On Algorithmic Equivalence of Instruction Sequences for Computing Bit String Functions

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Abstract. Every partial function from bit strings of a given length to bit strings of a possibly different given length can be computed by a finite instruction sequence that contains only instructions to set and get the content of Boolean registers, forward jump instructions, and a termination instruction. We look for an equivalence relation on instruction sequences of this kind that captures to a reasonable degree the intuitive notion that two instruction sequences express the same algorithm.

Keywords: structural algorithmic equivalence, structural computational equivalence, single-pass instruction sequence, bit string function.

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1 Introduction

In [9], it is among other things shown that a total function on bit strings whose result is a bit string of length 1 belongs to $P/poly$ iff it can be computed by polynomial-length instruction sequences that contain only instructions to set and get the content of Boolean registers, forward jump instructions, and a termination instruction. In [12], where instruction sequences are considered which contain backward jump instructions in addition to the above-mentioned instructions, it is among other things shown that there exist a total function on bit strings and an algorithm for computing the function such that the function can be computed according to the algorithm by quadratic-length instruction sequences without backward jump instructions and by linear-length instruction sequences with backward jump instructions.

With that, we implicitly assumed that the instruction sequences without backward jump instructions concerned and the instruction sequences with backward jump instructions concerned express the same algorithm. We considered this assumption acceptable because all the different views on what characterizes an algorithm lead to the conclusion that we have to do here with different expressions of the same algorithm. However, we cannot prove this due to the absence of a mathematically precise definition of an equivalence relation on the instruction sequences of the kind considered that captures the intuitive notion that two instruction sequences express the same algorithm.
Attempts have been made to define such an equivalence relation in other settings (see e.g. [10, 20, 25]), but there is still doubt if there exists an equivalence relation that completely captures the intuitive notion that two instruction sequences express the same algorithm (see e.g. [14, 15]). If such an equivalence relation would exist, algorithms could be defined as equivalence classes of programs with respect to this equivalence relation. We take the viewpoint that there may be different degrees to which programs express the same algorithm, these different degrees may give rise to different equivalence relations, and these different equivalence relations may be interesting despite the fact that they incompletely capture the intuitive notion that two instruction sequences express the same algorithm.

In this paper, we look for an equivalence relation on instruction sequences of the kind considered in [9] that captures to a reasonable degree the intuitive notion that two instruction sequences express the same algorithm. Because the existing viewpoints on what is an algorithm are diverse in character and leave loose ends, there is little that we can build on. Therefore, we restrict ourselves to what is virtually the simplest case. That is, we take two fixed but arbitrary natural numbers \( n, m \) and restrict ourselves to instruction sequences for computing partial functions from \( \{0, 1\}^n \) to \( \{0, 1\}^m \). If \( n \) and \( m \) are very large, this simple case covers, at a very low level of program and data representation, many non-interactive programs that are found in actual practice.

In [5], an attempt is made to approach the semantics of programming languages from the perspective that a program is in essence an instruction sequence. The groundwork for the approach is an algebraic theory of single-pass instruction sequences, called program algebra, and an algebraic theory of mathematical objects that represent the behaviours produced by instruction sequences under execution, called basic thread algebra. As in the previous works originating from this work on an approach to programming language semantics (see e.g. [8]), the work presented in this paper is carried out in the setting of program algebra and basic thread algebra. In this paper, we only give brief summaries of program algebra, basic thread algebra, and the greater part of the extension of basic thread algebra that is used. A comprehensive introduction, including examples, can among other things be found in [8].

This paper is organized as follows. First, we give a survey of program algebra and basic thread algebra (Section 2) and a survey of the greater part of the extension of basic thread algebra that is used in this paper (Section 3). Next, we introduce the remaining part of the extension of basic thread algebra that is used in this paper (Section 4) and present the instruction sequences that concern us in this paper (Section 5). After that, we give some background on the intuitive notion that two instruction sequences express the same algorithm (Section 6) and a picture of our intuition about this notion (Section 7). Then, we define an algorithmic equivalence relation on the instruction sequences introduced before that corresponds to this intuition (Section 8) and a related equivalence rela-

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1 In [5], basic thread algebra is introduced under the name basic polarized process algebra.
tion that happens to be too coarse to be an algorithmic equivalence relation (Section 9). Following this, we show how the algorithmic equivalence relation introduced before can be lifted to programs in a higher-level program notation (Section 10) and point out that we are still far from the definitive answer to the question “what is an algorithm?” (Section 11). Finally, we make some concluding remarks (Section 12).

Henceforth, we will regularly refer to the intuitive notion that two instruction sequences express the same algorithm as the notion of algorithmic sameness. Moreover, we will use the term algorithmic equivalence relation as a general name for equivalence relations that capture to some degree this intuitive notion.

The groundwork for the work presented in this paper has been laid in previous papers. Portions of this groundwork has been copied near verbatim or slightly modified in the current paper. The greater part of Sections 2 and 5 originate from [9] and Section 3 originates from [7]. Because self-plagiarism is currently a prevailing issue in some scientific disciplines, we will shortly explain in an appendix why we do not regard the above-mentioned practice as a matter of self-plagiarism.

2 Program Algebra and Basic Thread Algebra

In this section, we give a survey of PGA (ProGram Algebra) and BTA (Basic Thread Algebra) and make precise in the setting of BTA which behaviours are produced by the instruction sequences considered in PGA under execution. The greater part of this section originates from [9].

In PGA, it is assumed that there is a fixed but arbitrary set \( A \) of basic instructions. The intuition is that the execution of a basic instruction may modify a state and produces a reply at its completion. The possible replies are 0 and 1. The actual reply is generally state-dependent. The set \( A \) is the basis for the set of instructions that may occur in the instruction sequences considered in PGA. The elements of the latter set are called primitive instructions. There are five kinds of primitive instructions:

- for each \( a \in A \), a plain basic instruction \( a \);
- for each \( a \in A \), a positive test instruction \( +a \);
- for each \( a \in A \), a negative test instruction \( -a \);
- for each \( l \in \mathbb{N} \), a forward jump instruction \( \#l \);
- a termination instruction \( ! \).

We write \( \mathcal{I} \) for the set of all primitive instructions.

On execution of an instruction sequence, these primitive instructions have the following effects:

- the effect of a positive test instruction \( +a \) is that basic instruction \( a \) is executed and execution proceeds with the next primitive instruction if 1 is produced and otherwise the next primitive instruction is skipped and execution proceeds with the primitive instruction following the skipped one — if there is no primitive instruction to proceed with, inaction occurs;
Table 1. Axioms of PGA

| Axiom | Description |
|-------|-------------|
| $(X;Y);Z = X;(Y;Z)$ | PGA1 |
| $(X^n)\omega = X\omega$ | PGA2 |
| $X\omega;Y = X\omega$ | PGA3 |
| $(X;Y)\omega = X;(Y;X)\omega$ | PGA4 |

- the effect of a negative test instruction $-a$ is the same as the effect of $+a$, but with the role of the value produced reversed;
- the effect of a plain basic instruction $a$ is the same as the effect of $+a$, but execution always proceeds as if 1 is produced;
- the effect of a forward jump instruction $#l$ is that execution proceeds with the $l$th next primitive instruction — if $l$ equals 0 or there is no primitive instruction to proceed with, inaction occurs;
- the effect of the termination instruction $!$ is that execution terminates.

PGA has one sort: the sort $\text{IS}$ of instruction sequences. We make this sort explicit to anticipate the need for many-sortedness later on. To build terms of sort $\text{IS}$, PGA has the following constants and operators:

- for each $u \in \mathcal{I}$, the instruction constant $u : \rightarrow \text{IS}$;
- the binary concatenation operator $\cdot : \cdot : \text{IS} \times \text{IS} \rightarrow \text{IS}$;
- the unary repetition operator $\omega : \text{IS} \rightarrow \text{IS}$.

Terms of sort $\text{IS}$ are built as usual in the one-sorted case. We assume that there are infinitely many variables of sort $\text{IS}$, including $X,Y,Z$. We use infix notation for concatenation and postfix notation for repetition.

A closed PGA term is considered to denote a non-empty, finite or eventually periodic infinite sequence of primitive instructions. The instruction sequence denoted by a closed term of the form $t\cdot t'$ is the instruction sequence denoted by $t$ concatenated with the instruction sequence denoted by $t'$. The instruction sequence denoted by a closed term of the form $t\omega$ is the instruction sequence denoted by $t$ concatenated infinitely many times with itself.

Closed PGA terms are considered equal if they represent the same instruction sequence. The axioms for instruction sequence equivalence are given in Table 1. In this table, $n$ stands for an arbitrary natural number greater than 0. For each $n > 0$, the term $t^n$, where $t$ is a PGA term, is defined by induction on $n$ as follows: $t^1 = t$ and $t^{n+1} = t\cdot t^n$.

