EFFECTIVE EQUIDISTRIBUTION OF THE HOROCYCLE FLOW ON
GEOMETRICALLY FINITE HYPERBOLIC SURFACES

SAMUEL C. EDWARDS

ABSTRACT. We prove effective equidistribution of non-closed horocycles in the unit tangent
bundle of infinite-volume geometrically finite hyperbolic surfaces.

1. INTRODUCTION

1.1. Background. Let \( \mathcal{M} \) be a geometrically finite hyperbolic surface. \( \mathcal{M} \) may thus be realized
as a quotient \( \Gamma \backslash \mathbb{H} \), where \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) is the hyperbolic upper half-space
equipped with the standard Riemannian metric \( ds^2 = \frac{dx^2 + dy^2}{y^2} \) on which \( G = \text{PSL}(2, \mathbb{R}) \) acts
by orientation-preserving isometries in the form of Möbius transformations, and \( \Gamma \triangleleft G \) is a
finitely generated torsion-free discrete subgroup of \( G \). The unit tangent bundle \( T^1(\mathcal{M}) \) of \( \mathcal{M} \)
may be identified with the homogeneous space \( \Gamma \backslash G \). The group \( G \) acts naturally on \( \Gamma \backslash G \) by
right translation, that is

\[
g \cdot \Gamma h := \Gamma hg \quad \forall g \in G, \; \Gamma h \in \Gamma \backslash G.
\]

The goal of this article is to provide quantitative information about ergodic averages of orbits
(with respect to the above group action) of the horospherical subgroup \( N = \{ n_x = \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \} \subset G \), where \( G \) is the group \( \text{PSL}(2, \mathbb{R}) \).

A hyperbolic surface \( \mathcal{M} = \Gamma \backslash \mathbb{H} \) as above is said to have finite volume if any (and hence every)
fundamental domain \( \mathcal{F}_\Gamma \subset \mathbb{H} \) for \( \Gamma \) satisfies \( \mu_\mathbb{H}(\mathcal{F}_\Gamma) < \infty \), where \( \mu_\mathbb{H} \) is the \( G \)-invariant
Borel measure on \( \mathbb{H} \). If every fundamental domain \( \mathcal{F}_\Gamma \) has infinite \( \mu_\mathbb{H} \)-measure, then \( \mathcal{M} \) is said to be of infinite volume. If \( \Gamma \backslash \mathbb{H} \) has finite volume then \( \Gamma \) is said
to be a lattice.

The classification of \( N \)-invariant ergodic Radon measures on \( \Gamma \backslash G \) when \( \Gamma \) is a lattice
and \( N \) is a noncompact semisimple Lie group has a long history going back to Furstenberg \cite{Furstenberg},
Dani \cite{Dani1, Dani2} (amongst others), and culminating in the famous results of Ratner \cite{Ratner}.

Dani and Smillie \cite{DaniSmillie1, DaniSmillie2} were the first to prove equidistribution of \( N \)-orbits for general
lattices in \( G \); they proved that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\Gamma gn_t) \, dt = m_{\Gamma gN}(f) \quad \forall g \in \Gamma \backslash G, \; f \in C_c(\Gamma \backslash G),
\]

where either \( m_{\Gamma gN} \) is the unique \( G \)-invariant Borel probability measure on \( \Gamma \backslash G \), or \( \Gamma g \) a
periodic point for the \( N \)-action and \( m_{\Gamma gN} \) is the Lebesgue measure on \( \Gamma gN \) normalized so as to
be a probability measure. In more recent years, there has been interest in quantifying the convergence in (1), i.e. bounding
\[
\left| \frac{1}{T} \int_0^T f(\Gamma gn_t) \, dt - \mu_{\Gamma gN}(f) \right|
\]
by some explicit function depending on \( \Gamma g \), \( T \) (and \( f \)) that decays as \( T \to \infty \). Burger \cite{Burger}
proved effective equidistribution of horocycles in compact quotients \( \Gamma \backslash G \). More generally, one may use
Margulis’ thickening trick and exponential mixing of the geodesic flow, cf. e.g. Kleinbock and Margulis \cite{KleinbockMargulis}
Proposition 2.4.8 to prove a similar result for the action of horospherical subgroup acting on a
compact quotient of a semisimple Lie group. For non-compact \( \Gamma \backslash G \) (still with \( G = \text{PSL}(2, \mathbb{R}) \)
and \( \Gamma \) a lattice), effective equidistribution of horocycles was proved by Flaminio and Forni
\cite{FlaminioForni}, and Strömbergsson \cite{Strömbergsson}. See also Sarnak and Ubis \cite{SarnakUbis} for an alternative proof for
The limit set and critical exponent.

Before stating our results, we first recall some important aspects of dynamics on infinite-volume geometrically finite hyperbolic surfaces. We refer the reader to [39] for a more thorough exposition and further references for this material.

We start by recalling the definitions of the limit set and critical exponent of $\Gamma$. Let $\partial_\infty \mathbb{H}$ denote the geometric boundary of $\mathbb{H}$; i.e. $\partial_\infty \mathbb{H} = \mathbb{R} \cup \{\infty\}$. The action of $G$ has a unique continuous extension to $\partial_\infty \mathbb{H}$ given by

$$g \cdot z = \frac{az + b}{cz + d}, \quad \forall g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \ z \in \mathbb{H} \cup \partial_\infty \mathbb{H}.$$ 

The limit set of $\Gamma$ is denoted $\Lambda(\Gamma)$. This is the closed, $\Gamma$-invariant, subset of $\partial_\infty \mathbb{H}$ defined by

$$\Lambda(\Gamma) := \{ u \in \partial_\infty \mathbb{H} : \exists \{\gamma_n\}_{n=1}^\infty \subset \Gamma \text{ such that } \lim_{n \to \infty} \gamma_n \cdot i = u \}.$$
The metric on $\mathbb{H}$ induced from $ds$ is denoted $\text{dist}$; hence, given $z, w \in \mathbb{H}$, $\cosh(\text{dist}(z, w)) = 1 + \frac{|z-w|^2}{2 \text{Im}(z) \text{Im}(w)}$. Using this, we define the critical exponent $\delta_\Gamma$ of $\Gamma$ by

$$\delta_\Gamma := \inf \left\{ s \in \mathbb{R} : \sum_{\gamma \in \Gamma} e^{-s \text{dist}(\gamma, i)} < \infty \right\}.$$

If $\Lambda(\Gamma)$ consists of more than two points then it must in fact be an infinite set. We distinguish between these two cases by saying that $\Gamma$ is elementary if $\Lambda(\Gamma)$ consists of at most two points, and non-elementary otherwise. It will be the non-elementary groups $\Gamma$ that will be of most interest to us.

Beardon [3], Patterson [32], and Sullivan [47] all studied connections between $\Lambda(\Gamma)$ and $\Lambda_{pfp}(\Gamma)$ and the set of radial limit points is denoted $\Lambda_{rad}(\Gamma)$. As such,

$$\Lambda(\Gamma) = \Lambda_{pfp}(\Gamma) \cup \Lambda_{rad}(\Gamma),$$

where the union is disjoint. We observe that $\Lambda_{pfp}(\Gamma)$ and $\Lambda_{rad}(\Gamma)$ are both $\Gamma$-invariant.

Another subgroup of $G$ that will be of importance to us is

$$A = \left\{ a_y = \pm \left( \sqrt[1]{y} \begin{matrix} 0 & y \\ 0 & 1 \end{matrix} \right) : y \in \mathbb{R}_{>0} \right\}.$$

This subgroup is closely related to geodesics in $\mathbb{H}$: given $u_1 \neq u_2 \in \partial_\infty \mathbb{H}$, the geodesic from $u_1$ tending to $u_2$ is given by $\{ga_y \cdot i : y \in \mathbb{R}_{>0}\}$, where $g \in G$ is such that $g \cdot 0 = u_1$ and $g \cdot \infty = u_2$.

We also use $A$ to define the forward and backwards visual points of $g \in G$, $[g]^+$ and $[g]^-$, as follows:

$$[g]^+ := \lim_{y \to \infty} ga_y \cdot i \in \partial_\infty \mathbb{H}, \quad [g]^− := \lim_{y \to 0} ga_y \cdot i \in \partial_\infty \mathbb{H}.$$

Let $G_{rad}$ and $G_{pfp}$ denote the subsets of $G$ defined by

$$G_{rad} = \{ g \in G : [g]^+ \in \Lambda_{rad}(\Gamma) \}, \quad G_{pfp} = \{ g \in G : [g]^+ \in \Lambda_{pfp}(\Gamma) \}.$$

Since $\Lambda_{rad}(\Gamma)$ and $\Lambda_{pfp}(\Gamma)$ are both $\Gamma$-invariant, we may define subsets of $\Gamma \backslash G$ by

$$\Gamma \backslash G_{rad} := \{ \Gamma g \in \Gamma \backslash G : g \in G_{rad} \}, \quad \Gamma \backslash G_{pfp} := \{ \Gamma g \in \Gamma \backslash G : g \in G_{pfp} \},$$

$$\Gamma \backslash G_{wand} := \Gamma \backslash \left( \Gamma \backslash G_{rad} \cup \Gamma \backslash G_{pfp} \right).$$

The identity

$$a_y n t a_y^{-1} = n t \quad \forall y \in \mathbb{R}_{>0}, t \in \mathbb{R},$$

is of fundamental importance in the study of the dynamics of the $N$-action on $\Gamma \backslash G$. In particular, observe that

$$[gn_1]^+ = [g]^+ \quad \forall g \in G, \ t \in \mathbb{R},$$

so the sets $\Gamma \backslash G_{rad}$, $\Gamma \backslash G_{pfp}$, and $\Gamma \backslash G_{wand}$ are all $N$-invariant. These sets in fact characterize the $N$-orbits as follows:

1. $\Gamma g \in \Gamma \backslash G_{pfp} \iff \Gamma g$ is $N$-periodic, i.e., there exists $t_0 > 0$ such that $\Gamma gn_0 = \Gamma g$.
2. $\Gamma g \in \Gamma \backslash G_{wand} \iff \Gamma g$ is not $N$-periodic and $\overline{\Gamma g N} = \Gamma g N$.
3. $\Gamma g \in \Gamma \backslash G_{rad} \iff \overline{\Gamma g N} = \Gamma \backslash G_{rad} \cup \Gamma \backslash G_{pfp}.$
It is case (3) that we will be concerned with: the Burger-Roblin measure is supported on $\Gamma \backslash G_{\text{rad}} \cup \Gamma \backslash G_{\text{pfp}}$, and (as stated above) we intend to show the stronger statement that in this case, the $\mathbb{N}$-orbits become equidistributed in $\Gamma \backslash G_{\text{rad}} \cup \Gamma \backslash G_{\text{pfp}}$ in a quantifiable manner.

We conclude this section by recalling the definition of the convex core of $\Gamma \backslash G$ and convex cocompact $\Gamma$. Let $\text{hull}(\Gamma) \subset \mathbb{H}$ denote the convex hull of $\Lambda(\Gamma)$, that is: $\text{hull}(\Gamma)$ is the smallest (hyperbolic) convex subset of $\mathbb{H}$ containing all geodesics with both endpoints in $\Lambda(\Gamma)$. Since $\Lambda(\Gamma)$ is $\Gamma$-invariant, $\text{hull}(\Gamma)$ is as well. This allows us to define a subset $\text{core}(\mathcal{M}) \subset \mathcal{M} = \Gamma \backslash \mathbb{H}$ by

$$\text{core}(\mathcal{M}) := \Gamma \backslash \text{hull}(\Gamma).$$

Observe that if $[g]^+$ and $[g]^-$ are both in $\Lambda(\Gamma)$, then $g \cdot i \in \text{hull}(\Gamma)$. Since $\mathcal{M}$ is geometrically finite, $\text{core}(\mathcal{M})$ may be written as the (disjoint) union of a compact set and at most a finite number of cuspidal regions. If $\text{core}(\mathcal{M})$ has no cusps, then $\Gamma$ is said to be convex cocompact.

1.3. Main results. In this section, we state the main results of this paper: Theorems 4 and 2. In order to do this we first introduce some more notation.

Firstly, we let $\mathcal{Y}_\Gamma$ denote the invariant height function on $\Gamma \backslash G$. The stringent definition of $\mathcal{Y}_\Gamma$ will be given in Section 2, for now we simply state some of its properties. Our interest in $\mathcal{Y}_\Gamma$ comes from the fact that for $\Gamma g \in \Gamma \backslash G$, $\mathcal{Y}_\Gamma(\Gamma g)$ measures “how far” into a cusp of $\Gamma \backslash G$ the point $\Gamma g$ lies. This is made more precise as follows: $\mathcal{Y}_\Gamma$ is continuous and $\mathbb{R}_{\geq 1}$-valued. For convex cocompact $\Gamma$, we have $\mathcal{Y}_\Gamma(\Gamma g) = 1$ for all $\Gamma g \in \Gamma \backslash G$. For non-convex cocompact $\Gamma$, we use the hyperbolic metric $\text{dist}$ on $\mathbb{H}$ to define a metric $\text{dist}_{\Gamma \backslash G}$ on $\Gamma \backslash G$ by

$$\text{dist}_{\Gamma \backslash G}(\Gamma g, \Gamma h) := \inf_{\gamma \in \Gamma} \text{dist}(\gamma g \cdot i, h \cdot i) \quad \forall g, h \in G.$$ 

We then have (cf. Proposition 3): if $\mathcal{Y}_\Gamma(\Gamma g) > 1$, then $\Gamma g$ belongs to a cuspidal neighbourhood in $\Gamma \backslash G$, and there exist constants $0 < c_0 < c_1$ such that

$$c_0 e^{\text{dist}_{\Gamma \backslash G}(\Gamma g, \Gamma e)} \leq \mathcal{Y}_\Gamma(\Gamma g) \leq c_1 e^{\text{dist}_{\Gamma \backslash G}(\Gamma g, \Gamma e)}$$

for all $g \in G$ such that $\mathcal{Y}_\Gamma(\Gamma g) > 1$.

The invariant height function will be used to quantify the speed at which the $A$-action moves elements of $\Gamma \backslash G$ into the cusps. This quantity will in turn govern the rate of equidistribution of the horocycles. In connection with this, we need to introduce a norm that controls the growth of functions in the cusps of $\Gamma \backslash G$. For $\alpha \geq 0$, define $\| \cdot \|_{N^\alpha}$ by

$$\|f\|_{N^\alpha} := \sup_{x \in \Gamma \backslash G} \frac{|f(x)|}{\mathcal{Y}_\Gamma(x)^\alpha} \quad \forall f \in C(\Gamma \backslash G), \alpha \geq 0.$$ 

We let $B^\alpha$ denote the subspace of $C(\Gamma \backslash G)$ consisting of functions with finite $N^\alpha$-norm. Observe that if $\alpha_1 \leq \alpha_2$, then $\|f\|_{N^{\alpha_1}} \geq \|f\|_{N^{\alpha_2}}$. In addition to the norm $\| \cdot \|_{N^\alpha}$, we will also require Sobolev norms of functions on $\Gamma \backslash G$. Letting $K = \text{PSO}(2)$, we recall that we have the Iwasawa decompositions $G = NAK$ and $G = KAN$. The decomposition $G = NAK$ may be used to decompose the Haar measure $\mu_G$ on $G$ as $d\mu_G(n_x a_y k) = \frac{dx dy d\mu_K(k)}{y^2}$, where $\mu_K$ is the Haar probability measure on $K$. We denote the natural projection of $\mu_G$ on $\Gamma \backslash G$ by $\mu_{\Gamma \backslash G}$. Since $\mathcal{M}$ has infinite volume, $\mu_{\Gamma \backslash G}$ is an infinite measure. In Section 3.2 we define $L^2(\Gamma \backslash G) = L^2(\Gamma \backslash G, \mu_{\Gamma \backslash G})$-Sobolev norms $\| \cdot \|_{S^m(\Gamma \backslash G)}$ on functions on $\Gamma \backslash G$. The space of all functions $f$ on $\Gamma \backslash G$ such that $\|f\|_{S^m(\Gamma \backslash G)} < \infty$ is denoted $S^m(\Gamma \backslash G)$—this space essentially consists of all functions in $L^2(\Gamma \backslash G)$ with all Lie derivatives up to (and including) order $m$ also in $L^2(\Gamma \backslash G)$.

Another quantity that affects the rate of convergence is the spectral gap. We briefly recall some aspects of the spectral theory of the Laplace-Beltrami operator $\Delta = y^{-2}(\partial_y^2 + \partial_z^2)$ on $L^2(\mathcal{M})$ (the measure on $\mathcal{M}$ being the natural projection of $\mu_\mathbb{H}$ to $\mathcal{M}$), due to Patterson...
[31] [32]. Firstly, the spectrum of $-\Delta$ in the interval $[0, \frac{1}{4})$ consists of finitely many (discrete) eigenvalues, and denoting these by $\lambda_i$, $i = 0, \ldots, I$, we have

$$0 < \delta_T (1 - \delta_T) = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_I.$$  

We define $s_1 \in \left[\frac{1}{2}, \delta_T\right)$ by

$$s_1 := \begin{cases} \frac{1}{2}, & \text{if } I = 0 \\ \frac{1}{2} + \frac{1}{4} - \lambda_1, & \text{otherwise}. \end{cases}$$

Observe that $\frac{1}{2} \leq s_1 < \delta_T$. This will be important in Theorem 1.

Finally, we introduce notation for both three measures that appear in our equidistribution statements. Given a radial point $\Gamma g \in \Gamma \backslash G_{rad}$, the Patterson-Sullivan measure on $\Gamma g N$ is denoted $\mu_{PS}^{\Gamma g N}$; we give the precise definition of this in Section 4.2. Since $\Gamma g \in \Gamma \backslash G_{rad}$, the map from $\mathbb{R}$ to $\Gamma \backslash G$ given by $t \mapsto \Gamma g n_t$ is injective, allowing us to also view $\mu_{PS}^{\Gamma g N}$ as a measure on $\mathbb{R}$. This will be done throughout the article (often without comment). We let $B_T := \{ t \in \mathbb{R} : |t| \leq T \}$. Using the notation just introduced, we have

$$\mu_{PS}^{\Gamma g N}(B_T) = \mu_{PS}^{\Gamma g N}(\{ \Gamma g n_t : |t| \leq T \}).$$

The Burger-Roblin measure on $\Gamma \backslash G$ is denoted $m^{BR}_T$. Again, we postpone the precise definition of $m^{BR}_T$ until later, cf. Section 7.1. For now, we recall from Section 7.1 that $m^{BR}_T$ is the unique (up to scaling) $N$-invariant Radon measure on $\Gamma \backslash G$ that is not supported on a closed horocycle. The last measure we need is the Bowen-Margulis-Sullivan, or BMS-measure on $\Gamma \backslash G$. This measure is denoted $m^{BMS}_T$. We will actually not be required to carry out any calculations using the BMS-measure; it occurs solely as a normalizing factor in the main term of our equidistribution statements. The main fact we note about the BMS-measure on $\Gamma \backslash G$ is that it is finite: $m^{BMS}_T(\Gamma \backslash G) < \infty$.

We can now state our main theorem:

**Theorem 1.** Assume $\Gamma < G$ is geometrically finite and $\frac{1}{2} < \delta_T < 1$. Let $\Omega \subset \Gamma \backslash G$ be compact and $\alpha \in [0, \frac{1}{2})$. Then for all $\Gamma g \in \Omega \cap \Gamma \backslash G_{rad}$, $f \in S^1(\Gamma \backslash G) \cap B^\alpha$, and $T \gg \Omega$, 1,

$$\frac{1}{\mu_{PS}^{\Gamma g N}(B_T)} \int_{-T}^{T} f(\Gamma g n_t) \, dt = \frac{m^{BR}_T(f)}{m^{BMS}_T(\Gamma \backslash G)} + O_{\Omega, \alpha} \left( \| f \|_{S^1(\Gamma \backslash G)} \left( \left( \frac{\log t}{t} \right)^{\delta_T - \frac{1}{2}} \log^2 \left( 2 + \frac{T}{\delta_T} \right) \right) \right) + \| f \|_{N, \alpha} \left( \left( \frac{\log t}{t} \right)^{\delta_T - s_1} \right) \right).$$

We make some remarks:

**Remark 1.** The reason that this is an effective equidistribution statement for all radial starting points is that $\lim_{T \to \infty} \frac{\log t}{t} = 0$ for all $\Gamma g \in \Gamma \backslash G_{rad}$. This is due to the fact that if $\Gamma g \in \Gamma \backslash G_{rad}$ then the geodesic segment $\Gamma g a_y$ ($y \geq 1$) returns infinitely often to some compact subset of $\Gamma \backslash G$ (combined with Proposition 3 (2)). Theorem 1 thus shows that the speed of equidistribution of $\Gamma g B_T$ is governed by the cuspidal excursion rate of $\Gamma g a_T$; this is completely analogous to the situation for non-compact finite-volume quotients $\Gamma \backslash G$, cf. [44, Theorem 1]. We recall that excursion rates for geodesics are well-studied and related to approximation problems for $\Gamma$-orbits. For finite-volume $\Gamma \backslash G$, one has Sullivan’s logarithm law [16] and Melián and Pestana’s computation of the Hausdorff dimension of the set of directions in $T^1(\mathcal{M})$ around a given point of $\mathcal{M}$ with cuspidal excursion rate greater than a given number [25]. In the case that $\Gamma \backslash G$ has infinite volume, there exist corresponding results due to Stratmann and Velani [42] and Hill and Velani [18].

