Measurements of Nondegenerate Discrete Observables

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Abstract

Every measurement on a quantum system causes a state change from the system state just before the measurement to the system state just after the measurement conditional upon the outcome of measurement. This paper determines all the possible conditional state changes caused by measurements of nondegenerate discrete observables. For this purpose, the following conditions are shown to be equivalent for measurements of nondegenerate discrete observables: (i) The joint probability distribution of the outcomes of successive measurements depends affinely on the initial state. (ii) The apparatus has an indirect measurement model. (iii) The state change is described by a positive superoperator valued measure. (iv) The state change is described by a completely positive superoperator valued measure. (v) The output state is independent of the input state and the family of output states can be arbitrarily chosen by the choice of the apparatus. The implications to the measurement problem are discussed briefly.

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I. INTRODUCTION

Every measurement on a quantum system causes a quantum state reduction, a state transformation $\rho \rightarrow \rho_x$ from the system state $\rho$ just before the measurement to the system state $\rho_x$ just after the measurement conditional upon the outcome $x$ of measurement. In order to determine the quantum state reduction caused by a measurement of a given observable, von Neumann posed the repeatability hypothesis [1, p. 335]: If an observable is measured twice in succession in a system, then we get the same value each time. Then, a measurement of a discrete observable $A$ satisfies the repeatability hypothesis if and only if $\rho_x$ is a mixture of eigenstates of $A$ corresponding to the eigenvalue $x$. Thus, under this hypothesis the measurement of a nondegenerate discrete observable $A$ causes the unique quantum state reduction such that $\rho_x$ is the unique eigenstate corresponding to the eigenvalue $x$. However, von Neumann admitted that there are many quantum state reductions caused by measuring the degenerate observable $A$ even when the repeatability hypothesis holds [1, p. 348]. Thus, for degenerate observables further hypothesis were demanded. In order to characterize the least disturbing measurement, Lüders [2] posed the projection postulate: The measurement of a discrete observable $A$ in the state $\rho$ leaves the system in the state $\rho_x = E_x \rho E_x / \text{Tr}[E_x \rho]$, where $E_x$ is the eigenprojection corresponding to the eigenvalue $x$. Obviously, the projection postulate implies the repeatability hypothesis and determines the output state uniquely, even for the degenerate discrete observable $A$.

Despite the above attempts, Davies and Lewis [3] conjectured that no measurements of continuous observables satisfy the repeatability hypotheses and proposed abandoning the repeatability hypothesis. Actually, their conjecture was proved later in [4,5]: the essential part of the proof given in [3] shows that even in the measurement of a continuous observable the output state conditional upon the outcome can be still described by a density operator. Moreover, it can be readily seen that there are many ways of measuring the same observable without satisfying the repeatability hypothesis such as photon counting [3], which arises every optical experiment, and contractive state measurement [7–10], which beats the standard quantum limit for monitoring the free-mass position claimed in [11–13].

Once we abandon the repeatability hypothesis or the projection postulate, the problem of determining all the possible quantum state reductions caused by measurements of a given observable has a primary importance in quantum mechanics. Especially, the problem receives increasing interests recently not only from the foundational point of view but also from the technological point of view, since the measurement is used for preparing the state of the system in such processes as purification procedures in the field of quantum information [14,15].

The purpose of this paper is to give the complete solution to the above problem for the measurements of nondegenerate discrete observables. It will be shown that a surprisingly general condition for the measurement statistics suffices to determine all the possible quantum state reductions realized by indirect measurement models. It is shown that for measurements of nondegenerate discrete observables the output state is independent of the input state in any measurement and that the family of output states can be arbitrarily chosen by the choice of the apparatus. Moreover, all of them are shown to have indirect measurement models.

In order to obtain a mathematical description of quantum state reductions for the most
general class of measurements we consider the two requirements: one is necessary and the other is sufficient.

The necessary one is the mixing law of the joint probability that requires that the joint probability distribution of the outcomes of the successive measurement depends affinely on the input state. We require this condition as a necessary condition for every apparatus to satisfy. It will be shown that this is equivalent to the requirement that every apparatus has a normalized positive superoperator valued measure that satisfies the Davies-Lewis description of conditional state transformations [17]. The notion of normalized positive superoperator valued measures was first introduced by Davies and Lewis [3] to obtain a general description of conditional state transformation by unifying the notions of operations [17], effects [18], and probability operator valued measures [19,20]. Thus, the problem of determining possible quantum state reductions is reduced to the problem as to which normalized positive superoperator valued measure corresponds to an apparatus.

The sufficient condition is the unitary realizability condition that requires the existence of an indirect measurement model comprising of the probe preparation, the measuring interaction with unitary time evolution, and the probe detection. We require this condition as a sufficient condition so that if a normalized positive superoperator valued measure has an indirect measurement model then the corresponding apparatus exists. It was proved in [21,4] that this condition is equivalent to the condition that the normalized positive superoperator valued measure is completely positive.

According to the above approach, the class of possible quantum state reductions is included in the class of conditional state transformations satisfying the mixing law, i.e., the normalized positive superoperator valued measures, and includes the one satisfying the realizability condition, i.e., the normalized completely positive superoperator valued measures. These two classes are generally different.

Nevertheless, for the case where A is nondegenerate, this paper shows, the above two conditions are actually equivalent. Thus, both of them are necessary and sufficient and we reach a clear-cut conclusion. According to the analysis developed in this paper, for any apparatus A measuring a nondegenerate discrete observable $A = \sum_n a_n |\phi_n\rangle\langle \phi_n|$ there is a sequence $\{\rho_n\}$ of density operators independent of the input state $\rho$ such that the measurement leaves the system in the state $\rho_n$ with the probability $\langle \phi_n|\rho|\phi_n\rangle$, and conversely for any sequence $\{\rho_n\}$ of density operators such an apparatus exists.

II. MEASURING APPARATUSES

Let us consider the conventional quantum-mechanical description of the measurement of an observable represented by a self-adjoint operator $A$ with purely discrete spectrum on a separable Hilbert space $\mathcal{H}$. For any real number $x$ we shall denote by $E^A(x)$ the projection of $\mathcal{H}$ onto the subspace $\{\psi \in \mathcal{H} | A\psi = x\psi\}$. If $A$ has eigenvalues $a_1, a_2, \ldots$ then $E^A(a_n)$ is the spectral projection corresponding to $a_n$ and $E^A(x) = 0$ if $x$ is not an eigenvalue of $A$. If the state of the system at the instant before the measurement is given by the density operator $\rho$ on $\mathcal{H}$, then the measurement yields the outcome $a_n$ with the probability $\text{Tr}[E^A(a_n)\rho]$. If this measurement satisfies the projection postulate [2], then the state at the instant after the measurement is
\[ \rho_n = \frac{E^A(a_n)\rho E^A(a_n)}{\text{Tr}[E^A(a_n)\rho]} \] (1)

provided that the measurement leads to the outcome \( a_n \).

As it can be seen from the above description, every measuring apparatus \( A \) has the output variable \( x \) that takes the outcome in each measurement carried out by \( A \). Thus, the output variable is a random variable the probability distribution of which depends only on the input state, the state of the system at the instant just before the measurement. Throughout this paper, we assume that the output variable takes the values in a countable subset of the real line \( \mathbb{R} \). The probability distribution \( \text{Pr}\{x = x\parallel \rho\} \) of \( x \) in the input state \( \rho \) is called the output distribution of \( A \). The change from the unknown input state to the output distribution is called the objective state reduction. Depending on the input state \( \rho \) and the outcome \( x = x \), the state \( \rho_{\{x=x\}} \) just after the measurement is determined uniquely. The change from the unknown input state to the output state is called the quantum state reduction. The above two mathematical objects, the objective state reduction and the quantum state reduction, are called the statistical property of \( A \). Two apparatuses are called statistically equivalent if they have the same statistical property. In what follows, every apparatus is supposed to have its own distinctive output variable and we denote by \( A(x) \) the apparatus having the output variable \( x \).

In the above measurement of \( A \) satisfying the projection postulate, let us denote the measuring apparatus by \( A(a) \) where \( a \) stands for the output variable. The statistical property of \( A(a) \) is represented as follows.

\[
\begin{align*}
\text{output distribution: } & \quad \text{Pr}\{a = x\parallel \rho\} = \text{Tr}[E^A(x)\rho] \\
\text{output state: } & \quad \rho_{\{a=a\}} = \frac{E^A(a_n)\rho E^A(a_n)}{\text{Tr}[E^A(a_n)\rho]} 
\end{align*}
\] (2) (3)

In the above, \( a_n \) is an eigenvalue such that \( \text{Tr}[E^A(a_n)\rho] > 0 \).

