Pulsed Quantum Tunneling with Matter Waves

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Abstract

In this report we investigate the macroscopic quantum tunnelling of a Bose condensate falling under gravity and scattering on a Gaussian barrier that could model a mirror of far-detuned sheet of light. We analyze the effect of the inter-atomic interaction and that of a transverse confining potential. We show that the quantum tunneling can be quasi-periodic and in this way one could generate coherent Bose condensed atomic pulses. In the second part of the report, we discuss an effective 1D time-dependent non-polynomial nonlinear Schrödinger equation (NPSE), which describes cigar-shaped condensates. NPSE is obtained from the 3D Gross-Pitaevskii equation by using a variational approach. We find that NPSE gives much more accurate results than all other effective 1D equations recently proposed.

I. INTRODUCTION

A macroscopic signature of quantum properties of matter is the tunneling of a many-particle Bose condensate through a barrier. This subject has been investigated by various authors [1-2], in particular in connection to the formation of a Josephson current between two wells separated by a potential barrier [3-6].
The case we study is that of a falling condensate that scatters on a potential barrier that could model a mirror formed by a far-detuned sheet of light [7]. We find that the interatomic interaction and the geometrical aspects of the system, like the aspect ratio of the cloud or the fact the the cloud remains trapped in the transverse directions, have a strong effect on the tunneling probability. Moreover, we show that in our system macroscopic quantum tunneling (MQT) is a quasi-periodic phenomenon and it can be used to generate Bose condensed atomic pulses [8]. Finally, from the 3D Gross-Pitaevskii equation and using a variational approach, we derive an effective 1D wave-equation that describes the axial dynamics of a Bose condensate confined in an external potential with cylindrical symmetry. The trapping potential is harmonic in the transverse direction and generic in the axial one. Our equation, that is a time-dependent non-polynomial nonlinear Schrödinger equation (NPSE), can be used to model cigar-shaped condensates. In the limiting cases of weak and strong interaction, our approach gives rise to Schrödinger-like equations with different polynomial nonlinearities.

II. MQT OF A FALLING BOSE CONDENSATE

The Bose condensate of a dilute gas at zero temperature is well described by the 3D Gross-Pitaevskii equation (3D GPE)

\[ i\hbar \frac{\partial}{\partial t} \psi(r, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r) + g(N - 1)|\psi(r, t)|^2 \right] \psi(r, t), \tag{1} \]

where \( \psi(r, t) \) is the order parameter of the condensate in an external potential \( V_{\text{ext}}(r) \), and \( g = 4\pi\hbar^2a_s/m \), with \( a_s \) the s-wave scattering length. \( N \) is the number of condensed atoms.

The specific system we consider is a condensate falling under gravity and scattering on a Gaussian potential barrier. At \( t \leq 0 \) the condensate is confined by a harmonic trap and the external potential reads

\[ V_{\text{ext}}(\rho, z) = \frac{m}{2}(\omega_{\rho}^2\rho^2 + \omega_z^2(z - z_0)^2) + mgz + Ue^{-\frac{z^2}{\sigma^2}}, \tag{2} \]

where \( \rho = (x^2 + y^2)^{1/2} \). We use harmonic oscillator units: \( \omega_H = (\omega_{\rho}\omega_z)^{1/3} = 2\pi \times 100 \text{ Hz} \). \(^{23}\text{Na} \) atoms with \( a_H = (\hbar/(m\omega_H))^{1/2} = 27 \mu\text{m} \) and \( a_s = 3 \text{ nm} \).
We set $z_0 = 15a_H$ so that the condensate is initially far from the Gaussian potential barrier. The 3D GPE is numerically solved by using a predictor-corrector splitting method [8]. We calculate the ground-state wave-function in the external potential by means of the splitting method with imaginary time. Then, we switch off the harmonic potential and use the previous wave-function as initial condition. The total energy per particle of the condensate is about $180 \hbar \omega_H$ with $N$ ranging from 1 to $10^5$.

For large values of the energy barrier there is pure bouncing with interference. This effect have been experimentally observed by Bongs et al [7]. By reducing the energy barrier, in addition to bouncing there is MQT. We have found that, if initially the cloud is spherical under free fall, the tunneling fraction grows with the number of particles, i.e. the chemical potential. Moreover, as shown by Fig. 2, MQT is quasi-periodic phenomenon due to the quasi-periodic bouncing of a reflected part of the condensate, that is also expanding. Such a mechanism could be used to generate Bose condensed atomic pulses.

