Semileptonic decays of $\Lambda_c^+$ in dynamical approaches

C.Q. Geng$^{1,2,3,4}$, Chong-Chung Lih$^5$, Chia-Wei Liu$^3$ and Tien-Hsueh Tsai$^3$

$^1$School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, UCAS, Hangzhou 310024, China
$^2$International Centre for Theoretical Physics Asia-Pacific, Beijing/Hangzhou, China
$^3$Department of Physics, National Tsing Hua University, Hsinchu 300, Taiwan
$^4$Physics Division, National Center for Theoretical Sciences, Hsinchu 300, Taiwan
$^5$Department of Optometry, Central Taiwan University of Science and Technology, Taichung 406, Taiwan

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Abstract

We study the semileptonic decays of $\Lambda_c^+ \rightarrow \Lambda(n)\ell^+\nu_\ell$ in two relativistic dynamical approaches of the light-front constituent quark model (LFCQM) and MIT bag model (MBM). By considering the Fermi statistic between quarks and determining spin-flavor structures in baryons along with the helicity formalism in the two different dynamical models, we calculate the branching ratios ($\mathcal{B}s$) and averaged asymmetry parameters ($\alpha$s) in the decays. Explicitly, we find that $\mathcal{B}(\Lambda_c^+ \rightarrow \Lambda e^+\nu_e) = (3.43 \pm 0.57, 3.48\%)$ and $\alpha(\Lambda_c^+ \rightarrow \Lambda e^+\nu_e) = (-0.96 \pm 0.03, -0.83)$ in (LFCQM, MBM), in comparison with the data of $\mathcal{B}(\Lambda_c^+ \rightarrow \Lambda e^+\nu_e) = (3.6 \pm 0.4\%)$ and $\alpha(\Lambda_c^+ \rightarrow \Lambda e^+\nu_e) = -0.86 \pm 0.04$ given in the Particle Data Group, respectively. We also predict that $\mathcal{B}(\Lambda_c^+ \rightarrow \nu e^+\nu_e) = (2.15 \pm 0.41, 2.55) \times 10^{-3}$ and $\alpha(\Lambda_c^+ \rightarrow \nu e^+\nu_e) = (-0.97 \pm 0.01, -0.85)$ in (LFCQM, MBM), which could be observed by the ongoing experiments at BESIII, LHCb and BELLEII.
I. INTRODUCTION

Recently, the LHCb Collaboration has published the newest precision measurements on the anti-triplet charmed baryon lifetimes [1], given by

\[ \tau_{\Lambda^+_c} = 203.5 \pm 1.0 \pm 1.3 \pm 1.4 \text{ fs}, \]
\[ \tau_{\Xi^+_c} = 456.8 \pm 3.5 \pm 2.9 \pm 3.1 \text{ fs}, \]
\[ \tau_{\Xi^0_c} = 154.5 \pm 1.7 \pm 1.6 \pm 1.0 \text{ fs}. \] (1)

Surprisingly, the lifetime of \( \Xi^0_c \) measured by LHCb magnificently deviates from the previous value of \( \tau_{\Xi^0_c} = 112^{+13}_{-10} \text{ fs} \) in PDG [2]. Meanwhile, the Belle Collaboration has measured the absolute branching ratios of \( B(\Xi^0_c \to \Xi^- \pi^+) = (1.8 \pm 0.5)^\% \) [3] and \( B(\Xi^+_c \to \Xi^- \pi^+ \pi^+) = (2.86 \pm 1.21 \pm 0.38)^\% \) [4], which are the golden modes to determine other \( \Xi^0_c \) decay channels, respectively. It is clear that we are now witnessing a new era of charm physics. One can expect there will be more and more new experimental data and precision measurements in the future, which are also the guiding light for people to explore new physics beyond the standard model.

There have been recently many works discussing the anti-triplet charm baryon decays. Because of the complicated structures of these baryons with large non-perturbative effects of the quantum chromodynamic (QCD), it is very hard to calculate the decay amplitudes from first principles. In the literature, people use the flavor symmetry of \( SU(3)_f \) to analyze various charmed baryon decay processes, such as semi-leptonic, two-body and three-body non-leptonic decays, to obtain reliable results [5–25]. However, the \( SU(3)_f \) symmetry is an approximate symmetry, resulting in about 10% error for the predictions naturally. In order to more precision calculations, we need a dynamical QCD model to understand each process. To avoid other complicated problem like the non-factorizable effect, we only discuss the semi-leptonic processes, which are purely factorizable ones. In particular, we focus on the \( \Lambda^+_c \) semi-leptonic decays in this work. There are several theoretical analyses and lattice QCD calculations on the charmed baryon semi-leptonic decays with different models in the literature [26–31]. In this paper, we will mainly use the light-front (LF) formalism to study the decays and check the results in the MIT bag model (MBM) as comparisons.

The LF formalism is considered as a consistent relativistic approach, which has been very successful in the mesonic and light quark sectors [32, 33]. Due to this success, it has
been extended to other systems, such as those involving the heavy mesons, pentaquarks and so on [34–48]. In addition, the bottom baryon to charmed baryon nonleptonic decays in the LF approach have been done in Refs. [49, 50]. For a review on the non-perturbative nature in the equation of motion and QCD vacuum structure for the LF constituent quark model (LFCQM), one can refer to the article in Ref. [32]. The advantage of LFCQM is that the commutativity of the LF Hamiltonian and boost generators provide us with a good convenience to calculate the wave-function in different inertial frames because of the recoil effect. In addition, since the AdS/CFT correspondence [51] was proposed by Juan Maldacena in the late of 1997, the LF holography as a feature of the AdS/CFT duality has brought the LF QCD from a phenomenological theory to a more fundamental one [52].

