GLOBAL REGULARITY OF WAVE MAPS FROM $\mathbb{R}^{3+1}$ TO $\mathbb{H}^2$

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Abstract. We consider Wave Maps with smooth compactly supported initial data of small $H^{3/2}$-norm from $\mathbb{R}^{3+1}$ to hyperbolic 2-space and show that they stay smooth globally in time. Our methods are based on the introduction of a global Coulomb Gauge as in [3], followed by a dynamic separation as in [6]. We then rely on an adaptation of T.Tao’s methods used in his recent breakthrough result [16].

1. Introduction

Let $M$ be a Riemannian manifold with metric $(g_{ij}) = g$. A Wave Map $u : \mathbb{R}^{n+1} \to M$, $n \geq 2$ is by definition a solution of the Euler-Lagrange equations associated with the functional $u \to \int_{\mathbb{R}^{n+1}} < \partial_\alpha u, \partial^\alpha u >_g d\sigma$. Here the usual Einstein summation convention is in force, while $d\sigma$ denotes the volume measure on $\mathbb{R}^{n+1}$ with respect to the standard metric. In local coordinates, $u$ is seen to satisfy the equation

$$\square u_i + \Gamma_{jk}^i(u) \partial_\alpha u_j \partial^\alpha u_k = 0$$ (1)

where $\Gamma_{jk}^i$ refer to the Riemann-Christoffel symbols associated with the metric $g$. The relevance of this model problem arises from its connections with more complex nonlinear wave equations of mathematical physics: for example, Einstein’s vacuum equations under $U(1)$-symmetry attain the form of a Wave Maps equation coupled with additional elliptic equations.

We are interested in the well-posedness of the Cauchy problem for (1) with initial data $u[0] \times \partial_t u[0]$ at time $t = 0$ in $H^s \times H^{s-1}$. Classical theory relying on the energy inequality and Sobolev inequalities allows one to deduce local well-posedness in $H^s$ for $s > \frac{n}{2} + 1$.

Ideally, one would like to prove local well-posedness in $H^{\frac{n}{2}}$, as this would immediately imply global in time well-posedness. The reason for this is that $H^{\frac{n}{2}}$ is the Sobolev space invariant under the natural scaling associated with (1). Unfortunately, it is known that ”strong well-posedness” in the sense of analytic or even $C^2$-dependence on the initial data fails at the $H^{\frac{n}{2}}$-level, $n \geq 3$. Thus the best result to be hoped for is global regularity of Wave Maps with initial data of small $H^{\frac{n}{2}}$-norm.

In two space dimensions, the scale invariant Sobolev space coincides with the classical $H^1$, and numerical data as well as the conjectured non-concentration of energy
suggest global regularity for Wave Maps with arbitrary smooth initial data, provided the target is negatively curved. This underlines the importance of the hyperbolic plane as target manifold.

In the quest for reaching the critical $\frac{2}{n}$ regularity, local well-posedness for (1) with initial data in $\dot{H}^{\frac{2}{n}+\epsilon}$, $\epsilon > 0$ was proved for $n \geq 3$ by Klainerman and Machedon in [4], and for $n = 2$ in [3]. Later, Tataru established local well-posedness in the Besov space $\dot{B}^{\frac{2}{n}}_{n,1}$, [18], [19]. Note that $\dot{B}^{\frac{2}{n}}_{n,1}$ has the same scaling as $\dot{H}^{\frac{2}{n}}$, but unlike the latter controls $L^\infty$.

An important breakthrough with respect to global regularity was recently achieved by T.Tao in the case of Wave Maps to the sphere [15], [16], proving global regularity for smooth initial data small in $\dot{H}^{\frac{2}{n}}$: Tao’s work exemplifies the importance of taking the global geometry of the target into account, an aspect largely ignored by the local formulation (1). Embedding the target sphere in an ambient Euclidean space, the Wave Maps equation considered by Tao takes the form

$$\Box \phi = -\phi \partial_\alpha \phi^\alpha \partial^\alpha \phi = -(\phi \partial_\alpha \phi^\alpha - \partial_\alpha \phi^\alpha \partial^\alpha \phi)$$

$\alpha$ as usual runs over the space-time indices $0, 1, ... n$. The nonlinearity encodes both geometric (skew-symmetry of $\phi \partial_\alpha \phi^\alpha - \partial_\alpha \phi^\alpha$) as well as algebraic information (‘null-form’ structure). Tao manages to analyze all possible frequency interactions of the nonlinearity up to the case in which the derivatives fall on high frequency terms while the undifferentiated term has very low frequency. This bad case is then gauged away, using the skew-symmetric structure. With this method, which served as inspiration for the following developments, as well as sophisticated methods from harmonic analysis, Tao manages to go all the way to $n = 2$(note that the smaller the dimension, the more difficult the problem on account of the increasing scarcity of available Strichartz estimates).

After Tao, Klainerman and Rodnianski [7], extended this result to Wave Maps form $\mathbb{R}^{n+1}$, $n \geq 5$ to more general and in particular noncompact targets. More precisely, Klainerman and Rodnianski consider parallelizable targets which are well-behaved at infinity. Upon introducing a global orthonormal frame $\{e_i\}$, they define the new variables $\phi^i_\alpha$ defined by $u_\alpha \partial_\alpha = \phi^i_\alpha e_i$. It turns out that these satisfy the system of equations

$$\partial_\beta \phi^i_\alpha - \partial_\alpha \phi^i_\beta = C^i_{jk} \phi^j_\alpha \phi^k_\beta$$

$$\partial_\alpha \phi^k = -\Gamma^i_{jk} \phi^j_\beta \phi^k_\gamma m^{\beta \gamma}$$

$\text{[1]}$This has been proved by Struwe for Wave Maps to the hyperbolic plane which are corotationally symmetric.
where $m_{\beta\gamma}$ is the standard Minkowski metric on $\mathbb{R}^{n+1}$ and $C_{jk}^i, \Gamma_{jk}^i$ are defined as follows:

$$[e_j, e_k] = C_{jk}^i e_i$$

(5)

$$\nabla e_j e_k = \Gamma_{jk}^i e_i$$

(6)

There is again a skew-symmetric structure present in this formulation on account of $\Gamma_{jk}^i = -\Gamma_{kj}^i$. Moreover, by contrast with Tao’s formulation (2), the boundedness of $\phi$ is replaced here by the boundedness of the $C_{jk}^i, \Gamma_{jk}^i$. Klainerman and Rodnianski impose in addition the condition that all derivatives of these coefficients be bounded, or in their terminology that $M$ be ‘boundedly parallelizable.’ If one now passes to the wave equation satisfied by the vector $\phi_\alpha := \{\phi_\alpha^i\}$, one obtains

$$\square \phi_\alpha = -R_\mu \partial^\mu \phi_\alpha + E$$

(7)

where $R_\mu$ is skew-symmetric and moreover depends linearly on $\phi$, provided we assume the $C_{jk}^i, \Gamma_{jk}^i$ to be constant for simplicity’s sake. $E$ is a cubic polynomial in $\phi$. By contrast with (2), the leading term in the nonlinearity is ‘quadratic in $\phi$’. It is now possible to control all possible frequency interactions on the right hand side ($n \geq 5$) except when $R_\mu$ is localized to very low frequency while $\partial^\mu \phi$ is at large frequency. However, as Klainerman and Rodnianski observed, the curvature

$$\partial_\nu R_\mu - \partial_\mu R_\nu + [R_\mu, R_\nu]$$

(8)

when $R$ is reduced to low frequencies is ‘very small’, in the sense that it is quadratic in $\phi$, hence amenable to good Strichartz estimates. To take advantage of this, they introduce a Coulomb Gauge $\sum_{j=1}^3 \partial_j \tilde{R}_j = 0$, which allows one to replace the $R_\mu$ in (7) by $\tilde{R}_\mu$ which is ‘quadratic in $\phi$’, effectively replacing the nonlinearity by a term which is trilinear in $\phi$ and hence easily handled by Strichartz estimates. The general philosophy here is that the higher the degree of the nonlinearity, the more room is available to apply Strichartz estimates. Klainerman and Rodnianski’s method is thus similar to Tao’s in that it utilizes a microlocal Gauge Change to deal with specific bad frequency interactions.

The last result to be mentioned in this development is the simplification and extension of the previous arguments to include the case of $4 + 1$-dimensional Wave Maps to essentially arbitrary targets achieved by Shatah-Struwe [13] and (in more restrictive formulation) Uhlenbeck-Stefanov-Nahmod [11]. The former observed that
using a Coulomb Gauge, in a similar fashion as above, at the beginning without carrying out a frequency decomposition allows one to reduce the nonlinearity to a form directly amenable to Strichartz estimates. This allows them to avoid the microlocal Gauge Change of Tao and leads to a remarkable simplification of the argument. In addition, they are also able to treat the case of dimension 4 + 1.

The methods in [7] and [13] run into serious difficulties for 3 + 1-dimensional Wave Maps, and even more so for 2 + 1-dimensional Wave Maps. This can be seen intuitively as follows:

The global Coulomb Gauge puts the leading terms of the nonlinearity roughly into the form $D^{-1}(\phi^2)D\phi$. In dimensions 4 and higher, Shatah and Struwe can estimate such terms relying on the Strichartz type inequality for Lorentz spaces

$$||\phi||_{L^2_tL^{2,\sigma}_x} \leq C||\Box\phi||_{L^1_tH^\sigma} + C||\phi[0]||_{H^{\frac{\sigma}{2}-1}}$$

where $\sigma = \frac{n}{2} - 2$. This can be used to estimate the $L^1_tL^\infty_x$-norm of $D^{-1}(\phi^2)$. However, in three space dimensions, the above estimate fails. In order to handle the case when $D^{-1}(\phi^2)$ has much lower frequency than $D\phi$, one would have to use an endpoint $L^2_tL^\infty_x$-Strichartz estimate, which is false, even replacing the $L^\infty_x$-norm by $BMO$, see [17].

The present paper starts with the basic formulation (3), (4) of Klainerman and Rodnianski applied to the simplified context of $H^2$, but utilizes the Coulomb Gauge right at the beginning as do Shatah and Struwe. The main innovation over the preceding then is to introduce a special null-structure into the nonlinearity by way of what we term a dynamic separation

$$\psi_\alpha := e^{-i\Phi}\phi_\alpha$$

in complex notation for some potential function $\Phi$, and utilize the div-curl system satisfied by these to split them into a dynamic part, which has the form of a gradient, and an elliptic part, which satisfies an elliptic div-curl system. Substituting these components into the nonlinearity results in a fairly complicated trilinear null-structure, as well as error terms which are at least quintilinear in $\phi$.

In order to estimate the trilinear null-structure, we have to utilize the technical framework set forth in [16]. Also, as we are at the level of the derivative with respect to $u$ in (1) (and hence lose one degree of differentiability), we have to modify several key lemmata of Tao for our purposes in the case of high-high interactions. Moreover, we have to prove a Gauge Change estimate (Theorem 3.1) which is new for the spaces introduced in [16]. This Gauge Change result clearly breaks down for 2 + 1-dimensional Wave Maps, and so there is little hope to extend our result to that case by following the same route.

However, our result does extend to more general and in particular higher-dimensional

\footnote{Alternatively, as pointed out by Klainerman and Rodnianski, one can utilize an improved bilinear version of Strichartz estimates in [14] to handle these cases.}

\footnote{This terminology was suggested by S. Klainerman}

\footnote{This is to be contrasted with the null-structure in [6], which is bilinear}
targets. Details on this shall be contained in a forthcoming note.

Our main theorem is the following: Denote the 2-dimensional hyperbolic plane by $H^2 = \mathbb{R} \times \mathbb{R}_{>0}$. We utilize standard coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}_{>0}$, with respect to which the metric becomes $dg = \frac{dx^2 + dy^2}{y^2}$. A Wave Map will be described in terms of a pair of functions $(x, y)$ on $\mathbb{R}^{3+1}$ with range as specified in the preceding paragraph. Then we have the theorem

**Theorem 1.1.** Let $M$ be the hyperbolic 2-plane. There exists a number $\epsilon > 0$ such that for all compactly supported smooth initial data $(x, y)[0], (\partial_t x, \partial_t y)[0]$ with $\left\| \left( \nabla (x, y)[0], \partial_t (x, y)[0] \right) \right\|_{L^2} < \epsilon$, the WM-problem (1) with these initial data has a globally smooth solution.

**Acknowledgments:** The author would like to thank his Ph.D. advisor Sergiu Klainerman as well as Igor Rodnianski and Terence Tao for helpful suggestions and comments as well as reading the manuscript.

The research for this paper was conducted in the fall 2001.
2. Outline of the Argument

2.1. Basic formulation of the problem. This section will serve as outline for the rest of the paper.

Let us restate the main theorem, using the notation introduced at the end of last section:

**Theorem 2.2.** Let $M$ be the hyperbolic 2-plane. There exists a number $\epsilon > 0$ such that for all compactly supported smooth initial data $(x,y)[0], (\partial_t x, \partial_t y)[0]$ with $\|(\sum (x,y)[0], \partial_t (x,y)[0])\|_H^{\frac{1}{2}} < \epsilon$, the WM-problem \(\square\) with these initial data has a globally smooth solution.

