On Taylor series and Kapteyn series of the first and second type

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Abstract

We study the relation between the coefficients of Taylor series and Kapteyn series representing the same function. We compute explicit formulas for expressing one in terms of the other and give examples to illustrate our method.

Keywords: Kapteyn series, Taylor series, Bessel functions

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1. Introduction

Series of the form

\[ \sum_{n=0}^{\infty} \alpha_n^\nu J_{n+\nu} [(n + \nu) z], \quad (1) \]

and

\[ \sum_{n=0}^{\infty} \alpha_n^{\mu,\nu} J_{\mu+n} [(\mu + \nu + 2n) z] J_{\nu+n} [(\mu + \nu + 2n) z], \quad (2) \]

where \( \mu, \nu \in \mathbb{C} \) and \( J_n (\cdot) \) is the Bessel function of the first kind, are called Kapteyn series of the first kind and Kapteyn series of the second kind respectively.

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Kapteyn series have a long history, going back to Joseph Louis de Lagrange’s 1771 paper *Sur le Problème de Képler* [14], where he solved Kepler’s equation [5]

\[ M = E - \varepsilon \sin (E) , \tag{3} \]

using his method for solving implicit equations [13] (now called *Lagrange inversion theorem*) and obtained [7]

\[ E(M) = M + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{d^{n-1}}{dM^{n-1}} \sin^n (M) . \]

Here \( M \) is the mean anomaly (a parameterization of time) and \( E \) is the eccentric anomaly (an angular parameter) of a body orbiting on an ellipse with eccentricity \( \varepsilon \).

In 1819 Friedrich Wilhelm Bessel published his paper *Analytische Auflösung der Kepler’schen Aufgabe* [1], where he approached [3] using a different method. First of all he observed that the function \( g(M) = E(M) - M \) defined implicitly by \( g = \varepsilon \sin (g + M) \) is \( 2\pi \)-periodic and satisfies \( g(0) = 0 = g(\pi) \). Hence, \( g(M) \) can be expanded in a Fourier sine series

\[ g(M) = \sum_{n=1}^{\infty} b_n \sin (nM) , \]

where

\[ b_n = \frac{2}{\pi} \int_{0}^{\pi} g(M) \sin(nM) dM = \]

\[ = -\frac{2}{\pi} \left[ g(M) \frac{\cos(nM)}{n} \right]_{0}^{\pi} + \frac{2}{\pi n} \int_{0}^{\pi} \cos(nM) dg \]

\[ = \frac{2}{\pi n} \int_{0}^{\pi} \cos(nM) d(E - M) \]

\[ = \frac{2}{\pi n} \int_{0}^{\pi} \cos \left[ n (E - \varepsilon \sin E) \right] dE - \frac{2}{\pi n} \int_{0}^{\pi} \cos(nM) dM \]
and hence

\[ b_n = \frac{2}{\pi n} \int_0^\pi \cos (nE - n\varepsilon \sin E) \, dE. \]

He then introduced the function \( J_n(z) \) defined by

\[ J_n(z) = \frac{1}{\pi} \int_0^\pi \cos (nE - z \sin E) \, dE, \quad n \in \mathbb{Z} \quad (4) \]

which now bears his name and obtained

\[ E(M) = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\varepsilon) \sin (nM). \quad (5) \]

Bessel’s work on (4) was continued by other researchers including Lommel, who defined the Bessel function of the first kind by \[ J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left( \frac{z}{2} \right)^{\nu+2n}, \quad \nu \in \mathbb{C}, \quad (6) \]

where \( \Gamma(\cdot) \) is the Gamma function.

In 1817, Francesco Carlini \[2\] found an expression for the true anomaly \( v \) (an angular parameter), defined in terms of \( E \) and \( \varepsilon \) by

\[ \tan \left( \frac{v}{2} \right) = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \tan \left( \frac{E}{2} \right). \]

Carlini’s expression reads \[4\]

\[ v = M + \sum_{n=1}^{\infty} B_n \sin (nM), \]

where

\[ B_n = \frac{2}{n} J_n(n\varepsilon) + \sum_{m=0}^{\infty} \alpha^m \left[ J_{n-m}(n\varepsilon) + J_{n+m}(n\varepsilon) \right], \]

with \( \varepsilon = \frac{2\alpha}{1+\alpha^2} \). The problem considered by Carlini was to determine the asymptotic behavior of the coefficients \( B_n \) for large values of \( n \) \[9\]. The astronomer Johann Franz Encke drew Carl Gustav Jacob Jacobi’s attention to
the work of Carlini. In 1849, Jacobi published a paper improving and correct
ing Carlini’s article [10] and in 1850 Jacobi published a translation from
Italian into German [3], with critical comments and extensions of Carlini’s
investigation.

