A Time-Dependent Wave-Thermoelastic Solid Interaction

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This paper is dedicated to Wolfgang L. Wendland
on the occasion of his 80th birthday.

Abstract

This paper presents a combined field and boundary integral equation method for solving time-dependent scattering problem of a thermoelastic body immersed in a compressible, inviscid and homogeneous fluid. The approach here is a generalization of the coupling procedure employed by the authors for the treatment of the time-dependent fluid-structure interaction problem. Using an integral representation of the solution in the infinite exterior domain occupied by the fluid, the problem is reduced to one defined only over the finite region occupied by the solid, with nonlocal boundary conditions. The nonlocal boundary problem is analyzed with Lubich’s approach for time-dependent boundary integral equations. Using the Laplace transform in terms of time-domain data existence and uniqueness results are established. Galerkin semi-discretization approximations are derived and error estimates are obtained. A full discretization based on the Convolution Quadrature method is also outlined. Some numerical experiments are also included in order to demonstrate the accuracy and efficiency of the procedure.

Key words: Fluid-structure interaction, Coupling BEM-FEM, Kirchhoff representation formula, Retarded potential, Time-domain boundary integral equation, Variational formulation, Wave scattering, Convolution quadrature.

Mathematics Subject Classifications (1991): 35J20, 35L05, 45P05, 65N30.65N38

1 Introduction

The mathematical study of the thermodynamic response of a linearly elastic solid to mechanical strain dates back at least to Duhamel’s 1837 pioneering work [7] on thermoelastic...
materials where he proposed the constitutive relation linking the temperature variations and elastic strains with the thermoelastic stress now known as Duhamel-Neumann law [5, 28].

Kupradze’s encyclopedic works [19] can be considered the standard reference for a modern mathematical treatment of the purely thermoelastic problem. The dynamic problem is dealt with in more recent works like [30, 33] where the matrix of fundamental solutions for the dynamic equations is revisited, while [16, 17] provide generalized Kirchhoff-type formulas for thermoelastic solids.

In the case of the scattering of thermoelastic waves, major theoretical contributions have been made by Çakoni and Dassios in [4]. The unique solvability of a boundary integral formulation is established in [3] and the interaction of elastic and thermoelastic waves is explored for homogeneous materials in [5]. The study of time-harmonic interaction between a scalar field and a thermoelastic solid has been the subject of works like [23] where the interface is taken to be a plane, or [22, 18] where time-harmonic scattering by bounded obstacles is considered.

In this paper, we present a combined field and boundary integral method for a time-dependent fluid-thermoelastic solid interaction problem. The approach here is a generalization of the method employed by the authors for treating time-dependent fluid-structure interaction problems in [14, 13]. To our knowledge no attempt has been made to investigate with rigorous justifications the time-dependent acoustic scattering by a thermoelastic obstacle.

The paper is organized as follows. The problem setting is introduced in Section 2 along with the physical assumptions and the constitutive relation under consideration leading to the time-domain system of governing equations. The problem is then recast in Section 3 where the Laplace domain system is transformed into an equivalent integro-differential non-local problem that will be formulated variationally for discretization later on. The question of existence and uniqueness of the solutions to the non-local problem is dealt with on Section 4. Sections 5 and 6 discuss the computational considerations related to the numerical solution of the discrete problem, the implementation of Convolution Quadrature (CQ) and the coupling of boundary and finite elements. Convergence experiments are given for test problems in both frequency and time domains. In the latter case both second order backward differentiation formula (BDF2) and Trapezoidal Rule CQ are used, providing evidence that the approximation is stable and of second order globally. time-domain illustrative experiments using the proposed formulation are included.

## 2 Formulation of the problem

Consider a thermoelastic solid with constant density \( \rho \) in an undeformed reference configuration and at thermal equilibrium at temperature \( \Theta_0 \). Under the action of external forces the body will be subject to internal stresses that will induce local variations of temperature. Reciprocally, if a heat source induces a change in temperature, the body will react by dilating or contracting and this will create internal stresses and deformations. In the classical linear theory [19, 20], the coupling between the mechanic strain and the thermic gradient is modelled by the Duhamel-Neumann law which defines the
thermoelastic stress $\sigma(U, \Theta)$ and the thermoelastic heat flux $F(U, \Theta)$ (also known as free energy)

$$\sigma := C\varepsilon(U) - \zeta \Theta \mathbf{I},$$
$$F := -\eta \frac{\partial U}{\partial t} + \kappa \nabla \Theta.$$ 

In the previous expressions

$$\varepsilon(U) := \frac{1}{2}(\nabla U + (\nabla U)^t)$$

is the elastic strain tensor, $\mathbf{I}$ is the $3 \times 3$ identity matrix, $\kappa$ is the thermal conductivity coefficient, which from physical principles [11] is required to be positive, $\zeta$ is the thermal expansion coefficient, and $\eta$ is given by the relation

$$\eta = \Theta_0 \zeta / \kappa.$$

For the case of homogeneous isotropic material that we are considering, the elastic stiffness tensor $C$ is given by

$$C_{ijkl} := \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where the constants $\lambda$ and $\mu$ are Lamé’s second parameter and the shear modulus respectively, and $\delta_{ij}$ is Kronecker’s delta.

We are concerned with a time-dependent direct scattering problem in fluid-thermoelastic solid interaction, which can be simply described as follows: an acoustic wave propagates in a fluid domain of infinite extent in which a bounded thermoelastic body is immersed. Throughout the paper, we let $\Omega_-$ be the bounded domain in $\mathbb{R}^3$ occupied by the thermoelastic body with a Lipschitz boundary $\Gamma$ and we let $\Omega_+ := \mathbb{R}^3 \setminus \Omega_-$ be its exterior, occupied by a compressible fluid. The problem is then to determine the scattered velocity potential $V$ in the fluid domain, the deformation of the solid $U$ and the variation of the temperature $\Theta$ in the obstacle with respect to the equilibrium temperature $\Theta_0$. It is assumed that $|\Theta / \Theta_0| << 1$.

The governing equations of the displacement field $U$ and temperature field $\Theta$ are the thermo-elastodynamic equations:

$$\rho \frac{\partial^2 U}{\partial t^2} - \Delta^* U + \zeta \nabla \Theta = 0 \quad \text{in } \Omega_- \times (0, T),$$

$$\frac{1}{\kappa} \frac{\partial \Theta}{\partial t} - \Delta \Theta + \eta \frac{\partial}{\partial t}((\nabla \cdot U) = 0 \quad \text{in } \Omega_- \times (0, T),$$

where $T$ is a given positive final time, and as usual the symbol $\Delta^*$ is the Lamé operator defined by

$$\Delta^* U := \mu \Delta U + (\lambda + \mu) \nabla (\nabla \cdot U).$$

We remark that if the thermal effect is neglected ($\zeta = 0$) Duhamel-Neumann’s law reduces to the usual expression for Hooke’s law of the classical theory for an arbitrary isotropic medium (see, e.g. [5, 19]). In the thermoelastic medium, the given physical constants $\rho, \lambda, \mu, \zeta, \eta, \kappa$, are assumed to satisfy the inequalities:

$$\rho > 0, \quad \mu > 0, \quad 3\lambda + 2\mu > 0, \quad \zeta / \eta > 0, \quad \kappa > 0.$$
In the fluid domain $\Omega_+$, we consider a barotropic and irrotational flow of an inviscid and compressible fluid with density $\rho_f$ as in [14]. The formulation can be simplified in terms of a velocity potential $V = V(x,t)$ such that

$$V = -\nabla V \quad \text{and} \quad P = \rho_f \frac{\partial V}{\partial t}.$$  

Then we arrive at the wave equation

$$\frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \Delta V = 0 \quad \text{in} \quad \Omega_+ \times (0,T) \quad (2.3)$$

where $c$ is the sound speed.

On the interface $\Gamma$ between the solid and the fluid we have the transmission conditions

$$\sigma(U, \Theta)^- \cdot n = -\rho_f \left( \frac{\partial V}{\partial t} + \frac{\partial V^{inc}}{\partial t} \right)^+ \cdot n \quad \text{on} \quad \Gamma \times (0,T), \quad (2.4a)$$

$$\frac{\partial U^-}{\partial t} \cdot n = - \left( \frac{\partial V}{\partial n} + \frac{\partial V^{inc}}{\partial n} \right)^+ \quad \text{on} \quad \Gamma \times (0,T), \quad (2.4b)$$

$$\frac{\partial \Theta^-}{\partial n} = 0 \quad \text{on} \quad \Gamma \times (0,T), \quad (2.4c)$$

where $n$ is the exterior unit normal to $\Omega_-$, and $V^{inc}$ denotes the given incident field. Here and in the sequel, we adopt the notation that $q^\tau$ denotes the limit of the function $q$ on $\Gamma$ from $\Omega_-$ respectively. We assume the causal initial conditions

$$U(x,t) = \frac{\partial U(x,t)}{\partial t} = 0, \quad \Theta(x,t) = 0 \quad \text{for} \quad x \in \Omega_-, \ t \leq 0, \quad (2.5a)$$

$$V(x,t) = \frac{\partial V}{\partial t}(x,t) = 0, \quad \text{for} \quad x \in \Omega_+, \ t \leq 0. \quad (2.5b)$$

We term the time-dependent scattering problem as an initial-boundary transmission problem consisting of the partial differential equations (2.1)-(2.3) together with the transmission conditions (2.4a)-(2.4c) and the homogeneous initial conditions (2.5a)-(2.5b).

### 3 Reduction to a nonlocal problem

In order to apply Lubich’s approach as in the case of fluid-structure interaction [14] [13], we first need to transform the initial-boundary transmission problem (2.1)-(2.4c) in the Laplace domain. Then we reduce the corresponding problem to a nonlocal boundary problem. Throughout the paper let the complex plane be denoted by $\mathbb{C}$ and its positive half-plane denoted by

$$\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re} \ s > 0\}.$$  

We begin with the Laplace transform for an ordinary complex-valued function. Let $F: [0, \infty) \to \mathbb{C}$ be a complex-valued function with limited growth at infinity. We denote the Laplace transform of $F$ by

$$f(s) = \mathcal{L}\{F\}(s) := \int_0^\infty e^{-st}F(t)dt,$$
provided it exists. (We note that we break convention of using small letters in the time
domain and the corresponding capital in the Laplace domain, in order to avoid burdening
the sequel with too many capital letters.)

