UNORIENTED COBORDISM MAPS ON LINK FLOER HOMOLOGY

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Abstract. We study the problem of defining maps on link Floer homology induced by unoriented link cobordisms. We provide a natural notion of link cobordism, disoriented link cobordism, which tracks the motion of index zero and index three critical points. Then we construct a map on unoriented link Floer homology associated to a disoriented link cobordism. Furthermore, we give a comparison with Oszváth-Stipsicz-Szabó’s and Manolescu’s constructions of link cobordism maps for an unoriented band move.

Contents
1. Introduction 3
  1.1. The difference between unoriented cobordism and oriented cobordism 5
  1.2. Organization 6
  1.3. Further developments and possible applications 7
  1.4. Acknowledgement 7
2. Preliminaries 8
  2.1. Multi-pointed links 8
  2.2. Link Floer homology 8
  2.3. Unoriented link Floer homology 9
  2.4. Unoriented grid homology 10
3. Link categories 11
  3.1. Category 1: disoriented links 11
  3.2. Category 2: bipartite links 12
  3.3. Category 3: bipartite disoriented links 15
  3.4. Coloring and the relations between the three link categories 17
4. Heegaard diagrams for band moves 20
  4.1. Heegaard diagrams for bipartite links 20
  4.2. The existence of Heegaard triples subordinate to a band move 20
  4.3. Some topological facts about Heegaard triples subordinate to a band move 25
5. Bipartite link Floer homology 27
  5.1. Bipartite link Floer curved chain complex for null-homologous links 27
  5.2. Spin$^c$-structures 28
  5.3. Admissibility 29
  5.4. The curved chain complex $CFBL^-$ for bipartite links in $\#^g(S^1 \times S^2)$ 29
  5.5. Holomorphic triangles 31
6. Band moves and triangle maps 33
  6.1. Assumptions 33
  6.2. Distinguishing the top grading generators 33
  6.3. The relation of generators 35
| Section | Title | Page |
|---------|-------|------|
| 6.4     | A comparison with Ozsváth, Stipsicz and Szabó’s definition of band move maps | 38 |
| 6.5     | A comparison with Manolescu’s definition of unoriented band move maps | 39 |
| 6.6     | An example: a bipartite link cobordism from trefoil to unknot | 40 |
| 7       | Quasi-stabilizations and Disk-stabilizations | 42 |
| 7.1     | Topological facts about quasi-stabilizations | 42 |
| 7.2     | Choice of generators | 43 |
| 7.3     | Disk-stabilizations | 44 |
| 8       | Commutations | 46 |
| 8.1     | Commutation between $\alpha$-band moves and $\beta$-band moves | 46 |
| 8.2     | Commutation between $\beta$-band moves | 47 |
| 8.3     | Commutation between band moves and quasi-stabilizations | 51 |
| 8.4     | The relation between $\alpha$ and $\beta$-band moves | 57 |
| 8.5     | The relation between $\alpha$ and $\beta$-quasi-stabilizations | 60 |
| 9       | Functoriality | 63 |
| 9.1     | Ambient isotopies of disoriented link cobordism | 63 |
| 9.2     | Moves between regular forms of a disoriented link cobordism | 64 |
| 9.3     | Construction and invariance of disoriented link cobordism maps | 67 |
| References | | 69 |
1. Introduction

Heegaard Floer homology is an invariant of three-manifolds, introduced by Ozsváth and Szabó in [OS04b]. Knot Floer homology is a variation of Heegaard Floer homology, which was discovered by Ozsváth and Szabó [OS04a] and independently by Rasmussen [Ras03]. Later, it was generalized to link Floer homology in [OS08a] and proved to be a powerful and successful tool for studying knots and links in three-manifolds: it detects the Thurston norm of the link complement [OS08b]; it detects the fiberedness of a knot [Ni07]; one can extract a tau-invariant from it, and get a lower bound of four-ball genus of a knot or link [OS03]. There is also a version called unoriented link Floer homology that is independent of the link orientation, see [OSS15] and [OSS14].

While searching for applications of link Floer Homology, a natural question arises: whether an oriented (or unoriented resp.) link cobordism induces a map on link Floer homology (or unoriented link Floer homology resp.).

In [Juh06], Juhász introduced sutured Floer Homology. Later he provided a way to construct cobordism map for sutured manifolds, see [Juh14]. In particular, he built a notion of cobordism, the decorated link cobordism, which contains not only the link cobordism surface but also a family of one-manifolds on the surface. These one-manifolds provide extra data for defining the cobordism map. Juhász and Thurston also proved the naturality of link Floer homology in [JT12]. In [Juh16], Juhász showed that a decorated link cobordism induces a map on sutured Floer homology. In [Zem16b], Zemke generalized the idea in [Sar15] and established a way to study basepoints moving maps by using quasi-stabilization. Later, following Juhász’s framework, Zemke constructed link cobordism maps on link Floer homology and showed the invariance of this construction, see [Zem16a].

Notice that all of the above works are for oriented link cobordisms. For unoriented link cobordisms in [OSS15], Ozsváth, Stipsicz and Szabó construct maps on unoriented grid homology, which is the grid diagram version of unoriented link Floer homology. However, they did not prove the invariance of the maps. Inspired by these works, we will introduce a natural link cobordism notion and construct cobordism maps on unoriented link Floer homology.

The link category we study is made of the disoriented links. The objects of this category, disoriented links, are constructed as follows. Let $L$ be a link (with no assigned orientation) in a closed oriented three-manifold $Y$. Suppose that $p$ and $q$ are two sets of points which appear alternatively on each component of $L$. The set $L \setminus (p \cup q)$ consists of $2n$-arcs $I = \{l_1, \ldots, l_{2n}\}$, which we orient from $q$ to $p$, where $n$ is the number of $p$ points. A disoriented link is the quadruple $(L, p, q, I)$. The definition of disoriented links is inspired by the Morse function compatible with $L$. An example of disoriented link is shown in Figure 1.

Let $H = (\Sigma, \alpha, \beta, O)$ be the pointed Heegaard diagram induced by a Morse function $f$ compatible with $L$, where $O$ is the basepoints set $(\Sigma \cap L)$. As in [OSS15], we can construct a $\delta$-graded, unoriented link Floer chain complex $CFL(H)$ over the ring $\mathbb{F}_2[U]$. The differential $\partial$ acting on a generator $x \in T_\alpha \cap T_\beta$ is given by:

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \#(M(\phi) / \mathbb{R}) U^{n_\phi} y,$$

where $n_\phi$ is equal to the sum $\sum_{o_i \in O} n_{o_i}$. The relative $\delta$-grading between two generators $x$ and $y$ is given by

$$\delta(x, y) = \mu(\phi) - n_\phi.$$


If the link $L$ is **homologically even**, which means $[L] = 2a$, for some $a \in H_1(Y; \mathbb{Z})$, then the $\delta$-grading is a $\mathbb{Z}$-grading. By tracking the proof of the naturality of link Floer homology in [JT12], we know that the homology $HFL'$ of $CFL'$ is an invariant of the disoriented link $L$.

A disoriented link cobordism $\mathcal{W} = (W, F, A)$ from $L_0 = (L_0, p_0, q_0, l_0)$ in $Y_0$ to $L_1 = (L_1, p_1, q_1, l_1)$ in $Y_1$ contains two groups of data:

1. The data of the link cobordism, denoted by $(W, F)$: The manifold $W = (W, \partial W)$ is a cobordism from $Y_0$ to $Y_1$. The surface $F = (F, \partial F)$ is embedded in $(W, \partial W)$ with its boundary $\partial W$ identified with the links determined by $L_0$ in $Y_0$ and $L_1$ in $Y_1$.

2. The motion of the $p$ and $q$ points $A = (A, \partial A)$: This is an oriented one-manifold $A = (A, \partial A)$ properly embedded in $(F, \partial F)$. The boundary $\partial A$ identified with the zero manifold $q_0 - p_0 + p_1 - q_1$.

An example of a disoriented link cobordism is shown in Figure 2. A similar construction also appears in Khovanov homology, for details see Remark 3.7.

Similar to what is in [Zem16a], one can find a parametrized Kirby decomposition of a disoriented link cobordism. However, when defining maps induced by a four-dimensional two-handle attachment, we can not establish the correspondence between the Spin$^c$-structure for the four-manifold and the equivalence class of triangles which come from the Heegaard triple subordinate to the two-handle. For details, see Section 5.5.

To avoid this issue, we only consider surfaces inside $Y \times I$ and use the language of ambient isotopy of surfaces in four-manifolds instead of handle decompositions of the four-manifold. Our main result is the following:

**Theorem 1.1.** Suppose that the cobordism $W$ is a product $(Y \times I, \partial(Y \times I))$. Let $\mathcal{W} = (W, F, A)$ be a disoriented link cobordism from $(Y, L^0)$ to $(Y, L^1)$, such that the embedding $F$ induces a trivial map $F_* : H_2(F, \partial F) \to H_2(Y \times I, \partial(Y \times I))$. Then for a torsion Spin$^c$-structure $s$ of $Y \times I$, we can define a map:

$$F_{\mathcal{W}*} : HFL'(Y, L^0, s|_{Y \times \{0\}}) \to HFL'(Y, L^1, s|_{Y \times \{1\}}),$$

which is an invariant of $(\mathcal{W}, s)$. Furthermore, the map $F_{\mathcal{W}*}$ satisfies the composition law, i.e. if $\mathcal{W} = \mathcal{W}^1 \cup \mathcal{W}^2$, where the disoriented link cobordisms $\mathcal{W}^1$ and $\mathcal{W}^2$ satisfy the same conditions as $\mathcal{W}$, then

$$F_{\mathcal{W}^2*, s|_{\mathcal{W}^2}} \circ F_{\mathcal{W}^1*, s|_{\mathcal{W}^1}} = F_{\mathcal{W}*, s}.$$
Figure 2. A disoriented link cobordism from $\mathcal{L}^0$ to $\mathcal{L}^1$. The motion of $p$ and $q$ is marked as blue curves.

Note that a similar statement appears in\cite[Theorem A,B]{Zem16a} for arbitrary oriented link cobordisms (not only for cylinders $Y \times I$).

In order to construct the cobordism map $F_{W}$, we introduce another group of data (D3) on the surface $(F, \partial F)$, which allows us to extract Heegaard data. In detail, the data (D3) tracks the motion of basepoints: this is a one-manifold $A_\Sigma = (A_\Sigma, \partial A_\Sigma)$ embedded in $(F, \partial F)$. The boundary $\partial A_\Sigma$ are basepoints $O^0$ in $Y^0$ and $O^1$ in $Y^1$. Furthermore the one-manifold $A_\Sigma$ cut the surface $F$ into two parts, $F_\alpha$ and $F_\beta$, each of which is a collection of surfaces (can be non-orientable) embeded in $Y$.

The workflow of our construction is shown in Figure 3.

**Step 1:** we lift the disoriented link cobordism (containing data (D1) and (D2)) to a bipartite disoriented link cobordism (containing data (D1),(D2) and (D3)). For the definition of bipartite disoriented link cobordism see Section 3.3. This lifting is not unique, see Section 3.4.

**Step 2:** we construct cobordism maps for the bipartite disoriented link cobordism. In fact, we extract Heegaard data from (D1)+(D3). The groups of data (D1)+(D2) help us choose generators when defining maps induced by band moves and quasi-stabilizations.

**Step 3:** we show that the cobordism maps defined in Step 2 are independent of liftings (or the data (D3) in other words), at the level of $HFL'$.

1.1. **The difference between unoriented cobordism and oriented cobordism.** For an oriented band move, the number of link components will be changed. On the other hand, there is at least one pair of basepoints for each link component. Therefore, we deduce that, for a pointed Heegaard triple subordinate to an oriented band move, we need at least four basepoints on the Heegaard triple.

However, when the cobordism surface $(F, \partial F)$ is non-orientable, a band move may not change the number of link components. Furthermore, there exists a two-pointed Heegaard
Figure 3. Workflow of the construction: DLC (disoriented link cobordism), BLC (bipartite link cobordism), BDLC (bipartite disoriented link cobordism); the data (D1) is the link cobordism surface; the data (D2) is the motion of index zero/three critical points; the data (D3) is the motion of basepoints.

Figure 4. Band move from trefoil to unknot.

triple subordinate to an unoriented band move between two knots. One example of such a cobordism is given below.

We consider a band move from the trefoil to the unknot shown in Figure 4. This is an example of a band move of type I (see section 3.2 for the definition of type I and type II band move). For the Heegaard triple subordinate to a type I band move, the induced diagram $H_{\beta\gamma}$ no longer represents an unlink, but it is still a homologically even link in $\#^n(S^1 \times S^2)$. One can compare this fact for type I band moves with the fact about oriented band moves in [Zem16a, Lemma 6.6]. To deal with the link represented by $H_{\beta\gamma}$, it is necessary to build the unoriented link Floer homology theory for homologically even links.

1.2. Organization. In Section 2, we recall some notions of link Floer homology. In Section 3, we introduce three link categories: disoriented links, bipartite links and bipartite disoriented links. We will also discuss the relation between the three categories. In Section 4, we will construct the Heegaard triple subordinate to a band move of bipartite links. In Section 5, we will construct the bipartite link Floer curved chain complex. We focus our discussions on Spin$^c$-structure, admissibility and associativity. In Section 6, we will construct the cobordism map for band moves of bipartite disoriented links on unoriented link Floer chain complex. Particularly, we will compare our construction with the band move maps defined by Ozsváth, Stipsicz and Szabó in [OSS15] and the version of cobordism maps defined by Manolescu in [MOST07]. In Section 7, we define the
bipartite link cobordism maps induced by quasi-stabilizations/destabilizations and disk-stabilizations/destabilizations. In Section 8, we will prove the commutation between certain cobordism maps. We also provide relations between the cobordism maps induced by band moves (and quasi-stabilization/destabilizations resp.). In Section 9, we will prove Theorem 1.1.

1.3. **Further developments and possible applications.** As an application of the disoriented link cobordism theory, we will extend the involutive upsilon invariant defined by Hogancamp and Livingston [HL17] from knots to links. Furthermore, we will study the relation between involutive upsilon invariant and the unoriented four-ball genus for disoriented link cobordism. These discussions will appear in an upcoming paper [Fan18].

Another question one can think about is the following:

**Question 1.2.** Suppose we have a disoriented link cobordism $\mathfrak{W} = (W, F, A)$ from $(Y^0, L^0)$ to $(Y^1, L^1)$. Here we no longer require the four-manifold $W$ is cylindrical. Given a torsion $\text{Spin}^c$-structure $s$, can we still define a map $F_{\mathfrak{W}, s}: \text{HFL}'(Y^0, L^0, s|_{Y^0}) \to \text{HFL}'(Y^1, L^1, s|_{Y^1})$? If so, can we get a $\delta$-grading shifts formula for the map $F_{\mathfrak{W}, s}$?

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2. Preliminaries

2.1. Multi-pointed links. In this subsection, we recall the Heegaard diagrams for oriented multi-pointed links. For details, see [OS08a].

Definition 2.1. An oriented multi-pointed link is a triple \((Y, L, w, z)\), such that:

- \(Y\) is a closed three-manifold, \(L\) is an oriented link in \(Y\).
- The set \(w = \{w_1, \cdots, w_n\}\), \(z = \{z_1, \cdots, z_n\}\) are collection of basepoints on \(L\).
- On each component \(L_i\) of \(L\), there are basepoints \(w_{ij}, z_{ij}\) appear alternatively on \(L_i\), where \(j = 1, \cdots, n_i\).

We can construct a self-indexed Morse function \(f\) compatible with the triple \((Y, L, w, z)\), such that, the link \(L\) is a union of trajectories connecting index three and zero critical point of \(f\). A trajectory on \(L\) intersect with the surface \(\Sigma = f^{-1}(\frac{3}{2})\) at a point \(a\). If the direction of \(L\) agrees with the direction of this trajectory, we mark the intersection \(w\), and denote this trajectory a \(w\)-arc. Otherwise, we mark the intersection \(z\), and denote this trajectory a \(z\)-arc.

We say that a Heegaard diagram \(H = (\Sigma, \alpha, \beta, w, z)\) is compatible with a multi-pointed link \((Y, L, w, z)\), if it comes from a self-indexed Morse function \(f\) compatible with this multi-pointed link \((Y, L, w, z)\). In detail, this Heegaard diagram \(H\) should satisfy:

- The surface \(\Sigma\) is identified with \(f^{-1}(\frac{3}{2})\).
- The \(\alpha\)-curves on \(\Sigma\) are the intersections of \(\Sigma\) with the stable manifolds of all index one critical points of \(f\).
- The \(\beta\)-curves on \(\Sigma\) are the intersections of \(\Sigma\) and the unstable manifolds of all index two critical points of \(f\).
- The link \(L\) as a union of trajectories of \(f\) intersects with \(\Sigma\) at a \(w\) basepoint if this trajectory has the same direction with \(L\), at a \(z\) basepoint if it has the opposite direction.

Therefore, we implant the oriented link data \((L, w, z)\) into a 2n-pointed Heegaard diagram. Such a diagram \(H\) is a surface \(\Sigma\) of genus \(g\) equipped with two families of pairwise disjoint simple closed curves, \(\{\alpha_1, \cdots, \alpha_{g+n-1}\}\) and \(\{\beta_1, \cdots, \beta_{g+n-1}\}\) and two sets of basepoints \(w = \{w_1, \cdots, w_n\}\) and \(z = \{z_1, \cdots, z_n\}\), satisfying:

- The vector spaces \(\text{Span}(\alpha_1, \cdots, \alpha_{g+n-1})\) and \(\text{Span}(\beta_1, \cdots, \beta_{g+n-1})\) in \(H_1(\Sigma)\) is \(g\)-dimensional.
- The space \(\Sigma\setminus\alpha\) has \(n\) connected sets \(A_1, \cdots, A_n\). Similarly, the space \(\Sigma\setminus\beta\) also has \(n\) connected sets \(B_1, \cdots, B_n\).
- Each component \(A_i\) (or \(B_j\) resp.) contains exactly one pair of basepoints \((w, z)\) or \((z, w)\) respectively in \((\{w\}, \{z\})\) (or in \((\{z\}, \{w\})\) resp.).

2.2. Link Floer homology. In this section we recall the construction of various versions of the link Floer chain complex \(\text{CFL}^g\) from a 2n-pointed Heegaard diagram. For details, see [OS08a], [MO10], [Zem16b].

Consider a 2n-pointed Heegaard diagram \(H\) representing the oriented link \((L, w, z)\). Suppose \(L\) is null-homologous in \(Y\). The generators of \(\text{CFL}^g\) are the intersections of the two Lagrangian submanifolds \(T_\alpha\) and \(T_\beta\) inside \(\text{Sym}^{2g+n-1}(\Sigma)\). We view a generator as a \((g + n - 1)\)-tuple of points \(x = (x_1, \cdots, x_{g+n-1})\), where \(x_i\)'s are the intersections between \(\alpha\) and \(\beta\) curves on \(\Sigma\).

For each of the generator \(x \in T_\alpha \cap T_\beta\), we can construct a Spin^c-structure \(s_w(x)\) by picking a non-vanishing vector field on the complement of \(Y\) by removing neighborhoods
of flow lines through basepoints $w$ and the generator $x$. Similarly, we can construct Spin$^c$-structure map $s_x$ from $T_\alpha \cap T_\beta$ to Spin$^c(Y)$. Furthermore, the difference $s_w(x) - s_x(x)$ is equal to the Poincaré dual $PD[L]$ of the class $[L] \in H_1(Y)$. As the link $L$ we consider is null-homologous, there is no difference between the two Spin$^c$-structure $s_w(x)$ and $s_x(x)$.

(For details of the construction, see [OS04b, Section 2.6])

Suppose $s$ is an torsion Spin$^c$-structure of $Y$. We can associate a Maslov $\mathbb{Z}$-grading $gr_w$ for all generators $x$ with $s_w(x) = s$. As $L$ is null-homologous, we also have a Maslov $\mathbb{Z}$-grading $gr_z$ for generators in class $s$. The difference $gr_w(x) - gr_x(x)$ is equal to twice of the Alexander grading $A(x)$.

A **Heegaard Data** $\mathcal{H}$ is a pair $(H, J_1)$, where $J_1$ is a generic one parameter family of almost complex structure on $\text{Sym}^{g+n-1}(\Sigma)$. We assign to each basepoint $u_i$ a variable $U_i$ and to each basepoint $z_j$ a variable $V_j$. Given a strongly $s$-admissible Heegaard data for the link $(Y, L, w, z)$, we can define the a free $\mathbb{R} [U_1, \ldots, U_n, V_1, \ldots, V_n]$-module $CFL^-(\mathcal{H}, s)$ with the generators $x \in T_\alpha \cap T_\beta$ with $s_w(x)$ equal the given Spin$^c$-structure $s$. The module $CFL^\infty(\mathcal{H}, s)$, which contains elements of the form $\prod_{i,j} U_i^{s_w_i}(x) V_j^{s_z_j}(x)$, is the localization of $CFL^-(\mathcal{H}, s)$. We denote by $CFL^+(\mathcal{H}, s)$ the quotient module $CFL^\infty(\mathcal{H}, s)/CFL^-(\mathcal{H}, s)$.

There is an endomorphism

$$\partial : CFL^\circ(\mathcal{H}, s) \to CFL^\circ(\mathcal{H}, s),$$

which makes $CFL^\circ(Y, L, s)$ into a curved chain complex (see [Zem16b] and [Zem16a]). In detail, the endomorphism acts on a generator $x$ is given by:

$$(2.2) \quad \partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \#(\mathcal{M}(\phi)/\mathbb{R}) \prod_{i,j} U_i^{s_w_i}(x) V_j^{s_z_j}(x) y$$

From [Zem16b Lemma 2.1], we have

$$(2.3) \quad \partial^2 = \sum_i (U_{i,1} V_{i,2} + V_{i,2} U_{i,1} + \cdots + U_{i,n} V_{i,1} + U_{i,1} V_{i,1}).$$

Here $i$ refers the $i$-th component $L_i$ of $L$. We assign to each basepoint $w_{i,j}$ a variable $U_{i,j}$ and to each basepoint $z_{i,j}$ a variable $V_{i,j}$. The basepoints $w_{1,1}, z_{1,2}, \ldots, w_{i,n_i}, z_{i,1}$ appears clockwise on $L_i$.

**Remark 2.4.** Usually, the chain complex $CFL^-(\mathcal{H}, s)$ refers to the complex defined by setting all $V_j = 1$. When we say a curved chain complex $CFL^-(\mathcal{H}, s)$, we mean the module together with the endomorphism defined in (2.2).

### 2.3. Unoriented link Floer homology.

The unoriented link Floer chain complex $CFL'$ was first introduced in [OSS15] as a special case of the $t$-modified link Floer chain complex $tCFK$ by setting $t = 1$.

