Dynamical control of solitons in a parity-time-symmetric coupler by periodic management

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We consider a dual-core nonlinear waveguide with the parity-time ($PT$) symmetry, realized in the form of equal gain and loss terms carried by the coupled cores. To expand a previously found stability region for solitons in this system, and explore possibilities for the development of dynamical control of the solitons, we introduce “management” in the form of periodic sinusoidal variation of the loss-gain (LG) coefficients, along with synchronous variation of the inter-core coupling (ICC) constant. This system, which can be realized in optics (in the temporal and spatial domains alike), features strong robustness when amplitudes of the variation of the LG and ICC coefficients keep a ratio equal to that of their constant counterparts, allowing one to find exact solutions for $PT$-symmetric solitons. A stability region for the solitons is identified in terms of the management amplitude and period, as well as the soliton’s amplitude. In the long-period regime, the solitons evolve adiabatically, making it possible to predict their stability boundaries in an analytical form. The system keeping the Galilean invariance, collisions between moving solitons are considered too.

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1. Introduction

One of the most fundamental tenets in physics is the charge-parity-time ($CPT$) symmetry, which holds for all Lorentz-invariant systems obeying the causality principle \cite{1,2}. It implies invariance of the system with respect to the combined parity transformation, $P$, which reverses the coordinate axes; charge conjugation, $C$, which swaps particles and antiparticles; and time reversal, $T$. Its reduced forms, such as $PT$ and $CP$ symmetries, may be violated in specific situations, but they also play a profoundly important role in many physical theories. The usual proof of the presence of the latter symmetries applies to Hermitian Hamiltonians, whose eigenvalues are always real.

However, the invariance of the system with respect to the $PT$ and $CP$ transformations does not imply that the underlying Hamiltonian must be Hermitian. Indeed, it was known from some early examples \cite{3-7}, and was then discovered, in the systematic form, by Bender and Boettcher \cite{8,9} (see also review \cite{10} and book \cite{11}) that, in the most general case, Hamiltonians which commute with the $PT$ operator may include a dissipative (anti-Hermitian) term. Such $PT$-symmetric Hamiltonians often include a complex potential, $U(r)$, whose real and imaginary parts must be, respectively, even and odd functions of spatial coordinates ($r$), i.e.,

\begin{equation}
U(-r) = U^*(r),
\end{equation}

where * stands for the complex conjugate. Actually, $PT$-symmetric Hamiltonians may admit transformation into Hermitian ones \cite{12,13}. A well-established fact is that the spectrum of Hamiltonians with complex potentials subject to constraint (1) is real below a critical strength of the imaginary part of the potential, at which the $PT$ symmetry gets broken, making the system unstable \cite{14} (exceptions in the form of models with unbreakable $PT$ symmetry are known too \cite{15}).

Thus far, the $PT$ symmetry was not directly realized in quantum systems with complex potentials. On the other hand, a possibility to realize it was predicted for classical optical media with symmetrically inserted gain and loss \cite{16-30}. This possibility is based on the commonly known similarity between the quantum-mechanical Schrödinger equation and the propagation equation for optical waveguides, written in the paraxial approximation. Following these ideas, the $PT$ symmetry was experimentally implemented in various optical and photonic systems \cite{31-35}. Emulation of the $PT$ symmetry was also predicted in atomic Bose-Einstein condensates (BECs), assuming that the gain may be provided by elements working as matter-wave lasers \cite{36}.

As concerns the emulation of fundamental properties of quantum systems in terms of classical optics and in semi-classical BEC, (quasi-) $CP$ symmetries may be implemented too, in continuous \cite{37,39} and discrete \cite{40} media alike.
The $\mathcal{PT}$ symmetry in an optical waveguide (as well as its $\mathcal{CP}$ counterpart) may naturally combine with the material Kerr nonlinearity, giving rise to propagation models based on cubic nonlinear Schrödinger equations (NLSEs) with the complex potentials subject to condition (11). These models may generate $\mathcal{PT}$-symmetric solitons, which were addressed in many theoretical works [18, 23–28, 15] (see also reviews [11, 42]), and experimentally demonstrated too [24]. Although the presence of the gain and loss makes $\mathcal{PT}$-symmetric media dissipative, solitons exist in them in continuous families, similar to the commonly known situation in conservative models [13], while usual dissipative solitons exist as isolated solutions (attractors, if they are stable) [14].

