Distributed States Temporal Logic

Carlo Montangero  Laura Semini
Dipartimento di Informatica, Università di Pisa.
{monta, semini}@di.unipi.it

Abstract. We introduce a temporal logic to reason on global applications in an asynchronous setting. First, we define the Distributed States Logic (DSL), a modal logic for localities that embeds the local theories of each component into a theory of the distributed states of the system. We provide the logic with a sound and complete axiomatization. The contribution is that it is possible to reason about properties that involve several components, even in the absence of a global clock. Then, we define the Distributed States Temporal Logic (DSTL) by introducing temporal operators à la Unity. We support our proposal by working out a pair of examples: a simple secure communication system, and an algorithm for distributed leader election. The motivation for this work is that the existing logics for distributed systems do not have the right expressive power to reason on the systems behaviour, when the communication is based on asynchronous message passing. On the other side, asynchronous communication is the most used abstraction when modelling global applications.

1 Introduction

The current trend towards global computing needs software that works in an open, concurrent, distributed, high–latency, security–sensitive environment. Besides, this software must be reliable, scalable, and “shipped today”. Several trends are emerging in response to the challenges involved in the development of software with so demanding requirements.

On one side, there is an increasing interest in the seamless integration of asynchronous communication in programming, coordination, and specification languages, since message–passing, event–based programming, call–backs, continuations, dataflow models, workflow models etc. are ubiquitous in global computing. Notable examples in this direction can be found in the context of the Microsoft .NET initiative, like the introduction of support for the delegate–based asynchronous calling model in the libraries of the Common Language Runtime [25], and the proposal of chords in Polyphonic C# to accommodate asynchronous methods in C# [32]. We provide an example of coverage of asynchronous communication in coordination and specification languages in [29].

Another significant trend is represented by Component–Oriented Programming, that aims at producing software components for a software market and for late composition. Composers are third parties, possibly the end user, who are not able nor willing to modify components. This trend emphasizes the need for high quality specifications that put the composer into the position to decide what can be composed under which conditions. In a previous work with Oikos–adt1 [21,22], a specification language for distributed systems based on asynchronous communications, we showed how to accommodate asynchronous communication in the composition of distributed systems specifications.

A notable example of component programming in the context of global computing is offered by the Web Services [14], which leverage the standard representation of data provided by XML to foster the construction of new components (services) by the coordination of other services. Since the cooperation is based on asynchronous protocols, this is also an example of the convergence of asynchronous communications and component programming.

Formal methods can play a major role in global computing. Precisely because the actors are programmatically independent, they need to have reliable ways to share precise knowledge
of the artifacts they use or produce, independently of the particular technology (program- 
m Peng languages, middleware, . . . ) they rely on. Formal methods offer exactly this kind of 
independence and precision, since they provide abstract models to share when operating or 
developing with components. They can provide ways to make precise the specifications of the 
components and of their contextual dependencies, and to prove in advance global properties, 
i.e. that a composition will meet the specifications it addresses.

In this paper we define DSTL (Distributed States Temporal Logic), an extension of temporal 
logic to deal with distributed systems. In [23] we defined new modalities to name system 
components. Here, we introduce the operators to causally relate properties which might hold 
in distinguished components, in an asynchronous setting. A typical DSTL formula is:

\[ m p \xrightarrow{\text{leads\_to}} n q \land o r \]  

where the operator \text{leads\_to} is similar to Unity’s \( \rightarrow \) (leads to) [5], and \( m \), \( n \), and \( o \) express 
locality. Formula (1) says that a property \( p \) holding in component \( m \), causes properties \( q \) and 
\( r \) to hold in future states of components \( n \) and \( o \), respectively. An example is the computation 
below. Horizontal arrows denote the sequence of states of a component, oblique arrows denote 
the communications.

\[
\begin{array}{cccc}
  (n) & \rightarrow & \rightarrow & q \\
  (m) & \rightarrow & p & \rightarrow \\
  (o) & \rightarrow & r & \rightarrow 
\end{array}
\]

At this point a short philosophical note is needed. We tend to think that our operators 
express causality, even though, strictly speaking, they only define temporal relations, i.e. 
that their consequences (right hand side operands) hold after (or before, with past operators) 
their premises (left hand side operands). In fact, in our models, a state in a component is 
after one in another component only if there has been a communication between the two. 
Philosophically, this may not entail a causal relation, but our goal is to specify systems: 
it is natural to think that the communication carries the information needed to cause the 
intended effect. It is in this sense that we use the term causality.

A similar argument applies locally: the implementation will take care that a state satis-
fying the consequences is reached, after one satisfying the premises.

From a technical point of view, the usual choices to build a Kripke model for formulae like (1) 
are to consider the set of worlds \( W \) to be one of the following:

1. the set of the states of a computation, i.e. the union of all the states of the system 
   components, like the circles in the following figure.

\[
\begin{array}{cccc}
  (m) & \circ & \circ & \circ & \circ & \circ & \circ \\
  (n) & \circ & \circ & \circ & \circ & \circ & \circ 
\end{array}
\]

This choice was adopted in Oikos-\text{adtl} and has shown some problems. For instance, con-
sequence weakening, or, more in general, the possibility of reasoning on logical relations 
between formulae like the premises or the consequences of (1), is not part of the logic. In 
particular, a formula like

\[ (n q \land m r) \rightarrow n q \]  

(2)
which would permit to weaken the consequences of (1) would not be a legal formula, since no world can satisfy the conjunction $nq \land mr$.

2. the set of global states, or snapshots, of the system, where each world is a tuple of states, one for each component. These tuples must satisfy some constraints to be coherent with the communications between the subsystems. In the figure below, examples of worlds are $(s^i_m, s^n_j)_{0 \leq j \leq 2}$, while $(s^2_m, s^1_n)$ would not be a legal world.

\[
\begin{array}{c}
(m) \\
\begin{array}{c}
\cdots \\
\cdots \\
\bullet \\
\cdots \\
\bullet \\
\cdots \\
\bullet \\
\cdots \\
\bullet \\
\cdots \\
\end{array}
\end{array}
\begin{array}{c}
(n) \\
\begin{array}{c}
\cdots \\
\bullet \\
\cdots \\
\bullet \\
\cdots \\
\bullet \\
\cdots \\
\bullet \\
\cdots \\
\bullet \\
\cdots \\
\end{array}
\end{array}
\]

This choice, adopted in many logics for distributed systems (see Section 6) is not applicable in the case of asynchronous communication. Think of the case of property $p$ holding only in state $s^1_m$ and $q$ holding only in states $s^n_j$, for $0 \leq j \leq 4$. The formula

$$mp \rightarrow nq$$

would be valid in the model, inferring a remote instantaneous knowledge which is meaningless in an asynchronous setting. Moreover, it would be natural to say that world $\{s^2_m, s^3_n\}$ follows $\{s^1_m, s^2_n\}$. In this case, one could assert that $n p \leadsTo m q$ holds, if $p$ and $q$ hold in $s^n_2$ and $s^2_m$, respectively, even though not even a temporal relationship exists between these two states.

3. a third possibility would be to consider all the $k$–tuples of states (where $k$ is the number of the system components) as worlds. But then, formula (2) would be valid in the model above if $q$ holds in all the states of component $n$. Even if this is philosophically more acceptable, we claim that a better solution can be found. What is more, this choice is not adequate since if we let $p$ and $q$ hold in $s^1_m$ and $s^2_n$, respectively, we would like the computation above to be a model for $mp \leadsTo nq$. On the contrary, world $\{s^1_m, s^3_n\}$ satisfies the premise but is not followed by any state satisfying the consequence.

The first contribution of our work is to introduce the distributed state logic DSL, that carries over all meaningful propositional rules, like and simplification, so that they can be exploited orthogonally to any temporal operator. A major consequence of the introduction of DSL is that the exploitation of the local theories in the proofs of the distributed properties becomes smooth and robust.

The second part of the paper defines DSTL: we add the temporal operators, and the corresponding derivation rules. The semantic domain of DSL, the power–set of the set of all system states, even if chosen for technical reasons, makes the full logic DSTL a very expressive language, that meets the pragmatic expectations of a designer fully (see Section 6 for a discussion). The achievement is that it is possible to reason about properties that involve several components, even in the absence of a global clock, the typical assumption in an asynchronous setting.

Section 2 introduces the modal logic DSL, and its sound and complete axiomatization. Section 3 defines DSTL as an extension of DSL with the temporal operators. Sections 4 and 5 work out a pair of examples: a simple secure communication system, and an algorithm for the leader election problem. The last sections cover a discussion of the main design issues, related work and future perspectives.
2 DSL

We assume a countable set of propositional letters $P$, with $p, q, \ldots$ ranging over $P$. The DSL well-formed formulae over a finite set of components $\Sigma = \{m_1, m_2, \ldots, m_k\}$ are defined by:

$$F ::= p \mid \bot \mid \sim F \mid F \land F' \mid m_i F$$

where $\bot$ is the propositional constant false, and $m_i$ for $i = 1 \ldots k$ are unary location operators. With $\bar{m}_i$ we denote the dual of $m_i$, i.e., $\bar{m}_i F \equiv \sim m_i \sim F$. With $\top$ we denote true, i.e. $\top \equiv \sim \bot$.

2.1 Semantics

A model $\mathcal{M}$ for DSL formulae is a tuple $(W, R_1, \ldots, R_k, V)$. Let $u, v, w$ range over $W$, the reachability relations $R_i$ satisfy the following conditions:

$$(u, v) \in R_i \rightarrow (v, v) \in R_i$$

$$(u, v) \in R_i \rightarrow (v, w) \in R_j \text{ for } j \neq i$$

To help the intuition, $W$ can be thought as having $k$ disjoint subsets of worlds: we call these worlds leaves. Whenever $(u, v) \in R_i$, then $v$ is a leaf for relation $R_i$, namely an $i$–leaf. Condition (4) says that $R_i$ is reflexive on $i$–leaves, conditions (5) and (6) say that $i$–leaves are actually leaves: no other world can be reached. An example model is in Section 2.3, where the $i$–leaves are singleton sets, having as unique element a state of component $m_i$.

