Binding energy and root-mean-square radius are computed for the ground state of the lambda hypertriton ($T = 0, J^m = 1/2^+$). The computations are carried out using spin-averaged lambda-proton and lambda-neutron potentials restored from theoretical scattering phases through Gel’fand-Levitan-Marchenko theory. The lambda hypertriton is treated as a three-body system consisting of lambda-proton, lambda-neutron and proton-neutron subsystems. In coordinate space, the dynamics of the system is described using the Differential Faddeev Equations in hyperspherical variables. Faddeev amplitudes are represented as hyperradial wavefunctions on a basis of hyperspherical polynomials. These hyperradial wavefunctions are further expanded on a basis of associated Laguerre polynomials, resulting in a system of coupled hyperradial equations. By solving the eigenvalue problem derived from this system of coupled hyperradial equations, the binding energy and root-mean-square matter radius computed are -2.462 MeV and 7.00 fm respectively.

I. INTRODUCTION

Light hypernuclei play an important role as femtoscale laboratories for testing the accuracy of potentials developed for hyperon-nucleon and hyperon-hyperon interactions. In early helium bubble-chamber experiments and later emulsion experiments, a large number of lambda hypernuclei have been observed, compared to just one or two sigma and cascade (xi) hypernuclei [1, 2]. As a result of this large number of lambda hypernuclei, the lambda-nucleon interaction has received considerable attention over the last half-century, in contrast to the sigma-nucleon or cascade-nucleon interactions.

The lambda-nucleon potentials in common use have their origin in meson-exchange SU(3) theory [3–9], meson-exchange SU(6) theory [10–12] and Chiral Effective Field Theory [13–15]. In order to test their accuracy, these potentials have been used in calculations to compute some important structural properties. These properties include the binding energy, lifetime, and root-mean-square radius of a lambda hypertriton ($^3\Lambda\text{H}$). The lambda hypertriton plays a very important role in the development of new lambda-nucleon potentials. Its importance is similar to that of the deuteron and the triton for nucleon-nucleon potentials. Charge symmetry breaking and lambda-sigma conversion, which are very significant in the lambda-nucleon force, are also tested by computing lambda separation energies of isospin doublets such as helium-4-lambda ($^4\Lambda\text{He}$) and hydrogen-4-lambda ($^4\Lambda\text{H}$). Negligible, and sometimes significant, differences are observed between some of these computations and experimental observations. For example, lambda hypertriton lifetimes observed in experiments are about 30 - 50% shorter [17] than computed values [16].

The significant differences observed between theoretical predictions and experiments suggests that existing theories for the lambda-nucleon force are in need of some modifications, using data from more accurate experiments. These differences also suggest that new perspectives from alternative theories may be needed to complement existing theories. In line with the quest for alternative theories for the lambda-nucleon force, new lambda-proton and lambda-neutron potentials were developed in [18], through Gel’fand-Levitan-Marchenko theory. In the nonstrange sector, potentials from inverse scattering theory have been applied in one or two few-body calculations. For example, in [19] triton and alpha particle calculations were carried out using a nucleon-nucleon potential with a hybrid structure: the $^3\text{S}_0$ force was constructed through inverse scattering theory while the $^3\text{S}_1-^3\text{D}_1$ force has its origin in meson theory. In [20–22] potentials from inverse scattering theory were also used in calculations for triton binding energy with satisfactory results. The aim of this paper is to test the accuracy of the potentials developed in [18] by computing the binding energy and root-mean-square radius of a lambda hypertriton. The lambda hypertriton is treated as a proton+neutron+lambda three-body system, and the computations are done using the Differential Faddeev Equations in hyperspherical variables.

In Sections II and III of this paper, the Differential Faddeev Equations are presented in Jacobi coordinates and in hyperspherical variables, respectively. Coupled hyperradial equations are discussed in Section IV, and the expansion basis for the hyperradial wavefunctions are presented in Section V. In Section VI, Nonlinear Least Squares Fit is applied
on data from Gel’fand-Levitan-Marchenko theory to obtain two-body lambda-nucleon potentials and in Section VII some known aspects on the structure of a lambda hypertriton are outlined. Results and discussion on three-body calculations are presented in Section VIII while Section IX carries concluding remarks.