One of basic settings for the realization of the $\mathcal{PT}$ structure is provided by dual-core waveguides (couplers), with the gain and loss separately placed in parallel cores, which are coupled by tunnelling of the field (light, in optics, or matter waves, in BEC). Stable solitons in conservative couplers with the Kerr nonlinearity were predicted decades ago. These solitons may be symmetric or asymmetric with respect to the identical cores, the symmetry-breaking bifurcation happening at a critical value of the soliton’s total energy/norm (in terms of optics/BEC) [13–48], see also a review in Ref. [19].

A remarkable property of the model of the coupler which includes the cubic nonlinearity in each core, and the above-mentioned $\mathcal{PT}$-symmetric terms, in the form of the linear gain and loss in the two cores, is that $\mathcal{PT}$-symmetric and antisymmetric solitons not only can be found in an analytical form, but also their stability region can be identified in an exact form [24, 25] (this region is finite for the symmetric solitons, while antisymmetric ones are completely unstable). Unlike the conservative counterpart of the system, asymmetric solitons cannot exist in the presence of the gain and loss, because asymmetry between components of the soliton in the amplified and damped cores does not admit establishment of the balance between the gain and loss.

Expansion of the stability region for $\mathcal{PT}$-symmetric solitons and, more generally, developing methods for dynamical control of the solitons is a relevant problem. One potential possibility is suggested by the use of the “management” technique, i.e., periodic modulations of the loss-gain (LG) and inter-core-coupling (ICC) coefficients. In terms of the conservative model of the nonlinear coupler, the management scheme, which corresponds to $\gamma_0 = \gamma_1 = 0$ and $\delta > 0$ in Eq. (2), see below, was introduced in Ref. [50], where effects of the management on symmetric and asymmetric solitons and the transition between them were studied. Similar management schemes are well known to stabilize otherwise unstable or fragile solitons in other settings, such as the dispersion management applied to solitons in single-core waveguides (the local dispersion coefficient with a periodically flipping sign [51, 52]), or the stabilization of two-dimensional solitons (which are otherwise unstable against the critical collapse [53]) by means of periodic nonlinearity management [54–56]. In Ref. [57] a particular realization of the above-mentioned management format, implemented as periodic sign change of the LG and ICC coefficients, was applied to the stabilization of symmetric solitons in the $\mathcal{PT}$-supersymmetric coupler with the cubic intra-core nonlinearity and equal LG and ICC coefficients (the supersymmetry implies setting $\gamma_0 = 1$ in terms of Eq. (2) with $\gamma_1 = \delta = 0$, see below). In the supersymmetric coupler with constant parameters, all solitons are unstable (see Eq. (4) below), while the application of the management creates a stability region for them in the corresponding parameter space. Another application of the management to $\mathcal{PT}$-symmetric solitons was recently elaborated in terms of the single NLSE, with a localized complex potential, satisfying condition (11) and subject to cosinusoidal modulation [58].

The aim of the present work is to explore the stabilization and dynamical control of solitons in the $\mathcal{PT}$-symmetric nonlinear coupler by means of the management applied in a general form, which combines constant and periodically varying terms in the LG and ICC coefficients. Stability regions for $\mathcal{PT}$-symmetric solitons are identified by means of systematic simulations, and also in an analytical form, with the help of the adiabatic approximation, in the case of the long-period management format. Collisions between moving stable solitons are also considered, by dint of direct simulations.

The rest of the paper is arranged as follows. The model is introduced in Section 2. The main results, which produce stability regions for the $\mathcal{PT}$-symmetric solitons, are presented in Section 3. Collision between stable solitons are briefly considered in Section 4. The paper is concluded by Section 5.

2. The model

We consider the propagation of optical or matter waves in the dual-core system described by coupled NLSEs for wave amplitudes $u(z, t)$ and $v(z, t)$ in the cores which carry, severally, gain and loss:

$$
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial t^2} + |u|^2 u - i \left[ \gamma_0 + \gamma_1 \sin \left( \frac{2\pi}{L} z \right) \right] u + \left[ 1 + \delta \sin \left( \frac{2\pi}{L} z + \varphi \right) \right] v = 0,$$

$$
i \frac{\partial v}{\partial z} + \frac{1}{2} \frac{\partial^2 v}{\partial t^2} + |v|^2 v + i \left[ \gamma_0 + \gamma_1 \sin \left( \frac{2\pi}{L} z \right) \right] v + \left[ 1 + \delta \sin \left( \frac{2\pi}{L} z + \varphi \right) \right] u = 0.$$

(2)
In terms of optics, \( z \) is the propagation distance, while \( t \) is the reduced time in the optical model realized in the temporal domain, as a dual-core optical fiber \([43, 49]\), or the transverse coordinate in a dual-core planar waveguide, which represents the coupler in the spatial domain. The group-velocity-dispersion-Kerr coefficients in Eq. \( 2 \) are scaled to be one, assuming that the dispersion has the anomalous sign, which is necessary for maintaining bright solitons; in the spatial domain, the same term represents paraxial diffraction.