The semantics of the DSL formulae is given by:

$$(\mathcal{M}, u) \models \top$$

$$(\mathcal{M}, u) \models p \iff p \in V(u)$$

$$(\mathcal{M}, u) \models \sim F \iff \neg (\mathcal{M}, u) \models F$$

$$(\mathcal{M}, u) \models F \land F' \iff (\mathcal{M}, u) \models F \text{ and } (\mathcal{M}, u) \models F'$$

$$(\mathcal{M}, u) \models m_i F \iff \exists v. (u, v) \in R_i \text{ and } (\mathcal{M}, v) \models F$$

2.2 Axiom system

We propose the following axiomatization for DSL. For the sake of readability, we use $m$ and $n$, with $m \neq n$, instead of $m_i$ and $m_j$.

| PC   | axioms of the propositional calculus |
| K    | $\bar{m}(F \rightarrow F') \rightarrow (\bar{m}F \rightarrow \bar{m}F')$ |
| DSL1 | $\bar{m}(\bar{m}F) \leftrightarrow F$ |
| DSL2 | $\bar{m}\bot$ |
| MP   | $\frac{F \rightarrow F'}{F'}$ |
| Nec  | $\frac{F}{\bar{m}F}$ |

Theorem 1. The DSL axiom system is sound and complete.

Proof. The soundness of the axioms is easy to see. We prove completeness.

Let $(W^{DSL}, R_1^{DSL}, \ldots, R_k^{DSL}, V^{DSL})$ be the canonical model for DSL: worlds in $W^{DSL}$ are maximal consistent sets of DSL formulae (DSL–mcs in the following), and $(u, v) \in R_i^{DSL}$ if and only if $\bar{m}_i F \in u \rightarrow F \in v$. We need to show that, for all $i$, $R_i^{DSL}$ satisfies conditions (4)–(6).
Cond. 4: we prove that \((u, v) \in R_1^{DSL} \rightarrow (v, v) \in R_1^{DSL}\)

Suppose \(m_i F \in v, u\) is a DSL-MCS and hence (see DSL1) \(m_i (m_i F \rightarrow F) \in u\). But \((u, v) \in R_1^{DSL}\), hence \(m_i F \rightarrow F \in v\). Thus, by modus ponens, \(F \in v\).

Cond. 3: we prove that \((u, v) \in R_1^{DSL}\) and \((v, w) \in R_1^{DSL}\) imply \(v = w\)

It is sufficient to prove that \(v \subseteq w\). In fact, \(v\) and \(w\) are DSL-MCS and it is not the case that \(v \subseteq w\), thus \(v = w\). Let \(F \in v\), \(u\) is a DSL-MCS and hence (see DSL1) it includes \(m_i (F \rightarrow m_i F)\). But \((u, v) \in R_1^{DSL}\), hence \(F \rightarrow m_i F \in v\). Thus, by modus ponens, \(m_i F \in v\). As \((v, w) \in R_1^{DSL}\), we conclude that \(F \in w\).

Cond. 2: we prove that \((u, v) \in R_1^{DSL}\) implies \(\exists w. (v, w) \in R_1^{DSL}\), for \(j \neq i\)

Assume \((v, w) \in R_1^{DSL}\). As \(u\) is a DSL-MCS, it includes \(m_i m_j \perp\) (DSL2). As \((u, v) \in R_1^{DSL}\), then \(m_j \perp \in v\). As \((v, w) \in R_1^{DSL}\), then \(\perp \in w\), which is an absurd. \(\square\)

Example 2. The following formulae can be derived. Formulae are followed by the list of axioms or rules used in their proof. The proofs are in the appendix.

**axiom 4** \(mF \rightarrow mmF\) \([DSL1, K]\)

D1 \(mmF \rightarrow mF\) \([DSL1, K, PC]\)

D2 \(m(F \land F') \rightarrow (mF \land mF')\) \([PC, Nec, K]\)

D3 \(m(F \rightarrow F') \rightarrow (mF \rightarrow mF')\) \([Nec, K, MP, PC]\)

D4 \(mF \rightarrow (m \top \rightarrow mF)\) \([D3]\)

D5 \(m(mF \leftrightarrow F)\) \([DSL1, PC]\)

D6 \((m(F \rightarrow F') \land m(F' \rightarrow F'')) \rightarrow m(F \rightarrow F'')\) \([Nec, K]\)

D7 \(m(F \lor F') \leftrightarrow (mF \lor mF')\) \([D3, Nec, K, PC]\)

D8 \(m((mF \land mF') \rightarrow m(F \land F'))\) \([D5, D6, D7, Nec, K]\)

2.3 A frame of distributed states

Let \(S_i\) be the set of states of component \(m_i\), with \(S_i \cap S_j = \emptyset\) for \(i \neq j\), \(S = \bigcup_{i=1}^{j} S_i\), \(DS = 2^S\), and \(ds, ds' \in DS\). Let \((ds, ds') \in R_i\) if and only if \(ds'\) is a singleton set \(\{s\}\), with \(s \in S_i \cup ds\). The frame \((DS, R_1, \ldots, R_k)\), satisfies conditions 1–3 above. We call these frames **frames on DS**, and call DS the set of distributed states, from which the name of the logic DSL. The frames on DS play a central role in the paper, since they are used to build the models for DSTL formulae.

Some examples follow.

Example 3. Let the set \(DS\) be built on \(S_1 = \{s, s'\}\) and \(S_2 = \{s''\}\), then the frame on \(DS\) can be represented as:

![Diagram](image)

Note 4. For the sake of readability, when we use \(m\) and \(n\), we also use \(S_m\), and \(S_n\).
Example 5. If we take $s \in S_m$, $s' \in S_n$

$$(m) \quad \xrightarrow{s} \quad (n) \quad \xrightarrow{s'}$$

with $V(\{s\}) = \{p\}$, $V(\{s'\}) = \{q\}$, then the distributed state $\{s, s'\}$ satisfies $mp \land nq$.

Example 6. The implication $m(F \land F') \rightarrow mF \land mF'$ holds, while the converse does not. Indeed, for $ds = \{s, s'\} \subseteq S_m$

$$(m) \quad \xrightarrow{s} \quad (n) \quad \xrightarrow{s'}$$

and $V(\{s\}) = \{p\}$, $V(\{s'\}) = \{q\}$, we have $ds \models mp \land nq$, but not $ds \models m(p \land q)$. With an eye to the full logic DSTL, this non–equivalence is useful to specify that an event can have different future effects in a component, without constraining them to occur in the same state (see Section 6 for further discussion).

Example 7. The formula $mnF$ is false. In fact, $ds \models mnF$ if and only if there exists an $s \in S_n \cap S_m \cap ds$ such that $\{s\} \models F$, but no such $s$ can exist since $S_m$ and $S_n$ are disjoint. Conversely, $mmF$ is satisfiable, and it is equivalent to $mF$.

Example 8. The formula $m \top$ is satisfied by all the distributed states $ds$ such that $ds \cap S_m \neq \emptyset$.

3 DSTL

DSTL extends DSL adding temporal operators. Formulae are built as follows:

$$\phi ::= F \mid F \text{ leads-to } F' \mid F \text{ because } F' \mid F \text{ leads-to } c \mid F' \mid F \text{ because}_c F' \mid F \text{ unless } F' \mid \text{init } F$$

where $F, F' \in DSL$. Operator leads-to expresses a liveness condition, and is similar to Unity’s $\rightarrow$ (leads to): $F$ is surely followed by $F'$. Operator because expresses a safety condition, and says that $F$ must be preceded by $F'$.

Suffix $c$ stands for closely, leads-to $c$ requires formula $F'$ to hold in the distributed states in which $F$ holds, or in the next ones. Dually, because$_c$ says that $F'$ must hold in the states immediately preceding those satisfying $F$, or in the same ones.

Operator unless extends Unity’s unless, and init permits to describe the initial state. A special case of unless is stability:

$$\text{stable } F \overset{def}{=} F \text{ unless } \bot$$

3.1 Semantics

The models for DSTL formulae are built on structures like the one in the following figure, which describes the computation of a system with three components $(m, n, o)$. We call $s^i_m$ the $i^{th}$ state of component $m$. We call $ds^0$ the set of the initial states $\{s^0_m, s^0_n, s^0_o\}$.

$$(m) \quad s^0_m \xrightarrow{s_1^m} \xrightarrow{s_6^m} \xrightarrow{s_{12}^m} \quad (n) \quad s^0_n \xrightarrow{s_2^n} \xrightarrow{s_3^n} \xrightarrow{s_7^n} \xrightarrow{s_{15}^n} \quad (o) \quad s^0_o \xrightarrow{s_5^o} \xrightarrow{s_8^o}$$

In the figure, plain arrows denote atomic state transitions and communications, dotted arrows denote sequences of them.
Definition 9. \((R, R^=, R^*)\)

State transitions and communications define a next state relation \(R\), where \((s, s') \in R\) if and only if \(s\) and \(s'\) are states of the same component, with \(s'\) immediately following \(s\), or if there is a communication from \(s\) to \(s'\). For example, in the computation above, \((s^0_m, s^1_m), (s^1_m, s^3_m), (s^2_n, s^3_n) \in R\). The plain arrows between pairs of states denote relation \(R\).

We call \(R^=\) the reflexive closure of \(R\), and \(R^*\) its reflexive and transitive closure. For example, in the computation above, \((s^0_m, s^1_m), (s^0_m, s^3_m), (s^1_m, s^3_m), (s^2_n, s^3_m) \in R^*\). We say that \(s'\) causally depends on \(s\) when \((s, s') \in R^*\). Causal dependency has to be read as the partial order relationship between states of a distributed computation, defined by state transitions and communications \[15\]. If neither \((s, s') \in R^*\) nor \((s', s) \in R^*\), states \(s\) and \(s'\) are concurrent.

