Intensity Distribution in the Heads of Comets

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Abstract

In optical wavelengths, the light emitted by comets consists of sunlight, reflected by dust, and the emission features of atoms and molecules. In his paper from 1957, L. Haser develops the mathematical framework to describe the density distribution of cometary comas. When comet scientists refer to the Haser model, they mostly point to the analytic solution over the line of sight (column density) and considers two particular observation configurations of a nonhomogeneous coma. This translation of the paper, originally in French, stays as close to the source as possible, including using the original figures. Minor typesetting corrections were introduced and are annotated in the text. The Haser model is foundational in cometary atmosphere science, but Haser’s paper was not widely read due to language barriers and a lack of digital availability. This translation is presented in an effort to make Haser’s work accessible to the wider community. We would like to thank Drs Emmanuel Jehin (Univ. Liège) and Kumar Venkataramani (Auburn Univ.) for critically reading this translation.

Unified Astronomy Thesaurus concepts: Comets (280); Comet volatiles (2162); History of astronomy (1868)

1. Introduction

The radial distribution of molecules in the head of a comet is investigated theoretically. Formulae for comparison with the observations are given (Haser 1957).

Taking Whipple’s cometary model (Whipple 1951) as a basis, we have calculated the radial distribution of molecules in the head of a comet and the distribution of observable intensity that results.

We adopt the following hypotheses:

1. The comet has a spherical nucleus of radius \(r_0\), and its matter evaporates as a result of absorbing solar radiation. The molecules leave the surface of the nucleus in every direction with a radial speed \(v_0\).
2. The molecules emit light according to their resonant frequencies after being excited by sunlight.
3. The molecules are disintegrated by photodissociation following the law

\[ n = n_0 \cdot e^{-\frac{t}{\tau}}, \]

with \(n_0\) being the number of molecules present at time \(t = 0\), and \(\tau\) measures the average lifespan of a molecule.
4. Near the nucleus, the molecular density is \(D(r_0)\) molecules per cm\(^3\).

At a distance \(r\) from the center of the nucleus, the molecular density \(D(r)\) is given by the equation\(^3\), with the term \(\frac{1}{r}\) being the dilution factor, and \(e^{-\frac{r}{\tau}}\) the disintegration factor. With the substitution \(t = \frac{r}{v_0}\), this becomes

\[ D(r) = D(r_0) \cdot \left(\frac{r_0}{r}\right)^2 \cdot e^{-\frac{r}{\tau}}. \] (1)

valid for \(r \geq r_0\). Notice that if the emission mechanism responsible is fluorescence, we may interpret \(D(r)\) and \(D(r_0)\) as the number of quanta emitted per unit time and volume at a particular wavelength.

2. Distribution of Observable Intensity

We define the distribution of observable intensity as the projection of the radial distribution (Equation (1)) onto the celestial sphere along the line of sight (Figure 1):

\[ J(\rho) = D(r_0) \cdot r_0^2 \cdot e^{\frac{\rho}{\tau}} \int_{-\infty}^{\infty} \frac{1}{\rho^2 + z^2} \cdot e^{-\frac{\rho^2 + z^2}{\tau}} dz. \]

\[ J(\rho) = D(r_0) \cdot r_0^2 \cdot e^{\frac{\rho}{\tau}} \cdot I. \] (2)

We define

\[ \beta_0 = \frac{1}{v_0 \tau}. \]

Examining the integral

\[ I(\beta_0) = 2 \int_{0}^{\infty} \frac{1}{\rho^2 + z^2} \cdot e^{-\beta_0 \sqrt{\rho^2 + z^2}} dz. \] (3)

A derivative with respect to \(\beta_0\) gives

\[ \frac{\partial I}{\partial \beta_0} = -2 \int_{0}^{\infty} \frac{1}{\sqrt{\rho^2 + z^2}} \cdot e^{-\beta_0 \sqrt{\rho^2 + z^2}} dz = -2K_0(\rho \beta_0), \] (4)

with \(K_0\) denoting the modified Bessel function of the second kind.
We have

\[ I(\beta_0) - I(0) = -2 \int_0^{\beta_0} K_0(\rho \beta_0) d\beta_0 \]
\[ I(\beta_0) = \frac{2}{\rho} \left[ \pi - \int_0^{\rho \beta_0} K_0(y) dy \right]. \tag{5} \]

Where the observable intensity distribution can be written in the form

\[ J(\rho) = D(r_0) \cdot r_0^2 \cdot e^{\rho r_0} \cdot \left( \frac{\pi}{2} - \int_0^{\rho r_0} K_0(y) dy \right) \tag{6} \]
valid for \( \rho \geq r_0 \). The numerical values of the function \( \frac{\pi}{2} - \int_0^{\rho r_0} K_0(y) dy \) are found in the tables of Müller (1939).

In the case of comets, the distribution in the range \( \rho < r_0 \) has little effective importance (though it may be important in analogous problems like the solar chromosphere). We limit the line of sight to the nucleus as in Figure 2.

