The characterization of the graphs of bilinear \((d \times d)\)-forms over \(\mathbb{F}_2\)

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Abstract

The bilinear forms graph denoted here by \(Bil_q(e \times d)\) is a graph defined on the set of \((e \times d)\)-matrices \((e \geq d)\) over \(\mathbb{F}_q\) with two matrices being adjacent if and only if the rank of their difference equals 1.

In 1999, K. Metsch showed that the bilinear forms graph \(Bil_q(e \times d)\) is characterized by its intersection array if one of the following holds:
- \(q = 2\) and \(e \geq d + 4\),
- \(q \geq 3\) and \(e \geq d + 3\).

Thus, the following cases have been left unsettled:
- \(q = 2\) and \(e \in \{d, d+1, d+2, d+3\}\),
- \(q \geq 3\) and \(e \in \{d, d+1, d+2\}\).

In this work, we show that the graph of bilinear \((d \times d)\)-forms over the binary field, where \(d \geq 3\), is characterized by its intersection array. In doing so, we also classify locally grid graphs whose \(\mu\)-graphs are hexagons and the intersection numbers \(b_i, c_i\) are well-defined for all \(i = 0, 1, 2\).

1 Introduction

Let \(\mathbb{F}_q\) be the field with \(q\) elements. For integers \(e \geq d \geq 2\), define the bilinear forms graph \(Bil_q(e \times d)\), whose vertices are all \((e \times d)\)-matrices over \(\mathbb{F}_q\) with two matrices being adjacent if and only if the rank of their difference is equal to 1.
It is well known that $Bil_q(e \times d)$ is a $Q$-polynomial distance-regular graph with diameter $d$. (For definitions and notations see Section 2.)

Much attention has been paid to the problem of classification of all $Q$-polynomial distance-regular graphs with large diameter, which was suggested in the fundamental monograph by Bannai and Ito [1]. One of the steps towards the solution of this problem is a characterization of the known $Q$-polynomial distance-regular graphs by their intersection arrays. (The current status of the project can be found in the survey paper [11] by Van Dam, Koolen and Tanaka.)

As for the bilinear forms graphs, these graphs have been characterized, under some additional assumption (the so-called weak 4-vertex condition), for $e \geq 2d \geq 6$ and $q \geq 4$ by Huang [16], see also [14], and for $e \geq 2d+2 \geq 8$ and $q \geq 2$ by Cuypers [8], while the strongest result was obtained by Metsch in 1999 [21], who showed that the bilinear forms graph $Bil_q(e \times d)$, $d \geq 3$, can be uniquely determined as a distance-regular graph by its intersection array unless one of the following cases holds:

- $q = 2$ and $e \in \{d, d + 1, d + 2, d + 3\}$,
- $q \geq 3$ and $e \in \{d, d + 1, d + 2\}$.

In this work, we show that the graph of bilinear $(d \times d)$-forms, where $d \geq 3$, defined over the binary field is also characterized by its intersection array (see Theorem 1.3).

We remark that in the diameter two case there exist many non-isomorphic strongly regular graphs with the same parameters as $Bil_q(e \times 2)$. Indeed, the graph $Bil_q(e \times 2)$ has parameters

$$(v, k, \lambda, \mu) = (m^2, (m - 1)t, m - 2 + (t - 1)(t - 2), t(t - 1)),$$  

(1)

where $m = q^e$ and $t = q + 1$.

A strongly regular graph with parameters given by Eq. (1) is usually called a pseudo Latin square graph (see [5, Ch. 9.1.12]). A strongly regular Latin square graph can be constructed from $t - 2$ mutually orthogonal Latin $m \times m$-squares, and thus there exist exponentially many non-isomorphic strongly regular graphs with the same parameters given by Eq. (1), see [6] for the details.

Let us also briefly recall an idea, which was exploited in Metsch’s proof [21]. An incidence structure is a triple $(P, L, I)$ where $P$ and $L$ are sets (whose elements are called points and lines, respectively) and $I \subseteq P \times L$ is the incidence relation. We also assume that every line is incident with at least two points. An incidence structure is called semilinear (or a partial linear space) if there exists at most one line through any two points. The point graph of the incidence structure $(P, L, I)$ is a graph defined on $P$ as the vertex set, with two points being adjacent if they belong to the same line.

A semilinear incidence structure can be naturally derived from the bilinear forms graph $Bil_q(e \times d)$. For this purpose, we recall an alternative definition of $Bil_q(e \times d)$ [4, Chapter 9.5.A]. Let $V$ be a vector space of dimension $e + d$ over $\mathbb{F}_q$, $W$ be a fixed $e$-subspace of $V$. For an integer $i \in \{d - 1, d\}$, define

$$\mathcal{A}_i = \{U \subseteq V \mid \dim(U) = i, \dim(U \cap W) = 0\}.$$
Then \((\mathcal{A}_d, \mathcal{A}_{d-1}, \supseteq)\) is a semilinear incidence structure called the \((e, q, d)\)-attenuated space, while its point graph is isomorphic to \(Bil_q(e \times d)\). In other words, the vertices of \(Bil_q(e \times d)\) are the subspaces of \(\mathcal{A}_d\), with two subspaces from \(\mathcal{A}_d\) adjacent if and only if their intersection has dimension \(d - 1\).

Now it is easily seen that \(Bil_q(e \times d)\) has two types of maximal cliques. The maximal cliques of the first type are the collections of subspaces of \(\mathcal{A}_d\) containing a fixed subspace of dimension \(d - 1\), and each of them contains \(\left\lceil \frac{e + 1}{1} \right\rceil_q - \left\lceil \frac{e}{1} \right\rceil_q = q^e\) vertices, while the maximal cliques of the other type are the collections of subspaces of \(\mathcal{A}_d\) contained in a fixed subspace of dimension \(d + 1\), and each of them contains \(\left\lceil \frac{d + 1}{1} \right\rceil_q - \left\lceil \frac{d}{1} \right\rceil_q = q^d\) vertices, where \(\left[ \begin{array}{c} n \\ m \end{array} \right]_q\) denotes the \(q\)-ary Gaussian binomial coefficient. Note that the maximal cliques of the first type correspond to the lines of the semilinear incidence structure \((\mathcal{A}_d, \mathcal{A}_{d-1}, \supseteq)\).

Suppose now that a graph \(\Gamma\) is distance-regular with the same intersection array as \(Bil_q(e \times d)\). A key idea of the works by Huang [16] and Metsch [21] was as follows. Under certain conditions on \(e, d,\) and \(q\), it is possible to show that every edge of \(\Gamma\) is contained in a unique clique of size \(\sim q^e\), called a grand clique of \(\Gamma\). Hence \((V(\Gamma), L, \varepsilon)\) is a semilinear incidence structure, where \(L\) is the set of all grand cliques of \(\Gamma\). In order to show the existence of grand cliques, Huang used the so-called Bose-Laskar argument, which was valid for \(e \geq 2d \geq 6\), and Metsch applied its improved version [20], which was valid under weaker assumptions on \(e, q,\) and \(d\). One can then show that the semilinear incidence structure \((V(\Gamma), L, \varepsilon)\) satisfies some additional properties, and, in fact, it is a so-called \(d\)-net (see [16]). Finally, the result by Sprague [25] shows, for an integer \(d \geq 3\), every finite \(d\)-net is the \((e, q, d)\)-attenuated space for some prime power \(q\) and positive integer \(e\), and therefore \(\Gamma\) is isomorphic to \(Bil_q(e \times d)\).

For the cases remained open after the Metsch result, it seems that the Bose-Laskar type argument cannot be applied. Moreover, when \(e = d\), the maximal cliques of both families have the same size \(q^e = q^d\). Therefore, even if one can show that \(\Gamma\) contains such cliques, every edge is contained in two grand cliques. Thus, one has to prove that it is still possible to select a family of grand cliques that form lines of a semilinear incidence structure (when \(e \neq d\), we can easily distinguish between families of maximal cliques by their sizes). However, it is not possible in general, as for example, it is the case for the quotient of the Johnson graph \(J(2d, d)\), which has two families of maximal cliques of the same size, not being the point graph of any semilinear incidence structure, see [11 Proposition 2.7, Remark 2.8]).

In the present work, we will make use of a completely different approach, exploiting the \(Q\)-polynomiality of the bilinear forms graph. Namely, suppose that \(\Gamma\) is a \(Q\)-polynomial distance-regular graph with diameter \(D \geq 3\). In 1993, Terwilliger (see 'Lecture note on Terwilliger algebra' edited by Suzuki, [26]) showed that, for \(i = 2, 3, \ldots, D - 1\), there exists a polynomial \(T_i(\lambda) \in \mathbb{C}[\lambda]\) of degree \(4\) such that for any \(i\), any vertex \(x \in \Gamma\), and any non-principal eigenvalue \(\eta\) of the local graph \(\Gamma(x)\), one has

\[T_i(\eta) \geq 0.\]

We call \(T_i(\lambda)\) the Terwilliger polynomial of \(\Gamma\). In [15], the authors gave an explicit formula for this polynomial, and applied it to complete the classification of pseudo-partition graphs.
The Terwilliger polynomial depends only on the intersection array of \( \Gamma \) and its \( Q \)-polynomial ordering (note that the property 'being \( Q \)-polynomial' is determined by the intersection array). Thus, any two \( Q \)-polynomial distance-regular graphs with the same intersection array and \( Q \)-polynomial ordering have the same Terwilliger polynomial.

Using this fact, we first prove the following.

**Result 1.1** Let \( \Gamma \) be a distance-regular graph with the same intersection array as \( \text{Bil}_q(e \times d) \), \( e \geq d \geq 3 \). Let \( \eta \) be a non-principal eigenvalue of the local graph of a vertex \( x \in \Gamma \). Then \( \eta \) satisfies

\[-q - 1 \leq \eta \leq -1, \text{ or } q^d - q - 1 \leq \eta \leq q^e - q - 1.\]

For \( q = 2 \) and \( e = d \), we prove that this information is enough to show that the local graphs of \( \Gamma \) are the \((2^e - 1) \times (2^e - 1)\)-grids (see Lemma 4.2). Thus, \( \Gamma \) contains two families of maximal cliques of size \( 2^e \). By the remark above, we cannot immediately derive a semilinear incidence structure from \( \Gamma \).

Instead of this, applying a beautiful theorem by Munemasa and Shpectorov [22], we prove a more general result (Theorem 1.2), which requires distance-regularity of \( \Gamma \) up to distance 2 only.

**Theorem 1.2** Suppose that \( \Gamma \) is a graph with diameter \( d \geq 2 \) and with the following intersection numbers well-defined:

\[b_0 = nm, \quad b_1 = (n - 1)(m - 1), \quad b_2 = (n - 3)(m - 3), \quad \text{and} \quad c_2 = 6,\]

for some integers \( n, m, n \geq m \geq 3 \), and such that, for every vertex \( x \in \Gamma \), its local graph \( \Gamma(x) \) is the \((n \times m)\)-grid. Then \( \Gamma \) is isomorphic to the graph of bilinear \((e \times d)\)-forms over \( \mathbb{F}_2 \), where \((n, m) = (2^e - 1, 2^d - 1)\).

We recall that the problem of characterization of all locally grid graphs is the well known and rather difficult one, see [3]. In this context, we believe that Theorem 1.2 is of independent interest.

Finally, combining Lemma 4.2 and Theorem 1.2 gives our main result.

**Theorem 1.3** Suppose that \( \Gamma \) is a distance-regular graph with the same intersection array as \( \text{Bil}_2(d \times d) \), \( d \geq 3 \). Then \( \Gamma \) is isomorphic to \( \text{Bil}_2(d \times d) \).

2 Definitions and preliminaries

In this section we recall some basic theory of distance-regular graphs. For more comprehensive background on distance-regular graphs and association schemes, we refer the reader to [1], [4], [11], and [27].
2.1 Distance-regular graphs

All graphs considered in this paper are finite, undirected and simple. Suppose that $\Gamma$ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of adjacent vertices. The distance $d(x,y) := d_{\Gamma}(x,y)$ between any two vertices $x,y$ of $\Gamma$ is the length of a shortest path connecting $x$ and $y$ in $\Gamma$.

