Translation from Classical Two-Way Automata to Pebble Two-Way Automata

Viliam Geffert  Ľubomíra Ištoňová
Department of Computer Science – P.J. Šafárik University
Jesenná 5 – 040 01 Košice – Slovakia
viliam.geffert@upjs.sk  lubomira.istonova@upjs.sk

We study the relation between the standard two-way automata and more powerful devices, namely, two-way finite automata with an additional “pebble” movable along the input tape. Similarly as in the case of the classical two-way machines, it is not known whether there exists a polynomial trade-off, in the number of states, between the nondeterministic and deterministic pebble two-way automata. However, we show that these two machine models are not independent: if there exists a polynomial trade-off for the classical two-way automata, then there must also exist a polynomial trade-off for the pebble two-way automata. Thus, we have an upward collapse (or a downward separation) from the classical two-way automata to more powerful pebble automata, still staying within the class of regular languages. The same upward collapse holds for complementation of nondeterministic two-way machines.

These results are obtained by showing that each pebble machine can be, by using suitable inputs, simulated by a classical two-way automaton with a linear number of states (and vice versa), despite the existing exponential blow-up between the classical and pebble two-way machines.

1 Introduction

Relation between determinism and nondeterminism is one of the key topics in theoretical computer science. The most famous is the $\text{P} = \text{NP}$ question, but the oldest problem of this kind is $\text{DSPACE}(n^2) \supseteq \text{NSPACE}(n)$). Similarly, we do not know whether $\text{DSPACE}(\log n) \supseteq \text{NSPACE}(\log n)$. However, a positive answer for the $O(\log n)$ space would imply the positive answer for the $O(n)$ space, and hence the answers to these two questions are not independent. Analogically, a collapse for the $O(\log\log n)$ space would imply the same collapse for the $O(\log n)$ space. (For a survey and bibliography about such translations, see e.g. [7, 20].) Analogous upward translations can be derived for time complexity classes.

At first glance, the problem has been resolved for finite state automata. Even a two-way nondeterministic finite automaton (2NFA, for short) and hence any simpler device as well (e.g., its deterministic version, 2DFA) can recognize a regular language only. Thus, 2NFA’s can be converted into deterministic one-way automata. However, the problem reappears, if we take into account the size of these automata, measured in the number of states.

On one hand, eliminating nondeterminism in one-way $n$-state automata does not cost more than $2^n$ states (by the classical subset construction), and there exist witness regular languages for which exactly $2^n$ states are indeed required.

On the other hand, we know very little about eliminating nondeterminism in the two-way case: it was conjectured by Sakoda and Sipser [16] that there must exist an exponential blow-up for the conversion of 2NFA’s into 2DFA’s. Nevertheless, the best known lower bound is $\Omega(n^2)$ [4], while the best

*Supported by the Slovak Grant Agency for Science (VEGA) under contract 1/0035/09.
conversion uses about $2^{n^2}$ states (converting actually into deterministic one-way machines). Thus, it is not clear whether there exists a polynomial trade-off. The problem has been attacked several times by proving exponential lower bounds for restricted versions of 2DFA’s: Sipser [17] — for sweeping machines (changing the direction of the input head movement at the endmarkers only); Hromkovič and Schnitger [12] — for oblivious machines (moving the input head along the same trajectory on all inputs of the same length); Kapoutsis [14] — a computability separation for “moles” (seeing only a part of the input symbol thus traveling “in a network of tunnels” along the input). For machines accepting unary languages, a subexponential upper bound $2^{O((\log n))}$ has been obtained [9].

It was even observed [16] that there exists a family of regular languages $\{B_n : n \geq 1\}$ which is complete for the two-way automata, playing the same role as, e.g., the satisfiability of boolean formulas for the $P \neq NP$ question or the reachability in graphs for $DSPACE(\log n) \neq NSPACE(\log n)$: the trade-off between the 2NFA’s and 2DFA’s is polynomial if and only if $B_n$ can be accepted by a 2DFA with a polynomial number of states. (For 2NFA’s, $n$ states are enough to accept $B_n$.)

In the absence of a solution for the general case, it is quite natural to ask whether some properties of the two-way automata cannot be translated into more powerful machines, in perfect analogy with the corresponding results for the upward translation established for the classical space and time complexity classes. So far, the only result of this kind [1] is that if an exponential trade-off between 2NFA’s and 2DFA’s could be obtained already by using a subset of the original language that consists of polynomially long strings, then $DSPACE(\log n) \neq NSPACE(\log n)$.

In the same spirit, we shall study the relation between the standard two-way automata and more powerful devices, namely, two-way nondeterministic and deterministic finite automata equipped with a single additional “pebble”, movable along the input tape (pebble-2NFA, pebble-2DFA, respectively). Despite the fact that such pebble can be used to mark some input tape position, even a nondeterministic pebble machine cannot accept a nonregular language [2, 20]. However, measured in the number of states, the pebble machines are much more powerful. Converting a pebble-2DFA to a classical 2NFA may require an exponential blow-up, i.e., the loss of the pebble cannot be compensated economically by gaining nondeterminism. (See Theorem [1] below.)

