The Bohr-type operator on analytic functions and sections

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\textbf{ABSTRACT}

In this paper, first we prove two refined Bohr-type inequalities associated with area for bounded analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk. Later, we establish the Bohr-type operator on analytic functions and sections.

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\section{1. Preliminaries and some basic questions}

There has been an intensive research activity on Bohr’s phenomenon, examined first in 1914 by Bohr [1]. The main purpose of this article is to continue the investigation on the classical Bohr inequality in the refined formulation studied recently in [2,3] for the case of analytic functions bounded in the unit disk. See the recent survey articles [4–6] and [7, Chapter 8]. Bohr’s idea naturally extends to functions of several complex variables. The interest in the Bohr phenomena was revived in the 90s due to extensions to holomorphic functions of several complex variables and to more abstract settings. The Bohr radius for analytic functions from the unit disk into special domains (e.g. the punctured unit disk, the exterior of the closed unit disk and concave wedge domains) have been discussed in [8–11]. Ali et al. [12] considered Bohr’s phenomenon for even and odd analytic functions and for alternating series. This study was continued by Kayumov and Ponnusamy [13,14], which in turn settled one of the conjectures, proposed in [12], on Bohr radius for odd analytic functions. In continuation of the investigation on this topic, the authors in [15–17] concerned the Bohr radius for the class of all sense-preserving harmonic mappings and sense-preserving $K$-quasiconformal harmonic mappings. In [18,19], authors demonstrated the classical Bohr inequality using different methods of operators. Several other aspects and generalizations of Bohr’s inequality may be obtained from [13,20–23] and the references therein for some detailed account of work.
on this topic. In particular, after the appearance of the articles by Abu Muhanna et al. [4] and Kayumov and Ponnusamy [14], several investigations and new problems on Bohr’s inequality in the unit disk case have appeared in the literature (cf. [3,24–30]).

1.1. Classical inequality of H. Bohr

Let $\mathcal{A}$ denote that class of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{B} = \{f \in \mathcal{A} : |f(z)| \leq 1 \text{ in } \mathbb{D}\}$. For a fixed $z \in \mathbb{D}$, let $\mathcal{F}_z = \{f(z) : f \in \mathcal{A}\}$ and introduce the Bohr operator $B_r(f)$ on $\mathcal{F}_z$, $|z| = r$, by

$$B_r(f) := \sum_{k=0}^{\infty} |a_k|r^k.$$ 

In 1914, a classical result of Bohr [1] states the following:

**Theorem A (Bohr [1]):** If $f \in \mathcal{B}$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$B_r(f) \leq 1 \text{ for all } |z| = r \leq 1/3,$$

and constant $1/3$ cannot be improved.

Bohr actually obtained that the inequality (1) is true only when $r \leq 1/6$. Later Riesz, Schur and Wiener independently established the Bohr inequality (1) for $r \leq 1/3$ and that $1/3$ is the best possible constant which is called the Bohr radius for the space $\mathcal{B}$. Indeed for the function

$$\varphi_a(z) = \frac{a - z}{1 - az}, \quad a \in [0, 1),$$

it follows easily that $B(\varphi_a, r) > 1$ if and only if $r > 1/(1 + 2a)$, which for $a \to 1$ shows that $1/3$ is optimal. Bohr’s and Wiener’s proofs can be found in [1]. Other proofs of Bohr’s inequality may be found from [31,32]. Then it is worth pointing out that there is no extremal function in $\mathcal{B}$ such that the Bohr radius is precisely $1/3$ (cf. [7, Corollary 8.26]).

1.2. The Bohr inequality for bounded analytic functions

In what follows, we let $\|f\|_r^2 = \sum_{n=1}^{\infty} |a_n|^2 r^{2n}$ whenever $f(z) = \sum_{n=1}^{\infty} a_n z^n$ converges for $|z| < 1$ and $r < 1$. Recently, Ponnusamy et al. [30] established the following refined Bohr inequality.

**Theorem B ([30, Theorem 2]):** Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_0(z) = f(z) - a_0$. Then for $p = 1, 2$, the following sharp inequality holds:

$$|a_0|^p + \sum_{n=1}^{\infty} |a_n|r^n + \left(\frac{1}{1 + |a_0|} + \frac{r}{1 - r}\right)\|f_0\|_r^2 \leq 1 \text{ for } r \leq \frac{1}{1 + (1 + |a_0|)^2 - r^p}.$$ 

Besides these results, there are plenty of works about Bohr’s phenomenon. As a consequence of the development on this topic, the notion of Rogosinski’s inequality
and Rogosinski’s radius was investigated in [33–35]. Furthermore, Kayumov and Ponnusamy [36] introduced and studied the Bohr–Rogosinski inequality and found the Bohr–Rogosinski radius. Based on the initiation of [27], several forms of Bohr-type inequalities for the family $B$ were considered in [29] when the Taylor coefficients of classical Bohr inequality are partly or completely replaced by higher order derivatives of $f$. Let $S_r(f)$ denote the area of the image of the subdisk $|z| < r$ under the mapping $f$ and when there is no confusion, we let for brevity $S_r$ for $S_r(f)$. Let us now recall a couple of recent results for our reference.

**Theorem C ([37]):** Suppose that $f \in B$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_0(z) = f(z) - a_0$. Then for $p = 1, 2$, the following sharp inequality holds:

$$|a_0|^p + \sum_{n=1}^{\infty} |a_n|r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|_r^2$$

$$+ \left( \frac{8}{9} \right)^{3-2p} \frac{S_r}{\pi} \leq 1$$

for $r \leq \frac{1}{2 + (1 - |a_0|)^{p-1}}$.

Refined version of Theorem A, without the third term as in Theorem C were obtained in [2]. See also [2, Remarks 1, 2 and 3] and [2, Theorem 2] for the analog of this theorem with a replacement of the constant term by $|f(z)|^2$.