A typical model of PGA is the model in which:

- the domain is the set of all finite and eventually periodic infinite sequences over the set $\mathcal{I}$ of primitive instructions;
- the operation associated with $\cdot$ is concatenation;
- the operation associated with $\omega$ is the operation $\bar{\omega}$ defined as follows:

$2$ An eventually periodic infinite sequence is an infinite sequence with only finitely many distinct suffixes.
• if \( U \) is finite, then \( U^\omega \) is the unique infinite sequence \( U' \) such that \( U \) concatenated \( n \) times with itself is a proper prefix of \( U' \) for each \( n \in \mathbb{N} \);

• if \( U \) is infinite, then \( U^\omega \) is \( U \).

We confine ourselves to this model of PGA, which is an initial model of PGA, for the interpretation of PGA terms. In the sequel, we use the term PGA instruction sequence for the elements of the domain of this model, and we denote the interpretations of the constants and operators in this model by the constants and operators themselves. Below, we will use BTA to make precise which behaviours are produced by PGA instruction sequences under execution.

In BTA, it is assumed that a fixed but arbitrary set \( A \) of basic actions has been given. The objects considered in BTA are called threads. A thread represents a behaviour which consists of performing basic actions in a sequential fashion. Upon each basic action performed, a reply from an execution environment determines how the thread proceeds. The possible replies are the values 0 and 1.

BTA has one sort: the sort \( T \) of threads. We make this sort explicit to anticipate the need for many-sortedness later on. To build terms of sort \( T \), BTA has the following constants and operators:

– the inaction constant \( D : \to T \);
– the termination constant \( S : \to T \);
– for each \( a \in A \), the binary postconditional composition operator \( - \odot a \odot - : T \times T \to T \).

Terms of sort \( T \) are built as usual in the one-sorted case. We assume that there are infinitely many variables of sort \( T \), including \( x, y \). We use infix notation for postconditional composition. We introduce basic action prefixing as an abbreviation: \( a \odot t \), where \( t \) is a BTA term, abbreviates \( t \odot a \odot t \). We identify expressions of the form \( a \odot t \) with the BTA term they stand for.

The thread denoted by a closed term of the form \( t \odot a \odot t' \) will first perform \( a \), and then proceed as the thread denoted by \( t \) if the reply from the execution environment is 1 and proceed as the thread denoted by \( t' \) if the reply from the execution environment is 0. The thread denoted by \( S \) will do no more than terminate and the thread denoted by \( D \) will become inactive.

Closed BTA terms are considered equal if they are syntactically the same. Therefore, BTA has no axioms.

Each closed BTA term denotes a finite thread, i.e. a thread with a finite upper bound to the number of basic actions that it can perform. Infinite threads, i.e. threads without a finite upper bound to the number of basic actions that it can perform, can be defined by means of a set of recursion equations (see e.g. [7]). We are only interested in models of BTA in which sets of recursion equations have unique solutions, such as the projective limit model of BTA presented in [3]. We confine ourselves to this model of BTA, which has an initial model of BTA as a submodel, for the interpretation of BTA terms. In the sequel, we use the term BTA thread or simply thread for the elements of the domain of this model, and we denote the interpretations of the constants and operators in this model by the constants and operators themselves.
Table 2. Axioms for the thread extraction operator

| |a| = a ◦ D | |#l| = D |
|---|---|---|
| |a ; X| = a ◦ |X| | |#0 ; X| = D |
| |+a| = a ◦ D | |#1 ; X| = |X| |
| |+a ; X| = |X| ≤ a ≥ |#2 ; X| | |#l + 2 ; u| = D |
| |−a| = a ◦ D | |#l + 2 ; u ; X| = |#l + 1 ; X| |
| |−a ; X| = |#2 ; X| ≤ a ≥ |X| | || = S |
| |−a ; X| = |#2 ; X| ≤ a ≥ |X| | || = S |

Regular threads, i.e. finite or infinite threads that can only be in a finite number of states, can be defined by means of a finite set of recursion equations. The behaviours produced by PGA instruction sequences under execution are exactly the behaviours represented by regular threads, with the basic instructions taken for basic actions. The behaviours produced by finite PGA instruction sequences are the behaviours represented by finite threads.

We combine PGA with BTA and extend the combination with the thread extraction operator |·| : IS → T, the axioms given in Table 2 and the rule that |X| = D if X has an infinite chain of forward jumps beginning at its first primitive instruction. In Table 2 a stands for an arbitrary basic instruction from A, u stands for an arbitrary primitive instruction from I, and l stands for an arbitrary natural number from N. For each closed PGA term t, |t| denotes the behaviour produced by the instruction sequence denoted by t under execution.

Equality of PGA instruction sequence as axiomatized by the axioms of PGA is extensional equality: two PGA instruction sequences are equal if they have the same length and the nth instructions are equal for all n > 0 that are less than or equal to the common length. We define the function len that assigns to each PGA instruction sequence its length:

\[
\text{len}(u) = 1, \\
\text{len}(X ; Y) = \text{len}(X) + \text{len}(Y), \\
\text{len}(X^\omega) = \omega,
\]

and we define for each n > 0 the function \(i_n\) that assigns to each PGA instruction sequence its nth instruction if n is less than or equal to its length and #0 otherwise:

\[
\begin{align*}
\text{i}_1(u) &= u, \\
\text{i}_1(u ; X) &= u, \\
\text{i}_{n+1}(u) &= #0, \\
\text{i}_{n+1}(u ; X) &= \text{i}_n(X).
\end{align*}
\]

3 This rule, which can be formalized using an auxiliary structural congruence predicate (see e.g. [6]), is unnecessary when considering only finite PGA instruction sequences.
Let \( X \) and \( Y \) be PGA instruction sequences. Then we have by extensionality that
\[
X = Y \text{ iff } \text{len}(X) = \text{len}(Y) \text{ and } i_n(X) = i_n(Y) \text{ for all } n > 0.
\]
This means that each PGA instruction sequence \( X \) is uniquely characterized by \( \text{len}(X) \) and \( i_n(X) \) for all \( n > 0 \), which will be used several times in Section 8.

The depth of a finite thread is the maximum number of basic actions that the thread can perform before it terminates or becomes inactive. We define the function \( \text{depth} \) that assigns to each finite BTA thread its depth:

\[
\text{depth}(S) = 0, \\
\text{depth}(D) = 0, \\
\text{depth}(x \trianglelefteq a \triangleright y) = \max\{\text{depth}(x), \text{depth}(y)\} + 1.
\]

Let \( x \) be a thread of the form \( a_1 \circ \ldots \circ a_n \circ S \) or the form \( a_1 \circ \ldots \circ a_n \circ D \). Then \( \text{depth}(x) \) represents the only number of basic actions that \( x \) can perform before it terminates or becomes inactive. This will be used in Section 8.

3 Interaction of Threads with Services

Services are objects that represent the behaviours exhibited by components of execution environments of instruction sequences at a high level of abstraction. A service is able to process certain methods. The processing of a method may involve a change of the service. At completion of the processing of a method, the service produces a reply value. Execution environments are considered to provide a family of uniquely-named services. A thread may interact with the named services from the service family provided by an execution environment. That is, a thread may perform a basic action for the purpose of requesting a named service to process a method and to return a reply value at completion of the processing of the method. In this section, we extend BTA with services, service families, a composition operator for service families, and operators that are concerned with this kind of interaction. This section originates from [7].

In SFA, the algebraic theory of service families introduced below, it is assumed that a fixed but arbitrary set \( M \) of methods has been given. Moreover, the following is assumed with respect to services:

- a signature \( \Sigma_S \) has been given that includes the following sorts:
  - the sort \( S \) of services;
  - the sort \( R \) of replies;
and the following constants and operators:
  - the empty service constant \( \delta : \rightarrow S \);
  - the reply constants \( 0, 1, * : \rightarrow R \);
  - for each \( m \in M \), the derived service operator \( \partial_m : S \rightarrow S \);
  - for each \( m \in M \), the service reply operator \( \partial_m : S \rightarrow R \);
- a minimal \( \Sigma_S \)-algebra \( S \) has been given in which \( 0, 1, \) and \( * \) are mutually different, and
• $\bigwedge_{m \in \mathcal{M}} \frac{\partial}{\partial m}(z) = z \land \varrho_m(z) = * \Rightarrow z = \delta$ holds;

• for each $m \in \mathcal{M}$, $\frac{\partial}{\partial m}(z) = \delta \Leftrightarrow \varrho_m(z) = *$ holds.

