Bifurcation diagram of a one-parameter family of dispersive waves

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Abstract

The Korteweg de Vries (KdV) equation with small dispersion is a model for the formation and propagation of dispersive shock waves in one dimension. Dispersive shock waves in KdV are characterized by the appearance of zones of rapid modulated oscillations in the solution of the Cauchy problem with smooth initial data. The modulation in time and space of the amplitudes, the frequencies and the wave-numbers of these oscillations and their interactions is approximately described by the g-phase Whitham equations. We study the initial value problem for the Whitham equations for a one parameter family of monotone decreasing initial data. We obtain the bifurcation diagram of the number g of interacting oscillatory zones.

1 Introduction

The solution of the Cauchy problem for the KdV equation

\[
\begin{align*}
    u_t + 6u u_x + \epsilon^2 u_{xxx} &= 0, \quad t, x \in \mathbb{R} \\
    u(x, t = 0, \epsilon) &= u_0(x)
\end{align*}
\]

where \( \epsilon > 0 \) is a small parameter, is characterized by the appearance of zones of rapid modulated oscillations. These modulated oscillations are called dispersive shock waves and they are described in terms of the Whitham equations. For \( \epsilon > 0 \), no matter
how small, the solution of (1.1) with smooth initial data exists and remains smooth for all \( t > 0 \). For \( \epsilon = 0 \) (1.1) becomes the Cauchy problem for the Burgers equation \( u_t + 6uu_x = 0 \). If the initial data \( u_0(x) \) is decreasing somewhere the solution \( u(x,t) \) of the Burgers equation has always a point \((x_0,t_0)\) of gradient catastrophe where an infinite derivative develops.

Figure 1.1: The dashed line represents the formal solution of the Burgers equation after the time of gradient catastrophe \( t = t_0 \). The oscillations on the picture are close to a modulated periodic wave.

After the time of gradient catastrophe the solution \( u(x,t,\epsilon) \) of (1.1) develops an expanding region filled with rapid modulated oscillations as shown in Fig. 1.1.

The idea and first example of the description of the dispersive shock waves were proposed by the physicists Gurevich and Pitaevski (GP) [1]. They considered the oscillatory zone to be approximately described by a modulated periodic wave:

\[
    u(x,t,\epsilon) \simeq \frac{2a}{s} \, dn^2 \left( \frac{a}{6s} \frac{(x-Vt + x_0)}{\epsilon}, s \right) + \gamma, \tag{1.2}
\]

where \( dn(y,s) \) is the Jacobi elliptic function of modulus \( s \in (0,1) \), \( x_0 \) is a suitable phase, the quantities \( a, s, \gamma \) and \( V = \left[ 2a \frac{2-s}{3s} + \gamma \right] \) depend on \( x \) and \( t \). These quantities evolve according to the Whitham equations [2] which guarantee the validity of the approximate description (1.2). For constant values of the parameters \( a, s \) and \( \gamma \), \( u(x,t,\epsilon) \) is an exact periodic solution of KdV with amplitude \( a \), wave number \( k \) and frequency \( \omega \) given by the relations

\[
a = u_{max}(x,t,\epsilon) - u_{min}(x,t,\epsilon), \quad k = \frac{\pi}{\epsilon K(s)} \sqrt{\frac{a}{6s}}, \quad \omega = \frac{k}{V},
\]

where \( K(s) \) is the elliptic integral of the first kind of modulus \( s \).
Whitham introduced the Riemann invariants $u_1 > u_2 > u_3$ to write the equations for $a$, $s$ and $\gamma$ in diagonal form. These quantities are expressed in terms of $u_1 > u_2 > u_3$ by the relations

$$a = u_2 - u_3 \quad s = \frac{u_2 - u_3}{u_1 - u_3}, \quad \gamma = u_2 + u_3 - u_1.$$  (1.3)

The Whitham equations for the $u_i$, $i = 1, 2, 3$ read

$$\partial_t u_i(x, t) + \lambda_i(u_1, u_2, u_3)\partial_x u_i(x, t) = 0, \quad i = 1, 2, 3,$$  (1.4)

where

$$\lambda_i(u_1, u_2, u_3) = 2(u_1 + u_2 + u_3) + \frac{\prod_{j \neq i, j=1}^3 (u_i - u_j)}{u_i + \alpha_0},$$  (1.5)

and $E(s)$ is the complete elliptic integral of the second kind.

The variable $u_2$ can vary from $u_3$ to $u_1$. The oscillatory region is bounded on one side by the point $x^-(t)$ where $u_2(x, t) = u_3(x, t)$, and on the other side by the point $x^+(t)$ where $u_2(x, t) = u_1(x, t)$ (see Fig. 1.1 and Fig. 1.2). Outside the region $(x^-(t), x^+(t))$ the solution $u(x, t, \epsilon)$ of (1.1) is well approximated by the solution $u(x, t)$ of the Burgers equation.

In the theory of singularity a function behaves quadratically in the neighborhood of its generic singular point [3]. In a similar way, in the theory of dispersive shock waves the generic analytic monotone decreasing initial data $x = f(u|_{t=0})$ behaves (up to shifts and rescalings) like $x = -u^3$ in the neighborhood of its generic breaking point. Thus the behavior of the generic analytic initial data near a point of gradient catastrophe is

$$x - x_0 \simeq -c_3(u - u_0)^3 - c_4(u - u_0)^4 - \cdots - c_k(u - u_0)^k - \ldots.$$  (1.6)

GP kept only the first term of the above Taylor series namely they considered the initial data $x - x_0 = -(u - u_0)^3$ (up to rescaling) and solved the equations (1.4) numerically for such initial data. Potemin [4] obtained the analytic solution of the equations (1.4) for the same initial data and showed that $x^-(t) = x_0 - 12\sqrt{3}(t - t_0)^{\frac{2}{3}}$ and $x^+(t) = x_0 + 4/3\sqrt{5/3}(t - t_0)^{\frac{2}{3}}$ (see Fig. 1.2).

The higher order terms of the Taylor expansion (1.6) are negligible only for $t \ll t_0 + 2\sqrt{3}\sqrt{c_4}$ [5]. When one starts considering the higher order terms of the expansion
Figure 1.2: On picture a) the dashed line represents the formal solution of the Burgers equation, the continuous line represents the solution of the Whitham equations. The solution \( (u_1(x,t), u_2(x,t), u_3(x,t)) \) of the Whitham equations and the position of the boundaries \( x^-(t), x^+(t) \) of the oscillatory zone are to be determined from the condition

\[
u(x^-(t), t) = u_1(x^-(t), t), \quad u_x(x^-(t), t) = u_{1x}(x^-(t), t), \quad u(x^+(t), t) = u_3(x^+(t), t),
\]

\[
0 = u_x(x^+(t), t) = u_{3x}(x^+(t), t)
\]

where \( u(x,t) \) is the solution of the Burgers equation. Picture b) represents the functions \( x^-(t) \) and \( x^+(t) \) on the \( x - t \) plane.