Bessel’s research on series of the type (5) was continued by Ernst Meissel
[24] in his papers [19], [20] and by Willem Kapteyn (not to be confused with
his brother Jacobus Cornelius Kapteyn [23]) in the articles [11] and [12].
Most of the early work on Kapteyn series, together with their own results,
can be found in the books by Niels Nielsen [22, Chapter XXII] and George
Neville Watson [26, Chapter 17].

In recent years, there has been a renewed interest on Kapteyn series, par-
ticularly from researchers in the fields of Astrophysics and Electrodynamics
(see [17] for a review of current applications). In [15], Ian Lerche and Robert
C. Tautz studied the Kapteyn series of the second kind

$$S_1(a) = \sum_{n=1}^{\infty} n^4 J_n^2(na)$$

and derived the formula

$$S_1(a) = \frac{a^2 (64 + 592a^2 + 472a^4 + 27a^6)}{256 (1 - a^2)^{13/2}}.$$  

They continued their investigations in [16], where they outlined a way for
calculating more general Kapteyn series of the form

$$S_1(m, a) = \sum_{n=1}^{\infty} n^{2m} J_n^2(na), \quad m = 0, 1, \ldots \quad (7)$$

The purpose of this paper is to describe a method for computing the
coefficients in the Taylor series of functions defined by Kapteyn series of the
first (1) and second (2) kind. As an example, we will show a closed-form
expression of (7) valid for all values of $m$.

2. Kapteyn series of the first kind

We begin by considering functions expressed as Kapteyn series of the first
kind.
Theorem 1. Suppose that

$$f(z) = \sum_{m=0}^{\infty} b_m z^m$$  \hspace{1cm} (8)

and

$$z^\nu f(z) = \sum_{n=0}^{\infty} a_n^\nu J_{\nu+n} [(\nu + n) z],$$  \hspace{1cm} (9)

where both series converge absolutely for \(z\) in some domain \(\Omega\). Then, we have

$$a_n^\nu = \sum_{m=0}^{\infty} v_{s,m} b_{s-2m}^\nu$$  \hspace{1cm} (10)

and

$$b_s = \sum_{m=0}^{\infty} u_{s,m} a_{s-2m}^\nu$$  \hspace{1cm} (11)

for all values of \(\nu\), with

$$u_{n,k} = \frac{(-1)^k}{k! \Gamma (\nu + n - k + 1)} \left( \frac{\nu + n - 2k}{2} \right)^{\nu+n}$$

and

$$v_{n,k} = \frac{1}{2} \frac{(\nu + n - 2k)^2 \Gamma (\nu + n - k) \left( \frac{2}{\nu + n} \right)^{\nu+n-2k+1}}{k!}.$$  \hspace{1cm} (12)

Proof. Let \(z \in \Omega\). To prove (11), we use (8) and (9), to get

$$\sum_{m=0}^{\infty} b_m z^{\nu+m} = z^\nu f(z) = \sum_{n=0}^{\infty} a_n^\nu J_{\nu+n} [(\nu + n) z]$$

and from (6) we obtain

$$\sum_{s=0}^{\infty} b_s z^{s+\nu} = \sum_{n=0}^{\infty} a_n^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma (\nu + n + m + 1)} \left[ \frac{(\nu + n) z}{2} \right]^{\nu+n+2m}$$

or

$$\sum_{s=0}^{\infty} b_s z^s = \sum_{n=0}^{\infty} a_n^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma (\nu + n + m + 1)} \left( \frac{\nu + n}{2} \right)^{\nu+n+2m} z^{n+2m}. \hspace{1cm} (12)$$
Setting \( n + 2m = s \) on the right-hand side of (12), we have
\[
\sum_{s=0}^{\infty} b_s z^s = \sum_{s=0}^{\infty} z^s \sum_{m=0}^{s} \frac{(-1)^m a_{s-2m}^s}{m! \Gamma(\nu + s - m + 1)} \left( \frac{\nu + s - 2m}{2} \right)^{\nu + s},
\]
from which (11) follows.

