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A constructive method for convex solutions of a class of nonlinear Black-Scholes equations

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Abstract: In this work, we are concerned with the theoretical study of a nonlinear Black-Scholes equation resulting from market frictions. We will focus our attention on Barles and Soner’s model where the volatility is enlarged due to the presence of transaction costs. The aim of this paper is to give a constructive mathematical approach for proving the existence of convex solutions to a non degenerate fully nonlinear deterministic problem with nonlinear dependence upon the highest derivative. The existence of a strong solution to the original equation is shown by considering a monotone sequence satisfying an abstract Barenblatt equation and converging toward the solution of a limit problem.

Keywords: Fully nonlinear PDE, Black-Scholes equations, Barenblatt equation

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1 Introduction

In financial modeling, option pricing has gained popularity since the establishment of the Black-Scholes theory [1] in 1973. The authors assumed that the price $S(t)$ satisfies the stochastic differential equation

$$dS(t) = \mu S(t) + \sigma S(t)dB_t,$$

where $\mu$ is the expected rate of return, $\sigma$ is the constant volatility and $B_t$ is the standard Brownian motion on the underlying asset $S(t)$. This model gives us an estimate of the European call option price [1]. The aim of pricing derivative products, such as options is particularly interesting since they are used in order to minimize the damage caused by variations in the stock price. Using Itô’s lemma, the option price becomes the solution to the deterministic linear equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \quad S > 0, \quad t \in (0, T).$$

The unknown $V$ is the European Call option, $S$ is the underlying asset, $T$ is the maturity of the option and $r$ is the risk-free interest rate constant. Despite the success of F. Black and M. Scholes model, it is based on unrealistic assumptions. In fact, they assumed that the market is not subject to transaction costs; this is not true in the real world, as pointed by H. Leland [2]. Actually in this case the hedging of the portfolio becomes tremendously expensive since it requires an infinite number of adjustments. H. Leland had suggested in his work a discrete frequent revision of the portfolio. This idea was later developed by [3–6] and was studied...
numerically in [7, 8]. Another model was proposed by G. Barles and H. Soner in [9], they suggested a Black-Scholes extension with an enlarged volatility \( \tilde{\sigma} \) where

\[
\tilde{\sigma}(S, t, V_{SS})^2 = \sigma^2 \left( 1 + \Psi(e^{(T-t)a^2 S^2 V_{SS}}) \right).
\]  

(3)

The option price satisfies then the nonlinear PDE

\[
V_t + \frac{1}{2} \tilde{\sigma}(S, t, V_{SS})^2 S^2 V_{SS} + rSV_S - rV = 0, \quad S > 0, \quad t \in (0, T).
\]  

(4)

The constant \( a \) is a nonnegative parameter that measures transaction costs (see for an overview [2, 7, 10]) and \( \Psi \) denotes the solution to the nonlinear ordinary differential equation

\[
\begin{cases}
\Psi'(x) = \frac{\Psi(x) + 1}{2 \sqrt{x\Psi(x) - x}}, & x \neq 0, \\
\Psi(0) = 0.
\end{cases}
\]  

(5)

The analysis of (5) performed in [9] yields

\[
\lim_{x \to +\infty} \frac{\Psi(x)}{x} = 1, \quad \lim_{x \to -\infty} \Psi(x) = -1.
\]  

(6)

Equation (4) has been introduced in [9] by using the theory of stochastic optimal control; then the authors have proved that there exists a unique viscosity solution. Another theoretical study for (4) was given recently by D. Ševčovič in [11] by transforming (4) into a quasilinear parabolic equation. Since there is no exact solution to (4) many authors studied (4) numerically (for instance, see [7, 8, 12–14]). There are also some new studies in the literature in the case of pricing options under variable transaction costs (see [15], for more details). In this paper, we give an alternate and constructive theoretical approach of Barles and Soner’s model (4). By constructive method, we mean methods that lead naturally to numerical schemes to approximate the solution; this will appear in a forthcoming work. Besides, the variation of the hedging strategy is an important matter for option traders. This variation is measured by the Gamma parameter which is the second derivative of the option price \( \Gamma = V_{SS} \). In the sequel, we consider the case where we have a non-negativity constraint on the \( \Gamma \) (i.e \( \Gamma \geq 0 \)). This is a natural assumption since the delta hedge of a European Call or Put is increasing when the underlying is non decreasing. For the sake of simplicity, we restrict our attention to the case of \( \Psi(x) = x \), that is a suitable approximation of \( \Psi \) for large \( x \); the general case can be handled with similar but more lengthy methods (see Section 4.1 in the sequel). With this assumption, equation (4) becomes

\[
V_t + \frac{\sigma^2}{2}(1 + e^{(T-t)a^2 S^2 V_{SS}})S^2 V_{SS} + rSV_S - rV = 0.
\]  