Now, the following problem arises: Does every measuring apparatus for the observable \( A \) necessarily have the above statistical property? It is postulated by the Born statistical formula that the output distribution of the measurement of the observable \( A \) satisfies (2). Hence, every measuring apparatus for the observable \( A \) satisfies (2) by definition. The following argument will show, however, that the existence of an apparatus satisfying the projection postulate implies the existence of another apparatus which does not satisfy the projection postulate. Therefore, we cannot postulate that every measurement satisfies the projection postulate.

Suppose that the observable \( Y \) has degenerate eigenvalues and can be represented by

\[ Y = \sum_{n,m} y_{n,m} |n,m\rangle \langle n,m| \] (4)

for some orthonormal basis \( \{ |n,m\rangle \} \). Consider the following process of measuring \( Y \): (i) One measures the nondegenerate discrete observable

\[ X = \sum_{n,m} x_{n,m} |n,m\rangle \langle n,m| \] (5)
where \( x_{n,m} \) are all different. (ii) If the outcome \( x \) of the \( X \) measurement leads to the value \( x_{n,m} \) then the outcome \( y \) of the \( Y \) measurement is determined as \( y_n \). Then, even if the \( X \) measurement satisfies the projection postulate, the \( Y \) measurement does not satisfy it. In fact, with the probability \( \langle n,m|\rho|n,m\rangle \) the \( X \) measurement leads to the outcome \( x_{n,m} \) and leaves the system in the state \(|n,m\rangle\langle n,m|\) by the projection postulate. It follows that if the outcome is \( y_n \) then the state at the instant after the \( Y \) measurement is given by

\[
\rho_{\{y=y_n\}} = \frac{\sum_m \langle n,m|\rho|n,m\rangle |n,m\rangle\langle n,m|}{\sum_m \langle n,m|\rho|n,m\rangle}.
\]  

(6)

The above state depends on the choice of the orthonormal basis \( \{|n,m\rangle\} \). If \( Y \) is degenerate, there are infinitely many essentially different choices of \( \{|n,m\rangle\} \) and each choice gives a process of \( Y \) measurement which does not satisfy the projection postulate.

Generalizing the above, if two observables \( X, Y \) has the relation \( Y = f(X) \), then for any apparatus \( A(x) \) measuring \( X \) we have the apparatus \( A(f(x)) \) measuring \( Y \) that outputs the outcome \( f(x) = f(x) \) whenever \( A(x) \) outputs the outcome \( x = x \). In this case, even if \( A(x) \) satisfies the projection postulate, \( A(f(x)) \) does not necessarily satisfies the projection postulate. Therefore, the output distribution of \( Y \) measurement is unique but the quantum states reduction depends on the way of measuring the same observable \( Y \). More general construction of measuring apparatuses that do not satisfy the projection postulate will be discussed in Section VIII.

Can one determine all the possible quantum state reductions arising in measuring \( A \) that are allowed by the basic principles of quantum mechanics? This problem will be considered in the following sections.

III. SUCCESSIVE MEASUREMENTS

In order to clarify the operational meaning of the quantum state reduction, we shall generalize von Neumann’s idea on repeated measurements of the same observable [1, pp. 211–223] to arbitrary pair of measuring apparatuses and consider the joint probability distribution of the outcomes of the two measurements carried out in succession.

We suppose that the \( A \) measurement described in the preceding section is immediately followed by a measurement of a discrete observable \( B \) with eigenvalues \( b_m \). Then, the conditional probability of obtaining the outcome \( b_m \) at the \( B \) measurement is \( \text{Tr}[E^B(b_m)\rho_n] \) conditional upon having obtained \( a_n \) at the \( A \)-measurement. From (1), the joint probability of obtaining \( a_n \) at the \( A \) measurement and \( b_m \) at the \( B \) measurement is therefore

\[
p_{n,m} = \text{Tr}[E^B(b_m)\rho_n]\text{Tr}[E^A(a_n)\rho] = \text{Tr}[E^B(b_m)E^A(a_n)\rho E^A(a_n)].
\]

(7)

Generally speaking, if a measurement by the apparatus \( A(x) \) in the input state \( \rho \) is immediately followed by a measurement by the apparatus \( A(y) \), the joint probability distribution \( \Pr\{x = x, y = y|\rho\} \) of the output variables \( x \) and \( y \) depends only on the input state \( \rho \) of the first measurement and is given by

\[
\Pr\{x = x, y = y|\rho\} = \Pr\{y = y|\rho_{\{x=x\}}\} \Pr\{x = x|\rho\}.
\]

(8)
This joint probability distribution has the following significant property.

**Mixing law of the joint probability:** For any measuring apparatuses $A(x)$ and $A(y)$, if the input state $\rho$ is the mixture of $\rho_1$ and $\rho_2$ such that $\rho = \alpha \rho_1 + (1 - \alpha) \rho_2$ with $0 < \alpha < 1$ then the joint probability distribution of the outcomes of the successive measurement satisfies

$$\Pr\{x = x, y = y \parallel \rho\} = \alpha \Pr\{x = x, y = y \parallel \rho_1\} + (1 - \alpha) \Pr\{x = x, y = y \parallel \rho_2\}. \quad (9)$$

This is justified as follows. If the system is in the state $\rho_1$ with the probability $\alpha$ and in the state $\rho_2$ with the probability $1 - \alpha$ then the joint probability is their mixture in the right hand side. On the other hand, in this case the state of the system is described by the density operator $\rho$ and hence the above equality holds.

In the previous example, if the observable $B$ is measured by an apparatus $A(b)$ then from (7) we have

$$\Pr\{a = a_n, b = b_m \parallel \rho\} = \text{Tr}[E_B(b_m)E_A(a_n)\rho E_A(a_n)]. \quad (10)$$

Obviously, this joint probability satisfies the above mixing law.

In what follows, we require the mixing law of the joint probability. For an arbitrary apparatus $A(x)$ with the output distribution $\Pr\{x = x \parallel \rho\}$ and the output state $\rho_{\{x=x\}}$, we define the output operator $X(x, \rho)$ by

$$X(x, \rho) = \Pr\{x = x \parallel \rho\} \rho_{\{x=x\}}. \quad (11)$$

Then, $X(x, \rho)$ is a trace class operator [22] determined by the statistical property of the apparatus $A(x)$, the input state $\rho$, and the outcome $x = x$.

For the measuring apparatus $A(a)$, the output operator $X_a(x, \rho)$ is given by

$$X_a(x, \rho) = E^A(x)\rho E^A(x).$$

The above expression extends the definition of $X_a(x, \rho)$ to arbitrary trace class operators $\rho$. Then, it is easy to see that $X_a(x, \rho)$ has the following properties: (i) $X_a(x, \rho)$ is a positive operator if $\rho$ is positive, (ii) the correspondence $\rho \mapsto X_a(x, \rho)$ is linear, (iii) for any $\rho$ we have

$$\text{Tr}[\sum_x X_a(x, \rho)] = \text{Tr}[\rho].$$

In the following, we shall show that the output operator $X(x, \rho)$ of every apparatus $A(x)$ has the above properties.

Return to the joint probability distribution $\Pr\{x = x, y = y \parallel \rho\}$. If one measures the observable $B$ by the apparatus $A(b)$ instead of $A(y)$, from (8) and (11) we have

$$\Pr\{x = x, b = b_m \parallel \rho\} = \text{Tr}[E_B(b_m)\rho_{\{x=x\}} \Pr\{x = x \parallel \rho\}] = \text{Tr}[E_B(b_m)X(x, \rho)]. \quad (12)$$

Suppose that $\rho$ is the mixture $\rho = \alpha \rho_1 + (1 - \alpha) \rho_2$. From (9) we have
\[
\text{Tr}[E^B(b_m)X(x, \rho)]
\]
\[
= \alpha \text{Tr}[E^B(b_m)X(x, \rho_1)] + (1 - \alpha) \text{Tr}[E^B(b_m)X(x, \rho_2)]
\]
\[
= \text{Tr}[E^B(b_m)[\alpha X(x, \rho_1) + (1 - \alpha)X(x, \rho_2)]]
\]

Since \(B\) is arbitrary, we have
\[
X(x, \rho) = \alpha X(x, \rho_1) + (1 - \alpha)X(x, \rho_2).
\] (13)

In what follows, for any \(x\) let \(X(x)\) be the mapping that maps a density operator \(\rho\) to the trace class operator \(X(x, \rho)\). Since every trace class operator \(\sigma\) can be represented as the linear combination
\[
\sigma = \lambda_1 \sigma_1 - \lambda_2 \sigma_2 + i \lambda_3 \sigma_3 - i \lambda_4 \sigma_4
\] (14)
with four density operators \(\sigma_1, \ldots, \sigma_4\) and four positive numbers \(\lambda_1, \ldots, \lambda_4\), we can extend the mapping \(X(x)\) to a linear transformation on the space \(\tau c(\mathcal{H})\) of trace class operators on \(\mathcal{H}\) by
\[
X(x)\sigma = \lambda_1 X(x)\sigma_1 - \lambda_2 X(x)\sigma_2 + i \lambda_3 X(x)\sigma_3 - i \lambda_4 X(x)\sigma_4.
\] (15)

Since the decomposition (14) is not unique, in order for the extension (15) to be well-defined we need to show that the left hand side of (15) is uniquely determined independent of the decomposition of \(\sigma\). This can be proved from (13) and the proof will be shown in Appendix A.

