Finiteness properties of arithmetic groups over function fields

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Abstract

We determine when an arithmetic subgroup of a reductive group defined over a global function field is of type $FP_\infty$ by comparing its large-scale geometry to the large-scale geometry of lattices in real semisimple Lie groups.

1 Introduction

Throughout this paper, $K$ is a global function field, and $S$ is a finite nonempty set of pairwise inequivalent valuations on $K$. We let $\mathcal{O}_S \leq K$ be the corresponding ring of $S$-integers. We denote a reductive $K$-group by $G$.

In 1971 Serre proved that $G(\mathcal{O}_S)$ is of type $WFL$ if and only if the semisimple $K$-rank of $G$ equals 0; see Théorème 4 of [Se 1] and the following Compléments.

As type $FP_\infty$ is a weaker property than type $WFL$, an immediate consequence is that $G(\mathcal{O}_S)$ is of type $FP_\infty$ if the semisimple $K$-rank of $G$ equals 0. The converse of this statement had been believed since the late 1970’s and evidence had been collected to support it as a conjecture. However, it remained unresolved in general.

Our main result confirms this conjecture:

**Theorem A** The arithmetic group $G(\mathcal{O}_S)$ is of type $FP_\infty$ if and only if the semisimple $K$-rank of $G$ equals 0.

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1 see e.g. the final introductory paragraph of [St 2]
As a special case of our main result, $\text{SL}_n(\mathbb{F}_q[t])$ is not of type $FP_\infty$. Even this basic example was previously unknown in full generality; see Example below. Here, $\mathbb{F}_q[t]$ is the ring of polynomials with one indeterminate $t$ and coefficients in the finite field with $q$ elements, $\mathbb{F}_q$.

We will also give a more precise statement about the finiteness lengths of arithmetic groups; see Theorem B. As a special case of that result, $\text{SL}_n(\mathbb{F}_q[t])$ is not even of type $FP_{n-1}$.

**Historical remarks.** Interest in the finiteness properties of arithmetic groups over function fields was sparked in 1959 by Nagao’s proof that $\text{SL}_2(\mathbb{F}_q[t])$ is not finitely generated \[Na\].

Activities of the next 33 years completely determined which arithmetic subgroups of reductive groups over function fields are finitely generated, and which are finitely presented (the answers fit the form of Conjecture C below). Work on these results was carried out by Behr, Hurrelbrink, Keller, Kneser, Lubotzky, McHardy, Nagao, O’Meara, Rehmann-Soulé, Serre, Splitthoff, and Stuhler. See \[Be 1\], \[Be 2\], \[Be 3\], \[Hu\], \[Ke\], \[Lu\], \[McH\], \[OM\], \[Re-So\], \[Se 2\], \[Spl\], and \[St 1\].

Less understood are the higher finiteness properties for these groups, such as type $FP_n$ for $n \geq 3$. Aside from the result of Serre mentioned earlier, all of the work in this direction has been carried out with heavy restrictions on $G$ and $O_S$; see the papers of Abels, Abramenko, Behr, and Stuhler (\[Ab\], \[Abr 1\], \[Abr 2\], \[Abr 3\], \[Be 4\], and \[St 2\]).

Theorem A follows as a corollary of Theorem B below. Before presenting the statement of Theorem B, we introduce some notation.

**Type $FP_m$.** Recall that for a commutative ring $R$, we say a group $\Gamma$ is of type $FP_m$ over $R$ if there exists a projective resolution

$$P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow R \rightarrow 0$$

of finitely generated $R\Gamma$ modules, where the action of $R\Gamma$ on $R$ is trivial. If $\Gamma$ is of type $FP_m$ over $R$ for all nonnegative integers $m$, we say that $\Gamma$ is of type $FP_\infty$ over $R$. If $\Gamma$ is of type $FP_m$ (resp. $FP_\infty$) over $\mathbb{Z}$, we simply write that $\Gamma$ is of type $FP_m$ (resp. $FP_\infty$).

**Remark.** Every group is of type $FP_0$. Type $FP_1$ is equivalent to the property of finite generation. Every finitely presented group is of type $FP_2$, but the converse does not hold in general; see Bestvina-Brady’s Example 6.3(3) in \[Be-Br\].
Finiteness length. The homological finiteness length of $\Gamma$ over $R$ is defined to be the nonnegative integer

$$\phi(\Gamma; R) = \sup\{m \mid \Gamma \text{ is of type } FP_m \text{ over } R\}$$

For short, we write $\phi(\Gamma)$ in place of $\phi(\Gamma; Z)$.

Rank. For any field extension $L/K$, the $L$-rank of $G$, denoted $\text{rank}_L G$, is the dimension of a maximal $L$-split torus of $G$. The semisimple $L$-rank of $G$ is the $L$-rank of the derived subgroup of $G$.

If $\text{rank}_L(G) = 0$, we say $G$ is $L$-anisotropic. Otherwise, $G$ is $L$-isotropic.

Sum of local ranks. For a valuation $v$ of $K$, let $K_v$ be the completion of $K$ with respect to $v$. For any $K$-group $G$, we define the nonnegative integer

$$k(G, S) = \sum_{v \in S} \text{rank}_{K_v} G$$

We are now prepared to state

**Theorem B** If $H$ is a connected noncommutative absolutely almost simple $K$-isotropic $K$-group, then

$$\phi(H(O_S)) \leq k(H, S) - 1$$

That Theorem A follows from Theorem B is routine; see e.g. 2.6(c) of [Be 3].

**Example.** A special case of Theorem B is the inequality

$$\phi\left(\text{SL}_n(\mathbb{F}_q[t])\right) \leq n - 2$$

or more generally,

$$\phi\left(\text{SL}_n(O_S)\right) \leq |S|(n - 1) - 1$$

Indeed, for any field $L$, the number $\text{rank}_L \text{SL}_n$ equals the dimension of the diagonal subgroup in $\text{SL}_n$. Hence, for any $K$ and any $S$, we have

$$k(\text{SL}_n, S) = \sum_{v \in S} \text{rank}_{K_v} \text{SL}_n = |S|(n - 1)$$

This inequality is known to be sharp in some cases. For example, Stuhler showed that $\phi(\text{SL}_2(O_S)) = |S| - 1$ [St 2], and Abels andAbramenko
independently showed that $\phi(\text{SL}_n(\mathbb{F}_q[t])) = n - 2$ as long as $q \geq 2^{n-2}$ or $q \geq \left(\frac{n-2}{n-2}\right)$ respectively [Abi], [Abr 1].

Is the inequality sharp in general? Theorem B provides evidence for the following long-standing conjecture, which offers a striking relation between the two functions $\phi$ and $k$.

Conjecture C If $H$ is a connected noncommutative absolutely almost simple $K$-isotropic $K$-group, then

$$\phi(H(\mathcal{O}_S)) = k(H, S) - 1$$

See [Be 3] for other evidence.

Type $F_m$. Recall that a group $\Gamma$ is of type $F_m$ if there exists an Eilenberg-Mac Lane complex $K(\Gamma, 1)$ with finite $m$-skeleton. For $m \geq 2$, a group is of type $F_m$ if and only if it is finitely presented and of type $FP_m$. It then follows from [Be 3] that $FP_m$ and $F_m$ are equivalent conditions for the arithmetic groups considered in this paper. Thus, Theorems A and B, and Conjecture C, may be equivalently stated by substituting $F_m$ for $FP_m$.

Type WFL. Although we will make no further use of it, we recall the definition of type $WFL$ for completeness with respect to comments in the initial portion of the introduction: A group $\Gamma$ is of type $WFL$ if there exists a torsion-free finite-index subgroup of $\Gamma$, and if for any such subgroup $\Gamma'$, the ring $\mathbb{Z}$ admits a finite length resolution by finitely generated free $\mathbb{Z}\Gamma'$-modules.

Contrast with number fields. Our theorems are particular to the case of global fields of positive characteristic. In characteristic zero, we have the following

Theorem D (Raghunathan, Borel-Serre) Any $S$-arithmetic subgroup of a reductive group defined over a global number field contains a finite-index torsion-free subgroup $\Gamma$ that allows for a finite $K(\Gamma, 1)$.

In particular, any $S$-arithmetic subgroup of a reductive group in characteristic zero is finitely presented and of type $FP_\infty$. Examples of groups for which the above theorem applies include $\text{SL}_n(\mathbb{Z})$ and $\text{SL}_n(\mathbb{Z}[1/p])$. See [Ra 1] for the case of arithmetic groups, and [Bo-Se] for the case of $S$-arithmetic groups.
Idea behind the proof. Although Theorem A shows a difference between arithmetic groups in positive characteristic and those in characteristic zero, it is through the deep-rooted similarity of these two families that we shall find a proof of Theorem A.

