Partial Tail Correlation for Extremes

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Abstract

In order to understand structural relationships among sets of variables at extreme levels, we develop an extremes analogue to partial correlation. We begin by developing an inner product space constructed from transformed-linear combinations of independent regularly varying random variables. We define partial tail correlation via the projection theorem for the inner product space. We show that the partial tail correlation can be understood as the inner product of the prediction errors from transformed-linear prediction. We connect partial tail correlation to the inverse of the inner product matrix and show that a zero in this inverse implies a partial tail correlation of zero. We then show that under a modeling assumption that the random variables belong to a sensible subset of the inner product space, the matrix of inner products corresponds to the previously-studied tail pairwise dependence matrix. We develop a hypothesis test for partial tail correlation of zero. We demonstrate the performance in two applications: high nitrogen dioxide levels in Washington DC and extreme river discharges in the upper Danube basin.

1 Motivation

Describing, characterizing, and modeling high dimensional extremes is a formidable challenge. In the past few years, there has been a concerted effort to develop tools to understand extremal structure among pairs of variables and to use this learned structure to develop simplified models for high dimensional extremes. An exciting approach first studied by Gissibl and Klüppelberg [2018] constructs a max-linear model whose structure is given by a directed acyclic graph. This approach has been subsequently studied by Gissibl et al. [2021] who considered estimation for these models, Améndola et al. [2022] who studied conditional independence for max-linear networks, by Klüppelberg and Kraft [2021] who develop a method for understanding causal order in these models, and by Tran et al. [2021] who develop an efficient estimation algorithm. In another exciting approach, Engelke and Hitz [2020] develop the notion of conditional independence for a multivariate Pareto distribution and connect to a graphical representation. Engelke and Hitz [2020] focus on the Hüsler and Reiss [1989] model and show that its graphical structure coincides with sparsity in the inverse covariance matrices associated with this model.

In this work, we develop the notion of partial tail correlation, a novel method for characterizing and investigating structural relationships between the extremes of pairs of variables. We rely on multivariate regular variation on the positive orthant to describe extremal dependence in the upper tail. Starting with the inner product space defined in Lee and Cooley [2021], we develop the projection theorem for this space. Partial tail correlation is defined via the inner product of the prediction errors associated with transformed linear prediction. Similar to the Gaussian case, we connect partial tail correlation to the inverse of the inner product matrix, and show that a zero element in this inverse matrix implies a partial tail correlation of zero. A type of model sparsity follows as zero partial tail correlation implies two variables provide no additional linear information about one another, given the information provided by the other variables. Our approach differs from the max-linear approach of Gissibl and Klüppelberg [2018] in that the vector space

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we create is more closely linked to ideas from linear models in non-extreme statistics. Our approach is less model-based than [Engelke and Hitz 2020], as we do not specify a full model and instead only work from summaries of pairwise dependence.

The trade off for being less model based is that we cannot go so far as to say that two variables which have zero partial tail correlation are conditionally independent. Our work has similar implications as the notion of partial correlation in non-extreme statistics, which we briefly review here. When a model is specified to be Gaussian, conditional relationships can be completely specified since conditional distributions are obtainable and remain Gaussian. For Gaussian random vector $X$, if the $(i,j)$th element of the precision (inverse covariance) matrix is zero, then $X_i$ and $X_j$ are conditionally independent given $X_{\setminus(i,j)}$, all the remaining elements of $X$. When a distributional assumption is not made and one cannot fully characterize conditional distributions partial correlation provides a measure of the strength of the linear relationship between two variables after accounting for the information in the remaining variables. Consider a centered random vector $X$ with covariance matrix $\Sigma$. Partition to obtain $X = (X_1^\top, X_2^\top)^\top$ where $X_1 = (X_i, X_j)^\top$ and $X_2 = X_{\setminus(i,j)}$, and partition the covariance matrix accordingly

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}. $$

Partial correlation can be connected to the idea of prediction errors that are the orthogonal projections of $X_1$ onto the orthogonal space spanned by $X_2$ [Anderson 1962]. Letting $\Sigma_{1|2} = \mathbb{E}((X_1 - X_{1|2})(X_1 - X_1)^\top)$, where $X_{1|2} = (X_i, X_j)^\top$ is the vector of best linear predictors (in terms of mean squared prediction error) of $X_1$ and $X_j$ given $X\setminus(i,j)$, the partial correlation between $X_i$ and $X_j$ given $X\setminus(i,j)$ is given by

$$\rho_{ij} = \frac{\Sigma_{1|2|2}}{\sqrt{\Sigma_{1|2|1}} \Sigma_{2|2}}. $$

Note that $\rho_{ij} = 0$ if and only if $\Sigma_{1|2|2} = 0$. By matrix inversion, one can show that if the $(i,j)$th element in the precision matrix $\Sigma^{-1}$ is zero, then the partial correlation between $X_i$ and $X_j$ is also zero.

In the following we develop a similar parameter for describing structure between the extremes of pairs of variates. After reviewing multivariate regular variation and measures of tail dependence in Section 2, we present the inner product space and projection theorem in Section 3 and introduce partial tail correlation in Section 4. In Section 5, we argue that a subset of the inner product space provides a suitable setting for modeling and in this subspace the inner product matrix coincides with the tail pairwise dependence matrix (TPDM). In Section 6, we develop a test for the hypothesis that the partial tail correlation is zero. We investigate partial tail correlation in two applications: high nitrogen dioxide levels at five stations in Washington DC, and extreme river discharges at 31 gauging stations in the upper Danube basin.

2 Background

2.1 Multivariate Regular Variation

Our framework assumes multivariate regular variation [Resnick 2007], which is often used for extreme value modeling due to its close relationship to a characterization of the class of multivariate extreme value distributions [De Haan and Ferreira 2007, Appendix B]. A multivariate regularly varying distribution has a heavy tail. $X$ is a $p$-dimensional regularly varying random vector in $\mathbb{R}_+^p = [0, \infty)^p$ if there exists a normalizing function $b(s) \to \infty$ as $s \to \infty$ and a non-degenerate limit measure $\nu_X$ for sets in $E := [0, \infty)^p \setminus \{0\}$ such that

$$s \mathbb{P}(b(s)^{-1} X \in \cdot) \overset{\nu}{\to} \nu_X(\cdot)$$

as $s \to \infty$, where $\overset{\nu}{\to}$ indicates vague convergence in the space of non-negative Radon measures on $\mathbb{E}$. The normalizing function is of the form $b(s) = U(s)s^{1/\alpha}$ where $U(s)$ is slowly varying, and the tail index $\alpha$ determines the power law of the tail. We denote by $X \in RV_p^\alpha(\alpha)$.

Given any norm $\| \cdot \|$ and letting $(R, \mathcal{W}) = (\| X \|, X/\| X \|)$, regular variation implies

$$s \mathbb{P}((b(s)^{-1} R, W) \in \cdot) \overset{\nu_\alpha}{\to} cv \times H_X,$$

where $\nu_\alpha$ is a measure on $(0, \infty]$ such that $\nu_\alpha([x, \infty]) = cx^{-\alpha}$, and $H_X$ is a measure on the unit ball $\Theta_p^\alpha = \{ x \in \mathbb{E} : \| x \| = 1 \}$. The angular (or spectral) measure $H_x$ fully characterizes tail dependence in the limit; however, modeling $H_X$ is challenging in high dimensions.
2.2 Tail Pairwise Dependence Matrix

Rather than fully characterizing the angular measure, we instead rely on a matrix of bivariate summary measures which provides incomplete, but attainable and useful information about tail dependence. We assume $X \in RV^p_+(\alpha)$ and define $H_X$ via the $L_2$ norm, defining $\Theta^{\alpha-1}_+= \{x \in [0, \infty)^p \setminus \{0\} : ||x||_2 = 1\}$. The tail pairwise dependence matrix (TPDM) is $\Sigma_X = \{\sigma_{ij}\}_{i,j=1,\ldots,p,} \in \mathbb{R}^{p \times p}$, where

$$
\sigma_{ij} = \int_{\Theta^{\alpha-1}_+} w_i w_j dH_X(w).
$$

(3)