II. DIFFERENTIAL FADDEEV EQUATIONS IN JACOBI COORDINATES

Consider a three-body system with particles of masses \( m_1, m_2 \) and \( m_3 \) having space-fixed position vectors \( \vec{r}_1, \vec{r}_2 \) and \( \vec{r}_3 \in \mathbb{R}^3 \), respectively. The masses of the particles are in atomic mass units (a.m.u.). Furthermore, let reduced masses be defined as \( \mu_1 = m_1/m, \mu_2 = m_2/m \) and \( \mu_3 = m_3/m \), where \( m \) is a unit mass, taken here to be the mass of a nucleon. Mass-scaled Jacobi coordinates \((\vec{x}_i, \vec{y}_i)\), after elimination of centre of mass motion, are defined as follows [23]:

\[
\vec{x}_i = \sqrt{\frac{A_j A_k}{A_j + A_k}} (\vec{r}_j - \vec{r}_k) \quad (1)
\]

\[
\vec{y}_i = \sqrt{\frac{A_i (A_j + A_k)}{A_i + A_j + A_k}} (\vec{r}_i - \frac{A_j \vec{r}_j + A_k \vec{r}_k}{A_j + A_k}) \quad (2)
\]

where \( i, j, k \in (1, 2, 3) \). The three configurations of the Jacobi coordinates can be obtained through cyclic permutations of the indices \( i, j, k \).

Let \( \psi^J \) be the three-body wavefunction of the system. In the Faddeev formalism, this wavefunction can be written as a sum of two-body wavefunctions, \( \psi^J_i, i = 1, 2, 3 \), as follows:

\[
\psi^J = \sum_{i=1}^{3} \psi^J_i (\vec{x}_i, \vec{y}_i) \quad (3)
\]

where \( \psi^J_i (\vec{x}_i, \vec{y}_i) \) are Faddeev amplitudes. Ignoring three-body forces, the potential of the system can be written as a sum of two-body potentials \( V_{jk} \). In spectator notation, these two-body potentials are written as \( V_i \). With the Faddeev amplitudes and two-body potentials as specified, the dynamical behaviour of the system is described by the following system of coupled partial differential equations:

\[
(H^0_i + h + V_i - E)\psi^J_i (\vec{x}_i, \vec{y}_i) = -V_i(x_i) \sum_{j \neq i} \psi^J_j (\vec{x}_i, \vec{y}_i) \quad (4)
\]

where \( H^0_i \) is the kinetic energy operator for each set of Jacobi coordinates, and \( h \) is the sum of the intrinsic Hamiltonians of all three particles [23]. The kinetic energy can be written as the sum of Laplace operators in the variables \( x_i \) and \( y_i \) as follows [24]:

\[
H^0_i = -\frac{\hbar^2}{2m} (\nabla_{\vec{x}_i}^2 + \nabla_{\vec{y}_i}^2) \quad (5)
\]

where \( m \) is the nucleon mass. The nucleon mass appears here as a common factor because in the definition of the Jacobi coordinates, it was used in scaling the particle masses i.e. \( A_i = m_i/m \) \( (i = 1, 2, 3) \), as earlier presented. The coupled system in Equation (4) constitutes the Differential Faddeev Equations.

In order to carry out a partial wave decomposition of the Faddeev amplitudes, a coupling order for the orbital angular momenta and spins must first be established. For a given configuration of the Jacobi variables \( (x_i, y_i) \), let \( (\ell_{xi}, \ell_{yi}) \) be the respective orbital angular momentum coordinates. Furthermore, let \( s_i, s_j \) and \( s_k \) be the spins of the three particles, where \( i \) is the spectator and \( jk \) is the interacting pair. The isospin is left out in this formalism.

Let \( S_{xi} \) be the total spin of the interacting pair and \( L_i \) the total orbital angular momentum for the whole system. The orbital quantum numbers \( \ell_{xi} \) and \( \ell_{yi} \) are coupled together \( (L_i = \ell_{xi} + \ell_{yi}) \) while the spin quantum numbers of the interacting pair, \( s_j \) and \( s_k \), are coupled together \( (S_{xi} = s_j + s_k) \), leaving out the spin of the spectator. From \( J_i = L_i + S_{xi} \) the channel state is constructed as \( J_i + s_i \). The abbreviation \( \alpha_i \) is adopted for the coupling order of the quantum numbers i.e. \( \alpha_i = \{(\ell_{xi}, \ell_{yi})L_i, (s_j, s_k)S_{xi}\}J_i; s_i \). The Faddeev amplitudes can therefore be decomposed into partial waves as follows:

\[
\psi^J_i (\vec{x}_i, \vec{y}_i) = \sum_{\ell_{xi}, \ell_{yi}, L_i} \sum_{s_j, s_k, S_{xi}} \sum_{J_i, s_i} \psi^{i, L_i, S_{xi}, s_j, s_k, J}_i (x_i, y_i) |i:\{(\ell_{xi}, \ell_{yi})L_i, (s_j, s_k)S_{xi}\}J_i; s_i\rangle^J \quad (6)
\]

\[
\equiv \sum_{\alpha_i} \psi^{\alpha_i, J}_i (x_i, y_i) |i:\alpha_i\rangle^J \quad (7)
\]
In the following section, Jacobi coordinates are transformed into hyperspherical coordinates. The functions \( \psi_{n_i}^{j}(x_i, y_i) \), with the coupling order as specified, are next expanded on a hyperspherical basis.