Indeed, our motivating example for proving Theorem B was the proof of Epstein-Thurston that $\text{SL}_3(\mathbb{Z})$ is not combable; see Chapter 10 Section 4 of [Ep et al.]. Recall that their proof proceeds by creating an exponential Dehn function for $\text{SL}_3(\mathbb{Z})$ as follows. A family of closed curves with increasing lengths are constructed in a portion of the symmetric space $\text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$ that is a bounded distance from the subset $\text{SL}_3(\mathbb{Z})\text{SO}_3(\mathbb{R}) \subseteq \text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$. The discs in $\text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$ that fill these loops in the most metrically efficient manner have areas that are quadratic in the length of the loops that they bound. These efficient discs are not so useful in studying the large-scale geometry of $\text{SL}_3(\mathbb{Z})$ though, since the discs travel farther away from the subspace $\text{SL}_3(\mathbb{Z})\text{SO}_3(\mathbb{R})$ as the length of their boundary curves increase.

To better understand the geometry of $\text{SL}_3(\mathbb{Z})$, we only consider filling discs of the constructed loops that are contained in the original bounded neighborhood of $\text{SL}_3(\mathbb{Z})\text{SO}_3(\mathbb{R})$. What is shown in [Ep et al.] is that any such family of discs would be metrically inefficient in the sense that the discs would have areas that are at least exponential in the length of their boundary curves. The result is an exponential Dehn function for $\text{SL}_3(\mathbb{Z})$, which implies that $\text{SL}_3(\mathbb{Z})$ is not combable.

Our proof of Theorem B in the special case when $H(O_S) = \text{SL}_3(\mathbb{F}_q[t])$ proceeds by constructing an analogous family of loops in a bounded neighborhood of a given $\text{SL}_3(\mathbb{F}_q[t])$-orbit in the Euclidean building, $X$, associated to $\text{SL}_3(\mathbb{F}_q((t^{-1})))$ where $\mathbb{F}_q((t^{-1}))$ is a field of formal Laurent series. As with the case for $\text{SL}_3(\mathbb{Z})$, the closed curves have metrically efficient filling discs in $X$ whose areas are quadratic in the length of their boundary curves. Also like the case for $\text{SL}_3(\mathbb{Z})$, these filling discs travel farther away from the given orbit as the length of their boundary curves increase, so they are not helpful in learning about the large-scale geometry of $\text{SL}_3(\mathbb{F}_q[t])$. However, in this case, there does not exist a filling disc for any of our constructed loops that is contained in the fixed bounded neighborhood of the $\text{SL}_3(\mathbb{F}_q[t])$-orbit. Indeed, $X$ is a contractible 2-dimensional simplicial complex, so filling discs are essentially unique. We then apply K. Brown’s filtration criterion to conclude that $\text{SL}_3(\mathbb{F}_q[t])$ is not finitely presented.

Distortion dimension. The contrast between arithmetic groups over func-
tion fields with arithmetic groups over number fields diminishes if we consider a metric analogue of finiteness length.

Let us direct our attention for the moment to an irreducible lattice $\Gamma$ in a semisimple group over arbitrary nondiscrete locally compact fields; we can even allow for $\Gamma$ to be nonarithmetic. We let $X_\Gamma$ be the natural product of irreducible symmetric spaces and Euclidean buildings that $\Gamma$ acts on. Given a point $x \in X_\Gamma$ and a real number $r$, we define the space

$$X_\Gamma(r) = \{ y \in X_\Gamma \mid d(y, \Gamma x) \leq r \}$$

Using the Hurewicz theorem—as in Abels-Tiemeyer’s Theorem 1.1.4 of [A-T]—and recalling that type $FP_m$ and type $F_m$ are equivalent conditions for $\Gamma$ allows us to state K. Brown’s filtration criterion for $\Gamma$ to be of type $FP_m$ in terms of homotopy groups. Precisely, $\Gamma$ is of type $FP_m$ if and only if for any real number $r \geq 0$ there exists a real number $r' \geq r$ such that for any $k < m$ the homomorphism induced by inclusion

$$\pi_k(X_\Gamma(r), x) \to \pi_k(X_\Gamma(r'), x)$$

is trivial; see Theorems 2.2 and 3.2 [Br 1].

Hence, if $\Gamma$ acts cocompactly on $X_\Gamma$, it is of type $FP_\infty$. If $\Gamma$ does not act cocompactly, then Theorems A and D (along with Corollary 7.3 of [Lu]) characterize those $\Gamma$ contained in semisimple groups over function fields as precisely those which fail to be of type $FP_\infty$. (Recall that an arithmetic lattice $H(O_S)$ acts cocompactly on $X_H(O_S)$ if and only if the absolutely almost simple $K$-group $H$ is $K$-anisotropic.)

To include metric properties of the large-scale geometry of lattices, we define $\Gamma$ as being undistorted up to dimension $m$ if: given any $r \geq 0$, there exist real numbers $r' \geq r$, $K \geq 1$, and $C \geq 0$ such that for any $k < m$ and any Lipschitz $k$-sphere $s \subseteq X_\Gamma(r)$, there exists a Lipschitz $(k + 1)$-ball $B_\Gamma \subseteq X_\Gamma(r')$ with $\partial B_\Gamma = s$ and

$$\text{volume}(B_\Gamma) \leq K[\text{volume}(B_X)] + C$$

for all Lipschitz $(k + 1)$-balls $B_X \subseteq X$ with $\partial B_X = s$. We adopt the convention that $\Gamma$ is always undistorted up to dimension 0.

Now we define the distortion dimension of $\Gamma$ to be the nonnegative integer

$$\psi(\Gamma) = \sup\{m \mid \Gamma \text{ is undistorted up to dimension } m\}$$
Conjecture E If $\Gamma$ is an irreducible lattice in a semisimple group over nondiscrete locally compact fields, then $\psi(\Gamma) = \infty$ if and only if $\Gamma$ acts cocompactly on $X_\Gamma$.

That $\psi(\Gamma) = \infty$ when $\Gamma$ acts cocompactly is clear. The converse had been conjectured for lattices in real semisimple Lie groups following the Epstein-Thurston proof that $\psi(\text{SL}_n(\mathbb{Z})) \leq n - 2$, and a general proof seems approachable. (See 10.4 [Ep et al.] for $\text{SL}_n(\mathbb{Z})$.)

Less attention has been given to $S$-arithmetic lattices in characteristic zero, but the conjecture should not change in this setting. In positive characteristic, Conjecture E follows from Theorem A.

As Conjecture E extends Theorem A for absolutely almost simple $K$-groups $G$ into the context of arbitrary global fields, we are naturally led to speculate how Conjecture C might be broadened to include fields of characteristic zero. Thus, we define $\tau(X_\Gamma)$ to be the Euclidean rank of $X_\Gamma$, and we note that for $\Gamma = H(\mathcal{O}_S)$ as in Conjecture C, $\tau(X_\Gamma) = k(H,S)$. We ask

Question F Let $\Gamma$ be a noncocompact irreducible lattice in a semisimple group over nondiscrete locally compact fields. Is it true that

$$\psi(\Gamma) = \tau(X_\Gamma) - 1$$

If not, then can the definition of $\psi$ be reasonably modified so that the above formula is true?

This problem is daunting. For example, an affirmative answer to the first question implies Thurston’s claim that $\text{SL}_n(\mathbb{Z})$ has a quadratic Dehn function for $n \geq 4$. See also remarks from Gromov’s book (5.D.(5).(c) [Gr]).

We will just mention a few pieces of evidence for a positive answer. We note that $\psi(\Gamma) = 0$ if and only if either $\Gamma$ is not finitely generated or the word metric on $\Gamma$ is not quasi-isometric to the metric induced from its action on $X_\Gamma$; see the example in Section 2. Hence, it follows from work of Lubotzky (Corollary 7.3 of [L1]) and Lubotzky-Mozes-Raghunathan [L-M-R] that $\psi(\Gamma) = 0$ if and only if $\tau(X_\Gamma) = 1$. Note also that Leuzinger-Pittet [Le-Pi 1], Behr [Be 3], and (a generalization of) Taback (Lemma 4.2 of [T]), show that $\tau(X_\Gamma) = 2$ implies $\psi(\Gamma) = 1$.

For related material, see the papers of Drutu, Hattori, Leuzinger-Pittet, Noskov, and Pittet: [Dr 1], [Dr 2], [Hat], [Le-Pi 2], [No], and [Pi].
Possible generalizations to other rings of functions. In [Bu-Wo] we use techniques from this paper to give a geometric proof that $\text{SL}_2(\mathbb{Z}[t, t^{-1}])$ is not finitely presented—a fact first proved by Krstić-McCool [Kr-Mc].

It is likely that the ideas below can be used to do more in this direction of generalizing Theorem B to apply to a class of fields and rings that properly includes global function fields and their rings of $S$-integers; see the question in the introduction of [Bu-Wo].

**Outline of the paper.** We begin in Section 2 with a special case of our proof to motivate what follows. The proof of Theorem B is contained in Section 3. In the appendix we include the proofs of two well-known results for completeness: the existence of anisotropic maximal tori in semisimple groups defined over nondiscrete locally compact fields of positive characteristic, and the “if” implication of Theorem A.

