Self-organization by topological constraints: hierarchy of foliated phase space

Z. Yoshida
Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa, Japan

ABSTRACT
Topological constraints are the key to an understanding of how a macrosystem can be different from the simple sum of microelements. The emergence of a macrostructure is a reflection of reduced degrees of freedom, because the realization of all degrees of freedom, on an equal footing, maximizes the entropy and eliminates any inhomogeneity. Here, we formulate topological constraints as foliation of the phase space. A macrohierarchy is, then, a leaf (submanifold) embedded in the total phase space. A plasma confined in a magnetic field is invoked for explaining the organizing principle. In a magnetosphere, the plasma self-organizes to a state with a steep density gradient. The resulting nontrivial structure has maximum entropy in an effective phase space that is reduced by adiabatic invariants and the corresponding coarse-grained angle variables. Formally, the adiabatic invariants may be viewed as Casimir invariants, and the effective phase space is the Casimir leaf. Conversely, we may deem any Casimir invariant as an adiabatic invariant derived by separating a ‘micro’ conjugate variable; the topological constraint of the Casimir invariant can be unfrozen when the conjugate variable is recovered. We put this interpretation into the test in the context of ideal magnetohydrodynamics which is an infinite-dimensional Hamiltonian system obeying an infinite number of topological constraints. Diverse structures realized in plasmas are described as creations on different hierarchy of foliation.

CONTACT
Z. Yoshida yoshida@pp.k.u-tokyo.ac.jp

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1. Introduction

A macrosystem often manifests its capability of self-organizing interesting structures which are far beyond what we may imagine in a microworld. A biological body is a popular example of highly organized complex systems, in which hierarchical structures are programmed, by gene, to emerge. Diverse theories have been developed from the view point of nonequilibrium systems; yet, it is not clear whether some ‘mechanical’ models can hit the right nail on the understanding of life. Here, we direct our attention to simpler examples of self-organization, which are at the opposite pole of programmed systems. We discuss collective systems of particles (while the systems of our interest are purely ‘physical,’ they may yet reflect some interesting aspects of life, which are not programmed); the structures of our interest are vortexes which are ubiquitous in nature (appearing as tornadoes, hurricanes, planetary magnetospheres, galaxies, etc.), but the mechanisms of creation and evolution are still far from complete understanding. Their ‘blue prints’ are not found in the microelements, i.e. particles. Therefore, we are lead to an idea that such a structure is determined by the space, not by the materials composing the system. When the space is somehow ‘distorted,’ the motion or distribution of matter may be different from those occurring on a flat space.

Geometrically, a distorted space can be identified as a submanifold (often called a leaf) embedded in a larger space. To put it another way, topological constraints limit an ‘effective space’ to a subset of the total space of the microscopic degrees of freedom. Such topological constraints play the key role in the self-organizing phenomena.\(^1\) In fact, if all possible degrees of freedom are actualized on equal footing, the state of equipartition may bear no specific structure. By suppression of microscopic degrees of freedom, a macrosystem can be different from the simple sum of elements.

Let us see more explicitly how a distorted space (leaf) influences the dynamics. At micro, the energy (Hamiltonian) is usually equivalent to the norm, measuring the distance from the origin (vacuum). Here, we mean by ‘micro’ an elementary system; it does not necessarily correspond to a small scale. For example, the energy of a classical harmonic oscillator is the $L^2$ (Euclid or Lebesgue) norm of the phase space, that of a classical electromagnetic (EM) wave in vacuum is the $L^2$ norm of the EM field, or that of a quantum free particle is the $H^1$ (Sobolev) norm of the Schrödinger field. However, the energy (norm) on a curved leaf may have a nontrivial distribution; the ‘effective energy’ of the constrained system is deformed by the distortion of the effective space, bifurcating various nontrivial equilibrium points (see Figure 1).

We may also delineate the effect of distorted space in the context of entropy. The process of self-organization is often deemed as an antithesis of the maximum entropy principle. However, the entropy evaluated on the effective space may still increase while the entropy on a reference frame decreases. The invariant measure
Figure 1. Foliated phase space. The effective space on which the state vector can move is a leaf (submanifold) embedded in an a priori phase space (here $\mathbb{R}^3$, but it may be an arbitrary, even infinite-dimensional Hilbert space). A leaf is selected by the initial condition. The ‘effective energy’ (here the Euclid norm measuring the distance from the origin) of the constrained state vector is determined by the shape of the leaf. (b) and (c), respectively, shows the effective energy on the upper and lower leaves.

(metric) of the effective phase space may be distorted with respect to that of a reference frame (which is usually the $L^2$ measure), and then, the maximum-entropy (i.e. equipartition) distribution function on the effective phase space is multiplied by a Jacobian weight when we evaluate the corresponding distribution function on the reference frame. What appears as a structure is, therefore, the distortion of the effective space.

