Free coherent states and distributions on $p$-adic numbers

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Abstract

Free coherent states for a system with two degrees of freedom is defined. A linear map of the space of free coherent states to the space of distributions on 2-adic disc is constructed.

1 Construction of free (or Boltzmannian) coherent states

Free (or Boltzmannian) Fock space has been considered in some recent works on quantum chromodynamics [1], [2], [3] and noncommutative probability [4], [5].

The subject of this work is free coherent states. We will introduce free coherent states and investigate the space of coherent states corresponding to a fixed eigenvalue of the operator of annihilation. In the paper [3], the subset of the set of free coherent states was constructed. The main result of the paper [3] is the construction of the homeomorphism from the ring of integer 2-adic numbers to the subset of the space of free coherent states with topology defined by the Hilbert metric.

The result of the present paper is the generalization of the construction of the paper [3]. We introduce the space of coherent states. The main result of the present paper is the construction of the linear map of the space of coherent states to the space of distributions on the ring of integer 2-adic numbers. Coherent states introduced in the paper [3] correspond under this map to δ-functions.

We will consider the system with two degrees of freedom. The system with one degree of freedom was investigated in [3].

The free commutation relations are particular case of $q$-deformed relations

$$A_i A_j^\dagger = q A_j^\dagger A_i = \delta_{ij}$$

with $q = 0$. A correspondence of $q$-deformed commutation relations and non-archimedean (ultrametric) geometry was discussed in [3]. Non-archimedean mathematical physics was studied in [3].

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Free coherent states lies in the free Fock space. Free (or Boltzmannian) Fock space $F$ over a Hilbert space $H$ is the completion of the tensor algebra

$$F = \bigoplus_{n=0}^{\infty} H^\otimes n.$$ 

Creation and annihilation operators are defined in the following way:

$$A^\dagger (f_1 \otimes ... \otimes f_n) = f \otimes f_1 \otimes ... \otimes f_n$$

where $<f, g>$ is the scalar product in the Hilbert space $H$. Scalar product in the free Fock space is defined by the standard construction of the direct sum of tensor products of Euclidean spaces.

We consider the case $H = C \oplus C$, where $C$ is the field of complex numbers. In this case we have two creation operators $A^\dagger_0, A^\dagger_1$ and two annihilation operators $A_0, A_1$ with commutation relations

$$A_i A^\dagger_j = \delta_{ij}. \quad (1)$$

The vacuum vector $\Omega$ in the free Fock space satisfies

$$A_i \Omega = 0. \quad (2)$$

We define free coherent states in the following way.

Let us consider an infinite sequence of complex numbers $U = u_0 u_1 u_2 u_3 ..., u_i$ are complex numbers.

We introduce the free coherent state $X_U$ as the formal series

$$X_U = \sum_{k=0}^{\infty} \lambda^k X_U^k.$$ 

Here $X_0 = \Omega$ is vacuum and

$$X_U^{k+1} = \left( u_k A^\dagger_0 + (1 - u_k) A^\dagger_1 \right) X_U^k. \quad (3)$$

This formal series define a functional with dense domain in the free Fock space.

Free coherent states are formal eigenvectors of the annihilation operator

$$A_0 + A_1;$$

for the eigenvalue $\lambda$, i.e.

$$(A_0 + A_1) X_U = \lambda X_U \quad \forall U.$$ 

We define the linear space $X'$ of free coherent states (formal eigenvectors of $A_0 + A_1$) as the linear span of functionals $X_U$.

2 Construction of the isometry of a subset of the space of coherent states to 2-adic disc

In the present section the eigenvalue $\lambda \in (0, 1)$. The subject of the present section is the investigation of the set of coherent states $X_U$ where $U$ runs on all infinite sequences of 0 and 1. Degeneration of this eigenspace is parametrized by the set $\{U\}$ of infinite sequences of 0 and 1.

These sequences have a natural interpretation as 2-adic numbers. Every sequence $U$ is in one-to-one correspondence with a 2-adic number $U = \sum_{i=0}^{\infty} u_i 2^i$.

Let us consider the metric $\rho$ of free Fock space on the set of free coherent states. Let $U, V$ be arbitrary sequences of 0 and 1. These sequences coincide up to an element with number $k - 1$. 

