Violation of Luttinger’s theorem in strongly correlated electronic systems within a $1/N$ expansion

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We study the $1/N$ expansion of a generic, strongly correlated electron model ($SU(N)$ symmetric Hubbard model with $U = \infty$ and $N$ degrees of freedom per lattice site) in terms of $X$ operators. The leading order of the expansion describes a usual Fermi liquid with renormalized, stable particles. The next-to-leading order violates Luttinger’s theorem if a finite convergence radius for the $1/N$ expansion for a fixed and non-vanishing doping away from half-filling is assumed. We find that the volume enclosed by the Fermi surface, is at large, but finite $N$’s and small dopings larger than at $N = \infty$. As a by-product an explicit expression for the electronic self-energy in $O(1/N)$ is given which cannot be obtained by factorization or mode-coupling assumptions but contains rather sophisticated vertex corrections.

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1. INTRODUCTION

Luttinger’s theorem is one of the most fundamental theorems in solid state physics. It states that the volume enclosed by the Fermi surface is independent of the strength of the interaction between electrons. One definition of the Fermi surface is based on the momentum distribution function $n(k)$ of electrons. The Fermi surface is the set of all $k$-points where $n(k)$ or derivatives of it are singular. If the imaginary part of the self-energy $\Sigma(k, \omega)$ of the one-particle Green’s function $G(k, \omega)$ vanishes at $\omega = 0$ the Fermi surface can also be defined as the boundary of the set of $k$ points satisfying $ReG(k, \omega = 0) > 0$. If the electrons form a Fermi liquid $n(k)$ possesses a jump and $Im\Sigma(k,\omega) \sim \omega^2$ near the Fermi surface. Both definitions for the Fermi surface are then equivalent and Luttinger’s theorem also holds. Fermi liquid behavior of the electrons, however, is not a necessary prerequisite for the validity of Luttinger’s theorem: One-dimensional metals are Luttinger and not Fermi liquids, yet Luttinger’s theorem also holds. Fermi liquid behavior of the electrons, however, is not a necessary prerequisite for the validity of Luttinger’s theorem: One-dimensional metals are Luttinger and not Fermi liquids, yet Luttinger’s theorem also holds. Fermi liquid behavior of the electrons, however, is not a necessary prerequisite for the validity of Luttinger’s theorem: One-dimensional metals are Luttinger and not Fermi liquids, yet Luttinger’s theorem also holds. Fermi liquid behavior of the electrons, however, is not a necessary prerequisite for the validity of Luttinger’s theorem: One-dimensional metals are Luttinger and not Fermi liquids, yet Luttinger’s theorem also holds. Fermi liquid behavior of the electrons, however, is not a necessary prerequisite for the validity of Luttinger’s theorem: One-dimensional metals are Luttinger and not Fermi liquids, yet Luttinger’s theorem also holds.

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the $t-J$ model with $J = 0$, the so-called $t$ model, given, for instance, by the first term on the right-hand side of Eq.(1) of Ref\[15\]. One main ingredient of the calculation is the electron self-energy calculated in order $1/N$. Below we describe a more symmetric and transparent derivation of this quantity compared to that in Ref\[15\] which also allows to determine the $N$-dependence of the various contributions in a straightforward way. We also will discuss the change in the $n(\mu)$ relation ($n$ is the number of electrons per site, $\mu$ the chemical potential) if the self-energy contribution of $O(1/N)$ with $N = 2$ is used.

II. ELECTRONIC SELF-ENERGY IN $O(1/N)$

In this section we calculate the contribution of $O(1/N)$ to the electronic self-energy $\Sigma$. It represents the first correction beyond the mean-field approximation for $\Sigma$. We consider the $t-J$ model and use the general framework outlined in Ref\[15\]. Fermionic Hubbard operators, which annihilate or create an electron, will be denoted by $X_c(1)$ and $X_h(1)$, respectively. Writing out the arguments explicitly we would have $X^{0 \nu}(i_1 \tau_1)$ and $X^{\rho \nu}(i_1 \tau_1)$, respectively, where $p = 1, ..., N$ denotes an internal index and $i_1, \tau_1$ a lattice site and an imaginary time, respectively. The remaining bosonlike Hubbard operators will often be denoted by the letter $Y$. $Y(1)$ stands for $Y^{p_1 \nu}(i_1 \tau_1)$ with $p_1, q_1 = 0, ..., N$ with the cases $p_1 = 0, q_1 > 0$ and $p_1 > 0, q_1 = 0$ excluded. It will also be often convenient to abbreviate the index pair $i_1 \tau_1$ by $1$. All explicitly given formulas in the following refer to the $t-J$ model with $J = 0$. Using the rules discussed in Ref\[15\] it is straightforward to generate from them the corresponding contributions due to a spin-spin or a charge-charge interaction term.

We start from the equations of motion for $X$ operators,

$$\frac{\partial X_c(1)}{\partial \tau_1} = \int d3\mu_c(13)X_c(3) + \int d2d3t_c(123)Y(2)X_c(3),$$

$$\frac{\partial X_h(1)}{\partial \tau_1} = \int d3\mu_h(13)X_h(3) + \int d2d3t_h(123)Y(2)X_h(3).$$

The functions $\mu$ and $t$ in Eqs. (1) and (2) are defined by

$$\mu_c(12) = \mu(\bar{1} - 2)\delta_{q_1 q_2},$$

$$\mu_h(12) = -\mu(\bar{1} - 2)\delta_{p_1 p_2},$$

$$t_c(123) = \frac{t(\bar{1} - 3)}{N}\delta(\bar{1} - 2)[\delta_{p_2 0}\delta_{q_2 0}\delta_{q_1 q_3} + \delta_{q_1 q_2}\delta_{q_2 q_3}],$$

$$t_h(123) = -\frac{t(3 - \bar{1})}{N}\delta(\bar{1} - 2)[\delta_{p_2 0}\delta_{q_2 0}\delta_{p_1 p_3} + \delta_{p_1 p_2}\delta_{q_2 q_3}].$$

The electronic Green’s function $G$ is defined by

$$G(12) = -\langle TX_c(1)X_h(2) \rangle,$$

where $T$ is the time ordering operator. Dyson’s equation for $G$ reads:

$$G^{-1}(12) = \int d3Q^{-1}(13) [G_0^{-1}(32) + \mu_c(32) - \Sigma_c(32)].$$

The free Green’s function $G_0^{-1}$ is given by

$$G_0^{-1}(12) = -\delta(\bar{1} - 2)[\delta_{p_1 0}\delta_{p_2 0}\delta_{q_1 q_2} + \delta_{q_1 0}\delta_{q_2 0}\delta_{p_1 p_2}].$$

There is no need to specify any electron or hole index in $G_0^{-1}$ since $G_0^{-1}$ is identical in both cases. $Q$ is the spectral weight given by the expectation value of the equal-time anticommutator of the two fermionic operators of $G$, i.e., writing out explicitly the internal indices, by
Using Eq. (12) we apply the operator $G^{-1}$ in Eq. (11) to the left and act with it on the time-ordered product of Hubbard operators. Using Eqs. (8)–(10) and evaluating equal-time anticommutators we obtain

$$
\Sigma_c(12) = -\int d3d4d5t_c(1345)\langle TY(3)Y(4)\rangle Q^{-1}(52) + \int d3d4d5\lambda(1345)\langle X_h(3)X_c(4)\rangle Q^{-1}(52)
$$

(12)

where

$$
t(1342) = \frac{t(\bar{1} - 2)}{N}\delta(\bar{1} - 3)\delta(2 - 4)[\delta_{q_1,p_2}\delta_{2,0}\delta_{3,0}\delta_{4,0} + \delta_{q_1,q_4}\delta_{p_1,p_2}\delta_{p_3,0}\delta_{q_4,0} + \delta_{q_1,q_4}\delta_{p_1,p_2}\delta_{p_3,0}\delta_{q_4,0}]
$$

(13)

and

$$
\lambda(1342) = \frac{t(\bar{1} - 4)}{N}\delta(\bar{1} - 2)\delta(\bar{1} - 3)[\delta_{q_1,p_2}\delta_{p_1,q_4} - \delta_{q_1,q_4}\delta_{p_1,p_2}].
$$

(14)

In Eq. (12) we used the fact that reducible self-energy contributions cancel each other and thus can be dropped. Since this property of self-energy diagrams is not completely obvious in our case we give a proof of it in appendix A. The first two terms in Eq. (12) are frequency-independent and thus contribute only to the dispersion of quasi-particles. The third term in Eq. (12) depends on the frequency and determines the damping of the quasi-particles.