**Theorem D:** Suppose that $f \in B$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $S_r$ denotes the area of the image of the subdisk $|z| < r$ under the mapping $f$. Then

$$\sum_{k=0}^{\infty} |a_k|r^k + \frac{16}{9} \left( \frac{S_r}{\pi} \right) + \lambda \left( \frac{S_r}{\pi} \right)^2 \leq 1$$

for $r \leq \frac{1}{3}$

where

$$\lambda = \frac{4(486 - 261a - 324a^2 + 2a^3 + 30a^4 + 3a^5)}{81(1 + a)^3(3 - 5a)} = 18.6095 \ldots$$

and $a \approx 0.567284$, is the unique positive root of the equation $\psi(t) = 0$ in the interval $(0, 1)$, where

$$\psi(t) = -405 + 473t + 402t^2 + 38t^3 + 3t^4 + t^5.$$
1.3. The Bohr operator on analytic functions and sections

The notion closely related to the Bohr radius is the Rogosinski radius contained in the following result of Rogosinski [34].

**Theorem E ([34]):** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in A \) and \( |f(z)| \leq 1 \) for all \( |z| \leq r \), then for every \( k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) and \( 0 < r \leq 1 \), each section \( s_k(f) := s_k(z; f) = \sum_{n=0}^{k} a_n z^n \) of \( f \) satisfies the inequality

\[
|s_k(f)| \leq 1
\]

for \( |z| \leq r/2 \). The constant \( r/2 \) cannot be improved.

The number \( r/2 \) (stated actually in [34] with \( r = 1 \)) in Theorem E is known as the Rogosinski radius. In [38,39], Aizenberg et al. extended the Rogosinski phenomenon to holomorphic functions of several complex variables. Aizenberg [38] also studied the Rogosinski radius on Hardy spaces and Reinhardt domains. For more recent advances, see, e.g. [1,15,40].

In [25], Bhowmik and Das established the following result: if \( f \prec h \) in \( \mathbb{D} \), then

\[
B_r(f) \leq B_r(h), \quad 0 \leq r \leq \frac{1}{3}. \tag{2}
\]

Later in [15] this inequality has been extended for the family of quasi-subordinations which includes both subordination and majorization. Alternate proof of (2) has been obtained in [24]. More recently, Liu and Ponnusamy refined this result and in particular, they derived the following sharper version of (2):

**Theorem F ([41, Corollary 2]):** Suppose that \( f \prec h \) in \( \mathbb{D} \) and

\[
r_1(x) = \begin{cases} 
\sqrt{1-x} & \text{for } x \in [0, \frac{1}{2}), \\
\frac{1}{2} & \text{for } x = \frac{1}{2}, \\
\frac{1}{1+2x} & \text{for } x \in [\frac{1}{2}, 1].
\end{cases} \tag{3}
\]

Then we have

(a) \( B_r(f) \leq B_r(h) \) for \( r \leq r_1(|f'(0)|/h'(0)) \), when \( h'(0) \neq 0 \).
(b) \( B_r(f) \leq B_r(h) \) for \( r \leq 1/3 \), when \( h'(0) = 0 \).

Moreover, \( r_1(|f'(0)|/h'(0)) \) cannot be improved if \( |f'(0)|/h'(0)| \in [1/2, 1) \cup \{0\} \), and the constant \( 1/3 \) in (b) cannot be improved.

Note that \( r_1(x) \geq 1/3 \) for \( x \in [0, 1] \).

The paper is organized as follows. In Section 2, we present the main results of this paper. In Theorems 2.1 and 2.2, we present an affirmative answer to Problem 1.1. Also, we establish the Bohr-type operator on analytic functions and also on their sections. In Section 3, we state and prove several lemmas which are needed for the proofs of the following theorems. In Sections 4 and 5, we present the proofs of the main results.
2. Main results

Theorem 2.1: Suppose that $f \in B$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_0(z) = f(z) - a_0$, and $S_r$ denotes the area of the image of the subdisk $|z| < r$ under the mapping $f$. Then

$$A(r) := \sum_{n=0}^{\infty} |a_n| r^n + \left( \frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \|f_0\|^2_r + \frac{8}{9} \left( \frac{S_r}{\pi} \right) + \lambda \left( \frac{S_r}{\pi} \right)^2 \leq 1$$

for $r \leq \frac{1}{3}$

where

$$\lambda = \frac{-2673 + 2502a_* + 2025a_*^2 - 332a_*^3 - 255a_*^4 + 6a_*^5 + 7a_*^6}{162(a_* + 1)^2(5a_* - 3)} \approx 14.796883,$$

and $a_* \approx 0.587459$, is the unique root in $(0,1)$ of the equation $\psi(t) = 0$, where

$$\psi(t) = 2t^6 + 4t^5 + 14t^4 - 40t^3 - 1218t^2 - 1756t + 1458. \quad (4)$$

The equality in the last inequality is achieved for the function $f(z) = \frac{a-z}{1-az}$.

Theorem 2.2: Assume the hypotheses of Theorem 2.1. Then

$$B(r) := |a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left( \frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \|f_0\|^2_r + \frac{9}{8} \left( \frac{S_r}{\pi} \right) + \mu \left( \frac{S_r}{\pi} \right)^2 \leq 1$$

for $r \leq \frac{1}{3-|a_0|}$, where

$$\mu = \frac{-80919 + 119556a_* - 57591a_*^2 + 11664a_*^3 - 1620a_*^4}{8(a_* + 1)^2(a_* - 3)^3(9a_*^3 - 33a_* + 29a_* - 1)} = 13.966088 \cdots,$$

and $a_* \approx 0.638302$ is the unique root in $(0,1)$ of the equation $\phi(t) = 0$, where

$$\phi(t) = -1296t^8 + 17172t^7 - 154386t^6 + 798660t^5 - 2361960t^4$$

$$+ 4132944t^3 - 4244238t^2 + 2344464t - 524880. \quad (5)$$

The equality in the last inequality is achieved for the function $f(z) = \frac{a-z}{1-az}$.

In the next three theorems, we use the idea of [24] to establish some similar results for derivatives.