For a subset $X$ of the vertex set of $\Gamma$, we will also write $X$ for the subgraph of $\Gamma$ induced by $X$. For a vertex $x \in \Gamma$, define $\Gamma_i(x)$ to be the set of vertices that are at distance precisely $i$ from $x$ ($0 \leq i \leq D$), where $D := \max\{d(x,y) \mid x,y \in \Gamma\}$ is the diameter of $\Gamma$. In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset$. The subgraph induced by $\Gamma_1(x)$ is called the neighborhood or the local graph of a vertex $x$. We often write $\Gamma(x)$ instead of $\Gamma_1(x)$ for short, and we denote $x \sim y$ or simply $x \sim y$ if two vertices $x$ and $y$ are adjacent in $\Gamma$. For a set of vertices $\{x_1,x_2,\ldots,x_s\}$ of $\Gamma$, let $\Gamma(x_1,x_2,\ldots,x_s)$ denote $\cap_{i=1}^{s}\Gamma(x_i)$. In particular, for a pair of vertices $x,y$ of $\Gamma$ with $d(x,y) = 2$, the graph induced on $\Gamma(x,y)$ is called the $\mu$-graph (of $x$ and $y$).

For a graph $\Delta$, a graph $\Gamma$ is called locally $\Delta$ graph if the local graph $\Gamma(x)$ is isomorphic to $\Delta$ for all $x \in \Gamma$. A graph $\Gamma$ is regular with valency $k$ if the local graph $\Gamma(x)$ contains precisely $k$ vertices for all $x \in \Gamma$.

The eigenvalues of a graph $\Gamma$ are the eigenvalues of its adjacency matrix. If, for some eigenvalue $\eta$ of $\Gamma$, its eigenspace contains a vector orthogonal to the all ones vector, we say the eigenvalue $\eta$ is non-principal. If $\Gamma$ is regular with valency $k$, then all its eigenvalues are non-principal unless the graph is connected and then the only eigenvalue that is principal is its valency $k$.

Let $m_i$ denote the multiplicity of eigenvalue $\theta_i$, $0 \leq i \leq t$, of the adjacency matrix $A$ of a graph $\Gamma$, where $t$ is the number of distinct eigenvalues of $\Gamma$. Then, for a natural number $\ell$,

$$\sum_{i=0}^{t} m_i \theta_i^\ell = tr(A^\ell) = \text{ the number of closed walks of length } \ell \text{ in } \Gamma$$

(2)

where $tr(A^\ell)$ is the trace of matrix $A^\ell$.

Let $\Gamma$ be a graph with diameter $D$. For a pair of vertices $x,y \in \Gamma$ at distance $i = d(x,y)$, define

$$c_i(x,y) := |\Gamma(y) \cap \Gamma_{i-1}(x)|, \quad a_i(x,y) := |\Gamma(y) \cap \Gamma_i(x)|, \quad b_i(x,y) := |\Gamma(y) \cap \Gamma_{i+1}(x)|,$$

and we say that the intersection numbers $c_i$, $a_i$, or $b_i$ are well-defined, if $c_i(x,y)$, $a_i(x,y)$, or $b_i(x,y)$ respectively do not depend on the particular choice of vertices $x,y$ at distance $i$.

A connected graph $\Gamma$ with diameter $D$ is called distance-regular, if the intersection numbers $c_i$, $a_i$, and $b_{i-1}$ are well-defined for all $1 \leq i \leq D$. In particular, any distance-regular graph is regular with valency $k := b_0 = c_1 + a_1 + b_1$. We also define $k_i := \frac{b_{i-1} - b_i}{c_i - c_{i-1}}$, $1 \leq i \leq D$, and note that $k_i = |\Gamma_i(x)|$ for all $x \in \Gamma$ (so that $k = k_1$). The array $\{b_0,b_1,\ldots,b_{D-1};c_1,c_2,\ldots,c_D\}$ is called the intersection array of the distance-regular graph $\Gamma$.

A graph $\Gamma$ is distance-regular if and only if, for all integers $h,i,j$ ($0 \leq h,i,j \leq D$), and all vertices
x, y ∈ Γ with d(x, y) = h, the number
\[ p^h_{ij} := |\{z ∈ Γ | d(x, z) = i, d(y, z) = j\}| = |Γ_i(x) \cap Γ_j(y)| \]
does not depend on the choice of x, y. The numbers p^h_{ij} are called the intersection numbers of Γ.

Note that k_i = p^0_{ii}, c_i = p^1_{i,i-1}, a_i = p^1_{ii} (1 ≤ i ≤ D), and b_i = p^1_{i,i+1} (0 ≤ i ≤ D − 1).

Recall that the q-ary Gaussian binomial coefficient is defined by
\[ \left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}. \]

With this notation, the following result holds, see [4, Theorem 9.5.2].

**Result 2.1** The bilinear forms graph \( \text{Bil}_q(e \times d) \), \( e ≥ d \), is distance-regular with diameter \( d \), \( v = q^{de} \) vertices, and has intersection array given by (for \( 1 ≤ j ≤ d \))
\[ b_{j−1} = q^{2j−2}(q−1) \left[ \begin{array}{c} d−j+1 \\ 1 \end{array} \right]_q \left[ \begin{array}{c} e−j+1 \\ 1 \end{array} \right]_q, \tag{3} \]
\[ c_j = q^{j−1} \left[ \begin{array}{c} j \\ 1 \end{array} \right]_q. \tag{4} \]

A distance-regular graph with diameter 2 is called a strongly regular graph. We say that a strongly
regular graph Γ has parameters \( (v, k, \lambda, \mu) \), if \( v = |V(Γ)| \), \( k := b_0 \), \( \lambda := a_1 \), and \( \mu := c_2 \).

It is well known that a strongly regular graph has the three distinct eigenvalues usually denoted by
\( k \) (the valency), and \( r, s \), where \( r > 0 > s \), and \( r \) and \( s \) are the solutions of the following quadratic
equation:
\[ x^2 + (\mu − \lambda)x + (\mu − k) = 0. \]

An s-clique \( L \) of Γ is a complete subgraph (i.e., every two vertices of \( L \) are adjacent) of Γ with
exactly \( s \) vertices. We say that \( L \) is a clique if it is an s-clique for certain \( s \).

By the \( (n × m)\)-grid, we mean the Cartesian product of two complete graphs on \( n \) and \( m \) vertices.
The \( (n × n)\)-grid is a strongly regular graph with parameters \( (n^2, 2(n−1), n−2, 2) \). It is a well-known
fact that the \( (n × n)\)-grid has spectrum
\[ [2(n−1)]^1, [n−2]^{2(n−1)}, [-2]^{(n−1)^2}, \]
where \([θ]^m \) denotes that eigenvalue \( θ \) has multiplicity \( m \). Moreover, any graph with this spectrum is
the \( (n × n)\)-grid unless \( n = 4 \), as the Shrikhande graph is strongly regular with the same parameters
as the \( (4 × 4)\)-grid, see [24].
2.2 The Bose-Mesner algebra

Let $\Gamma$ be a distance-regular graph with diameter $D$. For each integer $i$ ($0 \leq i \leq D$), define the $i$th distance matrix $A_i$ of $\Gamma$ whose rows and columns are indexed by the vertex set of $\Gamma$, and, for any $x, y \in \Gamma$,

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x, y) = i, \\ 0 & \text{if } d(x, y) \neq i. \end{cases}$$

Then $A := A_1$ is just the adjacency matrix of $\Gamma$, $A_0 = I$ (the identity matrix), $A_i^\top = A_i$ ($0 \leq i \leq D$), and

$$A_iA_j = \sum_{h=0}^D p_{ij}^h A_h \quad (0 \leq i, j \leq D),$$

in particular,

$$AA_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (1 \leq i \leq D - 1),$$

$$AA_D = b_{D-1} A_{D-1} + a_D A_D,$$

and this implies that $A_i = p_i(A)$ for certain polynomial $p_i$ of degree $i$.

The Bose-Mesner algebra $M$ of $\Gamma$ is a matrix algebra generated by $A$ over $\mathbb{R}$. It follows that $M$ has dimension $D + 1$, and it is spanned by the set of matrices $A_0 = I, A_1, \ldots, A_D$, which form a basis of $M$.

Since the algebra $M$ is semi-simple and commutative, $M$ also has a basis of pairwise orthogonal idempotents $E_0 := \frac{1}{|\Gamma|} J, E_1, \ldots, E_D$ (the so-called primitive idempotents of $M$) satisfying:

$$E_iE_j = \delta_{ij} E_i \quad (0 \leq i, j \leq D), \quad E_i = E_i^\top \quad (0 \leq i \leq D),$$

$$E_0 + E_1 + \cdots + E_D = I,$$

where $J$ is the all ones matrix.

We recall that a distance-regular graph with diameter $D$ has $D + 1$ distinct eigenvalues exactly, which can be calculated from its intersection array, see [4, Section 4.1.B].

In fact, $E_j$ ($0 \leq j \leq D$) is the matrix representing orthogonal projection onto the eigenspace of $A$ corresponding to some eigenvalue, say $\theta_j$, of $\Gamma$. In other words, one can write

$$A = \sum_{j=0}^D \theta_j E_j,$$

where $\theta_j$ ($0 \leq j \leq D$) are the real and pairwise distinct scalars, which are exactly the eigenvalues of $\Gamma$ as defined above. We say that the eigenvalues (and the corresponding idempotents $E_0, E_1, \ldots, E_D$) are in natural order if $b_0 = \theta_0 > \theta_1 > \ldots > \theta_D$.

The Bose-Mesner algebra $M$ is also closed under entrywise (Hadamard or Schur) matrix multiplication, denoted by $\circ$. The matrices $A_0, A_1, \ldots, A_D$ are the primitive idempotents of $M$ with
respect to $\circ$, i.e., $A_i \circ A_j = \delta_{ij} A_i$, and $\sum_{i=0}^{D} A_i = J$. This implies that

$$E_i \circ E_j = \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D)$$

holds for some real numbers $q_{ij}^h$, known as the Krein parameters of $\Gamma$.

### 2.3 $Q$-polynomial distance-regular graphs

Let $\Gamma$ be a distance-regular graph, and $E$ be one of the primitive idempotents of its Bose-Mesner algebra. The graph $\Gamma$ is called $Q$-polynomial with respect to $E$ (or with respect to an eigenvalue $\theta$ of $A$ corresponding to $E$) if there exist real numbers $c_i^*, a_i^*, b_i^{i-1}$ $(1 \leq i \leq D)$ and an ordering of primitive idempotents such that $E_0 = \frac{1}{|V(\Gamma)|} J$ and $E_1 = E$, and

$$E_1 \circ E_i = b_i^{i-1} E_{i-1} + a_i^* E_i + c_i^{i+1} E_{i+1} \quad (1 \leq i \leq D-1),$$

$$E_1 \circ E_D = b_D^{D-1} E_{D-1} + a_D^* E_D.$$

We call such an ordering of primitive idempotents (and the corresponding eigenvalues of $\Gamma$) $Q$-polynomial. Note that a $Q$-polynomial ordering of the eigenvalues/idempotents does not have to be the natural one.

Further, the dual eigenvalues of $\Gamma$ associated with $E$ (or with its eigenvalue $\theta$) are the real scalars $\theta_i^*$ $(0 \leq i \leq D)$ defined by

$$E = \frac{1}{|V(\Gamma)|} \sum_{i=0}^{D} \theta_i^* A_i.$$

The Leonard theorem ([1, Theorem 5.1], [27, Theorem 2.1]) says that the intersection numbers of a $Q$-polynomial distance-regular graph have at least one of seven possible types: 1, 1$A$, 2, 2$A$, 2$B$, 2$C$, or 3.

We note that the bilinear forms graph $\text{Bil}_q(e \times d)$ is $Q$-polynomial (of type 1) with respect to the natural ordering of idempotents.

