\( \mathcal{N} = 1 \) Supersymmetric Double Field Theory

and the generalized Kerr-Schild Ansatz

Eric Lescano\(^\dagger\) and Jesús A. Rodríguez\(^*\)

\(^\dagger\) Instituto de Astronomía y Física del Espacio (IAFE-CONICET-UBA)

\textit{Ciudad Universitaria, Pabellón IAFE, 1428 Buenos Aires, Argentina}

\(^*\) Departamento de Física, FCEyN, Universidad de Buenos Aires (UBA)

\textit{Ciudad Universitaria, Pabellón 1, 1428 Buenos Aires, Argentina}

elescano@iafe.uba.ar, jarodriguez@df.uba.ar

Abstract

We construct the \( \mathcal{N} = 1 \) supersymmetric extension of the generalized Kerr-Schild ansatz in the flux formulation of Double Field Theory. We show that this ansatz is compatible with \( \mathcal{N} = 1 \) supersymmetry as long as it is not written in terms of generalized null vectors. Supersymmetric consistency is obtained through a set of conditions that imply linearity of the generalized gravitino perturbation and unrestricted perturbations of the generalized background dilaton and dilatino. As a final step we parametrize the previous theory in terms of the field content of the low energy effective 10-dimensional heterotic supergravity and we find that the perturbation of the 10-dimensional vielbein, Kalb-Ramond field and gravitino can be written in terms of a pair of null vectors, as expected.
1 Introduction

General Relativity is a very non-linear theory and many efforts were made in order to find exact solutions. The rotating black hole solution (the Kerr black hole) \cite{1} was the initial construction of a very simple and powerful ansatz called the Kerr-Schild ansatz \cite{2}. This ansatz consists in an exact and linear perturbation of a background metric tensor $g_{\mu\nu}$ of the form,

$$g_{\mu\nu} = g_{\mu\nu}^0 + \kappa l_\mu l_\nu,$$  \hspace{1cm} (1.1)

such that $\kappa$ is an arbitrary parameter that allows to quantify the order of the perturbation and $l_\mu$ is a null vector with respect to $g_{\mu\nu}$ and $g_{\mu\nu}^0$ i.e.

$$g^{\mu\nu}l_\mu l_\nu = g_{\mu\nu}^0 l_\mu l_\nu = 0.$$ \hspace{1cm} (1.2)

With this assumption, the exact inverse to (1.1) is

$$g^{\mu\nu} = g_{\mu\nu}^{0\nu} - \kappa l_\mu l_\nu.$$ \hspace{1cm} (1.3)

If we ask for linearity in the equation of motion of $g_{\mu\nu}$ \cite{3}, then $l_\mu$ is also a geodesic vector with respect to the background metric

$$g^{\mu\nu}l_\mu \nabla_{\nu\rho} l_\rho = 0,$$ \hspace{1cm} (1.4)

where $\nabla_\nu$ is a compatible and torsion-free covariant derivative using the Levi-Civita connection that depends on $g_{\mu\nu}$.

The Kerr-Schild formalism has been successful in different contexts of theoretical physics. It can be used to describe not only the Kerr black hole but also the Myers and Perry black hole \cite{4}, Einstein-Gauss-Bonnet gravity \cite{5}, Einstein-Lovelock gravity \cite{6}, a perturbative duality between gauge and gravity theories referred as Classical Double Copy \cite{7} and it has recently been applied \cite{8} in the context of Double Field Theory (DFT). In this work DFT \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13} is understood as a rewriting of a classical $d$-dimensional supergravity in a more general way such that the generalized version of the supergravity is manifestly invariant under the action of the $G = O(d, d|\mathbb{R})$ group. As $G$ is closely related to a symmetry of String Theory, DFT is often applied to reformulate...
supergravities whose bosonic field content includes a 2-form $b_{\mu\nu}$ (or Kalb-Ramond field) and a scalar field $\phi$ (or dilaton) in addition to the metric tensor. These fields conform the universal NS-NS sector of all the formulations of String Theory. The generalized field content of DFT can accommodate the supergravity field content in multiplets of the duality group and a geometry compatible with the invariant tensors of $G$ can be defined, usually referred to generalized geometry in the literature.

One of the most distinctive features of DFT is that the space-time coordinates of the $d$-dimensional supergravity must be doubled,

$$X^M = (x^\mu, \tilde{x}_\mu) ,$$ (1.5)

where $M = 0, \ldots, 2d-1$ and $X^M$ is a generalized coordinate that is in the fundamental of $G$. The addition of the new coordinates $\tilde{x}_\mu$ forces the appearance of the strong constraint,

$$\partial_M \star \partial^M \star = 0 , \quad \partial^M \partial_M \star = 0$$ (1.6)

where $\star$ means any combination of fields or parameters of the theory and the contractions are done with the $O(d,d)$ invariant metric $\eta_{MN}$. From a stringy point of view, the constraint (1.6) is related to the Fourier transformation of the Level Matching Condition when winding modes are admitted, and written in a duality covariant way. The dynamical background metric of DFT is the generalized metric $H_{oMN}$, which is a multiplet and an element of $G$, i.e.

$$H_{oMP}\eta^{PQ}H_{oNQ} = \eta_{MN} ,$$ (1.7)

parametrized by the background metric tensor $g_{o\mu\nu}$ and the background Kalb-Ramond field $b_{o\mu\nu}$.

The generalized Kerr-Schild ansatz was defined by K. Lee in [8] as an exact and linear perturbation of the generalized background metric with the following form

$$H_{MN} = H_{oMN} + \kappa \bar{K}_M K_N ,$$ (1.8)

where $\bar{K}_M = \bar{P}_M^N \bar{K}_N$ and $K_M = P_M^N K_N$ are a pair of generalized null vectors

$$\eta^{MN}\bar{K}_M \bar{K}_N = \eta^{MN}K_M K_N = \eta^{MN}\bar{K}_M K_N = 0$$ (1.9)
that satisfy
\[ \tilde{P}^{MN} \tilde{K}_M \nabla_{\alpha N} \tilde{K}_P = P^{MN} K_M \nabla_{\alpha N} K_P = 0, \]  
(1.10)
where \( \tilde{P}_{MN} = \frac{1}{2}(\eta_{MN} + H_{MN}) \), \( P_{MN} = \frac{1}{2}(\eta_{MN} - H_{MN}) \) and \( \nabla_{\alpha M} \) is a generalized covariant derivative. Relying on the previous conditions, the EOM of the generalized metric can be linearized in a similar fashion to (1.4). The ansatz (1.8) and the conditions (1.9) and (1.10) were proposed and analyzed in the semi-covariant formalism of DFT and a perturbation for the generalized dilaton \( d \) (parameterized by the 10-dimensional dilaton \( \phi \)) was also considered.

1.1 Main Results

The main goal of this work is to construct the \( \mathcal{N} = 1 \) supersymmetric extension of the ansatz (1.8) in the flux formalism of DFT [14], [15]. As we include generalized fermionic degrees of freedom, we are forced to work in the generalized background frame formalism and fix the space-time dimension. Particularly we consider \( d = 10 \). Since we are dealing with the same degrees of freedom as the supergravity limit of Heterotic String Theory, we let the inclusion of gauge fields in our setup and our starting point is a \( \mathcal{N} = 1 \) DFT with \( G = O(10,10 + n) \) invariance, where \( n = 496 \) is the dimension of the heterotic gauge group [12]. We consider the leading order terms in fermions and show that \( \mathcal{N} = 1 \) supersymmetry is compatible with the generalized Kerr-Schild ansatz as long as it is not written in terms of generalized null vectors.

