NONDEGENERACY AND THE JACOBI FIELDS OF ROTATIONALLY SYMMETRIC SOLUTIONS TO THE CAHN-HILLIARD EQUATION

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Abstract. In this paper we study rotationally symmetric solutions of the Cahn-Hilliard equation in $\mathbb{R}^3$ constructed in [14] by the authors. These solutions form a one parameter family analog to the family of Delaunay surfaces and in fact the zero level sets of their blowdowns approach these surfaces. Presently we go a step further and show that their stability properties are inherited from the stability properties of the Delaunay surfaces. Our main result states that the rotationally symmetric solutions are non degenerate and that they have exactly 6 Jacobi fields of temperate growth coming from the natural invariances of the problem (3 translations and 2 rotations) and the variation of the Delaunay parameter.

1. Introduction

1.1. Statement of the main result. In the classical van der Waals-Cahn-Hilliard theory that describes the process of phase separation of two components of a binary alloy one considers the Helmholtz free energy functional

$$E_\varepsilon(u) = \int_\Omega \left( F(u(x)) + \frac{1}{2}\varepsilon^2|\nabla u(x)|^2 \right) \, dx$$

in $H^{-1}(\Omega)$ subject to the average concentration to be constant, i.e.

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = m,$$

where $m \in [-1,1]$ (see [12], [13] for details) and the double-well potential $F(u)$ corresponds to the free energy density at low temperatures, which this paper we will take explicitly

$$F(u) = \frac{1}{4}(1-u^2)^2, \quad F'(u) = u^3 - u.$$  

From now on we will denote $F'(u) = -f(u)$. Note that constant functions $u \equiv \pm 1$ are minimizers of this functional subject to $m = \pm 1$. The Euler-Lagrange equation (with $f(u) = -F'(u)$) is

$$\varepsilon^2 \Delta u + f(u) = \delta_\varepsilon \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,$$

$$\frac{1}{|\Omega|} \int_\Omega u \, dx = m$$

where $\delta_\varepsilon$ is a Lagrange multiplier.

Using $\Gamma$-convergence approach Modica [33] showed that minimizers $u_\varepsilon$ of (1.1) subject to constraint (1.2) $\Gamma$-converge, as $\varepsilon \to 0$, to the function $1 - 2\chi_{A_0}$, where $\chi_{A_0}$ is the characteristic function of an open set $A_0 \subset \Omega$. Moreover $\partial A_0 \cap \Omega$ is locally a surface of constant mean curvature (CMC surface for short). Geometrically the set $A_0$ minimizes the perimeter functional $\text{Per}_\Omega(A)$ among the sets $A \subset \Omega$ whose volume is fixed. A generalisation of these results was given by Sternberg [37]. Furthermore Hutchinson and Tonegawa [15] studied limits of general critical points (1.1) and showed that their limits are locally minimal or CMC surfaces. On the other hand it is known [20] that if a set $A \subset \Omega$ is an isolated minimizer of the perimeter functional subject to the constant volume constraint then there exists a sequence of minimizers $u_\varepsilon$ of (1.1) which $\Gamma$-limit of $u_\varepsilon$...
converges to $A$. This result can be used to construct solutions to (1.3) at least in dimension $2$, see [9]. The most complete construction is due to Pacard and Ritoré [35] who proved the following: if $M$ is a compact Riemannian manifold and $N$ is a non degenerate minimal or CMC sub manifold of $M$ which divides $M$ into 2 disjoint components then for all sufficiently small $\varepsilon$ there exist critical points of (1.1) whose 0 level set converges to $N$. The counterpart of this theory for the time dependent problem was developed among others by Alkakos, Bates and Chen [2] who proved that as $\varepsilon \to 0$ the time evolution of interfaces is governed by the Helle-Shaw problem-of course CMC surfaces are stationary points of the flow. More detailed description of the Cahn-Hilliard flow and key spectral tools can be found for instance in [4], [6], [5], [3], [8] and the references therein. Additional examples of stationary solutions for the singular perturbation problem in a bounded domain have been constructed in [41], [40], [7].

In this paper we consider stationary solutions of the Cahn-Hilliard in the whole space, namely solutions to the following problem:

$$\Delta u + f(u) = \delta, \quad \text{in } \mathbb{R}^3, \quad (1.4)$$

It is convenient to rescale the equation (1.4) by dilatation of the independent variable by a (large) factor $\varepsilon^{-1} > 0$

$$x \mapsto \varepsilon^{-1}x,$$

and obtain an equivalent form of (1.4):

$$\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) = \ell_\varepsilon, \quad \text{in } \mathbb{R}^3, \quad (1.5)$$

where we have denoted $\frac{\varepsilon}{x} = \ell_\varepsilon$. Clearly, if $u_\varepsilon$ is a solution of (1.5) then $v(x) = u_\varepsilon(\varepsilon x)$ is a solution of (1.4). On the other hand, if $v$ is a solution of (1.4) then $u_\varepsilon(x) = v(\frac{x}{\varepsilon})$ is a solution of (1.5). In particular this means that while phase transition of the solutions of (1.4) are of order 1, for the solutions of (1.5) they are of order $\varepsilon$. Thus the latter are more "concentrated" and are blowdowns of the former. In the sequel we will focus on the form (1.5) of the stationary Cahn-Hilliard equation. From what we have said above about the singular perturbation problem it is clear that level sets of these solutions should converge, as $\varepsilon$ tends to 0, to CMC surfaces in $\mathbb{R}^3$.

Now we will describe a family of such solutions. Let $D_\tau, \tau(0,1)$ be a Delaunay unduloid and let $N_\tau$ be its normal vector field. Without loss of generality we may assume that $D_\tau$ is normalised in such a way that its mean curvature is 1. Let us notice that the surface $D_\tau$ divides the space into two disjoint components $\Omega^\pm_\tau$, such that $\mathbb{R}^3 \setminus D_\tau = \Omega^+_\tau \cup \Omega^-_\tau$, where $N_\tau$ points towards $\Omega^+_\tau$. By changing the orientation of $D_\tau$ if necessary we can chose $N_\tau$ in such a way that $\Omega^+_\tau$ contains the axis of symmetry of the surface. The following result is proven in [14]:

**Theorem 1.1.** For all $\tau \in (0,1)$ there exits $\varepsilon_\tau > 0$ such that for all $\varepsilon \in (0, \varepsilon_\tau)$ the problem

$$\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) = \ell_\varepsilon, \quad \text{in } \mathbb{R}^3, \quad (1.6)$$

has a solution $w_\tau$, which is one-periodic in the direction of the z-axis and rotationally symmetric with respect to rotations about the same axis. As $\varepsilon \to 0$ we have $\ell_\varepsilon = 1 + \mathcal{O}(\varepsilon)$, and $w_\tau$ satisfies

$$w_\varepsilon \to 1 \quad \text{as } \varepsilon \to 0 \quad \text{in } \Omega^+_\tau,$$

$$w_\varepsilon \to -1 \quad \text{as } \varepsilon \to 0 \quad \text{in } \Omega^-_\tau,$$

uniformly over compacts. Moreover $w_\tau$ is differentiable as a function of the parameter $\tau$.

The solution described in the above theorem can be translated in the direction of the coordinate axis and rotated about the $x$ and $y$ axis (accepting that the $z$ axis is its axis of the rotational symmetry). Additionally the parameter of the family $\tau \in (0,1)$ can be varied as well. The 5 symmetries and $\partial_\tau w_\tau$ determine 6 Jacobi fields of the linearized operator

$$L_{w_\tau} = \varepsilon \Delta + \frac{1}{\varepsilon} f'(w_\tau). \quad (1.7)$$

We will call them the geometric Jacobi fields. It is natural to ask whether all Jacobi fields come from these natural invariances. The answer is provided by our main result:
Theorem 1.2. \(\text{(i) For all } \tau \in (0, 1) \text{ and all small } \varepsilon \text{ the operator } L_{w_{\tau}} \text{ is nondegenerate in the sense that } \) 
\(\text{Ker } L_{w_{\tau}} = 0.\) 
\(\text{(ii) There exists } a > 0 \text{ such that the linear subspace of solutions of} \)
\[ L_{w_{\tau}} \phi = 0, \]
\(\text{with temperate growth in the direction of the axis of rotation of } w_{\tau} \text{ i.e. such that} \)
\[ \| e^{-a|z|} \phi \|_{L^2(\mathbb{R}^3)} < \infty, \]
\(\text{has dimension 6 and coincides with the linear subspace of the geometric Jacobi fields.}\)

We will see that any geometric Jacobi field is either bounded or it grows linearly in the direction of the \(z\) axis. Note however that our theorem does not exclude the possibility of existence of a solution of \(L_{w_{\tau}} \phi = 0\) such that \(\phi\) satisfies (1.8) with some large value of \(a.\)

To explain the importance of this result let us go back to the construction of Pacard and Ritoré [35]. They consider problem (1.5) but instead of the whole space on a smooth, compact manifold \(M.\) They assume that there exists \(N \subset M\) which is a smooth submanifold of constant mean curvature such that it is the nodal set of a smooth function on \(M\) for which 0 is a regular value. In particular it follows that \(N\) divides \(M\) into two disjoint components \(M^\pm(N),\) similarly as \(D_\tau\) divides \(\mathbb{R}^3.\) Furthermore it is assumed that \(N\) is non degenerate in the sense that the kernel of the Jacobi operator of \(N\)
\[ L_N = \Delta_N + |A_N|^2 + \text{Ric}_g(\nu_N, \nu_N) \]
is empty. Under these hypothesis it is shown in [35] that for any small \(\varepsilon\) there exists a solution of the Cahn-Hilliard equation (1.3) which converges uniformly to \(\pm 1\) over compact subsets of \(M^\pm(N).\) Our existence result in Theorem 1.1 also relies on the non degeneracy of the Delaunay surfaces, which in this case means that their Jacobi operator does not have kernel, and moreover it uses the fact that the Jacobi fields of these surfaces can be classified. Theorem 1.2 goes further since it provides a classification of the Jacobi fields of the family \(w_{\tau}\) of rotationally symmetric solutions of (1.5). This type of result is crucial if one wants to construct new solutions to (1.4) build upon more complicated CMC surfaces in \(\mathbb{R}^3,\) such as some of those constructed for instance in [19], [18], [29], [28], [26], [17], [16] (see also related construction in [23] for the Allen-Cahn equation on the plane).

To explain this let us recall that a non compact, Alexandrov embedded, complete CMC surface with finite topology outside of a compact set consists of finitely many half Delaunay surfaces ([32], [31], [22]) called Delaunay ends. In addition if the number of ends of such surface is \(k\) and this surface is non degenerate then set of nearby CMC surfaces is an analytic manifold of dimension \(3k.\) This was proven by Kusner, Mazzeo and Pollack in [24] and the argument of their paper is in many ways inspired by the similar result for the singular Yamabe problem [30]. One of the problems is to decide whether given CMC surface is non degenerate and this is rather difficult problem except for the Delaunay surface for which separation of variables and ODE methods can be used to prove non degeneracy (see also [21]). Pushing these arguments further one can also classify Jacobi fields with temperate growth [29] and show that all of them came from the natural invariances of the family of Delaunay surfaces. Starting from non degenerate Delaunay surface with \(k\) ends one can built more complicated examples by gluing to it either an extra end or another non degenerate surface and thus obtain CMC surfaces with arbitrary many ends. In some cases these new surfaces are also nondegenerate, see for instance [29], [28], [17], [16].

Theorem 1.2 is the precise analog of the result proven in [29] but in the case of the Cahn-Hilliard equation. Given what we said about the linear properties of the Delaunay surface its assertion is expected, which does not mean that the proof is equally obvious. Certainly what needs to be done is to connect the stability properties of the Delaunay surface \(D_\tau\) and the corresponding solution \(w_{\tau}\) of (1.5) and this can be achieved by expressing \(w_{\tau}\) in the Fermi coordinates of \(D_\tau\) (Section 2.2). While \(w_{\tau}\) is localized near \(D_\tau\) this kind of expression is only valid in a neighbourhood of the surface and this is what complicates the situation (see Section 2.3). In order to deal with this in this paper we replace the operator \(L_{w_{\tau}}\) with another operator \(L_{w_{\tau}}\) (Section 2.4), which locally agrees with the original one but which is easier to analyze. Using this idea in Section 3 we prove our theorem.
In this paper $C, c$ will stand for generic positive constants, $\delta$ will be a small, $\varepsilon$ independent constant and $\alpha \in (0, 1)$ will be a constant as well.

2. Preliminaries

2.1. The surfaces of Delaunay. As we have seen, our results are based in great part and in some sense they parallel the theory of CMC surfaces in $\mathbb{R}^3$ and because of this we begin by describing the basic geometric object in this paper which is the family of Delaunay surfaces. The following is a summary of what can be found for instance in [27] or [26]. The Delaunay unduloids $D_\tau$, $\tau \in (0, 1)$ are CMC surfaces of revolution in $\mathbb{R}^3$ and based on this one can easily parametrize them. Indeed, such parametrization has form

$$R_\tau(z, \theta) = (\rho_\tau(z)\theta, z), \quad (z, \theta) \in \mathbb{R} \times S^1,$$

where

$$\frac{2\rho_\tau}{\sqrt{1 + (\partial_z \rho_\tau)^2}} = \rho_\tau^2, \quad \rho_\tau(0) = 1 - \sqrt{1 - \tau^2}.$$

We can "normalize" the Delaunay surface and suppose that the mean curvature of $D_\tau$ is 1 for all $\tau \in (0, 1)$. A convenient way to parametrize Delaunay unduloids is to use the isothermal coordinates:

$$X_\tau(s, \theta) = \frac{1}{2}(\tau e^{\sigma_\tau(s)}\theta, \kappa_\tau(s)), \quad (s, \theta) \in \mathbb{R} \times S^1,$$

where functions $\sigma_\tau, \kappa_\tau$ are the unique solutions of the following system of ODEs:

$$\begin{aligned}
(\partial_s \sigma_\tau)^2 + \tau^2 \cosh^2 \sigma_\tau &= 1, \\
\partial_s \sigma_\tau(0) &= 0, \quad \sigma_\tau(0) < 0, \\
\partial_s \kappa_\tau - \tau^2 \cosh^2 \sigma_\tau &= 0, \quad \kappa_\tau(0) = 0.
\end{aligned}$$

We will now summarize some basic facts about the Delaunay surfaces and their isothermal parametrization (we reproduce here as well as elsewhere in this section the results proven in [29], [30]). We note first of all that $\sigma_\tau$ is periodic, and consequently the surfaces $D_\tau$ are one-periodic along the $z$-axis: namely, if $T_\tau$ denotes the minimal period then

$$D_\tau = D_\tau + T_\tau e_3.$$

Clearly we have the relation

$$T_\tau = \frac{1}{2} \kappa_\tau(s_\tau),$$

where $\kappa_\tau$ is the minimal period of $\sigma_\tau$.

The Jacobi operator $J_{D_\tau}$ of $D_\tau$ is defined by:

$$J_{D_\tau} := \Delta_{D_\tau} + |A_\tau|^2,$$

where $\Delta_{D_\tau}$ is the Laplace-Beltrami operator on $D_\tau$ and $|A_\tau|^2$ is the square of the norm of the second fundamental form of $D_\tau$. In the isothermal coordinates $(s, \theta) \in \mathbb{R} \times S^1$ its expression is given by:

$$J_{D_\tau} = \frac{1}{\tau^2 e^{2\sigma_\tau}} \{\partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma_\tau)\}.$$

The geometric Jacobi fields on $D_\tau$ solve $J_{D_\tau}\Phi = 0$ and are of three types:

1. The Jacobi fields arising from infinitesimal translations. For any $e \in \mathbb{R}^3$, $|e| = 1$ we define:

$$\Phi_{T_\tau e}^T = e \cdot N_\tau,$$

where $N_\tau$ is the unit normal vector to $D_\tau$. The coordinate vectors $e_j$, $j = 1, 2, 3$ generate three linearly independent Jacobi fields $\Phi_{T_\tau e_j}^T$ corresponding to translations of $D_\tau$ in the directions of the coordinate axis. We note that in the isothermal coordinates

$$\Phi_{T_\tau e_3}^T = \Phi_{T_\tau e_3}^T(s), \quad \Phi_{T_\tau e_j}^T = \Phi_{T_\tau e_j}^T(s, \theta), \quad j = 1, 2.$$

It is important to notice that the Jacobi fields $\Phi_{T_\tau e_j}^T$ are bounded.
The Jacobi fields arising from infinitesimal rotations. Let \( \mathbf{e} \in \mathbb{R}^3 \), \( |\mathbf{e}| = 1 \) be such that \( \mathbf{e} \cdot \mathbf{e}_3 = 0 \). The Killing vector field corresponding to the rotation about the vector \( \mathbf{e} \) is:
\[
\mathbf{x} \mapsto (\mathbf{x} \cdot \mathbf{e}) \mathbf{e}_3 - (\mathbf{x} \cdot \mathbf{e}_3) \mathbf{e}.
\]
We define the Jacobi field associated to this vector field by:
\[
\Phi^{R,e}_\tau = (|\mathbf{x} \cdot \mathbf{e}|) \mathbf{e}_3 - (|\mathbf{x} \cdot \mathbf{e}_3|) \mathbf{e} \cdot N_\tau.
\]
There are clearly two linearly independent Jacobi fields associated to the rotations. They are:
\[
\Phi^{R,\mathbf{e}_1}_\tau, \Phi^{R,\mathbf{e}_2}_\tau,
\]
and they correspond to rotations about the coordinate axis. Note that in isothermal coordinates functions \( \Phi^{R,\mathbf{e}_j}_\tau \), \( j = 1, 2 \) grow linearly as functions of \( s \).