**Remark 2.** The measure $m^{BR}_T$ is a priori only defined on $C_c(\Gamma \backslash G)$. However, (as will be seen in the proof of Theorem 1) it does have a (unique) extension as a distribution on $\Gamma \backslash G$ to a linear functional on $S^1(\Gamma \backslash G)$ (cf. [22, Theorem 7.3]).
Remark 3. An interesting feature of Theorem 1 is that it holds for quite general functions on \( \Gamma \setminus G \). Most previous equidistribution results for infinite-volume \( \Gamma \setminus G \) require the test functions to be bounded or have compact support.

Remark 4. The dependencies on the compact set \( \Omega \subset \Gamma \setminus G \) come solely from a lower bound on \( \mu_{\Gamma \setminus G}^N(B_T) \), cf. Proposition 13 and Corollary 10.

A key part of the proof of Theorem 1 consists of calculating integrals of the base eigenfunction along pieces of horocycles. The base eigenfunction is in \( L^2(\Gamma \setminus G) \) if and only if \( \delta_T > \frac{1}{2} \). This is the reason for the requirement \( \delta_T > \frac{1}{2} \) in Theorem 1. We recall that for \( \delta_T \leq \frac{1}{2} \), \( \Gamma \) is convex-cocompact. This allows us to use exponential mixing (we refer the reader to the beginning of Section 3 for a more thorough discussion of these matters) and Margulis’ thickening trick to also prove effective equidistribution of horocycles without the assumption \( \delta_T > \frac{1}{2} \).

Before stating our result in this direction we introduce some more spaces of functions. For a compact subset \( \Omega \subset \Gamma \setminus G \), let \( S^m(\Omega) \) denote the closure of

\[ \{ f \in C_c^\infty(\Gamma \setminus G) : \supp f \subset \Omega, \ f|_{\partial \Omega} = 0 \} \]

with respect to \( \| \cdot \|_{S^m(\Gamma \setminus G)} \).

Our effective equidistribution result for \( \Gamma \) with \( \delta_T \leq \frac{1}{2} \) reads

**Theorem 2.** Let \( \Gamma \) be non-elementary and convex cocompact. There exists \( \eta_\Gamma > 0 \) such that for any compact subset \( \Omega \subset \Gamma \setminus G \) and \( \Gamma g \in \Omega \cap \Gamma \setminus G_{\text{rad}} \),

\[
\frac{1}{\mu_{\Gamma \setminus G}^N(B_T)} \int_{-T}^T f(\Gamma g t) \, dt = \frac{m_{BR}^\Gamma(f)}{m_{BMS}^\Gamma(\Gamma \setminus G)} + O_{\Gamma, \Omega, \eta_\Gamma} \left( \| f \|_{S^1(\Gamma \setminus G)} T^{-m} \right)
\]

for all \( f \in S^1(\Omega) \), \( T \gg 1 \).

**Remark 5.** As in Theorem 1, the behavior of \( \Gamma g \) under the \( A \)-action affects the error term in the equidistribution statement. Here, it is the dependency of the implied constant on the starting point \( \Gamma g \) that is determined by properties of the \( A \)-orbit of \( \Gamma g \). Since \( \Gamma \) is convex cocompact, for every \( \Gamma g \in \Gamma \setminus G_{\text{rad}} \), the set \( \{ \Gamma g a_y : y \geq 1 \} \) is contained in a compact subset of \( \Gamma \setminus G \). It is the maximal distance of this set to some fixed basepoint that determines the implied constant’s dependency on the starting point, i.e. given \( r > 0 \), the implied constant can be made uniform over all \( \Gamma g \in \Omega \cap \Gamma \setminus G_{\text{rad}} \) such that \( \sup_{y \geq 1} \text{dist}_{\Gamma \setminus G}(\Gamma g a_y, \Gamma e) \leq r \). In particular, the implied constant can be made uniform over the set \( \{ \Gamma g : [g]^{\mp} \in \Lambda(\Gamma) \} \).

### 1.4. Overview of article.

The majority of the article (Sections 2-7) is devoted to the proof of Theorem 1. As mentioned above, to do this, we combine Strömbergsson’s effective equidistribution result [44, Theorem 1] with an effective equidistribution statement for the base eigenfunctions, Theorem 20. It is Theorem 20 that is the main technical result of the paper.

In Section 2, we define the invariant height function \( \eta_\Gamma \) and state a collection of its properties that will be used throughout the rest of the article. Section 3 consists of a recollection of a series of facts regarding harmonic analysis on \( \Gamma \setminus G \), in particular, the decomposition of \( L^2(\Gamma \setminus G) \) into irreducible unitary representations, as well as a couple of Sobolev inequalities.

The proof of Theorem 20 consists of a series of calculations using the Patterson-Sullivan density. In Section 4, we recall the definition of conformal densities on \( \partial_\infty \mathbb{H} \) and their properties. A key result here is Sullivan’s shadow lemma, which we use to bound the Patterson-Sullivan measures of certain sets in \( \partial_\infty \mathbb{H} \).

Having set up the necessary prerequisites, in Section 5 we state and prove Theorem 20. Strömbergsson’s effective equidistribution result is stated in Section 6 and combined with Theorem 20 in Section 7 to prove Theorem 1.

Section 8 is devoted to the proof of Theorem 2. We start by recalling results of Stoyanov [44] and Oh and Winter [30] on exponential mixing of the \( A \)-action on \( \Gamma \setminus G \). This is used to show effective equidistribution of expanding translates of pieces of horocycle orbits; the result we need is due to Mohammadi and Oh [27]. Theorem 2 is then proved by combining this result with Sullivan’s shadow lemma.
**Acknowledgements.** This research was funded by a scholarship from the Knut and Alice Wallenberg foundation. I would like to thank Hee Oh for asking me about this as well as for interesting and enlightening discussions, and Andreas Strömbergsson for some useful comments.

2. The Invariant Height Function

2.1. The invariant height function. Here we will define the invariant height function. Much of this section is similar to [15, Section 2], however since we deal only with the case $G = \text{PSL}(2, \mathbb{R})$, and [15] studies the general case $G = \text{SO}_0(n, 1)$, there are a number of simplifications. The primary reason for this is due to the fact that all cusps of $\Gamma \backslash \mathbb{H}$ have full rank, which is not necessarily the case in higher dimensions.

We start by recalling some properties regarding the action of $G$ on $\mathbb{H}$. For $\eta \in \partial_{\infty} \mathbb{H} \setminus \{\infty\}$, define the horoball of diameter $g$ based at $\eta$, $\mathcal{H}(\eta, \sigma) \subset \mathbb{H}$, by
\[
\mathcal{H}(\eta, \sigma) := \{z \in \mathbb{H} : |z - (\eta + i\frac{g}{\sigma})| < \frac{g}{\sigma}\}.
\]
We also define horoballs at infinity $\mathcal{H}(\infty, \sigma)$ by
\[
\mathcal{H}(\infty, \sigma) := \{z \in \mathbb{H} : \text{Im}(z) > \sigma\}.
\]
Observe that if $g \in G$ and $\eta \in \partial_{\infty} \mathbb{H}$, then for any $\sigma > 0$, there exists $\sigma_g > 0$ such that $g \cdot \mathcal{H}(\eta, \sigma) = \mathcal{H}(g \cdot \eta, \sigma_g)$.

Horoballs are important for studying the behaviour of functions in the cusps of $\Gamma \backslash G$. We will now define a function that captures the growth properties of functions in cusps in a succinct way. We follow [15, Section 2] and [44, Section 2]. Given a parabolic fixed point (henceforth abbreviated pfp) $\eta \in \partial_{\infty} \mathbb{H}$ of $\Gamma$, we define a subset $\mathcal{N}^{(G)}_{\eta} \subset G$ by
\[
\mathcal{N}^{(G)}_{\eta} := \{ h \in G : h \cdot \eta = \infty \text{ and } h \cdot \text{Stab}_\Gamma(\eta) h^{-1} = \pm \left(\frac{1}{1 \ 0} \right) \}.
\]
Note that given a pfp $\eta$ of $\Gamma$, we have $\text{Im}(h_1 \cdot z) = \text{Im}(h_2 \cdot z)$ for all $z \in \mathbb{H}$ and $h_1, h_2 \in \mathcal{N}^{(G)}_{\eta}$ (cf. [15 Lemma 2]). Another important property is that $\mathcal{N}^{(G)}_{\eta} g = \mathcal{N}^{(G)}_{g^{-1} \cdot \eta}$ (for all pfps $\eta$ of $\Gamma$ and $g \in G$). In particular, if $\eta$ is a pfp for $\Gamma$, then for all $\gamma \in \Gamma$, $\gamma \cdot \eta$ is also a pfp for $\Gamma$, and $\mathcal{N}^{(G)}_{\gamma \cdot \eta} = \mathcal{N}^{(G)}_{\eta} \gamma^{-1}$. We now define the invariant height function: let $\tilde{\mathcal{Y}}_{\Gamma} : \mathbb{H} \to \mathbb{R}_{>0}$ be defined by
\begin{equation}
(2) \quad \tilde{\mathcal{Y}}_{\Gamma}(z) := \sup_{\eta \in \partial_{\infty} \mathbb{H}} \text{Im}(h_\eta \cdot z) \quad (h_\eta \in \mathcal{N}^{(G)}_{\eta}),
\end{equation}
and
\[
\mathcal{Y}_{\Gamma}(z) := \max\{1, \tilde{\mathcal{Y}}_{\Gamma}(z)\}.
\]
We will see shortly that $\mathcal{Y}_{\Gamma}$ is well-defined, i.e. the supremum in the definition is finite for every $z \in \mathbb{H}$. Since $\Gamma$ is geometrically finite, the set of pfps for $\Gamma$ decomposes into a finite number $\kappa < \infty$ of $\Gamma$-orbits, cf. [6] Lemma 3.1.4], [7] Corollary 6.5]. Choosing a set of representatives $\eta_1, \ldots, \eta_\kappa$ for the $\Gamma$-orbits, we may use the equality $\mathcal{N}^{(G)}_{\eta} = \mathcal{N}^{(G)}_{\eta} \gamma^{-1}$ to express $\mathcal{Y}_{\Gamma}$ as
\[
\mathcal{Y}_{\Gamma}(z) = \max \left\{1, \max_{1 \leq i \leq \kappa} \text{Im}(h_{\eta_i} \gamma \cdot z) \right\} \quad (h_{\eta_i} \in \mathcal{N}^{(G)}_{\eta_i}).
\]
Observe that $\mathcal{Y}_{\Gamma}$ is left $\Gamma$-invariant; we may thus also view it as a function on $\Gamma \backslash \mathbb{H}$. Furthermore, we may view it as a left $\Gamma$-invariant and right $K$-invariant function on $G$ by the formula
\[
\mathcal{Y}_{\Gamma}(g) := \mathcal{Y}_{\Gamma}(g \cdot i) \quad \forall g \in G.
\]
The $\Gamma$-invariance allows us to also view $\mathcal{Y}_{\Gamma}$ as a function on $\Gamma \backslash G$. Note that $\mathcal{Y}_{\Gamma}(n_x a_y k) = \mathcal{Y}_{\Gamma}(x + iy)$ for all $x \in \mathbb{R}$, $y > 0$, $k \in K$. We will abuse notation slightly and use $\mathcal{Y}_{\Gamma}$ to denote the function on any of $\mathbb{H}$, $\Gamma \backslash \mathbb{H}$, $G$, and $\Gamma \backslash G$.

Several important properties of $\mathcal{Y}_{\Gamma}$ are captured in the following proposition:
Proposition 3.

(1) \( \gamma \Gamma \gamma g n z \leq \gamma \Gamma (g)(1 + |x|)^2 \) for all \( g \in G, x \in \mathbb{R} \).

(2) \( \gamma \Gamma (g) y q z \leq \gamma \Gamma (g) \max \{y, y^{-1}\} \) for all \( g \in G, y > 0 \).

(3) \( \gamma \Gamma (g) z = \gamma \Gamma g \gamma^{-1} \gamma g z \) for all \( g \in G, z \in \mathbb{H} \).

(4) The set \( \{ z \in \mathbb{H} : \gamma \Gamma (z) > 1 \} \) is a \( \Gamma \)-invariant disjoint union of horoballs based at the pfps of \( \Gamma \).

(5) There exist constants \( 0 < c_0 < c_1 \) such that
\[
e^{c_0 \text{dist}_{\gamma \Gamma} (\Gamma g, \Gamma e)} \leq \gamma \Gamma (g) \leq c_1 e^{c_1 \text{dist}_{\gamma \Gamma} (\Gamma g, \Gamma e)} \]for all \( g \in \{ h \in G : \gamma \Gamma (h) > 1 \} \).

Proof. These statements are all contained (either explicitly or implicitly) in \([15, \text{Section 2}]\) and \([14, \text{Section 2}]\) (cf. also \([14, \text{Lemma 5}]\)). For completeness, we give exact references and supplementary arguments. For (1) and (2), see \([14, (12), (13)]\), and the subsequent paragraph, p. 298. Item (3) follows from the fact that \( \lambda \eta \Gamma g = \lambda \gamma \eta^{-1} \Gamma g \).

To prove (4), we choose two pfps \( \eta_1 \neq \eta_2 \) of \( \Gamma \) and let \( \mathcal{H}(\eta_1, \sigma) \) be defined by \( \mathcal{H}(\eta_1, \sigma) = h_i^{-1} \mathcal{H}(\infty, 1) \), where \( h_i \in \lambda \eta \Gamma \), \( i = 1, 2 \). After possibly conjugating \( \Gamma \), we may assume that \( \eta_1 = \infty, h_1 = \pm \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \), and \( \Gamma \infty = \pm \left( \begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix} \right) \). Writing \( h_2 = \pm \left( \begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix} \right) \), if \( z = x + iy \in \mathcal{H}(\infty, 1) \cap \mathcal{H}(\eta_2, \sigma) \), then
\[
\text{Im}(h_2 \cdot z) = \frac{y}{(cx + d)^2 + (cy)^2} > 1.
\]

Since \( z \in \mathcal{H}(\infty, 1), y > 1 \). Observe also that since \( \eta_2 \neq \infty, h_2 \not\in \text{Stab}(\infty) \), hence \( c \neq 0 \), and thus
\[
1 < \text{Im}(h_2 \cdot z) \leq \frac{y}{(cy)^2} \leq \frac{1}{c^2 y}.
\]

We then have
\[
\mathcal{H}(\infty, 1) \cap \mathcal{H}(\eta_2, \sigma) \subset \{ z : 1 < \text{Im}(z) < \frac{1}{c^2 y} \}.
\]

Consider now the subgroup \( \Gamma' \subset \Gamma \) defined by
\[
\Gamma' = (\text{Stab}(\eta_1), \text{Stab}(\eta_2)) = (\pm \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), \pm h_2^{-1} \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)) h_2).
\]

Now, since \( h_2 = \left( \begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix} \right), h_2^{-1} \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) h_2 = \pm \left( \begin{smallmatrix} -1 & -2 \\ 0 & 1 \end{smallmatrix} \right) \). We now apply Shimizu’s lemma (cf. \([40, \text{Lemma 4}]\), \([26, \text{Lemma 1.7.3}]\)) to the discrete group \( \Gamma' \); if \( c^2 < 1 \), then \( h_2^{-1} \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) h_2 \in \Gamma \infty \).

Since \( \eta_2 \neq \infty, h_2^{-1} \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) h_2 \not\in \Gamma \infty \), and hence \( c^2 \geq 1 \), giving
\[
\mathcal{H}(\infty, 1) \cap \mathcal{H}(\eta_2, \sigma) \subset \{ z : 1 < \text{Im}(z) < \frac{1}{c^2 y} \} = \emptyset.
\]

This shows that \( \bigcup \text{pfps} \eta h_\eta^{-1} \mathcal{H}(\infty, 1) \) is in fact a disjoint union of horoballs. By (3), this is a \( \Gamma \)-invariant set. Consequently, \( \gamma \Gamma \) is well-defined: if \( z \in \mathbb{H} \setminus \bigcup \text{pfps} \eta h_\eta^{-1} \mathcal{H}(\infty, 1) \), then from (2), \( \gamma \Gamma \gamma (z) \leq 1 \), and if \( z \in h_\eta^{-1} \mathcal{H}(\infty, 1) \), then \( \gamma \Gamma (z) = \text{Im}(h_\eta \cdot z) > 1 \). Thus, \( \{ z \in \mathbb{H} : \gamma \Gamma (z) > 1 \} = \bigcup \text{pfps} \eta h_\eta^{-1} \mathcal{H}(\infty, 1) \).

To prove (5), we make use of the set \( \eta_1, \ldots, \eta_\kappa \) of \( \Gamma \)-inequivalent representatives for the set of all pfps. We assume that \( z = g \cdot i \) and \( \gamma \Gamma (z) > 1 \). By (4), \( z \in \mathcal{H}(\eta, \sigma) \) for some pfp \( \eta \) and \( \gamma \Gamma (z) = \text{Im}(h_\eta \cdot z) \). Using the \( \Gamma \)-invariance of \( \gamma \Gamma \) and \( \text{dist}_{\Gamma \Gamma} \gamma G \), we may assume that \( z \in \mathcal{H}(\eta_{j}, \sigma_{j}), 1 \leq j \leq \kappa \). We then have
\[
e^{\text{dist}_{\gamma \Gamma} (\Gamma g, \Gamma e)} \leq e^{\inf_{\gamma \in \Gamma} \text{dist}(\gamma z, i)} = e^{\inf_{\gamma \in \Gamma} \left( \text{dist}(h_{\eta_{j}} \gamma z, i) + \text{dist}(h_{\eta_{j}} \gamma i, i) \right)} \leq \left( e^{\inf_{\gamma \in \Gamma} \text{dist}(h_{\eta_{j}} \gamma z, i)} \right) \left( \max_{1 \leq i \leq \kappa} e^{\text{dist}(h_{\eta_{j}} \gamma i, i)} \right) \leq e^{\inf_{\gamma \in \Gamma} \text{dist}(h_{\eta_{j}} \gamma z, i)} e^{\text{dist}(h_{\eta_{j}} \gamma i, i)}.
\]

Now, since \( h_{\eta_{j}} \text{Stab}(\eta_{j}) h_{\eta_{j}}^{-1} = \pm \left( \begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix} \right) \), we can find \( \gamma \in \text{Stab}(\eta_{j}) \) such that
\[
h_{\eta_{j}} \gamma z = h_{\eta_{j}} \gamma h_{\eta_{j}}^{-1}(h_{\eta_{j}} \cdot z) = x_j + i \text{Im}(h_{\eta_{j}} \cdot z),
\]
with $|x_j| \leq \frac{1}{2}$. This gives $e^{\text{inf}_{\gamma \in \text{Stab}_\Gamma(\eta_j)} \text{dist}(\eta_j \gamma \cdot z, i)} \leq e^{\text{dist}(x_j + i \text{Im}(\eta_j \cdot z), i)} \ll \text{Im}(\eta_j \cdot z)$, and so $e^{\text{dist}(\Gamma g, \Gamma e)} \ll \text{Im}(\eta_j \cdot z) = \mathcal{Y}_\Gamma(z) = \mathcal{Y}_\Gamma(\Gamma g)$.

