Poisson approximation to the binomial distribution: extensions to the convergence of positive operators

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Abstract
The idea behind Poisson approximation to the binomial distribution was used in de la Cal and Luquin (J Approx Theory 68(3):322–329, 1992) and subsequent papers in order to establish the convergence of suitable sequences of positive linear operators. The proofs in these papers are given using probabilistic methods. We use similar methods, but in analytic terms. In this way we recover some known results and establish several new ones. In particular, we enlarge the list of the limit operators.

Keywords Poisson approximation to binomial distribution · Positive linear operators · Convergence · Limit operator

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1 Introduction

Let $X_n(p_n)$ be a binomial random variable with parameters $n \in \mathbb{N}$ and $p_n \in [0, 1]$. Its characteristic function is

$$g_n(s) = E \left( \exp(i s X_n(p_n)) \right) = \sum_{k=0}^{n} \binom{n}{k} p_n^k (1 - p_n)^{n-k} \exp(i s k)$$

$$= (1 - p_n + p_n e^{i s})^n, \quad s \in \mathbb{R},$$

where $E$ stands for mathematical expectation and $i$ is the imaginary unit.

Let $X(\lambda)$ be a Poisson variable with parameter $\lambda \geq 0$, having the characteristic function

$$h(s) = \sum_{k=0}^{\infty} e^{-\lambda \frac{k}{m}} \frac{\lambda^k}{k!} \exp(i s k) = e^{\lambda(e^{i s} - 1)}, \quad s \in \mathbb{R}.$$

Suppose that $\lim_{n \to \infty} n p_n = \lambda$. Then $\lim_{n \to \infty} g_n(s) = h(s), \quad s \in \mathbb{R}$, and hence $X_n(p_n)$ converges in law to $X(\lambda)$. This is the celebrated result about “Poisson approximation to the binomial distribution”, see, e.g., [20, 27]. It implies

$$\lim_{n \to \infty} E \varphi \left( X_n(p_n) \right) = E \varphi \left( X(\lambda) \right), \quad \varphi \in \mathcal{C}_b[0, \infty),$$

where $\mathcal{C}_b[0, \infty)$ is the space of all real valued, continuous and bounded functions on $[0, \infty)$.

Now let $x \geq 0, m \in \mathbb{N}, p_n = \frac{x}{n}, \lambda = mx$. From (1.1) we deduce

$$\lim_{n \to \infty} E \varphi \left( X_{mn} \left( \frac{x}{n} \right) \right) = E \varphi \left( X(mx) \right), \quad \varphi \in \mathcal{C}_b[0, \infty).$$

This can be written as

$$\lim_{n \to \infty} \sum_{k=0}^{mn} \binom{mn}{k} \left( \frac{x}{n} \right)^k \left( 1 - \frac{x}{n} \right)^{mn-k} \varphi \left( k \right) = \sum_{k=0}^{\infty} e^{-mx \frac{(mx)^k}{k!}} \varphi \left( k \right).$$

Let $f \in \mathcal{C}_b[0, \infty)$ and $\varphi \in \mathcal{C}_b[0, \infty), \varphi(t) = f \left( \frac{t}{m} \right), t \geq 0$. Now (1.2) becomes

$$\lim_{n \to \infty} \sum_{k=0}^{mn} \binom{mn}{k} \left( \frac{x}{n} \right)^k \left( 1 - \frac{x}{n} \right)^{mn-k} f \left( \frac{k}{m} \right) = \sum_{k=0}^{\infty} e^{-mx \frac{(mx)^k}{k!}} f \left( \frac{k}{m} \right).$$

In terms of Bernstein and Szász–Mirakyan operators this means that

$$\lim_{n \to \infty} B_{mn} \left( f(nt); \frac{x}{n} \right) = B_{m}^{[0]}(f(t); x), \quad f \in \mathcal{C}_b[0, \infty).$$

This result and similar ones involving other sequences of operators were obtained in [6–8, 16–18] using probabilistic methods and making references to several probability distributions. Rates of convergence in relations like (1.3) can be found in the same papers and the references therein.

In our paper we present a general result extending (1.3), using analytic terms and analytic methods. We apply it to several families of positive linear operators recovering some known results and establishing new ones.

