DEFINABLE EILENBERG–MAC LANE UNIVERSAL COEFFICIENT THEOREMS

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Abstract. We prove definable versions of the Universal Coefficient Theorems of Eilenberg–Mac Lane expressing the (Steenrod) homology groups of a compact metrizable space in terms of its integral cohomology groups, and the (Čech) cohomology groups of a polyhedron in terms of its integral homology groups. Precisely, we show that, given a compact metrizable space $X$, a (not necessarily compact) polyhedron $Y$, and an abelian Polish group $G$ with the division closure property, there are natural definable exact sequences

$$0 \to \text{Ext} \left( H^{n+1}(X), G \right) \to H^n(X;G) \to \text{Hom} \left( H^n(X), G \right) \to 0$$

and

$$0 \to \text{Ext} \left( H^{n-1}(Y), G \right) \to H^n(Y;G) \to \text{Hom} \left( H^n(Y), G \right) \to 0$$

which definably split, where $H^n(X;G)$ is the $n$-dimensional definable homology group of $X$ with coefficients in $G$ and $H^n(Y;G)$ is the $n$-dimensional definable cohomology group of $Y$ with coefficients in $G$.

Both of these results are obtained as corollaries of a general algebraic Universal Coefficient Theorem relating the cohomology of a cochain complex of countable free abelian groups to the definable homology of its $G$-dual chain complex of Polish groups.

1. Introduction

The integral (Steenrod) homology $H_\bullet(X)$ of a compact metrizable space $X$ was initially defined by Steenrod in the seminal paper [40]. Motivated by the problem of computing $H_\bullet(X)$ when $X$ is the diadic solenoid (a space homeomorphic to the Pontryagin dual group of $\mathbb{Z}[1/2]$), Eilenberg and Mac Lane proved in the groundbreaking paper [11] a formula relating the homology of a compact space to its (integral) cohomology $H^\bullet(X)$. More generally, Eilenberg and Mac Lane considered the homology $H_\bullet(X;G)$ of a compact metrizable space $X$ with coefficients in an abelian group $G$, and expressed it purely in terms of the cohomology of $X$ via the so-called Universal Coefficient Theorem; see also [25, Section 21.3].

The possibility of enriching the homology of a compact metrizable space $X$ was considered in [3]. It is shown therein that $H_\bullet(X)$ can be considered as a definable graded group (see Section 2.2). Furthermore, $H_\bullet(-)$ as a definable graded group is a complete invariant for torus-free solenoids up to homeomorphism, while the same does not hold when $H_\bullet(-)$ is regarded as a purely algebraic graded group.

In this paper we consider the definable homology of a compact metrizable space with coefficients in an abelian Polish group $G$ with the division closure property (see Definition 2.1). We prove in this context the natural definable version of the Universal Coefficient Theorem (UCT) of Eilenberg and Mac Lane for homology, including the case of pairs of spaces; see Theorem 2.7 and Theorem 2.10. In particular, this shows that the definable graded group $H_\bullet(X;G)$ only depends on the integral cohomology of the compact metrizable space $X$.

In [11], Eilenberg and Mac Lane also proved a Universal Coefficient Theorem expressing the cohomology of a (not necessarily compact) polyhedron in terms of its homology. In this paper, we consider definable cohomology of polyhedra with coefficients in a Polish group with the division closure property. In this context, we prove a definable version of the UCT of Eilenberg and Mac Lane for cohomology of (pairs of) polyhedra; see Theorem 3.6 and Theorem 3.9.

In fact, we obtain both the UCT for homology and the UCT for cohomology as particular instances of a general algebraic UCT, relating the cohomology of a cochain complex of countable groups to the homology of its $G$-dual complex.
chain complex of Polish groups. In particular, our approach can be seen as providing a new proof of the Universal Coefficient Theorems from [11], although inspired by the original proof of Eilenberg and Mac Lane.

The rest of this paper is divided as follows. In Section 2 we recall some fundamental notions concerning Polish groups and definable groups as introduced in [4, 23]. In Section 3 we present some basic facts about Polish chain complexes, and a purely algebraic UCT. The coherent category of towers of Polish chain complexes is introduced in Section 4, and the coherent category of inductive sequences of countable chain complexes is introduced in Section 5. Section 6 recalls some classical notions concerning simplicial complexes and their homological invariants. The UCT for (pairs of) compact metrizable spaces is inferred from the algebraic version in Section 7. Finally, the UCT for (pairs of) polyhedra is proved in Section 8. In what follows, we let \( \omega \) denote the set of positive integers including zero.

## 2. Polish and definable groups

In this section, we recall the definition and fundamental properties of Polish spaces and definable sets, as well as Polish groups and definable groups, as can be found in [2, 14, 17] and [4, 23].

### 2.1. Polish spaces and Polish groups

A Polish space is a topological space whose topology is second-countable and induced by some complete metric. Let \( X \) be a Polish space. The \( \sigma \)-algebra of Borel sets is the smallest \( \sigma \)-algebra of subsets of \( X \) that contains all open sets. A subset of \( X \) is Borel if it belongs to the Borel \( \sigma \)-algebra. A closed subset of a Polish space is a Polish space when endowed with the subspace topology. We regard every countable set as a Polish space endowed with the discrete topology. The product \( X \times Y \) of two Polish spaces is a Polish space when endowed with the product topology. Similarly, the product \( \prod_{n \in \omega} X_n \) of a sequence \( (X_n)_{n \in \omega} \) of Polish spaces is Polish when endowed with the product topology. The category of Polish spaces has Polish spaces as objects and continuous functions as morphisms. The Borel category of Polish spaces has Polish spaces as objects and Borel functions as morphisms. Let \( X, Y \) be Polish spaces, and \( f : X \to Y \) be a Borel function. In general the image of a Borel subset of \( X \) under \( f \) need not be a Borel subset of \( Y \). However, if \( f \) is an injective Borel function, then \( f \) maps Borel subsets of \( X \) to Borel subsets of \( Y \) [17, Theorem 15.1].

A Polish group is, simply, a group object in the category of Polish groups in the sense of [24, Section III.6]. In other words, a Polish group is a Polish space \( G \) endowed with a continuous group operation \( G \times G \to G \), and such that the function \( G \to G \) mapping each element to its inverse is also continuous. A subgroup \( H \) of a Polish group \( G \) is Polishable if it is Borel and can be endowed with a (necessarily unique) Polish topology that induces the Borel structure on \( H \) and turns \( H \) into a Polish group. This is equivalent to the assertion that \( H \) is the image of a continuous group homomorphism \( \varphi : \hat{G} \to G \) for some Polish group \( \hat{G} \). If \( G \) is a Polish group, and \( H \) is a closed subgroup, then \( H \) is a Polish group with the subspace topology. If \( H \) is closed and normal in \( G \), then \( G/H \) is a Polish group with the quotient topology.

A continuous action of a Polish group \( G \) on a Polish space \( X \) is an action \( G \curvearrowright X \) that is continuous as a function \( G \times X \to X \). A Polish \( G \)-space is a Polish space endowed with a continuous action of the Polish group \( G \). Following Steenrod [39] and Lefshetz [19], we consider the following notion.

**Definition 2.1.** An additively-denoted abelian Polish group \( G \) has the division closure property if, for every \( k \in \mathbb{N} \), \( kG = \{kg : g \in G\} \) is a closed subgroup of \( G \).

Clearly, every countable discrete group has the division closure property.

### 2.2. Definable sets and definable groups

Suppose that \( X \) is a Polish space. We regard an equivalence relation \( E \) on \( X \) as a subset of the product space \( X \times X \). Consistently, we say that \( E \) is Borel if it is Borel as a subset of \( X \times X \). We now recall the notion of idealistic equivalence relation as formulated in [23, Definition 1.6], which is slightly more generous than the original definition from [16]; see also [14, Definition 5.4.9] and [18].

**Definition 2.2.** Let \( C \) be a set. A nonempty collection \( \mathcal{F} \) of nonempty subsets of \( C \) is a \( \sigma \)-filter if it satisfies the following:

- \( S_n \in \mathcal{F} \) for every \( n \in \omega \) implies \( \bigcap_{n \in \omega} S_n \in \mathcal{F} \);
- if \( S \in \mathcal{F} \) and \( S \subseteq T \subseteq C \), then \( T \in \mathcal{F} \).

**Definition 2.3.** Let \( E \) be an equivalence relation on a Polish space \( X \). Then \( E \) is idealistic if there exist a Borel function \( s : X \to X \) and a function \( C \mapsto \mathcal{F}_C \) assigning to each \( E \)-class \( C \) a \( \sigma \)-filter \( \mathcal{F}_C \) of subsets of \( C \) such that:
(1) $x Es(x)$ for every $x \in X$;
(2) for every Borel subset $A$ of $X \times X$, the set
\[ \{ x \in X : \{ y \in [x]_E : (s(x), y) \} \in \mathcal{F}_{[x]_E} \} \]

is Borel, where
\[ [x]_E = \{ y \in X : (x, y) \in E \} \]

for $x \in X$.

One can equivalently reformulate the notion of idealistic equivalence relation in terms of $\sigma$-ideals, due to the duality between $\sigma$-ideals and $\sigma$-filters. As in [23], we let a definable set $X$ be a pair $(\hat{X}, E)$ where $\hat{X}$ is a Polish space and $E$ is a Borel and idealistic equivalence relation on $E$. We consider the pair $(\hat{X}, E)$ as an explicit presentation of $X$ as the quotient of a Polish space $\hat{X}$ by a “well-behaved” equivalence relation $E$. Consistently, we denote such a definable set also by $X = \hat{X}/E$. Every Polish space $\hat{X}$ is, in particular, a definable set $X = \hat{X}/E$ where $X = \hat{X}$ and $E$ is the identity relation on $\hat{X}$. Suppose that $X = \hat{X}/E$ and $Y = \hat{Y}/F$ are definable sets. A Borel subset $Z$ of $X$ is a subset of the form $\hat{Z}/E$ for some $E$-invariant Borel subset $\hat{Z}$ of $\hat{X}$. (Notice that $Z$ is itself a definable set.) Similarly, one defines a closed subset $Z$ of $X$ to be a subset of the form $\hat{Z}/E$ for some $E$-invariant closed subset $\hat{Z}$ of $X \hat{X}$. A function $\hat{f} : \hat{X} \to \hat{Y}$ is a lift of (or induces) the function $f : \hat{X}/E \to \hat{Y}/F$ if
\[ f([x]_E) = [\hat{f}(x)]_F \]

for every $x \in \hat{X}$.

**Definition 2.4.** Suppose that $X = \hat{X}/E$ and $Y = \hat{Y}/F$ are definable sets. A function $f : X \to Y$ is Borel-definable (or, briefly, definable) if it admits a Borel lift $\hat{f} : \hat{X} \to \hat{Y}$, and continuously-definable if it admits a continuous lift $\hat{f} : \hat{X} \to \hat{Y}$.

The category of definable sets has definable sets as objects and definable functions as morphisms. The continuous category of definable sets has definable sets as objects and continuously-definable functions as morphisms. Notice that the category of Polish spaces is a full subcategory of the continuous category of definable sets, and the Borel category of Polish spaces is a full subcategory of the category of definable sets.

The category of definable sets has nice properties, which generalize analogous properties of the Borel category of Polish spaces; see [23, Proposition 1.10]. For instance, if $f : X \to Y$ is an injective definable function between definable sets, then its image is a Borel subset of $Y$. Furthermore, if $f : X \to Y$ is a bijective definable function between definable sets, then its inverse $f^{-1} : X \to Y$ is also definable. The (continuous) category of definable sets has products, where the product of $X = \hat{X}/E$ and $Y = \hat{Y}/F$ is the definable set $X \times Y = (\hat{X} \times \hat{Y})/(E \times F)$, where the equivalence relation $E \times F$ on $\hat{X} \times \hat{Y}$ is defined by setting $(x, y)(E \times F)(x', y')$ if and only if $xEx'$ and $yFy'$. (It is easily seen that $E \times F$ is Borel and idealistic whenever both $E$ and $F$ are Borel and idealistic.)

A definable group is, simply, a group object in the category of definable sets. Thus, $G$ is a definable group if it is a definable set that is also a group, and such that the group operation $G \times G \to G$ and the function $G \to G$ that maps each element to its inverse are definable.

If $G$ is a Polish group and $\hat{X}$ is a Polish $G$-space, the corresponding orbit equivalence relation $E^X_G$ on $\hat{X}$ is defined by setting $xE^X_G y$ if and only if there exists $g \in G$ such that $g \cdot x = y$. Then one has that $E^X_G$ is idealistic. If $E^X_G$ is furthermore Borel, then $X := \hat{X}/E^X_G$ is a definable set. In particular, if $G$ is a Polish group and $H$ is a normal Polishable subgroup of $G$, then the coset equivalence relation $E^X_G$ of $H$ in $G$ is Borel and idealistic, and hence the quotient $G/H = G/E^X_G$ is a definable group, called a group with a Polish cover in [4].

**Definition 2.5.** Given a group with a Polish cover $G/H$, its corresponding weak Polish group is the quotient $G/\overline{H}$ of $G$ by the closure of $H$ in $G$, and its corresponding asymptotic group with a Polish cover is $\overline{\mathcal{T}}/H$. Notice that the assignment $G/H \to G/\overline{H}$ is a functor from the continuous category of groups with a Polish cover to the category of Polish groups. Similarly, the assignment $G/H \to \overline{\mathcal{T}}/H$ is a functor from the continuous category of groups with a Polish cover to itself.

Recall that a graded group $G_\bullet$ is a sequence $(G_n)_{n \in \mathbb{Z}}$ of groups indexed by $\mathbb{Z}$, where a (degree 0) homomorphism $f$ from $G_\bullet$ to $H_\bullet$ is a sequence $(f_n)_{n \in \mathbb{Z}}$ of group homomorphisms $f_n : G_n \to H_n$. More generally, for $k \in \mathbb{Z}$, a degree
k homomorphism $f$ from $G_\bullet$ to $H_\bullet$ is a sequence $(f_\eta)_{\eta \in \mathbb{Z}}$ of group homomorphisms $f_\eta : G_n \to H_{n+k}$. The notions of graded definable group and (continuously) definable degree $k$ homomorphism between graded definable groups are obtained by replacing groups with definable groups and group homomorphisms with (continuously) definable group homomorphisms.

2.3. **Group extensions.** In what follows, we will assume all the groups to be abelian and additively denoted. We will also assume that $G$ is a Polish group with the division closure property. Let also $A$ be a countable group. A (normalized 2-)cocycle on $A$ with values in $G$ is a function $c : A \times A \to G$ such that, for every $x, y, z \in A$:

- $c(x, 0) = 0$;
- $c(x, y) = c(y, x)$;
- $c(x, y) + c(x + y, z) = c(x, z) + c(x + z, y)$.

We regard the set $Z(A, G)$ of cocycles on $A$ with values in $G$ as a closed subgroup of the Polish group $G^{A \times A}$. A coboundary is a cocycle of the form

$$c_h(x, y) = h(x) + h(y) - h(x + y)$$

for some function $h : A \to G$ such that $h(0) = 0$. The set $\mathcal{B}(A, G)$ of coboundaries is a Polishable subgroup of $Z(A, G)$, being the image of the Polish group $\mathfrak{G} = \{h \in G^A : h(0) = 0\}$ under the continuous homomorphism $\mathfrak{G} \to Z(A, G)$, $h \mapsto c_h$. Two cocycles are cohomologous if they belong to the same $\mathcal{B}(A, G)$-coset. The definable group $\text{Ext}(A, G)$ is the group with a Polish cover $Z(A, G)/\mathcal{B}(A, G)$; see [4, Section 7]. We let $[c]$ be the element of $\text{Ext}(A, G)$ determined by the cocycle $c \in Z(A, G)$.

**Remark 2.6.** Given a cocycle $c$ on $A$ with values in $G$ one can then define by recursion on $n \geq 2$ a function $c : A^n \to G$ by setting

$$c(x_1, \ldots, x_{n+1}) = c(x_1, \ldots, x_n) + c(x_1 + \cdots + x_n, x_{n+1}).$$

Then one can prove by induction that this is a permutation-invariant function satisfying

$$c(x_1, \ldots, x_n, y_1, \ldots, y_m) = c(x_1, \ldots, x_n) + c(y_1, \ldots, y_n) + c(x_1 + \cdots + x_n, y_1 + \cdots + y_m)$$

for $n, m \geq 1$, where $c(x) = 0$ for $x \in A$.

The following lemma is also easily proved by induction.

**Lemma 2.7.** Suppose that $F$ is a countable free group, $G$ is a Polish group, and $c$ is a cocycle on $F$ with values in $G$. Consider for every $n \geq 1$ the corresponding function $c : F^n \to G$ as in Remark 2.6. For $x \in F$ and $k \geq 1$ define $x^{(k)}$ to be the $k$-tuple $(x, x, \ldots, x)$. Suppose that $B \subseteq F$ is a free $\mathbb{Z}$-basis of $F$. Define the function $\rho : F \to G$ by setting, for

$$x = (k_0b_0 + \cdots + k_\ell b_\ell) - (m_0d_0 + \cdots + m_s d_s) \in F,$$

where $k_0, \ldots, k_\ell, m_0, \ldots, m_s$ are strictly positive integers and $b_0, \ldots, b_\ell, d_0, \ldots, d_s$ are distinct elements of $B$,

$$\rho(x) := c(b_\ell^{(k_\ell)}, \ldots, b_0^{(k_0)}, -d_s^{(m_s)}, \ldots, -d_0^{(m_0)}) - \sum_{i=0}^s m_i c(d_i, -d_i).$$

Then $\rho$ satisfies, for every $x, x' \in F$ and $e \in B$:

1. $\rho(x) + c(x, e) = \rho(x + e)$ and $\rho(x) + c(x, -e) - c(e, -e) = \rho(x - e)$;
2. $\rho(x + x') = \rho(x) + \rho(x')$.