(7)
Equation (7) is supplemented with the boundary conditions

\[
\begin{align*}
V(t, 0) &= 0, \quad t \in (0, T), \\
V(t, S) &\sim S \quad (S \to +\infty),
\end{align*}
\]

and a final condition (Pay-off of the option \(V\)), given by \(V(T, S) = (S - K)^+\) where \(S \in [0, +\infty[\) and \(K\) is the strike price.

Fig. 2: The terminal value of the option \(V\).

In this article, we restrict our study to Call options, since in our opinion from the mathematical point of view they are more difficult to handle than Put options (we shall address this issue in a forthcoming work). This article is organized as follows. In Section 2, we set the mathematical framework, transforming the problem (7) into an abstract Barenblatt equation. We then introduce a suitable approximation of this equation by a time discretization. We prove existence and uniqueness of a solution to this approximation in Section 3. Then, we state our main convergence result in Section 4. Eventually we provide a conclusion and perspectives in a last section.

2 Mathematical framework

2.1 Preparing the equation

We first introduce the new function

\[ v(t, S) = a^2 \exp(r(T - t))V(t, Se^T). \]

This function is solution to

\[
\begin{align*}
v_t + \frac{\sigma^2}{2}(1 + S^2v_{SS})S^2v_{SS} &= 0, \quad S > 0, \quad t \in (0, T) \\
v(T, S) &= a^2e^{rT}(S - Ke^{-rT})^+, \quad S > 0 \\
v(t, 0) &= 0, \quad t \in (0, T) \\
v(t, S) &\sim a^2e^{rT}S \quad t \in (0, T).
\end{align*}
\]

Since we prefer to deal with a initial value problem, we perform the shift \(v(t, S) = v(t, s) - a^2Se^T\), the change of time \(t \mapsto T - t\), and a time dilation to get rid of \(\frac{\sigma^2}{2}\) (say \(V(t, S) = v(\frac{2(T-t)}{\sigma^2}, S)\)) to simplify (9) into

\[
\begin{align*}
V_t - (1 + S^2V_{SS})S^2V_{SS} &= 0, \quad S > 0, \quad t \in (0, T) \\
V(t, 0) &= 0, \\
V(0, S) &= a^2e^{rT}(S - Ke^{-rT})^+ - S, \\
\lim_{S \to +\infty} V_S(t, S) &= 0.
\end{align*}
\]
Let us emphasize that we supplement (10) with the constraint \( V_{SS} \geq 0 \), that is we seek convex solutions.

**Remark 1.**
- At this stage this is not clear why the far field boundary condition \( V(t, S) \sim S \) transforms to \( V_S \to 0 \). We will clarify this point in the sequel.
- Seeking a solution to equation (10) is equivalent to seek a solution to the original equation (7) since we can perform backward the change of variable. A variable of interest for (10) is the function \( V(\frac{S^2}{2}) \) that corresponds to the Call option at \( t = 0 \) for the original equation. We will prove below the convergence of this quantity to the corresponding initial Call option for the linear Black-Scholes equation when the transaction costs goes to 0.