In (1.3) a specific modification of the Bernstein operators, inspired by Poisson approximation to the binomial distribution, is presented. We will consider several other modifications
corresponding to suitable families of operators. As limit operator the Szász–Mirakjan operator appears in (1.3). It will have the same role in several other examples. But other operators will also appear as limit operators and we will present some of them.

Let $L_n$ and $L$ be positive linear operators associated with random variables. In Sect. 2 we show that the family of the complex exponentials $t \in \mathbb{R} \rightarrow e^{ist}$, indexed by $s \in \mathbb{R}$, can be used as test family for the convergence of sequences $L_n$ toward the operator $L$. This is presented in Theorem 2.1 which is the main result of our paper. We will apply Theorem 2.1 to several classical or new sequences of positive linear operators.

Sections 3–6 are devoted to sequences for which the limit operator is the classical Szász–Mirakjan operator. The Gamma operator, the Weierstrass operator and the Jakimovski–Leviatan operator appear as the limit operators in Sects. 7, 8, 9, respectively.

The necessary definitions and notations will be presented in the next sections.

2 The complex exponentials as test functions for convergence of operators

Let $I \subseteq \mathbb{R}$ be an interval and $C_b(I)$ the space of all real-valued continuous and bounded functions on $I$. Let $Z^x, Z_1^x, Z_2^x, \ldots$ be $I$-valued random variables whose probability distributions depend on a real parameter $x \in I$. Suppose that for each $f \in C_b(I)$ the functions $x \rightarrow Ef(Z^x)$ and $x \rightarrow Ef(Z_n^x), x \in I, n \geq 1$, are continuous on $I$.

Consider the positive linear operators

$$L_n : C_b(I) \rightarrow C_b(I), \quad n \geq 1,$$

and

$$L : C_b(I) \rightarrow C_b(I),$$

defined for $f \in C_b(I)$ and $x \in I$ by

$$L_n(f(t); x) := Ef(Z_n^x), \quad L(f(t); x) := Ef(Z^x). \quad (2.1)$$

For each $s \in \mathbb{R}$ we consider the function $t \rightarrow e^{ist}, t \in I$, and define

$$L_n(e^{ist}; x) = L_n(\cos(st); x) + i L_n(\sin(st); x)$$

and similarly for $L(e^{ist}; x)$.

**Theorem 2.1** Suppose that for each $s \in \mathbb{R}$ and $x \in I$,

$$\lim_{n \rightarrow \infty} L_n(e^{ist}; x) = L(e^{ist}; x). \quad (2.2)$$

Then,

$$\lim_{n \rightarrow \infty} L_n(f(t); x) = L(f(t); x) \quad (2.3)$$

for all $f \in C_b(I)$ and $x \in I$.

**Proof** Let $g_n^x(s)$ and $g^x(s)$ be the characteristic functions of $Z_n^x$ and $Z^x, n \geq 1, x \in I$. Then, according to (2.1)

$$g_n^x(s) := E(e^{isZ_n^x}) = L_n(e^{ist}; x),$$

$$g^x(s) := E(e^{isZ^x}) = L(e^{ist}; x).$$

Now (2.2) shows that

$$\lim_{n \rightarrow \infty} g_n^x(s) = g^x(s), \quad s \in \mathbb{R}.$$
Moreover, the function \( g^x(s) \) is uniformly continuous for any \( x \in I \). Indeed, we have
\[
|g^x(s + h) - g^x(s)| \leq E|\exp(ihZ^x) - 1| \to 0, \ h \to 0,
\]
as follows from the dominated convergence theorem.

Consequently we can apply the continuity theorem of Lévy (see [20, 27]) to conclude that the sequence \( (Z^x_n)_{n \geq 1} \) converges in law to \( Z^x \). Furthermore, this implies (see [20, 27])
\[
\lim_{n \to \infty} Ef(Z^x_n) = Ef(Z^x), \tag{2.4}
\]
and now (2.3) is a consequence of (2.1) and (2.4). \( \square \)

In the next sections we will be concerned with positive linear operators \( L_n \) and \( L \) for which it is easy to identify the corresponding random variables. See also [6–8, 16–18].