We translate this problem to the level of the derivative, utilizing the formulation (3), (4) with respect to the global orthonormal frame \(\{y\partial_x, y\partial_y\}\). More explicitly, we have

\[
\phi^1_\alpha = \frac{\partial_\alpha x}{y}, \quad \phi^2_\alpha = \frac{\partial_\alpha y}{y}
\]

The div-curl system satisfied by these quantities is then of the following form:

\[
\partial_\beta \phi^1_\alpha - \partial_\alpha \phi^1_\beta = \phi^1_\alpha \phi^2_\beta - \phi^1_\beta \phi^2_\alpha \quad (11)
\]

\[
\partial_\beta \phi^2_\alpha - \partial_\alpha \phi^2_\beta = 0 \quad (12)
\]

\[
\partial_\alpha \phi^{1\alpha} = -\phi^{1\alpha} \phi^{2\alpha} \quad (13)
\]

\[
\partial_\alpha \phi^{2\alpha} = \phi^{1\alpha} \phi^{1\alpha} \quad (14)
\]

$\alpha, \beta$ here vary over the space-time indices 0, 1, 2, 3, and Einstein’s summation convention is in force.

Once we can show that the $\phi^i_\alpha$ stay smooth globally in time, the actual Wave Map can be obtained by integration from \(\left(\frac{\partial_t x}{y}, \frac{\partial_t y}{y}\right) = (\phi^1_0, \phi^2_0)\).

Letting $\phi_\alpha$ denote the column vector with entries $\phi^1_\alpha, \phi^2_\alpha$, we obtain the following wave equations:

\[
\square \phi_\alpha = M_\alpha \partial^\nu \phi_\alpha + "\phi^{3n}" \quad (15)
\]
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where

$$M_\nu = \begin{pmatrix} 0 & -2\phi_\nu \\ 2\phi_\nu & 0 \end{pmatrix},$$

and " $\phi^3$ " refers to a vector with entries that are cubic polynomials in the $\phi^i$. The fine structure of these entries will actually be relevant later on, but we leave it out for the present discussion.

As explained in the introduction, this formulation does not lend itself to good estimates.

2.3. Introducing the global Coulomb Gauge. We now try to modify the matrix $M_\nu$ by adding a term of the form $2\partial_\nu A$, in such a way that the resulting matrix $\tilde{M}_\nu = M_\nu + 2\partial_\nu A$ has better properties. More precisely, we want this to depend 'quadratically' on $\phi$. This can be achieved by utilizing the Coulomb Gauge condition $\sum_{j=1}^3 \partial_j \tilde{M}_j = 0$, whence $A = -\frac{1}{2} \Delta^{-1} \sum_{j=1}^3 \partial_j M_j$.

Indeed, observe that the $\tilde{M}_\nu$ satisfy the following div-curl system:

$$\sum_{i=1}^3 \partial_i \tilde{M}_i = 0, \quad \partial_\mu \tilde{M}_\nu - \partial_\nu \tilde{M}_\mu = \begin{pmatrix} 0 & 2(\phi^1_\mu \phi^2_\nu - \phi^1_\nu \phi^2_\mu) \\ -2(\phi^1_\nu \phi^2_\mu - \phi^1_\mu \phi^2_\nu) & 0 \end{pmatrix},$$

whence

$$\tilde{M}_\nu = \begin{pmatrix} 0 & 2 \sum_{i=1}^3 \Delta^{-1} \partial_i (\phi^1_\nu \phi^2_i - \phi^1_i \phi^2_\nu) \\ -2 \sum_{i=1}^3 \Delta^{-1} \partial_i (\phi^1_\nu \phi^2_i - \phi^1_i \phi^2_\nu) & 0 \end{pmatrix},$$

or in a first approximation $\tilde{M}_\nu = "D^{-1}(\phi^2)"$.

We can now set $U = e^A$ and obtain

$$U^{-1} \Box (U \phi_\alpha) = U^{-1} \Box (U) \phi_\alpha + \tilde{M}_\nu \partial^\nu \phi_\alpha + "\phi^3"$$

Of course, we use the commutativity of the Gauge group for 2-dimensional target. The difference between this wave equation for $U \phi_\alpha$ and (13) is that the nonlinearity here consists of trilinear expressions. In particular, this modification suffices to handle the case of 4+1-dimensional Wave Maps. For this, observe for example that one can easily estimate the $L^1_t L^2_x$-norm of $\tilde{M}_\nu \partial^\nu \phi_\alpha$ since this is morally $D^{-1}(\phi^2)D\phi$ and

$$\|D^{-1}(\phi^2)D\phi\|_{L^1_t L^2_x} \leq C \|\phi\|_{L^3_t L^6_x}^2 \|\phi\|_{L^\infty_t H^1_x}$$

The right-hand terms are controlled by means of Strichartz' inequalities. Similarly, one can estimate the remaining terms of the nonlinearity in the $L^1_t L^2_x$-norm. This is Shatah and Struwe's method for $\mathbb{H}^2$. One can also estimate this term using the improved bilinear Strichartz estimate for $D^{-1}(\phi^2)$ in [10], as observed by Klainerman and Rodnianski.
For the 3-dimensional case, Strichartz’ estimates alone don’t seem sufficient. This can be seen by analyzing the case when \( D^{-1}(\phi^2) \) has very low frequency while \( D\phi \) has large frequency; in order to recoup the exponential loss caused by \( D^{-1} \), one seems to be forced to employ a \( L^2_t L_x^\infty \) Strichartz estimate, which unfortunately doesn’t exist. To proceed, we need to take into account more of the special structure of the nonlinear terms.

2.4. Implementing the dynamic separation. For convenience’s sake, and as we are dealing with 2-dimensional target, introduce complex notation:

Let \( \phi_\alpha = \phi_\alpha^1 + i \phi_\alpha^2 \), and define the twisted variables best suited for dynamic separation as follows:

\[
\psi_\alpha := \psi_\alpha^1 + i \psi_\alpha^2 = e^{-i\Phi} \phi_\alpha,
\]

where \( \Phi := \Delta^{-1} \sum_{k=1}^3 \partial_k \phi_k^1 \) (\( \Delta \) stands for \( \Delta_x \)). The \( \psi_\alpha \) satisfy the following wave equation:

\[
\Box \psi_\alpha = [-i\Box \Phi - \partial_\nu \Phi \partial^\nu \Phi] \psi_\alpha + 2ie^{-i\Phi} \Delta^{-1} \sum_{k=1}^3 \partial_k [\phi_k^1 \phi_\nu^2 - \phi_k^2 \phi_\nu^1] \partial^\nu \phi_\alpha + e^{-i\Phi} \phi_3 \tag{22}
\]

which of course is just (13) spelled out more completely. At first glance the nonlinear terms here do not seem to exhibit any straightforward null structure. To reveal it, we want to decompose the variables into dynamic and elliptic components. This allows us to rewrite the nonlinear terms as sums of trilinear terms with a well-defined null structure, and terms of higher degree of linearity.

The middle term on the right-hand side of (22) will turn out to be the most difficult, and indeed both low- and high-frequency interactions seem forbidding. Before implementing and using the ‘dynamic separation’ for the \( \psi_\alpha \), we need to express this middle term purely in terms of the \( \psi_\alpha \) (up to error terms). For this, observe that we have \( \phi_k^1 \phi_\nu^2 - \phi_k^2 \phi_\nu^1 = \Im(\phi_k \phi_\nu) = \Im(\psi_k \psi_\nu) \), whence we can rewrite it as

\[
\sum_{j=1}^3 \Delta^{-1} \partial_j [\psi_j^1 \psi_\nu^2 - \psi_j^2 \psi_\nu^1] \partial^\nu \phi_\alpha + \sum_{j=1}^3 i\Delta^{-1} \partial_j [\psi_j^1 \psi_\nu^2 - \psi_j^2 \psi_\nu^1] \partial^\nu \Phi \phi_\alpha \tag{23}
\]

The 2nd term in this equation, being quadrilinear, is to be considered an error term, and turns out to be actually fairly easily controllable once the first term is dealt with.

Now consider the div-curl system satisfied by the \( \psi_\alpha \):

\[
\partial_\alpha \psi_\beta - \partial_\beta \psi_\alpha = (\partial_\alpha \phi_\beta - \partial_\beta \phi_\alpha)e^{-i\Phi} - i \sum_{j=1}^3 (\phi_\beta \Delta^{-1} \partial_j \partial_\alpha \phi_j^1 - \phi_\alpha \Delta^{-1} \partial_j \partial_\beta \phi_j^1)e^{-i\Phi} \\
= i\psi_\beta \Delta^{-1} \partial_j (\psi_\alpha^1 \psi_j^2 - \psi_\alpha^2 \psi_j^1) - i\psi_\alpha \Delta^{-1} \partial_j (\psi_\beta^1 \psi_j^2 - \psi_\beta^2 \psi_j^1) \tag{24}
\]
\[
\partial_\nu \psi^\nu = \partial_\nu \phi^\nu e^{-i\Phi} - i\phi^\nu \Delta^{-1} \partial^\nu \partial_j \phi_j^1 e^{-i\Phi} = i\psi^\nu \Delta^{-1} \partial_j (\psi_j^1 \psi_j^1 - \psi_j^2 \psi_j^2)
\]

(25)

Thus the right-hand side consists of trilinear expressions in \(\psi\), by contrast with the bilinear expressions on the right-hand side of (11)–(14).

The dynamic separation for \(\psi_\alpha\) now consists in decomposing

\[
\psi_\nu = -R_\nu \sum_{k=1}^{3} R_k \psi_k + \chi_\nu := -R_\nu \Psi + \chi_\nu
\]

(26)

\(R_\nu\) here stands for the Riesz operator \((\sqrt{-\Delta_x})^{-1} \partial_\nu\), \(\nu = 0, 1, 2, 3\). The \(\chi_\nu\) ("elliptic part") in turn are determined via a simple elliptic div-curl system, namely

\[
\partial_j \chi_j = 0
\]

(27)

\[
\partial_i \chi_\nu - \partial_\nu \chi_i = \partial_\nu \psi_\nu - \partial_\nu \psi_i
\]

(28)

This in addition to (24) implies immediately that we can write

\[
\chi_\nu = i \sum_{i,j=1}^{3} \partial_i \Delta^{-1} (\psi_\nu \Delta^{-1} \partial_j (\psi_j^1 \psi_j^1 - \psi_j^2 \psi_j^2) - \psi_i \Delta^{-1} \partial_j (\psi_j^1 \psi_j^1 - \psi_j^2 \psi_j^2))
\]

(29)

Passing to real parts and imaginary parts, we can write \(\psi_\nu^1 = -R_\nu \Psi_1 + \Re \chi_\nu\), \(\psi_\nu^2 = -R_\nu \Psi_2 + \Im \chi_\nu\), where \(\Psi^a = \sum_{k=1}^{3} R_k \psi_k^a\).

Having implemented the dynamic separation, we can now see the null-structure, corresponding to substituting the dynamic parts \(-R_\nu \Psi^a\) for \(\psi_\nu^a\), \(a = 1, 2\) in \([,]\) of the first term of (23). There results

\[
\sum_{j=1}^{3} \Delta^{-1} \partial_j [R_j \Psi^1 \psi_\nu^1 - R_j \Psi^2 \psi_\nu^2] \partial^\nu \psi_\alpha
\]

(30)

A reason for calling this term a null-form, in addition to the fact that it appears to intertwine a \(Q_0\)-structure(corresponding to \(\partial_\nu \phi \partial^\nu \psi\) with a \(Q_{\nu j}\)-structure(corresponding to \(\partial_\nu \phi \partial_\nu \psi - \partial_\nu \phi \partial_\nu \psi\)), is most directly exemplified by the following elementary identity, which we state as an easily verified lemma:
Lemma 2.5. Let \( f, g, h \) be Schwartz functions. Then we have

\[
2 \sum_{j=1}^{3} \Delta^{-1} \partial_j [R_\nu f R_j g - R_j f R_\nu g] \partial^\nu h
= \sum_{j=1}^{3} \Box [\Delta^{-1} \partial_j [\nabla^{-1} f R_j g] h] - \sum_{j=1}^{3} \Delta^{-1} \partial_j [\nabla^{-1} f R_j g] h
- \sum_{j=1}^{3} \Delta^{-1} \partial_j [\nabla^{-1} f R_j g] \Box h - \nabla^{-1} f [\Box (\nabla^{-1} g) h]
+ \nabla^{-1} f [\Box (\nabla^{-1} g) h] + \nabla^{-1} f (\nabla^{-1} g) \Box h
\]

(31)

Remark: The bilinear null form in \( \Box \) exhibits similar structure, though our formulation, which avoids the Fourier transform, is more simple and explicit.

This identity will become useful once we have reduced all entries to small modulation, i.e. to have Fourier support close to the light cone, for then the \( \Box \)-operators kick in. However, when the expression has relatively large modulation (statements such as this one are always by comparison with frequencies and modulations of the entries), one can usually estimate it in the \( \dot{X}^{\frac{3}{2}, \frac{1}{2}} \) -norm. Moreover, restricting entries to 'large' modulation allows one to estimate them in \( L^2_t L^2_x \) (using \( \dot{X}^{\frac{3}{2}, \frac{1}{2}} \) -spaces) which is often strategically advantageous.

The embedded \( Q^{2j} \) structure will become useful when both entries in the expression \([R_j \Psi^1 R_\nu \Psi^2 - R_\nu \Psi^1 R_j \Psi^2]\) have very large frequency.

Unfortunately, in spite of the conceptual simplicity of this reasoning, filling in the details is still a formidable task, in part as one has so many cases to consider.