We come to a situation in which two or more oscillatory wave trains come in interaction or in the case in which the solution of (1.4) comes itself to a point of gradient catastrophe. We will see that such a phenomena necessarily occur in the simplest non trivial deformation of the cubic law which turns out to be the fifth order polynomial:

\[
x \simeq -c_3 (u - u_0)^3 - c_4 (u - u_0)^4 - c_5 (u - u_0)^5, \quad c_3 > 0, \ c_5 > 0.
\]  

(1.7)

Through the shift \( (x \to x + 6tu_0, u \to u + u_0) \) the above initial data becomes \( x \simeq -c_3 u^3 - c_4 u^4 - c_5 u^5 \). The parameter space \( (c_3, c_4, c_5) \) can be reduced exploiting the invariance of the KdV equation under the groups of transformations \( (x \to k^3 x, \ t \to k^2 t, \ u \to k u) \), \( k \neq 0 \) and \( (x \to \alpha x, \ t \to \alpha t) \), \( \alpha \neq 0 \). These transformations change, however, the value of the small parameter \( \epsilon \). Taking \( k = \sqrt{c_3/c_5} \) and \( \alpha = c_3 \), the initial data (1.7) can be reduced to the form

\[
x = -u^3 - c u^4 - u^5,
\]  

(1.8)
where the dimensionless parameter $c$ is chosen in the form

$$c = \frac{c_4}{\sqrt{c_3 c_5}} = \frac{\sqrt{5}}{2} \frac{f^{IV}(\xi)}{\sqrt{f^{''}(\xi)f^{V}(\xi)}}.$$ 

The monotonicity condition requires $c^2 \leq 15/4$. From the above considerations the polynomial (1.8) represents the generic one-parameter deformation of a monotone decreasing analytic initial data with cubic inflection point.

For cubic initial data the solution $u(x, t, \epsilon)$ of (1.1) is approximately given by equation (1.2) inside the oscillatory zone and by the solution of the Burgers equation outside this zone. For general initial data one have to use higher genera analogues of the Whitham equations (1.4) to approximately described the solution $u(x, t, \epsilon)$ of (1.1) [6],[7]. Lax and Levermore [6], and Venakides [7] described, for certain particular classes of initial data, the dispersive shock waves in the frame of the zero-dispersion asymptotics for the solution of the inverse scattering problem of KdV. According to their results, to the solution $u(x, t, \epsilon)$ as $\epsilon \to 0$ it corresponds a decomposition of the $(x, t)$ plane into a number of domains $D_g, g = 0, 1, \ldots$. In the domain $D_g$ the principal term of the asymptotics is given by the $g$-phase solution of the KdV equation [8]

$$u(x, t) \sim \Phi \left( \frac{S_1(x, t)}{\epsilon}, \ldots, \frac{S_g(x, t)}{\epsilon}; u_1(x, t), \ldots, u_{2g+1}(x, t) \right),$$

(1.9)

where the functions $S_j(x, t)$ satisfy the equations [10]

$$\frac{\partial S_j}{\partial x} = k_j(\vec{u}(x, t)), \quad \frac{\partial S_j}{\partial t} = \omega_j(\vec{u}(x, t)), \quad j = 1, \ldots, g,$$

(1.10)

and the formula

$$u(x, t) = \Phi (k_1 x + \omega_1 t + \phi_1, \ldots, k_g x + \omega_g t + \phi_g; u_1, \ldots, u_{2g+1})$$

for constant values of the parameters $u_1, \ldots, u_{2g+1}, k_j = k_j(\vec{u})$ and $\omega_j = \omega_j(\vec{u})$ and for arbitrary $\phi_j, j = 1, \ldots, g$, gives the family of the so-called $g$-phase exact solutions of KdV for $\epsilon = 1$ [8]. The wave parameters in (1.10) depend on the functions $u_1(x, t) > \cdots > u_{2g+1}(x, t)$ which satisfy the system of equations

$$\frac{\partial u_i}{\partial t} + \lambda_i(u_1, u_2, \ldots, u_{2g+1}) \frac{\partial u_i}{\partial x} = 0, \quad i = 1, \ldots, 2g + 1, \quad g \geq 0.$$ 

(1.11)

For a given $g$ the system (1.11) is called $g$-phase Whitham equations. The Whitham equations is the collection of all systems (1.11) for $g \geq 0$. The zero-phase Whitham
equation coincides with the Burgers equation. For \( g > 0 \) the speeds \( \lambda_i(u_1, u_2, \ldots, u_{2g+1}), i = 1, 2, \ldots, 2g+1 \), depend on \( u_1, \ldots, u_{2g+1} \), through complete hyperelliptic integrals of genus \( g \). For this reason the \( g \)-phase solution is also called solution of genus \( g \). When the \( g \)-phase solution comes to a point of gradient catastrophe, it should be continued by the \((g + 1)\)-phase solution. The main problem is to glue together the solutions of the Whitham equations for different \( g \) in order to produce a \( C^1 \)-smooth curve in the \( x - u \) plane evolving smoothly with \( t \). This problem is referred to as initial value problem of the Whitham equations. The genus \( g(x, t) \) is a piecewise constant on the \((x, t)\) plane (see Fig. 1.2). For generic initial data it is not whether the genus \( g(x, t) \) is a bounded function.

The algebraic geometric description of these equations for \( g > 1 \), was first derived in [9]. Dubrovin and Novikov developed a geometric Hamiltonian theory for the Whitham equations [10]. Based on this theory, Tsarev [11] was able to prove that, for each \( g \), equations (1.11) can be solved by a generalized method of characteristic. This method was put into an algebro-geometric setting by Krichever [12]. Recently Dubrovin proposed a variational principle for the Whitham equations. In [13] the minimizer of a functional defined on a certain infinite-dimensional space formally solves the initial value problem of the Whitham equations for each point of the \((x, t)\) plane.

We use this variational principle to arrive at the main result of this paper. This is a complete description of the bifurcation diagram in the \( x - t \) plane of the one-parameter family of initial value problems for (1.11). For the purpose we use the result in [14] where it is shown that for polynomial initial data of degree \( 2N + 1 \) the solution of the Whitham equations has genus at most equal to \( N \). In our case of fifth degree polynomial initial data, the solution of the initial value problem turns out to be glued from solutions of the Whitham equations of genera \( g = 0 \), \( g = 1 \) and \( g = 2 \). The various topological type of bifurcation diagrams are described in terms of the number of points of gradient catastrophe that appear in the solution of the Whitham equation and other particular points which we call double-leading edge, double trailing edge and leading-trailing edge.

This paper is organized as follows. On Sec. 2 we give some basic results about the theory of the Whitham equations and we describe Dubrovin’s variational principle for the Whitham equations. In Sec. 3 we describe the phase transition boundaries for general initial data. In Sec. 4 we study the function \( g = g(x, t) \) for the one parameter family of initial data (1.8).
2 Preliminaries on the theory of the Whitham equations

The $g$-phase Whitham equations (1.11) are constructed in the following way.

On the Riemann surface

$$S_g := \left\{ (P, \mu(r)), \mu^2 = \prod_{i=1}^{2g+1} (r - u_i) \right\}, \quad u_1 > u_2 > \cdots > u_{2g+1}, \quad u_i \in \mathbb{R}, \quad (2.1)$$

we define the abelian differentials of the second kind $\sigma_k$, $k \geq 0$ with a pole at infinity of order $2k+2$ [13]

$$\sigma_k^g(r) = \frac{P_k^g(r, \vec{u})}{2\mu} dr, \quad P_k^g(r, \vec{u}) = r^{k+g} + \alpha_{k+g-1}r^{k+g-1} + \cdots + \alpha_0, \quad (2.2)$$

where $\vec{u} = (u_1, \ldots, u_{2g+1})$ and the constants $\alpha_i = \alpha_i(\vec{u})$, $i = 0, 1, \ldots, k + g - 1$, are uniquely determined by the conditions:

$$\int_{u_{2j+1}}^{u_{2j}} \sigma_k^g(r) = 0, \quad j = 1, 2, \ldots, g, \quad \sigma_k^g(r) = \left[ \frac{r^{k-\frac{1}{2}}}{2} + O(r^{-\frac{3}{2}}) \right] dr \text{ for large } |r| \quad (2.3)$$