We let the complex $CFL'(\mathcal{H}, s)$ be the tensor product of the curved chain complex $CFL^-(\mathcal{H}, s)$ with the quotient ring $R = \mathbb{R}[U_1, \ldots, U_n, V_1, \ldots, V_n, U]/I$, where the ideal $I$ is generated by all $U_i - U$ and $V_j - U$. The endomorphism

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \#(\mathcal{M}(\phi)/\mathbb{R}) U^{n_w(x) + n_z(x)} y,$$

becomes a differential. Here $n_w = \sum_i n_{w_i}, n_z = \sum_i n_{z_i}$. 

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**UNORIENTED COBORDISM MAPS ON LINK FLOER HOMOLOGY 9**
As before, we assume the link $L$ is null-homologous and the Spin$^c$-structure $s$ is torsion. Then the two $\mathbb{Z}$-grading $gr_w(x)$ and $gr_z(x)$ are well-defined. The $\delta$-grading of the generator $x$ in $CFL'(\mathcal{H}, s)$ is defined to be:

$$\delta(x) = \frac{1}{2}(gr_w(x) + gr_z(x)).$$

We call the $\delta$-graded chain complex $CFL'(\mathcal{H}, s)$ the \textbf{unoriented chain complex}. The homology group $H_*(CFL'(\mathcal{H}, s))$ is called the \textbf{unoriented link Floer homology}. As we assign all basepoints $w$ and $z$ the same variable $U$, we lose the information of the orientation of $L$. Hence $H_*(CFL'(\mathcal{H}, s))$ is an invariant for unoriented link.

2.4. \textbf{Unoriented grid homology}. Grid diagrams provide us with a way to describe the Floer homology of oriented links in $S^3$ combinatorially, see [Man07], [MOS09]. In this subsection, we briefly talk about how to calculate the unoriented link Floer homology from a grid diagram. For details, see [OSS15].

Suppose $G$ is a special $2n$-pointed toroidal Heegaard diagram such that each connected component of $\Sigma - \alpha$ or $\Sigma - \beta$ is an annulus. Conventionally, we denote the $w$-basepoint as $X$ and $z$-basepoint as $O$ in the grid diagram.

Given such a grid diagram $G$, we can define grid chain complex $GC^0(G)$ and its grid homology $H_*(GC^0(G))$. A generator of grid chain complex $G$ is an $n$-tuple $x = \{x_1, \cdots, x_n\}$, such that each $\alpha$ and $\beta$-curves contain exactly one of the $x_i$’s. We call these generators \textbf{grid states} and denote the set of all grid states $S(G)$.

Let $\text{Rect}(x, y)$ be the set of rectangles from $x$ to $y$ and the weight $W(r)$ be equal to $\#(r \cap (X \cup O))$. The unoriented grid chain complex $GC'(G)$ is a $\delta$-graded $\mathbb{F}_2[U]$-module freely generated by grid states $x$, with differential

$$\partial x = \sum_{y \in S(G)} \sum_{r \in \text{Rect}(x, y)} U^{W(r)} y.$$

Given a grid diagram $G$, the delta grading of a generator $x$ can be calculated as follows. Let $P, Q$ be two finite subset of $\mathbb{R}^2$. The function $I(P, Q)$ count the number of pairs $p \in P$ and $q \in Q$, such that the vector $q - p$ lie in the first quadrant. We define a symmetric function:

$$J(P, Q) = \frac{I(P, Q) + I(Q, P)}{2}.$$

Then the $\delta$-grading is given by the formula:

$$\delta(x) = \frac{1}{2}(J(x - \emptyset, x - \emptyset) + J(x - X, x - X)) + \frac{n - l}{2} + 1.$$

We know that if $\mathcal{H}$ is a Heegaard diagram induced from a grid diagram $G$, then the chain complex $(CFK^-(\mathcal{H}, \partial \mathcal{H}^-))$ is isomorphic to $(GC^-(G), \partial_G^-)$. In fact, there is an identification of the bigrading in grid homology and link Floer homology. Similar result holds for unoriented chain complex, i.e. there is an isomorphism between the $\delta$-graded chain complexes $CFL'(\mathcal{H}, \partial)$ and $GC^-(G, \partial_G^-)$. 

3. Link categories

In this section we introduce three link categories: disoriented links, bipartite links, and bipartite disoriented links.

3.1. Category 1: disoriented links. The idea of this category comes from the Morse theory for links. In a disoriented link cobordism, we keep track of the motion of the index zero and index three critical points of the disoriented links.

**Definition 3.1.** A disoriented link is a link \( L \) in a closed oriented three-manifold \( Y \), together with two sets of points \( p = \{p_1, \ldots, p_n\} \) and \( q = \{q_1, \ldots, q_n\} \) on \( L \) such that \( p_i \) and \( q_j \) appear alternatively on each component of \( L \). These points cut the link \( L \) into \( 2n \)-arcs \( I = \{l_1, \ldots, l_{2n}\} \), which we orient from \( q \) to \( p \) such that:

\[
\partial I = \partial l_1 + \cdots + \partial l_{2n} = 2(p_1 + \cdots + p_n) - 2(q_1 + \cdots + q_n).
\]

We denote a disoriented link by \( \mathcal{L} = (L, p, q, I) \). We call the points \( p \) and \( q \) the **dividing set** of the disoriented link \( \mathcal{L} \). See Figure 1 for an example of disoriented link.

**Remark 3.2.** The idea of disoriented link comes from the construction of a Morse function \( f \) compatible with a given oriented link \((Y, L)\). We think of \( L \) as a union of trajectories \( I \) and forget the \( w \) markings and \( z \) markings (hence forget the orientation) of \( L \). The points \( p \) play the role of index zero critical points of \( f \). The points \( q \) play the role of index three critical points of \( f \).

**Definition 3.3.** A **surface with divides** is an embedding:

\[
\mathcal{A} : (A, \partial A) \hookrightarrow (F, \partial F)
\]

such that,

- The pair \((A, \partial A)\) is a compact, oriented one-manifold.
- The pair \((F, \partial F)\) is a compact surface and does not need to be orientable.
- The components of \( F \setminus A = \{F_1, \ldots, F_k\} \) are compact oriented surfaces with orientation induced from the one-manifold \((A, \partial A)\).

**Remark 3.4.** Our definition of surface with divides is a generalization of that is in [Juh16]. In [Juh16], for an oriented link cobordism, the orientation of the surface \( F \) induces the orientation on each piece \( F_i \). For \( F_i \) with \( w \) basepoints, the orientation agrees with the orientation induced by the oriented one manifold \( A \). For \( F_i \) with \( z \) basepoints, the orientation agrees with the opposite of the orientation induced by the oriented one manifold \( A \).

**Definition 3.5.** Suppose we have a disoriented link \( \mathcal{L}^0 = (L^0, p^0, q^0, I^0) \) in a closed oriented three-manifold \( Y^0 \), and a disoriented link \( \mathcal{L}^1 = (L^1, p^1, q^1, I^1) \) in a closed oriented three-manifold \( Y^1 \). A disoriented link cobordism from the disoriented link \((Y^0, \mathcal{L}^0)\) to \((Y^1, \mathcal{L}^1)\) is a triple \( \mathcal{W} = (W, \mathcal{F}, \mathcal{A}) \) such that:

- The pair \( W = (W, \partial W) \) is an oriented cobordism from \( Y^0 \) to \( Y^1 \).
- The map \( \mathcal{F} : (F, \partial F) \to (W, \partial W) \) is a smooth embedding of surface \((F, \partial F)\).
- The embedding \( \mathcal{A} : (A, \partial A) \to (F, \partial F) \) is a surface with divides.
- The boundary \( \partial A \) is the union of points \( q^0 - p^0 + p^1 - q^1 \). Furthermore, the intersection \((\partial A \cap Y^0)\) is \( q^0 - p^0 \) and the intersection \((\partial A \cap Y^1)\) is \( p^1 - q^1 \).
- The boundary \( \partial (F \setminus A) = \partial F_1 + \cdots + \partial F_n \) is \(-I^0 + I^1 + 2A\). Furthermore, the intersection \( \partial (F \setminus A) \cap -Y^0 \) is \(-I^0\) and \( \partial (F \setminus A) \cap Y^1 \) is \( I^1 \).


Remark 3.6. The orientation for the components of $F \setminus A$ is unique and determined by the disoriented link $L^0$ and $L^1$. If $F$ is orientable, then a disoriented link cobordism is equivalent to the decorated link cobordism in [Juh09] and [Zem16a].

Remark 3.7. In [CMW09], Clark, Morrison and Walker introduced disorientation in the link cobordism to make Khovanov homology functional with respect to link cobordisms. The disoriented links in our definition correspond to a special case of the ‘disoriented circle’ with the ‘disorientation number’ equal to zero, see [CMW09, Lemma 4.4].

Example 3.8. In Figure 2, we show a disoriented link cobordism between two disoriented link $L^0$ and $L^1$. The dividing set $A$ (bold blue line) cuts the surface $F$, which is non-orientable in this example, into three components $F_1, F_2, F_3$. Each surface $F_i$ is an oriented surface with orientation compatible with the orientation of $A$ (marked as the blue arrow). In the three-manifold $Y^1$, the orientation of $F_i$ agrees with the orientation of the oriented arcs $l^1$ in $Y^1$. In the three-manifold $Y^0$, the orientation of $F_i$ is the opposite of the orientation of $l^0$ in $Y^0$.

Definition 3.9. Suppose the four manifold $W$ is a product $Y \times I$. We call a disoriented link cobordism $(W, F, A)$ from $(Y, L^0)$ to $(Y, L^1)$ regular, if there exists a projection map $\pi : Y \times I \to I$ such that:

- The map $\pi|_F$ and $\pi|_A$ is a Morse function.
- If $a$ is a regular value for both $\pi|_F$ and $\pi|_A$, then the triple $(W, F, A) \cap \pi^{-1}([0, a])$ is a disoriented link cobordism from the disoriented link $(Y, L^0)$ to a disoriented link $(Y, L^a)$.
- The index one critical points of $\pi|_F$ do not lie on $A$.
- There is a sequence $\{a_1, \cdots, a_m\}$ of regular values for both $\pi|_F$ and $\pi|_A$ such that, there is only one critical points of $\pi|_A$ or index one critical point of $\pi|_F$ with its value in $(a_i, a_{i+1})$.

Remark 3.10. The condition (2) in Definition 3.9 guarantees the index two/zero critical points of $\pi|_F$ is included in the index one/zero critical points of $\pi|_A$.

We can decompose a regular cobordism into a composition of four types of elementary disoriented link cobordisms:

1. Isotopy of disoriented links.
2. Band move (saddle move) of disoriented link.
3. Disk-stabilization/destabilization of disoriented link.
4. Quasi-stabilization/destabilization of disoriented link.

Remark 3.11. By definition, it is easy to see:

1. An isotopy contains no critical points of $\pi|_F$ or $\pi|_A$.
2. A band move contains an index one critical point of $\pi|_F$.
3. A disk stabilization/destabilization contains an index two/zero critical point of $\pi|_F$.
4. A quasi-stabilization/destabilization contains an index one/zero critical points of $\pi|_A$.

3.2. Category 2: bipartite links. In a bipartite link cobordism, we keep track of the motion of the basepoints of the links.
Definition 3.12. A bipartite link is a link $L$ in a closed oriented three-manifold $Y$, together with $2n$-basepoints $O = \{o_1, \ldots, o_{2n}\}$ and two $n$-tuples of disjoint embedded arcs $L_\alpha = \{L_{\alpha,1}, \ldots, L_{\alpha,n}\}$ and $L_\beta = \{L_{\beta,1}, \ldots, L_{\beta,n}\}$ on $L$, such that:

- The ends $\partial L_\alpha = \partial (L_{\alpha,1} \cup \cdots \cup L_{\alpha,n})$ are identified with the ends $\partial L_\beta = \partial (L_{\beta,1} \cup \cdots \cup L_{\beta,n})$. Furthermore, the ends $\partial L_\alpha = \partial L_\beta$ are exactly the basepoints $O$ on $L$.
- The union of the two $n$-tuples of arcs $L_\alpha \cup L_\beta$ is the link $L$.

We denote a bipartite link by $L_{\alpha\beta} = (L, L_\alpha, L_\beta, O)$. See Figure 6 for an example of bipartite link.

Remark 3.13. Let $U_\alpha \cup_\Sigma U_\beta$ be a Heegaard splitting of a three-manifold $Y$. Here $U_\alpha$ and $U_\beta$ are two handlebodies, $\Sigma$ is a Heegaard surface. Suppose $L$ is a link in $Y$ and intersects $\Sigma$ transversely. Moreover, suppose the intersections of $L$ and $U_\alpha$ (or $U_\beta$ resp.) bound compressing disks to $\Sigma$. The $2n$-basepoints $O$ play the role of the intersections $L \cap \Sigma$. The $n$-tuple of disjoint embedded arcs $L_\alpha$ (or $L_\beta$ resp.) play the role of the intersection $U_\alpha \cap L$ (or $U_\beta \cap L$ resp.). We do not color the basepoints into $w$’s and $z$’s.

Definition 3.14. A bipartite link cobordism from a bipartite link $L_{\alpha\beta}^0$ in $Y^0$ to a bipartite link $L_{\alpha\beta}^1$ in $Y^1$ is a quintuple $(W, \mathcal{F}, F_\alpha, F_\beta, A_\Sigma)$, such that:

- The manifold $W = (W, \partial W)$ is an oriented cobordism from three-manifold $Y^0$ to $Y^1$. 

The map $\mathcal{F} : (F, \partial F) \to (W, \partial W)$ is an embedding of a compact surface $(F, \partial F)$ in $(W, \partial W)$.

The map $\mathcal{A}_\Sigma : (A_\Sigma, \partial A_\Sigma) \to (F, \partial F)$ is an embedding of a one-manifold $(A_\Sigma, \partial A_\Sigma)$ in $(F, \partial F)$.

The surface $F$ is decomposed along $A_\Sigma$ into two compact surfaces $F_\alpha$ and $F_\beta$. One side of $A_\Sigma$ on $F$ is belong to the interior of $F_\alpha$, the other side is belong to the interior of $F_\beta$.

The bipartite link $(Y^i, L^i_\alpha, L^i_\beta, O^i)$ is identified with $(Y^i, Y^i \cap \partial F_\alpha, Y^i \cap \partial F_\beta, Y^i \cap A_\Sigma)$, where $i = 0, 1$.

**Example 3.15.** Figure 7 shows a bipartite link cobordism from a bipartite link $L^0_{\alpha\beta}$ to a bipartite link $L^1_{\alpha\beta}$. The red curves forming $A_\Sigma$ cut the surface $F$ into four components. Two of the components are belong to $F_\alpha$, the other two are belong to $F_\beta$. One of the component of $F_\alpha$ is non-orientable.

We say that a bipartite link cobordism $(W, \mathcal{F}, F_\alpha, F_\beta, A_\Sigma)$ is **regular**, if it satisfies the same condition as in Definition 3.9 with $A_\Sigma$ plays the role of $A$ and bipartite links $L^i_{\alpha\beta}$.
play the role of disoriented links $L^i$. Similarly, we can still classify the elementary bipartite link cobordisms into four types: isotopies, band moves, disk-stabilizations/destabilizations, quasi-stabilizations/destabilizations.

Furthermore, we say that a critical point $p$ of $\pi|_{A\Sigma}$ or a saddle point of $\pi|_F$ is of $\alpha$-type if $(N_p \setminus p) \cap \pi|_\alpha^{-1}(\pi(p)) \subset F_\alpha$, otherwise; we say that the critical point $p$ is of $\beta$-type. Here $N_p$ is a small neighborhood of $p$ in $W$.

Suppose $\pi^{-1}[-a,a]$ contains only a saddle critical point $p$. Without loss of generality, suppose $p$ lies in a component $F^i_\alpha$ of $F_\alpha$. We call $p$ of Type I, if $\chi(F^i_\alpha \cap \pi^{-1}[c-\epsilon,c+\epsilon]) = 0$. If $\chi(F^i_\alpha \cap \pi^{-1}[c-\epsilon,c+\epsilon]) = 1$, we call $p$ of Type II, as shown in Figure 8.

Remark 3.16. Here, the surface $F_\alpha$ and $F_\beta$ may have some orientable components and non-orientable components. Consequently, the one-manifold $A\Sigma$ has no canonical orientation. Notice that the bipartite link cobordism is different from the decorated link cobordism [Juh09].

3.3. Category 3: bipartite disoriented links. Bipartite disoriented links combine the data from bipartite links and disoriented links. In a bipartite disoriented link cobordism, we keep track of the motion of index zero/three critical points and the basepoints.

Definition 3.17. A bipartite disoriented link $(L, O)$ is a disoriented link $L = (L, p, q, l)$ together with a set of basepoints $O$, consisting of a unique basepoint $o_i$ on the interior of each oriented arc $l_i \in 1$. See Figure 9 for an example of bipartite disoriented link.

Remark 3.18. A bipartite disoriented link $(L, O)$ determines a bipartite link as follows. The basepoints $O = \{o_1, \cdots, o_{2n}\}$ cut the link $L$ into $2n$-arcs. Let $L_\alpha$ be the collection of arcs which contain $p$-points and $L_\beta$ be the collection of arcs which contain $q$-points. Then $L_{\alpha\beta} = (L, L_\alpha, L_\beta, O)$ is a bipartite link.

Definition 3.19. A bipartite disoriented link cobordism from bipartite disoriented link $(L^0, O^0)$ in $Y^0$ to $(L^1, O^1)$ in $Y^1$ is a sextuple $\mathcal{W} = (\mathcal{W}, F, F_\alpha, F_\beta, A, A\Sigma)$ such that:

- The triple $(\mathcal{W}, F, A)$ is a disoriented link cobordism from the disoriented link $(Y^0, L^0)$ to $(Y^1, L^1)$.
The quintuple \((W, F, F_\alpha, F_\beta, A_\Sigma)\) is a bipartite link cobordism from bipartite link \((Y^0, L^0_{\alpha \beta})\) to \((Y^1, L^1_{\alpha \beta})\). Here \(L^i_{\alpha \beta}\) is the bipartite link determined by the bipartite disoriented link \((L^i, O^i), i = 0, 1\).

- The intersection \(F_\alpha \cap Y_0\) is a union of arcs containing all the \(p\) points.
- The intersection \(F_\beta \cap Y_0\) is a union of arcs containing all the \(q\) points.

Furthermore, if \(W = Y \times I\), we call a bipartite disoriented link cobordism \(W = (W, F, F_\alpha, F_\beta, A, A_\Sigma)\) regular, if it satisfies the following condition:

- The triple \((W, F, A)\) and the quintuple \((W, F, F_\alpha, F_\beta, A_\Sigma)\) are regular.
- The one-manifold \(A\) intersect \(A_\Sigma\) transversely. The intersection points \(A \cap A_\Sigma\) are exactly the critical points of \(\pi|_A\).

**Example 3.20.** Figure 10 shows an example of a bipartite disoriented link cobordism \((W, F, F_\alpha, F_\beta, A, A_\Sigma)\). The blue curves with arrows are the components of oriented one-manifold \(A\). The red curves without orientation are the components of \(A_\Sigma\). Clearly, the triple \((W, F, A)\) is the disoriented link cobordism shown in Figure 2. The quintuple \((W, F, F_\alpha, F_\beta, A_\Sigma)\) is the bipartite link cobordism shown in Figure 7.

If a decorated disoriented link cobordism is regular, we can decompose it into four types of elementary cobordism as regular disoriented link cobordisms. Suppose \((W, F, F_\alpha, F_\beta, A, A_\Sigma)\) is elementary. Furthermore, as we have defined the \(\alpha\)-type and \(\beta\)-type of the critical points of \(\pi|_A\) or saddle points of \(\pi|_F\), we have the following:

- If the elementary cobordism contains a saddle point \(p\) of \(\pi|_F\) in \(F_\alpha\), we call this elementary cobordism an \(\alpha\)-band move (or \(\alpha\)-saddle move), otherwise we call it a \(\beta\)-band move (or \(\beta\)-saddle move). See Figure 11.
- If the elementary cobordism contains a critical point \(p\) of \(\pi|_F\) and satisfies \((N_p \setminus p) \cap \pi_F^{-1}(\pi(p)) \subset F_\alpha\), we call it an \(\alpha\)-quasi-stabilization/destabilization. Here \(N_p\) is a small neighborhood of \(p\) in \(W\). If the critical point \(p\) of \(\pi|_{A_\Sigma}\) satisfying \((N_p \setminus p) \cap \pi_F^{-1}(\pi(p)) \subset F_\beta\), we call it a \(\beta\)-quasi-stabilization/destabilization. See Figure 12.

**Remark 3.21.** We can also classify band moves of bipartite disoriented link into Type I or Type II.
Figure 10. Bipartite disoriented link cobordism.

Figure 11. Two types of band move.

3.4. Coloring and the relations between the three link categories.

Definition 3.22. Let \( L_{\alpha\beta} \) be a bipartite link. A coloring of basepoints is a map

\[
\mathfrak{P} : \mathcal{O} = \{o_1, \ldots, o_n\} \rightarrow \{\pm 1\},
\]

such that the cardinality \(|\mathfrak{P}^{-1}(+1)|\) is equal to \(|\mathfrak{P}^{-1}(-1)|\). Furthermore, we say that a coloring is alternating, if and only if, for any pairs of adjacent basepoints \((o, o')\), the coloring \(\mathfrak{P}(o)\) is equal to \(-\mathfrak{P}(o')\).

Let \( w = \{w_1, \ldots, w_n\} \) be the set \(\mathfrak{P}^{-1}(+1)\), \( z = \{z_1, \ldots, z_n\} \) be the set \(\mathfrak{P}^{-1}(-1)\). We denote a bipartite link together with a coloring \(\mathfrak{P}\) by \((L_{\alpha\beta}, \mathfrak{P})\).

Remark 3.23. Given a bipartite link \( L_{\alpha\beta} \), there exists \(2^{|L|}\) different alternating colorings, where \(|L|\) is the number of link components of \(L\). Given a bipartite link together with a
alternating coloring, we can orient the arcs $L_\alpha$ from $z$ to $w$, and the arcs $L_\beta$ from $w$ to $z$. This assignment gives rise to a oriented link $L_\mathcal{P}$.

For convenience, we denote by $\mathcal{C}_{DL}$ the category of disoriented links, by $\mathcal{C}_{BL}$ the category of bipartite links, and by $\mathcal{C}_{BDL}$ the category of bipartite disoriented links. The relations of the three categories are shown below.

Let $\mathcal{W}$ be a bipartite disoriented link cobordism from bipartite disoriented link $L_0$ to $L_1$.

- The forgetful functor $F_B$ removes the basepoints on $L_i$ and the one-manifold $A_\Sigma$ on $W$.
- The forgetful functor $F_D$ removes the $p$ and $q$ points on $L_i$ and the oriented one-manifold $A$ on $W$. By definition 3.19 $F_D$ send $\mathcal{W}$ to a bipartite link cobordism $(W, F, F_\alpha, F_\beta, A_\Sigma)$ from $L_{\alpha\beta}^0 = F_D(\mathbb{L}^0)$ to $L_{\alpha\beta}^1 = F_D(\mathbb{L}^1)$.