The Jacobi field associated with the variation of the Delaunay parameter. We define:
\[
\Phi^D_\tau = -\partial_\tau X_\tau \cdot N_\tau.
\]
This Jacobi field is somewhat harder to write explicitly however it can be shown that the function \( \Phi^D_\tau(s) \) is linearly growing.

In summary, the Jacobi operator \( J_{D_\tau} \) has at least 6 explicit Jacobi fields which are either linearly growing or bounded. We know that these are all Jacobi fields with temperate growth.

2.2. The Fermi coordinates near Delaunay unduloids. Let \( D_\tau \) be a Delaunay unduloid as above and, let \( H_{D_\tau} \) denote its mean curvature. By \( N_\tau \), we will denote its inner unit normal. We will assume that there exists a tubular neighborhood \( N_\delta \) of \( D_\tau \) of width \( 2\delta \) in which we can introduce local system of coordinates (Fermi coordinates) \( (y_0, t) \in D_\tau \times (-\delta, \delta) \) setting:
\[
x \mapsto (y_0, t), \text{ where } x = y_0 + tN_\tau(y_0).
\]
We suppose that this map, which we denote by \( Y_\delta \), is in fact a diffeomorphism from \( N_\delta \) to \( D_\tau \times (-\delta, \delta) \) whenever \( \delta \) is taken sufficiently small. In the sequel we will use the inverse of this map
\[
Y^{-1}: D_\tau \times (-\delta, \delta) \rightarrow N_\delta
\]
\[
(y_0, t) \mapsto x.
\]
Given a function \( w: N_\delta \rightarrow \mathbb{R}^n \) we define its pullback \( Y^*w \) to \( D_\tau \times (-\delta, \delta) \) by the diffeomorphism \( Y_\delta \) as:
\[
Y^*w(y_0, t) = w \circ Y^{-1}(y_0, t).
\]
For technical reasons we will chose later the size of the tubular neighbourhood \( \delta \) depending on \( \epsilon \) but for now on we just take \( \delta \) small.

We will now derive formulas expressing the Laplace operator \( \Delta \) in \( \mathbb{R}^3 \) in terms of the Fermi coordinates \( (y_0, t) \in D_\tau \times (-\delta, \delta) \). We define for each \( t \in (-\delta, \delta) \)
\[
D_{\tau,t} = \{ x \in N_\delta \mid \text{dist}(D_\tau, x) = t \}.
\]
In other words \( D_{\tau,t} \) is the surface obtained from \( D_\tau \) by translation in the direction of the normal by \( t \). Then the well known formula gives:
\[
\Delta = \Delta_{D_{\tau,t}} + \partial_{tt} - H_{D_{\tau,t}}, \partial_t,
\]
where \( H_{D_{\tau,t}} \) denotes the mean curvature of \( D_{\tau,t} \). We need to expand these operators in terms of the variable \( t \). By \( g \) and \( g_t \), respectively, we will denote the metric on \( D_\tau, D_{\tau,t} \) (induced from \( \mathbb{R}^3 \)). Let us fix a point on \( D_\tau \) and some local parametrisation \( X(u) \), \( u \in \mathcal{U} \subset \mathbb{R}^2 \) of \( D_\tau \) in a neighbourhood of this point (\( X \) could be the isothermal coordinates but any parametrisation will do). In terms of these local coordinates we get the following relation:
\[
g_{z,ij} = g_{ij} + ta_{ij} + t^2 b_{ij},
\]
where
\[
g_{ij} = (\partial_{u_i} X \cdot \partial_{u_i} X), \quad a_{ij} = (\partial_{u_i} X \cdot \partial_{u_j} N_\tau) + (\partial_{u_i} X \cdot \partial_{u_j} N_\tau),
\]
\[
b_{ij} = (\partial_{u_i} N \otimes \partial_{u_j} N_\tau).
\]
Then, for the matrix \( g_t^{-1} = (g_t^{ij})_{i,j=1,...,2} \) we get, provided that \(|t|\) is sufficiently small:
\[
(2.4) \quad g_t^{-1} = g^{-1} + tM_1 + t^2M_2,
\]
where
\[
M_1 = M_1(u), \quad M_2 = M_2(u,t),
\]
are smooth matrix functions. The expression for the Laplace-Beltrami operator on \( D_\tau \) in local coordinates is:
\[
\Delta_{D_\tau} = \frac{1}{\sqrt{\det(g)}} \partial_{u_j} \left( \sqrt{\det(g)} g^{ij} \partial_{u_i} \right)
\]
\[
= g^{ij} \partial_{u_i,u_j} + \frac{1}{\sqrt{\det(g)}} \partial_{u_j} \left( \sqrt{\det(g)} g^{ij} \right) \partial_{u_i}
\]
\[
= g^{ij} \partial_{u_i,u_j} - g^{k\ell} \Gamma^i_{k\ell} \partial_{u_i},
\]
where \( \Gamma^i_{k\ell} \) are the Christoffel symbols. A similar formula holds for \( \Delta_{D_\tau,t} \). Using this we can write:
\[
\Delta_{D_\tau,t} = \Delta_{D_\tau} + c_{ij} \partial_{u_i,u_j} + d_i \partial_{u_i},
\]
where
\[
(2.5) \quad c_{ij} = g_t^{ij} - g^{ij}, \quad d_i = g_t^{k\ell} \left( \Gamma^i_{t,k\ell} - \Gamma^i_{k\ell} \right) + \Gamma^i_{k\ell} \left( g_t^{k\ell} - g^{k\ell} \right).
\]
Expressions in local coordinates for \( c_{ij}, d_i \) can be further derived using the above expansions, however their exact form is not crucial here. The point is that these functions are small in terms of \(|t|\):
\[
(2.6) \quad |c_{ij}(u,t)| + |d_i(u,t)| \leq C|t|.
\]
With a choice of local coordinates on \( D_\tau \) the constant in the above estimate does not depend on the point on \( D_\tau \).

Next, we will expand the mean curvature \( H_{D_\tau,x} \). To this end by \( k_j, j = 1, 2 \) we will denote the principal curvatures of \( D_\tau \). Then we have
\[
H_{D_\tau,x} = \sum_{j=1}^{2} \frac{k_j}{1 - tk_j}
\]
\[
= \sum_{j=1}^{2} k_j + t \sum_{j=1}^{2} k_j^2 + Q,
\]
\[
= H_{D_\tau} + t|A_{D_\tau}|^2 + Q,
\]
where
\[
(2.8) \quad Q(y,t) = t^2 \sum_{j=1}^{2} k_j^3 + t^3 \sum_{j=1}^{2} k_{\epsilon,j} + \ldots,
\]
and \( |A_{D_\tau}| \) is the norm of the second fundamental form on \( D_\tau \). Summarizing all this using (2.1) we can express the Laplace operator in Fermi coordinates as follows
\[
(2.9) \quad \Delta = \partial_{tt} + \Delta_{D_\tau} - \left( H_{D_\tau} + t|A_{D_\tau}|^2 + Q \right) \partial_t + \Lambda,
\]
where \( \Lambda \) is a differential operator whose coefficients are given in (2.5) and satisfy (2.6).

Next we introduce stretched Fermi coordinates
\[
\tau = \frac{t}{\varepsilon}, \quad y = y.
\]
As before we have a diffeomorphism \( Y_\varepsilon \) and its inverse \( Y_\varepsilon^{-1} : D_\tau \times (-\frac{4}{\varepsilon}, \frac{4}{\varepsilon}) \rightarrow N_\delta \), and for any function \( w : N_\delta \rightarrow \mathbb{R}^k \) we define its pullback by \( Y_\varepsilon \) by:
\[
Y_\varepsilon^* w(y, \tau) = w \circ Y_\varepsilon^{-1}(y, \tau).
\]
Taking into account formula (2.9) we get
\begin{equation}
\Delta = \varepsilon^{-2} \partial_{tt} - \varepsilon^{-1} (H_{D_r} + \varepsilon t |A_{D_r}|^2 + Q_\varepsilon) \partial_t + \Delta_{D_r} + \mathcal{A}_\varepsilon.
\end{equation}
where
\[ Q_\varepsilon(y, t) = Q(y, \varepsilon t), \quad \mathcal{A}_\varepsilon(y, t) = \mathcal{A}(y, \varepsilon t). \]

2.3. Two ended Delaunay solutions of the Cahn-Hilliard equation. Locally near the surface \( D_r \) the function \( w_r \), which is the solution of the Cahn-Hilliard equation described in Theorem 1.1 should, to main order, depend on the stretched Fermi variable \( t \) only. To find this first approximation of \( w_r \) we use (2.10) where we ignore terms of order \( o(\varepsilon) \). Arguing formally we are lead to solving the following problem:
\begin{align}
U''' - \varepsilon H_{D_r} U' + f(U) &= \varepsilon \ell_\varepsilon, & \text{ in } \mathbb{R}, \\
f(U(\pm \infty)) &= \varepsilon \ell_\varepsilon, \nonumber
\end{align}
where we also have to determine the Lagrange multiplier \( \ell_\varepsilon \). This function can be found by a straightforward perturbation argument assuming \( U = \Theta + \mathcal{O}(\varepsilon) \) where \( \Theta \) is the unique odd, monotonically increasing solution of the Allen-Cahn equation (equivalent to setting \( \varepsilon = 0 \) in (2.11)), see [14] for details. We have \( U(\pm \infty) = \pm 1 + \sigma_{\varepsilon}^\pm \), where
\begin{equation}
f(\pm 1 + \sigma_{\varepsilon}^\pm) = \varepsilon \ell_\varepsilon.
\end{equation}
Also,
\[ \ell_\varepsilon = \ell_0 + \mathcal{O}(\varepsilon), \quad \ell_0 = -\frac{1}{2} \int_{\mathbb{R}} \Theta'(s)^2 \, ds. \]

Now we will briefly summarise some of the tools and results in [14]. From the proof of Theorem 1.1 we can describe the local behaviour of \( w_r \) in more details. To this end it is useful to express \( w_r \) near \( D_r \) in the local stretched Fermi coordinates \((y, t)\) introduced above. We define weighted Hölder norms on \( D_r \times \mathbb{R} \) by:
\begin{align}
||u||_{C^0(\varepsilon)}(D_r \times \mathbb{R}) &= \sup_{t \in \mathbb{R}} (\cosh t)^\alpha ||u||_{C^0(\varepsilon)}(D_r \times (t-1, t+1)), \\
||u||_{C^1(\varepsilon)}(D_r \times \mathbb{R}) &= ||u||_{C^1(\varepsilon)}(D_r \times \mathbb{R}) + ||\nabla D_r \times \mathbb{R} u||_{C^0(\varepsilon)}(D_r \times \mathbb{R}), \\
||u||_{C^2(\varepsilon)}(D_r \times \mathbb{R}) &= ||u||_{C^2(\varepsilon)}(D_r \times \mathbb{R}) + ||\nabla^2 D_r \times \mathbb{R} u||_{C^0(\varepsilon)}(D_r \times \mathbb{R}) + ||\nabla^2 D_r \times \mathbb{R} u||_{C^0(\varepsilon)}(D_r \times \mathbb{R}).
\end{align}

With these definitions there exists \( \mu_0 > 0 \) and \( \alpha_0 > 0 \) such that for \( 0 < \mu < \mu_0 \) and \( 0 < \alpha < \alpha_0 \) it holds:
\begin{equation}
Y_{\varepsilon}^* w_r(y, t) = U(t) + \mathcal{O}_{C^2(\varepsilon)}(D_r \times \mathbb{R})(\varepsilon^{2-\alpha}).
\end{equation}

Above the symbol \( \mathcal{O}_{C^2(\varepsilon)}(D_r \times \mathbb{R})(\varepsilon^{2-\alpha}) \) denotes functions whose \( C^2(\varepsilon) \) norm is bounded by a constant times \( \varepsilon^{2-\alpha} \). This formula is valid in a tubular neighbourhood \( N_\delta(\varepsilon) \) of \( D_r \), where \( \delta(\varepsilon) = \mathcal{O}(\varepsilon^{2/3}) \). In local variables this means \( |t| \leq C \varepsilon^{-1/3} \). Outside of this neighbourhood we have
\begin{equation}
w_r = \pm 1 + \sigma_{\varepsilon}^\pm + \psi, \quad \text{where} \quad ||\psi||_{C^2(\mathbb{R}^3)} \leq C e^{-c/\varepsilon^{1/3}}.
\end{equation}

In fact more is true. We claim that \( w_r \) converges exponentially to constants \( \pm 1 + \sigma_{\varepsilon} \) away from \( D_r \). More precisely, if \( D_r \) is given as surface of revolution of the curve \( x_1 = \rho_r(z) \) then
\begin{equation}
|w_r(r, z) + 1 - \sigma_{\varepsilon}^-| \leq C \exp \left[ -\frac{c}{\varepsilon} (r - \rho_r(z)) \right], \quad r > \rho_r(z), \quad r = \sqrt{x_1^2 + x_2^2},
\end{equation}
with similar estimate when \( r < \rho_r(z) \). To prove this we note that (2.16) is valid in a tubular neighbourhood of \( D_r \) by (2.14) and the fact that \( t \) and \( \frac{\tau - \rho_r(z)}{\varepsilon} \) are comparable in this neighbourhood. Far from \( D_r \) we use the fact that \( w_r = \pm 1 + \sigma_{\varepsilon}^\pm + \psi \), where \( \psi \) is an exponentially small in \( \varepsilon \) function (see (2.15)) and a comparison argument. These estimates can be made more precise as far as the rate of exponential decay but we will not need such a precision here.

One property that we will need in the sequel is differentiability of \( w_r \) with respect to \( \tau \in (0, 1) \). Although this is not explicitly stated in [14] this property also follows from the proof of Theorem 1.1 by a rather standard argument using the version of the Banach fixed point theorem in [10]. We will omit the details.
pointing out only that the ansatz $U(t)$ is a differentiable function of $\tau$ since the Fermi coordinate is a smooth function of $\tau$. For future reference we note that we have on $D_\tau$ (i.e. $t = 0$)

$$\partial_\tau t = \frac{1}{\varepsilon} \partial_\tau x_\tau \cdot N_\tau = -\frac{1}{\varepsilon} \Phi^D_\tau,$$

where $\Phi^D_\tau$ is the Jacobi field on $D_\tau$ associated with the change of the Delaunay parameter.

2.4. The linearized operator near $D_\tau$. Our main objective in the next section will be to study the linearized operator of the Delaunay solution

$$L_{w_\tau} = \varepsilon \Delta + \frac{1}{\varepsilon} f'(w_\tau),$$

as an operator defined for functions on $\mathbb{R}^3$ and here we will introduce some basic observations and notations needed later.

