Chapter 1
Recognizing Concepts and Recognizing Musical Themes. A Quantum Semantic Analysis

Maria Luisa Dalla Chiara, Roberto Giuntini, Eleonora Negri, Giuseppe Sergioli

Abstract How are abstract concepts and musical themes recognized on the basis of some previous experience? It is interesting to compare the different behaviors of human and of artificial intelligences with respect to this problem. Generally, a human mind that abstracts a concept (say, table) from a given set of known examples creates a table-Gestalt: a kind of vague and out of focus image that does not fully correspond to a particular table with well determined features. A similar situation arises in the case of musical themes. Can the construction of a gestaltic pattern, which is so natural for human minds, be taught to an intelligent machine? This problem can be successfully discussed in the framework of a quantum approach to pattern recognition and to machine learning. The basic idea is replacing classical data sets with quantum data sets, where either objects or musical themes can be formally represented as pieces of quantum information, involving the uncertainties and the ambiguities that characterize the quantum world. In this framework, the intuitive concept of Gestalt can be simulated by the mathematical concept of positive centroid of a given quantum data set. Accordingly, the crucial problem “how can we classify a new object or a new musical theme (we have listened to) on the basis of a previous experience?” can be dealt with in terms of some special quantum similarity-relations. Although recognition procedures are different for human and

Maria Luisa Dalla Chiara
Dipartimento di Lettere e Filosofia, Università di Firenze. Via della Pergola 60, I-50121 Firenze, Italy; e-mail: dallachiara@unifi.it

Roberto Giuntini
Dipartimento di Pedagogia, Psicologia, Filosofia, Università di Cagliari. Via Is Mirrionis 1, I-09123 Cagliari, Italy; e-mail: giuntini@unica.it

Eleonora Negri
Scuola di Musica di Fiesole, Via delle Fontanelle 26, I-50014 Fiesole-Firenze, Italy; e-mail: eleonora.negri64@gmail.com

Giuseppe Sergioli
Dipartimento di Pedagogia, Psicologia, Filosofia, Università di Cagliari. Via Is Mirrionis 1, I-09123 Cagliari, Italy; e-mail: giuseppe.sergioli@gmail.com

arXiv:2202.10941v1 [cs.LG] 17 Feb 2022
for artificial intelligences, there is a common method of “facing the problems” that seems to work in both cases.

### 1.1 Introduction

There is a story about a king who was able to distinguish between two different pieces of music only: the “Royal March” and the “Non-Royal March”. One is dealing with an extreme case of a “fully non-musical personality”. Normally people behave differently; and children often show an early capacity of recognizing and repeating some simple songs they have listened to. Of course, such capacity generally depends on what is usually called the “musical talent” of each particular child. And, in the case of adult persons, one shall distinguish the behavior of professional musicians from the behavior of generic music-listeners who may have different degrees of musical culture.

Recognizing a musical theme is a cognitive operation that is very similar to what happens when we recognize an abstract concept that may refer either to concrete or to ideal objects (say, table, star, triangle,...). We know how quickly children learn to abstract concepts from the concrete objects they have met in their brief experience. On the basis of a small number of examples that have appeared in the environment where they are living, they easily recognize and use (mostly in a correct way) general concepts like table, house, toy.

How are abstract concepts formed and recognized on the basis of some previous experience? What happens in the human brain when we recognize either an abstract concept or a musical theme? These questions have been intensively investigated, with different methods, by psychologists, neuroscientists, artificial intelligence researchers, logicians, philosophers, musicians and musicologists. Some important researches in the field of neurosciences (which have used sophisticated brain-imaging techniques) have provided some partial answers. Interestingly enough, neuroscientific investigations have recently interacted with an important approach to psychology: the Gestalt-theory that had been proposed by Wertheimer, Kofka and Köhler in the early 20th century. As is well known, a basic idea of Gestalt-psychology is that human perception and knowledge of objects is essentially connected with our capacity of realizing a Gestalt (a form) of the objects in question: a holistic image that cannot be identified with the set of its component elements. The cognitive procedure goes from the whole to the parts, and not the other way around! These general ideas can be naturally applied to investigate the question “how are abstract concepts formed and recognized?” A human mind that abstracts the concept table from a given set of concrete examples, generally creates a table-Gestalt, a kind of vague and out of focus image that does not fully correspond to a particular table, with well determined features. When we ask different people the question: what do you see in

---

1 See, for instance. \[\text{5}\]
2 See, for instance. \[\text{3}\].
your mind when you hear the word “table”? We may receive different answers. An interesting answer that has been given by a person submitted to a psychological test is the following: “I see a table, with an indefinite color, floating in an indefinite space”. In this case the vagueness of the table-Gestalt has been expressively described by the metaphorical image of a “floating object”.

Creating a Gestalt associated to a given concept is a cognitive operation that is quite natural for human intelligences. But what happens in the case of artificial intelligences? Is it possible to find a mathematical definition for a Gestalt-like concept that could be taught to an intelligent machine? We will see how this intriguing question can be successfully investigated in the framework of a quantum inspired approach to pattern recognition and to machine learning.

From a logical point of view the relationship that connects gestaltic patterns with particular concrete or ideal objects can be analyzed by using the concept of similarity. Consider a child (let us call her Alice) who has recognized as a table a new object that has appeared in her environment. Apparently, her recognition is essentially based on a quick and probably unconscious comparison between the main features of the new object and the ideal table-Gestalt that Alice had previously stored in her memory.

Generally, any comparison involves the use of some similarity-relations, weak examples of relations that are

- reflexive: any object $a$ is similar to itself;
- symmetric: if $a$ is similar to $b$, then $b$ is similar to $a$;
- generally non-transitive: if $a$ is similar to $b$ and $b$ is similar to $c$, then $a$ is not necessarily similar to $c$.

Just the failure of the transitive property is one of the reasons why similarity-relations play an important role in many semantic and cognitive phenomena. A significant example is represented by metaphorical arguments, which frequently occur in natural languages as well in the languages of art. Metaphorical correlations generally involve some allusions that are based on particular similarity-relations. Ideas that are currently used as possible metaphors are often associated to concrete and visual features. Let us think, for instance, of a visual idea that is often used as a metaphor: the image of the sea, correlated to the concepts of immensity, of infinity, of pleasure or fear, of places where we may get lost and die.

In the tradition of scientific thought metaphorical arguments have often been regarded as “fallacious”. There is a deep logical reason that justifies such suspicion. Metaphors, based on particular similarity-relations, do not generally preserve the properties of the objects under consideration: if Alice is similar to Beatrix and Alice is clever, then Beatrix is not necessarily clever! Wrong extrapolations of properties from some objects to other similar objects are often used in rhetoric contexts, in order to obtain a kind of captatio benevolentiae. We need only think of the soccer-metaphors that are so frequently used by many politicians!