The dotted arrow $G_B$ and $G_D$ are not functors, but represent the processes of lifting objects and morphisms between the respective category. The processes depend on some choices, as detailed below.

For a disoriented link, $G_B$ add one basepoint to each of the oriented arcs $i = i_1, \cdots, i_{2n}$. To lift a disoriented link cobordism $\mathcal{W}$, we perturb $\mathcal{W}$ to regular positoin and decompose $\mathcal{W}$ into a composition of elementary cobordisms. Among the four types of elementary cobordisms, we are particularly interested in quasi-stabilizations and band moves. For isotopies and disk-stabilizations/destabilizations, the lifting is unique. A band move or quasi-stabilization/destabilization of disoriented links can be lifted to a bipartite disoriented link cobordism of either $\alpha$-type or $\beta$-type. See Figure 13 (a) for the lifting of band move, and Figure 13 (b) for the lifting of quasi-stabilization.
For a bipartite link, $G_D$ add one $p$-point to each component of $L_\alpha$ and one $q$-point to each component of $L_\beta$. To lift a bipartite link cobordism, we also perturb it to a regular position to decompose it into elementary cobordisms. Similarly, the lifting for isotopies and disk-stabilization/destabilization is unique. A type-I band move has a unique way to be lifted, as shown in Figure 14 (a); a type-II band move has two ways to be lifted, as shown in Figure 14 (b); a quasi-stabilization/destabilization has two ways to be lifted, as shown in Figure 14 (c).
4. Heegaard diagrams for band moves

In this section, we will relate bipartite link band moves to Heegaard triples. We focus our discussion on β-band moves. Similar results hold for α-band moves.

4.1. Heegaard diagrams for bipartite links. In this subsection we associate a Heegaard diagram to a bipartite link as follows.

Let \( L_{αβ} = (L, L_α, L_β, O) \) be a bipartite link in a closed oriented three-manifold \( Y \), where \( L_α = \{L_{α,1}, \cdots, L_{α,n}\} \), \( L_β = \{L_{β,1}, \cdots, L_{β,n}\} \), \( O = \{α_1, \cdots, α_{2n}\} \). We say that a Heegaard Diagram \( H_{αβ} = (Σ, (α, β, O)) \) is compatible with \( (Y, L_{αβ}) \), if:

- The surface \( Σ \) is closed, oriented and embedded in \( Y \). The genus of \( Σ \) is \( g \). The collection of curves \( α \) is a \((g + n - 1)\)-tuple \( \{α_1, \cdots, α_{g+n-1}\} \). The collection of curves \( β \) is a \((g + n - 1)\)-tuple \( \{β_1, \cdots, β_{g+n-1}\} \).
- The three-manifold \( Y \) is represented by \((Σ, (α, β))\), such that

\[
Y = Σ_g \cup (∪_{i=1}^{g+n-1} D_{α_i}) \cup (∪_{k=1}^n B_{α,k}) \cup (∪_{j=1}^n D_{β_j}) \cup (∪_{l=1}^n B_{β,l}).
\]

Here \( D_{α_i} \) (or \( D_{β_j} \) resp.) is a closed embedded disk with boundary identified with \( α_i \) (or \( β_j \) resp.) on \( Σ \), and \( B_{α,k} \) (or \( B_{β,l} \) resp.) is a closed three-ball embedded in \( Y \).
- The pair \((L_{α,k}, ∂L_{α,k})\) (or \((L_{β,l}, ∂L_{β,l})\) resp.) is unknotted embedded in \((B_{α,k}, Σ ∩ ∂B_{α,k})\) (or \((B_{β,l}, Σ ∩ ∂B_{β,l})\) resp.) for \( k = 1, \cdots, n \) (or \( l = 1, \cdots, n \) resp.).

For convenience, we denote by \( U_α \) the handlebody \( Σ_g \cup (∪_{i=1}^{g+n-1} D_{α_i}) \cup (∪_{k=1}^n B_{α,k}) \) and by \( U_β \) the handlebody \( Σ_g \cup (∪_{j=1}^n D_{β_j}) \cup (∪_{l=1}^n B_{β,l}) \). We say that the diagram \((Σ, (α, β, O))\) is compatible with the triple \((U_α, L_α, O)\), and the diagram \((Σ, β, O)\) is compatible with the triple \((U_β, L_β, O)\).

4.2. The existence of Heegaard triples subordinate to a band move. In [Zem16a, Definition 6.2], Zemke defines a Heegaard triple subordinate to an oriented band moves. Building on his work, in this subsection, we will construct a standard Heegaard Triple \( T \) subordinate to a band move \( B^3 \) from a bipartite link \((L, L_α, L_β, O)\) to \((L(B^3), L_α, L_β(B^3), O)\). Similar results hold for α-band move.

Let \( B^3 : [-ε, +ε] × [-1, 1] → Y \) be a band embedded in \( Y \) with two ends \( B^3([-ε, +ε] × \{±1\}) \) attached on \( L_β \), such that:

- The intersection \( B^3([-ε, +ε] × [-1, 1]) ∩ L \) is the ends of the band \( B^3([-ε, +ε] × \{±1\}) \).
- The union \( L_β(B^3) = (L_β \setminus B^3([-ε, +ε] × \{±1\})) ∪ B^3(\{±ε\} × [-1, 1]) \) is still a collection of arcs with boundary identified with \( O \).

Then the quadruple \((L(B^3), L_α, L_β(B^3), O)\) is still a a bipartite link in \( Y \), where \( L(B^3) \) is the link \( L_α ∪ L_β(B^3) \).

**Definition 4.1.** We say that a Heegaard triple \( T = (Σ, (α, β, γ, O)) \) is subordinate to a β-band move from a bipartite link \( L_{αβ} = (L, L_α, L_β, O) \) to \( L_{αγ} = (L(B^3), L_α, L_β(B^3), O) \), if it satisfies:
• The Heegaard diagram \( H_{\alpha\beta} = (\Sigma, \alpha, \beta, O) \) is compatible with the bipartite link \((Y, L_{\alpha\beta})\).

• The Heegaard diagram \( H_{\alpha\gamma} = (\Sigma, \alpha, \gamma, O) \) is compatible with the bipartite link \((Y, L_{\alpha\gamma})\).

Suppose the collection of curves \( \alpha \) (or \( \beta, \gamma \) resp.) has \( n \)-components, we say that the Heegaard Triple subordinate to a \( \beta \)-band move is \textit{standard} if it satisfies the following:

• The curves \( \gamma_1, \ldots, \gamma_{n-1} \) are small Hamiltonian isotopies away from any basepoints of \( \beta_1, \ldots, \beta_{n-1} \), with geometric intersection number \( |\beta_i \cap \gamma_j| = 2\delta_{ij} \).

• The curve \( \gamma_n \) is a Hamiltonian isotopy of \( \beta_n \) with intersection number \( |\beta_n \cap \gamma_n| = 2 \).

• There exists a disk region \( D \subset \Sigma \) contains two basepoints \( o \) and \( o' \) such that \( D \cap (\beta_n \cup \gamma_n) = 2 \). An example is shown in Figure 15.

**Theorem 4.2.** Let \((W, F, F_\alpha, F_\beta, A_\Sigma)\) be a \( \beta \)-band move from a bipartite link \((L, L_\alpha, L_\beta, O)\) to a bipartite link \((L(B^\beta), L_\alpha, L_\beta(B^\beta), O)\). Here both bipartite link are in same three-manifold \( Y \). Starting from a Heegaard diagram \( H_{\alpha\beta} \) of \((Y, L_\alpha, L_\beta, O)\), after a sequence of stabilization/destabilization and handleslides without crossing any basepoints, we can construct a Heegaard diagram \( H_{\alpha\beta}' = (\Sigma', \alpha', \beta', O) \) compatible with \((L, L_\alpha, L_\beta, O)\) together a disk region \( D \) on \( \Sigma' \), such that the local diagram \( D \cap H_{\alpha\beta}' \) is shown in either of the two top figures in Figure 16.

Furthermore, after a Hamiltonian perturbation of \( \beta_n \) across two basepoints \( o \) and \( o' \), and small Hamiltonian perturbation of \( \beta_1, \ldots, \beta_{n-1} \) without crossing any basepoints, we can get a standard Heegaard triple \( T \) subordinate to the \( \beta \)-band move, as shown in either of the two bottom figures in Figure 16.

**Proof.** The idea of the proof is to implant the data of the band \( B^\beta \) into a Heegaard Diagram compatible with the bipartite link \( L_{\alpha\beta} = (L, L_\alpha, L_\beta, O) \).

We lift the bipartite link \((Y, L_{\alpha\beta})\) to a bipartite disoriented link \((Y, L, O)\). Let \( f \) be a Morse function compatible with \((Y, L, O)\), such that its corresponding Heegaard diagram is \( H_{\alpha\beta} \). Then we have a Heegaard decomposition \( U_\alpha \cup U_\beta \) of \( Y \) with respect to \( f \). Without loss of generality, we assume the band \( B^\beta \) lies in the \( \beta \)-handlebody \( U_\beta \) and the ends of \( B^\beta \) lies in two oriented arc \( l \in 1 \) and \( l' \in 1 \) marked with \( o \) and \( o' \). Then we lift the bipartite link cobordism \((W, F, F_\alpha, F_\beta, A_\Sigma)\) to a bipartite disoriented link cobordism \((W, F, F_\alpha, F_\beta, A, A_\Sigma)\).
Step 1: Transversality assumption. Let $c_{o,o'}$ be the core of the $\beta$-band $B^3$. After a small perturbation of $f$, we can assume the core $c_{o,o'}$ of the $\beta$-band $B^3$ intersects the unstable manifolds of $f$ transversely. Particularly, $c_{o,o'}$ does not go through the critical points of $f$.

Now, we project the core $c_{o,o'}$ along the gradient flow of $f$ to the Heegaard surface $\Sigma$. The projection image $l_{o,o'}$ on $\Sigma$ is a path connecting $o,o'$. By the transversality assumption, the path $l_{w,z}$ intersects $\alpha,\beta$-curves transversely on $\Sigma$. Furthermore, the path $l_{o,o'}$ only has regular self-intersection points, and the interior of $l_{o,o'}$ does not go through any basepoints. See Figure 17.

Step 2: Resolving the self-intersection of the path $l_{o,o'}$. Suppose $p$ is a self-intersection of $l_{o,o'}$. We can find a disk neighborhood $D_p$ of $p$ such that $D_p \cap (O \cup \alpha \cup \beta) = \emptyset$. Let $\phi_t$ be the diffeomorphism induced by the flow $-\nabla f$. There is an embedded solid cylinder $C_p = D \times [0, \epsilon] \to U_\beta$ given by $C_p(d,t) = \phi_t(d)$. By choosing a big enough parameter $\epsilon$, we can assume the pair $(C_p, C_p \cap c_{o,o'})$ is shown in Figure 18.
Now we modify the Morse function $f$ inside the solid cylinder $C_p$ by adding a pair of index one and index two critical point. In other words, this is doing a stabilization of $H$ inside the solid cylinder $C$ shown in Figure 18.

By resolving all self-intersections of $l_{o,o'}$, we will get a new Morse function $f'$ and a corresponding balanced Heegaard diagram $H'$. If we projects the core $c_{o,o'}$ along the gradient flow of the modified function $f'$ on $\Sigma$, the projection image has no self-intersections.

**Step 3: Framing of the band.** By the transversality assumption, the gradient $-\nabla f|_{c_{o,o'}}$ induces a framing of the core $c_{o,o'}$. Heegaard Floer homology and integer surgeries on links. On the other hand the embedding of the $\beta$-band $B^2$ also induces a framing of the core. These two framings differed by $n \pm \frac{1}{2}$. In other words, consider the solid cylinder neighborhood of $c_{o,o'}$, if we identify the ends of the two neighborhood by the gradient framing, then the band will twist along the core of the solid torus by $(n \pm \frac{1}{2}) \times 2\pi$.

Suppose $c'_{o,o'}$ is a small perturbation of $c_{o,o'}$ as shown in Figure 19. The gradient framing of $c'_{o,o'}$ and $c_{o,o'}$ is differed by $\pm 1$. If we resolve the self-intersection of the projection image $l'_{o,o'}$ as in Step 2, the gradient framing of $c'_{o,o'}$ does not change. Therefore, by modifying the Morse function $f$ as in Step 2, we can modify the gradient framing of $c_{o,o'}$. Now we can assume that the difference between the gradient framing and band framing is $\pm \frac{1}{2}$.

**Step 4: Heegaard triple for band moves.** After resolving all self-intersections of $l_{o,o'}$, we can find a rectangle neighborhood $R : [-\epsilon, \epsilon] \times I \to \Sigma$ of $l_{o,o'}$, shown in Figure 20, such that:

- The image $R(0, I) = l_{o,o'}$ and $R(0, 0) = o$, $R(0, 1) = o'$.
- The intersection $(\alpha \cup \beta) \cap R = [-\epsilon, \epsilon] \times \{t_0, \ldots, t_m\}$, where $0 < t_0, \ldots, t_m < 1$. 
After $\alpha$ or $\beta$- isotopies as shown in Figure 20, we can assume, the intersection $\alpha \cap R = [-\epsilon, \epsilon] \times \{t_0, \ldots, t_s\}$ and $\beta \cap R = [-\epsilon, \epsilon] \times \{t_{s+1}, \ldots, t_m\}$. Then we stabilize the diagram as shown in Figure 21. Finally, by an $\alpha$ or $\beta$ isotopy, we get a diagram shown in Figure 22. The desired disk region is chosen to be the disk contains the two basepoint $o$ and $o'$ and one intersection of $\alpha$ and $\beta$. The two intersections determines two disk region.

Now we assume the $\alpha$ and $\beta$ curves in Figure 22 are $\alpha_n$ and $\beta_n$. Let the curve $\gamma_n$ be a Hamiltonian perturbation of $\beta_n$ as shown in Figure 16. We set $\gamma_1, \ldots, \gamma_{n-1}$ to be a small Hamiltonian perturbation of $\beta_1, \ldots, \beta_{n-1}$ without across any basepoints. Therefore, we construct a Heegaard Triple $T$. Clearly, the choice the disk determines whether the result
Figure 23. Links in $\#^n(S^1 \times S^2)$ generated by Type I band move

link is $L(B^\beta)$ or $L(B^\beta_{\pm 1})$. Here $B^\beta_{\pm 1}$ refers to the $\beta$-band which has the same core as $B^\beta$ but $\pm 1$-framing with respect to the framing of $B^\beta$.

4.3. Some topological facts about Heegaard triples subordinate to a band move.

**Lemma 4.3.** Any two Heegaard triples $T_1$ and $T_2$ subordinate to a $B^\beta$-band move can be connected by the following type of Heegaard moves:

- Ambient isotopies of $\Sigma$ which fix $L \cup B^\beta$.
- Isotopies and handle slides amongst $\alpha, \beta, \gamma$-curves without crossing any basepoints.
- Stabilization or destabilization of a standard Heegaard Triple on torus. Here a standard Heegaard triple on torus has only one $\alpha$, one $\beta$ and one $\gamma$-curve, with $\alpha$ intersecting $\beta$ and $\gamma$ once, and $\gamma$ is a small Hamiltonian isotopy of $\beta$ with intersection number $|\gamma \cap \beta| = 2$.

**Proof.** We stabilize the two Heegaard diagram sufficiently many times, then move the critical points of the three Morse functions compatible with the three pairs $(U_\alpha, L_\alpha),(U_\beta, L_\beta)$ and $(U_\gamma, L_\gamma)$ without changing the gradient-like flow in a small neighborhood of $L_\alpha, L_\beta, L_\gamma$.

**Lemma 4.4.** Suppose $T$ is a Heegaard triple subordinate to a $B^\beta$-band move from bipartite link $L_{\alpha\beta}$ to $L_{\alpha\gamma}$. Then the induced Heegaard diagram $H_{\beta\gamma}$ is a diagram for a bipartite link $L_{\beta\gamma}$ in $\#^g(S^1 \times S^2)$. In fact, we have

$$(\#^g(S^1 \times S^2), L_{\beta\gamma}) = \begin{cases} 
(\#^g(S^1 \times S^2), K)\#(S^3, U_{2}^{n-1}), & \text{if } B^\beta \text{ is of Type I.} \\
(\#^g(S^1 \times S^2), U_{4})\#(S^3, U_{2}^{n-2}), & \text{if } B^\beta \text{ is of Type II.}
\end{cases}$$

Here $K$ is a bipartite knot with two basepoint in $\#^g(S^1 \times S^2)$. The homology $[K]$ of the bipartite knot $K$ is equal to $(0, \cdots, 0, 2)$ in $H_1(\#^g(S^1 \times S^2))$. The bipartite link $U_{2k}$ is the bipartite unlink with $l$-components and $2k$-basepoints on each of these components.

**Proof.** Without loss of generality, we assume that the band $B^\beta$ is next to a pair of basepoints $(\alpha_1, \alpha_2)$. By Theorem 4.2, we get a Heegaard diagram, as shown in Figure 23 (or Figure 24 resp.) if $B^\beta$ is of Type I (or Type II resp.). The result follows directly from these two Heegaard diagrams.
Figure 24. Links in $\#(S^1 \times S^2)$ generated by Type II band move
5. Bipartite link Floer homology

In this section, we construct the bipartite link Floer homology for homologically even bipartite links.

5.1. Bipartite link Floer curved chain complex for null-homologous links. In this subsection, we construct a well-defined $\mathbb{Z}$-graded curved chain complex for a null homologous bipartite link $L_{\alpha\beta}$ in a closed oriented three-manifold $Y$.

Suppose $H_{\alpha\beta}$ is a Heegaard diagram for the bipartite link $L_{\alpha\beta}$. Let $J_t$ be a generic family of almost complex structures on $\Sigma$. We denote by $H_{\alpha\beta}$ the Heegaard data $(H_{\alpha\beta}, J_t)$. Now we can define a $\mathbb{F}_2[U_1, \ldots, U_{2n}]$-module $\text{CFBL}^-\left(H_{\alpha\beta}\right)$, together with an endomorphism $\partial$ as follows:

- The module $\text{CFBL}^-\left(H_{\alpha\beta}\right)$ is freely generated by the intersections of the two Lagrangians $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in the symplectic manifold $\text{Sym}^{g+1}(\Sigma)$.
- The endomorphism $\partial$ acting on a generator $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is given by:

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y), \mu(\phi) = 1} \#(M(\phi)/\mathbb{R}) \prod_{i=1}^{2n} t_i^{n_{\alpha}(\phi)} y.$$ 

Here $g$ is the genus of the Heegaard surface $\Sigma$, $2n$ is the number of basepoints on the bipartite link $L_{\alpha\beta}$.

Given an alternating coloring $\mathbf{P}$ of $L_{\alpha\beta}$, we can define a map from the set of generators of $\text{CFBL}^-\left(H_{\alpha\beta}\right)$ to the set of Spin$^c$-structure of $Y$ as in [OS04b, Section 2.6]. In detail, we set our Spin$^c$-structure map $s_{\mathbf{P}}(x)$ to be $s_{\mathbf{w}}(x)$, where $\mathbf{w}$ is the subset $\mathbf{P}^{-1}(+1)$ of basepoints. If $\mathbf{P}'$ is another alternating coloring of $L_{\alpha\beta}$, the difference between the two maps $s_{\mathbf{P}}$ and $s_{\mathbf{P}'}$ is:

$$s_{\mathbf{P}}(x) - s_{\mathbf{P}'}(x) = \frac{1}{2}(\text{PD}[L_{\mathbf{P}}] - \text{PD}[L_{\mathbf{P}'}]).$$

Here $L_{\mathbf{P}}$ and $L_{\mathbf{P}'}$ are the oriented links determined by the bipartite link $L_{\alpha\beta}$ and the coloring $\mathbf{P}$ and $\mathbf{P}'$, respectively in Remark 3.23. As we assume $L_{\alpha\beta}$ in $Y$ is null-homologous, the difference $s_{\mathbf{P}}(x) - s_{\mathbf{P}'}(x)$ is zero. Therefore, we have a map $s_\delta = s_{\mathbf{P}} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \text{Spin}^c(Y)$, which is independent of the choice of alternating coloring $\mathbf{P}$ of $L_{\alpha\beta}$. For the details of Spin$^c$-structure map, see [OS04b, Section 2.6] and [OS08a, Section 3.3]. The module $\text{CFBL}^-\left(H_{\alpha\beta}\right)$ now splits into a direct sum:

$$\text{CFBL}^-\left(H_{\alpha\beta}\right) = \bigoplus_{s \in \text{Spin}^c(Y)} \text{CFBL}^-\left(H_{\alpha\beta}, s\right),$$

where $\text{CFBL}^-\left(H_{\alpha\beta}, s\right)$ consists of generators $x$ whose image under $s_\delta$ is equal to $s$.

From now on, we assume $s$ is a torsion Spin$^c$-structure of $Y$. The diagram $H_{\alpha\beta}$ is weakly $s$-admissible, then we can get finite counts of moduli spaces. We will discuss admissibility in Section 5.3.

Based on the work in [OSS15], we have a relative $\delta$-grading for $\text{CFBL}^-\left(H_{\alpha\beta}, s\right)$ as follows. Let $x$ and $y$ be two generators in $\text{CFBL}^-\left(H_{\alpha\beta}, s\right)$ with $\pi_2(x, y)$ being non-empty. The relative $\delta$-grading between $x$ and $y$ is given by

$$\delta(x, y) = \mu(\phi) - n_{\alpha}(\phi),$$

where $n_{\alpha}$ denotes the sum $\sum_{i=1}^{2n} n_{\alpha}(\phi)$, and $\phi$ is an element in $\pi_2(x, y)$. Notice that there is a pair of basepoints $(\alpha, \alpha')$ on each component of $\Sigma_{\alpha}$ or $\Sigma_\beta$. Using Lipshitz’s
formula \cite[Corollary 4.3]{lip11}, we know that the Maslov index \(\mu(P)\) of a periodic domain \(P\) is equal to \(n_\beta(P)\). This implies that the relative \(\delta\)-grading is well-defined. Now we can set the \(\delta\)-grading of the variable \(U_i\) to be \(-1\) and extend the relative \(\delta\)-grading to the submodule \(CFBL^-(\mathcal{H}_{\alpha\beta}, s)\). Moreover, as the Spin\(^c\)-structure \(s\) is torsion, the \(\delta\)-grading is actually a \(\mathbb{Z}\)-grading on \(CFBL^-(\mathcal{H}_{\alpha\beta}, s)\).

As a corollary of \cite[Lemma 2.1]{zem16}, we have

\[
\partial^2 = \sum_i (U_{i,1}U_{i,2} + U_{i,2}U_{i,3} + \cdots + U_{i,k}U_{i,1}).
\]

Here \(i\) refers the \(i\)-th component \(L_i\) of \(L\), the variable \(U_{i,j}\) is the variable assigned to the basepoint \(o_{i,j}\). The basepoints \(o_{i,j}\) appear in order on the link component \(L_i\).