We have, therefore, shown that for every apparatus \(A(x)\) there exists a family \(\{X(x)\mid x \in \mathbb{R}\}\) of linear transformations on \(\tau c(\mathcal{H})\) such that for every density operator \(\rho\), we have
\[
X(x)\rho = \text{Pr}\{x = x\|\rho\} \rho_{\{x=x\}}.
\] (16)

The linear transformation \(X(x)\) defined above is called the operation of the apparatus \(A(x)\) for the outcome \(x = x\). The family \(\{X(x)\mid x \in \mathbb{R}\}\) is called the operational distribution of the apparatus \(A(x)\). It is obvious from (13) that by taking advantage of the operational distribution, the output distribution is represented by
\[
\text{Pr}\{x = x\|\rho\} = \text{Tr}[X(x)\rho]
\] (17)
and the output state by
\[
\rho_{\{x=x\}} = \frac{X(x)\rho}{\text{Tr}[X(x)\rho]},
\] (18)
where the outcome \(x = x\) is supposed to have positive probability.
IV. OPERATIONAL DISTRIBUTIONS

In order to explore mathematical properties of the operational distribution \( \{X(x) \mid x \in \mathbb{R}\} \) of the apparatus \( A(x) \), we shall provide relevant mathematical terminology. A linear transformation \( L \) on the space \( \tau c(\mathcal{H}) \) of trace class operators on \( \mathcal{H} \) is said to be \textit{bounded} if there is a constant \( K > 0 \) such that

\[
\|L\rho\|_{tr} \leq K\|\rho\|_{tr}
\]

for all \( \rho \in \tau c(\mathcal{H}) \), where \( \| \cdot \|_{tr} \) stands for the trace norm. Then, the norm of \( L \) is defined by

\[
\|L\|_{tr} = \sup_{\|\rho\|_{tr} \leq 1} \|L\rho\|_{tr}.
\]

A linear transformation \( M \) on the space \( \mathcal{L}(\mathcal{H}) \) of bounded operators on \( \mathcal{H} \) is said to be \textit{bounded} if there is a constant \( K > 0 \) such that

\[
\|MA\| \leq K\|A\|
\]

for all \( A \in \mathcal{L}(\mathcal{H}) \), where \( \| \cdot \| \) stands for the operator norm. Then, the norm of \( M \) is defined by

\[
\|M\| = \sup_{\|A\| \leq 1} \|MA\|.
\]

A bounded linear transformation on \( \tau c(\mathcal{H}) \) is called a \textit{superoperator}. For any superoperator \( L \) on \( \tau c(\mathcal{H}) \), its \textit{dual superoperator} \( L^* \) is the bounded linear transformation on \( \mathcal{L}(\mathcal{H}) \) defined by

\[
\text{Tr}[A(L\rho)] = \text{Tr}[(L^*A)\rho]
\]

for all \( A \in \mathcal{L}(\mathcal{H}) \) and \( \rho \in \tau c(\mathcal{H}) \). In this case, we have \( \|L\|_{tr} = \|L^*\| \). A superoperator or a dual superoperator is said to be \textit{positive} iff it maps positive operators to positive operators. Then, a superoperator \( L \) is positive if and only if so is its dual. A super operator or a dual superoperator is said to be \textit{contractive} iff it has the norm less than or equal to one. Then, a superoperator \( L \) is a contractive if and only if so is its dual. We have the following characterizations of positive contractive superoperators [23, p. 216], [16, p. 18].

**Theorem 1.** For a positive superoperator \( L \) the following conditions are all equivalent:

(i) \( L \) is a contractive superoperator.
(ii) \( L^* \) is a contractive dual superoperator.
(iii) \( 0 \leq \text{Tr}[L\rho] \leq 1 \) for all density operators \( \rho \).
(vi) \( 0 \leq L^*(I) \leq I \).

Moreover, a superoperator \( L \) is trace preserving, i.e.,

\[
\text{Tr}[L(\rho)] = \text{Tr}[\rho]
\]

for all \( \rho \in \tau c(\mathcal{H}) \) if and only if \( L^* \) is unital, i.e.,

\[
L^*(I) = I.
\]
Let us return to the operational distribution \( \{ X(x) \mid x \in R \} \) of the apparatus \( A(x) \). Let \( A(b) \) be an apparatus measuring a discrete observable \( B \) with eigenvalues \( b_m \) and let \( \rho \) be an arbtrary density operator. By the property of joint probability, we have

\[
0 \leq \Pr\{x = x, b = b_m\|\rho\} \leq 1.
\]

From (12) we have

\[
0 \leq \text{Tr}[E^B(b_m)X(x)\rho] \leq 1.
\]

Since \( B \) and \( \rho \) are arbitrary, the operation \( X(x) \) is a positive superoperator. Taking \( B = I \) and \( b_m = 1 \), we have

\[
0 \leq \text{Tr}[X(x)\rho] \leq 1.
\]

It follows from Thorem \[\[\]\] that the operation \( X(x) \) is a positive contractive superoperator. By the unicity of total probability, we have

\[
\sum_{x \in R} \Pr\{x = x\|\rho\} = 1.
\]

Hence, we have

\[
\text{Tr}[\sum_{x \in R} X(x)\rho] = 1.
\]

for all density operator \( \rho \). Let \( X(x)^* \) be the dual of the operation \( X(x) \). It follows that

\[
\sum_{x \in R} X(x)^*I = I. \quad (22)
\]

and that

\[
\text{Tr}[\sum_{x \in R} X(x)\rho] = \text{Tr}[\rho] \quad (23)
\]

for all \( \rho \in \tau c(cH) \). For any \( x \in R \), define the operator \( X(x) \) by

\[
X(x) = X(x)^*I. \quad (24)
\]

We call \( X(x) \) the effect of \( A(x) \) for the outcome \( x = x \). The family \( \{ X(x) \mid x \in R \} \) of the effects of \( A(x) \) is called the effect distribution of the apparatus \( A(x) \). From (17), the output distribution of the apparatus \( A(x) \) is determined by the effect as

\[
\Pr\{x = x\|\rho\} = \text{Tr}[X(x)\rho]. \quad (25)
\]

By the positivity of probability, we have \( \text{Tr}[X(x)\rho] \geq 0 \). Since the density operator \( \rho \) is arbitrary, \( X(x) \) is a positive operator. From (22), we have

\[
\sum_{x \in R} X(x) = I. \quad (26)
\]

In this case, \( X(x) = 0 \) except for countable number of \( x \).s. From (25), \( A(x) \) measures an observable \( A \) if and only if

\[
X(x) = E^A(x). \quad (27)
\]
Thus, \( \text{A}(\text{x}) \) measures an observable if and only if the effect distribution coincides with its spectral projections. Otherwise, the apparatus \( \text{A}(\text{x}) \) is interpreted to carry out a more general measurement such as an approximate measurement of an observable.

We define the positive superoperator \( \text{T} \) by

\[
\text{T} \rho = \sum_{x \in \mathbb{R}} \text{X}(x) \rho,
\]

(28)

where the sum is a countable sum since \( \text{X}(x) = 0 \) except for countable number of \( x \)s. This \( \text{T} \) is called the nonselective operation of the apparatus \( \text{A}(\text{x}) \). From (28) we have \( \text{T}^* \mathbb{I} = \mathbb{I} \) and hence \( \text{T} \) is a trace preserving positive superoperator.

