A COMPLEX SURFACE OF GENERAL TYPE
WITH $p_g = 0$, $K^2 = 4$, AND $\pi_1 = \mathbb{Z}/2\mathbb{Z}$

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Abstract. We construct a minimal complex surface of general type with $p_g = 0$, $K^2 = 4$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ using a rational blow-down surgery and a $\mathbb{Q}$-Gorenstein smoothing theory. In a similar fashion, we also construct a symplectic 4-manifold with $b_2^+ = 1$, $K^2 = 5$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

1. Introduction

In this paper we construct a new minimal complex surface of general type with $p_g = 0$ and $K^2 = 4$. It is a fundamental problem in the classification of complex surfaces to find a minimal complex surface of general type with $p_g = 0$. Recently simply connected complex surfaces of general type with $p_g = 0$ and $K^2 \leq 4$ are constructed; Y. Lee and J. Park [9], the author-J. Park-D. Shin [11, 12]. Also many families of non-simply connected complex surfaces of general type with $p_g = 0$ have been constructed; cf. BHPV [2, VII].

It is nevertheless an intriguing problem to find complex surfaces of general type with $p_g = 0$ and small nonzero fundamental groups, especially, $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ (the smallest nonzero group), because there are no known examples with $K^2 \geq 4$ and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. For instance, the first example with $p_g = 0$, $K^2 = 1$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ was constructed by Barlow [1]. It is very recent that examples with $p_g = 0$, $K^2 > 1$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ are constructed: D. Cartwright and T. Steger [3] constructed complex surfaces of general type with $p_g = 0$, $K^2 = 3$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. J. Keum and Y. Lee [6] constructed complex surfaces of general type with $p_g = 0$, $K^2 = 1, 2, 3$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$. However it is not known yet whether there are complex surfaces of general type with $p_g = 0$, $K^2 = 4$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

Motivated by the work of J. Keum and Y. Lee [6], we extend their result to the $K^2 = 4$ case in this paper. The main result of this paper is the following theorem.

Theorem 1.1. There exists a minimal complex surface of general type with $p_g = 0$, $K^2 = 4$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

The key ingredient of this paper is to construct a surface $Z$ with special configurations of rational curves by appropriately blowing-up several times starting with an Enriques surface. And then we apply a similar method developed in Y. Lee and J. Park [9] to the surface $Z$. That is, we contract the special chains of $\mathbb{C}\mathbb{P}^1$’s from the surface $Z$ so that we get a surface $X$ with permissible singular points. We prove that there is a global $\mathbb{Q}$-Gorenstein smoothing of the singular surface $X$. For this

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we show that the obstruction to a global $\mathbb{Q}$-Gorenstein smoothing is zero by using a similar strategy as in J. Keum and Y. Lee [6]. Then a general fiber of a global $\mathbb{Q}$-Gorenstein smoothing of the singular surface $X$ is a complex surface of general type with $p_g = 0$ and $K^2 = 4$. Finally we show that the surface has $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ by applying a Milnor fiber theory and a rational blow-down surgery.

We also construct a symplectic 4-manifold with $b^+_2 = 1$, $K^2 = 5$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ using the same technique.

Theorem 1.2. There exists a symplectic 4-manifold with $b^+_2 = 1$, $K^2 = 5$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

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2. Main construction

We start with a special elliptic fibration on an Enriques surface. According to Kondo [7], there is an Enriques surface $Y$ with an elliptic fibration over $\mathbb{P}^1$. In particular the Enriques surface $Y$ has an $I_9$-singular fiber and a nodal singular fiber $F$ which are not multiple fibers and two bisections $S_1$ and $S_2$ not passing through the node of the nodal fiber $F$. The configuration of the fibers and sections are as in Figure 1.

![Figure 1: An Enriques surface $Y$](image)

We blow up four times totally at the four marked points $\bullet$. We blow up again three times and eight times at the two marked points $\bigcirc$, respectively. We then get a surface $Z = Y\sharp 15\mathbb{P}^2$; Figure 2. There exist two disjoint linear chains of $\mathbb{C}\mathbb{P}^1$'s.
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By applying $\mathbb{Q}$-Gorenstein smoothing theory to the surface $Z$ as in [9, 11, 12], we construct a complex surface of general type with $p_g = 0$ and $K^2 = 4$. That is, we first contract the two chains of $\mathbb{CP}^1$’s from the surface $Z$ so that it produces a normal projective surface $X$ with two permissible singular points. In Section 3 we will show that the singular surface $X$ has a global $\mathbb{Q}$-Gorenstein smoothing. Let $X_t$ be a general fiber of the $\mathbb{Q}$-Gorenstein smoothing of $X$. Since $X$ is a singular surface with $p_g = 0$ and $K^2 = 4$, by applying general results of complex surface theory and a $\mathbb{Q}$-Gorenstein smoothing theory, one may conclude that a general fiber $X_t$ is a complex surface with $p_g = 0$ and $K^2 = 4$. Furthermore, it is not difficult to show that a general fiber $X_t$ is minimal by using a similar technique in [9, 11, 12]; hence $X_t$ is of general type. Finally it remains to show that $\pi_1(X_t) = \mathbb{Z}/2\mathbb{Z}$.