### 3 The Baskakov type operators \( B^{[c]}_n \)

Let \( c \in \mathbb{R}, n \in \mathbb{R}, n > c \) for \( c \geq 0 \) and \( -n/c \in \mathbb{N} \) for \( c < 0 \). Furthermore let \( I_c = [0, \infty) \) for \( c \geq 0 \) and \( I_c = [0, -1/c] \) for \( c < 0 \). Consider \( f : I_c \to \mathbb{R} \) given in such a way that the corresponding integrals and series are convergent.

The Baskakov-type operators are defined as follows (see [10, 21, 22])
\[
B^{[c]}_n(f; x) = \sum_{j=0}^{\infty} p^{[c]}_{n,j}(x) f \left( \frac{j}{n} \right),
\]
where
\[
p^{[c]}_{n,j}(x) = \begin{cases} 
\frac{n^j}{j!} x^j e^{-nx} & , c = 0, \\
\frac{n^{c+j}}{j!} x^j (1 + cx)^{-(\frac{n}{c}+j)} & , c \neq 0,
\end{cases} \tag{3.1}
\]
and \( a_{c,j} := \prod_{l=0}^{j-1}(a + cl) \), \( a_{c,0} := 1 \).

For \( c = -1 \) one recovers the well known Bernstein operators
\[
B^{[-1]}_n(f; x) := \sum_{j=0}^{n} f \left( \frac{j}{n} \right) p^{[-1]}_{n,j}(x), \text{ where } p^{[-1]}_{n,j}(x) := \binom{n}{j} x^j (1 - x)^{n-j}, \ x \in [0, 1].
\]

The classical Baskakov operators are obtained for \( c = 1 \) and are defined as follows
\[
B^{[1]}_n(f; x) := \sum_{j=0}^{\infty} f \left( \frac{j}{n} \right) p^{[1]}_{n,j}(x), \text{ where } p^{[1]}_{n,j}(x) := \binom{n+j-1}{j} x^j (1 + x)^{n+j}, \ x \in [0, \infty).
\]

The classical Szász–Mirakjan operators are Baskakov type operators with \( c = 0 \), defined by
\[
B^{[0]}_n(f; x) := \sum_{j=0}^{\infty} p^{[0]}_{n,j}(x) f \left( \frac{j}{n} \right), \text{ where } p^{[0]}_{n,j}(x) = e^{-nx} \frac{(nx)^j}{j!}, \ x \in [0, \infty). \tag{3.2}
\]

**Theorem 3.1** Let \( m \in \mathbb{N}, c \in \mathbb{R} \) and \( x \in I_c \) be given. Then
\[
\lim_{n \to \infty} B^{[c]}_{mn} \left( f(nt); \frac{x}{n} \right) = B^{[0]}_m(f(t); x), \ f \in C_b(I_c). \tag{3.3}
\]
Proof Consider first the case $c = 0$. Using elementary calculations we find

$$B_{mn}^{[0]} \left( f(nt); \frac{x}{n} \right) = B_{m}^{[0]} \left( f(t); x \right).$$

Now let $c \neq 0$. Then

$$B_{n}^{[c]} \left( e^{ist}; x \right) = \sum_{j=0}^{\infty} \frac{n^c j}{j!} x^j (1 + cx)^{-n/c-j} e^{isj/n}$$

$$= (1 + cx)^{-n/c} \sum_{j=0}^{\infty} \frac{n(n + c) \ldots (n + (j-1)c)}{j!} \left( \frac{x e^{is/n}}{1 + cx} \right)^j$$

$$= (1 + cx)^{-n/c} \sum_{j=0}^{\infty} \binom{n/c + j - 1}{j} \left( \frac{cx e^{is/n}}{1 + cx} \right)^j$$

$$= (1 + cx)^{-n/c} \left( 1 - \frac{cx e^{is/n}}{1 + cx} \right)^{-n/c},$$

and so

$$B_{n}^{[c]} \left( e^{ist}; x \right) = \left( 1 - cx(e^{is/n} - 1) \right)^{-n/c}. \quad (3.4)$$

