DENSITY FLUCTUATIONS FOR EXCLUSION PROCESSES WITH LONG JUMPS

PATRÍCIA GONÇALVES AND MILTON JARA

ABSTRACT. We show that the stationary density fluctuations of exclusion processes with long jumps, whose rates are of the form $c^\pm |y-x|^{-(1+\alpha)}$ where $c^\pm$ depends on the sign of $y-x$, are given by a fractional Ornstein-Uhlenbeck process for $\alpha \in (0, \frac{3}{2})$. When $\alpha = \frac{3}{2}$ we show that the density fluctuations are tight, in a suitable topology, and that any limit point is an energy solution of the fractional Burgers equation, previously introduced in [13] in the finite volume setting.

1. INTRODUCTION

A classical problem on the field of interacting particle systems corresponds to the derivation of a scaling limit for the stationary fluctuations of the conserved quantities of the system. The archetypical example is the exclusion process, which we describe as follows. The exclusion process is a system of particles on a given graph, on which each particle performs a continuous-time random walk with the restriction that each site on the graph is allowed to have at most one particle. Despite its simplicity, the richness of this process makes of it one of the favorite models on the realm of interacting particle systems. In these notes, we consider the exclusion process with long jumps on the one-dimensional lattice, introduced in [14]. In this case, the transition rates of the underlying random walk have a polynomial tail of the form $c^\pm |y-x|^{-(1+\alpha)}$ for some $\alpha \in (0, 2)$, where $c^\pm$ depends on the sign of $y-x$. The Bernoulli product measures $\mu_\rho$ of density $\rho \in [0, 1]$ on $\{0, 1\}^\mathbb{Z}$ are invariant under the evolution of this process, reflecting the fact that particles are neither created nor destroyed by the dynamics and the translation invariance of the transition rates. In these notes, we will study the stationary density fluctuations of the exclusion process with long jumps starting from $\nu_\rho$. For $\alpha \in (0, \frac{3}{2})$, we show that the scaling limit of the density fluctuations are given by the infinite-dimensional Ornstein-Uhlenbeck equation

$$d\mathbf{Y}_t = (\mathcal{L}_\rho)^\star \mathbf{Y}_t dt + \sqrt{2\rho_\rho(1-\rho)^{-1/2}} \mathbf{W}_t,$$

where $\mathbf{W}$ is a space-time white noise, $\mathcal{L}_\rho$ is the generator of an $\alpha$-stable, skewed Lévy process, $(\mathcal{L}_\rho)^\star$ is its adjoint and $\mathcal{L}_\rho^{1/2}$ is the symmetric part of $\mathcal{L}_\rho$. In the case $\alpha = \frac{3}{2}$ we prove that the density fluctuation field is tight and any limit point is an energy solution
of the fractional Burgers equation
\[ dY_t = (\mathcal{L}^\rho)^t Y dt + m \nabla Y^2 dt + \sqrt{2\rho(1-\rho)\mathcal{L}^{1/2}} d\hat{W}_t, \]
where \( m \) is the mean of the underlying transition rate. The notion of energy solution was introduced in [9] in the context of the KPZ equation
\[ \partial_t h = \Delta h + (\nabla h)^2 + d\hat{W}_t. \]
The fractional Burgers equation was introduced in [13] in finite volume and the notion of energy solutions was used to prove existence of solutions of this equation. In the past few years, a great deal of research around the KPZ/Burgers equation and its universality class has been done; see [7] for a review. A fundamental breakthrough on the mathematical understanding of the wellposedness of the KPZ equation has been given in [18, 19], settling on firm grounds questions about existence and uniqueness of solutions of the KPZ equation. However, the concept of energy solutions, based on more classical martingale problems, seems to be very elusive in the sense that the question of whether energy solutions of the KPZ are unique, has not been answered, neither in the positive nor in the negative. The uniqueness of energy solutions of the KPZ, combined with the results in [9], would imply a proof of the weak KPZ universality conjecture. However, the theory of regularity structures, in its current formulation, breaks down exactly at \( \alpha = 3/2 \), which is the parameter where our fractional Burgers equation appears.

The main motivation for these notes comes from the strong KPZ universality conjecture, which, roughly speaking, states that there is a universal object (the KPZ fixed point) that governs the fluctuations of stationary, non-equilibrium, conservative, one-dimensional stochastic models. Starting from various physical considerations, one important property of this universal object is its scale invariance with respect to the KPZ space-time scaling 1 : 2 : 3. The fractional Burgers equation is invariant under this scaling, and therefore it provides a candidate for, at least, the equation satisfied by the KPZ fixed point. As far as we know, this is the first example of a non-linear equation with a meaningful notion of solution, obtained as a scaling limit of a stochastic, conservative system, which is invariant under the KPZ scaling.

Our method of proof is an improvement over the proof carried out in [9], where the finite-range case is treated. The main technical novelty is the treatment of the non-local part of the drift, which requires a multiscale analysis which is different from the one introduced in [9] and similar to the one introduced in [8]. The idea taken from [9] is the following. Consider for simplicity a local observable of the dynamics, it could be for example the occupation number at the origin. Due to the conservation of the number of particles and the ergodicity of the dynamics, the local density of particles is the observable of the dynamics which takes more time to equilibrate. Therefore, if we look at the evolution on the right space-time scale, any observable of the dynamics should be asymptotically equivalent to a function of the density of particles on a block of, a macroscopical, small size around the support of the observable. The Boltzmann-Gibbs principle introduced in [3] states that, at first order, this function is linear on the density of particles; a claim supported by the equivalence of ensembles. The second-order Boltzmann-Gibbs principle introduced in [9], states that the second-order correction term is a quadratic function of the local density of particles. This allows to replace any local function of the dynamics by the corresponding function of the local density of particles. In these notes, the drift is a non-local function.

\[ \text{The fractional Burgers equation considered in [13] is defined in finite volume, and therefore the spatial scaling changes its domain.} \]
and the multiscale analysis introduced in [9] is not enough to handle this non-local function, so we introduce a second multiscale which, combined with the original one, allows to replace the drift by a quadratic function of the local density of particles. For $\alpha \leq 1$, this sophisticated method is not needed and the fluctuations can be obtained by means of classical methods. For $\alpha \in (1, 1 + \frac{2}{5 + \sqrt{33}})$, one step of the multiscale analysis of [9] is needed, so the proof is not very different from the case $\alpha \leq 1$. For $\alpha \in [1 + \frac{2}{5 + \sqrt{33}}, \frac{3}{2})$, the multiscale analysis shows that the drift term vanishes in the limit, which is the reason why the Ornstein-Uhlenbeck equation (1.1) is the limit in those cases. The division between the cases $\alpha \in (1, 1 + \frac{2}{5 + \sqrt{33}})$ and $\alpha \in [1 + \frac{2}{5 + \sqrt{33}}, \frac{3}{2})$ is rather artificial, and it is done just to emphasize that in order to obtain our result in full generality, it is necessary to introduce new ideas, which come, in these notes in the form of a different multiscale analysis. For $\alpha = \frac{3}{2}$ the drift makes its way up to the limit, in the form of a quadratic functional of the limiting field. This quadratic functional is extremely singular, and it is the source of trouble for the stochastic Burgers equation. Only after [18] we have been able to understand how to set up correctly a well-posed Cauchy problem for the (local) stochastic Burgers equation.

The theory of regularity structures works thanks to the following heuristic observation: the scaling of the nonlinearity of the equation is supercritical, with respect to the scaling of the linear part. Therefore, the theory of regularity structures makes possible to set up a Picard iteration scheme to solve it. This observation is no longer true for the fractional Burgers equation: the equation is critical, in the sense that, the nonlinear part and the linear part scale in the same way. Therefore, it is not surprising that we are not able to obtain a full convergence result for $\alpha = \frac{3}{2}$. Notice, however, that we have enough information about limit points to show that they are well defined as stochastic processes, that the nonlinearity is well defined in a strong sense, and that they solve a martingale formulation of the fractional Burgers equation.

These notes are organized as follows. In Section 2 we define the exclusion process with long jumps and we make precise formulations of the main results of the article. These formulations require a great deal of previous definitions, which are carried out along the section. In particular we need to define what do we understand by stationary solutions of the fractional Ornstein-Uhlenbeck equation and by stationary energy solutions of the fractional Burgers equation. A great deal of care is needed at this point. It is natural to consider the density fluctuation field as a distribution-valued stochastic process. Therefore, its action is well defined for test functions in the Schwartz space $\mathcal{S}(\mathbb{R})$. But $\mathcal{S}(\mathbb{R})$ is not left invariant by $\mathcal{L}^\rho$. In fact, for most functions $f \in \mathcal{S}(\mathbb{R})$, $\mathcal{L}^\rho f$ does not belong to $\mathcal{S}(\mathbb{R})$. This fact is easy to verify in Fourier space. But stationary processes are stochastically continuous in $L^2(\mathbb{R})$, which allows to define its action over functions in $L^2(\mathbb{R})$ through suitable approximations.

The general strategy of proof of the main results of these notes is not difficult to describe. Our definitions of solutions use martingale characterizations. Therefore, we will verify that the density fluctuations satisfy an approximate martingale problem, which, in the limit, becomes the martingale problem associated to the corresponding limiting process. The passage to the limit is allowed by tightness arguments, complemented by some uniform estimates on the errors on the approximation.

In Section 3 we collect various auxiliary results which will be needed in the proof of the main results. In Section 5.1 we define and compute various martingales associated to the density fluctuation fields, which will be used to show that the density fluctuation fields satisfy an approximated version of the martingale problem defined for the limiting processes. In Section 5.2 we state various tightness and convergence criteria that we will
use to show tightness of the density fluctuation fields. In Section 3.3 we state an estimate on the variance of additive functionals of the processes of Kipnis-Varadhan’s type and we use the spectral gap inequality in order to transform it into an effective estimate, stated as Proposition 3.8. We point out that once we have established Proposition 3.8 the text is completely independent of the Kipnis-Varadhan’s inequality or of the spectral gap inequality. In particular, if by some other means we were able to prove Proposition 3.8 the results of these notes would be proved without needing these inequalities. In Section 3.4 we state the form of the equivalence of ensembles that will be needed in these notes.

In Section 4 we prove the main results. The crucial part is to deal with the drift term $A_n^\alpha(f)$. In Section 4.1 we prove tightness of some terms in the martingale decomposition of the density fluctuation field, which works out for any $\alpha \in (0,2)$. In Section 4.1.1 we prove tightness in the case $\alpha \leq 1$. This case is a good warm-up to what follows next, since the standard proof found for example in Chapter 11 of [21] works well. In Section 4.1.2 we prove tightness in the case $1 < \alpha < 1 + \frac{2}{5 + \sqrt{33}}$. We wrote this section to show how to use Proposition 3.8 in order to get estimates on the variance of the drift term. We only have at our disposal a brute-force Cauchy-Schwarz estimate to deal with the tail part of the drift term. In this way we can show the asymptotic negligibility of jumps of size bigger than $n^{\frac{2\alpha-2}{2\alpha-1}}$. This jump size corresponds to macroscopical small jumps. The smaller jumps can be handled with Proposition 3.8. A single use of this proposition is enough to fill the gap up to $n^{\frac{2\alpha-2}{2\alpha-1}}$, only for $\alpha < 1 + \frac{2}{5 + \sqrt{33}}$, so a more refined argument is needed for the general case. In Section 4.1.3 we prove tightness for $\alpha < \frac{3}{2}$. In this case we need to introduce a multiscale analysis in order to use in an effective way Proposition 3.8. The idea is the following. Proposition 3.8 allows to estimate the variance of space-time additive functionals of the dynamics by their spatial variance, paying as a price the inverse of the spectral gap over the support of the spatial functions in consideration. Therefore, the largest the support of the functions we consider, the less effective Proposition 3.8 is. The current associated to big jumps has a very big support, but its variance decays with the distance as a power law. Therefore, there is a trade-off between the support and the intensity of a big jump. The right way to exploit this trade-off is through a multiscale analysis.

In Section 4.1.4 we prove tightness for $\alpha = \frac{3}{2}$. Although the multiscale analysis of Section 4.1.3 still makes big jumps negligible, very small jumps are no longer negligible and a new argument is needed. The multiscale analysis of Section 4.1.3 stops at size $n^{1-\delta}$ for some small $\delta > 0$ and it shows that the drift term is asymptotically equivalent to the square of the density on a box of size $n^{1-\delta}$. This is what is called the one-block estimate in the literature of interacting particle systems. Using the renormalization scheme introduced in [9], we show the two-blocks estimate, which states that the drift term is asymptotically equivalent to the square of the density on a box of size $n \epsilon$. This method shows a uniform $L^2$-bound for the difference between the drift term and the square of the density, which is good enough to prove tightness by means of the Kolmogorov-Centsov’s criterion stated in Proposition 3.3. In Section 4.2 we show Theorem 2.11 which states the convergence of the density fluctuation field to the stationary solution of the fractional Ornstein-Uhlenbeck equation. Once tightness is proved, the proof is standard and relies on the martingale characterization of such solutions. In Section 4.3 we show Theorem 2.16 which is also not very difficult to prove once the uniform bound (4.17) is obtained.

Section 5 contains a detailed discussion about how the main results of these notes are related to the so-called KPZ universality conjecture. In particular we formulate a conjecture, which, roughly speaking, says that the KPZ fixed point is a stationary energy solution.
of the fractional KPZ equation. We also explain for which kind of models we can expect to extend the results of these notes, and we give at least three lines of generalization that could be accomplished.

2. Notations and Results

2.1. The exclusion process. In this section we describe what it is known in the literature as the exclusion process on the one-dimensional lattice \( \mathbb{Z} \). We say that a function \( p : \mathbb{Z} \to [0, \infty) \) is a transition rate if \( p(0) = 0 \) and \( p^* = \sum_z p(z) < +\infty \). Let \( p(\cdot) \) be a transition rate. Let \( \Omega = \{0, 1\}^\mathbb{Z} \) be the state space of a Markov process. We consider on \( \Omega \) the product topology. We denote by \( \eta = \{\eta(x) : x \in \mathbb{Z}\} \) the elements of \( \Omega \). We say that \( x \in \mathbb{Z} \) is a site and that \( \eta \in \Omega \) is a configuration of particles. Let \( \eta \in \Omega \) be a configuration of particles. We say that there is a particle at the site \( x \) if \( \eta(x) = 1 \); otherwise we say that the site \( x \) is empty. Let us consider the following dynamics. Each particle waits an exponential time of rate \( p^* \) (for this reason we ask \( p^* \) to be finite), at the end of which it chooses a site \( y \in \mathbb{Z} \) with probability \( p(y-x)/p^* \), where \( x \) is the current position of the particle. If the chosen site is empty, the particle jumps into it. Otherwise it stays at its current position. In any case, a new exponential time starts afresh and the particle repeats the steps above.

The dynamics described above corresponds to a Markov process \( \{\eta_t : t \geq 0\} \) defined on \( \Omega \). If the number of particles is finite, it is not difficult to construct \( \eta_t \) for any \( t \geq 0 \). When the number of particles is infinite, a detailed construction of the process \( \{\eta_t : t \geq 0\} \) can be found in [23]. In particular, the derivation of the properties we will describe below can be found there. Notice that we are not assuming anything about \( p(\cdot) \) aside from \( p^* < +\infty \). In particular, we can assume that particles perform arbitrarily long jumps with positive probability, which is the case we are interested on these notes.

We say that a function \( f : \Omega \to \mathbb{R} \) is local if there exists \( A \subseteq \mathbb{Z} \) finite such that \( f(\eta) = f(\xi) \) whenever \( \eta(x) = \xi(x) \) for any \( x \in A \). We say that the smaller of such sets is the support of \( f \), and we denote it by \( \text{supp}(f) \). For \( \eta \in \Omega \) and \( x, y \in \mathbb{Z} \) we define \( \eta^{x,y} \in \Omega \) as

\[
\eta^{x,y}(z) = \begin{cases} 
\eta(y), & z = x \\
\eta(x), & z = y \\
\eta(z), & z \neq x, y.
\end{cases}
\]

For \( f : \Omega \to \mathbb{R} \) and \( x, y \in \mathbb{Z} \) we define \( \nabla_{x,y} f : \Omega \to \mathbb{R} \) as \( \nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta) \) for any \( \eta \in \Omega \). For a local function \( f : \Omega \to \mathbb{R} \) we define \( L f : \Omega \to \mathbb{R} \) as

\[
L f(\eta) = \sum_{x,y} p(y-x) \eta(x)(1 - \eta(y)) \nabla_{x,y} f(\eta) \quad \text{for any } \eta \in \Omega.
\]

Since \( f \) is a local function, we notice that the sum above has a finite number of non-zero entries, and, in particular, \( L f \) is well defined. The linear operator \( L \) defined in this way turns out to be closable with respect to the uniform topology on the space \( \mathcal{C}(\Omega) \) of continuous functions \( f : \Omega \to \mathbb{R} \). Moreover, its closure (also denoted by \( L \)) turns out to be the generator of the process \( \{\eta_t : t \geq 0\} \).

For \( \rho \in [0, 1] \), let \( \mu_{\rho} \) be the probability measure in \( \Omega \) given by

\[
\mu_{\rho}\{\eta \in \Omega : \eta(x_1) = 1, \ldots, \eta(x_t) = 1\} = \rho^t
\]

for any finite collection \( \{x_1, \ldots, x_t\} \) of sites in \( \mathbb{Z} \). The measures \( \{\mu_{\rho} : \rho \in [0, 1]\} \) are invariant under the evolution of \( \{\eta_t : t \geq 0\} \). If span \( \{x \in \mathbb{Z} : p(x) > 0\} = \mathbb{Z} \) (that is, if \( p(\cdot) \) is irreducible) the measures \( \{\mu_{\rho} : \rho \in [0, 1]\} \) are also ergodic under the evolution of \( \{\eta_t : t \geq 0\} \).

\[\text{From now on, if the set of indices of a sum is not specified, we assume that it is equal to } \mathbb{Z}.\]
2.2. Random walks with long jumps. Let $\alpha \in (0,2)$ and $c^+, c^- \geq 0$ be such that $c^+ + c^- > 0$. Let us define $p : \mathbb{Z} \to [0,\infty)$ as

$$p(z) = \frac{c(z)}{|z|^{1+\alpha}}, \text{ where } c(z) = \begin{cases} c^+; & z > 0 \\ 0; & z = 0 \\ c^-; & z < 0. \end{cases} $$

(2.2)

Notice that the condition $\alpha > 0$ ensures that $p(\cdot)$ is a transition rate. Let $\{x(t); t \geq 0\}$ be the continuous-time random walk on $\mathbb{Z}$ with transition rate $p(\cdot)$. The following is a classical result which can be found, for instance, in Chapter 1, Theorem 2.4 of [2].

