Schrödinger functional boundary conditions and improvement of the $SU(N)$ pure gauge action for $N > 3$

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Outline

1. Motivation
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Motivation

- Schrödinger functional boundary fields know only for \( N = 2, 3, 4 \)
  - More general analysis needed

- Boundary improvement for the gauge fields know for \( N = 2, 3 \)
  - Needed for reliable coupling constant measurements on the lattice

- Applications in beyond the standard model physics and large \( N \) limit
\( \mathcal{O}(a) \) improved SU\((N)\) gauge action in the Schrödinger functional scheme

\[
S = S_G + \delta S_{G,b} + S_{gf} + S_{FP},
\]

\[
S_G = \frac{1}{g_0^2} \sum_p \text{Tr}[1 - U(p)],
\]

\[
\delta S_{G,b} = \frac{1}{g_0^2} (c_t - 1) \sum_{p_t} \text{Tr}[1 - U(p_t)],
\]

\[
c_t = 1 + \left( c_t^{(1,0)} + N_F c_t^{(1,1)} \right) g_0^2 + \mathcal{O}(g_0^4)
\]

- For the specific form of \( S_{gf} \) and \( S_{FP} \), see\(^1\)

\(^1\)M. Lüscher, R. Narayanan, P. Weisz, U. Wolff, hep-lat/9207009v1
Theory: Schrödinger functional

Schrödinger functional boundary conditions

\[ U_k(t = 0, \vec{x}) = \exp[aC_k], \quad U_k(t = L, \vec{x}) = \exp[aC'_k] \]

\[ C_k = \frac{i}{L} \text{diag}(\phi_1(\eta), \ldots, \phi_n(\eta)), \quad C'_k = \frac{i}{L} \text{diag}(\phi'_1(\eta), \ldots, \phi'_n(\eta)) \]

- These boundary conditions induce a constant chromo-electric field

Effective action

\[ \Gamma = -\ln \left\{ \int D[\psi]D[\bar{\psi}]D[U]D[c]D[\bar{c}]e^{-S} \right\} = g_0^{-2}\Gamma_0 + \Gamma_1 + \mathcal{O}(g_0^2) \]
Theory: Step scaling

Running coupling

\[ g^2 = \frac{\partial \Gamma_0}{\partial \eta} \bigg/ \frac{\partial \Gamma}{\partial \eta} = g_0^2 - g_0^4 \frac{\partial \Gamma_1}{\partial \eta} \bigg/ \frac{\partial \Gamma_0}{\partial \eta} + \mathcal{O}(g_0^6) \]

Lattice step scaling function and its perturbative expansion

\[ \Sigma(u, s, L/a) = g^2(g_0, sL/a) \bigg|_{g^2(g_0, L/a)=u} = u + \left[ \Sigma_{1,0}(s, L/a) + \Sigma_{1,1}(s, L/a)N_F \right] u^2, \]

Definition of \( \delta_i \)

\[ \delta_i = \frac{\Sigma_{1,i}(2, L/a)}{\sigma_{1,i}(2)} = \frac{\Sigma_{1,i}(2, L/a)}{2b_{0,i} \ln 2}, \quad i = 0, 1. \]

\[ b_{0,0} = 11N_c/(48\pi^2), \quad b_{0,1} = N_f T_R/(12\pi^2). \]
**Fundamental domain**

Boundary fields $\phi$ and $\phi'$ are within the fundamental domain if

$$\phi_1 < \phi_2 < \ldots < \phi_n, \quad |\phi_i - \phi_j| < 2\pi, \quad \sum_{i=1}^{N} \phi_i = 0.$$ 

Vectors $\phi$ form a $N - 1$ simplex with vertices

$$X_1 = \frac{2\pi}{N} (-N + 1, 1, 1, \ldots, 1)$$
$$X_2 = \frac{2\pi}{N} (-N + 2, -N + 2, 2, \ldots, 2)$$
$$X_3 = \frac{2\pi}{N} (-N + 3, -N + 3, -N + 3, 3, \ldots, 3)$$

$$\vdots$$

$$X_{N-1} = \frac{2\pi}{N} (-1, -1, \ldots, -1, N - 1)$$
$$X_N = (0, 0, \ldots, 0).$$