A group extension of $A$ by $G$ is an exact sequence of Polish groups

$$0 \to G \xrightarrow{i} X \xrightarrow{\pi} A \to 0.$$

Such an extension determines $[c] \in \text{Ext}(A, G)$ defined by

$$c(x, y) = i^{-1}(\pi(t(x) + t(y) - t(x + y)))$$

for $x, y, z \in A$, where $t : A \to X$ is a right inverse for the function $\pi : X \to A$ such that $t(0) = 0$. Notice that $[c]$ does not depend on the choice of $t$. Conversely, given a definable cocycle $c$ on $A$ with values in $G$ one can consider the corresponding extension $X_c$ defined by

$$0 \to G \xrightarrow{i} X_c \xrightarrow{\pi} A \to 0.$$
where $X_c = A \times G$ is endowed with the product topology and the operation defined by

$$(x, y) + (x', y') = (x + x', c(x, x') + y + y').$$

Two cocycles $c, c'$ are cohomologous if and only if the corresponding extensions $X_c$ and $X_{c'}$ are equivalent \[35, \text{Section 7.2}], namely there exists a (topological) isomorphism $\psi : X_c \to X'_c$ that makes the following diagram commute.

$$
\begin{array}{ccc}
G & \longrightarrow & X_c \\
\downarrow & & \downarrow \psi \\
G & \longrightarrow & X'_c \\
\end{array}
$$

The assignments described above establish mutually inverse bijections between $\text{Ext}(A, G)$ and the set of equivalence classes of extensions of $A$ by $G$.

We let $B_c(A, G)$ be the closure of $\mathcal{B}(A, G)$ inside $Z(A, G)$. This is the closed subgroup of $Z(A, G)$ consisting of cocycles $c$ that are weak coboundaries, namely $c|_{S \times S}$ is a coboundary for every finite (or equivalently, for every finitely-generated) subgroup $S$ of $A$. Define then the pure Ext group (also known as phantom Ext group) $\text{PExt}(A, G)$ to be the group with a Polish cover $B_c(A, G)/B(A, G)$ \[7, 36\]. Thus, $\text{PExt}(A, G)$ is the asymptotic group with a Polish cover associated with $\text{Ext}(A, G)$; see Definition 2.5. Its name is due to the fact that the cocycles in $\text{PExt}(A, G)$ corresponds to extensions

$$0 \to G \xhookrightarrow{i} X \twoheadrightarrow F \to 0$$

of $A$ by $G$ that are pure, in the sense that $i$ (G) is a pure subgroup of $X$; see [13, Chapter V].

Following [11, Section 5], one can give an alternative description of $\text{Ext}(A, G)$ and $\text{PExt}(A, G)$, as follows. Suppose that $F$ is a free countable abelian group, and $R$ is a subgroup. Define:

- $\text{Hom}(R, G)$ to be the Polish group of homomorphisms $R \to G$, regarded as a closed subgroup of $G^R$,
- $\text{Hom}(F|R, G)$ to be the Polishable subgroup of $\text{Hom}(R, G)$ consisting of group homomorphisms $R \to G$ that extend to a group homomorphisms $F \to G$, and
- $\text{Hom}_f(F|R, G)$ to be the subgroup of $\text{Hom}(R, G)$ consisting of all group homomorphisms $R \to G$ that extend to a group homomorphism $F_0 \to R$ for every subgroup $F_0$ of $F$ containing $R$ as a finite index subgroup.

By [11, Lemma 5.1 and Lemma 5.2], we have that, for $\varphi \in \text{Hom}(R, G)$, the following assertions are equivalent:

- $\varphi \in \text{Hom}_f(F|R, G)$;
- for every subgroup $F_0$ of $F$ containing $R$ and such that $F_0/R$ is finitely-generated, $\varphi$ extends to a group homomorphism $F_0 \to G$;
- whenever $t \in F$ and $m \in \mathbb{Z}$ satisfy $mt \in R$, one has that $\varphi(mt) \in mG$.

This shows that, as $G$ is assumed to have the division closure property, $\text{Hom}_f(F|R, G)$ is a closed subgroup of $\text{Hom}(R, G)$, and in fact it is equal to the closure of the Polishable subgroup $\text{Hom}(F|R, G)$ of $\text{Hom}(R, G)$; see [11, Lemma 5.3 and Lemma 5.5].

Consider now the group with a Polish cover $\text{Hom}(R, G)/\text{Hom}(F|R, G)$. The following proposition is the definable version of [11, Theorem 10.1], and it can be proved in a similar fashion using Lemma 2.7.

**Proposition 2.8.** Suppose that $G$ is a Polish group with the division closure property, $F$ is a countable free group, and $R \subseteq F$ is a subgroup. Fix a right inverse $t : F/R \to F$ for the quotient map $F \to F/R$ in the category of sets such that $t(0) = 0$. Define then the cocycle $\zeta$ on $F/R$ with values in $R$ by setting $\zeta(x, y) = t(x) + t(0) - t(x + y)$ for $x, y \in F/R$. The function $\text{Hom}(R, G) \to \mathbb{Z}(F|R, G)$, $\theta \mapsto \theta \circ \zeta$ induces a natural isomorphism

$$\frac{\text{Hom}(R, G)}{\text{Hom}(F|R, G)} \cong \text{Ext}(F/R, G)$$

in the continuous category of definable groups, which restricts to a natural isomorphism

$$\frac{\text{Hom}_f(R, G)}{\text{Hom}(F|R, G)} \cong \text{PExt}(F/R, G).$$

The continuously-definable inverse

$$\text{Ext}(F/R, G) \cong \frac{\text{Hom}(R, G)}{\text{Hom}(F|R, G)}$$
is induced by the function $Z(F/R,G) \to \mathrm{Hom}(R,G)$, $\sigma \mapsto \rho_\sigma|_R$, where $\rho_\sigma : F \to G$ satisfies $\rho_\sigma(x) + \rho_\sigma(y) - \rho_\sigma(x+y) = \sigma(x + F, y + F)$, and can be obtained from $\sigma$ as in Lemma 2.7.

3. Chain complexes

In this section, we continue to assume all the groups to be abelian and additively denoted. Recall that we denote by $G$ a Polish group with the division closure property. We consider a duality between chain complexes and cochain complexes, and prove a general Universal Coefficient Theorem relating the cohomology of a cochain complex of free countable groups to the homology of its $G$-dual chain complex of Polish groups; see Theorem 3.11.

3.1. Polish chain complexes. A Polish chain complex is simply a chain complex in the additive category of Polish groups; see [35, Section 5.5]. Thus, a Polish chain complex is a sequence $A = (A_n, \partial_n)_{n \in \mathbb{Z}}$ such that, for every $n \in \mathbb{Z}$, $A_n$ is a Polish group, and the differential $\partial_n : A_n \to A_{n-1}$ is a continuous group homomorphism satisfying $\partial_{n-1} \circ \partial_n = 0$. In the following we will omit the homomorphisms $\partial_n$ from the notation for a Polish chain complex. We will also omit the subscripts in the homomorphisms $\partial_n$. The elements of $A_n$ are called $n$-chains of $A$.

Let $A$ and $B$ be Polish chain complexes. A continuous chain map $f : A \to B$ with source $s(f) = A$ and target $t(f) = B$ is a sequence $(f_n)_{n \in \mathbb{Z}}$ of continuous group homomorphisms $f_n : A_n \to B_n$ satisfying $\partial f_n - f_{n-1} \partial = 0$ for every $n \in \mathbb{Z}$. The composition $gf$ of $f : A \to B$ and $g : B \to C$ is defined by setting $(gf)_n = g_n f_n$ for $n \in \mathbb{Z}$. The identity chain map $1_A$ of $A$ is defined by $(1_A)_n = 1_{A_n}$ for every $n \in \mathbb{Z}$. We say that a Polish chain complex is countable (free countable, free finitely-generated, respectively) if for every $n \in \mathbb{Z}$, $A_n$ is a countable (free countable, free finitely-generated) group. One can regard Polish chain complexes as objects of a category, where morphisms are continuous chain maps. An isomorphism in this category is called a continuous chain isomorphism. The product of a sequence $(A^{(m)})_{m \in \omega}$ of chain complexes is the chain complex $B = \prod_m A^{(m)}$ defined by setting $B_n = \prod_m A^{(m)}_n$, where the map $\partial : B_n \to B_{n-1}$ is induced by the maps $\partial : A^{(m)}_n \to A^{(m)}_{n-1}$ for $m \in \omega$ by the universal property of the product of Polish groups.

Suppose that $A, B$ are Polish chain complexes, and $f, g : A \to B$ are continuous chain maps. A continuous chain homotopy $L : f \Rightarrow g$ from $f$ to $g$ is a sequence $(L_n)_{n \in \mathbb{Z}}$ of continuous group homomorphisms $L_n : A_n \to B_{n+1}$ such that $\partial L_n + L_{n-1} \partial = g_n - f_n$ for every $n \in \mathbb{Z}$. We say that $f$ is the source $s(L)$ of $L$ and $g$ is the target $t(L)$ of $L$. If $f, g, h : A \to B$ are continuous chain maps, and $L : f \Rightarrow g$ and $L' : g \Rightarrow h$ are continuous chain homotopies, then their composition (over 1-cells) $L' \circ_1 L : f \Rightarrow h$ is defined by setting $(L' \circ_1 L)_n = L'_n + L_n$ for $n \in \mathbb{Z}$. If $f, g : A \to B$, $h : A' \to A$, and $k : B \to B'$ are continuous chain maps, and $L : f \Rightarrow g$ is a continuous chain homotopy, then we let $Lh : fh \Rightarrow gh$ and $kL : kf \Rightarrow kg$ be the continuous chain homotopies defined by $(Lh)_n = L_nh_n$ and $(kL)_n = k_{n+1} L_n$ for $n \in \mathbb{Z}$. The identity chain homotopy $1_y$ of $y : A \to B$ is defined by setting $(1_y)_n = 0 : A_n \to B_{n+1}$ for $n \in \mathbb{Z}$.

It is observed in [22, Section 2] that Polish chain complexes form a strict $\omega$-category, with continuous chain maps as 1-cells, and continuous chain homotopies as 2-cells; see [20, Chapter 1]. In this paper, we will only refer to the notion of $k$-cells for $k \leq 3$. If $A, B$ are Polish chain complexes, $f, f' : A \to B$ are continuous chain maps, and $L, L' : f \Rightarrow f'$ are continuous chain homotopies, then a 3-cell $H : L \Rightarrow L'$ is a sequence of continuous group homomorphisms $H_n : A_n \to B_{n+2}$ for $n \in \mathbb{Z}$ satisfying $\partial H_n - H_{n-1} \partial = L'_n - L_n$ for $n \in \mathbb{Z}$.

We say that two continuous chain maps $f, g : A \to B$ are continuously chain homotopic if there is a continuous chain homotopy $h : f \Rightarrow g$. We let $[f]$ be the continuous chain homotopy class of the continuous chain map $f : A \to B$. We call $[f]$ a continuous chain $h$-map from $A$ to $B$. We define the homotopy category of Polish chain complexes to be the category that has Polish chain complexes as objects, and continuous chain $h$-maps as morphisms. Isomorphisms in this category are called continuous chain $h$-isomorphisms.

Let $A$ be a Polish chain complex. Then one defines:

- the (closed) subgroup of $n$-cycles $Z_n(A) = \ker(\partial_n) \subseteq A_n$;
- the (Polishable) subgroup of $n$-boundaries $B_n(A) = \text{ran}(\partial_{n+1}) \subseteq A_n$;
- the (closed) subgroup of weak $n$-boundaries $\overline{B}_n(A) \subseteq Z_n(A)$ equal to the closure of $B_n(A)$ inside $A_n$.

**Definition 3.1.** The $n$-th definable homology group of $A$ is the definable group $H_n(A) = Z_n(A)/B_n(A)$. The homology $H_n(A)$ of $A$ is the graded definable group $(H_n(A))_{n \in \mathbb{Z}}$. Given $a \in Z_n(A)$, we let $[a]$ be the corresponding element of $H_n(A)$. 
We call \( H^n_w(A) = Z_n(A)/B_n(A) \) the \( n \)-th weak homology group of \( A \), and \( H^n_\infty(A) = \overline{B_n(A)}/B_n(A) \) the \( n \)-th asymptotic homology group of \( A \). Since \( B_n(A) \) is a Borel subgroup of \( Z_n(A) \), and since a countable index Borel subgroup of a Polish group is clopen, \( H^n_\infty(A) \) is either trivial or uncountable.

**Definition 3.2.** A Polish chain complex \( A \) is proper if, for every \( n \in \mathbb{Z} \), \( B_n(A) \) is a closed subgroup of \( Z_n(A) \).

Notice that, if \( A \) is proper, then \( H_n(A) = H^n_w(A) \) is a graded Polish group.

If \( f \) is a continuous chain map from \( A \) to \( B \) then, \( f \) induces a continuously-definable homomorphism \( H_n(f) \) from \( H_n(A) \) to \( H_n(B) \). Furthermore, \( H_n(f) \) only depends on the continuous chain homotopy class of \( f \).

**Definition 3.3.** The definable homology functor for Polish chain complex is the functor \( A \mapsto H_*(A) \), \( [f] \mapsto H_*(f) \) from the homotopy category of Polish chain complexes to the continuous category of graded definable groups.

3.2. **Polish cochain complexes.** As in the case of chain complexes, a Polish cochain complex is simply a cochain complex in the additive category of Polish groups, namely a sequence \( A = (A^n, \delta^n)_{n \in \mathbb{Z}} \) where \( A^n \) is a Polish group, and the codifferential \( \delta^n : A^n \to A^{n+1} \) is a continuous homomorphism. We will often omit the superscripts from the maps \( \delta^n : A^n \to A^{n+1} \). We also denote the Polish cochain complex \( A \) as above simply by \( (A^n)_{n \in \mathbb{Z}} \). A Polish cochain complex \( A \) is countable, free countable, free finitely-generated, respectively, if for every \( n \in \mathbb{Z} \), \( A^n \) is countable, free countable, free finitely-generated, respectively. A *continuous cochain map* \( f : A \to B \) is a sequence \( (f^n)_{n \in \mathbb{Z}} \) of continuous group homomorphisms \( f^n : A^n \to B^n \) such that \( \delta f^n - f^{n+1} \delta = 0 \) for every \( n \in \mathbb{Z} \). A continuous cochain homotopy from a continuous cochain map \( f : A \to B \) to \( g : A \to B \) is a sequence \( (L^n)_{n \in \mathbb{Z}} \) of continuous group homomorphisms \( L^n : A^n \to B^{n-1} \) such that \( \delta L^n + L^{n+1} \delta = g^n - f^n \) for every \( n \in \mathbb{Z} \). Two continuous cochain maps are continuously cochain homotopic if there is a continuous cochain homotopy from one to the other. A continuous cochain h-map is a continuous cochain homotopy class of continuous cochain maps. The homotopy category of Polish cochain complexes is the category that has Polish cochain complexes as objects and continuous cochain h-maps as morphisms. As in the case of Polish chain complexes, one can also define 3-cells (or, more general, \( k \)-cells) in this context. If \( f, f' : A \to B \) are continuous cochain maps, and \( L, L' : f \Rightarrow f' \) are continuous cochain homotopies, then a 3-cell \( \hat{H} : L \Rightarrow L' \) is a sequence \((H^n)_{n \in \mathbb{Z}}\) of continuous group homomorphisms \( H^n : A^n \to B^{n-2} \) such that \( \delta H^n - H^{n+1} \delta = L^n - L^n \) for every \( n \in \mathbb{Z} \).

Clearly, there is a correspondence between Polish cochain complexes and Polish chain complexes. Indeed, if \((A_n, \partial_n)_{n \in \mathbb{Z}}\) is a Polish chain complex, then by setting \( A^n := A_{-n} \) and \( \delta^n := \partial_{-n} \) one defines a Polish cochain complex. It is easily seen that this establishes an isomorphism from the strict \( \omega \)-category of Polish chain complexes to the strict \( \omega \)-category of Polish cochain complexes. This perspective allows one to define the analogue for cochain complexes of the homology groups and any other notion about chain complexes. For instance, one defines:

- the (closed) subgroup of \( n \)-cocycles \( Z^n(A) = \ker(\delta^n) \subseteq A^n \);
- the (Polishable) subgroup of \( n \)-coboundaries \( B^n(A) = \operatorname{ran}(\delta^{n-1}) \subseteq A^n \);
- the (closed) subgroup of weak \( n \)-coboundaries defined as the closure \( \overline{B^n(A)} \) of \( B^n(A) \) inside \( A^n \).

**Definition 3.4.** The \( n \)-th definable cohomology group of \( A \) is the definable group \( H^n(A) = Z^n(A)/B^n(A) \). The homology \( H^*(A) \) of \( A \) is the graded definable group \((H^n(A))_{n \in \mathbb{Z}}\). Given \( a \in Z^n(A) \), we let \([a]\) be the corresponding element of \( H^n(A) \).

By definition, the weak group \( H^n_w(A) \) of \( H^n(A) \) is the Polish group \( \overline{Z^n(A)}/B^n(A) \), which we call the \( n \)-th weak cohomology group of \( A \). The asymptotic group of \( H^n(A) \) is the group with a Polish cover \( H^n_* = \overline{B^n(A)}/B^n(A) \), which we call the \( n \)-th asymptotic cohomology group of \( A \).

**Definition 3.5.** A Polish cochain complex \( A \) is *proper* if, for every \( n \in \mathbb{Z} \), \( B^n(A) \) is a closed subgroup of \( Z^n(A) \).

