Large–$N_c$ QCD and Spontaneous Chiral Symmetry Breaking

Eduardo de Rafael

aCentre de Physique Théorique, CNRS-Luminy, Case 907, F-13288 Marseille Cedex 9, France

I report on recent work done in collaboration with Marc Knecht [1] on patterns of spontaneous chiral symmetry breaking in the large–$N_c$ limit of QCD–like theories, and with Santi Peris and Michel Perrottet [2] concerning the question of matching long and short distances in large–$N_c$ QCD.

1. INTRODUCTION

The study of QCD in the limit of a large number of colours $N_c$ was suggested by ’t Hooft [3], soon after the discovery of asymptotic freedom, as a way to get an insight into non–perturbative properties of the theory. In spite of the efforts of many good theorists [2], QCD in the large–$N_c$ limit remains an unsolved problem as yet; though the research in this field has given rise to a series of remarkable theoretical developments like matrix models and two–dimensional quantum gravity, and is now coming back to QCD, as discussed by David Gross in this meeting, with promising perspectives.

It has been shown by Coleman and Witten [4] (see also refs. [5–8],) that if QCD with $N_c = 3$ confines, and if confinement persists in the large–$N_c$ limit then, in this limit, the chiral $U(n_f) \times U(n_f)$ invariance of the Lagrangian with $n_f$ flavours of massless quarks is spontaneously broken down to the diagonal $U(n_f)$ subgroup.

The hadronic spectrum of QCD in the large–$N_c$ limit, which we shall denote QCD($\infty$) for short, consists then of an infinite number of narrow states with specific quantum numbers. In this talk I shall discuss, within the framework of QCD($\infty$), how the ordering of states in the spectrum is related to the size of the local order parameters of spontaneous chiral symmetry breaking (S$\chi$SB). I shall show that the local order parameters of S$\chi$SB which govern the operator product expansion (OPE) and the non–local order parameters of S$\chi$SB which govern the couplings of the low–energy effective Lagrangian of chiral perturbation theory obey duality properties. Our discussion will focus on the left–right correlation function $\Pi_{LR}(Q^2)$, (see eqs. (1) to (3) below,) but the properties I discuss for this function are rather common to any correlation function which is an order parameter, though the details have to be discussed separately for each Green’s function.

1.1. The Left–Right Correlation Function

The correlation function $\Pi_{LR}(Q^2)$ is the invariant amplitude of the two–point function ($Q^2 \equiv -q^2 \geq 0$ for $q^2$ space–like)

$$\Pi_{LR}^{\mu\nu}(q^2) = 2i \int d^4x e^{iqx} \langle 0 | T \left( L^\mu(x) R^\nu(0) \right) | 0 \rangle , \quad (1)$$

with currents

$$R^\mu (L^\nu) = \bar{d}(x) \gamma^\mu \frac{1}{2} (1 \pm \gamma_5) u(x) . \quad (2)$$

In the chiral limit

$$\Pi_{LR}^{\mu\nu}(Q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_{LR}(Q^2) . \quad (3)$$

The function $\Pi_{LR}(Q^2)$ vanishes order by order in perturbation theory and is an order parameter of S$\chi$SB for all values of $Q^2$. It also governs the electromagnetic $\pi^+ - \pi^0$ mass difference [2]

$$m_{\pi^+ - \pi^0}^{\ EM} = \frac{\alpha}{\pi} \frac{3}{4f^2} \int_0^\infty dQ^2 (-Q^2 \Pi_{LR}(Q^2)) . \quad (4)$$
This integral converges in the ultraviolet region because \[13\]
\[
\lim_{Q^2 \to \infty} Q^2 \Pi_{LR}(Q^2) =
\left(-4\pi^2 \frac{a_s}{\pi} + O(\alpha_s^2)\right) \langle \bar{\psi} \psi \rangle^2 .
\]
This behaviour also entails the two Weinberg sum rules \[14\] in the chiral limit. Witten \[15\] has furthermore shown that, under rather general assumptions which are less restrictive than the large–Nc limit,
\[
- Q^2 \Pi_{LR}(Q^2) \geq 0 \quad \text{for} \quad 0 \leq Q^2 \leq \infty ,
\]
which in particular ensures the positivity of the integral in eq. \(15\) and thus the stability of the QCD vacuum with respect to small perturbations induced by electromagnetic interactions. The same two–point function \(\Pi_{LR}(Q^2)\) governs the full electroweak \(\pi^+ - \pi^0\) mass difference at the one–loop level in the electroweak interactions of the Standard Model and to lowest order in the chiral expansion \[14\].