This paper is organized as follows. We present our formal calculations of the baryonic transition form factors for LFCQM and MBM in Secs. II and III, respectively. We show our numerical results of the form factors, branching ratios and averaged asymmetry parameters in Sec. IV. We also compare our results with those in the literature. In Sec. V, we give our discussions and conclusions.

II. BARYONIC TRANSITION FORM FACTORS IN LFCQM

A. Vertex function of baryon

In LFCQM, a baryon with its momentum \( P \) and spin \( S \) as well the z-direction projection of \( S_z \) are considered as a bound state of three constitute quarks. As a result, the baryon state can be expressed by [32, 33, 42, 53–55]

\[ |B, P, S, S_z⟩ = \int \{d^3\tilde{p}_1\} \{d^3\tilde{p}_2\} \{d^3\tilde{p}_3\} 2(2\pi)^3 \frac{1}{\sqrt{P}} δ^3(\tilde{P} - \tilde{p}_1 - \tilde{p}_2 - \tilde{p}_3) \times \sum_{λ_1, λ_2, λ_3} \Psi^{SSz}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, λ_1, λ_2, λ_3) C^{αβγ} F_{abc} |q^a_α(\tilde{p}_1, λ_1) q^b_β(\tilde{p}_2, λ_2) q^c_α(\tilde{p}_3, λ_3)⟩ \]  

(2)

where \( \Psi^{SSz}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, λ_1, λ_2, λ_3) \) is the vertex function, which can be formally solved from Bethe-Salpeter equations by the Faddeev decomposition method, \( C^{αβγ} \) and \( F_{abc} \) are the color and flavor factors, \( λ_i \) and \( \tilde{p}_i \) with \( i = 1, 2, 3 \) are the LF helicities and 3-momentum of the on-mass-shell constituent quarks, defined as

\[ \tilde{p}_i = (p^+_i, p^i⊥), \quad p^i⊥ = (p^1_i, p^2_i), \quad p^+_i = \frac{m^2_i + p^2_i}{p_i^+}, \]  

(3)
and
\[ d^3\bar{p}_i \equiv \frac{dp_1^+ dp_2^+ dp_{i\perp}}{2(2\pi)^3}, \quad \delta^3(\bar{p}) = \delta(p^+ )\delta^2(p_{\perp}), \]
\[ |q_0(\bar{p}, \lambda)\rangle = d_{a}(\bar{p}, \lambda)|0\rangle, \quad \{d_{a}^i(\bar{p}, \lambda), d_{a}^j(\bar{p}, \lambda)\} = 2(2\pi)^3 \delta^3(\bar{p} - \bar{p})\delta_{\lambda\lambda'}\delta_{\alpha\alpha'}\delta^{\alpha\alpha'}. \tag{4} \]

To describe the internal motion of the constituent quarks, we introduce the kinematic variables \((q_\perp, \xi)\) and \((Q_\perp, \eta)\) and \(P_{tot}\), given by
\[ P_{tot} = \bar{P}_1 + \bar{P}_2 + \bar{P}_3, \quad \xi = \frac{p_1^+}{p_1^+ + p_2^+}, \quad \eta = \frac{p_1^+ + p_2^+}{P_{tot}^+}, \]
\[ q_\perp = (1 - \xi)p_{1\perp} - \xi p_{2\perp}, \quad Q_\perp = (1 - \eta)(p_{1\perp} + p_{2\perp}) - \eta p_{3\perp}, \tag{5} \]
where \((q_\perp, \xi)\) characterize the relative motion between the first and second quarks, while \((Q_\perp, \eta)\) the third quark and other two quarks. The invariant masses of \((q_\perp, \xi)\) and \((Q_\perp, \eta)\) systems are represented by:
\[ M_3^2 = \frac{q_\perp^2}{\xi(1 - \xi)} + \frac{m_1^2}{1 - \xi}, \]
\[ M_2^2 = \frac{Q_\perp^2}{\eta(1 - \eta)} + \frac{m_2^2}{1 - \eta}, \tag{6} \]
respectively. Unlike Refs. \[54, 55\] or Ref. \[53\], which treat the diquark as a point like object or spectator, we consider the three constituent quarks in the baryon independently with suitable quantum numbers satisfying Fermi statistics to have a correct baryon bound state system. The vertex function of \(\Psi_{SS_z}(\bar{p}_1, \bar{p}_2, \bar{p}_3, \lambda_1, \lambda_2, \lambda_3)\) in Eq. (2) can be written as:
\[ \Psi_{SS_z}(\bar{p}_1, \bar{p}_2, \bar{p}_3, \lambda_1, \lambda_2, \lambda_3) = \Phi(q_\perp, \xi, Q_\perp, \eta)\Xi_{SS_z}(\lambda_1, \lambda_2, \lambda_3), \tag{7} \]
where \(\Phi(q_\perp, \xi, Q_\perp, \eta)\) is the momentum distribution of constituent quarks and \(\Xi_{SS_z}(\lambda_1, \lambda_2, \lambda_3)\) represents the momentum-depended spin wave function, given by
\[ \Xi_{SS_z}(\lambda_1, \lambda_2, \lambda_3) = \sum_{s_1, s_2, s_3} \langle \lambda_1| R_1^\dagger |s_1\rangle \langle \lambda_2| R_2^\dagger |s_2\rangle \langle \lambda_3| R_3^\dagger |s_3\rangle \left( \frac{1}{2} s_1 \frac{1}{2} s_2 \frac{1}{2} s_3 |SS_z\rangle \right), \tag{8} \]
with \(\langle \frac{1}{2} s_1, \frac{1}{2} s_2, \frac{1}{2} s_3|SS_z\rangle\) the usual \(SU(2)\) Clebsch-Gordan coefficient, and \(R_i\) the well-known Melosh transformation, which corresponds to the \(i\)th constituent quark and can be expressed by
\[ R_M(x, p_\perp, m, M) = \frac{m + xM - \vec{i}\vec{\sigma} \cdot (\bar{n} \times \vec{q})}{\sqrt{(m + xM)^2 + q_\perp^2}}. \tag{9} \]
and

