DEVELOPMENTS IN 2D STRING THEORY

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1. Introduction

Recent years have witnessed a remarkable progress in 2d string theory and quantum gravity. Beginning with matrix models one found a new and computationally powerful description of the theory, free of mathematical complexities. The relevance of these models to string theory comes through a $1/N$ expansion where $1/N$ plays the role of a bare string coupling constant $g_{st}^0 = 1/N$. This classifies Feynman diagrams according to their topology; for a fixed topology the sum of all graphs in a dual picture becomes a sum of triangulated surfaces. The continuum theory is then approached by sending the value of lattice spacing to zero.

This heuristic picture was completely carried out in one dimension giving an exactly solvable theory of two-dimensional strings (for an earlier review see [1]). It lead first to a series of explicit results including the computation of free energy and correlation functions at any order in the loop expansion. The new formulation also offered a framework for non-perturbative investigations. It provided a new fundamental insight into the origin of metric fluctuations and the physical nature of the Liouville mode. Through a critical scaling limit a two-dimensional theory is generated where the logarithmic scaling violation is seen to be the origin of the extra dimension.

Most of the interesting features of 2d strings were clearly exhibited in the field-theoretic description achieved in terms of collective field theory. Starting from matrix models one builds a field theory describing the dynamics of observable (Wilson) loop
variables. The collective Hamiltonian describes the processes of joining and splitting of loops, giving a cubic interaction and a linear (tadpole) term were shown to successfully produce all tree and loop diagrams. The theory is naturally integrable and exactly solvable. Its integrable nature leads to understanding of a $w_\infty$ algebra as a space-time symmetry of the theory. This algebra acts in a nonlinear way on the basic collective field representing the tachyon. It is interpreted as a spectrum-generating algebra allowing to build an infinite sequence of discrete imaginary energy states which turn out to be remnants of higher string modes in two dimensions. The presence and interplay of discrete modes with the scalar tachyon are particularly interesting. The $w_\infty$ symmetry is seen to serve as an organizational principle specifying the dynamics.

Two-dimensional physics is made even richer by the existence of other nontrivial backgrounds. Most interesting is the black hole type classical solution described by an exact $SL(2,\mathbb{R})/U(1)$ sigma model. Its quantum mechanical interpretation is of major interest and was the object of various recent studies.

Even though there is a wealth of results coming from detailed studies of matrix models and conformal field theories a full understanding of the theory and its dynamics is still not available. In particular, a clear correspondence between the two fundamentally different methods is lacking. One has an (excellent) comparison of results and a pattern of similarities and analogies hinting at a more unified framework. Prospects for such a framework are particularly exciting since this would eventually represent a new formulation of string field theory.

A need for such a general framework is most clear already when addressing the question of the black hole. In general one would like to command sufficient insight to be able to go from one solution to another. This, at present, is also one of the fundamental challenges of string theory.

In this series of lectures we describe the progress already achieved. The emphasis is on a unified understanding of the subject. We will try to bridge the two major approaches: the matrix model and conformal field theory, as much as possible describing analogies and similarities that one has between them. In this process a dictionary emerges; it is most visible in the discussion of the infinite $w_\infty$ symmetry and the associated Ward identities. The question of incorporating the black hole background is then addressed and some preliminary results in this direction are described.

The selection of topics covered is as follows: In sect. 2 we give a summary of basic two-dimensional string theory (for a more detailed review see [2]). In sect. 3 we describe the matrix model and a transition to field theory. We discuss the integrability of the theory and the construction of exact states and their string interpretation. In sect. 4 the corresponding $w_\infty$ symmetry is described. A detailed comparison of Ward identities and a description of the agreement between matrix model and conformal field constructions is given. Sect. 5 contains the discussion of the $S$-matrix of the theory. The latter is described by an exact generating function, connection of which
to matrix model harmonic oscillator states we emphasize. In sect. 6 we discuss the black hole background.

2. String Theory in Two Dimensions

The conceptually simplest way to discuss the dynamics of strings is through a $\beta$-function approach which provides effective equations for low-lying fields. In the case of a closed string in two dimensions these are the $m^2 = 0$ scalar $T(X^\mu)$ (the would-be tachyon), the graviton $G_{\mu\nu}(X)$ and the dilaton $D(X)$. The leading $\beta$-function Lagrangian reads:

$$S_{\text{eff}} = \frac{1}{2\pi} \int d^2 X \sqrt{G} e^{-2D(X)} \left\{ \frac{1}{2} [\nabla_\mu T \nabla^\mu T + 2T^2 - V] + R + 4\nabla D \cdot \nabla D + \ldots \right\}.$$  \hspace{1cm} (2.1)

The tachyon potential $V(T)$ is not so well known and neither are the couplings to possibly higher-spin fields. But this effective Lagrangian exhibits several simple solutions which can serve as classical configurations of two-dimensional string theory.

Denoting $X^\mu \equiv (X^0 = t, X^1 = \varphi)$ one has the linear dilaton vacuum solution

$$T(X) = 0, \hspace{1cm} G_{\mu\nu}(X) = \eta_{\mu\nu}, \hspace{1cm} D(X) = -\sqrt{2} \varphi.$$  \hspace{1cm} (2.2)

The scalar (tachyon) effective Lagrangian in this linear dilaton background reads

$$S_{\text{eff}}(T) = \frac{1}{2} \int d^2 X e^{2\sqrt{2}\varphi} \left\{ \frac{1}{2} T \left( -\partial_t^2 + \partial_\varphi^2 + 2\sqrt{2} \partial_\varphi + 2 \right) T - V \right\}.$$  \hspace{1cm} (2.3)

Rescaling the scalar fields

$$e^{\sqrt{2}\varphi} T(t, \varphi) = \tilde{T}(t, \varphi)$$

yields a massless theory

$$S = \frac{1}{2} \int dt d\varphi \left\{ \frac{1}{2} \tilde{T} \left( -\partial_t^2 + \partial_\varphi^2 \right) \tilde{T} - e^{-\sqrt{2}\varphi} \frac{\tilde{T}^3}{3!} + \ldots \right\},$$

with a spatially dependent string coupling constant

$$g_{st}(\varphi) = e^{-\sqrt{2}\varphi}.$$  \hspace{1cm} (2.4)

(we have taken for simplicity a cubic interaction).
This coupling grows and becomes infinite at $\varphi \to -\infty$. This is usually taken as a signal that the linear dilaton vacuum should be modified (at least in the region $\varphi \to -\infty$). Indeed the linearized static tachyon equation
\[
\left( \partial_\varphi^2 + 2 \sqrt{2} \partial_\varphi + 2 \right) T_0(\varphi) = 0
\]
already has two linearly independent solutions $T_0(\varphi) = e^{-\sqrt{2} \varphi}, \varphi e^{-\sqrt{2} \varphi}$. This would imply that the correct vacuum is given by a tachyon condensate [3]. An (incomplete) analysis indicates that this vacuum is then described by a $c = 1$ conformal field theory coupled to a Liouville field:
\[
\mathcal{L} = \frac{1}{8\pi} \int d^2 z \left( \partial X \bar{\partial} X + \partial \varphi \bar{\partial} \varphi - 2\sqrt{2} \varphi(z, \bar{z}) R^{(2)} + \mu e^{-\sqrt{2} \varphi(z, \bar{z})} \right).
\]
Here the central charge $c_X = 1$ refers to the (matter) coordinate $X(z, \bar{z})$ while the Liouville field with $Q = 2\sqrt{2}$ carries a central charge $c_\varphi = 1 + 3Q^2 = 25$ leading to the required total of $c = c_X + c_\varphi = 26$. It is very interesting that in two dimensions one has another conformally invariant background, the WZW $SL(2, \mathbb{R})/U(1)$ sigma model representing a black hole (BH) [4–8]. Its physical properties are of major interest as is the general question of describing different string theory backgrounds in a single field-theoretic framework.

The presence of the cosmological term in the Liouville theory (and of the mass term in the black hole conformal field theory) leads to computational difficulties when evaluating the correlation functions (these actually become quite untractable for the BH case). It is a remarkable fact that the matrix model formulation succeeds in handling the first problem with ease and has some promise for addressing the second as well.

The spectrum of states is usually obtained by neglecting the nonlinear terms $\mu = 0$ (or $M = 0$ for the black hole) in which case one has a free field representation for the Virasoro generators. In the above limit the spectra of two theories are the same. They consist of a massless tachyon and an infinite sequence of discrete states.

We begin with the zero mode or tachyon states:
\[
(L_0 - 1) V_{k,\beta} = 0,
\]
\[
L_0 = \frac{1}{2} \left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial \varphi^2} + Q \frac{\partial}{\partial \varphi} \right),
\]
with two branches of solutions
\[
V_\pm = e^{ikX + \beta_\pm \varphi}, \quad \beta_\pm = -\sqrt{2} \pm |k|,
\]
following from the on-shell condition

\[ k^2 - \beta(Q + \beta) = 0. \quad (2.10) \]

Here we have taken an Euclidean (space) signature for \( X \) and \( \varphi \) which is a convention in conformal field theory discussions. One can take \( X \) to be the space variable and \( \varphi \) to be the (Euclidean) time variable. It will be more physical, and from the matrix model viewpoint more natural, to treat \( \varphi \) as a space coordinate and continue \( X \) to Minkowski time:

\[ X \to -it, \quad k \to ip. \quad (2.11) \]

In the context of full Liouville theory, the second branch with \( \beta_\pm = -\sqrt{2} - |k| \) has a questionable meaning since the wave functions grow at \( \varphi \to -\infty \) which is the location of the infinitely high Liouville wall \( \mu e^{-\sqrt{2} \varphi} \). These vertex operators are termed “wrongly” dressed. Operators with positive Liouville dressing have a clear meaning. Depending on the sign of the momentum, \( \pm = \text{sign } k \), these are either right- or left-moving waves. It is sensible to use them to compute scattering processes and denote them as

\[ T_k^\pm = e^{ikX + (\sqrt{2}\pm k)\varphi}, \quad \pm = \text{sign } k. \quad (2.12) \]

The Minkowskian continuation is \( k = \pm ip \) and

\[ T_p^+ = e^{ip(t+\varphi)}e^{-\sqrt{2}\varphi}, \]
\[ T_p^- = e^{-ip(t-\varphi)}e^{-\sqrt{2}\varphi}, \quad (2.13) \]

for \( p > 0 \) describe left- and right-moving waves, respectively.

In addition one has an infinite sequence of nontrivial discrete states specified by discrete (imaginary) values of energy and Liouville momenta [9]:

\[ ip_{\varphi} = -\sqrt{2}(1 - j), \quad ip = \sqrt{2}m, \quad (2.14) \]

with \( j = 0, \frac{1}{2}, 1, \ldots \) and \( m = -j, \ldots, j \). Clearly the states with \( m = j \) and \( m = -j \) are just special tachyon states. The simplest way then to reach the other states is to use the \( SU(2) \) generators as raising and lowering operators on the \( m = \pm j \) tachyon states. The \( SU(2) \) generators are given by

\[ t_+ = e^{i\sqrt{2}X(z)}, \]
\[ t_- = e^{-i\sqrt{2}X(z)}, \]
\[ t_3 = i\sqrt{2} \partial X(z). \quad (2.15) \]
Denoting now the highest weight state as
\[ W_{jj}^{(+)} = e^{i\sqrt{2}jX(z)} e^{-\sqrt{2}(1-j)\varphi(z)} \]
one gets the vertex operator for general discrete states
\[ W_{jm}^{(+)} = \left( \oint d\omega e^{-i\sqrt{2}X(\omega)} \right)^{j-m} W_{jj}^{(+)} , \quad -j \leq m \leq +j . \]
These can also be found in the Fock space where they solve the Virasoro conditions of the \( c = 1 \) theory
\[ (L_0 - 1)|jm\rangle = 0 , \]
\[ L_n |jm\rangle = 0 , \]
\[ |jm\rangle = \int dz W_{jm}(z) |0\rangle . \]
One also has operators with the opposite (negative) Liouville dressing
\[ W_{jm} = V_{jm}(X) e^{-\sqrt{2}(1+j)\varphi(z)} , \]
whose physical meaning is again questionable in the full Liouville theory. These states, however, turn out to play an important role as black hole mass perturbations.

Evaluation of correlation functions in the continuum approach is rather nontrivial and often relies on a number of educated guesses involving various analytic continuations. The problem lies in the nontrivial Liouville potential term. By separating and integrating out the zero mode \( \varphi(z, \bar{z}) = \varphi_0 + \tilde{\varphi} \) one finds, through a functional integral formulation, the representation
\[ \langle N \prod_{i=1}^{N} T_i \rangle = \frac{\mu}{\pi}^s \Gamma(-s) \left( \prod_i T_i \right) \left( \int d^2 z e^{\sqrt{2}\tilde{\varphi}} \right)^s \rangle_{\mu=0} . \]
Here the \( \Gamma \)-function is a result of \( \varphi_0 \) integration
\[ \int d\varphi_0 e^{Q\varphi_0} \left( \prod_i e^{\beta_i \varphi_0} \right) \exp \left( -\frac{\mu}{\hbar} e^{-\sqrt{2}\varphi_0} \right) \int e^{-\sqrt{2}\tilde{\varphi}} \), \]
and
\[ -\sqrt{2}s \equiv \sum_i \beta_i + Q . \]
The remaining correlation function is at \( \mu = 0 \) but has a nontrivial power of the Liouville term given by \( s \). It can be evaluated only for \( s = \text{integer} \) with the full result
to be obtained by some analytic continuation [10]. The $s = 0$ amplitude is termed a “bulk” amplitude since the condition $s = 0$ coincides with a Liouville momentum conservation. Nontrivial computation involving major cancellation between matter and Liouville contributions gives the simple result

$$T(k_1, k_2, \ldots, k_N) = (N - 3)! \prod_{i=1}^{N} \frac{\Gamma(-\sqrt{2}|k_i|)}{\Gamma(\sqrt{2}|k_i|)}.$$  

(2.23)

At $s = 0$ one has both energy and momentum conservation laws:

$$\sum_{i=1}^{N} k_i = 0, \quad \sum_{i=1}^{N} |k_i| = -2\sqrt{2}.$$  

(2.24)

Choosing $k_1, k_2, \ldots, k_{N-1} > 0$, one finds that the $N$-th particle momentum is totally determined

$$k_N = -\frac{N - 2}{\sqrt{2}},$$  

(2.25)

implying that the $N$-th leg factor diverges

$$\frac{\Gamma(-N + 2)}{\Gamma(N - 2)} \sim \frac{1}{0}.$$  

(2.26)

This is in agreement with the previous $\Gamma(0) \sim \frac{1}{0}$ divergence. This divergence is related to the length of the Liouville line $\int d\varphi_0$ and is only fully understood in the matrix description.