**Proposition 2.1.** $\pi_1(X_t) = \mathbb{Z}/2\mathbb{Z}$.

**Proof.** Let $Z_{73,19}$ be a rational blow-down 4-manifold obtained from $Z$ by replacing two disjoint configurations $C_{73,50}$ and $C_{19,13}$ with the corresponding rational balls $B_{73,50}$ and $B_{19,13}$, respectively. Then, since a general fiber $X_t$ of a $\mathbb{Q}$-Gorenstein smoothing of $X$ is diffeomorphic to the rational blow-down 4-manifold $Z_{73,19}$ by a Milnor fiber theory, we have $\pi_1(X_t) = \pi_1(Z_{73,19})$. Hence it suffices to show that $\pi_1(Z_{73,19}) = \mathbb{Z}/2\mathbb{Z}$.

We first decompose the surface $Z$ into

$$Z = Z_0 \cup \{C_{73,50} \cup C_{19,13}\}.$$ 

Then the 4-manifold $Z_{73,19}$ can be decomposed into

$$Z_{73,19} = Z_0 \cup \{B_{73,50} \cup B_{19,13}\}.$$
Let $\alpha$ and $\beta$ be normal circles of disk bundles $C_{73,50}$ and $C_{19,13}$ over the $(-2)$-curve and the $(-4)$-curve, respectively; Figure 3.

![Figure 3: Two normal circles on a surface $Z$](image)

Note that $\pi_1(Z) = \mathbb{Z}_2$ for $\pi_1(Y) = \mathbb{Z}_2$. Since $C_{73,50}$ and $C_{19,13}$ is simply connected, by Van Kampen theorem, we have

$$\mathbb{Z}_2 = \pi_1(Z) = \pi_1(Z_0)/\langle N_{\gamma_1} i_*(\alpha)^{-1}, N_{\gamma_2} j_*(\beta)^{-1} \rangle,$$

where $i_* : \mathbb{Z} \to \pi_1(Z_0)$ and $j_* : \mathbb{Z} \to \pi_1(Z_0)$ are the induced homomorphisms by the inclusions $i : C_{73,50} \cap Z_0 \to Z_0$ and $j : C_{19,13} \cap Z_0 \to Z_0$, respectively, and $\gamma_1$ (or $\gamma_2$) is a path connecting $\alpha$ (or $\beta$) and the reference point, respectively. Since two normal circles $\alpha$ and $\beta$ lie on the $(-1)$-sphere as in Figure 3, $\gamma_1^i i_*(\alpha)^{-1}$ and $\gamma_2^j j_*(\beta)^{-1}$ have the same order in $\pi_1(Z_0)$. However, since two integers 73 and 19 are relatively prime, we have $\langle N_{\gamma_1} i_*(\alpha)^{-1}, N_{\gamma_2} j_*(\beta)^{-1} \rangle = 1$. Therefore

$$\pi_1(Z_0) = \mathbb{Z}_2.$$

We now consider $\pi_1(Z_0 \cup B_{73,50})$. Note that the map $\pi_1(\partial B_{73,50}) \to \pi_1(Z_0)$ is given by

$$\mathbb{Z}_{73} \to \mathbb{Z}_2, \quad T \mapsto 0$$

and the map $\pi_1(\partial B_{73,50}) \to \pi_1(B_{73,50})$ is given by

$$\mathbb{Z}_{73} \to \mathbb{Z}_{73}, \quad T \mapsto T.$$

Therefore we have

$$\pi_1(Z_0 \cup B_{73,50}) = \pi_1(Z_0) *_{\pi_1(\partial B_{73,50})} \pi_1(B_{73,50}) = \mathbb{Z}_2 *_{\mathbb{Z}_{73}} \mathbb{Z}_{73} = \mathbb{Z}_2.$$

Similarly, we can conclude that

$$\pi_1(Z_{19,13}) = \pi_1(Z_0 \cup B_{19,13}) *_{\pi_1(\partial B_{19,13})} \pi_1(B_{19,13}) = \mathbb{Z}_2 *_{\mathbb{Z}_{19}} \mathbb{Z}_{19} = \mathbb{Z}_2. \quad \Box$$
3. Existence of a global $\mathbb{Q}$-Gorenstein smoothing

This section is devoted to a proof of the following theorem.