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2-adic distance of corresponding 2-adic numbers is \(||U - V||_2 = 2^{-k}\). Let \(X_U, X_V\) be corresponding free coherent states. We have:

\[
\rho(X_U, X_V)^2 = 2 \sum_{i=k}^{\infty} \lambda^{2i} = \frac{2}{1 - \lambda^2},
\]

Therefore the metric \(\rho\) is proportional to the degree of 2-adic metric.

The metric \(\rho\) is an ultrametric. This means that the metric \(\rho\) obeys the strong triangle inequality

\[
\rho(X_U, X_V) \leq \max(\rho(X_U, X_W), \rho(X_V, X_W))
\]

for arbitrary \(U, V, W\).

We get the following proposition.

**Proposition.** The map \(X_U \mapsto \sum_{i=0}^{\infty} u_i 2^i\) is the isometry of the set of free coherent states \(\{X_U\}\) parametrized by infinite sequences \(U\) of 0 and 1 with the metric \(\rho\) of scalar product to the ring of integer 2-adic numbers with the metric proportional to the degree of \(||U, V||_2\).

### 3 Construction of the map of the space of coherent states to the space of distributions on the 2-adic disc

In the present section the linear map of the space \(X'\) of coherent states to the space of distributions on the 2-adic disc will be constructed. In the present section we take \(\lambda \in (0, \sqrt{2})\).

Distributions on the 2-adic disc are linear functionals on the space of locally constant functions \(\mathcal{D}\). Therefore we have to construct the coherent state that corresponds to a locally constant function.

Let us consider the sequence \(W = w_0 w_1 w_2 ..., w_i\) are arbitrary complex numbers. Let us introduce the sequence

\[
W_k = w_0 w_1 w_2 ... w_{k-1} \underbrace{111...1}_{2^{k-1}} \underbrace{222...2}_{2^{k-1}} ...
\]

(4)

For \(\lambda \in (0, \sqrt{2})\) the coherent state \(X_{W_k}\) lies in the Hilbert space (the correspondent functional is bounded).

We will prove that an arbitrary locally constant function corresponds to the linear combination of coherent states \(X_{W_k}\). The linear span of the coherent states \(X_{W_k}\) will be denoted by \(X\). Every vector in \(X\) is a function of \(\lambda\). We will investigate the properties of the space \(X\) with the scalar product

\[
<X_U, X_V> = \lim_{\lambda \to \sqrt{2} - 0} \left(1 - \frac{\lambda^2}{2}\right) <X_U, X_V>.
\]

(5)

It is sufficient to find coherent states that correspond to locally constant functions of the type

\[
\theta_k(x - x_0) = \theta(2^k ||x - x_0||_2); \quad \theta(t) = 0, t > 1; \quad \theta(t) = 1, t \leq 1.
\]

Here \(x, x_0 \in \mathbb{Z}_2\) lies in the ring of integer 2-adic numbers and the function \(\theta_k(x - x_0)\) equals to 1 on the disc \(D(x_0, 2^{-k})\) of radius \(2^{-k}\) with the center in \(x_0\) and equals to 0 outside this disc.

Let us consider the sequences \(U = u_0 u_1 u_2 ..., u_i = 0, 1\) and \(V = v_0 v_1 v_2 ..., v_i = 0, 1\) that correspond to 2-adic numbers \(U = \sum_{i=0}^{\infty} u_i 2^i\) and \(V = \sum_{i=0}^{\infty} v_i 2^i\). We have the following lemma.

**Lemma 1.** The limit of the scalar product of coherent states \(X_{U_k}, X_{V_i}\) equals to the integral on 2-adic disc with respect to the Haar measure

\[
\lim_{\lambda \to \sqrt{2} - 0} \left(1 - \frac{\lambda^2}{2}\right) <X_{U_k}, X_{V_i}> =
\]

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Let us calculate the scalar product have a following form

$$D_k = \frac{1}{\mu(D(U, 2^{-k}))\mu(D(V, 2^{-l}))} \int_{\mathbb{Z}_2} \theta_k(x - U)\theta_l(x - V)dx.$$  

Here $\mu(D)$ is the Haar measure of the disc $D$.