A family \( \{ \text{W}(x) \mid x \in \mathbb{R} \} \) of positive superoperators is called a superoperator distribution iff

\[
\sum_{x \in \mathbb{R}} \text{W}(x)^* \mathbb{I} = \mathbb{I}.
\]

A family \( \{ \text{F}(x) \mid x \in \mathbb{R} \} \) of positive operators is called an operator distribution iff

\[
\sum_{x \in \mathbb{R}} \text{F}(x) = \mathbb{I}.
\]

The family \( \{ \text{E}^A(x) \mid x \in \mathbb{R} \} \) of spectral projections of a discrete self-adjoint operator \( A \) is an operator distribution. The family \( \{ \text{W}(x) \mid x \in \mathbb{R} \} \) of positive operators defined by

\[
\text{W}(x) = \text{W}(x)^* \mathbb{I}
\]

is an operator distribution and is called the operator distribution of \( \{ \text{W}(x) \mid x \in \mathbb{R} \} \). The superoperator \( \text{T} \) defined by

\[
\text{S} = \sum_{x \in \mathbb{R}} \text{W}(x)
\]

is a positive trace preserving superoperator and is called the total superoperator of \( \{ \text{W}(x) \mid x \in \mathbb{R} \} \).

We have shown under the mixing law of the joint probability that the operational distribution \( \{ \text{X}(x) \mid x \in \mathbb{R} \} \) of an apparatus \( \text{A}(\text{x}) \) is a superoperator distribution, the effect distribution of \( \text{A}(\text{x}) \) is the operator distribution \( \{ \text{X}(x) \mid \rho \} \), and the nonselective superoperator of \( \text{A}(\text{x}) \) is the total superoperator of \( \{ \text{X}(x) \mid x \in \mathbb{R} \} \). Conversely, if for given apparatuses \( \text{A}(\text{x}) \) and \( \text{A}(\text{y}) \) there are superoperator distributions \( \{ \text{X}(x) \mid x \in \mathbb{R} \} \) and \( \{ \text{Y}(y) \mid y \in \mathbb{R} \} \) satisfy (10) respectively, then the joint probability distribution of the outcomes of the successive measurements carried out by \( \text{A}(\text{x}) \) and \( \text{A}(\text{y}) \) satisfies

\[
\Pr\{ x = x, y = y \mid \rho \} = \text{Tr}[\text{Y}(y)\text{X}(x)\rho],
\]

and hence the mixing law of the joint probability holds.

From the arguments so far, we conclude that the mixing law of the joint probability is equivalent with the following requirement: \textit{For any measuring apparatus} \( \text{A}(\text{x}) \), \textit{there is a superoperator distribution} \( \{ \text{X}(x) \mid x \in \mathbb{R} \} \) \textit{such that the statistical property of} \( \text{A}(\text{x}) \) \textit{is represented as follows.}

\[
\text{output distribution: } \Pr\{ x = x \mid \rho \} = \text{Tr}[\text{X}(x)\rho]
\]

(29)

\[
\text{output state: } \rho_{(x=x)} = \frac{\text{X}(x)\rho}{\text{Tr}[\text{X}(x)\rho]}
\]

(30)
In (30) the outcome $x = x$ is supposed to have positive probability; henceforce, the analogous assumption will be required implicitly in the similar expressions on the output state.

It follows that the problem as to what statistical property is possible is reduced to the problem as to what superoperator distributions are the operational distributions of apparatuses.

V. DAVIES-LEWIS POSTULATE

For the case of the discrete output variables, the notion of superoperator distributions is equivalent to the notion of normalized positive superoperator valued measures introduced by Davies and Lewis [3]. A positive superoperator valued (PSV) measure is a mapping $\mathcal{E}$ which maps every Borel set $\Delta$ to a positive superoperator $\mathcal{E}(\Delta)$ such that if $\Delta_1, \Delta_2, \ldots$ is a countable Borel partition of $\Delta$, then we have

$$\mathcal{E}(\Delta)\rho = \sum_n \mathcal{E}(\Delta_n)\rho$$

for any $\rho \in \tau_c(\mathcal{H})$, where the sum is convergent in the trace norm. The PSV measure $\mathcal{E}$ is said to be normalized if it satisfies the further condition

$$\text{Tr}[\mathcal{E}(R)\rho] = \text{Tr}[\rho]$$

for any $\rho \in \tau_c(\mathcal{H})$. The equivalence is given below analogous to the case of discrete probability measures. If $\mathcal{E}$ is a normalized PSV measure, then the corresponding superoperator distribution $\{X(x)\mid x \in \mathbb{R}\}$ is given by

$$X(x) = \mathcal{E}(\{x\}), \quad (31)$$

where $\{x\}$ is the singleton set containing the point $x$. Conversely, if $\{X(x)\mid x \in \mathbb{R}\}$ is a superoperator distribution, then the corresponding normalized PSV measure is given by

$$\mathcal{E}(\Delta) = \sum_{x \in \Delta} X(x). \quad (32)$$

For the apparatus $A(x)$, the probability $\text{Pr}\{x \in \Delta\|\rho\}$ of obtaining the outcome in the Borel set $\Delta$ is given by

$$\text{Pr}\{x \in \Delta\|\rho\} = \sum_{x \in \Delta} \text{Pr}\{x = x\|\rho\} \quad (33)$$

and the output state of the ensemble of the samples with the outcome in the Borel set $\Delta$ is given by

$$\rho_{\{x \in \Delta\}} = \frac{\sum_{x \in \Delta} \text{Pr}\{x = x\|\rho\}\rho_{\{x = x\}}}{\text{Pr}\{x \in \Delta\|\rho\}}. \quad (34)$$

Davies and Lewis [3] proposed the following description of measurement statistics:

**Davies-Lewis postulate:** For any measuring apparatus $A(x)$, there is a normalized PSV measure $\mathcal{E}$ satisfying the following relations for any density operator $\rho$ and Borel set $\Delta$: 

Although the Davies-Lewis description of measurement is quite general, it is not clear by itself whether it is general enough to exhaust all the possible measurements. Our arguments are about to complete proving the following theorem that shows indeed it is the case.

**Theorem 2.** The Davies-Lewis postulate is equivalent to the mixing law of the joint probability.

In fact, under the Davies-Lewis postulate, we have the normalized PSV measures $\mathcal{E}_x$ and $\mathcal{E}_y$ for any apparatuses $A(x)$ and $A(y)$. By substituting (DL1) and (DL2) in (8), the joint probability is given by

$\Pr\{x = x, y = y \| \rho\} = \text{Tr}[\mathcal{E}_y(\{y\})\mathcal{E}_x(\{x\})\rho]$.

From the linearity of $\mathcal{E}_x(\{x\})$ and $\mathcal{E}_y(\{x\})$, the mixing law follows. Conversely, under the mixing law, we have shown that there is a superoperator distribution $\{X(x)\} | x \in \mathbb{R}$ satisfying (17) and (18). Now, it is easy to check that relations (31)–(34) leads to the Davies-Lewis description (DL1)–(DL2) and the proof is completed.

**VI. MEASUREMENTS OF DISCRETE OBSERVABLES**

For a given discrete self-adjoint operator $A$, a superoperator distribution $\{X(x)\} | x \in \mathbb{R}$ is called $A$-compatible iff $X(x)I = E^A(x)$ for all $x \in \mathbb{R}$. The operational distribution of an apparatus measuring the observable $A$ is an $A$-compatible superoperator distribution.

We have the following theorem [24]; a simplified proof will be given in Appendix B.

**Theorem 3.** Let $A$ be a discrete self-adjoint operator. Let $\{X(x)\} | x \in \mathbb{R}$ be an $A$-compatible superoperator distribution and $T$ its total superoperator. For any real number $x$ and trace class operator $\rho$, we have

\[ X(x)\rho = T[E^A(x)\rho] = T[\rho E^A(x)] = T[E^A(x)\rho E^A(x)]. \]

(35)

For any real number $x$ and bounded operator $B$, we have

\[ X(x)^*B = E^A(x)T^*(B) = T^*(B)E^A(x) = E^A(x)T^*(B)E^A(x). \]

(36)

From the above theorem, the operational distribution of an apparatus measuring an observable $A$ is determined uniquely by the nonselective operation. It follows from (36) that the range of $T^*$ consists of operators commuting with $A$. Let us define the commutant of $A$, denoted by $\{A\}'$, as the set of all bounded operators commuting with $A$. A trace
preserving positive superoperator $L$ on $\tau_c(\mathcal{H})$ is called $A$-compatible iff the range of its duel $L^*$ is included in the commutant $\{A\}'$ of $A$.

