Global well-posedness of the Cauchy problem for a fifth-order KP-I equation in anisotropic Sobolev spaces

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$$u_t + \partial_x^5 u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0.$$ 

Firstly, we establish the local well-posedness of the problem in the anisotropic Sobolev spaces $H^{s_1, s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{9}{8}$ and $s_2 \geq 0$. Secondly, we establish the global well-posedness of the problem in $H^{s_1, 0}(\mathbb{R}^2)$ with $s_1 > -\frac{4}{7}$. Our result improves considerably the results of Saut and Tzvetkov (J. Math. Pures Appl. 79(2000), 307–338.) and Li and Xiao (J. Math. Pures Appl. 90(2008), 338–352.) and Guo, Huo and Fang (J. Diff. Eqns. 263 (2017), 5696–5726).

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Abstract

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1. Introduction

This paper is devoted to studying the Cauchy problem for the fifth-order KP-I equa-

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tion

\[ u_t + \partial_x^5 u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0, \]  
\[ u(x, y, 0) = u_0(x, y) \]  
(1.1)  
(1.2)

in anisotropic Sobolev space \( H^{s_1, s_2}(\mathbb{R}^2) \).

(1.1) appears as a model describing certain long dispersive waves (see [1, 31, 32]). It is considered as the higher-order version of the following KP equation

\[ u_t + \alpha \partial_x^3 u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0, \]  
(1.3)

where the coefficient \( \alpha \) may be either positive or negative. The KP equation (1.3) occurs in physical contexts as models for the propagation of dispersive long waves with weak transverse effects and is regarded as the two-dimensional extensions of the Korteweg-de-Vries equation (see [30]). When \( \alpha < 0 \), (1.3) is known as the KP-I equation. When \( \alpha > 0 \), (1.3) is known as the KP-II equation.

Several people have studied its Cauchy problem for (1.3), see [4, 15, 16, 25–29, 37, 49–54] for the KP-II equation (1.3) with \( \alpha > 0 \), and see [3, 7, 8, 13, 17–19, 21–24, 33, 39–42, 44, 56] for the KP-I equation (1.3) with \( \alpha < 0 \).

For the KP-II equation, by using the Fourier restriction norm method, Bourgain [4] established the global well-posedness of its Cauchy problem in \( L^2(\mathbb{R}^2) \) and \( L^2(T^2) \). Takaokao and Tzvetkov [51] and Isaza and Mejía [25] established the local well-posedness of KP-II equation in \( H^{s_1, s_2}(\mathbb{R}^2) \) with \( s_1 > -\frac{1}{2} \) and \( s_2 \geq 0 \). Takaoka [49] established the local well-posedness of KP-II equation in \( H^{s_1, 0}(\mathbb{R}^2) \) with \( s_1 > -\frac{1}{2} \) with the assumption that

\[ \left\| \xi^{-\frac{1}{2} + \epsilon} \mathcal{F}_x u_0 \right\|_{L^2} < \infty \]

for the suitable chosen \( \epsilon \). By introducing some resolution spaces, Hadac et al. [16] established the small data global well-posedness and scattering result of KP-II equation in the homogeneous anisotropic Sobolev space \( \dot{H}^{-\frac{1}{2}, 0}(\mathbb{R}^2) \) defined in [16] and arbitrary large initial data local well-posedness in both homogeneous Sobolev space \( \dot{H}^{-\frac{1}{2}, 0}(\mathbb{R}^2) \) and inhomogeneous anisotropic Sobolev space \( H^{-\frac{1}{2}, 0}(\mathbb{R}^2) \). Recently, by using new bilinear
estimates, Koch and Li [37] established the global well-posedness and scattering for the KP-II equation in three space dimensions with small initial data.

For the KP-I equation, Kenig, Molinet, Saut and Tzvetkov studied its Cauchy problem and periodic boundary value problem and showed that the problems are globally well-posed in the second energy spaces on both $\mathbb{R}^2$ and $\mathbb{T}^2$ (see [33, 41, 42]). Molinet et al. [40] proved that the Picard iterative method does not work for the KP-I equation in standard Sobolev space and in anisotropic Sobolev space, since the flow map fails to be real-analytic at the origin in these spaces. Ionescu et al. [18] established the global well-posedness of KP-I in the natural energy space $E^1$ with the aid of some resolution spaces and bootstrap inequality and the energy estimates. Molinet et al. [43] established the local well-posedness of the Cauchy problem for the KP-I equation in $H^{s,0}(\mathbb{R}^2)$ with $s > \frac{3}{2}$. Guo et al. [13] established the local well-posedness of the Cauchy problem for the KP-I equation in $H^{1,0}(\mathbb{R}^2)$. Zhang [56] established the local well-posedness of the periodic KP-I initial value problem in the Besov type space $B^{\frac{3}{2}}_{2,1}(\mathbb{T}^2)$.

Saut and Tzvetkov [46] established the global well-posedness of the Cauchy problem for the fifth order KP-II equation

$$u_t - \partial_x^5 u + \alpha \partial_x^3 u + \partial_x^{-1} \partial_y^2 u + \frac{1}{2} \partial_x (u^2) = 0, \alpha \in \mathbb{R},$$

in $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{4}$, $s_2 \geq 0$. Isaza et al. [20] established the local well-posedness of the fifth-order KP-II equation in $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{5}{4}$, $s_2 \geq 0$ and globally well-posed in $H^{s_1,0}(\mathbb{R}^2)$ with $s_1 > -\frac{4}{7}$ with the aid of $I$-method.

By using the Fourier restriction norm method and the Cauchy-Schwartz inequalities as well as some calculus inequalities, Saut and Tzvetkov [47] established the global well-posedness of Cauchy problem for the fifth order KP-I equation (1.1) with initial data $u_0 \in L^2(\mathbb{R}^2)$ and finite energy. By using the Fourier restriction norm method and the dyadic decomposed Strichartz estimates, Chen et al. [6] established the local well-posedness of the problem (1.1)(1.2) in the interpolated energy space $E^s$ with $0 < s \leq 1$, where

$$E^s = \{ u_0 \in E^s : \| u_0 \|_{E^s} = \| (1 + |\xi|^2 + |\mu/\xi|)^s \mathcal{F}_{xy} u_0(\xi,\mu) \|_{L^2} < \infty \}.$$ 

In particular, Chen et al. established the global well-posedness of the problem (1.1)(1.2)
in the energy space $E^1$. By using the Fourier restriction norm method and sufficiently exploiting the geometric structure of the resonant set of (1.1) to deal with the high-high frequency interaction, Li and Xiao [38] established the global well-posedness of the Cauchy problem (1.1)(1.2) in $L^2(\mathbb{R}^2)$. Guo et al. [12] established the local well-posedness of the Cauchy problem for (1.1) in $H^{s,0}(\mathbb{R}^2)$ with $s \geq -\frac{3}{4}$. Yan et al. [55] proved that the Cauchy problem for (1.1) is locally well-posed in $H^{s,0}(\mathbb{R}^2)$ with $s > -\frac{6}{23}$ with the aid of $I$-method introduced in [9, 10]. The method of [55] establishing local well-posedness is different from the method of [12]. Saut and Tzvetkov [48] have proved that the Cauchy problem for (1.1) posed on $T \times \mathbb{R}$ is globally well-posed in the energy space. Compared to the fifth order KP-II equation, the structure of the fifth order KP-I equation is complicated. The reason is that the resonant function of the fifth-order KP-I equation does not possess the same good property as its of fifth-order KP-II equation. More precisely, the resonant function of the fifth order KP-I equation is

$$R_1(\xi_1, \xi_2, \mu_1, \mu_2) := \phi(\xi, \mu) - \phi(\xi_1, \mu_1) - \phi(\xi_2, \mu_2)$$

$$= \frac{\xi_1 \xi_2}{\xi} \left[ 5\xi^2(\xi_1^2 + \xi_1 \xi_2 + \xi_2^2) - \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right)^2 \right],$$

(1.5)

and the resonant function of the fifth order KP-II equation is

$$R_\Pi(\xi_1, \xi_2, \mu_1, \mu_2) := \phi(\xi, \mu) - \phi(\xi_1, \mu_1) - \phi(\xi_2, \mu_2)$$

$$= \frac{\xi_1 \xi_2}{\xi} \left[ 5\xi^2(\xi_1^2 + \xi_1 \xi_2 + \xi_2^2) + \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right)^2 \right].$$

(1.6)

We remark that $R_1(\xi_1, \xi_2, \mu_1, \mu_2) = 0$ gives a surface, while $R_\Pi(\xi_1, \xi_2, \mu_1, \mu_2)$ will never be zero away from the origin.

In this paper, motivated by [7, 20, 38, 47], by using the Fourier restriction norm method introduced in [2, 5, 36, 45] and developed in [34, 35], the Cauchy-Schwartz inequality and Strichartz estimates as well as suitable splitting of domains, we establish the local well-posedness of the Cauchy problem for the fifth-order KP-I equation in the anisotropic Sobolev spaces $H^{s_1, s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{9}{8}$ and $s_2 \geq 0$; combining the local well-posness result of this paper with the I-method introduced in [9, 10], we established the global well-posedness of the problem in $H^{s_1, 0}(\mathbb{R}^2)$ with $s_1 > -\frac{4}{7}$. Thus, our result considerably improves the result of [12, 38, 47].
We introduce some notations before presenting the main results. Throughout this paper, we assume that \( C \) is a positive constant which may vary from line to line. \( a \sim b \) means that there exist constants \( C_j > 0 (j = 1, 2) \) such that \( C_1 |b| \leq |a| \leq C_2 |b| \). \( a \gg b \) means that there exist a positive constant \( C' \) such that \( |a| > C' |b| \). \( 0 < \epsilon \ll 1 \) means that \( 0 < \epsilon < 10^{-4} \). We define

\[
\langle \cdot \rangle := 1 + |\cdot|,
\]
\[
\phi(\xi, \mu) := \xi^5 + \frac{\mu^2}{\xi},
\]
\[
\sigma := \tau + \phi(\xi, \mu), \sigma_j = \tau_j + \phi(\xi_j, \mu_j) (j = 1, 2),
\]
\[
\mathcal{F} u(\xi, \mu, \tau) := \frac{1}{(2\pi)^{3}} \int_{\mathbb{R}^3} e^{-ix\xi - iy\mu - i\tau t} u(x, y, t) dx dy dt,
\]
\[
\mathcal{F}_{xy} f(\xi, \mu) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi - iy\mu} f(x, y) dx dy,
\]
\[
\mathcal{F}^{-1} u(\xi, \mu, \tau) := \frac{1}{(2\pi)^{3}} \int_{\mathbb{R}^3} e^{ix\xi + iy\mu + i\tau t} u(x, y, t) dx dy dt,
\]
\[
D_x u(x, y, t) := \frac{1}{(2\pi)^{2}} \int_{\mathbb{R}^3} |\xi|^a \mathcal{F} u(\xi, \mu, \tau) e^{ix\xi + iy\mu + i\tau t} d\xi d\mu d\tau,
\]
\[
P^2 u(x, y, t) := \frac{1}{(2\pi)^{2}} \int_{|\xi| \geq 2} \int_{\mathbb{R}^2} \mathcal{F} u(\xi, \mu, \tau) e^{ix\xi + iy\mu + i\tau t} d\xi d\mu d\tau,
\]
\[
W(t) f := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi + iy\mu + i\phi(\xi, \mu)} \mathcal{F}_{xy} f(\xi, \mu) d\xi d\mu.
\]

Let \( \eta \) be a bump function with compact support in \([-2, 2] \subset \mathbb{R} \) and \( \eta = 1 \) on \((-1, 1) \subset \mathbb{R} \). For each integer \( j \geq 1 \), we define \( \eta_j(\xi) = \eta(2^{-j} \xi) - \eta(2^{1-j} \xi), \eta_0(\xi) = \eta(\xi), \eta_j(\xi, \mu, \tau) = \eta_j(\sigma) \), thus, \( \sum_{j \geq 0} \eta_j(\sigma) = 1 \). \( \psi(t) \) is a smooth function supported in \([0, 2] \) and equals 1 in \([0, 1] \). Let \( I \subset \mathbb{R}^d, \chi_I(x) = 1 \) if \( x \in I \); \( \chi_I(x) = 0 \) if \( x \) does not belong to \( I \).

We define

\[
\|f\|_{L^p_t L^q_x} := \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |f|^p dx dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}.
\]

We denote by \( H^{s_1, s_2}(\mathbb{R}^2) \) the anisotropic Sobolev space as follows:

\[
H^{s_1, s_2}(\mathbb{R}^2) := \left\{ u_0 \in \mathcal{S}'(\mathbb{R}^2) : \|u_0\|_{H^{s_1, s_2}(\mathbb{R}^2)} = \|\langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} \mathcal{F}_{xy} u_0(\xi, \mu)\|_{L^2_{\xi \mu}} \right\}.
\]

The Bourgain space \( X^{s_1, s_2}_b \) is defined by

\[
X^{s_1, s_2}_b := \left\{ u \in \mathcal{S}'(\mathbb{R}^3) : \|u\|_{X^{s_1, s_2}_b} = \|\langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} \langle \sigma \rangle^b \mathcal{F} u(\xi, \mu, \tau)\|_{L^2_{\xi \mu}} < \infty \right\}.
\]
The space $X_{b}^{s_{1},s_{2}}([0,T])$ denotes the restriction of $X_{b}^{s_{1},s_{2}}$ onto the finite time interval $[0,T]$ and is equipped with the norm

$$
\|u\|_{X_{b}^{s_{1},s_{2}}([0,T])} = \inf \left\{ \|g\|_{X_{b}^{s_{1},s_{2}}}, g \in X_{b}^{s_{1},s_{2}}, u(t) = g(t) \text{ for } t \in [0,T] \right\}.
$$

For $s < 0$ and $N \in \mathbb{N}^+, N \geq 20$, we define an operator $I_{N}$ by $\mathcal{F} I_{N} u(\xi,\mu,\tau) = M(\xi) \mathcal{F} u(\xi,\mu,\tau)$, where $M(\xi) = 1$ if $|\xi| < N$; $M(\xi) = (|\xi|/N)^{s}$ if $|\xi| \geq N$.

The main results of this paper are as follows.

**Theorem 1.1.** (Local well-posedness) Let $|\xi|^{-1} \mathcal{F}_{xy} u_{0}(\xi,\mu) \in \mathcal{S}'(\mathbb{R}^{2})$. Then, the Cauchy problem for (1.1) is locally well-posed in $H^{s_{1},s_{2}}(\mathbb{R}^{2})$ with $s_{1} > -9/8$, $s_{2} \geq 0$.

**Remark 1.** We only consider the case of $-9/8 < s_{1} < 0$, $s_{2} \geq 0$. For $s_{1} \geq 0$, $s_{2} \geq 0$ the local well-posedness is proved by Li and Xiao [38]. Lemmas 3.1 and 3.2 are the key ingredients in establishing the bilinear estimates in Lemmas 4.1 and 4.2. Once Lemma 4.1 is proven to be valid, then we can combine it and Lemma 2.6 with the fixed point argument to obtain the local wellposedness. Since the phase function $\phi(\xi,\mu)$ is singular at $\xi = 0$, to define the derivative of $W(t)u_{0}$, the requirement $|\xi|^{-1} \mathcal{F}_{xy} u_{0}(\xi,\mu) \in \mathcal{S}'(\mathbb{R}^{2})$ is necessary.

**Theorem 1.2.** (Global well-posedness) Let $|\xi|^{-1} \mathcal{F}_{xy} u_{0}(\xi,\mu) \in \mathcal{S}'(\mathbb{R}^{2})$. Then the Cauchy problem for (1.1) is globally well-posed in $H^{s_{1},0}(\mathbb{R}^{2})$ with $s_{1} > -4/7$.

**Remark 2.** We only consider the case of $-4/7 < s_{1} < 0$, $s_{2} \geq 0$. The case of $s_{1} \geq 0$, $s_{2} \geq 0$ is proved by Li and Xiao [38]. For the fifth order KP-II equation, Isaza, López and Mejía [29] obtained the same result about the global well-posedness, that is, the Cauchy problem for the fifth order KP-II equation is also globally well-posed in $H^{s_{1},s_{2}}(\mathbb{R}^{2})$ with $s_{1} > -\frac{4}{7}$, $s_{2} \geq 0$.

The rest of the paper is arranged as follows. In Section 2, we give some preliminaries. In Section 3, we establish two $L^{2}$ bilinear estimates. In Section 4, we establish three bilinear estimates. In Section 5, we prove the local well-posedness. In Section 6, we firstly prove Lemma 6.1 which is a variation of Theorem 1.1, then, we apply Lemmas 6.1, 4.2 and 2.6 to prove Theorem 1.2.

2. Preliminaries
This section is devoted to present Lemmas 2.1–2.6.

**Lemma 2.1.** Let \( b > |a| \geq 0 \). Then, we have

\[
\int_{-b}^{b} \frac{dx}{(x + a)^{\frac{1}{2}}} \leq C b^{\frac{1}{2}}, \tag{2.1}
\]

\[
\int_{\mathbb{R}} \frac{dt}{(t)^{\gamma}(t-a)^{\gamma}} \leq C (a)^{-\gamma}, \gamma > 1, \tag{2.2}
\]

\[
\int_{\mathbb{R}} \frac{dt}{(t)^{\gamma}|t-a|^{\gamma}} \leq C (a)^{-\frac{1}{2}}, \gamma \geq 1, \tag{2.3}
\]

\[
\int_{-K}^{K} \frac{dx}{|x|^{\frac{1}{2}}|a-x|^{\frac{1}{2}}} \leq C \frac{K^{\frac{1}{2}}}{|a|^{\frac{1}{2}}}. \tag{2.4}
\]

**Proof.** The conclusion of (2.1) is given in (2.4) of Lemma 2.1 in [20]. (2.2)-(2.3) can be seen in Proposition 2.2 of [47]. (2.4) can be seen in [15, Page 6562].

This completes the proof of Lemma 2.1.

**Lemma 2.2.** Let \( T \in (0, 1) \) and \( s_1, s_2 \in \mathbb{R} \) and \( -\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1 \). Then, for \( h \in X_{b'}^{s_1, s_2} \), we have

\[
\left\| \psi(t) S(t) \phi \right\|_{X_{b'}^{s_1, s_2}} \leq C \left\| \phi \right\|_{H_{s_1, s_2}}, \tag{2.5}
\]

\[
\left\| \psi \left( \frac{t}{T} \right) \int_{0}^{t} S(t - \tau) h(\tau) d\tau \right\|_{X_{b'}^{s_1, s_2}} \leq C T^{1+b'-b} \| h \|_{X_{b'}^{s_1, s_2}}. \tag{2.6}
\]

For the proof of Lemma 2.2, we refer readers to [5, 11, 34] and [14, Lemma 1.7 and Lemma 1.9].

**Lemma 2.3.** Let \( b > \frac{1}{2} \). Then,

\[
\left\| D_{x}^\frac{1}{2} u \right\|_{L_t^4 L_x^4(\mathbb{R}^3)} \leq C \| u \|_{X_{b}^{0,0}}. \tag{2.7}
\]

For the proof of Lemma 2.3, we refer readers to [15, Theorem 3.1].