Similarly as in the case of the classical machines, we do not know whether there exists a polynomial trade-off between the pebble-2NFA’s and pebble-2DFA’s. However, we shall show that these two models are related: if there exists a polynomial transformation from the classical 2NFA’s to 2DFA’s, then there must also exist a polynomial transformation, with the same degree of the polynomial, from the pebble-2NFA’s to pebble-2DFA’s. Thus, we have an upward collapse (and a downward separation) between the classical two-way automata and the much more powerful pebble model, within the class of regular languages.

A similar upward collapse holds for the trade-off between a two-way nondeterministic automaton accepting a language $L$ and a machine for the complement of $L$: if the trade-off is polynomial for the classical 2NFA’s, it must also be polynomial for the pebble-2NFA’s. (The problem is open for both these models.)

These results are obtained by showing that each pebble-2NFA (or pebble-2DFA) can be, by using suitable inputs, simulated by a classical 2NFA (or 2DFA, respectively) with only a linear number of states, despite the existing exponential blow-up between the classical and pebble machines. The same holds for the corresponding conversions from the classical machines to pebble machines.
2 Preliminaries

Here we introduce some basic notation and properties for the computational models we shall be dealing with. For a more detailed exposition and bibliography related to regular languages, the reader is referred to [11, 14, 15].

A two-way nondeterministic finite automaton (2NFA, for short) is a quintuple \( A = (Q, \Sigma, \delta, q_0, F) \), in which \( Q \) is the finite set of states, \( \Sigma \) is the finite input alphabet, \( \delta : Q \times (\Sigma \cup \{-, +\}) \rightarrow 2^Q \times \{-, +\} \) is the transition function, \( -, + \notin \Sigma \) are two special symbols, called the left and the right endmarker, respectively, \( q_0 \in Q \) is the initial state, and \( F \subseteq Q \) is the set of accepting (final) states.

The input is stored on the input tape surrounded by the two endmarkers. In one move, \( A \) reads an input symbol, changes its state, and moves the input head one cell to the right, left, or keeps it stationary, depending on whether \( \delta \) returns +1, −1, or 0, respectively. The input head cannot move outside the zone delimited by the endmarkers: transitions in the form \( \delta(q, -) \ni (q', +1) \) or \( \delta(q, +) \ni (q', -1) \) are not allowed. If \( |\delta(q, a)| = 0 \), the machine halts.

The machine accepts the input, if there exists a computation path starting in the initial state \( q_0 \) with the head on the left endmarker and reaching, anywhere along the input tape, an accepting state \( q \in F \).

The automaton \( A \) is said to be deterministic (2DFA), whenever \( |\delta(q, a)| \leq 1 \), for any \( q \in Q \) and \( a \in \Sigma \cup \{-, +\} \).

We also study a more powerful model, namely, a two-way finite automaton equipped with an additional “pebble” placed on the input tape. The action of the pebble machine depends on the current state, the currently scanned input tape symbol, and the presence of the pebble on this symbol. The action consists of changing the current state, moving the input head and, optionally, if the pebble is placed on the current state, moving also the pebble in the same direction.

Formally, a one-pebble two-way nondeterministic finite automaton (pebble-2NFA, for short) is \( A = (Q, \Sigma, \delta, q_0, F) \), where \( Q, \Sigma, q_0, F \) are defined as above, but the transition function is of the form \( \delta : Q \times (\Sigma \cup \{-, +\} \cup \{-*, +*\}) \rightarrow 2^Q \times \{-1, 0, +1, -1*, +*\} \). The presence of the pebble on the current input tape symbol \( a \in \Sigma \cup \{-, +\} \) is indicated by using \( a^* \in \Sigma^* \cup \{-*, +*\} \), while the new input head movements \( -1^*, +1^* \) are introduced to move the pebble. More precisely, a classical transition \( \delta(q, a) \ni (q', d) \), with \( a \in \Sigma \cup \{-, +\} \) and \( d \in \{-1, 0, +1\} \), is applicable only if the pebble in not placed on the current input tape symbol (change the current state from \( q \) to \( q' \) and move the input head in the direction \( d \)), while \( \delta(q, a^*) \ni (q', d^*) \) can be executed only if the pebble is placed on \( a \in \Sigma \cup \{-, +\} \) at the moment (move the input head in the direction \( d \), but leave the pebble in its original position). Finally, \( \delta(q, a^*) \ni (q', d^*) \), with \( d^* \in \{-1^*, +1^*\} \), moves also the pebble in the same direction \( d \), together with the input head. Transitions in the form \( \delta(q, a) \ni (q', d^*) \) are meaningless, and hence not allowed.

The machine \( A \) starts its computation in the initial state \( q_0 \) with both the input head and the pebble placed on the left endmarker, and accepts by reaching, anywhere along the input tape, a final state \( q \in F \). Similarly, the final position of the pebble is irrelevant for acceptance. A one-pebble two-way deterministic finite automaton (pebble-2DFA) is defined in the usual way.