Let then

\[ u := u(x, s) = \mathcal{L}\{U(x, t)\}, \quad \theta := \theta(x, s) = \mathcal{L}\{\Theta(x, t)\}, \quad v := v(x, s) = \mathcal{L}\{V(x, t)\}. \]

Then the initial-boundary transmission problem consisting of (2.1) - (2.4c) in the Laplace
transformed domain becomes the following transmission boundary value problem (TBVP):

\[
\begin{align*}
-\Delta^* u + \rho_\Sigma s^2 u + \zeta \nabla \theta &= 0 & \text{in} & \Omega_-, & (3.1a) \\
-\Delta \theta + \frac{s}{\kappa} \theta + s \eta \nabla \cdot u &= 0 & \text{in} & \Omega_-, & (3.1b) \\
-\Delta v + \frac{s^2}{c^2} v &= 0 & \text{in} & \Omega_+, & (3.1c) \\
\sigma(u, \theta) \cdot n + \rho_f s v^+ n &= -\rho_f s v^{inc} n & \text{on} & \Gamma, & (3.1d) \\
sv^- \cdot n + \frac{\partial v^+}{\partial n} &= -\frac{\partial v^{inc}}{\partial n} & \text{on} & \Gamma, & (3.1e) \\
\frac{\partial \theta^-}{\partial n} &= 0 & \text{on} & \Gamma, & (3.1f)
\end{align*}
\]

for \( Re s > 0 \), while \( u \equiv 0 \) and \( v \equiv 0 \) for \( Re s \leq 0 \) (note that \( \theta \) is causal by definition). We
remark that TBVP is an exterior scattering problem, and normally a radiation condition
is needed in order to guarantee the uniqueness of the solution of the problem. However,
in the present case no additional radiation condition is required and global \( H^1 \) behavior
at infinity suffices. (This is a Laplace domain reflection of Huygens’ principle.)

To derive a proper nonlocal boundary problem, as usual, we begin via Green’s third
identity with the representation of the solutions of (3.1c) in the form:

\[ v = D(s)\phi - S(s)\lambda \quad \text{in} \quad \Omega_+, \]

where \( \phi := v^+(s) \) and \( \lambda := \partial v^+/\partial n \) are the Cauchy data for \( v \) in (3.1c) and \( S(s) \) and
\( D(s) \) are the simple-layer and double-layer potentials, respectively defined by

\[
\begin{align*}
S(s)\lambda(x) &:= \int_{\Gamma} E_{s/c}(x, y) \lambda(y) d\Gamma_y, \quad x \in \mathbb{R}^3 \setminus \Gamma, & (3.3) \\
D(s)\phi(x) &:= \int_{\Gamma} \frac{\partial}{\partial n_y} E_{s/c}(x, y) \phi(y) d\Gamma_y, \quad x \in \mathbb{R}^3 \setminus \Gamma. & (3.4)
\end{align*}
\]

Here

\[ E_{s/c}(x, y) = \frac{e^{-s/c |x-y|}}{4\pi|x-y|} \]

is the fundamental solution of the operator in (3.1c). By standard arguments in potential
theory, we have the relations for the the Cauchy data \( \lambda \) and \( \phi \):

\[
\begin{pmatrix} \phi \\ \lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I + K(s) & -V(s) \\ -W(s) & (\frac{1}{2} I - K(s))' \end{pmatrix} \begin{pmatrix} \phi \\ \lambda \end{pmatrix} \quad \text{on} \quad \Gamma. \quad (3.5)
\]
Here \( V, K, K' \) and \( W \) are the four basic boundary integral operators familiar from potential theory [15] such that

\[
V(s)\lambda(x) := \int_{\Gamma} E_{s/c}(x, y)\lambda(y)d\Gamma_y \quad x \in \Gamma,
\]

\[
K(s)\phi(x) := \int_{\Gamma} \frac{\partial}{\partial n_y} E_{s/c}(x, y)\phi(y)d\Gamma_y \quad x \in \Gamma,
\]

\[
K'(s)\lambda(x) := \int_{\Gamma} \frac{\partial}{\partial n_x} E_{s/c}(x, y)\lambda(y)d\Gamma_y \quad x \in \Gamma,
\]

\[
W(s)\phi(x) := -\frac{\partial}{\partial n_x} \int_{\Gamma} \frac{\partial}{\partial n_y} E_{s/c}(x, y)\phi(y)d\Gamma_y \quad x \in \Gamma.
\]

By using the transmission condition (3.1e), we obtain from the second boundary integral equation in (3.5),

\[
- su \cdot n + W(s)\phi - (\frac{1}{2}I - K(s))'\lambda = \partial v_{inc} \quad \frac{\partial}{\partial n} \text{ on } \Gamma \quad (3.6)
\]

while the first boundary integral equation in (3.5) is simply

\[
(\frac{1}{2}I - K(s))\phi + V(s)\lambda = 0 \quad \text{on } \Gamma. \quad (3.7)
\]

With the Cauchy data \( \phi \) and \( \lambda \) as new unknowns, the partial differential equation (3.1c) in \( \Omega_+ \) may be eliminated. This leads to a nonlocal boundary problem in \( \Omega_- \) for the unknowns \( (u, \theta, \phi, \lambda) \) satisfying the partial differential equations (3.1a), (3.1b), and the boundary integral equations (3.6), (3.7) together with natural boundary conditions (3.1d) and (3.1f) on \( \Gamma \).

To be more precise, let us first consider the unknowns \( (u, \theta) \in H^1(\Omega_-) \times H^1(\Omega_-) \). Then multiplying (3.1a) by the test function \( v \) and integrating by parts, we obtain the weak formulation of (3.1a):

\[
a(u, v; s) - (\sigma(u, \theta)n, \gamma^-v)_{\Gamma} - \zeta(\theta, \nabla \cdot v)_{\Omega_-} = 0, \quad (3.8)
\]

where \( a(\cdot, \cdot; s) \) is the bilinear form (we will keep all brackets linear and use conjugates whenever needed) defined by

\[
a(u, v; s) := (C\varepsilon(u), \varepsilon(v))_{\Omega_-} + s^2\rho\varepsilon(u, v)_{\Omega_-}. \quad (3.9)
\]

In terms of the natural boundary condition (3.1d), we obtain from (3.8)

\[
a(u, v; s) - \zeta(\theta, \nabla \cdot v)_{\Omega_-} + \rho f s(\phi n, \gamma^-v)_{\Gamma} = -s \rho f (v_{inc} n, \gamma^-v)_{\Gamma}. \quad (3.10)
\]

Here and in the sequel, \( \gamma^\mp \) and \( \partial_n^\mp \) denote trace operators of the functions and their normal derivatives from inside and outside \( \Gamma \), respectively. Similarly, multiplying (3.1b) by the test function \( \vartheta \), integrating by parts and making use of the natural boundary condition (3.11), we have

\[
b(\theta, \vartheta; s) + s \eta(\nabla \cdot u, \vartheta)_{\Omega_-} = 0 \quad (3.11)
\]
with
\[ b(\theta, \vartheta; s) := (\nabla \theta, \nabla \vartheta)_{\Omega_{-}} + \frac{s}{\kappa} \theta \vartheta_{\Omega_{-}}. \] (3.12)

Now let \( A_{s} : H^{1}(\Omega_{-}) \rightarrow (H^{1}(\Omega_{-}))' \) and \( B_{s} : H^{1}(\Omega_{-}) \rightarrow (H^{1}(\Omega_{-}))' \) be the operators associated to the sesquilinear forms (3.9) and (3.12), respectively. Then from (3.10), where the data \((X, \theta, \phi, \lambda) \in X\) such that
\[
\begin{pmatrix}
  u \\
  \theta \\
  \phi \\
  \lambda
\end{pmatrix}
= \begin{pmatrix}
  A_{s} & -\zeta (\text{div})' & s \rho_{f} \gamma' & 0 \\
  s \eta \text{div} & 0 & 0 & 0 \\
  -s \mathbf{n}^{T} \gamma & 0 & W(s) & -(\frac{1}{2} I - K(s))' \\
  0 & 0 & \frac{1}{2} I - K(s) & V(s)
\end{pmatrix}
\begin{pmatrix}
  d_{1} \\
  d_{2} \\
  d_{3} \\
  d_{4}
\end{pmatrix},
\] (3.13)

where the data \((d_{1}, d_{2}, d_{3}, d_{4})\) is given by
\[ d_{1} = -s \rho_{f} \gamma' (\gamma + v^{inc} \mathbf{n}), \quad d_{2} = 0, \quad d_{3} = \partial_{n}^{+} v^{inc}, \quad d_{4} = 0. \] (3.14)

We have made use of the product spaces:
\[ X := H^{1}(\Omega_{-}) \times H^{1}(\Omega_{-}) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma), \]
\[ X' := (H^{1}(\Omega_{-}))' \times (H^{1}(\Omega_{-}))' \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \]
(i. e., \(X'\) is the dual of \(X\)). Our aim is to show that Equation (3.13) has a unique solution in \(X\). We will do this in the next section.

4 Existence and uniqueness results

Before considering the existence and uniqueness results, we first discuss the invertibility of the operator \(A\) in (3.13). We begin with the definitions of the following energy norms:
\[
\|u\|_{s; \Omega_{-}} := (\sigma(u), \varepsilon(u))_{\Omega_{-}} + \rho_{s} s \|u\|_{\Omega_{-}}^{2}, \quad u \in H^{1}(\Omega_{-}),
\] (4.1)
\[
\|\theta\|_{s; \Omega_{-}} := \|\nabla \theta\|_{\Omega_{-}} + \kappa^{-1} \|s \theta\|_{\Omega_{-}}^{2}, \quad \theta \in H^{1}(\Omega_{-}),
\] (4.2)
\[
\|u\|_{s; \Omega_{-}}^{2} := \|\nabla u\|_{\Omega_{-}}^{2} + \kappa^{-2} \|s u\|_{\Omega_{-}}^{2}, \quad u \in H^{1}(\Omega_{-}).
\] (4.3)

We also need the following inequalities for equivalent norms from these norms
\[
\sigma \|u\|_{1, \Omega_{-}} \leq \|u\|_{s; \Omega_{-}} \leq \frac{|s|}{\sigma} \|u\|_{1, \Omega_{-}},
\] (4.4)
\[
\sqrt{\pi} \|\theta\|_{1, \Omega_{-}} \leq \|\theta\|_{s; \Omega_{-}} \leq \sqrt{\frac{|s|}{\pi}} \|\theta\|_{1, \Omega_{-}},
\] (4.5)
\[
\sigma \|u\|_{1, \Omega_{+}} \leq \|u\|_{s; \Omega_{+}} \leq \frac{|s|}{\sigma} \|u\|_{1, \Omega_{+}},
\] (4.6)

which can be obtained from the inequalities:
\[
\min \{1, \sigma\} \leq \min \{1, |s|\}, \quad \max \{1, |s|\} \min \{1, \sigma\} \leq |s|, \quad \forall s \in \mathbb{C}_{+},
\]
with $\sigma := \Re s$. We remark that the norms $\|\theta\|_{1,\Omega_-}$ and $\|u\|_{1,\Omega_+}$ are equivalent to $\|\theta\|_{H^1(\Omega_-)}$ and $\|u\|_{H^1(\Omega_+)}$, respectively, and so is the energy norm $\|\mathbf{u}\|_{1,\Omega_-}$ equivalent to the $H^1(\Omega_-)$ norm of $\mathbf{u}$ by the second Korn inequality [3]. In the following, the $c_j$'s are generic constants independent of $s$ which may or may not be the same at different places.