The low \(Q^2\) behaviour of this self–energy function is governed by chiral perturbation theory:
\[
- Q^2 \Pi_{LR}(Q^2) = f_\pi^2 + 4L_{10} Q^2 + O(Q^4) ,
\]
where \(L_{10}\) is one of the Gasser–Leutwyler coupling constants \[14\] of the \(O(p^4)\) low energy effective chiral Lagrangian, i.e. the Lagrangian formulated in terms of Goldstone degrees of freedom and external local sources only.

In QCD(\(\infty\)) the spectral function associated with \(\Pi_{LR}(Q^2)\) consists of the difference of an infinite number of narrow vector states and an infinite number of narrow axial–vector states, together with the Goldstone pole of the pion:
\[
\frac{1}{\pi} \text{Im} \Pi_{LR}(t) = \sum_V f_V^2 M_V^2 \delta(t - M_V^2) - f_\pi^2 \delta(t) - \sum_A f_A^2 M_A^2 \delta(t - M_A^2) .
\]
Since \(\Pi_{LR}(Q^2)\) obeys an unsubtracted dispersion relation, it follows that
\[
- Q^2 \Pi_{LR}(Q^2) = f_\pi^2 + \sum_A f_A^2 M_A^2 \frac{Q^2}{M_A^2 + Q^2} - \sum_V f_V^2 M_V^2 \frac{Q^2}{M_V^2 + Q^2} .
\]

Furthermore, the two Weinberg sum rules that follow from eq. \(15\) constrain the couplings and masses of the narrow states as follows:
\[
\sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 = f_\pi^2
\]
and
\[
\sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0 ,
\]
ensuring the convergence of the integral in eq. \(15\) in QCD(\(\infty\)).

2. SPECTRAL CONSTRAINTS

In QCD(\(\infty\)) there exists an infinite number of Weinberg–like sum rules associated with the \(\Pi_{LR}(Q^2)\)–function. With the two constraints in eqs. \(16\) and \(17\) incorporated in the r.h.s. of eq. \(15\), the large–\(Q^2\) expansion of \(\Pi_{LR}(Q^2)\) becomes
\[
\Pi_{LR}(Q^2) = \left( \sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 \right) \frac{1}{Q^6} + \left( \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 \right) \frac{1}{Q^8} + \cdots .
\]

Matching this expansion in powers of \(1/Q^2\) with the corresponding OPE of \(\Pi_{LR}(Q^2)\) in QCD(\(\infty\)) leads to relations between hadronic parameters and the local order parameters of S\(\chi\)SB which appear as vacuum expectation values of composite operators in the OPE. For example, matching the \(1/Q^6\)–coefficient in eq. \(15\) with the result in eq. \(18\), we have that
\[
\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 = \left(-4\pi^2 \frac{a_s}{\pi} + O(\alpha_s^2)\right) \langle \bar{\psi} \psi \rangle^2 .
\]
[Notice that the negative sign in the r.h.s. above is certainly in accordance with Witten’s positivity constraint in eq. \(15\).] In full generality, positive moments of the \(\Pi_{LR}–\)spectral function correspond to local order parameters of S\(\chi\)SB \(\langle \Phi^{2n} \rangle\) of dimension \(2n\), \(n = 3, 4, \ldots\)
\[
\int_0^{\infty} dt t^{n-1} \left[ \frac{1}{\pi} \text{Im} \Pi_V(t) - \frac{1}{\pi} \text{Im} \Pi_A(t) \right] =
\]
\[ \sum_V f_V^2 M_V^{2n} - \sum_A f_A^2 M_A^{2n} = C_{2n} \langle \Phi^{2n} \rangle , \quad (15) \]

with \( C_{2n} \) the corresponding short–distance Wilson coefficient calculable in perturbative QCD \((\text{pQCD})\).