It is known [2] (see also Theorem 15.3.5 in [20]) that even nondeterministic Turing machines equipped with a single pebble and a worktape space of size \( o(\log \log n) \) can accept regular languages only. Since pebble-2NFA’s may be viewed as one-pebble Turing machines with \( O(1) \) worktape space, all models introduced above (2DFA, 2NFA, pebble-2DFA, pebble-2NFA) share the same expressible power — they all recognize the same class of regular languages.

However, if we take into account their number of states, the power is different. Converting a pebble-2DFA to a classical 2NFA may require an exponential blow-up. That is, the loss of the pebble cannot be
paid by gaining nondeterminism.

**Theorem 1.** For each $m \geq 1$, there exists a finite unary language $L_m$ that can be accepted by a pebble-2DFA with $O(m^2 \cdot \log m)$ states, but for which each 2NFA requires at least $2^{\Omega(m \cdot \log m)}$ states.

**Proof:** Let $M = p_1 \cdot p_2 \cdots \cdot p_m$, where $p_i$ denotes the $i$-th prime number, and let $L_m = \{1^\ell : \ell < M\}$.

The pebble machine $A$ recognizing $L_m$ utilizes the fact that $\ell < M$ if and only if no $x \in \{1, \ldots, \ell\}$ is a common multiple of $p_1, p_2, \ldots, p_m$. Therefore, $A$ repeatedly checks, for $x = 1, \ldots, \ell$, if $x$ is divisible by the primes $p_1, p_2, \ldots, p_m$. The value of $x$ is represented by the distance of the pebble from the left endmarker. In order to check if $p_i$ divides $x$, $A$ traverses between the pebble position and the left endmarker and counts modulo $p_i$ (alternating right-to-left traversals with left-to-right traversals for odd/even values of $i$). If $A$ finds a prime $p_i$ not dividing $x$, it does not check the next prime $p_{i+1}$ but, rather, enters the initial state $q_1$ in which it searches for the pebble and then moves the pebble one position to the right. After that, $A$ can start checking the next value of $x$ for divisibility by $p_1, p_2, \ldots, p_m$ or, if the pebble has reached the right endmarker, $A$ can halt in an accepting state $q_f$. Carefully implemented, $A$ uses only $2 + p_1 + p_2 + \cdots + p_m$ states. By the Prime Number Theorem (see, e.g., [5]), we have $p_i = (1 + o(1)) \cdot i \cdot \ln i$, which gives $2 + p_1 + p_2 + \cdots + p_m \leq O(m^2 \cdot \log m)$.

On the other hand, each classical 2NFA $A'$ recognizing $L_m$ must use at least $M - 1$ states. This can be seen by the use of $n \to n + n!$ method [3, 6]: On the input $1^{M-1}$, a machine $A'$ with fewer states than $M - 1$ cannot traverse the input tape from left to right without going into a loop, i.e., without repeating the same state after traveling some $h$ positions to the right, where $h \leq M - 1$. Thus, by iterating this loop $(M - 1)!/h = \prod_{i=1, i \neq h}^{M-1} i$ more times, we get a valid computation path traversing $(M - 1) + (M - 1)!$ positions to the right. Therefore, if $A'$ can get from a state $q_1$ to $q_2$ by traversing the entire input $1^{M-1}$, it can also get from $q_1$ to $q_2$ by traversing the entire input $1^{(M-1)+(M-1)!}$. The same holds for right-to-left traversals and also for U-turns, i.e., for computations starting and ending at the same endmarker. Thus, by induction on the number of visits at the endmarkers, we get that if $A'$ accepts the input $1^{M-1}$, it must also accept the input $1^{(M-1)+(M-1)!}$, which is a contradiction. Therefore, each 2NFA $A'$ recognizing $L_m$ must use at least $M - 1 = p_1 \cdot p_2 \cdots \cdot p_m - 1$ states. Since $p_1 \cdot p_2 \cdots \cdot p_m \geq m^{\Omega(m)}$ (see, e.g., Lemma 4.14 in [7]), we have $M - 1 \geq 2^{\Omega(m \cdot \log m)}$. \hfill $\Box$

## 3 Translation

In this section, we first show that each pebble-2NFA $M$ (or pebble-2DFA) can be, in a way, using a suitable encoding of the original input, simulated by a 2NFA $M'$ (or 2DFA, respectively) without a pebble. Then we show the corresponding conversions in the opposite direction, from the classical two-way machines to two-way machines with a pebble. The cost, in the number of states, will be linear for all these conversions, despite the exponential blow-up presented by Theorem [1]. After that, we shall derive some consequences of these translations.