The final result for these $s = 0$ bulk amplitudes is that they consist of purely external leg factors $\Delta = \Gamma(-\sqrt{2}|k|)/\Gamma(\sqrt{2}|k|)$ and that only $T_{+\ldots+}$ and $T_{-\ldots-}$ amplitudes contain a diverging factor playing the role of the Liouville volume. These bulk amplitudes can then lead to the full $s \neq 0$ amplitudes by an appropriate continuation [11]. This had to await developments given by the matrix model formalism. Concerning the full treatment of Liouville theory one has the interesting algebraic approach of [12,13].

3. Matrix Model and Field Theory

The manner in which a simple matrix dynamics gives rise to nonlinear two-dimensional string theory is rather interesting and is related to collective phenomena. The major tool employed is a field-theoretic representation given by collective field theory [14]. We shall now give the main features of the field-theoretic approach and describe its significance to string theory [15]. The field theory turns out to correctly
describe interactions of strings, it therefore represents a very simple string field theory. It provides some major insight into the physics of noncritical strings allowing the computation of scattering processes \[16\] and giving the exact $S$-matrix \[17\]. New higher space-time symmetries are seen to emerge \[18\] with further implications on general string field theory being likely.

The simple model that one considers is a Hermitian matrix $M^\dagger(t) = M(t)$ in one time dimension ($X^0 = t$) with a Lagrangian

$$L = \frac{1}{2} \text{Tr} \left( \dot{M}^2 - u(M) \right). \quad (3.1)$$

It has an associated $U(N)$ conserved (matrix) charge $J = i[M, \dot{M}]$. Restricting oneself to the singlet subspace $\langle \hat{J} \rangle = 0$ turns this model into a gauge theory. The matrix can be diagonalized: $M(t) \to \text{diag} (\lambda_i(t))$ with the eigenvalues describing a system of nonrelativistic fermions.

The collective variables of the model are the gauge invariant (Wilson) loop operators

$$\phi_k(t) = \text{Tr} \left( e^{ikM(t)} \right) = \sum_{i=1}^{N} e^{ik\lambda_i(t)}, \quad (3.2)$$

which, after a Fourier transform

$$\phi(x,t) = \int \frac{dk}{2\pi} e^{-ikx} \phi_k(t) = \sum_{i=1}^{N} \delta (x - \lambda_i(t)), \quad (3.3)$$

have a physical interpretation of a density field (of fermions). Introduction of a conjugate field $\Pi(x,t)$ with Poisson brackets

$$\{\phi(x), \Pi(y)\} = \delta(x - y) \quad (3.4)$$
gives a canonical phase space.

The dynamics of this field theory is directly induced from the simple dynamics of the matrix model variables $M(t)$ and $P(t) = \dot{M}(t)$. It is found to be given by the Hamiltonian

$$H_{\text{coll}} = \int dx \left\{ \frac{1}{2} \Pi_{,x} \dot{\phi} \Pi_{,x} + \frac{\pi^2}{6} \phi^3 + u(x)\phi \right\}, \quad (3.5)$$

where the first two terms come from the kinetic term of the matrix model $\text{Tr}(P^2/2)$ while the last term represents the potential (the latter can be easily seen through the
density representation):

\[
\frac{1}{2} \text{Tr} P^2 \rightarrow \frac{1}{2} \Pi_x \phi \Pi_x + \frac{\pi^2}{6} \phi^3 , \quad \text{Tr} u(M) \rightarrow \int dx u(x)\phi(x,t) . \tag{3.6}
\]

The Hamiltonian constructed in this way consists of a cubic (interaction) term and a linear (tadpole) term. In terms of basic loops (and strings) the cubic interaction has the effect of splitting and joining strings. The linear tadpole term represents a process of string annihilation into the vacuum. It contains the classical background potential. This potential is tuned to get a particular string theory background; the noncritical \( c = 1 \) string theory is obtained for example with an inverted oscillator potential. Two relevant facts are immediate in this transition to collective field theory:

(1) The field \( \phi(x,t) \) is two-dimensional with the extra spatial dimension \( x \) being related to the eigenvalue space \( \lambda_i \). The appearance of an extra dimension is the first sign that this theory will be describing \( D = 2 \) strings.

(2) The equations of motion for the induced fields are nonlinear while the matrix equations (in particular for the physically relevant oscillator potential \( u(M) = -\frac{M^2}{2} \)) are linear

\[
\ddot{M}(t) - M(t) = 0 . \tag{3.7}
\]

Through a nonlinear transformation, \( \phi(x,t) = \text{Tr} \delta(x - M(t)) \) the matrix model provides an exact solution to the nonlinear field theory. The feature of integrability and the collective transformation itself is very similar to the well known inverse scattering transformation in integrable field theories. Actually introduction of left- and right-moving chiral components \( \alpha_\pm(x,t) = \Pi_{,x} \pm \pi \phi(x,t) \) with Poisson brackets

\[
\{ \alpha_\pm(x), \alpha_\pm(y) \} = \pm 2\pi \partial_x \delta(x - y) \tag{3.8}
\]

brings the Hamiltonian to the form

\[
H_{\text{coll}} = \frac{1}{2} \int dx \left\{ \frac{1}{3} \left( \alpha_+^3 - \alpha_-^3 \right) - (x^2 - \mu) (\alpha_+ - \alpha_-) \right\} . \tag{3.9}
\]

The equations of motion

\[
\partial_t \alpha_\pm + \alpha_\pm \partial_x \alpha_\pm - \mu = 0 \tag{3.10}
\]

are then seen to be two copies of a large-wavelength KdV type equation with an external \((-x^2)\) potential. Collective field theory shares with some other field theories in two dimensions the feature of exact solvability. One can indeed write down an
infinite sequence of conserved commuting quantities (Hamiltonians). They are simply
given by [18]:

\[ H_n = \frac{1}{2\pi} \int dx \int_{\alpha_+}^{\alpha_-} d\alpha \left( \alpha^2 - x^2 \right)^n , \tag{3.11} \]

and are related to the matrix model quantities \( \text{Tr}(P^2 - M^2)^n \). In fact one has a
simple set of transition rules between the two descriptions. These are useful when
constructing exact eigenstates and symmetry generators of the theory.

One easily checks that the Poisson brackets vanish

\[ \{H_n, H_m\} = 0 \, , \tag{3.12} \]

and that these charges are formally conserved

\[ \frac{d}{dt} H_n = \int dx \partial_x (\alpha^2 - x^2) (\alpha^2 - x^2)^n = 0 \, . \tag{3.13} \]

This naturally is correct only up to surface terms which are present and will allow
particle production.

Before continuing with the integrability features of the theory one can study per-
turbation theory and small fluctuations to clarify at this simple level the connection
to string theory. The static (ground state) equation reads

\[ \frac{1}{2} (\pi \phi_0(x))^2 + u(x) = \mu_F , \tag{3.14} \]

where \( \mu_F \) is the Fermi energy introduced as a linear term in the Hamiltonian

\[ \Delta H = - \int dx \, \mu_F \phi(x,t) \, . \tag{3.15} \]

Denoting \( \pi \phi_0 = p_0 \) we see this as being simply the equation specifying the Fermi
surface: \( \frac{1}{2} p_0^2 + u(x) = \mu_F \) with the solution

\[ \pi \phi_0 = p_0(x) = \sqrt{2(\mu_F - u(x))} \, . \tag{3.16} \]

Introducing small fluctuations with a shift \( \phi(x,t) = \phi_0(x) + \frac{1}{\sqrt{\pi}} \partial_x \eta(x,t) \) the Hamil-
tonian becomes

\[ H = \int dx \left\{ (\pi \phi_0) \left( \frac{1}{2} \Pi^2 + \frac{1}{2} \eta_x^2 \right) + \frac{\pi^2}{6} (\eta_x)^3 \right\}, \tag{3.17} \]

with the quadratic term (in the Lagrangian form):

\[ L_2 = \int dt \int dx \left\{ \frac{1}{2} \left( \frac{\eta^2}{\pi \phi_0(x)} \right) - (\pi \phi_0) \eta_x^2 \right\}. \tag{3.18} \]

This is a free massless particle in an external gravitational background

\[ g_{\mu \nu}^0 = \left( \frac{1}{\pi \phi_0(x)} , \pi \phi_0(x) \right) \tag{3.19} \]

specified by our potential \( u(x) \). However, this metric is removable by a coordinate transformation. In terms of the time-of-flight coordinate

\[ \tau = \int x \frac{dx}{\pi \phi_0} \quad \text{or} \quad \frac{dx(\tau)}{d\tau} = p_0 \tag{3.20} \]

one has

\[ H = \int d\tau \left\{ \frac{1}{2} \left( \Pi^2 + (\partial_\tau \eta)^2 \right) + \frac{1}{6p_0} \left( (\partial_\tau \eta)^3 + 3\Pi^2(\partial_\tau \eta) \right) \right\}, \tag{3.21} \]

which describes a massless theory with a spatially dependent coupling constant

\[ g_{st}(\tau) = \frac{1}{p_0^2(\tau)}. \tag{3.22} \]

The continuum \( c = 1 \) string theory is approached for a special choice of the potential \( v(x) = -x^2/2 \). In this case one has a critical theory near \( \mu_F = -\mu \to 0 \). For the oscillator we have

\[ x(\tau) = \sqrt{2\mu} \cosh \tau, \quad p_0(\tau) = \sqrt{2\mu} \sinh \tau. \tag{3.23} \]

The length of the (physical) \( \tau \)-space diverges at the turning point \( x_0 = \sqrt{2\mu} \). The string coupling constant (3.22) is now

\[ g_{st}(\tau) = \frac{1}{2\mu \sinh^2 \tau}. \tag{3.24} \]

It depends on the Fermi level as \( g_{st} \sim 1/\mu \). This is in parallel with the dependence of the string coupling on the cosmological constant of the \( c = 1 \) string theory. We
also see that asymptotically $g_{\text{st}} \sim \frac{1}{\mu} e^{-2\tau}$ as $\tau \to +\infty$. Comparing it to the expected behavior in $c = 1$ string theory $g_{\text{st}} \sim \frac{1}{\mu} e^{-\sqrt{2}\varphi}$ one has the (asymptotic) identification

$$\tau \leftrightarrow \frac{1}{\sqrt{2}} \varphi, \quad t_M \leftrightarrow \frac{1}{\sqrt{2}} t_{c=1}.$$ (3.25)

One can now identify $\eta(\tau, t)$ with the tachyon field $T(\varphi, t)$. Remembering that $e^{\sqrt{2}\varphi} T(\varphi, t)$ was the field satisfying the massless Klein-Gordon equation, one also has the identification of the energy-momenta:

$$ip_{\tau} \leftrightarrow 2 + i\sqrt{2} p_{\varphi}, \quad i\epsilon \leftrightarrow i\sqrt{2} p,$$ (3.26)

where $\epsilon$ is the energy in the matrix model picture.

The above identification of the collective field $\eta(\tau, t)$ was only done asymptotically when the $\mu e^{-\sqrt{2}\varphi}$ term in the Liouville equation is ignored. A much more precise identification can be performed with the cosmological term also present.

In the matrix model the time-of-flight coordinate is introduced to bring the quadratic mass operator of the collective field into a Klein-Gordon form:

$$\left[ \partial_{\tau}^2 - \sqrt{x^2 - 2\mu} \partial_x \sqrt{x^2 - 2\mu} \partial_x \right] \eta = (\partial_{\tau}^2 - \partial_{\varphi}^2) \eta(t, \tau),$$ (3.27)

with $x = \sqrt{2\mu} \cosh \tau$. If we use a basis conjugate to $x$: $p = -i (\partial/\partial x)$ the spatial operator reads

$$\omega^2 = p^2 x^2 - 2\mu p^2,$$ (3.28)

and after a change of variables $p = \sqrt{2} e^{-\varphi/\sqrt{2}}$ this gives the Liouville operator

$$\hat{\omega}^2 = -\frac{1}{2} (\partial \varphi)^2 - 4\mu e^{-\varphi/\sqrt{2}} \equiv \mathcal{H}_L.$$ (3.29)

We see that the Liouville coordinate is to be identified more precisely [19] with the variable $p$ conjugate to the matrix eigenvalue $\lambda$. The conjugate basis is not unnatural in collective theory, it is associated with the (Wilson) loop operator itself

$$W(\ell, t) = \text{Tr} \left( e^{-\ell M} \right) = \int dx e^{-\ell x} \phi(x, t).$$ (3.30)

which at the linearized level

$$W(\ell, t) = \int_{0}^{\infty} d\tau e^{-\sqrt{2\mu} \ell \cosh \tau} \partial_{\tau} \eta,$$ (3.31)
is seen to obey
\[(\partial^2_t - \dot{\omega}^2) \dot{W}(\ell, t) = 0 , \quad (3.32)\]
with
\[
\dot{\omega}^2 = \partial^2_\tau \quad \Rightarrow \quad - (\ell \partial_\ell)^2 + 2\mu \ell^2 . \quad (3.33)
\]
After a change \(\ell = 2e^{-\varphi/\sqrt{2}}\) one has the Liouville operator \(\mathcal{H}_L\). In this conjugate momentum basis the connection to Liouville theory is therefore manifest. One could obviously write all equations in this representation but formulae are much simpler in terms of time-of-flight coordinate \(\tau\). The (Wilson) loop field and its natural connection to the Liouville picture will be relevant for defining the string theory \(S\)-matrix.