### III. DIFFERENTIAL FADDEEV EQUATIONS IN HYPERSPHERICAL COORDINATES

The two Jacobi coordinates, \( \vec{x}_i \) and \( \vec{y}_i \), are transformed into a 6-dimensional system of hyperspherical coordinates. These coordinates consist of one hyperradius \( \rho \) and 5 angles. These 5 angles are collectively labelled as \( \Omega_5 \). In this six-dimensional space, the kinetic energy is written as

\[
H^0_i = -\frac{\hbar^2}{2m} \left[ \frac{1}{\rho^5} \frac{\partial}{\partial \rho} \left( \rho^5 \frac{\partial}{\partial \rho} \right) \right. \left. - \frac{1}{\rho^2} C^2(\Omega_5) \right]
\]  

(8)

where \( C(\Omega_5) \) is the generalised angular momentum operator. The specific form of \( C^2(\Omega_5) \) depends on the parametrisation used for the the five angles \( \Omega_5 \). There are two common systems used in defining these angles, the Fock parametrisation (asymmetric) and the Smith parametrisation (symmetric) [25]. In this paper, the Fock parametrisation is used, and the 5 angular variables are \( \Omega_5 = (\theta_i, \nu_{x_i}, \nu_{y_i}, \omega_{x_i}, \omega_{y_i}) \). The variable \( \theta_i \in [0, \pi/2] \) is a hyperangle. The variables \( \nu_{x_i} \in [0, \pi] \) and \( \omega_{x_i} \in [0, 2\pi] \) are polar angles related to the Jacobi variable \( x_i \), while \( \nu_{y_i} \in [0, \pi] \) and \( \omega_{y_i} \in [0, 2\pi] \) are related to \( y_i \). The polar angles are defined such that \((x_i, \nu_{x_i}, \omega_{x_i})\) forms a spherical coordinate system and \((y_i, \nu_{y_i}, \omega_{y_i})\) forms another spherical coordinate system. The hyperradius and the hyperangle \( \theta_i \) are obtained through the following transformation relations:

\[
x_i = \rho \sin \theta_i
\]

\[
y_i = \rho \cos \theta_i
\]

(9)

(10)

From this transformation, \( \theta_i = \arctan(x_i/y_i) \) and \( \rho = (x_i^2 + y_i^2)^{1/2} \). The hyperradius defines the overall size of the system. It is invariant under permutation of particles and also invariant under all rotations and translations.

With the hyperspherical coordinates as defined, the kinetic takes the following form:

\[
H^0_i = -\frac{\hbar^2}{2m} \left[ \frac{1}{\rho^5} \frac{\partial}{\partial \rho} \left( \rho^5 \frac{\partial}{\partial \rho} \right) \right. \left. + \frac{1}{\sin^2 2\theta_i} \frac{\partial}{\partial \theta_i} \left( \sin^2 2\theta_i \frac{\partial}{\partial \theta_i} \right) - \frac{\tilde{\ell}_{x_i}^2}{\sin^2 \theta_i \cos^2 \theta_i} - \frac{\tilde{\ell}_{y_i}^2}{\sin^2 \theta_i \cos^2 \theta_i} \right]
\]

(11)

where the angular momentum squared operators \( \tilde{\ell}_{x_i}^2 \) and \( \tilde{\ell}_{y_i}^2 \) are associated with the angular variables \((\nu_{x_i}, \omega_{x_i})\) and \((\nu_{y_i}, \omega_{y_i})\), respectively. The operators \( \tilde{\ell}_{x_i}^2 \) and \( \tilde{\ell}_{y_i}^2 \) have eigenvalues \( \ell_{x_i}(\ell_{x_i} + 1) \) and \( \ell_{y_i}(\ell_{y_i} + 1) \), respectively.