Although recently Kiriliouk and Zhou [2022] generalized the TPDM for any $\alpha > 0$, by assuming $\alpha = 2$ and choosing the $L_2$ norm, $\Sigma_X$ has similar properties to a covariance matrix. Most importantly, $\Sigma_X$ can be shown to be positive semi-definite [Cooley and Thibaud [2019]]. Additionally, the diagonal elements $\sigma_{ii}$ imply the relative magnitudes of elements $X_i$ via tail probabilities, as $\lim_{s \to \infty} s P(b(s)^{-1} X_i > c) = c^{-2}\sigma_{ii}$. By letting $x = cU(s)s^{1/2}$, there is a corresponding slowly varying function $L$ such that the relation can be rewritten as

$$
\lim_{x \to \infty} \frac{P(X_i > x)}{x^{-2}L(x)} = \sigma_{ii}.
$$

(4)

The sum of diagonal elements is identical to the total mass of the angular measure since $\sum_{i=1}^p \sigma_{ii} = \int_{\Theta^{\alpha-1}_+} dH_X(w)$. Additionally, there are two other noteworthy properties of the TPDM. The variables $X_1$ and $X_2$ are asymptotically independent [Ledford and Tawn [1996] Resnick [2002]] if and only if $\sigma_{ij} = 0$. Also, $\Sigma_X$ is completely positive; that is, there exists some $q_* < \infty$ and a nonnegative $p \times q_*$ matrix $A$ such that $\Sigma_X = A_+A^*_+$ [Cooley and Thibaud [2019]].

3 Projection Theorem in Inner Product Space $\mathcal{V}^q$

3.1 Transformed Linear Operations and Inner Product Space $\mathcal{V}^q$

For the time being, we turn our attention to a subclass of $RV^p_+(\alpha)$ constructed from transformed linear combinations of independent regularly varying random variables. Cooley and Thibaud [2019] define transformed linear operations which, when applied to vectors in $\mathbb{R}^p_+$, remain in $\mathbb{R}^p_+$. They begin with a monotone bijection function $t$ mapping $\mathbb{R}$ to $\mathbb{R}_+$, understood to be componentwise when applied to vectors. For $x_1$ and $x_2 \in \mathbb{R}^p_+$, define $x_1 \oplus x_2 = t(t^{-1}(x_1) + t^{-1}(x_2))$, and $a \circ x_1 = t(a^{-1}(x_1))$ for $a \in \mathbb{R}$. Cooley and Thibaud [2019] further show that if $X_1, X_2 \in RV^p_+(\alpha)$, if $\lim_{y \to \infty} t(y)/y = \lim_{x \to \infty} t^{-1}(x)/x = 1$ and the components of $X_1, X_2$ meet a lower-tail condition specified by the behavior of $t$ as $y \to -\infty$, $X_1 \oplus X_2$ and $a \circ X_1$ are both in $RV^p_+(\alpha)$.

Additionally, Cooley and Thibaud [2019] consider a simple and useful model construction for $X \in RV^p_+(2)$. Applying a $p \times q$ matrix $A$ with $\max_{i=1,\ldots,p} a_{ij} > 0$ for all $j = 1,\ldots,q$, to a regularly varying random vector of independent elements $Z \in RV^q_+(2)$, they induce tail dependence in $X$. The angular measure consists of point masses $H_X = \sum_{j=1}^q \|a_{ij}\|^2 \delta_{a_{ij}^\circ} / \|a^\circ\|$, where $\delta_\cdot$ is a Dirac measure and the zero operation $a^{(0)} = \max(a, 0)$ is applied to vectors or matrices componentwise. The random vector $X$ has tail PDF $\Sigma_X = A^{(0)}A^{(0)\top}$, and the ‘scale’ of $X_i$ in terms of the generating noise $Z$ is $\sigma_{ii} = \lim_{z \to -\infty} P(X_i > z)/P(Z > z)$.

Lee and Cooley [2021] extend these ideas to construct a vector space of regularly varying random variables. Consider

$$
\mathcal{V}^q = \{X; X = a^\top \circ Z = a_1 \circ Z_1 \oplus \cdots \oplus a_q \circ Z_q\},
$$

(5)

where $a \in \mathbb{R}^q$, $Z = (Z_1,\ldots,Z_q)^\top$ and $Z_j \in RV^q_+(2), j = 1,\ldots,q$ with a common normalization $\lim_{z \to -\infty} P(Z_j > z)/P(Z_j > 0) = 1$. To make things specific, we define $t(y) = \log(\exp(y) + 1)$ and assume the lower tail condition $s P(Z_j \leq \exp(-kb(s))) \to 0$ as $s \to \infty$ is met for any $k > 0$, and for $j = 1,\ldots,q$. The vector space $\mathcal{V}^q$ differs from the one in Cooley and Thibaud [2019] which was not stochastic.

Lee and Cooley [2021] define the inner product for elements $X_1 = a_1^\top \circ Z$ and $X_2 = a_2^\top \circ Z$ in $\mathcal{V}^q$ as

$$
\langle X_1, X_2 \rangle := a_1^\top a_2 = \sum_{i=1}^q a_{1i} a_{2i}.
$$

The angle between $X_1$ and $X_2$ is $\theta = \cos^{-1}(\langle X_1, X_2 \rangle / (||X_1|| \cdot ||X_2||)) \in [0, \pi]$, and $X_1, X_2$ are orthogonal if $\langle X_1, X_2 \rangle = 0$. The norm of $X$ is defined as $\|X\|_{\mathcal{V}^q} = \sqrt{\langle X, X \rangle}$, and the subscript $\mathcal{V}^q$ is used to remind that
the norm is based on the coefficients which determine the random variable and to distinguish from the usual Euclidean norm based on a location in space.

For a random vector $\mathbf{X} = (X_1, \ldots, X_p)^T$ whose elements $X_i = a_i^T \circ Z \in \mathcal{V}$, we can denote $\mathbf{X} = A \circ Z$, where $A = (a_1, \ldots, a_p)^T$. We denote the inner product matrix by

$$\Gamma_X = (X_i, X_j)_{i,j=1,\ldots,p} = AA^T. \quad (6)$$

$\Gamma_X$ is defined only for $X_i \in \mathcal{V}$, but will be linked to the TPDM $\Sigma_X$ which exists for $\mathbf{X} \in RV^p_+(2)$ in section 5.

Lee and Cooley [2021] develop transformed-linear prediction of an unobserved $X_{p+1}$ given $\mathbf{X} = (X_1, \ldots, X_p)^T$ by finding $\hat{X}_{p+1} = b^T \circ \mathbf{X}$ that minimizes $\|X_{p+1} \ominus \hat{X}_{p+1}\|_\mathcal{V}$. While this metric is defined by the (unobserved) coefficients of the random variable $X_{p+1} \ominus \hat{X}_{p+1}$, Lee and Cooley [2021] show that it minimizes the estimable scale $D = \lim_{z \to \infty} \frac{P(\max(X_{p+1} \ominus \hat{X}_{p+1}, \hat{X}_{p+1} \ominus X_{p+1}) > z)}{P(Z > z)}$.

$\mathbf{3.2 \quad \text{Projection Theorem in } \mathcal{V}}$

Lee and Cooley [2021] focus on prediction as a minimization problem and only briefly mention the projection theorem as an alternative to obtain the same predictor; here, we present a more thorough development. Our development is similar to the presentation in Cline [1983] who considered standard linear combinations of symmetric regularly varying random variables, and more generally that of Brockwell et al. [1991]. We defer all proofs to the Appendix A and B.