It is convenient to introduce an external source field $K$ coupled to bosonic Hubbard operators for the evaluation of the expectation values in Eq. (12) similar as in Ref. [1]. The self-energy assumes then the form

$$
\Sigma_c(12) = -\int d3t_c(132)\langle Y(3)\rangle - \int d3d4d5t(1345)\frac{\delta\langle Y(4)\rangle}{\delta K(3)} Q^{-1}(52)
$$

$$
+ \int d3d4d5\lambda(1345)\langle X(3)X(4)\rangle Q^{-1}(52)
$$

$$
+ \int d3d4d5d6d7d8d9t_c(134)t_h(956)G(47)\frac{\delta\Gamma(78;5)}{\delta K(3)}G(86)Q^{-1}(92)
$$

$$
- \int d3d4d5d6d7d8d9d10d11t_c(134)t_h(1156)G(47)\Gamma(78;5)G(89)\Gamma(910;3)G(106)Q^{-1}(112)
$$

$$
- \int d3d4d5d6d7t_c(134)t_h(756)\frac{\delta\langle Y(5)\rangle}{\delta K(3)}G(46)Q^{-1}(72).
$$

(15)

$\Gamma$ is a vertex function defined by

$$
\Gamma(12;3) = \frac{\delta G^{-1}(12)}{\delta K(3)}.
$$

(16)

We tacitly assumed in Eq. (15), and will do so also in the following, that the external source $K$ is put to zero once the functional derivatives have been carried out. The evaluation of the fourth term in Eq. (15) is somewhat more involved than that of the other terms and is therefore deferred to appendix B. In the following we often will abbreviate this term by $\delta \Sigma$. Summing over the internal indices or exhibiting them explicitly Eq. (15) becomes

$$
\Sigma^{\sigma\sigma}(12) = -\frac{t(\bar{1} - 2)}{N}(\langle Y^{00} \rangle + \langle Y^{\sigma\sigma} \rangle) + (N - 1)\frac{t(\bar{1} - 3)}{N}g(31)+\delta(\bar{1} - 2)
$$

$$
+ \frac{t(\bar{1} - 3)}{N}g(34)\frac{t(\bar{4} - 2)}{N} - \frac{t(\bar{1} - 2)}{N}(\langle Y^{00} \rangle + \langle Y^{\sigma\sigma} \rangle) \times
$$
\[
\left[ -\frac{\delta (Y^{\sigma_2 \sigma_1} (\bar{2}))}{\delta K^{00} (1)} + \frac{\delta (Y^{\sigma_2 \sigma_1} (\bar{2}))}{\delta K^{00} (1)} - \frac{\delta (Y^{\sigma_2 \sigma_1} (\bar{2}))}{\delta K^{\sigma_2 \sigma_1} (1)} + \frac{\delta (Y^{\sigma_2 \sigma_1} (\bar{2}))}{\delta K^{\sigma_2 \sigma_1} (1)} \right]
\]

\[
+ \frac{t(1 - 3)}{N} g(34) g(56) g(78) \frac{t(8 - 2)}{N} [\langle Y^{00} \rangle + \langle Y^{\sigma_2 \sigma_1} \rangle^2 [\Gamma^{00}_{\sigma_2 \sigma_1} (45; \bar{2}) \Gamma^{00}_{\sigma_2 \sigma_1} (67; \bar{1}) - \Gamma^{\sigma_2 \sigma_1}_{\sigma_2 \sigma_1} (45; \bar{2}) \Gamma^{\sigma_2 \sigma_1}_{\sigma_2 \sigma_1} (67; \bar{1})]
\]

\[
+ \Gamma^{\sigma_2 \sigma_1}_{\sigma_2 \sigma_1} (45; \bar{2}) \Gamma^{\sigma_2 \sigma_1}_{\sigma_2 \sigma_1} (67; \bar{1}) + \Gamma^{\sigma_2 \sigma_1}_{\sigma_2 \sigma_1} (45; \bar{2}) \Gamma^{\sigma_2 \sigma_1}_{\sigma_2 \sigma_1} (67; \bar{1})] + \delta \Sigma^{\sigma_2} (12). \quad (17)
\]

In Eq. (17) we have renamed the internal indices \(p, p_1, \ldots\) by \(\sigma, \sigma_1, \ldots\) where \(\sigma, \sigma_1, \ldots\) assume only values between 1 and \(N\). We also used the fact that in equilibrium \(\Sigma\) and \(g\) are diagonal and independent of spin indices, \(Y\) is diagonal and in \(\Gamma\) either the rows or the columns have the same spin indices (the upper two indices in \(\Gamma\) result from the two fermionic, the lower two indices from the bosonic Hubbard operators). Spin indices on the right-hand side of Eq. (17) which do not appear on the left-hand side of this equation are summed over. In accordance with that we explicitly show the index \(\sigma\), for instance, at the self-energy, in spite of the fact that \(\Sigma\) is independent of spin labels in equilibrium. In Eq. (17) we also have introduced a normalized Green’s function by means of

\[
G(12) = \int d3g_e(13)Q(32) = \int d3Q(13)g_h(32). \quad (18)
\]

In equilibrium we have \(g_e = g_h\) so we simply can use the letter \(g\) in Eq. (17) for convenience. Let us introduce normalized vertex functions by

\[
\gamma_e(12; 3) = \frac{\delta g_e^{-1}(12)}{\delta K(3)}, \quad (19)
\]

\[
\gamma_h(12; 3) = \frac{\delta g_h^{-1}(12)}{\delta K(3)}. \quad (20)
\]

Unlike \(g_e\) and \(g_h\), \(\gamma_e\) is not equal to \(\gamma_h\) in equilibrium. The previously defined vertex function \(\Gamma\) can now be expressed as

\[
\Gamma(12; 3) = \int d4Q^{-1}(14)\gamma_e(42; 3) + \int d4\frac{\delta Q^{-1}(14)}{\delta K(3)} g_e^{-1}(42)
\]

\[
= \int d4\gamma_h(14; 3)Q^{-1}(42) + \int d4g_h^{-1}(14)\frac{\delta Q^{-1}(42)}{\delta K(3)}. \quad (21)
\]

We stress that the two expressions for \(\Gamma\) in terms of \(e\)- or \(h\)- quantities are identical, so that we may take the more convenient one of the two whenever we use Eq. (21).