Now consider the error terms arising upon substituting at least one \( \chi_\nu \) for \( \psi_\nu \) in the first term of (23). One thereby obtains

\[
\nabla^{-1} (\nabla^{-1} (\psi^2) \psi) R_\nu \Psi^2) \nabla_{x,t} \psi
\]

(32)

\[
\nabla^{-1} (\nabla^{-1} (\psi^2) \psi) \nabla^{-1} (\nabla^{-1} (\psi^2) \psi) \nabla_{x,t} \psi
\]

(33)

the terms being written schematically, and we have referred to (23). These terms will be totally straightforward, invoking the \( L^1_t L^2_x \) -Strichartz estimate.

Therefore, most of the work for this paper will go into proving an estimate for the crucial term (30), as well as proving that the 'twisted variables' \( \psi_\alpha \) are controlled, and in turn allow one to control the \( \phi_\alpha \), a nontrivial task on account of the complex Banach spaces used for our argument, viz. the following sections.
Finally, having disposed of the middle term in (22), we also have to control the remaining terms. Each of these has its own flavor, but there are connections between all of them. For example, the term \((\Box \Phi) \psi_\alpha\) calls for expanding \(\Box \Phi\) by means of the wave equation (13), and then invoking the dynamic separation associated with the basic div-curl system satisfied by the \(\phi_\alpha\) in order to introduce a simple \(Q_0\)-type null-structure.

This null-structure is of course already present in the next term \(\partial_\nu \Phi \partial_\nu \Phi \psi_\alpha\) in (22), where one has to use it only to deal with high-frequency interactions.

Finally, as to the term "\(\phi^{3n} e^{-i\phi}\)" this term will be trivial to estimate provided all entries are at low frequencies. Otherwise, one again has to apply dynamic separation, this time to the variables \(\phi_\alpha\), in order to split it into a \(Q_0\)-type null-form and quadrilinear error terms, which turn out to be altogether elementary. Here the fine structure of "\(\phi^{3n}\) will of course play a crucial role.

This then summarizes the basic strategy for estimating (22).

2.6. The Bootstrapping argument. In order to prove the global regularity of the \(\phi_\alpha\), we utilize a bootstrapping argument, quite similar to the one in (17). More precisely, we introduce certain translation invariant Banach spaces \(S[k]([-T, T] \times \mathbb{R}^3), N[k]([-T, T] \times \mathbb{R}^3), k \in \mathbb{Z}, T > 0\) which enjoy a list of remarkable properties.

The norms \(|| \cdot ||_{S[k]([-T, T] \times \mathbb{R}^3)}\) will be used to estimate the components at frequency \(\sim 2^k\) of the \(\phi_\alpha\), which are known to be smooth on the time interval \([-T, T]\), while the norms \(|| \cdot ||_{N[k]([-T, T] \times \mathbb{R}^3)}\) will be used to estimate the components at frequency \(\sim 2^k\) of the nonlinearity, again restricted to and smooth on the time interval \([-T, T]\). Of course, \(|| \cdot ||_{S[k]}\) will have to majorize the energy \(|| \cdot ||_{H^{1/2}}\) as well as a certain range of Strichartz norms, all applied to functions microlocalized at frequency \(\sim 2^k\).

Our goal will be to bootstrap each of the norms \(|| P_k \phi_\alpha ||_{S[k]([-T, T] \times \mathbb{R}^3)}\). As a matter of fact, we will only have to bootstrap \(|| P_k \phi_\alpha ||_{S[0]([-T, T] \times \mathbb{R}^3)}\), because the \(S[k]\) will be constructed compatible with 'dilations' compatible with the div-curl system (11)-(14): denoting \(\phi_\lambda := 2^k \phi(x/2^k)\), we will have \(|| P_{k+\lambda} \phi_\lambda ||_{S[k+\lambda]([-T, T] \times \mathbb{R}^3)} = || P_k \phi ||_{S[k]([-T, T] \times \mathbb{R}^3)}\).

The \(S[k]\) and \(N[k]\) (leaving out the time-parameter \(T\) for simplicity’s sake) will be related by the fundamental energy inequality:

\[
|| P_k \phi ||_{S[k]([-T, T] \times \mathbb{R}^3)} \leq C \left( || \Box P_k \phi ||_{N[k]([-T, T] \times \mathbb{R}^3)} + || P_k \phi [0] ||_{H^{1/2} \times H^{-1/2}} \right)
\]

(34)

where \(C\) is independent of \(T\). In order to use this inequality, we need to estimate the \(N[k]\)-norm of the nonlinearity. For this, it will be important to us amongst other things that there are

1. null-form estimates of the form

\[
|| P_0 \left[ R_{A_0} P_{k_1} \phi \partial \nu P_{k_2} \psi \right] ||_{N[0]} \leq C 2^{-\delta \max\{k_1, 0\}} || P_{k_1} \phi ||_{S[k_1]} || P_{k_2} \psi ||_{S[k_2]}, \quad \delta > 0
\]

(35)
2. Bilinear estimates that make up for the missing $L^2_t L^\infty_x$-estimates. These come about by using null frame spaces, and have roughly the form
\[ ||P_{k_1} \phi P_{k_2} \psi||_{L^2_t L^2_x} \leq C 2^{\frac{k_1}{2} - \frac{k_2}{2}} ||P_{k_1} \phi||_{S[k_1]} ||P_{k_2} \psi||_{S[k_2]} \]
provided $\phi, \psi$ are microlocalized on small caps whose distance is at least comparable to their radius, and provided their Fourier support lives fairly closely to the cone.

3. Trilinear estimates: the preceding observations will play a role in establishing the genuine trilinear estimate
\[ ||P_0 \sum_{j=1}^3 \triangle^{-1} \partial_j [R_\nu P_{k_1} \psi_1 R_{\nu} P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_j P_{k_2} \psi_2] \partial^\nu P_{k_3} \psi_3||_{N[0]} \]
\[ \leq C 2^{-\delta_1} 2^{-\delta_2} \prod ||P_{k_j} \phi_j||_{S[k_j]}, \delta_1, \delta_2 > 0 \]
This will be the crux of the paper.

4. The $S[k]$ have to be well-behaved under the Gauge Change. In particular, we need an assertion of the form that provided $||P_k \phi||_{S[k]}$ are small in a suitable sense, then so are $||P_k |f(\nabla^{-1} \partial)\phi||_{S[k]}$, where $\nabla^{-1}$ stands for a linear combination of operators of the form $\triangle^{-1} \partial_j$, and $f(x)$ is a smooth function all of whose derivatives are bounded.

3. Technical preparations

The spaces $S[k], N[k]$ and many of their properties were considered in Tao’s seminal paper [14], although their origins can be traced back to Tataru’s [19]. Most of this section(except the trilinear inequality and the Gauge Change result) is due to these 2 authors; we will therefore be rather brief with the definitions.

First, we introduce Tao’s concept of frequency envelope, as in [15], [16]: for any section(except the trilinear inequality and the Gauge Change result) is due to these authors; we will therefore be rather brief with the definitions. For any Schwartz function $\psi$ on $\mathbb{R}^3$, we consider the quantities
\[ c_\sigma := \left( \sum_{k \in \mathbb{Z}} 2^{-\sigma |a-k|} ||P_k \psi||_{\dot{H}^{\frac{\sigma}{2}}}^2 \right)^{\frac{1}{2}} \]
Here $P_k, k \in \mathbb{Z}$ are the standard Littlewood-Paley operators that localize to frequency $\sim 2^k$, i.e. they are given by Fourier multipliers $m_k(|\xi|) = m_0(\frac{|\xi|}{2^k})$, where $m_0(\lambda)$ is a smooth function compactly supported within $\frac{1}{2} \leq \lambda \leq 2$ with $\sum_{k \in \mathbb{Z}} m_0(\frac{1}{2^k}) = 1, \lambda > 0$.
The $\sigma > 0$ is chosen to be smaller than any of the exponential decays occurring later in the paper: E.g. $\frac{1}{1000}$ would suffice. We note that all of the generic constants $C$ occurring in the sequel depend at most on this envelope exponent $\sigma$.

Note that
\[ c_k 2^{-\sigma |a-k|} \leq c_\sigma \leq 2^{\sigma |a-k|} c_k \]
as well as $\sum_{k \in \mathbb{Z}} c_k^2 \leq C ||\psi||^2_{\dot{H}^{\frac{\sigma}{2}}}$. One reason for why this concept is useful is that provided we know that the frequency localized components $P_k \rho$ for some other Schwartz function $\rho$ on $\mathbb{R}^3$ (think: the time-evolved Wave Map) have $\dot{H}^{\frac{\sigma}{2}}$-norm bounded by a multiple $C c_k$, we can immediately bound the $\dot{H}^{\frac{\sigma}{2} + \epsilon}$-norm of $\rho$ for $\epsilon > 0$ small enough. This will allow us...
later to continue the Wave Map, by referring to local well-posedness of the div-curl system \( 1 \) in \( H^{\frac{5}{2}} \), and finite speed of propagation.

We introduce the following norms on frequency localized Schwartz functions on \( \mathbb{R}^{3+1} \) for our bootstrapping argument: for every \( l > 10 \), choose a covering \( K_l \) of \( S^2 \) by finitely overlapping caps \( \kappa \) of radius \( 2^{-l} \). This is to be chosen such that the set of concentric caps with half the radius still covers the sphere. Now let

\[
||| \psi \||_{S[k]} := ||| \nabla_{t,x} \psi |||_{L_t^\infty L_x^2} + ||| \nabla_{t,x} \psi |||_{X_k^{-\frac{3}{2},\frac{1}{2}}} + \sup_{l > 10} \sup_{\kappa \in K_l} \sum_{\kappa \in K_l} ||| \tilde{P}_k, \pm \kappa Q_{<k-2l}^\pm \psi |||_{S[k, \kappa]}^2 \right)^{\frac{1}{2}} \tag{40}
\]

where it is understood that \( \psi \) lives at frequency \( \sim 2^k, k \in \mathbb{Z} \). The operators \( \tilde{P}_k, \kappa \) are given by symbols \( \tilde{m}_k(|\xi|) a_\kappa(\frac{\xi}{\kappa}) \), where \( a : S^2 \to \mathbb{R} \) is a smooth function with support contained in the concentric cap inside \( \kappa \) with half the radius of \( \kappa \), and \( \tilde{m}_k \) localizes frequency to size \( \sim 2^k \) and satisfies \( \tilde{m}_k m_k = m_k \), where \( m_k \) is the multiplier chosen above. We also require that \( \sum_{\kappa \in K_l} \tilde{P}_k, \kappa = \tilde{P}_k \), the latter being defined in the obvious way.

\( Q^\pm_{<k-2l} \) localizes the modulation, i.e. \( ||| \tau - |\xi| || \), to size \( < 2^{k-2l} \) and also restricts the Fourier support to \( \tau > 0 \), i.e. to the upper or lower half-space. More precisely, it is given by the multiplier \( \sum_{i < k - 2l} m_i (||| \tau - |\xi| ||) \chi_{\kappa} (\pm \tau) \).

The norm \( ||| \phi \||_{X_k^{-\frac{3}{2},\frac{1}{2}}} \) refers to \( 2^{-\frac{k}{2}} \sum_{j \in \mathbb{Z}} 2^j \| Q_j \phi \|_{L_t^2 L_x^2} \).

As we are in 3 space dimensions, we strengthen \( S[k, \kappa] \) with respect to Tao’s original definition to also contain Strichartz norms such as \( ||| \cdot |||_{L_t^1 L_x^1}, 2^\frac{k}{2} ||| \cdot |||_{L_t^6 L_x^6}, 2^\frac{k}{2} ||| \cdot |||_{L_t^4 L_x^4} \). Hence we define

\[
||| \psi \||_{S[k, \kappa]} := 2^{\frac{k}{2}} ||| \psi |||_{NFA^*_{\kappa}} + |\kappa|^{-\frac{1}{2}} 2^{-\frac{k}{2}} ||| \psi |||_{PW_{\kappa}} + 2^k ||| \psi |||_{L_t^\infty L_x^2} + \sup_{\frac{2k}{p} + \frac{k}{q} \leq \frac{5}{2}, p > 2 + \mu} 2^{k(\frac{2k}{p} + \frac{k}{q} - 1)} ||| \psi |||_{L_t^p L_x^q} \tag{41}
\]

for some very small \( \mu > 0 \), e.g. \( \mu < \frac{1}{100} \) will do. The definitions of the individual ingredients are as follows:

\[
||| \psi \||_{NFA^*_{\kappa}} := \sup_{\omega \notin 2\kappa} dist(\omega, \kappa) ||| \phi |||_{L_t^\infty L_x^2} \tag{42}
\]

Here \((t_\omega, x_\omega)\) refer to null-frame coordinates, i.e. \( t_\omega = (t, x) \cdot \frac{1}{\sqrt{2}} (1, \omega), x_\omega = (t, x) - t_\omega \frac{1}{\sqrt{2}} (1, \omega) \).
Also, define $||.||_{PW[k]}$ to be the norm associated with the atomic Banach space whose atoms are the set $A$ of all Schwartz functions $\psi$ with $||\psi||_{L^2_TL^2_x} \leq 1$ for some $\omega \in \kappa$. In other words,

$$||\psi||_{PW[k]} = \inf \{ \lambda | \exists \{0 \leq \lambda_i \leq 1\}, \{ \psi_i \} \subset A, 1 \leq i \leq N \text{ s.t. } \sum \lambda_i = 1, \lambda \sum \psi_i = \psi \} \quad (43)$$

Of course, the Banach space $S[k]$ is obtained by completing the Schwartz functions on $\mathbb{R}^{3+1}$ with respect to $||.||_{S[k]}$.