### 2.4 Classical parameters

We say that a distance-regular graph $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ if the diameter of $\Gamma$ is $D$, and the intersection numbers of $\Gamma$ satisfy

$$c_i = \left[ \begin{array}{c} i \\ 1 \end{array} \right] \left( 1 + \alpha \left[ \begin{array}{c} i - 1 \\ 1 \end{array} \right] \right),$$

so that, in particular, $c_2 = (b + 1)(\alpha + 1)$,

$$b_i = \left( \left[ \begin{array}{c} D \\ 1 \end{array} \right] - \left[ \begin{array}{c} i \\ 1 \end{array} \right] \right)(\beta - \alpha \left[ \begin{array}{c} i \\ 1 \end{array} \right]),$$

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where
\[ j \begin{bmatrix} 1 \\ j \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}. \]

Note that a distance-regular graph with classical parameters is \( Q \)-polynomial, see \cite{4} Corollary 8.4.2. By \cite{4} Table 6.1, we have the following result.

**Result 2.2** The bilinear forms graph \( Bil_q(e \times d) \), \( e \geq d \), has classical parameters
\[(D, b, \alpha, \beta) = (d, q, q - 1, q^e - 1).\]

### 2.5 The Terwilliger polynomial

In this section, we briefly recall the concept of the Terwilliger polynomial, which was introduced in 1993 (see 'Lecture note on Terwilliger algebra' edited by Suzuki, \cite{26}), and recently studied in our paper \cite{15}. We refer the reader to \cite{15} for further details.

Let \( \Gamma \) be a distance-regular graph. Let \( V = \mathbb{R}^{V(\Gamma)} \) denote the vector space of columns, whose coordinates are indexed by the set \( V(\Gamma) \), and endowed with the standard dot product \( \langle \cdot, \cdot \rangle \), where
\[ \langle u, v \rangle = u^\top v \quad (u, v \in V). \]

Fix any vertex \( x \in \Gamma \). For each integer \( i \) (\( 0 \leq i \leq D \)), let \( E_i^* := E_i^*(x) \) denote a diagonal matrix with rows and columns indexed by \( V(\Gamma) \), and defined by
\[ (E_i^*)_{y,y} = (A_i)_{x,y} \quad (y \in \Gamma). \]

Note that \( E_i^* E_j^* = \delta_{ij} E_i^* \), \( \sum_{i=0}^D E_i^* = I \). These matrices span the dual Bose-Mesner algebra \( \mathcal{M}^* := \mathcal{M}^*(x) \) with respect to vertex \( x \), which is called the base vertex. The Terwilliger (or subconstituent) algebra \( \mathcal{T} := \mathcal{T}(x) \) with respect to \( x \) is the matrix algebra generated by \( \mathcal{M} \) and \( \mathcal{M}^*(x) \), see \cite{27}. (Note that \( \mathcal{T}(x) \) may depend on the base vertex \( x \), as for example it is the case for the twisted Grassmann graphs, see \cite{2}.) That this algebra is semi-simple is a standard observation.

Pick any 3-tuple \( xyz \) of vertices of \( \Gamma \) such that \( y \) and \( z \) are neighbours of \( x \). We consider \( x \) as a base vertex, and let \( \mathcal{T} = \mathcal{T}(x) \) denote the Terwilliger algebra with respect to \( x \).

Let \( [\ell, m, n] := [\ell, m, n]_{x,y,z} \) denote the number of vertices \( u \) of \( \Gamma \) such that \( u \) is at distances \( \ell \) from \( x \), \( m \) from \( y \), and \( n \) from \( z \), respectively, i.e., \( [\ell, m, n] \) denotes a triple intersection number. In general, \( [\ell, m, n] \) depends on the choice of \( x, y, \) and \( z \).

It is known that vanishing of some of the Krein parameters often leads to non-trivial relations between triple intersection numbers, see, for example, \cite{7, 18, 29} and \cite{11} Section 6.3.

Most of the Krein parameters vanish when \( \Gamma \) is \( Q \)-polynomial. Assuming that \( \Gamma \) has a \( Q \)-polynomial structure and its diameter \( D \) is at least 3, Terwilliger \cite{28} and Dickie \cite{13} showed that \([i, i-1, i-1]\)
Consider the scalar product of \( w \) an eigenvector if, for any of its vertices be an irreducible \( T \) particular, any non-zero vector \( W \) is a one-dimensional subspace of endpoint 0, called the 1 where

\[ \tilde{A} = \begin{pmatrix} N & O \\ O & O' \end{pmatrix} \]

and the principal submatrix \( N \) is, in fact, the adjacency matrix of \( \Gamma(x) \), the local graph of the base vertex \( x \), and \( O, O' \) are the all zeros blocks.

We now recall some facts about irreducible \( T \)-modules, see [26], cf. [17]. A \( T \)-module is a subspace \( W \subset \mathcal{V} \) such that \( Tw \in W \) for any \( T \in T, w \in W \). A non-trivial \( T \)-module is irreducible if it does not properly contain a non-zero \( T \)-module. Since \( T \) is semi-simple, each \( T \)-module is a direct sum of irreducible \( T \)-modules, and \( \mathcal{V} \) decomposes into an orthogonal direct sum of irreducible \( T \)-modules.

Let \( W \) be an irreducible \( T \)-module. We define the endpoint of \( W \) by \( \min \{ i : E_i^wW \neq 0 \} \). An irreducible \( T \)-module \( W \) is called thin if \( \dim(E_i^wW) \leq 1 \) for all \( i = 0, 1, \ldots, D \); the graph \( \Gamma \) is called thin if, for any of its vertices \( x \), each irreducible \( T(x) \)-module is thin. There is a unique irreducible \( T \)-module of endpoint 0, called the trivial module; it is thin and has basis \( \{ E_i^w1 \mid 0 \leq i \leq D \} \), where 1 is the all ones vector.

Let \( U_1^* \) be the subspace of \( E_1^* \mathcal{V} \), which is orthogonal to 1 (so that \( w^*Jw = 0 \) for any \( w \in U_1^* \)). Let \( W \) be an irreducible \( T \)-module of endpoint 1. Then \( E_1^*W \) is a one-dimensional subspace of \( U_1^* \); in particular, any non-zero vector \( w \in E_1^*W \) is an eigenvector of \( \tilde{A} \), and \( W = Tw \). Conversely, for an eigenvector \( w \in U_1^* \) of \( \tilde{A} \), the subspace \( W = Tw \) is an irreducible \( T \)-module of endpoint 1. Let \( a_0(W) \) denote the corresponding eigenvalue of \( \tilde{A} \). Note that \( a_0(W) \) is a non-principal eigenvalue of the local graph of \( x \).

Further, for a vector \( w \in E_1^*W \), we define

\[ w_i^+ = E_i^*A_{i-1}E_1^*w, \quad w_i^- = E_i^*A_{i+1}E_1^*w \quad (2 \leq i \leq D - 1). \]

Consider the scalar product of \( w_i^\epsilon \) and \( w_i^\delta \), where \( \epsilon, \delta \in \{ +, - \} \):

\[ \langle w_i^\epsilon, w_i^\delta \rangle = (w_i^\epsilon)^T E_1^*A_{i\epsilon 1}E_i^*E_i^*E_{i\delta 1}E_1^*w = w_i^\epsilon E_1^*A_{i\epsilon 1}E_i^*A_{i\delta 1}E_1^*w, \]
and hence
\[ \langle w_i^+, w_i^- \rangle = w^\top (\alpha \delta \tilde{J} + p_i^\delta (A)) w = \|w\|^2 p_i^\delta (a_0(W)). \] (8)

The determinant of the Gram matrix of \( w_i^+, w_i^- \) is non-negative:
\[ \det \left( \begin{array}{cc}
\langle w_i^+, w_i^+ \rangle & \langle w_i^+, w_i^- \rangle \\
\langle w_i^-, w_i^+ \rangle & \langle w_i^-, w_i^- \rangle
\end{array} \right) \geq 0, \] (9)

and it follows from \[26, \text{Lemma 63, Lecture 34-4}\] that \( W \) is thin if and only if \( w_i^+, w_i^- \) are linearly dependent for every \( i, 2 \leq i \leq D - 1 \).

Define
\[ T_i(\lambda) := p_i^{++}(\lambda)p_i^{--}(\lambda) - p_i^{+-}(\lambda)^2 \] (10)
so that, by Eq. (8),
\[ \det \left( \begin{array}{cc}
\langle w_i^+, w_i^+ \rangle & \langle w_i^+, w_i^- \rangle \\
\langle w_i^-, w_i^- \rangle & \langle w_i^-, w_i^+ \rangle
\end{array} \right) = \|w\|^4 T_i(a_0(W)). \] (11)

Combining Eqs. (9) and (11) gives the following result \[15, \text{Theorem 4.2}\].

**Theorem 2.3** Let \( \Gamma \) be a \( Q \)-polynomial distance-regular graph with diameter \( D \geq 3 \). Then, for any \( i = 2, 3, \ldots, D - 1 \), for any vertex \( x \in \Gamma \) and any non-principal eigenvalue \( \eta \) of the local graph of \( x \), \( T_i(\eta) \geq 0 \) holds, with equality if and only if \( W := T(x)w \) is a thin irreducible \( T(x) \)-module of endpoint 1, where \( w \in E_1^*V \) is an eigenvector of \( \tilde{A} \) with eigenvalue \( \eta = a_0(W) \).

The polynomial \( T_i(\lambda) \) defined by Eq. (10) depends only on the intersection numbers of \( \Gamma \) and its \( Q \)-polynomial ordering.

We will call the polynomial \( T_i(\lambda) \) the **Terwilliger polynomial** of \( \Gamma \).

The following result (see \[15, \text{Proposition 4.3}\]) gives the roots of \( T_i(\lambda) \) for distance-regular graphs with classical parameters.

**Proposition 2.4** Let \( \Gamma \) be a \( Q \)-polynomial distance-regular graph with classical parameters \( (D, b, \alpha, \beta) \) and with diameter \( D \geq 3 \).

Then the sign of the leading term coefficient of \( T_i(\lambda) \) is equal to the sign of
\[ -(2 \left[ \begin{array}{c} i + 1 \\ 1 \end{array} \right] - (1 + b^i)) (2 \left[ \begin{array}{c} i \\ 1 \end{array} \right] - (1 + b^{i-1})), \]
and the four roots of \( T_i(\lambda) \) (for all \( i = 2, 3, \ldots, D - 1 \)) are
\[ \beta - \alpha - 1, \quad -1, \quad -b - 1, \quad \alpha b b^{D-1} - 1 \]
where \( b - 1 - 1 \).
2.6 The Munemasa-Shpectorov theorem

In this section, we recall the Munemasa-Shpectorov theorem (see Theorem 2.6 below), which was shown in [22, Section 7].

Let us recall some definitions from [22]. We define a path in a graph Γ as a sequence of vertices \( x_0, x_1, \ldots, x_s \) such that \( x_i \) is adjacent to \( x_{i+1} \) for \( 0 \leq i < s \), where \( s \) is the length of the path. A subpath of the form \( (y, x, y) \) is called a return. We do not distinguish paths, which can be obtained from each other by adding or removing returns. This gives an equivalence relation on the set of all paths of \( Γ \). Equivalence classes of this relation are in a natural bijection with paths without returns.

A closed path or a cycle is a path with \( x_0 = x_s \). For cycles, we also do not distinguish the starting vertex, i.e., two cycles obtained from one another by a cyclic permutation of vertices are considered as equivalent.

Given two cycles \( \hat{x} = (x_0, x_1, \ldots, x_s = x_0) \) and \( \hat{y} = (y_0, y_1, \ldots, y_t = y_0) \) satisfying \( x_0 = y_0 \), we define a cycle \( \hat{x} \cdot \hat{y} = (x_0, x_1, \ldots, x_s, y_1, \ldots, y_t) \).

Iterating this process, we say that a cycle \( \hat{x} \) can be decomposed into a product of cycles \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_\ell \), whenever there are cycles \( \hat{x}' \) and \( \hat{x}'_1, \hat{x}'_2, \ldots, \hat{x}'_\ell \), equivalent to \( \hat{x} \) and \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_\ell \), respectively, such that \( \hat{x}' = \hat{x}'_1 \cdot \hat{x}'_2 \cdot \ldots \cdot \hat{x}'_\ell \).

A graph is called triangulable, if each of its cycles can be decomposed into a product of triangles (i.e., cycles of length 3). The following lemma (see [22, Lemma 6.2]) gives necessary conditions for a graph to be triangulable.

