Does DLCQ S-matrix have a covariant continuum limit?

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We develop a systematic DLCQ perturbation theory and show that DLCQ S-matrix does not have a covariant continuum limit for processes with $p^+ = 0$ exchange. This implies that the role of the zero mode is more subtle than ever considered in DLCQ and hence must be treated with great care also in non-perturbative approach. We also make a brief comment on DLCQ in string theory.
Classically, or putting aside the operator ordering first, we can straightforwardly solve the zero-mode constraint perturbatively and we obtain,

\[
\phi_0(x^+ , x_-) = -i \lambda \int d^d y \Delta_0^{xy} \frac{\phi^3(y)}{3!} + O(\lambda^3), \quad (3)
\]

where \( \Delta_0^{xy} \) is the Feynman propagator for the zero mode.

\[
\Delta_0^{xy} = \frac{1}{2L} \int \frac{d^d k}{(2\pi)^{d-1}} \frac{e^{-i k \cdot (x^+ - y^+) + i k \cdot (x_- - y_-)}}{m^2 + k^2},
\]

\[
= \frac{1}{2iL} \frac{1}{m^2 - \partial^2_{x^+}} \delta(x^+ - y^+) \delta^{(d-2)}(x_- - y_-). \quad (4)
\]

Plugging eq.(3) into the Hamiltonian (1), we obtain the effective interaction Hamiltonian,

\[
- i \int dx^+ H_{\text{int}} = -i \lambda \int d^d y \frac{\phi^3(y)}{4!} + \frac{1}{2!} (-i \lambda)^2 \int d^d x d^d y \left( \frac{\phi^3(y)}{3!} \Delta_0^{xy} \frac{\phi^3(y)}{3!} \right) + \frac{3}{3!} (-i \lambda)^3 \int d^d x d^d y d^d z \left( \frac{\phi^3(y)}{2!} \Delta_0^{xy} \frac{\phi^3(z)}{3!} \right) + O(\lambda^4). \quad (5)
\]

The first three terms on the r.h.s. of eq.(5) correspond to the diagrams in Fig.1 respectively.

Now we comment on the operator ordering. In the ordinary quantization, the equal-time commutator \([\phi(x), \phi(y)]_{x=0} = 0 \) due to the micro-causality, while in the light-cone quantization, the equal light-cone time commutator \([\varphi(x), \varphi(y)]_{x^+ = y^+} \) in general has a non-zero value. Thus we must fix the operator ordering in eq.(3). Here we choose the Weyl ordering for the equal light-cone time operators. Actually, it is easily checked that the Weyl ordering is consistent with the Euler-Lagrange equation.

Having fixed the ordering of the equal light-cone time operators, we can perform the perturbative expansion (Dyson expansion) with the effective interaction Hamiltonian (5) systematically. From the Schrödinger equation of the light-cone time evolution, \( i (\partial / \partial x^+) |x^+ \rangle = H_{\text{int}}(x^+) |x^+ \rangle \) in the interaction picture, we obtain the S-operator, \( S = T \exp \left[ -i \int_{-\infty}^{\infty} dx^+ H_{\text{int}}(x^+) \right] \), where \( T \)-symbol means the light-cone time ordered product. It should be noted that micro-causality is not necessary in the scattering when we take the continuum limit \( (L \to \infty) \). Naively, one might consider that eq.(3) would agree with the covariant answer in the continuum limit. However, in DLCQ.
the continuum limit should be taken after the whole calculations are done with finite $L$. Hence we first perform the momentum integrals in (4). Using the usual parameter integral formula, $\int_{0}^{\infty} d\alpha e^{-i\alpha(X-ie)}$, we obtain,

$$A = \frac{i}{2L} \sum_{l} \int \frac{dk}{(2\pi)^d-1} \int_{0}^{\infty} \frac{d\alpha_{1} d\alpha_{2}}{L} \times \exp \left[-i\alpha_{1} \left( m_{\gamma}^{2} - 2k \left( \frac{l\pi}{L} + (k_{\perp})^{2} \right) \right) \right] \times \exp \left[-i\alpha_{2} \left( m_{\gamma}^{2} - 2k \left( \frac{l-n_{2}-n_{3}}{L} + (k_{\perp})^{2} \right) \right) \right].$$