For the invertibility of $\mathbf{A}$, let us introduce the diagonal matrix $\mathbf{X} = \mathbf{X}(s)$:

$$
\mathbf{X} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \zeta(\eta|s|)^{-1} & 0 & 0 \\
0 & 0 & \rho_f & 0 \\
0 & 0 & 0 & \rho_f
\end{pmatrix}
$$

and consider the modified operator

$$
\mathbf{B} := \mathbf{X} \mathbf{A} = \begin{pmatrix}
\mathbf{A}_s & -\zeta (\text{div})' & s \rho_f \gamma' & 0 \\
s|s|^{-1} \zeta \text{div} & \zeta (\eta |s|)^{-1} B_s & 0 & 0 \\
-\rho_f \mathbf{n}^\top \gamma^- & 0 & \rho_f (W(s) + C_T(s)) & 0 \\
0 & 0 & \rho_f V(s) & \rho_f (\frac{1}{2} I - K(s))
\end{pmatrix}.
$$

It will be clear that the invertibility of $\mathbf{A}$, it suffices to discuss the invertibility of $\mathbf{B}$. By the Gaussian elimination procedure (as in [21]), a simple computation shows that $\mathbf{B}$ can be decomposed in the form:

$$
\mathbf{B} = \mathbf{P} \mathbf{C} \mathbf{P}^{-1},
$$

where

$$
\mathbf{C} = \begin{pmatrix}
\mathbf{A}_s & -\zeta (\text{div})' & s \rho_f \gamma' & 0 \\
\zeta (\eta |s|)^{-1} B_s & 0 & 0 & 0 \\
-\rho_f \mathbf{n}^\top \gamma^- & 0 & \rho_f (W(s) + C_T(s)) & 0 \\
0 & 0 & \rho_f V(s) & \rho_f (\frac{1}{2} I - K(s))
\end{pmatrix},
$$

$$
\mathbf{P}' = \begin{pmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & V^{-1}(s) (\frac{1}{2} I - K(s))
\end{pmatrix}
$$

and $\mathbf{P}^{-1}$ is invertible, if $\mathbf{C}$ does. As in time-dependent fluid-structure interaction [14], the operator matrix $\mathbf{C}$ is indeed invertible. In fact, it is not difficult to show that the operator matrix $\mathbf{C}$ is strongly elliptic (15, 29) in the sense that

$$
\text{Re}\left\{ Z(s) (\mathbf{C}(\mathbf{v}, \vartheta, \psi, \chi), (\mathbf{v}, \vartheta, \psi, \chi)) \right\} \geq C(\zeta, \eta, \rho_f) \frac{\sigma \sigma^3}{|s|^2} \|\mathbf{v}, \vartheta, \psi, \chi\|_{X}^2
$$

for all $(\mathbf{v}, \vartheta, \psi, \chi) \in X$, where $C(\zeta, \eta, \rho_f)$ is a constant depending only on the physical parameters and on the geometry of $\Omega_-$, and $Z(s)$ is the matrix defined by

$$
Z(s) := \begin{pmatrix}
\bar{s}/|s| & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \bar{s}/|s| & 0 \\
0 & 0 & 0 & s/|s|
\end{pmatrix}.
$$
As for the proof, we only want to point out that
\[
\text{Re}\left\{ \frac{s}{|s|}(-\zeta (\text{div}')\theta, \nabla)\Omega_+ + \frac{s}{|s|}(s\rho_f \gamma^{-}\psi, \mathbf{n}, \mathbf{v})\Omega_+ \right. \\
+ (s|s|^{-1}\zeta \text{div} \mathbf{v}, \partial)\Omega_+ + \frac{s}{|s|}(-s\rho_f \mathbf{n}^+ \gamma^{-}, \varphi, \gamma)\Omega_+ \right\} = 0. 
\] (4.11)

Details are omitted, since a similar proof will be repeated when we discuss the existence and uniqueness results for the solution of the nonlocal problem (3.13).

We now return to the solutions of the modified system of equations (4.8) from (3.8):
\[
\mathcal{B} \begin{pmatrix} \mathbf{u} \\ \theta \\ \phi \\ \lambda \end{pmatrix} := \mathcal{L} \mathcal{A} \begin{pmatrix} \mathbf{u} \\ \theta \\ \phi \\ \lambda \end{pmatrix} = \mathcal{L} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} \zeta/\eta |s|^{-1} d_2 \\ \rho_f d_3 \\ \rho_f d_4 \end{pmatrix}. 
\] (4.12)

Suppose that \((\mathbf{u}, \theta, \phi, \lambda) \in X\) is a solution of (4.12). Let
\[
u := \mathcal{D}(s)\phi - \mathcal{S}(s)\lambda \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma. 
\] (4.13)

Then \(u \in H^1(\mathbb{R}^3 \setminus \Gamma)\) is the solution of the transmission problem:
\[- \Delta u + (s/c)^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma 
\] (4.14)
satisfying the following jump relations across \(\Gamma\),
\[
[\gamma u] := \gamma^+ u - \gamma^- u = \phi \in H^{1/2}(\Gamma), \quad [\partial_n u] := \partial^+_n u - \partial^-_n u = \lambda \in H^{-1/2}(\Gamma). 
\]

First, from (4.14) we see that
\[
\mathbf{A}_s \mathbf{u} - \zeta (\text{div}')\theta + s \rho_f \gamma^{-} [\gamma u] \mathbf{n} = d_1 \quad \text{in} \quad \Omega_-, 
\] (4.15)
\[
+ s|s|^{-1}\zeta \text{div} \mathbf{u} + \zeta/\eta |s|^{-1} B_* \theta = \zeta/\eta |s|^{-1} d_2 \quad \text{in} \quad \Omega_-, 
\] (4.16)
\[-s \rho_f \mathbf{n} \cdot \gamma^- \mathbf{u} - \rho_f \partial^+_n u = \rho_f d_3 \quad \text{on} \quad \Gamma, 
\] (4.17)
\[-\rho_f \gamma^- u = \rho_f d_4 \quad \text{on} \quad \Gamma. 
\] (4.18)

Since \(d_4 = 0\), this means that \(u\) is a solution of the homogeneous Dirichlet problem for the partial differential equation (4.14) in \(\Omega_-\). Hence by the uniqueness of the solution, we obtain \(u \equiv 0\) in \(\Omega_-\). Consequently, we have
\[
[\gamma u] = \gamma^+ u = \phi \quad \text{and} \quad [\partial_n u] = \partial^+_n u = \lambda. 
\] (4.19)

Next, we consider the variational formulation of the problem for equations (4.14), (4.15), and (4.16) together with the boundary conditions (4.17) and (4.18). We seek a solution
\[
(\mathbf{u}, \theta, u) \in \mathcal{H} = H^1(\Omega_-) \times H^1(\Omega_-) \times H^1(\Omega_+) 
\]
with the corresponding test functions \((\mathbf{v}, \varphi, \vartheta)\) in the same function space. To derive the variational equations, we should keep in mind that the variational formulation should be
formulated not in terms of the Cauchy data $\phi$ and $\lambda$ directly, but only through the jumps $[\gamma u]$ and $[\partial_n u]$ as indicated.

We begin with the first Green formula for the equation (4.14). Let $(u, v) \in H^1(\Omega_+) \times H^1(\Omega_+)$.

Then

$$-\langle \partial_n^+ u, \gamma^+ v \rangle_\Gamma = \int_{\Omega_+} (\nabla u \cdot \nabla v + (s/c)^2 u v) \, dx$$

$$= c_{\Omega_+}(u, v; s) = (C_{\Omega_+} u, v)_{\Omega_+}.$$

Note that in the definitions of the bilinear form $c_\Omega(u, v; s)$ and the associated operator $C_{\Omega}$ the domain of integration is indicated explicitly. From condition (4.17), we obtain

$$\rho_f(C_{\Omega_+} u, v)_{\Omega_+} = -\rho_f(\partial_n^+ u, \gamma^+ v)$$

$$= \rho_f(d_3, \gamma^+ v)_{\Gamma} + \rho_f(s \gamma u \cdot n, \gamma^+ v)_{\Gamma}. \quad (4.20)$$

Together with the weak formulations of (4.15) and (4.16), we arrive at the following variational formulation: Find $(u, \theta, u) \in \mathbf{H}$ satisfying

$$(A_s u, v)_{\Omega_-} - \zeta(\theta, \div v)_{\Omega_+} + s \rho_f(\gamma^+ u n, \gamma^- v)_{\Gamma} = (d_1, v)_{\Omega_-} \quad \forall v \in H^1(\Omega_-),$$

$$\frac{\zeta}{|s|\eta}(B_s \theta, \partial)_{\Omega_+} + \frac{s}{|s|} \zeta(\div v, \partial)_{\Omega_+} = \frac{\zeta}{|s|\eta}(d_2, \partial)_{\Omega_+} \quad \forall \partial \in H^1(\Omega_-),$$

$$\rho_f(C_{\Omega_+} u, v)_{\Omega_+} - s \rho_f(\gamma^- u n, \gamma^+ v)_{\Gamma} = \rho_f(d_3, \gamma^+ v)_{\Gamma} \quad \forall v \in H^1(\Omega_+). \quad (4.21)$$

We remark that by the construction, it can be shown that as in [31] this variational problem is equivalent to the transmission problem defined by (4.14), (4.15), (4.16), and (4.17). The latter is equivalent to the nonlocal problem defined by (4.12), which is equivalent to (4.13). Consequently, the variational problem (4.21) is equivalent to the nonlocal problem (4.13). Hence for the existence of the solution of (4.13), it is sufficient to show the existence of the solution of (4.21).

We have the following basic results.

**Theorem 4.1.** The variational problem (4.21) has a unique solution $(u, \theta, u) \in \mathbf{H}$. Moreover, the following estimate holds:

$$\left( \|u\|^2_{s, \Omega_-} + \|\theta\|^2_{s, \Omega_-} + \|u\|^2_{s, \Omega_+} \right)^{1/2} \leq c_0 \frac{|s|^{3/2}}{|\sigma|^{3/2}} \|0, d_3, 0\|_{X'}, \quad (4.22)$$

where $c_0$ is a constant depending only on the physical parameters $\rho_f, \zeta, \eta$.