The management format, applied to the LG and ICC coefficients, includes constant terms, with the constant part of the ICC parameter scaled to be 1, and \( \gamma_0 < 1 \) being the constant part of the ICC. The LG and ICC modulation amplitudes are, respectively, \( \gamma_1 \) and \( \delta \), which may be introduced with a phase shift, \( \varphi \), and \( L \) is the modulation period. The dissipative coefficients and \( \delta \) are defined to be non-negative, without the loss of generality:

\[
\gamma_0 \geq 0, \; \gamma_1 \geq 0, \; \delta \geq 0
\]  

(values \( \delta < 0 \) and \( \varphi \) are tantamount to \(-\delta \) and \( \varphi + \pi \)). Note, however, that the full local LG and ICC coefficients, i.e., \( \gamma_0 + \gamma_1 \sin (2\pi z/L) \) and \( 1 + \delta \sin (2\pi z/L) \), respectively, may take negative values – in particular, because we will consider, among others, the cases of \( \gamma_0 = 0 \) and \( \delta > 1 \).

In the absence of the management, \( \gamma_1 = \delta = 0 \), a family of exact soliton solutions to Eq. \( 2 \) can be easily found \([24, 25]\), provided that \( \gamma_0 \leq 1 \):

\[
v(z,t) = \left(i\gamma_0 \pm \sqrt{1 - \gamma_0^2}\right) u(z,t),
\]

\[
u(z,t) = A \exp \left[i \left(A^2/2 \pm \sqrt{1 - \gamma_0^2}\right) z\right]\ \text{sech}(At),
\]

where \( A \) is an arbitrary amplitude, and signs + and − correspond to the \( \mathcal{PT} \)-symmetric and antisymmetric solitons, respectively. Note that solution \( 4 \) implies

\[
|v(z,t)|^2 = |u(z,t)|^2,
\]

which maintains equilibrium between the gain and loss.

The stability region for the exact symmetric solitons in the static model (\( \gamma_1 = \delta = 0 \)), given by Eqs. \( 4 \) and \( 5 \), can also be found in an exact form \([24, 25]\): they are stable if the squared amplitude takes values

\[
A^2 \leq A_{\text{crit}}^2 (\gamma_0) = (4/3) \sqrt{1 - \gamma_0^2},
\]

while the antisymmetric solitons are completely unstable (although their instability may be very weak, depending on the parameters). For this reason, antisymmetric solitons are not considered below.

As mentioned above, an essential difference of the \( \mathcal{PT} \)-symmetric system \( 2 \) from its conservative counterpart, with \( \gamma_0 = 0 \), is that, at \( A^2 > A_{\text{crit}}^2 \), unstable \( \mathcal{PT} \)-symmetric solitons are not replaced by stable asymmetric ones (cf. Refs. \([45, 46]\), where asymmetric solitons are considered in the conservative system), because asymmetric states, that do not obey condition \( 6 \), cannot maintain the LG balance (i.e., asymmetric solitons cannot exist in the case of \( \gamma_0 > 0 \)). As a result, unstable \( \mathcal{PT} \)-symmetric soliton suffer blowup, similar to what is shown below in Fig. \( 2 \)c).

3. Stabilization and dynamical control of solitons by the management

3.1. Numerical results

We focus on the case of zero phase shift between the variations of the LG and ICC coefficients, i.e., \( \varphi = 0 \) in Eq. \( 2 \), as the management format with \( \varphi \neq 0 \) (in particular, \( \varphi = \pi \), which, as mentioned above, is tantamount to taking \( \delta < 0 \)) leads to strong instability. The stability of the solitons under the action of the management was identified from sufficiently long direct simulations, with the input taken in the form of Eqs. \( 4 \) and \( 5 \) at \( z = 0 \) and a given value of \( \gamma_0 \), while \( A \) was varied, to collect systematic results for the stability of the solitons with different amplitudes, cf. Eq. \( 7 \). A rigorous study of the stability against small perturbations, which makes it necessary to solve linearized equations around the periodically varying solution, is a challenging task, which we do not tackle here.