**Proof**

The formula (3) for the coherent states $X_{U_k}$ and $X_{V_l}$ have the following form

$$X_{U_k} = \sum_{i=0}^{k-1} \lambda^i X_{U_k}^i + \lambda^k \sum_{i=0}^{\infty} \lambda^i \left( \frac{1}{2} A_0 + \frac{1}{2} A_1 \right)^i X_{U_k}^k;$$

$$X_{V_l} = \sum_{i=0}^{l-1} \lambda^i X_{V_l}^i + \lambda^l \sum_{i=0}^{\infty} \lambda^i \left( \frac{1}{2} A_0 + \frac{1}{2} A_1 \right)^i X_{V_l}^l.$$  

Let $k \leq l$.

If the first $k$ indices of the sequence $U$ coincide with the first $k$ indices of the sequence $V$ then the scalar product have a following form

$$(X_{U_k}, X_{V_l}) = \sum_{i=0}^{k-1} \lambda^i X_{U_k}^i + \sum_{i=k}^{l-1} \lambda^i \left( \frac{1}{2} \right)^{i-k} + \sum_{i=l}^{\infty} \lambda^i \left( \frac{1}{2} \right)^i.$$  

If the first $k$ indices of the sequence $U$ do not coincide with the first $k$ indices of the sequence $V$ then the series for scalar product $(X_{U_k}, X_{V_l})$ contains only the finite number of terms. Therefore the limit $\lim_{\lambda \to \sqrt{2}} \lim_{l \to \infty} (X_{U_k}, X_{V_l}) = \min(2^k, 2^l)$ if one of the discs $D(U, 2^{-k})$ and $D(V, 2^{-l})$ contains another and equals to 0 if these discs do not intersect.

Therefore the limit of the scalar product of vectors $X_{U_k}, X_{V_l}$ in the free Fock space corresponds to the integral on 2-adic disc with respect to the Haar measure. Coherent state $X_{U_k}$ corresponds to the locally constant function $\frac{1}{\mu(D(U, 2^{-k}))} \theta_k(x - U)$.

Let us investigate functionals on the space $X$ of locally constant functions.

Coherent states $X_U$ where sequences $U$ consist of 0 and 1 correspond to $\delta$-functions.

Let $U$ and $V$ be sequences of 0 and 1. We have the following lemma.

**Lemma 2.** The limit of the action of the functional $X_U$ on the vector $X_{V_l}$ have the following form

$$\lim_{\lambda \to \sqrt{2}} \lim_{l \to \infty} \left( 1 - \frac{\lambda^2}{2} \right) (X_{U}, X_{V_l}) = \frac{1}{\mu(D(V, 2^{-l}))} \int_{\mathbb{Z}_2} \delta(x - U)\theta_l(x - V)dx.$$  

Therefore the coherent state $X_U$ corresponds to the $\delta$-function $\delta(x - U)$.

**Lemma 3.** Vectors $X_{W_k}$ lie in the domain of the functional $X_U$ for arbitrary sequence $U$ of complex numbers.

**Proof**

The formula (3) for the coherent state $X_{W_k}$ have the following form

$$X_{W_k} = \sum_{i=0}^{k-1} \lambda^i X_{W_k}^i + \lambda^k \sum_{i=0}^{\infty} \lambda^i \left( \frac{1}{2} A_0 + \frac{1}{2} A_1 \right)^i X_{W_k}^k.$$  

The action of the functional $X_U$ is defined by the following formal series

$$(X_U, X_{W_k}) = \sum_{i=0}^{\infty} \lambda^i (X_U^i, X_{W_k}^i).$$  

(6)

Let us calculate

$$(X_{U^k}^{k+i}, X_{W_k}^{k+i}) = (X_U^{k+i-1}, (u_{k+i-1}^* A_0 + (1 - u_{k+i-1}) A_1) \left( \frac{1}{2} A_0 + \frac{1}{2} A_1 \right) X_{W_k}^{k+i-1}).$$
for \( i > 0 \). We have

\[
(u^*_{k+i-1}A_0 + (1 - u^*_{k+i-1})A_1) \left( \frac{1}{2} A_0^\dagger + \frac{1}{2} A_1^\dagger \right) = \frac{1}{2}(u^*_{k+i-1} + 1 - u^*_{k+i-1}) = \frac{1}{2}
\]

Therefore

\[
(X_{U}^{k+i}, X_{W_k}^{k+i}) = \frac{1}{2}(X_{U}^{k+i-1}, X_{W_k}^{k+i-1});
\]

we have \( \frac{\lambda^2}{2} < 1 \) and the series (3) converges.