We have
\[
\sum_{j=k}^{p} u_{2j,j-k} v_{2s,s-j} \sum_{m=0}^{s} \left( \frac{(-1)^m 2^{\nu + 2p}}{m! \Gamma(s - p)} \left( \frac{\nu + 2k}{\nu + 2s} \right)^{\nu + 2p + 1} \right) \times \frac{(\nu + 2k)}{(s - k)(\nu + s + k)(p - k)}
\]
and hence
\[
\sum_{j=k}^{s} u_{2j,j-k} v_{2s,s-j} = 0, \quad k \neq s.
\]
When \( k = s \), we get
\[
\frac{1}{\Gamma(\nu + 2s + 1)} \left( \frac{\nu + 2s}{2} \right)^{\nu + 2s} \times (\nu + 2s) \Gamma(\nu + 2s) \left( \frac{2}{\nu + 2s} \right)^{\nu + 2s} = 1
\]
and therefore
\[
\sum_{j=k}^{s} u_{2j,j-k} v_{2s,s-j} = \delta_{k,s}.
\]

Thus,
\[
\sum_{m=0}^{s} v_{2s,m} b_{2(s-m)} = \sum_{j=0}^{s} v_{2s,s-j} b_{2j} = \sum_{j=0}^{s} v_{2s,s-j} \sum_{m=0}^{j} u_{2j,m} a_{2(s-j-m)}^s
\]
\[
= \sum_{j=0}^{s} v_{2s,s-j} \sum_{k=0}^{j} u_{2j,j-k} a_{2k}^s = \sum_{k=0}^{s} a_{2k}^s \sum_{j=k}^{s} u_{2j,j-k} v_{2s,s-j} = \sum_{k=0}^{s} a_{2k}^s \delta_{k,s} = a_{2s}^\nu
\]
and a similar computation holds for \( b_{2k+1} \) and \( a_{2k+1}^\nu \), proving (10).
Remark 2. Formula (10) appeared in [26, 17.5 (6)], but it contains a small mistake because there is a factor of \(\frac{1}{2}\) missing in the denominator. The result was also published in [8, 7.10.2 (29)], but there is also a misprint there, since the factor \(b_{n-2m}\) is missing in the sum.

In [6], we computed formulas for \(b_s\) and \(a^\nu_n\) when \(\nu = 0\).

In order to find the coefficients \(b_s\) for a particular choice of \(\nu = 0\) and \(a^\nu_n\), we need the following result.

Lemma 3. Let \(r, m \in \mathbb{N}\). Then, we have

\[
\sum_{k=0}^{r} \binom{r}{k} (-1)^k (r-2k)^m = 2^r \left. \frac{d^m \sinh^r(t)}{dt^m} \right|_{t=0}.
\] (13)

Proof. We have

\[
\sinh^r(t) = \left(\frac{e^t - e^{-t}}{2}\right)^r = \frac{1}{2^r} \sum_{k=0}^{r} \binom{r}{k} (-1)^k e^{(r-2k)t}.
\]

Since

\[
\frac{d^m e^{at}}{dt^m} = a^m e^{at},
\]

we obtain,

\[
\left. \frac{d^m \sinh^r(t)}{dt^m} \right|_{t=0} = \frac{1}{2^r} \sum_{k=0}^{r} \binom{r}{k} (-1)^k (r-2k)^m.
\]

Example 4. Let’s consider the special case \(\nu = 0\) and \(a^\nu_n = n^{2p}\). Then, (11) gives

\[
b_s(p) = \frac{1}{s!2^{s}} \sum_{k=0}^{\lfloor s/2 \rfloor} (-1)^k \binom{s}{k} (s - 2k)^{s+2p}.
\] (14)

Since the terms in the sum are symmetric with respect to \(k = \lfloor s/2 \rfloor\), we can write

\[
b_s(p) = \frac{1}{s!2^{s}} \frac{1}{2^s} \sum_{k=0}^{s} (-1)^k \binom{s}{k} (s - 2k)^{s+2p},
\]

unless \(s = 0 = p\), in which case we have

\[b_0(0) = 1.\]
Thus, we have

\[ b_s(p) = \varepsilon_{s,p} \frac{1}{s!} \frac{1}{2^s} \sum_{k=0}^{s} (-1)^k \binom{s}{k} (s - 2k)^{s+2p}, \]

with

\[ \varepsilon_{s,p} = \begin{cases} 1, & s = 0 = p \\ \frac{1}{2}, & \text{otherwise} \end{cases}. \]  

