Convex Obstacles from Travelling Times

Lyle Noakes∗  Luchezar Stoyanov†

Abstract: A construction is given for the recovery of a disjoint union of strictly convex smooth planar obstacles from travelling-time information. The obstacles are required to be such that no Euclidean line meets more than two of them.

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1 Introduction

For some $n \geq 1$, let $K_1, K_2, \ldots, K_n$ be disjoint closed convex subsets of Euclidean 2-space $E^2 \cong \mathbb{R}^2$, with each boundary $\partial K_k$ a $C^\infty$ strictly convex Jordan curve. Let $K := \bigcup_{k=1}^n K_k$ be contained in the interior of the bounded component $B$ of $E^2 \setminus C$, where $C \subset E^2$ is also a strictly convex Jordan curve.

By a geodesic in the closure $M$ of $E^2 \setminus K$ we mean a piecewise-affine constant-speed curve $x : \mathbb{R} \to M$ whose junctions are points of reflection on $\partial K$. The restriction of $x$ to an interval is also called a geodesic, and the set of all geodesics $x : [0,1] \to M$ with $x(0), x(1) \in C$ is denoted by $\mathcal{X}$. Then $x \in \mathcal{X}$ is critical for the length functional

$$J(x) := \int_0^1 \|\dot{x}(t)\| \, dt$$

over constant-speed piecewise-$C^1$ curves in $M$ satisfying $x(0) = x_0, x(1) = x_1$.

By Theorem 1.1 of [4], $K$ is uniquely determined by its travelling-time data

$$\mathcal{T} := \{(x(0), x(1), J(x)) : x \in \mathcal{X}\}.$$.

Similar results are proved in [5] for obstacles in $E^m$ where $m > 2$. Unfortunately the proof of Theorem 1.1 in [4] is not constructive: all that is shown is that $\mathcal{T}$ is different for different convex obstacles $K$. When $n = 1$ it is straightforward to

∗Department of Mathematics, University of Western Australia, Crawley WA 6009, Australia (lyle.noakes@uwa.edu.au)
†Department of Mathematics, University of Western Australia, Crawley WA 6009, Australia (luchezar.stoyanov@uwa.edu.au)
calculate \( K \) from \( T \), and with a little more effort \( K \) can also be reconstructed when \( n = 2 \) (see Sect. 4 in \([6]\)). More interestingly\(^1\) Theorem 1.1 of \([10]\) allows to compute the area of \( K \) from \( T \). Importantly, the application of the result in \([10]\) is made possible by the fact (proved in \([4]\)) that the set of points generating trapped trajectories in the exterior of obstacles \( K \) considered in this paper has Lebesgue measure zero. Constructing \( K \) is equivalent to constructing \( \partial K \), but this seems difficult for \( n \geq 3 \). In the present paper we show how to construct \( \partial K \) from \( T \) when \( K \) is in general position, namely when no line meets more than two connected components of \( K \).

Inverse problems concerning metric rigidity have been studied for a long time in Riemannian geometry: we refer to \([8]\), \([1]\) and their references for more information. In the last 20 years or so similar problems have been considered for scattering by obstacles, where the task is to recover geometric information about an obstacle from its scattering length spectrum \([9]\), or from travelling times of scattering rays in its exterior \([6]\).

In general, an obstacle in the Euclidean space \( E^m \cong \mathbb{R}^m \) \((m \geq 2)\) is a compact subset \( K \) of \( \mathbb{R}^m \) with a smooth (e.g. \( C^3 \)) boundary \( \partial K \) such that \( \Omega_K = \mathbb{R}^m \setminus K \) is connected. The scattering rays in \( \Omega_K \) are generalized geodesics (in the sense of Melrose and Sjöstrand \([2]\), \([3]\)) that are unbounded in both directions. Most of these scattering rays are billiard trajectories with finitely many reflection points at \( \partial K \). When \( K \) is a finite disjoint union of strictly convex domains, then all scattering rays in \( \Omega_K \) are billiard trajectories, namely geodesics of the type described above.

It turns out that some kinds of obstacles are uniquely recoverable from their travelling times spectra. For example, as mentioned above, this was proved in \([5]\) for obstacles \( K \) in \( \mathbb{R}^m \) \((m \geq 3)\) that are finite disjoint unions of strictly convex bodies with \( C^3 \) boundaries. The case \( m = 2 \) requires a different proof, given recently in \([4]\).

The set of the so called trapped points (points that generate trajectories with infinitely many reflections) plays a rather important role in various inverse problems in scattering by obstacles, and also in problems on metric rigidity in Riemannian geometry. As an example of M. Livshits shows (see e.g. Figure 1 in \([7]\) or \([10]\)), in general the set of trapped points may contain a non-trivial open set. In such a case the obstacle cannot be recovered from travelling times. In dimensions \( m > 2 \) examples similar to that of Livshits were given in \([7]\).

The layout of the paper is as follows.

In \([2]\) we collect some simple observations about linear (non-reflected) geodesics. This leads to the construction of \( 4(n^2 - n) \) so-called vacuous arcs \( \beta_j \) in \( T \), and then \( 2(n^2 - n) \) initial arcs in \( \partial K \). Our plan is to build on the initial arcs, using travelling-time data from reflected rays to construct incremental arcs in \( \partial K \),

\(^1\)Indeed \([10]\) applies in a much more general setting, where the obstacles are not necessarily convex, and \( E^2 \) is replaced by a Riemannian manifold of any finite dimension.
until eventually the whole of $\partial K$ is found. In §6 we describe an inductive step for constructing incremental arcs from previously determined arcs, and from observations of $\mathcal{T}$. To make the relevant observations we need to understand some of the mathematical structure of $\mathcal{T}$.

