Affine Projection Tensor Geometry: Decomposing the Curvature Tensor When the Connection is Arbitrary and the Projection is Tilted

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Abstract

This paper constructs the geometrically natural objects which are associated with any projection tensor field on a manifold with any affine connection. The approaches to projection tensor fields which have been used in general relativity and related theories assume normal projection tensors of co-dimension one and connections which are metric compatible and torsion-free. These assumptions fail for projections onto lightlike curves or surfaces and other situations where degenerate metrics occur as well as projections onto two-surfaces and projections onto spacetime in the higher dimensional manifolds of unified field theories. This paper removes these restrictive assumptions. One key idea is to define two different "extrinsic curvature tensors" which become equal for normal projections. In addition, a new family of geometrical tensors is introduced: the cross-projected curvature tensors. In terms of these objects, projection decompositions of covariant derivatives, the full Riemann curvature

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tensor and the Bianchi identities are obtained and applied to perfect fluids, timelike curve congruences, string congruences, and the familiar 3+1 analysis of the spacelike initial value problem of general relativity.

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I. INTRODUCTION

Applications of general relativity and related theories can often be stated in terms of projection tensor fields. Familiar examples are the co-moving frame projections which are central to hydrodynamics\footnote{\cite{1,2,3}} and the 3-surface projections which arise in initial value problems\footnote{\cite{4}}. In situations where projection tensors are not traditionally used, they often provide an improved description. For example, the embedding in a metric space of a submanifold with co-dimension higher than one is usually described by a non-unique set of normal vectors\footnote{\cite{5}} but is better described by the unique normal projection into the subspace tangent to the submanifold.

Aside from elegance and improved invariance properties, projection tensor techniques offer another advantage: They always lead to the same operations and those operations are always simplified by the construction of the same geometrical objects regardless of the nature of the system which is being described. Thus, I am led to describe a projection tensor geometry which contains results of wide applicability. The resulting geometry greatly enlarges the scope of projection tensor methods because it does not assume projections onto surfaces or normal (i.e. perpendicular) projections, or even the existence of a metric tensor.

When only one projection-tensor field is considered, this geometry is modeled on traditional surface embedding theory and generalizes the intrinsic and extrinsic curvature of a surface to the case of a projection-tensor field which need not be surface-forming. Just as in surface embedding theory, the main result is a decomposition of the Riemannian curvature tensor in terms of the projection curvatures. When a projection-tensor field is hyper-surface forming, the curvature decomposition includes the Gauss-Codazzi equations\footnote{\cite{6}} which have become familiar to relativists as the foundation of the 3+1 decomposition of the space-like hypersurface initial value problem in general relativity. In familiar cases where the projection-tensor field is not surface-forming — fluid flow for example — the projection curvatures turn out to be composed of such well-known quantities as the shear, divergence, and vorticity of fluid flow and the curvature decomposition leads to such familiar results as
the Raychaudhuri equation. In less familiar cases such as two-dimensional projections in a four-dimensional manifold, projections in the higher dimensional manifolds of unified field theories, and projections onto lightlike curves and surfaces, the curvature decomposition introduces geometrical objects and relationships which I, at least, have not seen before.

A previous paper on this subject introduced a compact ‘decorated index’ formalism for describing a single projection tensor field and applied it to the hydrodynamics and thermodynamics of a perfect fluid in general relativity. That paper imposed two major restrictions on the situations which it could cover: (1) There had to be a metric tensor which was at least invertible, thus excluding a projection-tensor approach to spacetime perturbation theory which was developed in my earlier papers as well as any discussion of unified field theories which use non-metric-compatible connections. (2) Projections in null or light-like directions were not allowed, thus excluding a projection-tensor approach to the propagation of radiation and the characteristic initial value problem. This paper removes those restrictions.

Although most of the applications which I have in mind involve spaces with zero torsion, I carry the torsion tensor throughout. As has often been observed, particularly in connection with the ECSK theory of gravity, differential geometry is a far more elegant and symmetrical theory with the torsion tensor present than without it. Here, I find it useful to define generalized torsion tensors associated with a projection tensor field in order to produce projected structure equations which are simple and symmetrical.

The compact index notation of the previous paper is not easily generalized to multiple projection tensor fields. Since multiple projection tensors often arise in applications, this paper will mostly use the familiar, unadorned index notation of tensor analysis. Although clumsy in some ways, this notation is one which we can all understand without explanations which might obscure the essential points which I am trying to make. I depart slightly from the notation in my previous papers by using an operator notation for covariant derivatives: \( \nabla_\delta M^{\alpha\beta}_{\mu\nu} = M^{\alpha\beta}_{\mu\nu;\delta} \). Notice that this operator notation does not change the convention that the differentiating index is added to the end of the list of tensor indexes. Also notice
the distinction between a covariant derivative, which increases the rank of a tensor, and a
directional derivative (used in my previous papers) which does not. The spacetime signature
is taken to be $- + ++$ and my conventions on the torsion and curvature tensors may be
seen in Eqs. (25,32). 

Section II of this paper reviews the basic properties of projection tensor fields and defines
the new geometrical structures and operations which become natural when a projection
tensor field is present. Section III introduces a few of the many situations in which projection
tensor fields play central roles. The key formal results of the paper are contained in Section
IV which defines the generalized projection curvatures and in Section V which presents the
projection decompositions of the metricity and torsion tensors as well as the Riemann and
Ricci tensors. This section introduces several new tensor fields — cross-projected torsion
and curvature tensors — which are needed for a full analysis of the way that a projection
tensor field interacts with a connection. A corresponding set of projected Bianchi identities
obeyed by these tensors is also worked out. Section VI shows how these results are used
in two familiar situations, fluid dynamics and the spacelike initial value problem of general
relativity. The applications considered here are taken just far enough to demonstrate and
provide a familiar context for the techniques developed in this paper. I expect to return in
later papers to the new applications which these techniques make possible.

II. PROJECTION TENSOR DEFINITIONS

A. Projection Tensor Fields

A projection tensor-field $H$ assigns to each point $P$ of a manifold a linear map of the
tangent space $H(P): T_P \rightarrow T_P$ such that

$$H^2 = H.$$  \hspace{1cm} (1)

It follows from this definition and the basic properties of a vector space that the projection
tensor $H$ acts as an identity operator on the projection subspace $HT_P$. 

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If $H$ is a projection tensor field, and $I$ is the identity map, the tensor field

$$V = I - H$$  \hspace{1cm} (2)

is also a projection tensor field which will be called the \textit{complement} of $H$. An immediate consequence of Eqs. (1,2) is $VH = HV = 0$. The complement of an expression involving projection tensor fields is obtained by replacing each projection tensor by its complement. As was discussed in an earlier paper, complementation is a valuable feature because the complement of a definition or identity is always another valid definition or identity, thus cutting in half the work of stating or deriving expressions.

The natural action of a projection tensor on the cotangent spaces to a manifold is defined by its pull-back $H^*$. In terms of components, if $\beta$ is a one-form with components $\beta_\alpha$ and $u$ is a vector with components $u^\alpha$

$$(Hu)^\alpha = H^\alpha_\rho u^\rho, \quad (H^*\beta)_\alpha = H^\rho_\alpha \beta_\rho.$$  

Since higher rank tensors may always be regarded as linear functions of forms and vectors, projection tensors can act on them by acting on their arguments. For example, the tensor with components $M^{\alpha\beta}_{\mu\gamma}$ would have a number (sixteen to be precise) of projections by the tensor $H$ and its complement, including the projection

$$M \left[ V^\nu H^\mu V \right]^{\alpha\beta}_{\mu\gamma} = V^\sigma H^\beta_\rho H^\gamma_\mu M^{\sigma\rho}_{\tau\nu} V^\nu \gamma.$$  \hspace{1cm} (3)

Notice the way in which I have named this tensor projection. I will often use this sort of naming convention for projected tensors and tensor subspaces so that one can see at a glance which projections have been performed on which index positions.

\textbf{B. Projection Subspaces and Projection Identities}

A projection tensor field such as $H$ splits each tensor space into \textit{projection subspaces}. The tangent space $T_P$ at the point $P$ is split into the projection subspaces $HT_P$ and $VT_P$. The cotangent space $\hat{T}_P$ is split into the projection subspaces $H^*\hat{T}_P$ and $V^*\hat{T}_P$. Similarly a
tensor space such as $T_P \otimes T_P \otimes \hat{T}_P \otimes \hat{T}_P$ is split into sixteen projection subspaces, including
the one inhabited by the example discussed above:

$$T \left[ V^H_H V \right]_P = VT_P \otimes HT_P \otimes H^* \hat{T}_P \otimes H^* \hat{T}_P. \quad (4)$$

An object which lies entirely in a projection subspace at each point of a manifold is said to be fully projected and is said to obey a projection identity. For example, a vector $v$ which is in the projection subspace $HT_P$ obeys the identity $Hv = v$. Similarly, a fully projected tensor $M^{VH}_{HV}$ which is in the projection subspace $T \left[ V^H_H V \right]_P$ defined in Eq. (4) obeys the projection identity

$$M^{VH}_{HV} \left[ V^H_H V \right]_{\alpha\beta}^{\mu\gamma} = V^{\alpha}_\sigma H^{\beta}_\rho H^\tau_\mu M^{VH}_{HV} M^{\sigma\rho}_{\tau\nu} V^\nu_\gamma = M^{VH}_{HV} M^{VH}_{HV}.$$

Notice the distinction between a tensor $M^{VH}_{HV}$ which inhabits a particular projection subspace and a tensor $M \left[ V^H_H V \right]$ which is the result of projecting a tensor $M$ into that subspace. The subscripts and superscripts which identify a fully projected tensor such as $M^{VH}_{HV}$ will be called projection labels. Often these labels will be replaced by variable projection labels which stand for possible choices of projections. For example, $M^{XY}_{ZW}$ with the choices $(X, Y, Z, W) = (H, V, V, H)$ would stand for the tensor $M^{VH}_{HV}$. In order to avoid confusion between projection labels and tensor indexes, the labels are grouped together in a block immediately after the symbol for the tensor and the first tensor index is positioned to the right of the last projection label as in $R^{V}_{H\alpha}^{\beta \mu \nu}$.

**C. Adapted Frames**

For each projection tensor field $H$ one can define an adapted reference frame which assigns the vectors $e_a (P), e_A (P)$ to each point $P$ with the vectors $\{e_a\}$ spanning the subspace $HT_P$ and the vectors $\{e_A\}$ spanning the subspace $VT_P$. Because $H$ acts as an identity operation on $HT_P$ and $V$ similarly acts as an identity on $VT_P$, the two sets of basis vectors can be characterized by

$$He_a = e_a, \quad Ve_A = e_A.$$
and the non-zero adapted-frame projection tensor components are

\[ H^a_b = \delta^a_b, \quad V^A_B = \delta^A_B. \]

In an adapted frame, the fully projected tensor \( M^{YH}_{HV} \), which I have been using as an example would have non-zero components \( M^{YH}_{HV} AbcD \) with all other components, such as \( M^{YH}_{HV} abcd \) equal to zero.

### III. SETTINGS OF PROJECTION TENSOR GEOMETRY

#### A. Normal Projection Tensors

So long as there is a regular metric tensor, one can define a normal projection tensor field to be a projection \( H \) such that the kernel of \( H \) is orthogonal to \( Hv \) for any vector \( v \). In other words, one requires \( g(Hv, u) = 0 \) whenever \( Hu = 0 \) or

\[ H_{\beta\alpha} v^\alpha u^\beta = 0 \text{ whenever } H^{\alpha\beta} u^\beta = 0. \]

It is easy to see that a sufficient condition for \( H \) to be normal is \( H_{\alpha\beta} = H_{\beta\alpha} \). To show that the condition is also necessary, notice that a reference frame which is adapted to a normal projection tensor satisfies \( g(e_a, e_A) = g_{aA} = 0 \) so that the non-zero adapted frame components of \( H_{\alpha\beta} \) are \( H_{ab} = g_{ar} H^r_b = g_{ar} \delta^r_b = g_{ab} \) and are manifestly symmetric.

A remark is needed at this point: Many discussions of projection tensor applications begin with the relation \( H_{ab} = g_{ab} \). This relation is correct only for normal projection tensors. As will be seen in the next section, it cannot be imposed on null projection tensors or the projection tensors associated with spacetime deformations.

Normal projection tensors are useful because they are uniquely determined from the projection subspace \( HT_P \) of the tangent space \( T_P \) at each point of a manifold. For example, given a spacelike hypersurface, there are two (past and future-pointing) unit normal vectors \( n^\alpha \) at each point. For either one, the projection which takes arbitrary vectors into vectors tangent to the surface is just
\[ H^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta. \] (5)

For a more general submanifold, one would choose an orthonormal basis on the subspace of vectors normal to the surface at each point and use this set of normal vectors to construct a unique projection.