A combination of panels displayed in Fig. \( 1 \) show stability areas for the solitons with different values of the constant part of the LG coefficient, \( \gamma_0 \), for a fixed management period, \( L = \pi/3 \), and two different values of amplitude \( A \) in the input expression \( 5 \). Naturally, the stability areas are larger for smaller \( A \) (similar to what is predicted by Eq. \( 7 \) in the absence of the management), and they shrink with the increase of the modulation amplitudes, \( \gamma_1 \) and \( \delta \). It
is clearly seen that the strongest stability is provided by the management scheme in which the ratio of $\delta$ and $\gamma_1$ is the same as the ratio of their counterparts in the constant parts of the ICC and LG coefficients (in other words, the management is applied \textit{coherently} with the static part of the system):

$$\frac{\delta}{\gamma_1} = \frac{1}{\gamma_0}.$$  \hspace{1cm} (8)

This finding is explained by the fact that, when relation (8) holds, Eqs. (2) admit \textit{exact solutions} for $\mathcal{PT}$-symmetric and antisymmetric solitons:

$$u(z,t) = \exp \left[ i \left( \frac{A^2}{2} \pm \sqrt{1 - \frac{\gamma_1^2}{\gamma_0^2}} \right) z \mp i \frac{L \gamma_1}{2\pi \gamma_0} \cos \left( \frac{2\pi}{L} z \right) \right] \text{sech} (At),$$  \hspace{1cm} (9)

cf. Eq. (5), with $v(z,t)$ given by exactly the same equation (4) as above. Of course, stability conditions for these exact solutions cannot be found in the same simple form (7) which is valid in the absence of the management.

A principally different case is one shown in Fig. 1(a), which corresponds to $\gamma_0 = 0$ (i.e., the static system is the conservative one). In this case, relation (5) does not exist and, accordingly, the shape of the stability area is completely different from those displayed in panels (b)-(f).

The evolution of a stable soliton which satisfies constraint (5) is displayed in Fig. 2(a), showing that it keeps a constant shape, in exact agreement with Eqs. (9) and (4). On the other hand, in the case when the management parameters deviate from condition (8), a typical example of the evolution of stable solitons shows small but visible fluctuations in Fig. 2(b). On the other hand, unstable solitons eventually blow up due to the failure of the LG balance, as shown in Fig. 2(c).

Similar results, pertaining to a larger management period, $L = \pi$, are presented in Fig. 3. It is seen that they are qualitatively similar to those in Fig. 1 but the large period supports smaller stability areas, both in the case of $\gamma_0 = 0$ and $\gamma_0 > 0$. Note that the same condition (5) determines the condition of the optimum stability in this case too, when exact $\mathcal{PT}$-symmetric solitons are given by Eqs. (4) and (9).

Another essential summary of the results is presented in Fig. 4, in the form of stability maps in the plane of $(\gamma_1, A)$, while $\delta$ is linked to $\gamma_1$ by the optimum-stability condition (8). At $\gamma_1 = 0$ (hence $\delta = 0$ too), i.e., in the absence the management, the largest values of $A$ admitting the stability in Figs. 4(a) and (b) are precisely the same as predicted analytically by Eq. (7) for the static $\mathcal{PT}$-symmetric coupler.

A natural trend evidenced by Figs. 4(a) and (b) is that the stability limit, given by the largest value of $A$ up to which the solitons persist, decreases with the increase of the modulation strength, $\gamma_1$. A noteworthy feature observed by Fig. 4(b) is that, for a relatively large management period, $L = \pi$, the stability boundary for larger $\gamma_0$ may be located higher, in terms of $A$, than its counterpart for smaller $\gamma_0$; see, for instance, the boundaries for $\gamma_0 = 0.15$ and $\gamma_0 = 0.5$. This feature is counter-intuitive, as the exact result (7) for the solitons in the static model demonstrates monotonic decay of $A_{\text{crit}}$ with the increase of $\gamma_0$. An explanation for this point is that, at smaller $\gamma_0$, the stability is more sensitive to changes of period $L$. The analysis of the situation for the long-period modulations with large $L$ is presented in the next section, with the help of the adiabatic approximation.