To further clarify the identification of the Liouville mode let us write the transformation between the matrix eigenvalue \(x = \lambda\) and the time-of-flight coordinate \(\tau\) as a point canonical transformation:

\[
x = \sqrt{2\mu} \cosh \tau , \\
p = \frac{1}{\sqrt{2\mu} \sinh \tau} \ p_\tau , \quad (3.34)
\]
where \(p\) and \(p_\tau\) are the conjugates: \(\{x, p\} = \{\tau, p_\tau\} = 1\). Introducing \(p = \sqrt{2} e^{-\varphi/\sqrt{2}}\) we have

\[
p_\varphi = \sqrt{2\mu} e^{-\varphi/\sqrt{2}} \cosh \tau , \\
p_\tau = \sqrt{2} \sqrt{2\mu} e^{-\varphi/\sqrt{2}} \sinh \tau , \quad (3.35)
\]
as a canonical transformation between the Liouville and time-of-flight coordinates. The property of this transformation is that

\[
\frac{1}{2} \omega^2 = \frac{1}{2} p_\tau^2 = p_\varphi^2 - 2\mu e^{-\varphi/\sqrt{2}} .
\]
Now in Liouville theory one also usually deals with two alternate descriptions and two different fields: the original Liouville field \(\varphi(z, \bar{z})\) and a free field \(\psi(z, \bar{z})\). They are related by a canonical (Bäcklund) transformation

\[
\dot{\varphi} = \psi' + \sqrt{2\mu} e^{-\varphi/\sqrt{2}} \cosh(\psi/\sqrt{2}) , \\
\dot{\psi} = \varphi' + \sqrt{2\mu} e^{-\varphi/\sqrt{2}} \sinh(\psi/\sqrt{2}) , \quad (3.36)
\]
where the two derivatives correspond to the two-dimensional space \(z = \sigma + i\xi\). The above transformation relates the Liouville action to the action of a free field \(\psi(z, \bar{z})\).
Clearly for the center of mass mode ($\phi' = \psi' = 0$) one sees

\[
\Pi_\phi = \sqrt{2\mu} e^{-\phi/\sqrt{2}} \cosh(\psi/\sqrt{2}) , \quad \Pi_\psi = \sqrt{2\mu} e^{-\phi/\sqrt{2}} \sinh(\psi/\sqrt{2}) .
\] (3.37)

The transformation between $\phi$ and $\psi$ is identical to the one in the matrix model.

We have then the fact that the time-of-flight coordinate $\tau$ is to be identified with the free field zero mode $\psi_0 : \psi_0 = \sqrt{2}\tau$. In most of the vertex operator construction it is the free field which is used.

We shall now continue and discuss the exact classical solution of the theory and exhibit its integrability. Consider first the physical meaning of the component fields $\alpha_\pm$ and the nature of boundary conditions at the turning point or wall $\tau = 0$. Shifting by the classical solution, $\alpha_\pm = \pm p_0 + \epsilon_\pm$, the equations of motion linearize to

\[
\partial_\pm \epsilon_\pm \pm (p_0 \partial_x + \partial_x p_0) \epsilon_\pm = 0 .
\] (3.38)

Denoting $\epsilon_\pm \equiv \mp \frac{1}{p_0} \psi_\mp$ we have

\[
(\partial_t \pm \partial_\tau) \psi_\mp = 0 .
\] (3.39)

So indeed, $\psi_\pm = \psi_\pm(t \pm \tau)$ are left- and right-moving waves, respectively. There is however a nontrivial boundary condition in the theory which comes in as follows: The eigenvalue density $\phi = \frac{1}{2\pi} (\alpha_+ - \alpha_-)$ gives the conserved fermion number

\[
\dot{N} = \int dx \dot{\phi} = \frac{1}{2\pi} \int dx (\alpha_+^2 - \alpha_-^2) = 0 .
\] (3.40)

At the boundary point for $x$ (or $\tau = 0$), this implies

\[
(\alpha_+^2 - \alpha_-^2) \bigg|_{\text{boundary}} = 0 ,
\] (3.41)

so that there is no leakage into the region under the barrier (this may have to be given up in nonperturbative discussion [21]). For the small fluctuations we then have

\[
\psi_+(x) = \psi_-(x) ,
\] (3.42)

implying Dirichlet boundary conditions. In terms of Fourier modes

\[
\psi_\pm = \int_{-\infty}^{\infty} dk \alpha_k^\pm e^{ik(t \pm \tau)}
\] (3.43)

with $\alpha_{-k} = \alpha_k^+$ our boundary condition implies that one has only one set of oscillators with positive momenta

\[
\alpha_k^+ = \alpha_k^- = \alpha_k , \quad k > 0 .
\] (3.44)

This is appropriate for a theory defined on a half-line $\tau \in [0, \infty)$. 
A very simple form for the exact solution of the collective equations was given by Polchinski [17]. At the classical level one has a phase space picture of the eigenvalues $\lambda(\sigma,t)$ and their momenta $p(\sigma,t) = \dot{\lambda}$. They obey the classical equations of motion

$$\dot{p}(\sigma,t) = -u'(\lambda(\sigma,t)) .$$

(3.45)

The information that the particles are fermions is contained in the statement that the equation $x = \lambda(\sigma,t)$ is invertible: $\sigma = \sigma(x,t)$ so that for each $\sigma$ there is only one particle (actually there is a degeneracy corresponding to the upper and lower Fermi surface). Consider in particular the inverted oscillator: the solution is immediately written as

$$x = a(\sigma) \cosh(t - \sigma),$$

$$p = a(\sigma) \sinh(t - \sigma) .$$

(3.46)

Here $a(\sigma)$ is an arbitrary function giving an arbitrary initial condition. The simplest configuration is obtained for $a(\sigma) = \sqrt{\mu} = \text{const.}$ and we have

$$p_{\pm} = p(\sigma_{\pm}, t) = \pm \sqrt{x^2 - \mu} .$$

(3.47)

This is recognized as the static ground state collective field configuration $\pi \phi_0(x)$. It is easy to see that the general configuration leads to the solution of collective equations. The collective field is identified with the Fermi momentum densities

$$\alpha_{\pm}(x,t) \equiv p_{\pm} = p(\sigma_{\pm}(x,t), t) .$$

(3.48)

Conversely, $p(\sigma,t) = \alpha(x(\sigma,t), t)$. Using the chain rule

$$\frac{\partial p}{\partial t} = \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x} \frac{\partial x}{\partial t} = -u'(x) ,$$

(3.49)

and the equation of motion obeyed by $p(\sigma,t)$, there follows the equation

$$\frac{\partial \alpha}{\partial t} = -u'(x) - \alpha \partial_x \alpha .$$

(3.50)

These are the decoupled quadratic equations for the collective fields $\alpha_{\pm}(x,t)$ associated with the cubic Hamiltonian.

Knowledge of the exact solution can be directly used to determine scattering amplitudes. One considers and follows the time evolution of an incoming left-moving lump. A point parametrized by $\sigma$ which passes through $x$ at some (early) time $t$ will reflect on the boundary and pass through the same point $x$ at some later time $t'$ as a right-moving lump. The time evolution of the particle coordinates is known
explicitly (3.46) so one can determine the relationship between the two times \( t \) and \( t' \). Consider the exact solution given by Eq. (3.46), at a distance \( \tau \) large enough one has

\[
x = e^{\tau} = \begin{cases} 
  a(\sigma) e^{-(t-\sigma)}, & t \to -\infty \\
  a(\sigma) e^{+(t'+\sigma)}, & t' \to +\infty
\end{cases}
\]

from where

\[
t' - \tau = t + \tau + \ln a^2(\sigma) .
\] (3.51)

On the other hand \( a^2(\sigma) \) is related to \( \alpha_{\pm} \):

\[
a^2 = x^2 - p^2 = x^2 - \alpha_{\pm}^2 \approx 1 + \psi_{\pm} .
\] (3.52)

The outgoing particle momentum \( p_+(t',\sigma) \) is equal in magnitude (but opposite in sign) to the incoming momentum of the particle \( p_-(t,\sigma) \):

\[
p_+(t',\sigma) = -p_-(t,\sigma) .
\] (3.53)

This elementary relationship provides a relationship between the incoming and outgoing wave and therefore yields the \( S \)-matrix. Collecting the above formulas we have

\[
\psi_-(z) = \psi_+(z - \ln(1 + \psi_-(z))) .
\] (3.54)

This functional equation determines the relation between the left- and right-moving (incoming and outgoing fields). It represents a nonlinear version of our Dirichlet boundary conditions and is characteristic of scattering problems involving a wall. An expansion in power series can be performed determining explicitly the outgoing modes in terms of the incoming ones. This is then sufficient to give the scattering amplitudes. We shall return to this subject in sect. 5.

In general, all features of the exactly solvable matrix model translate into string theory. More precisely there is a direct translation of matrix model quantities into the collective field theory which itself is then completely integrable as we have emphasized. We end this section by summarizing the set of translation rules between the matrix model and collective field theory representations.

At the classical level one thinks of matrix variables as coordinates in a fermionic phase space \( M \to \lambda, P \to p \). Collective field theory represents a second quantization according to \( p \to \alpha(x,t) \). So we have the correspondences:

\[
\begin{align*}
  M & \leftrightarrow \lambda \leftrightarrow x , \\
  P & \leftrightarrow p \leftrightarrow \alpha(x,t) .
\end{align*}
\] (3.55)

The \( U(N) \) trace becomes a phase space integration in the fermionic picture and in
the collective representation:

\[ \text{Tr} \{ \} \rightarrow \int \frac{dx}{2\pi} \int_{\alpha_{-}(x,t)}^{\alpha_{+}(x,t)} d\alpha \{ \} , \tag{3.56} \]

where \( \alpha_{\pm}(x, t) \) are the chiral components of the scalar field density. For example the collective Hamiltonian comes out as follows:

\[ \frac{1}{2} \text{Tr} (P^2 - M^2) \rightarrow \frac{1}{2}(p^2 - x^2) \rightarrow \int \frac{dx}{2\pi} \int d\alpha \frac{1}{2}(\alpha^2 - x^2) = \frac{1}{2} \int \frac{dx}{2\pi} \left[ \frac{\alpha^3}{3} - x^2 \alpha \right]^{+}. \tag{3.57} \]

The above transition rules summarize the statement that the Poisson brackets of single particle quantities in the Fermi (or matrix) phase space

\[ \{ f_1(x, p), f_2(x, p) \}_{\text{P.B.}}. \tag{3.58} \]

remain preserved in the field theory. For example, the field-theoretic operator inferred from the oscillator states is

\[ B_{n}^{\pm} = \int \frac{dx}{2\pi} \int d\alpha \ (\alpha \pm x)^{n}. \tag{3.59} \]

We can now use the \( \alpha \)-field Poisson brackets \( \{ \alpha_{\pm}(x), \alpha_{\pm}(y) \} = \mp 2\pi i\delta'(x - y) \) to verify that indeed

\[ \{ H_{\text{coll}}, B_{n}^{\pm} \} = \pm n B_{n}^{\pm}. \tag{3.60} \]

This represents an eigenstate of the collective field theory Hamiltonian. At the quantum level a normal ordering prescription is used to completely define the operators.

The outlined string field theory gives a systematic perturbation theory in the string coupling constant. The Feynman rules that are constructed are characterized by a nontrivial cubic vertex exhibiting discrete poles in the momenta. Most importantly a fully quantized Hamiltonian is achieved through normal ordering with the counter-terms being supplied by the original collective formalism. So what one has is a totally finite string field theory capable of reproducing string theory diagrams to all orders. It works at loop level without further counter-terms giving a single covering of modular space. This, as is well known, has always been quite nontrivial in a string-theoretic framework. For more details of the quantum theory and explicit calculations at the loop level the reader should consult [16].
4. $\omega_\infty$ Symmetry

The matrix model description has the virtue of great simplicity: it is linear and trivially exactly solvable. For the matrix Hamiltonian

$$H = \frac{1}{2} \text{Tr} (P^2 - M^2)$$  \hspace{1cm} (4.1)

one can write down exact creation–annihilation operators

$$B_n^\pm = \text{Tr} (P \pm M)^n , \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (4.2)

creating imaginary energy eigenstates

$$[ H, B_n^\pm ] = \mp inB_n^\pm , \quad \epsilon_n = \pm in \hspace{1cm} (4.3)$$

The whole point here is to be able to translate this exact information into physical results which, as we have emphasized, is achieved through collective field theory. The direct connection of the space-time string field theory with the matrix model leads then further insight. The simple oscillator structure with its creation–annihilation basis implies the presence of a similar structure in the field theory and therefore string theory.