**Theorem 3.1.** The singular surface $X$ has a global $\mathbb{Q}$-Gorenstein smoothing.

The following proposition tells us a sufficient condition for the existence of a global $\mathbb{Q}$-Gorenstein smoothing of $X$.

**Proposition 3.2** (Y. Lee and J. Park [9]). Let $X$ be a normal projective surface with singularities of class $T$. Let $\pi : V \rightarrow X$ be the minimal resolution and let $A$ be the reduced exceptional divisor. Suppose that $H^2(T_V(-\log A)) = 0$. Then there is a global $\mathbb{Q}$-Gorenstein smoothing of $X$.

Since the contraction map $Z \rightarrow X$ is the minimal resolution of the singular surface $X$, the existence of a global $\mathbb{Q}$-Gorenstein smoothing of $X$ follows from the vanishing of the cohomology $H^2(T_Z(-\log A))$, where $A$ is the divisor on $Z$ consisting of the two linear chains of $\mathbb{C}^2$’s contracted to the two singular points of $X$. On the one hand we have the following well-known result.

**Proposition 3.3** (Flenner and Zaidenberg [5, §1]). Let $V$ be a nonsingular surface and let $A$ be a simple normal crossing divisor in $V$. Let $f : V' \rightarrow V$ be a blowing up of $V$ at a point $p$ of $A$. Set $A' = f^{-1}(A)_{\text{red}}$. Then $h^2(T_{V'}(-\log A')) = h^2(T_V(-\log A))$.

Let $\tau : V \rightarrow Y$ be the blowing-up at the node of the nodal singular fiber $F$ and let $E$ be the exceptional divisor of $\tau$. We denote again by $F$ the proper transforms of the nodal singular fibers $F$ on $V$. Let $D_1, \ldots, D_7$ be a part of the $I_9$-singular fiber; Figure 4. Let

$$D = D_1 + \cdots + D_7 + S_1 + S_2 + F \in \text{Div}(V).$$

By Proposition 3.3, we have

$$h^2(T_Z(-\log A)) = h^2(T_V(-\log D)).$$

(3.1)

Theorem 3.1 follows from (3.1) and the following proposition:

**Proposition 3.4.** $H^2(T_V(-\log D)) = H^0(\Omega_V(-\log D)(K_V)) = 0$.

In order to prove Proposition 3.4, we follow the same strategy as in J. Keum and Y. Lee [6]. That is, we consider a K3 surface $Y$ blown-up two times which is a double covering of the surface $V = Y \# \mathbb{CP}^2$ and then we use the push-forward map of the double covering for proving that $H^0(\Omega_V(-\log D)(K_V)) = 0$.

We first construct a double covering of the surface $V = Y \# \mathbb{CP}^2$. According to Kondo [7], there is an unramified double covering $\phi : \overline{Y} \rightarrow Y$ from a K3 surface $\overline{Y}$ to the Enriques surface $Y$. The K3 surface $\overline{Y}$ has two $I_9$-singular fiber, two nodal singular fiber $F_1$ and $F_2$, and four sections $S_1, \ldots, S_4$ such that $\phi(F_1) = \phi(F_2) = F$, $\phi(S_1) = \phi(S_3) = S_1$, and $\phi(S_2) = \phi(S_4) = S_2$; Figure 5. Let $\tau : \overline{V} \rightarrow \overline{Y}$ be the blowing-up at the two nodes of the two nodal singular fibers $F_1, F_2$ and let $E_1, E_2$ be the exceptional divisors of $\tau$. We denote again by $F_1, F_2$ the proper transforms of the nodal singular fibers $F_1, F_2$ on $\overline{V}$; Figure 6. It is clear that there is an induced unramified double covering $\psi : \overline{V} \rightarrow V$. We denote

$$\Delta = D_1 + \cdots + D_7 + S_1 + \cdots + S_4 + F_1 \in \text{Div}(\overline{V}).$$

Note that $\Delta \leq \psi^*D$ and $\psi_* \Delta = D$.

