On the Dependence of Quantum States on the Value of Planck’s Constant

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Abstract

We begin by discussing known theoretical results about the sensitivity of quantum states to changes in the value of Planck’s constant $h$. These questions are related to positivity issues for self-adjoint trace class operators, which are not yet fully understood. We thereafter briefly discuss the implementation of experimental procedure to detect possible fluctuations of $h$.

1 Introduction

There has been for a long time an ongoing and controversial debate on whether the fine structure constant $\alpha$ is really constant; its non-constancy would imply that at least one of the related quantities $e, c, h$ could also be variable; see for instance [1, 2, 6]. So far all attempts to test the variability of physical constants have relied on experimental evidence. In the present paper we propose a theoretical approach for detecting possible variations of Planck’s constant; in principle this method would be able to detect such a variation no matter how small it is. In the present Letter we examine some

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of the consequences of the possible non-constancy of \( h \) would have for pure mixed and quantum states; the mathematical theory is far from complete, and related to difficult questions involving positivity issues, so we will limit ourselves mainly to the case of Gaussian pure or mixed states which is well understood. We thereafter suggest experimental procedures in the form of \textit{Gedankenexperiments}.

2 \hspace{1cm} \textbf{The Density Matrix and the KLM Conditions}

As soon as one deepens the elementary description of mixed states one is confronted with difficult (and yet not totally solved) mathematical problems which are usually ignored in experimental physics. This difficulty is due to the relation between density matrices and their associated Wigner distributions, and which is not as straightforward as it could seem at first sight. It lies in the verification of positivity issues, and these very much depend on the value which is given to Planck’s constant. Recall that a mixed quantum state in \( \mathbb{R}^n \) is characterized by its density matrix

\[
\hat{\rho} = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j| \tag{1}
\]

where \( \lambda_j \geq 0, \sum_j \lambda_j = 1, \) and \( \langle \psi_j | \psi_j \rangle = 1 \). The operator \( |\psi_j\rangle \langle \psi_j| \) is the orthogonal projection in \( L^2(\mathbb{R}^n) \) on the state \( |\psi_j\rangle \). The datum of \( \hat{\rho} \) is equivalent to that of Wigner distribution function (WDF)

\[
W_{\hat{\rho}}(x,p) = \left( \frac{1}{2\pi \hbar} \right)^n \int e^{-ipy/\hbar} (x + \frac{1}{2} y |\hat{\rho}| x - \frac{1}{2} y) d^n y \tag{2}
\]

(see e.g. Hillery et al. [11], Littlejohn [16]) that is, using formula (1)

\[
W_{\hat{\rho}}(x,p) = \rho(x,p) = \sum_j \lambda_j W_{\psi_j}(x,p) \tag{3}
\]

where \( W_{\psi_j} \) is the usual Wigner transform of \( \psi_j \):

\[
W_{\psi_j}(x,p) = \left( \frac{1}{2\pi \hbar} \right)^n \int e^{-ipy/\hbar} \psi_j(x + \frac{1}{2} y) \psi_j^*(x - \frac{1}{2} y) d^n y. \tag{4}
\]

We address the following question:

\textit{Suppose we are given a real function \( \rho \) on \( \mathbb{R}^{2n} \). How can we know whether this function is the Wigner distribution of some density operator \( \hat{\rho} \), and what happens if we replace \( \hbar \) with another real number \( \hbar' > 0 \), that is, if we allow Planck’s constant to vary?}
The key to an understanding of this problem lies in the following remark: in mathematical terms a density operator (or matrix) on $L^2(\mathbb{R}^n)$ is a self-adjoint trace class operator $\hat{\rho}$ with unit trace, and which is in addition positive semidefinite: $\hat{\rho} \geq 0$ (we will say for short “positive”). The last condition means that we have

$$\langle \psi | \hat{\rho} | \psi \rangle \geq 0 \text{ for all } \psi \in L^2(\mathbb{R}^n)$$  \hfill (5)

(this condition implies the self-adjointness of $\hat{\rho}$ since $L^2(\mathbb{R}^n)$ is an infinite-dimensional Hilbert space, but we keep it since we are going to deal with self-adjoint trace-class operators $\hat{\rho}$ which are not necessarily positive). Such operators $\hat{\rho}$ are compact, and the spectral theorem then implies that we can always write $\hat{\rho}$ in the form (1). The definition (3) of the Wigner distribution then shows that if we view $\hat{\rho}$ as a Weyl operator (which always is possible [7, 8]) then $\rho = W\hat{\rho}$ is just the Weyl symbol of $\hat{\rho}$ divided by $(2\pi\hbar)^n$ (loc.cit.).