Theorem 2.3: Suppose $f \prec h$ in $\mathbb{D}$. Then for each $k$ sections of $f'$ and $h'$, we have the following

1. $|s_k(f')| \leq B_r(s_k(h'))$, for $0 \leq r \leq \frac{1}{2}(1 - \sqrt{\frac{2}{3}})$

2. $B_r(s_k(f')) \leq B_r(s_k(h'))$ for $0 \leq r \leq \frac{1}{3}(1 - \sqrt{\frac{2}{3}})$.

Theorem 2.4: Suppose $h$ is analytic in $\mathbb{D}$. If $0 \leq r \leq 1 - \sqrt{\frac{2}{3}}$, then

$$B_r((h \circ \varphi)' \leq B_r(h')$$

for every Schwarz function $\varphi$. 
Theorem 2.5: Suppose the analytic functions $f, g$ and $h$ satisfy $f(z) = g(z)h(\phi(z))$, $z \in \mathbb{D}$, for some Schwarz function $\phi$. Further suppose that $|g(z)| \leq b$ for $|z| < 1$. Then

$$B_r(f') \leq b \left( 2B_r(h) + B_r(h') \right)$$

holds for $|z| = r \leq r_2$, where $r_2 = 1 - \sqrt{\frac{2}{3}}$.

3. Key lemmas and their proofs

In order to establish our main results, we need the following lemmas.

Lemma 3.1 ([37]): Suppose that $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then for any $N \in \mathbb{N}$, the following inequality holds:

$$\sum_{n=N}^{\infty} |a_n| r^n + \text{sgn}(t) \sum_{n=1}^{t} |a_n|^2 \frac{r^n}{1-r} + \left( \frac{1}{1+|a_0|} \right) \sum_{n=t+1}^{\infty} |a_n|^2 r^{2n} \leq (1 - |a_0|^2) \frac{r^N}{1-r}$$

for $r \in [0, 1)$, where $t = [(N - 1)/2]$.

Lemma 3.2 ([36, Lemma 1]): Suppose that $f \in \mathcal{B}, f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $S_r$ denotes the area of the image of the subdisk $|z| < r$ under the mapping $f$. Then the following sharp inequality holds:

$$\frac{S_r(f)}{\pi} := \sum_{n=1}^{\infty} n |a_n|^2 r^{2n} \leq r^2 \frac{(1 - |a_0|^2)^2}{(1 - |a_0|^2 r^2)^2}$$

for $0 < r \leq 1/\sqrt{2}$.

In [42], Bhowmik et al. established the Bohr phenomenon for the derivative of an analytic self map $\phi$ of $\mathbb{D}$, where $\phi(0) = 0$. The next result is stated with radius dependence on the initial coefficient of $\phi$ and the proof is on the lines of the proof of [42].

Lemma 3.3: Let $\phi \in \mathcal{B}$ such that $\phi(0) = 0$, i.e. $\phi$ is a Schwarz function. Then

$$B_r(\phi') \leq 1 \quad \text{for } |z| = r \leq r_0(|\phi'(0)|), \quad r_0(x) = 1 - \sqrt{\frac{1+x}{2+x}}$$

Here $r_0(x) \geq r_0(1) = 1 - \sqrt{\frac{2}{3}}$ for all $x \in [0, 1]$ and the number $r_0(1)$ is optimal.

Proof: We may write $\phi(z) = z\omega(z)$, where $\omega$ is an analytic self map of $\mathbb{D}$ with $\omega(0) = \phi'(0)$. Let $\omega(z) = \sum_{n=0}^{\infty} w_n z^n$. Then $|w_n| \leq 1 - |w_0|^2$ for $n \geq 1$ and $\phi'(z) = z\omega'(z) + \omega(z)$ so that

$$B_r(\phi') = |w_0| + \sum_{n=1}^{\infty} (n+1)|w_n|r^n$$
\[
\leq |w_0| + (1 - |w_0|^2) \left[ \frac{1}{(1 - r)^2} - 1 \right]
= 1 - (1 - |w_0|^2) \left[ \frac{2 + |w_0|}{1 + |w_0|} - \frac{1}{(1 - r)^2} \right],
\]
which is less than or equal to 1 provided \( r \leq r_0(|w_0|) = r_0(|\varphi'(0)|) \). It is a simple exercise to see that for \( \zeta(z) = z(\frac{z-a}{1-az}) \), \( 0 \leq a < 1 \), we have

\[
B_r(\zeta') = a + (1 - a^2) \frac{2r - ar^2}{(1 - ar)^2} > 1 \quad \text{whenever } r > \frac{1}{a} \left[ 1 - \sqrt{\frac{1 + a}{1 + 2a}} \right].
\]

Allowing \( a \to 1 \), we see that \( r_0(1) = 1 - \sqrt{\frac{2}{3}} \) is optimal. \( \blacksquare \)

**Remark 3.1:** The function \( r_0(x) (x \in [0,1)) \) is strictly decreasing from \( 1 - \sqrt{\frac{1}{2}} \) to \( 1 - \sqrt{\frac{2}{3}} \).

**Lemma 3.4:** If \( \varphi \) is a Schwarz function, then \( B_r(\varphi(z)) \leq |z| \) for \( |z| = r \leq r_1(|\varphi'(0)|) \), where \( r_1(x) \) is defined by (3) and \( r_1(|\varphi'(0)|) \) cannot be improved if \( |\varphi'(0)| \in [1/2, 1) \cup \{0\} \).

**Proof:** We may write \( \varphi(z) = z\omega(z) \), where \( \omega \in B \) and \( \omega(0) = \varphi'(0) \). According to [41, Theorem 1],

\[
B_r(\omega) \leq 1 \quad \text{for } |z| \leq r_1(|\omega(0)|) = r_1(|\varphi'(0)|),
\]
and \( r_1(x) \) is as in the statement. This implies that

\[
B_r(\varphi(z)) \leq B_r(\omega(z))B_r(z) \leq |z|
\]
for \( |z| \leq r_1(|\omega(0)|) = r_1(|\varphi'(0)|) \). Note that \( |\varphi'(0)| \in [0,1] \) and the function \( r_1(x) (x \in [0,1)) \) is strictly decreasing from \( 1/\sqrt{2} \) to \( 1/3 \). The sharpness part follows as in [41, Theorem 1]. \( \blacksquare \)

Clearly, Lemma 3.4 refines [24, Lemma 1].