In what follows, we shall need a function $P$ that maps each input $w$ of the given pebble automaton $M$ into a new word $P(w)$ providing all possible positions of the pebble in $w$. This image $P(w)$ can be used as an input for a classical automaton $M'$ (no pebble), such that $M'$ accepts $P(w)$ if and only if $M$ accepts $w$. Let $P : \Sigma^* \to (\Sigma \cup \{>, \langle\} \cup \Sigma^c)^*$ maps a word $w = a_1 \ldots a_k$ as follows:

$$P(a_1 \ldots a_k) = a_1 \ldots a_k \langle \langle a_1^2 a_2 \ldots a_k \langle \langle a_1 a_2^2 \ldots a_k \langle \langle \ldots \langle a_1 \ldots a_{k-1}^2 a_k \langle a_1 \ldots a_{k-1} a_k^2 \langle \langle a_1 \ldots a_k,$$

(1)
where $\triangleright, \triangleleft$ are new symbols and $\Sigma' = \{a^2 : a \in \Sigma\}$, that is, $\Sigma'$ simply denotes the letters of the original alphabet marked by some box.

Thus, $P(a_1 \ldots a_k)$ consists of $k + 2$ segments, enumerated from 0. The $p$-th segment, for $p = 0, \ldots, k + 1$, will be used by $M'$ to simulate $M$ in situations when $M$ has the pebble placed on the $p$-th position of the input tape. For these reasons, the $p$-th segment is of type $\triangleright a_1 \ldots a_p \triangleleft a_{p+1} \ldots a_k \triangleleft$, that is, the $p$-th symbol is marked by the box. (Except for $p = 0$ and $p = k + 1$, there is exactly one such “pseudo pebble” in each segment.) The symbols $\triangleright$ and $\triangleleft$ are “stoppers”, imitating the left and right endmarkers of the original input tape. The first and last segments are of special kind, representing the situations when $M$ has the pebble placed on the left or right endmarker, respectively, with no letters marked by the box.

As an example, if $w = a_1 a_2 a_3$, then $P(a_1 a_2 a_3) = a_1 a_2 a_3 \triangleleft a_1 a_2 a_3 \triangleright a_1 a_2 a_3 \triangleleft a_1 a_2 a_3 \triangleright$. Thus, taking also into account the endmarkers, the input tape for the pebble automaton $M$ is in the form $\vdash a_1 a_2 a_3 \vdash$ while the input tape for a classical automaton $M'$ (no pebble) in the form $\vdash a_1 a_2 a_3 \triangleleft a_1 a_2 a_3 \triangleright a_1 a_2 a_3 \triangleright$. Similarly, for $w = \varepsilon$, we have $P(\varepsilon) = \varepsilon \triangleleft \varepsilon \triangleright \varepsilon$, that is, the input tapes for $M$ and $M'$ are $\vdash$ and $\vdash$, respectively. Therefore, the left and right endmarkers can be handled by $M'$ as if marked by the “pseudo pebble” box, that is, the symbols $\vdash, \triangleright$ can be viewed as if equal to $\vdash, \triangleright$,

**Theorem 2.** (a) For each pebble-2NFA $M = (Q, \Sigma, \delta, q_f, F)$ with $m$ states, there exists a classical 2NFA $M' = (Q', \Sigma', \delta', q'_f, F')$ with at most $3 \cdot m$ states such that, for each input $w \in \Sigma^*$, $M'$ accepts $w' = P(w) \in \Sigma'^*$ if and only if $M$ accepts $w$. Here $\Sigma' = \Sigma \cup \{\triangleright, \triangleleft\} \cup \Sigma^2$ and $P$ is the mapping function defined by \([1]\).

(b) Moreover, if $M$ is deterministic, then so is $M'$.

**Proof:** Note that $M'$ does not have to check whether its input $w' \in \Sigma'^*$ is indeed a valid image obtained by the use of $P$, i.e., if $w' = P(w)$, for some $w \in \Sigma^*$. Assuming that $w' = P(w)$, $M'$ simply checks whether $M$ accepts $w$. If this assumption is wrong, the answer of $M'$ can be quite arbitrary. In general, an input $w' \in \Sigma'^*$ does not necessarily have the structure described by \([1]\), for any $w \in \Sigma^*$.

The basic idea is as follows. If, during the simulation, $M$ has its pebble placed on the $p$-th position of $w$, $M'$ works within the $p$-th segment of $P(w)$. The simulation is quite straightforward and $M'$ does not have to leave this segment until the moment when $M$ moves its pebble. Recall that $M'$ relies on the assumption that the $p$-th segment contains one exact copy of $w$, correctly enclosed in between the symbols $\triangleright$ and $\triangleleft$, and that the current pebble position of $M$ is clearly marked inside this segment, i.e., there is exactly one symbol marked with the box, namely, the symbol on the $p$-th position. If this never-verified assumption were wrong, the simulation could turn out to be wrong. Using this idea, we start our construction of $\delta'$, the transition function for the automaton $M'$, as follows.