**Lemma 2.5** Let \( Γ \) be a graph. Suppose that, for any vertex \( x \in Γ \), and \( y_1, y_2 \in Γ_j(x), j \geq 2 \), the following holds.

(i) The graph induced by \( Γ_{j-1}(y_1) \cap Γ(x) \) is connected.

(ii) If \( y_1 \) and \( y_2 \) are adjacent, then \( Γ_{j-1}(y_1) \cap Γ_{j-1}(y_2) \cap Γ(x) \neq \emptyset \).

Then \( Γ \) is triangulable.

We show in Section 3.3 that the bilinear forms graph \( Bil_q(e \times d) \) satisfies the conditions of Lemma 2.5, i.e., \( Bil_q(e \times d) \) is triangulable.

Let \( Γ \) and \( \tilde{Γ} \) be two graphs. Let \( x \) and \( \tilde{x} \) be vertices of \( Γ \) and \( \tilde{Γ} \), respectively. An isomorphism between the local graphs at \( x \) and \( \tilde{x} \), say,

\[
\varphi : \{x\} \cup \tilde{Γ}(\tilde{x}) \to \{x\} \cup Γ(x)
\] (12)
is called *extendable* if there is a bijection

\[ \varphi' : \{ \tilde{x} \} \cup \tilde{\Gamma}(\tilde{x}) \cup \tilde{\Gamma}_2(\tilde{x}) \rightarrow \{ x \} \cup \Gamma(x) \cup \Gamma_2(x), \]

mapping edges to edges, such that \( \varphi' \mid_{\{ \tilde{x} \} \cup \tilde{\Gamma}(\tilde{x})} = \varphi \). In this case, \( \varphi' \) is called the extension of \( \varphi \).

We say that \( \Gamma \) has distinct \( \mu \)-graphs if \( \Gamma(x, y_1) = \Gamma(x, y_2) \) for \( y_1, y_2 \in \Gamma_2(x) \) implies that \( y_1 = y_2 \).

**Theorem 2.6** Let \( \Gamma \) and \( \tilde{\Gamma} \) be two graphs. Assume that \( \Gamma \) has distinct \( \mu \)-graphs and the following holds.

(i) There exists a vertex \( x \) of \( \Gamma \) and a vertex \( \tilde{x} \) of \( \tilde{\Gamma} \), and an extendable isomorphism \( \varphi \) defined by Eq. (12).

(ii) If \( x, \tilde{x} \) are vertices of \( \Gamma \) and \( \tilde{\Gamma} \), respectively, \( \varphi \) is an extendable isomorphism defined by Eq. (12), \( \varphi' \) its extension, and \( \tilde{y} \in \tilde{\Gamma}(\tilde{x}) \), then

\[ \varphi' \mid_{\{ \tilde{y} \} \cup \tilde{\Gamma}(\tilde{y})} : \{ \tilde{y} \} \cup \tilde{\Gamma}(\tilde{y}) \rightarrow \varphi(\{ \tilde{y} \}) \cup \Gamma(\varphi(\tilde{y})) \]

is an extendable isomorphism.

(iii) \( \tilde{\Gamma} \) is triangulable.

Then the graph \( \Gamma \) is covered by \( \tilde{\Gamma} \).

We will use Theorem 2.6 in the proof of Theorem 1.2.

**2.7 Semi-partial geometries**

In this section we briefly recall the notion of a semi-partial geometry and one characterization of a class of semi-partial geometries with certain parameters. For the details, we refer the reader to [12].

A *semi-partial geometry* with parameters \((s, t, \alpha, \mu)\) is a finite incidence structure \( S = (P, B, I) \) for which the following properties hold:

(SPG1) if \( x \) and \( y \) are two distinct points, then there exists at most one line incident with \( x \) and \( y \);

(SPG2) any line is incident with \( s + 1 \) points, \( s \geq 1 \);

(SPG3) any point is incident with \( t + 1 \) lines, \( t \geq 1 \);

(SPG4) if a point \( x \) and a line \( L \) are not incident, then there exist 0 or \( \alpha \) (with \( \alpha \geq 1 \)) points \( x_i \), and, respectively, 0 or \( \alpha \) lines \( L_i \) such that \((x, L_i) \in I, (x_i, L_i) \in I, (x_i, L) \in I \) for all \( i = 1, \ldots, \alpha \).
If two points $x$ and $y$ are collinear, then we write $x \sim y$. If $x$ and $y$ are two distinct collinear points of $S$, then $L_{x,y}$ denotes the line of $S$, which is incident with $x$ and $y$.

A semi-partial geometry $S = (P, B, I)$ satisfies the diagonal axiom if and only if, for any elements $x, y, z, u \in P$, with $x \neq y$, $x \sim y$, and $L := L_{x,y}$, the following implication holds:

$$(z, L) \notin I, (u, L) \notin I, z \sim x, z \sim y, u \sim x, u \sim y \Rightarrow z \sim u.$$ 

A semi-partial geometry is called partial if $\mu = (t + 1)\alpha$ holds.

In Section 3 we will make use of the following result proven in [12, Section 10].

**Theorem 2.7** Let $S = (P, B, I)$ be a semi-partial geometry with parameters $(s, t, \alpha, \mu)$ with $\alpha > 1$ and $\mu = \alpha(\alpha + 1)$, which is not a partial geometry and which satisfies the diagonal axiom. Then $S$ is isomorphic to the structure formed by:

- the lines of an $n$-dimensional projective space $PG(n, q)$, $n \geq 4$, that have no point in common with a given $(n-2)$-dimensional subspace $PG(n-2, q)$ of $PG(n, q)$,
- the planes of $PG(n, q)$ that have exactly one point in common with $PG(n-2, q)$,

and the natural incidence relation, so that

$$s = q^2 - 1, \quad t = \frac{q^{n-1} - 1}{q - 1} - 1, \quad \alpha = q, \quad \mu = q(q + 1).$$

Recall that the bilinear forms graph $Bil_q(e \times d)$ can be defined (see [4, Chapter 9.5.A]) on the set of $d$-dimensional subspaces of the $(e + d)$-dimensional vector space over $F_q$ that are skew to given $e$-dimensional subspace, with two such $d$-subspaces adjacent if their intersection has dimension $d - 1$. Taking into account this definition, we obtain the following direct consequence of Theorem 2.7.

**Corollary 2.8** Let $S = (P, B, I)$ be a semi-partial geometry with parameters $(s, t, \alpha, \mu)$ with $\alpha > 1$ and $\mu = \alpha(\alpha + 1)$, which is not a partial geometry and which satisfies the diagonal axiom. Then $t = q^{(e-1)} - 1$ holds for some prime power $q$ and natural number $e \geq 3$, and the point graph of $S$ is isomorphic to the bilinear forms graph $Bil_q(e \times 2)$.

### 3 Locally grid graphs with hexagons as $\mu$-graphs

In this section, we prove Theorem 1.2. For the rest of the section, we assume that $\Gamma$ is a graph satisfying Theorem 1.2, i.e., $\Gamma$ has diameter $D \geq 2$ and the following intersection numbers are
well-defined:
\[ b_0 = nm, \quad b_1 = (n-1)(m-1), \quad b_2 = (n-3)(m-3), \quad \text{and} \quad c_2 = 6, \] (13)
for some integers \( n, m \), \( n \geq m \geq 3 \), and, for every vertex \( x \in \Gamma \), the local graph \( \Gamma(x) \) is the \((n \times m)\)-grid.

We first need the following simple lemma, which explains the title of this section.

**Lemma 3.1** For every pair of vertices \( y, z \in \Gamma \) with \( d(y, z) = 2 \), the graph induced on \( \Gamma(y, z) \) is a 6-gon.

**Proof:** Let \( y, z \in \Gamma \) be a pair of vertices at distance 2. Let \( x \in \Gamma(y, z) \). As \( \Delta := \Gamma_1(x) \) is the \((n \times m)\)-grid, we see that \( \Delta(y, z) \) is a coclique of size 2. This means that \( \Gamma(y, z) \) is a triangle-free graph with valency 2, on \( c_2 = 6 \) vertices. Thus, \( \Gamma(y, z) \) is a hexagon. \( \blacksquare \)

### 3.1 Embedding of the bilinear forms graphs of diameter 2 into \( \Gamma \)

Pick a vertex \( x \in \Gamma \). According to the assumption of Theorem 1.2, let \( \{w_{ij}\}_{i=1,j=1}^{n,m} \) denote the vertex set of \( \Gamma(x) \), where \( w_{ij} \sim w_{i'j'} \) holds if and only if either \( i = i' \) or \( j = j' \).

Denote by \( L_i \) the maximal \( m \)-clique of \( \Gamma(x) \) that contains the vertices \( w_{ij} \) for \( j = 1, \ldots, m \), and by \( L_j^\top \) the maximal \( n \)-clique of \( \Gamma(x) \) that contains the vertices \( w_{ij} \) for \( i = 1, \ldots, n \).

Let \( z \in \Gamma_2(x) \). Without loss of generality, we may assume that \( \Gamma(x, z) \subset L_1 \cup L_2 \cup L_3 \), say:

\[
\Gamma(x, z) = \{w_{11}, w_{12}, w_{22}, w_{23}, w_{33}, w_{31}\} \subset (\bigcup_{i=1}^{3} L_i) \cap (\bigcup_{j=1}^{3} L_j^\top). \tag{14}
\]

Define a subgraph \( \Sigma \) of \( \Gamma \) induced by the following set of vertices:

\[
\{x\} \cup L_1 \cup L_2 \cup L_3 \cup \{y \in \Gamma_2(x) \mid \Gamma(x, y) \subset L_1 \cup L_2 \cup L_3\}, \tag{15}
\]
so that \( z \in \Sigma, \quad \Sigma(x) = L_1 \cup L_2 \cup L_3 \), and the local graph induced on \( \Sigma(x) \) is the \((3 \times m)\)-grid.

Similarly, one can define a subgraph \( \Sigma^\top \) of \( \Gamma \) induced on:

\[
\{x\} \cup L_1^\top \cup L_2^\top \cup L_3^\top \cup \{y \in \Gamma_2(x) \mid \Gamma(x, y) \subset L_1^\top \cup L_2^\top \cup L_3^\top\}. \tag{16}
\]

The aim of this section is to show the following lemma.

**Lemma 3.2** There exist natural numbers \( e \geq 2 \) and \( d \geq 2 \) such that:
(1) $m = 2^d - 1$ holds, and the graph $\Sigma$ is isomorphic to the bilinear forms graph $\text{Bil}_2(d \times 2)$,

(2) $n = 2^e - 1$ holds, and the graph $\Sigma^T$ is isomorphic to the bilinear forms graph $\text{Bil}_2(e \times 2)$.

We only prove the first statement of Lemma 3.2 since exactly the same argument shows the second one. We first show some claims.

Claim 3.3 With vertices $x$ and $z$ chosen as above, the following holds:

$$
\Gamma_2(z) \cap \Gamma(x) = \left( \bigcup_{i=1}^{3} (L_i \cup L_i^T) \right) \setminus \Gamma(x, z),
$$

(17)

$$
\Gamma_3(z) \cap \Gamma(x) = \Gamma(x) \setminus \left( \bigcup_{i=1}^{3} (L_i \cup L_i^T) \right).
$$

(18)

Proof: Clearly, we see that $\Gamma_3(z) \cap \Gamma(x) \subseteq \Gamma(x) \setminus \left( \bigcup_{i=1}^{3} (L_i \cup L_i^T) \right)$. Now Eqs. (17) and (18) follow as $|\Gamma(x) \setminus \left( \bigcup_{i=1}^{3} (L_i \cup L_i^T) \right)| = (n-3)(m-3)$, and $b_2 = (n-3)(m-3)$ holds by Eq. (13).

