Small oscillations of test particles with angular momentum in Reissner-Nordstrom field

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The oscillations of the test particles on circular orbits in field of super-extremal charged central object are researched. The formulae for oscillation frequencies are obtained in cases of circular orbits perturbations caused by small change in energy of the particle or by the small change in mass of the central object.

1. Introduction

Radial motion of a charged test particle with orbital angular momentum in the field of super-extreme charged central body have been considered in [1]. The classification motion types of test particles have been done and their circular orbits have been described. In this paper we consider oscillations of a particle caused by a small perturbation of a circular orbit. The perturbations can be caused by small change of any of the system parameters \( \{ M, Q, L, E, m \} \). We consider perturbations that are caused by the change in energy of the particle and the change in mass of the central object.

The Hamiltonian of the system, the equation of motion and the effective potential have the form

\[
H = \sqrt{Fm^2 + F^2P_R^2 + F\frac{L^2}{R^2}}, \quad F = 1 - \frac{2M}{R} + \frac{Q^2}{R^2}, \quad \frac{m}{dt}R^2 = -U_V = E^2 - W_\varepsilon^2, \quad W_\varepsilon^2 = F \left( m^2 + \frac{L^2}{R^2} \right), \quad (1.1)
\]

where \( M < Q \) is the mass and charge of the central body, \( m, L, E \) - the mass, the orbital angular momentum and energy of the test particle. Minimum of the effective potential \( W_\varepsilon^2 \) determines the circular orbit \( R = R_{extr} \) of a particle.

2. Perturbations caused by changes in the particle energy

In this case, small deviations of the particle’s world line can be reduced to a canonical transformation \( R = R_{extr} + r, \quad P_R = P_r \). Expanding the new Hamiltonian near an equilibrium position, and taking into
account that the first derivatives and mixed derivatives equal zero, we obtain

\[ H(P_r, R) \rightarrow H(P_r, r) = E_{\text{min}} + \frac{1}{2} H_{PP} P_r^2 + \frac{1}{2} H_{rr} r^2, \]  
(2.1)

\[ H_{PP} = \left( \frac{\partial^2 H}{\partial P_r^2} \right)_{r=0, P_r=0} = \left( \frac{F^2}{E_{\text{min}}} \right), \]  
(2.2)

\[ H_{rr} = \left( \frac{\partial^2 H}{\partial r^2} \right)_{r=0, P_r=0} = \left( \frac{d^2 W_\varepsilon}{dR^2} \right)_{R=R_{\text{extr}}}. \]  
(2.3)

Comparing this Hamiltonian with a classical oscillator Hamiltonian, we obtain the expression for the oscillation frequency

\[ H_{PP} = \frac{1}{m_{\text{eff}}}, \quad k = H_{rr}, \quad \omega = \sqrt{\frac{k}{m_{\text{eff}}}} = \sqrt{H_{rr} H_{PP}}, \]  
(2.4)

\[ \omega = \frac{1}{\sqrt{2}} \left( \frac{F}{E_{\text{min}}} \sqrt{\frac{d^2 W_\varepsilon}{dR^2}} \right)_{R=R_{\text{extr}}}. \]  
(2.5)

The formula for \( \omega \) is not given in terms of \( E_{\text{min}}, R_{\text{extr}} \) in order to reduce the paper length.

### 3. Perturbations caused by changes in mass of the central object

The dynamics of test particles with orbital angular momentum can also be studied using the mass potential [2]. The velocity potential has the form

\[ U_V(M, Q, E, m, L) = -E^2 + \left( 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \right) \left( m^2 + \frac{L^2}{R^2} \right). \]  
(3.1)

Definition of the mass potential is as follows \( U_V(U_M, Q, E, m, L) = 0 \). Hence, the mass potential is

\[ U_M = \frac{1}{2R} \left[ (R^2 + Q^2) - \frac{R^2 E^2}{m^2 + \frac{L^2}{R^2}} \right]. \]  
(3.2)