1. If $\delta(q, a) \ni (q', d)$, for some $q, q' \in Q$, $a \in \Sigma$, and $d \in \{-1, 0, +1\}$, then $\delta'(q, a) \ni (q', d)$.
2. If $\delta(q, \vdash) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{0, +1\}$, then $\delta'(q, \vdash) \ni (q', d)$.
3. If $\delta(q, \dashv) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{-1, 0\}$, then $\delta'(q, \dashv) \ni (q', d)$.
4. If $\delta(q, a^*) \ni (q', d)$, for some $q, q' \in Q$, $a \in \Sigma$, and $d \in \{-1, 0, +1\}$, then $\delta'(q, a^*) \ni (q', d)$.
5. If $\delta(q, \vdash^*) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{0, +1\}$, then $\delta'(q, \vdash^*) \ni (q', d)$.
6. If $\delta(q, \dashv^*) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{-1, 0\}$, then $\delta'(q, \dashv^*) \ni (q', d)$. 


As soon as $M$ moves its pebble from the $p$-th position to the right, $M'$ has to travel from the $p$-th segment to the next, i.e., the $(p+1)$-st segment, and find the symbol marked by the box within this segment. Assuming that the input is in the form $w' = P(w)$, for some $w \in \Sigma^*$, this only requires to find the next symbol marked by the box lying to the right of the current input position. Recall that the $(p+1)$-st segment has, by assumption, the same structure; the only difference is in the position of the symbol marked with the box, which corresponds exactly to the changed position of the pebble for $M$. Thus, after finding the marked symbol within the neighboring segment, $M'$ can resume the simulation.

7. If $\delta(q, a^*) \ni (q', +1^*)$, for some $q, q' \in Q$ and $a \in \Sigma$, we add the following instructions:
   - $\delta'(q, a^*) \ni (q'_{+1}, +1)$, where $q'_{+1}$ is a passing-through state — a new copy of $q'$,
   - $\delta'(q'_{+1}, x) \ni (q'_{-1}, +1)$, for each $x \in \Sigma \cup \{>, <\}$,
   - $\delta'(q'_{+1}, x^*') \ni (q', 0)$, for each $x^* \in \Sigma^* \cup \{-\}$.

8. If $\delta(q, \bot^*) \ni (q', +1^*)$, for some $q, q' \in Q$, then
   - $\delta'(q, \bot) \ni (q'_{+1}, +1)$.
   - Transitions for $q'_{+1}$ are defined in the same way as in the item (7).

If $M$ moves the pebble to the left, $M'$ travels to the previous, i.e., the $(p-1)$-st segment. This is resolved symmetrically with the items (7) and (8), replacing, respectively, $-, \bot$ and $+1$ by $\bot, -, 1$ and $-1$, thus using $q'_{-1}$ (another passing-through copy of the state $q'$) instead of $q'_{+1}$.

From the above construction, we get $Q' = Q \cup Q_{+1} \cup Q_{-1}$, where $Q$ is the set of the original states in $M$ and $Q_{+1}, Q_{-1}$ represent the sets of two new copies of states in $Q$, used for traversing to the neighboring segments, introduced as $q'_{+1}$ and $q'_{-1}$. The initial state and the final states do not change: $q'_1 = q_i$ and $F' = F$. This completes the definition of $M'$. It is also easy to see that the construction preserves determinism.

**Claim.** On the input $w$, $M$ can get from its initial configuration, i.e., from the state $q_i$ with both the input head and the pebble at the left endmarker, to a state $q \in Q$ with the input head at a position $h$ and the pebble at a position $p$ if and only if, on the input $P(w)$, $M'$ can get from its initial configuration, i.e., from the state $q'_1$ with the input head at the left endmarker, to the same state $q \in Q$ with the input head at the $h$-th position of the $p$-th segment.

The argument for the “⇒” part is shown by induction on the number of computation steps executed by $M$, while the “⇐” part uses an induction on the number of times the machine $M'$ is in a state $q \in Q$, i.e., not in a passing-through state $q \in Q_{+1} \cup Q_{-1}$ (instead of induction on single computation steps).

As a consequence of this Claim, $M$ has an accepting computation path on the input $w \in \Sigma^*$, i.e., $M$ can reach an accepting state $q$ on the input $w$ if and only if $M'$, on the input $P(w)$, can reach the same state $q \in F' = F$ (not a passing-through state in $M'$), i.e., if and only if $M'$ has an accepting computation path on the input $w' = P(w) \in \Sigma^*$. □

Now we shall show a linear translation in the opposite direction.

**Theorem 3.** (a) For each classical 2NFA $N = (Q, \Sigma', \delta, q_i, F)$ with $n$ states, there exists a pebble-2NFA $N' = (Q', \Sigma, \delta', q'_1, F')$ with at most $5 \cdot n$ states such that, for each input $w \in \Sigma^*$, $N'$ accepts $w$ if and only if $N$ accepts $w' = P(w) \in \Sigma^*$. Here $\Sigma' = \Sigma \cup \{>, <\} \cup \Sigma^*$ and $P$ is the mapping function defined by (7).

(b) Moreover, if $N$ is deterministic, then so is $N'$.
\textbf{Proof}: Note that $N'$ does not have to be capable of simulating $N$ on all strings $w' \in \Sigma^\ast$. $N'$ simulates $N$ only on inputs in the form $w' = P(w)$, where $w \in \Sigma^\ast$ is its own input. Thus, $N'$ can utilize the fact that the string $P(w)$ has the structure described by (1).