Next, we will place frequency localized pieces of the nonlinearity into the following spaces $N[k]$, again introduced by Tao: they are the atomic Banach spaces whose atoms are

1. Schwartz functions $F$ at frequency between $2^{k-4}$ and $2^{k+4}$ with $||F||_{L^1_TL^2_x} \leq 2^{\frac{k}{2}}$.
2. Schwartz functions $F$ with frequency between $2^{k-4}$ and $2^{k+4}$ and modulation between $2^{j-5}$ and $2^{j+5}$ such that $||F||_{L^2_TL^2_x} \leq 2^{\frac{j+6}{2}}$.
3. Schwartz functions $F$ for which there exists a number $l > 10$ and Schwartz functions $F_\kappa$ with Fourier support in the region $\{(\tau, \xi) | \pm \tau > 0, ||\tau|-|\xi|| \leq 2^{k-2l}, 2^{k-4} \leq |\xi| \leq 2^{k+4}, \Theta \leq \frac{1}{2}\kappa \}$ such that $F = \sum_{\kappa \in K_l} F_\kappa$ and $(\sum_{\kappa \in K_l} ||F_\kappa||_{NFA[k]}^2)^{\frac{1}{2}} \leq 2^{\frac{k}{2}}$. Here $\Theta = \frac{r_2}{|\tau||\xi|}$ and $NFA[k]$ is the dual space of $NFA[k]^*$, i.e.

$$||\psi||_{NFA[k]} = \inf_{\omega \in 2\kappa} \frac{1}{\text{dist}(\omega, \kappa)} ||\psi||_{L^1_TL^2_x} \quad (44)$$

The reason for introducing these spaces is that they allow one to establish essential bilinear estimates, in particular the following:

$$||\phi \psi||_{L^2_TL^2_x} \leq C \frac{2^{\frac{k}{2}}|\kappa'|^{\frac{1}{2}}}{\text{dist}(\kappa, \kappa')} ||\phi||_{S[k, \kappa]} ||\psi||_{S[k', \kappa']} \quad (45)$$

$$||\phi \psi||_{NFA[k]} \leq C \frac{2^{\frac{k}{2}}|\kappa'|^{\frac{1}{2}}}{\text{dist}(\kappa, \kappa')} ||\phi||_{L^2_TL^2_x} ||\psi||_{S[k', \kappa']} \quad (46)$$

These follow immediately from the definitions, or as in $\Box$.

Another reason for including the null-frame version $L^2_TL^\infty_x$ in $S[k, \kappa]$ instead of the forbidden $L^2_TL^\infty_x$ is that we have the following inequality valid for all Schwartz functions with Fourier support localized along $\kappa$ and at frequency $\sim 2^k$:

$$||\phi||_{S[k, \kappa]} \leq C |||\phi||_X^{\frac{1}{2}, \frac{1}{4}} \quad (47)$$

Of course, in light of the ingredients of $S[k]$, this is a substitute for the missing $L^2_TL^\infty_x$-Strichartz estimate.
Moreover, it is easily seen that the $S[k]$-norm of $P_k \phi$ as defined above majorizes $\sup_{\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, K > p > 2 + \mu} 2^{k\left(\frac{1}{2} + \frac{1}{q} - 2\right)} ||\nabla_{x,t} P_k \phi||_L^q L^p_x$, $K$ an arbitrarily large constant, up to multiplication with a constant (depending on $K$). Indeed, splitting $P_0 \phi = P_0 Q_{<-100} \phi + P_0 Q_{\geq -100} \phi$, we can control the 2nd high-modulation summand by invoking the fact that

$$||P_0 Q_{< -100} \phi||_L^p L^q_t \leq C 2^{j\frac{1}{2} - \frac{1}{4}} ||P_0 Q_j \phi||_L^q L^2_x \leq 2^{-\frac{1}{4}} C ||P_0 Q_j \phi||_{X^0_{\frac{1}{2} + \frac{1}{4}, \infty}} \quad (48)$$

for $p \leq q$ by Bernstein’s inequality, which gives a uniform exponential gain for $j \geq -100$ provided $p \leq q$, $K > p > 2 + \mu$, and interpolating between the inequality thus gotten for $L^p_t L^q_x$ and the inequality $||P_0 Q_j \phi||_{L^p_t L^q_x} \leq C 2^{j\frac{1}{2}} ||P_0 Q_j \phi||_L^q L^2_x$. For $P_0 Q_{<-100} \phi$, we split the unit sphere into caps of size $2^{-50}$, say, observe that

$$||P_0 Q_{< -100})||_L^p L^q_t = ||(P_0 Q_{< -100} \phi)^2||_L^p L^q_t = \left( \sum_{\kappa \in K_{-50}} ||P_{0,\kappa} Q_{<-100} \phi||_L^p L^q_x \right)^2 \leq C \left( \sum_{\kappa \in K_{-50}} ||P_{0,\kappa} Q_{<-100} \phi||_L^p L^q_x \right)^2 \quad (49)$$

Since we will be implementing a bootstrapping argument, we can only assume the a priori existence of a solution on a finite time interval $[-T, \bar{T}]$. We therefore need to localize the above (frequency-localized) norms to this interval. To wit

$$||P_k \phi||_{S[k]([-T, \bar{T}] \times \mathbb{R}^3)} := \inf_{\psi \in S(\mathbb{R}^{3+1}), \psi|_{[-T, \bar{T}]} = \phi} ||P_k \psi||_{S[k](\mathbb{R}^{3+1})} \quad (50)$$

$$||P_k \phi||_{N[k]([-T, \bar{T}] \times \mathbb{R}^3)} := \inf_{\psi \in S(\mathbb{R}^{3+1}), \psi|_{[-T, \bar{T}]} = \phi} ||P_k \psi||_{N[k](\mathbb{R}^{3+1})} \quad (51)$$

We can now formulate the following energy inequality, which is the essential link between the $N[k]$ and $S[k]$-norm that will allow us to finish the bootstrapping argument:

$$||P_k \phi||_{S[k]([-T, \bar{T}] \times \mathbb{R}^3)} \leq C \left( ||\Box P_k \phi||_{N[k]([-T, \bar{T}] \times \mathbb{R}^3)} + ||\phi[0]||_{H^{\frac{1}{2}} \times H^{\frac{1}{4}} \times \mathbb{R}^{3+1}} \right) \quad (52)$$

where $C$ is independent of $T$. This is proved as in [10]; the only difference between our $S[k, \kappa]$ norm and Tao’s $S[k, \kappa]$-norm (other than the different scaling, which doesn’t affect the proof) is the addition of $\sup_{\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, p > 2 + \mu} 2^{k\left(\frac{1}{2} + \frac{1}{q} - 1\right)} ||P_k \phi||_L^q L^p_x$. However, this is majorized by $||\phi||_{N_{\frac{3}{2}, \frac{1}{2}, \mathbb{R}^{3+1}}}$, and moreover well behaved under multiplication with $L^\infty$ functions, which is all that is required to prove the above inequality, viz. Tao’s proof.

It is important that the $S[k]([-T, \bar{T}] \times \mathbb{R}^3)$-norms of a Schwartz function are in a sense uniformly lower semicontinuous with
respect to $T$, as demonstrated in [16]. In particular, we may assume that $T > 0$ has been chosen such that the component functions $\phi$ of our Wave Map satisfy

$$
\|P_k \phi\|_{[S[k],[\cdot,T,T] \times \mathbb{R}^3]} \leq C c_k \tag{53}
$$

where $c_k$ is a frequency envelope associated with the initial conditions $\phi[0] \times \partial_t \phi[0]$ as above, i.e.,

$$
c_k := \left( \sum_{k'} 2^{-\delta k' - k} (\|P_{k'} \phi\|_{H^{-\frac{1}{2}}} + \|P_k \partial_k \phi\|_{H^{-\frac{1}{2}}}^2)^{\frac{1}{2}} \right)^{\frac{1}{2}} \tag{54}
$$

Moreover, since we assume that $\phi$ is rapidly decaying in space directions, we can construct a Schwartz function $\tilde{\phi}$ with $\tilde{\phi}|_{[-T,T]} = \phi$ and such that $\|P_k \tilde{\phi}\|_{S[k]} \leq 2 C c_k$.

This is achieved by using a partition of unity. We will always substitute $\tilde{\phi}$ for $\phi$ when making actual estimates.

**Notation** The Riesz operators $R_\nu$, $\nu \in \{0, 1, 2, 3\}$, refer to operators $\partial_\nu (\sqrt{-\Delta_x})^{-1}$. We usually omit the subscript for operators like $\nabla_x$, $\Delta_x$, understanding that they refer only to space variables.

The symbol $\nabla^{-1}$ is either a shorthand for an operator $\Delta^{-1} \partial_i$, or else refers to $(\sqrt{-\Delta})^{-1}$, depending on the context.

We use the notation $P_{k+O(1)} = \sum_{k_1=k+O(1)} P_{k_1}$, $Q_{j+O(1)} = \sum_{j_1=j+O(1)} Q_{j_1}$. Also, $\|\phi\|_{S[k+O(1)]} = \sum_{k_1=k+O(1)} \|P_{k_1} \phi\|_{S[k_1]}$ etc.

The following terminology, introduced by T.Tao in [16], shall be useful in the future: we call a Fourier multiplier disposable if it is given by convolution with a translation invariant measure of mass $\leq O(1)$. In particular, operators such as $P_k$, $P_k Q_{<j}$ where $j \geq k + O(1)$ are disposable, see above reference. By contrast, $Q_j$ is not disposable. However, it acts boundedly on Lebesgue spaces of the form $L^p_t L^2_x$.

Whenever we consider an expression of the form $P_0(AB|CD|)$, for example, we shall refer to $A$, $B$, $C$, $D$ as inputs and the whole expression as output. Also, when referring to $[\cdot]$, we mean $|CD|$, while $(\cdot)$ would refer to $P_0(AB|CD|)$; thus the shape of brackets matters in the discussion. When considering a part of the whole expression such as $|CD|$, we may also refer to this as output, and $C$, $D$ as inputs, depending on the context.
Two Important Inequalities

Bernstein’s inequality in the form \(||P_k \phi||_{L^p_x} \leq C 2^{\frac{2p}{p-1} - \frac{2k}{p}} ||P_k \phi||_{L^2_x}\) or variations thereof will be frequently used in the sequel.

Moreover, the following improvement of Bernstein’s inequality which is a consequence of Strichartz’ inequality shall be used later on:

\[ ||P_k Q_j \phi||_{L^2_t L^\infty_x} \leq C 2^{\frac{1}{2} \min(0, j-k)} 2^{\frac{3k}{p}} ||P_k Q_j \phi||_{L^2_x} \]

For this see [16]. The intuition here is that Strichartz estimates in the form \(2^k(\frac{2}{p} + \frac{q}{q-1}) ||P_k Q_j \phi||_{L^p_t L^q_x} \leq C ||P_k Q_j \phi||_{X^{\frac{1}{p} + \frac{1}{q-1} - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}}_x}\) for a Strichartz pair \((p, q)\), \(p \leq q\) can be interpreted as an improvement over Bernstein’s inequality; indeed, one gains exponentially in \(j - k\), if this is small.

Summary of the key properties satisfied by these spaces

One difficulty in the present approach consists in showing that the ”renormalized \(\tilde{\phi}\)”, i.e. \(\psi := e^{-i \Delta^{-1} \sum_{j=1}^3 \partial_j \tilde{\phi}_j} \tilde{\phi}\) is under approximately the same frequency envelope as \(\phi\), where \(\tilde{\phi}, \tilde{\phi}_j\) are (real-valued) \(\mathcal{S}(\mathbb{R}^{3+1})\) functions agreeing with \(\phi, \phi_j\) on \([-T, T]\) and for which the \(S[k]\)-norms of the frequency localized pieces sit under approximately the same frequency envelope, and later on translate the improved estimates for \(\psi\) back to \(\phi\). This step is trivial in 4 dimensions and higher, as one doesn’t have to invoke the Fourier transform. However, it becomes rather delicate in our present setting. For this, we have the following theorem:

**Theorem 3.1.** Let \(\tilde{\phi}_j\) be as above, and \(\psi \in \mathcal{S}(\mathbb{R}^{3+1})\) satisfy \(||P_k \psi||_{S[k]} \leq \mu c_k\). Then we have

\[ ||P_k (e^{i \sum_{j=1}^3 \Delta^{-1} \partial_j \tilde{\phi}_j}) \psi||_{S[k]} \leq C \mu c_k \]

This theorem will follow immediately from the following natural generalization:

**Proposition 3.2.** Let \(f(x)\) be a smooth function with all derivatives up to and including fourth order bounded. Then, under the same assumptions as in the theorem, we have

\[ ||P_k (f(\Delta^{-1} \sum_{j=1}^3 \partial_j \tilde{\phi}_j)) \psi||_{S[k]} \leq C \mu c_k \]

The proof of this Proposition, as well as the following technical lemmata and propositions, shall be relegated to a later section.