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SF boundary conditions and $O(a)$ improvement
Figure: Fundamental domain of $SU(4)$
Define a mapping $R_{i,j}(\phi)$ s.t. it reflects points in FD w.r.t. $(N - 2)$ d hyperplane

- Goes through vertices $X_k, k \neq i, j$
- Intersects line connecting $X_i$ and $X_j$ at the middle

**Composite mapping from FD to itself**

$$M(\phi) = (R_{1,N-1} \circ R_{2,N-2} \circ \cdots \circ R_{\lfloor N/2 \rfloor,N-\lfloor N/2 \rfloor})(\phi)$$

- $\phi'$ derived using above mapping

**Transformation rule for components of $\phi'$ and $\phi$**

$$\phi'_i = \phi_{N-i+1}$$

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$R_{i,i}(\phi)$ is the identity mapping and $\lfloor x \rfloor$ means the integer part of $x$
Fundamental domain $N = 4$

Figure: All possible $R_{i,j}(\phi)$ hyperplanes on FD of SU(4)
Conjecture: Signal to noise maximized if $\phi$ and $\phi'$ chosen s.t.

- they are as far from the edges and each other as possible
- $\phi$ and $\phi'$ transformed to each other using the previous transformation
- We choose $\phi$ to be in the middle of a line connecting $X_1$ and the centeroid of FD
- We associate flow $^3 \ t(\eta)$ to direction which is mirrored by $R_{1,N-1}(\phi)$ and points outside from FD

$$t(\eta) = \frac{\eta^N}{2\pi(N-2)} \left( X_1 - X_{N-1} \right)$$

$$= \left( -\eta, \frac{2\eta}{N-2}, \cdots, \frac{2\eta}{N-2}, -\eta \right)$$

$^3$The normalization is chosen to match the standard case of SU(3)
Fundamental domain $N > 3$

| Example of the boundary fields |
|--------------------------------|
| SU(4)  | $\phi = \begin{cases} -\eta - 9\pi/8 \\ \eta + \pi/8 \\ \eta + 3\pi/8 \\ -\eta + 5\pi/8 \\ -\eta - 6\pi/5 \\ 2\eta/3 \\ 2\eta/3 + \pi/5 \\ 2\eta/3 + 2\pi/5 \\ -\eta + 3\pi/5 \\ -\eta - 5\pi/4 \\ \eta/2 - \pi/12 \\ \eta/2 + \pi/12 \\ \eta/2 + \pi/4 \\ \eta/2 + 5\pi/12 \\ -\eta + 7\pi/12 \end{cases}$ |
| $\phi' = \begin{cases} \eta - 5\pi/8 \\ -\eta - 3\pi/8 \\ -\eta - \pi/8 \\ -\eta + 9\pi/8 \\ \eta - 3\pi/5 \\ -2\eta/3 - 2\pi/5 \\ -2\eta/3 - \pi/5 \\ -2\eta/3 \\ \eta + 6\pi/5 \\ \eta - 7\pi/12 \\ -\eta/2 - 5\pi/12 \\ -\eta/2 - \pi/4 \\ -\eta/2 - \pi/12 \\ -\eta/2 + \pi/12 \\ \eta + 5\pi/4 \end{cases}$ |
| SU(5)  | $\phi = \begin{cases} 2\eta/3 + \pi/5 \\ 2\eta/3 + 2\pi/5 \\ -\eta + 3\pi/5 \\ -\eta - 5\pi/4 \\ \eta/2 - \pi/12 \\ \eta/2 + \pi/12 \\ \eta/2 + \pi/4 \\ \eta/2 + 5\pi/12 \\ -\eta + 7\pi/12 \end{cases}$ |
| $\phi' = \begin{cases} \eta - 3\pi/5 \\ -2\eta/3 - 2\pi/5 \\ -2\eta/3 - \pi/5 \\ -2\eta/3 \\ \eta + 6\pi/5 \\ \eta - 7\pi/12 \\ -\eta/2 - 5\pi/12 \\ -\eta/2 - \pi/4 \\ -\eta/2 - \pi/12 \\ -\eta/2 + \pi/12 \\ \eta + 5\pi/4 \end{cases}$ |
| SU(6)  | $\phi = \begin{cases} 2\eta/3 + \pi/5 \\ 2\eta/3 + 2\pi/5 \\ -\eta + 3\pi/5 \\ -\eta - 5\pi/4 \\ \eta/2 - \pi/12 \\ \eta/2 + \pi/12 \\ \eta/2 + \pi/4 \\ \eta/2 + 5\pi/12 \\ -\eta + 7\pi/12 \end{cases}$ |
| $\phi' = \begin{cases} \eta - 3\pi/5 \\ -2\eta/3 - 2\pi/5 \\ -2\eta/3 - \pi/5 \\ -2\eta/3 \\ \eta + 6\pi/5 \\ \eta - 7\pi/12 \\ -\eta/2 - 5\pi/12 \\ -\eta/2 - \pi/4 \\ -\eta/2 - \pi/12 \\ -\eta/2 + \pi/12 \\ \eta + 5\pi/4 \end{cases}$ |
First approximation of signal strength is $\partial_\eta \Gamma_0 = \partial_\eta g_0^2 S[V]