**Proof.** Starting with the system (4.21), a simple computation shows that

$$\text{Re} \left\{ \frac{s}{|s|} (A_s u, \overline{u})_{\Omega_+} + \frac{\zeta}{|s|\eta}(B_s \theta, \overline{\partial})_{\Omega_-} + \frac{s}{|s|} \rho_f(C_{\Omega_+} u, \overline{u})_{\Omega_+} \right\}$$

$$= \text{Re} \left\{ \frac{s}{|s|} (d_1, \overline{u})_{\Omega_-} + \frac{\zeta}{|s|\eta} (d_2, \overline{\partial})_{\Omega_-} + \frac{s}{|s|} \rho_f(d_3, \overline{\gamma^+ u})_{\Gamma} \right\}. \quad (4.23)$$
On the other hand, it is not hard to verify that
\[
\text{Re}\left\{ \frac{s}{|s|} (\mathcal{A}_s u, \overline{u})_{\Omega_-} \right\} = \frac{\sigma}{|s|} \|u\|^2_{s|s|, \Omega_-},
\]
(4.24)
\[
\text{Re}\left\{ \frac{\zeta/\eta}{|s|} (B_s \theta, \overline{\theta})_{\Omega_-} \right\} \geq \zeta/\eta \frac{\sigma}{|s|^2} \|\theta\|^2_{s|s|, \Omega_-},
\]
(4.25)
\[
\text{Re}\left\{ \frac{s}{|s|} \rho_f (C_s \Omega_+, u, \overline{u})_{\Omega_+} \right\} = \rho_f \frac{\sigma}{|s|} \|u\|^2_{s|s|, \Omega_+}.
\]
(4.26)
Therefore, combining (4.21) - (4.26), substituting into (4.23) and recalling that \(d_2 = 0\), it follows that
\[
\frac{\sigma}{|s|} \left( \|u\|^2_{s|s|, \Omega_-} + \rho_f \|u\|^2_{s|s|, \Omega_+} + \zeta/\eta \frac{1}{|s|} \|\theta\|^2_{s|s|, \Omega_-} \right) \leq \|d_1, \overline{u}\rangle_{\Omega_-} \leq \|d_1, \overline{u}\rangle_{\Omega_-} + \langle d_1, \overline{\gamma} + u \rangle_{\Gamma}. \]
However, from the definition \(\sigma := \min\{1, \sigma\}\) we see that the left hand side of this expression satisfies
\[
\frac{\sigma}{|s|} \left( \|u\|^2_{s|s|, \Omega_-} + \zeta/\eta \|\theta\|^2_{s|s|, \Omega_-} + \rho_f \|u\|^2_{s|s|, \Omega_+} \right) \leq \text{LHS}.
\]
Consequently, we have the estimate
\[
\left( \|u\|^2_{s|s|, \Omega_-} + \|\theta\|^2_{s|s|, \Omega_-} + \|u\|^2_{s|s|, \Omega_+} \right)^{1/2} \leq c_0 \frac{|s|^{3/2}}{\sigma^{3/2}} \|d_1, 0, d_3, 0\|_{X'},
\]
(4.27)
where \(c_0\) is a constant depending only on the physical parameters \(\rho_f, \zeta, \eta\). In deriving the estimate (4.27), we have tacitly applied the relations (4.4), (4.5) and (4.6).

As we will see the estimate (4.27) will lead us to show the invertibility of the operator \(\mathcal{A}\) in (3.13) (or (4.12) rather). In fact, the following result holds for the operator \(\mathcal{A}\) of (4.12).

**Theorem 4.2.** Let
\[
X := H^1(\Omega_-) \times H^{1/2}(\Omega_-) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma),
\]
\[
X' := (H^1(\Omega_-))' \times (H^{1/2}(\Omega_-))' \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma),
\]
\[
X_0 := \{(d_1, d_2, d_3, d_4) \in X' \mid d_2 = 0 \text{ and } d_4 = 0\}.
\]
Then \(\mathcal{A} : X \to X_0\) is invertible. Moreover, we have the estimate:
\[
\|\mathcal{A}^{-1}|_{X_0'}\|_{X', X} \leq c_0 \frac{|s|^2}{\sigma^3},
\]
(4.28)
where \(c_0\) is a constant independent of \(s\) and \(\sigma\).

**Proof.** It was pointed out in (4.19) that
\[
\gamma^+ u = [\gamma u] = \phi \in H^{1/2}(\Gamma), \quad \partial_n^+ u = [\partial_n u] = \lambda \in H^{-1/2}(\Gamma).
\]
Then we have the estimates (see, e.g. [14]):

$$
\|\phi\|_{H^{1/2}(\Gamma)}^2 = \|\gamma^+ u\|_{H^{1/2}(\Gamma)}^2 \leq c_1 \|u\|_{I,\Omega^+}^2 \leq c_1 \frac{1}{\sigma^2} \|u\|_{s,\Omega^+}^2
$$

(4.29)

Similarly, we have

$$
\|\lambda\|_{H^{-1/2}(\Gamma)} = \|\partial_n^+ u\|_{H^{-1/2}(\Gamma)} = \sup \frac{\langle \partial_n^+ u, v \rangle}{\|v\|_{H^{1/2}(\Gamma)}} \leq \frac{\|u\|_{s,\Omega^+} \|v\|_{s,\Omega^+}}{\|v\|_{H^{1/2}(\Gamma)}} \leq c_2 \|u\|_{s,\Omega^+}^2
$$

In the last line we have used Bamberger and Ha-Duong’s optimal lifting [1, 2] to estimate the norms of \(v\). Thus, we can conclude that

$$
\|\lambda\|_{H^{-1/2}(\Gamma)}^2 \leq c_2 \frac{s^2}{\sigma^2} \|u\|_{s,\Omega^+}^2
$$

(4.30)

From (4.29) and (4.30), we obtain the estimates

$$
\frac{1}{2} \left( \frac{1}{c_1} \sigma^2 \|\phi\|_{H^{1/2}(\Gamma)}^2 + \frac{\sigma}{c_2} \|\lambda\|_{H^{-1/2}(\Gamma)}^2 \right) \leq \|u\|_{s,\Omega^+}^2
$$

(4.31)

As a consequence of (4.22), it follows that

$$
\sigma^2 \|u\|_{I,\Omega^-}^2 + \sigma \|\theta\|_{I,\Omega^-}^2 + \sigma \left( \frac{\sigma^2}{|s|} \|\phi\|_{H^{1/2}(\Gamma)}^2 + \frac{|s|}{\sigma^2} \|\lambda\|_{H^{-1/2}(\Gamma)}^2 \right) \leq \left( c_0 \frac{|s|^{3/2}}{\sigma^{3/2}} \|d_1, 0, d_3, 0\|_{\mathcal{X}'} \right)^2.
$$

A simple manipulation shows that the left hand side of the previous expression is such that

$$
\frac{3}{|s|} \left( \|u\|_{I,\Omega^-}^2 + \|\theta\|_{I,\Omega^-}^2 + \|\phi\|_{H^{1/2}(\Gamma)}^2 + \|\lambda\|_{H^{-1/2}(\Gamma)}^2 \right) \leq LHS,
$$

which implies that

$$
\left( \|u\|_{I,\Omega^-}^2 + \|\theta\|_{I,\Omega^-}^2 + \|\phi\|_{H^{1/2}(\Gamma)}^2 + \|\lambda\|_{H^{-1/2}(\Gamma)}^2 \right)^{1/2} \leq c_0 \frac{|s|^2}{\sigma^3} \|d_1, 0, d_3, 0\|_{\mathcal{X}'}
$$

with constant \(c_0\) independent of \(s\) and \(\sigma\).

We remark that in view of (3.2), we see that \(u, \theta\) and \(v\) are solutions of the system

$$
\begin{pmatrix}
\mathcal{A}^{-1} \\
\mathcal{B} \\
\mathcal{C}
\end{pmatrix}
\begin{pmatrix}
u \\
\theta \\
\omega
\end{pmatrix} = 
= 
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \mathcal{D}(s) & -\mathcal{S}(s)
\end{pmatrix}
\begin{pmatrix}
d_1 \\
0 \\
d_3 \\
0
\end{pmatrix}
$$

(4.32)
With the properties of solutions available in the transformed domains, we are now in a position to estimate the corresponding properties of solutions in the time domain based on Lubich’s Convolution Quadrature \cite{26} introduced in the early 90’s for treating time-dependent boundary integral equations of convolution type. An essential feature of this approach is that estimates of properties of solutions in the time domain are obtained without applying directly the Laplace inverse transformation to solutions in the transformed domain. Instead, the crucial result on Proposition 4.3 below is employed for this purpose.

In order to make the notation for the time-domain estimates more compact, we introduce the regularity spaces

\[ W^k_+(\mathcal{H}) := \left\{ w \in C^{k-1}(-\infty, 0), w^{(k)} \in L^1(\mathbb{R}; \mathcal{H}) \mid w \equiv 0 \text{ in } (-\infty, 0) \right\} \tag{4.33} \]

where \( \mathcal{H} \) denotes a Banach space.

**Proposition 4.3** (see, e.g. \cite{31, 21}). Let \( A = \mathcal{L}\{a\} \in A(\mu, \mathcal{B}(X, Y)) \) with \( \mu \geq 0 \) and let

\[ k := \lfloor \mu + 2 \rfloor \quad \varepsilon := k - (\mu + 1) \in (0, 1]. \]

If \( g \in W^k_+(\mathbb{R}, X) \), then \( a * g \in \mathcal{C}(\mathbb{R}, Y) \) is causal and

\[ \| (a * g)(t) \|_Y \leq 2^{\mu+1}C_\varepsilon(t)C_A(t^{-1}) \int_0^t \| (P_k g)(\tau) \|_X \, d\tau, \]

where

\[ C_\varepsilon(t) := \frac{t^\varepsilon}{\pi \varepsilon}, \quad \text{and} \quad (P_k g)(t) = \sum_{\ell=0}^k \left( \begin{array}{c} k \\ \ell \end{array} \right) g^{(\ell)}(t). \]

Some explanations for the notation adopted in Proposition 4.3 are warranted. For Banach spaces \( X \) and \( Y \), let \( \mathcal{B}(X, Y) \) denote the set of bounded linear operators from \( X \) to \( Y \). Then the elements of the class \( A \in A(\mu, \mathcal{B}(X, Y)) \) are analytic functions \( A : \mathbb{C}_+ \to \mathcal{B}(X, Y) \) for which there exists a \( \mu \in \mathbb{R} \) such that

\[ \| A(s) \|_{X, Y} \leq C_A(\text{Res}) |s|^{\mu} \quad \text{for} \quad s \in \mathbb{C}_+, \]

where \( C_A : (0, \infty) \to (0, \infty) \) is a non-increasing function such that

\[ C_A(\sigma) \leq \frac{c}{\sigma^m}, \quad \forall \quad \sigma \in (0, 1] \]

for some \( m \geq 0 \).

As an immediate consequence of this result, we see from Theorem 4.2 that \( \mathcal{A}^{-1}|_{X'_0} \in \mathcal{A}(2, \mathcal{B}(X', X)) \). Moreover, \( k = 4 \) and \( \varepsilon = 1 \) and we thus have the estimate:

**Theorem 4.4.** Let \( \mathbf{D}(t) := \mathcal{L}^{-1}\{(d_1, 0, d_3, 0)^\top\} \) belong to \( W^4_+(\mathbb{R}, X') \). Then

\[ (\mathbf{U}, \Theta, \mathcal{L}^{-1}\{\phi\}, \mathcal{L}^{-1}\{\lambda\})^\top \in \mathcal{C}([0, T], X) \]

and there exists \( c > 0 \) depending only on the geometry such that

\[ \| (\mathbf{U}, \Theta, \mathcal{L}^{-1}\{\phi\}, \mathcal{L}^{-1}\{\lambda\})^\top(t) \|_X \leq ct^2 \max\{1, t^3\} \int_0^t \| (P_4 \mathbf{D})(\tau) \|_{X'} \, d\tau. \tag{4.34} \]
Similarly, in view of Theorem 4.4 applying Proposition 4.3 with \( \mu = 3/2, k = 3, \varepsilon = 1/2 \) to the elastic, thermal, and potential fields leads to:

**Theorem 4.5.** Let \( \mathbb{H} := H^1(\Omega_-) \times H^1(\Omega_-) \times H^1(\Omega_+) \) and \( \mathbb{D}(t) := \mathcal{L}^{-1}\{(d_1, 0, d_3, 0)^\top\}(t) \in W^2_0(\mathbb{R}, X') \).

