Duality functors for triple vector bundles\footnote{Mathematics Subject Classification (MSC2000): 53D17 (primary), 18D05, 18D35, 20E99, 55R99, 70H50 (secondary).} \footnote{Keywords: double vector bundles, triple vector bundles, iterated tangent bundles, duality of multiple vector bundles, Lie algebroids, Poisson structures, extensions of symmetric groups, groups of order 96.}

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Abstract

We calculate the group of dualization operations for triple vector bundles, showing that it has order 96 and not 72 as given in Mackenzie’s original treatment. The group is a nonsplit extension of $S_4$ by the Klein group. Dualization operations are interpreted as functors on appropriate categories and are said to be equal if they are naturally isomorphic. The method set out here will be applied in a subsequent paper to the case of $n$-fold vector bundles.

Introduction.

The duality of finite-rank vector bundles $E$ is involutive, $(E^*)^* \cong E$, and so may be said to have group $\mathbb{Z}_2$. In [7] and [9], Mackenzie initiated the study of duality for multiple vector bundles, showing that the group of dualization operations of a double vector bundle is the symmetric group of order six, and stating that the corresponding group for triple vector bundles has order 72.

Shortly before [9] went to press, Gracia-Saz convinced Mackenzie that the order of this group is 96. In the present paper we prove that this is indeed so, by a method which we will subsequently extend to $n$-fold vector bundles [5]. The approach in [9] considered only the triple vector bundles themselves; in this paper, we show that the correct question to pose is when two dualization operations are equal. The crux of
the method is to regard dualization operations as functors on appropriate categories, and to regard two dualization operations as equal if they are naturally isomorphic as functors.

One of the motivating examples at the inception of category theory [6] was the distinction between the isomorphisms which can be defined between a (finite-dimensional) vector space $V$ and its dual by use of a basis, and the natural isomorphism which exists between $V$ and $(V^*)^*$. Our method extends this familiar idea to multiple vector bundles. For ordinary vector bundles, local trivializations are transformed into each other by maps into a general linear group, and dualizing the bundle corresponds to taking the transpose in the group. Multiple vector bundles can always be globally decomposed — that is, they are always isomorphic to a combination of pullbacks of ordinary vector bundles — and there is a corresponding group of what we call statomorphisms (see Definition 3.3). In order to distinguish between two dualization operations for triple vector bundles it is not sufficient to consider the action on the constituent bundles, as was done in [9] — it is necessary to study their effect on the statomorphism group.

Double vector bundles have been used for many years in some accounts of connection theory [11] and in some approaches to theoretical mechanics [13]. Their general theory was initiated by Pradines [12]. In the 1990s double and triple vector bundles began to arise in Poisson geometry, as a result of the relationships between Poisson structures and Lie algebroids. At the simplest level, the dual of a Lie algebroid $A$ has a Poisson structure, the linearity properties of which can be expressed as the condition that the Poisson anchor $\pi^#$ is a morphism of double vector bundles $T^*A^* \to TA^*$. Secondly, given a Lie bialgebroid $(A,A^*)$, the cotangent double $T^*A^* \cong T^*A$ can play a role corresponding to that of the classical Drinfel’d double of a Lie bialgebra. More generally, the duality of double vector bundles, and the triple vector bundles associated with them, are crucial to the compatibility conditions of [10] between the Lie algebroid structures in a double Lie algebroid. The results of the present paper will be applied elsewhere to the triple Lie algebroids which arise from double Lie algebroids, and in particular to understanding the duals of thrice-iterated tangent bundles $T^3M$. No familiarity with Lie algebroids or Poisson structures is needed in this paper.

The duality group for double vector bundles $(D;A,B;M)$ is the symmetric group $S_3$ and duality operations in this case can be identified by their action on the three "building bundles": the side bundles $A$ and $B$ and the core dual $C$. Indeed the duality group for double vector bundles can be interpreted as those rotations of the cotangent triple $T^*D$ which preserve $T^*D$ and $M$ [9]. In the triple case the corresponding result does not hold: there are three nontrivial duality operations which preserve the building bundles but are not equivalent to the identity. The significance of these operations deserves to be investigated further.

In §1 we briefly review the basic notions for double vector bundles; some readers
may prefers to start with [2] To introduce the methods of this paper and of [5], we begin by illustrating them on the known case of double vector bundles. The duality theory in this case was set out in [9, §1–§3] but we reformulate it in [2] to clarify the problems which arise in the case of triple vector bundles and to introduce the functorial point of view which we use in this paper and in [5]. In Definition 2.8 we define the group $\mathcal{DF}_2$, the group of duality functors modulo natural equivalence, which we usually refer to simply as the duality group for double vector bundles.

In [3] we set up a notation for triple vector bundles that will extend readily to the $n$-fold case. The main work of the paper is in [4]. This is the calculation of the duality group $\mathcal{DF}_3$ for triple vector bundles, in terms of its action on the group of statomorphisms (Definition 3.3). The final [5] provides some further information on the structure of $\mathcal{DF}_3$. Throughout the paper we consider smooth vector bundles of finite rank over the reals.

Multiple vector bundles in the setting of supergeometry are being developed by Voronov [14] and Grabowski [4].

Our work on this paper began after Mackenzie had spoken on [9] at Poisson 2004 in Luxembourg. Gracia–Saz’s research was partially supported by fellowships from the Secretaría de Estado de Universidades e Investigación del Ministerio Español de Educación y Ciencia and from the Japanese Society for the Promotion of Science. Gracia–Saz would also like to thank the Pure Mathematics department of the University of Sheffield for its hospitality during various visits, and both authors thank the London Mathematical Society for funding one of these visits. The authors also very much appreciate the comments of the referees.

1 Review of double vector bundles

A double vector bundle $(D;A,B;M)$, as shown on the left of Figure 1, consists firstly of a manifold $D$ together with two vector bundle structures, on bases $A$ and $B$, each of which is itself a vector bundle on base $M$, such that for each structure on $D$, the structure maps (projection, addition, scalar multiplication) are vector bundle morphisms with respect to the other structure. A morphism of double vector bundles from $(D;A,B;M)$ to $(D';A',B';M')$ is a system of maps $\varphi: D \to D'$, $\varphi_A: A \to A'$, $\varphi_B: B \to B'$, $\varphi_M: M \to M'$ such that each of $(\varphi, \varphi_A), (\varphi, \varphi_B), (\varphi_A, \varphi_M)$ and $(\varphi_B, \varphi_M)$ is a morphism of vector bundles.

Two examples to keep in mind, also shown in Figure 1 are the tangent prolongation of an ordinary vector bundle $A$, and the double vector bundle $A \ast B$ which is formed from two vector bundles $A$ and $B$ on $M$ by giving the pullback manifold $A \times_M B$ the pullback vector bundle structures $q_A^! B$ and $q_B^! A$. (We denote the pullback of $B$ over $q_A$ by $q_A^! B$ instead of $q_A^* B$, so as not to confuse the symbol $\ast$ with the many duals that appear in this paper.) We use the notation $A \ast B$ to distinguish this double vector bundle from the Whitney sum $A \oplus B$; as manifolds they are the same, but we
regard $A \oplus B$ as a vector bundle over the base $M$, and $A \ast B$ as a vector bundle over the bases $A$ and $B$. The tangent prolongation vector bundle $TA \to TM$ is formed by applying the tangent functor to the structure maps of $A \to M$.

\[
\begin{array}{cccc}
D & \xrightarrow{q_B} & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{q_A} & M \\
\end{array}
\hspace{1cm}
\begin{array}{cccc}
TA & \to & TM \\
\downarrow & & \downarrow \\
A & \xrightarrow{q_A} & M \\
\end{array}
\hspace{1cm}
\begin{array}{cccc}
A \ast B & \to & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{q_A} & M \\
\end{array}
\]

Figure 1.

We also need a further condition, which was not made explicit in [9] or [8].

**Definition 1.1.** A double vector bundle $(D; A, B; M)$ satisfies the splitting condition if there is a morphism of double vector bundles $\Sigma: A \ast B \to D$ which preserves $A$ and $B$; such a map is a splitting of $D$.

Equivalently, a splitting is a map $\Sigma: A \times_M B \to D$ which is right-inverse to the combination of the two projections $D \to A \times_M B$ and, when regarded as $q_A^1B \to D$, is linear over $A$ and, when regarded as $q_B^1A \to D$, is linear over $B$. We note below that the splitting condition is equivalent to the existence of a decomposition, and as such it is the counterpart of requesting local triviality in the definition of vector bundle. We could request only a local splitting condition, but a standard Čech cohomology argument shows that this implies the global splitting condition. It can be shown that for double vector bundles the splitting condition is implied by the rest of the definition [4]. In this paper, we include it as part of the definition.

It is standard that connections in a vector bundle $A \to M$ correspond to maps $A \ast TM \to TA$ which are linear in each variable and preserve $A$ and $TM$ (see Example 2.11). It is easy to check (see below Definition 2.2) that the duals of a double vector bundle which satisfies the splitting condition, also satisfy the condition.

We briefly recall the basic constructions with double vector bundles as used in [8, Chap. 9]. In terms of $(D; A, B; M)$ as in Figure 1 we refer to $A$ and $B$ as the side bundles of $D$, and to $M$ as the double base. In the two side bundles the addition, scalar multiplication and subtraction are denoted by the usual symbols $+$, juxtaposition, and $-$. We distinguish the two zero-sections, writing $0^A: M \to A$, $m \mapsto 0^A_m$, and $0^B: M \to B$, $m \mapsto 0^B_m$. We may denote an element $d \in D$ by $(d; a, b; m)$ to indicate that $a = q^A_A(d)$, $b = q^B_B(d)$, $m = q_B(q^B_B(d)) = q_A(q^A_A(d))$.