### 2.2 Functional analysis

Throughout this paper, we use the following notations. For \( \Omega = \mathbb{R}^+ \), the set \( D(\Omega) \) is the space of smooth compactly supported functions in \( \Omega \) and \( D'(\Omega) \) its dual space. We set \( C_0(\Omega) \) for the set of continuous functions \( V \) such that \( V(0) = 0 \). We denote by \( L^p_{\text{loc}}(\Omega) \) the space of locally integrable functions in \( \Omega \) with \( 1 \leq p < \infty \). Finally we denote by \( L^p_S(\Omega) \) the weighted Lebesgue space with a weight decaying at infinity

\[
L^p_S(\Omega) = \left\{ V \in L^p_{\text{loc}}(\Omega) ; \quad \frac{V(S)^p}{S^2} dS < \infty \right\}, \quad (1 \leq p < \infty).
\]

We also define the Hilbert space

\[
\mathcal{V} = \left\{ V \in L^1_{\text{loc}}(\Omega) ; \quad \int_{\Omega} V^2(S) dS < \infty \right\},
\]

as the closure of \( D(\Omega) \) for the norm \( \| V \|_\mathcal{V} = (\int_{\Omega} V^2(S) dS)^{\frac{1}{2}} \).

**Lemma 2.1.** The set \( \mathcal{V} \) is a subset of \( L^2_S(\Omega) \cap C_0(\Omega) \). Moreover if \( V \) belongs to \( \mathcal{V} \) then for any \( S > 0 \) the inequality \( V(S)^2 \leq 8 S \| V \|_\mathcal{V}^2 \) is valid.

**Proof.** The proof of \( \mathcal{V} \subset C(\Omega) \) is standard and then omitted. For a test function \( V \) in \( D(\Omega) \) we have that, appealing Cauchy-Schwarz inequality

\[
\int_{\Omega} \frac{V^2(S)}{S^2} dS = -2 \int_{\Omega} V(S) V_S(S) \frac{dS}{S} \leq 2 \| V \|_\mathcal{V} \left( \int_{\Omega} \frac{V^2(S)}{S^2} dS \right)^{\frac{1}{2}}.
\]

We then have the Hardy inequality,

\[
\int_{\Omega} \frac{V^2(S)}{S^2} dS \leq 4 \| V \|_\mathcal{V}^2,
\]

and eventually the embedding \( \mathcal{V} \subset L^2_S(\Omega) \) is valid thanks to a density argument. Moreover the identity

\[
\frac{V^2(S)}{S} = \int_S^\infty \left( \frac{V(s)^2}{s^2} - \frac{2V(s)V_S(s)}{s} \right) ds \leq 8 \| V \|_\mathcal{V}^2
\]

completes the proof of Lemma 2.1.

Let us now define the unbounded operator \( A \) in \( L^2_S(\Omega) \) defined as \( AV = -S^2 V_{SS} \). In other words \( A \) is defined by

\[
(AV, W)_{L^2_S(\Omega)} = \int_{\Omega} V_S(s) W_S(s) ds.
\]
For later use we state a maximum principle for the operator $A$.

**Lemma 2.2.** Consider $V$ in $\mathbb{V}$ such that $AV \leq 0$ in $\mathbb{V}^\prime$. Then $V(S) \leq 0$ for any $S \geq 0$.

**Proof.** We would like to point out that the boundary condition respectively at $S = 0$ and at $S \sim +\infty$ are hidden in the assumption $V \in \mathbb{V}$. We prove the result for smooth $V$ and we appeal a density argument to conclude. Assume then that $V_{SS} \geq 0$. Then $V$ is a convex function that satisfies for $s < S$ the inequality

$$V(S) \geq V(s) + V_S(s)(S - s).$$

We infer from this inequality, dividing by $S$ and letting $S$ diverge to $+\infty$, appealing Lemma 2.1, that $V_S \leq 0$. Then the result. $\square$

**Remark 2.** Let us assume that $V$ satisfies the assumptions of Lemma 2.2. Since $V(S)^2 \leq CS$ one can prove that actually $\lim_{S \to +\infty} V_S(S) = 0$. Then the condition at $S \sim +\infty$ in (10) makes sense.