(15)

From (13), we obtain

\[ b_s(p) = \varepsilon_{s,p} \frac{d^{s+2p} \sinh^s(t)}{dt^{s+2p}} \bigg|_{t=0} \]

\[ = \varepsilon_{s,p} (s + 2p)! [t^{s+2p}] \sinh^s(t) \]

or

\[ b_s(p) = \varepsilon_{s,p} (s + 1)_{2p} [t^{2p}] \left( \frac{\sinh(t)}{t} \right)^s, \]  

(16)

where \([t^r] G(t)\) denotes the coefficient of \(t^r\) in the Maclaurin series of \(G(t)\).

From (16) we get

\[ b_s(0) = \varepsilon_{s,0}, \quad b_s(1) = \frac{1}{2} (s + 1)_{2} \frac{s}{6}, \]

\[ b_s(2) = \frac{1}{2} (s + 1)_{4} \frac{s (5s - 2)}{360}, \]

\[ b_s(3) = \frac{1}{2} (s + 1)_{6} \frac{s (35s^2 - 42s + 16)}{45360}, \]

\[ b_s(4) = \frac{1}{2} (s + 1)_{8} \frac{s (5s - 4) (35s^2 - 56s + 36)}{5443200}, \ldots \]

If we define

\[ \sum_{n=0}^{\infty} n^{2p} J_n(nz) = f_p(z) = \sum_{s=0}^{\infty} b_s(p) z^s, \]

it follows from (16) that

\[ f_p(z) = [t^{2p}] \sum_{s=0}^{\infty} \varepsilon_{s,p} (s + 1)_{2p} \left( \frac{\sinh(t)}{t} \right)^s \]

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\[ f_p(z) = \left[ t^{2p} \right] \left\{ \frac{1}{2} + \frac{(2p)!}{2} \left[ 1 - \frac{\sinh(t)}{t} \right]^{-(2p+1)} \right\}. \] (17)

Using (17), we get

\[
\begin{align*}
  f_0(z) &= \frac{2 - z}{2(1 - z)}, \quad f_1(z) = \frac{z}{2(1 - z)^2}, \\
  f_2(z) &= \frac{z(9z + 1)}{2(1 - z)^3}, \quad f_3(z) = \frac{z(255z^2 + 54z + 1)}{2(1 - z)^4}, \\
  f_4(z) &= \frac{z(11025z^3 + 4131z^2 + 243z + 1)}{2(1 - z)^5}, \ldots
\end{align*}
\]

and in general

\[ f_n(z) = \frac{z}{2(1 - z)^{3n+1}} P_{n-1}(z), \]

where \( P_0(z) = 1 \) and \( P_n(z) \) is a polynomial of degree \( n \) for \( n = 1, 2, \ldots \).

Using the relation \([26, 17.33]\)

\[ f_{n+1}(z) = \frac{1}{1 - z^2} \left( z \frac{d}{dz} \right)^2 f_n(z), \]

we find that the polynomials \( P_n(z) \) satisfy

\[
\begin{align*}
P_{n+1}(z) &= \frac{z^2(z - 1)^2}{z + 1} P_n''(z) + \frac{z}{z + 1} \left[ (1 - 6n) z^2 + (6n - 4) z + 3 \right] P_n'(z) \\
&\quad + \frac{1}{z + 1} \left[ 9n^2 z^2 + (9n + 1) z + 1 \right] P_n(z).
\end{align*}
\]

3. Kapteyn series of the second kind

We now consider functions expressed as Kapteyn series of the second kind.