Lastly, Fig. 4(d) demonstrates that, in the case of $\gamma_0 = 0$, when the constant term is absent in the LG coefficient (hence Eq. (8) is irrelevant), the increase of the modulation amplitudes of both the LG and ICC terms, i.e., $\gamma_1$ and $\delta$, naturally causes shrinkage of the stability area.

\subsection*{3.2. The adiabatic approximation}

The long-period management may be considered as adiabatic under the condition that the period is much larger than the intrinsic period of phase oscillations of soliton (5):

$$L \gg 4\pi/A^2.$$  \hspace{1cm} (10)

The adiabatic limit allows one to approximately transform Eq. (2) into equations with constant coefficients, by defining

$$t \equiv \frac{\tilde{t}}{\sqrt{1 + \delta \sin \left( \frac{2\pi}{L} z \right)}},$$  \hspace{1cm} (11)

$$z \equiv \frac{\tilde{z}}{1 + \delta \sin \left( \frac{2\pi}{L} z \right)},$$  \hspace{1cm} (12)

$$(u,v) \equiv (\tilde{u}, \tilde{v}) \sqrt{1 + \delta \sin \left( \frac{2\pi}{L} z \right)},$$  \hspace{1cm} (13)
FIG. 1: (Color online) Stability charts for $\mathcal{PT}$-symmetric solitons under the action of the management with a fixed value of the period, $L = \pi/3$. In panels (a) to (f), the constant part of the LG coefficient is $\gamma_0 = 0, 0.15, 0.25, 0.5, 0.6, 0.9$, respectively. The solitons, with initial amplitudes $A = 0.7$ and 1 in Eq. (5), are stable, respectively, in the blue and pink areas in the plane of the modulation amplitudes, $(\gamma_1, \delta)$ (in the pink areas, both $A = 0.7$ and 1 produce stable solitons). The straight dashed line denotes the optimum-stability relation (8), see the text. The simulations produce unstable evolution in the ambient gray region. In panel (f), there is no stability area for $A = 1$, in agreement with Eq. (7), which yields, in this case, $A_{\text{crit}}^2(0.9) \approx 0.58$.

FIG. 2: (Color online) Typical examples of the evolution of stable and unstable solitons under the action of the management. Parameters $(L, \gamma_0, \gamma_1, A) = (\pi/3, 0.5, 0.3, 1)$ are fixed. In panel (a), $\delta = 0.6$, which satisfies the optimum-stability relation (8), the soliton’s shape remains unchanged in the course of the propagation, in agreement with Eq. (9). In panel (b), $\delta = 0.64$, which does not meet condition (8) (this $\delta$ corresponds to $\gamma_0 (\delta/\gamma_1) \approx 1.07$, instead of 1), but belongs to an edge of the stability area in Fig. 1(d). In this case, shape oscillations are small but visible. Panel (c) is an example of the evolution of an unstable soliton with $\delta = 0.1$, which lies deep in the gray area in Fig. 1(d). This soliton blows up due to the imbalance between the gain and loss (note the difference in the vertical scales between panels (a,b) and (c)).

where it is assumed that $\gamma_1$ and $\delta$ are related by Eq. (8), to address the case most relevant for the stability analysis. The substitution of variables (11)-(13) leads, in the first approximation, to the following equations replacing Eq. (2)
FIG. 3: (Color online) The same as in Fig. 1 but for the management period which is three times as large, $L = \pi$. The structure of the stability domains remains similar, but their overall size essentially decreases, in comparison with the case of $L = \pi/3$. In particular, there is no stability area for $A = 1$ in panel (f), for the same reason as in Fig. 1(f).

in the first approximation:

$$i \frac{\partial \tilde{u}}{\partial z} + \frac{1}{2} \frac{\partial^2 \tilde{u}}{\partial t^2} + |\tilde{u}|^2 \tilde{u} - i \gamma_0 \tilde{u} + \tilde{v} = 0,$$

$$i \frac{\partial \tilde{v}}{\partial z} + \frac{1}{2} \frac{\partial^2 \tilde{v}}{\partial t^2} + |\tilde{v}|^2 \tilde{v} + i \gamma_0 \tilde{v} + \tilde{u} = 0.$$

This approximate transformation is relevant under condition

$$\delta < 1,$$

which is necessary to secure condition $1 + \delta \sin (2\pi z/L) > 0$; otherwise, the transformation given by the becomes singular. Taking into regard the currently imposed relation $\delta < 1$, Eq. (15) may also be written as $\gamma_1 < \gamma_0$.