The first step towards this understanding is made in §3, where some simple facts about (typically non-reflected) geodesics are recalled. These facts, including a known result for computing initial directions of geodesics, are applied in §4 to investigate the structure of travelling-time data of nowhere-tangent geodesics. In particular, cusps in so-called telegraphs of $\mathcal{T}$ correspond to geodesics that are tangent to $\partial K$.

The family of all such cusps is studied in §5, where the augmented travelling-time data $\tilde{T}$ is shown to be the closure of a countable family of disjoint open $C^\infty$ arcs $\tilde{\beta}_j$. As described in §6, the property of extendibility can be checked for each $\tilde{\beta}_j$. When $\tilde{\beta}_j$ is extendible it yields an incremental arc in $\partial K$. When $\tilde{\beta}_j$ is not extendible, a trick using general position replaces $\tilde{\beta}_j$ by an extendible $\tilde{\beta}_j$ yielding an incremental arc as previously.

### 2 Linear Geodesics and Vacuous Arcs

From now on let $K = \bigcup_{k=1}^n K_i$ be an obstacle in $E^2$, where $K_1, K_2, \ldots, K_n$ are disjoint closed convex subsets of $E^2$ with boundaries that are $C^\infty$ strictly convex Jordan curves. As before, assume that $K$ is contained in the interior of the bounded component $B$ of $E^2 \setminus C$, where $C \subset E^2$ is also a strictly convex Jordan curve. We also assume that $K$ is in general position.

**Lemma 1.** Geodesics in $\mathcal{X}$ are not tangent to $\partial K$, except perhaps at the first or last points of contact with $\partial K$ (either or both).

**Proof:** If tangency was at an intermediate point of contact, the tangent line would have common points with at least 3 connected components of $K$, contradicting general position.

We begin by investigating travelling times of linear geodesics, namely geodesics in $\mathcal{X}$ that do not reflect at all. The travelling time data from linear geodesics is

$$\mathcal{T}_0 := \mathcal{T} \cap \{(x_0, x_1, \|x_1 - x_0\|) : x_0, x_1 \in C\}$$

from which we find

$$\partial \mathcal{T}_0 := (C \times C \times (0, \infty)) \cap \partial \mathcal{T}_0 = \bigcup_{q \geq 1} \mathcal{T}_0^q$$

where $\mathcal{T}_0^q$ is defined as the travelling-time data from geodesics meeting $\partial K$ exactly $q$ times tangentially and nowhere else. By Lemma 1, $\mathcal{T}_0^q = \emptyset$ for $q \geq 3$. Unlike the initial arcs, there are countably many incremental arcs, yielding diminishing additional information from ever-increasing amounts of precisely known data. In practice, insufficient data and limited computing power makes it difficult to carry out more than a few inductive steps, and $\partial K$ is found only approximately.
namely \( \partial T_0 = T_0^1 \cup T_0^2 \). In the simplest case where \( n = 1 \), \( T_0^2 \) is empty and \( \partial K \) is constructed as the envelope of the line segments \([x_0, x_1]\) where \((x_0, x_1, \|x_1 - x_0\|) \in \partial T_0\). Suppose \( n \geq 2 \) from now on.

**Proposition 1.** \( T_0^1 \) is a union of \( 4(n^2 - n) \) nonintersecting bounded open \( C^\infty \) arcs \( \beta_j \) whose boundaries in \( \partial T_0 \) comprise \( T_0^2 \) which is finite of size \( 4(n^2 - n) \).

**Proof:** For \( 1 \leq k \neq k' \leq n \) there are 8 directed Euclidean line segments (linear bitangents) tangent to both \( \partial K_k \) and \( \partial K_{k'} \). Each directed linear bitangent is an endpoint of two maximal open arcs of directed line segments that are singly-tangent. The travelling-time data for the linear bitangents is \( T_0^2 \). The travelling-time data for the open arcs \( \beta_j \) are the path components of \( T_0^1 \). \( \square \)

So \( n \) is found from \( T_0^1 \).

**Definition 1.** For \( 1 \leq j \leq 4(n^2 - n) \) the conjugate \( \tilde{j} \) is defined to be \( j + 2(n^2 - n) \) or \( j - 2(n^2 - n) \) according as \( 1 \leq j \leq 2(n^2 - n) \) or \( 2(n^2 - n) + 1 \leq j \leq 4(n^2 - n) \). \( \square \)

Evidently \( \tilde{\tilde{j}} = j \). Order the arcs \( \beta_j \) in \( T_0^1 \) so that \((x_0, x_1, t) \in \beta_j \Leftrightarrow (x_1, x_0, t) \in \beta_j \). The initial arcs in \( \partial K \) are the \( 2(n^2 - n) \) nonempty disjoint connected open subsets of \( \partial K \) found as the envelopes of the \([x_0, x_1]\), where \((x_0, x_1, t) \in \beta_j \) for \( 1 \leq j \leq 2(n^2 - n) \). For each \( 1 \leq k \leq n \) there are \( 2(n - 1) \) initial arcs in \( \partial K_k \).

### 3 Nonlinear Geodesics

In order to construct envelopes of other singly-tangential geodesics, we shall identify the travelling-time data \( T^q \) of \( q \)-times tangential geodesics in \( X \), especially \( q = 1 \). Whereas \( T_0^1 \) is found by simple inspection of \( T \), some effort is required to isolate \( T^1 \). We first recall some known results about directions of geodesics and travelling-times.