While $N$ works within the same segment of $P(w)$, the simulation by $N'$ is quite straightforward: the endmarkers $\vdash, \dashv$ surrounding $w$ are interpreted as stoppers $\vdash, \dashv$ in $P(w)$, and the presence of the pebble on a symbol $a \in \Sigma \cup \{-, \}^\ast$ scanned by the input head of $N'$ indicates that $N$ reads $a^2 \in \Sigma^\ast$ or the corresponding endmarker $\vdash, \dashv$. Thus, the pebble placed at a position $p$ reflects the fact that $N'$ simulates, at the moment, $N$ working within the $p$-th segment of $P(w)$:

1. If $\delta(q, a) \ni (q', d)$, for some $q, q' \in Q$, $a \in \Sigma$, and $d \in \{-1, 0, +1\}$, then $\delta'(q, a) \ni (q', d)$.
2. If $\delta(q, \vdash) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{0, +1\}$, then $\delta'(q, \vdash) \ni (q', d)$.
3. If $\delta(q, a^\ast) \ni (q', d)$, for some $q, q' \in Q$, $a \in \Sigma$, and $d \in \{-1, 0, +1\}$, then $\delta'(q, a^\ast) \ni (q', d)$.
4. If $\delta(q, \vdash) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{0, +1\}$, then $\delta'(q, \vdash) \ni (q', d)$.
5. If $\delta(q, \dashv) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{-1, 0\}$, then $\delta'(q, \dashv) \ni (q', d)$.
6. If $\delta(q, \dashv) \ni (q', d)$, for some $q, q' \in Q$ and $d \in \{0, +1\}$, then $\delta'(q, \dashv) \ni (q', d)$.

Each time $N$ leaves the current segment, e.g., if it moves its input head from the symbol $\vdash$ to the right (that is, in the next step, $N$ will read the symbol $\vdash$ belonging to the next segment), $N'$ does not try to move its input head to the right from the right endmarker but, rather, it temporarily interrupts the simulation and enters a passing-through routine in which it traverses the entire input $w$ from right to left and, during this traversal, it moves the pebble one position to the right. After that, with the input head at the left endmarker of $w$, $N'$ is ready to resume the simulation on the next segment of $P(w)$. Note that the instructions defined in the item (7) cover also three special subcases, namely, migration of the pebble from the left endmarker to the first input symbol, from the last input symbol to the right endmarker, or, for $w = \varepsilon$, from the left endmarker directly to the right endmarker:

7. If $\delta(q, \vdash) \ni (q', +1)$, for some $q, q' \in Q$, we add the following instructions:
   - $\delta'(q, \dashv) \ni (q'_{-1}, -1)$, where $q'_{-1}$ is a new copy of $q'$ — a passing-through state searching for the pebble to the left,
   - $\delta'(q'_{-1}, x) \ni (q'_{-1}, -1)$, for each $x \in \Sigma$,
   - $\delta'(q'_{-1}, a^\ast) \ni (q'_{-2}, +1^\ast)$, for each $x \in \Sigma \cup \{-\}$, where $q'_{-2}$ is another copy of $q'$ — a passing-through state searching for the left endmarker,
   - $\delta'(q'_{-2}, a^\ast) \ni (q'_{-2}, -1)$, for each $x \in \Sigma \cup \{-\}$,
   - $\delta'(q'_{-2}, x) \ni (q'_{-2}, -1)$, for each $x \in \Sigma$,
   - $\delta'(q'_{-2}, \dashv) \ni (q', 0)$.

Similarly, each time $N$ leaves the current segment for the previous segment by moving from the symbol $\vdash$ to the left (after which it will read $\vdash$), $N'$ interrupts the simulation and enters a routine traversing the entire input $w$ from left to right and, during this traversal, it moves the pebble one position to the left. This is resolved symmetrically with the item (7), replacing, respectively, $\vdash, \dashv, \vdash, \dashv$ and $+1, -1$ by $\vdash, \dashv, \vdash, \dashv$ and $-1, +1$, thus using $q'_{+1}, q'_{+2}$, some passing-through counterparts of $q'_{-1}, q'_{-2}$.

Thus, $Q' = Q \cup Q_{-1} \cup Q_{-2} \cup Q_{+1} \cup Q_{+2}$, where $Q$ is the original set of states and $Q_{-1}, Q_{-2}, Q_{+1}, Q_{+2}$ are four new copies of $Q$. Finally, $q'_0 = q_0$ and $F' = F$. 