It is easily seen that each of the following sets of vertices:

$$
M_1 := \{w_{11}, w_{31}\} \cup \{y \in \Gamma(z) \cap \Gamma_2(x) \mid \{w_{11}, w_{31}\} \subseteq \Gamma(x, y, z)\},
$$

(19)

$$
M_2 := \{w_{12}, w_{22}\} \cup \{y \in \Gamma(z) \cap \Gamma_2(x) \mid \{w_{12}, w_{22}\} \subseteq \Gamma(x, y, z)\},
$$

(20)

$$
M_3 := \{w_{23}, w_{33}\} \cup \{y \in \Gamma(z) \cap \Gamma_2(x) \mid \{w_{23}, w_{33}\} \subseteq \Gamma(x, y, z)\}
$$

(21)

induces a maximal $m$-clique of $\Gamma(z)$.

Claim 3.4 Let $y$ be a vertex of $\Gamma(z) \cap \Gamma_2(x)$. Then $\Gamma(y, x) \subset L_1 \cup L_2 \cup L_3$ holds if and only if $y \in M_i \setminus \Gamma(x, z)$ for some $i \in \{1, 2, 3\}$.

Proof: We first prove the “if” part. Suppose that $y$ is a vertex of $\Gamma(z) \cap \Gamma_2(x)$ such that, without loss of generality, $\{w_{11}, w_{31}\} \subseteq \Gamma(x, y, z)$, i.e., $y \in M_1$ by Eq. (19). Note that the graph induced on $\Gamma(x, y, z)$ consists of an edge or of two disjoint edges.

Suppose that $\Gamma(x, y, z) = \{w_{11}, w_{31}\}$. Then $y \sim w_{1j}$ for some $j > 1$ and $j \neq 2$. If $j > 3$ then $\Gamma(x, y) \subset L_1 \cup L_2 \cup L_3$ holds by $\Gamma(x, y) \subset \Gamma(z) \cup \Gamma_2(z)$ and Eq. (13). If $j = 3$ then $y \sim w_{3i}$ for some $i > 3$, and, further, $y \sim w_{2}$ and $y \sim w_{32}$. But then the subgraph induced by $\Gamma(y, w_{12})$ contains a 2-claw $\{w_{11}; z, w_{13}\}$ and an edge $\{w_{32}, w_{12}\}$, which is disjoint from the 2-claw. This contradicts the fact that $\Gamma(y, w_{12})$ induces a 6-gon by Lemma 3.1.

The case when $\Gamma(x, y, z)$ consists of two disjoint edges can be considered in the same manner. (In this case we also have $\Gamma(x, y) \subset \left( \bigcup_{i=1}^{3} L_i \right) \cap \left( \bigcup_{i=1}^{3} L_i^T \right)$.)

Now the “only if” part is easily seen if we interchange $z$ and $x$, and assuming that $y \in \Gamma(x) \cap \Gamma_2(z)$. Then the assertion follows from Claim 3.3 and the argument proving the “if” part.
Claim 3.5 The graph induced on \( \Sigma(z) \) is the \((3 \times m)\)-grid.

Proof: It follows immediately from Claim 3.4

Claim 3.6 The graph induced on \( \Sigma(z, w) \) is a 6-gon for all \( w \in \Sigma(x) \) such that \( w \not\sim z \).

Proof: Suppose that \( w \in L_i \) for some \( i \in \{1, 2, 3\} \). Applying Claim 3.4 to the tuple \((w, z, x, L_i)\) in the role of \((y, x, z, M_i)\), we obtain that \( \Gamma(w, z) \subset M_1 \cup M_2 \cup M_3 = \Sigma(z) \), i.e., \( \Gamma(w, z) = \Sigma(w, z) \), and the claim follows.

Claim 3.7 The graph induced on \( \Sigma(y, z) \) is a 6-gon for all \( y \in \Sigma \), \( y \not\sim z \).

Proof: By Claim 3.6 we may assume that \( y \in \Sigma_2(x) \) and \( y \not\sim z \). Note that \( \Gamma(x, y, z) \) consists of mutually non-adjacent vertices (as otherwise, for some vertex \( w \in \Gamma(x, y, z) \), the subgraph induced by \( \Gamma(w) \) contains a 3-claw, which is impossible). Thus, \( 0 \leq |\Gamma(x, y, z)| \leq 3 \).

If \( |\Gamma(x, y, z)| = 3 \), then it easily seen that \( \Gamma(z, y) \subset M_1 \cup M_2 \cup M_3 = \Sigma(z) \), since \( \Gamma(z, y) \) contains a vertex from each of cliques \( M_1, M_2, M_3 \).

Suppose that \( |\Gamma(x, y, z)| \in \{0, 1, 2\} \). Then there is an edge, say \( \{w, w'\} \subset \Gamma(x, y) \setminus \Gamma(x, z) \) such that \( w \in L_p \), \( w' \in L_q \) for some distinct \( p, q \in \{1, 2, 3\} \).

It follows from Claim 3.6 that

\[
\Gamma(z, w) = \Sigma(z, w) \subset M_1 \cup M_2 \cup M_3 \quad \text{and} \quad \Gamma(z, w') = \Sigma(z, w') \subset M_1 \cup M_2 \cup M_3.
\]

Let \( N_1, N_2, N_3 \) be three maximal and pairwise disjoint \( m \)-cliques of \( \Gamma(w) \) such that \( \Gamma(z, w) \subset N_1 \cup N_2 \cup N_3 \) and, say, \( N_p = L_p \cup \{x\} \setminus \{w\} \), where \( L_p \ni w \).

Applying Claim 3.4 to the tuple \((z, w, \{M_i\}_{i=1}^3, \{N_i\}_{i=1}^3, w')\) in the role of \((x, z, \{L_i\}_{i=1}^3, \{M_i\}_{i=1}^3, y)\) shows that \( w' \in N_\ell \) for some \( \ell \in \{1, 2, 3\} \setminus \{p\} \).

As the local graphs of \( \Gamma \) are the \((n \times m)\)-grids, the vertex \( w' \) belongs to two maximal cliques of sizes \( n \) and \( m \) of the local graph \( \Gamma(w) \), one of which contains \( y \), and the other one contains \( x \) (and has size \( n \)). The latter is distinct from \( N_p \), and it intersects \( N_p \) in \( x \). Hence the former is \( N_\ell \), and thus \( y \in N_\ell \). Now applying Claim 3.4 to the tuple \((z, w, \{M_i\}_{i=1}^3, \{N_i\}_{i=1}^3, y)\) in the role of \((x, z, \{L_i\}_{i=1}^3, \{M_i\}_{i=1}^3, y)\) shows that \( \Gamma(y, z) \subset M_1 \cup M_2 \cup M_3 \) and \( \Gamma(z, y) = \Sigma(z, y) \). This proves the claim.

Proof of Lemma 3.2 Claims 3.4, 3.5, 3.6, 3.7 show that \( \Sigma \) is a geodetically closed subgraph of \( \Gamma \) with diameter 2, and \( |\Sigma(y, z)| = 6 \) for every pair of non-adjacent vertices \( y, z \in \Sigma \), and, for every vertex \( z \in \Sigma \), the local graph \( \Sigma(z) \) is the \((3 \times m)\)-grid. Therefore \( |\Sigma(y, z)| = m + 1 \) for every
pair of adjacent vertices \( y, z \in \Sigma \). This yields that \( \Sigma \) is a strongly regular graph with parameters 
\((k, \lambda, \mu) = (3m, m + 1, 6)\).

If \( m = 3 \), then \( \Sigma \) has parameters \((16, 9, 4, 6)\). There are only two graphs with this parameter set
(see [24]), namely, the complement to the \((4 \times 4)\)-grid, and the complement to the Shrikhande
graph. The latter one has \( \mu \)-graphs that are not hexagons. The former one is isomorphic to the
bilinear forms graph \( Bil_2(2 \times 2) \). Hence, in this case, \( \Sigma \) is isomorphic to
\( Bil_2(2 \times 2) \).

Further, we assume that \( m > 3 \). Let \( P \) denote the vertex set of \( \Sigma \), \( B \) denote the set of all
maximal 4-cliques of \( \Sigma \). Then \( G = (P, B, \in) \) is a semi-partial geometry with parameters
\((s, t, \alpha, \mu) = (3m - 1, 2, 6)\), which is not a partial geometry, as \( m > 3 \). Moreover, one can see that \( G \)
satisfies the diagonal axiom. Therefore, by Theorem 2.7 and Corollary 2.8, we have that
\( s = q^2 - 1, \ t = \frac{q^d - 1}{q - 1} - 1 \) (for some \( d \geq 3 \)), \( \alpha = q, \mu = q(q + 1) \),
thus, \( q = 2 \), and the point graph of \( G \), i.e., the graph \( \Sigma \), is isomorphic to the bilinear forms graph
\( Bil_2(d \times 2) \). The lemma is proved.

### 3.2 Balls of radius 2 in \( \Gamma \)

In this section, we follow the notation from Section 3.1. Using Lemma 3.2, we shall show that any
ball of radius 2 in \( \Gamma \) is isomorphic to a ball of radius 2 in the bilinear forms graph \( \tilde{\Gamma} := Bil_2(e \times d) \).

**Lemma 3.8** The graphs induced on \(
\{x\} \cup \Gamma(x) \cup \Gamma_2(x) \) and on \( \{\tilde{x}\} \cup \tilde{\Gamma}(\tilde{x}) \cup \tilde{\Gamma}_2(\tilde{x}) \) are isomorphic,
for any vertices \( x \in \Gamma \) and \( \tilde{x} \in \tilde{\Gamma} \).

We first prove some preliminary claims. By Lemma 3.2, we have that \( m = 2^d - 1 \) and \( n = 2^e - 1 \).
As in Section 3.1, we pick a vertex \( x \in \Gamma \), and let \( \{L_i\}_{i=1}^{2^e-1}, \{L_j^\top\}_{j=1}^{2^d-1} \) be the sets of the maximal
cliques of \( \Gamma(x) \), and \( \Gamma(x) = \left\{ w_{ij}\right\}_{i=1, j=1}^{2^e-1, 2^d-1} \), where \( \left\{ w_{ij}\right\} = L_i \cap L_j^\top \).

Recall that, by Lemma 3.1, for a vertex \( y \in \Gamma_2(x) \), the subgraph induced by \( \Gamma(x, y) \) is a 6-
gon, say, \( \Gamma(x, y) = \left\{ w_{ij} : t = 1, 2, \ldots, 6 \right\} \). Let \( \mu_x(y) \) denote the set of pairs \( (i, j) \) such that
\( \left\{ w_{ij} : (i, j) \in \mu_x(y) \right\} = \Gamma(x, y) \).

We shall show that the adjacency between vertices in \( \Gamma_2(x) \) is determined by the intersection of
their images under the mapping \( \mu_x \). We further show that, for any vertex \( x \in \Gamma \) and any vertex
\( \tilde{x} \in \tilde{\Gamma} \), the mappings \( \mu_x \) and \( \mu_{\tilde{x}} \) can be chosen in such a way that the sets of their images coincide,
which in turn implies Lemma 3.8.

**Claim 3.9** For \( y, z \in \Gamma_2(x) \), \( y \sim z \) holds if and only if
\[ \Gamma(x, y, z) := \left\{ w_{ij} : (i, j) \in \mu_x(y) \cap \mu_x(z) \right\} \]
induces either an edge or two disjoint edges in \( \Gamma(x) \).
is isomorphic to the bilinear forms graph $\text{Bil}$.

We note that the intersection $\Sigma$ and $\Sigma'$ is such that $\Gamma(x, z) \subset L_i \cup L_j \cup L_\ell$ or $\Gamma(x, z) \subset L_i^T \cup L_j^T \cup L_\ell^T$, respectively. By $\mathcal{B}_{\Gamma, x}$ ($\mathcal{B}_{\Gamma, x}^T$, respectively) we denote the set of all ($\mathcal{T}$-)blocks.

For a block $\{i, j, \ell\}$ and a $\mathcal{T}$-block $\{r, s, t\}$, by $H_{\{r, s, t\}}^{i, j, \ell}$ we denote the set of all sets $\sigma$ consisting of pairs $(i, j)$ such that the set $\{w_{ij} \mid (i, j) \in \sigma\}$ induces a 6-gon in the $(3 \times 3)$-grid induced by $(L_i \cup L_j \cup L_\ell) \cap (L_i^T \cup L_j^T \cup L_\ell^T)$.

