A DEFORMED HERMITIAN YANG-MILLS FLOW

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ABSTRACT. We study a new deformed Hermitian Yang-Mills flow on a compact Kähler manifold. We first show the existence of the longtime solution of the flow. We then show that under the Collins-Jacob-Yau’s condition on the subsolution, the longtime solution converges to the solution of the deformed Hermitian Yang-Mills equation, which was solved by Collins-Jacob-Yau [5] by the continuity method. Moreover, as an application of the flow, we show that on a compact Kähler surface, if there exists a semi-subsolution of the deformed Hermitian Yang-Mills equation, then the flow converges smoothly to a singular solution to the deformed Hermitian Yang-Mills equation away from a finite number of curves of negative self-intersection. Such a solution can be viewed as a boundary point of the moduli space of the deformed Hermitian Yang-Mills solutions for a given Kähler metric.

1. Introduction

Let \((M, \omega)\) be a compact Kähler manifold of dimension \(n\) and \(\chi\) be a real closed \((1, 1)\) form. Motivated by mirror symmetry [24], Jacob-Yau [22] initiated to study the deformed Hermitian Yang-Mills (dHYM) equation:

\[
\text{Re}(\chi_u + \sqrt{-1}\omega)^n = \cot \theta_0 \text{Im}(\chi_u + \sqrt{-1}\omega)^n,
\]

where \(\theta_0\) is the argument of the complex number \(\int_M (\chi + \sqrt{-1}\omega)^n\) and \(\chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u\) for a real smooth function \(u\) on \(M\).

Let \(\lambda = (\lambda_1, \ldots, \lambda_n)\) be the eigenvalues of \(\chi_u\) with respect to \(\omega\), and if necessary we denote \(\lambda\) by \(\lambda(\chi_u)\) and \(\lambda_i(\chi_u)\) for each \(1 \leq i \leq n\). Then by Jacob-Yau [22] the dHYM equation has an equivalent form

\[
\theta_\omega(\chi_u) := \sum_{i=1}^n \arccot \lambda_i = \theta_0.
\]

It is called supercritical if \(\theta_0 \in (0, \pi)\) and hypercritical if \(\theta_0 \in (0, \frac{\pi}{2})\).

1.1. Previous results. The dHYM equation has been extensively studied by many mathematicians ([2], [3], [5], [4], [6], [7], [16], [17], [18], [20], [22], [25], [26], [28]).

We first introduce the related results in the elliptic case. When \(n = 2\), Jacob-Yau [22] solved the equation by translating it into the complex Monge-Ampère equation which was solved by Yau [36]. When \(n \geq 3\), Collins-Jacob-Yau [5] solved the dHYM equation for
the supercritical case by assuming the existence of a subsolution \( u \) and an extra condition on \( u \). For convenience, for a smooth function \( v \) on \( M \) we define

\[
A_0(v) := \max_M \max_{1 \leq j \leq n} \sum_{i \neq j} \arccot \lambda_i(\chi v)
\]

and

\[
B_0(v) = \max_M \theta_\omega(\chi v).
\]

A smooth function \( u \) on \( M \) is called a subsolution of dHYM equation (1.2) if \( u \) satisfies the inequality

\[
A_0(u) < \theta_0.
\]

The extra condition on \( u \) is

\[
B_0(u) < \pi.
\]

To be precise, Collins, Jacob and Yau proved the following

**Theorem 1.1** (Collins-Jacob-Yau [5]). Let \((M, \omega)\) be a compact Kähler manifold of dimension \( n \) and \( \chi \) a closed real \((1, 1)\) form on \( M \) with \( \theta_0 \in (0, \pi) \). Suppose there exists a subsolution \( u \) of dHYM equation (1.2) in the sense of (1.3) and \( u \) also satisfies inequality (1.4). Then there exists a unique smooth solution of dHYM equation (1.2).

Without condition (1.4) Pingali [28] then solved the equation for \( n = 3 \) and Lin [26] solved it for \( n = 3, 4 \). On the other hand, Lin [25] generalized Collins-Jacob-Yau’s result to the Hermitian case \((M, \omega)\) with \( \partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0 \); Huang-Zhang-Zhang [20] also considered the solution on a compact almost Hermitian manifold for the hypercritical case.

For the parabolic flow method, there are also several results. More precisely, Jacob-Yau [22] and Collins-Jacob-Yau [5] solved the line bundle mean curvature flow (LBMCF)

\[
\begin{cases}
  u_t = \theta_0 - \theta_\omega(\chi u) \\
  u(0) = u_0
\end{cases}
\]

(1.5)

under the assumptions:

(1) \( \theta_0 \in (0, \frac{\pi}{2}) \);

(2) the existence of a subsolution \( u \) in the sense of (1.3); and

(3) \( \theta_\omega(\chi u_0) \in (0, \frac{\pi}{2}) \).

Takahashi [32] proved the existence and convergence of the tangent Lagrangian phase flow (TLPF)

\[
\begin{cases}
  u_t = \tan(\theta_0 - \theta_\omega(\chi u)) \\
  u(0) = u_0
\end{cases}
\]

(1.6)

under the same assumptions (1) and (2) of flow (1.5) and the assumption:

(3') \( \theta_\omega(\chi u_0) - \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \).
Another important problem raised by Collins-Jacob-Yau [5] is to find a sufficient and necessary geometric condition on the existence of a solution of the dHYM equation. There are some important progresses made by Chen [2] and Chu-Lee-Takahashi [4].

1.2. Our results. Motivated by the concavity of \( \cot \theta_\omega(\chi_u) \) by Chen [2], we consider a dHYM flow:

\[
\begin{cases}
    u_t = \cot \theta_\omega(\chi_u) - \cot \theta_0, \\
    u(x, 0) = u_0(x).
\end{cases}
\]

Assume \( u_0 \) satisfies

\[ B_0(u_0) < \pi. \]

This condition is the same as \((1.4)\) if \( u_0 = u \).

We first prove an existence theorem of the longtime solution of flow \((1.7)\).

**Theorem 1.2.** Let \((M, \omega)\) be a compact Kähler manifold and \( \chi \) a closed real \((1, 1)\) form with \( \theta_0 \in (0, \pi) \). If \( u_0 \) satisfies inequality \((1.8)\), then dHYM flow \((1.7)\) has a unique smooth longtime solution \( u \).

Next we consider the convergence of longtime solution of flow \((1.7)\). Now we need to assume the dHYM equation has a subsolution \( u \) which also satisfies inequality \((1.4)\). The first main result of this paper is

**Theorem 1.3.** Let \((M, \omega)\) be a compact Kähler manifold of dimension \( n \) and \( \chi \) a closed real \((1, 1)\) form with \( \theta_0 \in (0, \pi) \). Suppose dHYM equation \((1.2)\) has a subsolution \( u \) in the sense of \((1.3)\) which also satisfies \((1.4)\). If \( u_0 \) satisfies \((1.8)\), then the longtime solution \( u(x, t) \) of dHYM flow \((1.7)\) converges to a smooth solution \( u^\infty \) to the dHYM equation:

\[ \theta_\omega(\chi_{u^\infty}) = \theta_0. \]

The extra condition \((1.4)\) in our result is the same as the one in Theorem 1.1 which is therefore reproved. Our proof here looks like simpler than the one in [5]. On the other hand, compared with the results in [22] and [32], we only need \( \theta_0 \in (0, \pi) \). Moreover, condition \((3)\) of flow \((1.5)\) or \((3')\) of flow \((1.6)\) is stronger than condition \((1.4)\).

In addition to the concavity of \( \cot \theta_\omega(\chi_u) \), our flow has two advantages: The first one is the imaginary part of the Calabi-Yau functional (see the definition in Section 2) is constant along the flow, which is the key to do the \( C^0 \) estimate; The second one is a subsolution \( u \) of equation \((1.2)\) satisfying \((1.4)\) is also a subsolution of flow \((1.7)\), which allows us to use Lemma 3 in Phong-Tô [27] to do higher order estimates. If we can establish the similar lemma without extra condition \((1.4)\) of \( u \), we then can relax condition \((1.4)\).

The second motivation of this paper is to look for applications of flow \((1.7)\). A smooth function \( u \) is called a semi-subsolution of the dHYM equation if

\[ A_0(u) \leq \theta_0. \]
In the 2-dimensional case, this condition is equivalent to
\[(1.10) \quad \chi_u - \cot \theta_0 \omega.\]

Now we restrict ourselves to this case.

Assume there exists a semi-subsolution \(u\) of the dHYM equation and replace \(\chi_u\) by \(\chi\), i.e., assume that \(u = 0\) is a semi-subsolution. For any \(B_1 \in (0, \pi)\), define the set
\[(1.11) \quad \mathcal{H}_{B_1} = \{v \in C^\infty(M, \mathbb{R}) : \theta_0(\chi_v) \in (0, B_1)\}.\]

Then if \(\theta_0 \in (0, \frac{\pi}{2})\), the set \(\mathcal{H}_{B_1}\) for any \(B_1 \in (2\theta_0, \pi)\) is non-empty, for example, \(0 \in \mathcal{H}_{B_1}\); if \(\theta_0 \in [\frac{\pi}{2}, \pi)\), we can prove that the set \(\mathcal{H}_{B_1}\) for any \(B_1 \in (\theta_0, \pi)\) is also non-empty, see Lemma 5.2.

We take a function in \(\mathcal{H}_{B_1}\) for any \(B_1 \in (\theta_0, \pi)\) as \(u_0\) in flow (1.7). We can state the second main theorem of the paper.

**Theorem 1.4.** Let \((M, \omega)\) be a compact Kähler surface and \(\chi\) a closed real \((1, 1)\) form. Assume \(\theta_0 \in (0, \pi)\) and \(\chi \geq \cot \theta_0 \omega\). Then there exist a finite number of curves \(E_i\) of negative self-intersection on \(M\) such that the solution \(u(x, t)\) of dHYM flow (1.7) converges to a bounded function \(u^\infty\) in \(C^\infty_{\text{loc}}(M \setminus \bigcup_i E_i)\) as \(t\) tends to \(\infty\) with the following properties.

1. \(\chi + \sqrt{-1} \partial \bar{\partial} u^\infty - \cot \theta_1 \omega\) is a Kähler current which is smooth on \(M \setminus \bigcup_i E_i\);
2. \(u^\infty\) satisfies the dHYM equation on \(M \setminus \bigcup_i E_i\);
3. \(\chi_{u^\infty}\) converges to \(\chi_{u^\infty}\) and \(u^\infty\) satisfies (1.12) on \(M\) in the sense of currents.

We note that by assuming \(\theta_0 \in (0, \frac{\pi}{2})\) and \(B_1 \leq \frac{\pi}{2}\), Takahashi \[33\] proved the same convergence result of the LBMCF. A similar result of the J-flow was studied in Fang-Lai-Song-Weinkove\[12\]. As done by \[12, 33\], we need the singular solution of the degenerate complex Monge-Ampère equation (5.4) by Eyssidieux-Guedj-Zeriahi \[10\], which will be used in the \(C^0\) estimate. We establish a similar lemma, i.e., Lemma 5.7 as Lemma 3 in \[27\] by the semi-subsolution condition to do the gradient estimate and the second order estimate. As to the convergence of \(u_t\), the key point is that along the dHYM flow the real part of the Calabi-Yau functional is uniformly bounded. In this way we can prove Theorem 1.4.

As an application of Theorem 1.4, we have the lower bound of the \(\mathcal{J}\)-functional on certain spaces, see the definition in Section 2.

**Corollary 1.5.** Let \((M, \omega)\) be a compact Kähler surface and \(\chi\) a closed real \((1, 1)\) form. Assume that \(\theta_0 \in (0, \pi)\) and \(\chi \geq \cot \theta_0 \omega\). The \(\mathcal{J}\)-functional is bounded from below in \(\mathcal{H}_{B_1}\) for any \(B_1 \in (\theta_0, \pi)\).

If \(\theta_0 \in (0, \frac{\pi}{2})\), Takahashi proved that \(\mathcal{J}\) is bounded from below in \(\mathcal{H}_{\frac{\pi}{2}}\).

We have mentioned that for 2 dimensional case, along the dHYM flow the real part of the Calabi-Yau functional is uniformly bounded. We believe that the same conclusion for
the higher dimension also holds. Hence the real part of the Calabi-Yau functional plays the similar role as the Donaldson functional defined on the space of Hermitian metrics on a holomorphic vector bundle. We expect that we can use our flow to study the moduli space of solutions of the dHYM equation on a compact Kähler manifold \((M, \omega)\).

The rest of this paper is arranged as follows. In Section 2, we give some preliminary results on the linearized operator on the dHYM flow, the convexity of \(\cot \theta(\lambda)\), the parabolic subsolution, and the Calabi-Yau functional. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.3, including the \(C^0\) estimate, the gradient estimate and the second order estimate. In Section 5, we prove Theorem 1.4 and Corollary 1.5.

**Notations:** In this paper a closed real \((1, 1)\) form \(\chi\) is fixed. We will use the constant \(C\) in the generic sense which is dependent on \(\omega\), \(\chi\), \(u\), \(u_0\) and \(n\). If necessary, we will use \(C_i\) as a specific constant.

Notations of covariant derivatives are used. For example, \(u_{i\bar{j}k}\) represents the third order covariant derivative of function \(u\), \(\alpha_{i\bar{j}}\) represents the covariant derivative of \((1,1)\) form \(\alpha\).

We use Einstein summation convention if there is no confusion.

### 2. Preliminary results

#### 2.1. The linearized operator

Note

\[
\cot \theta(\lambda) = \frac{\text{Re}(\chi u + \sqrt{-1}w)^n}{\text{Im}(\chi u + \sqrt{-1}w)^n}.
\]  

We manipulate the linearized operator \(\mathcal{P}\) of dHYM flow (1.7) in the following lemma.

**Lemma 2.1.** The operator \(\mathcal{P}\) has the form:

\[
\mathcal{P}(v) = v_t - \csc^2 \theta(\lambda)(wg^{-1}w + g)^{ij}v_{ij},
\]

where \(g = (g_{ij})_{n\times n}\), \(w = (w_{ij})_{n\times n}\) for \(w_{ij} = \chi_{ij} + u_{ij}\), and \(D^{ij} := (D^{-1})_{ij}\) for an invertible Hermitian symmetric matrix \(D\).