The intuition concerning $\frac{\partial}{\partial m}$ and $\varrho_m$ is that on a request to service $s$ to process method $m$:

- if $\varrho_m(s) \neq *$, $s$ processes $m$, produces the reply $\varrho_m(s)$, and then proceeds as $\frac{\partial}{\partial m}(s)$;
- if $\varrho_m(s) = *$, $s$ is not able to process method $m$ and proceeds as $\delta$.

The empty service $\delta$ itself is unable to process any method.

It is also assumed that a fixed but arbitrary set $\mathcal{F}$ of foci has been given. Foci play the role of names of services in a service family.

SFA has the sorts, constants and operators from $\Sigma_S$ and in addition the sort $\text{SF}$ of service families and the following constant and operators:

- the empty service family constant $\emptyset : \rightarrow \text{SF}$;
- for each $f \in \mathcal{F}$, the unary singleton service family operator $\cdot f : \text{S} \rightarrow \text{SF}$;
- the binary service family composition operator $\cdot \oplus : \text{SF} \times \text{SF} \rightarrow \text{SF}$;
- for each $F \subseteq \mathcal{F}$, the unary encapsulation operator $\partial_F : \text{SF} \rightarrow \text{SF}$.

We assume that there are infinitely many variables of sort $\text{S}$, including $z$, and infinitely many variables of sort $\text{SF}$, including $u, v, w$. Terms are built as usual in the many-sorted case (see e.g. [22,24]). We use prefix notation for the singleton service family operators and infix notation for the service family composition operator. We write $\bigoplus_{i=1}^n t_i$, where $t_1, \ldots, t_n$ are terms of sort $\text{SF}$, for the term $t_1 \oplus \ldots \oplus t_n$.

The service family denoted by $\emptyset$ is the empty service family. The service family denoted by a closed term of the form $f.t$ consists of one named service only, the service concerned is the service denoted by $t$, and the name of this service is $f$. The service family denoted by a closed term of the form $t \oplus t'$ consists of all named services that belong to either the service family denoted by $t$ or the service family denoted by $t'$. In the case where a named service from the service family denoted by $t$ and a named service from the service family denoted by $t'$ have the same name, they collapse to an empty service with the name concerned. The service family denoted by a closed term of the form $\partial_F(t)$ consists of all named services with a name not in $F$ that belong to the service family denoted by $t$.

The axioms of SFA are given in Table 3. In this table, $f$ stands for an arbitrary focus from $\mathcal{F}$ and $F$ stands for an arbitrary subset of $\mathcal{F}$. These axioms simply formalize the informal explanation given above.

For the set $\mathcal{A}$ of basic actions, we now take the set $\{f.m \mid f \in \mathcal{F}, m \in \mathcal{M}\}$. Performing a basic action $f.m$ is taken as making a request to the service named $f$ to process method $m$.

We combine BTA with SFA and extend the combination with the following constants and operators:

- the binary abstracting use operator $\cdot \backslash : \text{T} \times \text{SF} \rightarrow \text{T}$;
Table 3. Axioms of SFA

| Axiom | Symbol | Description |
|-------|--------|-------------|
| $u \oplus \emptyset = u$ | SFC1 | $\delta_F(\emptyset) = \emptyset$ |
| $u \oplus v = v \oplus u$ | SFC2 | $\delta_F(f.z) = \emptyset$ if $f \in F$ |
| $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ | SFC3 | $\delta_F(f.z) = f.z$ if $f \notin F$ |
| $f.z \oplus f.z' = f.\delta$ | SFC4 | $\delta_F(u \oplus v) = \delta_F(u) \oplus \delta_F(v)$ |
| $\partial F(\emptyset) = \emptyset$ | SFE1 | |
| $\partial F(f.z) = \emptyset$ if $f \in F$ | SFE2 | |
| $\partial F(f.z) = f.z$ if $f \notin F$ | SFE3 | |
| $\partial F(u \oplus v) = \partial F(u) \oplus \partial F(v)$ | SFE4 | |

Table 4. Axioms for the abstracting use operator

| Axiom | Symbol | Description |
|-------|--------|-------------|
| $S \cdot u = u$ | AU1 | |
| $D \cdot u = \emptyset$ | AU2 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot \partial_{t(j)}(u) = (x \cdot \partial_{t(j)}(u)) \trianglelefteq f.m \trianglerighteq (y \cdot \partial_{t(j)}(u))$ | AU3 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot (f.t \oplus \partial_{t(j)}(u)) = x \cdot (f \frac{\partial}{\partial m}t \oplus \partial_{t(j)}(u))$ if $\mu_m(t) = 1$ | AU4 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot (f.t \oplus \partial_{t(j)}(u)) = y \cdot (f \frac{\partial}{\partial m}t \oplus \partial_{t(j)}(u))$ if $\mu_m(t) = 0$ | AU5 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot (f.t \oplus \partial_{t(j)}(u)) = D$ if $\mu_m(t) = *$ | AU6 | |

Table 5. Axioms for the apply operator

| Axiom | Symbol | Description |
|-------|--------|-------------|
| $S \cdot u = u$ | A1 | |
| $D \cdot u = \emptyset$ | A2 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot \partial_{t(j)}(u) = \emptyset$ | A3 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot (f.t \oplus \partial_{t(j)}(u)) = x \cdot (f \frac{\partial}{\partial m}t \oplus \partial_{t(j)}(u))$ if $\mu_m(t) = 1$ | A4 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot (f.t \oplus \partial_{t(j)}(u)) = y \cdot (f \frac{\partial}{\partial m}t \oplus \partial_{t(j)}(u))$ if $\mu_m(t) = 0$ | A5 | |
| $(x \trianglelefteq f.m \trianglerighteq y) \cdot (f.t \oplus \partial_{t(j)}(u)) = \emptyset$ if $\mu_m(t) = *$ | A6 | |

The thread denoted by a closed term of the form $t \triangleright t'$ and the service family denoted by a closed term of the form $t \cdot t'$ are the thread and service family, respectively, that result from processing the method of each basic action performed by the thread denoted by $t$ by the service in the service family denoted by $t'$ with the focus of the basic action as its name if such a service exists. When the method of a basic action performed by a thread is processed by a service, the service changes in accordance with the method concerned and the thread reduces to one of the two threads that it can possibly proceed with dependent on the reply value produced by the service.
Table 6. Axioms for the tracking use operator

| Axiom | Description |
|-------|-------------|
| $S \not\triangleleft u = S$ | PAU1 |
| $D \not\triangleleft u = D$ | PAU2 |
| $(x \triangleleft f.m \triangleright g y) \not\triangleleft (x \not\triangleleft (f(m \triangleright g y)) = (x \not\triangleleft f.m \triangleright g y)$ | PAU3 |
| $(x \triangleleft f.m \triangleright g y) \not\triangleleft (f.t \oplus \partial(f(t))) = f.m!1 \circ (x \not\triangleleft (f.t \oplus \partial(f(t))))$ if $g_m(t) = 1$ | PAU4 |
| $(x \triangleleft f.m \triangleright g y) \not\triangleleft (f.t \oplus \partial(f(t))) = f.m!0 \circ (y \not\triangleleft (f.t \oplus \partial(f(t))))$ if $g_m(t) = 0$ | PAU5 |
| $(x \triangleleft f.m!r \triangleright g y) \not\triangleleft f.m!r \circ (x \not\triangleleft u)$ | PAU6 |
| $(x \triangleleft f.m!r \triangleright g y) \triangleright u = f.m!r \circ (x \triangleright u)$ | AU7 |
| $(x \triangleleft f.m!r \triangleright g y) \bullet u = x \bullet u$ | A7 |

4 The Tracking Use Operator

In this section, we extend the combination of BTA with SFA further with a tracking use operator. Abstracting use does not leave a trace of what has taken place during the interaction between thread and service family, whereas tracking use leaves a detailed trace. The tracking use operator has been devised for its usefulness for the work presented in this paper.

For the set $A$ of basic actions, we now take the set $\{f.m \mid f \in \mathcal{F}, m \in \mathcal{M}\} \cup \{f.m!r \mid f \in \mathcal{F}, m \in \mathcal{M}, r \in \{0, 1, *\}\}$. A basic action $f.m!r$ represents the successful handling of a request to the service named $f$ to process method $m$ with reply $r$ if $r \neq *$, and the unsuccessful handling of such a request otherwise.

We extend the combination of BTA with SFA further with the following operator:

- the binary tracking use operator $\not\triangleleft \cdot : \mathcal{T} \times \mathcal{SF} \to \mathcal{T}$;

and the axioms given in Tables 6. In this table, $f$ stands for an arbitrary focus from $\mathcal{F}$, $m$ stands for an arbitrary method from $\mathcal{M}$, $r$ stands for an arbitrary constant of sort $\mathcal{R}$, and $t$ stands for an arbitrary term of sort $\mathcal{S}$. The axioms PAU1–PAU7 formalize the informal explanation given below. The axioms AU7 and A7 stipulate what is the result of abstracting use and apply if basic actions of the form $f.m!r$ are involved. We use infix notation for the tracking use operator.