We use the notation

$$dp_g(r) = \sigma_0^g(r), \quad dq_g(r) = 12 \sigma_1^g(r). \quad (2.4)$$

In literature the differentials $dp_g(r)$ and $dq_g(r)$ are called quasi momentum and quasi-energy respectively [10]. The speeds $\lambda_i(\vec{u})$ of the $g$-phase Whitham equations (1.11) are given by the ratio [3]:

$$\lambda_i(\vec{u}) = \left. \frac{dq_g(r)}{dp_g(r)} \right|_{r=u_i}, \quad i = 1, 2 \ldots 2g+1. \quad (2.5)$$

In the case $g = 0$, we have $dp_0(r) = \frac{dr}{\sqrt{r-u}}$ and $dq_0(r) = \frac{12r-6u}{\sqrt{r-u}} dr$ respectively, so that the zero-phase Whitham equation (1.11) coincides with the Burgers equation $u_t + 6uu_x = 0$. We consider monotone decreasing polynomial initial data of the form

$$x = f(u) = c_0 + c_1u + \cdots + c_{2N}u^{2N} + c_{2N+1}u^{2N+1}. \quad (2.6)$$
For such initial data the solution of the Burgers equation is obtained by the method of characteristic \[2\] and is given by the expression

\[ x = 6t \, u + f(u), \quad (2.7) \]

This solution is globally well defined only for \(0 \leq t < t_c\) where the time \(t_c = \frac{1}{6} \min_{u \in \mathbb{R}} [-f'(u)]\) is the time of gradient catastrophe of the solution. The breaking is cause by an inflection point in the initial data. For \(t > t_c\) the solution of the Whitham equations is obtained gluing together \(C^1\)-smoothly solutions of different genera. The \(g\)-phase solution \(u_1 > u_2 > \cdots > u_{2g+1}\) which is attached \(C^1\) smoothly to the \((g + 1)\)-phase solution is obtained by a generalization of the method of characteristic and reads \[11,17\]

\[ x = \lambda_i(\bar{u}) \, t + w_i(\bar{u}), \quad w_i(\bar{u}) = \left. \frac{ds_g(r)}{dp_g(r)} \right|_{r=u_i}, \quad i = 1, \ldots, 2g + 1, \quad (2.8) \]

where \(\lambda_i(\bar{u})\) has been defined in \((2.3)\) and the differential \(ds_g(r)\) is given by the expression

\[ ds_g(r) = \frac{2^{N+1}}{(2k - 1)!!} \frac{2^k k!}{c_k \sigma^g(r)}, \quad (2.9) \]

where \(\sigma^g\) has been defined in \((2.2)\) and the \(c_k\)'s have been defined in \((2.6)\).

The solution \((2.8)\) can be written in the equivalent algebro-geometric form. Consider the differential

\[ \Omega_g(r) = xdp_g(r) - tdq_g(r) - ds_g(r), \quad (2.10) \]

where \(dp_g, dq_g\) and \(ds_g\) have been defined in \((2.4)\) and \((2.9)\) respectively. Then the \(g\)-phase solution is given by the relation \[12,17\]

\[ \Omega_g(r) \mid_{r=u_i} = 0, \quad i = 1, 2, \ldots, 2g + 1. \quad (2.11) \]

The solution \(u_1 > u_2 > \cdots > u_{2g+1}\) of the \(g\)-phase Whitham equations \([1.11]\) is implicitly defined as a function of \(x\) and \(t\) by the equations \((2.8)\) or \((2.11)\). The solution is uniquely defined only for those \(x\) and \(t\) such that the functions \(u_i(x,t)\) are real and \(\partial_x u_i(x,t), i = 1, \ldots, 2g + 1,\) are not vanishing.
### 2.1 Variational principle for the Whitham equations

The solution of the Whitham equations, for a given initial data, can be written as the
minimizer of a functional defined on a certain infinite-dimensional space [13]. Let us
first consider the zero-phase equation. The characteristic equation
\[ x = 6tu + f(u), \]
where \( f(u) \) is the polynomial (2.6), can be consider as the minimum of the function
\[ G^0_{[x,t,c]}(u) = xu - 3tu^2 - F(u) \]  
(2.12)
where \( x, t \) and \( \vec{c} = (c_0, c_1, \ldots, c_{2N+1}) \) are parameters and \( F'(u) = f(u) \). The minimum
is unique only for \( t \leq t_c \). In [13] the function of type (2.12) is extended to a functional
onto the moduli space of all hyperelliptic Riemann surfaces with real branch points
in such a way that the absolute minimum of this functional gives the solution of the
initial value problem of the Whitham equations (1.11) for the initial data (2.6).

First we define the restriction of this functional on the hyperelliptic Riemann sur-
faces of genus \( g \) with real branch points \( u_1 > u_2 > \cdots > u_{2g+1} \). The restriction is a
function depending on the branch points. This function is build as follows. Consider
the asymptotic expansion of the quasi-momentum \( dp_g \) (see e.g. [8])
\[ dp_g(r) = \left[ \frac{1}{2\sqrt{r}} - \frac{1}{2\sqrt{r}} \sum_{k=0}^{\infty} \frac{(2k+1)I_k}{2^{2k+1}r^{k+1}} \right] dr. \]  
(2.13)
The coefficients \( I_k = I_k(u_1, u_2, \ldots, u_{2g+1}) \), are the so called KdV integrals and are
smooth functions of the parameter \( u_1 > u_2 > \cdots > u_{2g+1} \).

**Theorem 2.1** [13] On the Riemann surface \( S_g \) consider the function
\[ G^g_{[x,t,c]}(\vec{u}) = -xI_0(\vec{u}) + 3tI_1(\vec{u}) - \sum_{k=0}^{2N+1} \frac{k!}{2^{k}(2k-1)!!}c_kI_k(\vec{u}), \]  
(2.14)
depending on the variable \( u_1, u_2, \ldots, u_{2g+1} \) and on the parameters \( x, t \) and \( \vec{c} \). Then the equations
\[ \frac{\partial}{\partial u_i} G^g_{[x,t,c]}(\vec{u}) = 0, \quad i = 1, 2, \ldots, 2g + 1, \]  
(2.15)
are equivalent to the equations \( \Omega_g(r) |_{r=u_i} = 0, \quad i = 1, \ldots, 2g + 1 \), where \( \Omega_g(r) \) has
been defined in (2.10).
Remark 2.2 The function \( G_{x,t,\vec{c}}^g(\vec{u}) \) can also be written in the form
\[
G_{x,t,\vec{c}}^g(\vec{u}) = \text{Res}_{z=\infty} \left[ dp_g(z)(\mathcal{F}(z) - 2\sqrt{z}x) \right], \quad \mathcal{F}(z) = \int_0^z \frac{6t\xi + f(\xi)}{\sqrt{z - \xi}} d\xi, \tag{2.16}
\]
and \( \text{Res}_{z=\infty} \) is the residue evaluate at infinity.

To extend the function (2.14) defined on the hyperelliptic surfaces \( S_{g \geq 0} \) to a functional on the infinite dimensional space \( M \) of all hyperelliptic Riemann surfaces \( S_{g \geq 0} \) and their degeneration, we follow [13].