**Proposition 2.1.** The process $\{x^n_t; t \geq 0\}$ given by

$$x^n(t) = \frac{1}{n} \left( x(tn^\alpha) - m^\alpha_n t \right), \text{ where } m^\alpha_n = \begin{cases} 0; & \alpha < 1 \\ n\sum_{|z| \leq n} xp(x); & \alpha = 1 \\ n^\alpha \sum_{|z|} xp(x); & \alpha > 1 \end{cases}$$

(2.3)

converges in distribution, as $n \to \infty$, to a Markov process $\{Z_t; t \geq 0\}$.

The generator $\mathcal{L}$ of the process $\{Z_t; t \geq 0\}$ is given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}} \frac{c(y)}{|y|^{1+\alpha}} \left( f(x+y) - f(x) - \theta^\alpha(y) f'(x) \right) dy,$$

(2.4)

where

$$\theta^\alpha(y) = \begin{cases} 0; & \alpha < 1 \\ y; & \alpha = 1 \\ y \mathbf{1}(|y| \leq 1); & \alpha > 1. \end{cases}$$

(2.5)

Notice that the generator of the process $\{x^n_t; t \geq 0\}$ is given by

$$\mathcal{L}_n f \left( \frac{x}{n} \right) = n^{\alpha} \sum_y p(y) \left( f \left( \frac{x+y}{n} \right) - f \left( \frac{x}{n} \right) \right) - \frac{m^\alpha_n}{n} f' \left( \frac{x}{n} \right).$$

(2.6)

The generator acts on functions $f : \mathbb{R} \to \mathbb{R}$. We have denoted real numbers as $\frac{x}{n}$ to emphasize that, aside from a constant drift, the operator $\mathcal{L}_n$ is discrete in nature. When $c^+ = c^-$, that is, when the transition rate $p(\cdot)$ is symmetric, the operator $\mathcal{L}$ is a constant multiple of the fractional Laplacian $-(-\Delta)^{\alpha/2}$.

Another, more analytical, way to face a result like the one in Proposition 2.1 is through the convergence of generators. In fact, we have the following result.

**Proposition 2.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $C^2(\mathbb{R})$. Then,

$$\lim_{n \to \infty} \sup_{x \in \mathbb{Z}} \left| \mathcal{L}_n f \left( \frac{x}{n} \right) - \mathcal{L} f \left( \frac{x}{n} \right) \right| = 0,$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_x \left| \mathcal{L}_n f \left( \frac{x}{n} \right) - \mathcal{L} f \left( \frac{x}{n} \right) \right| = 0.$$  

(2.7)

This result is classical and a proof may be found on Appendix A. For $x \in \mathbb{Z}$, let us define the symmetric part $s(x)$ and the antisymmetric part $a(x)$ of $p(x)$ as

$$s(x) = \frac{1}{2} \left( p(x) + p(-x) \right), \quad a(x) = \frac{1}{2} \left( p(x) - p(-x) \right).$$

(2.7)

Let us notice that

$$s(x) = \frac{c^+ + c^-}{2|x|^{1+\alpha}}, \quad a(x) = \frac{c^+ - c^-}{2|x|^{1+\alpha}} \text{ sgn}(x).$$
An important functional associated to the operator $\mathcal{L}$ is the Dirichlet form defined as
\[
\mathcal{E}(f) = -\int_{\mathbb{R}} f(x) \mathcal{L} f(x) dx = \frac{c^+ + c^-}{4} \iint_{\mathbb{R}^2} \frac{(f(y) - f(x))^2}{|y - x|^{1+\alpha}} dxdy.
\] (2.8)

The discrete counterpart of this functional is given by
\[
\mathcal{E}_n(f) = \frac{n^{\alpha-1}}{2} \sum_{xy} s(y - x) (f(x_n) - f(y_n))^2.
\] (2.9)

For that purpose notice that the previous sum is the Riemann sum of the integral $\mathcal{E}(f)$, since it can be written as
\[
\mathcal{E}_n(f) = \frac{c^+ + c^-}{4n^2} \sum_{xy} \frac{n^{1+\alpha}}{|y - x|^{1+\alpha}} (f(x_n) - f(y_n))^2.
\]

We have the following

**Proposition 2.3.** Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then,
\[
\lim_{n \to \infty} \mathcal{E}_n(f) = \mathcal{E}(f).
\]

This proposition is a simple consequence of Proposition 2.2 or simply by noticing the limit of the Riemann sum to the double integral.

2.3. The spectral gap inequality. A classical problem in the theory of Markov chains is the study of the time that the chain needs to reach the equilibrium. In the case of a (continuous time) finite state ergodic Markov chain it is known that the convergence to equilibrium is exponentially fast. Therefore the relevant question is the exponential rate at which this happens. Let $\{x(t); t \geq 0\}$ be an ergodic Markov chain on a finite state space $E$. Let $\mu$ be its unique invariant measure. For $f : E \to \mathbb{R}$ and $x \in E$, let $P_tf(x) = \mathbb{E}[f(x(t)) | x(0) = x]$. Let $\langle \cdot \rangle_\mu$ denote the expectation with respect to $\mu$. Then we define
\[
\lambda = -\sup_{f : E \to \mathbb{R}} \limsup_{t \to \infty} \frac{1}{t} \log(\|P_tf - \langle f \rangle_\mu\|_{L^2(\mu)}).
\]

The number $1/\lambda$ is known as the relaxation time of the chain $\{x(t); t \geq 0\}$. In the case on which the chain $\{x(t); t \geq 0\}$ is reversible with respect to $\mu$, the number $\lambda$ is equal to the spectral gap of the generator $A$ of the chain $\{x(t); t \geq 0\}$, that is, the absolute value of the largest non-zero eigenvalue of $A$. In that case, we have the variational formula
\[
\lambda^{-1} = \sup_{\langle f \rangle_\mu = 0} \frac{(f^2)_\mu}{(-Af)_\mu}.
\] (2.10)

When the chain is not reversible, this variational formula provides an upper bound for $\lambda^{-1}$. For this reason, a natural question in the theory of Markov chains is to estimate the spectral gap of a Markov chain, or of a family of Markov chains of increasing complexity.

For the symmetric simple random walk on $\{1, \ldots, n\}$ it is well known that $\lambda^{-1} = O(n^2)$. It turns out that this property of the random walk over finite intervals, by means of a computation of Nash type, allows one to show that in the case of the symmetric simple random walk on $\mathbb{Z}$,
\[
\|P_tf - \langle f \rangle_\mu\|_{L^2(\mu)} = o\left(\frac{1}{t}\right) \text{ for any } a < \frac{1}{2},
\]
and therefore a sharp upper bound on the spectral gap of finite-state Markov chains gives valuable information even in the case of chains on infinite state spaces.
Let us write \( \Lambda_\ell = \{1, \ldots, \ell\} \). In the case of random walks with long jumps we have the following result:

**Proposition 2.4.** Let \( p(\cdot) \) be given by (2.2). There exists \( \kappa > 0 \) such that
\[
\sum_{x \in \Lambda_\ell} f(x)^2 \leq \kappa \ell^\alpha \sum_{x,y \in \Lambda_\ell} p(y-x)(f(y) - f(x))^2
\]  
(2.11)
for any \( \ell \in \mathbb{N} \) and any \( f : \Lambda_\ell \rightarrow \mathbb{R} \) such that
\[
\sum_{x \in \Lambda_\ell} f(x) = 0.
\]  
(2.12)

**Remark 2.5.** This proposition is telling us that the spectral gap of the Markov chain \( \{x(t) : t \geq 0\} \), restricted to the interval \( \Lambda_\ell \), is bounded from below by \( \frac{1}{\kappa \ell^\alpha} \). In addition, pairing together the two terms involving \( x \) and \( y \), we see that only the behavior of the symmetric part of \( p(\cdot) \) is relevant for this proposition.

The proof of this proposition is in fact very simple. For that purpose notice that
\[
\sum_{x,y \in \Lambda_\ell} p(y-x)(f(y) - f(x))^2 = \sum_{x,y \in \Lambda_\ell} 2s(y-x)(f(y) - f(x))^2.
\]  
(2.13)
To conclude, use the fact that for \( f \) satisfying (2.12), it holds:
\[
\sum_{x,y \in \Lambda_\ell} (f(y) - f(x))^2 = 2\ell \sum_{x \in \Lambda_\ell} f(x)^2,
\]
together with \( s(y-x) \geq \frac{c}{2\ell^\alpha} \) for any \( x, y \in \Lambda_\ell \).

As a corollary of Proposition 2.4 we can obtain a lower bound for the spectral gap of the exclusion process with transition rate \( p(\cdot) \):

**Corollary 2.6.** Let \( p(\cdot) \) be defined by (2.2). Let \( f : \Omega \rightarrow \mathbb{R} \) be a local function with \( \text{supp}(f) \subseteq \Lambda_\ell \). Assume that \( \int f d\mu_\sigma = 0 \) for any \( \sigma \in [0,1] \). Then,
\[
\int f^2 d\mu_\sigma \leq \kappa \ell^\alpha \sum_{x,y \in \Lambda_\ell} p(y-x) \int (\nabla_x f)^2 d\mu_\sigma
\]
for any \( \sigma \in [0,1] \).

The simplest way to prove this corollary is by means of the Aldous’ conjecture, proved in [16], which says that the spectral gap of an exclusion process with symmetric rates is equal to the spectral gap of the random walk with the same rates. Another proof using a comparison principle can be found in [16].

2.4. **The Ornstein-Uhlenbeck process.** Let \( \mathcal{C}^\infty_c(\mathbb{R}) \) be the set of infinitely differentiable functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) with compact support. For \( f \in \mathcal{C}^\infty_c(\mathbb{R}) \) and \( k, \ell \in \mathbb{N} \), we define
\[
\|f\|_{k,\ell,\infty} = \sup_{x \in \mathbb{R}} (1 + x^2)^{k/2} |f^{(\ell)}(x)|.
\]
The Schwartz space of test functions is defined as the closure of \( \mathcal{C}^\infty_c(\mathbb{R}) \) with respect to the metric
\[
d(f,g) = \sum_{k,\ell} \frac{1}{2^{k+\ell}} \min \{1, \|f-g\|_{k,\ell,\infty}\}.
\]
This space is denoted by \( \mathcal{S}(\mathbb{R}) \) and it coincides with the space of infinitely differentiable functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \|f-g\|_{k,\ell,\infty} < +\infty \) for any \( k, \ell \in \mathbb{N} \). The space \( \mathcal{S}'(\mathbb{R}) \) of

\[5\] We write \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \).
tempered distributions} is defined as the topological dual of \( \mathcal{S}(\mathbb{R}) \). We will consider in \( \mathcal{S}'(\mathbb{R}) \) the weak-* topology. We denote by

\[
\|f\| = \left( \int_{\mathbb{R}} f(x)^2 \, dx \right)^{1/2}
\]

the \( L^2(\mathbb{R}) \)-norm of \( f \) and we denote by \( \langle f, g \rangle = \int f(x)g(x) \, dx \) the inner product between \( f \) and \( g \) in \( L^2(\mathbb{R}) \).

One of the simplest examples of \( \mathcal{S}'(\mathbb{R}) \)-valued random variables is the so-called white noise. We say that an \( \mathcal{S}'(\mathbb{R}) \)-valued random variable \( \omega \) is a white noise of variance \( \chi \) if for any \( f \in \mathcal{S}(\mathbb{R}) \) the real-valued random variable \( \omega(f) \) has a Gaussian distribution of mean zero and variance \( \chi \|f\|^2 \).

Let \( T > 0 \) be a fixed number. This number \( T \) will be fixed up to the end of these notes. For a given topological space \( E \) we denote by \( \mathcal{C}([0,T];E) \) the space of continuous functions from \([0,T] \) to \( E \) and by \( \mathcal{D}([0,T];E) \) the space of càdlàg trajectories from \([0,T] \) to \( E \).

We say that an \( \mathcal{S}'(\mathbb{R}) \)-valued process \( \{ \mathcal{W}_i; t \in [0,T] \} \) is a (standard) Brownian motion if for any function \( f \in \mathcal{S}(\mathbb{R}) \) the real-valued process \( \{ \mathcal{W}_i(f); t \in [0,T] \} \) is a Brownian motion of variance \( \|f\|^2 \). For more details about the construction of this and other distribution-valued processes we refer to \cite{25}.

Let us recall the definition of the operator \( \mathcal{L} \) given in \cite{24}. We notice that \( \mathcal{L} \) is determined by the constants \( c^+, c^- \) and \( \alpha \). We will use the notation \( \mathcal{L}(c^+), c^-; \alpha \) whenever we need to stress this dependence. Let \( \mathcal{L}^* \) be the adjoint of \( \mathcal{L} \) in \( L^2(\mathbb{R}) \) and let \( \mathcal{I} = \frac{1}{2}(\mathcal{L} + \mathcal{L}^*) \) be the symmetric part of \( \mathcal{L} \). We notice that

\[
\mathcal{L}^* = \mathcal{L}(c^-, c^+; \alpha) \quad \text{and} \quad \mathcal{I} = \mathcal{L}(\frac{c^+ - c^-}{2}, \frac{c^+ + c^-}{2}; \alpha).
\]

Now we want to define what we understand as a stationary solution of the infinite-dimensional Ornstein-Uhlenbeck equation

\[
d\mathcal{W}_t = \mathcal{L}^* \mathcal{W}_t \, dt + \sqrt{2\chi}(-\mathcal{I} \mathcal{W}_t) \, d\mathcal{W}_t, \tag{2.14}
\]

where \( \{ \mathcal{W}_i; t \in [0,T] \} \) is an \( \mathcal{S}'(\mathbb{R}) \)-valued Brownian motion and \( \chi > 0 \) is fixed. A first naïve definition could be the following. We say that an \( \mathcal{S}'(\mathbb{R}) \)-valued process \( \{ \mathcal{Y}_i; t \in [0,T] \} \) is a solution of the martingale problem associated to \( \mathcal{L} \) if for any differentiable trajectory \( f : [0,T] \to \mathcal{S}(\mathbb{R}) \) the process

\[
\mathcal{Y}_i(f_t) - \mathcal{Y}_0(f_0) - \int_0^t \mathcal{Y}_s((\partial_s + \mathcal{L}) f_s) \, ds
\]

is a martingale of quadratic variation

\[
2\chi \int_0^t \langle f_s, -\mathcal{I} f_s \rangle \, ds
\]

with respect to the natural filtration associated to \( \{ \mathcal{Y}_i; t \in [0,T] \} \). This formulation has a serious problem: for general test functions \( f \), \( \mathcal{L} f \) does not belong to \( \mathcal{S}(\mathbb{R}) \) and therefore \( \mathcal{Y}_i(\mathcal{L} f) \) is not defined. This is not just a problem of choosing test functions such that \( \mathcal{L} f \in \mathcal{S}(\mathbb{R}) \). The solution passes through the following property:

**Proposition 2.7.** The operator \( \mathcal{L} : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R}) \) is continuous. Moreover, \( \mathcal{L} f \) is bounded and infinitely differentiable for any \( f \in \mathcal{S}(\mathbb{R}) \).

Notice that, by the definition of \( \mathcal{L} \) given in \cite{24}, for any \( f \in \mathcal{S}(\mathbb{R}) \), \( \mathcal{L} f' = (\mathcal{L} f)' \) and in particular, from the previous proposition, \( (\mathcal{L} f)^{(\ell)} \) is bounded for any \( \ell \in \mathbb{N}_0 \).
Definition 2.8. We say that an $\mathcal{F}^t(\mathbb{R})$-valued process $\{\mathcal{Y}_t ; t \in [0,T]\}$ defined on some probability space $(X, \mathcal{F}, P)$ is stationary if for any $t \in [0,T]$ the $\mathcal{F}^t(\mathbb{R})$-valued random variable $\mathcal{Y}_t$ is a white noise of variance $\chi$.

The constant $\chi$ above will be the same appearing in (2.14). An important property of a stationary process is that $\mathcal{Y}(f)$ can be extended, by continuity, to any $f \in L^2(\mathbb{R})$. In particular, for any $f \in \mathcal{F}(\mathbb{R})$, the random variable $\mathcal{Y}(\mathcal{L} f)$ makes sense by Proposition 2.7. In a more precise way, let $\psi \in \mathcal{F}(\mathbb{R})$ be given by $\psi(x) = e^{-x^2}$ for any $x \in \mathbb{R}$ and define $\psi_{\epsilon} \in \mathcal{F}(\mathbb{R})$ as $\psi_{\epsilon}(x) = \psi(\epsilon x)$ for any $x \in \mathbb{R}$. Then $\psi_{\epsilon} \mathcal{L} f \in \mathcal{F}(\mathbb{R})$ for any $\epsilon > 0$ and any $f \in \mathcal{F}(\mathbb{R})$. Moreover, $\psi_{\epsilon} \mathcal{L} f \to \mathcal{L} f$ in $L^2(\mathbb{R})$, as $\epsilon \to 0$, from where we conclude that $\mathcal{Y}(\psi_{\epsilon} \mathcal{L} f)$ converges in $L^2(P)$, as $\epsilon \to 0$, to a random variable which we call $\mathcal{Y}(\mathcal{L} f)$.