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**2 An example**

The first piece of evidence for Theorem A was:

**Theorem 2(a) (Nagao)** The group $\text{SL}_2(\mathbb{F}_q[t])$ is not finitely generated.

In this section we will see how our proof of Theorem B applies to this special case. For motivation, we will first review some of the geometry of $\text{SL}_2(\mathbb{Z})$, a mathematical cousin of $\text{SL}_2(\mathbb{F}_q[t])$. 
Consider the action of $\text{SL}_2(\mathbb{Z})$ on the hyperbolic plane $\mathbb{H}^2$. The diagram shows the upper half-plane model. There is a distinguished point $\infty$ at the top of the diagram that no $\text{SL}_2(\mathbb{Z})$-orbit accumulates on. Specifically, it is well-known that the orbit of the complex number $i$ avoids the open horoball $B$ that is centered at $\infty$ and consists of all complex numbers with imaginary parts greater than 1. The boundary of this horoball is approximated by the points $n + i$ for $n \in \mathbb{Z}$. (Notice that $n + i = \binom{1}{n}1$.)

The geodesic joining $i$ and $1 + i$ travels into the horoball $B$. The geodesic between $i$ and $2 + i$ travels farther into the horoball, the geodesic between $i$ and $3 + i$ farther still, and so on. Continuing this process, we see that no metric neighborhood of the orbit $\text{SL}_2(\mathbb{Z})i \subseteq \mathbb{H}^2$ is convex in $\mathbb{H}^2$. Sufficiently large metric neighborhoods of $\text{SL}_2(\mathbb{Z})i$ are however connected, as $\text{SL}_2(\mathbb{Z})$ is finitely generated:

**Lemma 2(a)** Suppose a finitely generated group $\Gamma$ acts on a geodesic metric space $X$. Then, for any point $x \in X$, there is a number $R > 0$ such that the $R$-neighborhood $\text{Nbhd}_R(\Gamma x) \subseteq X$ is connected.

**Proof:** Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a finite generating set for $\Gamma$. Choose $R$ such that the ball $B_R(x)$ contains all translates $\gamma_i x$. Then $\Gamma B_R(x) = \text{Nbhd}_R(\Gamma x)$ is connected.

Now let’s look at a proof of Nagao’s theorem along the lines of our proof for Theorem B. This is not Nagao’s original proof, rather it is a simplified form of Stuhler’s argument [St 2].

The locally compact field $\mathbb{F}_q((t^{-1}))$ contains $\mathbb{F}_q[t]$ as a discrete subring. Thus, $\text{SL}_2(\mathbb{F}_q[t])$ is a discrete subgroup of $\text{SL}_2(\mathbb{F}_q((t^{-1})))$.

There is a natural nonpositively curved space that $\text{SL}_2(\mathbb{F}_q((t^{-1})))$ acts on: the regular $(q + 1)$-valent tree $T$. If $\mathbb{F}_q[[t^{-1}]] \subseteq \mathbb{F}_q((t^{-1}))$ is the ring of Taylor series, then this well-known action is obtained by identifying the vertices of $T$ with homothety classes of spanning $\mathbb{F}_q[[t^{-1}]]$-submodules of a
2-dimensional vector space over $\mathbb{F}_q((t^{-1}))$ that are free and of rank 2. This is in analogy to the identification of the unit tangent bundle of $\mathbb{H}^2$ with the unit tangent bundle of the Teichmüller space of 2-dimensional Euclidean tori with volume equal to 1, or equivalently, with homothety classes of spanning $\mathbb{Z}$-submodules of $\mathbb{R}^2$ that are free and of rank 2.

Just as the boundary of $\mathbb{H}^2$ is a circle, or $\mathbb{P}^1(\mathbb{R})$, the boundary of $T$ can be identified with $\mathbb{P}^1(\mathbb{F}_q((t^{-1})))$. We use the standard identification of $\mathbb{P}^1(\mathbb{F}_q((t^{-1})))$ with $\mathbb{F}_q((t^{-1})) \cup \{\infty\}$. The group $\text{SL}_2(\mathbb{F}_q((t^{-1})))$ has two induced actions on $\mathbb{P}^1(\mathbb{F}_q((t^{-1})))$: one from its action on $T$, and one from its action on the 2-dimensional vector space $\mathbb{F}_q((t^{-1}))^2$. These actions coincide.

In another analogy with the situation for $\text{SL}_2(\mathbb{Z})$, any ray from any point $x \in T$ towards $\infty$ escapes every metric neighborhood of the orbit $\text{SL}_2(\mathbb{F}_q[t]):x$ (one can see this using Mahler’s compactness criterion). The diagram on the right has $x$ contained in the geodesic joining the two boundary points 0 and $\infty$. We write $f \ast x \in T$ as shorthand for the point $\left(\begin{smallmatrix} 1 & f \hfill 1 \\
0 & 1 \end{smallmatrix}\right)x$. The geodesic segment between $x$ and $t^n \ast x$ is the portion of the geodesic joining the boundary points 0 and $t^n$ that lies at or above the level of $x$ in the diagram. These segments contain increasing subsets of the geodesic ray from $x$ to $\infty$ as $n \in \mathbb{N}$ grows. Hence, no metric neighborhood of the orbit $\text{SL}_2(\mathbb{F}_q[t]):x \subseteq T$ is convex.

The comparison with $\text{SL}_2(\mathbb{Z})$ stops here since, in $T$, convexity is equivalent to connectedness. Using Lemma 2(a), we see that $\text{SL}_2(\mathbb{F}_q[t])$ is not finitely generated. Our proof is complete.

3 Proof of Theorem B

In what follows, we let $p$ equal the characteristic of $K$. Rather than proving Theorem B directly, our goal will be to prove a slightly stronger claim:

**Proposition B* If $H$ is a connected noncommutative absolutely almost simple $K$-isotropic $K$-group, then

$$\phi(H(O_S); \mathbb{F}_p) \leq k(H, S) - 1$$

**Proof** Theorem B follows since a group $\Gamma$ is of type $FP_m$ over $\mathbb{F}_p$ if it is of type $FP_m$ over $\mathbb{Z}$: just tensor a projective resolution for $\mathbb{Z}$ by finitely generated $\mathbb{Z}\Gamma$-modules.
modules with $\mathbb{F}_p$ to obtain a projective resolution for $\mathbb{F}_p\Gamma$-modules.

**3.1 Method of proof for Proposition B***

We define the ring

$$K_S = \prod_{v \in S} K_v$$

so that

$$H(K_S) = \prod_{v \in S} H(K_v)$$

Let $X$ be the Euclidean building corresponding to $H(K_S)$, that is the product of the irreducible Euclidean buildings for $H(K_v)$. Recall that $X$ has dimension $k(H, S)$.

We fix a base point $e \in X$ (to be specified later) and consider closed metric neighborhoods of the orbit $H(O_S)e$. That is, for each number $r \geq 0$, we set

$$X(r) = \{ x \in X \mid d(x, H(O_S)e) \leq r \}$$

We will find a number $r_0 > 0$ and construct, for each $r \geq r_0$, a cycle in $X(r_0)$ that represents a nontrivial element in the reduced homology group with coefficients in $\mathbb{F}_p$

$$\tilde{H}_{k(H,S)-1}(X(r); \mathbb{F}_p)$$

This shows that the inclusions $X(r_0) \subseteq X(r)$ induce nontrivial homomorphisms

$$\tilde{H}_{k(H,S)-1}(X(r_0); \mathbb{F}_p) \longrightarrow \tilde{H}_{k(H,S)-1}(X(r); \mathbb{F}_p)$$

In view of K. Brown’s filtration criterion (see Theorem 2.2 and the following remark in [Br 1]), the existence of this family of nontrivial homomorphisms together with the following standard facts about the action of $H(O_S)$ on $X$ implies Proposition B*: (i) $X$ is contractible; (ii) $H(O_S)$ acts on $X$ with finite cell stabilizers; and (iii) the subspaces $X(r)$ are $H(O_S)$-invariant and compact modulo $H(O_S)$.

**Excluding a tree.** For the remainder of this paper, we will assume that $k(H, S) > 1$. That is, we assume that $X$ is not a tree. This assumption is made only to avoid complications in our exposition; the philosophy of the proof still applies to the case when $k(H, S) = 1$ as is shown in Section 2.

**3.2 An apartment coarsely separated by $H(O_S)$**
We will find an apartment in $X$ that “coarsely intersects” an $H(O_S)$-orbit in a hyperplane. (Later, we will use this $(k(H, S) - 1)$-dimensional hyperplane and its translates to construct the $(k(H, S) - 1)$-cycles mentioned above.) Since apartments in $X$ correspond to products of maximal $K_t$-split tori in $H$, this problem reduces to algebra.

We begin by choosing a parabolic group that will follow with us throughout our proof. In what follows, we are assuming that the reader has a basic knowledge of the structure of parabolic subgroups of reductive groups relative to fields that are not algebraically closed, as can be found for example in 21.11 and 21.12 of $[B]$. 

Since $H$ is $K$-isotropic, there exists a nontrivial maximal $K$-split torus of $H$. We let $\Phi_K$ be the roots of $H$ with respect to this torus. Choose an ordering on $\Phi_K$, and let $\Delta_K \subseteq \Phi_K$ denote the corresponding collection of simple roots.

Choose, and fix throughout, a root $\alpha_0 \in \Delta_K$. We define the 1-dimensional $K$-split torus

$$T_1 = \left( \bigcap_{\alpha \in \Delta_K - \alpha_0} \ker(\alpha) \right)^\circ$$

The above superscript $\circ$ denotes the connected component of the identity. We let $Z_H(T_1)$ be the centralizer of $T_1$ in $H$.