In the opposite direction, note that if $\gamma \not\in \text{Stab}_\Gamma(\eta_j)$, then $\text{Im}(\eta_j \gamma h^{-1} \cdot i) \leq 1$ (see the proof of (4)). This gives

$$\text{dist}(\gamma \cdot z, i) = \text{dist}(h_{\eta_j} \gamma \cdot z, h_{\eta_j} \cdot i) \geq (\text{dist}(h_{\eta_j} \gamma \cdot z, i) - \text{dist}(h_j \cdot i, i))$$

$$\geq \text{dist}(h_{\eta_j} \gamma \cdot z, i) - \left(\max_{1 \leq l \leq \kappa} \text{dist}(h_l \cdot i, i)\right)$$

$$= \text{dist}(h_{\eta_j} \cdot z, h_{\eta_j} \gamma h^{-1} \cdot i) - \left(\max_{1 \leq l \leq \kappa} \text{dist}(h_l \cdot i, i)\right).$$

Since $\text{Im}(h_{\eta_j} \cdot z) > 1$ and $\text{Im}(h_{\eta_j} \gamma h^{-1} \cdot i) \leq 1$,

$$\text{dist}(h_{\eta_j} \cdot z, h_{\eta_j} \gamma h^{-1} \cdot i) \geq \log \left(\frac{\text{Im}(h_{\eta_j} \cdot z)}{\text{Im}(h_{\eta_j} \gamma h^{-1} \cdot i)}\right) \geq \log(\text{Im}(h_{\eta_j} \cdot z)).$$

This gives

$$\text{dist}(\gamma \cdot z, i) \geq \log(\text{Im}(h_{\eta_j} \cdot z)) - \left(\max_{1 \leq l \leq \kappa} \text{dist}(h_l \cdot i, i)\right).$$

For $\gamma \in \text{Stab}_\Gamma(\eta_j)$, $h_{\eta_j} \gamma h^{-1} \in \mathbb{Z}$, hence

$$\text{dist}(\gamma \cdot z, i) = \text{dist}(h_{\eta_j} \gamma \cdot z, i) - \text{dist}(h_j \cdot i, i)$$

$$\geq \left(\inf_{n \in \mathbb{Z}} \text{dist}(h_{\eta_j} \cdot z + n, i)\right) - \left(\max_{1 \leq l \leq \kappa} \text{dist}(h_l \cdot i, i)\right)$$

$$\geq \log \left(\text{Im}(h_{\eta_j} \cdot z)\right) - \left(\max_{1 \leq l \leq \kappa} \text{dist}(h_l \cdot i, i)\right).$$

In conclusion,

$$e^{\text{dist}(\Gamma g, \Gamma e)} = e^{\text{inf}_{\gamma \in \Gamma} \text{dist}(\gamma \cdot z, i)} \geq e^{\log \left(\text{Im}(h_{\eta_j} \cdot z)\right) - \left(\max_{1 \leq l \leq \kappa} \text{dist}(h_l \cdot i, i)\right)} \gg \text{Im}(h_{\eta_j} \cdot z) = \mathcal{Y}_\Gamma(z).$$

\[ \square \]

3. Decomposition of $L^2(\Gamma \backslash G)$ and Sobolev Inequalities

3.1. Unitary representations. Recall the notation from Section 1—$\Delta$ ($\Delta$ is the Laplace-Beltrami operator acting on $L^2(\mathcal{M})$) has finitely many eigenvalues $\lambda_0, \ldots, \lambda_I$ in $[0, \frac{1}{4})$: $0 < \delta_{\mathcal{M}}(1 - \delta_{\mathcal{M}}) = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_I < \frac{1}{4}$, and we write $\lambda_i = s_i(1 - s_i)$ with $s_i \in (\frac{1}{2}, 1)$, $i = 0, \ldots, I$ (note thus that $s_0 = \delta_{\mathcal{M}}$).

We now recall the decomposition of the unitary representation $(\rho, L^2(\Gamma \backslash G))$ into tempered and non-tempered parts; here $\rho$ denotes right translation, i.e. $(\rho(g)f)(\Gamma h) = f(\Gamma hg)$ for all $g \in G$, $f \in L^2(\Gamma \backslash G)$, and $\Gamma h \in \Gamma \backslash G$. Letting $H$, $X_+$, and $X_-$ denote the following elements of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ of $G$:

$$H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

the Casimir element $C$ of $\mathfrak{g}$ may be expressed as $C = H^2 - H + X_+ X_-$. Identifying $L^2(\mathcal{M})$ with the subspace $L^2(\Gamma \backslash G)_K \subset L^2(\Gamma \backslash G)$ of $\rho(K)$-invariant vectors, one observes that $C$ acts on $L^2(\Gamma \backslash G)_K$ as $\Delta$; this allows one to combine the spectral theory of $\Delta$ on $L^2(\mathcal{M})$ with the classification of the unitary dual of $G$ to obtain the following:

**Proposition 4.** (cf. [22] Theorem 3.1)

$$(\rho, L^2(\Gamma \backslash G)) = \bigoplus_{i=0}^{I} \rho_i C_i \oplus (\rho, L^2(\Gamma \backslash G)_{\text{temp}}),$$
where each \((p, C_i)\) is a complementary series representation which \(C\) acts on the smooth vectors of by \(s_i(s_i - 1)\), and \((p, L^2(\Gamma G)_{\text{temp}})\) is tempered.

### 3.2. Sobolev inequalities.

We start by recalling the definition of the Sobolev norms that we need. Fix a basis \(X_1, X_2, X_3\) of \(g\), and for \(m \in \mathbb{N}\), define

\[
\|f\|_{S^m(\Gamma G)} := \sqrt{\sum_{U} \|Uf\|_{L^2(\Gamma G)}^2}, \quad \forall f \in C^\infty(\Gamma G) \cap L^2(\Gamma G),
\]

where the sum runs over all monomials \(U\) in the \(X_i\) of order not greater than \(m\) (this includes the element “1” of order zero). We let \(S^m(\Gamma G) \subset L^2(\Gamma G)\) denote the closure (with respect to \(\|\cdot\|_{S^m(\Gamma G)}\)) of the elements \(f\) of \(L^2(\Gamma G) \cap C^\infty(\Gamma G)\) with \(\|f\|_{S^m(\Gamma G)} < \infty\). Also, define \(S^\infty(\Gamma G) := \bigcup_{m \in \mathbb{N}} S^m(\Gamma G)\).

Using an automorphic Sobolev inequality of Bernstein and Reznikov \cite[Proposition B.2]{4}, we may use \(Y_{1, \Gamma}\) and Sobolev norms to express the following pointwise bound on functions in \(S^2(\Gamma G)\):

**Lemma 5.**

\[
|f(\Gamma g)| \ll_{\Gamma} \|f\|_{S^2(\Gamma G)} Y_{1,\Gamma}(\Gamma g)^{1/2} \quad \forall f \in S^2(\Gamma G), \ g \in G.
\]

**Proof.** This is \cite[Proposition 6]{15}. Observe that “\(Y_{1, \Gamma}\)” in \cite{15} is equal to “\(\tilde{Y}_{1, \Gamma}\)” (cf. (2)) here. \(\square\)

For “smooth enough” functions in the subrepresentations \(C_i\), we have the following stronger pointwise bound:

**Lemma 6.** Given \(i \in \{0, \ldots, I\}\) and \(s_i\) as in Proposition \cite{3},

\[
|f(\Gamma g)| \ll_{\Gamma} \|f\|_{S^{s_i}(\Gamma G)} Y_{1,\Gamma}(\Gamma g)^{1-s_i} \quad \forall f \in C_i \cap S^3(\Gamma G), \ g \in G.
\]

**Proof.** This is \cite[Lemma 16]{44}. Observe that the proof there essentially follows from “constant term” calculations in the cusps of \(\Gamma \backslash G\). For \(G = \text{SL}(2, \mathbb{R})\) and \(\Gamma\) geometrically finite, the cusps have the same structure as for the cusps in the case \(\Gamma\) is a lattice (that is to say: all cusps have full rank). This enables the proof given in \cite{44} to be carried over without modification. \(\square\)

### 4. Patterson-Sullivan Densities and Measures

Here we recall the definitions of the Patterson-Sullivan densities on \(\partial_\infty \mathbb{H}\) and measures on \(N\)-orbits in \(\Gamma \backslash G\). Since we will require these construction for conjugations \(g^{-1} \Gamma g \ (g \in G)\) as well as for \(\Gamma\), we will be (perhaps overly) careful with expressing dependencies on \(\Gamma\).

#### 4.1. Conformal densities.

We start by recalling the definition of a conformal density. Let \(H\) be a subgroup of \(G\). An \(H\)-invariant conformal density of dimension \(\delta\) is a collection \(\{\mu_z\}_{z \in \mathbb{H}}\) of finite Borel measures on \(\partial_\infty \mathbb{H}\) that satisfy

\[
\left(\frac{d\mu_{w}}{d\mu_{z}}\right)(u) = e^{-\delta \beta_{u}(w,z)}, \quad h_{\ast} \mu_{z} = \mu_{h \ast z} \quad \forall z, w \in \mathbb{H}, \ u \in \partial_\infty \mathbb{H}, \ h \in H.
\]

We recall the (standard) notation used here: for a measure \(\mu\) on \(\mathbb{H} \cup \partial_\infty \mathbb{H}\) and \(g \in G\), the measure \(g_{\ast} \mu\) is defined via \((g_{\ast} \mu)(A) = \mu(g^{-1} \cdot A)\) for suitable \(A \subset (\mathbb{H} \cup \partial_\infty \mathbb{H})\). Also, \(\beta_{u}(w,z)\) denotes the Busemann cocycle, i.e., for \(u \in \partial_\infty \mathbb{H}\),

\[
\beta_{u}(w,z) := \lim_{t \to \infty} \text{dist}(w,\xi_{t}) - \text{dist}(z,\xi_{t}) \quad \forall w, z \in \mathbb{H},
\]

where \(\xi_{t}\) is any geodesic ray in \(\mathbb{H}\) tending to \(u\).

There exists a unique up to scaling \(\Gamma\)-invariant conformal density of dimension \(\delta_{\Gamma}\), called the Patterson-Sullivan density (cf. \cite{32, 45}). Given \(w \in \mathbb{H}\), we may realize this conformal density as the collection \(\{\nu_{z}^{(\Gamma,w)}\}_{z \in \mathbb{H}}\), where each \(\nu_{z}^{(\Gamma,w)}\) is defined via the weak limit

\[
\nu_{z}^{(\Gamma,w)} := \lim_{s \to \delta_{\Gamma}^{+}} \sum_{\gamma \in \Gamma} e^{-s \\text{dist}(\gamma \cdot w, z)} \sum_{\gamma \in \Gamma} e^{-s \\text{dist}(\gamma \cdot w, w)} \delta_{\gamma \cdot w} \quad \forall z, w \in \mathbb{H}.
\]
Proof. Using the observation
denote the unit mass at \( \zeta \in \mathbb{H} \). We recall that all the measures in the Patterson-Sullivan density are supported on \( \Lambda(\Gamma) \) and are non-atomic; we may thus also view it as a collection of measures on \( \mathbb{R} = \partial_{\infty} \mathbb{H} \setminus \{ \infty \} \).

Since the Patterson-Sullivan density is unique up to scaling, there exists a function \( \mathcal{P}_\Gamma : \mathbb{H} \to \mathbb{R}_{>0} \) such that
\[
(5) \quad \nu_z(\Gamma, w) = \mathcal{P}_\Gamma(w) \nu_z(\Gamma, i) \quad \forall z, w \in \mathbb{H}.
\]
Note that it follows from (1) that \( \mathcal{P}_\Gamma(\gamma \cdot w) = \mathcal{P}_\Gamma(w) \) for all \( \gamma \in \Gamma, w \in \mathbb{H} \).

**Lemma 7.**
\begin{enumerate}[i)]  
\item \( \nu_z(\Gamma, w)(A) \leq e^{\delta_1 \text{dist}(z,v)} \nu_v(\Gamma, w)(A) \quad \forall z, w \in \mathbb{H}, A \subset \partial_{\infty} \mathbb{H} \) measurable.  
\item \( e^{-\delta_1 \text{dist}(w,\Gamma \cdot i)} \leq \mathcal{P}_\Gamma(w) \leq e^{\delta_1 \text{dist}(w,\Gamma \cdot i)} \quad \forall w \in \mathbb{H}. \)
\end{enumerate}

**Proof.** Using the observation \(|\beta_u(z,v)| \leq \text{dist}(z,v)\) and (3), i) is proved as follows:
\[
\nu_z(\Gamma, w)(A) = \int_A d\nu_z(\Gamma, w)(A) = \int_A e^{-\delta_1 \text{dist}(z,v)} d\nu_v(\Gamma, w)(A) \leq \mathcal{P}_\Gamma(w) \nu_v(\Gamma, w)(A).
\]

For ii), note that from the definition that each \( \nu_z(\Gamma, w) \) is a probability measure, hence (again using \(|\beta_u(z,v)| \leq \text{dist}(z,v)\) and (3))
\[
1 = \int_{\partial_{\infty} \mathbb{H}} d\nu_z(\Gamma, w)(u) = \mathcal{P}_\Gamma(w) \int_{\partial_{\infty} \mathbb{H}} d\nu_v(\Gamma, i)(u) = \mathcal{P}_\Gamma(w) \int_{\partial_{\infty} \mathbb{H}} e^{-\delta_1 \text{dist}(u,\Gamma \cdot i)} d\nu_v(\Gamma, i)(u),
\]
so
\[
e^{-\delta_1 \text{dist}(w,\Gamma \cdot i)} \mathcal{P}_\Gamma(w) \leq 1 \leq e^{\delta_1 \text{dist}(w,\Gamma \cdot i)} \mathcal{P}_\Gamma(w).
\]

Now using the \( \Gamma \)-invariance of \( \mathcal{P}_\Gamma \), we have
\[
\left( \inf_{\gamma \in \Gamma} e^{\delta_1 \text{dist}(w,\gamma i)} \right)^{-1} \mathcal{P}_\Gamma(w) \leq 1 \leq \left( \inf_{\gamma \in \Gamma} e^{\delta_1 \text{dist}(w,\gamma i)} \right) \mathcal{P}_\Gamma(w).
\]

Using (4) we obtain the following transformation rule:

**Lemma 8.** For a geometrically finite group \( \Gamma \subset G \) and \( g \in G \), the Patterson-Sullivan densities of \( \Gamma \) and \( g^{-1} \Gamma g \) satisfy
\[
\nu_z(g^{-1} \Gamma g, w) = (g^{-1})_\ast \nu_y(\Gamma, g^{-1}w) \quad \forall z, w \in \mathbb{H}.
\]

4.2. **Patterson-Sullivan measures on N-orbits.** For any \( g \in G \), recall that the forward and backward visual maps, \([g]^+\) and \([g]^−\), of \( g \) are defined by
\[
[g]^+ := \lim_{y \to \infty} g a_y \cdot i \in \partial_{\infty}\mathbb{H} \quad [g]^− := \lim_{y \to 0} g a_y \cdot i \in \partial_{\infty}\mathbb{H}.
\]

Let \( \Gamma g \in \Gamma \backslash G_{\text{rad}} \), that is \([g]^+ \in \Lambda_{\text{rad}}(\Gamma) \). The map from \( N \) to \( \Gamma \backslash G \) given by
\[
n \mapsto \Gamma gn \quad \forall n \in N
\]
is then injective. This allows us to “lift” measures in the Patterson-Sullivan density to a measure on \( \Gamma g N \subset \Gamma \backslash G \) by
\[
d\mu_{\Gamma g N}^PS(\Gamma gn) := e^{\delta_1 \text{dist}(g^{-1}n, i)} d\nu_z(\Gamma, i)([gn]^{-}) \quad \forall n \in N,
\]
where \( z \in \mathbb{H} \). Since \( \Gamma g N \leftrightarrow \mathbb{R} \), we may view this as a measure on \( \mathbb{R} \) (or \( N \)) via
\[
d\mu_{\Gamma g N}^PS(x) = d\mu_{\Gamma g N}^PS(n_x) := e^{\delta_1 \text{dist}(g^{-1}n, i)} d\nu_z(\Gamma, i)([gn_x]^{-}) = e^{\delta_1 \text{dist}(z,g^{-1}(z+x)) \nu_z(\Gamma, i)(g \cdot x)}.
\]
The properties in (3) show that \( \mu_{\Gamma g N}^PS \) is well-defined, i.e. independent of the chosen representative of \( \Gamma g \) and basepoint \( z \in \mathbb{H} \). Furthermore, by [28] Lemma 2.4, \( \mu_{\Gamma g N}^PS \) is an infinite measure (on \( \mathbb{R} \) alt. \( N \)). Recall that \( B_T = \{ t \in \mathbb{R} : |t| \leq T \} \).
Lemma 9.
\[ \mu_{(g^{-1}T\Gamma g)\in N}(B_T) = \frac{\mu_{(g^{-1}T\Gamma g)\in N}(B_T)}{\mathcal{P}_T(g \cdot i)} \quad \forall g \in G_{\text{rad}}, \ T \geq 0. \]

Proof. Using the definition of \( \mu_{(g^{-1}T\Gamma g)\in N} \), (3), (5), and Lemma 8 (as well as the fact that \( \delta_{g^{-1}T\Gamma g} = \delta_T \)),
\[
\mu_{(g^{-1}T\Gamma g)\in N}(B_T) = \int_T - T \mu_{(g^{-1}T\Gamma g)\in N}(x) = \int_T - T e^{\delta_T \beta_x(z,x+z)} dv_{g^{-1}T\Gamma g,i}(x) \\
= \int_T - T e^{\delta_T \beta_x(z,x+z)} d((g^{-1})_i \nu_2^{(T,g,i)})(x) = \mathcal{P}_T(g \cdot i) \int_T - T e^{\delta_T \beta_x(z,x+z)} dv_{g^{-1}T\Gamma g,i}^{(T,i)}(g \cdot x) \\
= \mathcal{P}_T(g \cdot i) \int_T - T e^{\delta_T \beta_x(z,x+z)} e^{-\delta_T \beta_g (z',z)} dv_{g^{-1}T\Gamma g,i}^{(T,i)}(g \cdot x).
\]

Since \( g \) acts as an isometry on \( \mathbb{H} \), \( \beta_x(z,x+z) = \beta_{g^{-1}x}(g \cdot z, g \cdot (x+z)) \) for all \( g \in G, \ z \in \mathbb{H} \), \( x \in \partial_\infty \mathbb{H} \). This, combined with the cocycle property of \( \beta \), gives
\[
\beta_x(z,x+z) - \beta_{g^{-1}x}(g \cdot z, z) = \beta_{g^{-1}x}(g \cdot z, g \cdot (x+z)) - \beta_{g^{-1}x}(g \cdot z, z) = \beta_{g^{-1}x}(z, g \cdot (x+z)),
\]
and so (once again using the definition of \( \mu_{(g^{-1}T\Gamma g)\in N} \))
\[
\frac{\mu_{(g^{-1}T\Gamma g)\in N}(B_T)}{\mathcal{P}_T(g \cdot i)} = \mathcal{P}_T(g \cdot i) \int_T - T e^{\delta_T \beta_x(z,x+z)} dv_{g^{-1}T\Gamma g,i}^{(T,i)}(g \cdot x) = \mathcal{P}_T(g \cdot i) \mu_{(g^{-1}T\Gamma g)\in N}(B_T).
\]

Remark 6. Observe that since \( \mathcal{P}_T(g \cdot i) = \mathcal{P}_T(g \cdot i) \), both sides of the equation in Lemma 9 are therefore independent of the representative chosen from \( \Gamma g \). We will henceforth also view \( \mathcal{P}_T \) as a function on \( \Gamma \backslash G \) by defining \( \mathcal{P}_T(\Gamma g) := \mathcal{P}_T(g \cdot i) \). Note that Lemma 8 (ii) then gives
\[
e^{-\delta_T \text{dist}_{\Gamma \backslash G}(\Gamma g, \Gamma e)} \leq \mathcal{P}_T(\Gamma g) \leq e^{\delta_T \text{dist}_{\Gamma \backslash G}(\Gamma g, \Gamma e)}.
\]

Lemma 8 will be used together with the following observation: if \( \infty \in \Lambda_{\text{rad}}(\Gamma) \), then \( \Gamma e \) is radial, and
\[
\mu_{e \in N}(B_T) = \int_T - T e^{\delta_T \beta_x(i,x+i)} dv_{i}^{(T,i)}(x) = \int_T - T (1 + z^2) e^{\delta_T} dv_{i}^{(T,i)}(x).
\]

We make one final observation regarding \( \mu_{(g^{-1}T\Gamma g)\in N} \), which is proved using calculations similar to those in the proof of Lemma 9.

Lemma 10. For all \( T > 0 \) and \( \mathcal{I} \subset \mathbb{R} \) measurable,
\[
\mu_{(g^{-1}T\Gamma g)\in N}(\mathcal{I}) = T^{\delta_T} \mu_{(g^{-1}T\Gamma g)\in N}(\mathcal{T}),
\]
where \( \mathcal{T} = \{ T \in \mathbb{R} : x \in \mathcal{I} \} \).