The thread denoted by a closed term of the form $t \not\triangleleft t'$ differs from the thread denoted by a closed term of the form $t \triangleright t'$ as follows: when the method of a basic action performed by a thread is processed by a service, the thread reduces to one of the two threads that it can possibly proceed with dependent on the reply value produced by the service prefixed by the basic action $f.m!r$, where $f$ is the name of the processing service, $m$ is the method processed, and $r$ is the reply value produced. Thus, the resulting thread represents a trace of what has taken place.
5 Instruction Sequences Acting on Boolean Registers

The basic instructions that concern us in the remainder of this paper are instructions to set and get the content of Boolean registers. We describe in this section services that make up Boolean registers, introduce special foci that serve as names of Boolean registers, and describe the instruction sequences on which we will define an algorithmic equivalence relation. The greater part of this section originates from [9].

First, we describe services that make up Boolean registers. It is assumed that set:0, set:1, get ∈ M. These methods are the ones that Boolean register services are able to process. They can be explained as follows:

- set:0: the contents of the Boolean register becomes 0 and the reply is 0;
- set:1: the contents of the Boolean register becomes 1 and the reply is 1;
- get: nothing changes and the reply is the contents of the Boolean register.

For Σ, we take the signature that consists of the sorts, constants and operators that are mentioned in the assumptions with respect to services made in Section 3 and constants BR₀ and BR₁.

For S, we take a minimal ΣS-algebra that satisfies the conditions that are mentioned in the assumptions with respect to services made in Section 3 and the following conditions for each b ∈ {0, 1}:

\[
\begin{align*}
\partial_{\text{set}:0} (BR_b) &= BR_0, & \partial_{\text{set}:1} (BR_b) &= BR_b, \\
\partial_{\text{get}} (BR_b) &= BR_b, & \partial_{\text{set}:0} (BR_b) &= BR_0, & \partial_{\text{set}:1} (BR_b) &= BR_1, & \partial_{\text{get}} (BR_b) &= BR_b, & \partial_{\text{set}:0} (BR_b) &= BR_0, & \partial_{\text{set}:1} (BR_b) &= BR_1, & \partial_{\text{get}} (BR_b) &= BR_b, & \partial_{\text{set}:0} (BR_b) &= BR_0, & \partial_{\text{set}:1} (BR_b) &= BR_1, & \partial_{\text{get}} (BR_b) &= BR_b, & \partial_{\text{set}:0} (BR_b) &= BR_0, & \partial_{\text{set}:1} (BR_b) &= BR_1, & \partial_{\text{get}} (BR_b) &= BR_b, & \partial_{\text{set}:0} (BR_b) &= BR_0, & \partial_{\text{set}:1} (BR_b) &= BR_1, & \partial_{\text{get}} (BR_b) &= BR_b.
\end{align*}
\]

In the instruction sequences which concern us in the remainder of this paper, a number of Boolean registers is used as input registers, a number of Boolean registers is used as auxiliary registers, and a number of Boolean register is used as output register. It is assumed that, for each i ∈ N⁺, in:i, aux:i, out:i ∈ F. These foci play special roles:

- for each i ∈ N⁺, in:i serves as the name of the Boolean register that is used as ith input register in instruction sequences;
- for each i ∈ N⁺, aux:i serves as the name of the Boolean register that is used as ith auxiliary register in instruction sequences;
- for each i ∈ N⁺, out:i serves as the name of the Boolean register that is used as ith output register in instruction sequences.

We define the following sets:

\[
\begin{align*}
F^n_{\text{in}} &= \{\text{in}:i \mid 1 \leq i \leq n\}, & F^n_{\text{aux}} &= \{\text{aux}:i \mid i \geq 1\}, & F^n_{\text{out}} &= \{\text{out}:i \mid 1 \leq i \leq n\}, \\
F^m_{\text{br}} &= F^n_{\text{in}} \cup F^n_{\text{aux}} \cup F^m_{\text{out}}, & M_{\text{br}} &= \{\text{set}:0, \text{set}:1, \text{get}\}, \\
A^m_{\text{br}} &= \{\text{f.get} \mid f \in F^n_{\text{in}} \cup F^n_{\text{aux}}\} \cup \{\text{f.set:b} \mid f \in F^n_{\text{aux}} \cup F^m_{\text{out}} \land b \in \{0, 1\}\}.
\end{align*}
\]
We write $\mathcal{I}_{n,m}$ for the set of primitive instructions in the case where $\mathcal{A}_{n,m}$ is taken for the set $\mathcal{A}$ of basic instructions, and we write $\mathcal{IS}_{n,m}$ for the set of all finite PGA instruction sequences in the case where $\mathcal{I}_{n,m}$ is taken for the set $\mathcal{I}$ of primitive instructions. Moreover, we write $\mathcal{T}_{n,m}$ for the set of all finite BTA threads in the case where $\mathcal{A}_{n,m}$ is taken for the set $\mathcal{A}$ of basic actions.

Let $n, m \in \mathbb{N}$, let $f : \{0, 1\}^n \to \{0, 1\}^m$ and let $X \in \mathcal{IS}_{n,m}$. Then $X$ computes $f$ if there exists an $l \in \mathbb{N}$ such that, for all $b_1', \ldots, b_l' \in \{0, 1\}$:

- for all $b_1, \ldots, b_n, b_1', \ldots, b_l' \in \{0, 1\}$ with $f(b_1, \ldots, b_n) = b_1', \ldots, b_l'$:
  
  $$
  (|X| \mm ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'}))) \bullet (\bigoplus_{i=1}^m \text{out}:i.BR_0)
  = \bigoplus_{i=1}^m \text{out}:i.BR_{b_i'}.
  $$

- for all $b_1, \ldots, b_n \in \{0, 1\}$ with $f(b_1, \ldots, b_n)$ undefined:
  
  $$
  (|X| \mm ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'}))) \bullet (\bigoplus_{i=1}^m \text{out}:i.BR_0)
  = \emptyset.
  $$

We know from Theorem 1 in [9] that, for each $n, m \in \mathbb{N}$, for each $f : \{0, 1\}^n \to \{0, 1\}^m$, there exists an $X \in \mathcal{IS}_{n,m}$ such that $X$ computes $f$. It is easy to see from the proof of the theorem that this result generalizes from total functions to partial functions.

6 Background on the Notion of Algorithmic Sameness

In section 8, we will define an equivalence relation that is intended to capture to a reasonable degree the intuitive notion that two instruction sequences express the same algorithm. In this section, we give some background on this notion in order to put the definition that will be given into context.

In [12], where instruction sequences are considered which contain backward jump instructions in addition to instructions to set and get the content of Boolean registers, forward jump instructions, and a termination instruction, it is shown that the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system can be computed according to a minor variant of the long multiplication algorithm by quadratic-length instruction sequences without backward jump instructions and by linear-length instruction sequences with backward jump instructions.

With that, we implicitly assumed that the instruction sequences without backward jump instructions concerned and the instruction sequences with backward jump instructions concerned express the same algorithm. We have asked ourselves the question why this is an acceptable assumption and what this says about the notion of an algorithm. We considered it an acceptable assumption because all the different views on what characterizes an algorithm lead to the

---

4 We write as usual $f : \{0, 1\}^n \to \{0, 1\}^m$ to indicate that $f$ is partial function from $\{0, 1\}^n$ to $\{0, 1\}^m$. 

conclusion that we have to do here with different expressions of the same algorithm. However, we cannot prove this due to the absence of a mathematically precise definition of an equivalence relation on the instruction sequences of the kind considered that captures the intuitive notion that two instruction sequences express the same algorithm.

The cause of this absence is the general acceptance of the exact mathematical concept of a Turing machine and equivalent mathematical concepts as adequate replacements of the intuitive concept of an algorithm. Unfortunately, for bit strings of any given length, we can construct at least two different Turing machines for the minor variant of the long multiplication algorithm referred to above: one without a counterpart of a for loop and one with a counterpart of a for loop. This means that, like programs, Turing machines do not enforce a level of abstraction that is sufficient for algorithms. Moreover, Turing machines are quite remote from anything related to actual programming. Therefore, we doubt whether the mathematical concept of a Turing machine is an adequate replacement of the intuitive concept of an algorithm. This means that we consider a generally accepted mathematically precise definition of the concept of an algorithm still desirable.