We put the aforementioned picture of self-organization to the test on some different examples of physical systems. We start with a short review of Hamiltonian formalism (Section 2), whereby mechanical constraints are represented by degeneracy of Poisson algebra [2], and the macrohierarchy is formulated as Casimir leaves. In Section 3, we invoke a particle picture (a finite-dimensional Hamiltonian mechanics) to find prototype topological constraints in adiabatic invariants [4]. By drawing attention to the adiabatic invariants, the notion of scale enters into our view [5]. Casimir leaves are identified as leaves of adiabatic invariants, where the microdegrees of freedom are coarse-grained by averaging over periodic orbits [6,7]. The spontaneous confinement of plasma in a magnetosphere [8] is shown to be a natural process under the constraints on adiabatic invariants of magnetized particles [9]. In Section 4, we discuss vortical structures in plasmas. The determinant topological constraints are the helicities [10–12]. The Taylor relaxed states [13,14] and nonlinear Alfvén waves [15,16], as well as other structures with higher topological complexities are discussed, on a common basis, as creations on different kinds of Casimir leaves [17].
2. Hamiltonian formalism of macrohierarchy

Let $X$ be a phase space, which is assumed to be a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$. We denote the state vector by $\mathbf{z} \in X$ ($X$ may be a function space, and then we denote the state vector by $\mathbf{u}$). An observable is a real-valued smooth function (functional) on $X$. We define a bilinear product of observables such that

$$\{F, G\} := \langle \partial_z F(\mathbf{z}), \mathcal{J} \partial_z G(\mathbf{z}) \rangle, \quad (1)$$

where $\partial_z$ is the gradient in $X$, and $\mathcal{J}$ is the Poisson operator. We allow $\mathcal{J}$ to be a function of $\mathbf{z}$ on $X$, and then write it as $\mathcal{J}(\mathbf{z})$. We assume that $\{\cdot,\cdot\}$ is antisymmetric and satisfies the Jacobi identity; then it is a Poisson bracket. The Poisson algebra of observables will be denoted by $C^\infty(\{\cdot,\cdot\})(X)$. Given a Hamiltonian $H(\mathbf{z}) \in C^\infty(\{\cdot,\cdot\})(X)$, the evolution of an observable $F(\mathbf{z})$ obeys

$$\frac{d}{dt} F = \{F, H\}. \quad (2)$$

Equivalently, the state vector $\mathbf{z}$ obeys Hamilton’s equation of motion

$$\frac{d}{dt} \mathbf{z} = \mathcal{J}(\mathbf{z}) \partial_z H(\mathbf{z}). \quad (3)$$

A canonical Hamiltonian system is endowed with a symplectic Poisson operator where

$$\mathcal{J}_c := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (4)$$

However, our interest is in noncanonical systems endowed with Poisson operators that are degenerate, i.e. $\text{Ker}(\mathcal{J}(\mathbf{z}))$ contains nonzero elements (and its dimension may change as a function of $\mathbf{z}$). Suppose that $\mathcal{J}(\mathbf{z}) \xi = 0$, i.e. $\xi \in \text{Ker}(\mathcal{J}(\mathbf{z}))$. Since $\mathcal{J}(\mathbf{z})$ is antisymmetric, $0 = \langle \mathcal{J}(\mathbf{z}) \xi, \xi \rangle = -\langle \xi, \mathcal{J}(\mathbf{z}) \xi \rangle \forall \xi$, i.e. $\xi \in \text{Coker}(\mathcal{J}(\mathbf{z}))$. Therefore, $\langle \xi, \mathcal{J}(\mathbf{z}) \partial_z H \rangle = 0$ for an arbitrary Hamiltonian $H$, implying that the state vector, obeying the equation of motion (3), cannot go to the direction of $\xi$. A Hamiltonian system that has a nontrivial $\text{Ker}(\mathcal{J})$ is said to be noncanonical.

A functional $C(\mathbf{z})$ such that $\{G, C\} = 0 \forall G$ is called a Casimir invariant; by the definition of the Poisson bracket (1), this means that $\partial_z C(\mathbf{z}) \in \text{Ker}(\mathcal{J}(\mathbf{z})).$ From (2), such $C(\mathbf{z})$ is a constant of motion. The level-sets of $C(\mathbf{z})$, which we call Casimir leaves, foliate the phase space $X$. The intersection of the Casimir leaves is the effective phase space, on which the state vector $\mathbf{z}$ can move.

Representing a topological constraint by a scalar (Casimir invariant) brings about a number of advantages because we may deal with it on a Hamiltonian (or an action). Multiplying a Casimir invariant by a constant (Lagrange multiplier) and adding it to the Hamiltonian, we may define an energy-Casimir function [2,3,17–20], and apply it to the study of bifurcation and stability (see Section 4.2). Or, we can view a Casimir invariant as a momentum (or an action variable)
and supply a conjugate coordinate (or an angle) variable to make a canonical pair. By including the new conjugate variable into the Hamiltonian, we can ‘unfreeze’ the Casimir invariant, and remove the topological constraint [6,7].

3. Magnetic confinement and adiabatic invariants

3.1. Magnetosphere

There are two different types of structures created by particles; one is epitomized by clusters and the other is by vortexes. Beneath them are the two different types of forces acting on particles. The potential forces (gravity and electrostatic forces) and magnetic forces, respectively, correspond to the two classes (centrifugal force is a cousin of gravity, while the Coriolis force belongs to the latter). Gravity confines particles and creates star (as described by the virial theorem). Magnetic fields also confine plasmas and create, for example, stellar magnetospheres. However, the mechanism of ‘magnetic confinement’ is not so simple.

Since the magnetic force directs perpendicular to the velocity of a charged particle, it does not change the energy; hence, the Boltzmann distribution, for instance, is independent of the magnetic field. Then, how can the magnetic confinement be possible? The essential role must be played by the space, not by the energy.