In order to give rigorous meaning to the Ornstein-Uhlenbeck equation (2.14) in a proper sense, we need to define the following object:

Lemma 2.9. Let $\{\mathcal{Y}_t ; t \in [0,T]\}$ be a stationary process. Let $f : [0,T] \to \mathcal{F}(\mathbb{R})$ be differentiable. Then the process $\{\mathcal{A}(f) : t \in [0,T]\}$ given by

$$\mathcal{A}(f) = \lim_{\epsilon \to 0} \int_0^t \mathcal{Y}(\psi_{\epsilon} \mathcal{L} f_s)ds$$

is well defined.

Proof. It is enough to observe that $\psi_{\epsilon} \mathcal{L} f_s \to \mathcal{L} f_s$, as $\epsilon \to 0$, in $L^2(\mathbb{R})$, uniformly in $s$ and to notice that $\mathcal{Y}$ is a linear functional and a white noise with variance $\chi$.

The previous lemma explains how to define the integral term on the martingale problem associated to the equation (2.14). Let $\{\mathcal{Y}_t ; t \in [0,T]\}$ be a stationary process and let $f : [0,T] \to \mathcal{F}(\mathbb{R})$ be differentiable. We define

$$\int_0^t \mathcal{Y}(\mathcal{L} f_s)ds = \mathcal{A}(f).$$

We say that a stationary process $\{\mathcal{Y}_t ; t \in [0,T]\}$ is a stationary solution of equation (2.14) if for any differentiable function $f : [0,T] \to \mathcal{F}(\mathbb{R})$ the process

$$\mathcal{Y}(f_t) - \mathcal{Y}(f_0) = \int_0^t \mathcal{Y}(\partial_s f_s + \mathcal{L} f_s)ds$$

is a continuous martingale of quadratic variation

$$2 \chi \int_0^t \langle f_s, -\mathcal{A} f_s \rangle ds.$$

Notice that $\langle f_s, -\mathcal{A} f_s \rangle = \mathcal{C}(f_s)$, where $\mathcal{C}(\cdot)$ is defined in (2.8). The following proposition explains in which sense the stationary solutions of (2.14) are unique.

Proposition 2.10. Two stationary solutions of (2.14) have the same distribution.

The proof of this proposition is standard, and for completeness we have included it in Appendix B.

---

6 This property assures us that the application $f \mapsto \mathcal{Y}(f)$ from $\mathcal{F}(\mathbb{R}) \subset L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ is uniformly continuous, and equicontinuous on $t$. Any other property that would ensure this equicontinuity would serve as a substitute to stationarity; nonetheless stationarity will be a consequence of other hypotheses needed to prove our main results, so we will not give too much attention to this point.

7 Notice that $\mathcal{Y}(\mathcal{L} f)$ is well defined up to a set of null probability. This set depends on the choice of the function $f$ and therefore we can not think about $\mathcal{Y}(\mathcal{L} f)$ as a distribution-valued random variable.
2.5. **The density fluctuation field.** Let \( p(\cdot) \) be given by (2.2) and let \( \rho \in (0, 1) \). The density \( \rho \) and the transition rate \( p(\cdot) \) will be fixed from now on and up to the end of these notes. Let \( \{\eta_t; t \geq 0\} \) be an exclusion process with jump rate \( p(\cdot) \) and initial distribution \( \mu_0 \). Since \( \mu_0 \) is invariant, \( \eta_t \) has distribution \( \mu_t \) for any \( t \geq 0 \) and in particular \( E_{\mu_t}[\eta_t(x)] = \rho \) for any \( t \geq 0 \) and any \( x \in \mathbb{Z} \). Let \( n \in \mathbb{N} \) be a scaling parameter. We define \( \eta^n_t = \eta_{tn} \) for \( t \in [0, T] \) and \( n \in \mathbb{N} \). We call \( \{\eta^n_t; t \in [0, T]\} \) the rescaled process. We will use the notation 
\[
\tilde{\eta}^n_t(x) = \eta^n_t(x) - \rho.
\]
We denote by \( \mathbb{P}_n \) the distribution on \( \mathcal{D}([0, T]; \Omega) \) of \( \{\eta^n_t; t \in [0, T]\} \) starting from \( \mu_0 \) and we denote by \( \mathbb{P}_n \) the expectation with respect to \( \mathbb{P}_n \). The density fluctuation field is defined as the \( \mathcal{S}^1(\mathbb{R}) \)-valued process \( \{\mathbb{Y}^n_t; t \in [0, T]\} \) given by 
\[
\mathbb{Y}^n_t(f) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \tilde{\eta}^n_t(x)f\left(\frac{x-\sqrt{1-2\rho}(\sqrt{n}t)}{n}\right)
\]
for any \( t \in [0, T] \), any \( n \in \mathbb{N} \) and any \( f \in \mathcal{S}^1(\mathbb{R}) \). Notice the Galilean transformation embedded into this definition. Recall the definition of the constant \( m_n^\alpha \) given in (2.3). The factor \( (1-2\rho)m_n^\alpha \) is the characteristic velocity of the process \( \{\eta^n_t; t \in [0, T]\} \). For this reason we say that we observe the fluctuations on Lagrangian coordinates.

The main objective of these notes is to identify the limit, as \( n \to \infty \), of the fluctuation field \( \{\mathbb{Y}^n_t; t \in [0, T]\} \). We have restricted ourselves to a finite size time window in order to avoid uninteresting topological considerations. Notice that the process \( \{\mathbb{Y}^n_t; t \in [0, T]\} \) has trajectories in \( \mathcal{D}([0, T]; \mathcal{S}^1(\mathbb{R})) \). Notice as well that for any \( f \in \mathcal{S}^1(\mathbb{R}) \) the real-valued random variable \( \mathbb{Y}^n_t(f) \) converges in distribution, as \( n \to \infty \), to a Gaussian distribution of mean zero and variance \( \rho(1-\rho)\|f\|^2 \). In other words, for any \( t \in [0, T] \) the sequence \( \{\mathbb{Y}^n_t\}_{n \in \mathbb{N}} \) converges in distribution, as \( n \to \infty \), to a white noise of variance \( \rho(1-\rho) \). Notice that \( \mathbb{Y}^n_t \) can be understood as a random signed measure. However, the white noise can not be constructed as a random measure, which makes more appropriate to think about \( \mathbb{Y}^n_t \) as a random distribution.

2.5.1. **The case \( \alpha < 3/2 \): the Ornstein-Uhlenbeck equation.** Let \( c^+, c^- \) be the constants associated to the transition rate \( p(\cdot) \) and let \( \mathcal{L}^\rho \) be the operator given by 
\[
\mathcal{L}^\rho f(x) = \int_{\mathbb{R}} \frac{c^\rho(y)}{|y|^{\alpha+1}} \left( f(x+y) - f(x) - \theta^\alpha(y)f'(x) \right) dy,
\]
where 
\[
c^\rho(x) = \begin{cases} 
c^+(1-\rho) + c^-\rho; & x \geq 0 \\
c^+\rho + c^-(1-\rho); & x < 0.
\end{cases}
\]
In other words, \( \mathcal{L}^\rho = \mathcal{L}(c^+(1-\rho) + c^-\rho; c^+\rho + c^-(1-\rho); \alpha) \). Notice that \( (\mathcal{L}^\rho)^* = \mathcal{L}^{1-\rho} \) and that the symmetric part of \( \mathcal{L}^\rho \) is equal to \( \mathcal{L}^{1/2} \). In particular, the symmetric part of \( \mathcal{L}^\rho \) does not depend on \( \rho \).

Now we have at our disposal all the definitions needed to state the first main result of these notes.

**Theorem 2.11.** Let \( p(\cdot) \) be as in (2.2). Assume that \( \alpha < 3/2 \) and that \( \eta^n_t \) has distribution \( \mu_0 \). Then, the sequence of processes \( \{\mathbb{Y}^n_t; t \in [0, T]\} \) converges in distribution, as \( n \to \infty \), with respect to the \( J_1 \)-Skorohod topology of \( \mathcal{D}([0, T]; \mathcal{S}^1(\mathbb{R})) \) to the stationary solution of the infinite-dimensional Ornstein-Uhlenbeck process given by 
\[
d\mathbb{Y}_t = (\mathcal{L}^\rho)^* \mathbb{Y}_t dt + \sqrt{2\rho(1-\rho)}(-\mathcal{L}^{1/2})d\mathbb{W}_t,
\]
where \( \mathcal{Y}; t \in [0, T] \) is an \( \mathcal{F}^t(\mathbb{R}) \)-valued Brownian motion.

2.5.2. The case \( \alpha = 3/2 \): the fractional Burgers equation. The Galilean transformation used in (2.15) has as a consequence that in the limit equation (2.17), in spite of the transition rate \( p(\cdot) \) being asymmetric, there is no transport term. It turns out that when \( \alpha \) is exactly equal to \( 3/2 \), the second-order correction of the transport term of the dynamics, non-linear in nature, has the same strength that the linear part of the dynamics. In this case, the limiting process corresponds to what we call the fractional Burgers equation, introduced in [13] in the case of the circle as spatial state. To define what we understand as an energy solution of the fractional Burgers equation, we need to introduce various definitions.

**Definition 2.12.** We say that an \( \mathcal{F}^t(\mathbb{R}) \)-valued process \( \{\mathcal{Y}; t \in [0, T]\} \) is uniformly stochastically continuous in \( L^2(\mathbb{R}) \) (USC) if there exists a finite constant \( K_0 \) such that

\[
E[\mathcal{Y}(f)^2] \leq K_0 \|f\|^2 \tag{2.18}
\]

for any \( f \in \mathcal{F}(\mathbb{R}) \) and any \( t \in [0, T] \).

The USC property is satisfied by a stationary process, and the stationary case is the only one that will be considered in these notes. Notice that USC is a static property, in the sense that involves only one time instant. Observe also that the USC property allows to apply Lemma [2.9] and therefore for any process satisfying USC, the integral

\[
\int_0^T \mathcal{Y}(t) \mathcal{L}^t f \, ds
\]

is well defined for any differentiable trajectory \( f : [0, T] \to \mathcal{F}(\mathbb{R}) \).

Now we will describe a property that involves the time evolution of the process \( \{\mathcal{Y}; t \in [0, T]\} \). Let \( \{t_\varepsilon; \varepsilon \in (0, 1)\} \) be an approximation of the identity. An example is

\[
t_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/2\varepsilon^2},
\]

or in general \( t_\varepsilon(x) = \frac{1}{\varepsilon^2} h(t/\varepsilon) \), where \( h \in \mathcal{F}(\mathbb{R}) \) is positive and \( \int_{\mathbb{R}} h(x) dx = 1 \). If the process \( \{\mathcal{Y}; t \in [0, T]\} \) is USC, the function \( h \) can even be in \( L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \) instead of \( \mathcal{F}(\mathbb{R}) \).

**Definition 2.13.** Let \( \{\mathcal{Y}; t \in [0, T]\} \) be a given process and let us define for \( \varepsilon \in (0, 1) \), \( s < t \in [0, T] \) and \( f \in \mathcal{F}(\mathbb{R}) \),

\[
\mathcal{A}^\varepsilon_{s,t}(f) = \int_s^t \int_{\mathbb{R}} \mathcal{Y}(s) * t_\varepsilon(x)^2 f'(x) dx ds'.
\tag{2.19}
\]

We say that \( \{\mathcal{Y}; t \in [0, T]\} \) satisfies an energy estimate (EE) if there exist \( \kappa_0 > 0, \beta \in (0, 1) \) such that

\[
E \left[ (\mathcal{A}^\varepsilon_{s,t}(f) - \mathcal{A}^\varepsilon_{s,s}(f))^2 \right] \leq \kappa_0 \varepsilon (t-s)^\beta \|f'\|^2 \tag{2.20}
\]

for any \( f \in \mathcal{F}(\mathbb{R}) \), any \( 0 < \beta < \varepsilon < 1 \) and any \( 0 \leq s < t \leq T \).

Notice that the energy estimate implies the existence of the limit

\[
\mathcal{A}^\varepsilon_{s,s}(f) = \lim_{\varepsilon \to 0} \mathcal{A}^\varepsilon_{s,s}(f)
\tag{2.21}
\]

in \( L^2(\mathbb{P}) \) for any \( 0 \leq s < t \leq T \) and any \( f \in \mathcal{F}(\mathbb{R}) \). In fact, we can say more about this limit process.

**Proposition 2.14.** Let \( \{\mathcal{Y}; t \in [0, T]\} \) be a process satisfying (2.18) and (2.20) (that is, \( \{\mathcal{Y}; t \in [0, T]\} \) is a USC process satisfying an energy estimate). Let \( \{\mathcal{A}^\varepsilon_{s,t}(f); s < t \in [0, T]\} \) be the random variables obtained in (2.21). Then, there exists an \( \mathcal{F}(\mathbb{R}) \)-valued process \( \{\mathcal{A}; t \in [0, T]\} \) with continuous trajectories such that
i) There exists a finite constant $C$ such that for any $0 \leq s < t \leq T$ and any $f \in \mathcal{S}(\mathbb{R})$, 
\[ E \left[ \left( \mathcal{A}_t(f) - \mathcal{A}_s(f) \right)^2 \right] \leq C|t-s|^{1+\beta/2}\|f\|^2 \]  
(2.22)

and in particular $\{ \mathcal{A}_t; t \in [0, T] \}$ is a.s. $\gamma$-Hölder continuous for any $\gamma < \frac{\beta}{4}$.

ii) $\mathcal{A}_{s,t}(f) = \mathcal{A}_t(f) - \mathcal{A}_s(f)$ a.s. for any $f \in \mathcal{S}(\mathbb{R})$ and any $0 \leq s < t \leq T$.

This proposition corresponds to Theorem 2.2 of [9] for the case $\beta = 1$. The proof extends easily to the case $\beta \in (0, 1)$.

Finally, we can define what we understand by an energy solution of the fractional Burgers equation.

**Definition 2.15.** We say that an $\mathcal{S}'(\mathbb{R})$-valued process $\{ \mathcal{Y}_t; t \in [0, T] \}$ is an energy solution of the fractional Burgers equation if
\[
d\mathcal{Y}_t = (\mathcal{L}^p)^\ast \mathcal{Y}_t dt + m\nabla \mathcal{Y}_t^2 dt + \sqrt{2p(1-p)}(-\mathcal{L}^{1/2})d\mathcal{W}_t, \tag{2.23}
\]

where $\{ \mathcal{W}_t; t \in [0, T] \}$ is an $\mathcal{S}'(\mathbb{R})$-valued Brownian motion, if:

a) the process $\{ \mathcal{Y}_t; t \in [0, T] \}$ is USC and satisfies an energy estimate,

b) for any differentiable trajectory $f : [0, T] \to \mathcal{S}(\mathbb{R})$, the process
\[
\mathcal{M}_t(f) = \mathcal{Y}_t(f) - \mathcal{Y}_0(f) - \int_0^t \mathcal{Y}_s((\partial_s + \mathcal{L}^p)f) ds - m\mathcal{A}_t(f) \tag{2.24}
\]
is a continuous martingale of quadratic variation
\[
2p(1-p) \int_0^t \langle f_s, -\mathcal{L}^{1/2}f_s \rangle ds.
\]

If the process $\{ \mathcal{Y}_t; t \in [0, T] \}$ is, in addition, stationary, we say that $\{ \mathcal{Y}_t; t \in [0, T] \}$ is a stationary energy solution of the fractional Burgers equation. This notion of solution was proposed in [9] in the context of the usual KPZ equation (that is, with $\mathcal{L}^p$ replaced by $\Delta$). In [13] it was shown the existence of energy solutions for $\mathcal{L} = -(-\Delta)^{\alpha/2}$ if $\alpha > 1$ and their uniqueness if $\alpha > 9/4$.

The second main result of these notes is the following:

**Theorem 2.16.** Let $p(\cdot)$ be given by (2.2). Let us assume that $\alpha = 3/2$ and that $\eta_0^\alpha$ has distribution $\mu_\alpha$. Then the sequence of processes $\{ \mathcal{Y}_t^\alpha; t \in [0, T] \}_{n \in \mathbb{N}}$ is tight with respect to the $J_1$-Skorohod topology of $\mathcal{S}(\mathcal{S}(\mathbb{R}); \mathcal{S}'(\mathbb{R}))$ and any of its limit points is a stationary energy solution of the fractional Burgers equation (2.23).

**Remark 2.17.** A consequence of this theorem is the existence of energy solutions of (2.24). The method used in [13] restrict ourselves to finite volume.

**Remark 2.18.** In a formal way, equation (2.23) is invariant under the KPZ scaling $1 : 2 : 3$.

**Remark 2.19.** The dependence of $\mathcal{L}^p$ on the density $\rho$ is a new feature, not observed before in the literature.