The most general linear perturbation of the generalized frame is,

\[ E_M \tilde{A} = E_{\alpha M} \tilde{A} + \frac{1}{2} \kappa E_{\alpha M} \tilde{B} \Delta_B \tilde{A} \]
\[ E_M \Delta = E_{\alpha M} \Delta - \frac{1}{2} \kappa E_{\alpha M} \tilde{\eta} \Delta \tilde{B} \]
(1.11)

where \( A = (A, \tilde{A}) \) are indices in \( O(9,1)_L \times O(1,9 + n)_R \) respectively and \( \Delta_{A^B} \) is a mixed-projected perturbation that satisfies,

\[ \Delta_{\tilde{A}\tilde{B}} = \Delta_{\tilde{A}B} = 0, \]
\[ \Delta \eta^{-1} \Delta = 0, \]
(1.12)
in order to be consistent with the constraints of DFT. We find that (1.11) cannot be written in terms of generalized null vectors when supersymmetry is considered and therefore conditions (1.10) are not available to simplify the perturbation of the generalized Ricci scalar and/or the EOM of the generalized frame. We perturb the generalized background dilaton, gravitino and dilatino in the following way,

\[ d = d_0 + \kappa f, \quad f = \sum_n \kappa^n f_n \]  
\[ \Psi_A = \Psi^0_A + \kappa \Theta_A, \quad \Theta_A = \sum_n \kappa^n \Theta^n_A \]  
\[ \rho = \rho_0 + \kappa g, \quad g = \sum_n \kappa^n g_n, \]

where \( n \geq 0 \). With the previous setup we find that \( \mathcal{N} = 1 \) supersymmetry only restricts the generalized gravitino expansion,

\[ \Theta_n = 0, \quad n > 1, \]

while the perturbations of the generalized dilatino and dilaton remain unrestricted. Condition (1.16) forces the following supersymmetric consistency conditions,

\[ \delta^{(n)} \Theta_1 = \delta^{(n)} \Theta_2 = 0, \]

with \( \delta \) a generic symmetry transformation.

As a final step we parametrize the generalized perturbations in terms of the heterotic supergravity field content and we find,

\[ g_{\mu\nu} = g_{\mu\nu} + \kappa l_a \bar{l}_b e_{a\mu}^{\ a} e_{b\nu}^{\ b} \]  
\[ b_{\mu\nu} = b_{\mu\nu} - \kappa l_a \bar{l}_b e_{a\mu}^{\ a} e_{b\nu}^{\ b} \]  
\[ \phi = \phi_o + \kappa f \]  
\[ \psi_a = \psi^0_a - \frac{\kappa}{2} l_a \bar{l}_b \psi^b, \]  
\[ \lambda = \lambda_o + \frac{\kappa}{2} g, \]

where \( e_{\mu a} \) is a 10-dimensional vielbein, \( \bar{l}_a = e^\mu_{\ a} \bar{l}_\mu \) and \( l_a = e^\mu_{\ a} l_\mu \) are a pair of null vectors (we consider \( g^{ab} l_a \bar{l}_b = 0 \) for simplicity) and \( \psi_a \) and \( \lambda \) are the 10-dimensional gravitino and dilatino of the effective heterotic supergravity. The indices \( \mu = 0 \ldots 9 \) and
a = 0 . . . 9 are space-time and \( O(1, 9) \) Lorentz indices respectively. In (1.18) \( n \) is not fixed by supersymmetry as happens in DFT. The ordinary Kerr-Schild ansatz is recovered when \( l_a = \bar{l}_a \). The remaining fields of the effective heterotic supergravity cannot be perturbed

\[
A_{\mu i} = A_{\mu i}^o \\
\chi = \chi^o,
\]

where \( A_{\mu i} \) is a 10-dimensional gauge connection and \( \chi \) is a 10-dimensional gaugino. For the parametrization of the generalized perturbations we consider

\[
\Delta_{\alpha\beta} = (2l_a \bar{l}_b \delta_{\alpha\beta}^a, \Delta_{ai} \delta_{ij}^{ai})
\]

\[
\Theta_{0i} = (\frac{-1}{2} l_a \bar{l}_b \psi_{0i} \delta_a^b, \Theta_{0i}^{0i})
\]

and \( O(10, 10 + n) \) invariance forces \( \Delta_{ai} = \Theta_{0i} = 0 \).

A very interesting aspect of (1.20) is that the supersymmetric extension of the generalized Kerr-Schild formalism can be parametrized in terms of a pair of null vectors. The supersymmetric consistency constraints of DFT (1.17) can be understood as some extra conditions on the expansion of \( \psi_a \). The transformation rule of \( l_a \) and \( \bar{l}_a \) is

\[
\delta l_a = \xi^\mu \partial_\mu l_a + l_b \Lambda_a^b
\]

\[
\delta \bar{l}_a = \xi^\mu \partial_\mu \bar{l}_a + \bar{l}_b \Lambda_a^b
\]

where \( \Lambda_{ab} \) parametrizes an \( O(1, 9) \) Lorentz symmetry and \( \xi_\mu \) parametrizes 10-dimensional diffeomorphisms. The previous conditions are stronger than the usual geodesic equation, but in this case the equation of motion of \( g_{\mu\nu} \) is no more linear in \( \kappa \) due to the 10-dimensional dilatonic and fermionic perturbations.

This work is organized as follows: In section 2 we introduce the field content, the symmetries and the action principle of \( \mathcal{N} = 1 \) DFT for background fields. Section 3 is dedicated to explore the supersymmetric extension of the generalized Kerr-Schild ansatz. First we include finite perturbations on the background field content. Then we discuss the supersymmetric consistency conditions and write schematically the action principle and the equations of motion. In section 4 we parametrize the theory in terms of the
field content of the 10-dimensional heterotic supergravity and find the extra supersymmetric conditions that are necessary for consistency. Finally, in section we present the conclusions of the work and some future directions to explore.

2 \( \mathcal{N} = 1 \) Supersymmetric Double Field Theory for background fields

2.1 Field content and symmetries

\( \mathcal{N} = 1 \) supersymmetric DFT is defined on a double space with coordinates \( X^M \) which transforms under the fundamental representation of the symmetry group \( G = O(10, 10 + n) \), with \( M = 0, \ldots, 19 + n \), and \( n \) the dimension of the gauge group. For instance \( n = 496 \) if we want to encode a low energy description of heterotic supergravity in a T-duality covariant framework. The theory is invariant under a global \( G \) symmetry which infinitesimally reads

\[
\delta_G V_M = V_N h^N_M , \tag{2.1}
\]

where \( V_M \) is a generic \( G \)-multiplet and \( h \in O(10, 10+n) \) is the \( G \)-parameter. The invariant metric of \( G \) is \( \eta_{MN} \in G \) and \( G \)-invariance imposes

\[
h_{MN} = -h_{NM} , \tag{2.2}
\]

where we use \( \eta \) and \( \eta^{-1} \) in order to lower and raise all the \( G \)-indices.

Another symmetry of the theory are generalized diffeomorphisms, generated infinitesimally by \( \xi^M \) through the generalized Lie derivative, defined by

\[
\hat{\mathcal{L}}_\xi V_M = \xi^N \partial_N V_M + (\partial_M \xi^N - \partial^N \xi_M)V_N + f_{MNP} \xi^N V^P + t \partial_M \xi^M , \tag{2.3}
\]

where \( V_M \) is an arbitrary generalized tensor, \( t \) is a weight constant and \( f_{MNP} \) are the generalized version of the structure constants that satisfy

\[
f_{MNP} = f_{[MNP]} , \quad f_{[MN]} R^P f_P R^Q = 0 . \tag{2.4}
\]
The theory is also invariant under a local double Lorentz \( \mathcal{H} = O(9,1)_L \times O(1,9+n)_R \) symmetry generated infinitesimally by a generalized parameter \( \Gamma_{AB} \) where \( A = (\underline{A}, \overline{A}) = (a, \overline{a}, \overline{7}) \) splitting into \( O(9,1)_L \) and \( O(1,9+n)_R \) vector indices, \( \underline{A} = 0, \ldots , 9 \) and \( \overline{A} = 0, \ldots , 9+n \), i.e.,

\[
\delta_{\mathcal{H}} V_A = V_B \Gamma^B_A ,
\]

for a generic \( \mathcal{H} \)-vector. The \( \mathcal{H} \)-invariance of \( \eta_{AB} \) imposes \( \Gamma_{AB} = -\Gamma_{BA} \).

Supersymmetry is parameterized by an infinitesimal generalized Majorana fermion \( \epsilon \) which behaves as a spinor of \( O(9,1)_L \). We work at leading order in fermions, such that supersymmetric transformations of bosons are at most quadratic in fermions, and supersymmetric transformations of fermions are linear in fermions. The explicit transformation rules will be discussed later.