In the vertical bundle structure on $D$ with base $A$ the vector bundle operations are denoted $+_A$, $\cdot_A$, $-A$, with $0^A: A \to D$, $a \mapsto 0^A_a$, for the zero-section. In the horizontal bundle structure on $D$ with base $B$ we likewise write $+_B$ and so on. For $m \in M$ the double zero $0^A_m = 0^B_m$ is denoted $\circ_m$ or $0^2_m$. 

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The map \( D \to A \ast B \) which combines the two bundle projections is a morphism of double vector bundles and the set of elements of \( D \) which map to the double zeros of \( A \ast B \) is denoted \( C \) and called the core of \( D \). Thus

\[
C = \{ c \in D \mid \exists m \in M \text{ such that } q_B^D(c) = 0^B_m, \ q_A^D(c) = 0^A_m \}.
\]

The core is an embedded submanifold of \( D \) and has a well-defined vector bundle structure on base \( M \). The projection \( q_C \) is the restriction of \( q_B \circ q_B^D = q_A \circ q_A^D \) and the addition and scalar multiplication are the restrictions of either of the operations on \( D \). This is the core vector bundle of \( D \).

As an ordinary vector bundle, \( D \to B \) has a dual which we denote \( D^X \) or \( D|B \). There is a second vector bundle structure on \( D^X \), with base \( C^* \), and projection \( \zeta \) given by

\[
\langle \zeta(\Psi), c \rangle = \langle \Psi, \bar{0}_c^B + Bc \rangle
\]

where \( c \in C_m \), \( \Psi : (q_B^D)^{-1}(b) \to \mathbb{R} \) and \( b \in B_m \). The addition \( +_{C^*} \) and scalar multiplication in \( D^X \to C^* \) are defined by

\[
\langle \Psi +_{C^*} \Psi', d +_A d' \rangle = \langle \Psi, d \rangle + \langle \Psi', d' \rangle
\]

\[
\langle t_{-C^*} \Psi, t_A d \rangle = t \langle \Psi, d \rangle,
\]

for suitable elements. The zero above \( \kappa \in C^*_m \) is denoted \( \bar{0}_c^B \) and is defined by

\[
\langle \bar{0}_c^B, \bar{0}_a^A + Bc \rangle = \langle \kappa, c \rangle
\]

where \( a \in A_m, \ c \in C_m \). The core of \( D^X \) is \( A^* \) with \( \varphi \in A^*_m \) defining \( \varphi \in D^X \) by

\[
\langle \varphi, \bar{0}_a^A + Bc \rangle = \langle \varphi, a \rangle.
\]

With these structures \((D^X; C^*, B; M)\) is a double vector bundle with core \( A^* \). Likewise the dual of \( D \to A \), denoted \( D^Y \) or \( D|A \), has a vector bundle structure over \( C^* \), making \((D^Y; A, C^*; M)\) a double vector bundle with core \( B^* \). These are displayed in Figure 2.

Observe that the notation \((D; A, B; M)\), and the diagram in the Figure, indicate that a choice has been made between the two structures on \( D \). We may say that such a double vector bundle has been placed, and that the flip of \( D \), namely \((D^f; B, A; M)\) in which the two structures on \( D \) have been interchanged, has the opposite placement.

There is a non-degenerate pairing between \( D^X \) and \( D^Y \) given, with the conventions of [8], by

\[
\left[ \Phi, \Psi \right] = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B
\]

where \( \Phi \in D^Y \) and \( \Psi \in D^X \) project to the same element of \( C^* \) and \( d \) is any element of \( D \) which can be paired with \( \Phi \) over \( A \) and with \( \Psi \) over \( B \). This definition depends on the placement of \( D \); the pairing for \( D^f \) is the negative of (5).
The pairing induces an isomorphism of double vector bundles

\[ Q : D^{XYX} \rightarrow D^f. \]  

This isomorphism is natural in the usual informal sense of the word and preserves \( A \) and \( B \) but it induces minus the identity on the core. One can modify \( Q \) so that it will induce minus the identity on one of the side bundles instead, but not so that it will induce the identity on all of \( A, B, \) and \( C \). Accordingly, \([9]\) does not identify \( D^{XYX} \) with \( D^f \). At the end of this section we will show that although there does exist an isomorphism between \( D^{XYX} \) and \( D^f \) which induces the identity on \( A, B, \) and \( C \) there is no canonical one. This shows that \( XYX \) and \( f \) are different as functors.

In \([9]\) the group of dualization operations, there denoted \( \text{VB}_2 \), is taken to be the group generated by \( X \) and \( Y \), subject to \( X^2 = I \) and \( Y^2 = I \) and the identifications which follow from the existence of the non-degenerate pairing between the two duals. Here we denote this group by \( \mathcal{D}_2 \), for duality functors, and we provide a formal definition for it in \( \underline{2.8} \).

## 2 Duality functors

In this section we will illustrate the methods which we will use in the triple and \( n \)-fold cases by showing that \( XYX \) and \( YXY \) are equal as functors, but that \( XYX \) and \( f \) are not. Indeed we will see that \( f \) is not an element of the group of duality functors.

**Definition 2.1.** Consider two double vector bundles \( D \) and \( D' \) which have the same side bundles \( A \) and \( B \), and the same core bundle \( C \). A statomorphism \( \varphi : D \rightarrow D' \) is a morphism of double vector bundles which preserves \( A, B \) and \( C \).

A statomorphism is necessarily an isomorphism \( D \rightarrow D' \). All dualizations of a statomorphism are also statomorphisms.

The role played for vector spaces by the choice of a basis, or equivalently the choice of an isomorphism \( V \rightarrow \mathbb{R}^n \), is played for multiple vector bundles by the notion of decomposition. First recall the notion of a decomposed double vector bundle.
Let $A$, $B$ and $C$ be vector bundles over $M$. Consider their fibered product

$$A \ast B \ast C = \{(a, b, c) \in A \times B \times C \mid q^A(a) = q^B(b) = q^C(c) \in M\}.$$ 

This has a pullback vector bundle structure over $A$:

$$(a, b, c) + A(a, b', c') := (a, b + b', c + c')$$

and a pullback vector bundle structure over $B$:

$$(a, b, c) + B(a', b, c') := (a + a', b, c + c')$$

With these structures $A \ast B \ast C$ is called the \textit{decomposed double vector bundle} with side bundles $A$ and $B$ and core bundle $C$ and denoted $A \ast B \ast C$. The structures on $A \ast B \ast C$ are $q^A_1(B \oplus C)$ and $q^B_1(A \oplus C)$. These need to be distinguished from the Whitney sum bundle $A \oplus B \oplus C$ on base $M$.

In general, if $D$ is a double vector bundle with side bundles $A$ and $B$ and core bundle $C$, we say that $A \ast B \ast C$ is the \textit{decomposed double vector bundle associated to $D$} and write $D := A \ast B \ast C$.

\textbf{Definition 2.2.} A decomposition of $D$ is a statomorphism $P: D \to \overline{D}$.

The set of decompositions of the double vector bundle $D$ will be denoted $\text{Dec}(D)$.

\textbf{Remark 2.3.} Let $U$ be an open subset of $M$ such that the restrictions of the side bundles and the core over $U$ are trivializable. If we compose a decomposition of $D$ restricted over $U$ with trivializations of $A$, $B$, and $C$, we obtain a statomorphism onto:

$$\begin{array}{ccc}
U \times V_A \times V_B \times V_C & \longrightarrow & U \times V_B \\
\downarrow & & \downarrow \\
U \times V_A & \longrightarrow & U
\end{array}$$

where $V_A$, $V_B$, and $V_C$ are the fiber types of $A$, $B$, and $C$ respectively. This produces convenient local coordinates for the double vector bundle $D$. Thanks to a Čech cohomology argument, the existence of local coordinates is equivalent to the existence of a decomposition (which, as explained below, is equivalent to the splitting condition).

Given a decomposition $P$, there is a splitting $\Sigma(a, b) = P^{-1}(a, b, 0)$ and conversely given a splitting $\Sigma$, a decomposition is defined by $P^{-1}(a, b, c) = \Sigma(a, b) + A(c + B \overline{0}_b)$.

Denote by $\text{Stat}(A \ast B \ast C)$ the group of statomorphisms from $A \ast B \ast C$ to itself; that is, $\text{Stat}(A \ast B \ast C) = \text{Dec}(A \ast B \ast C)$. Each element of $\text{Stat}(A \ast B \ast C)$ is of the form

$$\varphi_\lambda(a, b, c) = (a, b, c + \lambda(a, b))$$

where $\lambda: A \otimes B \to C$ is a linear map. (Throughout the paper we will usually write tuples in place of tensor elements, when they are the argument of a linear map.)
Accordingly Stat\((A \ast B \ast C)\) is a group with composition \(\varphi_\lambda \circ \varphi_{\lambda'} = \varphi_{\lambda + \lambda'}\) and can be identified with the additive group \(\Gamma(A^* \otimes B^* \otimes C)\). (Here and elsewhere in the paper, the tensor products are over \(M\).) We do not consider the \(C^\infty(M)\)-module structure on \(\Gamma(A^* \otimes B^* \otimes C)\).