The self-organization of magnetospheric plasmas (the naturally occurring ones such as the planetary magnetospheres [8,24,25], as well as their laboratory simulations [9,26–29]) is a process driven by some spontaneous fluctuations that violate the constancy of angular momentum. In a strong enough dipole magnetic field, the canonical angular momentum \( P_\theta \) is dominated by the magnetic part \( q \psi \): the charge multiplied by the flux function (in the \( r-\theta-z \) cylindrical coordinates, \( \psi = rA_\theta \), where \( A_\theta \) is the \( \theta \) component of the vector potential). Therefore, the conservation of \( P_\theta \approx q \psi \) restricts the particle motion to the magnetic surface (level-set of \( \psi \)). It is only via random fluctuations that the particles can diffuse across magnetic surfaces [30–33]. Although the diffusion is usually a process that diminishes gradients, a steep gradient is created in the density distribution. In Figure 2, we show an experimental evidence, given by a laboratory magnetosphere’ created on the RT-1 device [9], of spontaneous generation of density gradient, resulting in self-organized confinement of high-temperature plasma. While the increase in density gradient is seemingly contradicting the entropy principle, it is consistent with the entropy maximization if the entropy is evaluated on a proper frame. To formulate the appropriate entropy, we have to start with the Hamiltonian mechanics to identify the relevant phase space.

3.2. Topological constraints imposed by adiabatic invariants

Here, we study the Hamiltonian mechanics of magnetized particles, and demonstrate how topological constraints emerge to constrain (foliate) the phase space. The Hamiltonian of a charged particle is a sum of the kinetic energy and the
Figure 2. Experimental observation of self-organized confinement in the laboratory magnetosphere RT-1. The electron density profiles (measured by interferometers [34]) are plotted on a poloidal cross section of the magnetic dipole system: (a) In the initial phase of discharge (plasma is produced by electron-cyclotron heating by 8.2 GHz microwave), density has a relatively flat distribution. (b) After 1.5 ms, the electrons diffuse inward to create a ‘radiation belt.’ At $r = 0.25$ m, $z = 0$ m, a superconducting ring magnet is levitated to produce a dipole magnetic field. Electrons have a temperature $\sim 10$ keV, and the local beta value (the ratio of the thermal energy density to the magnetic energy density) is approximately 1.

Potential energy:

$$H = \frac{m}{2} v^2 + q\phi,$$

where $v := (P - qA)/m$ is the velocity, $P$ is the canonical momentum, $(\phi, A)$ is the electromagnetic four-potential, $m$ is the particle mass and $q$ is the charge. Here, we may treat electrons and ions equally. In later discussion, we will neglect $\phi$ assuming charge neutrality, but generalization to a nonneutral plasma derives interesting results [27,35]). Denoting by $v_\parallel$ and $v_\perp$ the parallel and perpendicular (with respect to the local magnetic field) components of the velocity, we may write

$$H = \frac{m}{2} v_\perp^2 + \frac{m}{2} v_\parallel^2 + q\phi.$$

The velocities are related to the mechanical momentum as $p := mv$, $p_\parallel := mv_\parallel$, and $p_\perp := mv_\perp$.

In a strong magnetic field, $v_\perp$ can be decomposed into a small-scale cyclotron motion $v_c$ and a macroscopic guiding-center drift motion $v_d$. The periodic cyclotron motion $v_c$ can be quantized to write $(m/2)v_c^2 = \mu \omega_c(x)$ in terms of the magnetic moment $\mu$ and the cyclotron frequency $\omega_c(x)$; the adiabatic invariant $\mu$ and the gyration phase $\vartheta_c := \omega_c t$ constitute an action-angle pair.

In the standard interpretation, in analogy with the Landau levels in quantum theory, $\omega_c$ is the energy level and $\mu$ is the number of quasi-particles (quantized periodic motions) at the corresponding energy level.

A dipole magnetic field can be written as $B = \nabla \psi \times \nabla \theta$ ($\psi$ is the flux function, and $\theta$ is the toroidal angle). Adding a longitudinal coordinate $\zeta$ along magnetic field lines, we define a magnetic coordinate system $(\psi, \zeta, \theta)$. The macroscopic part of the perpendicular kinetic energy is expressed as $mv_\perp^2/2 = (P_\theta -$
\( q_\psi^2/(2m r^2) \), where \( P_\theta \) is the canonical angular momentum in the \( \theta \) direction and \( r \) is the radius from the geometric axis. In terms of the canonical-variable set \( z := (\vartheta_c, \mu, \zeta, p_\parallel, \theta, P_\theta) \) the Hamiltonian of the guiding center (or, the quasi-particle) becomes

\[
H_c = \mu \omega_c + \frac{1}{2m} p_\parallel^2 + \frac{1}{2m} \left( \frac{P_\theta - q_\psi}{r^2} \right)^2 + q \phi.
\] (7)

Notice that the energy of the cyclotron motion has been written as \( \mu \omega_c(x) \) by coarse-graining the gyro-phase \( \vartheta_c \).

Now, we formulate the \textit{macrohierarchy} on which the thermal equilibrium creates a structure [4]. The adiabatic invariance of the magnetic moment \( \mu \) imposes a topological constraint on the motion of particles. The scale hierarchy is equivalent to a foliation of the phase space by this constraint. To explain the connection between the adiabatic invariant and Casimir leaves, we start with the general (micro–macro total) formulation, and then separate the microscopic action-angle pair \( \mu - \vartheta_c \); the \textit{macrophase space} is the remaining submanifold immersed in the general phase space, which we delineate as a leaf of \( \mu \) viewed as a \textit{Casimir invariant}.