As any $\mathbf{X} \in \mathcal{V}$ is uniquely identifiable by its vector of coefficients $a$, $\mathcal{V}$ is isomorphic to $\mathbb{R}^q$ with the same inner product, and thus is a Hilbert space [Lee 2022]. Let $\mathbf{X} = A \circ Z$ as in Section 3.1 and assume $p < q$. We consider the closed subspace $\mathcal{V}_A = \{b^T \circ \mathbf{X} ; b \in \mathbb{R}^q\}$, that is, the space spanned by $\{X_1, \ldots, X_p\}$. Define the orthogonal complement of $\mathcal{V}_A$, $\mathcal{V}_A^\perp = \{\mathbf{X} \in \mathcal{V} ; \langle X, Y \rangle = 0, \forall Y \in \mathcal{V}_A\}$; that is, $\mathcal{V}_A^\perp$ is the set of all elements of $\mathcal{V}$ which are orthogonal to all elements of $\mathcal{V}_A$.

**Theorem 3.1. (Projection theorem)** Let $\mathcal{V}_A$ be the previously defined subspace of the Hilbert space $\mathcal{V}$ and $\mathbf{X} \in \mathcal{V}$.

1. There exists a unique element $\hat{X} \in \mathcal{V}_A$ such that

$$||X \ominus \hat{X}||_\mathcal{V} = \inf_{Y \in \mathcal{V}_A} ||X \ominus Y||_\mathcal{V}, \text{ and}$$

2. $\hat{X} \in \mathcal{V}_A$ such that $||X \ominus \hat{X}||_\mathcal{V} = \inf_{Y \in \mathcal{V}_A} ||X \ominus Y||_\mathcal{V}$ if and only if $\hat{X} \in \mathcal{V}_A$ and $(X \ominus \hat{X}) \in \mathcal{V}_A^\perp$.

Define the transformed linear projection mapping $P_{\mathcal{V}_A}$ from $\mathcal{V}$ onto $\mathcal{V}_A$ to be

$$P_{\mathcal{V}_A} \mathbf{X} = \{\hat{X} \in \mathcal{V}_A \text{ such that } ||X \ominus \hat{X}||_\mathcal{V} = \inf_{Y \in \mathcal{V}_A} ||X \ominus Y||_\mathcal{V}\}.$$  

One can think of this mapping as taking the argument $\mathbf{X} \in \mathcal{V}$ and returning $\hat{X} \in \mathcal{V}_A$ which is closest to $\mathbf{X}$. By Theorem 3.1, $\hat{X}$ has the property

$$\langle X \ominus \hat{X}, Y \rangle = 0 \quad (7)$$

for all $Y \in \mathcal{V}_A$. By definition, $\hat{X} = b^T \circ \mathbf{X}$, where $b$ is such that $||X \ominus (b^T \circ X_p)||_\mathcal{V}$ is minimized. Lee and Cooley [2021] consider transformed linear predictors of $X_{p+1}$ given the vector $\mathbf{X}$ and use the fact that (7) holds for $X_1, \ldots, X_p$ to find the optimal weights $b$.

The following proposition gives two useful properties of projection mappings.

**Proposition 3.1.** Let $P_{\mathcal{V}_A}$ be the projection mapping of $\mathcal{V}$ onto a subspace $\mathcal{V}_A$.

1. $P_{\mathcal{V}_A}(\alpha \circ X \ominus \beta \circ Y) = \alpha \circ P_{\mathcal{V}_A} X \ominus \beta \circ P_{\mathcal{V}_A} Y, \quad X, Y \in \mathcal{V}, \quad \alpha, \beta \in \mathbb{R}$. [That is, the projection mapping $P_{\mathcal{V}_A}$ is a linear mapping.]

2. For every $\mathbf{X} \in \mathcal{V}$, there exists an element of $\mathcal{V}_A$ and an element of $\mathcal{V}_A^\perp$ such that

$$\mathbf{X} = P_{\mathcal{V}_A} \mathbf{X} \ominus (I - P_{\mathcal{V}_A}) \mathbf{X},$$

where $I$ is the identity mapping on $\mathcal{V}$, and this decomposition is unique.
3.3 Inner Product Matrix of Prediction Errors

Rather than considering predicting $X_{p+1}$ based on observed $X$, we partition $X = (X_1^T, X_2^T)^T$, where $X_1$ has dimension $p_1 < p$ and $X_2$ has dimension $p - p_1$. Without loss of generality, $X$ can be reordered so that $X_1$ is any subvector of elements of $X$. Partitioning $A$ yields $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. The matrix of inner products of $(X_1^T, X_2^T)^T$ is

$$
\Gamma_X = \begin{bmatrix} A_1 A_1^T & A_1 A_2^T \\ A_2 A_1^T & A_2 A_2^T \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}.
$$

(8)

We assume the elements of $X$ are linearly independent; that is, that no element is a transformed-linear combination of the others, and consequently $\Gamma_X$ is full rank. We now consider the problem of finding $P_{\mathcal{V}_{A_2}} X_1 = (P_{\mathcal{V}_{A_2}} X_1, \ldots, P_{\mathcal{V}_{A_2}} X_p)^T$. To simplify notation, denote $X_1 = P_{\mathcal{V}_{A_2}} X_1$ and $X_q = P_{\mathcal{V}_{A_2}} X_q$ for $q = 1, \ldots, p_1$ and let $\hat{B} \in \mathbb{R}^{p_1 \times (p-p_1)}$ be such that $\hat{X}_1 = \hat{B} \circ X_2$. As equation (7) implies \( \langle X_q \ominus \hat{X}_q, X_r \rangle = 0 \) for $q = 1, \ldots, p_1$ and $r = p_1 + 1, \ldots, p$, we obtain $A_1 - \hat{B} A_2 = 0$. $\hat{B} = \Gamma_{12} \Gamma_{22}^{-1}$. The transformed linear predictor in [Lee and Cooley 2021] is the case when $p_1 = 1$.

With this best linear predictor, we can consider the vector of prediction errors $X_1 \ominus \hat{X}_1 = (A_1 - \hat{B} A_2) \circ Z$. The inner product matrix of these prediction errors is given by the Schur complement

$$
\Gamma_{1/2} = (A_1 - \hat{B} A_2) (A_1 - \hat{B} A_2)^T = \Gamma_{11} - \Gamma_{12} \Gamma_{22}^{-1} \Gamma_{21}.
$$

(9)

Although $\Gamma_{1/2}$ is positive definite, it is not necessarily completely positive.

4 Partial Tail Correlation

4.1 Partial Tail Correlation via the Projection Theorem

We now turn attention to developing the notion of partial tail correlation between pairs of elements of a vector $X$, where $X_i \in \mathcal{V}^q$ for $i = 1, \ldots, p$. Let $X_1 = (X_i, X_j)^T$ and $X_2 = X_\setminus(i,j)$. With this partition specific to the $(i,j)$th elements of $X$, we denote the $(1,2)$ element of the $2 \times 2$ Schur complement matrix from (9) as $\gamma_{ij}$ and also denote by $\gamma_{ii}$ and $\gamma_{jj}$ the $(1,1)$ and $(2,2)$ elements, respectively. The partial tail correlation between $X_i$ and $X_j$ given $X_2$, can be seen as the cosine of the angle between the prediction errors.

**Definition 4.1.** Let $X_i \in \mathcal{V}^q$ for $i = 1, \ldots, p$. Denote by $\mathcal{V}_{A_2}$ the space spanned by the set of variables $X_2 = X_\setminus(i,j)$. Let $X_i \ominus P_{\mathcal{V}_{A_2}} X_i$ and $X_j \ominus P_{\mathcal{V}_{A_2}} X_j$ be prediction errors obtained after projecting $X_i$ and $X_j$ onto the space $\mathcal{V}_{A_2}$, respectively. Then, the partial tail correlation between $X_i$ and $X_j$ given $X_2$ is defined as

$$
\rho_{ij}^E = \frac{\langle X_i \ominus P_{\mathcal{V}_{A_2}} X_i, X_j \ominus P_{\mathcal{V}_{A_2}} X_j \rangle}{\|X_i \ominus P_{\mathcal{V}_{A_2}} X_i\|_V \|X_j \ominus P_{\mathcal{V}_{A_2}} X_j\|_V} = \frac{[\Gamma_{1/2}]_{ij}}{\sqrt{[\Gamma_{1/2}]_{ii} [\Gamma_{1/2}]_{jj}}} = \frac{\gamma_{ij}}{\sqrt{\gamma_{ii} \gamma_{jj}}}.
$$

(10)

The superscript $E$ in $\rho_{ij}^E$ stands for "extreme". $\langle X_i \ominus P_{\mathcal{V}_{A_2}} X_i, X_j \ominus P_{\mathcal{V}_{A_2}} X_j \rangle = 0$ iff $\rho_{ij}^E = 0$, which we denote by $X_i \perp X_j | X_\setminus(i,j)$.