Let us introduce some short notations that will be useful later:

\[
\Pi \left( \begin{array}{cc} \sigma_2 & \sigma_3 \\ \bar{2} & \bar{1} \end{array} \right) = \frac{\delta (Y^{\sigma_2 \sigma_3} (\bar{2}))}{\delta K^{\sigma_2 \sigma_3} (1)}, \quad (22)
\]

\[
\Gamma \left( \begin{array}{cc} \sigma_2 & \sigma_3 \\ \bar{2} & \bar{3} \end{array} \right) = \Gamma^{\sigma_2 \sigma_3}_{\sigma_2 \sigma_3} (\bar{2}; \bar{3}), \quad (23)
\]

\[
\gamma \left( \begin{array}{cc} \sigma_2 & \sigma_3 \\ \bar{2} & \bar{3} \end{array} \right) = \gamma_{\sigma_2 \sigma_3}^{\sigma_2 \sigma_3} (\bar{2}; \bar{3}). \quad (24)
\]

The \(1/N\) expansion can be simplified by noting that

\[
O \left( \frac{\delta A^{\alpha \beta}}{\delta K^{\gamma \delta} \mid_{K=0}} \right) \leq O \left( A^{\alpha \beta} \mid_{K=0} \right), \quad (25)
\]

where \(A^{\alpha \beta}\) is a functional of \(g\). \(O(\ldots)\) on the left- and right-hand sides of Eq. (25) denotes always the largest order occurring in one of the tensor elements. The functional derivative on the left-hand side of Eq. (25) acts on a Green’s function \(g^{\sigma_2 \sigma_3}\) and can be written as
\[
\frac{\delta g_{\sigma_1 \sigma_2}}{\delta K_{\sigma_3 \sigma_4}} = -g_{\sigma_1 \sigma_3} g_{\sigma_2 \sigma_4},
\]

(26)

omitting space and time labels, zeros in fermionic index pairs and taking finally all quantities in equilibrium. If \( \sigma_3 = \sigma_4 \) in \( K \) the number of internal sums on the two sides of Eq. (25) is equal. Since the orders \( O(g) \) and \( O(\gamma) \) are at most 1 Eq. (25) holds. If \( \sigma_3 \neq \sigma_4 \) in \( K \) the left-hand side of Eq. (25) may contain first one sum more than the right-hand side of this equation. However, because of Eq. (26) and because of the selection rules for \( \gamma \), this additional sum does not survive when finally going to equilibrium so that Eq. (25) again holds. Using Eq. (25) the expansion of various quantities in powers of \( 1/N \) read

\[
\Sigma(\bar{1}\bar{2}) = \sum_{i=0}^{\infty} \Sigma_i(\bar{1}\bar{2}),
\]

(27)

\[
g(\bar{1}\bar{2}) = \sum_{i=0}^{\infty} g_i(\bar{1}\bar{2}),
\]

(28)

\[
\langle Y^{00} \rangle = \sum_{i=-1}^{\infty} \langle Y^{00} \rangle_i,
\]

(29)

\[
\langle Y^{\sigma \sigma} \rangle = \sum_{i=0}^{\infty} \langle Y^{\sigma \sigma} \rangle_i,
\]

(30)

\[
\Pi(\bar{2};\bar{1}) = \sum_{i=0}^{\infty} \Pi_i(\bar{2};\bar{1}),
\]

(31)

\[
\Gamma(\bar{1}\bar{2};3) = \sum_{i=1}^{\infty} \Gamma_i(\bar{1}\bar{2};3),
\]

(32)

\[
\gamma(\bar{1}\bar{2};3) = \sum_{i=0}^{\infty} \gamma_i(\bar{1}\bar{2};3).
\]

(33)

Carrying out the \( 1/N \) expansion, we find for the leading order of the self-energy:

\[
\Sigma_0^{\sigma \sigma}(\bar{1}\bar{2}) = -\frac{\langle Y^{00} \rangle}{N} - t(\bar{1} - 2) + t(\bar{1} - 3)g_0(3\bar{1}^+)\delta(\bar{1} - 2).
\]

(34)

Eq. (34) reads in Fourier space

\[
\Sigma_0(k) = -\frac{\langle Y^{00} \rangle}{N} - t(k) + \lambda_0,
\]

(35)

where

\[
\lambda_0 = \sum_k t(k)g_0(k)e^{i\omega_n 0^+},
\]

(36)

and

\[
g_0(k, i\omega_n) = \frac{1}{i\omega_n + (\langle Y^{00} \rangle N) t(k) + \mu - \lambda_0}.
\]

(37)

In Eq. (36) and in the following equations the short notation \( \sum_k \) stands for \( (T/V) \sum_{k,\omega_n} \) where \( T \) is the temperature and \( V \) the volume of the crystal.
For the order $O(1/N)$ of $\Sigma$ we obtain:

$$
\Sigma^{\sigma\sigma}_1(\bar{1}\bar{2}) = -\frac{t(\bar{1}-2)}{N}\langle Y^{00} \rangle_0 + \langle Y^{\sigma\sigma} \rangle_0 + t(\bar{1}-3) \left[ g_1(3\bar{1}\bar{1}^+) - \frac{g_0(\bar{3}\bar{1}\bar{1}^+)}{N} \right] \delta(\bar{1}-2)
$$

$$
+ \frac{t(\bar{1}-2)}{N} \left[ \frac{t(\bar{1}-3)}{N} g_0(\bar{3}4) t(4-2) \right] \left[ \Pi_0 \left( \frac{a a}{2} : 00 : 1 \right) + \Pi_0 \left( \frac{a a : \sigma}{1} : \sigma \right) - \Pi_0 \left( \frac{\sigma a : \sigma}{1} \right) \right]
$$

$$
+ \frac{t(\bar{1}-3)}{N} g_0(\bar{3}4) g_0(5\bar{6}) g_0(\bar{7}8) t(\bar{8}-2) \langle Y^{00} \rangle_2 \times
$$

$$
\left[ \Gamma_1 \left( \frac{a a : 00}{2} : \sigma \right) \Gamma_1 \left( \frac{a a : \sigma}{1} : \sigma \right) + \Gamma_1 \left( \frac{a a : \sigma}{1} : \sigma \right) \Gamma_1 \left( \frac{a a : \sigma}{1} : \sigma \right) \right]
+ \Gamma_1 \left( \frac{a a : \sigma}{1} : \sigma \right) \Gamma_1 \left( \frac{a a : \sigma}{1} : \sigma \right) \Gamma_1 \left( \frac{\sigma a \sigma}{1} : \sigma \right)
$$

$$
+ \delta \Sigma^{\sigma\sigma}_1(\bar{1}\bar{2}).
$$

(38)

We are now evaluating analytically all the unknown quantities in Eq. (38) using the relation:

$$
\langle Y^{\sigma\sigma}(\bar{1}) \rangle_{K=0} = g \left( \frac{\sigma}{\bar{1} \bar{1}+} \right) - \Pi \left( \frac{\sigma\sigma}{1} : 00 \right) \frac{1}{\langle Y^{00} \rangle} + \left[ g \left( \frac{\sigma}{1 \bar{1}+} \right) - 1 \right] \frac{\langle Y^{\sigma\sigma} \rangle}{\langle Y^{00} \rangle},
$$

(39)

and the constraint

$$
Y^{00} + \sum_{\sigma=1}^{N} Y^{\sigma\sigma} = \frac{N}{2}.
$$

(40)

Performing explicitly the $1/N$ expansion, we obtain

$$
\langle Y^{00} \rangle_{-1} = \frac{N}{2} - N \langle Y^{\sigma\sigma} \rangle_0,
$$

(41)

$$
\langle Y^{00} \rangle_0 = -N \langle Y^{\sigma\sigma} \rangle_1,
$$

(42)

$$
\langle Y^{\sigma\sigma} \rangle_0 = g_0 (\bar{1}\bar{1}^+),
$$

(43)

$$
\langle Y^{\sigma\sigma} \rangle_1 = g_1 (\bar{1}\bar{1}^+) + \frac{1}{\langle Y^{00} \rangle_{-1}} \left[ -\Pi_0 \left( \frac{\sigma\sigma}{1} : 00 \right) g_0^2 (\bar{1}\bar{1}^+) \right],
$$

(44)

$$
\Pi_0 \left( \frac{a a}{2} : 00 \right) = -g_0(2\bar{3}) \gamma_{e,0} \left( \frac{a a}{3 \bar{4}} : 1 \right) g_0(2\bar{2}^+),
$$