Construct the space \( M \) inductively starting from \( M_0 = \mathbb{R} \). We denote \( u \) the coordinate in \( M_0 \). Define now
\[
M_g = M_0^g \cup L_{g-1} \cup T_{g-1},
\]
where
\[
M_0^g = \left\{ (u_1, u_2, \ldots, u_{2g+1}) \in \mathbb{R}^{2g+1} \mid u_1 > u_2 > \cdots > u_{2g+1} \right\},
\]
\[
L_{g-1} = \bigcup_{j=1}^g L_{g-1}^j, \quad T_{g-1} = \bigcup_{j=1}^g T_{g-1}^j.
\]
Each space \( L_{g-1}^j \) is attached to the component of the boundary of \( M_0^g \) where
\[
\lim_{j \to j+1} f_{g+1} \to 0, \quad j = 1, 2, \ldots, g;
\]
each space \( T_{g-1}^j \) is attached to the component of the boundary of \( M_0^g \) where
\[
\lim_{j \to j+1} f_{g+1} \to 0, \quad j = 1, 2, \ldots, g.
\]

For reasons that will become clear later, we call leading edge each boundary component \( L_{g-1}^j, j = 1, \ldots, g \) and trailing edge each boundary component \( T_{g-1}^j, j = 1, \ldots, g \).

A smooth functional \( f \) on \( M \) is a sequence of function \( f_g \) defined on every \( M_g \) and having certain good analytic properties on the boundaries of the space \( M_g \) (for details cfr [13]). These properties are illustrated in the following example.

Example. Let us consider the Riemann surface of genus \( g + 1 \)
\[
\tilde{\mu}^2 = (r - v - \sqrt{\delta})(r - v + \sqrt{\delta}) \prod_{j=1}^{2g+1} (r - u_j), \quad v \in \mathbb{R}, \quad u_1 > u_2 > \cdots > u_{2g+1}.
\]
where \( v \neq u_j, \ j = 1, \ldots, 2g + 1 \) and \( 0 < \delta \ll 1 \). We denote with \( dp_{g+1}(r, \delta) \), the quasi momentum defined on this surface and with \( G_{[x,t,c]}^q(\vec{u}, \delta) \) the function equivalent to \( \eqref{2.14} \) defined on this surface and depending on the variables \( v - \sqrt{\delta}, v + \sqrt{\delta}, u_1, \ldots, u_{2g+1} \). We study the behavior of \( dp_{g+1}(r, \delta) \) in the limit \( \delta \to 0 \).

First we consider a trailing edge of the boundary of the space \( M_{g+1} \), namely the case in which \( v \in (u_{2j}, u_{2j-1}), 1 \leq j \leq g, u_{2g+2} = -\infty \). We have that \( \eqref{18} \)

\[
dp_{g+1}(r, \delta) = dp_g(r) + \frac{\delta}{2} dp_g(v) \frac{\partial}{\partial v} \omega_v(r) + O(\delta^2) \tag{2.17}
\]

where \( dp_g(v) = \left. \frac{dp_g(r)}{dr} \right|_{r=v} \) and \( O(\delta^2)/\delta^2 \) is a meromorphic differential with zero residues. The differential \( \omega_v(r) \) is a normalized Abelian differential of the third kind with poles at the points \( P^\pm (v) = (v, \pm \mu(v)) \) with residue \( \pm 1 \) respectively, namely

\[
\omega_v(r) = \frac{dr}{\mu(r) - v} - \sum_{k=1}^g \frac{r^{g-k}dr}{\mu(r)} B_k(v), \quad \mu^2(r) = \prod_{i=1}^{2g+1} (r - u_i). \tag{2.18}
\]

The constants \( B_k(v) \) are uniquely determined by the normalization conditions

\[
\int_{u_{2k+1}}^{u_{2k}} \omega_v(r) = 0, \quad k = 1, \ldots, g.
\]

When \( v \in (u_{2j+1}, u_{2j}), \ j = 0, \ldots, g, \) and \( u_0 = +\infty \), we are considering one of the leading edge of the space \( M_{g+1} \).

In this case the expansion of the differential \( dp_{g+1}(r, \delta) \) for \( \delta \to 0 \) reads \( \eqref{18} \)

\[
dp_{g+1}(r, \delta) = dp_g(r) + \rho \omega_v(r) \int_{p^- (v)}^{p^+ (v)} dp_g(\xi) + O(\rho^2) \tag{2.19}
\]

where \( \omega_v(r) \) has been defined in \( \eqref{2.18} \) and \( \rho = -\frac{1}{\log \delta} \). The correction in the right hand side of \( \eqref{2.17} \) contains also exponentially small terms like \( \exp(-1/\rho) \).

Combining equations \( \eqref{2.11}, \eqref{2.16} \) and \( \eqref{2.17} \) we obtain the expressions of the function \( G_{[x,t,c]}^{q+1}(\vec{u}, \delta) \) near the trailing edges of the space \( M_{g+1} \), namely

\[
G_{[x,t,c]}^{q+1}(\vec{u}, \delta) = G_{[x,t,c]}^q(\vec{u}) + \frac{\delta}{2} dp_g(v) \Omega_g(v) + O(\delta^2), \tag{2.20}
\]

where the differential \( \Omega_g(v) \) has been defined in \( \eqref{2.10} \) and here and below \( \Omega_g(v) = \left. \frac{\Omega_g(r)}{dr} \right|_{r=v} \). 

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The behavior of the function \( G^{g+1}_{[x,t,c]}(\bar{u}, v, \delta) \) near the leading edges of the space \( M_{g+1} \), is obtained from (2.11), (2.16) and (2.19):

\[
G^{g+1}_{[x,t,c]}(\bar{u}, v, \delta) \simeq G^g_{[x,t,c]}(\bar{u}) + \rho \int P^+(v) \Omega_g(r) dr + \rho \int P^-(v) \Omega_g(r) dr.
\]

The function \( G^g_{[x,t,c]}(u_1, u_2, \ldots, u_{2g+1}) \) defined on \( M_0 \) can be extended to a smooth functional on the space \( M \). In [13] the extension is built proving that the \( I_k(u_1, \ldots, u_{2g+1}), k \geq 0 \), can be extended to smooth functionals on \( M \). We state the following theorem

**Theorem 2.3** [13] The functional

\[
G_{[x,t,c]} = -xI_0 + 3tI_1 - \sum_{k=0}^{2N+1} \frac{k!}{2k(2k-1)!!} c_k I_k
\]

is a \( C^\infty \) smooth functional on \( M \). Its absolute minimum is a \( C^1 \)-smooth multi-valued function of \( x \) depending \( C^1 \)-smoothly on the parameters \( t, c_1, \ldots, c_{2(N+1)} \).

If the absolute minimum \((u_1(x,t), \ldots, u_{2g+1}(x,t))\) belongs to \( M_0 \) for certain values of the parameters, then it satisfies the \( g \)-phase Whitham equations.

### 3 Study of the function \( g = g(x,t) \)

In this section we study the behaviour of the genus \( g = g(x,t) \) of the solution of the Whitham equations (1.11) for the one-parameter family of initial data (1.8). In [14] it was shown that for polynomial initial data of degree \( 2N+1 \) the solution of the Whitham equations has genus at most equal to \( N \). Hence in the case of initial data (1.8) the solution will have genus at most equal to 2. For this reason we need only to describe the locus of points of the \( x - t \) plane where the genus \( g(x,t) \) increases from zero to one or two and from one to two. To this end we study the behaviour of the functional (2.22) on the phase transition boundaries of the space \( M_0, M_1 \) and \( M_2 \). The restriction of the functional (2.22) on \( M_0 \) has the form (2.12). Thus the solution of the Burgers equation (i.e. the Whitham equation for \( g = 0 \))

\[
x = 6tu + f(u)
\]

minimizes the functional (2.22) for all \( x \) and \( t < t_c \) where \( t_c \) is the time of gradient catastrophe. Let us find for which \( x \) and \( t \) the solution of (3.1) is also minimal along directions transversal to \( M_0 \subset M \). For the purpose we consider the embedding of \( M_0 \)
as the component of the boundaries of $M_1$ ($T_0$, $L_0$) and we study the behaviour of the functional (2.22) near the trailing edge and leading edge of the space $M_1$.