3 Gaussian States

Consider first the simple example of a centered normal probability distribution on $\mathbb{R}^2$ defined by

$$\rho_{X,P}(x, p) = \frac{1}{2\pi\sigma_X\sigma_P} \exp \left[ -\frac{1}{2} \left( \frac{x^2}{\sigma_X^2} + \frac{p^2}{\sigma_P^2} \right) \right]$$  \hfill (6)

where $\sigma_X, \sigma_P > 0$; the variances are $\sigma_X^2$ and $\sigma_P^2$ and the covariance is $\sigma_{XP} = 0$. It is known [20, 22] that $\rho_{X,P}$ is the Wigner distribution of a density operator $\hat{\rho}_{X,P}$ if and only if it satisfies the Heisenberg inequality $\sigma_X \sigma_P \geq \frac{1}{2}\hbar$. If we have $\sigma_X \sigma_P = \frac{1}{2}\hbar$ (which we assume from now on) then $\rho_{X,P}$ is the Wigner distribution of the coherent state

$$\psi_X(x) = (2\pi\sigma^2)^{-1/4} e^{-x^2/2\sigma_X^2}$$

and $\hat{\rho}_{X,P}$ is then just the pure-state density operator $|\psi_X\rangle\langle\psi_X|$, whose Wigner distribution is precisely $\rho_{X,P} = W\psi_X$. Suppose now that we replace $\hbar$ with a number $\hbar' > 0$ playing the role of a “new” Planck’s constant. If we still want $\rho_{X,P}$ to qualify as the Wigner distribution of a quantum state the spreading $\sigma_X$ and $\sigma_P$ must satisfy the new Heisenberg inequality $\sigma_X \sigma_P \geq \frac{1}{2}\hbar'$; since the product $\sigma_X \sigma_P$ is fixed from the beginning as being $\frac{1}{2}\hbar$ this implies that we must have $\hbar' \leq \hbar$. This means that if we decrease the value of Planck’s constant, then $\hat{\rho}_{X,P}$ becomes the density operator of a now mixed quantum state, but if we increase its value, then the Gaussian
\( \rho_{X,P} \) cannot be a Wigner distribution, but is a probability density representing a classical state. (Intuitively, the decrease of Planck’s constant has the effect of making the Gaussian \( \rho \) too sharply peaked around the origin, which causes the violation of the Heisenberg inequality.)

The discussion above extends without difficulty to generalized Gaussians

\[
\rho_\Sigma(z) = (2\pi)^{-n} \sqrt{|\det \Sigma^{-1}|} e^{-\frac{1}{2} \Sigma^{-1} z^2}
\]

in 2n-dimensional phase space; \( \Sigma \) is a positive-definite symmetric 2n \( \times \) 2n matrix which is identified with the covariance matrix:

\[
\Sigma = \int (z - \bar{z})(z - \bar{z})^T \rho(z) d^n z, \quad \bar{z} = \int z \rho(z) d^n z
\]  

(7)

(we use the notation \( z = (x,p) \), \( x \) and \( p \) being the generalized coordinate vectors \((x_1,...,x_n)\) and \((p_1,...,p_n)\) viewed as column vectors in all calculations).

A necessary and sufficient condition for \( \rho_\Sigma \) to be the Wigner distribution of a density matrix is the following (see [20, 21, 22]):

\[
\Sigma + \frac{i}{2} \hbar J \text{ is positive semidefinite}
\]  

(8)

(for short: \( \Sigma + \frac{i}{2} \hbar J \geq 0 \)) where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) is the standard symplectic matrix. It can be restated as a condition on the eigenvalues of \( J \Sigma \): noting that these are the same as the eigenvalues of the antisymmetric matrix \( \Sigma^{1/2} J \Sigma^{1/2} \) they must be of the type \( \pm i \lambda_1, ..., \pm i \lambda_n \) with \( \lambda_j > 0 \); the numbers \( \lambda_1, ..., \lambda_n \) are the symplectic eigenvalues of \( \Sigma \). Condition (8) is then equivalent to \( \lambda_j \geq \frac{1}{2} \hbar \) for \( j = 1, 2, ..., n \), that is to

\[
\lambda_{\text{min}} \geq \frac{1}{2} \hbar
\]  