\[ R_1 = R_M(\eta, Q_\perp, M_3, M)R_M(\xi, q_\perp, m_1, M_3), \]
\[ R_2 = R_M(\eta, Q_\perp, M_3, M)R_M(1 - \xi, -q_\perp, m_2, M_3), \]
\[ R_3 = R_M(1 - \eta, -Q_\perp, m_3, M), \]

where \( \sigma_i \) is the Pauli matrix and \( \vec{n} = (0, 0, 1) \). This is the generalization of the Melosh transformation from two-particle systems, which can be derived from the transformation property of angular momentum operators \[33, 57\]. We further represent the LF kinematic variables \((\xi, q_\perp)\) and \((\eta, Q_\perp)\) in the forms of the ordinary 3-momenta \(q\) and \(Q\):

\[ E_{1(2)} = \sqrt{q^2 + m_1^2}, \quad E_{12} = \sqrt{Q^2 + M_3^2}, \quad E_3 = \sqrt{Q^2 + m_3^2}, \]
\[ q_z = \frac{\xi M_3}{2} - \frac{m_1^2 + q_\perp^2}{2M_3\xi}, \quad Q_z = \frac{\eta M}{2} - \frac{M_3^2 + Q_\perp^2}{2M_\eta}, \]

(11)

to get more clear physical pictures of the momentum distribution wave functions.

It is known that the exact momentum wave function cannot be solved from the QCD first principle currently due to the lack of knowledge in the QCD effective potential in the three-body system. Hence, we choose the phenomenological Gaussian type wave function with suitable shape parameters to including the diquark clustering effects in \(\Lambda_c^+\) and \(\Lambda\) baryons \[33, 53\]. The baryon spin-flavor-momentum wave function \(F_{abc}\Psi^{SSz}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \lambda_1, \lambda_2, \lambda_3)\) should be totally symmetric under any permutations of quarks to keep the Fermi statistics. The spin-flavor-momentum wave functions of \(\Lambda_c^+, \Lambda\) and the neutron are given by

\[ |\Lambda_c\rangle = \frac{1}{\sqrt{6}}[\phi_3 \chi_3^{\alpha_3}(|duc\rangle - |udc\rangle) + \phi_2 \chi_2^{\alpha_2}(|dcu\rangle - |ucd\rangle) + \phi_1 \chi_1^{\alpha_1}(|cdu\rangle - |cud\rangle)], \]
\[ |\Lambda\rangle = \frac{1}{\sqrt{6}}[\phi_3 \chi_3^{\alpha_3}(|duc\rangle - |uds\rangle) + \phi_2 \chi_2^{\alpha_2}(|dsu\rangle - |usd\rangle) + \phi_1 \chi_1^{\alpha_1}(|sdu\rangle - |sud\rangle)], \]
\[ |n\rangle = \frac{1}{\sqrt{3}}[\chi_{\lambda_3}^{\lambda_3}|ddu\rangle + \chi_{\lambda_2}^{\lambda_2}|dud\rangle + \chi_{\lambda_1}^{\lambda_1}|udd\rangle], \]

(12)

respectively, where

\[ \chi_\uparrow^{\alpha_3} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle), \quad \chi_\uparrow^{\lambda_3} = \frac{1}{\sqrt{6}}(|\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle - 2|\uparrow\uparrow\downarrow\downarrow\rangle), \]
\[ \phi_3 = \mathcal{N} \sqrt{\frac{\partial q_z}{\partial \xi} \frac{\partial Q_z}{\partial \eta} e^{-\frac{q_z^2}{2\delta_q^2} - \frac{Q_z^2}{2\delta_Q^2}}}, \]

(13)

and \(\phi_{1(2)}\) has the form by replacing \((q, Q)\) with \((q_{1(2)}, Q_{1(2)})\) in \(\phi_3\), with \(\mathcal{N} = 2(2\pi)^3(\beta_q \beta_Q \pi)^{-3/2}\) and \(\beta_q, \beta_Q\) being the normalized constant and shape parameters, respectively. Explicitly, \(q_{1(2)}\)
and $Q_{1(2)}$ are given by

\begin{align*}
\xi_{1(2)} &= \frac{p_{2(3)}^+}{p_{2(3)}^+ + p_{(1)}^+}, \quad \eta_{1(2)} = 1 - \frac{P_{(2)}^+}{P_{tot}^+}, \\
q_{1(2)} &= (1 - \xi_{1(2)})p_{2(3)} - \xi_{1(2)}p_{3(1)}^\perp, \\
Q_{1(2)} &= (1 - \eta_{1(2)})(p_{2(3)} + p_{3(1)}^\perp) - \eta_{1(2)}p_{1(2)}^\perp.
\end{align*}