**Proof.** We only need to deal with the variation of \(\cot \theta(\lambda)\). Indeed, let \(u(s)\) be a variation of the function \(u\) and \(\frac{du(s)}{ds}|_{s=0} = v\). Let \(A(s) := g^{-1}w(s) + \sqrt{-1}I\) with \(w(s)\) being the local matrix of \(\chi_{u(s)}\). Then

\[
A(s)^{-1} = (g^{-1}w(s) - \sqrt{-1}I)((g^{-1}w(s))^2 + I)^{-1}.
\]

For simplicity, we write \(A\) instead of \(A(s)\). By (2.1) we have

\[
\delta(\cot \theta(\lambda)) = \frac{\text{Re}(\delta \det A)}{\text{Im}(\det A)} - \frac{\text{Re}(\det A)\text{Im}(\delta \det A)}{(\text{Im}(\det A))^2}.
\]
Since \( \delta(\det A) = (\det A)\delta(\log \det A) \), if we write \( \det A = a_1 + \sqrt{-1}a_2 \) and \( \delta(\log \det A) = b_1 + \sqrt{-1}b_2 \), then
\[
\delta(\cot \theta(\chi_u)) = \frac{a_1b_1 - a_2b_2}{a_2} - \frac{a_1(a_1b_2 + a_2b_1)}{a_2^2}
= -\frac{a_1^2 - a_2^2}{a_2^2}b_2 = -\csc^2 \theta(\chi_u)b_2.
\]

On the other hand, by (2.2) we have
\[
b_2 = \text{Im} \delta(\log \det A) = -\text{tr}((w^{-1}g + g)^{-1}\delta w(s)|_{s=0}) = -(w^{-1}g + g)^{ij}v_{ij}.
\]
Hence
\[
(2.3) \quad \delta(\cot \theta(\chi_u)) = \csc^2 \theta(\chi_u)(w^{-1}g + g)^{ij}v_{ij}.
\]

We denote
\[
F^{ij} := \csc^2 \theta(\chi_u)(w^{-1}g + g)^{ij}
\]
and hence
\[
\mathcal{P}(v) = v_i - F^{ij}v_{ij}.
\]

The following lemma is useful in the gradient and second order estimates.

**Lemma 2.2.** Let \( u \) be a solution of dHYM flow (1.7). Then

(2.5) \quad \quad \quad \quad u_{tp} - F^{ij}w_{ij,p} = 0,

and

\[
u_{tp} - F^{ij}w_{ij,pp} - F^{ik}w_{kj,p}w_{ij,lp} = -(w^{-1}g + g)^{ij}w_{ij,lp}(-w_{km}g^{rh}w_{ml,ij} + w_{km}g^{rh}w_{ml,ij})
+ 2\csc \theta(\chi_u)F^{ij}w_{ij,p}(w^{-1}g + g)^{kl}w_{kl,p}.
\]

**Proof.** Similar as the proof of (2.3), differentiating equation (1.7) leads to (2.5) directly:
\[
u_{tp} = \csc^2 \theta(\chi_u)(w^{-1}g + g)^{ij}w_{ij,lp} = F^{ij}w_{ij,lp}.
\]

Differentiating the equation twice, we have
\[
u_{tp} = \csc^2 \theta(\chi_u)(w^{-1}g + g)^{ij}w_{ij,lp} + (\csc^2 \theta(\chi_u))_{lp}(w^{-1}g + g)^{ij}w_{ij,lp}
- \csc^2 \theta(\chi_u)(w^{-1}g + g)^{ij}(w^{-1}g + g)^{kl}w_{kl,p}(w^{-1}g + g)^{ij}w_{ij,lp},
\]
where
\[
(\csc^2 \theta(\chi_u))_{lp} = 2\csc \theta(\chi_u)(\cot \theta(\chi_u))_{lp} = 2\cot \theta(\chi_u)F^{kl}w_{kl,p}
\]
and
\[
(w^{-1}g + g)^{kl,p} = (w_{km}g^{rh}w_{ml} + g_{kl})_{lp} = w_{km}g^{rh}w_{ml} + w_{km}g^{rh}w_{ml,p}.
\]
Hence identity (2.6) follows. \( \square \)

2.2. The concavity of \( \cot \theta(\lambda) \) in \( \Gamma_\tau \) for \( \tau \in (0, \pi) \). Here

\[
\theta(\lambda) := \sum_{i=1}^{n} \arccot \lambda_i \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n
\]

and

\[
\Gamma_\tau := \{ \lambda \in \mathbb{R}^n \mid \theta(\lambda) < \tau \} \subset \mathbb{R}^n \quad \text{for } \tau \in (0, \pi).
\]

We have the following useful facts.

**Lemma 2.3** (Yuan [37], Wang-Yuan [35]). If \( \theta(\lambda) \leq \tau \in (0, \pi) \) for \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), then the following inequalities holds.

1. \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \cot \frac{\pi}{\tau} > 0 \), and \( \lambda_{n-1} \geq |\lambda_n| \);
2. \( \lambda_1 + (n-1)\lambda_n \geq 0 \).

Moreover, \( \Gamma_\tau \) is convex for any \( \tau \in (0, \pi) \).

**Lemma 2.4** (Chen [2]). For any \( \tau \in (0, \pi) \), the function \( \cot \theta(\lambda) \) on \( \Gamma_\tau \) is concave.

**Proof.** For completeness, we give an elementary proof here. When \( n = 1 \), \( \cot \theta(\lambda) = \lambda_1 \) is obviously concave. We now assume \( n \geq 2 \). By definition (2.8) we have

\[
\frac{\partial^2 \cot \theta(\lambda)}{\partial \lambda_i \partial \lambda_j} = -\frac{\partial}{\partial \lambda_j} \left( \csc^2 \theta(\lambda) \frac{\partial \theta(\lambda)}{\partial \lambda_i} \right) = \frac{\partial}{\partial \lambda_j} \left( \frac{\csc \theta(\lambda)}{1 + \lambda_i^2} \right)
\]

\[
\frac{1}{1 + \lambda_i^2} - \frac{1}{1 + \lambda_j^2} = \frac{\lambda_i^2 - \lambda_j^2}{(1 + \lambda_i^2)(1 + \lambda_j^2)}.
\]

Hence the function \( \cot \theta(\lambda) \) on \( \Gamma_\tau \) is concave if and only if the matrix

\[
\Lambda = (\lambda_i \delta_{ij} - \cot \theta(\lambda))_{n \times n}
\]

is positive definite. Without loss of generality, we assume \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Since \( \theta(\lambda) \in (0, \pi) \), by Lemma 2.3 (1), we have \( \lambda_{n-1} > 0 \).

By the definition of \( \theta(\lambda) \), for any \( 1 \leq j_1 < j_2 < \cdots < j_k, 1 \leq k \leq n - 1 \), we have \( \sum_{i=1}^{k} \arccot \lambda_{j_i} < \theta(\lambda) \). Hence

\[
\text{Re} \left( \prod_{i=1}^{k} (\lambda_{j_i} + \sqrt{-1}) \right) - \cot \theta(\lambda) \text{Im} \left( \prod_{i=1}^{k} (\lambda_{j_i} + \sqrt{-1}) \right) > 0.
\]

Let \( \sigma_i(\lambda_{j_1,j_2,\ldots,j_k}) \) for \( 1 \leq i \leq k \) be the \( i \)-th elementary symmetric polynomial of \( \lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_k} \). Then (2.10) can be written as

\[
\sum_{i=0}^{[k/2]} (-1)^i \sigma_{k-2i}(\lambda_{j_1,j_2,\ldots,j_k}) - \cot \theta(\lambda) \sum_{i=0}^{[k-1]/2} (-1)^i \sigma_{k-1-2i}(\lambda_{j_1,j_2,\ldots,j_k}) > 0.
\]
Denote by $D_k$ the $k$-th leading principal minor of the matrix $\Lambda$. We need to prove $D_k > 0$ for any $1 \leq k \leq n$. When $k = 1$, $D_1 = \lambda_1 - \cot \theta(\lambda) > 0$. When $2 \leq k \leq n$, by direct computation, we have

\[ D_k = \sigma_k(\lambda_{12..k}) - \cot \theta(\lambda) \sigma_{k-1}(\lambda_{12..k}). \]

Hence by (2.11), we have

\[ D_k > - \sum_{i=1}^{[k/2]} (-1)^i \sigma_{k-2i}(\lambda_{12..k}) + \cot \theta(\lambda) \sum_{i=1}^{[(k-1)/2]} (-1)^i \sigma_{k-1-2i}(\lambda_{12..k}) \]

\[ =: E_{k-2}(\lambda_{12..k}) \]

We prove $E_{k-2}(\lambda_{12..k}) > 0$ for any $2 \leq k \leq n$.

We use the well-known formula

\[ \sigma_i(\lambda_{12..k}) = \sigma_i(\lambda_{2..k}) + \lambda_1 \sigma_{i-1}(\lambda_{2..k}) \]

for $1 \leq i \leq k - 1$ to deduce that

\[ E_{k-2}(\lambda_{12..k}) = F_{k-2}(\lambda_{2..k}) + \lambda_1 E_{k-3}(\lambda_{2..k}), \]

where

\[ F_{k-2}(\lambda_{2..k}) = - \sum_{i=1}^{[k/2]} (-1)^i \sigma_{k-2i}(\lambda_{2..k}) + \cot \theta(\lambda) \sum_{i=1}^{[(k-1)/2]} (-1)^i \sigma_{k-1-2i}(\lambda_{2..k}) \]

\[ = \sum_{j=0}^{[(k-2)/2]} (-1)^j \sigma_{k-2-2j}(\lambda_{2..k}) - \cot \theta(\lambda) \sum_{j=0}^{[(k-3)/2]} (-1)^j \sigma_{k-3-2j}(\lambda_{2..k}) \]

and

\[ E_{k-3}(\lambda_{2..k}) = - \sum_{i=1}^{[k/2]} (-1)^i \sigma_{k-2i-1}(\lambda_{2..k}) + \cot \theta(\lambda) \sum_{i=1}^{[(k-1)/2]} (-1)^i \sigma_{k-2i-1}(\lambda_{2..k}) \]

\[ = \sum_{j=0}^{[(k-2)/2]} (-1)^j \sigma_{k-3-2j}(\lambda_{2..k}) - \cot \theta(\lambda) \sum_{j=0}^{[(k-3)/2]} (-1)^j \sigma_{k-4-2j}(\lambda_{2..k}). \]

By (2.12) we compute directly to get

\[ F_{k-2}(\lambda_{2..k}) = \text{Re} \left( \prod_{j=3}^{k} (\lambda_j + \sqrt{-1}) \right) - \cot \theta(\lambda) \text{Im} \left( \prod_{j=3}^{k} (\lambda_j + \sqrt{-1}) \right) + \lambda_2 F_{k-3}(\lambda_{3..k}). \]

Hence

\[ F_{k-2}(\lambda_{2..k}) > \lambda_2 F_{k-3}(\lambda_{3..k}). \]

From this we deduce that

\[ F_{k-2}(\lambda_{2..k}) > \lambda_2 \lambda_3 \cdots \lambda_{k-2} F_1(\lambda_{(k-1)k}) = \lambda_2 \lambda_3 \cdots \lambda_{k-2} (\lambda_{k-1} + \lambda_k - \cot \theta(\lambda)) > 0. \]
Combined with (2.13), we have

\[ E_{k-2}(\lambda_{12k}) > \lambda_1 E_{k-3}(\lambda_{2k}). \]

Hence for any \(2 \leq k \leq n\) we have

\[ E_{k-2}(\lambda_{12k}) > \lambda_1 \lambda_2 \cdots \lambda_{k-3} E_1(\lambda_{k-2}(\lambda_{k-1} + \lambda_k - \cot \theta(\lambda))) > 0. \]

In summary, we finish the proof of the lemma. \(\square\)

2.3. **Parabolic subsolution.** B. Guan [14] introduced the definition of a subsolution of fully non-linear equations. G. Székelyhidi [31] gave a weaker version of a subsolution and Collins-Jacob-Yau [15] used it to the dHYM equation which is equivalent to (1.3). These two notions are equivalent for the type 1 cones by the appendix in [15]. On the other hand, Phong-Tô [27] modified the definition in [14] and [31] to the parabolic case. We use their definition to the parabolic dHYM equation.

**Definition 2.5.** A smooth function \(u(x, t)\) on \(M \times [0, T)\) is called a subsolution of parabolic dHYM equation (1.7) if there exists a constant \(\delta > 0\) such that for any \((x, t) \in M \times [0, T)\), the subset of \(\mathbb{R}^{n+1}\)

\[ S_\delta(x, t) = \{ (\mu, \tau) \in \mathbb{R}^n \times \mathbb{R} | \cot \theta(\lambda(\chi_{u(x, t)} + \mu)) - u(x, t) + \tau = \cot \theta_0, \mu_i > -\delta \text{ for } 1 \leq i \leq n \text{ and } \tau > -\delta \} \]

is uniformly bounded, i.e., it is contained in the ball \(B^\circ_{K}(0)\) in \(\mathbb{R}^{n+1}\) with radius \(K\), a uniform constant.

We have the following observation.

**Lemma 2.6.** If \(u\) is a subsolution of dHYM equation (1.1) with \(B_0(u) < \pi\), then the function \(\tilde{u}(x, t) = u(x)\) on \(M \times [0, \infty)\) is also a subsolution of parabolic dHYM equation (1.7).

**Proof.** We want to find a constant \(\delta\) in Definition 2.5. If such a \(\delta\) exists, we let \((\mu, \tau) \in S_\delta(x, t)\) for \((x, t) \in M \times [0, \infty)\). Since \(\mu_i > -\delta\) for each \(1 \leq i \leq n\), by the definition of \(B_0(u)\) in (1.4) we have

\[ 0 < \theta(\lambda(\chi_{\tilde{u}(x)}) + \mu) \leq \theta_\omega(\chi_{\tilde{u}(x)}) + n\delta \leq B_0(u) + n\delta. \]

Hence if \(0 < \delta \leq \frac{\pi - B_0(u)}{2n}\), then

\[ 0 < \theta(\lambda(\chi_{\tilde{u}(x)}) + \mu) < \frac{\pi + B_0(u)}{2} < \pi, \]

and by the definition of \(S_\delta(x, t)\), \(\tau\) is bounded from above:

\[ \tau = \cot \theta_0 - \cot \theta(\lambda(\chi_{\tilde{u}(x)}) + \mu) \leq \cot \theta_0 - \cot \left( \frac{\pi + B_0(u)}{2} \right). \]
Since also \( \mu_i > -\delta \) for each \( 1 \leq i \leq n \), by subsolution condition (1.3) we have
\[
\sum_{i \neq j} \arccot(\lambda_j(\chi_{u(s)}) + \mu_i) \leq \sum_{i \neq j} \arccot(\lambda_i(\chi_{u(s)}) + (n-1)\delta) \leq A_0(u) + (n-1)\delta.
\]
If \( 0 < \delta \leq \frac{\theta_0 - A_0(u)}{2(n+1)} \), then
\[
\sum_{i \neq j} \arccot(\lambda_i(\chi_{u(s)}) + \mu_i) \leq \frac{\theta_0 + A_0(u)}{2}.
\]
Since \( \tau > -\delta \), by the definition of \( S_\delta(x,t) \) we have for each \( j \)
\[
\arccot(\lambda_j(\chi_{u(s)}) + \mu_j) = \arccot(\cot \theta_0 - \tau) - \sum_{i \neq j} \arccot(\lambda_i(\chi_{u(s)}) + \mu_i)
\geq \arccot(\cot \theta_0 + \delta) - \sum_{i \neq j} \arccot(\lambda_i(\chi_{u(s)}) + \mu_i)
\geq \theta_0 - \delta - \frac{\theta_0 + A_0(u)}{2} \geq \frac{n(\theta_0 - A_0(u))}{2(n+1)} > 0.
\]
Hence we have
\[
\mu_j \leq \max_M |\lambda(\chi_{u(s)})| + \cot\left(\frac{n(\theta_0 - A_0(u))}{2(n+1)}\right).
\]
Therefore, if we choose \( \delta = \min\{\frac{\pi - B_0(u)}{2n}, \frac{\theta_0 - A_0(u)}{2(n+2)}\} \), then for any \((x,t) \in M \times [0,\infty) \) and \((\mu,\tau) \in S_\delta(x,t) \), we have
\[
|\mu| + |\tau| \leq K := 2n \left(\delta + \max_M |\lambda(\chi_u)| + \cot \theta_0 - \cot\left(\frac{\pi + B_0(u)}{2}\right) + \cot\left(\frac{n(\theta_0 - A_0(u))}{2(n+1)}\right)\right).
\]