For any trace preserving positive superoperator $L$, let

$$L'\rho = \sum_{x \in \mathbb{R}} L[E^A(x)\rho E^A(x)].$$

Then $L'$ is an $A$-compatible positive superoperator. Obviously, $L$ itself is $A$-compatible if and only if $L' = L$.

From (36), the total superoperator of an $A$-compatible superoperator distribution is an $A$-compatible positive superoperator. Conversely, for any $A$-compatible positive superoperator $T$, let $X(x)\rho = T[E^A(x)\rho]$ for all $\rho \in \tau_c(\mathcal{H})$. Then $\{X(x)|x \in \mathbb{R}\}$ is an $A$-compatible superoperator distribution and $T$ is its total superoperator. From the above argument, we have obtained the following theorem.

**Theorem 4.** Let $A$ be a discrete self-adjoint operator on $\mathcal{H}$. The relation

$$X(x)\rho = T[E^A(x)\rho]$$

for all real number $x$ and trace class operator $\rho$ sets up a one-to-one correspondence between the $A$-compatible superoperator distribution $\{X(x)|x \in \mathbb{R}\}$ and the $A$-compatible positive superoperators $T$.

From the above theorem, we conclude the following: For any apparatus $A(x)$ measuring a discrete observable $A$, there is an $A$-compatible positive superoperator $T$ such that the statistical property of $A(x)$ is represented as follows.

- output distribution: $\Pr\{x = x|\rho\} = \text{Tr}[E^A\rho]$ (38)
- output state: $\rho_{\{x=x\}} = \frac{T[E^A(x)\rho]}{\text{Tr}[E^A(x)\rho]}$ (39)

It follows that the problem of determining all the possible quantum state reductions $\rho \rightarrow \rho_{\{x=x\}}$ arising in the apparatus measuring $A$ is reduced to the following problems: (i) Does every $A$-compatible positive superoperator have the corresponding measuring apparatus? (ii) If not, what condition does ensure the existence of the corresponding measuring apparatus?

**VII. MEASUREMENTS OF NONDEGENERATE DISCRETE OBSERVABLES**

In this section, we confine our attention to the observables with nondegenerate eigenvalues. In this case, the projection $E^A(a_n)$ is of rank 1 and is the density operator representing the eigenstate, so that we have

$$E^A(a_n)\rho E^A(a_n) = \text{Tr}[E^A(a_n)\rho]E^A(a_n).$$

Let $T$ be an $A$-compatible positive superoperator. From (35), we have

$$T[E^A(a_n)\rho] = \text{Tr}[E^A(a_n)\rho]T[E^A(a_n)].$$

(40)
We define a sequence \( \{ \varrho_n \} \) of density operators by
\[
\varrho_n = T[E^A(a_n)].
\] (41)
Then, we have
\[
T(\rho) = \sum_n \text{Tr}[E^A(a_n)\rho] \varrho_n.
\] (42)
Conversely, for any sequence \( \{ \varrho_n \} \) of density operators, we define the positive superoperator \( T \) on \( \tau_c(\mathcal{H}) \) by (42). Then, \( T \) is an \( A \)-compatible positive superoperator satisfying (41). Thus, we have proved the following theorem.

**Theorem 5.** Let \( A \) be a nondegenerate discrete self-adjoint operator. The relation
\[
T(\rho) = \sum_n \text{Tr}[E^A(a_n)\rho] \varrho_n,
\]
where \( \rho \in \tau_c(\mathcal{H}) \), sets up a one-to-one correspondence between the families \( \{ \varrho_x \mid x \in \mathbb{R} \} \) of density operators and the \( A \)-compatible positive superoperators \( T \) on \( \tau_c(\mathcal{H}) \).

Let \( \rho_{x=a_n} \) be the output state of an apparatus measuring \( A \) for the input state \( \rho \). Then, there is an \( A \)-compatible positive superoperator \( T \) satisfying (39) and there is a sequence \( \{ \varrho_n \} \) of density operators satisfying (41), so that we have
\[
\rho_{x=a_n} = \frac{T[E^A(a_n)\rho]}{\text{Tr}[E^A(a_n)\rho]} = T[E^A(a_n)] = \varrho_n.
\]
It follows that the output state for the output \( x = a_n \) is given by
\[
\rho_{x=a_n} = \varrho_n.
\] (43)

From the above argument we conclude the following: For any apparatus \( A(x) \) measuring a nondegenerate discrete observable \( A = \sum_n a_n |\phi_n\rangle\langle \phi_n| \), there is a sequence \( \{ \varrho_n \} \) of density operators such that the statistical property of \( A(x) \) is represented as follows.

output distribution: \( \Pr\{x = a_n \mid \rho\} = \langle \phi_n | \rho | \phi_n \rangle \) (44)
output state: \( \rho_{x=a_n} = \varrho_n \) (45)

It follows that the problem of determining all the possible quantum state reductions arising in the measurement of a nondegenerate discrete observable \( A \) is reduced to the problem as to what sequence \( \{ \varrho_n \} \) of states can be obtained from the measurement of \( A \). In order to obtain the answer to this question, in the next section we shall consider indirect measurement models and ask what sequences can be obtained from those models.

It should be noted here that the apparatus satisfies the projection postulate if and only if we have
\[
\varrho_n = E^A(a_n)
\]
for all \( n \). Von Neumann [1, pp. 439–442] showed that this case can be obtained from an indirect measurement model.
VIII. INDIRECT MEASUREMENT MODELS

In general, if a measurement on the object in the input state $\rho$ by the apparatus $A(x)$ is immediately followed by a measurement of the observable $B$ by the apparatus $A(b)$, the joint probability distribution of their output variables is given by (12). Now, consider the marginal probability

$$\text{Pr}\{x \in \mathbb{R}, b = b_m\|\rho\} = \sum_{x \in \mathbb{R}} \text{Pr}\{x = x, b = b_m\|\rho\}. \quad (46)$$

Then, this represents the probability of obtaining the outcome $b = b_m$ after interacting the apparatus $A(x)$ with the object without reading out the outcome of the $x$ measurement. Such a process is called a nonselective measurement. Let $T$ be the nonselective operation of the apparatus $A(x)$. Then, by (12), we have

$$\text{Pr}\{x \in \mathbb{R}, b = b_m\|\rho\} = \text{Tr}[E^B(b_m)T\rho]. \quad (47)$$

Thus, the nonselective measurement transforms the input state $\rho$ to the output state $\rho_{\{x \in \mathbb{R}\}} = T\rho$.

Let us call any interaction between the object and the apparatus caused by a measurement as the measuring interaction. Then, the superoperator $T$ is determined by the measuring interaction. In what follows we shall examine the properties of the measuring interaction.

Since the nonselective measurement transforms the input state $\rho$ to the output state $T\rho$, there should be an interaction during finite time interval when the object changes from $\rho$ to $T\rho$. Moreover, the object should be free from the apparatus before and after the interaction. Thus, we suppose that the measuring interaction turns on from the time $t$ just before the measurement to the time $t + \Delta t$ just after the measurement where $\Delta t > 0$, and that the object is free from the apparatus before the the time $t$ and after the time $t + \Delta t$. It follows that if the second measurement on the same object follows immediately after the above measurement, the time just before the second measurement coincides with the time $t + \Delta t$ just after the first measurement. In this way, the temporal boundary of the measuring interaction is determined as a fixed domain from time $t$ to $t + \Delta t$.

Next, in order to determine the spatial boundary of the measuring interaction, we consider the smallest subsystem of the measuring apparatus such that the composite system of the object and the subsystem is isolated from the time $t$ to the time $t + \Delta t$. We call the above subsystem as the prove.

The effect of the measuring interaction is given by the change of an observable $M$, called the probe observable, from $t$ to $t + \Delta t$. From the minimality of the probe, it is natural to assume that the interaction Hamiltonian excludes any macroscopic part of the measuring apparatus such as the macroscopic pointer position. It follows that the measuring interaction is a quantum mechanical interaction and the state change can be described by the unitary time evolution of the composite system of the object and the probe.

On the other hand, in order to transduce the microscopic change in the probe observable $M$ to the macroscopic change such as the change of the position of the pointer, we need an amplification process in the apparatus after $t + \Delta t$. This transduction from a microscopic observable to a macroscopic observable corresponds to the direct measurement of the probe.
observable \( M \) at the time \( t + \Delta t \). The problem of describing this process as a dynamical process belongs to the so-called measurement problem. Within quantum mechanics, the Born statistical formula gives the probability distribution of the outcome of the \( M \) measurement. Let \( t + \Delta t + \tau \) be the time just after this amplification process where \( \tau > 0 \). This time is called the time of read-out.