First we need the following theorem.

**Theorem 3.1** [22] On the solution $u_1(x, t) > u_2(x, t) > \cdots > u_{2g+1}$ of the $g$-phase Whitham equations, the differential $\Omega_g(r)$ defined in (2.10) reads

$$\Omega_g(r) = -\mu(r)[\partial_r \Psi_g(r, \bar{u}) + \sum_{i=1}^{2g+1} \partial_{u_i} \Psi_g(r, \bar{u})]dr \quad (3.2)$$

where the function $\Psi_g(r, \bar{u})$ is given by the expression

$$\Psi_g(r, \bar{u}) = -\text{Res}_{z=\infty} \left[ \frac{\mathcal{F}(z)dz}{\mu(z)(z-r)} \right], \quad \mathcal{F}(z) = \int_0^z \frac{6t\xi + f(\xi)}{\sqrt{z-\xi}} d\xi, \quad (3.3)$$

and $\text{Res}_{z=\infty}$ is the residue evaluate at infinity.

Using (2.20), (3.2) and (3.3) we can simplify the functional (2.22) near the trailing edge of the space $M_1$ to the form

$$G^1_{[x, t, \bar{c}]}(u, v, \delta) = G^0_{[x, t, \bar{c}]}(u) + \frac{\delta}{2}(-6t - \partial_v q(u, v) - \partial_u q(u, v)) + O(\delta^2), \quad (3.4)$$

where $v < u$, $G^0_{[x, t, \bar{c}]}(u)$ has been defined in (2.12) and

$$q(u, v) = -\text{Res}_{z=\infty} \left[ \frac{\left( \int_0^z f(\xi) \frac{d\xi}{\sqrt{z-\xi}} \right) dz}{(z-v)\sqrt{z-u}} \right] = \frac{1}{4\sqrt{2}} \int_{-1}^{1} \frac{f\left( \frac{1+m}{2}v + \frac{1-m}{2}u \right)}{\sqrt{1-m}} dm. \quad (3.5)$$

Analogously on the leading edge of the space $M_1$ we obtain

$$G^1_{[x, t, \bar{c}]}(u, v, \delta) = G^0_{[x, t, \bar{c}]}(u) + 16\rho(v-u)^2(-2t - \partial_v q(u, v)) + O(\rho^2). \quad (3.6)$$

If, for fixed $(u, t)$, the $\delta$-correction of (3.4) is positive for every $v \leq u$, then the minimizer belongs to $M_0$. If it is negative for some values of $v \leq u$, then the minimizer belongs to the inner part of $M_1$. The points $v < u$ belong to the trailing edge of the space $M_1$ if the triple $(t, u, v)$ is a zero and a minimum with respect to $v$ of the $\delta$ correction of (3.4). From these considerations we obtain the following lemma.
Lemma 3.2 The points $v$ and $u$, $v < u$, belong to the boundary $T_0$ of the space $M_1$ if $(t, u, v)$ satisfy the system

$$
\begin{aligned}
6t + \partial_v q(u, v) + \partial_u q(u, v) &= 0 \\
\partial_v (\partial_v q(u, v) + \partial_u q(u, v)) &= 0 \\
(\partial_v)^2(\partial_v q(u, v) + \partial_u q(u, v)) &< 0.
\end{aligned}
$$

The curve $x = x^-(t)$ on the $x-t$ plane where the genus increases from zero to one is determined solving the above system together with the zero-phase solution $x = 6tu + f(u)$. We will call the curve $x^-(t)$ trailing edge.

Analogous consideration can be done for the leading edge.

Lemma 3.3 The points $v$ and $u$, $v > u$, belong to the boundary $L_0$ iff $(t, u, v)$ satisfy the system

$$
\begin{aligned}
-2t + \partial_u q &= 0 \\
\partial_v \partial_u q &= 0 \\
(\partial_v)^2 \partial_u q &< 0.
\end{aligned}
$$

The curve $x = x^+(t)$ on the $x-t$ plane where the genus increases from zero to one is determined solving the above system together with the zero-phase solution $x = 6tu + f(u)$. We will call the curve $x^+(t)$ leading edge.

Remark 3.4 Both systems (3.7) and (3.8) in the limit $v \to u$ become

$$
\begin{aligned}
6t + f'(u) &= 0 \\
f''(u) &= 0 \\
f'''(u) &< 0
\end{aligned}
$$

where $f(u)$ is the initial data (2.6). If the above system admits a real solution $(u, t)$ such that

$$
\begin{aligned}
2t + \partial_v q &\leq 0 \quad \forall \ v \neq u \\
6t + \partial_v q + \partial_u q &\leq 0 \quad \forall \ v \neq u,
\end{aligned}
$$

then $u$ belongs to the boundary $T_0 \cap L_0$ of the space $M_1$. The corresponding point $(x, t, u)$ is a point of gradient catastrophe of the solution of the Burgers equations.
3.1 Boundary between the one-phase solution and the two-phase solution

In the following, we write explicitly the equations determining the phase transition boundary between the one-phase solution and the two phase solution.

Let be $dp_1(r)$ the quasi-momentum restricted to the inner part of $M_1$,

$$dp_1(r) = \frac{r + \alpha_0}{\sqrt{(r - u_1)(r - u_2)(r - u_3)}} dr, \quad \alpha_0 = -\frac{\int_{u_3}^{u_2} \frac{r dr}{\sqrt{(r - u_1)(r - u_2)(r - u_3)}}}{\int_{u_3}^{u_2} \frac{dr}{\sqrt{(r - u_1)(r - u_2)(r - u_3)}}} \quad (3.11)$$

From (2.16) the restriction on the inner part of $M_1$ of the functional $G_{[x,t,c]}$ reads

$$G_{[x,t,c]}(u_1, u_2, u_3) = \text{Res}_{z=\infty} [dp_1(z)(F(z) - 2\sqrt{zx})]. \quad (3.12)$$

**Proposition 3.5** The critical points of (3.12) on the space of elliptic curves are given by the equations for $u_1 > u_2 > u_3$,

$$x = \lambda_i t + w_i, \quad i = 1, 2, 3 \quad (3.13)$$

where $\lambda_i = \lambda_i(u_1, u_2, u_3)$ has been defined in (1.5) and

$$w_i(u_1, u_2, u_3) = \frac{\prod_{j \neq i}^3 (u_i - u_j)}{\alpha_0 + u_i} \sigma(u_1, u_2, u_3) \quad (3.14)$$

$$\sigma(u_1, u_2, u_3) = -\text{Res}_{z=\infty} \left[ \frac{\left( \int_0^z f(\xi) d\xi \right) dz}{\sqrt{(z - u_1)(z - u_2)(z - u_3)}} \right].$$

**Proof:** Using the formula [13]

$$\frac{\partial}{\partial u_i} \alpha_0 = -\frac{1}{2} + \frac{1}{2} \prod_{j \neq i}^3 (u_i - u_j)^2, \quad i = 1, 2, 3 \quad (3.15)$$

equations (3.13) and (3.14) are recovered straightforward. □

The trailing edge of the phase transition boundary between the one-phase solution and the two-phase solution is determined from (2.20), (3.2) and (3.3). When $g = 1$ (3.3) reads

$$\Psi_1(r, \bar{u}) = -\text{Res}_{z=\infty} \left[ \frac{F(z) dz}{(z - r)\sqrt{(z - u_1)(z - u_2)(z - u_3)}} \right], \quad (3.16)$$
so that (2.20) simplifies to the form

$$G^2_{[x,t,\vec{c}]}(\vec{u}, v, \delta) \approx G^1_{[x,t,\vec{c}]}(\vec{u}) + \frac{\delta}{2}(v + \alpha_0)(-\partial_v \Psi_1(v, \vec{u}) - \sum_{i=1}^{3} \partial_{u_i} \Psi_1(v, \vec{u}))$$