The Poisson bracket on the total phase space, spanned by the canonical variables \( z := (\vartheta_c, \mu, \zeta, p_\parallel, \theta, P_\theta) \), is \( \{F, G\} := \langle \partial_z F, J \partial_z G \rangle \), where \( J \) is the canonical symplectic matrix:

\[
J := \begin{pmatrix} J_c & J_c \\ J_c & -J_c \end{pmatrix}, \quad J_c := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\] (8)

Vacant entries are zeros. Liouville’s theorem determines the invariant measure \( d^6 \vartheta \), by which we obtain the standard Boltzmann distribution. To extract the macrohierarchy, we separate the microscopic variable \( \vartheta_c \) and reduce the state vector \( z \) to \( z_{nc} = (\mu, \zeta, p_\parallel, \theta, P_\theta) \). The gradient \( \partial_{z_{nc}} F \) is now a five-dimensional vector. We define a degenerate \( 5 \times 5 \) Poisson matrix

\[
J_{nc} := \begin{pmatrix} 0 & J_c \\ J_c & J_c \end{pmatrix}.
\] (9)

The modified Poisson bracket \( \{F, G\}_{nc} = \langle \partial_{z_{nc}} F, J_{nc} \partial_{z_{nc}} G \rangle_{nc} \) determines the kinematics on the macrohierarchy; the corresponding kinetic equation \( \partial_t f + \{H_c, f\}_{nc} = 0 \) reproduces the familiar drift-kinetic equation (for example [36]).

The nullity of \( J_{nc} \) makes the Poisson bracket \( \{, \}_{nc} \) noncanonical. Evidently, \( \mu \) is a Casimir invariant (more generally \( C = g(\mu) \) with \( g \) being any smooth function). The level-set of \( \mu \), a leaf of the Casimir foliation, identifies what we may call the \textit{macrohierarchy}. By applying Liouville’s theorem to the Poisson bracket
Figure 3. Density distribution (contours) and the magnetic field lines (level-sets of $\psi$) in the neighborhood of a point dipole. (a) The equilibrium on the leaf of $\mu$-foliation. (b) The equilibrium on the leaf of $\mu$ and $J_{\parallel}$-foliation.

\[ \{, \}_n, \] the invariant measure on the macrohierarchy is \( d^4z = d^6z/(2\pi d\mu) \), the total phase-space measure modulo the microscopic measure.

### 3.3. Thermal equilibrium on macrohierarchy

The most probable state on the macroscopic ensemble must maximize the entropy with respect to this invariant measure. The variational principle is set up with the help of Lagrange multipliers; we maximize the entropy \( S = -\int f \log f d^6z \) for a given particle number \( N = \int f d^6z \), a quasi-particle number \( M = \int \mu f d^6z \), and an energy \( E = \int H_c f d^6z \), to obtain the distribution function

\[
f = f_\alpha := Z^{-1} e^{-(\beta H_c + \alpha \mu)}, \tag{10}
\]

where \( \alpha, \beta, \) and \( \log Z - 1 \) are, respectively, the Lagrange multipliers on \( M, E, \) and \( N \). In this grand-canonical distribution function, \( \alpha/\beta \) is the chemical potential associated with the quasi-particles.\(^3\)

The factor \( e^{-\alpha \mu} \) in \( f_\alpha \) yields a direct \( \omega_c \) dependence of the configuration-space density:

\[
\rho = \int f_\alpha \frac{2\pi \omega_c}{m} d\mu d\nu d\nu_{\parallel} \propto \frac{\omega_c(x)}{\beta \omega_c(x) + \alpha}. \tag{11}
\]

Notice that the Jacobian \( (2\pi \omega_c/m)d\mu \) multiplying the macroscopic measure \( d^4z \) reflects the distortion of the macrophase space (Casimir leaf) caused by the inhomogeneous magnetic field. Figure 3(a) shows the density distribution and the magnetic field lines.

The same scenario applies when we further separate the bounce action-angle variables assuming the constancy of the bounce action \( f_{\parallel} \)\(^4\); see Figure 3(b). The scale hierarchy is selected by the space–time scale of driving fluctuations; an adiabatic invariance holds when the corresponding frequency is higher than the frequency range of the fluctuations.\(^4\)
The diffusion process on the Casimir leaf can be formulated on the basis of ergodic hypothesis. The key is the identification of the proper coordinates on which an appropriate diffusion operator must be formulated. By the separation (coarse-graining) of the adiabatic action-angle variables, the Casimir leaf has a natural symplectic structure, by which we can derive a Fokker-Planck equation [39].

4. Vortical structures and helicity

4.1. Topological constraints on ideal plasma

Here we consider an ideal MHD plasma obeying

\begin{align}
\partial_t \rho &= - \nabla \cdot (\rho \mathbf{V}), \\
\partial_t \mathbf{V} &= - (\nabla \times \mathbf{V}) \times \mathbf{V} + \rho^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla (V^2/2 + h), \\
\partial_t \mathbf{B} &= \nabla \times (\mathbf{V} \times \mathbf{B}),
\end{align}

where \( \rho \) is the density, \( \mathbf{V} \) is the velocity, \( \mathbf{B} \) is the magnetic field, and \( h \) is the specific enthalpy. We assume a barotropic relation \( h = h(\rho) \). All variables are written in the standard Alfvén units. The plasma is contained in a smoothly bounded domain \( \Omega \subset \mathbb{R}^3 \), satisfying the boundary conditions \( \mathbf{n} \cdot \mathbf{V} = 0 \) and \( \mathbf{n} \cdot \mathbf{B} = 0 \) (\( \mathbf{n} \) is the unit normal vector onto the boundary). When \( \Omega \) is multiply connected, we assume that the magnetic flux in each handle is a constant of motion.

