Two-dimensional Fermi Gas Revisited

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A number of authors have taken issue with the demonstration that the 2D Fermion gas with short-range repulsive interactions (and, of course, including spin) cannot be consistently treated as a renormalised quasiparticle system. This paper shows that the arguments given in some of these papers are invalid or irrelevant.

A number of papers have addressed the two-dimensional Fermion gas from the viewpoint of "Fermi liquid theory" in the sense of perturbation theory based on renormalised single-particle excitations. The first of these was by P. Bloom in 1975[1], the most recent by Disertori and Rivasseau in 2000[2], based on earlier work by Trubowitz et al.[3], using the quantum renormalisation group methods pioneered by Anderson, Yuval and Hamann[4]. I have argued, on the other hand, that there is strong evidence indicating fundamental difficulties with the theory. The point is made here that (a) the restriction on the lower cutoff in energy accepted by refs. [2] and [3] excludes specifically the regions of parameter values where the effects calculated in my work [2] occur so that refs. [2] and [3] are not relevant to the correctness of [2]; and (b) that the calculations in reference [3] (the only attempt at a rigorously controlled calculation, to my knowledge, that might have settled the matter if correct) suffer from an incorrect solution of the fundamental integral equation.

Some authors [5], [6] have accepted my calculation of the interaction vertex and tried to reconcile it with quasiparticle theory; I also (c) give a new argument demonstrating the failure of this approach.

The method of [3] in fact follows the reasoning of Bloom in [1], who adapted to two dimensions the scheme developed independently by Galitskii and by Huang, Lee and Yang[7, 8], for hard-core systems in three dimensions. This is the "pseudopotential method" which was demonstrated to give an expansion in powers of the density, hence is a rigorously controlled approximation. The basis of the expansion is the idea that encounters of more than two particles within each others’ range of interaction, a, occupy a volume of configuration space which is smaller by the factor $n a^D \approx (k_F a)^D$ than binary encounters. In diagrammatic terms, diagrams with extra hole lines are smaller by this factor than those with a single hole line and can be neglected.

Some criticisms have proposed (e.g. [5]) that the particle-particle ladder diagrams responsible for the effects in [3] could be cancelled by particle-hole loops. The latter, however, have a different density dependence and are irrelevant in the low-density limit.

I should add two remarks about this method. One is that although the specific calculations in the original references are carried out for hard spheres or discs, they are valid a fortiori for any short-range repulsive potential, as Bloom shows in an appendix to his paper. The second is that it can be made very plausible that the restriction to binary interactions is qualitatively generalisable to any density, as is for instance the case in the fermi liquid theory itself, since the Landau functional contains only binary quasiparticle interactions.

The plan of the method of Galitskii and Huang et al (as explained in the textbook of Fetter and Walecka[10], for instance) is first to derive the pseudopotential or renormalised vertex $\Gamma(P; q, q')$ which is the sum of ladder diagrams, that is the solution of the Bethe-Salpeter equation for the repeated scattering of two particles. Here $P$ is the total momentum of the incoming particles of momenta $p$ and $p'$, and $q$ their relative momentum, i.e. $P = p + p'$ and $q = p - p'$, while $q'$ is the relative momentum of the outgoing particles, their total momentum being $P$. The ladder diagrams are the only ones which survive at this order. We note, of course, that the important interactions are only between opposite-spin particles, since the odd angular momenta are down by at least a factor $a^2$. Once having obtained $\Gamma$, the diagrams for self-energy, chemical potential etc become simple hartree loops and Bloom, at least, applies well-known formulae to evaluate these quantities. (I am not sure whether the conventional formulas are quite correct for these essentially on-shell scatterings, but this is not germane to the present argument.)

