A note on Harnack and Transportation inequalities For Stochastic Differential Equations with reflections.

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Abstract

We establish transportation cost inequalities, with respect to the uniform and $L_2$-metric, on the path space of continuous functions, for laws of solutions of stochastic differential equations with reflections. We also consider the case of stochastic differential equations involving local times. Harnack inequalities for the associated semigroups are also established.

Keywords: Reflected diffusion; Local time; Girsanov transformation; Transportation inequality; Harnack inequality.

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1. Introduction

Let $(E,d)$ be a metric space equipped with a $\sigma$–field $\mathcal{B}$ such that $d(\cdot,\cdot)$ is $\mathcal{B} \times \mathcal{B}$ measurable. Given $p \geq 1$ and two probability measures $\mu$ and $\nu$ on $E$, we define the Wasserstein distance of order $p$ between $\mu$ and $\nu$ by

$$W_d^p(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{E \times E} d(x,y)^p d\pi(x,y) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on the product space $E \times E$ with marginals $\mu$ and $\nu$. The relative entropy of $\nu$ with respect to $\mu$ is defined as

$$H(\nu/\mu) = \begin{cases} \int_E \ln \frac{d\nu}{d\mu} d\nu, & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise.} \end{cases}$$

The probability measure $\mu$ satisfies the $L^p$–transportation inequality on $(E,d)$ if there exists a constant $C \geq 0$, such that for any probability measure $\nu$,

$$W_d^p(\mu, \nu) \leq \sqrt{2CH(\nu/\mu)}.$$ 

We shall write $\mu \in T_p(C)$ for this relation. It’s well known that the cases "$p = 1$" and "$p = 2$" are the most interesting cases. $T_1(C)$ is related to concentration of measure phenomenon and well characterized, as it was shown by Djellout \textit{et al.} (2004) using preliminary results obtained in Bobkov and Götze. (1999). Since Talagrand’s paper (see Talagrand, 1996), where $T_2(C)$ inequality has been established for Gaussian measure,
several works have emerged. Feyel and Üstünel (2002) and (2004) generalized Talagrand’s inequality to an abstract Wiener space. Moreover, many different arguments have been developed to establish the transportation inequalities. The most used method is the Girsanov transformation argument, introduced in Talagrand (1996) and efficiently applied by many authors, see, e.g., Wu and Zhang (2000) for infinite-dimensional dynamical systems, Üstünel (2012) for multi-valued SDEs and singular SDEs, Saussereau (2012) for SDEs driven by a fractional Brownian motion and Li and Luo (2015) for stochastic delay evolution equations driven by fractional Brownian motion. Recently, Riedel (2017) investigated the transportation inequality for the law of SDE driven by general Gaussian processes by using Lyons’ rough paths theory. The first author studied in Boufoussi and Hajji (2018) $T^2_2(C)$ inequality, with respect to $L^2$-metric, for the stochastic heat equation driven by space-time white noise and driven by fractional noise.

Even if $T^2_2(C)$ is not well characterized it has many interesting properties. $T^2_2(C)$ is stronger than $T^1_1(C)$ and it has the dimension free tensorization property. The property $T^2_2(C)$ is also intimately linked to many other functional properties such as concentration of measure phenomen, Poincaré inequality, logarithmic Sobolev inequality and Hamilton-Jacobi equations. In their famous paper Otto and Villani (2000) showed that in a smooth Riemannian setting, the logarithmic Sobolev inequality implies $T^2_2(C)$, whereas $T^2_2(C)$ implies the Poincaré’s inequality.

In this paper, we first consider the following reflected stochastic differential equations (RSDEs):

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t) - d\eta(t); \quad X(0) = x \in \mathcal{O},$$

where $\eta(t)$ is a process with bounded variations forcing $X$ to stay inside a given regular domain $\mathcal{O} \subset \mathbb{R}^d$. There is a rich literature on RSDEs. They arise as a model of some phenomena with constraints, and are useful in a variety domains of applications, such as control theory, games theory and financial mathematics, see for example Soner and Shreve (1989), KruK (2000), Ramasubramanian (2006,...). Otherwise, they allow to give probabilistic representations for elliptic and parabolic partial differential equations with Neumann type and/or mixed boundary conditions, (see Freidlin (1985), Talay (1996) and Brillinger et al (2002), etc.). The existence and uniqueness of solutions for RSDEs of type (1) were first investigated by Skorohod (see e.g. Skorhod (1962)). After, many works related to reflected solutions to SDEs have been done. Among others we cite the works of Tanaka (1971), Menaldi (1982), Stroock and Varadhan (1971), ...etc.

Our first aim is to investigate the properties $T^1_1(C)$ and $T^2_2(C)$ w.r.t. uniform and $L^2$—metrics for the solutions of such equations. As a consequence, we deduce a useful concentration inequality satisfied by the law of the solution $X$. And under suitable regularity assumptions on the coefficients, we give an estimation for the Wasserstein distance between the transition density of the solution $X$ and its associated stationary distribution. Furthermore, by means of the coupling and Girsanov transformation arguments, we show a log-Harnack and Harnack inequalities for the operator semigroup

$$P_t f(x) = \mathbb{E} f(X^x(t)), \quad t \geq 0,$$

where $X^x$ is the solution of (1) with $X^x(0) = x$, and $f$ is a bounded positive measurable function.
On the other hand, the second part of this paper concerns stochastic differential equations involving local times (SDELs):
\[
\text{d}X(t) = b(X(t))\text{d}t + \sigma(X(t))\text{d}B(t) + \int_{\mathbb{R}} \nu(dx) \text{d}L^x_t(X),
\]
(2)
where \(\nu\) is a bounded measure on \(\mathbb{R}\) and \(L^x_t(X)\) is the symmetric local time at \(x \in \mathbb{R}\) of the unknown process \((X_t)_{t \geq 0}\). These equations appeared first in the work of Stroock and Yor (1981) and were subsequently developed by many other authors. The necessary and sufficient conditions for pathwise uniqueness property of SDELs are given in Le Gall (1984) (see also Engelbert and Schmidt (1989-1991)). We know that in some special situations (when \(\sigma\) and \(b\) are smooth and \(\nu = \beta \delta_0\)), the solutions of these equations are related, by mean of Feynman-Kac formula, to parabolic differential equations with transmission conditions. An interesting case is obtained when \(b = 0\), \(\sigma = 1\) and \(\nu = \beta \delta_0\) with \(\beta \in (-1, 1)\), the solution \(X\) becomes the so-called Skew Brownian motion introduced and studied in Harrison and Shepp (1981).

Our second goal in this work is to prove transportation cost inequalities for the solutions of SDELs, Moreover, as for the RSDE case we give, under a dissipativity condition, an estimation of the Wasserstein distance between the transition density of \(X\) and its unique invariant measure. We prove also a Harnack inequality for the corresponding semigroup. We would like to point out here that the skew Brownian is not covered by our result and that the investigation of the inequality \(T_2\) remains open and interesting in mathematical point of view.