**Lemma 2.4.** Let

\[
|\sigma - \sigma_1 - \sigma_2| = \left| \xi \xi_1 \xi_2 (5 \xi^2 - 5 \xi_1 \xi + 5 \xi_1^2) - \frac{\xi_1 \xi_2}{\xi} \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right) \right|
\geq \left| \xi \xi_1 \xi_2 (5 \xi^2 - 5 \xi_1 \xi + 5 \xi_1^2) \right| \geq \frac{1}{4}.
\]
\[ F P_\mathcal{H}^4(u_1, u_2)(\xi, \mu, \tau) = \int_{\mathbb{R}^3} \chi_{|\xi| \leq \frac{1}{4}}(\xi, \mu_1, \tau_1, \xi, \mu, \tau) \prod_{j=1}^2 \mathcal{F} u_j(\xi_j, \mu_j, \tau_j) d\xi_1 d\mu_1 d\tau_1. \]

For \( b > \frac{1}{2} \), we have
\[
\left\| P_\mathcal{H}^4(u_1, u_2) \right\|_{L^2_{\xi \mu \tau}} \leq C \left\| D_x^{\frac{1}{2}} u_1 \right\|_{X^{\sigma,0}_b} \left\| D_x^{-1} u_2 \right\|_{X^{0,0}_b}. \tag{2.8}
\]

**Proof.** Let
\[
f_1(\xi_1, \mu_1, \tau_1) = |\xi_1|^{-\frac{1}{2}} (\sigma_1)^b \mathcal{F} u_1(\xi_1, \mu_1, \tau_1), f_2(\xi_2, \mu_2, \tau_2) = |\xi_2|^{-1} (\sigma_2)^b \mathcal{F} u_2(\xi_2, \mu_2, \tau_2).
\]

To obtain (2.8), it suffices to prove that
\[
\left\| \int_{\mathbb{R}^3} \frac{|\xi_1|^{-\frac{1}{2}} |\xi_2| f_1(\xi_1, \mu_1, \tau_1) f_2(\xi_2, \mu_2, \tau_2)}{2 \prod_{j=1}^2 (\sigma_j)^b} d\xi_1 d\mu_1 d\tau_1 \right\|_{L^2_{\xi \mu \tau}} \leq C \prod_{j=1}^2 \| f_j \|_{L^2_{\xi \mu \tau}}. \tag{2.9}
\]

To obtain (2.9), by duality, it suffices to prove that
\[
\left\| \int_{\mathbb{R}^3} \frac{|\xi_1|^{-\frac{1}{2}} |\xi_2| f(\xi, \mu, \tau) f_1(\xi_1, \mu_1, \tau_1) f_2(\xi_2, \mu_2, \tau_2)}{2 \prod_{j=1}^2 (\sigma_j)^b} d\xi_1 d\mu_1 d\tau_1 \right\|_{L^2_{\xi \mu \tau \tau}} \leq C \| f \|_{L^2_{\xi \mu \tau \tau}} \prod_{j=1}^2 \| f_j \|_{L^2_{\xi \mu \tau}}. \tag{2.10}
\]

We define
\[
I(\xi, \mu, \tau) := \int_{\mathbb{R}^3} \frac{|\xi_1|^{-1} |\xi_2|^2}{2 \prod_{j=1}^2 (\sigma_j)^b} d\xi_1 d\mu_1 d\tau_1. \tag{2.11}
\]

For fixed \((\xi, \mu, \tau)\), we make the change of variables \( L : (\xi_1, \mu_1, \tau_1) \rightarrow (\Delta, \sigma_1, \sigma_2) \), where
\[
\Delta := \xi \xi_1 \xi_2 (5 \xi^2 - 5 \xi_1 + 5 \xi_2), \\
\sigma_1 := \tau_1 + \phi(\xi_1, \mu_1), \sigma_2 := \tau_2 + \phi(\xi_2, \mu_2).
\]
By using a direct computation, since \( \sigma = \tau + \phi(\xi, \mu) \), we have
\[
\sigma_1 + \sigma_2 - \sigma = -\Delta + \frac{(\xi_1 \mu_2 - \mu_1 \xi_2)^2}{\xi_1 \xi_2}.
\] (2.12)

Thus, we have the Jacobian determinant equals
\[
\frac{\partial (\Delta, \sigma_1, \sigma_2)}{\partial (\xi_1, \mu_1, \tau_1)} = -10 \left( \xi_1^2 - \xi_2^2 \right) \left( \xi_1^2 + \xi_2^2 \right) \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right)
= -10 \left( \xi_1^2 - \xi_2^2 \right) \left( \xi_1^2 + \xi_2^2 \right) \left( \sigma_1 + \sigma_2 - \sigma + \Delta \right)^{\frac{1}{2}} \left( \frac{\xi}{\xi_1 \xi_2} \right)^{\frac{1}{2}}.
\] (2.13)

Notice that it is possible to divide the integration into a finite number of open subsets \( W_i \) such that \( L \) is an injective \( C^1 \)-function in \( W_i \) with non-zero Jacobian determinant.

From (2.13), since \( |\xi_1| \leq \frac{\xi_1}{4} \) and \( |\Delta| \sim |\xi_1| |\xi_2| \), we have
\[
\left| \frac{\partial (\Delta, \sigma_1, \sigma_2)}{\partial (\xi_1, \mu_1, \tau_1)} \right| = 10 \left| \left( \xi_1^2 - \xi_2^2 \right) \left( \xi_1^2 + \xi_2^2 \right) \left( \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right) \right|
= 10 \left| \left( \xi_1^2 - \xi_2^2 \right) \left( \xi_1^2 + \xi_2^2 \right) \left( \sigma_1 + \sigma_2 - \sigma + \Delta \right)^{\frac{1}{2}} \left( \frac{\xi}{\xi_1 \xi_2} \right)^{\frac{1}{2}} \right|
\sim |\xi_1|^{-1} |\xi_2|^2 |\Delta|^{\frac{1}{2}} |\sigma_1 + \sigma_2 - \sigma + \Delta|^{\frac{1}{2}}.
\] (2.14)

Since \( |\sigma_1 + \sigma_2 - \sigma| \geq \frac{|\Delta|}{4} \), by using the change of variables \( (\xi_1, \mu_1, \tau_1) \rightarrow (\Delta, \sigma_1, \sigma_2) \) and (2.4), we have
\[
I(\xi, \mu, \tau) := \int_{\mathbb{R}^3} \chi_{|\xi_1| \leq \frac{\xi_1}{4}} \frac{|\xi_2|^2 |\xi_1|^{-1} d\xi_1 d\mu_1 d\tau_1}{\prod_{j=1}^{26} (\sigma_j)^{2b}}
\leq C \int_{\mathbb{R}^3} \frac{\chi_{|\Delta| \leq 4|\sigma_1 + \sigma_2 - \sigma| \Delta d\sigma_1 d\sigma_2}}{|\Delta|^{\frac{1}{2}} |\sigma_1 + \sigma_2 - \sigma + \Delta|^{\frac{1}{2}} 2 \prod_{j=1}^{26} (\sigma_j)^{2b}}
= \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} \frac{\chi_{|\Delta| \leq 4|\sigma_1 + \sigma_2 - \sigma| \Delta}}{|\Delta|^{\frac{1}{2}} |\sigma_1 + \sigma_2 - \sigma + \Delta|^{\frac{1}{2}} 2 \prod_{j=1}^{26} (\sigma_j)^{2b}} d\sigma_1 d\sigma_2 \right)
\leq C \int_{\mathbb{R}^2} \frac{d\sigma_1 d\sigma_2}{\prod_{j=1}^{26} (\sigma_j)^{2b}} \leq C.
\] (2.15)

11
Combining (2.10) with (2.15), by using the Cauchy-Schwartz inequality twice, we have

\[
\int_{\mathbb{R}^3} \chi_{|\xi_1| \leq \frac{|\xi_2|}{4}} |\xi_2||\xi_1|^{-\frac{3}{2}}f_1(\xi_1, \mu_1, \tau_1)f_2(\xi_2, \mu_2, \tau_2)f(\xi, \mu, \tau) d\xi_1 d\mu_1 d\tau_1 \leq C \left[ \sup_{\xi, \mu, \tau} I(\xi, \mu, \tau) \right]^\frac{1}{2} \|f\|_{L^2_{\tau \mu}} \prod_{j=1}^2 \|f_j\|_{L^2_{\tau \mu}}
\]

\[
\leq C \|f\|_{L^2_{\tau \mu}} \prod_{j=1}^2 \|f_j\|_{L^2_{\tau \mu}}. \tag{2.16}
\]

This completes the proof of Lemma 2.4.

**Lemma 2.5.** Let

\[
|\sigma - \sigma_1 - \sigma_2| = \left| \frac{\xi \xi_1 \xi_2 (5\xi^2 - 5\xi_1 + 5\xi_1^2) - \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}}{4} \right| \geq \frac{|\xi_1 \xi_2 (5\xi^2 - 5\xi_1 + 5\xi_1^2)|}{4},
\]

\(b > \frac{1}{2}\) and \(G(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) = f_1(\xi_1, \mu_1, \tau_1)f_2(\xi - \xi_1, \mu - \mu_1, \tau - \tau_1)f(\xi, \mu, \tau)\), we have

\[
\int_{\mathbb{R}^6} \left| \frac{|\xi|^{-\frac{1}{2}}|\xi_2| G(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau)}{\sigma(\sigma)^b} \right| d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau \leq C \|f\|_{L^2_{\tau \mu}} \prod_{j=1}^2 \|f_j\|_{L^2_{\tau \mu}}, \tag{2.17}
\]

\[
\int_{\mathbb{R}^6} \left| \frac{|\xi|^{-\frac{1}{2}}|\xi_2| G(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau)}{\sigma(\sigma)^b} \right| d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau \leq C \|f\|_{L^2_{\tau \mu}} \prod_{j=1}^2 \|f_j\|_{L^2_{\tau \mu}}, \tag{2.18}
\]

\[
\int_{\mathbb{R}^6} \left| \frac{|\xi|^{-\frac{1}{2}}|\xi_2| G(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau)}{\sigma(\sigma)^b} \right| d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau \leq C \|f\|_{L^2_{\tau \mu}} \prod_{j=1}^2 \|f_j\|_{L^2_{\tau \mu}}, \tag{2.19}
\]

\[
\int_{\mathbb{R}^6} \left| \frac{|\xi|^{-\frac{1}{2}}|\xi_2| G(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau)}{\sigma(\sigma)^b} \right| d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau \leq C \|f\|_{L^2_{\tau \mu}} \prod_{j=1}^2 \|f_j\|_{L^2_{\tau \mu}}. \tag{2.20}
\]

**Proof.** We firstly prove (2.17). When \(\frac{|\xi_1|}{4} \geq |\xi_2|\), from Lemma 2.4, we have (2.17) is valid. When \(\frac{|\xi_1|}{4} < |\xi_2|\), since \(|\xi|^{-\frac{1}{2}}|\xi_2| \leq C|\xi_1|^{-\frac{1}{2}}|\xi_2|\), from Lemma 2.3, we know that (2.17) is valid. Let \(\xi_1 = \xi', \mu_1 = \mu', \tau_1 = \tau'\) and \(-\xi_2 = \xi', -\mu_2 = \mu'\) and \(-\xi = \xi' - \xi_1, -\mu = \mu' - \mu_1, -\tau = \tau' - \tau_1\) and \(\sigma_2 = \sigma_2' = \phi(\xi_2, \mu_2), \sigma_1 = \sigma_1' = \phi(\xi_1, \mu_1)\). Thus, \(-\sigma = \sigma_2', \sigma_1 = \sigma_1'\). Let

\[
H(\xi_1', \mu_1', \tau_1', \xi', \mu', \tau') = f_1(\xi_1', \mu_1', \tau_1')f_2(-\xi', -\mu', -\tau')f(-\xi_2', -\mu_2', -\tau_2').
\]
To obtain (2.18), it suffices to prove that

\[
\left| \int_{\mathbb{R}^6} \frac{|\xi^1|^2 |\xi^2|}{\langle \sigma^1 \rangle b \langle \sigma^2 \rangle b} H(\xi^1, \mu^1, \tau^1, \xi^2, \mu^2, \tau^2) d\xi^1 d\mu^1 d\tau^1 d\xi^2 d\mu^2 d\tau^2 \right|
\leq C \|f\|_{L^2_{\xi \mu}} \prod_{j=1}^2 \|f_j\|_{L^2_{\xi \mu}}.
\]

(2.21)

Obviously, (2.21) follows from (2.17). By using a proof similar to (2.18), we obtain that (2.19)-(2.20) are valid.

This ends the proof of Lemma 2.5.

**Lemma 2.6.** Let \(0 < b_1 < b_2 < \frac{1}{2}\). Then, we have

\[
\|\chi_1(\cdot) u\|_{X^0,b_1} \leq C \|u\|_{X^0,b_2},
\]

(2.22)

\[
\|\chi_1(\cdot) u\|_{X^0,b_2} \leq C \|u\|_{X^0,b_1}.
\]

(2.23)

For the proof of Lemma 2.6, we refer the readers to Lemma 3.1. of [26].

### 3. \(L^2\)-bilinear estimates

Inspired by the idea of Lemma 5.1 of [18], we give the proof of Lemma 3.1. For \(k \in \mathbb{Z}\) and \(l, j \in \mathbb{R}\), we define

\[
D_{k,l,j} := \{ (\xi, \mu, \tau) : |\xi| \in [2^{k-1}, 2^{k+1}], |\mu| \leq 2^l, |\tau + \phi(\xi, \mu)| < 2^j \},
\]

\[
D_{k,\infty,j} := \bigcup_{l \in \mathbb{Z}} D_{k,l,j}.
\]

**Lemma 3.1.** Assume \(\alpha \in \mathbb{R}\) and \(k_1, k_2, k_3 \in \mathbb{Z}\), \(k_{\max} = \max \{k_1, k_2, k_3\}\) and \(k_{\min} = \min \{k_1, k_2, k_3\}\) and \(j_1, j_2, j_3 \in \mathbb{Z}_+; j_{\max} = \max \{j_1, j_2, j_3\}\) and \(f_i : \mathbb{R}^3 \to \mathbb{R}\) are \(L^2\) functions supported in \(D_{k,\infty,j_i}(i=1, 2, 3)\). We assume that

\[
R_l(\xi_1, \xi_2, \mu_1, \mu_2) \leq 2^{k_1+k_2+k_3+2k_{\max}-60},
\]

(3.1)

\[
j_{\max} \leq k_1 + k_2 + k_3 + 2k_{\max} - 60.
\]

(3.2)

(1) If \(|k_1 - k_2| \leq 5, k_1 \geq 20\), then, we have

\[
\int_{\mathbb{R}^3} (f_1 * f_2) f_3 d\xi d\mu d\tau \leq C 2^{\frac{j_3 + j_2 + j_1}{2}} 2^{-\frac{7}{4}(k_1 + k_2) + \frac{k_3}{2}} \prod_{j=1}^3 \|f_j\|_{L^2}.
\]

(3.3)
(2) If \( k_2 - 10 \geq k_1 \) and \(|k_2 - k_3| \leq 5, k_2 \geq 20, \) then, we have

\[
\int_{\mathbb{R}^3} (f_1 * f_2) f_3 d\xi_1 d\mu_1 d\tau_1 \leq C 2^{\frac{1}{2}+\frac{k_1}{2}} 2^{-\frac{7}{2}(k_2+k_3)-\frac{k_1}{2}} \prod_{j=1}^{3} \|f_j\|_{L^2}. \tag{3.4}
\]

**Proof.** First we prove (3.3). From (5.4) of [18], we have

\[
\int_{\mathbb{R}^3} (f_1 * f_2) f_3 = \int_{\mathbb{R}^3} (\tilde{f}_1 * \tilde{f}_3) f_2 = \int_{\mathbb{R}^3} (\tilde{f}_2 * \tilde{f}_3) f_1. \tag{3.5}
\]

where \( \tilde{f}_i = f_i(-\xi, -\mu, -\tau)(i = 1, 2). \) Due to the symmetry in (3.5), without loss of generality, we may assume \( j_3 = \max \{j_1, j_2, j_3\}. \) We define

\[
f_i^\#(\xi_i, \mu_i, \theta_i) := f_i(\xi_i, \mu_i, \theta_i - \phi(\xi_i, \mu_i))(i = 1, 2, 3).
\]

Obviously, \( \|f_i^\#\|_{L^2} = \|f_i\|_{L^2}. \) The left-hand side of (3.3) can be rewritten as follows:

\[
\int_{\mathbb{R}^6} \left( \prod_{i=1}^{2} f_i^\#(\xi_i, \mu_i, \theta_i) \right) f_3^\#(\xi_1 + \xi_2, \mu_1 + \mu_2, R_1(\xi_1, \mu_1, \xi_2, \mu_2))d\xi_1 d\xi_2 d\mu_1 d\mu_2 d\theta_1 d\theta_2, \tag{3.6}
\]

where \( R_1(\xi_1, \xi_2, \mu_1, \mu_2) \) is the resonant function defined as in (1.5). The functions \( f_i^\# \) \((i = 1, 2)\) are supported in the sets

\[
\left\{ (\xi_i, \mu_i, \theta_i) : \xi_i \in \tilde{I}_{ki}, \mu_i \in \mathbb{R}, |\theta_i| \leq 2^{j_i} \right\}
\]

and \( f_3^\# \) is supported in the set

\[
\left\{ (\xi_3, \mu_3, \theta_3) : \xi_3 \in \tilde{I}_{k_3}, \mu_3 \in \mathbb{R}, |\theta_3| \leq 2^{j_3} \right\}.
\]

We will prove that if \( g_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) are \( L^2 \) functions supported in \( \tilde{I}_{ki} \times \mathbb{R}(i = 1, 2) \) and \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) are \( L^2 \) functions supported in \( \tilde{I}_{k} \times \mathbb{R} \times [-2^j, 2^j] \), then, we have

\[
\int_{\mathbb{R}^3} \left( \prod_{j=1}^{2} g_j(\xi_j, \mu_j) \right) g(\xi_1 + \xi_2, \mu_1 + \mu_2, R_1(\xi_1, \mu_1, \xi_2, \mu_2))d\xi_1 d\xi_2 d\mu_1 d\mu_2
\]

\[
\leq C 2^\frac{1}{2} 2^{-\frac{7}{2}(k_1+k_2)+\frac{k_1}{2}} \|g\|_{L^2} \prod_{j=1}^{2} \|g_j\|_{L^2}. \tag{3.7}
\]

When \( j \leq k_1 + k_2 + k_3 + 2k_{\text{max}} - 60 \) is valid, we may assume that the integral in the left-hand side of (3.7) is taken over the set

\[
\mathcal{R}_{++} = \left\{ (\xi_1, \mu_1, \xi_2, \mu_2) : \xi_1 + \xi_2 \geq 0, \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \geq 0 \right\} \tag{3.8}
\]
since other case can be proved similarly to case (3.8). We make the changes of variables

\[ \mu_1 = \sqrt{5}\xi_1^3 + \beta_1 \xi_1, \mu_2 = -\sqrt{5}\xi_2^3 + \beta_2 \xi_2. \]  

(3.9)

From (3.9), we have

\[ \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} = \sqrt{5}\xi_1 + \beta_1 - \beta_2 = \sqrt{5}(\xi_1^2 + \xi_2^2) + \beta_1 - \beta_2 \geq 0. \]  