To understand the physical meaning of the (oscillator) states recall that in the collective field theoretic description we have another spatial quantum number in addition to the energy. This feature arose as a consequence of scaling invariance. The coordinate and the fields transform as

$$x \rightarrow ax \hspace{0.5cm} \text{and} \hspace{0.5cm} \alpha(x,t) \rightarrow \frac{1}{a} \alpha(ax,t),$$  \hspace{1cm} (4.4)

and the Hamiltonian, without the chemical potential term, $-\mu\alpha$, scales as

$$H \rightarrow \frac{1}{a^4} H \hspace{1cm} (4.5)$$

The classical equations of motion are consequently scale invariant. One then defines the scaling momentum as

$$ip_s = -4 + s \hspace{1cm} (4.6)$$

where $s$ is the naive scaling dimension $s[x] = s[\alpha] = 1$. The creation–annihilation operators

$$\tilde{T}_n^\pm = \frac{1}{n} \int \frac{dx}{2\pi} \frac{(\alpha \pm x)^{n+1}}{n+1} \left\vert \alpha_+ \right\vert \left\vert \alpha_- \right\rangle$$  \hspace{1cm} (4.7)

consequently have the following energy-momentum:

$$i\epsilon = n \hspace{0.5cm} \text{and} \hspace{0.5cm} ip_s = -2 + n \hspace{1cm} (4.8)$$

We find these to be in precise agreement with the discrete tachyon vertex operator.
states since there
\[ i\sqrt{2}p = \pm 2j, \quad i\sqrt{2}p\phi = -2 + 2j, \quad (4.9) \]
and we have already noted the relations \( \sqrt{2}p = \epsilon, \sqrt{2}p\phi = p_s \). We then have a one-to-one correspondence between oscillator states of the matrix model and discrete tachyon vertex operators of the conformal description of \( c = 1 \) string theory
\[ B_n^\pm = \text{Tr}(P \pm M)^n \leftrightarrow T_p^{(\pm)} = e^{\pm i\sqrt{2}jX}e^{-\sqrt{2}(1-j)\phi} \quad (4.10) \]
with \( n = 2j \) or \( j = n/2 \).

An analytic continuation of discrete imaginary momenta to real values \((n = i\kappa, p_s = 2i - \kappa)\) gives the scattering operators
\[ B_{-i\kappa}^- = \text{Tr}(P - M)^{-i\kappa} \sim e^{-i\kappa(t+\tau)}, \]
\[ B_{-i\kappa}^+ = \text{Tr}(P + M)^{-i\kappa} \sim e^{-i\kappa(t-\tau)}, \quad (4.11) \]
describing left- and right-moving waves, respectively. These operators can be used to construct the in- and out-states of scattering theory
\[ \text{Tr}(P - M)^{-i\kappa}|0\rangle = |\kappa; \text{in}\rangle, \]
\[ \text{Tr}(P + M)^{+i\kappa}|0\rangle = |\kappa; \text{out}\rangle. \quad (4.12) \]
Namely, for an in-state, one needs a left-moving wave while the out-state is necessarily given by a right-moving one. Here we have used the picture where the wall is at \( \tau = -\infty \) corresponding to the physical space being defined on the right semi-axis \( x = e^\tau \geq 0 \). Had we chosen to define the theory on the other side of the barrier, the states
\[ \text{Tr}(P + M)^{ik}|0\rangle = e^{ik(t+\tau)}, \]
\[ \text{Tr}(P - M)^{ik}|0\rangle = e^{ik(t-\tau)}, \quad (4.13) \]
would be physical since they have the meaning of a right-moving in-wave and a left-moving out-wave. Hence there is a one-to-one correspondence between the scattering operators in the matrix model and the string theory vertex operators
\[ \text{Tr}(P \pm M)^{-i\kappa} \leftrightarrow T^\pm = e^{\pm i\phi \sqrt{2}}e^{-\sqrt{2}(1-j)\phi}. \quad (4.14) \]
A typical transition amplitude reads
\[ S = \langle \text{out}|\text{in} \rangle = \langle 0 | \text{Tr}(P + M)^{\hat{\kappa}} \text{Tr}(P - M)^{\hat{\kappa}} | 0 \rangle. \quad (4.15) \]
It only contains operators with the same (Liouville)-exponential dressing. This is in total agreement with the continuum string theory situation.
In addition to the tachyon states, the matrix oscillator description immediately allows a construction of an infinite sequence of discrete states \([20,18]\). They are created by the operators

\[
B_{n,\bar{n}} = \text{Tr} \left( (P + M)^n (P - M)^{\bar{n}} \right),
\]

with energies and momenta given by

\[
i\epsilon = n - \bar{n}, \quad ip_s = -2 + (n + \bar{n}).
\]

Comparing this with the discrete spectrum of the string theory given by

\[
i\sqrt{2}p = 2m, \quad i\sqrt{2}p_\varphi = -2 + 2j,
\]

we find the correspondence

\[
m = \frac{n - \bar{n}}{2}, \quad j = \frac{n + \bar{n}}{2}.
\]

These are indeed half-integers once \(n, \bar{n}\) are integers. The field theory operators

\[
B_{jm} = \int \frac{dx}{2\pi} \frac{\alpha_s}{\alpha_-} \int d\alpha (\alpha + x)^{j+m} (\alpha - x)^{j-m}
\]

can be shown (again by using the Poisson brackets or the commutators) to generate discrete imaginary energy eigenstates of the Hamiltonian

\[
[H, B_{jm}] = -2imB_{jm}.
\]

This commutator shows that the operators \(B_{jm}\) are spectrum-generating operators for the Hamiltonian \(H\); but it also signals the existence of a large symmetry algebra which operates in this theory [18,19,22,23,24].

First we had the sequence of conserved quantities

\[
H_l = \text{Tr} \left( P^2 - M^2 \right)^{l+1}
\]

commuting among themselves

\[
[H_l, H_l'] = 0.
\]

These are particular cases of the spectrum-generating operators \(B_{jm}\). One is then lead to consider the complete algebra of all the operators. Introducing the more
standard notation

\[ O_{JM} = (p + x)^{J+M+1}(p - x)^{J-M+1}, \]

with the associated collective field realization

\[ O_{JM} = \int \frac{dx}{2\pi} \int d\alpha \ (\alpha + x)^{J+M+1}(\alpha - x)^{J-M+1}, \]

one checks that they obey the \( w_\infty \) commutation relations

\[ [O_{J_1 M_1}, O_{J_2 M_2}] = 4i \left( (J_2 + 1)M_1 - (J_1 + 1)M_2 \right) O_{J_1 + J_2, M_1 + M_2}. \]

We note that this commutator results if no special ordering is taken for the noncommuting factors. At the full operator quantization level, field theory requires special normal ordering. It is likely that this modifies the simple \( w_\infty \) algebra to a \( W_{1+\infty} \) algebra.

Recalling the form of tachyon operators \( T_n^{\pm} = \text{Tr}(P \pm M)^n \) one sees a special relationship between the tachyon operators and the \( w_\infty \) generators. A simple computation shows that

\[ O_{JM} = \frac{1}{2i} \frac{1}{(J + M + 2)(J - M + 2)} \left[ T_{J+M+2}^{+}, T_{J-M+2}^{-} \right]. \]

This first sheds some light on the nature of higher discrete modes in the collective formalism: they are composite states of the tachyon. More importantly, one then expects a simple realization of the \( w_\infty \) algebra on the tachyon sector.

To understand the role played by the scalar collective field with respect to the \( w_\infty \) algebra one can first look at the following Virasoro subalgebra:

\[ O_l \equiv O_{l \frac{1}{2} \frac{1}{2}} = \int \frac{dx}{2\pi} \int d\alpha \ (\alpha + x)^{l+1}(\alpha - x), \]

with

\[ [O_l, O_{l'}] = 2i(l - l') O_{l+l'}. \]

We can then determine the transformation property of the tachyon field under this subalgebra. Actually, the exact tachyon creation operator \( T_n \) can itself be written as an extension of the whole algebra

\[ \tilde{T}_n^+ = \frac{1}{n} \int \frac{dx}{2\pi} \frac{(\alpha + x)^{n+1}}{n+1} = \frac{1}{n} O_{\frac{n}{2} - 1, \frac{n}{2}}, \]

and this determines the commutator (with the indices extended outside the standard range \(|m| \leq j\)). Alternatively one can also directly use the basic commutation
relations to find
\[ [O_l, \tilde{T}^+_n] = 2i(n + l) \tilde{T}^+_n, \tag{4.31} \]
which shows that the tachyon transforms as a field of conformal weight 1. This is understood to be a space-time and not a world sheet feature. The fact that an infinite space-time symmetry appears in the collective field theory explains many similarities that it has with conformal field theory. The $w_\infty$ generators act in a nonlinear way on the tachyon field. This implies that this symmetry can be used to write down Ward identities for correlation functions and the $S$-matrix.

Let us study in more detail the nonlinearity involved in the collective representation (here we summarize the results achieved in [25]). One is in general interested in comparison with similar nonlinearities (and Ward identities) obtainable in the world sheet conformal field theory analysis. The latter is only performed in the approximation neglecting the cosmological constant term ($\mu \to 0$) which represents the strong coupling regime of the field theory ($g_{st} = 1/\mu \to \infty$). In this limit one simply expands
\[ \alpha_\pm(x, t) = \pm x + \frac{1}{2x} \hat{\alpha}_\pm, \tag{4.32} \]
which is an approximate form when
\[ \pi \phi_0(x) = \sqrt{x^2 - \mu} \to x = \frac{e^\tau}{2}. \tag{4.33} \]
The exact tachyon operators reduce in the leading (linear) approximation to
\[ \tilde{T}^\pm_n = \frac{1}{n} \int \frac{d\tau}{2\pi} e^{n\tau} \hat{\alpha}_\pm, \tag{4.34} \]
This is as it should be since they are to describe left- and right-moving waves, respectively. Consider now the $w_\infty$ generators in the same approximation. With the above background shift one easily finds that they reduce to
\[ O_{JM} = \frac{1}{J - M + 2} \int \frac{d\tau}{2\pi} e^{2M\tau} \hat{\alpha}_+^{J-M+2} + \frac{(-1)^{J-M}}{J + M + 2} \int \frac{d\tau}{2\pi} e^{-2M\tau} \hat{\alpha}_-^{J+M+2}. \tag{4.35} \]
Here we see that the operator $O_{JM}$ behaves as the $(J - M + 2)$th power of the right-moving tachyon $\alpha_+$ and also the $(J + M + 2)$th power of the left-moving tachyon $\alpha_-$. These are the leading polynomial powers in the left- and right-moving components of the tachyon; even in this strong coupling limit the theory is nonlinear and one has further higher order terms. Concerning these one can go to the in (out) fields (where
it is likely that only leading terms remain). The in (out) fields are simply limits of the component fields $\alpha_{\pm}$:

$$\alpha_{\text{out}}(t - \tau) = \lim_{t \to +\infty} \alpha_+, \quad \alpha_{\text{in}}(t + \tau) = \lim_{t \to -\infty} \alpha_- , \quad (4.36)$$

Since the operators $O_{JM}$ are conserved (up to a phase), looking at the $t \to \pm \infty$ limit of $e^{2Mt} O_{JM}$ we obtain an identity (between the in- and out-representation):

$$O_{JM} = \frac{1}{J - M + 2} \int \frac{dz}{2\pi} \alpha_{\text{out}}^{J-M+2}(z) = \frac{(-1)^{J-M}}{J + M + 2} \int \frac{dz}{2\pi} \alpha_{\text{in}}^{J+M+2}(z) . \quad (4.37)$$

Introducing creation–annihilation operators

$$\alpha_{\text{in}}(z) = \int dz e^{-ikz} \alpha(k), \quad \alpha_{\text{out}}(z) = \int dz e^{-ikz} \beta(k) , \quad (4.38)$$

with $\alpha(k) = a(k)$ and $\alpha(-k) = ka(k)^\dagger$; $\beta(k) = b(k)$ and $\beta(-k) = kb(k)^\dagger$ we have the expressions after a continuation $k \to ik$:

$$O_{J,-M} = \frac{1}{J - M + 2} \int dk_1 \ldots dk_{J-M+2} \alpha(k_1) \ldots \alpha(k_{J-M+2}) \delta(\sum k_i + 2M)$$

$$= \frac{(-1)^{J-M}}{J + M + 2} \int dp_1 \ldots dp_{J+M+2} \beta(p_1) \ldots \beta(p_{J+M+2}) \delta(\sum p_i + 2M) . \quad (4.39)$$

These representations can be compared with analogue expressions found in conformal field theory [24].