4.3. The Lebesgue density. In Sections 7.1 and 8 we will also require the Lebesgue density. This is a \( G \)-invariant density of dimension one, and denoted \( \{ m_z \}_{z \in \mathbb{H}} \). Each \( m_z \) is non-atomic, again allowing us to view them as measures on \( \mathbb{R} \). Defining a measure \( \tilde{\mu} \) on \( \mathbb{R} \) by
\[
d\tilde{\mu}(u) = (1 + u^2) dm_i(u) = e^{\beta_u(i,u+i)} dm_i(u) \quad \forall u \in \mathbb{R},
\]
we obtain that for all \( y \in \mathbb{R}_{>0}, \ x, u \in \mathbb{R} \),
\[
d\tilde{\mu}(yu + x) = d\tilde{\mu}(nxu^{-1}g \cdot u) = e^{\beta_u(nz^-1,i,yu+x+i)} dm_i(nz^-1u^{-1}g \cdot u) \\
= e^{\beta_u(nz^-1,i,yu+x+i)} dm_i(nz^-1u^{-1}g \cdot u) = e^{\beta_u(nz^-1,i,yu+x+i)} dm_i(u) \\
= e^{\beta_u(i,u+i)} dm_i(u) = y(1 + u^2) dm_i(u) = y d\tilde{\mu}(u).
\]

The measure \( \tilde{\mu} \) must therefore be a scalar multiple of the Lebesgue measure. This allows us to therefore assume that the density \( \{ m_z \}_{z \in \mathbb{H}} \) has been scaled so that \( dm_i(u) = \frac{du}{1+u^2} \).
4.4. The shadow lemma. We will use a version of Sullivan’s Shadow Lemma to obtain (both upper and lower) bounds for the $\nu^i_{\Gamma}(g)$-measures of certain subsets of $\partial_{\infty}\mathbb{H}$. We start by recalling the definition of the base eigenfunction $\phi_0 \in L^2(\Gamma \backslash G)$, cf. [32, 45]. This is a $\rho(K)$-invariant function in $C_{\infty} \cap S^\infty(\Gamma \backslash G)$ (cf. Proposition [1], and is given by the formula

$$\phi_0(\Gamma g) = N_G \int_{\partial_{\infty}\mathbb{H}} e^{-\delta r \beta_u(g \cdot i)} \nu^i_{\Gamma}(u),$$

where the constant $N_G \in \mathbb{R}_{>0}$ is chosen so that $\|\phi_0\|_{L^2(\Gamma \backslash G)} = 1$. Observe that $\phi_0(\Gamma g) > 0$ for all $g \in G$. Since $\phi_0 \in S^\infty(\Gamma \backslash G) \cap C_{\infty}$, by Lemma [6]

$$|\phi_0(\Gamma g)| \ll \|\phi_0\|_{S^\infty(\Gamma \backslash G)} \mathcal{Y}_T(\Gamma g)^{1-\delta r} \ll \mathcal{Y}_T(\Gamma g)^{1-\delta r},$$

(recall that $s_0 = \delta r$).

For $w \in \mathbb{H}$ and $r > 0$, let $B_r(w)$ denote the open (hyperbolic) ball of radius $r$ around $w$. Given another point $z \in \mathbb{H}$, we let $O_z(w, r) \subset \partial_{\infty}\mathbb{H}$ denote the shadow of $B_r(w)$ seen from $z$; this is the set of points $u \in \partial_{\infty}\mathbb{H}$ with the property that the geodesic segment from $z$ to $u$ intersects $B_r(w)$. Observe that since $G$ acts by isometry on $\mathbb{H}$, $g \cdot O_z(w, r) = O_{g.z}(g \cdot w, r)$.

We have the following result, due to Sullivan, cf. [47, Section 7]:

**Lemma 11.** For all $z, w \in \mathbb{H}$, $r > 0$,

$$\nu^i_{\Gamma}(O_z(w, r)) \ll \mathcal{Y}_T(\Gamma w)^{1-\delta r}. $$

**Proof.** Using (7), (3), and writing $w = x + iy$, we have

$$\phi_0(\Gamma n_x a_y) = N_T \int \mathbb{R} e^{-\delta r \beta_u(x+iy, i)} \nu^i_{\Gamma}(u) = N_T \int \mathbb{R} e^{-\delta r \beta_u(w, z)} \nu^i_{\Gamma}(u) \geq N_T \int_{O_z(w, r)} e^{\delta r \beta_u(z, w)} \nu^i_{\Gamma}(u).$$

Now, for all $u \in O_z(w, r)$,

$$\text{dist}(z, w) - 2r \leq \beta_u(z, w) \leq \text{dist}(z, w),$$

hence

$$\phi_0(\Gamma n_x a_y) \geq N_T \int_{O_z(w, r)} e^{\delta r \beta_u(z, w)} \nu^i_{\Gamma}(u) \geq N_T e^{\delta r (\text{dist}(z, w) - 2r)} \nu^i_{\Gamma}(O_z(w, r)).$$

By (6), we then have

$$\nu^i_{\Gamma}(O_z(w, r)) \leq N_T^{-1} e^{-\delta r (\text{dist}(z, w) - 2r)} \phi_0(\Gamma n_x a_y) \ll \mathcal{Y}_T(\Gamma w)^{1-\delta r}. $$

The following is a more or less straightforward consequence of Lemmas 6 and 11.

**Lemma 12.**

$$\nu^{(g^{-1} \Gamma g, i)}\{ x \in \mathbb{R} : |x| \geq T \} \ll \mathcal{P}_T(\Gamma g)T^{-\delta r} \mathcal{Y}_T(\Gamma ga_T)^{1-\delta r} \quad \forall g \in G, T \geq 1.$$

**Proof.** Observe that $O_i(iT, \text{arcsinh}(1)) = \{ x \in \mathbb{R} : |x| \geq T \} \cup \{ \infty \}$. By Lemmas 6 and 11 we have

$$\nu^{(g^{-1} \Gamma g, i)}\{ x \in \mathbb{R} : |x| \geq T \} = \nu^{(g^{-1} \Gamma g, i)}(O_i(iT, \text{arcsinh}(1)))$$

$$= (g^{-1} \nu^{(g, i)}_{\Gamma}(O_i(iT, \text{arcsinh}(1))) = \mathcal{P}_T(\Gamma g)\nu^{(i)}_{\Gamma}(g \cdot O_i(iT, \text{arcsinh}(1)))$$

$$= \mathcal{P}_T(\Gamma g)\nu^{(i)}_{\Gamma}(O_{g.i}(g \cdot iT, \text{arcsinh}(1)))$$

$$\ll \mathcal{P}_T(\Gamma g)e^{-\delta r (\text{dist}(g^{-1}g, g \cdot iT))} \mathcal{Y}_T(\Gamma g)^{1-\delta r}$$

$$= \mathcal{P}_T(\Gamma g)e^{-\delta r \text{dist}(\Gamma g, \Gamma g a_T)^{1-\delta r}}.$$
Lemma 13.
\[
\nu_i^{(g^{-1}G,g,i)}\{u \in \mathbb{R} : (1 - \epsilon)T \leq |u| \leq (1 + \epsilon)T\} \ll T \gamma_T(\Gamma g a_T)^{1 - \delta_T}
\]
for all \(g \in G\), \(T \geq 2\), \(0 \leq \epsilon \leq \frac{1}{2}\).

Proof. We prove the bound for the interval \([(1 - \epsilon)T, (1 + \epsilon)T]\); the negative interval is dealt with in a completely symmetric manner. Given \(r > 0\) such that
\[
(1)\quad [(1 - \epsilon)T, (1 + \epsilon)T] \subset O_i(T + i\epsilon T, r),
\]
by Lemmas 8 and 11, we then have
\[
\nu_i^{(g^{-1}G,g,i)}\{(1 - \epsilon)T, (1 + \epsilon)T\} \leq \nu_i^{(g^{-1}G,g,i)}(O_i(T + i\epsilon T, r)) = (g^{-1})_i \nu_i^{(G,g,i)}(O_i(T + i\epsilon T, r))
\]
\[
= \mathcal{P}(\Gamma g) \nu_i^{(G,g,i)}(g \cdot O_i(T + i\epsilon T, r)) = \mathcal{P}(\Gamma g) \nu_i^{(G,g,i)}(O_i(g \cdot (T + i\epsilon T), r))
\]
\[
\ll \gamma_T(\Gamma g a_T)^{2\delta_T - \delta_i \text{dist}(g \cdot (T + i\epsilon T), r)} Y_T(g \cdot (T + i\epsilon T))^{1 - \delta_i}
\]
\[
= \mathcal{P}(\Gamma g) e^{2\delta_T - \delta_i \text{dist}(i, T + i\epsilon T)} Y_T(\Gamma g a_T n_1 a_e)^{1 - \delta_i}.
\]
By (1) and (2) of Proposition 3, \(Y_T(\Gamma g a_T n_1 a_e)^{1 - \delta_i} \ll e^{\delta_i - 1} Y_T(\Gamma g a_T)^{1 - \delta_i}\). Furthermore,
\[
e^{-\delta_i \text{dist}(i, T + i\epsilon T)} = \left(\frac{\sqrt{T^2 + (\epsilon T - 1)^2} + \sqrt{T^2 + (\epsilon T + 1)^2}}{2\epsilonT}\right)^{-2\delta_T} \leq \left(\frac{T}{\sqrt{cT}}\right)^{-2\delta_T} = e^{\delta_i T - \delta_T}.
\]
We thus have
\[
\nu_i^{(g^{-1}G)}\{(1 - \epsilon)T, (1 + \epsilon)T\} \ll e^{2\delta_T - \delta_i \text{dist}(i, T + i\epsilon T)} e^{2\delta_T - 1} Y_T(\Gamma g a_T)^{1 - \delta_i}.
\]
In order to complete the proof, we need to find an \(r > 0\) satisfying (3). Observe that \(B_r(T + i\epsilon T)\) is a Euclidean ball centred at \(T + i \cosh(r)\epsilon T\) with radius \(\sinh(r)\epsilon T\). The points on the geodesic rays from \(i\) to \((1 \pm \epsilon)T\) are given by
\[
G^\pm_{T,\epsilon} \left\{ z \in \mathbb{H} : \left| z - \frac{T(1 \pm \epsilon) - i\epsilon T}{2} \right| = \frac{T(1 \pm \epsilon) + i\epsilon T}{2} \right\},
\]
respectively. If \(G^\pm_{T,\epsilon}\) have non-empty intersections with \(B_r(T + i\epsilon T)\), then \([T(1 - \epsilon), T(1 + \epsilon)]\) is contained in \(O_i(T + i\epsilon T, r)\), i.e. if the following two inequalities are satisfied:
\[
\left| T + i\epsilon T \cosh(r) - \frac{T(1 - \epsilon) - i\epsilon T}{2} \right| < \epsilon T \sinh(r) + \frac{T(1 - \epsilon) + i\epsilon T}{2},
\]
\[
\left| T + i\epsilon T \cosh(r) - \frac{T(1 + \epsilon) - i\epsilon T}{2} \right| + \epsilon T \sinh(r) > \frac{T(1 + \epsilon) + i\epsilon T}{2}.
\]
These inequalities are fulfilled if
\[
1 - \frac{2\epsilon(1 + \epsilon)}{T^2(1 \pm \epsilon)^2} < \sinh(r),
\]
so taking \(r = \arcsinh(\frac{\epsilon}{2})\) suffices for all relevant \(T\) and \(\epsilon\).
\(\square\)

Since we normalize the integral over \(B_T\) in Theorem 1 by \(\nu_i^{PS}(B_T)\), we will require a lower bound on \(\nu_i^{PS}(B_T)\).

We first introduce some more notation: for \(u \in \partial_\infty \mathbb{H}\) and \(t \geq 0\), let \(h_t(u)\) be the point on the geodesic segment from \(i\) tending to \(u\) at distance \(t\) from \(i\). Let \(S(u, t) \subset \partial_\infty \mathbb{H}\) denote the set of points whose orthogonal projection onto the geodesic from \(i\) to \(u\) lie between \(h_t(u)\) and \(u\). Observe that since \(K = \text{Stab}_C(i)\), we have \(k \cdot h_t(u) = h_t(k \cdot u)\) and \(k \cdot S(u, t) = S(k \cdot u, t)\) for all \(k \in K\) and \(u \in \partial_\infty \mathbb{H}\).
**Theorem 14.** (cf. [24, Theorem 2], [36, Theorem 3.2]) There exist $0 < c_0 < c_1$ such that

$$c_0 e^{-\delta r} \mathcal{Y}_r(h_t(\eta))^{1-\delta r} \leq \nu_i^{(\Gamma, i)}(S(\eta, t)) \leq c_1 e^{-\delta r} \mathcal{Y}_r(h_t(\eta))^{1-\delta r} \quad \forall t \geq 0, \eta \in \Lambda.$$

**Remark 7.** Here we have simply used Proposition 3 (5) to simply express the results from [36, 42] using the invariant height function.

**Proposition 15.** There exist continuous functions $C_r, D_r : G \to \mathbb{R}_{>0}$ such that

$$\mu_{\Gamma \cap \Delta N}(B_T) \geq C_r(g) T^{\delta r} \mathcal{Y}_r(\Gamma \cap \Delta T)^{1-\delta r}$$

for all $g \in G_{\text{rad}}$ and $T \geq D_r(g)$.

**Proof.** Using Lemma 8, we have

$$\mu_{\Gamma \cap \Delta N}(B_T) = \frac{\mu_{\Gamma \cap \Delta N}(B_T)}{P_r(\Gamma g)}.$$

Now, $\infty \in \Lambda_{\text{rad}}(g^{-1} \Gamma g)$, so

$$\mu_{\Gamma \cap \Delta N}(B_T) = \int_{\{u : \frac{T}{2} \leq |u| \leq T\}} (1 + u^2)^{\delta r} \, d\nu_i^{(g^{-1} \Gamma g, i)}(u).$$

We now choose some $R \geq 2$ (depending on $\Gamma g$ and later to be specified further), and note that by (8)

$$\mu_{(g^{-1} \Gamma g) \cap N}(B_T) \geq \int_{\{u : \frac{R}{2} \leq |u| \leq T\}} (1 + u^2)^{\delta r} \, d\nu_i^{(g^{-1} \Gamma g, i)}(u)$$

$$\geq \left(1 + \left(\frac{T}{R}\right)^2\right)^{\delta r} \int_{\{u : \frac{R}{2} \leq |u| \leq T\}} d\nu_i^{(g^{-1} \Gamma g, i)}(u)$$

$$= \left(1 + \left(\frac{T}{R}\right)^2\right)^{\delta r} \left(\nu_i^{(g^{-1} \Gamma g, i)}(\{u \in \mathbb{R} : |u| \geq \frac{T}{R}\}) - \nu_i^{(g^{-1} \Gamma g, i)}(\{u \in \mathbb{R} : |u| \geq T\})\right).$$

Let $g = k a_y n_x$. Then by Lemma 8 for any $S \geq 1$, we have

$$\nu_i^{(g^{-1} \Gamma g, i)}(\{u \in \mathbb{R} : |u| \geq S\}) = \mathcal{P}_r(\Gamma g) \nu_i^{(\Gamma, i)}(k \cdot \{y(x + u) \in \mathbb{R} : |u| \geq S\}).$$

Assuming

$$y(x - S) \leq -1 < 1 \leq y(x + S)$$

(i.e. $|x| \leq S - \frac{1}{2}$), we let

$$S_- := \min\{|y(x - S)|, |y(x + S)|\} = y(S - |x|), \quad S_+ := \max\{|y(x - S)|, |y(x + S)|\} = y(S + |x|).$$

We then have

$$k \cdot \{u \in \mathbb{R} : |u| \geq S_+\} \subset g \cdot \{u \in \mathbb{R} : |u| \geq S\} \subset k \cdot \{u \in \mathbb{R} : |u| \geq S_-\}.$$

Observe now that $k \cdot \infty = [g]^+ \in \Lambda_{\text{rad}}(\Gamma)$. Furthermore, $\{u \in \mathbb{R} : |u| \geq S\} = S_i(\infty, \log S)$ (for all $S \geq 1$), hence

$$k \cdot \{u \in \mathbb{R} : |u| \geq S_\pm\} = k \cdot S_i(\infty, \log S_\pm) = S_{k \cdot \infty}(\log S_\pm).$$

Returning to (11), we now have

$$\mathcal{P}_r(\Gamma g) \nu_i^{(\Gamma, i)}(S_i([g]^+, \log S_+)) \leq \nu_i^{(g^{-1} \Gamma g, i)}(\{u \in \mathbb{R} : |u| \geq S\}) \leq \mathcal{P}_r(\Gamma g) \nu_i^{(\Gamma, i)}(S_i([g]^+, \log S_-)),$$

and so Lemma 7 gives

$$\mathcal{P}_r(\Gamma g) e^{-\delta r \cdot \text{dist}(g^{-1}, i)} \nu_i^{(\Gamma, i)}(S_i([g]^+, \log S_+)) \leq \nu_i^{(g^{-1} \Gamma g, i)}(\{u \in \mathbb{R} : |u| \geq S\}) \leq \mathcal{P}_r(\Gamma g) e^{\delta r \cdot \text{dist}(g^{-1}, i)} \nu_i^{(\Gamma, i)}(S_i([g]^+, \log S_-)).$$
Keeping the notation \( g = ka_{g}n_{x} \), we assume that \( \frac{T}{n} \) satisfies the conditions placed on the variable \( S \) in (12). Note that \( T \) then also fulfills these assumptions. Combining (13), (11), and (10), we have

\[
\mu_{y \ast n_{y}}^{\text{PS}}(B_{T}) \geq (1 + (\frac{T}{n})^2)^{\delta_{T}} \left( e^{-\delta_{T} \cdot \text{dist}(g \ast i, i)} \left( \mathcal{S}_{i}([g]^{+}, \log(\frac{T}{n})^{+}) \right) 
- e^{\delta_{T} \cdot \text{dist}(g \ast i, i)} \left( \mathcal{S}_{i}([g]^{-}, \log(T)^{-}) \right) \right),
\]

where

\[
\{ \frac{T}{n} \}^{+} = y(\frac{T}{n} + |x|)
\{ T \}^{-} = y(T - |x|).
\]

Now let \( 0 < c_{0} < c_{1} \) be the constants from Theorem 14. Using both the upper and lower bounds from the same theorem, we obtain

\[
\mu_{y \ast n_{y}}^{\text{PS}}(B_{T}) \geq (1 + (\frac{T}{n})^2)^{\delta_{T}} \left( c_{0} e^{-\delta_{T} \cdot \text{dist}(g \ast i, i)} \left( \{ \frac{T}{n} \}^{+} \right) \right)
- c_{1} e^{\delta_{T} \cdot \text{dist}(g \ast i, i)} \left( \{ T \}^{-} \right).
\]

Since \([g]^{+} = k \cdot \infty, h_{i}([g]^{+}) = ka_{g} \ast i \), and so

\[
h_{i} \log(\frac{T}{n})^{+}([g]^{+}) = ka_{g} \log(\frac{T}{n})^{+} \ast i = ga_{T}n_{x}a_{T}^{-1}a_{\{ \frac{T}{n} \}^{+}} \ast i = ga_{T}n_{x}a_{T}^{-1}a_{\{ \frac{T}{n} \}^{+}} \ast (yT) \ast i.
\]

By Proposition 3 (1) and (2), for all \( Y > 0 \),

\[
\frac{\mathcal{Y}_{i}(\Gamma g a_{T})}{(1 + \frac{|x|}{T})^{2} \max\{ \frac{Y}{yT}, \frac{yT}{T} \}} \leq \mathcal{Y}_{i}(ga_{T}n_{x}a_{(yT) \ast i}) \leq \mathcal{Y}_{i}(\Gamma g a_{T})(1 + \frac{|x|}{T})^{2} \max\{ \frac{Y}{yT}, \frac{yT}{T} \}.
\]

In particular,

\[
\mathcal{Y}_{i}(h_{i} \log(\frac{T}{n})^{+}([g]^{+})) \geq \frac{\mathcal{Y}_{i}(\Gamma g a_{T})}{(1 + \frac{|x|}{T})^{2} \max\{ \frac{\{ \frac{T}{n} \}^{+}}{yT}, \frac{yT}{\{ \frac{T}{n} \}^{+}} \}},
\]

and

\[
\mathcal{Y}_{i}(h_{i} \log(\{ T \}^{-}([g]^{+})) \leq \mathcal{Y}_{i}(\Gamma g a_{T})(1 + \frac{|x|}{T})^{2} \max\{ \frac{\{ T \}^{-}}{yT}, \frac{yT}{\{ T \}^{-}} \}.
\]

Using these bounds in (14), we have

\[
\mu_{y \ast n_{y}}^{\text{PS}}(B_{T}) \geq T^{\delta_{T}} \mathcal{Y}_{i}(\Gamma g a_{T})^{1-\delta_{T}} \times (*),
\]
where “(*)” equals

$$\frac{T^\delta r}{R^{2\delta r}} \left( \frac{c_0 e^{-\delta r \text{dist}(g_{i,i})} \left( \frac{T}{R} \right)^{-\delta r}}{(1 + \frac{|x|}{T})^{2-2\delta r}} \max \left\{ \frac{T}{yT}, \frac{T}{yT} \right\}^{1-\delta r} \right)$$

$$- c_1 e^{\delta r \text{dist}(g_{i,i})} \left( \frac{T}{yT} \right)^{\delta r} \max \left\{ \frac{T}{yT}, \frac{T}{yT} \right\}^{1-\delta r}$$

(15)

$$= \frac{e^{\delta r \text{dist}(g_{i,i})}}{R^{2\delta r}} \left( \frac{c_0 e^{-\delta r \text{dist}(g_{i,i})} \left( \frac{T}{R} \right)^{-\delta r}}{(1 + \frac{|x|}{T})^{2-2\delta r}} \max \left\{ \frac{T}{yT}, \frac{T}{yT} \right\}^{1-\delta r} \right)$$

$$- c_1 \left( \frac{T}{yT} \right)^{\delta r} \max \left\{ \frac{T}{yT}, \frac{T}{yT} \right\}^{1-\delta r}.$$
Observe that $e^{\delta r \text{dist}(g \cdot i)} \geq y^{\delta r}$, hence
\[
\left( \frac{e^{\delta r \text{dist}(g \cdot i)2^{-2\delta r}c_1}}{y^{\delta r} (2 + R_0)^{2\delta r}} \right) \geq \frac{1}{(2 + R_0)^{2\delta r}}
\]
\[
\geq \left( 2 + \frac{e^{6-4\delta r}c_1}{c_0} \right)^{\frac{1}{2\delta r-1}} \frac{1}{e^{2\delta r-1}2^{\delta r} \text{dist}(g \cdot i)} \geq \gamma e^{-\frac{4\delta ^2}{2\delta r-1} \text{dist}(g \cdot i)}
\]
giving
\[
\mu_{\Gamma \cap N}^{PS}(B_T) \gg \gamma e^{-\frac{4\delta ^2}{2\delta r-1} \text{dist}(g \cdot i)} T^{\delta r} \mathcal{Y}_T(\Gamma g a_T)^{1-\delta r}.
\]
This bound is proved under the assumption $\frac{T}{R} \geq |x| + \frac{1}{y}$ (cf. [22]), i.e.
\[
T \geq (|x| + \frac{1}{y}) \left( 2 + \left( \frac{e^{6-4\delta r}c_1}{c_0} \right)^{\frac{1}{2\delta r-1}} \frac{1}{e^{2\delta r-1}2^{\delta r} \text{dist}(g \cdot i)} \right).
\]

**Corollary 16.** Let $\Omega \subset \Gamma \setminus G$ be compact. Then
\[
\mu_{\Gamma \cap N}^{PS}(B_T) \gg \Omega T^{\delta r} \mathcal{Y}_T(\Gamma g a_T)^{1-\delta r} \quad \forall \Gamma g \in \Omega \cap \Gamma \setminus G_{\text{rad}}, \ T \gg 1.
\]

5. **Effective Equidistribution of the Base Eigenfunctions**

We will now prove the effective equidistribution of the base eigenfunctions $\phi_n \ (n \in \mathbb{Z})$. Recall that each $\phi_n$ is a unit vector in $L^2(\Gamma \setminus G)$ of $K$-type $2n$. As a starting point, we will use expressions for the $\phi_n$ in terms of integrals against a measure in the Patterson-Sullivan density. The explicit formulas we need have been developed by Lee and Oh in [22 Section 3]. For $\theta \in \mathbb{R}/\pi \mathbb{Z}$, let $k_\theta = \left( \frac{-\sin \theta}{\cos \theta \cos \theta} \right)$.