Using (2.10) we find expression of $L_{w_\tau}$ in stretched Fermi coordinates in $\mathcal{N}_3$:

$$(2.17) \quad Y_\varepsilon' L_{w_\tau} u = \varepsilon^{-1} \left[ \partial_\tau t u - \varepsilon (H_{D_\tau} + \varepsilon t |A_{D_\tau}|^2 + Q_\varepsilon) \partial_\tau u + f'(w_\tau) u \right] + \varepsilon \Delta D_\tau u + \varepsilon \Phi_{\varepsilon} u,$$

where with some abuse of notation we write $u$ and $w_\tau$ instead of $Y_\varepsilon^* u$ and $Y_\varepsilon^* w_\tau$ (we will consistently abuse notation this way whenever it is unambiguous). One technical problem we will have to face in this paper is the fact that while the operator $L_{w_\tau}$ is defined in $\mathbb{R}^3$ its expression in local coordinates $Y_\varepsilon^* L_{w_\tau}$, makes sense only in $\mathcal{N}_3$ and not as we would like in $D_\tau \times \mathbb{R}$. There are possibly many ways to extend $Y_\varepsilon^* L_{w_\tau}$ and we will chose one of them for the rest of the paper. Let $\chi(s)$ be a smooth nonnegative cut-off function equal to 1 for $|s| \leq 1$ and equal to 0 for $|s| > 2$. We set

$$(2.19) \quad \chi_{\varepsilon/\delta}(t) = \chi \left( \frac{\varepsilon t}{\delta} \right).$$

We need to extend the function $Y_\varepsilon^* w_\tau$ in such a way that it is defined outside of $\mathcal{N}_3$. To this end we set

$$w_\tau = \chi_{\varepsilon/\delta}(t) Y_\varepsilon^* w_\tau + (1 - \chi_{\varepsilon/\delta}(t)) U(t).$$

Next we define the extension of the operator $Y_\varepsilon^* L_{w_\tau}$ by

$$(2.20) \quad L_{w_\tau} u = \varepsilon^{-1} \left[ \partial_\tau t u - \varepsilon \left( H_{D_\tau} + \varepsilon t \chi_{\varepsilon/\delta}(t) |A_{D_\tau}|^2 + \chi_{\varepsilon/\delta}(t) Q_\varepsilon \right) \partial_\tau u + f'(w_\tau) u \right] + \varepsilon \Delta D_\tau u + \varepsilon \Phi_{\varepsilon/\delta}(t) \Phi_{\varepsilon} u.$$

As we will see $L_{w_\tau}$ resembles the operator

$$\mathcal{L} u = \frac{1}{\varepsilon} |\partial_\tau t u + f'(\Theta) u| + \varepsilon \left[ \Delta D_\tau u + |A_{D_\tau}|^2 u \right]$$

whose kernel is fairly easy to determine by separation of variables. Indeed, taking $u = \Theta'(t) \psi(y)$ we get

$$(2.21) \quad \mathcal{L}(\Theta' \psi) = \varepsilon \Theta' \left[ \Delta D_\tau \psi + |A_{D_\tau}|^2 \psi \right]$$

and therefore the Jacobi fields of $D_\tau$ determine the Jacobi fields of $\mathcal{L}$. Let us explain in what sense $L_{w_\tau}$ and $\mathcal{L}$ are similar. To do this we will use the operator $L_{w_\tau}$ (our theory of the operator $L_{w_\tau}$ is based on exploiting this link). First we need a function which will play a role of $\Theta'(t)$. Since our proof is based on a perturbation argument there is no unique way to define such a function but a natural candidate seems to be $\partial_\tau Y_{\varepsilon/\delta}(t) w_\tau$. An important observation to make is that $\partial_\tau$ and $\Delta$ do not commute so we do not have $\mathcal{L}_{w_\tau} \partial_\tau w_\tau = 0$ and as we will see below the commutator $[\Delta, \partial_\tau]$ gives rise to the term $|A_{D_\tau}|^2$ in $\mathcal{L}$. Since $\partial_\tau Y_{\varepsilon/\delta}(t) w_\tau$ is defined only in $\mathcal{N}_3$ we define the extension of this function to $D_\tau \times \mathbb{R}$ by

$$(2.22) \quad V = \chi_{\varepsilon/\delta}(t) \partial_\tau Y_{\varepsilon/\delta}^* w_\tau + (1 - \chi_{\varepsilon/\delta}(t)) U'(t),$$

where $U$ is the solution of (2.11). Note that $V$ depends on $t \in \mathbb{R}$ and $y \in D_\tau$ but using (2.14) we get

$$(2.23) \quad V(y, t) = U'(t) + \mathcal{O}_{C^1}(D_\tau \times \mathbb{R})(\varepsilon^{2-\alpha}),$$

globally on $D_\tau \times \mathbb{R}$, which means that the dependence on $y$ is mild. Next we calculate

$$(2.24) \quad \mathcal{L}_{w_\tau} V = \chi_{\varepsilon/\delta} \mathcal{L}_{w_\tau} \partial_\tau Y_{\varepsilon/\delta}^* w_\tau + (1 - \chi_{\varepsilon/\delta}) \mathcal{L}_{w_\tau} U' + \left[ \mathcal{L}_{w_\tau}, \chi_{\varepsilon/\delta} \partial_\tau Y_{\varepsilon/\delta}^* w_\tau + [\mathcal{L}_{w_\tau}, 1 - \chi_{\varepsilon/\delta}] U' \right].$$
The first term above is the most complicated. For brevity let us denote \( v_\tau = \partial_\tau Y_\tau^* h_0 w_\tau \). With this notation differentiating the equation satisfied by \( w_\tau \) in \( \mathcal{N}_\delta \) with respect to \( \tau \) we have

\[
L_{w_\tau} v_\tau - \varepsilon |A_{D_\tau}|^2 v_\tau = \varepsilon \left[ Q_\varepsilon, \partial_\tau \right] v_\tau + \varepsilon \left[ A_\varepsilon, \partial_\tau \right] Y_\tau^* w_\tau,
\]

where \([A, B] = AB - BA\). By definition of \( Q_\varepsilon \) we see that

\[
[Q_\varepsilon, \partial_\tau] v_\tau = -\varepsilon^2 \left( 2t \sum_{j=1}^{2} k_j^3 + 3\varepsilon t \sum_{j=1}^{2} k_j^4 + \ldots \right) v_\tau = \mathcal{O}_{C^{1,0}_\varepsilon(D_\tau \times \mathbb{R})}(\varepsilon^2)
\]

The differential operator \( A_\varepsilon \) contains derivatives in \( y \in D_\tau \) only while \( Y_\tau^* w_\tau \) is, up to order \( \mathcal{O}_{C^{2,0}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{3-\alpha}) \), a function of \( \tau \). This gives

\[
\varepsilon \left[ A_\varepsilon, \partial_\tau \right] Y_\tau^* w_\tau = \varepsilon A_\varepsilon \partial_\tau Y_\tau^* w_\tau - \varepsilon \partial_\tau A_\varepsilon(y, \varepsilon \tau) Y_\tau^* w_\tau = \mathcal{O}_{C^{0,0}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{3-\alpha}).
\]

It follows that in \( \mathcal{N}_\delta \) we get

\[
L_{w_\tau} v_\tau - \varepsilon |A_{D_\tau}|^2 v_\tau = \mathcal{O}_{C^{0,0}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{3-\alpha})
\]

Considering other terms in (2.24) from the fact that \( \chi_{\varepsilon/\delta} = \chi_{\varepsilon/\delta}(t) \) and (2.14) we get

\[
[L_{w_\tau}, \chi_{\varepsilon/\delta}] \partial_\tau Y_\tau^* h_0 w_\tau = \mathcal{O}_{C^{0,0}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{3-\alpha}).
\]

Similar estimates hold for terms involving \( U'(t) \). In summary we get

(2.25) \[ L_{w_\tau} v - \varepsilon |A_{D_\tau}|^2 v = \mathcal{O}_{C^{0,0}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{3-\alpha}). \]

Now let \( \psi \in C^{2,0}(D_\tau) \) be fixed. Using (2.25) we get

(2.26) \[ L_{w_\tau}(\psi v) = \psi L_{w_\tau} v + \varepsilon V(\Delta_{D_\tau} + (\varepsilon/\delta) A_\varepsilon) \psi + \varepsilon \left[ \Delta_{D_\tau} + (\varepsilon/\delta) A_\varepsilon, V \right] \psi = \psi \left( L_{w_\tau} v - \varepsilon |A_{D_\tau}|^2 v \right) + \varepsilon V(\Delta_{D_\tau} + |A_{D_\tau}|^2 + (\varepsilon/\delta) A_\varepsilon) \psi + \varepsilon \left[ \Delta_{D_\tau} + (\varepsilon/\delta) A_\varepsilon, V \right] \psi = \varepsilon V(J_{D_\tau} + (\varepsilon/\delta) A_\varepsilon) \psi + \varepsilon \left[ \Delta_{D_\tau} + (\varepsilon/\delta) A_\varepsilon, V \right] \psi + \mathcal{O}_{C^{0,0}_\mu(D_\tau \times \mathbb{R})}(\varepsilon^{3-\alpha}) \psi,
\]

where \( J_{D_\tau} \) the Jacobi operator on \( D_\tau \). For future reference we note that

(2.27) \[ \left\| \left[ \Delta_{D_\tau} + (\varepsilon/\delta) A_\varepsilon, V \right] \psi \right\|_{C^{0,0}_\mu(D_\tau \times \mathbb{R})} \leq C \varepsilon^{2-\alpha} \| \psi \|_{C^{1,0}(D_\tau)}.
\]

Observe that formula (2.26) is quite similar to (2.21) and in particular it is clear that if \( L_{w_\tau}(\psi V) \approx 0 \) then \( \psi \) should be a Jacobi field on \( D_\tau \), and as a consequence we should get an approximate Jacobi field of \( w_\tau \). Indeed we can easily find describe explicit. Jacobi fields of the two ended Delaunay solution \( w_\tau \) which are approximately of the form \( \psi V \). Let \( h = \sum_{i=1}^{3} h_0 e_i \) be a vector, \( R_{\vartheta}(x) = R_{\vartheta_1, \vartheta_2}(x) \) be a rotation in \( \mathbb{R}^3 \), where \( \vartheta_i \) is the angle of the rotation about the \( x_i \) axis \( i = 1, 2, \) and \( \eta \) be a number such that \( |\eta| \) is small. Then the function

\[
\Phi_{h, \vartheta, \eta}(w_\tau) = (w_\tau + \eta \circ R_{\vartheta})(x + h),
\]

is also a solution of the Cahn-Hilliard equation (1.5). In particular, taking derivatives of \( \Phi_{h, \vartheta, \eta}(w_\tau) \) with respect to the parameters we get

\[
L_{w_\tau} \partial_h \Phi_{h, \vartheta, \eta}(w_\tau) \big|_{h, \vartheta, \eta = 0} = 0, \quad i = 1, 2, 3,
\]

\[
L_{w_\tau} \partial_\vartheta \Phi_{h, \vartheta, \eta}(w_\tau) \big|_{h, \vartheta, \eta = 0} = 0, \quad i = 1, 2,
\]

\[
L_{w_\tau} \partial_\eta \Phi_{h, \vartheta, \eta}(w_\tau) \big|_{h, \vartheta, \eta = 0} = 0,
\]

and hence the 6 dimensional linear space

(2.28) \[ \mathcal{I}_{w_\tau} = \text{span} \{ \partial_h \Phi_{h, \vartheta, \eta}(w_\tau) \big|_{h, \vartheta, \eta = 0}, \partial_\vartheta \Phi_{h, \vartheta, \eta}(w_\tau) \big|_{h, \vartheta, \eta = 0}, \partial_\eta \Phi_{h, \vartheta, \eta}(w_\tau) \big|_{h, \vartheta, \eta = 0} \}.
\]

These are the geometric Jacobi fields of \( L_{w_\tau} \) introduced in already in the introduction. For future use we state the following lemma:
Lemma 2.1. With the above notations the following formulas hold in a tubular neighbourhood \( N_\delta(\varepsilon) \), \( \delta(\varepsilon) = \mathcal{O}(\varepsilon^3) \):

\[
\begin{align*}
Y_\varepsilon^* \partial_i \Phi_{h,\vartheta,\eta}(w_\tau) \big|_{h,\vartheta,\eta=0} &= \varepsilon^{-1} \Phi_T^0, t, \varepsilon + \mathcal{O}_{C_\mu^1, \alpha}(D_\tau \times \mathbb{R})(1), \\
Y_\varepsilon^* \partial_\vartheta \Phi_{h,\vartheta,\eta}(w_\tau) \big|_{h,\vartheta,\eta=0} &= \varepsilon^{-1} \Phi_T^0, t, \varepsilon + \mathcal{O}_{C_\mu^1, \alpha}(D_\tau \times \mathbb{R})(1), \\
Y_\varepsilon^* \partial_\eta \Phi_{h,\vartheta,\eta}(w_\tau) \big|_{h,\vartheta,\eta=0} &= \varepsilon^{-1} \Phi_T^0, t, \varepsilon + \mathcal{O}_{C_\mu^1, \alpha}(D_\tau \times \mathbb{R})(1).
\end{align*}
\]

(2.29)

Proof. We recall that by (2.14) in \( N_\delta(\varepsilon) \) we have

\[
Y_\varepsilon^* w_\tau(y, t) = U(t) + \mathcal{O}_{C_\mu^2, \alpha}(D_\tau \times \mathbb{R})(\varepsilon^{2-\alpha}).
\]

(2.30)

In \( N_\delta(\varepsilon) \) we can write explicitly using the isothermal coordinates on \( D_\tau \):

\[
x = X_\tau(s, \theta) + \varepsilon t N_\tau(s, \theta).
\]

(2.31)

Now, fix a unit vector \( e \in \mathbb{R}^3 \) and denote \( x_h = x + he \). Taking derivative in \( h \) of (2.31) and evaluating at \( h = 0 \) we get:

\[
e = \varepsilon \partial_{x_1} t N_\tau + \partial_{x_2} s \partial_\vartheta X_\tau + \varepsilon t \partial_\vartheta N_\tau + \partial_{x_3} \vartheta \partial_\eta X_\tau + \varepsilon t \partial_\eta N_\tau.
\]

Taking the scalar product with \( N_\tau, \partial_\tau X_\tau \) and \( \partial_\vartheta X_\tau \) we find expression for \( \partial_e t, \partial_e s, \partial_e \vartheta \). Note in particular that \( \partial_{x_1} t = -1 \varepsilon e_1 \cdot N_\tau = \varepsilon^{-1} \Phi_T^0, t, e \). Then, taking derivatives \( \partial_e t \) of (2.30) we get the first formula in (2.29). We follow a similar argument to show the two remaining identities. \( \square \)

3. Proof of Theorem 1.2

3.1. A functional analytic setting for \( L_{w_\tau} \). We will introduce suitable weighted Sobolev norms to study the invertibility theory for \( L_{w_\tau} \). Let \( \text{dist}(x, D_\tau) \) denote the signed distance function, where we chose the orientation of \( D_\tau \) in such a way that the sign of \( \text{dist}(x, D_\tau) \) agrees with that of \( \rho_\tau(z) - r \). We have globally

\[
|\text{dist}(x, D_\tau)| \leq |\rho_\tau(z) - r|,
\]

and the two quantities are comparable near \( D_\tau \). Recall that above we have denoted \( t = \frac{1}{\varepsilon} \text{dist}(x, D_\tau) \) as long as \( |\text{dist}(x, D_\tau)| \leq \delta \).