In spite of their possible “dangers”, metaphors have sometimes played an important role even in exact sciences. An interesting example in logic is the current use of the metaphor of possible world, based on a general idea that had been deeply investigated by Leibniz. In some situations possible worlds, that correspond to special
examples of semantic models, can be imagined as a kind of “ideal scenes”, where abstract objects behave as if they were playing a theatrical play. And a “theatrical imagination” has sometimes represented an important tool for scientific creativity, also in the search of solutions for logical puzzles and paradoxes.

The classical concept of possible world is characterized by a strong logical determinism: due to the semantic excluded middle principle, any sentence that refers to a given possible world shall be either true or false. Quantum information and quantum computation theories have recently inspired the development of a new form of quantum-logical semantics: classical possible worlds have been replaced by vague possible worlds, where events are generally uncertain and ambiguous, as happens in the case of microobjects[3] In the next Sections we will see how recognition-processes that may concern either abstract concepts or musical themes can be naturally investigated in the framework of this quantum-semantic approach.

1.2 A quantum semantics inspired by quantum information theory

For the readers who are not familiar with quantum mechanics it may be useful to recall some basic concepts of the theory that play an important semantic role. Suppose a physicist is studying a quantum physical system S (say, an electron) at a given time. His (her) information about S can be identified with a particular mathematical object that represents the state of S at that time. In the happiest situations our physicist might have about S a maximal information that cannot be consistently extended to a richer knowledge. In such a case, the information in question is called a a pure state of the system. An observer who has assigned a pure state to a given system knows about this system all that even a hypothetical omniscient mind would know. Following a happy notation introduced by Paul Dirac, quantum pure states are usually denoted by the expressions $|\psi\rangle, |\varphi\rangle, \ldots$ (where $| \ldots \rangle$ represent the so called “ket-brackets”). Mathematically, any pure state $|\psi\rangle$ is a vector (with length 1) living in a special abstract space, called a Hilbert space (usually indicated by the symbol $\mathcal{H}$).[4] A strange logical feature of the quantum formalism is the following: although representing a maximal piece of information, a pure state $|\psi\rangle$ cannot decide all physical properties that may hold for a quantum system described by $|\psi\rangle$. Due to the celebrated Heisenberg’s uncertainty principle some basic properties (that may concern, for instance, either the position or the velocity) turn out to be essentially indeterminate.

Generally, a quantum pure state $|\psi\rangle$ can be represented as a superposition (a vector-sum) of other pure states $|\psi_i\rangle$:

---

[3] See, for instance, [2, 4].

[4] Hilbert spaces are special examples of vector spaces that represent generalizations of geometric Euclidean spaces. A simple example of a Hilbert space is the geometric plane, whose set of points corresponds to the set of all possible ordered pairs of real numbers.
Recognizing Concepts and Recognizing Musical Themes

\[ |\psi\rangle = \sum_i c_i |\psi_i\rangle. \]

where \( c_i \) are complex numbers (called amplitudes). The physical interpretation of this formal representation is the following: a quantum system whose state is \( |\psi\rangle \) might verify the properties that are certain for a system in state \( |\psi_i\rangle \) with a probability-value that depends on the number \( c_i \).

From an intuitive point of view one can say that a superposition-state seems to describe a kind of ambiguous cloud of possibilities: a set of potential properties that are, in a sense, all co-existent for a given quantum object.

Special examples of superpositions that play an important role in quantum information are represented by qubits. As is well known, in classical information theory information is measured in terms of bits. One bit represents the information-quantity that is transmitted (or received) when one answers either “Yes” or “No” to a given question. The two bits are usually indicated by the natural numbers 0 (corresponding to the answer “No”) and 1 (corresponding to the answer “Yes”). On this basis, complex pieces of information are represented by sequences of many bits, called registers. For instance, a register consisting of 8 bits represents one byte.

In quantum information theory the quantum counterpart of the classical notion of bit is the concept of qubit. In this framework, the two classical bits still exist and are represented as two particular pure states (usually indicated by \( |0\rangle \) and \( |1\rangle \)) that live in a special two-dimensional Hilbert space, based on the set of all ordered pairs of complex numbers. On this basis the concept of qubit is then defined as any pure state \( |\psi\rangle \) (living in this space) that is a possible superposition of the two bits \( |0\rangle \) and \( |1\rangle \). Thus, the typical form of a qubit is the following:

\[ |\psi\rangle = c_0 |0\rangle + c_1 |1\rangle. \]

From an intuitive point of view, the qubit \( |\psi\rangle \) can be interpreted as a probabilistic information: the answer (to a given question) might be “No” with a probability value that depends on the number \( c_0 \) and might be “Yes” with a probability value that depends on the number \( c_1 \).

Not all states of quantum systems are pure. More generally, a piece of quantum information may correspond to a non-maximal knowledge: a mixed state (or mixture), that is mathematically represented as a special Hilbert-space operator called density operator. Quantum mixed states give rise to a kind of second degree of ambiguity: while any pure state verifies with certainty some specific quantum properties, a mixed

---

\[ More \text{ precisely, this probability value is represented by the real number } |c_i|^2 \text{ (the squared modulus of } c_i). \text{ Since the length of } |\psi\rangle \text{ is 1, we have: } \sum_i |c_i|^2 = 1. \]

\[ More \text{ precisely, the probability of the answer “No” is the number } |c_0|^2, \text{ while the probability of the answer “Yes” is the number } |c_1|^2. \]

6 In this space (usually indicated by the symbol \( \mathbb{C}^2 \)) the two classical bits \( |0\rangle \) and \( |1\rangle \) are identified with the two number-pairs (1, 0) and (0, 1), respectively.

7 More precisely, the probability of the answer “No” is the number \( |c_0|^2 \), while the probability of the answer “Yes” is the number \( |c_1|^2 \).
state may leave indeterminate all non-trivial quantum properties. At the same time, all pure states correspond to special examples of density operators. Complex pieces of quantum information (which may involve many qubits) are supposed to be stored by composite quantum systems (say, systems of many electrons). Thus, the quantum theoretic representation of composite systems comes into play, giving rise to one of the most mysterious features of the quantum world: *entanglement*, a phenomenon that had been considered “potentially paradoxical” by some of the founding fathers of quantum theory (for instance, by Einstein and by Schrödinger).

Consider a quantum composite system

\[ S = S_1 + S_2 \]

(say, a system consisting of two electrons). Any state of \( S \) shall live in a particular Hilbert space \( \mathcal{H} \) that is a special product (called *tensor product*) of the two spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), associated to the subsystems \( S_1 \) and \( S_2 \), respectively. It is customary to write:

\[ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \]

where \( \otimes \) indicates the tensor product.