**Lemma 3.5:** Let \( \varphi \) be a Schwarz function, \( j \in \mathbb{N} \), \( r_0(x) \) and \( r_1(x) \) be as in Lemma 3.3 and (3), respectively. Then the following inequalities hold:

\[
\sup_{|z| \leq r} \left| s_k \left( \varphi'(z)\varphi^j(z) \right) \right| \leq |z|^j \quad \text{for } r \leq \frac{1}{2} r_0(|\varphi'(0)|) \quad (6)
\]
and

\[
B_r \left( s_k \left( \varphi'(z)\varphi^j(z) \right) \right) \leq |z|^j \quad \text{for } r \leq r_0(|\varphi'(0)|)r_1(|\varphi'(0)|^j). \quad (7)
\]
Proof: First we remark once again that \( r_0(|\varphi'(0)|) \geq r_0(1) = 1 - \sqrt{\frac{7}{3}} \) and \( r_1(|\varphi'(0)|) \geq r_1(1) = 1/3). Next, we observe (by Lemma 3.3) that

\[
|\varphi'(|\varphi/z)| = |\varphi'| |\varphi/z| \leq B_r (\varphi') |\varphi/z| \leq |\varphi/z| \leq 1
\]

for \(|z| = r \leq r_0(|\varphi'(0)|)). According to Theorem E, we obtain (6). Again

\[
B_r (\varphi'(|\varphi/z|) \leq B_r (\varphi') B_r ((\varphi/z)) \leq B_r ((\varphi/z)) \quad \text{for } 0 \leq r \leq r_0(|\varphi'(0)|).
\]

which in turn implies (cf. [41, Theorem 1]) that

\[
B_r (\varphi'(|\varphi/z|)) \leq 1 \quad \text{for } 0 \leq r \leq r_0(|\varphi'(0)|)r_1(|\varphi'(0)|).
\]

As \( B_r(s_k(\varphi'(|\varphi/z|)) \leq B_r(\varphi'(|\varphi/z|)), \) the desired conclusion follows from the last inequality.

Lemma 3.6: There is a unique root \( a_* \approx 0.587459 \) in \((0, 1)\) of the equation \( \psi(a) = 0 \), where \( \psi \) is given by (4).

Proof: Firstly, we begin by a direct computation that \( \psi(0) = 1458 > 0 \) and \( \psi(1) = -1536 < 0 \). Also,

\[
\begin{align*}
\psi'(a) &= 12a^5 + 20a^4 + 56a^3 - 120a^2 - 2436a - 1756, \\
\psi''(a) &= 60a^4 + 80a^3 + 168a^2 - 240a - 2436, \\
\psi^{(3)}(a) &= 240a^3 + 240a^2 + 336a - 240, \\
\psi^{(4)}(a) &= 720a^2 + 480a + 336.
\end{align*}
\]

Since the discriminant of the equation \( \psi^{(4)}(a) = 0 \) is less than 0 and \( \psi^{(4)}(0) = 336 > 0 \), we obtain that \( \psi^{(4)}(a) > 0 \) for \( 0 \leq a \leq 1 \). This implies that \( \varphi^{(3)}(a) \) is an increasing function of \( a \) in \([0, 1]\). We note that

\[
\psi^{(3)}(0) = -240 < 0 \quad \text{and} \quad \psi^{(3)}(1) = 576 > 0,
\]

and therefore, we obtain that \( \psi^{(3)}(a) = 0 \) has a unique zero \( a_1 \in (0, 1) \). Thus we find that \( \psi^{(3)}(a) < 0 \) for \( 0 < a < a_1 \) and \( \psi^{(3)}(a) > 0 \) for \( a_1 < a < 1 \). Therefore \( \psi^{(3)}(a) \) is a decreasing function of \( a \in [0, a_1] \) and is an increasing function of \( a \in (a_1, 1] \). Since

\[
\psi''(0) = -2436 < 0 \quad \text{and} \quad \psi''(1) = -2368 < 0,
\]

we obtain that \( \psi''(a) < 0 \) for \( a \in [0, 1] \), which shows that \( \psi'(a) \) is decreasing in \([0, 1]\) so that

\[
\psi'(a) \leq \psi'(0) = -1756 < 0 \quad \text{for } a \in [0, 1].
\]

We conclude that \( \psi(a) \) is a decreasing function of \( a \) in \([0, 1]\), with \( \psi(0) = 1458 > 0 \) and \( \psi(1) = -1536 < 0 \) which yields that the equation \( \psi(a) = 0 \) has a unique positive root in \((0, 1)\). Through the calculation of Mathematica or Maple, we obtain that the root is \( a^* \approx 0.587459 \).
Lemma 3.7: There is a unique root $a_{**} \approx 0.638302$ in $(0, 1)$ of the equation $\phi(a) = 0$, where $\phi$ is given by (5).

Proof: Firstly we observe that $\phi(0) = -524880 < 0$ and $\phi(1) = 6480 > 0$. By differentiating $\phi(a)$, we easily obtain

\[
\phi'(a) = -10368a^7 + 120204a^6 - 926316a^5 + 3993300a^4 - 9447840a^3 + 12398832a^2 - 8488476a + 23446464,
\]

\[
\phi''(a) = -72576a^6 + 721224a^5 - 4631580a^4 + 15973200a^3 - 28343520a^2 + 24797664a - 8488476,
\]

\[
\phi^{(3)}(a) = -435456a^5 + 3606120a^4 - 18526320a^3 + 47919600a^2 - 56687040a + 24797664,
\]

\[
\phi^{(4)}(a) = -2177280a^4 + 14424480a^3 - 55578960a^2 + 95839200a - 56687040,
\]

\[
\phi^{(5)}(a) = -8709120a^3 + 43273440a^2 - 111157920a + 95839200,
\]

\[
\phi^{(6)}(a) = -26127360a^2 + 86546880a - 111157920.
\]