By the above discussion, we know that the \(\mathbb{F}_2[U_1, \cdots, U_{2n}]\)-module \(CFBL^-(\mathcal{H}_{\alpha\beta}, s)\) together with the endomorphism \(\partial\) and the \(\delta\)-grading is a well-defined \(\mathbb{Z}\)-graded curved chain complex.

**Remark 5.3.** Given an alternating coloring \(\mathcal{P}\) of \(L_{\alpha\beta}\), we can get the curved chain complex \(CFL^U_{UV}(\mathcal{H}_{\alpha\beta})\) defined by Zemke in \cite{zem16} from \(CFBL^-(\mathcal{H}_{\alpha\beta})\) by setting the variable of \(z\)-basepoints to be \(V_i\)'s. Furthermore, the average of the two gradings \(gr_w\) and \(gr_z\) gives the \(\delta\)-grading on the curved chain complex \(CFL^U_{UV}\). The curved chain complex \(CFL^-(\mathcal{H}_{\alpha\beta})\) together with a coloring \(\mathcal{P}\) has the same amount of data as \(CFL^U_{UV}(\mathcal{H}_{\alpha\beta})\).

### 5.2. Spin\(^c\)-structures

In this section, we will discuss the relation between the Spin\(^c\)-structures of \(Y\) and a Spin\(^c\)-structure map induced by homologically even bipartite links.

Consider the Heegaard diagram \(H_{\beta\gamma}\) of a bipartite link \(L_{\beta\gamma}\) induced from a standard Heegaard triple \(T\) subordinate to a \(\beta\)-band move of type I as in Lemma 4.4 and Figure 23. Notice that \(L_{\beta\gamma}\) is not null-homologous. Let \(\mathcal{P}\) be an alternating coloring of \(L_{\beta\gamma}\). Then all the generators in \(T_{\beta} \cap T_{\gamma}\) belong to the same equivalence class \(s\). We have the following equality for the Spin\(^c\) structure map \(\mathcal{s}_{\mathcal{P}}:\)

\[
\mathcal{s}_{\mathcal{P}}(x) - \mathcal{s}_0 = \mathcal{s}_0 - \mathcal{s}_{-\mathcal{P}}(x) = \pm \text{PD}[\beta_n],
\]

where \(\mathcal{s}_0\) is the only torsion Spin\(^c\)-structure on \(\#^n(S^1 \times S^2)\). The grading \(gr_{\mathcal{P}}\) (or \(gr_{-\mathcal{P}}\)) on the equivalence class \(s\) is a \(\mathbb{Z}_2\)-grading, which can not be lifted to a \(\mathbb{Z}\)-grading. Therefore, it is necessary to find a sufficient condition for which the \(\delta\)-grading on the equivalence class \(s\) is a \(\mathbb{Z}\)-grading.

**Definition 5.5.** We say that a bipartite link \(L_{\alpha\beta}\) is **homologically even**, if the homology class \([L_{\mathcal{P}}] \in H_1(Y)\) is divisible by two, where \(\mathcal{P}\) is an alternating coloring of \(L_{\beta\gamma}\).

**Lemma 5.6.** Let \(L_{\beta\gamma}\) be a homologically even bipartite link in a closed oriented three-manifold \(Y\). Then, the map \(\mathcal{s}_{\delta}: T_{\alpha} \cap T_{\beta} \to \text{Spin}^c(Y)\) defined by

\[
\mathcal{s}_{\delta}(x) \doteq \mathcal{s}_{\mathcal{P}}(x) - \frac{1}{2} \text{PD}[L_{\mathcal{P}}],
\]

is independent the choice of \(\mathcal{P}\). Furthermore, if \(x\) and \(y\) are two generators in \(T_{\alpha} \cap T_{\beta}\) with \(\pi_2(x, y)\) being non-empty, then \(\mathcal{s}_{\delta}(x) = \mathcal{s}_{\delta}(y)\). Therefore, we have the following decomposition

\[
CFBL^-(\mathcal{H}_{\alpha\beta}) = \bigoplus_{s \in \text{Spin}^c(Y)} CFBL^-(\mathcal{H}_{\alpha\beta}, s),
\]

where \(CFBL^-(\mathcal{H}_{\alpha\beta}, s)\) consists of generators \(x\) with \(\mathcal{s}_{\delta}(x) = s\).
Proof. Let $\mathcal{W}'$ be another alternating coloring of $L_{\beta\gamma}$. By Equation 5.1, the difference between the two Spin$^c$-structure $s_{\mathcal{W}}(x) - s_{\mathcal{W}'}(x) = \text{PD}[L_{\mathcal{W}\mathcal{W}'}]$. Here the link $L_{\mathcal{W}\mathcal{W}'}$ is a sublink of $L_{\mathcal{W}}$ consisting of components $L_1$ of $L$ with $\mathcal{W}(L_1) = -\mathcal{W}'(L_1)$. On the other hand, by the construction in Remark 3.23, we have $2\text{PD}[L_{\mathcal{W}\mathcal{W}'}] = \text{PD}[L_{\mathcal{W}}] - \text{PD}[L_{\mathcal{W}'}]$. Combining these two equalities, we get that the Spin$^c$-structure map $s_{\mathcal{W}}(x) - \frac{1}{2}\text{PD}[L_{\mathcal{W}}]$ is equal to $s_{\mathcal{W}'}(x) - \frac{1}{2}\text{PD}[L_{\mathcal{W}'}]$.

If $\pi_2(x,y)$ is non-empty, we get $s_{\mathcal{W}}(x) = s_{\mathcal{W}'}(y)$. Therefore the map $s_{\delta}$ sends the equivalence classes of generators in $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$ to the Spin$^c$-structures of $Y$. □

From now on, we assume all the bipartite link we discuss satisfies the same condition in Lemma 5.6 i.e. its homology class is divisible by two.

5.3. Admissibility. Although one can assign an alternating coloring to a bipartite link and define the admissibility with respect to the $w$-basepoints. This admissibility with respect to $w$-basepoints can not help us define a $\mathbb{Z}$-graded complex for some null-homologous links in $\#(S^1 \times S^2)$. In this subsection, we will introduce the admissibility with respect to Spin$^c$-structure map $s_{\delta}$.

Definition 5.8. Suppose $s$ is a Spin$^c$-structure over $Y$, we say that a Heegaard diagram $H_{\alpha\beta} = (\Sigma, \alpha, \beta, O)$ is $s$-realized, if there is a point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and a $n$-tuple of points $q$ on $\Sigma \setminus (\alpha \cup \beta)$, such that:

$$s_{\delta}(x) = s_q(x) = s.$$

Furthermore, we say that an $s$-realized Heegaard diagram $H_{\alpha\beta}$ is weakly $s$-admissible (or strongly $s$-admissible resp.), if the diagram $(\Sigma, \alpha, \beta, q)$ is weakly $s$-admissible (or strongly $s$-admissible resp.).

Lemma 5.9. Given a bipartite link $L_{\beta\gamma}$ in a closed three-manifold $Y$, together with a fixed Spin$^c$-structure $s$, we can construct a weakly $s$-admissible (or strongly $s$-admissible resp.) Heegaard diagram $H_{\beta\gamma}$ compatible with the pair $(Y, L_{\beta\gamma})$.

Proof. This follows directly from the proof of [OS04b, Lemma 5.2] and the proof of [OS04b, Lemma 5.4]. Here, we require the finger moves of $\beta$ and $\gamma$ curves do not across the basepoints $O$. □

Lemma 5.10. Any two weakly $s$-admissible (or strongly $s$-admissible resp.) Heegaard diagram $H_{\beta\gamma}$ compatible with the pair $(Y, L_{\beta\gamma})$ can be connected by a sequence of Heegaard moves without crossing basepoints $O$. Furthermore, each intermediate Heegaard diagram is weakly $s$-admissible (or strongly $s$-admissible resp.).

Proof. This follows from the proof of [OS04b, Lemma 5.6]. See also [OS08a, Section 3.4], and [JT12]. Again, we require the Heegaard moves do not across basepoints $O$. □

5.4. The curved chain complex $CFBL^{-}$ for bipartite links in $\#(S^1 \times S^2)$.

Lemma 5.11. Let the Heegaard data $H_{\beta\gamma}$ for the bipartite link $L_{\beta\gamma}$ be weakly $s$-admissible. Here $s$ is a torsion Spin$^c$-structure of $Y$. Then the curved chain complex $CFBL^{-}(H_{\beta\gamma}, s)$ is a well-defined $\mathbb{Z}$-graded curved chain complex, with grading defined as in Equation 5.2.

Proof. Let $x, y$ be two generators in $CFBL^{-}(H_{\beta\gamma}, s)$. We want to show the grading:

$$\delta(x, y) = \mu(\phi) - n_{O}(\phi)$$
is well-defined. It suffice to show the following equality:
\[ \mu(\psi) = n_\O(\psi) \]
holds for every class \( \psi \in \pi_2(\mathbf{x}, \mathbf{x}) \) of \( H_{\alpha\beta} \). Because \( \mathbf{s} \) is torsion, we have a relative \( \mathbb{Z} \)-grading:
\[ gr_\mathbf{q}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_\mathbf{q}(\phi), \]
where \( \phi \) is a class in \( \pi_2(\mathbf{x}, \mathbf{y}) \). Its Maslov index of \( \psi \in \pi_2(\mathbf{x}, \mathbf{x}) \) is
\[ \mu(\psi) = 2n_\mathbf{q}(\psi). \]

Given an alternating coloring \( \mathfrak{P} \) of \( L_{\beta\gamma} \), we denote by \( \mathbf{w} = \{ w_1, \ldots, w_n \} \) the \( n \)-tuple of basepoints satisfying \( \mathfrak{P}(w_i) = 1 \) and by \( \mathbf{z} = \{ z_1, \ldots, z_n \} \) the \( n \)-tuple of basepoints satisfying \( \mathfrak{P}(z_i) = -1 \). By [OS04b, Lemma 2.18], we have the following equalities:
\[ n_\mathbf{w}(\psi) - n_\mathbf{q}(\psi) = (H(\psi), a^*) = n_\mathbf{q}(\psi) - n_\mathbf{z}(\psi). \]
Here, \( H(\psi) \in H_2(Y; \mathbb{Z}) \) is the homology class belonging to the periodic class, \( a^* \) is a cohomology class in \( H^1(Y) \) determined by the relative position between \( \mathbf{w} \) and \( \mathbf{q} \). Combining the Equation 5.12 and Equation 5.13, we have Maslov grading
\[ \mu(\psi) = 2n_\mathbf{q}(\psi) = n_\mathbf{w}(\psi) + n_\mathbf{z}(\psi) = n_\O(\psi). \]
This implies the relative \( \delta \)-grading which is defined by
\[ \delta(\mathbf{x}, \mathbf{y}) = \mu(\phi) - n_\O(\phi) \]
is a \( \mathbb{Z} \)-grading on \( CFBL^-(\mathcal{H}_{\beta\gamma}, \mathbf{s}) \).

Moreover, as \( n_\O(\psi) = 2n_\mathbf{q}(\psi) \), the finiteness of counting follows from the admissibility with respect to \( \mathbf{q} \).

By Lemma 5.11, the Heegaard diagram \( H_{\beta\gamma} \) is a diagram for the bipartite link \( L_{\beta\gamma} \) in \( \#^g(S^1 \times S^2) \). Let \( \mathbf{s} \) be the unique torsion Spin\(^c\)-structure of \( \#^g(S^1 \times S^2) \). Then, the \( \delta \)-graded curved chain complex \( CFBL^-(\mathcal{H}_{\alpha\beta}, \mathbf{s}) \) is a \( \mathbb{Z} \)-graded curved chain complex. Furthermore, the span of the top grading generators in \( HFL'(\mathcal{H}_{\alpha\beta}, \mathbf{s}) \) is a two-dimensional vector space.

Applying the above results to the bipartite links in \( \#^g(S^1 \times S^2) \), we have the following.

**Corollary 5.14.** Suppose \( \mathcal{T} \) is a Heegaard triple subordinate to a \( \beta \)-band move \( B^\beta \), and \( H_{\beta\gamma} \) is the induced Heegaard diagram for bipartite link \( L_{\beta\gamma} \) in \( \#^g(S^1 \times S^2) \). Let \( \mathbf{s} \) be the unique torsion Spin\(^c\)-structure of \( \#^g(S^1 \times S^2) \). Then, the \( \delta \)-graded curved chain complex \( CFBL^-(\mathcal{H}_{\alpha\beta}, \mathbf{s}) \) is a \( \mathbb{Z} \)-graded curved chain complex. Furthermore, the span of the top grading generators in \( HFL'(\mathcal{H}_{\alpha\beta}, \mathbf{s}) \) is a two-dimensional vector space.

**Proof.** By Lemma 4.4, the Heegaard diagram \( H_{\beta\gamma} \) is a diagram for the bipartite link \( \#^g(S^1 \times S^2), K \#(S^3, \mathbb{U}^{n-1}) \) if \( B^\beta \) is of type I, and for \( \#^g(S^1 \times S^2), U_4 \#(S^3, \mathbb{U}^{n-2}) \) if \( B^\beta \) is of type II. Here \( n \) is the number of \( \alpha \)-circles on \( \mathcal{T} \), \( K \) is a knot with homology equal to twice of the dual of \( \beta_n \). In either of these two cases, the homology class \( [L_{\mathbf{q}}] \) for an alternating coloring \( \mathfrak{P} \) of the bipartite link \( L_{\beta\gamma} \) is divisible by two. Applying Lemma 5.11, we get that the \( \delta \)-grading for curved chain complex \( CFBL^-(\mathcal{H}_{\alpha\beta}, \mathbf{s}) \) is a \( \mathbb{Z} \)-grading.

Without loss of generality, suppose \( \mathcal{T} \) is a standard Heegaard triple subordinate to the band move \( B^\beta \) from \( L_{\alpha\beta} \) to \( L_{\alpha\gamma} \). By Theorem 4.2 as \( \mathcal{T} \) is standard, the curves \( \gamma_1, \cdots, \gamma_{n-1} \) are a small Hamiltonian isotopies of \( \beta_1, \cdots, \beta_{n-1} \) without crossing any basepoints. The curve \( \gamma_n \), which is a Hamiltonian isotopy of \( \beta_n \) crossing basepoints, intersect \( \beta \)-circles at two points (See Figure 23 and Figure 24). Therefore, we have \( 2^n \) generators for the module \( CFBL^-(H_{\beta\gamma}, \mathbf{s}) \). Clearly, there are two top grading generators in the kernel of \( \partial \) for \( CFL'(H_{\beta\gamma}, \mathbf{s}) \). \( \square \)
5.5. Holomorphic triangles. Recall from [OS04b, Section 8] that, given a Heegaard triple $T$, we can classify the homotopy classes of Whitney triangles as follows. Suppose $\psi \in \pi_2(x,y,v)$ and $\psi' \in \pi_2(x',y',v')$ are two Whitney triangles. We say that $\psi$ and $\psi'$ are equivalent, if there exists classes $\phi_1 \in \pi_2(x,x')$ and $\phi_2 \in \pi_2(y,y')$ and $\phi_3 \in \pi_2(v,v')$, such that:

$$\psi' = \psi + \phi_1 + \phi_2 + \phi_3.$$ 

We denoted by $S_{\alpha\beta\gamma}(T)$ the set of equivalence classes of the Whitney triangles of $T$.

Suppose $q$ is a $n$-tuple of basepoints on $T$, such that the induced $n$-pointed diagrams $H_{\alpha\beta}, H_{\beta\gamma}$ and $H_{\alpha\gamma}$ are well-defined pointed Heegaard diagrams for pointed closed oriented three-manifolds $Y_{\alpha\beta}, Y_{\beta\gamma}$ and $Y_{\alpha\gamma}$. By [OS04b, Proposition 8.5], we have a one-to-one map:

$$s_q : S_{\alpha\beta\gamma}(T) \to \text{Spin}^c(X_{\alpha\beta\gamma}).$$

Here $X_{\alpha\beta\gamma}$ is an oriented four-manifold constructed from the Heegaard triple $T$, whose boundary $\partial X_{\alpha\beta\gamma}$ is the union of three-manifolds $-Y_{\alpha\beta} \sqcup -Y_{\beta\gamma} \sqcup Y_{\alpha\gamma}$.

Suppose the Heegaard triple $T$ is subordinate to a band move from a bipartite link $(Y,L_{\alpha\beta})$ to $(Y,L_{\alpha\gamma})$. The three-manifold $Y_{\alpha\beta}$ and $Y_{\alpha\gamma}$ are diffeomorphic to $Y$, the three-manifold $Y_{\beta\gamma}$ is diffeomorphic to $#^n(S^1 \times S^2)$ and the four-manifold $X_{\alpha\beta\gamma}$ is actually $(Y \times I) \setminus N(U_3 \times \{1/2\})$.

Notice that, we can always construct an almost complex structure over $Y \times I$. Choices of almost complex structure establish a one-to-one correspondence between the Spin$^c$-structure of $Y \times I$ and $H^2(Y \times I; \mathbb{Z}) \cong H^2(Y; \mathbb{Z})$. For a Spin$^c$-structure $s$ on $Y \times I$, its restriction $s_{\alpha\beta}$ on $Y_{\alpha\beta}$ is equal to $s_{\alpha\gamma}$ on $Y_{\alpha\gamma}$, and its restriction $s_{\beta\gamma}$ should be $s_0$ on $#^n(S^1 \times S^2)$. Conversely, we claim that a Spin$^c$-structure $s$ on $Y_{\alpha\beta}$ and $Y_{\alpha\gamma}$, together with the unique Spin$^c$-structure $s_0$ on $#^n(S^1 \times S^2)$, can be uniquely extended to a Spin$^c$-structure $s_{\alpha\beta\gamma}$ on $X_{\alpha\beta\gamma}$. Otherwise, we suppose $s^1_{\alpha\beta\gamma}$ and $s^2_{\alpha\beta\gamma}$ are two possible extensions. As $s^i_{\alpha\beta\gamma}$ restricted on $#^n(S^1 \times S^2)$ is $s_0$, we can further extend $s^i_{\alpha\beta\gamma}$ to a Spin$^c$-structure $\tilde{s}^i_{\alpha\beta\gamma}$ over $Y \times I$. By previous discussion, we know $\tilde{s}^1_{\alpha\beta\gamma}$ and $\tilde{s}^2_{\alpha\beta\gamma}$ have the same restriction on $Y_{\alpha\beta}$ and $Y_{\alpha\gamma}$. This implies $\tilde{s}^1_{\alpha\beta\gamma} - \tilde{s}^2_{\alpha\beta\gamma}$ is not in $H^2(Y \times I)$. Hence the difference $s^1_{\alpha\beta\gamma} - s^2_{\alpha\beta\gamma}$ is 0 in $H^2(X_{\alpha\beta\gamma})$.

**Definition 5.15.** We suppose that the Heegaard triple $T$ is a $2n$-pointed Heegaard triple subordinate to a band move from $(Y,L_{\alpha\beta})$ to $(Y,L_{\alpha\gamma})$. We also fix a Spin$^c$-structure $s_{\alpha\beta\gamma}$ over $X_{\alpha\beta\gamma}$ which comes from the restriction of a Spin$^c$-structure over $Y \times I$, we say that the Heegaard triple $T$ is $s$-**realized**, if there exists points $x \in T_{\alpha} \cap T_{\beta}$, $\theta \in T_{\beta} \cap T_{\gamma}$ and $y \in T_{\alpha} \cap T_{\gamma}$ and a $n$-tuple of points $q \in \Sigma \setminus (\alpha \cup \beta)$, such that:

- there is a unique point $q_i$ in each component of $\Sigma \setminus \alpha$ and $\Sigma \setminus \beta$;
- the map $s_q(x) = s_{\alpha\beta}(x) = s_{\alpha\beta}$, where $s_{\alpha\beta}$ is the restriction of $s_{\alpha\beta\gamma}$ on the boundary three-manifold $Y_{\alpha\beta}$;
- the map $s_q(\theta) = s_{\beta\gamma}(\theta) = s_{\beta\gamma}$, where $s_{\beta\gamma}$ is the restriction of $s_{\alpha\beta\gamma}$ on the boundary three-manifold $Y_{\beta\gamma}$;
- the map $s_q(y) = s_{\gamma}(y) = s_{\beta\gamma}$, where $s_{\gamma}$ is the restriction of $s_{\alpha\beta\gamma}$ on the boundary three-manifold $Y_{\alpha\gamma}$.

Furthermore, we say that an $s$-realized Heegaard triple $T = (\Sigma, \alpha, \beta, \gamma, 0)$ is weakly $s$-admissible (or strongly $s$-admissible resp.), if the pointed Heegaard triple $(\Sigma, \alpha, \beta, \gamma, q)$ is weakly $s$-admissible (or strongly $s$-admissible resp.).
Remark 5.16. If $\mathcal{T}$ is $s$-realized, we claim that there is a unique class $[\Delta] \in S_{\alpha\beta\gamma}$ of triangges with $s_q([\Delta]) = s_{a\beta\gamma}$ for some $n$-tuple of points $q$ satisfying the condition in Definition 5.15. Otherwise, we suppose there is another class $[\Delta]'$ in $S_{\alpha\beta\gamma}$ with $s_q([\Delta]') = s_{a\beta\gamma}$. We know that $s_q([\Delta])$ and $s_q([\Delta]')$ have the same restriction on boundary and can be extend to a Spin$^c$-structure over $Y \times I$. Hence we have $s_q([\Delta]) = s_{a\beta\gamma}$. As the map $s_q$ is one-to-one, we get $[\Delta] = [\Delta]'$.

If the Heegaard triple $\mathcal{T}$ is subordinate to a four-dimensional two-handle attachment, one may not find a well-defined map. Recall that if $(F, \partial F)$ is orientable, we can choose $s_w$ or $s_z$ as the one-to-one map from $S_{\alpha\beta\gamma}$ to Spin$^c(X_{\alpha\beta\gamma})$. If $(F, \partial F)$ is non-orientable, we have no canonical choice of $n$-tuple of basepoints $q$ on $\Sigma$. The choice of $q$ may affects the map $s_q$, i.e. in some cases, there exist $q$ and $q'$, both of which satisfy the condition in Definition 5.15, but $s_q \neq s_{q'}$. If this happens, we will not know which class of triangle in $S_{\alpha\beta\gamma}$ should be associated to certain Spin$^c$-structure $s$.

In light of the proof in [OS04b] Lemma 5.2], by finger moves of $\alpha, \beta$ and $\gamma$ curves along their dual curves on $\Sigma$ without crossing the basepoints $O$, we can isotope a Heegaard triple $\mathcal{T}$ subordinate to a band move to a weakly $s$-admissible (or strongly $s$-admissible resp.) Heegaard triple, where $s$ is a Spin$^c$-structure for $Y \times I$.