---

8 The bound (2.22) shows that we can define
\[
\mathcal{A}_t(f) = \lim_{n \to \infty} \sum_{i=0}^{n-1} \frac{1}{n} \left( \mathcal{A}_{s_{i+1}}(f_q) - \mathcal{A}_{s_i}(f_q) \right).
\]

9 The theory of regularity structures of [9] provides a uniqueness criterion for the stochastic Burgers equation (the case $\alpha = 2$) and in principle this criterion could be extended to $\alpha$ strictly larger than $3/2$, at least in finite volume. The case $\alpha = 3/2$, which is the relevant one for these notes, seems to be out of the reach of the current state of this theory.
3. AUXILIARY DEFINITIONS

3.1. The associated martingales. Since the limit processes on Theorems 2.11 and 2.16 are characterized by martingale problems, it is natural to begin defining various martingales related to the processes \( \{ \mathcal{Y}_t^n; t \in [0, T] \} \). Dynkin’s formula tells us that for well behaved functions \( F : [0, T] \times \Omega \to \mathbb{R} \), the process

\[
F(t, \eta^n_t) - F(0, \eta^n_0) - \int_0^t (\partial_s + n^\alpha L) F(s, \eta^n_s)ds
\]

is a martingale, whose quadratic variation is given by

\[
n^\alpha \int_0^t \{LF(s, \eta^n_s)^2 - 2F(s, \eta^n_s)LF(s, \eta^n_s)\} ds.
\]

We will use this formula for functions of the form

\[
\frac{f}{\alpha}
\]

where \( f : [0, T] \to \mathcal{S}(\mathbb{R}) \) is smooth and \( f_1(\frac{x}{n}) = f((1 - 2\rho)n^\alpha \theta) \). Let us define the martingale \( \{ \mathcal{M}^n_t(f); t \in [0, T] \} \) as

\[
\mathcal{M}^n_t(f) = \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t (\partial_s + n^\alpha L) \mathcal{Y}_s^n(f)ds.
\]

The quadratic variation of this martingale is equal to

\[
\mathcal{M}^n_t(f) = \int_0^t n^{\alpha - 1} \sum_{x,y} p(y-x) (\eta^n_s(y) - \eta^n_s(x))^2 (f_1(\frac{x}{n}) - f_1(\frac{y}{n}))^2 ds.
\]

The integral part of \( \mathcal{M}^n_t(f) \) can be written in a more explicit way. For a smooth function \( f : \mathbb{R} \to \mathbb{R} \) we define \( \mathcal{L}_n^\theta f : \mathbb{R} \to \mathbb{R} \) as

\[
\mathcal{L}_n^\theta f(\frac{x}{n}) = n^\alpha \sum_{y \in \mathbb{Z}} (\{((1 - \rho)p(y-x) + \rho p(x-y))(f(\frac{x}{n}) - f(\frac{y}{n}))\}
\]

\[
- (1 - 2\rho)\frac{m^\theta}{n} f'(\frac{x}{n})
\]

A simple computation also shows that

\[
\mathcal{L}_n^\theta f(\frac{x}{n}) = n^\alpha \sum_{y \in \mathbb{Z}} \{((1 - \rho)p(y-x) + \rho p(x-y))(f(\frac{x}{n}) - f(\frac{y}{n})) - \theta^\alpha (\frac{y-x}{n}) f'(\frac{x}{n})\},
\]

where \( \theta^\alpha \) was defined in (2.5). We also define the process \( \{ A_t^n(f); t \geq 0 \} \) as

\[
A_t^n(f) = \int_0^t n^{\alpha - 1/2} \sum_{x,y} p(y-x) \eta^n_s(y) \eta^n_s(x) (f_1(\frac{x}{n}) - f_1(\frac{y}{n})) ds.
\]

By symmetry we can replace \( p(\cdot) \) by \( a(\cdot) \) in this formula. For that purpose write the sum above as twice its half and in one of the parcels exchange \( x \) with \( y \) to have

\[
\frac{1}{2} \sum_{x,y} p(y-x) \eta^n_s(y) \eta^n_s(x) (f_1(\frac{x}{n}) - f_1(\frac{y}{n})) + \frac{1}{2} \sum_{x,y} p(x-y) \eta^n_s(y) \eta^n_s(x) (f_1(\frac{x}{n}) - f_1(\frac{y}{n})) = \sum_{x} a(x-y) \eta^n_s(y) \eta^n_s(x) (f_1(\frac{x}{n}) - f_1(\frac{y}{n}))
\]
We have that
\[
\int_0^t (\partial_x + n^\alpha L) \mathcal{Y}_n^a(f) \, ds = \int_0^t \mathcal{Y}_n^a (\mathcal{L}_n^p f) \, ds - A_n^a(f),
\]
so that
\[
\mathcal{M}_n^a(f) = \mathcal{Y}_n^a(f) - \mathcal{Y}_0^a(f) - \int_0^t \mathcal{Y}_n^a (\mathcal{L}_n^p f) \, ds + A_n^a(f), \tag{3.4}
\]
with
\[
A_n^a(f) = \int_0^t n^{\alpha - 1/2} \sum_{x,y} a(y-x) \bar{\eta}_n^a(y) \bar{\eta}_n^a(x) \left( f_x \left( \frac{x}{n} \right) - f_x \left( \frac{y}{n} \right) \right) \, ds. \tag{3.5}
\]
Equation (3.4) is deduced with details in Appendix C.

Notice that Proposition 2.2 also holds for the operators \( \mathcal{L}_n^p \) and \( \mathcal{L}^p \), therefore \( \mathcal{L}_n^p \) converges to \( \mathcal{L}^p \), as \( n \to \infty \), in the sense described there. We will use indistinctly the symbol \( f \) for a function \( f \in \mathcal{S}(\mathbb{R}) \) and for the function from \([0, T]\) to \( \mathcal{S}(\mathbb{R}) \) with constant value equal to \( f \).

3.2. Tightness. The proof of Theorems 2.11 and 2.16 follows the classical structure of convergence in distribution of stochastic processes. First we prove tightness with respect to the corresponding topologies, then we prove that any limit point is a solution of the corresponding martingale problem. In the case of Theorem 2.11 the uniqueness criterion of Proposition 2.10 allows to conclude the desired result. In the case of Theorem 2.16 the lack of an uniqueness criterion as the one stated in Proposition 2.10 restrict ourselves to convergence through subsequences. In this section we will detail the tightness criteria we will need in order accomplish the first step in the proof of Theorems 2.11 and 2.16. We start with the so-called Mitoma’s criterion:

**Proposition 3.1** Mitoma’s criterion \([24]\). A sequence \( \{\mathcal{Y}_n^a ; t \in [0, T]\}_{n \in \mathbb{N}} \) of stochastic processes with trajectories in \( \mathcal{D}([0, T]; \mathcal{S}'(\mathbb{R})) \) is tight with respect to the J₁-Skorohod topology if, and only if, the sequence of real-valued processes \( \{\mathcal{Y}_n^a(f); t \in [0, T]\}_{n \in \mathbb{N}} \) is tight with respect to the J₁-Skorohod topology of \( \mathcal{D}([0, T]; \mathbb{R}) \) for any \( f \in \mathcal{S}(\mathbb{R}) \).

In other words, Mitoma’s criterion reduces the proof of tightness of distribution-valued processes to the proof of tightness for real-valued processes. Therefore, we need now a tightness criteria for real-valued processes. In the case of martingales, we will use the following convergence criterion (see Theorem 2.1 of \([26]\)):

**Proposition 3.2.** A sequence \( \{M_n^a; t \in [0, T]\}_{n \in \mathbb{N}} \) of square-integrable martingales converges in distribution, with respect to the J₁-Skorohod topology of \( \mathcal{D}([0, T]; \mathbb{R}) \), as \( n \to \infty \), to a Brownian motion of variance \( \sigma^2 \) if:

i) Asymptotically negligible jumps: for any \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq T} |M_n^a - M_n^a| > \varepsilon \right) = 0,
\]

ii) Convergence of quadratic variations: for any \( t \in [0, T] \),
\[
\lim_{n \to \infty} \langle M_n^a \rangle = \sigma^2 t,
\]
in distribution.

If we have access to uniform moment bounds for the sequence of processes we are interested in, the following criterion is very useful:
Proposition 3.3 (Kolmogorov-Centsov’s tightness criterion). A sequence of continuous processes \( \{X^n_t : t \in [0,T]\} \) is tight, with respect to the uniform topology of \( \mathcal{C}([0,T];\mathbb{R}) \), if there exist constants \( K, a, b > 0 \) such that
\[
E \left[ \left| X^n_t - X^n_s \right|^a \right] \leq K|t-s|^{1+b}
\]  
(3.6)
for any \( s, t \in [0,T] \) and any \( n \in \mathbb{N} \). If the processes \( \{X^n_t : t \in [0,T]\} \) are stationary, it is enough to verify that
\[
E \left[ \left| X^n_t \right|^a \right] \leq Kt^{1+b}.
\]
Most of the processes we will consider in these notes are semimartingales. Therefore, it is useful to have a tightness criterion for integral processes:

Proposition 3.4. A sequence of processes of the form \( \left\{ \int_0^t x_n(s) ds ; t \in [0,T] \right\} \) is tight, with respect to the uniform topology in \( \mathcal{C}([0,T];\mathbb{R}) \), if
\[
\lim_{n \to \infty} \sup_{0 \leq s \leq t} E\left[|x_n(s)|^2\right] < +\infty.
\]
This proposition is an immediate application of the Kolmogorov-Centsov’s criterion: it is enough to observe that, by the Cauchy-Schwarz inequality we have that
\[
E \left[ \left( \int_s^t x_n(s') ds' \right)^2 \right] \leq K|t-s|^2.
\]

3.3. Variance of additive functionals of the dynamics. For each \( \sigma \in [0,1] \), let \( L^2(\mu_\sigma) \) be the Hilbert space associated to the measure \( \mu_\sigma \), that is, the space of functions \( f : \Omega \to \mathbb{R} \) such that \( \int f^2 d\mu_\sigma < +\infty \). We denote by \( \langle f, g \rangle_\sigma \) the inner product in \( L^2(\mu_\sigma) \). For \( f \in L^2(\mu_\sigma) \) we define
\[
\|f\|_2 = \sup_{g \text{ local}} \left\{ 2\langle f, g \rangle_\sigma - \langle g, -Lg \rangle_\sigma \right\}.
\]
Notice that \( \|f\|_2 = +\infty \) if \( \int f d\mu_\sigma \neq 0 \). The relevance of this quantity is given by the following inequality:

Proposition 3.5 (Kipnis-Varadhan inequality). Assume that \( \eta_0^\alpha \) has distribution \( \mu_\sigma \) and let \( f : [0,T] \to L^2(\mu_\sigma) \). Then,
\[
E\left[ \left( \sup_{0 \leq s \leq T} \int_0^s f(s, \eta_s) ds \right)^2 \right] \leq k \int_0^T \|f(t, \cdot\rangle_2^2 dt.
\]
This kind of inequality was introduced in \cite{22} in the context of stationary, reversible Markov chains. A proof of this inequality in the version stated above can be found in \cite{3}. In order to make effective use of Proposition 3.5 we need to know how to estimate \( \|f\|_2 \) at least for a class of functions big enough. This is the context of the following proposition:

Proposition 3.6. Let \( m \in \mathbb{N} \) and let \( k_0 < \cdots < k_m \) be a sequence on \( \mathbb{Z} \). Let \( \{f_1, \ldots, f_m\} \) be a sequence of local functions such that \( \text{supp}(f_i) \subseteq \{k_{i-1} + 1, \ldots, k_i\} \) for any \( i \in \{1, \ldots, m\} \). Let us define \( \ell_i = k_i - k_{i-1} \) for \( i = 1, \ldots, m \).
Assume that \( \int f_i d\mu_\sigma = 0 \) for any \( \sigma \in [0,1] \) and any \( 1 \leq i \leq m \). Then, for any \( \sigma \in [0,1] \)
\[
\left\| f_1 + \cdots + f_m \right\|_2 \leq \kappa \sum_{i=1}^m \ell_i \int f_i^2 d\mu_\sigma.
\]
\(^{10}\) Notice that the support of \( f_i \) has at most diameter \( \ell_i \).
When \( p(\cdot) \) is the jump rate of a simple random walk, that is \( \alpha = 2 \), this proposition is exactly Proposition 7 in [9]. In our case, the proof is practically identical to the proof of that proposition, therefore we omitted it.

Combining Propositions 3.5 and 3.6 we obtain the following inequality:

**Proposition 3.7.** Let \( \{f_1, \ldots, f_m\} \) be as in Proposition 3.6. Then, for \( f = f_1 + \ldots + f_m \),

\[
E \left[ \left( \sup_{0 \leq t \leq T} \int_0^t f(s, \eta_s) ds \right)^2 \right] \leq 14 \kappa \int_0^T \sum_{i=1}^m \ell_i^\alpha \int f_i(s, \eta) d\mu_\sigma.
\]

If we take into account the time scale of \( \eta \), then last proposition can be rewritten as

**Proposition 3.8.** Let \( \{f_1, \ldots, f_m\} \) be as in Proposition 3.6. Then, for \( f = f_1 + \ldots + f_m \),

\[
E \left[ \left( \sup_{0 \leq t \leq T} \int_0^t f(s, \eta_n^\ell s) ds \right)^2 \right] \leq 14 \kappa \int_0^T \sum_{i=1}^m \ell_i^\alpha \int f_i(s, \eta) d\mu_\sigma.
\]

Throughout these notes we will use repeatedly Proposition 3.8; and we notice that Propositions 3.5 and 3.6 are needed only to prove this proposition.

### 3.4. Equivalence of ensembles.

For any \( \ell \in \mathbb{N}, \ell \geq 2, \) any \( x \in \mathbb{Z} \) and any \( \eta \in \Omega \) we define

\[
\eta^\ell(x) = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \eta(x + i).
\]

Do not confuse the function \( \eta^\ell(x) : \Omega \to \mathbb{R} \) with the process \( \eta^\ell_n(x) \). We will not use both notations together; the risk of confusion will be minimal. Fix \( \rho \in (0, 1) \) and let us define \( \psi^\ell : \Omega \to \mathbb{R} \) as

\[
\psi^\ell(\eta) = E \left[ \bar{\eta}(x) \bar{\eta}(x + 1) | \eta^\ell(x) \right],
\]

where the conditional expectation is taken with respect to the measure \( \mu_\rho \). An explicit computation shows that

\[
\psi^\ell(\eta) = \frac{\ell}{\ell - 1} \left\{ \left( \bar{\eta}^\ell(x) \right)^2 - \frac{\rho(1 - \rho)}{\ell} \right\} + \frac{2\rho - 1}{\ell - 1} \bar{\eta}^\ell(x),
\]

and in particular

\[
\int \psi^\ell(\eta)^2 d\mu_\rho \leq \frac{c(\rho)}{\ell^2},
\]

\[
\int \left( \psi^\ell(\eta) - (\eta^\ell(x) - \rho)^2 + \frac{\rho(1 - \rho)}{\ell} \right)^2 d\mu_\rho \leq \frac{c(\rho)}{\ell^3}.
\]  

### 4. Proof of Theorems 2.11 and 2.16

The proofs of Theorems 2.11 and 2.16 are not very different among them. The main difference is that we will show that \( A_n^\alpha(f) \) converges to 0, as \( n \to \infty \), if \( \alpha < 3/2 \) and that it is asymptotically equivalent to a functional of the fluctuation field \( \mathcal{Y}_n^\alpha \), if \( \alpha = 3/2 \). We start by proving tightness.
4.1. Tightness.

By Mitoma’s criterion - Proposition 3.1 - in order to prove tightness of \( \{ \mathcal{Y}^n_t; t \in [0, T] \}_{n \in \mathbb{N}} \) it is enough to show tightness of the sequence of real-valued processes \( \{ \mathcal{Y}^n_{t}(f); t \in [0, T] \}_{n \in \mathbb{N}} \) for any \( f \in \mathcal{S}(\mathbb{R}) \). According to (3.4), it is enough to show that the processes

\[
\{ \mathcal{Y}^n_{0}(f) \}_{n \in \mathbb{N}}, \quad \left\{ \int_0^t \mathcal{Y}^n_s(\mathcal{L}^n_0 f) ds; t \in [0, T] \right\}_{n \in \mathbb{N}}, \quad \{ A^n_t(f); t \in [0, T] \}_{n \in \mathbb{N}},
\]

are tight, and that the martingales \( \{ \mathcal{M}^n_t(f); t \in [0, T] \}_{n \in \mathbb{N}} \) are convergent.

We start with \( \mathcal{Y}^n_{0}(f) \). Computing the characteristic function of \( \mathcal{Y}^n_{0}(f) \), we can check that it converges in distribution, as \( n \to \infty \), to a Gaussian law of mean 0 and variance \( \rho (1 - \rho) \| f \|^2 \). In particular, \( \{ \mathcal{Y}^n_{0}(f) \}_{n \in \mathbb{N}} \) is tight.

In order to prove the convergence of the martingales \( \{ \mathcal{M}^n_t(f); t \in [0, T] \}_{n \in \mathbb{N}} \) we will use Proposition 3.2. First we notice that the jumps of \( \mathcal{M}^n_t(f) \) are the same of \( \mathcal{Y}^n_t(f) \); while the other terms in (3.4) are continuous. Therefore,

\[
\sup_{t \in [0, T]} | \mathcal{M}^n_t(f) - \mathcal{M}^n_t(f) | \leq \frac{\| f \|}{\sqrt{n}}
\]

and the jumps of \( \{ \mathcal{M}^n_t(f); t \in [0, T] \} \) are asymptotically negligible. Notice that

\[
\mathbb{E}_n \left[ n^{\alpha - 1} \sum_{x,y} p(y-x) (\eta^n_y - \eta^n_x)^2 (f(\frac{x}{n}) - f(\frac{y}{n}))^2 \right]
\]

\[
= \frac{n^{\alpha - 1}}{2} \sum_{x,y} (p(y-x) + p(x-y)) 2\rho (1 - \rho) (f(\frac{x}{n}) - f(\frac{y}{n}))^2
\]

\[
+ \frac{n^{\alpha - 1}}{2} \sum_{x,y} (p(y-x) + p(x-y)) 2\rho (1 - \rho) (\frac{x}{n} - f(\frac{y}{n}))^2
\]

\[
= n^{\alpha - 1} \sum_{y} s(y-x) 2\rho (1 - \rho) (f(\frac{x}{n}) - f(\frac{y}{n}))^2
\]

\[
= 4\rho (1 - \rho) \mathcal{E}_n(f) \xrightarrow{n \to \infty} 4\rho (1 - \rho) \mathcal{E}(f),
\]

while the corresponding variance is equal to \( 1 \)

\[
c_1(\rho) n^{2\alpha - 2} \sum_{x,y} s(y-x)^2 (f(\frac{x}{n}) - f(\frac{y}{n}))^4 +
\]

\[
+ c_2(\rho) n^{2\alpha - 2} \sum_{x,y} \left( \sum_{y} s(y-x) (f(\frac{x}{n}) - f(\frac{y}{n}))^2 \right)^2.
\]

When multiplied by \( n^2 \), the first sum becomes a Riemann sum of a finite integral, namely, \( \int_{\mathbb{R}^2} \frac{(f(y) - f(x))^4}{|y-x|^{\alpha + 2\alpha}} dxdy \), therefore it is of order \( n^{2\alpha - 5} \), and vanishes as \( n \to \infty \). The second sum above, when multiplied by \( n \), becomes a Riemann sum of a finite integral, namely, \( \int (f(y) - f(x))^2 |y-x|^{\alpha + 2\alpha} dy \), therefore, it is of order \( n^{\alpha - 3} \), and vanishes as \( n \to \infty \). Therefore, we have shown that

\[
\lim_{n \to \infty} \mathbb{E}_n \left[ (\mathcal{M}^n_t(f)) - 4\rho (1 - \rho) \mathcal{E}(f)t \right] = 0.
\]

\[\text{Along these notes, we will denote by } c_1(\rho) \text{ various constants which depend only on } \rho. \text{ The exact value of these constants will not be important; only the fact that they depend only on } \rho.\]
We conclude that the martingales $\{M^n_t(f) : t \in [0,T]\}_{n \in \mathbb{N}}$ converge in distribution, as $n \to \infty$, to a Brownian motion of variance $4\rho(1-\rho)\mathcal{L}(f)$.

