Fuzzy Picard’s method for derivatives of second order

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Abstract. In this paper we proposed an iterative scheme for the solution of second order fuzzy differential equations with fuzzy initial conditions using fuzzy picard method. The convergence is also discussed under generalized H – differentiability. Finally this method evidenced is illustrated by solving some examples to show its efficiency

Key Words: generalized H – differentiability, Fuzzy derivatives, Fuzzy picard’s method.

AMS Subject Classifications: 65L05, 65L06

1. Introduction
Some quantities like initial values which are uncertain are modeled by fuzzy numbers (or) fuzzy functions in real world problems. The solution of the model also may be fuzzy function. We workout with the initial values for fuzzy differential equations, if suppose the equations of the model has differential. Chang and Zadeh [5] were the first persons who introduced the fuzzy derivative. Then, based on the extension principle Dubois and Prade[7] presented the fuzzy derivative concept. In 1987, Kandel and Byatt[8] introduced the fuzzy differential equations. Buckley and Feuring [9] treated the theory of fuzzy differential equations. And then Puri and Ralesu[12] made Hukuhara derivative. Following a generalized H-differentiability of fuzzy functions was introduced by Bede and Gal[3,4]. Ming Ma et.al[11] solved the fuzzy differential equations in numerical method. Under generalized H-differentiability by picard method S. S. Behzadi and T. Allahviranloo [1] followed them to solve the system of first order fuzzy differential equations with fuzzy initial conditions. For modeling uncertain non-linear systems Raheleh Jafari et al.[2] constructed neural models with the structure of these ordinary differential equations using picard method. Shadan Sadigh Behzadi [13] initialized the Cauchy reaction-diffusion equation of first order fuzzy differential equations with initial values under generalized H-differentiability by picard method. S. Narayambamourthy et al. [10] studied the differential systems containing fuzzy valued functions and interaction with discrete time controller which it named hybrid fuzzy differential equations (HFDEs).

The rest of the paper is organized as follows: In section 2, the fundamental concepts on fuzzy derivations and integrations are presented. In section 3, we introduce second order
fuzzy picard method and the existence and uniqueness of the solution and convergence of the proposed method are discussed in detail. We give some numerical example on picard method in section 4. In the last we present our conclusion.

2. Preliminaries

Definition 2.1. A fuzzy number is expressed as a fuzzy set defined a fuzzy interval in the real number R.

(i). $\tilde{a}$ is convex i.e $a(tx + (1-t)y) \geq \min\{a(x), a(y)\}, \forall t \in [0,1], x, y \in R$

(ii). Normalized fuzzy set (i.e) there exists $x_0 \in R$ with $a(x_0) = 1$

(iii). Supp A is bounded. Where Supp A = \{x \in R | u(x) > 0\}

Definition 2.2. A trapezoidal fuzzy number $\tilde{A}$ is a fuzzy number specified by $\tilde{A} = (a_1, a_2, a_3, a_4)$ with their membership function

$$\tilde{A}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4, \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.3. A triangular fuzzy number $\tilde{A}$ is a fuzzy number specified by $\tilde{A} = (a_1, a_2, a_3)$ with their membership function

$$\tilde{A}(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & x \geq a_3. \end{cases}$$

2.1.1 Arithmetic operations on fuzzy numbers

The fuzzy number $\tilde{a} \in [0,1]$ can also be represented as a pair $\tilde{a} = (a(r), \overline{a}(r))$ for $0 \leq r \leq 1$ which satisfies

(i). $a(r)$ is a bounded monotonic increasing right continuous function.

(ii). $\overline{a}(r)$ is a bounded monotonic decreasing left continuous function.

(iii). $\overline{a}(r) - a(r), 0 \leq r \leq 1.$
The derivative $x'(t)$ of the fuzzy process $x$ is defined by $[x'(t)]_r = [x(t; r), \tilde{x}(t; r)]$, $t \in R$,  $r \in (0,1]$, provided that this equation determines the fuzzy number, according to Seikkala [11] the set of all nonempty compact subsets $k$ of $R$ and by $k_c$ the subset of $k$ consisting of nonempty convex compact sets. We know that $\rho (x, A) = \min_{a \in A} \|x - a\|$ is a distance of the point $x \in R$ from $A \in k$ and that the Hausdorff separation $\rho (A, B)$ of $A, B \in k$ is defined as $\rho (A, B) = \max_{a \in A} \rho (a, B)$.