Since the discriminant of the equation $\phi^{(6)}(a) = 0$ is less than 0 and the coefficient of $a^2$ in $\phi^{(6)}(a)$ is negative, we obtain that $\phi^{(6)}(a) < 0$ for $a \in [0, 1]$. Thus $\phi^{(5)}(a)$ is a decreasing function of $a$ for $a \in [0, 1]$, and therefore,

\[
\phi^{(5)}(a) \geq \phi^{(5)}(1) = 19245600 > 0 \quad \text{for } a \in [0, 1],
\]

showing that $\phi^{(4)}(a)$ is increasing in $[0, 1]$ so that

\[
\phi^{(4)}(a) \leq \phi^{(4)}(1) = -4179600 < 0 \quad \text{for } a \in [0, 1].
\]

This implies that $\phi^{(3)}(a)$ is a decreasing function of $a$ for $a \in [0, 1]$, so that

\[
\phi^{(3)}(a) \geq \phi^{(3)}(1) = 674568 > 0 \quad \text{for } a \in [0, 1].
\]

We conclude that $\phi''(a)$ is an increasing function of $a$ in $[0, 1]$ and therefore,

\[
\phi''(a) \leq \phi''(1) = -44064 < 0 \quad \text{for } a \in [0, 1],
\]

which shows that $\phi'(a)$ is decreasing in the interval $[0, 1]$. As $\phi'(0) > 0$ and $\phi'(1) = -16200 < 0$, there exists a unique $a_4 \in (0, 1)$ (in fact $a_4 \approx 0.853918$) such that $\phi'(a) > 0$ when $0 < a < a_4$, and $\phi'(a) < 0$ when $a_4 < a < 1$. Thus $\phi(a)$ is an increasing function of $a \in (0, a_4)$ and is a decreasing function of $a \in (a_4, 1)$ with $\phi(0) < 0$ and $\phi(1) = 6480 > 0$. Thus the equation $\phi(a) = 0$ has a unique positive root $a_{**}$ in $(0, 1)$. We find that $a_{**} \approx 0.638302$ as desired. ■
4. Bohr-type inequalities for bounded analytic functions

4.1. Proof of Theorem 2.1

Let $|a_0| = a \in (0, 1)$. By Lemma 3.1 with $N = 1$, and Lemma 3.2, we have

$$A(r) \leq a + \frac{(1 - a^2) r}{1 - r} + \frac{8 (1 - a^2)^2 r^2}{9 (1 - a^2 r^2)} + \frac{\lambda (1 - a^2)^4 r^4}{(1 - a^2 r^2)^2} := A_1(r)$$

Since $A_1(r)$ is an increasing function of $r$, we have for $r \leq 1/3$,

$$A(r) \leq A_1 \left( \frac{1}{3} \right) = a + \frac{1 - a^2}{2} + \frac{8 (1 - a^2)^2}{(9 - a^2)^2} + 81 \lambda \frac{(1 - a^2)^4}{(9 - a^2)^4} A_2(a),$$

where

$$A_2(a) = 5265 + 2673a - 1251a^2 - 675a^3 + 83a^4 + 51a^5 - a^6 - a^7 + 162 \lambda (-1 - 3a - 2a^2 + 2a^3 + 3a^4 + a^5),$$

$$= 5265 + 2673a - 1251a^2 - 675a^3 + 83a^4 + 51a^5 - a^6 - a^7 - 162 \lambda (1 - a^2)(1 + a)^3.$$

Next, we will verify that the function $A_2(a)$ has exactly one stationary point $a_* = 0.587459 \cdots$ in $[0, 1]$, which is the unique root in $[0, 1]$ of the equation $A_2(a) = 0$. We remark that $A'_2(a) = 0$ is equivalent to

$$162 \lambda (a + 1)^3 (5a - 3) = -2673 + 2502a + 2025a^2 - 332a^3 - 255a^4 + 6a^5 + 7a^6,$$

from which we obtain the value of $\lambda$ mentioned in the statement of the theorem.

In fact, through the calculation of Mathematica or Maple, we obtain that the number $a_* \approx 0.587459$ is the unique root in $(0, 1)$ of the equation $A'_2(a) = 0$, where

$$A'_2(a) = 2673 - 2502a - 2025a^2 + 332a^3 + 255a^4 - 6a^5 - 7a^6 + 162 \times 14.796883(a + 1)^3 (5a - 3).$$

(In fact, the equation $A'_2(a) = 0$ has six real roots, and the other five roots are out of the interval $[0, 1]$.)

Now we plug the value of $\lambda$ into the expression for $A_2(a_*)$. This gives

$$A_2(a_*) = \frac{1}{5a_* - 3} \cdot (a_* + 3)(a_* - 3) \psi(a_*),$$

where $\psi(a)$ is given by (4). By Lemma 3.6, we know that $a_*$ is the unique root in $(0, 1)$ of the equation $\psi(a) = 0$. Thus $A_2(a_*) = 0$ and $A'_2(a_*) = 0$. Besides this observation, we have $A_2(0) \approx 2867.904954 > 0$ and $A_2(1) = 6144 > 0$. Consequently, $A_2(a) \geq 0$ in the interval $(0, 1)$, which proves that $A(r) \leq 1$ for $r \leq 1/3$. 

Finally, to prove that the constant $\lambda$ is sharp, we consider the function $f$ given by
\[ f(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{n=1}^{\infty} a^{n-1} z^n, \quad z \in \mathbb{D}, \tag{8} \]
where $a \in (0, 1)$.