Perturbations of the mass of the central object can be reduced to the transformation \( U_M = M + \mu \). Velocity and acceleration of a particle in terms of the mass potential are

\[ \left( \frac{dR}{ds} \right)^2 = \frac{2}{R} (M - U_M), \]  
(3.3)

\[ \frac{d^2 R}{ds^2} = -\frac{1}{R^2} \left( M - U_M + R \frac{dU_M}{dR} \right). \]  
(3.4)

Let us expand the mass potential in the vicinity of \( R_{\text{min}} \), \( R = R_{\text{min}} + r \),

\[ U_M(R) \rightarrow U_M(r) = U_M(0) + r \left( \frac{dU_M}{dr} \right)_{r=0} + \frac{r^2}{2} \left( \frac{d^2 U_M}{dr^2} \right)_{r=0}, \]  
(3.5)

\[ U_M''(0) = \left( \frac{d^2 U_M}{dr^2} \right)_{r=0} = \left( \frac{d^2 U_M}{dR^2} \right)_{R=R_{\text{min}}} = \frac{1}{R} \left[ \frac{Q^2}{R^2} - \frac{L^2}{R^2} \left( \frac{E^2}{m^2 + \frac{L^2}{R^2}} \right)^2 \left( \frac{4L^2}{m^2 + \frac{L^2}{R^2}} - 1 \right) \right]. \]  
(3.6)

Substituting this expansion into formulas for velocity and acceleration of the particle and neglecting all second and higher orders infinitesimals we obtain the equation of the oscillator and the oscillation frequency
as follows

\[
\frac{d^2 r'}{ds^2} + \frac{U''_{M0}}{R_{\min}} r' = 0, \quad r' = r + \frac{\mu}{U''_{M0} R_{\min}}, \tag{3.7}
\]

\[
\dot{\omega} = \sqrt{\frac{U''_{M0}}{R_{\min}}} \frac{1}{R_{\min}} \left[ \frac{Q^2}{R_{\min}^2} - \frac{L^2}{R_{\min}^2} \frac{F_{\min}^2}{m^2 + L^2 R_{\min}} \left( \frac{4 L^2}{R_{\min}^2} - 1 \right) \right]. \tag{3.8}
\]

Coordinate transformation \( r \rightarrow r' \) is interpreted as the shift of the center of particle oscillations caused by the change in mass of the central object.

Note that the frequency of small oscillations of the particles \( \omega \) (see (2.5)) is measured relative to the time an infinitely distant observer \( T \), whereas the frequency \( \tilde{\omega} \) (see (3.8)) - relative to the proper time. The oscillation frequency \( \omega_\infty \) of a source, measured by a distant observer, and the oscillation frequency \( \tilde{\omega} \) of the same source relative to the proper time are connected by the relation

\[
\omega_\infty = \tilde{\omega} \sqrt{g_{00}} = \text{const.}
\]

Therefore, in the case of small oscillations of the particles around the equilibrium position, we have the following relation between frequencies (2.5) and (3.8)

\[
\omega = \omega \sqrt{F_{\text{extr}}} = \tilde{\omega} \sqrt{1 - \frac{2M}{R_{\text{extr}}} + \frac{Q^2}{R_{\text{extr}}^2}}.
\]

4. Tables of the oscillation frequencies for the substantially different types of system behavior

Types of system behavior can be divided by type of roots of circular orbit equation; it has been done in [1]. There have also been highlighted regions of parameters for each type of behavior and calculated circular orbits. In this work we obtained the values of oscillation frequencies for the two types of perturbations for each of the parameters sets with significantly different behavior of the system.

Following [1] we use dimensionless quantities

\[
\varepsilon = \frac{E}{mc^2}, \sigma = \frac{|Q| m \sqrt{k}}{c |L|}, \mu = \frac{Mmk}{c |L|}, \zeta = \frac{Rmc}{|L|}, \Omega = \frac{|L| mc^2}{\omega}, \bar{\Omega} = \frac{|L| mc^2}{\tilde{\omega}}.
\]

Thus the frequency is represented in units of \(|L| mc^2\). We can estimate this value; for example, for a central body with mass of the sun this value is measured in kilohertz. The \( \Omega \) is the frequency for small deviations of \( \varepsilon \), and the \( \bar{\Omega} \) is the frequency for small deviations of \( \mu \).