(3.10)

From (3.10), we know that

\[ \beta_1 - \beta_2 \geq -\sqrt{5}(\xi_1^2 + \xi_2^2). \]  

(3.11)

By using the assumption upon \( g \) and the definition of \( R_1(\xi_1, \mu_1, \xi_2, \mu_2) \), we infer that

\[
\left| (\beta_1 - \beta_2)^2 + 2\sqrt{5}(\beta_1 - \beta_2)(\xi_1^2 + \xi_2^2) + 5\xi_1\xi_2(4\xi_1\xi_2 - 3\xi^2) \right| \leq 2^{j+k-1-k_1-k_2+3}. 
\]

(3.12)

The left hand side of (3.7) can be bounded by

\[
C2^{k_1+k_2} \int_S h_1(\xi_1, \sqrt{5}\xi_1^3 + \beta_1 \xi_1)h_2(\xi_2, -\sqrt{5}\xi_2^3 + \beta_2 \xi_2) \\
\times h(\xi_1 + \xi_2, \sqrt{5}(\xi_1^3 - \xi_2^3) + \beta_1 \xi_1 + \beta_2 \xi_2, R_1(\xi_1, \beta_1, \xi_2, \beta_2)d\xi_1 d\xi_2 d\beta_1 d\beta_2. 
\]

(3.13)

where

\[
S = \{(\xi_1, \beta_1, \xi_2, \beta_2) : \xi_1 + \xi_2 \geq 0, \beta_1 - \beta_2 \text{ satisfies (3.11) - (3.12)} \}. 
\]

(3.14)

and

\[
\tilde{R}_1(\xi_1, \beta_1, \xi_2, \beta_2) \\
= (\beta_1 - \beta_2)^2 + 2\sqrt{5}(\beta_1 - \beta_2)(\xi_1^2 + \xi_2^2) + 5\xi_1\xi_2(4\xi_1\xi_2 - 3\xi^2). 
\]

(3.15)

We define the functions \( h_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) supported in \( \tilde{I}_k \times \mathbb{R} \) \((i = 1, 2)\)

\[
h_1(\xi_1, \beta_1) = 2^k g_1(\xi_1, \sqrt{5}\xi_1^3 + \beta_1 \xi_1), 
\]

(3.16)

\[
h_2(\xi_2, \beta_2) = 2^k g_2(\xi_2, -\sqrt{5}\xi_2^3 + \beta_2 \xi_2). 
\]

(3.17)

with \( \|h_i\|_{L^2} \approx \|g_i\|_{L^2} \) \((i = 1, 2)\).
To prove (3.7), it suffices to prove that
\[
2^{k_1+k_2} \int_\mathcal{S} h_1(\xi_1, \beta_1)h_2(\xi_2, \beta_2) \\
\times h(\xi_1 + \xi_2, \sqrt{5}(\xi_1^3 - \xi_2^3) + \beta_1 \xi_1 + \beta_2 \xi_2, \tilde{R}_1(\xi_1, \beta_1, \xi_2, \beta_2) d\xi_1 d\xi_2 d\beta_1 d\beta_2 \\
\leq C 2^{\frac{k_1+k_2}{2}} 2^{-\frac{7}{2}(k_1+k_2)+\frac{k_1}{2}} \|h\|_{L^2} \prod_{j=1}^2 \|h_j\|_{L^2}. \tag{3.18}
\]

Combining (3.11) with (3.12), we have
\[
\sqrt{5}(B_1 - B_2) \leq \beta_1 - \beta_2 \\
\leq \frac{\sqrt{5}(B_1 + B_2)}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} + B_2}. \tag{3.19}
\]
where
\[
B_1 = \xi_1 \xi_2 \left[3\xi_1^2 + 2\xi_1 \xi_2 + 3\xi_2^2\right], \quad B_2 = \frac{2^{j+k-k_1-k_2+3}}{5}. \tag{3.20}
\]

Now we claim that the following inequality is valid
\[
\left| \beta - \frac{\sqrt{5}B_1}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)}} \right| \leq B_3, \tag{3.21}
\]
where
\[
B_3 = 2^j 3^{(k_1+k_2)+10}. \tag{3.22}
\]

When \(\xi_1 \xi_2 \geq 0\), we have
\[
\sqrt{\xi^2(\xi^2 - \xi_1\xi_2 - \frac{3\alpha}{5})} \pm B_2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \geq \xi^2. \tag{3.23}
\]

By using a direct computation, since
\[
B_1 \leq 3\xi^4, \quad \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2 \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \geq \frac{B_1}{3}, \tag{3.24}
\]
we have
\[
\left| \frac{B_1}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)}} - \frac{B_1}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2} \right| \\
= \frac{B_1 B_2}{\left[\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2\right] \left[\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2\right]} \\
\times \frac{1}{\left[\sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \right]} \\
\leq \frac{B_3}{10}. \tag{3.25}
\]
By using a direct computation, we have
\[
\frac{B_2}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2} \leq \frac{B_3}{10}. \tag{3.26}
\]
Combining (3.25) with (3.26), we have (3.21) is valid.

When \(\xi_1\xi_2 \leq 0\), we have
\[
\sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2 \geq |\xi||\xi_1\xi_2|^{1/2}. \tag{3.27}
\]
Since
\[
\left[\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2\right]
\left[\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2\right] \geq \frac{B_1}{4}, \tag{3.28}
\]
by a direct computation we have
\[
\left|\frac{B_1}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2} - \frac{B_1}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2}\right|
\leq \frac{B_3}{10}. \tag{3.29}
\]
By a direct computation, we have
\[
\frac{B_2}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)} \pm B_2} \leq \frac{B_3}{10}. \tag{3.30}
\]
By (3.23)-(3.30), we see that (3.21) is valid.

To obtain (3.18), we make the change of variable \(\beta_1 = \beta_2 + \beta\). Thus, (3.11)-(3.12) can be rewritten as follows:
\[
\beta \geq -\sqrt{5(\xi_1^2 + \xi_2^2)},
\left|\beta^2 + 2\sqrt{5}\beta(\xi_1^2 + \xi_2^2) + 5(4\xi_1^2\xi_2^2 - 3\xi_1^2\xi_2)\right| \leq 2^{j+k-1-2^{k+1}}. \tag{3.31}
\]
Since \(|\xi_j| \in [2^{k-1}, 2^{k+1}] (j = 1, 2)\), we can assume that \(\xi_j = a_j2^{k_j} (j = 1, 2)\), where \(1 \leq |a_j| \leq 2\). Consequently, (3.21) can be rewritten as follows:
\[
\left|\beta - \sqrt{5}f_1(a_1, a_2, k_1, k_2)\right| \leq B_3, \tag{3.32}
\]
17
where
\[ f_1(a_1, a_2, k_1, k_2) = \frac{a_1 a_2 2^{k_1+k_2} (3a_1^2 4^{k_1} + a_1 a_2 2^{k_1+k_2+1} + 3a_2^2 4^{k_2})}{a_1^2 4^{k_1} + a_2^2 4^{k_2} + \sqrt{(a_1 2^{k_1} + a_2 2^{k_2})^2 (a_1^2 4^{k_1} + a_1 a_2 2^{k_1+k_2} + a_2^2 4^{k_2})}} \frac{B_1}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2 (\xi^2 - \xi_1 \xi_2)}}. \] (3.33)

Thus, the left hand side of (3.18) can be bounded by
\[ 2^{k_1+k_2} \int_{\tilde{S}} h_1(\xi_1, \beta + \beta_2) h_2(\xi_2, \beta_2) \chi_{[-1, 1]} \left( \frac{\beta - \sqrt{5} f_1(a_1, a_2, k_1, k_2)}{B_3} - m \right) \times h(\xi_1, \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta)) d\xi_1 d\xi_2 d\beta d\beta_2, \] (3.34)

where
\[ \tilde{S} = \{(\xi_1, \xi_2, \beta, \beta_2) \in \mathbb{R}^4, \xi_1 + \xi_2 \geq 0, \beta \text{ satisfies (3.21)}\}. \] (3.35)
\[ A(\xi_1, \xi_2, \beta) = \sqrt{5} [\xi_1^3 - \xi_2^3] + \beta \xi_1, \] (3.36)
\[ B(\xi_1, \xi_2, \beta) = \frac{\xi_1 \xi_2}{\xi} \left( \beta^2 + 2\sqrt{5} \beta (\xi_1^2 + \xi_2^2) + 5 \xi_1 \xi_2 (4 \xi_1 \xi_2 - 3 \xi^2) \right). \] (3.37)

Let \( j' = j - \frac{3(k_1+k_2)}{2} + 10 \). Decompose
\[ h_i(\xi', \beta') = \sum_{m \in \mathbb{Z}} h_i^m(\xi', \beta') \chi_{[0, 1]} \left( \frac{\beta' - \sqrt{5} f_1(a_1, a_2, k_1, k_2)}{2^{j'}} - m \right) = \sum_{m \in \mathbb{Z}} h_i^m(\xi', \beta'), \quad i = 1, 2 \] (3.38)

for all \( a_j \in R, \frac{1}{2} \leq |a_j| \leq 2 (j = 1, 2) \). Obviously, if \( m_1, m_2 \in \mathbb{Z}, m_1 \neq m_2 \), then
\[ \prod_{i=1}^{2} h_i^m(\xi', \beta') = 0. \]

Thus, for \( m_1, m_2 \in \mathbb{Z}, m_1 \neq m_2 \), we have \( \prod_{i=1}^{2} h_i^m(\xi', \beta') = 0 \). Consequently, we have
\[ \|h_i(\xi', \beta')\|_{L_{x'}^2} = \left\| \sum_{m \in \mathbb{Z}} h_i^m(\xi', \beta') \right\|_{L_{x'}^2} = \left( \sum_{m \in \mathbb{Z}} \|h_i^m\|_{L_{x'}^2}^2 \right)^{\frac{1}{2}}. \] (3.39)

Thus, (3.34) is controlled by
\[ 2^{k_1+k_2} \sum_{|m-m'| \leq 4} \int_{\tilde{S}} h_1^m(\xi_1, \beta + \beta_2) h_2^{m'}(\xi_2, \beta_2) \times h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta)) d\xi_1 d\xi_2 d\beta d\beta_2. \] (3.40)
To prove (3.18), it suffices to prove that

$$2^{k_1+k_2} \int_S h_1^m(\xi_1, \beta + \beta_2) h_2^{m'}(\xi_2, \beta_2) \chi_{[m',m+1]} \left( \frac{\beta_2 - \sqrt{5} f_1(a_1, a_2, k_1, k_2)}{2^{j'}} \right)$$

$$\times h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta)) \, d\xi_1 d\xi_2 d\beta d\beta_2$$

$$\leq C 2^{\frac{j}{4}(k_1+k_2)+\frac{3k_3}{4}} \|h\|_{L^2} \|h_1^m\|_{L^2} \|h_2^{m'}\|_{L^2}. \quad (3.41)$$

If (3.41) is valid, by using the Cauchy-Schwartz inequality, we have

$$2^{k_1+k_2} \sum_{|m-m'| \leq 4} \int_S h_1^m(\xi_1, \beta + \beta_2) h_2^{m'}(\xi_2, \beta_2)$$

$$\times h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta)) \, d\xi_1 d\xi_2 d\beta d\beta_2$$

$$\leq C 2^{\frac{j}{4}(k_1+k_2)+\frac{3k_3}{4}} \|h\|_{L^2} \sum_{|m-m'| \leq 4} \|h_1^m\|_{L^2} \|h_2^{m'}\|_{L^2}$$

$$= C 2^{\frac{j}{4}(k_1+k_2)+\frac{3k_3}{4}} \|h\|_{L^2} \left[ \sum_{m \in \mathbb{Z}} \|h_1^m\|_{L^2} \left( \sum_{m-4 \leq m' \leq m+4} \|h_2^{m'}\|_{L^2} \right) \right]$$

$$\leq C 2^{\frac{j}{4}(k_1+k_2)+\frac{3k_3}{4}} \|h\|_{L^2} \left[ \sum_{m \in \mathbb{Z}} \|h_1^m\|_{L^2}^2 \right]^{\frac{1}{2}} \left[ \sum_{m \in \mathbb{Z}} \left( \sum_{m-4 \leq m' \leq m+4} \|h_2^{m'}\|_{L^2}^2 \right) \right]^{\frac{1}{2}}$$

$$\leq C 2^{\frac{j}{4}(k_1+k_2)+\frac{3k_3}{4}} \|h\|_{L^2} \left( \sum_{m \in \mathbb{Z}} \sum_{m-4 \leq m' \leq m+4} \|h_2^{m'}\|_{L^2}^2 \right)$$

$$\leq C 2^{\frac{j}{4}(k_1+k_2)+\frac{3k_3}{4}} \|h\|_{L^2} \prod_{i=1}^2 \|h_i\|_{L^2}. \quad (3.42)$$

To prove (3.41), without loss of generality, we assume that $|\xi_1| \leq |\xi_2|$. To prove (3.41), by using the Minkowski inequality with respect to variables $(\xi_1, \xi_2, \beta)$ with

$$S' = \{(\xi_1, \xi_2, \beta) \in \mathbb{R}^3, \xi_1 + \xi_2 \geq 0, \beta \text{ satisfies } (3.21)\}, \quad (3.43)$$

the left-hand side of (3.40) is controlled by

$$2^{k_1+k_2} \int_{\mathbb{R}} \chi_{[m,m+1]} \left( \frac{\beta_2 - \sqrt{5} f_1(a_1, a_2, k_1, k_2)}{2^{j'}} \right)$$

$$\times \left( \int_S |h_1^m(\xi_1, \beta + \beta_2) h_2^{m'}(\xi_2, \beta_2)|^2 d\xi_1 d\xi_2 d\beta d\beta_2 \right)^{\frac{1}{2}}$$

$$\times \left( \int_S |h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))|^2 d\xi_1 d\xi_2 d\beta d\beta_2 \right)^{\frac{1}{2}} d\beta_2. \quad (3.44)$$
From (3.44), it suffices to prove that
\[
\left( \int_{S''} |h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))|^2 \, d\xi_1 d\xi_2 d\beta \right)^{\frac{1}{2}} \leq C 2^{-\frac{k_1 + k_2}{2}} \|h\|_{L^2}.
\] (3.45)

If (3.45) is valid, by using the Cauchy-Schwartz inequality with respect to $\beta_2$, we have (3.44) is controlled by
\[
C 2^{-(k_1 + k_2) + \frac{k_2}{2}} \int_{\mathbb{R}} \chi(m', m'+1) \left( \frac{\beta_2 - \sqrt{5} f_1(a_1, a_2, k_1, k_2)}{2^{r'}} \right) \|h_1^m\|_{L^2} \|h_2^{m'}(\cdot, \beta_2)\|_{L^{k_2}} \, dh \, d\beta_2
\]
\[
= C 2^{-(k_1 + k_2) + \frac{k_2}{2}} \int_{\mathbb{R}} \chi(m'+1)^{2^{r'}} + \sqrt{5} f_1(a_1, a_2, k_1, k_2) \|h_1^m\|_{L^2} \|h_2^{m'}(\cdot, \beta_2)\|_{L^{k_2}} \, dh \, d\beta_2
\]
\[
\leq C 2^{-(k_1 + k_2) + \frac{k_2}{2}} \frac{2^{r'}}{2^{r}} \|h\|_{L^2} \|h_1^m\|_{L^2} \|h_2^{m'}\|_{L^2}
\]
\[
\leq C 2^{\frac{k_1 + k_2}{2}} \frac{2^{r'}}{2^{r}} \|h\|_{L^2} \|h_1^m\|_{L^2} \|h_2^{m'}\|_{L^2}. \quad \text{(3.46)}
\]

To prove (3.45), we may assume that $\beta_2 = 0$ and make the change of variable $\beta = \sqrt{5} \xi_1 \xi_2 \nu$. From (3.21), we have
\[
\left| \nu - \frac{3\xi_1^2 + 2\xi_1 \xi_2 + 3\xi_2^2}{\xi_1^2 + \xi_2^2 + \sqrt{5} (\xi_1^2 - \xi_2^2)} \right| \leq 2^{-20}. \quad \text{(3.47)}
\]

The left hand side of (3.44) is controlled by
\[
C 2^{\frac{k_1 + k_2}{2}} \left( \int_{S''} |h(\xi_1 + \xi_2, H_1(\xi_1, \xi_2, \nu), H_2(\xi_1, \xi_2, \nu))| \, d\xi_1 d\xi_2 d\nu \right)^{\frac{1}{2}}, \quad \text{(3.48)}
\]
where
\[
S'' = \left\{(\xi_1, \xi_2, \nu) \in \mathbb{R}^3 : \xi_i \in \tilde{I}_k, \nu \text{ satisfies (3.45)} \right\}, \quad \text{(3.49)}
\]
\[
H_1(\xi_1, \xi_2, \nu) = \sqrt{5} (\xi_1^3 - \xi_2^3 + \nu \xi_1^2 \xi_2), \quad \text{(3.50)}
\]
\[
H_2(\xi_1, \xi_2, \nu) = \frac{5(\xi_1 \xi_2)^2}{\xi} \left( \xi_1 \xi_2 \nu^2 + 2\nu (\xi_1^3 + \xi_2^3) + 4\xi_1 \xi_2 - 3\xi^2 \right). \quad \text{(3.51)}
\]

We consider $\xi_1 \xi_2 \geq 0, \xi_1 \xi_2 \leq 0$, respectively.

Firstly, we consider $\xi_1 \xi_2 \geq 0$. We define the function
\[
G(\xi, x) = 2^{-k_3 + 2k_1 + 2k_2} \left| h \left( \xi, \sqrt{5} \left[ \xi_1^3 - \xi_2^3 + x \right], \frac{5(\xi_1 \xi_2)^2}{\xi} \left[ y + 4\xi_1 \xi_2 - 3\xi_2^2 \right] \right) \right|^2, \quad \text{(3.52)}
\]

\[
20
\]
where

\[ x = \xi_1^2 \xi_2 \nu, \quad y = \xi_1 \xi_2 \nu^2 + 2(\xi_1^2 + \xi_2^2) \nu. \]  

(3.53)

Obviously, \( \|G\|_{L^1} = \|h\|_{L^2}^2 \). From (3.52), we have (3.48) can be bounded by

\[ C 2^{-k_1 + k_2 - k_3} \left( \int_{S''} |G(\xi_1 + \xi_2, \xi_1^2 \xi_2 \nu, \xi_1 \xi_2 \nu^2 + 2(\xi_1^2 + \xi_2^2) \nu)|d\xi_1 d\xi_2 d\nu \right)^{\frac{1}{2}}. \]  

(3.54)

We make the change of variables \((\xi_1, \xi_2, \nu) \rightarrow (\xi_1 + \xi_2, \xi_1^2 \xi_2 \nu, \xi_1 \xi_2 \nu^2 + 2(\xi_1^2 + \xi_2^2) \nu)\), thus the absolute value of the Jacobi determinant equals

\[ |\nu \xi_1| |\nu \xi_1 \xi_2 (\xi_1 - 3 \xi_2) + 2 (\xi_1^3 - \xi_1 \xi_2^2 - 2 \xi_2^3)|. \]  

(3.55)

By using a direct computation, we have (3.55) equals

\[ |\nu ^2 \xi_1 (\xi_1 - 3 \xi_2)| \left| \nu + \frac{2(\xi_1^3 - \xi_1 \xi_2^2 - 2 \xi_2^3)}{\xi_1 \xi_2 (\xi_1 - 3 \xi_2)} \right|, \]  

(3.56)

where \( \nu \) satisfies (3.47).