Unlike the case of classical physics, quantum composite systems (say, a system \( S = S_1 + S_2 \)) have a peculiar holistic behavior: the state of the global system (\( S \)) determines the states of its component parts (\( S_1, S_2 \)), and generally not vice versa. Thus, the procedure goes from the whole to the parts, and not the other way around. Entanglement-phenomena arise in the case of particular examples of composite systems that are characterized by the following properties:

- the state of the composite system is a pure state \( |\psi\rangle \) (a maximal information);
- this state determines the states of the parts, which (owing to the peculiar mathematical form of \( |\psi\rangle \)) cannot be pure. One is dealing with mixed states that might be indistinguishable from one another.

Thus, the information about the whole turns out to be more precise than the information about the parts. Consequently, against the classical compositionality-principle, the information about the whole cannot be determined as a function of the pieces of information about the parts. Metaphorically, we might think of a strange puzzle that, once broken into its component pieces, cannot be reconstructed again, recreating its original image.

---

8 Any density operator \( \rho \) of a given Hilbert space can be represented (in a non-unique way) as a weighted sum of some projection-operators, having the form: \( \rho = \sum_i w_i P_{|\phi_i\rangle\langle\phi_i|} \), where the weights \( w_i \) are positive real numbers such that \( \sum_i w_i = 1 \), while each \( P_{|\phi_i\rangle\langle\phi_i|} \) is the projection operator that projects over the closed subspace determined by the vector \( |\phi_i\rangle \). Thus, any pure state \( |\psi\rangle \) corresponds to a special example of a density operator: the projection \( P_{|\psi\rangle\langle\psi|} \). The physical interpretation of a mixed state \( \rho = \sum_i w_i P_{|\phi_i\rangle\langle\phi_i|} \) is the following: a quantum system in state \( \rho \) might be in the pure state \( P_{|\phi_i\rangle\langle\phi_i|} \) with probability value \( w_i \).
Let us now briefly recall the basic ideas of the semantics that has been suggested by quantum information theory. In this semantics (which is often called quantum computational semantics) linguistic expressions (sentences, predicates, individual names, ...) are supposed to denote pieces of quantum information: possible pure or mixed states of quantum systems that are storing the information in question. At the same time, logical connectives are interpreted as quantum logical gates: special operators that transform the pieces of quantum information under consideration in a reversible way. Consequently, logical connectives acquire a dynamic character, representing possible computation-actions. Any semantic model of a quantum computational language assigns to any sentence a meaning that lives in a Hilbert space whose dimension depends on the linguistic complexity of the sentence in question. In this way, meanings turn out to preserve, at least to a certain extent, the “memory” of the logical complexity of the sentences under consideration. In accordance with the quantum-theoretic formalism, quantum computational models are holistic: generally, the meaning of a compound expression (say, a sentence) determines the contextual meanings of its well-formed parts. Thus, the procedure goes from the whole to the parts, and not the other way around, against the compositionality-principle, that had represented a basic assumption of classical semantics (strongly defended by Frege). In some interesting situations it may happen that the meaning of a sentence (say, Bob loves Alice) is an entangled pure state, while the contextual meanings of the component expressions (the names Bob, Alice and the predicate loves) are proper mixtures. In such a case the parts of our sentence turn out to be more vague and ambiguous than the sentence itself. One is dealing with a semantic situation that often occurs in the case of natural languages as well in the languages of art. This is one of the reasons why quantum computational semantics gives rise to some natural and interesting applications to fields that are far apart from microphysics.

1.3 A quantum approach to pattern recognition and to machine learning

How are abstract concepts formed and recognized on the basis of some previous experience? This question can be successfully investigated in the framework of a quantum inspired approach to pattern recognition and to machine learning, which has been intensively developed in recent years.

Consider an agent (let us call her Alice) who is interested in a given concept C that may refer either to concrete or to abstract objects. The name Alice may denote either a human or an artificial intelligence. We will use Alice_H for a human mind and Alice_M for an intelligent machine. Alice will then correspond either to Alice_H or to Alice_M.

---

9 Technical details can be found in [2].
10 See, for instance, [6], [7] and [9].
We suppose that Alice (on the basis of her previous experience) has already recognized and classified a given set of objects for which the question “does the object under consideration verify the concept \( C \)?” can be reasonably asked. And we assume that the possible answers to this question are:

- “YES!”
- “NO!”
- “PERHAPS!”

As an example, Alice might be a child who has already recognized in the environment where she is living:

- the objects that are tables;
- the objects that are not tables.

At the same time, she might have been doubtful about the right classification of some particular objects. For instance, she might have answered: “PERHAPS!” to the question “is this food-trolley a table?”.

While Alice may have seen the objects under consideration, seeing is of course more problematic for Alice. Thus, generally, one shall make recourse to a theoretic representation that faithfully describes the objects in question. As happens in physics, one can use some convenient mathematical objects that represent object-states.

In the classical approach to pattern recognition and to machine learning an object-state is usually represented as a vector

\[
\mathbf{x} = (x_1, \ldots, x_d),
\]

that belongs to the real space \( \mathbb{R}^d \) (the set of all ordered sequences consisting of \( d \) real numbers, where \( d \geq 1 \)). Every component \( x_i \) of the vector \( \mathbf{x} \) is supposed to correspond to a possible value of an observable quantity (briefly, observable) that is considered relevant for recognizing the concept \( C \); while \( d \) represents the number of the relevant observables that are taken into consideration. Each number \( x_i \) is usually called a feature of the object represented by the vector \( \mathbf{x} \). As an example, suppose we are referring to a class of flowers and let \( C \) correspond to a particular kind of flower (say, the rose). We can assume that each flower-instance is characterized by two features: the petal length and the petal width. In such a case, any object-state will be a vector \( \mathbf{x} = (x_1, x_2) \) that belongs to the space \( \mathbb{R}^2 \).

The basic idea of a quantum approach to pattern recognition can be sketched as follows: replacing classical object-states with pieces of quantum information: possible states of quantum systems that are storing the information in question. From a semantic point of view, these quantum object-states can be regarded as possible meanings of individual names of a convenient quantum computational language.

In some pattern-recognition situations it may be useful to start with a classical information represented by an object-state \( \mathbf{x} \). Then, the transition to a quantum pure state \( |\psi\rangle \) can be realized by adopting an encoding procedure:

\[
\mathbf{x} \implies |\psi\rangle_\mathbf{x},
\]
where $|\psi\rangle_\mathbf{x}$ represents the quantum pure state into which the encoding procedure has transformed the classical object-state $\mathbf{x}$. An example of a “natural” encoding is the so called amplitude encoding, which is defined as follows.

**Definition 1** Amplitude encoding
Consider an object-state $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

The quantum-amplitude encoding of $\mathbf{x}$ is the following unit vector that lives in the (real) Hilbert space $\mathbb{R}^{(d+1)}$:

$$|\psi\rangle_\mathbf{x} = \frac{(x_1, \ldots, x_d, 1)}{||(x_1, \ldots, x_d, 1)||}$$

(though $||(x_1, \ldots, x_d, 1)||$ is the length of the vector $(x_1, \ldots, x_d, 1)$).