According to the above description, the process from the time just before the measurement to the time of read-out is divided into the measuring interaction and the amplification. It should be noted that just after the measuring interaction, the object is free from the apparatus so that it is possible to start the interaction with the second apparatus. It follows that in the successive measurement experiment the time just before the second measurement is considered to be the time just after the measuring interaction rather than the time of read-out. The above description of measuring process is called an indirect measurement description.

Let \( \mathcal{H} \) be the state space of the object \( S \), and \( \mathcal{K} \) the state space of the probe \( P \). The state of the object at the time \( t \) of measurement is the input state \( \rho \). The probe \( P \) is supposed to be prepared in the fixed state \( \sigma \) at the time of measurement. Thus, the state of the composite system at the time \( t \) is

\[
\rho_{S+P}(t) = \rho \otimes \sigma.
\]

If the time evolution of the composite system \( S + P \) from \( t \) to \( t + \Delta t \) is represented by the unitary operator \( U \), the composite system is in the state

\[
\rho_{S+P}(t + \Delta t) = U(\rho \otimes \sigma)U^\dagger
\]

at \( t + \Delta t \). Suppose that the \( A(x) \) measurement in \( \rho \) is followed immediately by a measurement of an observable \( B \) carried out by \( A(b) \). Then, the observable \( B \) is measured at the time \( t + \Delta t \) and the outcome is recorded by \( b \). On the other hand, the probe observable \( M \) is also measured actually at the time \( t + \Delta t \) and the outcome is recorded by \( x \). Since the two measurements are carried out locally, it follows from the local measurement theorem [25,26] that the joint probability distribution of the outcomes of the above two measurements satisfies

\[
\Pr\{x = x, b = b_m\|\rho\} = Tr[(E^B(b_m) \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger]
\]

\[
= Tr[E^B(b_m)Tr_{\mathcal{K}}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger]],
\]

where \( Tr_{\mathcal{K}} \) is the partial trace over the Hilbert space \( \mathcal{K} \). Thus, from (12) we have

\[
X(x)\rho = Tr_{\mathcal{K}}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger].
\]

Hence, the statistical property of the apparatus \( A(x) \) is given as follows.

output distribution:

\[
\Pr\{x = x\|\rho\} = Tr[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger]
\]

output state:

\[
\rho_{x=x} = \frac{Tr_{\mathcal{K}}[(I \otimes E^A(x))U(\rho \otimes \sigma)U^\dagger]}{Tr[(I \otimes E^A(x))U(\rho \otimes \sigma)U^\dagger]}
\]
From (28) and (50), the nonselective operation of $A(x)$ is given by
\[ T \rho = \text{Tr}_K[U(\rho \otimes \sigma)U^\dagger]. \] (53)

From (25) and (51), the effect distribution of $A(x)$ is given by
\[ X(x) = \text{Tr}_K[U^\dagger(I \otimes E^M(x))U(I \otimes \sigma)]. \] (54)

In general, a four tuple $(K, \sigma, U, M)$ is called an \textit{indirect measurement model} iff it consists of a separable Hilbert space $K \cong B(\mathcal{H})$ a density operator $\sigma$ on $K$, a unitary operator $U$ on $\mathcal{H} \otimes K$, and a self-adjoint operator $M$ on $K$. So far we have not posed any sufficient condition for the existence of an apparatus except that every observable has at least one apparatus to measure it. Here, we pose the following hypothesis.

**Unitary realizability hypothesis:** For any indirect measurement model $(K, \sigma, U, M)$, there is an apparatus $A(x)$ with the following statistical property:

output distribution:
\[ \text{Pr}\{x = x\|\rho\} = \text{Tr}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger] \]

output state:
\[ \rho\{x=x\} = \frac{\text{Tr}_K[(I \otimes E_A(x))U(\rho \otimes \sigma)U^\dagger]}{\text{Tr}[(I \otimes E^A(x))U(\rho \otimes \sigma)U^\dagger]} \]

A superoperator distribution $\{X(x)\mid x \in \mathbb{R}\}$ is said to be \textit{realized} by an indirect measurement model $(K, \sigma, U, M)$ iff (50) holds for any $\rho \in \tau_c(\mathcal{H})$, and in this case it is called \textit{unitarily realizable}. Under the unitary realizability hypothesis, unitarily realizable superoperator distributions are operational distributions of some apparatuses. In the next section, we shall give an intrinsic characterization of the unitarily realizable superoperator distributions.

**IX. COMPLETE POSITIVITY**

Let $D = \tau_c(\mathcal{H})$ or $D = \mathcal{L}(\mathcal{H})$. A linear transformation $L$ on $D$ is called \textit{completely positive (CP)} iff for any finite sequences of bounded operators $A_1, \ldots, A_n \in D$ and vectors $\xi_1, \ldots, \xi_n \in \mathcal{H}$ we have
\[ \sum_{ij} \langle \xi_i | L(A_i^\dagger A_j) | \xi_j \rangle \geq 0. \]

The above condition is equivalent to that $L \otimes I$ maps positive operators in the algebraic tensor product $D \otimes \mathcal{L}(\mathcal{K})$ to positive operators in $D \otimes \mathcal{L}(\mathcal{K})$ for any Hilbert space $\mathcal{K}$. Obviously, every CP superoperators are positive. A superoperator is CP if and only if its dual superoperator is CP. A superoperator distribution $\{X(x)\mid x \in \mathbb{R}\}$ is called \textit{completely positive} iff every $X(x)$ is CP. It can be seen easily from (50) that unitarily realizable superoperator distributions are CP. Conversely, the following theorem, proved in [21,4] for an even more general formulation, asserts that every CP superoperator distribution is unitarily realizable.
Theorem 6. For any CP superoperator distribution \( \{X(x)\mid x \in \mathbb{R}\} \), there is a separable Hilbert space \( \mathcal{K} \), a unit vector \( \Phi \) in \( \mathcal{K} \), a unitary operator \( U \) on \( \mathcal{H} \otimes \mathcal{K} \), and a discrete self-adjoint operator \( M \) on \( \mathcal{K} \) satisfying the relation

\[
X(x)\rho = \text{Tr}_\mathcal{K}[(I \otimes E^M(x))U(\rho \otimes \sigma)U^\dagger].
\]

for all \( \rho \in \tau(cH) \).

For any trace preserving CP superoperator \( T \), we have a CP superoperator distribution \( \{X(x)\mid x \in \mathbb{R}\} \) such that \( X(0) = T \) and that \( X(x) = 0 \) for all \( x \neq 0 \). Applying the above theorem to this family, we obtain the following representation theorem to this family, we obtain the following representation theorem of trace preserving CP superoperators, which was proved independently by Kraus [27] and the present author [21].

Theorem 7. For any trace preserving CP superoperator \( T \), there is a separable Hilbert space \( \mathcal{K} \), a unit vector \( \Phi \) in \( \mathcal{K} \), a unitary operator \( U \) on \( \mathcal{H} \otimes \mathcal{K} \), such that \( T \) satisfies the relation

\[
T\rho = \text{Tr}_\mathcal{K}[U(\rho \otimes |\Phi\rangle\langle\Phi|)U^\dagger].
\]

for all \( \rho \in \tau(\mathcal{H}) \).

From Theorem 6, every \( A \)-compatible superoperator distribution \( \{X(x)\mid x \in \mathbb{R}\} \) satisfies the relation \( X(x)\rho = T[E^A(x)\rho E^A(x)] \). Thus, if \( \{X(x)\mid x \in \mathbb{R}\} \) is CP, then the total map \( T = \sum_{x \in \mathbb{R}} X(x) \) is CP, since the sum of CP superoperators is CP. Conversely, if \( T \) is an \( A \)-compatible CP superoperator, then the corresponding \( A \)-compatible superoperator distribution \( \{X(x)\mid x \in \mathbb{R}\} \) is CP, since the superoperator \( \rho \mapsto E^A(x)\rho E^A(x) \) is CP and the composition of any CP superoperators is CP. Thus we have the following:

Theorem 8. Let \( A \) be a nondegenerate discrete self-adjoint operator. Then, an \( A \)-compatible superoperator distribution is CP if and only if its total superoperator is CP.

From the above theorem, we conclude the following [4]: The statistical equivalence classes of apparatuses \( A(x) \) measuring a discrete observable \( A \) with indirect measurement models are in one-to-one correspondence with the \( A \)-compatible CP superoperators, where the statistical property is represented by \( \{33\} \) and \( \{38\} \).