The existing viewpoints on what is an algorithm are diverse in character. The viewpoint that algorithms are equivalence classes of programs was already taken in [19]. This viewpoint was recently also taken in [25], but a rather strange twist is that constructions of primitive recursive functions are considered to be programs. In [20], algorithms are viewed as isomorphism classes of tuples of recursive functionals that can be defined by repeated application of certain schemes. This viewpoint is somewhat reminiscent of the viewpoint taken in [25]. On the other hand, in [13], which is concerned with algorithms on Kahn-Plotkin’s concrete data structures, algorithms are viewed as pairs of a function and a computation strategy that resolves choices between possible ways of computing the function. This viewpoint is quite different from the other viewpoints mentioned above.

In [17], it is claimed that the only algorithms are those expressed by Kolmogorov machines and that therefore the concept of a Kolmogorov machine can be regarded as an adequate formal characterization of the concept of an algorithm (see also [23]). With this the concept of a Kolmogorov machine is actually qualified as a replacement of the concept of an algorithm. In [16], an algorithm is defined as an object that satisfy certain postulates. The postulates concerned seem to be devised with the purpose that Gurevich’s abstract state machines would satisfy them. Be that as it may, they are primarily postulates for models of computation of a certain kind, i.e. replacements of the concept of an algorithm.

In [14], it is argued that the intuitive notion that two programs express the same algorithm cannot be captured by an equivalence relation. This is also argued in the philosophical discussion of the view that algorithms are mathematical objects presented in [15]. Quite a few of the given arguments are biased towards current patterns of thinking within subfields of theoretical computer science like the analysis of algorithms and computational complexity theory. An important such pattern is the following: if we have proved a result concerning programs or
abstract machines, then we may formulate it as a result concerning algorithms. This pattern yields, among other things, a biased view on what are the properties that two programs expressing the same algorithm must have in common.

Moreover, the existence of different opinions and subjective judgments concerning the question whether two programs express the same algorithm is also a weak argument. Different opinions and subjective judgments are inevitable in the absence of a mathematically precise definition of an equivalence relation that captures the intuitive notion that two programs express the same algorithm. All this means that the arguments given in [14,15] are no reason for us to doubt the usefulness of looking for an equivalence relation that captures to a reasonable degree the intuitive notion that two instruction sequences express the same algorithm.

7 Intuition about the Notion of Algorithmic Sameness

In this section, we give a picture of our intuition about the notion that two instruction sequences express the same algorithm. In section 8, we will define in a mathematically precise way an equivalence relation corresponding to this intuition.

We would like to ground our intuition in the general thinking on the concept of an algorithm. However, because the existing viewpoints on what is an algorithm are diverse in character and leave loose ends, there is little that we can build on. Therefore, we restrict ourselves to what is virtually the simplest case. That is, we take two fixed but arbitrary natural numbers \( n, m \) and restrict ourselves to instruction sequences for computing partial functions from \( \{0, 1\}^n \) to \( \{0, 1\}^m \). If \( n \) and \( m \) are very large, this simple case covers, at a very low level of program and data representation, many non-interactive programs that are found in actual practice. This simple case has the advantage that data representation is hardly an issue in the expressions of algorithms.

In the case that we restrict ourselves to instruction sequences for computing partial functions from \( \{0, 1\}^n \) to \( \{0, 1\}^m \), taking into account the experience gained in [10,11,12] with expressing algorithms by instruction sequences, we consider the following to be a first rough approximation of a definition of the concept of an algorithm: “an algorithm is an equivalence class of instruction sequences from \( IS_{n,m} \) with respect to an equivalence relation that completely captures the intuitive notion that two instruction sequences express the same algorithm”.

However, the equivalence relation to be defined in Section 8 is likely to incompletely capture this notion of algorithmic sameness. Because we want to capture it to a reasonable degree, we have made a serious attempt to establish its main characteristics in the simple case under consideration. The main characteristics found are:

- each instruction sequence is algorithmically the same as each instruction sequence that produces the same behaviour under execution;

\[ \text{We regard total functions as special cases of partial functions. Henceforth, total functions are shortly called functions.} \]
– each instruction sequence is algorithmically the same as the instruction sequence obtained from it by consistently exchanging 0 and 1 as far as an auxiliary Boolean register is concerned;
– each instruction sequence is algorithmically the same as each instruction sequence obtained from it by renumbering the auxiliary Boolean registers used;
– each instruction sequence is algorithmically the same as each instruction sequence obtained from it by transposing the basic instructions in two always successively executed primitive instructions if their combined effect never depends on the order in which they are executed.

The first characteristic expresses that algorithmic sameness is implied by behavioural equivalence. It is customary to ascribe this characteristic to the notion of algorithmic sameness. The second characteristic expresses that algorithmic sameness is implied by mutual step-by-step simulation of behaviour with 0 represented by 1 and 1 represented by 0 as far as intermediate results are concerned. It is customary to ascribe a characteristic of which this one is a special case to the notion of algorithmic sameness. Usually, the characteristic has a more general form because the data of interest are not restricted to bit strings.

The third characteristic can be paraphrased as follows: algorithmic sameness identifies two instruction sequences if they only differ in the choice of Boolean registers used for storing the different intermediate results. It is customary to ascribe a characteristic of which this one is a special case to the notion of algorithmic sameness in those cases where the programs concerned are of a concrete form. The fourth characteristic can be paraphrased as follows: algorithmic sameness identifies two instruction sequences if they both express the same parallel algorithm. It should be fairly customary to ascribe this characteristic to the notion of algorithmic sameness, but to our knowledge it is seldom made explicit. A remarkable exception is [18], a book that describes a theoretical framework for optimization of sequential programs by parallelization.

We remark that it is not easy to think of what may be additional main characteristics of the intuitive notion of algorithmic sameness for instruction sequences of the kind considered here. Because we restrict ourselves to algorithms for computing partial functions from \( \{0, 1\}^n \) to \( \{0, 1\}^m \), for fixed \( n, m \), and simple instruction sequences without advanced features, it is not difficult to gain a comprehensive view of what is within the bounds of the possible with regard to the notion of algorithmic sameness.

However, there remain doubtful cases. An instruction sequence may contain primitive instructions that are superfluous in the sense that their execution cannot contribute to the partial function that it computes. For example, if a primitive instruction of a form other than \( \text{out} : i \rightarrow \text{set} : b \) or \#l with \( l \neq 1 \) is immediately followed by a termination instruction, then the former instruction cannot contribute to the partial function that the instruction sequence computes. We are doubtful whether an instruction sequence that contains such a superfluous instruction is algorithmically the same as the instruction sequence obtained from it by replacing the superfluous instruction by \#1. To our knowledge, this does
not correspond to any characteristic ever ascribed to the notion of algorithmic sameness.

As mentioned before, we restrict ourselves in this paper to algorithms for computing partial functions from \( \{0,1\}^n \) to \( \{0,1\}^m \), for fixed \( n, m \). We could restrict ourselves further to algorithms for computing partial functions from \( \{0,1\}^n \) to \( \{0,1\}^m \) that can handle their restriction to \( \{0,1\}^k \) for each \( k < n \) if sufficiently many leading zeros are added. This means that an instruction sequence that computes a partial function from \( \{0,1\}^n \) to \( \{0,1\}^m \) can also be used to compute its restriction to \( \{0,1\}^k \) for each \( k < n \). These functions include, for instance, all functions that model the restriction of an operation on natural numbers to the interval \([0, 2^n - 1]\) on their representation in the binary number system.

8 The Structural Algorithmic Equivalence Relation

In this section, we define an algorithmic equivalence relation that corresponds to the intuition about the notion of algorithmic sameness described in Section 7. Preceding that, we introduce the way in which we will characterize instruction sequences several times in this section.

For each of the main characteristics of the intuitive notion of algorithmic sameness mentioned in Section 7, there is a corresponding equivalence relation which partially captures the notion of algorithmic sameness. Because some of these equivalence relations may well be interesting as they are, we first define them and then define an algorithmic equivalence relation in terms of them.