4.2 Partial Tail Correlation and Transformed Linear Prediction

We return temporarily to the problem of predicting one variable $X_{p+1}$ given $X = A \circ Z \in RV^P_E(2)$, and state an important relationship between the partial tail correlation and the prediction coefficients. In the proposition below, we investigate without loss of generality the partial correlation between $X_1$ and $X_{p+1}$ given $X_2, \ldots, X_p$.

**Proposition 4.1.** Let $\mathcal{V}_A$ be the previously defined subspace of the Hilbert space $\mathcal{V}^q$. Assume $X_i \in \mathcal{V}_A$, $i = 1, \ldots, p+1$. Then the partial tail correlation between $X_{p+1}$ and $X_1$ is zero if and only if the first coefficient of $b$ in the best transformed-linear predictor $P_{\mathcal{V}_A} X_{p+1} = \hat{b}^\top \circ X = b_1 \circ X_1 + \cdots + b_p \circ X_p$ is zero.

**Proof.** By the projection theorem, the space $\mathcal{V}_A$ can be decomposed into two orthogonal subspaces $\mathcal{V}_{A_2}$ spanned by $(X_2, \ldots, X_p)$ and $\mathcal{V}_{A_2}'$ spanned by $(X_1 \ominus P_{\mathcal{V}_{A_2}} X_1)$. Thus, the projection of $X_{p+1}$ onto the space $\mathcal{V}_A$ can also be split into two parts,

$$
P_{\mathcal{V}_A} X_{p+1} = P_{\mathcal{V}_{A_2}} X_{p+1} \ominus P_{\mathcal{V}_{A_2}'} X_{p+1} = P_{\mathcal{V}_{A_2}} X_{p+1} \ominus (X_1 \ominus P_{\mathcal{V}_{A_2}} X_1),
$$

(11)
where \( c = \frac{(X_{p+1}, X_1 \otimes P_{V_{A_2}} X_1)}{||X_1 \otimes P_{V_{A_2}} X_1||^2} = \frac{(X_{p+1} \otimes P_{V_{A_2}} X_{p+1}, X_1 \otimes P_{V_{A_2}} X_1)}{||X_1 \otimes P_{V_{A_2}} X_1||^2} \) since \( P_{V_{A_2}} X_{p+1} \perp X_1 \otimes P_{V_{A_2}} X_1 \). We show that \( c \) is related to the partial tail correlation between \( X_1 \) and \( X_{p+1} \). To find the form of \( c \), we note that the projection of any variable in \( V_i \) onto the space \( V_{A_2} \) is represented by the transformed-linear combination of the remaining variables \( \{ X_2, \cdots, X_p \} \). The projection of \( X_1 \) onto \( V_{A_2} \) is \( P_{V_{A_2}} X_1 = \bigoplus_{i=1}^{p-1} d_i \circ X_i + 1 \) and the projection of \( X_{p+1} \) onto \( V_{A_2} \) is \( P_{V_{A_2}} X_{p+1} = \bigoplus_{i=1}^{p-1} e_i \circ X_i + 1 \). Substituting these projections into \( \rho^E \), \( P_{V_{A_2}} X_{p+1} = c \circ X_1 + \left( \sum_{i=1}^{p-1} (d_i - e_i) \circ X_i + 1 \right) \), from which we obtain \( c = b_1 \). Thus,

\[
b_1 = \frac{(X_{p+1} \otimes P_{V_{A_2}} X_{p+1}, X_1 \otimes P_{V_{A_2}} X_1)}{||X_1 \otimes P_{V_{A_2}} X_1||^2} = \rho^E_{X_1, X_{p+1}} \| X_{p+1} \otimes P_{V_{A_2}} X_{p+1} \|_{V_i},
\]

and \( b_1 \) is zero if and only if the partial tail correlation between \( X_{p+1} \) and \( X_1 \) is zero.

The implication of this theorem is if \( X_i \perp \perp X_j \mid X_{(i,j)} \) then transformed linear prediction of \( X_j \) is not improved by including \( X_i \) when already predicting on \( X_{(i,j)} \).

### 4.3 Relation between Partial Tail Correlation and the Inverse Inner Product Matrix

In the non-extreme setting, the partial correlation between \( X_i \) and \( X_j \) given all other elements of \( X_{(i,j)} \) is related to the \((i, j)\)th element of the precision matrix (the inverse of the covariance matrix). Specifically, \( \Sigma^{-1}_{i,j} = 0 \Leftrightarrow X_i \perp \perp X_j \mid X_{(i,j)} \). Analogously, we connect the idea of partial tail correlation to the inverse of the inner product matrix.

Let \( X \in RV_p^+ (2) \) be a \( p \)-dimensional regularly varying random vector where \( X_i \in V^q, i = 1, \ldots, p \). As in Section 4.1, partition \( X \) into two subvectors \( X_1 := (X_i, X_j)^\top \) and \( X_2 = X_{(i,j)} \). Recall the block form of the inner product matrix of \( X = (X_1, X_2)^\top \)

\[
\Gamma_X := \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} \\
\Gamma_{21} & \Gamma_{22}
\end{bmatrix},
\]

(12)

By block matrix inversion,

\[
\Gamma^{-1}_X = \begin{bmatrix}
\Gamma^{-1}_{11} & -\Gamma^{-1}_{12}
\\
-\Gamma^{-1}_{21} \Gamma^{-1}_{11} + \Gamma^{-1}_{22} & \Gamma^{-1}_{22} + \Gamma^{-1}_{21} \Gamma^{-1}_{12} \Gamma^{-1}_{11} \Gamma^{-1}_{21}
\end{bmatrix}.
\]

(13)

Since \( \Gamma_{1|2} \) and \( \Gamma^{-1}_{1|2} \) are \( 2 \times 2 \) matrices, the inverse is shown by

\[
\Gamma^{-1}_{1|2} = \frac{1}{| \Gamma_{1|2} |} \left[ \begin{bmatrix} | \Gamma_{1|2} |_{22} & -| \Gamma_{1|2} |_{12} \\
-| \Gamma_{1|2} |_{21} & | \Gamma_{1|2} |_{11} \end{bmatrix} \right].
\]

(14)

Writing the partial tail correlation in terms of the elements of \( \Gamma^{-1}_{1|2} \) yields

\[
\rho^E_{ij} = \frac{-| \Gamma^{-1}_{1|2} |_{12}}{\sqrt{| \Gamma^{-1}_{1|2} |_{11} | \Gamma^{-1}_{1|2} |_{22}}} = \frac{-| \Gamma^{-1}_{X} |_{ij}}{\sqrt{| \Gamma^{-1}_{X} |_{ii} | \Gamma^{-1}_{X} |_{jj}}},
\]

(15)

where the last expression is in terms of the original vector \( X \) before reordering its elements to obtain the partition \( X_1 \) and \( X_2 \). Clearly, \( \rho^E_{ij} = 0 \) if and only if the \((i, j)\)th element of \( \Gamma^{-1}_X \) is zero. We also note that the direction of the partial tail correlation is of the opposite sign of \( | \Gamma^{-1}_{1|2} |_{12} \). If \( | \Gamma^{-1}_{1|2} |_{12} < 0 \), it implies that \( X_i \) and \( X_j \) given \( X_{(i,j)} \) are partially uncorrelated in terms of tail behavior.