2.4. The Calabi-Yau Functional. Recall the definition of the Calabi-Yau functional by Collins-Yau [8]: for any \( v \in C^2(M, \mathbb{R}) \),
\[
\text{CY}_C(v) := \frac{1}{n+1} \sum_{i=0}^{n} \int_M v(\chi_v + \sqrt{-1} \omega)^i \wedge (\chi + \sqrt{-1} \omega)^{n-i}.
\]

The \( \mathcal{J} \)-functional is defined by
\[
\mathcal{J}(v) := \text{Im}(e^{-\sqrt{-1} \theta_0} \text{CY}_C(v)).
\]

Let \( v(s) \in C^{2,1}(M \times [0, T], \mathbb{R}) \) be a variation of the function \( v \), i.e., \( v(0) = v \). The integration by parts gives
\[
\frac{d}{ds} \text{CY}_C(v(s)) = \int_M \frac{\partial v(s)}{\partial s}(\chi_{v(s)} + \sqrt{-1} \omega)^n,
\]
\[
\frac{d}{ds} \mathcal{J}(v(s)) = \int_M \frac{\partial v(s)}{\partial s} \text{Im}(e^{-\sqrt{-1} \theta_0}(\chi_{v(s)} + \sqrt{-1} \omega)^n).
\]
Lemma 2.7. Let $u(x, t)$ be a solution of dHYM flow (1.7). Then

(2.16) $\text{Im}(\nabla_C(u(\cdot, t))) = \text{Im}(\nabla_C(u_0)),$

(2.17) $\frac{d}{dt} \text{Re}(\nabla_C(u(\cdot, t))) = \int_M \left( \frac{\partial u(t)}{\partial t} \right)^2 \text{Im}(\chi_u + \sqrt{-1}\omega)^n,$

(2.18) $\frac{d}{dt} J(u(\cdot, t))) \leq 0.$

Proof. Denote by $u(t) := u(x, t)$ for simplicity. Then we have

$$
\frac{d}{dt} \text{Im}(\nabla_C(u(t))) = \int_M \frac{\partial u(t)}{\partial t} \text{Im}(\chi_u + \sqrt{-1}\omega)^n
$$

$$= \int_M \left( \text{Re}(\chi_{u(t)} + \sqrt{-1}\omega)^n - \cot \theta \right) \text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n
$$

$$= \int_M \text{Re}(\chi + \sqrt{-1}\omega)^n - \cot \theta \int_M \text{Im}(\chi + \sqrt{-1}\omega)^n
$$

$$= 0,$$

where each equality is successively by (2.14), (1.7) and (2.1), Stokes’ theorem, and the definition of $\theta_0$. Hence (2.16) holds as $u(0) = u_0.$

Then we can also prove (2.17).

$$
\frac{d}{dt} \text{Re}(\nabla_C(u(t))) = \int_M \frac{\partial u(t)}{\partial t} \text{Re}(\chi_{u(t)} + \sqrt{-1}\omega)^n
$$

$$= \int_M \frac{\partial u(t)}{\partial t} \cot \theta \text{Re}(\chi_{u(t)}) \text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n
$$

$$= \int_M \frac{\partial u(t)}{\partial t} \left( \frac{\partial u(t)}{\partial t} + \cot \theta \right) \text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n
$$

$$= \int_M \left( \frac{\partial u(t)}{\partial t} \right)^2 \text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n,$$

where the last equality follows from (2.16).
Locally
\[
\frac{\partial u(t)}{\partial t} \text{Im}(e^{-\sqrt{-1} \theta_0}(\chi_u + \sqrt{-1} \omega)^n)
\]
\[
= \prod_{i=1}^{n}(1 + \lambda_i^2)(\cot \theta_0(\chi_u(t)) - \cot \theta_0) \sin(\theta_0(\chi_u(t)) - \theta_0) \omega^n
\]
\[
= - \prod_{i=1}^{n}(1 + \lambda_i^2) \frac{\sin^2(\theta_0(\chi_u(t)) - \theta_0)}{\sin \theta_0(\chi_u(t)) \sin \theta_0} \omega^n \leq 0,
\]
where the last inequality follows from \(\theta_0(\chi_u(t)) \in (0, \pi)\) by (3.2). Hence \(J\) is decreasing and (2.18) follows.

Next we prove that along the dHYM flow the real part of the Calabi-Yau functional can be controlled by \(|u|_{L^\infty}\) without the subsolution condition.

**Proposition 2.8.** Let \(u(x, t)\) be a solution of dHYM flow (1.7) with the initial data satisfying (1.8). Then there exists a uniform constant \(C\) such that
\[
\text{Re} (\text{CY}_C(u)) \leq C|u|_{L^\infty}.
\]
**Proof.** By the definition of the Calabi-Yau functional, we only need to prove that for any \(0 \leq k, l \leq n\) with \(0 \leq k + l \leq n\)
\[
(2.19) \quad \left| \int_M u \chi_u^k \wedge \chi^l \wedge \omega^{n-k-l} \right| \leq C|u|_{L^\infty}.
\]
We prove the above estimates by inductive argument on \(k\). When \(k = 0\), it obviously holds. Now assume inequality (2.20) holds for \(k \leq m\) with \(0 \leq k + l \leq n\). We prove inequality (2.20) holds for \(k = m + 1\). Indeed, since along the flow by (3.2) \(\chi_u \geq - \cot B_0(u_0) \omega\), there exists a constant \(C_0 > 0\) such that \(\chi_u + C_0 \omega > 0\) and \(\chi + C_0 \omega > 0\). We write
\[
\int_M u \chi_u^{m+1} \wedge \chi^l \wedge \omega^{n-m-l-1} = \int_M u(\chi_u + C_0 \omega)^{m+1} \wedge (\chi + C_0 \omega)^l \wedge \omega^{n-m-l-1}
\]
\[
- \sum_{p=0}^{m} \sum_{q=0}^{l} C_{pq} \int_M u \chi_u^p \wedge \chi^q \wedge \omega^{n-p-q}
\]
for some constants \(C_{pq}\). Now
\[
\left| \int_M u \ (\chi_u + C_0 \omega)^{m+1} \wedge (\chi + C_0 \omega)^l \wedge \omega^{n-m-l-1} \right|
\]
\[
\leq |u|_{L^\infty} \left| \int_M (\chi_u + C_0 \omega)^{m+1} \wedge (\chi + C_0 \omega)^l \wedge \omega^{n-m-l-1} \right|
\]
\[
= |u|_{L^\infty} \left| \int_M (\chi + C_0 \omega)^{m+l+1} \wedge \omega^{n-m-l-1} \right|
\]
\[
\leq C_1 |u|_{L^\infty}
\]
(2.21)
and then by inductive assumption, inequality (2.20) follows. □

3. The existence of the longtime solution and proof of Theorem 1.2

In this section we prove Theorem 1.2, i.e. the following

Theorem 3.1. Let \((M, \omega)\) be a compact Kähler manifold and \(\chi\) a closed real \((1,1)\) form with \(\theta_0 \in (0, \pi)\). If \(u_0\) satisfies inequality (1.8), then dHYM flow (1.7) has a unique smooth longtime solution \(u\).

We assume that \(u\) is the solution of dHYM flow (1.7) in \(M \times [0, T]\), where \(T\) is the maximal existence time. By showing the uniform a priori estimates in the following subsections, we can prove \(T = \infty\).

3.1. The \(u_t\) estimate.

Lemma 3.2. Let \(u(x,t)\) be a solution of dHYM flow (1.7) with the initial data satisfying (1.8). For any \((x,t) \in M \times [0, T)\),

\[
\min_M u_t|_{t=0} \leq u_t(x,t) \leq \max_M u_t|_{t=0},
\]

in particular,

\[
0 < \min_M \cot \theta_\omega(\chi_{u_0}(x)) \leq \theta_\omega(\chi_{u(x,t)}) \leq B_0(u_0) < \pi.
\]

Proof. The \(u_t\) satisfies the equation:

\[
(u_t)_t = F^{ij}(u_t)_{ij}.
\]

By the maximum principle, \(u_t\) attains its maximum and minimum on the initial time, i.e., inequality (3.1) holds, i.e.,

\[
\min_M \cot \theta_\omega(\chi_{u_0}) \leq u_t(x,t) + \cot \theta_0 \leq \max_M \cot \theta_\omega(\chi_{u_0}),
\]

or

\[
\min_M \cot \theta_\omega(\chi_{u_0}) \leq \cot \theta_\omega(\chi_{u(x,t)}) \leq \max_M \cot \theta_\omega(\chi_{u_0}).
\]

Thus we obtain

\[
0 < \min_M \theta_\omega(\chi_{u_0}) \leq \theta_\omega(\chi_{u(x,t)}) \leq \max_M \theta_\omega(\chi_{u_0}) = B_0(u_0).
\]

□

We have a useful corollary of the above lemma.

Corollary 3.3. Let \(\lambda_n(x,t)\) be the smallest eigenvalue of \(\chi_u\) with respect to the metric \(\omega\) at \((x,t)\). Then

\[
\max_{M \times [0,T]} |\lambda_n| \leq A_1 \quad \text{for} \quad A_1 := |\cot B_0(u_0)| + \left| \cot \left( \frac{\min_M \theta_\omega(\chi_{u_0})}{n} \right) \right|.
\]
Proof. By Lemma 3.2, we have
\[ 0 < \frac{\min_{M} \theta_{\omega}(x_{u_{0}})}{n} \leq \frac{\theta_{\omega}(x_{u_{0}})}{n} \leq \arccot \lambda_{n} \leq B_{0}(u_{0}) < \pi. \]
Hence we have
\[ \cot B_{0}(u_{0}) \leq \lambda_{n} \leq \cot \left( \frac{\min_{M} \theta_{\omega}(x_{u_{0}})}{n} \right). \]

3.2. The complex Hessian estimate. For any \( T_{0} < T \), we have proved \( u_{t} \) is uniformly bounded and thus \( |u| \leq CT_{0} + |u_{0}|e^{\omega} \) in \( M \times [0, T_{0}] \). We next prove the complex Hessian estimate.

**Proposition 3.4.** Let \( u(x, t) \) be a solution of dHYM flow (1.7) with the initial data satisfying (1.8). There exists a uniform constant \( C \) such that
\[ \sup_{M \times [0, T_{0}]} |\partial \tilde{u} u_{\omega}| \leq Ce^{CT_{0}}. \]

**Proof.** Denote \( w_{ij} := \chi_{ij} + u_{ij} \) as before. Denote \( S(T^{1,0}M) := \bigcup_{x \in M} \{ \xi \in T^{1,0}M \ | \ |\xi|_{\omega} = 1 \} \). Consider on \( S(T^{1,0}M) \times [0, T_{0}] \) the auxiliary function
\[ \tilde{Q}(x, t, \xi(x)) = \log(w_{ij}(\xi(\bar{\xi}))) - K_{0}t, \]
where \( K_{0} \) is a uniformly large constant to be chosen later.

Suppose the function \( \tilde{Q} \) attains its maximum at \( (x_{0}, t_{0}) \) along the direction \( \xi_{0} = \xi(x_{0}) \). We will prove that \( t_{0} = 0 \) and thus the estimate follows. If \( t_{0} > 0 \), we choose holomorphic coordinates near \( x_{0} \) such that
\[ g_{ij}(x_{0}) = \delta_{ij}, \quad \partial_{k}g_{ij}(x_{0}) = 0, \text{ and} \]
\[ w_{ij}(x_{0}, t_{0}) = \lambda_{1} \delta_{ij} \text{ with } \lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{n} \]
which forces \( \xi_{0} = \frac{\alpha_{1}}{\partial_{x_{1}}} \). We extend \( \xi_{0} \) near \( x_{0} \) as \( \tilde{\xi}_{0}(x) = (g_{11})^{-1/2} \frac{\alpha_{1}}{\partial_{x_{1}}} \). Then the function \( Q(x, t) = \tilde{Q}(x, t, \tilde{\xi}_{0}(x)) \) on \( M \times [0, T_{0}] \) attains its maximum at \( (x_{0}, t_{0}) \).

By the maximum principle, we have at \((x_{0}, t_{0})\)
\[ 0 \leq Q_{t} = \frac{u_{11}}{w_{11}} - K_{0}, \]
\[ 0 = Q_{t} = \frac{w_{11, i}}{w_{11}}, \]
\[ 0 \leq -Q_{\bar{i}} = -\frac{w_{11, \bar{i}}}{w_{11}} + \frac{|w_{11}|^{2}}{w_{11}} = -\frac{w_{11, \bar{i}}}{w_{11}}. \]
Hence we have
\[ 0 \leq Q_{t} - F_{\bar{i}}Q_{\bar{i}} = \lambda_{1}^{-1}(u_{11} - F_{\bar{i}}w_{11, \bar{i}}) - K_{0}. \]
Since $d\chi = 0$, by covariant derivative formulae, we have

$$w_{11,\hat{t}} = w_{\hat{i},11} + (\lambda_i - \lambda_j)R_{11\hat{i}}.$$  

(3.5)

On the other hand, by (2.6), we have

$$u_{t11} - F^\hat{i}w_{\hat{i},11} = - F^\hat{i}(1 + \lambda_i^2)^{-1}(\lambda_i + \lambda_j)|w_{\hat{i},1}|^2$$

$$+ 2 \cot \theta_\omega(\chi_u) \csc^2 \theta_\omega(\chi_u)(1 + \lambda_i^2)^{-1}w_{\hat{i},11}$$

$$= - \sum_{i\neq j} F^\hat{i}(1 + \lambda_i^2)^{-1}(\lambda_i + \lambda_j)|w_{\hat{i},1}|^2 - 2F^\hat{i}\lambda_i(1 + \lambda_i^2)^{-1}|w_{\hat{i},1}|^2$$

(3.6)

$$+ 2 \cot \theta_\omega(\chi_u) \csc^2 \theta_\omega(\chi_u)(1 + \lambda_i^2)^{-1}w_{\hat{i},1}w_{\hat{i},11}.$$ 

However since $\cot \theta(\lambda)$ is concave, by (2.9)

$$u_{t11} - F^\hat{i}w_{\hat{i},11} \leq - \sum_{i\neq j} F^\hat{i}(1 + \lambda_i^2)^{-1}(\lambda_i + \lambda_j)|w_{\hat{i},1}|^2 \leq 0,$$

(3.7)

since $\lambda_i + \lambda_j > 0$ for any $i \neq j$.