Note that the notation is consistent, so $\rho (a, B) = \rho (\{a\}, B)$. Now, $\rho (A, B) = 0$ is not a metric. In fact, if and only if $A \subseteq B$. The Hausdorff metric $d_H$ on $k$ is defined by

$$d_H (A, B) = \max \{\rho (A, B), \rho (B, A)\}.$$  

The metric $d_H$ on $E$ is as follows: $d_H (u, v) = \sup \{d_H ([u], [v]) : 0 \leq r \leq 1\}$, $u, v \in E$.

In this paper, we use an arbitrary fuzzy number with compact support by a pair of functions $(\alpha (r), B(r))$, $\beta (r)$. According to Zedeh’s extension principle, if $u, v \in E$ and $\lambda \in R$ then $[u + v][\alpha] = [u][\alpha] + [v][\alpha]$ and $[\lambda u][\alpha] = \lambda [u][\alpha] \forall \alpha \in [0,1]$.

Let $E$ be the space of fuzzy numbers. It is easy to see that $D_H$ is a metric in $E$ and has the following properties

(i) $D_H (a + c, b + c) = D_H (a, b), \forall a, b, c \in R$,

(ii) $D_H (ka, kb) = kD_H (a, b), \forall k \in E, a, b \in R$,

(iii) $D_H (a + b, c + d) \leq D_H (a, c) + D_H (b, d), \forall a, b, c, d \in R$, and $(D_H, R)$ is a complete metric space.

**Definition 2.4** Let be $u, v \in R$. If there exists $w \in R$ such that $u = v \oplus w$, then $w$ is called the H-Difference of $u$ and $v$ and is denoted by $u \ominus v$.

**Proposition 2.5** If $f : (a, b) \to E$ is a continuous fuzzy-valued function then

$$g(x) = \int_a^x f(t) \, dt$$ is differentiable, with derivative $g'(x) = f(x).$ [3]

**Definition 2.6** (Generalized Fuzzy Derivative)

Let $F : (a, b) \to E$ and $t_0 \in (a, b)$. we say that $F$ is generalized differentiable at $t_0$ (Bede-Gal differentiability) if there exists an element $F'(t_0) \in R$ such that

(i). For $h > 0$ sufficiently small $\exists F(t_0 + h) \oplus F(t_0), F(t_0) \ominus F(t_0 - h)$, and the limits satisfy

$$\lim_{h \to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0).$$

(ii). For $h > 0$ sufficiently small $\exists F(t_0) \ominus F(t_0 + h), F(t_0 - h) \ominus F(t_0)$, and the limits satisfy

$$\lim_{h \to 0} \frac{F(t_0) \ominus F(t_0 + h)}{(-h)} = \lim_{h \to 0} \frac{F(t_0 - h) \ominus F(t_0)}{(-h)} = F'(t_0) \ h \text{ and } (-h) \text{ at denominators mean } \frac{1}{h} \text{ and } -\frac{1}{h} \text{ respectively.}$$
Definition 2.7 Let $F':(a,b)\to E$ and $t_0 \in (a,b)$, we say that $F'$ is strongly generalized differentiable at $t_0$ if there exists an element $F'(t_0) \in E$ such that

(i). For $h > 0$ sufficiently small $\exists F'(t_0 + h) \Theta F'(t_0), F'(t_0) \Theta F'(t_0 - h),$ and the limits satisfy

$$\lim_{h \to 0} \frac{F'(t_0 + h) \Theta F'(t_0)}{h} = \lim_{h \to 0} \frac{F'(t_0) \Theta F'(t_0 - h)}{h} = F'(t_0)$$

(ii). For $h > 0$ sufficiently small $\exists F'(t_0) \Theta F'(t_0 + h), F'(t_0) \Theta F'(t_0 - h),$ and the limits satisfy

$$\lim_{h \to 0} \frac{F'(t_0) \Theta F'(t_0 + h)}{(-h)} = \lim_{h \to 0} \frac{F'(t_0) \Theta F'(t_0 - h)}{(-h)} = F'(t_0)$$

All the limits are taken in the metric space $(E,D)$, and at the end points of $t_0 \in (a,b)$ and we consider only one–sided derivatives.