The rest of this paper is organized as follows, In section 2, we recall a result of existence and uniqueness for RSDEs via penalization method, we investigate inequalities \(T_1(C)\) and \(T_2(C)\) w.r.t. uniform and \(L_2\)-metric for laws of the solutions and we present a Harnack inequality for the associated semigroup. In the last section, by using a stability argument of transportation inequalities, we prove the property \(T_2(C)\) for the law of the solution of the equation (2) with respect to the \(L_2\) and uniform distance on \(C([0, T], \mathbb{R})\). The Harnack inequality is also proved.

2. Reflected stochastic differential equations

Let \((B_t, t \geq 0)\) be a standard \(d\)-Brownian motion \((d \geq 1)\), defined on a filtrated probability space \(\left(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, P\right)\) satisfying the usual conditions. Let \(\mathcal{O}\) be a bounded convex domain in \(\mathbb{R}^d\) and \(\bar{\mathcal{O}}\) denotes its closure. Consider the normal reflected diffusion on \(\bar{\mathcal{O}}\) described as:
\[
\begin{aligned}
\begin{cases}
\text{d}X(t) = b(X(t))\text{d}t + \sigma(X(t))\text{d}B(t) - d\eta(t), \\
X(0) = x
\end{cases}
\end{aligned}
\]
(3)
where \(x \in \bar{\mathcal{O}}, \ b : \mathbb{R}^d \rightarrow \mathbb{R}^d\) and \(\sigma : \mathbb{R}^d \rightarrow \mathcal{M}_{d \times d}(\mathbb{R})\) are Borel measurable functions. For a fixed horizon time \(T > 0\), we have:

**Definition 1** A strong solution of the equation (3) on \([0, T]\) is a pair of adapted continuous processes \((X, \eta)\) such that:
1. \( X \) takes values in the closure \( \overline{O} \) and \( \eta \) has locally bounded variation with \( \eta(0) = 0 \).

2. For every adapted continuous process \( Y(t) \) taking values in the closure \( \overline{O} \) we have

\[
\int_0^t <X(s) - Y(s), d\eta(s)> \geq 0
\]

and

\[
X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s) - \eta(t)
\]

We will make use of the following assumptions:

\( H(1) \) \( b \) and \( \sigma \) are locally Lipschitz on \( \mathbb{R}^d \).

\( H(2) \) There exists a constant \( M > 0 \) such that for every \( x \in \mathbb{R}^d \)

\[
|b(x)|^2 + |\sigma(x)|^2 \leq M(1 + |x|^2),
\]

where \( |\cdot| \) denotes the Euclidean norm on \( \mathbb{R}^d \). The stochastic variational inequality (3) can be approximated by means of a classic penalization method applied to a stochastic differential equation, defined on the whole space \( \mathbb{R}^d \). Without loss of generality, one can assume that the coefficients \( b \) and \( \sigma \) are defined on the whole space \( \mathbb{R}^d \), even if they need to be defined only on the closure \( \overline{O} \).

Define the penalty function \( \beta(x) := x - \mathcal{P}_O(x) \), where \( \mathcal{P}_O \) is the orthogonal projection on \( \overline{O} \), and for every \( \varepsilon > 0 \), consider the stochastic differential equation:

\[
\begin{cases}
    dX_\varepsilon(t) = b(X_\varepsilon(t))dt + \sigma(X_\varepsilon(t))dB(t) - \frac{1}{\varepsilon}\beta(X_\varepsilon(t))dt, \\
    X(0) = x.
\end{cases}
\]  

(5)

Since \( \beta \) is Lipschitz continuous, under assumptions \( H(1) \) and \( H(2) \) there exists a unique strong solution \( X_\varepsilon \) of (5). According to Menaldi and Robin \( \text{[1985]} \) we have the following convergence result

**Theorem 1** Under the assumptions \( H(1) \) and \( H(2) \), there exists a unique solution \( (X(t), \eta(t)) : t \in \left[0, T\right] \) of the stochastic variational inequality as described by Definition \( \text{[1]} \). Moreover for every \( T > 0 \), the following convergence holds in probability

\[
\sup_{0 \leq t \leq T} \{|X_\varepsilon(t) - X(t)| + |\eta_\varepsilon(t) - \eta(t)|\} \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]

(6)

where \( X_\varepsilon \) is the solution of the stochastic differential equation (5), and

\[
\eta_\varepsilon(t) := \frac{1}{\varepsilon} \int_0^t \beta(X_\varepsilon(s))ds.
\]

**Remark 2** When the convex domain \( O \) is unbounded, Theorem 1 is still valid under an additional technical assumption, namely, there exists a point \( a \in \mathbb{R}^d \) and a constant \( c > 0 \) such that

\[
<x - a, \beta(x)> \geq c|\beta(x)|, \quad \forall x \in \mathbb{R}^d.
\]

(7)

Note that if \( O \) is bounded then the inequality (7) is satisfied, we will use it in the proof of Theorem 1.
We recall a stability property of $T_p(C)$ under the weak convergence of measures, which will be useful to prove the property $T_1(C)$ for Equation (3) (see Djellout et al. (2004)).

**Lemma 1** Let $(E, d)$ be a metric separable and complete space, and $(\mu_n, \mu)_{n \in \mathbb{N}}$ a family of probability measures on $E$. Assume that $\mu_n \in T_p(C)$ for all $n \in \mathbb{N}$ and $\mu_n \rightarrow \mu$ weakly. Then $\mu \in T_p(C)$. We make the following assumptions: there exist $A, B > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \|\sigma(x)\|_{HS} \leq A, \quad <y - x, b(y) - b(x)> \leq B (1 + |y - x|^2), \quad \forall x, y \in \mathbb{R}^d,$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm.

**Theorem 3** Suppose that $H(1), H(2)$ and the conditions (8) hold. Let $\mathbb{P}_X$ be the law of the solution $X$ of (3), with initial point $X_0 = x \in \bar{O}$. Then for each $T > 0$ there exists some constant $C = C(T, A, B)$ independent of $x$ such that $\mathbb{P}_X$ satisfies $T_1(C)$, on the space $C([0,T], \mathbb{R}^d)$ equipped with the uniform metric

$$d_{\infty}(\gamma_1, \gamma_2) = \sup_{t \in [0,T]} |\gamma_1(t) - \gamma_2(t)|.$$

**Proof:** Since by the inequality (7) we have $<x - y, \beta_1(x) - \beta_1(y)> \geq 0 \forall x, y, \forall \varepsilon$, it is clear that the coefficients of the pannellized equation (5) satisfy the assumptions of Corollary 4.1 in Djellout et al. (2004), which ensures that for any $\varepsilon > 0$, the law $\mathbb{P}_{X_{\varepsilon}}$ of $X_{\varepsilon}$ satisfies $T_1(C)$ for some constant $C = C(T, A, B)$ independent of $\varepsilon$. By the stability argument of $T_1(C)$ under the weak convergence of measures, we conclude that $\mathbb{P}_X \in T_1(C)$.