Define

$$L_n(f(t); x) := B_{mn}^{[c]} \left( f(nt); \frac{x}{n} \right), \quad (3.5)$$

$$L(f(t); x) := B_{m}^{[0]} \left( f(t); x \right). \quad (3.6)$$

From (3.4), (3.5), (3.6) we obtain

$$\lim_{n \to \infty} L_n(e^{ist}; x) = \lim_{n \to \infty} B_{mn}^{[c]} \left( e^{isnt}; \frac{x}{n} \right)$$

$$= \lim_{n \to \infty} \left( 1 - c \frac{x}{n} \left( e^{is/m} - 1 \right) \right)^{-mn/c} = e^{mx(e^{is/m} - 1)}$$

$$= B_{m}^{[0]} \left( e^{ist}; x \right) = L(e^{ist}; x).$$

Thus (2.2) is satisfied and (3.3) is a consequence of (2.3), (3.5) and (3.6). □

4 The $k$th order Kantorovich modification of the Baskakov type operators

The $k$th order Kantorovich modifications of the operators $B_{n}^{[c]}$ are defined by (see [21] and the references therein)

$$B_{n}^{[c](k)} := D^k \circ B_{n}^{[c]} \circ I_k, \quad k \in \mathbb{N},$$

where $D^k$ denotes the $k$th order ordinary differential operator and

$$I_k f = f, \quad \text{if } k = 0 \quad \text{and} \quad (I_k f)(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t) dt, \quad \text{if } k \in \mathbb{N}.$$
We have \( B_n^{[c](k)} (f(t); x) := \frac{d^k}{dx^k} B_n^{[c]}((I_k f)(t); x) \). Using (3.4) we get

\[
B_n^{[c](k)}(e^{ist}; x) = \frac{1}{(is)^k} \frac{d^k}{dx^k} B_n^{[c]}(e^{ist}; x) = n^{c, K} \left( \frac{e^{is/n} - 1}{is} \right)^k \left( 1 - c x \left( e^{is/n} - 1 \right)^{-n/c-k} \right), \; c \neq 0,
\]

(4.1)

\[
B_n^{[c](k)}(e^{ist}; x) = \left( \frac{n(e^{is/n} - 1)}{is} \right)^k e^{nx(e^{is/n} - 1)}, \; c = 0.
\]

(4.2)

The image of the constant function 1 under \( B_n^{[c](k)} \) is obtained with \( s = 0 \) and is the constant function \( n^{c, K} \). Therefore, the operators

\[
V_n^{[c](k)} := (n^k/n^{c, K}) B_n^{[c](k)}
\]

preserve the constant function 1.

**Theorem 4.1** For \( m \in \mathbb{N}, c \in \mathbb{R}, x \in I_c, \) and \( f \in C_b(I) \), we have

\[
\lim_{n \to \infty} V_n^{[c](k)} \left( f(nt); \frac{x}{n} \right) := V_m^{[0](k)} (f(t); x).
\]

(4.4)

**Proof** According to Theorem 2.1, it suffices to show that (4.4) holds for \( f(t) := e^{ist}, s \in \mathbb{R} \).

Let \( c \neq 0 \). Using (4.3) and (4.1) we have

\[
\lim_{n \to \infty} V_n^{[c](k)} \left( e^{ist}; \frac{x}{n} \right) = \lim_{n \to \infty} \left( \frac{m}{is} (e^{is/m} - 1) \right)^k \left( 1 - c \frac{x}{n} (e^{is/m} - 1) \right)^{-mn/c-k} = V_m^{[0](k)} (f(t); x).
\]

Using (4.3) and (4.2) it is easy to check that (4.4) holds for \( f(t) = e^{ist}, s \in \mathbb{R} \), also for \( c = 0 \). Now (4.4) for \( f \in C_b(I) \) is a consequence of Theorem 2.1.

\( \square \)

**Remark 4.1** Since \( \lim_{n \to \infty} n^k/n^{c, K} = 1 \), from (4.3) and (4.4) we derive

\[
\lim_{n \to \infty} B_n^{[c](k)} \left( f(nt); \frac{x}{n} \right) = B_m^{[0](k)} (f(t); x).
\]

5 The Bleimann–Butzer–Hahn operators

Let

\[
H_n(f(t); x) := (1 + x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k f \left( \frac{k}{n - k + 1} \right)
\]

(5.1)

be the Bleimann–Butzer–Hahn operators [11]. Using the probabilistic approach it was proved in [16, Theorem 2 a)] that

\[
\lim_{n \to \infty} H_{mn} \left( f \left( \frac{nt}{1+t} \right); \frac{x}{n} \right) = B_m^{[0]} (f(t); x).
\]

(5.2)
Here is an analytic proof based on Theorem 2.1. In fact, all that we have to prove is that (5.2) holds for \( f(t) := e^{ist}, s \in \mathbb{R} \).