(9)

where \( \lambda_{\text{min}} \) is the smallest symplectic eigenvalue. Notice that (9) reduces to the Heisenberg inequality \( \sigma_X \sigma_P \geq \frac{1}{2} \hbar \) when

\[
\Sigma = \begin{pmatrix} \sigma_{x_j} & 0 \\ 0 & \sigma_{p_j} \end{pmatrix}
\]

as in the example above. More generally, writing \( \Sigma \) in block-form \( \begin{pmatrix} \Sigma_{xx} & \Sigma_{xp} \\ \Sigma_{px} & \Sigma_{pp} \end{pmatrix} \)

where \( \Sigma_{xx} = (\sigma_{x_j,x_k})_{1 \leq j,k \leq n}, \Sigma_{xp} = (\sigma_{x_j,p_k})_{1 \leq j,k \leq n} \) and so on, one can
show [7, 8] that the conditions (8), (9) are equivalent to the Robertson–Schrödinger inequalities (RSI)

\[ \sigma_{x_j}^2 \sigma_{p_j}^2 \geq \sigma_{x_j,p_j}^2 + \frac{1}{4} \hbar^2. \]  

(10)

Assume that the inequalities in (9) all become equalities: \( \lambda_1 = \cdots = \lambda_n = \frac{1}{2} \hbar \). The RSI are then saturated, that is

\[ \sigma_{x_j}^2 \sigma_{p_j}^2 = \sigma_{x_j,p_j}^2 + \frac{1}{4} \hbar^2 \]  

(11)

and the state is now a generalized Gaussian (squeezed coherent state); one can show its Wigner distribution is a phase space Gaussian [7, 8, 16]. One can apply the same arguments as above to discuss the effect of a variation of Planck’s constant: if \( \hbar' > \hbar \) then (11) becomes

\[ \sigma_{x_j}^2 \sigma_{p_j}^2 \leq \sigma_{x_j,p_j}^2 + \frac{1}{4} \hbar'^2 \]  

(12)

and the RSI are thus violated: there exists no quantum state, pure or mixed, whose WDF is \( \rho \). If we replace \( \hbar \) with \( \hbar'' < \hbar \) then (11) becomes

\[ \sigma_{x_j}^2 \sigma_{p_j}^2 \geq \sigma_{x_j,p_j}^2 + \frac{1}{4} \hbar''^2 \]

indicating that \( \rho \) is now the WDF of a mixed Gaussian state.

### 4 Arbitrary Quantum States

The dependence of arbitrary density matrices on the values of Planck’s constant is an open mathematical problem, and a difficult one. Consider an arbitrary phase space function \( \rho \) such that

\[ \int \rho(z)dz = 1 \]

and let us ask the question: “how can I know whether \( \rho \) is the Wigner distribution of a (pure, or mixed) quantum state?” Defining the covariance matrix as above one can prove (Narcowich [20, 21], Narcowich and O’Connell [22]) that the condition

\[ \Sigma + \frac{i \hbar}{2} J \geq 0 \]

is necessary for \( \Sigma \) to be the covariance matrix of a quantum state, but it is not sufficient: see the discussion and the counterexample given in de Gosson and Luef [10]. The study of the difficult problem to determine whether a
function $\rho$ represents a quantum state is closely related to the mathematical work of Kastler [12] and Loupias and Miracle-Sole [17, 18] in the end of the 1960s, where the notion of function of $\hbar$-positive type was defined using the machinery of $C^*$-algebras. This leads to a quantum version of a classical theorem of Bochner’s characterizing classical probability densities which is difficult to use even numerically since it involves the simultaneous verification of an uncountable infinity of inequalities. Outside Gaussian functions, the only case which has been successfully addressed so far is that of a pure state $|\psi\rangle$. Dias and Prata [4] have shown, using techniques from the theory of complex variables, that if $W\psi$ is the Wigner distribution of a non-Gaussian pure state $|\psi\rangle$ then any variation of $\hbar$ will destroy this property. We conjecture – but this has yet to be proven – that a decrease of $\hbar$ will turn the pure state $|\psi\rangle$ into a mixed one (as in the case of Gaussian pure states), while an increase of $\hbar$ will not even lead to a classical state since the Wigner transform of a non-Gaussian function fails to be positive.