In $^4$ Lucini et.al. used boundary condition

$$\phi = \begin{cases} 
-\eta/2 - \sqrt{2}\pi/4 \\
-\eta/2 - (2 - \sqrt{2})\pi/4 \\
\eta/2 + (2 - \sqrt{2})\pi/4 \\
\eta/2 + \sqrt{2}\pi/4 
\end{cases} \quad \phi' = \begin{cases} 
\eta/2 - (2 + \sqrt{2})\pi/4 \\
\eta/2 - (4 - \sqrt{2})\pi/4 \\
-\eta/2 + (4 - \sqrt{2})\pi/4 \\
-\eta/2 + (2 + \sqrt{2})\pi/4 
\end{cases}$$

$\partial_\eta \Gamma_0[\text{Us}] = 48L^2 \sin((2\eta + \pi/2)/L^2)$ vs. $\partial_\eta \Gamma_0[\text{Lucini}] = 24L^2 \sin((\eta - \pi/2)/L^2)$

Enhancement by a factor of 2

$^4$B. Lucini and G. Moraitis, hep-lat/0805.2913
Boundary improvement with $N > 3$

- Improvement coefficient $c_t^{(1,0)}$ previously known to one loop only for $N = 2, 3$
  
  $$c_t^{(1,0)}(SU(2)) = -0.0543(5), \quad c_t^{(1,0)}(SU(3)) = -0.08900(5)$$

- Can similarly\(^5\) be calculated for $N > 3$

- Preliminary results

| $N$ | $c_t^{(1,0)}$  | $\delta c_t^{(1,0)}$ |
|-----|----------------|-----------------------|
| 2   | $-0.0543$      | 0.0002                |
| 3   | $-0.088$       | 0.005                 |
| 4   | $-0.1220$      | 0.0002                |
| 5   | $-0.154$       | 0.004                 |
| 6   | $-0.1859$      | 0.0008                |
| 7   | $-0.218$       | 0.004                 |
| 8   | $-0.249$       | 0.004                 |

\(^5\)M. Lüscher, R. Narayanan, P. Weisz, U. Wolff, hep-lat/9207009v1
Figure: The unimproved (green) and improved (blue) one loop lattice step scaling function normalized to the continuum limit as a function of $a/L$ for $SU(N)$ pure gauge with $2 \leq N \leq 8$
We expect
\[ c_t^{(1,0)} = AC_2(R) + BC_2(G) = \left(\frac{A}{2} + B\right)N - \frac{A}{(2N)} \]

Figure: \( c_t^{(1,0)} \) as a function of \( N \) and \( C_1N + C_2/N \) fit to the data
Thank you!