Definition 10. (Models, \(\leq, \leq_c\))

A model \(\mathcal{M}\) is a tuple \((DS, R_1, \ldots, R_k, \leq, \leq_c, V)\), where:

\[
\begin{align*}
\text{ds} \leq \text{ds}' & \iff \forall s \in \text{ds}, \exists s' \in \text{ds}'. (s, s') \in R^* \text{ and } \forall s' \in \text{ds}', \exists s \in \text{ds}. (s, s') \in R^* \\
\text{ds} \leq_c \text{ds}' & \iff \forall s \in \text{ds}, \exists s' \in \text{ds}'. (s, s') \in R^= \text{ and } \forall s' \in \text{ds}', \exists s \in \text{ds}. (s, s') \in R^=
\end{align*}
\]

We also ask that the valuation function \(V : DS \to 2^P\) satisfies \(V(\text{ds}) = \bigcap_{s \in \text{ds}} V\{s\}\).

Definition 11. (Semantics)

Let \(\mathcal{M}\) be a model, and \(\text{ds}^0\) the set of its initial states. We define:

\[
\begin{align*}
\mathcal{M} \models_T F & \iff \forall \text{ds}. \text{ds} \models F \\
\mathcal{M} \models_T F \text{ LEADS} \rightarrow F' & \iff \forall \text{ds}. \text{ds} \models F \text{ implies } \exists \text{ds}' \geq \text{ds}. \text{ds}' \models F' \\
\mathcal{M} \models_T F \text{ BECAUSE} F' & \iff \forall \text{ds}. \text{ds} \models F \text{ implies } \exists \text{ds}' \leq \text{ds}. \text{ds}' \models F' \\
\mathcal{M} \models_T F \text{ LEADS} \rightarrow_c F' & \iff \forall \text{ds}. \text{ds} \models F \text{ implies } \exists \text{ds}' \geq_c \text{ds}. \text{ds}' \models F' \\
\mathcal{M} \models_T F \text{ BECAUSE} \rightarrow_c F' & \iff \forall \text{ds}. \text{ds} \models F \text{ implies } \exists \text{ds}' \leq_c \text{ds}. \text{ds}' \models F' \\
\mathcal{M} \models_T F \text{ UNLESS} F' & \iff \forall \text{ds}. \text{ds} \models F \text{ implies } \exists \text{ds}' \geq \text{ds} \models F' \\
\mathcal{M} \models_T \text{INIT} F & \iff \text{ds}^0 \models F
\end{align*}
\]

where \(\models\) is the DSL satisfiability relation.

The next section discusses this definition using some examples. In particular, the side condition \(\text{ds}' \nsubseteq \text{ds}\) for \text{UNLESS} is illustrated in Example \[15\].

3.2 Examples

To exemplify the definition of the DSTL semantics, we choose some formulae and discuss whether they are satisfied or not by a model \(\mathcal{M}\) (a computation of a system made up of two components, \(m\) and \(n\)). In the examples we can only present the initial fragments, the discussion on satisfiability is done with respect to the given fragment. From now on, we label the states with the predicates holding in them instead of a name.

We recall that, according to the definition in Section \[2\], a distributed state is any set of states. This means that when we have to check a condition like \(\forall \text{ds} \ldots \exists \text{ds}' \ldots\), we need to consider all possible sets of states as \(\text{ds}\). This may lead to counter-intuitive choices, like picture (c) of Table \[3\] to reason on the first formula of Example \[13\]. We consider these choices in the examples to clarify the semantic details. However, the specifier can be safely guided by the natural interpretation of the operators. Anyhow, our definition of distributed state is exactly what was needed to overcome the problems with the existing models, as discussed in the introduction.
Table 1. We provide various representations of a computation, to outline the distributed states of interest for the examples.
Example 12. (Invariants.) We consider, as model \( M \), the computation in Table 3.2. We refer to picture (a), and call \( s \) and \( s' \) the states outlined by the circle and the rectangle, respectively. We show that \( w \rightarrow t \), \( n(w \rightarrow t) \), and \( n \top \rightarrow n(w \rightarrow t) \) are invariants of the computation, while \( n(w \rightarrow t) \) is not invariant.

\[ M \models_T w \rightarrow t. \]  
This formula reads: in any distributed state of the computation, \( w \rightarrow t \) holds.

State \( s \) is the only one satisfying \( w \). Take \( ds = \{ s \} \), then \( ds \models w \land t \), and thus \( ds \models w \rightarrow t \).

For any distributed state \( ds' \neq ds \) we have that \( ds' \not\models w \) (even though \( s \in ds' \)), and thus \( ds' \models w \rightarrow t \).

\[ M \models_T \bar{n}(w \rightarrow t). \]  
This formula reads: in any distributed state \( w \rightarrow t \) holds in all the states of \( n \), or, in short, \( w \rightarrow t \) holds in any state of \( n \).

We have to show that for all \( ds \), \( ds \models \bar{n}(w \rightarrow t) \), that is for all \( s_n \in ds \cap S_n \), \( \{ s_n \} \models w \rightarrow t \).

This, in turn, holds since \( \{ s \} \models w \land t \), and for all \( s_n \neq s \), \( \{ s_n \} \not\models w \). By the way, this result follows by \( \text{Nec} \) from the previous one.

\[ M \not\models_T n \top \rightarrow n(w \rightarrow t). \]  
This formula reads: in any distributed state of the computation that contains at least one state of \( n \), there is a state of \( n \) where \( w \rightarrow t \) holds.

Any distributed state \( ds \) satisfying the premise \( n \top \) includes a state in \( S_n \), and all states in \( S_n \) satisfy \( w \rightarrow t \). So \( ds \models n(w \rightarrow t) \).

\[ M \not\models_T n(w \rightarrow t). \]  
The formula reads: in any distributed state of the computation, there is a state of \( n \) where \( w \rightarrow t \) holds, and it is false in \( M \).

For \( M \models_T n(w \rightarrow t) \) to be true, we would need that for all \( ds \), \( ds \models n(w \rightarrow t) \), which is true only if a state \( s_n \in ds \cap S_n \) exists, and satisfies \( w \rightarrow t \). However, there are distributed states not including any state \( s_n \in S_n \), e.g. \( \{ s' \} \). In the practice, formulae like \( mF \) are used only as subformulae of larger formulae, e.g. as premises and conclusions of an implication.

Example 13. (Temporal operators.) In the example, we refer to pictures (b)–(h) in Table 3.2.

The distributed state \( ds \) satisfying the premise is the set of states outlined with a circle, and the distributed state \( ds' \) satisfying the consequence is the set of states outlined with a rectangle.

\[ M \models_T nu \text{ leads}_\top mu. \]  
It is enough to consider those distributed states that contain the last state of \( u \) where \( u \) holds. Pictures (b) and (c) show two relevant cases: in the second case we need to consider a larger distributed state to evaluate the consequence, just to satisfy the “follows” relation.

Picture (c) also shows that DSTL overcomes the problems discussed at point 3 in the introduction: a distributed state satisfying the consequence and following \( ds \) exists.

\[ M \models mp \land nv \text{ leads}_\top mz \land nt. \]  
See picture (d): the distributed state satisfying \( mp \land nv \) is followed by a distributed state satisfying \( mz \land nt \).

\[ M \models mq \text{ leads}_\top nu. \]  
See picture (e): the distributed state satisfying the premise includes states which are irrelevant with respect to property \( mq \), for them we only need to check that the “follows” relation is satisfied. The state satisfying \( z \) belongs both to \( ds \) and \( ds' \).

\[ M \models mp \land nu \text{ leads}_\top q. \]  
See picture (f): the state where \( q \) holds immediately follows the one satisfying \( p \). Then any state equal or immediately following the one satisfying \( v \) is fine to build the distributed state satisfying the consequence, and the “closely” relation.

\[ M \models nw \text{ because } np \land nu. \]  
Here it is enough to consider those distributed states that contain the first state of \( n \) where \( w \) holds. Then, in the example model, we show two distributed states that satisfy the consequence: see pictures (g) and (h).

\[ M \models nw \text{ because } n(p \land u). \]  
See picture (g). Note that we need a singleton state satisfying both \( p \) and \( q \). Hence, in this case, the distributed state \( ds' \) in picture (h) does not satisfy the consequence.
Table 2. We provide a pair of representations of a computation, to outline the distributed states of interest for example }

Example 14. (unless formulae.) We consider, as a model, the computation in Table 3. We take singleton sets for $ds$ and $ds'$, and outline with a sequence of circles the sequence of distributed states satisfying the formula premise, and use a rectangle to outline the distributed state satisfying the formula consequence.

$M \models np \text{ unless } nt$. See picture (i): we take singleton sets for $ds$ and $ds'$, and outline with a sequence of circles the sequence of distributed states satisfying the formula premise, and use a rectangle to outline the distributed state satisfying the formula consequence.

$M \models p \text{ unless } q \lor t$. The sequence of distributed states in picture (i) provides a first demonstration. We also consider, in picture (l), the distributed states in the sequence to be pairs of states: each distributed state is made of the two states related by a dotted line, circles outline the states satisfying the formula premise, rectangles the states satisfying the formula consequence. For instance, the initial state is the first distributed state we consider, followed by the set \{first state of $m$, second state of $n$\}, and so on.

Example 15. (ds' $\not\supseteq ds$ in the definition of the semantics of unless.) Assume we did not require condition $ds' \not\supseteq ds$ in the definition of the semantics of unless, then the following computation would have been a model for $np \text{ unless } nq$, in discrepancy with the intended meaning for unless. We consider the sequence $ds$, $ds'$, $ds''$, $ds'''$, ... of distributed states, where $ds$ contains the first state of component $n$, $ds'$ contains the first two states of component $n$, $ds''$ contains the first three, and so on: all these distributed states satisfy $np$.