A slightly different example shows that normal projection tensors need not be projections onto the tangent spaces of submanifolds. A fluid can be described by giving the four-velocity \( u^\alpha \) at each place and time. The constraint \( u_\alpha u^\alpha = -1 \) then ensures that the tensor

\[ V^\alpha_\beta = -u^\alpha u_\beta \]

obeys the requirement \( V^2 = V \) and is a projection tensor. In this case, \( V \) projects onto the tangent spaces to the fluid world-lines. However the story is different for the complementary projection tensor \( H = I - V \) or

\[ H^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta \]

which takes the space components of vectors in the local rest frame of the fluid. When the fluid has twist or vorticity, these local rest frames cannot be integrated to give a global rest-frame. Thus, the normal projection tensor \( H \) may not be surface-forming.

One idea which will be revisited in section \( \text{VI} \) is Barry Collins’ notion of the intrinsic geometry which is associated with a non-surface-forming projection tensor field. A similar idea, the quotient geometry which results from a single Killing vector field, was developed by Geroch in a framework very similar to the one used here.

For these normal projection tensors, the techniques which are developed in this paper reduce to familiar calculations. However, there are some new insights to be gained from seeing these old calculations in this more general setting, so this paper reviews them in section \( \text{VI} \).
B. Null Projection Tensors

In general, a null projection tensor $H$ is characterized by a projected metric tensor $g_{[HH]}$ (or projected inverse metric tensor $g^{-1}_{[HH]}$) which is non-invertible and thus fails to define a metric on the subspace $HT_P$. In spacetime, a null projection tensor is one which preserves exactly one null ray.

For a projection onto a null hypersurface, $H$ could take the form

$$H^\alpha_\beta = -\ell^\alpha n_\beta + e_1^\alpha e_{1\beta} + e_2^\alpha e_{2\beta}$$

where $\ell$ is a null vector tangent to the hypersurface and $n$ is a null vector pointing out of the surface. I will not go into detail here, but it is evident that such a projection tensor is not symmetric and cannot be normal. Thus, the usual definitions of surface curvature do not work. One is left to guess whether the extrinsic curvature of such a surface should be built from derivatives of $\ell$ or from derivatives of $n$. The case for $n$ is that it points out of the surface like the normal to a spacelike surface does. The case for $\ell$ is that an attempt to approximate a null surface by a sequence of Lorentz-boosted spacelike surfaces shows their normal rays approaching the null ray defined by $\ell$.

The next section will show that there are two, equally natural, definitions of the curvature of a projection-tensor field. For normal projection tensor fields, they coincide. For null projection tensor fields they do not coincide and one turns out to be the $n$ version of the extrinsic curvature while the other is the $\ell$ version. Thus, the general projection-tensor geometry developed in this paper adapts easily to the peculiarities of null projection tensors.

One situation which is naturally described by a system of null projection tensors is radiation originating from a compact source. Far from the source, slice spacetime by null hypersurfaces which look like future light cones. Each of these null hypersurfaces contains a congruence of null geodesics which correspond, in the optical limit, to world-lines of the radiation. Within this setting one formulates the characteristic initial value problem to describe the propagation of radiation. This null-surface formulation is particularly useful
for numerical computations because it lends itself to a conformal transformation which permits a finite evaluation grid to reach future null infinity where precise definitions of the amount of gravitational radiation flux are available.

The usual approaches to the characteristic initial value problem introduce spinors, null tetrads, or pairs of null congruences and lead to equations of motion for the corresponding connection coefficients and components of the Weyl tensor. A projection-tensor approach would begin with the observation that there is not just one projection tensor field in this system but a nested pair of them — the projections onto the level surfaces of the optical function and projections onto the null geodesics which lie in those level surfaces. This nested set of projections with their corresponding intrinsic and extrinsic curvatures is not fully exploited in any analysis that I am aware of.

This paper extends the definitions of intrinsic and extrinsic curvatures so that they can be applied to a null-hypersurface analysis of radiation. That analysis will be developed in a later paper.

C. Deformation Geometries

Given a family of manifolds, each carrying a metric and a connection, and each labeled by a set of parameters \( \{ \epsilon^A \} \), regard each manifold \( S (\epsilon) \) as a submanifold of a larger manifold, \( M \), and regard the parameters as functions on that larger manifold. When the submanifolds are spacetimes, I call the larger manifold a *spacetime deformation*. Geroch used this concept to provide a geometrical framework for discussing limits of spacetime sequences. I have used it as a geometrical framework for spacetime perturbation theory.

At each point of the larger manifold \( M \) choose a basis \( \{ e_\alpha \} \) for the subspace tangent to the submanifold \( S (\epsilon) \). Let \( g^{\alpha\beta} \) be the corresponding components of the spacetime metric \( S (\epsilon) \) and construct the tensor field \( g^{-1} = g^{\alpha\beta} e_\alpha \otimes e_\beta \) on \( M \). This tensor field plays the role of an inverse metric tensor on \( M \). It can be used to map forms into vectors according to

\[
g^{-1} (\mu) = g^{\alpha\beta} e_\alpha \otimes e_\beta (\mu) = e_\alpha g^{\alpha\beta} \mu_\beta = e_\alpha \mu^\alpha
\]
and of course it provides an inner product for forms according to

\[ \mu \cdot \nu = g^{-1}(\mu) \cdot \nu = g^{\alpha \beta} \mu_\alpha \nu_\beta. \]

However, when this tensor is applied to the differential forms \( d\epsilon A \), it maps them to zero. Thus it provides a \textit{degenerate inner product} and cannot be inverted to give a metric on the larger manifold.

The geometry of a spacetime deformation is incomplete unless one adds some structure to it. The approach which might at first seem obvious, completing the metric tensor by adding terms corresponding to the parameter directions is not very useful. It adds arbitrary parameter-space structure which has nothing to do with understanding the spacetimes in the deformation. Geroch’s approach is to add a vector field which maps the points of a spacetime to the points of its neighbors. My own work imposes a projection tensor field \( H \) which basically does the same job as Geroch’s vector field. I require the projected tangent space \( HT_P \) at each point of the deformation to be the tangent space to the spacetime which passes through \( P \). Since there is no regular metric on the deformation, this requirement leaves some freedom to choose the projection tensor field – a freedom which corresponds to Geroch’s choice of vector field and to the choice of gauge in perturbation theory.

A major point of my approach to deformation theory is that it can be modeled on a projection-tensor formulation of surface embedding theory. However, without a regular metric on the deformation, a direct link with the Gauss-Weingarten theory of surface embedding is lacking. This paper provides that direct link.

\section*{D. Geometrical Symmetry Breaking in Unified Field Theories}

Many promising approaches to unified field theory involve manifolds with more than four dimensions which evolve in such a way that all but four of the dimensions "collapse" or else fail to expand. Regardless of the details of the theory or the precise symmetry-breaking mechanism, the final state is best described by a \textit{spacetime projection-tensor field} which...
projects to zero those vectors which point in "collapsed" directions. When this description is used, all of the dynamical fields which appear in the resulting four-dimensional spacetime become components of the geometrical objects which are discussed in this paper.

All of the identities and field equations which can arise in this type of unified field theory are implicit in the projection tensor geometry developed by this paper. Further, they are worked out in a unified geometrical framework which should provide new insights into these theories. I expect to explore some of these insights in later papers.

IV. GENERALIZED PROJECTION CURVATURES

A. Projection Curvatures Without a Metric

Given a projection tensor $H$ and a connection, the tensor

$$h^H_{\alpha \gamma \delta} = H^\rho_\gamma H^\sigma_\delta \nabla_\sigma H^\alpha_\rho$$

is defined to be the curvature of $H$. In the familiar case of a spacelike hypersurface with a well-defined normal vector $n^\alpha$, Eq. (4) can be substituted into this definition to relate it to the familiar expression for the extrinsic curvature or second fundamental form $k_{\gamma \delta}$ of the surface.\[29\,\text{–}\,31\,\text{,} \,26\]

$$h^H_{\alpha \gamma \delta} = H^\rho_\gamma H^\sigma_\delta n^\alpha \nabla_\sigma n_\rho = n^\alpha k_{\gamma \delta}.$$  

The definition of the projection curvature tensor is projected explicitly on two of its three indexes. However, it is not difficult to show that it obeys the full set of projection identities

$$h^H_{\left[ V_H H \right] \alpha \gamma \delta} = V^\rho_\gamma h^H_\tau \rho H^\rho_\gamma H^\sigma_\delta = h^H_{\alpha \gamma \delta}.$$  

This result is obtained by taking the covariant derivative of Eq. (4), the defining requirement for $H$ to be a projection tensor, and then projecting the result.

Another way to project the covariant derivative of a projection tensor yields the tensor

$$h^T_{\alpha \gamma \delta} = H^\alpha_\rho H^\sigma_\delta \nabla_\sigma H^\rho_\gamma.$$  

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When a metric is available for raising and lowering indexes, this tensor is the curvature associated with the transpose of $H$. Because of this association with the transposed projection, this tensor will be called the \textit{transpose curvature} of $H$. For a normal projection tensor, it is exactly the same as the tensor $h_H$ with the indexes appropriately raised and lowered. By differentiating and projecting Eq. (9), it is a straightforward matter to show that $h^T_H$ obeys the projection identity

$$h^T_H \left[ V^H_H \right]^{\alpha \gamma \delta} = h_H^{T \alpha \gamma \delta}$$

(10)

In addition to the two curvature tensors $h_H$ and $h^T_H$ associated with the projection tensor $H$, the same definitions yield a projection curvature $h_V$ and transpose curvature $h^T_V$ associated with the complementary tensor $V$. These curvature tensors obey projection identities which are simply the complements of Eqs. (8,10):

$$h_V \left[ H^V_V \right]^{\alpha \gamma \delta} = h_V^{\alpha \gamma \delta}, \quad h^T_V \left[ H^V_V \right]^{\alpha \gamma \delta} = h_V^{T \alpha \gamma \delta}.$$  

(11)

In the familiar case where $H$ projects onto a family of spacelike hypersurfaces with a unit timelike normal vector field $n$, these two curvatures are the same and are easily found by using $V^\alpha_{\beta} = -n^\alpha n_\beta$ in place of $H^\alpha_{\beta}$ in Eq. (7).

$$h^T_V \gamma \delta = h_V^{\alpha \gamma \delta} = V^\rho_{\gamma V} V^\sigma_{\delta V} \nabla_\sigma V^\alpha_{\rho} = -a^\alpha V_{\gamma \delta}$$

where $a^\alpha = n^\sigma \nabla_\sigma n^\alpha$ is the acceleration (or curvature) of the hypersurface-orthogonal world lines.

\textbf{B. Decomposition of the Projection Gradient}

The covariant derivative $\nabla H$ of a projection tensor $H$ will arise whenever one takes the covariant derivative of a tensor which obeys projection identities. Since I am engaged in expressing everything in terms of tensors which obey projection identities, the projection gradient $\nabla H$ will arise often. Thus, my first task is to express the projection gradient in terms of fully projected tensors. The resulting expression will be the key to everything else
in this paper, so I will include more of the details of its derivation than I would for a result of lesser significance.

Use the decomposition of the identity tensor $I = H + V$ to force a decomposition of $\nabla H$ which I write symbolically as

$$\nabla H = \nabla H \begin{bmatrix} I & I \end{bmatrix} = \nabla H \begin{bmatrix} H + V & H + V \end{bmatrix}$$

In this same abbreviated notation, the various projection curvatures (with $\delta$ the differentiating index) are

$$h_H^{\alpha \gamma \delta} = \nabla H \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} H \end{bmatrix}^{\alpha \gamma \delta}, \quad h_T^{\alpha \gamma \delta} = \nabla H \begin{bmatrix} H & I \end{bmatrix} \begin{bmatrix} I \end{bmatrix}^{\alpha \gamma \delta}$$

$$h_V^{\alpha \gamma \delta} = -\nabla H \begin{bmatrix} I & V \end{bmatrix} \begin{bmatrix} V \end{bmatrix}^{\alpha \gamma \delta}, \quad h_T^{\alpha \gamma \delta} = -\nabla H \begin{bmatrix} H & I \end{bmatrix} \begin{bmatrix} V \end{bmatrix}^{\alpha \gamma \delta}$$

and the decomposition becomes

$$\nabla_\delta H^{\alpha \gamma} = h_H^{\alpha \gamma \delta} + \nabla H \begin{bmatrix} H & V \end{bmatrix} \begin{bmatrix} I \end{bmatrix}^{\alpha \gamma \delta} + h_T^{\alpha \gamma \delta} - h_V^{\alpha \gamma \delta}$$

$$+ h_H^{\alpha \gamma \delta} + h_T^{\alpha \gamma \delta} - \nabla H \begin{bmatrix} V & H \end{bmatrix}^{\alpha \gamma \delta} - h_V^{\alpha \gamma \delta}$$

The two terms which have not been expressed in terms of curvatures can be shown to vanish by projecting the covariant derivative of Eq. (11). Half of the remaining terms vanish because of the projection identities given in Eqs. (8,10,11). The remaining projections have no effect because of the same projection identities. The resulting decomposition is just

$$\nabla_\delta H^{\alpha \gamma} = h_H^{\alpha \gamma \delta} - h_V^{\alpha \gamma \delta} + h_T^{\alpha \gamma \delta} - h_V^{\alpha \gamma \delta}$$

(12)

The decomposition of the complementary projection gradient $\nabla V$ is given by the complement of this expression — Exchange $H$ and $V$ everywhere.