There exists a maximal proper $K$-parabolic subgroup of $H$, denoted $P^+$, with the following Levi decomposition:

$$P^+ = R_u(P^+) \rtimes Z_H(T_1)$$

In the above, $R_u(P^+)$ is the unipotent radical of $P^+$.

We can expand the Levi decomposition to a Langlands decomposition by noting that $Z_H(T_1)$ is an almost direct product of $T_1$, the derived group $Z_H(T_1)_{der}$, and $D_a$ for some $K$-anisotropic diagonalizable group $D_a$. Thus:

$$P^+ = R_u(P^+) \rtimes T_1 D_a Z_H(T_1)_{der}$$

Before proceeding with the existence of the torus and the apartment that is our goal in this section, we record the following well-known result.

**Proposition 3.2(a)** Let $G$ be a reductive $K$-group. Then for any finite nonempty set $S'$ of pairwise inequivalent valuations and any family $\{A_v\}_{v \in S'}$
of maximal $K_v$-tori of $G$, there is a maximal $K$-torus $A_\pi$ of $G$ and group elements $g_v \in G(K_v)$ such that

$$A_\pi = g_v A_v$$

for all $v \in S'$, where $g_v A_v$ denotes $A_v$ conjugated by $g_v$.

**Proof:** There is a proof of this proposition in Section 7.1 Corollary 3 of [Pl-Ra] for the case when $K$ is a global number field. The proof also applies for global function fields after replacing the argument for the $K$-rationality of the variety of maximal tori in $G$ with the proof of Theorem 7.9 in [Bo-Sp].

We will make use of the above proposition in the proof of the proposition below.

**Proposition 3.2(b)** There exists a maximal $K$-torus $A \leq H$ such that:

(i) The maximal $K$-split torus of $A$ is $T_1$, and

(ii) $A$ contains a maximal $K_v$-split torus of $H$ for all $v \in S$.

**Proof:** For each $v \in S$, let $A_v$ be a maximal $K_v$-torus of $Z_H(T_1)_{\text{der}}$ such that $A_v$ contains a maximal $K_v$-split torus of $Z_H(T_1)_{\text{der}}$.

Then choose a valuation of $K$, call it $w$, that is inequivalent to any of the valuations of $S$, and let $A_w$ be a maximal $K_w$-torus in $Z_H(T_1)_{\text{der}}$ that is $K_w$-anisotropic. The existence of such a torus is well-known; see the appendix for a proof.

Now apply Proposition 3.2(a) to $G = Z_H(T_1)_{\text{der}}$ and $S' = S \cup \{w\}$. Since $A_\pi$ is $K_w$-anisotropic, it is necessarily $K$-anisotropic. Therefore, part (i) is satisfied by

$$A = T_1 D_a \circ A_\pi$$

To verify part (ii), note that $T_1$ is contained in a maximal $K_v$-split torus of $H$. Hence,

$$\text{rank}_{K_v}(H) = \text{rank}_{K_v}(Z_H(T_1))$$

$$= \text{rank}_{K_v}(T_1 D_a) + \text{rank}_{K_v}(Z_H(T_1)_{\text{der}})$$

$$= \text{rank}_{K_v}(T_1 D_a) + \text{rank}_{K_v}(A_\pi)$$

$$= \text{rank}_{K_v}(A)$$

$\blacksquare$
Since $A$ contains a maximal $K_v$-split torus for all $v \in S$, there is an apartment $\Sigma \subseteq X$ that $A(K_S)$ acts on properly and cocompactly as a translation group of maximal rank, $\dim(\Sigma) = k(H, S)$. By Dirichlet’s units theorem (see Theorem 5.12 [Pl-Ra]) and the preceding proposition, the arithmetic group $A(O_S)$ is a finitely generated abelian group of rank

$$\left( \sum_{v \in S} \text{rank}_{K_v}(A) \right) - \text{rank}_K(A) = k(H, S) - 1$$

Choose a point $e \in \Sigma$. Since $A(O_S) \leq A(K_S)$ acts properly on $\Sigma$, the base point $e$ is contained in an affine hyperplane $V \subseteq \Sigma$, of dimension $k(H, S) - 1$, that $A(O_S)$ acts on cocompactly. This point $e \in \Sigma$ is the point we specify for our definition in Section 3.1 of the spaces $X(r) \subseteq X$.

**Example.** In the case when $K = \mathbb{F}_q(t)$, $O_S = \mathbb{F}_q[t]$, and $H = \text{SL}_3$, the torus $T_1$ can be taken as the group of matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}$$

Then the parabolic group $P^+$ can be taken to be the determinate 1 matrices of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

The resulting group $R_u(P^+)$ would be the 2-dimensional commutative group

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

This would leave the semisimple group $Z_H(T_1)_{\text{der}}$ to be the copy of $\text{SL}_2$ that sits in the upper left corner of $\text{SL}_3$, and $D_a$ would be trivial.

The group $A(O_S)$ in this example can be taken to be the group generated by the matrix

$$\begin{pmatrix} t^2 + 1 & t & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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With the notation from the proof of Proposition 3.2(b), the Zariski closure of $A(\mathcal{O}_S)$ would equal $A_\pi \leq Z_{H(T_1)_{\text{der}}}$.  

3.3 A space for the manufacture of cycles: choosing $r_0$

Let $D \subseteq V$ be a fundamental domain for the $A(\mathcal{O}_S)$-action on $V$. By Behr-Harder reduction theory (e.g. Satz 3 of [Be 1]), there is a compact set $C \subseteq R_u(\mathbb{P}^+)(K_S)$ such that $R_u(\mathbb{P}^+)(K_S) = R_u(\mathbb{P}^+)(\mathcal{O}_S)C$. Since $A(\mathcal{O}_S)$ normalizes $R_u(\mathbb{P}^+)(K_S)$, we have:

$$R_u(\mathbb{P}^+)(K_S)V \subseteq R_u(\mathbb{P}^+)(K_S)A(\mathcal{O}_S)D \subseteq A(\mathcal{O}_S)R_u(\mathbb{P}^+)(K_S)D \subseteq A(\mathcal{O}_S)R_u(\mathbb{P}^+)(\mathcal{O}_S)CD \subseteq H(\mathcal{O}_S)CD$$

Since the region $CD \subseteq X$ is bounded, we can choose a number $r_0 > 0$ such that

$$R_u(\mathbb{P}^+)(K_S)V \subseteq H(\mathcal{O}_S)CD \subseteq X(r_0)$$

It is inside the space $R_u(\mathbb{P}^+)(K_S)V$ where we shall produce cycles that remain nontrivial in the homology of $X(r)$ for $r \geq r_0$.

3.4 A direction away from $X(r_0)$

Recall our choice of $\alpha_0 \in \Delta_K$ from the beginning of Section 3.2. This root is nontrivial when restricted to $T_1$, so for any $v \in S$, the set

$$\{ a \in T_1(K_v) \mid |\alpha_0(a)|_v > 1 \}$$

is nonempty and open in the Hausdorff topology induced by the metric $| \cdot |_v$ on $K_v$ that arises from $v$.

Since $T_1$ is $K$-isomorphic to an affine line with a point removed, it satisfies the weak approximation property with respect to $S$. That is, the diagonal embedding

$$T_1(K) \rightarrow \prod_{v \in S} T_1(K_v)$$

has a dense image. Therefore, there exists some $a_+ \in T_1(K)$ such that

$$|\alpha_0(a_+)|_v > 1$$

for all $v \in S$.  

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It will be important for us later to have a direction in $\Sigma$ that leads away from every $X(r)$. The direction we will use is given by the sequence $(a^n_+ e)_{n \in \mathbb{N}}$. Note that the above condition on $a_+$ assures us that the sequence $(a^n_+ e)_{n \in \mathbb{N}}$ is not contained in any compact subset of $H(K_S)$. Therefore, $(a^n_+ e)_{n \in \mathbb{N}}$ does specify a direction in $\Sigma$.

Let $X^\infty$ be the visual boundary of $X$. It can be identified in a natural way with the spherical Tits building for $H(K_S)$. Note that $X^\infty$ is the spherical join of the spherical buildings for the groups $H(K_v)$ with $v \in S$. We let $\Sigma^\infty \subseteq X^\infty$ be the apartment corresponding to $\Sigma \subseteq X$, and we let $a^\infty_+ \in \Sigma^\infty$ be the accumulation point of $(a^n_+ e)_{n \in \mathbb{N}}$.

We let $\Pi^\infty_+$ be the unique simplex in $X^\infty$ that is maximal among all simplices stabilized by the action of $P^+(K_S)$. Note that $\Pi^\infty_+$ is the spherical join over $S$ of the simplices associated with $P^+(K_v)$ in the spherical buildings for $H(K_v)$.

The conditions on our choice of $a_+$ were imposed to insure that the orbit of $e$ under its iterates would accumulate inside $\Pi^\infty_+$. Specifically, we have:

**Lemma 3.4(a)** The point $a^\infty_+ \in \Sigma^\infty$ is contained in $\Pi^\infty_+$.

**Proof:** Using the definition of spherical joins, we can reduce to the case when $S$ contains a single valuation $v$. What follows is routine; see e.g. 2.4 [Pr].