**Remark 4** We can deduce several consequences of Theorem 3

1. For any Lipschitzian function $F : C([0,T], \mathbb{R}^d) \rightarrow \mathbb{R}$ we have, see Theorem 1.1 in Djellout et al. (2004),

$$\mathbb{P}_X(F - E_{\mathbb{P}_X}F > r) \leq \exp\left(-\frac{r^2}{2C\|F\|_{Lip}^2}\right)$$

where,

$$\|F\|_{Lip} = \sup_{\gamma_1 \neq \gamma_2} \frac{|F(\gamma_1) - F(\gamma_2)|}{d_{\infty}(\gamma_1, \gamma_2)}.$$

2. We have the following concentration inequality (see Gozlan and Léonard, 2011). For all measurable $A \subset C([0,T], \mathbb{R}^d)$ with $\mathbb{P}_X(A) \geq 1/2$,

$$\mathbb{P}_X(A^c) \geq 1 - \exp\left(-\frac{r - r_0}{2C}\right), \quad r \geq r_0 = \sqrt{2C\log(2)}$$

where $A^c$ is defined by

$$A^c := \{\gamma \in C([0,T], \mathbb{R}^d); d_{\infty}(\gamma, A) \leq r\}, \quad r \geq 0.$$

In the sequel, let $\mathcal{F}_t := \sigma(B(s), s \leq t) \vee \mathcal{N}$, where $\mathcal{N}$ is the class of all $\mathbb{P}$-negligible sets. We will show that the law $\mathbb{P}_X$ of the solution $X$ of the equation (3) satisfies $T_2(C)$ on the space $C([0,T], \mathbb{R}^d)$ with respect to the two metrics

$$d_2(\gamma_1, \gamma_2) = \left(\int_0^T |\gamma_1(t) - \gamma_2(t)|^2 dt\right)^{1/2} \quad \text{and} \quad d_{\infty}(\gamma_1, \gamma_2) = \sup_{t \in [0,T]} |\gamma_1(t) - \gamma_2(t)|.$$

For this end, we assume the following dissipative condition:
There is $\delta > 0$ such that:

$$\|\sigma(x) - \sigma(y)\|_{HS}^2 + 2 < x - y, b(x) - b(y) \geq -2\delta|x - y|^2, \forall x, y \in \mathbb{R}^d,$$

where $M^t$ and $tr(M)$ denote respectively the transpose and the trace of a matrix $M$, and $|.|$ stands for the Euclidean norm on $\mathbb{R}^d$. We will note by $\|\sigma\|_\infty = \sup_{x \in \mathbb{R}^d} \sup_{\|z\| \leq 1} |\sigma(x) z|$.

**Theorem 5** Suppose that $H(1), H(2), H(3)$ hold and $\|\sigma\|_\infty < \infty$. Then for any initial point $X_0 = x \in \bar{O}$ and $T > 0$, the law $\mathbb{P}_X$ of the solution $X$ satisfies $T_2(\frac{\|\sigma\|^2}{\delta})$ on $C([0, T], \mathbb{R}^d)$ with respect to the metric $d_2$.

**Proof:** The first part of the proof follows the argument of [Djellout et al. (2004)](Djellout(2004)). The idea is to express the finiteness of the entropy by means of the energy of the drift arising from the Girsanov transform of a well-chosen probability. Let $Q$ be a probability measure on $C([0, T], \mathbb{R}^d)$ such that $Q \ll \mathbb{P}_X$, we assume that $H(Q/\mathbb{P}_X) < \infty$ and we consider

$$\tilde{Q} := \frac{dQ}{d\mathbb{P}_X}(X)\mathbb{P}_X$$

Clearly $\tilde{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ and

$$H(\tilde{Q}/\mathbb{P}) = \int_\Omega \ln \left( \frac{dQ}{d\mathbb{P}} \right) d\tilde{Q} = \int_\Omega \ln \left( \frac{dQ}{d\mathbb{P}_X}(X) \right) d\tilde{Q} = \int_{C([0, T], \mathbb{R}^d)} \ln \left( \frac{dQ}{d\mathbb{P}_X} \right) d\tilde{Q} = H(Q/\mathbb{P}_X).$$

The principal key of the proof is the following result (see [Djellout et al. (2004)](Djellout(2004))): There exists a predictable process $\rho = (\rho^1(t), ..., \rho^d(t))_{0 \leq t \leq T}$, such that

$$H(\tilde{Q}/\mathbb{P}) = H(Q/\mathbb{P}_X) = \frac{1}{2} \mathbb{E}_Q \int_0^T |\rho(t)|^2 dt.$$

By Girsanov’s theorem the process defined by:

$$\tilde{B}(t) := B(t) - \int_0^t \rho(s) ds$$

is a Brownian motion under $\tilde{Q}$, consequently $X$ verifies

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))d\tilde{B}(t) + \sigma(X(t))\rho(t)dt - d\eta_X(t), \\ X(0) = x. \end{cases} \tag{10}$$

We consider the solution $Y$ of the following equation

$$\begin{cases} dY(t) = b(Y(t))dt + \sigma(Y(t))d\tilde{B}(t) - d\eta_Y(t), \\ Y(0) = x. \end{cases} \tag{11}$$
Under $\tilde{Q}$ and by the uniqueness argument, the law of the process $(Y(t))_{t \in [0,T]}$ is exactly $\mathbb{P}_X$. Then, under $\tilde{Q}$, $(X,Y)$ is a coupling of $(Q,\mathbb{P}_X)$, then it follows that

$$\left[ W^d_2 \left( \mathcal{Q}, \mathbb{P}_X \right) \right]^2 \leq \mathbb{E}_Q \left( d_2(X,Y)^2 \right) = \mathbb{E}_Q \left( \int_0^T |X(t) - Y(t)|^2 \, dt \right).$$

Now, we estimate the distance on $C([0,T],\mathbb{R}^d)$ between $X$ and $Y$ with respect to the distance $d_2$. With the notations:

$$\tilde{X}(t) := X(t) - Y(t) \quad \tilde{\sigma}(t) := \sigma(X(t)) - \sigma(Y(t))$$

$$\tilde{b}(t) := b(X(t)) - b(Y(t)) \quad \tilde{\eta}(t) := \eta_X(t) - \eta_Y(t),$$

the process $\tilde{X}$ satisfies the following Itô equation

$$d\tilde{X}(t) = \tilde{b}(t)dt + \tilde{\sigma}(t)d\tilde{B}(t) + \sigma(X(t))\rho(t)dt - d\tilde{\eta}(t). \quad (12)$$