For this function, straightforward calculations show that
\[
A_{\lambda_1}(r) := \sum_{n=0}^{\infty} |a_n| r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + \frac{8}{9} \left( \frac{Sr}{\pi} \right) + \lambda_1 \left( \frac{Sr}{\pi} \right)^2
\]
\[
= a + \frac{(1 - a^2) r}{1 - ar} + \frac{1 + ar}{(1 + a)(1 - r)} \frac{(1 - a^2)^2 r^2}{1 - a^2 r^2}
\]
\[
+ \frac{8(1 - a^2)^2 r^2}{9(1 - a^2 r^2)^2} + \lambda_1 \frac{(1 - a^2)^4 r^4}{(1 - a^2 r^2)^4}
\]
\[
= a + \frac{(1 - a^2) r}{1 - r} + \frac{8(1 - a^2)^2 r^2}{9(1 - a^2 r^2)^2} + \lambda_1 \frac{(1 - a^2)^4 r^4}{(1 - a^2 r^2)^4}.
\]

For $r = 1/3$, the last expression becomes
\[
A_{\lambda_1}(1/3) = a + \frac{1 - a^2}{2} + \frac{8(1 - a^2)^2}{9(9 - a^2)^2} + 81\lambda \frac{(1 - a^2)^4}{(9 - a^2)^4} + 81(\lambda_1 - \lambda) \frac{(1 - a^2)^4}{(9 - a^2)^4}.
\]
Choose $a$ as the unique root $a_*$ in $(0, 1)$ of the equation $\psi(a) = 0$. As a consequence, we see that
\[
A_{\lambda_1}(1/3) = 1 + 81(\lambda_1 - \lambda) \frac{(1 - a^2)^4}{(9 - a^2)^4}
\]
which is bigger than 1 in case $\lambda_1 > \lambda$. This proves the sharpness assertion and the proof of Theorem 2.1 is complete.

4.2. **Proof of Theorem 2.2**

Let $|a_0| = a \in (0, 1)$. By Lemma 3.1 with $N = 1$, and Lemma 3.2, we have
\[
B(r) \leq a^2 + \frac{(1 - a^2) r}{1 - r} + \frac{9(1 - a^2)^2 r^2}{8(1 - a^2 r^2)^2} + \frac{\mu(1 - a^2)^4 r^4}{(1 - a^2 r^2)^4} := B_1(r).
\]
Since $B_1(r)$ is an increasing function of $r$, we have for $r \leq 1/(3 - a)$ that
\[
B(r) \leq B_1\left(\frac{1}{3 - a}\right) = a^2 + \frac{1 - a^2}{2 - a} + \frac{(3 - a)(1 - a^2)^2}{8(3 - 2a)^2} + \frac{\mu(3 - a)^4(1 - a^2)^4}{81(3 - 2a)^4}
\]
\[
= 1 - \frac{(1 + a)(1 - a)^3}{648(2 - a)(3 - 2a)^4} B_2(a),
\]
where
\[ B_2(a) = 39366 - 80919a + 59778a^2 - 19197a^3 + 2916a^4 - 324a^5 \]
\[ - 8\mu (162 + 27a - 378a^2 + 30a^3 + 290a^4 - 108a^5 - 62a^6 + 50a^7 - 12a^8 + a^9). \]

Following the idea of the proof of Theorem 2.1, we show that the function \( B_2(a) \) has exactly one stationary point \( a_{**} = 0.638302 \ldots \) in \([0, 1]\), which is the unique root in \([0, 1]\) of the equation \( B_2(a) = 0 \).

In fact, through the calculation of Mathematica or Maple, we obtain that the number \( a_{**} = 0.638302 \ldots \) is the unique root in \((0, 1)\) of the equation \( B'_2(a) = 0 \), where
\[ B'_2(a) = -80919 + 119556a - 57591a^2 + 11664a^3 - 1620a^4 \]
\[ - 8 \times 13.966088(a + 1)^2(a - 3)^3(9a^3 - 33a^2 + 29a - 1). \]

(In fact, the equation \( B'_2(a) = 0 \) has eight real roots, and the other seven roots are out of the interval \([0, 1]\).

Now we plug the value of \( \mu \) defined in the statement of the theorem into the expression for \( B_2(a_{**}) \). This gives
\[ B_2(a_{**}) = \frac{1}{9a_{**}^3 - 33a_{**}^2 + 29a_{**} - 1} \phi(a_{**}). \]

By Lemma 3.7, we know that \( a_{**} \) is the unique root in \((0, 1)\) of the equation \( \phi(a) = 0 \). Thus, \( B_2(a_{**}) = 0 \) and \( B'_2(a_{**}) = 0 \), by the choice of the value of \( \mu \). Besides this observation, we have \( B_2(0) \approx 21265.989952 > 0 \) and \( B_2(1) = 1620 > 0 \). Consequently, \( B_2(a) \geq 0 \) in the interval \((0, 1)\), which proves that \( B(r) \leq 1 \) for \( r \leq 1/(3 - a) \).

To prove that the constant \( \mu \) is sharp, we consider the function \( f \) given by \((8)\) and compute for this function the value of \( B_{\mu_1}(r) \), which is obtained from \( B(r) \) by replacing \( \mu \) by \( \mu_1 \). Indeed, straightforward calculations as before show that
\[ B_{\mu_1}(r) = a^2 + \frac{(1 - a^2)r}{1 - r} + \frac{9}{8} \frac{(1 - a^2)^2 r^2}{(1 - a^2 r)^2} + \mu_1 \frac{(1 - a^2)^4 r^4}{(1 - a^2 r)^4}. \]

For \( r = 1/(3 - a) \), we have
\[ B_{\mu_1}\left(\frac{1}{3 - a}\right) = a^2 + \frac{1 - a^2}{2 - a} + \frac{(3 - a)^2 (1 - a)^2}{8(3 - 2a)^2} + \frac{\mu_1 (3 - a)^4 (1 - a)^2}{81 (3 - 2a)^4} \]
\[ + \frac{1}{81} (\mu_1 - \mu) \frac{(3 - a)^4 (1 - a^2)^4}{(3 - 2a)^4}. \]

Choose \( a \) as the unique root \( a_{**} \) in \((0, 1)\) of the equation \( \phi(a) = 0 \). As a consequence, we find that
\[ B_{\mu_1}(1/(3 - a)) = 1 + \frac{1}{81} (\mu_1 - \mu) \frac{(3 - a)^4 (1 - a^2)^4}{(3 - 2a)^4} \]
which is bigger than 1 in case \( \mu_1 > \mu \). Thus the proof of Theorem 2.2 is complete.
5. The Bohr-type operator on classes of subordination