Then \( \mathbf{U} = (U, \Theta, V) \) belongs to \( C([0, T], \mathbb{H}) \) and there holds the estimate

\[
\|(U, \Theta, V)(t)\|_{\mathbb{H}} \leq C_0 t^{3/2} \max\{1, t^{5/2}\} \int_0^t \|(\mathcal{P}_3 \mathbb{D})(\tau)\|_{X'} \, d\tau
\]

for some constant \( C_0 > 0 \).

5 Computational aspects

In this section, we give a brief description concerning the discretization of (3.13). We begin with the Galerkin semidiscretization in space of the system of equations. Let

\[
V_h \subset H^1(\Omega_-), \quad W_h \subset H^1(\Omega_-), \quad X_h \subset H^{-1/2}(\Gamma), \quad Y_h \subset H^{1/2}(\Gamma)
\]

be families of finite dimensional subspaces. We say \((u^h, \theta^h, \phi^h, \lambda^h) \in V_h \times W_h \times X_h \times Y_h\) is a Galerkin solution of (3.13) if it satisfies the Galerkin equations:

\[
\left(\mathcal{A}(u^h, \theta^h, \phi^h, \lambda^h)^\top, (v, \vartheta, \psi, \xi)^\top\right) = \left((d_1, d_2, d_3, d_4)^\top, (v, \vartheta, \psi, \xi)^\top\right) \quad (5.1)
\]

for all \((v, \vartheta, \psi, \xi) \in V_h \times W_h \times X_h \times Y_h\). Again, we multiply (5.1) by the diagonal matrix \( \mathbb{D}(s) \) in (4.7) and consider the Galerkin equations for the modified equation (4.12):

\[
\left(\mathcal{B}(u^h, \theta^h, \phi^h, \lambda^h)^\top, (v, \vartheta, \psi, \xi)^\top\right) = \left((d_1, \zeta/\eta \, |s|^{-1} d_2, \rho f d_3, \rho f d_4)^\top, (v, \vartheta, \psi, \xi)^\top\right) \quad (5.2)
\]

for all \((v, \vartheta, \psi, \xi) \in V_h \times W_h \times Y_h \times X_h\). Solutions of Galerkin equations of (5.2) can be established in the same manner as the exact solutions of (4.7). We will not repeat the process and consider only the error estimates here.

We note if \((u^h, \theta^h, \phi^h, \lambda^h) \in V_h \times W_h \times X_h \times Y_h\) is a Galerkin solution of (5.2), then

\[
u^h := \mathcal{D}(s)\phi^h - \mathcal{S}(s)\lambda^h \in H^1(\mathbb{R}^3 \setminus \Gamma) \quad (5.3)
\]

satisfies

\[
-\Delta u^h + (s/c)^2 u^h = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \quad (5.4)
\]

\[
[\gamma u^h] = \phi^h \quad \phi^h \in Y_h \subset H^{1/2}(\Gamma), \quad [\partial u^h] = \lambda^h \quad \lambda^h \in X_h \subset H^{-1/2}(\Gamma).
\]

Now set

\[
V_h := \{v \in H^1(\mathbb{R}^3 \setminus \Gamma) : [\gamma v] \in Y_h, \gamma^- v \in X_h^0\},
\]
where $X_h^o$ denotes the annihilator of $X_h$, i.e.,

$$X_h^o := \{ v \in X_h^\prime : \langle w, v \rangle = 0 \quad \forall w \in X_h \}. $$

Applying Green’s formula to (5.4), we obtain for $v \in V_h$,

$$\int_\Gamma \partial^+_n u^h [\gamma v] \, d\Gamma = -c_{\mathbb{R}^3} \langle u^h, v \rangle - \int_\Gamma [\partial_n u^h] \gamma v \, d\Gamma. \quad (5.5)$$

As a consequence, we see that $(u^h, \theta^h, u^h) \in V_h \times W_h \times V_h$ satisfies the variational equations

$$a(u^h, v; s) - \zeta(\theta^h, div v)_{\Omega_-} + s \rho_f([\gamma u^h]n, \gamma v)_{\Gamma} = (d_1, v)_{\Omega_-} \quad \forall \ v \in V_h,$n

$$\frac{s}{|s|} \zeta(div u^h, \partial)_\Omega + \frac{1}{|s|} b(\theta^h, \partial; s) = \zeta(1, \partial)_{\Omega_-} \quad \forall \ \partial \in W_h, \quad (5.6)$$

$$-s \rho_f(\gamma u^h, [[\gamma v]])_{\Gamma} + \rho f_{\mathbb{R}^3} (u^h - v, s) = \rho f(3, [\gamma v])_{\Gamma} \quad \forall v \in V_h.$$n

For the error estimate, we need to compare $(u^h, \theta^h, u^h)$ with the exact solution $(u, \theta, u)$ of the transmission problem. The exact solution $(u, \theta, u) \in H^1(\Omega_-) \times H^1(\Omega_-) \times H^1(\mathbb{R}^3 \setminus \Gamma)$ satisfies the variational equation (5.21) which can be put in the same form as (5.6), replacing test functions by the ones in the subspaces. This implies that

$$a(u^h - u, v; s) - \zeta(\theta^h - \theta, div v)_{\Omega_-} + s \rho_f([\gamma (u^h - u)]n, \gamma v)_{\Gamma} = 0 \quad \forall v \in V_h,$n

$$\frac{s}{|s|} \zeta(div(u^h - u), \partial)_{\Omega_-} + \frac{1}{|s|} b(\theta^h - \theta, \partial; s) = 0 \quad \forall \partial \in W_h, \quad (5.7)$$

$$-s \rho_f(\gamma (u^h - u), [[\gamma v]])_{\Gamma} + \rho f_{\mathbb{R}^3} (u^h - u, s) + \rho f(\partial_n (u^h - u), \gamma v)_{\Gamma} = 0 \quad \forall v \in V_h.$$n

The significance of (5.6) is that it indicates that the Galerkin solutions are the best possible approximations of the exact solution in the finite dimensional subspaces with respect to the inner products defined by the underlying bilinear forms. However, it is worth emphasizing that the errors $(u^h - u)$, $(\theta^h - \theta)$ are not in the approximate function spaces. In order to justify (5.6) as a proper variational formulation, we will make use of the rigid motion spaces

$$\mathcal{R}_u := \{ m \in H^1(\Omega_-) : (\sigma(m), \varepsilon(m))_{\Omega_-} = 0 \}, \quad (5.8)$$

$$\mathcal{R}_\theta := \{ m \in H^1(\Omega_-) : (\nabla m, \nabla m)_{\Omega_-} = 0 \}, \quad (5.9)$$

which in what follows will always be assumed to be contained on the discrete subspaces $V_h$ and $W_h$ respectively. We now define the elliptic projections:

$$P_h : H^1(\Omega_-) \rightarrow V_h \subset H^1(\Omega_-) \quad (5.10)$$

$$\sigma(P_h u, \varepsilon(\varepsilon^h))_{\Omega_-} = (\sigma(u), \varepsilon(\varepsilon^h))_{\Omega_-} \quad \forall \ v \in V_h,$n

$$(P_h u, m)_{\Omega_-} = (u, m)_{\Omega_-} \quad \forall m \in \mathcal{R}_u,$n

$$Q_h : H^1(\Omega_-) \rightarrow W_h \subset H^1(\Omega_-) \quad (5.11)$$

$$\nabla(Q_h \theta), \nabla \varepsilon^h)_{\Omega_-} = (\nabla \theta, \nabla \varepsilon^h)_{\Omega_-} \quad \forall \theta \in W_h,$n

$$(Q_h \theta, m)_{\Omega_-} = (\theta, m)_{\Omega_-} \quad \forall m \in \mathcal{R}_\theta.$$
In terms of the elliptic projection $P_h$, we can define
\[ e^h_u := u^h - P_h u, \quad r^h_u := P_h u - u, \]
so that the error function $u^h - u$ can be decomposed as
\[ u^h - u = (u^h - P_h u) + (P_h u - u) = e^h_u + r^h_u. \]
As a consequence
\[ a(u^h - u, v^h; s) = a(e^h_u, v^h; s) + s^2 \rho_\Sigma(r^h_u, v^h)_{\Omega_\ast}. \]
We may decompose the error $\theta^h - \theta$ in a similar manner by letting
\[ e^h_\theta := \theta^h - Q_h \theta, \quad r^h_\theta := Q_h \theta - \theta, \]
so that
\[ \theta^h - \theta = (\theta^h - Q_h \theta) + (Q_h \theta - \theta) = e^h_\theta + r^h_\theta. \]
\[ b(\theta^h - \theta, \vartheta^h; s) = b(e^h_\theta, \vartheta^h; s) + (s/\kappa)(r^h_\theta, \vartheta^h)_{\Omega_\ast}. \]
Finally, we define
\[ e^h_u := D(s)(\varphi - \phi) - S(s)(\lambda^h - \lambda) \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma. \]
This leads to the variational formulation for the error functions $(e^h_u, e^h_\theta, e^h_u) \in V_h \times W_h \times H^1(\mathbb{R}^3 \setminus \Gamma)$.

**Theorem 5.1.** The error functions $(e^h_u, e^h_\theta, e^h_u) \in V_h \times W_h \times H^1(\mathbb{R}^3 \setminus \Gamma)$ satisfy the variational formulation
\[ (\gamma e^h_u, [\gamma e^h_u] + \phi, [\partial_n e^h_u] + \lambda) \in X^\circ_h \times Y_h \times X_h, \]
\[ \mathcal{A}((e^h_u, e^h_\theta, e^h_u), (v, \vartheta, v); s) = \ell((v, \vartheta, v); s), \quad \forall (v, \theta, v) \in V_h \times W_h \times V_h, \]
where the bilinear form $\mathcal{A}$ is defined by
\[ \mathcal{A}((e^h_u, e^h_\theta, e^h_u), (v, \vartheta, v); s) := a(e^h_u, v; s) + \frac{s}{|s|}\zeta(div e^h_u, \vartheta)_{\Omega_\ast} - s\rho_f(\gamma^{-} e^h_u, [\gamma v]n)_{\Gamma} \]
\[ + \frac{\zeta}{\eta} \frac{1}{|s|} b(e^h_\theta, \vartheta; s) - \zeta(e^h_\theta, div v)_{\Omega_\ast} \]
\[ + \rho_f c_{\mathbb{R}^3 \setminus \Gamma}(e^h_u, v; s) + s\rho_f([\gamma^+ e^h_u] n, \gamma^{-} v)_{\Gamma}, \quad (5.12) \]
and the functional $\ell$ is given by
\[ \ell((v, \vartheta, v); s) := -s^2 \rho_\Sigma(r^h_u, v)_{\Omega_\ast} + \zeta(r^h_\theta, div v)_{\Omega_\ast} - \frac{s}{|s|}\zeta(div r^h_u, \vartheta)_{\Omega_\ast} \]
\[ + \frac{\zeta}{\eta} \frac{1}{|s|} (r^h_\theta, \vartheta)_{\Omega_\ast} + s\rho_f(\gamma^{-} r^h_u, [\gamma^+ v]n)_{\Gamma} + \rho_f(\lambda, \gamma^{-} v)_{\Gamma}. \quad (5.13) \]
Proof. The sesquilinear form $A$ follows easily from the left hand side of (5.6) replacing $(u^h - u, \theta - \theta, u^h - u)$ by $(e^h_u + r^h_u, e^h_\theta + r^h_\theta, e^h_u)$ and taking special care of the term $e^h_\theta$.