For illustration, we briefly consider the transformed-linear model

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\phi & 1 & 0 & 0 \\
\phi^2 & \phi & 1 & 0 \\
\phi^3 & \phi^2 & \phi & 1
\end{bmatrix} \circ \begin{bmatrix}
Z_1 \\
Z_2 \\
Z_3 \\
Z_4
\end{bmatrix}
\]

(16)

where \( \{ Z_i \} \) is a sequence of independent regularly varying \( \alpha = 2 \) with unit scale. Set \( X = (X_1, \ldots, X_4)^\top \). Assume \( \phi \in (0, 1) \) to induce a positive dependence in the \( \{ X_i \} \). The relationship between the \( X_i \)'s is most easily seen via the sequential generating equation

\[
X_i = \phi \circ X_{i-1} \oplus Z_i, \text{ for } i = 1, 2, 3, 4,
\]
and where \( X_0 = 0 \ a.s. \). The inner product matrix is not sparse, but the inverse

\[
\Gamma^{-1} = \begin{bmatrix}
1 & -\phi & 0 & 0 \\
-\phi & 1 + \phi^2 & -\phi & 0 \\
0 & -\phi & 1 + \phi^2 & -\phi \\
0 & 0 & -\phi & 1
\end{bmatrix}.
\]

conveys the relationship shown by the generating equation as the partial tail correlation between \( X_i \) and \( X_j \) is zero if \(|i-j| > 1\). In terms of transformed linear prediction, \( \hat{X}_1 = 0 \oplus X_1 \oplus 0 \oplus X_2 \oplus \phi \circ X_3 \). These optimized weights imply that given \( X_3 \), knowledge of \( X_1 \) or \( X_2 \) does not provide additional information about \( X_4 \) in terms of tail behavior.

## 5 Positive Subset \( V_+^q \) as a Modeling Framework

Thus far, we have considered random vectors \( X \) with elements \( X_i \in \mathcal{V}_q \) for \( i = 1, \ldots, p \); that is, \( X = A \circ Z \) for some \( A \in \mathbb{R}^{p \times q} \) and \( Z \) as in Section 3.1, and \( \Gamma_X = AA^\top \). While allowing negative elements in \( A \) is necessary to obtain an inner product space, it may feel largelyacademic as negative elements do not contribute to the tail behavior of \( X \). Consequently, \( \Gamma_X \) is not estimable from observations of \( X \). [Lee and Cooley, 2021] make the case for restricting attention to the subset \( V_+^q = \{ X; X = a^\top \circ Z = a_1 \circ Z_1 + \cdots + a_q \circ Z_q \} \), where \( a_j \in [0, \infty), j = 1, \ldots, q \) in practice, and give an example showing that transformed-linear prediction is more interpretable with this assumption. Below we briefly explain why restricting attention to \( V_+^q \) is a sensible modeling assumption.

Consider the random vectors \( X = A \circ Z \) and \( X_+ = A^{(0)} \circ Z \) having elements in \( \mathcal{V}_q \) and \( V_+^q \) respectively. Assuming \( A \) has negative elements, \( X \) and \( X_+ \) are distinct and they have different inner product matrices. However, these random vectors are indistinguishable by their tail behavior as they both share the same angular measure: \( H_X = H_{X_+} = \sum_{j=1}^q ||a_j^{(0)}||^2 \delta_{a_j^{(0)}} / ||a||_1 \), and the same TPDM \( \Sigma_X = \Sigma_{X_+} = A^{(0)} A^{(0)\top} \). In terms of modeling, it seems there is nothing lost by assuming the elements of \( X \) are in \( V_+^q \). With this assumption we gain the ability to estimate the inner product matrix as \( \Sigma_{X_+} = \Gamma_{X_+} \), and the TPDM is estimable.

Even if one does not believe that \( X_i, i = 1, \ldots, p \), actually arise from transformed linear combinations, [Lee and Cooley, 2021] argue that assuming \( X_i \in V_+^q \) is a useful modeling assumption and not overly restrictive. The motivating setting is that \( X \in RV_+^q(2) \), and that \( p \) is large enough that fully modeling \( H_X \) is infeasible. The TPDM can be used to summarize the tail behavior of \( X \), and itself does not require any assumption beyond \( X \in RV_+^q(2) \). A natural concern to further assuming \( X_i \in V_+^q \) is that it implies the angular measure \( H_X \) consists of \( q \) point masses. However, angular measures arising from \( p \times q \) non-negative matrices \( A \) are dense in the class of angular measures for \( p \)-dimensional regularly varying random vectors as \( q \to \infty \) [Cooley and Thibaud, 2019]. More importantly, \( q \) is latent and does not need to be specified for inference. One is not required to model \( H_X \) as some \( q \) point masses, one only needs the TPDM to perform transformed linear prediction as in [Lee and Cooley, 2021] or to investigate tail relationships via partial tail correlation as we do here.

## 6 Hypothesis Testing for Zero Elements in the Inverse TPDM

### 6.1 Asymptotic Normality of TPDM Estimates

We aim to develop a hypothesis test for \( H_0 : \rho_{ij}^E = 0 \) versus \( H_1 : \rho_{ij}^E \neq 0 \). We first review asymptotic normality results for elements of the sample TPDM \( \hat{\Sigma}_X \) given by Resnick [2004] and Larsson and Resnick [2012]. Let \( X \in RV_+^q(2) \) have angular measure \( H_X \). Let \( m = H_X(\Theta_p^{(1)}) \), \( R = ||X||_2 \), and \( W = X/R \). Equivalent to (3), we write \( \sigma_{ij} = \lim_{x \to \infty} m \mathbb{E}[W_i W_j | R > x] \), whose form suggests a natural estimator. Let \( x_\ell \) for \( \ell = 1, \ldots, n \) be iid copies of \( X \). Letting \( r_\ell = ||x_\ell|| \) and \( w_\ell = r_\ell^{-1} x_\ell \), define

\[
\hat{\sigma}_{ij} = \frac{n}{k} \sum_{\ell=1}^n w_{\ell,i} w_{\ell,j} \mathbb{I}[r_\ell > r_{(k)}],
\]

where \( \hat{n} \) is an estimate of \( H_X(\Theta_p^{(1)}) \), \( k := k(n) \) is such that \( k \to \infty \) and \( k/n \to 0 \) as \( n \to \infty \), and \( r_{(k)} \) is the \( k^{th} \) upper order statistic.

Using the fact that the radial and angular components become independent in the limit, Larsson and Resnick [2012] give a condition for which

\[
\sqrt{k}(\hat{\sigma}_{ij} - \sigma_{ij}) \sim N(0, \tau_{ij}^2),
\]
where $\tau_{ij}^2 = m^2 \text{Var}(W_t W_j)$, and here $W$ is a random vector with distribution $m^{-1} H_X$. The condition states that the dependence between $(W_t W_j)$ and $R_t$ decays fast enough as $k \to \infty$, details can be found in [Larsson and Resnick, 2012, Theorem 1]. This condition is not able to be observed, and is instead assumed. The estimate $\hat{\tau}_{ij}^2 = \frac{1}{k-1} \sum_{t=1}^k (m w_t w_{tj} - \bar{a}_{ij})^2 [r_t > r_{(k)}].$

6.2 Residuals and Hypothesis Testing for $\rho_{ij}^E$

Our hypothesis test will actually have null hypothesis that $\gamma_{ij} = 0$, as this implies $\rho_{ij}^E = 0$. In order to develop our hypothesis test, we derive the asymptotic normality and variance of elements of the sample Schur complement matrix, $\hat{\Gamma}_{1|2}$. We assume $X \in \mathcal{V}_q$ so that its inner product matrix coincides with its TPDM. As in Section 4, we partition such that $X_1 = (X_i, X_j)^\top$ and $X_2 = X_{(i,j)}$.