(14)

Here, the baryon state is normalized as

\[ \langle B, P', S', S'_z | B, P, S, S_z \rangle = 2(2\pi)^3 P^+ \delta^3(\vec{P}' - \vec{P}) \delta_{S_z' S_z}, \]

(15)

resulting in the normalization of the momentum wave function, given by

\[ \frac{1}{2^2(2\pi)^6} \int d\xi_{1(2)} d\eta_{1(2)} d^2q_{1(2)} d^2Q_{1(2)} |\phi_{3(1,2)}|^2 = 1. \]

(16)

We emphasize that the momentum wave functions of $\phi_i$ with the different shape parameters of $\beta_q$ and $\beta_Q$ describe the scalar diquark effect in $\Lambda_{(c)}$. For the neutrons, the momentum distribution functions are the same, i.e. $\phi = \phi_3(\beta_q = \beta_Q)$, for any spin-flavor state due to the isospin symmetry. Note that there is no $SU(6)$ spin-flavor symmetry in $\Lambda_{(c)}$ even though the forms of these states are similar to those with the $SU(6)$ spin-flavor wave functions.

### B. Transition form factors

The baryonic transition form factors of the $V - A$ weak current are defined by

\[ \langle B_f, P', S', S'_z | \bar{q}\gamma^\mu(1 - \gamma_5)c | B_i, P, S, S_z \rangle = \frac{2(2\pi)^3 P^+ \delta^3(\vec{P}' - \vec{P}) \delta_{S_z' S_z}}{2^2(2\pi)^6} \]

resulting in the normalization of the momentum wave function, given by

\[ \frac{1}{2^2(2\pi)^6} \int d\xi_{1(2)} d\eta_{1(2)} d^2q_{1(2)} d^2Q_{1(2)} |\phi_{3(1,2)}|^2 = 1. \]

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matrix elements through the relations

\[ f_1(k^2) = \frac{1}{2P^+} \langle B_f, P', \uparrow | q^+ c | B_i, P, \uparrow \rangle, \]
\[ f_2(k^2) = \frac{1}{2P^+} \frac{M_{B_i}}{k_\perp} \langle B_f, P', \uparrow | q^+ c | B_i, P, \downarrow \rangle, \]
\[ g_1(k^2) = \frac{1}{2P^+} \langle B_f, P', \uparrow | q^+ \gamma_5 c | B_i, P, \uparrow \rangle, \]
\[ g_2(k^2) = \frac{1}{2P^+} \frac{M_{B_i}}{k_\perp} \langle B_f, P', \uparrow | q^+ \gamma_5 c | B_i, P, \downarrow \rangle. \]

Note that \( f_3 \) and \( g_3 \) cannot be obtained when \( k^+ = 0 \), but they are negligible because of the suppressions of the \( k^2 \) factors. With the help of the momentum distribution functions and the Melosh transformation matrix, the transition matrix elements can be expressed as

\[
\langle B_f, P', S', S'_3 | q^+ c | B_i, P, S, S_3 \rangle = \frac{1}{2^2(2\pi)^6} \int d\xi d\eta d^2 q_\perp d^2 Q_\perp \Phi(q'_\perp, \xi, Q'_\perp, \eta) \Phi(q_\perp, \xi, Q_\perp, \eta) F_{def}^{(a) (b)} F_{abc}^{(a) (b)} \]
\[ \times \sum_{s_1, s_2, s_3} \sum_{s'_1, s'_2, s'_3} \langle S', S'_3 | s'_1, s'_2, s'_3 \rangle \langle s_1, s_2, s_3 | S, S_3 \rangle \langle s'_1 | R'_1 R'_1 | s_1 \rangle \langle s'_2 | R'_2 R'_2 | s_2 \rangle \]
\[ \times 2P^+ \sum_{\lambda'_3 \lambda_3} \langle s'_3 | R'_3 | \lambda'_3 \rangle (\delta_{q'q} 3 \delta_{q'q} \lambda'_3 \lambda_3 \delta_{\sigma'\sigma} (\lambda_3 | R'_3 | s_3) \rangle, \]

\[
\langle B_f, P', S', S'_3 | q^+ \gamma_5 c | B_i, P, S, S_3 \rangle = \frac{1}{2^2(2\pi)^6} \int d\xi d\eta d^2 q_\perp d^2 Q_\perp \Phi(q'_\perp, \xi, Q'_\perp, \eta) \Phi(q_\perp, \xi, Q_\perp, \eta) F_{def}^{(a) (b)} F_{abc}^{(a) (b)} \]
\[ \times \sum_{s_1, s_2, s_3} \sum_{s'_1, s'_2, s'_3} \langle S', S'_3 | s'_1, s'_2, s'_3 \rangle \langle s_1, s_2, s_3 | S, S_3 \rangle \langle s'_1 | R'_1 R'_1 | s_1 \rangle \langle s'_2 | R'_2 R'_2 | s_2 \rangle \]
\[ \times 2P^+ \sum_{\lambda'_3 \lambda_3} \langle s'_3 | R'_3 | \lambda'_3 \rangle (\delta_{q'q} 3 (\sigma_z) \lambda'_3 \lambda_3 \delta_{\sigma'\sigma} (\lambda_3 | R'_3 | s_3) \rangle. \]
Using Eqs. (18), (19) and (20), we find that