Remark 2.8 A function that is generalized H- differentiability as in cases (i) and (ii) of definition 2.6., will be referred as (i) - differentiable or as (ii) - differentiable, respectively

Lemma 2.9 For $x_0 \in R$, the fuzzy differential equation

$$y' = f(x,y), y(x_0) = y_0 \in E, where f: R \to E, is supposed to be continuous, if equivalent to one of the integral equations [1]$$

$$\tilde{y}(x) = \tilde{y}_0 \Theta \int_{x_0}^{x} f(t, \tilde{y}(t))dt, \quad \forall x \in [x_0, x_1],$$

$$\tilde{y}(x) = \tilde{y}_0 \Theta (-1) \int_{x}^{x_0} f(t, \tilde{y}(t))dt, \quad \forall x \in [x_0, x_1].$$

On some interval $(x_0, x_1) \subset R$, under the differentiability condition, (i) or (ii), respectively.

3. Picard Method

In solving a second order fuzzy ordinary differential equations of the form

$$y'' = f(t, y, y') \text{ subject to } y(t_0) = y_0, \ y'(t_0) = y'_0 \quad (1)$$

Equation (1) can be reduced to simultaneous fuzzy differential equations as follows:

$$\begin{align*}
  y' &= f(t, y, y'), \quad t \in [t_0, t_n] \\
  x' &= g(t, y, x),
\end{align*}$$

$$\begin{align*}
  y(t_0) &= y_0, & y'(t_0) &= y'_0,
\end{align*}$$

$$\begin{align*}
  y(t_n) &= y_n, \quad y'(t_n) &= y'_n,
\end{align*}$$

where $y_0$ and $y_0$ are triangular fuzzy numbers

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by

$$[Y(t_n)]_r = [\overline{y}(t_n; r), \underline{y}(t_n; r)] \quad \text{and} \quad [y(t_n)]_r = [\overline{y}(t_n; r), \underline{y}(t_n; r)]$$

The exact solution and the approximate solutions of first differential $t_n, 0 \leq n \leq N$ are denoted by
\[ Y'(t_n) = \bar{Y}(t_n, r), \quad Y(t_n) = \bar{Y}(t_n, r) \] and \[ y'(t_n) = \bar{y}(t_n, r), \quad y(t_n) = \bar{y}(t_n, r) \]

Picard method is an iterative method that obtains the approximate solution of fuzzy differential equations.

Let us consider the following general fuzzy differential equation

\[ y'(t) = f(x, y, t), \quad y(t_0) = y_0 \quad \text{and} \quad x'(t) = g(x, y, t), \quad x(t_0) = x_0 \]

Now we integrate above equation from \( s = 0 \) to \( s = t \) and apply the Picard method for fuzzy initial value problem

\[ y_{n+1}(t) = y_n(t) + \int_0^t (f(s, y(s), x(s))ds \quad \text{and} \quad x_{n+1}(t) = x_n(t) + \int_0^t (f(s, x(s), y(s))ds \]

We can also be represented as a parametric form of the Picard method for higher fuzzy differential equations

\[ y_{n+1}(t, r) = y_n(t, r) + \int_0^t f(s, y(s, r), x(s))ds \quad \text{and} \quad x_{n+1}(t, r) = x_n(t, r) + \int_0^t f(s, x(s, r), y(s))ds \]

This gives the general iterative formula for \( x \) and \( y \).

3.1.1 Existence and convergence analysis

**Result 3.1**

\[ D(\bar{B}X, \bar{B}X^*) = D(\sum_{j=1}^n \bar{b}_j \bar{x}_j, \sum_{j=1}^n \bar{b}_j \bar{x}_j^*) \leq D(\bar{b}_1 \bar{x}_1, \bar{b}_1 \bar{x}_1^*) + D(\bar{b}_2 \bar{x}_2, \bar{b}_2 \bar{x}_2^*) + \ldots + D(\bar{b}_n \bar{x}_n, \bar{b}_n \bar{x}_n^*) \]

\[ \leq |\bar{b}_i| D(\bar{x}_i, \bar{x}_i^*) + \ldots + |\bar{b}_n| D(\bar{x}_n, \bar{x}_n^*) = \sum_{j=1}^n |\bar{b}_j| D(\bar{x}_j, \bar{x}_j^*), \quad 1 \leq i \leq n. \]

**Lemma 3.2** If \( \bar{x}, \bar{y}, \bar{z} \in E^n \) and \( \lambda \in R \), then,

(i) \( D(\bar{x} \Theta \bar{y}, \bar{x} \Theta \bar{z}) = D(\bar{y}, \bar{z}) \),

(ii) \( D(\Theta \bar{x}, \Theta \bar{y}, \Theta \bar{z}) = |\lambda| D(\bar{x}, \bar{y}) \).