By Itô formula, we have

$$d|\tilde{X}(t)|^2 = \left[ 2 < \tilde{X}(t), \tilde{\sigma}(t) \tilde{X}(t) > + \text{tr}(\tilde{\sigma}(t)\tilde{\sigma}(t)^T) \right] dt + 2 < \tilde{X}(t), \tilde{\sigma}(t)d\tilde{B}(t) > -2 < \tilde{X}(t), d\tilde{\eta}(t) > . \quad (13)$$

Using assumption H(3) and the condition $\mathbf{H}$, we get

$$|\tilde{X}(t)|^2 \leq -2\delta \int_0^t |\tilde{X}(s)|^2 \, ds + 2 \int_0^t < \tilde{X}(s), \sigma(X(s))\rho(s) > ds + 2 \int_0^t < \tilde{X}(s), \tilde{\sigma}(s)d\tilde{B}(s) > .$$

By using a localization argument and the Cauchy-Schwartz inequality, we obtain for each $\lambda > 0$

$$\mathbb{E}_Q |\tilde{X}(t)|^2 \leq (\lambda - 2\delta) \int_0^t \mathbb{E}_Q |\tilde{X}(s)|^2 \, ds + \frac{\|\sigma\|^2}{\lambda} \mathbb{E}_Q \int_0^t \|\rho(s)\|^2 \, ds.$$

Gronwall's lemma entails

$$\mathbb{E}_Q |\tilde{X}(t)|^2 \leq \frac{\|\sigma\|^2}{\lambda} \mathbb{E}_Q \int_0^t e^{(\lambda - 2\delta)(t-s)} \|\rho(s)\|^2 \, ds. \quad (14)$$

Thus,

$$\left[ W^d_2 \left( \mathcal{Q}, \mathbb{P}_X \right) \right]^2 \leq \mathbb{E}_Q \int_0^T |\tilde{X}(t)|^2 \, dt$$

$$\leq \frac{\|\sigma\|^2}{\lambda} \mathbb{E}_Q \int_0^T \int_0^t e^{(\lambda - 2\delta)(t-s)} |\rho(s)|^2 \, ds \, dt$$

$$\leq \frac{\|\sigma\|^2}{\lambda} \frac{1 - e^{(\lambda - 2\delta)T}}{2\delta - \lambda} \mathbb{E}_Q \int_0^T |\rho(s)|^2 \, ds$$

Choosing $\lambda = \delta$, we get $\mathbb{P}_X$ verifies $T_2(\|\sigma\|^2/\delta)$. 

**Remark 6** It is not surprising to obtain the same constant $C$ as in the case of non reflected diffusions (see Theorem 5.6 of [Djellout et al. (2004)]). It is due to the fact that the reflection term $\eta_X$ satisfies

$$< X - Y, \eta_X - \eta_Y > \geq 0, \text{ a.s.};$$

for two solutions $X, Y$ of the RSDE.
Remark 7 We have notable consequences of $T_2(C)$ such as:

1. For any smooth cylindrical function $F$ on $C([0,T],\mathbb{R}^d) \subset G := L^2([0,T],\mathbb{R}^d; dt)$, that is

   $$F \in S = \{ f(<\gamma, h_1>,..., <\gamma, h_n>); n \geq 1, h \in H, f \in C^\infty_b(\mathbb{R}^n) \}$$

   where $H$ is Cameron-Martin space and $<\gamma_1, \gamma_2> = \int_0^T \gamma_1(t) \gamma_2(t) \, dt$, we have

   $$\text{Var}_\mathbb{P}_X(F) \leq \frac{\|\sigma\|^2}{\delta^2} \int_{C([0,T],\mathbb{R}^d)} \|\nabla F(\gamma)\|^2 \, d\mathbb{P}_X,$$

   (15)

   where $\text{Var}_\mathbb{P}_X(F)$ is the variance of $F$ under the law $\mathbb{P}_X$, and $\nabla F(\gamma) \in G$ is the gradient of $F$ at $\gamma$.

2. Let $K$ be a nonempty subset in $G$ such that $Z(\gamma) = \sup_{h \in K} <\gamma, h> \in L^1(\mathbb{P}_X)$, then we have the following Tsvirson’s type concentration inequality

   $$\int \exp \left[ \frac{\delta^2}{\|\sigma\|^2 \sup_{h \in K} \left[ <\gamma, h> - \frac{\|h\|^2}{2} \right] } \right] \, d\mathbb{P}_X \leq \exp \left( \frac{\delta^2}{\|\sigma\|^2} \mathbb{E}_\mathbb{P}_X Z \right)$$

   (16)

   Let $(P_t(x,.))_{t \geq 0}$ be the transition probability kernels of the solution $X(t)$ with initial point $X(0) = x \in \bar{O}$. In the following we derive an estimation of the Wasserstein-distance between the invariant measure of $X^x$ and its associated transition distributions. More precisely, we have

Theorem 8 Under the same assumptions of Theorem [5] the following holds true: $P_t$ admits a unique invariant probability measure $\mu$, and

$$W_2(P_t(x,.), \mu) \leq e^{-\delta t} \left( \int |x - y|^2 \, d\mu(y) \right)^{\frac{1}{2}}, \quad \forall x \in \bar{O}, \, t > 0.$$  

(17)

Proof: Let $X^x(t)$ and $X^y(t)$ be the solutions of equation [3] with initial point $x$, $y \in \bar{O}$ respectively. We use Itô formula, assumption $H(3)$, condition [11] and Gronwall’s lemma to obtain

$$\mathbb{E}[X^x(t) - X^y(t)]^2 \leq |x - y|^2 e^{-2\delta t}, \quad \forall t \geq 0,$$

which gives rise, by a classic coupling argument (see for example [Wu, 2010]), to the existence of a unique invariant probability measure of $(P_t)$ on $\mathbb{R}^d$ satisfying [17].