The ideal MHD equations can be casted into the Hamiltonian form (3) in terms of the state vector \( u = (\rho, \mathbf{V}, \mathbf{B}) \) (the phase space \( X \) is the \( L^2 \) Hilbert space) with the Poisson operator and Hamiltonian [2,40]:

\[
\mathcal{J}_{\text{MHD}} := \begin{pmatrix}
0 & -\nabla \cdot \\
-\nabla & -\rho^{-1} (\nabla \times \mathbf{V}) \times \mathbf{V} & \rho^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} & 0 \\
0 & \nabla \times (\mathbf{V} \times \mathbf{B}) & \rho^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} & 0
\end{pmatrix},
\]

\[
H := \int_{\Omega} \left[ \rho \left( \frac{1}{2} |\mathbf{V}|^2 + U(\rho) \right) + \frac{1}{2} |\mathbf{B}|^2 \right] \, d^3x,
\]

where \( U(\rho) \) is the thermal energy \( (h(\rho) = d(\rho U(\rho)/d\rho \text{ is the enthalpy}).

The Poisson operator \( \mathcal{J}_{\text{MHD}} \) defines three independent Casimir invariants:

\[
C_1 := \int_{\Omega} \rho \, d^3x,
\]

\[
C_2 := \int_{\Omega} \mathbf{V} \cdot \mathbf{B} \, d^3x,
\]

\[
C_3 := \frac{1}{2} \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, d^3x,
\]

where \( \mathbf{A} \) is the vector potential of \( \mathbf{B} \) (the gauge is arbitrarily fixed). Physically, \( C_1 \) is the total mass, \( C_2 \) is the cross helicity (measuring the total circulation along
magnetic field lines [7]), and $C_3$ is the magnetic helicity (measuring the total link, twist, and writhe of magnetic field lines [41,42]); cf. Remarks 1 and 2.

The minimum-energy point in the total phase space is the ‘vacuum’ $u = \{\rho, V, B\} = 0$. However, if we assume a finite magnetic flux in the handles of $\Omega$, $B = 0$ is no longer possible; then a vacuum magnetic field ($\nabla \times B = 0$) minimizes the energy $H$. The flux-constraint is a kind of topological constraint (in fact, the flux is a Casimir invariant [7]), which ‘creates’ nontrivial equilibrium points. In the following subsections, we will observe rich examples of equilibrium points created on the Casimir leaves.

**Remark 1:** (topological charge) In this article, we use the term ‘topology’ in general, basic meaning; topology is a ‘system of differences’ that gives identity to each element of a set. If we call an element of the set (space) a ‘particle’ (or a pure state), we may consider a variety of definitions of particles, depending on which topology is of our subject. A topological constraint on dynamics means the constancy of some identity of particles throughout the evolution, i.e. a homotopy of the map relating different times. Being a ‘Hamiltonian system’ is already a strong topological constraint by the demand that the symplectic 2-form must be conserved – Liouville’s theorem (invariant measure) is a direct consequence, which implies the constancy of a ‘particle’ as a point in the phase space. We may give a similar meaning of ‘particle conservation’ to a Casimir invariant; the attribute represented by a Casimir invariant is often called a charge. Remember that the adiabatic invariant can be regarded as the number of quasi-particles (Section 3.2). The helicity is the total charge of links; a ‘particle’ that bears such a topological charge is not a simple point, but is a pair of linked unit-flux loops co-moving with the fluid (by Kelvin’s circulation law, the flux is constant) [43]. Not only the total helicity (the integral over the total domain) but also every local helicity, defined by an integral over an arbitrary co-moving domain containing the loops, is constant. See also Remark 2 for an alternative interpretation of the helicity.

**Remark 2:** (canonization and symmetry) By representing the vector fields $V$ and $B$ in terms of Clebsch parameters, we can rewrite the MHD and Hall MHD systems as canonical Hamiltonian systems [44–49]. Then, the Casimir invariants become the Noether charges corresponding to the gauge symmetries of the Clebsch parameterization [50]. Another well-known method of canonization is the usage of Lagrangian variables (initial position of each fluid element) to represent the dynamics of fluid elements. Formulating an action by a Lagrangian of Newtonian-mechanical displacement vectors, we obtain a canonical systems of Hamilton’s equation of motion [3,51,52]. In the canonized system of Lagrangian variables, the Casimir invariants are translated differently. The total mass $C_1$ and the magnetic helicity $C_3$ are trivially conserved as the integrals of the attributes of fluid elements. Only the cross helicity $C_2$ is a nontrivial first integral; it is the Noether charge pertaining to the relabeling symmetry [53,54].
4.2. Taylor relaxed state

A dramatic finding about ‘self-organization’ was made in the purgatory of plasma turbulence that thwarted the attempt of confining high-temperature plasmas for fusion energy. Following a violent phase of turbulence, the quiescent state emerged with a strikingly simple structure of magnetic field [55]. The self-organized magnetic field turns out to be an eigenfunction of the curl operator [56], which we call a Beltrami field.