By using a direct computation, we have

\[ \left| \frac{2(\xi_1^3 - \xi_1 \xi_2^2 - 2 \xi_2^3)}{\xi_1 \xi_2 (\xi_1 - 3 \xi_2)} \right| \geq 2. \]  

(3.57)

From (3.47), we have

\[ 1 - 2^{-20} \leq |\nu| \leq \left| \frac{3 \xi_1^2 + 2 \xi_1 \xi_2 + 3 \xi_2^2}{\xi_1^2 + \xi_2^2 + \sqrt{\xi_2^2 (\xi^2 - \xi_1 \xi_2)}} \right| + 2^{-20} \leq \frac{3}{2} + 2^{-20}. \]  

(3.58)

Combining (3.57) with (3.58), we have

\[ |\nu| \left| \nu + \frac{2(\xi_1^3 - \xi_1 \xi_2^2 - 2 \xi_2^3)}{\xi_1 \xi_2 (\xi_1 - 3 \xi_2)} \right| \geq \frac{1}{4}. \]  

(3.59)

Combining (3.56) with (3.59), we have

\[ |\nu \xi_1| |\nu \xi_1 \xi_2 (\xi_1 - 3 \xi_2) + 2 (\xi_1^3 - \xi_1 \xi_2^2 - 2 \xi_2^3)| \geq C \xi_1^2 \xi_2^2. \]  

(3.60)

Combining (3.54) with (3.60), we have (3.45) can be bounded by

\[ C 2^{-\frac{3(k_1 + k_2)}{2} + k_3} \|G\|_{L^1} \leq C 2^{-\frac{3(k_1 + k_2)}{2} + k_3} \|h\|_{L^2}^2. \]  

(3.61)
Now we consider $\xi_1 \xi_2 \leq 0$. We define
\[
G(\xi, x, y) = 2^{-k_3 + 7k_1} \left| h \left( \xi, \sqrt{5} \left[ \xi_1^3 - \xi_2^3 + \xi_1^2 \xi_2 \right], \frac{5(\xi_1 \xi_2)^2}{\xi} \left[ y + 4\xi_1 \xi_2 - 3\xi_2^2 \right] \right) \right|^2, \quad (3.62)
\]
where
\[
x = \nu, \quad y = \xi_1 \xi_2 \nu^2 + 2(\xi_1^2 + \xi_2^2) \nu. \quad (3.63)
\]
Obviously, $\|G\|_{L^1} \approx \|h\|_{L^2}^2$. From (3.62), we have (3.48) can be bounded by
\[
C 2^{-\frac{5k_1 - k_3}{2}} \left( \int_{S^*} |G(\xi_1 + \xi_2, \nu, \xi_1 \xi_2 \nu^2 + 2(\xi_1^2 + \xi_2^2) \nu)|d\xi_1 d\xi_2 d\nu \right)^\frac{1}{2}.
\quad (3.64)
\]
We make the change of variables $(\xi_1, \xi_2, \nu) \rightarrow (\xi_1 + \xi_2, \nu, \xi_1 \xi_2 \nu^2 + 2(\xi_1^2 + \xi_2^2) \nu)$, thus the absolute value of the Jacobi determinant equals
\[
|\xi_1 - \xi_2| |\nu(4 - \nu)|. \quad (3.65)
\]
From (3.47), we have
\[
1 - 2^{-20} \leq \frac{3\xi_1^2 + 2\xi_1 \xi_2 + 3\xi_2^2}{\xi_1^2 + \xi_2^2 + \sqrt{\xi_2^2(\xi_1^2 - \xi_1 \xi_2)}} - 2^{-20}
\leq |\nu| \leq \frac{3\xi_1^2 + 2\xi_1 \xi_2 + 3\xi_2^2}{\xi_1^2 + \xi_2^2 + \sqrt{\xi_2^2(\xi_1^2 - \xi_1 \xi_2)}} + 2^{-20} \leq 3 + 2^{-20}. \quad (3.66)
\]
Combining (3.65) with (3.66), we have
\[
|\xi_1 - \xi_2| |\nu(4 - \nu)| \sim 2^{k_1}. \quad (3.67)
\]
Combining (3.64) with (3.67), we have (3.44) can be bounded by
\[
C 2^{-\frac{3(\xi_1^2 + \xi_2^2)}{2}} \left( \int_{S^*} |G(\xi_1, \beta_1)h_1(\xi_1, \beta_1)h_2(\xi_2, \beta_2) \times h(\xi_1 + \xi_2, \sqrt{5}(\xi_1^3 - \xi_2^3) + \beta_1 \xi_1 + \beta_2 \xi_2, \beta_1 \xi_1 + \beta_2 \xi_2, d\xi_1 d\xi_2 d\beta_1 d\beta_2 \leq 2^{2\frac{1}{2} \frac{5k_1 - 5k_2}{2}} \|h\|_{L^2} \prod_{j=1}^{2} \|h_j\|_{L^2}. \quad (3.68)
\]
Therefore the proof of (3.3) is completed.

Now we prove (3.4). From (3.5)-(3.17), we know that it suffices to prove
\[
2^{\frac{k_1 + k_2}{2}} \int_{S^*} h_1(\xi_1, \beta_1)h_2(\xi_2, \beta_2) \times h(\xi_1 + \xi_2, \sqrt{5}(\xi_1^3 - \xi_2^3) + \beta_1 \xi_1 + \beta_2 \xi_2, \beta_1 \xi_1 + \beta_2 \xi_2, d\xi_1 d\xi_2 d\beta_1 d\beta_2 \leq C 2^{2\frac{1}{2} \frac{5k_1 - 5k_2}{2}} \|h\|_{L^2} \prod_{j=1}^{2} \|h_j\|_{L^2}. \quad (3.69)
\]

where \(h_i(i = 1, 2)\) are defined as in (3.16)-(3.17) and \(S\) is defined as in (3.14) and \(\tilde{R}_1(\xi_1, \beta_1, \xi_2, \beta_2)\) is defined as in (3.15) and \(\beta_1 - \beta_2\) satisfies (3.19)-(3.20). To obtain (3.69), we make the change of variable \(\beta_1 = \beta_2 + \beta\). Now we claim that the following inequality is valid

\[
\left| \beta - \frac{\sqrt{5}B_1}{\xi_1^2 + \xi_2^2 + \sqrt{\xi^2(\xi^2 - \xi_1\xi_2)}} \right| \leq B_4, \tag{3.70}
\]

where \(B_1\) is defined as in (3.21) and

\[
B_4 = 2^{j - 3k_2 + 10}. \tag{3.71}
\]

(3.70) can be proved similarly to (3.21). Since \(|\xi_j| \in [2^{kj-1}, 2^{kj+1}](j = 1, 2)\), we can assume that \(\xi_j = a_j 2^kj(j = 1, 2)\), where \(\frac{1}{2} \leq |a_j| \leq 2\). Consequently, (3.70) can be rewritten as follows:

\[
\left| \beta - \sqrt{5}f_1(a_1, a_2, k_1, k_2) \right| \leq B_4, \tag{3.72}
\]

where \(f_1(a_1, a_2, k_1, k_2)\) is defined as in (3.33).

Thus, the left hand side of (3.68) can be bounded by

\[
2^{k_1 + k_2} \int_{\tilde{S}} h_1(\xi_1, \beta + \beta_2)h_2(\xi_2, \beta_2)\chi_{[-1,1]} \left( \frac{\beta_2 - \sqrt{5}f_1(a_1, a_2, k_1, k_2)}{B_4} - m \right) \times h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))d\xi_1d\xi_2d\beta d\beta_2, \tag{3.73}
\]

where \(\tilde{S}\) satisfies (3.35) and \(A(\xi_1, \xi_2, \beta)\) satisfies (3.36) and \(B(\xi_1, \xi_2, \beta)\) satisfies (3.37).

Let \(j'' = j - 3k_2 + 10\). Decompose

\[
h_i(\xi', \beta') = \sum_{m \in \mathbb{Z}} h_i(\xi', \beta')\chi_{[0,1]} \left( \frac{\beta' - \sqrt{5}f_1(a_1, a_2, k_1, k_2)}{2^{j''}} - m \right)
= \sum_{m \in \mathbb{Z}} h_i^m(\xi', \beta'), \tag{3.74}
\]

\(i = 1, 2\). Obviously, \(\|h_i\|_{L^2} = \left( \sum_{m \in \mathbb{Z}} \|h_i^m\|_{L^2}^2 \right)^{\frac{1}{2}}\). Thus, (3.73) is controlled by

\[
2^{k_1 + k_2} \sum_{|m - m'| \leq 4} \int_{\tilde{S}} h_i^m(\xi_1, \beta + \beta_2)h_2^{m'}(\xi_2, \beta_2) \times h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))d\xi_1d\xi_2d\beta d\beta_2. \tag{3.75}
\]
To prove (3.69), it suffices to prove that
\[
2^{\frac{k_1+k_2}{2}} \int_{S} h_1^m(\xi_1, \beta + \beta_2)h_2^{m'}(\xi_2, \beta_2)\chi_{[m',m'+1]} \left( \frac{\beta_2 - \sqrt{5}f_1(a_1, a_2, k_1, k_2)}{2^{\gamma'}} \right)
\times h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))d\xi_1d\xi_2d\beta d\beta_2
\leq C 2^{\frac{k_1+k_2}{2}} \|h\|_{L^2} h_1^m \|h_2^{m'}\|_{L^2}.
\tag{3.76}
\]

From (3.66), it suffices to prove that
\[
\left( \int_{S'} |h(\xi_1 + \xi_2, A(\xi_1, \xi_2, \beta) + \beta_2(\xi_1 + \xi_2), B(\xi_1, \xi_2, \beta))|^2 d\xi_1d\xi_2d\beta \right)^{\frac{1}{2}} 
\leq C 2^{-\frac{3k_2}{2}} \|h\|_{L^2}.
\tag{3.77}
\]

If (3.77) is valid, by using the Cauchy-Schwartz inequality with respect to \( \beta_2 \), we have (3.75) is controlled by
\[
C 2^{-\frac{k_1+k_2}{2}} \int_{\mathbb{R}} \chi_{[m',m'+1]} \left( \frac{\beta_2 - \sqrt{5}f_1(a_1, a_2, k_1, k_2)}{2^{\gamma'}} \right)
\times \|h_1^m\|_{L^2} \|h_2^{m'}(\cdot, \beta_2)\|_{L^2_2} \|h\|_{L^2} d\beta_2 
= C 2^{-\frac{k_1+k_2}{2}} \int_{\mathbb{R}} \left( \frac{\beta_2 - \sqrt{5}f_1(a_1, a_2, k_1, k_2)}{2^{\gamma'}} \right)
\times \|h_1^m\|_{L^2} \|h_2^{m'}(\cdot, \beta_2)\|_{L^2_2} \|h\|_{L^2} d\beta_2 
\leq C 2^{-\frac{k_1+k_2}{2}} \|h\|_{L^2} \|h_1^m\|_{L^2} \|h_2^{m'}\|_{L^2}.
\tag{3.78}
\]

To prove (3.77), we may assume that \( \beta_2 = 0 \) and make the change of variable \( \beta = \sqrt{5}\xi_1\xi_2\nu \). The left hand side of (3.77) is controlled by
\[
C 2^{k_1+k_2} \left( \int_{S''} |h(\xi_1 + \xi_2, H_1(\xi_1, \xi_2, \nu), H_2(\xi_1, \xi_2, \nu))|^2 d\xi_1d\xi_2d\nu \right)^{\frac{1}{2}},
\tag{3.79}
\]
where \( S'' \) is defined as in (3.49) and \( H_1(\xi_1, \xi_2, \nu) \) is defined as in (3.50) and \( H_2(\xi_1, \xi_2, \nu) \) is defined as in (3.51).

We define the function \( G(\xi, x, y) \) as in (3.52) and \( x, y \) as in (3.53). Obviously, \( \|G\|_{L^1} \approx \|h\|_{L^2}^2 \). Thus, we have (3.79) can be bounded by
\[
C 2^{-\frac{k_1}{2}} \left( \int_{S''} |G(\xi_1 + \xi_2, \xi_1^2\xi_2\nu, \xi_1\xi_2\nu^2 + 2(\xi_1^2 + \xi_2^2)\nu)|d\xi_1d\xi_2d\nu \right)^{\frac{1}{2}}.
\tag{3.80}
\]

24
We make the change of variables \((\xi_1, \xi_2, \nu) \rightarrow (\xi_1 + \xi_2, \xi_1^2 \xi_2 \nu, \xi_1 \xi_2 \nu^2 + 2(\xi_1^2 + \xi_2^2)\nu)\), thus the absolute value of the Jacobi determinant equals

\[
2|\nu(\xi_1^3 - \xi_1 \xi_2^2 - 2\xi_2^3)| \left| \frac{\xi_1 \xi_2(\xi_1 - 3\xi_2)\nu}{2(\xi_1^3 - \xi_1 \xi_2^2 - 2\xi_2^3)} + 1 \right|. \tag{3.81}
\]

In this case, by using a direct computation, we have

\[
|\nu(\xi_1^3 - \xi_1 \xi_2^2 - 2\xi_2^3)| \left| \frac{\xi_1 \xi_2(\xi_1 - 3\xi_2)\nu}{2(\xi_1^3 - \xi_1 \xi_2^2 - 2\xi_2^3)} + 1 \right| \leq \frac{1}{32}. \tag{3.82}
\]

From (3.43), we have

\[
1 - 2^{-20} \leq \left| \frac{3\xi_1^2 + 2\xi_1 \xi_2 + 3\xi_2^2}{\xi_1^2 + \xi_2^2 + \sqrt{\xi_1^2 (\xi_2^2 - \xi_1 \xi_2)}} \right| - 2^{-20} \leq |\nu| \leq \left| \frac{3\xi_1^2 + 2\xi_1 \xi_2 + 3\xi_2^2}{\xi_1^2 + \xi_2^2 + \sqrt{\xi_1^2 (\xi_2^2 - \xi_1 \xi_2)}} \right| + 2^{-20} \leq 3 + 2^{-20}. \tag{3.83}
\]

Combining (3.82) with (3.83), we have

\[
|\nu(\xi_1^3 - \xi_1 \xi_2^2 - 2\xi_2^3)| \left| \frac{\xi_1 \xi_2(\xi_1 - 3\xi_2)\nu}{2(\xi_1^3 - \xi_1 \xi_2^2 - 2\xi_2^3)} + 1 \right| \sim |\xi_1(\xi_1^3 - \xi_1 \xi_2^2 - 2\xi_2^3)| \sim 2^{k_1 + 3k_2}. \tag{3.84}
\]

Combining (3.84) with the fact that \(\|G\|_{L^1} \approx \|h\|_{L^2}^2\), we have (3.78) can be bounded by

\[
C2^{-\frac{3k_3}{2} - k_1} \|G\|_{L^1} \leq C2^{-\frac{3k_3}{2} - k_1} \|h\|_{L^2}^2. \tag{3.85}
\]

This completes the proof of Lemma 3.1.

Inspired by the idea of Lemma 5.2 of [18], we give the proof of Lemma 3.2.

**Lemma 3.2.** Assume \(\alpha \in \mathbb{R}\) and \(k_1, k_2, k_3 \in \mathbb{Z}\), \(k_{\text{max}} = \max \{k_1, k_2, k_3\} \geq 20\) and \(k_{\text{min}} = \min \{k_1, k_2, k_3\}\) and \(j_1, j_2, j_3 \in \mathbb{Z}_+\), \(j_{\text{max}} = \max \{j_1, j_2, j_3\}\) and \(f_i : \mathbb{R}^3 \to \mathbb{R}\) are \(L^2\) functions supported in \(\mathcal{D}_{k_i, \infty, j_i, i = 1, 2, 3}\). Then, \(k_{\text{max}} \geq 20\), we have

\[
\int_{\mathbb{R}^3} (f_1 * f_2) f_3 d\xi d\mu d\tau \leq C2^{\frac{j_1 + j_2 + j_3}{2}} 2^{-\frac{k_{\text{max}} - k_{\text{min}}}{2}} \prod_{j=1}^3 \|f_j\|_{L^2}. \tag{3.86}
\]

**Proof.** From (3.79), we assume that \(j_3 = \max \{j_1, j_2, j_3\}\). Then, we have

\[
\int_{\mathbb{R}^3} (f_1 * f_2) f_3 d\xi d\mu d\tau \leq C\|f_3\|_{L^2} \|f_1 * f_2\|_{L^2} \leq C\|f_3\|_{L^2} \prod_{j=1}^2 \|\mathcal{F}^{-1}(f_j)\|_{L^4}. \tag{3.87}
\]
From Theorem 3.1 of [15], we have
\[ \left\| \int \chi^\frac{1}{2} f_j(\xi, \mu) e^{ix\xi + iy\mu} e^{it\phi(\xi, \mu)} d\xi d\mu \right\|_{L^4_{x,y,t}} \leq C \| f_j \|_{L^2_{\xi,\mu}} (j = 1, 2). \] (3.88)

From (3.88), by using the Cauchy-Schwartz inequality with respect to \( \theta \), we have
\[ \left\| \int |\xi|^{\frac{1}{4}} f_j^\#(\xi, \mu, \tau) e^{ix\xi + iy\mu} e^{it\phi(\xi, \mu)} d\xi d\mu d\tau \right\|_{L^4_{x,y,t}} \leq C \int |f_j^\#(\xi, \mu, \theta)|^{\frac{1}{2}} d\theta \leq C 2^{\frac{k}{2}} \| f_j^\#(\xi, \mu, \theta) \|_{L^2_{\xi,\mu,\theta}}. \] (3.89)

Here \( f_j^\#(j = 1, 2) \) are defined as in Lemma 3.1. From (3.89), we have
\[ \| \mathcal{F}^{-1}(f_j) \|_{L^4} \leq C 2^{-\frac{k}{2}} 2^{\frac{k}{2}} \| f_j(\xi, \mu, \tau) \|_{L^2_{\xi,\mu,\tau}}. \] (3.90)

Inserting (3.90) into (3.87) yields
\[ \int_{\mathbb{R}^3} (f_1 * f_2) f_3 \leq C \| f_3 \|_{L^2} \| f_1 * f_2 \|_{L^2} \]
\[ \leq C 2^{\frac{j_1 + j_2}{2}} 2^{-\frac{k_{\max} + k_{\min}}{2}} \| f \|_{L^2} \prod_{j = 1}^2 \| \mathcal{F}^{-1}(f_j) \|_{L^4}. \] (3.91)

Combining the fact with \( j_3 = \max \{ j_1, j_2, j_3 \} \) with (3.91), we have (3.86) is valid.

This completes the proof of Lemma 3.2.

4. Bilinear estimates

This section is devoted to establishing Lemmas 4.1-4.3. Lemma 4.1 is used to establish Theorems 1.1. Lemma 4.2 is used to establish the almost conservation. Lemma 4.3 is used to establish Lemma 6.1 which is the variant of Theorem 1.1.