**Proposition 17.** ([22] Theorem 3.3])
\[
\phi_n(\Gamma g x a_g k_\theta) = \frac{N_{\Gamma}}{\mathcal{P}_\Gamma(\Gamma g)} \frac{e^{2i n \theta}}{\sqrt{\Gamma(\theta)^{1+|n|+1-\delta r}}} \int_{\mathbb{R}} \left( \frac{u^{2+i} + 1}{\Gamma(\theta)^{1+|n|+1-\delta r}} \right) \Gamma \mu_i(\Gamma g i)(u)
\]
for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $y > 0$, $\theta \in \mathbb{R}/\pi \mathbb{Z}$.

**Remark 8.** The constant $N_{\Gamma}$ (cf. [17]) does not appear in the formula given in [22]. This is due to the fact that we require $\nu_i(\Gamma g i)$ to be a probability measure, whereas this is not the case in [22]. We thus obtain that “$\nu_j$” in [22] equals our $N_{\Gamma} \nu_i(\Gamma g i)$.

**Corollary 18.**
\[
\phi_n(\Gamma g x a_g k_\theta) = \frac{N_{\Gamma} e^{2i n \theta}}{\mathcal{P}_\Gamma(\Gamma g)} \frac{1}{\sqrt{\Gamma(\theta)^{1+|n|+1-\delta r}}} \int_{\mathbb{R}} \left( \frac{u^{2+i} + 1}{\Gamma(\theta)^{1+|n|+1-\delta r}} \right) \Gamma \mu_i(\Gamma g i)(u)
\]
for all $g \in G$, $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $y > 0$, $\theta \in \mathbb{R}/\pi \mathbb{Z}$.

**Proof.** For all $g, h \in G$, using [13] and Lemma [5] we have
\[
\phi_0(\Gamma g h) = \frac{N_{\Gamma}}{\mathcal{P}_\Gamma(\Gamma g)} \int \frac{e^{-\delta r \beta_i(\text{gh}h \cdot i)} d\mu_i(g \cdot i)}{\Gamma(\Gamma g)}(u) = \frac{N_{\Gamma}}{\mathcal{P}_\Gamma(\Gamma g)} \int \frac{e^{-\delta r \beta_i(\text{gh}h \cdot i)} e^{-\delta r \beta_i(\text{gh}h \cdot i)} d\mu_i(g \cdot i)}{\Gamma(\Gamma g)}(u)
\]
\[
= \frac{N_{\Gamma}}{\mathcal{P}_\Gamma(\Gamma g)} \int \frac{e^{-\delta r \beta_i(\text{gh}h \cdot i)} d\mu_i(g \cdot i)}{\Gamma(\Gamma g)}(u) = \frac{N_{\Gamma}}{\mathcal{P}_\Gamma(\Gamma g)} \int \frac{e^{-\delta r \beta_i(\text{gh}h \cdot i)} d\mu_i(g \cdot i)}{\Gamma(\Gamma g)}(g \cdot u)
\]
For $h = n_x a_g k_\theta$, $e^{-\delta r \beta_i(n_x a_g k_\theta \cdot i)} = \left( \frac{(u^{2+i} + 1)}{\Gamma(\theta)^{1+|n|+1-\delta r}} \right) \Gamma \mu_i(\Gamma g i)$, so the formula holds for $n = 0$. Following the proof of [22] Theorem 3.3], the remaining cases follow from applying the raising and lowering operators to the function $h \mapsto e^{-\delta r \beta_i(h \cdot i)}$ on $G$. \qed
It follows from the formulas above that $|\phi_n(\Gamma g)| \ll \phi_0(\Gamma g)$ for all $n \in \mathbb{Z}, \, g \in G$.

Before stating the main result of this section, we make some auxiliary definitions: let

$$c_n(\delta_T) := \frac{\sqrt{T(1-\delta_T)iN(|n|+\delta_T)}}{\sqrt{T(\delta_T)I(|n|+1-\delta_T)}} \quad \forall n \in \mathbb{Z},$$

$$\psi_n(t) := \left(\frac{1}{t^2 + 1}\right)^{\delta_T} \left(\frac{t - i}{t + i}\right)^n \quad \forall t \in \mathbb{R}, \, n \in \mathbb{Z},$$

and

$$\kappa_n(\delta_T) := \int_{\mathbb{R}} \psi_n(t) \, dt = \frac{4^{1-\delta_T} \pi (-1)^n \Gamma(2\delta_T - 1)}{\Gamma(\delta_T + n) \Gamma(\delta_T - n)} = \frac{4^{1-\delta_T} \pi \Gamma(2\delta_T - 1) \Gamma(|n| + 1 - \delta_T)}{\Gamma(\delta_T) \Gamma(1 - \delta_T) \Gamma(|n| + \delta_T)} = \frac{\kappa_0(\delta_T) \Gamma(|n| + 1 - \delta_T)}{\Gamma(1 - \delta_T) \Gamma(|n| + \delta_T)} = \frac{\kappa_0(\delta_T)}{c_n(\delta_T)^2} \quad \forall n \in \mathbb{Z}.$$

Observe that $c_n(\delta_T) \ll \kappa_0(\delta_T)$ and $|\kappa_n(\delta_T)| \leq \kappa_0(\delta_T)$. Using the $c_n$s and $\kappa_n$s, we define the following functional on $S^1(\Gamma \backslash G)$:

$$\mathcal{M}_T(f) := \sum_{n \in \mathbb{Z}} \mathcal{N}_T c_n(\delta_T) \kappa_n(\delta_T) \langle f, \phi_n \rangle_{L^2(\Gamma \backslash G)} = \sum_{n \in \mathbb{Z}} \frac{\mathcal{N}_T \kappa_0(\delta_T)}{c_n(\delta_T)} \langle f, \phi_n \rangle_{L^2(\Gamma \backslash G)} \quad \forall f \in S^1(\Gamma \backslash G).$$

We also have the following basic fact that will be used without comment throughout the proof of the main result of this section:

**Lemma 19.**

$$\int_{-R}^R \psi_n(t) \, dt = \kappa_n(\delta_T) + O_{\delta_T}(R^{1-2\delta_T}) \quad \forall R > 0$$

and

$$\int_{\{|t| \geq R\}} \psi_n(t) \, dt = O_{\delta_T}(R^{1-2\delta_T}) \quad \forall R > 0.$$

Both implied constants are independent of $n$.

We now come to the main result of this section, which is essentially an effective equidistribution statement for the base eigenfunctions:

**Theorem 20.** For all $\Gamma g \in \Gamma \backslash G_{\text{rad}}, \, T \geq 4, \, n \in \mathbb{Z},$

$$\int_{-T}^T \phi_n(\Gamma g n) \, dt = \mu_{g,T}^{PS}(B_T) \mathcal{N}_T c_n(\delta_T) \kappa_n(\delta_T) + O_T \left(\frac{\mu_{g,T}^{PS}(B_T)}{N(\Gamma g n)} T^{\frac{3}{2} - \delta_T} + \mathcal{Y}_T(\Gamma g a_T)^{-\delta_T} T^{\frac{3}{2}}\right).$$

**Proof.** Using Corollary 18, Lemma 9 and 10, we have

$$\int_{-T}^T \phi_n(\Gamma g n) \, dt - \mathcal{N}_T c_n(\delta_T) \kappa_n(\delta_T) \mu_{g,T}^{PS}(B_T)$$

$$= \frac{\mathcal{N}_T c_{\kappa}(n)}{\mathcal{P}_T(\Gamma g)} \left( \int_{-T}^T \int_{\mathbb{R}} \left(\frac{u^2 + 1}{(t-u)^2 + 1}\right)^{\delta_T} \left(\frac{t-n+i}{t+n+i}\right)^n \, du \, d\psi_i(t-u) \, dt \right)$$

$$- \kappa_n(\delta_T) \int_{-T}^T (1 + u^2)^{\delta_T} \, d\psi_i(t-u)$$

$$= \frac{\mathcal{N}_T c_{\kappa}(n)}{\mathcal{P}_T(\Gamma g)} \left( \int_{-T}^T \psi_n(t-u) \, dt - \kappa_n(\delta_T) 1_{[-T,T]}(u) \right) (1 + u^2)^{\delta_T} \, d\psi_i(t-u) $$

$$= \frac{\mathcal{N}_T c_{\kappa}(n)}{\mathcal{P}_T(\Gamma g)} \left( \int_{-T}^T \psi_n(t) \, dt - \kappa_n(\delta_T) 1_{[-T,T]}(u) \right) (1 + u^2)^{\delta_T} \, d\psi_i(t-u) $$

where $\mathcal{P}_T(\Gamma g)$ is the constant associated with the measure $\mu_{g,T}^{PS}(B_T)$.
We bound each of these four integrals in turn:

I: \{u : |u| \leq (1 - \epsilon)T\}. Since \(1_{[-T,T]}(u) \equiv 1\), the integral we are interested in is

\[
\int_{-(1-\epsilon)T}^{(1-\epsilon)T} \psi_n(t) dt - \kappa_n(\delta_T) \left(1 + u^2\right) \delta_T \nu_i^{(g^{-1} \Gamma_{g,i})}(u).
\]

Using \(|u| \leq (1 - \epsilon)T\),

\[
\int_{-u-T}^{T-u} \psi_n(t) dt = \int_{-\epsilon T}^{\epsilon T} \psi_n(t) dt + \int_{\epsilon T}^{T-\epsilon T} \psi_n(t) dt + \int_{T-\epsilon T}^{-u-T} \psi_n(t) dt = \kappa_n(\delta_T) + O((\epsilon T)^{1-2\delta_T}),
\]

hence

\[
\tag{18}
\frac{\mathcal{N}_{\Gamma g,i}(n)}{\mathcal{P}_T(\Gamma g)} \int_{-(1-\epsilon)T}^{(1-\epsilon)T} \left(\int_{-u-T}^{T-u} \psi_n(t) dt - \kappa_n(\delta_T) \right) \left(1 + u^2\right) \delta_T \nu_i^{(g^{-1} \Gamma_{g,i})}(u)
\]

\[
= O_T \left(\frac{(T\epsilon)^{1-2\delta_T}}{\mathcal{P}_T(\Gamma g)} \int_{-(1-\epsilon)T}^{(1-\epsilon)T} \left(1 + u^2\right) \delta_T \nu_i^{(g^{-1} \Gamma_{g,i})}(u)\right)
\]

\[
= O_T \left(\frac{(T\epsilon)^{1-2\delta_T}}{\mathcal{P}_T(\Gamma g)} \mu_{PS}^{(g^{-1} \Gamma_{g,N})}(B_{(1-\epsilon)T})\right)
\]

\[
= O_T \left(\frac{(T\epsilon)^{1-2\delta_T}}{\mathcal{P}_T(\Gamma g)} \mu_{PS}^{(g^{-1} \Gamma_{g,N})}(B_{(1-\epsilon)T})\right)
\]

where Lemma[9] and [10] were again used.

II: \{u : (1 - \epsilon)T \leq |u| \leq (1 + \epsilon)T\}. Here we use the bound

\[
\left| \int_{-u-T}^{T-u} \psi_n(t) dt - \kappa_n(\delta_T) 1_{[-T,T]}(u) \right| \leq 2\kappa_0(\delta_T) = O_T(1).
\]

Assuming \(\epsilon \leq \frac{1}{2}\), we now use Proposition [13]

\[
\tag{19}
\frac{\mathcal{N}_{\Gamma g,i}(n)}{\mathcal{P}_T(\Gamma g)} \int_{\{u : (1-\epsilon)T \leq |u| \leq (1+\epsilon)T\}} \left(\int_{-u-T}^{T-u} \psi_n(t) dt - \kappa_n(\delta_T) \right) \left(1 + u^2\right) \delta_T \nu_i^{(g^{-1} \Gamma_{g,i})}(u)
\]

\[
= O_T \left(\frac{T^{2\delta_T}}{\mathcal{P}_T(\Gamma g)} \nu_i^{(g^{-1} \Gamma_{g,i})}(\{u \in \mathbb{R} : (1 - \epsilon)T \leq |u| \leq (1 + \epsilon)T\})\right)
\]

\[
= O_T \left(\frac{T^{2\delta_T}}{\mathcal{P}_T(\Gamma g)} \nu_i^{(g^{-1} \Gamma_{g,i})}(\{u \in \mathbb{R} : (1 - \epsilon)T \leq |u| \leq (1 + \epsilon)T\})\right)
\]

III: \{u : (1 + \epsilon)T \leq |u| \leq 2T\}. For \(u\) in this range we have

\[
\left| \int_{-u-T}^{T-u} \psi_n(t) dt - \kappa_n(\delta_T) 1_{[-T,T]}(u) \right| = \left| \int_{-u-T}^{T-u} \psi_n(t) dt \right|
\]

\[
\leq \int_{-u-T}^{T-u} \psi_0(t) dt = \int_{|u|-T}^{T+|u|} \psi_0(t) dt \leq \int_{\{u \geq T\}} \psi_0(t) dt = O((\epsilon T)^{1-2\delta_T}).
\]
Lemma 2 gives

\[
(20) \quad \frac{N_{\Gamma}c_{\delta}(n)}{p_{\Gamma}(\Gamma g)} \int_{\{u:(1+\epsilon)|T|\leq |u|<2T\}} \left( \int_{-u-T}^{T} \psi_n(t) dt - \kappa_n(\delta_T) \right) (1 + u^2)^{\delta_T} d\nu_i^{(g^{-1}\Gamma g_i)}(u) = O_T \left( \frac{1}{p_{\Gamma}(\Gamma g)} T^{2\delta_T} (eT)^{1-2\delta_T} \nu_i^{(g^{-1}\Gamma g_i)}(\{x \in \mathbb{R} : |x| \geq (1 + \epsilon)T\}) \right) \\
= O_T \left( \frac{1}{p_{\Gamma}(\Gamma g)} T e^{1-2\delta_T} \nu_i^{(g^{-1}\Gamma g_i)}(\{x \in \mathbb{R} : |x| \geq T\}) \right) \\
= O_T \left( T^{1-\delta_T} e^{1-2\delta_T} \gamma_T(\Gamma g a_T)^{1-\delta_T} \right).
\]

IV: \{u : |u| \geq 2T\}. For the final integral, we use dyadic decomposition:

\[
\frac{N_{\Gamma}c_{\delta}(n)}{p_{\Gamma}(\Gamma g)} \int_{\{u: |u| \geq 2T\}} \left( \int_{-u-T}^{T} \psi_n(t) dt - \kappa_n(\delta_T) \right) (1 + u^2)^{\delta_T} d\nu_i^{(g^{-1}\Gamma g_i)}(u) \\
= \frac{N_{\Gamma}c_{\delta}(n)}{p_{\Gamma}(\Gamma g)} \sum_{m=1}^{\infty} \int_{\{u: 2^m T \leq |u| < 2^{m+1} T\}} \left( \int_{-u-T}^{T} \psi_n(t) dt (1 + u^2)^{\delta_T} d\nu_i^{(g^{-1}\Gamma g_i)}(u) \right) \\
= O_T \left( \frac{1}{p_{\Gamma}(\Gamma g)} \sum_{m=1}^{\infty} (T 2^m)^{2\delta_T} \int_{\{u: 2^m T \leq |u| < 2^{m+1} T\}} \left( \int_{-u-T}^{T} \psi_n(t) dt d\nu_i^{(g^{-1}\Gamma g_i)}(u) \right) \right).
\]

For \(u\) such that \(2^m \leq |u| \leq 2^{m+1}\), \(m \geq 1\), we have

\[
\int_{-u-T}^{T} \psi_n(t) dt \leq \frac{T}{(1 + \min_{t \in [-T,T]} |t|^2)^{\delta_T}} \leq \frac{T}{(2^{m+1} - 1)^{2\delta_T}} \leq T^{1-2\delta_T} 2^{-2\delta_T(m-1)} \ll T^{1-2\delta_T} 2^{-2\delta_T m},
\]

so

\[
\sum_{m=1}^{\infty} (T 2^m)^{2\delta_T} \int_{\{u: 2^m T \leq |u| < 2^{m+1} T\}} \left( \int_{-u-T}^{T} \psi_n(t) dt d\nu_i^{(g^{-1}\Gamma g_i)}(u) \right) = O \left( T \nu_i^{(g^{-1}\Gamma g_i)}(\{x \in \mathbb{R} : |x| \geq 2T\}) \right).
\]

We use Lemma 2 again to obtain

\[
(21) \quad \frac{N_{\Gamma}c_{\delta}(n)}{p_{\Gamma}(\Gamma g)} \int_{\{u: |u| \geq 2T\}} \left( \int_{-u-T}^{T} \psi_n(t) dt - \kappa_n(\delta_T) \right) (1 + u^2)^{\delta_T} d\nu_i^{(g^{-1}\Gamma g_i)}(u) \\
= O_T \left( T^{1-\delta_T} \gamma_T(\Gamma g a_T)^{1-\delta_T} \right).
\]

Combining (18), (19), (20), and (21) gives

\[
\int_{-T}^{T} \phi_n(\Gamma gn) dt = \mu_{T g N}(B_T) N_{\Gamma}c_{\delta}(\delta_T) \kappa_n(\delta_T) \\
+ O_T \left( (T e)^{1-2\delta_T} \mu_{T g N}(B_T) + \gamma_T(\Gamma g a_T)^{1-\delta_T} (T^{\delta_T} e^{2\delta_T-1} + T^{1-\delta_T} e^{1-2\delta_T} + T^{1-\delta_T}) \right).
\]

Now choosing \(\epsilon = T^{-\frac{1}{2}}\) completes the proof (this is permitted since \(T \geq 4\), and the only requirement placed on \(\epsilon\) is \(0 < \epsilon \leq \frac{1}{2}\)).

\[\square\]

**Corollary 21.** Let \(\Omega \subset \Gamma \backslash G\) be compact. Then

\[
\frac{1}{\mu_{T g N}(B_T)} \int_{-T}^{T} \phi_n(\Gamma gn) dt = N_{\Gamma}c_{\delta}(\delta_T) \kappa_n(\delta_T) + O_{\Gamma,\Omega} \left( T^{\frac{1}{2} - \delta_T} \right)
\]

for all \(g \in \Omega \cap \Gamma \backslash G_{\text{rad}}, T \gg_\Omega 1, n \in \mathbb{Z}\).

**Proof.** Divide both sides of (17) by \(\mu_{T g N}(B_T)\) and apply Corollary 18. \(\square\)
6. Effective Equidistribution in the Orthogonal Complement of $H_{δ_Γ}$

Let $H_1$ denote the orthogonal complement in $L^2(Γ \backslash G)$ of $C_0$, i.e.

$$H_1 = \bigoplus_{i=1}^l C_i \oplus L^2(Γ \backslash G)_{\text{temp}}$$

(cf. Proposition 3).