\vspace{1cm}

V. Geffert, L. Ištoňová 137
The argument showing that $N'$ accepts $w \in \Sigma^*$ if and only if $N$ accepts $P(w) \in \Sigma^*$ is very similar to that of Theorem 2, by induction on the number of steps executed by $N$ and by induction on the number of times $N'$ is in a state $q \in Q$ (i.e., not in a passing-through state), we can prove the claim saying that, on the input $w$, $N'$ can reach a state $q \in Q$ with the input head at a position $h$ and the pebble at a position $p$ if and only if, on the input $P(w)$, $N$ can reach the same state $q$ with the input head at the $h$-th position of the $p$-th segment. It is also easy to see that the construction preserves determinism. \hfill $\square$

Now we are ready to draw some consequences of the above translations.

**Theorem 4.** If, for some function $f(n)$, each 2NFA with $n$ states can be replaced by an equivalent 2DFA with at most $f(n)$ states (no pebbles), then each pebble-2NFA with $m$ states can be replaced by an equivalent pebble-2DFA having no more than $5 \cdot f(3m)$ states.

In particular, if $f(n) \leq O(n^k)$, that is, if there exists a polynomial transformation from nondeterministic to deterministic classical two-way automata, then there must also exist a polynomial transformation, with the same degree of the polynomial, from nondeterministic to deterministic two-way automata equipped with a pebble, since $5 \cdot O(3m)^k = (5 \cdot 3^k) \cdot m^k \leq O(m^k)$.

**Proof:** By Theorem 2(a), each pebble-2NFA $M$ with $m$ states accepting a language $L \subseteq \Sigma^*$ can be replaced by a classical 2NFA $M'$ with at most $3 \cdot m$ states, accepting some other language $L' \subseteq \Sigma^*$. However, for each input $w \in \Sigma^*$, $M$ accepts $w$ if and only if $M'$ accepts $P(w) \in \Sigma^*$. Here $P$ denotes the mapping function defined by (1). By assumption, $M'$ can be replaced by a classical 2DFA $N$, with at most $f(3m)$ states, equivalent to $M'$. Among others, $M'$ accepts $P(w)$ if and only if $N$ accepts $P(w)$. Now, by Theorem 3(b), we can replace $N$ by a pebble-2DFA $N'$ with no more than $5 \cdot f(3m)$ states, such that $N$ accepts $P(w) \in \Sigma^*$ if and only if $N'$ accepts $w \in \Sigma^*$. Thus, for each input $w \in \Sigma^*$, $M$ accepts $w$ if and only if $N'$ accepts $w$, and hence these two pebble machines are equivalent. \hfill $\square$

The situation for complementing nondeterministic machines is very similar.

**Theorem 5.** If, for some function $f(n)$, each 2NFA with $n$ states can be replaced by a 2NFA with at most $f(n)$ states recognizing the complement of the original language (no pebbles), then each pebble-2NFA with $m$ states can be replaced by a pebble-2NFA with no more than $5 \cdot f(3m)$ states recognizing the complement.

In particular, if $f(n) \leq O(n^k)$, that is, if there exists a polynomial transformation for complementing nondeterministic classical two-way automata, then there must also exist a polynomial transformation, with the same degree of the polynomial, for complementing nondeterministic two-way automata equipped with a pebble.

The argument is very similar to the proof of Theorem 4, using Theorems 2(a) and 3(a) instead of Theorems 2(a) and 3(b).

**Corollary 6.** For each pebble-2DFA with $m$ states, there exists a pebble-2DFA with at most $60 \cdot m$ states recognizing the complement of the original language.

This time we use Theorems 2(b) and 3(b), together with the fact that an $n$-state 2DFA can be complemented with no more than $4 \cdot n$ states \cite{10}.

It was known that a pebble-2DFA can be made halting on every input, and hence a machine for the complement can be obtained by exchanging accepting with rejecting states \cite{2, 3, 18, 20}. This would give a pebble-2DFA with $O(m \cdot 2^s)$ states, where $m$ is the original number of states and $s$ the size of the input alphabet. This way, a linear upper bound is obtained for languages over a fixed input alphabet, but not in the general case, where the alphabet size $s$ can grow exponentially in $m$ (see, e.g., \cite{17}). The construction using Corollary 6 does not depend on the size of the input alphabet. (However, we are convinced that the upper bound $60 \cdot m$ is quite high and can be improved.)
4 Conclusion

Already in 1978, it was conjectured by Sakoda and Sipser [16] that there must exist an exponential blow-up, in the number of states, for the transformation of the classical 2NFA’s into 2DFA’s. Nevertheless, this problem is still open. We have shown, by Theorem 4 above, that such blow-up could possibly be derived by proving an exponential gap between pebble-2NFA’s and pebble-2DFA’s. Even showing a less impressive lower bound for the pebble-2NFA versus pebble-2DFA trade-off, say, $\Omega(n^k)$ with some $k \geq 3$, would imply the same lower bound $\Omega(n^k)$ for the classical 2NFA versus 2DFA conversion. (To the best of authors’ knowledge, the highest lower bound obtained so far is $\Omega(n^2)$ [4].) Since a pebble automaton is a different computational model, the argument might use some different witness languages.