Because each projection curvature tensor has two indexes which project into the same subspace, one can either contract those two indexes or else extract symmetric and antisymmetric parts. In the absence of a metric, one can define the divergence form

$$\theta_T^{\alpha \gamma \delta} = h_H^{\alpha \gamma \delta} - h_V^{\alpha \gamma \delta}$$
the twist or vorticity tensor

$$\omega_H^{\alpha}_{\mu\nu} = \frac{1}{2} (h_H^{\alpha}_{\mu\nu} - h_H^{\alpha}_{\nu\mu})$$

and the expansion rate tensor

$$\theta_H^{\alpha}_{\mu\nu} = \frac{1}{2} (h_H^{\alpha}_{\mu\nu} + h_H^{\alpha}_{\nu\mu}).$$

For fluid flow characterized by a tangent vector field $u^\alpha$, and $H = \delta_\beta^\alpha + u^\alpha u_\beta$, these definitions are closely related to the usual definitions of vorticity, divergence, and expansion rate:

$$\omega_H^{\alpha}_{\mu\nu} = u^\alpha \omega_{\mu\nu}, \quad \theta_H^{\alpha}_{\mu\nu} = u^\alpha \theta_{\mu\nu}, \quad \theta_T^{\alpha}_{\gamma} = u_\gamma \theta.$$

When a metric tensor is present, it can be used to raise or lower indexes and define the additional quantities: $\theta_H^{\alpha}$, $\omega_T^{\alpha}_{\mu\nu}$, $\theta_T^{\alpha}_{\mu\nu}$, as well as the shear tensors

$$\sigma_H^{\alpha}_{\mu\nu} = \theta_H^{\alpha}_{\mu\nu} - \frac{1}{d_H} \theta_H^{\alpha} H_{\mu\nu}, \quad \sigma_T^{\alpha}_{\mu\nu} = \theta_T^{\alpha}_{\mu\nu} - \frac{1}{d_H} \theta_T^{\alpha} H_{\mu\nu}$$

where $d_H = H^\rho_\rho$ is the dimensionality of the projected subspace $HT_P$.

C. The Projected Connection

1. Projected and Anti-projected Covariant Derivatives

Consider a vector field $v$ such that $v(P) \in HT_P$ for every point $P$ in a manifold. This vector field obeys the identity $Hv = v$. The covariant derivative $\nabla v$ of this field can be thought of as having a part which primarily reflects the behavior of the projection tensor and a part which reflects the behavior of $v$ within the projected subspaces. The part of the covariant derivative which reflects how $v$ changes within the projected subspaces is the projected covariant derivative $Dv$ with components

$$D_\delta v^\alpha = H^\alpha_\rho \nabla_\delta v^\rho. \quad (13)$$

The part of the covariant derivative which ignores how $v$ changes within the projected subspace is the anti-projected covariant derivative $\bar{D}v$ with components
\[ D_\delta v^\alpha = V_\rho^\alpha \nabla_\delta v^\rho \] (14)

More generally, if \( M \) is a fully projected tensor obeying the projection identities \( OM = M \), the projected covariant derivative takes the form \( DM = O (\nabla M) \) and the anti-projected tensor takes the form \( \bar{D}M = \bar{O} (\nabla M) = (I - O) (\nabla M) \). For example, if \( M \in T [V^H H V]_P \), then \( M \) obeys the projection identity

\[ V^\sigma H^\beta \rho H^\tau_\mu M^\sigma_\rho \tau_\nu V^\nu_\gamma = M^{\alpha\beta}_{\mu\gamma} \]

and its projected covariant derivative \( DM \) has components

\[ D_\delta M^{\alpha\beta}_{\mu\gamma} = V^\sigma H^\beta \rho H^\tau_\mu \nabla_\delta M^\sigma_\rho \tau_\nu V^\nu_\gamma \] (15)

while its antiprojected covariant derivative \( \bar{D}M \) has components

\[ \bar{D}_\delta M^{\alpha\beta}_{\mu\gamma} = \nabla_\delta M^{\alpha\beta}_{\mu\gamma} - V^\sigma H^\beta \rho H^\tau_\mu \nabla_\delta M^\sigma_\rho \tau_\nu V^\nu_\gamma \]

A point about notation: The projected and anti-projected covariant derivatives \( D, \bar{D} \) carry no indication of what projection tensor fields are to be used for evaluating them. They act only on tensor fields which are identified as belonging to particular projected subspaces and inherit the information about what projections to make from the object which is being differentiated. One consequence of this inheritance property is that these derivatives obey the product rule only for products of fully projected tensors.

From the decomposition of the projection gradient given in Eq. (12) and the projection identities obeyed by the projection curvatures given in Eqs. (8,10) it is easy to show that the projected derivatives give zero when they act on the projection tensor fields themselves.

\[ D_\delta H^{\alpha_\beta} = 0, \quad D_\delta V^{\alpha_\beta} = 0. \] (16)

To relate the projected covariant derivative to the ordinary covariant derivative, just take the covariant derivative of the projection identity which the fully projected tensor obeys.

For the simple case of a projected vector field \( v \in HT_P \), the projection identity is

\[ v^\alpha = H^\alpha_\rho v^\rho \]
which becomes
\[ \nabla_\delta v^\alpha = (\nabla_\delta H^\alpha_{\rho}) v^\rho + H^\alpha_{\rho} \nabla_\delta v^\rho. \]

Use the decomposition of the projection gradient eq.(12), the definition of the projected derivative, eq.(13), and the projection identities obeyed by the projection curvature tensors (Eqs. (8,10)) together with the one obeyed by \( v \) to obtain the relation
\[ \nabla_\delta v^\alpha = D_\delta v^\alpha + \left( h_{H^\alpha_{\rho} \delta} - h_{V^\rho_{\delta}} \right) v^\rho. \]  

(17)

For a projected one-form field \( \phi (P) \in H^*\hat{T}_P \) the same procedure yields
\[ \nabla_\delta \phi_\beta = D_\delta \phi_\beta + \phi_\rho \left( h_{H^\beta_{\rho} \delta} - h_{V^\rho_{\beta} \delta} \right) \]  

(18)

For vectors and forms obeying projection identities with the complementary projection \( V \), just take the complements of these results.

In terms of the anti-projected covariant derivative, these last results take the form:
\[ \bar{D}_\delta v^\alpha = \left( h_{H^\alpha_{\rho} \delta} - h_{V^\rho_{\delta}} \right) v^\rho, \quad \bar{D}_\delta \phi_\beta = \phi_\rho \left( h_{H^\beta_{\rho} \delta} - h_{V^\rho_{\beta} \delta} \right) \]  

(19)

It is evident that the anti-projected derivative ignores how the vector field \( v \) changes within the subspaces \( HT_P \). More generally, if a gauge-field assigns a transformation \( \Lambda (P) : HT_P \to HT_P \) to each point \( P \) of a manifold, and \( v (P) \in HT_P \) then the antiprojected derivative has the property:
\[ \bar{D} (\Lambda v) = \Lambda \left( \bar{D} v \right) \]

Thus, the gauge-transformation field \( \Lambda \) does not get differentiated and acts purely locally. This gauge-locality property of the anti-projected covariant derivative makes it a natural ingredient in any theory which possesses a gauge group.

2. Intrinsic and Extrinsic Projected and Anti-projected Covariant Derivatives

The projected and anti-projected covariant derivatives are not quite what I want because they are not fully projected. The desired fully projected objects are the *intrinsic projected*
covariant derivative $D_H v$ with components

$$D_{H \delta} v^\alpha = H^\rho_\delta D_\rho v^\alpha$$

the extrinsic projected covariant derivative $D_V v$ with components

$$D_{V \delta} v^\alpha = V^\rho_\delta D_\rho v^\alpha$$

the intrinsic anti-projected covariant derivative $\bar{D}_H v$ with components

$$\bar{D}_{H \delta} v^\alpha = H^\rho_\delta \bar{D}_\rho v^\alpha$$

and the extrinsic anti-projected covariant derivative $\bar{D}_V v$ with components

$$\bar{D}_{V \delta} v^\alpha = V^\rho_\delta \bar{D}_\rho v^\alpha$$

Projecting Eqs. (17, 18) and using the projection identities obeyed by the projection curvatures (equations (8, 10)) yields the following full decomposition of the covariant derivative:

$$H^\rho_\delta \nabla_\rho v^\alpha = D_{H \delta} v^\alpha + v^\rho h_{H \rho \delta}^\alpha$$

$$V^\rho_\delta \nabla_\rho v^\alpha = D_{V \delta} v^\alpha - v^\rho h_{V \rho \delta}^T_\alpha$$

$$H^\rho_\delta \nabla_\rho \phi_\beta = D_{H \delta} \phi_\beta + \phi_\rho h_{H \beta \rho \delta}^T$$

$$V^\rho_\delta \nabla_\rho \phi_\beta = D_{V \delta} \phi_\beta - \phi_\rho h_{V \rho \beta \delta}^T$$

These expressions refer to a vector field $v$ with $v(P) \in HT_P$ and a form-field $\phi$ with $\phi(P) \in H^*\hat{T}_P$. The expressions for $v(P) \in VT_P$ and $\phi(P) \in V^*\hat{T}_P$ can be obtained by taking the complements — exchange $H$ and $V$ everywhere. The net result of these exchanges is just to exchange $h_{H \rho \delta}^\alpha$ and $-h_{H \rho \delta}^T$ and similarly exchange $h_V$ and $-h_V^T$ in the above expressions.

The anti-projected versions of these results are just

$$\bar{D}_{H \delta} v^\alpha = v^\rho h_{H \rho \delta}^\alpha, \quad \bar{D}_{V \delta} v^\alpha = -v^\rho h_{V \rho \delta}^T$$

$$\bar{D}_{H \delta} \phi_\beta = \phi_\rho h_{H \beta \rho \delta}^T, \quad \bar{D}_{V \delta} \phi_\beta = -\phi_\rho h_{V \rho \beta \delta}^T$$
3. Fully Projected Decompositions of Covariant Derivatives

For a general, fully projected tensor, the decomposition of the covariant derivative has the same structure as the vector and form decomposition shown above, but with a correction term for each index of the tensor. For example, if \( M \in T^{V^H H^V}_\rho \), then the covariant derivative has the decomposition

\[
H^\alpha_\delta \nabla_\rho M^{\alpha\beta}_{\mu\nu} = D_{H\delta} M^{\alpha\beta}_{\mu\nu} - M^{\alpha\beta}_{\mu\nu} h^{T}_H^{\alpha_\delta} + M^{\alpha\rho}_{\mu\nu} h_H^{\beta_\rho_\delta} \\
+ M^{\alpha\beta}_{\mu\nu} h^{T}_H^{\rho_\delta} - M^{\alpha\beta}_{\mu\nu} h_H^{\rho_\delta}
\]

\[
V^\rho_\delta \nabla_\rho M^{\alpha\beta}_{\mu\nu} = D_{V\delta} M^{\alpha\beta}_{\mu\nu} + M^{\alpha\beta}_{\mu\nu} h_V^{\alpha_\rho_\delta} - M^{\alpha\rho}_{\mu\nu} h_V^{T\alpha_\rho_\delta} \\
- M^{\alpha\beta}_{\mu\nu} h_V^{\rho_\delta} + M^{\alpha\beta}_{\mu\nu} h_V^{T\rho_\delta}
\]

The projection correction terms in this sort of decomposition can be written with the help of just three rules:

- (1) Contract each tensor index in turn with the first or second index of one of the four projection curvature tensors \( h_H, h^{T}_H, h_V, h^{T}_V \). Set the last index on the projection curvature equal to the differentiating index. Set the remaining index equal to the tensor index which is being corrected.

- (2) Choose the projection curvature tensor whose indexes are in the right positions (up or down) and have the correct projection properties to yield a consistent non-zero term — For each index, only one choice will work. (Do not raise or lower indexes.)

- (3) When the corrected tensor index obeys a projection identity complementary to the one obeyed by the differentiating index, the correction term has a minus sign. Otherwise it has a plus sign.