Example 16. (stable.) The following computation is a model for $stable \ p$.

Notice that, unlike in Unity, $p$ is not an invariant of the computation, even though $init \ p$ and $stable \ p$ hold. In the next section, we provide the correct derivation rule (SE) that can be used in DSTL.
3.3 Axioms and Rules

We present the most useful axioms and rules of the logic. Among them, temporal operators introduction, strengthening of premises and weakening of consequences, transitivity.

**Necessitation.** First, we observe that the definition for $\mathcal{M} \models^T F$ entails that a necessitation rule holds (we use $\vdash_T$ for the sake of comprehension).

\[
\vdash F \\
\frac{}{\vdash_T F} \text{Nec}
\]

**Operators introduction and elimination.** Rules and axioms $\text{LcI}$, $\text{BcI}$, $\text{LI}$, $\text{BI}$, $\text{UI}$, $\text{InI}$, $\text{SI}$ introduce $\text{leads}_\text{TO}_\mathcal{C}$, $\text{because}_\mathcal{C}$, $\text{leads}_\text{TO}$, $\text{because}$, $\text{unless}$, $\text{init}$, $\text{stable}$ respectively. Rule $\text{SE}$ eliminates $\text{stable}$.

- **LcI:** $F \text{ leads}_\text{TO}_\mathcal{C} F$
- **BcI:** $F \text{ because}_\mathcal{C} F$
- **LI:** $F \text{ leads}_\text{TO} G$
- **BI:** $F \text{ because } G$
- **UI:** $F \text{ unless } F$
- **InI:** $F \text{ init } F$
- **SI:** $F \text{ stable } F$
- **SE:** $\neg F\text{ stable } F$

**Transitivity.** $\text{LTR}$ and $\text{BTR}$ are the rules for $\text{leads}_\text{TO}$ and $\text{because}$ transitivity.

\[
\frac{F \text{ leads}_\text{TO} F' \quad F' \text{ leads}_\text{TO} G}{F \text{ leads}_\text{TO} G} \text{ LTR}
\]
\[
\frac{F \text{ because } F' \quad F' \text{ because } G}{F \text{ because } G} \text{ BTR}
\]

No transitivity rule holds for $\text{leads}_\text{TO}_\mathcal{C}$ and $\text{because}_\mathcal{C}$. In the case of $\text{unless}$, there is a weaker result (a weak form of the rule called cancellation in [5]):

\[
\frac{mF \text{ unless } mF' \quad mF' \text{ unless } mG}{mF \lor mF' \text{ unless } mG} \text{ UC}
\]

**Premises and consequences strengthening and weakening.** $\ast\text{SW}$ permits the strengthening of the premise, and the weakening of the consequences, and $\ast\text{PD}$ and $\ast\text{CC}$ stay for premise disjunction and consequence conjunction, respectively. Actual rules $\text{LSW}$, $\text{LPD}$ and $\text{LCC}$ are obtained by substituting $\text{op}$ with $\text{leads}_\text{TO}$. Similarly, $\text{BSW}$, $\text{BPD}$, and $\text{BCC}$ are obtained by substituting $\text{op}$ with $\text{because}$; $\text{LeSW}$, $\text{LePD}$, and $\text{LeCC}$ are obtained by substituting $\text{op}$ with $\text{leads}_\text{TO}_\mathcal{C}$; $\text{BeSW}$, $\text{BePD}$, and $\text{BeCC}$ are obtained by substituting $\text{op}$ with $\text{because}_\mathcal{C}$.

\[
\frac{G \to F \quad F \text{ op } F' \quad F' \to G'}{G \text{ op } G'} \ast\text{SW}
\]
\[
\frac{F \text{ op } G \quad F' \text{ op } G}{F \lor F' \text{ op } G} \ast\text{PD}
\]
\[
\frac{G \text{ op } F \quad G \text{ op } F'}{G \text{ op } F \land F'} \ast\text{CC}
\]

In the case of $\text{unless}$ and $\text{init}$:

\[
\frac{F \text{ unless } F' \quad F' \to G}{F \text{ unless } G} \text{ UCW}
\]
\[
\frac{F \text{ unless } F' \quad G \text{ unless } G'}{F \lor G \text{ unless } F' \lor G'} \text{ UD}
\]
\[
\frac{\text{init } F \quad F \to G}{\text{init } G} \text{ IW}
\]
**Notification.** Some future remote assertions can be made on the bases of a message received.

\[
\begin{align*}
F \quad \text{BECAUSE} \quad G \quad \text{LEADS_TO} \quad mG' \\
\text{STABLE} \quad mG' \\
\hline
F \land m\top \quad \text{LEADS_TO} \quad mG'
\end{align*}
\]

Notif

Explicit reference to the name \( m \) of the component where the remote effect \( G' \) takes place, and the extra premise \( m\top \) are needed to guarantee that the state satisfying the consequence follows the state satisfying the premise, even in the absence of a communication towards \( m \).

To help the intuition, we consider an instance of the rule:

\[
\begin{align*}
np \quad \text{BECAUSE} \quad mq \quad \text{LEADS_TO} \quad mr \\
\text{STABLE} \quad mr \\
\hline
np \land m\top \quad \text{LEADS_TO} \quad mr
\end{align*}
\]

condition \( p \) can be established in \( n \) only if previously \( q \) has held in \( m \). The second and the third premises guarantee that if \( q \) holds somewhere in \( m \), then eventually \( r \) will hold, and it will continue holding forever. Thus, for any \( ds \) satisfying \( np \land m\top \) we can find a state \( s_m \) of \( S_m \), such that \( \{s_m\} \geq ds \) and \( \{s_m\} \models mr \). Conversely, in the absence of communications from \( n \) to \( m \), if we take a state \( s_n \) of \( S_n \) such that \( \{s_n\} \models np \), we cannot find any distributed state following \( \{s_n\} \) and including a state of \( S_m \), as needed to satisfy \( mr \).

**Confluence.** The converse of DSL axiom D2 holds, under appropriate stability conditions:

\[
\begin{align*}
\text{STABLE} \quad mF' \\
\text{STABLE} \quad mF' \\
\hline
mF \land mF' \rightarrow m(F \land F')
\end{align*}
\]

Conf

**Properties of the initial state.** The following rules are a consequence of the fact that the initial distributed state \( ds^0 \) contains exactly one state for each component.

\[
\begin{array}{ccc}
I_1: & \text{INIT} \quad m\top & \text{INIT} \quad mF \\
I_2 & \text{INIT} \quad mF & \text{INIT} \quad mF \\
I_3 & \text{INIT} \quad mF & \text{INIT} \quad mF
\end{array}
\]

Example 17. (SE) The following computation satisfies \( \text{INIT} \quad mp \) and \( \text{STABLE} \quad mp \).

\[
\begin{align*}
(m) & \quad p \rightarrow p \rightarrow p \rightarrow p, r, z \rightarrow p, u, z \rightarrow p, z \rightarrow p, z \rightarrow p, z \rightarrow p, z \\
(n) & \quad p, t \rightarrow u \rightarrow u, p \rightarrow u \rightarrow p \rightarrow u \rightarrow w, l \rightarrow p, t \rightarrow p, t \rightarrow
\end{align*}
\]

Hence, applying rule SE, we obtain that the computation satisfies \( \text{\overline{mp}} \), i.e. that \( p \) is invariantly true in component \( m \).

It is also interesting to discuss why the cancellation rule

\[
\begin{align*}
F \quad \text{UNLESS} \quad F' \quad \text{F' UNLESS} \quad G \\
\hline
F \lor F' \quad \text{UNLESS} \quad G
\end{align*}
\]

does not hold in general. We consider, as rule premises, \( mp \quad \text{UNLESS} \quad mp \land nq \) and \( mp \land nq \quad \text{UNLESS} \quad mr \land ns \). The following computation is a model of the premises, but not of the consequence \( mp \lor (mp \land nq) \quad \text{UNLESS} \quad mr \land ns \).

\[
\begin{align*}
(m) & \quad p \rightarrow p \rightarrow p \rightarrow p \rightarrow p \rightarrow u, z \rightarrow z \rightarrow z \rightarrow z \rightarrow z \\
(n) & \quad p, t \rightarrow u \rightarrow q \rightarrow s \rightarrow w, l \rightarrow p, t \rightarrow p, t \rightarrow
\end{align*}
\]
**Theorems.** We introduce two rules we need in the case study of Section 5. They are derived by the rules above, as shown in the appendix.

\[
\begin{align*}
F \vdash G \lor G' & \quad G \vdash F' \\
\hline
\therefore \quad F \vdash F' \lor G'
\end{align*}
\]

\[
\begin{align*}
F \vdash G \lor G' & \quad G \vdash F' \\
\hline
\therefore \quad F \vdash F' \lor G'
\end{align*}
\]

**Correctness and completeness.** The soundness of the DSTL proof system can be immediately proved applying Def. 11. In the appendix we provide the proof of the most complex rules, namely Notif and Conf.

Unfortunately, the proof system is not complete. Let us consider a system satisfying \( \bar{m}_i p \), for all \( i \). The system also satisfies \( p \), as a consequence of the property \( V(ds) = \bigcap_{s \in ds} V(\{s\}) \), but we cannot find a general rule to derive it. Indeed, the rule

\[
\frac{\bar{m}_1 F \quad \bar{m}_2 F \quad \ldots \quad \bar{m}_k F}{F}
\]

is not correct. It holds for \( F = p \), or \( F = p \land q \), but not, for instance, for \( F = p \lor q \). In fact, consider a very simple system composed of a unique component \( m \), with states \( s_0, s_1, s_2, \ldots \), and \( p \in V(s_0), q \in V(s_1), q \in V(s_2), \ldots \). All distributed states satisfy \( \bar{m}(p \lor q) \), while the distributed states including \( s_0 \) do not satisfy \( p \lor q \). Take \( ds = \{s_0, s_1\} \), we have that \( ds \models p \lor q \) if \( ds \models p \) or \( ds \models q \), if \( p \in V(ds) \) or \( q \in V(ds) \), if \( p \in V(s_0) \cap V(s_1) \) or \( q \in V(s_0) \cap V(s_1) \).

