An extension of an asymptotic result of Tricomi concerning a definite integral

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Abstract
We consider the expansion of an integral considered by F.G. Tricomi given by
\[ \int_{-\infty}^{\infty} x e^{-x^2} \left( \frac{1}{2} + \frac{1}{2} \text{erf} x \right)^m \, dx \]
as \( m \to \infty \). The procedure involves a suitable change of variable and the inversion of the complementary error function \( \text{erfc} x \). Numerical results are presented to demonstrate the accuracy of the expansion.

A second part examines an extension of an integral arising in airfoil theory.

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1. Introduction

In a short note published in the Accademia dei Lincei [4], F.G. Tricomi obtained the leading asymptotic behaviour of the integral arising in probability theory
\[ \int_{-\infty}^{\infty} x e^{-x^2} \left( 1 + \text{erf} \frac{x}{2} \right)^m \, dx \]
as \( m \to \infty \), where \( \text{erf} x \) is the error function defined by
\[ \text{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt. \]
The factor in brackets in the above integral satisfies \( 0 < \frac{1}{2} + \frac{1}{2} \text{erf} x < 1 \) for \( x \in (-\infty, \infty) \), with the upper and lower bounds approached as \( x \to \pm \infty \), respectively. By a suitable transformation of the integration variable, Tricomi showed that the leading behaviour of this integral is given by
\[ \frac{\sqrt{\pi \log m}}{m} \quad (m \to \infty). \] (1.1)

In the first part of this paper we extend the result in (1.1) by obtaining higher-order terms in the large-\( m \) expansion of the integrals
\[ I_{n,m} = \int_{-\infty}^{\infty} x^n e^{-x^2} \left( \frac{1 + \text{erf} x}{2} \right)^m \, dx \quad (n = 0, 1, 2), \]
the case \( n = 0 \) being trivial. The second part considers the evaluation of an extension of an integral arising in airfoil theory.
2. Extension of Tricomi’s analysis

Following Tricomi [3], we make the change of variable

\[ e^{-t} = \frac{1 + \text{erf} x}{2}, \quad \frac{e^{-x^2}}{\sqrt{\pi}} \frac{dx}{dt} = -e^{-t} \]  

(2.1)

to yield

\[ I_{n,m} = -\int_0^\infty x^n e^{-x^2} \frac{dx}{dt} e^{-mt} dt = \sqrt{\pi} \int_0^\infty x^n e^{-st} dt, \quad s := m + 1. \]  

(2.2)

When \( n = 0 \), we then have the trivial result

\[ I_{0,m} = \frac{\sqrt{\pi}}{m + 1}. \]

To deal with the cases \( n = 1 \) and \( n = 2 \), we require the inversion of the first relation in (2.1) written in the form

\[ \text{erfc} x = 2 - 2e^{-t} \]  

(2.3)

to yield \( x \equiv x(t) \), which is finite and continuous for \( 0 < t < \infty \). From the well-known asymptotic behaviour [2, p. 164]

\[ \text{erfc} x \sim \frac{e^{-x^2}}{\sqrt{\pi x}}, \quad \text{erfc}(-x) \sim 2 - \frac{e^{-x^2}}{\sqrt{\pi x}} \quad (x \to +\infty), \]

it is seen that \( x(t) \sim \sqrt{-\log t} \) as \( t \to 0 \) and \( x(t) \sim -\sqrt{t} \) as \( t \to +\infty \), with \( x(t) = 0 \) when \( t = \log 2 \).

To proceed we split the integral into two parts: \([0, \alpha]\) and \((\alpha, \infty)\), where \( \alpha \) is finite and bounded away from zero. Then, with the change of variable \( t = u/s \), we have

\[ I_{n,m} = \frac{\sqrt{\pi}}{s} \int_0^\alpha x^n e^{-u/s} du + \frac{\sqrt{\pi}}{s} \int_{\alpha}^\infty x^n e^{-u} du \]  

(2.4)

where \( x \equiv x(u/s) \). Since \( x(t) > -K\sqrt{t} \) for some suitable positive constant \( K \), the second integral is bounded by

\[ \frac{1}{s} \int_{\alpha}^\infty |x(u/s)|n e^{-u} du < \frac{K^n}{s^{n/2+1}} \int_{\alpha}^\infty u^{n/2} e^{-u} du = \frac{K^n}{s^{n/2+1}} \Gamma\left(\frac{1}{2}n + 1, \alpha s\right) = O(s^{-1}e^{-\alpha s}), \]

as \( s \to \infty \), where we have employed the result for the (upper) incomplete gamma function \( \Gamma(a, z) \sim z^{a-1}e^{-z} \) as \( z \to +\infty \).