(3.17)

where $G^1_{[x,t,\vec{c}]}(\vec{u})$ has been defined in (3.12) and $\alpha_0$ has been defined in (3.11). The point $v \in (-\infty, u_3)$ or $v \in (u_2, u_1)$. In this case $v + \alpha_0 \neq 0$. From the $\delta$ correction of (3.17) we obtain the equations determining the trailing edge, namely

$$\begin{cases}
\partial_v \Psi_1(v, \vec{u}) + \sum_{i=1}^{3} \partial_{u_i} \Psi_1(v, \vec{u}) = 0 \\
\partial_v (\partial_v \Psi_1(v, \vec{u}) + \sum_{i=1}^{3} \partial_{u_i} \Psi_1(v, \vec{u})) = 0 \\
(\partial_v)^2[(v + \alpha_0)(\partial_v \Psi_1(v, \vec{u}) + \sum_{i=1}^{3} \partial_{u_i} \Psi_1(v, \vec{u})) < 0
\end{cases}$$

(3.18)

where $\vec{u} = (u_1(x,t), u_2(x,t), u_3(x,t))$ is determined from (3.13). A similar system can be obtained for the leading edge using (3.6).

In the limit $v \to u_l$, $1 \leq l \leq 3$, system (3.18) reads

$$\begin{cases}
\partial_{u_l} (\partial_{u_1} \sigma + \partial_{u_2} \sigma + \partial_{u_3} \sigma) = 0 \\
(\partial_{u_l})^2(\partial_{u_1} \sigma + \partial_{u_2} \sigma + \partial_{u_3} \sigma) = 0 \\
(\partial_{u_l})^3[(\alpha_0 + u_l)(\partial_{u_1} \sigma + \partial_{u_2} \sigma + \partial_{u_3} \sigma)] < 0
\end{cases}$$

(3.19)

where $\sigma = \sigma(u_1, u_2, u_3)$ has been defined in (3.14). If the above system admit a solution $x^*, t^*$, then the one-phase solution $u_1(x^*, t^*) > u_2(x^*, t^*) > u_3(x^*, t^*)$ determined by (3.13) has on the $u_l$ branch a point of gradient catastrophe, namely $\frac{\partial}{\partial x} u_l(x,t) \bigg|_{i=t^*} \to \infty$ and $\frac{\partial^2}{\partial x^2} u_l(x,t) \bigg|_{i=t^*} \to \infty$.

**Theorem 3.6** If the initial data (2.6) satisfies the condition $f'''(u) < 0$, then the solution of the one-phase Whitham equations has no point of gradient catastrophe.

**Proof:** Using the following expression for $\sigma(u_1, u_2, u_3)$ [19]

$$\sigma(u_1, u_2, u_3) = \frac{1}{2\sqrt{2\pi}} \int_{-1}^{1} \int_{-1}^{1} f \left( \frac{1+s}{2} \frac{1+t}{2} u_1 + \frac{1-s}{2} \frac{1+t}{2} u_2 + \frac{1-t}{2} u_3 \right) \frac{dt \, ds}{\sqrt{(1-t)(1-s^2)}}$$

(3.20)
it can be easily checked that the second equation of (3.19) cannot be satisfied for any real \( u_1, u_2 \) and \( u_3 \).

**Remark 3.7** Using a different approach Tian has proved [19] that for smooth monotone decreasing initial data satisfying the condition \( f'''(u) < 0 \), the solution of the one phase Whitham equations exists for all \( t > 0 \).

### 4 Bifurcation diagrams

We study the bifurcation diagram of the solution of the Whitham equations for the one parameter family of initial data

\[
f_c(u) = -(u^3 + cu^4 + u^5), \quad c^2 \leq \frac{15}{4}.
\]  

(4.21)

For such initial data, the functional (2.22) reads

\[
G_{[x,t,c]} = -xI_0 + 3tI_1 - \frac{1}{20} I_3 - \frac{1}{70} cI_4 - \frac{1}{252} I_5.
\]  

(4.22)

The restriction of this functional on \( M_0 \) (that is on the curve \( \mu^2 = r - u, \ u \in \mathbb{R} \)) has the form

\[
G^0_{[x,t,c]}(u) = xu - 3tu^2 + \frac{u^4}{4} + c\frac{u^5}{5} + \frac{u^6}{6},
\]  

(4.23)

thus the minimizer given by

\[
x = 6ut - u^3 - cu^4 - u^5
\]  

(4.24)

solves the Burgers equation until the time of gradient catastrophe \( t_0 = 0 \). At later times the minimizer of \( G_{[x,t,c]} \) may belong to \( M_0, M_1 \) or \( M_2 \).

The Burgers equation has another point of gradient catastrophe if (3.9) and (3.10) with the initial data (4.21) are satisfied, that is

\[
\begin{align*}
-6t + 3u^2 + 4cu^3 + 5u^4 &= 0 \\
6u + 12cu^2 + 20u^3 &= 0 \\
6 + 24cu + 60u^2 &> 0 \\
6t + \partial_v q_c + \partial_u q_c &\leq 0 \ \forall \ v \neq u, \ v \in \mathbb{R} \\
2t + \partial_u q_c &\leq 0 \ \forall \ v \neq u, \ v \in \mathbb{R},
\end{align*}
\]  

(4.25)
where

\[
q_c(u, v) = -\frac{1}{35} (5 u^3 + 6 u^2 v + 8 u v^2 + 16 v^3) - \frac{c}{315} (35 u^4 + 40 u^3 v + 64 u^2 v^2 + 128 v^4) - \frac{1}{693} (63 u^5 + 70 u^4 v + 80 u^3 v^2 + 96 u^2 v^3 + 128 u v^4 + 256 v^5).
\]

The solutions of (4.25) are obviously \( u_0 = 0, \ t_0 = 0 \) and

\[
\begin{cases}
  u_1 = -\frac{3}{10} (c + \sqrt{c^2 - \frac{10}{3}}) & \text{for } c > 0, \\
  u_1 = -\frac{3}{10} (c - \sqrt{c^2 - \frac{10}{3}}) & \text{for } c < 0, \\
  t_1 = \frac{1}{6} (3u_1^2 + 4cu_1^3 + 5u_1^4) > 0,
\end{cases}
\]

with the constraints

\[
-\frac{\sqrt{15}}{2} \leq c \leq -\frac{1}{6} \sqrt{5(13 + 5\sqrt{7})}, \quad \frac{1}{2} \sqrt{\frac{1}{11}(75 + 21\sqrt{15})} \leq c \leq \frac{\sqrt{15}}{2},
\]

which are obtained from the last two inequalities of (4.25). We put

\[
\nu_1 = -\frac{1}{6} \sqrt{5(13 + 5\sqrt{7})} \approx -1.90863 \quad (4.29)
\]

\[
\nu_4 = \sqrt{\frac{1}{44}(75 + 21\sqrt{15})} \approx 1.88494 \quad (4.30)
\]

because these numbers occur frequently in the following. The \( x_1 \) coordinate relative to \( (t_1, u_1) \) is recovered from the equation (4.24). For \( c \) positive \( x_1 > 0 \) and for \( c \) negative \( x_1 < 0 \).