The key to the hyperspherical method are the hyperspherical harmonics, \( \mathcal{Y}(\Omega_5) \), which are the eigenvectors of the operator \( C(\Omega_5) \). These hyperspherical harmonics are constructed as follows:

\[
\mathcal{Y}^{\ell_{x_i}, \ell_{y_i}}_{K_i, m_{x_i}, m_{y_i}}(\Omega_5) = \phi^{\ell_{x_i}, \ell_{y_i}}_{K_i}(\theta_i) Y_{m_{x_i}}^{\ell_{x_i}}(\nu_{x_i}, \omega_{x_i}) Y_{m_{y_i}}^{\ell_{y_i}}(\nu_{y_i}, \omega_{y_i})
\]

(12)

where \( \phi^{\ell_{x_i}, \ell_{y_i}}_{K_i}(\theta_i) \) are hyperspherical polynomials, with \( Y_{m_{x_i}}^{\ell_{x_i}}(\nu_{x_i}, \omega_{x_i}) \) and \( Y_{m_{y_i}}^{\ell_{y_i}}(\nu_{y_i}, \omega_{y_i}) \) being spherical harmonics.

The hyperangular functions, \( \phi^{\ell_{x_i}, \ell_{y_i}}_{K_i}(\theta_i) \) are known to be the solutions of the following hyperangular equation [24, 26]:

\[
\left[ \frac{1}{\sin^2 2\theta_i} \frac{d}{d\theta_i} \left( \sin^2 2\theta_i \frac{d}{d\theta_i} \right) - \frac{\ell_{x_i}(\ell_{x_i} + 1)}{\sin^2 \theta_i} - \frac{\ell_{y_i}(\ell_{y_i} + 1)}{\cos^2 \theta_i} \right] \phi^{\ell_{x_i}, \ell_{y_i}}_{K_i}(\theta_i) = -K_i(K_i + 4) \phi^{\ell_{x_i}, \ell_{y_i}}_{K_i}(\theta_i)
\]

(13)

where \(-K_i(K_i + 4)\) are separation constants and \( K_i \) are hyperangular momenta. The solutions of this equation are given by

\[
\phi^{\ell_{x_i}, \ell_{y_i}}_{K_i}(\theta_i) = N^{\ell_{x_i}, \ell_{y_i}}_{K_i}(\sin \theta_i)^{\ell_{x_i}}(\cos \theta_i)^{\ell_{y_i}} P_{n_i}^{(a, b)}(\cos 2\theta_i)
\]

(14)

where \( P_{n_i}^{(a, b)} \) is a Jacobi polynomial of degree \( n_i \), with \( a = \ell_{x_i} + 1/2 \) and \( b = \ell_{y_i} + 1/2 \). The degree \( n_i \) is constrained to values given by the relation \( K_i = \ell_{x_i} + \ell_{y_i} + 2n_i \). The quantities \( N^{\ell_{x_i}, \ell_{y_i}}_{K_i} \) are normalisation constants which are obtained from the orthogonality of the hyperangular functions with respect to the weights \( \sin^2 \theta_i \cos^2 \theta_i \) on \([0, \pi/2] \):

\[
\int_0^{\pi/2} \phi^{\ell_{x_i}, \ell_{y_i}}_{K_m}(\theta_i) \phi^{\ell_{x_i}, \ell_{y_i}}_{K_n}(\theta_i) \sin^2 \theta_i \cos^2 \theta_i d\theta_i = \delta_{K_m K_n}
\]

(15)
An orthonormal basis potential energy matrix elements and the kinetic energy matrix elements are then also computed on this same basis.

V. ASSOCIATED LAGUERRE POLYNOMIAL BASIS FOR HYPERRADIAL WAVEFUNCTIONS

The orthonormality of the hyperspherical polynomials, $\phi_{K_i}^{\ell_x, \ell_y, \ell_z} (\theta_i)$, suggests that the Faddeev amplitudes $\psi_{\alpha_i}^{i,J}(x_i, y_i)$ can be expanded on these hyperangular basis functions as follows:

$$\psi_{\alpha_i}^{i,J}(x_i, y_i) = \sum_{K_i = K_{\min}}^{K_{\max}} \frac{\chi_{\alpha_i, K_i}^{i,J}(\rho)}{\rho^{5/2}} \phi_{K_i}^{\ell_x, \ell_y, \ell_z} (\theta_i)$$

The hyperangular solutions in Equation (14) are the same for any given potential. The task that is left is to find the hyperradial solutions $\chi_{\alpha_i, K_i}^{i,J}(\rho)$, which depend on a given potential, so that the Faddeev amplitudes can be completely known.