**Definition 3.6.** The definable cohomology functor for Polish cochain complexes is the functor \( A \mapsto H^*(A) \), \( [f] \mapsto H^*(f) \) from the homotopy category of Polish cochain complexes to the continuous category of graded definable groups.

3.3. **Duality.** If \( A \) is a countable group, define the \( G \)-dual \( A^* \) of \( A \) to be the Polish group \( \operatorname{Hom}(A, G) \) of homomorphisms \( A \to G \), regarded as a closed subgroup of \( G^A \). We can then consider a \( G \)-valued pairing \( A^* \times A \to G, \langle \varphi, a \rangle \mapsto \varphi(a) \).
The assignment $A \mapsto A^*$ and $f \mapsto f^*$, where $f^*(\varphi) = \varphi \circ f$, defines a contravariant functor from the category of countable groups to the category of Polish groups.

**Definition 3.7.** Let $A$ be a countable cochain complex. The $G$-dual complex $A^* = \text{Hom}(A, G)$ is the Polish chain complex defined by setting

$$A^*_n := (A^n)^*$$

and

$$\partial_n := (\delta^{n-1})^* : A^*_n \to A^*_{n-1}$$

for $n \in \mathbb{Z}$.

**Remark 3.8.** Notice that, in Definition 3.7, if $A$ is a free finitely-generated cochain complex, then $A^*$ is a proper Polish chain complex since $G$ is assumed to satisfy the division closure property. This easily follows from the structure of group homomorphisms between free finitely-generated groups given by the Smith Normal Form for integer matrices [33].

Suppose that $A$ and $B$ are countable chain complexes. A cochain map $f : A \to B$ induces a continuous chain map $f^* : B^* \to A^*$ obtained by setting

$$f_n^* = (f_n^n)^* : B^*_n \to A^*_n$$

for $n \in \mathbb{Z}$. Similarly, a cochain homotopy $L : f \Rightarrow g$ induces a continuous chain homotopy $L^* : f^* \Rightarrow g^*$ by setting

$$L_n^* = (L_n^{n+1})^* : B^*_n \to A^*_{n+1}.$$  

A 3-cell $H : L \Rightarrow L'$ similarly induces a 3-cell $H^* : L'^* \Rightarrow L^*$ by setting

$$H_n^* = (H_n^{n+2})^* : B^*_n \to A^*_{n+2}.$$

**Definition 3.9.** If $A$ is a free countable chain complex, define $A \otimes G$ to be the Polish chain complex $\text{Hom}(\text{Hom}(A, Z), G)$. Notice that $A \otimes G$ is proper if $A$ is a free finitely-generated chain complex by Remark 3.8.

### 3.4. An algebraic Universal Coefficient Theorem

Suppose that $A$ is a free countable cochain complex. We let $A^*$ be the $G$-dual Polish chain complex of $A$. Consider for $n \in \mathbb{Z}$ the closed subgroups

$$\text{Ann} (Z^n(A)) = \{z \in A_n^* : \forall a \in Z^n(A), \langle z, a \rangle = 0\} \subseteq Z_n^*(A^*)$$

and

$$^\circ H_n (A^*) := \text{Ann} (Z^n(A)) / B^n(A) \subseteq H_n (A^*).$$

The pairing between $A^*$ and $A$ induces the index homomorphism $\text{Index} : H_\bullet (A^*) \to \text{Hom} (H^\bullet (A), G)$. This is the natural continuously-definable homomorphism obtained by setting $\text{Index} ([z]) ([a]) = \langle z, a \rangle$ for $[z] \in H_n (A^*)$ and $[a] \in H^n (A)$. We also define a co-index homomorphism

$$\text{coIndex} : \frac{\text{Hom} (B^{n+1}(A), G)}{\text{Hom} (Z^{n+1}(A) | B^{n+1}(A), G)} \to H_\bullet (A^*).$$

This is the natural continuously-definable homomorphism obtained by setting

$$\text{coIndex} ([f]) = [(-1)^n (f \circ \delta^n)]$$

for $n \in \mathbb{Z}$ and $f \in \text{Hom} (B^{n+1}(A), G)$.

**Lemma 3.10.** Suppose that $A$ is a free countable cochain complex, and let $A^*$ be its $G$-dual Polish chain complex.

1. The index homomorphism

$$\text{Index} : H_\bullet (A^*) \to \text{Hom} (H^\bullet (A), G)$$

is a natural continuously-definable homomorphism with kernel equal to $^\circ H_\bullet (A^*)$, and it is a split epimorphism in the continuous category of graded definable groups.

2. The co-index homomorphism

$$\text{coIndex} : \frac{\text{Hom} (B^{n+1}(A), G)}{\text{Hom} (Z^{n+1}(A) | B^{n+1}(A), G)} \to H_\bullet (A^*)$$

is a natural continuously-definable group homomorphism with image equal to $^\circ H_\bullet (A^*)$, and it is a split monomorphism in the continuous category of graded definable groups.
Proof. (1) It is easy to see that Index is a natural continuously-definable homomorphism with kernel equal to $^\circ H_\bullet (A^*)$. We now prove that Index is a split epimorphism in the continuous category of graded definable groups. Recall that $A$ is a free countable cochain complex. Since subgroups of free abelian groups are free abelian, and free abelian groups are projective, we can fix a left inverse $i: A^\circ \to Z^n(A)$ for the inclusion map $Z^n(A) \to A^n$ in the category of groups. Let also $p^n : Z^n(A) \to H^n(A)$ be the quotient map. We can then consider the continuously-definable homomorphism

$$\text{Hom}(H^n(A), G) \to H_\bullet (A^*), \; \varphi \mapsto \left[ \varphi \circ p^n \circ i^n \right]$$

for $\varphi \in \text{Hom}(H^n(A), G)$. We claim that this is a well-defined right inverse for Index. Indeed, if $n \in \mathbb{Z}$, $\varphi \in \text{Hom}(H^n(A), G)$, and $z_{\varphi} := \varphi \circ p^n \circ i^n$, then we have that, for every $a \in A^{n-1}$,

$$\langle \partial z_{\varphi}, a \rangle = \langle z_{\varphi}, \delta a \rangle = (\varphi \circ p^n)(\delta a) = 0.$$

Hence, $\partial z_{\varphi} = 0$ and $z_{\varphi} \in Z^n(A^*)$. Furthermore, we have that for $a \in Z^n(A)$,

$$\text{Index} \left( [z_{\varphi}] \right) ([a]) = \langle z_{\varphi}, a \rangle = \left( \varphi \circ p^n \circ i^n \right)(a) = \varphi([a]).$$

This shows that $\text{Index} \left( [z_{\varphi}] \right) = \varphi$, concluding the proof.

(2) It is easy to see that coIndex is a natural continuously-definable homomorphism. We now prove that coIndex is a split monomorphism in the continuous category of graded definable groups with image equal to $^\circ H_\bullet (A^*)$. For $n \in \mathbb{Z}$, consider a right inverse $\delta^n: B^{n+1}(A) \to A^n$ for the split epimorphism $\delta^n: A^n \to B^{n+1}(A)$ in the category of groups. Then we can consider the continuously-definable homomorphism

$$H_\bullet (A^*) \to \frac{\text{Hom}(B^{n+1}(A), G)}{\text{Hom}(Z^{n+1}(A), B^{n+1}(A), G)}, \; [z] \mapsto \left[ (-1)^n \left( z \circ \delta^n \right) \right]$$

for $z \in H_n (A^*)$. It is immediate to verify that this is a right inverse for coIndex. It remains to show that the image of coIndex is equal to $^\circ H_\bullet (A^*)$. If $f \in \text{Hom}(B^{n+1}(A), G)$ then we have that $(-1)^n f \circ \delta^n \in \text{Ann}(Z^n(A))$, and hence the image of coIndex is contained in $^\circ H_\bullet (A^*)$. Conversely, if $z \in \text{Ann}(Z^n(A))$, then $[z]$ is equal to the image of $\left[ (-1)^n \left( z \circ \delta^n \right) \right]$ under coIndex. Indeed, for every $a \in A^n$,

$$a - (\delta^n \circ \delta^n)(a) \in Z^n(A)$$

and hence, since $z \in \text{Ann}(Z^n(A))$,

$$z(a) = (z \circ \delta^n \circ \delta^n)(a) = \text{coIndex} \left( \left[ (-1)^n \left( z \circ \delta^n \right) \right] \right) ([a]).$$

This concludes the proof. \qed

Considering the natural definable isomorphism

$$\frac{\text{Hom}(B^{n+1}(A), G)}{\text{Hom}(Z^{n+1}(A), B^{n+1}(A), G)} \cong \text{Ext}(H^{n+1}(A), G)$$

in the continuous category of definable groups as in Proposition 2.8, we can consider the co-index homomorphism as a natural continuously-definable isomorphism

$$\text{coIndex} : \text{Ext}(H^{n+1}(A), G) \to ^\circ H_\bullet (A^*).$$

From Lemma 3.10 we immediately obtain the following.

**Theorem 3.11.** Suppose that $G$ is a Polish group with the division closure property. Let $A$ be a free countable cochain complex, and let $A^*$ be its $G$-dual Polish chain complex. Then we have a natural continuously-definable exact sequence of definable graded groups

$$0 \to \text{Ext}(H^{n+1}(A), G) \xrightarrow{\text{coIndex}} H_\bullet (A^*) \to \text{Index} \text{Hom}(H^n(A), G) \to 0$$

which continuously-definably splits, called the UCT exact sequence for $A$. 
3.5. Exact sequences of chain complexes. A (short) locally split exact sequence of Polish chain complexes \( 0 \to A \to B \to C \to 0 \) consists of Polish chain complexes \( A, B, C \) and continuous chain maps \( i : A \to B \) and \( \pi : B \to C \) such that, for every \( n \in \mathbb{Z} \), the sequence

\[
0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{\pi_n} C_n \to 0
\]

is split exact in the category of Polish groups. (Notice that this is weaker than asking that \( 0 \to A \to B \to C \to 0 \) be split exact in the category of Polish chain complexes.) The notion of continuous chain map between locally split exact sequences of Polish chain complexes is defined in the obvious way, as it is the notion of continuous chain homotopy between continuous chain maps. The (homotopy) category of locally split exact sequences of Polish chain complexes has (continuous chain homotopy classes of) continuous chain maps as arrows. The proof of the following proposition is a standard diagram-chase argument; see [35, Theorem 6.10].

**Proposition 3.12.** Let \( 0 \to A \to B \to C \to 0 \) be a locally split exact sequence of Polish chain complexes. For every \( n \in \mathbb{Z} \), let \( \pi_n^i : C_n \to B_n \) be a right inverse for \( \pi_n : B_n \to C_n \) in the category of Polish groups, and let \( i_n^\dagger : B_n \to A_n \) be right inverses for \( i_n : A_n \to B_n \) in the category of Polish groups, such that \( i_n^\dagger \pi_n^i = 0 \). Define \( d_n := (i_n^\dagger \circ \partial \circ \pi_n^i) : C_n \to A_{n-1} \). Then one has that \( d_n \) for \( n \in \mathbb{Z} \) induce a natural degree \(-1\) continuously-definable homomorphism

\[
d_\ast : H_\ast(C) \to H_{\ast-1}(A),
\]

called connecting homomorphism, which does not depend on the choice of \( \pi_n^i \) and \( i_n^\dagger \). Furthermore, \( d_n \) fits into the exact sequence

\[
\cdots \to H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(\pi)} H_n(C) \xrightarrow{d_n} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} H_{n-1}(B) \to \cdots
\]

Notice that one can equivalently state Proposition 3.12 in terms of cochain complexes.

If \( 0 \to A \to B \to C \to 0 \) is a locally split exact sequence of free countable cochain complexes, then its \( G \)-dual \( 0 \to C^* \to B^* \to A^* \to 0 \) is a locally split exact sequence of Polish chain complexes. By Proposition 3.12, we have a connecting homomorphism \( d^\bullet : H^\bullet(C) \to H^{\bullet+1}(A) \) and a connecting homomorphism \( d^\bullet_\ast : H_\ast(A^*) \to H_{\ast-1}(C^*) \). In turn, \( d^\bullet \) induces natural continuously-definable homomorphisms \( \text{Ext}(d^\bullet, G) : \text{Ext}(H^{\bullet+1}(A), G) \to \text{Ext}(H^\bullet(C), G) \) and \( \hom(d^\bullet, G) : \hom(H^\bullet(A), G) \to \hom(H^{\bullet-1}(C), G) \).

**Lemma 3.13.** Suppose that \( 0 \to A \to B \to C \to 0 \) is a locally split exact sequence of free countable cochain complexes, and \( 0 \to C^* \to B^* \to A^* \to 0 \) is its \( G \)-dual locally split exact sequence of Polish chain complexes. The diagram

\[
\begin{array}{ccc}
\text{Ext}(H^{\bullet+1}(A), G) & \xrightarrow{\text{colIndex}} & H_\ast(A^*) \\
\downarrow \text{Ext}(d^\bullet, G) & & \downarrow \text{Index} \\
\text{Ext}(H^\bullet(C), G) & \xrightarrow{\text{colIndex}} & H_{\ast-1}(C^*) \\
\end{array}
\]

commutes.

**Proof.** Fix, for \( n \in \mathbb{Z} \), a right inverse \( \pi_n^i : C_n \to B^n \) for the map \( \pi_n : B^n \to C^n \) and a left inverse \( i_n^\dagger : B^n \to A^n \) for the map \( i_n : A^n \to B^n \) in the category of groups such that \( i_n^\dagger \pi_n^i = 0 \). Define then \( i_n^\dagger := (\pi_n^i)^* : B_n^* \to C_n^* \) and \( \pi_n := (i_n)^* : B_n^* \to A_n^* \), and observe that they are left inverses in the category of Polish groups for \( i_n = (\pi_n)^* : C_n^* \to B_n^* \) and \( \pi_n = (i_n)^* : B_n^* \to A_n^* \), respectively, such that \( i_n^\dagger \pi_n^i = 0 \). Define then

\[
\hat{d}^n := i_n^{\dagger+1} \delta^n \pi_n^i : C^n \to A^{n+1}
\]

and

\[
\hat{d}_n := i_n^{\dagger} \partial_n \pi_n^i : A_n^* \to C_{n-1}^*.
\]

By definition, we have that \( \hat{d}_n = (\hat{d}^{n-1})^* \), which easily gives commutativity of the right square.

Via the natural isomorphisms as in Proposition 2.8 for \( H^{n+1}(A) = \mathbb{Z}^{n+1}(A)/\mathbb{B}^{n+1}(A) \) and \( H^\bullet(C) = \mathbb{Z}^\bullet(C)/\mathbb{B}^\bullet(C) \), the map \( \text{Ext}(d^\bullet, G) \) corresponds to the map

\[
\frac{\hom(B^{n+1}(A), G)}{\hom(Z^{n+1}(A)/B^{n+1}(A), G)} \to \frac{\hom(B^\bullet(C), G)}{\hom(Z^\bullet(C)/B^\bullet(C), G)}
\]
holds. Indeed, 
\[ (-1)^n \delta^n \hat{a}^{n-1} = (-1)^{n-1} \hat{d}^n \delta^{n-1} \]

This concludes the proof. \(\square\)

As an immediate consequence of Lemma 3.13, we obtain the following.

**Theorem 3.14.** Let \( G \) be a Polish group with the division closure property. Suppose that \( 0 \to A \to B \to C \to 0 \) is a locally split exact sequence of free countable cochain complexes, and \( 0 \to C^* \to B^* \to A^* \to 0 \) is its \( G \)-dual locally split exact sequence of Polish chain complexes. Then the connecting homomorphisms for \( 0 \to A \to B \to C \to 0 \) and \( 0 \to C^* \to B^* \to A^* \to 0 \) induce a natural degree \(-1\) homomorphism from the UCT exact sequence for \( A \) to the UCT exact sequence for \( C \).

### 4. Towers of Polish chain complexes

In this section, we present the coherent category of towers of Polish chain complexes from [22, Section 3], and the notion of homotopy limit of a tower. This category can be seen as the algebraic counterpart of the coherent category of towers of topological spaces from strong shape theory, which underpins the approach to Steenrod homology in that context; see [25]. Recall that we let \( G \) denote a Polish group with the division closure property.

#### 4.1. Morphisms.

A tower of Polish chain complexes is an inverse sequence \( A = (A^{(m)}, p^{(m,m+1)}) \) of Polish chain complexes and continuous chain maps \( p^{(m,m+1)} : A^{(m+1)} \to A^{(m)} \). We then define \( p^{(m_0,m_1)} \) to be the continuous chain map \( p^{(m_0,m_0+1)} \circ \cdots \circ p^{(m_1-1,m_1)} \) from \( A^{(m_1)} \) to \( A^{(m_0)} \) for \( m_0 < m_1 \), and \( p^{(m_0,m_0)} = 1_{A^{(m_0)}} \). We now recall the notion of coherent morphisms between towers of Polish chain complexes.

**Definition 4.1.** Let \( A = (A^{(m)}, p^{(m,m+1)}) \) and \( B = (B^{(k)}, p^{(k,k+1)}) \) be towers of Polish chain complexes. A 1-cell from \( A \) to \( B \) is given by a sequence \( f = (m_k, f^{(k)}, f^{(k,k+1)}) \) such that:

- \( (m_k) \) is an increasing sequence in \( \omega \),
- \( f^{(k)} : A^{(m_k)} \to B^{(k)} \) is a continuous chain map from \( A^{(m_k)} \) to \( B^{(k)} \), and
- \( f^{(k,k+1)} : p^{(k,k+1)} f^{(k)} \to f^{(k+1)} p^{(m_{k+1},m_k+1)} \) is a continuous chain homotopy.