For \((x_0, v_0) \in (E^2 - K) \times E^2\) let \( x_{x_0, v_0} : \mathbb{R} \to M \) be the geodesic satisfying \( x_{x_0, v_0}(0) = x_0 \) and \( x_{x_0, v_0}(1) = v_0 \). The endpoint map \( E : (E^2 - K) \times E^2 \to M \) is the continuous function given by \( E(x_0, v_0) := x_{x_0, v_0}(1) \).

**Lemma 2.** For \( x_0 \in E^2 - K \), suppose that \( x_{x_0, v_0}[0, 1] \) is nowhere tangent to \( \partial K \), and that \( x_{x_0, v_0}(1) \not\in \partial K \). Then \( E \) is \( C^\infty \) near \((x_0, v_0) \in (E^2 - K) \times E^2\), and the restriction of its derivative \( dE_{x_0, v_0} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) to \( \{0\} \times \mathbb{R}^2 \) is a linear isomorphism.

**Proof:** Because the \( K_k \) are disjoint and strictly convex, the endpoints of \( x_{x_0, v_0}[0, 1] \) are nonconjugate. \( \square \)

**Lemma 3.** For \( x_0 \neq x_1 := E(x_0, v_0) \) and with the hypotheses of Lemma 2, there exists an open neighbourhood \( U_0 \) of \( x_0 \) in \( E^2 - K \), and a unique \( C^\infty \) function
\[ \phi := \phi_{x_0,v_0} : U_0 \to (0, \infty) \text{ whose gradient } \nabla \phi \text{ is everywhere of unit length, satisfying} \]
\[ \mathcal{E}(\tilde{x}_0, -\phi(\tilde{x}_0)\nabla \phi(\tilde{x}_0)) = x_1 \]

for all \( \tilde{x}_0 \in U_0 \). Here \( \phi(x_0) = \|v_0\| \) with \( \nabla \phi(x_0) = -v_0/v_0 \).

**Proof:** By Lemma 2 and the implicit function theorem, there exist a unique \( C^\infty \) function \( X : U_0 \to E^2 \) satisfying \( \mathcal{E}(\tilde{x}_0, X(\tilde{x}_0)) = x_1 \) for all \( \tilde{x}_0 \in U_0 \). Because \( x_0 \neq x_1, X \) is never-zero for \( U_0 \) sufficiently small. Then the geodesic \( x_{\tilde{x}_0,X(\tilde{x}_0)} : [0,1] \to M \) joining \( \tilde{x}_0, x_1 \), has length
\[ \phi(\tilde{x}_0) := \|X(\tilde{x}_0)\| = J(x_{\tilde{x}_0,X(\tilde{x}_0)}). \]

Differentiating with respect to \( \tilde{x}_0 \in U_0 \) in the direction of \( \delta \in \mathbb{R}^2 \), we find \( d\phi_{\tilde{x}_0}(\delta) = -\langle X(\tilde{x}_0)/\|X(\tilde{x}_0)\|, \delta \rangle \), because geodesics are critical for \( J \) when variations have fixed endpoints. \( \square \)

The order \( o(x) \) of a geodesic \( x \) is the number of intersections with \( \partial K \). Write \( \mathcal{X}_r := \{x \in \mathcal{X} : o(x) = r\} \).

Let \( d_K \) be the minimum distance between obstacles. Writing \( \tau(x) := t \) for the travelling time (length) of \( x \in \mathcal{X}_r \),
\[ o(x)d_K \leq J(x) \leq o(x)diam(C). \quad (1) \]

## 4 Arcs and Generators for Nowhere-Tangent Geodesics

Let \( \mathcal{X}^0 \) be the space of geodesics \( x \in \mathcal{X} \) that are exactly \( q \)-times tangent to \( \partial K \), and set
\[ \mathcal{T}^q := \{(x(0), x(1), J(x)) : x \in \mathcal{X}^q\} \subset \mathcal{T}. \]

Then \( \mathcal{T}^0 \subset \mathcal{T}^q \) and, by Lemma \( \square \) \( \mathcal{T}^q = \emptyset \) for \( q \geq 3 \). We have constructed \( \mathcal{T}^q_0 \) from \( \mathcal{T} \), but not yet \( \mathcal{T}^q \) for \( q \leq 2 \). For \( x_1 \in C \) define
\[ \mathcal{T}_{x_1} := \{(x_0,t) : (x_0, x_1, t) \in \mathcal{T} \} \subset C \times \mathbb{R}, \mathcal{T}^q_{x_1} := \{(x_0,t) : (x_0, x_1, t) \in \mathcal{T}^q \} \subset \mathcal{T}_{x_1}. \]

Likewise \( \mathcal{X}^q_{x_1} \subset \mathcal{X} \) is the set of geodesics \( x : [0,1] \to M \) with \( x(1) = x_1 \). Write \( \mathcal{X}^0_{x_1} := \mathcal{X}_{x_1} \cap \mathcal{X}^q_{x_1} \).