In the first integral in (2.4) the dominant contribution will arise from the neighbourhood of \( u = 0 \) as \( m \to \infty \). We therefore require the inversion of (2.3) as \( t \to 0 \) expressed in the form

\[ \text{erfc} x = \frac{2u}{s} \left(1 - \frac{u}{2s} + \cdots\right) \quad (u \to 0). \]  

(2.5)

In the appendix it is shown that (see A.1)

\[ x^2 = \log s \left(1 + \frac{A(u)}{L} + \frac{B(u)}{2L^2} + \frac{C(u)}{4L^3} + O(L^{-4}, (sL)^{-1})\right), \]  

(2.6)

where we have defined

\[ L = \log s, \quad L_1 = \log \sqrt{\log s}, \quad a = 2\sqrt{\pi}, \]
and the first few coefficients are
\[ A(u) = -\log au - L_1, \quad B(u) = -A(u) - 1, \quad C(u) = A^3(u) + 3A(u) + \frac{7}{2}. \]

We define the integrals
\[ \lambda_n := \int_0^{\infty} (\log u)^n e^{-u} du, \quad (n = 0, 1, 2, \ldots), \]
so that \( \lambda_0 = 1 \) and
\[ \lambda_1 = -\gamma, \quad \lambda_2 = \gamma^2 + \frac{\pi^2}{6}, \quad \lambda_3 = -\gamma^2 - \frac{\pi^2 \gamma}{2} - 2\zeta(3), \]
where \( \gamma = 0.57721 \ldots \) is the Euler-Mascheroni constant and \( \zeta \) is the Riemann zeta function.

Then, in the case \( n = 2 \), we have upon extending the upper limit of integration to \( \infty \) (thereby introducing an exponentially small error of \( O(e^{-s\pi}) \))
\[
I_{2,m} \sim \sqrt{\pi \log s} \frac{s}{s} \int_0^{\infty} \left( 1 + \frac{A(u)}{L} + \frac{B(u)}{2L^2} + \frac{C(u)}{4L^3} + \cdots \right) e^{-u} du
\]
\[ = \sqrt{\pi \log s} \left\{ 1 + \frac{A_2}{L} + \frac{B_2}{2L^2} + \frac{C_2}{4L^3} + \cdots \right\} \quad (m \to \infty), \quad (2.7) \]
where
\[ A_2 = G - L_1, \quad B_2 = -G + L_1 - 1, \]
\[ C_2 = G^2 + (3 - 2L_1)G + L_1^2 - 3L_1 + \frac{\pi^2}{6} + \frac{7}{2} \]
with \( G := \gamma - \log a \).

In Tricomi’s case with \( n = 1 \), we find from (2.6) that
\[ x = \sqrt{\log s} \left( 1 + \frac{A(u)}{2L} + \frac{2B(u) - A^2(u)}{8L^2} + \frac{2C(u) - 2A(u)B(u) + A^3(u)}{16L^3} + O(L^{-4}, (sL)^{-1}) \right). \]
Hence we obtain the expansion
\[
I_{1,m} \sim \sqrt{\pi \log s} \frac{s}{s} \left\{ 1 + \frac{A_1}{2L} - \frac{B_1}{8L^2} + \frac{C_1}{16L^3} + \cdots \right\} \quad (m \to \infty), \quad (2.8) \]
where
\[ A_1 = G - L_1, \quad B_1 = G^2 + 2(1 - L_1)G + L_1^2 - 2L_1 + \frac{\pi^2}{6} + 2, \]
\[ C_1 = G^3 + (4 - 3L_1)G^2 + (3L_1^2 - 8L_1 + 8 + \frac{\pi^2}{6})G + 4L_1^2 - 8L_1 - L_1^3 + \frac{\pi^2}{6}(4 - 3L_1) + 2\zeta(3) + 7 \]
and we recall that \( s = m + 1 \). It is seen that the leading term in this expansion agrees with the result stated in (1.1).