The duals of \(A \ast B \ast C\) are the decomposed double vector bundles \(C^* \ast B \ast A^*\) and \(A \ast C^* \ast B^*\). Again, the statomorphisms from \(C^* \ast B \ast A^*\) to itself form an abelian group isomorphic to \(\Gamma(C \otimes B^* \otimes A^*)\), and those from \(A \ast C^* \ast B^*\) to itself form an abelian group \(\Gamma(A^* \otimes C \otimes B^*)\). In what follows we will identify these three groups without comment and denote them by \(G_2\).

Let \(D\) be a double vector bundle with side bundles \(A\) and \(B\) and core bundle \(C\). Then \(G_2 = \text{Stat}(A \ast B \ast C)\) acts on \(\text{Dec}(D)\) to the left, and the action is simple and transitive. Thus \(\text{Dec}(D)\) is a (non–empty) \(G_2\)-torsor.

We now wish to consider the effect of the dualization operations on morphisms between double vector bundles, and specifically on decompositions. We first define the relevant category.

**Definition 2.4.** Let \(\mathcal{C}\) be the category whose objects are double vector bundles, and whose morphisms are statomorphisms of double vector bundles.

For the rest of this section, fix three vector bundles \(A, B\) and \(C\) over \(M\) and, for the moment, denote them by \(E_1 := A, E_2 := B\) and \(E_0 := C^*\). (In the triple case we will use an extension of this notation exclusively.) Let \(S_3\) be the group of permutations of the set \(\{0, 1, 2\}\). We will also write \(\mathcal{C}^{op}\) or \(\mathcal{C}^{-1}\) for the opposite category to \(\mathcal{C}\).

We can now think of \(X, Y\) and \(f\) as functors:

\[
\mathcal{C} \xrightarrow{X} \mathcal{C}^{op}, \quad \mathcal{C} \xrightarrow{Y} \mathcal{C}^{op}, \quad \mathcal{C} \xrightarrow{f} \mathcal{C}^{op}
\]  

(8)

The effect of \(X\) on objects has already been defined and the action on morphisms is as expected: Let \(\varphi: D_1 \rightarrow D_2\) be a morphism in the category \(\mathcal{C}\). Then \(\varphi\) is a statomorphism of double vector bundles. In particular \(\varphi: D_1 \rightarrow D_2\) is a morphism of vector bundles over \(B\) and so can be dualized to \(\varphi^X: D_1^X \rightarrow D_2^X\); this is also a statomorphism of double vector bundles \(D_2^X \rightarrow D_1^X\) and is thus a morphism \(\varphi^X: D_1^X \rightarrow D_2^X\) in the category \(\mathcal{C}^{op}\).

Note that, since every morphism in \(\mathcal{C}\) is invertible, we can regard \(X\) and \(Y\) as functors \(\mathcal{C} \rightarrow \mathcal{C}^{op}\) or as functors \(\mathcal{C}^{op} \rightarrow \mathcal{C}\); hence, we can compose them, and we can talk of the group of functors generated by \(X\) and \(Y\). Denote this group by \(\mathcal{W}\). Now for any \(W \in \mathcal{W}\) there exists a unique permutation \(\sigma \in S_3\) such that, if \(D\) is a double vector bundle with side bundles \(E_1\) and \(E_2\), and with core bundle \(E_0^*\), then \(D^W\) is a double vector bundle with side bundles \(E_{\sigma(1)}^*\) and \(E_{\sigma(2)}^*\), and with core bundle \(E_{\sigma(0)}^*\). Define \(\pi(W) := \sigma\), and define \(\varepsilon_W = \pm 1\) to be the signature of \(\pi(W)\). Now \(\mathcal{C} \xrightarrow{W} \mathcal{C}^{\varepsilon_W}\) is a functor.
The flip operation \( f \) is most naturally considered as a covariant functor; it can be defined as such for all morphisms of double vector bundles, not only for statomorphisms.

Finally, we recall the definition of natural isomorphism.

**Definition 2.5.** Let \( \text{Cat}_i, i = 1, 2, \) be two categories, and let \( F, G : \text{Cat}_1 \to \text{Cat}_2 \) be two functors. A natural transformation \( s : F \to G \) is a collection of morphisms \( s(O) : F(O) \to G(O) \) in \( \text{Cat}_2 \) for every object \( O \) in \( \text{Cat}_1 \) such that, given any morphism \( f : O \to O' \) in \( \text{Cat}_1 \), the following diagram is commutative:

\[
\begin{array}{ccc}
F(O) & \xrightarrow{F(f)} & F(O') \\
\downarrow{s(O)} & & \downarrow{s(O')}
\end{array}
\]

\[
\begin{array}{ccc}
G(O) & \xrightarrow{G(f)} & G(O') \\
\downarrow{s(O)} & & \downarrow{s(O')}
\end{array}
\]

If, in addition, \( s(O) \) is an isomorphism in \( \text{Cat}_2 \) for every object \( O \) in \( \text{Cat}_1 \), the natural transformation \( s \) is called a natural isomorphism.

Consider \( W \in \mathcal{W}_2 \) such that \( \pi(W) = 1 \in S_3 \). Thus \( W \) is a word in \( X \) and \( Y \), and as a functor, \( W \) is \( \mathcal{C} \to \mathcal{C} \). We want criteria for a natural isomorphism to exist between \( W \) and the identity functor. We resume the notation of Figure 2 so as to facilitate relating this section to [8].

Let \( D \) be a double vector bundle with side bundles \( A \) and \( B \) and core \( C \). Since \( \pi(W) = 1 \), \( D^W \) is also a double vector bundle with side bundles \( A \) and \( B \) and core \( C \). Choose a decomposition \( P : D \to D \) and apply the functor \( W \) to it. We obtain a decomposition of \( D^W \):

\[
P^W : D^W \to D^W = D = A ? B ? C.
\]

Hence we obtain a statomorphism \((P^W)^{-1} \circ P : D \to D^W \). This is our candidate for a natural isomorphism between the identity functor and the functor \( W \). In order for it to succeed, it should at least be independent of the choice of decomposition. The next two theorems together show that this is also a sufficient condition.

The following result is valid for any element of \( \mathcal{W}_2 \). Part (i) has been stated already and is included here for reference.

**Theorem 2.6.** Let \( W \in \mathcal{W}_2 \) and let \( D \) be a double vector bundle with side bundles \( A \) and \( B \), and core \( C \).

(i). The spaces of decompositions \( \text{Dec}(D) \) and \( \text{Dec}(D^W) \) are \( G_2 \)-torsors.

(ii). Choose a decomposition \( P_0 \in \text{Dec}(D) \). Then the bijection \( \vartheta_W : \text{Dec}(D) \to \text{Dec}(D^W) \) defined by

\[
\vartheta_W(P) = (P^W)^{wW},
\]
induces a group automorphism \( \theta_W : G_2 \to G_2 \) such that

\[
\theta_W(\varphi \circ P_0) = \theta_W(\varphi) \circ \theta_W(P_0)
\]

for \( \varphi \in G_2 \). Moreover, \( \theta_W \) does not depend on the choice of \( P_0 \) (that is, \( \theta_W \) is a morphism of \( G_2 \)-torsors) nor on the choice of the double vector bundle \( D \).

(iii). The map \( \mathcal{H}_2 \times G_2 \to G_2 \), \( (W, \lambda) \mapsto \theta_W(\lambda) \), is a group action.

**Proof.** (ii) It is enough to prove the result for \( W = X \), since the same proof will hold for \( Y \), and then for any word \( W \) in \( X \) and \( Y \).

Consider \( P \mapsto (P^X)^{-1}, \text{Dec}(D) \to \text{Dec}(D^X) \). For \( \varphi \in \text{Stat}(A \ast B \ast C) \) we have, by functoriality,

\[
\vartheta_X(\varphi \circ P) = (\varphi^X)^{-1} \circ (P^X)^{-1} = (\varphi^X)^{-1} \circ \vartheta_X(P),
\]

so that \( \theta_X(\varphi) = (\varphi^X)^{-1} \).

Let us write \( \varphi = \varphi_\lambda \) for some \( \lambda : A \otimes B \to C \) as in (7). Similarly, let us write \( \theta_X(\varphi) = \theta_\lambda \) for some \( \lambda' : C' \otimes B' \to A' \). Notice that with our abuse of notation we can also write \( \theta_W(\lambda) = \lambda' \). Consider the following diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{P} & D^X \\
\downarrow{\varphi_\lambda \circ P} & & \downarrow{\varphi_\lambda} \\
D & \xrightarrow{(\varphi_\lambda \circ P)^X} & D^X
\end{array}
\]

If \( d \in D_b \) and \( \delta \in D_b^X = (DB)_b \), we can calculate the pairing of \( d \) and \( \delta \) in any decomposition. That is:

\[
\langle \delta | d \rangle = \langle (P^X)^{-1}(\delta) | P(d) \rangle = \langle \varphi_\lambda \circ (P^X)^{-1}(\delta) | (\varphi_\lambda \circ P)(d) \rangle
\]

Let us write \( P(d) = (a, b, c) \in A \ast B \ast C \) and \( (P^X)^{-1}(\delta) = (\gamma, b, \alpha) \in C^* \ast B \ast A^* \). Then:

\[
\langle (P^X)^{-1}(\delta) | P(d) \rangle = \langle (\gamma, b, \alpha) | (a, b, c) \rangle = \langle \alpha | a \rangle + \langle \gamma | c \rangle
\]

and

\[
\langle \varphi_\lambda : ((P^X)^{-1}(\delta)) | \varphi_\lambda (P(d)) \rangle = \langle (\gamma', b, \alpha + \lambda(\gamma, b)) | (a, b, c + \lambda(a, b)) \rangle
\]

\[
= \langle \alpha | a \rangle + \langle \gamma | c \rangle + \lambda(a, b, \gamma) + \lambda'(a, b, \gamma).
\]

In order for these to be equal we need \( \lambda = -\lambda' \). We have proved that \( \theta_X \) is minus the identity on \( G_2 \), and that \( \varphi^X = \varphi \). This completes the proof of (ii).