Remark 9

1– Similar arguments as those used in [Djellout et al., 2004] for the case of diffusions without reflections are valid to derive that

$$P_T(x,. \in T_2(\frac{\|\sigma\|^2}{2\delta})].$$

Since $P_T(x,. \rightarrow \mu$ as $T \rightarrow \infty$, we get by Lemma [11] that the invariant measure $\mu$ satisfies also $T_2(\frac{\|\sigma\|^2}{2\delta}).$

2– The following Poincaré inequality holds (see Theorem 5.6, [Bobkov et al., 2001]), for any $g \in C^\infty_b(\mathbb{R}^d)$

$$\text{Var}_{P_T(x,.)}(g) \leq \frac{\|\sigma\|^2}{2\delta} \int_{\mathbb{R}^d} \|\nabla g(y)\|^2 \, P_T(x, dy).$$
Similarly to the proof of Theorem 3, we have with the same notations
\[ d\hat{X}(t) = \hat{b}(t)dt + \hat{\sigma}(t)d\tilde{B}(t) + \sigma(X(t))\rho(t)dt - d\tilde{\eta}(t). \]

By Itô formula, we get
\[ d|\hat{X}(t)|^2 = \left[ 2 < \hat{X}(t), \hat{b}(t) + \sigma(X(t))\rho(t) > + \text{tr}(\hat{\sigma}(t)\hat{\sigma}(t)^T) \right] dt + 2 < \hat{X}(t), \hat{\sigma}(t)d\tilde{B}(t) > - 2 < \hat{X}(t), d\tilde{\eta}(t) >. \]

By virtue of H(3), the condition (14) and Cauchy-Schwartz inequality, we achieve for each \( \lambda > 0 \)
\[ \sup_{s \leq t} |\hat{X}(s)|^2 \leq (\lambda - 2\delta) \int_0^t |\hat{X}(s)|^2 ds + \frac{\|\sigma\|_{\infty}^2}{\lambda} \int_0^t |\rho(s)|^2 ds + \sup_{s \leq t} 2 \left| \int_0^s < \hat{X}(u), \hat{\sigma}(u)d\tilde{B}(u) > \right|. \quad (18) \]

Burkholder-Davies-Gundy inequality gives, for any \( \alpha > 0 \)
\[
\mathbb{E}_\tilde{Q} \sup_{s \leq t} 2 \left| \int_0^s < \hat{X}(u), \hat{\sigma}(u)d\tilde{B}(u) > \right| \leq \left[ 6 \mathbb{E}_\tilde{Q} \left( \int_0^t |\hat{\sigma}(s)\hat{X}(s)|^2 ds \right) \right]^{\frac{1}{2}} \\
\leq \left[ 6 \|\sigma\|_{Lip} \mathbb{E}_\tilde{Q} \left( \int_0^t |\hat{X}(s)|^4 ds \right) \right]^{\frac{1}{2}} \\
\leq \left[ 6 \|\sigma\|_{Lip} \mathbb{E}_\tilde{Q} \left( \sup_{s \leq t} |\hat{X}(s)|^2 \int_0^t |\hat{X}(s)|^2 ds \right) \right]^{\frac{1}{2}} \\
\leq \left[ 3 \|\sigma\|_{Lip} \mathbb{E}_\tilde{Q} \left( \alpha \sup_{s \leq t} |\hat{X}(s)|^2 + \frac{1}{\alpha} \int_0^t |\hat{X}(s)|^2 ds \right) \right].
\]

Therefore, by (15), together with the last inequality we obtain
\[
\mathbb{E}_\tilde{Q} \sup_{s \leq t} |\hat{X}(s)|^2 \leq 3 \|\sigma\|_{Lip} \alpha \mathbb{E}_\tilde{Q} \sup_{s \leq t} |\hat{X}(s)|^2 + (\lambda - 2\delta + \frac{3\|\sigma\|_{Lip}}{\alpha}) \int_0^t \mathbb{E}_\tilde{Q} \sup_{u \leq s} |\hat{X}(u)|^2 du \\
+ \frac{\|\sigma\|_{\infty}^2}{\lambda} \mathbb{E}_\tilde{Q} \int_0^t |\rho(s)|^2 ds.
\]

Thus, choosing \( 0 < \alpha < \frac{1}{3\|\sigma\|_{Lip}} \), we get
\[
\mathbb{E}_\tilde{Q} \sup_{s \leq t} |\hat{X}(s)|^2 \leq \frac{\lambda - 2\delta + \frac{3\|\sigma\|_{Lip}}{\alpha}}{1 - 3\alpha\|\sigma\|_{Lip}} \int_0^t \mathbb{E}_\tilde{Q} \sup_{u \leq s} |\hat{X}(u)|^2 du + \frac{\|\sigma\|_{\infty}^2}{\lambda(1 - 3\alpha\|\sigma\|_{Lip})} \mathbb{E}_\tilde{Q} \int_0^t |\rho(s)|^2 ds.
\]

Let
\[
C_1 = \frac{\lambda - 2\delta + \frac{3\|\sigma\|_{Lip}}{\alpha}}{1 - 3\alpha\|\sigma\|_{Lip}} \quad \text{and} \quad C_2 = \frac{\|\sigma\|_{\infty}^2}{\lambda(1 - 3\alpha\|\sigma\|_{Lip})}.
\]
Gronwall’s lemma implies
\[ E_{\tilde{Q}} \sup_{t \leq T} |\tilde{X}(t)|^2 \leq C E_{\tilde{Q}} \int_0^T |\rho(s)|^2 \, ds \]
Thus,
\[ \left[ W^{d_\infty}(\mathbb{Q}, \mathbb{P}_X) \right]^2 \leq C E_{\tilde{Q}} \int_0^T |\rho(s)|^2 \, ds. \]
Which proves that \( \mathbb{P}_X \in T_2(C) \), where \( C = C_2 e^{C_1 T}. \)

**Remark 11** The property \( T_2(C) \) with respect to the uniform metric is stronger than \( T_2(C) \) with respect to \( L^2 \)–metric. However this gain have two costs, the first one is the globally Lipschitzian property of \( \sigma \), the second one is the loss of sharpness of the constant as in Theorem 4. Remark that if we choose \( \lambda = 2\delta \), the optimality is obtained with the constant \( C = \frac{||\sigma||_{\infty}}{\delta} e^{3\delta ||\sigma||_{Lip} T}. \)

**Remark 12** The following result was established in Bobkov et al. (2001) on \( \mathbb{R}^d \) and extended after to \( C([0, T], \mathbb{R}^d) \) (see Villani (2003)): Let \( F \) be a lower bounded measurable function on \( C([0, T], \mathbb{R}^d) \), and consider
\[ Q_C F(\gamma) := \inf_{h \in \mathcal{C}([0, T], \mathbb{R}^d)} \left( F(\gamma + h) + \frac{1}{2C} ||h||^2_{\infty} \right) \]
the inf-convolution on \( C([0, T], \mathbb{R}^d) \) with respect to metric \( d_\infty \). Then \( T_2(C) \) in Theorem 10 implies
\[ E_{\mathbb{P}_X} \exp \left( Q_C F \right) \leq \exp \left( E_{\mathbb{P}_X} F \right). \]
If in addition \( F \) is Lipschitzian, since \( Q_C F \geq F - \frac{C}{2} ||F||^2_{Lip} \), we have the following concentration inequality
\[ E_{\mathbb{P}_X} \exp \left( F - E_{\mathbb{P}_X} F \right) \leq \exp \left( \frac{C}{2} ||F||^2_{Lip} \right) \quad (19) \]
which is similar to the one obtained by the property \( T_1(C) \), but here the constant \( C \) is explicit. By Chebyshev’s inequality and an optimization argument we obtain
\[ \mathbb{P}_X \left( F - E_{\mathbb{P}_X} F > r \right) \leq \exp \left( -\frac{r^2}{2C ||F||^2_{Lip}} \right), \quad \forall \ r > 0, \]
which is also valid, by an approximation argument, for unbounded Lipschitzian function \( F \).

