, (i) for the case of a pressureless $p = 0$ (dust) fluid and (ii) for the cases in which $F_T(R, T) = 0$.

In what follows we will see how all the previous ideas can be applied to a FLRW dust universe and so, the divergence of the energy momentum tensor in equation (18) is null. It is worth noting that this condition on the energy-momentum tensor for many applications needs to be zero, including applications to the universe at any epoch. In other words, one should impose the null divergence of the energy momentum tensor for any real physical system.

IV. COSMOLOGICAL APPLICATIONS

For an isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) universe, the interval $ds$ is given by [see e.g. 13]:

$$ds^2 = c^2 \left( ds \right)^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),$$  \hspace{1cm} (21)

where $a(t)$ is the scale factor of the universe normalised to unity, i.e. $a(t_0) = 1$, at the present epoch $t_0$, and the angular displacement $d\Omega^2 := d\theta^2 + \sin^2 \theta d\varphi^2$ for the polar $d\theta$ and azimuthal $d\varphi$ angular displacements with a comoving coordinate distance $r$. From now on, we assume a null space curvature $k = 0$ at the present epoch in accordance with observations and deal with the expansion of the universe dictated by the field equations (13), avoiding any form of dark unknown component. Since we are interested on the compatibility of this cosmological model with SNIa observations, in what follows we assume a dust $p = 0$ model for which the covariant divergence of the energy-momentum tensor vanishes, and so as discussed in section III the trajectories of test particles are geodesic, i.e. the divergence of the energy-momentum tensor is zero. As mentioned at the end of the previous section, note also that this condition has to be satisfied for any equation of state and not only for a dust universe.

For simplicity, let us consider that the function $f(\chi)$ obeys a power-law relation:
\[ f(\chi) = \chi^b, \quad (22) \]

with a constant exponent \( b \). Let us now rewrite the field equations \[14\] inspired by the approach first introduced by Capozziello and Fung \[8\] (see also Capozziello and Faraoni \[7\]) as follows:

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} \left\{ \left( 1 + \frac{c^4}{8\pi G} F_T \right) \frac{T_{\nu\mu}}{F_R} + T_{\mu\nu}^{\text{curv}} \right\}, \quad (23) \]

where the Einstein tensor is given by its usual form:

\[ G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad (24) \]

and

\[ T_{\mu\nu}^{\text{curv}} := \frac{c^4}{8\pi G F_R} \left[ \left( \frac{1}{2} (F - RF_R) - \Delta F_R \right) g_{\mu\nu} + \nabla_\mu \nabla_\nu F_R \right], \quad (25) \]

represents the "energy-momentum" curvature tensor. Since \( T_{00} = \rho c^2 \), then it will be useful the identification \( T_{00} := \rho_{\text{curv}} c^2 \). With this last definition and using the fact that the Laplace-Beltrami operator applied to a scalar field \( \psi \) is given by \[\text{cf. 8}\]:

\[ \Delta \psi = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \psi \right), \quad (26) \]

then

\[ \rho_{\text{curv}} = \frac{c^2}{8\pi G F_R} \left[ \frac{1}{2} (RF_R - F) - \frac{3H dF_R}{c^2 dt} \right], \quad (27) \]

where \( H := \dot{a}(t)/a(t) \) represents Hubble’s constant.

With the above definitions and using the 00 component of the field’s equations \[23\] and the relation \[\text{cf. 8}\]

\[ R = -\frac{6}{c^2} \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{\kappa c^2}{a^2} \right], \quad (28) \]

between Ricci’s scalar and the derivatives of the scale factor for a FLRW universe, then the dynamical Friedman-like equation for a dust flat universe is:

\[ H^2 = \frac{8\pi G}{3} \left[ \left( 1 + \frac{c^4}{8\pi G} F_T \right) \frac{\rho}{F_R} + \rho_{\text{curv}} \right], \quad (29) \]

The energy conservation equation is given by the null divergence of the energy-momentum tensor:

\[ \left( \frac{8\pi G}{c^4} + F_T \right) \left( \dot{\rho} + 3H\rho \right) = -\rho \frac{dF_T}{dt} = 0, \quad (30) \]

and so

\[ \dot{\rho} + 3H\rho = 0. \quad (31) \]