In the following we will use an approach inspired by [16]. Let us define $\beta: \mathbb{R}^+ \mapsto \mathbb{R}^+$ as the inverse function of $y \mapsto y^2 + y$, that is

$$\beta^2(y) + \beta(y) = y \quad \text{i.e.} \quad \beta(y) = \sqrt{y + \frac{1}{4} - \frac{1}{2}}.$$  \hfill (16)

Equation (10) becomes

$$V_i - \beta^{-1}(-AV) = 0.$$  \hfill (17)

Rather than solving (17) with respect to the boundary conditions in (10) we solve

$$\begin{aligned}
\beta(V_i) + AV &= 0, \quad S > 0, \quad t \in (0, T) \\
V(t, 0) &= 0, \\
V(0, S) &= a^2 e^{\tau T} \left( (S - Ke^{-\tau T})^+ - S \right), \\
\lim_{S \to +\infty} V_S(t, S) &= 0.
\end{aligned}$$  \hfill (18)

The equation above belongs to the class of Barenblatt’s equations. To handle $\beta(V_i)$ we need another functional space. Hence we introduce a suitable Orlicz space (see [17, 18]). Let us consider the function $J$ defined by

$$J: \mathbb{R}^+ \mapsto \mathbb{R}^+$$

$$y \mapsto \beta(y)y.$$  

**Proposition 1.** The function $J$ is convex and nondecreasing on $\mathbb{R}^+$. Moreover for $y \geq 0$ we have $2J(y) \leq J(2y) \leq 4J(y)$.

**Proof.** Deriving $J$ implies

$$J'(y) = \sqrt{y + \frac{1}{4}} - \frac{1}{2} + \frac{y}{2\sqrt{y + \frac{1}{4}}} \geq 0,$$

and

$$J''(y) = \frac{1}{2\sqrt{y + \frac{1}{4}}} + \frac{y + \frac{1}{2}}{4(y + \frac{1}{4})^{\frac{3}{2}}} \geq 0.$$  

Hence $J$ is convex and monotone increasing on $\mathbb{R}^+$. Since $J$ is convex and $J(0) = 0$ then $2J(y) \leq J(2y)$. Since $\beta$ is concave $\beta(2y) \leq 2\beta(y)$ and then $J(2y) \leq 4J(y)$. $\square$

**Definition 1.** The weighted Orlicz space associated to $J$ is the vector space generated by $\nu \in L^1_{loc}(\Omega)$ that satisfies

$$\int_{\Omega} \frac{J(|\nu(S)|)}{S^2} dS = \int_{\Omega} \beta(|\nu(S)|) \frac{|\nu(S)|}{S^2} dS < +\infty,$$  \hfill (19)
this space is equipped with the Luxemburg norm

\[ L_j(v) = \inf \left\{ \lambda > 0 \mid \int_\Omega \beta \left( \frac{|v(S)|}{\lambda} \right) \frac{|v(S)|}{\lambda} dS \leq 1 \right\}. \]  

(20)

Actually, since \( 2f(y) \leq f(2y) \leq 4f(y) \) it is standard to prove that there exists \( c > 0 \) such that

\[ c \min(||v||_{L_j}, ||v||_{L_j}^2) \leq \int_\Omega f(|v(S)|) \frac{dS}{\sqrt{\lambda}} \leq \frac{1}{c} \max(||v||_{L_j}, ||v||_{L_j}^2). \]

**Proposition 2.** Actually this weighted Orlicz space \( L_j \) is \( L_{0,2}^2(\Omega) + L_{1,2}^2(\Omega) \) that is the dual space of \( L_{0,2}^2(\Omega) \cap L_{1,2}^2(\Omega) \). Moreover \( V \) belongs to \( L_j \) if and only if \( \min(V^2, |V|^2) \) belongs to \( L_{0,2}^2(\Omega) \). 

**Proof.** We know that \( \beta(y) = \sqrt{y + \frac{1}{4} - \frac{1}{2}} = \frac{y}{\sqrt{y + \frac{1}{4} + \frac{1}{2}}} \). Hence \( \beta(y) \leq \min(y, \sqrt{y}) \). Conversely we have \( y = \beta(y) + \beta^2(y) \), thus \( y \leq 2 \max(\beta(y), \beta^2(y)) \). Either \( \beta(y) \geq 1 \) and thus \( y \leq 2\beta^2(y) \) or \( \beta(y) \leq 1 \) and thus \( y \leq 2\beta(y) \). Then, there exists \( c > 0 \) such that \( c \min(y, \sqrt{y}) \leq \beta(y) \). Therefore \( V \) belongs to \( L_j \) if and only if \( \min(V^2, |V|^2) \) belongs to \( L_{0,2}^2(\Omega) \). To prove that \( L_j \) is \( L_{0,2}^2(\Omega) + L_{1,2}^2(\Omega) \) is then standard. \[ \square \]

### 2.3 Main results

**Theorem 2.3.** There exists a unique convex solution to (10) that belongs, for any \( T > 0 \) to \( C(0, T; V) \) and such that \( V_t \) belongs to the dual space of \( L^2([0, T] \times \Omega) \cap L^1(0, T \times \Omega) \). 