Then, the integral over $k_{\perp}$ gives a $\delta$-function,

$$A = \frac{i}{4\pi} \sum_{l} \int \frac{dl}{(2\pi)^{d-2}} \int_{0}^{\infty} \frac{d\gamma}{L} \delta(l-\gamma(n_{2}-n_{3})) \times \exp \left[-i\alpha \left( m_{\gamma}^{2} - \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right) \right],$$

where we have changed the variables $(k_{\perp}, \alpha_{1}, \alpha_{2}) \rightarrow (k_{\perp}, \gamma, \gamma) \equiv (k_{\perp} + \gamma(p_{2} - p_{3}), \alpha_{1} + \alpha_{2}/(\alpha_{1} + \alpha_{2}))$. Wick-rotating the variable $\alpha \rightarrow -i\alpha$, we finally get,

$$A = \frac{1}{2\pi(2\pi)^{d-2}} \sum_{l} \int_{0}^{\infty} \frac{d\gamma}{L} \delta(l-\gamma(n_{2}-n_{3})) \times \exp \left[-\alpha \left( m_{\gamma}^{2} - \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right) \right].$$

One comment is in order: If eq.(8) had included a zero-mode loop contribution ($l = 0, n_{2} = n_{3}$), which is actually absent in $\sum' \gamma$, it would suffer from a pathological divergence of $\delta(0)$. This is, in fact, the divergence in ref. in $L^{3}$. In DLCQ, however, this problem is absent from the outset.

Now we are ready to examine the continuum limit $(L \rightarrow \infty)$. First, we consider the $n_{2} \neq n_{3}$ case in eq.(8). Performing the $\gamma$-integration in eq.(8), we have [13].

$$A = \frac{1}{2\pi(2\pi)^{d-2}} \int_{0}^{\infty} \frac{d\gamma}{L} \exp \left( \alpha \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right) \left( \frac{1}{|n_{2}-n_{3}|} \right)$$

where $\gamma = l/|n_{2}-n_{3}|$ and we have used the prescription $\int_{0}^{\infty} d\gamma f(\gamma) = (1/2) f(0)$. The first term “1” in the brace is the sum of $l = 0$ and $l = |n_{2}-n_{3}|$ contributions which correspond to the diagram having a zero-mode propagator (Fig.2(b)).

Now it is easy to take the continuum limit, $|n_{2}-n_{3}| \rightarrow \infty$, of eq.(9). In fact, we obtain,

$$A = \frac{1}{2\pi(2\pi)^{d-2}} \int_{0}^{\infty} \frac{d\gamma}{L} \exp \left( -\alpha \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right) \left( \frac{1}{|n_{2}-n_{3}|} \right) \left( \frac{1}{|n_{2}-n_{3}|} \right) \left( \frac{1}{|n_{2}-n_{3}|} \right),$$

which coincides with the covariant result. Note that in the continuum limit, the contributions of the diagram in Fig.2(b) vanish $(1/|n_{2}-n_{3}| \rightarrow 0)$ [13].

Next we consider the $n_{2} = n_{3}$ case in eq.(8). Such a case is precisely what was missing in [13] and actually yields non-covariant continuum limit. For clarity, we consider a diagram in two dimensions (d = 2) [22] where the external momenta $p_{i}$ ($i = 1, \cdots, 4$) satisfy $p_{1} = p_{4}$ and $p_{2} = p_{3}$ in Fig.2(a), i.e., the forward scattering. Note that there is no $p_{t}$-momentum exchange in this process and hence the diagram of Fig.2(b) does not exist. From eq.(9), the amplitude $A_{F}$ is given by,

$$A_{F} = \frac{2}{2\pi(2\pi)^{d-2}} \int_{0}^{\infty} d\gamma \delta(\delta) \exp \left( -\alpha \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right).$$

Furthermore, changing the variable $\alpha \rightarrow \beta = 2(l\pi/L)\alpha$ for later convenience, we obtain,

$$A_{F} = \frac{1}{2\pi(2\pi)^{d-2}} \int_{0}^{\infty} d\gamma \beta \delta(\beta) \exp \left( -\beta \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right).$$

This becomes zero due to $\int_{0}^{\infty} d\gamma \beta \delta(\beta) \exp \left( -\beta \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right) = 0$. Thus the scattering amplitude of this process is zero in the continuum limit $(L \rightarrow \infty)$.