**Claim 3.10** The following holds:

$$\{\mu_\chi(y) \mid y \in \Gamma_2(x)\} = \{H_{\{r, s, t\}}^{i, j, \ell} \mid \{i, j, \ell\} \in \mathcal{B}_{\Gamma, x}, \{r, s, t\} \in \mathcal{B}_{\Gamma, x}^T\}.$$ 

**Proof:** We note that the intersection $\Sigma' := \Sigma_1 \cap \Sigma_2$ of the graphs induced by 

$$\Sigma_1 := \{x \cup L_i \cup L_j \cup L_\ell \cup \{y \in \Gamma_2(x) \mid \Gamma(x, y) \subset L_i \cup L_j \cup L_\ell\}$$

and  

$$\Sigma_2 := \{x \cup L_i^T \cup L_j^T \cup L_\ell^T \cup \{y \in \Gamma_2(x) \mid \Gamma(x, y) \subset L_i^T \cup L_j^T \cup L_\ell^T\}$$

is isomorphic to the bilinear forms graph $\text{Bil}_2(2 \times 2)$, which has parameters $(16, 9, 4, 6)$ and is locally the $(3 \times 3)$-grid graph. The $(3 \times 3)$-grid contains exactly six 6-gons, and there are exactly $16 - 9 - 1 = 6$ vertices of $\Sigma'$ at distance 2 from $x$. For a vertex $y \in \Sigma'$ at distance 2 from $x$, the set of common neighbours of $x$ and $y$ in $\Sigma'$ clearly coincides with $\Gamma(x, y)$ and therefore induces a 6-gon. On the other hand, the set $\mu_\chi(y)$ uniquely determines a block $\{i, j, \ell\}$ and a $\mathcal{T}$-block $\{r, s, t\}$ such that $\mu_\chi(y) \in H_{\{r, s, t\}}^{i, j, \ell}$. This shows the claim.

We now pick a vertex $\bar{x} \in \Gamma$, and let $\{\bar{L}_i\}_{i=1}^{2^{e-1}-1}, \{\bar{L}_j^T\}_{j=1}^{2^d-1}$ be the sets of the maximal cliques of $\bar{\Gamma}(\bar{x})$. As above, we define the sets $\mathcal{B}_{\Gamma, \bar{x}}$ and $\mathcal{B}_{\Gamma, \bar{x}}^T$.

**Claim 3.11** There exist permutations $\pi$ acting on the set $\{1, 2, \ldots, 2^e - 1\}$ and $\pi^\tau$ acting on the set $\{1, 2, \ldots, 2^d - 1\}$ such that 

$$\pi(\mathcal{B}_{\Gamma, \bar{x}}) = \mathcal{B}_{\Gamma, x}, \text{ and } \pi^\tau(\mathcal{B}_{\Gamma, \bar{x}}^T) = \mathcal{B}_{\Gamma, x}^T,$$

where $\pi(\mathcal{B}_*) = \{\pi(b) \mid b \in \mathcal{B}_*\}$.

**Proof:** Without loss of generality, we may assume that $\{1, 2, 3\}$ is an element of all four sets $\mathcal{B}_{\Gamma, \bar{x}}$, $\mathcal{B}_{\Gamma, x}$, $\mathcal{B}_{\Gamma, \bar{x}}^T$, and $\mathcal{B}_{\Gamma, x}^T$. By Lemma 3.2, the graphs induced by 

$$\Sigma := \{x \cup L_1 \cup L_2 \cup L_3 \cup \{y \in \Gamma_2(x) \mid \Gamma(x, y) \subset L_1 \cup L_2 \cup L_3\}$$

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and
\[ \Sigma := \{ \bar{x} \} \cup \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \cup \{ \bar{y} \in \bar{\Gamma}_2(\bar{x}) | \bar{\Gamma}(\bar{x}, \bar{y}) \subset \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \} \]
are isomorphic. Since every subgraph induced on \( \Gamma(x, y) \) for \( y \in \Gamma_2(x) \) (or on \( \bar{\Gamma}(\bar{x}, \bar{y}) \) for \( \bar{y} \in \bar{\Gamma}_2(\bar{x}) \)) uniquely determines a block and a \( \top \)-block, the isomorphism between \( \Sigma \) and \( \bar{\Sigma} \) defines the permutation \( \pi_{\top} \). The same argument applied to \( \{1, 2, 3\} \) as a \( \top \)-block shows the existence of \( \pi \), and thus the claim follows.

\[ \text{Proof of Lemma 3.8:} \]
By Claims 3.11 and 3.10, we may assume that
\[ \{ \mu_x(y) | y \in \Gamma_2(x) \} = \{ \mu_{\bar{x}}(\bar{y}) | \bar{y} \in \bar{\Gamma}_2(\bar{x}) \} \quad (22) \]
holds. The lemma now follows from Claim 3.9.

Now we can precisely describe an extendable (in the sense of Section 2.6) isomorphism \( \varphi \) between the local graphs at \( x \) and \( \bar{x} \):
\[ \varphi : \{ \bar{x} \} \cup \bar{\Gamma}(\bar{x}) \to \{ x \} \cup \Gamma(x) \]
with its extension \( \varphi' \), i.e., a bijection:
\[ \varphi' : \{ \bar{x} \} \cup \bar{\Gamma}(\bar{x}) \cup \bar{\Gamma}_2(\bar{x}) \to \{ x \} \cup \Gamma(x) \cup \Gamma_2(x) , \]
mapping edges to edges, such that \( \varphi' |_{\{ \bar{x} \} \cup \bar{\Gamma}(\bar{x})} = \varphi \). In fact, it follows from Lemma 3.8 that \( \varphi' \) is an isomorphism.

We may simply assume that \( \varphi \) sends a unique vertex of \( \bar{L}_i \cap \bar{L}_j \) to \( w_{ij} \) (and, clearly, \( \bar{x} \) to \( x \)). By Claims 3.11 and 3.10 we may assume that Eq. (22) holds. We then let \( \varphi' \) send a vertex \( \bar{y} \in \bar{\Gamma}_2(\bar{x}) \) to a unique vertex \( y \in \Gamma_2(x) \) such that \( \mu_x(y) = \mu_{\bar{x}}(\bar{y}) \).

### 3.3 Triangulability of the bilinear forms graphs

Our result in this section is the following proposition.

**Proposition 3.12** The bilinear forms graph \( \text{Bil}_q(e \times d) \) is triangulable.

**Proof:** We will make use of an alternative definition of \( \text{Bil}_q(e \times d) \) \[4, Chapter 9.5.A\]. Let \( V \) be a vector space of dimension \( e + d \) over \( \mathbb{F}_q \), \( W \) be a fixed \( e \)-subspace of \( V \). Then the vertices of \( \text{Bil}_q(e \times d) \) are the \( d \)-dimensional subspaces of \( V \) skew to \( W \), with two such subspaces \( X, Y \) adjacent if and only if \( \text{dim}(X \cap Y) = d - 1 \).

Recall that the number of \( m \)-dimensional subspaces of a \( k \)-dimensional vector space over \( \mathbb{F}_q \) that contain a given \( \ell \)-dimensional subspace is equal to
\[ \begin{bmatrix} k - \ell \\ m - \ell \end{bmatrix}_q . \]
Claim 3.13 The graph $\text{Bil}_q(e \times d)$ satisfies (i) of Lemma 2.5.

Proof: Let $X$ and $Y_1$ be two $d$-dimensional subspaces corresponding to vertices $x$ and $y_1$ at distance $j \geq 2$ of the bilinear forms graph $\text{Bil}_q(e \times d)$, i.e., $\dim(X \cap Y_1) = d - j$, $\dim(X \cap W) = \dim(Y_1 \cap W) = 0$. We are interested in the subgraph of $\text{Bil}_q(e \times d)$ induced by the $d$-subspaces $U$ of $V$ satisfying

\[ \dim(U \cap X) = d - 1, \quad \dim(U \cap Y_1) = d - (j - 1), \]  

(23)

and $\dim(U \cap W) = 0$.

Note that any $d$-subspace $U$ satisfying Eq. (23) contains $X \cap Y_1$. Hence any such subspace can be formed by choosing $(j - 1)$-dimensional subspace in $X/(X \cap Y_1)$ and 1-dimensional subspace in $Y_1/(X \cap Y_1)$. Thus, the number of $d$-subspaces $U$ of $V$ satisfying Eq. (23) (however, note that some of these subspaces may not satisfy $\dim(U \cap W) = 0$) is equal to

\[
\left[ \frac{d - (d - j)}{1} \right]_q \times \left[ \frac{d - (d - j)}{j - 1} \right]_q = \left[ \frac{j}{1} \right]_q \times \left[ \frac{j}{j - 1} \right]_q = \left[ \frac{j}{1} \right]_q \times \left[ \frac{j}{1} \right]_q .
\]

The graph $\Lambda$ induced by the set of $d$-subspaces satisfying Eq. (23) with two such subspaces adjacent if their intersection has dimension $d - 1$ is the $\left( \left[ \frac{j}{1} \right]_q \times \left[ \frac{j}{1} \right]_q \right)$-grid, whose maximal $\left[ \frac{j}{1} \right]_q$-cliques consist of all $d$-dimensional subspaces containing a given $(j - 1)$-dimensional subspace from $X/(X \cap Y_1)$ or a given 1-dimensional subspace from $Y_1/(X \cap Y_1)$.

Now we need to exclude from our consideration $d$-subspaces satisfying Eq. (23), but intersecting $W$ non-trivially, and then to show that the graph $\Lambda'$ obtained from $\Lambda$ by removing the corresponding vertices is still connected.

Let $A$ be an 1-dimensional subspace in $Y_1/(X \cap Y_1)$. Then the subspace $Y$ generated by $A$ and $X$ has dimension $d + 1$, and thus $Y$ intersects $W$ in an 1-dimensional subspace, say, $P$. Hence the number of $d$-subspaces of $Y$ satisfying Eq. (24) (i.e., containing $X \cap Y_1$), containing $A$, and intersecting $W$ non-trivially (in $P$), is equal to

\[
\left[ \frac{(d + 1) - (d - j + 2)}{d - (d - j + 2)} \right]_q = \left[ \frac{j - 1}{j - 2} \right]_q = \left[ \frac{j - 1}{1} \right]_q .
\]

Therefore, from every maximal clique of $\Lambda$ we need to remove precisely $\left[ \frac{j - 1}{1} \right]_q$ vertices. Note that the number of vertices left in $\Lambda'$ equals

\[ |\Lambda'| = \left[ \frac{j^2}{1} \right]_q - \left[ \frac{j}{1} \right]_q \left[ \frac{j - 1}{1} \right]_q = q^{j-1} \left[ \frac{j}{1} \right]_q = c_j, \]

compare with Eq. (4).

Suppose now that the graph $\Lambda'$ is disconnected. Let $z_1$ and $z_2$ be two vertices of $\Lambda$ belonging to different connected components of $\Lambda'$. Note that $\Lambda_1(z_1)$, the local graph of $z_1$ in $\Lambda$, is the disjoint
union of two cliques, say \( L_1 \) and \( L_2 \), each of size \( \left\lfloor \frac{j}{q} \right\rfloor - 1 \). Denote by \( L_i^-= i \in \{1,2\} \), the set of vertices from \( \Lambda(z_1) \setminus \Lambda_1'(z_1) \) belonging to the clique \( L_i \subset \Lambda_1(z_1) \), i.e., \( |L_i^-| = \left\lfloor \frac{j-1}{q} \right\rfloor \).

Then \( z_2 \) belongs to the subgraph of \( \Lambda \) induced by
\[
\{ w \in \Lambda \mid \exists (w_1,w_2) \in L_1^- \times L_2^- \text{ and } w \sim w_1 \text{ and } w \sim w_2 \},
\]
and, moreover, \( \Lambda' \) does not contain vertices from any of the following sets:
\[
\{ w \in \Lambda \mid \exists (w_1,w_2) \in (\{z_1\} \cup L_1 \setminus L_1^-) \times L_2^- \text{ and } w \sim w_1 \text{ and } w \sim w_2 \},
\]
\[
\{ w \in \Lambda \mid \exists (w_1,w_2) \in L_1^- \times (\{z_1\} \cup L_2 \setminus L_2^-) \text{ and } w \sim w_1 \text{ and } w \sim w_2 \},
\]
as otherwise there is a path in \( \Lambda' \) from \( z_2 \) to \( z_1 \) through a vertex in one of these sets. This yields that
\[
|\{z_1\} \cup (L_1 \setminus L_1^-)| = \left\lfloor \frac{j}{q} \right\rfloor - \left\lfloor \frac{j-1}{q} \right\rfloor \leq \left\lfloor \frac{j-1}{q} \right\rfloor,
\]
which is impossible.