**Lemma 4.1.** Let \( -\frac{9}{8} + 16\epsilon \leq s_1 < 0, s_2 \geq 0 \) and \( u_j \in X^{s_1, s_2} \frac{1}{2 + \epsilon} (j = 1, 2) \). Then, we have
\[ \| \partial_x (u_1 u_2) \|_{X^{s_1, s_2} \frac{1}{2 + \epsilon}} \leq C \prod_{j = 1}^2 \| u_j \|_{X^{s_1, s_2} \frac{1}{2 + \epsilon}}. \] (4.1)
Proof. To derive (4.1), by duality, it suffices to show that
\[
\left| \int_{\mathbb{R}^3} \tilde{u} \partial_x (u_1 u_2) dx dy dt \right| \leq C \| u \|_{X^{-s_1, -s_2}_{\frac{3}{2} - 2\epsilon}} \prod_{j=1}^{2} \| u_j \|_{X^{s_1, s_2}_{\frac{3}{2} + \epsilon}}. \tag{4.2}
\]
for \( u \in X^{-s_1, -s_2}_{\frac{3}{2} - 2\epsilon} \). Let
\[
F(\xi, \mu, \tau) = \langle \xi \rangle^{-s_1} \langle \mu \rangle^{-s_2} \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \mathcal{F} u(\xi, \mu, \tau),
\]
and
\[
F_j(\xi_j, \mu_j, \tau_j) = \langle \xi_j \rangle^{s_1} \langle \mu_j \rangle^{s_2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon} \mathcal{F} u_j(\xi_j, \mu, \tau_j) (j = 1, 2), \tag{4.3}
\]
and
\[
D := \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in \mathbb{R}^6, \xi = \sum_{j=1}^{2} \xi_j, \mu = \sum_{j=1}^{2} \mu_j, \tau = \sum_{j=1}^{2} \tau_j \right\}.
\]
To derive (4.2), from (4.3), it suffices to show that
\[
\int_{D} \frac{|\xi| \langle \xi \rangle^{s_1} \langle \mu \rangle^{s_2} F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j)}{\langle \sigma_j \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \mu_j \rangle^{s_2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} dV \leq C \| F \|_{L^2_{\xi, \mu}} \prod_{j=1}^{2} \| F_j \|_{L^2_{\xi, \mu}}, \tag{4.4}
\]
where \( dV := d\xi d\mu d\tau d\xi d\mu d\tau \). Without loss of generality, by using the symmetry, we assume that \( |\xi_1| \geq |\xi_2| \) and \( F(\xi, \mu, \tau) \geq 0, F_j(\xi_j, \mu_j, \tau_j) \geq 0 (j = 1, 2) \) and
\[
D^* := \{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D, |\xi_2| \geq |\xi_1| \}.
\]
Let
\[
\Omega_1 = \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_2| \leq |\xi_1| \leq 80\},
\]
\[
\Omega_2 = \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 80, |\xi_1| \gg |\xi_2|, |\xi_2| \leq 20\},
\]
\[
\Omega_3 = \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 80, |\xi_1| \gg |\xi_2|, |\xi_2| > 20\},
\]
\[
\Omega_4 = \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 80, 4|\xi| \leq |\xi_1| \sim |\xi_2|, |\xi| \leq 20, \xi_1 \xi_2 < 0\},
\]
\[
\Omega_5 = \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 80, 4|\xi| \leq |\xi_1| \sim |\xi_2|, |\xi| > 20, \xi_1 \xi_2 < 0\},
\]
\[
\Omega_6 = \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 80, |\xi_2| \sim |\xi_1|, \xi_1 \xi_2 < 0, |\xi| \geq |\xi_2| / 4\},
\]
\[
\Omega_7 = \{(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| \geq 80, |\xi_2| \sim |\xi_1|, \xi_1 \xi_2 > 0\}.
\]
Obviously, \( D^* \subset \bigcup_{j=1}^{7} \Omega_j \). We define
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) := \frac{|\xi| \langle \mu \rangle^{s_2} \langle \xi \rangle^{s_1}}{\langle \sigma_j \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \mu_j \rangle^{s_2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \tag{4.5}
\]
and

\[
\text{Int}_j := \int_{\Omega_j} K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) F(\xi, \mu, \tau) \prod_{j=1}^{2} F_j(\xi_j, \mu_j, \tau_j) d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau, \tag{4.6}
\]

\[1 \leq j \leq 7, j \in \mathbb{N}. \]

Since \( s_2 \geq 0 \) and \( \mu = \sum_{j=1}^{2} \mu_j \), we have \( \langle \mu \rangle^{s_2} \leq \prod_{j=1}^{2} \langle \mu_j \rangle^{s_2} \), thus, we have

\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{\frac{1}{2}-2\epsilon} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{1}{2}+\epsilon}}. \tag{4.6}
\]

Now we estimate the integrals \( \text{Int}_j \) over the above seven regions one by one.

(I) Region \( \Omega_1 \). In this region \(|\xi| \leq |\xi_1| + |\xi_2| \leq 160 \), this case can be proved similarly to case \( \text{low} + \text{low} \rightarrow \text{low} \) of Pages 344–345 of Theorem 3.1 in [38].

(II) Region \( \Omega_2 \). In this region, we have \(|\xi| \sim |\xi_1|\).

By using the Cauchy-Schwartz inequality with respect to \( \xi_1, \mu_1, \tau_1 \), from (4.6), we have

\[
\text{Int}_2 \leq C \int_{\mathbb{R}^3} \frac{|\xi|}{\langle \sigma \rangle^{\frac{1}{2}-2\epsilon}} \left( \int_{\mathbb{R}^3} \langle \sigma_1 \rangle^{-(1+2\epsilon)} \langle \sigma_2 \rangle^{-(1+2\epsilon)} d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} \times \left( \int_{\mathbb{R}^3} \prod_{j=1}^{2} |F_j(\xi_j, \mu_j, \tau_j)|^2 d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} F(\xi, \mu, \tau) d\xi d\mu d\tau. \tag{4.7}
\]

By using (2.2), we have

\[
\frac{|\xi|}{\langle \sigma \rangle^{\frac{1}{2}-2\epsilon}} \left( \int_{\mathbb{R}^3} \langle \sigma_1 \rangle^{-(1+2\epsilon)} \langle \sigma_2 \rangle^{-(1+2\epsilon)} d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} \leq C \frac{|\xi|}{\langle \sigma \rangle^{\frac{1}{2}-2\epsilon}} \left( \int_{\mathbb{R}^2} \frac{d\xi_1 d\mu_1}{\langle \tau + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) \rangle^{1+2\epsilon}} \right)^{\frac{1}{2}}. \tag{4.8}
\]

Let \( \nu = \tau + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) \) and \( \Delta = \xi_1 \xi_2 (5\xi_2^2 - 5\xi_1 + 5\xi_1^2) \), since \(|\xi_1| \gg |\xi_2| \), then we have the absolute value of Jacobian determinant equals

\[
\left| \frac{\partial(\Delta, \nu)}{\partial(\xi_1, \mu_1)} \right| = 2 \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right| |5(\xi_1^4 - \xi_2^4) - 3\alpha(\xi_1^2 - \xi_2^2)|
= 2 |\sigma - \nu - \Delta|^{\frac{1}{2}} \left| \frac{\xi}{\xi_1 \xi_2} \right| \frac{1}{2} |5(\xi_1^4 - \xi_2^4) - 3\alpha(\xi_1^2 - \xi_2^2)|
\sim |\sigma - \nu + \delta|^{\frac{1}{2}} \left| \frac{\xi}{\xi_1 \xi_2} \right| |\xi_1|^4. \tag{4.9}
\]

28
Inserting (4.9) into (4.8), using (2.3), we have

\[
\frac{\|\xi\|}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \left( \int_{\mathbb{R}^3} \langle \sigma_1 \rangle^{-(1+2\epsilon)} \langle \sigma_2 \rangle^{-(1+2\epsilon)} d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}}
\]

\[
\leq C \frac{\|\xi\|}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \left( \int_{\mathbb{R}^2} \frac{d\xi_1 d\mu_1}{\langle \tau + \phi(\xi_1, \mu_1) + \phi(\xi_2, \mu_2) \rangle^{1+2\epsilon}} \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C}{\|\xi\| \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \left( \int_{|\Delta| < 20|\xi|^4} \frac{d\Delta}{\langle \Delta - \sigma \rangle^{\frac{1}{2}}} \right)^{\frac{1}{2}}. \tag{4.10}
\]

When $|\sigma| < 20|\xi|^4$, combining (4.10) with (2.1), we have

\[
\frac{C}{\|\xi\| \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \left( \int_{|\Delta| < 20|\xi|^4} \frac{d\Delta}{\langle \Delta - \sigma \rangle^{\frac{1}{2}}} \right)^{\frac{1}{2}} \leq C \frac{|\xi|}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \leq C. \tag{4.11}
\]

When $|\sigma| \geq 20|\xi|^4$, from (4.10), we have

\[
\frac{C}{\|\xi\| \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \left( \int_{|\Delta| < 20|\xi|^4} \frac{d\Delta}{\langle \Delta - \sigma \rangle^{\frac{1}{2}}} \right)^{\frac{1}{2}} \leq C \frac{|\xi|^2}{\|\xi\| \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \leq C \frac{|\xi|}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \leq C. \tag{4.12}
\]

Combining (4.8) with (4.9)-(4.12), we have

\[
\frac{|\xi|}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \left( \int_{\mathbb{R}^3} \langle \sigma_1 \rangle^{-(1+2\epsilon)} \langle \sigma_2 \rangle^{-(1+2\epsilon)} d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} \leq C. \tag{4.13}
\]

Inserting (4.13) into (4.7), by using the Cauchy-Schwartz inequality with respect to $\xi, \mu, \tau$, we have

\[
\text{Int}_2 \leq C \int_{\mathbb{R}^3} \frac{|\xi|}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon}} \left( \int_{\mathbb{R}^3} \langle \sigma_1 \rangle^{-(1+2\epsilon)} \langle \sigma_2 \rangle^{-(1+2\epsilon)} d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_{\mathbb{R}^3} 2 \prod_{j=1}^2 |F_j(\xi_j, \mu_j, \tau_j)|^2 d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} F(\xi, \mu, \tau) d\xi d\mu d\tau
\]

\[
\leq C \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} 2 \prod_{j=1}^2 |F_j(\xi_j, \mu_j, \tau_j)|^2 d\xi_1 d\mu_1 d\tau_1 \right)^{\frac{1}{2}} F(\xi, \mu, \tau) d\xi d\mu d\tau
\]

\[
\leq C \|F\|_{L^2_{\xi,\mu}} \prod_{j=1}^2 \|F_j\|_{L^2_{\xi,\mu}}. \tag{4.14}
\]

(III) Region $\Omega_3$. In this region, we have $|\xi| \sim |\xi_1|$. In this region, we consider

\[
|\sigma - \sigma_1 - \sigma_2| = \left| \xi_1 \xi_2 (5\xi_2^2 - 5\xi_1 + 5\xi_1^2) - \frac{\xi_1 \xi_2}{\xi} \left| \frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right|^2 \right|
\]

\[
\geq \frac{|\xi_1 \xi_2 (5\xi_2^2 - 5\xi_1 + 5\xi_1^2)|}{2^{70}} \tag{4.15}
\]
Thus, combining (2.17) with (4.20), we have

\[ |\sigma - \sigma_1 - \sigma_2| = \left| \xi_1 \xi_2 (5\xi^2 - 5\xi_1 + 5\xi_2^2) - \frac{\xi_1 \xi_2}{\xi} \left( \frac{\mu_1}{\xi} - \frac{\mu_2}{\xi_2} \right)^2 \right| < \frac{|\xi_1 \xi_2 (5\xi^2 - 5\xi_1 + 5\xi_2^2)|}{2^{70}}, \tag{4.16} \]

respectively.

When (4.15) is valid, we have one of the following three cases must occur:

\[ |\sigma| := \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C \left| \xi_1 \xi_2 (5\xi^2 - 5\xi_1 + 5\xi_2^2) \right|, \tag{4.17} \]

\[ |\sigma_1| := \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C \left| \xi_1 \xi_2 (5\xi^2 - 5\xi_1 + 5\xi_2^2) \right|, \tag{4.18} \]

\[ |\sigma_2| := \max \{ |\sigma|, |\sigma_1|, |\sigma_2| \} \geq C \left| \xi_1 \xi_2 (5\xi^2 - 5\xi_1 + 5\xi_2^2) \right|. \tag{4.19} \]

When (4.17) is valid, since \(-\frac{q}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{1+8\epsilon} \xi_2 |^{-s_1 - \frac{1}{2} + 2\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{1+8\epsilon} \xi_2 |^{-s_1 - \frac{1}{2} + 2\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \leq \frac{|\xi|^{\frac{1}{2}} |\xi_2|^{-\frac{1}{2}}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}}. \tag{4.20} \]

Thus, combining (2.17) with (4.20), we have

\[ |\text{Int}_3| \leq C \| F \|_{L^2_{\xi_\mu}} \prod_{j=1}^{2} \| F_j \|_{L^2_{\xi_\mu}}. \]

When (4.18) is valid, since \(-\frac{q}{8} + 16\epsilon \leq s_1 < 0\) and

\[ \langle \sigma \rangle^{\frac{1}{2} + 2\epsilon} \langle \sigma_1 \rangle^{\frac{1}{2} - \epsilon} \leq \langle \sigma \rangle^{\frac{1}{2} + 2\epsilon}, \]

we have

\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{1+8\epsilon} \xi_2 |^{-s_1 - \frac{1}{2} + 2\epsilon}}{\langle \sigma \rangle^{1+2\epsilon} \langle \sigma_2 \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{\frac{1}{2}} |\xi_2|^{-\frac{1}{2}}}{\langle \sigma \rangle^{\frac{1}{2} + \epsilon} \langle \sigma_2 \rangle^{\frac{1}{2} + \epsilon}}. \tag{4.21} \]

Thus, combining (2.19) with (4.21), we have

\[ |\text{Int}_3| \leq C \| F \|_{L^2_{\xi_\mu}} \prod_{j=1}^{2} \| F_j \|_{L^2_{\xi_\mu}}. \]
Thus, combining (2.20) with (4.22), we have

\[ \langle \sigma \rangle^{-\frac{1}{2} + 2 \epsilon} \langle \sigma_2 \rangle^{-\frac{1}{2} - \epsilon} \leq \langle \sigma \rangle^{-\frac{1}{2} - \epsilon} \langle \sigma_2 \rangle^{-\frac{1}{2} + 2 \epsilon}, \]

we have

\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{\langle \xi \rangle^2}{\langle \sigma \rangle^{\frac{1}{2} - 2 \epsilon} \prod_{j=1}^{2} \langle \xi_j \rangle^2 \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{1 + 8 \epsilon} |\xi_1|^{-s_1 - \frac{1}{2} + 2 \epsilon}}{|\sigma_1|^{\frac{1}{2} + \epsilon} |\sigma|^{\frac{1}{2} + \epsilon}}
\]

\[
\leq C \frac{|\xi_1|^{\frac{1}{2} - 14 \epsilon} |\xi|^{-1 + 8 \epsilon}}{|\sigma_1|^{\frac{1}{2} + \epsilon} |\sigma|^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{-\frac{1}{2}} |\xi|}{|\sigma|^{\frac{1}{2} + \epsilon} |\sigma|^{\frac{1}{2} + \epsilon}}.
\]

(4.22)

Thus, combining (2.20) with (4.22), we have

\[ |\text{Int}_3| \leq C \|F\|_{L^2_{2\xi_\mu}} \prod_{j=1}^{2} \|F_j\|_{L^2_{2\xi_\mu}}. \]

When (4.16) is valid, we consider the following two cases respectively.

\[
\max \{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{|\xi_1 \xi_2 (5 \xi_2^2 - 5 \xi_1 \xi + 5 \xi_1^2)|}{2^{80}}, \quad (4.23)
\]

\[
\max \{|\sigma|, |\sigma_1|, |\sigma_2|\} < \frac{|\xi_1 \xi_2 (5 \xi_2^2 - 5 \xi_1 \xi + 5 \xi_1^2)|}{2^{80}}, \quad (4.24)
\]

We dyadically decompose the spectra as

\[ \langle \sigma \rangle \sim 2^j, \quad \langle \sigma_1 \rangle \sim 2^{j_1}, \quad \langle \sigma_2 \rangle \sim 2^{j_2}, \quad |\xi| \sim 2^k, \quad |\xi_1| \sim 2^{k_1}, \quad |\xi_2| \sim 2^{k_2}. \]

We define

\[ f_{k,m,j_m} := |\eta_k(\xi_m) \eta_{j_m}(\sigma_m) F(\xi_m, \mu_m, \tau_m)|, \quad (m = 1, 2), \]

\[ f_{k,j} := |\eta_k(\xi) \eta_j(\sigma) F(\xi, \mu, \tau)|. \]

When (4.23) is valid, by using Lemma 3.2, since \(-\frac{a}{8} + 16 \epsilon \leq s_1 < 0\), we have

\[
\text{Int}_3 \leq C \sum_{k,k_1,k_2 > 0} \sum_{j_1,j_2,j \geq 0} 2^{-j(\frac{1}{2} - 2 \epsilon) - (j_1 + j_2)(\frac{1}{2} + \epsilon) - k_2 s_1 + k} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV \]

\[
\leq C \sum_{k,k_1,k_2 > 0} \sum_{j_1,j_2,j \geq 0} 2^{2j_1 - (j_1 + j_2) \epsilon - \max \frac{k}{2} - k_2 (s_1 + \frac{1}{2}) + \frac{16}{\epsilon}} \|f_{k,j}\|_{L^2} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2}; \quad (4.25)
\]
when \( j = j_{\text{max}} \), from (4.25), since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_3 \leq C \sum_{k,k_1,k_2>0} \sum_{j_1,j_2,j>0} 2^{-j(\frac{7}{2} - 2\epsilon) - (j_1 + j_2)\epsilon - k_2(s_1 + \frac{1}{4}) + \frac{3k}{4}} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV 
\leq C \sum_{k,k_1,k_2>0} 2^{-k_2(s_1 + \frac{3}{4} - 8\epsilon) - k(\frac{7}{4} - 8\epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2} 
\leq C \sum_{k,k_1,k_2>0} 2^{-k_2(s_1 + \frac{3}{4} - 10\epsilon) - k(\frac{7}{4} - 5\epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2} 
\leq C \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2};
\] (4.26)

when \( j_1 = j_{\text{max}} \), from (4.25), since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_3 \leq C \sum_{k,k_1,k_2>0} \sum_{j_1,j_2,j>0} 2^{2j\epsilon - (j_1 + j_2)\epsilon - \frac{4m\epsilon}{s_1} - k_2(s_1 + \frac{1}{4}) + \frac{3k}{4}} \|f_{k,j}\|_{L^2} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2} 
\leq C \sum_{k,k_1,k_2>0} 2^{-k_2(s_1 + \frac{3}{4} - \epsilon) - k(\frac{7}{4} - \epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2} 
\leq C \sum_{k,k_1,k_2>0} 2^{-k_2(s_1 + \frac{3}{4} - 5\epsilon) - k(\frac{7}{4} - 5\epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2} 
\leq C \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2};
\] (4.27)

when \( j_2 = j_{\text{max}} \), this case can be proved similarly to case \( j_1 = j_{\text{max}} \).