On the other hand, inverse moments of the spectral function associated with \( \Pi_{LR} \), with the pion pole removed, (this is the meaning of the tilde symbol on top of \( \text{Im} \Pi_A(t) \) in eq. (16) below,) determine the couplings of the low–energy effective chiral Lagrangian. For example,

\[
\int_0^\infty dt \frac{1}{t} \left[ \frac{1}{\pi} \text{Im} \Pi(t) - \frac{1}{\pi} \text{Im} \Pi_A(t) \right] = \sum_V f_V^2 - \sum_A f_A^2 = -4L_{10}. \quad (16)
\]

Moments with higher inverse powers of \( t \) are associated with couplings of composite operators of higher dimension in the chiral Lagrangian.

### 2.1. Patterns of \( S \chi SB \)

In ref. [4] we have shown how the ordering of vector and axial vector states in the hadronic spectrum of QCD \((\text{QCD})\) is correlated to the size of the local order parameters \( \langle \Phi^{2n} \rangle \) of \( S \chi SB \). Quite generally, we have shown that in QCD \((\text{QCD})\) spontaneous chiral symmetry breaking à la Nambu–Goldstone with \( f_\pi^2 \neq 0 \) necessarily implies the existence of non–zero local order parameters which transform according to the representation \((n_f, \bar{n}_f) + (\bar{n}_f, n_f)\) of the chiral group. This is in a way the converse theorem\(^3\) to the Coleman–Witten theorem \([3]\) stated in the Introduction.

The minimal pattern of a spectrum compatible with the short–distance properties of the \( \Pi_{LR} \)–function in QCD \((\text{QCD})\) with \( n_f = 3 \), is one which besides the Goldstone pseudoscalar nonet has a vector nonet of states and an axial–vector nonet of states. The required ordering is then \( M_V < M_A \). Implicit here, of course, is the assumption that the sum of the infinite number of narrow vector states and the sum of the infinite number of narrow axial–vector states with masses higher than the highest mass explicitly considered (here \( M_A \)) are already dual to their respective pQCD \((\text{pQCD})\) continuum. Their contributions to the spectral function \( \text{Im} \Pi_{LR}(t) \) cancel then each other and, therefore, they vanish in the Weinberg sum rules, as well as in the generalized Weinberg sum rules discussed above. This minimal pattern is also the one that the authors of ref. [3] considered in their evaluation of the electromagnetic pion mass difference, which gives as a result: \( \Delta m_\pi = 5.2 \text{ MeV} \) remarkably close to the experimental result: \( \Delta m_\pi |_{\text{exp.}} = 4.59 \text{ MeV} \).

As shown in ref. [4], the extreme version of the so called generalized \( \chi \)PT proposed by J. Stern et al. [3], where \( \langle \psi \bar{\psi} \rangle = 0 \), is incompatible with this phenomenologically successful minimal pattern of a hadronic low–energy spectrum with only one \( V \)–state and only one \( A \)–state.

Another interesting feature discussed in ref. [4] is the possible existence of low–energy particle spectra in vector–like gauge theories with a rather different structure than the one observed in the QCD hadronic spectrum. For example, the minimal pattern required to have a negative electroweak \( S \) parameter (the equivalent of \(-4L_{10}\) in an underlying technicolour–like model of electroweak breaking) is a spectrum with two axial–vector states \( A_1 \) and \( A_3 \) and a vector state \( V_2 \) with an increasing ordering of masses: \( M(A_1) < M(V_2) < M(A_3) \).