Fig. 3 shows that the tunneling fraction is reduced by the interatomic interaction if the initially spherical falling condensate is under a strong enough transverse confinement. Instead, for a cigar-shaped quasi-1D Bose condensate the interatomic interaction enhances the tunneling rate, in agreement with the theoretical predictions of Ref. [3].

III. EFFECTIVE 1D EQUATION FOR CIGAR-SHAPED CONDENSATES

As previously stated, the 3D Gross-Pitaevskii equation (3D GPE) is accurate to describe a Bose condensate of dilute bosons near zero temperature, where thermal excitations can be neglected. An interesting problem is to find a reliable mapping from the 3D GPE to an effective 1D equation that describes cigar-shaped Bose condensates. This problem is non trivial due to the nonlinearity of the GPE. In this section we illustrate a new variational approach which gives very accurate results.

First we observe that the 3D GPE can be obtained by minimizing the following action functional
\[ S = \int dt dr \psi^*(r, t) \left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 - V_{\text{ext}}(r) - \frac{1}{2}gN|\psi(r, t)|^2 \right] \psi(r, t). \] (3)

Note that we now use \( N \) instead of \( N - 1 \). We analyze a trapping potential that is generic in the axial direction and harmonic in the radial one:

\[ V_{\text{ext}}(r) = \frac{1}{2}m\omega_{\perp}^2(x^2 + y^2) + V(z) \] (4)

We minimize the action functional \( S \) by choosing the following trial wavefunction [9]:

\[ \psi(r, t) = \phi(x, y, t; \eta(z, t)) f(z, t), \] (5)

where both \( \phi \) and \( f \) are normalized and \( \phi \) is represented by a Gaussian [10]:

\[ \phi(x, y, t; \eta(z, t)) = \frac{e^{-\frac{(x^2 + y^2)}{2a^2}}}{\pi^{1/2} \eta(z, t)}. \] (6)

The variational functions \( \eta(z, t) \) and \( f(z, t) \) will be determined by minimizing the action functional after integration in the \((x, y)\) plane.

From the two Euler-Lagrange equations, one eventually obtains

\[ i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V + \frac{gN}{2\pi a_{\perp}^2} \frac{|f|^2}{\sqrt{1 + 2a_s N|f|^2}} \right. \]
\[ + \frac{\hbar \omega_{\perp}}{2} \left( \frac{1}{\sqrt{1 + 2a_s N|f|^2}} + \sqrt{1 + 2a_s N|f|^2} \right) \left] \right. f. \] (7)

This equation is a time-dependent non-polynomial nonlinear Schrödinger equation (NPSE) [10]. Note that from NPSE in certain limiting cases one recovers familiar results. In the weakly-interacting limit \( a_s N|f|^2 << 1 \) the previous equation reduces to

\[ i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V + \frac{gN}{2\pi a_{\perp}^2} |f|^2 \right] f, \] (8)

where the constant \( \hbar \omega_{\perp} \) has been omitted because it does not affect the dynamics. This equation is a 1D Gross-Pitaevskii equation (1D GPE), whose nonlinear coefficient is \( g' = g/(2\pi a_{\perp}^2) \). This ansatz has been used by various authors, for example in Ref. [11].

In the strongly-interacting limit \( a_s N|f|^2 >> 1 \) the NPSE becomes
\[
i\hbar \frac{\partial}{\partial t} f = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V + \frac{3}{2} \frac{gN^{1/2}}{2\pi a_s^2 \sqrt{2\pi a_s}} |f| \right] f.
\]

In this limit, and in the stationary case, the kinetic term can be neglected (Thomas-Fermi approximation) and one finds

\[
|f(z)|^2 = \frac{2}{9} \frac{1}{(\hbar\omega_{\perp})^2 a_s N} (\mu' - V(z))^2,
\]

where \(\mu'\) is the chemical potential, fixed by the normalization condition. This 1D Thomas-Fermi density profile is quadratic in the term \(\mu' - V(z)\). The same quadratic dependence is obtained starting from the Thomas-Fermi approximation of the 3D stationary GPE, i.e. neglecting the spatial derivatives in Eq. (1), and then integrating along \(x\) and \(y\) variables. In this way one finds a formula that differs from the previous one only for the numerical factor which is \(1/4\) instead of \(2/9\).