(iii) is clear now that we know the map \( \theta \) is well defined.

The action of \( \mathcal{H}_2 \) on \( G_2 \) is not faithful. However, if we restrict it to the kernel of \( \pi : \mathcal{H}_2 \to S_3 \) then it becomes faithful, as the next theorem shows.
Theorem 2.7. For $W \in \mathcal{W}_2$ with $\pi(W) = 1$, the following are equivalent:

(i). $\theta_W$ is the identity on $G_2$.

(ii). There exists a natural isomorphism between the identity functor and $W$.

**Proof.** (i) $\implies$ (ii) For every double vector bundle $D$ in $\mathcal{C}$ we want to obtain a statomorphism $s(D) : D \to D^W$. Choose a decomposition $P : D \to A*B*C$. We also have $P^W : D^W \to A*B*C$. Compose them to define $s(D) := (P^W)^{-1} \circ P$.

We check first that $s(D)$ defined in this way does not depend on the choice of $P$. If $P_1, P_2 \in \text{Dec}(D)$, there is some $\lambda \in G_2$ such that $P_2 = \varphi_\lambda \circ P_1$. We then have $(P_2)^W = \varphi_{\theta_W(\lambda)} \circ (P_1)^W$. Since $\theta_W = \text{id}_{G_2}$, it follows that $\varphi_{\theta_W(\lambda)} = \varphi_\lambda$, and the following diagram is commutative:

![Diagram](image)

which proves that $s(D)$ is well defined.

We need to check that $s$ is a natural transformation. Let $f : D_1 \to D_2$ be a statomorphism of double vector bundles. Choose a decomposition $P_2 : D_2 \to A*B*C$ of $D_2$ and consider the decomposition $P_1 := P_2 \circ f$ of $D_1$. Then the following diagram is commutative:

![Diagram](image)

(ii) $\implies$ (i) Reciprocally, let $s$ be a natural transformation from the identity functor to $W$. Let $D$ be a double vector bundle in $\mathcal{C}$ and let $D = D^W = D^W$ be the decomposed double vector bundle. Let $P_1, P_2 \in \text{Dec}(D)$ be two decompositions of $D$ such that $P_2 = \varphi_\lambda \circ P_1$ for $\lambda \in G_2$. Then the following diagram is commutative:

![Diagram](image)

Hence $\varphi_{\theta_W(\lambda)} = (\varphi_\lambda)^W = \varphi_\lambda$ and $\theta_W(\lambda) = \lambda$. $\square$

We can now give a precise definition of $\mathcal{D}_G$ and calculate it.
Definition 2.8. The group $\mathcal{DF}_2$ is the group $\mathcal{W}_2$ of functors generated by $X$ and $Y$, quotiented over natural isomorphism in $\mathcal{C}$.

From (8) we have $\sigma_1 := \pi(X) = (01)$ and $\sigma_2 := \pi(Y) = (02)$ and it follows that $\pi: \mathcal{DF}_2 \rightarrow S_3$ is surjective. Define $K_3$ to be the kernel of $\pi$. The following lemma is an easy exercise in algebra.

Lemma 2.9. Let $f: G \rightarrow S$ be a surjective group homomorphism. Let $g_1, \ldots, g_n$ be a set of generators of $G$ and write $\sigma_i := f(g_i)$. Let $\{R_j(\sigma_1, \ldots, \sigma_n) \mid j = 1, \ldots, m\}$ be a set of relations for a presentation of $S$ with generators $\sigma_1, \ldots, \sigma_n$. Then the kernel of $f$ is the normal subgroup of $G$ generated by $\{R_j(g_1, \ldots, g_n) \mid j = 1, \ldots, m\}$.

A presentation of $S_3$ with generators $\sigma_1, \sigma_2$ is
\[
\langle \sigma_1, \sigma_2 \mid \sigma_1^2, \sigma_2^2, (\sigma_1\sigma_2)^3 \rangle.
\]
Hence, $K_3 = \ker \pi$ is the normal subgroup of $\mathcal{DF}_2$ generated by
\[
\{X^2, Y^2, (XY)^3\}.
\]

In the proof of Theorem 2.6 we showed that $\theta_X = \theta_Y = -\text{id}$. From this we obtain
\[
\theta_{X^2} = \theta_{Y^2} = \theta_{(XY)^3} = \text{id}.
\]
By Theorem 2.7 it follows that $X^2 = Y^2 = (XY)^3 = 1$ in $\mathcal{DF}_2$. Thus $K_3$ is the trivial group and $\mathcal{DF}_2 = S_3$, as in [9]. In the following sections we will extend this method to determine the group $\mathcal{DF}_3$.

We end the present section by demonstrating that, for any double vector bundle $D$, a choice of statomorphism between $D^XYX$ and $D^f$ is equivalent to choosing a decomposition of $D$.

First, let $P: D \rightarrow A \ast B \ast C$ be a decomposition. Applying the functors $XYX$ and $f$ we get statomorphisms
\[
P^{XYX}: B \ast A \ast C \rightarrow D^{XYX} \quad \text{and} \quad P^f: D^f \rightarrow B \ast A \ast C.
\]
Hence $\Phi(S) := P^{XYX} \circ P^f: D^f \rightarrow D^{XYX}$ is a statomorphism. The interesting result is that those are all the statomorphisms.

Proposition 2.10. Let $\mathcal{S}$ denote the set of all statomorphisms $D^f \rightarrow D^{XYX}$. The map
\[
\Phi: \text{Dec}(D) \rightarrow \mathcal{S}
\]
defined above is a bijection.
Proof. Let \( h : D^f \rightarrow D^{XYX} \) be a statomorphism. We want to prove that there is a unique \( P \in \text{Dec}(D) \) such that \( \Phi(P) = h \). First, using notation similar to that in the proof of Theorem 2.6, we note that the groups of statomorphisms of the decomposed double vector bundles \( A \ast B \ast C \) and \( B \ast A \ast C \) are canonically isomorphic to \( G_2 \).

\[
\begin{align*}
\bar{\phi} : G_2 & \longrightarrow \text{Stat}(B \ast A \ast C) \\
\lambda & \mapsto \bar{\phi}_\lambda : B \ast A \ast C \longrightarrow B \ast A \ast C \\
(b,a,c) & \mapsto (b,a,c + \lambda(a,b))
\end{align*}
\]

We also know from the proof of Theorem 2.6(ii) that \( (\phi_\lambda)^f = (\phi_\lambda)^{XYX} = \bar{\phi}_\mu \).

Let us pick a decomposition \( P_1 \in \text{Dec}(D) \). Then \( ((P_1)^f)_{-1} \circ h \circ ((P_1)^{XYX})_{-1} \) is an automorphism of \( B \ast A \ast C \), and hence equals \( \bar{\phi}_\mu \) for some \( \mu \in G_2 \).

\[
\begin{array}{ccc}
D^f & \xrightarrow{h} & D^{XYX} \\
\downarrow (P_1)^f & & \downarrow (P_1)^{XYX} \\
B \ast A \ast C & \xrightarrow{\bar{\phi}_\mu} & B \ast A \ast C
\end{array}
\]

We know that every decomposition of \( D \) is of the form \( \phi_\lambda \circ P_1 \) for a unique \( \lambda \in G_2 \). So we want to prove that there is a unique \( \lambda \in G_2 \) such that \( h = \Phi(\phi_\lambda \circ P_1) \). Noting that

\[
h = (P_1)^{XYX} \circ \bar{\phi}_\mu \circ (P_1)^f
\]

we have that

\[
\Phi(\phi_\lambda \circ P_1) = (\phi_\lambda \circ S_1)^{XYX} \circ (\phi_\lambda \circ P_1)^f = (P_1)^{XYX} \circ (\phi_\lambda)^{XYX} \circ (\phi_\lambda)^f \circ (P_1)^f
\]

\[
= (P_1)^{XYX} \circ \bar{\phi}_\lambda \circ (P_1)^f = (P_1)^{XYX} \circ \bar{\phi}_{2\lambda} \circ (P_1)^f
\]

and so \( h = \Phi(\phi_\lambda \circ P_1) \) if and only if \( 2\lambda = \mu \). This completes the proof. \( \square \)

To summarize: there is a canonical isomorphism between \( D^{XYX} \) and \( D^f \), but it is not a statomorphism; there are statomorphisms between \( D^{XYX} \) and \( D^f \), but not canonical ones, as choosing one such statomorphism is equivalent to choosing a decomposition; the functors \( XYX \) and \( f \) are not naturally isomorphic through statomorphisms. Because of this, we regard \( XYX \) and \( f \) as distinct functors.

Example 2.11. It is worthwhile to consider the significance of decompositions for the tangent prolongation of an ordinary vector bundle.

Let \( A \xrightarrow{q} M \) be a vector bundle. Applying the tangent functor, we obtain a double vector bundle:

\[
\begin{array}{ccc}
TA & \xrightarrow{Tq} & TM \\
\downarrow \pi_A & & \downarrow \pi_M \\
A & \xrightarrow{q} & M
\end{array}
\]
with side bundles $A$, and $TM$. The core consists of those $X \in TA$ which are annulled by both $T(q)$ and $\pi_A$; that is, $C$ is the set of vertical vectors along the zero section, and may be identified with $A$.