**Theorem 5.** Suppose that

\[ f(z) = \sum_{m=0}^{\infty} b_m z^m \] (18)
and
\[ z^{\mu+\nu} f(z) = \sum_{n=0}^{\infty} (a_n^{\mu,\nu} + zc_n^{\mu,\nu}) J_{\mu+n} [(\mu + \nu + 2n) z] J_{\nu+n} [(\mu + \nu + 2n) z], \]  
(19)
where both series converge absolutely for \( z \) in some domain \( \Omega \). Then, we have
\[ a_0^{\mu,\nu} = \sum_{k=0}^{s} \alpha_{s,k}^{\mu,\nu} b_{2k}, \quad c_0^{\mu,\nu} = \sum_{k=0}^{s} \alpha_{s,k}^{\mu,\nu} b_{2k+1} \]  
(20)
and
\[ b_{2s} = \sum_{k=0}^{s} \beta_{s,k}^{\mu,\nu} a_k^{\mu,\nu}, \quad b_{2s+1} = \sum_{k=0}^{s} \beta_{s,k}^{\mu,\nu} c_k^{\mu,\nu}, \]  
(21)
for all \( \mu, \nu \), with
\[ \alpha_{s,k}^{\mu,\nu} = \frac{(\mu + \nu + 2k)^2 \Gamma (\mu + k + 1) \Gamma (\nu + k + 1)}{(\mu + \nu + s + k) (\mu + \nu + 2s)} \times \left( \frac{\mu + \nu + s + k}{s - k} \right) \left( \frac{2}{\mu + \nu + 2s} \right)^{\mu+\nu+2k}, \]
and
\[ \beta_{s,k}^{\mu,\nu} = \frac{(-1)^{s+k}}{\Gamma (\mu + s + 1) \Gamma (\nu + s + 1)} \left( \frac{\mu + \nu + 2s}{s - k} \right) \left( \frac{\mu + \nu + 2k}{2} \right)^{\mu+\nu+2s}. \]

**Proof.** Let \( z \in \Omega \). Using the formula [26, 5.41]
\[ J_{\mu} (z) J_{\nu} (z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma (\mu + k + 1) \Gamma (\nu + k + 1)} \left( \frac{\mu + \nu + 2k}{k} \right) \frac{(z \sqrt{2})^{\mu+\nu+2k}}{\mu+\nu+2k}, \]
in (19), we get
\[ z^{\mu+\nu} f(z) = \sum_{n=0}^{\infty} (a_n^{\mu,\nu} + zc_n^{\mu,\nu}) \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma (\mu + n + k + 1) \Gamma (\nu + n + k + 1)} \frac{(\mu + \nu + 2n + 2k)}{k} \left[ \frac{(\mu + \nu + 2n) z}{2} \right]^{\mu+\nu+2n+2k}, \]
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or

\[
\begin{align*}
  f(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (a_n^{\mu,\nu} + zc_n^{\mu,\nu}) \frac{(-1)^k}{\Gamma(\mu + n + k + 1) \Gamma(\nu + n + k + 1)} \\
  &\quad \times \left( \frac{\mu + \nu + 2n + 2k}{k} \right) \frac{\mu + \nu + 2n + 2k}{2} z^{2(n+k)}.
\end{align*}
\]

Setting \( n = s - k \), we have

\[
\begin{align*}
  f(z) &= \sum_{s=0}^{\infty} \sum_{k=0}^{s} (a_{s-k}^{\mu,\nu} + zc_{s-k}^{\mu,\nu}) \frac{(-1)^k}{\Gamma(\mu + s + 1) \Gamma(\nu + s + 1)} \\
  &\quad \times \left( \frac{\mu + \nu + 2s}{k} \right) \frac{\mu + \nu + 2s - 2k}{2} z^{2s},
\end{align*}
\]

or

\[
\begin{align*}
  f(z) &= \sum_{s=0}^{\infty} \frac{z^{2s}}{\Gamma(\mu + s + 1) \Gamma(\nu + s + 1)} \sum_{k=0}^{s} (-1)^k \left( \frac{\mu + \nu + 2s}{k} \right) \frac{\mu + \nu + 2s - 2k}{2} a_{s-k}^{\mu,\nu} \\
  &\quad + \sum_{s=0}^{\infty} \frac{z^{2s+1}}{\Gamma(\mu + s + 1) \Gamma(\nu + s + 1)} \sum_{k=0}^{s} (-1)^k \left( \frac{\mu + \nu + 2s}{k} \right) \frac{\mu + \nu + 2s - 2k}{2} c_{s-k}^{\mu,\nu}.
\end{align*}
\]

Comparing with (18), we conclude that

\[
\begin{align*}
  b_{2s} &= \frac{1}{\Gamma(\mu + s + 1) \Gamma(\nu + s + 1)} \sum_{k=0}^{s} (-1)^k \left( \frac{\mu + \nu + 2s}{k} \right) \frac{\mu + \nu + 2s - 2k}{2} a_{s-k}^{\mu,\nu}, \\
  b_{2s+1} &= \frac{1}{\Gamma(\mu + s + 1) \Gamma(\nu + s + 1)} \sum_{k=0}^{s} (-1)^k \left( \frac{\mu + \nu + 2s}{k} \right) \frac{\mu + \nu + 2s - 2k}{2} c_{s-k}^{\mu,\nu}.
\end{align*}
\]

from which (21) follows.