We will define the weighted Sobolev norms we will use in the sequel. First, let us consider Sobolev spaces \( L^2(D_\tau \times \mathbb{R}) \) and \( H^s(D_\tau \times \mathbb{R}) \). Since functions in these spaces can be expressed in terms of the isothermal coordinates \( (s, \theta) \) and the Fermi coordinate \( t \) we define

\[
L^2_{s, \gamma}(D_\tau \times \mathbb{R}) = \cos^{-s}(s) \cos^{-\gamma}(t) L^2(D_\tau \times \mathbb{R}), \quad H^s_{s, \gamma}(D_\tau \times \mathbb{R}) = \cos^{-s}(s) \cos^{-\gamma}(t) H^s(D_\tau \times \mathbb{R}).
\]

Second, let us consider the subspace \( H^{\ell}(\mathbb{R}^3)_+ \) (respectively \( H^{\ell}(\mathbb{R}^3)_- \)) of \( H^{\ell}(\mathbb{R}^3) \) which consists functions supported in the set \( \{ z \geq -1 \} \) (respectively \( \{ z \leq 1 \} \)). We define weighted Sobolev norms in these subspaces as follows

\[
\| u \|_{H^\ell_{s, \gamma}(\mathbb{R}^3)_+} = \sum_{|\alpha| = 0}^\ell \| e^{az} e^{\gamma \left( \frac{z - \rho_\tau(z)}{\varepsilon} \right)} D^\alpha u \|_{L^2(\mathbb{R}^3)_+} , \quad \text{and}
\]

(3.1)

\[
\| u \|_{H^\ell_{a, \gamma}(\mathbb{R}^3)_-} = \sum_{|\alpha| = 0}^\ell \| e^{-az} e^{\gamma \left( \frac{z - \rho_\tau(z)}{\varepsilon} \right)} D^\alpha u \|_{L^2(\mathbb{R}^3)_-} ,
\]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is a multi index and derivatives are taken with respect to \( x_j, j = 1, 2, \) and \( z \) (which we will identify with \( x_3 \) when convenient). Note that \( \gamma \) measures the rate of decay or growth of the solution in the transversal direction to \( D_\tau \) and \( a \) measures the rate of decay or growth along the axis of \( D_\tau \) in the positive (respectively negative) direction. Finally, we define

\[
H^\ell_{a, \gamma}(\mathbb{R}^3) = H^\ell_{a, \gamma}(\mathbb{R}^3)_+ \oplus H^\ell_{a, \gamma}(\mathbb{R}^3)_-.
\]

With these definitions when \( \gamma > 0, a > 0 \) our spaces consist of exponentially decaying functions, in the opposite case they are exponentially increasing. Combinations of signs for \( \gamma \) and \( a \) are of course allowed.
The norms $H^s_{a,\gamma}(D_r \times I_{\delta/\varepsilon})$, where $I_{\delta/\varepsilon} = (-\delta/\varepsilon, \delta/\varepsilon)$, and $H^s_{a,\gamma}(\mathbb{R}^3 \cap \{\text{dist } (x, D_r) < \delta\})$ are equivalent in the following sense:

\begin{align}
\|\hat{\phi}\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{\text{dist } (x, D_r) < \delta\})} & \leq C\varepsilon^{1/2}\|Y^*_\varepsilon \hat{\phi}\|_{L^2_{a,\gamma}(D_r \times I_{\delta/\varepsilon})}, \\
\|\hat{\phi}\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{\text{dist } (x, D_r) < \delta\})} & \geq C\varepsilon^{1/2}\|Y^*_\varepsilon \phi\|_{L^2_{a,\gamma}(D_r \times I_{\delta/\varepsilon})},
\end{align}

where in general constants $a, \gamma$, $a^*, \gamma^*$ and $a_*, \gamma_*$ are different. In addition, relating the norms of gradients and second derivatives we expect to lose powers of $\varepsilon$. For instance

\begin{align}
\|\nabla \hat{\phi}\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{\text{dist } (x, D_r) < \delta\})} & \leq C\varepsilon^{-1/2}\|\nabla Y^*_\varepsilon \phi\|_{L^2_{a^*,\gamma^*}(D_r \times I_{\delta/\varepsilon})}, \\
\|\nabla \hat{\phi}\|_{L^2_{a,\gamma}(\mathbb{R}^3 \cap \{\text{dist } (x, D_r) < \delta\})} & \geq C\varepsilon^{-1/2}\|\nabla Y^*_\varepsilon \phi\|_{L^2_{a^*,\gamma^*}(D_r \times I_{\delta/\varepsilon})}.
\end{align}

Similar estimates hold for the second derivatives. We will use these estimates later on.

3.2. The Fourier-Laplace transform of $L_{w_r}$. We will consider the linear operator $L_{w_r}$ acting on the space $L^2_{a,\gamma}(\mathbb{R}^3)$ with dense domain $D(L_{w_r}) = H^2_{a,\gamma}(\mathbb{R}^3)$ defined by

$$L_{w_r} : H^2_{a,\gamma}(\mathbb{R}^3) \rightarrow L^2_{a,\gamma}(\mathbb{R}^3),$$

$$u \mapsto L_{w_r} u.$$

The important property of the operator $L_{w_r}$ is the fact that it is periodic in $z$. This will allow us to define the Fourier-Laplace transform of $L_{w_r}$ (this idea was originated by Taubes [39], [38] and developed in the form that we adopt here in [30] and [27]).

To begin we define the Fourier-Laplace transform for functions on $\mathbb{R}$ by:

\begin{equation}
\hat{h}(\sigma, \zeta) = \mathcal{F}(h) = \sum_{-\infty < k < \infty} e^{-i(k+\sigma)\zeta} h(\sigma + k), \quad \sigma \in [0, 1), \quad \zeta = \mu + iv.
\end{equation}

Observe that with this definition we have

\begin{equation}
\hat{h}(\sigma + 1, \zeta) = \sum_{-\infty < k < \infty} e^{-i(k+1+\sigma)\zeta} h(\sigma + k + 1) = \hat{h}(\sigma, \zeta).
\end{equation}

Note that the definition we adopt here is slightly different from the one in [30]-the two differ by a factor $e^{-i\zeta \nu}$.

The Fourier-Laplace transform can be inverted and the inverse is given by an explicit formula. To state it let $s \in \mathbb{R}$ be given and denote the fractional part of $s$ by $s \mod 1$. With this notation we have

\begin{equation}
h(s) = \mathcal{F}^{-1}(\hat{h})(s) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{is\zeta} \hat{h}(s \mod 1, \zeta) \, d\mu,
\end{equation}

where we integrate along the line $\text{Im } \zeta = \nu$, $\zeta = \mu + iv$ (see [39]). The Fourier-Laplace transform is well defined in the Schwarz class $\mathcal{S}$ and, by Cauchy’s theorem, the value of the integral in the inversion formula does not depend on $\nu$, since the segment along which we integrate can be vertically shifted. However, for our purpose it is convenient to consider the class of functions which are allowed to grow exponentially at $+\infty$ (or at $-\infty$). Suppose for instance that $h$ is a continuous function, supported in $[-1, \infty)$ and such that $|e^{as} h(s)| < \infty$. Then the series in (3.4) is well defined as long as $\text{Im } \zeta = \nu < a$. Likewise, we can define the transform on a subspace:

$$H^f(\mathbb{R})_+ = e^{-as} H^f(\mathbb{R}),$$

of the Sobolev space $H^f(\mathbb{R})$ consisting of functions supported in $[-1, \infty)$, where $a$ is the rate of exponential decay or growth. In a similar way we define the subspace $H^f(\mathbb{R})_-$ of $H^f(\mathbb{R})$ consisting of exponentially decaying or growing functions supported in $(-\infty, 1]$. As long as $\text{Im } \zeta = \nu \leq a$ the Fourier-Laplace transform of $h \in H^f_a(\mathbb{R})_+$ is well defined. Moreover the function $h$ can be recovered from $\hat{h}(\cdot, \zeta)$ if the path of integration in the formula (3.6) is taken in the lower half plane $\mathbb{H}^-_a = \{\text{Im } (\zeta) = \nu \leq a\}$. The situation is similar when instead we consider the Fourier-Laplace transform in the space of functions $H^f_a(\mathbb{R})_-$, except that now the transform is defined in the upper half plane $\mathbb{H}^+_a = \{\text{Im } (\zeta) = \nu \geq a\}$. 


We observe that from Plancherel’s formula:

\begin{equation}
\int_{\mu=0}^{2\pi} \int_0^1 |\hat{h}(\sigma, \zeta)|^2 \, d\sigma \, d\mu = \int_{\mathbb{R}} |e^{\nu s} h(s)|^2 \, ds, \quad \zeta = \mu + i\nu,
\end{equation}

it follows that \( L^2 \) norms of the Fourier-Laplace transforms are equal to the exponentially weighted \( L^2 \) norm of functions. This property is crucial for our purpose.

Note that if \( u(\sigma, \zeta) \) is an \( L^2([0, 1]) \) function which is analytic as a function of \( \zeta \) with values in \( L^2([0, 1]) \) in the lower half plane \( \mathbb{H}^- \), then, by Cauchy’s theorem, the path of integration in the inversion formula (3.6) can be shifted down to any path \( \zeta = \mu + i\nu, \nu < a \). If in addition \( u(\cdot, \zeta) \) is bounded by \( e^{-\nu} \) along such paths then the inverse transform \( \mathcal{F}^{-1} u(s) \) is supported in \([-1, \infty)\). This explains the reason we have paid so much attention to functions defined on a half-line. On the other hand Fourier-Laplace transforms of functions in \( H^1_0(\mathbb{R})_+ \) have the property described above.

The Fourier-Laplace transform plays a similar role as the Fourier transform in the theory of linear PDEs with constant coefficients when the differential operator at hand is periodic with respect to the independent variable. To fix attention on a concrete example let us suppose that \( A(s): L^2_\sigma(\mathbb{R})_+ \to L^2_\sigma(\mathbb{R})_+, s \in \mathbb{R} \) is a family of densely defined, linear operators. Then it is natural to define

\begin{equation}
\hat{A}(\sigma, \zeta) \hat{h} = \hat{A}(s) \hat{h}(\sigma, \zeta).
\end{equation}

Now, let us suppose that \( A \) is periodic with period 1, i.e. \( A(s) = A(s + 1) \). We have

\[ \hat{A}(\sigma, \zeta) \hat{h} = \sum_{-\infty < k < \infty} e^{-i(k+\sigma)\zeta} A(\sigma + k) h(\sigma + k) = e^{-i\zeta\sigma} A(\sigma) e^{i\zeta\sigma} \hat{h}, \]

hence explicitly

\begin{equation}
\hat{A}(\sigma, \zeta) = e^{-i\zeta\sigma} A(\sigma) e^{i\zeta\sigma}.
\end{equation}

With our definition of the Fourier-Laplace transform we have \( \hat{\hat{h}}(\sigma) = \hat{h}(\sigma + 1) \) and also \( \hat{A}(\sigma, \zeta) = \hat{A}(\sigma + 1, \zeta) \). It follows that the operator \( \hat{A}(\sigma, \zeta) \) is naturally defined on functions in the space of \( L^2 \) functions defined on \( S^1 \). Through the identification \( u(\sigma) = \hat{u}(e^{2\pi i\sigma}) \) we consider this as a space of periodic functions on \([0, 1]\) and denote it by \( L^2_{\text{per}}([0, 1]) \).

Often one has to deal with operators that are periodic with period \( T > 0 \) that is not necessarily equal to 1. It is elementary to modify our definitions of the Fourier-Laplace transform of a function and a linear operator in this case. For a given function \( h \) and \( T > 0 \) our objective is to obtain define the Fourier-Laplace transform of \( h \) which is periodic of period \( T \). We set \( h_T(x) = h(Tx) \) and let naturally \( \hat{h}(\xi, \zeta) = h_T(\xi/T, \zeta) \) so that

\[ \hat{h}(\xi, \zeta) = \sum_{-\infty < k < \infty} e^{-i(\xi +Tk)\zeta/T} \hat{h}(\xi + Tk), \quad h(x) = \frac{1}{2\pi} \int_{\mu=0}^{2\pi} e^{i\xi\zeta/T} \hat{h}(x \mod T, \zeta/T) d\mu, \]

the Plancherel’s formula is

\[ \int_{\mu=0}^{2\pi} \int_0^T |\hat{h}(\sigma, \zeta)|^2 d\sigma \, d\mu = \int_{\mathbb{R}} |e^{\nu s/T} h(s)|^2 \, ds, \quad \zeta = \mu + i\nu, \]

and the Fourier-Laplace transform of a \( T \) periodic operator \( A \) is

\[ \hat{A}(\xi, \zeta) = e^{-i\xi\zeta/T} A(\xi) e^{i\xi\zeta/T}. \]

The operator \( \hat{A}(\sigma, \zeta) \) acts on a space of functions \( L^2_{\text{per}}([0, T]) \). Note that from the Plancherel’s formula we see that if \( h \in L^2_\sigma(\mathbb{R})_+ \) and its Fourier-Laplace transform is \( T \) periodic then it is natural to take \( \zeta = \mu + i\nu = \mu + iTa, \mu \in (0, 2\pi) \) as the path of integration.

In many applications, and this will be in particular the case in our context, the family of operators \( \hat{A}(\sigma, \zeta) \) is Fredholm and depends holomorphically on the variable \( \zeta \). If this is the case one can use the analytic Fredholm theorem to conclude that either \( \hat{A}(\sigma, \zeta) \) is nowhere invertible or it is invertible in the set of all admissible \( \zeta \) except possibly a discrete set. If the latter happens then in order to solve the equation

\[ A(x)h = g, \]
we can pass to the Fourier-Laplace transform

$$
\hat{A}(\xi, \zeta) \hat{h}(\xi, \zeta) = \hat{g}(\xi, \zeta) \implies h(x) = \frac{1}{2\pi} \int_{\mu=0}^{2\pi} e^{i\xi \zeta/T} (\hat{A}^{-1} \hat{g})(x \mod T, \zeta/T) \, d\mu,
$$

where in the last integral the path of integration should avoid the poles of $\hat{A}^{-1}(x \mod T, \zeta)$. If between two such paths there is no pole of $\hat{A}^{-1}(x \mod T, \zeta)$ then the path of integration can be shifted from one of the paths to the other horizontally without changing the value of the integral. This follows by Cauchy’s theorem, since the integrals over the vertical segments cancel out due to (3.5). This means for instance that we can get the inverse of $A(x)$ in a space of functions $L^2_\gamma(\mathbb{R})$ whenever $\hat{A}^{-1}(\xi, \zeta)$ is analytic in some neighbourhood of the segment $\zeta = \mu + iT\epsilon, \mu \in [0, 2\pi]$. Alternatively, this means that $\hat{A}^{-1}(\xi, \zeta)$ is well defined in the space $L^2_\text{per}(\mathbb{R}, T)$, for $\zeta = \mu + iT\nu, |\nu - a| < \kappa$ with some $\kappa > 0$. It may however happen that $\hat{A}^{-1}(\xi, \zeta)$ is analytic along two paths $\zeta_j = \mu + iT\nu_j, j = 1, 2, \mu \in [0, 2\pi]$ and $\nu_1 < \nu_2$, but it has a pole at some $\zeta^* = \mu^* + iT\nu^*$, with $\nu_1 < \nu^* < \nu_2, \mu^* \in (0, 2\pi)$. In this case formula (3.10) would give two solutions $h_1$ and $h_2$ (by integrating over the paths $\zeta = \mu + iT\nu_j, j = 1, 2$) which would differ by an element of the kernel of $A(x)$. This corresponds to the residue of $\hat{A}^{-1}(\xi, \zeta), \zeta^* = \mu^* + iT\nu^*$.

3.3. Mapping properties of $L_w$ in weighted Sobolev spaces. Going back to our context, we see that since $L_w$ is $T_r$ periodic in the $z$ variable, and so is it induces a family of operators on $L^2_{\gamma,\text{per}}(\mathbb{R}^2 \times [0, T_r])$, which is densely defined and holomorphic, as a function of $\zeta$, in a neighbourhood of the segment $[0, 2\pi]$.

In working with the family $\hat{A}(\xi, \zeta)$, we use the following:

$$
L_w(\zeta) = \{\zeta(\xi \mid r_\gamma, \zeta(\xi = \sqrt{x_1^2 + x_2^2}, \xi \in [0, T_r]) \} \text{ and let } M > 0 \text{ be such that } f'(w_r) < -\frac{\sqrt{2}}{M} \gamma \text{ in } \mathbb{R}^2 \times [0, T_r] \setminus A_{M}. \text{ There exists } \delta_r > 0 \text{ such that for all } \zeta = \mu + i\nu, \mu \in [0, 2\pi], \text{ and } \gamma \text{ such that } a^2 + \gamma^2 < \delta_r, \text{ and all sufficiently small } \varepsilon, \text{ it holds}
$$

$$
(3.11) \quad \varepsilon\frac{\gamma}{\sqrt{2}} \frac{\partial f'}{\partial \varepsilon} = \frac{\partial f'}{\partial \gamma} \text{ for any function } \phi \in H^2_{\gamma,\text{per}}(\mathbb{R}^2 \times [0, T_r]). \text{ The constant } C \text{ above depends on } \zeta, M \text{ and } \gamma.
$$

Proof of Lemma 3.1. This type of estimate is well known and it can be found for instance in [1]. We will outline the proof here (following the proof of a similar result in [11]). We agree that $\Gamma$ is one of the functions

$$
\Gamma = e^{\gamma \frac{\gamma f'(w_r)}{\sqrt{2}}}, \quad \Gamma = \cos(\gamma \frac{r_\gamma}{\sqrt{2}}).
$$
We take a cutoff function $\chi_{\varepsilon M}$ which is supported in the complement of the set $A_{\varepsilon M/2}$ and is identically equal to 1 in the complement of the set $A_{\varepsilon M}$. Let us denote

$$\phi_\zeta = e^{i\zeta/T_r} \phi,$$

so that

$$\tilde{L}_{w_r} \phi = e^{-i\zeta/T_r} \left[ \varepsilon \Delta + \frac{1}{\varepsilon} f'(w_r) \right] \phi_\zeta = \phi.$$

Multiply the left hand side of the last equation by $\phi \Gamma \chi_{\varepsilon M}^2$ and integrate by parts. This gives

$$\int_{\mathbb{R}^2 \times [0, T_r]} \tilde{L}_{w_r} (\zeta) \phi \Gamma \chi_{\varepsilon M}^2 = - \varepsilon \int_{\mathbb{R}^2 \times [0, T_r]} [\nabla \phi_\zeta]^2 + \varepsilon^2 |\phi_\zeta|^2 \Gamma \chi_{\varepsilon M}^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2 \times [0, T_r]} f'(w_r) |\phi_\zeta|^2 \Gamma \chi_{\varepsilon M}^2$$

$$- \varepsilon \int_{\mathbb{R}^2 \times [0, T_r]} \nabla \phi_\zeta \nabla (\Gamma \chi_{\varepsilon M}^2).$$

Using notation of Lemma 3.1 let $w_r(\zeta) = \varepsilon^2 \int_{\mathbb{R}^2 \times [0, T_r]} |\phi_\zeta|^2 \Gamma \chi_{\varepsilon M}^2 + C \varepsilon \int_{\mathbb{R}^2 \times [0, T_r]} \phi_\zeta^2 |\nabla \chi_{\varepsilon M}|^2.$

Young’s inequality gives for example

$$\varepsilon |\nabla \phi_\zeta \cdot \nabla \Gamma \phi_\zeta| \leq \varepsilon \kappa |\nabla \phi_\zeta|^2 \Gamma + \frac{\varepsilon}{4\kappa} |\phi_\zeta|^2 |\nabla \Gamma|^2 \leq \varepsilon \kappa |\nabla \phi_\zeta|^2 \Gamma + \frac{C \gamma}{4\varepsilon \kappa} |\phi_\zeta|^2 \Gamma.$$

Combining similar manipulations and adjusting the constants in the Young’s inequality and the exponent $\gamma$ suitably we find

$$\varepsilon \int_{\mathbb{R}^2 \times [0, T_r]} |\nabla \phi_\zeta|^2 \Gamma \chi_{\varepsilon M}^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2 \times [0, T_r]} |\phi_\zeta|^2 \Gamma \chi_{\varepsilon M}^2 \leq C \int_{\mathbb{R}^2 \times [0, T_r]} |g|^2 \Gamma \chi_{\varepsilon M}^2 + C \varepsilon \int_{\mathbb{R}^2 \times [0, T_r]} |\phi_\zeta|^2 \nabla \chi_{\varepsilon M}|^2.$$

As $|\nabla \chi_{\varepsilon M}| = O(\varepsilon^{-1})$ and

$$|\phi_\zeta| = e^{\varepsilon \text{Im} \zeta/T_r} |\phi|, \quad |\nabla \phi_\zeta| = |\nabla (e^{-i\zeta/T_r} \phi_\zeta)| \leq e^{\varepsilon \text{Im} \zeta/T_r} |\nabla \phi_\zeta| + \frac{|\zeta|}{T_r} e^{\varepsilon \text{Im} \zeta/T_r} |\phi_\zeta|,$$

the Lemma follows from this.