One then defines for \( k_0 \leq k_1 \) the continuous chain homotopy
\[ f^{(k_0,k_1)} : p^{(k_0,k_1)} f^{(k_1)} \Rightarrow f^{(k_0)} p^{(m_{k_0},m_{k_1})} \]
by setting
\[ f^{(k_0,k_0)} = 1_{f^{(k_0)}} \]
and, for \( k_0 < k_1 \),
\[ f^{(k_0,k_1)} = \sum_{k=k_0}^{k_1-1} p^{(k_0,k)} f^{(k,k+1)} p^{(m_{k+1},m_{k_1})} \]

One can also define \( f^{(k_0,k_1)} \) recursively by setting
\[ f^{(k_0,k_1+1)} = f^{(k_0,k_1)} p^{(m_{k_1},m_{k_1+1})} \circ \hat{d}_{f^{(k_0,k_1)}} f^{(k_1,k_1+1)}. \]

The identity 1-cell \( 1_A \) of \( A \) is the 1-cell \( (m_k, f^{(k)}, f^{(k,k+1)}) \) where \( m_k = k, f^{(k)} = 1_{A^{(k)}} \), and \( f^{(k,k+1)} = 1_{p^{(k,k+1)}} \).
We now define composition of 1-cells.

Definition 4.2. Suppose that $A$, $B$, and $C$ are towers of Polish chain complexes, and $f = (m_k, f(\cdot)^k, f(\cdot,k+1))$ and $g = (k_t, g(\cdot)^t, g(\cdot,t+1))$ are 1-cells from $A$ to $B$ and from $B$ to $C$, respectively. We define

$gf = g \circ_0 f = \left( m_k, g(\cdot)^t f(\cdot,k_t), g(\cdot) f(\cdot,k_t+1) \circ_1 g(\cdot,t+1) f(\cdot,k_t+1) \right)$.

Remark 4.3. Composition of 1-cells is not associative. However, it induces a well-defined associative operation on the set of equivalence classes of 1-cells, where we identify 1-cells that are equal up to 2-cells. This leads us to consider the following notion of 2-cell between 1-cells.

Definition 4.4. Let $A = (A^{(m)}, p^{(m)})$ and $B = (B^{(k)}, p^{(k)})$ be towers of Polish chain complexes. Let $f = (m_k, f(\cdot)^k, f(\cdot,k+1))$ and $f' = (m_k', f'(\cdot)^k, f'(\cdot,k+1))$ be 1-cells from $A$ to $B$. A 2-cell from $f$ to $f'$ is a sequence $L = (\tilde{m}_k, L^{(k)}, L^{(k,k+1)})$ such that:

- $(\tilde{m}_k)$ is an increasing sequence in $\omega$ such that $\tilde{m}_k \geq \max \{m_k, m_k'\}$,
- $L^{(k)} : f(\cdot)^k p^{(m\cdot,\tilde{m}_k)} \Rightarrow f'(\cdot)^k p^{(m\cdot,\tilde{m}_k)}$ is a continuous chain homotopy,
- for every $k \in \omega$, $L^{(k,k+1)} : f'(\cdot,k+1)^k p^{(m\cdot,k+1,\tilde{m}_k+1)} \circ_1 p^{(k+1)}(\cdot)^k L^{(k,k+1)} \Rightarrow L^{(k)} p^{(\cdot,\tilde{m}_k,k+1)}(\cdot)^k f(\cdot,k+1)^k p^{(m\cdot+1,\tilde{m}_k+1)}$ is a 3-cell.

Let $A$ and $B$ be two towers of Polish chain complexes. We say that two 1-cells $f, f' : A \to B$ represent the same coherent morphism if there exists a 2-cell $f \Rightarrow f'$. This defines an equivalence relation on the set of 1-cells from $A$ to $B$. Given a 1-cell $f$ we let $[f]$ be its equivalence class, and we call $[f]$ the coherent morphism from $A$ to $B$ represented by $f$. Setting $[g] \circ [f] = [gf]$ gives a well-defined associative operation between coherent morphisms. The identity morphism of $A$ is $[1_A]$ where $1_A$ is the identity 1-cell of $A$. This yields a category which we call the coherent category of towers of Polish chain complexes; see [22, Theorem 3.6].

4.2. The homotopy limit of a tower. We associate as in [25, Section 17] with a tower of Polish chain complexes $A$ a Polish chain complex $\operatorname{holim} A$, called its homotopy limit; see also [22, Section 4] and [8]. For $n \in \mathbb{Z}$, define $C_n(A) = (\operatorname{holim} A)_n$ to be the closed subgroup of

\[
\prod_{m \in \omega} A_n^{(m)} \oplus \prod_{m_0 \leq m_1} A_{n+1}^{(m_0)}
\]

consisting of those elements $z = (z_m, z_{m_0,m_1})$ such that

$z_{m_0,m_1} + p_n^{(m_0,m_1)}(z_{m_1,m_2}) = z_{m_0,m_2}$

for every $m_0 \leq m_1 \leq m_2$. The differential $d_n : C_{n+1}(A) \to C_n(A)$ is defined by setting, for $z \in C_n(A)$,

$$(d_n z)_m = \partial_n z_m$$

and

$$(d_n z)_{m_0,m_1} = \partial_n z_{m_0,m_1} + (-1)^n (p_n^{(m_0,m_1)}(z_{m_1}) - z_{m_0})$$

for every $m \in \omega$ and $m_0 \leq m_1$. This defines a Polish chain complex

$$\cdots \to C_{n+1}(A) \xrightarrow{d_{n+1}} C_n(A) \xrightarrow{d_n} C_{n-1}(A) \to \cdots$$

which we denote by $\operatorname{holim} A$.

Remark 4.5. If $z \in C_n(A)$, then one has that

$z_{m_0,m_0} = 0$

and

$z_{m_0,m_1} = \sum_{m=m_0}^{m_1-1} p_n^{(m_0,m)}(z_{m,m+1})$.
for every $m_0 < m_1$. Thus the values $z_{m, m+1}$ for $m \in \omega$ determine the values $z_{m_0, m_1}$ for every $m_0 \leq m_1$.

Suppose that $A$ and $B$ are towers of Polish chain complexes, and $f = (m_k, f^{(k)}, f^{(k, k+1)})$ is a 1-cell from $A$ to $B$. We define a continuous chain map $f^{(\infty)} = \text{holim} f$ from $\text{holim} A$ to $\text{holim} B$, called the (homotopy) limit of $f$, as follows. For $n \in \mathbb{Z}$, $z \in C_n(A)$, we define $f^{(\infty)}_n(z) \in C_n(B)$, by setting

$$f^{(\infty)}_n(z)_k = f^{(k)}_n(z_{m_k})$$

and

$$f^{(\infty)}_n(z)_{k, k+1} = f^{(k)}_n(z_{m_k, m_{k+1}}) + (-1)^n f^{(k, k+1)}_n(z_{m_{k+1}}).$$

for every $k \in \omega$.

One can show that the continuous chain homotopy class of $\text{holim} f$ only depends on the coherent morphism represented by $f$. Furthermore, the assignment $A \mapsto \text{holim} A$, $[f] \mapsto \text{holim} [f] := [\text{holim} f]$ yields a functor from the coherent category of towers of Polish chain complexes to the continuous category of graded definable groups; see [22, Theorem 4.2]. Thus, the assignment $A \mapsto H_*(\text{holim} A)$ is a functor from the coherent category of towers of Polish chain complexes to the continuous category of graded definable groups.

Recall that a Polish chain complex $A$ is proper if, for every $n \in \mathbb{Z}$, $B_n(A)$ is a closed subset of $A_n$. This implies that $H_*(A)$ is a graded Polish group. Suppose that $A = (A^{(k)})$ is a tower of proper Polish chain complexes. For every $k \in \omega$, we have a continuous cochain map

$$\text{holim} A \to A^{(k)}, z \mapsto z_k.$$  

This induces a continuously-definable homomorphism

$$H_*(\text{holim} A) \to H_*(A^{(k)}).$$

These homomorphisms for $k \in \omega$ induce a natural continuously-definable homomorphism

$$\text{Local} : H_*(\text{holim} A) \to \prod_{k \in \omega} H_*(A^{(k)}).$$

**Lemma 4.6.** Let $A = (A^{(k)})$ be a tower of proper Polish chain complexes. Then the image of the natural continuously-definable homomorphism

$$\text{Local} : H_*(\text{holim} A) \to \prod_{k \in \omega} H_*(A^{(k)})$$

is equal to the (inverse) limit $\text{holim} H_*(A^{(k)})$, while the kernel is equal to $H_\infty^*(\text{holim} A)$.

**Proof.** The first assertion is easy to prove, so we just prove the second assertion. We have that the kernel of $\text{Local}$ is a closed subgroup of $H_*(\text{holim} A)$. Thus, it suffices to prove that the kernel of $\text{Local}$ is contained in $H_\infty^*(\text{holim} A)$. Suppose that $z \in \mathbb{Z}_n(\text{holim} A)$ is such that $[z]$ belongs to the kernel of $\text{Local}$. Then we have that $z_{m} \in B_n(A^{(m)})$ for $m \in \omega$.

Fix $m_0 \in \omega$. Set $w_k = 0$ for $k > m_0$ and $w_{k, k+1} = 0$ for every $k \in \omega$. Since $z_{m_0} \in B_n(A^{(m_0)})$ we can choose $w_{m_0} \in A^{(m_0)}_{n+1}$ such that $\partial w_{m_0} = z_{m_0}$. We then define recursively $w_k$ for $0 \leq k < m_0$ by setting

$$w_k := (-1)^n z_{k, k+1} + p^{(k, k+1)}_{n+1} (w_{k+1}) \in A^{(k)}_{n+1}.$$ 

This defines an element $w \in C_{n+1}(A)$. We show by induction that, for $0 \leq k \leq m_0$, $(d_{n+1} w)_k = \partial w_k = z_k$. For $k = m_0$ this holds for the choice of $w_{m_0}$. Suppose that $0 \leq k < m_0$ is such that the conclusion holds for $k + 1$. Then we have that

$$(d_{n+1} w)_k = \partial w_k = \partial (-1)^n z_{k, k+1} + p^{(k, k+1)}_{n+1} (w_{k+1})$$

$$= z_{k} - p^{(k, k+1)}_{n+1} (z_{k+1}) + p^{(k, k+1)}_{n+1} (z_{k+1}) = z_k.$$

Fix now $0 \leq k < m_0$. Then we have that

$$(d_{n+1} w)_{k, k+1} = \partial w_{k, k+1} + (-1)^{n+1} p^{(k, k+1)}_{n+1} (w_{k+1}) - w_k$$

$$= (-1)^{n+1} p^{(k, k+1)}_{n+1} (w_{k+1}) - (-1)^n z_{k, k+1} + p^{(k, k+1)}_{n+1} (w_{k+1}))$$

$$= z_{k, k+1}.$$
Since \( m \in \omega \) is arbitrary, this concludes the proof that \( \varepsilon \in B_\omega (\text{holim} A) \) and \( [\varepsilon] \in H^\infty (\text{holim} A) \).

**Remark 4.7.** Although we will not need this fact, one can similarly show that, if \( A = (A^{(k)}) \) is a tower of proper Polish chain complexes, then \( H^\infty (\text{holim} A) \) is naturally isomorphic in the category of definable graded groups to \( \lim^1 H^\infty (A^{(k)}) \). Here, the \( \lim^1 \) of a tower of Polish groups is regarded as a group with a Polish cover as in [4, Section 5].

### 5. Inductive Sequences of Countable Cochain Complexes

In this section we introduce the coherent category of inductive sequences of cochain complexes, and the homotopy colimit functor. Via the obvious duality between inductive sequences of cochain complexes and towers of chain complexes, homotopy colimits correspond to homotopy limits. This allows one to obtain from Theorem 3.11 a definable Universal Coefficient Theorem relating the homology of the homotopy colimit of an inductive sequence to complexes, homotopy colimits correspond to homotopy limits. This allows one to obtain from Theorem 3.11 a definable Universal Coefficient Theorem relating the homology of the homotopy colimit of an inductive sequence to the cohomology of the homotopy limit of its dual tower; see Theorem 5.5.

#### 5.1. Morphisms

An *inductive sequence* of cochain complexes is a sequence \( A = (A_\ell, \eta_{\ell+1, \ell}) \) of cochain complexes, and cochain maps \( \eta_{\ell+1, \ell} : A_\ell \to A_{\ell+1} \). Then we set \( \eta_{\ell_0, \ell_0} = 1_{A_{\ell_0}} \) and \( \eta_{\ell_0, \ell_1} = \eta_{\ell_0, \ell_1-1} \circ \cdots \circ \eta_{\ell_0+1, \ell_0} : A_{\ell_0} \to A_{\ell_1} \) for \( \ell_0 < \ell_1 \). The notion of 1-cell between inductive sequences is the natural dual version of the notion of 1-cell between towers of chain complexes; see Section 4.1.

**Definition 5.1.** Let \( A = (A_\ell, \eta_{\ell+1, \ell}) \) and \( B = (B_\ell, \eta_{\ell+1, \ell}) \) be inductive sequences of countable cochain complexes. A 1-cell from \( A \) to \( B \) is given by a sequence \( f = (t_\ell, f_\ell, f'_{\ell+1, \ell}) \) such that:

- \( (t_\ell) \) is an increasing sequence in \( \omega \),
- \( f_\ell : A_\ell \to B_{t_\ell} \) is a cochain map from \( A_\ell \) to \( B_{t_\ell} \) for every \( \ell \in \omega \), and
- \( f'_{\ell+1, \ell} : f_{\ell+1} \eta_{\ell+1, \ell} \circ f_{\ell}(\ell) \) is a cochain homotopy.

One then defines \( f_{\ell+1, \ell_0} = f_{\ell+1, \ell_1} \eta_{\ell_1, \ell_0} \circ f_{\ell}(\ell_1) \eta_{\ell_1, \ell_0} \circ f_{\ell}(\ell_0) \) for every \( \ell_0 \leq \ell_1 \) recursively by setting \( f_{\ell, \ell} = 1_{f_{\ell}} \) for \( \ell \in \omega \) and

\[
\eta_{\ell_1+1, \ell_0} = f_{\ell_1+1, \ell_1} \eta_{\ell_1, \ell_0} \circ f_{\ell_1}(\ell_1) \eta_{\ell_1, \ell_0} \circ f_{\ell_0}(\ell_0)
\]

The identity 1-cell of \( A \) is the 1-cell \( (t_\ell, f_\ell, f'_{\ell+1, \ell}) \) where \( t_\ell = \ell, f_\ell = 1_{A_\ell} \) and \( f'_{\ell+1, \ell} = 1_{\eta_{\ell+1, \ell}} \).

In a similar fashion, one can define the notion of 2-cell between 1-cells.

**Definition 5.2.** Let \( A = (A_\ell, \eta_{\ell+1, \ell}) \) and \( B = (B_\ell, \eta_{\ell+1, \ell}) \) be inductive sequences of countable cochain complexes. Let \( f = (t_\ell, f_\ell, f'_{\ell+1, \ell}) \) and \( f' = (t'_\ell, f'_\ell, f'_{\ell+1, \ell}) \) be 1-cells from \( A \) to \( B \). A 2-cell from \( f \) to \( f' \) is a sequence \( L = (\hat{t}_\ell, L_\ell) \) such that:

- \( (\hat{t}_\ell) \) is an increasing sequence in \( \omega \) such that \( \hat{t}_\ell \geq \max \{ t_\ell, t'_\ell \} \) for \( \ell \in \omega \),
- \( L_\ell : \eta_{\hat{t}_\ell, t_\ell} f_\ell \Rightarrow \eta_{\hat{t}_\ell, t_\ell} f'_\ell \) is a cochain homotopy, and
- for every \( \ell \in \omega \),

\[
L_{\ell+1, \ell} : \eta_{\hat{t}_{\ell+1}, \hat{t}_\ell} L_\ell \circ_1 \eta_{\hat{t}_{\ell+1}, t_{\ell+1}} f_{\ell+1, \ell} \Rightarrow \eta_{\hat{t}_{\ell+1}, t_{\ell+1}} f'_{\ell+1, \ell} \circ_1 L_{\ell+1, \ell} \eta_{\ell+1, \ell}
\]

is a 3-cell.

We define composition of 1-cells as follows.

**Definition 5.3.** Let \( A, B, C \) be inductive sequences of countable cochain complexes. Let \( f = (t_\ell, f_\ell, f'_{\ell+1, \ell}) : A \to B \) and \( g = (r_\ell, g_\ell, g'_{\ell+1, \ell}) : B \to C \) be 1-cells. We define their composition \( g f = g \circ_0 f \) to be the 1-cell \( (r_\ell, g_\ell f_\ell, g_\ell f'_{\ell+1, \ell}) \) such that:

\[
L_{\ell+1, \ell} : \eta_{\hat{t}_{\ell+1}, \hat{t}_\ell} L_\ell \circ_1 \eta_{\hat{t}_{\ell+1}, t_{\ell+1}} f_{\ell+1, \ell} \Rightarrow \eta_{\hat{t}_{\ell+1}, t_{\ell+1}} f'_{\ell+1, \ell} \circ_1 L_{\ell+1, \ell} \eta_{\ell+1, \ell}.
\]

Letting two 1-cells \( f, g : A \to B \) between inductive sequences of countable cochain complexes be equivalent if there exists a 2-cell \( L : f \Rightarrow g \) defines an equivalence relation. Denoting by \( [f] \) the equivalence class of \( f \), and defining \( [g] \circ [f] = [g f] \), one obtains a category with inductive sequences of countable cochain complexes as objects and equivalence classes of 1-cells as arrows. We refer to this category as the *coherent category* of inductive sequences of countable cochain complexes, and to its arrows as *coherent morphisms*.