Thus, $|\psi\rangle_\mathbf{x}$ is a quantum pure state that preserves all features described by the classical object-state $\mathbf{x}$.

Of course, one could also directly “reason” in a quantum-theoretic framework, avoiding any reference to a (previously known) classical object-state $\mathbf{x}$. In such a case, one will assume, right from the outset, that an object-state is represented by a quantum pure state $|\psi\rangle$ living in a given Hilbert space.

We can now discuss in a quantum framework the problem: how is a concept $C$ recognized on the basis of a previous experience? Suppose that (at a given time $t_0$) Alice is interested in the concept $C$. Her previous experience concerning $C$ can be described by the formal notion of quantum $C$-data set according to the following definition.

**Definition 2** Quantum $C$-data set
A quantum $C$-data set is a sequence

$$^CDS = (^C\mathcal{H}, ^CSt$, $^CSt^+, ^CSt^-$, $^CSt^?),$$

where:

1. $^C\mathcal{H}$ is a finite-dimensional Hilbert space associated to $C$.
2. $^CSt$ is a finite set of pure states $|\psi\rangle$ of $^C\mathcal{H}$ for which the question “does the object described by $|\psi\rangle$ verify the concept $C$?” can be reasonably asked.
3. $^CSt^+$ is a subset of $^CSt$, consisting of all states that have been positively classified with respect to the concept $C$. The elements of this set are called the positive instances of the concept $C$.
4. $^CSt^-$ is a subset of $^CSt$, consisting of all states that have been negatively classified with respect to the concept $C$. The elements of this set are called the negative instances of the concept $C$.
5. $^CSt^?$ is a (possibly empty) subset of $^CSt$, consisting of all states that have been considered problematic with respect to $C$. The elements of this set are called the indeterminate instances of the concept $C$. 
6. The three sets $C S_i^+$, $C S_i^-$, $C S_i^?$ are pairwise disjoint. Furthermore, $C S_i^+ \cup C S_i^- \cup C S_i^? = C S_i$.

We will indicate by $n^+, n^-, n^?$ the cardinal numbers of the sets $C S_i^+$, $C S_i^-$, $C S_i^?$, respectively.

Suppose now that at a later time ($t_1$) Alice “meets” a new object described by the object-state $|\varphi\rangle$. Alice shall find a rule that allows her to answer the question “does the object described by $|\varphi\rangle$ verify the concept $C$?” And this answer shall be based on her previous knowledge that is represented by the quantum $C$-data set

$$C DS = (C H, C S_i, C S_i^+, C S_i^-, C S_i^?)$$

A winning strategy is based on the use of two special concepts: the quantum positive centroid and the quantum negative centroid of a quantum $C$-data set.

**Definition 3** Positive and negative centroids

Consider a quantum $C$-data set $C DS = (C H, C S_i, C S_i^+, C S_i^-, C S_i^?)$.

1. The quantum positive centroid of $C DS$ is the following density operator of the space $C H$:

$$\rho^+ = \sum_i \left\{ \frac{1}{n^+} P_{|\psi_i\rangle} : |\psi_i\rangle \in C S_i^+ \right\}$$

2. The quantum negative centroid of $C DB$ is the following density operator of the space $C H$:

$$\rho^- = \sum_i \left\{ \frac{1}{n^-} P_{|\psi_i\rangle} : |\psi_i\rangle \in C S_i^- \right\}$$

The concept of quantum positive centroid seems to represent a “good” mathematical simulation for the intuitive idea of Gestalt. Both the quantum positive centroid and the intuitive idea of Gestalt describe an imaginary object, representing a vague, ambiguous idea that Alice has obtained as an abstraction from the “real” examples she had met in her previous experience. As happens in the case of the intuitive idea of Gestalt, the quantum positive centroid, represented by the density operator $\rho^+ = \sum_i \left\{ \frac{1}{n^+} P_{|\psi_i\rangle} : |\psi_i\rangle \in C S_i^+ \right\}$, ambiguously alludes to the concrete positive instances that Alice had previously met (which are mathematically represented by the pure states $|\psi_i\rangle \in C S_i^+$)

It is worthwhile noticing that the characteristic ambiguity of quantum positive centroids is not shared by the notion of positive centroid that is defined in many classical approaches to pattern recognition. In the classical case, a positive centroid

---

11 We recall that $P_{|\psi_i\rangle}$ indicates the projection operator that projects over the closed subspace determined by the vector $|\psi_i\rangle$: a special example of a density operator that corresponds to the pure state represented by the vector $|\psi_i\rangle$. According to the canonical physical interpretation of mixtures, $\rho^+$ represents a state that ambiguously describes a quantum system that might be in the pure state $|\psi_i\rangle$ with probability-value $\frac{1}{n^+}$. 
represents an exact object-state, that is obtained by calculating the average values of the values that all positive instances assign to the observables under consideration.

Thus, unlike the quantum case, classical positive centroids turn out to describe imaginary objects that are characterized by precise features, without any “cloud” of ambiguity.

As we have noticed, human recognitions and classifications are usually performed by means of a quick and mostly unconscious comparison between the main features of some new objects we have met and a gestaltic pattern that we had previously constructed in our mind. We also know that any comparison generally involves the use of some similarity-relations that are mostly grasped in a vague and intuitive way by human intelligences.

Similarity-relations play a relevant role in the quantum theoretic formalism. Important examples of quantum similarities can be defined in terms of a special function, called fidelity. In the case of pure states this function is defined as follows.

**Definition 4** Fidelity
Consider a Hilbert space $\mathcal{H}$. The fidelity-function on $\mathcal{H}$ is the function $F$ that assigns to any pair $|\psi\rangle$ and $|\varphi\rangle$ of pure states of $\mathcal{H}$ the real number

$$F(|\psi\rangle, |\varphi\rangle) = \langle\psi | \varphi\rangle^2$$

(where $\langle\psi | \varphi\rangle$ is the inner product of $|\psi\rangle$ and $|\varphi\rangle$).

From an intuitive point of view, the number $F(|\psi\rangle, |\varphi\rangle)$ can be interpreted as a measure of the degree of closeness between the two states $|\psi\rangle$ and $|\varphi\rangle$.

The definition of fidelity can be easily generalized to the case of density operators, which may represent either pure or mixed states. Thus, in the general case we will write: $F(\rho, \sigma)$.

It is interesting to recall the main properties of this function, which play an important role in many applications:

1. $F(\rho, \sigma) \in [0, 1]$.
2. $F(\rho, \sigma) = F(\sigma, \rho)$.
3. $F(\rho, \sigma) = 0$ iff $\rho \sigma$ is the null operator.
4. $F(\rho, \sigma) = 1$ iff $\rho = \sigma$.

From a physical point of view, the fidelity-function can be regarded as a form of symmetric conditional probability: $F(\rho, \sigma)$ represents the probability that a quantum system in state $\rho$ can be transformed into a system in state $\sigma$, and vice versa.