This immediately follows from

\[
\lim_{n \to \infty} H_{mn} \left( e^{isnt/(1+t)}; \frac{x}{n} \right) = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{-mn} \sum_{k=0}^{mn} \binom{mn}{k} \left( \frac{x}{n} \right)^k e^{isnk/(mn+1)}
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{-mn} \left( 1 + \frac{x}{n} e^{isn/(mn+1)} \right)^{mn}
\]

\[
= e^{mx(e^{is/m} - 1)} = B_m^{(0)}(e^{ist}; x).
\]

**Example 5.1** It was proved in [6, Corollary 3] that

\[
\lim_{n \to \infty} H_{mn} \left[ f \left( \frac{(mn + 1)t}{m(1 + t)} \right); \frac{x}{n} \right] = B_m^{(0)}(f(t); x).
\]

(5.3)

Here is the proof based on Theorem 2.1. We have to prove (5.3) for \( f(t) = e^{ist}, s \in \mathbb{R} \).

\[
\lim_{n \to \infty} H_{mn} \left[ \exp \left( is \frac{(mn + 1)t}{m(1 + t)} \right); \frac{x}{n} \right]
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{-mn} \sum_{k=0}^{mn} \binom{mn}{k} \left( \frac{x}{n} \right)^k \exp \left( is \frac{(mn + 1)k}{m(1 + k/(mn+k+1))} \right)
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{-mn} \sum_{k=0}^{mn} \binom{mn}{k} \left( \frac{x}{n} \right)^k \left( e^{is/m} \right)^k
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{-mn} \left( 1 + \frac{x}{n} e^{is/m} \right)^{mn}
\]

\[
= e^{mx(e^{is/m} - 1)} = B_m^{(0)}(e^{ist}; x).
\]

This ends the proof.

Concerning this example see also [6, Remark 2, p. 496], where it is proved that

\[
\left| H_{mn} \left( f \left( \frac{(mn + 1)t}{m(1 + t)} \right); \frac{x}{n} \right) - H_{mn} \left( f \left( \frac{nt}{1 + t} \right); \frac{x}{n} \right) \right| \leq 2\omega \left( f; \frac{x}{mn} \right),
\]

\( \omega \) being the first modulus of continuity.

### 6 Lototsky–Schnabl operators

Let \( \lambda : [0, 1] \to [0, 1] \) be continuous, \( \lambda(0) = 1 \). For \( f \in C[0, 1], a \in [0, 1], x \in [0, 1] \), we consider the function

\[
f_{x,a}(t) := f(at + (1 - a)x), \ t \in [0, 1].
\]

Using the Bernstein operators \( B_r^{[-1]} \) one defines the Lototsky–Schnabl operators \( A_{n,\lambda} : C[0, 1] \to C[0, 1] \),

\[
A_{n,\lambda} (f(t); x) := \sum_{r=0}^{n} \binom{n}{r} \lambda(x)^r (1 - \lambda(x))^{n-r} B_r^{[-1]} \left( f_{x,r}(t); x \right).
\]

For details see [8].
Theorem 6.1 Given \( f \in C_b[0, \infty), x \in [0, \infty), m \geq 1 \), we have
\[
\lim_{n \to \infty} A_{mn, \lambda} \left( f(nt); \frac{x}{n} \right) = B_m^{[0]} \left( f(t); x \right). \tag{6.1}
\]