The next term on the list is $\{\int_0^T \mathcal{Y}^n_t(\mathcal{L}^D f) dt : t \in [0,T]\}_{n \in \mathbb{N}}$. We will use Proposition 3.4. First, notice that

$$E_n[\mathcal{Y}^n_t(\mathcal{L}^D f)^2] = \frac{(1-\rho)}{n} \sum_x (\mathcal{L}_n^D f(\frac{x}{n}))^2.$$  

By Proposition 2.2, this sum converges to $\rho(1-\rho)\int_\mathbb{R}(\mathcal{L}^D f)^2(x)dx$, and therefore the hypotheses of Proposition 3.4 are satisfied. We notice that Proposition 2.2 is stated for different operators but the same result holds in the cases considered here. Therefore, we conclude that $\{\int_0^T \mathcal{Y}^n_t(\mathcal{L}^D f) dt : t \in [0,T]\}_{n \in \mathbb{N}}$ is tight.

Notice that so far the previous results hold for $\alpha \leq 3/2$.

The really difficult term is $\{A^n_t(f) : t \in [0,T]\}_{n \in \mathbb{N}}$. The problem comes from the fact that the spatial normalization is $n^{a-3/2}$ instead of $n^{-1/2}$. Therefore, for $\alpha > 1$ we need to make efficient use of the time integration in order to show the tightness of this term. For $\alpha < 3/2$ we will see that this term is asymptotically negligible, while for $\alpha = 3/2$ we will show that it is asymptotically equivalent to a function of the density fluctuation field. Since the arguments to prove tightness of $\{A^n_t(f) : t \in [0,T]\}_{n \in \mathbb{N}}$ depend on the regime of $\alpha$ we devote separate sections for them, namely, $\alpha \leq 1$, $\alpha \in (1,3/2)$ and $\alpha = 3/2$.

### 4.1.1. Tightness of $\{A^n_t(f) : t \in [0,T]\}_{n \in \mathbb{N}}$: the case $\alpha \leq 1$.

Recall the definition of $A^n_t(f)$ from (3.3). Notice that

$$E_n\left[\left(\sum_{x,y} a(y-x) \mathcal{Y}^n_t(x) \mathcal{Y}^n_t(y) (f\left(\frac{x}{n}\right) - f\left(\frac{y}{n}\right))\right)^2\right] = c_3(\rho)n^{2\alpha-1} \sum_{x,y} a(y-x)^2 (f\left(\frac{x}{n}\right) - f\left(\frac{y}{n}\right))^2.$$ 

Aside from a factor $\frac{c_3(\rho)}{n}$, this last sum is a Riemann sum of the integral

$$\int_{\mathbb{R}^2} \frac{(f(y) - f(x))^2}{|y-x|^{2+2\alpha}} dxdy.$$ 

This integral is convergent if $\alpha < 1/2$, and therefore the expectation above is of order $O(\frac{1}{n})$. If $\alpha \geq 1/2$, the integral is divergent. The order of divergence of the sum is $n^{2\alpha-1}$ (log$(n)$ for $\alpha = 1/2$). This divergence is easy to guess if we observe that the sum above is actually a Riemann sum for the integral over the set $\{\max\{|x|,|y|\} \geq \frac{1}{n}\}$. Therefore the expectation above is of order $o(1)$ for $\alpha < 1$ and bounded for $\alpha = 1$. In any case, by the compactness criterion of Proposition 3.4, we conclude that

**Lemma 4.1.** For any $\alpha \leq 1$ and any $f \in \mathcal{S}(\mathbb{R})$, the sequence $\{A^n_t(f) : t \in [0,T]\}_{n \in \mathbb{N}}$ is tight.

This ends the proof of tightness of $\{\mathcal{Y}^n_t : t \in [0,T]\}_{n \in \mathbb{N}}$ in the case $\alpha \leq 1$.

### 4.1.2. Tightness of $\{A^n_t(f) : t \in [0,T]\}_{n \in \mathbb{N}}$: the case $1 < \alpha < 1 + \frac{2}{5+\sqrt{33}}$.

Below, unless explicitly stated, we do not assume $1 < \alpha < 1 + \frac{2}{5+\sqrt{33}}$ on the computations made in this section. We start by noticing that refining the computations made in the
two reasons. First, the supports of the functions¯
Proposition 3.8 is. Proposition 3.8 does not apply directly to the integral (4.1) for
the point is that the smaller the support of the functions involved in, the more
can be solved dividing the sum in (4.1) into
As above, to prove last result it is enough to see the previoussum as the Riemann sum of
Define
where
\( R_n(x, y) \) is bounded and smooth. We can verify that
\[
\lim_{n \to \infty} \mathbb{E}_n \left[ \left( n^{\alpha-1/2} \sum_{|y-x| \leq K} a(y-x)(f(\frac{x}{n}) - f(\frac{y}{n})) \hat{\eta}^n(x) \hat{\eta}^n(y) \right)^2 \right] = 0
\]
as soon as \( K_n \gg n^{2\alpha-2} \). For that purpose notice that the previous sum is the Riemann sum of
an integral of order \( \frac{n^{2\alpha-2}}{K_n} \), and we can restrict the sum in the definition of \( A_n^{\alpha}(f) \) to \( |y-x| \leq K_n \); the rest of the sum is tight by Proposition 3.3. In order to simplify the notation, we will
drop the subscript \( n \) from \( K_n \). Notice that \( K_n = o(n) \). Therefore, for \( x, y \in \mathbb{Z} \) such that
\( |y-x| \leq K \) we have that
\[
f(\frac{x}{n}) - f(\frac{y}{n}) = \frac{x-y}{n} f'(\frac{x}{n}) + \frac{(x-y)^2}{n^2} R_n(x, y),
\]
where
\[
\lim_{n \to \infty} \mathbb{E}_n \left[ \left( n^{\alpha-1/2} \sum_{|y-x| \leq K} a(y-x) \hat{\eta}^n(x) \hat{\eta}^n(y) \frac{(y-x)^2 n}{n^2} R_n(x, y) \right)^2 \right] = 0.
\]
As above, to prove last result it is enough to see the previous sum as the Riemann sum of
an integral of order \( \frac{n^{2\alpha-2}}{K_n} \). Since \( K \gg n^{2\alpha-2} \), the expectation vanishes as \( n \to \infty \). Therefore
we only need to prove tightness for
\[
\int_0^t n^{\alpha-3/2} \sum_{|y-x| \leq K} (y-x)a(y-x) \hat{\eta}^n(x) \hat{\eta}^n(y) f'(\frac{x}{n}) ds.
\]
Up to now, the idea was to reduce \( A_n^{\alpha}(f) \) to a sum of variables with the smallest possible
support. The point is that the smaller the support of the functions involved in, the more
effective Proposition 3.8 is. Proposition 3.8 does not apply directly to the integral (4.1) for
two reasons. First, the supports of the functions \( \hat{\eta}^n(x) \hat{\eta}^n(y) \) are intertwined. This problem
can be solved dividing the sum in (4.1) into \( K \) sums of functions with disjoint supports.
And second, the functions \( \hat{\eta}^n(x) \hat{\eta}^n(y) \) do not have mean zero for all invariant measures
\( \mu_\sigma \). A strategy to solve the second problem is to add and subtract the function \( \psi^K_\alpha(\eta) \).
Define \( \mathcal{Z}^n = \{Kz + j; z \in \mathbb{Z}\} \). Then, by the inequality \( (x+y)^2 \leq 4x^2 + 4y^2 \), the variance
of (4.1) is bounded by
\[
2\mathbb{E}_n \left( \int_0^t n^{\alpha-3/2} \sum_{x \in \mathcal{Z}^n} \sum_{j=1}^{K-1} ya(y) \left\{ \hat{\eta}^n(x) \hat{\eta}^n(x+y) - \psi^K_\alpha(\eta^n) \right\} f'(\frac{x}{n}) ds \right)^2
\]
\[
+ 2\mathbb{E}_n \left( \int_0^t n^{\alpha-3/2} \sum_{x \in \mathcal{Z}^n} \sum_{j=1}^{K-1} ya(y) \psi^K_\alpha(\eta^n) f'(\frac{x}{n}) ds \right)^2. \tag{4.2}
\]
By the Cauchy-Schwarz inequality and by (3.8), the second expectation is bounded by
\[
Kr^2 n^{2\alpha-3} \left( \sum_{y=1}^{K-1} ya(y) \right)^2 \sum_x f'(\frac{x}{n})^2 \int \psi^K_\alpha(\eta)^2 \mu_\rho(d\eta) \leq C(f, \rho) \frac{r^2 n^{2\alpha-2}}{K}.
\]
Above we used the fact that \( \sum_{y=1}^{K-1} ya(y) < \infty \), since we are in the regime \( \alpha > 1 \). Using
Proposition 3.8 and by independence, we see that the first expectation in (4.2) is bounded
by
\[ c_4(\rho)tK^{\alpha/\alpha_n}n^{2\alpha - 3} \sum_x f'((\frac{1}{n})^2) \sum_{y=1}^{K-1} y^2 \sigma_y(y)^2 \leq C(f, \rho)K^{1+\alpha}t n^{2\alpha - 1}. \]

Above, we used the fact that \( \sum_{y=1}^{K-1} y^2 \sigma_y(y)^2 < \infty. \) At this point we collect the estimates we have so far on \( K. \) Then we have \( K \gg n^{2\alpha/\alpha}, \) \( K \gg n^2 \alpha^{-2} \) and \( K \ll n^{2\alpha/3}. \) Since \( \alpha > 1 \) we are therefore reduced to \( n^{2\alpha - 2} \ll K \ll n^{2\alpha/3}. \) Now, if \( \alpha < 1 + \frac{2}{3 + \sqrt{33}} \), there exists \( \alpha \) such that
\[ 2\alpha - 2 < \gamma < \frac{2 - \alpha}{1 + \alpha}. \]

Notice that the value \( 1 + \frac{2}{3 + \sqrt{33}} \), comes from the fact that we need to have \( \alpha > 1 \) such that
\[ 2\alpha - 2 < \frac{2 - \alpha}{1 + \alpha}. \] Finally, looking at the bounds \( \frac{1}{n^{2\alpha}} \) and \( \frac{1}{n^{2\alpha - 2}} \), we take \( K_n = n^{5\alpha/3} \) to conclude that the variance of (4.1) is bounded by
\[ C(f, \rho)K^{1+\alpha}t \theta, \]
where \( \theta = \frac{2\alpha^2 - \alpha - 4}{3 + \sqrt{33}}. \) Notice that \( \theta < 0 \) if \( \alpha < 1 + \frac{2}{3 + \sqrt{33}}. \) By the Cauchy-Schwarz inequality and performing similar computations to those before Lemma 4.1, we have a rough bound for the variance of (4.1) as
\[ C(f, \rho)t^2 n^{2\alpha - 2}. \]

Now, the following lemma will be useful.

**Lemma 4.2.** For any \( a, b > 0, \) there exist \( C, \delta, \epsilon > 0 \) such that
\[ \min \{ \frac{l}{m}, l^2 n^b \} \leq \frac{Ct^{1+\delta}}{n^\epsilon}. \]

The proof is elementary and we omit it. Using this lemma we conclude that, for \( 1 < \alpha < 1 + \frac{2}{3 + \sqrt{33}}, \) the variance of (4.1) is bounded by \( C t^{1+\delta} n^{-\epsilon}. \) By Proposition 3.3 we conclude two things. First that \( \{ A_n^\alpha(f); t \in [0, T] \}_{n \in \mathbb{N}} \) is tight and second that any of its limit points are identically null. In order to cross the barrier \( \alpha < 1 + \frac{2}{3 + \sqrt{33}} \), we will need to perform a multiscale analysis.

**4.1.3. Tightness of \( \{ A_n^\alpha(f); t \in [0, T] \}_{n \in \mathbb{N}} \): the case \( 1 + \frac{2}{3 + \sqrt{33}} \leq \alpha < \frac{3}{2}. \)**

In (4.1) we reduced the sum defining \( A_n^\alpha(f) \) to a sum over \(|y - x| \leq K, \) where \( K \gg n^{2\alpha/3}. \) The first term in (4.2) converges to 0 if \( K \ll n^{2\alpha/3}. \) We need to see what can we say about the sum for \( y \) between the two sets: \( K \gg n^{2\alpha/3} \) and \( K \ll n^{2\alpha/3}. \) In order to do that, let \( L < K \) be given, and let us estimate the second moment of
\[ \int_0^t n^{\alpha - 3/2} \sum_{x \in \mathbb{Z}^2} \sum_{j=1}^{K-1} y a(y) \left( \hat{\eta}^n_x(x) \hat{\eta}^n_x(y) - \psi^n_x(\eta^n_x) \right) f'(\frac{1}{n}) ds. \] (4.3)

By Proposition 3.3 the second moment of this expression is bounded above by
\[ c(\rho)Kn^{2\alpha - 3} \sum_x f'(\frac{1}{n})^2 \sum_{y=1}^{K-1} \frac{1}{y^{2\alpha}} \leq \frac{C(f, \rho)K^{1+\alpha}}{n^{2\alpha - 2} L^{2\alpha - 1}}. \] (4.4)
If we take $L = n^T$ and $K = n^T$, this last quantity goes to 0 as soon as
\[ \gamma < \frac{2 - \alpha}{1 + \alpha} \frac{2\alpha - 1}{1 + \alpha}. \]

Now, we notice that if we plug this estimate into (4.4), we bound it by $t$, which is not enough for our purposes. Then we take $\delta > 0$ such that
\[ \delta + \gamma(1 + \alpha) \leq 2 - \alpha + \gamma(2\alpha - 1). \]

Now we get the bound $tn^{-\delta}$ for (4.4), which vanishes as $n \to \infty$. If we think about this inequality as a recurrence, we see that it has an attractive fixed point at $\gamma = 1$. Therefore, we have the following

**Lemma 4.3.** For any $0 < \delta < 2\alpha$ there exists a finite sequence $\{\gamma_0, \gamma_1, \ldots, \gamma_l\}$ such that $\gamma_0 = 0$, $\gamma_l < 1 - \delta$ and
\[ \frac{\delta}{1 + \alpha} + \gamma \leq \frac{2 - \alpha}{1 + \alpha} + \gamma(2\alpha - 1) \]
for any $i = 1, \ldots, l$.

The multiscale analysis of $A_n^\ell(f)$ goes by fixing $0 < \delta < \frac{1}{2\alpha - 1}$ and defining the scales $K_i = K_i^\ell = n_i$ for $i = 1, \ldots, \ell$, where the sequence $\{\gamma_0, \gamma_1, \ldots, \gamma_l\}$ is given by Lemma 4.3. Choosing $L = K_i^\ell$ and $K = K_i^{\ell+1}$, we see that the variance of (4.3) is bounded by $C(f,\rho)tn^{-\delta}$. Let us write $\Psi_i^\ell = \psi_{\alpha\ell}^i$. By the Minkowski inequality and the previous estimate, we have that the variance of
\[ \int_0^t n^{\alpha - 3/2} \sum_{x} \sum_{y=1}^{K_i^{\ell-1}} ya(y)\bar{\eta}_x^n(x)\bar{\eta}_y^n(y) f'(\frac{s}{n}) ds \]
\[ = \int_0^t n^{\alpha - 3/2} \sum_{x} \sum_{y=1}^{K_i^{\ell-1}} ya(y)\left(\frac{\Psi_i^n(x)\bar{\eta}_x^n(y) - \Psi_i^n(\bar{\eta}_x^n)}{\alpha}_y\right) f'(\frac{s}{n}) ds \]
is bounded by
\[ \sum_{i=1}^{\ell} B_i^{n\ell}(f), \]

where
\[ B_i^{n\ell}(f) = \int_0^t n^{\alpha - 3/2} \sum_{x} m^\ell\Psi_i^n(\bar{\eta}_x^n) f'(\frac{s}{n}) ds, \]
and $m^\ell = \sum_{y=K_i^{\ell-1}}^{K_i^{\ell-1}} ya(y)$.