Since

\[
\sum_{k=j}^{p} a_{s,k}^{\mu,\nu} b_{k,j}^{\mu,\nu} = (-1)^{j+p} \left( \frac{\mu + \nu + s + p}{s - p} \right) \left( \frac{\mu + \nu + 2p}{p - j} \right) \\
\times \left( \frac{\mu + \nu + 2j}{\mu + \nu + 2s} \right)^{\mu + \nu + 2p + 2} \frac{(p-s)(\mu + \nu + 2s)}{(j-s)(\mu + \nu + s + j)},
\]

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we see that
\[\sum_{k=j}^{s} \alpha_{s,k}^{\mu,\nu} \beta_{k,j}^{\mu,\nu} = 0, \quad j \neq s,\]
while for \(j = s\), we have
\[\alpha_{s,s}^{\mu,\nu} \beta_{s,s}^{\mu,\nu} = \Gamma (\mu + s + 1) \Gamma (\nu + s + 1) \left(\frac{2}{\mu + \nu + 2s}\right)^{\mu+\nu+2s} \times \frac{1}{\Gamma (\mu + s + 1) \Gamma (\nu + s + 1)} \left(\frac{\mu + \nu + 2s}{2}\right)^{\mu+\nu+2s} = 1.\]

Therefore,
\[\sum_{k=0}^{s} \alpha_{s,k}^{\mu,\nu} b_{2k} = \sum_{k=0}^{s} \alpha_{s,k}^{\mu,\nu} \sum_{j=0}^{k} \beta_{k,j}^{\mu,\nu} a_{j}^{\mu,\nu} = \sum_{j=0}^{s} a_{j}^{\mu,\nu} \sum_{k=j}^{s} \alpha_{s,k}^{\mu,\nu} \beta_{k,j}^{\mu,\nu} = \sum_{j=0}^{s} a_{j}^{\mu,\nu} \delta_{j,s} = a_{s}^{\mu,\nu}.\]

The same calculation holds for \(b_{2k+1}\) and \(c_{k}^{\mu,\nu}\), proving (20). ■

**Remark 6.** Nielsen [21], defined Kapteyn series of the second type as
\[\left(\frac{z}{2}\right)^{\frac{\mu + \nu}{2}} F (z) = \sum_{n=0}^{\infty} a_{n}^{\mu,\nu} J_{\frac{\mu + \nu}{2} + n} \left[\left(\frac{\mu + \nu}{2} + n\right) z\right] J_{\frac{\nu}{2} + n} \left[\left(\frac{\mu + \nu}{2} + n\right) z\right]\]
and assuming that
\[F (z) = \sum_{m=0}^{\infty} B_{m} \left(\frac{z}{2}\right)^{m},\]
he obtained
\[a_{s}^{\mu,\nu} = \sum_{k=0}^{\left\lfloor \frac{s}{2} \right\rfloor} \left(\frac{\mu + \nu}{2} + s - 2k\right) \Gamma \left(\frac{\mu + s}{2} - k + 1\right) \Gamma \left(\frac{\nu + s}{2} - k + 1\right) \times \left(\frac{\mu + \nu}{2} + s - k - 1\right) \left(\frac{2}{\mu + \nu + 2s}\right)^{\frac{\mu+\nu}{2}+s-k+1} B_{s-2k}.\]

We now have all the necessary tools to compute (7).
Example 7. Let 

\[ a_n^{\mu,\nu} (p) = n^{2p}, \quad c_n^{\mu,\nu} = 0, \quad \mu = \nu = 0. \]

Then, \( b_{2s+1} = 0 \) and we have

\[ b_{2s}(p) = \frac{(-1)^s}{(s!)^2} \sum_{k=0}^{s} (-1)^k \binom{2s}{s-k} k^{2(s+p)}, \]

or

\[ b_{2s}(p) = \frac{1}{(s!)^2} \sum_{k=0}^{s} (-1)^k \binom{2s}{k} (s-k)^{2(s+p)}. \]