Proof Let \( L_n(f(t); x) := A_{mn, \lambda} \left( f(nt); \frac{x}{n} \right) \). Remark that
\[
f_{\frac{x}{n}, \frac{r}{mn}}(nt) = f \left( \frac{rt}{m} + \left( 1 - \frac{r}{mn} \right) \frac{x}{n} \right)
\]
and
\[
B_r^{-1} \left( f_{\frac{x}{n}, \frac{r}{mn}}(nt); \frac{x}{n} \right) = \sum_{k=0}^{r} \binom{k}{r} \left( \frac{x}{n} \right)^k \left( 1 - \frac{x}{n} \right)^{r-k} f \left( \frac{k}{m} + \left( 1 - \frac{r}{mn} \right) \frac{x}{n} \right).
\]
With \( f(t) := e^{ist} \) we have
\[
B_r^{-1} \left( f_{\frac{x}{n}, \frac{r}{mn}}(nt); \frac{x}{n} \right) = \left( 1 - \frac{x}{n} + \frac{x}{n} e^{is/m} \right)^r e^{is \left( 1 - \frac{r}{mn} \right) \frac{x}{n}}.
\]
Therefore,
\[
L_n(e^{ist}; x) = \sum_{r=0}^{mn} \binom{mn}{r} \lambda \left( \frac{x}{n} \right)^r \left( 1 - \lambda \left( \frac{x}{n} \right) \right)^{mn-r} \left( 1 - \frac{x}{n} + \frac{x}{n} e^{is/m} \right)^r e^{is \left( 1 - \frac{r}{mn} \right) \frac{x}{n}}
= e^{ix/n} \left[ 1 - \lambda \left( \frac{x}{n} \right) + \lambda \left( \frac{x}{n} \right) \left( 1 + \frac{x}{n} \left( e^{is/m} - 1 \right) \right) e^{-ix/mn^2} \right]^{mn}.
\]
By straightforward calculations we get
\[
\lim_{n \to \infty} L_n \left( e^{ist}; x \right) = e^{mx(e^{is/m} - 1)} = B_m^{[0]} \left( e^{ist}; x \right).
\]

Now Theorem 2.1 shows that
\[
\lim_{n \to \infty} A_{mn, \lambda} \left( f(nt); \frac{x}{n} \right) = \lim_{n \to \infty} L_n \left( f(t); x \right) = B_m^{[0]} \left( f(t); x \right),
\]
and this concludes the proof. \( \square \)

Remark 6.1 An estimate of the rate of convergence in (6.1) can be found in [8, Theorem 4].

7 Meyer-König and Zeller operators

The operators of Meyer-König and Zeller [24] in the slight modification of Cheney and Sharma [12] are defined for \( f \in C[0, 1] \) as follows
\[
M_n(f(t); x) = \begin{cases} 
\sum_{k=0}^{\infty} \binom{n+k}{k} x^k (1-x)^{n+1} f \left( \frac{k}{n+k} \right), & x \in [0, 1), \\
f(1), & x = 1.
\end{cases}
\tag{7.1}
\]
Let us observe that
\[
M_m \left[ f \left( \frac{t}{n(1-t)} \right); \frac{nx}{1+nx} \right] = \left( 1 - \frac{nx}{1+nx} \right)^{m+1} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{k}{n(1-k/m+k)} \frac{(m+k)(nx)^k}{m+1}.
\]

In particular,
\[
\lim_{n \to \infty} M_m \left[ e^{is/n(1-t)}; \frac{nx}{1+nx} \right] = \lim_{n \to \infty} \frac{1}{(1+nx)^{m+1}} \sum_{k=0}^{\infty} \frac{k}{mn} \frac{(m+k)(nx)^k}{m+1} \frac{1}{\left( 1 - nx e^{is/mn} \right)^{m+1}}.
\]

Now consider the Gamma operator
\[
G_m (f(t); x) := \frac{x^{-m}}{(m-1)!} \int_0^\infty f \left( \frac{t}{m} \right) t^{m-1} e^{-t/x} dt, \quad f \in C_b, x \in \mathbb{R}.
\]

where \( f \in C_b[0, \infty), x \in [0, \infty) \). (If \( x = 0 \), \( G_m (f(t); 0) = f(0) \)).

Using the well known characteristic function of Gamma distribution it is easy to infer that
\[
G_m \left( e^{is}; x \right) = \left( 1 - \frac{isx}{m} \right)^{-m}, \quad s \in \mathbb{R}.
\]

Using (7.2), (7.4) and Theorem 2.1 we obtain

**Theorem 7.1** If \( f \in C_b[0, \infty), x \in [0, \infty) \) and \( m \geq 1 \), then
\[
\lim_{n \to \infty} M_m \left[ f \left( \frac{t}{n(1-t)} \right); \frac{nx}{1+nx} \right] = G_{m+1} \left( f(t); \frac{m+1}{m} x \right).
\]