Below we define the equivalence relation \( \equiv_b \) on \( IS_{br}^{n,m} \). This relation associates each instruction sequence from \( IS_{br}^{n,m} \) with the instruction sequences from \( IS_{br}^{n,m} \) that produce the same behaviour under execution. The *behavioural equivalence* relation \( \equiv_b \) on \( IS_{br}^{n,m} \) is defined by

\[
X \equiv_b Y \iff |X| = |Y| .
\]

Below we define the equivalence relation \( \equiv_x \) on \( IS_{br}^{n,m} \). This relation associates each instruction sequence from \( IS_{br}^{n,m} \) with the instruction sequences from \( IS_{br}^{n,m} \) obtained from it by consistently exchanging 0 and 1 as far as an auxiliary Boolean register is concerned. The *equivalence under bit exchange* relation \( \equiv_x \) on \( IS_{br}^{n,m} \) is defined as the smallest relation on \( IS_{br}^{n,m} \) such that for all \( X, Y \in IS_{br}^{n,m} \):

- if there exists a finite \( I \subset \mathbb{N}^+ \) such that \( \chi_I(X) = Y \), then \( X \equiv_x Y \);

where, for each finite \( I \subset \mathbb{N}^+ \), \( \chi_I \) is the unique function on \( IS_{br}^{n,m} \) such that \( \text{len}(\chi_I(X)) = \text{len}(X) \) and \( i_n(\chi_I(X)) = \chi_I(i_n(X)) \) for all \( n > 0 \), where the function \( \chi_I \) on \( IS_{br}^{n,m} \) is defined as follows:

\[
\begin{align*}
\chi_I(f.m) &= f.\chi''(m) & \text{if } f \in \{ \text{aux} : i \mid i \in I \} , \\
\chi_I(f.m) &= f.m & \text{if } f \notin \{ \text{aux} : i \mid i \in I \} , \\
\chi_I(+f.m) &= -f.\chi''(m) & \text{if } f \in \{ \text{aux} : i \mid i \in I \} , \\
\chi_I(+f.m) &= +f.m & \text{if } f \notin \{ \text{aux} : i \mid i \in I \} ,
\end{align*}
\]

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χ′_i(-f.m) = +f.χ''(m) if f ∈ {aux:i | i ∈ I},
χ′_i(-f.m) = -f.m if f ∉ {aux:i | i ∈ I},
χ′_i(#l) = #l,
χ′_i(!) = !,
where the function χ'' on M is defined as follows:

χ''(set:0) = set:1,
χ''(set:1) = set:0,
χ''(get) = get.

Below we define the equivalence relation ≡_r on IS^n,m. This relation associates each instruction sequence from IS^n,m with the instruction sequences from IS^n,m obtained from it by renumbering the auxiliary Boolean registers. The equivalence under register renumbering relation ≡_r on IS^n,m is defined as the smallest relation on IS^n,m such that for all X,Y ∈ IS^n,m:

- if there exists a bijection r on N such that ρ_r(X) = Y, then X ≡_r Y;

where, for each bijection r on N, the function ρ_r is the unique function on IS^n,m such that len(ρ_r(X)) = len(X) and i_n(ρ_r(X)) = ρ'_r(i_n(X)) for all n > 0, where the function ρ'_r on IS^n,m is defined as follows:

ρ'_r(f.m) = ρ''_r(f).m,
ρ'_r(+f.m) = +ρ''_r(f).m,
ρ'_r(-f.m) = -ρ''_r(f).m,
ρ'_r(#l) = #l,
ρ'_r(!) = !,

where the function ρ''_r on T^n,m is defined as follows:

ρ''_r(in:i) = in:i,
ρ''_r(aux:i) = aux:r(i),
ρ''_r(out:i) = out:i.

Below we define the equivalence relation ≡_t on IS^n,m. This relation associates each instruction sequence from IS^n,m with the instruction sequences from IS^n,m obtained from it by transposing the basic instructions in two always successively executed primitive instructions if their combined effect never depends on the order in which they are executed. The equivalence under instruction transposition relation ≡_t on IS^n,m is defined by

X ≡_t Y ⇔ |X| ≡_t |Y|,

where ≡_t is the smallest relation on T^n,m such that for x,y,z ∈ T^n,m and a,b ∈ A^n,m:
Then, it is sufficient to prove that for all $\chi$ be such that $\equiv$ and $X$ sufficient to prove the theorem with $\chi$, $\equiv_t x$;

- if $x \equiv_t y$ and $y \equiv_t z$, then $x \equiv_t z$;

- if $x \equiv_t x'$ and $y \equiv_t y'$, then $x \leq a \geq y \equiv_t x' \leq a \geq y'$.

It is easy to check that the relations $\equiv_b$, $\equiv_x$, $\equiv_r$, and $\equiv_t$ are actually equivalence relations.

Now we are ready to define the algorithmic equivalence relation on $\mathcal{IS}_{br}^{n,m}$ that corresponds to the intuition about the notion of algorithmic sameness described in Section 7 in terms of the equivalence relations $\equiv_b$, $\equiv_x$, $\equiv_r$, and $\equiv_t$. The structural algorithmic equivalence relation $\equiv_{sa}$ on $\mathcal{IS}_{br}^{n,m}$ is defined as the smallest relation on $\mathcal{IS}_{br}^{n,m}$ such that for all $X, Y, Z \in \mathcal{IS}_{br}^{n,m}$:

- if $X \equiv_b Y$ or $X \equiv_x Y$ or $X \equiv_r Y$ or $X \equiv_t Y$, then $X \equiv_{sa} Y$;

- if $X \equiv_{sa} Y$ and $Y \equiv_{sa} Z$, then $X \equiv_{sa} Z$.

It is easy to check that the relation $\equiv_{sa}$ is actually an equivalence relation.

If $X$ computes a partial function from $\{0, 1\}^n$ to $\{0, 1\}^m$ and $X \equiv_{sa} Y$, then $X$ and $Y$ compute the same partial function in the same number of steps. This is made precise in the following theorem.

**Theorem 1.** For all $X, Y \in \mathcal{IS}_{br}^{n,m}$, $X \equiv_{sa} Y$ only if there exists an $l \in \mathbb{N}$ such that, for all $b_1', \ldots, b_l' \in \{0, 1\}$, there exist $b_1'', \ldots, b_l'' \in \{0, 1\}$ such that, for all $b_1, \ldots, b_n \in \{0, 1\}$:

$$
(|X| \parallel ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'}))) \bullet (\bigoplus_{i=1}^m \text{out}:i.BR_0) = (|Y| \parallel ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'}))) \bullet (\bigoplus_{i=1}^m \text{out}:i.BR_0)
$$

and

$$
depth(|X|) ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'})) / (\bigoplus_{i=1}^m \text{out}:i.BR_0) = depth(|Y|) ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'})) / (\bigoplus_{i=1}^m \text{out}:i.BR_0).
$$

**Proof.** By the definition of $\equiv_{sa}$ and elementary logical reasoning rules, it is sufficient to prove the theorem with $\equiv_{sa}$ replaced by $\equiv_b$, $\equiv_x$, $\equiv_r$, and $\equiv_t$. The case where $\equiv_{sa}$ is replaced by $\equiv_b$ is trivial.

The case where $\equiv_{sa}$ is replaced by $\equiv_x$ is proved as outlined below. Let $l \in \mathbb{N}^+$ be such that $\chi_f(X) = Y$ (such an $l$ exists according to the definition of $\equiv_x$).

Then, it is sufficient to prove that $X \equiv_x Y$ only if there exists an $l \in \mathbb{N}$ such that, for all $b_1', \ldots, b_l' \in \{0, 1\}$, for all $b_1, \ldots, b_n \in \{0, 1\}$:

$$
(|X| \parallel ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'}))) = (|Y| \parallel ((\bigoplus_{i=1}^n \text{in}:i.BR_{b_i}) \oplus (\bigoplus_{i=1}^l \text{aux}:i.BR_{b_i'})))
$$

---

6 Here we write $\text{focus}(a)$ for the unique $f \in \mathcal{F}_{br}^{n,m}$ for which there exists an $m \in \mathcal{M}_{br}$ such that $a = f.m$. 

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and
\[
\text{depth}(|X| \downarrow (\bigoplus_{i=1}^{n} \text{in} \cdot BR_{b_i}) \uplus (\bigoplus_{i=1}^{l} \text{aux} \cdot BR_{b'_i})))
\]
\[
= \text{depth}(|Y| \downarrow (\bigoplus_{i=1}^{n} \text{in} \cdot BR_{b_i}) \uplus (\bigoplus_{i=1}^{l} \text{aux} \cdot BR_{b'_i})))
\]
where, for each \(i\) with \(1 \leq i \leq l\), \(b'_i = \overline{b'}_i\) if \(i \in I\) and \(b''_i = b'_i\) if \(i \notin I\). This is easily proved by induction on \(\text{len}(X)\) and case distinction on the possible forms of the first primitive instruction in \(X\). The case where \(\equiv_{sa}\) is replaced by \(\equiv_t\) is proved similarly.

The case where \(\equiv_{sa}\) is replaced by \(\equiv_t\) is easily proved by induction on the construction of \(\equiv_t'\).

In Theorem \(\Box\) “only if” cannot be replaced by “if and only if”. Take as an example:

\[
X = \text{out:1.set:0} ; \ldots ; \text{out:m.set:0} ; \text{aux:1.set:0} ; !,
\]
\[
Y = \text{out:1.set:0} ; \ldots ; \text{out:m.set:0} ; \text{aux:1.set:1} ; !.
\]

Then we do not have that \(X \equiv_{sa} Y\), but \(X\) and \(Y\) compute the same function from \(\{0, 1\}^n\) to \(\{0, 1\}^m\) in the same number of steps.