The Ward identities essentially follow from the in–out representations of the generators in terms of the tachyon field. A typical $S$-matrix element is given by

$$S(\{k_i\}; \{p_j\}) = \langle 0 | \prod_j \beta(p_j) \prod_i \alpha(-k_i) | 0 \rangle . \quad (4.40)$$

Consider a general matrix element of the $w_{\infty}$ generator $O_{JM}$,

$$\langle 0 | \beta O_{JM} \alpha | 0 \rangle .$$

It can be evaluated by commuting to the left or to the right using alternatively the above in–out representations. The two different evaluations give the identity

$$\langle 0 | [\beta, O_{JM}] \alpha | 0 \rangle = \langle 0 | \beta [O_{JM}, \alpha] | 0 \rangle , \quad (4.41)$$

which summarizes the general Ward identities. These, when written out explicitly using the representations for $O_{JM}$, have the form of recursion relations reducing the
$N$-point amplitude to lower point ones. Specifically, the one creation operator term of the $\alpha$-representation for $O_{JM}$ gives:

$$O_{M+1,-M} a^\dagger(k_1) a^\dagger(k_2)|0\rangle = 4\pi(k_1 + k_2 + 2M) a^\dagger(k_1 + k_2 + 2M)|0\rangle,$$

(4.42)

which turns a two-particle state into one-particle state. Generally,

$$O_{M+N,-M} a^\dagger(k_1) \ldots a^\dagger(k_{N+1})|0\rangle = 2\pi^N (N + 1)! (\sum k_i + 2M) a^\dagger(\sum k_i + 2M)|0\rangle,$$

(4.43)

showing a reduction of the $(N + 1)$-particle state into a single-particle state.

As an example let us calculate the $3 \rightarrow 1$ amplitude

$$S_{3,1} = \langle 0|b(p) a^\dagger(k_1) a^\dagger(k_2) a^\dagger(k_3)|0\rangle.$$

(4.44)

The energy-momentum conservation laws imply (recall that for $B^\pm_n$, $\epsilon = \pm n, p_s = -2 + n$):

$$k_1 + k_2 + k_3 - p = 0,$$

$$(-2 + k_1) + (-2 + k_2) + (-2 + k_3) + (-2 + p) = -4.$$

The latter is a specific case of the general Liouville conservation (or bulk condition):

$$\sum_{i=1}^N = p_s^i = -4.$$

(4.45)

The energy-momentum relations specify the momentum of the 4th particle

$$p = 2 \quad \text{or} \quad k_1 + k_2 + k_3 = 2.$$

Use now the operator

$$O_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{\mu}} \int dk_1 dk_2 dk_3 \; k_1 a^\dagger(k_1) a(k_2) a(k_3) \delta(k_1 - k_2 - k_3 + 1)$$

$$= \sqrt{\mu} \int dp_1 dp_2 \; p_1 b^\dagger(p_1) b(p_2) \delta(p_1 - p_2 + 1)$$

to deduce

$$S_{3,1} = \langle b(2) a^\dagger(k_1) a^\dagger(k_2) a^\dagger(k_3) \rangle$$

$$= \frac{\pi}{\mu} (k_1 + k_2 - 1) \langle b(1) a^\dagger(k_1 + k_2 - 1) a^\dagger(k_3) \rangle$$

$$+ \frac{\pi}{\mu} (k_2 + k_3 - 1) \langle b(1) a^\dagger(k_2 + k_3 - 1) a^\dagger(k_1) \rangle$$

$$+ \frac{\pi}{\mu} (k_1 + k_3 - 1) \langle b(1) a^\dagger(k_1 + k_3 - 1) a^\dagger(k_2) \rangle.$$
Taking the normalized three-point function to be \( S_3 = 1/\mu \), the result
\[
S_{3,1} = \frac{\pi}{\mu^2} \left( 2(k_1 + k_2 + k_3) - 3 \right) = \frac{\pi}{\mu^2}
\]
follows. One can iteratively repeat the same reduction for higher point amplitudes and find
\[
S_{N,1} = \frac{\pi^{N-1}}{(N-2)!} \mu^{-N+1}, \quad (4.46)
\]
which is the \((N + 1)\)-point “bulk” scattering amplitude.

In describing the infinite symmetry we have followed the matrix model approach where the appearance of the symmetry structure is most natural. The features described arise also in the continuum conformal field theory language where the Ward identities take a particularly elegant form.

Of crucial importance in establishing continuum quantities that are analogous with those of the matrix model is Witten’s identification of the ground ring \([23]\). This consist of ghost number zero, conformal spin zero operators \(O_{JM}\) which are closed under operator products \(O' \cdot O'' \sim O''\) (up to BRST commutators). The basic generators are
\[
O_{0,0} = 1, \quad O_{\frac{1}{2}, \pm \frac{1}{2}} = \left[ cb \pm \frac{i}{\sqrt{2}} \partial X - \frac{1}{\sqrt{2}} \partial \varphi \right] e^{(\pm iX + \varphi)/\sqrt{2}}. \quad (4.47)
\]
The suggestion (of Witten) was that \(O_{\frac{1}{2}, \pm \frac{1}{2}}\) are the variables which correspond to the phase space coordinates of the matrix model
\[
O_{\frac{1}{2}, + \frac{1}{2}} = a_+ \equiv p + x, \\
O_{\frac{1}{2}, - \frac{1}{2}} = a_- \equiv p - x, \quad (4.48)
\]
with \(O_{0,0} = 1\) being the cosmological constant operator. Once the (fermionic) matrix eigenvalue coordinates have been identified one could study the action of discrete states vertex operators upon them. They turn out to act as vector fields on the scalar ring
\[
\Psi_{JM} = \frac{\partial h}{\partial a_+} \frac{\partial}{\partial a_-} - \frac{\partial h}{\partial a_-} \frac{\partial}{\partial a_+}, \quad (4.49)
\]
with the familiar matrix model form \(h_{JM} = a^J_+ a^{J-M}_-\). In the continuum approach the \(w_\infty\) generators are integrals of conserved currents which are for closed string
theory constructed as

\[ Q_{JM} = \oint \frac{dz}{2\pi i} W_{JM}(z, \bar{z}) , \]  

(4.50)

\[ W_{JM}(z, \bar{z}) = \Psi_{J+1,M}^+(z) O_{JM}(\bar{z}) . \]

One can study the action of these operators on the the tachyon vertex operators. A formula derived by Klebanov [26] reads

\[ Q_{M+N-1} T_{k_1}^+(0) \int T_{k_2}^+ \cdots \int T_{k_N}^+ = F_{N,M}(k_1, \ldots, k_N) T_{-\sum k_i, M}^+ , \]  

(4.51)

where

\[ F_{N,M}(k_1, \ldots, k_N) = 2\pi^{N-1} N! k \frac{\Gamma(2k)}{\Gamma(1-2k)} \prod_{i=1}^N \frac{\Gamma(1-2k_i)}{\Gamma(2k_i)} , \]  

(4.52)

with a similar formula for the action of \( Q_{-M+N-1} \) on \( N \) oppositely moving \( T^- \) tachyons. These representations of the \( w_\infty \) generators on tachyon vertex operators are clearly comparable to the direct representation obtained in the matrix model (or more precisely collective field formalism). The comparison and agreement of these representations is the closest one comes in being able to identify the two approaches.

The conformal (vertex operator) formalism gives a very elegant summary of Ward identities in the form of general (master) equation. We end this section with a short description of this equation [27,28,29]. It follows from BRST invariance of the discrete state vertex operators

\[ \{ Q_{BRST}, c(z) W_{JM}(z) \} = 0 , \]  

(4.53)

which implies that for general tachyon correlation function

\[ \langle \{ Q_{BRST}, c W_{JM} \} V_{k_1}^{\pm} \cdots V_{k_n}^{\pm} \rangle = 0 . \]  

(4.54)

Changing to operator formalism

\[ \sum_{\text{perm}} \langle V_{k_1}^{\pm} \{ Q_{BRST}, c W_{JM} \} \Delta V_{k_1}^{\pm} \cdots \Delta V_{k_n}^{\pm} V_{k_n}^{\pm} \rangle = 0 , \]  

(4.55)

allows one to eliminate \( Q_{BRST} \). The vertex operators are all BRST invariant while the propagator is essentially the inverse of \( Q \):

\[ [Q, \Delta] = \Pi_{L_0-\bar{L}_0} b^-_0 \]  

(4.56)

where the \( \Pi \) projects on the subspace \((L_0 - \bar{L}_0)|\Phi_i\rangle = 0\). The final form of the Ward
identity then follows

\[ \sum_{\text{partitions}} \langle V_{i_1} \ldots V_{i_m} \Phi \rangle \langle \Phi V_{j_1} \ldots V_{j_{m'}} cW_{JM} \rangle = 0 . \]  

(4.57)

Most of the considerations of this section and most of the studies of the \( w_\infty \) symmetry are performed in the extreme limit where the cosmological term is ignored. This is particularly the case for the continuum, conformal field theory approach. Some attempts to extensions and inclusion of the nontrivial cosmological constant effect were made however. In the matrix model the cosmological term is introduced in a simple and elegant way corresponding to nonzero Fermi energy

\[ h_0 = \frac{1}{2} (p^2 - x^2) \quad \rightarrow \quad h_\mu = \frac{1}{2} (p^2 - x^2) + \mu . \]  

(4.58)

Since the ground ring generators \( a_\pm \) were identified to be analogues to \( p \pm x \) it is then expected that an equivalent deformation from \( a_+ a_- = 0 \) can be established in the continuum conformal field theory approach. This is seen by considering the action of a ground ring on tachyons. For \( \mu = 0 \) it reads

\[ a_+ c \bar{c} \tilde{T}_k^+ = c \bar{c} \tilde{T}_{k+1}^+ , \]

\[ a_- c \bar{c} \tilde{T}_k^+ = 0 . \]  

(4.59)

The effect of cosmological perturbation \( \mu T_{k=0}^+ \) is found by evaluating the first order perturbation theory contribution

\[ a_- c \bar{c} \tilde{T}_k^+ = -a_- c \bar{c} \tilde{T}_{k}^+(0) \left( \mu \int d^2 z , \tilde{T}_{k=0}^+(z) \right) . \]  

(4.60)

On the right-hand side \( a_- \) essentially fuses the two tachyon operators into one giving (to first order)

\[ a_- \tilde{T}_k^+ = -\mu \tilde{T}_{k-1}^+ . \]  

(4.61)

This replaces the second relation above and now the nonzero Fermi level condition \( a_+ a_- = -\mu \) results. To first order \([23,30]\) one then has an agreement with the matrix model. This is encouraging and one would clearly like to establish the complete agreement at an exact level.
5. S-matrix

Let us now describe the complete tree-level $S$-matrix of the $c = 1$ theory. In the previous sections we have seen the “bulk” scattering amplitudes which follow from the Ward identities or are computed in conformal field theory. The complete $N$-point scattering amplitude $S_N = \langle T_{p_1} T_{p_2} \ldots T_{p_N} \rangle$ takes the (factorized) form

$$S_N = \prod_{i=1}^{N} (-\mu^{ip_i}) \frac{\Gamma(-ip_i)}{\Gamma(+ip_i)} A_{\text{coll}}(p_1, \ldots, p_N). \quad (5.1)$$

The external leg factors are associated with a field redefinition [31] of vertex operators

$$T_k^\pm = \frac{\Gamma(\mp k)}{\Gamma(\pm k)} \tilde{T}_k^\pm. \quad (5.2)$$

It is the redefined tachyon vertex operator $\tilde{T}$ that found its natural role in collective field theory.

The external leg factors of the full $S$-matrix have a very relevant physical meaning which we now discuss. In Minkowski space-time, $(k = ip)$, one has

$$\Delta = \mu^{\mp ip} \frac{\Gamma(\pm ip)}{\Gamma(\mp ip)}. \quad (5.3)$$

So, the factors $\Delta = e^{i\theta_p}$ are pure phases. As such they give no contributions to the actual transition amplitudes and could be ignored. The fact is however that they carry physical information on the nature of tachyon background. The factors exhibit poles at discrete imaginary energy

$$p\sqrt{2} = in, \quad n = 1, 2, 3, \ldots \quad (5.4)$$

If we consider a process with an incoming tachyon and $N$ outgoing ones, the discrete imaginary value of the incoming momenta signifies the resonant on-shell process in which a certain number $r$ of Liouville exponentials participate

$$\langle T_-(\mu e^{-\sqrt{2}r})^r T_+ \ldots t_+ \rangle. \quad (5.5)$$

The on-shell condition in this case indeed gives

$$i\sqrt{2}p = -(r + N - 1), \quad (5.6)$$

in agreement with the discrete imaginary energy poles noted above.
In collective field theory the external leg factors are associated with a field redefinition given by an integral transformation. The transformation comes from the change of coordinates between the Liouville and the time-of-flight variables. It is the later that appears naturally in the collective field formalism and as we have seen provides a simple description of the theory. Let us recall the basic (Wilson) loop operator of the matrix model with its Laplace transform

\[ \hat{W}(\ell,t) \equiv \text{Tr} (e^{-\ell M}) = W_0 + \int dx e^{-\ell x} \partial_x \eta. \]  \hspace{1cm} (5.7)

After the change to the time of flight coordinate \( x = \sqrt{2\mu} \cosh \tau \) and the explicit identification of the Liouville \( \ell = 2e^{-\varphi/\sqrt{2}} \) the integral transformation results

\[ \hat{W}(\ell,t) = \int_0^\infty d\tau \exp \left[ -2\sqrt{2\mu} e^{-\varphi/\sqrt{2}} \cosh \tau \right] \partial_\tau \eta(\tau,t), \]  \hspace{1cm} (5.8)

We have seen in our earlier study of the linearized theory that this integral transformation takes the Liouville operator into a Klein-Gordon operator

\[ (\partial^2_t - \partial^2_\tau) \eta \iff (\partial^2_t - \frac{1}{2} \partial^2_\varphi + 4\mu e^{-\varphi/\sqrt{2}}) \hat{W}. \]  \hspace{1cm} (5.9)

The integral transformation therefore expresses the tachyon field in terms of a simple Klein-Gordon field \( \eta(\tau,t) \):

\[ T(\varphi,X) \equiv e^{-\sqrt{2} \varphi} \hat{W}(\ell,t) = \int_0^\infty d\tau \exp \left[ -2\sqrt{2\mu} e^{-\varphi/\sqrt{2}} \cosh \tau \right] \partial_\tau \eta, \]  \hspace{1cm} (5.10)

with the expected relation between the matrix model and string theory times \( X = \sqrt{2} t \). The correlation functions of the tachyon field \( T \) are then expressible in terms of correlation functions of the collective field \( \eta \). The transformation described takes plane wave solutions of the Klein-Gordon equation

\[ \eta(\tau,t) = \int_{-\infty}^\infty \frac{dp}{p} \tilde{\eta}(p) e^{-ipt} \sin(p\tau) \]  \hspace{1cm} (5.11)

into Liouville solutions

\[ T(\varphi,t) = \int dp e^{-ipt} \gamma(p) K_{ip}(2\sqrt{\mu} e^{-\varphi/\sqrt{2}}) \tilde{\eta}(p). \]  \hspace{1cm} (5.12)

The above redefinition of the in–out fields does have an effect in supplying external
leg factors. The asymptotic behavior of \( T(\varphi, t) \) reads
\[
T \sim \int dp e^{-ipt} \left( \Gamma(ip)\mu^{-ip/2} e^{ip\varphi/\sqrt{2}} + \Gamma(-ip)\mu^{ip/2} e^{-ip\varphi/\sqrt{2}} \right) ,
\]
giving the reflection coefficient
\[
R(p) = -\mu^{ip} \frac{\Gamma(-ip)}{\Gamma(ip)}
\]
for each external leg of the \( S \)-matrix.