**8 Convergence toward Weierstrass operator**

Let \( a > 0 \). The Weierstrass operator is defined by (see [9, (5.2.78)])
\[
W_a (f(t); x) := \frac{1}{\sqrt{2\pi a}} \int_{\mathbb{R}} f(t) e^{-\frac{(t-x)^2}{2a}} dt, \quad f \in C_b(\mathbb{R}), x \in \mathbb{R}.
\]

The same definition can be found in [19, (9.1.14)] with notation \( a = 1/n \). The following results were briefly presented in [28, (24.25)] and [25]. Here we give detailed proofs based on Theorem 2.1.
Let 0 < u < 1 be given, and let \( b_n > 0 \) such that \( \lim_{n \to \infty} nb_n^2 = a. \) Then

\[
\lim_{n \to \infty} B_n^{[-1]} \left( f \left( x + \frac{nb_n}{\sqrt{u(1-u)}} (t-u) \right); u \right) = W_a(f(t); x),
\]

(8.1)

where \( B_n^{[-1]} \), \( n \geq 1 \), are the classical Bernstein operators.

**Proof** Let

\[
L_n (f(t); x) := B_n^{[-1]} \left( f \left( x + \frac{nb_n}{\sqrt{u(1-u)}} (t-u) \right); u \right).
\]

(8.2)

Then, by a straightforward calculation we get

\[
L_n \left( e^{ist}; x \right) := B_n^{[-1]} \left( \exp \left( is \left( x + \frac{nb_n}{\sqrt{u(1-u)}} (t-u) \right) \right); u \right)
\]

\[
= e^{ist} \sum_{j=0}^{n} \binom{n}{j} u^j (1-u)^{n-j} \exp \left( is \frac{nb_n}{\sqrt{u(1-u)}} \left( j - u \right) \right)
\]

\[
= e^{ist} \left[ (1-u) \exp \left( \frac{-i\text{sub}_n}{\sqrt{u(1-u)}} \right) + u \exp \left( \frac{i(1-u)\text{bn}_n}{\sqrt{u(1-u)}} \right) \right] n.
\]

Remark that

\[
\lim_{n \to \infty} L_n \left( e^{ist}; x \right) = e^{ists} \exp \left( \lim_{n \to \infty} \left( \frac{(1-u) e^{i\frac{-u}{\sqrt{n}}(1-u)}}{b_n^2} + u e^{i(1-u)\frac{u}{\sqrt{n}}(1-u)} - 1 \right) \right).
\]

Since \( nb_n^2 \to a \) we have \( b_n \to 0 \) and it readily follows that

\[
\lim_{n \to \infty} L_n (e^{ist}; x) = e^{ists - \frac{1}{2}as^2}.
\]

(8.3)

On the other hand, using the well-known characteristic function of a Gaussian random variable with mean \( x \) and variance \( a \), we can write

\[
W_a(e^{ist}; x) = \frac{1}{\sqrt{2\pi a}} \int_{\mathbb{R}} e^{ists - \frac{(t-x)^2}{2a}} dt = e^{ists - \frac{1}{2}as^2}.
\]

(8.4)

From (8.3), (8.4) and Theorem 2.1 we infer that

\[
\lim_{n \to \infty} L_n (f(t); x) = W_a(f(t); x).
\]

(8.5)

Now (8.1) is a consequence of (8.2) and (8.5).

We consider again a sequence \( b_n > 0 \), \( n \geq 1 \), such that \( \lim_{n \to \infty} nb_n^2 = a > 0 \). The following Bernstein–Schnabl type operators were introduced in [26] and investigated in [1–3, 5, 25, 28]:

\[
S_{n,b_n} (f(t); x) := \frac{1}{(2nb_n)^n} \int_{x-nb_n}^{x+nb_n} \cdots \int_{x-nb_n}^{x+nb_n} f \left( \frac{u_1 + \cdots + u_n}{n} \right) du_1 \cdots du_n.
\]

**Theorem 8.2** Let \( f \in C_b(\mathbb{R}) \) and \( x \in \mathbb{R} \). Then

\[
\lim_{n \to \infty} S_{n,b_n} (f(t); x) = W_{a/3}(f(t); x).
\]

(8.6)
**Proof** Using Theorem 2.1 it suffices to show that
\[ \lim_{n \to \infty} S_n, b_n(e^{ist}; x) = W_{a/3}(e^{ist}; x). \] (8.7)