Since the terms in the sum are symmetric with respect to \( k = s \), we can write

\[ b_{2s}(p) = \frac{1}{2} \frac{1}{(s!)^2} \sum_{k=0}^{s} (-1)^k \binom{2s}{k} (s-k)^{2(s+p)}, \]

unless \( s = 0 = p \), in which case we have

\[ b_0(0) = 1. \]

Thus, we have

\[ b_{2s}(p) = \varepsilon_{s,p} \frac{1}{(s!)^2} \sum_{k=0}^{2s} (-1)^k \binom{2s}{k} (s-k)^{2(s+p)} \]

\[ = \varepsilon_{s,p} \frac{1}{(s!)^2} \frac{1}{4^{s+p}} \sum_{k=0}^{2s} (-1)^k \binom{2s}{k} (2s-2k)^{2(s+p)}, \]

where \( \varepsilon_{s,p} \) was defined in (15). Using (13), we get

\[ b_{2s}(p) = \frac{\varepsilon_{s,p}}{4^p (s!)^2} \frac{d^{2(s+p)}}{dt^{2(s+p)}} \sinh^{2s}(t) \bigg|_{t=0}, \]

or

\[ b_{2s}(p) = \frac{\varepsilon_{s,p}}{4^p (s!)^2} (2s + 2p)! \left[ t^{2p} \right] \left[ \frac{\sinh(t)}{t} \right]^{2s}, \quad (22) \]
and therefore

\[
b_{2s}(0) = \varepsilon_{s,0} \frac{(2s)!}{(s!)^2}, \quad b_{2s}(1) = \frac{1}{2} \frac{(2s + 2)!}{4(s!)^2} s, \\
b_{2s}(2) = \frac{1}{2} \frac{(2s + 4)!}{4^2(s!)^2} \frac{s(5s - 1)}{90}, \\
b_{2s}(3) = \frac{1}{2} \frac{(2s + 6)!}{4^3(s!)^2} \frac{s(35s^2 - 21s + 4)}{5670}, \\
b_{2s}(4) = \frac{1}{2} \frac{(2s + 8)!}{4^4(s!)^2} \frac{s(5s - 2)(35s^2 - 28s + 9)}{340200}, \ldots.
\]

Let \( g_p(z) \) be defined by

\[
\sum_{n=0}^{\infty} n^{2p} J_2^2(2nz) = g_p(z) = \sum_{s=0}^{\infty} b_{2s}(p) z^{2s}.
\]

Then, since

\[
(p + 1)_s \left( p + \frac{1}{2} \right)_s = \frac{(2s + 2p)!}{(2p)! 4^s},
\]

we get from (22) that

\[
g_p(z) = \frac{(2p)!}{4^p} \left[ t^{2p} \right] \sum_{s=0}^{\infty} \varepsilon_{s,p} \frac{(p + 1)_s \left( p + \frac{1}{2} \right)_s}{(s!)^2} \left[ \frac{2 \sinh (t) z}{t} \right]^{2s}
\]

\[
= \frac{(2p)!}{4^p} \left[ t^{2p} \right] \left\{ \frac{1}{2} + \frac{1}{2} z \right\} F_1 \left[ \begin{array}{c} p + 1, p + \frac{1}{2} \\ 1 + \frac{1}{2} \end{array} \right| 4 \frac{\sinh^2 (t) z^2}{t^2} \right\}.
\]  
\text{(23)}

Using (23), we obtain

\[
g_0(z) = \frac{1}{2} + \frac{1}{2 \sqrt{1 - 4z^2}}, \quad g_1(z) = \frac{z^2 (1 + z^2)}{(1 - 4z^2)^{\frac{3}{2}}}, \\
g_2(z) = \frac{z^2 (1 + 37z^2 + 118z^4 + 27z^6)}{(1 - 4z^2)^{\frac{5}{2}}}, \\
g_3(z) = \frac{z^2 (1 + 217z^2 + 5036z^4 + 23630z^6 + 22910z^8 + 2250z^{10})}{(1 - 4z^2)^{\frac{7}{2}}}, \ldots.
\]
Remark 8. Formulas for $g_0(z)$ and $g_1(z)$ were computed by George Augustus Schott in [25]. They were reproduced by Watson in [26, 17.6 (2)-(3)], but there is a typographical mistake in the equation for $g_1(z)$, since the denominator should contain a $\frac{7}{2}$ power, instead of the $\frac{1}{2}$ printed.