In this part we establish a Harnack inequality for the semigroup of the reflected RSDE. We shall use the same technics as those used in Wang (2011), which consist in constructing a coupling under a new probability measure by Girsanov transformation. We need to suppose the following additional assumptions:

\[ H(4) : \quad \sigma(x)^t \sigma(x) \geq \lambda I, \quad \forall x \in \mathbb{R}^d, \]
\[ H(5) : \quad |< \sigma(x) - \sigma(y), x - y>| \leq k|x - y|, \quad \forall x, y \in \mathbb{R}^d, \]
where \( \lambda, k > 0 \) are two real constants.

**Theorem 13** 1. If \( H(3) \) and \( H(4) \) are satisfied, then log-Harnack inequality holds, for all \( f \geq 1, x, y \in \mathcal{O} \)
\[ P_T \log f(y) \leq \log P_T f(x) + \frac{-\delta |x - y|}{\lambda^2 (1 - e^{2\delta T})}. \]
2. If $H(3)$, $H(4)$ and $H(5)$ hold, then for $p > (1 + \frac{k}{2})^2$ and $c_p = \max\{k, \frac{1}{2}((\sqrt{p} - 1))$, then the Harnack inequality

\[ (P_Tf(y))^p \leq (P_Tf(x))^p \exp \left[ -\delta \sqrt{p}(\sqrt{p} - 1)|x - y| \right] \]

holds for all $T > 0, x, y \in \mathcal{O}$ and $\delta$ bounded positive function.

**Proof:** Exploiting in many places the fact that for two solutions $X, Y$ we have

\[ <X - Y, \eta_X - \eta_Y> \geq 0, \ a.s.; \]

the proof follows exactly the same lines as for the non-reflected diffusions case. So, we only give some ideas of the proof and refer to Wang (2011) for more details.

Let $x, y \in \mathcal{O}, T > 0$ and $p > (1 + \frac{k}{2})^2$ be fixed such that $x \neq y$. We set

\[ \theta_T := \frac{2k}{(\sqrt{p} - 1)\lambda} \in (0, 2) \]  

(20)

For $\theta \in (0, 2)$, we consider

\[ \xi_t = \frac{\theta - 2}{2\delta}(1 - e^{-2\lambda(t-T)}), t \in [0, T]. \]

Then $\xi$ is smooth and strictly positive on $[0, T)$ such that

\[ 2 + 2\delta \xi_t + \xi'_t = \theta, \ t \in [0, T]. \]  

(21)

We consider the coupling

\[
\begin{align*}
\frac{dX(t)}{dt} &= b(X(t))dt + \sigma(X(t))dB(t) - d\eta_X(t), \quad X(0) = x, \\
\frac{dY(t)}{dt} &= b(Y(t))dt + \sigma(Y(t))dB(t) - d\eta_Y(t) + \frac{1}{\xi_t}\sigma(X(t))\sigma(X(t))^{-1}(X(t) - Y(t))dt, \quad Y(0) = y.
\end{align*}
\]

$(X(t), Y(t))$ is a well defined continuous process for $t < T \wedge \zeta$, where $\zeta = \lim_n \xi_n$ for

\[ \zeta_n := \inf\{t \in [0, T) : |Y(t)| \geq n\}, \]

with convention $\inf \emptyset = T$. Let

\[ R_t := \exp \left[ -\int_0^{t\wedge \zeta} \frac{1}{\xi_s} <\sigma(X(s))^{-1}(X(s) - Y(s)), dB(s) > -\frac{1}{2} \int_0^{t\wedge \zeta} \frac{1}{\xi_s^2}|\sigma(X(s))^{-1}(X(t) - Y(t))|^2ds \right] \]

$t \in [0, T)$.

$(R_t)_{t \in [0, T]}$ is a uniformly integrable martingale (see Wang (2011)), and then the process

\[ dB(t) = dB(t) + \frac{1}{\xi_t}\sigma(X(t))^{-1}(X(t) - Y(t))dt \]

is a Brownian motion under the new probability $Q = R_T P$. Consequently the processes $X$ and $Y$ satisfy under $Q$

\[
\begin{align*}
\frac{dX(t)}{dt} &= b(X(t))dt + \sigma(X(t))dB(t) - d\eta_X(t) - \frac{X(t) - Y(t)}{\xi_t}dt, \quad X(0) = x, \\
\frac{dY(t)}{dt} &= b(Y(t))dt + \sigma(Y(t))dB(t) - d\eta_Y(t), \quad Y(0) = y.
\end{align*}
\]
Taking into account that $X(T) = Y(T) \text{ Q-a.s.}$, Young inequality gives rise for any $f \geq 1$

$$P_T \log f(y) = \mathbb{E}_Q \log f(Y_T) = \mathbb{E} \left[ R_T \log f(X_T) \right]$$

$$\leq \mathbb{E} R_T \log R_T + \log \mathbb{E} f(X(T)).$$

Now, using the following estimation given by Lemma 2.1 in Wang (2011)

$$\mathbb{E} R_T \log R_T \leq -\delta |x - y|^2$$

and taking $\theta = 1$, we complete the proof of the first inequality.

On other hand, let $\theta = \theta_T$, since $X(T) = Y(T) \text{ Q-a.s.}$, we have

$$(P_T f(y))^p = (\mathbb{E}_Q f(Y(T)))^p = (\mathbb{E} R_T f(X(T)))^p$$

$$\leq P_T f^p(x) \left( \mathbb{E} R_T^{p/(p-1)} \right)^{p-1}.$$

By equality in (20) we see that $\frac{p}{p-1} = 1 + \frac{\lambda^2 \theta_T^2}{4c_p(c_p + \theta_T \lambda)} = 1 + r_T$. Lemma 2.2 in Wang (2011), yields

$$\left( \mathbb{E} R_T^{p/(p-1)} \right)^{p-1} = \left( \mathbb{E} R_T^{1 + r_T} \right)^{p-1}$$

$$\leq \exp \left[ \frac{-(p-1) \theta_T \delta (2k + \theta_T \lambda) |x - y|^2}{4k^2 (2 - \theta_T)(k + \theta \lambda)(1 - e^{2\delta T})} \right]$$

$$= \exp \left[ \frac{-\delta \sqrt{p} (\sqrt{p} - 1) |x - y|^2}{2c_p ([\sqrt{p} - 1] \lambda - c_p) (1 - e^{2\delta T})} \right].$$

This completes the proof of the second inequality.