Bloom follows Galitskii in solving first for the case of vanishing density, and then iterating the integral equation for $\Gamma$. At vanishing density this first step of course is equivalent to solving the scattering equation for two free particles of momenta $p$ and $p'$. Two free particles of opposite spins with momenta $pa$, $p'a << 1$ scatter, to order $a^2$, in the $l=0$ channel only, with amplitude

$$T(q) = -2\exp(i\phi)\sin\phi,$$

where $\phi$ is the $l=0$ “phase shift” for relative momentum and $T$ is the conventional scattering matrix. We have chosen the normalisation used by Bloom. The interpretation of $\phi$ as a phase shift is simply a statement of the boundary condition which the interaction enforces on the wave function of the two particles at $(r - r') \to 0$, and does not directly make any statement about the imaginary part of any self-energy, or the like. For free particles,

$$\phi_0 = \frac{2}{t/U + |\ln qa|}, \tag{2}$$
which is essentially the result of the appendix to reference 1. Here, to make the model more specific, I have used a tight-binding Hubbard model of lattice constant a and interaction $U$, but at low density this result is model-independent.

It is important to realise that $\phi_0$ in [2] vanishes as $q \to 0$. This is because $T_0$ satisfies an integral equation with a singular kernel,

$$T_0(q, q') = U(q - q') + \int \frac{d^2 p}{(2\pi)^2} \frac{U(p - q) T_0(p, q')}{p^2 - q^2 + i\eta}$$

whose kernel can and does diverge, leading to a logarithmic singularity in $q$ for the integral term. From $T$ the vertex function $\Gamma$ can be derived via an integral representation given in references (1) and (10), but in fact its value on the energy shell is identical to the scattering amplitude $T$.

When the density is finite the integral equation for the scattering amplitude $T$ is modified for the effects of the exclusion principle. The effect of interactions with other particles may be neglected in our low density approximation, but the statistical "interactions" may not be ignored because they are effectively of infinite range. The integral equation [3] is modified by including a factor in the kernel to take account of the exclusion of intermediate states,

$$N(P, q^*) = 1 - n\{(P + q^*)/2\} - n\{(P - q^*)/2\}$$

Here $n$ is the Fermi function, =1 when its argument is less than the Fermi momentum and 0 otherwise. As has been shown repeatedly [4], if the momenta $p$ and $p'$ are on the Fermi surface or below it, the factor $N$ eliminates the singular kernel in the integral equation and its solution becomes finite at $q=0$. $T$ is given by [5] with

$$\phi = \frac{2}{t/U + \ell n k_F a}$$

This is the point at which we can dispose of point [1] of the opening paragraph. References [2] and [3] use finite temperature techniques and are based on continuation from a weak coupling expansion. It is necessary (for reasons I do not quite understand) for them to restrict the temperature to

$$T/T_F \geq \exp(-\frac{t}{U})$$

It seems that with this as a cutoff for the Fermi functions in [4], it would not be possible to distinguish between the phase shifts [2] and [3]. If $t/U$ is sufficiently small that the logarithmic factor in the denominator dominates, $T/T_F$ will be of order unity and there will be no sharp fermi surface to exhibit Fermi liquid properties. On the other hand, if $U$ is small the cutoff will set in before the logarithmic factor dominates, and again one cannot tell whether or not the Fermi surface is sharp.

As I said, the reasons for this restriction are not clear. It cannot really be to avoid the Kohn-Luttinger divergence in the Cooper channel [11] because that involves higher $l$ amplitudes which bring in higher powers of $a$ or of $1/U$. My suspicion is that it may be precisely the divergence of scattering length implied by [5] which these works have run up against.

Now let me discuss point (b). Bloom follows Galitskii in solving first for the integral equation equivalent to [3] for the vertex function $\Gamma_0$, and then deriving an integral equation for the difference between the true $\Gamma$ including the effects of exclusion, i.e. the factor $N$ from [4], and $\Gamma_0$. This procedure is acceptable in 3 dimensions because, although the kernel has a pole, the resulting singularity is harmless, amounting to $\int \frac{dk}{q^2}$. In 2 dimensions, however, there is a true singularity in the kernel, which is precisely the reason for the $1/lnq$ behavior of $\Gamma_0$ at $q=0$. The offending integral equation is equation [4] of reference [6], which we reproduce here:

$$\Gamma(p, p'; P) = \Gamma_0 + \int \frac{d^2 k}{(2\pi)^2} \Gamma_0(p, k; P) f(P, k) \Gamma(k, p'; P),$$

where

$$f(P, k) = \left[ \frac{N(P, k)}{\epsilon - k^2 + i\delta N(P, k)} - \frac{1}{\epsilon - k^2 + i\delta} \right]$$