We assume that we observe iid copies of $X$ from which we obtain $\Sigma$. A straightforward estimator of the Schur complement matrix is

$$\hat{\Gamma}_{1|2} = \hat{\Sigma}_{11} - \hat{\Sigma}_{12} \hat{\Sigma}_{22}^{-1} \hat{\Sigma}_{21},$$

(18)

where $\Sigma_{ij}$ for $i, j = 1, 2$ are sample block matrices in (12). However, the distribution of $\hat{\Gamma}_{1|2}$ is not straightforward to obtain from (18). As $\hat{\Gamma}_{1|2}$ is the inner product matrix of prediction errors, we will use ‘residuals’ to understand the properties of the Schur complement matrix. However, careful consideration of residuals is required.

The prediction errors $X_1 \odot \hat{X}_1 = (A_1 - \hat{B} A_2) \circ Z$ are in $\mathcal{V}_q$ but cannot be assumed to be in $\mathcal{V}_q$. Unlike the original data where we can assume away the importance of any negative coefficients as described in Section 5, negative coefficients for the errors are consequential, as they contribute to the magnitude of the error $\|X_1 \odot \hat{X}_1\|_\infty$. Observed prediction errors or their TPDM would mask these negative coefficients. The TPDM of the prediction errors, $\Sigma_{X_1 \odot \hat{X}_1} = (A_1 - \hat{B} A_2)(0)(A_1 - \hat{B} A_2)^0 \neq \Gamma_{1|2},$

and $\Gamma_{1|2}$ is not necessarily completely positive.

To account for negative coefficients, we consider the preimages of the prediction errors. Define

$$U_{ij} := t^{-1} (X_1 \odot \hat{X}_1) = (A_1 - \hat{B} A_2)^{-1}(Z).$$

As $U_{ij} \in RV^2(2)$ [Cooley and Thibaud, 2019, Lemma A4], we let $H_{U_{ij}}$ denote its angular measure which has support on $\Theta^1 = \{y \in \mathbb{R}^2 : \|y\|_2 = 1\}$. Similar to the TPDM but for $U_{ij} \in RV^2(2)$, we can summarize tail dependence

$$\Sigma_{U_{ij}} = \int_{\Theta^1} v_i v_j dH_{U_{ij}}(v) = \left. [i,j=1,2]. \right.$$

$H_{U_{ij}}$ consists of $q$ point masses and consequently $\Sigma_{U_{ij}} = (A_1 - \hat{B} A_2)(A_1 - \hat{B} A_2)^\top = \Gamma_{1|2}$. Importantly, the elements of $\Sigma_{U_{ij}}$ have estimators whose properties can be developed. Beginning with iid observations $x_{t}, \ell = 1, \ldots, n$, partition to obtain $x_{t1} = (x_{t1}, x_{t2})^\top$ and $x_{t2} = x_t \setminus \{x_{t1}, x_{t1}\}$. Use $\hat{\Sigma}_X$ to obtain $\hat{B}$, calculate the predicted values $\hat{x}_{t1} = \hat{B} \circ x_{t2}$, and calculate the observed residuals $u_t = t^{-1}(x_{t1} \odot \hat{x}_{t1}) - t^{-1}(x_{t1}) - t^{-1}(\hat{x}_{t1})$. We focus on estimating the $(1,2)$ element of $\Sigma_{U_{ij}}$, which corresponds to $\gamma_{ij}$. Similarly to (17), let

$$\hat{\gamma}_{ij} = \hat{\Sigma}_{U_{ij}} = \frac{1}{m} \sum_{i=1}^n v_{t1} v_{t2} I[r_t > r_{(k)}],$$

(19)

where $r_t = \|u_t\|_2$, $v_t = u_t / r_t$, $r_{(k)}$ is the $k^{th}$ upper order statistic, and $\hat{\Sigma}_{U_{ij}}$ is an estimate of $m^{-1} H_{U_{ij}}(\Theta_1)$.

The estimate $\hat{\Sigma}_{U_{ij}}$ can be obtained from the trace of $\hat{\Gamma}_{1|2}$ in [15]. Assuming the condition in [Larsson and Resnick, 2012, Theorem 1],

$$\sqrt{k}(\hat{\gamma}_{ij} - \gamma_{ij}) \sim N(0, \tau_{ij}^2),$$

where $\tau_{ij}^2 = m^2 \text{Var}(V_1 V_2)$ and $V$ is a random vector with distribution $m^{-1} H_{U_{ij}}$. $\tau_{ij}^2$ can be estimated in the same manner as $\tau_{ij}^2$. Under the null hypothesis $H_0 : \rho_{ij}^E = 0$ (equivalently $\gamma_{ij} = 0$),

$$\frac{\hat{\gamma}_{ij}}{\sqrt{\tau_{ij}^2 / k}} \sim T_{k-1},$$

(20)

where $T_{k-1}$ denotes a $t$-distribution with $k - 1$ degrees of freedom. With this asymptotic result, we can construct confidence intervals and perform a hypothesis test for zero elements in the inverse TPDM. A simulation study, available in [Lee, 2022], shows that the hypothesis test yielded appropriate coverage rates.
7 Applications

7.1 Nitrogen Dioxide Air Pollution

We apply the idea of partial tail correlation NO\textsubscript{2} pollution data from 5163 days at five stations in the Washington DC metropolitan area (see Figure 1). These data were analyzed by Lee and Cooley [2021] who first detrended the data and then performed a marginal transformation so that \( X_t = (X_{t,1}, \ldots, X_{t,5}) \) can be assumed to be iid and in \( RV_5^\wedge(2) \).

![Figure 1: Left: An outline of Washington, D.C. with locations of five NO\textsubscript{2} monitors. Center: The extremal graph induced by partial tail correlation for five stations. The thickness of edges corresponds to absolute values of test statistics being greater than 4.797. Right: A scatter plot of comparison between \( D_{null} \) versus \( D_{Arl} \) after transformation to the original scale of the NO\textsubscript{2} data.](image)

To test whether or not extreme NO\textsubscript{2} levels between each pair of stations exhibit significant partial tail correlation, we first estimate the TPDM. Let \( x_t \) denote the observed daily NO\textsubscript{2} level on day \( t \). For each \( i \neq j \), let \( r_{t,ij} = ||(x_{t,i}, x_{t,j})||_2 \) and \( (w_{t,i}, w_{t,j}) = (x_{t,i}, x_{t,j})/r_{t,ij} \). We let \( \hat{\sigma}_{ij} = 2k^{-1} \sum_{t=1}^{n} w_{t,i} w_{t,j} I[r_{t,ij} > r_{ij}^\wedge], \) where \( k = \sum_{t=1}^{n} I[r_{t,ij} > r_{ij}^\wedge] \) is the number of exceedances. We set \( r_{ij}^\wedge \) as the 0.95 quantile for radial components. The total mass 2 arises from the fact that each \( X_t \) has the unit scale after preprocessing. From \( \Sigma_{X} \), we can compute its inverse. In turn, we can calculate the partial tail correlation estimates, which are shown in the upper triangle of Table 1. Values near zero indicate that the pair of stations have a weak tail relationship after accounting for the information in the other stations.

Table 1: Upper triangle are the partial tail correlations estimated for each pair of stations. Lower triangle are the test statistics computed for each pair. Test statistics found to be significant are in bold.

|       | Alx | Mc  | Rt  | Tak | Arl |
|-------|-----|-----|-----|-----|-----|
| Alx   | -   | 0.10| 0.18| 0.09| 0.46|
| Mc    | 1.69| -   | 0.23| 0.38| 0.13|
| Rt    | 1.69| 6.18| -   | 0.11| 0.25|
| Tak   | 2.37| 7.83| 2.42| -   | 0.23|
| Arl   | 9.89| 3.27| 4.50| 5.31| -   |

Our hypothesis test actually assesses whether \( \hat{\tau}_{ij} \) is significantly different from zero. We use the prediction residuals as in Section 6 to estimate the variance of these estimates. For each \( i \neq j \) for \( i, j = 1, \ldots, 5 \), let \( x_{t,1} = (x_{t,i}, x_{t,j})^T \) and \( x_{t,2} = x_t \setminus \{x_{t,i}, x_{t,j}\} \). Given the estimated TPDM \( \Sigma_{X} \), we obtain \( \hat{x}_{t,1} = B \circ x_{t,2} \), where \( B = \Sigma_{X} x_{t,2} \Sigma_{X}^{-1} \). Then we find residual vectors \( (u_{t,i}, u_{t,j}) = t^{-1}(x_{t,1}) - t^{-1}(\hat{x}_{t,1}) \). Let \( r_{t,ij} = ||(u_{t,i}, u_{t,j})||_2 \) and \( (v_{t,i}, v_{t,j}) = (u_{t,i}, u_{t,j})/r_{t,ij} \). Following Larsson and Resnick [2012], we let \( \hat{\tau}_{U,ij}^2 = \frac{1}{k-1} \sum_{t=1}^{n} (\hat{m}_{U,ij} v_{t,i} v_{t,j} - \hat{\tau}_{ij}^2) / r_{t,ij} \), where \( k = \sum_{t=1}^{n} I[r_{t,ij} > r_s] \). We choose \( r_s \) as the 0.98 quantile for radial components.