Inserting (3.5) and (3.7) into (3.4), we have

$$0 \leq Q_t - F^\hat{i}Q_{\hat{i}} \leq 2|\text{Rm}|_{C^0} \sum_{i=1}^n F^\hat{i} - K_0.$$  

(3.8)

Noting that $\sin \theta_\omega(\chi_u) \geq \min \{\sin B_0(u_0), \sin (\min_M \theta_\omega(\chi_u))\}$, for any $1 \leq i \leq n$ we have

$$F^\hat{i} = \frac{1}{\sin^2 \theta_\omega(\chi_u)(1 + \lambda_i^2)} \leq \frac{1}{\min \{\sin^2 B_0(u_0), \sin^2 (\min_M \theta_\omega(\chi_u))\}} := A_2.$$ 

Inserting the above into (3.8) and choosing $K_0 = 2nA_2|\text{Rm}|_{C^0} + 1$, we have

$$0 \leq Q_t - F^\hat{i}Q_{\hat{i}} \leq 2nA_2|\text{Rm}|_{C^0} - K_0 = -1,$$

(3.9)

which is a contradiction. Therefore $t_0 = 0$ and then for any $t \in [0, T_0]$, it holds

$$w_{ij}\xi_i \bar{\xi}_j(x, t)e^{-K_0t} \leq w_{11}(x, 0) = u(0)_{11} + \chi_{11} \leq C.$$ 

□

3.3. Proof of Theorem 3.1. Since we have proved the $u_t$ estimate, the $C^0$ estimate and the complex Hessian estimate, by the concavity of the flow (1.7), we can apply the Evans-Krylov theory to get the higher order estimates of the solution.

If the maximal existence time $T < \infty$, then $u$ is uniformly $C^k$-bounded (for any $k \geq 0$) in $M \times [0, T]$ and then there exists $\epsilon > 0$ such that the flow exists on $M \times [0, T + \epsilon_0]$, which is a contradiction since $T$ is the maximal existence time. Thus $T = \infty$. 

4. Convergence of Longtime Solution and Proof of Theorem 1.3

In this section, we prove Theorem 1.3, i.e., the following

**Theorem 4.1.** Let \((M, \omega)\) be a compact Kähler manifold of dimension \(n\) and \(\chi\) a closed real \((1, 1)\) form with \(\theta_0 \in (0, \pi)\). Suppose \(\text{dHYM}\) equation (1.2) has a subsolution \(u\) in the sense of (1.3) which also satisfies (1.4). If \(u_0\) satisfies (1.8), then the longtime solution \(u(x, t)\) of \(\text{dHYM}\) flow (1.7) converges to a smooth solution \(u^\infty\) to the \(\text{dHYM}\) equation:

\[
\theta_0(\chi_{u^\infty}) = \theta_0.
\]

4.1. The \(C^0\) estimate. We first prove a Harnack type inequality along the \(\text{dHYM}\) flow.

**Lemma 4.2.** Let \(u\) be the solution of the \(\text{dHYM}\) flow on \(M \times [0, \infty)\). Then for any \(T_0 < \infty\) we have the following Harnack type inequality:

\[
\sup_{M \times [0, T_0]} u(x, t) \leq C \left( - \inf_{M \times [0, T_0]} (u(x, t) - u_0(x)) + 1 \right).
\]

**Proof.** For any \(t \in [0, T_0]\), we have \(\theta_\omega(\chi_{u(t)}) \leq B_0(u_0) < \pi\) by Lemma 3.2. Then by the convexity of \(\Gamma_{\omega, B_0(u_0)} := \{\alpha \in \Lambda^{1,1}(M, \mathbb{R}) \mid \theta_\omega(\alpha) < B_0(u_0)\}\) in Lemma 2.4, we have

\[
\theta_\omega(\chi_{su(t)+(1-s)u_0}) \leq B_0(u_0) < \eta_0 < \pi,
\]

where \(\eta_0 = B_0(u_0)/6 + 5\pi/6\) for convenience. Hence,

\[
\frac{\text{Im}(\chi_{su(t)+(1-s)u_0} + \sqrt{-1}\omega)^n}{\omega^n} \\
= \prod_{k=1}^{n} (1 + \lambda_k^2 (\chi_{su(t)+(1-s)u_0}))^{\frac{\eta_0}{n}} \sin \theta_\omega(\chi_{su(t)+(1-s)u_0}) \\
\geq \begin{cases} 
\sin \eta_0, & \text{if } \theta_\omega(\chi_{su(t)+(1-s)u_0}) \geq \frac{\pi}{6} \\
\sqrt{1 + \lambda_1^2 \sin \text{arccot} \lambda_1} = 1, & \text{if } \theta_\omega(\chi_{su(t)+(1-s)u_0}) < \frac{\pi}{6}
\end{cases}
\]

(4.1)

where \(c_0 := \sin \eta_0\).

By Lemma 2.7, the imaginary part of the Calabi-Yau functional is constant along the flow. Hence,

\[
0 = \text{Im}(CY_C(u(t))) - \text{Im}(CY_C(u_0)) \\
= \int_0^t \frac{d}{ds} \text{Im}(CY_C(su(t) + (1-s)u_0)) ds \\
= \int_0^t \int_M (u(t) - u_0) \text{Im}(\chi_{su(t)+(1-s)u_0} + \sqrt{-1}\omega)^n ds \\
= \int_M (u(t) - u_0) \left( \int_0^t \text{Im}(\chi_{su(t)+(1-s)u_0} + \sqrt{-1}\omega)^n ds \right).
\]

(4.2)
Thus we have

\[
\int_M (u - u_0) \omega^n = \int_M (u - u_0) \omega^n - \frac{1}{c_0} \int_M (u - u_0) \left( \int_0^1 \text{Im} (\chi_{su(t)+\chi_{u_0}} + \sqrt{-1} \omega)^n ds \right)
\]

\[
= \frac{1}{c_0} \int_M -(u - u_0) \left( -c_0 \omega^n + \int_0^1 \text{Im} (\chi_{su(t)+\chi_{u_0}} + \sqrt{-1} \omega)^n ds \right)
\]

This term is nonnegative by (4.1)

\[
\leq \frac{\inf_{M \times [0, T_0]} (u - u_0)}{c_0} \int_M \left( -c_0 \omega^n + \int_0^1 \text{Im} (\chi_{su(t)+\chi_{u_0}} + \sqrt{-1} \omega)^n ds \right)
\]

\[
- \frac{\inf_{M \times [0, T_0]} (u - u_0)}{c_0} \int_M \omega^n + \int_0^1 \text{Im} \int_M (\chi_{su(t)+\chi_{u_0}} + \sqrt{-1} \omega)^n ds
\]

\[
\leq c_0^{-1} \text{Im} \int_M (\chi + \sqrt{-1} \omega)^n \left( - \inf_{M \times [0, T_0]} (u - u_0) \right)
\]

\[
= C \left( - \inf_{M \times [0, T_0]} (u - u_0) \right),
\]

where \( C = c_0^{-1} \text{Im} \int_M (\chi + \sqrt{-1} \omega)^n \). Therefore we have

(4.3) \[
\int_M u(x, t) \omega^n \leq C \left( - \inf_{M \times [0, T_0]} (u(x, t) - u_0(x)) + 1 \right).
\]

On the other hand, let \( G(x, z) \) be Green’s function of the metric \( \omega \) on \( M \). Then for any \((x, t) \in M \times [0, T_0]\),

\[
u(x, t) = \left( \int_M \omega^n \right)^{-1} \int_M u(z, t) \omega^n - \int_{z \in M} \Delta_\omega u(z, t) G(x, z) \omega^n.
\]

Since \( \Delta_\omega u > -\text{tr}_\omega \chi > -C_0 \) and \( G(x, y) \) is bounded from below, there exits a uniform constant \( C \) such that

(4.4) \[
u(x, t) \leq \left( \int_M \omega^n \right)^{-1} \int_M u(z, t) \omega^n + C.
\]

Combing (4.3) with (4.4), we obtain the desired estimate. \( \square \)

Now we can prove the \( C^0 \) estimate similar as Phong-Tô [27].

**Proposition 4.3.** Along the dHYM flow, there exists a uniform constant \( M_0 \) independent of \( T \) such that

\[
|u|_{C^0(M \times [0, \infty) \times \mathcal{O})} \leq M_0.
\]
Proof. Combining (4.2) with (4.1) implies for any \( t \in [0, \infty) \),
\[
\sup_{x \in M} (u(x, t) - u_0(x)) \geq 0.
\]
Combing the above inequality with the concavity of the equation, we can apply Lemma 1 by Phong-Tô [27]: there exists a uniform constant \( C_1 \) such that
\[
\inf_{M \times [0, T_0]} (u - u) \geq -C_1 \quad \text{for any } T_0 > 0.
\]
Then combing this estimate with the Harnack type inequality in Lemma 4.2, we have
\[
\sup_{M \times [0, T_0]} u \leq C.
\]
Since \( T_0 \) is arbitrary, the result follows. □

4.2. The gradient estimate. We can use the following lemma by Phong-Tô which plays an important role in the gradient and second order estimates. In fact, it follows from the concavity of the function \( \cot \theta(\chi_u) \).

Lemma 4.4. Let \( \delta \) and \( K \) be two constants in Definition 2.5. There exists a constant \( \kappa_0 \) depending only on \( \delta, K, u, (M, \omega) \), and \( \chi \) such that if
\[
1 + \lambda_1^2 > \max \{ (K + \max_M |\lambda(\chi_u)| + 1)^2, \kappa_0^{-1}(1 + A_1^2) \},
\]
then
\[
(4.5) \quad u_t - \sum F^{\bar{j}}(u_{\bar{j}} - u_\bar{j}) \geq \kappa_0 \sum F^i.
\]
We prove the gradient estimate following the argument in the elliptic case by Collins-Yau [8].

Proposition 4.5. Let \( u \) be the solution of dHYM flow (1.7). There exists a uniform constant \( M_1 \) such that
\[
\max_{M \times [0, \infty)} |\nabla u|_\omega \leq M_1.
\]

Proof. Without loss of generality, we assume \( u = 0 \); otherwise we write \( \chi_u = \chi_u + i\partial \bar{\partial}(u - u) \) and replace \( \chi \) by \( \chi_u \) and \( u \) by \( u - u \).

We consider the function
\[
\tilde{G} = |\nabla u|^2 \exp \psi(u)
\]
where
\[
\psi(u) = -D_0 u + (u + M_0 + 1)^{-1}
\]
where \( M_0 \) is from Proposition 4.3 and \( D_0 \) is a constant to be determined later.

For any fixed time \( T_0 < \infty \), assume the function \( \tilde{G} \) on \( M \times [0, T_0] \) attains its maximum at \((x_0, t_0)\). If \( t_0 = 0 \), we have the desired estimate directly. Hence we assume \( t_0 > 0 \).
The function \( G := \log \tilde{G} = \log |\nabla u|^2 + \psi(u) \) also attains its maximum at \((x_0, t_0)\). By the maximum principle, we have \( \mathcal{P}G(x_0, t_0) \geq 0 \).

Take the holomorphic coordinates (3.3) near \( x_0 \). By (2.4)

\[
F^{ij}(x_0, t_0) = \frac{\csc^2 \theta(\lambda)}{1 + \lambda^2} \delta_{ij}.
\]

We take the manipulation at \((x_0, t_0)\):

\[
G_i = \frac{u_{ki}u_k + u_ku_{ki}}{|\nabla u|^2} + \psi' u_i,
\]

\[
G_i = \frac{u_{ki}u_k + u_ku_{ki}}{|\nabla u|^2} + \psi' u_i = 0,
\]

\[
G_{ij} = \frac{u_{ki}u_k + u_ku_{ki} + u_{kj}u_j + u_ku_{kj}}{|\nabla u|^2} - \frac{(u_{ki}u_k + u_ku_{ki})(u_{ij}u_j + u_iu_{ij})}{|\nabla u|^4} + \psi' u_{ij} + \psi'' u_iu_j.
\]

Hence

\[
0 \leq \mathcal{P}G = G_i - F^{\tilde{a}}G_{\tilde{a}}
\]

\[
= \frac{(u_{ki} - F^{\tilde{a}}u_{\tilde{a}k})u_k + (u_{ki} - F^{\tilde{a}}u_{\tilde{a}k})u_k}{|\nabla u|^2} (\text{denoted by (I)})
\]

\[
- \frac{F^{\tilde{a}}(u_{ki}u_{\tilde{a}j} + u_{ki}u_{\tilde{a}j})|\nabla u|^2 - F^{\tilde{a}}|\nabla |\nabla u|^2|^2}{|\nabla u|^4} (\text{denoted by (II)})
\]

\[
+ \psi'(u_i - F^{\tilde{a}}u_{\tilde{a}i}) - \psi'' F^{\tilde{a}}|u_i|^2.
\]

We first estimate term (I). By covariant derivatives formula and (2.5), we have

\[
(I) \leq \frac{(u_{ki} - F^{\tilde{a}}u_{\tilde{a}k})u_k + (u_{ki} - F^{\tilde{a}}u_{\tilde{a}k})u_k + 2F^{\tilde{a}}|Rm||\nabla u|^2}{|\nabla u|^2}
\]

\[
\leq \frac{(u_{ki} - F^{\tilde{a}}w_{\tilde{a}k})u_k + (u_{ki} - F^{\tilde{a}}w_{\tilde{a}k})u_k + F^{\tilde{a}}(|\nabla u| + 2|Rm||\nabla u|)|\nabla u|}{|\nabla u|^2}
\]

\[
= \frac{F^{\tilde{a}}(|\nabla u| + 2|Rm||\nabla u|)}{|\nabla u|} \leq C_2.
\]