The concept of fidelity allows us to define in any Hilbert space $\mathcal{H}$ a special class of similarity-relations, called $r$-similarities, where $r$ is any real number in the interval $[0, 1]$.

**Definition 5** $r$-similarity
Let $\rho$ and $\sigma$ be two density operators of a Hilbert space $\mathcal{H}$ and let $r \in [0, 1]$. The state $\rho$ is called $r$-similar to the state $\sigma$ (briefly, $\rho \leq_r \sigma$) iff $r \leq F(\rho, \sigma)$. 

One can easily check that (owing to the main properties of the fidelity function) this relation is reflexive, symmetric and generally non-transitive.

Now Alice has at her disposal the mathematical tools that allow her to face the classification-problem. Suppose that Alice’s information about a concept $C$ is the quantum $C$-data set

$$C^{DS} = (\mathcal{C}H, CSt, CSt^+, CSt^-, CSt?)$$

whose positive and negative centroids are the states $\rho^+$ and $\rho^-$, respectively. And let $r^*$ be a threshold-value in the interval $(\frac{1}{2}, 1]$, that is considered relevant for $C^{DS}$. The main goal is defining a classifier function, that assigns to every state $\sigma$ (which describes an object that $Alice_M$ may meet)

- either the value $+$ (corresponding to the answer “YES!”);
- or the value $-$ (corresponding to the answer “NO!”);
- or the value $?$ (corresponding to the answer “PERHAPS!”).

**Definition 6 Classifier function**

Let $C^{DS} = (\mathcal{C}H, CSt, CSt^+, CSt^-, CSt?)$ be a quantum $C$-data set and let $r^*$ be a threshold-value for $C^{DS}$. The classifier function determined by $C^{DS}$ and $r^*$ is the function $Cl_{[C^{DS}, r^*]}$ that satisfies the following condition for any state $\sigma$ of the space $\mathcal{H}$:

$$Cl_{[C^{DS}, r^*]}(\sigma) = \begin{cases} +, & \text{if } \sigma \nleq_{r^*} \rho^+ \text{ and not } \sigma \nleq_{r^*} \rho^- \text{.} \\ -, & \text{if } \sigma \nleq_{r^*} \rho^- \text{ and not } \sigma \nleq_{r^*} \rho^+ \text{.} \\ ?, & \text{otherwise.} \end{cases}$$

In other words, $\sigma$ is “sufficiently similar” to the positive centroid and is not “sufficiently similar” to the negative centroid.

This definition of the classifier function turns out be quite suitable for $Alice_M$, who can now perform a simple computation in order to decide whether a new object submitted to her verifies a given concept. Apparently, $Alice_M$’s algorithmic procedure corresponds to what is intuitively grasped by $Alice_H$, when she quickly compares her description of a new object with a gestaltic pattern stored in her memory. Even if recognition-procedures are different for human and for artificial intelligences, there is a common method of “facing the problems” that seems to work in both cases.

### 1.4 Musical themes and musical similarities

Although music and quantum theory belong to two far apart worlds, the semantics suggested by quantum information theory can be successfully applied to a formal
analysis of music. We will see how this special form of quantum musical semantics represents a useful tool for investigating the intriguing question of musical recognitions.

Any musical composition (a sonata, a symphony, a lyric opera,...) is generally determined by three basic elements:

1. a score;
2. a set of performances;
3. a set of musical ideas (or musical thoughts), which represent possible meanings for systems of musical phrases written in the score.

Scores represent the syntactical component of musical compositions: systems of signs that are, in a sense, similar to the formal systems of scientific theories. Performances are, instead, physical events, that occur in space and time. As is well known, not all pieces of music are associated with a score. We need only think of folk songs or of jazz music. However, in classical Western music compositions are usually equipped with a score that has been written by a composer.

Musical ideas represent a more mysterious element. One could ask: is it reasonable to assume the existence of such ideal objects that are similar to the intensional meanings investigated by logic? We give a positive answer to this question. In fact, a musical composition cannot be simply reduced to a score and to a system of sound-events. Between a score and the sound-events created by a performance there is something intermediate: the world of the musical thoughts that underlie the different performances. This is the ideal world where normally live composers and conductors, who are often accustomed to study scores without any help of a material instrument.

The basic principle of quantum musical semantics is that musical ideas can be formally represented as special cases of pieces of quantum information, which may have the characteristic form of quantum superpositions. Accordingly, we can conventionally write:

\[ |\mu\rangle = \sum_i c_i |\mu_i\rangle, \]

where:

- \(|\mu\rangle\) is an abstract object representing a musical idea that alludes to other ideas \(|\mu_i\rangle\) that are all co-existent:
- the number \(c_i\) measures the “importance” of the component \(|\mu_i\rangle\) in the context \(|\mu\rangle\).

The use of the superposition-formalism is a powerful abstract tool that allows us to represent, in a natural way, the allusions and the ambiguities that play an essential role in music. And in some special cases musical ideas can be even represented as peculiar mixtures, that are characterized by a deeper degree of ambiguity. In accordance with the quantum-theoretic formalism we will use the symbols \(|\mu\rangle, |\mu_1\rangle, |\mu_2\rangle, \ldots\) for pure musical ideas that behave as quantum pure states. At the same time...
time, generic musical ideas that may behave either as pure or as mixed states will be indicated by the symbols \( \mu, \mu_1, \mu_2, \ldots \).

As is well known, an important feature of music is the capacity of evoking some *extra-musical meanings*: subjective feelings, situations that are vaguely imagined by the composer or by the interpreter or by the listener, real or virtual theatrical scenes (which play an important role in the case of lyric operas and of *Lieder*). The interplay between *musical ideas* and *extra-musical meanings* can be naturally represented in the framework of our quantum musical semantics: *extra-musical meanings* can be dealt with as examples of *vague possible worlds*, where *events* are generally ambiguous, as happens in the quantum world.

Musical scores are characteristic *two-dimensional syntactical objects* that can be formally represented as special kinds of *matrices* (with *rows* and *columns*). Each column contains symbols for notes or pauses that shall be performed at the same time; while each row is a sequence of symbols corresponding to notes or pauses that shall be performed in succession. In this way, *chords* can be represented as fragments of columns, while *melodies* can be represented as fragments of rows. The two-dimensional configuration of scores clearly reflects, in the musical notation, the role played by parallelism in music. As an example we can refer to the celebrated *incipit* of Beethoven’s Fifth Symphony (Fig.1).

![Fig. 1.1 The incipit of Beethoven’s Fifth Symphony](image)

Any score is subdivided in complex systems of *musical phrases*. Generally, a phrase may be either a *monodic phrase*, represented by a one-dimensional *horizontal* fragment of the score; or a *polyphonic phrase* (with *horizontal* and *vertical* components), represented by a two-dimensional fragment of the score. Unlike the
notion of sentence of a formal scientific language, the concept of musical phrase does not generally represent a rigid notion. The subdivision of a score in musical phrases may also depend on the interpreter’s choices. And it is not by chance that one often speaks of the “phrasing” that characterizes the interpretation of a given performer.