\begin{equation}
    f_1(k^2) = \frac{3}{2^2(2\pi)^6} \int d\xi d\eta d^2Q_\perp \Phi(q'_\perp, \xi, Q'_\perp, \eta)\Phi(q_\perp, \xi, Q_\perp, \eta)(F^{def} F_{abc} \delta_{qf} \delta_{cq} \delta^a \delta_b) \\
    \times \sum_{s_1,s_2,s_3} \sum_{s'_1,s'_2,s'_3} \langle s', \uparrow | s'_1, s'_2, s'_3 \rangle \langle s_1, s_2, s_3 | S, \uparrow \rangle \prod_{i=1,2,3} \langle s_i | R'_i R^i_i | s_i \rangle ,
\end{equation}

\begin{equation}
    g_1(k^2) = \frac{3}{2^2(2\pi)^6} \int d\xi d\eta d^2Q_\perp \Phi(q'_\perp, \xi, Q'_\perp, \eta)\Phi(q_\perp, \xi, Q_\perp, \eta)(F^{def} F_{abc} \delta_{qf} \delta_{cq} \delta^a \delta_b) \\
    \times \sum_{s_1,s_2,s_3} \sum_{s'_1,s'_2,s'_3} \langle s', \uparrow | s'_1, s'_2, s'_3 \rangle \langle s_1, s_2, s_3 | S, \uparrow \rangle \prod_{i=1,2,3} \langle s_i | R'_i R^i_i | s_i \rangle ,
\end{equation}

\begin{equation}
    f_2(k^2) = \frac{3}{2^2(2\pi)^6} \frac{M_{B_1}}{k_\perp} \int d\xi d\eta d^2Q_\perp \Phi(q'_\perp, \xi, Q'_\perp, \eta)\Phi(q_\perp, \xi, Q_\perp, \eta)(F^{def} F_{abc} \delta_{qf} \delta_{cq} \delta^a \delta_b) \\
    \times \sum_{s_1,s_2,s_3} \sum_{s'_1,s'_2,s'_3} \langle s', \uparrow | s'_1, s'_2, s'_3 \rangle \langle s_1, s_2, s_3 | S, \downarrow \rangle \prod_{i=1,2,3} \langle s_i | R'_i R^i_i | s_i \rangle ,
\end{equation}

\begin{equation}
    g_2(k^2) = \frac{3}{2^2(2\pi)^6} \frac{M_{B_1}}{k_\perp} \int d\xi d\eta d^2Q_\perp \Phi(q'_\perp, \xi, Q'_\perp, \eta)\Phi(q_\perp, \xi, Q_\perp, \eta)(F^{def} F_{abc} \delta_{qf} \delta_{cq} \delta^a \delta_b) \\
    \times \sum_{s_1,s_2,s_3} \sum_{s'_1,s'_2,s'_3} \langle s', \uparrow | s'_1, s'_2, s'_3 \rangle \langle s_1, s_2, s_3 | S, \downarrow \rangle \prod_{i=1,2,3} \langle s_i | R'_i R^i_i | s_i \rangle \langle s'_i | R'_3 R^3_i | s'_3 \rangle .
\end{equation}

### III. Baryonic Transition Form Factors in MBM

The formalism and other details for MBM can be found in Ref. [31]. In the calculation of MBM, we take the same notations as those in Ref. [31]. In this approach, the current quark masses are used, given by

\begin{equation}
    m_{u,d} = 0.005 \text{ GeV}, \quad m_s = 0.28 \text{ GeV}, \quad m_c = 1.5 \text{ GeV}, \quad R = 5 \text{ GeV}^{-1},
\end{equation}

where $R$ corresponds to the bag size. Note that the form factors can be only evaluated at $k = 0$ ($k^2 = \Delta M^2$) due to the assumption of the static bag. The form factors are decomposed as follows:

\begin{align}
    f_1 &= \nu_0 - \nu_M \Delta M^2 / M_{12} - \nu_V \Delta M, \\
    f_2 &= (-\nu_0 + \nu_M M_{12} + \nu_V \Delta M) M_1 / M_{12}, \\
    f_3 &= \nu_V M_1 + \nu_M M_1 \Delta M / M_{12}, \\
    g_1 &= (1 - \Delta M^2 / 2 M_{12}^2) A_s + (A_T \Delta M - A_0) 4 M_1 M_2 \Delta M / M_{12}, \\
    g_2 &= (A_T \Delta M - A_0 \Delta M / 8 M_1 M_2 - A_0) 4 M_1^2 M_2 / M_{12}, \\
    g_3 &= (A_s / 2 + A_T M_1 M_2) M_1 / M_{12},
\end{align}

(26)
with $\Delta M = M_1 - M_2$, $M_{12} = M_1 + M_2$ and

$$V_0 = AR^3 \left( W^+_1 W^+_1 I_{00} + W^+_1 W^+_1 I_{11} \right),$$

$$V_e = AR^3 \left( W^-_1 W^+_1 I_{10} - W^+_1 W^+_1 I_{01} \right) (R/3),$$

$$V_M = AR^3 \left( W^-_1 W^+_1 I_{10} + W^+_1 W^-_1 I_{01} \right) (R/3),$$

$$A_0 = AR^3 \left( W^+_1 W^+_1 I_{10} - W^+_1 W^-_1 I_{01} \right) (R/3),$$

$$A_f = AR^3 \left( W^+_1 W^+_1 - W^-_1 W^-_1 I_{11}/3 \right),$$

$$A_T = AR^3 W^+_1 W^-_1 J_{11} (-2R^2/15),$$

(27)