In the next section we handle the proof of the theorem.

### 3 Proof of the main theorem

#### 3.1 Approximation scheme

Now we approximate (18) by a discrete-time scheme, and then we use the monotone iterative technique [19]. Given \( V^n \) in a set that will be defined below and \( \tau > 0 \), we solve recursively the problem, with \( V^0(S) = a^2 e^{\tau T}((S - Ke^{-\tau T})^+ - S) \).

\[
\left\{ \begin{array}{lcl}
\beta\left( \frac{V^{n+1} - V^n}{\tau} \right) - S^2 V_{SS}^{n+1} &=& 0 \quad S > 0, \\
V^{n+1}(0) &=& 0, \\
\lim_{S \to +\infty} V_S^{n+1}(S) &=& 0.
\end{array} \right.
\]

(21)

**Proposition 3.** Let us consider the convex set \( K_n \) that reads

\[ K_n = \{ V \in V^\prime/V^n \leq V \leq 0, V - V^n \in L_j \}. \]

(22)

For all \( n \geq 0 \), there is a unique \( V^{n+1} \in K_n \) solution of (21).

**Proof.** We assume that \( V^n \) is known and we prove the existence of \( V^{n+1} \). Let us observe that the Lipschitz constant of \( \beta \) is 1. We solve recursively in \( V \) the equation

\[-\tau S^2 W_{SS}^{m+1} + W^{m+1} = W^m - \tau \beta\left( \frac{W^m - V^n}{\tau} \right), \]

(23)

with initial data \( W^0 = V^n \). Assume that \( W^m - V^n \) belongs to \( L_j \). Then the right hand side of (23) belongs to \( V^\prime \) and (23) can be solved by Lax-Milgram Theorem. Since the map \( G(v) = v - \tau \beta\left( \frac{v - W^n}{\tau} \right) \) is increasing, we can prove
recursively by the Maximum Principle (see Proposition 2.2) that \( W^m \leq W^{m+1} \). Moreover \( W^m \leq 0 \) since 0 is a supersolution above \( V^0 \). Hence the Perron sub and supersolution method converges and \( V^{n+1} = \sup_m W^m \) is a weak solution to (21). Besides, we know that \( W^m - V^n = (W^m - V^0) - (V^n - V^0) \). The function \( V^n - V^0 \) is in \( L^1 \) and the non negative function \( W^m - V^0 \) that is bounded by above by \(-V^0 \) is also in \( L^1 \). Then \( V^{n+1} - V^n \) is in \( L^1 \) and the right hand side of (21) is in \( V^\prime \). Then \( V^{n+1} \) is in \( V^\prime \).

We prove the uniqueness as follows. Consider \( \tilde{V}^{n+1} \) another solution. Then

\[
\tau \int_\Omega \left( \beta \left( \frac{V^{n+1} - V^n}{\tau} \right) - \beta \left( \frac{\tilde{V}^{n+1} - V^n}{\tau} \right) \right) \left( \frac{V^{n+1} - V^n}{\tau} - \frac{\tilde{V}^{n+1} - V^n}{\tau} \right) \, dS + (A(V^{n+1} - V^n), (V^{n+1} - \tilde{V}^{n+1})) = 0.
\]

Since the two terms in the left hand side of (24) are non negative, the conclusion follows promptly.