Interpreting a given score (say, Beethoven’s Fifth Symphony) means assigning to every system of musical phrases written in the score a convenient musical idea that evolves in time. And, as happens in the quantum computational semantics, musical meanings have a characteristic holistic behavior: generally, the meaning of a global phrase-system determines the contextual meanings of all its parts (and not the other way around).

Let us consider again the incipit of Beethoven’s Fifth Symphony. And let us briefly indicate this first phrase of the symphony by Phr$^{\text{Fifth1}}$. Every interpretation of Phr$^{\text{Fifth1}}$ realizes a particular musical idea. As is well known, different conductors have proposed different interpretations of this famous musical phrase; and every interpretation may depend on a particular choice of the dynamics or of the tempo. In the framework of our quantum musical semantics a musical idea that represents a possible interpretation of the phrase Phr$^{\text{Fifth1}}$ can be conventionally indicated as follows:

$$|\mu^{\text{Fifth1}}\rangle.$$ 

One could ask: what kind of abstract object is a musical meaning? Does the ket-notation, used in our quantum musical semantics, simply play a “metaphorical” role”? In fact, we could be technically more precise, representing a musical meaning (say, $|\mu^{\text{Fifth1}}\rangle$) as a piece of quantum information that, in principle, could be stored by a quantum composite system S. In this representation, each column of the score should be associated to a particular subsystem of S. However, as we can imagine, the details of such a technical representation would not be interesting for the aims of a musical analysis. What is important is regarding musical meanings as special examples of intensional meanings that behave according to the general rules of the quantum computational semantics.

A critical question concerns the relationship between musical ideas and musical themes. But what exactly are musical themes? The term “theme” has been used for the first time in a musical sense by Gioseffo Zarlino, in his Le istitutioni harmoniche (1558), as a melody that is repeated and varied in the course of a musical work. In the framework of our semantics the concept of musical theme cannot be simply identified with a musical idea that represents a possible interpretation of a particular musical phrase (written in a given score). We have seen how any interpretation of the Fifth Symphony associates to the first phrase of the symphony a musical idea:

$$|\mu^{\text{Fifth1}}\rangle.$$ 

At the same time, what is usually called the main theme of the Fifth Symphony’s first movement is something more abstract that neglects a number of musical parameters, which may concern, for instance, the pitch or the timbre. Musically cultivated people
generally recognize this famous Beethoven’s theme when it is played by different instruments, in different (low or high) registers and in different (minor) tonalities.

Which are the characteristic features of this theme that represent some invariant parameters, that cannot be neglected? First of all, a particular sequence of melodic intervals and pauses. Then, a particular meter ($\frac{2}{4}$) and a peculiar rhythmic structure, which is independent of the notes that shall be played. Thus, generally, a theme can be regarded as a highly abstract musical idea that is determined by a sequence of melodic intervals and pauses, embedded in a given rhythmic structure. This suggests to consider an abstraction from a given phrase written in a score. We can introduce the concept of abstract musical theme, which represents an invariant with respect to possible timbre and pitch-transformations.

As an example, let us refer again to the first phrase of the Fifth Symphony. We will briefly represent the abstract theme associated to this phrase by the notation described in Fig.2. The symbolic convention assumed in this notation is the following:

- the squared brackets mean that we are abstracting from the “real notes” written in the score-fragment inside the brackets. What we are referring to is a particular sequence of melodic intervals and pauses, embedded in a rhythmic structure (which is determined by the score-fragment under consideration).[[13]
- The ket-brackets mean that we are representing our abstract theme as a particular example of a pure musical idea $|\mu\rangle$ dealt with as a meaning in the framework of the quantum musical semantics.[[14]

---

Fig. 1.2 A notation for the main theme of the Fifth Symphony’s first movement

---

[[13] The use of the squared brackets is suggested by a notation often used in mathematics, where operations involving an abstraction are frequently indicated by the brackets $[\ldots \ldots]$.[[13]

[[14] Of course, we might use a “more mathematical” notation, indicating all melodic intervals by convenient arithmetical expressions. However, this kind of notation (which plays an important role in the framework of computer music) would be too heavy and hardly interesting for the aims of our semantic approach.}
We can assume that just this \textit{abstract theme}, briefly indicated by $|\mu^{Fifth\text{1}}\rangle$, can formally represent what is usually called the \textit{main theme} of the Fifth Symphony’s first movement.

Of course, \textit{abstract themes} represent special examples of \textit{musical ideas}. On this basis we can say that: the \textit{musical idea}

$$|\mu^{Fifth\text{1}}\rangle$$

that represents a possible interpretation of the phrase Phr$^{Fifth\text{1}}$ expresses the \textit{abstract theme}

$$|[\mu^{Fifth\text{1}}]\rangle.$$

As expected, one could also directly associate \textit{abstract themes} to \textit{phrases}, asserting that a given phrase expresses a corresponding \textit{abstract theme}.

So far, we have considered examples of \textit{monodic abstract themes}, which correspond to fragments of \textit{rows} in the formal representation of a given score. However, in some cases it may be interesting to consider also examples of \textit{polyphonic abstract themes}, that correspond to polyphonic phrases of the score in question. For instance, we might consider the complex abstract theme that corresponds to the global first phrase of the Fifth Symphony (represented in Fig. 1). In such a case, the \textit{harmonic relationships} between the different parts of our theme would come into play. Notice that what is usually called “The Theme” of a composition having the form \textit{Theme and Variations} is generally represented by a \textit{polyphonic musical idea} that, in turn, can include a particular \textit{monodic abstract theme}. And just this monodic theme represents the “main character” which all \textit{Variations} allude to, according to modalities that may appear more or less hidden.

Let us now turn to the intriguing question that concerns musical recognitions. What does \textit{recognizing a melody or a musical theme} mean? Recognition processes in music seem to be quite similar to what happens when we recognize an \textit{abstract concept}, which may refer either to concrete or to ideal objects. In the case of music, the role played by abstract concepts can be naturally replaced by \textit{musical themes}. We suppose that \textit{Alice} is interested in a particular musical theme (say, the theme expressed by the \textit{incipit} of Beethoven’s Fifth Symphony). The musical idea that \textit{Alice’s} mind associates to this theme may be quite vague (depending, of course, on \textit{Alice’s} musical preparation). What is important is that \textit{Alice} can use a \textit{label (a name)} that, in principle, can refer to a particular \textit{abstract musical theme}; in this case, the theme that we have previously indicated by the notation $|[\mu^{Fifth\text{1}}]\rangle$. Accordingly, in our musical applications of pattern recognition-methods we will write $T$ (theme), instead of $C$ (concept).