where $A$ is the normalized factor for the baryon, corresponding to the baryon spin-flavor structures as given by Table. II in Ref. [31]. $W^q_\pm$ are associated with the normalized factors for quarks, given by

$$W^q_\pm \equiv \left( \frac{\omega^q \pm m_q}{\omega^q} \right)^{1/2}$$

(28)

with $\omega^q$ representing the quark energy, and $I$ and $J$ stand for the overlapping factors for the quark wave functions, defined by

$$I_{nm} \equiv \int_0^1 dt t^j j_n (tx^i_0) j_m (tx^i_0), \quad n = 0, 1$$

$$I_{nm} \equiv \int_0^1 dt t^j j_n (tx^i_0) j_m (tx^i_0), \quad n, m = 0, 1 \ (n \neq m)$$

$$J_{11} \equiv \int_0^1 dt t^j j_1 (tx^i_0) j_1 (tx^i_0),$$

(29)

with $j_n$ the Bessel function and $x^q_0$ the lowest root of the transcendental equation of

$$\tan(x^q) = \frac{x^q}{1 - m_q R - [(x^q)^2 + (m_q R)^2]^{1/2}}.$$

(30)

**IV. NUMERICAL RESULTS**

In section II, we have derived the baryonic transition form factors in LFCQM. The form factors can be evaluated only in the space-like region ($k^2 = -k^2$) because of the condition $k^+ = 0$. Thus, we follow the standard procedures in Refs. [41, 42, 54] to extract the information of the form factors in the time-like region. These procedures have widely been tested and discussed in the mesonic sector [58, 59]. We fit $f_{1(2)}(k^2)$ and $g_{1(2)}(k^2)$ with some analytic functions in the space-like region, which are analytically continued to the physical
TABLE I. Values of the constituent quark masses ($m_i$) and shape parameter ($\beta_{qB}$ and $\beta_{QB}$) in units of GeV.

| $m_c$ | $m_s$ | $m_d$ | $m_c$ | $\beta_{q\Lambda_c}$ | $\beta_{Q\Lambda_c}$ | $\beta_{q\Lambda}$ | $\beta_{Q\Lambda}$ | $\beta_{q\Lambda}$ | $\beta_{Q\Lambda}$ |
|-------|-------|-------|-------|------------------------|------------------------|-------------------|-------------------|-------------------|-------------------|
| 1.3   | 0.4   | 0.26  | 0.26  | 0.89                   | 0.75                   | 0.53              | 0.52              | 0.44              | 0.44              |

time-like region ($k^2 > 0$). We employ the numerical values of the constituent quark masses and shape parameters in Table. I. The values of the shape parameters can be determined approximately by the calculations in the mesonic sectors [53, 60]. By assuming that the Coulomb-like potential is dominant in the quark-quark strong interaction, one can deduce the shape parameter of quark pairs to be $\sqrt{2}$ greater than those from the mesonic sectors because the interaction is about twice stronger between the quark-quark pair than quark-anti-quark one [53]. Since the reciprocals of the shape parameters are related to the sizes of systems, we adopt $\beta_{q\Lambda_c} \simeq 2(\sqrt{2}\beta_{ud})$ and $\beta_{q\Lambda} \simeq 1.2(\sqrt{2}\beta_{ud})$, where the factors of 2 and 1.2 come from the effects of the diquark clusterings, respectively, which make the light quark pairs to be more compact. By using Eqs. (21)-(24), we compute totally 32 points for all form factors from $k^2 = 0$ to $k^2 = -9.7$ GeV$^2$. With the MATLAB curve fitting toolbox, we present our results of $\Lambda_c^+ \rightarrow \Lambda$ in Figs. 2 and 3 and $\Lambda_c^+ \rightarrow n$ in Figs. 4 and 5 with 95% confidence bounds in Appendix. To fit the $k^2$ dependences of the form factors, we use the form

$$F(k^2) = \frac{p_1 k^4 + p_2 k^2 + F(0)}{1 - q_1 k^2 + q_2 k^4}.$$  \hspace{1cm} (31)

In the $\Lambda_c^+ \rightarrow \Lambda$ transition, we choose $p_1 = 0$, $p_2 = 0$ for $f_{1,2}$ and $g_1$, but only $c$ $q_1 = 0$ for $g_2$. On the other hand, in the $\Lambda_c^+ \rightarrow n$ transition, we take $p_1 = p_2 = 0$ for all form factors to fit the numerical values in the space-like region. We present our fitting results in Table. III.

For MBM, we use the Lorentzian type functions for the $k^2$ dependences of the form factors, given by

$$f_i(k^2) = \frac{(1 + d_f) f_i(0)}{(1 - k^2 M_V^2)^2 + d_f}$$  \hspace{1cm} (32)

$$g_i(k^2) = \frac{(1 + d_g) g_i(0)}{(1 - k^2 M_A^2)^2 + d_g}$$  \hspace{1cm} (33)

where $M_V = 2.112 \ (2.010)$, $M_A = 2.556 \ (2.423)$, $d_f = 0.2 \ (0.1)$ and $d_g = 0.1 \ (0.05)$ for $c \rightarrow s(d)$ processes, respectively. We list $f_i(0) = f_i$ and $g_i(0) = g_i$ in Table. III.
In order to calculate the decay branching ratios and other physical quantities, we introduce the helicity amplitudes of $H_{V(A)}^{V(A)}$, which give more intuitive physical pictures and simpler expressions when discussing the asymmetries of the decay processes, such as the integrated (averaged) asymmetry, also known as the longitudinal polarization of the daughter...
baryon. Relations between the helicity amplitudes and form factors are given by