The proof of Proposition 3.4 begins with the following two results.
Figure 4: A surface $V = \mathbb{P}^2 \mathbb{C}P^2$

Figure 5: A K3 surface $\overline{\mathcal{Y}}$

**Proposition 3.5** (Esnault and Viehweg [4, §2]). Let $A = \sum_{i=1}^r A_i$ be a reduced normal crossing divisor on an algebraic manifold $W$. One has the exact sequence

$$0 \to \Omega_W(\log A) \to \Omega_W(\log(A - A_1))(A_1) \to \Omega_{A_1}(\log(A - A_1)|_{A_1})(A_1) \to 0.$$  

**Lemma 3.6.** $H^0(\Omega_{\mathcal{Y}}(\log \Delta)(K_{\mathcal{Y}})) = 0$.

**Proof.** Note that $K_{\mathcal{Y}} = \mathcal{E}_1 + \mathcal{E}_2$. By Proposition 3.5 we have an exact sequence

$$0 \to \Omega_{\mathcal{Y}}(\log(\Delta + \mathcal{E}_1))(\mathcal{E}_2) \to \Omega_{\mathcal{Y}}(\log \Delta)(\mathcal{E}_1 + \mathcal{E}_2) \to \Omega_{\mathcal{E}_1}(\log \Delta|_{\mathcal{E}_1})(\mathcal{E}_1 + \mathcal{E}_2) \to 0.$$
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Since $E_1^2 = -1$ and $E_1E_2 = 0$, we have $H^0(\Omega_{\mathcal{T}_1}(\log \Delta|_{\mathcal{T}_1})(E_1 + E_2)) = 0$. Therefore it follows that

$$H^0(\Omega_{\mathcal{T}}(\log \Delta)(K_{\mathcal{T}})) = H^0(\Omega_{\mathcal{T}}(\log(\Delta + E_1))(E_2)).$$

Applying Proposition 3.5 again, we have an exact sequence

$$0 \to \Omega_{\mathcal{T}}(\log(\Delta + E_1))(E_2) \to \Omega_{\mathcal{T}}(\log(\Delta + E_1))(E_2) \to \Omega_{\mathcal{T}}(\log(\Delta + E_1)(E_2)) \to 0.$$

Since $H^0(\Omega_{\mathcal{T}}(\log(\Delta + E_1))(E_2)) = 0$, we have

$$H^0(\Omega_{\mathcal{T}}(\log(\Delta))(K_{\mathcal{T}})) = H^0(\Omega_{\mathcal{T}}(\log(\Delta + E_1))(E_2)) \to H^0(\Omega_{\mathcal{T}}(\log(\Delta + E_1 + E_2))). \quad (3.2)$$

Therefore we need to show that $H^0(\Omega_{\mathcal{T}}(\log(\Delta + E_1 + E_2))) = 0$.

We now consider the following exact sequence:

$$0 \to \Omega_{\mathcal{T}} \to \Omega_{\mathcal{T}}(\log(\Delta + E_1 + E_2)) \to \bigoplus_{i=1}^7 \mathcal{O}_{\mathcal{T}_{i}} \oplus \bigoplus_{i=1}^4 \mathcal{O}_{\mathcal{S}_{i}} \oplus \mathcal{O}_{\mathcal{F}_1} \oplus \bigoplus_{i=1}^2 \mathcal{O}_{\mathcal{E}_{i}} \to 0.$$

Note that the connecting homomorphism

$$\delta : \bigoplus_{i=1}^7 H^0(\mathcal{O}_{\mathcal{T}_{i}}) \oplus \bigoplus_{i=1}^4 H^0(\mathcal{O}_{\mathcal{S}_{i}}) \oplus H^0(\mathcal{O}_{\mathcal{F}_1}) \oplus \bigoplus_{i=1}^2 H^0(\mathcal{O}_{\mathcal{E}_{i}}) \to H^1(\Omega_{\mathcal{T}})$$

is the first Chern class map. Since the intersection matrix consisting of the intersection numbers of $\mathcal{T}_{i}$ ($i = 1, \ldots, 7$), $\mathcal{S}_{j}$ ($j = 1, \ldots, 4$), and $\mathcal{F}_1$, $\mathcal{E}_{k}$ ($k = 1, 2$) is invertible, their images by the map $\delta$ are linearly independent. Therefore the map $\delta$ is injective. Furthermore $H^0(\Omega_{\mathcal{T}}) = 0$. Hence, we have