Below we define basic algorithms as equivalence classes of instruction sequences with respect to the algorithmic equivalence relation \(\equiv_{sa}\) defined above. Because it is quite possible that \(\equiv_{sa}\) does not completely captures the intuitive notion that two instruction sequences express the same algorithm, the concept of a basic algorithm introduced below is considered to be merely a reasonable approximation of the intuitive concept of an algorithm. The prefix “basic” is used because we will introduce in Section \(\Box\) the concept of a \(PN\)-oriented algorithm (a concept parameterized by a program notation \(PN\)) and that concept is basically built on the one introduced here.

Let \(f : \{0, 1\}^n \rightarrow \{0, 1\}^m\). Then a basic algorithm for \(f\) is an \(A \in IS_{br}^{n,m} / \equiv_{sa}\) such that, for all \(X \in A\), \(X\) computes \(f\). A basic algorithm is an \(A \in IS_{br}^{n,m} / \equiv_{sa}\) for which there exists an \(f : \{0, 1\}^n \rightarrow \{0, 1\}^m\) such that \(A\) is a basic algorithm for \(f\). Let \(A\) be a basic algorithm and \(X \in IS_{br}^{n,m}\). Then we say that \(X\) expresses \(A\) if \(X \in A\).

9 The Structural Computational Equivalence Relation

In this section, we define an equivalence relation on \(IS_{br}^{n,m}\) that is coarser than the structural algorithmic equivalence relation defined in Section \(\Box\). The equivalence relation in question is called the structural computational equivalence relation on \(IS_{br}^{n,m}\). Although it was devised hoping that it would capture the intuitive notion of algorithmic sameness to a higher degree than the structural algorithmic equivalence relation, this coarser equivalence relation turns out to make equivalent instruction sequences of which it is inconceivable that they are considered to express the same algorithm. Therefore, any equivalence relation

\[\text{Here, we write as usual } \overline{b} \text{ for the complement of } b.\]
\[\text{Here, we write as usual } IS_{br}^{n,m} / \equiv_{sa} \text{ for the quotient set of } IS_{br}^{n,m} \text{ by } \equiv_{sa}.\]
that captures the notion of algorithmic sameness to a higher degree than the structural algorithmic equivalence relation must be finer than the structural computational equivalence relation defined in this section.

The structural computational equivalence relation on $IS_{br}^{n,m}$ will be defined in terms of the equivalence relations $\equiv_x$, $\equiv_r$, and $\equiv_t$ defined in Section 8, and an equivalence relation $\equiv_{ct}$ replacing $\equiv_b$.

Below we define the equivalence relation $\equiv_{ct}$ on $IS_{br}^{n,m}$. This relation associates each instruction sequence from $IS_{br}^{n,m}$ with the instruction sequences from $IS_{br}^{n,m}$ that produce the same behaviour under execution for all possible contents of the input Boolean registers. The computational trace equivalence relation $\equiv_{ct}$ on $IS_{br}^{n,m}$ is defined by

$$X \equiv_{ct} Y \iff \forall b_1, \ldots, b_n \in \{0,1\} \cdot |X| \not\vdash (\bigoplus_{i=1}^n \text{in:}BR_{b_i}) = |Y| \not\vdash (\bigoplus_{i=1}^n \text{in:}BR_{b_i}).$$

It is easy to check that the relation $\equiv_{sc}$ is actually an equivalence relation.

Now we are ready to define the equivalence relation $\equiv_{sc}$ on $IS_{br}^{n,m}$ in terms of the equivalence relations $\equiv_{ct}$, $\equiv_x$, $\equiv_r$, and $\equiv_t$. The structural computational equivalence relation $\equiv_{sc}$ on $IS_{br}^{n,m}$ is defined as the smallest relation on $IS_{br}^{n,m}$ such that for all $X, Y, Z \in IS_{br}^{n,m}$:

- if $X \equiv_{ct} Y$ or $X \equiv_x Y$ or $X \equiv_r Y$ or $X \equiv_t Y$, then $X \equiv_{sc} Y$;
- if $X \equiv_{sc} Y$ and $Y \equiv_{sc} Z$, then $X \equiv_{sc} Z$.

**Theorem 2.** $\equiv_{sa} \subset \equiv_{sc}$.

**Proof.** Let $\varphi$ be a proposition containing proposition variables $v_1, \ldots, v_n$. Let $f : \{0,1\}^n \to \{0,1\}^m$ be such that $f(b_1, \ldots, b_n) = 1^m$ if $\varphi$ is satisfied by the valuation that assigns $b_1$ to $v_1$, $\ldots$, $b_n$ to $v_n$ and $f(b_1, \ldots, b_n) = 0^m$ otherwise. Let $X \in IS_{br}^{n,m}$ be such that $X$ computes $f$. Let $X' \in IS_{br}^{n,m}$ be obtained from $X$ by replacing, for each $f \in F_{\text{set}}$, all occurrences of the basic instruction $f.\text{set:}1$ by the basic instruction $f.\text{set:}0$. It follows immediately that not $X \equiv_{sa} X'$. Now suppose that $\varphi$ is not satisfiable. Then $X \equiv_{sc} X'$. Hence, $\equiv_{sa} \subset \equiv_{sc}$. □

The proof of Theorem 2 does not only show that there exist $X$ and $X'$ such that $X \equiv_{sc} X'$ and not $X \equiv_{sa} X$'. It also shows that there exist $X$ and $X'$ such that $X \equiv_{sc} X'$ whereas it is inconceivable that $X$ and $X'$ are considered to express the same algorithm. The point is that an algorithm may take alternatives into account that will not occur and there is abstracted from such alternatives in the case of the structural computational equivalence relation $\equiv_{sc}$.

**10 On Algorithmic Equivalence of Higher-Level Programs**

In most program notations used for actual programming, programs are more advanced than the instruction sequences from $IS_{br}^{n,m}$. In this section, we show that the algorithmic equivalence relation on $IS_{br}^{n,m}$ defined in Section 8 can
easily be lifted to programs in a higher-level program notation if the approach of projection semantics (see below) is followed in giving the program notation concerned semantics.

PGA instruction sequences has not been designed to play a part in actual programming. In fact, they are less suitable for actual programming than the instruction sequences that are found in low-level program notations such as assembly languages. However, even high-level program notations with advanced features such as conditional constructs, loop constructs, and subroutines that may call themselves recursively can be given semantics by means of a mapping from the programs in the notation concerned to PGA instruction sequences and a service family. This approach to the semantics of program notations, which is followed in [4, 5], is called projection semantics.

We define a program notation as a triple \((L, \varphi, S)\), where \(L\) is a set of programs, \(\varphi\) is a mapping from \(L\) to the set of all PGA instruction sequences, and \(S\) is a service family. The mapping \(\varphi\) is called a projection. The behaviour of each program \(P\) in \(L\) is determined by \(\varphi\) and \(S\) as follows: the behaviour of \(P\) is the behaviour represented by the thread \(|\varphi(P)|/S\).

For certain program notations, it is fully sufficient that \(S\) is the empty service family. An example is the program notation in which programs are finite instruction sequences that differ from finite PGA instruction sequences in that they may contain backward jump instructions (see e.g. [5]). For certain other program notations, it is virtually or absolutely necessary that \(S\) is a non-empty service family. Examples are program notations with features such as subroutines that may call themselves recursively (see e.g. [4]). It is clear that \((\mathcal{IS}_{br}^{n,m}, \iota, \emptyset)\), where \(\iota\) is the identity mapping on \(\mathcal{IS}_{br}^{n,m}\), is the proper program notation for \(\mathcal{IS}_{br}^{n,m}\).

Because we build on the algorithmic equivalence relation \(\equiv_{sa}\) on \(\mathcal{IS}_{br}^{n,m}\) defined in Section 8, we restrict ourselves to program notations \((L, \varphi, S)\) that satisfy the following conditions:

- \(\partial_{\mathcal{IS}_{br}^{n,m}}(S) = S\);
- for each \(P \in L\), there exists an \(X \in \mathcal{IS}_{br}^{n,m}\) such that \(|\varphi(P)|/S = |X|\);
- for each \(X \in \mathcal{IS}_{br}^{n,m}\), there exists a \(P \in L\) such that \(|\varphi(P)|/S = |X|\).

The first condition is a healthiness condition. The other two conditions entail a restriction to program notations with the same computational power as \((\mathcal{IS}_{br}^{n,m}, \iota, \emptyset)\).