\(^5\)Note that the \(S[k], N[k]\) are conjugation invariant. Thus we can always find real-valued extensions of our component functions with the required properties.
In addition to the preceding "Gauge-Change"-estimate, we need to utilize slightly modified versions of bilinear estimates proved in \[10\], and most importantly the following version of Tao’s lemma (14.1):

**Lemma 3.3.** (T.Tao) Let \( j \leq \min(k_1,k_2) + O(1) \). Then
\[
||P_k(F\psi)||_{N[k]} \leq C2^{-\delta_1|k-\max(k_1,k_2)|}2^{-\delta_2|j-\min(k_1,k_2)|}||F||_{\dot{X}^{\frac{1}{2}}_{k_1} \times X^{\frac{1}{2}}_{k_2}} ||\psi||_{S[k_2]}
\] (58)

for all Schwartz functions \( F \) with Fourier support at frequency \( 2^{k_1} \) and modulation \( 2^j \) while \( \psi \) is at frequency \( 2^{k_2} \), \( \delta_1, \delta_2 > 0 \). Moreover, we also have
\[
||\nabla P_k(F\psi)||_{N[k]} \leq C2^{-\delta_1|k-\max(k_1,k_2)|}2^{-\delta_2|j-\min(k_1,k_2)|}||F||_{\dot{X}^{\frac{1}{2}}_{k_1} \times X^{\frac{1}{2}}_{k_2}} ||\nabla \psi||_{S[k_2]}
\] (59)

The only difference here is that our spaces \( S[k], N[k] \) are scaled-down versions of Tao’s spaces. In particular, for high-high interactions, the above version as expressed in the first inequality is slightly stronger than the original version. Nevertheless, the proof follows Tao’s original proof almost identically, and only deviates in one small detail.

The intuition behind this rather technical estimate is that it allows one for example to play the (small) size of the modulation of \( F \) against the frequency of \( \psi \). This shall be important for estimating the trilinear null-form \( \langle 50 \rangle \), in particular in the delicate case when all inputs have relatively low modulation but still too large to employ successfully the full algebraic null-structure, see e.g. the proof of lemma 5.3 in the last section.

This lemma entails the following fundamental null-form estimate, due to Tao, which we again modify for our scaled-down spaces \( S[k] \).

**Lemma 3.4.** (T.Tao) Let \( \phi, \psi \) be Schwartz functions on \( \mathbb{R}^{3+1} \). Then letting \( k_1 = k_2 + O(1) \geq O(1) \), we have
\[
||P_0[R_\nu P_k \phi \partial^\nu P_2 \psi]||_{N[0]} \leq C2^{-\delta k_1}||P_{k_1} \phi||_{S[k_1]}||P_{k_2} \psi||_{S[k_2]}
\] (60)

for some \( \delta > 0 \).

In addition to the above "high-high" version, we will also need the following "low-high" version, proved by T.Tao:

**Lemma 3.5.** Let \( k_1 = O(1), k_2 < O(1) \). Then
\[
||P_0 \nabla_x [R_\nu P_k \phi R^\nu P_2 \psi]||_{N[0]} \leq C||P_{k_1} \phi||_{S[k_1]}||P_{k_2} \psi||_{S[k_2]}
\] (61)

We will need the following fundamental bilinear inequality, which is again a slight modification of an inequality of Tao:

**Lemma 3.6.** Let \( \phi, F \) be Schwartz functions, and \( k_1 = k_2 + O(1) \). Then we have
\[
||P_0(P_{k_1} \phi P_{k_2} F)||_{N[0]} \leq C2^{-\delta k_1}||P_{k_1} \phi||_{S[k_1]}||\nabla_x (P_{k_2} F)||_{N[k_2]}
\] (62)
for some $\delta > 0$. Moreover, we have the estimate
\begin{equation}
\|P_0 \nabla_x (\phi P_{k_2} F)\|_{N[0]} \leq C(\|\phi\|_{L^\infty_t L^\infty_x} + \sup_k \|P_k \nabla_x \phi\|_{S[k]})\|\nabla_x (P_{k_2} F)\|_{N[k_2]}
\end{equation}

**Proof** The proof of this is again almost identical to Tao’s proof; in particular, for the first inequality, one simply has to substitute the above version of Tao’s lemma (14.1) in the appropriate places.

Finally, we have the important trilinear inequality

**Proposition 3.7.** Let $\psi_l, l = 1, 2, 3$ be Schwartz functions on $\mathbb{R}^{3+1}$. We then have the estimate
\begin{equation}
\|P_0 \left( \sum_{j=1}^{3} \Delta^{-1} \partial_j [R_{1j} P_{k_1} \psi_1 R_{j} P_{k_2} \psi_2 - R_{j} P_{k_1} \psi_1 R_{1j} P_{k_2} \psi_2] \partial_3 P_{k_3} \psi_3] \right)\|_{N[0]}
\leq C2^{-\delta_1 |k_1 - k_2|} 2^{-\delta_2 |k_3|} \prod_{l=1}^{3} \|P_k \psi_l\|_{S[k_l]}
\end{equation}

for appropriate constants $\delta_1, \delta_2 > 0$. 
4. Proof of Theorem 2.2

As explained in section 2, the proof of the Theorem can be reduced to the proof of global regularity of the $\phi_\alpha$, or $\phi$ for short. In order to formulate a 'bootstrapping Proposition' for $\phi$, we first need to verify that smallness of the initial conditions as stated in Theorem 2.2 implies a corresponding statement for the initial conditions of $\phi$.

**Lemma 4.1.** Under the conditions of Theorem 2.2, we have $||\phi[0] \times \partial_t \phi[0]||_{H^{1/2} \times H^{-1/2}} < C\epsilon$.

**Proof:** Note that the wave equation satisfied by $(x, y)$ is

$$\Box x = \frac{2}{y} \partial_x \partial_y x, \Box y = -\frac{1}{y} (\partial_x \partial_y x - \partial_y \partial_x y)$$  \hspace{1cm} (65)

We need to show that $||\partial_t (\frac{\partial x}{y})||_{H^{-1/2}}$ is small, the other cases being similar or simpler. Now

$$\partial_t (\frac{\partial x}{y}) = \sum_{i=1}^{3} \partial_t (\frac{\partial x}{y}) - \sum_{i=1}^{3} \frac{\partial_x x \partial_y y}{y^2}$$  \hspace{1cm} (66)

Clearly, the first term has small $H^{-1/2}$-norm by our assumptions. As to the 2nd, this is a simple consequence of $||fg||_{H^{-1/2}} \leq C||f||_{H^{1/2}} ||g||_{H^{1/2}}$.

We now formulate the bootstrapping argument implying global regularity of the $\phi$ for suitably small initial conditions:

**Proposition 4.2.** Let the smooth functions $\phi_\alpha$ satisfy the system (11)-(14) on $[-T, T] \times \mathbb{R}^3$, with initial conditions as in the preceding lemma. Assume that

$$||P_k \phi_\alpha||_{S[k([-T, T] \times \mathbb{R}^3)} \leq M c_k$$  \hspace{1cm} (67)

for a frequency envelope $c_k$ covering the initial data. Then we have

$$||P_k \phi_\alpha||_{S[k([-T, T] \times \mathbb{R}^3\}} \leq \frac{M}{2} c_k$$  \hspace{1cm} (68)

provided that $\epsilon$ is small enough, independently of $T$, and $M \gg 1$.

**Remark:** Global regularity of $\phi$ follows now via the local result in [3]. See also [3].
Global regularity of wave maps from $\mathbb{R}^{3+1}$ to $\mathbb{H}^2$

Proof Recall the quantities $\psi_\alpha := e^{-i\Phi}(\phi_\alpha^1 + i\phi_\alpha^2)$ introduced in section 2, where $\Phi = \Delta^{-1}\sum_{j=1}^{3}\partial_j\phi_j^1$. They satisfy the equation

$$\square\psi_\alpha = I + II + III$$

where

$$I = [-i\square\Phi - \partial_\nu\Phi\partial^\nu\Phi]\psi_\alpha$$

$$II = 2i(\Delta^{-1}\sum_{j=1}^{3}\partial_j[\psi_j^1\psi_\nu^2 - \psi_j^2\psi_\nu^1]\partial^\nu\psi_\alpha + i\sum_{j=1}^{3}\Delta^{-1}\partial_j[\psi_j^1\psi_\nu^2 - \psi_j^2\psi_\nu^1]\partial^\nu\Phi\psi_\alpha)$$

$$III = e^{-i\Phi}[\phi_\alpha^1(\phi_\nu^1\phi_\alpha^1 + \phi_\nu^2\phi_\alpha^2) + 2i\phi_\nu^1(\phi_\nu^1\phi_\alpha^2 - \phi_\alpha^1\phi_\nu^2)]$$

We now consider each of these in turn, starting with the most difficult and pivotal term II. Recall from last section that we have to localize to frequency $\sim 2^0$ and evaluate the $N[0]$-norm. The atomic nature of this space implies that we can estimate in any of the norms $L^1_tL^4_x$, $X_0^{-\frac{1}{2},-\frac{1}{2}}$, as well as the complicated norm involving null-frame spaces, i.e. the third class of atoms defining $N[0]$.

In order to make estimates, we have to substitute Schwartz functions for the inputs of the nonlinearity coinciding with them on the time interval $[-T,T]$, as well as having frequency localized components whose $S[k]$-norms are bounded by a multiple of the frequency envelope $M\nu_k$. In the sequel, we shall not make a distinction between these extensions and the local Wave Map, implicitly carrying out the above substitution immediately before estimating and after having reformulated the individual ingredients of the nonlinearity to become amenable to estimates.

4.3. Treating II: First term of II. Recall from section 2 that using dynamic separation, we have reduced this to estimating (32), (33) as well as the difficult (30). Note that on account of the $L^4_xL^4_x$-boundedness of $\nabla^{-1}(\nabla^{-1}(\psi^2)\psi)$ (which of course follows from Sobolev’s inequality as well as passing from frequency localized pieces to the full function via the fundamental theorem of Littlewood-Paley theory), we can immediately reduce control of (33) to estimating (32).

As to the latter, we distinguish between low-high, high-high and high-low interactions, i.e. reducing $\nabla_{x,t}\psi$ to medium, high and low frequency:

Low-High interactions: we use here the fact that $L^4_xL^4_x$, the latter a Lorentz space. Also, we utilize the improved Hardy-Littlewood-Sobolev inequality for these spaces:
\[ ||P_0||_{O(1)} \nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\psi^2)R_\nu \Psi^a)P_{O(1)} \nabla_{x,t} \psi)||_L^1L_x^2 \leq C||P_{O(1)} \nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\psi^2)\psi)R_\nu \Psi^a)\nabla_{x,t} \psi||_L^pL_x^2 \leq C||\nabla^{-1}(\psi^2)\psi||_{L_t^1L_x^2}^2 ||R_\nu \Psi^a||_{L_t^1L_x^2} ||P_{O(1)} \nabla_{x,t} \psi||_L^\infty L_x^2 \leq C||\nabla^{-1}(\psi^2)||_{L_t^1L_x^2} \leq ||P_{O(1)} \nabla_{x,t} \psi||_L^\infty L_x^2 \]

High-High interactions: This is more elementary. Note that we utilize the exponential decay provided by \( ||P_k \psi||_{L_x^2} \leq C2^{-\frac{k}{2}} ||P_k \phi||_{H_x^\frac{3}{2}} \), provided \( k > O(1) \), as well as Bernstein’s inequality to majorize \( L_t^1L_x^2 \) of the output by a multiple of \( L_t^1L_x^2 \) of the output, which lives at frequency \( \sim 1 \):

\[
\begin{align*}
|| \sum_{k \geq O(1)} P_0(\nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\psi^2)\psi)R_\nu \Psi^a)\nabla_{x,t} P_k \psi)||_L^1L_x^2 & \leq C \sum_{k \geq O(1)} 2^{-k} ||P_{k+O(1)}(\nabla^{-1}(\nabla^{-1}(\psi^2)\psi)R_\nu \Psi^a)||_{L_t^1L_x^2} ||\nabla_{x,t} \psi||_L^\infty L_x^2 \\
& \leq C \sum_{k \geq O(1)} M^{5} 2^{-\frac{k}{2}} c_k \leq CM^{5} c_0
\end{align*}
\]

High-Low interactions: this is even more elementary as an extra derivative \( (\nabla_{x,t}) \) falls on a low frequency function, while the first \( \nabla^{-1} \) is harmless. We therefore skip this case.

Now we estimate the null-form, which is straightforward in light of Proposition 8.7, the exponential gains in the statement of this Proposition allow one to counteract the 'exponential loss' inherent in the frequency envelope, provided its exponent \( \sigma \) is chosen small enough:

\[
\begin{align*}
||P_0(3 \sum_{j=1} \Delta^{-1} \partial_j (R_\nu \Psi^1 R_j \Psi^2 - R_j \Psi^1 R_\nu \Psi^2)\partial^\nu \psi_\alpha)||_N[0] & \leq \sum_{k_1,k_2,k_3} ||P_0(3 \sum_{j=1} \Delta^{-1} \partial_j (R_\nu P_{k_1} \Psi^1 R_j P_{k_2} \Psi^2 - R_j P_{k_1} \Psi^1 R_\nu P_{k_2} \Psi^2)\partial^\nu P_{k_3} \psi_\alpha)||_N[0] \\
& \leq \sum_{k_1,k_2,k_3} C2^{-\delta_1|k_1-k_2|}|2^{-\delta_2|k_3|} M^3 \prod_{a=1}^{3} c_{k_a} \\
& \leq CM^3 \sum_{k} c_k^2 c_0 \leq CM^3 \epsilon c_0
\end{align*}
\]
provided we choose the $\sigma$ in the definition of the frequency envelope (viz. section 3) much smaller than $\min\{\delta_1, \delta_2\}$.