As a numerical verification of the expansions (2.7) and (2.8) we present\(^4\) in Table 1 the asymptotic values of \( I_{n,m} \) \((n = 1, 2)\) for different \( m \) compared with the values obtained by high-precision evaluation of (1.2). In Table 2 we show the absolute relative error in the computation of \( I_{1,m} \) for different \( m \) as a function of the truncation index \( k \) in the expansion (2.8).

\(^4\)In the tables we write \( x \times 10^y \) as \( x(y) \).
evaluate the integral up to \( \epsilon \)

\[ x \leq \text{middle of the airfoil}, \]

we have

\[ \Gamma \] is the circulation and the

\[ w \]

\( w \)

wingspan induced by a single vortex filament is [3, p. 201]

In airfoil theory, the expression for the downward velocity

\[ w(a) = -\frac{1}{4\pi} \int_{-1}^{1} \frac{1}{x-a} \frac{d\Gamma}{dx} dx \quad (-1 < a < 1), \]

where \( \Gamma \) is the circulation and the \( x \)-coordinate is normalised to the wingspan so that \(-1 \leq x \leq 1\). For an elliptical distribution \( \Gamma = \Gamma_0(1-x^2)^{1/2} \), where \( \Gamma_0 \) is the circulation in the middle of the airfoil, we have

\[ w(a) = \frac{\Gamma_0}{4\pi} \int_{-1}^{1} \frac{x}{x-a} \frac{dx}{\sqrt{1-x^2}} = \frac{\Gamma_0}{4\pi} \left\{ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} + a \int_{-1}^{1} \frac{1}{x-a} \frac{dx}{\sqrt{1-x^2}} \right\} = \frac{\Gamma_0}{4\pi} \left\{ \pi + a \int_{-1}^{1} \frac{dx}{x-a} \frac{1}{\sqrt{1-x^2}} \right\}. \]

(3.1)

It is clear that \( w(-a) = w(a) \) so that it is sufficient to consider \( 0 < a < 1 \).

The integral appearing in (3.1) must be treated as a Cauchy principal value; that is, we evaluate the integral up to \( \epsilon \) either side of \( x = a \) and then let \( \epsilon \to 0 \). Thus we have, with \( R(w) := (1-a^2-2aw-w^2)^{1/2} \),

\[ \int_{-1}^{1} \frac{1}{x-a} \frac{dx}{\sqrt{1-x^2}} = \int_{\epsilon}^{1-a} \frac{dw}{wR(w)} + \int_{1+a}^{\epsilon} \frac{dw}{wR(-w)} \]

\[ = \frac{1}{\sqrt{1-a^2}} \left\{ \log \left[ \frac{w}{2(1-a^2-aw+\sqrt{1-a^2R(w)})} \right]_{\epsilon}^{1-a} + \log \left[ \frac{w}{2(1-a^2+aw+\sqrt{1-a^2R(-w)})} \right]_{1+a}^{\epsilon} \right\} = -\frac{2ae}{(1-a^2)^{3/2}} + O(e^2) \]

### Table 1: The values of \( I_{n,m} \) (\( n = 1,2 \)) compared with the expansions (2.7) and (2.8) for different \( m \).

| \( m \) | \( I_{1,m} \) \((2.7)\) | \( I_{2,m} \) \((2.6)\) |
|---|---|---|
| \( 10^2 \) | 3.116097\((-2)\) | 3.119672\((-2)\) | 5.694564\((-2)\) | 5.695077\((-2)\) |
| \( 10^3 \) | 4.058838\((-3)\) | 4.060226\((-3)\) | 9.413132\((-3)\) | 9.414250\((-3)\) |
| \( 10^4 \) | 4.826833\((-4)\) | 4.827422\((-4)\) | 1.322800\((-3)\) | 1.322857\((-3)\) |
| \( 10^5 \) | 5.494877\((-5)\) | 5.495164\((-5)\) | 1.710063\((-4)\) | 1.710084\((-4)\) |
| \( 10^6 \) | 6.094732\((-6)\) | 6.094889\((-6)\) | 2.101178\((-5)\) | 2.101184\((-5)\) |

### Table 2: The absolute relative error in \( I_{1,m} \) using the expansion (2.8) for different truncation index \( k \).

| \( k \) | \( m = 10^4 \) | \( m = 10^5 \) | \( m = 10^6 \) |
|---|---|---|---|
| 0 | 1.143\((-1)\) | 9.447\((-2)\) | 8.094\((-2)\) |
| 1 | 5.526\((-3)\) | 3.686\((-3)\) | 2.656\((-3)\) |
| 2 | 1.361\((-5)\) | 1.013\((-4)\) | 7.440\((-5)\) |
| 3 | 1.220\((-5)\) | 5.214\((-5)\) | 2.581\((-5)\) |