If \( A = (A_\ell, \eta_{\ell+1, \ell}) \) is an inductive sequence of free finitely-generated cochain complexes, we let \( A^* \) be the tower of proper Polish chain complexes \( (A^*_\ell, \eta^*_{\ell+1, \ell}) \), where \( A^*_\ell \) is the \( G \)-dual Polish chain complex of \( A_\ell \). We call \( A^* \)
the $G$-dual tower of $A$. The assignment $A \mapsto A^*$ defines a functor from the coherent category of inductive sequences of finitely-generated chain complexes to the coherent category of towers of the proper Polish chain complexes.

5.2. Homotopy colimits. We associate with an inductive sequence $A$ of countable cochain complexes a countable cochain complex $\text{hocolim} A$, called the \textit{homotopy colimit} of $A$. For $n \in \mathbb{Z}$, define $C^n (A) = (\text{hocolim} A)^n$ to be the countable group

$$
\bigoplus_{m \in \omega} A^{n+1}_{(m)} \oplus \bigoplus_{k \in \omega} A^n_{(k)}.
$$

For $\ell \in \omega$ and $z \in A^n_{(\ell)}$, we let $z e_\ell \in C^n (A)$ be the element where all the coordinates are 0 apart from the one indexed by $k = \ell$. If $\ell \in \omega$ and $z \in A^{n+1}_{(\ell)}$, then we let $z e_{\ell, \ell+1} \in C^n (A)$ be the element where all the coordinates are 0 apart from the one indexed by $m = \ell$. We also set $e_{\ell, 0} = 0$ and, for $\ell_0 < \ell_1$, and $z \in A^{n+1}_{(\ell_0)}$:

$$
z e_{\ell_0, \ell_1} := \sum_{\ell = \ell_0}^{\ell_1} \eta(\ell, \ell_0) (z) e_{\ell, \ell_1}.
$$

We define the codifferential $d^n : C^n (A) \to C^{n+1} (A)$ to be the group homomorphism such that, for $\ell \in \omega$, $z \in A^n_{(\ell)}$, and $w \in A^{n+1}_{(\ell)}$,

$$
d^n (z e_\ell) = (\delta^n z) e_\ell
$$

and

$$
d^n (w e_{\ell, \ell+1}) = (\delta^{n+1} w) e_{\ell, \ell+1} + (-1)^n (w e_\ell - \eta(\ell+1, \ell) (w) e_{\ell+1}).
$$

Notice that this definition implies that, for $\ell_0 < \ell_1$, and $w \in A^{n+1}_{(\ell_0)}$,

$$
d^n (w e_{\ell_0, \ell_1}) = (\delta^{n+1} w) e_{\ell_0, \ell_1} + (-1)^n (w e_{\ell_0} - \eta(\ell_1, \ell_0) (w) e_{\ell_1}).
$$

We define $\text{hocolim} A$ to be the countable cochain complex

$$
\cdots \to C^{n-1} (A) \xrightarrow{d^{n-1}} C^n (A) \xrightarrow{d^n} C^{n+1} (A) \to \cdots
$$

Suppose now that $A$ and $B$ are inductive sequences of countable cochain complexes, and $f = (f_\ell, f_{(\ell, \ell+1)})$ is a 1-cell from $A$ to $B$. We define a cochain map $f_{(\infty)} : = \text{hocolim} (f)$ from $\text{hocolim} A$ to $\text{hocolim} B$, as follows. For $n \in \mathbb{Z}$, $\ell \in \omega$, $z \in A^n_{(\ell)}$, and $w \in A^{n+1}_{(\ell)}$, we define

$$
f^n_{(\infty)} (z e_\ell) = f^n_{(\ell)} (z) e_\ell
$$

and

$$
f^n_{(\infty)} (w e_{\ell, \ell+1}) = f^n_{(\ell)} (w) e_{\ell, \ell+1} + (-1)^n f^{n+1}_{(\ell+1, \ell)} (w) e_{\ell+1}.
$$

This defines a cochain map $f_{(\infty)} : \text{hocolim} A \Rightarrow \text{hocolim} B$. One can show that the cochain homotopy class $[f] := [\text{hocolim} f]$ of $\text{hocolim} f$ only depends on the coherent morphism $[f]$ represented by $f$. Furthermore, the assignment $A \mapsto \text{hocolim} A$, $[f] \mapsto [\text{hocolim} f]$ defines a functor from the coherent category of inductive limits of countable cochain complexes to the homotopy category of countable cochain complexes.

Suppose that $A$ is an inductive sequence of countable cochain complexes. Then for every $\ell \in \omega$ we have a cochain map

$$
A_{(\ell)} \to \text{hocolim} A, z \mapsto z e_\ell.
$$

This induces a homomorphism

$$
H^\bullet (A_{(\ell)}) \to H^\bullet (\text{hocolim} A).
$$

Such homomorphisms for $\ell \in \omega$ induce a natural homomorphism

$$
\text{colim}_\ell H^\bullet (A_{(\ell)}) \to H^\bullet (\text{hocolim} A),
$$

which can be seen to be an isomorphism. We thus have the following.

**Proposition 5.4.** Suppose that $A$ is an inductive sequence of countable cochain complexes. Then $H^\bullet (\text{hocolim} A)$ and $\text{colim}_\ell H^\bullet (A_{(\ell)})$ are naturally isomorphic.
5.3. Duality. We now observe that the construction of the homotopy colimit is dual to the construction of the homotopy limit. Suppose that $A = (A_{(t)}, \eta_{(t+1,t)})$ is an inductive sequence of free finitely-generated cochain complexes. Recall that we let $A^* = (A^*_{(m)}, \eta^*_{(m+1,m)})$ be its $G$-dual tower of proper Polish chain complexes. For every $n \in \mathbb{Z}$, we have a pairing

$$C_n(A^*) \times C^n(A) \to G$$

defined by

$$\langle z, a e_\ell \rangle = \langle z_\ell, a \rangle$$
$$\langle z, b e_{\ell+1} \rangle = \langle z_{\ell+1}, b \rangle$$

for $z \in C_n(A^*)$, $\ell \in \omega$, $a \in A^n_{(\ell)}$, and $b \in A^{n+1}_{(\ell+1)}$. This pairing establishes a natural isomorphism

$$C_n(A^*) \cong C^n(A)^*, \quad z \mapsto \langle z, \cdot \rangle.$$

It is easy to verify that, via these isomorphisms, the differential $d_n : C_n(A^*) \to C_n(A^*)$ corresponds to the dual $(d^n)^* : C^n(A) \to C^{n-1}(A)$. Thus, such isomorphisms for $n \in \mathbb{Z}$ induces a natural isomorphism of Polish chain complexes

$$\text{holim}(A^*) \cong (\text{hocolim}A)^*.$$

We can therefore infer from Theorem 3.11 the following result. Recall the definition of the index and co-index homomorphisms as in Section 3.4, and the homomorphism Local as in Section 4.2.

**Theorem 5.5.** Let $G$ be a Polish group with the division closure property. Suppose that $A$ is an inductive sequence of free finitely-generated cochain complexes, and let $A^*$ be its $G$-dual tower of proper Polish chain complexes. Define $A := \text{hocolim}A$. Then there is a natural continuously definable exact sequence

$$0 \to \text{Ext}(H^{*+1}(A), G) \xrightarrow{\text{coIndex}} H_*(\text{holim}(A^*); G) \xrightarrow{\text{Index}} \text{Hom}(H^*(A), G) \to 0$$

which continuously-definably splits, and a natural continuously definable exact sequence

$$0 \to \text{PExt}(H^{*+1}(A), G) \xrightarrow{\text{coIndex}} H_*(\text{holim}(A^*)) \xrightarrow{\text{Local}} \lim_{\ell} H_*(A^*_{(\ell)}) \to 0$$

where $\text{coIndex}_\infty : \text{PExt}(H^{*+1}(A), G) \xrightarrow{\sim} H_\infty^*(\text{holim}(A^*))$ is the restriction of $\text{coIndex}$.

**Proof.** By the above remarks, we can identify $\text{holim}(A^*)$ with $A^*$. The first exact sequence is simply the UCT exact sequence for $A$ as in Theorem 3.11. It remains to prove the second assertion. Since $\text{coIndex}$ is continuously-definable, it restricts to a homomorphism $\text{coIndex}_\infty : \text{PExt}(H^{*+1}(A), G) \to H_\infty^*(A^*)$. By Lemma 4.6,

$$\text{Local} : H_*\! (A^*) \to \lim_{\ell} H_*(A^*_{(\ell)})$$

is surjective with kernel $H_\infty^*(A^*)$. It thus remains to prove that the image of $\text{coIndex}_\infty$ contains $H_\infty^*(A^*)$.

For $\ell \in \omega$, the canonical cochain map $A_{(\ell)} \to A$ induces a continuous chain map $A^* \to A^*_{(\ell)}$ and a continuously-definable homomorphism

$$\text{Ext}(H^*(A), G) \to \text{Ext}(H^*(A_{(\ell)}), G).$$

Consider the co-index isomorphism

$$\text{coIndex}_{(\ell)} : \text{Ext}(H^*(A_{(\ell)}), G) \to H_*(A_{(\ell)}^*)$$

associated with $A_{(\ell)}$. By naturality, the following diagram commutes

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Ext}(H^*(A), G) \xrightarrow{\text{coIndex}} H_*(A^*) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ext}(H^*(A_{(\ell)}), G) \xrightarrow{\text{coIndex}_{(\ell)}} H_*(A^*_{(\ell)})
\end{array}$$

We have that

$$H_\infty^*(A^*) = \text{Ker}(H_*(A) \xrightarrow{\text{Local}} \lim_{\ell} H_*(A^*_{(\ell)})) = \bigcap_{\ell \in \omega} \text{Ker}(H_*(A^*) \to H_*(A^*_{(\ell)})).$$
Furthermore, since $H^\bullet(A_\ell)$ is finitely-generated for every $\ell \in \omega$, we have by definition that

$$\text{PExt} (H^\bullet(A), G) = \bigcap_{\ell \in \omega} \text{Ker} \left( \text{Ext} (H^\bullet(A_\ell), G) \to \text{Ext} (H^\bullet(A_{\ell + 1}), G) \right).$$

This shows that the image of coIndex_{\infty} contains to $H^\bullet_{\infty}(A^\bullet)$, concluding the proof. □

**Remark 5.6.** In view of Remark 4.7, the second continuously-definable exact sequence in Theorem 5.5 can be seen as the Milnor exact sequence

$$0 \to \lim^1 H^\bullet_{\kappa + 1}(A^\bullet_\ell) \to H^\bullet_{\kappa}(\text{holim} A^\bullet) \to \lim H^\bullet_{\kappa}(A^\bullet_\ell) \to 0;$$

see [26, Theorem 5]. Furthermore, $\lim^1 H^\bullet_{\kappa + 1}(A^\bullet_\ell)$ is naturally definably isomorphic to $\lim^1 \text{Hom} (H^\bullet_{\kappa + 1}(A_\ell), G)$. This follows from the (definable) six-term exact sequence relating lim and $\lim^1$ [36, Proposition 3.3(5)] applied to the UCT exact sequences for $A^\bullet_\ell$ for $\ell \in \omega$, considering that $\lim^1 \text{Ext} (H^\bullet(A_\ell), G) = 0$ by Roos’ theorem [36, Theorem 6.2].

6. **Simplicial complexes**

In this section we recall the combinatorial notion of simplicial complex, simplicial map, and simplicial carrier as can be found in [12]. We then describe the classical homological invariants of simplicial complexes, including their chain complexes and homology groups, and a correspondence between simplicial carriers and chain maps.

6.1. **Simplicial complexes and carriers.** A simplicial complex $K$ is a family of nonempty finite sets that is closed downwards, i.e., $\sigma \subseteq \tau \in K \Rightarrow \sigma \in K$. A simplex of $K$ is any element $\sigma \in K$. If $\sigma, \sigma'$ are simplices of $K$ such that $\sigma \subseteq \sigma'$, then $\sigma$ is a face of $\sigma'$. A vertex of $K$ is any element $v$ of $\text{dom}(K) := \bigcup \{K\}$. A subcomplex $K'$ of a simplicial complex $K$ is a subfamily of $K$ that is itself a simplicial complex. A subcomplex $K'$ of $K$ is full if its simplices are precisely the simplices $\sigma$ of $K$ such that $\sigma \subseteq \text{dom}(K')$. We identify a simplex $\sigma$ of $K$ with the full subcomplex of $K$ whose simplices are the faces of $\sigma$. Let $v$ be a vertex of $K$. The (closed) star $\text{St}_K(v)$ of $K$ is the subcomplex of $K$ consisting of the simplices $\sigma$ of $K$ such that $\sigma \cup \{v\}$ is a simplex of $K$. Notice that, if $\tau$ is a simplex of $K$ and $v$ is a vertex of $\tau$, then $\tau = \text{St}_\tau(v)$.

Let $K, L$ be two simplicial complexes. A simplicial map $f : K \to L$ is any function $f : \text{dom}(K) \to \text{dom}(L)$ so that $\{f(v_0), \ldots, f(v_n)\}$ a simplex of $L$ for every simplex $\{v_0, \ldots, v_n\}$ of $K$. A carrier $\kappa : K \to L$ is a function $\kappa$ from simplices of $K$ to subcomplexes of $L$ with the property that, whenever $\sigma, \sigma'$ are simplices of $K$ such that $\sigma \subseteq \sigma'$, one has that $\kappa(\sigma) \subseteq \kappa(\sigma')$ [12, Chapter VI, Definition 5.5]. A star carrier is a carrier $\kappa : K \to L$ such that for every simplex $\sigma$ of $K$ there is a vertex $w(\sigma)$ of $\kappa(\sigma)$ such that $\text{St}_{\kappa(\sigma)}(w(\sigma)) = \kappa(\sigma)$. In this case, we call $\sigma \mapsto w(\sigma)$ a choice function for $\kappa$. A simplicial carrier is a carrier $\kappa$ such that, for every simplex $\sigma$ of $K$, $\kappa(\sigma)$ is a simplex of $L$. Every simplicial carrier is, in particular, a star carrier. We identify a simplicial map $f$ with the simplicial carrier $\{v_0, \ldots, v_n\} \mapsto \{f(v_0), \ldots, f(v_n)\}$. We define an order relation among carriers by setting $\kappa \leq \kappa'$ if and only if $\kappa(\sigma) \subseteq \kappa'(\sigma)$ for every simplex $\sigma$. We let the category of simplicial complexes have simplicial complexes as objects and simplicial carriers as arrows, with composition defined as follows.

**Definition 6.1.** Suppose that $K$ and $L$ are simplicial complexes. The identity simplicial carrier $1_K : K \to K$ is the identity function on the set of simplices of $K$. If $\kappa : K \to L$ and $\tau : L \to T$ are simplicial carriers, then their composition $\tau \kappa = \tau \circ \kappa$ is defined by

$$(\tau \kappa)(\sigma) = \tau(\kappa(\sigma))$$

for every simplex $\sigma$ of $K$.

A simplicial complex is *finite* if it has finitely many vertices, *countable* if it has countably many vertices, and *locally finite* if every vertex belongs to finitely many simplices. In this paper, we will assume all the simplicial complexes to be countable and locally finite.

One can associate with a (countable, locally finite) simplicial complex $K$ a locally compact Polish space, called its geometric realization, as follows. Let $\Xi$ be the separable Hilbert space with orthonormal basis $(e_v)_{v \in \text{dom}(K)}$ indexed by the set of vertices of $K$. For each simplex $\sigma = \{v_0, \ldots, v_n\}$ of $K$ define

$$|\sigma| = \{t_0 e_{v_0} + \cdots + t_n e_{v_n} : t_0, \ldots, t_n \in [0,1], t_0 + \cdots + t_n = 1\} \subseteq \Xi.$$
Then $|K|$ is defined to be the closed subspace of $\Xi$ obtained as the union of $|\sigma|$ where $\sigma$ ranges among the simplices of $K$. A polyhedron is a locally compact Polish space that is obtained in this fashion from some simplicial complex [32, Chapter 1]; see also [21, Section II.3, Proposition 3.6]. A polyhedron is compact if and only if it is the geometric realization of a finite simplicial complex.

Suppose that $\kappa : K \to L$ is a simplicial carrier. A continuous map $\alpha : |K| \to |L|$ has support $\kappa$ if $\alpha(|\sigma|) \subseteq |\kappa(\sigma)|$ for every simplex of $\sigma$. Any two maps $|K| \to |L|$ with support $\kappa$ are homotopic. Assigning to a simplicial carrier $\kappa : K \to L$ the homotopy class of continuous maps $|K| \to |L|$ with support $\kappa$, one obtains a functor from the category of simplicial carriers to the homotopy category $\text{Ho}(\mathbf{P})$ of polyhedra. Such a category has polyhedra as objects and homotopy classes of continuous maps as arrows.

Suppose that $K$ is a simplicial complex. The barycentric subdivision $\text{Sd}(K)$ of $K$ is the simplicial complex with $\text{dom}(\text{Sd}(K))$ equal to the set of simplices of $K$. A simplex in $\text{Sd}(K)$ is a set $\{\sigma_0, \ldots, \sigma_n\}$ of simplices of $K$ that is linearly ordered by inclusion. Notice that a simplicial carrier $\kappa : K \to \text{Sd}(K)$ is simplicial. Then $\text{Ho}(\mathbf{P})$ with support $\kappa$. Thus, $K \mapsto \text{Sd}(K)$ is a functor from the category of simplicial complexes to itself.