**Second term of $\Pi$**

Applying dynamic separation to $[,]$ as explained in section 2.4, we decompose this term into a term with null-structure, as well as error terms of high linearity. Of course one is tempted here to simply use the $L^4_t L^4_x$-Strichartz estimate; however, in a sense our strategy of controlling each frequency localized piece is backfiring here, as we ‘have to recover the initial frequency envelope’ in our estimates, which requires in particular obtaining exponential gains for high-frequency interactions. The $L^4_t L^4_x$-estimate is too weak for this, and we have to invoke the hidden null-structure. It goes without saying that being able to prove a genuine trilinear null-form estimate on the level of Lebesgue spaces for (30) would render these convoluted considerations unnecessary, although the overall complexity of the argument would barely improve.

Hence we reduce the 2nd term of $\Pi$ to the following:

**Null-form:**

\[ \sum_{j=1}^{3} (\triangle^{-1} \partial_j [R_j \Psi^1 R_\nu \Psi^2 - R_j \Psi^2 R_\nu \Psi^2] \partial^\nu \Phi) \psi \alpha \]  
\[ \text{(75)} \]

**Error terms:** these are of the rough form

\[ \nabla^{-1} (R_\mu \Psi^a \nabla^{-1} (\nabla^{-1} (\psi^2) \psi)) \nabla^\nu \Phi \psi \]  
\[ \text{(76)} \]

\[ \nabla^{-1} (\nabla^{-1} (\psi^2) \psi) \nabla^{-1} (\nabla^{-1} (\psi^2) \psi)) \nabla^\nu \Phi \psi, \]  
\[ \text{(77)} \]

where $\Psi^a$, $a = 1, 2$ is as in section 2.4.

These error terms can be handled as before, utilizing e.g. Lorentz spaces for low-high frequency interactions. As this doesn’t offer anything new, it is left out.

As to the null-form (75), we decompose this into various frequency interactions of $\psi_\alpha$ and ( , ). Most cases are elementary, with the exception of the high-high case. However, we can then profit from Proposition 3.7.
Estimation of the null-form (73):

1): Both $\langle , \rangle$ and $\psi_\alpha$ at frequency $< O(1)$:

$$\|P_0(P_{< O(1)}(\Delta^{-1}\partial_j[R_j \Psi^1 R_\nu \Psi^2 - R_\nu \Psi^1 R_j \Psi^2]\partial^{\nu} \Phi)P_{< O(1)}\psi_\alpha)\|_{L^2_t L^2_x}$$

$$\leq C\|\nabla^{-1}[R_j \Psi^1 R_\nu \Psi^2 - R_\nu \Psi^1 R_j \Psi^2]\|_{L^2_t L^2_x}\|\partial^{\nu} \Phi\|_{L^2_t L^2_x}\|P_{< O(1)}\psi_\alpha\|_{L^2_t L^2_x}$$

(78)

where we have used the fact that $\|P_{< O(1)}\psi_\alpha\|_{L^2_t L^2_x} \leq C c_0$, as is easily verified.

2.1): High-High interactions of $\langle , \rangle$ and $\psi_\alpha$: $\psi_\alpha$ at frequency $2^{k_2}$, $k_2 >> 1$, $\Phi$ at frequency $< 2^{k_2-100}$. This implies that $\langle , \rangle$ is at frequency $2^{k_2+O(1)}$. Hence we can estimate this by

$$\|P_0(\Delta^{-1}\partial_j P_{k_2+O(1)}[R_j \Psi^1 R_\nu \Psi^2 - R_\nu \Psi^1 R_j \Psi^2]P_{< k_2-100} \partial^{\nu} \Phi P_{k_2}\psi_\alpha)\|_{L^2_t L^2_x}$$

$$\leq C 2^{-\delta k_2}\|\nabla x L \nabla^{-1} \Psi^1\|_{L^2_t L^2_x}\|\nabla x L \nabla^{-1} \Psi^2\|_{L^2_t L^2_x}\|\partial^{\nu} \Phi\|_{L^2_t L^2_x}\|P_{k_2}\psi_\alpha\|_{L^2_t L^2_x}$$

(79)

This can be summed over $k_2 >> 1$, to give a bound of the form $C M^4 \epsilon c_0$.

2.2): High-High interactions of $\langle , \rangle$ and $\psi_\alpha$: $\psi_\alpha$ at frequency $2^{k_2}$, $k_2 >> 1$, $\Phi$ at frequency $\geq 2^{k_2-100}$. As mentioned before, this case does not appear covered by the Strichartz estimates, on account of the possibility that all inputs could have large frequencies while there is 'no exponential gain for free', as before. The easiest way around this difficulty is to refer to the already proved estimate for the null-form (80), in addition to the estimate lemma 3.6. This gives the estimate

$$\|P_0(\sum_{j=1}^{3} \Delta^{-1}\partial_j [R_j \Psi^1 R_\nu \Psi^2 - R_\nu \Psi^1 R_j \Psi^2]P_{\geq k_2-100} \partial^{\nu} \Phi P_{k_2}\psi_\alpha)\|_{N[0]}$$

$$\leq C 2^{-\delta k_2}\|\nabla(\sum_{j=1}^{3} \Delta^{-1}\partial_j [R_j \Psi^1 R_\nu \Psi^2 - R_\nu \Psi^1 R_j \Psi^2]P_{\geq k_2-100} \partial^{\nu} \Phi P_{k_2})\|_{N[0]+O(1)}$$

(80)

$$\|P_{k_2}\psi_\alpha\|_{S[k_2]}$$

$$\leq C M^4 \epsilon 2^{-\delta k_2} c k_2$$

This can be summed over $k_2 >> 1$ to yield a bound of the required form, provided the $\sigma$ in the definition of the frequency envelope is much smaller than $\delta_2$. We are
done with the estimation of $II$.

### 4.4. Treating $I$: First term of $I$

Recall the definition of $\Phi$: $\Phi = \sum_{k=1}^{3} \Delta^{-1} \partial_k \phi_k^1$.

Thus upon using the wave equation satisfied by $\phi$, the first term of $I$ is seen to reduce to

$$
\sum_{j=1}^{3} \Delta^{-1} \partial_j \left( - \phi^1_\nu \partial^\nu \phi_j^2 + \nabla^\nu (\phi_j^2) \right) \psi_\alpha
$$

(81)

In spite of the trilinear character of this expression, it is not yet amenable to good estimates, as can be seen when for example $\partial^\nu \phi_j^2$ is at very large frequency. We therefore employ the div-curl system and corresponding dynamic separation for the $\phi_k^1$ in order to introduce a null-structure plus error terms. There results the expression

$$
\sum_{j,l}^{3} \Delta^{-1} \partial_j \left[ (R_\nu R_l \phi_1^3) + \nabla^{-1} (\phi_j^2) \right] \partial^\nu \phi_j^2 + \nabla^\nu \phi_j^3 \right] \psi_\alpha
$$

(82)

Here $\nabla^{-1} (\phi_j^2)$ refers to $\sum_{\tau=1}^{3} \Delta^{-1} \partial_\tau (\phi_1^1 \phi_\nu^2 - \phi_\nu^1 \phi_1^2)$. We estimate each of these terms in turn:

**A**: $\Delta^{-1} \partial_j (\phi^3) \psi_\alpha$: this term is entirely elementary, for the simple reason that high-high interactions of $(,)$ and $\psi_\alpha$ imply an automatic exponential gain in the frequency on account of the $\Delta^{-1} \partial_j$. For completeness’ sake, we provide the simple details:

**A.1**: $\psi_\alpha$ at frequency $O(1)$, $(,)$ at frequency $< O(1)$:

$$
\|P_0 \left( \sum_{k< O(1)} \Delta^{-1} \partial_j P_k (\phi^3) P_{O(1)} \psi_\alpha \right) \|_{L^1_t L^2_x} \\
\leq C \sum_{k< O(1)} C^2 \|\phi^3\|_{L^4_t L^4_x} \|P_{O(1)} \psi_\alpha\|_{L^4_t L^4_x}
$$

(83)

where we have employed Bernstein’s inequality estimating $L^4_x$ in terms of $L^4_x$ for the term $\phi^3$.

**A.2**: $\psi_\alpha$ at frequency $< O(1)$, $(,)$ at frequency $O(1)$:
\[
\|P_0[P_{O(1)} \triangle^{-1} \partial_j (\phi^3) \sum_{k < O(1)} P_k \psi_\alpha]\|_{L^1_t L^2_x} \\
\leq C \sum_{k < O(1)} \|\phi^3\|_{L^4_t L^4_x} \|P_k \psi_\alpha\|_{L^4_t L^4_x} \\
\leq \sum_{k < O(1)} C M^4 2^{k c} c_k \leq CM^4 \epsilon c_0, \tag{84}
\]

where we have applied Bernstein’s inequality estimating \(L^2_x\) in terms of \(L^4_x\) to the output.

A.3): \(\psi_\alpha\) at frequency \(2^k, k >> 1\), \(\phi\) at frequency \(2^{k+O(1)}\): we have

\[
\sum_{k >> 1} \|P_0[\triangle^{-1} \partial_j P_{k+O(1)} (\phi^3) P_k \psi_\alpha]\|_{L^1_t L^2_x} \\
\leq \sum_{k >> 1} C 2^{-k} \|\phi^3\|_{L^4_t L^4_x} \|P_k \psi_\alpha\|_{L^4_t L^4_x} \\
\leq C \sum_{k >> 1} M^4 2^{-k} c c_k \leq CM^4 \epsilon c_0 \tag{85}
\]

This then disposes of this term.

B: \(\triangle^{-1} \partial_j (R_\nu R_\lambda \phi^1_\mu \phi^2_\nu) \psi_\alpha\): this term of course has an obvious null-structure inherent in it. We will split the term into various frequency interactions between \((\), \(\)) and \(\psi_\alpha\):

B.1): Low -high interactions: \((\), \(\)) at frequency \(< O(1), \psi_\alpha\) at frequency \(O(1)\): we can’t simply put all inputs into \(L^4_t L^6_x\), as the inputs of \((\), \(\)) could have very large frequency. In this bad case, we will revert to lemma 3.4 estimating \((\), \(\)) in \(N[k]\) and gaining exponentially in the difference between the logarithm of the frequency of \((\), \(\)) and its (large frequency) inputs. Then we can apply lemma 3.6 to conclude. Of course, when the inputs of \((\), \(\)) have frequency at most comparable to the frequency of \((\), \(\)), we put all inputs into \(L^4_t L^6_x\). Writing the details:
where the $\delta$ in the preceding comes from lemma 3.4.

**B.2): High-low interactions**: these are dealt with in identical fashion, using lemma 3.4 and lemma 3.6 for the high-low interaction case (note that we have $\|P_{< O(1)} \psi_\alpha\|_{L^\infty_t L^\infty_x} \leq Cc_0$), hence left out.

**B.3): High-high interactions**: (.) at frequency $2^{k_1}$, $k_1 \gg 1$, $\psi_\alpha$ at frequency $2^{k_2}$ with $k_2 = k_1 + O(1)$: we distinguish between the case when both inputs of (.) have frequency at least comparable to $2^{k_1}$, and its complement, i.e. one input of (.) at much lower frequency than (.). The former case is dealt with as in the preceding cases, using lemmata 3.4, 3.6, hence not further elaborated. The treatment of the latter case, i.e. both inputs of (.) have at most frequency comparable with $2^{k_1}$, is subsumed under the following lemma:

**Lemma 4.5.**: Let $1 \ll k_3$, $k_2 \leq k_1 - 10 \leq k_3$. Also, let $\psi_j$, $j = 1, 2, 3$ be Schwartz functions. Then the following estimates hold:

\[
\|P_0 [\nabla^{-1} (R_\nu R_k \psi_2 P_k \psi_3 \partial^{\nu} \psi_1) P_{k_3} \psi_3] \|_{N[0]} \leq C 2^{-\delta_1 k_3} 2^{\delta_2 (k_2 - k_1)} \prod_{a=1}^{3} \|P_{k_a} \psi_a\|_{S[k_a]} \tag{87}
\]

\[
\|P_0 [R_\nu P_k \psi_2 \nabla^{-1} P_{k_1} \psi_1 \partial^{\nu} P_{k_3} \psi_3] \|_{N[0]} \leq C 2^{-\delta_1 k_3} 2^{\delta_2 (k_2 - k_1)} \prod_{a=1}^{3} \|P_{k_a} \psi_a\|_{S[k_a]} \tag{88}
\]

In the first inequality, $\nabla^{-1}$ refers to any operator of the form $\triangle^{-1} \partial_j$, $j = 1, 2, 3$. Also, if $k_1 \gg 1$, $k_2 \leq k_1$, then
\[ \|P_0[R_\nu P_{k_2} \psi_2 P_{k_1} R' \psi_1 \psi_3]\|_{N[0]} \leq C 2^{-\delta_1 k_1 2^{\delta_2 \min\{0,k_2\}}} \prod_{a=1}^{2} \|P_{k_a} \psi_a\| S[k_a] \sup_k \{\|P_k \psi_3\| S[k]\} \quad (89) \]

**Proof**: The proofs of these three assertions are essentially identical, hence we will only deal with the first of them: the idea of the proof consists in splitting into the case of moderately small \(k_2\), namely \(k_2 \geq -k_1\), and very small \(k_2\), or \(k_2 < -k_1\). Of course, \(k_1 = k_3 + O(1)\), hence is large. In the first case, we utilize lemma 3.5 and lemma 3.6, the latter furnishing an exponential decay accounting for both factors on the right-hand side of the inequality.