Now we have to bound the variance of
\[ \sum_{i=1}^{\ell} B_i^{n\ell}(f) - \int_0^t n^{\alpha - 3/2} \sum_{x} \sum_{y=1}^{K_i^{\ell-1}} ya(y)\Psi_i^n(\bar{\eta}_x^n) f'(\frac{s}{n}) ds. \]
Since, \( \sum_{y=1}^{m^d-1} y a(y) = \sum_{i=1}^{\ell} m^i \), we rewrite (4.6) as
\[
\sum_{i=1}^{\ell} B_i^m(f) - \int_0^t n^{\alpha-3/2} \sum_{x=1}^{\ell} \sum_{i=1}^{\ell} K_i^{m-1} y a(y) \Psi_i^y(\eta_x^y) f'(\frac{x}{n}) ds
= \int_0^t n^{\alpha-3/2} \sum_{x=1}^{\ell} \sum_{i=1}^{\ell} m^i \Psi_i^y(\eta_x^y) f'(\frac{x}{n}) ds - \int_0^t n^{\alpha-3/2} \sum_{x=1}^{\ell} \sum_{i=1}^{\ell} m^i \Psi_i^y(\eta_x^y) f'(\frac{x}{n}) ds
= \int_0^t n^{\alpha-3/2} \sum_{x=1}^{\ell} \sum_{i=1}^{\ell} m^i \left( \Psi_i^y(\eta_x^y) - \Psi_i^{y+1}(\eta_x^y) \right) f'(\frac{x}{n}) ds
= \int_0^t n^{\alpha-3/2} \sum_{x=1}^{\ell} \sum_{i=1}^{\ell} m^i \sum_{j=1}^{m-1} \left( \Psi_i^j(\eta_x^y) - \Psi_i^{j+1}(\eta_x^y) \right) f'(\frac{x}{n}) ds.
\]
(4.7)

Notice that sum of the constants \( m^i \) is bounded by \( m = \sum_{y=1}^{\infty} y a(y) < \infty \) since \( \alpha > 1 \). At this point we have to compute the variance of
\[
\int_0^t n^{\alpha-3/2} \sum_{x} \left( \Psi_i^y(\eta_x^y) - \Psi_i^{y+1}(\eta_x^y) \right) f'(\frac{x}{n}) ds.
\]
(4.8)

By Proposition 3.8 and by \( (3.8) \), the variance of the previous term is bounded by
\[
C(\rho, f) t n^{2\alpha-2} \frac{(K^{i+1})^{\alpha-1}}{n^{\alpha}} \int \Psi_i^y(x) ds \leq C(f, \rho) t \frac{(K^{i+1})^{\alpha-1}}{(K^{i+1})^{\alpha-1}}.
\]
(4.9)

But this bound is not sufficient for us. Therefore, we will use the multiscale structure introduced in (11) in order to get a better bound for the variance of (4.8), more precisely, we will prove that
\[
E_n \left[ \left( \int_0^t n^{\alpha-3/2} \sum_{x} \left( \Psi_i^y(\eta_x^y) - \Psi_i^{y+1}(\eta_x^y) \right) f'(\frac{x}{n}) ds \right)^2 \right] \leq C(f, \rho) t \frac{(K^{i+1})^{\alpha-1}}{n^{\alpha-1}}.
\]
(4.10)

For that purpose, let \( j \) be fixed and take the sequence of boxes \( \ell_0 = K^j, \ell_1 = 2\ell_0 \) and for \( \rho \geq 2, \ell_p = 2^p \ell_0 \). Suppose that there exists \( m \) sufficiently big such that \( 2^m \ell_0 = K^{i+1} \). Then, performing a telescopic sum, and using the Minkowski inequality together with the previous estimate, the expectation on the left hand side of (4.10) is bounded from above by
\[
C(f, \rho) t \left( \sum_{p=0}^{m} \frac{(2^{p+1}\ell_0)^{\alpha-1}}{n^{\alpha-1}} \right)^2 \leq C(f, \rho) t \frac{(2^m \ell_0)^{\alpha-1}}{n^{\alpha-1}}.
\]
This proves (4.10) for the case \( K^{j+1} = 2^m \ell_0 \). In the other cases, we take \( m \) sufficiently big such that \( 2^m \ell_0 \leq K^{j+1} \leq 2^{m+1} \ell_0 \). Let \( \Psi_i^y = \Psi_i^{2\ell_0} \). Then, by using the inequality \((x+y)^2 \leq 2x^2 + y^2\), the expectation on the left hand side of (4.10) is bounded from above by
\[
2E_n \left[ \left( \int_0^t n^{\alpha-3/2} \sum_{x} \left( \Psi_i^y(\eta_x^y) - \Psi_i^{y+1}(\eta_x^y) \right) f'(\frac{x}{n}) ds \right)^2 \right]
+ 2E_n \left[ \left( \int_0^t n^{\alpha-3/2} \sum_{x} \left( \Psi_i^y(\eta_x^y) - \Psi_i^{y+1}(\eta_x^y) \right) f'(\frac{x}{n}) ds \right)^2 \right]
\]
(4.11)

From the previous estimate, the first expectation in (4.11) is bounded from above by
\[
C(f, \rho) t \frac{(2^m \ell_0)^{\alpha-1}}{n^{\alpha-1}},
\]
while from (4.9) the second expectation is bounded from above by
\[
C(f, \rho) t \frac{(2^{m+1}\ell_0)^{\alpha-1}}{(2^m \ell_0)^{\alpha-1}} \leq C(f, \rho) t \frac{(2^{m+1}\ell_0)^{\alpha-1}}{(2^m \ell_0)^{\alpha-1}}.
\]
Putting together the two previous estimates, the proof of (4.10) ends.

Now we return to (4.7) which, by Fubini’s, can be written as
\[ \int_0^\ell n^{2/3} y \sum_{j=0}^{j-1} \left( \Psi^{K_j} - \Psi^{K_{j+1}} \right) \sum_{i=0}^{i-1} m^i \left( \frac{\bar{y}}{2} \right) ds, \]
and whose variance is bounded by
\[ C(f, \rho) \ell \sum_{j=0}^{j-1} \sum_{i=0}^{i-1} m^i \frac{2 (K_j^{\ell+1})^{\alpha-1}}{n^{2\alpha}} \leq C(f, \rho) \ell^2 m^2 n^{-(3-2\alpha)-2(\alpha-1)}. \]

Last bound is obtained by our choice of \( \gamma \) given in Lemma 4.3. Since \( \alpha \leq 3/2 \), the exponent of \( n \) in this last expression is negative. Summarizing the estimates we have proved so far and writing \( \varepsilon = \delta(\alpha - 1) \), we have just showed that

**Lemma 4.4.** For any \( 0 < \varepsilon < (2 - \alpha)(\alpha - 1) \) there exist \( C = C(f, \rho) \) and \( \ell = \ell(\varepsilon) \) such that
\[ \mathbb{E}_n \left[ \int_0^\ell n^{2/3} \sum_{|y-x| \leq K_n} a(y-x) y \left( \tilde{\Psi}^{K_n} - \Psi^{K_n} \right) \left( \frac{\bar{y}}{2} \right) ds \right] \leq \frac{c t^2}{n^{2\alpha}}, \]
where \( K_n = n^{1-\delta} \) and \( \delta = \frac{\varepsilon}{a-1} \).

Finally, using the Cauchy-Schwarz inequality we see that
\[ \mathbb{E}_n \left[ \int_0^\ell n^{2/3} \sum_{|y-x| \leq K_n} a(y-x) y \left( \tilde{\Psi}^{K_n} - \Psi^{K_n} \right) \left( \frac{\bar{y}}{2} \right) ds \right] \leq c t^2 n^{2\alpha-2} \sum_{i=0}^{i-1} K_i f \left( \frac{\bar{y}}{2} \right) ds \]
\[ \leq C(f, \rho) t^2 n^{2\alpha-3.5}. \]

It is exactly on the last line the only place where we need to assume that \( \alpha < 3/2 \). If \( \alpha < 3/2 \), we have just proved that for any \( 0 < \delta < 3 - 2\alpha \), there exists a constant \( C = C(f, \rho, \delta, T) \) such that the variance of
\[ \int_0^\ell n^{2/3} \sum_{|y-x| \leq K_n} a(y-x) y \left( \tilde{\Psi}^{K_n} - \Psi^{K_n} \right) \left( \frac{\bar{y}}{2} \right) ds \]
is bounded by \( C t n^{-\delta} \). Recall the rough bound \( C t n^{2\alpha-2} \) for the variance of (4.12) obtained in Section 4.1.1. By Lemma 4.2, we conclude that there exist \( C, \varepsilon, \delta > 0 \) such that the variance of (4.12) is bounded by \( C t^{1+\delta} n^{-\varepsilon} \). By Proposition 3.3, we conclude that the sequence \( \{A_t^n; t \in [0, T]\} \) is tight for any \( \alpha < 3/2 \), and moreover any limit point is identically zero.

4.1.4. **Tightness of \( \{A_t^n; t \in [0, T]\} \): the case \( \alpha = 3/2 \).**

In the previous sections we showed, for \( \alpha < 3/2 \), that the sequence of processes \( \{A_t^n; t \in [0, T]\} \) is tight and that any limit point is identically zero. For \( \alpha = 3/2 \), the limit points are given by a non-trivial function of the density of particles and in particular there is no reason to believe that they are identically zero. In this section we will show tightness of
\( \{A^n(f) : t \in [0, T]\}_{n \in \mathbb{N}} \) for \( \alpha = 3/2 \). Notice that in the previous section we showed that \( A^n(f) \) is asymptotically equivalent to

\[
m \int_0^t \sum_x \psi^n_x(\eta^n) f'(\frac{\eta^n}{n}) \, ds \]  \quad (4.13)

where \( K_n = n^{1-\delta} \) for some \( \delta > 0 \) small enough, in the sense that the difference converges to 0 in distribution with respect to the Skorohod topology, as \( n \to \infty \). Therefore, it is enough to prove tightness of this process. By the equivalence of ensembles \( \{K_n(i) \} \) we know that \( \psi^n_x(\eta) \) is well approximated by the square of the number of particles on a box of size \( K_n \) around \( x \). If this box were of size \( \varepsilon n \), then it would be a function of the fluctuation field \( \mathcal{F}^n \). Therefore, our mission now will be to go from a block of size \( n^{1-\delta} \) to a block of size \( \varepsilon n \). This step is what we call the \emph{two-blocks estimate}. The proof we will present here was introduced in \(9\) (see also \(10,12\)). Let \( M \) be given. Then, by Proposition 5.8 we have that

\[
\mathbb{E}_n \left[ \left( \int_0^t \sum_x (\psi^n_x(\eta^n) - \psi^n_{x+1}(\eta^n)) f'(\frac{\eta^n}{n}) \, ds \right)^2 \right] \leq C(f,\rho) t \sqrt{\frac{M}{n}}.
\]

Defining \( M_0 = M \) and \( M_i = 2^i M \) for \( i \in \mathbb{N} \), by writing a telescopic sum, by Minkowski’s inequality and the previous estimate we see that

\[
\mathbb{E}_n \left[ \left( \int_0^t \sum_x (\psi^n_x(\eta^n) - \psi^n_{x+1}(\eta^n)) f'(\frac{\eta^n}{n}) \, ds \right)^2 \right] = \mathbb{E}_n \left[ \left( \int_0^t \sum_x \sum_{i=0}^{n-1} (\psi^n_x(\eta^n) - \psi^n_{x+1}(\eta^n)) f'(\frac{\eta^n}{n}) \, ds \right)^2 \right] \leq C(f,\rho) t \sqrt{\frac{M}{n}}.
\]

Taking \( M = n^{1-\delta} \) and \( M_i = \varepsilon n \) we conclude that

\[
\mathbb{E}_n \left[ \left( \int_0^t \sum_x (\psi^n_x(\eta^n) - \psi^n_{x+1}(\eta^n)) f'(\frac{\eta^n}{n}) \, ds \right)^2 \right] \leq C(f,\rho) t \sqrt{\varepsilon}.
\]  \quad (4.14)

By the Cauchy-Schwarz inequality together with (3.8), we have that

\[
\mathbb{E}_n \left[ \left( \int_0^t \sum_x \psi^n_x(\eta^n)f'(\frac{\eta^n}{n}) \, ds \right)^2 \right] \leq C(f,\rho) t^2 \frac{n}{K} \frac{\varepsilon}{n}\]  \quad (4.15)

for any \( K \in \mathbb{N} \). Choosing \( K = K_n \) and \( K = \varepsilon n \), from the previous estimates, we obtain the bound

\[
\mathbb{E}_n \left[ \left( \int_0^t \sum_x \psi^n_x(\eta^n)f'(\frac{\eta^n}{n}) \, ds \right)^2 \right] \leq C(f,\rho) \min\{t^2 n^{\delta} ; t \sqrt{\varepsilon} + t^2 \varepsilon^{-1}\}.
\]

Notice that above, the first bound comes from (4.15) taking \( K_n = n^{1-\delta} \), the second bound comes from (4.14) and the last bound comes from (4.15) with \( K = n \varepsilon \). If we optimize over \( \varepsilon \) in the second and third bounds, by taking \( \varepsilon = t^\theta \), we see that \( \theta = 2/3 \) and the expectation is bounded from above by \( C(f,\rho) t^{4/3} \). However, for that to hold we have the restriction \( \varepsilon \geq n^{-\delta} \), which imposes \( t \geq n^{-3\delta/2} \). For \( t \leq n^{-3\delta/2} \), the first bound also gives a bound of

\[\text{(Note: This is a continuation of the text.)}\]

\[\text{(Continued on the next page.)}\]
the form $C(f, \rho)t^{4/3}$. By Proposition 3.3, we conclude that (4.13) is tight, as we wanted to show.

4.2. Convergence: the case $\alpha < 3/2$.

In Sections 4.1.1-4.1.3 we showed that the sequence $\{Y^n_t; t \in [0,T]\}_{n \in \mathbb{N}}$ is tight for $\alpha < 3/2$. Let $\{Y_t; t \in [0,T]\}$ be a limit point. For simplicity, up to the end of this section we denote by $n$ the subsequence along which $\{Y^n_t; t \in [0,T]\}_{n \in \mathbb{N}}$ converges to $\{Y_t; t \in [0,T]\}$. We will pass to the limit in equation (3.1) to obtain a martingale characterization in terms of the processes $\{A^n_t; t \in [0,T]\}$ and $\{M^n_t(f); t \in [0,T]\}$, the martingale decomposition (3.1) reads

$$Y^n_t(f) = Y^n_0(f) + \int_0^t Y^n_s(L^n_0 f)ds - A^n_t(f) + M^n_t(f).$$

(4.16)

Without loss of generality we can assume that the real-valued martingale processes $\{M^n_t(g); g \in \mathcal{B}(\mathbb{R})\}n \in \mathbb{N}$ converge to $\{M_t(g); t \in [0,T]\}$ for any $g \in \mathcal{B}(\mathbb{R})$, as $n \to \infty$. Notice that, by the definition of the density fluctuation field given in (2.15), the function $f$ in (4.16) is a trajectory and therefore the previous result does not apply to our setting.

Recall from Section 4.1 that the initial distribution $Y^n_0$ converges to a white noise of variance $\rho(1 - \rho)$. In fact, for any $t \in [0,T]$ the same affirmation is true: the $\mathcal{B}(\mathbb{R})$-valued random variables $Y^n_t$ converge in distribution to a white noise of variance $\rho(1 - \rho)$. Therefore, the limit process $\{Y_t; t \in [0,T]\}$ is stationary.

Now we turn into the terms $A^n_t(f), M^n_t(f)$. These terms are not quite covered by the computations of Sections 4.1.1-4.1.3 since the function $f$ was constant there. The martingale term is not difficult to deal with: for $t \in [0,T]$ and $\ell \in \mathbb{N}$ define $L = \lfloor \ell^\infty \rfloor, t_1 = \frac{L}{\ell^2}$ and

$$M_t(f) = \lim_{\ell \to \infty} \frac{1}{\ell} \sum_{i=0}^{\ell-1} (M_{t+i}(f_{t+i}) - M_i(f_{t+i}))$$

Using the orthogonal increments property, we can show that $\{M_t(f); t \in [0,T]\}$ is a martingale. The same approximation procedure can be done for $\{M^n_t(f); t \in [0,T]\}_{n \in \mathbb{N}}$ and the limit, as $n \to \infty$, is uniform in $t$. Therefore, we conclude that

$$M_t(f) = \lim_{n \to \infty} M^n_t(f)$$

in $L^2(\mathbb{P})$. This is sufficient to take the limit in (4.16) in what respects to the martingale term. The corresponding quadratic variation is equal to

$$\langle M_t(f) \rangle = 2\rho(1 - \rho) \int_0^t \mathcal{E}(f_s)ds.$$

Repeating the computations made in Sections 4.1.1-4.1.3 for smooth trajectories $f : [0,T] \to \mathcal{S}(\mathbb{R})$, one can see that at each appearance of the constant $C(f, \rho)t$, we can replace it by $\int_0^t C(f_s, \rho)ds$, which, again, can be replaced by $C(f, \rho)t$, meaning this time that the constant $C(f, \rho)$ depends on $\rho$ and on the whole trajectory After this observation we conclude that $A^n_t(f)$ converges to 0 in $L^2(\mathbb{P})$, as $n \to \infty$.

Finally, using Proposition 2.2 we can change $L^n_0 f_s$ by $L_0^\rho f_s$ in (4.16). Therefore, we are left to prove the convergence of the integral term

$$\int_0^t Y^n_s(L_0^\rho f)ds \quad \text{to} \quad \int_0^t Y_s(L_0^\rho f)ds,$$

13 In order to avoid heavy notation, we decided to restrict the computations in previous sections to functions not depending on time.
as } n \to \infty \text{.} \) Recall that this last integral is defined through a limiting procedure, approximating } L^p f_i \text{ by } \psi_i L^p f_i \text{. We can check that the approximation of } E^n L^p f_i \text{ by } E^n (\psi_i L^p f_i) \text{ is uniform in } n, \text{ and therefore the convergence of the integral term is guaranteed.}

Putting all these elements together, we conclude that for any smooth trajectory } f : [0, T] \to \mathcal{Y}(\mathbb{R}) \text{ we have }

\[ \mathcal{Y}_i(f) = \mathcal{Y}_0(f) + \int_0^T \mathcal{Y}_s(L^p f)\, ds + \mathcal{M}_i(f), \]

where \( \{ \mathcal{M}_i(f) : t \in [0, T] \} \) is a continuous martingale of quadratic variation

\[ \langle \mathcal{M}_i(f) \rangle = 2\rho (1 - \rho) \int_0^T \mathcal{E}(f_s)\, ds. \]

In other words, \( \{ \mathcal{Y}_i : t \in [0, T] \} \) is a stationary solution of (2.17). By Proposition 2.10 the distribution of \( \{ \mathcal{Y}_i : t \in [0, T] \} \) is uniquely determined. We conclude that the sequence \( \{ E^n : t \in [0, T] \}_{n \in \mathbb{N}} \) has a unique limit point, and therefore it actually converges to this limit point. This ends the proof of Theorem 2.11.