**Remark 3.1.** Estimate (3.12) is of separate interest and it and its variants will be used for instance when we analyse the operator $L_{w_r}$ below. In particular we will need such a variant in the proof Lemma 3.4 (to follow). To explain this let us suppose that the weight function $\Gamma$ depends on $\varepsilon$ as well, say $\Gamma = (\cosh \varepsilon^2 z)^{a \varepsilon}$ and consider the problem

$$L_{w_r} \phi = g,$$

where $\phi, g \in L^2_{\text{per}}(\mathbb{R}^3)$. Choosing the cutoff function $\chi_{\varepsilon M}$ as above (understood now as a function on $\mathbb{R}^3$) and multiplying by $\phi \Gamma \chi_{\varepsilon M}^2$ we see that the term we need to control is of the form

$$\varepsilon |\nabla \phi \cdot \nabla \Gamma \phi| \leq \varepsilon \kappa |\nabla \phi_\zeta|^2 \Gamma + \frac{C \gamma}{4\varepsilon \kappa} |\phi_\zeta|^2 \Gamma,$$

where the last inequality follows since we still have

$$\frac{|\nabla \Gamma|}{\Gamma} \leq C \varepsilon^{-1}.$$

As a consequence we get an estimate of the same type as (3.12) but with integrals taken over the whole space $\mathbb{R}^3$.

**Lemma 3.2.** The operator $\tilde{L}_{w_r} (\zeta)$ acting on $H^2_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_r])$ is Fredholm.

**Proof of Lemma 3.2.** We need to show that $\tilde{L}_{w_r} (\zeta)$ has finite dimensional kernel, closed range and that codimension of the range is also finite. To see that the dim Ker $\tilde{L}_{w_r} (\zeta)$ is finite we argue by contradiction. Using notation of Lemma 3.1 let

$$B_1 = \left\{ \phi \in H^2_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_r]) \mid \tilde{L}_{w_r} (\zeta) \phi = 0, \quad ||\phi||_{L^2(A_{\varepsilon M})} = 1 \right\}.$$ 

By Lemma 3.1 we know that set $B_1$ is bounded in $H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_r])$ and then by Sobolev embedding it is compact in $L^2(A_{\varepsilon M})$ and thus it must be finite dimensional. To show that $\tilde{L}_{w_r} (\zeta)$ has finite range we argue
similarly (see for instance [34] for a detailed proof). To show that the codimension of the range is finite we use the fact that \( \dim \ker \tilde{L}_{w,\gamma}(\zeta) = \text{codim} \text{Range}(\tilde{L}_{w,\gamma}(\zeta)) \), by duality (the dual of \( L^2_{\gamma,\text{per}}(\mathbb{R}^2 \times [0, T_r]) \) being \( L^2_{\gamma,\text{per}}(\mathbb{R}^2 \times [0, T_r]) \)).

We will use this in proving:

**Proposition 3.1.** There exists \( \delta_r > 0 \) and a finite set \( S_0 \), such that for all \( a, \gamma \) with \( a^2 + \gamma^2 < \delta_r \), \( \gamma \notin S_0 \), for all sufficiently small \( \varepsilon \) and for all \( g \in L^2_{a,\gamma}(\mathbb{R}^3) \) there exists a solution of the problem

\[
L_{w,\gamma} \phi = g,
\]

where \( \phi \in H^2_{[a],\gamma}(\mathbb{R}^3) \).

Note that even if the right hand side of (3.13) is decaying as \( z \to \pm \infty \) (i.e. \( a > 0 \)) we get a solution which in general may be increasing as \( z \to \pm \infty \) at the exponential rate proportional to \( e^{|a|z} \).

**Proof of Proposition 3.1.** The idea of the proof is to show that \( \tilde{L}_{w,\gamma}(\zeta) \) is an isomorphism for \( \zeta \) in some neighbourhood of \([0, 2\pi]\), except possibly a finite set of points, and then use the parametrix formula to solve (3.13). Since \( \tilde{L}_{w,\gamma}(\zeta) \) is a Fredholm family of holomorphic operators in an open set \( U \subset \mathbb{C} \) with \([0, 2\pi] \subset U \) it is either non invertible everywhere in \( U \) or it is invertible except a discrete subset of \( U \) [36]. In particular, if we consider \( \zeta \in [0, 2\pi] \) (note that the operator \( L_{w,\gamma}(\zeta) \) is self adjoint for \( \zeta \in \mathbb{R} \)) and are able to show that it is injective there except possibly a discrete set of points then we will conclude that it is invertible in \([0, 2\pi]\) except the discrete set and then the same will be true at least in a neighbourhood \( U \) of this segment.

To carry out this plan we consider \( \tilde{L}_{w,\gamma} \), taken with respect to variable \( z \). This operator is defined on the space of functions in \( H^2_{\gamma,\text{per}}(\mathbb{R}^2 \times [0, T_r]) \) which consists of functions which are periodic with period \( T_r \). Recall that we have

\[
\tilde{L}_{w,\gamma}(\zeta) = e^{-i\xi \zeta / T_r} \{ \varepsilon \Delta + e^{-1} f'(w_r) \} e^{i\xi \zeta / T_r}.
\]

We want to express \( \tilde{L}_{w,\gamma} \) in terms of the stretched Fermi co-ordinates in \( N_\delta \). Let \( \tilde{D}_r \) be the one period piece of \( D_r \) (i.e. \( 0 \leq z \leq T_r \)) with the top and the bottom identified. The natural domain for the expression of the Fourier-Laplace transforms of functions in \( L^2_{\gamma,\text{per}}(\mathbb{R}^2 \times [0, T_r]) \) in the stretched Fermi coordinates is \( \tilde{D}_r \times [-\delta/z, \delta/z] \). For example from the definition of the shifted Fermi coordinates we see that

\[
(Y^*_r \xi)(y, t) = [y + \varepsilon t N_r(y)] \cdot \mathbf{e}_3.
\]

It is convenient to extend this function from \( \tilde{D}_r \times [-\delta/z, \delta/z] \) to \( \tilde{D}_r \times \mathbb{R} \). We will use this for the purpose the cutoff function \( \chi_{\delta/z} \) defined in (2.19) and set

\[
\xi^* = \chi_{\delta/z}(Y^*_r \xi) + (1 - \chi_{\delta/z}) y \cdot \mathbf{e}_3
\]

for the extension of \( (Y^*_r \xi) = \xi^* \), understanding that this is a function of \((y, t)\).

We use the operator \( L_{w,\gamma} \) (see (2.20)) to define also a natural extension of \( \tilde{L}_{w,\gamma} \) to \( \tilde{D}_r \times \mathbb{R} \)

\[
\tilde{L}_{w,\gamma}(\zeta) = e^{-i\xi^*/T_r} L_{w,\gamma} e^{i\xi^*/T_r}.
\]

The operator \( \tilde{L}_{w,\gamma}(\zeta) \) is "almost" the Fourier-Laplace transform of \( L_{w,\gamma} \). Note that \( \tilde{L}_{w,\gamma}(0) = L_{w,\gamma} \). The strategy of the proof is to show first that the operator \( \tilde{L}_{w,\gamma}(\zeta) \) is injective and then conclude from this that \( \tilde{L}_{w,\gamma}(\zeta) \) is injective.

We will study the kernel of \( \tilde{L}_{w,\gamma}(\zeta) \) in the space of functions \( L^2_{\gamma,\text{per}}(\tilde{D}_r \times \mathbb{R}) \). Let us suppose that for some \( \gamma, |\gamma| < \delta_r \), and \( \zeta \in [0, 2\pi] \) there exists a function \( \phi \in H^2_{\gamma,\text{per}}(\tilde{D}_r \times \mathbb{R}) \)

\[
\tilde{L}_{w,\gamma}(\zeta) \phi_0 = L_{w,\gamma}(e^{i\xi^*/T_r} \phi_0) = L_{w,\gamma} \phi_0 = 0,
\]

where we have denoted

\[
\phi_0 = e^{i\xi^*/T_r} \phi_0.
\]

We can normalize \( \| \phi_0 \|_{L^2_{\gamma}(\tilde{D}_r \times \mathbb{R})} = 1 \) and then by elliptic estimates for any \( M > 0 \) in the set \( \tilde{D}_r \times (-M, M) \) the function \( \phi_0 \) is bounded (we bound the real and imaginary parts of \( \phi_0 \) separately). Take \( M \) large so that \( f'(w_r) < -2 + \eta \) with some small \( \eta > 0 \). Take \( \delta_r \) in the statement of the Proposition small so that
\( \gamma \in (-\sqrt{2-\eta}, \sqrt{2-\eta}) \). Using the comparison principle for the operator \( \tilde{L}_w \), it is then easy to show that in fact
\[
\phi_0(\gamma, t) = |\phi_0| \leq Ce^{-\sqrt{2-\eta}t}
\]
and therefore \( \phi_0 \in H^2(\tilde{D}_r \times \mathbb{R}) \).

For complex valued functions \( \phi_1, \phi_2 \in L^2(\tilde{D}_r \times \mathbb{R}) \) we define Hermitian inner product
\[
\langle \phi_1, \phi_2 \rangle = \int_{\tilde{D}_r \times \mathbb{R}} \phi_1 \overline{\phi_2} dV_{\tilde{D}_r} dt
\]

Above \( \nabla_{\tilde{D}_r} \) and \( dV_{\tilde{D}_r} \) are respectively the gradient and the volume element on \( \tilde{D}_r \). We introduce an orthogonal decomposition in \( L^2(\tilde{D}_r \times \mathbb{R}) \) as follows: Let \( V \) be the function defined in (2.22) (we recall that it is an extension of \( \partial_t Y^*_r w_r \)). Given a function \( \phi \in L^2(\tilde{D}_r \times \mathbb{R}) \) we denote \( \phi = e^{i\zeta^*/T_r} \phi \) and decompose
\[
\phi = \phi_{\zeta} + \psi V,
\]
where
\[
\phi_{\zeta} \in X_0 := \{ \phi \in L^2_{\gamma, \text{per}}(\tilde{D}_r \times \mathbb{R}) \mid \int_\mathbb{R} \phi V dt = \int_\mathbb{R} \tilde{\phi} V dt = 0 \}, \quad \psi \in \frac{f_0 \phi V dt}{\int_\mathbb{R} V^2 dt}.
\]
In particular for \( \phi_0 \in \text{Ker} \tilde{L}_w, (\zeta) \) we have
\[
\ll_{\text{per}} \phi_0 = \text{L}_w \phi_0 + \text{L}_w, (\psi \phi_0) = 0
\]
and
\[
\langle -\text{L}_{\text{per}}, \phi_0 \parallel, \parallel \phi_0 \parallel \rangle = -\langle \ll_{\text{per}}, \psi \phi_0 \parallel, \parallel \phi_0 \parallel \parallel \rangle.
\]
We will use this identity to estimate \( \phi_0 \parallel \parallel \phi_0 \parallel \) in terms of suitable norm of \( \psi_0 \parallel \). To do so we need:

**Lemma 3.3.** It holds
\[
\langle -\ll_{\text{per}}, \phi_0 \parallel, \parallel \phi_0 \parallel \parallel \rangle \geq C \varepsilon \left( \| \partial_t \phi_0 \parallel^2_{L^2(\tilde{D}_r \times \mathbb{R})} + \| \phi_0 \parallel^2_{L^2(\tilde{D}_r \times \mathbb{R})} \right) + C \varepsilon \| \nabla_{\tilde{D}_r}, \phi_0 \parallel^2_{L^2(\tilde{D}_r \times \mathbb{R})}.
\]

**Proof of Lemma 3.3.** We recall the well known fact: with \( \Theta(x) = \tanh \left( \frac{x}{\sqrt{2}} \right) \) the bilinear form
\[
\int_\mathbb{R} |\psi'|^2 - f'(\Theta)\psi^2
\]
is positive definite on the space of functions \( L^2(\mathbb{R}) \) orthogonal to \( \Theta'(x) \). Consider a quadratic form
\[
B(\phi, \phi) = \frac{1}{\varepsilon} \int_{\tilde{D}_r \times \mathbb{R}} |\partial_t \phi|^2 + \varepsilon^2 |\nabla_{\tilde{D}_r}, \phi|^2 - f'(\Theta)|\phi|^2
\]
for \( \phi \in X_0 \). Write
\[
\phi = \phi_1 + \phi_2 \Theta', \quad \text{where} \quad (\phi_1, \Theta') = 0, \quad \phi_2 = \frac{\phi, \Theta'}{\Theta', \Theta'},
\]
and where we have denoted
\[
(\phi, \psi) = \int_\mathbb{R} \phi \overline{\psi} dt
\]
We have
\[
0 = (\phi, \psi) = (\phi, \Theta') + (\phi, \psi - \Theta') = (\Theta', \Theta') + (\phi, \psi - \Theta'),
\]
and also
\[
0 = \nabla_{\tilde{D}_r} (\phi, \psi) = \left( \nabla_{\tilde{D}_r}, \phi, \psi \right) + \left( \phi, \nabla_{\tilde{D}_r}, \psi \right) = \nabla_{\tilde{D}_r}, \phi_2 (\Theta', \Theta')^2 + \left( \nabla_{\tilde{D}_r}, \phi, \psi - \Theta' \right) + \left( \phi, \nabla_{\tilde{D}_r}, \psi \right)
\]
Since
\[
v - \Theta' = U' - \Theta' + \mathcal{O}_{\mathcal{C}^{1,\alpha}_{\tilde{D}_r \times \mathbb{R}}} (\varepsilon^{2-\alpha}) = \mathcal{O}_{\mathcal{C}^{1,\alpha}_{\tilde{D}_r \times \mathbb{R}}} (\varepsilon), \quad \nabla_{\tilde{D}_r}, v = \mathcal{O}_{\mathcal{C}^{1,\alpha}_{\tilde{D}_r \times \mathbb{R}}} (\varepsilon^{2-\alpha})
\]
we get
\[
\|\phi_2 \Theta'\|_{H^1(\tilde{D}_r \times \mathbb{R})} \leq C \varepsilon \|\phi\|_{H^1(\tilde{D}_r \times \mathbb{R})}.
\]
By (3.19) 
\[ B(\phi_1, \phi_1) \geq \frac{C}{\varepsilon} \left( \|\phi_1\|^2_{L^2(D_0 \times \mathbb{R})} + \|\partial_t \phi_1\|^2_{L^2(D_0 \times \mathbb{R})} + \varepsilon^2 \|\nabla \phi_1\|^2_{L^2(D_0 \times \mathbb{R})} \right), \]
hence from (3.20) 
\[ B(\phi, \phi) \geq \frac{C}{\varepsilon} \left( \|\phi\|^2_{L^2(D_0 \times \mathbb{R})} + \|\partial_t \phi\|^2_{L^2(D_0 \times \mathbb{R})} + \varepsilon^2 \|\nabla \phi\|^2_{L^2(D_0 \times \mathbb{R})} \right), \]
for any \( \phi \in \mathcal{X}_\gamma. \) 