(45)

$$
\Pi_0 \left( \frac{a a : \sigma}{1} : \sigma \right) = -g_0(2\bar{3}) \gamma_{e,0} \left( \frac{a a : \sigma}{3 \bar{4}} : 1 \right) g_0(2\bar{2}^+),
$$

(46)

$$
\Pi_0 \left( \frac{\sigma a}{2} : \sigma \right) = -g_0(2\bar{3}) \gamma_{e,0} \left( \frac{\sigma a : \sigma}{3 \bar{4}} : 1 \right) g_0(2\bar{2}^+),
$$

(47)

$$
\gamma_{e,0} \left( \frac{\sigma}{\bar{1} \bar{2}} : 00 \right) = -\delta(\bar{1} - 2) \delta(\bar{1} - 3) + \left[ \delta(\bar{1} - 2) t(\bar{1} - 4) + t(\bar{1} - 2) \delta(\bar{1} - 4) \right]
$$

$$
\times g_0(4\bar{5}) \gamma_{e,0} \left( \frac{\sigma}{\bar{5} \bar{6}} : 00 \right) g_0(6\bar{1}^+),
$$

(48)

$$
\gamma_{h,0} \left( \frac{\sigma}{\bar{1} \bar{2}} : 00 \right) = -\delta(\bar{1} - 2) \delta(\bar{1} - 3) + g_0(2\bar{4}) \gamma_{h,0} \left( \frac{\sigma}{\bar{4} \bar{5}} : 00 \right) g_0(5\bar{6}^+)
$$

$$
\times \left[ \delta(\bar{1} - 2) t(\bar{5} - 2) + t(\bar{1} - 2) \delta(\bar{5} - 2) \right],
$$

(49)

$$
\gamma_{e,0} \left( \frac{\sigma a}{3 \bar{3}} : \sigma \right) = \delta(\bar{1} - 2) \delta(\bar{1} - 3),
$$

(50)
the expectation values are given by Eqs. (43) and (44). Writing

$$E_{\gamma,0} \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) = 0, \quad (51)$$

$$\Gamma_1 \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) = \frac{1}{(Y_{00})_{-1}} E_{\gamma,0} \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) + \frac{N}{(Y_{00})_{-1}^2} \Pi_0 \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) g_{0^{-1}}(12)$$

$$= \frac{1}{(Y_{00})_{-1}} E_{\gamma,0} \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) + \frac{N}{(Y_{00})_{-1}^2} g_{0^{-1}}(12) \Pi_0 \left( \frac{\sigma}{2}, \frac{\sigma}{3} \right), \quad (52)$$

$$\Gamma_1 \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) = \frac{1}{(Y_{00})_{-1}} E_{\gamma,0} \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right), \quad (53)$$

$$\Gamma_1 \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) = + \frac{N}{(Y_{00})_{-1}} \Pi_0 \left( \frac{\sigma}{12}, \frac{\sigma}{3} \right) g_{0^{-1}}(12) = 0. \quad (54)$$

Furthermore $g_1$ is related to $g_0$ by

$$g_1(12) = g_0(13) \Sigma_1(34) g_0(42), \quad (55)$$

which yields a self-consistent equation for $g_1$, or, equivalently, for $\Sigma_1$.

Using Eqs. (53), we obtain for $\Sigma_1^{\sigma \sigma}$:

$$\Sigma_1^{\sigma \sigma}(12) = \frac{1}{\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta}} \sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

$$\sum_{\gamma} \sum_{\alpha} \sum_{\beta} \sum_{\delta} \left[ \gamma_{00} \left( \frac{\gamma}{3}, \frac{\gamma}{3} \right) g_{0^{-1}}(12) \right]$$

In Eq. (56) we have used the explicit expression for the term $\delta \Sigma_1^{\sigma \sigma}(12)$ calculated in appendix B where the function $h(45;12)$ is also defined.

The expectation value of the particle number operator with spin $\sigma$ is $n_\sigma = \left( Y_{00} \right)_{0}^{\sigma} + \left( Y_{00} \right)_{1}^{\sigma} = n_{\sigma} + n_{1\sigma}$, where the expectation values are given by Eqs. (13) and (14). Writing $\gamma(12;3) = \gamma(1-2,1-3)$, $\Pi(12) = \Pi(1-2)$ and $h(12;34) = h(1-2,4-1,1-3)$, performing Fourier transforms, and adding the contributions of $O(1)$ and $O(1/N)$ the self-energy becomes

$$\Sigma^{\sigma \sigma}(k) = \frac{N-1}{N} \lambda_0 + \delta \lambda_1 + \left[ -1 + \frac{N-1}{N} \sum_{q}^{2} t(k+q) \Delta \Pi(q) - \frac{1}{N} \sum_{q}^{2} t^2(k+q) g_0(k+q) \Delta \Pi(q) + \frac{1}{N} \sum_{q}^{2} t(k+q) \Delta \Pi(q) \right]$$

$$\sum_{q}^{2} t(k+q) \Delta \Pi(q)$$

$$\sum_{q}^{2} t(k+q) \Delta \Pi(q)$$

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where $\Delta \Pi(q) = \Pi_{00}^{\sigma \sigma}(q) - \Pi_{00}^{\sigma \sigma}(q)$ and

$$\delta \lambda_1 = \sum_{k} t(k) g_1(k) e^{i \omega_0 t} = \sum_{k} t(k) g_0^2(k) \Sigma_1(1). \quad (58)$$

Here we wrote the second index pair of $\Pi(q)$ as a subscript to simplify the notation. Explicit expressions for the functions $\gamma$, $\Pi$, $h$ are
\[ \gamma_{e,0}(k, q) = -\frac{1 + b(q) - a(q)t(k)}{[1 + b(q)][1 + b(-q)] - a(q)c(q)}, \quad (59) \]

\[ \gamma_{h,0}(k, q) = -\frac{1 + b(-q) - a(q)t(k + q)}{[1 + b(q)][1 + b(-q)] - a(q)c(q)}, \quad (60) \]

\[ \Pi_{00}^\sigma(q) = -\frac{a(q)}{[1 + b(q)][1 + b(-q)] - a(q)c(q)}, \quad (61) \]

\[ \Pi_{00}^{\sigma\alpha}(q) = a(q), \quad (62) \]

\[ h(k_1, q, k_2) = \gamma_{e,0}(k_1, q)g_0(k_2) \left[ t(k_2) - \frac{c(q)}{1 + b(q)} \right], \quad (63) \]

and for the susceptibilities \( a, b, c \)

\[ a(q) = -\sum_k g_0(k)g_0(k + q), \quad (64) \]

\[ b(q) = -\sum_k t(k)g_0(k)g_0(k + q), \quad (65) \]

\[ c(q) = -\sum_k t(k)t(k + q)g_0(k)g_0(k + q). \quad (66) \]

Note that \( \gamma_{e,0}(p, q) = \gamma_{h,0}(p + q, -q) \).

Finally, the relation between the particle number \( n_\sigma \) and the chemical potential becomes, using Eq.(64),

\[ n_\sigma = n_{0\sigma}\left(1 - \frac{2 - 2n_{0\sigma}}{N(1 - 2n_{0\sigma})}\right) - \frac{2}{N(1 - 2n_{0\sigma})} \sum_q \Pi_{00}^{\sigma\sigma}(q) + \delta n_{1\sigma}, \quad (67) \]

where \( \delta n_{1\sigma} \) is defined by

\[ \delta n_{1\sigma} = \sum_k g_1(k)e^{i\omega_n0^+} = \sum_k g_0^2(k)\Sigma_1^{\sigma\sigma}(k). \quad (68) \]

Eq.(67) gives an explicit expression for the self-energy taking into account the orders \( O(1) \) and \( O(1/N) \) of the \( 1/N \) expansion. The first two terms represent renormalizations of the chemical potential. The third and fourth terms are frequency-independent and thus contribute only to the dispersion of quasi-particles. Terms 5-7 are frequency-dependent, contribute to the damping of quasi-particles, and cannot simply be obtained by a factorization or mode-coupling assumption applied to Eq.(1).