As happens in the case of concepts, we suppose that, at a given time $t_0$, \textit{Alice} has already classified a (finite) set of \textit{pure musical ideas}

$$M1d = \{|\mu_1\rangle, \ldots, |\mu_n\rangle\}$$

with respect to the theme $T$. In other words, for every musical idea $|\mu_i\rangle$ in the set $M1d$, \textit{Alice} has answered either “YES!” or “NO!” or “PERHAPS!” to the question
“does the musical idea $|\mu_i\rangle$ express the theme $T$?” As expected, the elements of the set $M1d$ (which are formally dealt with as pieces of quantum information) represent musical thoughts (stored in Alice’s memory) corresponding to pieces of music that Alice might have either listened to or performed as a musician.

On this basis, we can naturally introduce the notion of quantum musical $T$-data set, identified with a system

$$T MDS = (M1d, M1d^+, M1d^-, M1d^?)$$

where:

1. $M1d$ is a finite set of pure musical ideas $|\mu_i\rangle$ for which the question “does the musical idea $|\mu_i\rangle$ express the theme $T$?” can be reasonably asked.
2. $M1d^+, M1d^-$, $M1d^?$ represent, respectively, the sets of the positive, of the negative and of the indeterminate instances for the theme $T$.

We assume that any quantum musical $T$-data set satisfies the same conditions that we have required in the case of quantum $C$-data sets (Definition 2).

The concepts of positive centroid and of negative centroid of a quantum musical $T$-data set can be now naturally defined as we have done in the case of quantum $C$-datasets. The positive centroid is determined as a musical idea $\kappa^+$ that is a mixture of all positive instances $|\mu_1^+\rangle$, $|\mu_2^+\rangle$, $\ldots$, $|\mu_n^+\rangle$ of $T MDS$. Each weight occurring in $\kappa^+$ is identified with the number $\frac{1}{n^+}$ (where $n^+$ is the number of all positive instances).

Thus, from an intuitive point of view, $\kappa^+$ represents an ambiguous musical idea that vaguely alludes to the pure musical ideas $|\mu_1^+\rangle$, $|\mu_2^+\rangle$, $\ldots$, $|\mu_n^+\rangle$.

In a symmetric way, one can define the negative centroid of $T MDS$ as the musical idea $\kappa^-$ that is a mixture of all negative instances of $INF_T^T$.

Suppose now that at a later time $t_1$ Alice listens to a new musical phrase, say the incipit of Beethoven’s Fifth Symphony, which might be performed by a regular orchestra or by a piano or even simply sung by someone. And suppose that Alice ask herself “is this piece of music the main theme of the Fifth Symphony’s first movement?” Of course, Alice’s answer should be based on her previous knowledge, which is formally represented by the quantum musical $T$-data set $T MDS$.

In such situation, our natural wish would be trying and applying the same classifier function that we have successfully used for concept-recognitions. But is this procedure possible in the case of music? We have seen how, in the case of concepts, our definition of the classifier function has been essentially based on the notion of $r$-similarity, which admits a precise mathematical definition in the framework of quantum information theory. To what extent can this strategy be reasonably extended to musical recognitions?

It is well known that similarity-relations play a very important role in the structure of music. Musical themes are normally transformed in different ways in the framework of a given composition. And all variation-phenomena (which some authors have described as the Urprinzip of music) are characterized by the occurrence
of some similarity-relations. We may only think of the structure of Fugues, of the Sonata form and of the Theme and Variations-form, where abstract themes often appear as a kind of “ghosts” in a somewhat mysterious way.

It is customary to distinguish different kinds of musical similarities: melodic, rhythmic, harmonic, timbric,..... In the case of tonal music some important similarity-relations are often connected with a mode-transformation: from a major tonality to a minor tonality or vice versa. As an example, let us refer to the first movement of Beethoven’s piano sonata op.10 n.1, in C minor (the same tonality of the Fifth Symphony). The primary theme of the movement is proposed at the very beginning (Fig. 3). One is dealing with an ascending phrase, based on the three elements of the triad of the C minor key (C, E flat, G). The dynamic indication (forte) as well as

the peculiar rhythmic structure (a dotted rhythm) seem to suggest a strong statement (a kind of “act of will”). Soon after, this theme is suddenly transformed into a major mode (the C major key) (Fig. 4), restating (in a major version) the same strong idea that had been asserted before.
Unlike the case of the Sonata Op. 10 n.1, the main theme of the Fifth Symphony’s first movement is never directly transformed into a major version. The phrase represented in Fig. 5 does not belong to the symphony’s score. There is, however, a new theme that appears very soon (at bars 59-62): a phrase played fortissimo by the horns (in the B flat major tonality) that is naturally perceived as very close to the idea expressed by the main theme (Fig. 6).

In the framework of our quantum musical semantics it is interesting to study how Alice lets interact different forms of musical similarities. For the sake of simplicity, we will now restrict our attention to two particular examples of similarity (which play an essential role in the structure of music): melodic and rhythmic similarities. Let \( \mu_1 \) and \( \mu_2 \) represent two musical ideas. When \( \mu_1 \) is considered melodically similar to \( \mu_2 \), we will briefly write: \( \mu_1 \sim_{M} \mu_2 \).

And we will write: \( \mu_1 \sim_{R} \mu_2 \),

when \( \mu_1 \) is considered rhythmically similar to \( \mu_2 \).
Recognizing Concepts and Recognizing Musical Themes

Melodic and rhythmic similarities can be *logically combined* in different ways. In some cases it may be interesting to consider the *conjunction* between a melodic similarity $Sim^M$ and a rhythmic similarity $Sim^R$. This gives rise to a new relation, that can be called *strong similarity*. We assume (by definition) that:

two musical ideas $\mu_1$ and $\mu_2$ are strongly similar if and only if they are melodically similar and rhythmically similar at the same time.

In some other cases it may be interesting to consider the *disjunction* of the two relations $Sim^M$ and $Sim^R$. This gives rise to a different relation, that can be called *weak similarity*. We assume (by definition) that:

two musical ideas $\mu_1$ and $\mu_2$ are weakly similar if and only if they are either melodically similar or rhythmically similar.

As we have done in the case of concepts, we can distinguish different degrees of *musical similarity*. Although musicologists do not normally speak of $r$-similarity relations, musical analyses often use some locutions like “highly similar”, “somewhat similar”, “slightly similar”, that can be conventionally associated to some particular numerical values (in the interval $[0, 1]$).

Let $Sim$ represent any form of musical similarity. When a musical idea $\mu_1$ is considered *$r$-similar* to a musical idea $\mu_2$, we will briefly write:

$$\mu_1 Sim_r \mu_2.$$  

And as happens in the case of concepts, there are musical situations, where it may be useful to choose a particular threshold-value $r^*$ (in the interval $[\frac{1}{2}, 1]$), that is considered relevant for the musical context under consideration. Suppose, for instance, that $\mu_1 Sim_{r^*} \mu_2$, while $r^*$ is “very close” to 1 (say, $r^* = 0.9$). In such a case, it seems reasonable to conclude that:

$\mu_1$ and $\mu_1$ are highly similar.