**Remark 1.** \( \mathcal{T}_{x_1} \) is found directly from \( \mathcal{T} \), but \( \mathcal{T}^q_{x_1} \) is yet to be determined for \( q \leq 2 \). \( \square \)

**Remark 2.** \( \mathcal{T}^0_{x_1} \) is open and dense in \( \mathcal{T}_{x_1}, \mathcal{T}^1_{x_1} \) is open and dense in \( \mathcal{T}^1_{x_1} \cup \mathcal{T}^2_{x_1}, \) and \( \mathcal{T}^2_{x_1} \) is discrete. \( \square \)

**Proposition 2.** For \( x_1 \in C \) we have \( \mathcal{T}^q_{x_1} = \cup_{i \geq 1} \alpha_{i,x_1} \), where
1. the \( \alpha_i := \alpha_{i,x_1} \) are pairwise-transversal \( C^\infty \) open bounded arcs in \( C \times \mathbb{R} \),
2. \( \cup_{q \geq 1} \mathcal{T}^q_{x_1} = \cup_{i \geq 1} \partial \alpha_i \).
3. Each $\alpha_i$ has a generator, namely a $C^\infty$ function $\phi_i : U_i \to \mathbb{R}$ with $U_i \subset E^2$ open, such that

- $U_i \cap C$ is an open arc in $C$, and $\tilde{x}_0 \mapsto (\tilde{x}_0, \phi_i(\tilde{x}_0))$ is a diffeomorphism from $U_i \cap C$ onto $\alpha_i$,
- $x_{\tilde{x}_0, v_i(\tilde{x}_0)}(0,1) \in X_{x_1}^0$, where $\nu_i(\tilde{x}_0) := -\partial_i(\tilde{x}_0) \cdot \nabla \phi_i(\tilde{x}_0)$ for $\tilde{x}_0 \in U_i \cap C$.

**Proof:** For $(x_0,t) \in T_{x_1}^0$, there exists $x \in X_{x_1}$ with $J(x) = t$. Then $x_1 = \mathcal{E}(x_0, v_0)$ where $v_0 = \dot{x}(0)$. By Lemma 3 for some open neighbourhood $U_0$ of $x_0$ in $E^2$, there is a unique $C^\infty$ function $\phi = \phi_{x_0,v_0} : U_0 \to (0, \infty)$ with $\phi(x_0) = t$ and $\nabla \phi_{x_0} = -v_0/\|v_0\|$, such that $\mathcal{E}(\tilde{x}_0, -\phi(\tilde{x}_0) \nabla \phi(\tilde{x}_0)) = x_1$ for all $\tilde{x}_0 \in U_0$. In particular the last equation holds for $\tilde{x}_0 \in U_0 \cap C$, namely $\tilde{x}_0 \mapsto (\tilde{x}_0, \phi(\tilde{x}_0))$ embeds $U_0 \cap C$ in $T_{x_1}^0$. By continuation, the embedding extends uniquely in both directions around $C$, until just before $x$ is tangent to some $\partial K_i$, which must eventually happen. So $T_{x_1}$ is a countable union of $C^\infty$ embedded arcs $\alpha_i$.

Pairwise transversality is proved by contradiction as follows. Suppose $\alpha_i \neq \alpha_i'$ meet tangentially at $(x_0, t) \in T_{x_1}^0$. Then $\nabla \phi_{x_0,v_i}(x_0) = \nabla \phi_{x_0,v_i'}(x_0)$, where $\|v_i\| = t = \|v_i'\|$, and $w \neq 0$ is tangent to $C$ at $x_0$. By Lemma 3, $\nabla \phi_{x_0,v_i'}(x_0)$ and $\nabla \phi_{x_0,v_i}(x_0)$ point out from $B$ at $x_0$. So $-v_i = t \nabla \phi_{x_0,v_i'}(x_0) = t \nabla \phi_{x_0,v_i'}(x_0) = -v_i$ by Lemma 3, contradicting $\alpha_i \neq \alpha_i'$.

Because $T_{x_1}^0$ is open and dense in $T_{x_1} \cup \bigcup_{i \geq 1} \partial \alpha_i$, □

By continuity, the orders $o(\alpha_i)$ of the $x_{\tilde{x}_0, v_i(\tilde{x}_0)} \in \mathcal{X}_0$ are independent of $\tilde{x}_0 \in U_i \cap C$. From (1) we obtain, for all $\tilde{x}_0 \in U_i \cap C$,

$$o(\alpha_i) d_K \leq \phi_i(\tilde{x}_0) \leq o(\alpha_i) \text{diam}(C), \quad (2)$$

and the arcs $\alpha_i$ are similarly bounded. For any $i$, the closures $\bar{\alpha}_i$ and $\bar{\alpha}_i'$ in $T_{x_1}$ are disjoint for all but finitely many $i'$, where $i, i' \geq 1$. The generator $\phi_i$ defines $x_{\tilde{x}_0, v_i(\tilde{x}_0)} \in \mathcal{X}_{x_1}^0$ for every $\tilde{x}_0 \in U_i \cap C$. For $(x_0, t) \in \partial \alpha_i$, define

$$\nu_i(x_0) := \lim_{\tilde{x}_0 \to x_0} \nu_i(\tilde{x}_0) \in E^2.$$

Then $x_{\tilde{x}_0, v_i(\tilde{x}_0)} \in \mathcal{X}_{x_1}^1 \cup \mathcal{X}_{x_1}^2$, and $(x_0, t) \in T_{x_1}^1 \cup T_{x_1}^2$ where $t = \|\nu_i(x_0)\|$.