It was conjectured by Taylor [13,14] that the magnetic helicity $C_3$ is the most robust constant of motion, and hence the plasma relaxes into the minimum energy state under the constraint on $C_3$; the minimizer is called the Taylor relaxed state. The variational principle is the minimization of $H_\mu := H - \mu C_3$ ($\mu$ is a Lagrange multiplier), and the corresponding Euler-Lagrange equation reads the eigenvalue problem of the curl operator:

$$\nabla \times A = \mu A,$$  \hspace{1cm} (20)

The target functional $H_\mu = H - \mu C_3$ is nothing but the energy-Casimir functional. Therefore, the Taylor relaxed state is the equilibrium point on the Casimir leaf of $C_3$. This ‘re-interpretation’ puts the Taylor relaxed state into the perspective of Hamiltonian mechanics, by which we may develop bifurcation and stability theories [17]. Since the transformation $H \mapsto H_\mu$ does not change the dynamics (2), we may view $H_\mu$ as an equivalent Hamiltonian which may have nonmonotonic profile on some leaves (as depicted in Figure 1). The magnetic helicity $C_3$ (thus, the Lagrange parameter $\mu$) is the bifurcation parameter; see Figure 4. As shown in (20), the bifurcation problem is attributed to the spectral theory of the curl operator; the rigorous theory of [56] is readily applied for arbitrary topology, three-dimensional domain $\Omega$.

In the neighborhood of the equilibrium point $u_\mu$ of $H_\mu$, we may approximate

$$H_\mu(u_\mu + \tilde{u}) \approx H_\mu(u_\mu) + \frac{1}{2} \langle \mathcal{H}_\mu \tilde{u}, \tilde{u} \rangle$$  \hspace{1cm} (21)

with a linear operator $\mathcal{H}_\mu$. If the coercivity condition

$$c \| \tilde{u} \|^2 \leq \langle \mathcal{H}_\mu \tilde{u}, \tilde{u} \rangle$$  \hspace{1cm} (22)

holds ($c$ is a positive constant and $\| \tilde{u} \|$ is some appropriate norm of $\tilde{u}$; here, the $L^2$-norm), the quadratic form $\langle \mathcal{H}_\mu \tilde{u}, \tilde{u} \rangle$ plays the role of a Lyapunov function bounding the norm of $\tilde{u}$, i.e. the equilibrium $u_\mu$ is stable [20]. Again, the spectral theory of curl applies to the estimate (22).

4.3. Alfvén waves

Let us investigate structures created by other Casimir invariants. The stationary point of the energy-Casimir functional $H - \mu C_1 - \lambda C_2$ is given by $\rho V = \lambda B$,
Figure 4. Bifurcation of Taylor relaxed states (the figures plot the magnetic surfaces). The helicity $C_3$ plays the role of a bifurcation parameter. By changing $C_3$, we seek the equilibrium points (the stationary point of the Hamiltonian) on different Casimir leaves. For a sufficiently large helicity, bifurcated equilibrium points coexist on a common leaf (as depicted in Figure 1). (a) helical equilibrium. (b) simple-geometry equilibrium.

\[ B = \lambda V, \] and Bernoulli’s relation $\rho V^2/2 + h(\rho) = \mu$. Nontrivial solutions are given by $\lambda^2 = 1$, $\rho = \text{constant} = 1$ (by the normalization), and

\[
\begin{align*}
V &= \pm B, \\
|V| &= \text{constant}.
\end{align*}
\]  

(23)  

(24)

We can convert these equilibrium solutions to *Alfvén waves* propagating on a homogeneous ambient magnetic field $B_0$ (which can be arbitrarily chosen) [16]. Let us rewrite $B$ and $V = \pm B$ as

\[
B = B_0 + b, \quad V = \pm B_0 + v.
\]  

(25)

Boosting the coordinate $x \rightarrow x \mp B_0 t$, we find that the decomposed component (which is the wave component) satisfies

\[
\begin{align*}
\partial_t v &= - (\nabla \times v) \times v + (\nabla \times b) \times (b + B_0) - \nabla (V^2/2 + h), \\
\partial_t b &= \nabla \times [v \times (b + B_0)],
\end{align*}
\]  

(26)  

(27)

which are exactly the Alfvén wave equations with an ambient field $B_0$. Notice that the wave component $b$ and $v$ propagate with the Alfvén velocity $\pm B_0$.

The determining Equations (23) and (24) have a large set of exact nonlinear solutions, implying that Alfvén waves, propagating on a homogeneous ambient field, have a large degree of freedom; as far as the relations (23) and (24) are
satisfied, arbitrarily shaped waves propagate keeping the wave form. This is, however, an artifact created by the neglect of the dispersion effect in the ideal MHD model. In a more appropriate model, the dispersion effect will bring about a fundamental change of the picture. Here, we invoke the Hall MHD model to study nonlinear Alfvén waves under the dispersive effect.

The Hall effect deforms the ‘geometry’ of the phase space; we replace \( J_{\text{MHD}} \) by

\[
J_{\text{HMHD}} := \begin{pmatrix} 0 & -\nabla \cdot \rho \mathbf{v} & \rho \mathbf{v} \times \mathbf{B} \\ -\nabla \cdot \rho \mathbf{v} & 0 & -\rho \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} \\ \rho \mathbf{v} \times \left( \mathbf{B} \times \mathbf{B} \right) & -\rho \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} & 0 \end{pmatrix},
\]

where \( \epsilon \) is the ion skin depth (\( \epsilon = 0 \) yields the ideal MHD). With the same Hamiltonian (16), we obtain the Hall MHD equations which modifies (14) with the Hall term \( -\epsilon \nabla \times \left[ \rho \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} \right] \) to be added on the right-hand side [49, 57].

The Hall MHD Hamiltonian system has Casimir invariants \( C_1, C_3 \) (same as those of MHD) and, modifying \( C_2 \),

\[
C'_2 := \frac{1}{2} \int_{\Omega} \mathbf{P} \cdot \nabla \times \mathbf{P} \, d^3 x,
\]

where \( \mathbf{P} = \mathbf{V} + \epsilon^{-1} \mathbf{A} \) is the ion canonical helicity [58]. In what follows, we put \( \epsilon = 1 \) by normalizing the length scale by the ion skin depth.