We get 
\[ \left\langle -\mathbb{L}_{w_\varepsilon}, \phi_\varepsilon^\parallel \right\rangle = B(\phi_\varepsilon^\parallel, \phi_\varepsilon^\parallel) = \frac{1}{\varepsilon} \int_{D_0 \times \mathbb{R}} \left[ f'(w_\varepsilon) - f'(\Theta) \right] |\phi_\varepsilon^\parallel|^2 \, dV_{D_0} \, dt \]
\[ + \frac{\varepsilon}{2} \int_{D_0 \times \mathbb{R}} \partial_t \left( \varepsilon \frac{1}{\delta} |A_{D_0}|^2 |\phi_\varepsilon^\parallel|^2 \, dV_{D_0} \, dt + \langle \chi_{\varepsilon/\delta} \mathcal{A}_{\varepsilon}, \phi_\varepsilon^\parallel, \phi_\varepsilon^\parallel \rangle + \varepsilon \langle \chi_{\varepsilon/\delta} \mathcal{A}_{\varepsilon}, \phi_\varepsilon^\parallel, \phi_\varepsilon^\parallel \rangle \]
\[ = B(\phi_\varepsilon^\parallel, \phi_\varepsilon^\parallel) + \mathcal{O}(1) + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta^2) \left( \|\phi_\varepsilon^\parallel\|^2_{L^2(D_0 \times \mathbb{R})} + \|\nabla \phi_\varepsilon^\parallel\|^2_{L^2(D_0 \times \mathbb{R})} \right) + \mathcal{O}(\varepsilon^2) \|\nabla \phi_\varepsilon^\parallel\|^2_{L^2(D_0 \times \mathbb{R})}. \]

Since \( \delta \) can be taken as small as we wish the assertion of the Lemma follows. \( \square \)

Now, we need to control the mixed term in (3.17) 
\[ \left\langle -\mathbb{L}_{w_\varepsilon}, (\psi_0 \mathcal{V}), \phi_\varepsilon^\parallel \right\rangle = -\varepsilon \left\langle \mathcal{J}_{D_0} + \chi_{\varepsilon/\delta} \mathcal{A}_{\varepsilon}, \psi_0 \mathcal{V}, \phi_\varepsilon^\parallel \right\rangle - \varepsilon \left\langle \left[ \Delta_{D_0} + \chi_{\varepsilon/\delta} \mathcal{A}_{\varepsilon}, \mathcal{V} \right], \psi_0 \mathcal{V}, \phi_\varepsilon^\parallel \right\rangle \]
\[ + \mathcal{O}(C^\alpha_{D_0 \times \mathbb{R}}) \|\psi_0 \mathcal{V}\|_{H^1(D_0 \times \mathbb{R})} \|\phi_\varepsilon^\parallel\|_{H^1(D_0 \times \mathbb{R})} \]
\[ + \mathcal{O}(C^\alpha_{D_0 \times \mathbb{R}}) \|\nabla \mathcal{V}\|_{L^2(D_0 \times \mathbb{R})} \|\phi_\varepsilon^\parallel\|_{L^2(D_0 \times \mathbb{R})}. \]

where the last equality follows because the coefficients of the operator \( \mathcal{A}_{\varepsilon} \) are bounded by \( \varepsilon t \) and \( \mathcal{V} \) is exponentially decaying in \( t \). By the Cauchy-Schwarz inequality for any some \( \eta > 0 \) small we get 
\[ (3.21) \quad \left\langle -\mathbb{L}_{w_\varepsilon}, (\psi_0 \mathcal{V}), \phi_\varepsilon^\parallel \right\rangle \leq \eta \left( \varepsilon^{-1} \|\phi_\varepsilon^\parallel\|^2_{H^1(D_0 \times \mathbb{R})} + \varepsilon \|\phi_\varepsilon^\parallel\|^2_{H^1(D_0 \times \mathbb{R})} \right) + C \eta \varepsilon^3 \|\psi_0 \mathcal{V}\|^2_{H^1(D_0 \times \mathbb{R})}. \]

It follows from (3.17) and Lemma 3.3 
\[ \frac{1}{\varepsilon} \left( \|\partial_t \phi_\varepsilon^\parallel\|^2_{L^2(D_0 \times \mathbb{R})} + \|\phi_\varepsilon^\parallel\|^2_{L^2(D_0 \times \mathbb{R})} \right) + \varepsilon \|\nabla \phi_\varepsilon^\parallel\|^2_{L^2(D_0 \times \mathbb{R})} \leq C \varepsilon^3 \|\psi_0 \mathcal{V}\|^2_{H^1(D_0 \times \mathbb{R})}. \]

Now consider the orthogonal complement of \( \mathcal{X}_\gamma. \) From (2.26) we obtain: 
\[ \mathbb{L}_{w_\varepsilon}(\psi_0 \mathcal{V}) = \varepsilon \left\langle \mathcal{J}_{D_0} + \chi_{\varepsilon/\delta} \mathcal{A}_{\varepsilon}, \psi_0 \mathcal{V} + \varepsilon \left[ \Delta_{D_0} + \chi_{\varepsilon/\delta} \mathcal{A}_{\varepsilon}, \mathcal{V} \right], \psi_0 \mathcal{V} + \mathcal{O}(\varepsilon^{-\alpha}) \psi_0 \mathcal{V}, \psi_0 \mathcal{V} \right\rangle \]

Using this and projecting (3.16) onto \( \mathcal{V} \) and integrating over \( \mathbb{R} \) we get 
\[ (3.22) \quad \mathcal{J}_{D_0}, \psi_0 \mathcal{V} = T(\psi_0 \mathcal{V}, \phi_0 \mathcal{V}) \]
where 
\[ \left\| T(\psi_0 \mathcal{V}, \phi_0 \mathcal{V}) \right\|_{L^2(D_0 \times \mathbb{R})} \leq C \delta \|\psi_0 \mathcal{V}\|_{H^1(D_0 \times \mathbb{R})} + C \left( \varepsilon^{-1} \|\phi_0 \mathcal{V}\|^2_{L^2(D_0 \times \mathbb{R})} + \varepsilon^2 \|\phi_0 \mathcal{V}\|_{H^1(D_0 \times \mathbb{R})} \right) \]
\[ \leq C \left( \delta \|\psi_0 \mathcal{V}\|_{H^1(D_0 \times \mathbb{R})} + \varepsilon \|\psi_0 \mathcal{V}\|_{H^1(D_0 \times \mathbb{R})} \right) \]
We claim that from this it follows that for any \( \zeta \in (0, 1) \) there exists \( \varepsilon_\zeta > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_\zeta) \) we have \( \psi_0 \mathcal{V} = 0 \) and hence \( \phi_0 = 0. \) To show this claim we note that by definition 
\[ \psi_0 \mathcal{V} = \frac{(\psi_0 \mathcal{V}, \mathcal{V})}{(\mathcal{V}, \mathcal{V})} = \frac{(e^{iCt \mathcal{V}}/T, \mathcal{V})}{(\mathcal{V}, \mathcal{V})} = \frac{(e^{iCt \mathcal{V}}/T, \mathcal{V})}{(\mathcal{V}, \mathcal{V})} = e^{iCt \mathcal{V}}/T \]
where \( \tilde{\psi}_0 \mathcal{V} \) is periodic in \( y_3 \) with period \( T \). We see that \( \psi_0 \mathcal{V} \) satisfies 
\[ \psi_0 \mathcal{V}(y_1, y_2, y_3 + T) = e^{iCt} \psi_0 \mathcal{V}(y_1, y_2, y_3), \quad y = (y_1, y_2, y_3) \in D_0, \]
with similar relation for $\partial_\gamma \psi_0 \zeta$. By Proposition 4.2 in [27] we know that the operator $J_{\delta_\tau}$ is invertible in the space of functions satisfying these conditions as long as $\zeta \in (0, 2\pi)$ with an inverse whose norm depends on $\tau$. The claim now follows from (3.22) and (3.23).

In particular we conclude that the operator $\tilde{L}_{\omega_r}(\zeta)$ is injective for $\zeta \in (0, 2\pi)$ and by the same argument for $\zeta \in (-2\pi, 0)$ (note that $\tilde{L}_{\omega_r}(\zeta) = \tilde{L}_{\omega_r}(-\zeta)$). A version of Lemma 3.1 for $\tilde{L}_{\omega_r}(\zeta)$ shows that this operator is Fredholm, depends analytically on $\zeta$ and, as a consequence, it is invertible in a neighbourhood of $[0, 2\pi]$ except for a discrete set.

Now let us suppose that for some $\zeta \in (0, 2\pi)$ there exists a function $\phi_0 \in H^2_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])$, with some $\gamma$, $|\gamma|$ small, such that $\tilde{L}_{\omega_r} \phi_0 = 0$. Since $\phi_0$ is bounded locally near $\tilde{D}_r$ we can use comparison principle to show that $\phi_0$ is decaying away from $\tilde{D}_r$ at least like $e^{-\sqrt{|\zeta - \pi| - \frac{\pi}{2} \delta_{2\tau}}}$ (the argument is similar to the one leading to (3.15)). Using Lemma 3.1 we get

$$
\| \phi_0 \|_{H^2_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])} \leq C \varepsilon^{-1/2} \| \phi_0 \|_{L^2_{\tau, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])}.
$$

We normalize $\| \phi_0 \|_{H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])} = 1$ and set $\tilde{\phi}_0 = \chi_{\varepsilon/\delta} \psi \phi_0$. With this notation (see (3.2))

$$
\| \tilde{\phi}_0 \|_{L^2(\tilde{D}_r \times \mathbb{R})} \sim \varepsilon^{-1/2} \| \phi_0 \|_{L^2(\{ \text{dist}(x, \tilde{D}_r) < \delta \})}
$$

since $dt dV_{\tilde{D}_r} \sim \varepsilon^{-1} dx$. Similarly, we have

$$
\| \tilde{\phi}_0 \|_{H^1(\tilde{D}_r \times \mathbb{R})} \sim \varepsilon^{-1/2} \| \phi_0 \|_{H^1(\{ \text{dist}(x, \tilde{D}_r) < \delta \})}.
$$

Next, we observe that since $\phi_0$ is decaying exponentially away from $\tilde{D}_r$ we have by (3.24)

$$
\| \tilde{\phi}_0 \|^2_{H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])} = \| \phi_0 \|^2_{H^1_{\gamma, \text{per}}(\{ \text{dist}(x, \tilde{D}_r) < \delta \})} + \| \phi_0 \|^2_{H^1_{\gamma, \text{per}}(\{ \text{dist}(x, \tilde{D}_r) > \delta \})}
$$

$$
\leq \| \phi_0 \|^2_{H^1_{\gamma, \text{per}}(\{ \text{dist}(x, \tilde{D}_r) < \delta \})} + \mathcal{O}(e^{-\delta/\varepsilon}) \| \phi_0 \|^2_{H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])} \cap H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])
$$

$$
\leq \| \phi_0 \|^2_{H^1_{\gamma, \text{per}}(\{ \text{dist}(x, \tilde{D}_r) < \delta \})} + \mathcal{O}(e^{-\delta/\varepsilon}) \| \phi_0 \|^2_{H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])}.
$$

hence

$$
\| \tilde{\phi}_0 \|^2_{H^1_{\gamma, \text{per}}(\{ \text{dist}(x, \tilde{D}_r) < \delta \})} \geq \frac{1}{2} \| \phi_0 \|^2_{H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])} = \frac{1}{2}.
$$

Given all this we claim that we can find a nontrivial function $\tilde{\phi} = \tilde{\phi}_0 + \tilde{\phi}_1$, $\psi \in L^2(\tilde{D}_r \times \mathbb{R})$, such that $\tilde{L}_{\omega_r} \tilde{\phi} = 0$, by solving

$$
\tilde{L}_{\omega_r}(\zeta) \tilde{\phi}_1 = -\left[ \chi_{\varepsilon/\delta}, \tilde{L}_{\omega_r}(\zeta) \right] \tilde{\phi}_0 - e^{-i\varepsilon \zeta^{\tau}/T_\tau} \chi_{\varepsilon/\delta} \left( 1 - \chi_{\varepsilon/\delta} \right) \left[ (e^\varepsilon |A_{\delta_r}|^2 + Q_{\varepsilon}) \partial_t + \varepsilon A_{\delta_r} \right] e^{i\varepsilon \zeta^{\tau}/T_\tau} \tilde{\phi}_0 := R_{\varepsilon/\delta}(y, t).
$$

In fact, since $R_{\varepsilon/\delta}$ is supported in the set $\delta/2 \varepsilon \leq |t| \leq \delta \varepsilon$ therefore

$$
\| R_{\varepsilon/\delta} \|_{L^2(\tilde{D}_r \times \mathbb{R})} \leq C e^{-\delta / \varepsilon} \| \phi_0 \|_{H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])} \cap H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau]).
$$

Next we decompose $\tilde{\phi}_1 = \tilde{\phi}_1^1 + \tilde{\psi}_1 V$ and use (with only slight modifications) the argument that we have used to show that $\tilde{L}_{\omega_r}(\zeta)$ is injective to get:

$$
\| \tilde{\phi}_1 \|_{H^1(\tilde{D}_r \times \mathbb{R})} \leq C \varepsilon^{-1} \| R_{\varepsilon/\delta} \|_{L^2(\tilde{D}_r \times \mathbb{R})} \leq C e^{-\delta / \varepsilon} \| \phi_0 \|_{H^1_{\gamma, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])}.
$$

From (3.25) it now follows

$$
\| \tilde{\phi}_1 \|_{H^1(\tilde{D}_r \times \mathbb{R})} \geq \| \tilde{\phi}_1 \|_{H^1(\tilde{D}_r \times \mathbb{R})} - \| \tilde{\phi}_1 \|_{H^1(\tilde{D}_r \times \mathbb{R})}
$$

$$
\geq C \varepsilon^{-1/2} \| \phi_0 \|_{H^1_{\gamma, \text{per}}(\{ \text{dist}(x, \tilde{D}_r) < \delta \})} + \mathcal{O}(e^{-\delta / \varepsilon}) \| \phi_0 \|_{L^2_{\tau, \text{per}}(\mathbb{R}^2 \times [0, T_\tau])} > 0,
$$

for $\varepsilon$ sufficiently small. This contradicts the fact that $\tilde{L}_{\omega_r}(\zeta)$ is injective. Taking this into account we see that $\tilde{L}_{\omega_r}(\zeta)$ is invertible at least for $\zeta \in (0, 2\pi)$, and thus by the Fredholm alternative is invertible for all $\zeta$ such that $|\text{Im} \zeta| < \delta_{2\tau}$, expect possibly a finite set where $\tilde{L}_{\omega_r}(\zeta)$ has poles. We claim that the required properties of $L_{\omega_r}$ follow now by taking the inverse Fourier-Laplace transform at any $a$ for which $\tilde{L}_{\omega_r}(\zeta)$ is well defined.
for \( \zeta = \mu + iT, \alpha, \mu \in [0, 2\pi] \). Indeed, given \( g \in L^2_{2, \gamma}(\mathbb{R}^3) \) with \( a^2 + \gamma^2 < \delta_\tau \) and cutoff functions \( \chi^\pm(z) \) such that \( \chi^+(z) + \chi^-(z) = 1 \) and \( \text{supp} \chi^+ = (-1, \infty) \) we can solve

\[
L_{w, r} \phi^\pm = \chi^\pm g.
\]

To do this we let \( \hat{g}^\pm \) to be the Fourier-Laplace transforms of \( g^\pm \). Then we solve

\[
\hat{L}_{w, r}(\zeta) \hat{\phi}^\pm = \hat{g}^\pm \implies \hat{\phi}^\pm = \hat{L}_{w, r}(\zeta)^{-1} \hat{g}^\pm,
\]

and by taking the inverse of the Fourier-Laplace transform \( \mathcal{F} \) we determine

\[
\phi^\pm = \mathcal{F}^{-1}(\hat{L}_{w, r}(\zeta)^{-1} \hat{g}^\pm),
\]

and define

\[
\phi = G_{w, r}(g) := \phi^- + \phi^+.
\]

This ends the proof.