After this redefinition the problem is reduced to calculating amplitudes in collective field theory: \( A_{\text{coll}}(p_1, \ldots, p_N) \). There, as we have already seen, one can derive an exact relationship between the in- and out-field which contains the complete information about the \( S \)-matrix. The solution to the scattering problem can also be directly deduced from the exact oscillator states. It is this procedure that turns out to be the most straightforward and we now describe it in detail.

Consider the exact (tachyon) creation–annihilation operators in collective field theory
\[
B_{\pm ip} = \int dx \frac{1}{2\pi} \left\{ \frac{(\alpha_+ \pm x)^{1\pm ip}}{1 \pm ip} - \frac{(\alpha_- \pm x)^{1\pm ip}}{1 \pm ip} \right\} .
\]
(5.15)

Previously we have seen that at fixed time these operators serve as exact creation–annihilation operators of the nonlinear collective Hamiltonian. Let us now follow the time-dependent formalism (we describe here the derivation given in [25]). The exact creation–annihilation operators have a simple time evolution
\[
B_{\pm ip}(t) = e^{-ipt} B_{\pm ip}(0) .
\]
(5.16)

Consequently the quantity
\[
\hat{B}_{\pm ip} = e^{ipt} B_{\pm ip}(t)
\]
(5.17)
is time-independent. We can simply look at the operator \( \hat{B}_{\pm ip} \) at asymptotic times \( t = \pm \infty \) and obtain a relationship between the in and out fields (4.36). The operator
\[
B_{\pm ip} = \int dx \frac{1}{2\pi} \left\{ \frac{(\alpha_+ \pm x)^{1\pm ip}}{1 \pm ip} - \frac{(\alpha_- \pm x)^{1\pm ip}}{1 \pm ip} \right\}
\]
(5.18)
contains contributions from both \( \alpha_+ \) and \( \alpha_- \). At \( t = \pm \infty \) only one of the terms survives and we have an identity (recall that \( \hat{B} \) is time-independent), \( \hat{B}(+\infty) = \hat{B}(-\infty) \), which reads
\[
\int dx \frac{1}{2\pi} (\alpha_\pm \pm x)^{1\pm ip} = \int dx \frac{1}{2\pi} (\alpha_\pm \pm x)^{1\pm ip} .
\]
(5.19)

It relates in and out fields (\( \alpha_\pm \) now represent the asymptotic fields). This is the scattering equation. It contains the full specification of the \( S \)-matrix.
One can evaluate and expand the left- and right-hand side of the Eq. (5.19). Shifting by the static background

\[ \alpha_{\pm}(t, x) \approx \pm (x - \frac{1}{2x}) + \frac{1}{2x} \hat{\alpha}_{\pm}(t \mp \tau) , \]  

the left-hand side becomes

\[ L = \int dx (-2x)^{1\pm ip} \left\{ 1 - \frac{(1 \pm ip)}{4x^2} (1 \pm \hat{\alpha}_{\pm}) + O\left(\frac{1}{x^4}\right) \right\} . \]  

After a change of integration variable \( x = \cosh \tau \approx e^\tau/2 \) the \( O(1/x^4) \) terms are seen to decay away exponentially and what remains is only the term linear in \( \hat{\alpha}_{\pm} \). For the right-hand side we simply find

\[ R = \int dx \left( -\frac{1}{2x} \right)^{1\pm ip} (1 \pm \hat{\alpha}_{\mp})^{1\pm ip} . \]  

The scattering equation then becomes

\[ \int dz e^{-ipz} \frac{1}{\mu} \alpha_{\pm}(z) = \frac{1}{1 \pm ip} \int dz e^{-ipz} \left\{ \left( 1 \pm \frac{1}{\mu} \alpha_{\mp} \right)^{1\mp ip} - 1 \right\} , \]  

giving the solution for the in-field as a function of the out-field and vice versa. We have also explicitly restated the string coupling constant \( g_{st} = 1/\mu \). This solution was originally obtained [32] by explicitly solving the functional relationships between the left and right collective field components \( \alpha_+ \) and \( \alpha_- \) given earlier. We see here that it directly follows from the exact oscillator states.

Before proceeding with the consideration of the \( S \)-matrix let us note that the solution found has a reasonable strong coupling limit. Indeed, for \( \mu \to 0 \) one is lead to choose \( ip \) to be an integer, \( ip = N \), and the strong coupling relation

\[ \int dz e^{-Nz} \frac{1}{\mu} \alpha_+(z) = \frac{1}{1 + N} \int dz e^{-Nz} \left( \frac{1}{\mu} \alpha_- \right)^{N+1} \]  

results. It is recognized as a statement specifying the bulk amplitude where the \( S_{1,N} \) and \( S_{N,1} \) amplitudes were nonzero with the momenta of the first (last) particle being equal to \(-1 + N\). We have already used relations of the above type (and their \( w_\infty \) generalizations) in our discussion of the Ward identities at (strong coupling) \( \mu = 0 \).
One can explicitly perform the series expansion in \( g_{st} = 1/\mu \), it reads
\[
\hat{\alpha}_\pm(z) = \sum_{l=1}^{\infty} \frac{(-g_{st})^{l-1}}{l!} \frac{\Gamma(\mp \vartheta + 1)}{\Gamma(\mp \vartheta + 2 - l)} \hat{\alpha}_\pm(z) .
\] (5.25)

The \( S \)-matrix is defined in terms of momentum space creation–annihilation operators
\[
\pm \alpha_\pm(z) = \int \frac{dp}{2\pi} e^{-ipz} \tilde{\alpha}_\pm(p) , \quad [\tilde{\alpha}_\pm(p), \tilde{\alpha}_\pm(p')] = p \delta(p + p') ,
\] (5.26)

with \( \tilde{\alpha}_-(p) \) and \( \tilde{\alpha}_+(p) \) being the in–out creation operators, respectively (\( \alpha(p) \) and \( \beta(p) \) in our earlier notation). In momentum space
\[
\tilde{\alpha}_\mp(p) = \sum_{l=1}^{\infty} \frac{(-g_{st})^{l-1}}{l!} \frac{\Gamma(1 \pm ip)}{\Gamma(2 \pm ip - l)} \int dp_i \delta(p - \sum p_i) \tilde{\alpha}_\mp(p_1) \ldots \tilde{\alpha}_\mp(p_l) ,
\] (5.27)

and the \( n \to m \) \( S \)-matrix element is defined by
\[
A_{\text{coll}}(\{p_i\} \to \{p'_{j}\}) = \langle 0 | \prod_{j=1}^{m} \tilde{\alpha}_+(p'_{j}) \prod_{i=1}^{n} \tilde{\alpha}_-(p_i) | 0 \rangle .
\] (5.28)

Consider for example \( n = 1, m = 3 \) which is the four-point amplitude
\[
A_{1,3} = \langle 0 | \alpha_+(p'_{1}) \alpha_+(p'_{2}) \alpha_+(p'_{3}) \alpha_-(p_1) | 0 \rangle .
\] (5.29)

It is given by the cubic term \( \mathcal{O}(g_{st}^2) \) in the expansion of (5.27) which equals
\[
\alpha_-(p_1) = \frac{1}{3!} g_{st}^2 \frac{\Gamma(1 - ip_1)}{\Gamma(-1 - ip_1)} \int dp'_1 \delta(p_1 - \sum p'_i) \alpha_+(p'_1) \alpha_+(p'_2) \alpha_+(p'_3)
\] (5.30)

and
\[
A_{1,3}(p_1; p'_1, p'_2, p'_3) = i g_{st}^2 p_1 p'_1 p'_2 p'_3 (1 + ip_1) .
\] (5.31)

In general, an arbitrary amplitude is given in [32] to read
\[
A_{n,m} = i (-g_{st})^{n+m-2} \left( \prod_{i=1}^{n} p_i \right) \left( \prod_{j=1}^{m} p'_j \right) \frac{\Gamma(-ip_n)}{\Gamma(1 - m - ip_n)} \frac{\Gamma(1 - m - i\Omega)}{\Gamma(-3 - n - m - i\Omega)}
\] (5.32)

where \( \Omega = \sum_{i=1}^{n} p_i \) and the result is valid in the kinematic region \( p_n > p'_k > \sum_{j=1}^{n-1} p_j \).

This completes the derivation of the collective tree-level amplitudes.
6. Black Hole

Two-dimensional string theory possesses another interesting curved space solution taking the form of a black hole. It is described exactly by the \( SL(2, \mathbb{R})/U(1) \) nonlinear \( \sigma \)-model

\[
S_{WZW} = \frac{k}{8\pi} \int d^2 z \, \text{Tr} \left( g^{-1} \partial g^{-1} \tilde{\partial} g \right) - ik \Gamma_{WZW} + \text{Gauge},
\]

with \( k = \frac{9}{4} \). This then gives the required central charge \( c = \frac{3k}{k-2} - 1 = 26 \). As such the model should be thought of as a different classical solution of the same theory. We have in the earlier lectures seen that the flat space-time string theory is very nicely and very completely described by a matrix model. The black hole solution is however markedly different from the \( c = 1 \) theory. It is characterized by the absence of tachyon condensation and a nontrivial metric and dilaton field:

\[
T(X) = 0,
\]

\[
(ds)^2 = -\frac{k}{2 M - uv} \frac{dudv}{M - uv},
\]

\[
D = \log(M - uv).
\]

Here a particular \( SL(2, \mathbb{R}) \) parametrization is chosen: \( g = \left( \begin{array}{cc} \alpha & u \\ -v & \beta \end{array} \right), \alpha \beta + uv = 1 \) and \( M \) is the black hole mass. There is actually a parametrization (related to the \( c = 1 \) theory) in which the \( \sigma \)-model Lagrangian reads

\[
S_{\text{eff}} = \frac{1}{8\pi} \int d^2 z \left\{ \left( \partial X' \right)^2 + \left( \partial \varphi' \right)^2 - 2 \sqrt{2} \varphi' R^{(2)} \\
+ M \left| \frac{1}{2\sqrt{2}} \partial \varphi' + i \sqrt{\frac{k}{2}} \partial X' \right|^2 e^{-2\sqrt{2} \varphi'} \right\}.
\]

This parametrization corresponds to a linear dilaton but in contrast to the \( c = 1 \) theory, one has a black hole mass term perturbation represented by a gravitational vertex operator instead of the cosmological constant term given by a tachyon operator \( e^{-\sqrt{2} \varphi} \). One of the surprising facts is however that there exists a classical duality transformation that can be used to relate the two \( \sigma \)-models to each other [33]. From this there arises a hope that one could possibly be able to describe the black hole by a matrix model also. More generally from a string field theory viewpoint one would hope to be able to describe different classical solutions in the same setting. In what follows we will present some joint work done with T. Yoneya on this subject [34]. For other different attempts see [35, 36].