From Green’s formula (5.5), we have

$$\langle \partial^+ e^h_u, [\gamma v] \rangle = -c_{R^3 \Gamma} (e^h_u, v; s) - \langle [\partial e^h_u], \gamma^{-1} v \rangle.$$ 

But equations (4.17) and (4.18) imply

$$-s n \cdot (e^h_u + r^h_u) - \partial e^h_u \in Y^\circ_h, \quad \text{and} \quad \gamma^{-1} e^h \in X^\circ_h.$$ 

Hence,

$$-s P \langle \gamma^{-1} e^h_u, [\gamma v] n \rangle + \rho \gamma c_{R^3 \Gamma} (e^h_u, v; s) = s P \langle \gamma^{-1} r^h_u, [\gamma v] n \rangle - \rho \langle \partial e^h_u, \gamma^{-1} v \rangle.$$ 

We can rewrite the last term on the right hand side as

$$-\rho \langle \partial e^h_u, \gamma^{-1} v \rangle = -\rho \langle \partial e^h_u + \lambda - \lambda, \gamma^{-1} v \rangle = \rho \langle \lambda, \gamma^{-1} v \rangle.$$ 

Where we have used that $\lambda^h = [\partial e^h_u] + \lambda \in X_h$, and $\gamma^{-1} v \in X^\circ_h$ but $\lambda \notin X_h$. This completes the proof. \( \square \)

Following arguments similar as those employed in the proof of Theorem 4.1, we can obtain the error estimate. In the following, for simplicity, let

$$\| (e^h_u, e^h_\theta, e^h_u) \|_s := \| e^h_u \|^2_{s, \Omega_+} + \| e^h_\theta \|^2_{s, \Omega_-} + \| e^h_u \|^2_{s, R^3 \setminus \Gamma}.$$ 

**Theorem 5.2.** For $(u, \theta, \phi, \lambda) \in H^1(\Omega_+) \times H^1(\Omega_-) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, there holds the error estimate:

$$\| (e^h_u, e^h_\theta, e^h_u) \|_s \leq C \left( \| \lambda \|^2_{-1/2, \Gamma} + \| s r^h_u \|^2_{1, \Omega_+} + \| s r^h_\theta \|^2_{1, \Omega_-} + \| r^h_u \|^2_{1, \Omega_+} + \| r^h_\theta \|^2_{1, \Omega_-} \right)$$

where the constant $C$ depends only on the geometry, and physical parameters.

**Proof.** It is easy to see from the definition of the bilinear form $A$ in (5.12) that there is a constant $C$ depending only on the geometry and physical parameters such that

$$|A(e^h_u, e^h_\theta, e^h_u), (v, \theta, v); s) \leq C \frac{|s|}{\sigma} \| (e^h_u, e^h_\theta, e^h_u) \|_1 \| (v, v, v) \|_s.$$  \( (5.13) \)

We also need the estimate for the functional

$$|f((v, v, v); s) \leq C \left( \| \lambda \|^2_{-1/2, \Gamma} + \| s r^h_u \|^2_{1, \Omega_+} + \| s r^h_\theta \|^2_{1, \Omega_-} + \| r^h_u \|^2_{1, \Omega_+} + \| r^h_\theta \|^2_{1, \Omega_-} \right) \| (v, v, v) \|_s.$$ \( (5.15) \)

For $\phi \in H^{1/2}(\Gamma)$, we pick a lifting $u_\phi \in H^1(\mathbb{R}^3 \setminus \Gamma)$ such that $\gamma^+ u_\phi = \phi, \gamma^- u_\phi = 0$. Thus,

$$\| u_\phi \|_{1, \mathbb{R}^3 \setminus \Gamma} \leq C \| \phi \|_{1/2, \Gamma}.$$
Corollary 5.3. Let $(e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi) \in V_h \times W_h \times V_h$, it follows from equations (4.24)-(4.26) that

\[
\left\| \frac{\sigma}{2s} \| (e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi) \|_s \right\|^2 \leq \mathcal{A}((e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi), (e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi); s) \]
\[
= \| \ell((e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi); s) + \mathcal{A}((0, 0, u_\phi), (e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi); s) \| \leq \frac{C}{s} \| (e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi) \|_s (\| s^2 r^h_u \|_{1, \Omega} + \| s r^h_{\theta} \|_{1, \Omega} + \| r^h_u \|_{1, \Omega} + \| r^h_{\theta} \|_{1, \Omega})
\]
\[
\leq \frac{C}{s^2} \| (e^h_{\theta}, e^h_{\phi}, e^h_u + u_\phi) \|_s (\| s^2 r^h_u \|_{1, \Omega} + \| s r^h_{\theta} \|_{1, \Omega} + \| r^h_u \|_{1, \Omega} + \| r^h_{\theta} \|_{1, \Omega}).
\]

And the result follows from this relation and the observation that

\[
\| (0, 0, u_\phi) \| \leq \frac{C}{s} \| s \phi \|_{1/2, \Gamma}.
\]

As a consequence of Theorem 5.2 and estimates (4.31), we have the results.

**Corollary 5.3.** Let $(u, \theta, \phi, \lambda) \in H^1(\Omega_-) \times H^1(\Omega_-) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ be the unique solution of the problem (4.12) and let $(u^h, \theta^h, \phi^h, \lambda^h)$ be their corresponding Galerkin solutions of (5.6), then we have the estimates

\[
\| (e^h_u, e^h_{\theta}, e^h_{\phi}) \|_1 + \| \phi^h - \phi \|_{1/2, \Gamma} \leq C \frac{|s|^2}{\sigma^2} \left( \| s \phi \|_{1/2, \Gamma} + \| \lambda \|_{-1/2, \Gamma} \right)
\]
\[
+ \| s^2 r^h_u \|_{1, \Omega} + \| s r^h_{\theta} \|_{1, \Omega} + \| r^h_u \|_{1, \Omega} + \| r^h_{\theta} \|_{1, \Omega}.
\]

\[
\| \lambda^h - \lambda \|_{-1/2, \Gamma} \leq C \frac{|s|^{5/2}}{\sigma^{7/2}} \left( \| s \phi \|_{1/2, \Gamma} + \| \lambda \|_{-1/2, \Gamma} \right)
\]
\[
+ \| s^2 r^h_u \|_{1, \Omega} + \| s r^h_{\theta} \|_{1, \Omega} + \| r^h_u \|_{1, \Omega} + \| r^h_{\theta} \|_{1, \Omega}.
\]

By applying Proposition 4.3 to Corollary 5.3, we may obtain estimates in the time domain.

**Corollary 5.4.** If the exact solution quadruple satisfies

\[
(\mathbf{U}, \Theta, \mathcal{L}^{-1}\{\phi\}, \mathcal{L}^{-1}\{\lambda\}) \in W^4_+(H^1(\Omega_-)) \times W^4_+(H^1(\Omega_-)) \times W^5_+(H^{1/2}(\Gamma)) \times W^4_+(H^{-1/2}(\Gamma))
\]

then

\[
\mathcal{L}^{-1}\{(e^h_u, e^h_{\theta}, e^h_{\phi})\} \in \mathcal{C}(\mathbb{R}; H^1(\Omega_-) \times H^1(\Omega_-) \times H^1(\mathbb{R}^3 \setminus \Gamma))
\]
is causal and for $t \geq 0$

\[
\| \mathcal{L}^{-1}\{(e^h_u, e^h_{\theta}, e^h_{\phi})\} \|_1 + \| \mathcal{L}^{-1}\{\phi^h - \phi\} \|_{1/2, \Gamma} \leq C t^2 \max\{1, t^4\} g_h(t),
\]
\[
\| \mathcal{L}^{-1}\{\lambda^h - \lambda\} \|_{-1/2, \Gamma} \leq C t^{3/2} \max\{1, t^{7/2}\} g_h(t),
\]

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where
\[
g_h(t) := \int_0^t \left( \|P_4(L^{-1}\{\dot{\phi} - \Pi_h \phi\})(\tau)\|_{1/2,\Gamma} + \|P_4(L^{-1}\{\lambda - \Pi_{X_h} \lambda\})(\tau)\|_{-1/2,\Gamma} \right) d\tau \\
+ \int_0^t \left( \|P_4(\bar{U} - P_h \bar{U})(\tau)\|_{1,\Omega} + \|P_4(\bar{U} - P_h \bar{U})(\tau)\|_{1,\Omega} \right) d\tau \\
+ \int_0^t \left( \|P_4(U - P_h U)(\tau)\|_{1,\Omega} + \|P_4(U - P_h U)(\tau)\|_{1,\Omega} \right) d\tau.
\]

It should be pointed out that we have inserted the best approximation operators
\[
\Pi_{X_h} : H^{-1/2}(\Gamma) \mapsto X_h, \quad \text{and} \quad \Pi_{Y_h} : H^{1/2}(\Gamma) \mapsto Y_h
\]
into the right hand side of the bound in Corollary 5.4. This can be done since the error produced by computing the exact solution and by the difference of the exact solution with their best approximations are the same.

Next, we present a general procedure describing the full discretization of time-dependent coupling of finite element and boundary integral variational equations such as (5.2) using Lubich’s multistep-based Convolution Quadrature [24, 25], which was designed originally only for treating convolutional boundary integral equations (see, e.g., [26], [27]). We proceed as follows. Suppose that
\[
\dim V_h = \dim W_h = N, \quad \text{and} \quad \dim X_h = \dim Y_h = M,
\]
Let
\[
\{\mu_j\}_{j=1}^N, \{\theta_j\}_{j=1}^N, \{\lambda_j\}_{j=1}^M \quad \text{and} \quad \{v_j\}_{j=1}^M
\]
be the basis functions of the spaces for \(V_h, W_h, X_h\) and \(Y_h\), respectively. We choose a time-step \(\Delta t > 0\), and let us consider the uniform grid in time \(t_n := n\Delta t\), for \(n \geq 0\). We define the stiffness matrix of equation (5.2) as a function of \(s \in \mathbb{C}_+\) with matrix-valued \(A(s) \in \mathbb{C}^{2(N+M),2(N+M)}\) such that
\[
A(s) := \begin{bmatrix}
(B(\mu_1), \mu_1)_{N \times N} & (B(\mu_1), \theta_1)_{N \times N} & (B(\mu_1), \varphi_1)_{N \times N} & 0_{N \times M} \\
(B(\theta_1), \mu_1)_{N \times N} & (B(\theta_1), \theta_1)_{N \times N} & 0_{N \times M} & 0_{N \times M} \\
(B(\varphi_1), \mu_1)_{M \times N} & 0_{M \times N} & (B(\varphi_1), \varphi_1)_{N \times M} & (B(\varphi_1), \lambda_1)_{N \times M} \\
0_{N \times M} & 0_{M \times N} & (B(\lambda_1), \varphi_1)_{N \times M} & (B(\lambda_1), \lambda_1)_{N \times M}
\end{bmatrix}
\]
The data are sampled in time and tested to define vectors $f_n \in \mathbb{R}^{2(N+M)}$:

- $f_{n,i} := (D_1(t_n), \mu_i)_{\Omega_-}$, $i = 1, \ldots, N$
- $f_{n,i} := (D_2(t_n), \vartheta_i)_{\Omega_-}$, $i = N + 1, \ldots, 2N$
- $f_{n,i} := (D_3(t_n), v_i)_{\Gamma}$, $i = 2N + 1, \ldots, 2N + M$
- $f_{n,i} := (D_4(t_n), \lambda_i)_{\Gamma}$, $i = 2N + M + 1, \ldots, 2(N + M)$,

where $D_i(t) = \mathcal{L}\{d_i\}$, $i = 1, \ldots, 4$ in Theorem 4.4. The CQ discretization of (5.2) starts with the Taylor expansion

$$A \left( \frac{\gamma(z)}{\Delta t} \right) = \sum_{n=0}^{\infty} A_n(\Delta t) z^n; \quad \gamma(z) = \frac{\alpha_0 + \cdots + \alpha_k z^{-k}}{\beta_0 + \cdots + \beta_k z^{-k}},$$

(5.16)

where $\gamma(z)$ characterizes the underlying $k$-multistep method, and is, therefore, usually referred to as the characteristic function of the linear multistep method. For the discretization of (5.2), we seek the sequence of vectors $b_n \in \mathbb{R}^{2(N+M)}$ given by the recurrence:

$$A_0(\Delta t) b_n = f_n - \sum_{m=1}^{n} A_m(\Delta t) b_{n-m}, \quad n \geq 0.$$  

(5.17)

If the solution of (5.17) $b_n$ assumes the form $b_n = (b_{n,1}, \ldots, b_{n,2(N+M)})$, then the Galerkin solutions of (5.2) at $t_n$ are given by

$$u^h_n = \sum_{j=1}^{N} b_{n,j} \mu_j, \quad \theta^h_n = \sum_{j=N+1}^{2N} b_{n,j} \theta_j$$

(5.18)

$$\phi^h_n = \sum_{j=2N+1}^{2N+M} b_{n,j} v_j, \quad \lambda^h_n = \sum_{j=2N+M+1}^{2(N+M)} b_{n,j} \lambda_j.$$ 

(5.19)

In order to speed up computations of solving the linear system (5.17), it is a common practice to decouple the computations of finite element and boundary element solutions via a Schur complement strategy as done in [12].

In closing, we remark that for homogeneous thermoelastic solid medium, a pure boundary integral equation formulation may be adapted as in the fluid-structure interaction problem [13]. We will pursue these investigations in a separate communication.

### 6 Numerical Experiments

In order to test numerically the formulations of the previous sections and to explore the case when the Lamé parameters or the thermal diffusivity and expansion are non constant tensors, computational convergence studies were performed in both frequency and time domains. The interior domain $\Omega_-$ where the thermoelastic equations were imposed was the general polygon depicted in Figure [1]. The domain was generated and meshed using Matlab’s pdetool and the refinements were done using the refinement capabilities of the pde toolbox.
The linear system arising from the discretization has a structure that can be depicted by the block matrix

\[
\begin{bmatrix}
\text{FEM}(s) & s\rho_f(\mathbb{N}^\Gamma)^t_h \\
-s\rho_f(\mathbb{N})_h & \text{BEM}(s)
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}^h \\
\mathbf{\phi}^h
\end{bmatrix}
= \begin{bmatrix}
-s\rho_f\Gamma^t_h \beta^h \\
\eta^h \\
0 \\
\rho_f\alpha^h
\end{bmatrix},
\]

where the sparse Finite Element block

\[
\text{FEM}(s) := s^2 \begin{bmatrix}
(\rho_{\Sigma}u_j, v_i)_{\Omega_-} & 0 \\
0 & 0
\end{bmatrix} + s \begin{bmatrix}
0 \\
-\eta u_j, \nabla v_i)_{\Omega_-} & (\theta_j, v_i)_{\Omega_-} \\
(\zeta \theta_j, \varepsilon(v_i))_{\Omega_-} & (\kappa \nabla \theta_j, \nabla v_i)_{\Omega_-}
\end{bmatrix}
\]

contains mass and stiffness matrices as well as first order terms related to the elastic and thermal unknowns. The boundary element block \(\text{BEM}(s)\) contains the Galerkin discretization of the operators of the acoustic Calderón calculus and the coupling trace matrix \(\mathbb{N}^\Gamma_h\) is the discretization of the bilinear form arising from the duality pairing \(\langle \mathbf{u}^h \cdot \mathbf{\nu}, \chi^h \rangle_{\Gamma}\).

**Physical parameters.** The following values of the physical parameters are functions only of space and were used equally for both series of experiments. They are chosen for validation and expository purposes only and do not correspond with any relevant physical material. For the entries of the tensors we make use of the symmetries and of Voigt’s notation \([10]\) to shorten the subscripts.

1. Density of the elastic solid and Lamé parameters

\[
\rho_{\Sigma} = 5 + \sin(x) \sin(y), \quad \lambda = 2, \quad \mu = 3.
\]

2. Thermal expansion tensor \(\zeta\):

\[
\zeta_1 \leftrightarrow \zeta_{11} = \sin(x) + \cos(y), \quad \zeta_2 \leftrightarrow \zeta_{22} = -\sin(y), \quad \zeta_3 \leftrightarrow \zeta_{12} = \zeta_{21} = \cos(x).
\]

![Figure 1: Interior geometry used in the numerical experiments for both frequency-domain and time-domain studies. The domain was generated and meshed using Matlab’s pdetool and refined uniformly using pde tool’s refinement capabilities.](image-url)
3. Thermal diffusivity tensor $\kappa$:

$$
\kappa_1 \leftrightarrow \kappa_{11} = 10 + x^2, \quad \kappa_2 \leftrightarrow \kappa_{22} = 10 + y, \quad \kappa_3 \leftrightarrow \kappa_{12} = \kappa_{21} = 0. \quad (6.3)
$$

4. The reference temperature was chosen to be $\theta_0 = 1$ and the tensor $\eta$ was determined through the relation

$$
\eta = \theta_0 \kappa^{-1} \zeta. \quad (6.4)
$$

**Convergence studies in the frequency domain.** We first verify the results in the frequency domain. We proceed by the method of manufactured solutions using the functions

$$
u := \left( x^3 + xy + y^3, \sin(x) \cos(y) \right), \quad \theta := \sin^2(\pi x) \sin^2(y), \quad (6.5a)
$$

$$
v := i^4 H_0^{(1)}(isr), \quad r = \sqrt{x^2 + y^2}, \quad (6.5b)
$$

together with the parameters defined in (6.1) through (6.5). Right-hand side load vectors and boundary conditions were constructed accordingly.

Lagrangian $P_k$ finite elements were used for the elastic and thermal unknowns, while Galerkin $P_k/P_{k-1}$ continuous/discontinuous Boundary Elements were used for the acoustic potential $v$. Tables 1 to 3 and Figure 2 show the results of the time harmonic experiments with $s = 2.8i$ and successive refinements of the grid shown in Figure 1 for polynomial degrees $k = 1, 2, 3$.

**Convergence studies in the time domain.** In a way analogous to the previous section, the numerical experiments were carried out using the physical parameters and coefficients given in (6.1) through (6.5) and with manufactured solutions using the functions

$$
u := T(t)(x^3 + xy + y^3, \sin(x) \cos(y)), \quad \theta := T(t) \sin^2(\pi x) \sin^2(y), \quad (6.6a)
$$

$$
v := \mathcal{L}^{-1} \left\{ i H_0^{(1)}(isr) \mathcal{L}\{ \mathcal{H}(t) \sin(3t) \} \right\}, \quad r = \sqrt{x^2 + y^2}, \quad (6.6b)
$$

where $\mathcal{L}\{\cdot\}$ is the Laplace transform, the time factor $T(t)$ is given by

$$
T := \mathcal{H}(t)(t^2 + 2t), \quad (6.6c)
$$

and $\mathcal{H}(t)$ is the $C^5$ approximation to Heaviside’s step function

$$
\mathcal{H}(t) := t^5(1 - 5(t - 1) + 15(t - 1)^2 - 35(t - 1)^3 + 70(t - 1)^4 - 126(t - 1)^5) \chi_{[0,1]}(t) + \chi_{[1,\infty]}(t).
$$

The experiments were carried out using the same geometry as in the frequency domain with a fixed spatial mesh, namely the second level of refinement used for the frequency-domain experiments. Starting with 40 time steps for time discretization and polynomial degree $k = 1$ for space discretization, the number of time steps was doubled and the polynomial degree increased by one in every successive refinement. The $L^2(\Omega_-)$ and $H^1(\Omega_-)$ errors were measured for a final time $t = 1.5$.

The performance of BDF2 and Trapezoidal Rule Convolution Quadrature was compared, Table 4 shows the error and estimated convergence rates for BDF2 based time
stepping, while Table 5 shows the results of the experiment using Trapezoidal Rule. The convergence graphs of the experiments is shown in Figure 3.

**Examples.** We now present a couple of illustrative examples simulation the interaction of an incident acoustic wave with thermoelastic obstacles. The first example shows the interaction between the plane wave

\[ v^{inc} = 3 \chi_{[0,0.3]}(88\tau) \sin(88\tau), \quad \tau := t - r \cdot d, \quad r := (x,y), \quad d := (1,5)/\sqrt{26}, \]

and a pentagonal scatterer with mass density given by

\[ \rho_\Sigma = 15 + 40e^{-49r^2}, \quad r := \sqrt{x^2 + y^2}. \]

The values of the elastic parameters, thermic diffusivity \( \kappa \), thermoelastic expansion tensors \( \zeta \) and \( \eta \) were the same as those used for the convergence experiments in the previous paragraphs and given in equations (6.1)-(6.5). The simulation used \( P_2 \) Lagrangian finite elements on a grid with mesh parameter \( h = 7 \times 10^{-3} \) and 36096 elements. The inherited boundary element grid had 496 panels and a grid parameter of \( h = 9.1 \times 10^{-3} \), and \( P_2/P_1 \) continuous/discontinuous Galerkin boundary elements were used. Trapezoidal rule-based discretization was applied in time with a relatively coarse time step \( \kappa = 1 \times 10^{-2} \). Some snapshots of the simulation are shown in Figures 4-6.

The second example is a trapping geometry with density \( \rho_\Sigma = 20+|x|+|y| \) and physical parameters given by (6.1)-(6.3). For this example \( P_5 \) Lagrangian elements were used on a grid with 2992 elements and mesh parameter \( h = 1.72 \times 10^{-2} \), the acoustic equations were discretized with \( P_5/P_4 \) continuous/discontinuous Galerkin boundary elements on a mesh with 236 panels and mesh parameter \( h = 2.5 \times 10^{-2} \). For time discretization trapezoidal rule-based CQ was used with a time step size of \( \kappa = 2 \times 10^{-3} \). Figures 7-9 show snapshots of the acoustic, elastic and temperature fields.