### 3. An extension of an integral arising in airfoil theory

In airfoil theory, the expression for the downward velocity

\[ w(a) \]

\( w \)
Then the extension of the integral (3.1) we consider is

\[ \int_{-1}^{1} \frac{x}{x - a - \sqrt{1 - x^2}} dx = \pi \]  

(3.2)

independent of \( a \) [3, p. 202]. Hence \( w(a) = \Gamma_0/4 \), which means that in the case of an elliptical distribution of the circulation the induced downward velocity is constant across the wing.

The situation when \( dI/dx = x^{2n+1}\Gamma_0/\sqrt{1-x^2} \), \( n = 0, 1, 2, \ldots \), corresponds to a flattening of the basic elliptic profile \( (n = 0) \). For example, when \( n = 1, 2 \) we have the profiles

\[ \Gamma = \frac{\Gamma_0}{2} (2 + x^2) \sqrt{1 - x^2} \quad (n = 1), \quad \Gamma = \frac{\Gamma_0}{15} (8 + 4x^2 + 3x^4) \sqrt{1 - x^2} \quad (n = 2); \]

as \( n \) increases the profile becomes progressively flatter in the central portion of the wing. Then the extension of the integral (3.4) we consider is

\[ J_n(a; \mu) := \int_{-1}^{1} \frac{x^{2n+1} - a^{2n+1}}{x - a} \frac{dx}{(1 - x^2)^{\mu}} + a^{2n+1} \int_{-1}^{1} \frac{dx}{x - a} (1 - x^2)^{\mu} = I_1 + a^{2n+1} I_2. \]

(3.3)

In the first integral we employ the expansion

\[ \frac{x^{m+1} - a^{m+1}}{x - a} = \sum_{r=0}^{m} a^{m-r} x^r \quad (m = 0, 1, 2, \ldots) \]  

(3.4)

to find

\[ I_1 = \sum_{r=0}^{n} a^{2n-2r} \int_{-1}^{1} \frac{x^{2r}}{(1 - x^2)^{\mu}} dx = \frac{a^{2n} \sqrt{\pi} \Gamma(1 - \mu)}{\Gamma(\frac{3}{2} - \mu)} \sum_{r=0}^{n} (\frac{1}{2} - \mu)^{-r} \]

\[ = \frac{a^{2n} \sqrt{\pi} \Gamma(1 - \mu)}{\Gamma(\frac{3}{2} - \mu)} F_1(\frac{1}{2}, 1; \frac{3}{2} - \mu; a^{-2}) |n, \]

where the subscript \( n \) on the Gauss hypergeometric function \( F_1 \) denotes that only the first \( n + 1 \) terms are to be taken.

The second integral is

\[ I_2 = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \int_{-1}^{1} \frac{x^{2k}}{x - a} dx = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \left\{ a^{2k} \int_{-1}^{1} \frac{dx}{x - a} + \int_{-1}^{1} \frac{x^{2k} - a^{2k}}{x - a} dx \right\} \]

\[ = \frac{1}{(1 - a^2)^{\mu}} \log \left( \frac{1 - a}{1 + a} \right) + \sum_{k=1}^{\infty} \frac{(\mu)_k}{k!} \int_{-1}^{1} \frac{x^{2k} - a^{2k}}{x - a} dx. \]

Use of (3.3) in the above integral shows that

\[ S := \sum_{k=1}^{\infty} \frac{(\mu)_k}{k!} \int_{-1}^{1} \frac{x^{2k} - a^{2k}}{x - a} dx = \frac{1}{a} \sum_{k=1}^{\infty} \frac{(\mu)_k}{k!} \sum_{r=0}^{k-1} a^{2k-2r} \int_{-1}^{1} \frac{x^{2r}}{x - a} dx \]
\[
= 2a \sum_{k=0}^{\infty} \frac{(\mu)_{k+1} a^{2k}}{(k+1)!} \sum_{r=0}^{k} \frac{a^{-2r}}{2r+1} = 2a \sum_{r=0}^{\infty} \frac{a^{-2r}}{2r+1} \sum_{k=r}^{\infty} \frac{(\mu)_{k+r+1} a^{2k}}{(k+r+1)!}
\]