The expression in Equation (6) is replaced by the following:

\[ J(\rho) = D(r_0) \cdot r_0^2 \cdot e^{\rho r_0} \cdot \int \frac{1}{r^2 + z^2} \cdot e^{-\frac{\pi}{2} \rho \sqrt{r^2 + z^2}} \cdot dz. \tag{7} \]

It seems that the result cannot be expressed in terms of known functions, but we may instead expand it in a series:

\[ J(\rho) = D(r_0) \cdot r_0^2 \cdot e^{\rho r_0} \cdot e^X \left[ F_e(X) + \frac{1}{2} E_e(X) + \frac{1}{24} dF_e(X) + \ldots \right]. \tag{8} \]

where

\[ X = \frac{r_0}{v_0 r_0}, \quad \alpha = \frac{\rho}{r_0}, \quad E_\alpha(X) = \int_1^{\infty} \frac{1}{u^\alpha} \cdot e^{- Xu} \cdot du. \]

The numerical values of the functions \( E_\alpha(X) \) are found in Mocknatsch (1938).

### 3. Distribution of Dissociation Products

Let \( N(r_0) \) be the number of molecules that escape the nucleus per second per cm², and \( N(X) \) the same quantity at distance \( X \) from the nucleus. The total number of molecules that cross a sphere of radius \( X \) is therefore

\[ 4\pi X^2 \cdot N(X) = 4\pi r_0^2 \cdot N(r_0) \cdot e^{-\frac{\pi}{2} \rho \sqrt{r_0^2 + z^2}}. \]

and the rate of production of disintegration products is

\[ -\frac{d}{dX} [4\pi X^2 N(X)] = 4\pi r_0^2 \cdot N(r_0) \cdot \frac{1}{v_0 r_0} \cdot e^{-\frac{\pi}{2} \rho \sqrt{r_0^2 + z^2}}. \]

It is assumed that the molecules are produced by successive disintegrations of parent molecules.

When the molecules produced at a distance \( X \) from the nucleus reach the distance \( r \), their number is reduced by a factor
of $e^{-\frac{r_0}{r}}$, and we then have

$$4\pi r_0^2 N(r_0) \frac{1}{v_0\gamma_0} \cdot e^{-\frac{r_0}{r}} \cdot e^{-\frac{r}{r_0}} \cdot dX$$

molecules arriving at the distance $r$ that originate from a layer of thickness $dX$ at a distance $X$ from the nucleus.

The total number of molecules arriving per cm$^2$ at a distance $r$ is therefore

$$N_1(r) = N(r_0) \cdot \left(\frac{r_0}{r}\right)^2 \cdot \frac{1}{v_0\gamma_0} \cdot \int_0^r e^{-\frac{r}{r_0}-\frac{x}{r}} \cdot dX.$$  

If we use the notation

$$N(r_0) = v_0D(r_0) \quad N_1(r) = v_1D_1(r) \quad \beta_0 = \frac{1}{v_0\gamma_0} \quad \beta_1 = \frac{1}{v_1\gamma_1},$$

we obtain

$$D_1(r) = D(r_0) \frac{v_0\left(\frac{r_0}{r}\right)^2}{v_1} \beta_0 \cdot \frac{e^{-\beta_0(r-r_0)} - e^{-\beta_1(r-r_0)}}{\beta_1 - \beta_0}. \quad (9)$$

And the observable intensity distribution is then

$$J_1(\rho) = \int_{-\infty}^{\infty} D_1(r)dz.$$  

Taking into account the results of the preceding section,

$$J_1(\rho) = D_1(\rho) \cdot r_0^2 \cdot \beta_0 \cdot \frac{v_0}{v_1} \frac{B_0 - B_1}{\beta_1 - \beta_0}. \quad (10)$$

With

$$B_0 = \frac{\pi}{2} - \int_0^{\rho_0} K_0(y)dy \cdot e^{\rho_0\beta_0}$$

$$B_1 = \frac{\pi}{2} - \int_0^{\rho_1} K_0(y)dy \cdot e^{\rho_1\beta_1},$$

valid for $\beta$ $> r_0$.

This calculation could be applied to disintegration processes such as

$$\text{NH}_2 \rightarrow \text{NH}$$

$$\text{C}_3 \rightarrow \text{C}_2.$$

4. Nonspherical Density Distribution

If the nucleus of the comet is not rotating and its heat conductivity is small, there is evaporation only on the illuminated part (Figure 3). The distribution of energy is then given by

$$I = I_0 \cos \varphi.$$  

Where $\varphi = 0$ corresponds to the comet-Sun axis.

The density distribution is not spherical and the appearance of the comet is a function of the Sun–comet–Earth phase angle $\alpha$. We consider the cases $\alpha = \frac{\pi}{2}$ and $\alpha = 0$.

Assuming $calable$ is a typo for $valable$.