We then deal with term (II). Since \( G_i = 0 \) for each \( 1 \leq i \leq n \), we have

\[
|\nabla |\nabla u|^2|^2 = \left| \sum u_{ki}u_k \right|^2 + \left| \sum u_{ki}u_{ki} \right|^2
\]

\[
= \left| \sum u_{ki}u_k \right|^2 + \left| \sum u_{ki}u_{ki} \right|^2 + 2Re\left( \sum u_{ki}u_k \sum u_ku_{ki} \right)
\]

\[
= \left| \sum u_{ki}u_k \right|^2 + \left| \sum u_{ki}u_{ki} \right|^2 + 2Re\left( - \sum u_ku_{ki} |\nabla u|^2 \psi' u_i \sum u_iu_{ki} \right)
\]

\[
= \left| \sum u_{ki}u_k \right|^2 - \left| \sum u_{ki}u_{ki} \right|^2 - 2|\nabla u|^2 \psi' \Re\left( u_i \sum u_ku_{ki} \right).
\]
Hence

$$\begin{align*}
(II) &= -|\nabla u|^2 F_{ii} \left( \sum |u_{ki}|^2 + \sum |u_{kj}|^2 \right) + |\nabla u|^{-4} F_{ii} \sum u_{ki} u_{kj}^2 \\
&\quad - |\nabla u|^{-4} F_{ii} \sum u_{ki} u_{kj}^2 - 2|\nabla u|^{-2} \psi' F_{ii} \Re(u_{i} \sum u_{k} u_{kj}) \\
&\leq -2|\nabla u|^{-2} \psi' F_{ii} \Re(u_{i} \sum u_{k} u_{kj})
\end{align*}$$

where the last inequality holds by the Cauchy-Schwarz inequality:

$$|\sum u_{ki} u_{kj}|^2 \leq \sum |u_{ki}|^2 |\nabla u|^2.$$ 

Since $u_{ki} = w_{ki} - \chi_{ki} = \lambda_{i} \delta_{ki} - \chi_{ki}$, by the Cauchy-Schwarz inequality again, we have

$$\begin{align*}
(II) &\leq -2|\nabla u|^{-2} \psi' F_{ii} |u_{i}|^2 \lambda_{i} + 2|\nabla u|^{-2} \psi' F_{ii} \Re(u_{i} \sum u_{k} \chi_{ki}) \\
&\leq 2|\psi'| |\nabla u|^{-1} \left( \sum F_{ii} |u_{i}|^2 \right)^{1/2} \left( \sum F_{ii} \lambda_{i}^2 \right)^{1/2} + 2|\chi| |\psi'| |\nabla u|^{-1} \left( \sum F_{ii} |u_{i}|^2 \right)^{1/2} \left( \sum F_{ii} \right)^{1/2}.
\end{align*}$$

Clearly $\max \{ \sum F_{ii}, \sum F_{ii} \lambda_{i}^2 \} \leq n \max \limits_{\theta_{\omega}(\chi_{in})} \csc^{2} \theta_{\omega}(\chi_{in})$ by (3.2). If we take

$$C_{3} := 4n \max \limits_{\lambda} \csc \theta_{\omega}(\chi_{in})(1 + \max \limits_{\lambda} |\lambda|),$$

then

$$\begin{align*}
(II) \leq C_{3} |\psi'| |\nabla u|^{-1} \left( \sum F_{ii} |u_{i}|^2 \right)^{1/2}.
\end{align*}$$

Inserting the estimates of (I) and (II) into (5.23), we obtain

$$0 \leq \mathcal{P} \leq -\psi' (-u_{i} + F_{ii} u_{i}) - \psi'' F_{ii} |u_{i}|^2 + C_{3} |\psi'| |\nabla u|^{-1} \left( F_{ii} |u_{i}|^2 \right)^{1/2} + C_{2}.$$ 

We use the argument of Collins-Yau [8] and consider the two cases. Let $\epsilon_{0}$ be a positive constant satisfying

$$\begin{align*}
\epsilon_{0} &< \min \left\{ (K + \max \limits_{\lambda} |\lambda(\chi_{in})| + 1)^{-1}, \kappa_{0}^{1/2}(1 + A_{k}^2)^{-1/2}, \frac{1}{2} C_{3}^{-1} \kappa_{0}(1 + A_{k}^2)^{-1} \right\}.
\end{align*}$$

**Case 1:** $\sum \limits_{i=1}^{n} F_{ii} |u_{i}|^2 \geq \epsilon_{0}^{2} |\nabla u|^2$. By the definition of $\psi$, $D_{0} \leq -\psi' \leq D_{0} + 1$ and $\psi'' = 2(u - \inf \limits_{\lambda} u + 1)^{-3}$. Hence, by (4.8)

$$\begin{align*}
0 &\leq -\frac{2\epsilon_{0}^{2} |\nabla u|^2}{(u + M_{0} + 1)^{3}} + (D_{0} + 1) \left( |u_{i}| + \frac{\csc^{2} \theta(\lambda)}{1 + \lambda_{i}^2} |\lambda_{i} - \chi_{ii}| \right) + C_{3} (D_{0} + 1) \csc \theta(\lambda) + C_{2} \\
&\leq -\frac{2\epsilon_{0}^{2} |\nabla u|^2}{(u + M_{0} + 1)^{3}} + C(D_{0} + 1).
\end{align*}$$

Thus we obtain

$$|\nabla u|^2 \leq C(D_{0} + 1) \epsilon_{0}^{-2} (u + M_{0} + 1)^{3}.$$
Case 2: $\sum_{i=1}^{n} F_{ii} |u_i|^2 \leq \varepsilon_0^2 |\nabla u|^2$. In this case, since $\psi'' > 0$, inequality (4.8) implies

(4.11) $0 \leq -\psi'(-u_t + F_{ii} u_{ii}) + C_3(-\psi')\varepsilon_0 + C_2$.  

On the other hand, since $F_{111} \leq F_{\tilde{u}}$, we have

$$\varepsilon_0^2 |\nabla u|^2 \geq F_{111} |\nabla u|^2 = \csc^2 \theta(\lambda) |\nabla u|^2 \frac{1}{1 + \lambda_1^2}.$$  

Hence we get

$$1 + \lambda_1^2 \geq \varepsilon_0^{-2} \csc^2 \theta(\lambda) \geq \varepsilon_0^{-2} > (K + \max_M |\lambda(\chi_w) + 1)|^2, \kappa_0^{-1}(1 + A_1^2).$$  

Now we apply the key Lemma 4.4 to get

$$u_t - F^{ij} u_{ij} \geq \kappa_0 \sum_{i=1}^{n} F^i.$$  

Combined with (4.11), we get

(4.12) $0 \leq \psi' \kappa_0 \sum F_{ii} + C_3(-\psi')\varepsilon_0 + C_2$.  

Since $\sum F_{ii} > F_{\bar{u}} = \frac{\csc^2 \theta_0(\lambda_0)}{1 + \lambda_0^2} \geq (1 + A_1^2)^{-1}$ by Corollary 3.3, and $\varepsilon_0 \leq \frac{1}{2} C_3^{-1} \kappa_0 (1 + A_1^2)^{-1}$ by (4.9), the sum of one half of the first term and the second term in (4.12) is non-positive. Hence if we choose $D_0 > 2\kappa_0^{-1} C_2 (1 + A_1^2)$, we obtain the following contradiction.

$$0 \leq \frac{1}{2} \psi' \kappa_0 \sum F_{ii} + C_2 \leq -\frac{D_0}{2} \kappa_0 (1 + A_1^2)^{-1} + C_2 < 0.$$  

Therefore if we choose $\varepsilon_0$ satisfying (4.9) and $D_0 = 2\kappa_0^{-1} C_2 (1 + A_1^2) + 1$, we really obtain the desired estimate (4.10). \hfill \Box

4.3. Second order estimates. In the elliptic case, Collins-Jacob-Yau [5] used an auxiliary function containing the gradient term which modifies the one in Hou-Ma-Wu [19]. Here our auxiliary function does not contain the gradient term.

Proposition 4.6. There exists a uniform constant $M_2$ such that

$$\sup_{M \times [0, \infty)} |\partial \tilde{\omega} u| \leq M_2.$$  

Proof. Without loss of generality, we assume that $\theta = 0$. Denote $w_{ij} := \psi_{ij} + u_{ij}$ as before. For any fixed $T_0 < \infty$, we consider the auxiliary function on $S(T^{1,0}M) \times [0, T_0]$:

$$\tilde{H}(x, t, \xi(x)) = \log(w_{ij} \xi^i \xi^j) + \psi(u)$$  

where $\psi(u) = -D_1 u + u^2/2$ with $D_1$ to be determined later. Recall $M_0$ is the uniform bound of $|u|$ in Lemma 4.3. Hence we have

(4.13) $-D_1 - M_0 \leq \psi' \leq -D_1 + M_0$ and $\psi'' = 1.$
Suppose the function $\tilde{H}$ attains its maximum at $(x_0, t_0)$ along the direction $\xi_0 = \xi(x_0)$. If $t_0 = 0$, the estimate clearly holds. Hence we assume $t_0 > 0$. Take holomorphic coordinates (3.3) near $x_0$ which forces $\xi_0 = \frac{\partial}{\partial q_1}$. Extend $\xi_0$ near $x_0$ as $\tilde{\xi}_0(x) = (g_{11})^{-\frac{1}{2}}\frac{\partial}{\partial q_1}$. Then the function $H(x, t) = \tilde{H}(x, t, \tilde{\xi}_0(x))$ on $M \times [0, T_0]$ attains its maximum at $(x_0, t_0)$.

By the maximum principle, we have at $(x_0, t_0)$

$$0 \leq H_t = \frac{u_{11}}{w_{11}} + \psi' u_t, \quad (4.14)$$

$$0 = H_i = \frac{w_{11,i}}{w_{11}} + \psi' u_i, \quad (4.15)$$

Hence we have

$$0 \leq H_t - F^{\tilde{H}} H_{\tilde{u}} = \lambda_1^{-1}(u_{11} - F^{\tilde{H}} w_{11,\tilde{u}}) + \lambda_1^{-2} F^{\tilde{H}} |w_{11,\tilde{u}}|^2$$

(4.16) and by (1),

$$0 \leq u_{11} - F^{\tilde{H}} w_{11,\tilde{u}} = u_{11} - F^{\tilde{H}} w_{\tilde{u},11} - F^{\tilde{H}}(\lambda_1 - \lambda_1 R_{11}) \lambda_1^{-1} \lambda_1^{-2} F^{\tilde{H}} |w_{11,\tilde{u}}|^2. \quad (4.17)$$

Since $\lambda_1 + \lambda_2 > 0$ for any $i \neq j$, the above inequality implies

$$u_{11} - F^{\tilde{H}} w_{\tilde{u},11} \leq - \lambda_1^{-1} \lambda_1^{-2} F^{\tilde{H}} |w_{11,\tilde{u}}|^2$$

(4.18) and (4.14), we can estimate (I) as follows.

\[
(I) \leq - \lambda_1^{-1} \sum_{i=2}^{n} \frac{\lambda_1 + \lambda_i}{1 + \lambda_1^2} F^{\tilde{H}} |w_{11,i}|^2 + \lambda_1^{-1} \sum_{i=1}^{n} F^{\tilde{H}} |w_{11,i}|^2 + C_4
\]

\[
= \lambda_1^{-2} \sum_{i=2}^{n} F^{\tilde{H}} |w_{11,i}|^2 \left(1 - \frac{\lambda_1 \lambda_i}{1 + \lambda_1^2}\right) + \lambda_1^{-2} F^{11} |w_{11,1}|^2 + C_4
\]

\[
= \psi^2 \sum_{i=2}^{n} F^{\tilde{H}} |u_t|^2 \left(1 - \frac{\lambda_1 \lambda_i}{1 + \lambda_1^2}\right) + \psi^2 F^{11} |u_t|^2 + C_4.
\]

By Lemma 2.3, we have $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \cot(B_0(u_0)/2)$, and without loss of generality we assume $\lambda_1 > 1/\cot(B_0(u_0)/2)$. Hence for $2 \leq i \leq n - 1, 1 - \lambda_1 \lambda_i < 0$. For $i = n$, since
\(|\lambda_n| \leq A_1\), we have
\[
\frac{1 - \lambda_1 \lambda_n}{1 + \lambda_n^2} \leq \frac{1 + A_1}{\lambda_1}.
\]

Hence
\[
(4.19) \quad (I) \leq \psi'^2 F_1^2 \overline{\nabla u}^2 + \psi'^2 (1 + A_1) \lambda_1^{-1} F_{n|u_n|^2} + C_4.
\]

Inserting (4.19) into (4.15), we have
\[
0 \leq (-1 + (1 + A_1) \psi'^2 \lambda_1^{-1}) F_{n|u_n|^2} + \psi'^2 F_{11} \overline{\nabla u}^2 + \psi' (u_t - F_{\overline{u}}) + C_4
\]
\[
\leq (-1 + (1 + A_1) (D_1 + M_0)^2 \lambda_1^{-1}) F_{n|u_n|^2}
\]
\[
+ (D_1 + M_0)^2 M_1^2 \csc^2 \theta(1 + \lambda_1^2)^{-1} + \psi' (u_t - F_{\overline{u}}) + C_4.
\]

The first term is negative if we assume
\[
\lambda_1 > 2(1 + A_1) (D_1 + M_0)^2.
\]

We further assume
\[
(4.22) \quad 1 + \lambda_1^2 > \max \{(K + \max_M |\nabla u|) + 1, \kappa_0^{-1} (1 + A_1^2)\}.
\]

Then by Lemma 4.4, we have
\[
u_t - F_{\overline{u}} \geq \kappa_0 \sum_{i=1}^{n} F_{\overline{u}} \geq \kappa_0 \frac{\csc \theta(\lambda)}{1 + \lambda_n^2} \geq \kappa_0 \frac{\csc \theta(\lambda)}{1 + \lambda_1^2}.
\]

Hence if \(D_1 > M_0\), (4.20) becomes
\[
0 \leq (D_1 + M_0)^2 M_1^2 \csc^2 \theta(1 + \lambda_1^2)^{-1} - (D_1 - M_0) \csc^2 \theta(1 + \lambda_1^2)^{-1} + C_4
\]

or
\[
((D_1 - M_0) \kappa_0^{-1} (1 + A_1^2)^{-1} - C_4)(1 + \lambda_1^2) \leq (D_1 + M_0)^2 M_1^2.
\]

We choose
\[
D_1 = (1 + C_4) \kappa_0^{-1} (1 + A_1^2) + M_0.
\]

Then we have
\[
(4.23) \quad \lambda_1 \leq (D_1 + M_0) M_1.
\]

Combining (4.21), (4.22) and (4.23), we have \(\lambda_1 < C\) and then can obtain the desired \(C^2\) estimate. \(\square\)
4.4. **Proof of Theorem 4.1.** The proof is the similar as the one in Phong-Tô [27]. We sketch it for completeness. We have proved the uniform a priori estimates up to the second order. By the concavity of \(\theta_{\omega}(\chi_u)\), we have the uniform \(C^{2,\alpha}\) estimates and then the higher estimates hold.

Since \(u_t\) is uniformly bounded, there exists a constant \(C\) such that \(v(x, t) := u_t(x, t) + C > 0\). Since \(v\) satisfies \(v_t = (u_t)_t = F^{ij}(u_t)_{ij} = F^{ij}v_{ij}\) and \(F^{ij}\) is uniformly elliptic, we can apply the differential Harnack inequality (Cao [1] and Gill [13]) to get the following estimates

\[
\max_{M} u_t(\cdot, t) - \min_{M} u_t(\cdot, t) = \max_{M} v(\cdot, t) - \min_{M} v(\cdot, t) \leq Ce^{-Ct},
\]

where \(C\) is a uniform constant.