where we have put \(k \to k+r\) in the last sum. From the identity \((\mu)_{k+r+1} = (\mu)_{r+1}(\mu+r+1)_k\), we then obtain

\[
S = 2a \sum_{r=0}^{\infty} \frac{(\mu)_{r+1}}{(2r+1)(r+1)!} \sum_{k=0}^{\infty} \frac{(\mu+r+1) a^{2k}}{(r+2)k}
\]

\[
= 2a \sum_{r=1}^{\infty} \frac{(\mu)_r}{(2r-1)r!} \, _2F_1(1, \mu; r+1; a^2)
\]

\[
= \frac{2a}{1-a^2} \sum_{r=1}^{\infty} \frac{(\mu)_r}{(2r-1)r!} \, _2F_1\left(1, 1-\mu; r+1; -\frac{a^2}{1-a^2}\right)
\]

upon use of the well-known Euler transformation \([2] (15.8.1)\) for the hypergeometric function.

Then we have the result

\[
J_n(a; \mu) = \frac{a^{2n} \sqrt{\pi} \Gamma(1-\mu)}{\Gamma(\frac{1}{2}-\mu)} \, _2F_1\left(\frac{1}{2}, 1; \frac{1}{2} - \mu; a^{-2}\right)_n + \frac{a^{2n+1} \log\left(1-a\right) (1-a^2)^{\mu}}{1+a^2}
\]

\[
+ \frac{2a^{2n+2}}{1-a^2} \sum_{r=1}^{\infty} \frac{\mu)_r}{(2r-1)r!} \, _2F_1\left(1, 1-\mu; r+1; -\frac{a^2}{1-a^2}\right)
\]

valid for \(\mu < 1\) and non-negative integer \(n\). We remark that the convergence of the infinite sum in (3.5) is slow, being controlled by \(r^{\mu-2}\) as \(r \to \infty\). A simple means of accelerating the convergence is given in the sub-section below.

### 3.2 Improved convergence of (3.5)

We can write the hypergeometric function appearing in the infinite sum in (3.5) in the form

\[
_2F_1(1, 1-\mu; r+1; -X) = 1 - \frac{(1-\mu)X}{r+1} + \frac{(1-\mu)_2X^2}{(r+1)_2} \, _2F_1(1, 3-\mu; r+3; -X),
\]

where, for convenience, we have put \(X := a^2/(1-a^2)\). Then

\[
S = \frac{2a}{1-a^2} \left(\sigma_0 - (1-\mu)_X \sigma_1 + (1-\mu)_2X^2 \sigma_2\right)
\]

\[
+ \frac{2a}{1-a^2} (1-\mu)_2X^2 \sum_{r=1}^{\infty} \frac{(\mu)_r}{(2r-1)(r+2)!} \left\{ _2F_1(1, 3-\mu; r+3; -X) - 1 \right\}, \quad (3.6)
\]

where

\[
\sigma_0 = \sum_{r=1}^{\infty} \frac{(\mu)_r}{(2r-1)r!} = 1 - \sqrt{\pi} \frac{\Gamma(1-\mu)}{\Gamma(\frac{1}{2}-\mu)},
\]

\[
\sigma_1 = \sum_{r=1}^{\infty} \frac{(\mu)_r}{(2r-1)(r+1)!} = \frac{2-3\mu}{3(1-\mu)} - \frac{2\sqrt{\pi} \Gamma(1-\mu)}{3\Gamma(\frac{1}{2}-\mu)},
\]

\[
\sigma_2 = \sum_{r=1}^{\infty} \frac{(\mu)_r}{(2r-1)(r+2)!} = \frac{16 - 25\mu + 15\mu^2}{30(1-\mu)(2-\mu)} - \frac{4\sqrt{\pi} \Gamma(1-\mu)}{15 \Gamma(\frac{1}{2}-\mu)}.
\]
The convergence of the final sum in \((3.6)\) is now controlled by terms of \(O(r^{\mu-5})\) as \(r \to \infty\), which is an improvement on that in \((3.5)\). This procedure can be continued to produce higher rates of convergence, but at the expense of the evaluation of higher-order sums \(\sigma_m, m \geq 2\).