IV. COUPLED HYPERRADIAL EQUATIONS

The hyperradial equation is obtained by substituting Equation (7), making use of Equation (17), into the Differential Faddeev Equations. After hyperangular integration, and making use of the orthonormality of the hyperangular basis, the following system of coupled equations for the hyperradial wavefunctions $\chi_{\alpha_i, K_i}^{i,J}(\rho)$ emerges:

$$\left\{-\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + \frac{\hbar^2}{2m\rho^2} \mathcal{L}_{K_i} (\mathcal{L}_{K_i} + 1) - E\right\} \chi_{\alpha_i, K_i}^{i,J}(\rho) = - \sum_{j \neq i, K_j} V_{\alpha_i K_i, \alpha_j K_j}^{i,j} \chi_{\alpha_i, K_i}^{i,J}(\rho)$$

where $\mathcal{L}_{K_i} = K_i + 3/2$. The term $\mathcal{L}_{K_i} (\mathcal{L}_{K_i} + 1)/\rho^2$ is a centrifugal barrier. The kinetic energy is diagonal while the two-body potentials, $V_{\alpha_i K_i, \alpha_j K_j}^{i,j}$, are given by the following hyperangular integration of the two-body potentials:

$$V_{\alpha_i K_i, \alpha_j K_j}^{i,j} = \langle \phi_{K_j}^{\ell_x, \ell_y, \ell_z} (\theta_j) | (V_{12} + V_{23} + V_{13}) | \phi_{K_i}^{\ell_x, \ell_y, \ell_z} (\theta_i) \rangle$$

These hyperradial couplings are computed by making use of the Raynal-Revai transformation coefficients in [23, 31]. The hyperangular integration is done through Gauss-Jacobi quadrature on $N_{jac}$ grid points. The number of coupled hyperradial equations in Equation (18) is equal to the number of terms used in the expansion in Equation (17). Some of the values of $K_j$ in $[K_{\min}, K_{\max}]$ may be excluded by the Pauli principle. This one-dimensional coupled system of equations is solved on the domain $\rho \in [0, \infty)$, using the usual boundary conditions for bound states:

$$\chi_{\alpha_i, K_i}^{i,J}(\rho) \to 0, \quad \rho \to 0$$

$$\chi_{\alpha_i, K_i}^{i,J}(\rho) \to 0, \quad \rho \to \infty$$

V. ASSOCIATED LAGUERRE POLYNOMIAL BASIS FOR HYPERRADIAL WAVEFUNCTIONS

In order to solve the Equation (18), the first step is to expand the wavefunction $\chi_{\alpha_i, K_i}^{i,J}(\rho)$ on a suitable basis. The potential energy matrix elements and the kinetic energy matrix elements are then also computed on this same basis. An orthonormal basis $B = \{ R_{\alpha}(\rho) \}$ is used, with elements defined as follows:

$$R_{\alpha}(\rho) = \rho^{5/2} \left[ \frac{\Gamma(n+1)}{\Gamma(n+6)} \right]^{1/2} L^\alpha_n(z) \exp \left( -\frac{z}{2} \right)$$

where $z = \rho/\rho_0$ is a dimensionless hyperradial variable and $\rho_0$ is the scaling hyperradius. The function $L^\alpha_n(x)$ is an associated Laguerre polynomial, defined recursively as follows:

$$L^\alpha_0(x) = 1$$

$$L^\alpha_1(x) = -x + q + 1$$

$$L^\alpha_{k+1}(x) = \frac{(2k + 1 + q - x)L^\alpha_k(x) - (k + q)L^\alpha_{k-1}(x)}{k + 1}, \quad k \geq 1$$
In the calculations carried out here, associated Laguerre polynomials with \(q = 5\) are used. On the basis \(\{R_n(\rho)\}\), the wavefunctions \(\chi_{\alpha_i, K_i}(\rho)\) are expanded as follows:

\[
\chi_{\alpha_i, K_i}(\rho) = \sum_{n=0}^{N_b} a_{K_i, \alpha_i}^{n, J} R_n(\rho) \tag{26}
\]

where \(N_b\) is the size of the model space i.e. the number of elements in the basis \(B\). The expansion coefficients \(a_{K_i, \alpha_i}^{n, J}\) are determined by solving a linear system. Using the expansion of the hyperradial wavefunctions in the coupled hyperradial equations, the following linear system is obtained \([23]\):

\[
Ha = aE \tag{27}
\]

where \(H\) is the coefficient matrix. The computation of matrix elements of the kinetic energy operator on \(B = \{R_n(\rho)\}\) is outlined in \([23]\). The potential matrix elements on \(B = \{R_n(\rho)\}\) are given by hyperradial integrals, and these are computed using Gauss-Laguerre quadrature on \(N_{tag}\) grid points. The eigenvalues for the three-body system are obtained by solving Equation (27) using the numerical method outlined in \([23]\).