Similarly, by Theorem 5 proving an exponential gap for the complementation of the pebble-2NFA’s would imply the same exponential gap for the complementation of the classical 2NFA’s. This, in turn, would imply the exponential gap for the trade-off between 2NFA’s and 2DFA’s, and also between pebble-2NFA’s and pebble-2DFA’s, since the complementation for the deterministic two-way machines is linear (namely, $4 \cdot n$ states for 2DFA’s, by [10], and at most $60 \cdot n$ states for pebble-2DFA’s, by Corollary 6 above).

The most natural related open problem is whether the translation results presented in Theorems 4 and 5 cannot be generalized to two-way automata equipped with more than one pebble placed on the input tape. More precisely, we do not know whether a polynomial trade-off between nondeterministic and deterministic $k$-pebble two-way automata implies the polynomial trade-off for automata equipped with $k + 1$ pebbles. The same question can be asked about complementation of multi-pebble 2NFA’s. (The argument might be quite difficult, since such machines can accept nonregular languages: as an example, already with 2 pebbles we can easily recognize $L = \{a^n b^n c^n : n \geq 0\}$.) Nevertheless, the answers to these questions might bring a deep insight into the world of $O(\log n)$ space bounded computations, since the multi-pebble 2NFA’s and 2DFA’s correspond to the complexity classes $\text{NSPACE}(\log n)$ and $\text{DSPACE}(\log n)$, respectively (see Section 3.2 in [20]). For example, we know that $\text{NSPACE}(\log n)$ is closed under complement, by the inductive counting [13, 19], but the inductive counting technique increases the number of pebbles.

Similarly, we need translation(s) among other computational models that are weak enough to stay within the class of regular languages but strong enough to provide, in some cases, a more succinct representation than the classical models. (As an example, we do not know too much about complexity of two-way automata equipped with a pushdown of a constant height [3].)

References

[1] J. Berman & A. Lingas (1977): On the complexity of regular languages in terms of finite automata. Technical Report 304, Polish Academy of Sciences.

[2] J. H. Chang, O. H. Ibarra, M. A. Palis & B. Ravikumar (1986): On pebble automata. Theoret. Comput. Sci. 44, pp. 111–121.

[3] R. Chang, J. Hartmanis & D. Ranjan (1991): Space bounded computations: Review and new separation results. Theoret. Comput. Sci. 80, pp. 289–302.

[4] M. Chrobak (1986): Finite automata and unary languages. Theoret. Comput. Sci. 47, pp. 149–158. (Corrigendum: ibid., 302, 497–498, 2003).

[5] W. Ellison & F. Ellison (1985): Prime Numbers. John Wiley & Sons.

[6] V. Geffert (1991): Nondeterministic computations in sublogarithmic space and space constructibility. SIAM J. Comput. 20, pp. 484–498.
Translation from Classical Two-Way Automata to Pebble Two-Way Automata

[7] V. Geffert (1998): Bridging across the log(n) space frontier. Inform. & Comput. 142, pp. 127–158.
[8] V. Geffert, C. Mereghetti & B. Palano (2008): More concise representation of regular languages by automata and regular expressions. In: Proc. Develop. Lang. Theory, LNCS 5257. Springer-Verlag, pp. 359–370.
[9] V. Geffert, C. Mereghetti & G. Pighizzini (2003): Converting two-way nondeterministic unary automata into simpler automata. Theoret. Comput. Sci. 295, pp. 189–203.
[10] V. Geffert, C. Mereghetti & G. Pighizzini (2007): Complementing two-way finite automata. Inform. & Comput. 205, pp. 1173–1187.
[11] J. Hopcroft, R. Motwani & J. Ullman (2001): Introduction to Automata Theory, Languages, and Computation. Addison-Wesley.
[12] J. Hromkovič & G. Schnitger (2003): Nondeterminism versus determinism for two-way nondeterministic automata: Generalizations of Sipser’s separation. In: Proc. Internat. Colloq. Automata, Languages, & Programming, LNCS 2719. Springer-Verlag, pp. 439–451.
[13] N. Immerman (1988): Nondeterministic space is closed under complementation. SIAM J. Comput. 17, pp. 935–938.
[14] Ch. A. Kapoutsis (2007): Deterministic moles cannot solve liveness. J. Automat. Lang. Combin. 12, pp. 215–235.
[15] C. Mereghetti & G. Pighizzini (2001): Optimal simulations between unary automata. SIAM J. Comput. 30, pp. 1976–1992.
[16] W. Sakoda & M. Sipser (1978): Nondeterminism and the size of two-way finite automata. In: Proc. ACM Symp. Theory of Comput. pp. 275–286.
[17] M. Sipser (1979): Lower bounds on the size of sweeping automata. In: Proc. ACM Symp. Theory of Comput. pp. 360–364.
[18] M. Sipser (1980): Halting space bounded computations. Theoret. Comput. Sci. 10, pp. 335–38.
[19] R. Szelepcsényi (1988): The method of forced enumeration for nondeterministic automata. Acta Inform. 26, pp. 279–84.
[20] A. Szepietowski (1994): Turing Machines with Sublogarithmic Space, LNCS 843. Springer-Verlag.