3.3 The special case \(n = 0, \mu = \frac{1}{2}\)

We demonstrate that the result in \((3.5)\) reduces to the value given in \((3.2)\) when \(n = 0\) and \(\mu = \frac{1}{2}\). We have from \((3.5)\) with \(X = a^2/(1 - a^2)\)

\[
J_0(a; \frac{1}{2}) = \pi + \frac{a}{\sqrt{1 - a^2}} \log \left( \frac{1 - a}{1 + a} \right) + 2X \sum_{r=1}^{\infty} \frac{(\mu r)}{(2r - 1)r!} {}_2F_1 \left( \frac{1}{2}, 1; r + 1; -X \right). 
\]

The sum can be written in the form

\[
T := X \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_{r+1}}{(r + \frac{1}{2})(r + 1)!} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-X)^k}{(r + 2)_k} = X \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-X)^k \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_{r+1}}{(r + \frac{1}{2})(r + k + 1)!}}{(k + 1)!} 
\]

\[
= X \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-X)^k}{(k + 1)!} 2F_1 \left( \frac{1}{2}, 1; k + 2; 1 \right). 
\]

Application of Gauss’ summation theorem series [2, (15.4.20)] shows that the \(2F_1(1)\) series has the value \((k + 1)\Gamma(k + \frac{1}{2})/\Gamma(k + \frac{3}{2})\), whence

\[
T = 2X \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (\frac{1}{2})_k}{(k)_k k!} (-X)^k = 2X \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-X)^k}{(k + 1)!} 2F_1 \left( \frac{1}{2}, 1; \frac{1}{2}; \frac{3}{2}; -X \right)
\]

\[
= \frac{2X}{\sqrt{1 + X}} 2F_1 \left( \frac{1}{2}, 1; \frac{3}{2}; \frac{X}{1 + X} \right) = \frac{a}{\sqrt{1 - a^2}} \log \left( \frac{1 + a}{1 - a} \right)
\]

upon use of Euler’s transformation followed by evaluation of the resulting hypergeometric function by [2 (15.4.2)]. Hence, we recover the result

\[
J_0(a; \frac{1}{2}) = \pi 
\]

as given in \((3.2)\).

Appendix: The inversion of \((2.5)\)

We require the inversion of the expression

\[
erfc x = \frac{2u}{s} \left( 1 - \frac{u}{2s} + \cdots \right) \quad (u \to 0). 
\]

Since the limit \(u \to 0\) corresponds to \(x \to +\infty\), we employ the asymptotic expansion [2 (7.12.1)]

\[
erfc x = \frac{e^{-x^2}}{\sqrt{\pi x}} \left[ 1 - \frac{1}{2x^2} + \frac{3}{4x^4} + O(x^{-6}) \right] \quad (x \to +\infty) 
\]

to yield

\[
xe^{x^2} = \frac{s(1 - \frac{1}{2}x^{-2} + \frac{3}{4}x^{-4} - \cdots)}{au(1 - u/(2s) + \cdots)}, \quad a = 2\sqrt{\pi}.
\]

Taking logarithms, we obtain

\[
x^2 = \log s \left( 1 - \log \frac{au}{L} - \frac{\log x}{L} - \frac{1}{2x^2L} + \frac{5}{8x^4L} + \cdots + O((sL)^{-1}) \right), \quad (A.1)
\]
where we set $L = \log s$, $L_1 = \log \sqrt{\log s}$.

Following the iterative procedure described in [1, pp. 25–26] we have as a first approximation $x^2 = \log s$, whence

$$x^2 = \log s \left(1 + \frac{A(u)}{L}\right), \quad A(u) = -\log au - L_1.$$

Then with

$$\frac{\log x}{L} = \frac{L_1}{L} + \frac{A(u)}{2L^2} + \ldots,$$

we find a second approximation given by

$$x^2 = \log s \left(1 + \frac{A(u)}{L} + \frac{B(u)}{2L^2} + \ldots\right), \quad B(u) = -A(u) - 1.$$

This last result produces

$$\frac{\log x}{L} = \frac{L_1}{L} + \frac{A(u)}{2L^2} + \frac{B(u) - A^2(u)}{4L^3} + \ldots,$$

so that we obtain the third approximation

$$x^2 = \log s \left(1 + \frac{A(u)}{L} + \frac{B(u)}{2L^2} + \frac{C(u)}{4L^3} + O(L^{-4},(sL)^{-1})\right), \quad (A.2)$$

where

$$C(u) = A^2(u) + 3A(u) + \frac{7}{2}.$$

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