For completeness, we write down the correspondent generalisation of Raychadhuri’s equation for a dust flat universe:

\[ 2\ddot{a} + H^2 = -\frac{8\pi G \rho_{\text{curv}}}{c^2}, \quad (32) \]

where the “curvature-pressure”

\[ \rho_{\text{curv}} := \omega c^2 \rho_{\text{curv}}, \quad (33) \]

and

\[ w = \frac{c^2 (F - RF_R)/2 + d^2 F_R/dt^2 + 3HdF_R/dt}{c^2 (RF_R - F)/2 - 3HdF_R/dt}. \quad (34) \]

On the other hand, note that the mass \( M \) that appears on the length \( L_M \) must be the causally connected mass at a certain cosmic time \( t \), since particles beyond Hubble’s (or particle) horizon with respect to a given fundamental observer do not have any gravitational influence on him. In other words:

\[ M = \int_0^{r_H} \rho r^2 dr = \frac{4}{3} \pi \rho c^3 H^3, \quad (35) \]

where

\[ r_H := \frac{c}{H(t)} \quad (36) \]

is the Hubble radius or the distance of causal contact at a particular cosmic epoch \[15\]. With such an election for the mass \( M(t) \), it follows that at any fixed cosmic epoch the field’s action \[11\] is the same for all fundamental observers. Using equation \[36\], the length \[8\] can be written as:

\[ L_M = \zeta \left( \frac{\rho c^3 G}{c^4_0} \right)^{3/4} \rho^{3/4} H^{9/4}, \quad (37) \]

and so, by using relation \[22\] then:

\[ \frac{dF_R}{dt} = b(b-1)R^{b-1} L_M^{2(b-1)} H \left[ j - q - \frac{2}{1 - q} \left( \beta + \frac{3}{2} \right) \right], \quad (38) \]

\[ \frac{dF_T}{dt} = \frac{3}{2} (b-1) \frac{R^b L_M^{2b-2}}{\rho c^2}, \quad (39) \]
of the speed of light

tion (43) at the present epoch, yielding:

\[
q(t) := -\frac{1}{a(t)} \frac{d^2 a}{dt^2}, \quad \text{and} \quad j := \frac{1}{a(t)} \frac{d^3 a}{dt^3} H^{-3},
\]
are the deceleration parameter and the jerk respectively.

As it is usually done, let us find power law solutions satisfying

\[
a(t) = a(t_0) \left( \frac{t}{t_0} \right) ^\alpha, \quad \rho(t) = \rho_0 \left( \frac{a(t)}{a(t_0)} \right) ^\beta,
\]
for the unknown indices \(\alpha\) and \(\beta\). With these and the value of \(L_M\) from equation (37), the curvature density (27) is

\[
\rho_{\text{curv}} = \frac{3H^2}{8\pi G} (b - 1) \left[ (1 - q) - \frac{j - q - 2}{1 - q} - \frac{3}{2} \left( \beta + \frac{3}{\alpha} \right) \right].
\]

Substitution of the previous relations on Friedmann’s equation (29) it follows that:

\[
H^2 = \frac{8\pi G \rho}{3 Z}, \quad \text{where}
\]

\[
Z := 1 + (b - 1) \left[ \frac{j - q - 2}{1 - q} - \frac{4(1 - q)}{b} + \frac{3}{2} \left( \beta + \frac{3}{\alpha} \right) \right],
\]
is a dimensionless function.

An important result can be obtained evaluating equation (33) at the present epoch, yielding:

\[
a_0 = \frac{9}{4} \zeta^4 (1 - q_0)^2 (bZ_0)^{2/(b-1)} \left( \Omega_{\text{matt}}^{(0)/(3b-5)/(b-1)} \right) c H_0,
\]
where the density parameter \(\Omega_{\text{matt}}^{(0)}\) at the present epoch has been defined by it’s usual relation:

\[
\Omega_{\text{matt}}^{(0)} := \frac{3H^2 \rho}{8\pi G}.
\]

In other words, the value of Milgrom’s acceleration constant \(a_0\) at the current cosmic epoch is such that

\[
a_0 \approx c \times H_0.
\]

The numerical coincidence between the value of Milgrom’s acceleration constant \(a_0\) and the multiplication of the speed of light \(c\) by the current value of Hubble’s constant \(H_0\) has been noted since the early development of MOND [see e.g. 3, and references therein]. Note that equation (47) means that this coincidence relation occurs at approximately the present cosmic epoch in complete agreement with the results by Bernal et al. 2 where it is shown that \(a_0\) shows no cosmological evolution and hence it can be postulated as a fundamental constant of nature.