Hence, since \( V(s_0) \cap V(s_1) = \emptyset \), we have that \( ds \not\models p \lor q \).

Thus, a complete proof system, if any, would likely be unmanageable, and we do not pursue the issue further. On the other side, the consequence of relaxing the constraint on the valuation function, would be as unpractical as explicitly specifying the truth value of all predicated on all distributed states.

4 An Example: Private Keys

Consider the system \( \{b, t, u\} \), where \( b \) is a component that broadcasts the encrypted version of a message to all the other components in the system, i.e. \( t \) (trusted) and \( u \) (untrusted). We assume that these components try to decrypt the message. We represent with predicate \( p \) the fact that the message is readable, and with predicate \( dep \) the fact that a decryption has been attempted. However, the decryption yields \( p \) if and only if the key is held. Predicate \( key \) represents the property of holding the key.

4.1 Reasoning on distributed states: DSL

The properties of the distributed states of the system are described by the following DSL formulae:

\[
(\sim b \top) \rightarrow ((key \land dep) \leftrightarrow p) \tag{7}
\]

\[
\check{t} \sim key \tag{8}
\]

\[
\check{u} \sim key \tag{9}
\]

Formula (7) tells that in all components, with the exception of \( b, p \) and \( dep \) are equivalent only if the key is held. Indeed, if (7) holds, as required, in all \( ds \in DS \), it holds in particular
in all $ds$ which are singleton sets. So, it holds for all $\{s_t\}$ and $\{s_u\}$. Since in these states the premise of (7) is satisfied, so it is the conclusion, i.e. in all states of $t$ and $u$: $(key \land dep) \leftrightarrow p$.

We derive the property for $t$:

\[
\frac{(\sim b \top) \to ((key \land dep) \to p)}{t((key \land dep) \to p)} \quad \text{Nec, K} \\
\frac{\bar{t}(\sim b \top) \to \bar{t}((key \land dep) \leftrightarrow p)}{\bar{t}(key \to (dep 	o p))} \quad \text{Nec, K, MP} \\
\]

Component $t$ holds the key $s_t$, while component $u$ does not $s_u$. We derive that $t$ is able to correctly decrypt the message. We pick one of the implications, i.e. $(key \land dep) \to p$ and prove that $t (dep \to p)$ (the top leftmost formula is a tautology of the propositional calculus):

\[
\frac{(key \land dep) \to p \to (key \to (dep \to p))}{t(key \to (dep \to p))} \quad \text{Nec, K, MP} \\
\frac{\bar{t}key \to \bar{t}(dep \to p)}{\bar{t}(dep \to p)} \quad \text{K, MP} \\
\]

We now consider component $u$ and prove that $\bar{u} \sim p$ holds, i.e. that the untrusted component is not able to read the message. We consider the implication $p \to (key \land dep)$ (the top leftmost formula is a tautology of the propositional calculus):

\[
\frac{(p \to (key \land dep)) \to (\sim key \to \sim p)}{\bar{u}(\sim key \to \sim p)} \quad \text{Nec, K, MP} \\
\frac{\bar{u}(key \land dep) \leftrightarrow p}{\bar{u} \sim key} \quad \text{K, MP} \\
\]

4.2 Reasoning on distributed computations: DSTL

We now add some constraints on the temporal behaviour of the private keys system: as soon as the message is readable in $b$, $b$ broadcasts its encrypted version (10); $t$ and $u$ try to decrypt the message (11, 12).

\[
b \; \L \; t ep \land u ep \quad (10) \\
t ep \; \L \; t dep \quad (11) \\
u ep \; \L \; u dep \quad (12) \\
\]

To prove that $u$ will not correctly decrypt the message, we need to prove that $\bar{u} \sim p$. This is immediately obtained by applying \text{Nec} to the corresponding DSL formula derived in Section 4.1. We prove that $b \; \L \; t p$. We exploit the conclusion of Section 4.1 $\bar{t}(dep \to p)$.

\[
\frac{b \; \L \; t p \land u ep \quad (t ep \land u ep) \to t ep}{b \; \L \; t p} \quad \text{LSW} \\
\frac{t ep \; \L \; t dep}{t ep \; \L \; t dep} \quad \text{LTR} \\
\frac{b \; \L \; t dep}{b \; \L \; t dep} \quad \text{D3} \\
\frac{t dep \to t p}{t dep \to t p} \quad \text{LSW} \\
\frac{b \; \L \; t p}{b \; \L \; t p} \\
\]

5 An Example: Leader Election

The leader election problem is a typical example of distributed consensus. It is well known that in an asynchronous setting, no algorithm can guarantee that a distributed consensus is reached (see, for instance [26]). The solution we discuss here leads to the election of a leader, or to the agreement that no leader has been chosen, in this case a new election round can take place.

Initially all the $k$ participants are eligible. They toss a coin: those who get head are no longer eligible and acknowledge the other participants; those who get tail toss the coin again. The election round ends when either only one participant is still eligible and becomes the leader, or nobody is eligible.

Predicate $e_i$ says that participant $i$ is still eligible: initially all participants agree that they are all eligible; each participant falsify his $e_i$ when acknowledged that participant $i$ got a head.

In the following we list the local properties satisfied by each participant and derive the global property of the proposed solution: eventually all participants agree that either nobody is eligible, i.e. $\sim e_i$ holds for all $i$ and for all participants, or only one participant is still eligible, i.e. there exists a $j$ such that for all participants $e_j$ holds while $e_k$ is false for all $k \neq j$. Formally:

$$\bigwedge_i m_i \top \text{leads to} \quad \bigwedge_i m_i \bigwedge_j ~ e_j \lor \bigvee_j \bigwedge_i m_i (e_j \land \bigwedge_{k \neq j} ~ \sim e_k)$$

In the case of two participants:

$$m_1 \top \land m_2 \top \text{leads to} \quad m_1 (\sim e_1 \land \sim e_2) \land m_2 (\sim e_1 \land \sim e_2) \quad \text{(no leader elected)}$$

$$\lor \quad m_1 (e_1 \land \sim e_2) \land m_2 (e_1 \land \sim e_2) \quad \text{(e_1 elected)}$$

$$\lor \quad m_1 (e_2 \land \sim e_1) \land m_2 (e_2 \land \sim e_1) \quad \text{(e_2 elected)}$$

The local properties follow.

1. Fairness of the toss up: nobody can spin the coin infinite times and nether get a head. So, either a participant eventually stops spinning the coin or he gets a head. For all $i$:

$$m_i \top \text{leads to} m_i (\text{stop} \lor h)$$

2. Participant $i$ stops if and only if the other participants are no longer eligible:

$$\bar{m}_i (\text{stop} \leftrightarrow \bigwedge_{j \neq i} \sim e_j)$$

3. When participant $i$ gets a head, he sends an ack to all participants, who declare $i$ non eligible.

$$m_i h \text{leads to} \quad \bigwedge_j m_j \sim e_i$$

4. A participant can be declared non eligible only if he got a head:

$$m_i \sim e_j \text{ because } m_j h$$

5. Initially all participants are eligible.

$$\text{init} \quad \bigwedge_i m_i (\bigwedge_j e_j \land \sim h)$$
6. Non eligibility is stable:

\[ \text{STABLE } m_i \sim e_j \]

We prove that the global property holds in the case of two participants. The proofs for the other cases are similar. In the first step of the proof, we exploit properties 1 and 2:

\[
\begin{align*}
& m_1 \top \text{ LEADS}_\text{TO } m_1 (\text{stop} \lor h) \quad \text{D7} \\
& m_1 \top \text{ LEADS}_\text{TO } m_1 \text{stop} \lor m_1 h \quad \text{D3} \\
& m_1 \top \text{ LEADS}_\text{TO } m_1 \text{stop} \lor m_1 h \quad \text{LSW}
\end{align*}
\]

The same holds for \( m_2 \). We apply \( \text{LSW} \) and \( \text{LCC} \) and obtain:

\[
\begin{align*}
& m_1 \top \land m_2 \top \quad \text{LEADS}_\text{TO} \\
& m_1 h \land m_2 h \quad \text{(13)} \\
& \lor m_1 h \land m_2 \sim e_1 \quad \text{(14)} \\
& \lor m_1 \sim e_2 \land m_2 h \quad \text{(15)} \\
& \lor m_1 \sim e_2 \land m_2 \sim e_1 \quad \text{(16)}
\end{align*}
\]

In the remaining part of the section we prove that:

\[
\begin{align*}
& \text{[18] LEADS}_\text{TO } m_1 (\sim e_1 \land \sim e_2) \land m_2 (\sim e_1 \land \sim e_2) \\
& \quad \text{[19] LEADS}_\text{TO } m_1 (\sim e_1 \land e_2) \land m_2 (\sim e_1 \land e_2) \quad \text{(no leader elected)} \\
& & \lor m_1 (\sim e_1 \land \sim e_2) \land m_2 (\sim e_1 \land \sim e_2) \quad \text{(e2 elected or no leader)} \\
& & \lor m_1 (\sim e_1 \land e_2) \land m_2 (\sim e_1 \land e_2) \quad \text{(e1 elected or no leader)} \\
& & \lor m_1 (\sim e_1 \land \sim e_2) \land m_2 (\sim e_1 \land \sim e_2) \quad \text{(no leader elected)}
\end{align*}
\]

So, we can apply \( \text{Cor1} \) and conclude.