On the contrary, the covariant amplitude of the same process $A_{F}^{cov}$ is given by,

$$A_{F}^{cov} = \int \frac{dk^{0}dk^{+}}{(2\pi)^{d+1}} \frac{1}{(m_{\gamma}^{2} - (k^{0})^{2} + (k^{+})^{2})^{2}}$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} d\gamma e^{-m_{\gamma}^{2} \alpha} = \frac{1}{4\pi m_{\gamma}^{2}}.$$

This is non-zero and finite! The continuum limit of $A_{F}$ ($= 0$) does not coincide with $A_{F}^{cov}$.

To see why this discrepancy has occurred, we consider the continuum light-front amplitude of the same process. The amplitude $A_{F}^{LC}$ is given by,

$$A_{F}^{LC} = \int \frac{dk^{+}dk^{-}}{(2\pi)^{d}} \exp \left( -\beta \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right).$$

Similarly to the above, we obtain,

$$A_{F}^{LC} = \int_{0}^{\infty} \frac{dk^{+}}{(2\pi)^{d}} \int_{0}^{\infty} d\beta \delta(\beta) \exp \left( -\beta \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right).$$

Note that if we discretize the light-cone coordinate, $k^{+} \rightarrow (l\pi/L)$ and $(1/2\pi) \int_{0}^{\infty} dk^{+} \rightarrow (1/2L) \sum_{l>0}$, then $|l|$ becomes (11). At first sight, (14) seems to be zero similarly to (11) due to $\int_{0}^{\infty} d\beta \delta(\beta) \exp \left( -\beta \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right) = 0$. However, this is not the case. The reason is that once we put $\int_{0}^{\infty} d\beta \delta(\beta) \exp \left( -\beta \gamma(1-\gamma)(p_{2}-p_{3})^{2} \right)$ in eq.(14) zero, the remaining integral $\int_{0}^{\infty} dk^{+} \exp \left( 4\pi \beta(1-\gamma)(p_{2}-p_{3})^{2} \right)$ diverges, i.e., we shall suffer from $\infty \times 0$ in such a calculation of (14). The proper procedure is, in fact, to perform $k^{+}$-integral first and then $\beta$-integral afterward. Changing the variable $k^{+} \rightarrow k = 1/k^{+}$, we can easily carry out the procedure,
\( A_F^{LC} = \int_0^\infty d\beta \delta(\beta) \int_0^\infty \frac{dk}{4\pi} \exp \left( -\frac{\beta m^2 k}{2} \right) \)
\[ = \frac{1}{2\pi m^2} \int_0^\infty d\beta \delta(\beta) = \frac{1}{4\pi m^2}, \tag{15} \]

where we have used the prescription \( \int_0^\infty d\beta \delta(\beta)f(\beta) = (1/2)f(0) \) as before. This exactly coincides with the covariant result \( (12) \).

It is now clear why the discrepancy between \( A_F \) and \( A_F^{cov} \) has occurred even in the continuum limit. In contrast to eq.\( (14) \), since \( (1/2L) \sum_{l>0} [2(\pi/L)^2]^{-1} = L/24 \) is finite, we can put \( \int_0^\infty d\beta \delta(\beta) \exp \left( -\beta m^2 / (2\pi/L) \right) = 0 \) in \( (14) \), and hence we have \( A_F = 0 \). In DLCQ, since the continuum limit should be taken after the whole calculations are done with finite \( L \), we cannot obtain any non-zero result in eq.\( (11) \) even in the continuum limit and the covariant result \( (13) \) can never be reproduced in DLCQ. Thus the continuum limit of the DLCQ S-matrix does not coincide with the covariant one in the same order \( [10] \), which wipes off such a discrepancy appeared in field theory.

However, since in the integral over the scattering angle, the region which contributes to the discrepancy has measure zero in the configuration space of external momenta, this explicit result nevertheless implies that the role of the zero modes is more subtle than ever thought in DLCQ. Thus various DLCQ calculations, perturbative or non-perturbative, should be done with great care on the zero modes.

Finally, we make a brief comment on DLCQ in string theory. Due to the s-t channel duality, a string scattering diagram of a certain order in string perturbation contains all the possible field theoretical Feynman diagrams of the same order \( [10] \), which wipes off such a discrepancy appeared in field theory.

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