**Proposition 3.** If $(x_0, t) \in T_{x_1}^1$ then \{$(x_0, t) \} = (\partial \alpha_i) \cap (\partial \alpha_i')$ for some unique $i, i'$ with $o(\alpha_i') = o(\alpha_i) + 1$. Then $U_i \cap C$ and $U_i' \cap C$ are disjoint on the same side of $x_0$ in $C$ and, for $\tilde{x}_0 \in U_i \cap U_i' \cap C$, $\phi_i(\tilde{x}_0) > \phi_i'(\tilde{x}_0)$. We also have

$$\lim_{\tilde{x}_0 \to x_0} \phi_i(\tilde{x}_0) = \lim_{\tilde{x}_0 \to x_0} \phi_i'(\tilde{x}_0) = t \quad \text{and} \quad \lim_{\tilde{x}_0 \to x_0} \nabla \phi_i(\tilde{x}_0) = \lim_{\tilde{x}_0 \to x_0} \nabla \phi_i'(\tilde{x}_0).$$

**Proof:** We can write $t = J(x_{\tilde{x}_0, v_0})$ where $x_{\tilde{x}_0, v_0} \in \mathcal{X}_{x_1}^1$ and $\|v_0\| = t$. Suppose the last (respectively first) segment of $x_{\tilde{x}_0, v_0}$ is not tangent to $\partial K$. Then, by general position, the first (last) segment is tangent. Perturbing the last segment while maintaining the endpoint $x_1$, gives two arcs of nowhere-tangent geodesics,
whose initial points \( \tilde{x}_0 \) lie on the same side of \( x_0 \) in \( C \). Along one arc the first (last) segment remains linear and the order decreases by 1. Along the other arc, the first (last) segment breaks into two linear segments, maintaining the order and increasing the travelling time.

For \( \tilde{x}_0 \) near \( x_0 \), the two arcs of geodesics define arcs \( \{(\tilde{x}_0, \phi_i(\tilde{x}_0))\} \) in \( T^0_{x_0} \) contained in maximal arcs \( \alpha_i, \alpha_{i'} \), labelled so that \( o(\alpha_{i'}) = o(\alpha_i) + 1 \). Then \( \nu_i(x_0) = v_0 = \nu_i(\tilde{x}_0) \), and \( \phi_{i'}(\tilde{x}_0) > \phi_i(x_0) \). We also have \( \lim_{\tilde{x}_0 \to x_0} \phi_i(\tilde{x}_0) = \|v_0\| = \lim_{\tilde{x}_0 \to x_0} \phi_{i'}(\tilde{x}_0) \), and \( \lim_{\tilde{x}_0 \to x_0} \nabla \phi_i(\tilde{x}_0) = v_0/\|v_0\| = \lim_{\tilde{x}_0 \to x_0} \nabla \phi_{i'}(\tilde{x}_0). \)

Since \( T^1_1 \) is dense in \( T^1_1 \cup T^2_1 \), Proposition 3 has the

**Corollary 1.** \( T^1_1 \cup T^2_1 \) is the closure in \( T_{x_0} \) of the set of all points \((x_0, t) \in T_{x_0} \) where \( T_{x_0} \) has an isolated cusp. □

A \( C^\infty \) embedding \( \epsilon \) of \( T_{x_1} \) in \( E^2 \) is given by \( \epsilon(x_0, t) := x_0 + tv(x_0) \), with \( v : C \to E^2 \) some constant-length nonzero outward-pointing normal field. Cusps in \( T_{x_1} \) are found by inspecting the telegraph at \( x_1 \), defined as \( \epsilon(T_{x_1}) \subset E^2 \).

**Example 1.** Figure 7 displays part of \( \epsilon(T_{x_1}) \) with \( x_1 = (0.4, 4) \) with \( n = 2 \) and \( C \) the circle of radius 4 and centre \((0, 0)\). The telegraph is mainly smooth, but different arcs (light-blue, yellow-green and red) meet in cusps, and the first (last) segment breaks into two linear segments, maintaining the order decreases by 1. Along the other arc, the first (last) segment remains linear and the order decreases by 1. Along the other arc, the first (last) segment breaks into two linear segments, maintaining the order and increasing the travelling time.

Next we augment \( T_{x_1} \) and \( T \) to data sets \( \tilde{T}_{x_1} \) and \( \tilde{T} \) that include initial velocities of geodesics. We first exclude points of intersection of the open arcs \( \alpha_i = \alpha_i, x_1 \) in Proposition 2 (these points are reinserted later), by defining \( \alpha^*_i = \alpha^*_i, x_1 := \alpha_i - \cup_{i' \neq i} \alpha_{i'} \).

**Remark 3.** Any \( \alpha_i \) intersects at most finitely many \( \alpha_{i'} \). Because intersections of \( \alpha_i \) and \( \alpha_{i'} \) are transversal for \( i \neq i' \), \( T^0 \) is dense in \( T^0 := \cup_{i \in I} \alpha^*_i, x_1 \), and

\[
\begin{align*}
T^{0*} := \{(x_0, x_1, t) : (x_0, t) \in T^0, x_1 \in C\} & \text{ is dense in } T^0 := \{(x_0, x_1, t) : (x_0, t) \in T^0, x_1 \in C\}.
\end{align*}
\]

□

**Remark 4.** The \( \alpha^*_i, x_1 \) partition \( T^{0*} \). The generators \( \phi_i \) restrict to \( C^\infty \) functions on the open subsets

\[ D_{i, x_1} := \{c \in C : (c, t) \in \alpha^*_i, x_1 \} \]

of \( C \). □

For \( (x_0, t) \in T^{0*} \) define \( u_0 = u_{x_0, t, x_1} \) to be the unit vector \(-\nabla \phi_i(x_0)\) pointing inwards from \( C \). Then set

\[
\begin{align*}
\tilde{T}^{0*}_{x_1} := \{(x_0, u_0, t) : (x_0, t) \in T^{0*}_{x_1}\} & \text{ and } \tilde{T}^{0*} := \{(x_0, u_0, x_1, t) : (x_0, u_0, t) \in \tilde{T}^{0*}_{x_1}\}
\end{align*}
\]
To reinsert the excluded points, define $\tilde{T}^0$ to be the closure of $\tilde{T}^{0*}$ in
\[ \{(x_0, u_0, x_1, t) : (x_0, x_1, t) \in T^0 \text{ with } u_0 \in S^1\}, \]
and $\tilde{T}^0_{x_1} := \{(x_0, u_0, t) : (x_0, u_0, x_1, t) \in \tilde{T}^0\}$. Define $\tilde{T}$ be the closure of $\tilde{T}^0$
in
\[ \{(x_0, u_0, x_1, t) : (x_0, x_1, t) \in T \text{ with } u_0 \in S^1\}, \]
and $\tilde{T}_{x_1} := \{(x_0, u_0, t) : (x_0, u_0, x_1, t) \in \tilde{T}\}$. For $q \geq 1$ define
\[ \tilde{T}^q := \{(x_0, u_0, x_1, t) \in \tilde{T} : (x_0, x_1, t) \in T^q\} \]