First insight into the black hole problem is gained by considering the linearized tachyon [37] field in the external background. In the conformal field theory this is
given by the zero mode Virasoro condition. The Virasoro operator $L_0(u,v)$ consists of two parts,

$$L_0 = -\Delta_0 + \frac{1}{4} (u\partial_u - v\partial_v)^2 , \quad (6.4)$$

where $\Delta_0$ is the Casimir operator of $SL(2, \mathbb{R})$. The Virasoro condition for the linear tachyon field (vertex operator) reads:

$$L_0(u,v)T \equiv \frac{1}{k - 2} \left[ (1 - uv)\partial_u\partial_v - \frac{1}{2}(u\partial_u + v\partial_v) - \frac{1}{2k}(u\partial_u - v\partial_v)^2 \right] T = T . \quad (6.5)$$

The on-shell tachyon corresponds to the continuous representation of $SL(2, \mathbb{R})$ which has eigenvalues $\Delta_0 = -\lambda^2 - \frac{1}{4} \quad (\lambda = \text{real})$ and $-i\partial_t = 2i\omega$ with the on-shell condition $\lambda^2 = 9\omega^2$ at $k = 9/4$. The above equation can be interpreted as corresponding to a covariant Laplacian $L_0 = -\frac{1}{2\epsilon^\mu \sqrt{G}} \partial^\mu \epsilon^D \sqrt{G} G_{\mu\nu} \partial_\nu$ in the background space-time metric $G_{\mu\nu}$ and dilaton $D$, which can be read off from Eq. (6.5)

$$ds^2 = \frac{k - 2}{2} [dr^2 - \beta^2(r) d\bar{t}^2] ,$$

$$D = \log \left( \frac{r}{\beta(r)} \right) + a , \quad (6.6)$$

$$\beta(r) = 2 (\coth^2 \frac{r}{2} - \frac{2}{k})^{-1/2} .$$

These are then candidates for the “exact” background. Here the new coordinate $r$ and time $\bar{t}$ are defined by

$$u = \sinh \frac{r}{2} e^{\bar{t}} , \quad v = -\sinh \frac{r}{2} e^{-\bar{t}} . \quad (6.7)$$

These variables describe the static exterior region outside the event horizon located at $r = 0$. The constant $a$ determines the mass of the black hole

$$M_{\text{bh}} = \sqrt{\frac{2}{k - 2}} e^a . \quad (6.8)$$

The exact metric can be shown to be free of curvature singularity. However, one still has a “dilaton singularity” at $uv = 1$ where the string coupling $g_{\text{st}} \sim e^{-D/2}$ diverges. In terms of the variables $u$ and $v$, the dilaton reads

$$D = \log \left[ 4(-uv(1 - uv))(-\frac{1 - uv}{uv} - \frac{2}{k})^{1/2} \right] + a , \quad (6.9)$$

and the region $uv > 1$ corresponds to a disjoint region with a naked singularity.
The free parameter $a$ can be eliminated by a scale transformation

$$u \to M^{-1/2} u, \quad v \to M^{-1/2} v, \quad M \equiv e^a. \quad (6.10)$$

This introduces the black hole mass parameter in more explicit way, where one replaces $(1 - uv)$ by $(M - uv)$ in the expressions for the dilaton and the metric. An important relation is the connection of the string coupling constant with the parameter $a$, or rather the black hole mass. In general, the dilaton field determines the string coupling constant and in the present case one obtains

$$g_{st} (r = 0) \propto e^{-a/2} = M^{-1/2}. \quad (6.11)$$

This is to be compared with the dependence of $g_{st} \propto \mu^{-1}$ on the cosmological constant in flat space-time. One notes the different power which comes from the different scaling dimensions of the two parameters. The two backgrounds become identical in the asymptotic region. Consider the asymptotic behavior of the Virasoro operator and the dilaton when $r \to \infty$. Using $u \sim e^{\frac{r}{2} + \bar{t}}$, $v \sim e^{\frac{r}{2} - \bar{t}}$ one finds

$$L_0 \sim \frac{1}{4(k-2)} (\partial^2_r + \partial_r) + \frac{1}{4k} \partial^2_{\bar{t}},$$

$$D \sim r + a - \log 4. \quad (6.12)$$

This is the form of Virasoro operator in the linear dilaton case, the parameters $r, \bar{t}$ are identified asymptotically with the $\varphi$ and $t$ for the linear dilaton background as

$$\bar{t} \leftrightarrow \sqrt{\frac{1}{2k}} t = \frac{\sqrt{2}}{3} t, \quad (6.13)$$

$$r \leftrightarrow \sqrt{\frac{2}{k-2}} \varphi = 2\sqrt{2} \varphi.$$

For the conjugate momentum and energy, the correspondence is then

$$ip_{\varphi} = -\sqrt{2} + i2\sqrt{2} \lambda = -\sqrt{2} + \frac{i}{\sqrt{2}} p_\tau,$$

$$ip = \frac{2\sqrt{2}}{3} \omega = \frac{i}{\sqrt{2}} p_t. \quad (6.14)$$

This implies a one-to-one correspondence of tachyon states in the black hole and linear dilaton backgrounds. There is also a correspondence between the discrete states spectra in the two theories.
In the Minkowski metric, the spectrum of the discrete states for the black hole is isomorphic to that in the linear dilaton background. In particular, the first nontrivial discrete state with zero energy \( (j = 1, m = 0 \text{ or } ip_\varphi = -2\sqrt{2}, p = 0) \) is identified with the operator associated with the mass of black hole, as can be seen from the first correction to the asymptotic behavior of the exact space-time metric

\[
d s^2 \sim k^{-2} \left[ d r^2 - \frac{4k}{k - 2} \left( 1 - \frac{4k}{k - 2} e^{-r} + \mathcal{O}(e^{-2r}) \right) d t^2 \right].
\]  

(6.15)

It is important to note that the \( \varphi \) momentum is twice that of the operator corresponding to tachyon condensation.

The solutions of the tachyon Virasoro conditions describe the scattering of a single tachyon on the black hole. It represents one of the few quantities that has been rigorously computed in black hole string theory [37]. The amplitude provides some nontrivial physical insight and is obtained as follows. One writes an integral representation for the solution with definite energy \( \omega \) and momentum \( \lambda \) as

\[
\int_C \frac{d x}{x} x^{-2i \omega} (\sqrt{M - u v + \frac{u}{x}})^{-\nu_-} (\sqrt{M - u v - v x})^{-\nu_+},
\]  

(6.16)

with \( \nu_\pm = \frac{1}{2} - i(\lambda \pm \omega) \). In general, one has four different contours of integration with two linearly independent solutions corresponding, for example, to the contours \( C_2 \equiv [u \sqrt{M - u v}, 0], C_4 \equiv (-\infty, \nu^{-1} \sqrt{M - u v}] \) as \( y \equiv u v = -\sinh^2 \frac{r}{2} \):

\[
T_{C_2} = U^\lambda_\omega = e^{-2i \omega \tilde{t}} F^\lambda_\omega(y),
\]

\[
T_{C_4} = V^\lambda_\omega = e^{-2i \omega \tilde{t}} F^\lambda_{-\omega}(y),
\]  

(6.17)

where

\[
F^\lambda_\omega(y) = (-y)^{-i \omega} B(\nu_+, \nu_-) F(\nu_+, \nu_-, 1 - 2i \omega, y).
\]  

(6.18)

The asymptotic behaviors of the solutions are, for \( r \rightarrow 0 \) (horizon):

\[
U^\lambda_\omega \sim \beta(\lambda, \omega) \left( \frac{u}{\sqrt{M}} \right)^{-2i \omega},
\]

\[
V^\lambda_\omega \sim \beta(\lambda, -\omega) \left( -\frac{v}{\sqrt{M}} \right)^{-2i \omega}
\]  

(6.19)

while for null-infinity \( r \rightarrow \infty \):

\[
F^\lambda_\omega \sim \alpha(\lambda, \omega)(-y)^{-\frac{1}{2} + i \lambda} + \alpha(-\lambda, \omega)(-y)^{-\frac{1}{2} - i \lambda},
\]  

(6.20)
where
\[
\alpha(\lambda, \omega) = \frac{\Gamma(\nu_+) \Gamma(\bar{\nu}_- - \nu_+)}{\Gamma(\bar{\nu}_-)} ,
\]
\[
\beta(\lambda, \omega) = B(\nu_+, \bar{\nu}_-) .
\]

We see that \( U^\lambda_\omega \) describes a wave coming from past null-infinity scattering on the black hole, while \( V^\lambda_\omega \) describes a wave emitted by the white hole crossing the past event horizon. The solution \( U^\lambda_\omega \) gives the \( S \)-matrix elements of tachyons, incoming from the asymptotic flat region at past null-infinity and scattered out to future null-infinity. On-shell \( \omega = 3 \lambda (> 0) \), and this solution gives the reflection and transmission coefficients as ratios of the coefficients appearing in the above asymptotic forms:
\[
R_B(\lambda) = \frac{\alpha(\lambda, \omega)}{\alpha(-\lambda, \omega)} ,
\]
\[
T_B(\lambda) = \frac{\beta(\lambda, \omega)}{\alpha(-\lambda, \omega)} .
\]

The reflection and absorption coefficients satisfy the unitarity relation
\[
|R_B|^2 + \frac{\omega}{\lambda} |T_B|^2 = 1 .
\]

This describes the two-point correlation function and it is of major interest to formulate a full quantum field theory in the presence of a black hole which would be capable of giving general \( N \)-point scattering amplitudes and correlation functions. One is also very interested in being able to evaluate loop effects and even to discuss formation and evaporation of black holes in the general field theoretic framework. In the absence of a general theory one can try to follow the analogy with the \( c = 1 \) theory and attempt to guess the structure required for the black hole. This is what was done in [34]. Pursuing the above analogy we can postulate again that string theory in the black hole background is described by a factorized \( S \)-matrix. It is then reasonable to expect that the external leg factors of the full \( S \)-matrix are again determined through a non-local field redefinition whose role is to connect the Virasoro equations in the black hole background with the free massless Klein-Gordon equation. The main part of the \( S \)-matrix is then to be determined. The suggestion based on the analogy with the \( c = 1 \) theory is that one again has a description in terms of a matrix model and the associated collective field theory. To simulate the black hole background the matrix model is expected to include a deformation from the standard inverted oscillator potential. There as yet exist no general principles for constructing the theory but one can make certain concrete suggestions on the eventual form of the matrix model. Let us describe first the expected form for the external leg factors. These are supplied by a field redefinition whose purpose is to reduce the black hole background Virasoro...
condition to the scalar free field equation. We have summarized the black hole Virasoro equation and its solutions in detail, so let us consider the integral representation (6.16) with the contour \( C_2 \) which is appropriate for the scattering problem in the exterior region \((u > 0, v < 0)\):

\[
U_{\lambda}^{\omega}(u, v) = \int_{C_2} \frac{dx}{x} x^{-2i\omega} (\sqrt{M} - uv + \frac{u}{x})^{-\nu_1 - \nu_2} (\sqrt{M} - uv - vx)^{-\nu_2} . \tag{6.24}
\]

Since the spectrum of the on-shell solution has a one-to-one correspondence through (6.14) with that of the free Klein-Gordon equation, it is natural to make the following change of integration variable:

\[
(\sqrt{M} - uv + \frac{u}{x})^{-1} (\sqrt{M} - uv - vx) = e^{-4t/3} ,
(\sqrt{M} - uv + \frac{u}{x}) (\sqrt{M} - uv - vx) = e^{-4\tau} . \tag{6.25}
\]

The integral formula for the solution takes now the form

\[
U_{\lambda}^{\omega} = \int_{-\infty}^{\infty} dt \int_{0}^{\infty} d\tau \delta\left(\frac{ue^{-2t/3} + \nu e^{2t/3}}{2} - \sqrt{M} \cosh 2\tau\right) e^{-4i\omega t/3} \cos 4\lambda \tau . \tag{6.26}
\]

This is seen to be an integral transform of a Klein-Gordon plane wave with momentum and energy

\[
p_\tau = 4\lambda , \quad p_t = \frac{4}{3} \omega . \tag{6.27}
\]

Since the plane waves are recognized as natural eigenstates of the linearized collective field terms of time-of-flight variable we have the candidate for the non-local field redefinition

\[
T(u, v) = \int_{-\infty}^{\infty} dt \int_{0}^{\infty} d\tau \delta\left(\frac{ue^{-2t/3} + \nu e^{2t/3}}{2} - \sqrt{M} \cosh 2\tau\right) \gamma(i\partial_t) \partial_\tau \eta(t, \tau) , \tag{6.28}
\]

where \( \gamma(i\partial_t)^* = \gamma(-i\partial_t) \) is an arbitrary weight function to be fixed by normalization condition.

In terms of the Fourier decomposition

\[
\eta(t, \tau) = \int_{-\infty}^{\infty} \frac{dp}{p} \tilde{\eta}(p) e^{-ipt} \sin p\tau , \tag{6.29}
\]
it reads
\[ T(u, v) = \int_{-\infty}^{\infty} dp \, \tilde{\eta}(p) \, \gamma(p) \, U_{\omega(p)}(u, v), \]
(6.30)
with \( \omega(p) = 3p/2, \lambda(p) = p/2 \). In particular, the asymptotic behavior for \( y \to \infty \) is
\[ T(u, v) = \int_{-\infty}^{\infty} dp \, \tilde{\eta}(p) \, \gamma(p) \left[ (-y)^{-\frac{1}{2} + i\lambda(p)} \alpha(\lambda(p), \omega(p)) + (-y)^{-\frac{1}{2} - i\lambda(p)} \alpha(-\lambda(p), \omega(p)) \right] e^{-2i\omega(p)t}. \]
(6.31)
This shows that an asymptotic wave packet of \( \eta \) field is transformed into a deformed wave packet of the tachyon field. The integral transformation that we have given supplies the leg factors of the conjectured black hole \( S \)-matrix. Even if the factorization becomes only an approximate feature of the full theory one could expect that the factorization holds near the poles of the \( S \)-matrix.

Let us now study the possible resonance poles produced by the external leg factors. It turns out that studying the location of these poles gives useful and nontrivial constraints on the full \( S \)-matrix. From the asymptotic behavior of (6.20) and the associated reflection coefficient (6.22), we see that the positions of the resonance poles are
\[ i4\lambda = i\frac{4}{3}\omega = i\sqrt{2}p_t = -2, -4, -6, \ldots. \]
(6.32)
This contrasts with the case of the usual \( c = 1 \) model where we have poles at all negative integers of the corresponding energy. On the other hand, if we consider an amplitude for an incoming tachyon with producing \( N - 1 \) outgoing tachyons, the energy and momentum conservation laws are satisfied when the energy of incoming tachyon obeys
\[ i\sqrt{2}p_t = -(2r + N - 2), \]
(6.33)
where \( r \) now counts the number of insertions of the black hole mass operator. The factor 2 multiplying \( r \) comes about because the momentum carried by the black hole mass is twice that of the tachyon condensation. Comparing next the two expressions for the location of the poles we see that these are consistent only if \( N \) is even. More precisely, only the even \( N = 2k \) point amplitudes are to be nonzero while the odd \( N = 2k + 1 \) point amplitudes should vanish. This represents a strong requirement on the form of the complete theory.