By Lemma 2.7 and inequality (4.1) we know that for any \(t \in (0, \infty)\), there exists a point \(x_0(t)\) such that \(u_t(x_0(t), t) = 0\). Therefore, for any \((x, t) \in M \times (0, \infty)\), by (4.24), we have

\[
|u_t(x, t)| = |u_t(x, t) - u_t(x_0(t), t)| \leq Ce^{-Ct},
\]

and thus \(u(x, t)\) converges exponentially to a function \(u^\infty\). By the uniform \(C^k\) estimates of \(u(x, t)\) for all \(k \in \mathbb{N}\), \(u(x, t)\) converges to \(u^\infty\) in \(C^\infty\) and \(u^\infty\) satisfies

\[
\theta_{\omega}(\chi_{u^\infty}) := \sum_{i=1}^{n} \arccot \lambda_i(\chi_{u^\infty}) = \theta_0.
\]

5. **The convergence result on Kähler surface with the semi-subsolution condition**

In this section, we consider the compact Kähler surface case when \(\chi\) satisfies the semi-subsolution condition i.e. \(\chi - \cot \theta_0 \omega \geq 0\). We prove Theorem 1.4, i.e.,

**Theorem 5.1.** Let \((M, \omega)\) be a compact Kähler surface and \(\chi\) a closed real \((1, 1)\) form. Assume \(\theta_0 \in (0, \pi)\) and \(\chi \geq \cot \theta_0 \omega\). Then there exist a finite number of curves \(E_i\) of negative self-intersection on \(M\) such that the solution \(u(x, t)\) of dHYM flow (1.7) converges to a bounded function \(u^\infty\) in \(C^\infty_{\text{loc}}(M \cup \bigcup_i E_i)\) as \(t\) tends to \(\infty\) with the following properties.

1. \(\chi + \sqrt{-1} \partial \bar{\partial} u^\infty - \cot \theta_1 \omega\) is a Kähler current which is smooth on \(M \setminus \bigcup_i E_i\);
2. \(u^\infty\) satisfies the dHYM equation on \(M \setminus \bigcup_i E_i\)
3. \(\chi_{u^\infty}\) converges to \(\chi_{u^\infty}\) and \(u^\infty\) satisfies (5.1) on \(M\) in the sense of currents.

Here \(u_0\) is a function in \(\mathcal{H}_{B_1}\) for any \(B_1 \in (\theta_0, \pi)\). If \(\theta_0 \in (0, \frac{\pi}{2})\), we have \(0 \in \mathcal{H}_{B_1}\) for any \(B_1 \in (2\theta_0, \pi)\). If \(\theta_0 \in [\frac{\pi}{2}, \pi)\), we first show that the semi-subsolution condition implies the non-empty of \(\mathcal{H}_{B_1}\) for any \(B_1 \in (\theta_0, \pi)\).

**Lemma 5.2.** Let \((M, \omega)\) be a compact Kähler surface. Assume \(\chi \geq \cot \theta_0 \omega\) and \(\theta_0 \in [\frac{\pi}{2}, \pi)\). Then for any \(B_1 \in (\theta_0, \pi)\), there exists a smooth function \(u\) such that \(u \in \mathcal{H}_{B_1}\).
Proof. Let $\chi_\varepsilon := \chi - \varepsilon \omega$ with $\varepsilon > 0$ sufficiently small. Define $\theta_0(\varepsilon)$ as the principal argument of $\int_M (\chi_\varepsilon + \sqrt{-1} \omega)^2$. Then by definition,

$$\cot \theta_0(\varepsilon) = \frac{\int_M \text{Re}(\chi_\varepsilon + \sqrt{-1} \omega)^2}{\int_M \text{Im}(\chi_\varepsilon + \sqrt{-1} \omega)^2}.$$  

Since $\theta_0 \in (0, \pi)$, for any $\varepsilon > 0$ sufficiently small we have $\theta_0(\varepsilon) \in (0, \pi)$ and thus $\text{Im} \int_M (\chi_\varepsilon + \sqrt{-1} \omega)^2 = 2 \int_M \chi_\varepsilon \wedge \omega > 0$. By direct manipulation, we have

$$\cot \theta_0(\varepsilon) = \frac{\int_M (\chi^2 - \omega^2 + \varepsilon^2 \omega^2 - 2 \varepsilon \chi \wedge \omega)}{2 \int_M (\chi \wedge \omega - \varepsilon \omega^2)} = \cot \theta_0 - \varepsilon + \varepsilon \left( \cot \theta_0 - \frac{\varepsilon}{2} \right) \frac{\int_M \omega^2}{\int_M (\chi \wedge \omega - \varepsilon \omega^2)} < \cot \theta_0 - \varepsilon.$$

This shows $\chi_\varepsilon \geq \cot \theta_0 \omega - \varepsilon \omega > \cot \theta_0(\varepsilon) \omega$. Hence by Jacob-Yau [22] there exists a smooth function $u_\varepsilon$ solving

$$\text{Re}(\chi_\varepsilon + \sqrt{-1} \partial \bar{\partial} u_\varepsilon + \sqrt{-1} \omega)^2 = \cot \theta_0(\varepsilon) \text{Im}(\chi_\varepsilon + \sqrt{-1} \partial \bar{\partial} u_\varepsilon + \sqrt{-1} \omega)^2.$$  

Thus for any $B_1 \in (\theta_0, \pi)$, we have

$$\theta_0(\chi_\varepsilon) < \theta_0(\chi_\varepsilon + \sqrt{-1} \partial \bar{\partial} u_\varepsilon) = \theta_0(\varepsilon) < B_1$$

when $\varepsilon$ is sufficiently small since $\theta_0(\varepsilon)$ attends to $\theta_0$ as $\varepsilon$ goes to 0. □

We will use the following proposition proved by Song-Weinkove [29].

**Proposition 5.3 (Song-Weinkove [29]).** Let $M$ be a Kähler surface with a Kähler class $\beta \in H^{1,1}(M, \mathbb{R})$. If $\alpha \in H^1(M, \mathbb{R})$ satisfies $\alpha^2 > 0$ and $\alpha \cdot \beta > 0$, then either $\alpha$ is Kähler or there exists a positive integer $m$, curves of negative self-intersection $E_i, 1 \leq i \leq m$ and positive numbers $a_i, 1 \leq i \leq m$ such that

$$\alpha - \sum_{i=1}^{m} a_i [E_i]$$

is a Kähler class.

Since $2 \cot \theta_0[\chi] \cdot [\omega] = [\chi]^2 - [\omega]^2$, if we let $\tilde{\chi} = \chi - \cot \theta_0 \omega$, then we have

$$[\tilde{\chi}]^2 = [\chi]^2 - 2 \cot \theta_0[\chi] \cdot [\omega] + \cot^2 \theta_0[\omega]^2 = (1 + \cot^2 \theta_0)[\omega]^2 > 0.$$

Since $\tilde{\chi} \geq 0$, we also have

$$[\tilde{\chi}] \cdot [\omega] > 0,$$
otherwise $\tilde{\chi} \equiv 0$ which contradicts with (5.2). Hence we can apply Proposition 5.3 to get that there exists a finite number $m \geq 0$, curves of negative self-intesection $E_i$, $1 \leq i \leq m$ and positive numbers $a_i$, $1 \leq i \leq m$ such that $[\tilde{\chi}] - \sum_{i=1}^{m} a_i[E_i]$ is a Kähler class.

Let $h_i$ be the hermitian metric on $[E_i]$ and $s_i$ be a holomorphic section of $[E_i]$ which vanishes along $E_i$ to order 1. Define

$$S := \sum_{i=1}^{m} a_i \log |s_i|_{h_i}^2,$$

then

$$(5.3) \quad \tilde{\chi} + \sqrt{-1}\partial\bar{\partial}S > 0.$$

Similar as the argument in Section 2 in [12] which is based on [10], [34] and [38], we get the following result.

**Lemma 5.4.** Let $(M, \omega)$ be a compact Kähler surface. Assume $\tilde{\chi} := \chi - \cot \theta_0 \omega \geq 0$ and $\theta_0 \in (0, \pi)$. Then there exists a unique (up to adding a constant) bounded $\tilde{\chi}$-PSH function $v$ on $M$ and $v \in C^\infty_{\text{loc}}(M \cup_i E_i)$ satisfying

$$(5.4) \quad (\tilde{\chi} + \sqrt{-1}\partial\bar{\partial}v)^2 = \csc^2 \theta_0 \omega^2,$$

in the sense of currents.

### 5.1. The uniform $C^0$-estimate

We have proved the uniform $u_t$ estimate and thus along the flow we have $\theta_\omega(\chi_u) \in (\min_M \theta_\omega(\chi_{u(0)}), B_1)$.

**Proposition 5.5.** Assume the same conditions hold as in Theorem 1.4. Then there exists a uniform constant $M_0$ such that for any $(x, t) \in M \times [0, \infty)$

$$(5.5) \quad |u(x, t)| \leq M_0.$$

**Proof.** For any $T_0$, we will prove $\sup_{M \times [0, T_0]} |u(x, t)| \leq M_0$. We use similar auxiliary functions by Fang-Lai-Song-Weinkove [12] for the J-flow and Takahashi [33] for the LBMCF.

We first prove the upper bound of $u$ using the solution $v$ in Lemma 5.4. Consider

$$w_\varepsilon(x, t) := u - (1 + \varepsilon)v + \varepsilon S - C_0 \varepsilon t,$$

where $C_0$ is a large constant to be determined later. Since $w_\varepsilon(x, t)$ is upper semi-continuous on $M \times [0, T_0]$ with $w_\varepsilon = -\infty$ in $\cup_i E_i$, $w_\varepsilon$ attains its maximum on $M \times [0, T_0]$ at $(x_0, t_0)$ with $x_0 \in M \setminus \cup_i E_i$. Our goal is to show $t_0 = 0$.

At $(x_0, t_0)$, we have

$$0 \geq \sqrt{-1}\partial\bar{\partial}w_\varepsilon = \sqrt{-1}\partial\bar{\partial}u - (1 + \varepsilon) \sqrt{-1}\partial\bar{\partial}v + \varepsilon \sqrt{-1}\partial\bar{\partial}S$$

$$= \tilde{\chi}_u - (1 + \varepsilon)\tilde{\chi}_v + \varepsilon(\tilde{\chi} + \sqrt{-1}\partial\bar{\partial}S)$$

$$(5.6) \quad \geq \tilde{\chi}_u - (1 + \varepsilon)\tilde{\chi}_v,$$
where in the last inequality we use inequality (5.3). Let \( \lambda = (\lambda_1, \lambda_2) \) and \( \mu = (\mu_1, \mu_2) \) be the eigenvalues of \( \chi_a(x_0, t_0) \) and \( \chi_a(x_0, t_0) \) with respect to the metric \( \omega \) respectively. Then \( \lambda_i = \mu_i + \cot \theta_0 \). Without loss of generality, we assume \( \lambda_1 \geq \lambda_2 \). By direct manipulation, we have

\[
\frac{dw_e}{dt}(x_0, t_0) = \frac{du}{dt}(x_0, t_0) - C_0 \varepsilon = \cot \theta(\chi_a(x_0, t_0)) - \cot \theta_0 - C_0 \varepsilon = \frac{\lambda_1 \lambda_2 - 1}{\lambda_1 + \lambda_2} - \cot \theta_0 - C_0 \varepsilon
\]

(5.8)

\[
= \frac{\mu_1 \mu_2 - \csc^2 \theta_0}{\lambda_1 + \lambda_2} - C_0 \varepsilon.
\]

**Case 1:** \( \mu_1 \geq 0 \) and \( \mu_2 \geq 0 \). By (5.6) and (5.4), we have

\[
\mu_1 \mu_2 \leq (1 + \varepsilon) \frac{\chi_a^2}{\omega^2} = (1 + \varepsilon) \csc^2 \theta_0.
\]

Inserting (5.9) into (5.8), we obtain

\[
\frac{dw_e}{dt}(x_0, t_0) \leq \frac{\csc^2 \theta_0}{\lambda_1 + \lambda_2} \varepsilon - C_0 \varepsilon \leq \frac{\csc^2 \theta_0}{\cot \left(\frac{\lambda_1}{2} - \cot B_1\right)} \varepsilon - C_0 \varepsilon
\]

(5.10)

where we use \( \lambda_1 + \lambda_2 \geq \cot \left(\frac{\lambda_1}{2} - \cot B_1\right) > 0 \) and choose \( C_0 = \frac{\csc^2 \theta_0}{\cot \left(\frac{\lambda_1}{2} - \cot B_1\right)} + 1. \)

**Case 2:** \( \mu_1 \geq 0 \) and \( \mu_2 \leq 0 \). By (5.8), \( \frac{dw_e}{dt}(x_0, t_0) < -C_0 \varepsilon < 0. \)

**Case 3:** \( \mu_1 \leq 0 \) and \( \mu_2 \leq 0 \). Then \( \lambda_1 = \mu_1 + \cot \theta_0 \leq \cot \theta_0 \) and we get \( \cot \theta(\chi_a(x_0, t_0)) = \lambda_1 - \frac{1 + \varepsilon}{\lambda_1 + \lambda_2} < \cot \theta_0. \) Thus by (5.7), we have \( \frac{dw_e}{dt}(x_0, t_0) = \frac{du}{dt}(x_0, t_0) - C_0 \varepsilon < 0. \)

From the above three cases, we conclude \( \frac{dw_e}{dt}(x_0, t_0) < 0 \) and thus \( t_0 = 0. \) Thus for any \( \varepsilon > 0 \) and \( (x, t) \in (M \setminus \cup E_i) \times [0, T_0] \), we have

\[
u(x, t) \leq u_0(x_0) - (1 + \varepsilon)v(x_0) + \varepsilon S(x_0) + (1 + \varepsilon)v(x) - \varepsilon S(x) + C_0 \varepsilon t.
\]

Fix \( (x, t) \in (M \setminus \cup E_i) \times [0, T_0] \) and let \( \varepsilon \) tend to 0, since \( S \) is upper bounded, we have \( u(x, t) \leq \max u_0 + 2 \max |v|, \) which also holds for any \( (x, t) \in M \times [0, T_0] \) by continuity of \( u(x, t). \) Since \( T_0 \) is arbitrary, \( u \leq \max u_0 + 2 \max |v| \) in \( M \times [0, \infty]. \)