6.1. Effective equidistribution. Strömbergsson’s proof of [44, Theorem 1] carries over to our setting of infinite covolume geometrically finite $Γ$, giving the following effective equidistribution result for functions in $H_1$:

**Theorem 22.** For all $f \in S^4(Γ \backslash G) \cap H_1 \cap B_α$, $0 \leq α < \frac{1}{2}$, $T \gg 1$,

$$\frac{1}{2T} \int_{-T}^T f(Γgn_t) \, dt = O(Δ) \left( \|f\|_{S^4(Γ \backslash G)} \left( \frac{Y_{Γ}(ΓgαT)}{T} \right)^{1-s_1} + \left( \frac{Y_{Γ}(ΓgαT)}{T} \right)^{\frac{1}{2}} \log^3 \left( 2 + \frac{T}{Y_{Γ}(ΓgαT)} \right) \right)$$

$$+ \|f\|_{N_α} \left( \frac{Y_{Γ}(ΓgαT)}{T} \right)^{\frac{1}{2}}.$$

**Discussion of Proof.** It is assumed throughout [44] that $Γ$ is a lattice. However, by following the proofs of [44, Proposition 3.1 and Theorem 1], one obtains the statement above. (The only place in the aforementioned proofs where the fact that $Γ \backslash G$ has finite volume is used is [44, bottom of p. 304]. We do not claim (or require) as precise a statement as [44, Theorem 1]-in particular, we do not distinguish between the cuspidal and non-cuspidal parts of the tempered spectrum. One may thus replace the arguments of [44] regarding the tempered cuspidal spectrum on [44, pp. 304-305] with the treatment of the continuous spectrum given on [44, pp. 302-303].) Indeed, the results of [44] are based on a representation-theoretic method first developed by Burger in [8] in order to classify the $N$-invariant ergodic Radon measures on $Γ \backslash G$ for $Γ$ convex-cocompact (possibly of infinite covolume) with $δ_Γ > \frac{1}{2}$. In [44], Strömbergsson combined this method with properties of the invariant height function $Y_Γ$ to show the effective equidistribution of dense horocycles in any finite-volume $Γ \backslash G$. As noted previously, due to the fact that the cusps of geometrically finite hyperbolic surfaces with infinite volume have the same structure as those of finite volume surfaces, their invariant height functions share essentially the same properties, allowing the same treatment to work here.

The following follows from Theorem 22 (and Corollary 16) in the same way that Corollary 21 follows from Theorem 20.

**Corollary 23.** Let $Ω \subseteq Γ \backslash G$ be compact. Then for all $Γg \in Ω \cap Γ \backslash G_{\text{rad}}$, $f \in S^4(Γ \backslash G) \cap H_1 \cap B_α$, $0 \leq α < \frac{1}{2}$, and $T \gg 1$,

$$\frac{1}{μ_{Γ}^N(B_T)} \int_{-T}^T f(Γgn_t) \, dt = O(Δ) \left( \|f\|_{S^4(Γ \backslash G)} \left( \frac{Y_{Γ}(ΓgαT)}{T} \right)^{δ_Γ-s_1} + \|f\|_{N_α} \left( \frac{Y_{Γ}(ΓgαT)}{T} \right)^{δ_Γ-\frac{1}{2}} \right)$$

$$+ \|f\|_{S^4(Γ \backslash G)} \left( \frac{Y_{Γ}(ΓgαT)}{T} \right)^{δ_Γ-\frac{1}{2}} \log^3 \left( 2 + \frac{T}{Y_{Γ}(ΓgαT)} \right)$$

7. Proof of Theorem 1

Before proving our main result, Theorem 1, we first recall the definition of the Burger-Roblin measure associated to $N$ on $Γ \backslash G$, denoted $m^\text{BR}_Γ$ (and referred to as the BR-measure for short).
7.1. The Burger-Roblin measure. Using the Iwasawa decomposition $G = KAN$, we define a left $\Gamma$-invariant (cf. [3]) and right $N$-invariant measure $\tilde{m}_1^{BR}$ on $G$ by
\[\tilde{m}_1^{BR}(f) = \int_{KAN} f(ka_n x) g^{\delta_T-1} dx \, dy \, d\nu_1^{(G,\gamma)}(k \cdot \infty) \quad \forall f \in C_c(G).\]
We may also express this in terms of the Patterson-Sullivan and Lebesgue densities as follows: firstly, observe that the map
\[g \mapsto ([g]_+, [g]_-, \beta_{[g]} + (i, g \cdot i))\]
is a bijection from $G$ to $((\partial_\infty \mathbb{H} \times \partial_\infty \mathbb{H}) \setminus \{(u, u) : u \in \partial_\infty \mathbb{H}\}) \times \mathbb{R}$. We may then write the BR-measure as
\[\tilde{m}_1^{BR}(f) = \int_G f(g) e^{\delta_T \beta_{[g]+}(i, g \cdot i)} e^{\beta_{[g]-}(i, g \cdot i)} \, dm([g]_-) \, d\nu_1^{(G,\gamma)}([g]_-) \, d\nu_1^{(G,\gamma)}([g]_+) \quad \forall f \in C_c(G),\]
where $r = \beta_{[g]+}(i, g \cdot i)$. In a similar manner, we define the so-called $BR_*$-measure $\tilde{m}_1^{BR_*}$ on $G$ by
\[\tilde{m}_1^{BR_*}(f) = \int_G f(g) e^{\delta_T \beta_{[g]+}(i, g \cdot i)} e^{\beta_{[g]-}(i, g \cdot i)} \, dm([g]_+) \, d\nu_1^{(G,\gamma)}([g]_+) \, ds \quad \forall f \in C_c(G),\]
where $s = \beta_{[g]+}(i, g \cdot i)$. Observe that $\tilde{m}_1^{BR_*}$ is right $U$-invariant, where $U$ is the subgroup of $G$ defined by
\[U = \{u_k = (\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}) : u \in \mathbb{R}\}.\]
The surjective map $\pi : C_c(G) \to C_c(G \backslash \Gamma)$ given by $[\pi(f)](\Gamma g) := \sum_{\gamma \in \Gamma} f(\gamma g)$ allows us to then define the measure $m_1^{BR}$ on $\Gamma \backslash G$ by
\[m_1^{BR}(\pi(f)) := \tilde{m}_1^{BR}(f) \quad \forall f \in C_c(G)\]
(the left $\Gamma$-invariance of $\tilde{m}_1^{BR}$ ensures that $m_1^{BR}$ is well-defined). The measure $m_1^{BR}$ is defined in a completely analogous way. Note that both $m_1^{BR}$ and $m_1^{BR_*}$ are infinite measures on $\Gamma \backslash G$.

**Proof of Theorem 1.** Without loss of generality, we may assume that $1 - \delta_T \leq \alpha < \frac{1}{2}$. We now write $f$ as the orthogonal sum $f = f_0 + f_1$, where $f_0 \in C_0 \cap S^4(\Gamma \backslash G)$ and $f_1 \in H_1 \cap S^4(\Gamma \backslash G)$. By Lemma 4, $f_0 \in B_\alpha$, hence $f_1 = f - f_0 \in B_\alpha$. This allows us to apply Corollary 23 to $f_1$, which, after noting that $||f_1||_{L^4(\Gamma \backslash G)} \leq ||f||_{L^4(\Gamma \backslash G)}$ and $||f_1||_{N^\alpha} = ||f - f_0||_{N^\alpha} \leq ||f||_{N^\alpha} + ||f||_{N^\alpha} \ll ||f_0||_{S^4(\Gamma \backslash G)} + ||f||_{N^\alpha} \leq ||f||_{S^4(\Gamma \backslash G)} + ||f||_{N^\alpha}$, gives
\[\frac{1}{\mu_{PS}^{(G,\gamma)}} \int_{-T}^T f(\Gamma g_n t) \, dt = \frac{1}{\mu_{PS}^{(G,\gamma)}} \int_{-T}^T f_0(\Gamma g_n t) \, dt + O_T,\alpha \left( \frac{\left| Y_t(\Gamma g_n t) \right|}{T^{\frac{1}{2}} - \delta_T} \right) \delta_T^{s_1} + \frac{||f||_{S^4(\Gamma \backslash G)}}{T^{\frac{1}{2} - \delta_T}} \log^3 \left( 2 + \frac{T}{\left| Y_t(\Gamma g_n t) \right|} \right).
\]
To complete the proof, it now suffices to prove that
\[\frac{1}{\mu_{PS}^{(G,\gamma)}} \int_{-T}^T f_0(\Gamma g_n t) \, dt = \frac{m_1^{BR}(f)}{m_1^{BR}(\Gamma \backslash G)} + O_T,\alpha \left( ||f||_{S^4(\Gamma \backslash G)} T^{\frac{1}{2} - \delta_T} \right) \]
We observe that $f_0 = \sum_{n \in \mathbb{Z}} \langle f, \phi_n \rangle L^2(\Gamma \backslash G) \phi_n$. Using Proposition 3 (1), Lemma 6 and the bound $|\phi_n(\Gamma h)| \ll \phi_0(\Gamma h)$, we have
\[\sum_{n \in \mathbb{Z}} |\langle f, \phi_n \rangle L^2(\Gamma \backslash G) \phi_n(\Gamma g t)| \ll \left| Y_t(\Gamma g t) \right|^{1-\delta_T} \sum_{n \in \mathbb{Z}} |\langle f, \phi_n \rangle L^2(\Gamma \backslash G)| \ll \left| 1 + T \right|^{2-2\delta_T} \left| Y_t(\Gamma g t) \right|^{1-\delta_T} \quad \forall t \in B_T, \forall g \in \Gamma \backslash G.
This permits us to write \( \int_{-T}^{T} f_{0}(\Gamma g_{n_{t}}) dt = \sum_{n \in \mathbb{Z}} \langle f, \phi_{n} \rangle_{L^{2}(\Gamma \backslash G)} \int_{-T}^{T} \phi_{n}(\Gamma g_{n_{t}}) dt \), and so Corollary \[24\] gives

\[
\frac{1}{\mu_{G}^{PS}(B_{F})} \int_{-T}^{T} f_{0}(\Gamma g_{n_{t}}) dt = \sum_{n \in \mathbb{Z}} \langle f, \phi_{n} \rangle_{L^{2}(\Gamma \backslash G)} \int_{-T}^{T} \phi_{n}(\Gamma g_{n_{t}}) dt
\]

\[
= \sum_{n \in \mathbb{Z}} \langle f, \phi_{n} \rangle_{L^{2}(\Gamma \backslash G)} \left( \mathcal{N}_{\Gamma} c_{n}(\delta_{\Gamma}) \kappa_{n}(\delta_{\Gamma}) + O_{\Gamma, \Omega} \left( T^{\frac{1}{2} - \delta_{\Gamma}} \right) \right)
\]

\[
= \left( \sum_{n \in \mathbb{Z}} \mathcal{N}_{\Gamma} c_{n}(\delta_{\Gamma}) \kappa_{n}(\delta_{\Gamma}) \langle f, \phi_{n} \rangle_{L^{2}(\Gamma \backslash G)} \right) + O_{\Gamma, \Omega} \left( T^{\frac{1}{2} - \delta_{\Gamma}} \sum_{n \in \mathbb{Z}} |\langle f, \phi_{n} \rangle_{L^{2}(\Gamma \backslash G)}| \right)
\]

\[
= \mathcal{M}_{\Gamma}(f) + O_{\Gamma, \Omega} \left( T^{\frac{1}{2} - \delta_{\Gamma}} \|f\|_{S^{1}(\Gamma \backslash G)} \right)
\]

(cf. \[10\]).

Now, \[22\] and \[23\] show that \( \lim_{T \to \infty} \frac{1}{\mu_{G}^{PS}(B_{F})} \int_{-T}^{T} f_{0}(\Gamma g_{n_{t}}) dt = \mathcal{M}_{\Gamma}(f) \). However, \[28\] Theorem 1.5| or \[24\] Theorem 1.1] gives \( \frac{1}{\mu_{G}^{PS}(B_{F})} \int_{-T}^{T} \psi_{0}(\Gamma g_{n_{t}}) dt = \frac{m_{BMS}(\psi)}{m_{BMS}(\Gamma)} \) for all \( \psi \in C_{c}(\Gamma \backslash G) \) (note that \( \mu_{G}^{PS} \) and \( m_{BMS}^{BR} \) are scaled with a factor \( \mathcal{N}_{\Gamma} \) compared with those of \[25\], this enables us to use the cited result). Observing that \( |\mathcal{M}_{\Gamma}(f)| \lesssim_{\Gamma} \|f\|_{S^{1}(\Gamma \backslash G)} \), we obtain the claimed extension of \( f \mapsto m_{BMS}^{BR}(f) \).

\[]

Remark 9. Since \( C_{c}^{\infty}(\Gamma \backslash G) \subset L^{1}(\Gamma \backslash G, m_{BMS}^{BR}) \cap S^{1}(\Gamma \backslash G) \), we obtain the following identity for the BR-measure:

\[
m^{BR}_{\Gamma}(f) = m^{BMS}_{\Gamma}(\Gamma \backslash G) \mathcal{M}_{\Gamma}(f) = \sum_{n \in \mathbb{Z}} \frac{m_{BMS}^{BR}(\Gamma \backslash G) \mathcal{N}_{\Gamma} \kappa_{0}(\delta_{\Gamma})}{c_{n}(\delta_{\Gamma})} \langle f, \phi_{n} \rangle_{L^{2}(\Gamma \backslash G)} \forall f \in C_{c}^{\infty}(\Gamma \backslash G).
\]

A similar identity is obtained in \[22\] Theorem 7.3]. At a first glance, our formula appears to be different from that given in \[22\]; the identities do not appear to give the same value even up to scaling. A closer inspection reveals that this is due to a small typo in \[22\]: in the case \( n = 2 \), the formula given in \[22\] Theorem 4.6] should read

\[
\phi_{1}^{N}(a_{y}) = c_{2}(0) \frac{\Gamma(\delta) \Gamma(1 - \delta + i)}{\Gamma(1 - \delta) \Gamma(\delta + i)} y^{1 - \delta}.
\]

After making a subsequent correction to \[22\] (6.1), p. 610], it is straightforward to verify that \[24\] agrees with \[22\] Theorem 7.3] (at least up to scaling).

8. Convex-Cocompact \( \Gamma \backslash G \)

We will now restrict our attention to convex cocompact \( \Gamma \) and demonstrate how one can deduce effective equidistribution of non-closed horocycles from the exponential mixing of the diagonal action with respect to the Bowen-Margulis-Sullivan measure (abbreviated as the BMS-measure) without the assumption that \( \delta_{\Gamma} > \frac{1}{2} \). As such, throughout this section \( \Gamma \) is non-elementary and convex cocompact. As previously noted, if \( \delta_{\Gamma} \leq \frac{1}{2} \) then \( \Gamma \) is necessarily convex cocompact.

8.1. Exponential mixing. The key result which we need is exponential mixing of the diagonal subgroup of \( G \). This was first obtained by Stoyanov with respect to the BMS-measure for convex cocompact \( \Gamma \). In \[30\] Section 5.2, Oh and Winter show how to obtain an exponential mixing statement for the Haar measure from that for the BMS-measure. It is this result that will be the main ingredient in the proof of Theorem 2.

Before giving the precise statement, we recall some of the terminology introduced in Section 1 for \( \Omega \subset \Gamma \backslash G \), we let \( S^{m}(\Omega) \) denote the closure of \( \{ f \in C_{c}^{\infty}(\Omega) : f|_{\partial \Omega} = 0 \} \) with respect to
the norm \( \| \cdot \|_{S^m(\Gamma \backslash G)} \). Similarly, we let \( \| \cdot \|_{W^m} \) denote the standard \( L^2 \)-Sobolev norm of order \( m \) on \( \mathbb{R} \), and for an interval \( I \subset \mathbb{R} \), we let \( W^m(I) \) denote the closure of \( \{ C^\infty_c(I) : f|_I = 0 \} \) with respect to \( \| \cdot \|_{W^m} \).

Combining [41] Corollary 1.5 with [30] Theorem 5.8 gives

**Theorem 24.** There exists \( \eta_0 > 0 \) such that for any compact subset \( \Omega \subset \Gamma \backslash G \),

\[
\int_{\Gamma \backslash G} f_1(\Gamma g a y) f_2(\Gamma g) \, d\mu_{\Gamma \backslash G}(\Gamma g) = \frac{m_{1}^{BR} m_{1}^{BR} (f_2)}{m_{1}^{BR}(\Gamma \backslash G)} y^{1-\delta r} + O_{\Gamma, \Omega} (y^{1-\delta r + \eta_0} \| f_1 \|_{S^1(\Gamma \backslash G)} \| f_2 \|_{S^1(\Gamma \backslash G)})
\]

for all \( 0 < y \leq 1 \), \( f_1, f_2 \in S^3(\Omega) \).

**Remark 10.** Observe that \( y \to \infty \) in [30] Theorem 5.8. Using the \( G \)-invariance of \( \mu_{\Gamma \backslash G} \) and the fact that our definitions of \( m_{1}^{BR} \) and \( m_{1}^{BR} \) are interchanged compared with those in [30], we obtain the main term stated here. To obtain our error term from that of [30] Theorem 5.8, we simply use the Sobolev inequality \( \| f \|_{C^1} \ll \| f \|_{S^1(\Gamma \backslash G)} \) (cf. Lemma 5).

8.2. Effective equidistribution of expanding translates. Since \( \Gamma \) is convex cocompact, there is a uniform lower bound on the injectivity radius at each point of \( \Gamma \backslash G \). This allows us to deduce the effective equidistribution of non-closed horocycles from the effective equidistribution of expanding translates of compact pieces of horospherical orbits. This result in turn follows from the exponential mixing of the diagonal subgroup via the classical “Margulis thickening trick” see e.g. Kleinbock and Margulis [19] Proposition 2.4.8 for the proof in the general finite-volume setting.

For infinite volume \( \Gamma \backslash G \), the result we require is due to Mohammadi and Oh [27] Theorem 5.13. The main complication compared with the finite volume setting is that the Lebesgue and Haar measures can (in general) give much greater mass to subsets than those given by the PS- and BR-measures. One must thus avoid bounding any approximations of functions until making use of the exponential mixing from Theorem 24. Since there are slight variations in our notation and setting compared with [27] (as well as the fact that we will also require similar estimates in the proof of Theorem 2), we closely follow [27] Section 5 and reproduce the key steps of their proof. We refer the reader to [27] Section 5 for more details.

We start by recalling the Bruhat NAU decomposition of \( G \): NAU is an open neighbourhood of the identity in \( G \) and \( G = NAU \) (cf. [20] Proposition 8.45)). This allows us to make the following decomposition of the BR-\( * \)-measure (cf. [27] (5.3), p. 868): 

**Lemma 25.** Let \( B_1 \subset N \), \( B_2 \subset A \), \( B_3 \subset U \) be open neighbourhoods of the identity (in the respective subgroups) and let \( g \in G \). Then for any \( f \in C_c(G) \) with \( \text{supp}(f) \subset gB_1B_2B_3 \),

\[
\tilde{m}_{1}^{BR}(f) = \int_{\{n_x \in B_1\}} \int_{\{a_y \in B_2\}} \int_{\{n_u \in B_3\}} f(gn_x a_y n_u^*) e^{\delta r \beta_{[gn_x]}(i, gn_x) y^{1-\delta r}} \, du \, d\nu_i^{\Gamma, i}([gn_x]) .
\]

**Proof.** Using the definition from Section 7.1

\[
\tilde{m}_{1}^{BR}(f) = \int_G f(h) e^{\delta r s} e^{\beta_{[h]}(i, h \cdot i)} \, ds \, dm_i([h]^+) \, d\nu_i^{\Gamma, i}([h]^+)
\]

where \( s = \beta_{[h]}(i, h \cdot i) \). Writing \( h = gn_x a_y n_u^* \), we observe that

\[
[g_n x a_y n_u^*]^- = [gn_x^-]
\]

\[
s = \beta_{[gn_x]}(i, gn_x a_y n_u^*) = \beta_{[gn_x]}(i, gn_x) + \beta_0(i, a_y n_u^*) - \beta_{[gn_x]}(i, gn_x) - \log y
\]

\[
e^{\beta_{[gn_x]}(i, gn_x) - \beta_0(i, a_y n_u^*)} \, dm_i([gn_x a_y n_u^*]^+) = e^{\beta_{[gn_x a_y]^{-1} i, n_u^*}} \, dm_i([gn_x a_y]^{-1} i, [n_u^*]) = e^{\beta_{[n_u^*]}(i, n_u^*)} \, dm_i([n_u^*])
\]

\[
= (u^2 + 1) \frac{d(h)}{1 + u^2} .
\]
This gives \( e^{\beta_{\frac{1}{2}}(i,n_x^*)} \) \( dm_1(\frac{1}{u}) \) \( ds = y \, du \, dy \), and so

\[
\bar{m}_{\Gamma}^{BR}(f) = \int_{\{ gn_x a_y n_u^* \in B_{123} \}} f(gn_x a_y n_u^*) \, e^{\delta_r \beta_{\frac{1}{2}g_{(x)}}(i,n_x^*) - \delta_r \beta_{\frac{1}{2}}(i,n_x^*)} \, dm_1(\frac{1}{u}) \, ds \, dv_{1,(g)}(\{ gn_x \})
\]

\[
= \iint_{B_{123}} f(gn_x a_y n_u^*) \, e^{\delta_r \beta_{g_{(x)}}(i,n_x^*) - \delta_r \beta_{(i,n_x^*)}} \, du \, dy \, dv_{1,(g)}(\{ gn_x \}).
\]

\( \square \)

Let \( \text{dist}_G \) denote the Riemannian metric on \( G \) induced from the Killing form on \( g \) and \( B_r \) to denote the open ball of radius \( r \) around the identity in \( G \). The corresponding norm on \( g \) is denoted by \( | \cdot | \). We now choose \( r_H \leq 1 \) small enough so that the exponential map is surjective onto \( B_{r_H} \), and for each \( g \in \Gamma \backslash G \), the map from \( B_{r_H} \) to \( \Gamma \backslash G \) given by \( h \mapsto \Gamma gh \) is injective.