5 Singly-Tangent Geodesics

Summarising so far, for $x_1 \in C$,
• $T_{x_1}$ is read directly from $T$,
• $T_{x_1}^+ := T_{x_1}^1 \cup T_{x_1}^2$ (respectively $T_{x_1}^0$) is the non-smooth (respectively smooth) part of $T_{x_1}$,
• we have seen how to find arcs $\alpha_{i,x_1}$ and generators $\phi_i$ for $T_{x_1}^0$,
• $\tilde{T}_{x_1} := \tilde{T}_{x_1}^1 \cup \tilde{T}_{x_1}^2$ and $\tilde{T}_{x_1}^0$ are obtained using the $\phi_i$,
• $\tilde{T}^+ := \tilde{T}^1 \cup \tilde{T}^2$ and $\tilde{T}^0$ are found by varying $x_1$.

To distinguish $\tilde{T}^1$ from $\tilde{T}^2$ we need Proposition 4, which is a structural result, analogous to Proposition 2. A geodesic $x^* \in X$ is said to be \textit{bitangent} when it has two points of tangency to $\partial K$. We call $x^*$ \textit{linear} when it has no other points of contact with $\partial K$.

**Proposition 4.** For some countable locally finite family $\mathcal{B} = \{\tilde{\beta}_j : j \geq 1\}$ of disjoint bounded open $C^\infty$ arcs in $\tilde{T}^+$,

1. $\tilde{T}^1 = \bigcup_{j \geq 1} \tilde{\beta}_j$,
2. for $1 \leq j \leq 4(n^2 - n)$, $\tilde{\beta}_j = \{(x_0,(x_1 - x_0)/t,x_1,t) : (x_0,x_1,t) \in \beta_j\}$, where the $\beta_j$ are the vacuous arcs in $T_{x_1}^1$, defined in 4,
3. for every $j \geq 1$ there is a diffeomorphism $\psi_j : V_j \to \tilde{\beta}_j$ where $V_j$ is an open arc in $C$, and $\psi_j(x_0) \in \{x_0\} \times S^1 \times C \times (0, \infty)$ for all $x_0 \in V_j$,
4. each $(x_0^*,u_0^*,x_1^*,t^*) \in \tilde{T}^2$ is an endpoint of four open arcs $\tilde{\beta}_j, \tilde{\beta}_j', \tilde{\beta}_j'', \tilde{\beta}_j'''$, where three of $V_j, V_j', V_j'', V_j'''$ are on one side of $x_0^* \in C$, and one is on the other side.
5. $\tilde{T}^2 = \bigcup_{j \geq 1} \partial \tilde{\beta}_j$.

**Proof:** For $(x_0,t_0,x_1,t) \in \tilde{T}^1$, we have $x_{x_0,t_0} \in X^1$ and $x_{x_0,t_0}(1) = x_1$. Now $x_{x_0,t_0}$ is tangent to $\partial K$ at precisely one point. By Lemma 1 this is either the first or last point of contact with $\partial K$.

If the tangency is first then, perturbing the point of tangency in $\partial K$ gives a small open $C^\infty$ arc around $(x_0,t_0,x_1,t)$ contained in $\tilde{T}^1$. Similarly, if the tangency is last, an open $C^\infty$ arc in $\tilde{T}^+$ is given by perturbing the point of tangency in $\partial K$. So the path components $\tilde{\beta}_j$ of $\tilde{T}^1$ in $\tilde{T}^+$ are connected smooth 1-dimensional submanifolds of $C \times S^1 \times C \times \mathbb{R}$. They are bounded, nonclosed and, for $1 \leq j \leq 4(n^2 - n)$, can be listed as augmentations of the $\beta_j$. Then 1. and 2. hold.

For $(x_0^*,u_0^*,x_1^*,t^*) \in \tilde{T}^2$ the geodesic $x^* = x_{x_0^*,t^*,u_0^*}$ is tangent to $\partial K$ at both first and last points of contact, and nowhere else. Nearby geodesics in $X^1$ are obtained by maintaining tangency either at a variable first point of contact, or

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3Including possibly $j > 4(n^2 - n)$. 

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at a variable last point of contact with \( \partial K \). The tangencies at first (respectively last) points of contact generate arcs \( \hat{\beta}_j, \hat{\beta}'_j \) (respectively \( \hat{\beta}''_j, \hat{\beta}'''_j \)) in \( \mathcal{T}^1 \), separated by \((x^*_0, u^*_0, x^*_1, t^*)\).