Summarizing, we are in the following situation:

- **If we are in presence of a Gaussian state any decrease of $\hbar$ will yield a new Gaussian state; if the original state is a pure Gaussian it will become a mixed Gaussian state. Any increase in $\hbar$ will transform the state into a classical (Gaussian) state;**

- **A pure non-Gaussian state does not remain pure under any variation of Planck’s constant. It is not known (but however conjectured) that it becomes a mixed state if $\hbar$ decreases.**

5 Experimental Issues

Recently one of us has shown [13, 15], using GPS data, that modern measurement techniques do not allow to prove the constancy of $\hbar$ within an error of 0.7% (also see the rejoinder [3], and the answer [14]. This result opens the door to the possibility of a variable Planck’s constant. Using the sensitivity of density matrices to changes of Planck’s constant there are several possible experimental scenarios to test this hypothesis. If one wants to study possible the dependence of $\hbar$ on location one of them would be the following: An atom in an excited state propagates from B to A in a direction exactly perpendicular to the gradient of $\hbar$. After some time, the atom decays and emits energy. The direction of the relaxation emission depends on the type of transition. In the case where there is no variation in $\hbar$, the vector momentum of the atom will change in response to the direction and energy
of the emission. In the case where there is a variation in $h$, the atom will incur an additional deflection related to how the emission aligns with the gradient of $h$. The size of this effect would be vanishingly small. Meaningful detection would require high energies and long propagation distances. High energy cosmic rays may be an avenue to explore the validity of this prediction. This thought experiment predicts that, if $h$ is varying, there will exist an anisotropy in the trajectories of high energy particles undergoing a decay process that depends on their initial velocity vector. If $h$ is not varying, but rather some dimensionless constant such as the fine structure constant were to vary, then this anisotropy would not exist. Another possible experimental setup is the following: imagine a quantum state teleportation scheme that occurs between two regions that have different values of Planck's constant. The actor “Bob” prepares a maximally entangled particle pair locally in region B (where Planck’s constant is of value $h_B$) and the actor “Alice” receives one of the particles and measures it against a local Bell state prepared in region A (where Planck’s constant is of value $h_A$). As the transmitted particle traverses the spacetime interval between B and A, its quantum state will modulate into a superposition of the available local states. When the particle is received by Alice in some final state (A), it will not be a pure Bell state in Basis set A. However, the local half of the entangled pair held by Bob will not be affected by this. Therefore, the teleportation process will be impossible without some knowledge of the effective rotation operator associated with propagation through the $\partial h/\partial z$ spacetime. This predicted result is in direct opposition to the results expected if $h$ could not vary between A and B, but some dimensionless constant such as the fine structure constant could vary between A and B.

As for the case of a time-varying $h$ one could envisage the following scenario: suppose that after the Big Bang, during the “Planck epoch”, where quantum theory as presently understood becomes applicable, the Early Universe had a smaller Planck constant $h_0 < h$ as today. This Early Universe would then have been much more “quantum” than the current Universe; assuming a steady increase of $h_0$ to its present value $h$ this would mean that the Universe becomes more “classical” with time. It would be interesting to analyze (both theoretically and experimentally) what such an evolution implies at a macroscopic level.
6 Remarks

The topic of “variability” of physical “constants”, which started with Dirac’s Nature paper [5] has always been a controversial one; it is often argued that one should only test the non-constancy of dimensionless parameters (Duff [6]) such as, for instance, the fine-structure constant $\alpha$ (see however the answer of one of us (MM) to the objections in [3]). The variability of $\alpha$ is problematic, but many experimental results seem to point towards a non-constant value of $\alpha$. There is however one thing which is not controversial: these are the mathematical truths exposed in this paper. If one accepts – as most quantum physicists do – that mixed quantum states are represented by density matrices and their Wigner distributions, there is no way to refute the conclusions of any experiment leading to a proof of the variability of Planck’s constant based on the mathematical dependence of states on $\hbar$ as exposed here.

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