**Remark 14** As it is shown in Wang (2011) for non-reflected diffusions case, we can apply Theorem 13 to get some Harnack inequalities and contractivity properties for transition probabilities of reflected diffusion semigroups.

### 3. Stochastic differential equations involving local times

As was pointed out in the introduction, in this section we consider the following stochastic differential equation (SDEL):

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t) + \int_R \nu(da) dL^a_t(X)$$

$$X(0) = x,$$

where $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions and $\nu$ is a bounded signed measure on $\mathbb{R}$, such that $|\nu(a)| < 1, \forall a \in \mathbb{R}$. The process $L^a_t(X)$ stands for the symmetric local time of the unknown process $X$ at a point $a$ and $B_t$ is a real Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by $B_t$. SDEs of type (22) have been studied previously by many authors. We will refer essentially to the paper of Le Gall (1984) where the author gives necessary and sufficient conditions for pathwise uniqueness, and together with results on the existence of weak solutions he also proves results on strong uniqueness.
Consider the function $f_\nu$ given by:

$$f_\nu(x) = \exp \left( -2\nu^c([-\infty, x]) \right) \prod_{y \leq x} \left( \frac{1 - \nu(y)}{1 + \nu(y)} \right), \quad x \in \mathbb{R}$$

where $\nu^c$ is the continuous part of $\nu$. The following lemma appears in Le Gall (1984).

**Lemma 2** Let $\nu$ be a bounded signed measure on $\mathbb{R}$. Then we have:

1. $f_\nu$ is of bounded variation on $\mathbb{R}$,
2. $f_\nu$ is right continuous,
3. there exist constants $m, M > 0$ such that $m \leq f_\nu \leq M$,
4. the function $f_\nu$ satisfies $f_\nu'(dx) + \left( f_\nu(x) + f_\nu(x-) \right) \nu(dx) = 0$, with $f_\nu'(dx)$ denotes the bounded measure associated with $f_\nu$ and $f_\nu(x-)$ denotes the left-limit of $f_\nu$ at a point $x$.

We consider the function $F(x) = \int_0^x f_\nu(u) \, du, \, x \in \mathbb{R}$. It is easy to show that $F$ is one to one and that $F$ and $F^{-1}$ are Lipschitz functions. An appeal to Itô-Tanaka formula provides us the following lemma (see Le Gall (1984))

**Lemma 3** A process $X$ is a solution of equation (22) if and only if $Y := F(X)$ is a solution of:

$$dY(t) = \bar{b}(Y(t)) \, dt + \bar{\sigma}(Y(t)) \, dB(t), \quad Y(0) = F(x), \quad (23)$$

where

$$\bar{b}(x) = (bf_\nu) \circ F^{-1}(x) \quad \text{and} \quad \bar{\sigma}(x) = (sf_\nu) \circ F^{-1}(x).$$

In the literature, existence and uniqueness results of strong solutions for equations of type (22) and (23) were obtained under weaker conditions than we will assume here. But, the proof of $T_2(C)$ inequality requires some strong conditions on the coefficients (even in ordinary SDEs framework). In the rest of this section we assume that there is a unique strong solution of (23).

To prove our results we need the following stability property of $T_2(C)$ (see Lemma 2.1 in Djellout et al. (2004)):

**Lemma 4** Let $(E, d_E)$ and $(F, d_F)$ be two metric spaces and $\psi : (E, d_E) \to (F, d_F)$ is a Lipschitz application ($\alpha > 0$), such that for an $\alpha > 0$

$$d_F(\psi(x), \psi(y)) \leq \alpha d_E(x, y), \quad \forall x, y \in E.$$  

If $\mu \in T_p(C)$ on $(E, d_E)$, then $\tilde{\mu} := \mu \circ \psi^{-1} \in T_p(\alpha^2 C)$ on $(F, d_F)$, for any $p \geq 1$.

The next theorem is the main result of this section. We need to suppose that $\bar{b}$ and $\bar{\sigma}$ are globally Lipschitz functions, that is we make the following assumptions on the coefficients of equation (22):

**H(6)** \[ |\bar{b}(x) - \bar{b}(y)| \vee |\bar{\sigma}(x) - \bar{\sigma}(y)| \leq k |x - y|, \quad \forall x, y \in \mathbb{R}. \]
We also suppose the following dissipativity assumption:

\[ H(7) \quad \text{there exists } \delta > 0 \text{ such that, for all } x, y \in \mathbb{R} \text{ we have} \]
\[
\left( \bar{\sigma}(x) - \bar{\sigma}(y) \right)^2 + 2(x - y)(\bar{b}(x) - \bar{b}(y)) \leq -2\delta |x - y|^2.
\]

**Theorem 15** Suppose that \( H(6) \) and \( H(7) \) hold and \( ||\bar{\sigma}||_{\infty} < \infty \). Let \( P_X \) be the law of \( X \), the solution of the stochastic differential equation (22) with initial condition \( X(0) = x \in \mathbb{R} \), then we have

1. The probability measure \( P_X \) satisfies \( T_2(\frac{||\bar{\sigma}||^2}{m^2}) \) on the metric space \( C([0, T], \mathbb{R}) \) equipped with the metric \( d_2 \).
2. There exists some constant \( C = C(T, ||\bar{\sigma}||_{\infty}, m, k) > 0 \) such that \( P_X \in T_2(C) \) on \( C([0, T], \mathbb{R}) \) with respect to \( d_\infty \)-metric.

**Proof:** By notations of Lemma 3, let us consider the following ordinary stochastic differential equation:

\[
\begin{cases}
    dY(t) = \bar{b}(Y(t)) \, dt + \bar{\sigma}(Y(t)) \, dB(t), \\
    Y(0) = F(x).
\end{cases}
\]  

(24)

Following the same arguments as those used in section 2, we can prove under assumptions of Theorem 15 that \( P_Y \in T_2(\frac{||\bar{\sigma}||^2}{m^2}) \) with respect to the \( L^2 \) norm (we can also see Djellout et al. (2004)). We now consider the application \( \Psi \) defined by:

\[
\Psi : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})
\]

\[ \gamma \mapsto \psi(\gamma) = F^{-1} \circ \gamma. \]

It’s clear that for each \( \gamma_1, \gamma_2 \in C([0, T], \mathbb{R}) \)

\[
d_2(\Psi(\gamma_1), \Psi(\gamma_2)) \leq \frac{1}{m}d_2(\gamma_1, \gamma_2),
\]

thus, the map \( \Psi \) is \( \frac{1}{m} \)-Lipschitzian, where \( m \) is provided by Lemma (2).

In other hand, we have

\[ P_X = P_Y \circ \Psi^{-1}, \]

and by stability property of \( T_2(C) \) under Lipschitzian maps (Lemma (7)), we conclude that \( P_X \in T_2(\frac{||\bar{\sigma}||^2}{m^2}) \) on \( C([0, T], \mathbb{R}), d_2 \). Which ends the proof of the first assertion. The second point uses the same arguments.