Eq.(67) represents the relation between the particle number and the chemical potential. As can be seen from Eqs.(63) and (64), \( n_\sigma \) is not only determined by the zeroth and first orders of the normalized Green’s function \( g \) but by additional terms. The appearance of such additional terms is not unexpected because \( n_\sigma \) is given by the normalized Green’s function \( g(\Pi^+) \) at \( N = \infty \) and by the Green’s function \( G(\Pi^+) \) at \( N = 2 \) so that the \( 1/N \) expansion must interpolate between these two cases.

### III. VIOLATION OF LUTTINGER’S THEOREM IN O(1/N)

Eq.(67) shows that the imaginary part of the self-energy vanishes at the chemical potential. There is thus a Fermi surface \( \{ k_F \} \) determined by the condition

\[ \mu = \Sigma(k_F, \omega = 0, \mu), \quad (69) \]
where the chemical potential $\mu$ is determined by the particle number $n = Nn_\sigma$ per site. $n_\sigma$ does not depend on the spin direction $\sigma$ in the normal state, of course. Nevertheless, it is convenient to denote the particle number per spin direction by $n_\sigma$ to distinguish it from the total particle number $n$. In Eq. (69) we have written explicitly the dependence of $\Sigma$ on the wave vector $k$, the frequency $\omega$ and the chemical potential $\mu$ and omitted unnecessary spin labels.

In the unconstrained, free case the self-energy $\Sigma$ in Eq. (59) should be replaced by $-t(k)/N$. Each state $k$ can be occupied by $N$ particles. The set of Hamiltonians with different $N$’s used for the $1/N$ expansion should correspond to the same $n_\sigma$. As a result the free Fermi surface $\{k_F\}$ is independent of $N$ though the corresponding chemical potential $\mu$ and the electron dispersion depend on $N$. Turning on the interaction between electrons the Fermi surface depends in general on $N$. Luttinger’s theorem states, however, that the volume enclosed by the Fermi surface is independent of $N$ and also of the interaction strength.

In the constrained case one expands for a fixed $n_\sigma$ the quantities $\mu$, $\Sigma$, and $k_F$ in powers of $1/N$ similar as in Eqs. (27)-(33). Using Eqs. (34) and (41) the zeroth order of Eq. (69) is

$$\mu_0 = -\left(\frac{1}{2} - n_{0\sigma}\right)t(k_0F) + \lambda_0(\mu_0).$$

(70)

Eqs. (57) and (13) yield at $T = 0$

$$n_{0\sigma} = \sum_k \Theta(\mu_0 - \lambda_0 + \frac{1}{2} - n_{0\sigma})t(k).$$

(71)

For a fixed $n_{0\sigma}$ Eq. (71) determines $\mu_0 - \lambda_0$ and Eq. (70) $k_{0F}$. These equations are the same as in the unconstrained, free case if we choose $(\frac{1}{2} - n_{0\sigma})t(k)$ for the dispersion and renormalize the chemical potential appropriately. This means that at $N = \infty$ the Fermi surface is identical to that of the free, unconstrained case and that, in particular, Luttinger’s theorem holds. This result is not very surprising because at $N = \infty$ the particles are renormalized but do not interact with each other.

The first-order of Eq. (69) in $1/N$ is given by

$$\frac{\partial \Sigma_0(k_{0F}, \mu_0)}{\partial \mu_0}, \mu_1 + \frac{\partial \Sigma_0(k_{0F}, \mu_0)}{\partial k_{0F}} \cdot k_{1F} + \Sigma_1(k_{0F}, \omega = 0, \mu_0).$$

(72)

Here $k_{1F}$ is assumed to be parallel to the vector $\partial \Sigma_0/\partial k_{0F}$ with $k_{0F}$ as origin. Because $\Sigma_0$ is independent of frequency we have dropped the frequency argument in $\Sigma_0$ for convenience. The change $\mu_1$ in the chemical potential is to be calculated for a fixed $n_\sigma$, i.e., from

$$\left(\frac{\partial n_{0\sigma}}{\partial \mu_0}\right) \cdot \mu_1 + n_{1\sigma}(\mu_0) = 0.$$  

(73)

Eq. (72) must hold for any point $k_{0F}$ on the Fermi surface. This equation thus determines $k_{1F}$ as a function of $k_{0F}$ and thus the change of the Fermi surface in $O(1/N)$. Note that due to the last term in Eq. (72) the shape of the Fermi surface will in general change by the interaction between the particles.

Eq. (72) can be simplified in the following way. $\Sigma$ depends on $\mu$ only via $\mu - \lambda$ where $\lambda$ is again a function of $\mu$. Defining a renormalized chemical potential $\tilde{\mu}$ by

$$\tilde{\mu} = \mu - \lambda,$$

(74)

we can drop $\lambda$ in $\Sigma$ everywhere replacing $\mu$ by $\tilde{\mu}$. Eq. (73) also holds if $\mu_0$ and $\mu_1$ are replaced by $\tilde{\mu}_0$ and $\tilde{\mu}_1$, respectively, yielding

$$\left(\frac{\partial n_{0\sigma}}{\partial \tilde{\mu}_0}\right) \cdot \tilde{\mu}_1 + n_{1\sigma}(\tilde{\mu}_0) = 0.$$  

(75)

It is then easy to see that the first term in Eq. (74), written in terms of $\tilde{\mu}$, cancels the term $n_{1\sigma}t(k)$ in Eq. (74). Eq. (72) thus becomes

$$\tilde{\mu}_1 = \frac{\partial \Sigma_0(k_{0F}, \tilde{\mu}_0)}{\partial k_{0F}} \cdot k_{1F} + \tilde{\Sigma}_1(k_{0F}, \omega = 0, \tilde{\mu}_0),$$

(76)

where $\tilde{\Sigma}_1$ is given by Eq. (57), with all terms in the first line on the right-hand side dropped except for the term $-n_{0\sigma}t(k)/N$. According to Eq. (71) the derivative $\partial n_{0\sigma}/\partial \tilde{\mu}_0$ gives rise to two contributions, one in which the derivative acts on $\tilde{\mu}_0$ and one, where it acts on $n_{0\sigma}$. We also split $n_{1\sigma}$, given by Eq. (67), into three contributions,
\[ n_{1\sigma} = \bar{n}_{1\sigma} + \delta \bar{n}_{1\sigma} + \delta n_{1\sigma}, \]  
\[ \bar{n}_1 = \frac{4n_{0\sigma}(1-n_{0\sigma})}{N(1-2n_{0\sigma})}(1 + \sum_q \Pi_{00}^{\text{two}}(q)), \]  
\[ \delta \bar{n}_{1\sigma} = \sum_k g_0^2(k) \Sigma_1^{\sigma\sigma}(k), \]  
\[ \delta n_{1\sigma} = \sum_k g_0^2(k) n_{1\sigma} t(k). \]