### 4.3. Convergence along subsequences: the case } \alpha = 3/2 \text{.}

In Section 4.1.4 we showed that the sequence of processes \( \{ E^n : t \in [0, T] \}_{n \in \mathbb{N}} \) is tight for } \alpha = 3/2 \text{. As in the previous section, let } \{ \mathcal{Y}_i : t \in [0, T] \} \text{ be one of its limit points. For simplicity we call } n \text{ the subsequence over which } \{ E^n : t \in [0, T] \}_{n \in \mathbb{N}} \text{ converges to } \{ \mathcal{Y}_i : t \in [0, T] \}. \text{ The treatment of the initial field, the martingale and the integral term in (3.1) remains the same as in the previous section. The difference between the case } \alpha < 3/2 \text{ and } \alpha = 3/2 \text{ comes from the term } A^n(f). \text{ We showed in Section 4.1.3 that } A^n(f) \text{ is asymptotically equivalent to }

\[ m \int_0^T \sum_x \psi^n x (\eta^n_x) f'(x)\, dx. \]

In Section 4.1.4 we showed in (4.14) that

\[ \mathbb{E}_n \left[ \left( \int_0^T \sum_x (\psi^n_x(\eta^n_x) - \psi^n_x(\eta^n_x)) f'(x)\, dx \right)^2 \right] \leq C(f, \rho) t \sqrt{\varepsilon}. \tag{4.17} \]

This bound is uniform in } n, \text{ so if we are able to show that

\[ \int_0^T \sum_x \psi^n_x(\eta^n_x) f'(x)\, dx \]

is asymptotically equivalent to a function of the process \( E^n : t \in [0, T] \) we will be close to prove Theorem 2.16. In Section 2.5.2 we introduced a general approximation of the identity } t_{\varepsilon}. \text{ In this section we use the particular choice } t_{\varepsilon}(x) = \frac{1}{\varepsilon} 1\{ x \in (0, \varepsilon] \}. \text{ This is specially convenient because of the identity

\[ \frac{1}{\varepsilon \sqrt{n}} \sum_{i=1}^n \bar{\eta}^n_i (x + i) = E^n * t_{\varepsilon}(\frac{x}{\varepsilon}). \]

Notice that last identity is a consequence of the fact that } E^n * t_{\varepsilon}(\frac{x}{\varepsilon}) = \frac{1}{\sqrt{n}} \sum y \in [0, \varepsilon] t_{\varepsilon}(y - x) \bar{\eta}^n_i(y). \text{ In terms of this notation the equivalence of ensembles (3.8) gives

\[ \mathbb{E}_n \left[ \left( \psi^n_x(\eta^n_x) - \frac{1}{\varepsilon} \left( E^n * t_{\varepsilon}(\frac{x}{\varepsilon}) \right)^2 + \rho (1 - \rho) \frac{1}{\varepsilon n} \right)^2 \right] \leq \frac{c(\rho)}{\varepsilon n}. \tag{4.18} \]
Using this bound we can see that
\[
\mathbb{E}_n \left[ \left( \int_t^T \sum_x \left( \mathcal{E}_x^m(\eta^n_x) - \frac{1}{n} \mathcal{E}_x^m * \mathcal{L}_n \left( \frac{x}{n} \right) \right) f'_n(x) ds \right]^2 \right] \leq C(f, \rho, t) \left( \frac{1}{n^2} + \frac{1}{n^2} \right). \tag{4.19}
\]
We notice that the previous bound follows from the following computation: first sum and subtract \(\frac{1}{n} \sum f'_n(x) \) inside the time integral, use the inequality \((x + y)^2 \leq 2x^2 + 2y^2\); the first error term comes from (4.18), and the second one comes from the approximation of the integral by the Riemann sum. Therefore, we have just proved that
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \mathbb{E}_n \left[ \left( A_n^\varepsilon(f) - m \int_0^t \sum_x \left( \mathcal{E}_x^m * \mathcal{L}_n \left( \frac{x}{n} \right) \right) f'_n(x) ds \right]^2 \right] = 0. \tag{4.20}
\]
Now we are in position to prove Theorem 2.16. First we notice that (4.17) implies the bound
\[
\mathbb{E}_n \left[ \left( \int_t^T \sum_x \left( \mathcal{E}_x^m(\eta^n_x) - \mathcal{E}_x^m(\eta^n_x) \right) f'_n(x) ds \right]^2 \right] \leq C(f, \rho, t) \varepsilon \sqrt{T}
\]
for any \(\delta < \varepsilon\). Recall (2.19). Passing to the limit in the previous expression, after using (4.19), we can prove that
\[
\mathbb{E} \left[ \left( \mathcal{A}_t^\varepsilon(f) - \mathcal{A}_t^\varepsilon(\mathcal{A}_t^\delta(f)) \right)^2 \right] \leq C(f, \rho, t) \varepsilon \sqrt{T}.
\]
A careful checking of the constants \(C(f, \rho)\) shows that for each \(n\) we can choose \(C(f, \rho) = C_n(f, \rho)\) in such a way that
\[
\limsup_{n \to \infty} C_n(f, \rho) = c(\rho) \int_0^t \|f'_s\|^2 ds.
\]
Noticing that the process \(\{\mathcal{A}_t; t \in [0, T]\}\) is stationary, from Definition 2.13, the previous bound shows that \(\{\mathcal{A}_t; t \in [0, T]\}\) satisfies an energy estimate with \(\kappa_0 = c(\rho)\) and \(\beta = \frac{1}{2}\). Therefore, the process \(\{\mathcal{A}_t; t \in [0, T]\}\) given by
\[
\mathcal{A}_t(f) = \lim_{\varepsilon \to 0} \mathcal{A}_t^\varepsilon(f)
\]
is well defined and by (4.20) we have that
\[
\lim_{n \to \infty} A_n^\varepsilon(f) = m \mathcal{A}_t(f).
\]
Let us recall what we have proved about the process \(\{\mathcal{A}_t; t \in [0, T]\}\). In the previous section, we showed that \(\{\mathcal{A}_t; t \in [0, T]\}\) is stationary, and in particular it is USC, see the comments below 4.16. We just proved that it satisfies an energy estimate for \(\beta = \frac{1}{2}\). We proved that the discrete process \(\{A_n^\varepsilon; t \in [0, T]\}\) converges in distribution to the process \(\{\mathcal{A}_t; t \in [0, T]\}\), which is well defined in virtue of the energy condition. The arguments of the previous section shows that
\[
\mathcal{A}_t(f) - \mathcal{A}_0(f) - \int_0^t \mathcal{A}_s(\mathcal{L}^0 f) ds + m \mathcal{A}_t(f)
\]
is a continuous martingale of quadratic variation
\[
2 \rho (1 - \rho) \int_0^t \mathcal{E}(f_s) ds.
\]
This is exactly what we called a stationary energy solution of the Burgers equation 2.23. This ends the proof of Theorem 2.16.
5. DISCUSSION AND REMARKS

5.1. The KPZ universality conjecture. The stochastic Burgers equation

\[ dY_t = \Delta Y_t dt + \lambda \nabla Y_t^2 dt + \nabla dW_t \]

has received a lot of attention in recent years. In a groundbreaking work, [18] developed a meaningful notion of solution for this equation, and proved uniqueness of such solutions. In a very different line of research, the relation of this equation with stochastic integrable systems allows to describe in a very precise way various one-dimensional marginals of the solutions of this equation, see [7] for a review. The stochastic Burgers equation and its integrated counterpart, namely, the KPZ equation, are conjectured to describe the height fluctuations of growing, one-dimensional flat interfaces, or more generally fluctuation phenomena of one-dimensional stochastic systems near a stationary, non-equilibrium state. Those system are supposed to belong to the so-called KPZ universality class, which is characterized by the scaling exponents 1 : 2 : 3. From the point of view of non-rigorous renormalization group theory, the strong KPZ universality conjecture states that there exists a unique process (the so-called KPZ fixed point) which is an attractive fixed point of the space-time renormalization group with exponents 1 : 2 : 3. The so-called weak KPZ universality conjecture states that the unstable manifold of this unique fixed point is one-dimensional, and it is composed by the stationary solutions of the Burgers equation, parametrized by the real number \( \lambda \). A proof of the weak KPZ conjecture for a restricted class of models has been announced [20]. Up to now, the best results in the direction of the proof of the weak KPZ universality are the ones in [9, 12]. In the language of the definition of this article, the results in [9, 12] state that the unstable manifold of the KPZ fixed point is contained on the set of stationary, energy solutions of the stochastic Burgers equation. Therefore, a uniqueness result for energy solutions of the stochastic Burgers equation would imply the weak KPZ universality conjecture.

One of the main motivations for this work is to give support to the following conjecture:

**Conjecture 5.1.** The derivative of the KPZ fixed point is an energy solution of the fractional Burgers equation.

Let us describe in a more precise way what do we understand by the space-time renormalization group of exponents 1 : 2 : 3. For \( \lambda > 0 \) and for a given process \( \{ Y_t; t \in [0, \infty) \} \) with values in \( \mathcal{S}'(\mathbb{R}) \) we define the process \( \{ T_\lambda Y_t; t \in [0, \infty) \} \) as

\[
T_\lambda Y_t (f) = \frac{1}{\lambda} T_{\lambda^2} Y_t (f),
\]

where \( T_{\lambda^2} f(x) = f(\frac{x}{\lambda^2}) \). The family of transformations \( \{ T_\lambda; \lambda > 0 \} \) is what we call the space-time renormalization group of exponents 1 : 2 : 3. In connection with this definition, energy solutions of the fractional Burgers equation have the following property:

**Theorem 5.2.** The space of stationary, energy solutions of (2.23) is invariant under the space-time renormalization group of exponents 1 : 2 : 3.

The proof is elementary, so we omit it. The point we wanted to stress is that the fractional Burgers equation is left invariant by \( T_\lambda \), making it a candidate for the equation satisfied by the KPZ fixed point. The fact that it appears as the scaling limit of a microscopic dynamics makes this candidate natural in some sense. The fact that particles have non-local interactions make this claim less natural. However, it has been recently proved that the fractional operator \( \mathcal{L}_\alpha \) with \( \alpha = 3/2 \) appears in the scaling limit of fluctuations of
one-dimensional conservative systems \[1\]-\[17\]. The connection between these results and the fractional Burgers equation is yet to be found.

5.2. Weakly (a)symmetric systems. In \[13\], a family of fractional Burgers equations was introduced. More precisely, the concept of stationary energy solutions of

\[ d\mathcal{Y}_t = (\mathcal{L}^{1/2})\mathcal{Y}_t dt + \lambda \nabla \mathcal{Y}_t^2 dt + \sqrt{-\mathcal{L}^{1/2}} d\mathcal{W}_t \]  

(5.1)

was introduced, although in finite volume. Existence was shown for \( \alpha > 1 \) and uniqueness was shown for \( \alpha > 9/4 \). Introducing weak (a)symmetries into the system, we can obtain the equations as scaling limits of long-range exclusion processes. More precisely, consider the family of transition rates \( \{p_n(\cdot); n \in \mathbb{N}\} \) given by

\[ p_n(z) = \frac{c}{|z|^{1+\alpha}} + \lambda n^{3/2-\alpha} 1(z = 1). \]

For \( \alpha > 3/2 \) this model is weakly asymmetric in the sense that the asymmetric part of the rate vanishes, as \( n \to \infty \), and for \( \alpha < 3/2 \) this model is weakly symmetric in the sense that the asymmetric part of the transition rate grows to infinity, as \( n \to \infty \). For \( \alpha = 3/2 \), the asymmetric and symmetric parts of the transition rate are perfectly balanced. The interested reader may verify that the proof of Theorem \[2.16\] carries through this family of transition rates, and the result stated there holds for the fractional Burgers equation (5.1).

5.3. Normal domains of attraction. We say that a transition rate \( p(\cdot) \) is in the normal domain of attraction of an \( \alpha \)-stabilizable if there exist constants \( c^+, c^- \geq 0 \) such that \( c^+ + c^- > 0 \) and

\[ \lim_{x \to +\infty} x^\alpha \sum_{y \geq x} p(y) = c^+. \]

It is well known that the random walk with transition rate \( p(\cdot) \) converges to a non-trivial Markov process under the scaling of Proposition \[2.1\] if, and only if, \( p(\cdot) \) belongs to the domain of normal attraction of an \( \alpha \)-stable law. If the transition rate \( p(\cdot) \) is symmetric, it can be checked that Theorem \[2.11\] holds for any \( \alpha \in (0, 2) \). Notice that in this case the process \( A^\alpha_t(f) \) is identically null. In the case of non-symmetric transition rates \( p(\cdot) \), the model is truly non-linear and we need the full power of Proposition \[2.8\] in order to handle \( A^\alpha_t(f) \). In order to prove Proposition \[2.8\] we need to prove the corresponding spectral gap inequality. It turns out that this is a non-trivial question, which is answered in \[16\]. With the spectral gap inequality at our disposal, we can check whether the proofs of Theorems \[2.11\] and \[2.16\] can be generalized to transition rates on the normal domain of attraction of \( \alpha \)-stabilizable laws. It turns out that the proofs can be generalized without any extra assumption on the symmetric part \( s(\cdot) \) of the rate \( p(\cdot) \). However, some additional technical condition on the asymmetric part \( a(\cdot) \) of the transition rate is needed to repeat the proof. This technical condition does not allow to handle any transition rate on the normal domain of attraction of an \( \alpha \)-stable law, but it is close to optimal. The proof becomes extremely technical without adding any insight on the models, and therefore we decided to omit it.

5.4. Generalization to other models. A natural question related to the question of universality is whether the results of these notes can be generalized to other models. The scheme of proof presented here can be applied for models in which the symmetric part of the dynamics satisfies the gradient condition with local functions. Roughly speaking, a model satisfies the gradient condition if the current of particles between two sites \( x, y \) can be written as \( (\tau_y - \tau_x)h \), where \( h \) is a local function and \( \tau_x, \tau_y \) are the shifts to \( x, y \). In the exclusion process, the current is equal to \( \eta(y) - \eta(x) \), so the gradient condition is satisfied. It is very difficult to find interacting particle systems satisfying this property. In \[15\] it is
observed that the zero-range process also satisfies this property, which is used to obtain the hydrodynamic limit of such a model. More examples can be constructed using the family of misanthrope processes introduced in [6], but even among this class of models, the gradient condition is very restrictive.

In the context of stochastically perturbed Hamiltonian dynamics it is very easy to construct models on which the techniques of these notes would allow to prove similar results. Just to give a simple example, consider the Markovian dynamics in \( \mathbb{R}^d \) generated by \( L = S + A \), where

\[
S = \sum_{x,y} s(y-x)e^{\frac{d}{2}\sum \eta(c)^2} (\partial_y - \partial_x)e^{-\frac{d}{2}\sum \eta(c)^2} (\partial_y - \partial_x),
\]

\[
A = \sum_{x,y} a(y-x)e^{\frac{d}{2}\sum \eta(c)^2} (b_y \partial_y - b_x \partial_x)e^{-\frac{d}{2}\sum \eta(c)^2},
\]

where \( b \) is some local function and \( b_y \) is the shift of \( b \) to site \( y \). For this dynamics, the spectral gap over boxes of finite sites is well understood, and product invariant measures are readily guaranteed by the construction of the dynamics.

**ACKNOWLEDGEMENTS**

P.G. thanks CNPq (Brazil) for support through the research project “Additive functionals of particle systems”, Universal n. 480431/2013-2, also thanks FAPERJ “Jovem Cientista do Nosso Estado” with the grant E-25/203.407/2014 and the Research Centre of Mathematics of the University of Minho, for the financial support provided by “FEDER” through the “Programa Operacional Factores de Competitividade COMPETE” and by FCT through the research project PEst-OE/MAT/UI0013/2014.

M.J. was funded by FAPERJ “Jovem Cientista do Nosso Estado” with the grant E-26/103.051/2012.