**Proof of [18] LEADS TO no leader elected**

We exploit hypothesis 3:

\[
\begin{align*}
& m_1 h \land m_2 h \quad \text{LEADS}_\text{TO} \\
& m_1 \sim e_1 \land m_2 \sim e_1 \land m_1 \sim e_2 \land m_2 \sim e_2
\end{align*}
\]

We apply \( \text{Conf} \):

\[
\begin{align*}
& \text{STABLE } m_1 \sim e_1 \quad \text{STABLE } m_1 \sim e_2 \\
& m_1 \sim e_1 \land m_1 \sim e_2 \quad \rightarrow \quad m_1 (\sim e_1 \land \sim e_2)
\end{align*}
\]

A similar implication holds for \( m_2 \), hence:

\[
\begin{align*}
& m_1 h \land m_2 h \quad \text{LEADS}_\text{TO} \\
& m_1 (\sim e_1 \land \sim e_2) \land m_2 (\sim e_1 \land \sim e_2)
\end{align*}
\]

**Proof of [19] LEADS TO e2 elected or no leader elected (the case for [18] is symmetric).**

We exploit again hypothesis 3 and obtain, using \( \text{Cor2} \), that:

\[
\begin{align*}
& m_1 h \land m_2 \sim e_1 \quad \text{LEADS}_\text{TO} \\
& m_1 \sim e_1 \land m_2 \sim e_1
\end{align*}
\]
Now, since we don’t know anything on the truth of \( e_2 \), we need to consider all the possibilities:

\[
\begin{align*}
    m_1 \sim e_1 \land m_2 \sim e_1 & \iff m_1(\sim e_1 \land e_2) \land m_2(\sim e_1 \land e_2) & (17) \\
    \lor m_1(\sim e_1 \land e_2) \land m_2(\sim e_1 \land e_2) & (18) \\
    \lor m_1(\sim e_1 \land e_2) \land m_2(e_1 \land e_2) & (19) \\
    \lor m_1(e_1 \land e_2) \land m_2(\sim e_1 \land e_2) & (20)
\end{align*}
\]

In case (17) an agreement is reached that \( e_2 \) is the leader. In case (20) the participants agree that no leader has been elected. The other two cases are symmetric: we consider case (18) and show that it leads to a state where no leader has been elected. We first show that a state is reached where participant 2 agrees that he cannot be the leader:

\[
\begin{align*}
    m_1 \sim e_1 \land \neg m_2 & \text{ because } m_2 \text{ h} \\
    m_1(\sim e_1 \land e_2) & \text{ because } m_2 \text{ h} \\
    m_2 \text{ LEADS_TO } m_2 \sim e_2 & \text{ Stable } m_2 \sim e_2 \\
    \text{ Notif} \\
    \text{ BSW} \\
    \text{ LSW}
\end{align*}
\]

where the last step (LSW) exploits the following implication:

\[
\begin{align*}
    \sim e_1 \land e_2 \rightarrow \top & \text{ Nec,D3} \\
    m_2(\sim e_1 \land e_2) \rightarrow m_2 \top
\end{align*}
\]

We carry on some calculation:

\[
\begin{align*}
    m_1(\sim e_1 \land e_2) \land m_2(\sim e_1 \land e_2) \text{ LEADS_TO } m_2 \sim e_2 & \text{ LCC(F LEADS_TO F)} \\
    m_1(\sim e_1 \land e_2) \land m_2(\sim e_1 \land e_2) \text{ LEADS_TO } m_1(\sim e_1 \land e_2) \land m_2(\sim e_1 \land e_2) \land m_2 \sim e_2 & \text{ D2, LSW (twice)}
\end{align*}
\]

We now apply \textit{Conf} and conclude:

\[
\begin{align*}
    m_1(\sim e_1 \land e_2) \land m_2(\sim e_1 \land e_2) \text{ LEADS_TO } m_1(\sim e_1 \land e_2) \land m_2(\sim e_1 \land e_2)
\end{align*}
\]

\textbf{Proof of (18) LEADS_TO no leader elected}

We apply the proof schema above (\textit{Notif} and then \textit{Conf}) twice and conclude.

\section{Discussion and Related Work}

\textbf{The semantic domain of DSL.} The choice of \( 2^S \) as a semantic domain of the distributed state logic formulae, and the non–equivalence between \( \mathbf{m}(F \land F) \) and \( \mathbf{m} F \land \mathbf{m} F' \) are useful to specify that a given condition can have different future effects, without constraining them to occur in the same state. Similarly, we can express complex preconditions in a temporal formula. For instance, assume we want to specify and reason on the delivery of credit cards to customers. The bank, for security reasons, sends the card and the code separately. Once the customer has got both of them, he is allowed to withdraw money from an ATM machine:

\[
\begin{align*}
    \text{bank new card \ LEADS_TO user receive card} \land \text{user receive code} & \text{ (21)} \\
    \text{user can withdraw \ BECAUSE user receive card} \land \text{user receive code} & \text{ (22)}
\end{align*}
\]
The equivalence between \( m(F \land F') \) and \( mF \land mF' \) would have required the following specification, somewhat less intuitive:

\[
\begin{align*}
\text{bank new card} & \Rightarrow \text{user receive card} \\
\text{bank new card} & \Rightarrow \text{user receive code} \\
\text{user can withdraw} & \Rightarrow \text{user receive card} \\
\text{user can withdraw} & \Rightarrow \text{user receive code}
\end{align*}
\]  

(23) \( \text{bank new card} \Rightarrow \text{user receive card} \)

(24) \( \text{bank new card} \Rightarrow \text{user receive code} \)

(25) \( \text{user can withdraw} \Rightarrow \text{user receive card} \)

(26) \( \text{user can withdraw} \Rightarrow \text{user receive code} \)

since (21), (22) would be too restrictive, asking for card and code to be received at the same time.

Last, but not least, with an eye to a 1st order extension, a formula like (21) makes it easier to bind variables in card and code than with the unrelated formulae (23), (24).

**Classical Logic.** Another point of discussion is why we need a modality (m) rather than a distinguished propositional symbol here\(_m\), to replace systematically each sub-formula \( mF \) with here\(_m \land F \). One motivation is that we do not want the equivalence between \( m(F \land F') \) and \( mF \land mF' \), as discussed previously. On the contrary, the two translations here\(_m \land F \land F'\) and here\(_m \land F \land \text{here}_m \land F'\) would be equivalent.

More importantly, \((mF \land nF') \land \text{leads}_m \Rightarrow \text{oG}\) would be translated in a formula with a false premise, namely \((\text{here}_m \land F \land \text{here}_n \land F') \Rightarrow \text{leads}_m \Rightarrow (\text{here}_o \land G)\).

**Hybrid Logic.** Hybrid logic allows the specifier to directly refer to specific points (states) in the model, through the use of nominals \([1]\). A nominal \(i\) is an atom which is true at exactly one point in any model. The operator \(@_i\) permits to jump to the point named by nominal \(i\). We might consider defining an hybrid signature including distinguished sets of state variables, one for each component, and translate \( mF \) in \( \exists x. @_x F \), where \(x\) is a state variable in the appropriate set. Likely, the resulting setting would be more complex than that offered by DSTL.

**Metric and Layered Temporal Logic.** Some similarities can be found between our location operator and the MLTL operators defined in [20], that make it possible to compose formulae associated with different time granularities and to switch from one granularity to another. Time instants are organized in temporal domains, and the set of temporal domains is totally ordered with respect to the coarseness of the domain elements. To look for an embedding of DSTL, we can consider three domains: system, with a unique element; components, whose elements are the components \(m_1, \ldots, m_k\); states, the domain of the states.

Then the formulae are translated using an appropriate combination of MLTL operators. For instance, the translation of \( mF \) should be \( \Diamond_{\text{components}} \Delta_m \exists \alpha \Delta_{\text{states}} \alpha \land F \). Since the full expressive power of MLTL is likely not needed, the simpler framework of DSTL is of pragmatical interest.

**Other logics for distributed systems.** Various extensions of temporal logic have been defined in the literature to deal with distributed systems.

In TTL [18], for each local state of the system, a visibility function specifies which remote information is accessible. The visibility function is defined on the basis of a relation among states which is symmetric in the case of states belonging to distinguished components.

A trace based extension of linear time temporal logic, called TrPTL, has been defined in [30] (see also [31]). The logic has been designed to be interpreted over infinite traces, i.e.,
labelled partial orders of actions, which respect some dependence relations associated to the alphabet of actions.

In [13], a temporal logic, StepTL, is defined and interpreted over multistep transition systems. These are a well known extension of transition systems, permitting to describe as concurrent the steps of computation that can actually be executed in parallel. A multistep transition system thus contains transitions of the form $s A s'$, where $A$ is a set of actions, instead of a single one.

Three distinguished logics are presented in [28] to describe systems composed of sets of communicating agents. The logics differentiate on the amount of information each agent can have on the other agents running on the system, but share a common setting: agents communicate via common actions. The models for these logics are runs of networks of synchronizing automata. The logics $D_0$ and $D_1$ presented in [6] are based on a similar approach.

In all these proposals, components communicate via some form of synchronization, and logic formulae are interpreted on models shaping:

\[
\begin{array}{cccccc}
  (m) & \xrightarrow{a} & \xrightarrow{b} & \xrightarrow{c} & \\
  (n) & \xrightarrow{d} & \xrightarrow{e} & \xrightarrow{f} & \\
\end{array}
\]

Therefore, in any logic defined over these models, it is not possible to express the asymmetric nature of causality we are interested in when modelling the behaviour of agents communicating asynchronously by message passing. Indeed, in the previous model we can both assert that $a \text{ leads_to } f$ and that $d \text{ leads_to } c$.

A logic closer to DSTL is proposed in [16], where a branching time temporal logic for asynchronously communicating sequential agents (ACSAs) is defined. ACSAs communicate asynchronously via message passing. The logic contains temporal modalities indexed with a local point of view of one agent and allows an agent “i” to refer to local properties of another agent “j” according to the latest message received: an agent can gain information about another agent by receiving messages but not by sending them. We allow agents to make remote future assertions: therefore it is easier to express global liveness properties.