Using these results Eq. (77) can be rewritten as
\[ \frac{\partial n_{0\sigma}}{\partial \mu_0} |_{\text{ex}} \cdot \bar{\mu}_1 + \sum_k g_0^2(k) \left( \frac{\partial n_{0\sigma}}{\partial \mu_0} \right) t(k) \bar{\mu}_1 + \delta \bar{n}_{1\sigma} + \delta n_{1\sigma} = 0, \]  
where \( |_{\text{ex}} \) means that the derivative should be taken only with respect to the explicit dependence on \( \mu_0 \). From Eq. (77) and (80) it follows that the second and fifth terms on the left-hand side of Eq. (81) cancel each other. Solving Eq. (81) for \( \bar{\mu}_1 \) we insert its solution into Eq. (76) and average the resulting equation over the zeroth-order Fermi surface and obtain
\[ -\bar{n}_{1\sigma} - \delta \bar{n}_{1\sigma} = \sum_k \delta (\epsilon_k - \bar{\mu}_0) \frac{\partial \bar{\mu}_1}{\partial k} \cdot k_{1F} + \sum_k \delta (\epsilon_k - \bar{\mu}_0) \bar{\Sigma}_1(k, \omega = 0, \bar{\mu}_0), \]

where \( \epsilon_k \) denote the zeroth-order one-particle energies. The first term on the right-hand side of Eq. (82) is equal to the Fermi surface integral \( \int d\mathbf{S}_F \cdot k_{1F} = V_{1F} \), where \( \mathbf{S}_F \) is a directed element of the Fermi surface and \( V_{1F} \) is the change in \( O(1/N) \) of the volume enclosed by the Fermi surface. Eq. (77) becomes finally
\[ V_{1F} = -\bar{n}_{1\sigma} - \sum_k g_0^2(k) \left( \bar{\Sigma}_1(k, \bar{\mu}_0) - \bar{\Sigma}_1(k, \omega = 0, \bar{\mu}_0) \right). \]

The first term on the right-hand side of Eq. (83) is due to the unusual relation between the particle occupation \( n_{0\sigma} \) and the chemical potential \( \bar{\mu}_0 \) specified in Eq. (17). In Fermi liquid theory \( n_{0\sigma} \) is given by the one-particle Green’s function integrated over all momenta and over frequency up to \( \bar{\mu}_0 \). Such a relation holds in our case only at \( N = \infty \) and \( N = 2 \) with \( g \) and \( G \) to be used as Green’s function, respectively. The \( 1/N \) expansion must interpolate between these two cases which is the origin of the more involved nature of Eq. (83). Thus the first term in Eq. (83) is intimately linked to the \( 1/N \) expansion and of fundamental nature. In Fermi liquid theory it is, of course, absent.

In the second term on the right-hand side of Eq. (83) any frequency-independent contribution to \( \bar{\Sigma}_1 \) drops out. This term has the same form as the term in Fermi liquid theory where the product of the frequency derivative of the self-energy and the Green’s function is integrated over (second term in the square bracket in Eq. (19.12) in Ref. [17]). The only difference is that in our case the involved Green’s function is the zeroth-order Green’s function \( g_0 \) and \( \Sigma \) is a functional of \( g_0 \). This term vanishes in Fermi liquids for general reasons yielding Luttinger’s theorem. Though we are presently unable to prove something similar for the second term in Eq. (83) it is easy to see that it cannot cancel in general the first term on the right-hand side of Eq. (83). From Eq. (83) follows that \( \bar{n}_1 \) diverges in the limit \( n_{0\sigma} \to 1/2 \). In contrast to that the second term in Eq. (83) is clearly regular in that limit. From this one concludes that there exists at least a finite, \( N \)-independent interval in doping near half-filling where the first term in Eq. (83) dominates and makes \( V_{1F} \) non-zero. Assume now that \( V_{1F} \) is zero in an arbitrarily small interval of the doping \( \delta = 1 - 2n_{0\sigma} \). This implies that the regular part of \( V_{1F} \) is equal to \( A/\delta \) with some constant \( A \) throughout this interval. Performing an analytic continuation in the variable \( \delta \) in \( \Sigma_1 \) and \( V_{1F} \) shows that the regular part should be given by \( A/\delta \) in the whole interval \( 0 < \delta < 1 \) if there is no phase transition present making the interchange of sums over \( k \) and the derivation in \( \delta \) impossible. The regular part in \( k_{1F} \) would then develop a singularity at \( \delta = 0 \) in contradiction to our assumptions. Since detailed calculations show that at \( U = \infty \) and large \( N \)’s there are no phase transitions one concludes that the Luttinger theorem must be violated for dopings throughout the entire interval \( 0 < \delta < 1 \).

Fig. 1 shows the dependence of the sum of \( n_0 \) and \( n_1 \) as a function of \( n_0 \) for \( N = \infty \) (dashed line) and \( N = 16, 8, 4, \) and \( 2 \) (solid lines). \( n_1 \) is always negative which means that for a given \( n_0 \) or \( \bar{\mu}_0 \) less particles can be accommodated at
FIG. 1. Sum of the $O(1)$ and $O(1/N)$ contributions $n_0$ and $n_1$, respectively, to the site occupancy as a function of $n_0$ for different $N$'s. Only terms which diverge near half-filling have been considered in $n_1$.

a site if electronic correlations are taken into account compared to the uncorrelated case. This effect is very small at small $n_0$’s but increases monotonously towards half-filling where $n_1$ diverges. For $N = 2$ a maximal site occupation of only about 0.4 is possible. For larger $\mu_0$ the $n(\mu_0)$ relation has a maximum and then a negative slope signalizing an unstable homogenous state. This result clearly shows that a calculation which includes only $O(1)$ and $O(1/N)$ contributions cannot be used to describe the physical case of $N = 2$ near half-filling. With increasing $N$ the maximal site occupancy increases monotonously towards 1 and the maximum of the curves is shifted towards $n_0 = 1$.

Fig. 2 shows the total volume $V_F$ enclosed by the Fermi surface and calculated in $O(1)$ and $O(1/N)$ keeping the singular first term in Eq.(83). The dashed line corresponds to $N = \infty$ and represents exactly the volume predicted by the Luttinger theorem. The solid lines describe the cases $N = 16, 8, 4,$ and 2. The Figure shows that electronic correlations, taken into account in $O(1/N)$, always increase $V_F$. This increase is small at small electron concentrations but becomes larger and larger at small dopings. Our results are asymptotically exact at sufficiently large $N$’s and in a finite, $N$-independent doping interval near half-filling. As argued above this implies that Luttinger’s theorem is violated throughout the entire doping interval $0 < \delta < 1$. This means that the solid curves for large $N$’s are never coincident with the dashed line in Fig. 2 and that these lines are at least qualitatively a good approximation for all dopings. Fig. 2 illustrates also the fact that increasing electronic correlations increases the Luttinger volume $V_F$, i.e., $V_F$ is positive. The solid lines in Fig. 2 diverge towards half-filling due to the $1/N$ expansion of the reciprocal spectral weight $Q^{-1}$. The convergence radius of this expansion is determined by $1/N < \delta/n_0$ and thus becomes very small at small dopings. This means that for a fixed $N$ the power expansion in $1/N$ breaks down at small dopings. The larger $N$ is the smaller is the critical value for the doping where this expansion breaks down.

FIG. 2. Sum of the $O(1)$ and $O(1/N)$ contributions to the Luttinger volume $V_F$ as a function of the doping $\delta = 1 - n_0$ keeping only divergent terms in the $O(1/N)$ contribution. The dashed line corresponds to $N = \infty$ and obeys Luttinger’s theorem.
IV. CONCLUSION

Using a $1/N$ expansion we have derived explicit expressions for the $O(1)$ and $O(1/N)$ contributions, $\Sigma_0$ and $\Sigma_1$, respectively, for the self-energy of a $SU(N)$ symmetric Hubbard model with $N$ degrees of freedom and $U=\infty$. Using previously derived rules these results can immediately be generalized to the case of a finite, but large $U$. We find that the frequency-dependent part of $\Sigma$ is regular in the doping $\delta$, whereas the frequency-independent part develops a singularity in $\delta$ due to the expansion of the inverse spectral weight $Q^{-1}$ in powers of $1/N$. This means that near half-filling large $N$’s are required to obtain quantitative results. As an example we find that the $n(\mu)$ curve, calculated with $\Sigma_0$ and $\Sigma_1$ and $N=2$, becomes unreliable already near and below a doping $\delta \sim 0.20$. Our derivation of $\Sigma$ also shows that $\Sigma_1$ involves rather sophisticated vertex corrections and that usual many-body assumptions (neglect of vertex corrections, lowest-order iterations of the functional equations for the self-energy and the vertex) cannot be used in order not to miss cancellation effects. These features are connected to the fact that the hopping $t$ cannot be considered as a small parameter in the perturbation expansion in the kinetic energy of the electrons, at least in the metallic state. This is related to the fact that $t$ appears not only in the numerator but also in the denominator of the perturbation series because the dispersion of the one-particle energies is also determined by $t$.