### 2.2. Duality Properties of \( \Pi_{LR}(Q^2) \)

It is instructive to reconsider the two–point function \( \Pi_{LR}(Q^2) \) in the simple case of a minimal spectrum with one vector state \( V \) and one axial–vector state \( A \). In this case \( \Pi_{LR}(Q^2) \) reduces to a very simple form

\[
-Q^2 \Pi_{LR}(Q^2) = f_\pi^2 \frac{1}{1 + \frac{Q^2}{M_V^2} \left( 1 + \frac{Q^2}{M_A^2} \right)}
\]

\[
= f_\pi^2 \frac{M_V^2 M_A^2}{Q^4} \frac{1}{1 + \frac{M_V^2}{Q^2} \left( 1 + \frac{M_A^2}{Q^2} \right)}. \quad (17)
\]

This equation shows explicitly a remarkable short–distance \( \equiv \text{long–distance} \) symmetry. Indeed, with \( g_A \) defined so that \( M_A^2 = g_A M_V^2 \) and \( z \equiv \frac{Q^2}{M_V^2} \), then

\[
-Q^2 \Pi_{LR}(Q^2) \equiv f_\pi^2 \mathcal{H}(z; g_A), \quad (18)
\]
and we find that 
\[ \mathcal{H}(z; g_A) = \frac{1}{z^2} \frac{1}{g_A} \mathcal{H} \left( \frac{1}{z}, \frac{1}{g_A} \right). \] (19)

This means that, in the minimal pattern spectrum, the non-local order parameters corresponding to the long-distance expansion for \( z \to 0 \), which correspond to couplings of the effective chiral Lagrangian i.e.,
\[ -Q^2 \Pi_{LR}(Q^2) = f^2 \frac{1}{g_A} \frac{1}{z^2} \times \left\{ 1 - \left( 1 + \frac{1}{g_A} \right) \frac{1}{z} + \left( 1 + \frac{1}{g_A} + \frac{1}{g_A^2} \right) \frac{1}{z^2} + \cdots \right\}, \] (20)

are exactly correlated to the local order parameters of the short-distance OPE for \( z \to \infty \) in a very simple way:
\[ -Q^2 \Pi_{LR}(Q^2) = f^2 \frac{1}{g_A} \frac{1}{z^2} \times \left\{ 1 - \left( 1 + \frac{1}{g_A} \right) \frac{1}{z} \right\}; \] (21)
in other words, knowing the expansion at large \( z \) we can reconstruct the corresponding expansion at small \( z \) and vice versa.

3. APPROXIMATED QCD(∞)

The minimal pattern of an acceptable hadronic spectrum in QCD(∞) turns out to be rather successful in describing global features of low-energy hadron phenomenology. Partly inspired by the traditional successes of “vector meson dominance” in predicting, e.g., the low-energy constants of the effective chiral Lagrangian [19], we have recently proposed [2] to consider the approximation to QCD(∞) which restricts the hadronic spectrum in the channels with \( J^P \) quantum numbers 0−, 1−, 0+ and 1+ to the lowest energy state and treats the rest of the narrow states as a pQCD(∞) continuum, the onset of the continuum being fixed by consistency constraints from the operator product expansion; (like the absence of \( d = 2 \) operators.) We have shown that there exists a useful effective Lagrangian description of this well defined lowest meson dominance (LMD) approximation to QCD(∞). The degrees of freedom in the effective Lagrangian are then a nonet of pseudoscalar Goldstone particles which are collected in a unitary matrix \( U(x) \), and nonets of vector fields \( V(x) \), scalar fields \( S(x) \) and axial-vector fields \( A(x) \) associated with the lowest energy states of the hadronic spectrum which are retained. We have derived the effective Lagrangian by implementing successive requirements on an extended Nambu–Jona-Lasinio (ENJL)–type Lagrangian [21, 24] which we have chosen as the initial ansatz. The first requirement is to eliminate the effects of non-confining QQ discontinuities (\( Q \) denotes the constituent quark field) by introducing an infinite number of appropriate local operators with couplings which can be fixed in terms of the three parameters of the starting ENJL–Lagrangian itself, i.e. the coupling constants \( G_S, G_V \) and the scale \( \Lambda_\chi \). We have shown that the matching of the two-point functions of this effective Lagrangian to their QCD(∞) short-distance behaviour can be systematically implemented. In particular, the first and second Weinberg sum rules are automatically satisfied.