We let $Q$ be the $K_v$-parabolic subgroup of $H$ with

$$Q(K_v) = \{ g \in H(K_v) \mid ga^\infty_+ = a^\infty_+ \}$$

The proof of this lemma amounts to showing that

$$P^+(K_v) \leq Q(K_v)$$

Note that $ga^\infty_+ = a^\infty_+$ if and only if

$$d(ga^n_+ e, a^n_+ e) = d(a^{-n} ga^n_+ e, e)$$

is a bounded sequence. Since distance from $e$ is a proper function, and because the action of $H(K_v)$ on $X$ is proper, we can alternatively characterize $Q(K_v)$ as the group

$$\{ g \in H(K_v) \mid (a^{-n} ga^n_+)_{n \in \mathbb{N}} \subseteq H(K_v) \text{ is precompact} \}$$
Let $u$ and $\mathfrak{h}$ be the Lie algebras of $\mathbf{R}_u(P^+)$ and $H$ respectively. We denote the set of positive roots given by our ordering of $\Phi_K$ in Section 3.2 as $\Phi^+_K \subseteq \Phi_K$, and we write the set of roots that are linear combinations of elements in $\Delta_K - \alpha_0$ as $[\Delta_K - \alpha_0]$.

If $T$ is the maximal $K$-split torus in $H$ that was chosen to produce the roots $\Phi_K$, then our choice of $P^+$ from Section 3.2 implies that

$$u = \bigoplus_{\alpha \in \Phi^+_K - [\Delta_K - \alpha_0]} \{ v \in \mathfrak{h} \mid \text{Ad}_t(v) = \alpha(t)v \text{ for all } t \in T \}$$

Note that $\Phi^+_K - [\Delta_K - \alpha_0]$ is exactly the subset of $\Phi_K$ consisting of sums of the form $\sum_{\alpha_i \in \Delta_K} n_i \alpha_i$ with $n_i \geq 0$ for all $i$ and $n_0 \geq 1$. By our definition of $T_1 \leq T$ as being contained in the kernel of every root in $\Delta_K - \alpha_0$, and as $a_+ \in T_1$, we can express $u$ as a finite direct sum

$$u = \bigoplus_{n \geq 1} \{ v \in u \mid \text{Ad}_{a_+}(v) = n\alpha_0(a_+)v \}$$

Since $|\alpha_0(a_+)| > 0$, we see that for any $u \in \mathbf{R}_u(P^+)(K_v)$,

$$a_+^{-n}ua_+^n \to 1$$

as $n \to \infty$. Hence, if $u \in \mathbf{R}_u(P^+)(K_v)$ and $z \in Z_H(T_1)(K_v)$, then

$$a_+^{-n}uza_+^n = a_+^{-n}ua_+^nz \to z$$

In particular, the above sequence is precompact. As a consequence, $uz$, and thus all of

$$P^+(K_v) = \mathbf{R}_u(P)(K_v) \rtimes Z_H(T_1)(K_v)$$

is contained in $Q(K_v)$.

With a little more effort, it can be shown that $a_+^\infty \in \Pi^\infty - \partial \Pi^\infty$, but we will not need this fact.

Now we know the direction of $a_+^\infty$. Our last point of business in this section is to see that this direction leads away from the orbit $H(\mathcal{O}_S)e$. This argument is standard.

**Lemma 3.4(b)** For any $r > 0$, there exists an $n \in \mathbb{N}$ such that $a_+^n e \notin X(r)$.
**Proof:** Choose any nontrivial $\gamma \in R_u(P^+)(O_S)$. As in the proof of the preceding lemma, $a_+^{-n}\gamma a_+^n \to 1$.

From Theorem I.1.12 of [Ra 2], the sequence $(a_+^n)_{n \in \mathbb{N}} \subseteq H(K_S)$ induces a sequence in the quotient space $H(O_S) \setminus H(K_S)$ that is not contained in any compact set. The lemma follows.

3.5 A blueprint at infinity

In this section we will construct a cycle inside $X^\infty$ in the direction given by the sequence $(a_+^n e)_{n \in \mathbb{N}}$. This is the direction in $\Sigma$ that is opposite to $a_+^\infty$. In Section 3.7, translates of this cycle will be “coned off” from points of the form $a_+^ne$. Then, these cones will be intersected with $R_u(P^+)(K_S)V$ to produce cycles in $X(r_0)$.

We let $\Pi_\infty^-$ be the simplex opposite to $\Pi_\infty^+$ in the spherical apartment $\Sigma^\infty$. This simplex is the unique maximal simplex in $X^\infty$ that is fixed under the action of $P^-(K_S)$, where $P^-$ is the maximal proper $K$-parabolic subgroup of $H$ that contains $Z_H(T_1)$ and is opposite to $P^+$. We let $\Delta^\infty_-$ be the simplicial star of $\Pi_\infty^-$ in the apartment $\Sigma^\infty$. That is, $\Delta^\infty_-$ is the union of all simplices in $\Sigma^\infty$ that contain $\Pi_\infty^-$. The description of a chain in the boundary.

Let $\sigma$ be a codimension 1 simplex in $\Sigma^\infty$ that is contained in the boundary of $\Delta^\infty_-$. The geodesic continuation of $\sigma$ in $X^\infty$ is a great sphere, that is, the boundary of a closed simplicial hemisphere $R_\alpha \subseteq \Sigma^\infty$ (called a root space). Among the two possible hemispheres, $R_\alpha$ and $R_{-\alpha}$, in $\Sigma^\infty$ that contain $\sigma$ in their boundary (called opposite root spaces), we fix notation so that $R_{-\alpha}$ contains $\Pi_\infty^-$. Theorem 3.5(a) There exists a group element $u_{-\alpha} \in R_u(P^-)(K_S)$ fixing $R_{-\alpha}$ pointwise and satisfying the condition

$$\Sigma^\infty \cap u_{-\alpha}\Sigma^\infty = R_{-\alpha}$$

**Proof:** We may assume that $S$ consists of a single valuation $v$. The general case follows from the definition of the spherical join.

Let $Q$ be the minimal $K_v$-parabolic subgroup of $H$ corresponding to the chamber containing $\sigma$ and $\Pi_\infty^-$. Let $\Phi_{K_v}$ be the set of roots of $H$ with respect to the maximal $K_v$-split torus in $A$, let $\Phi^\text{nd}_{K_v} \subseteq \Phi_{K_v}$ be the set of nondivisible
roots, and let $\Delta_{K_v} \subseteq \Phi_{K_v}^{\text{nd}}$ be the set of simple roots associated with our choice of $Q$.

As explained in 5.6 of [Ti], there is a root $-\alpha \in \Phi_{K_v}^{\text{nd}}$ such that any nontrivial element $u_{-\alpha} \in R_{-\alpha}$ of the root group $U_{(-\alpha)}(K_v) \leq H(K_v)$ fixes $R_{-\alpha}$ pointwise and satisfies $\Sigma^\infty \cap u_{-\alpha}\Sigma^\infty = R_{-\alpha}$. (A similar statement holds by replacing $-\alpha$ throughout with $\alpha$, where the root $\alpha \in \Phi_{K_v}^{\text{nd}}$ is the negative of $\alpha$.)

Note that all we have left to show is $U_{(-\alpha)} \leq R_u(P^-)$.

Recall the standard correspondence that assigns to any subset $I \subseteq \Delta_{K_v}$ a $K_v$-parabolic subgroup of $H$ containing $Q$, denoted $Q_I$; see e.g. 21.12 [Bo]. Since $\sigma$ is of codimension 1 in $\Sigma^\infty$, the $K_v$-parabolic subgroup of $H$ corresponding to $\sigma$ is of the form $Q_{\{\beta\}}$ for a single simple root $\beta \in \Delta_{K_v}$. We also have that $U_{(-\alpha)}(K_v) \leq Q_{\{\beta\}}(K_v)$ and $U_{(\alpha)}(K_v) \leq Q_{\{\beta\}}(K_v)$ since $\sigma \subseteq R_{-\alpha} \cap R_\alpha$ is fixed by $U_{(-\alpha)}(K_v)$ and $U_{(\alpha)}(K_v)$. It follows from 21.12 of [Bo] that either $-\alpha = \beta$ or $\alpha = \beta$.

Since $U_{(-\alpha)}(K_v)$ fixes $R_{-\alpha}$ pointwise, the chamber corresponding to $Q$ is also fixed under the action of $U_{(-\alpha)}(K_v)$. Hence, $U_{(-\alpha)} \leq Q$ implying that $-\alpha$ is positive under the ordering on $\Phi_{K_v}$ consistent with $\Delta_{K_v}$. Now it must be that $-\alpha = \beta$.

Since $\Pi^\infty \notin \sigma$, we have $Q_{\{-\alpha\}} = Q_{\{\beta\}} \notin P^-$. Therefore, if we assume $J \subseteq \Delta_{K_v}$ is such that $Q_J = P^-$, then $-\alpha \notin J$. It follows that $U_{(-\alpha)} \leq R_u(P^-)$ as desired.