Thus, \( \Lambda' \) is connected, and \( Bil_q(e \times d) \) satisfies (i) of Lemma 2.5, which shows the claim. \( \blacksquare \)

**Claim 3.14** The graph \( Bil_q(e \times d) \) satisfies (ii) of Lemma 2.5.

**Proof:** Let \( X, Y_1, Y_2 \) be \( d \)-dimensional subspaces of \( V \) corresponding to vertices \( x, y_1, y_2 \) of the bilinear forms graph \( Bil_q(e \times d) \) and satisfying \( \dim(X \cap Y_1) = \dim(X \cap Y_2) = d - j \), where \( j \geq 2 \), \( \dim(Y_1 \cap Y_2) = d - 1 \), and \( \dim(X \cap W) = \dim(Y_1 \cap W) = \dim(Y_2 \cap W) = 0 \). We shall show that there exists a \( d \)-subspace \( U \) of \( V \) satisfying
\[
\dim(U \cap X) = d - 1, \quad \dim(U \cap Y_1) = \dim(U \cap Y_2) = d - (j - 1), \quad \text{and} \quad \dim(U \cap W) = 0. \quad (24)
\]

We first consider a partial case when \( j = d \), the diameter of \( Bil_q(e \times d) \). Let \( A \) be an 1-dimensional subspace of \( Y_1 \cap Y_2 \). Then the subspace \( Y \) generated by \( A \) and \( X \) has dimension \( d + 1 \), and thus \( Y \) intersects \( W \) in an 1-dimensional subspace, say, \( P \).

Further, the number of \( d \)-subspaces in \( Y \) that contain \( A \) is
\[
\left\lfloor \frac{d + 1 - 1}{d - 1} \right\rfloor_q,
\]
while the number of \( d \)-subspaces in \( Y \) that contain both \( A \) and \( P \) is
\[
\left\lfloor \frac{d + 1 - 2}{d - 2} \right\rfloor_q = \left\lfloor \frac{d - 1}{d - 2} \right\rfloor_q.
\]

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Thus, the number of \( d \)-subspaces \( U \) of \( Y \) that do not contain \( P \), but contain \( A \) (and hence \( U \) satisfies Eq. (24)) is equal to

\[
\binom{d}{d-1} q - \binom{d-1}{d-2} q > 0,
\]

which shows the claim in given partial case.

We now turn to the general case. Note that if \( \dim(X \cap Y_1 \cap Y_2) = d - j \), then we may consider the bilinear forms graph \( Bil_q(e \times d) \) defined on \( V/(X \cap Y_1 \cap Y_2) \), and the claim follows from the previous partial case \( j = d \). Therefore we may assume that \( \dim(X \cap Y_1 \cap Y_2) = d - j - 1 \).

Again, considering (if necessarily) the bilinear forms graph defined on \( V/(X \cap Y_1 \cap Y_2) \), we may assume that \( j = d - 1 \), \( \dim(X \cap Y_1 \cap Y_2) = 0 \), and \( A = X \cap Y_1 \cap Y_2 \) are 1-dimensional subspaces. Let \( C \) be an 1-dimensional subspace of \( Y_1 \cap Y_2 \). Then the subspace \( Y \) generated by \( C \) and \( X \) has dimension \( d + 1 \), \( A, B, C \subset Y \), and thus \( Y \) intersects \( W \) in an 1-dimensional subspace, say, \( P \). As above, we count the number of \( d \)-subspaces of \( Y \) that contain \( \langle A, B, C \rangle \), but do not contain \( \langle A, B, C, P \rangle \) as

\[
\binom{d + 1 - 3}{d - 3} q - \binom{d + 1 - 4}{d - 4} q > 0,
\]

and it is the number of \( d \)-subspaces \( U \) satisfying Eq. (24). This shows the claim.

Proposition 3.12 follows from Claims 3.13, 3.14 and Lemma 2.5.

### 3.4 Proof of Theorem 1.2

We are now in a position to prove Theorem 1.2.

In the notation of Theorem 2.6, we take the bilinear forms graph \( Bil_q(e \times d) \), \( e \geq d \geq 2 \), as \( \bar{\Gamma} \), and \( \Gamma \) as a graph satisfying Theorem 1.2, i.e., \( \Gamma \) is locally the \( (n \times m) \)-grid, with diameter \( D \geq 2 \), and the intersection numbers given by Eq. (13) are well-defined.

**Proof of Theorem 1.2**: By Lemma 3.8, the graph \( \Gamma \) has distinct \( \mu \)-graphs, and the graphs \( \Gamma \) and \( \bar{\Gamma} \) satisfy (i) of Theorem 2.6 with the extendable isomorphism \( \varphi \) defined in Section 3.2. By Proposition 3.12, the graph \( \bar{\Gamma} \) satisfies (iii) of Theorem 2.6.

Thus, what is left is to show that the graphs \( \Gamma \) and \( \bar{\Gamma} \) satisfy (ii) of Theorem 2.6. We will follow the notation of Section 3.2. Let \( \bar{y} \in \bar{\Gamma}(\bar{x}) \). Our goal is to show that

\[
\varphi'\big|_{(\bar{y}) \cup \bar{\Gamma}(\bar{y})}: \{\bar{y}\} \cup \bar{\Gamma}(\bar{y}) \rightarrow \varphi(\{\bar{y}\}) \cup \Gamma(\varphi(\bar{y}))
\]

is an extendable isomorphism.

As in Section 3.2, we may similarly define the sets \( B^\top_{\bar{\Gamma},\bar{y}} \) and \( B^\top_{\bar{\Gamma},\bar{y}} \) of all blocks and \( \top \)-blocks, respectively, of the \( (n \times m) \)-grid \( \bar{\Gamma}(\bar{y}) \).
Without loss of generality, we assume that \( \{1, 2, 3\} \in B_{\tilde{\Gamma}, \tilde{x}} \), and \( \{\tilde{y}\} = \tilde{L}_1 \cap \tilde{L}_2^\top \). Pick a \( \top \)-block \( \{i, j, \ell\} \in B_{\tilde{\Gamma}, \tilde{x}}^\top \), and choose a vertex \( \tilde{z} \in \tilde{\Gamma}_2(\tilde{x}) \cap \tilde{\Gamma}_2(\tilde{y}) \) such that \( \mu_{\tilde{z}}(\tilde{z}) \in H^{\{1,2,3\}}_{\{i,j,\ell\}} \), i.e., \( \tilde{z} \) is at distance 2 from \( \tilde{y} \), and \( \tilde{\Gamma}(\tilde{x}, \tilde{y}, \tilde{z}) \) is a pair of vertices containing in \( \tilde{L}_1 \). As the \( \mu \)-graphs in \( \tilde{\Gamma} \) are hexagons, this yields that \( \tilde{y} \) and \( \tilde{z} \) have four more common neighbours in \( \tilde{\Gamma}_2(\tilde{x}) \), and, in particular, these four common neighbours determine three maximal \( n \)-cliques of \( \tilde{\Gamma}(\tilde{y}) \). Thus, the \( \top \)-block \( \{i, j, \ell\} \in B_{\tilde{\Gamma}, \tilde{x}}^\top \) uniquely determines a \( \top \)-block of \( B_{\tilde{\Gamma}, \tilde{y}}^\top \). In the same manner, it follows that each of blocks of \( B_{\tilde{\Gamma}, \tilde{x}} \) uniquely determines a block of \( B_{\tilde{\Gamma}, \tilde{y}} \).

The same argument applied to the graph \( \Gamma \), the vertex \( x \) and the vertex \( y = \varphi(\{\tilde{y}\}) \), and the proof of Lemma 3.8 show that the isomorphism \( \varphi' |_{\{\tilde{y}\} \cup \tilde{\Gamma}(\tilde{y})} \) is extendable. The theorem is proved.

4 Main result

In this section we prove our main result, Theorem 1.3.

Let \( \Gamma \) be a distance-regular graph with the same intersection array as \( Bil_2(d \times d), d \geq 3 \). Using Theorem 2.3 and Proposition 2.4, in Section 4.1 we show that \( \Gamma \) has the same local graphs as \( Bil_2(d \times d) \). Theorem 1.3 then follows from Theorem 1.2.

4.1 Local graphs of \( \Gamma \)

In this section, we assume that \( \Gamma \) is a distance-regular graph with the same intersection array as \( Bil_q(e \times d), e \geq d \geq 3 \). Let \( \Delta := \Gamma_1(x) \) denote the local graph for a vertex \( x \in \Gamma \), and let \( \eta \) be a non-principal eigenvalue of \( \Delta \).

The following lemma shows Result 1.4

**Lemma 4.1** The eigenvalue \( \eta \) satisfies

\[-q - 1 \leq \eta \leq -1, \text{ or } q^d - q - 1 \leq \eta \leq q^e - q - 1.\]

**Proof:** The result follows immediately from Result 2.2, Theorem 2.3 and Proposition 2.4.

Now we show that the spectrum of \( \Delta \) is uniquely determined if \( e = d \) and \( q = 2 \).

**Lemma 4.2** If \( q = 2 \) and \( e = d \), then \( \Delta \) has spectrum

\[\{2(2^d - 2)\}, \{2^d - 3\}^{2(2^d - 2)}, \{-2\}^{(2^d - 2)^2},\]

and \( \Delta \) is the \((2^d - 1) \times (2^d - 1)\)-grid.
Proof: We first need the following claim.

**Claim 4.3** The graph \( \Delta \) has integral non-principal eigenvalues only, i.e., \( \eta \in \{-3, -2, -1, 2^d - 3\} \).

**Proof:** Recall that the eigenvalues of a graph are algebraic integers and their product is an integer. Suppose that \( \eta_1, \ldots, \eta_s \) are all non-integral (i.e., irrational) eigenvalues of \( \Delta \). Then, by Lemma 4.1, we see that 
\(-3 < \eta_i < -1 \) holds for all \( i = 1, \ldots, s \), and \( \prod_{i=1}^{s} \eta_i \) is an integer. Therefore \( \prod_{i=1}^{s} (\eta_i + 2) \) is an integer.

As \(-3 < \eta_i < -1 \) holds, it follows that \( |\eta_i + 2| < 1 \) and thus \( \prod_{i=1}^{s} (\eta_i + 2) = 0 \), which shows the claim.

We may assume now that \( \Delta \) has the following distinct eigenvalues 
\( \eta_0 = a_1 = 2(2^d - 2), \; \eta_1 = 2^d - 3, \; \eta_2 = -1, \; \eta_3 = -2, \; \eta_4 = -3, \)
and let \( f_i \) denote the multiplicity of \( \eta_i, \; i = 0, \ldots, 4 \).

Note that \( \Delta \) is a connected graph, as otherwise \( \eta_0 \) must be a non-principal eigenvalue of \( \Delta \), which contradicts Lemma 4.1. Hence \( f_0 = 1 \).

We now consider the system of linear equations with respect to unknowns \( f_1, f_2, f_3, f_4 \):

\[
\begin{align*}
    f_1 + f_2 + f_3 + f_4 &= (2^d - 1)^2 - 1, \\ 
    (2^d - 3) f_1 - f_2 - 2 f_3 - 3 f_4 &= -2(2^d - 2), \\ 
    (2^d - 3)^2 f_1 + f_2 + 4 f_3 + 9 f_4 &= 2(2^d - 2)(2^d - 1)^2 - 4(2^d - 2)^2,
\end{align*}
\]

following from Eq. (2) for \( \ell = 0, 1, 2 \).