\[ H^V_{\pm 1} = \sqrt{2K_\pm} \left( -f_1 - \frac{M_{B_i} + M_{B_f}}{M_{B_i}} f_2 \right), \]

\[ H^V_{\pm 1} = \frac{\sqrt{K_-}}{\sqrt{k^2}} \left( (M_{B_i} + M_{B_f}) f_1 + \frac{k^2}{M_{B_i}} f_2 \right), \]

\[ H^V_{\pm 1} = \frac{\sqrt{K_+}}{\sqrt{k^2}} \left( (M_{B_i} + M_{B_f}) f_1 + \frac{k^2}{M_{B_i}} f_3 \right), \]

\[ H^A_{\pm 1} = \sqrt{2K_+} \left( g_1 - \frac{M_{B_i} - M_{B_f}}{M_{B_i}} g_2 \right), \]

\[ H^A_{\pm 0} = \frac{\sqrt{K_+}}{\sqrt{k^2}} \left( -(M_{B_i} - M_{B_f}) g_1 + \frac{k^2}{M_{B_i}} g_2 \right), \]

\[ H^A_{\pm 0} = \frac{\sqrt{K_+}}{\sqrt{k^2}} \left( -(M_{B_i} - M_{B_f}) g_1 + \frac{k^2}{M_{B_i}} g_3 \right), \]

where \( K_\pm = (M_{B_i} - M_{B_f})^2 - k^2 \). We note that both \( f_3 \) and \( g_3 \) have been set to be 0 in LFCQM.

The differential decay width and asymmetries can be expressed in the analytic forms in terms of the helicity amplitudes, which can be found in our previous work of Ref. [22]. In our numerical calculations, we use the center value of \( \tau_{\Lambda^0_c} = 203.5 \times 10^{-15} \text{s} \) in Eq. (1) [1]. Our predictions of the decay branching ratios (\( Brs \)) and asymmetries (\( \alpha s \)) are listed in Table. IV In Table. V we compare our results with the experimental data and those in various calculations in the literature.

### TABLE IV. Predictions of branching ratio and asymmetry parameters

|                | LFCQM          | MBM            |
|----------------|----------------|----------------|
|                | \( Br(\%) \)  | \( \alpha \)  | \( Br(\%) \)  | \( \alpha \)  |
| \( \Lambda_c^+ \to \Lambda e^+ \nu_e \) | 3.43 ± 0.57    | -0.96 ± 0.03   | 3.48           | -0.83          |
| \( \Lambda_c^+ \to \Lambda \mu^+ \nu_\mu \) | 3.30 ± 0.56    | -0.96 ± 0.03   | 3.38           | -0.82          |
| \( \Lambda_c^+ \to n e^+ \nu_e \) | 0.215 ± 0.041  | -0.97 ± 0.01   | 0.255          | -0.85          |
| \( \Lambda_c^+ \to n \mu^+ \nu_\mu \) | 0.209 ± 0.041  | -0.97 ± 0.01   | 0.250          | -0.85          |

In LF [30] and HQET [26], the authors use a specific spin-flavor structure of \( c(ud - du) \chi_{s_z}^{\rho_3} \) for the charmed baryon state, in which only the permutation relation is considered between light quarks. In addition, they assume that the diquarks from the light quark pairs are spectator and structureless. These simplifications in Refs. [30] and [26] make their results
TABLE V. Our results in comparisons with the experimental data and those in various calculations in the literature.

|               | $\Lambda^+_c \rightarrow \Lambda^+\nu_e$ | $\Lambda^+_c \rightarrow \eta^+\nu_e$ |
|---------------|----------------------------------------|----------------------------------------|
| $Br(\%)$      | $\alpha$                               | $Br(\%)$                              |
| LFCQM         | $3.43 \pm 0.57$                        | $-0.96 \pm 0.03$                      |
| MBM           | $3.48$                                 | $-0.83$                               |
| Data [2]      | $3.6 \pm 0.4$                          | $-0.86 \pm 0.04$                      |
| $SU(3)$ [22]  | $3.4 \pm 0.3$                          | $-0.86 \pm 0.04$                      |
| HQET [26]     | $1.42$                                 | -                                      |
| LF [30]       | $1.63$                                 | $0.201$                               |
| MBM a (NRQM)  | $2.6$ (3.2)                            | $0.20$ (0.30)                         |
| LQCD [27, 28] | $3.80 \pm 0.22$                        | $-0.410 \pm 0.029$                    |
| RQM [29]      | $3.25$                                 | $0.268$                               |

* Although the values of $f_i$ and $g_i$ are the same at the zero recoil point ($q = 0$), we use the Lorentzian type of the $k^2$ dependences for the form factors instead of the dipole ones in this work.