**APPENDIX A. PROOF OF PROPOSITION 2.2**

We do the proof for the case \( \alpha > 1 \), the others being analogous. The proof of the proposition is elementary, but very tedious. Recall the definition of \( \mathcal{L}^+ \) and \( \mathcal{L}^- \) from (2.6) and (2.4), respectively. First notice that by the definition of \( m_n^\alpha \) in (2.3), for this regime of \( \alpha \) we rewrite \( \mathcal{L}^- f(x/n) = n^{\alpha-1} \sum \psi_{y,n}(x/n) \) and \( \mathcal{L}^+ f(x) = \int_0^\infty p(x+y) \psi_{y,n}(x) dy \) where \( \psi_{y}(v) = f(u+v) - f(u) - v R_1(u) \), for \( f \in C^1(\mathbb{R}) \). Second, notice that \( \mathcal{L}^+_n f(x/n) = n^{\alpha} \sum_{y \geq 1} p(y) \psi_{y,n}(x/n) \) and \( \mathcal{L}^+ f(x) = \int_0^\infty \int_{-\infty}^\infty \psi_s(y) dy \) are well-defined and that it is enough to show (2.7) for \( \mathcal{L}^+_n \) and \( \mathcal{L}^+ \). For \( x \in \mathbb{N} \), define \( P(x) = \sum_{y \geq 1} p(y) \) and \( a(x) = x^\alpha p(x) - \frac{1}{\alpha x^{\alpha-1}} \). Notice that \( a(x) \) tends to 0, as \( x \to \infty \). The idea is to perform an integration by parts in the formula of \( \mathcal{L}^+_n f \) in order to work with the more regular object \( P(y) \). By writing \( p(y) = P(y) - P(y+1) \), performing a summation by parts and a Taylor expansion on \( \psi_s \), we see that \( \mathcal{L}^+_n f(x/n) = n^{\alpha-1} \sum_{y \geq 1} P(y) \psi_{y,n}(x/n) + R_1^n(x) \), where \( R_1^n(x) \) is an error term which satisfies \( |R_1^n(x)| \leq ||\psi'_{y/n}||_\infty n^{\alpha-2} \sum_{y \geq 1} p(y) \). Notice that \( ||\psi'_{y/n}||_\infty = ||f'||_\infty \), which does not depend on \( x \). This last sum is equal to \( \sum_{y \geq 1} y p(y) < +\infty \) and since \( \alpha < 2 \), \( R_1^n(x) \) vanishes, as \( n \to \infty \). For \( y \geq 1 \), let \( A(y) = \sup_{z \geq y} |a(z)| \). We have that

\[
\mathcal{L}^+_n f(x/n) = n^{\alpha-1} \sum_{y \geq 1} \frac{c^+}{\alpha y^{\alpha}} \psi_{y/n}^+(x/n) + n^{\alpha-1} \sum_{y \geq 1} \frac{a(y)}{y^{\alpha}} \psi_{y/n}^+(x/n).
\]

(A.1)
Notice that $\psi'_\alpha$ is bounded and that $\psi_\alpha(0) = \psi'_\alpha(0) = 0$. Therefore, there exists a constant $K$ such that $|\psi'_\alpha(v)| \leq K v$ for any $v \in [0, 1]$ and such that $|\psi'_\alpha(v)| \leq K$ for any $v > 1$. In fact, $K = \max\{2||f'||_\infty, ||f''||_\infty\}$. Therefore, the second term on the right hand side of (A.1) is bounded by

$$
|n^{\alpha-1} \sum_{y=1}^{n} \frac{a(y)}{y^{\alpha}} \psi'_{\alpha/\sqrt{n}}(\frac{y}{n})| + |n^{\alpha-1} \sum_{y\geq n+1} \frac{a(y)}{y^{\alpha}} \psi'_{\alpha/\sqrt{n}}(\frac{y}{n})|
$$

$$
\leq Kn^{\alpha-2} \left(A(1) \sum_{y=1}^{k} y^{1-\alpha} + A(k+1) \sum_{y=k+1}^{n} y^{1-\alpha} + Kn^{\alpha-1} A(n) \sum_{y\geq n+1} \frac{1}{y^{\alpha}}\right)
$$

$$
\leq \frac{K}{2-\alpha} \left(A(1)(\frac{k}{n})^{2-\alpha} + A(k+1)\right) + KA(n+1)\frac{1}{\alpha - 1}
$$

for any $k < n$. Choosing, for example, $k = \sqrt{n}$, the last sums vanish, as $n \to \infty$. Moreover, the first term on the right hand side of (A.1), after an integration by parts, is just a Riemann sum for $\mathcal{L}^{+}f(y)$. Finally, since for $0 < y < z$, we have that $|\psi'(y) - \psi'(z)| \leq C(y-z)$ for some constant $C$ which depends only on $||f''||_\infty$, we conclude that

$$
|n^{\alpha-1} \sum_{y=1}^{n} \frac{c^{1}}{y^{\alpha}} \psi'_{\alpha/\sqrt{n}}(\frac{y}{n}) - \int_{1/n}^{\infty} \frac{c^{1}}{y^{\alpha}} \psi'(y) dy| \leq Cn^{\alpha-2} \sum_{y\geq 1} \frac{1}{y^{\alpha}}
$$

Since the last sum is finite and $\alpha < 2$, we have just shown that

$$
\lim_{n \to \infty} |\mathcal{L}^{+}f(\frac{1}{n}) - \mathcal{L}^{+}f(\frac{1}{z})| = 0.
$$

Moreover, since all the constants above do not depend on $x$, the limit is uniform in $x$, showing the first half of the proposition. The second half can be proved in a similar way.

**APPENDIX B. PROOF OF PROPOSITION 2.10**

For the reader’s convenience, we repeat here various definitions introduced in Section 2.4. Let $\mathcal{L}$ be a generator of a Lévy process in $\mathbb{R}$. Let $\{W_t; t \geq 0\}$ be a Brownian motion on $L^2(\mathbb{R})$ and let $\mathcal{S} = \frac{1}{2}(\mathcal{L} + \mathcal{L}^*)$ be the symmetric part of the operator $\mathcal{L}$. We say that a stochastic process $\{\mathcal{H}_t; t \geq 0\}$ is a stationary solution of the infinite-dimensional Ornstein-Uhlenbeck equation

$$
d\mathcal{H}_t = \mathcal{L}^{+} \mathcal{H}_t dt + \sqrt{-2\chi} \mathcal{S} \mathcal{H}_t dt
$$

if for each $t \in [0, T]$ the random variable $\mathcal{H}_t$ is a white noise of variance $\chi$ and for any differentiable function $f : [0, T] \to \mathcal{S}(\mathbb{R})$ the process

$$
\mathcal{H}_t(f_t) - \mathcal{H}_0(f_0) - \int_0^t \mathcal{H}_s((\partial_t + \mathcal{L})f_s) ds
$$

is a martingale of quadratic variation $2\chi \int_0^t (f_s, -\mathcal{L} f_s) ds$. We will prove following result:

**Proposition B.1.** Two stationary solutions of (B.1) have the same distributions.

**Proof.** Let $f$ be a function in $\mathcal{S}(\mathbb{R})$ and take $t \geq 0$. Let $\{P_t; t \geq 0\}$ the semigroup associated to the generator $\mathcal{L}$, that is, $P_t = e^{t\mathcal{L}}$ for any $t \geq 0$. Since $\mathcal{L}$ is the generator of a Lévy process, $\{P_t; t \geq 0\}$ is a strongly continuous, contraction semigroup on $C_b(\mathbb{R})$. In particular $f_t = P_t f$ is a differentiable trajectory on $C_b(\mathbb{R})$ satisfying $\frac{d}{dt} f_t = -\mathcal{L} f_t$ for any $s \leq t$ and $f_t = f$. Since $\{P_t; t \geq 0\}$ is also contractive in $L^1(\mathbb{R})$, it is a contraction in $L^2(\mathbb{R})$. Notice that $\{f_t; s \leq t\}$ is not a legitimate test function, since although $P_{t-s}$ is infinitely differentiable, it does not satisfy the decay properties needed to be in $\mathcal{S}(\mathbb{R})$. However, $\{f_t; s \leq t\}$
can be approximated in $L^2$ by differentiable functions $f_k : [0, t] \to \mathcal{M}(\mathbb{R})$, justifying the use of \{f_k : s \leq t\} as a test function. Since $(\partial_t + \mathcal{L})f_k = 0$, we conclude that

$$\mathcal{M}_{t,s}^2(f) = \mathcal{B}(P_{t-s}f) - \mathcal{B}_0(P_t f)$$

is a martingale of quadratic variation (with respect to $s$)

$$2\int_0^t \langle P_s f, -\mathcal{P}_s f \rangle ds.$$ 

Since the quadratic variation of $\mathcal{M}_{t,s}(f)$ is deterministic, \{\mathcal{M}_{t,s}(f); s \in [0, t]\} and in consequence \{\mathcal{B}_t; t \in [0, T]\} are Gaussian processes. Notice that

$$\frac{d}{dt} \mathcal{B}_t(f, P_t f) = 2\langle P_t f, \mathcal{P}_t f \rangle = -2\langle P_t f, -\mathcal{P}_t f \rangle.$$

Therefore

$$2\int_0^t \langle P_s f, -\mathcal{P}_t f \rangle = \chi((f, f) - \langle P_t f, P_t f \rangle).$$

We conclude that $\mathcal{B}_t(f)$ can be written as the sum of two independent Gaussian variables: $\mathcal{B}_t(f, P_t f)$, which depends only on the initial distribution and $\mathcal{M}_{t,s}(f)$, which is independent of $\mathcal{B}_t$. Since \{\mathcal{B}_t; t \in [0, T]\} is a Gaussian process, it is characterized by its covariance structure. By stationarity, the computations above show that for any $0 \leq s \leq t \leq T$ and any $f, g \in \mathcal{M}(\mathbb{R})$,

$$E[\mathcal{B}_t(f) \mathcal{B}_s(g)] = E[\mathcal{B}_{t-s}(f) \mathcal{B}_0(g)] = E[(\mathcal{B}_0(P_{t-s}f) + \mathcal{M}_{t,s-t}(f)) \mathcal{B}_0(g)]$$

$$= \chi \langle P_{t-s} f, g \rangle,$$

which shows uniqueness in distribution of the process \{\mathcal{B}_t; t \in [0, T]\}.

\begin{appendix}

\section*{Appendix C. Computations Involving the Auxiliary Martingales}

In this section we compute the integral part of $\mathcal{M}_t^\alpha f$ in \eqref{equation:1} for $f \in \mathcal{M}(\mathbb{R})$.

Recall that $\mathcal{M}_t^\alpha f = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \hat{\eta}^\alpha_y(x) f(\frac{x}{n})$, where $f(\frac{x}{n}) = f(\frac{1-(1-2\rho)\alpha^2}{n})$.

A simple computation shows that

$$L \eta^\alpha(x) = \sum_{y \in \mathbb{Z}} \left(p(x-y) \eta^\alpha_y(y)(1-\eta^\alpha_y(x)) - p(y-x) \eta^\alpha_y(x)(1-\eta^\alpha_y(y))\right),$$

therefore

$$n^\alpha L \mathcal{M}^\alpha f = \frac{n^\alpha}{\sqrt{n}} \sum_{x,y \in \mathbb{Z}} \left(f(\frac{x}{n}) - f(\frac{y}{n})\right) p(x-y) \eta^\alpha_y(x)(1-\eta^\alpha_y(y)). \quad \text{ (C.1)}$$

By centering and writing

$$\eta^\alpha_y(x)(1-\eta^\alpha_y(y)) - \rho(1-\rho) = (1-\rho) \tilde{\eta}^\alpha_y(x) - \rho \tilde{\eta}^\alpha_y(y) - \tilde{\eta}^\alpha_y(x) \tilde{\eta}^\alpha_y(y),$$

we can rewrite the right hand side of \eqref{equation:C.1} as

$$-\frac{n^\alpha}{\sqrt{n}} \sum_{x,y \in \mathbb{Z}} \left(f(\frac{x}{n}) - f(\frac{y}{n})\right) p(y-x)\left((1-\rho) \tilde{\eta}^\alpha_y(x) - \rho \tilde{\eta}^\alpha_y(y) - \tilde{\eta}^\alpha_y(x) \tilde{\eta}^\alpha_y(y)\right)$$

$$= \frac{n^\alpha}{\sqrt{n}} \sum_{x,y \in \mathbb{Z}} \left(f(\frac{x}{n}) - f(\frac{y}{n})\right) \left((1-\rho)p(y-x) + \rho p(x-y)\right) \eta^\alpha_x(y)$$

$$- \frac{n^\alpha}{\sqrt{n}} \sum_{x,y \in \mathbb{Z}} \left(f(\frac{x}{n}) - f(\frac{y}{n})\right) p(y-x) \tilde{\eta}^\alpha_y(x) \tilde{\eta}^\alpha_y(y)$$

\end{appendix}
On the other hand,
\[ \partial_t \mathcal{M}_n(f) = -\langle 1 - 2\rho \rangle \frac{m^\alpha}{n^{\alpha}} \sum_{x \in \mathbb{Z}} f'_x(\bar{z}) \tilde{\eta}_n^\alpha(x) \]
and a simple computation shows that the right hand side of the previous expression equals to
\[ -\frac{n^\alpha}{\sqrt{n}} \sum_{x,y \in \mathbb{Z}} ((1 - \rho)p(y-x) + \rho p(x-y)) \theta^\alpha \left( \frac{y-x}{n} \right) f'_x(\bar{z}) \tilde{\eta}_n^\alpha(x). \tag{C.2} \]
Putting together the previous computations, the integral part of the martingale can be written as
\[
\int_0^t \frac{n^\alpha}{\sqrt{n}} \sum_{x,y \in \mathbb{Z}} \left( f_x(\bar{z}) - f_y(\bar{z}) \right) - \theta^\alpha \left( \frac{y-x}{n} \right) f'_x(\bar{z}) \times \left( (1 - \rho)p(y-x) + \rho p(x-y) \right) \tilde{\eta}_n^\alpha(x) \, ds \]
which coincides with
\[ \int_0^t \mathcal{Y}_n^\alpha \left( \mathcal{L}_n^\alpha f \right) \, ds - A^\alpha_n(f), \]
where \( \mathcal{L}_n^\alpha \) was defined in (3.2) and \( A^\alpha_n \) was defined in (3.3). Finally we obtain
\[ \mathcal{M}_n^\alpha(f) = \mathcal{Y}_n^\alpha(f) - \mathcal{Y}_0^\alpha(f) - \int_0^t \mathcal{Y}_n^\alpha \left( \mathcal{L}_n^\alpha f \right) \, ds + A^\alpha_n(f). \tag{C.3} \]

REFERENCES
[1] C. Bernardin, P. Gonçalves and M. Jara: 3/4-Fractional superdiffusion in a system of harmonic oscillators perturbed by a conservative noise, accepted for publication in ARMA and online at arxiv (2015).
[2] A. N. Borodin and I. A. Ibragimov: Limit Theorems for Functionals of Random Walks, Proceedings of the Steklov Institute of Mathematics, American Mathematical Society, 195 n. 2, (1995).
[3] T. Brox and H. Rost: Equilibrium fluctuations of stochastic particle systems: the role of conserved quantities, Ann. Prob., 12 no. 3, 742–759 (1984).
[4] P. Caputo, T. Liggett and T. Richthammer: Proof of Aldous’ spectral gap conjecture, J. Amer. Math. Soc., 23, 831–851 (2010).
[5] C. C. Chang, C. Landim and S. Olla: Equilibrium fluctuations of asymmetric simple exclusion processes in dimension d ≥ 3, Prob. Theory Related Fields, 119 no. 3, 381–409 (2001).
[6] C. Cocozza-Thivent: Processus des misanthropes, Prob. Theo. Relat. Fields, 70 no. 4, 509–523 (1985).
[7] I. Corwin: The Kardar-Parisi-Zhang equation and universality class, Random Matrices: Theory and Applications, 1, 76 pages (2012).
[8] P. Gonçalves: Central Limit Theorem for a Tagged Particle in Asymmetric Simple Exclusion, Stoch. Proc. Appl., 118, 474–502 (2008).
[9] P. Gonçalves and M. Jara: Nonlinear fluctuations of weakly asymmetric interacting particle systems, Archive for Rational Mechanics and Analysis, 212 no. 2, 597–644 (2014).
[10] P. Gonçalves and M. Jara: Crossover to the KPZ equation, Annales Henri Poincaré, 13 no. 4, 813–826 (2012).
[11] P. Gonçalves and M. Jara: Scaling limits of additive functionals of interacting particle systems, Comm. Pure Appl. Math, 66 no. 5, 649–677 (2013).
[12] P. Gonçalves, M. Jara and S. Sethuraman: A stochastic Burgers equation from a class of microscopic interactions, Ann. Prob., 43 no. 1, 286–338 (2015).
[13] M. Gubinelli and M. Jara: Regularization by noise and stochastic Burgers equations, Stochastic Partial Differential Equations: Analysis and Computations, 1 no. 2, 325–350 (2013).
[14] M. Jara: Non-equilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps, Comm. Pure Appl. Math, 62 no. 2, 198–214 (2009).
[15] M. Jara: Current and density fluctuations for interacting particle systems with anomalous diffusive behavior, arXiv:0901.0229.

[16] M. Jara: Spectral gap for random walks with long jumps and applications, in preparation.

[17] M. Jara, T. Komorowski, S. Olla: Superdiffusion of energy in a chain of harmonic oscillators with noise, arXiv:1402.2988.

[18] M. Hairer: Solving the KPZ equation, Annals of Maths, 178 no. 2, 559–664 (2013).

[19] M. Hairer: A theory of regularity structures Invent. Math. 198 no. 2, 269–304 (2014).

[20] Private communication.

[21] C. Kipnis, C. and C. Landim: Scaling Limits of Interacting Particle Systems, Springer-Verlag, New York (1999).

[22] C. Kipnis and S. R. S. Varadhan: Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, Comm. Math. Phys., 104 no. 1, 1–19 (1986).

[23] T. M. Liggett: Interacting particle systems, Springer-Verlag, Berlin (2005).

[24] I. Mitoma: Tightness of probabilities on $C([0,1];\mathcal{S}')$ and $D([0,1];\mathcal{S}')$, Ann. Prob., 11 no. 4, 989–999 (1983).

[25] J. B. Walsh: An Introduction to Stochastic Partial Differential Equations, cole d’ete de probabilites de Saint-Flour, XIV 1984, 265-439, Lecture Notes in Math, 1180, Springer, Berlin (1986).

[26] W. Whitt: Proofs of the martingale FCLT, Probability Surveys, 4, 268–302 (2007).

DEPARTAMENTO DE MATEMÁTICA, PUC-RIO, RUA MARQUÊS DE SÃO VICENTE, NO. 225, 22453-900, RIO DE JANEIRO, RJ-BRAZIL AND CMAT, CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO MINHO, CAMPO DE GUALtar, 4710-057 BRAGA, PORTUGAL.

E-MAIL: patg@math.uminho.pt and patricia@mat.puc-rio.br

MILTON JARA, IMPA, ESTRADA DONA CASTORINA 110, JARDIM BOTÂNICO, CEP 22460-340, RIO DE JANEIRO, BRAZIL.

E-MAIL: mjara@impa.br