**Remark 6.2.** There is a natural simplicial carrier $\pi_K : \text{Sd}(K) \to K, \{\sigma_0, \ldots, \sigma_n\} \mapsto \sigma_n$, where $\{\sigma_0 \subseteq \cdots \subseteq \sigma_n\}$. We also have star carrier $\text{Sd}_K : K \to \text{Sd}(K)$, $\sigma \mapsto \text{St}_{\text{Sd}(K)}(\sigma)$. Notice that $\pi_{\text{Sd}(K)} \circ \text{Sd}_K$ is the identity simplicial carrier of $K$, while $\text{Sd}_K \circ \pi_K : \text{Sd}(K) \to \text{Sd}(K)$ is a star carrier such that $1_{\text{Sd}(K)} \leq \text{Sd}_K \circ \pi_K$.

### 6.2. The chain complex of a simplicial complex

We now recall the definition of the classical homology invariants of a simplicial complex. Recall that we assume all simplicial complexes to be countable and locally finite. We adopt the notation and terminology from [12, Chapter VI]. For $n \geq 0$, an elementary $n$-chain is any tuple $(v_0, \ldots, v_n) \in \text{dom}(K)^{n+1}$ with $(v_0, \ldots, v_n)$ a simplex of $K$. Let $C_n(K)$ be the free abelian group generated by the set of elementary $n$-chains for $n \geq 0$, and $C_n = \{0\}$ for $n < 0$ [12, Definition 2.3]. Elements of $C_n(K)$ are called (ordered) $n$-chains of $K$. The differential $\partial : C_n(K) \to C_{n-1}(K)$ is defined by

$$\partial(v_0, \ldots, v_n) = \sum_{i=0}^{n} (-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_n),$$

where $\hat{v}_i$ denotes the omission of $v_i$. This gives rise to a free countable chain complex $C_\bullet(K)$ which we call the chain complex of $K$. The chain complex of $K$ with coefficients in $G$ is $C_\bullet(K; G) := C_\bullet(K) \otimes G$; see Definition 3.9.

The cochain complex $C^\bullet(K)$ of $K$ is the $\mathbb{Z}$-dual cochain complex of $C_\bullet(K)$. More generally, the cochain complex $C^\bullet(K; G)$ of $K$ with coefficients in $G$ is the $G$-dual cochain complex of $C_\bullet(K)$; see Definition 3.7. Notice that $C^\bullet(K; G)$ and $C^\bullet(K; G)$ are proper whenever $K$ is finite.

The index homomorphism is the group homomorphism $\text{Ind} : C_0(K) \to \mathbb{Z}$ defined by $\text{Ind}(v_0) = 1$ for every vertex $v_0$ of $K$. If $K, L$ are simplicial complexes, then we say that a chain map $f : C_\bullet(K) \to C_\bullet(L)$ is augmentation-preserving if $\text{Ind} \circ f_0 = \text{Ind}$.

**Definition 6.3.** Suppose that $K$ and $L$ are simplicial complexes, and that $\kappa : K \to L$ is a carrier function.

- For $n, m \in \mathbb{Z}$, a homomorphism $\varphi : C_n(K) \to C_m(L)$ has support $\kappa$ if $\varphi(C_n(\sigma)) \subseteq C_m(\kappa(\sigma))$ for every simplex $\sigma$ of $K$.

- An augmentation-preserving chain map $f : C_\bullet(K) \to C_\bullet(L)$ has support $\kappa$ if $f_n$ has support $\kappa$ for every $n \in \mathbb{Z}$.

- Suppose that $f, f' : C_\bullet(K) \to C_\bullet(L)$ are augmentation-preserving chain maps with support $\kappa$, and $F : f \Rightarrow f'$ is a chain homotopy. Then $F$ has support $\kappa$ if $F_n$ has support $\kappa$ for every $n \in \mathbb{Z}$.

Suppose that $K$ is a simplicial complex, and $v$ is a vertex of $K$. We define the group homomorphism $C_n(\text{St}_K(v)) \to C_{n+1}(K)$, $x \mapsto v \wedge x$ by setting

$$v \wedge (v_0, \ldots, v_n) = (v, v_0, \ldots, v_n)$$

for every elementary $n$-chain $(v_0, \ldots, v_n)$. Then one has that, for $n \in \omega$ and $x \in C_n(\text{St}_K(v))$,

$$\partial(v \wedge x) = \begin{cases} x - \text{Ind}(x)v & \text{if } n = 0, \\ x - (v \wedge \partial x) & \text{if } n > 0; \end{cases}$$

see also [12, Chapter VI, Lemma 5.1]. One can easily prove the following lemma by induction on $n \geq 0$.

**Lemma 6.4.** Suppose that:
\begin{itemize}
\item $K$ and $L$ are simplicial complexes;
\item $\kappa : K \to L$ is a star carrier and $w$ is a choice function for $\kappa$;
\item $f, f' : C_\bullet(K) \to C_\bullet(L)$ are augmentation-preserving chain maps with support $\kappa$;
\item $F, F' : f \Rightarrow f'$ are chain homotopies with support $\kappa$.
\end{itemize}

(1) Define by recursion on $n \in \omega$ group homomorphisms $f_n^{(\kappa, w)} : C_n(K) \to C_{n+1}(L)$ by setting $f_n^{(\kappa, w)} = 0$ for $n < 0$ and

$$f_n^{(\kappa, w)}(v_0, \ldots, v_n) = w(\{v_0, \ldots, v_n\}) \circ (f_n^{(\kappa, w)} - f_{n-1}^{(\kappa, w)}) (v_0, \ldots, v_n)$$

for $n \geq 0$ and elementary $n$-chain $(v_0, \ldots, v_n) \in C_n(K)$. Then $f_n^{(\kappa, w)} : C_\bullet(K) \to C_\bullet(L)$ is an augmentation-preserving chain map with support $\kappa$.

(2) Define by recursion on $n \in \omega$ group homomorphisms

$$F_n^{(\kappa, w)} : C_n(K) \to C_{n+1}(L)$$

with support $\kappa$ by setting $F_n^{(\kappa, w)} = 0$ for $n < 0$ and

$$F_n^{(\kappa, w)}(v_0, \ldots, v_n) = w(\{v_0, \ldots, v_n\}) \circ ((F_n^{\kappa} - F_n) (v_0, \ldots, v_n)$$

for $n \geq 0$ and elementary $n$-chain $(v_0, \ldots, v_n) \in C_n(K)$. Then $F_n^{(\kappa, w)} : f \Rightarrow f'$ is a chain homotopy with support $\kappa$.

(3) Define by recursion on $n \in \omega$ group homomorphisms

$$E_n^{(\kappa, w)} : C_n(K) \to C_{n+2}(L)$$

with support $\kappa$ by setting $E_n^{(\kappa, w)} = 0$ for $n < 0$ and

$$E_n^{(\kappa, w)}(v_0, \ldots, v_n) = w(\{v_0, \ldots, v_n\}) \circ (E_n^{\kappa} - E_n) (v_0, \ldots, v_n)$$

for $n \geq 0$ and elementary $n$-chain $(v_0, \ldots, v_n) \in C_n(K)$. Then $E_n^{(\kappa, w)} : F \Rightarrow F'$ is a 3-cell.

If $K, L$ are simplicial complexes, and $f : K \to L$ is a simplicial map, then $f$ induces in the obvious way an augmentation-preserving chain map $C_\bullet(K) \to C_\bullet(L)$ with support $f$, obtained by setting $f_n (v_0, \ldots, v_n) = (f(v_0), \ldots, f(v_n))$ for $n \in \omega$. More generally, we have the following result, which is an immediate consequence of Lemma 6.4.

**Lemma 6.5.** Suppose that $\kappa : K \to L$ is a star carrier. Then

1. There exists an augmentation-preserving chain map $f : C_\bullet(K) \to C_\bullet(L)$ with support $\kappa$;
2. If $f, f' : C_\bullet(K) \to C_\bullet(L)$ are augmentation-preserving chain maps with support $\kappa$, then there exists a chain homotopy $F : f \Rightarrow f'$ with support $\kappa$.
3. If $f, f' : C_\bullet(K) \to C_\bullet(L)$ are augmentation-preserving chain maps with support $\kappa$, and $F, F' : f \Rightarrow f'$ are chain homotopies with support $\kappa$, then there exists a 3-cell $E : F \Rightarrow F'$.

By Lemma 6.5 the assignment $K \mapsto C_\bullet(K), \kappa \mapsto [f_\kappa]$ where $f_\kappa$ is a chain map with support $\kappa$, is a functor from the category of simplicial complexes to the homotopy category of free countable chain complexes. Thus, the assignment $K \mapsto C_\bullet(K; G)$ defines a functor from the category of simplicial complexes to the homotopy category of Polish chain complexes, and the assignment $K \mapsto C^\bullet(K; G)$ defines a contravariant functor from category of simplicial complexes to the homotopy category of Polish cochain complexes.

**Definition 6.6.** Let $K$ be a simplicial complex. The definable homology $H_\bullet(K; G)$ of $K$ with coefficients in $G$ is the definable homology of the Polish chain complex $C_\bullet(K; G)$, and the definable cohomology $H^\bullet(K; G)$ of $G$ is the definable cohomology of the Polish cochain complex $C^\bullet(K; G)$.

**Remark 6.7.** Notice that if $K$ is a simplicial complex, and $f_K : C_\bullet(Sd(K)) \to C_\bullet(K)$ is a chain map with support the canonical simplicial carrier $\pi_K : Sd(K) \to K$, then $f_K$ is a natural chain homomorphism by Lemma 6.5 and Remark 6.2; see also [12, Section VI.7].

**Definition 6.8.** A tower of finite simplicial complexes is a sequence $K = (K^{(m)}, \rho^{(m,m+1)})$ where $K^{(m)}$ is a finite simplicial complex and $\rho^{(m,m+1)} : K^{(m+1)} \to K^{(m)}$ is a simplicial carrier. We assign to $K$ the tower of free finitely-generated chain complexes $C_\bullet(K) = (C_\bullet(K^{(m)}))_{m \in \omega}$, where the bonding map $C_\bullet(K^{(m+1)}) \to C_\bullet(K^{(m)})$...
is an augmentation-preserving chain map with support \( p^{(m,m+1)} \). We also assign to \( K \) the tower of polyhedra \( (|K^{(m)}|)_{m \in \omega} \), where the bonding map \( |K^{(m+1)}| \to |K^{(m)}| \) is a continuous map with support \( p^{(m,m+1)} \).

6.3. Pairs of simplicial complexes. A pair of simplicial complexes is a pair \((K, K')\) such that \( K \) is a simplicial complex and \( K' \) is a subcomplex of \( K \). One can regard pairs of simplicial complexes as objects of a category, where a morphism from \((K, K')\) to \((L, L')\) is a simplicial carrier \( K \to L \) mapping simplices of \( K' \) to simplices of \( L' \). Suppose that \((K, K')\) is a pair of simplicial complexes. For every \( n \in \mathbb{Z} \), an elementary \( n \)-chain for \( K' \) is also an elementary \( n \)-chain for \( K \). Thus, there is a canonical chain map \( C_\bullet(K') \to C_\bullet(K) \) such that, for every \( n \in \mathbb{Z} \), \( C_n(K') \to C_n(K) \) is a split monomorphism in the category of groups. Whence, one can define a free countable chain complex \( C_\bullet(K, K') \) such that

\[
0 \to C_\bullet(K') \to C_\bullet(K) \to C_\bullet(K, K') \to 0
\]

is a locally split short exact sequence of free countable chain complexes. Correspondingly, defining \( C_\bullet(K, K'; G) := C_\bullet(K, K') \otimes G \) and \( C^\bullet(K, K'; G) := \text{Hom}(C_\bullet(K, K'); G) \), we have a locally split short exact sequence of Polish chain complexes

\[
0 \to C_\bullet(K'; G) \to C_\bullet(K; G) \to C_\bullet(K, K'; G) \to 0.
\]

and a locally split short exact sequence of Polish cochain complexes

\[
0 \to C^\bullet(K'; G) \to C^\bullet(K; G) \to C^\bullet(K, K'; G) \to 0.
\]

Notice that \( C_\bullet(K, K'; G) \) and \( C^\bullet(K, K'; G) \) are proper when \( K \) is finite.

7. Homology of compact metrizable spaces

In this chapter, we assume all the spaces to be compact and metrizable.

7.1. Covers. Let \( X \) be a space. A \textit{cover} of \( X \) is a family \( \mathcal{U} = (U_i^\mathcal{U})_{i \in \omega} \) of open subsets of \( X \) such that \( X \) is the union of \( \{U_i : i \in \omega\} \). The \textit{nerve} \( N(\mathcal{U}) \) of a cover \( \mathcal{U} \) of \( X \) is the simplicial complex with

\[
\text{dom}(N(\mathcal{U})) = \text{supp}(\mathcal{U}) := \{i \in \omega : U_i^\mathcal{U} \neq \emptyset\}
\]

and \( \sigma = \{v_0, \ldots, v_n\} \) a simplex of \( N(\mathcal{U}) \) if and only if

\[
U^\mathcal{U}_{v_0} \cap \cdots \cap U^\mathcal{U}_{v_n} \neq \emptyset.
\]

We say that \( \mathcal{U} \) is finite or locally finite if \( N(\mathcal{U}) \) is finite or locally finite, respectively. A cover \( \mathcal{U} \) of \( X \) is a \textit{refinement} of a cover \( \mathcal{V} \) of \( X \) if for every \( i \in \omega \) there exists \( j \in \text{supp}(\mathcal{V}) \) such that \( U^\mathcal{U}_i \subseteq U^\mathcal{V}_j \). In this case we set \( \mathcal{V} \leq \mathcal{U} \), and define the \textit{refinement carrier} \( \kappa^{(\mathcal{U}, \mathcal{V})}_X : N(\mathcal{U}) \to N(\mathcal{V}) \) by setting

\[
\kappa^{(\mathcal{U}, \mathcal{V})}_X(\sigma) = \{j \in \omega : U^\mathcal{U}_\sigma \subseteq U^\mathcal{V}_j\}
\]

for every simplex \( \sigma \) of \( N(\mathcal{U}) \). Notice that this is a simplicial carrier. The relation \( \leq \) renders the set \( \text{cov}(X) \) of \textit{finite covers} of \( X \) an upward directed ordering of countable cofinality. Throughout this section, we assume all the covers to be finite.

**Definition 7.1.** Let \( X \) be a compact metrizable space. A \textit{cofinal sequence} (of finite open covers) for \( X \) is a cofinal increasing sequence \( \mathcal{U} = (\mathcal{U}_m)_{m \in \omega} \) in \( \text{cov}(X) \). A covered space is a pair \((X, \mathcal{U})\) where \( X \) is a space and \( \mathcal{U} \) is a cofinal sequence for \( X \). A continuous map \( f : (X, \mathcal{U}) \to (Y, \mathcal{W}) \) between covered compact metrizable spaces is a continuous map \( f : X \to Y \). The \textit{category of covered spaces} has covered spaces as objects and continuous maps as morphisms.

Suppose that \((X, \mathcal{U}), (Y, \mathcal{V})\) are covered spaces, and \( \alpha : X \to Y \) is a continuous map. Let us say that \( \alpha \) is \((\mathcal{U}, \mathcal{V})\)-continuous if for every \( i \in \omega \) there exists \( j \in \omega \) such that \( \alpha(U^\mathcal{U}_i) \subseteq U^\mathcal{V}_j \). We can then define the simplicial carrier \( \kappa^{(\mathcal{U}, \mathcal{V})}_\alpha : N(\mathcal{U}) \to N(\mathcal{V}) \) associated with \( \alpha \) by setting

\[
\kappa^{(\mathcal{U}, \mathcal{V})}_\alpha(\sigma) = \{j \in \omega : \alpha(U^\mathcal{U}_\sigma) \subseteq U^\mathcal{V}_j\}
\]

for every simplex \( \sigma \) of \( N(\mathcal{U}) \). This subsumes the notion of refinement carrier, which corresponds to the case when \( X = Y \) and \( \alpha \) is the identity map of \( X \).
7.2. Homology of spaces. Suppose now that \((X, \mathcal{U})\) is a covered space. We let \(K(X, \mathcal{U})\) be the tower of finite simplicial complexes \((N(\mathcal{U}_m)))_{m \in \omega}\), where the bonding map \(N(\mathcal{U}_{m+1}) \to N(\mathcal{U}_m)\) is the refinement carrier \(\kappa_X^{(\mathcal{U}_{m+1}, \mathcal{U}_m)}\).

Define then \(C_\bullet(X, \mathcal{U})\) to be the tower of free finitely-generated chain complexes associated with \(K(X, \mathcal{U})\) as in Definition 6.8.

**Definition 7.2.** Suppose that \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) are covered spaces, and \(\alpha : X \to Y\) is a continuous map. A 1-cell \((m_k, f^{(k)}, f^{(k,k+1)}): C_\bullet(X, \mathcal{U}) \to C_\bullet(X, \mathcal{U})\) has support \(\alpha\) if, for every \(k \in \omega\), \(\alpha\) is \((\mathcal{U}_{m_k}, \mathcal{V}_k)-\)continuous, \(f^{(k)}\) has support \(\kappa_\alpha^{(\mathcal{U}_{m_k}, \mathcal{V}_k)}\), and \(f^{(k,k+1)}\) has support \(\kappa_\alpha^{(\mathcal{U}_{m_{k+1}}, \mathcal{V}_k)}\).

The following lemma is an immediate consequence of Lemma 6.5.

**Lemma 7.3.** Suppose that \((X, \mathcal{U}), (Y, \mathcal{V}), (Z, \mathcal{W})\) are covered spaces, and \(\alpha : X \to Y\) and \(\beta : Y \to Z\) are continuous maps.