We consider an energy-Casimir functional \( H - \mu_1 C_1 - \mu_2 C_2' - \mu_3 C_3 \). The equilibrium points are given by \( |\mathbf{V}|^2/2 + h(\rho) = \mu_1, \rho \mathbf{V} = \mu_2 (\nabla \times \mathbf{V} + \mathbf{B}), \) and \( \nabla \times \mathbf{B} - \rho \mathbf{V} = \mu_3 \mathbf{B} \). Nontrivial solutions are obtained with \( \rho = 1, |\mathbf{V}|^2 = \text{constant}, \mathbf{V} = \mu_2 (\nabla \times \mathbf{V} + \mathbf{B}), \) and \( \nabla \times \mathbf{B} - \mathbf{V} = \mu_3 \mathbf{B} \).

Combining the last two equations, we obtain

\[
\nabla \times \nabla \times \mathbf{B} - (\mu_3 + \mu_2^{-1}) \nabla \times \mathbf{B} + (1 + \mu_3/\mu_2) \mathbf{B} = 0.
\]

The same equation must be satisfied by \( \mathbf{V} \). We may rewrite (30) as

\[
(\text{curl} - \lambda_0)(\text{curl} - \lambda_1) \mathbf{B} = 0,
\]

where eigenvalues \( \lambda_0 \) and \( \lambda_1 \) are determined by \( \lambda_0 + \lambda_1 = \mu_2^{-1} + \mu_3 \) and \( \lambda_0 \lambda_1 = 1 + \mu_3/\mu_2 \). The general solution of (31) is given by a linear combination of two eigenfunctions of the curl operator [59, 60]: with \( \mathbf{G}_\ell \) such that \( (\text{curl} - \lambda_\ell) \mathbf{G}_\ell = 0 \) and arbitrary constants \( a_\ell (\ell = 0, 1) \),

\[
\mathbf{B} = a_0 \mathbf{G}_0 + a_1 \mathbf{G}_1,
\]

\[
\mathbf{V} = a_0 (\lambda_0 - \mu_3) \mathbf{G}_0 + a_1 (\lambda_1 - \mu_3) \mathbf{G}_1.
\]
From the stationary solutions, we can derive Alfvén waves. First we assume $\mu_2 = -\mu_3$ to make $\lambda_0 = 0$. Then, we may put $B_0 = a_0 G_0 = e_z$ and let it play the role of an ambient magnetic field. The residual component $b = a_1 G_1$ will become a wave by boosting the coordinate. The other eigenvalue is $\lambda_1 = \mu_2^{-1} - \mu_2$, and then, $V = \mu_2 e_z + \mu_2^{-1} b$. By Galilean-boost $(x, y, z) \mapsto (x, y, z - \mu_2 t)$, the flow field becomes $v = \mu_2^{-1} b$, which, together with the wave magnetic field $b = a_1 G_1$, satisfies the Alfvén wave Equations (26) and (27) with the Hall term $-\nabla \times [(B_0 + b) \times (\nabla \times b)]$ added on the right-hand side. Explicitly, $G_1 = (\sin (\lambda_1 \xi), \cos (\lambda_1 \xi), 0)$, which is a circularly-polarized sinusoidal wave with the wave number (Beltrami eigenvalue) $\lambda_1$ (see also [61]). The wave propagation velocity is given by $\mu_2 = (\lambda_1 \pm \sqrt{\lambda_1^2 + 4})/2$. Although the wave is a sinusoidal function, and the dispersion relation is the same as the linear wave theory, it satisfies the fully nonlinear equation with an arbitrary amplitude $a_1$; this remarkable structure is a manifestation of the nonlinearity-dispersion interplay, which is somewhat different from that of solitons.

4.4. Hierarchy of topological constraints

The ideal plasma/liquid dynamics is constrained by an infinite number of topological constraints. They produce infinitely many equilibrium solutions. Then, which solution does the physics choose as the self-organized particular structure? Or, equivalently, which constraints are robust in the natural selection? In the example of Section 3, the topological constraints are imposed by adiabatic invariants, thus, the robustness is attributed to the accuracy of adiabatic invariance. Here, we may apply the same argument by interpreting Casimir invariants as adiabatic invariants; we deem a Casimir invariant as an adiabatic invariant associated with a hidden ‘microscopic’ angle variable [7]. By recovering the angle variable, the Casimir invariant can be unfrozen when the Hamiltonian is perturbed by the angle variable. Providing such a perturbation with physical meaning, this formalism elucidates how the selection rule organizes a hierarchy of topological constraint. A general guideline for evaluating the robustness is the space–time scale included in the Casimir invariant. Suppose that a Casimir invariant $C$ includes a term $\nabla p u$ ($u$ is a field variable). Then, a perturbation $\delta u$ with a length scale $\ell$ causes $\delta C \sim \ell^{-p} \delta u$, implying that only a very small-scale ($\ell \ll 1$) perturbation can influence the variation of $C$ when $p$ is small. The magnetic helicity $C_3$ is a rugged one, because it includes $A = \text{curl}^{-1} B \sim \ell B$.