Lemma 5.17. Let $\mathcal{T}$ be a strongly $s$-admissible Heegaard triple subordinate to a band move from $(Y, L_{\alpha\beta})$ to $(Y, L_{\alpha\gamma})$. If $s$ is torsion, we have a triangle chain map:

$$f_{\alpha\beta\gamma} : CFL'(H_{\alpha\beta}, s_{\alpha\beta}) \otimes CFL'(H_{\beta\gamma}, s_{\beta}^0) \to CFL'(H_{\alpha\gamma}, s_{\alpha\gamma})$$

whose restriction on generators are defined by:

$$f_{\alpha\beta\gamma}(x \otimes \theta; s) \triangleq \sum_{y \in \pi_2(x, \theta, y)} \sum_{\{\psi \in \pi_2(x, \theta, y) | \mu(\psi) = 0, s_q(\psi) = s\}} (#M(\psi)) U^{\alpha}(\psi) y.$$

Here $q$ is $n$-tuple of points being used to define $s$-admissibility in Definition 5.15.

Proof. Let $\psi, \psi' \in \pi_2(x, \theta, y)$, then the difference

$$\psi - \psi' = \phi_{\alpha\beta} + \phi_{\beta\gamma} + \phi_{\alpha\gamma},$$

where $\phi_{\alpha\beta}$ is a class in $\pi_2(x, x)$, $\phi_{\beta\gamma}$ is a class in $\pi_2(\theta, \theta)$, $\phi_{\alpha\gamma}$ is a class in $\pi_2(y, y)$.

As in the proof of Lemma 5.11, we have three equalities, $n_{O}(\phi_{\alpha\beta}) = 2n_{q}(\phi_{\alpha\beta})$, $n_{O}(\phi_{\beta\gamma}) = 2n_{q}(\phi_{\beta\gamma})$, and $n_{O}(\phi_{\alpha\gamma}) = 2n_{q}(\phi_{\alpha\gamma})$, the finiteness of counting follows from the strongly $s$-admissibility for diagram $T_{q} = (\Sigma, \alpha, \beta, \gamma, q)$.

Furthermore, by tracking the proof of [OS04b] Theorem 8.16], we have CFL$'$ flavor of associativity as described below.

Lemma 5.19. Given a strongly $\mathcal{G}$-admissible Heegaard quadruple $(\Sigma, \alpha, \beta, \gamma, \delta, O)$ with induced Heegaard triples $\mathcal{T}_{\alpha\beta\gamma}$ and $\mathcal{T}_{\alpha\gamma\delta}$ satisfying the same condition in Lemma 5.17, then we have the following equality

$$\sum_{s \in \mathcal{G}} F_{\alpha\gamma\delta}(F_{\alpha\beta\gamma}(\theta_{\alpha\beta} \otimes \theta_{\beta\gamma}; s_{\alpha\beta\gamma}) \otimes \theta_{\gamma\delta}; s_{\alpha\gamma\delta})$$

$$= \sum_{s \in \mathcal{G}} F_{\alpha\beta\delta}(\theta_{\alpha\beta} \otimes F_{\beta\gamma\delta}(\theta_{\beta\gamma} \otimes \theta_{\gamma\delta}; s_{\beta\gamma\delta}); s_{\alpha\beta\delta}).$$

Here $\theta_{\alpha\beta}, \theta_{\beta\gamma}$ and $\theta_{\gamma\delta}$ lie in HFL$'(Y_{\alpha\beta}),$ HFL$'(Y_{\beta\gamma})$ and HFL$'(Y_{\gamma\delta})$ respectively, and $\mathcal{G}$ is a $\delta H^1(Y_{\beta\gamma}) + \delta H^1(Y_{\alpha\gamma})$ orbit of a fixed Spin$^c$-structure over $X_{\alpha\beta\gamma\delta}$. 


6. Band moves and triangle maps

6.1. Assumptions. In this section, we assume that the bipartite link cobordisms or bipartite disoriented link cobordisms satisfy the following conditions:

- The four-manifold $W$ is a product $Y \times I$.
- The inclusion $\mathcal{F} : (F, \partial F) \to (W, \partial W)$ induces a trivial map $\mathcal{F}_* : H_2(F, \partial F) \to H_1(W, \partial W)$. Consequently, the bipartite links in the boundary three-manifolds are null-homologous.

**Lemma 6.1.** Given a $\beta$-band move $B^\beta$ from bipartite link $L_{\alpha\beta}$ in $Y$ to a bipartite link $L_{\alpha\gamma}$ in $Y$, there exists a strongly $s$-admissible Heegaard triple (or standard Heegaard triple) subordinate to this band move $B^\beta$. Here $s$ is a Spin$^c$-structure over $X_{\alpha\beta\gamma}$ such that the restriction $s_{\alpha\beta}$ on $Y_{\alpha\beta} \cong Y$ is equal to its restriction on $Y_{\alpha\gamma} \cong Y$.

**Proof.** Recall that in the final step of the construction of standard Heegaard triple in Theorem 4.2, we can do finger moves for the diagram $H'_{\alpha\beta}$ without crossing the basepoints $O$ and away from the disk neighborhood $D$, such that the diagram $H_{\alpha\beta}$ is strongly $s$-admissible. Then the result follows.

As a corollary of Lemma 4.3 and [OS04b, Proposition 7.2], we have the following lemma.

**Lemma 6.2.** Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two strongly $s$-admissible Heegaard triples subordinate to the same band move $B^\beta$ from a bipartite link $L_{\alpha\beta}$ to a bipartite link $L_{\alpha\gamma}$. Then these two triples can be connected by a sequence moves in Lemma 4.3, such that in each intermediate step, Heegaard triple is strongly $s$-admissible.

Suppose the bipartite disoriented link cobordism $(W, \mathcal{F}, F_\alpha, F_\beta, A, A_\Sigma)$ is a $B^\beta$-band move next to basepoints $(o_i, o_j)$ (cf. Figure 25). From the discussion in Section 3.4 we know that it determines a unique bipartite link cobordism $(W, \mathcal{F}, F_\alpha, F_\beta, A, A_\Sigma)$. By Theorem 4.2, there exists a Heegaard triple $\mathcal{T}$ subordinate to this bipartite link cobordism. As the bipartite link $L_{\alpha\beta}$ and $L_{\alpha\gamma}$ in $Y_{\alpha\beta} \cong Y$ and $Y_{\alpha\gamma} \cong Y$ are null-homologous, we can associate them with two $\mathbb{Z}$-graded curved chain complex $CFBL^-(\mathcal{H}_{\alpha\beta}, s)$ and $CFBL^-(\mathcal{H}_{\alpha\gamma}, s)$ respectively, where $s$ is a torsion Spin$^c$-structure of $Y$. For the bipartite link $L_{\beta\gamma}$ in $Y_{\beta\gamma} \cong \#^n(S^1 \times S^2)$, by Corollary 5.14, we have also has a $\mathbb{Z}$-graded curved chain complex $CFBL^-(\mathcal{H}_{\beta\gamma}, s_0)$.

6.2. Distinguishing the top grading generators. Let $B^\beta$ be a band move from bipartite link $L_{\alpha\beta}$ to $L_{\alpha\gamma}$. If the surface $(F, \partial F)$ is orientable, for the diagram $(\Sigma, \beta, \gamma, w, z)$, the top $\delta$-grading generators of $CFL'_{\Sigma}(\mathcal{H}_{\beta\gamma}, s_0)$ can be distinguished by using $gr_w$ and $gr_z$. Clearly, the choice of generators depends on the coloring data. See [Zem16a, Section 6] for details.

Generically, as the surface $(F, \partial F)$ can be non-orientable, we do not have $gr_w$ or $gr_z$-grading. Therefore we need extra data to distinguish the top grading generators in the homology $HFL'(\mathcal{H}_{\beta\gamma}, s_0)$. Actually, the extra data we need are included in $A$. In fact, the band $B^\beta$ can either be next to a pair of basepoints $(o_i, o_j)$ or the other pair $(o'_i, o'_j)$ (See Figure 25). Here, we denoted a $\beta$-band by $B^{\beta, o_i, o_j}$ if it is next to the pair of basepoints $(o_i, o_j)$.

Recall that, for the construction of Heegaard triple in Theorem 4.2, we set the band $B^\beta$ being next to a pair of basepoints $(o_i, o_j)$ of $L_{\alpha\beta}$. If $B^\beta$ is of type II, we can choose the other pair $(o'_i, o'_j)$ and construct a triple $\mathcal{T}'$. Then both of the triples $\mathcal{T}$ are subordinate to the band move $B^\beta$, and can be connected by Heegaard moves in Lemma 4.3.
Lemma 6.3. Given a band move $B_{o_i,o_j}^{\beta}$ of a bipartite disoriented link and a strongly $s$-admissible Heegaard triple $\mathcal{T}$ subordinate to it, there is a chain complex $CFL'_{o_i,o_j}(\mathcal{H}_{\beta\gamma}, s_0)$, such that the top $\delta$-grading $\mathbb{F}_2$-submodule of its homology is one-dimensional.

Proof. We define the module

$$CFL'_{o_i,o_j}(\mathcal{H}_{\beta\gamma}, s_0) = CFBL(\mathcal{H}_{\beta\gamma}, s_0) \otimes_{\mathbb{F}_2[U_1, \ldots, U_{2n}]} \mathbb{F}_2[U,U_1, \ldots, U_{2n}] / I,$$

where $I$ is an ideal generated by $(U_k - U_i)$, $k \neq i,j$. By [Zem16a, Lemma 2.1], the endomorphism $\partial$ is a differential. Without loss of generality, if the triple is standard, we have

$$\partial \theta = (U_i + U_j) \theta'$$
$$\partial \theta' = 0,$$

where $\theta$ and $\theta'$ are two top grading generators of $CFL'_{o_i,o_j}(\mathcal{H}_{\beta\gamma}, s_0)$ or $CFBL'_{o_i,o_j}(\mathcal{H}_{\beta\gamma}, s_0)$. $\square$

Let $\theta^{o_i,o_j}$ be the generator $\theta$ in the proof of Lemma 6.2. We have the following theorem.

Theorem 6.4. Given a $B^3$-band move next to a pair of basepoints $(o_i,o_j)$ of a bipartite disoriented link $L_{o\beta}$ in $Y$, and a torsion Spin$^c$-structure $s$ of $Y$, we can construct a $\mathbb{Z}$-filtered chain map

$$\sigma^{o_i,o_j} : CFL'(Y,L_{o\beta}, s) \to CFL'(Y,L_{o\beta}, s),$$

which is well-defined up to $\mathbb{Z}$-filtered chain homotopy. This construction is independent of the choices of Heegaard triples. Therefore, it gives rise to a map on homology:

$$\sigma^{o_i,o_j} : HFL'(Y,L_{o\beta}, s) \to HFL'(Y,L_{o\beta}, s),$$

which is an invariant of this bipartite disoriented link cobordism.

Proof. We define the chain map $\sigma^{o_i,o_j} : CFL'(\mathcal{H}_{o\beta}, s) \to CFL'(\mathcal{H}_{o\beta}, s)$ by

$$\sigma^{o_i,o_j} = f_{o\beta\gamma}(x \otimes \theta^{o_i,o_j} ; s_{o\beta\gamma}).$$

Here $\theta^{o_i,o_j}$ is the generator defined in the proof of Lemma 6.3, and the Spin$^c$-structure $s_{o\beta\gamma}$ restricts on $Y$ and on $\#^3(S^1 \times S^2)$ are $s$ and $s_0$ respectively.

Similar to the proof of [Juh16, Theorem 6.9], using the associativity which we proved in Lemma 5.19 the map $\sigma^{o_i,o_j}$ is well-defined up to $\delta$-filtered chain homotopy, i.e. it is independent of the choices of Heegaard triple $\mathcal{T}$. Thus, it gives rise to a homomorphism on homology level. $\square$
6.3. The relation of generators. Suppose \( B^{\beta,o_i,o_j} \) is a band move from bipartite disoriented link \( (\mathcal{L}_0, O) \) to bipartite disoriented link \( (\mathcal{L}_1, O) \). Then there exists an inverse band move \( (B^{\beta,o_i,o_j})^{-1} \) from \( (\mathcal{L}_1, O) \) to \( (\mathcal{L}_0, O) \). See Figure 26 for an example of a composition of two type II band move with surface \((F, \partial F)\) orientable.

Let \( T_{\alpha\beta\gamma} = (\Sigma, \alpha, \beta, \gamma, O) \) be a strongly \( s\)-admissible Heegaard triple subordinate to the band move \( B^{\beta,o_i,o_j} \). We set the curves \( \delta \) to be small Hamiltonian isotopies of the curves \( \alpha \). Then the triple \( T_{\alpha\gamma\delta} = (\Sigma, \alpha, \gamma, \delta, O) \) is subordinate to the inverse band move \( (B^{\beta,o_i,o_j})^{-1} \).

We have the following lemma for the induced Heegaard triple \( T_{\beta\gamma\delta} \).

**Lemma 6.5.** Suppose \((\Sigma, \alpha, \beta, \gamma, \delta, O)\) is a strongly \( s\)-admissible Heegaard quadruple, with the induced Heegaard triples \( T_{\alpha\beta\gamma} \) subordinate to \( B^{\beta,o_i,o_j} \) and \( T_{\alpha\gamma\delta} \) subordinate to \((B^{\beta,o_i,o_j})^{-1}\). For the top grading generators \( \Theta_{\beta\gamma} \in HFL'(H_{\beta\gamma}), \Theta_{\beta\gamma} \in HFL'(H_{\beta\gamma}) \) and \( \Theta_{\gamma\delta} \in HFL'(H_{\gamma\delta}) \), we have the following relation:

\[
(6.6) \quad f_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}) = U \cdot \Theta_{\beta\delta}
\]

**Proof.** Without loss of generality, we assume the Heegaard triples \( T_{\beta\gamma\delta} \) is a standard Heegaard triple as shown in Figure 27. The three small black dots are in the generators...
\(\Theta_{\beta\gamma}, \Theta_{\beta\delta}\) and \(\Theta_{\gamma\delta}\) respectively. The shaded area in Figure 27 represents a triangle \(\Delta\) in \(\pi_2(\Theta_{\beta\gamma}, \Theta_{\gamma\delta}, \Theta_{\beta\delta})\) with Maslov index equal to zero. Similar to the arguments in [OS04b, Section 9], by checking the grading of those generators and periodic domains, there is no other triangle with two vertices \(\Theta_{\beta\gamma}, \Theta_{\gamma\delta}\) and Maslov index zero. As \(n_\Theta(\Delta) = 1\), we get a \(U\) before \(\Theta_{\beta\delta}\) in Equation 6.6.

**Lemma 6.7.** Let \(\sigma_{o_i}^{o_j}\) be the chain map induced by \(B^{\beta,o_i,o_j}_\beta\) defined in Theorem 6.4. Then, there exists a chain map induced by the inverse of \(B^{\beta}_\beta\):

\[\tau_{o_i}^{o_j} : CFL'(Y, L_{\alpha\gamma}, s) \to CFL'(Y, L_{\alpha\beta}, s),\]

such that

- the composition \(\tau_{o_i}^{o_j} \circ \sigma_{o_i}^{o_j}\) is chain homotopic to:
  \[U : CFL'(Y, L_{\alpha\beta}, s) \to CFL'(Y, L_{\alpha\beta}, s)\]

- the composition \(\sigma_{o_i}^{o_j} \circ \tau_{o_i}^{o_j}\) is chain homotopic to:
  \[U : CFL'(Y, L_{\alpha\gamma}, s) \to CFL'(Y, L_{\alpha\gamma}, s)\]

**Proof.** Consider a strongly \(s\)-admissible Heegaard quadruple constructed from the composition \((B^{\beta,o_i,o_j}_\beta)^{-1} \circ B^{\beta,o_i,o_j}_\beta\) of cobordisms as in Lemma 6.5. The composition is cobordism from \(L_{\alpha\gamma}\) to itself. The map \(\sigma_{o_i}^{o_j}\) is defined to be \(f_{\alpha\beta\gamma}(\cdot \otimes \Theta_{\beta\gamma})\), where \(\Theta_{\beta\gamma}\) is the generator defined in Lemma 6.3. Similarly, we define \(\tau_{o_i}^{o_j}\) to be \(f_{\alpha\gamma\delta}(\cdot \otimes \Theta_{\gamma\delta})\), where \(\Theta_{\beta\delta}\) is also the generator defined in Lemma 6.3.

Then the composition,

\[\tau_{o_i}^{o_j} \circ \sigma_{o_i}^{o_j} = f_{\alpha\gamma\delta}(\cdot \otimes (\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}))\]

\[= f_{\alpha\beta\delta}(\cdot \otimes (f_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta})))\]  (by Lemma 5.19)

\[= f_{\alpha\beta\delta}(\cdot \otimes U \cdot \Theta_{\beta\delta})\]  (by Lemma 6.5)

\[= U f_{\alpha\beta\delta}(\cdot \otimes \Theta_{\beta\delta})\]

From the construction of the quadruple, we know that the curves \(\delta\) are just small Hamiltonian isotopy of \(\beta\) without crossing any basepoints. As the small triangle map \(f_{\alpha\beta\delta}(\cdot \otimes \Theta_{\beta\delta})\) is chain homotopic to the nearest point map, the composition \(\tau_{o_i}^{o_j} \circ \sigma_{o_i}^{o_j}\) is actually \(\mathbb{Z}\)-filtered chain homotopic to the map \(U\).

**Remark 6.8.** If the band \(B^{\beta,o_i,o_j}_\beta\) is of type II, we can construct the other composition of cobordism \((B^{\beta,o_i,o_j}_\beta)^{-1} \circ B^{\beta,o_i,o_j}_\beta\), where the basepoints \(o_i\) and \(o_j\) are connected by a component of \(L_{\beta}\), the basepoints \(o_i\) and \(o_j\) are connected by another component of \(L_{\beta}\). See Figure 28 for an example of the above composition of cobordisms with surface \((F, \partial F)\) orientable. Then this cobordism induces a map:

\[\tau_{o_i}^{o_j} \circ \sigma_{o_i}^{o_j} = f_{\alpha\beta\delta}(\cdot \otimes \Theta_{\beta\delta}')\).

Here the generator \(\Theta_{\beta\delta}'\), which has the same \(\delta\)-grading as \(U \Theta_{\beta\delta}\), is shown in Figure 29.
Figure 28. The composition of cobordisms $(B_{\beta,o_i,o_j})^{-1} \circ B_{\beta,o_i,o_j}$

Figure 29. The composition of cobordisms $(B_{\beta,o_i,o_j})^{-1} \circ B_{\beta,o_i,o_j}$

Figure 30. The standard grid move for the band move $F_{B_{\beta,o_i,o_j}}$
6.4. A comparison with Ozsváth, Stipsicz and Szabó’s definition of band move maps. In this subsection, we compare our band move maps $F_{B^{\beta,o_i,o_j}}$ with the band move maps defined by Ozsváth, Stipsicz and Szabó in [OSS15].

**Definition 6.9.** We say that a band $B$ for links (or bipartite links, bipartite disoriented links resp.) is an **oriented band**, if the number of link components changes after the band move $B$. Otherwise, we say that $B$ is an **unoriented band**.

Recall that in [OSS15], for a band move from a link in $S^3$ to a link in $S^3$, they construct a standard grid move as shown in Figure 30. The band move is represented by switching the markings in a grid diagram. Furthermore, we can require that switching corresponds to the unoriented resolution of a positive crossing.

To establish the relation between the standard grid move above and the Heegaard triple subordinate to a band move, we need the following lemma.

**Lemma 6.10.** Let $B^{\beta,o_i,o_j}$ be a band move from a bipartite disoriented link $(L, O)$ to $(L', O)$. After a sequence of quasi-stabilizations and Heegaard moves without changing or crossing the basepoints, we can find a standard Heegaard triple $T_{\alpha \beta \gamma}$ such that:

- the induced Heegaard diagram $H_{\alpha \beta}, H_{\alpha \gamma}$ are grid diagrams.
- the grid moves switching the markings $o_i$ and $o_j$ corresponds to a resolution of a positive crossing, as shown in Figure 30.

**Proof.** Recall that in Theorem 4.2, we can require the moves $H_{\alpha \beta}$ to $H_{\alpha \gamma}$ be as shown in the bottom right of Figure 17. The remaining part of the proof follows from Lemma 4.3. □

We call the grid move in Figure 30 the **standard grid move** representing $B^{\beta,o_i,o_j}$. Recall that in [OSS15], the band move maps are defined as follows. Let $A$ and $B$ be the subset of the curve $\beta_n$ as shown in Figure 30. For a grid state $x \in T_{\alpha} \cap T_{\beta}$, the map $\nu(x)$ is

$$\nu(x) = \begin{cases} U \cdot x & \text{if } x \cap A \neq \emptyset \\ x & \text{if } x \cap A = \emptyset. \end{cases}$$
For our definition, the band map $F_{B,\beta_1,\beta_2}^a(\cdot)$ is defined to be the triangle map $F_{a,b_1,b_2}(\cdot \otimes \Theta_{b_1,b_2})$, where $\Theta_{b_1,b_2}$ is the top grading generator determined by $B_{a,b_2,o_1,o_2}$ as shown in Figure 31.

For a intersection $x \in B_0 \cap a$, we let $x'$ be the closest intersection to $x$ in $B_1 \cap a$. We have the following observation:

- if $x \in A$, the small triangle with endpoints $x,x',\Theta_{b_1,b_2}$ intersects the basepoint set $O$ once;
- if $x \notin A$, the small triangle with endpoints $x,x',\Theta_{b_1,b_2}$ does not intersect any basepoints.

Based on these observations, we have the following proposition.

**Proposition 6.11.** For a standard grid move representing an unoriented band $B = B_{a,b_1,b_2}$, we have the following:

- The map $\nu$ defined in [OSS15] agrees with the triangle map $F_B$.
- The map $\nu'$ defined in [OSS15] agrees with the triangle map $F_{B^{-1}}$.

**Proof.** As $\gamma_n$ is a small isotopy of $\beta_n$ crossing basepoints $a_1,a_2$, we can identify the map $\nu$ with the continuation map induced by the Hamiltonian isotopy. This follows from the proof of [OS10, Theorem 6.6]. The only difference is that, in our case, the continuation map may cross the basepoints set $O$, and we keep track of that with variable $U$. On the other hand, the equivalence between the continuation map and triangle maps in [Lip06, Proposition 11.4] also works in our case.

Therefore, based on these two results and the observations above, we conclude that the map $\nu$ is filtered chain homotopic to the triangle map $F_B$. The equivalence between $\nu'$ and $F_{B^{-1}}$ is similar.