The fundamental background fields of the theory consist in a generalized frame \( E_{oM}^A \) parameterizing the coset \( \frac{G}{\mathcal{H}} = \frac{O(10,10+n)}{O(9,1)_L \times O(1,9+n)_R} \), and a generalized dilaton field \( d_o \). The action of the symmetry groups on these fields is

| \( G \) | \( \mathcal{H}_L \) | \( \mathcal{H}_R \) | Diff |
|---|---|---|---|
| \( E_{oM}^A \) | G-vector | \( \mathcal{H}_L \)-vector | \( \mathcal{H}_R \)-vector | tensor |
| \( d_o \) | G-invariant | \( \mathcal{H}_L \)-invariant | \( \mathcal{H}_R \)-invariant | scalar \((t = -\frac{1}{2})\) |

Consistency of the construction requires constraints which restrict the coordinate dependence of fields and gauge parameters. The strong constraint

\[
\partial_M \partial^M \star = 0 , \quad \partial_M \star \partial^M \star = 0 , \quad f_{MN}^P \partial_P \star = 0 ,
\]

where \( \star \) refers to products of fields, will be assumed throughout. This constraint locally removes the field dependence on \( 10+n \) coordinates, so that fermions can be effectively defined in a 10-dimensional tangent space.

The frame-formulation of DFT demands the existence of two constant, symmetric and invertible \( \mathcal{H} \)-invariant metrics \( \eta_{AB} \) and \( H_{AB} \). The former is used to raise and lower the indices that are rotated by \( \mathcal{H} \) and the latter is constrained to satisfy

\[
H_A^C H_C^B = \delta_A^B .
\]
The generalized background frame \( E^M_o A \) is constrained to relate the metrics \( \eta_{AB} \) and \( \eta_{MN} \) and defines a generalized background metric \( H_{oMN} \) from \( H_{AB} \)

\[
\eta_{AB} = E^M_o A \eta_{MN} E^N_o B , \quad H_{oMN} = E^A_o H_{AB} E^B_o N .
\]

\( H_{oMN} \) is also an element of \( O(10,10+n) \), i.e.

\[
H_{oMP} \eta^{PQ} H_{oQN} = \eta_{MN} .
\]

It is convenient to introduce the projectors

\[
P_{oMN} = \frac{1}{2} (\eta_{MN} - H_{oMN}) \quad \text{and} \quad \overline{P}_{oMN} = \frac{1}{2} (\eta_{MN} + H_{oMN}) ,
\]

which satisfy the usual properties

\[
\overline{P}_{oMQ} P_{oQ} N = P_{oMN} , \quad P_{oMQ} \overline{P}_{oQ} N = P_{oMN} ,
\]

\[
P_{oMQ} \overline{P}_{oQ} N = P_{oMQ} P_{oQ} N = 0 , \quad \overline{P}_{oMN} + P_{oMN} = \eta_{MN} ,
\]

and the same can be done with \( \eta_{AB} \) and \( H_{AB} \) to define \( P_{oAB} , \overline{P}_{oAB} \). We use the convention that \( P_{oAB} , \overline{P}_{oAB} \) and their inverse lower and raise projected indices. Since \( \eta_{AB} \) and \( H_{AB} \) are invariant under the action of \( \hat{L} , G \) and \( H \) we find, \( \Gamma_{AB} = 0 \), where \( \Gamma_{AB} \) was defined in \( [2,5] \), and

\[
\Gamma_{AB} = \overline{P}_{A} C P_{B} D \Gamma_{CD} .
\]

A crucial object for the consistency of the theory is the Lorentz covariant derivative. Acting on a generic vector this derivative is defined as

\[
\nabla_{oA} V_B = E^A_{oA} V_B + \omega_{oAB} C V_C
\]

where \( E_{oA} \equiv \sqrt{2} E^A_o M \partial_M \) and \( \omega_{oAB} C \) is a spin connection that satisfies

\[
\omega_{oABC} = -\omega_{oACB} \quad \text{and} \quad \omega_{oAB} C = \omega_{oAB} C ,
\]

in order to be compatible with \( \eta_{AB} \) and \( H_{AB} \) respectively.

Unlike general relativity, DFT consists of a generalized notion of geometry and there are not enough compatibility conditions to fully determine the generalized spin connection.
Only the totally antisymmetric and trace parts of $\omega_{oABC}$ can be determined in terms of $E_{oM}^A$ and $d_o$, i.e.

$$\omega_{o[ABC]} = -E_{o[A}E_o^{N}B E_{oNC]} - \frac{\sqrt{2}}{3} f_{MNP}E_{o}^{M}A E_{o}^{N}B E_{o}^{P}C \equiv -\frac{1}{3}F_{oABC} \,, \quad (2.14)$$

$$\omega_{oBA}^B = -\sqrt{2}e^{2d_o} \partial_M \left( E_{o}^{M}A e^{-2d_o} \right) \equiv -F_{oA} \,, \quad (2.15)$$

the latter arising from partial integration with the dilaton density.

The $\mathcal{N} = 1$ supersymmetric extension of DFT is achieved by adding a couple of generalized background spinor fields that act as supersymmetric partners of the bosonic fields: the generalized gravitino $\Psi_{oA}$ and the generalized dilatino $\rho_o$. Under the action of the symmetry groups these fields behave as

|     | $G$    | $\mathcal{H}_L$     | $\mathcal{H}_R$     | Diff         |
|-----|--------|---------------------|---------------------|--------------|
| $\Psi_{oA}$ | G-invariant | $\mathcal{H}_L$-spinor | $\mathcal{H}_R$-vector | scalar($t = 0$) |
| $\rho_o$     | G-invariant | $\mathcal{H}_L$-spinor | $\mathcal{H}_R$-invariant | scalar($t = 0$) |

The covariant derivative of spinor fields acquires an additional term in order to derive the spinor indices. For instance, the covariant derivative of the generalized background gravitino and generalized background dilatino are

$$\nabla_{oA}\Psi_{oB} = E_{oA}\Psi_{oB} + \omega_{oAB}\overline{\Psi}_{oC} - \frac{1}{4}\omega_{oABC}\gamma^{BC}\Psi_{oB} \,,
$$

$$\nabla_{oA}\rho_o = E_{oA}\rho_o - \frac{1}{4}\omega_{oABC}\gamma^{BC}\rho_o \,. \quad (2.16)$$

The gamma matrices satisfy a Clifford algebra for $\mathcal{H}$

$$\{ \gamma^A, \gamma^B \} = -2\sigma^{AB} \,, \quad (2.17)$$

and we use the standard convention for antisymmetrization of $\gamma$-matrices $\gamma^{A \ldots B} = \gamma^{[A \ldots B}$.

The generalized supersymmetry transformations of the fundamental fields are parameterized by an infinitesimal Majorana fermion $\epsilon$, that is a spinor of $O(1,9)_L$. These
transformations can be written as

\[ \delta E^A_{oM} = \tau \gamma^{[B} \Psi_{o}^{A]} E_{oM}^{B}, \]
\[ \delta \Psi_{oA} = \nabla_{oA} \epsilon, \]
\[ \delta d_{o} = -\frac{1}{4} \tau \rho_{o}, \]
\[ \delta \rho_{o} = -\gamma^{A} \nabla_{oA} \epsilon. \] (2.18)

If we now include all the symmetries described in the previous subsection, the background fields transform as

\[ \delta E^A_{oM} = \xi^P \partial_P E^A_{oM} + (\partial^M \xi_P - \partial_P \xi^M) E^P_{oA} + E^M_{oB} \Gamma^B_{AB} \xi^M_{oA} - \frac{1}{2} \xi^M \xi^N E^M_{oB} E^N_{oA}, \]
\[ \delta E^M_{oA} = \xi^P \partial_P E^M_{oA} + (\partial^M \xi_P - \partial_P \xi^M) E^P_{oA} + E^M_{oB} \Gamma^B_{AB} \xi^M_{oA} + \frac{1}{2} \xi^M \xi^N \xi^B_{oA} E^M_{oB}, \]
\[ \delta d_{o} = \xi^P \partial_P d_{o} - \frac{1}{2} \partial_P \xi^P - \frac{1}{4} \tau \rho_{o}, \] (2.19)
\[ \delta \Psi_{oA} = \xi^M \partial_M \Psi_{oA} + \xi^P \Gamma^P_{A} \Psi_{oB} + \frac{1}{4} \Gamma^B_{AC} \Psi_{oA} + \nabla_{oA} \epsilon, \]
\[ \delta \rho_{o} = \xi^M \partial_M \rho_{o} + \frac{1}{4} \Gamma^B_{AC} \rho_{o} - \gamma^{A} \nabla_{oA} \epsilon. \]