**Lemma 26.**

\[
|f(\Gamma gh) - f(\Gamma g)| \ll r \| f \|_{S^3(\Gamma \backslash G)} \quad \forall 0 \leq r \leq r_\Gamma, \ g \in G, \ h \in B_r, \ f \in S^3(\Gamma \backslash G).
\]

**Proof.** Given \( h \) in such a \( B_r \), there exists \( X \in g \) such that \( h = \exp(X) \) and \( |X| \ll r \). We then have

\[
|f(\Gamma gh) - f(\Gamma g)| \leq \int_0^1 |Xf(\Gamma g \exp(sX))| \, ds \ll \| Xf \|_{S^3(\Gamma \backslash G)} \ll r \| f \|_{S^3(\Gamma \backslash G)}.
\]

\( \square \)

We also let \( \epsilon_\Gamma \leq r_\Gamma \) be small enough so that

\[
\{ n_x a_y n_u^* : \max \{ |x|, \log y, |u| \} < \epsilon_\Gamma \} \subset B_{r_\Gamma/2}.
\]

**Theorem 27.** There exists \( \eta_1 > 0 \) such that for any compact subset \( \Omega \subset \Gamma \backslash G \),

\[
\int_{-\epsilon_\Gamma}^{\epsilon_\Gamma} f(\Gamma gn_t a_y) \phi(t) \, dt = \frac{m^{PS}_{\Gamma \backslash G}(\phi)}{m^{PS}_{\Gamma \backslash G}(\phi)} y^{1-\delta_r} + O_{\Gamma, \Omega}(y^{1-\delta_r + \eta_1} \| f \|_{S^3(\Gamma \backslash G)} \| \phi \|_{S^3(\Gamma \backslash G)}^{\delta_r + \eta_1})
\]

for all \( \Gamma g \in \Omega \), and non-negative \( f \in S^3(\Omega), \phi \in C_c^{\infty}((-\epsilon_\Gamma, \epsilon_\Gamma)). \)

**Remark 11.** We have previously only defined the measures \( m^{PS}_{\Gamma \backslash G} \) for radial points \( \Gamma g \). While we will only need Theorem 27 for the radial points, we note that since \( \Gamma \) is convex-cocompact, the map from \( N \) to \( \Gamma \backslash G \) given by \( n \mapsto \Gamma gn \) is injective for all \( \Gamma g \in \Gamma \backslash G \); the definition given in Section 12 therefore still works for all \( \Gamma g \in \Gamma \backslash G \). It is in the case that \( \Gamma \) is not convex-cocompact that more care is required in the definition; this is due to the presence of periodic horocycles around the cusps of \( \Gamma \backslash G \), cf. [28, Section 2].

**Proof.** We start by defining, for \( \epsilon \leq \epsilon_\Gamma \), functions \( f^+_{\epsilon} \) and \( f^-_{\epsilon} \) by

\[
f^+_{\epsilon}(\Gamma g) := \sup_{h \in B_\epsilon} f(\Gamma gh), \quad f^-_{\epsilon}(\Gamma g) := \inf_{h \in B_\epsilon} f(\Gamma gh).
\]

Observe that \( f^+_{\epsilon} \in S^3(\Omega B_\epsilon) \) and by Lemma 26 \( |f(\Gamma g) - f^+_{\epsilon}(\Gamma g)| \ll \epsilon \epsilon_\Gamma \epsilon \| f \|_{S^3(\Gamma \backslash G)}. \)

By [11] Lemma 2.4.7, given \( \epsilon > 0 \), there exists \( \rho_\epsilon \in B_\epsilon \cap C_c^{\infty}(AU) \) such that:

\[
\rho_\epsilon(a_\epsilon n_u^*) \geq 0 \quad \forall v \in \mathbb{R}_{>0} \ u \in \mathbb{R}, \quad \iint_{\mathbb{R}_{>0}} \rho_\epsilon(a_\epsilon n_u^*) \frac{du \, dv}{v^2} = 1.
\]

We now define a function \( \Phi_\epsilon \in C_c^{\infty}(\Gamma \backslash G) \) by

\[
\Phi_\epsilon(\Gamma h) = \begin{cases} \phi(t) \rho_\epsilon(a_\epsilon n_u^*) & \text{if } \Gamma h = \Gamma gn_t a_\epsilon n_u^* \\ 0 & \text{otherwise.} \end{cases}
\]
Observe that $\Phi_\epsilon$ is well-defined is due to the uniqueness of the \textit{NAU} decomposition, and that $\epsilon \leq \epsilon_\Gamma \leq \frac{\pi}{2}$ (which is less than the injectivity radius of $\Gamma \backslash G$); $\Phi_\epsilon$ is thus supported on $\mathcal{G}_B_{\epsilon, \Gamma} \subset \Omega B_{\epsilon, \Gamma}$. Using this definition, we have

$$
\int_{-\epsilon}^{\epsilon} f(\Gamma g n_t a_y) \phi(t) \, dt = \int_{-\epsilon}^{\epsilon} f(\Gamma g n_t a_y) \phi(t) \, dt \left( \int_{\mathbb{R}_0^+} \int_{\mathbb{R}} \rho_\epsilon(a_v n_u^*) \frac{da \, du}{v^2} \right)
$$

$$
= \int_{-\epsilon}^{\epsilon} \int_{\mathbb{R}_0^+} \int_{\mathbb{R}} f(\Gamma g n_t a_y) \phi(t) \rho_\epsilon(a_v n_u^*) \left( \frac{dt \, da \, du}{v^2} \right)
$$

$$
= \int_{-\epsilon}^{\epsilon} \int_{\mathbb{R}_0^+} \int_{\mathbb{R}} f(\Gamma g n_t a_v n_u^* a_y(a_v n_u^*)^{-1}) \Phi_\epsilon(\Gamma g n_t a_v n_u^*) \left( \frac{dt \, da \, du}{v^2} \right).
$$

Since $y \leq 1$ and $a_v n_u^* \in B_c$, $(a_v n_u^*)^{-1} \in B_c$, hence

$$
f_\epsilon^-(\Gamma g n_t a_v n_u^* a_y) \leq f(\Gamma g n_t a_v n_u^* a_y(a_v n_u^*)^{-1}) \leq f_\epsilon^+(\Gamma g n_t a_v n_u^* a_y).
$$

Now, $d\mu_G(n_x a_v n_u^*) = \frac{de \, dv}{\pi}$; we may thus bound the integral we are concerned with as follows:

$$
\int_{B_c} f_\epsilon^-(\Gamma h a_y) \Phi_\epsilon(\Gamma h) \, d\mu_G(h) \leq \int_{-\epsilon}^{\epsilon} f(\Gamma g n_t a_y) \phi(t) \, dt \leq \int_{B_c} f_\epsilon^+(\Gamma h a_y) \Phi_\epsilon(\Gamma h) \, d\mu_G(h).
$$

By Theorem 24

$$
\int_{B_c} f_\epsilon^+(\Gamma h a_y) \Phi_\epsilon(\Gamma h) \, d\mu_G(h) = \int_{\mathcal{G}_B_{\epsilon, \Gamma}} f_\epsilon^+(\Gamma h a_y) \Phi_\epsilon(\Gamma h) \, d\mu_{\Gamma \backslash G}(\Gamma h)
$$

$$
= \frac{m_{\text{BR}}^g(f_\epsilon^+) m_{\text{BR}, \epsilon}(\Phi_\epsilon)}{m_{\text{BR}, \epsilon}(\Gamma \backslash G)} y^{1-\delta_\Gamma} + O_{\Gamma, \Omega B_{\epsilon, \Gamma}} \left( y^{1-\delta_\Gamma + \eta_1} \| f_\epsilon^+ \|_{S^1(\Gamma \backslash G)} \| \Phi_\epsilon \|_{S^1(\Gamma \backslash G)} \right).
$$

We have $\| f_\epsilon^+ \|_{S^1(\Gamma \backslash G)} \ll \| f \|_{S^1(\Gamma \backslash G)}$ (cf. [24] (5.8), p. 868)). Also, again appealing to Lemma 2.4.7 gives the bound $\| \Phi_\epsilon \|_{S^1(\Gamma \backslash G)} \ll \| \phi \|_{W^3} \epsilon^{-4}$, hence

$$
\text{(25)} \quad O_{\Gamma, \Omega B_{\epsilon, \Gamma}} \left( y^{1-\delta_\Gamma + \eta_1} \| f_\epsilon^+ \|_{S^1(\Gamma \backslash G)} \| \Phi_\epsilon \|_{S^1(\Gamma \backslash G)} \right) = O_{\Gamma, \Omega} \left( y^{1-\delta_\Gamma + \eta_1} \| f \|_{S^1(\Gamma \backslash G)} \| \phi \|_{W^3} \epsilon^{-4} \right).
$$

Since $m_{\text{BR}}^g$ is locally finite, Lemma 29 gives

$$
\text{(26)} \quad m_{\text{BR}}^g(f_\epsilon^+) = m_{\text{BR}}^g(f) + m_{\text{BR}}^g(f_\epsilon^--f) = m_{\text{BR}}^g(f) + O_{\Gamma, \Omega}(\epsilon \| f \|_{S^1(\Gamma \backslash G)}).
$$

We now use Lemma 24 to compute $m_{\text{BR}, \epsilon}(\Phi_\epsilon)$:

$$
m_{\text{BR}, \epsilon}(\Phi_\epsilon) = m_{\text{BR}, \epsilon}(h \mapsto \Phi_\epsilon(\Gamma g h))|_{B_{\epsilon, \Gamma}}
$$

$$
= \int_{-\epsilon}^{\epsilon} \int_{\mathcal{G}_B_{\epsilon, \Gamma}} \phi(t) \rho_\epsilon(a_y n_u^*) e^{\delta_\epsilon g n_x} \left( (g n_x) \right) \left( (g n_x) \right) \frac{dv \, du}{v^2} \left( (g n_x) \right) \left( (g n_x) \right).
$$

For $v \in B_c$, $| \log v | \ll \epsilon$, so $v^{3-\delta_\Gamma} = 1 + O_\Gamma(\epsilon)$, hence

$$
m_{\text{BR}, \epsilon}(\Phi_\epsilon) = m_{\text{BR}, \epsilon}(\phi) \times \int_{\mathcal{G}_B_{\epsilon, \Gamma}} \rho_\epsilon(a_y n_u^*) \left( 1 + O_\Gamma(\epsilon) \right) \frac{dv \, du}{v^2}
$$

$$
= m_{\text{BR}, \epsilon}(\phi) \left( 1 + O_\Gamma(\epsilon) \right).
$$
This, together with \(26\), gives
\[
\begin{align*}
\text{Lemma 28.} & \quad m_{\text{T}}^{\text{BR}}(f_{\epsilon}^\pm) m_{\text{T}}^{\text{BR}}(\Phi_e) = (m_{\text{T}}^{\text{BR}}(f) + O_{\nu}(\epsilon\|f\|_{S^5(\Gamma^\text{T})})) \mu_{g_N}^{\text{PS}}(\phi)(1 + O(1)) \\
& = m_{\text{T}}^{\text{BR}}(f) \mu_{g_N}^{\text{PS}}(\phi) + \nu_{\epsilon}(\mu_{g_N}^{\text{PS}}(\phi) + \epsilon\|f\|_{S^5(\Gamma^\text{T})} \mu_{g_N}^{\text{PS}}(\phi)) \\
& = m_{\text{T}}^{\text{BR}}(f) \mu_{g_N}^{\text{PS}}(\phi) + \nu_{\epsilon}(\epsilon\mu_{g_N}^{\text{PS}}(\phi) \|f\|_{S^5(\Gamma^\text{T})}).
\end{align*}
\]
Combining this expression with \(25\) yields
\[
\int_{\mathcal{E}_t} f_{\epsilon}(\Gamma t y) \Phi_e(\Gamma h) d\mu_G(h) = \frac{m_{\text{T}}^{\text{BR}}(f) \mu_{g_N}^{\text{PS}}(\phi)}{m_{\text{T}}^{\text{BR}}(\Gamma^\text{T})} y^{1-\epsilon_t} + O_{\nu}(\epsilon \|f\|_{S^5(\Gamma^\text{T})} \{ \epsilon \mu_{g_N}^{\text{PS}}(\phi) + y^\eta \epsilon^{-4} \|\phi\|_{W^3}\}).
\]

Since \(\int_{\mathcal{E}_t} f(\Gamma gn_t a_y) \phi(t) dt\) is bounded from above and below by the integrals in the right-hand side of this expression, the same must hold for it. Choosing \(\epsilon = y^\eta\) then completes the proof, with \(\eta_1 = \frac{\eta_0}{\eta}\).

8.3. **The shadow lemma.** The final step before proceeding with the proof of Theorem 2 involves adapting the results of Section 4.4 to the case \(\delta_T \leq \frac{1}{2}\). For \(\delta_T \leq \frac{1}{2}\), the integral in \(7\) still defines an eigenfunction of \(-\Delta\) on \(\Gamma \backslash \mathbb{H}\) with eigenvalue \(\delta_T (1 - \delta_T)\) (cf. \([31][32]\)), however it is no longer in \(L^2(\Gamma \backslash \mathbb{H})\); we thus define
\[
\tilde{\phi}_0(\Gamma g) := \int_{\partial_{\mathbb{H}} \mathbb{H}} e^{-\delta_T \beta_a(g,i)} d\nu_{\epsilon}(\Gamma g, i, u)
\]
(i.e. we remove the constant \(N_T\) from the definition given in \(7\) since it is not well-defined for \(\delta_T \leq \frac{1}{2}\)). We note, however, that \(\tilde{\phi}_0\) is bounded:

**Lemma 28.** Let \(\Gamma\) be convex cocompact. Then \(\tilde{\phi}_0 \in L^\infty(\Gamma \backslash G)\).

In fact, \(\tilde{\phi}_0\) decays outside the convex core of \(\Gamma \backslash G\), cf. \([27]\) Proposition 4.2], though for \(\delta_T \leq \frac{1}{2}\) not fast enough so that \(\tilde{\phi}_0 \in L^2(\Gamma \backslash G)\).

Since \(\tilde{\phi}_0(\Gamma g) \ll_{\Gamma} 1 = \mathcal{Y}_T(\Gamma g)\), the results of Section 4.4 all hold even without the assumption \(\delta_T > \frac{1}{2}\). Moreover, simplifications occur due to the fact that we no longer have to take \(\mathcal{Y}_T\) into account. Lemmas 14, 12, and 13 in the convex cocompact setting read as follows:

**Lemma 29.** For all \(z, w \in \mathbb{H}, r > 0\),
\[\nu_{z}(\Gamma g)(O_{z}(w, r)) \ll_{\Gamma} e^{2\delta_T r - \delta_T \text{dist}(z,w)}\]

**Lemma 30.**
\[\nu_{\epsilon}(\Gamma g^\text{T})(\{x \in \mathbb{R} : |x| \geq T\}) \ll_{\Gamma} P_{\Gamma}(T) e^{\delta_T} T^{-\delta_T} \quad \forall g \in G, T \geq 1.\]

**Lemma 31.**
\[\nu_{\epsilon}(\Gamma g^\text{T})(\{u \in \mathbb{R} : (1 - \epsilon)T \leq |u| \leq (1 + \epsilon)T\}) \ll_{\Gamma} P_{\Gamma}(T) e^{\delta_T} T^{-\delta_T} \quad \forall g \in G, T \geq 2, 0 \leq \epsilon \leq \frac{1}{2}.\]

Noting that Theorem 14 also holds for convex cocompact \(\Gamma\) without the assumption \(\delta_T > \frac{1}{2}\), cf., e.g., \([29]\) Theorem 4.6.2], Proposition 15 thus also holds, as well as Corollary 16, which in the current setting reads as

**Corollary 32.** Let \(\Omega \subset \Gamma \backslash G\) be compact. Then
\[\mu_{g_N}^{\text{PS}}(B_T) \gg_{\Omega} T^{\delta_T} \quad \forall \Gamma g \in \Omega \cap \Gamma \backslash G_{\text{rad}}, T \gg_{\Omega} 1.\]
8.4. Proof of Theorem 2. We start by assuming that $f$ is $\mathbb{R}_{\geq 0}$-valued. For $r > 0$, we have

$$\int_{-T}^T f(\Gamma g \tau) \, dt = \frac{T}{r} \int_{-r}^r f(\Gamma g T / r) \, dt = \frac{T}{r} \int_{-r}^r f(\Gamma g T / r, n_1 a_T^{-1} / r) \, dt. \tag{27}$$

By [19] Lemma 2.4.7, given $\epsilon > 0$, there exists $\psi_\epsilon \in C_c^\infty((-\epsilon, \epsilon))$ such that:

$$\psi_\epsilon(x) \geq 0 \forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} \psi_\epsilon(x) \, dx = 1, \quad \|\psi_\epsilon\|_{W^3} \ll \epsilon^{-\frac{2}{3}}.$$  

For $\epsilon < \epsilon_\Gamma$ (\(\epsilon_\Gamma\) being as in Section 8.2), let $\chi_\epsilon = \psi_{\epsilon/2} \ast 1_{[-\epsilon_\Gamma + \epsilon, \epsilon_\Gamma - \epsilon/2]}$, i.e.

$$\chi_\epsilon(x) = \int_{-\epsilon_\Gamma + \frac{\epsilon}{2}}^{\epsilon_\Gamma - \frac{\epsilon}{2}} \psi_\epsilon(x - u) \, du.$$  

Observe that $0 \leq \chi_\epsilon(x) \leq 1$ and

$$\chi_\epsilon(x) = \begin{cases} 1 & \text{if } |x| \leq \epsilon_\Gamma - \epsilon \\ 0 & \text{if } |x| \geq \epsilon_\Gamma \end{cases}$$

for all $x \in \mathbb{R}$. Note also that

$$\int_{\mathbb{R}} \left| \frac{d}{dx} \chi_\epsilon(x) \right|^2 \, dx = \int_{-\epsilon_\Gamma}^{\epsilon_\Gamma} \left| \frac{d}{dx} \psi_{\epsilon/2}(x - u) \right|^2 \, dx \leq 2\epsilon_\Gamma \int_{-\epsilon_\Gamma}^{\epsilon_\Gamma} |\psi_{\epsilon/2}^{(j)}(u)|^2 \, du,$$

so $\|\chi_\epsilon\|_{W^2} \ll \epsilon_\Gamma \|\psi_\epsilon\|_{W^2} \ll \epsilon^{-\frac{2}{3}}$. This choice of $\chi_\epsilon$ and the fact that $f(\Gamma h) \geq 0$ for all $\Gamma h \in \Gamma \setminus G$ gives

$$\int_{-T}^T f(\Gamma g \tau) \, dt = \frac{T}{\epsilon_\Gamma} \int_{-\epsilon_\Gamma}^{\epsilon_\Gamma} f(\Gamma g a_T / \epsilon_\Gamma, n_1 a_T^{-1} / \epsilon_\Gamma) \, dt \geq \frac{T}{\epsilon_\Gamma} \int_{-\epsilon_\Gamma}^{\epsilon_\Gamma} f(\Gamma g a_T / \epsilon_\Gamma, n_1 a_T^{-1} / \epsilon_\Gamma) \chi_\epsilon(t) \, dt, \tag{28}$$

and

$$\int_{-T}^T f(\Gamma g \tau) \, dt = \frac{T}{\epsilon_\Gamma - \epsilon} \int_{-\epsilon_\Gamma}^{\epsilon_\Gamma - \epsilon} f(\Gamma g a_T / (\epsilon_\Gamma - \epsilon), n_1 a_T^{-1} / (\epsilon_\Gamma - \epsilon)) \, dt \leq \frac{T}{\epsilon_\Gamma - \epsilon} \int_{-\epsilon_\Gamma}^{\epsilon_\Gamma} f(\Gamma g a_T / (\epsilon_\Gamma - \epsilon), n_1 a_T^{-1} / (\epsilon_\Gamma - \epsilon)) \chi_\epsilon(t) \, dt. \tag{29}$$