Being tantamount in their form to Eq. (2), equations (7) produce solutions in the form of (4) and (5), which, in turn, are stable under the accordingly transformed criterion (7). The critical condition corresponds to the largest amplitude, in terms of the transformed fields $\tilde{A}$, which is $A^2_{\text{crit}} = A^2 / (1 - \delta)$, attained at $\sin (2\pi z/L) = -1$. In terms of the original notation, the respective approximate stability criterion for the slowly varying solitons takes the form of

$$A_0^2 < (A_0^2)_{\text{crit}} = \frac{4}{3} \sqrt{1 - \gamma^2_0 (1 - \delta)} \equiv \frac{4}{3} \sqrt{1 - \gamma^2_0} \left(1 - \frac{\gamma_1}{\gamma_0}\right).$$

(Eq. (8) is used to write the right-hand side of Eq. (16) in two equivalent forms).

The global picture of the transformation of the stability boundary in the plane of $(\gamma_1, A)$ is illustrated in Fig. 5, by showing it for six different management periods, which cover the range of four order of magnitude (from $L = \pi/20$ to $L = 1000\pi$), and two values $\gamma_0 = 0.25, 0.5$. It is observed that, in each case, there is a critical value, $L_c$, such that the stability boundary is monotonous at $L < L_c$ and non-monotonous at $L > L_c$.

Further, the analytical prediction given by Eq. (10) is compared to the numerically found stability boundaries, for a very large period, $L \equiv 1000\pi$, in Fig. 6(a). It is seen that the prediction is close to the numerical counterparts in the region of $\gamma_1 < \gamma_0$. At $\gamma_1 = \gamma_0$ the analytically predicted critical amplitude vanishes in Eq. (10), and it ceases to
FIG. 4: (Color online) Panels (a,b,c): imposing the optimum-stability condition (8), i.e., $\delta = \gamma_1/\gamma_0$, stability boundaries for $\mathcal{P}\mathcal{T}$-symmetric solitons are displayed in the plane of $(\gamma_1, A)$ (recall $A$ is the input’s amplitude, according to Eq. (5)). The solitons are stable beneath boundaries shown in the panels. In panels (a) and (b), the management period is fixed, respectively, at $L = \pi/3$ and $L = \pi$, and the results are presented for a set of different values of $\gamma_0$, cf. Figs. 1 and 3. In panel (c), $\gamma_0 = 0$ is fixed, and a set of the stability boundaries are displayed for different periods $L$. Panel (d) is plotted for $\gamma_0 = 0$, for which condition (8) is irrelevant. In this case, the stability boundaries are presented for fixed $L = \pi/3$ and three different values of the amplitude of the ICC modulation: $\delta = 0, 0.5$, and 1.

exist at $\gamma_1 > \gamma_0$, i.e., in the case when the sign of the total LG periodically changes, according to Eq. (2). On the contrary to that, the numerically found critical amplitude does not vanish at point $\gamma_1 = \gamma_0$, but, instead, it attains a finite minimum value. At $\gamma_1 > \gamma_0$, the stability area still exists, and actually slowly expands with the increase of $\gamma_1$. At all values of $\gamma_0$, the minimum of $A_{\text{crit}}$ is attained exactly at $\gamma_1 = \gamma_0$, in full agreement with the analytical prediction. This fact is confirmed by the numerically generated value of $\gamma_0/\gamma_1$ at the minimum point, which is shown, versus $L$, by the dashed line in Fig. 6(b). As concerns value $A_{\text{min}}$ of the critical amplitude at the minimum point, it slowly decreasing with the increase of $L$, as is also shown in Fig. 6(b).

4. Collisions of solitons

Because Eq. (2) keeps its Galilean invariance, a stable soliton can be set in motion by applying kick $\eta$ to it, i.e., multiplying the quiescent solution $\{u_0(z, t), v_0(z, t)\}$ by $\exp(i \eta t)$. As a result, the it will be transformed into a moving one,

$$\{u_\eta, v_\eta\} = \{u_0(z, t - cz), v_0(z, t - cz)\} \exp\left(-\frac{i}{2} c^2 z\right), \ c = \eta. \quad (17)$$

This, in turn, suggests to simulate collisions between initially separated solitons, boosted in opposite directions by kicks $\pm \eta$, cf. Ref. [24].