A splitting of $TA$ is a connection in $A$ in the usual sense; see, for example, [1]. Given a splitting

$$\Sigma: TM \times_M A \to TA,$$

and a vector field $x$ on $M$, define a vector field $\widetilde{x}$ on $A$ by

$$\widetilde{x}(e) = \Sigma(x(q(e)), e).$$

It follows from the properties of $\Sigma$ that $x \mapsto \widetilde{x}$ is a connection in $A$. Conversely a connection in $A$ induces a decomposition, and Dec($TA$) can be identified with the space of connections in $A$.

As is well known, connections in $A$ form an affine space with model space $\Gamma(T^*M \otimes \text{End}(A))$. This includes the $G_2$-torsor structure.

The dual over $TM$, namely $(TA)^X$, can be identified with the tangent double vector bundle $T(A^*)$ [8, 9.3.2] and so Theorem 2.6(ii) includes the correspondence between connections in $A$ and connections in $A^*$.

3 Triple vector bundles

We begin by recalling the definition of a triple vector bundle. See [9] for fuller details. We use a notation that extends readily to the $n$-fold case.

Definition 3.1. A triple vector bundle $E$ is a system as on the left of Figure 3 where each arrow represents a vector bundle structure, such that each face is a double vector bundle, and which satisfies the splitting condition stated below.
A morphism of triple vector bundles \( \varphi : E \rightarrow F \) is a system of maps \( \varphi_I : E_I \rightarrow F_I \), for all subsets \( I \) of \( \{1, 2, 3\} \), such that for all nonempty subsets \( I \) and each \( k \in I \), \( \varphi_I \) is a morphism of vector bundles over \( \varphi_I(\{k\}) \).

We always read figures as in Figure 3 with \( (E_{1,2,3}; E_{1,2}, E_{2,3}; E_2) \) at the rear and \( (E_{1,3}; E_1, E_3; M) \) coming out of the page toward the reader. Notice that when a nonempty subscript has \( k \) commas, \( 0 \leq k \leq 2 \), the space which it denotes has \( k + 1 \) vector bundle structures. We sometimes denote \( M \) by \( E_0 \) for uniformity.

The three structures of double vector bundles on \( E_{1,2,3} \) are the upper double vector bundles, and \( E_{1,2}, E_{2,3}, E_3, 1 \) are the lower double vector bundles. We refer to \( (E_{1,2}; E_1, E_2; M) \) as the floor of \( E \) and to \( (E_{1,2,3}; E_{1,3}, E_2, E_3; M) \) as the roof of \( E \), with left, right, front and back having their usual meanings.

The cores of the lower double vector bundles are denoted \( E_{12}, E_{23}, E_{31} \), and the cores of the upper double vector bundles are denoted \( E_{312}, E_{123}, E_{231} \). The latter are vector bundles over the former, as indicated on the right of Figure 3 and form three core double vector bundles, \( (E_{123}; E_1, E_{23}; M), (E_{231}; E_2, E_{31}; M), \) and \( (E_{312}; E_3, E_{12}; M) \). Each of the three core double vector bundles has the same core, denoted \( E_{123} \) and called the ultracore. The seven vector bundles \( E_1, E_2, E_3, E_{12}, E_{23}, E_{31}, E_{123} \) are called the building bundles of \( E \). We denote them collectively by \( E_\bullet \).

Example 3.2. The tangent prolongation of a double vector bundle \( (D; A, B; M) \) is the triple vector bundle \( TD \) shown in Figure 4(a). The three core double vector bundles are \( (D; A, B; M), (D; A, B; M) \) and \( (TC; C.TM; M) \), where \( C \) is the core of \( D \). The ultracore is \( C \).

![Figure 4](image)

Suppose given three double vector bundles arranged as in Figure 4(b) and a vector bundle \( E_{123} \) on \( M \). Define the manifold \( E'_{1,2,3} \) to be the pullback of the diagram; that is, \( E'_{1,2,3} \) consists of triples \( (e_{1,2}, e_{1,3}, e_{2,3}) \in E_{1,2} \times E_{1,3} \times E_{2,3} \) such that

\[
q_1^{1,2}(e_{1,2}) = q_1^{1,3}(e_{1,3}), \quad q_2^{1,2}(e_{1,2}) = q_2^{2,3}(e_{2,3}), \quad q_1^{1,3}(e_{1,3}) = q_3^{1,3}(e_{1,3}),
\]
using an obvious notation for the projections $e_k^{j,l}$. Finally, define $E_{1,2,3}$ to be the manifold of tuples $(e_{1,2}, e_{1,3}, e_{2,3}, e_{123}) \in E'_{1,2,3} \times E_{123}$ such that $(e_{1,2}, e_{1,3}, e_{2,3})$ projects to the same point of $M$ as does $e_{123}$. Define a vector bundle structure on $E_{1,2,3}$ with base $E_{1,2}$ by

$$(e_{1,2}, e_{1,3}, e_{2,3}, e_{123}) = (e_{1,2}, e_{1,3} + e'_{1,3}, e_{2,3} + e'_{2,3}, e_{123} + e'_{123}),$$

and scalar multiplication analogously. With the similar structures over $E_{1,3}$ and $E_{2,3}$, this makes $E_{1,2,3}$ a triple vector bundle with ultracore $E_{123}$. Thus any diagram of the form Figure 4(b) can be completed to a triple vector bundle with an arbitrary ultracore.

Given an indexed set of vector bundles $E_\bullet = \{E_1, E_2, E_3, E_{12}, E_{23}, E_{31}, E_{123}\}$ over base $M$, performing this construction with the decomposed double vector bundles $E_i,j = E_i \ast E_j \ast E_{ij}$ yields the decomposed triple vector bundle $E_\bullet$ with the $E_\bullet$ as building bundles. (Note that the bar here means something slightly different from the notation $\overline{D}$ for a double vector bundle $D$. In both cases the bar means that we build a decomposed $n$-fold vector bundle, but in one case we start with a general $n$-fold vector bundle, while in the other we start with a set of building bundles.)

**Definition 3.3.** Let $E$ and $F$ be triple vector bundles with the same building bundles $E_\bullet$. A statomorphism from $E$ to $F$ is an isomorphism $\varphi: E \to F$ which induces the identity on each of the building bundles.

A triple vector bundle $E$ satisfies the splitting condition if there is a statomorphism of triple vector bundles $\varphi: E \to E_\bullet$; such a map is a splitting of $E$.

**Proposition 3.4.** If a double vector bundle $(D;A,B,M)$ satisfies the splitting condition, then its tangent prolongation $TD$ does also.

**Proof.** Taking the tangent of a decomposition of $D$ we have a diffeomorphism $TD \to TA \times_{TM} TB \times_{TM} TC$. Using decompositions of $TA, TB$ and $TC$, this gives a diffeomorphism

$$TD \cong (A \ast A \ast TM) \times_{TM} (B \ast B \ast TM) \times_{TM} (C \ast C \ast TM) \cong A \ast A \ast B \ast B \ast C \ast C \ast TM,$$

where $*$ denotes pullback over $M$. Setting $E_1 = A, E_2 = B, E_3 = TM, E_{12} = C, E_{13} = A, E_{23} = B, E_{123} = C$ we have a decomposition of $TD$. □

The duals of a triple vector bundle are set out in [9, §7]; see Figure 5. Here we need a different notation to use for the duals of decomposed triple vector bundles. For any subset $I \subseteq \{0, 1, 2, 3\}$, for the duals of building bundles, write

$$E_i^* = E_F,$$
where \( I^c \) us the complement of \( I \). In particular \( E_0 = E_{123}^* \) is the dual of the ultracore. The effect of dualization on the building bundles is shown in Table 1. In this table the middle column shows the duals of the ultracores, rather than the ultracores themselves.

\[
E^X = E_{123} \rightarrow E_{23} \rightarrow E_3
\]

\[
E_{231} \rightarrow E_2 \rightarrow E_0 \rightarrow M
\]

\[
E_{13} \rightarrow E_3 \rightarrow E_{23} \rightarrow E_02
\]

\[
E_{123} \rightarrow E_{23} \rightarrow E_{03}
\]

**Figure 5.** The \( X \) dual of a general triple vector bundle.

Define \( R_3 \) to be the group of functors generated by \( X, Y, Z \) up to natural isomorphism, that is, two functors are considered the same if there is a natural isomorphism between them through statomorphisms. (We will normally denote elements of \( R_3 \) by representatives.) As in the double case, there is a group homomorphism \( \pi: R_3 \rightarrow S_4 \), defined by the action of \( X, Y \) and \( Z \) on \( E_1, E_2, E_3 \) and \( E_0 \). We have

\[
\pi(X) = (01), \quad \pi(Y) = (02), \quad \pi(Z) = (03).
\]

|   | \( E_1 \) | \( E_2 \) | \( E_3 \) | \( E_{12} \) | \( E_{23} \) | \( E_{31} \) |
|---|---|---|---|---|---|---|
| \( X \) | \( E_0 \) | \( E_2 \) | \( E_3 \) | \( E_1 \) | \( E_{02} \) | \( E_{23} \) | \( E_{03} \) |
| \( Y \) | \( E_1 \) | \( E_0 \) | \( E_3 \) | \( E_2 \) | \( E_{01} \) | \( E_{03} \) | \( E_{31} \) |
| \( Z \) | \( E_1 \) | \( E_2 \) | \( E_0 \) | \( E_3 \) | \( E_{12} \) | \( E_{02} \) | \( E_{01} \) |

**Table 1.** The action on the building bundles, with the ultracores replaced by their duals.

Write \( \sigma_1, \sigma_2, \sigma_3 \) for the images of \( X, Y \) and \( Z \). These generate \( S_4 \) and so \( \pi \) is surjective. Write \( K_4 \) for the kernel of \( \pi \). A presentation for \( S_4 \) in terms of these generators is [2]:

\[
\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_i^2, (\sigma_i\sigma_j)^3, (\sigma_i\sigma_j\sigma_k)^2 \rangle,
\]

17
where \(i, j, \) and \(k\) are distinct. Hence, by Lemma 2.9 \(K_4\) is the normal subgroup generated by the elements in the list

\[
X^2, Y^2, Z^2, (XY)^3, (YZ)^3, (ZX)^3, (XYXZ)^2, (YXYZ)^2, (ZXZY)^2.
\]

(9)

A priori, and according to our presentation, the list should contain further elements, but they are not needed since \(XYX = YXY\) implies \((XYXZ)^2 = (YXYZ)^2\) and similar conditions.