**Remark 6.12.** In a similar vein, we can also identify the oriented band move maps $\sigma$ and $\mu$ in [OSS15] with $F_B$ and $F_{B^{-1}}$. Furthermore, the formulas $\nu' \circ \nu = U, \nu \circ \nu' = U$ in [OSS15, Proposition 5.7] (and similarly, the formulas $\sigma \circ \mu = U$ and $\sigma \circ U = U$ in [OSS15, Proposition 5.1]) agree with the formulas in Lemma 6.7.

### 6.5. A comparison with Manolescu’s definition of unoriented band move maps.

Recall that, some band maps were constructed in the proofs of the unoriented skein exact triangle in [Man07] using special Heegaard diagrams. In the construction, the band move is actually a type-II $\beta$-band move, and the Heegaard triple $T_{a,b_1,b_2}$ is standard. The Heegaard triple shown in [Man07, Figure 6] matches the Figure 39.

Let $\Theta,\Theta'$ be the two top $\delta$-grading generator of $T_a \cap T_b$. In [Man07], the map $f_0 : C\text{F}L(L_0) \to C\text{F}L(L_1)$ is defined by the triangle map:

$$f_0(\cdot) = f_{a,b_1,b_2}(\cdot \otimes (\Theta + \Theta')).$$

The map $f_0$ induces a map $f'_0$ on the unoriented link Floer chain complex. In fact, $f'_0 : C\text{F}L'(L_0) \to C\text{F}L'(L_1)$ is given by:

$$f'_0(\cdot) = f_{a,b_1,b_2}(\cdot \otimes (\Theta + \Theta')).$$

**Proposition 6.15.** Suppose a type $II$ $\beta$-band move $B^\beta$ is next to either the pair of basepoints $(a_1,a_2)$ or the pair of basepoints $(a_3,a_4)$. For the band map defined in [6.14] we have:

$$f'_0 = F_{B^\beta,a_1,a_2} + F_{B^\beta,a_3,a_4}.$$
Figure 32. A Heegaard triple subordinate to a saddle from trefoil to unknot

Figure 33. Local diagram for a Heegaard triple

Proof. Immediate from the definition.

Remark 6.16. Notice that, the definition of \( f'_0 \) does not depend on the one-manifold \( A \) on the bipartite disoriented link cobordism. Therefore, the map \( f_0 \) is a well-defined map for band moves \( B^3 \) between bipartite links.

6.6. An example: a bipartite link cobordism from trefoil to unknot. Let \( T \) shown in Figure 32 be a Heegaard triple subordinate a type I \( \beta \)-band move. The Heegaard diagram \( H_{\alpha\beta} \) is compatible with the trefoil \( K_0 \subset S^3 \); the diagram \( H_{\beta\gamma} \) is compatible with a homologically even bipartite link \( K_1 \subset S^1 \times S^2 \); the diagram \( H_{\alpha\gamma} \) is compatible with the unknot \( U \) in \( S^3 \).

Let \( a, a', b, b', c, c' \) be the intersections shown in Figure 33, which is a local diagram of Figure 32.

For the chain complex \( CFL'(H_{\alpha\beta}) \), we have:
\[
\partial a = U(b + c) \quad \text{and} \quad \partial b = \partial c = 0.
\]

For the chain complex \( CFL'(H_{\alpha\gamma}) \), we have:
\[
\partial a' = (b' + c') \quad \text{and} \quad \partial b = \partial c = 0.
\]

It is easy to check that, the cobordism map \( \sigma \) acting on \( a, b, c \) are given by the following:
\[
\sigma(a) = Ua', \sigma(b) = b' \quad \text{and} \quad \sigma(c) = c'.
\]
Furthermore, by computation, the map $\tau : CFL'(U) \to CFL'(K_0)$ defined in Lemma 6.7 sends $a'$ to $a$, $b'$ to $Ub$ and $c'$ to $Uc$. Therefore, the composition $\sigma \circ \tau$ (or $\tau \circ \sigma$ resp.) is exactly the map $U$. 
Figure 34. An example of quasi-stabilization $S_{+}^{\beta,o_i}$. The dividing point $q_i$ is in the center of the arc connecting $o_i$ and $o_j$. The four new points $(q_s, o, p_s, o')$ appear in order on the arc between $o_i$ and $q_i$.

7. Quasi-stabilizations and Disk-stabilizations

In this subsection, we recall some facts about quasi-stabilization from [MO10], [Zem16b] and [Zem16a]. For convenience, we focus on $\beta$-quasi-stabilizations. Similar results hold for $\alpha$-quasi-stabilizations.

7.1. Topological facts about quasi-stabilizations. Recall that in Section 3.3, a quasi-stabilization $W_{qs}$ from bipartite disoriented link $(L_0, O_0)$ to a bipartite disoriented link $(L_1, O_1)$ is an elementary bipartite link cobordism such that:

- The bipartite disoriented links $(L_0, O_0)$ and $(L_1, O_1)$ determine the same link $L$ in $Y$.
- The dividing set $(p_1, q_1)$ of $L_1$ is the union $(p_0 \cup p_s, q \cup q_s)$, where $(p_0, q_0)$ is the dividing set of $L_0$.
- The basepoints set $O_1$ is the union $O_0 \cup \{o, o'\}$.

We denote a quasi-stabilization by $S_{+}^{\beta,o_i}$ (or in short, by $S^{\beta,o_i}$), if the four new points $(q_s, o, p_s, o')$ appear in order on $L$ and are between the point $(o_i, q_i)$ in $U_\beta$.

Starting from a Morse function $f_0$ which is compatible with $(L_0, O_0)$, we can modify $f_0$ inside a three-ball $D \subset Y$ to get a Morse function $f_1$ compatible with $(L_1, O_1)$. In fact, we can pick up a three-ball $D$ inside $Y \setminus \text{Crit}(f)$, such that:

- The three ball $D$ contains the arc connecting $(q_s, o')$.
- The intersection $D \cap \Sigma$ is a disk contained in $\Sigma \setminus (\alpha \cup \beta)$.

We add a pair of index zero and three critical points at $p_s$ and $q_s$ respectively. Then we replace the disk $D \cap \Sigma$ with a disk $\overline{D}$ which intersects $L$ at the two basepoints $(o, o')$. We also need to introduce an index two critical point inside $U_\beta \cap D$ and an index one critical point inside $U_\alpha \cap D$, such that the gradient flow of the modified Morse function $f_1$ agrees with $f$ on the boundary $\partial D$ of the three ball $D$. The unstable manifold of the new index two critical points intersects $\partial \overline{D}$ at two point. The $\alpha$-circle corresponded to the new index one critical point is inside the disk $\overline{D}$. This modification from $f_0$ to $f_1$ is shown in Figure 34.

Now we consider this construction at the level of Heegaard diagram. Given a Heegaard diagram $H = (\Sigma, \alpha, \beta, O_0)$ compatible with $(L_0, O_0)$, we choose a point $p$ on $\Sigma \setminus (\alpha \cup \beta)$ together with a small disk neighborhood $D_p$ and a circle $\beta_s$ through $p$ without intersecting
\( \beta \). We replace \( D_p \) by another disk \( \overline{D} \) and introduce two basepoint \((o, o')\) on \( \overline{D} \). Next, we extend the \( \beta_s \) into the disk \( \overline{D} \) and get a closed curve separating \((o, o')\) on \( \Sigma = (\Sigma \setminus D_p) \cup \overline{D} \).

From the Morse function viewpoint, as we require the curve \( \beta_s \) intersect the projection of \( L \) at one point, this extension of \( \beta_s \) over \( \overline{D} \) is unique up to isotopy. Then we set the new alpha circle \( \alpha_s \) is parallel to \( \partial \overline{D} \) and get a Heegaard diagram \( \overline{H} = (\Sigma, \alpha \cup \alpha_s, \beta \cup \beta_s, O \cup \{o, o'\}) \), where \( \beta_s \) is the extension of \( \beta_s \) over \( \Sigma \). See Figure 34 for this construction.

**Remark 7.1.** The construction of Heegaard diagram \( \overline{H} \) does not depend on the curve \( A \) of \( \mathbb{W}_{qs} \).

### 7.2. Choice of generators

In this subsection, we will define a \( \mathbb{Z} \)-filtered curved chain homomorphism:

\[
F_{\mathbb{W}_{qs}} : CFBL(L_0, O_0) \to CFBL(L_1, O_1).
\]

As in Section 7.1, suppose that the quasi-stabilization \( \mathbb{W}_{qs} \) is \( S_+^{\beta, \alpha_i} \). In other words, the four new points \((q_s, o, p_s, o')\) appear in order on \( L \) and are between the point \((o_i, q_i)\) in \( U_{\beta} \).

For the Heegaard diagram \( \overline{H} \), the \( \alpha_s \) intersects \( \beta_s \) at two points. If we close the diagram \( (\overline{D}, \alpha_s, \beta_s \cap \overline{D}, o, o') \), we will get a diagram \( (S^2, \alpha_s, \beta_s \cap \overline{D}, o, o') \). We denote by \( x^+ \) the intersection with the highest \( gr_o \)-grading, and by \( x^- \) the other intersection. The generators of \( CFBL(\overline{H}) \) are of the form \( x \times x^\pm \), where \( x \) is a generator of \( CFBL(\overline{H}) \).

We define \( F_{S^{\beta, \alpha_i}} : CFBL(L_0, O_0) \to CFBL(L_1, O_1) \) by

\[
F_{S^{\beta, \alpha_i}}(x) = x \times x^+.
\]

If we reverse a quasi-stabilization \( S_+^{\beta, \alpha_i} \), we will get a quasi-destabilization \( S_-^{\beta, \alpha_i} \) from \((L_1, O_1)\) to \((L_0, O_0)\). Then we define \( F_{S_-^{\beta, \alpha_i}} : CFBL(L_1, O_1) \to CFBL(L_0, O_0) \) by

\[
F_{S_-^{\beta, \alpha_i}}(x) = x \times x^-.
\]

\[
F_{S_-^{\beta, \alpha_i}}(x) = x.
\]

**Remark 7.5.** The construction of \( F_{S^{\beta, \alpha_i}} \) is similar to Zemke’s definition \( S_+^{\beta, \alpha} \) and \( T_+^{\beta, \alpha} \) in [Zem16b]. Instead of colorings, the curves \( A \) of \( \mathbb{W}_{qs} \) will provide us extra data to distinguish the generators.

We have the following lemma about the invariance of \( F_{S_+^{\beta, \alpha_i}} \) and \( F_{S_-^{\beta, \alpha_i}} \).

**Lemma 7.6.** The map \( F_{S_+^{\beta, \alpha_i}} \) and \( F_{S_-^{\beta, \alpha_i}} \) is well-defined up to \( \mathbb{Z} \)-filtered curved chain homotopy and is independent of the choices of Heegaard diagram \( \overline{H} \).

**Proof.** One could prove the following invariance by tracking the proof of [Zem16b Theroem A].

**Proposition 7.7.** Suppose that \( \mathbb{W}^1 = S^{\beta, \alpha_1} \circ S^{\beta, \alpha_1} \) and \( \mathbb{W}^2 = S^{\beta, \alpha_1} \circ S^{\beta, \alpha_2} \) are two bipartite disoriented link cobordism from \((L, \emptyset)\) to \((L, \emptyset)\), where \( \alpha_1 \) and \( \alpha_2 \) are two adjacent basepoints. See Figure 35 for the local picture of \( \mathbb{W}^1 \) and \( \mathbb{W}^2 \). For the two cobordism maps \( F_{\mathbb{W}^1} : CFL(L, \emptyset) \to CFL(L, \emptyset) \), we have

\[
F_{\mathbb{W}^1} \cong 0
\]

and \( F_{\mathbb{W}^2} \cong \text{Id} \).
7.3. Disk-stabilizations. Before we provide the proof for Theorem 1.1, we recall the definition of the map induced by a disk-stabilization/destabilization from [Zem16a, Section 7.3].

Let \((L, \mathcal{O})\) be a bipartite disoriented link in \(Y\) and \((\mathcal{U}, o_1 \cup o_2)\) be a bipartite disoriented unknot with two basepoints. Suppose \(\mathcal{U}\) bounds a disk \(D\) such that \((D \cap L) = \emptyset\). Then we can form a new bipartite disoriented link \((L, \mathcal{O}) \cup (\mathcal{U}, o_1 \cup o_2)\). We call this process a disk-stabilization and its inverse a disk-destabilization (its cobordism picture is shown in Figure 5). For convenience, we denote by \(D^+\) a disk-stabilization and by \(D^-\) a disk-destabilization.

We can pick a Heegaard diagram \((\Sigma, \alpha, \beta, \mathcal{O})\) for \((L, \mathcal{O})\) such that \((D \cap \Sigma) \cap (\alpha \cup \beta) = \emptyset\). By replacing a small disk neighborhood of \((D \cap \Sigma)\) on \(\Sigma\) with the picture shown in Figure 36, we get a Heegaard diagram \(\hat{H} = (\Sigma, \alpha \cup \alpha_0, \beta \cup \beta_0, \mathcal{O} \cup o_1 \cup o_0)\) for \((L, \mathcal{O}) \cup (\mathcal{U}, o_1 \cup o_2)\). The generators of \(CFL'(\hat{H})\) are of the form \(x \times \theta^+\) or \(x \times \theta^-\), where \(x \in CFL'(H)\), \(\theta^+\) and \(\theta^-\) are the two new intersection of \(\alpha_0\) and \(\beta_0\). We define the map \(f_{D^+} : CFL'(H) \to CFL'(\hat{H})\) by setting

\[
f_{D^+}(x) = x \times \theta^+,
\]
and the map \(f_{D^-} : CFL'(\hat{H}) \to CFL'(H)\) by setting

\[
f_{D^-}(x \times \theta^+) = 0 \quad \text{and} \quad f_{D^-}(x \times \theta^-) = x.
\]
From the discussion [Zem16a, Section 7.1], we know that $f_{D^+}$ and $f_{D^-}$ induce well-defined maps $F_{D^+}$ and $F_{D^-}$ on unoriented link Floer homology $HFL'$. 
8. COMMUTATIONS

We divide this section into two parts. In the first part (Subsection 8.1 and 8.2), we show that changing the ordering of two critical points of $\pi|_F$ or $\pi|_A$ does not affect the cobordism maps at the level of $CFL'$. The results from this part will be used to show the well-definedness of the cobordism maps on $CFL'$ constructed from a given bipartite disoriented link cobordism.

In the second part (Subsection 8.4 and 8.5), we provide a relation between the maps induced by $\alpha$-band moves and $\beta$-band moves, and a relation between the maps induced by $\alpha$-quasi-stabilizations and $\beta$-quasi-stabilizations. These two relations will be used to show that the cobordism maps on $CFL'$ constructed from bipartite dioriented link cobordisms are independent of the motion of basepoints (the data (D3)). Hence we get well-defined maps on $CFL'$ for disoriented link cobordism.

8.1. Commutation between $\alpha$-band moves and $\beta$-band moves. Based on Zemke’s work in [Zem16a, Section 7.3], by constructing a Heegaard quadruple, we will show the commutation between $\alpha$-band moves and $\beta$-band moves at the level of $CFL'$.

Suppose $B^{\alpha,i,o}_{j,o'}$ and $B^{\beta,i,o}_{j,o}$ are two disjoint band on bipartite disoriented link $(L_1, O)$. Then we have four elementary bipartite disoriented link cobordisms:

- the band move $B^{\alpha,i,o}_{j,o'}$ from $(L_0, O)$ to $(L_1, O)$.
- the band move $B^{\alpha,i,o}_{j,o'}$ from $(L_2, O)$ to $(L_3, O)$.
- the band move $B^{\beta,i,o}_{j,o}$ from $(L_0, O)$ to $(L_2, O)$.
- the band move $B^{\beta,i,o}_{j,o}$ from $(L_1, O)$ to $(L_3, O)$.

The two composition of bipartite disoriented link cobordisms $B^{\beta,i,o}_{j,o} \circ B^{\alpha,i,o}_{j,o'}$ and $B^{\alpha,i,o}_{j,o'} \circ B^{\beta,i,o}_{j,o}$ are isomorphic. Both of the composition are from the bipartite disoriented link $(L_0, O)$ to $(L_3, O)$.

**Lemma 8.1.** There exists a Heegaard quadruple $(\Sigma, \alpha', \alpha, \beta, \beta', O)$ such that,

- the induced Heegaard triple $T_{\alpha',\alpha,\beta}$ is subordinate to the band move $B^{\alpha,i,o}_{j,o'}$ from $(L_0, O)$ to $(L_1, O)$.
- the induced Heegaard triple $T_{\alpha',\alpha,\beta'}$ is subordinate to the band move $B^{\alpha,i,o}_{j,o'}$ from $(L_2, O)$ to $(L_3, O)$.
- the induced Heegaard triple $T_{\alpha',\alpha',\beta}$ is subordinate to the band move $B^{\beta,i,o}_{j,o}$ from $(L_0, O)$ to $(L_2, O)$.
- the induced Heegaard triple $T_{\alpha',\alpha',\beta'}$ is subordinate to the band move $B^{\beta,i,o}_{j,o}$ from $(L_1, O)$ to $(L_3, O)$.

**Proof.** This construction can be done in two steps.

**Step 1:** There exists a Heegaard decomposition such that the $\alpha$-band lies in $U_\alpha$ and the $\beta$-band lies in $U_\beta$.

Suppose we have a Morse function $f$ compatible with the bipartite disoriented link $(L_0, O)$. This induces a Heegaard decomposition $U_\alpha \cup_\Sigma U_\beta$ of the three manifold $Y$. The core of the band $B^{\alpha,i,o}_{j,o'}$ intersects $U_\beta$ with some arcs. For each of these arcs, one can replace the two disk neighborhoods of the two endpoints of the arc with a surface with two holes which does not intersects $L_0$ and any bands and get a new surface $\Sigma$. Furthermore, one can require $\Sigma$ cut the manifold into two handlebodies. One example of this process is shown in Figure 37. When we perform this for both bands, the $\alpha$ band will lie in $U_\alpha$ and $\beta$ band lies in $U_\beta$. 
Step 2: Using the techniques described in the proof of Theorem 4.2, we further stabilize the diagram $H_{\alpha\beta}$ of $(L_0, O)$ to implant the data of the two bands $B^{\beta,\alpha_i,\alpha_j}$ and $B^{\alpha_i,\alpha_j'}$ into the Heegaard diagram. Notice that, these two implantation processes (stabilizations and handleslides) happened in different handlebodies. Therefore, the two process are independent to each other. Finally, after Hamiltonian isotopies, we will get the desired Heegaard quadruple.

Similar to the proof of [Zem16a, Proposition 7.7], we have the following lemma.

Lemma 8.2. Let $B^{\alpha_i,\alpha_j'}$ and $B^{\beta,\alpha_i,\alpha_j}$ be the two disjoint band on bipartite disoriented link $(L_0, O)$. For the induced $\mathbb{Z}$-filtered chain maps $F_{B^{\beta,\alpha_i,\alpha_j}}$ and $F_{B^{\alpha_i,\alpha_j'}}$ at the level of $CFL'$, we have the following commutation:

$$F_{B^{\beta,\alpha_i,\alpha_j}} \circ F_{B^{\alpha_i,\alpha_j'}} \simeq F_{B^{\alpha_i,\alpha_j'}} \circ F_{B^{\beta,\alpha_i,\alpha_j}}.$$ 

Proof. Let’s consider the Heegaard quadruple constructed in Lemma 8.1. By the associativity we proved in Lemma 5.19, we have the following equality:

$$F_{B^{\beta,\alpha_i,\alpha_j}} \circ F_{B^{\alpha_i,\alpha_j'}} = f_{\alpha_i \beta' \alpha}(f_{\alpha_i \alpha' \beta}(\Theta_{\alpha \alpha'} \otimes -) \otimes \Theta_{\beta \beta'})$$

$$\simeq f_{\alpha_i \beta' \alpha}(\Theta_{\alpha \alpha'} \otimes f_{\alpha \beta' \beta'}(- \otimes \Theta_{\beta \beta'}))$$

$$= F_{B^{\alpha_i,\alpha_j'}} \circ F_{B^{\beta,\alpha_i,\alpha_j}}.$$ 

8.2. Commutation between $\beta$-band moves. In this subsection we will generalize the results in [Zem16a, Section 7.5] and show the commutation of the triangle maps induced by two distinct $\beta$-band move. As we introduce type I band moves, the proof will be slightly different from the proofs in [Zem16a, Section 7.5]. See Figure 38 for an example of the commutation between a type I $\beta$-band move and type II $\beta$-band move.

Suppose $B_1^\beta$ is a type II $\beta$-band on bipartite link $L_{\alpha \beta}$. Recall from Section 3.2 that, the set $L_\beta = L_{\beta,1}, \ldots, L_{\beta,n}$ is a $n$-tuple of arcs lie in $\beta$-handlebody. The endpoints of these arcs are exactly all the basepoints $O$ of $L_{\alpha \beta}$. As the band $B_1^\beta$ is of type II, the two ends of the bands should lie in two components $L_{\beta,i}, L_{\beta,j}$ of $L_\beta$. We call the ends of $L_{\beta,i}, L_{\beta,j}$ the four nearest basepoints adjacent to $B_1^\beta$. Following the proof of Theorem 4.2, we have a lemma below.

Lemma 8.3. Suppose $B^\beta$ is a type II $\beta$-band on bipartite link $L_{\alpha \beta}$. Let $T$ be a standard Heegaard triple subordinate to $B^\beta$. We can do $\beta$ and $\gamma$ handleslides of the triple without
crossing any basepoints $O$ to get a simplified Heegaard triple $T'$, with a disk neighborhood $D$ on the surface $\Sigma$ such that $D \cap H'_{\beta\gamma}$ is shown in Figure 39. Here, the four basepoints $o_1, o_2, o_3, o_4 \in O$ are the four nearest basepoints adjacent to the band $B^3$ the diagram $H'_{\beta\gamma}$ is the Heegaard diagram induced from $T'$.

**Proof.** Straightforward. \qed

**Remark 8.4.** Notice that the top right basepoint in Figure 39 can be any of the four nearest basepoints adjacent to band $B^3$.

We say that two type II $\beta$-band move $B^1_1$ and $B^2_2$ are away from each other if the two ends of $B^1_1$ and the two ends of $B^2_2$ lie in four different components of $L_{\beta}$.

**Lemma 8.5.** Suppose $B_1 = B^1_{1, o_i, o_j}$ and $B_2 = B^2_{1, o_i', o_j'}$ are two distinct bands on $L_{\alpha\beta}$, such that the composition of cobordisms $B_2 \circ B_1$ is isomorphic to $B_1 \circ B_2$. There exists a Heegaard quintuple $(\Sigma, \alpha, \beta, \gamma, \gamma', \delta)$ such that,

- The induced Heegaard triple $T_{\alpha\beta\gamma}$ is subordinate to the band move $B_1$ from $(L_0, O)$ to $(L_1, O)$. 
The induced Heegaard triple $\mathcal{T}_{\alpha\beta\gamma}$ is subordinate to the band move $B_2$ from $(L_2, O)$ to $(L_3, O)$.