A typical set of collisions, simulated for a set of different values of the kicks, is displayed in Fig. 7. Panels(a) to (f) show different outcomes of the collisions, produced by increasing values of $\eta$. In all the cases, the colliding solitons eventually separate. However, in panels (a) and (b) slowly moving soliton pairs form a quasi-bound state, in which they perform several oscillations before re-emerging with larger values of inverse velocities $\pm c$ (see Eq. (17)) than they had prior to the collision. The formation of the intermediate bound state resembles the effect which was previously
FIG. 5: (Color online) Stability boundaries in the plane of \((γ_1, A)\) under condition (9), for six very different values of the management period, from \(L = π/20\) to \(L = 1000π\), and two different values of \(γ_0\). The stability area shrinks but does not vanish with the increase of \(L\). The shape of the stability boundary becomes non-monotonous (the minimum point appears on it) at \(L ≥ L_c\). The solid and dashed lines designate the non-monotonous and monotonous stability boundaries, respectively.

FIG. 6: (Color online) (a) Stability boundaries in the plane of \((γ_1, A)\) under condition (8), at a very large fixed management period, \(L = 1000π\), and three different values of \(γ_0\). Solid and dashed lines represent, severally, numerical results, and analytical ones predicted by Eq. (16). (b) Value \(A_{\text{crit}}\) of the amplitude at the minimum point vs. \(L\), at three different values of \(γ_0\). The horizontal dashed line shows the value of \(γ_0/γ_1\) at the minimum point, confirming that it exactly corresponds to \(γ_1 = γ_0\), as predicted by the analytical approximation.

found in simulations of soliton-soliton collisions in other nonintegrable models [59–61]. Fast moving solitons, boosted by stronger kicks, pass through each other elastically (panels (d,e,f)), which is typical for collisions between solitons in conservative systems [61], and remains true in the present \(\mathcal{PT}\)-symmetric one.

Lastly, Fig. 8 displays simulations of the collisions with the same values of \(L, γ_0,\) and \(A\), under the action of the same kicks as in the top row in Fig. 7 but in the absence of the management, i.e., for \(γ_1 = δ = 0\). It is clearly seen that the collisions are completely elastic (similar to those simulated in [24]), and the intermediate quasi-bound states do not emerge. Thus, the presence of the management accounts for the creation of the those states. For larger kicks, corresponding to the bottom row in Fig. 7, the collisions remain the same (elastic) as displayed in Fig. 7.

5. Conclusion

The objective of this work is to generalize the known model of the \(\mathcal{PT}\)-symmetric coupler, based on linearly coupled waveguides with the intrinsic cubic nonlinearity and equal gain and loss coefficients carried by the guiding cores. The generalization introduces “management”, which makes the LG (loss-gain) and ICC (inter-core-coupling) coefficients periodically varying functions of the evolutional variable. The model may be realized in optics, in the temporal
FIG. 7: (Color online) Typical examples of collisions between identical stable $\mathcal{PT}$-symmetric solitons moving in opposite directions under the action of kicks $\pm \eta$, see Eq. (17). The solitons’ parameters are $(L, \gamma_0, \gamma_1, \delta, A) = (\pi/3, 0.5, 1.205, 0.5)$. From panel (a) to (f), the kicks applied to the soliton pairs are $\eta = \pm 0.008, \pm 0.01, \pm 0.02, \pm 0.3, \pm 1.5, \pm 3$, respectively.

FIG. 8: (Color online) The same as in the top row of Fig. 7 but in the absence of the management, i.e., with $\gamma_1 = \delta = 0$, other parameters, including the kicks, keeping the same values.

and spatial domains alike. Stability of $\mathcal{PT}$-symmetric solitons and possibilities of applying the dynamical control to them by means of the management are explored by means of systematic simulations, and also analytically in the adiabatic approximation, which corresponds to the long-period limit. The stability is strongest when the ratio of the amplitudes of the modulation of the LG and ICC coefficients is equal to its counterpart for the constant parts of the same coefficients. In the latter case, an exact solution is found for $\mathcal{PT}$-symmetric solitons. Collisions between moving solitons were briefly considered too.

A challenging possibility for the development of the present analysis is to develop analysis of the management for solitons in two-dimensional $\mathcal{PT}$-symmetric systems.
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