3.2 Passing to the limit

Let us fix a \( \varepsilon > 0 \). For \( \tau > 0 \) let \( N \) be such as \( N\tau = \varepsilon \) and let us consider the function

\[
V^\tau(t, S) = \sum_{n=0}^{\infty} \sigma \left( \frac{t}{\tau} - n \right) V^n(S),
\]

where \( \sigma \) is defined by \( \sigma(t) = \max(0, 1 - |t|) \). We plan to let \( N \to +\infty \) (and so \( \tau \to 0 \)). Note that in the time interval \( I_n = [n\tau, (n+1)\tau] \) we have

\[
V^\tau(t, S) = (n + 1 - \frac{t}{\tau}) V^n(S) + (\frac{t}{\tau} - n) V^{n+1}(S),
\]

\[
\frac{dV^\tau}{dt}(t, S) = \frac{V^{n+1}(S) - V^n(S)}{\tau}.
\]

Lemma 3.1. The following assertions hold true

1. The sequence \( \frac{dV^\tau}{dt} \) is bounded in \( L^2(\Omega \times [0, \varepsilon]) \).
2. The sequence \( V^\tau \) is bounded in \( L^\infty([0, \varepsilon]; L^2(\Omega)) \).
3. The sequence \( \beta \frac{dV^\tau}{dt} \) is bounded in \( L^2(\Omega \times [0, \varepsilon]) \cap L^3(\Omega \times [0, \varepsilon]) \).
4. The sequence \( V^\tau \) is bounded in \( L^\infty(\Omega \times [0, \varepsilon]) \).

Proof. 1. Summing (24) for \( \tilde{V}^{n+1} - V^n \) with respect to \( n \) yields

\[
\sum_{n=0}^{\infty} \int_\Omega \beta \left( \frac{V^{n+1} - V^n}{\tau} \right) \left( \frac{V^{n+1} - V^n}{\tau} - \frac{\tilde{V}^{n+1} - V^n}{\tau} \right) \, dS + \frac{1}{2} \int_\Omega |V^n| \, dS \leq \frac{1}{2} \int_\Omega |V^n| \, dS.
\]

Hence 1. and 2. of the Lemma are proved. The point 3. is straightforward due to the properties of \( \beta \) that maps \( L^2 + L^3 \) into \( L^2 \cap L^3 \). Besides, since \( V^\tau \) is a convex combination of \( V^n \) and \( V^{n+1} \) in \( I_n \) then \( V^0 \leq V^\tau \leq 0 \) everywhere. This completes the proof of the Lemma.

We know pass to the limit when \( \tau \to 0 \) using a monotonicity argument [19].

For \( (t, S) \) in \( I_n \times \Omega \), we have

\[
\beta \left( \frac{dV^\tau}{dt} \right) - S^2 V^\tau_{SS} = S^2(n + 1 - \frac{t}{\tau})(V^{n+1} - V^n). \tag{28}
\]

We first consider the scalar product in \( L^2(\Omega) \) by a test function \( \phi \in D(\Omega) \); then we have

\[
\int_\Omega \left( \beta \left( \frac{dV^\tau}{dt} \right) - S^2 V^\tau_{SS} \right) \frac{\phi(S)}{S^2} \, dS = \epsilon_n. \tag{29}
\]
where
\[
\epsilon_n = (n + 1 - \frac{t}{\tau}) \int_\Omega \left( \frac{V_n^{n+1} - V_n}{\tau} \right) \tau \phi_{S^2}.
\]

Let us observe that
\[
\epsilon_n \leq C \tau \| \frac{dV^T}{dt} \|_{L^2(\Omega)} \| \phi_{S^2} \|_{L^2(\Omega)} \| \phi_{S^2} \|_{L^2(\Omega)},
\]
and then this term converges towards 0 when \( \tau \to 0 \). Considering now a subsequence \( V^T \) that converges towards \( V \) in \( L^\infty(0, \tau; \Omega) \) and such that \((\frac{dV^T}{dt}, \beta(\frac{dV^T}{dt}))\) converges weakly towards \((\frac{dV}{dt}, \chi)\) in \( L^1_T \times L^1_T \), we have that \( \chi = S^2 V_{S^2} \).