Naively iterating this equation allows Bloom to obtain a series for gamma in powers of $1/lnq$, with each term becoming successively more innocuous. We note, however, that the kernel in this equation subtracts away the same singularity that one has in the zero-density case, while the added term is regular for any incoming momenta which are on or below the fermi surface. That is, the iteration is not justified because the solution for finite density is qualitatively different from that for zero density. In fact, as we have already shown elsewhere, the solution for finite density is perfectly regular and does not vanish at $q = 0$; it is given by equations [3] and [5]. This is then the mathematical mistake which led Bloom to conclude that the 2 dimensional Fermi gas behaves conventionally.

What is the source of the difficulties of quasiparticle theory that this result entails? That these difficulties are plausible is clear on the face of it. The 3-dimensional theory of Huang et al and Galitskii brings one straightforwardly to the traditional "scattering length" theory, where the scattering length $a$ (not to be confused with the interaction range) is defined by

$$\lim(q \to 0) \phi = qa$$

The cross-section, for instance, for elastic scattering is $\propto a^2$ in 3D. Even for zero density, in 2D the scattering length diverges for forward scattering; that is, the concept has no real meaning; but Bloom showed that one could nonetheless just get by if it diverged only as $1/\sqrt{\ln q}$. But in fact it diverges as $1/q$, which means that
the scattering length is of the same order as the size of the system: like statistical interactions, the range is infinite. Qualitatively, there are always other particles with the same momentum and opposite spin which cannot live at the same momentum value because they are always within each others’ effective scattering length.

Some sets of authors have accepted the demonstration in reference [3] of the divergent scattering length, but argued that in one way or another one may revive quasiparticle theory nonetheless [13 for example]. A paper with Kveshchenko [12] gives a bosonisation technique for disproving this, but bosonisation is neither familiar nor totally rigorous. Therefore I give here a rather more straightforward demonstration.

What we have demonstrated above is that there is a finite phase shift for elastic, on-shell scattering in the \( l = 0 \) channel for opposite-spin particles with zero relative momentum. If such a thing were to happen for free particles, it would violate Levinson’s theorem, but there is no proof that Levinson’s theorem holds in the presence of the exclusions due to the gas of other particles.

Surprisingly, the appropriate divergences appear in the conventional formulas for self-energy as presented in references [6] and [7]. They come from the extension of the scattering amplitude off the energy shell, i.e. from the virtual transitions which cause modification of the wave functions rather than real scattering. It is not surprising that they do not contain the exclusion factor \( N(P, q) \) for this reason. If one simply takes the formula used by Bloom for the second-order self-energy, his equation [8c], it is easy to derive that the derivative with respect to momentum at fixed energy is logarithmically divergent, and hence the effective mass is infinite. The appropriate term in Bloom’s equation [8c] is

\[
\Sigma_b(p) = \int \int \frac{d^3k d^3k'}{(2\pi)^4} \left\{ T^2(p, k')N(k)P \frac{1}{k'^2 - q^2} \right\} + \text{inelastic terms}
\]

Here \( q = 1/(2(p - k)) \). Clearly, setting \( T \) equal to the finite constant \( E \) and taking the derivative with respect to \( p \) under the integrals leads to a logarithmic divergence where the denominator vanishes. Another indication of this anomaly in \( \partial \Sigma_b / \partial p \) is the fact that the effect of the exclusion factor changes radically from \( P < 2k_F \) to \( P > 2k_F \), so that there is an infinity in \( \partial \phi(q) / \partial q \).

Another source of difficulties derives from the optical theorem as applied to \( T \). The existence of a finite scattering amplitude in the \( l = 0 \) channel leads to a finite forward scattering, which comes through as a delta-function in the effective interaction for \( p = p' \). This changes the character of zero sound—it essentially becomes the tomographic boson of [12].

With a little more effort, the divergence of the renormalisation constant \( Z \) to zero could be shown. The divergence of the scattering length should however be suffi-

\[\textbf{References}\]

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