When (4.24) is valid, by using (2) of Lemma 3.1, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_3 \leq C \sum_{k,k_1,k_2>0} \sum_{j_1,j_2,j>0} 2^{-j(\frac{7}{2} - 2\epsilon) - (j_1 + j_2)\epsilon - k_2s_1 + k} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV 
\leq C \sum_{k,k_1,k_2>0} \sum_{j_1,j_2,j>0} 2^{2j\epsilon - (j_1 + j_2)\epsilon - k_2(s_1 + \frac{1}{2}) - \frac{3k}{2}} \|f_{k,j}\|_{L^2} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2} 
\leq C \sum_{k,k_1,k_2>0} 2^{-k_2(s_1 + \frac{3}{2}) - \frac{3k}{2}} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2} 
\leq C \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}.
\] (4.28)

(IV) Region \( \Omega_4 \). In this case, we consider (4.15), (4.16), respectively.

When (4.15) is valid, one of (4.17)-(4.19) must occur.
When (4.17) is valid, since \(-\frac{3}{8} + 16\epsilon \leq s_1 < 0\), we have
\[
\begin{align*}
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) &\leq \frac{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} 2 \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} 
&\leq C \frac{\langle \xi \rangle^{\frac{1}{2} + 2\epsilon} \langle \xi_2 \rangle^{\frac{1}{2} - 2\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} 
&\leq C \frac{\langle \xi \rangle^{\frac{1}{2} + 2\epsilon} \langle \xi_2 \rangle^{\frac{1}{2} - 2\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}}. \tag{4.29}
\end{align*}
\]
Thus, combining (2.17) with (4.29), we have
\[
|\text{Int}_4| \leq C \|F\|_{L^2_{\xi \mu}} \prod_{j=1}^{2} \|F_j\|_{L^2_{\xi \mu}}.
\]

When (4.18) is valid, since \(-\frac{3}{8} + 16\epsilon \leq s_1 < 0\) and
\[
\langle \sigma \rangle^{-\frac{1}{2} - 2\epsilon} \langle \sigma_1 \rangle^{-\frac{1}{2} - \epsilon} \leq \langle \sigma \rangle^{-\frac{1}{2} - \epsilon} \langle \sigma_1 \rangle^{-\frac{1}{2} - 2\epsilon},
\]
we have
\[
\begin{align*}
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) &\leq \frac{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} 2 \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} 
&\leq C \frac{\langle \xi \rangle^{\frac{1}{2} + 2\epsilon} \langle \xi_2 \rangle^{\frac{1}{2} - 2\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} 
&\leq C \frac{\langle \xi \rangle^{\frac{1}{2} + 2\epsilon} \langle \xi_2 \rangle^{\frac{1}{2} - 2\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}}. \tag{4.30}
\end{align*}
\]
Thus, combining (2.19) with (4.30), we have
\[
|\text{Int}_4| \leq C \|F\|_{L^2_{\xi \mu}} \prod_{j=1}^{2} \|F_j\|_{L^2_{\xi \mu}}.
\]

When (4.19) is valid, this case can be proved similarly to (4.18).

When (4.16) is valid, we consider (4.23) and (4.24), respectively.

We dyadically decompose the spectra as
\[
\langle \sigma \rangle \sim 2^j, \quad \langle \sigma_1 \rangle \sim 2^{j_1}, \quad \langle \sigma_2 \rangle \sim 2^{j_2}, \quad |\xi| \sim 2^k, \quad |\xi_1| \sim 2^{k_1}, \quad |\xi_2| \sim 2^{k_2}.
\]

We define
\[
\begin{align*}
f_{km,jm} := &\ |\eta_{km}(\xi_m)\eta_{jm}(\sigma_m)F_l(\xi_m, \mu_m, \tau_m)|, \quad (m = 1, 2), \\
f_{k,j} := &\ |\eta_k(\xi)\eta_j(\sigma)F(\xi, \mu, \tau)|, \quad dV = d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau.
\end{align*}
\]
When (4.23) is valid, by Lemma 3.2, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_4 \leq C \sum_{k_1, k_2 > 0} \sum_{k, j_1, j_2, j \geq 0} 2^{-j(\frac{1}{2} - 2\epsilon) - (j_1 + j_2)(\frac{1}{4} + \epsilon) - 2k_2s_1 + k} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{km,jm} dV
\]

\[
\leq C \sum_{k_1, k_2 > 0} \sum_{k, j_1, j_2, j \geq 0} 2^{2j\epsilon - (j_1 + j_2)\epsilon - k_2(2s_1 + \frac{1}{4}) + \frac{3k}{2}} \text{\|} f_{k,j} \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_{km,jm} \text{\|}_{L^2}; \quad (4.31)
\]

when \(j = j_{\text{max}}\), from (4.31), since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_4 \leq C \sum_{k_1, k_2 > 0} \sum_{k, j_1, j_2, j \geq 0} 2^{-j(\frac{1}{2} - 2\epsilon) - (j_1 + j_2)(\frac{1}{4} + \epsilon) - 2k_2s_1 + k} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{km,jm} dV
\]

\[
\leq C \sum_{k_1, k_2 > 0} 2^{-k_2(2s_1 + \frac{7}{4} - 8\epsilon) + k(\frac{1}{4} + \epsilon)} \text{\|} f \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_m \text{\|}_{L^2}
\]

\[
\leq C \text{\|} f \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_m \text{\|}_{L^2}; \quad (4.32)
\]

when \(j_1 = j_{\text{max}}\), from (4.31), since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_4 \leq C \sum_{k_1, k_2 > 0} \sum_{k, j_1, j_2, j \geq 0} 2^{2j\epsilon - (j_1 + j_2)\epsilon - k_2(2s_1 + \frac{1}{4}) + \frac{3k}{2}} \text{\|} f_{k,j} \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_{km,jm} \text{\|}_{L^2}
\]

\[
\leq C \sum_{k_1, k_2 > 0} 2^{-k_2(2s_1 + \frac{7}{4} - 8\epsilon) + k(\frac{1}{4} + \epsilon)} \text{\|} f \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_m \text{\|}_{L^2}
\]

\[
\leq C \text{\|} f \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_m \text{\|}_{L^2}; \quad (4.33)
\]

when \(j_2 = j_{\text{max}}\), this case can be proved similarly to case \(j_1 = j_{\text{max}}\).

When (4.24) is valid, by using (1) of Lemma 3.1, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_4 \leq C \sum_{k_1, k_2 > 0} \sum_{k, j_1, j_2, j \geq 0} 2^{-j(\frac{1}{2} - 2\epsilon) - (j_1 + j_2)(\frac{1}{4} + \epsilon) - 2k_2s_1 + k} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{km,jm} dV
\]

\[
\leq C \sum_{k_1, k_2 > 0} \sum_{k, j_1, j_2, j \geq 0} 2^{2j\epsilon - (j_1 + j_2)\epsilon - k_2(2s_1 + \frac{7}{2}) + \frac{3k}{2}} \text{\|} f_{k,j} \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_{km,jm} \text{\|}_{L^2}
\]

\[
\leq C \sum_{k_1, k_2 > 0} 2^{-k_2(2s_1 + \frac{7}{2} - 8\epsilon) + k(\frac{3}{2} + \epsilon)} \text{\|} f \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_m \text{\|}_{L^2}
\]

\[
\leq C \text{\|} f \text{\|}_{L^2} \prod_{m=1}^{2} \text{\|} f_m \text{\|}_{L^2}; \quad (4.34)
\]

(V) Region \(\Omega_5\). In this region, we consider (4.15), (4.16), respectively.
When (4.15) is valid, one of (4.17)-(4.19) must occur.

When (4.17) is valid, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^2 \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{\frac{5}{8} + 14\epsilon} |\xi_j|^{\frac{1}{2} - 24\epsilon}}{\langle \sigma \rangle^{\frac{1}{2} + \epsilon} \langle \sigma_j \rangle^{\frac{1}{2} - 24\epsilon}} \leq C \frac{|\xi|^{\frac{7}{8} - \frac{3}{2} |\xi_j|}}{\langle \sigma \rangle^{\frac{1}{2} + \epsilon} \langle \sigma_j \rangle^{\frac{1}{2} - 24\epsilon}}.
\] (4.35)

Thus, combining (2.17) with (4.35), we have
\[
|\text{Int}_5| \leq C \|F\|_{L^2_{\xi_1,\mu}} \prod_{j=1}^2 \|F_j\|_{L^2_{\xi_j,\mu}}.
\]

When (4.18) is valid, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\) and
\[
\langle \sigma \rangle^{-\frac{1}{2} + 2\epsilon} \langle \sigma_1 \rangle^{-\frac{1}{2} - \epsilon} \leq \langle \sigma \rangle^{-\frac{1}{2} - \epsilon} \langle \sigma_1 \rangle^{-\frac{1}{2} + 2\epsilon},
\]
we have
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq \frac{|\xi| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^2 \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{|\xi|^{\frac{5}{8} + 14\epsilon} |\xi_j|^{\frac{1}{2} - 24\epsilon}}{\langle \sigma \rangle^{\frac{1}{2} + \epsilon} \langle \sigma_j \rangle^{\frac{1}{2} - 24\epsilon}} \leq C \frac{|\xi|^{\frac{7}{8} - \frac{3}{2} |\xi_j|}}{\langle \sigma \rangle^{\frac{1}{2} + \epsilon} \langle \sigma_j \rangle^{\frac{1}{2} - 24\epsilon}}.
\] (4.36)

Thus, combining (2.19) with (4.36), we have
\[
|\text{Int}_5| \leq C \|F\|_{L^2_{\xi_1,\mu}} \prod_{j=1}^2 \|F_j\|_{L^2_{\xi_j,\mu}}.
\]

When (4.19) is valid, this case can be proved similarly to (4.18) with the aid of (2.20).

When (4.16) is valid, consider (4.23), (4.24), respectively.

We dyadically decompose the spectra as
\[
\langle \sigma \rangle \sim 2^j, \quad \langle \sigma_1 \rangle \sim 2^{j_1}, \quad \langle \sigma_2 \rangle \sim 2^{j_2}, \quad |\xi| \sim 2^k, \quad |\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}.
\]

We define
\[
f_{k_m,j_m} := |\eta_{k_m}(\xi_m)\eta_{j_m}(\sigma_m)F_1(\xi_m, \mu_m, \tau_m)| (m = 1, 2),
\]
\[
f_{k,j} := |\eta_{k}(\xi)\eta_{j}(\sigma)F(\xi, \mu, \tau)|, \quad dV = d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau.
\]
When (4.23) is valid, we use Lemma 3.2 to deal with this case. Thus, by Lemma 3.2, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

$$\text{Int}_5 \leq C \sum_{k_1, k_2 > 0, k_1, j_2 \geq 0} \sum_{2} 2^{-j\left(\frac{3}{4} - 2\epsilon\right) - (j_1 + j_2)(\frac{3}{4} + \epsilon) - 2k_2s_1 + k(1 + s_1)} \left(\int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} \, dV\right) \leq C \sum_{k_1, k_2 > 0, k_1, j_2 \geq 0} 2^{2^j e - (j_1 + j_2)\epsilon - \max_k - k_2(2s_1 + \frac{3}{4}) + k\left(\frac{3}{4} + s_1\right)} \left\| f_{k,j} \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{k_m,j_m} \right\|_{L^2}. \quad (4.37)$$

When \(j = j_{\text{max}}\), from (4.37), if \(\frac{9}{8} + 16\epsilon \leq s_1 < -\frac{3}{4}\), we have

$$\text{Int}_5 \leq C \sum_{k_1, k_2 > 0, k_1, j_2 \geq 0} \sum_{2} 2^{2^j e - (j_1 + j_2)\epsilon - \max_k - k_2(2s_1 + \frac{3}{4}) + k\left(\frac{3}{4} + s_1\right)} \left\| f_{k,j} \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{k_m,j_m} \right\|_{L^2} \leq C \sum_{k_1, k_2 > 0, k} 2^{-k_2(2s_1 + \frac{9}{8} - 8\epsilon) + k\left(\frac{3}{4} + 2\epsilon\right)} \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_m \right\|_{L^2} \leq C \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_m \right\|_{L^2}. \quad (4.38)$$

When \(j = j_{\text{max}}\), from (4.37), if \(-\frac{3}{4} \leq s_1 < 0\), we have

$$\text{Int}_5 \leq C \sum_{k_1, k_2 > 0, k_1, j_2 \geq 0} \sum_{2} 2^{2^j e - (j_1 + j_2)\epsilon - \max_k - k_2(2s_1 + \frac{3}{4}) + k\left(\frac{3}{4} + s_1\right)} \left\| f_{k,j} \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{k_m,j_m} \right\|_{L^2} \leq C \sum_{k_1, k_2 > 0} 2^{-k_2(2s_1 + \frac{9}{8} - 8\epsilon)} \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_m \right\|_{L^2} \leq C \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_m \right\|_{L^2}. \quad (4.39)$$

When \(j_1 = j_{\text{max}}\), from (4.38), if \(-\frac{9}{8} + 16\epsilon \leq s_1 < -\frac{3}{4}\), we have

$$\text{Int}_5 \leq C \sum_{k_1, k_2 > 0, k_1, j_2 \geq 0} \sum_{2} 2^{2^j e - (j_1 + j_2)\epsilon - \max_k - k_2(2s_1 + \frac{3}{4}) + k\left(\frac{3}{4} + s_1\right)} \left\| f_{k,j} \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{k_m,j_m} \right\|_{L^2} \leq C \sum_{k_1, k_2 > 0, k} 2^{-k_2(2s_1 + \frac{9}{8} - 4\epsilon)} \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_m \right\|_{L^2} \leq C \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_m \right\|_{L^2}. \quad (4.40)$$
When \( j_2 = j_{\text{max}} \), from (4.38), if \(-\frac{3}{4} \leq s_1 < 0\), we have

\[
\text{Int}_5 \leq C \sum_{k_1, k_2 > 0} \sum_{j_1, j_2, j_3 \geq 0} 2^{2j_\epsilon - (j_1 + j_2)\epsilon - \frac{3}{2}k_1 + k_2 s_1} \left\| f_{k,j} \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{k_m,j_m} \right\|_{L^2}

\leq C \sum_{k_1, k_2 > 0} 2^{-k_2 (s_1 + \frac{3}{4} - 4\epsilon)} \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{m} \right\|_{L^2}

\leq C \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{m} \right\|_{L^2}.
\] (4.41)

When \( j_2 = j_{\text{max}} \), this case can be proved similarly to case \( j_1 = j_{\text{max}} \).

When (4.24) is valid, by using (1) of Lemma 3.1, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
\text{Int}_5 \leq C \sum_{k, k_1, k_2 > 0} \sum_{j_1, j_2, j_3 \geq 0} 2^{-j(\frac{3}{4} - 2\epsilon - (j_1 + j_2)\epsilon - 2k s_1) + k_1 (1 + s_1)} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV

\leq C \sum_{k, k_1, k_2 > 0} \sum_{j_1, j_2, j_3 \geq 0} 2^{-j(\frac{3}{4} - 2\epsilon - (j_1 + j_2)\epsilon - 2k s_1) + k_1 (1 + s_1)} \left\| f_{k,j} \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{k_m,j_m} \right\|_{L^2}

\leq C \sum_{k, k_1, k_2 > 0} 2^{-k_2 (s_1 + \frac{3}{8} - 4\epsilon) + k_1 (1 + s_1)} \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{m} \right\|_{L^2}

\leq C \sum_{k, k_1, k_2 > 0} 2^{-k_2 (s_1 + 2 - 10\epsilon)} \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{m} \right\|_{L^2}

\leq C \left\| f \right\|_{L^2} \prod_{m=1}^{2} \left\| f_{m} \right\|_{L^2}.
\] (4.42)

(VI) Region \( \Omega_6 \). In this region, we consider (4.15), (4.16), respectively.

When (4.15) is valid, one of (4.17)-(4.19) must occur.

When (4.17) is valid, since \(-\frac{9}{8} + 16\epsilon \leq s_1 < 0\), we have

\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{\left| \xi \right| \langle \xi \rangle^{s_1}}{\langle \sigma \rangle^{\frac{3}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \xi_j \rangle^{s_1} \langle \sigma_j \rangle^{\frac{3}{2} + \epsilon}} \leq C \frac{\left| \xi_2 \right|^{\frac{3}{8} - s_1 + 10\epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{3}{2} + \epsilon}} \leq C \frac{\left| \xi_1 \right| \langle \xi \rangle^{s_1} \langle \xi \rangle^{\frac{3}{2} + \epsilon}}{\prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{3}{2} + \epsilon}}.
\] (4.43)

Thus, combining (2.17) with (4.43), we have

\[
|\text{Int}_6| \leq C \left\| F \right\|_{L^2_{\xi_1\mu}} \prod_{j=1}^{2} \left\| F_j \right\|_{L^2_{\xi_2\mu}}.
\]
When (4.18) is valid, since $-\frac{9}{8} + 16\epsilon \leq s_1 < 0$ and
\[
\langle \sigma \rangle^{-\frac{1}{2} + 2\epsilon} \langle \sigma_1 \rangle^{-\frac{1}{2} - \epsilon} \leq \langle \sigma \rangle^{-\frac{1}{2} - \epsilon} \langle \sigma_1 \rangle^{-\frac{1}{2} + 2\epsilon},
\]
we have
\[
K_1(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{\langle \sigma \rangle^{\frac{3}{2} - s_1 + 10\epsilon}}{\langle \sigma \rangle^{\frac{1}{2} + \epsilon} \langle \sigma_2 \rangle^{\frac{1}{2} + \epsilon}} \leq C \frac{\langle \xi \rangle^s_1}{\langle \sigma \rangle^{\frac{1}{2} + \epsilon} \langle \sigma_2 \rangle^{\frac{1}{2} + \epsilon}}.
\]

Thus, combining (2.19) with (4.44), we have
\[
|\text{Int}_6| \leq C \|F\|_{L^2_{r,\xi,\mu}} \prod_{j=1}^2 \|F_j\|_{L^2_{r,\xi,\mu}}.
\]

When (4.19) is valid, this case can be proved similarly to (4.18) with the aid of (4.20).

When (4.16) is valid, this case can be proved similarly to case (4.16) of Region $\Omega_5$.

(VII) Region $\Omega_7$. This case can be proved similarly to Region $\Omega_6$.

This completes the proof of Lemma 4.1.

Remark 3. In the case (4.23) of Region $\Omega_4$ it leads to the requirement $-\frac{9}{8} < s_1 < 0$.