Any $K_v$-parabolic subgroup of $H$ that is contained in $P^-$ must contain $R_u(P^-)$. Thus, $u_{-\alpha} \in R_u(P^-)(K_S)$ fixes $\Delta_{K_v}^\infty$ pointwise. Therefore, $\Delta_{K_v}^\infty \subseteq R_{-\alpha}$ which, in turn, implies that $\Delta_{K_v}^\infty \cap R_\alpha$ is the union of some codimension 1 simplices in the boundary of $\Delta_{K_v}^\infty$ (including $\sigma$). We name this union $F_\alpha$ and call it a geodesically continued face of $\Delta_{K_v}^\infty$. We take a minimal (hence finite) family of root spaces $\{R_\alpha\}_{\alpha \in A}$ which exhaust the boundary of $\Delta_{K_v}^\infty$ as the union of the corresponding geodesically continued faces of $\Delta_{K_v}^\infty$.

Applying Lemma 3.5(a) to the opposite parabolic and opposite root space, we have that for each for each $\alpha \in A$, there is a group element $u_\alpha \in R_u(P^+)(K_S)$ that fixes $R_\alpha$ pointwise and satisfies $\Sigma^\infty \cap u_\alpha \Sigma^\infty = R_\alpha$. Hence, $\Delta_{K_v}^\infty \cap u_\alpha \Delta_{K_v}^\infty = F_\alpha$.

We define the group $U \leq R_u(P^+)(K_S)$ to be generated by the finite set of $u_\alpha$ as above. As it will be useful in Section 3.7, we also choose our $u_\alpha$.
to fix the point $e$. This can always be arranged by replacing the $u_\alpha$ with conjugates by elements of $A(K_S)$.

It is well known that every element of $R_u(P^+)(K_S)$ has order a power of $p$ (see e.g. 4.1 [Bo]), so $U$ is a $p$-group. Generalizing Schur’s work on the generalized Burnside problem, Kaplansky showed that any finitely generated linear torsion group is finite (see e.g. Theorem 9.9 [La]). We conclude that $U$ is a finite $p$-group.

By abuse of notation, we shall denote the formal sum of chambers in $\Delta_\infty$ simply by $\Delta_\infty$. Now we form the $(k(H, S) − 1)$-chain $\sum_{u \in U} u \Delta^\infty$.

**Properties of the chain in the boundary.** In the remainder of this section, we will show that $\sum_{u \in U} u \Delta^\infty$ is a cycle describing a simplicial decomposition of $U \Delta^\infty = \bigcup_{u \in U} u \Delta^\infty$.

**Lemma 3.5(b)** If $u \in U$ is nontrivial and $\mathcal{C}^\infty \subseteq \Delta^\infty$ is a chamber, then $u \mathcal{C}^\infty \not\subseteq \Delta^\infty$.

**Proof:** Suppose $u \mathcal{C}^\infty \subseteq \Delta^\infty$. Then we have $\Pi^\infty_\mathcal{C} \subseteq \mathcal{C}^\infty \cap u \mathcal{C}^\infty$ by the definition of $\Delta^\infty$. As the action of $H(K_S)$ on $X^\infty$ is type preserving, $u \Pi^\infty_\mathcal{C} = \Pi^\infty_\mathcal{C}$. This implies that $u \in P^-(K_S) \cap R_u(P^+)(K_S) = 1$.

**Lemma 3.5(c)** The chain $\sum_{u \in U} u \Delta^\infty$ is a cycle over $\mathbb{F}_p$.

**Proof:** Suppose that $u \in U$ is nontrivial and that $\Delta^\infty \cap u \Delta^\infty$ contains an interior point $x$ of a maximal simplex of a geodesically continued face of $\Delta^\infty$, say $F_\alpha$. We begin by verifying that $u$ fixes $F_\alpha$ pointwise, and that $F_\alpha = \Delta^\infty \cap u \Delta^\infty$.

Indeed, $u$ fixes pointwise a simplex of $F_\alpha$ that contains $x$, since $u$ acts by type preserving simplicial automorphisms on $X^\infty$. The antipodal point of $x$ in $\Sigma^\infty$ is contained in the boundary of a chamber of $\Sigma^\infty$ containing $\Pi^\infty_\mathcal{C}$; we call this chamber $\mathcal{C}^\infty$. As in the comment immediately following proof of Lemma 3.5(a), we see that $\mathcal{C}^\infty$ is fixed by $u \in R_u(P^+)(K_S)$.

The hemisphere $R_\alpha$ is the convex hull spanned by the simplex of $F_\alpha$ that contains $x$ and the chamber $\mathcal{C}^\infty$. Therefore, $u$ fixes every point in $R_\alpha \supseteq F_\alpha$.

For the remaining claim that $F_\alpha = \Delta^\infty \cap u \Delta^\infty$: If there was a point $y \in \Delta^\infty \cap u \Delta^\infty$ outside of $R_\alpha$, then $u$ would have to fix $y$ since $R_\alpha$ is fixed pointwise by $u$ and the action is by isometries. Hence, $u$ fixes pointwise the
convex hull of \( R_\alpha \) and \( y \). But that is all of \( \Sigma^\infty \), and any \( u \in R_u(\mathbf{P}^+)(K_S) \) fixing \( \Sigma^\infty \) pointwise is the identity. So we have verified our claims.

We are now prepared to show that the homological boundary of \( \sum_{u \in U} u\Delta^\infty \) is 0 modulo \( p \). Applying the boundary homomorphism yields:
\[
\partial \left( \sum_{u \in U} u\Delta^\infty \right) = \sum_{u \in U} \partial(u\Delta^\infty) = \sum_{u \in U} \sum_{\alpha \in A} uF_\alpha
\]
where, again stretching notation slightly, \( F_\alpha \) denotes the formal sum of all simplices in the geodesically continued face.

The claims we verified above show that, for \( u,v \in U \) and all \( \alpha \in A \), either \( uF_\alpha \cap vF_\alpha \) is contained in the topological boundary of \( uF_\alpha \) or alternatively, \( uF_\alpha \) and \( vF_\alpha \) are equal as chains. Thus, we choose a complete set \( \{f_1, f_2, \ldots, f_n\} \) of representatives for the chains in \( \{uF_\alpha\}_{u \in U, \alpha \in A} \) so that
\[
\sum_{u \in U} \sum_{\alpha \in A} uF_\alpha = \sum_{i=1}^n |U_i|f_i
\]
where \( U_i \leq U \) is the stabilizer of \( f_i \). Since \( U \) is a finite \( p \)-group, \(|U_i|\) is a power of \( p \). Moreover, since each \( F_\alpha \) is stabilized by a nontrivial \( u_\alpha \in U \), each group \( U_i \) is nontrivial. Therefore,
\[
\partial \left( \sum_{u \in U} u\Delta^\infty \right) \equiv 0 \pmod{p}
\]

**Observation.** By the preceding lemmas, \( U\Delta^\infty \) represents a class in the homology group \( \tilde{H}_k(H,S)_{-1}(U\Delta^\infty; \mathbb{F}_p) \).

**3.6 A line of communication from infinity to \( X(r_0) \)**

In the next section, we will build cycles in \( X(r_0) \) by transferring the topological data from \( U\Delta^\infty \) into \( X(r_0) \) by method of “casting shadows” of \( U\Delta^\infty \) on \( R_u(\mathbf{P}^+)(K_S)V \). For the shadow to contain the same topological data as \( U\Delta^\infty \), it is important, for example, to have the shadow of \( \Delta^\infty \) in \( V \) be compact. The purpose of this section is to establish that fact, although we state this problem below using different language.

Recall that \( \Sigma^\infty \) can be regarded as the space of all geodesic rays in \( \Sigma \) based at \( e \). We let \( V^\infty \subseteq \Sigma^\infty \) be the set of all geodesic rays contained in \( V \) emanating from \( e \). Note that \( V^\infty \) is an equatorial sphere in \( \Sigma^\infty \).
We call a point in $\Sigma^\infty$ rational if it is represented by a geodesic ray based at $e$ that passes through another (and hence infinitely many) points of $A(K_S)e$. Let $\Sigma^\infty_Q$ denote the set of rational points in $\Sigma^\infty$. Since $A(O_S)$ acts on $\Sigma$ as a lattice of translations of full rank $k(H, S) = \dim(\Sigma)$, the set $\Sigma^\infty_Q$ is dense in $\Sigma^\infty$.

Similarly, we let $V^\infty_Q$ denote the set of those points in $V^\infty$ that can be joined to $e$ by a geodesic ray passing through infinitely many points of $A(O_S)e$. From our choice of $V$ before the Example in Section 3.2, it is also clear that $V^\infty_Q$ is dense in $V^\infty$.

Lemma 3.6(a) We have $V^\infty_Q = V^\infty \cap \Sigma^\infty_Q$.

Proof: The action of $A(O_S)$ factors through the inclusion $A(O_S) \hookrightarrow A(K_S)$. Since $A(O_S)$ acts on $V$ as a lattice of maximum rank $k(H, S) - 1 = \dim(V)$, the affine lattices $A(O_S)e$ and $V \cap A(K_S)e$ are commensurable. Hence, they define identical rational structures at infinity. ■

The goal of this section is:

Lemma 3.6(b) We have $\Delta^\infty \cap V^\infty = \emptyset$.