Now, let \( A \) be a nondegenerate discrete observable and let \( T \) be an \( A \)-compatible positive superoperator. Then, \( T \) is of the form \( \{12\} \). Let \( \sigma_1, \ldots, \sigma_n \in \tau(\mathcal{H}) \) and \( \xi_1, \ldots, \xi_n \in \mathcal{H} \). Then, we have

\[
\sum_{ij} \langle \xi_i | T(\sigma_i^\dagger \sigma_j) | \xi_j \rangle = \sum_n \sum_{ij} \text{Tr}[E^A(a_n)\sigma_i^\dagger \sigma_j] \langle \xi_i | \xi_j \rangle \geq 0,
\]

where the last inequality follows from the fact that the trace of the product of two positive definite matrices \( \text{Tr}[E^A(a_n)\sigma_i^\dagger \sigma_j] \) and \( \langle \xi_i | \xi_j \rangle \) is nonnegative. It follows that \( T \) is a CP superoperator. Thus, every \( A \)-compatible superoperator is CP. Since every \( A \)-compatible superoperator distribution is obtained from an \( A \)-compatible superoperator by Theorem 4, it follows from Theorem 8 that every \( A \)-compatible superoperator distribution is CP. We have therefore obtained the following statements.
Theorem 9. Let \( A \) be a nondegenerate discrete self-adjoint operator. Every \( A \)-compatible positive superoperator is completely positive. Every \( A \)-compatible superoperator distribution is completely positive.

From the above theorem and Theorem 3, we conclude: Every apparatus measuring \( A \) is statistically equivalent to the one having an indirect measurement model.

Every sequence \( \{ \rho_n \} \) of density operators defines an \( A \)-compatible positive superoperator by Theorem 5, and it is automatically completely positive so that it is realized by an indirect measurement model. Thus, we have reached the answer to the question what sequence of states can be obtained from an apparatus measuring \( A \) that every sequence can. Thus, we conclude the following: The statistical equivalence classes of apparatuses \( A(x) \) measuring a nondegenerate discrete observable \( A \) are in one-to-one correspondence with the sequences \( \{ \rho_n \} \) of density operators, where the statistical property is represented by (44) and (45).

Given any sequence \( \{ \rho_n \} \), an indirect measurement model with the quantum state reduction \( \rho \mapsto \rho_{\{x=x_n\}} = \rho_n \) is constructed explicitly as follows. Let \( \{ \phi_n \} \) be an orthonormal basis of \( \mathcal{H} \) consisting of the eigenvectors of \( A \). Let \( \mathcal{K} = \mathcal{H} \otimes \mathcal{H} \). Let

\[
\rho_n = \sum_j \lambda_{nj} |\eta_{nj}\rangle \langle \eta_{nj}|
\]

be the spectral decomposition of \( \rho_n \). Then, there exists a unitary operator \( U \) on \( \mathcal{H} \otimes \mathcal{K} \) satisfying

\[
U |\phi_n \otimes \phi_0 \otimes \phi_0\rangle = \sum_j \sqrt{\lambda_{nj}} |\eta_{nj} \otimes \phi_j \otimes \phi_0\rangle.
\]

Now, we define the density operator \( \sigma \) on \( \mathcal{K} \) by \( \sigma = |\phi_0 \otimes \phi_0\rangle \langle \phi_0 \otimes \phi_0| \) and define a self-adjoint operator \( M \) on \( \mathcal{K} \) by \( M = I \otimes A \). Then, we have the indirect measurement model \( (\mathcal{K}, \sigma, U, M) \) such that the statistical property of its apparatus satisfies (44) and (45).

**X. CONCLUSIONS**

Let \( A(x) \) be an apparatus with the discrete output variable \( x \). Then, depending on the input state \( \rho \) and the outcome \( x \), the apparatus \( A(x) \) determines the output probability \( \Pr\{ x = x \| \rho \} \) and the output state \( \rho_{\{x=x\}} \). The transformation from the input state \( \rho \) to the output distribution \( \Pr\{ x = x \| \rho \} \) is called the objective state reduction and the one from the input state \( \rho \) to the output states \( \rho_{\{x=x\}} \) is called the quantum state reduction. The pair of the objective state reduction and the quantum state reduction is called the statistical property of the apparatus \( A(x) \). Two apparatuses with the same statistical property is said to be statistically equivalent. In order to obtain a mathematical description of quantum state reductions for the most general class of measurements we have considered two requirements: one is necessary and the other is sufficient.

The necessary one is the mixing law of the joint probability. Suppose that a measurement carried out by an apparatus \( A(x) \) in the input state \( \rho \) is followed immediately by another...
measurement carried out by another apparatus $A(y)$. The joint probability distribution of the outcomes $x$ and $y$ is determined by their statistical properties as follows.

$$\Pr\{x = x, y = y\|\rho\} = \Pr\{y = y\|\rho_{\{x=x\}}\} \Pr\{x = x\|\rho\}. $$

This joint probability distribution is considered to respect the mixture of input states and the mixing law of the joint probability requires that this is the case for any apparatuses $A(x)$ and $A(y)$. Under this hypothesis, any apparatus $A(x)$ has a superoperator distribution $\{X(x)\mid x \in \mathbb{R}\}$, called the operational distribution of $A(x)$, satisfying

$$X(x)\rho = \Pr\{x = a\|\rho\}\rho_{\{x=x\}}. \quad (55)$$

The sufficient condition is the unitary realizability condition. The apparatus $A(x)$ is said to have an indirect measurement model $(\mathcal{K}, \sigma, U, M)$ iff the statistical property of $A(x)$ is given as follows.

output distribution:

$$\Pr\{x = x\|\rho\} = \text{Tr}[(I \otimes E^M(x))U(\rho \otimes I)U^\dagger]$$

output state:

$$\rho_{\{x=x\}} = \frac{\text{Tr}_K[(I \otimes E^M(x))U(\rho \otimes I)U^\dagger]}{\text{Tr}[(I \otimes E^M(x))U(\rho \otimes I)U^\dagger]}$$

In general, an apparatus has an indirect measurement model if and only if its operational distribution is completely positive. The unitary realizability hypothesis states that every indirect measurement model defines an apparatus with the above statistical property. It follows that the statistical equivalence classes of apparatuses with indirect measurement models are in one-to-one correspondence with the CP superoperator distributions $\{X(x)\mid x \in \mathbb{R}\}$, under the relation (55).

Let $A$ be a discrete observable. A trace preserving positive superoperator $L$ is called $A$-compatible iff the range of its dual $L^*$ is included in the commutant $\{A\}'$ of $A$. The statistical property of an apparatus measuring an observable $A$ is represented by an $A$-compatible positive superoperator $T$ as follows.

output distribution:

$$\Pr\{x = x\|\rho\} = \text{Tr}[E^A\rho] \quad (56)$$

output state:

$$\rho_{\{x=x\}} = \frac{T[E^A(x)\rho]}{\text{Tr}[E^A(x)\rho]} \quad (57)$$

In particular, the statistical equivalence classes of apparatuses with indirect measurement models measuring $A$ are in one-to-one correspondence with the $A$-compatible completely positive superoperators $T$, under the above description.

According to the above, the class of possible quantum state reductions is included in the class of conditional state transformations satisfying the mixing law, i.e., the general superoperator distributions, and includes the one satisfying the unitary realizability condition, i.e., the completely positive superoperator distributions. Since these two classes are generally different, there seems to be still a room for the debate in measurement theory on what class between them is the true class of all the possible quantum state reductions.
Nevertheless, for the case where $A$ is nondegenerate, this paper shows, the above two conditions are actually equivalent. Thus, both of them are necessary and sufficient and we reach a clear-cut conclusion. In fact, if $A$ is nondegenerate, all the $A$-compatible positive superoperators $T$ are completely positive and they are in one-to-one correspondence with the sequences $\{q_n\}$ of density operators, under the relation $T(E^A(a_n)) = q_n$ where $\{a_n\}$ is the sequence of the eigenvalues of $A$. In this case, every apparatus measuring $A$ is statistically equivalent with the one with an indirect measurement model. The statistical equivalence classes of the apparatuses measuring $A$ are, therefore, in one-to-one correspondence with the sequences $\{q_n\}$ of density operators and their statistical properties are represented as follows.

output distribution: $\Pr\{x = a_n|\rho\} = \text{Tr}[E^A(a_n)\rho]$  \hspace{1cm} (58)
output states: $\rho_{x=a_n} = q_n$  \hspace{1cm} (59)