It is straightforward to show that the previous transformation close off-shell\(^1\) with the following parameters

\[ \xi_{12} = [\xi_1, \xi_2]_{(C_f)} - \frac{1}{\sqrt{2}} E^M_{oA} \Gamma^B_{AB} \xi^M_{oA}, \]
\[ \Gamma_{12AB} = 2 \xi^P_{1} \partial_P \Gamma_{2|AB} - 2 \Gamma_{1A}^{C} \Gamma_{2|CB} + E_{oA} \left( \tau \gamma^{B} \epsilon_{2} \right) - \frac{1}{2} \left( \tau \gamma^{B} \epsilon_{2} \right) F_{oABC}, \] (2.20)
\[ \epsilon_{12} = -\frac{1}{2} \Gamma_{1BC} \gamma_{BC} \epsilon_{2} + 2 \xi^P_{1} \partial_P \epsilon_{2}, \]

where the \(C_f\)-bracket is defined as

\[ [\xi_1, \xi_2]_{(C)} = 2 \xi^P_{1} \partial_P \xi^M_{2} - \xi^N_{1} \partial^M \xi_{2|N} + f_{PQ}^{M} \xi^P_{1} \xi^Q_{2}. \] (2.21)

\(^1\)In case of considering the full-order fermion transformations, the closure is given only on-shell.
2.2 The invariant action

The transformation rules of the background fields discussed in the previous subsection leave the following action invariant (up to leading order terms in fermions)

\[
S_{N=1} = \int d^{20}X e^{-2d_0} \left( R_o + L_{oF} \right)
= \int d^{20}X e^{-2d_0} \left( R_o + \overline{\Psi}_o \gamma^B \nabla_{o B} \Psi_{o A} - \overline{\rho}_o \gamma^A \nabla_{o A} \rho_o + 2 \overline{\Psi}_o \nabla_{o A} \rho_o \right), \tag{2.22}
\]

where \( L_{oF} \) is the fermionic part of the Lagrangian and \( R_o \) is the generalized Ricci scalar,

\[
R_o = 2 E_{o A} F^A_o + F_{o A} F^A_o - \frac{1}{6} F_{o A B C} F^{A B C}_o - \frac{1}{2} F_{o A B C} F^{A B C}_o. \tag{2.23}
\]

We can notice that the previous expression is written in terms of determined components of the generalized spin connection, even when it is obtained from a T-duality invariant curvature tensor \( R_{ABCD} \) which is not fully determined. Moreover, the covariant derivatives appearing in \( L_{oF} \) are also fully determined and therefore the full \( N = 1 \) action is fully determined.

The \( \mathcal{N} = 1 \) DFT action is invariant under \( G, \mathcal{H} \), generalized diffeomorphisms and supersymmetry. The equations of motion obtained from (2.22), up to leading order terms in fermions, are

\[
R_{o B A} + \overline{\Psi}^C \gamma_B E^A_{\overline{C}} \Psi C - \overline{\rho}_{B} \gamma^A \nabla_{o A} \rho_o - 2 \overline{\Psi}_A E_{B} \rho = 0,
\]

\[
R_o + L_F = 0,
\]

\[
\gamma^{B} \nabla_{o B} \Psi_{o A} + \nabla_{o A} \rho_o = 0,
\]

\[
\gamma^{A} \nabla_{o A} \rho_o + \nabla_{o A} \overline{\Psi}_o = 0, \tag{2.24}
\]

where \( R_{o B A} \) is the bosonic part of the EOM of the generalized frame.

Up to this point, we have described the basics of \( \mathcal{N} = 1 \) DFT for generalized background fields. In the next section we perturb these background fields, asking for a linear perturbation of the generalized frame. This perturbation is compatible with \( \mathcal{N} = 1 \) supersymmetry and reduces to a generalized Kerr-Schild ansatz when supersymmetry is turned off. Then we inspect how \( \mathcal{N} = 1 \) supersymmetry is accomplished in the other fields of the theory.
3 The $\mathcal{N} = 1$ supersymmetric generalized Kerr-Schild ansatz

3.1 Finite perturbations on the background fields

We consider the most general linear perturbation for the generalized frame in the flux formalism of DFT. We start defining,

\[
E_M^\nabla = E_oM^\nabla + \frac{1}{2} \kappa E_oM^B \Delta^B_E^\nabla
\]

\[
E_M^A = E_oM^A - \frac{1}{2} \kappa E_oM^\nabla \Delta^A_B \nabla^B
\]

(3.1)

with $\kappa$ an arbitrary parameter and $\Delta^A_B$ a mixed-projected perturbation that satisfies

\[
\Delta^A_B = 0,
\]

\[
\Delta^A_B = 0,
\]

\[
\Delta^A_B = 0,
\]

(3.2)

(3.3)

in order to be consistent with the constraints of DFT. There is no ambiguity in the contractions in (3.3). The inclusion of a finite perturbation on the generalized background frame satisfying (3.3) and (3.3) only deforms the curved version of the projectors,

\[
P_{MN} = E_M^\nabla E_N^\nabla = P_{oMN} - \kappa E_oM^A \Delta^B_E^\nabla E_{oN}^B
\]

\[
P_{MN} = E_M^\nabla E_N^\nabla = \bar{P}_{oMN} + \kappa E_oM^\nabla \Delta^A_B E_{oN}^B
\]

\[
P_{AB} = E_M^\nabla E_M^B = \bar{P}_{oAB}
\]

\[
P_{AB} = E_M^\nabla E_M^\nabla = \bar{P}_{oAB}.
\]

(3.4)

The ansatz (3.1) is compatible with $\mathcal{N} = 1$ supersymmetry and reduces to the generalized Kerr-Schild ansatz introduced in [8] when one considers

\[
\Delta^A_B = K_M \bar{K}_N E_M^\nabla E_N^\nabla.
\]

(3.5)

The perturbation $\Delta_{AB}$ is a $G$-singlet, $\mathcal{H}$-vector and a generalized scalar with weight $t = 0$ with respect to generalized diffeomorphisms. The generalized background dilaton can be
perturbed with a generic $\kappa$ expansion,

$$d = d_o + \kappa f, \quad f = \sum_{n=0}^{\infty} \kappa^n f_n,$$

(3.6)

with $n \geq 0$. The function $f$ is a $G$-singlet, a $\mathcal{H}$-invariant and a scalar with weight $t = 0$ under generalized diffeomorphisms. The previous expansion was introduced in [8] in the context of Heterotic DFT but supersymmetry was not considered. As we are interested in this last point, we mimic the structure of the generalized perturbation of the generalized dilaton and propose

$$\Psi_A = \Psi_o A + \kappa \Theta_A, \quad \Theta_A = \sum_{n=0}^{\infty} \kappa^n \Theta_{nA},$$

(3.7)

and

$$\rho = \rho_o + \kappa g, \quad g = \sum_{n=0}^{\infty} \kappa^n g_n.$$

(3.8)

Using the conventions of the previous section it is possible to find that $\Theta_A$ is a $G$-singlet, a spinor of $O(9,1)_L$, a vector of $O(1,9+n)_R$ and a scalar with weight $t = 0$ under generalized diffeomorphisms, and $g$ is a $G$-singlet, a spinor of $O(9,1)_L$, an invariant of $O(1,9+n)_R$ and a scalar with weight $t = 0$ under generalized diffeomorphisms.

In the next part of this work we explicitly show how supersymmetry truncates the $\kappa$ expansions for some of the the generalized background fields in order to be consistent with the supersymmetric extension of the generalized Kerr-Schild ansatz defined in (3.1).