VI. TWO-BODY POTENTIALS

As a three-body system, the lambda hypertriton is treated as a \(p+n+\Lambda\) system. There are therefore three distinct subsystems: the \(\Lambda + n\), \(\Lambda + p\) and \(n+p\) subsystems. At the introduction, it was stated that the aim of this paper is to introduce lambda-proton and lambda-neutron potentials, developed through Gel’fand-Levitan-Marchenko theory, into few-body hypernuclear physics. These potentials were restored through the application of inverse scattering theory on sub-threshold theoretical scattering phases \([18]\). In order to render these potentials easy to use in this paper and elsewhere, the data representing the effective potentials from \([18]\) were fitted using a statistical model. This model, which is a sum of three Gaussians, is shown in Equation (28).

\[
V_{\Lambda N}(r) = \sum_{i=1}^{3} V_i \exp\left\{\frac{-(r - \mu_i)^2}{\sigma_i^2}\right\} \tag{28}
\]

For the lambda-proton and lambda-neutron effective potentials from \([18]\), the parameters \(V_i, \mu_i\), and \(\sigma_i\) were determined through a nonlinear Least Squares Fit. Minimization of the objective functional was carried out using the Levenberg-Marquardt algorithm. After convergence, the estimated parameters obtained are displayed in Table I with the error (uncertainty) in each estimate indicated. The effective potentials with these estimated parameters are displayed in Figures 1 and 2. For ease of reference, these effective lambda-nucleon potentials shall be called GLM-YN0 potentials.

| \(V_i\) /MeV | \(\mu_i\) / fm | \(\sigma_i\) / fm |
|--------------|---------------|-----------------|
| \(i = 1\)   | 45.88 ± 0     | 0.1148 ± 0.0006601 | -0.3932 ± 0.0008502 |
| \(i = 2\)   | 8.106 ± 07 ± 0 | -1.193 ± 0.001948 | 0.3575 ± 0.0005306 |
| \(i = 3\)   | -47.04 ± 0    | 0.3748 ± 0.0001386 | 0.1667 ± 0.0002179 |

Table I. Estimates of fit parameters of \(\Lambda\)-proton and \(\Lambda\)-neutron effective potentials, \(V_{\Lambda p}\) and \(V_{\Lambda n}\), respectively. The error in each estimate is indicated.

\[
V_{np}(r) = \sum_{i=1}^{2} \frac{V_i}{r} \exp(-\beta_i r) \tag{29}
\]

For the neutron-proton partition, the spin-averaged Malfliet-Tjon potential (MTV) was used \([32]\). The radial form factor for the MTV potential is a sum of Yukawa functions, as shown in Equation (29). The MTV potential, with parameters \(V_i\) and \(\beta_i\) from \([32]\) (see Table II), is shown in Figure 3.
FIG. 1. Three-Gaussian sum fit to spin-averaged (effective) Λp two-body potential. The fitting parameters, with the errors in their estimates, are shown in Table II.

FIG. 2. Three-Gaussian sum fit to spin-averaged (effective) Λn two-body potential. The fitting parameters, with the errors in their estimates, are shown in Table II.

TABLE II. Parameters for spin-averaged Malfliet-Tjon potential, MTV ($V_{np}$). These parameters are from Zabolitzky [33].

| i  | $V_i$/MeV.fm | $\beta_i$/fm$^{-1}$ |
|----|--------------|---------------------|
| 1  | 1458.05      | 3.11                |
| 2  | -578.09      | 1.55                |

VII. STRUCTURE OF A LAMBDA HYPERTRITON

The lambda hypertriton, $pn\Lambda$, is channel of the $NN\Lambda$ system for which the total isospin, $T$ is zero ($T = 0$). Another interesting channel of this system is $nn\Lambda$, for which $T = 1$. The bound state that was conjectured for $nn\Lambda$ [34] has not been conclusively found in any theoretical studies up this moment [35–41]. Since the $pn\Lambda$ state is only weakly bound, a $pp\Lambda$ bound state is not expected to exist because of the Coulomb repulsion between the two protons. In a separate paper, the $nn\Lambda$ channel will be investigated within Gel’fand-Levitan-Marchenko theory.