Here and below all the numerical results are obtained using Mathematica 3.0 for Solaris Copyright 1988-97 Wolfram Research, Inc.

### 4.1 Trailing edges

The equations determining the trailing edge of the space \( M_1 \) are given by system (3.7), namely

\[
\begin{cases}
  6t + \partial_v q_c + \partial_u q_c = 0 \\
  \partial_v (\partial_v q_c + \partial_u q_c) = 0 \\
  (\partial_v)^2 (\partial_v q_c + \partial_u q_c) < 0,
\end{cases}
\]

where \( q_c(u, v) \) has been defined in (4.26). The above system can be easily solved numerically getting \( u = u(t) > v = v(t) \). Substituting the value of \( u = u(t) \) in the
zero phase solution \( [4.24] \) we get, in the \( x - t \) plane, the curve \( x^-(t) \) which describes the phase transition from the \( g = 0 \) solution to the \( g = 1 \) solution. For some values of \( c \) and \( t \) system \( [4.31] \) has two different solutions \( u_1(t) > v_1(t) \) and \( u_2(t) > v_2(t) \) and correspondingly we have two curves \( x_1(t) \) and \( x_2(t) \) in the \( x - t \) plane. This occurs for \( c \) in the intervals

\[
-\frac{\sqrt{15}}{2} \leq c < \nu_1, \quad \nu_4 < c \leq \frac{\sqrt{15}}{2}
\]

where \( \nu_1 \) and \( \nu_4 \) have been defined in \([4.29]\) and \([4.30]\) respectively.

A particular solution of system \([4.31]\) is the one which describes the double-trailing edge (cfr. below). System \([4.31]\) determines the zeros which are also a maxima with respect to \( v \) of the polynomial \( 6t + \partial_u q_c + \partial_v q_c \). Since \( \partial_u q_c + \partial_v q_c \) is a fourth degree polynomial in the variable \( v \) with negative leading coefficients, it cannot have more than two maxima. When system \([4.31]\) admits such two maxima \((\tilde{t}, \tilde{u}, \tilde{v}) \) and \((\tilde{t}, \tilde{u}, \tilde{w}) \) with \( \tilde{t} > 0, \tilde{v} \leq \tilde{u} \) and \( \tilde{w} \leq \tilde{u} \), we are in a situation of a double trailing edge (see Figure 4.1). The degenerate Riemann surface describing this situation is given by the equation \( \mu^2 = (r - \tilde{u})(r - \tilde{v})^2(r - \tilde{w})^2 \) and represents the boundary \( T_1^1 \cap T_2^1 \) of the space \( M_2 \). The curve \( x^-(t) \) describing the trailing edge in the \( x - t \) plane looses the \( C^1 \)-smoothness at the point \( t = \tilde{t} \). Such two solutions exists for \( c \) in the intervals

\[
-\frac{\sqrt{15}}{2} \leq c \leq \nu_1, \quad \nu_3 \leq c \leq \frac{\sqrt{15}}{2}
\]

Figure 4.1: Double trailing edge.
where \( \nu_1 \) has been defined in (4.29) and

\[
\nu_3 = \sqrt{\frac{5}{2} + \frac{5}{18} \left( \frac{25}{18} \right)^\frac{3}{2} \left( (27 - 7\sqrt{21})^\frac{3}{2} + (27 + 7\sqrt{21})^\frac{3}{2} \right)} \simeq 1.78167. \tag{4.32}
\]

For \( c = \nu_3 \) the two solutions \((\hat{t}, \hat{u}, \hat{v})\) and \((\bar{t}, \bar{u}, \bar{w})\) become coincident, namely \( \hat{v} = \bar{w} \).

For \( c = \nu_1 \) the solution \((\hat{t}, \hat{u}, \hat{w})\) satisfies \( \bar{w} = \hat{u} \). The point \((\hat{v}, \bar{w} = \hat{u})\) belongs to the component \( L_1^1 \cap T_1^1 \cap T_2^1 \) of the boundary of \( M_2 \) and the corresponding degenerate Riemann surface is \( \mu^2 = (r - \hat{u})^3(r - \hat{v})^2 \). The solution of the Burgers equation has in \((\hat{x}, \hat{t}, \hat{w} = \hat{u})\) a point of gradient catastrophe in correspondence of the trailing edge.

For \( c \) in the interval \( \nu_1 < c < \nu_3 \) there exists just one real solution \( v(t, c) < u(t, c) \) of system \((4.31)\) for all \( t > 0 \). Substituting \( u(c, t) \) in the zero-phase equation we get a curve \( x(t) \) in the \( x - t \) plane which is smooth for all \( t > 0 \).

### 4.2 Leading edges

The leading edges of the space \( M_1 \) are determined from system \((3.8)\) namely

\[
\begin{cases}
2t + \partial_u q_c(u, v) = 0 \\
\partial_t \partial_u q_c(u, v) = 0 \\
(\partial_v)^2 \partial_u q_c(u, v) < 0
\end{cases}
\tag{4.33}
\]

where \( q_c(u, v) \) has been defined in (4.26). For some values of \( t > 0 \) and for \( c \) in the intervals

\[-\frac{\sqrt{15}}{2} \leq c < \nu_1, \quad \nu_4 < c \leq \frac{\sqrt{15}}{2}\]

system \((4.33)\) has two different solutions \( u_1(t) < v_1(t) \) and \( u_2(t) < v_2(t) \). Correspondingly we have two leading edges \( x_1^+(t) \) and \( x_2^+(t) \) in the \( x - t \) plane which describe the phase transition from the \( g = 0 \) solution to the \( g = 1 \) solution.

We also consider the situation of a double leading edge as shown in Fig.4.2. That is we study for which values of \( c \) system \((4.33)\) has two real solutions \((\hat{t}, \hat{u}, \hat{w})\) and \((\bar{t}, \bar{u}, \bar{v})\) satisfying the constraints \( \hat{v} \geq \bar{v} \), \( \bar{w} \geq \hat{w} \) and \( \bar{t} > 0 \). These solutions belong to the boundary component \( L_1^1 \cap L_2^1 \) of the space \( M_2 \) which is described by the degenerate Riemann surface \( \mu^2 = (r - \hat{u})(r - \hat{v})^2(r - \hat{w})^2 \). This double solution occurs only for \( c \) in the intervals

\[\nu_4 \leq c \leq \sqrt{\frac{15}{4}}, \quad -\sqrt{\frac{15}{4}} \leq c < \nu_2,\]
where \( \nu_4 \) has been defined in (4.30) and

\[
\nu_2 = -\sqrt{\frac{5}{2} + \frac{35}{11} \cos \frac{\pi - \theta}{3}} \approx -1.85585, \quad \cos \theta = \frac{11}{14}.
\]  

(4.34)

For \( c = \nu_4 \), these two solutions become coincident, namely \( \tilde{v} = \tilde{w} \).

For \( c = \nu_2 \) the solution \((\tilde{t}, \tilde{u}, \tilde{w})\) satisfies the relation \( \tilde{w} = \tilde{u} \). The corresponding point \((\tilde{x}, \tilde{t}, \tilde{w} = \tilde{u})\) is a point of gradient catastrophe of the Burgers equation and it occurs at the leading edge.

When \( c \in (\nu_2, \nu_4) \), there exists just one real solution \( v(t, c) > u(t, c) \) of system (4.31) for all \( t > 0 \). In this case there exists just a single leading edge for all \( t > 0 \).