At some risk of taking excessive poetic license, I will refer to these rules as the generalized Gauss-Weingarten Relations. When all of the differences in notation and point of view have been swept away, these rules do the essential job of the Gauss-Weingarten relations: They relate the full connection to the projected connection.
One consequence of these rules is that when the differentiating index has been projected with $H$, then the correction terms can only be constructed from $h_H, h^T_H$. Similarly when the differentiating index has been projected with $V$, then the correction terms can only be constructed from $h_V, h^T_V$. Note the projection identities obeyed by the projection curvatures (Eqs. (10,11)). A useful consequence of these identities is that the correction term which is associated with a given tensor index is always projected in a manner complementary to that of the original tensor index. This consequence can be used as a consistency check. It also means that the correction terms which are associated with contracted indexes often vanish.

A final point about notation: The operators $D_H, D_V, \bar{D}_H, \bar{D}_V$ specify the projection which is to be performed on the differentiating index of the tensors which they produce. However, just like the projected derivative $D$, they do not specify the projections which are to be performed on the other indexes. Those projections are "inherited" from the tensor fields which are being differentiated. Thus, although the situations which have been considered so far involve only combinations of a single projection tensor field, $H$, and its complement, $V$, there will be cases (such as null projections) where more than one projection tensor field is present. In those cases, one may have an operation $D_H T$ which projects the indexes inherited from the tensor $T$ with a projection tensor field which is neither $H$ nor its complement $V$. One may also have an object which belongs to more than one projection subspace so that it needs to be assigned a "home subspace" for its projected derivative operators to be defined.

V. GEOMETRICAL STRUCTURE DECOMPOSITIONS

A. Metricity

When a form-metric with components $g^{\mu\nu}$ exists, the metricity tensor is just the covariant derivative of the metric which can be decomposed by the rules of the previous section. Here, I will take $g^{\mu\nu}$ to be an arbitrary tensor field which need not have all of the properties which
we usually associate with a metric. For example, the vector-metric \( g_{\mu\nu} \) may not exist. First, decompose this ‘metric’ into fully projected tensors:

\[
g_{XY \mu \nu} = g_{[XY]} \mu \nu = g^{\rho \sigma} X_{\rho} Y_{\sigma}
\]

\[
g^{\mu \nu} = g^{HH \mu \nu} + g^{HV \mu \nu} + g^{VH \mu \nu} + g^{VV \mu \nu}
\]

where the projection labels \( X, Y \) stand for either \( H \) or \( V \)

express the metricity, \( Q^{\mu \nu \rho}_{\delta} = -\nabla_{\rho} g^{\mu \nu} \), in terms of these:

\[
-Q^{\mu \nu \rho}_{\delta} H^\rho = D_{H \delta} g^{HH \mu \nu} + g^{HV \rho \nu} h^{H \mu \rho \delta} + g^{HH \mu \rho} h^{H \nu \rho \delta}
\]

\[
+D_{H \delta} g^{HV \mu \nu} + g^{HV \rho \nu} h^{H \mu \rho \delta} - g^{HV \mu \rho} h^{H T \mu \nu \rho \delta}
\]

\[
+D_{H \delta} g^{VH \mu \nu} - g^{VH \rho \nu} h^{H \mu \rho \delta} + g^{VH \mu \rho} h^{H \nu \rho \delta}
\]

\[
+D_{H \delta} g^{VV \mu \nu} - g^{VV \rho \nu} h^{T \mu \rho \delta} - g^{VV \mu \rho} h^{T \nu \rho \delta}
\]

The decomposition of the projection \( Q^{\mu \nu \rho}_{\delta} V^\rho \) is then obtained by taking the complement of this result.

When \( H \) projects onto surfaces, the quantity \( D_{H \delta} g^{HH \mu \nu} \) is the metricity of the intrinsic geometry on those surfaces. In general, I define the intrinsic metricity to be the projected intrinsic derivative

\[
Q^{HH \mu \nu \rho}_{\delta} = -D_{H \delta} g^{HH \mu \nu}
\]

and, to complete the decomposition of the metricity, I define cross-projected metricities

\[
Q^{XY \mu \nu \rho}_{\delta} = -D_{H \delta} g^{XY \mu \nu}
\]

as well as the complements of these objects. My earlier caution (see Section IV C 1) about the product rule for projected derivatives of tensor products comes into play here. If the connection is metric compatible, one might suspect that the intrinsic and cross-projected metricities would automatically vanish as a consequence of Eq. (16). As is shown next, they do not necessarily vanish.

Project out the different components of the metricity:

\[
Q^{\left[ HH \right]}_{H} \mu \nu \rho \delta = Q^{HH \mu \nu \rho}_{\delta} + g^{HV \mu \rho} h^{T \mu \nu \delta} + g^{VH \rho \nu} h^{T \mu \rho \delta}
\]

(21)
\[
\begin{align*}
Q^{[HH\nu]}_{\mu\delta} &= Q^{HH\nu}_{\mu\delta} - g^{HH\nu\rho} h^{\nu\rho}_{\delta} - g^{VH\rho\nu} h^{\rho}_{\nu\mu\delta} \\
Q^{[HV\nu]}_{HH\mu\delta} &= Q^{HV\nu}_{HH\mu\delta} - g^{HH\nu\rho} h^{\nu}_{HH\rho\mu\delta} + g^{VV\rho\nu} h^{T}_{HV\rho\mu\delta} \\
Q^{[HV\nu]}_{HV\mu\delta} &= Q^{HV\nu}_{HV\mu\delta} + g^{HH\nu\rho} h^{T}_{HV\rho\mu\delta} - g^{VV\nu\rho} h^{\nu}_{VV\rho\mu\delta}
\end{align*}
\] (22)

Ordinarily the connection is metric compatible so that the metricity tensor vanishes and all of the above equations have zero on their left-hand sides. For a normal projection, the cross-projected metric \(g^{HV}\) is zero. When both these conditions hold, the above equations simply say that both the intrinsic and the cross-projected metricity tensors vanish and the two types of projection curvature are the same: \(h = h^T\). If, however, the connection is metric compatible but the projection is not normal, the above equations yield interesting results including:

\[
Q^{[HH\nu]}_{HV\mu\delta} = -g^{HV\nu\rho} h^{T}_{HV\rho\mu\delta} - g^{VH\rho\nu} h^{T}_{HV\rho\mu\delta}.
\]

The intrinsic and cross-projected metricities do not necessarily vanish for non-normal projection tensor fields even if the connection is metric compatible.

**B. Torsion**

1. **Definition and Projection**

The torsion tensor \(S^{\rho}_{\mu\nu}\) is defined by the relation

\[
[\nabla_\nu, \nabla_\mu] \phi = S^{\rho}_{\mu\nu} \nabla_\rho \phi.
\] (25)

for any function \(\phi\) on the manifold. To decompose this relation into fully projected parts, begin by decomposing the gradient

\[
\nabla_\alpha \phi = D_{H\alpha} \phi + D_{V\alpha} \phi
\]

so that the definition of torsion becomes

\[
\nabla_\nu D_{H\mu} \phi + \nabla_\nu D_{V\mu} \phi - \nabla_\mu D_{H\nu} \phi - \nabla_\mu D_{V\nu} \phi = S^{\rho}_{\mu\nu} \nabla_\rho \phi.
\]
Project the two free indexes \( \mu, \nu \) with \( H \) and use the definition of the intrinsic projected derivative as well as Eq. (20) to obtain the \( HH \)-projection

\[
[D_{H\nu}, D_{H\mu}] \phi = 2\omega_H^{\rho \mu \nu} D_{V\rho} \phi + S'_{H H} [H H \rho \mu \nu] \nabla \phi
\]

of the torsion definition. Project one free index with \( H \) and the other with \( V \) and proceed as before to obtain the \( HV \)-projection

\[
[D_{V\nu}, D_{H\mu}] \phi = -h_T^{H\nu \rho \mu} D_{H\rho} \phi + h_T^{V\nu \rho \mu} D_{V\rho} \phi
\]

or

\[
=S'_{H V} [H V \rho \mu \nu] \nabla \phi
\]

of the torsion definition.

2. Intrinsic and Cross Torsions

When \( H \) projects onto surfaces, the quantity \([D_{H\nu}, D_{H\mu}] \phi\) is simply related to the torsion of the intrinsic geometry on those surfaces. In general, define the intrinsic and cross torsion tensors \( S_{H H}, S_{H V}, S_{H V}, S_{V V} \) and their complements by

\[
[D_{Y\nu}, D_{X\mu}] \phi = S_{X Y} [H H \rho \mu \nu] D_{H\rho} \phi + S_{X Y} [H V \rho \mu \nu] D_{V\rho} \phi
\]

From the decompositions (Equations (26, 27)) above,

\[
S'_{H H} [H H \rho \mu \nu] \nabla \phi + S'_{H V} [H H \rho \mu \nu] \nabla \phi + 2\omega_H^{\rho \mu \nu} D_{V\rho} \phi
\]

\[
=S_H^{\rho \mu \nu} D_{H\rho} \phi + S_V^{\rho \mu \nu} D_{V\rho} \phi
\]

\[
S'_{H V} [H V \rho \mu \nu] \nabla \phi + S'_{V V} [H V \rho \mu \nu] \nabla \phi + h_T^{H\nu \rho \mu} D_{H\rho} \phi - h_T^{V\nu \rho \mu} D_{V\rho} \phi
\]

\[
=S_H^{\rho \mu \nu} D_{H\rho} \phi + S_V^{\rho \mu \nu} D_{V\rho} \phi
\]

which give the decompositions:

\[
S'_{H H} [H H \rho \mu \nu] \nabla \phi = S_H^{\rho \mu \nu}
\]

\[
S'_{H V} [H V \rho \mu \nu] \nabla \phi = S_H^{\rho \mu \nu} - 2\omega_H^{\rho \mu \nu}
\]

\[
S'_{V V} [H V \rho \mu \nu] \nabla \phi = S_H^{\rho \mu \nu} - h_T^{H\nu \rho \mu}
\]

and their complements.
3. Surface Formation: Frobenius Theorem

When a projection tensor field $H$ yields projected tangent spaces $HT_P$ which are the tangent spaces to a system of submanifolds, each submanifold has its own, fully self-contained *intrinsic geometry.* When these intrinsic geometries exist, they provide powerful computational tools and important insights as in the 3+1 decomposition of the initial value problem of general relativity for example. Ordinarily, however, the subspaces and intrinsic derivative operations associated with a given projection tensor field $H$ do not form fully self-contained intrinsic geometries. What conditions on a projection tensor field are sufficient for intrinsic geometries to exist?

In an adapted frame, the $HH$-projection of the torsion definition (Eq. (26) above) takes the form

$$\left\{ [e_n, e_m] - \left( 2\Gamma^r_{[mn]} + S^{H}_{HH} r_{mn} \right) e_r - S^{V}_{HH} R_{mn} e_R \right\} \phi = 0. \quad (31)$$

while the $HV$-projection is

$$\left\{ [e_E, e_d] - \left( \Gamma^r_{dE} + S^{H}_{HV} r_{dE} \right) e_r + \left( \Gamma^R_{Ed} - S^{V}_{HV} R_{dE} \right) e_R \right\} \phi = 0$$

The $VV$-projection can be obtained by taking the complement of Eq. (31) – replace $H$ by $V$ and switch upper and lower case indexes everywhere. Notice how these results simplify when expressed in terms of the cross-torsion tensor components given in equations (29,30) above.

The $HH$-projection of the torsion tensor definition given by Eq. (31) above provides the needed relation. This result shows that two vector fields with values in the subspaces $HT_P$ have a commutator which lies in the same subspaces if and only if the cross-projected torsion tensor $S^{V}_{HH}$ is zero. The Frobenius theorem then guarantees that the subspaces are tangent to a system of submanifolds. Thus, the vanishing of the tensor $S^{V}_{HH}$ is a necessary and sufficient condition for a projection tensor field to yield subspaces which are tangent to submanifolds.
In a torsion-free geometry, the cross-projected torsion $S^V_{HH}$ is not always zero. From Equation (29) it is related to the twist or vorticity tensor by $S^V_{HH}{}^{\mu\nu} = 2\omega_{H}{}^{\mu\nu}$. For this reason, I will call this particular cross-torsion tensor, the *generalized twist* of a projection tensor field.