**Remark 16** If we take, \( b = 0, \sigma = 1 \) and \( \nu = \beta \delta_0 \), where \( |\beta| < 1 \), we recognize the famous Skew Brownian motion for which \( H(7) \) is no longer valid, and we cannot get the property \( T_2(C) \) (or \( T_1(C) \)) for this process form Theorem 15. To our knowledge, this question has not yet been adressed in the literature. The only related result we found is in Abakirova (2014) where some Poincaré and log-Sobolev type inequalities have been highlighted.
Remark 17 1. An appeal to Jensen inequality yields that $T_2(C) \Rightarrow T_1(C)$, then the property $T_1(C)$ holds for the probability measure $\mathbb{P}_X$ and we have for any Lipschitzian function $G : C([0, T], \mathbb{R}) \rightarrow \mathbb{R}$

$$
\mathbb{P}_X(G - \mathbb{E}_{\mathbb{P}_X} G > r) \leq \exp\left(-\frac{r^2}{2C\|G\|_{Lip}^2}\right), \quad \forall \ r > 0.
$$

Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitzian function, such that $\|V\|_{Lip} \leq \alpha$. We define $F_V$ and $F_\infty$ on $C([0, T], \mathbb{R})$ by

$$
F_V(\gamma) = \frac{1}{T} \int_0^T V(\gamma(t)) \, dt,
$$

$$
F_\infty(\gamma) = \sup_{t \in [0, T]} |\gamma(t) - \gamma(0)|.
$$

The function $F_V$ is $\alpha$-Lipschitzian with respect to $d_\infty$. As for $F_\infty$, it’s $1$-Lipschitzian map with respect to $d_\infty$. Using (25) we have the following Hoeffding-type inequalities for the solution $X$ of (22) on the metric space of continuous functions, endowed with the metric $d$. For all $r > 0$ we have

$$
\mathbb{P}\left(\frac{1}{T} \int_0^T V(X(t)) - \mathbb{E}V(X(t)) \, dt > r\right) \leq \exp\left(-\frac{r^2}{2C\alpha^2}\right),
$$

and using the functional $F_\infty$ we get

$$
\mathbb{P}\left(\sup_{t \in [0, T]} |X(t) - x| - \mathbb{E}\left[\sup_{t \in [0, T]} |X(t) - x|\right] > r\right) \leq \exp\left(-\frac{r^2}{2C}\right).
$$

2. The estimates (20) and (24) are well adapted to the study of small and large time asymptotics of the solution of equation (22).

The solution $X$ of SDEL (22) is a strong Markov process. Let $(P_t)$ be the semigroup of transition probability kernels of our diffusion. The next proposition shows the existence of a unique invariant measure.

Proposition 1 Under the assumption $H(7)$ there exist a unique invariant measure $\mu$ for $(P_t)$, and we have the following exponential convergence in sense of Wasserstein distance:

$$
W_2(P_t(x, \cdot), \mu) \leq \frac{M}{m} e^{-\delta t} \left(\int |x - y|^2 \mu(dy)\right)^{\frac{1}{2}}, \forall x \in \mathbb{R}, \ t > 0.
$$

Proof: Let $Q_t(x, \cdot)$ denote the transition kernels associated to the Markov process solution of equation (22), by Djellout et al. (2004), $(Q_t)$ admit a unique invariant measure $\bar{\mu}$, thanks to the transformation $X_t = F^{-1}(Y_t)$, we have

$$
P_t f(x) = Q_t(f \circ F^{-1})(F(x)).
$$

Then it’s easy to check that $\mu := \bar{\mu} \circ F$ is the unique measure invariant for $(P_t)$. Again by Djellout et al. (2004), we get

$$
W_2(P_t(x, \cdot), \mu) = W_2(Q_t(F(x), F(\cdot)), \bar{\mu} \circ F)
\leq \frac{1}{m} W_2(Q_t(F(x), \cdot), \bar{\mu})
\leq \frac{1}{m} e^{-\delta t} \left(\int |F(x) - y|^2 \bar{\mu}(dy)\right)^{\frac{1}{2}}
\leq \frac{1}{m} e^{-\delta t} \left(\int |F(x) - F(F^{-1}(y))|^2 \bar{\mu}(dy)\right)^{\frac{1}{2}}
\leq \frac{M}{m} e^{-\delta t} \left(\int |x - y|^2 \bar{\mu}(dy)\right)^{\frac{1}{2}}.
$$
Which completes the proof.

The next theorem shows a Harnack inequality for the semigroup of $X$, which is a consequence of the Theorem 1.1 proven in [Wang (2011)] for ordinary stochastic differential equations under the additional assumptions:

$$H(8) \quad \text{There exist } \lambda, \beta > 0, \text{ s.t. } \bar{\sigma}(x)^2 \geq \lambda , \quad \text{and } \quad |(\bar{\sigma}(x) - \bar{\sigma}(y))(x - y)| \leq \gamma |x - y|, \quad x, y \in \mathbb{R},$$

**Theorem 18** If $H(7)$ and $H(8)$ hold, then for $p > (1 + \frac{2}{p})^2$ and $\beta_p = \max\{\gamma, \frac{\lambda}{\sqrt{p} - 1}\}$, the Harnack inequality

$$(P_T f(y))^p \leq (P_T f^p(x)) \exp \left[ -\frac{\delta M \sqrt{p}(\sqrt{p} - 1)}{2\gamma_p(\sqrt{p} - 1)\lambda - \gamma_p(1 - e^{\delta T})} |x - y|^2 \right]$$

holds for all $T > 0$, $x, y \in \mathbb{R}$ and $f$ positive bounded measurable function on $\mathbb{R}$.

**Proof:** According to Theorem 1.1 in [Wang (2011)], the semigroup $Q$ associated to the transformed equation (23) satisfies the Harnack inequality, and by the relation $P_t f(x) = Q_t(f \circ F^{-1})(F(x))$ we deduce the desired inequality.

**Example 1** For $\delta > 0$, $0 < \beta < 1$, we consider Equation (22) driven by the coefficients:

$$\sigma(x) = I_{x<0} + \frac{1 + \beta}{1 - \beta} I_{x\geq 0}; \quad b(x) = -\delta x I_{x<0} - \delta x \frac{1 + \beta}{1 - \beta} I_{x\geq 0}$$

and the measure $\nu = \beta \delta_0$.

We get easily $\bar{\sigma}(x) = 1$, $\bar{b}(x) = -\delta x$, $m = \frac{1 - \beta}{1 + \beta}$ and $M = 1$. The corresponding process $Y$ solution of Equation (22) is an Ornstein-Uhlenbeck process and that $\bar{\sigma}$, $\bar{b}$ satisfy the conditions of Theorem 14 and Theorem 18.

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