From Table 1 it is clear that the action of \(X, Y, Z\) on the set of building bundles is determined by the action on the set \(\{0, 1, 2, 3\}\) of indices, and so acts as \(S_4\). In particular, \(W\) acts trivially on the set of building bundles if and only if \(\pi(W) = 1\). As a consequence, since two triple vector bundles are statomorphic if and only if they have the same building bundles, we conclude that \(W \in K_4\) if and only if \(E\) and \(E^W\) are statomorphic for every triple vector bundle \(E\) (albeit not canonically, unless \(W = 1\)).

**Example 3.5.** For the tangent prolongation of a double vector bundle \((D; A, B; M)\) with core \(C\), as shown in Figure 4(a), two duals are shown in Figure 6. The first is the cotangent double of \(D\) and is described in detail in [9]. In (b), \(T^* A D \rightarrow TA\) denotes the dual of the tangent prolongation bundle \(TD \rightarrow TA\) and likewise \(T^* C \rightarrow TM\) is the dual of \(TC \rightarrow TM\).

The duality between \(C\) and \(C^*\) induces a duality between \(TC \rightarrow TM\) and \(T(C^*) \rightarrow TM\) and induces an isomorphism \(T(C^*) \rightarrow T^* C\) [11]. Similarly there is an isomorphism \(T(D|A) \rightarrow T^* D\). These combine into an isomorphism from the triple vector bundle in (b) to the tangent prolongation of \((D|A; A, C^*; M)\).

In the case where \(D\) is the double tangent bundle \(T^2 M\), the \(X, Y\) and \(Z\) duals of \(T^3 M\) may be canonically identified with \(T^*(T^2 M)\), \(T(T^* T M)\) and \(T^2(T^* M)\). There are also canonical isomorphisms from the first of these to the second and third, and to the three triple vector bundles \(T^*(T^* T M)\), \(T^*(T T^* M)\) and \(T(T^* T^* M)\). These isomorphisms are of a different character to those treated in this paper and will be studied elsewhere.

![Figure 6](image-url)
4 Decompositions of triple vector bundles

We know from \[ \text{II} \] that the first six elements listed in \[ \text{III} \] are equal to the identity. In addition, the group generated by the last three elements is normal in \( \mathcal{F}_3 \). This can be proven by a direct calculation: using the fact that the first six elements are equal to the identity, it follows that:

\[ X(XXYZ)^2X^{-1} = (ZXY)^{-2}, \]
\[ Y(XXYZ)^2Y^{-1} = (YXY)^{-2}, \]
\[ Z(XXYZ)^2Z^{-1} = (XYX)^{-2}. \]  

Hence, \( K_4 \) is the subgroup generated by the last three elements in \( \text{III} \). To calculate it we use a triple analogue of 2.7.

We start by determining the statomorphisms from a decomposed triple vector bundle to itself. Denote the set of decompositions of the triple vector bundle \( E \) by \( \text{Dec}(E) \).

**Proposition 4.1.** Each statomorphism \( \varphi \) from \( E \) to itself is of the form

\[
\varphi(e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}) = (e_1, e_2, e_3, \\
e_{12} + \gamma(e_1, e_2), e_{13} + \beta(e_1, e_3), e_{23} + \alpha(e_2, e_3), \\
e_{123} + \nu(e_3, e_{12}) + \lambda(e_1, e_{23}) + \mu(e_2, e_{13}) + \rho(e_1, e_2, e_3))
\]  

where

\[
\gamma : E_1 \otimes E_2 \to E_{12}, \quad \beta : E_1 \otimes E_3 \to E_{13}, \quad \alpha : E_2 \otimes E_3 \to E_{23}, \\
\lambda : E_1 \otimes E_{23} \to E_{123}, \quad \mu : E_2 \otimes E_{13} \to E_{123}, \quad \nu : E_3 \otimes E_{12} \to E_{123},
\]

and \( \rho : E_1 \otimes E_2 \otimes E_3 \to E_{123} \) are linear maps.

Let \( G_3 \) denote the set of all such \( (\gamma, \beta, \alpha, \lambda, \mu, \nu, \rho) \). This is a group under the composition

\[
(\gamma', \beta', \alpha', \lambda', \mu', \nu', \rho') (\gamma, \beta, \alpha, \lambda, \mu, \nu, \rho) = (\gamma'', \beta'', \alpha'', \lambda'', \mu'', \nu'', \rho'')
\]  

where \( \gamma'' = \gamma' + \gamma, \ldots, \nu'' = \nu' + \nu \) and

\[
\rho''(e_1, e_2, e_3) = \rho(e_1, e_2, e_3) + \rho'(e_1, e_2, e_3) \\
+ \lambda'(e_1, \alpha(e_2, e_3)) + \mu'(e_2, \beta(e_1, e_3)) + \nu'(e_3, \gamma(e_1, e_2)).
\]

Given two decompositions \( P \) and \( P_0 \) of a triple vector bundle \( E \), there is a unique statomorphism \( \varphi \in G_3 \) such that \( P = \varphi \circ P_0 \). Thus \( \text{Dec}(E) \) is a simply transitive \( G_3 \)-space, or torsor.
As in the double case, we define \( \mathcal{C} \) to be the category whose objects are triple vector bundles, and whose morphisms are statomorphisms. We denote by \( \mathcal{C}^{op} \) or by \( \mathcal{C}^{-1} \) its opposite category. The dualization operations \( X, Y \) and \( Z \) can now be regarded as functors \( \mathcal{C} \rightarrow \mathcal{C}^{op} \) or as functors \( \mathcal{C}^{op} \rightarrow \mathcal{C} \).

Let \( W \in \mathcal{D}\mathcal{F}_3 \). There is a unique permutation \( \sigma \in S_4 \) such that, if \( E \) is a triple vector bundle with building bundles \( E_\ast \), then \( E^W \) is a triple vector bundle with building bundles

\[
E_{\sigma(1)}, E_{\sigma(2)}, E_{\sigma(3)}, E_{\sigma(1)\sigma(2)}, E_{\sigma(1)\sigma(3)}, E_{\sigma(2)\sigma(3)}, E_{\sigma(1)\sigma(2)\sigma(3)}.
\]

Define \( \pi(W) := \sigma \) and define \( \epsilon_W = \pm 1 \) to be the signature of \( \pi(W) \). We can now look at \( W \) as a functor \( \mathcal{C} \rightarrow \mathcal{C}^{\epsilon_W} \). We now give the triple analogue of Theorem 2.6.

We know that \( \text{Dec}(E) \) is a torsor modelled on \( G_3 \), which is (as a set) the sum of the seven \( C^\infty(M) \)-modules

\[
\Gamma(E^1_1 \otimes E^2_1 \otimes E_{12}), \quad \Gamma(E^1_2 \otimes E^2_1 \otimes E_{13}), \quad \Gamma(E^1_3 \otimes E^2_1 \otimes E_{23}),
\]

\[
\Gamma(E^1_2 \otimes E^2_2 \otimes E_{12}), \quad \Gamma(E^1_3 \otimes E^2_2 \otimes E_{13}), \quad \Gamma(E^1_3 \otimes E^2_3 \otimes E_{12}), \quad \Gamma(E^1_2 \otimes E^2_3 \otimes E_{13}),
\]

In the same way it is easy to establish that \( \text{Dec}(E^X) \) is a torsor modelled on a group which is the sum of the modules

\[
\Gamma(E^0_0 \otimes E^2_0 \otimes E_{02}), \quad \Gamma(E^0_1 \otimes E^2_1 \otimes E_{03}), \quad \Gamma(E^0_2 \otimes E^2_2 \otimes E_{23}),
\]

\[
\Gamma(E^0_2 \otimes E^2_2 \otimes E_{12}), \quad \Gamma(E^0_1 \otimes E^2_2 \otimes E_{03}), \quad \Gamma(E^0_3 \otimes E^2_3 \otimes E_{02}), \quad \Gamma(E^0_3 \otimes E^2_3 \otimes E_{02}),
\]

(see Table 1). Up to rearrangements of the tensor products, this is a permutation of the first list: for example, \( E^1_1 \otimes E^2_2 \otimes E_{02} \), the first space in the second list, is \( E_{123} \otimes E^2_2 \otimes E^1_1 \), a rearrangement of the fifth space in the first list. A similar result holds for \( E^Y \) and \( E^Z \) and thus for \( E^W \) for any word \( W \in \mathcal{D}\mathcal{F}_3 \).

In this way one proves the following result.

**Proposition 4.2.** Let \( W \in \mathcal{D}\mathcal{F}_3 \) and let \( E \) be a triple vector bundle with building bundles \( E_\ast \). Then there is a canonical representation of the space of decompositions \( \text{Dec}(E^W) \) as a torsor modelled on \( G_3 \).