$$H^0(\Omega_{\mathcal{T}}(\log(\Delta + E_1 + E_2))) = 0.$$

Therefore it follows from (3.2) that

$$H^0(\Omega_{\mathcal{T}}(\log(\Delta))(K_{\mathcal{T}})) = H^0(\Omega_{\mathcal{T}}(\log(\Delta + E_1 + E_2))) = 0. \quad \square$$
Proof of Proposition 3.4. Since $K_{\Phi} = \psi^* K_V$, it follows from the projection formula that
\[ \psi_*(\Omega_{\Phi}(\log \Delta)(K_{\Phi})) = \psi_*(\Omega_{\Phi}(\log \Delta))(K_V). \]
On the one hand, by the choice of $\Delta$, we have
\[ \Omega_V(\log D) \subset \psi_*(\Omega_{\Phi}(\log \Delta)). \]
Therefore there is an injection
\[ 0 \to \Omega_V(\log D)(K_V) \to \psi_*(\Omega_{\Phi}(\log \Delta)(K_{\Phi})). \]
Hence it follows by Lemma 3.6 that
\[ H^0(V, \Omega_V(\log D)(K_V)) = 0. \]

4. A symplectic 4-manifold with $b_2^+ = 1$, $K^2 = 5$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$

In this section we construct a symplectic 4-manifold with $b_2^+ = 1$, $K^2 = 5$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ using a rational blow-down surgery.

We consider again the Enriques surface $Y$ used in Section 2. We blow up five times totally at the five marked points $\bullet$. We blow up again three times and four times at the two marked points $\bigcirc$, respectively. We then get a surface $Z = Y_{12} \# \mathbb{CP}^2$; Figure 8. There exist two disjoint linear chains of $\mathbb{CP}^1$’s in $Z$:

$C_{4,1} : \quad -6 -2 -2$

$C_{151,31} : \quad -5 -8 -6 -2 -3 -2 -2 -2 -3 -2 -2 -2 -2 -2 -2 -2$

We now perform a rational blow-down surgery of the surface $Z = Y_{12} \# \mathbb{CP}^2$. By the results of Symington [13, 14], the rational blow-down $Z_{151,4}$ is a symplectic 4-manifold. Thus we get a symplectic 4-manifold $Z_{151,4}$ with $b_2^+ = 1$ and $K^2 = 5$. By applying a similar method in the proof of Proposition 2.1, one can easily show that $\pi_1(Z_{151,4}) = \mathbb{Z}/2\mathbb{Z}$. 

Figure 7: An Enriques surface $Y$
The rational blow-down $Z_{151,4}$ of the surface $Z$ in the construction above is a symplectic 4-manifold with $b_2^+ = 1$, $K^2 = 5$, and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.

**Remarks.**
1. One can prove that the symplectic 4-manifold $Z_{151,4}$ constructed in this section is minimal by using a technique in Ozsváth and Szabó [10].
2. It is an intriguing question whether the symplectic 4-manifold $Z_{151,4}$ admit a complex structure. Since the cohomology $H^2(T_X^0)$ is not zero in this case, it is hard to determine whether there exists a global $\mathbb{Q}$-Gorenstein smoothing. We leave this question for future research.
3. As a corollary, we can reconstruct a simply connected symplectic 4-manifold with $b_2^+ = 3$ and $K^2 = 10$ from the symplectic 4-manifold $Z_{151,4}$. We briefly sketch the construction. We consider the unramified double covering $\phi: \overline{Y} \to Y$ from the K3 surface $\overline{Y}$ to the Enriques surface $Y$ in the proof of Proposition 3.4. Whenever we blow up $Y$ in the above construction of the surface $Z = Y\sharp 12\mathbb{CP}^2$, we blow up twice $Y$ at the preimages of $\pi$. Then we obtain an unramified double covering $\overline{\pi}: \overline{Z} \to Z$ from a complex surface $\overline{Z} = \overline{Y}\sharp 24\mathbb{CP}^2$ which has four disjoint linear chains of $\mathbb{CP}^1$'s: two $C_{4,1}$'s and two $C_{151,31}$'s. We perform a rational blow-down surgery of the surface $\overline{Z}$. We then get a symplectic 4-manifold $\overline{Z}_{151,4}$ with $b_2^+ = 3$ and $K^2 = 10$. Since $\pi_1(Z_{151,4}) = \mathbb{Z}/2\mathbb{Z}$ and there is an unramified double covering $\overline{Z}_{151,4} \to Z_{151,4}$ induced by the double covering $\pi: \overline{Z} \to Z$, the symplectic 4-manifold $\overline{Z}_{151,4}$ is simply connected.

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