5.1. Proof of Theorem 2.3

Given \( f \prec h \). Then \( f(z) = h(\varphi(z)) \) for some Schwarz function \( \varphi \). We may let \( h(z) = \sum_{n=0}^{\infty} b_n z^n \). It follows that

\[
f'(z) = \varphi'(z) h'(\varphi(z)) = \sum_{n=1}^{\infty} n b_n \varphi^{n-1}(z) \varphi'(z) \quad \text{and} \quad s_k f' = \sum_{n=1}^{k+1} n b_n s_k \left( \varphi^{n-1}(z) \varphi'(z) \right).
\]

Now by (6), we find that

\[
|s_k f'| \leq \sum_{n=1}^{k+1} n |b_n| |s_k \left( \varphi(z) \varphi^{n-1}(z) \right)| \leq \sum_{n=1}^{k+1} n |b_n| |z|^{n-1} = B_r \left( s_k h' \right)
\]

for \( |z| \leq \frac{1}{2} r_0(|\varphi'(0)|) \), which in particular proves the first part of the theorem, since \( r_0(|\varphi'(0)|) \geq 1 - \sqrt{\frac{2}{3}} \).

For the second part, by the subadditivity of \( B_r \) and (7), we have

\[
B_r \left( s_k f' \right) \leq \sum_{n=1}^{k+1} n |b_n| B_r \left( s_k \left( \varphi \varphi^{n-1}(z) \right) \right) \leq \sum_{n=1}^{k+1} n |b_n| |z|^{n-1} = B_r \left( s_k h' \right)
\]

for \( |z| \leq r_0(|\varphi'(0)|) r_1(|\varphi'(0)|) \), which in particular proves the second part of the theorem, since \( r_0(|\varphi'(0)|) r_1(|\varphi'(0)|) \geq \frac{1}{3} (1 - \sqrt{\frac{2}{3}}) \).

5.2. Proof of Theorem 2.4

Let \( h(z) = \sum_{n=0}^{\infty} a_n z^n \). By Lemma 3.3, we have

\[
B_r \left( (h \circ \varphi)' \right) \leq B_r \left( h' \circ \varphi \right) B_r \left( \varphi' \right) \leq B_r \left( h' \circ \varphi \right) \leq \sum_{n=1}^{\infty} n |a_n| B_r(\varphi^{n-1})
\]

for \( 0 \leq r \leq 1 - \sqrt{\frac{2}{3}} \).

Since \( |\varphi(z)/z|^{n-1} \leq 1 \) in \( D \) for each \( n \in \mathbb{N}_+ \), Bohr’s theorem or Lemma 3.4 readily shows that

\[
B_r(\varphi^{n-1}) \leq B_r((\varphi(z)/z)^{n-1}) B_r(z^{n-1}) \leq r^{n-1}
\]

for \( |z| = r < 1/3 \). Note that \( 1 - \sqrt{\frac{2}{3}} \leq \frac{1}{3} \), we get that

\[
B_r \left( (h \circ \varphi)' \right) \leq \sum_{n=1}^{\infty} n |a_n| B_r(\varphi^{n-1}) \leq \sum_{n=1}^{\infty} n |a_n| r^{n-1} = B_r \left( h' \right)
\]

for \( 0 \leq r \leq 1 - \sqrt{\frac{2}{3}} \).
5.3. Proof of Theorem 2.5

By assumption \( f(z) = g(z)h(\varphi(z)) \). Then

\[
 f'(z) = g'(z)h(\varphi(z)) + g(z)h'(\varphi(z))\varphi'(z).
\]

By Theorems F and 2.4, we have

\[
 B_r(f') \leq B_r(g') B_r(h \circ \varphi) + B_r(g) B_r(h' \circ \varphi) B_r(\varphi')
\]

\[
 \leq B_r(g') B_r(h) + B_r(g) B_r(h')
\]

for \( 0 \leq r \leq 1 - \sqrt{\frac{2}{3}} \).

Let \( \phi(z) = \frac{g(z) - g(0)}{2b} \). Since \( \phi(z) \) is an analytic self map on \( \mathbb{D} \) and \( \phi(0) = 0 \), Lemma 3.3 shows that

\[
 B_r(g') = 2bB_r(\phi') \leq 2b \quad \text{for } |z| = r \leq r_0(|\varphi'(0)|), \quad r_0(x) = 1 - \sqrt{\frac{1+x}{2+x}}.
\]

Therefore,

\[
 B_r(f') \leq b(2B_r(h) + B_r(h')) \quad \text{for } |z| = r \leq r_2,
\]

where \( r_2 = \min\{r_1(|g'(0)|), r_0(|\varphi'(0)|)\} \geq \min\{\frac{1}{3}, 1 - \sqrt{\frac{2}{3}}\} = 1 - \sqrt{\frac{2}{3}} \).

Data availability statement

The authors declare that this research is purely theoretical and does not associate with any data.

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References

[1] Bohr H. A theorem concerning power series. Proc Lond Math Soc. 1914;2(13):1–5.
[2] Ismagilov AA, Kayumov IR, Ponnusamy S. Sharp Bohr type inequality. J Math Anal Appl. 2020;486(1):124147.
[3] Ponnusamy S, Wirths K-J. Bohr type inequalities for functions with a multiple zero at the origin. Comput Meth Funct Theor. 2020;20:559–570.
[4] Abu-Muhanna Y, Ali RM, Ponnusamy S. On the Bohr inequality. In: Progress in approximation theory and applicable complex analysis. Govil NK, Mohapatra RN, Qazi, M et al., editors. 2016. p. 265–295. (Springer Optim Appl; 117).
[5] Ismagilov AA, Kayumova AV, Kayumov IR, et al. Bohr inequalities in some classes of analytic functions. (Russian) Complex analysis (Russian). Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI); Moscow: 2018. p. 69–83. (Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz.; 153).

[6] Kayumova A, Kayumov IR, Ponnusamy S. Bohr’s inequality for harmonic mappings and beyond. Mathematics and computing. Singapore: Springer; 2018. p. 245–256. (Commun. Comput. Inf. Sci.; 834).