For the second case, we place the low-frequency term into a Strichartz space slightly worse than \(L^2_t L^\infty_x\), while the high-frequency terms are placed into small perturbations of the neutral (as far as exponential gains or losses are concerned) \(L^1_t L^2_x\). The assumption on the smallness of the low frequency allows to play this successfully against the high frequencies, resulting in the desired exponential gain.

Case 1): \(k_1 - 10 \geq k_2 \geq -k_1\): use lemma 3.5, lemma 3.6 to conclude that

\[ \|P_0[\nabla^{-1} (R_\nu P_{k_2} \psi_2 P_{k_1} \partial^\nu \psi_1) P_{k_3} \psi_3]\|_{N[0]} \leq C 2^{-\delta k_3} \|\nabla^{-1} (R_\nu P_{k_2} \psi_2 P_{k_1} \partial^\nu \psi_1)\|_{N[k_3 + O(1)]} \|P_{k_3} \psi_3\| S[k_3] \quad (90) \]

\[ \leq C 2^{-\delta k_3} 2^{\delta (k_2 - k_1)} \prod_{a=1}^{3} \|P_{k_a} \psi_a\| S[k_a] \]

Case 2): \(k_2 < -k_1\):

\[ \|P_0[\nabla^{-1} \partial^\nu P_{k_1} \psi_3 P_{k_2} R_\nu \psi_2] P_{k_3} \psi_3\|_{N[0]} \leq C 2^{-k_3} \|P_{k_1} \partial^\nu \psi_3\|_{L^1_t L^{1+}_{x'}} \|P_{k_2} R_\nu \psi_2\|_{L^2_t L^2_{x'}} \|P_{k_3} \psi_3\|_{L^1_t L^{1+}_{x'}} \quad (91) \]

where we set \(\frac{2}{4+} + \frac{1}{\frac{M}{2}} = 1, \frac{2}{4+} + \frac{1}{\frac{M}{4+}} = 1, \frac{1}{2+} + \frac{1}{\frac{M}{2+}} = \frac{1}{2} \cdot \frac{1}{4+} = \frac{1}{4} \). Using this we can estimate the above by

\[ C 2^{4 \lambda k_1} 2^{\delta (4 \lambda - 4 \lambda) k_2} \prod_{a=1}^{3} \|P_{k_a} \psi_a\| S[k_a] \quad (92) \]

\[ \leq C 2^{\delta (\frac{1}{8} - \lambda) (k_2 - k_1)} 2^{\delta (4 \lambda - (\frac{1}{4} - 2 \lambda)) k_1} \prod_{a=1}^{3} \|P_{k_a} \psi_a\| S[k_a] \]

We can now choose \(\lambda = \frac{1}{100}\) in order to obtain the required estimate. Note that the Strichartz estimates invoked here are still controlled by our choice of \(S[k]\). \(\blacksquare\)
C: \( P_0(\sum_{j=1}^3 \Delta^{-1} \partial_j [\sum_{r=1}^3 \Delta^{-1} \partial_r (\phi', \phi'' - \phi', \phi'') \partial^r \phi_j^2]|\psi_0) \) This is very similar to the 2nd term of II: we have simply traded an extra \( \nabla^{-1} \) for an extra \( \nabla \) (in \( \partial^r \phi_j^2 \)), recalling that \( \partial^r \Phi \) is morally equivalent (as far as scaling is concerned) to \( \phi \). As before, the quadrilinear structure appears insufficient to render this term amenable to a simple energy estimate. The difficulty arises as usual for certain high-high interactions. By now however, enough estimates are available to dispose of this case rather quickly, invoking in particular the estimates for II:

C.1): Low-high interactions: \([,\) at frequency \(< O(1), \psi_\alpha \) at frequency \( O(1)\):

\[
\|P_0(\sum_{j=1}^3 \Delta^{-1} \partial_j \sum_{k<O(1)} P_k(\sum_{r=1}^3 \Delta^{-1} \partial_r (\phi', \phi'' - \phi', \phi'') \partial^r \phi_j^2)|P_O(\psi_\alpha)\|_{L^4_t L^2_x} \\
\leq C \sum_{j=1}^3 \sum_{k<O(1)} 2^{-k} \|P_k(P_{<k+O(1)} \nabla^{-1} (\phi^2) P_{<k+O(1)} \partial^r \phi_j^2)|P_O(\psi_\alpha)\|_{L^4_t L^2_x} \\
+ C \sum_{j=1}^3 \sum_{k<O(1)} \sum_{a>k+O(1)} 2^{-k} \|P_k(P_{a} \nabla^{-1} (\phi^2) P_{a+O(1)} \partial^r \phi_j^2)|P_O(\psi_\alpha)\|_{L^4_t L^2_x} \\
\leq C \sum_{j=1}^3 \sum_{k<O(1)} 2^{-k} 2^{\frac{j}{2}} \|P_{<k+O(1)} \nabla^{-1} (\phi^2)|_{L^2_t L^2_x} \|P_{<k+O(1)} \partial^r \phi_j^2\|_{L^4_t L^2_x} \\
\|P_O(\psi_\alpha)\|_{L^4_t L^2_x} \\
+ C \sum_{j=1}^3 \sum_{k<O(1)} C 2^{\frac{j}{2}a} \|P_{\geq a+O(1)} \phi\|_{L^2_t L^3_x} \|P_{a+O(1)} \partial^r \phi_j^2\|_{L^4_t L^2_x} \|\phi\|_{L^4_t L^2_x} \\
\|P_O(\psi_\alpha)\|_{L^4_t L^2_x} \\
\leq \sum_{k<O(1)} CM^4 2^{\frac{j}{2}} c_0 + C \sum_{k<O(1)} \sum_{a \geq k+O(1)} 2^{\frac{j}{2}} c_0^2 c_0 \\
\leq CM^4 c_0
\]

We have repeatedly used Bernstein’s inequality, as well as Sobolev’s inequality and the fact that \( P_{a} (\phi^2) = P_{a} (P_{\geq a-10} \phi \phi) + P_{a} (P_{<a-10} \phi P_{\geq a-10} \phi) \).

C.2): High-high interactions: \([,\) and \( \psi_\alpha \) at high frequencies. Here we utilize the estimates proved for II:
\[ \left\| P_0 \sum_{j=1}^{3} \Delta^{-1} \partial_j P_0 \left( \sum_{r=1}^{3} \Delta^{-1} \partial_r (\phi_{r1}^1 \phi_{r2}^2 - \phi_{r1}^2 \phi_{r2}^1) \partial'' \phi_j^2 \right) P_{a+O(1)} \psi_\alpha \right\|_{N[0]} \]

\[ \leq \sum_{a \geq O(1)} C2^{-\delta a} \left\| \nabla \sum_{j=1}^{3} \Delta^{-1} \partial_j P_0 \left( \sum_{r=1}^{3} \Delta^{-1} \partial_r (\phi_{r1}^1 \phi_{r2}^2 - \phi_{r1}^2 \phi_{r2}^1) \partial'' \phi_j^2 \right) \right\|_{N[a]} \]

\[ \left\| P_{a+O(1)} \psi_\alpha \right\|_{S[a+O(1)]} \]

\[ \leq \sum_{a \geq O(1)} CM^42^{-\delta a} c_0^2 \leq CM^4 \epsilon c_0 \]

We have used lemma 3.3 for the second step.

C.3): High-low interactions: \([,]\) at frequency \(O(1)\), \(\psi_\alpha\) at frequency < \(O(1)\): this case is handled exactly as the preceding one, utilizing lemma 3.6, hence left out.

This finishes the treatment of the first term \((\Box \Phi)\psi_\alpha\) of I.

**Second term of I**

We only need to look at the cases when at least two inputs of \(P_0 [\partial_\gamma \Phi \partial'' \Phi \psi_\alpha]\) have large frequency, since otherwise, one can simply place all inputs into \(L^1_t L^6_x\).

However, one can then simply use the estimate provided by lemma 1.3, in particular the third inequality, to handle this case: in detail

\[ \left\| P_0 \sum_{k_1 > 10, k_2 \leq k_1} [\partial_\gamma P_{k_1} \Phi \partial'' P_{k_2} \Phi \psi_\alpha] \right\|_{N[0]} \]

\[ + \left\| P_0 \sum_{k_2 > 10, k_1 < k_2} [\partial_\gamma P_{k_1} \Phi \partial'' P_{k_2} \Phi \psi_\alpha] \right\|_{N[0]} \]

\[ \leq \sum_{k_1 > 10, k_2 \leq k_1} C2^{-\delta k_1} 2^{\delta k_2 \min\{0, k_2\}} M^3 \epsilon c_{k_1} c_{k_2} \] (94)

\[ + \sum_{k_2 > 10, k_1 < k_2} 2^{-\delta k_2} 2^{\delta k_1 \min\{0, k_1\}} M^3 \epsilon c_{k_1} c_{k_2} \]

\[ \leq CM^3 \epsilon c_0 \]

4.6. **Treating III.** Utilizing the Proposition 3.1, we see that we need to estimate \(\left\| P_0 (\phi_{11}^{12} \phi_{12}^{12} \psi) \right\|_{N[0]}\), where \(\psi\) is a Schwartz function whose frequency localized components have \(S[k]\)-norms majorized by the frequency envelope \(CMc_k\) for appropriate \(C\). The other terms in III are entirely analogous.

Of course, as before, this only requires consideration if at least one (and then of
course at least two) inputs have very large frequency. Our strategy shall again be to employ dynamic separation to render a hidden $Q_0$-structure visible. We need to estimate the following expression:

$$\| P_0 \left[ \sum_{k_1 > 10} P_{k_1} \phi_1^1 \phi^{1\nu} P_{< k_1 + O(1)} \psi \right] \|_{N[0]} \quad (95)$$

We apply dynamic separation to $\phi_1^1$, referring to (3), (4), and decompose the term within $\| \|$ into the following two terms:

1): $$\sum_{k,l=1}^{3} P_0 [ P_{k_1} R_{k} R_{l} \phi_1^1 \phi^{1\nu} P_{< k_1 + O(1)} \psi ] \quad (98)$$

2): $$\sum_{k_1 > 10} P_0 [ P_{k_1} \nabla^{-1} (\phi^2)^n \phi^{1\nu} P_{< k_1 + O(1)} \psi ] \quad (99)$$

The 2nd term in the immediately preceding is easy to estimate, placing all inputs into $L^4_t L^4_x$ and profiting from the exponential gain $2^{-k_1}$ coming from $\nabla^{-1}$. In order to estimate the first term, we need to apply another dynamic separation. This results in the terms

First case: $(.,.)$ at frequency $> 2^{k_1-10}$: Simply place all entries into $L^4_t L^4_x$ and estimate $L^1_t L^1_x$ in terms of $L^1_t L^1_x$ for the output, as well as
using the exponential decay $2^{-k_1}$ coming from $\partial_t \triangle^{-1} P_{>k_1-10}$.

Second case: $(\cdot)$ at frequency $\leq 2^{k_1-10}$; this can be rewritten as (leaving out $\sum_{k,r=1}^{3} P_{k,r}$ for simplicity’s sake):

\[
P_0[P_{k_1} R_\nu \phi^1_k \triangle^{-1} \partial_r P_{\leq k_1-10}(\phi^1_\nu \phi^2_\nu - \phi^2_\nu \phi^1_\nu) P_{<k_1+O(1)} \psi] \\
= \sum_{k=1}^{k_1} P_0[P_{k_1+O(1)}(P_{k_1} R_\nu \phi^1_k \triangle^{-1} \partial_r P_{\leq k_1-10}(\phi^1_\nu \phi^2_\nu - \phi^2_\nu \phi^1_\nu)) P_{<k_1+O(1)} \psi] \\
= \sum_{k=1}^{k_1} P_0[P_{k_1+O(1)}(P_{k_1} R_\nu \phi^1_k \triangle^{-1} \partial_r (\phi^1_\nu \phi^2_\nu - \phi^2_\nu \phi^1_\nu)) P_{<k_1+O(1)} \psi]
\]

(100)

The 2nd term in the last equality can be treated exactly as in the first case, of course. As to the first term, we use lemma 3.6 and the estimates proved in II to conclude that

\[
||P_0[P_{k_1+O(1)}(P_{k_1} R_\nu \phi^1_k \triangle^{-1} \partial_r (\phi^1_\nu \phi^2_\nu - \phi^2_\nu \phi^1_\nu)) P_{<k_1+O(1)} \psi]||_{N[0]} \\
\leq C 2^{-\delta k_1} ||P_{k_1+O(1)}(P_{k_1} R_\nu \phi^1_k \triangle^{-1} \partial_r (\phi^1_\nu \phi^2_\nu - \phi^2_\nu \phi^1_\nu)) P_{<k_1+O(1)} \psi||_{S[0]} \\
\leq CM^4 2^{-\delta k_1} c\]

(101)

This can be easily summed over $k_1 > 10$ to yield the required bound $CM^4\epsilon c_0$.

The proof of Proposition 4.2 is now completed by fixing $M >> 1$, then choosing $\epsilon > 0$ small enough such that using (52) and Theorem 3.1, as well as the estimates proved in this section, we can conclude $||P_0 \phi||_{S[0]} \leq \frac{M}{T} c_0$. 