The induced Heegaard triple $\mathcal{T}_{\alpha\beta\delta}$ is subordinate to the band move $B_2$ from $(L_0, O)$ to $(L_2, O)$.

The induced Heegaard triple $\mathcal{T}_{\alpha\gamma\delta}$ is subordinate to the band move $B_1$ from $(L_1, O)$ to $(L_3, O)$.

Furthermore, if $B_1$ and $B_2$ are of type II and away from each other, then there exists two distinct disk neighborhoods $D_1$ and $D_2$ on $\Sigma$, such that the four local diagram $D_1 \cap \mathcal{T}_{\beta\gamma\delta}$, $D_2 \cap \mathcal{T}_{\beta\gamma\delta}$, $D_1 \cap \mathcal{T}_{\beta\gamma'\delta}$, $D_2 \cap \mathcal{T}_{\beta\gamma'\delta}$ are the four diagrams shown in Figure 40 and Figure 41.

**Proof.** Recall in the proof of Theorem 4.2, we start from the Heegaard diagram of the bipartite link $L_{\alpha\beta}$ determined by $(L_0, O)$ and implant the data of the core of the bands by stabilize the Heegaard diagram. As the two $\beta$-band are distinct bands, the stabilizations and Dehn twists for implanting the data $B_1$ is independent to the stabilizations and Dehn
Figure 42. Construction of Heegaard quintuple. Here, $c_1$ and $c_2$ are the cores of the band $B_1$ and $B_2$. The dashed line $l_1$ and $l_2$ are the projection image on $\Sigma$ of $c_1$ and $c_2$. For convenience, we didn’t draw the curves $\gamma'$ and some other curves.

twists for implanting the data of $B_2$. Then we can get a desired Heegaard quintuple. One example of this process is shown in Figure 42.

Furthermore, if the two bands are of type II and away from each other, then we can start from the Heegaard triple subordinate to $B_1$ shown in Lemma 8.3, we have a disk region $D_1$ shown in Figure 39.

Now we can require that the stabilization and Dehn twist for $B_2$ happens away from the region $D_1$. After handleslide, we will get the desired Heegaard quintuple with the two disk region $D_1$ and $D_2$ shown in Figure 40 and Figure 41.

The following lemma is a result from the triangle computations of the bipartite disoriented links in $\#^n(S^1 \times S^2)$ induced by the above Heegaard quintuple.

**Lemma 8.6.** Suppose $B_1 = B_1^{\alpha_i, \alpha_j}$ and $B_2 = B_2^{\alpha_i', \alpha_j'}$ are two type II $\beta$-band away from each other. Then the induced map $F_{B_1}$ and $F_{B_2}$ (defined in Theorem 6.4) commutes with each other.

**Proof.** We construct a Heegaard quintuple by Lemma 8.5. We denote by $\Theta_{\beta \gamma}$ (and $\Theta_{\gamma' \delta}$ resp.) the generator determined by $B_1$ and by $\Theta_{\beta \gamma'}$ (and $\Theta_{\gamma \delta}$ resp.) the generator determined by $B_2$.

By the triangle calculation shown in Figure 40, we have:

$$f_{\beta \gamma \delta}(\Theta_{\beta \gamma} \otimes \Theta_{\gamma \delta}) = \Theta_{\beta \delta}.$$ 

Here $\Theta_{\beta \delta}$ is a top grading generator of $CFL'(H_{\beta \delta}, \delta)$. Similarly, by the triangle calculation shown in Figure 41, we have:

$$f_{\beta \gamma' \delta}(\Theta_{\beta \gamma'} \otimes \Theta_{\gamma' \delta}) = \Theta_{\beta \delta}.$$ 

Notice that the generator $\Theta_{\beta \delta}$ in Figure 40 is exactly the one shown in Figure 41.

By associativity, we have:
Lemma 8.7. Suppose $B_1$ and $B_2$ are two distinct $\beta$-bands on bipartite disoriented link $(\mathcal{L}, \mathcal{O})$. If the bipartite disoriented link cobordism $B_2 \circ B_1$ is isomorphic to $B_1 \circ B_2$, then the two maps $F_{B_1}$ and $F_{B_2}$ commute with each other.

Proof. We postpone the proof to the end of the Section 8.4. □

8.3. Commutation between band moves and quasi-stabilizations. In this subsection, we will show the commutation between the maps on $CFL'$ associated to band moves and quasi-stabilizations. The discussions about commutations are based on Manolescu’s work in [10] and Zemke’s work in [Zem16b] and [Zem16a].

The bipartite link cobordism for the commutation between a band move $B_\beta$ and quasi-stabilization $S_\beta$ contains two critical points. One is the saddle critical point of the link cobordism surface $F$, and the other one is the critical point for one-manifold $A_\Sigma$. We classify the cobordism of the commutation into two cases:

Case 1 (band move type changes): as shown in Figure 43, the critical point of $A_\Sigma$ and the saddle critical point of $F$ lie in the same component of $F_\beta$. Furthermore, the type of band move changes from type I to type II after quasi-stabilization. Notice that, this case can never happen for an oriented link cobordism. For convenience, we call this a special commutation between $B_\beta$ and $S_\beta$.

Case 2 (band move type does not change): we further classify the cobordism into the following three subcases.

(a) As shown in Figure 44 (a), the critical point of $A_\Sigma$ and the saddle critical point of $F$ lie in the same component of $F_\beta$. The band move $B_\beta$ is of type I.
Figure 44. Case 2: Band move type does not change.

(b) As shown in Figure 44 (b), the critical point of $A_{\Sigma}$ and the saddle critical point of $F$ lie in the same component of $F_{\beta}$. The band move $B^\beta$ is of type II.

(c) The critical point of $A_{\Sigma}$ and the saddle critical point of $F$ lie in different components of $F_{\beta}$.

Next, we translate the cobordism data into Heegaard diagrams data.

**Definition 8.8.** Suppose we have a bipartite link cobordism made of a band move $B^\beta$ and a quasi-stabilization $S^\beta$, such that $B^\beta \circ S^\beta \cong S^\beta \circ B^\beta$. We say that $T = (\Sigma, \alpha \cup \{\alpha_s\}, \beta) \cup \{\beta_s\}, \gamma \cup \{\gamma_s\}, O \cup \{o_s, o'_s\}$ is a **stabilized Heegaard triple subordinate to the commutation**, if it satisfies the following diagram:

\[
\begin{array}{ccc}
H_{\alpha\beta} & \xrightarrow{B^\beta} & H_{\alpha\gamma} \\
S^\beta \downarrow & & \downarrow S^\beta \\
H_{\alpha\beta} & \xrightarrow{B^\beta} & H_{\alpha\gamma}
\end{array}
\]

Here, the Heegaard diagrams above are described below:

- the triple $T = (\Sigma, \alpha, \beta, \gamma, O)$ (or the $T$ resp.) is subordinate to the band move $B^\beta$.
- the stabilized Heegaard diagram $H_{\alpha\gamma}$ (or $H_{\alpha\beta}$ resp.) is subordinate to $S^\beta$.

Next, we will construct a stabilized Heegaard triple for the commutation between $B^\beta$ and $S^\beta$. This construction depends on the type of $\beta$-band moves and the relative position between $B^\beta$ and $S^\beta$.

**Lemma 8.9.** Suppose $W$ is a bipartite link cobordism for band move $B^\beta$ and quasi-stabilization $S^\beta$. If the band move type changes after quasi-stabilization, then we can construct a stabilized Heegaard triple subordinate to $W$ as shown in Figure 45. If the band move type does not change, then we can construct a free stabilized Heegaard triple subordinate to $W$ as shown in Figure 46.

**Proof.** The construction of the stabilized Heegaard triple relies on the bipartite link cobordism.

For case 1: Let $T$ be a standard Heegaard triple subordinate to the band move $B^\beta$. Suppose that $\gamma_n$ is a small Hamiltonian isotopy crossing the basepoints $(o_i, o_j)$. As shown
in Figure 47, we choose a parallel copy $\beta_s$ of $\beta_n$, and we set $\gamma_s$ be a small Hamiltonian isotopy of $\beta_s$ without crossing any basepoints $O$.

For case 2: Based on the construction for case 1, we can handleslide $\beta_s$ such that $\beta_s$ is contractible on the Heegaard surface $\Sigma$. See Figure 48 for the construction of case 2b. We show the stabilized Heegaard triple for case 2a,2b,2c in Figure 46.
Remark 8.10. By [Zem16b, Lemma 6.3], the Heegaard diagram \((\Sigma, \beta \cup \{\beta_s\}, \gamma \cup \{\gamma_s\})\), which comes from the Heegaard triples we constructed in Lemma 8.9, are strongly positive. See [Zem16b, Definition 6.2] for the definition of strongly positive.

Now we lift the bipartite link cobordism for the commutation between the band move \(B_\beta\) and quasi-stabilization \(S_\beta\) to a bipartite disoriented link cobordism. Consequently, the band move \(B_\beta\) and quasi-stabilization \(S_\beta\) will also be lifted (they should be adjacent to certain basepoints with respect to different lifting). Let’s focus our discussion on the lifting for the commutation of case 1.

For the commutation of case 1, we have two possible lifting:

- the commutation \(S^{\beta, o_i} \circ B^{\beta, o_i, o_j} \simeq B^{\beta, o_i, o_j} \circ S^{\beta, o_i, o_j}\), as shown in the Figure 49 (a);
- the commutation \(S^{\beta, o_j} \circ B^{\beta, o_i, o_j} \simeq B^{\beta, o_i, o_j} \circ S^{\beta, o_i, o_j}\), as shown in the Figure 49 (b).

Lemma 8.11. For the commutation of bipartite disoriented link cobordism \(S^{\beta, o_i} \circ B^{\beta, o_i, o_j} \simeq B^{\beta, o_i, o_j} \circ S^{\beta, o_i}\) in case 1, we have the following commutation on the unoriented link Floer chain complex up to chain homotopy:

\[
F_{S^{\beta, o_i}} \circ F_{B^{\beta, o_i, o_j}} \simeq F_{B^{\beta, o_i, o_j}} \circ F_{S^{\beta, o_i}}.
\]

A similar result holds for the commutation \(S^{\beta, o_j} \circ B^{\beta, o_i, o_j} \simeq B^{\beta, o_i, o_j} \circ S^{\beta, o_i}\).

By Lemma 8.9, we have a Heegaard triple (see Figure 45) subordinate to this commutation. We have to show the following commutation diagram:
Figure 50. Connected sum of two Heegaard triples.

\[ \text{CFL}'(\Pi_{\alpha\beta}) \xrightarrow{B^{\beta,o_i,o_j}} \text{CFL}'(\Pi_{\alpha\gamma}) \xrightarrow{S^{\beta,o_i}} \text{CFL}'(H_{\alpha\beta}) \xrightarrow{B^{\beta,o_i,o_j}} \text{CFL}'(H_{\alpha\gamma}) \]

Here, we suppose that:

- the map \( F_{S^{\beta,o_i}} \) send the generator \( x \) to \( x \times x^+ \), where \( x^+ \) is an intersection of \( \alpha_s \) and \( \beta_s \) determined by \( S^{\beta,o_i} \);
- the map \( F_{S^{\beta,o_j}} \) send the generator \( z \) to \( z \times z^+ \), where \( z^+ \) is an intersection of \( \alpha_s \) and \( \gamma_s \) determined by \( S^{\beta,o_j} \);
- the map \( F_{B^{\beta,o_i,o_j}} \) send the generator \( x \) to \( F_T(x \otimes \Theta_{\beta\gamma}) \), where the generator \( \Theta_{\beta\gamma} \) is determined by \( B^{\beta,o_i,o_j} \).
- the map \( F_{B^{\beta,o_i,o_j}} \) send a generator of the form \( x \times x^+ \) to \( F_T((x \times x^+) \otimes (\Theta_{\beta\gamma} \times \theta^+)) \), where \( \theta^+ \) is the intersection of \( \beta_s \) and \( \gamma_s \) determined by \( B^{\beta,o_i,o_j} \).

Now we consider the triple \( T \) as the connected sum of the triple \( T^1 = (\Sigma^1, \alpha, \beta \cup \beta_s, \gamma \cup \gamma_s) \) and the triple \( T^2 = (S^2, \alpha_s, \beta_s, \gamma_s) \) as shown in Figure 50. Notice that, if we remove the \( \beta_s \) and \( \gamma_s \) curve form \( T^1 \) we will get the triple \( T \).

Notice that the intersection \( x \in T_{\alpha \cup \{\alpha_s\}} \cap T_{\beta \cup \{\beta_s\}} \) for \( H_{\alpha\beta} \), can be uniquely written as \( x = x^1 \times x^2 \). Here the intersection \( x^1 \) belongs to \( T_{\alpha \cap T_{\beta}} \) for \( H_{\alpha\beta} \), the intersection \( x^2 \) belongs to \( T_{\alpha_s \cap T_{\beta_s}} \) for \( (S^2, \alpha_s, \beta_s, \gamma_s) \).

Similar to [MO10, Lemma 6.9], we have a restriction map

**Lemma 8.12.** For the stabilized Heegaard triple \( T \) (as shown in Figure 43) subordinate to a special commutation, there is a well-defined restriction map:

\[ \sigma : \pi_2(x,y,z) \to \pi_2(x^2,y^2,z^2) \]
Now we define an equivalence class for the triple $(\phi_1, P_\beta, P_\gamma)$, as follows. We say that the triple $(\phi_1, P_\beta, P_\gamma)$ is equivalent to $(\phi_1', P_\beta', P_\gamma')$, if we have the equality,

$$\phi_1 + P_\beta + P_\gamma = \phi_1' + P_\beta' + P_\gamma'.$$

Here, $\phi_1$ is in $\pi_2((x_1, y_1, z_1))$, $P_\beta$ is in $H_2(\Sigma_1, \beta \cup \{\beta_s\})$, and $P_\gamma$ is in $H_2(\Sigma_1, \gamma \cup \{\gamma_s\})$.

**Lemma 8.13.** Suppose $\phi_2 \in \pi_2(x^2, \Theta^+, z^2)$ the restriction of $\phi \in \pi_2(x, \Theta \times \theta^+, z)$ on $(S^2, \alpha_s, \beta_s, \gamma_s)$. Let $(\phi_1, P_\beta, P_\gamma)$ be the equivalence class determined by $\phi$. We have the following Maslov index formula:

$$\mu(\phi) = \mu(\phi_1) + \mu(P_\beta) + \mu(P_\gamma) - m_1(\phi) - m_2(\phi) + \mu(\phi_2).$$

Here, $m_1(\phi)$, $m_2(\phi)$, $a$ and $a'$ are the multiplicities of the regions of $\phi$ shown in Figure 51.

**Proof.** By the Sarkar’s Maslov index formula for holomorphic triangles in [Sar06], we have:

$$\mu(\phi) = \mu(\phi^1) + \mu(P_\beta) + \mu(P_\gamma) - \frac{1}{2}(m_1(\phi) + m_2(\phi) + a + a') + \mu(\phi_2).$$

On the other hand, as the intersection $\theta^-$ (shown in Figure 45) does not belong to the generator, so we have $m_1 + m_2 = a + a'$. Hence, we get the Maslov index formula. □

**Lemma 8.14.** For a class $\phi_2 \in \pi_2(x^2, \Theta^+, z^2)$, where $\theta^+$ is the intersection shown in Figure, we have the following Maslov index formula:

$$\mu(\phi_2) = m_1 + m_2 + m_3 + m_4$$

**Proof.** Straightforward. □

**Proof of Lemma 8.11.** We use Lipshitz’s cylindrical formulation to count the holomorphic triangles, see [Lip06]. Let $\phi \in \pi_2(x, y, z)$ with Maslov index zero. By Lemma 8.13 and Lemma 8.14 we have:

$$\mu(\phi) = \mu(\phi^1) + \mu(P_\beta) + \mu(P_\gamma) - m_1(\phi) - m_2(\phi) + \mu(\phi_2) = \mu(\phi^1) + m_3 + m_4 + \mu(P_\beta) + \mu(P_\gamma) = 0.$$

Hence, we have $\mu(P_\beta) = \mu(P_\gamma) = 0$, the multiplicity $m_3 = m_4 = \mu(\phi^1) = 0$ and $m_1 = m_2 = a = a' = m$. Therefore we have $\phi^1 \in \pi_2(x, \theta, y)$ or $\phi_1 \in \pi_2(x', \theta, y')$. This is determined by the bipartite disoriented link cobordism, shown in Figure 49.

**Figure 51.** Local picture of Heegaard triple for Maslov index calculations.
On the other hand, we consider the connected sum

$$\Sigma(T) = (\Sigma^1 - D^1)\#([-T - 1, T + 1] \times S^1)\#(S^2 - D^2).$$

Here, $D^1$ is a small disk neighborhood of $p^1$ on $\Sigma^1$, and $D^2$ is a small disk neighborhood of $p^2$ on $S^2$, see Figure 51. We pick up an almost complex structure $J_1 \in \Sigma \times \Delta$ and $J_2$ on $\Sigma^2 \times \Delta$. We construct an almost complex structure $J(T)$ on $\Sigma(T) \times \Delta$. By Remark 8.10, we know that, when $T \to \infty$, the sequence of holomorphic triangles will have a subsequence converging to some holomorphic objects on $\Sigma^1$ and on $S^2$. On $S^2$, the objects are some broken triangles; on $\Sigma$, the objects are some annoying $\beta$ degenerations, annoying $\gamma$ degenerations and broken triangles (see [MO10] Lemma 6.3 for the definition of annoying curves). As we have $m_3 = m_4 = 0$, the objects should be of the form $(\phi^1, 0, 0)$. Hence, for sufficient large necklength, no boundary point of $S$ can be mapped to $p^2$ under the projection $\pi_\Delta \circ u$. Here $u : S \to \Sigma$ is a holomorphic representative of $\phi$. As in [MO10], Proposition 6.15, we can identify the moduli spaces $\mathcal{M}(\phi)$ with some fiber product $\mathcal{M}(\phi^1) \times_{\text{Sym}^m} \mathcal{M}(\phi^2)$. The counting of holomorphic triangle argument is similar to that in the proof of [MO10] Proposition 6.2].

**Proposition 8.15.** Suppose we have a bipartite disoriented link cobordism made of a band move $B^3$ and a quasi-stabilization $S^3$, such that $B^3 \circ S^3 \cong S^3 \circ B^3$. Then the maps associated to $B^3 \circ S^3$ and $S^3 \circ B^3$ on unoriented link Floer chain complex $\text{CFL}'$ are chain homotopic to each other.

**Proof.** For special commutation, see Lemma 8.11. For the remaining cases, as the triples shown in Figure 46 are free stabilizations, we can apply [MO10] Proposition 6.2] to show the commutation at the level of $\text{CFL}'$. □

8.4. The relation between $\alpha$ and $\beta$-band moves. Let $B^{3,0,0}$ be a $\beta$-band move from a bipartite disoriented link $(L^0, O)$ to a bipartite disoriented link $(L^1, O)$. We consider a gradient flow $\phi_t$ induced by a Morse function $f$ compatible with $(L^0, O)$. There exist a $t_0$ such that the band $\phi_{t_0}(B^{3,0,0})$ lies in the $\alpha$-handlebody with respect to the Heegaard decomposition induced by $f$. For convenience, we denote by $\beta^3,0,0$ the $\alpha$-band $\phi_{t_0}(B^{3,0,0})$. The band move $B^{\alpha,0,0}$ changes the bipartite disoriented link $(L^0, O)$ to a bipartite disoriented link $(L^2, O)$. By an isotopy $\sigma$ of $(L^2, O)$ supported only on a three-ball $D$, we get a bipartite disoriented link $(\sigma(L^2), \sigma(O))$ with the following properties:

- the disoriented link $\sigma(L^2)$ agrees with $L^1$,
- the (ordered) set of basepoints $\sigma(O)$ and $O$ differ by switching $o_i$ and $o_j$.

Similar to [Zem16a], Proposition 7.8], we have the following relation between the band move maps induced by the two band moves $B^{3,0,0}$, and $B^{\alpha,0,0}$.

**Proposition 8.16.** Let the band moves $B^{3,0,0}$ and $B^{\alpha,0,0}$ be as described above. We have the two sequences of moves between bipartite disoriented links as shown in Figure 52.

$$(MM \ 1): \ (L^0, O) \xrightarrow{B^{3,0,0}} (L^1, O)$$

$$(MM \ 2): \ (L^0, O) \xrightarrow{B^{\alpha,0,0}} (L^2, O) \xrightarrow{\sigma} (\sigma(L^2) = L^1, \sigma(O)) \xrightarrow{\text{switching } o_i \text{ and } o_j} (L^1, O).$$

At the level of unoriented link Floer chain complex $\text{CFL}'$, the maps induced by the above two sequence of moves are chain homotopic to each other. In other words, we have:

$$F_{B^{3,0,0}} \cong \text{Id}_{\text{renum}} \circ \sigma \circ F_{B^{\alpha,0,0}}.$$
Here the map $\text{Id}_{\text{renum}}$ on $CFL'$ is the identity map induced by switching the basepoints $o_i$ and $o_j$, and the map $\sigma_*$ is the canonical isomorphism induced by the isotopy $\sigma$.

**Proof.** By Theorem 4.2, we consider a standard Heegaard Triple $T_{\alpha\beta\gamma}$ subordinate to $B_{\beta,o_i,o_j}$. On the same Heegaard surface, we construct the $\delta$-curves such that,

1. the curves $\delta_1, \cdots, \delta_{n-1}$ are small Hamiltonian isotopies without crossing any basepoints,
2. the curve $\delta_n$ is a small Hamiltonian isotopy of $\alpha_n$ crossing the basepoints $o_i$ and $o_j$. The geometric intersection $|\delta_n \cap \alpha| = 2 = |\delta_n \cap \delta \cap D|$.

The Heegaard triples $T_{\alpha\beta\gamma}$ and $T_{\delta\alpha\beta}$ are shown in Figure 53. It is easy to see that $T_{\delta\alpha\beta}$ is subordinate to the band move $B_{\alpha,o_i,o_j}$.

Without loss of generality, we can require the isotopy $\sigma$ to satisfy the following:

1. $\sigma$ preserves the Heegaard surface $\Sigma$ setwise,
Figure 54. Sequence of Heegaard moves

(2) $\sigma|_{(\Sigma\setminus D)} = \Id|_{(\Sigma\setminus D)}$,
(3) $\sigma(o_i) = o_j$, $\sigma(o_j) = o_i$ and $\sigma(o_k) = o_k$ if $k \neq i, j$.
(4) after switching the markings $o_i$ and $o_j$, the curves $\sigma(\delta)$ are small Hamiltonian isotopies of $\alpha$ without crossing basepoints and the curves $\sigma(\beta)$ are small Hamiltonian isotopies of $\gamma$.

The two sequences of movie moves in Figure 52 induce two sequences of Heegaard moves shown in Figure 54.