Knowledge Logic. A logic to reason on asynchronous message passing systems is proposed in [7]. The language used, $L^U_n$, is obtained by extending their language of knowledge with the modal operators $U$ and $\Box$. Formulae in $L^U_n$ permit to express how the $n$ agents in a system gain knowledge over time. A set of characteristic formulae valid in the logic are presented, but a sound and complete axiom system is not defined. The authors focus their attention on systems based un-reliable communications, while only state that properties of reliable communications can be expressed. A major difference with our work relies on the models used to interpret $L^U_n$ formulae. Even if the knowledge of the agents is limited to their current local histories, i.e. sequences of messages sent or received and of internal actions, interpretation structures are based on global time and state.

Partial Order Temporal Logics. Partial Order Temporal Logics (POTL) [27] permits to deal with the causal relationships between the events of a set of processes executing concurrently. The Interleaving Set Temporal Logic (ISTL) [12] extends POTL with features form linear temporal logic and branching temporal logic. The Kripke structures for both logics are very different from the one defined in this paper.

We are addressing a specific class of systems that we consider very relevant nowadays, that is distributed systems with asynchronous message passing. These systems have a few
notable characteristics: there is no global state, and interactions among components occur only via messages. As a consequence, a specification is essentially devoted to describing the causal relationships among the components. We think that these characteristics are so important that the designer working on a specification will greatly benefit if they are naturally embedded in the basic model he is using. Hence, the investigation in Kripke’s structures presented in this paper.

**Logics for Mobile Systems.** Often mobile systems are specified using a process calculus with primitives for mobility, and some logics have been defined, tailored for these calculi. This is the case, for instance, of the Ambient Logic [3], studied for the Ambient Calculus [4], the logic for Klaim [17], and the Spatial Logic forConcurrency [2], whose underlying computational model is the asynchronous \( \pi \)-calculus. These logics include modalities for describing the evolution over time and the location of the system processes. They are inspired by the Hennessy–Milner logic: they are conceived for model checking rather than for specifying and reasoning on the system properties.

In particular, the Spatial Logic for Concurrency can express properties of freshness, secrecy, structure, and behavior of concurrent systems. Spatial operations correspond to composition, local name restriction, and a primitive fresh name quantifier. The logical treatment of the notion of freshness can prove useful in extending DSTL to reason on the dynamic creation of components.

A linear–time logic for specifying mobile systems is MTLA [19], which extends Lamport’s Temporal Logic of Actions with spatial modalities to deal with mobile systems. The main difference with DSTL is that a synchronous computational model is assumed.

**Oikos–adtl.** The work reported here stems from our experience with Oikos–adtl, a specification language for distributed systems based on asynchronous communications, designed to support the composition of specifications [21]. Oikos–adtl is intended to give designers a language to express the properties of interest in a natural way, and it is associated with a refinement method which supports the gradual introduction of details, as design proceeds. It has been used to specify software architectures and patterns [21] and to analyse security issues in mobile systems [11,9,8]. It is supported by a proof assistant, Mark [10], that deploys a number of proof strategies that partially automate property verification.

Coming back to our motivating example in the introduction, in Oikos–adtl it is possible to weaken the consequences of a formula like (1) including operator leads-to, but the rule shapes

\[
\frac{m \; p \; \text{leads-to} \; n \; q \land o \; r}{m \; p \; \text{leads-to} \; n \; q}
\]

since a formula like (2) is not part of the logic. So, the price is writing one rule for each possible weakening relation.

7 Conclusions

To reason on global applications, we have introduced the temporal logic DSTL. Models for DSTL are space–time diagrams describing the behaviour of a set of components communicating asynchronously. The logic has been introduced in two steps. First, we have defined DSL, a modal logic for localities that embeds the theories describing the local states of each component into a theory of the distributed states of the system. No notion of time or state
transition is present at this stage. To support reasoning in the logic, we have presented a sound and complete axiom system. Then, we have added the temporal operators, and the corresponding derivation rules. The contribution is that it is possible to reason about properties that involve several components, even in the absence of a global clock, which is a meaningless notion in an asynchronous setting. The logic has been used to reason on the properties of a simple secure communication system and on an algorithm for the leader election.

Future work includes the extension of DSTL to predicate logic, the introduction of an event operator, the study of compositionality results, and a revision of the theorem prover Mark. We foresee that formulae in the 1st order extension will shape \( mp(x) \text{ leads}_\text{TO} n q(x, y) \), and be interpreted as \( \forall x. [ mp(x) \text{ leads}_\text{TO} \exists y. n q(x, y)] \). This way, the semantics should smoothly extend that of DSTL. Compositionality results will permit to derive the properties satisfied by a system from the properties satisfied by its components when executed in isolation. This requires reasoning on the possible interferences due to communications from the added components.

References

1. C. Areces. Logic Engineering. The Case of Description and Hybrid Logics. PhD thesis, ILLC, University of Amsterdam, The Netherlands., 2000.
2. L. Caires and L. Cardelli. A Spatial Logic for Concurrency (Part I and II). Part 1: To appear in Journal of Information and Computation special issue on TACS 2001. Part 2: Proc. CONCUR’2002, LNCS 2421, Springer-Verlag.
3. L. Cardelli and A. Gordon. Anytime, anywhere — modal logics for mobile ambients. In Proceedings of POPL ’00, pages 365-377. ACM, 2000.
4. L. Cardelli and A.D. Gordon. Mobile ambients. Theoretical Computer Science, 240(1):177–213, 2000. An extended abstract appeared in Proceedings of FoSSaCS ’98, number 1378 of Lecture Notes in Computer Science, pages 140-155, Springer, 1998.
5. K.M. Chandy and J. Misra. Parallel Program Design: A Foundation. Addison-Wesley, Reading Mass., 1988.
6. H.-D. Ehrich, C. Calcier, A. Sernadas, and G. Denker. Logics for specifying concurrent information systems. In J. Chomicki and G. Saake, editors, Logic for Databases and Information Systems, pages 167–198. Kluver Academic Publishers, 1998.
7. R. Fagin, J. Halpern, Y. Moses, and M. Vardi. Reasoning About Knowledge. MIT Press, 1995.
8. G. Ferrari, C. Montangero, L. Semini, and S. Semprini. Refinement Calculus for Mobility: expressing security policies. In J. Vitek, editor, Workshop on Distributed Object Security, Denver, Co, 1999.
9. G. Ferrari, C. Montangero, L. Semini, and S. Semprini. Ad Hoc Network Applications: Specification, Design, and Verification in mobadtl. In G.-C. Roman and G.P. Picco, editors, Proc. ICSE’01 Workshop on Software Engineering and Mobility, Toronto, CA, 13,14 May 2001.
10. G. Ferrari, C. Montangero, L. Semini, and S. Semprini. Mark, a reasoning kit for mobility. Automated Software Engineering, 9(2):137–150, Apr 2002.
11. G. Ferrari, C. Montangero, L. Semini, and S. Semprini. Multiple Security Policies in Mobadtl. In Proc. Work. on Issues in the Theory of Security, Geneva, 2000.
12. S. Katz and D. Peled. Interleaving set temporal logic. Theoretical Computer Science, 75(3):263-287, 1990.
13. K. Lodaya, R. Parikh, R. Ramanujam, and P.S. Thiiagarajan. A Logical Study of Distributed Transition Systems. Information and Computation, 119(1):91–118, ‘95.
14. H. Kreger. Web Services Conceptual Architecture. IBM Software Group, May 2001. www-4.ibm.com/software/solutions/webservices/pdf/WSCA.pdf.
15. L. Lamport. Time, clocks, and the ordering of events in a distributed system. Communications of the ACM, 21(7):558–565, Jul 1978.
16. Lodaya, Ramamujam, and Thiiagarajan. Temporal logics for communicating sequential agents: I. Int. Journal of Found. of Computer Science, 3(2):117–159, ‘92.
17. M. Loreti and R. De Nicola. A modal logic for mobile agents. ACM Transaction on Computational Logic (to appear).
18. A. Masini and A. Maggiolo-Schettini. TTL: A formalism to describe local and global properties of distributed systems. Informatique théorique et Applications/Theoretical Informatics and Applications, 26(2):115–149, 1992.
19. S. Merz, M. Wirsing, and J. Zappe. A spatio–temporal logic for the specification and refinement of mobile systems. In M. Pezzè, editor, *Proc. FASE 2003*, volume 2621 of *Lecture Notes in Computer Science*, pages 87–101. Springer-Verlag, 2003.

20. A. Montanari. *Metric and Layered Temporal Logic for Time Granularity*. PhD thesis, University of Amsterdam, 1996. Chapter 3.

21. C. Montangero and L. Semini. Refining by Architectural Styles or Architecting by Refinements. In *2nd Int. Software Architecture Workshop, Proc. of the SIGSOFT ’96 Workshops*, pages 76–79, San Francisco, CA, Oct 1996. ACM Press.

22. C. Montangero and L. Semini. Software specification and design: from formal methods to standard middleware. In *Proc. 6th ERCIM Inter. Workshop on Formal Methods for Industrial Critical Systems (FMICS’01)*, 2001.

23. C. Montangero and L. Semini. Distributed states logic. In *9th International Symposium on Temporal Representation and Reasoning*, Manchester, UK, July 2002. IEEE CS Press.

24. C. Montangero and L. Semini. Composing Specifications for Coordination. In P. Ciancarini and A. Wolf, editors, *Proc. COORDINATION 99*, volume 1594 of *LNCS*, pages 118–133, Amsterdam, April 99. Springer-Verlag.