C. Riemann Curvature

1. Definition and Projection

The curvature tensor is defined by the equation

$$v^\rho R_{\rho}{}^{\gamma}{}_{\alpha\beta} = ([\nabla_{\beta}, \nabla_{\alpha}] - S_{\alpha\beta}^\rho \nabla_{\rho}) v^\gamma. \quad (32)$$

By letting the torsion definition act on the function $\phi_{\gamma} v^\gamma$ one finds that this definition may be restated in terms of derivatives acting on one-forms:

$$([\nabla_{\beta}, \nabla_{\alpha}] - S_{\alpha\beta}^\rho \nabla_{\rho}) \phi_{\gamma} = -R_{\gamma}{}^{\rho}{}_{\alpha\beta} \phi_{\rho}. \quad (33)$$

Either form of the definition can be decomposed by using the rules given in section IV C. Start with a vector $v$ such that $Hv = v$ and decompose the first derivative

$$\nabla_{\alpha} v^\gamma = D_{H\alpha} v^\gamma + v^\rho h_{H}{}^{\gamma}{}_{\rho\alpha} + D_{V\alpha} v^\gamma - v^\rho h_{V}{}^{\gamma}{}_{\rho\alpha}$$

and then, in a straightforward calculation, the second derivative, $\nabla_{\beta} \nabla_{\alpha} v^\gamma$. Use this result to evaluate Eq. (32) and form all of its independent projections. The results become simple and symmetrical when they are expressed in terms of the intrinsic and cross torsion tensors.

$$v^\rho R_{\rho}{}^\gamma{}_{\alpha\beta} \left[ H^H_{HH} \right] = \left( [D_{H\beta}, D_{H\alpha}] - S_{H\beta H}^H{}^{\rho}{}_{\alpha\beta} D_{H\rho} - S_{H\alpha H}^V{}^{\rho}{}_{\alpha\beta} D_{V\rho} \right) v^\gamma$$

$$+ v^\rho \left( -h_{H}^T{}_{\gamma}{}_{\beta} h_{H}{}^{\sigma}{}_{\rho\alpha} + h_{H}^T{}_{\gamma}{}_{\alpha} h_{H}{}^{\sigma}{}_{\rho\beta} \right) \quad (34)$$

$$v^\rho R_{\rho}{}^\gamma{}_{\alpha\beta} \left[ H^H_{HV} \right] = \left( [D_{V\beta}, D_{H\alpha}] - S_{HV H}^H{}^{\rho}{}_{\alpha\beta} D_{H\rho} - S_{HV V}^V{}^{\rho}{}_{\alpha\beta} D_{V\rho} \right) v^\gamma$$

$$+ v^\rho \left( h_{V}^{\sigma}{}_{\rho\alpha} h_{V}{}^{\gamma}{}_{\sigma\beta} - h_{H}^T{}_{\gamma}{}_{\sigma} h_{V}^T{}_{\sigma}{}_{\rho\beta} \right) \quad (35)$$
The rest of the projections of the curvature tensor obey equations which are the complements of these.

\[ v^\rho R \left[ H^H V^V \right]_{\rho}^{\gamma \alpha \beta} = \left( [D_{\gamma \beta}, D_{\alpha \alpha}] - S_{V^V}^{H} \rho \alpha \beta D_{H \rho} - S_{H^H}^{V} \rho \alpha \beta D_{V \rho} \right) v^\gamma \]
\[ + v^\rho \left( h^H_{\gamma \sigma \alpha} h^T_{V \rho} \sigma \beta - h^V_{\gamma \sigma \beta} h^T_{V \rho} \sigma \alpha \right) \]
\[ = v^\rho \left( \frac{D_{H \beta}}{H \rho \alpha} - \frac{D_{H \alpha}}{H \gamma \rho \beta} - h^H_{\gamma \rho \sigma} S_{H H}^{H} \sigma \alpha \beta + \frac{h^T_{V \rho \gamma \sigma}}{V} S_{V}^{H} \sigma \alpha \beta \right) \]
(37)

\[ v^\rho R \left[ H^V H^H \right]_{\rho}^{\gamma \alpha \beta} \]
\[ = v^\rho \left( \frac{D_{V \beta}}{V \gamma \rho \alpha} - \frac{D_{H \alpha}}{H \gamma \rho \beta} - h^H_{\gamma \rho \sigma} S_{H H}^{H} \sigma \alpha \beta + \frac{h^T_{V \rho \gamma \sigma}}{V} S_{V}^{H} \sigma \alpha \beta \right) \]
(38)

\[ v^\rho R \left[ H^V V^V \right]_{\rho}^{\gamma \alpha \beta} \]
\[ = v^\rho \left( - \frac{D_{V \beta}}{V \rho \rho \alpha} + \frac{D_{V \alpha}}{V \gamma \rho \beta} - h^H_{\gamma \rho \sigma} S_{V}^{V} \sigma \alpha \beta + \frac{h^T_{V \rho \gamma \sigma}}{V} S_{V}^{V} \sigma \alpha \beta \right) \]
(39)

The rest of the projections of the curvature tensor obey equations which are the complements of these.

2. Intrinsic and Cross-projected Curvature Tensors

Because each of these equations is an identity, the expressions on their right-hand sides must be strictly local in the vector field \( v \). Thus, the combinations of intrinsic and extrinsic projected derivatives which appear lead to the identification of several new tensors. Six new tensors are defined as follows: For any vector field \( v \) such that \( Zv = v \),

\[ v^\rho R_{XY \rho}^{Z} \gamma \alpha \beta = \left( [D_{Y \beta}, D_{X \alpha}] - S_{XY}^{H} \rho \alpha \beta D_{H \rho} - S_{XY}^{V} \rho \alpha \beta D_{V \rho} \right) v^\gamma \]
(40)

where each of the projection labels \( X, Y, Z \) can be either \( H \) or \( V \). Since the right-hand side of each of these equations gives zero for a vector field \( v \) such that \( \tilde{Z}v = v \), it is clear that each of these tensors belongs to a projected subspace: \( R_{XY}^{Z}(P) \in T_{P} \left[ Z_{XY}^{Z} \right] \). Notice that the first two arguments of these tensors are always in the same projected subspace. It is essential that the generalized commutation operators which define these tensors map projected subspaces into themselves.
A notation which will be used later replaces the upper projection label on the tensor $R^Z_{XY}$ by the product of two projection tensors. Thus, $R^{ZW}_{XY}$ is a tensor whose upper label is the result of the product $ZW$ when that product is a projection tensor. When $ZW$ is zero, the tensor $R^{ZW}_{XY}$ is also zero. For example:

$$R^{HH}_{HV} = R^H_{HV}, \quad R^{VV}_{HV} = R^V_{HV}, \quad R^{HV}_{HV} = 0$$

With this notation, $R^{ZW}_{XY}(P) \in T_P[z^W_{XY}]$ and there is a (partly trivial) correspondence between the projection labels and the indexes of the tensor.

Each of these tensors may also be defined for one-form fields in the same way as the full curvature tensor. For example, $R^H_{VV}$ can be defined by requiring for any $\eta$ with $\eta(P) \in H^*T_P$,

$$-\eta_\rho R^{H}_{VV\gamma}{}^{\rho}{}_{a\beta} = ([D_{V\beta}, D_{V\alpha}] - S_{VV}^H{}^{\rho}{}_{a\beta}D_{H\rho} - S_{VV}^V{}^{\rho}{}_{a\beta}D_{V\rho}) \eta_\gamma.$$  

These one-form versions of the definitions can be obtained from the vector forms by repeating the usual argument for the full curvature tensor — Let the torsion definition act on the function $\eta_{\rho}v^\rho$ where $\eta(P)$ and $v(P)$ are restricted to $H^*T_P$ and $HT_P$. Alternatively, one can decompose the one-form version of the curvature definition given by Eq. (33).

Two of these new tensors are familiar: When the projection $H$ is surface-forming, the generalized twist tensor $S^V_{HH}$ vanishes and $R^H_{HH}$ is clearly the intrinsic curvature tensor of the surface. In an adapted frame, the components of $R^H_{HH}$ are given by the familiar-looking expression

$$R^H_{HHr}{}^{c}{}_{ab} = e_b (\Gamma^c_{ra}) - e_a (\Gamma^c_{rb})$$
$$+\Gamma^s_{ra} \Gamma^c_{sb} - \Gamma^s_{rb} \Gamma^c_{sa} - \Gamma^c_{rs} \left(2\Gamma_s{}^{[ab]} - S^s{}_{ab}\right) - S^s_{ab} \Gamma^c_{rs}$$  

For the general case, we define this tensor to be the intrinsic curvature tensor of the projection tensor field $H$. Similarly, $R^V_{VV}$ is the intrinsic curvature of the projection tensor field $V$ and has an adapted frame expression which is the complement of Eq. (41).

For non-surface-forming projection-tensor fields, the intrinsic curvature tensor is still an object which has been seen before, although not in the generality which is presented here. It was developed by MacCallum in the context of three-dimensional projections into the
rest-frame subspaces of a fluid in spacetime. Collaboration between Collins and Szafron developed a scheme for classifying fluid-containing spacetimes by their intrinsic curvature tensors, and showed that the Szekeres solutions of Einstein’s equations are examples of a restricted class in this scheme.

The remaining four tensors, $R^H_{HV}, R^H_{VH}, R^V_{VH}, R^V_{HH}$ are not so familiar. They are essentially the commutators of intrinsic and extrinsic derivatives. I will call them cross-projected curvature tensors. In an adapted frame, these cross-projected curvature tensors have the expressions:

$$R^H_{HV} = e_E (\Gamma_{rd}) - e_d (\Gamma_{rE}) + \Gamma_{rd} \Gamma_{sE} - \Gamma_{sE} \Gamma_{sd}$$

$$R^H_{VH} = e_A (\Gamma_{rA}) - e_A (\Gamma_{rB}) + \Gamma_{rA} \Gamma_{sB} - \Gamma_{sB} \Gamma_{sA}$$

with $R^V_{VH}, R^V_{HH}$ given by the complement expressions. Notice that the key ingredients in these tensors are precisely those mixed adapted-frame connection coefficients $\Gamma_{sB}$ which are not identified as projection curvatures. Just as the intrinsic curvature $R^H_{HH}$ is the simplest tensor which can be constructed from the intrinsic connection coefficients $\Gamma_{rd}$, the cross-curvatures are the simplest tensors which can be constructed from the remaining connection coefficients.

The simple applications which are described in section reveal one reason that the cross-projected curvatures are unfamiliar: The traditional applications of projection tensor methods all involve situations where $VT_P$ is one-dimensional so that all of these cross-projected curvatures either vanish identically or can be expressed in terms of the projection curvatures via the Bianchi identities.

3. Full Curvature Decomposition

In terms of the intrinsic and cross curvature tensors, the decomposition of the Riemannian curvature tensor becomes:
\[R\left[H^H_{HH}\right] = R^H_{HH} \gamma_{\rho\alpha\beta} - h^T_{HH} \gamma_{\rho\alpha} h^\sigma_{\beta\rho} + h^T_{H\sigma} \gamma_{\alpha h^\sigma_{\rho\beta}} (42)\]

\[R\left[H^H_{HV}\right] = R^H_{HV} \gamma_{\rho\alpha\beta} + h^\rho_{\rho\beta} h^\gamma_{\gamma\sigma} - h^T_{H\sigma} \gamma_{\alpha h^\sigma_{\rho\beta}} (43)\]

\[R\left[H^H_{VV}\right] = R^H_{VV} \gamma_{\rho\alpha\beta} + h^\rho_{\rho\beta} h^T_{V\sigma} - h^T_{V\sigma} \gamma_{\alpha h^T_{V\sigma}} (44)\]

\[R\left[H^V_{HH}\right] = D_{H\beta} h^\gamma_{\rho\alpha} - D_{H\alpha} h^\gamma_{\rho\beta} - h^\gamma_{\rho\sigma} S^H_{HH} \gamma_{\alpha\beta} + h^T_{V\sigma} \gamma S^V_{HH} \alpha\beta (45)\]

\[R\left[H^V_{HV}\right] = D_{V\beta} h^T_{H\rho} \gamma_{\alpha\beta} + h^\rho_{\rho\sigma} S^H_{HV} \gamma_{\alpha\beta} + h^T_{V\sigma} \gamma S^V_{HV} \alpha\beta (46)\]

\[R\left[H^V_{VV}\right] = -D_{V\beta} h^T_{V\rho} \gamma_{\alpha\beta} + h^\rho_{\rho\sigma} S^V_{VV} \gamma_{\alpha\beta} + h^T_{V\sigma} \gamma S^V_{VV} \alpha\beta (47)\]

Some of these projections of the curvature tensor are familiar: Equations (42,45) are the generalizations of the Gauss-Codazzi relations. The complement of Equation (47)

\[R\left[V^H_{HH}\right] \gamma_{\rho\alpha\beta} = -D_{H\beta} h^T_{H\rho} \gamma_{\alpha\beta} + h^\rho_{\rho\sigma} S^H_{HV} \gamma_{\alpha\beta} + h^T_{V\sigma} \gamma S^V_{HV} \alpha\beta (48)\]

is almost the same as Equation (45). When the projection is normal and the connection is metric compatible, this last relation is just Equation (45) with the first two indexes reversed. In general, however, it is a necessary and independent addition to the generalized Gauss-Codazzi relations.