Lemma 4.2. Let $-1 + 10\epsilon \leq s < 0$. Then, we have
\[
\|\partial_x [I_N(u_1 u_2) - I_N u_1 I_N u_2]\|_{X^{0,0}_{\frac{1}{2} + 2\epsilon}} \leq C N^{-2 + 10\epsilon} \prod_{j=1}^2 \|I_N u_j\|_{X^{0,0}_{\frac{1}{2} + \epsilon}}.
\]

Proof. To prove (4.45), by duality, it suffices to prove that
\[
\left| \int_{\mathbb{R}^3} \bar{h} \partial_x [I_N(u_1 u_2) - I_N u_1 I_N u_2] \, dxdydt \right| \leq C N^{-2 + 10\epsilon} \|h\|_{X^{0,0}_{\frac{1}{2} - 2\epsilon}} \prod_{j=1}^2 \|I_N u_j\|_{X^{0,0}_{\frac{1}{2} + \epsilon}}.
\]

for $h \in X^{0,0}_{\frac{1}{2} - 2\epsilon}$. Let
\[
F(\xi, \mu, \tau) = \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} M(\xi) \Phi h(\xi, \mu, \tau),
\]
\[
F_j(\xi_j, \mu_j, \tau_j) = M(\xi_j) \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon} \Phi u_j(\xi_j, \mu, \tau_j) \quad (j = 1, 2).
\]
To obtain (4.46), from (4.47), it suffices to prove
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G(\xi_1, \xi_2) F(\xi_1, \xi_2, \mu_1, \mu_2) \prod_{j=1}^2 F_j(\xi_j_1, \mu_j, \tau_j) \frac{|\xi|}{2} \frac{d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau}{(\sigma_j)^{\frac{1}{2}-2\epsilon} \prod_{j=1}^2 (\sigma_j)^{\frac{1}{2}+\epsilon}} \leq C N^{-1+10\epsilon} \|F\|_{L^2_{\xi\mu\tau}} \prod_{j=1}^2 \|F_j\|_{L^2_{\xi\mu\tau}},
\]
(4.48)

where
\[
G(\xi_1, \xi_2) = \frac{M(\xi_1)M(\xi_2) - M(\xi)}{M(\xi_1)M(\xi_2)}.
\]

Without loss of generality, we assume that \(F(\xi, \mu, \tau) \geq 0, F_j(\xi_j, \mu_j, \tau_j) \geq 0\) \((j = 1, 2)\). By symmetry, we can assume that \(|\xi_1| \geq |\xi_2|\).

We define
\[
A_1 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_2| \leq |\xi_1| \leq \frac{N}{2} \right\},
\]
\[
A_2 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, |\xi_2| \leq 2A \right\},
\]
\[
A_3 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, 2A < |\xi_2| \leq N \right\},
\]
\[
A_4 = \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in D^*, |\xi_1| > \frac{N}{2}, |\xi_1| \geq |\xi_2|, |\xi_2| > N \right\}.
\]

Here \(D^*\) is defined as in Lemma 3.1. Obviously, \(D^* \subset \bigcup_{j=1}^4 A_j\). We define
\[
K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) := \frac{|\xi| G(\xi_1, \xi_2)}{\langle \sigma_j \rangle^{\frac{1}{2}-2\epsilon} \prod_{j=1}^2 \langle \sigma_j \rangle^{\frac{1}{2}+\epsilon}}
\]
(4.49)

and
\[
J_k := \int_{A_j} K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) F(\xi_1, \mu_1, \tau_1) \prod_{j=1}^2 F_j(\xi_j_1, \mu_j, \tau_j) d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau,
\]

\(1 \leq k \leq 4, k \in \mathbb{N}\).

We consider (4.15) and (4.16), respectively.

When (4.15) is valid, one of (4.17)-(4.19) must occur, from [20, Lemma 1.4], we have
\[
\sum_{k=1}^4 J_k \leq C N^{-2(2-10\epsilon)} \|F\|_{L^2_{\xi\mu\tau}} \prod_{j=1}^2 \|F_j\|_{L^2_{\xi\mu\tau}},
\]

39
Thus, we only consider the case (4.16).

Now we consider the integrals over the above four regions one by one.

(I) Region \(A_1\). In this case, since \(M(\xi_1, \xi_2) = 0\), thus we have \(J_1 = 0\).

(II) Region \(A_2\). From [20, Page 902], we have

\[
G(\xi_1, \xi_2) \leq C \frac{|\xi_2|}{|\xi_1|}. 
\] (4.50)

Inserting (4.46) into (4.47) yields

\[
K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{|\xi| G(\xi_1, \xi_2)}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{4}{3} + \epsilon}} \leq \frac{C \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{4}{3} + \epsilon}}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j=1}^{2} \langle \sigma_j \rangle^{\frac{4}{3} + \epsilon}}. 
\] (4.51)

We dyadically decompose the spectra as

\[
\langle \sigma \rangle \sim 2^j, \quad \langle \sigma_1 \rangle \sim 2^{i_1}, \quad \langle \sigma_2 \rangle \sim 2^{i_2}, \quad |\xi| \sim 2^k, \quad |\xi_1| \sim 2^{k_1}, \quad |\xi_2| \sim 2^{k_2}.
\]

We define

\[
f_{k_m,j_m} := \eta_{k_m}(\xi_m) \eta_{j_m}(\sigma_m) F_j(\xi_m, \mu_m, \tau_m) (m = 1, 2),
\]

\[
f_{k,j} := \eta_k(\xi) \eta_j(\sigma) |F(\xi, \mu, \tau)|. 
\]

Thus, by using (2.4), we have

\[
J_2 \leq C \sum_{k_1, k_2 \geq 0} \sum_{j_1, j_2 \geq 0} 2^{-j(\frac{5}{2} - 2\epsilon) - (j_1 + j_2)(\frac{4}{3} + \epsilon) + k_2} \int_{\mathbb{R}^d} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV. 
\] (4.52)

In this case, we consider (4.23), (4.24), respectively.

When (4.23) is valid, we consider \(j = j_{\text{max}}, j_1 = j_{\text{max}}, j_2 = j_{\text{max}}\), respectively.

When \(j = j_{\text{max}}\) is valid, from (4.52), we have

\[
J_2 \leq C \sum_{k_1, k_2 \geq 0} \sum_{j_1, j_2} 2^{-k(\frac{5}{2} - 8\epsilon) - (j_1 + j_2)\epsilon + k_2(\frac{4}{3} + 2\epsilon)} \|f\|_{L^2} \prod_{j=1}^{2} \|f_j\|_{L^2}
\]

\[
\leq CN^{-(\frac{5}{2} - 8\epsilon)} \|f\|_{L^2} \prod_{j=1}^{2} \|f_j\|_{L^2}. 
\] (4.53)
When \( j_1 = j_{\max} \) is valid, from (4.52), we have

\[
J_2 \leq C \sum_{k_1, k_2 > 0, \ k_1, j_1, j_2, j_3 \geq 0} 2^{2j_3 - 2(j_1 + j_2)\epsilon} \sum_{k_1, k_2 > 0, \ k_1, j_1, j_2, j_3 \geq 0} \frac{2^{j_3} - j_3}{k_3} \frac{2^{j_3} + j_3}{k_3} \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}
\]

\[
\leq C \sum_{k_1, k_2 > 0, \ k_1, j_1, j_2 \geq 0} 2^{2j_3 - 2(j_1 + j_2 - j_3)\epsilon} \sum_{k_1, k_2 > 0, \ k_1, j_1, j_2 \geq 0} \frac{2^{j_3} - j_3}{k_3} \frac{2^{j_3} + j_3}{k_3} \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}
\]

\[
\leq C \sum_{k_1, k_2 > 0, \ k} 2^{( \frac{2}{3} - 4\epsilon)} k_1 + k_2 (\frac{1}{3} + \epsilon) \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}
\]

\[
\leq C N^{-\left( \frac{2}{3} - 4\epsilon \right)} \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}.
\] (4.54)

When \( j_2 = j_{\max} \) is valid, this case can be proved similarly to \( j_1 = j_{\max} \) of Region \( A_2 \).

When (4.24) is valid, by using (2) of Lemma 3.1, we have

\[
J_2 \leq C \sum_{k_1, k_2 > 0, \ k_1, j_1, j_2, j_3 \geq 0} 2^{2j_3 - 2(j_1 + j_2)\epsilon} \sum_{k_1, k_2 > 0, \ k_1, j_1, j_2, j_3 \geq 0} \frac{2^{j_3} - j_3}{k_3} \frac{2^{j_3} + j_3}{k_3} \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}
\]

\[
\leq C \sum_{k_1, k_2 > 0, \ k_1, j_1, j_2 \geq 0} 2^{-(2 - 8\epsilon)k_1 + 3k_2(\frac{1}{3} + \epsilon)} \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}
\]

\[
\leq C \sum_{k_1, k_2 > 0, \ k} 2^{( \frac{1}{2} - 8\epsilon)k_1 + k_2 (\frac{1}{2} + \epsilon)} \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}
\]

\[
\leq C N^{-\left( \frac{1}{2} - 8\epsilon \right)} \| f \|_{L^2} \prod_{j_1}^2 \| f_j \|_{L^2}.
\] (4.55)

(III) Region \( A_3 \). From of [20, Page 902], we know that (4.50) is valid. Combining (4.50) with (4.49), we have

\[
K_2(\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \leq C \frac{\min \{ |\xi_1|, |\xi_1|, |\xi_2| \}}{\langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \prod_{j_1}^2 \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon}}.
\] (4.56)

We dyadically decompose the spectra as

\[
\langle \sigma \rangle \sim 2^j, \quad \langle \sigma_1 \rangle \sim 2^{j_1}, \quad \langle \sigma_2 \rangle \sim 2^{j_2}, \quad |\xi| \sim 2^k, |\xi_1| \sim 2^{k_1}, \quad |\xi_2| \sim 2^{k_2}.
\]

We define

\[
f_{km,jm} := \eta_{km}(\xi_m) \eta_{jm}(\sigma_k) F_m(\xi_m, \mu_m, \tau_m) \quad (m = 1, 2),
\]

\[
f_{k,j} := \eta_k(\xi) \eta_j(\sigma) |F(\xi, \mu, \tau)|.
\]
Thus, we have

\[
J_3 \leq C \sum_{m_1, m_2 > 0, m_1, j_2 \geq 0} 2^{-j_{1/2} - 2\epsilon} (j_1 + j_2) (1/2 + \epsilon) + \min\{k, k_1, k_2\} \\
\times \int_{\mathbb{R}^6} f_{m,j} \prod_{m=1}^{2} f_{k_m,j_m} dV.
\] (4.57)

In this case, we consider (4.23), (4.24), respectively.

When (4.23) is valid, by using Lemma 3.2, we have

\[
J_3 \leq C \sum_{k, k_1, k_2 > 0} 2^{-j_{1/2} - 2\epsilon} (j_1 + j_2) (1/2 + \epsilon) + \frac{k}{2} + \min\{k, k_1, k_2\} \\
\times \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV.
\] (4.58)

When \( j = j_{\text{max}} \), from (4.58), we have

\[
J_3 \leq C \sum_{k_1, k_2 > 0} 2^{-j_{1/2} - 2\epsilon} (j_1 + j_2) (1/2 + \epsilon) + \frac{k}{2} + \min\{k, k_1, k_2\} \|f_{k,j}\|_{L^2} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2} \\
\leq C N^{-2+10\epsilon} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}. \tag{4.59}
\]

When \( j_1 = j_{\text{max}} \), from (4.59), we have

\[
J_3 \leq C \sum_{k_1, k_2 > 0} 2^{j_2 - (1/2 - 2\epsilon)} (j_1 + j_2) (1/2 + \epsilon) + \frac{k}{2} + \min\{k, k_1, k_2\} \|f_{k,j}\|_{L^2} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2} \\
\leq C \sum_{k_1, k_2 > 0, k} 2^{-k_2 (1/4 - 4\epsilon) + \min\{k, k_1, k_2\} (1/2 + \epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2} \\
\leq C N^{-2+5\epsilon} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}. \tag{4.60}
\]

When \( j_2 = j_{\text{max}} \), this case can be proved similarly to the case \( j_1 = j_{\text{max}} \).
When (4.24) is valid, combining (2) of Lemma 3.1 with (4.60), we have

\[ J_3 \leq C \sum_{k,k_1,k_2>0,j_1,j_2,j_3 \geq 0} 2^{-j_1(\frac{3}{2} - 2\epsilon) - (j_1 + j_2)(\frac{3}{2} + \epsilon) + \min\{k,k_1,k_2\}} \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV \]

\[ \leq C \sum_{k,k_1,k_2>0,j_1,j_2,j_3 \geq 0} 2^{2j_1(2 \epsilon) - 2k_1} \frac{\|f_{k,j}\|_{L^2}}{} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2} \]

\[ \leq C \sum_{k,k_1,k_2>0} 2^{-k_1(2 - 10\epsilon)} \frac{\|f\|_{L^2}}{} \prod_{m=1}^{2} \|f_{m}\|_{L^2} \]

\[ \leq C N^{-2 + 11\epsilon} \frac{\|f\|_{L^2}}{} \prod_{m=1}^{2} \|f_{m}\|_{L^2}. \] (4.61)

(IV) Region $A_4$. In this case, we have

\[ M(\xi_1, \xi_2) \leq C N^{2\delta_1} (|\xi_1|, |\xi_2|)^{-\delta_1}. \] (4.62)

We dyadically decompose the spectra as

\[ \langle \sigma \rangle \sim 2^j, \quad \langle \sigma_1 \rangle \sim 2^{j_1}, \quad \langle \sigma_2 \rangle \sim 2^{j_2}, \quad |\xi| \sim 2^k, \quad |\xi_1| \sim 2^{k_1}, \quad |\xi_2| \sim 2^{k_2}. \]

We define

\[ f_{k_m,j_m} := \eta_{k_m} (\xi_m) \eta_{j_m} (\sigma_m) F_m (\xi_m, \mu_m, \tau_m), \quad (m = 1, 2), \]
\[ f_{k,j} := \eta_k (\xi) \eta_j (\sigma) |F(\xi, \mu, \tau)|. \]

Thus, we have

\[ J_4 \leq C N^{2s} \sum_{m_1,m_2>0,m_{j_1,j_2,j_3 \geq 0}} 2^{-j_1(\frac{1}{2} - 2\epsilon) - (j_1 + j_2)(\frac{1}{2} + \epsilon) + \min\{k,k_1,k_2\} + k} \]
\[ \times \int_{\mathbb{R}^6} f_{k,j} \prod_{m=1}^{2} f_{k_m,j_m} dV. \] (4.63)

In this case, we consider (4.23), (4.24), respectively.

When (4.23) is valid, by using Lemma 3.2, from (4.63), we have

\[ J_4 \leq C N^{2s} \sum_{m_1,m_2>0,m_{j_1,j_2,j_3 \geq 0}} 2^{2\epsilon(j_1 + j_2)\epsilon - \frac{\max}{2} + (m_1 + m_2)s - \frac{k_1 - k_2}{4} \min\{k,k_1,k_2\} + k} \]
\[ \times \|f\|_{L^2} \prod_{m=1}^{2} \|f_{m}\|_{L^2}. \] (4.64)
In this case, we consider \( k = k_{\text{min}}, k_2 = k_{\text{min}}, \) respectively.

When \( k = k_{\text{min}} \) is valid, we consider \( j = j_{\text{max}}, j_1 = j_{\text{max}}, j_2 = j_{\text{max}}, \) respectively.

When \( j = j_{\text{max}} \), from (4.64), since \( s_1 \geq -1 + 6\epsilon \), we have

\[
\begin{align*}
J_4 \leq C N^{2s_1} \sum_{k_1, k_2 > 0, k} \sum_{j_1, j_2, j \geq 0} 2^{-j(\frac{1}{2} - 2\epsilon) - (j_1 + j_2)\epsilon - k(2s_1 + \frac{1}{4}) + \frac{3}{4}k} \| f_{k,j} \|_{L^2} \prod_{m=1}^{2} \| f_{k_m, j_m} \|_{L^2} \\
\leq C \sum_{k_1, k_2 > 0, k} 2^{-k(2s_1 + \frac{1}{4} - 8\epsilon) + k(\frac{1}{4} + 2\epsilon)} \| f \|_{L^2} \prod_{m=1}^{2} \| f_m \|_{L^2} \\
\leq C \sum_{k_1 > 0} 2^{-k(2s_1 + 2 - 10\epsilon)} \| f \|_{L^2} \prod_{m=1}^{2} \| f_m \|_{L^2} \\
\leq C N^{-2 + 10\epsilon} \| f \|_{L^2} \prod_{m=1}^{2} \| f_m \|_{L^2}. \tag{4.65}
\end{align*}
\]

When \( j_1 = j_{\text{max}} \) is valid, from (4.64), since \( s_1 \geq -1 + 6\epsilon \), we have

\[
\begin{align*}
J_4 \leq C \sum_{k_1, k_2 > 0, k} \sum_{j_1, j_2 \geq 0} 2^{-\left(\frac{1}{2} - \epsilon\right)j_1 - j_2\epsilon - (k_1 + k_2)s_1 - k\frac{1}{4} + \frac{3}{4}k} \| f_{k,j} \|_{L^2} \prod_{m=1}^{2} \| f_{k_m, j_m} \|_{L^2} \\
\leq C \sum_{k_1, k_2 > 0, k} 2^{-k(2s_1 + \frac{1}{4} - 4\epsilon) + k(\frac{1}{4} + \epsilon)} \| f \|_{L^2} \prod_{m=1}^{2} \| f_m \|_{L^2} \\
\leq C \sum_{k_1 > 0} 2^{-k(2s_1 + 2 - 5\epsilon)} \| f \|_{L^2} \prod_{m=1}^{2} \| f_m \|_{L^2} \\
\leq C N^{-2 + 11\epsilon} \| f \|_{L^2} \prod_{m=1}^{2} \| f_m \|_{L^2}. \tag{4.66}
\end{align*}
\]

When \( j_2 = j_{\text{max}} \) is valid, this case can be proved similarly to \( j_1 = j_{\text{max}} \).

When \( k_2 = k_{\text{min}} \) is valid, we consider \( j = j_{\text{max}}, j_1 = j_{\text{max}}, j_2 = j_{\text{max}}, \) respectively.

\[
44
\]
When \( j = j_{\text{max}} \), from (4.64), since \( s_1 \geq -1 + 10\epsilon \), we have

\[
J_4 \leq CN^{2s_1} \sum_{k_1, k_2 > 0, k} \sum_{j_1, j_2 \geq 0} 2^{-(\frac{1}{2} - 2\epsilon)(j_1 + j_2)\epsilon - k_1(s_1 - \frac{1}{4}) - k_2(s_1 + \frac{1}{2})} \times \|f_{k,j}\|_{L^2} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2}
\]

\[
\leq CN^{2s_1} \sum_{k_1, k_2 > 0, k} 2^{-(\frac{1}{2} - 2\epsilon)(j)\epsilon - k_1(s_1 - \frac{1}{4}) - k_2(s_1 + \frac{1}{2})} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}
\]

\[
\leq CN^{2s_1} \sum_{k_1, k_2 > 0} 2^{-(\frac{1}{2} - 2\epsilon)(j)\epsilon - k_1(s_1 + \frac{1}{2} - 8\epsilon) - k_2(s_1 + \frac{1}{2} - 2\epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}
\]

\[
\leq CN^{2s_1} \sum_{k} 2^{-k_1(s_2 + \frac{1}{4} - 4\epsilon) - k_2(s_2 + \frac{1}{2} + \epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}
\]

\[
\leq CN^{-2+10\epsilon} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}. \tag{4.67}
\]

When \( j_1 = j_{\text{max}} \) is valid, from (4.64), since \( s_1 \geq -1 + 10\epsilon \), we have

\[
J_4 \leq CN^{2s_1} \sum_{k_1, k_2 > 0, k} \sum_{j_1, j_2 \geq 0} 2^{-(\frac{1}{2} - \epsilon)j_1 - (j_2)\epsilon - k_1(s_1 - \frac{1}{4}) - k_2(s_1 + \frac{1}{2})} \|f_{k,j}\|_{L^2} \prod_{m=1}^{2} \|f_{k_m,j_m}\|_{L^2}
\]

\[
\leq CN^{2s_1} \sum_{k_1, k_2 > 0, k} 2^{-(\frac{1}{2} - \epsilon)j_1 - (j_2)\epsilon - k_1(s_1 - \frac{1}{4}) - k_2(s_1 + \frac{1}{2})} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}
\]

\[
\leq CN^{2s_1} \sum_{k} 2^{-k_1(s_2 + \frac{1}{2} - 4\epsilon) - k_2(s_2 + \frac{1}{2} + \epsilon)} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}
\]

\[
\leq CN^{-2+5\epsilon} \|f\|_{L^2} \prod_{m=1}^{2} \|f_m\|_{L^2}. \tag{4.68}
\]

When \( j_2 = j_{\text{max}} \) is valid, this case can be proved similarly to \( j_1 = j_{\text{max}} \).