Next we prove the lower bound of \( u. \) Consider

\[
\hat{w}_\varepsilon := u - (1 - \varepsilon)v - \varepsilon S + C_0 \varepsilon t,
\]

where \( C_0 \) is a constant as above. Since \( \hat{w}_\varepsilon(x, t) \) is lower semi-continuous with \( \hat{w}_\varepsilon = +\infty \) in \( \cup E_i, \) \( \hat{w}_\varepsilon \) attains its minimum in \( M \times [0, T_0] \) at \( (x_1, t_1) \) with \( x_1 \in M \setminus \cup E_i. \)
At \((x_1, t_1)\), we have
\[
0 \leq \sqrt{-1} \bar{\partial} \bar{\partial} \tilde{w}_\varepsilon = \sqrt{-1} \bar{\partial} \bar{\partial} u - (1 - \varepsilon) \sqrt{-1} \bar{\partial} \bar{\partial} v - \varepsilon \sqrt{-1} \bar{\partial} \bar{\partial} S
\]
\[
= \tilde{x}_u - (1 - \varepsilon) \tilde{x}_v - \varepsilon (\tilde{x} + \sqrt{-1} \bar{\partial} \bar{\partial} S)
\]
\[
\leq \tilde{x}_u - (1 - \varepsilon) \tilde{x}_v.
\]
(5.11)
This implies
\[
\mu_1 \mu_2 \geq (1 - \varepsilon) \tilde{x}_u^2 = (1 - \varepsilon) \csc^2 \theta_0.
\]
Hence
\[
\frac{d\tilde{w}_\varepsilon}{dt}(x_1, t_1) = \frac{\mu_1 \mu_2 - \csc^2 \theta_0}{\lambda_1 + \lambda_2} + C_0 \varepsilon
\]
\[
\geq - \frac{\csc^2 \theta_0}{\lambda_1 + \lambda_2} \varepsilon + C_0 \varepsilon > 0.
\]
Thus \(\tilde{w}_\varepsilon\) attains its minimum at \(t_1 = 0\) and the lower bound of \(u\) follows. \(\square\)

Combining the above uniform estimate and Proposition 2.8 yields

**Corollary 5.6.** *Along the dHYM flow, there exists a uniform constant \(C\) such that*

\[
(5.12) \quad \text{Re}(CY_c(u)) \leq C.
\]

5.2. **\(C^k\)-estimate in compact set** \(K \subset M \setminus \bigcup_i E_i\). Since \(\chi - \cot \theta_0 \omega_0\) is only nonnegative, we could not apply Lemma 4.4 directly. But we can prove a similar type inequality as in Lemma 4.4. In fact, we consider \(\tilde{u} = u - S\). Since \(\chi - \cot \theta_0 \omega \geq 0\) and all \(E_i, 1 \leq i \leq m\) are negative self-intersection, we have \(\chi - \cot \theta_0 \omega + \sqrt{-1} \bar{\partial} \bar{\partial} S > 0\), and thus there exists a small constant \(\delta > 0\) such that

\[
(5.13) \quad \chi + \sqrt{-1} \bar{\partial} \bar{\partial} S \geq (\cot \theta_0 + \delta) \omega.
\]

We can prove the following useful inequality which is the key for us to prove the gradient estimate and the complex Hessian estimate.

**Lemma 5.7.** *Assume the same conditions hold as in Theorem 5.1. There exist uniform constants \(K_0 > 0\) and \(c_0 > 0\) such that if \(|\lambda(\chi_u)| > K_0\), then*

\[
u_t - F^{ij}(u_{ij} - S_{ij}) \geq c_0.
\]

**Proof.** Choose the normal coordinates at \((x, t)\) as before. By (5.13) we have
\[
u_t - F^{ij}(u_{ij} - S_{ij}) = \cot \theta_0 (\chi_u) - \cot \theta_0 - F^{\tilde{u}}(w_{\tilde{u}} - \chi_{\tilde{u}} - S_{\tilde{u}})
\]
\[
\geq \cot \theta_0 (\chi_u) - \cot \theta_0 - F^{\tilde{u}} w_{\tilde{u}} + (\delta + \cot \theta_0) \sum_{i=1}^2 F^{\tilde{u}}.
\]
(5.14)
By (3.3), we have $|\lambda_2| \leq A_1$. Recall that $\cot \theta_\omega(\chi_u) = \frac{\lambda_1 + \lambda_2}{(\lambda_1 + \lambda_2)^2}$ and $\csc^2 \theta_\omega(\chi_u) = 1 + \cot^2 \theta_\omega(\chi_u) = \frac{(1+\lambda_1^2)}{(\lambda_1 + \lambda_2)^2}$. Hence we have
\[
\cot \theta_\omega(\chi_u) - F_{ii} w_{ii} = \frac{\lambda_1 \lambda_2 - 1}{\lambda_1 + \lambda_2} - \frac{(1 + \lambda_1^2) \lambda_2}{(\lambda_1 + \lambda_2)^2} - \frac{(1 + \lambda_2^2) \lambda_1}{(\lambda_1 + \lambda_2)^2}
\]
(5.15)
\[
= -\frac{2}{\lambda_1 + \lambda_2} \geq -C\lambda_1^{-1}.
\]

For the other terms in (5.14), we have
\[
-cot \theta_0 + (\delta + cot \theta_0) \sum_{i=1}^{2} F_{ii} \geq cot \theta_0 \left( \frac{\csc^2 \theta_\omega(\chi_u)}{1 + \lambda_2^2} - 1 \right) + \delta \sum_{i=1}^{2} F_{ii} - C\lambda_1^{-1}
\]
\[
= cot \theta_0 \left( \frac{1 + \lambda_1^2}{(\lambda_1 + \lambda_2)^2} - 1 \right) + \delta \sum_{i=1}^{2} F_{ii} - C\lambda_1^{-1}
\]
\[
\geq -C\lambda_1^{-1} + \delta \sum_{i=1}^{2} F_{ii}
\]
\[
\geq -C\lambda_1^{-1} + \delta \frac{(1 + \lambda_1^2)}{(\lambda_1 + \lambda_2)^2}
\]
(5.16)
\[
\geq \frac{\delta}{2},
\]
where we assume $\lambda_1 \geq K_0$ and choose $K_0$ sufficiently large.

Inserting (5.15) and (5.16) into (5.14), we obtain
\[
u_t - F_{ij}(u_{ij} - S_{ij}) \geq \frac{\delta}{2} - C\lambda_1^{-1} \geq \frac{\delta}{4}.
\]
\]
\]

The following lemma is useful for us to prove the gradient estimate and the complex Hessian estimate.

**Lemma 5.8.** There exist uniform positive constants $\Lambda_0 := \min_i |a_i^{-1}|$ and $\Lambda_1$ such that for any $x \in M \setminus \cup_i E_i$, it holds
\[
e^{\Lambda_0 S(x)} \left( |S(x)|^3 + |\nabla S|^2(x) \right) \leq \Lambda_1.
\]
(5.17)

**Proof.** Since $S = \sum_{i=1}^{m} a_i \log |s_i|^2$, there exists a uniform constant $C > 0$ such that
\[
|\nabla S|^2 \leq C(1 + \sum_{i=1}^{m} |s_i|^{-2}).
\]
(5.18)

Then we have (5.17). \qed
\textbf{Proposition 5.9.} There exist uniform constants $D_0$ and $M_1$ such that for any $(x, t) \in M \setminus \cup_i E_i \times [0, \infty)\] \begin{equation} \tag{5.19} |\nabla u|_\infty(x, t) \leq M_1 \prod \limits_i |s_i|^{-D_0 u_i}(x). \end{equation} \]

\textit{Proof.} Since $S$ is upper semi-continuous, there exists a uniform constant $S_0$ such that \(\sup_M S \leq S_0\). We consider the function \(G = \log |\nabla u|^2 + \psi(\bar{u}),\) where \(\bar{u} = u - S\) and

\[
\psi(\bar{u}) = -D_0 \bar{u} + (\bar{u} + S_0 + M_0 + 1)^{-1},
\]

where \(D_0 > \Lambda_0 := \min\{a_t^{-1}\}\) is a uniform constant to be determined later.

Since $S$ is upper semi-continuous, we know that $G$ is also upper semi-continuous. Suppose that \(\max\limits_{M \times [0, T_0]} G(x, t) = G(x_0, t_0)\). Since $S = -\infty$ on $\cup_i E_i$, we have $G(x, t) = -\infty$ on $\cup_i E_i$ and then $x_0 \in M \setminus \cup_i E_i$.

If $t_0 = 0$, we have for any $(x, t) \in M \setminus \cup_i E_i \times [0, T_0]$

\[
\tag{5.20} e^{G(x,t)} \leq e^{G(x_0,0)} \leq |\nabla u_0|^2 e^{D_0 S_0 + D_0 M_0 + S_0 + M_0 + 1} \leq M_{1,0},
\]

where we used $S \leq S_0$ and $M_{1,0} := \max_M |\nabla u_0|^2 e^{(D_0 + 1)(S_0 + M_0) + 1}$. This gives the estimate (5.9).

In the following, we always assume $t_0 > 0$.

If $|\nabla u|(x_0, t_0) \leq 2 |\nabla S|(x_0, t_0)$, by Lemma 5.8, we get the desired estimate as follows

\[
\tag{5.21} e^{G(x_0,t_0)} \leq C |\nabla u|^2(x_0, t_0) e^{D_0 S(x_0)} \leq 4C |\nabla S|^2(x_0, t_0) e^{D_0 S(x_0,t_0)} \leq M_{1,1}.
\]

Thus in the following proof, we always assume $|\nabla u|(x_0, t_0) \geq 2 |\nabla S|(x_0, t_0)$ and then we have

\[
\tag{5.22} \frac{1}{2} |\nabla u|(x_0, t_0) \leq |\nabla \bar{u}|(x_0, t_0) \leq 2 |\nabla u|(x_0, t_0).
\]

Taking the manipulation at $(x_0, t_0)$, we have

\[
G_i = \frac{u_{i0}u_k + u_k u_{i0}}{|\nabla u|^2} + \psi' u_i,
\]

\[
G_i = \frac{u_{i0}u_k + u_k u_{i0}}{|\nabla u|^2} + \psi' \bar{u}_i = 0,
\]
and

\[ 0 \leq PG = G_t - F^\bar{u}G_{\bar{u}} \]

\[
\begin{aligned}
&= \frac{(u_{i\bar{i}} - F^\bar{u}u_{i\bar{i}})u_k + (u_{i\bar{k}} - F^\bar{u}u_{i\bar{k}})u_k}{|\nabla u|^2} \\
&\quad - \frac{F^{\bar{u}}(u_{k\bar{i}} + u_{\bar{i}k})|\nabla u|^2 - F^{\bar{u}}|\nabla |\nabla u|^2|^2}{|\nabla u|^4} \\
&+ \psi'(u_t - F^\bar{u}u_{\bar{u}}) - \psi''F^\bar{u}|u_{\bar{u}}|^2. 
\end{aligned}
\] (5.23)

(denoted by (I))

We divide two cases to do the estimate. Let

\[ \psi = \psi(\bar{u}) = F^\bar{u}u_{\bar{u}}, \]

(5.24)

By the same estimate as that in Proposition 4.5, we have

(II) \leq C.

We then deal with term (II). Since \( G_t = 0 \) for each \( 1 \leq i \leq 2 \), we have

\[
|\nabla |\nabla u|^2|^2 = \left| \sum u_{k\bar{i}} \right|^2 + \left| \sum u_{i\bar{k}} \right|^2 + 2Re\left( \sum u_{k\bar{i}} \sum u_{i\bar{k}} \right)
\]

\[
\quad = \left| \sum u_{k\bar{i}} \right|^2 + \left| \sum u_{i\bar{k}} \right|^2 + 2Re\left( \sum u_{k\bar{i}} |\nabla u|^2 |\nabla |\nabla u|^2| \right) \sum u_{i\bar{k}}
\]

\[
\quad = \left| \sum u_{k\bar{i}} \right|^2 - \left| \sum u_{i\bar{k}} \right|^2 - 2|\nabla u|^2 \psi'Re\left( \bar{u}_i \sum u_{i\bar{k}} \right).
\]

Hence

(II) \leq - 2|\nabla u|^{-2}\psi'F^\bar{u}Re(\bar{u}_i \sum u_{i\bar{k}}).

Similar as the estimate in Proposition 4.5, we have

(II) \leq C|\psi'||\nabla u|^{-1}\left( \sum F^\bar{u} |u_{\bar{u}}|^2 \right)^{\frac{1}{2}}.

Inserting the estimates of (I) and (II) into (5.23), we obtain

\[ 0 \leq PG \leq -\psi'(-u_t + F^\bar{u}u_{\bar{u}}) - \psi''F^\bar{u}|u_{\bar{u}}|^2 + C|\psi'||\nabla u|^{-1}(F^\bar{u}|u_{\bar{u}}|^2)^{\frac{1}{2}} + C. \] (5.24)

We divide two cases to do the estimate. Let

\[ \epsilon_0 = \min\left\{ K_0^{-\frac{1}{2}} \sum c, \frac{\epsilon_0}{2c_{\min}(\sin \theta, |\chi_{u_0}|)} \right\} \]

where \( K_0 \) is the uniform constant in Lemma 5.7 and \( C \) is the constant in (5.24).

**Case 1:** \( \sum F_i |u_i|^2 \geq \epsilon_0^2 \left| \nabla u \right|^2 \). Since \( D_0 \leq -\psi' \leq D_0 + 1 \) and \( \psi'' = 2(\bar{\nu} + S_0 + M_0 + 1)^{-3} \), by (5.24), we have

\[
0 \leq -\frac{2\epsilon_0^2 \left| \nabla u \right|^2}{\left( \bar{\nu} + S_0 + M_0 + 1 \right)^3} + (D_0 + 1)(|u_t| + \max_M \csc^2 \theta_{u_0}(\chi_{u_0})) \]

\[
\quad + (D_0 + 1) \max_M \csc \theta_{u_0}(\chi_{u_0}) \left| \nabla u \right| \left| \nabla u \right|^{-1} + C.
\]

From the above inequality, by (5.22), we have

\[ \left| \nabla u \right|^2 \leq C(2M_0 + S_0 + 1 - S)^3. \] (5.25)
By Lemma 5.8, we obtain
\((5.26)\quad G(x_0, t_0) = |\nabla u|^2(x_0, t_0)e^{\psi(\tilde{u}(x_0, t_0))} \leq C_1(2M_0 + S_0 + 1 + |S|(x_0, t_0))^2e^{D_0S} \leq M_{1,2}.\)

**Case 2:** \(2 \sum_{i=1}^2 F_i^{|\tilde{u}|^2} \leq \epsilon_0^2|\nabla u|^2.\) In this case, since \(\psi'' > 0,\) by inequality (5.24), we have
\((5.27)\quad 0 \leq -\psi'(-u_t + F_i^{|u|} + C \max_M \csc \theta(\chi_0)|(-\psi')e_0 + C.\)

On the other hand, since \(F_1 \leq F_2,\) we have
\(\epsilon_0^2|\nabla u|^2 \geq F_1^{|u|^2} \geq 1 + \frac{\lambda^2_2}{(\lambda_1 + \lambda_2)^2}|\tilde{u}|^2 \geq \frac{1}{4\lambda_1^2}|\tilde{u}|^2.\)

From this inequality and (5.22), we get
\(\lambda_1 \geq \frac{1}{4\epsilon_0^{-1}} = K_0.\)

Then we can apply our Lemma 5.7 to get
\(-u_t + F_i^{|u|} \leq -c_0.\)

Inserting the above inequality into (5.27), we get
\(0 \leq \psi'c_0 + \epsilon_0C \max_M \csc \theta(\chi_0)|(-\psi') + C\)
\(\leq D_0(-c_0 + \epsilon_0C \max_M \csc \theta(\chi_0)|) + C.\)

(5.28)

Since \(\epsilon_0C \max_M \csc \theta(\chi_0)| \leq \frac{c_0}{2},\) if we choose \(D_0 = 2c_0^{-1}(C + 1),\) we get the following contradiction
\((5.29)\quad 0 \leq -D_0 \frac{c_0}{2} + C = 1.\)

Thus this case cannot occur.