Under the null hypothesis that \( \rho_{ij}^2 = 0 \), for each \( i \neq j \), we calculate test statistics \( T_{ij} = \sqrt{k} (\hat{\tau}_{ij} / \hat{\tau}_{U,ij}) \). We employ the Tukey’s exact procedure to adjust for multiple comparisons because the Tukey’s exact procedure is well-suited for all pairwise comparisons where the number of exceedances is equal across all pairwise comparisons. We have the total number of observations \( N = 103 \times 10 = 1030 \) where each pairwise comparison has the equal number of
threshold exceedances of 103 and there are 10 pairwise comparisons. The degrees of freedom is \( df = N - 10 = 1020 \). Having a critical value of \( t_{crit} = 4.797 \), we summarize test statistics in the lower triangle of Table 1. If \( |T_{ij}| < 4.797 \), then we fail to reject the null hypothesis that \( \rho_{ij} = 0 \). To visualize, we create an undirected graphical model for five stations given in Figure 1, where nodes \( i \) and \( j \) are connected if \( \rho_{ij} \) is found to be significantly different from zero. The thickness of lines in Figure 1 is proportional to the test statistics, and describes the strength of the tail dependence after accounting for all other stations.

Lee and Cooley [2021] considered predicting large NO\(_2\) levels at the Alexandria station given large observed values at the four other stations. The extremal graph in the center panel in Figure 1 suggests that the predicted values at the Alexandria station based solely on large value at the Arlington station, may be comparable to those obtained using large values from all four stations. To compare results, we compare the predicted values obtained using all four stations \( \hat{X}_{All,t} \), with those obtained using only the Arlington station \( \hat{X}_{Arl,t} \). We calculate \( D_{All,t} = \max \{ \hat{X}_{All,t} \ominus X_t, X_t \ominus \hat{X}_{All,t} \} \) and \( D_{Arl,t} = \max \{ \hat{X}_{Arl,t} \ominus X_t, X_t \ominus \hat{X}_{Arl,t} \} \) for 226 days when both \( \hat{X}_{All,t} \) and \( \hat{X}_{Arl,t} \) exceed their 0.95 quantiles. Values are transformed back to the scale of the original pollution data for interpretability. The average value of \( D_{All,t} \) and \( D_{Arl,t} \) on the original scale is 55 and 54.5, respectively, indicating that prediction based on Arlington alone is comparable to prediction based on all four stations. The right panel of Figure 1 shows a scatter plot comparing \( D_{All,t} \) and \( D_{Arl,t} \) on the original scale. Points fall near the diagonal line, further suggesting comparable prediction.

7.2 Danube River Basin

We also employ the notion of partial tail correlation to investigate tail relationships between pairs of river discharge measurements in the upper Danube basin. We analyze average daily river discharges from 31 gauging stations for 1960-2009 obtained from the Bavarian Environmental Agency\(^{2}\). Figure 2 gives a graphical representation of the upper Danube’s flow network where the path \( 10 \rightarrow \cdots \rightarrow 1 \) is the main channel and the 21 other locations are on tributaries. This data set was first studied by Asadi et al. [2015] who fit a spatial extremes model for these 31 stations. With the similar aim of identifying essential pairwise relationships, Engelke and Hitz [2020] fit an extremal undirected graphical model based on the Hüsler-Reiss model to identify conditional independence in this data.

Like Engelke and Hitz [2020] and Asadi et al. [2015], we only consider June, July, and August to eliminate seasonality and because extreme flooding occurs in these summer months. It results in \( n = 50 \times 92 = 4600 \) daily river discharges where all gauging stations have measurements. Extreme discharges for each station occur in clusters because extreme discharges at downstream may occur a few days later from upstream stations. To remove temporal dependence, Engelke and Hitz [2020] and Asadi et al. [2015] set nonoverlapping timewindows of length \( p = 9 \) days and take the largest value within each window, resulting in a declustered time series of \( n = 428 \) independent data from the original data. We instead use the whole sample of size \( n = 4600 \). Ignoring temporal dependence could imply we underestimate uncertainty of parameter estimates, and could also miss dependence between stations best seen in lagged measurements.

\(^2\)http://www.gkd.bayern.de
We proceed as in Section 7.1 by first performing a marginal transformation to obtain observations \( x_t = (x_{t,1}, \ldots, x_{t,31})^\top \) which can be assumed to arise from random variables in \( V_t \). We estimate the TPDM using the top 5%, and obtain an estimate for \( \Sigma_{X}^{-1} \). We calculate the best transformed linear predictors \( \hat{x}_t \), use these to obtain residuals, and use the residuals to estimate \( \tau_{U_{i,j}}^2 \) for all \( i \neq j \).

Applying our hypothesis testing procedure with 31 stations proves to be more challenging than in the 5-dimensional setting of Section 7.1, as there are \( \binom{31}{2} = 465 \) possible pairs to consider. Rather than considering all possible pairs, we implement a "backward" iterative algorithm. Let \( G_t \) denote the graph resulting from the initial hypothesis test. For each variable \( X_i, i = 1, \ldots, 31 \), we perform a second round of hypothesis tests considering only the variables which were found to exhibit significant partial tail correlation. For example, we perform hypothesis tests pairing \( X_1 \) only with the 15 variables listed above. These subsequent tests only invert a submatrix of the TPDM, and residuals are based on a smaller number of predictors. Stepping through each variable, we collect the list of pairs which exhibit significant partial tail correlation. The graph \( G_t \) displaying pairs found to be significant is given in Figure 3.

8 Summary and Discussion

We define partial tail correlation via the projection theorem applied to the inner product space of random variables constructed from transformed linear combinations first proposed by [Lee and Cooley] [2021]. Partial tail correlation between two variables can be understood to be the inner product of the residuals from prediction on the other variables. Similar to (non-extreme) partial correlation, sparsity in the inverse inner product matrix corresponds to zero partial tail correlation and also implies the pair of variables add no additional value over the remaining variables in terms of transformed linear prediction. Thus, knowledge of when two variables have zero partial tail correlation provides information for simplified structure in the tail dependence among the variables in the random vector. In the air pollution application, it was shown that simplified prediction based on structure estimated from partial tail correlation performed similarly to prediction based on data at all available stations.
Figure 3: The extremal graph constructed from partial tail correlation via a backward variable selection for each variable.

spaces using the max-linear arithmetic. A difference between the Hušler-Reiss model based approach of [Engelke and Hitz (2020)] and more recently [Röttger et al. (2021)] is that our approach does not need the model to be fully specified, only that the random variables are can be assumed to be in $Y^\mathbb{Q}_+$. However, without a fully specified model, we cannot go so far as to say that zero partial tail correlation implies conditional independence.