When the bitangent geodesic \( x^* \) is linear, there is an open arc \( V_j \subset C \) of initial points of perturbations initially tangent to \( \partial K_k \), and another open arc \( V'_{j'} \subset C \) of initial points of perturbations initially tangent to \( \partial K_p \), as in Figure 2 where \( x^*_0, V_j, V_{j'} \) appear on the right of the illustration. Perturbations whose initial points are in \( V_j \) (green) and \( V'_{j'} \) (red) have no other points of contact with \( \partial K \). There are also two unlabelled open arcs \( V''_{j''}, V'''_{j'''}, C \) bordered by \( x^*_0 \), consisting of initial points of geodesics whose first points of contact are nontangent to \( \partial K \), and whose second points of contact are tangent to \( \partial K_k \) (green) or \( \partial K_p \) (red) respectively.\(^4\)

Evidently \( j \neq j' \), because \( V_j \) and \( V_{j'} \) are on opposite sides of \( x^*_0 \), and similarly \( j' \neq j'', j''' \) in Figure 2. Indeed, from the geometry of perturbations of \( x^* \), all of \( j, j', j'', j''' \) are distinct.

![Figure 2: A linear bitangent (proof of Proposition 4)](image)

In Figure 3 the nonlinear bitangent geodesic \( x^* \) is tangent to \( \partial K_k \) and \( \partial K_p \) at the first and last points of contact respectively. It is not tangent anywhere else to \( \partial K \), but is reflected at other points of contact, as suggested by the illustration. As before, the nonlinear bitangent is perturbed while maintaining tangency either with \( \partial K_k \) (green) or with \( \partial K_p \) (red), but now the first and

\(^4\)Similarly, the green and red arrows on the left of Figures 2, 3 indicate intervals of terminal points of perturbations.
last points of contact remain on $\partial K_k$ and $\partial K_p$ respectively. The initial points of perturbations tangent to $\partial K_k$ sweep out open arcs $V_j, V_j' \subset C$ (green) on either side of $x_0^*$. Initial points of perturbations tangent to $\partial K_p$ give the other intervals $V_j'', V_j'''$ on one side of $x_0^*$, as indicated by the two red arrows on the left of Figure 3. Again $j, j', j'', j'''$ are distinct.

Figure 3: A nonlinear bitangent geodesic (proof of Proposition 4)

An element $(x_0, u_0, x_1, t)$ of $\tilde{\beta}_j$ corresponds precisely to the point of tangency (first or last contact) of $x_{x_0, t u_0}$ with $\partial K$. Because there is only one point of tangency it corresponds diffeomorphically to $x_0$. This proves 3.

So $(x_0^*, u_0^*, x_1^*, t^*)$ is an endpoint of precisely 4 open arcs and 4. is proved.

Because $\tilde{T}^1$ is open in $\tilde{T}^+, \bigcup_{j \geq 1} \partial \tilde{\beta}_j = \partial \tilde{T}^1 \subseteq \tilde{T}^2$. Because $\tilde{T}^1$ is dense in $\tilde{T}^+$, $\tilde{T}^2 = \bigcup_{j \geq 1} \partial \tilde{\beta}_j$, proving 5. $\square$

**Corollary 2.** $\tilde{T}^1$ is the smooth part of the 1-dimensional space $\tilde{T}^+$. $\square$

**Remark 5.** For $j \geq 1$ the $o(x_{x_0, t u_0})$ for $(x_0, u_0, x_1, t) \in \tilde{\beta}_j$ depend only on $\tilde{\beta}_j$. So we may write them as $o(\tilde{\beta}_j)$. $\square$

We need the following definitions:

- The open arcs $\tilde{\beta}_j \in \tilde{B}$ where $1 \leq j \leq 4(n^2 - n)$ are said to be vacuous.
- For $(x_0, u_0) \in E^2 \times S^1$ denote the undirected line through $x_0$ parallel to $u_0$ by $\lambda(x_0, v_0)$. 11
For \( j \geq 1 \) define \( \lambda_j : V_j \to \mathbb{R}P^2 \) by \( \lambda_j(x_0) = \lambda(x_0, u_0) \) where \( \psi_j(x_0) = (x_0, u_0, x_1, t) \).

- Denote the envelope of \( \lambda_j \) by \( \Lambda_{\bar{\beta}_j} : V_j \to E^2 \).

## 6 Extendible Arcs and the Inductive Step

At the end of the proof of Proposition 5, the travelling-time data \( T \) is used to find \( 4(n^2 - n) \) open arcs \( \beta_j \subset \mathbb{T}^1 \). Each of these is augmented, as described in Proposition 4, to a vacuous open arc \( \tilde{\beta}_j \subset \mathbb{T}^1 \). From the definition in 4 of the conjugate \( j \) of \( j \), for \( 1 \leq j \leq 4(n^2 - n) \),

\[
\Lambda_{\tilde{\beta}_j}(V_j) = \Lambda_{\bar{\beta}_j}(V_j).
\]  

We also obtain \( C^\infty \) parameterisations \( \psi_j : V_j \to \tilde{\beta}_j \). More generally (inductively) suppose we have this kind of information where possibly \( j > 4(n^2 - n) \).

In precise terms, suppose we are given a \( C^\infty \) parameterisation \( \psi_j : V_j \to \tilde{\beta}_j \) of some possibly nonvacuous arc \( \tilde{\beta}_j \in \mathcal{B} \). Here \( V_j \subset C \) is a maximal open arc with the property that, for all \( x_0 \in V_j \) and \( (x_0, u_0, x_1, t) := \psi_j(x_0) \), the first segment of the geodesic \( x_{u_0,t_{u_0}} \) is tangent to \( \partial K_k \). The inductive step extends the open arc \( \Lambda_{\bar{\beta}_j}(V_j) \subset \partial K \) by adjoining another such arc to its clockwise endpoint, as follows.