Contractions and anti-symmetric parts of these projections provide other useful results. From Eq. (46) one easily finds the divergence integrability condition:

\[R\left[H^V_{HV}\right] \gamma_{\rho\alpha\beta} = D_{V\beta} h^T_{V\rho} \gamma_{\alpha\beta} + D_{H\alpha} h^T_{H\rho} \gamma_{\beta\rho} - h^\gamma_{\rho\sigma} S^H_{HV} \gamma_{\alpha\beta} + h^T_{V\sigma} \gamma S^V_{HV} \alpha\beta (49)\]

4. Projections of Contracted Curvatures

Obtain the HH-projection of the Ricci curvature tensor from Eqs. (42,46) and take its complement to obtain a result which has been the basis for singularity theorems in general relativity:
\[ R[HH]_{\alpha \beta} = R_{HH \alpha \beta}^H + D_V h_H^{\sigma \alpha \beta} + D_H h_\beta \theta_V^T \]
\[ + h_H^{\rho \alpha \sigma} \left( h_T H^{\sigma \beta} - S_{HV}^{\sigma \beta \rho} \right) - h_H^{\sigma \alpha \beta} \theta_H^{\sigma} + h_V^{\rho \alpha \sigma} S_{HV}^{\sigma \beta \rho} \]

\[ R[VV]_{\alpha \beta} = R_{VV \alpha \beta}^V + D_V h_V^{\sigma \alpha \beta} + D_V h_\beta \theta_V^T \]
\[ + h_V^{\rho \alpha \sigma} \left( h_V^{\sigma \beta} - S_V^{\sigma \beta \rho} \right) - h_V^{\sigma \alpha \beta} \theta_V^{\sigma} + h_V^{\rho \alpha \sigma} S_{VV}^{\sigma \beta \rho} \]

Here \( R_{VV \rho \alpha}^V \) is the intrinsic Ricci curvature which is associated with the projection tensor field \( V \).

By contracting this last result with the projected metric tensor \( g_{VV}^{\alpha \beta} \) one finds the generalized Raychaudhuri equation:

\[ g_{VV}^{\alpha \beta} R[VV]_{\alpha \beta} = R_{VV}^V + D_V h_V^{\sigma \alpha \beta} + D_V h_\beta \theta_V^T - \theta_V^{\sigma} \theta_V^{\sigma} + h_V^{\rho \alpha \sigma} S_{VV}^{\sigma \beta \rho} \]
\[ + Q_V^{\alpha \rho \alpha \sigma} \theta_H^T + Q_V^{\alpha \rho \beta} h_V^{\sigma \beta \rho} - h_V^{\rho \alpha \sigma} S_{VV}^{\sigma \beta \rho} + h_V^{\rho \alpha \sigma} S_{VV}^{\sigma \beta \rho} \]

which governs the evolution of geodesic congruences.

The mixed projection can be obtained from Eqs. (53,47)

\[ R[HV]_{\alpha \beta} = R_{HV \alpha \beta}^H - D_V h_V^{\sigma \alpha \beta} + D_V h_\beta \theta_V^T \]
\[ - h_H^{\rho \alpha \sigma} \left( h_V^{\sigma \beta} + S_{VV}^{\sigma \beta \rho} \right) + \theta_H^{\sigma} h_V^{\alpha \beta} + h_V^{\rho \alpha \sigma} S_{VV}^{\sigma \beta \rho} \]

where \( R_{HV \rho \alpha}^H = R_{VV \rho \alpha}^V \) is the cross-projected Ricci tensor. The projections \( R[HH] \) and \( R[VH] \) may be obtained by taking the complements of these results.

Forming the scalar curvature requires the use of a tensor \( g^{\alpha \beta} \). This tensor plays the role of a metric on one-forms. However, it need not be invertible or covariantly constant. In terms of the intrinsic and cross-projected metric tensors, the scalar curvature is
\[
R = R_{HH}^H + R_{HV}^H + R_{HV}^V + R_{VV}^V
\]
\[+ g^{HH\alpha\beta} \left[ D_{H\beta} \theta^T_{V\alpha} + D_{V\sigma} h_{H\sigma\alpha\beta} - h_{H\rho\alpha\sigma} \left( S_{HV}^H \beta\rho - h_{HV} \sigma \right) \right]
\]
\[+ h_{H\sigma\alpha\beta} \theta_{H\sigma} + h_{V\alpha\rho} S_{VV}^V \sigma \rho \]
\[+ g^{HV\alpha\beta} \left[ D_{V\beta} \theta^T_{V\alpha} - D_{V\sigma} h_{V\sigma\alpha\beta} - h_{V\rho\alpha\sigma} \left( S_{HV}^V \beta\rho + h_{HV} \sigma \right) \right]
\]
\[+ \theta_{V\alpha\rho} S_{VV}^V \sigma \rho \]
\[+ g^{HH\alpha\beta} \left[ D_{H\beta} \theta^T_{H\alpha} - D_{H\sigma} h_{H\sigma\alpha\beta} - h_{VV \rho\alpha\sigma} \left( S_{HV}^H \beta\rho + h_{HV} \sigma \right) \right]
\]
\[+ \theta_{V\rho\sigma} S_{HH}^H \rho \sigma \]
\[+ g^{VV\alpha\beta} \left[ D_{V\beta} \theta^T_{V\alpha} + D_{H\sigma} h_{V\sigma\alpha\beta} - h_{VV \rho\alpha\sigma} \left( S_{HV}^V \beta\rho - h_{HV} \sigma \right) \right]
\]
\[+ h_{V\alpha\beta} \theta_{V\sigma} + h_{H\alpha\rho} S_{VV}^V \rho \sigma \]

\[(54)\]

D. Projections of Curvature Identities

1. Unprojected Identities

The curvature tensor obeys the usual identities. From the Jacobi identity which ensures consistency of covariant derivatives acting on functions and their gradients, one finds the torsion Bianchi identity:

\[ R_{[\gamma\rho\mu\nu]} + \nabla_{[\gamma} S_{\rho\mu\nu]} + S_{\rho\sigma[\gamma} S_{\sigma\rho\mu\nu]} = 0. \]

Here, I am using the usual index bracket notation to indicate total antisymmetrization. Because the curvature and torsion are already antisymmetric in their last two indexes, the antisymmetrization just generates three terms with the indexes cyclically permuted. The Jacobi identity for covariant derivatives acting on vector or form fields yields the curvature Bianchi identity in the form

\[ \nabla_{[\alpha} R_{\rho][\gamma\mu\nu]} + R_{\rho\gamma\sigma[\alpha} S_{\sigma\rho\mu\nu]} = 0. \]

Again I am using brackets to indicate antisymmetrization with vertical bars around indexes which are not included. The definition of metricity and the definition of curvature provide
still another identity

\[ R^{(\gamma\rho)}_{\alpha\beta} = \nabla_{[\alpha} Q^{\gamma\rho}_{\beta]} + \frac{1}{2} Q^{\gamma\rho}_{\sigma} S^\sigma_{\alpha\beta}. \]

The parentheses indicate symmetrization.

2. Projected Torsion Bianchi Identities

The above identities could be analyzed by projecting each identity in all possible ways and expressing the results in terms of the intrinsic and cross-projection objects which have been introduced so far. Contemplate this task briefly and notice that it will generate a very large number of terms, many of which will eventually cancel. A much more efficient procedure is to start over with the Jacobi identities for the intrinsic and extrinsic projected derivatives of functions and vectors and proceed directly to find the identities obeyed by the intrinsic and cross torsions and curvatures. All of the resulting identities turn out to have a common structure so that it is easiest to write a single general expression with variable projection labels before discussing where the individual identities come from. For projection tensors \(X, Y, Z, W\) we will establish the identity:

\[
R^{ZW}_{XY\gamma} \rho \mu \nu + R^{YW}_{ZX\nu} \rho \gamma \mu + R^{XW}_{YZ\mu} \rho \nu \gamma + DZ \gamma S^{W}_{XY} \rho \mu \nu + DY \nu S^{W}_{ZX} \rho \gamma \mu + DX \mu S^{W}_{YZ} \rho \nu \gamma \\
- S^{W}_{ZH} \gamma \sigma S_{XY} \sigma \mu \nu - S^{W}_{YH} \rho \sigma S_{ZH} \gamma \mu - S^{W}_{XH} \mu \sigma S_{YZ} \sigma \nu \gamma \\
- S^{W}_{ZV} \gamma \sigma S_{XY} \sigma \mu \nu - S^{W}_{YV} \rho \sigma S_{ZV} \gamma \mu - S^{W}_{XV} \mu \sigma S_{YV} \sigma \nu \gamma = 0
\]  

(55)

The Jacobi identity for intrinsic projected covariant derivatives acting on a function \(\phi\) establishes the consistency of the torsion definition and yields two identities. From the coefficient of \(D_{H\rho} \phi\) comes the intrinsic projected torsion Bianchi identity corresponding to \((X, Y, Z, W) = (H, H, H, H)\) in the above general expression. From the coefficient of \(D_{V\rho} \phi\) comes another identity corresponding to \((X, Y, Z, W) = (H, H, H, V)\). This expression is worth writing out and commenting on.

\[
D_{H\gamma} S^{V}_{HH} \rho \mu \nu + D_{H\nu} S^{V}_{HH} \rho \gamma \mu + D_{H\mu} S^{V}_{HH} \rho \nu \gamma \\
- S^{V}_{HH} \rho \gamma \sigma S_{HH} \sigma \mu \nu - S^{V}_{HH} \rho \sigma \gamma \mu - S^{V}_{HH} \mu \sigma S^{V}_{HH} \sigma \nu \gamma \\
- S^{V}_{HV} \gamma \sigma S_{HH} \sigma \mu \nu - S^{V}_{HV} \rho \sigma S_{HV} \gamma \mu - S^{V}_{HV} \mu \sigma S^{V}_{HH} \sigma \nu \gamma = 0
\]  

(56)
When one specializes the general expression to this case, all of the intrinsic and cross-projected curvature terms are missing because this choice of projection labels gives $XW = YW = ZW = 0$. Notice that the expression is linear and homogeneous in the generalized twist tensor $S^V_{HH} \rho_{\mu\nu}$ and can be viewed as a consequence of that object’s definition in terms of the antisymmetric derivative of $H$. The complements of these identities yield projected torsion Bianchi identities corresponding to $(X, Y, Z, W) = (V, V, V, V), (V, V, V, H)$.

Similarly, the Jacobi identity for mixed intrinsic and extrinsic derivatives acting on a function yields the cross-projected torsion Bianchi identities which correspond to the cases $(X, Y, Z, W) = (H, H, V, H), (H, H, V, V)$ the remaining identities, corresponding to $(X, Y, Z, W) = (V, V, H, V), (V, V, H, H)$ can be found by taking the complements of these. Since the identities are symmetric under cyclic permutations of the labels $X, Y, Z$, the proof of the general expression is complete.

3. Projected Curvature Bianchi Identities

The simplest way to decompose the curvature Bianchi identities is to start with the Jacobi identity for projected covariant derivatives acting on vectors or forms and use the definitions of the intrinsic and cross curvature tensors to evaluate all of the commutators. From the Jacobi identity for the intrinsic projected derivatives $D_{H\nu}$ acting on a form $\eta$ with $\eta(P) \in H^*T_P$, one finds three identities, two of which are torsion Bianchi identities which were found above and one is new — the intrinsic projected curvature Bianchi identity:

$$D_{H\gamma}R^H_{HH\delta} \rho_{\mu\nu} + S^H_{HH} \sigma_{\mu\nu} R^H_{HH} \delta^\rho_{\sigma\gamma} + S^V_{HH} \sigma_{\mu\nu} R^H_{VH} \delta^\rho_{\sigma\gamma}$$

$$+ D_{H\nu} R^H_{HH\delta} \rho_{\gamma\mu} + S^H_{HH} \gamma_{\mu} R^H_{HH} \delta^\rho_{\sigma\nu} + S^V_{HH} \gamma_{\mu} R^H_{VH} \delta^\rho_{\sigma\nu}$$

$$+ D_{H\mu} R^H_{HH\delta} \rho_{\nu\gamma} + S^H_{HH} \nu_{\gamma} R^H_{HH} \delta^\rho_{\sigma\mu} + S^V_{HH} \nu_{\gamma} R^H_{VH} \delta^\rho_{\sigma\mu} = 0$$

The complement of this expression yields the corresponding identity for the projection tensor $V$.