We now prove that \( \chi = \beta \left( \frac{dV}{dt} \right) \). We then compute
\[
0 \leq Q_\tau = \int_0^\tau < \beta \left( \frac{dV^T}{dt} \right), \frac{dV^T}{dt} - v >_{L^2_T},
\]
where \( v \) is a test function. By weak convergence we have
\[
\int_0^\tau < \beta \left( \frac{dV^T}{dt} \right), \frac{dV^T}{dt} >_{L^2_T} < \beta(v), \frac{dV^T}{dt} >_{L^2_T} \to \int_0^\tau < \chi, \frac{dV}{dt} >_{L^2_T}.
\]

Moreover, observing that
\[
\int_0^\tau < \beta \left( \frac{dV^T}{dt} \right), \frac{dV^T}{dt} >_{L^2_T} = \sum_n \int_\Omega V_{S^2}^{n+1} (V_n^{n+1} - V_n) dS
\]
\[
\leq -\frac{1}{2} \int_\Omega |V_{S^2}(T)|^2 dS + \frac{1}{2} \int_\Omega |V_{S^2}(0)|^2 dS,
\]
we have, by weak convergence
\[
\lim_{\tau \to 0} \int_0^\tau < \beta \left( \frac{dV^T}{dt} \right), \frac{dV^T}{dt} >_{L^2_T} \leq -\frac{1}{2} \int_0^\tau |V_{S^2}(T)|^2 dS + \frac{1}{2} \int_0^\tau |V_{S^2}(0)|^2 dS.
\]

Therefore
\[
\lim_{\tau \to 0} \int_0^\tau \beta \left( \frac{dV^T}{dt} \right) \frac{dV^T}{dt} \leq -\frac{1}{2} \int_0^\tau \frac{d}{dt} |V_{S^2}|^2 dt = \int_0^\tau < \chi, \frac{dV}{dt} >_{L^2_T}.
\]

Thus we have
\[
0 \leq \lim_{\tau \to 0} Q_\tau \leq \int_0^\tau < \beta(v) - \chi, v - \frac{dV}{dt} >_{L^2_T}.
\]

We choose \( v = \frac{dV}{dt} + \epsilon \phi \) where \( \phi \) is a given test function; then we have
\[
0 \leq \epsilon \int_0^\tau < \beta \left( \frac{dV}{dt} \right), \phi >_{L^2_T} + o(\epsilon).
\]

By changing \( \phi \) to \(-\phi\) and tending \( \epsilon \) to 0+ we obtain
\[
\beta \left( \frac{dV}{dt} \right) = \chi \text{ a.e.}
\]
This completes the proof of the existence of a solution $V$ that belongs to $L^\infty(0, \Omega)$ such that $\frac{dV}{dt}$ belongs to $L^2(\Omega \times (0, \Omega)) + L^\frac{3}{2}(\Omega \times (0, \Omega))$. To prove that $V$ is continuous in $t$ with values in $V$ is a consequence of

$$||V(t) - V(s)||_V \leq ||\frac{dV}{dt}||_{L^2(\Omega \times (0, \Omega)) + L^\frac{3}{2}(\Omega \times (0, \Omega))} \max(|t-s|^\frac{1}{2}, |t-s|^\frac{1}{3}).$$

### 3.3 Uniqueness of the solution

Let $V$ and $\tilde{V}$ be two solutions of (18), and $W = V - \tilde{V}$. We have

$$\left< \beta \left( \frac{dV}{dt} \right) - \beta \left( \frac{d\tilde{V}}{dt} \right), \frac{dW}{dt} \right>_S \geq \frac{1}{2} \frac{d}{dt} ||W||_V^2 = 0.$$ (36)

Thus $V = \tilde{V}$.

### 4 Miscellaneous results

#### 4.1 Handling the original Barles-Soner equation

For the sake of simplicity of exposure, we have chosen to set $\Psi(x) = x$ in this article. We know indicate how to handle the original Barles-Soner equation. The main difference is the very definition of the Orlicz space that is defined through a function $\tilde{\beta}(y) = \tilde{J}(\beta(y))y$ where $\tilde{\beta}(y)$ is solution to $(1 + \Psi(\beta(y)))\tilde{\beta}(y) = y$. This Orlicz space has similar properties than $L^2(\Omega) + L^{\frac{3}{2}}(\Omega)$ because $\tilde{\beta}$ and $\beta$ have the same behavior at $y \sim 0$ and $y \sim \infty$, but computations are more lengthy.