[7] Garcia SR, Mashreghi J, Ross WT. Finite Blaschke products and their connections. Cham: Springer; 2018.

[8] Abu-Muhanna Y. Bohr’s phenomenon in subordination and bounded harmonic classes. Complex Var Elliptic Equ. 2010;55(11):1071–1078.

[9] Abu-Muhanna Y, Ali RM. Bohr’s phenomenon for analytic functions into the exterior of a compact convex body. J Math Anal Appl. 2011;379(2):512–517.

[10] Abu-Muhanna Y, Ali RM. Bohr’s phenomenon for analytic functions and the hyperbolic metric. Math Nachr. 2013;286(11–12):1059–1065.

[11] Abu-Muhanna Y, Ali RM, Ng ZC, et al. Bohr radius for subordinating families of analytic functions and bounded harmonic mappings. J Math Anal Appl. 2014;420(1):124–136.

[12] Ali RM, Barnard RW, Solynin A Yu. A note on the Bohr’s phenomenon for power series. J Math Anal Appl. 2017;449(1):154–167.

[13] Kayumov IR, Ponnusamy S. Bohr inequality for odd analytic functions. Comput Meth Funct Theo. 2017;17(4):679–688.

[14] Kayumov IR, Ponnusamy S. Bohr’s inequality for analytic functions with lacunary series and harmonic functions. J Math Anal and Appl. 2018;465(2):857–871.

[15] Alkhaleefah SA, Kayumov IR, Ponnusamy S. On the Bohr inequality with a fixed zero coefficient. Proc Amer Math Soc. 2019;147(12):5263–5274.

[16] Kayumov IR, Ponnusamy S, Shakirov N. Bohr radius for locally univalent harmonic mappings. Math Nachr. 2018;291:1757–1768.

[17] Liu ZH, Ponnusamy S. Bohr radius for subordination and K-quasiconformal harmonic mappings. Bull Malays Math Sci Soc. 2019;42:2151–2168.

[18] Paulsen VI, Popascu G, Singh D. On Bohr’s inequality. Proc Lond Math. 2002;85(2):493–512.

[19] Paulsen VI, Singh D. Extensions of Bohr’s inequality. Bull Lond Math Soc. 2006;38(6):991–999.

[20] Defant A, Frerick L, Ortega-Cerdà J, et al. The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive. Ann Math. 2011;174(2):512–517.

[21] Evdoridis S, Ponnusamy S, Rasila A. Improved Bohr’s inequality for locally univalent harmonic mappings. Indag Math (N.S.). 2019;30:201–213.

[22] Liu MS, Ponnusamy S, Wang J. Bohr’s phenomenon for the classes of quasi-subordination and K-quasiregular harmonic mappings. RACSAM. 2020;114:1–15. Art. 115, 15 pages. Available from: https://doi.org/10.1007/s13398-020-00844-0.

[23] Paulsen VI, Singh D. Bohr’s inequality for uniform algebras. Proc Amer Math Soc. 2004;132:3577–3579.

[24] Abu-Muhanna Y, Ali RM, Lee SK. The Bohr operator on analytic functions and sections. Available from: arXiv:1912.11787v1.

[25] Bhowmik B, Das N. Bohr phenomenon for subordinating families of certain univalent functions. J Math Anal Appl. 2018;462(2):1087–1098.

[26] Huang Y, Liu MS, Ponnusamy S. Refined bohr-type inequalities with area measure for bounded analytic functions. Anal Math Phys. 2020;10(4):1–21. Article 50.

[27] Kayumov IR, Ponnusamy S. Improved version of Bohr’s inequality. C R Math Acad Sci Paris. 2018;356(3):272–277.

[28] Liu G, Ponnusamy S. On harmonic ψ-Bloch and ψ-Bloch-type mappings. Results Math. 2018;73(90). Available from: https://doi.org/10.1007/s00025-018-0853-2

[29] Liu MS, Shang YM, Xu JF. Bohr-type inequalities of analytic functions. J Inequal Appl. 2018;2018:1–13. Art. 345, 13 pages.
[30] Ponnusamy S, Vijayakumar R, Wirths K-J. New inequalities for the coefficients of unimodular bounded functions. Results Math. 2020;75(107): Available from: https://doi.org/10.1007/s00025-020-01240-1.

[31] Sidon S. Über einen Satz von Herrn Bohr. Math. Z. 1927;26(1):731–732.

[32] Tomić M. Sur un théorème de H. Bohr. Math. Scand. 1962;11:103–106.

[33] Landau E, Gaier D. Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. Third edition. Springer-Verlag, Berlin, 1986. xi+201 pp.

[34] Rogosinski W. Über Bildschranken bei Potenzreihen und ihren Abschnitten. Math Z. 1923;17:260–276.

[35] Schur I, Szegö G. Üdie Abschnivse einer im Einheitskreise beschränkten Potenzreihe. Sitz -Ber Preuss Acad Wiss Berlin Phys-Math Kl. 1925;0:545–560.

[36] Kayumov IR, Ponnusamy S. Bohr-Rogosinski radius for analytic functions. Available from: arXiv: 1708.05585v1.

[37] Liu G, Liu ZH, Ponnusamy S. Refined Bohr inequality for bounded analytic functions. Bulletin des sciences mathématiques. To appear; See also Available from: arXiv:2006.08930v1.

[38] Aizenberg L. Remarks on the Bohr and Rogosinski phenomena for power series. Anal Math Phys. 2012;2(1):69–78.

[39] Aizenberg L, Elin M, Shoikhet D. On the Rogosinski radius for holomorphic mappings and some of its applications. Studia Math. 2005;168(2):147–158.

[40] Boas HP, Khavinson D. Bohr’s power series theorem in several variables. Proc Amer Math Soc. 1997;125(10):2975–2979.

[41] Liu G, Ponnusamy S. Improved Bohr inequality for harmonic mappings. Math Nachr. 2021. p. 14. Available from: arXiv:submit/3671842.

[42] Bhowmik B, Das N. A note on the Bohr inequality. Available from: arXiv:1911.06597v1.