Using the above results we have studied the $1/N$ expansion for the Luttinger volume $V^F = V_0^F + V_1^F + ...$ where $V^F$ denotes the volume enclosed by the Fermi surface. The $O(1)$ contribution $V_0^F$ obeys Luttinger’s theorem but this theorem is violated already in $O(1/N)$. We have shown that $V_1^F \neq 0$ throughout the entire doping interval $0 < \delta < 1$, if the normal state is stable, and that $V_1^F > 0$ at least in a finite, $N$-independent interval near half-filling. Keeping only terms which are leading at small dopings our calculations suggest that $V_1^F > 0$ for all dopings and that $V_1^F$ becomes very small at large dopings. There are also good reasons to believe that the normal state to which our discussions is restricted, can be considered as the stable state at large $N$’s throughout the interval $0 < \delta < 1$. The largest transition temperatures to superconducting states behave roughly as $\sim U^{-2}$ at large $U$’s. Strictly speaking, this is not entirely true because extremely weak superconducting instabilities have been found even for $U = \infty$. The corresponding transition temperatures are, however, astronomically small so that these instabilities can be neglected for our purposes. In a similar way all the other instabilities of the $t - J$ model at large $N$’s such as flux phases, bond-order or charge density waves, vanish in the limit $J \rightarrow 0$ or $U \rightarrow \infty$. This means that we have indeed dealt with the most stable ground state of our system.

Finally we want to point out that our results agree with two recent numerical studies of the Luttinger volume in the Hubbard model. In Ref.\(^3\) Quantum Monte-Carlo simulations for the 2D Hubbard model with $U/t = 8$ and $T/t = 0.33$ have been preformed and an increase in $V_F$ beyond the value predicted by the Luttinger theorem has been found for $\delta \leq 0.2$. Since at these high temperatures structures due to the quasi-particle peak and the incoherent background cannot be distinguished the interpretation of the results is not unique: a) $V_F$ may be the volume enclosed by the true Fermi surface or b) $V_F$ is a volume enclosing a “spectral weight Fermi surface” and thus be related to the spectral distribution of the whole Green’s function. Similar results, subject to a similar uncertainty in the interpretation, have been found in Ref.\(^4\) using high-temperature expansions and considering the momentum distribution function. In our case it is clear that only the case a) can apply and that $V_F$ denotes the volume enclosed by a constant energy contour of quasi-particles.

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APPENDIX A: CANCELLATION OF REDUCIBLE SELF-ENERGY CONTRIBUTIONS

In Eq. (15) we have omitted the following reducible self-energy contributions:

\[
\Sigma_{\text{red}}^e(12) = -\int d1'd2'd3'd4't_e(132')(Y(3))G(2'1')(Y(3'))\mu_h(43'1')Q^{-1}(12) = \int d1'd2'd3'd4'd5'd6't_e(132')(Y(3))G(2'1')\Gamma(1'4'; 3')G(4'5')t_h(63'5')Q^{-1}(12)
\]

\[
+ \int d1'd2'd3'd4'd5'd6't_e(134)G(45)\Gamma(52'; 3)G(2'1')\Gamma(1'4'; 3')G(4'5')t_h(63'5')Q^{-1}(12)
\]

\[
- \int d1'd2'd3'd4'd5'd6't_e(134)G(45)\Gamma(52'; 3)G(2'1')\Gamma(1'4'; 3')G(4'5')t_h(63'5')Q^{-1}(12)
\]

\[
- \int d1'd2'd3'd4'd5'd6't_e(134)G(45)\Gamma(52'; 3)G(2'6)Q^{-1}(61')\Sigma_e(1'2)
\]

\[
+ \int d1'd2'd3'd4'd5'd6't_e(132')(Y(3))G(2'4)Q^{-1}(61')\Sigma_e(1'2).
\]

(A1)

In equilibrium the factors \( Q^{-1}(12) \) yield a delta function \( \delta(\hat{t} - \hat{\bar{t}}) \) and a constant factor which can be taken outside of the integrals. Writing the expectation value of three Hubbard operators in Eq. (15) in terms of a functional derivative and using Eq. (10) we obtain from Eq. (11)

\[
\Sigma_e(12) = -\int d3t_e(132')(Y(3)) + \int d3d4d5t_e(134)G(45)\Gamma(52; 3).
\]

(A2)

An alternative procedure to calculate \( G \) uses the equation of motion Eq. (2) instead of Eq. (1). Eqs. (8) is then replaced by the equivalent equation

\[
G^{-1}(12) = \int d3[G_0^{-1}(13) - \mu_h(13) + \Sigma_h(13)]Q^{-1}(32),
\]

(A3)

and Eq. (A2) by

\[
\Sigma_h(12) = -\int d3t_h(231)(Y(3)) + \int d3d4d5\Gamma(14; 3)G(45)t_h(235).
\]

(A4)

In view of Eqs. (4) and (5) Eqs. (8) and (A4) yield \( \Sigma_h(12) = -\Sigma_e(12) \). Using this identity as well as Eqs. (A2) and (A4) Eq. (A1) can be written as

\[
\Sigma_{\text{red}}^e(12) = -Q^{-1}\int d1'd2'\Sigma_e(12')G(2'1')\Sigma_h(1'2) - Q^{-1}\int d1'd2'\Sigma_e(12')G(2'1')\Sigma_e(1'2) = 0.
\]

(A5)

We thus have shown that the reducible self-energy terms cancel each other and do not contribute to the self-energy.
APPENDIX B: EVALUATION OF $\delta \Sigma$

Let us define a function $F$ by

$$F(12,34) = \frac{\delta^2 G^{-1}(12)}{\delta K(3)\delta K(4)} = \frac{\delta \Gamma(12;3)}{\delta K(4)} = \frac{\delta \Gamma(12;4)}{\delta K(3)}.$$  \hfill (B1)

The equilibrium contribution $\delta \Sigma$ can then be written as

$$\delta \Sigma^{\sigma\sigma}(12) = -\int \frac{d^3d4d5d6}{N} \frac{t(1-3)}{N} g(34)g(56) \frac{t(6-2)}{N} \times [(Y^{00}) + (Y^{\sigma\sigma})]$$

$$\left[ F\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{\sigma}{0} \frac{0}{1}\right) + F\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{\sigma}{1} \frac{0}{2}\right) + F\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{0}{0} \frac{\sigma}{0}\right) + F\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{\sigma}{2} \frac{0}{1}\right) \right].$$  \hfill (B2)

where we have written out the spin labels and summation over indices which do not appear on the left-hand side of this equation is assumed. It is also convenient to introduce the function

$$f(12,34) = \frac{\delta^2 g^{-1}(12)}{\delta K(3)\delta K(4)}.$$  \hfill (B3)

F can then be expressed as

$$F(12;34) = Q^{-1}(1)f(12;34) + \frac{\delta Q^{-1}(1)}{\delta K(3)} \gamma_0(12;4)$$

$$+ \frac{\delta Q^{-1}(1)}{\delta K(4)} \gamma_0(12;3) + \frac{\delta Q^{-1}(1)}{\delta K(3)\delta K(4)} \gamma_0^{-1}(12).$$  \hfill (B4)