### 3.2 Supersymmetric consistency conditions

We start analyzing the supersymmetric transformation of $\Delta_{\underline{A\overline{B}}}$: Considering

$$\delta_\epsilon E_{MA} = \bar{\epsilon} \gamma^{[B} \Psi^{A]} E_{MB}$$

(3.9)

and proposing the $\kappa$ expansions discussed in the previous section we find,

$$\delta_\epsilon \Delta_{\underline{A\overline{B}}} = \bar{\epsilon} \gamma \Delta \Theta_{n\overline{B}},$$

(3.10)

where we have used

$$\delta^{(n)}(\Delta \eta^{-1} \Delta) = 0,$$

(3.11)
with $\delta$ a generic transformation. The expression (3.10) forces

$$\Theta_n = 0 \quad , \quad n \geq 1 .$$ (3.12)

On the other hand (3.10) is correct up to a generalized Lorentz transformation that can be reabsorbed in the generalized Lorentz parameter. Let us observe that the decomposition of $\Delta_A^B$ in terms of null vectors $K_M, \bar{K}_M$ is not allowed since (3.10) cannot be solved for both vectors.

The supersymmetric transformation of $\Theta_\alpha$ is

$$\delta_\epsilon \Theta_\alpha = \frac{1}{2} \Delta_B^A E_B^\epsilon + \frac{1}{4\kappa} \tilde{F}_{\alpha BC} \gamma^{BC} \epsilon$$ (3.13)

where

$$F_{\alpha BC} = F_{\alpha BC} + \tilde{F}_{\alpha BC} .$$ (3.14)

Since the perturbations on the fluxes are cubic in $\kappa$, we need to impose some supersymmetric consistency constraint on the generalized gravitino transformation. Explicitly we have,

$$\delta_\epsilon \Theta_0 = \frac{1}{2} \Delta_B^A E_B^\epsilon + \frac{1}{4} \left( \frac{1}{2} \Delta_B^A F_{o BC} - \Delta_B^\bar{D} F_{o D A C} + E_C (\Delta_B^\alpha) \right)$$

$$+ \frac{1}{\sqrt{2}} f_{DBC} \Delta_D^A \epsilon + \left( \frac{1}{2} \Delta_B^\bar{D} (\Delta_B^\alpha) + \sqrt{2} f_{MNP} \Delta_B^M \Delta_C^N \chi^P \right) \gamma^{BC} \epsilon ,$$ (3.15)

$$\delta_\epsilon \Theta_1 = \left( \frac{1}{4} \Delta_B^\bar{D} \Delta_C^\epsilon F_{o D A E} - \frac{1}{4} \Delta_B^\bar{D} \Delta_C^\epsilon \Delta_D^A \Delta_A^D \right)$$

$$+ \frac{1}{2} \Delta_B^\bar{D} (E_D^A \Delta_E^C) + \sqrt{2} f_{ADE} \Delta_B^\bar{D} \Delta_C^\epsilon \gamma^{BC} \epsilon ,$$ (3.16)

$$\delta_\epsilon \Theta_2 = \left( \frac{1}{8} \Delta_B^\bar{D} \Delta_C^\epsilon \Delta_D^A F_{o D E F} + \sqrt{2} f_{DEF} \Delta_D^A \Delta_B^E \Delta_C^F \right) \gamma^{BC} \epsilon ,$$ (3.17)

where we have used the following notation $f_{ABC} = f_{MNP} E_M^A E_N^B E_P^C$. Therefore we impose the following supersymmetric consistency constraints,

$$\delta^{(n)} \Theta_1 = \delta^{(n)} \Theta_2 = 0 ,$$ (3.18)
in order to reproduce a linear $\kappa$ expansion for the generalized perturbed gravitino. This requirement cannot be solved invoking (3.11) and thus (3.18) must be treated as extra constraints on the theory.

By a similar argument we seek constraints in the generalized background dilatino transformation,

$$\delta g = -\frac{1}{2\kappa} \Delta_A \bar{\Delta}_B \bar{E} \bar{\epsilon} - \frac{1}{12\kappa} \bar{F}_{ABC} \gamma^{ABC} \epsilon - \frac{1}{2\kappa} \bar{F}_B \gamma^B \epsilon$$

(3.19)

where

$$F_{ABC} = F_{oABC} + \bar{F}_{ABC}$$

$$F_A = F_{oA} + \bar{F}_A,$$

(3.20)

and

$$\bar{F}_{ABC} = -\frac{3\kappa}{2} (\Delta_A \bar{\Delta}_B F_{0DE|BC} + \sqrt{2} f_{D[A} \Delta_{BC]})$$

$$+ \frac{3\kappa^2}{4} (\Delta_A \bar{\Delta}_B \Delta^E F_{0DE|C} + \sqrt{2} f_{D[A} \Delta_{BC]})$$

$$- \frac{3\kappa^3}{8} (\Delta_A \bar{\Delta}_B \Delta^E F_{0DEF} + \sqrt{2} f_{DEF} \Delta_A \bar{\Delta}_B)$$

(3.21)

Because of the appearance of $f$ in the last expression, we have an infinite $\kappa$ expansion for the generalized dilatino that can be solved once the generalized dilaton is solved. The previous statement means that the $\kappa$ expansion of these fields are not restricted by supersymmetry.

### 3.3 Perturbed action and equations of motion

Up to this point, we have perturbed the field content of $\mathcal{N} = 1$ DFT in a consistent way. The action of the perturbed theory must be of the same form as (2.22), i.e.

$$S_{\mathcal{N} = 1} = \int d^{20} X e^{-2d} \mathcal{R} + \bar{\Psi}^\mathcal{T} \gamma^B \nabla_B \Psi - \bar{\Psi} \gamma^\mathcal{T} \nabla_{\mathcal{T}} + 2 \bar{\Psi}^\mathcal{T} \nabla_{\mathcal{T}}.$$

(3.22)
and the equations of motion up to leading order terms in fermions, are

\[ R_{BA} + \bar{\Psi} \gamma_B E_A \Psi_C - \bar{\rho} \gamma_B E_A \rho - 2 \bar{\Psi} E_B \rho = 0, \]
\[ \mathcal{R} + L_F = 0, \]
\[ \gamma^B \nabla_o B \Psi_o A + \nabla_A \rho = 0, \]
\[ \gamma^A \nabla_A \rho + \nabla_A \Psi_A = 0. \] (3.23)

Since the generalized geodesic equations introduced in (1.10) cannot be defined in terms of \( \Delta_{AB} \), \( R_{BA} \) has cubic contributions of the perturbation parameter \( \kappa \) coming from the generalized fluxes \( ^2 \). As a consequence, the generalized equations of motions are no longer linear in \( \kappa \) even if \( f = g = 0 \), unlike the result obtained in [8].

In the next section we proceed to parametrize the previous field content and find the necessary conditions to obtain the \( \mathcal{N} = 1 \) supersymmetric extension of the ordinary Kerr-Schild ansatz in the context of the low energy effective heterotic field theory. We start reviewing the parametrization of the background field content and then we go straightforwardly to the perturbative theory.

## 4 Reduction to \( \mathcal{N} = 1 \) supergravity

### 4.1 Parameterization of the background field content

We start by splitting the \( G \) and \( \mathcal{H} \) indices as \( M = (\mu, \nu, i) \) and \( A = (\mathcal{A}, \overline{\mathcal{A}}) \) with \( \mathcal{A} = a, \overline{\mathcal{A}} = (\bar{a}, \bar{i}) \), respectively, \( \mu, \nu, a, \bar{a}, \bar{i} = 0, \ldots, 9, i, \bar{i} = 1, \ldots, n \). The parametrization of the fundamental background fields of \( \mathcal{N} = 1 \) DFT must respect all the constraints of the theory. The generalized background frame is an \( O(d, d+n) \) element, so it is parametrized in the following way,

\[
E_{\nu A}^M = \begin{pmatrix}
E_{\nu a} & E_{\nu i}^a & E_{\nu i}^i \\
E_{\nu \overline{\alpha}} & E_{\nu \overline{i}}^\alpha & E_{\nu \overline{i}}^i \\
E_{\nu \overline{\alpha}} & E_{\nu \overline{i}}^\alpha & E_{\nu \overline{i}}^i 
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-\epsilon_{\nu a} - C_{\nu \alpha} \epsilon_{\alpha a} & \epsilon_{\nu i}^a & -A_{\nu i} \epsilon_{\alpha a} \\
\bar{e}_{\nu a} - C_{\nu \alpha} \bar{e}_{\alpha a} & \bar{e}_{\nu i}^a & -A_{\nu i} \bar{e}_{\alpha a} \\
\sqrt{2} A_{\nu i} \epsilon_{\alpha i} & 0 & \sqrt{2} \epsilon_{\alpha i}
\end{pmatrix}, \quad (4.1)
\]