Following tables contain computed oscillation frequencies \( \Omega, \bar{\Omega} \) for the data \((z_{\min}, \varepsilon_{\min})\) obtained in [1] for given system parameters \((\sigma, \mu)\). Complex values of oscillation frequencies \( \Omega, \bar{\Omega} \) correspond to the non-stable circular orbits.

| Table 1. Region \( D_1^{(3)} \) |
|----------------------------------|
| \( D_1^{(3)} \), system parameters \( \sigma^2 = 0.1, \mu^2 = 0.088 \) |
| Radius of the orbit \( z_m \) | Energy \( \varepsilon_{\min} \) | Frequency of small oscillations \( \Omega \) | Frequency of small oscillations \( \bar{\Omega} \) |
| \( z_m_1 \) | 2.6839 | 0.9502 | 0.08354 | 0.09382 |
| \( z_m_2 \) | 0.40706 | 1.0134 | 0.5133 | 1.3434 |
| \( z_m_3 \) | 0.61713 | 0.61713 | 0.5854 \( i \) | 0.62998 \( i \) |
Table 2. Region $D_2^{(3)}$

| $z_m$ | $\varepsilon_{min}$ | $\Omega$ | $\tilde{\Omega}$ |
|-------|---------------------|---------|-----------------|
| 1.6325 | 0.91449             | 0.1012  | 0.1298          |
| 0.55539 | 0.89857              | 0.3681  | 0.8435          |
| 0.95681 | 0.91902             | 0.1551 $I$ | 0.2441 $I$ |

Table 3. Region $\Sigma_+^{(2)}$

| $z_m$ | $\varepsilon_{min}$ | $\Omega$ | $\tilde{\Omega}$ |
|-------|---------------------|---------|-----------------|
| 0.50947 | 0.84792             | 0.4534  | 1.1785          |
| 1.296 | 0.90913             | 0           | 0.002125 $I$ |

Table 4. Region $\Sigma_-^{(2)}$

| $z_m$ | $\varepsilon_{min}$ | $\Omega$ | $\tilde{\Omega}$ |
|-------|---------------------|---------|-----------------|
| 2.7691 | 0.95211             | 0.08108 | 0.09054         |
| 0.4971 | 1.0742              | 0.005439 $I$ | 0.009074 $I$ |

Table 5. Region $\Gamma^{(1)}$

| $z_m$ | $\varepsilon_{min}$ | $\Omega$ | $\tilde{\Omega}$ |
|-------|---------------------|---------|-----------------|
| 1     | $2/\sqrt{5}$       | 0.1       | 0               |

Table 6. Region $D_1^{(1)}$

| $z_m$ | $\varepsilon_{min}$ | $\Omega$ | $\tilde{\Omega}$ |
|-------|---------------------|---------|-----------------|
| 2.1114 | 0.90309             | 0.1351  | 0.1657          |

Table 7. Region $D_2^{(1)}$

| $z_m$ | $\varepsilon_{min}$ | $\Omega$ | $\tilde{\Omega}$ |
|-------|---------------------|---------|-----------------|
| 0.44716 | 0.64921             | 0.4874  | 1.8389          |
Table 8. Region $D_3^{(1)}$

|              | Radius of the orbit $z_m$ | Energy $\varepsilon_{min}$ | Frequency of small oscillations $\Omega$ | Frequency of small oscillations $\tilde{\Omega}$ |
|--------------|----------------------------|-----------------------------|------------------------------------------|------------------------------------------|
| $z_m$        | 2.7781                     | 0.95074                     | 0.08264                                  | 0.09238                                  |

References

[1] Gladush V.D., Kulikov D.A. Classification for the radial component of particles motion in the field of a super-extremely charged object. arXiv:1110.3179 [gr-qc].

[2] Gladush V.D., Galadgyi M.V. Some peculiarities of motion of neutral and charged test particles in the field of a spherically symmetric charged object in general relativity. Gen Relativ Gravit. 43, 1347–1363 (2011).