We view the hypothesis test we develop as the beginning of a research path on estimating sparse tail structure via partial tail correlation. We feel our test based on asymptotic properties of the estimator is a natural starting point. However, it is clear from the Danube application that applying hypothesis tests in a high dimensional setting is challenging due to multiple testing issues and possible numerical issues arising from inverting a larger matrix. We were quite pleased that the simple-to-understand backward approach described in Section 7.2 resulted in an estimated graph with a relatively small number of connections for each node and an apparent ability to learn the channel and tributary structure from the data. But we do not think this approach is the end of the estimation discussion. Although beyond the scope of this paper, the backward approach warrants further investigation, and there are other conceivable ways of approaching the estimation problem, including investigating regularization of the inverse TPDM.

Acknowledgements

We are aware of independent and parallel work by Gong, Zhong, Opitz, and Huser which also investigates partial tail correlation for extremes. Our understanding is that inference in their work is done from a perspective of model selection rather than hypothesis testing.

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Theorem A.1. (Projection theorem) Let $\mathcal{V}_A$ be the previously defined subspace of the Hilbert space $\mathcal{V}^q$ and $X \in \mathcal{V}^q$. Let $X_i = \sum_{j=1}^q a_{ij} \circ Z_j \in \mathcal{V}^q$, $i = 1, \ldots, p$, and let $X = \sum_{j=1}^q a_j^* \circ Z_j \in \mathcal{V}^q$. Then, we have the following.

1. $\hat{X} := P_{\mathcal{V}_A} X$ ($\hat{X}$ is the projection of $X$ onto $\mathcal{V}_A$) has a unique element in $\mathcal{V}_A$ such that
   \[ ||X \otimes \hat{X}||_{\mathcal{V}^q} = \inf_{Y \in \mathcal{V}_A} ||X \otimes Y||_{\mathcal{V}^q}, \]
   and

2. $\hat{X} \in \mathcal{V}_A$ such that $||X \otimes \hat{X}||_{\mathcal{V}^q} = \inf_{Y \in \mathcal{V}_A} ||X \otimes Y||_{\mathcal{V}^q}$ if and only if $\hat{X} \in \mathcal{V}_A$ and $(X \otimes \hat{X}) \in \mathcal{V}_A^r$.

Proof.  

1. Consider $X_i = \sum_{j=1}^q a_{ij} \circ Z_j$, $i = 1, \ldots, p$, and $X = \sum_{j=1}^q a_j^* \circ Z_j$ in $\mathcal{V}^q$. For $X = (X_1, \ldots, X_p)^T$, consider $b^T \circ X \in \mathcal{V}_A$. \[ ||X \otimes (b^T \circ X)||_{\mathcal{V}^q}^2 = \sum_{j=1}^q (a_j^* - b^T a_j)^2 \]
   where $a_j$ is the $j$th column vector of $A$. We assume $\text{Rank}(A) = p$. Let $S_j = \{b \in \mathbb{R}^p \text{ such that } b^T a_j = a_j^* \}$ and $f_j(b) = (a_j^* - b^T a_j)^2$. For $b \notin S_j$, \[ \frac{\partial f_j(b)}{\partial b} = 2a_j^* (b^T a_j - a_j^*) \] and \[ \frac{\partial^2 f_j(b)}{\partial b \partial b^T} = 2a_j a_j^T (b a_j - a_j^*). \] As $a_j a_j^T$ is nonnegative definite, $f_j$ is convex off of $S_j$. Since $f_j$ is minimized on $S_j$, $f_j$ is convex everywhere. Thus for $b_1$ and $b_2$ and any $w \in (0, 1)$,
   \[ w f_j(b_1) + (1 - w) f_j(b_2) \geq f_j(w b_1 + (1 - w) b_2), \]
   where equality above implies $b_1^T a_j = b_2^T a_j$. Equality does not hold for every $j$. \[ ||X \otimes (b^T \circ X)||_{\mathcal{V}^q}^2 = \sum_{j=1}^q f_j \]
   is strictly convex since $A$ is full rank. \[ ||X \otimes (b^T \circ X)||_{\mathcal{V}^q} \to \infty \text{ as } \max_{1 \leq j \leq p} |a_j^*| \to \infty. \] Thus, \[ ||X \otimes (b^T \circ X)||_{\mathcal{V}^q} \text{ must have a unique minimum.} \]
2. Suppose $\hat{X} \in V_A$ and $(A \odot \hat{X}) \in V_A^\perp$. For any $Y \in V_A$,
\[
\|X \odot Y\|_{V_A}^2 = ((X \odot \hat{X}) \oplus (\hat{X} \odot Y), (X \odot \hat{X}) \oplus (\hat{X} \odot Y)) \\
= \|X \odot \hat{X}\|_{V_A}^2 + \|\hat{X} \odot Y\|_{V_A}^2 \\
\geq \|X \odot \hat{X}\|_{V_A}^2,
\]
with equality iff $Y = \hat{X}$. Thus, $\hat{X}$ is such that $\|X \odot \hat{X}\|_{V_A} = \inf_{Y \in V_A} \|X \odot Y\|_{V_A}$.
Conversely if $\hat{X} \in V_A$ and $(X \odot \hat{X}) \notin V_A$, then $\hat{X}$ is not the element of $V_A$ closest to $X$ since there exists $\hat{X} = \hat{X} \oplus a \circ Y/\|Y\|_{V_A}^2$ closer to $X$ where $Y$ is any element of $V_A$ such that $(X \odot \hat{X}, Y) \neq 0$ and $a = \langle X \odot \hat{X}, Y \rangle$.
\[
\|X \odot \hat{X}\|_{V_A}^2 = (X \odot \hat{X} \odot X \odot \hat{X}, X \odot \hat{X} \odot \hat{X} \odot \hat{X}) \\
= \|X \odot \hat{X}\|_{V_A}^2 + 2(X \odot \hat{X}, \hat{X} \odot \hat{X}) \\
= \|X \odot \hat{X}\|_{V_A}^2 - a \circ 1/\|Y\|_{V_A}^2 \\
< \|X \odot \hat{X}\|_{V_A}^2.
\]
\[\square\]

B Property of Projection Mappings

**Proposition B.1.** (Property of Projection Mappings) Let $P_{V_A}$ be the projection mapping of $V_A$ onto a subspace $V_A$. Then, we have the following.

1. $P_{V_A}(\alpha \circ X \oplus \beta \circ Y) = \alpha \circ P_{V_A}X \oplus \beta \circ P_{V_A}Y, \quad X, Y \in V_A, \quad \alpha, \beta \in \mathbb{R}$. [That is, the projection mapping $P_{V_A}$ is a linear mapping.]

2. For every $X \in V_A$, there exist an element of $V_A$ and an element of $V_A^\perp$ such that
\[
X = P_{V_A}X \oplus (I - P_{V_A})X,
\]
where $I$ is the identity mapping on $V_A$, and this decomposition is unique.

**Proof.** 1. $(\alpha \circ P_{V_A}X) \oplus (\beta \circ P_{V_A}Y) \in V_A$ since $V_A$ is a linear subspace of $V_A$. In addition,
\[
\alpha \circ X \oplus \beta \circ Y \ominus (\alpha \circ P_{V_A}X \oplus \beta \circ P_{V_A}Y) = \alpha \circ (X \ominus P_{V_A}X) \oplus \beta \circ (Y \ominus P_{V_A}Y) \in V_A^\perp
\]
since $V_A^\perp$ is a linear subspace of $V_A$. Thus, these two properties indicate $\alpha \circ P_{V_A}X \oplus \beta \circ P_{V_A}Y$ is the projection of $P_{V_A}(\alpha \circ X + \beta \circ Y)$. We note that this linear mapping is not necessarily true when $\alpha \neq 2$.

2. To show uniqueness of decomposition, let $X = Y \oplus Z, \quad Y \in V_A, \quad Z \in V_A^\perp$ be another decomposition, then
\[
Y \oplus P_{V_A}X \oplus Z \ominus (I - P_{V_A})X = 0.
\]
By taking inner products of each side with $Y \ominus P_{V_A}$, $\|Y \ominus P_{V_A}X\|_{V_A}^2 = 0$ since $Z \ominus (I - P_{V_A})X \in V_A^\perp$. Hence $Y = P_{V_A}X$ and $Z = (I - P_{V_A})X$.
\[\square\]