Proof: We proceed by contradiction. So assume $\Delta^\infty \cap V^\infty \neq \emptyset$. Our first step will be to show that $\Delta^\infty \cap V^\infty_Q \neq \emptyset$. There are two cases. First, $V^\infty_Q$ contains an interior point of $\Delta^\infty_Q$. Then the intersection $\Delta^\infty \cap V^\infty_Q$ is open in $V^\infty$ and contains a rational point since these are dense in $V^\infty_Q$. That is $\Delta^\infty \cap V^\infty_Q \neq \emptyset$. Second, $V^\infty_Q$ contains a boundary simplex of $\Delta^\infty_Q$. Since the affine lattice $A(K_S)e \subseteq \Sigma$ is commensurable to the affine lattice of vertices in the Euclidean Coxeter complex underlying the apartment $\Sigma$, rational points are dense in every simplex in $\Sigma^\infty$. Therefore, $V^\infty \cap (\Sigma^\infty_Q \cap \Delta^\infty) \neq \emptyset$. Using Lemma 3.6(a), we again find a point in $\Delta^\infty \cap V^\infty_Q$.

Now choose $b \in A(O_S)$ such that $b^ne$ converges to a point $b^\infty \in \Delta^\infty \cap V^\infty_Q$ as $n \to \infty$.

Recall that for each $v \in S$, the group $R_u(P^-)$ is contained in any minimal $K_v$-parabolic subgroup of $H$ that is contained in $P^-$. Therefore, $R_u(P^-)(K_S)$ fixes $\Delta^\infty$ pointwise and, consequently, fixes the point $b^\infty \in \Delta^\infty$. As in the
proof of Lemma 3.4(a),

\[ R_u(P^-)(O_S) \leq R_u(P^-)(K_S) \]
\[ \leq \{ g \in H(K_S) \mid gb^\infty = b^\infty \} \]
\[ = \{ g \in H(K_S) \mid (b^{-n}gb^n)_{n \in \mathbb{N}} \text{ is precompact} \} \]

Therefore, for any \( \gamma \in R_u(P^-)(O_S) \), the sequence \( (b^{-n}\gamma b^n)_{n \in \mathbb{N}} \subseteq H(O_S) \) is both discrete and precompact. Hence, it is finite. We conclude that

\[ b^{-n}\gamma b^n = b^{-m}\gamma b^m \]

for distinct \( n \) and \( m \). Now, \( \gamma \) centralizes \( b^{n-m} \).

Let \( D \) be the subgroup of \( A \) that is the Zariski closure of the group generated by \( b^{n-m} \). Then, \( \gamma \) centralizes \( D \). Note that \( b \in D \), so \( D \) is also the Zariski closure of the group generated by \( b \). Thus, \( D \) is independent of our choice of \( \gamma \in R_u(P^-)(O_S) \). Hence, \( R_u(P^-)(O_S) \) centralizes \( D \).

Since \( b^v e \to b^\infty \), iterates of \( b \) define an unbounded sequence in \( D(K_v) \) for at least one \( v \in S \). It follows that \( D \) contains a nontrivial \( K_v \)-split torus \( D_d \). Indeed, if \( D \) were \( K_v \)-anisotropic, then \( D(K_v) \) would be compact.

We denote the centralizer of \( D_d \) in \( H \) by \( L \). Therefore, \( L \) is a Levi subgroup of a \( K_v \)-parabolic subgroup of \( H \) (20.4 [Bo]). It is clear that \( A \leq L \).

We have shown that

\[ R_u(P^-)(O_S) \leq L(K_v) \]

As \( R_u(P^-) \) is \( K \)-isomorphic as a variety to affine space (see 21.20 [Bo]), \( R_u(P^-)(O_S) \) is Zariski dense in \( R_u(P^-) \) (use 3.1.1.ii [Mar]). Thus,

\[ R_u(P^-) \leq L \]

Since \( L(K_v) \) fixes \( b^\infty \in \Delta^\infty \), there is a minimal \( K_v \)-parabolic subgroup of \( H \), which we will write as \( Q \), such that \( A \leq Q \leq P^- \) and such that \( L \) is a Levi subgroup for a \( K_v \)-parabolic subgroup containing \( Q \).

Let \( \Phi_{K_v} \) be the set of roots of \( H \) with respect to a maximal \( K_v \)-split torus in \( A \), and let \( \Phi_{K_v}^+ \) and \( \Delta_{K_v} \) be the sets of positive and simple roots respectively that correspond to \( Q \).

In the notation of 21.11 [Bo], there is a proper subset of simple roots \( I \subseteq \Delta_{K_v} \) such that

\[ U_{\Phi_{K_v}^+[I]} = R_u(P^-) \]

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where \([I]\) is the subset of \(\Phi_K\), consisting of linear combinations of elements in \(I\). There is also a subset of simple roots \(J \subseteq \Delta_K\) such that

\[
[J] = \{ \alpha \in \Phi_K \mid U(\alpha) \leq L \}
\]

If \(L\) is a proper subgroup of \(H\), then \(J\) is a proper subset of \(\Delta_K\).

Since \(H\) is absolutely almost simple, \(\Phi_K\) is irreducible. Hence, there exists a “highest root” \(\tilde{\alpha} \in \Phi^+_K\) such that \(\tilde{\alpha} = \sum_{\alpha \in \Delta_K} n_\alpha \alpha\) with \(n_\alpha \geq 1\) for all \(\alpha \in \Delta_K\); see e.g. VI.1.8 [Bou]. Thus, \(\tilde{\alpha} \in \Phi^+_K - [I]\) so

\[
U(\tilde{\alpha}) \leq U_{\Phi^+_K-[I]}
\]

Stringing together the group equalities and inclusions we have collected so far yields

\[
U(\tilde{\alpha}) \leq U_{\Phi^+_K-[I]} = R_u(P) \leq L
\]

It follows from our description of the relationship between \([J]\) and the root groups contained in \(L\) that \(\tilde{\alpha} \in [J]\). Hence, \(J\) is not a proper subset of \(\Delta_K\), and we are left with \(L = H\). That is, the center of \(H\) contains the infinite group generated by \(b\). This is our contradiction.

\[\square\]

### 3.7 Cycle assembly in \(X(r_0)\)

Let \(C_\Delta \subseteq \Sigma\) be the cone of all geodesic rays contained in \(\Sigma\), based at \(e\), and limiting to points in \(\Delta^\infty\).

Recall our choice of \(a_+ \in H(K_S)\) as a translation of \(\Sigma\) such that

\[
a_+^n e \to a_+^\infty \in \Pi^\infty
\]

Recall also that \(\Pi^\infty \subseteq \Sigma^\infty\) is the collection of antipodal points for points in \(\Pi^\infty \subseteq \Sigma^\infty\) and that \(\Delta^\infty\) is the union of chambers in \(\Sigma^\infty\) containing \(\Pi^\infty\).

Therefore, Lemma 3.6(b) implies for \(n \geq 1\) that any geodesic ray emanating from \(a_+^n e\) and limiting to \(\Delta^\infty\) is separated by \(V\). Hence, there is a well-defined geodesic projection toward \(a_+^n e\) that gives rise to a homeomorphism

\[
\Delta_\Delta \to V \cap a_+^n C_\Delta
\]
Recall that we chose $U$ to fix $e$. Thus, $a_+^n U a_+^{-n}$ fixes $a_+^n e$. It follows that for all $u \in U$, there are well-defined geodesic projections toward $a_+^n e$ that give rise to homeomorphisms
\[ a_+^n u \Delta_- = a_+^n u a_+^{-n} \Delta_- \to a_+^n u a_+^{-n} (V \cap a_+^n C_{\Delta}) \]
Note that these maps piece together to give a continuous surjection
\[ \pi_n : a_+^n U \Delta_- \to a_+^n U a_+^{-n} (V \cap a_+^n C_{\Delta}) \]
whose image is contained in $R_u(P^+(K_S)V \subseteq X(r_0)$. The collection of $\sum_{u \in U} \pi_n(a_+^n u \Delta_-)$ are the cycles we have been searching for throughout this paper.

**Lemma 3.7(a)** There is a point $s \in \Sigma$, a chamber $s \subseteq \Sigma$, and a sector $\mathcal{G} \subseteq C_{\Delta}$ such that:

(i) $s \in s \subseteq \mathcal{G}$; and

(ii) For each nontrivial $u \in U$,
\[ \mathcal{G} \cap g_{\Sigma,s}(uC_{\Delta}) = \emptyset \]
where $g_{\Sigma,s} : X \to \Sigma$ is the building retraction for the pair $(\Sigma, s)$.