According to the definition we have
\[ S_n, b_n(e^{ist}; x) = \frac{1}{(2n b_n)^n} \int_{-\infty}^{x} \int_{-\infty}^{x} e^{is(u_1 + \cdots + u_n)/n} du_1 \cdots du_n \] (8.8)
\[ = \frac{1}{(2n b_n)^n} \left( \int_{-\infty}^{x} e^{isu/n} du \right)^n = e^{ixx} \left( \frac{\sin sb}{sb} \right)^n. \] (8.9)

Since \( nb_n^2 \to a \) we have \( b_n \to 0 \) and therefore
\[ \lim_{n \to \infty} S_n, b_n(e^{ist}; x) = e^{ixx} \exp \left( \lim_{n \to \infty} \left( n \frac{\sin sb_n - sb_n}{sb_n} \right) \right) = e^{ixx} \exp \left( \frac{s^2(nb_n^2)}{(sb_n)^3} \sin sb_n - sb_n \right). \]

It follows that
\[ \lim_{n \to \infty} S_n, b_n(e^{ist}; x) = e^{ixx - \frac{1}{6}ax^2}. \] (8.10)

Now (8.7) is a consequence of (8.4) and (8.10). Thus (8.6) is proved.

\[ \square \]

### 9 Convergence toward Jakimovski–Leviatan operator

The Szász–Mirakyan operator appeared as limit operator in previous examples. This section presents an instance where a Jakimovski–Leviatan operator appears as limit operator.

The Jakimovski–Leviatan operators are generalizations of Szász–Mirakjan operators (see [4, 14, 23]). To define them we need a sequence of Appell polynomials \( (A_n(x))_{n \geq 0} \). If they have the explicit representation
\[ A_n(x) = \alpha_n + \binom{n}{1} \alpha_{n-1} x + \binom{n}{2} \alpha_{n-2} x^2 + \cdots + \alpha_0 x^n, \quad n = 0, 1, \ldots, \] (9.1)
one associates the power series
\[ a(t) = \alpha_0 + \frac{t}{1!} \alpha_1 + \frac{t^2}{2!} \alpha_2 + \cdots + \frac{t^n}{n!} \alpha_n + \cdots, \quad \alpha_0 \neq 0, \] (9.2)
called the generating function.

In fact, the sequence and its generating function are related by
\[ a(t) e^{sx} = A_0(x) + \frac{t}{1!} A_1(x) + \frac{t^2}{2!} A_2(x) + \cdots + \frac{t^n}{n!} A_n(x) + \cdots \] (9.3)
For details see, e.g., [13–15].

The Jakimovski–Leviatan type operators associated with \( (A_n(x))_{n \geq 0} \) are defined by
\[ \Psi_n f(x) := \frac{e^{-nx}}{a(1)} \sum_{k=0}^{\infty} \frac{1}{k!} A_k(nx) f \left( \frac{k}{n} \right), \quad n \geq 1, \] (9.4)
where \( f \in C[0, \infty) \) is a function for which the series is convergent for all \( x \in [0, \infty) \).
**Example 9.1** Let \( p \in [0, \infty) \) and \( a(t) := e^{bt} \), \( t \in \mathbb{R} \). Using (9.3) we find the associated Appell polynomials \( A_{k,p}(x) = (x + p)^k \), \( k \geq 0 \). The Jakimovski–Leviatan operators are

\[
\Psi_{n,p}(f(t); x) = e^{-nx-p} \sum_{k=0}^{\infty} \frac{(nx + p)^k}{k!} f \left( \frac{k}{n} \right), \quad f \in C_b(0, \infty).
\]

Let us remark that \( \Psi_{n,p}(f(t); x) \) are the Szász–Mirakyan–Schurer operators presented, e.g. in [9, p. 338].

Let \( B_{[1]}^n \) be the classical Baskakov operators, and

\[
L_n(f(t); x) := B_{mn}^{[1]} \left( f(nt); \frac{x + p/m}{n} \right).
\]

Then \( \lim_{n \to \infty} L_n(e^{ist}; x) = e^{(mx+p)(e^{ist/m}-1)} = \Psi_{m,p}(e^{ist}; x) \) and consequently

\[
\lim_{n \to \infty} B_{mn}^{[1]} \left( f(nt); \frac{x + p/m}{n} \right) = \Psi_{m,p}(f(t); x). \tag{9.5}
\]

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