1. There exists a 1-cell \(C_\bullet(X, \mathcal{U}) \to C_\bullet(Y, \mathcal{V})\) with support \(\alpha\);
2. If \(f, f' : C_\bullet(X, \mathcal{U}) \to C_\bullet(Y, \mathcal{V})\) are 1-cells with support \(\alpha\), then there is a 2-cell \(L : f \Rightarrow f'\);
3. If \(f : C_\bullet(X, \mathcal{U}) \to C_\bullet(Y, \mathcal{V})\) and \(g : C_\bullet(Y, \mathcal{V}) \to C_\bullet(Z, \mathcal{W})\) are 1-cells with support \(\alpha\) and \(\beta\), respectively, then \(g \circ f\) has support \(\beta \circ \alpha\).

It follows from Lemma 7.3 that choosing, for each continuous map \(\alpha : (X, \mathcal{U}) \to (Y, \mathcal{V})\) a 1-cell \(f_\alpha : C_\bullet(X, \mathcal{U}) \to C_\bullet(Y, \mathcal{V})\) with support \(\alpha\) defines a functor \((X, \mathcal{U}) \to C_\bullet(X, \mathcal{U}), \alpha \mapsto [f_\alpha]\) from the category of covered spaces to the coherent category of towers of free finitely-generated complexes.

Suppose that \(X\) is a compact space. Then one can associate with \(X\) a tower \((X^{(m)})_m\) of compact polyhedra, called **polyhedral resolution** of \(X\), such that \(X\) is homeomorphic to the inverse limit \(\lim_m X^{(m)}\); see [27, Theorem 7, page 61]. The assignment \(X \mapsto (X^{(m)})_m\) is a functor from the category of spaces to the category of towers \(\text{tow}(\text{Ho}(\mathbf{P}))\) associated with the homotopy category of polyhedra \(\text{Ho}(\mathbf{P})\) [27, Corollary 6, page 67]. (See [10, Section 2.1] for the notion of category of towers \(\text{tow}(\mathcal{C})\) associated with a category \(\mathcal{C}\).) One can find a cofinal sequence \(\mathcal{U}^X = (\mathcal{U}^X_m)_{m \in \omega}\) for \(X\) such that, setting \(K^{(m)} := N(\mathcal{U}^X_m)\) for \(m \in \omega\), the tower \((|K^{(m)}|)_{m \in \omega}\) as in Definition 6.8 is isomorphic to \((X^{(m)})_{m \in \omega}\) in \(\text{tow}(\text{Ho}(\mathbf{P}))\); see [12, Section IX.9]. Define then \(K(X) := K(X, \mathcal{U}^X)\).

The tower of free finitely-generated chain complex \(C_\bullet(X, G)\) can be taken to be \(C_\bullet(X, \mathcal{U}^X)\). Consider the corresponding tower of proper Polish chain complexes \(C_\bullet(X; G) := C_\bullet(X) \otimes G\) (Definition 3.9), and the \(\mathbb{Z}\)-dual inductive sequence \(C_\bullet(X) := \text{Hom}(C_\bullet(X), \mathbb{Z})\) of free finitely-generated cochain complexes (Definition 3.7). This defines a functor \(X \mapsto C_\bullet(X; G)\) from the category of spaces to the coherent category of towers of proper Polish chain complexes, which can be seen to be homotopy invariant, and a functor \(X \mapsto C_\bullet^*(X)\) from the category of spaces to the coherent category of inductive sequences of free finitely-generated cochain complexes. Set also \(C_\bullet(X; G) := \text{holim}(C_\bullet(X; G))\), and \(C_\bullet^*(X) := \text{hocolim}(C_\bullet^*(X))\). Notice that, by definition, \(C_\bullet(X; G)\) is the \(G\)-dual tower of \(C_\bullet^*(X)\).

**Definition 7.4.** Let \(X\) be a compact metrizable space, and \(G\) be a Polish group with the division closure property. The **definable homology** of \(X\) with coefficients in \(G\) is the definable homology of the Polish chain complex \(C_\bullet(X; G)\). The **integral cohomology** of \(X\) is the cohomology of the free countable cochain complex \(C_\bullet^*(X)\).

One defines the asymptotic homology \(H_*^\infty(X; G)\) to be the asymptotic group with a Polish cover associated with \(H_*^w(X; G)\), and the weak homology group \(H^w(X; G)\) to be the weak Polish group associated with \(H_*^w(X; G)\); see Definition 2.5.

**Remark 7.5.** The strong homology \(H_*^s(X; G)\) is defined in [25, Section 21] as the homology of the chain complex \(C_\bullet(X; G) = \text{holim}(C_\bullet(X; G))\). Here, \(C_\bullet(X; G)\) is the tower of chain complexes \((C_\bullet(K^{(m)}); G))_{m \in \omega}\) where \((K^{(m)})_m\) is the tower of finite simplicial complexes \(K(X)\) defined as above, and \(\hat{C}_\bullet(K^{(m)}; G)\) is the singular chain complex of the geometric realization of \(K^{(m)}\). The canonical inclusions \(\hat{C}_\bullet(K^{(m)}; G) \to C_\bullet(K^{(m)}; G)\) are quasi-isomorphisms (i.e., they induce an isomorphism at the level of homology) by [15, Theorem 2.27] and the Künneth formula [41, Theorem 3.6.1], and hence they induce a natural quasi-isomorphism \(C_\bullet(X; G) \to C_\bullet(X; G)\). This shows that \(H_*^s(X; G)\) and \(H_*^w(X; G)\) are naturally isomorphic graded groups.

A similar argument shows that \(H_*^s(X; G)\) is naturally isomorphic to the Steenrod homology of \(X\) as in [10, Chapter 8], which is defined as above by replacing the Čech nerve with the Vietoris nerve of covers. To see this, it
suffices to notice that the geometric realization of the Vietoris nerve of a cover is naturally homotopy equivalent to the geometric realization of the Čech nerve [10, Theorem 8.2.10].

Since $H_\bullet(-; G)$ is a homotopy-invariant functor [25, Corollary 19.2], one obtains from these observation an alternative proof, not using the results about the coherent category of towers of chain complexes from [22], that $H_\bullet(-; G)$ is a homotopy-invariant functor.

**Lemma 7.6.** Suppose that $K$ is a finite simplicial complex. Then $H_\bullet(K; G)$ and $H_\bullet(|K|; G)$ are naturally isomorphic graded Polish groups.

**Proof.** Since $H_\bullet(K)$ is finitely-generated, we have that $\mathsf{PExt}(H_\bullet+1(K), G) = \{0\}$. One can choose a cofinal sequence $\mathcal{U}^K_\omega$ for $|K|$ such that $K (|K|, \mathcal{U}^K_\omega)$ is the tower of simplicial complexes $(K^{(m)})_\omega$, where $K^{(0)} = K$, and for $m \geq 0$, $K^{(m+1)} = \text{Sd}(K^{(m)})$ is the $(m+1)$-st barycentric subdivision of $K$ [12, Section II.6], and $K^{(m+1)} \to K^{(m)}$ is the canonical simplicial carrier $\pi_{\text{Sd}(K^{(m)})} : \text{Sd}(K^{(m)}) \to K^{(m)}$. Thus, by Theorem 5.5, we have that $H_\bullet(|K|; G)$ is naturally isomorphic to $\lim_m H_\bullet(K^{(m)}; G)$. The bonding maps $H_\bullet(K^{(m+1)}; G) \to H_\bullet(K^{(m)}; G)$ are isomorphisms by Remark 6.7. Therefore, we have that $\lim_m H_\bullet(K^{(m)}; G)$ is naturally isomorphic to $H_\bullet(K; G)$. This concludes the proof. \qed

In view of Lemma 7.6, as a particular instance of Theorem 5.5 we obtain the following.

**Theorem 7.7.** Let $X$ be a compact metrizable space with polyhedral resolution $(X^{(m)})_\omega$, and let $G$ be a Polish group with the division closure property. Then there exist a natural continuously-definable exact sequence

$$0 \longrightarrow \mathsf{Ext}(H_\bullet+1(X), G) \xrightarrow{\text{coIndex}} H_\bullet(X; G) \xrightarrow{\text{Index}} \mathsf{Hom}(H_\bullet(X), G) \longrightarrow 0$$

which continuously-definably splits, called the UCT exact sequence for $X$, and a natural continuously definable exact sequence

$$0 \longrightarrow \mathsf{PExt}(H_\bullet+1(X), G) \xrightarrow{\text{coIndex}} H_\bullet(X; G) \xrightarrow{\text{Local}} \lim_m H_\bullet(X^{(m)}; G) \longrightarrow 0$$

where $\text{coIndex}_\infty : \mathsf{PExt}(H_\bullet+1(X), G) \xrightarrow{\sim} H_\bullet^\infty(X; G)$ is the restriction of $\text{coIndex}$.

**Remark 7.8.** Considering Remark 5.6, the second exact sequence in Theorem 7.7 can be seen as the Milnor exact sequence for homology

$$0 \to \lim^1_m H_\bullet+1(X^{(m)}; G) \to H_\bullet(X; G) \to \lim_m H_\bullet(X^{(m)}; G) \to 0$$

from [31], where $\lim^1_m H_\bullet+1(X^{(m)}; G)$ is definably isomorphic to $\lim^1_m \mathsf{Hom}(H_\bullet+1(X^{(m)}), G)$.

**7.3. Pairs of spaces.** A pair of spaces is a pair $(X, X')$ such that $X$ is a compact metrizable space, and $X'$ is a closed subspace. We identify a single space $X$ with the pair $(X, \emptyset)$. Pairs of spaces form a category, where a morphism from $(X, X')$ to $(Y, Y')$ is a continuous function $X \to Y$ mapping $X'$ to $Y'$. If $\mathcal{U}$ is a cover of $X$, define $\mathcal{U}|_{X'}$ to be the cover $(U_i \cap X')_{i \in \omega}$ of $X'$. One can associate with a pair of spaces $(X, X')$ a tower $(X^{(m)}, X'^{(m)})$ of pairs of compact polyhedra, called *polyhedral resolution* of $(X, X')$, such that $(X, X')$ is isomorphic to the (inverse) limit $\lim_m (X^{(m)}, X'^{(m)})$ in the category of pairs of spaces [27, Section I.3]. As in the previous section, it is easy to see that there is a cofinal sequence $\mathcal{U}(X, X') = (U_m^{(X, X')})_m$ for $X$ such that:

- $(U_m^{(X, X')})_m$ is a cofinal sequence for $X'$, and
- setting $K^{(m)} = \mathsf{N}(U_m^{(X, X')})$ and $K'^{(m)} = \mathsf{N}(U_m^{(X, X')}|_{X'})$, one has that $(|K^{(m)}|, |K'^{(m)}|)_m$ is isomorphic to $(X^{(m)}, X'^{(m)})$ in the category of towers associated with the homotopy category of pairs of compact polyhedra.

One then defines $C_\bullet(X, X'; G)$ to be the homotopy limit of the tower of proper Polish chain complexes

$$(C_\bullet(K^{(m)}, K'^{(m)}; G))_m,$$

and $C_\bullet(X, X')$ to be the homotopy colimit of the tower of free finitely-generated cochain complexes

$$(C_\bullet(K^{(m)}, K'^{(m)}))_m.$$
Definition 7.9. Let \((X, X')\) be a pair of compact metrizable spaces, and let \(G\) be a Polish group with the division closure property. The definable homology of \((X, X')\) with coefficients in \(G\) is the definable homology of the Polish chain complex \(C_\bullet(X, X')\). The integral cohomology \(H^\bullet(X, X')\) of \((X, X')\) is the cohomology of the countable chain complex \(C^\bullet(X, X')\).

As a consequence of Theorem 5.5, Proposition 3.12, and Theorem 3.14, one obtains the following.

Theorem 10. Let \((X, X')\) be a pair of compact metrizable spaces with polyhedral resolution \((X^{(m)}, X'^{(m)})_{m \in \omega}\), and let \(G\) be a Polish group with the division closure property. Then there is a continuously-definable exact sequence

\[ 0 \rightarrow \text{Ext} (H^{\bullet+1}(X, X'), G) \rightarrow \text{colim}_{\text{strict}} \text{Hom} (H_{\bullet}(X, X'), G) \rightarrow \text{Hom} (H^{\bullet}(X, X'), G) \rightarrow 0 \]

which continuously-definably splits, called the UCT exact sequence for \((X, X')\), and a natural continuously-definable exact sequence

\[ 0 \rightarrow \text{PExt} (H^{\bullet+1}(X, X'), G) \rightarrow \text{colim}_{\text{local}} \text{lim}_m \text{Hom} (H_{\bullet}(X_m, X'_m), G) \rightarrow 0 \]

where \(\text{colim}_{\text{local}} : \text{PExt} (H^{\bullet+1}(X, X'), G) \rightarrow \text{colim}_{\text{strict}} H^{\bullet+1}(X', G)\), while the restriction of the coindex to the UCT exact sequence for \((X, X')\) is the restriction of \(\text{colim}_{\text{local}}\).

Furthermore, there are a natural continuously-definable degree \(-1\) homomorphism \(d_* : H_{\bullet}(X, X'; G) \rightarrow H_{\bullet-1}(X'; G)\) and a natural continuously-definable degree \(1\) homomorphism \(d^* : H^{\bullet}(X') \rightarrow H^{\bullet+1}(X, X')\), called connecting homomorphisms, that fit into the exact sequences

\[ \cdots \rightarrow H_n (X'; G) \xrightarrow{H_n(i)} H_n (X; G) \xrightarrow{H_n(j)} H_n (X, X'; G) \xrightarrow{d_*} H_{n-1} (X'; G) \xrightarrow{H_{n-1}(i)} H_{n-1} (X; G) \rightarrow \cdots \]

and

\[ \cdots \rightarrow H^n (X, X') \xrightarrow{H^n(i)} H^n (X) \xrightarrow{H^n(j)} H^n (X'; G) \xrightarrow{d^*} H^{n+1} (X, X') \xrightarrow{H^{n+1}(i)} H^{n+1} (X) \rightarrow \cdots \]

where \(i : X' \rightarrow X\) and \(j : X \rightarrow (X, X')\) are the inclusion maps. The connecting homomorphisms \(d_*\) and \(d^*\) induce a natural homomorphism from the UCT exact sequence for \((X, X')\) to the UCT exact sequence for \(X'\).

8. Cohomology of polyhedra

In this section, we discuss limits of towers of Polish chain complexes, and colimits of inductive sequences of countable chain complexes. We then prove a definable version of the Eilenberg–Mac Lane Universal Coefficient Theorem for (pairs of) polyhedra; see Theorem 8.6 and Theorem 8.9.

8.1. Towers of Polish cochain complexes. Suppose that \(A = (A^{(k)})_{k \in \omega}\) is a tower of Polish cochain complexes. Then one can define its limit \(\text{lim} A\) to be the Polish cochain complex obtained by setting \((\text{lim} A)^n = \text{lim}_k (A^{(k)})^n\) with codifferentials induced by the codifferentials of \(A^{(k)}\) for \(k \in \omega\). This defines a functor \(A \mapsto \text{lim} A\) from the strict category of towers of Polish cochain complexes to the category of Polish chain complexes. An arrow in the strict category of Polish cochain complexes (strict morphism) is a coherent morphism that is represented by a 1-cell \((m_k, f^{(k)})\) such that \(p^{(k,k+1)} f^{(k+1)} = f^{(k)} p^{(m_k, m_{k+1})}\) for every \(k \in \omega\).

As in the case of homotopy limits, if \(A\) is a tower of Polish cochain complexes, then we have a natural continuously-definable homomorphism

\[ \text{Local} : H^\bullet (\text{lim} A) \rightarrow \prod_{k \in \omega} H^\bullet (A^{(k)}). \]

We say that a tower of proper Polish cochain complexes \(A = (A^{(k)})\) with bonding maps \(p^{(k,k+1)} : A^{(k+1)} \rightarrow A^{(k)}\) is split epimorphic if, for every \(k \in \omega\) and \(n \in \mathbb{Z}\), \(p^{(k,k+1)} n : (A^{(k+1)})^n \rightarrow (A^{(k)})^n\) is a split epimorphism in the category of Polish groups. Notice that we do not require \(p^{(k,k+1)}\) to be a split epimorphism in the category of Polish chain complexes. A similar proof as for Lemma 4.6 shows the following.

Lemma 8.1. Let \(A\) be a split epimorphic tower of proper Polish cochain complexes. Then the image of the natural continuously-definable homomorphism

\[ \text{Local} : H^\bullet (\text{lim} A) \rightarrow \prod_{k \in \omega} H^\bullet (A^{(k)}) \]

is equal to the (inverse) limit \(\text{lim}_k H_{\bullet} (A^{(k)})\), while the kernel is equal to \(H^\bullet_\infty (\text{lim} A)\).
Remark 8.2. Although we will not use this fact, one can also show that, if $A$ is a split epimorphic tower of proper Polish cochain complexes, then $H^*_\omega (\lim A)$ is naturally isomorphic to $\lim^1 H^{*-1}(A^{(\ell)})$, where the $\lim^1$ of a tower of Polish groups is regarded as a group with a Polish cover as in [4, Section 5].