To test the accuracy of the NPSE, we take \(V(z) = (1/2)m\omega_z^2 z^2\), and compare the results of 4 different procedures: 1) 3D GPE: numerical solution of 3D GPE; 2) 1D GPE: numerical solution of 1D GPE. 3) CGPE: numerical solution of 1D GPE, where the nonlinear term is found by imposing that the 1D wave-function has the same chemical potential of the 3D one [12]; 4) NPSE: numerical solution of NPSE.

As shown in Fig. 5, CGPE gives better results than 1D GPE for large values of scattering length but in any case NPSE is superior. Our calculations (for details see [10]) show that the ground-state and also the dynamics of the condensate are accurately described by NPSE.

**CONCLUSIONS**

We have studied the dynamics of Bose condensates falling under gravity and scattering on a Gaussian barrier that models a mirror of light. Apart the pure bouncing with interference, that can be seen for very large values of the energy barrier, we have investigated quantum tunneling. In our system quantum tunneling is a quasi-periodic phenomenon: it can be used to generate Bose-Einstein condensed atomic pulses. In addition, we have found that geometrical aspects are important in tunneling: 1D results do not always reflect the behavior
of 3D systems. Finally, we have verified that the 1D non-polynomial nonlinear Schrödinger equation we have obtained by using a Gaussian variational ansatz from the 3D Gross-Pitaevskii action functional is quite reliable in describing cigar-shaped condensates. The agreement with the results of the 3D Gross-Pitaevskii equation is very good for both ground-state and dynamics of the condensate. This equation will be useful for detailed numerical analysis of the dynamics of cigar-shaped condensates, particularly when the local density may undergo sudden and large variations. In conclusion, we observe that Pulsed MQT of matter waves can be generated also with other mechanisms. For example, we are currently studying the stimulated MQT of matter waves produced by a Bose condensate trapped in a potential with a periodically modulated energy barrier.

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Fig. 1: Density profile $|\psi(\rho = 0, z)|^2$ of the free falling Bose condensate at $t = 0$, $t = 2$ and $t = 4$. Gaussian energy barrier $U e^{z^2/\sigma^2}$, with $U = 200$ and $\sigma = 1$. Initial chemical potential: $\mu = 184.23$. $N = 10^5$ condensed atoms. Length in units $a_H = 27 \, \mu m$, time in units $\omega_H^{-1} = 1.6$ ms and energy in units $\hbar \omega_H$. 
Fig. 2: Tunneling fraction $P_T$ as a function of time $t$. Comparison between radially-free ($\omega_\rho = 0$) and under radial confinement ($\omega_\rho = 0.5$) falling condensates. $N = 10^5$ condensed atoms and initial position $z_0 = 15$. Gaussian energy barrier $U e^{z^2/\sigma^2}$, with $\sigma = 1$. Units as in Fig 1.
Fig. 3: Tunneling fraction $P_T$ as a function of the initial chemical potential $\mu$ for different frequencies $\omega_\rho$ of radial confinement. The points in each line correspond to a sequence of numbers $N$ of atoms. From left to right: $N = 1, 10^2, 10^3, 10^4, 5 \times 10^4, 10^5$. Gaussian energy barrier $U e^{z^2/\sigma^2}$, with $U = 200$ and $\sigma = 1$. Units as in Fig 1.
Fig. 4: Density profile $|\psi(\rho = 0, z)|^2$ at $t = 0$ for various values of the number $N$ of condensed atoms. Gaussian energy barrier $U e^{z^2/\sigma^2}$, with $U = 200$ and $\sigma = 1$. Radially-confined ($\omega_\rho = 1$) falling condensate with cigar-shaped ($\omega_\rho/\omega_z = 10$) initial condition. $P_T$ is the tunneling fraction. Units as in Fig 1.
Fig. 5: Normalized density profile $\rho(z) = |f(z)|^2$ along the axial direction $z$ of the Bose condensate in harmonic potential. Number of Bosons: $N = 10^4$ and trap anisotropy: $\omega_\perp/\omega_z = 10$. Four different procedures: 3D GPE (solid line), 1D GPE (dashed line), CGPE (long-dashed line) and NPSE (dotted line). From top to bottom: $a_s/a_z = 10^{-4}$, $a_s/a_z = 10^{-3}$, $a_s/a_z = 10^{-2}$. Length $z$ in units of $a_z = \sqrt{\hbar/(m\omega_z)}$. 