In $pn\Lambda$, the $np$ subsystem is bound and it is the well-known deuteron ($d$). A $^3S_1$ weakly bound state was reported for the $\Lambda n$ subsystem, with a binding energy of $-0.05$ MeV [42]. No bound state has been observed for the $\Lambda p$
The experimental value of the binding energy of the lambda hypertriton \((J = 1/2^+)\) is \(-2.35 \pm 0.05 \text{ MeV}\) (emulsion experiment) while that for its \(pn\) subsystem or the deuteron \((J = 1^+)\) is \(-2.224575(9) \text{ MeV}\) \(^{13}\). These numbers reveal the fact that more than 95\% of the binding energy of the lambda hypertriton goes into binding its deuteron subsystem. In other words, the \(\Lambda\) hyperon is very loosely attached to the deuteron. This accounts for the very large root-mean-square (r.m.s.) radius of the lambda hypertriton. Some theoretical predictions for the r.m.s. matter radius of the hypertriton are 4.9 fm \(^{14}\) and 5.48 fm \(^{15}\). In analogy to neutron and proton halo nuclei, the \(\Lambda\) hypertriton is sometimes described as \(\Lambda\) halo around a deuteron core \(^2,^{45}\). The triton \((pnn)\) has a ground state \((J = 1/2^+)\) with a binding energy of \(-8.481798 \pm 0.000002 \text{ MeV}\) \(^{46}\). By comparing this binding energy with that of the lambda hypertriton, it can be seen that the lambda hypertriton is a very loosely bound system. Due to its stable deuteron core, the hypertriton can also be accurately treated as a two-body \(\Lambda d\) system \(^{47}\). Some aspects of the \(\Lambda\) hypertriton structure discussed here are illustrated in the following section, where results of three-body calculations are presented.

**VIII. RESULTS AND DISCUSSION**

In this section, the results of three-body Faddeev calculations for the ground state of the hypertriton \((J = 1/2^+)\), using the potentials in Section \[\text{VI}\] are reported. In these calculations, the masses used for the proton and neutron are \(m_p = 1.007276466 \text{ a.m.u.}\) and \(m_n = 1.008664915 \text{ a.m.u.}\), respectively \(^{48}\). The mass of the lambda hyperon is calculated from its energy equivalence i.e. \(m_\Lambda = 1115.683/931.5 = 1.198 \text{ a.m.u.}\). These masses enter the computation through the Jacobi coordinates, which are transformed into hyperspherical variables, as shown in Section \[\text{V}\]. The mass parameter, \(\hbar^2/2m\), in Equation \[\text{18}\] was computed as follows:

\[
\frac{\hbar^2}{2m} = \frac{(\hbar c)^2}{2mc^2} = \frac{(197.3 \text{ MeV.fm})^2}{2(939.0 \text{ MeV})} = 20.7281 \text{ MeV.fm}^2
\]

where \(mc^2 = 939.0 \text{ MeV}\) is the energy equivalence of the nucleon mass.

The computational parameter \(N_{\text{lag}}\) defines the hyperradial grid size used in Gauss-Laguerre quadrature while \(N_{\text{jac}}\) is for the hyperangular grid size used Gauss-Jacobi quadrature. \(\rho_0\) is a scaling constant for the hyperradial grid. The values used for these grid parameters are \(N_{\text{lag}} = 180\), \(N_{\text{jac}} = 180\) and \(\rho_0 = 0.3 \text{ fm}\). The values used for \(N_{\text{lag}}\) and \(N_{\text{jac}}\) are sufficiently large in order to ensure that the detailed behaviour of the lambda-nucleon potentials are captured in the computation, especially the repulsive core, where changes in the potentials are sharper. The maximum values of the quantum numbers \(K_i, S_{zi}, \ell_{xi}, \ell_{yi}\) used in defining the channels are displayed in Table \[\text{III}\].

With the grid sizes fixed and the channels constructed, the dimension of the model space \(N_b\) is increase until convergence is achieved. The three-body computations, done through an inverse iteration, found a \(J = 1/2^+\) bound subsystem \(^1\).
TABLE III. Maximum values of quantum numbers used in constructing partial waves.

| $K_{\text{max}}$ | $S_{\text{max}}$ | $\ell_{x,\text{max}}$ | $\ell_{y,\text{max}}$ |
|-----------------|-----------------|-----------------|-----------------|
| 8               | 1.0             | 2               | 2               |

state of $-2.462$ MeV and a r.m.s. matter radius of 7.00 fm for the hypertriton. Table IV and Figure 4 show the convergence behaviour of the binding energy. The $\Lambda$ separation energy ($B_\Lambda$) is computed as follows:

$$B_\Lambda(\Lambda H) = E(\Lambda H) - E^{\exp}(2H)$$

(30)

where $E^{\exp}(2H) = -2.224575$ is the experimental binding energy of the deuteron. Using this relation, it is found that $B_\Lambda(\Lambda H) = 0.237$ MeV. The convergence behaviour shown in these results is identical to that in [49], where the Non-Symmetrized Hyperspherical Harmonics method was used in computing the binding energy of a triton. Using lambda-nucleon Gaussian potentials from [50], the computed hypertriton binding energy in [49] also showed a comparable convergence behaviour. In some applications of the hyperspherical harmonics method, convergence is usually accelerated by including either a pair correlation factor or a Jastrow correlation factor in Equation (17). A comparison of the results just presented with those from experimental studies and other theoretical predictions is shown in Table V.