\( \square \)

**Remark 3.2.** We will describe a useful consequence of local elliptic estimates. Let us suppose that we know a priori \( \phi, g \in L^2_{a, \gamma}(\mathbb{R}^3) \) where

\[
\Delta \phi = g.
\]

The goal is to obtain weighted Sobolev estimates for the derivatives of \( \phi \). First, consider a cube \( Q_r(x_0) \) centred at \( x_0 \in \mathbb{R}^3 \) and with its sides equal to \( r \). Standard elliptic estimates show

\[
r \| D^2 \phi \|_{L^2(Q_r(x_0))} + \| \nabla \phi \|_{L^2(Q_r(x_0))} \leq C \| g \|_{L^2(Q_{2r}(x_0))} + Cr^{-1} \| \phi \|_{L^2(Q_{2r}(x_0))},
\]

If \( r = \varepsilon \) then we get from this

\[
\varepsilon \| D^2 \phi \|_{L^2_{\gamma}(\mathbb{R}^3)} + \| \nabla \phi \|_{L^2_{\gamma}(\mathbb{R}^3)} \leq C \| g \|_{L^2_{\gamma}(\mathbb{R}^3)} + C \varepsilon^{-1} \| \phi \|_{L^2_{\gamma}(\mathbb{R}^3)},
\]

since the exponential weights are comparable on the sets with diameters proportional to \( \varepsilon \). Arranging now a countable collection of cubes \( \{ Q_{\varepsilon}(x_j) \} \in \mathbb{N} \) in such a way that for each \( x_j \) the number of cubes \( Q_{\varepsilon}(x_j') \), \( j' \neq j \), whose intersection with \( Q_{\varepsilon}(x_j) \) is nonempty is finite and bounded independently on \( j \), while at the same time \( \mathbb{R}^3 = \bigcup_{j \in \mathbb{N}} Q_{\varepsilon}(x_j) \), we see that above local estimates can be summed up to yield:

\[
\varepsilon \| D^2 \phi \|_{L^2_{\gamma}(\mathbb{R}^3)} + \| \nabla \phi \|_{L^2_{\gamma}(\mathbb{R}^3)} \leq C \| g \|_{L^2_{\gamma}(\mathbb{R}^3)} + C \varepsilon^{-1} \| \phi \|_{L^2_{\gamma}(\mathbb{R}^3)}.
\]

**Lemma 3.4.** Let \( \phi \in L^2_{a, \gamma}(\mathbb{R}^3) \) be a solution of \( L_{w, r} \phi = g \) with \( g \in L^2_{a, \gamma}(\mathbb{R}^3) \) where \( \gamma > 0 \), \( \gamma' < \gamma \) and \( a^2 + \gamma^2 < \delta_\tau \), \( a^2 + \gamma'^2 < \delta_\tau \). Then \( \phi \in L^2_{a, \gamma}(\mathbb{R}^3) \). An analogous statement holds when we assume that \( \gamma < 0 \) and \( \gamma' < \gamma \).

**Proof.** We follow the proof of a similar result in [11]. Let \( \chi \) be a cutoff function supported in the set \( \varepsilon M < r - \rho_\tau(z) \), \( r \equiv x_1^2 + x_2^2 \), and such that \( \chi \equiv 1 \) in the set \( r - \rho_\tau(z) > 2\varepsilon M \) where \( M \) is chosen so that \( f'(w_\tau) < -1 \) for \( r - \rho_\tau(z) > \varepsilon M \). We calculate

\[
L_{w, r}(\chi \phi) = \chi g + [\varepsilon \Delta, \chi] \phi \equiv g_1, \quad [\varepsilon \Delta, \chi] \phi = \varepsilon \Delta(\chi \phi) - \varepsilon \chi \Delta \phi.
\]

We have \( \varepsilon \nabla \chi = O(1) \) and \( \varepsilon \Delta \chi = O(\varepsilon^{-1}) \). Moreover, by local elliptic estimates applied to the equation

\[
\Delta \phi = \varepsilon^{-1} g + \varepsilon^{-2} f'(w_\tau) \phi
\]

we can show that (see Remark 3.2)

\[
\| \varepsilon \nabla \chi \cdot \nabla \phi \|_{L^2_{\gamma}(\mathbb{R}^3)} \leq C \varepsilon^{-1} \| g \|_{L^2_{\gamma}(\mathbb{R}^3)} + C \varepsilon^{-2} \| \phi \|_{L^2_{\gamma}(\mathbb{R}^3)}.
\]

We find from this

\[
\| \varepsilon \nabla \chi \cdot \nabla \phi \|_{L^2_{\gamma}(\mathbb{R}^3)} + \| \varepsilon \phi \Delta \chi \|_{L^2_{\gamma}(\mathbb{R}^3)} \leq C \varepsilon^{-1} \| g \|_{L^2_{\gamma}(\mathbb{R}^3)} + C \varepsilon^{-2} \| \phi \|_{L^2_{\gamma}(\mathbb{R}^3)}.
\]

Above, we use the fact that the weighted norms \( \| \cdot \|_{L^2_{a, \gamma}(\mathbb{R}^3)} \) and \( \| \cdot \|_{L^2_{a, \gamma}(\mathbb{R}^3)} \) are comparable in the set \( r - \rho_\tau(z) \in [\varepsilon M, 2\varepsilon M] \). From this we obtain

\[
\| g_1 \|_{L^2_{\gamma}(\mathbb{R}^3)} \leq C \varepsilon^{-1} \| g \|_{L^2_{\gamma}(\mathbb{R}^3)} + C \varepsilon^{-2} \| \phi \|_{L^2_{\gamma}(\mathbb{R}^3)}.
\]
Now we solve the problem

\[ L_w \phi_{1,R} = g_1, \quad \text{in } \Omega_{\varepsilon M,R}, \]
\[ \phi_{1,R} = 0, \quad \text{on } \Omega_{\varepsilon M,R}. \]

in a bounded set \( \Omega_{\varepsilon M,R} = \{ \varepsilon M < r - \rho_r(z) < R, |z| < R \} \). Using similar argument as the one leading to (3.12) in the proof of Lemma 3.1 we get

\[
\| \phi_{1,R} \|_{H^2_{\alpha,\gamma}(\Omega_{\varepsilon M,R})} \leq C \varepsilon^{-1/2} \| g_1 \|_{L^2_{\alpha,\gamma}(\mathbb{R}^3)} \]
\[
\leq C \varepsilon^{-3/2} \| g \|_{L^2_{\alpha,\gamma}(\mathbb{R}^3)} + C \varepsilon^{-5/2} \| \phi \|_{L^2_{\alpha,\gamma}(\mathbb{R}^3)}. \]

Note the that in the first of the above inequalities only the right hand side of the equation appears, which is due to the fact that we assumed homogeneous Dirichlet boundary conditions on \( \phi_{1,R} \) and we do not need to introduce the cut off function \( \chi_{\varepsilon M} \) in proving a version of Lemma 3.1 needed here. Letting \( R \to \infty \) we get a solution \( \phi_1 \) of the equation \( L_w \phi_1 = g_1 \) but now in the set \( \varepsilon M < r - \rho_r(z) \), such that

\[
\| \phi_1 \|_{H^2_{\alpha,\gamma}(\mathbb{R}^3)} \leq C \varepsilon^{-3/2} \| g \|_{L^2_{\alpha,\gamma}(\mathbb{R}^3)} + C \varepsilon^{-5/2} \| \phi \|_{L^2_{\alpha,\gamma}(\mathbb{R}^3)}. \]

We also have \( L_w(\chi \phi - \phi_1) = 0 \) and \( \phi_1 = \chi \phi = 0 \) along the surface \( \varepsilon M = r - \rho_r(z) \). Then, an estimate similar to (3.12), shows that actually \( \chi \phi = \phi_1 \). Similar argument applied in the set \( -\varepsilon M > r - \rho_r(z) \) ends the proof. \( \square \)

3.4. The deficiency space and the kernel of \( L_w \). Let us summarize our results so far. Let \( g \in L^2_{\alpha,\gamma}(\mathbb{R}^3) \) with \( a^2 + \gamma^2 < \delta_\varepsilon \), and cutoff functions \( \chi^\pm(z) \) such that \( \chi^+(z) + \chi^-(z) = 1 \) and \( \text{supp } \chi^\pm = [-1, \infty) \) be given.

(i) As in Proposition 3.1 we can solve

\[ L_w \phi_{\pm} = \chi^\pm g \]

where \( \phi^\pm \in H^2_{|a|,\gamma}(\mathbb{R}^3) \) (except for a finite set of \( a \)).

(ii) If we have \( L_w, \phi = g, \phi \in L^2_{a,\gamma}(\mathbb{R}^3), g \in L^2_{\alpha,\gamma}(\mathbb{R}^3) \) with \( \gamma > 0 \) and \( \gamma' < \gamma \) then \( \phi \in H^2_{a,\gamma}(\mathbb{R}^3) \). In particular if \( g \) is decaying exponentially away from the surface \( D_r \), so that we have \( g \in L^2_{a,\gamma}(\mathbb{R}^3) \) then \( \phi \in H^2_{a,\gamma}(\mathbb{R}^3) \cap H^2_{a,\gamma}(\mathbb{R}^3) \). This means that the decay rate of the solution away from the nodal set improves together with the rate of decay of the right hand side.

(iii) When the right hand side decays both along the nodal set and in the direction transversal to it, for example \( g \in L^2_{a,\gamma}(\mathbb{R}^3) \cap L^2_{a,\gamma}(\mathbb{R}^3) \), with \( a > 0 \), then we can use the parametrix to solve the equation \( L_w \phi^+ = \chi^+ g \) and determine a solution \( \phi^+ \) such that \( \chi^+ \phi^+ \in L^2_{a,\gamma}(\mathbb{R}^3) \cap L^2_{a,\gamma}(\mathbb{R}^3) \). At the same time we can find another solution \( \phi^- \), such that \( \chi^- \phi^- \in L^2_{a,\gamma}(\mathbb{R}^3) \cap L^2_{a,\gamma}(\mathbb{R}^3) \) and we get the following decomposition:

\[ \phi^+ = \sum_{j=1}^{k} Z^+_j + \phi^+_1, \]

where \( Z^+_j \) are in the kernel of the operator \( L_w \). Then we have

\[ \phi^+ = \chi^+ \phi^+ + \chi^- \phi^+ = \chi^+ \phi^+ + \chi^- \phi^+_1 + \sum_{j=1}^{k} Z^+_j \chi^- \]

where \( (\chi^+ \phi^+ + \chi^- \phi^+_1) \in H^2_{a,\gamma}(\mathbb{R}^3) \). Of course we can argue similarly for the equation \( L_w \phi^- = \chi^- g \) and thus at the end we get the following formula

\[ \phi = \phi_0 + \sum_{j=1}^{k} \chi^- Z^+_j + \sum_{j=1}^{k} \chi^+ Z^-_j, \]

where \( \phi_0 \in H^2_{a,\gamma}(\mathbb{R}^3) \). This is the so called linear decomposition formula. It says that any solution to \( L_w \phi = g \) can be decomposed into an exponentially decaying part and and a linear combination of \( 2k \) functions which are related to the residues of \( \hat{L}_w^{-1}(\zeta) \) at its poles. We say that these functions belong to the deficiency space. Clearly the elements of the kernel of \( L_w \) (which is \( k \) dimensional) belong to the deficiency space and thus removing them from it we obtain a space on which \( L_w \) is an isomorphism (see Lemma (3.5) below).
Before stating precisely the next Lemma we introduce weighted Sobolev spaces
\[ L_{a,\gamma}(\mathbb{R}^3) := L_{a,\gamma}^2(\mathbb{R}^3) \cap L_{a,-\gamma}^2(\mathbb{R}^3), \quad H_{a,\gamma}(\mathbb{R}^3) := H_{a,\gamma}^2(\mathbb{R}^3) \cap H_{a,-\gamma}^2(\mathbb{R}^3). \]

Note that \( \phi \in L_{a,\gamma}(\mathbb{R}^3) \) decays away from \( D_r \) as \( \cosh^{-\gamma} \left( \frac{r - r_0(z)}{\varepsilon} \right) \) if \( \gamma > 0 \), and decays (for \( a > 0 \)) or grows (for \( a < 0 \)) along \( D_r \) at the rate \( \cosh^{-a} z \). Based on observations (i)–(iii) we have:

**Lemma 3.5.** Let \( \gamma > 0, a > 0 \), with \( a^2 + \gamma^2 < \delta \) and let us define the deficiency space

\[ D_{w_r} := \text{span} \{ \chi^+ Z_j, \chi^- Z_j, j = 1, \ldots, k; Z_j \in \text{Ker} L_{w_r} \}. \]

We further decompose \( D_{w_r} = \mathcal{K}_{w_r} \oplus \mathcal{E}_{w_r} \), where \( \mathcal{K}_{w_r} = \text{Ker} L_{w_r} \). Then the operator

\[ L_{w_r} : H_{a,\gamma}^2(\mathbb{R}^3) \oplus \mathcal{E}_{w_r} \rightarrow \tilde{L}_{a,\gamma}^2(\mathbb{R}^3) \]

\[ \phi \mapsto L_{w_r} \phi. \]

is an isomorphism.

Note that \( \dim \mathcal{E}_{w_r} = k = \dim \mathcal{K}_{w_r} \) and that we know already that \( k \geq 6 = \dim \mathcal{I}_{w_r} \) where the linear subspace \( \mathcal{I}_{w_r} \) was defined in (2.28). We will show next that indeed \( k = 6 \).

**Proposition 3.2.** We have \( \mathcal{K}_{w_r} = \mathcal{I}_{w_r} \).

**Proof of Proposition 3.2.** The idea of the proof is to relate the kernel of the operator \( L_{w_r} \) with the space of the Jacobi fields of the operator \( J_{D_r} \), that is explicitly known and in particular its dimension is 6. Let us consider a \( \phi \in \mathcal{K}_{w_r} \). \textit{A priori} it may happen that \( \phi \) is exponentially increasing in the \( z \) variable but we know already (see the argument leading to (3.24) and also Lemma 3.4 and Remark 3.1) that it must be decaying at least like \( \cosh^{-\gamma} \left( \frac{r - r_0(z)}{\varepsilon} \right) \) with some \( \gamma > 0 \). In particular all integrations with respect to the transversal direction to \( D_r \) that will appear below are justified.

Next, we note that formula (2.26) suggests that near the surface \( D_r \) the elements of \( \mathcal{K}_{w_r} \) should be proportional, asymptotically as \( \varepsilon \to 0 \), to \( V \) times a function on \( D_r \). To make this rigorous we first prove the following:

**Lemma 3.6.** Let \( \phi \in \mathcal{K}_{w_r} \) be such that

\[ \int_{\mathbb{R}^2} (V \phi)(y, t) V(y, t) \chi_{\varepsilon/3}(t) \, dt = 0, \quad \forall y \in D_r. \]

Then we have \( \phi \equiv 0 \).