This completes the proof of Lemma 4.2.

**Lemma 4.3.** Let \( s \geq -\frac{9}{8} + 16\epsilon, s_2 \geq 0 \) and \( u_j \in X^{s_1, s_2}_{\frac{1}{2}+\epsilon} \). Then, we have

\[
\|\partial_x I(u_1 u_2)\|_{X_{\frac{1}{2}+\epsilon}^{0,-2\epsilon}} \leq C \prod_{j=1}^{2} \|I u_j\|_{X_{\frac{1}{2}+\epsilon}^{0,0}}. \tag{4.69}
\]

**Proof.** To prove (4.69), by duality, it suffices to prove that

\[
\left| \int_{\mathbb{R}^3} \bar{u} \partial_x I(u_1 u_2) dx dy dt \right| \leq C \|u\|_{X_{\frac{1}{2}+\epsilon}^{0,0}} \prod_{j=1}^{2} \|I u_j\|_{X_{\frac{1}{2}+\epsilon}^{0,0}}. \tag{4.70}
\]

for \( u \in X_{\frac{1}{2}+\epsilon}^{0,0} \). Let

\[
F(\xi, \mu, \tau) = \langle \sigma \rangle^{\frac{1}{2} - 2\epsilon} \mathcal{F} u(\xi, \mu, \tau),
\]

\[
F_j(\xi_j, \mu_j, \tau_j) = M(\xi_j) \langle \sigma_j \rangle^{\frac{1}{2} + \epsilon} \mathcal{F} u_j(\xi_j, \mu, \tau_j)(j = 1, 2), \tag{4.71}
\]
and
\[ D := \left\{ (\xi_1, \mu_1, \tau_1, \xi, \mu, \tau) \in \mathbb{R}^6, \xi = \sum_{j=1}^{2} \xi_j, \mu = \sum_{j=1}^{2} \mu_j, \tau = \sum_{j=1}^{2} \tau_j \right\}. \]

To obtain (4.70), from (4.71), it suffices to prove that
\[
\int_D |\xi| M(\xi) F(\xi, \mu, \tau) F_j(\xi_j, \mu_j, \tau_j) d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau
\leq C\|F\|_{L^2_b}^2 \prod_{j=1}^{2} \|F_j\|_{L^2_b}. \quad (4.72)
\]

From (2.4) of [26], we have
\[
\frac{M(\xi)}{\prod_{j=1}^{2} M(\xi_j)} \leq C \frac{\|\xi\|^s}{\prod_{j=1}^{2} (\xi_j)^s}. \quad (4.73)
\]

Inserting (4.73) into the left hand side of (4.72), we have
\[
\int_D \frac{|\xi|\|\xi\|^s F(\xi, \mu, \tau) F_j(\xi_j, \mu_j, \tau_j)}{(\sigma_j)^{1+2\epsilon}} \prod_{j=1}^{2} (\xi_j)^s (\sigma_j)^{1+2\epsilon} d\xi_1 d\mu_1 d\tau_1 d\xi d\mu d\tau. \quad (4.74)
\]

By using (4.4), we have (4.74) can be bounded by
\[
C\|F\|_{L^2_b}^2 \prod_{j=1}^{2} \|F_j\|_{L^2_b}. \quad (5.3)
\]

This completes the proof of Lemma 4.3.

5. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1.

We define
\[
\Phi_1(u) := \psi(t) W(t) u_0 + \frac{1}{2} \psi \left( \frac{t}{\tau} \right) \int_0^t W(t-\tau) \partial_x(u^2) d\tau, \quad (5.1)
\]
\[
B_1(0, 2C\|u_0\|_{H^{s+2}}) := \left\{ u : \|u\|_{X^{s+2}_{\delta+2}} \leq 2C\|u_0\|_{H^{s+2}} \right\}. \quad (5.2)
\]

Combining Lemmas 2.2, 4.1 with (5.1)-(5.2), we derive that
\[
\|\Phi_1(u)\|_{X^{s+2}_{\delta+2}} \leq \|\eta(t) W(t) u_0\|_{X^{s+2}_{\delta+2}} + \left\| \frac{1}{2} \eta \left( \frac{t}{\tau} \right) \int_0^t W(t-\tau) \partial_x(u^2) d\tau \right\|_{X^{s+2}_{\delta+2}}
\leq C\|u_0\|_{H^{s+2}} + CT^\epsilon \|\partial_x(u^2)\|_{X^{s+2}_{\delta+2}}
\leq C\|u_0\|_{H^{s+2}} + CT^\epsilon \|u\|_{X^{s+2}_{\delta+2}}^2
\leq C\|u\|_{H^{s+2}} + 4C^3 T^\epsilon \|u_0\|_{H^{s+2}}^2. \quad (5.3)
\]
We define
\[ T^\varepsilon := \left[ 16C^2(\|u_0\|_{H^{s_1,s_2}} + 1) \right]^{-1}. \] (5.4)

From (5.3)-(5.4), we have
\[ \|\Phi_1(u)\|_{X^{s_1,s_2}} \leq C\|u_0\|_{H^{s_1,s_2}} + C\|u_0\|_{H^{s_1,s_2}} = 2C\|u_0\|_{H^{s_1,s_2}}. \] (5.5)

Thus, \( \Phi_1 \) maps \( B_1(0, 2C\|u_0\|_{H^{s_1,s_2}}) \) into \( B_1(0, 2C\|u_0\|_{H^{s_1,s_2}}) \). Combining Lemmas 2.2, 4.1 with (5.4)-(5.5), we have
\[ \|\Phi_1(u_1) - \Phi_1(u_2)\|_{X^{s_1,s_2}} \leq C \left( \frac{1}{2} \eta \left( \frac{t}{T} \right) \int_0^t W(t-\tau) \partial_x (u_1^2 - u_2^2) d\tau \right) \leq CT^\varepsilon \|u_1 - u_2\|_{X^{s_1,s_2}} \leq 4C^2T^\varepsilon \|u_0\|_{H^{s_1,s_2}} \|u_1 - u_2\|_{X^{s_1,s_2}} \leq \frac{1}{2} \|u_1 - u_2\|_{X^{s_1,s_2}}. \] (5.6)

Thus, \( \Phi_1 \) is a contraction mapping in the closed ball \( B_1(0, 2C\|u_0\|_{H^{s_1,s_2}}) \). Consequently, \( u \) is the fixed point of \( \Phi \) in the closed ball \( B_1(0, 2C\|u_0\|_{H^{s_1,s_2}}) \). Then \( v := u|_{[0,T]} \in X^{s_1,s_2}_{\frac{T}{T+\varepsilon}}([0,T]) \) is a solution to the Cauchy problem for (1.1) with the initial data \( u_0 \) in the interval \([0,T]\). For the facts that uniqueness of the solution and the solution to the Cauchy problem for (1.1) is continuous with respect to the initial data, we refer the readers to Theorems II, III of [24].

This completes the proof of Theorem 1.1.

6. Proof of Theorem 1.2

We firstly prove Lemma 6.1 which is a variant of Theorem 1.1, then we apply Lemma 6.1 to prove Theorem 1.2.

**Lemma 6.1.** Let \( s_1 > -\frac{9}{8} \) and \( R := \frac{1}{8(C+1)^{\gamma}}, \) where \( C \) is the largest of those constants which appear in (2.7)-(2.8), (4.42), (4.66). Then, the Cauchy problem for (1.1) is locally well-posed for data satisfying
\[ \|I_N u_0\|_{L^2} \leq R. \] (6.1)

Moreover, the solution to the Cauchy problem for (1.1) exists on a time interval \([0,1]\).
**Proof.** We define \( v := I_N u \). Let \( u \) be the solution to the Cauchy problem for (1.1), then \( v \) is the solution to the following equations

\[
v_t + \partial_x^5 v + \partial_x^{-1} \partial_y^2 v + \frac{1}{2} I_N \partial_x (I_N^{-1} v)^2 = 0. \tag{6.2}\]

Thus, \( v \) satisfies the following equations

\[
v = W(t)v_0 + \frac{1}{2} \int_0^t W(t - \tau) I_N \partial_x (I_N^{-1} v)^2 \, d\tau. \tag{6.3}\]

We define

\[
\Phi_2(v) := \psi(t) W(t) I_N u_0 + \frac{1}{2} \psi(t) \int_0^t W(t - \tau) I_N \partial_x (I_N^{-1} v)^2 \, d\tau. \tag{6.4}\]

Combining Lemma 2.2 with 4.3, we have

\[
\|\Phi_2(v)\|_{X^{0,0}} \leq \|\psi(t) W(t) I_N u_0\|_{X^{0,0}} + C \|\psi(t) \int_0^t W(t - \tau) I_N \partial_x (I_N^{-1} v)^2 \|_{X^{0,0}} \]

\[
\leq C \|I_N u_0\|_{L^2} + C \|I_N \partial_x (I_N^{-1} v)^2\|_{X^{0,0}} \]

\[
\leq C \|I_N u_0\|_{L^2} + C \|I_N \partial_x (I_N^{-1} v)^2\|_{X^{0,0}} \]

\[
\leq C \|I_N u_0\|_{L^2} + C \|v\|_{X^{0,0}}^2 \]

\[
\leq CR + C \|v\|_{X^{0,0}}^2. \tag{6.5}\]

We define

\[
B_2(0, 2CR) := \left\{ v : \|v\|_{X^{0,0}} \leq 2CR \right\}. \tag{6.6}\]

Combining (6.5)-(6.6) with the definition of \( R \), we have

\[
\|\Phi_2(v)\|_{X^{0,0}} \leq CR + 4C^3 R^2 = 2CR. \tag{6.7}\]

Thus, \( \Phi_2 \) is a map from \( B_2(0, 2CR) \) to \( B_2(0, 2CR) \). We define

\[
v_j := I_N u_j \ (j = 1, 2), \quad w_1 = I_N^{-1} v_1 - I_N^{-1} v_2, \quad w_2 := I_N^{-1} v_1 + I_N^{-1} v_2. \tag{6.8}\]

Combining Lemmas 2.2, 3.1, 3.2, (6.5)-(6.6) with the definition of \( R \), we have

\[
\|\Phi_2(v_1) - \Phi_2(v_2)\|_{X^{0,0}} \leq C \|\psi(t) \int_0^t W(t - \tau) \partial_x I_N \left[(I_N^{-1} v_1)^2 - (I_N^{-1} v_2)^2\right] d\tau\|_{X^{0,0}} \]

\[
\leq C \|\partial_x I_N (w_1 w_2)\|_{X^{0,0}} \]

\[
\leq C \|v_1 - v_2\|_{X^{0,0}} \left(\|v_1\|_{X^{0,0}} + \|v_2\|_{X^{0,0}}\right) \]

\[
\leq 4C^2 R \|v_1 - v_2\|_{X^{0,0}} \leq \frac{1}{2} \|v_1 - v_2\|_{X^{0,0}}. \tag{6.9}\]

48
Thus, $\Phi_2$ is a contraction mapping from $B_2(0, 2CR)$ to $B_2(0, 2CR)$. Consequently, $u$ is the fixed point of $\Phi_2$ in the closed ball $B_2(0, 2CR)$. Then $v := Iu_{|[0,1]} \in X^{0,0}_{\frac{1}{2}+\epsilon}([0,1])$ is a solution to the Cauchy problem for (5.3) with the initial data $I_Nu_0$ on the interval $[0,1]$. For the uniqueness of the solution and the fact that the solution is continuous with respect to the initial data, we refer the readers to Theorem II, III of [24].

This completes the proof of Lemma 6.1.

Now we apply the idea of [20] and Lemmas 2.7, 4.2, 6.1 to prove Theorem 1.2.

For $\lambda > 0$, we define

$$u_\lambda(x, y, t) := \lambda^{\frac{4}{s+2}} u \left( \lambda^{\frac{1}{s} + \frac{2}{s+2}} x, \lambda^{\frac{1}{s} + \frac{2}{s+2}} y, \lambda t \right), \quad u_{0\lambda}(x, y) := \lambda^{\frac{4}{s+2}} u \left( \lambda^{\frac{1}{s} + \frac{2}{s+2}} x, \lambda^{\frac{1}{s} + \frac{2}{s+2}} y \right).$$

(6.10)

Thus, $u_\lambda(x, y, t) \in X^{s+1,0}_{\frac{1}{2}+\epsilon}([0, \frac{T}{\lambda}])$ is the solution to

$$
\begin{align*}
\partial_t u_\lambda + \partial_x^2 u_\lambda + \partial_x^{-1} \partial_y^2 u_\lambda + u_\lambda \partial_x u_\lambda &= 0, \\
u_\lambda(x, y, 0) &= u_{0\lambda}(x, y),
\end{align*}
$$

(6.11)

if and only if $u(x, y, t) \in X^{s,0}_{\frac{1}{2}+\epsilon}([0, T])$ is the solution to the Cauchy problem for (1.1) in $[0, T]$ with the initial data $u_0$. By using a direct computation, for $\lambda \in (0, 1)$, we have

$$
\|I_N u_{0\lambda}\|_{L^2} \leq C N^{-s} \lambda^{\frac{2}{s+2} + \frac{s}{s+2}} \|u_0\|_{H^{s,0}}.
$$

(6.13)

For $u_0 \neq 0$ and $u_0 \in H^{s,0}(\mathbb{R}^2)$, we choose $\lambda, N$ such that

$$
\|I_N u_{0\lambda}\|_{L^2} \leq C N^{-s} \lambda^{\frac{2}{s+2} + \frac{s}{s+2}} \|u_0\|_{H^{s,0}} := \frac{R}{4}.
$$

(6.14)

Then there exists $w_3$ which satisfies that $\|w_3\|_{X^{s,0}_{\frac{1}{2}+\epsilon}} \leq 2CR$ such that $v := w_3 \mid_{[0,1]}$ is a solution to the Cauchy problem for (6.11) with $u_{0\lambda}$. Multiplying (6.11) by $2I_Nu_{\lambda}$ and integrating with respect to $x, y$ yield

$$
\frac{d}{dt} \int_{\mathbb{R}^2} (I_N u)^2 \, dx \, dy + \int_{\mathbb{R}^2} I_N u \partial_x I_N \left[ (u)^2 \right] \, dx \, dy = 0.
$$

(6.15)

Inserting

$$
\int_{\mathbb{R}^2} I_N u \partial_x \left[ (I_N u)^2 \right] \, dx \, dy = 0
$$

into (6.15) yields

$$
\frac{d}{dt} \int_{\mathbb{R}^2} (I_N u)^2 \, dx \, dy = - \int_{\mathbb{R}^2} I_N u \partial_x \left[ I_N \left( (u)^2 \right) \right] \, dx \, dy.
$$

(6.16)
Combining (6.16) with Lemmas 2.6, 4.2, we have

\[
\int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 dx dy - \int_{\mathbb{R}^2} (I_N u_0)^2 dx dy = -\int_{0}^{1} \int_{\mathbb{R}^2} I_N u_\lambda \partial_x \left[ I_N ((u_\lambda)^2) - (I_N u_\lambda)^2 \right] dx dy dt
\]

\[
\leq C \left\| \chi_{[0,1]}(t) I_N u_\lambda \right\|_{X^{0,0}_\frac{1}{2}+\epsilon} \left\| \chi_{[0,1]}(t) \partial_x \left[ I_N ((u_\lambda)^2) - (I_N u_\lambda)^2 \right] \right\|_{X^{0,0}_\frac{1}{2}+\epsilon}
\]

\[
\leq C \| I_N u_\lambda \|_{X^{0,0}_\frac{1}{2}+\epsilon} \left\| \partial_x \left[ I_N ((u_\lambda)^2) - (I_N u_\lambda)^2 \right] \right\|_{X^{0,0}_\frac{1}{2}+2\epsilon}
\]

\[
\leq CN^{-2+10\epsilon} \| I_N u_\lambda \|_{X^{0,0}_\frac{3}{4}+\epsilon}^3. \tag{6.17}
\]

From (6.14) and (6.15) and the definition of \( R \), we have

\[
\int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 dx dy \leq \frac{R^2}{16} + CN^{-2+10\epsilon} \| I_N u_\lambda \|_{X^{\frac{3}{4}+\epsilon}}^{3}
\]

\[
\leq \frac{R^2}{16} + 8C^4 N^{-2+10\epsilon} R^3 \leq \frac{R^2}{16} + CN^{-2+10\epsilon}. \tag{6.18}
\]

Let \( N \) be sufficiently large such that such that \( 8C^4 N^{-2+10\epsilon} R^3 \leq \frac{3}{4} R^2 \), then

\[
\left[ \int_{\mathbb{R}^2} (I_N u(x, y, 1))^2 dx dy \right]^\frac{1}{2} \leq R. \tag{6.19}
\]

We consider \( I_N u(x, y, 1) \) as the initial data and repeat the above argument, from Lemma 6.1, we obtain that (6.11)-(6.12) possess a solution in \( \mathbb{R}^2 \times [1, 2] \). In this way, we can extend the solution to (6.11)-(6.12) to the time interval \([0, 2]\). The above argument can be repeated \( L \) steps, where \( L \) is the maximal positive integer such that

\[
CN^{-2+10\epsilon} L \leq \frac{3}{4} R^2. \tag{6.20}
\]

More precisely, the solution to (6.11)-(6.12) can be extended to the time interval \([0, L]\). Thus, we can prove that (6.11)-(6.12) are globally well-posed in \([0, T_\lambda]\) if

\[
L \geq \frac{T_\lambda}{\lambda}. \tag{6.21}
\]

From (6.20), we know that

\[
L \sim N^{2-10\epsilon}. \tag{6.22}
\]

50
We know that (6.21) is valid provided that the following inequality is valid

\[ CN^{2-10\epsilon} \geq \frac{T}{\lambda} \sim CTN^{-\frac{5\epsilon}{2s}}. \]  

(6.23)

In fact, (6.23) is valid if

\[ N^2 > N^{-\frac{5\epsilon}{2s}} \]  

(6.24)

which is equivalent to \(-\frac{4}{7} < s < 0\).

This completes the proof of Theorem 1.2.

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