TABLE IV. Convergence of hypertriton ground state binding energy and root-mean-square matter radius with size of model space.

| $N_b$ | $E$ / MeV | R.m.s matter radius / fm |
|------|-----------|-------------------------|
| 06   | -6.235987 | 3.668                   |
| 08   | -0.585377 | 4.376                   |
| 10   | -2.852280 | 5.575                   |
| 12   | -2.245663 | 6.328                   |
| 14   | -2.410353 | 6.724                   |
| 16   | -2.450869 | 6.903                   |
| 18   | -2.459966 | 6.969                   |
| 20   | -2.461853 | 6.990                   |
| 22   | -2.462224 | 6.996                   |
| 24   | -2.462294 | 6.998                   |
| 26   | -2.462307 | 6.998                   |
| 28   | -2.462309 | 6.998                   |
| 30   | -2.462309 | 6.998                   |
| 32   | -2.462310 | 6.998                   |

TABLE V. Hypertriton binding energy from our three-body calculation, compared with results from other three-body studies and from experiments. The $\Lambda$-nucleon potentials used are indicated in parentheses.

|                          | $E$ / MeV | $B_\Lambda$ / MeV |
|--------------------------|-----------|-------------------|
| Experiment 1 [51, 1] (Emulsion) | $-2.35 \pm 0.05$ | $0.13 \pm 0.05$ |
| Experiment 2 [52] (Helium bubble chambers) | $-2.47 \pm 0.31$ | $0.25 \pm 0.31$ |
| This paper (GLM-YN0) | **-2.462** | **0.237** |
| Fujiwara et al. [53] (FSS) | $-3.134$ | 0.878 |
| Fujiwara et al. [53] (fss2) | $-2.514$ | 0.289 |
| Fujiwara et al. [54] (fss2, modified) | $-2.487$ | 0.262 |
| Ferrari et al. [55] (NSC97f) | $-2.41(2)$ | $0.17(2)$ |
| Tominaga & Ueda [56, 57] (Ehime 00A, single) | $-2.35$ | — |
| Miyagawa et al. [58] (NSC97f) | $-2.37$ | — |
| Polinder et al. [59, 60] (χEFT LO) | $-2.34 \pm -2.36$ | — |
| Haidenbauer [61, 62] (χEFT NLO) | $-2.31 \pm -2.34$ | — |
| Haidenbauer [63] (NSC97f) | $-2.30$ | — |
| Miyagawa et al. [64] (Jülich '04) | $-2.27$ | — |
| Miyagawa & Glöckle [65] (Jülich A) | Unbounded | — |

It is important to mention that the hyperradial behaviour is obtained from the contribution of all channels. Each channel is identified by the quantum numbers $\alpha = \{K, L, S_x, \ell_x, \ell_y\}$, in that order. In the lambda hypertriton
FIG. 4. Convergence of hypertriton ground state binding energy ($E$) and root-mean-square radius with size of model space ($N_b$)

computations just presented, there are a total of 60 channels. Some of these channels make a negligible contribution. For the channel with the dominant contribution, the first four hyperradial wavefunctions are shown in Figure 5. As one progresses within this channel, these hyperradial wavefunctions are observed to become increasingly oscillatory.

IX. CONCLUSIONS

The ground state binding energy and root-mean-square radius of the lambda hypertriton were computed through the Differential Faddeev Equations in hyperspherical variables. The lambda hypertriton was treated as a proton+neutron+lambda three-body system. The lambda-proton and lambda-neutron potentials used have their roots in Gel’fand-Levitan-Marchenko theory. The results obtained are -2.462 MeV and 7.00 fm for the binding energy and root-mean-square radius, respectively. This prediction for the binding energy is equal to the value observed in an earlier helium bubble chamber experiment, $-2.47 \pm 0.31$ MeV. The binding energy reported from an emulsion experiment is $-2.35 \pm 0.05$ MeV. The convergence of the few-body calculations using these GLM-YN0 lambda-nucleon potentials were also observed to be good. These computations are significant because they represent the first application of hyperon-nucleon potentials from Gel’fand-Levitan-Marchenko theory in hypernuclear few-body physics. The results
show that inverse scattering theory can play a useful role as a complement to meson theory and chiral effective field theory in probing the hyperon-nucleon force. Further computations are required to assess these new lambda-proton and lambda-neutron potentials for conformity with other known features of the lambda-nucleon force. For example, Charge Symmetry Breaking can be verified by computing lambda separation energies of isospin doublets such as helium-4-lambda and hydrogen-4-lambda.

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