\(^2\)Higher order terms are identically null.
where $e_{o\mu a}$ and $\overline{e}_{o\mu a}$ satisfy
\[ e_{o\mu a} \eta^{ab} e_{o\nu b} = \overline{e}_{o\mu a} \eta^{ab} \overline{e}_{o\nu b} = g_{o\mu\nu}, \quad (4.2) \]
with $\eta_{ab}$ the ten dimensional flat metric, $a, b = 0, \ldots, 9$, $C_{o\mu\nu} = b_{o\mu\nu} + \frac{1}{2} A^i_{o\mu} A_{o\nu i}$, with $A^i_{o\mu}$ being the gauge connection. The invariant projectors of DFT are parametrized in the following way
\[ P_{\bar{a}b} = -\eta_{ab} \delta^a_{\bar{a}} \delta^b_{\bar{b}}, \quad \overline{P}_{\bar{a}b} = \eta_{ab} \delta^a_{\bar{a}} \delta^b_{\bar{b}}, \quad \overline{P}_{ij} = e^{i}_\tau \kappa_{ij} e^{j}_\tau = \kappa_{i\bar{j}}, \quad (4.3) \]
where $\kappa_{ij}$ and $\kappa_{i\bar{j}}$ are the Cartan-Killing metrics associated with the $SO(32)$ or $E_8 \times E_8$ heterotic gauge group. The gauge fixing (4.2) imposes
\[ \delta e_{o\mu a} = \delta \overline{e}_{o\mu a}, \quad (4.4) \]
and therefore the parametrization of the components of the generalized Lorentz parameters are not independent
\[ \Gamma_{\bar{a}b} \delta^{\bar{a}}_{\bar{a}} \delta_{\bar{b}} \quad \Gamma_{\bar{a}b} \delta^{\bar{a}}_{\bar{a}} \delta_{\bar{b}} = -\Lambda_{ab} + \varepsilon [a \psi_{ob}], \quad (4.5) \]
where $\Lambda_{ab}$ denotes the generator of the $O(1, 9)$ transformations. We also impose $\delta E^a_\tau = 0$ and $\delta E^{\mu}_{-\tau} = 0$ which leads to
\[ \Gamma_{ij} = f_{ijk} \xi^k \delta^i_{\bar{i}} \delta^j_{\bar{j}} \quad \text{and} \quad \Gamma_{ai} = -\Gamma_{\bar{a}i} = \frac{1}{2\sqrt{2}} \gamma_a \chi_{ai} \delta^a_{\bar{a}} \delta^i_{\bar{i}}, \quad (4.6) \]
where we have parameterized the generalized gravitino field as
\[ \Psi_{oA} = (0, \epsilon^{\mu}_{o\mu} \psi_{o\nu}, \frac{1}{\sqrt{2}} e^i \chi_{o\nu}). \quad (4.7) \]
The structure constants are trivially incorporated,
\[ f_{MN}^P = \begin{cases} f_{ij}^k & \text{for } M, N, P = i, j, k \\ 0 & \text{otherwise.} \end{cases} \quad (4.8) \]
In addition we parameterize
\[ \xi^M = (\xi^\mu, \lambda^i, \xi^i), \quad (4.9) \]
where the parameter $\xi^\mu$ is associated with the usual Lie derivative, defined as

$$\mathcal{L}_\xi v^\mu = \xi^\nu \partial_\nu v^\mu + (\partial_\nu \xi^\mu) v^\nu ,$$

(4.10)

with $v^\mu$ a generic tensor. The parameter $\lambda_\mu$ parameterizes the abelian gauge symmetry of the background Kalb-Ramond field,

$$\delta \lambda_\mu = 2 \partial_\mu \lambda_v ,$$

(4.11)

while $\xi_i$ is the non-abelian gauge parameter. On the other hand, the parametrizations of the generalized background dilaton and dilatino are

$$d = \phi_o - \frac{1}{2} \log \sqrt{-g_o} ,$$

$$\rho = 2 \lambda_o + \gamma^a \psi_{oa} .$$

(4.12)

The $\gamma$-functions $\gamma^a = \gamma^a \delta^a{}_{ab}$ verify the Clifford algebra

$$\{ \gamma^a , \gamma^b \} = 2 \eta^{ab}$$

(4.13)

and the supersymmetric transformation rules of the background field content are

$$\delta \varepsilon_{o\mu}^a = \frac{1}{2} \varepsilon \gamma^a \psi_{o\mu} ,$$

$$\delta \psi_{o\mu} = \partial_\mu \varepsilon - \frac{1}{4} w_{oab}^{(+) \gamma^{ab} \varepsilon} ,$$

$$\delta \mu = 2 \phi_o + \frac{1}{2} \lambda_\mu \lambda_o,$$

$$\delta \lambda_o = \frac{1}{2} \gamma^a \partial_\alpha \phi_o \varepsilon + \frac{1}{24} H_{oabc} \gamma^{abc} \varepsilon ,$$

$$\delta \epsilon^i_o = \frac{1}{2} \varepsilon \gamma^{oi} \chi_o^i ,$$

$$\delta \chi_o^i = - \frac{1}{4} \gamma^{ab} \varepsilon$$

(4.14)

where

$$w_{oab}^{(+) \gamma^{ab} \varepsilon} = - e^\mu_a e_\alpha^c \partial_\mu e_{o\alpha c} + e^\mu_\alpha e_{o\beta c} \partial_\mu e_{o\alpha \beta} + e^\mu_\beta e_{o\alpha c} \partial_\mu e_{o\alpha \beta} \pm H_{o\mu \rho \sigma} e_\rho^a e_\sigma^b ,$$

$$F_{o\mu \nu} = 2 \partial_\nu A_{o\mu} - f^i_{jk} A_{o\mu} A_{o\nu}^k ,$$

$$H_{oabc} = 3 e^\mu_a e^\nu_b e^\rho_c \left( \partial_\mu b_{o\nu \rho} - A_{o\nu}^i \partial_\nu A_{o\rho}^j + \frac{1}{3} f^i_{jk} A_{o\mu} A_{o\nu}^j A_{o\rho}^k \right) .$$

The transformations (4.14) leave the low energy effective heterotic action invariant

$$S = \int d^{10}x \ e^{-2 \phi_o} \left[ R - \frac{1}{12} H_o \wedge H_o + 4 (\partial \phi_o)^2 - \frac{1}{4} \mbox{Tr} (F_o \wedge F_o) + L_F \right] ,$$

19
where \( L_F \) depends on the fermionic content of the theory. A detailed form of this is given in \[15\]. The conventions for the Riemann tensor are

\[
R^\rho_{\sigma\mu\nu} = e^\rho_o e^o_\sigma R_{\mu\nu}^{ab} = e^\rho_o e^o_\sigma \left( -2\partial_{[\mu} w_{\nu]\alpha]ab + 2w_{[\mu|\alpha} e_{\nu]\alpha]|cb} \right),
\]

and therefore the Ricci scalar is

\[
R = R_{\mu\nu}^{ab} e_a\epsilon^\mu b c^o\epsilon^o_{o\nu}.
\]

### 4.2 Parameterization of the perturbations

In section (3) we introduce the supersymmetric extension of the generalized Kerr-Schild ansatz in the flux formalism of DFT. Now we proceed with the parametrization of the perturbations of the generalized fields.