To be not good compared with the experimental data as shown in Table V. Based on the Fermi statistics, the overall spin-flavor-momentum structures are determined, from which the parameters like quark masses, baryon masses and shape parameters can recover the spin-flavor symmetry. In LFCQM, we consider the different diquark clustering effect in different baryons. We expect that this effect is stronger if the mass of the third quark is greater than others, which is encoded in the shape parameter of $\beta_{qB}$. There is an interesting observation that the shape parameters $\beta_{QB}$ and $\beta_{qB}$ in our study are almost the same in each baryon, which implies the totally symmetric momentum distribution of three constituent quarks in the baryon. In addition, the flavor symmetry breaking effect due to the quark masses seems to get canceled due to the clustering effect of the shape parameters in the momentum distribution functions. Our numerical results indicate that the form factors follow the Lorentzian functions of $F(k^2) = F(0)/(1 - q_1k^2 + q_2k^4)$ except $g_2(k^2)$ in the $\Lambda^+_c \rightarrow \Lambda$ processes. Our results of $f_i(k^2) \neq g_i(k^2)$ show that the lowest order of the heavy quark symmetry is failure because the constituent charm quark mass is not heavy enough.

From Table V, we predict that $B(\Lambda^+_c \rightarrow \Lambda^+\nu_e) = (3.43 \pm 0.57) \times 10^{-2}$ and $B(\Lambda^+_c \rightarrow$
ne^+\nu_e) = (2.15 \pm 0.41) \times 10^{-3}$, and $\alpha(\Lambda_c^+ \to \Lambda e^+\nu_e) = -0.96 \pm 0.03$ and $\alpha(\Lambda_c^+ \to ne^+\nu_e) = -0.97 \pm 0.01$ in LFCQM, in which the value of $B(\alpha)$ for the mode of $\Lambda_c^+ \to \Lambda e^+\nu_e$ is lower (higher) than but acceptable by the experimental one $(3.6 \pm 0.4) \times 10^{-2}$ $(-0.86 \pm 0.04)$ in PDG [2]. The errors in our results mainly come from the numerical fits of the MATLAB curve fitting toolbox in Appendix, in which the 95% confidence bounds are broadened and tightened in the time-like space-like regions, respectively. Our results are also consistent with those in the Lattice QCD (LQCD) [27, 28] and relativistic quark model (RQM) [29]. For MBM, Although the semi-leptonic processes have been fully studied in Ref. [31], their results are mismatched with the current data. By using the same formalism with the same input parameters, we are able to get the same values of the form factors at the zero recoil point. By taking the Lorentzian $k^2$ dependences for the form factors, inspired from our LF calculations, we obtain much better results as shown in Table. V. It is interesting to see that our results for $\Lambda_c^+ \to ne^+\nu_e$ are consistent with other calculations except those from $SU(3)_F$ and LQCD.

V. CONCLUSIONS

We have studied the semi-leptonic decays of $\Lambda_c^+ \to \Lambda(n)\ell^+\nu_\ell$ in the two dynamical approaches of LFCQM and MBM. We have used the Fermi statistics to determine the overall spin-flavor-momentum structures and recover the spin-flavor symmetry with the quark and baryon masses and shape parameters. We have found that $B(\Lambda_c^+ \to \Lambda e^+\nu_e) = (3.43 \pm 0.57)\%$ and 3.48% in LFCQM and MBM, which are consistent with the experimental data of $(3.6 \pm 0.4) \times 10^{-2}$ [2], as well as the values predicted by $SU(3)_F$ [22], LQCD [27, 28] and RQM [29], but about a factor of two larger than those in HQET [26] and LF [30]. We have also obtained that $\alpha(\Lambda_c^+ \to \Lambda e^+\nu_e) = (-0.96 \pm 0.03)$ and $-0.83$ in LFCQM and MBM, which are lower and higher than but acceptable by the experimental data of $-0.86 \pm 0.04$ [2], respectively. We have also predicted that $B(\Lambda_c^+ \to ne^+\nu_e) = (2.15 \pm 0.41, 2.55) \times 10^{-3}$ and $\alpha(\Lambda_c^+ \to ne^+\nu_e) = (-0.97 \pm 0.01, -0.85)$ in (LFCQM, MBM), in which our results of $B(\Lambda_c^+ \to ne^+\nu_e)$ in LFCQM and MBM as well as that in RQM [29] are consistent with each other, but about two times smaller than those in $SU(3)_F$ [22] and LQCD [27, 28]. It is clear that our predicted values for the decay branching ratio and asymmetry in $\Lambda_c^+ \to ne^+\nu_e$ could be tested in the ongoing experiments at BESIII, LHCb and BELLEII. Finally, we re-
mark that our calculations in LFCQM and MBM can be also extended to the other charmed baryons, such as $\Xi_c^+$, $\Xi_c^0$, and even $b$ baryons.

VI. APPENDIX

We now show our numerical results for the form factors in Eqs. (21)-(24) in LFCQM. In Fig. 2, we plot the vector form factors of $f_{1,2}$ with respect to the transfer momentum $k^2$ in unit of GeV$^2$ for $\Lambda_c^+ \to \Lambda$, where the symbol of “•” denotes the value calculated by Eqs. (21) and (23) from $k^2 = 0$ to $-9.7$ GeV$^2$ with Mathematica, while the blue line corresponds to the fitted function by the MATLAB curve fitting toolbox and the dashed line represents the 95% confidence bound of the fit. Similarly, we depict the axial-vector form factors of $g_{1,2}$ in Fig. 3. The corresponding results for $\Lambda_c^+ \to n$ are given in Figs. 4 and 5.

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FIG. 2. Vector form factors of $f_{1,2}$ with respect to the transfer momentum $k^2$ in unit of GeV$^2$ for $\Lambda_c^+ \to \Lambda$.

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FIG. 4. Legend is the same as Fig. 2 but for $\Lambda_c^+ \to n$.

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FIG. 5. Legend is the same as Fig. 3 but for $\Lambda_c^+ \to n$.

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