As an example, let us refer again to the first movement of Beethoven’s Sonata op.10 n.1. Suppose that $\mu_1$ is a musical idea that corresponds to the movement’s primary theme (Fig.3), while $\mu_2$ corresponds to its major transformation (Fig.4). Apparently, the primary theme and its major transformation have exactly the same rhythmic structure. Thus, it seems reasonable to conclude that:

$$\mu_1 Sim^R_{r^*} \mu_2.$$  

In other words, our two musical ideas are *maximally similar* from the rhythmic point of view.

The situation changes if we refer to melodic similarity. Clearly, the primary theme and its major transformation do not have the same melodic structure, since in the major variant the minor triad ($C, E$ flat, $G$) has been replaced by the major triad ($C, E, G$). However, a musical Alice$_H$, who is familiar with tonal music, “perceives” the two musical ideas $\mu_1$ and $\mu_2$ as “very close” to each other. Hence, by a convenient choice of the threshold-value $r^*$ (for instance, by choosing $r^* = 0, 9$), it seems reasonable to conclude that:

$$\mu_1 Sim^M_{r^*} \mu_2.$$
In other words, \( \mu_1 \) and \( \mu_2 \) are melodically very similar.

A quite different situation arises if we consider two musical ideas \( \mu_1 \) and \( \mu_2 \) that correspond to the main theme of the first movement of Beethoven’s Fifth Symphony (Fig.2) and to its major variant (Fig.6), respectively. In such a case, both the melodic structure and the rhythmic structure of \( \mu_1 \) and \( \mu_2 \) are different. In spite of this, we perceive a deep relationship that connects the musical thoughts expressed by \( \mu_1 \) and \( \mu_2 \). The major variant seems to restate, in a more incisive and permanent way, the strong assertion that had been proposed by the main theme (in a minor tonality).

Musical similarities are often grasped in an intuitive and rapid way by human listeners who are familiar with classical Western music. At the same time, trying to analyze, by abstract methods, the relationships between different forms and different degrees of similarity is not an easy task. We have seen that in the case of concepts, \( r \)-similarity relations (defined in terms of the fidelity-function) allow us to define a classifier function that has an objective and universal behavior. This precise mathematical situation can be hardly reproduced in the case of music, where any particular choice of a similarity-relation seems to depend, at least to a certain extent, on subjective preferences. What we can do is referring to a class of possible musical similarity-relations, admitting that, in different contexts, we have the freedom of choosing some special elements in this class.

Once chosen a particular similarity-relation \( \text{Sim} \), associated to a threshold-value \( r^* \), it will be possible to apply the same method used for recognizing concepts. Suppose that \( \text{Alice} \)’s information about a given abstract theme \( T \) is represented by the quantum musical \( T \)-data set

\[
\text{\(T\)MDS} = (M1d, M1d^+, M1d^-, M1d^?),
\]

associated to a threshold value \( r^* \), and let \( \kappa^+ \) and \( \kappa^- \) be, respectively, the positive centroid and the negative centroid. As happens in the case of concepts, a musical classification function \( \text{MCI} \) (based on \( \text{\(T\)MDS} \) and on \( r^* \)) shall assign to any musical idea \( \nu \) (which may represent a new musical example that \( \text{Alice} \) has listened to) either the value + or the value – or the value ?. On this basis, we can assume by definition that:

1. \( \text{MCI}(\nu) = + \), if

\[\nu \text{ Sim}_{r^*} \kappa^+ \text{ and not } \nu \text{ Sim}_{r^*} \kappa^-\]

In other words, \( \nu \) is sufficiently similar to the positive centroid \( \kappa^+ \) and is not sufficiently similar to the negative centroid \( \kappa^- \).

2. \( \text{MCI}(\nu) = – \), if

\[\nu \text{ Sim}_{r^*} \kappa^- \text{ and not } \nu \text{ Sim}_{r^*} \kappa^+\]

In other words, \( \nu \) is sufficiently similar to the negative centroid \( \kappa^- \) and is not sufficiently similar to the positive centroid \( \kappa^+ \).

3. \( \text{MCI}(\nu) = ? \), otherwise.

The application of pattern-recognition methods to musical problems has confirmed the interest of investigating by abstract tools the intriguing concept of musi-
cal similarity, which has been analyzed, with different perspectives and methods, by musicians, musicologists as well by researchers in the field of musical informatics.

Finally, we would like to conclude with the following general remark: adopting a quantum approach to pattern recognition has allowed us to obtain some natural simulations for artificial intelligences of the intrinsic vagueness and ambiguity that characterize many human cognitive behaviors. And, interestingly enough, one has shown that a systematic use of quantum uncertainties gives rise to significant improvements of the accuracy and of the algorithmic efficiency of some machine-learning procedures\[15\] These results seem to confirm that a quantum inspired investigation of ambiguity-phenomena can represent a powerful resource both for theoretic and for technological achievements.

References

1. Dalla Chiara, M.L., Giuntini, R., Luciani, R., Negri, E., From Quantum Information to Musical Semantics, College Publications, London (2012).
2. Dalla Chiara, M.L., Giuntini, R., Leporini, R., Sergioli, S., Quantum Computation and Logic. How Quantum Computers have inspired Logical Investigations, Springer (2018).
3. Ehrenstein, W.H., Spillmann, L., Sarris, W., Gestalt Issues in Modern Neuroscience, Axiomathes, 13, 433–458 (2003).
4. Dalla Chiara, M.L., Giuntini, R., Sergioli, G., Leporini, R. A many-valued approach to quantum computational logics, Fuzzy Sets and Systems, 335, 94–111, Springer (2018).
5. Honing, H., Musical Cognition. A Science of Listening, Transaction Publishers, London (2009).
6. Schuld, M., Petruccione, F., Supervised Learning with Quantum Computers, Springer (2018).
7. Sergioli, G., G.M. Bosyk, G.M. Santucci, R. Giuntini, A Quantum-inspired version of the classification problem, International Journal of Theoretical Physics (2017).
8. Sergioli, G., Russo, G., Santucci, E., Stefano, A., Torrisi, S.E., Palmucci, S., Vancheri, C., Giuntini, R. Quantum-inspired minimum distance Classification in Biomedical Context, International Journal of Quantum Information (2018).
9. Sergioli, G., Giuntini, R., Freytes, G. A new quantum approach to binary classification, PLoS ONE 14(5): e0216224 (2019).
10. Sergioli, G., Quantum and Quantum-like machine learning. A note on similarities and differences, Soft Computing (2019).
11. Sergioli G., Militello C., Rundo L., Minafra L., Torrisi F., Russo G., Chow K.L., Giuntini, R. A quantum-inspired classifier for clonogenic assay evaluations. Scientific Reports, Nature 11:28301, ISSN 2405-2322 (2021).

\[15\] See, for instance, [8] and [11].