In closing, we remark that for interested reader details of all the computational results are available in the recent dissertation by Sánchez-Vizuet.

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\[ k = 1 \]

| \( h \) | \( E_h^v \) | e.c.r. | \( E_h^u \) | e.c.r. | \( E_h^{u,a} \) | e.c.r. | \( E_h^\theta \) | e.c.r. | \( E_h^\kappa \) | e.c.r. |
|-------|-----|--------|-----|--------|--------|-----|--------|--------|-----|--------|
| 1 E-1 | 1.782 E-2 | — | 3.999 E-2 | — | 2.015 E-3 | — | 2.011 E-1 | — | 7.430 E-2 | — |
| 5.016 E-2 | 7.292 E-3 | 1.293 | 1.675 E-2 | 1.255 | 6.397 E-4 | 1.656 | 8.733 E-2 | 1.203 | 3.746 E-2 | 0.988 |
| 2.508 E-2 | 2.272 E-3 | 1.683 | 5.344 E-3 | 1.648 | 1.837 E-4 | 1.799 | 3.297 E-2 | 1.405 | 1.876 E-2 | 0.976 |
| 1.254 E-2 | 6.099 E-4 | 1.897 | 1.447 E-3 | 1.885 | 4.824 E-5 | 1.929 | 1.314 E-2 | 1.327 | 9.383 E-3 | 0.996 |
| 6.27 E-3 | 1.556 E-4 | 1.971 | 3.703 E-4 | 1.966 | 1.223 E-4 | 1.980 | 0.961 E-3 | 1.141 | 4.692 E-3 | 1.000 |

Table 1: The experiments were run using \( \mathcal{P}_k \) Lagrangian finite elements and \( \mathcal{P}_k/\mathcal{P}_{k-1} \) boundary elements. This table shows the relative errors and estimated convergence rates in the frequency domain for \( k = 1 \). The maximum length of the panels used to discretize the boundary is denoted by \( h \).

\[ k = 2 \]

| \( h \) | \( E_h^v \) | e.c.r. | \( E_h^u \) | e.c.r. | \( E_h^{u,a} \) | e.c.r. | \( E_h^\theta \) | e.c.r. | \( E_h^\kappa \) | e.c.r. |
|-------|-----|--------|-----|--------|--------|-----|--------|--------|-----|--------|
| 1 E-1 | 7.926 E-5 | — | 1.284 E-4 | — | 9.742 E-5 | — | 3.514 E-3 | — | 6.446 E-3 | — |
| 5.016 E-2 | 6.676 E-6 | 3.570 | 1.181 E-5 | 3.442 | 3.514 E-6 | 3.004 | 8.708 E-4 | 2.013 | 1.630 E-4 | 1.983 |
| 2.508 E-2 | 5.590 E-7 | 3.578 | 1.207 E-6 | 3.290 | 1.417 E-6 | 3.000 | 3.297 E-5 | 2.003 | 4.093 E-5 | 1.993 |
| 1.254 E-2 | 4.630 E-8 | 3.594 | 1.331 E-7 | 3.004 | 2.172 E-7 | 3.000 | 5.426 E-6 | 2.001 | 5.426 E-6 | 1.997 |
| 6.27 E-3 | 3.793 E-9 | 3.609 | 1.550 E-9 | 3.341 | 1.837 E-9 | 3.1 60 | 7.892 E-8 | 3.000 | 2.566 E-8 | 3.000 |

Table 2: The experiments were run using \( \mathcal{P}_k \) Lagrangian finite elements and \( \mathcal{P}_k/\mathcal{P}_{k-1} \) boundary elements. This table shows the relative errors and estimated convergence rates in the frequency domain for \( k = 2 \). The maximum length of the panels used to discretize the boundary is denoted by \( h \).

\[ k = 3 \]

| \( h \) | \( E_h^v \) | e.c.r. | \( E_h^u \) | e.c.r. | \( E_h^{u,a} \) | e.c.r. | \( E_h^\theta \) | e.c.r. | \( E_h^\kappa \) | e.c.r. |
|-------|-----|--------|-----|--------|--------|-----|--------|--------|-----|--------|
| 1 E-1 | 6.847 E-7 | — | 1.726 E-6 | — | 4.564 E-6 | — | 9.540 E-6 | — | 4.018 E-4 | — |
| 5.016 E-2 | 3.869 E-8 | 4.145 | 9.804 E-8 | 4.138 | 2.886 E-7 | 3.983 | 7.701 E-7 | 3.631 | 5.044 E-5 | 2.994 |
| 2.508 E-2 | 2.279 E-9 | 4.086 | 5.794 E-9 | 4.081 | 1.810 E-8 | 3.995 | 7.600 E-8 | 3.441 | 6.312 E-6 | 2.998 |
| 1.254 E-2 | 1.375 E-10 | 4.051 | 3.502 E-10 | 4.048 | 1.132 E-9 | 3.999 | 5.426 E-9 | 3.983 | 5.426 E-9 | 3.000 |
| 6.27 E-3 | 8.468 E-12 | 4.021 | 2.141 E-11 | 3.012 | 7.970 E-11 | 4.000 | 1.011 E-9 | 3.972 | 9.866 E-8 | 3.000 |

Table 3: The experiments were run using \( \mathcal{P}_k \) Lagrangian finite elements and \( \mathcal{P}_k/\mathcal{P}_{k-1} \) boundary elements. This table shows the relative errors and estimated convergence rates in the frequency domain for \( k = 3 \). The maximum length of the panels used to discretize the boundary is denoted by \( h \).

| BDF2 | \( \kappa/\text{Ndef} \) | \( E_h^v \) | e.c.r. | \( E_h^u \) | e.c.r. | \( E_h^{u,a} \) | e.c.r. | \( E_h^\theta \) | e.c.r. | \( E_h^\kappa \) | e.c.r. |
|-------|--------|-----|--------|-----|--------|--------|-----|--------|--------|-----|--------|
| 3.75 E-2 / 108 | 7.793 E-3 | — | 1.231 E-2 | — | 5.184 E-3 | — | 2.975 E-1 | — | 2.222 E-1 | — |
| 1.875 E-2 / 394 | 2.775 E-3 | 1.489 | 7.225 E-4 | 3.994 | 3.275 E-4 | 3.984 | 1.258 E-2 | 4.563 | 1.940 E-2 | 3.518 |
| 9.375 E-3 / 859 | 7.955 E-4 | 1.803 | 1.980 E-4 | 1.964 | 4.061 E-5 | 3.012 | 1.916 E-3 | 2.715 | 1.265 E-3 | 3.938 |
| 4.687 E-3 / 1503 | 2.072 E-4 | 1.941 | 5.035 E-5 | 1.975 | 9.408 E-6 | 2.110 | 4.905 E-4 | 1.966 | 1.125 E-4 | 3.489 |
| 2.344 E-3 / 2326 | 5.258 E-5 | 1.978 | 1.267 E-5 | 1.991 | 7.329 E-6 | 2.014 | 4.592 E-5 | 1.988 | 2.055 E-5 | 2.259 |
| 1.172 E-3 / 3328 | 1.323 E-5 | 1.991 | 3.175 E-6 | 1.996 | 3.795 E-7 | 2.007 | 3.100 E-5 | 1.995 | 5.825 E-6 | 2.015 |

Table 4: Time domain convergence results for BDF2-based CQ. The experiments were run with a fixed mesh using \( \mathcal{P}_k \) Lagrangian finite elements and \( \mathcal{P}_k/\mathcal{P}_{k-1} \) boundary elements. In every successive refinement level the size of the time step was halved and the polynomial degree of the space refinement increased by one. The table shows the relative errors and estimated convergence rates measured for a final time \( t = 1.5 \).
Figure 2: Convergence studies in the frequency domain for polynomial degrees $k = 1, 2, \text{ and } 3$. The mesh was refined uniformly on every successive iteration.
Table 5: Time domain convergence results for Trapezoidal Rule-based CQ. The experiments were run with a fixed mesh using $P_k$ Lagrangian finite elements and $P_k/P_{k-1}$ boundary elements. In every successive refinement level the size of the time step was halved and the polynomial degree of the space refinement increased by one. The table shows the relative errors and estimated convergence rates measured for a final time $t = 1.5$.

| $\kappa / \text{Ndof}$ | $L^2(\Omega)$ | $H^1(\Omega)$ |
|---------------------|----------------|----------------|
|                     | $n$ | e.e.r. | $n$ | e.e.r. | $n$ | e.e.r. | $n$ | e.e.r. | $n$ | e.e.r. |
| 3.75 E-2 / 108      | 5.620 E-3 | 1.218 E-2 | 5.213 E-3 | 2.976 E-1 | 2.221 E-1 |
| 1.875 E-2 / 394     | 8.283 E-4 | 2.762 E-4 | 5.489 E-4 | 4.151 E-4 | 4.805 E-4 | 1.044 E-2 | 3.522 |
| 9.375 E-3 / 859     | 2.107 E-4 | 1.975 E-5 | 5.085 E-5 | 2.416 E-5 | 1.660 E-5 | 4.144 E-4 | 4.424 | 4.299 |
| 4.687 E-3 / 1503    | 5.278 E-5 | 1.997 E-5 | 1.272 E-5 | 2.349 E-6 | 2.821 E-6 | 1.242 E-4 | 1.997 | 6.549 | 4.206 |
| 2.344 E-3 / 2326    | 1.320 E-5 | 1.996 E-6 | 3.184 E-6 | 1.999 E-7 | 2.026 E-7 | 3.107 E-5 | 1.999 | 6.286 E-6 | 3.381 |
| 1.172 E-3 / 3328    | 3.300 E-6 | 2.000 E-7 | 7.956 E-7 | 2.000 E-7 | 1.422 E-7 | 2.001 E-7 | 2.000 | 1.451 E-6 | 2.115 |

Figure 3: time-domain convergence studies for Trapezoidal Rule (Left) and BDF2 (Right) based convolution quadrature. In every refinement the number of time steps was doubled and the polynomial degree of the spatial discretization increased by one.
Figure 4: Snapshots of the total acoustic field at times $t = 0.25, 0.6, 0.95, 1.3, 1.65, 2$. The interior domain shows the norm of the elastic displacement.
Figure 5: Close up of the norm of the elastic displacement for times $t = 0.25, 0.6, 0.95, 1.3, 1.65, 2$. Black represents no displacement.
Figure 6: Close up of the norm of the temperature variations with respect to the reference configuration for times $t = 0.25, 0.6, 0.95, 1.3, 1.65, 2$. Black represents zero, whereas shades of red and blue represent positive and negative variations respectively.
Figure 7: Snapshots of the total acoustic field at times $t = 0.3, 0.6, 0.9, 1.2, 1.5, 1.8$. The interior domain shows the norm of the elastic displacement.
Figure 8: Close up of the norm of the elastic displacement for times $t = 0.3, 0.6, 0.9, 1.2, 1.5, 1.8$. Black represents no displacement.
Figure 9: Close up of the norm of the temperature variations with respect to the reference configuration for times $t = 0.3, 0.6, 0.9, 1.2, 1.5, 1.8$. Black represents zero, whereas shades of red and blue represent positive and negative variations respectively.