Define $\Omega_{\Gamma g} := \Omega \cup \{\Gamma g g y : y \geq 1\}$. Since $\Gamma g \in \Gamma \setminus \Gamma_{\text{rad}}$ and $\Gamma$ is convex cocompact, $\Omega_{\Gamma g}$ is compact. Assuming $T \geq \epsilon_\Gamma$ then allows us to apply Theorem 27 for $r \leq \epsilon_\Gamma$,

$$\frac{T}{r} \int_{-\epsilon_\Gamma}^{\epsilon_\Gamma} f(\Gamma g a_T / r, n_1 a_T^{-1} / r) \chi_\epsilon(t) \, dt = \frac{T}{r} \left( \frac{m^{BR}(f) \mu_{\Gamma g T / r \mathcal{N}(\chi_\epsilon)}}{m^{PS}(\Gamma \setminus G)} \right)^{\frac{T}{r}} 1_{-\delta_T} + O_{\Gamma, \Omega_{\Gamma g}} \left( \frac{T}{r} \right)^{1-\delta_T + \eta_1} \|f\|_{S^1(\Gamma \setminus G)} \left\{ \|\chi_\epsilon\|_{W^3} + \mu_{\Gamma g T / r \mathcal{N}(\chi_\epsilon)} \right\} \right)$$

$$= \frac{m^{BR}(f) \mu_{\Gamma g T / r \mathcal{N}(\chi_\epsilon)}}{m^{PS}(\Gamma \setminus G)} \left( \frac{T}{r} \right)^{\delta_T} + O_{\Gamma, \Omega_{\Gamma g}} \left( \frac{T}{r} \right)^{\delta_T - \eta_1} \|f\|_{S^1(\Gamma \setminus G)} \left\{ \epsilon^{-\frac{2}{3}} + \mu_{\Gamma g T / r \mathcal{N}(\chi_\epsilon)} \right\}. \tag{28}$$

We now observe that

$$\mu_{\Gamma g T / r \mathcal{N}(\chi_\epsilon)} = \mu_{\Gamma g T / r \mathcal{N}(B_T)} + O\left( \mu_{\Gamma g T / r \mathcal{N}(\{t \in \mathbb{R} : \min\{\epsilon_\Gamma - \epsilon, r\} \leq |t| \leq \epsilon_\Gamma\})} \right)$$

and so Lemma 111 gives

$$\mu_{\Gamma g T / r \mathcal{N}(\chi_\epsilon)} = \left( \frac{T}{r} \right)^{-\delta_T} \left( \mu_{\Gamma g T / r \mathcal{N}(B_T)} + O\left( \mu_{\Gamma g T / r \mathcal{N}(\{t \in \mathbb{R} : \frac{\min\{\epsilon_\Gamma - \epsilon, r\}}{r} \leq |t| \leq \frac{\epsilon_\Gamma}{r} T\})} \right) \right).$$

Since our choices of $r$ are $r = \epsilon_\Gamma$ and $r = \epsilon_\Gamma - \epsilon$, in both cases we have

$$\mu_{\Gamma g T / r \mathcal{N}(\{t \in \mathbb{R} : \frac{\min\{\epsilon_\Gamma - \epsilon, r\}}{r} T \leq |t| \leq \frac{\epsilon_\Gamma}{r} T\})} \leq \mu_{\Gamma g T / r \mathcal{N}(\{t \in \mathbb{R} : (1 - \frac{\epsilon_\Gamma}{r}) T \leq |t| \leq (1 + \frac{\epsilon_\Gamma}{r}) T\})}. \tag{29}$$
Assuming \( \epsilon \leq \min\{ \frac{1}{7}, \frac{1}{\epsilon_T} \} \), by the definition of \( \mu^{PS}_{\Gamma gN} \) and Lemma \[\text{[31]}\] we have

\[
\mu^{PS}_{\Gamma gN}(\{t \in \mathbb{R} : (1 - \frac{\epsilon}{\epsilon_T})T \leq |t| \leq (1 + \frac{\epsilon}{\epsilon_T})T\}) \\
\ll g_y T^{2\delta r} \times t^{(\frac{\epsilon}{\epsilon_T})g}\{\{t \in \mathbb{R} : (1 - \frac{\epsilon}{\epsilon_T})T \leq |t| \leq (1 + \frac{\epsilon}{\epsilon_T})T\}\}
\]

This gives

\[
\frac{1}{\mu^{PS}_{\Gamma gN}(B_T)} \int_{-T}^{T} \frac{f(\Gamma g_{\eta_T})}{m^{\mu\theta}_{\Gamma gN}} \chi_{\epsilon}(t) \, dt = \frac{m^{\mu\theta}(f)}{m^{\mu\theta}_{\Gamma gN}(G)} + O_{\Gamma, \Omega, \bar{\eta}} \left( \|f\|_{S^3(\Gamma \setminus G)} \right) \left\{ T^{\delta r - \eta_T} e^{-\frac{\eta_T}{2}} + T^{\delta r - \eta_T} e^{-\epsilon_T} \right\}.
\]

Since \( \int_{-T}^{T} f(\Gamma g_{\eta_T}) \, dt \) is bounded from above and below by the integrals in the right-hand side of this expression (cf. \[\text{[23]}\] and \[\text{[29]}\]), the same must hold for it. Dividing by \( \mu^{PS}_{\Gamma gN}(B_T) \) and using the bounds \( m^{\mu\theta}(f) \ll_{\Gamma, \Omega, \bar{\eta}} \|f\|_{S^3(\Gamma \setminus G)} \) and \( \mu^{PS}_{\Gamma gN}(B_T) \gg_{\Gamma, \Omega, \bar{\eta}} T^{\delta r} \) then yields

\[
\frac{1}{\mu^{PS}_{\Gamma gN}(B_T)} \int_{-T}^{T} f(\Gamma g_{\eta_T}) \, dt = \frac{m^{\mu\theta}(f)}{m^{\mu\theta}_{\Gamma gN}(G)} + O_{\Gamma, \Omega, \bar{\eta}} \left( \|f\|_{S^3(\Gamma \setminus G)} \right) \left\{ \epsilon_T + T^{-\frac{\eta_T}{2}} e^{-\frac{\eta_T}{2}} + T^{\delta r - \eta_T} \right\}.
\]

Choosing \( \epsilon = T^{-\frac{\eta_T}{2}} \) gives

\[
\frac{1}{\mu^{PS}_{\Gamma gN}(B_T)} \int_{-T}^{T} f(\Gamma g_{\eta_T}) \, dt = \frac{m^{\mu\theta}(f)}{m^{\mu\theta}_{\Gamma gN}(G)} + O_{\Gamma, \Omega, \bar{\eta}} \left( \|f\|_{S^3(\Gamma \setminus G)} \right) T^{-\frac{\eta_T}{2}},
\]

where \( \bar{\eta}_T = \frac{\delta r - \eta_T}{2} \). Theorem \[\text{[2]}\] is thus proved for non-negative functions.

In order to generalize to all functions in \( S^3(\Omega) \), we first that note that if \( f \in S^3(\Omega) \), then \( \text{Im}(f), \text{Re}(f) \in S^3(\Omega) \), and \( \|\text{Re}(f)\|_{S^3(\Gamma \setminus G)} \ll \|f\|_{S^3(\Gamma \setminus G)} \) and \( \|\text{Im}(f)\|_{S^3(\Gamma \setminus G)} \ll \|f\|_{S^3(\Gamma \setminus G)} \), so by considering the real and imaginary parts it suffices to to extend \[\text{[30]}\] to \( \mathbb{R} \)-valued \( f \in S^3(\Omega) \).

By Lemma \[\text{[26]}\] there exists \( C = C(\Gamma) \) such that if \( \text{dist}_G(h, \epsilon) < \epsilon \), then \( \|f(\Gamma gh) - f(\Gamma g)\|_{S^3(\Gamma \setminus G)} \leq C\epsilon\|f\|_{S^3(\Gamma \setminus G)} \). Using this, we assume now that \( f \) is \( \mathbb{R} \)-valued, and for \( \epsilon > 0 \), define sets \( \Omega^+_{\epsilon}(f), \Omega^-_{\epsilon}(f) \subset \Omega \) by

\[
\Omega^+_{\epsilon}(f) = \{ \Gamma h \in \Omega : (\pm 1)f(\Gamma h) > C\epsilon\|f\|_{S^3(\Gamma \setminus G)} \},
\]

We now turn again to \[\text{[19]}\] Proposition 2.4.7]: for all \( 0 < \epsilon < 1 \) there exists \( \rho_{\epsilon} \in C^\infty_c(\mathcal{B}_\epsilon) \) such that

\[
\rho_{\epsilon}(h) \geq 0 \quad \forall h \in G, \quad \int_{\mathcal{B}_\epsilon} \rho_{\epsilon}(h) \, d\mu_G(h) = 1, \quad \|\rho_{\epsilon}\|_{S^m(G)} \ll \epsilon^{-(m+3/2)},
\]

where \( S^m(G) \) denotes the \( m \)-th order \( L^2 \)-Sobolev norm on \( G \) (defined analogously to \( S^m(\Gamma \setminus G) \)). Define functions \( \varphi_{f,\epsilon}^\pm \) on \( \Gamma \setminus G \) by

\[
\varphi_{f,\epsilon}^\pm(\Gamma h) = \mathbb{1}_{\Omega^\pm_{\epsilon}(f)} \ast \rho_{\epsilon/2}(\Gamma h) = \int_{\mathcal{B}_{\epsilon/2}} \mathbb{1}_{\Omega^\pm_{\frac{\epsilon}{2}}(\Gamma hh')\rho_{\epsilon/2}(h'^{-1})} \, d\mu_G(h').
\]

This definition gives \( \text{supp}(\varphi_{f,\epsilon}^\pm) \subset \Omega^\pm_{\epsilon/2}(f) \subset \Omega \) and \((\pm 1)f(\Gamma h) > C\epsilon\|f\|_{S^3(\Gamma \setminus G)} \Rightarrow \varphi_{f,\epsilon}^\pm(\Gamma h) = 1 \). Note also that

\[
\|\varphi_{f,\epsilon}^\pm\|_{S^m(\Gamma \setminus G)} \ll_{\Gamma, \Omega} \epsilon^{-(m+3/2)}.
\]
We now use $[30]$: 
\[
\frac{1}{\mu_{\Gamma g N}(B_T)} \int_{-T}^{T} f(\Gamma g n_t) dt = \frac{1}{\mu_{\Gamma g N}(B_T)} \int_{-T}^{T} \varphi_{f, \epsilon}^+(\Gamma g n_t) f(\Gamma g n_t) dt \\
- \frac{1}{\mu_{\Gamma g N}(B_T)} \int_{-T}^{T} \varphi_{f, \epsilon}^- f(\Gamma g n_t) dt \\
+ \frac{1}{\mu_{\Gamma g N}(B_T)} \int_{-T}^{T} [\Omega - \varphi_{f, \epsilon}^+ - \varphi_{f, \epsilon}^-] (\Gamma g n_t) f(\Gamma g n_t) dt
\]

\[
= \frac{m^{BR}(f)}{m^{BMS}(\Gamma \setminus G)} + O_{\Gamma, \Omega, \Gamma g} \left( m^{BR}(\Omega) + \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} + \| f \varphi_{f, \epsilon}^- \|_{S^1(\Gamma \setminus G)} \right) T^{-\eta}\]

The terms in the “$O_{\Gamma, \Omega, \Gamma g}$” are dealt with individually:

\[
| m^{BR}(\Omega) | \leq C \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon^{3/2}.
\]

To bound \( \frac{1}{\mu_{\Gamma g N}(B_T)} \int_{-T}^{T} [\Omega - \varphi_{f, \epsilon}^+ - \varphi_{f, \epsilon}^-] (\Gamma g n_t) f(\Gamma g n_t) dt \), we note that

\[
| [\Omega - \varphi_{f, \epsilon}^+ - \varphi_{f, \epsilon}^-] (\Gamma g n_t) f(\Gamma g n_t) | \leq C \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq C \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq C \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon^{3/2}.
\]

Now, \( \supp(\Omega * \rho) = \Omega B_\epsilon, \| \Omega * \rho \|_{L^\infty(\Gamma \setminus G)} = 1 \), and \( \Omega * \rho \|_{S^1(\Gamma \setminus G)} \leq \varepsilon^{-1/2} \). We apply $[30]$ again:

\[
\frac{1}{\mu_{\Gamma g N}(B_T)} \int_{-T}^{T} [\Omega - \varphi_{f, \epsilon}^+ - \varphi_{f, \epsilon}^-] (\Gamma g n_t) f(\Gamma g n_t) dt \\
\leq \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon \| f \varphi_{f, \epsilon}^+ \|_{S^1(\Gamma \setminus G)} \leq \varepsilon^{3/2}.
\]

In total, we have

\[
\frac{1}{\mu_{\Gamma g N}(B_T)} \int_{-T}^{T} f(\Gamma g n_t) dt = \frac{m^{BR}(f)}{m^{BMS}(\Gamma \setminus G)} + O_{\Gamma, \Omega, \Gamma g} \left( \varepsilon + \varepsilon^{3/2} T^{-\eta} \right).
\]

Letting \( \varepsilon = T^{-\eta} \) completes the proof, with \( \eta = \frac{2\eta}{11} = \frac{2\eta}{11} \). \qed

References

[1] J. Aaronson, On the ergodic theory of non-integrable functions and infinite measure spaces, Israel Journal of Mathematics, Vol. 27 (1977) pp. 163-173.
[2] J. Aaronson, Z. Kosloff, B. Weiss, Symmetric Birkhoff sums in infinite ergodic theory, Ergodic Theory and Dynamical Systems, 37 (2017), pp. 2394-2416.
[3] A. F. Beardon, The Hausdorff dimension of singular sets of properly discontinuous groups, Amer. J. Math., 88 (1966), pp. 722-736.
[4] J. Bernstein and A. Reznikov, Analytic continuation of representations and estimates of automorphic forms, Ann. of Math., Vol. 150 (1999), pp. 329-352.
[5] D. Borthwick, Spectral theory of infinite-area hyperbolic surfaces, Birkhäuser, 2007.
[6] B. H. Bowditch. Geometrical finiteness for hyperbolic groups. J. Funct. Anal., Vol. 113(2) (1993), pp. 245-317.
[7] B. H. Bowditch, Geometrical finiteness with variable negative curvature, Duke Math. J. 77(1) (1995), pp. 229-274.
[8] M. Burger, Horocycle flow on geometrically finite surfaces, Duke Math. J., Vol. 61, (1990), pp. 779-803.
[9] M. Burger and R. Canary, A lower bound on $\lambda_0$ for geometrically finite hyperbolic manifolds, Journal für die reine und angewandte Mathematik, 454 (1994), pp. 37-57.
[10] S. G. Dani, Invariant measures of horospherical flows on noncompact homogeneous spaces, Invent. Math. 47 (1978), no. 2, pp. 101-138.
[11] S. G. Dani, Invariant measures and minimal sets of horospherical flows, Invent. Math. 64 (1981), pp. 357-385.
[12] S. G. Dani, On uniformly distributed orbits of certain horocycle flows, Ergodic Theory and Dynamical Systems, Vol. 2 (1982), pp. 139-158.
[13] S. G. Dani and J. Smillie, Uniform distribution of horocycle orbits for Fuchsian groups, Duke Math. J. 51 (1984), no. 1, pp. 185-194.
[14] S. C. Edwards, On the rate of equidistribution of expanding horospheres in finite-volume quotients of $SL(2,\mathbb{C})$, J. Mod. Dyn., Vol. 11 (2017), pp. 155-188.
[15] S. C. Edwards, Effective equidistribution of horospheres in infinite-volume quotients of $SO(n,1)$ by geometrically finite groups, preprint (2018).
[16] L. Flaminio and G. Forni, Invariant distributions and time averages for horocycle flows, Duke Math. J. 119 (2003), No. 3, pp. 465-526.
[17] H. Furstenberg, The unique ergodicity of the horocycle flow, Recent Advances in Topological Dynamics (Proc. Conf., Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund), Lecture Notes in Math., vol. 318, Springer-Verlag, Berlin, 1973, pp. 95-115.
[18] R. Hill and S. Velani, The Jarník-Besicovitch theorem for geometrically finite Kleinian groups, Proceedings of the LMS, 77 (1998), pp. 524-550.
[19] D. Kleinbock and G. A. Margulis, Bounded orbits of nonquasiunipotent flows on homogeneous spaces, Sinai’s Moscow Seminar on Dynamical Systems, Amer. Math. Soc. Transl. Ser. 2, 171, Amer. Math. Soc., Providence, RI (1996), pp. 141-172.
[20] A. Knapp, Lie Groups Beyond an Introduction Second Edition, Birkhäuser, 2002.
[21] P.D. Lax and R.S. Phillips, The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces, Journal of Functional Analysis, Vol. 46 No. 3 (1982), pp. 280-350.
[22] M. Lee and H. Oh, Effective circle count for Apollonian circle packings and closed horospheres, Geom. Funct. Anal., Vol. 23 (2013), pp. 580-621.
[23] F. Maucourant, B. Schapira, Distribution of orbits in $\mathbb{R}^2$ of a finitely generated group of $SL(2,\mathbb{R})$, Amer. J. of Math., vol. 136, no. 6 (2014), pp. 1497-1542.
[24] M. V. Melián and D. Pestana, Geodesic excursions into cusps in finite-volume hyperbolic manifolds, Michigan Math. J., 40 (1993), pp. 77-93.
[25] T. Miyake, Modular Forms, Springer, Berlin, 1989.
[26] A. Mohammadi and H. Oh, Matrix coefficients, counting and primes for orbits of geometrically finite groups, J. Eur. Math. Soc., Vol. 17 (2015), pp. 837-897.
[27] A. Mohammadi, H. Oh, Classification of joinings for Kleinian groups, Duke Math. J., Vol. 165 (2016), pp. 2155-2223.
[28] P. Nicholls, The ergodic theory of discrete groups, London Mathematical Society Lecture Note Series, Cambridge University Press, 1989.
[29] H. Oh and D. Winter, Uniform exponential mixing and resonance free regions for convex co-compact congruence subgroups of $SL_2(\mathbb{Z})$, Journal of the American Mathematical Society, Vol. 29 (2016) pp. 1069-1115.
[30] S. J. Patterson, The Laplacian operator on a Riemann surface, Compositio Math., Vol. 31 (1975), pp. 83-107.
[31] S. J. Patterson, The limit set of a Fuchsian group, Acta Math. 136 (1976), pp. 241-273.
[32] M. Ratner, On Raghunathans measure conjecture, Ann. of Math. (2) 134 (1991), no. 3, pp. 545-607.
[33] T. Roblin, Ergodicité et équidistribution en courbure négative. Mém. Soc. Math. Fr. (N.S.), No. 95 (2003), vi+96, pp. 1-96.
[34] P. Sarnak and A. Ubis, The horocycle flow at prime times, J. Math. Pures Appl. Vol. 103 (2015), pp. 575-618.
[35] B. Schapira, Lemme de l’ombre et non divergence des horocycles dune variété géométriquement finie, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 4, pp. 939-987.
[37] B. Schapira, Equidistribution of the horocycles of a geometrically finite surface, International Mathematics Research Notices, Issue 40 (2005), pp. 2447-2471.

[38] B. Schapira, Density and equidistribution of halfhorocycles on a geometrically finite hyperbolic surface, Journal of the London Mathematical Society, Vol. 84 (2011), pp. 785-806.

[39] B. Schapira, Dynamics of Geodesic and Horocyclic Flows, in: B. Hasselblatt (eds), Ergodic Theory and Negative Curvature, Lecture Notes in Mathematics, Vol. 2164, Springer 2017, pp. 129-155.

[40] H. Shimizu, On discontinuous groups operating on the product of the upper half planes, Ann. of Math. (2) 77 (1963), pp. 33-71.

[41] L. Stoyanov, Spectra of Ruelle transfer operators for axiom A flows, Nonlinearity 24 (2011), no. 4, pp. 1089-1120.

[42] B. Stratmann and S. Velani, The Patterson measure for geometrically finite groups with parabolic elements, new and old, Proc. of the London Math. Soc., Vol. s3-71, Issue 1 (1995), pp. 197-220.

[43] A. Strömbergsson, On the uniform equidistribution of long closed horocycles, Duke Math. J., Vol. 123 (2004), pp. 507-547.

[44] A. Strömbergsson, On the Deviation of Ergodic Averages for Horocycle Flows, J. Mod. Dyn., Vol. 7 (2013), pp. 291-328.

[45] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Pub. Math. de IHES, (50) (1979), pp. 171-202.

[46] D. Sullivan, Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics, Acta Math., 149 (1982), pp. 215-237.

[47] D. Sullivan, Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups, Acta Math. Vol. 153 (1984), pp. 259-277.

[48] W. A. Veech, Unique ergodicity of horospherical flows, Amer. J. of Math. 99 (1977), no. 4, pp. 827-859.

[49] D. Winter, Mixing of frame flow for rank one locally symmetric spaces and measure classification, Israel Journal of Mathematics, Vol. 210 (2015) pp. 467-507.

Department of Mathematics, Yale University, New Haven 06511 CT, USA

E-mail address: samuel.edwards@yale.edu