**Theorem 3.3** Let \( 0 < r < 1 \), then equations (1,2), have a unique solution when \( \bar{x}_1 \) is (i)-differentiable and \( \bar{x}_2 \) is (ii)-differentiable respectively.

**Theorem 3.4** The solution \( \bar{x}_n(t, x, y) \) obtained from the relation (3) using Picard method converges to the exact solution \( \bar{X}(t, x, y) \) of the problems (1,2) when \( \exists 0 < r < 1 \).

4. Numerical Examples

**Example 4.1** Consider the second order fuzzy initial value problem

\[ y'' - 2y' = 0 \quad t \in [0, 0.5] \quad y(0) = (r, 2 - r) \quad y'(0) = (3 + r, 4) \]

**Solution:** Put \( y' = x(t) \), \( y(0) = (r, 2 - r) \) and \( x'(t) = 2x : x(0) = y'(0) = (3 + r, 4) \). The exact solutions are
\begin{align*}
y(t) &= \frac{1}{2} t - \frac{3}{2} + \frac{1}{2} (3 + r) e^{2t} \\
\bar{x}(t) &= (3 + r) e^{2t} \quad \text{and} \quad \bar{x}(t) = 4 e^{2t}
\end{align*}

Picard method is required to find that particular solution which assumes the value \( y_0, x_0 \) when \( t = t_0 \).

\[
y_{n+1} (t, r) = y_n (t, r) + \int_0^t f (s, y_n (s, r)) ds \quad \text{and} \quad \bar{y}_{n+1} (t, r) = \bar{y}_n (s, r) + \int_0^t f (s, \bar{y}_n (s, r)) ds
\]

\[
x_{n+1} (t, r) = x_n (s, r) + \int_0^t f (s, x_n (s, r)) ds \quad \text{and} \quad \bar{x}_{n+1} (t, r) = \bar{x}_n (s, r) + \int_0^t f (s, \bar{x}_n (s, r)) ds
\]

This is an integral equation equivalent to equation for it contains the unknown \( y, x \) under the integral sign.

In this example, the exact and the approximate solution of the equation and the first differential for \( s = 0.4 \) have been shown in Table 1 and 2 respectively and it come coincidence with exact solution.

### Table 1.
| \( r \) | \( X \) | \( \bar{X} \) | \( x \) | \( \bar{x} \) |
|--------|--------|--------|--------|--------|
| 0      | 8.15484 | 10.87313 | 8.15484 | 10.87313 |
| 0.2    | 8.69850 | 10.87313 | 8.69850 | 10.87313 |
| 0.4    | 9.24215 | 10.87313 | 9.24215 | 10.87313 |
| 0.6    | 9.78581 | 10.87313 | 9.78581 | 10.87313 |
| 0.8    | 10.32947 | 10.87313 | 10.32947 | 10.87313 |
| 1      | 10.87312 | 10.87313 | 10.87312 | 10.87313 |

### Table 2.
| \( r \) | \( Y \) | \( \bar{Y} \) | \( y \) | \( \bar{y} \) |
|--------|--------|--------|--------|--------|
| 0      | 6.67662 | 8.602164 | 6.67662 | 8.602164 |
| 0.2    | 7.12173 | 8.902164 | 7.12173 | 8.902164 |
| 0.4    | 7.56684 | 8.902164 | 7.56684 | 8.902164 |
| 0.6    | 8.01194 | 8.902164 | 8.01194 | 8.902164 |
| 0.8    | 8.45705 | 8.902164 | 8.45705 | 8.902164 |
| 1      | 8.90216 | 8.902164 | 8.90216 | 8.902164 |

**Remark 4.2:**
Picard method is used in numerical analysis when discussing fixed point iteration for find a numerical successive approximation to the equation. It is considerable theoretical values that
lead to approximate solutions and of controlling the error involved. This method consists of constructing a sequence of functions which will get closer to closer to the desired solution. It uses in computation to generate a sequence of number which converges to a solution. But it can be applied only to a limited class of equations in which the successive integrations can be performed easily.

5 Conclusion
As a conclusion for this paper, picard method of fuzzy second order differential equation is established and proved under generalized H-differentiability. For the efficiency of the proposed method illustrated by giving examples, in the future research we can apply the picard method to solve a large class of FDEs.

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