8.2. Inductive sequences of countable chain complexes. Suppose that $A = (A_{(\ell)})_{\ell \in \omega}$ is an inductive sequence of countable chain complexes. Then one can define its colimit $\text{colim} A$ by setting $(\text{colim} A)_n = \text{colim}_\ell \left((A_{(\ell)})_n\right)$ for $n \in \mathbb{Z}$, with differential maps induced by the differential maps of $A_{(\ell)}$ for $\ell \in \omega$. This defines a functor $A \mapsto \text{colim} A$ from the strict category of inductive sequences of countable chain complexes to the category of countable chain complexes. An arrow in the strict category (strict morphism) is a coherent morphism that is represented by a 1-cell $(t, f_{(\ell)}, f_{(\ell+1)})$ such that $\eta_{(\ell+1, \ell)} f_{(\ell+1)} = f_{(\ell+1)} \eta_{(\ell+1, \ell)}$ for every $\ell \in \omega$. For $\ell \in \omega$, we have a chain map $A_{(\ell)} \to \text{colim} A$, which induces a homomorphism $H_\omega (A_{(\ell)}) \to H_\omega (\text{colim} A)$. In turn, these homomorphisms induce a natural homomorphism $\text{colim}_\ell H_\omega (A_{(\ell)}) \to H_\omega (\text{colim} A)$, which can be seen to be an isomorphism.

We say that an inductive sequence $A = (A_{(\ell)})$ of countable chain complexes with bonding maps $\eta_{(\ell+1, \ell)} : A_{(\ell)} \to A_{(\ell+1)}$ is split monomorphic if, for every $\ell \in \omega$ and $n \in \mathbb{Z}$, $(\eta_{(\ell+1, \ell)})_n : (A_{(\ell)})_n \to (A_{(\ell+1)})_n$ is a split monomorphism in the category of countable groups. We do not require $\eta_{(\ell+1, \ell)}$ to be a split monomorphism in the category of countable chain complexes. Notice that, if $A$ is a split monomorphic inductive sequence of free finitely-generated chain complexes, then its $G$-dual tower $A^*$ is a split epimorphic tower of proper Polish cochain complexes. Observe furthermore that $\text{lim}(A^*)$ is naturally isomorphic to $(\text{colim} A)^*$. Thus, the same proof as in Theorem 5.5 allows one to infer from Theorem 3.11 (where one reverses the roles of chain complexes and cochain complexes) and Lemma 8.1 the following.

Theorem 8.3. Let $G$ be a Polish group with the division closure property. Suppose that $A$ is a split monomorphic inductive sequence of free finitely-generated chain complexes, and let $A^*$ be its $G$-dual split epimorphic tower of proper Polish cochain complexes. Define $A := \text{colim} A$. Then there is a natural continuously-definable exact sequence $0 \longrightarrow \text{Ext}(H_{\omega-1}(A), G) \overset{\text{colIndex}}{\longrightarrow} H^*(\text{lim}(A^*), G) \overset{\text{Index}}{\longrightarrow} \text{Hom}(H_\omega (A); G) \longrightarrow 0$ which continuously-definably splits, and a natural continuously-definable exact sequence $0 \longrightarrow \text{PExt}(H_{\omega-1}(A), G) \overset{\text{colIndex}}{\longrightarrow} H^*(\text{lim}(A^*)) \overset{\text{Local}}{\longrightarrow} \text{lim}_\omega H^*(A^{(\ell)}) \longrightarrow 0$ where $\text{colIndex}_\omega : \text{PExt}(H_{\omega-1}(A), G) \overset{\text{lim}}{\longrightarrow} H^*_\omega (\text{lim}(A^*))$ is the restriction of colIndex.

Remark 8.4. In view of Remark 5.6 and Remark 8.2, the second continuously-definable exact sequence in Theorem 8.3 can be seen as the Milnor exact sequence $0 \longrightarrow \lim^1 H^{*-1}(A^{(\ell)}) \rightarrow H^*(\text{lim}(A^*)) \rightarrow \lim^1 H^*(A^{(\ell)}) \rightarrow 0$ where $\lim^1 H^{*-1}(A^{(\ell)})$ is definably isomorphic to $\lim^1 \text{Hom}(H_{\omega-1}(A_{(\ell)}), G)$; see [41, Theorem 3.5.8].

8.3. Cohomology of polyhedra. In this section we assume all the simplicial complexes to be countable and locally finite, but not necessarily finite. Recall that a polyhedron is a topological space $X$ that is the geometric realization $|K|$ of a simplicial complex $K$. If $|K|$ is a polyhedron, then one defines its corresponding chain complex $C_\omega(|K|)$ to be the free countable chain complex $C_\omega(K)$ associated with $K$ as in Section 6. If $\alpha : |K| \rightarrow |L|$ is a continuous function between polyhedra, then one defines a corresponding chain h-map $[\alpha] : C_\omega(K) \rightarrow C_\omega(L)$, as follows. (Recall that a chain h-map is a morphism in the homotopy category of chain complexes.) One fixes a subdivision $K'$ of $K$, a simplicial map $f : K' \rightarrow L$ that is a simplicial approximation of $\alpha$, and a simplicial map $g : K' \rightarrow K$ that is a simplicial approximation of the identity map $|K'| \rightarrow |K|$. Here, the geometric realization of the subdivision $K'$ of $K$ is identified with the geometric realization of $K$ as in [32, Section 15]). Let $\kappa : K \rightarrow K'$ be the carrier defined by letting $\kappa(\sigma)$ be the full subcomplex of $K'$ consisting of the simplices of $K'$ whose vertices belong to $|\sigma|$. One then lets $\tilde{f} : C_\omega(K') \rightarrow C_\omega(L)$ be an augmentation-preserving chain map with support $f$, and $\tilde{g} : C_\omega(K') \rightarrow C_\omega(K)$ be an augmentation preserving chain map with support $g$. One has that $\tilde{g}$ is a chain h-isomorphism [32, Theorem 17.2], and the chain homotopy class $[\tilde{g}] := [\tilde{f}] \circ [\tilde{g}]^{-1}$ only depends on $\alpha$ [32, Section 18]. Furthermore, the assignment $[K] \rightarrow C_\omega(K)$, $\alpha \mapsto [\tilde{f}]$ defines a homotopy-invariant functor from the category of polyhedra to the homotopy category of free countable chain complexes [32, Theorem 18.1, Theorem 19.2].
If \(|K|\) is a polyhedron, then one defines the corresponding cochain complex \(C^*(|K|; G)\) to be the \(G\)-dual of \(C_*(|K|)\); see [32, Section 42]. The assignment \(|K| \mapsto C^*(|K|; G)\) defines a homotopy-invariant functor from the category of polyhedra and continuous maps to the homotopy category of Polish cochain complexes.

**Definition 8.5.** Suppose that \(X\) is a polyhedron, and \(G\) is a Polish group with the division closure property. The (simplicial) definable cohomology \(H^* (X; G)\) of \(X\) with coefficients in \(G\) is the definable cohomology of the Polish cochain complex \(C^*(X; G)\). The integral homology \(H_*(X)\) of \(X\) is the homology of the countable chain complex \(C_* (X)\); see [32, Chapter 5].

One similarly defines the asymptotic definable cohomology \(H^* (X; G)\) and the weak definable cohomology \(H^* (X; G)\) of a polyhedron \(X\). Notice that Lemma 7.6 asserts that Definition 7.4 and Definition 8.5 agree in the case where they overlap, namely for compact polyhedra.

**8.4. The Universal Coefficient Theorem for polyhedra.** Suppose that \(X = |K|\) is a polyhedron. A polyhedral cofiltration of \(X\) is an increasing sequence \((|K_m|)_{m \in \omega}\) of compact subspaces of \(X\) such that \((K_m)_{m \in \omega}\) is an increasing subsequence of finite subcomplexes of \(K\) such that \(K\) is the union of \(\{K_m : m \in \omega\}\). One can then consider the corresponding split monomorphic inductive sequence of free finitely-generated chain complexes \((C_* (|K_m|))_{m \in \omega}\).

Notice that \(C_* (|K|)\) is naturally chain h-isomorphic to \(\text{colim}_m C(|K_m|)\). Thus, \(H_* (|K|)\) is naturally isomorphic to \(\text{colim}_m H_* (|K_m|)\). As a particular instance of Theorem 8.3, we obtain the Universal Coefficient Theorem for cohomology of polyhedra.

**Theorem 8.6.** Let \(X\) be a polyhedron with polyhedral cofiltration \((X_m)_{m \in \omega}\) and let \(G\) be a Polish group with the division closure property. Then there exists a natural continuously-definable exact sequence

\[
0 \rightarrow \text{Ext}(H_{*-1} (X), G) \xrightarrow{\text{colim}_{\text{Index}}} H^*(X; G) \xrightarrow{\text{Index}} \text{Hom}(H_* (X), G) \rightarrow 0
\]

which continuously-definably splits, called the UCT exact sequence for \(X\), and a natural continuously definable exact sequence

\[
0 \rightarrow \text{PExt} (H_{*-1} (X), G) \xrightarrow{\text{colim}_{\text{Index}}} H^*(X; G) \xrightarrow{\text{loc}} \lim_m H^*(X_m; G) \rightarrow 0
\]

where \(\text{colim}_{\text{Index}} : \text{PExt} (H_{*-1} (X); G) \xrightarrow{\sim} H^*_{\omega} (X; G)\) is the restriction of \(\text{colim}_{\text{Index}}\).

**Remark 8.7.** In view of Remark 8.4, the second exact sequence from Theorem 8.6 can be seen as the Milnor exact sequence for cohomology

\[
0 \rightarrow \lim_m H^{* -1} (X_m; G) \rightarrow H^* (X; G) \rightarrow \lim_m H^* (X_m; G) \rightarrow 0
\]

from [30], where \(\lim_m H^{* -1} (X_m; G)\) is definably isomorphic to \(\lim_m \text{Hom} (H_{*-1} (X_m), G)\).

**8.5. Cohomology of pairs of polyhedra.** A pair of polyhedra is a pair \((|K|, |K'|)\) such that \((K, K')\) is a pair of simplicial complexes. Morphisms between pairs of polyhedra are defined as in the case of pairs of compact metrizable spaces. One defines \(C^* (|K|, |K'|; G) := C^* (K, K'; G)\) and \(C_* (K, K') := C_* (K, K')\); see Section 6.3.

**Definition 8.8.** Let \((X, X')\) be a pair of polyhedra, and \(G\) be a Polish group with the divisor closure property. The definable cohomology of \((X, X')\) with coefficients in \(G\) is the definable homology of the Polish cochain complex \(C^* (X, X'; G)\). The homology of \((X, X')\) is the homology of the countable chain complex \(C_* (X, X')\).

Similarly, one defines the weak and asymptotic homology of a pair of polyhedra \((|K|, |K'|)\). A polyhedral cofiltration for \((|K|, |K'|)\) is a sequence \((|K_m|, |K'_m|)_{m \in \omega}\) such that \(|K_m|\) is a polyhedral cofiltration of \(|K|\), \(|K'_m|\) is a polyhedral cofiltration of \(|K'|\), and \(K'_m\) is a subcomplex of \(K_m\) for \(m \in \omega\). As a consequence of Theorem 5.5, Proposition 3.12, and Theorem 3.14, one obtains the following.

**Theorem 8.9.** Let \((X, X')\) be a pair of polyhedra, with polyhedral cofiltration \((X_m, X'_m)_{m \in \omega}\), and \(G\) be a Polish group with the division closure property. Then there exists a natural continuously-definable exact sequence

\[
0 \rightarrow \text{Ext} (H_{*-1} (X, X'), G) \xrightarrow{\text{colim}_{\text{Index}}} H^* (X, X'; G) \xrightarrow{\text{Index}} \text{Hom} (H_* (X, X'), G) \rightarrow 0
\]

which continuously-definably splits, called the UCT exact sequence for \((X, X')\), and a natural continuously definable exact sequence

\[
0 \rightarrow \text{PExt} (H_{*-1} (X, X'), G) \xrightarrow{\text{colim}_{\text{Index}}} H^* (X, X'; G) \xrightarrow{\text{loc}} \lim_m H^* (X^{(m)}, X'^{(m)}; G) \rightarrow 0
\]
where \( \mathrm{colIndex}_{\infty} : \mathrm{PExt}(H_{\bullet-1}(X,X'), G) \cong H^\bullet_{\infty}(X,X';G) \) is the restriction of colIndex.

Furthermore, there are a natural continuously-definable degree 1 homomorphism \( d^* : H^* (X'; G) \to H^{*+1} (X', X'; G) \) and a natural continuously-definable degree \(-1\) homomorphism \( d_* : H_*(X,X') \to H_{*-1} (X') \), called connecting homomorphisms, that fit into the exact sequences

\[
\cdots \to H^n (X, X'; G) \xrightarrow{H^n (j)} H^n (X; G) \xrightarrow{H^n (i)} H^n (X'; G) \xrightarrow{d^n} H^{n+1} (X, X'; G) \xrightarrow{H^{n+1} (j)} H^{n+1} (X; G) \to \cdots
\]

and

\[
\cdots \to H_n(X') \xrightarrow{H_n (i)} H_n (X) \xrightarrow{H_n (j)} H_n(X, X') \xrightarrow{d_n} H_{n-1} (X') \xrightarrow{H_{n-1} (i)} H_{n-1} (X) \to \cdots
\]

where \( i : X' \to X \) and \( j : X \to (X, X') \) are the inclusion maps. The connecting homomorphisms \( d^* \) and \( d_* \) induce a natural homomorphism from the UCT exact sequence for \((X, X')\) to the UCT exact sequence for \(X\).

### 8.6. Cohomology of homotopy polyhedra.

By homotopy invariance, one can extend Definition 8.5 to the class of homotopy polyhedra, namely spaces that are homotopy equivalent to a polyhedron. Indeed, if \( X \) is homotopy equivalent to a polyhedron \(|K_X|\), as witnessed by continuous maps \( s_X : X \to |K_X| \) and \( t_X : |K_X| \to X \), then one can set \( H^* (X; G) := H^* (|K_X|; G) \) and \( H_* (X) := H_* (|K_X|) \). If \( f : X \to Y \) is a continuous map, then the corresponding continuously-definable homomorphisms \( H^* (Y; G) \to H^* (X; G) \) and \( H_* (X) \to H_* (Y) \) are the ones associated with the continuous map \( \alpha \circ s_X \circ t_X : |K_X| \to |K_Y| \).

The class of homotopy polyhedra contains all countable CW complexes [1, Section 1.5]. A CW complex is countable if it is obtained by attaching countably many cells. Every polyhedron is a countable simplicial complex. Conversely, every countable CW complex is a homotopy polyhedron by [29, Theorem 1], [21, Section IV.6, Theorem 6.1], [42, Section I.9, Theorem 13]. Similar considerations apply to absolute neighborhood retract (ANR) [27, Section I.3]: every polyhedron is an ANR and, conversely, every ANR is a homotopy polyhedron [27, Section I.4, Theorem 5].

We also have that a second countable paracompact space with a good cover is a homotopy polyhedron. Recall that a second countable topological spaces is paracompact if every cover has a countable, locally finite refinement. A good cover of a paracompact space \( X \) is a countable, locally finite cover \( U \) of \( X \) such that, for every \( \sigma \in N(U) \), \( U^\sigma \) is contractible. In this case, one has that \( X \) is homotopy equivalent to the polyhedron \(|N(U)|\) [15, Corollary 4G.3].

The class of second countable paracompact spaces with a good cover includes all locally compact Polish spaces admitting a basis that (1) is closed under intersections, and (2) consists of precompact contractible open sets. In particular, all second countable Riemannian manifolds have this property; see [38, Chapter 11], [6, Theorem 5.1], [34, proof of Theorem 89], [9, Section 3.4], [28, Remark after Lemma 10.3].

Analogous considerations apply to the case of pairs of spaces.

### 8.7. Definable cohomology and homotopical definable cohomology.

Suppose that \( G \) is a countable abelian group. A homotopical approach to definable cohomology with coefficients in \( G \) is considered in [5]. Fix \( n \geq 1 \). Let \( (P, *) \) be a pointed polyhedron that is an Eilenberg–MacLane space of type \((G, n)\) [37, Section 8.1], and let \( (X, X') \) be a pair of polyhedra. It is shown in [5] that the set \([(X, X'), (P, *)]\) of homotopy classes of continuous maps \( (X, X') \to (P, *) \) is a definable set, regarded as the quotient of the Polish space of continuous maps \( (X, X') \to (P, *) \) endowed with the compact-open topology. Furthermore, the H-space structure on \((P, *)\) [37, Section 1.5] induces a definable group structure on \([(X, X'), (P, *)]\), which we call the \( n \)-th homotopical definable cohomology group \( H^n_{\eta}(X, X'; G) \) of \((X, X')\) with coefficients in \( G \).

We have that \( H_n (P, *) \cong \pi_n (P, *) \cong G \) by the Hurewicz isomorphism theorem [37, Theorem 7.5.4]. Therefore, by the Universal Coefficient Theorem for cohomology of pairs of polyhedra, there exists \( v \in H^n (P, *; G) \) such that \( \text{Index}(v) \in \text{Hom}(H_n (P, *), G) \) is an isomorphism. Such an element is called \( n \)-characteristic for \((P, *)\); see [37, Section 8.1].

Fix an \( n \)-characteristic element \( v \in H^n (P, *; G) \). Given a pair of polyhedra \((X, X')\), one can consider a natural definable group homomorphism \( \varphi_X : H^n_{\eta}(X, X'; G) \to H^n(X, X'; G) \), \( f \mapsto [H^n(f)(v)] \) where \( f : (X, X') \to (P, *) \) is a continuous map and \( H^n(f) : H^n (P, *; G) \to H^n(X, X'; G) \) is the homomorphism induced by \( f \). By [37, Theorem 8.1.8], \( \varphi_X \) is a group isomorphism. Thus, by [23, Proposition 1.10(3)], \( \varphi_X \) is an isomorphism in the category of definable groups. It follows that, for a pair of polyhedra \((X, X')\), the homotopical definable cohomology
group $H^i_n(X, X'; G)$ and the definable cohomogy group $H^n(X, X'; G)$ as in Definition 8.8 are naturally definably isomorphic.

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