It suffices to show that, 

$$\Phi^{\alpha\gamma}_{\sigma(\delta)\sigma(\beta)} \circ \sigma_\ast \circ F_{B^0,o_i,o_j} \simeq F_{B^0,o_i,o_j}.$$

Here the map $\Phi^{\alpha\gamma}_{\sigma(\delta)\sigma(\beta)}$ is induced by small isotopy without crossing any basepoints.

For the Heegaard triple $T_{\alpha\beta\gamma}$ we have the nearest point map $N^{\beta\gamma}_{\alpha}$, see [OS08b]. It sends a generator $x = (x_1, \ldots, x_n)$ to the unique generator $z = (z_1, \ldots, z_n)$ where the crossing $x_i$ and $y_i$ are vertices of a small triangle. Notice from Figure 53 that the shaded small triangles going over with basepoints $o_i$ or $o_j$. Hence, in our case the map $N^{\beta\gamma}_{\alpha}$ is no longer a chain map. Therefore, we define a chain map $\Psi^{\beta\gamma}_{\alpha}$ as

$$\Psi^{\beta\gamma}_{\alpha}(x) = \begin{cases} U \cdot N^{\beta\gamma}_{\alpha}, & \text{if } x_s \in x, \\ N^{\beta\gamma}_{\alpha} & \text{if } x_s \notin x. \end{cases}$$

Similarly to the proof of Proposition 6.11, the map $\Psi^{\beta\gamma}_{\alpha}$ is chain homotopic to $F_{B^0,o_i,o_j}$.

On the other hand, for the Heegaard triple $T_{\delta\alpha\beta}$ we also have a nearest point map $N^{\beta\gamma}_{\delta\alpha}$. It sends a generator $x = (x_1, \ldots, x_n) \in T_{\alpha} \cap T_{\beta}$ to the unique generator $z = (z_1, \ldots, z_n) \in T_{\delta} \cap T_{\beta}$ where the crossings $x_i$ and $z_i$ are vertices of a small triangle.

We also define a chain map $\Psi^{\beta\gamma}_{\delta\alpha}$ as

$$\Psi^{\beta\gamma}_{\delta\alpha}(x) = \begin{cases} U \cdot N^{\beta\gamma}_{\delta\alpha}, & \text{if } x_s \in x, \\ N^{\beta\gamma}_{\delta\alpha} & \text{if } x_s \notin x. \end{cases}$$

which is chain homotopic to $F_{B^0,o_i,o_j}$.
By our construction, after switching the markings $o_i$ and $o_j$, the curve $\sigma(\delta)$ and $\alpha$ (or $\sigma(\beta)$ and $\gamma$ resp.) are differed by a small isotopy without crossing any basepoints. Hence the map $\Phi_{\sigma(\beta)\sigma(\gamma)}^{\alpha}$ is chain homotopic to a composition of a sequence of nearest point map. Hence the nearest point maps $\Psi_{\alpha,\beta}$ and $\Phi_{\sigma(\beta)\sigma(\gamma)}^{\alpha}$ are chain homotopic. This implies that $\Phi_{\sigma(\beta)\sigma(\gamma)}^{\alpha}$ is chain homotopic to $F_{B^{\alpha,o_i,o_j}}$.

Proof of Lemma 8.7. We denote by $\mathcal{W}^1$ the bipartite disoriented link cobordism $B^1 \circ B^2$ and by $\mathcal{W}^2$ the bipartite disoriented link cobordism $B_2 \circ B_1$. Both cobordism $\mathcal{W}^1$ and $\mathcal{W}^2$ are from the bipartite disoriented link $(L_1, O_1)$ to $(L_1, O_2)$. Without loss of generality, we assume $B_2 = B_1^{\alpha,o_i,o_j}$ and the bipartite disoriented link cobordisms $\mathcal{W}^1$ and $\mathcal{W}^2$ are as shown in the top of Figure 55.

We consider another two bipartite disoriented link cobordism $\mathcal{W'}^1$ and $\mathcal{W'}^2$ from $(L_1, O'_1)$ to $(L_1, O'_2)$. Here, the bipartite disoriented links $(L_1, O'_1)$ and $(L_1, O_1)$ are differ by a basepoints-moving map $\sigma_1$ which moves $o'_i$ and $o_j'$ but fix all the other basepoints; the bipartite disoriented links $(L_1, O_2)$ and $(L_1, O'_2)$ are differ by switching $o'_i$ and $o_j'$ and a basepoints-moving map $\sigma_2$ which moves $o'_i$ and $o_j'$ but fix all the other basepoints. See the bottom of Figure 55 for $\mathcal{W'}^1$ and $\mathcal{W'}^2$.

We denote by $B^\alpha$ the $\alpha$-band $B_1^{\alpha,o_i,o_j}$. By Proposition 8.16, we know that

$$F_{\mathcal{W}^1} = \sigma_2 \circ F_{\mathcal{W'}^1} \circ \sigma_1 = \sigma_2 \circ F_{B_1^\alpha} \circ F_{B_2} \circ \sigma_1$$

$$F_{\mathcal{W}^2} = \sigma_2 \circ F_{\mathcal{W'}^2} \circ \sigma_1 = \sigma_2 \circ F_{B_2^\alpha} \circ F_{B_1} \circ \sigma_1.$$

As the band move maps $F_{B_1^\alpha}$ commute with $F_{B_2^\alpha}$ by Lemma 8.2, we conclude that $F_{\mathcal{W}^1}$ agree with the map $F_{\mathcal{W}^2}$.

8.5. The relation between $\alpha$ and $\beta$-quasi-stabilizations. Let $S^{\beta,o_i}$ be a $\beta$-quasi-stabilization from $(L^0, O^0)$ to $(L^1, O^1)$. We also have an $\alpha$-quasi-stabilization $S^{\alpha,o_i}$, which adds a pair of basepoints $(o_i, o'_i)$ to the same bipartite disoriented link $(L^0, O^0)$ and gets a bipartite disoriented link $(L^2, O^2)$. We provide a local picture for $S^{\alpha,o_i}$ and $S^{\beta,o_i}$ in Figure 57. We can find a basepoint moving map $\sigma$ (an isotopy preverses the link $L$), which satisfies the following,

- the disoriented link $\sigma(L^2)$ agrees with $L^1$,
- the (ordered) basepoint sets $\sigma(O^2)$ and $O^1$ differ by changing the order of the three basepoint $(o_i, o'_i, o_l)$,
- if a basepoint $o_l$ is not $o_i, o'_i$, or $o_l$ in $O^2$, then $\sigma(o_l) = o_l$.

At the level of $CFL'$, we have the following relation about the maps induced by $S^{\alpha,o_i}$ and $S^{\beta,o_i}$.

Proposition 8.17. Let $S^{\beta,o_i}$ and $S^{\alpha,o_i}$ be two quasi-stabilizations described above. We consider the following two sequences of moves, as shown in Figure 57.

\begin{align*}
\text{(MM 1)}: \quad (L^0, O^0) & \xrightarrow{S^{\beta,o_i}} (L^1, O^1), \\
\text{(MM 2)}: \quad (L^0, O^0) & \xrightarrow{S^{\alpha,o_i}} (L^2, O^2) \xrightarrow{\sigma} (\sigma(L^2) = L^1, \sigma(O^2)) \xrightarrow{\text{Renumbering}} (L^1, O^1).
\end{align*}

At the level of $CFL'$, the maps induced by these two sequence of movie moves are chain homotopic. In other words,

$$F_{S^{\beta,o_i}} \cong \sigma \circ F_{S^{\alpha,o_i}}.$$
If we reverse the two sequence of movie moves, similar results hold for quasi-destabilization.

**Proof.** The proof is similar to the proof of [Zem16a, Lemma 3.23]. We compare the Heegaard diagrams shown in the left and right of Figure 57. As we can require $\sigma$ to change the diagram $H_2$ in the right hand side to the Heegaard diagram $H_1$ in the left hand side (in fact, $\sigma(H_2)$ and $H_1$ differ by reordering $(o, o', o_i)$), it is easy to check the map $F_{S^0, o_i}$ agrees with $\sigma \circ F_{S^\beta, o_i}$. \qed
Figure 56. α and β-quasi-stabilizations

Figure 57. α and β-quasi-stabilizations
9. Functoriality

In this section, we assume that the cobordism $W$ between three-manifolds is the product $Y \times I$. We will show that a disoriented link cobordism can be isotoped to a regular form. We also show that any two regular forms of a given disoriented link cobordism can be connected by certain moves. For the construction of the map, we isotope the disoriented link cobordism to a regular form and then lift it to a bipartite disoriented link cobordism. We construct the map from this disoriented link cobordism. Finally, we prove the following:

- the map is independent of liftings,
- the link cobordism map is invariant under the moves between regular forms of the disoriented link cobordism.

These two results imply the invariance of our construction.

9.1. Ambient isotopies of disoriented link cobordism. In this section, we define an equivalence relation between disoriented link cobordism analogous to the equivalence relation of knotted surface in $\mathbb{R}^4$, see [CS98, Chapter 2].

Recall that a disoriented link cobordism $\mathcal{W} = (\mathcal{W}, \mathcal{F}, \mathcal{A})$ from $(Y, L_0)$ to $(Y, L_1)$ contains the data of two maps:

- the embedding of an oriented one-manifold $\mathcal{A} : (A, \partial A) \hookrightarrow (F, \partial F)$,
- the embedding of a surface $\mathcal{F} : (F, \partial F) \hookrightarrow (W, \partial W)$.

**Definition 9.1.** We say that two disoriented link cobordism $\mathcal{W}_i, i = 0, 1$ are equivalent or ambient isotopic, if there exists a pair of smooth maps $h : (W \times I) \to W$ and $g : (F \times I) \to F$, such that

1. the map $h_t = h(-,t) : (W, \partial W) \to (W, \partial W)$ is a diffeomorphism with $h_0 = id_W$ and $h_1 \circ F^0 = F^1$,
2. the map $g_t = g(-,t) : (F, \partial F) \to (F, \partial F)$ is a diffeomorphism with $g_0 = id_F$ and $g_1 \circ A^0 = A^1$.

**Remark 9.2.** In fact, one can also define the equivalence relation between disoriented link cobordism by using isotopy, i.e. a one-parameter family of embeddings $(\mathcal{F}^t, \mathcal{A}^t)$. In our case, as the manifolds are all compact, the isotopy extension theorem implies these two definitions of equivalence of disoriented link cobordism are equivalent, see [Hir76, Chapter 8.1].

As we require the cobordism $W$ to be a product $Y \times I$, there is a natural projection $\pi : Y \times I \to I$. Recall that in section 3.1, we said that a disoriented link cobordism is in regular form, if $(\mathcal{F}, \mathcal{A})$ satisfies the following:

(R1) The surface $\mathcal{F}(F, \partial F)$ is in generic position with respect to the natural projection $\pi$. In other words, $\pi \circ \mathcal{F}$ is a Morse function for $(F, \partial F)$.
(R2) The one-manifold $\mathcal{A}(A, \partial A)$ is in generic position with respect to the natural projection $\pi$. In other words, $\pi \circ \mathcal{F} \circ \mathcal{A}$ is a Morse function for $(A, \partial A)$.
(R3) The index two/zero critical points of the Morse function $\pi \circ \mathcal{F}$ on $F$ are included in the image of index one critical points of $(A, \partial A)$ under the map $\mathcal{A}$.
(R4) The index one critical points of the Morse function $\pi \circ \mathcal{F}$ do not lie on the one-manifold $\mathcal{A}(A, \partial A)$.
(R5) The critical values of $\pi \circ \mathcal{F} \circ \mathcal{A}$ and the critical values of $\pi \circ \mathcal{F}$ corresponding to index one critical points are distinct on $I$.
(R6) For a regular value \( a \in I \), \((\pi^{-1}(a) \cap \mathcal{F}(F, \partial F), \pi^{-1}(a) \cap \mathcal{A}(A, \partial A))\) is a disoriented link in \( \pi^{-1}(a) \cong Y \).

**Theorem 9.3.** For any disoriented link cobordism \((F, A)\), there exists a regular disoriented link cobordism \((F', A')\) which is isotopic to \((F, A)\).

**Proof.** By linear perturbations of the embedding \(F\) in local charts, we can find an embedding \(F'\) which is close to \(F\) with respect to some metric and satisfies (R1). Similar to the proof in [Ros04, Proposition 4.5], we know that as \(F'\) is close to \(F\), they should be isotopic to each other.

Now we fix \(F\) and move the embedding \(A\) such that \(A(\mathcal{A}(A, \partial A))\) go through all index zero and index two critical points of \(\pi \circ F\) on \((F, \partial F)\). As each piece of \(F\), \(\mathcal{A}(A)\) has a boundary component in \(\mathcal{A}(A)\), hence we move \(\mathcal{A}(A)\) such that for each link component \(L_i\) of \((\pi \circ F)^{-1}(a)\), the intersection \(L_i \cap \mathcal{A}(A)\) is non-empty. By a further ambient isotopy which fixes the small neighborhood of all index two and index zero critical points of \(\pi \circ F\), we get a desired embedding \(A'\) satisfies (R2) to (R6). See Figure 58 for an example. □

**9.2. Moves between regular forms of a disoriented link cobordism.** Recall that in Cerf theory, given a path connecting two Morse functions \(f_0\) and \(f_1\), one can perturb and get a path of functions \(f_t\) such that \(f_t\) is Morse function except at a finite set of \(t \in (0, 1)\). Similarly, we will show that given a path of smooth embeddings \((\mathcal{F}', \mathcal{A}')\) or equivalently an ambient isotopy connecting two regular disoriented link cobordisms \((\mathcal{F}_0, \mathcal{A}_0)\) and \((\mathcal{F}_1, \mathcal{A}_1)\), one can perturb the path of embeddings such that \((\mathcal{F}', \mathcal{A}')\) is in regular form except a finite subset \(E = \{t_1, t_2, \cdots, t_k\} \subset (0, 1)\). We say that a path of pair of embedding satisfy the above condition is **nice**.

To describe this path of embeddings, we pick up a finite subset of time \(R = \{a_1, \cdots, a_r\} \subset I \setminus E\) where \(a_1 = 0\), \(a_r = 1\) and \(a_i < a_j\) if \(i < j\), such that for all component of \(I \setminus E\) there exist at least one \(a_i\) in \(R\). Then, the path of embedding are represented by the **move** from the embedding \((\mathcal{F}^{a_i}, \mathcal{A}^{a_i})\) to \((\mathcal{F}^{a_{i+1}}, \mathcal{A}^{a_{i+1}})\).

We sort these move into 11 cases and provide examples of local pictures of these moves in Figure 59 and Figure 60.

**M 1:** Ambient isotopies that do not change the level of all critical points of \(\pi \circ F\) and \(\pi \circ A\).

**M 2:** Cancellation/birth of a pair of index zero/two and index one critical points of \(\pi \circ F\).

**M 3:** Cancellation/birth of a pair of index one/zero critical points of \(\pi \circ F \circ A\).
Figure 59. Moves between regular forms (Part I)

(M 4): Ambient isotopies of \((A, \partial A)\) on \((F, \partial F)\) which go across an index two/zero critical point of \(\pi \circ \mathcal{F}\) and switch the height of two critical points of \(\pi \circ \mathcal{F} \circ A\).

(M 5): Ambient isotopies of \((A, \partial A)\) on \((F, \partial F)\) which go across an index one critical point of \(\pi \circ \mathcal{F}\).

(M 6): Switching height of a pair of index one/zero critical points of \(\pi \circ \mathcal{F} \circ A\).

(M 7): Switching the height of two index one critical points of \(\pi \circ \mathcal{F}\).

(M 8): Switching the height of two index zero/two critical points of \(\pi \circ \mathcal{F}\).

(M 9): Switching the height of an index one critical points and a index two/zero critical point of \(\pi \circ \mathcal{F}\).

(M 10): Switching the height of an index zero/one critical points of \(\pi \circ \mathcal{F} \circ A\) and a index one critical point of \(\pi \circ \mathcal{F}\).

(M 11): Switching the height of an index zero/one critical points of \(\pi \circ \mathcal{F} \circ A\) and a distant index two/zero critical point of \(\pi \circ \mathcal{F}\).

**Theorem 9.4.** Any two regular forms of a disoriented link cobordism can be connected by (M 1) to (M 11).
Proof. Let \((F^0, A^0)\) and \((F^1, A^1)\) be two regular forms of a disoriented link cobordism. By definition, there exists a path of embeddings \((F^t, A^t)\) connecting them.

Similar to the proofs in [Ros04, Proposition 5.4], the path of embeddings \(F^t\) can be approximated by another path \(F^t_{\text{reg}}\) such that \(\pi \circ F^t_{\text{reg}}\) is a Morse function except a finite number of \(t \in [0, 1]\). The path of the pair of embeddings \((F^t_{\text{reg}}, A^0)\) may not be nice, but we can find another path of embedding \(A^t_{\text{mid}}\) starting from \(A^0\) such that the path of pairs of embeddings \((F^t_{\text{reg}}, A^t_{\text{mid}})\) can be represented by \((M 1),(M 2),(M 7),(M 8)\) and \((M 9)\).

The embedding \(A^t_{\text{mid}}\) may not be \(A^1\), therefore, we find the path of embedding \(A^t_{\text{fin}}\) from \(A^t_{\text{mid}}\) to \(A^1\) such that the path of pair of embeddings \((F^1, A^t_{\text{fin}})\) is nice. This path can be represented by \((M 1), (M 3), (M 4), (M 5), (M 6), (M 10)\) and \((M 11)\).

Finally, we concatenate the two paths \((F^t_{\text{reg}}, A^t_{\text{mid}})\) and \((F^1, A^t_{\text{fin}})\) to get a desired isotopy. See Figure 61 for an example of the isotopy.

\[\square\]
9.3. Construction and invariance of disoriented link cobordism maps.

Proof of Theorem 1.1. The proof include six steps: Regular form, lifting, elementary pieces, composition of maps, independence of liftings and moves between regular forms.

Step 1-Regular Form: By theorem 9.3 we can perturb $\mathcal{W}$ and get a disoriented link cobordism $\mathcal{W}_{\text{reg}}$ in regular form.

Step 2-Lifting: By the discussion in Section 3.4, we can lift $\mathcal{W}_{\text{reg}}$ to a bipartite disoriented link cobordism $\mathcal{W}_{\text{reg}}^\text{bipartite}$.

Step 3-Elementary pieces: Then we cut $\mathcal{W}_{\text{reg}}^\text{bipartite}$ into elementary pieces $\mathcal{W}_{\text{reg}}^\text{bipartite}_i$. These elementary pieces are categorized into four types: isotopies, disk-stabilization and destabilizations, quasi-stabilization and destabilizations, band moves. We constructed a map $F_{\mathcal{W}_{\text{reg}}^\text{bipartite}_i}$ on the unoriented link Floer homology for each type of these elementary cobordism. For disk-stabilization/destabilization see the discussion in the beginning of Section 9.3. For the definition of band move maps, see Theorem 6.4. For the definition of maps induced by quasi-stabilization/destabilizations, see Section 7.

Step 4-Composition of maps: We define the map $F_{\mathcal{W}}$ to be the composition of the map $F_{\mathcal{W}_{\text{reg}}^\text{bipartite}}$. The decomposition corresponds to a division of the interval $I$. If we have another decomposition of $\mathcal{W}$, we can find a common subdivision of $I$ for the two decomposition. By the invariance of the maps induced by elementary pieces, we know that subdivision induce the same map. Hence the map $F_{\mathcal{W}}$ is independent of the decomposition.
**Step 5-Independence of liftings:** Recall that in the process of lifting $\mathcal{W}_{\text{reg}}$, we lift each elementary piece $\mathcal{W}_i$ to $\mathcal{W}_i$ and glue them together. For an isotopy, a disk-stabilization or a disk-destabilization, we have a unique way to lift. For a band move or a quasi-stabilization/destabilization, we have two ways to lift. Therefore it suffices to check that for a band move or a quasi-stabilization/destabilization, the map on unoriented link Floer homology we defined is independent of different liftings. By the result in Proposition 8.16 and Proposition 8.17, we get a well-defined map $F_{\mathcal{W}_{\text{reg}}} = F_{\mathcal{W}_{\text{reg}}}$.

**Step 6-Moves between regular forms:** As one may perturb $\mathcal{W}$ to different regular forms $\mathcal{W}_{\text{reg}1}$ or $\mathcal{W}_{\text{reg}2}$ (by a small ambient isotopy), we need to verify the map is independent of the perturbations. By Theorem 9.4, we know that the two regular forms $\mathcal{W}_{\text{reg}1}$ and $\mathcal{W}_{\text{reg}2}$ are connected by a sequence of moves. It suffices to check the $F_{\mathcal{W}_{\text{reg}}}$ is invariant under the moves.

By definition, a move between regular forms can be realized as an ambient isotopy $g_t$ of $\mathcal{F}$, where $g_t$ is a path of diffeomorphism of $Y \times I$. In [Zem16a], from a path of Morse functions $\pi \circ g_t$ of $Y \times I$, we get a sequence of moves between parametrized Kirby decompositions. We have the following table comparing the moves between regular forms and the moves between parametrized Kirby decomposition described in [Zem16a].

| Regular forms | Parametrized Kirby decompositions |
|---------------|----------------------------------|
| (M 1)         | Move(1)                          |
| (M 2)         | Move(7)                          |
| (M 3)         | Move(10)                         |
| (M 4) (M 5)   | Move(14)                         |
| (M 6)         | Move(11)                         |
| (M 7) (M 8) (M 9) | Move(12)                  |
| (M 10) (M 11) | Move(13)                         |

If the surface $F$ is orientable, the moves between regular forms can be described by the moves in [Zem16a]. Hence we only need to verify (M 5) (M 7) (M 9) and (M 10) which can involve an unoriented band move.

For (M 4), it is easy to verify the invariance by constructing Heegaard diagrams for the two cobordisms (both are composition of a disk-stabilization and a quasi-stabilization, so the Heegaard diagrams can be very simple) and keep track of the top grading generator. For (M 5), one can still apply the results in [Zem16a] (invariance of map under Move(14)) to unorientable band moves and show that (M 5) does not change the map $F_{\mathcal{W}}$. Actually, we can also lift the two disoriented link cobordisms in (M 5) to bipartite disoriented link cobordisms. By the discussion in Section 8.3, we can construct a Heegaard triple subordinate to the band moves and show that the maps for the two bipartite disoriented link cobordisms are the same. Similarly, the invariance under (M 9) can be directly verified by constructing certain Heegaard triples as in Section 8.3 or in [Zem16a]. For (M 7), we lift $\mathcal{W}_{\text{reg}}$ to bipartite disoriented link cobordisms (see Figure 48 for an example) and apply the results in Lemma 8.2 (the commutation between a $\alpha$ and a $\beta$-band move) and Lemma 8.7 (the commutation between two $\beta$-band moves) to show the invariance. For (M 10), we lift the two disoriented link cobordism to bipartite disoriented link cobordisms (see Figure 49 for an example) and apply Proposition 8.15.

□
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