The Jacobi identities for mixed intrinsic and extrinsic projected covariant derivatives each yield one new projected Bianchi identity. All of these identities follow the same pattern as the
intrinsic identity above. In terms of the variable projection labels $X, Y, Z, W$ the identities are:

$$D_X R_{ZY}^W \delta^{\rho \mu} + S_{Z}^{H, \sigma} R_{H X}^W \delta^{\rho \sigma} + S_{Z}^{Y} \mu \nu R_{X}^W \delta^{\rho} \sigma \gamma + D_Y R_{XZ}^W \delta^{\rho} \gamma \mu + S_{X}^{H, \sigma} R_{H Y}^W \delta^{\rho} \sigma \nu + S_{X}^{V, \sigma} \nu \gamma R_{Y}^W \delta^{\rho} \sigma \mu + D_Z R_{YX}^W \delta^{\rho} \nu \gamma + S_{Y}^{H} \mu \gamma R_{H Z}^W \delta^{\rho} \sigma \nu + S_{Y}^{V} \nu \gamma R_{V}^W \delta^{\rho} \sigma \mu = 0$$

From the identity for two intrinsic derivatives and one extrinsic derivative acting on a form-field which assigns forms in $H^* T_P$, one finds the cross-projected curvature Bianchi identity corresponding to the projection labels $(X, Y, Z, W) = (H, H, V, H)$. The Jacobi identity for two extrinsic derivatives and one intrinsic derivative acting on a form-field yields the cross-projected curvature Bianchi identity corresponding to $(X, Y, Z, W) = (H, V, V, H)$ and so on. There are six such cross-projected curvature Bianchi identities.

### 4. Projected Curvature-Metricity Identities

Begin with the intrinsic curvature definition for vectors $v \in H T_P$

$$v^\rho R_{HH}^H \gamma^{\alpha \beta} = \left( [D_H, D_H] - S_{HH}^{H, \rho \alpha \beta} D_H^\rho - S_{HH}^{V, \rho \alpha \beta} D_V^\rho \right) v^\gamma$$

and take $v^\rho = g^{\rho \delta} \xi_H = g^{HH \rho \delta} \xi_{H^\delta} + g^{HV \rho \delta} \xi_{V^\delta}$. Use the product rule for the derivatives to obtain an expression in which the curvature operator acts directly on the forms $\xi_{H^\delta}$ and $\xi_{V^\delta}$. Equating the coefficients of $\xi_{H^\delta}$ yields the identity

$$R_{HH \rho}^H \gamma^{\alpha \beta} g^{HH \rho \delta} + R_{HH \rho}^H \delta^{\alpha \beta} g^{HH \gamma \rho} = D_H \xi_H^H \gamma^H \delta^{\beta} - D_H \xi_H^H \gamma^H \delta^{\alpha} + S_{HH}^{H, \rho \alpha \beta} \xi_H^H \gamma^H \rho + S_{HH}^{V, \rho \alpha \beta} \xi_V^H \gamma^H \rho \quad (57)$$

While the coefficients of $\eta_{V^\delta}$ yield

$$R_{HH \rho}^H \gamma^{\alpha \beta} g^{HV \rho \delta} + R_{HH \rho}^H \delta^{\alpha \beta} g^{HV \gamma \rho} = D_H \xi_H^{HV} \gamma^H \delta^{\beta} - D_H \xi_H^{HV} \gamma^H \delta^{\alpha} + S_{HH}^{H, \rho \alpha \beta} \xi_H^{HV} \gamma^H \rho + S_{HH}^{V, \rho \alpha \beta} \xi_V^{HV} \gamma^H \rho \quad (58)$$

By beginning with the cross-projected curvature definitions, one finds still more identities of this sort. There are sixteen such identities in all. All of the definitions and operations which go into deriving these identities have exactly the same structure, differing only in the
projection labels. In terms of variable projection labels, \( X, Y, Z, W \) all of the identities may be obtained from the expression:

\[
R^Z_{XY\rho} \gamma_{\alpha\beta} g^{ZW\rho\delta} + R^W_{XY\rho} \delta_{\alpha\beta} g^{ZW\gamma\rho} \\
= D_Y a Q^Z_{X\beta} - D_X b Q^Z_{Y\alpha} + S^H_{XY\rho} \alpha_{\beta} Q^Z_{W\rho\gamma\delta} + S^V_{XY\rho} \alpha_{\beta} Q^Z_{V\rho\gamma\delta}
\]

The structural similarity of these curvature-metricity identities disappears when assumptions are made about the metric, the metricity, or the torsion. For example, normal projection tensors are characterized by \( g^{VH} = 0, \quad Q^{VH} = 0, \quad Q^{VH}_V = 0 \) so that the identities which correspond to \( (Z, W) = (V, H) \) become empty.

VI. FAMILIAR APPLICATIONS

A. Perfect Fluid Thermodynamics

An earlier paper discussed projection tensor fluid dynamics, so I will not go into much detail here. However, the earlier paper made very restrictive assumptions about the geometry — a normal projection tensor in a torsion and metricity-free spacetime. It is interesting to note that those assumptions are unnecessary and do not even simplify the discussion.

A perfect fluid has the stress-energy tensor

\[
T^{\mu\nu} = p_H H^{\mu\nu} + p_V V^{\mu\nu}
\]

where \( p_H = p \) is the pressure and \( p_V = -\rho \) where \( \rho \) is the mass-energy density. The projection tensor \( H \) and its complement \( V \) are normal projection tensors as described in section or this paper. However, this fact is not needed. A projection decomposition of the conservation law \( \nabla_\mu T^{\mu\nu} = 0 \) follows directly from the contraction of Eq. (12) which, with the projection identities obeyed by the projection curvatures, yields

\[
\nabla_\mu H^{\mu\nu} = \theta^T_{H\nu} - \theta^T_{V\nu}
\]

and its complement so that

\[
\nabla_\mu T^{\mu\nu} = D_{H\nu} p_H + p_H \left( \theta^T_{H\nu} - \theta^T_{V\nu} \right) + D_{V\nu} p_V + p_V \left( \theta^T_{V\nu} - \theta^T_{H\nu} \right)
\]
The conservation law then implies the equation

\[ D_{H\nu} p_H + (p_V - p_H) \theta^T_{V\nu} = 0 \]

and its complement. As was discussed in the earlier paper, these equations are indeed Euler’s equation and the equation of continuity for fluid flow.

A curious result of the earlier paper is that the complementary pairing of pressure and energy density could be extended to a pairing of all the thermodynamic potentials by insisting that the ”Tds” equations of thermodynamics should be invariant with respect to the complement operation. The resulting pairs are \( T_H = \text{temperature} \), \( T_V = \text{baryon density} \), \( s_H = \text{entropy density} \), \( s_V = \text{chemical potential} \). In terms of these definitions, the chemical potential is defined by the relation

\[ p_H - p_V = T_H s_H - T_V s_V \]

and the first law of thermodynamics is

\[ dp_V = s_V dT_V - T_H ds_H. \]

The law of baryon conservation has the same form as the continuity equation, but without a pressure term:

\[ D_{V\nu} T_V - T_V \theta^T_{H\nu} = 0. \]

Complementation symmetry takes on the appearance of magic at this point because the complement of this last equation is recognizable as another valid thermodynamic relation, the general relativistic thermal equilibrium condition — the red-shifted temperature is a constant. As the earlier paper discussed, other thermodynamic relations may also be derived in complementary pairs.

I am a bit surprised that the simple projection-geometry form of the fluid thermodynamic equations does not depend on the use of normal projection tensors to construct the fluid stress-energy. In fact, it is also unaffected by the properties of the connection — metricity and torsion make no difference at all. Evidently, the geometrical definition of the divergence
$\theta^T_{H\nu}$ of a projection tensor field corresponds closely to what the physics of fluids requires and other aspects of the geometry do not play direct roles.

**B. Normal Projection Tensor Fields and Torsionless Metric-compatible Connections**

1. **The Simplifications**

The curvature decomposition and the various torsion and curvature identities are strongly affected by the normality of the projection tensor field as well as by the properties of the connection. For a torsion-free connection, the intrinsic torsion is zero but the cross-projected torsions are not:

$$S^V_{H\rho} = 2\omega^V_{\rho\nu}, \quad S^H_{HV\rho} = h^T_{H\nu \rho}$$

For a normal projection tensor field, $h = h^T$, and $g^V = g^H = 0$. With these specializations, the divergence integrability condition and the generalized Raychaudhuri equation become:

$$0 = D_{H[a}\theta_V^{\rho]} + D_V\beta\omega^\beta_{\rho\alpha} - h_{H\beta}^\sigma [a h_{H\rho\sigma}^\beta - h_{V[\rho | \sigma]} h_{V\alpha]}^\sigma_{\beta}$$

$$g^{VV\alpha\rho} R_{[VV]}^\alpha_{\rho} = R^V_{VV} + D_{V\alpha} \theta^\alpha_H + D_{V\beta} \theta^\beta_V - \theta_V^\sigma \theta^\sigma_V - h_{H\alpha}^\sigma h_{H\beta}^\sigma$$

The projections of the Riemann and Ricci tensors as well as the various projected Bianchi identities also simplify.

2. **Timelike Geodesic Congruences: Dust Clouds**

A cloud of freely falling particles is represented by a congruence of timelike geodesics — the world-lines of the particles. I will have nothing new to say about this well-understood system. However, its very familiarity makes it a useful illustration of projection tensor geometry.

Begin with the projection tensor field $V^\alpha = -u^\alpha u_\beta$ where $u^\alpha$ is the four-velocity of the particles. The accelerations of the particles are described by the curvature $h_V$ of this
projection tensor field. For particles in free fall,

\[ h^I_\gamma\delta = h_{\gamma\delta}^\alpha = V_{\gamma\delta}a^\alpha = 0. \]

The vanishing of the entire projection curvature tensor is a consequence of the one-dimensional nature of the projection subspaces \( V_T P \). The projection curvature associated with \( H \) does not vanish and is related to the usual twist, shear, and divergence as follows:

\[ h^T_H \gamma\alpha\beta = h_{\gamma\alpha\beta} = k_{\alpha\beta}u^\gamma. \]  \hfill (61)

\[ k_{\alpha\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\theta H_{\alpha\beta} \]  \hfill (62)

In this situation, the divergence integrability condition, Eq. (49), becomes just

\[ D_{V \beta}\omega^\beta_{H \rho\alpha} = h_{H \beta}^\sigma[a h_{H \rho\sigma}] \]

Substitute \( h_H = \theta_H + \omega_H \) on the right-side of this equation and obtain a result which is manifestly linear in the vorticity:

\[ D_{V \beta}\omega^\beta_{H \rho\alpha} = \theta_{H \beta}^\sigma[a \omega_{H \rho\sigma} + \omega_{H \beta}^\sigma[a \theta_{H \rho\sigma}] \]  \hfill (63)

The projected derivative \( D_{V \beta} \) generates just the usual Fermi derivative along the particle world-lines so this result is just the usual evolution equation for the vorticity. In detail, the definition of the projected derivative as well as Eqs. (61,62) give the result

\[ H^\sigma_{\rho} H^\tau_{\alpha} V^\beta_{\mu} V^\delta_{\beta} \nabla_{\delta} (\omega_{\sigma\tau}u^\mu) = \frac{1}{2}\left(k_{\alpha\beta}k_{\rho\sigma}u^\beta - k_{\rho\sigma}k_{\alpha\beta}u^\beta\right) \]

which collapses to

\[ H^\sigma_{\rho} H^\tau_{\alpha} u^\mu \nabla_{\mu} \omega_{\sigma\tau} = \frac{1}{2}\left(k_{\alpha\sigma}k_{\rho\tau} - k_{\rho\tau}k_{\alpha\sigma}\right) \]

or, recognizing the Fermi derivative on the left and using \( k = \theta + \omega \) on the right,

\[ \dot{\omega}_{[H_H]}_{\alpha\beta} = \theta_{\alpha\sigma}\omega^\sigma_{\rho} + \omega_{\alpha\sigma}\theta^\sigma_{\rho} \]  \hfill (64)

One remarkable (and well-known) thing about this evolution equation for the vorticity is that it depends only on the local anisotropic expansion rate of the cloud. There is no direct
dependence on the spacetime geometry. Looking back at the projection of the Riemann
tensor which gave rise to this result, Eq. (46), it can be seen that the antisymmetrization
in Eq. (49) eliminates the curvature term when it has the usual index symmetries. In the
presence of torsion, the Riemann tensor would not have all of the usual index symmetries
and there would be a direct contribution of the spacetime geometry to the evolution of the
vorticity.

Another remarkable thing about this evolution equation is that its essential structure can
be read from the general projection tensor form in equation (63). Specializing the equation
still further to obtain Eq. (64) does not yield any new insights.