For Green’s functions beyond two-point functions, the removal of the non-confining QQ discontinuities produced by the initial ENJL ansatz is however not enough to guarantee in general the correct matching to the leading QCD short-distance behaviour and further local operators have to be included. We have discussed this explicitly in the case of the VPP and VPA three-point functions, and shown that the matching with the QCD short-distance leading behaviour which follows from the OPE restricts the initial three free parameters of the ENJL–Lagrangian ansatz to just one mass scale \( M_\rho \) and a dimensionless constant \( M_\rho^2/\Lambda_\chi^2 \). The resulting low-energy Lagrangian in the vector and axial–vector sector, and to \( O(p^4) \) in the chiral expansion, coincides with the class of phenomenological Lagrangians discussed in ref. [21] which also have two free parameters \( f^2_\rho \) and \( f^2_\rho/M_\rho^2 \). In this respect, this explains the relation to QCD of the phenomenological VMD Lagrangians discussed in ref. [21]; to \( O(p^4) \), they can be viewed as the effective low-energy Lagrangians of the LMD approximation to QCD(∞). On the other hand, the fact that the resulting low-energy Lagrangian coincides with the phenomenological VMD La-
grangians discussed in ref. [20] demystifies to a large extent the rôle of the ENJL–Lagrangian itself as a fundamental step in deriving the low–energy effective Lagrangian of QCD. The ENJL–Lagrangian turns out to be already a very good ansatz to describe in terms of quark fields degrees of freedom the LMD approximation to QCD(∞) and this is why it is already quite successful at the phenomenological level; but, when the non–confining $Q\bar{Q}$ discontinuities are systematically removed, the phenomenological predictions improve even more.

There is an advantage, however, in starting with the ENJL–Lagrangian as an ansatz, and that is that in this description of the LMD approximation to QCD(∞), all the couplings to all orders in $\chi$PT are clearly correlated to the same two free parameters. For example the $L_5$ and $L_8$ constants, as well as part of the contribution to the $L_3$ constant which result from scalar exchanges, are now proportional to a universal dimensionless parameter, while in a purely phenomenological description in terms of chiral effective Lagrangians which include resonances as discussed e.g. in refs. [19] and references therein, these low–energy constants require new phenomenological input.

Finally, we wish to insist on the phenomenological successes of the LMD approximation to QCD(∞) as demonstrated by the results presented in ref. [2]. In particular the Gasser–Leutwyler coupling constants of the $O(p^4)$ chiral Lagrangian which do not trivially vanish in the large–$N_c$ limit, are all fixed by the ratio $f_\pi^2/M_V^2$:

$$6L_1 = 3L_2 = \frac{8}{7}L_3 = 4L_5 = 8L_8$$

$$= \frac{3}{4}L_9 = -L_{10} = \frac{3}{8} \frac{f_\pi^2}{M_V^2}.$$  \hspace{1cm} (22)

These results show that the LMD approximation to QCD(∞) is indeed a very good approximation to full fledged QCD. The deep reason for that may very well be correlated to the size of $\Lambda^{(3)}_{\overline{MS}}$. Indeed, the onset of the pQCD continuum associated with the LMD approximation in the case, e.g., of the vector two–point function is at a $t$ value [3]:

$$t_0 \simeq 1.5 \text{GeV}^2,$$

which is already sufficiently high for pQCD(∞) to be applicable if $\Lambda^{(3)}_{\overline{MS}} \simeq 400 \text{MeV}$ as known phenomenologically.

It seems now worthwhile to apply the LMD approximation to QCD(∞) to the calculation of couplings of $O(p^6)$ and $O(e^2p^2)$ as well. One can also, at last, reconsider non–leptonic weak interactions in the light of this effective Lagrangian framework with some hope of success, since within this approach we expect to be able to show a rather good matching between the long–distance evaluation of matrix elements of four–quark operators and the short–distance pQCD logarithmic dependence of the Wilson coefficients.

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