We start by considering that both components of the generalized frame

\[
E_{\hat{M}\hat{A}} = E_{o\hat{M}\hat{A}} + \frac{\kappa}{2} \Delta_{\hat{M}\hat{A}} E_{0\hat{M}} \hat{A}
\]

\[
E_{M\hat{A}} = E_{0M\hat{A}} - \frac{\kappa}{2} \Delta_{\hat{M}\hat{B}} E_{0M} \hat{B}
\]

are \( O(10, 10 + n) \) elements. So we can parametrize them as

\[
E^M_A = \begin{pmatrix} E_{\mu a} & E^\mu_a & E^i_a \\ E_{\mu\pi} & E^{\mu\pi} & E^i_{\pi} \\ E_{\mu\iota} & E^{\mu\iota} & E^i_{\iota} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e_{\mu a} - C_{\rho\mu} e^\rho_a & e^\mu_a & -A^i_{\rho i} e^\rho_a \\ \overline{e}_{\mu a} - C_{\rho\mu} \overline{e}^\rho_a & \overline{e}^\mu_a & -\overline{A}^i_{\rho i} \overline{e}^\rho_a \\ \sqrt{2} A_{\mu i} e^i_{\iota} & 0 & \sqrt{2}\overline{e}^i_{\iota} \end{pmatrix}
\]

where \( e_{\mu a} \) and \( \overline{e}_{\mu a} \) satisfy

\[
e_{\mu a} \eta^{ab} e_{\nu b} = \overline{e}_{\mu a} \eta^{ab} \overline{e}_{\nu b} = g_{\mu\nu}.
\]

Condition (4.4) forces

\[
\Delta_{\Delta_{\hat{B}}} = (\Delta_{ab} \delta_{\rho b}^{\rho a}, \Delta_{ai} \delta_{\omega i}^{\omega a}),
\]

where \( \Delta_{ab} \) is a symmetric perturbation that verifies

\[
\Delta_{ab} g^{bd} \Delta_{cd} + \Delta_{ai} \kappa^{ij} \Delta_{cj} = 0,
\]
and

\[ \Delta_{ai}g^{ab}\Delta_{bj} = 0, \]
\[ \Delta_{ai}g^{ab}\Delta_{bc} = 0, \] (4.22)
\[ \Delta_{ai}g^{ab}\Delta_{bj} = 0. \] (4.23)

The previous parametrization can be decomposed in the following way,

\[ \Delta_{ab} = \frac{1}{2}l_a\bar{l}_b + \frac{1}{2}\bar{l}_a l_b \] (4.24)
\[ \Delta_{ai} = 0 \] (4.25)

where \( l_a = e^{\mu}_a l_\mu \) is the rotation of the null vector associated to the perturbation of the metric (1.1), that satisfies

\[ l_a \eta^{ab} l_b = 0, \] (4.26)

and \( \bar{l}_a \) is an auxiliary null vector,

\[ \bar{l}_a \eta^{ab} \bar{l}_b = 0. \] (4.27)

In (4.24) \( \Delta_{ai} = 0 \) is a requirement in order to work with the following gauge fixing

\[ e_i^\bar{i} = e_{ai}^\bar{i} \] (4.28)

and we impose \( \eta^{ab}l_a\bar{l}_b = 0 \) for simplicity.

A very interesting aspect of (4.25) is that the supersymmetric extension of the Kerr-Schild formalism can be done in terms of a pair of null vectors, as we are going to verify. Using (4.18) and recalling that the generalized frame is an element of \( O(10, 10 + n) \) it is straightforward to find,

\[ g_{\mu\nu} = g_{o\mu\nu} + \kappa l_a \bar{l}_b \epsilon_{o\mu}^a \epsilon_{o\nu}^b \]
\[ b_{\mu\nu} = b_{o\mu\nu} - \kappa l_a \bar{l}_b \epsilon_{o\mu}^a \epsilon_{o\nu}^b \]
\[ A_{\mu i} = A_{o\mu i}. \] (4.29)

From the previous expression we note that the standard Kerr-Schild ansatz can be obtained in the case \( l_a = \bar{l}_a \). On the other hand, the perturbation of the 10-dimensional gravitino is

\[ \psi_a = \psi_{oa} + \kappa \Theta_{0a}. \] (4.30)
The supersymmetric transformation of \( l_a \) and \( \bar{l}_a \) in terms of \( \Delta_{ab} \) can be read from (3.10). When we parametrize it we find,

\[
\delta_\epsilon \Delta_{ab} = \bar{\epsilon} \gamma_a \Theta_{0b} + \frac{1}{2} \Delta^c_b \bar{\epsilon} \gamma_a \psi_{oc},
\]

(4.31)

where the second term comes from the gauge fixing (4.5) of the double Lorentz parameters. In this point we identify

\[
\Theta_{0a} = -\frac{1}{4} (\bar{l}_a l_b + \bar{l}_b l_a) \psi_b^b,
\]

(4.32)

to finally obtain

\[
\delta_\epsilon \Delta_{ab} = 0.
\]

(4.33)

Therefore we have showed that it is possible to decompose \( \Delta_{ab} \) in terms of a pair of null vectors transforming as,

\[
\delta l_a = \xi^\mu \partial_\mu l_a + l_b \Lambda_b^b,
\]

(4.34)

\[
\delta \bar{l}_a = \xi^\mu \partial_\mu \bar{l}_a + \bar{l}_b \Lambda_b^a.
\]

(4.35)

In the previous expressions we recognise a scalar transformation with respect to diffeomorphisms and a local Lorentz transformation. The identification (4.32) is consistent as long as both sides transform in the same way, which includes some extra supersymmetric consistency conditions at this level.

Let us observe that all the supersymmetric consistency conditions found in this work are stronger than the usual requirements of the ordinary Kerr-Schild ansatz but the EOM for \( g_{\mu\nu} \) is not linear in \( \kappa \) due to the fermionic and dilatonic degrees of freedom. As we have seen in the previous section, the perturbation of the dilaton is not constrained by supersymmetry,

\[
\phi = \phi_0 + \kappa f,
\]

(4.36)

and consistently, the perturbation of the dilatino has to be,

\[
\lambda = \lambda_0 + \frac{\kappa}{2} g.
\]

(4.37)

Finally, as the bosonic gauge sector is unperturbed, we have

\[
\Theta_{0i} = 0
\]

(4.38)

and therefore \( \chi_i = \chi_{0i} \).
5 Conclusions

In this work we present the supersymmetric extension of the Generalized Kerr-Schild ansatz in the flux formulation of $\mathcal{N} = 1$ supersymmetric DFT. This ansatz is compatible with $\mathcal{N} = 1$ supersymmetry as long as it is not written in terms of generalized null vectors. We find that imposing a set of supersymmetric consistency conditions the perturbation of the generalized gravitino is linear in $\kappa$. The perturbations of the generalized dilaton and dilatino have no restrictions.

When we parametrize the theory in terms of the field content of the low energy effective heterotic supergravity, we find linear perturbations for the 10-dimensional vielbein, Kalb-Ramond field and gravitino in terms of a pair of null vectors and an unrestricted perturbation for the 10-dimensional dilaton and dilatino. The remaining fields of the theory, $A_{0\mu}$ and $\chi_0$ cannot be perturbed. Moreover, the supersymmetric conditions found in the $\mathcal{N} = 1$ DFT framework must be supplemented with extra consistency conditions. However linearity in the EOM of $g_{\mu\nu}$ cannot be achieved when supersymmetry is turned on.

The present results open the door to future directions:

- Finding all the 2-derivative deformations to the DFT action was addressed in [16] and then fully studied in several works [17]. In [16], a biparametric family of duality covariant theories was introduced. Some of them are low energy effective field theories of string theories but some of them are not (the main example is the so-called HSZ theory [18]). Exploring the Generalized Kerr-Schild ansatz in all these theories is straightforward with the results of this work.

- Extended Kerr-Schild (xKS) [19] is a possible deformation of the Kerr-Schild anzast which consists in a linear perturbation using 2 null vectors and the inverse metric tensor receives an exact and second-order perturbation. Implementing this kind of more general but exact ansatz in the context of $\mathcal{N} = 1$ DFT would allow to describe a wide range of heterotic supergravity solutions in a duality covariant way.
Acknowledgements

We sincerely thank C.Nunez, D.Marques, S.Iguri and T.Codina for many useful comments. Support by CONICET is also gratefully acknowledged.

References

[1] R. P. Kerr, 'Gravitational field of a spinning mass as an example of algebraically special metrics,' Phys. Rev. Lett.11 (1963) 237.

[2] R. P. Kerr and A. Schild, 'A new class of vacuum solutions of the Einstein field equations,' Proc. Symp. Appl. Math.17(1965), 199.

R. P. Kerr and A. Schild, 'Some algebraically degenerate solutions of Einsteins gravitational field equations,' Proc. Symp. Appl. Math.17, 199 (1965).

G. C. Debney, R. P. Kerr and A. Schild, 'Solutions of the Einstein and Einstein-Maxwell Equations,' J. Math. Phys.10 , 1842 (1969).