We now calculate \( \theta_W : G_3 \rightarrow G_3 \). First, we explain some notation. For any \( I \subseteq \{0,1,2,3\} \), we write \( e_I \) for a generic element of \( E_I \). Given a linear map \( \gamma: E_1 \otimes E_2 \rightarrow E_{12} \), we can also think of it, for instance, as a map \( \gamma: E_{03} \otimes E_2 \rightarrow E_{023} \); we can also think of it as simply \( \gamma \in \Gamma(E^1_1 \otimes E^2_2 \otimes E^3_3) \). With this abuse of notation, we can write:

\[
\langle e_{03} | \gamma(e_1, e_2) \rangle = \langle e_1 | \gamma(e_2, e_{03}) \rangle = \gamma(e_1, e_2, e_{03}).
\]
If, in addition, \( v \in \Gamma(E_0^* \otimes E_1^* \otimes E_2^* \otimes E_3^* \otimes E_{12}^* \), there is a linear map \( \gamma v \in \Gamma(E_0^* \otimes E_1^* \otimes E_2^* \otimes E_3^* ) \) defined by:

\[
\gamma v(e_0, e_1, e_2, e_3) := \langle v(e_0, e_3) | \gamma(e_1, e_2) \rangle = \langle e_3 | v(e_0, \gamma(e_1, e_2)) \rangle = \ldots
\]

Let \( g = (\gamma, \beta, \alpha, \lambda, \mu, \nu, \rho) \) be an element of \( G_3 \) and let \( \varphi = \varphi_g : E_\gamma \to E_\gamma \) be the corresponding statomorphism. Then \( \theta_X(\varphi) : E_\gamma^X \to E_\gamma^X \) is another statomorphism. We can write \( \theta_X(\varphi) = \varphi_g \) for some element \( \tilde{g} = (\tilde{\mu}, \tilde{\nu}, \tilde{\alpha}, \tilde{\lambda}, \tilde{\beta}, \tilde{\rho}) \) in \( G_3 \). In order to describe \( \theta_X \), we want to write \( \tilde{g} \) in terms of \( g \).

As we explained with double vector bundles, if \( d \in E_\gamma \) and \( \delta \in E_\gamma^X = E_\gamma^X \mid E_{2,3} \) are two elements that project to the same element in \( E_{2,3} \) (so that they can be paired over \( E_{2,3} \)), then we shall have \( \langle \delta | d \rangle = \langle \theta_X(\varphi)(\delta) | \varphi(d) \rangle \). Let us write

\[
d = (e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}), \quad \delta = (e_0, e_2, e_3, e_{02}, e_{03}, e_{23}, e_{023}).
\]

Then \( \varphi(d) \) is given by (11), whereas

\[
\theta_X(\varphi)(\delta) = (e_0, e_2, e_3, e_{02} + \tilde{\mu}(e_0, e_2), e_{03} + \tilde{\nu}(e_0, e_3), e_{23} + \tilde{\alpha}(e_2, e_3),
\]

\[
e_{023} + \tilde{\beta}(e_0, e_{02}) + \tilde{\lambda}(e_0, e_{23}) + \tilde{\gamma}(e_2, e_{03}) + \tilde{\rho}(e_0, e_2, e_3))
\]

First we notice that, in order to be able to pair \( \theta_X(\varphi)(\delta) \) and \( \varphi(d) \) over \( E_{2,3} \), we need to have \( \tilde{\alpha} = \alpha \). Now:

\[
\langle \delta | d \rangle = \langle e_{023} | e_1 \rangle + \langle e_{03} | e_{12} \rangle + \langle e_{02} | e_{13} \rangle + \langle e_0 | e_{123} \rangle,
\]

and \( \langle \theta_X(\varphi)(\delta) | \varphi(d) \rangle \) is equal to

\[
\langle e_{023} | e_1 \rangle + \tilde{\lambda}(e_0, e_{23}, e_1) + \tilde{\gamma}(e_{03}, e_1, e_1) + \tilde{\beta}(e_{02}, e_3, e_1) + \tilde{\rho}(e_0, e_2, e_3, e_1)
\]

\[
+ \langle e_{03} | e_{12} \rangle + \tilde{\nu}(e_{03}, e_2, e_1) + \gamma(e_0, e_3, e_{12}) + \gamma \nu(e_0, e_2, e_3, e_1) + \langle e_{02} | e_{13} \rangle
\]

\[
+ \tilde{\mu}(e_{02}, e_3, e_1) + \beta(e_0, e_{02}, e_{13}) + \beta \tilde{\mu}(e_0, e_2, e_3, e_1) + \langle e_0 | e_{123} \rangle
\]

\[
+ \tilde{\lambda}(e_0, e_{23}, e_1) + \mu(e_0, e_{02}, e_{13}) + \nu(e_0, e_3, e_{12}) + \rho(e_0, e_2, e_3, e_1).
\]

For these two expressions to be equal we must have:

\[
\lambda + \tilde{\lambda} = 0, \quad \mu + \tilde{\mu} = 0, \quad \nu + \tilde{\nu} = 0,
\]

\[
\beta + \tilde{\beta} = 0, \quad \gamma + \tilde{\gamma} = 0, \quad \rho + \tilde{\rho} + \gamma \nu + \beta \tilde{\mu} = 0.
\]

To conclude we solve these equations and we obtain the action of \( \theta_X \) on \( G_3 \), which is summarized in the first row of Table 2. The actions of \( Y \) and \( Z \) are obtained in the same way. Recall that the notation \( \gamma \nu + \beta \mu - \rho \) indicates the map \( \tilde{\rho} \) given by

\[
\tilde{\rho}(v_0, e_2, e_3) = \gamma(\nu(v_0, e_3), e_2) + \beta(\mu(v_0, e_2), e_3) - \rho(v_0, e_2, e_3).
\]

The next two results can now be proved in the same way as in (11)
\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$g$ & $\gamma$ & $\beta$ & $\alpha$ & $\lambda$ & $\mu$ & $\nu$ & $\rho$ \\
\hline
$\theta_X(g)$ & $-\mu$ & $-\nu$ & $\alpha$ & $-\lambda$ & $-\gamma$ & $-\beta$ & $\gamma \nu + \beta \mu - \rho$ \\
\hline
$\theta_Y(g)$ & $-\lambda$ & $\beta$ & $-\nu$ & $-\gamma$ & $-\mu$ & $-\alpha$ & $\alpha \lambda + \gamma \nu - \rho$ \\
\hline
$\theta_Z(g)$ & $\gamma$ & $-\lambda$ & $-\mu$ & $-\beta$ & $-\alpha$ & $-\nu$ & $\alpha \lambda + \beta \mu - \rho$ \\
\hline
\end{tabular}
\end{table}

Table 2. The action of $X, Y, Z$ on $G_3$. In the final column, $i, j, k$ indicates that the domain of $\rho$ is $E_i \otimes E_j \otimes E_k$; this is needed in the proof of Proposition 4.5.

**Theorem 4.3.** Let $W \in \mathcal{DF}_3$ and let $E$ be a triple vector bundle in $\mathcal{C}$.

(i). The bijection $\vartheta_W : \text{Dec}(E) \rightarrow \text{Dec}(E^W)$ defined by

$$\vartheta_W(P) = (P^W)^{\vartheta_W},$$

is a map of torsors and the associated group automorphism $\theta_W : G_3 \rightarrow G_3$ does not depend on the choice of triple vector bundle $E$.

(ii). The map $\mathcal{DF}_3 \times G_3 \rightarrow G_3, \ (W, g) \mapsto \theta_W(g)$, is a group action.

**Theorem 4.4.** For $W \in \mathcal{DF}_3$ with $\pi(W) = 1$, the following are equivalent:

(i). $\theta_W$ is the identity on $G_3$.

(ii). There exists a natural isomorphism between the identity functor and $W$.

Using Table 2 we can determine which words act as the identity.

**Proposition 4.5.** The words $(XYX)^2$, $(YZX)^2$, $(ZXY)^2$ have order 2 in $\mathcal{DF}_3$, and

$$(YZX)^2(YXY)^2 = (ZXY)^2.$$ (13)

**Proof.** Consider $(XYX)^2$. From Table 2 the first six columns of Table 3 are easily established.

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$g$ & $\gamma$ & $\beta$ & $\alpha$ & $\lambda$ & $\mu$ & $\nu$ \\
\hline
$X$ & $-\mu$ & $-\nu$ & $\alpha$ & $-\lambda$ & $-\gamma$ & $-\beta$ & $\gamma \nu + \beta \mu - \rho$ \\
\hline
$YX$ & $\mu$ & $\alpha$ & $-\nu$ & $\gamma$ & $\lambda$ & $-\beta$ & $\rho - \beta \mu - \gamma \nu$ \\
\hline
$XYX$ & $-\gamma$ & $\alpha$ & $\beta$ & $-\mu$ & $-\lambda$ & $\nu$ & $-\rho$ \\
\hline
$XYXZ$ & $-\gamma$ & $\mu$ & $\lambda$ & $-\alpha$ & $-\beta$ & $-\nu$ & $\rho - \alpha \lambda - \beta \mu$ \\
\hline
$(XYX)^2$ & $\gamma$ & $-\beta$ & $-\alpha$ & $-\lambda$ & $-\mu$ & $\nu$ & $\rho$ \\
\hline
\end{tabular}
\end{table}

Table 3. Calculation of the action of $(XYX)^2$. We omit $\theta$ from the notation.
Now consider the final column. The operation of $X$ sends $\rho$ to $\gamma\nu + \beta\mu - \rho$, considered as a map $E_0 \otimes E_2 \otimes E_3 \to E_{023}$. Applying $Y$ to this we get

$$(-\lambda)(-\alpha) + \beta(-\mu) - (\alpha\lambda + \gamma\nu - \rho),$$

which should be considered as a map $E_0 \otimes E_1 \otimes E_3 \to E_{013}$.