25. Microsoft .NET. at [http://research.microsoft.com](http://research.microsoft.com).

26. C. Palamidessi. Comparing the expressive power of the synchronous and the asynchronous pi-calculus. *Mathematical Structures in Computer Science*. To appear. [http://www.cse.psu.edu/~catuscia/papers/pi_calc/mscs.ps](http://www.cse.psu.edu/~catuscia/papers/pi_calc/mscs.ps)

27. Pinter and Wolper. A temporal logic for reasoning about partially ordered specifications. In *Proc. 3rd ACM Principles of Distributed Computing*, pages 28–37, Vancouver, B.C., 1984.

28. R. Ramanujam. Locally linear time temporal logic. In *Proc. 11th IEEE Symp. on Logic In Computer Science*, pages 118–127. IEEE Computer Society, 1996.

29. L. Semini and C. Montangero. A Refinement Calculus for Tuple Spaces. *Science of Computer Programming*, 34:79–140, 1999.

30. P. S. Thiagarajan. A trace based extension of linear time temporal logic. In *Proceedings, Ninth Annual IEEE Symposium on Logic in Computer Science*, pages 438–447, Paris, Jul 1994. IEEE Computer Society Press.

31. P. S. Thiagarajan and J. G. Henriksen. Distributed versions of linear time temporal logic: A trace perspective. In W. Reisig and G. Rozenberg, editors, *Lectures on Petri Nets I: Basic Models, Advances in Petri Nets*, volume 1491 of *Lecture Notes in Computer Science*, pages 643–681. Springer-Verlag, 1998.

32. C. Wille. *Presenting C#*. SAMS Publishing, 2000.
Appendix

Proofs from Section 2.2

Axiom 4

\[
\frac{\bar{m}(\bar{m}F \iff F)}{\text{K}}
\]

\[
\frac{\bar{m}mF \iff \bar{m}F}{\text{K}}
\]

D1

\[
\frac{\bar{m}(\bar{m} \sim F \iff \sim F)}{\text{K}}
\]

\[
\frac{\bar{m}\bar{m} \sim F \iff \bar{m} \sim F}{\text{K}}
\]

\[
\frac{\sim \bar{m} \sim \bar{m} \sim F \iff \sim \bar{m} \sim F}{\text{K}}
\]

\[
\frac{\sim \bar{m} \sim \bar{m} \sim F \iff \bar{m} \sim \bar{m} \sim F}{\text{K}}
\]

D2 We show that \(m(F \wedge F') \rightarrow mF\), \(m(F \wedge F') \rightarrow mF'\) is proved analogously.

\[
\frac{F \wedge F' \rightarrow F}{\text{Nec}}
\]

\[
\frac{\sim (F \wedge F') \iff \sim F}{\text{Nec}}
\]

\[
\frac{\bar{m}(\sim (F \wedge F') \iff \sim F)}{\text{K}}
\]

\[
\frac{\sim \bar{m} \sim (F \wedge F') \iff \sim \bar{m} \sim F}{\text{K}}
\]

\[
\frac{\sim \bar{m} \sim (F \wedge F') \iff \sim \bar{m} \sim F}{\text{K}}
\]

D3

\[
\frac{(F \rightarrow F') \rightarrow (\sim F' \rightarrow \sim F)}{\text{Nec, K}}
\]

\[
\frac{m(F \rightarrow F') \rightarrow (\sim F' \rightarrow \sim F)}{\text{K}}
\]

\[
\frac{m(F \rightarrow F') \rightarrow (\sim F' \rightarrow \sim F)}{\text{K}}
\]

\[
\frac{\bar{m}(\sim F' \rightarrow \sim F)}{\text{K}}
\]

\[
\frac{\bar{m} \sim F' \rightarrow \bar{m} \sim F}{\text{Def}}
\]

\[
\frac{\sim \bar{m}F' \iff \sim mF'}{\text{PC}}
\]

D5

\[
\frac{\bar{m}(\sim m \sim F \iff \sim F)}{\text{K}}
\]

\[
\frac{\bar{m}(\sim m \sim F \iff \sim F)}{\text{K}}
\]

\[
\frac{\bar{m}(mF \iff F)}{\text{K}}
\]
D6 For $A = F \rightarrow F'$, $B = F' \rightarrow F''$, and $C = F \rightarrow F''$

\[
\begin{array}{c}
A \land B \rightarrow C \\
\sim A \lor \sim B \lor C \\
A \rightarrow (B \rightarrow (A \land B)) \\
\bar{m}A \rightarrow \bar{m}(B \rightarrow (A \land B)) \quad \text{Nec, K} \\
\bar{m}A \rightarrow (\bar{m}B \rightarrow \bar{m}(A \land B)) \quad K \\
\sim \bar{m}A \lor \sim \bar{m}B \lor \bar{m}(A \land B) \\
\sim (\bar{m}A \land \bar{m}B) \lor \bar{m}(A \land B) \\
(\bar{m}A \land \bar{m}B) \lor \bar{m}(A \land B) \\
\end{array}
\]

D7 We prove $\bar{m}(F \lor F') \rightarrow (\bar{m}F \lor \bar{m}F')$ on the left, and $\bar{m}(F \lor F') \leftarrow (\bar{m}F \lor \bar{m}F')$ on the right.

\[
\begin{array}{c}
(A \land B) \rightarrow (A \land B) \\
\sim A \lor \sim B \lor (A \land B) \\
A \rightarrow (B \rightarrow (A \land B)) \\
\bar{m}A \rightarrow \bar{m}(B \rightarrow (A \land B)) \quad \text{Nec, K} \\
\bar{m}A \rightarrow (\bar{m}B \rightarrow \bar{m}(A \land B)) \quad K \\
\sim \bar{m}A \lor \sim \bar{m}B \lor \bar{m}(A \land B) \\
\sim (\bar{m}A \land \bar{m}B) \lor \bar{m}(A \land B) \\
(\bar{m}A \land \bar{m}B) \lor \bar{m}(A \land B) \\
\end{array}
\]

D8 If we prove $\bar{m}((mF \land mF') \rightarrow (F \land F'))$ and $\bar{m}((F \land F') \rightarrow m(F \land F'))$ then we can apply D6 and conclude. The second formula is an instance of D5, we prove the first one:

\[
\begin{array}{c}
(mF \rightarrow F \land mF' \rightarrow F') \rightarrow (mF \land mF' \rightarrow F \land F') \\
\bar{m}(mF \rightarrow F \land mF' \rightarrow F') \rightarrow \bar{m}(mF \land mF' \rightarrow F \land F') \quad \text{Nec, K} \\
\end{array}
\]

\[
\begin{array}{c}
\bar{m}(mF \rightarrow F) \land \bar{m}(mF' \rightarrow F') \quad \text{D5} \\
\bar{m}(mF \rightarrow F) \land \bar{m}(mF' \rightarrow F') \quad \text{D7} \\
\bar{m}(mF \land mF' \rightarrow F \land F') \quad \text{MP, with the conclusions of the previous derivation}
\end{array}
\]
Proof of the Notification Rule

\[
\begin{align*}
F & \text{ because } G \\
G \text{ leads to } mG' & \text{ stable } mG' \\
\hline
F \land m\top & \text{ leads to } mG' \\
\end{align*}
\]

Notif

Let \( ds \) be a distributed state satisfying \( F \land m\top \), we have that:

\[
\begin{align*}
\forall ds \mid F \land m\top \Rightarrow ds \mid F \\
\Rightarrow \exists ds' \leq ds. \; ds' \mid G \quad (\text{since } F \text{ because } G) \\
\Rightarrow \exists ds'' \geq ds'. \; ds'' \mid mG' \quad (\text{since } G \text{ leads to } mG')
\end{align*}
\]

Summing up, \( \forall ds \mid F \; \exists ds'' \mid mG' \).

Now, \( ds'' \mid mG' \) implies that \( \exists s \in ds'' \cap S_m \) with \( \{s\} \mid mG' \). Stability of \( mG' \) guarantees that for any state \( s' \in S_m \) that follows \( s \), \( \{s'\} \mid mG' \). So, we can build a distributed state which follows any \( ds \) satisfying \( F \land m\top \) and satisfies \( mG' \).

Proof of the Confluence Rule

\[
\begin{align*}
\text{stable } mF & \quad \text{stable } mF' \\
\hline
mF \land mF' & \rightarrow m(F \land F') \\
\end{align*}
\]

Conf

Let \( ds \) be a distributed state satisfying \( mF \land mF' \):

\[
\begin{align*}
ds \mid mF \land mF' \Leftrightarrow ds \mid mF \text{ and } ds \mid mF' \\
\Leftrightarrow \exists s \in ds \cap S_m. \; \{s\} \mid F \text{ and } \exists s' \in ds \cap S_m. \; \{s'\} \mid F'
\end{align*}
\]

Let \( \{s\} \geq \{s'\} \) (the case \( \{s\} \leq \{s'\} \) is symmetric), for the stability of \( F' \) we have that also \( \{s\} \) satisfies \( F' \):

\[
\begin{align*}
\{s\} \mid F \text{ and } \{s\} \mid F' \Leftrightarrow \{s\} \mid F \land F' \\
\Leftrightarrow ds \mid m(F \land F')
\end{align*}
\]

Proof of Cor1 and Cor2

\[
\begin{align*}
F \text{ leads to } F'' & \quad F'' \rightarrow F' \lor G' \\
\hline
G \text{ leads to } F' \lor G' & \text{ LSW } \\
\end{align*}
\]

\[
\begin{align*}
G' \text{ leads to } G' \lor G'' & \text{ LSW } \\
\hline
G' \lor G'' \text{ leads to } F' \lor G'' & \text{ LPD } \\
\end{align*}
\]

\[
\begin{align*}
F \text{ leads to } F' \lor G' & \text{ LTR } \\
\hline
\end{align*}
\]

\[
\begin{align*}
F \text{ leads to } G & \text{ LSW } \\
\hline
G \text{ leads to } F' & \text{ LTR } \\
F \text{ leads to } G & \text{ LSW } \\
\hline
F \text{ leads to } F' \land G' & \text{ LCC }
\end{align*}
\]