In conclusion, for any \((x, t) \in M \setminus \cup_i E_i,\) we have \(G(x, t) \leq G(x_0, t_0) \leq M_{1,0} + M_{1,1} + M_{1,2}\) and then we obtain the desired estimate
\((5.30)\quad |Du|^2(x, t) \leq M_1^2e^{D_0S(x)} = M_1 \prod_i |s_i|^{-2D_0u_i}(x).\)

□

**Proposition 5.10.** There exist uniform constant \(D_1\) and \(M_2\) such that for any \((x, t) \in M \setminus \cup_i E_i \times [0, \infty)\)
\((5.31)\quad |\partial \tilde{u}|_{\omega}(x, t) \leq M_2 \prod_i |s_i|^{-2D_1u_i}(x, t).\)
Proof. We consider
\[ \tilde{H}(x, t, \xi(x)) = \log(w_{ij} \xi^i \bar{\xi}^j) + \psi(\tilde{u}) \]
where \( \psi(\tilde{u}) = -D_1 \tilde{u} + (\tilde{u} + M_0 + S_0 + 1)^{-1} \) and \( \tilde{u} = u - S \). Recall \( M_0 \) is the uniform bound of \( |u| \) in Lemma 4.3 and \( S_0 \) is the upper bound of \( S \). Hence we have

(5.32) \[ D_1 \leq -\psi' \leq D_1 + 1 \quad \text{and} \quad \psi'' = 2(\tilde{u} + M_0 + S_0 + 1)^{-3}. \]

For any \( T_0 \in (0, \infty) \), suppose the function \( \tilde{H} \) which is upper semi-continuous attains its maximum on \( M \times [0, T_0] \) at \( (x_0, t_0) \) along the direction \( \xi_0 = \tilde{\xi}(x_0) \). Since \( \tilde{H} = -\infty \) on \( \cup_i E_i \), we have \( x_0 \in M \setminus \cup_i E_i \). If \( t_0 = 0 \), the estimate holds since \( S \) is upper bounded. Hence in the following we assume \( t_0 > 0 \).

Take holomorphic coordinates near \( x_0 \) such that (3.3) holds. Then the function \( H(x, t) = \tilde{H}(x, t, \tilde{\xi}_0(x)) \) attains its maximum on \( M \times [0, T_0] \) at \( (x_0, t_0) \).

At \( (x_0, t_0) \), we have

(5.33) \[ \begin{align*}
0 \leq & H_t = \frac{u_{t_{11}}}{w_{11}} + \psi' u_t, \\
0 = & H_t = \frac{w_{11,i}}{w_{11}} + \psi' \tilde{u}_t,
\end{align*} \]

and

(5.34) \[ 0 \leq H_t - F^{\tilde{u}} H_{\tilde{u}} = \lambda_1^{-1} (u_{t_{11}} - F^{\tilde{u}} w_{11,\tilde{u}}) + \lambda_1^{-2} F^{\tilde{u}} |w_{11,\tilde{u}}|^2 \quad \text{(denoted by (I))} \]

By the same argument as that in section 4, (I) has the following estimate

(5.35) \[ (I) \leq \psi'^2 F^{11} |\nabla \tilde{u}|^2 + \psi'^2 (1 + A_1) \lambda_1^{-1} F^{22} |\tilde{u}_2|^2 + C. \]

Inserting (5.35) into (5.34), by (5.32), we have

(5.36) \[ \begin{align*}
0 \leq & (-\psi'' + (1 + A_1) \psi'^2 \lambda_1^{-1}) F^{22} |\tilde{u}_2|^2 + \psi'^2 F^{11} |\nabla \tilde{u}|^2 + \psi' (u_t - F^{\tilde{u}} \tilde{u}_t) + C \\
\leq & (-2(S + 2M_0 + S_0 + 1)^{-3} + (1 + A_1)(D_1 + 1)^2 \lambda_1^{-1} F^{22} |\tilde{u}_2|^2 \\
+ & (D_1 + 1)^2 |\nabla \tilde{u}|^2 \frac{1 + A_2}{(\lambda_1 + A_2)^2} + \psi' (u_t - F^{\tilde{u}} \tilde{u}_t) + C. \end{align*} \]

The first term is negative if we assume

(5.37) \[ \lambda_1 > (1 + A_1)(D_1 + 1)^2 (-S + 2M_0 + S_0 + 1)^3, \]

We further assume

(5.38) \[ \lambda_1 > 2K_0. \]

Then by Lemma 5.7 and (5.32), we have

\[ \psi' (u_t - F^{\tilde{u}} \tilde{u}_t) \leq -c_0 D_1. \]
Hence (5.36) becomes
\[ 0 \leq (D_1 + 1)^2|\nabla \bar{u}|^2 \frac{1 + A_1^2}{(\lambda_1 - A_1)^2} - c_0 D_1 + C \]
or
\[(c_0 D_1 - C)(\lambda_1 - A_1)^2 \leq (D_1 + 1)^2(1 + A_1^2)|\nabla \bar{u}|^2.\]

We choose
\[ D_1 > c_0^{-1}(C + 1). \]
Then we have
\[
\lambda_1 \leq (D_1 + 1)(1 + A_1^2)|\nabla \bar{u}| + A_1 \\
\leq (D_1 + 1)(1 + A_1^2)(|\nabla u| + |\nabla S|) + A_1 \\
\leq (D_1 + 1)(1 + A_1^2)(M_1 \prod_i |s_i|_{h_i}^{-M_1} + |\nabla S|) + A_1, \]
where in the last inequality we use (5.9).

By (5.37), (5.38) and (5.40), we obtain
\[
\alpha_1 e^{d(h)} \leq C\alpha_1 e^{D_1} (M_1 \prod_i |s_i|_{h_i}^{-M_1} + |\nabla S| + (-S + 2M_0 + S_0 + 1)^3) + C.
\]

If we choose \( D_1 > (M_1 + 1) \min_i \{a_i^{-1}\} \), the above inequality has a uniform upper bound and thus we obtain the estimate (5.10).

**Proposition 5.11.** For any compact set \( K \subset M \setminus \bigcup_i E_i \) and positive integer \( k \), there exists a uniform constant \( C_{k,K} \) such that
\[ |u|_{C^k(K)} \leq C_{k,K}. \]

**Proof.** By the complex Hessian estimate in Proposition 5.10, the dHYM flow is uniformly parabolic. Since \( \cot \theta_\omega(\chi_{\omega}) \) is concave, by the Evans-Krylov theory [9, 23], we obtain the higher order estimates in \( K \).

As an application of Proposition 5.11, we first show

**Proposition 5.12.** For any compact set \( K \subset M \setminus \bigcup_i E_i \), \( \frac{\partial u}{\partial t} \) uniformly converges to 0 in \( K \) as \( t \) tends to \( \infty \).
Proof. We first prove that \( \frac{\partial u}{\partial t} \) pointwisely converges to 0 in \( M \setminus \bigcup_j E_j \). Since

\[
(5.42) \quad \text{Re}(\text{CY}_C(u(t))) - \text{Re}(\text{CY}_C(u(0))) = \int_0^t \int_M \left( \frac{\partial u}{\partial t} \right)^2 \text{Im}(\chi_{u(s)}) + \sqrt{-1} \omega)^2 ds,
\]

by Corollary 5.6 we have

\[
\int_0^\infty \int_M \left( \frac{\partial u}{\partial t} \right)^2 \text{Im}(\chi_u + \sqrt{-1} \omega)^2 dt \leq C.
\]

Since along the flow \( \text{Im}(\chi_u + \sqrt{-1} \omega)^2 \geq c_0 \omega^2 > 0 \), the above inequality gives

\[
(5.43) \quad \int_0^\infty \int_M \left( \frac{\partial u}{\partial t} \right)^2 \omega^2 dt \leq c_0^{-1} C.
\]

If there exists \( x_0 \in K \) such that \( \lim_{i \to \infty} \frac{\partial u}{\partial t}(x_0, t) \neq 0 \), then there exists \( \epsilon_0 > 0 \) and a sequence \( \{t_i\} \) which tends to \( \infty \) such that

\[
(5.44) \quad \left| \frac{\partial u}{\partial t}(x_0, t_i) \right| \geq \epsilon_0.
\]

Let \( U \) be a small neighborhood of \( x \) such that \( U \subset M \setminus \bigcup_j E_j \). Then by Proposition 5.11, \( \frac{\partial u}{\partial t} \) and its time and space derivative are uniformly bounded in \( U \times [0, \infty) \) and thus by (5.44), there exist a small neighborhood \( V \subset U \) of \( x_0 \) and a uniform constant \( \delta > 0 \) such that

\[
\left| \frac{\partial u}{\partial t} \right| \geq \frac{\epsilon_0}{2} \text{ for any } (x, t) \in V \times [t_i, t_i + \delta].
\]

This implies

\[
\int_0^\infty \int_M \left( \frac{\partial u}{\partial t} \right)^2 \omega^2 dt \geq \sum_{i=1}^\infty \int_{t_i}^{t_{i+\delta}} \int_V \left( \frac{\partial u}{\partial t} \right)^2 \omega^2 dt \geq \sum_{i=1}^\infty \delta \frac{\epsilon_0^2}{4} \text{vol}_\omega(V) = \infty,
\]

which contradicts with (5.43). Hence \( \frac{\partial u}{\partial t} \) pointwisely converges to 0 in \( M \setminus \bigcup_j E_j \).

Let \( K \subset \bigcup_{j=1}^\infty B_r(x_j) \subset M \setminus \bigcup_j E_j \). We can apply the differential Harnack inequality for \( \frac{\partial u}{\partial t} \) in every \( B_r(x_j) \) to prove that \( \frac{\partial u}{\partial t} \) converges in any compact subset \( K \) uniformly to 0. \( \square \)

5.3. Proof of Theorem 5.1. Similarly as the proof by Fang-Lai-Song-Weinkove [12] and Takahashi [33], we have

**Lemma 5.13.** Let \( \{u_i\} \) be a sequence of smooth functions satisfying \( \chi_{u_i} - \cot B_1 \omega > 0 \) and \( |u_i|_{C^0} \leq C \) for \( C > 0 \). Let \( u^\infty \) be a bounded \( (\chi - \cot B_1 \omega) \)-PSH function on \( M \). Let \( Y \) be a proper subvariety of \( M \). Assume that \( u_i \) converges to \( u^\infty \) in \( C^0_{\text{loc}}(M \setminus Y) \) as \( j \to \infty \). Then \( \text{CY}_C(u^\infty) \) and \( \mathcal{J}(u^\infty) \) are well-defined. Moreover,

\[
\lim_{i \to \infty} \text{Im}(\text{CY}_C(u_i)) = \text{Im}(\text{CY}_C(u^\infty)),
\]

\[
\lim_{i \to \infty} \text{Re}(\text{CY}_C(u_i)) = \text{Re}(\text{CY}_C(u^\infty)),
\]

\[
\lim_{i \to \infty} \mathcal{J}(u_i) = \mathcal{J}(u^\infty).
\]
Proof of Theorem 5.1. By the $C^0$ estimate proved in Proposition 5.5, there exists a sequence $\{t_i\}$ such that $u(\cdot, t_i)$ converges to a function $u^\infty \in L^\infty(M)$. By the $C^k$ estimates in Proposition 5.11, by passing a subsequence (for convenience we still denote by $t_i$), $u(\cdot, t_i)$ smoothly converges to $u^\infty$ in any compact subset of $M \setminus \cup_i E_i$ and thus $u^\infty \in C^\infty(M \setminus \cup_i E_i)$. Since $\chi_u > \cot B_1 \omega$, then $\chi_{u^\infty} - \cot B_1 \omega$ is a Kähler current and is smooth in $M \setminus \cup_i E_i$. By Lemma 5.13 and Lemma 2.7, we have $\text{Im}(\text{CY}_C(u^\infty)) = \text{Im}(\text{CY}_C(u_0))$.

By Proposition 5.12, $u^\infty$ satisfies (5.1) in $M \setminus \cup_i E_i$ and then $\theta_0(\chi_{u^\infty}) = \theta_0$ on $M \setminus \cup_i E_i$. We can define $\chi_{u^\infty}^2$ and $\chi_{u^\infty} \wedge \omega$ as finite measures on $M$ such that they do not charge pluripolar subsets. Thus $(\chi_{u^\infty}^2 + \sqrt{-1} \omega)^2$ is well-defined and $u^\infty$ satisfies the equation (5.1) on $M$ in the sense of currents. Moreover, $u^\infty$ is $\tilde{\chi}$-PSH on $M$ and satisfies the equation (5.4) in the sense of currents.

Finally, by the $C^\infty_{loc}(M \setminus \cup_i E_i)$ uniform estimate of $u(t)$ and the uniqueness of the equation (5.4), similar as the argument in [12], we have $u(t)$ converges smoothly to $u^\infty$ on $M \setminus \cup_i E_i$. \hfill $\Box$

5.4. $\mathcal{I}$ functional. As an application of our dHYM flow, we prove the lower bound of the $\mathcal{I}$-functional in the following set.

$$\mathcal{H}_{B_1} = \{ w \in C^\infty(M, \mathbb{R}) : \theta_0(\chi_w) \in (0, B_1) \}.$$ 

Corollary 5.14. Let $(M, \omega)$ be a compact Kähler surface and $\chi$ a closed real $(1, 1)$ form. Assume that $\theta_0 \in (0, \pi)$ and $\chi \geq \cot \theta_0 \omega$, the $\mathcal{I}$-functional is bounded from below in $\mathcal{H}_{B_1}$ for any $B_1 \in (\theta_0, \pi)$.

Proof. For $u_0 \in \mathcal{H}_{B_1}$, let $u(t)$ be the solution of the dHYM flow $u_t = \cot \theta_0(\chi_u) - \cot \theta_0$ with $u(0) = u_0$. By Theorem 5.1, $u(t)$ converges to a bounded function $u^\infty$ solving (5.4). Since $\mathcal{I}$ is decreasing along the flow, we have

$$\mathcal{I}(u_0) \geq \lim_{t \to \infty} \mathcal{I}(u(t)) = \mathcal{I}(u^\infty).$$

Let $v$ be a weak solution of (5.4) in Lemma 5.4. By the uniqueness, there exists a constant $c_0$ such that $u^\infty = v + c_0$. Since $\mathcal{I}(u^\infty) = \mathcal{I}(v)$, we have

$$\mathcal{I}(u_0) \geq \mathcal{I}(v).$$

$\Box$

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