For \( x_0^* \subset C \) the clockwise terminal limit of \( x_0 \in V_j \), set

\[
(x_0^*, u_0^*, x_1^*, t^*) := \lim_{x_0 \to x_0^*} \psi_j(x_0) \in \mathbb{T}^2.
\]

By Proposition 4 there are three other open arcs \( \tilde{\beta}_j, \tilde{\beta}_j', \tilde{\beta}_j'' \subset \tilde{\beta} \) adjacent to \( \tilde{\beta}_j \subset \mathbb{T}^1 \) at \( (x_0^*, u_0^*, x_1^*, t^*) \), and the unordered set \( \mathcal{B}_j := \{ \tilde{\beta}_j, \tilde{\beta}_j', \tilde{\beta}_j'' \} \subset \tilde{\beta} \) is found by inspecting \( \mathbb{T}^+ \). In the proof of Proposition 4 the arcs \( \tilde{\beta}_j, \tilde{\beta}_j' \) (respectively \( \tilde{\beta}_j'', \tilde{\beta}_j''' \)) are generated by geodesics whose first (respectively last) segments are tangent to \( \partial K \). Construct

\[
\mathcal{B}_j^+ := \{ \tilde{\beta} \in \mathcal{B}_j : V \cap V_j = \emptyset \}
\]

where \( V := \{ x_0 : (x_0, u_0, x_1, t) \in \tilde{\beta} \} \).

**Definition 2.** \( \tilde{\beta} \in \mathcal{B}_j^+ \) is an extension of \( \tilde{\beta}_j \) when the closure \( \Lambda_{\tilde{\beta}_j} \) of \( \Lambda_{\bar{\beta}_j}(V_j) \cup \Lambda_{\bar{\beta}_j}(V) \) is a \( C^\infty \) strictly convex arc in \( E^2 \). When an extension of \( \tilde{\beta}_j \) exists, the arc \( \tilde{\beta}_j \) is said to be extendible (otherwise nonextendible).

**Proposition 5.** If \( \tilde{\beta}_j \) is extendible the extension \( \tilde{\beta} \in \mathcal{B}_j^+ \) is unique, and \( \Lambda_{\tilde{\beta}}(V) \) is an arc in \( \partial K_k \). If \( \tilde{\beta}_j \) is nonextendible then \( \tilde{\beta}_j \) is vacuous and \( \tilde{\beta}_j^* \) is extendible.

\( \text{From the proof of Proposition 4, } \mathcal{B}_j^* \text{ has size 1 or 3.} \)
Proof: By continuity of $\psi_j$, the bitangent $x_{x^*_0, t^* u_0^*}$ is tangent to $\partial K_k$ at some $q := x_{x^*_0, t^* u_0^*}(t_k)$ where $0 < t_k < 1$, and $q$ is a limit of points of first tangency and first contact with $\partial K_k$. By general position $q$ is either the first point of contact of the bitangent with $\partial K$ or the second point of contact.

If $q$ is the first point of contact then $\tilde{\beta}_j$ is extended by $\tilde{\beta}_j'$ whose associated geodesics maintain tangency to $\partial K_k$. Evidently $\Lambda_{\tilde{\beta}_j'}(V_j')$ is an arc in $\partial K_k$.

For $j^* = j''$ or $j^* = j'''$, and $(\tilde{x}_0, \tilde{u}_0, \tilde{x}_1, \tilde{t}) \in \tilde{\beta}_{j^*}$ near $(x^*_0, u^*_0, x^*_1, t^*)$, the last points of contact of $x_{\tilde{x}_0, \tilde{u}_0}$ are tangent to $\partial K$ near $q' \in \partial K_{k'}$ where $q' \neq q$. By the argument in §3 of [4], the $\lambda(\tilde{x}_0, \tilde{u}_0)$ are not all tangent to a $C^\infty$ strictly convex arc, namely $\Lambda_{\beta_{j^*}}$ is not strictly convex, and $\tilde{\beta}_{j^*}$ does not extend $\tilde{\beta}_j$. So the extension $\tilde{\beta} = \tilde{\beta}_j'$ is unique.

If alternatively $q$ is the second point of contact, then the first point of tangency is at $q' := x_{x^*_0, t^* u_0^*}(s') \in \partial K_{k'}$ where $0 < t_{k'} < t_k$ with $k' \neq k$. By Lemma [4], $q'$ is the first point of contact of the bitangent with $\partial K$. By Lemma [4], and because $q$ is the second point of tangency, the bitangent is linear with $q, q'$ the only points of contact with $\partial K$. So $1 \leq j \leq 4(n^2 - n)$, and $q'$ is the first point of contact of the linear bitangent $x_{x^*_1, -t^* u_0^*}$ with $\partial K$. Then $\tilde{\beta}_j$ is extended by requiring tangency to $\partial K_k$ of the associated geodesics. □

The arc $\Lambda_{\tilde{\beta}_j}(V_j)$ in $\partial K_k$ is therefore extended by an incremental arc $\Lambda_{\tilde{\beta}}(V)$, where $\tilde{\beta}$ is an extension either of $\tilde{\beta}_j$ or of $\tilde{\beta}_j'$. This completes the inductive step.

Now the construction of $\partial K$ proceeds as follows. First $\tilde{\beta}_j$ is chosen with $1 \leq j \leq 4(n^2 - n)$, and the inductive step is carried out repeatedly with $\tilde{\beta}$ replacing $\beta_j$ after each step, until the incremental arcs $\Lambda_{\tilde{\beta}}(V)$ in $\partial K$ are acceptably small. Then another vacuous arc is used to restart the iterative process. This is repeated until all the vacuous arcs are used. Finally $\partial K$ is the union of the closures of all the arcs (initial and incremental) in $\partial K$.

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*Countably many repetitions would be needed for perfect reconstruction.*
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