Calculating the reduced row echelon form of this system gives:

\[
\begin{align*}
    f_1 + \frac{2}{(2^d - 1)(2^d - 2)} f_4 &= 2(2^d - 2), \\ 
    f_2 - \frac{2^d}{2^d - 2} f_4 &= 0, \\ 
    f_3 + \frac{2^d + 1}{2^d - 1} f_4 &= (2^d - 2)^2,
\end{align*}
\]

As all \( f_i \)’s are non-negative integers, one can see from Eq. (28) that if \( f_4 \neq 0 \) then \( f_4 \geq (2^d - 1)(2^d - 2)/2 \) and then \( f_2 \geq 2^d(2^d - 1)/2 \) follows from Eq. (29). Thus, \( f_2 + f_4 \geq (2^d - 1)^2 \), and Eq. (25) yields \( f_1 + f_3 \leq -1 \), a contradiction. Therefore, \( f_4 = f_2 = 0 \), and \( \Delta \) has spectrum 
\( [2(2^d - 2)]^1, \; [2^d - 3]^{2(2^d - 2)}, \; [-2]^{(2^d - 2)^2} \).

This yields that \( \Delta \) is strongly regular with the same parameters as the \( (2^d - 1) \times (2^d - 1) \)-grid. As \( d \geq 3 \) holds, and the \( (m \times m) \)-grid is uniquely determined by its parameters whenever \( m \neq 4 \) (see [24]), the lemma and Theorem 1.3 follow.

\( \Box \)
5 Concluding remarks

In this paper, we showed that the bilinear forms graph $\text{Bil}_q(e \times d)$, with $q = 2$ and $e = d \geq 3$, is uniquely determined by its intersection array. Of course, the main challenge is to generalize this result for any prime power $q$ and $e \in \{d, d + 1, d + 2\}$ (and $e = d + 3$ if $q = 2$). An attempt to prove it in the same manner as we did would require to modify almost all steps of the proof of Theorem 1.3, including the spectral characterization of the $(q - 1)$-clique extensions of grids. The latter problem seems to be a highly non-trivial one even for $q = 3$: in Yang et al. [30] they showed that there exists a positive integer $T$ such that the 2-clique extension of the $(t \times t)$-grid is characterized by its spectrum if $t \geq T$.

We also wonder whether the characterization of the bilinear forms graphs (for all $q$, $e$ and $d$) in the spirit of Theorem 1.2 is possible, and, in particular, whether we really need to assume that the intersection number $b_2$ is well-defined. Note that the same assumption (that the intersection number $a_2$ is well-defined) was used in [23] for local characterization of the graphs of alternating forms and the graphs of quadratic forms over $\mathbb{F}_2$, where the authors hoped that the condition would be shown superfluous in further research. We are aware of only one such attempt, see [19], which requires that $a_2(x, y)$ does not exceed some number (equivalently, $b_2(x, y)$ is not less than some number), for any pair of vertices $x, y \in \Gamma$ at distance 2.

Finally, we would like to close our paper with one more result and an open problem. One may check that the intersection array

$$\{7(M - 1), 6(M - 2), 4(M - 4); 1, 6, 28\} \tag{31}$$

is feasible (in the sense of [4, Chapter 4.1.D]) for all integers $M \geq 6$. The only known graphs with this array are the bilinear forms graphs $\text{Bil}_2(m \times 3)$, where $M = 2^m$.

By the result of Metsch, see [21, Corollary 1.3(d)], if a distance-regular graph $\Gamma$ with intersection array given by Eq. (31) is not the bilinear forms graph, then $M \leq 133$.

The case when $M = 6$ was ruled out in [18], the proof was based on counting some triple intersection numbers. Here we present an alternative proof for this result.

**Proposition 5.1** There exists no distance-regular graph with intersection array $\{35, 24, 8; 1, 6, 28\}$.

**Proof:** The graphs with intersection array given by Eq. (31) are $Q$-polynomial with diameter $D = 3$ and classical parameters $(D, b, \alpha, \beta) = (3, 2, 1, M - 1)$.

Let $\Gamma$ be a graph with intersection array given by Eq. (31) with $M = 6$, i.e., $\{35, 24, 8; 1, 6, 28\}$. By Proposition 2.4, the Terwilliger polynomial of $\Gamma$ has the following four roots:

$$3, -1, -3, 5,$$

while the sign of its leading term coefficient is negative.
This yields that, for a vertex \( x \in \Gamma \) and a non-principal eigenvalue \( \eta \) of the local graph \( \Delta := \Gamma(x) \), one has:

\[-3 \leq \eta \leq -1 \quad \text{or} \quad 3 \leq \eta \leq 5.\]

Moreover, by [1, Theorem 4.4.3], we have that \( \eta \leq -1 - \frac{b_1}{\theta_{D+1}} \), where the smallest eigenvalue \( \theta_D \) of \( \Gamma \) is equal to \(-7\). Thus, \( \eta \leq 3 \). Now, in the same manner, as in the proof of Lemma 4.2, one can show that the local graph \( \Delta \) may only have integer eigenvalues, i.e., \( \eta \in \{3, -1, -2, -3\} \), including the principal eigenvalue equal to \( a_1 = 10 \), whose multiplicity \( f_0 \) equals 1.

We may assume that \( \Delta \) has the following distinct eigenvalues

\[ \eta_0 = a_1 = 10, \quad \eta_1 = 3, \quad \eta_2 = -1, \quad \eta_3 = -2, \quad \eta_4 = -3, \]

and let \( f_i \) denote the multiplicity of \( \eta_i, i = 0, \ldots, 4 \).

Eq. (2) gives the following system of linear equations with respect to unknown multiplicities \( f_1, f_2, f_3, f_4 \):

\[
\begin{align*}
  f_1 + f_2 + f_3 + f_4 &= 34, \\
  3f_1 - 2f_2 - 3f_3 - 3f_4 &= -10, \\
  9f_1 + f_2 + 4f_3 + 9f_4 &= 250,
\end{align*}
\]

which has the only solution in non-negative integers: \( f_1 = 13, \ f_2 = 7, \ f_3 = 0, \ f_4 = 14 \), and hence \( \Delta \) has spectrum

\[ [10]^1, \ [3]^{13}, \ [-1]^7, \ [-3]^{14}. \]

As the graph \( \Delta \) is regular and has the four distinct eigenvalues, it follows that the number of triangles through a given vertex \( y \) is independent of \( y \), and equals (see, for instance, [11, Section 3.1])

\[
\frac{1}{2 \cdot 35} (10^3 + 13 \cdot 3^3 + 7 \cdot (-1)^3 + 14 \cdot (-3)^3) = \frac{966}{70},
\]

which is impossible. Therefore there exists no graph \( \Delta \) with given spectrum, and the proposition follows.

Now let \( \Gamma \) be a graph with intersection array given by Eq. (31) with \( M = 7 \), i.e., \( \{42, 30, 12; 1, 6, 28\} \). Similarly to the proof of Proposition 5.1, one can show that, for a vertex \( x \in \Gamma \), the local graph \( \Delta := \Gamma(x) \) of \( x \) has spectrum

\[ [11]^1, \ [4]^{12}, \ [-1]^{14}, \ [-3]^{15}, \]

however, this time the number of closed walks of length \( \ell \) through a vertex of \( \Delta \) given by:

\[
\frac{1}{2 \cdot 42} (11^\ell + 12 \cdot 4^\ell + 14 \cdot (-1)^\ell + 15 \cdot (-3)^\ell)
\]

is integer for all \( \ell \).

We challenge the reader to solve whether a distance-regular graph with intersection array \( \{42, 30, 12; 1, 6, 28\} \) does exist.
References

[1] Bannai E., Ito T. Algebraic combinatorics. I. Association schemes. *The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA*, 1984. xxiv+425 pp.

[2] Bang S., Fujisaki T., Koolen J.H. The spectra of the local graphs of the twisted Grassmann graphs. *European J. Combin.*, 30(3):638–654, 2009.

[3] Blokhuis A., Brouwer A.E. Locally 4-by-4 grid graphs. *J. Graph Theory*, 13:229–244, 1989.

[4] Brouwer A.E., Cohen A.M., Neumaier A. Distance-regular graphs. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, (3), 18. Springer-Verlag, Berlin, 1989. xviii+495 pp.

[5] Brouwer A.E., Haemers W.H. Spectra of Graphs. Springer, Heidelberg, 2012.

[6] Cameron P. Strongly regular graphs. in *Selected Topics in Algebraic Graph Theory* (eds. L.W. Beineke and R.J. Wilson), Cambridge Univ. Press, 2004.

[7] Coolsaet K., Jurišić A. Using equality in the Krein conditions to prove nonexistence of certain distance-regular graphs. *J. Combin. Theory Ser. A*, 115(6):1086–1095, 2008.

[8] Cuypers H. Two remarks on Huang’s characterization of the bilinear forms graphs. *Europ. J. Combinatorics*, 13(1):33–37, 1992.

[9] Cuypers H. The dual of Pasch’s axiom. *Europ. J. Combinatorics*, 13(1):15–31, 1992.

[10] Van Dam E. Regular graphs with four eigenvalues *Linear Algebra Appl.*, 226-228:139–162, 1995.

[11] Van Dam E., Koolen J.H., Tanaka H. Distance-regular graphs. *Preprint*, August 2013 version.

[12] Debroey I. Semi partial geometries satisfying the diagonal axiom *J. of Geometry*, 13(2):171–190, 1979.

[13] Dickie G. Twice Q-polynomial distance-regular graphs. *J. Combin. Theory Ser. B*, 68(1):161–166, 1996.

[14] Fu T.-S., Huang T. A Unified Approach to a Characterization of Grassmann Graphs and Bilinear Forms Graphs. *Europ. J. Combinatorics*, 15(4):363–373, 1994.

[15] Gavril'yuk A.L., Koolen J.H. The Terwilliger polynomial of a Q-polynomial distance-regular graph and its application to pseudo-partition graphs. *Linear Algebra Appl.*, 466(1):117–140, 2015.

[16] Huang T. A characterization of the association schemes of bilinear forms. *Europ. J. Combinatorics*, 8:159–173, 1987.

[17] Hobart S., Ito T. The structure of nonthin irreducible T-modules of endpoint 1: ladder bases and classical parameters. *J. Algebraic Combin.*, 7(1):53–75, 1998.

[18] Jurišić A., Vidali J. Extremal 1-codes in distance-regular graphs of diameter 3. *Des. Codes Cryptogr.*, 65(1–2):29–47, 2012.
[19] Makhnev A.A., Paduchikh D.V. Characterization of graphs of alternating and quadratic forms as covers of locally Grassman graphs. *Doklady Mathematics*, 79(2):158–162, 2009.

[20] Metsch K. Improvement of Bruck’s completion theorem. *Des. Codes Cryptogr.*, 1:99–116, 1991.

[21] Metsch K. On a Characterization of Bilinear Forms Graphs. *Europ. J. Combinatorics*, 20:293–306, 1999.

[22] Munemasa A., Shpectorov S.V. A local characterization of the graph of alternating forms. in *Finite Geometry and Combinatorics* (Ed. F. de Clerck and J. Hirschfeld.), Cambridge Univ. Press, 289–302, 1993.

[23] Munemasa A., Pasechnik D.V., Shpectorov S.V. A local characterization of the graphs of alternating forms and the graphs of quadratic forms over $GF(2)$ in *Finite Geometry and Combinatorics* (Ed. F. de Clerck and J. Hirschfeld.), Cambridge Univ. Press, 303–318, 1993.

[24] Shrikhande S.S. The uniqueness of the $L_2$ association scheme. *Ann. Math. Statist.*, 30:781–798, 1959.

[25] Sprague A. Incidence structures whose planes are nets. *Europ. J. Combinatorics*, 2:193–204, 1981.

[26] Terwilliger P. Lecture note on Terwilliger algebra (edited by H. Suzuki), 1993.

[27] Terwilliger P. The subconstituent algebra of an association scheme, I. *J. Algebraic Combin.*, 1(4):363–388, 1992.

[28] Terwilliger P. Kite-free distance-regular graphs. *Europ. J. Combinatorics*, 16:405–414, 1995.

[29] Urlep M. Triple intersection numbers of $Q$-polynomial distance-regular graphs. *European J. Combin.*, 33(6):1246–1252, 2012.

[30] Yang Q.Q., Abiad A., Koolen J.H. Spectral characterization of the 2-clique extension of the square grid. *Preprint*, 2015.