![Figure 4.2: Double leading edge.](image)

![Figure 4.3: Leading-Trailing edge.](image)

### 4.3 Leading-trailing edge

We call leading-trailing edge the situation in which a leading edge and a trailing edge have the same \((x, t)\) coordinates as shown in Fig.4.3. In other words a leading trailing edge is determined by a solution \((\tilde{t}, \tilde{u}, \tilde{w})\) of system (4.33) and \((\tilde{t}, \tilde{u}, \tilde{v})\) of system (4.31) where \( \tilde{w} \geq \tilde{v} \). We check numerically that such two solutions exist for \( c \) in the intervals

\[-\sqrt{\frac{15}{4}} \leq c < \nu_1, \quad \nu_4 < c \leq \sqrt{\frac{15}{4}}.\]
4.4 Point of gradient catastrophe of the one-phase solution

The solution of the one-phase Whitham equation with the initial data (4.21) has a point of gradient catastrophe if the correspondent system (3.19) has a solution. From Theorem 3.6 it follows that for $c^2 < \frac{5}{2}$, the solution of the one-phase Whitham equations with initial data (4.21) has no point of gradient catastrophe.

We solve numerically systems (3.19) and (3.13) for the initial data (4.21) restricting $c$ in the intervals $-\sqrt{\frac{15}{2}} \leq c \leq -\sqrt{\frac{5}{2}}$ and $\sqrt{\frac{5}{2}} \leq c \leq \sqrt{\frac{15}{2}}$.

We find that there exists a point of gradient catastrophe for the solution of the one-phase Whitham equations on the $u_1$-branch for $c \in [\nu_1, \nu_2)$, where $\nu_1 \simeq -1.90863$ and $\nu_2 \simeq -1.85585$ have been defined in (4.29) and (4.34) respectively, (see figure 4.4).

There is a point of gradient catastrophe in the solution of the one-phase Whitham equations on the $u_2$-branch for $c$ in the intervals $[-\sqrt{\frac{15}{2}}, \nu_2)$ and $(\nu_3, \sqrt{\frac{15}{2}})$, see figure 4.4.

There is a point of gradient catastrophe in the solution of the one phase Whitham equations on the $u_3$-branch for $c \in (\nu_3, \nu_4]$, where $\nu_3 \simeq 1.78167$ and $\nu_4 \simeq 1.88494$ have been defined in (4.29) and (4.30) respectively, see figure 4.4.

**Remark 4.1** A point of gradient catastrophe appears in the solution of the one-phase Whitham equations whenever the solution is changing genus. If the point of gradient catastrophe appears on the $u_1$ or $u_3$-branch, the corresponding solution will increase genus by one near this point.

If the point of gradient catastrophe appears in the one-phase solution on the $u_2$-branch it means that the two-phase solution has just disappeared. We have checked numerically this fact solving system (3.18) in a neighborhood of each point of gradient catastrophe. System (3.18) determines the phase transition boundary between the one-phase solution and the two-phase solution. When a point of gradient catastrophe $(x^*, t^*)$ appears on the $u_1$ or $u_3$-branch, then system (3.18) admits a solution for $t > t^*$. Namely a two-phase oscillatory zone is developing. On the contrary if a point of gradient catastrophe $(x^*, t^*)$ appears on the $u_2$ branch, then system (3.18) does not have admissible solutions for $t > t^*$.

4.5 Bifurcation diagrams in the $x - t$ plane

We draw in the $(x, t)$ plane the various topological types of bifurcation diagrams of the solution of the Whitham equations with initial data $x = -u^3 - c u^4 - u^5$, $c^2 \leq \frac{15}{4}$. We draw with a solid line the points of the $(x, t)$ plane where the genus increases from zero to one and with a dashed line the points where the genus increases from one to two. We have several cases.
1) $\nu_4 < c \leq \sqrt{\frac{15}{4}}$; there exists a second breakpoint for the zero-phase solution in $x_1 > 0$, $t_1 > 0$; there exists a point of gradient catastrophe on the $u_2$-branch of the one-phase solution for $t > t_1$; there are a double trailing edge, a leading-trailing edge and a double leading edge hence the bifurcation diagram of the genus $g(x,t)$ is

point of gradient catastrophe of the zero-phase solution

2a) $c = \nu_4$; the Burgers equation has a second point of gradient catastrophe in correspondence of the leading edge, there is a point of gradient catastrophe on the $u_2$-branch of the one-phase solution and there is a double trailing edge.

2b) $c = \nu_1$; the solution of the Burgers equation has a second point of gradient catastrophe $(x_1, t_1)$ in correspondence of the trailing edge, there exists a point of gradient catastrophe on the $u_2$-branch of the one-phase solution for $t > t_1$ and there is a double leading edge.
3a) \( \nu_1 < c < \nu_2 \); there are points of gradient catastrophe in the one-phase solution on the \( u_1 \)-branch and \( u_2 \)-branch for \( t > 0 \) and there is double leading edge.

3b) \( \nu_3 < c < \nu_4 \); there are points of gradient catastrophe in the one-phase solution on the \( u_3 \)-branch and one on the \( u_2 \)-branch and there is a double trailing edge.

4) \( \nu_2 \leq c \leq \nu_3 \); there is only the point \((x = 0, t = 0, u = 0)\) of gradient catastrophe of the solution of the Burgers equation. The curves \( x^-(t) \) and \( x^+(t) \) are \( C^1 \)-smooth for all \( t > 0 \).

For \( c = \nu_3 \) the curve \( x^-(t) \) looses the \( C^1 \)-smoothness for a single value of \( t > 0 \). For \( c = \nu_2 \) the curve \( x^+(t) \) looses the \( C^1 \)-smoothness for a single value of \( t > 0 \).

5) \(-\sqrt{\frac{15}{4}} \leq c < \nu_1 \); there is a second breakpoint in the solution of the Burgers equation for \( x_1 < 0 \) and \( t_1 > 0 \); there is a point of gradient catastrophe in the one-phase solution on the \( u_2 \)-branch for \( t > t_1 \); there are a double leading edge, a leading-trailing edge and a double trailing edge.
4.6 Conclusion

In this paper we have studied the bifurcation diagram of the solution of the Whitham equations for a one-parameter family of initial data. On the $x - t$ plane we have characterized the bifurcation diagrams in terms of particular singular points which we have called double leading edge, double trailing edge, leading-trailing edge and points of gradient catastrophe. The $x - t$ plane is split in regions where the solution of the Whitham equations has genus $g = 0, 1, 2$. The genus $g = 2$ solution survives only for a finite time. This result is in agreement with the result in [22], [20] where it is shown that the solution of the Whitham equations with monotone polynomial initial data has a universal one-phase self-similar asymptotics. This implies that the large time behaviour of the bifurcation diagram of the solution of the Whitham equations can be described analytically. Indeed, as follows from [22], for all times $t > T$, where $T > 0$ is a sufficiently large time, the solution of the Whitham equations for the one parameter family of initial data (1.8) is of genus one inside the interval $x^-(t) < x < x^+(t)$, where $x^-(t) \simeq -16.82t^{\frac{4}{5}}$ and $x^+(t) \simeq 1.58t^{\frac{4}{5}}$. It is of genus zero outside this interval.

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Figure 4.4: On the plots a) and b) the time of gradient catastrophe of the one-phase solution is plotted as a function of $c$. The circles represent the points of gradient catastrophe on the $u_2$ branch, the triangles are the points of gradient catastrophe on the $u_3$ branch and the squares are the points of gradient catastrophe on the $u_1$ branch. The solid lines represent the time of gradient catastrophe of the zero-phase solution as a function of $c$. 