To proceed based on this scenario, we have to represent topological constraints in terms of Casimir invariants. However, almost all of them are not integrable as Casimir invariants. There are infinitely many local constraints (as epitomized by Kelvin’s circulation law) that prevent topological changes of chains co-moving with the fluid. Recently proposed systematic method enables us to integrate such topological constraints as cross helicities on an extended Poisson manifold; the cross helicities describe the link of the constrained fields and the newly introduced phantoms that co-move with the fluid (cf. Remark 1) [7]. For example,
the tearing-mode instability [62–64] emerges when the topological constraint on the helical magnetic flux (or the circulation of the magnetic field along a ‘resonant’ helical loop) is removed by a finite parallel electric field [6,17]. Such a local topological constraint, represented by a cross helicity (measuring the resonant helical flux) is a fragile singular functional, which is easily removed to allow the plasma to relax into a lower energy state on a leaf of robust Casimir invariant such as $C_3$.

5. Summary

We have built a formulation of a macrohierarchy as phase-space foliation by Casimir invariants. By putting the guiding-center motion of magnetized particles into the perspective of noncanonical Hamiltonian formalism (Section 3), we found a prototype of Casimir invariant in the magnetic moment, i.e. the adiabatic invariant pertinent to the cyclotron motion. Coarse-graining the microscopic gyromotion, the model of particle is reduced to that of a ‘quasi-particle’ representing the guiding-center motion, which resides on the leaf of the adiabatic invariant. Conversely, by recovering the coarse-grained angle variable, the adiabatic invariant can become dynamical. Such unfreezing of an adiabatic invariant occurs when sufficiently small-scale perturbations violate the adiabatic condition.

Based on these observations, we propose to interpret every Casimir invariant as an adiabatic invariant. Adding (retrieving) a conjugate variable, we can formulate a larger system in which the topological constraint of the target Casimir invariant is unfrozen. In the examples of Section 4, however, the origin of the topological constraints are not yet fully identified. A fluid-mechanical model is some ‘closure’ of the moment hierarchy of the velocity distribution function, thus the reduction is, in principle, caused by the averaging over the particle velocity. However, the relation between the Casimir invariants (especially the helicities) and the velocity-space averaging is not evident. It is a tall order to retrieve the hidden microscopic variable corresponding to each Casimir invariant. In fact, mathematical methods of canonization are not unique (cf. Remark 2). Perturbative consideration will guide us to seek a physically reasonable route. For example, the magnetic helicity can be unfrozen by introducing a finite electric field in the direction parallel to the magnetic field. The helical magnetic flux (a cross helicity in an extended phase space; Section 4.4) is more easily unfrozen by a local (resonant) electric field. We can define a hierarchy of topological constraints in accordance with the robustness of Casimir invariants. Unfreezing of fragile Casimir invariants will lead to relaxation to a less-constrained lower-energy state on a relatively robust Casimir leaves.

Notes

1. The idea of ‘constrained relaxation’ has been argued in various ways (cf. [1]). Here we investigate the problem from an angle of geometry, and put it into the perspective of
Hamiltonian mechanics [2,3]. A more rigorous and general meaning of ‘topological constraint’ will be discussed in Remark 1.

2. For \( \zeta \in \text{Ker}(\mathcal{J}(z)) \) to be represented as a ‘gradient’ \( \partial_z C(z) \) of a scalar function \( C(z) \), \( \zeta \) must be an exact 1-form. It is not always the case, i.e. topological constraints are not necessarily ‘integrable’ to define Casimir invariants. Moreover, the point where \( \text{Rank}(\mathcal{J}(z)) \) changes is the singularity from which singular Casimir invariants are generated. They are beyond the scope of this short review; cf. [7,21–23]. Here we concentrate on simple topological constraints that are representable by regular Casimir invariants.

3. One may interpret (10) as a Boltzmann distribution with two different energies \( H \) and \( \mu \) (with the corresponding inverse temperatures \( \beta \) and \( \alpha \)). We remind the pioneering work of Nambu [37], in which a similar grand canonical distribution function was derived for a ‘generalized Hamiltonian system’ with two Hamiltonians on an so (3) configuration space.

4. In the RT-1 experiment [9] (Figure 2), the electron cyclotron frequency is of order 1 GHz, the bounce frequency is of order 1 MHz, and the azimuthal drift frequency is of order 1 kHz, while the density fluctuation is ranged up to the order of 1 MHz. Hence the constancy of the bounce action is marginal [38].

5. The original idea of the use of the magnetic helicity as the constraint in relaxation goes back to the pioneering work of Woltjer [11]. Among many other topological constraints, Taylor chose \( C_3 \) as the determinant constant of motion. This idea was extended for other systems as the model of selective decay; see [1].

6. The variations with respect to \( V \) and \( \rho \) yield \( V = 0 \) and \( \rho = \text{constant} \).

7. The ‘relaxation process’ that sends the state vector to an equilibrium point needs a finite dissipation. Note, however, that the dissipative mechanism is not the subject of the present argument. Instead, we delineate the distribution of the energy (Hamiltonian) on the effective phase space, by which we identify stable equilibrium points that can be the candidates of the relaxed state. In Section 4.4, we discuss ‘robustness’ of topological constraints, which is related to the argument of selective decay (see note 5).

8. If we consider the full set of Casimir invariants, the stationary point of \( H - \mu C_1 - \lambda C_2 - \kappa C_3 \) gives a modified Taylor relaxed state with a finite flow \( V \) aligned to \( B \). We find that \( \kappa = 0 \) is a singularity at which the Beltrami condition (20) is removed to inflate the set of solutions infinitely.

9. If we use the incompressible model, the condition (24) is omitted, and we obtain a much larger set of solutions; arbitrary profiles of \( V \) are allowed as far as they are accompanied by \( B = \pm V \).

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