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References

[1] F. W. Bessel. Analytische Auflösung der Keplerschen Aufgabe. Abh. Preuß. Akad. Wiss. Berlin, XXV:49–55, 1819.

[2] F. Carlini. Ricerche sulla convergenza della serie che serve alla soluzione del problema di Keplero. Effem. Astron. (Milano), 44(Appendice):3–48, 1817.

[3] F. Carlini. Untersuchungen über die Convergenz der Reihe durch welche das Kepler'sche Problem gelöst wird. Schumacher Astron. Nachr., 30(14):197–212, 1850.

[4] P. Colwell. Bessel functions and Kepler’s equation. Amer. Math. Monthly, 99(1):45–48, 1992.

[5] P. Colwell. Solving Kepler’s equation over three centuries. Willmann-Bell Inc., Richmond, VA, 1993.

[6] D. Dominici. A new Kapteyn series. Integral Transforms Spec. Funct., 18(5-6):409–418, 2007.

[7] J. Dutka. On the early history of Bessel functions. Arch. Hist. Exact Sci., 49(2):105–134, 1995.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. Higher transcendental functions. Vol. II. Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981.

[9] N. Fröman and P. O. Fröman. Physical problems solved by the phase-integral method. Cambridge University Press, Cambridge, 2002.
[10] C. G. J. Jacobi. Ueber die annähernde Bestimmung sehr entfernter Glieder in der Entwicklung der elliptischen Coordinaten nebst einer Ausdehnung der Laplaceschen Methode zur Bestimmung der Functionen gerader Zahlen. *Schumacher Astron. Nachr.*, 28(17):257–272, 1849.

[11] W. Kapteyn. Recherches sur les fonctions de Fourier-Bessel. *Ann. Sci. École Norm. Sup. (3)*, 10:91–122, 1893.

[12] W. Kapteyn. On an expansion of an arbitrary function in a series of Bessel functions. *Messenger of Math.*, 35:122–125, 1906.

[13] J. L. Lagrange. Nouvelle méthode pour résoudre les équations littérales par le moyen des séries. *Mém. de l'Acad. des Sci. Berlin*, XXIV:251–326, 1768.

[14] J. L. Lagrange. Sur le Problème de Képler. *Mém. de l'Acad. des Sci. Berlin*, XXV:204–233, 1771.

[15] I. Lerche and R. C. Tautz. A note on summation of Kapteyn series in astrophysical problems. *Astrophys. J.*, 665(2):1288–1291, 2007.

[16] I. Lerche and R. C. Tautz. Kapteyn series arising in radiation problems. *J. Phys. A*, 41(3):035202, 10, 2008.

[17] I. Lerche, R. C. Tautz, and D. S. Citrin. Terahertz-sideband spectra involving Kapteyn series. *J. Phys. A*, 42(36):365206, 9, 2009.

[18] E. C. J. v. Lommel. *Studien über die Bessel'schen Funktionen*. Leipzig, 1868.

[19] E. Meissel. Neue Entwicklungen über die Bessel’schen Functionen. *Astron. Nachr.*, 129(3089):281–284, 1892.

[20] E. Meissel. Weitere Entwicklungen über die Bessel’schen Functionen. *Astron. Nachr.*, 130(3116):363–368, 1892.

[21] N. Nielsen. Recherches sur les séries de fonctions cylindriques dues à C. Neumann et W. Kapteyn. *Ann. Sci. École Norm. Sup. (3)*, 18:39–75, 1901.

[22] N. Nielsen. *Handbuch der Theorie der Zylinderfunktionen*. Druck und Verlag von B. G. Teubner, Leipzig, 1904.
[23] E. R. Paul. The life and works of J. C. Kapteyn. *Space Science Reviews*, 64(1-2):93–174, 1993.

[24] J. Peetre. Outline of a scientific biography of Ernst Meissel (1826–1895). *Historia Math.*, 22(2):154–178, 1995.

[25] G. A. Schott. *Electromagnetic Radiation*. Cambridge University Press, Cambridge, 1912.

[26] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.