The above measurement statistics has the following two remarkable features: (i) The output states are independent of the input state. (ii) The family of output states can be arbitrarily chosen by the choice of the apparatus. The possibility of this kind of generalized measurements was first pointed out in part by Gordon and Louisell [28] relative to the measurement of an overcomplete family of states generalizing the conventional measurement of an orthonormal basis. Yuen [7] generalized the Gordon-Louisell description to the following measurement described by the set of operators $\{|\Psi_x\rangle\langle\Phi_x|\}$, where $\{\Phi_x\}$ is an overcomplete family of vectors and $\{\Psi_x\}$ is a Borel family of state vectors, as follows.

output distribution: $\Pr\{x \in dx|\rho\} = \langle\Phi_x|\rho|\Phi_x\rangle dx$
output states: $\rho_{x=x} = |\Psi_x\rangle\langle\Psi_x|$  \hspace{1cm} (60)

The unitary realizability of the above measurement statistics was assumed by Yuen [7] to claim the realizability of the contractive state measurement and proved rigorously in [29]; see [10] for survey. We can see that for the nondegenerate discrete observable $A = \sum_n a_n|\Phi_n\rangle\langle\Phi_n|$ and the output states $q_n = |\Psi_n\rangle\langle\Psi_n|$ the measurement statistics given in (58) and (59) corresponds to the (discrete version of) measurement described by $\{|\Psi_n\rangle\langle\Phi_n|\}$. The present paper has proved rigorously, even without assuming the unitary realizability, that every measurement of a nondegenerate discrete observable is always of this form.

Along with the analogous arguments, it can be shown that the statistical equivalence classes of the apparatuses measuring a nondegenerate (but not necessarily discrete) observable including the position or the momentum observable are in one-to-one correspondence with the Borel families of density operators (modulo the spectral measure). Since the precise mathematical formulation for that result is beyond the scope of this paper, we shall discuss the nondiscrete case in a separate article.

Therefore, we can conclude that as long as the statistical properties of measurements of nondegenerate observables are concerned, we can always assume that the measuring process are described by an indirect measurement model in which the interaction between the object and the apparatus is described by a unitary operator. For measurements of degenerate observables and even for measurements of general probability operator valued measures, it appears to be an important question whether every apparatus is statistically equivalent with the one having the indirect measurement model that has the unitary measuring interaction.
Since in this case there are many superoperator distributions (or normalized PSV measures) that are not completely positive \([30]\), we need further physical requirements to settle this problem.

Following von Neumann \([1]\), some authors appear to support the hypothesis that every apparatus has an indirect measurement model, the converse of the unitary realizability hypothesis. If this is the case, the description of measuring processes will be simplified considerably as shown in Section \([\textbf{VII}]\). In particular, we have an instant of time at which the measuring process is divided into the measuring interaction and the amplification process (including the so-called decoherence process) and the output state has been prepared for the next measurement before the amplification mode of the first measurement \([24]\). It is also interesting whether non-conventional quantum mechanics such as nonlinear quantum mechanics will provide a different measurement statistics from the unitarily realizable ones.

**APPENDIX A: LINEAR EXTENSION OF THE QUANTUM STATE REDUCTION**

For any \(x \in \mathbb{R}\) and any density operator \(\rho\), the trace class operator \(X(x, \rho)\) is defined by \([1]\). In this section, we shall prove that the mapping \(X(x) : \rho \mapsto X(x, \rho)\) defined on the space of density operators can be extended uniquely to a linear transformation on the space \(\tau_c(\mathcal{H})\) of trace class operators on \(\mathcal{H}\). By the linearity of the extension, for any trace class operator \(\sigma\) with decomposition \([14]\) it is necessary for \(X(x)\sigma\) to be defined by \([15]\). Since the decomposition \([14]\) is not unique, in order for the extension \([15]\) to be well-defined we need to show that the right hand side of \([14]\) is uniquely determined independent of the decomposition of \(\sigma\). Namely, we need to prove that if \(\sigma\) has another decomposition

\[
\sigma = \lambda'_1 \sigma'_1 - \lambda'_2 \sigma'_2 + i\lambda'_3 \sigma'_3 - i\lambda'_4 \sigma'_4, \tag{A1}
\]

then we have

\[
\begin{align*}
\lambda_1 X(x)\sigma_1 - \lambda_2 X(x)\sigma_2 + i\lambda_3 X(x)\sigma_3 - i\lambda_4 X(x)\sigma_4 \\
= \lambda'_1 X(x)\sigma'_1 - \lambda'_2 X(x)\sigma'_2 + i\lambda'_3 X(x)\sigma'_3 - i\lambda'_4 X(x)\sigma'_4.
\end{align*}
\tag{A2}
\]

The proof runs as follows \([31]\). By equating the right hand sides of \([14]\) and \([A1]\) and comparing the real and imaginary parts in both sides, we have

\[
\begin{align*}
\lambda_1 \sigma_1 + \lambda'_2 \sigma'_2 &= \lambda'_1 \sigma'_1 + \lambda_2 \sigma_2 \tag{A3} \\
\lambda_3 \sigma_3 + \lambda'_4 \sigma'_4 &= \lambda'_3 \sigma'_3 + \lambda_4 \sigma_4. \tag{A4}
\end{align*}
\]

Taking the trace of both sides of \([A3]\), we have

\[
\lambda_1 + \lambda'_2 = \lambda'_1 + \lambda_2. \tag{A5}
\]

By dividing both sides of \([A3]\) by this value, we have

\[
\alpha \sigma_1 + (1 - \alpha) \sigma'_2 = \beta \sigma'_1 + (1 - \beta) \sigma_2,
\]
where we define $\alpha$ and $\beta$ by

$$0 < \alpha = \frac{\lambda_1}{\lambda_1 + \lambda_2} < 1$$

$$0 < \beta = \frac{\lambda'_1}{\lambda'_1 + \lambda_2} < 1.$$  

Thus, from (13) we have

$$\alpha X(x)\sigma_1 + (1 - \alpha)X(x)\sigma'_2 = \beta X(x)\sigma'_1 + (1 - \beta)X(x)\sigma_2.$$  

Multiplying both sides by the value of (A5), we have

$$\lambda_1 X(x)\sigma_1 - \lambda_2 X(x)\sigma_2 = \lambda'_1 X(x)\sigma'_1 - \lambda'_2 X(x)\sigma'_2.$$  

By the similar manipulations for (A4), we have

$$i\lambda_3 X(x)\sigma_3 - i\lambda_4 X(x)\sigma_4 = i\lambda'_3 X(x)\sigma'_3 - i\lambda'_4 X(x)\sigma'_4.$$  

Thus, we have proved equation (A2). It is concluded, therefore, that $X(x)\sigma$ is defined uniquely for every $\sigma$ by (15).

**APPENDIX B: PROOF OF THEOREM 3**

Let $\{X(x)\mid x \in \mathbb{R}\}$ be an $A$ compatible family of positive maps and $T$ its total map. Let $C$ be a bounded operator such that $0 \leq C \leq I$ and let $x \in \mathbb{R}$. We define

$$A_{11} = X(x)^*C,$$

$$A_{12} = X(x)^*(I - C),$$

$$A_{21} = \sum_{y \neq x} X(y)^*C,$$

$$A_{22} = \sum_{y \neq x} X(y)^*(I - C),$$

$$P_1 = E^A(x),$$

$$P_2 = I - E^A(x),$$

$$Q_1 = T^*(C),$$

$$Q_2 = I - T^*(C).$$

Then $0 \leq A_{ij} \leq P_i$, so that $[A_{ij}, P_i] = [A_{ij}, P_j] = 0$. It follows that $Q_j = A_{ij} + A_{2j}$ commutes with $P_1$ and $P_2$ as well. Thus,

$$A_{ij} = P_i A_{ij} \leq P_i Q_j.$$  

On the other hand, we have $\sum_{i} A_{ij} = I$ and $\sum_{i} P_i Q_j = I$, whence $A_{ij} = P_i Q_j$. It follows that $X(x)^*C = E^A(x)T^*(C)$. Since any bounded operator $B$ can be represented by $B = \sum_{n=0}^{\infty} i^n \lambda_n C_n$ with positive operators $0 \leq C_n \leq I$ and positive reals $\lambda_n$, we have $X(x)^*B = E^A(x)T^*(B)$ for any real number $x$ and bounded operator $B$. Since $[E^A(x), T^*(B)] = 0$, other assertions follow immediately.
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