Therefore an arbitrary coherent state \( X_U \) corresponds to a distribution on the 2-adic disc.

As a corollary of the Lemma 3 we introduce the linear map \( \phi \):

\[
\phi : X' \rightarrow D'(\mathbb{Z}_2);
\]

\[
\phi : X \rightarrow D(\mathbb{Z}_2);
\]

where \( D'(\mathbb{Z}_2) \) is the space of distributions on the 2-adic disc \( \mathbb{Z}_2 \) and \( D(\mathbb{Z}_2) \) is the space of locally constant functions on the 2-adic disc \( \mathbb{Z}_2 \). We have the following corollary of Lemma 3.

**Corollary.** The formula

\[
\lim_{\lambda \rightarrow \sqrt{2}-0} \left( 1 - \frac{\lambda^2}{2} \right) (X_U, X_W) = \frac{1}{\mu(D(V,2^{-j}))} \int_{\mathbb{Z}_2} \theta_i(x-U)\theta_j(x-V)dx;
\]

defines the action of the distribution \( \phi(X_U) \) on the locally constant function \( \phi(X_W) \).

We get the following theorem.

**Theorem.** The map \( \phi \)

\[
X_{V_j} \mapsto \frac{1}{\mu(D(V,2^{-j}))} \theta_j(x-V)
\]

where \( V \) is an arbitrary sequence of 0 and 1, \( j = 0, 1, 2... \) extends to the isomorphism of the space \( X \) of coherent states of a type (2) to the space \( D(\mathbb{Z}_2) \) of locally constant functions on the ring of integer 2-adic numbers with the scalar product defined by the Haar measure.

The scalar product of locally constant functions in \( L_2 \) with respect to the Haar measure equals to the limit of the scalar product in the free Fock space of corresponding coherent states

\[
\lim_{\lambda \rightarrow \sqrt{2}-0} \left( 1 - \frac{\lambda^2}{2} \right) (X_{U'}, X_{V'}) = \frac{1}{\mu(D(U,2^{-i}))\mu(D(V,2^{-j}))} \int_{\mathbb{Z}_2} \theta_i(x-U)\theta_j(x-V)dx;
\]

here \( U, V \) are arbitrary sequences of 0 and 1.

An arbitrary coherent state \( X_U \) corresponds to a distribution on the ring of integer 2-adic numbers with the action on locally constant functions defined by the formula

\[
(\phi(X_U), \phi(X_W)) = \lim_{\lambda \rightarrow \sqrt{2}-0} \left( 1 - \frac{\lambda^2}{2} \right) (X_U, X_W).
\]

4 Conclusion

Coherent states introduced in the present paper can be interpreted in the spirit of noncommutative geometry.

For example let us consider the real plane \( \mathbb{R}^2 \) with coordinates \( x, y \). Coordinates \( x, y \) we will consider as operators acting on functions on \( \mathbb{R}^2 \). Let us introduce ”coherent states” \( f \) as eigenvectors of the operator \( x + y \):

\[
(x + y)f = f.
\]
It is easy to see that these eigenvectors are distributions with support on the line \( x + y - 1 = 0 \).

An arbitrary algebraic manifold with equation \( F(x, y) = 0 \) can be described in the same way. The distribution \( \eta(x, y) \) on this manifold is a distribution on the plane \( R^2 \) that is annihilated by the polynom \( F(x, y) \):

\[
F(x, y)\eta(x, y) = 0.
\]

A point on the algebraic manifold with equation \( F(x, y) = 0 \) will correspond to the \( \delta \)-function with support in the point \( (x_0, y_0) \) such that \( F(x_0, y_0) = 0 \).

For the case of free (Boltzmannian) coherent states we have a noncommutative algebraic manifold with equation \( A_0 + A_1 - \lambda = 0 \). The result of the present paper can be interpreted in the following form: the limit for \( \lambda \rightarrow \sqrt{2} - 0 \) of the manifold with equation \( A_0 + A_1 - \lambda = 0 \) is 2-adic disc.

This result illustrates the conjecture of the paper [8]: noncommutative algebras can be used for a deformation of real manifolds to \( p \)-adic manifolds.

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