We are than lead to the main problem of specifying the full dynamics in the form of a generalized matrix model. In the limit of vanishing black hole mass, the black
hole background reduces to the linear dilaton vacuum. This is a singular limit in the sense that the string coupling diverges, corresponding to the $c = 1$ matrix model with vanishing 2d cosmological constant $\mu = 0$, or zero Fermi energy. Since, according to our hypothesis, the deformation corresponding to non-vanishing black hole mass cannot be described by the usual matrix model, we have to seek for other possible deformations than the one given by the Fermi energy. We assume that the Fermi energy is kept exactly at zero, while the Hamiltonian itself is modified.

From the earlier analysis we have several hints or constraints which the modified Hamiltonian should obey. The first is that there is a double scaling limit and that the resulting string coupling constant squared should be given by the black hole mass $M$. The second constraint is the required vanishing of all odd $N$-point amplitudes. Finally, in agreement with the world sheet description of the black hole string theory one should have a natural $SL(2, \mathbb{R})$ symmetry.

Consider a general modification of the inverted oscillator Hamiltonian

$$ h(p, x) \rightarrow h_M(p, x) = \frac{1}{2} (p^2 - x^2) + M\delta h(p, x) . $$

(6.34)

We have assumed that the deformation is described by a term linear in $M$. The first requirement for $\delta h$ is a scaling property to ensure that the string coupling is proportional to $M^{-1/2}$. In collective field theory after a shift by the classical ground state one has that the string coupling generally proportional to $(\frac{dx}{d\tau})^{-2}$. Thus the above requirement is satisfied if the deformation operator $\delta h$ scales as $\delta h(p, x) \rightarrow \rho^{-2} \delta h(p, x)$ under scale transformations $(p, x) \rightarrow (\rho p, \rho x)$. This leads to

$$ \delta h(p, x) = \frac{1}{2x^2} f\left(\frac{p}{x}\right) . $$

(6.35)

To further specify the general function $f(p/x)$, one invokes the requirement of $SL(2, \mathbb{R})$ symmetry. We have seen in sect. 4, that the usual $c = 1$ Hamiltonian $h = (p^2 - x^2)/2$ allows a set of eigenoperators $O_{j,m}$ satisfying the the $w_\infty$ algebra (4.26). The origin of this algebraic structure, which is supposed to encode the extended nature of strings, can be traced to existence of an $SL(2, \mathbb{R})$ algebra consisting of

$$ L_1 = \frac{1}{4} (p^2 - x^2) = h(p, x) , $$

$$ L_2 = -\frac{1}{4} (px + xp) , $$

$$ L_3 = \frac{1}{4} (p^2 + x^2) . $$

(6.36)

The eigenoperators satisfying the $w_\infty$ algebra are constructed in terms of the $SL(2, \mathbb{R})$
operators according to

\[ O_{j,m} = L_{j^2}^{i+m} L_{-j^2}^{-i-m} , \quad L_{\pm} = L_3 \pm L_2 , \quad (6.37) \]

which close under the Poisson bracket since the Casimir invariant has a fixed value

\[ L_1^2 + L_2^2 - L_3^2 = \frac{3h}{16}. \quad (6.38) \]

(the Planck constant indicates the effect of operator ordering).

Since the spectrum of discrete states in the black hole background is expected to be the same as that of the usual \( c = 1 \) model in the Minkowski metric, it is natural to require that the deformed model should also share a similar algebraic structure.

There is actually a very simple model with the above structure. It is given for \( f = 1 \) in which case one has the extra term represented by a well known singular potential. One has the \( SL(2, \mathbb{R}) \) generators of the form:

\[
L_1(M) = \frac{1}{2} h_M(p, x) = \frac{1}{4} (p^2 - x^2 + \frac{M}{x^2}) , \\
L_2(M) = -\frac{1}{4} (px + xp) , \\
L_3(M) = \frac{1}{4} (p^2 + x^2 + \frac{M}{x^2}) ,
\]

which satisfy

\[ L_1(M) + L_2^2(M) - L_3^2(M) = -\frac{M}{2} + \frac{3h}{16}. \quad (6.40) \]

We note that because of different constraint for the Casimir invariant the algebra of eigenoperators is now modified in an \( M \)-dependent way. The algebraic properties of the model with the singular potential have been investigated in detail in [39]. The model is exactly solvable and possesses some features characteristic of black hole background.

Let us proceed to describe the properties of the deformed model:

\[ h_M(p, x) = \frac{1}{2} (p^2 - x^2) + \frac{M}{2x^2}. \quad (6.41) \]

We assume here that \( M > 0 \). Then the genus zero free energy in the limit of vanishing scaling parameter, \( \bar{M} \rightarrow 0 \), behaves like \( F \sim \frac{N^2}{8\pi\sqrt{2}} \bar{M} \log \frac{\bar{M}}{\sqrt{2}} \). The double scaling limit is thus the limit \( \bar{M} \rightarrow 0, N \rightarrow \infty \) with \( M \equiv N^2 \bar{M} \) being kept fixed. After the usual rescaling, \( x \equiv \sqrt{N} \times \) matrix eigenvalue, the system is reduced to the free fermion system with the one-body potential \( -\frac{1}{2} x^2 + \frac{M}{2x^2} \). Note that in the limit \( M \rightarrow 0 \) the potential approaches the usual inverted harmonic oscillator potential with a repulsive \( \delta \)-function-like singularity.
The solution of the classical equations with energy $\epsilon$ reads

$$x^2(t) = -\epsilon + \sqrt{M + \epsilon^2} \cosh 2t . \tag{6.42}$$

The ground state corresponding to zero Fermi energy is obtained by setting $\epsilon = 0$ and replacing the time variable $t$ by the time-of-flight coordinate $\tau$, $x^2 = \sqrt{M} \cosh 2\tau$. This is recognized as precisely the quantity appearing in the integral transformation (6.28). The $\delta$-function present in the transformation gives a relation between the black hole and the matrix model variables. It serves to identify the matrix eigenvalue as

$$x^2 = \left(ue^{-2t/3} + ve^{2t/3}\right)/2 .$$

The string coupling is now space dependent

$$g(\tau) \equiv \frac{\sqrt{\pi}}{12} \left(\frac{dx}{d\tau}\right)^{-2} = \frac{1}{48} \sqrt{\frac{\pi}{M}} \left(\frac{1}{\sinh^2 \tau} + \frac{1}{\cosh^2 \tau}\right) , \tag{6.43}$$

with the required relation with the black hole mass and the asymptotic behavior at large $\tau$.

The tree level scattering amplitudes are generally obtained from the exact solution of the classical equations. The exact solution to the collective equations has the following parametrized form

$$x(t, \sigma) = \left[-a(\sigma) + \sqrt{M + a^2(\sigma)} \cosh 2(\sigma - t)\right]^{1/2} ,$$

$$\alpha(t, \sigma) = \frac{1}{x(t, \sigma)} \sqrt{M + a^2(\sigma)} \sinh 2(\sigma - t) . \tag{6.44}$$

It contains an arbitrary function $a(\sigma)$ describing the deviation of the Fermi surface from its ground state form. The asymptotic behavior for large $x$, of the profile function reads

$$\alpha_{\pm}(t, \tau) = \pm x(\tau) \left(1 - \frac{\psi_{\pm}(t \pm \tau)}{x^2(\tau)}\right) + O\left(\frac{1}{x^2}\right) . \tag{6.45}$$

The functions $\psi_{\pm}(t \pm \tau)$ represent incoming and outgoing waves, respectively. In terms of the $\eta$ field, we have

$$(\partial_t \pm \partial_\tau)\eta = \pm \frac{1}{\sqrt{\pi}} \psi_{\pm}(t \pm \tau) . \tag{6.46}$$

for $t \to \mp \infty$. 
A nonlinear relation between incoming and outgoing fields $\psi_+$ and $\psi_-$ can be established by studying the time delay. Take the times at which a parametrized point $\sigma$ is passed by the incoming and outgoing waves at a fixed value of large $\tau$ be $t_1 (\to -\infty)$ and $t_2 (\to \infty)$, respectively. From (6.44) we have then

$$(M + a^2(\sigma))^{1/4} e^{\sigma-t_1} = M^{1/4} e^\tau,$$

$$(M + a^2(\sigma))^{1/4} e^{t_2-\sigma} = M^{1/4} e^\tau. \tag{6.47}$$

This implies

$$t_1 + \tau = t_2 - \tau + \frac{1}{2} \log\left(1 + \frac{a^2(\sigma)}{M}\right), \tag{6.48}$$

and hence

$$a(\sigma) = \psi_+(t_1 + \tau) = \psi_-(t_2 - \tau). \tag{6.49}$$

This then gives functional scattering equations connecting the incoming and outgoing waves

$$\psi_\pm(z) = \psi_\mp \left(z \mp \frac{1}{2} \log \left(1 + \frac{1}{M} \psi_\pm^2(z)\right)\right). \tag{6.50}$$

The result is similar in form to that of the usual $c = 1$ model. However, one notes a crucial difference that Eq. (6.50) is even, i.e. it is invariant under the change of sign of $\psi_\pm \to -\psi_\pm$. This ensures that the number of particles participating in the scattering is even. All the odd point amplitudes do vanish in the deformed model.

The explicit power series solution of (6.50) is

$$\psi_\pm(z) = \sum_{p=0}^{\infty} \frac{M^{-p}}{p! (2p + 1)} \frac{\Gamma(1 \pm \frac{1}{2} \partial_z)}{\Gamma(1 - p \pm \frac{1}{2} \partial_z)} \psi_\mp^{2p+1}(z), \tag{6.51}$$

which shows that the amplitudes are essentially polynomial with respect to the momenta without any singularity.

The scattering equation (6.50) can also be derived using directly the exact states [40, 41], as was done in sect. 5 for the $c = 1$ model. First, one recalls the symmetry structure of the collective theory with Hamiltonian (6.41) given in [39]:

$$\left[ O_{j_1, m_1}^{a_1}, O_{j_2, m_2}^{a_2} \right] = -4i(j_1 m_2 - m_1 j_2) O_{j_1+j_2-2, m_1+m_2}^{a_1 + a_2 + 1} - 4i(a_1 m_2 - m_1 a_2) O_{j_1+j_2, m_1+m_2}^{a_1 + a_2 - 1} \tag{6.52}.$$
where

$$O_{j,m}^{a} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left( \left( \alpha^2 - x^2 \right)^a \left( \left( \alpha + x \right)^2 + \frac{M}{x^2} \right)^{\frac{j+m}{2}} \right) \left( \left( \alpha - x \right)^2 + \frac{M}{x^2} \right)^{\frac{j-m}{2}}. \quad (6.53)$$

The operators $T^{(-)}_{ip}$ ($T^{(+)}_{ip}$) which create exact tachyon in (out) states are obtained by analytic continuation $j \rightarrow \pm ip/2$ of some special operators (6.53):

$$O_{j,j}^{a=0} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left[ (\alpha + x)^2 + \frac{M}{x^2} \right]^j \equiv T_{2j}^{(+)};$$

$$O_{j,-j}^{a=0} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left[ (\alpha - x)^2 + \frac{M}{x^2} \right]^j \equiv T_{2j}^{(-)}. \quad (6.54)$$

Eq. (6.50) then easily follows from an asymptotic expansion of

$$T_{ip,+}^{(+)} = -T_{ip,-}^{(+)};$$

for large $\tau$, where $T_{ip,+}^{(+)}$ and $T_{ip,-}^{(+)}$ are defined by

$$T_{ip}^{(+)} = T_{ip,+}^{(+)} - T_{ip,-}^{(+)}. \quad (6.55)$$

The scattering equation can also be rewritten in terms of energy-momentum tensor

$$T_{\pm\pm}(z) = \frac{1}{2\pi} \psi_{\pm}^2(z), \quad (6.56)$$

as

$$\int dz \ e^{i\omega z} T_{\pm\pm}(z) = \frac{M}{2\pi} \int dz \ e^{i\omega z} \frac{1}{1 \pm \frac{i\omega}{2M}} \left[ \left( 1 + \frac{2\pi}{M} T_{\mp\mp}(z) \right)^{1\pm \frac{i\omega}{2M}} - 1 \right]. \quad (6.57)$$

One can easily check that this defines a canonical transformation by confirming that the Virasoro algebra (at the level of Poisson bracket) is preserved by this transformation. This relation for the energy momentum tensor is very similar to the one obtained recently by Verlinde and Verlinde [42] for the $S$-matrix of the $N = 24$ dilaton gravity. A slight difference is that in the case of dilaton gravity one has the derivative of the energy momentum tensor participating in the equation.
In conclusion, the framework presented above gives some initial picture of a black hole in the matrix model. It contains some basic requirements for a consistent formalism. In particular, the scaling properties of the black hole mass deformation are in agreement with the corresponding vertex operators (see also [43]). The particular singular matrix model studied has an interesting double scaling limit with an $SL(2, \mathbb{R})$ algebraic structure. This clearly is not enough to completely describe black hole and further generalizations and studies are likely to lead to further interesting results.

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