[3] M. Gurses and F. Gursey, 'Lorentz Covariant Treatment of the Kerr-Schild Geometry,' J. Math. Phys.16, 2385 (1975)

[4] R. C. Myers and M. J. Perry, 'Black Holes In Higher Dimensional Space-Times,' Annals Phys. 172, 304 (1986).

S.W. Hawking, C.J. Hunter, M. Taylor, Phys. Rev. D59 064005 (1999).

G.W. Gibbons, H. Lu, D.N. Page and C.N. Pope, J. Geom. Phys. 53, 49 (2005)

[5] A. Anabalon, N. Deruelle, Y. Morisawa et al., 'Kerr-Schild ansatz in Einstein-Gauss-Bonnet gravity: An exact vacuum solution in five dimensions,' Class. Quant. Grav. 26, 065002 (2009), arXiv:0812.3194 [hep-th].

[6] B. Ett and D. Kastor, 'KerrSchild Ansatz in Lovelock Gravity,' JHEP 1104 (2011) 109, arXiv:1103.3182 [hep-th].

[7] R. Monteiro, D. OConnell and C. D. White, 'Black holes and the double copy,' JHEP 1412 (2014) 056 arXiv:1410.0239 [hep-th].
A. K. Ridgway and M. B. Wise, 'Static Spherically Symmetric Kerr-Schild Metrics and Implications for the Classical Double Copy,' Phys. Rev. D 94 (2016) no.4, 044023 [arXiv:1512.02243 [hep-th]].

M. Carrillo-Gonzalez, R. Penco and M. Trodden, 'The classical double copy in maximally symmetric spacetimes,' JHEP 1804 (2018) 028 [arXiv:1711.01296].

R. Alawadhi, D. S. Berman, B. Spence, D. P. Veiga “S-duality and the double copy”, [arXiv:1911.06797 [hep-th]].

[8] K. Lee, 'Kerr-Schild Double Field Theory and Classical Double Copy,’ JHEP 1810(2018) 027.

W. Cho, K. Lee, 'Heterotic Kerr-Schild Double Field Theory and Classical Double Copy,’ JHEP 07(2019) 030.

[9] W. Siegel, 'Two vierbein formalism for string inspired axionic gravity,’ Phys. Rev. D 47 (1993) 5453 [hep-th/9302036].

W. Siegel, ‘Superspace duality in low-energy superstrings,’ Phys. Rev. D 48 (1993) 2826 [hep-th/9305073].

W. Siegel, 'Manifest duality in low-energy superstrings,' In *Berkeley 1993, Proceedings, Strings 93* 353-363, and State U. New York Stony Brook - ITP-SB-93-050 (93,rec.Sep.) 11 p. (315661) [hep-th/9308133].

[10] C. Hull and B. Zwiebach, 'Double Field Theory,’ JHEP 0909 (2009) 099 [arXiv:0904.4664 [hep-th]].

O. Hohm, C. Hull and B. Zwiebach, 'Generalized metric formulation of Double Field Theory,’ JHEP 1008 (2010) 008 [arXiv:1006.4823 [hep-th]].

[11] I. Jeon, K. Lee and J. H. Park, 'Stringy differential geometry, beyond Riemann,’ Phys. Rev. D 84 (2011) 044022 [arXiv:1105.6294 [hep-th]].

I. Jeon, K. Lee and J. H. Park, 'Differential geometry with a projection: Application to Double Field Theory,’ JHEP 1104 (2011) 014 [arXiv:1011.1324 [hep-th]].
[12] O. Hohm and S. K. Kwak, 'Double Field Theory Formulation of Heterotic Strings,' JHEP **1106** (2011) 096 [arXiv:1103.2136 [hep-th]].

[13] G. Aldazabal, D. Marques and C. Nunez, 'Double Field Theory: A Pedagogical Review,' Class. Quant. Grav. **30**, 163001 (2013) [arXiv:1305.1907 [hep-th]].

O. Hohm, D. Lust and B. Zwiebach, 'The Spacetime of Double Field Theory: Review, Remarks, and Outlook,' Fortsch. Phys. **61**, 926 (2013) [arXiv:1309.2977 [hep-th]].

D. S. Berman and D. C. Thompson, 'Duality Symmetric String and M-Theory,' Phys. Rept. **566**, 1 (2014) [arXiv:1306.2643 [hep-th]].

[14] O. Hohm and S. K. Kwak, 'Frame-like Geometry of Double Field Theory,' J. Phys. A **44** (2011) 085404 [arXiv:1011.4101 [hep-th]].

D. Geissbuhler, D. Marques, C. Nunez and V. Penas, “Exploring Double Field Theory,” JHEP **1306** (2013) 101 doi:10.1007/JHEP06(2013)101 [arXiv:1304.1472 [hep-th]].

[15] A. Coimbra, C. Strickland-Constable and D. Waldram, “Supergravity as Generalised Geometry I: Type II Theories,” [arXiv:1107.1733 [hep-th]].

I. Jeon, K. Lee and J. H. Park, “Incorporation of fermions into double field theory,” JHEP **1111**, 025 (2011) [arXiv:1109.2035 [hep-th]].

I. Jeon, K. Lee and J. H. Park, “Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity,” Phys. Rev. D **85**, 081501 (2012) Erratum: [Phys. Rev. D **86**, 089903 (2012)] [arXiv:1112.0069 [hep-th]].

O. Hohm and S. K. Kwak, 'N=1 Supersymmetric Double Field Theory,' JHEP **1203**, 080 (2012) [arXiv:1111.7293 [hep-th]].

D. S. Berman and K. Lee, “Supersymmetry for Gauged Double Field Theory and Generalised Scherk-Schwarz Reductions,” Nucl. Phys. B **881**, 369 (2014) [arXiv:1305.2747 [hep-th]].

E. Lescano, C. Nuñez and A. Rodriguez, to appear.

[16] “T-duality and α’-corrections,” JHEP **1510**, 084 (2015) [arXiv:1507.00652 [hep-th]].
[17] A. Coimbra, R. Minasian, H. Triendl and D. Waldram, “Generalised geometry for string corrections,” JHEP 1411 (2014) 160 [arXiv:1407.7542 [hep-th]].

K. Lee, ‘Quadratic α'-corrections to heterotic double field theory,’ Nucl. Phys. B 899, 594 (2015) [arXiv:1504.00149 [hep-th]].

W. H. Baron, J. J. Fernandez-Melgarejo, D. Marques and C. Nunez, “The Odd story of α’-corrections,” JHEP 1704, 078 (2017) [arXiv:1702.05489 [hep-th]].

W. H. Baron, E. Lescano and D. Marques, “The generalized Bergshoeff-de Roo identification,” JHEP 1811, 160 (2018) [arXiv:1810.01427 [hep-th]].

[18] O. Hohm, W. Siegel and B. Zwiebach, “Doubled α'-geometry,” JHEP 1402, 065 (2014) doi:10.1007/JHEP02(2014)065 [arXiv:1306.2970 [hep-th]].

O. Hohm, U. Naseer and B. Zwiebach, On the curious spectrum of duality invariant higher-derivative gravity, JHEP 1608 (2016) 173 doi:10.1007/JHEP08(2016)173 [arXiv:1607.01784 [hep-th]].

E. Lescano and D. Marques, “Second order higher-derivative corrections in Double Field Theory,” JHEP 1706, 104 (2017) [arXiv:1611.05031 [hep-th]].

[19] A. N. Aliev and D. K. Ciftci, A note on rotating charged black holes in Einstein-Maxwell-Chern-Simons theory, Phys. Rev. D79 (2009) 044004, [arXiv:0811.3948 [hep-th]].

B. Ett and D. Kastor, An extended Kerr-Schild ansatz, Class. Quant. Grav. 27 (2010) 185024, [arXiv:1002.4378 [hep-th]].

T. Malek, Extended Kerr-Schild spacetimes: General properties and some explicit examples, Class. Quant. Grav. 31, 185013 (2014) [arXiv:1401.1060 [hep-th]].

Kwangeon Kim, Kanghoon Lee, Ricardo Monteiro, Isobel Nicholson, David Peinador Veigac, ”The Classical Double Copy of a Point Charge”, [arXiv:1912.02177 [hep-th]].