4.1. Case 1

We first consider the case $\alpha = \frac{\pi}{2}$. The radial distribution of molecules is again

$$D(r) = \mathbb{D}(r_0) \cdot \left(\frac{r_0}{r}\right)^2 \cdot e^{-\beta_1(r-r_0)}.$$  

But $\mathbb{D}$ is now a function of $\varphi$:

$$\mathbb{D}(r_0) = \frac{D(r_0)}{\cos \delta} = D(r_0) \cos \varphi \cos \eta$$

and

$$\cos \eta = \frac{\rho}{r} = \frac{\rho}{\sqrt{\rho^2 + z^2}},$$

and the distribution function is then

$$D(r) = D(r_0) \cdot r_0^2 \cos \varphi \cdot e^{\beta_0\rho} \cdot \frac{\rho}{r} \cdot e^{\beta_1r}. \quad (11)$$

In this case, the observable distribution is the projection of this distribution function parallel to the line of sight onto the celestial sphere:

$$J(\rho, \varphi) = 2A \cos \varphi \int_0^{\infty} \rho \cdot e^{-\beta_0\sqrt{\rho^2 + z^2}}dz. \quad (12)$$

With the transformation $\frac{z}{\rho} = \sinh t$, we have

$$J(\rho, \varphi) = 2A \cdot \cos \varphi \cdot \frac{\cos \varphi}{\rho} \int_0^{\infty} e^{-b\cosh t}dt.$$  

Taking two derivatives with respect to $b = \rho\beta_0$ we find

$$\frac{\partial^2 J}{\partial b^2} = 2A \cdot \cos \varphi \cdot \frac{\cos \varphi}{\rho} \int_0^{\infty} e^{-b\cosh t}dt$$

$$= 2A \cdot \cos \varphi \cdot K_0(b).,$$

with $K_0(b)$ again being the Bessel function.
Consequently,

\[ J(\rho, \varphi) = 2A \cdot \cos \varphi \rho \int_{\rho_0}^{\rho} \int_{Y_0}^{\infty} K_0(y) \cdot dY \cdot dX. \]

The integration may be done following a method given by Müller (1939), and we get as a result

\[ J(\rho, \varphi) = 2A \cdot \beta_0 \cos \varphi \left\{ K_1(\rho \beta_0) - \int_{\rho \beta_0}^{\infty} K_0(y) \cdot dy \right\}. \quad (13) \]

The numerical values of \( K_1 \) and \( \int K_0(y) \cdot dy \) are found in Watson (1922) and Müller (1939), respectively.

### 4.2. Case 2

In the case \( \alpha = 0 \), the radial density distribution is once again (Figure 4)

\[ D(r) = D(r_0) \cdot \left( \frac{r_0}{r} \right)^2 \cdot \cos \varphi \cdot e^{\beta_0 r_0} \cdot e^{-\beta_0 r}. \]

According to Figure 4, \( \cos \varphi = \frac{r}{\rho} \) and we have

\[ D(r) = A \cdot \frac{z}{r^3} \cdot e^{-\beta_0 r}. \]

The observable intensity distribution is obtained by projecting the preceding expression parallel to the line of sight:

\[ J(\rho) = A \int_0^{\infty} z \cdot e^{-\beta_0 \sqrt{r^2 + z^2}} \cdot dz \cdot dY. \]

Making the substitutions

\[ \frac{z}{\rho} = \sinh t \quad \rho \beta_0 = b, \]

and taking two derivatives with respect to \( b \) gives

\[ \frac{\partial^2 J}{\partial b^2} = A \cdot \frac{1}{\rho} \int_0^{\infty} \sinh t \cdot e^{-\rho \beta_0 \cosh t} \cdot dt \]

\[ = \frac{A}{\rho} \cdot \frac{e^{-b}}{b}. \]

from which we obtain

\[ J(\rho) = A \cdot \frac{1}{\rho} \cdot \int_{\rho \beta_0}^{\infty} E_1(y) \cdot dy. \quad (15) \]

To express this integral in terms of functions with known numerical values, we consider the general case

\[ E_n(x) = \int_{\frac{1}{1}}^{\infty} t^{-n} \cdot e^{-st} \cdot dx. \]

Which has recurrence relations

\[ \frac{dE_n(x)}{dx} = -E_{n-1}(x) \]

\[ (n - 1)E_n(x) = e^{-x} - xE_{n-1}(x). \]

We eliminate \( E_{n-1} \) to find

\[ x \frac{dE_n}{dx} = (n - 1)E_n + e^{-x} = 0 \]

and integrate over the interval \((b, \infty)\):

\[ \int_b^{\infty} E_n(x) dx = \int_0^{\infty} [-bE_n(b) + e^{-b}] \cdot dx. \]

Using this expression, the observable intensity distribution may be written in the form

\[ J(\rho) = A \beta_0 [E_1(\rho \beta_0) - E_1(\rho \beta_0)]. \quad (16) \]

The profiles we have deduced may be compared with the intensity distribution from photos or spectra to determine the relevant physical parameters.

Our calculations differ from those published by Mocknatsch (1938). They also seem to differ from those of L. V. Wallace, of which only a short summary has been published (Wallace 1956).

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