For the power law (22) and the assumptions made above, it follows that the energy conservation equation (33) is given by:

\[
(\dot{\rho} + 3H\rho) + \frac{c^2}{8\pi G} \left( A\dot{\rho} + B H \right) R^6 L_M^{(b-1)} = 0, \quad \text{(48)}
\]
where:

\[
A := \frac{9}{4} (b - 1)^2, \quad B := \frac{9 b - 1}{b} + \frac{27}{4} \frac{(b - 1)^2}{\alpha} + \frac{3 b (b - 1)(j - q - 2)}{2(1 - q)}.
\]

Direct substitution of the density power law (41) into relation (48) gives a constraint equation between \(\alpha\), \(\beta\) and \(b\):

\[
\beta = \frac{1}{\alpha} \left( \frac{9 - 5b}{3b - 5} \right).
\]

Let us now proceed to fix the so far unknown parameters of the theory \(\alpha\), \(\beta\) and \(b\). To do so, we need reliable observational data and as such, we use the redshift-magnitude SNIIa data obtained by Riess et al. [20] and the following well known standard cosmological relations [see e.g. 13]:

\[
1 + z = a(t_0)/a(t), \quad \mu(z) = 5 \log_{10} [H_0 d_L(z)] - 5 \log_{10} h + 42.38, \quad \text{(50)}
\]
\[
d_L(z) = (1 + z) \int_0^z \frac{c}{H(z)} \, dz, \quad \text{(51)}
\]
for the cosmological redshift \(z\), the distance modulus \(\mu\), the Luminosity distance \(d_L\) and where the normalised Hubble constant \(h\) at the present epoch is given by \(h := H_0 / (100 \, \text{km/s/Mpc})\). Also, from equation (41) it follows that

\[
H(\alpha) = H_0 \left( \frac{a}{a(t_0)} \right)^{-1/\alpha} = H_0 (1 + z)^{1/\alpha}, \quad \text{(53)}
\]
and the substitution of this into equation (52) gives the distance modulus \(d_L\) as a function of the redshift \(z\). This means that the redshift magnitude relation (51) is a function that depends on the values of the current Hubble constant \(H_0\) and the value of \(\alpha\). Figure 1 shows the best fit to the redshift-magnitude relation of SNIIa observed by Riess et al. [20], yielding \(a = 1.359 \pm 0.139\) and \(h = 0.64 \pm 0.009\). The best fit presented on the figure was
FIG. 1. Redshift magnitude plot for SNIa showing the distance modulus $\mu$ as a function of the redshift $z$ for SNIa as presented by Riess et al. The dotted red line shows the best fit to the data with the flat dust FLRW universe (see text) with no dark components. The continuous blue line represents the best fit according to the standard concordance dust $\Lambda$CDM model.

\[ h = 0.64 \pm 0.009, \quad q_0 = -0.2642 \pm 0.075, \quad j_0 = -0.1246 \pm 0.004. \] (54)

V. DISCUSSION

The obtained value $b \approx 3/2$ is a completely expected result due to the following arguments. As explained by Mendoza et al. [10], a gravitational system for which its characteristic size $r$ is such that $x := l_M / r \lesssim 1$ is in the MONDian regime of gravity. For the case of the universe, $x \sim a$ few, and as such if not totally in the MONDian regime of gravity, then it is far away from the regime of Newtonian gravity. The relativistic version of this means that the universe is close to the regime for which $f(\chi) = \chi^{3/2}$ and so $b = 3/2$. This is a very important result since, seen in this way, the accelerated expansion of the universe is due to an extended gravity theory deviating from general relativity. It is quite interesting to note that the function $f(\chi) = \chi^{3/2}$ at which its non-relativistic limit is capable of predicting the correct dynamical behaviour of many astrophysical phenomena, is also able to explain the behaviour of the current accelerated expansion of the universe.

Seen in this way, the behaviour of gravity towards the past (for sufficiently large redshifts $z$) will differ from $f(\chi) = \chi^{3/2}$ and eventually converge to $f(\chi) = \chi$, i.e. the gravitational regime of gravity is general relativity for sufficiently large redshifts. A very detailed investigation into this needs to be done at different levels in order to be coherent with many different cosmological observations [see e.g. [14]].

It is quite remarkable that a metric extended theory of gravity is able to reproduce phenomena from mass and length scales associated to the solar system up to cosmological scales. Many other cosmological applications of the theory will be addressed elsewhere.

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