**Proof of Lemma 3.6.** As we have pointed out it is not hard to show that \( \phi \) decays exponentially like \( \cosh^{-\gamma} \left( \frac{r - r_0(z)}{\varepsilon} \right) \) and so we can compute

\[ \int_{\mathbb{R}^2} \phi^2(x, z) \, dx = h(z), \quad x = (x_1, x_2). \]

Direct calculation shows:

\[ \frac{\varepsilon}{2} \frac{d^2 h}{dz^2} = \int_{\mathbb{R}^2} [\varepsilon |\nabla_x \phi|^2 - \frac{1}{\varepsilon} f'(w_r) \phi^2] \, dx + \varepsilon \int_{\mathbb{R}^2} |\partial_x \phi|^2 \, dx. \]

We claim that the orthogonality condition (3.28) implies

\[ \int_{\mathbb{R}^2} [\varepsilon |\nabla_x \phi|^2 - \frac{1}{\varepsilon} f'(w_r) \phi^2] \, dx \geq \frac{\kappa}{\varepsilon} \int_{\mathbb{R}^2} \phi^2(x, z) \, dx = \frac{\kappa}{\varepsilon} h, \]

with some constant \( \kappa > 0 \). To prove this claim we need:

**Lemma 3.7.** There exists a constant \( \kappa > 0 \) such that for any sufficiently large \( R \) and any \( v \in H^1((-R, R)) \) it holds

\[ \int_{-R}^{R} |v'|^2 - f'(\Theta)v^2 \geq \kappa \int_{-R}^{R} v^2 \quad \text{whenever} \quad \int_{-R}^{R} v \Theta' \chi_R = 0, \]

where \( \chi_R \) is a smooth cutoff function supported in \((-R, R)\) such that \( \chi_R(x) = 1 \) in \((-R/2, R/2)\).
A proof of this Lemma (using for instance (3.19) as a point of departure) is omitted.
Changing to Fermi coordinates we have in \( \mathcal{N}_3 \):

\[
|\nabla_x \phi|^2 = \frac{1}{\varepsilon^2} |\partial_t Y_*^* \phi|^2 + \mathcal{O}(\varepsilon) |\partial_t Y_*^* \phi|^2 + \mathcal{O}(\varepsilon) |\partial_\theta Y_*^* \phi|^2.
\]

Next, for a fixed \( z \) we consider a diffeomorphism \((x_1, x_2) \mapsto (\theta, t)\) defined by

\[
x_j = (X(s, \theta) + \varepsilon t N_r(s, \theta)) \cdot e_j,
\]

where \( s = s(\theta, t; z) \) is determined from

\[
z = (X(s, \theta) + \varepsilon t N_r(s, \theta)) \cdot e_3.
\]

The Jacobian matrix of this map can be calculated explicitly but for our purpose it is enough to note that

\[
dx_1 dx_2 = \varepsilon \mu_0(\theta) d\theta dt + \varepsilon^2 \mu_1(\theta, t) d\theta dt,
\]

where \( \mu_0, \mu_1 \) are positive densities and

\[
|\mu_1(\theta, t)| \leq C.
\]

From this we find

\[
\int_{\mathbb{R}^2} |\nabla_x \phi|^2 - \frac{1}{\varepsilon^2} f'(w_r) \phi^2\ dx \geq \int_0^{2\pi} \left\{ \int_{|t| \leq \delta/\varepsilon} \left[ |\partial_t Y_*^* \phi|^2 - f' (\Theta(t)) |Y_*^* \phi|^2 \right] dt \right\} \mu_0 d\theta
\]

\[
+ \int_{\mathbb{R}^2 \setminus \mathcal{N}_3} \left[ |\nabla_x \phi|^2 + \frac{1}{\varepsilon^2} \phi^2 \right] dx - K(\delta + \varepsilon) \int_{\mathbb{R}^2} \left[ |\nabla_x \phi|^2 + \frac{1}{\varepsilon^2} \phi^2 \right] dx
\]

(3.32)

The potential \( f'(w_r) \) in the first line on the left can be replaced by \( f'(\Theta) \) on the right of this line since \( Y_*^* w_r = \Theta + \mathcal{O}(\varepsilon) \). The term in the second line above appears because \( f'(w_r) < -2 + \eta \) in the complement of \( \mathcal{N}_3 \). Finally, all the other terms are of smaller size and can be controlled by the integral in the third line times \( K(\delta + \varepsilon) \), where \( K \) is a constant. Using Lemma 3.7 and going back to the original variables we get

\[
\int_0^{2\pi} \left\{ \int_{|t| \leq \delta/\varepsilon} \left[ |\partial_t Y_*^* \phi|^2 - f' (\Theta(t)) |Y_*^* \phi|^2 \right] dt \right\} \mu_0 d\theta \geq \frac{C}{\varepsilon} \int_{\mathcal{N}_3} \phi^2\ dx.
\]

It follows

\[
\int_{\mathbb{R}^2} |\nabla_x \phi|^2 - \frac{1}{\varepsilon^2} f'(w_r) \phi^2\ dx \geq \frac{C}{\varepsilon} \int_{\mathbb{R}^2} \phi^2\ dx - K(\varepsilon + \delta) \int_{\mathbb{R}^2} \left[ |\nabla_x \phi|^2 + \frac{1}{\varepsilon^2} \phi^2 \right] dx,
\]

hence

\[
[1 + K(\varepsilon + \delta)] \int_{\mathbb{R}^2} |\nabla_x \phi|^2 - \frac{1}{\varepsilon^2} f'(w_r) \phi^2\ dx \geq \frac{C}{\varepsilon} \int_{\mathbb{R}^2} \phi^2\ dx - \frac{K(\varepsilon + \delta)}{\varepsilon} \int_{\mathbb{R}^2} [1 + f'(w_r)] \phi^2\ dx
\]

which gives (3.30) provided that \( \varepsilon \) and \( \delta \) are small enough.

From (3.30) and (3.29) we find

\[
\frac{\varepsilon}{2} \frac{d^2 h}{dz^2} - \frac{\kappa}{\varepsilon} h > 0.
\]

By Lemma 3.5 we know a priori that \( \phi \), hence \( h \), is growing in \( z \) at \( \pm \infty \) at some exponential rate which is independent on \( \varepsilon \). Applying the comparison principle we see that \( h \), and hence \( \phi \), is actually decaying as \( z \to \pm \infty \), at some exponential rate proportional to \( \varepsilon^{-1} \). Using again orthogonality condition (3.28) we calculate

\[
\langle -L_w, \phi \rangle = \int_{\mathbb{R}^3} \left[ \varepsilon |\nabla \phi|^2 - \frac{1}{\varepsilon^2} f'(w_r) \phi^2 \right] dx dz \geq c \varepsilon^{-1} \| \phi \|_{L^2(\mathbb{R}^3)}^2,
\]

hence \( \phi \equiv 0 \) as claimed. This ends the proof of the Lemma.

\[\square\]
We continue with the proof of the Proposition. For a given \( \phi \in K_{w_r} \), we define
\[
\varphi = (Y_\varepsilon^* \phi) \chi_{\delta/\varepsilon}.
\]
The function \( \varphi \) is a cutoff of \( Y_\varepsilon^* \phi \) and is supported in \( N_\delta \). Since \( \phi \in H^2_{a,\gamma} (\mathbb{R}^3) \) with some \( a \in \mathbb{R} \), and \( \gamma > 0 \) both small (\( \phi \) decays or grows in \( z \) like \( \cosh^{-a} z \), and it decays like \( \cosh^{-\gamma} (\frac{r-\rho(z)}{\varepsilon}) \) away from \( D_r \)) we have that \( \varphi \in H^2_{a,\gamma} (D_r \times \mathbb{R}) \) with some \( a_s \in \mathbb{R} \) and \( \gamma_s > 0 \) both small. We also have
\[
\| \varphi \|_{L^2_{a,\gamma,\gamma_s,\Gamma} (D_r \times \mathbb{R})} \leq C \varepsilon^{-1/2} \| \phi \|_{L^2_{a,\gamma} (\mathbb{R}^3)},
\]
with similar estimates for other Sobolev norms. Note that since \( \phi \) decays like \( \cosh^{-\gamma} (\frac{r-\rho(z)}{\varepsilon}) \) away from \( D_r \) then \( \varphi \) decays at least like \( \cosh^{-\tilde{\gamma}} t \) with some \( \tilde{\gamma} > 0 \). Above estimate holds then for any \( \gamma_s < \tilde{\gamma} \) and we will consider only \( \gamma_s \) restricted this way.

In what follows we will argue by contradiction and we will assume that \( \dim K_{w_r} > 6 \). Since we know explicitly six linearly independent elements in \( K_{w_r} \), which are the geometric Jacobi fields spanning the subspace \( \mathcal{I}_{w_r} \), defined in (2.28) we can find a function \( \phi \in K_{w_r} \), such that \( \phi \notin \mathcal{I}_{w_r} \) and in particular we can assume
\[
(3.33) \quad \int_{D_r \times \mathbb{R}} \chi_{\delta/\varepsilon} (Y_\varepsilon^* \phi) (Y_\varepsilon^* \Phi) \cosh^{a_s} (s) \, dV_{D_r} \, dt = 0, \quad \forall \Phi \in \mathcal{I}_{w_r}.
\]

We decompose
\[
\varphi = \psi \mathcal{V} + \varphi^\parallel, \quad \int_{\mathbb{R}} \chi_{\delta/\varepsilon} \varphi^\parallel (y, t) \mathcal{V}(y, t) \, dt = 0.
\]

From Lemma 3.6 we know that \( \psi \neq 0 \) and therefore we can assume \( \| \psi \|_{L^2_{a,\gamma,\Gamma} (D_r)} = 1 \) (indeed we expect \( \| \varphi^\parallel \|_{L^2_{a,\gamma,\Gamma} (D_r \times \mathbb{R})} = o(1) \)). We compute
\[
\| \psi \|_{L^2_{a,\gamma,\Gamma} (D_r \times \mathbb{R})} = \| \psi \|_{L^2_{a,\gamma,\Gamma} (D_r \times \mathbb{R})} = 1
\]

where, more explicitly,
\[
(Y_\varepsilon^* L_{w_r}) \varphi = \varepsilon^{-1} ((Y_\varepsilon^* \phi) \partial_t \chi_{\delta/\varepsilon} + 2 \partial_t (Y_\varepsilon^* \phi) \partial_t \chi_{\delta/\varepsilon}) - (H_{D_r} + \varepsilon \tau |A_{D_r}|^2 + Q_\varepsilon) Y_\varepsilon^* \phi \partial_t \chi_{\delta/\varepsilon}.
\]

It is not hard to see that
\[
\| \chi_{\delta/\varepsilon} \Phi^\parallel \|_{L^2_{a,\gamma,\Gamma} (D_r \times \mathbb{R})} \leq O (e^{-c_\delta/\varepsilon}) \| \Phi^\parallel \|_{H^2_{a,\Gamma} (D_r \times \mathbb{R})},
\]
since \( a_s < \tilde{\gamma} \). Using this we can calculate
\[
\int_{\mathbb{R}} \chi_{\delta/\varepsilon} \Phi^\parallel \mathcal{V} \, dt = \int_{\mathbb{R}} \chi_{\delta/\varepsilon} \Phi^\parallel \mathcal{V} \, dt = T (\varphi, \psi),
\]

where \( T \) is a linear operator satisfying
\[
(3.35) \quad \| T (\varphi^\parallel, \psi) \|_{L^2_{a,\Gamma} (D_r)} \leq C \varepsilon^{-1} \| \varphi^\parallel \|_{L^2_{a,\Gamma} (D_r \times \mathbb{R})} + C \varepsilon \| \varphi^\parallel \|_{H^2_{a,\Gamma} (D_r \times \mathbb{R})} + C \varepsilon^{1-a} \| \psi \|_{H^2_{a,\Gamma} (D_r)} + C \delta \| \psi \|_{H^2_{a,\Gamma} (D_r)},
\]

with some \( \alpha \in (0, 1) \). Next we will estimate \( \varphi^\parallel \). Since this argument is similar to that of Proposition 3.1 we will outline the main points omitting some tedious but straightforward calculations. Let \( K > 0 \) be a large constant and \( \chi^\pm : \mathbb{R} \rightarrow \mathbb{R}_{+} \) be smooth cutoff functions such that \( \chi^+ + \chi^- \equiv 1 \), \( \chi^+(s) = 1 \) when \( s > 1 \) and \( \chi^+(s) = 0 \) when \( s < -K \) and additionally \( K |\chi^\pm| + |K^2| |\chi^\parallel| \leq C \).

We define \( \varphi^\parallel = \chi^\parallel \varphi^\parallel \). Taking the Fourier-Laplace transform (with respect to \( s \)) we get
\[
(3.34) \quad (\mathbb{L}_{w_r} \varphi^\parallel)^\wedge = (\chi^\pm \mathbb{L}_{w_r} \varphi^\parallel)^\wedge + \left( [\mathbb{L}_{w_r}, \chi^\pm] \varphi^\parallel \right)^\wedge = (\chi^+ \mathbb{g})^\wedge = (\chi^+ \mathbb{L}_{w_r} (\psi \mathcal{V}))^\wedge + \left( [\mathbb{L}_{w_r}, \chi^\pm] \varphi^\parallel \right)^\wedge.
\]
We can project
\[
\int_{[0,T_r] \times [0,2\pi] \times \mathbb{R}} \xi^\pm (L_{\psi \chi}, \Phi^\pm) = \int_{[0,T_r] \times [0,2\pi] \times \mathbb{R}} \xi^\pm (\chi^\pm g) \\
(3.36)
- \int_{[0,T_r] \times [0,2\pi] \times \mathbb{R}} \xi^\pm (\chi^\pm L_{w' \psi \chi \psi \chi}) \\
+ \int_{[0,T_r] \times [0,2\pi] \times \mathbb{R}} \xi^\pm \left( [L_{w' \psi} \chi^\pm] \Phi^\pm \right)
\]
Since we have
\[
\int_{\mathbb{R}} \chi_{\varepsilon} \xi^\pm \Lambda^\pm \psi \bar{\psi} \, dt = 0,
\]
therefore the bilinear form on the left hand side in (3.36) is positive definite and by an argument similar to the one in Proposition 3.1 we get
\[
\left| \int_{[0,T_r] \times [0,2\pi] \times \mathbb{R}} \xi^\pm (L_{\psi \chi}, \Phi^\pm) \right| \geq C \| \xi^\pm \|_{L^2([0,T_r] \times [0,2\pi] \times \mathbb{R})}^2 \geq \frac{C}{\varepsilon} \| \xi^\pm \|_{L^2_{\psi \chi \psi \chi}}^2 + \varepsilon \| D^2 \xi^\pm \|_{L^2_{\psi \chi \psi \chi}}^2
\]
where the last inequality follows from Plancherel's identity. Using Cauchy-Schwarz inequality and Plancherel identity again on the right hand side of (3.36) we find
\[
\varepsilon^{-1} \| \xi^\pm \|_{L^2_{\psi \chi \psi \chi}} \leq C \left( \| \chi^\pm g \|_{L^2_{\psi \chi \psi \chi}} + \| \chi^\pm L_{w' \psi \chi \psi \chi} \|_{L^2_{\psi \chi \psi \chi}} + \left\| \left[ L_{w' \psi} \chi^\pm \right] \Phi^\pm \right\|_{L^2_{\psi \chi \psi \chi}} \right)
\]
Using an argument similar to the one indicated in Remark 3.1 and Remark 3.2 we can show from this
\[
R \equiv \left( \| \chi^\pm g \|_{L^2_{\psi \chi \psi \chi}} + \| \chi^\pm L_{w' \psi \chi \psi \chi} \|_{L^2_{\psi \chi \psi \chi}} + \left\| \left[ L_{w' \psi} \chi^\pm \right] \Phi^\pm \right\|_{L^2_{\psi \chi \psi \chi}} \right)
(3.37)
\]
We have
\[
\| \chi^\pm g \|_{L^2_{\psi \chi \psi \chi}} \leq O(e^{-\varepsilon g}) \left( \| \xi^\pm \|_{H^2_{\psi \chi \psi \chi}} + \| \psi \|_{H^2_{\psi \chi \psi \chi}} \right)
\]
\[
\| \chi^\pm L_{w' \psi \chi \psi \chi} \|_{L^2_{\psi \chi \psi \chi}} \leq C \| \xi^\pm \|_{H^2_{\psi \chi \psi \chi}}
\]
Combining these inequalities we get from (3.37)
\[
(3.38) \quad \varepsilon^{-1} \| \xi^\pm \|_{L^2_{\psi \chi \psi \chi}} + \varepsilon \| D^2 \xi^\pm \|_{L^2_{\psi \chi \psi \chi}} \leq C \| \psi \|_{H^2_{\psi \chi \psi \chi}}
\]
This and estimate (3.35) imply
\[
\| T(\psi^\pm, \psi) \|_{L^2_{\psi \chi \psi \chi}} \leq C \| T(\psi^\pm, \psi) \|_{L^2_{\psi \chi \psi \chi}} \leq C \| \psi \|_{H^2_{\psi \chi \psi \chi}} + C \| \psi \|_{H^2_{\psi \chi \psi \chi}}.
\]
Decomposing \( \psi = \psi^+ + \psi^- \), we can use the Fourier-Laplace transform to show that
\[
\psi = \psi^0 + \psi^1
\]
where \( \psi^1 \) is a linear combination of the geometric Jacobi fields and
\[
\| \psi^1 \|_{H^2_{\psi \chi \psi \chi}} \leq C \| T(\psi^\pm, \psi) \|_{L^2_{\psi \chi \psi \chi}} \leq C \| \psi \|_{H^2_{\psi \chi \psi \chi}} + C \| \psi \|_{H^2_{\psi \chi \psi \chi}}.
(3.39)
\]
At the same time from (3.33), (3.38) and Lemma 2.1 we see that \( \psi \) satisfies
\[
\int_{D_r} \psi \Phi^\pm \cosh a^\pm s dV_{D_r} = 0 \implies \int_{D_r} \psi_0 \Phi^\pm \cosh a^\pm s dV_{D_r} = - \int_{D_r} \psi_1 \Phi^\pm \cosh a^\pm s dV_{D_r}
\]
for each geometric Jacobi field \( \Phi^\pm \) of \( J_{D_r} \). It follows that
\[
\| \psi \|_{L^2_{\psi \chi \psi \chi}} \leq C \| \psi \|_{L^2_{\psi \chi \psi \chi}}
\]
which, together with (3.39), implies \( \psi^1 \equiv 0 \) hence \( \psi \equiv 0 \), which is a contradiction. The proof of the proposition is complete.
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