For each program notation \((L, \varphi, S)\) that satisfies these conditions, there exists a program notation \((L, \varphi', \emptyset)\) such that, for each \(P \in L\), \(|\varphi(P)|/S = |\varphi'(P)|/\emptyset\). This means that we may restrict ourselves to program notations \((L, \varphi, S)\) with \(S = \emptyset\). However, this restriction leads in some cases to a complicated projection \(\varphi\). Take, for example, a program notation with subroutines that may call themselves recursively. Owing to the restriction to program notations with the same computational power as \((\mathcal{IS}_{br}^{n,m}, \iota, \emptyset)\), the recursion depth is bounded. Using a service that makes up a bounded stack of natural numbers is convenient and explanatory in the description of the behaviour of the programs concerned, but it is not necessary.
Below we lift the algorithmic equivalence relation $\equiv_{sa}$ defined in Section 7 from $\mathcal{IS}_{br}^{n,m}$ to programs in a higher-level program notation. The lifted equivalence relation is uniformly defined for all program notations. For each program notation, it captures the notion of algorithmic sameness to the same degree as $\equiv_{sa}$.

Let $PN = (L, \varphi, S)$ be a program notation. Then we define the structural algorithmic equivalence relation $\equiv_{PN} \subseteq L \times L$ for $PN$ as follows: $P \equiv_{PN} Q$ iff there exist $X, Y \in IS_{br}^{n,m}$ such that $|\varphi(P)|//S = |X|$, $|\varphi(Q)|//S = |Y|$, and $X \equiv_{sa} Y$.

Suppose that the projection $\varphi$ in the program notation $PN = (L, \varphi, S)$ is optimizing in the sense that it removes or replaces in whole or in part that which is superfluous. Because of this, certain programs would be structurally algorithmically equivalent that would not be so otherwise. This may very well be considered undesirable (cf. the discussion about the replacement of superfluous instructions in Section 7). Fortunately, projections like the supposed one are excluded by the third condition that must be satisfied by the program notations to which we restrict ourselves.

The concept of a basic algorithm can easily be lifted to programs in a higher-level program notation as well. The lifted concept is uniformly defined for all program notations. For each program notation, it is of course still merely an approximation of the intuitive concept of an algorithm.

Let $PN = (L, \varphi, S)$ be a program notation. Then the concept of a $PN$-oriented algorithm is essentially the same as the concept of a basic algorithm in the sense that there exists a surjection $\psi$ from $L$ to $\mathcal{IS}_{br}^{n,m}$ such that, for all $A \equiv_{PN} \subseteq L/\equiv_{sa}^{PN}$, $A$ is a $PN$-oriented algorithm iff $\psi(P)$ is a $PN$-oriented algorithm for $f$.

In [12], instruction sequences without backward jump instructions and instruction sequences with backward jump instructions were given which were assumed to express the same minor variant of the long multiplication algorithm. If we take the program notation used as $PN$, then according to the definition of $\equiv_{sa}$ given above, the instruction sequences without backward jump instructions and the instruction sequences with backward jump instructions are structurally algorithmically equivalent, which indicates that they express the same algorithm.

Two or more different program notations as considered in this section can be combined into one. This is done in the obvious way if the sets of programs to be combined are mutually disjoint and the sets of foci that serve as names in the service families to be combined are mutually disjoint. Otherwise, sufficient renaming must be applied first. The case of a structural algorithmic equivalence relation on the programs from two or more different program notations is covered by the definition given above as well because of this possibility to combine several program notations into one.

22
11 Discussion on What is an Algorithm

In this section, we point out that we are still far from the definitive answer to the question “what is an algorithm?”.

When we consider an $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$, we are faced with the question what is an algorithm $A$ for computing $f$. We may assume the existence of a class $ALG^{n,m}$ of algorithms for partial functions from $\{0, 1\}^n$ to $\{0, 1\}^m$, in which case we may assume that $A \in ALG^{n,m}$. However, as reported in Section 6, the computer science literature gives little to go on with regard to $ALG^{n,m}$. What we know about algorithms for computing $f$ is that they can be expressed by one or more instruction sequences from $IS_{br}^{n,m}$. Moreover, different algorithms for computing $f$ may differ in comprehensibility and efficiency, i.e. the number of steps in which they compute $f$ and the number of auxiliary Boolean registers that they need to compute $f$.

Suppose that $\Gamma_A(X)$ is a real number in the interval $[0, 1]$ which represents the degree to which algorithm $A$ is expressed by instruction sequence $X$. What we can say about $ALG^{n,m}$ based on the work presented in this paper is that:

- $ALG^{n,m}$ is approximated by the class of equivalence classes of instruction sequences from $IS_{br}^{n,m}$ with respect to the structural algorithmic equivalence relation $\equiv_{sa}$;
- an instruction sequence $X \in IS_{br}^{n,m}$ expresses an $A \in ALG^{n,m}$ if $X \in A$;
- $\Gamma_A$ is the characteristic function of $A$.

Perhaps we can say more about $ALG^{n,m}$ if we allow $\Gamma_A$ to yield values other than 0 and 1 to deal with uncertainty about whether an instruction sequence expresses an algorithm. This seems to fit in with actual practice where an algorithm $A$ is an idea in the mind of a programmer, say $P_1$, and $\Gamma_A$ comprises judgments by $P_1$. If a colleague of $P_1$ tries to get the “idea of $A$”, then questioning $P_1$ about his or her judgments $\Gamma_A$ may be the best option available to the colleague. If the judgments of all members of a group of programmers are the same, then $A$ is given in the group by the shared $\Gamma_A$.

Returning to the restricted setting considered in this paper, we realize that important questions are still unanswered. Among them are:

- how can structural algorithmic equivalence be generalized from finite PGA instruction sequences to finite and eventually periodic infinite PGA instruction sequences;
- how can algorithms be represented directly, rather than indirectly via a representative of an equivalence class;
- what is the exact connection between algorithms and efficiency of computation?

12 Concluding Remarks

We have looked for an equivalence relation on instruction sequences that captures to a reasonable degree the intuitive notion that two instruction sequences express
the same algorithm. Restricting ourselves to algorithms for computing partial functions from \( \{0, 1\}^n \) to \( \{0, 1\}^m \), for fixed \( n, m \), we have pictured our intuition about the notion that two instruction sequences express the same algorithm, defined an algorithmic equivalence relation corresponding to this intuition, and defined the concept of a basic algorithm using this equivalence relation. We have also shown how this algorithmic equivalence relation can be lifted to programs in a higher-level program notation, i.e., a program notation with advanced features such as conditional constructs, loop constructs, and subroutines that may call themselves recursively.

We have further defined an equivalence relation whose relevance is that any equivalence relation that captures the notion that two instruction sequences express the same algorithm to a higher degree than the algorithmic equivalence relation defined in this paper must be finer than this equivalence relation. We have also pointed out that we are still far from the definitive answer to the question “what is an algorithm?”.

We leave it for future work to show how the algorithmic equivalence relation defined in this paper can be generalized to the case where programs compute partial functions on data of a higher level than bit strings. In case the usual viewpoint is taken that the data may be differently represented in algorithmically equivalent programs, defining such a generalization in a mathematically precise way is nontrivial. The issue is that this viewpoint, although intuitively clear, leaves a loose end: it remains vague about which inescapable differences between programs due to different data representations must be considered inessential for algorithmic equivalence. It is mainly the tying up of this loose end what makes defining the generalization of the algorithmic equivalence relation nontrivial.

**Appendix: Self-Plagiarism**

Self-plagiarism is currently a prevailing issue in some scientific disciplines. This is why we have asked ourselves the question whether our practice to copy portions of the groundwork for the work presented in this paper verbatim or slightly modified from previous papers may be considered a matter of self-plagiarism. In the case of a paper for which the groundwork is laid in previously published papers, we are used to copy portions of the groundwork near verbatim or slightly modified from previously published papers and to make explicit reference to the papers in question. Moreover, we are used to copy not more than necessary to understand the new paper. However, we are unused to employ quotation marks or indentation to delineate copied text if it is awkward to do so.

In our opinion, the practice outlined above is not a matter of self-plagiarism. However, because self-plagiarism became a prevailing issue recently in some scientific disciplines, we have sought recent confirmation for our opinion within the discipline of computer science. A clear recent confirmation is found in the Plagiarism Policy of the Association for Computing Machinery [1, Section 1]:

Note that self-plagiarism does not apply to publications based on the author’s own previously copyrighted work (e.g., appearing in a conference.
proceedings) where an explicit reference is made to the prior publication. Such reuse does not require quotation marks to delineate the reused text but does require that the source be cited.

Recent confirmations can also be found within other disciplines. For example, within the discipline of psychology, the following confirmation is found in [2, page 16]:

The general view is that the core of the new document must constitute an original contribution of knowledge, and only the amount of previously published material necessary to understand that contribution should be included, primarily in the discussion of theory and methodology.

An interesting legal view on self-plagiarism, published in a computer science magazine, is found in [21, page 25].

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