The term $\lambda\alpha$ in $YX$ when considered as a map $E_0 \otimes E_1 \otimes E_3 \to E_{013}$ is

$$\langle (\lambda\alpha)(e_0, e_1, e_3) \mid e_2 \rangle = \langle \lambda(e_1, \alpha(e_2, e_3)) \mid e_0 \rangle.$$

On the other hand, the term $\alpha\lambda$ in $YX$ when considered as a map $E_0 \otimes E_1 \otimes E_3 \to E_{013}$, is

$$\langle (\alpha\lambda)(e_0, e_1, e_3) \mid e_2 \rangle = \langle \alpha(\lambda(e_0, e_1), e_3)) \mid e_2 \rangle.$$

These are equal and so we may briefly write $\alpha\lambda = \lambda\alpha$. Thus (14) simplifies to $\rho - \beta\mu - \gamma\nu$.

The remaining three entries in the $\rho$ column are obtained in the same way, and it is now clear that $(XYXZ)^2$ has order 2.

For reference, we list the actions of all three elements in Table 4.

| $(XYXZ)^2$ | $\gamma$ | $\beta$ | $\alpha$ | $\lambda$ | $\mu$ | $\nu$ | $\rho$ |
|------------|----------|----------|----------|-----------|-------|-------|-------|
| $(YZYX)^2$ | $-\gamma$ | $-\beta$ | $\alpha$ | $\lambda$ | $-\mu$ | $-\nu$ | $\rho$ |
| $(ZXZY)^2$ | $-\gamma$ | $-\beta$ | $-\alpha$ | $-\lambda$ | $\mu$ | $-\nu$ | $\rho$ |

Table 4. Actions on $G_3$ of the nonidentity elements of $K_4$.

We prove (13) by applying $XYX = YXY$ and its conjugates repeatedly to the LHS. Eventually we arrive at $(YZXZ)^2$. Since this has order 2 and each of $X, Y, Z$ have order 2, this is equal to $(ZXZY)^2$.

We remark that, given the action of $X$ and the action of $Y$ on $G_3$, there are two ways to understand how the action of the composition $YX$ should be calculated. They are duals of each other, and they correspond to thinking in terms of “frames” or thinking in terms of “coordinates”. If the reader is obtaining different calculations, this may be the reason. They are equivalent, however, and they both should agree on $K_4$.

From $(XYXZ)^2 \neq I$ it follows that $XYX$ and $Z$ do not commute. Notice however that the actions of $XYX$ and $Z$ on the set of building bundles do commute. The action of $XYX$ on the decomposed triple vector bundle, as distinguished from the action on the statomorphisms, takes place entirely in the roof double vector bundle (in terms of Figure 3) and preserves the floor double vector bundle. The action on $G_3$ however shows that the operations of $XYX$ and $Z$ are ‘entangled’.
From \cite{4.5} it follows that $K_4$ is the Klein 4-group. As an immediate consequence we have:

**Theorem 4.6.** The order of $DF_3$ is 96.

In \cite{9} the order of $DF_3$ (there denoted $VB_3$) was given as 72. Further, it was stated that $(XYZ)^4 = I$. Applying Table 2 the action of $(XYZ)^4$ on $G_3$ is as shown in Table 5 so $(XYZ)^4$ has order 2. More specifically, $(XYZ)^4 = (ZXZ)^2$ is a non-trivial element of $K_4$.

| $(XYZ)^4$ | $\gamma$ | $\beta$ | $\alpha$ | $\lambda$ | $\mu$ | $\nu$ | $\rho$ |
|-----------|-----------|---------|---------|---------|-------|-------|-------|
|           | $-\gamma$ | $\beta$ | $-\alpha$ | $-\lambda$ | $\mu$ | $-\nu$ | $\rho$ |

Table 5. Action of $(XYZ)^4$ on $G_3$.

**Theorem 4.7.**

(i). The group $DF_3$ is an extension of $S_4$ by $K_4$; that is, there is a short exact sequence $K_4 \longrightarrow DF_3 \longrightarrow S_4$.

(ii). As an $S_4$-module, $K_4$ is isomorphic to the normal subgroup of $S_4$

\{1, (12)(30), (23)(10), (13)(20)\}

with action by conjugation.

(iii). The extension $K_4 \longrightarrow DF_3 \longrightarrow S_4$ is not split.

**Proof.** (i) has already been established, and (ii) follows from \cite{10}. We now prove (iii).

Assume the extension is split. Let $f: DF_3 \rightarrow K_4 \rtimes S_4$ be an isomorphism, where $\rtimes$ represents semidirect product. Let $\sigma = (12)(30)$. Then $f(XYZ) = (u, \sigma)$ for some element $u \in K_4$. Using the semidirect product rule:

$f((XYZ)^2) = (u, \sigma)^2 = (u(\sigma \cdot u), \sigma^2) = (u^2, 1) = (1, 1),$

where we have used the fact that $\sigma$ acts trivially on $K_4$. We have reached a contradiction. \hfill $\square$

The three nontrivial elements of $K_4$ are in some ways comparable to the dualization of an ordinary vector bundle $A$: isomorphisms between $A$ and $A^*$ exist but are not natural, and likewise there are statomorphisms between a triple vector bundle $E$ and $E^W$ where $W \in K_4, W \neq 1$, but there is no canonical statomorphism. However, whereas ordinary dualization is contravariant, the elements of $K_4$ are covariant functors.

For arbitrary $n$–fold vector bundles, (i) of Theorem 4.7 will still be true \cite{5}. Instead of (ii) we will give a combinatorial description of $K_{n+1}$. The corresponding extension will be split if and only if $n$ is even.
5 The group $\mathcal{D}\mathcal{F}_3$

In this section we identify $\mathcal{D}\mathcal{F}_3$ as a semi-direct product and give its normal and conjugacy class structure.

First, consider the words $a = (ZXY)^2$, $b = (XYZ)^2$, and $c = (YZX)^2$. Let $H$ be the subgroup of $\mathcal{D}\mathcal{F}_3$ generated by $a$, $b$, and $c$. These three elements have order 4, commute with each other, and satisfy $abc = 1$. Thus $H \cong \mathbb{Z}_4 \times \mathbb{Z}_4$. A direct calculation shows that $H$ is a normal subgroup of $\mathcal{D}\mathcal{F}_3$. Second, consider $\mathcal{D}\mathcal{F}_2 \cong S_3$ as a subgroup of $\mathcal{D}\mathcal{F}_3$ generated by $X$ and $Y$. Then $S_3$ acts on $H$ by permuting $a$, $b$, and $c$. In addition, $H \cap \mathcal{D}\mathcal{F}_2 = \{1\}$, producing:

**Theorem 5.1.** $\mathcal{D}\mathcal{F}_3$ is isomorphic to the semidirect product $(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes S_3$, as described above.

All normal subgroups of $\mathcal{D}\mathcal{F}_3$ can be obtained with help of the homomorphism $\pi: \mathcal{D}\mathcal{F}_3 \to S_4$.

**Theorem 5.2.** The normal subgroups of $\mathcal{D}\mathcal{F}_3$, apart from the trivial ones and $K_4 = \ker \pi$, are

- $\pi^{-1}(A_4)$, which has index 2. This consists of the words in $X$, $Y$, and $Z$ that have an even number of letters. Alternatively, this consists of the duality functors which are covariant.

- The subgroup $H$ described above, which has index 6, and which is also $\pi^{-1}(V)$, where $V$ is the normal subgroup of $S_4$ of order 4.

**Proof.** It is clear that these are normal subgroups. We obtained that there are no others by use of GAP [3].

Also from GAP, or equally by hand calculation, we find that there are nine non-trivial conjugacy classes. In Table 6, we have listed their size, representative elements, their orders, and the action of these elements on $G_3$. We can read all the normal subgroups from this table. Apart from the identity, $K_4$ consists of the first conjugacy class, $H$ consists of the first four conjugacy classes, and $\pi^{-1}(A_4)$ consists of the first five conjugacy classes.

To conclude, there is a faithful linear representation of $\mathcal{D}\mathcal{F}_3$ on $\mathbb{R}^6$. Neglecting the $\rho$ column of Table 2, the action of $X, Y, Z \in \mathcal{D}\mathcal{F}_3$ on $G_3$ defines a linear action on $\mathbb{R}^6$ by matrices with entries 0, +1 or −1. From Table 6, the only element of $\mathcal{D}\mathcal{F}_3$ to fix all of $\gamma, \ldots, \nu$ is the identity.

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| s  | o | α  | β  | γ  | λ  | μ  | ν  | ρ  |
|----|---|----|----|----|----|----|----|----|
| (XYZ)^2 | 2 | 32 | | | | | | |
| (XYZ)^2 | 4 | 32 | | | | | | |
| (ZYX)^2 | 4 | 32 | | | | | | |
| XZXY | 4 | 32 | | | | | | |
| XY | 2 | 32 | | | | | | |
| Z | 2 | 32 | | | | | | |
| XYZXYZ | 2 | 32 | | | | | | |
| XYZ | 2 | 32 | | | | | | |
| ZYX | 2 | 32 | | | | | | |

Table 6. Conjugacy class structure of \( R\hat{F}_3 \). The leftmost column gives a representative element of each class, followed by the size of the class \( s \), the order of any element of the class \( o \), and the action of the representative on \( G_3 \).

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