**Proof:** Let $\mathcal{G}' \subseteq C_{\Delta}$ and $\mathcal{T} \subseteq X$ be sectors that do not contain a common subsector. Consider an apartment $\Sigma_0 \subseteq X$ that contains disjoint subsectors $\mathcal{G}_0 \subseteq \mathcal{G}'$ and $\mathcal{T}_0 \subseteq \mathcal{T}$. For any chamber $c \subseteq \mathcal{G}_0$, the retraction $g_{\Sigma,c}$ restricts to an isometry from $\Sigma_0$ to $\Sigma$ that fixes $\mathcal{G}_0$ pointwise. Thus, we have
\[ \mathcal{G}_0 \cap g_{\Sigma,c}(\mathcal{T}_0) = \emptyset \]

Choose $D \geq 0$ such that $\mathcal{T}$ is contained within the closed metric $D$-neighborhood of $\mathcal{T}_0$. Now choose $\mathcal{G} \subseteq \mathcal{G}_0$ such that the closed metric $D$-neighborhood of $\mathcal{G}$ in $\Sigma$ is completely contained within $\mathcal{G}_0$. Then for any two chambers $s \subseteq \mathcal{G}$ and $t \subseteq \mathcal{T}$, the distance from $g_{\Sigma,s}(t)$ to $g_{\Sigma,s}(\mathcal{T}_0)$ is at most $D$ since $g_{\Sigma,s}$ does not increase distances. As the distance from $g_{\Sigma,s}(\mathcal{T}_0)$ to $\mathcal{G}$ is at least $D$, we find
\[ \mathcal{G} \cap g_{\Sigma,s}(\mathcal{T}) = \emptyset \]
By Lemma 3.5(b), $uC_\Delta$ can be covered by finitely many $\mathcal{T}$ as above for any nontrivial $u \in U$. Thus, we can assume, after perhaps passing to a subsector of $\mathcal{S}$, that $\mathcal{S} \cap \mathcal{V}_x(\mathcal{T}) = \emptyset$ for all such $u$ and $\mathcal{T}$. Hence, the lemma is satisfied for any choice of $s \in \mathfrak{s}$.

We fix $s, \mathfrak{s},$ and $\mathcal{S}$ as above, and for every $n \in \mathbb{N}$ we let
\[ \phi_n : \Sigma - \{a_+^n s\} \to \Sigma^\infty \]
be the visual projection to the boundary from the point $a_+^n s$.

**Lemma 3.7(b)** For every $r \geq r_0$, the inclusion $X(r_0) \hookrightarrow X(r)$ induces a nontrivial homomorphism
\[ \widetilde{H}_k(H, \mathcal{S}) - 1(a_+^n s \cup \Delta^\infty \cup X(r_0) ; \mathbb{F}_p) \longrightarrow \widetilde{H}_k(H, \mathcal{S}) - 1(\Sigma^\infty ; \mathbb{F}_p) \]

**Proof:** Choose $n \in \mathbb{N}$ such that $V$ separates $a_+^n \mathcal{S}$ into a compact component (containing $s$) and a noncompact component, and such that
\[ a_+^n e \notin X(r + d(e, s)) \]
The latter condition can be arranged by Lemma 3.4(b), and it implies that
\[ \mathcal{V}_x(a_+^n s) = \{a_+^n s\} \notin X(r) \]
where $\mathcal{V}_x(a_+^n s)$ is the retraction corresponding to the pair $(\Sigma, a_+^n s)$. Therefore, the following composition is well defined:
\[ a_+^n U \Delta^\infty \to X(r_0) \hookrightarrow X(r) \to \Sigma \to \Sigma^\infty \]
where the map on the left is $\pi_n$, the map second from the right is $\mathcal{V}_x(a_+^n s)$, and the map on the far right is $\phi_n$.

Since $\mathcal{V}_x(a_+^n s)$ is simply $\mathcal{V}_x(s)$ conjugated by $a_+^n$, Lemma 3.7(a) implies that there is an open neighborhood of $\phi_n(V \cap a_+^n \mathcal{S}) \subseteq \Delta^\infty$ that has 1-point pre-images of points under the above composition. Hence, using excision—as in determining degrees of maps between spheres (see e.g. Proposition 2.30 of [Ha])—one sees that the induced homomorphism
\[ \widetilde{H}_k(H, \mathcal{S}) - 1(a_+^n U \Delta^\infty ; \mathbb{F}_p) \longrightarrow \widetilde{H}_k(H, \mathcal{S}) - 1(\Sigma^\infty ; \mathbb{F}_p) \]
Our result follows as the above homomorphism factors through
\[ \tilde{H}_{k(H,S)-1}(X(r_0); \mathbb{F}_p) \longrightarrow \tilde{H}_{k(H,S)-1}(X(r); \mathbb{F}_p) \]

Our proof of Proposition B* is complete.

4 Appendix

For completeness, in this section we record two results.

Existence of anisotropic tori. The following lemma is well-known, though we do not know of a reference for it. We thank Stephen DeBacker for communicating the following proof to us.

Lemma 4(a) Any semisimple group defined over \( \mathbb{F}_q((t)) \) contains a maximal \( \mathbb{F}_q((t)) \)-torus that is \( \mathbb{F}_q((t)) \)-anisotropic.

Proof: Let \( k = \mathbb{F}_q((t)) \), let \( k_s \) be a separable closure of \( k \), and let \( \overline{\mathbb{F}}_q \) be the algebraic closure of \( \mathbb{F}_q \) in \( k_s \).

We define \( \tau_q : \overline{\mathbb{F}}_q \rightarrow \overline{\mathbb{F}}_q \) as the Frobenius automorphism \( x \mapsto x^q \), and we extend \( \tau_q \) to an automorphism \( \hat{\tau}_q : \overline{\mathbb{F}}_q((t)) \rightarrow \overline{\mathbb{F}}_q((t)) \) by applying \( \tau_q \) to the coefficients of the Laurent series. Finally, we choose some \( \tau \) in the Galois group \( \text{Gal}(k_s/k) \) that extends \( \hat{\tau}_q \). Below we will write \( k_u \) for the field \( \mathbb{F}_q((t)) \) so that we have \( k < k_u < k_s \).

Suppose \( G \) is a connected semisimple \( k \)-group, and let \( T \leq G \) be a maximal \( k_u \)-split torus that is defined over \( k \). Such a torus exists by a result of Bruhat-Tits (Corollaire 5.1.12 of [Br-Ti]). We denote the character group of \( T \) by \( X(T) \).

Let \( N \) be the normalizer of \( T \) in \( G \), and let \( Z \) be the centralizer of \( T \) in \( G \). Note that \( Z \) is defined over \( k \), and \( Z \) is a torus since \( G \) is necessarily \( k_u \)-quasisplit (see e.g. Lemma 4.7 [Sp 2]). Choose an \( n \in N(k_u) \) such that \( n \circ \tau \) acts as a so-called “twisted Coxeter element” on the vector space \( X(T) \otimes \mathbb{R} \) (see e.g. Lemma 7.4(i) [Sp 1]); thus, \( n \circ \tau \) has no nontrivial fixed points in \( X(T) \otimes \mathbb{R} \).

By Lang’s theorem, there exists a \( g \in G(k_u) \) such that \( g^{-1} \tau(g) = n \) (see e.g. the proof of 16.4 [Bo]). We claim that \( {}^gZ \) satisfies the lemma.

First, to show that \( {}^gZ \) is defined over \( k \), we need to verify that \( ({}^gZ)^\varphi = {}^gZ \) for all \( \varphi \in \text{Gal}(k_s/k) \). Since \( {}^gZ \) is defined over \( k_u \), it suffices to check that
Thus we see that $gZ$ is defined over $k$:

$$(gZ)^\tau = \tau(g)(Z) = g^nZ = gZ$$

Second, note that showing $gZ$ is $k$-anisotropic reduces to verifying that $gT$ is $k$-anisotropic since the maximal $k_u$-anisotropic torus of $Z$ remains $k_u$-anisotropic (and hence $k$-anisotropic) after conjugating by $g$.

To show that $gT$ is $k$-anisotropic, it suffices to check that there are no nontrivial fixed points of $\tau$ on the character group $X(gT)$. As shown in Proposition 3.3.4(i) of [Ca], there is a bijection $X(gT) \to X(T)$ that creates a correspondence between the $\tau$-action on $X(gT)$ and the $(n \circ \tau)$-action on $X(T)$. Hence, the result follows from our choice of $n$.

The “if” implication of Theorem A. We sketch the proof of this result as can be found in Théorème 4 of [Se 1].

We may assume that $G$ is connected. The arithmetic group $G(O_S)$ is a direct product “up to finite groups” of $G_{der}(O_S)$ and $Z(G)(O_S)$, where $Z(G)$ is the center of $G$. By Dirichlet’s units theorem, $Z(G)(O_S)$ is a finitely generated abelian group, hence of type $FP_{\infty}$.

Since $G_{der}$ is $K$-anisotropic, Behr-Harder reduction theory yields that $G_{der}(O_S)$ is a cocompact lattice in a semisimple group. Hence, $G_{der}(O_S)$ acts properly and cocompactly on a Euclidean building. Therefore, $G_{der}(O_S)$ is of type $FP_{\infty}$ as well. We conclude that $G(O_S)$ is of type $FP_{\infty}$. (More is true: since $G_{der}(O_S)$ is residually finite, and there are, up to conjugacy, only finitely many stabilizer subgroups of $G_{der}(O_S)$ for the action on the building, $G_{der}(O_S)$ contains a finite-index torsion-free subgroup $\Gamma$ that acts freely and cocompactly. Thus, there is a finite $K(\Gamma, 1)$.)

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