Next we expand the functions $F$, $f$ in powers of $1/N$ similar as in Eqs. (27)-(33). Considering the $O(1/N)$ contribution of $\delta \Sigma$, i.e., $\delta \Sigma_1$, the first term in the square bracket in Eq. (B2) drops out and we obtain

$$\delta \Sigma_1^{\sigma\sigma}(12) = -\int \frac{d^3d4d5d6}{N} \frac{t(1-3)}{N} g(34)g(56) \frac{t(6-2)}{N} (Y^{00})_{-1} \times$$

$$\left[ F_1\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{\sigma}{0} \frac{0}{1}\right) + F_1\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{0}{0} \frac{\sigma}{0}\right) + \frac{1}{(Y^{00})_{-1}} f\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{0}{0} \frac{1}{2}\right) \right].$$  \hfill (B5)

In the last term in the square bracket in Eq. (B3) the $O(1)$ contribution is to be taken and there is a sum over $\sigma_1$ whereas $\alpha$ is a fixed generic index $\alpha \neq \sigma$. Straightforward considerations show that

$$f_0\left(\frac{\sigma}{1} \frac{\alpha}{2} \frac{\sigma}{3} \frac{0}{4}\right) = f_0\left(\frac{\alpha}{1} \frac{\sigma}{2} \frac{0}{0} \frac{\alpha}{4}\right) = 0,$$  \hfill (B6)

$$F_1\left(\frac{\sigma}{1} \frac{\alpha}{2} \frac{\sigma}{3} \frac{0}{4}\right) = F_1\left(\frac{\sigma}{1} \frac{\alpha}{2} \frac{0}{0} \frac{\sigma}{4}\right) = \frac{N}{(Y^{00})_{-1}^2} \Pi_0\left(\frac{\sigma}{3} \frac{0}{4}\right) \delta(1-2) \delta(1-3).$$  \hfill (B7)

For example, consider Eq. (B4). We rewrite the functional derivative $\delta/\delta K^{(00)}(\frac{a}{3})$ as a derivative with respect to $g$ and obtain in leading order

$$f_0\left(\frac{\sigma}{1} \frac{\alpha}{2} \frac{\sigma}{3} \frac{0}{4}\right) = \frac{\delta^2 \gamma\left(\frac{\sigma}{1} \frac{\alpha}{2} \frac{\sigma}{3} \frac{0}{4}\right)}{\delta K(\frac{a}{3})} = -\sum_\beta \int d\bar{t}d\bar{\epsilon} \delta \gamma\left(\frac{\bar{t}}{2} \frac{\bar{\epsilon}}{3} \frac{0}{4}\right) \cdot g_0(\bar{t}^2)\cdot g_0(\bar{\epsilon}^2),$$  \hfill (B8)

where all quantities are to be taken in equilibrium once the derivatives have been carried out. The $O(1)$ term of $\gamma$ is independent of $g$ and thus drops out. Since in the generic case $\beta \neq \sigma$, $\beta \neq \alpha$, one internal sum in the $O(1/N)$ contribution for $\gamma$ is removed by the derivative $\delta/\delta g^{(\frac{a}{3})}$, but the sum over $\beta$ gives a factor $N$. This means that the right-hand side of Eq. (B8) is of order $O(1/N)$ and yields Eq. (B5).

Finally, the last term in the square bracket of Eq. (B5) can be evaluated as follows. First we rewrite it in a similar way as $f_0$ in Eq. (B8) and obtain

$$f\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{0}{0} \frac{1}{2}\right) = \frac{\delta^2 \gamma\left(\frac{\sigma}{4} \frac{\alpha}{5} \frac{0}{0} \frac{1}{2}\right)}{\delta K(\frac{a}{2})} = \int d\bar{t}d\bar{\epsilon} \delta \gamma\left(\frac{\bar{t}}{2} \frac{\bar{\epsilon}}{3} \frac{0}{4}\right) \cdot g_0(\bar{t}^2)\cdot g_0(\bar{\epsilon}^2),$$  \hfill (B9)
The quantity \( \delta \gamma / \delta g \) in Eq. (B9) is very similar to the quantity \( g^{(2)} \) in Eq. (34) of Ref. 18 and can also be calculated in a similar way. One obtains

\[
\Sigma_{\sigma_1} \frac{\delta \gamma (\sigma_1 \bar{\sigma}; \bar{\sigma})}{\delta g (\bar{\sigma} \bar{\bar{\sigma}} \sigma)} = \delta (\bar{\sigma} - \bar{\bar{\sigma}}) \cdot h(\bar{\sigma}; \bar{\bar{\sigma}}),
\]

(B10)

where \( h \) satisfies the equation

\[
h(\bar{\sigma}; \bar{\bar{\sigma}}) = \int d \bar{t} (\bar{\bar{\sigma}} - \bar{\sigma}) \gamma_{e,0}(\bar{\bar{\sigma}}; \bar{\bar{\sigma}}) g_{0}(\bar{\bar{\sigma}} - \bar{\sigma})
\]

\[
+ \int d \bar{t} d \bar{t} (\bar{\bar{\bar{\sigma}}}; \bar{\sigma}) g_{0}(\bar{\bar{\bar{\sigma}}}; \bar{\bar{\bar{\sigma}}}) h_{0}(\bar{\sigma}; \bar{\bar{\bar{\sigma}}}).
\]

(B11)

Writing \( h(\bar{\sigma}; \bar{\bar{\sigma}}) = h_{0}(\bar{\bar{\bar{\sigma}}}; \bar{\sigma} \bar{\sigma}) \), and performing Fourier transformations, Eq. (B11) becomes

\[
h(k_1, q, k_2) = t(k_2) g_0(k_2) \gamma_{e,0}(k_1, q) + g_0(k_2) \sum_p t(p) g_0(p) h(k_1, q, p + q),
\]

(B12)

with the solution

\[
h(k_1, q, k_2) = \gamma_{e,0}(k_1, q) g_0(k_2) \left[ t(k_2) - \frac{c(q)}{1 + b(q)} \right].
\]

(B13)

Then, inserting Eqs. (B7) and (B8) into (B3), and making use of Eqs. (B9)-(B11), we obtain

\[
\delta \Sigma_{\sigma}(k) = - \frac{2}{(V_{00})^{-1}} \lambda_0 \sum_q t(k + q) g_0(k + q) \Pi_{00}^{\sigma}(q)
\]

\[
+ \frac{1}{N} \sum_{p, q} t(p) g_0(k + q) g_0(p + q) g_0(p) \gamma_{e,0}(p, q) \frac{c(q)}{1 + b(q)}.
\]

(B14)

The first term in Eq. (B14) cancels with a similar contribution in the total self-energy (38) coming from the terms \( \Gamma_1 (\bar{\sigma}; \bar{\bar{\sigma}}) \Gamma_1 (\bar{\bar{\bar{\sigma}}}; \bar{\bar{\bar{\sigma}}}) \) and \( \Gamma_1 (\bar{\bar{\bar{\sigma}}}; \bar{\sigma} \bar{\sigma}) \Gamma_1 (\bar{\sigma}; \bar{\bar{\sigma}}) \) once the corresponding explicit expressions Eq. (52) for the charge vertex are used.
FIG. 3. Sum of the $O(1)$ and $O(1/N)$ contributions $n_0$ and $n_1$, respectively, to the site occupancy as a function of $n_0$ for different $N$’s. Only terms which diverge near half-filling have been considered in $n_1$. 
FIG. 4. Sum of the $O(1)$ and $O(1/N)$ contributions to the Luttinger volume $V^F$ as a function of the doping $\delta = 1 - n_0$ keeping only divergent terms in the $O(1/N)$ contribution. The dashed line corresponds to $N = \infty$ and obeys Luttinger’s theorem.