#### 4.2 Comparison with the solution of the linear equation

Rescaling the solution $V$ of Theorem 2.3 to $V^a = a^{-2} V$ we then have a solution to the equation

$$V^a_t = (1 + a^2 S^2 V^a_{SS}) S^2 V^a_{SS},$$

with initial condition $V^a(S) = e^{rT} (S - Ke^{rT})_+ - S$. We will compare this solution to the solution of the linear Barles-Soner equation $V^l$

$$V^l_t = S^2 V^l_{SS},$$

supplemented with the same initial data. Actually, if the transaction costs are small, then $a \to 0$. To begin with we prove a result that implies that the so-called Gamma parameter $V^a_{SS}$ remains bounded in some space when $a$ converges to 0.

**Lemma 4.1.** There exists a constant $C$ that depends on the data $r$, $K$, $T$ but that is independent of $a$ such that

$$\int_0^t \int_\Omega (S^2 V^a_{SS})^2 \frac{dS}{S^2} d\tau \leq C.$$ (37)

**Proof.** We know that

$$\beta_a \left( \frac{dV_a}{dt} \right) = \Delta_a V_a,$$
where $\beta_{a}(y)$ is the positive solution to $a^2 \beta_{a} + \beta_{a} - y = 0$. Multiplying this equation by $S^{-2} V_{a}$ and integrating in $\Omega \times [0, t]$ leads to

$$
\int_{0}^{t} \int_{\Omega} \beta_{a}(dV_{a}) dV_{a} dS d\tau \leq \frac{1}{2} \int_{\Omega} V_{a}^{0} dS = C.
$$

(38)

On the other hand

$$
\Delta_{S} V_{a} \leq \Delta_{S} V_{a} + a^2 (\Delta_{S} V_{a}) = \frac{dV_{a}}{dt},
$$

and then $(\Delta_{S} V_{a})^2 \leq \beta_{a}(dV_{a}) dV_{a}$ that completes the proof of the Lemma.

Proposition 4. We have that $V^{a} \geq V^{l}$ everywhere, and that there exists a constant $C$ that depends on $K, r, T$ such that

$$
||V^{a} - V^{l}||_{L^1(\Omega)} \leq C a^2.
$$

Proof. Set $W = V^{a} - V^{l}$. Since $W_{t} - S^{-2} W_{SS} \geq 0$ then by the maximum principle $W \geq 0$. On the other hand

$$
W_{t} - S^{-2} W_{SS} \leq a^2 (S^{-2} V_{a}^{SS})^{2}.
$$

Multiplying by $S^{-2}$ and integrating in $\Omega$ yields

$$
dt ||V^{a} - V^{l}||_{L^1(\Omega)} = -W_{S}(0) + a^2 \int_{\Omega} S^{2}(V_{SS}(a))^{2}(S) dS,
$$

since $W_{S}(S) \to 0$ when $S$ diverges towards $+\infty$ due to Lemma 2.1. On the other hand, since $W(0) = 0$ and $W(S) \geq 0$ for $S > 0$ we then have $-W_{S}(0) \leq 0$. Then

$$
dt ||V^{a} - V^{l}||_{L^1(\Omega)} \leq a^2 \int_{\Omega} S^{2}(V_{SS}(a))^{2}(S) dS,
$$

Then integrating in time concludes, thanks to Lemma 4.1, concludes the proof of the Proposition.

The result of Proposition 4 means that the option price when transaction costs occur is more expensive than the option price in the linear Black-Scholes framework. This is an expected result since pricing in this case implies additional costs, in fact the higher the parameter $a$ is, the greater option price becomes. Proposition 4 can also be seen formally as a deterministic way of proving that hedging errors vanish when transaction costs converge toward $0$ where the contingent claim can be replicated with the delta-hedge strategy; see for instance [3, 20].

4.3 Conclusion

We have provided here a deterministic approach to construct a solution of a nonlinear Black-Scholes equation with an enlarged volatility. We have transformed the problem into a Barenblatt equation in order to overcome the difficulty coming from the nonlinearity. Our method gives a constructive approximation for convex solutions in a suitable Orlicz space, in contrast to the general method of viscosity solutions [9]. We then have compared the solution to Barles and Soner’s equation to solution of the linear Black-Scholes model. For future work we will be interested in performing the numerics to approximate the solution of the Barles-Soner equation using our approximation procedure. We are also interested in extending our theory for the two dimensional problem of (4) where the option depends on two underlying assets.
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