Evolution equations for the shear and divergence can be obtained from the projections
of the Ricci tensor and Einstein’s field equations in their trace-reversed and projected form:

\[ R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \]

For example, the generalized Raychaudhuri equation needs the projection

\[ R[V V]_{\alpha\beta} = \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) u^\mu u^\nu u_\alpha u_\beta = -\frac{1}{2} \left( \rho + \sum p_i \right) V_{\alpha\beta} \]

where \( \rho \) is the total mass-energy density of all the matter present and \( \sum p_i \) is the sum of the
principal pressures. With all of the specializations which apply to a cloud of free particles,
the generalized Raychaudhuri equation reduces to

\[ -\frac{1}{2} \left( \rho + \sum p_i \right) = D_{V\alpha} \theta H^\alpha - h H^\alpha_{\beta} h H^{\alpha\beta}_{\sigma} \]

Notice that the one-dimensional nature of \( V \) has been used to eliminate the term \( R_{V V} \). A
short calculation gives this equation in the usual form

\[ \dot{\theta} = -\frac{1}{2} \left( \rho + \sum p_i \right) - \sigma^2 - \frac{1}{3} \theta^2 + \omega^2 \]

which shows that \( \dot{\theta} < 0 \) whenever the cloud has no vorticity and the strong energy condition
is satisfied — The cloud tends to collapse because gravitation is attractive.
3. String Clouds

In an earlier paper\(^7\), I noted that a projection-tensor formulation of fluid dynamics can be applied directly to string fluids by a simple change in the dimensionality of the projection tensors which are used. Here I apply the same technique to freely falling strings. Instead of a one-dimensional projection onto the world-lines of particles, let \( V \) be a two-dimensional projection onto the timelike world-sheets of freely falling strings. Consider a cloud of such strings in which the world-sheets do not cross each other and repeat the analysis of the previous section to determine how such a cloud can evolve.

First, think about ordinary particles again. When \( V \) projects onto the world-lines of freely falling particles, the corresponding projection curvature is zero. This result can be obtained by noting that the divergence form \( \theta_{V \alpha} \) can be expressed as the rate of change of the line element along a world-line as one moves from one dust-particle to another. The particles move so as to extremize the lengths of their world lines, which leads directly to the condition

\[
\theta_{V \alpha} = 0. \tag{65}
\]

Writing this condition in terms of the usual variables, it becomes just \( a_\alpha = 0 \) — the particles are unaccelerated. Thus, the vanishing of the projection divergence captures the essential equation of motion of a cloud of freely falling particles. The additional consequence that the entire projection curvature \( h_{V \mu \nu \alpha} \) vanishes is an accidental consequence of the low dimensionality of the projection tensor \( V \).

Now turn to freely falling strings. In this case, \( V \) projects onto a timelike two-dimensional surface. Keep Eq. (65) as the essential dynamical condition and see if it makes sense. Here, the divergence \( \theta_{V \alpha} \) expresses the rate of change of the timelike area element from one string to the next. Thus, Eq. (65) corresponds to strings which move so as to have extremal area world-sheets — just what is usually assumed for Goto-Nambu bosonic strings and certainly a reasonable generalization of timelike geodesics.\(^{34-36}\) Because the projection
curvature tensor $h_{\nu \alpha}^\mu$ now has more components than its divergence, it will not necessarily vanish. However, there is an additional requirement because the strings are assumed to be extended objects which hold together and sweep out surfaces. From the projection-tensor version of Frobenius’s Theorem, this requirement means that

$$\omega_V^{\alpha \mu \nu} = 0.$$  \hspace{1cm} (66)

Combining the two requirements (Eqs. (65, 66)) yields the most general form which the $V$ projection curvature of a freely falling string-cloud can have:

$$h_{\nu \alpha}^\mu = \sigma_V^{\alpha \mu \nu}. \hspace{1cm} (67)$$

The divergence integrability condition, Eq. (49), for this string cloud becomes

$$D_V \omega_H^\beta_\rho_\sigma - h_{\nu \sigma}^\alpha_\rho_\sigma h_{H \beta}^\rho_\rho - h_{V [\rho \sigma]}^\beta_H \sigma_{\alpha \beta} = 0$$

The last term in this expression vanished for particle clouds because the $V$-projection curvature vanished in that case. Here, the term is again zero because of Eq. (66). As a result, we simply get equation (63) again. The interpretation of the equation is slightly different because the first index on the vorticity tensor $\omega_H^{\alpha \mu \nu}$ can now take two different values.

To interpret the projection curvature of a string-cloud, choose an adapted orthonormal coordinate system with the spacelike basis vector $e_1 = s$ and a timelike vector $e_0 = u$ tangent to the string world-sheets. The projection curvature $H$ is then found to have components

$$h_H^0_{\gamma \delta} = k_{\gamma \delta}, \hspace{1cm} h_H^1_{\gamma \delta} = -b_{\gamma \delta}$$

where $k_{\gamma \delta} = \nabla u_{[HH]}_{\gamma \delta}$ is the familiar projected gradient of the timelike fluid flow vector field $u$ while $b_{\gamma \delta} = \nabla s_{[HH]}_{\gamma \delta}$ is the corresponding object — the projected gradient of the spacelike string tangent vector field $s$ — for a $t = \text{const.}$ snapshot of the string-cloud. Thus, there is both a spacelike curl $b_{[\mu \nu]} = -\omega_H^1_{\mu \nu}$ and a timelike vorticity $k_{[\mu \nu]} = \omega_{\mu \nu} = \omega_H^0_{\mu \nu}$ and these two tensors do not have separate evolution equations. Inspecting equation (63) reveals that it is an evolution equation for the string vorticity and has a term proportional to
the gradient of the string curl. Physically, this makes perfect sense: The strings can change their vorticity by winding and unwinding.

The generalized Raychaudhuri equation does not prove to be quite so useful for strings as it is for particles. For a string cloud, it takes the form

\[ V^{\alpha\rho} R_{\alpha\rho} = R_{VV}^V + D_{V\alpha} \theta_H^\alpha - h_{H\alpha}^\sigma h_{H}^{\alpha\beta}. \]

In the adapted orthonormal holonomic frame on string world-sheets, the expression becomes

\[ \dot{\theta}^0 = -p_2 - p_3 - R_{VV}^V - n \cdot \nabla \theta^1 - [\sigma^0]^2 - \frac{1}{2} (\theta^0)^2 + [\omega^0]^2 \]

\[ + [\sigma^1]^2 + \frac{1}{2} (\theta^1)^2 - [\omega^1]^2 \]

Unlike the particle case, there are no reasonable conditions under which the timelike divergence \( \theta^0 \) is guaranteed to be decreasing. The terms in the expression do make physical sense, however. Gravity continues to be purely attractive, tending to collapse the string cloud, but acts only through the transverse principal pressures. Positive intrinsic curvature of the strings tend to collapse the cloud. If the spatial divergence, \( -\theta^1 \) of the strings decreases as one moves in the positive direction along the strings, then the string tension tends to collapse the cloud just as elementary Newtonian physics would suggest. As with a particle cloud, timelike shear and divergence collapse the cloud while timelike vorticity tends to expand it.

4. Spacelike 3+1 Initial Value Analysis

Take \( H \) to be the normal projection onto a spacelike hypersurface \( \Sigma \) and project Einstein’s equations

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \]

in all possible ways. The curvature projection equations in this paper make this a relatively straightforward process.

The only unfamiliar feature of the calculation is the appearance of cross-projected curvature terms such as \( R_{HV}^H \). Some of these terms are easily disposed of by using the one-dimensional nature of \( VT_p \). The definition of the cross-projected curvatures, yield \( R_{VV}^X = 0 \)
and the curvature-metricity relations in Eq. (53) with the metricity set to zero yield $R^V_{XY} = 0$ in this one-dimensional case. To dispose of the cross-projected curvature $R^H_{HV}$, turn to the projected torsion Bianchi identity in Eq. (55) with $(X, Y, Z, W) = (V, H, H, H)$. With the unprojected torsion set to zero, and some help from Eq. (60), this identity becomes

$$R^H_{HV} \gamma^\rho_{\mu \nu} + R^H_{HV} \rho^\gamma_{\mu \nu} = D_H h^H_{\rho \mu} - D_H h^H_{\rho \nu}$$

Contract this identity and notice that the curvature-metricity identity requires $R^H_{HV} \gamma^\rho_{\mu \nu} = 0$. The resulting identity shows how to express the cross-Ricci-curvature in terms of projection curvatures.

$$R^H_{V H \gamma \mu} = D_H \theta^H_{\gamma \mu} - D_H \theta^H_{\rho \mu} + h^H_{\sigma \rho} h^V_{\sigma \gamma \mu} - \theta^H_{\rho \sigma} h^V_{\gamma \sigma \mu}$$

Once the cross-curvature terms have been eliminated from the projections of the Ricci curvature tensor, the rest of the task is familiar. It is important to realize, however, that the relation

$$h^T = h^V \theta^V$$

is needed to produce the usual simple results. This relation as well as the simplifications already used to express the cross-projected curvature terms depend on the one-dimensional nature of the projected tangent space $VT_P$. Because these relations are obviously not symmetrical between $H$ and $V$, the resulting expressions will not have complementation symmetry.

The scalar curvature expression in Eq. (54) simplifies to just

$$R = R^H_{HH} + 2D_V \theta^V_{\sigma} + 2D_H \theta^H_{\sigma} - 2 \theta^V_{\sigma} \theta^V_{\sigma}$$

and the Einstein tensor projections follow from the Ricci tensor projections given in Eqs. (50, 51, 53). The results are the familiar ones in an only slightly unfamiliar form:

$$G_{[HV]}^{\alpha \beta} = -D_H p^\beta_{\alpha}$$
\[ g^{\alpha \beta} G \left[ V V \right]_{\alpha \beta} = -\frac{1}{2} \left( R^H_{HH} + p^{\beta \rho} p_\beta^{\sigma} p_\rho^{\sigma} - \frac{1}{4} p^\sigma p_\sigma \right) \]

\[ G \left[ HH \right]_{\alpha \beta} = D V \rho^{\sigma} p_\rho^{\alpha} p_\sigma^{\beta} + U_{\alpha \beta} - U H_{\alpha \beta} + \frac{1}{2} p_{\sigma} p^{\sigma} \]

\[ + \frac{1}{2} \left( p^{\rho} \rho^{\sigma} p_\sigma^{\tau} - \frac{1}{2} p^{\tau} p^{\tau} \right) H_{\alpha \beta} + G^H_{HH} \alpha \beta \]  

(68)

where I define

\[ p_{\gamma}^{\alpha \beta} = h_{H \gamma}^{\alpha \beta} - H^{\alpha \beta} \theta_{H \gamma} \]

and

\[ U_{\alpha \beta} = D_{H \beta} \theta_{V \alpha} - \theta_{V \alpha} \theta_{V \beta}. \]

It is interesting to note that the divergence integrability condition, equation (49), ensures that the tensor \( U_{\alpha \beta} \) is symmetric and that \( \theta_{V \alpha} \) is the gradient of a scalar potential.

As has been discussed in many places, using many different approaches, it is evident that four of Einstein’s equations contain no timelike derivatives of \( p_{\alpha \beta \gamma} \) and serve only to constrain the initial value data while the remaining six can be regarded as providing the time derivatives which are needed to evolve the field. I will not complete the projection-tensor geometry version of the discussion here. Instead, I will just note what is left to be done at this point: (1) Express the projection curvatures in terms of Lie derivatives of the intrinsic metric along a timelike curve congruence. (2) Make explicit the dependence on the arbitrary choice of curve congruence (The ADM approach uses the "lapse function" and the "shift vector" for this purpose). Here we have the freedom to let \( V \) project directly onto the curve congruence by relaxing the restriction to normal projections. In that case, \( g^{VV} \) plays the role of the lapse and \( g^{HV} \) contains the shift vector. (3) Organize the resulting equations of motion into one or another constrained Hamiltonian form. (4) Discuss conditions which can be imposed in order to constrain the choice of timelike curve congruence.

**VII. DISCUSSION**

Although projection tensor techniques have often been used in general relativity, they have always been restricted in peculiar ways: Useful in the spacelike initial value problem
of general relativity but not in the characteristic initial value problem; Easily applied to the spacelike initial value Einstein equations but notoriously difficult to apply to the remaining, dynamical Einstein equations\[39\]; Useful for spacelike projections in fluids and initial value problems but not for timelike projections; Useful when the co-dimension is one but less useful otherwise. This paper shows that the restrictions have been the result of several "missing puzzle pieces" which are needed to perform straightforward calculations in projection tensor geometry. These pieces are:

- The transpose projection curvature tensor.
- The cross-projected torsion and curvature tensors.

To see what can happen, compare the calculation of Eq. (68) with the calculations which appear in the literature\[39\]. The calculation here is made easy by using the cross-projected torsion Bianchi identities to eliminate the cross-projected curvature tensor terms. Even though the cross-projected torsion and curvature do not occur in the final answer, they play an essential role in getting there and it is easy to see why the result can be tedious to obtain without them.

In addition to filling in missing pieces, this paper has extended the projection tensor approach to new situations and given it increased flexibility in familiar situations. Thus, null hypersurfaces and the characteristic initial value problem can now be fitted into the same geometrical framework that has been used for spacelike hypersurfaces. The intrinsic geometry classification of fluid-containing spacetimes developed by Collins and Szafron\[6\] can now be extended to higher dimensional cases and perhaps developed further in other ways. Even the familiar 3+1 calculations can now be done in new ways without resorting to the explicit use of coordinates. I expect to exploit some of these opportunities in later papers.
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