5. Proof of the technical estimates

5.1. The Trilinear Estimate.

Proposition 5.2. Let $\psi_l$, $l = 1, 2, 3$ be Schwartz functions on $\mathbb{R}^{3+1}$. We then have the estimate

$$
\left\| P_0 \left( \sum_{j=1}^{3} \Delta^{-1} \partial_j \left[ R_\nu P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_\nu P_{k_2} \psi_2 \right] \partial^\nu P_{k_3} \psi_3 \right) \right\|_{N[0]} \leq C 2^{-\delta_1 |k_1 - k_2|} 2^{-\delta_2 |k_3|} \prod_{l=1}^{3} \left\| P_{k_l} \psi_l \right\|_{S[k_l]}
$$

(102)

for appropriate constants $\delta_1, \delta_2 > 0$.

Proof The proof is split into estimating the various frequency interactions of $\left[ \cdot, \cdot \right]$ and $\partial^\nu \psi_3$. The author apologizes in advance for the extremely technical and mechanical nature of these estimates. Many of the following cases are similar and almost write themselves, yet we have opted for as complete an account as possible. This section may be treated as an appendix to the paper.

1) Low-High Interactions: $\left[ \cdot, \cdot \right]$ at frequency $< 2^{-10}$, $\partial^\nu \psi_3$ at frequency between $2^{-10}$ and $2^{10}$, i.e. $k_3 = O(1)$.

We distinguish between the cases

A: $k_1 < k_2 - 10$

B: $k_1 > k_2 + 10$

C: $|k_1 - k_2| \leq 10$

On account of the apparent symmetry between A and B, only one of them has to be treated. We estimate the corresponding terms by restricting the Fourier supports of the inputs of the trilinear form further. I.e., in addition to frequency localization, we also localize in modulation, corresponding to the quantity $||\tau| - |\xi||$. The general strategy in the estimates to follow is to reduce output and inputs to small modulation, as it is only then that the algebraic structure as exemplified by (31) becomes useful. However, when the output has "large modulation", it can usually be easily estimated in the $X^{\frac{\lambda}{2} - \frac{1}{2} - 1}_0$-norm. Similarly, whenever an input has "large modulation", it can be placed into $L^2_t L^2_x$, which allows one more flexibility for the other inputs.
1.A): Estimation of A

Either output or third input at large modulation:

1.A.1): Output reduced to modulation > \(2^{k_2-100}\): we have (denoting \((\sqrt{-\Delta})^{-1}\) as \(\nabla^{-1}\))

\[
||P_0 Q_{> k_2-100} (\sum_{j=1}^{3} \Delta^{-1} \partial_j P_{< -10} R_v P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_v P_{k_2} \psi_2)\|_{N[0]}
\]

\[
P_{k_3} \partial^\nu \psi_3
\]

\[
\leq 3 \sum_{j=1}^{3} ||P_0 Q_{> k_2-100}(P_{k_2+O(1)} \Delta^{-1} \partial_j [R_v P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_v P_{k_2} \psi_2])||_{L^2 L_x^\infty} \]

\[
||P_{k_3} \partial^\nu \psi_3||_{L^\infty L_x^2}
\]

\[
\leq C ||\nabla_{x,t} \nabla^{-1} P_{k_1} \psi_1||_{L^2 L_x^\infty} ||\nabla_{x,t} \nabla^{-1} \psi_2||_{L^2 L_x^\infty} ||\nabla_{x,t} P_{k_3} \psi_3||_{L^2 L_x^\infty}
\]

\[
\leq C 2^{-\frac{k_2-k_1}{2}} \prod_{l=1}^{3} ||P_{k_l} \psi_l||_{S[k_l]}
\]

1.A.2): \(\partial^\nu P_{k_3} \psi_3\) reduced to modulation > \(2^{k_2-100}\), output reduced to modulation \(\leq 2^{k_2-100}\):

\[
||P_0 Q_{\leq k_2-100} (\sum_{j=1}^{3} \Delta^{-1} \partial_j P_{< -10} R_v P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_v P_{k_2} \psi_2)\|_{N[0]}
\]

\[
P_{k_3} Q_{> k_2-100} \partial^\nu \psi_3
\]

\[
\leq ||P_0 Q_{\leq k_2-100} (\sum_{j=1}^{3} P_{k_2+O(1)} \Delta^{-1} \partial_j [R_v P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_v P_{k_2} \psi_2])||_{L^2 L_x^\infty}
\]

\[
||P_{k_3} Q_{> k_2-100} \partial^\nu \psi_3||_{L^2 L_x^\infty}
\]

\[
\leq 3 \sum_{j=1}^{3} C 2^{-k_2} ||R_v P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_v P_{k_2} \psi_2||_{L^2 L_x^\infty}
\]

\[
||P_{k_3} Q_{> k_2-100} \partial^\nu \psi_3||_{L^2 L_x^\infty}
\]

\[
\leq C 2^{-\frac{k_2-k_1}{6}} \prod_{l=1}^{3} ||P_{k_l} \psi_l||_{S[k_l]}
\]
Output and third input at small modulation. Part of null-structure becomes useful:

1.A.3): Both output and $\partial^\nu P_k \psi_3$ are reduced to modulation $\leq 2^{k_2-100}$; notice that this implies in particular that $[\cdot]$ can be restricted to modulation $\leq 2^{k_2+O(1)}$. We use the elementary identity

$$2 \sum_{j=1}^{3} \triangle^{-1} \partial_j (R_j f R_\nu g - R_\nu f R_j g) \partial^\nu h$$

(105)

valid for all Schwartz functions, say. We consecutively estimate each of the terms, with $f$ etc. replaced by the appropriately microlocalized components of $P_k \psi$ etc.:

1.A.3.a):

$$\|P_0 Q \leq k_2-100[\triangle^{-1} \partial_j (P_{k_1} R_j \psi_1 \nabla^{-1} P_{k_2} \psi_2) P_{k_3} Q \leq k_2-100 \psi_3]\|_{N[0]}$$

$$\leq \|P_0 Q \leq k_2-100[\triangle^{-1} \partial_j P_{k_2+O(1)} (P_{k_1} R_j \psi_1 \nabla^{-1} P_{k_2} \psi_2) P_{k_3} Q \leq k_2-100 \psi_3]\|_{X^{0, \frac{2}{3}, \frac{2}{3}}_0}$$

$$\leq C 2^{-\frac{k_2}{2}} \|P_{k_2+O(1)} (P_{k_1} R_j \psi_1 \nabla^{-1} P_{k_2} \psi_2)\|_{L^2_t L^\infty_x} \|P_{k_3} Q \leq k_2-100 \psi_3\|_{L^\infty_t L^2_x}$$

(106)

$$\leq C 2^{-\frac{k_2}{2}} \prod_{l=1}^{3} \|P_{k_l} \psi_l\|_{S[k_l]}$$

1.A.3.b):

$$\|P_0 Q \leq k_2-100[\Box \triangle^{-1} \partial_j (R_j P_{k_1} \psi_1 \nabla^{-1} P_{k_2} \psi_2) P_{k_3} Q \leq k_2-100 \psi_3]\|_{N[0]}$$

$$\leq C \|P_0 Q \leq k_2-100[\Box \triangle^{-1} \partial_j P_{k_2+O(1)} Q \leq k_2+O(1) (R_j P_{k_1} \psi_1 \nabla^{-1} P_{k_2} \psi_2) P_{k_3} Q \leq k_2-100 \psi_3]\|_{L^1_t L^2_x}$$

(107)

$$\leq C 2^{-\frac{k_2}{2}} \prod_{l=1}^{3} \|P_{k_l} \psi_l\|_{S[k_l]}$$

as is seen by placing all entries into $L^2_t L^6_x$. Of course this is even better than the estimate required for the lemma, since $k_1, k_2 < O(1)$.

1.A.3.c):
\[ ||P_0 Q_{\leq k_2 - 100} [\triangle^{-1} \partial_j (R_j P_{k_1} \psi_{k_1} \nabla^{-1} P_{k_2} \psi_{k_2}) \Box P_{k_3} Q_{\leq k_2 - 100} \psi_3] ||_{N[0]} \leq C ||P_{k_2 + O(1)} [\triangle^{-1} \partial_j (R_j P_{k_1} \psi_{k_1} \nabla^{-1} P_{k_2} \psi_{k_2}) ||_{L^6_t L^6_x} || \Box P_{k_3} Q_{\leq k_2 - 100} \psi_3 ||_{L^2_t L^2_x} \leq C 2^{\frac{k_1 - k_2}{6}} \prod_{l=1}^3 ||P_{k_l} \psi_l||_{S[k_l]} \] (108)

again by placing \( P_{k_1} \psi_1, P_{k_2} \psi_2 \) into \( L^3_t L^6_x, L^6_t L^3_x \) respectively.

1.A.3.d):

\[ ||P_0 Q_{\leq k_2 - 100} [R_{\nu} P_{k_1} \psi_1 \nabla^{-1} P_{k_2} \psi_2 \partial_{\nu} P_{k_3} Q_{\leq k_2 - 100} \psi_3] ||_{N[0]} \]

This term is more difficult and will be estimated with the aid of the 'Q0-structure' inherent in it (viz. the complete expansion (31)). It really is a consequence of a deep trilinear inequality proved by T. Tao, but we prove it here for completeness' sake, and also since the proof in 3 dimensions is somewhat simpler: first, we get rid of the multiplier \( Q_0 \leq k_2 - 100 \) in front of it as well as in front of the third input; for example:

\[ ||P_0 Q_{\leq k_2 - 100} [R_{\nu} P_{k_1} \psi_1 \nabla^{-1} P_{k_2} \psi_2 \partial_{\nu} P_{k_3} Q_{\leq k_2 - 100} \psi_3] ||_{N[0]} \leq C 2^{-\frac{k_1 - k_2}{6}} \prod_{l=1}^3 ||P_{k_l} \psi_l||_{S[k_l]} \] (109)

Let us restate what we are trying to prove

**Lemma 5.3.** Under the hypotheses of case 1.A), we have the estimate

\[ ||P_0 [R_{\nu} P_{k_1} \psi_1 \nabla^{-1} P_{k_2} \psi_2 \partial_{\nu} P_{k_3} \psi_3] ||_{N[0]} \leq C 2^{\delta (k_1 - k_2)} \prod_{l=1}^3 ||P_{k_l} \psi_l||_{S[k_l]} \] (110)

for some \( \delta > 0 \).

**Proof:** The proof is again by considering many different cases. However, the cases are somewhat more involved than in the preceding; in particular, we will have to invoke lemma 3.3 for some ”intermediate cases” where inputs have relatively small modulation but still too large to invoke the algebraic identity. We first consider the case when \( P_{k_1} \psi_1 \) has relatively large modulation, i.e. modulation \( \geq 2^{k_1 + O(1)} \). We
cannot yet invoke the null-structure in this case. However, Lemma 3.3 allows us to
play the modulation of \( P_{k_1} \psi_1 \) against the frequency of \( P_{k_2} \psi_2 \):

1): \( P_{k_1} \psi_1 \) restricted to modulation \( \geq 2^{k_1 + 100} \).

1.1.1) \( P_{k_1} \psi_1 \) at modulation \( 2^r \) with \( k_1 + 100 \leq r \leq k_2 + 100 \), \( P_{k_3} \psi_3 \) at modulation
\( < 2^{r-100} \): From elementary geometrical considerations, we see that we in
this case \( R_x \psi_1 P_{k_3} \psi_3 \) has Fourier support at distance approximately \( 2^r \) from
the light cone. Hence using Lemma 3.3, we can play this modulation against
the frequency \( 2^{k_2} \) of the third term \( P_{k_2} \psi_2 \). Of course, we then have to also play
\( r \) against \( k_1 \), which is possible by placing \( P_{k_1} \psi_1 \) into \( L_t^2 L_x^\infty \). The upshot is an
exponential gain in \( k_1 - k_2 \). In detail:

\[
\sum_{k_2 + 100 \geq r \geq k_1 + 100} P_0(R_x \psi_1 P_{k_3} \psi_3) \nabla^{-1} P_{k_2} \psi_2 \delta \nu \nu P_{k_3} \psi_3 \| \|_{N[0]}
\leq C 2^{(r-k_2)} \| P_{O(1)} \psi_1 P_{k_3} \psi_3 \|_{L_t^\infty L_x^\infty} \| P_{k_2} \psi_2 \|_{S[k_2]}
\leq \sum_{k_2 + 100 \geq r \geq k_1 + 100} C 2^{\frac{r}{2} - \frac{k_1}{2}} \| R_x P_{k_1} \psi_1 \|_{L_t^2 L_x^\infty} \| P_{k_3} \psi_3 \|_{L_t^\infty L_x^\infty} \| P_{k_2} \psi_2 \|_{S[k_2]}
\leq \sum_{k_2 + 100 \geq r \geq k_1 + 100} C 2^{\frac{r}{2} - \frac{k_1}{2}} 2^{k_1 - r} \| P_{k_1} \psi_1 \|_{S[k_1]}^3
\leq C 2^{\frac{r}{2} - \frac{k_1}{2}} \| P_{k_1} \psi_1 \|_{S[k_1]}^3
\]

1.1.2.a) \( P_{k_1} \psi_1 \) at modulation \( 2^r \) with \( k_1 + 100 \leq r \leq k_2 + 100 \), \( P_{k_3} \psi_3 \) at modulation
\( \geq 2^{r-100} \) but \( \leq 2^{k_2 + 100} \), \( P_{k_2} \psi_2 \) at modulation \( < 2^{r-100} \): we need to group the
terms differently, again using Lemma 3.3. Note that the \( \tilde{X}_k^{\frac{1}{2}, -\frac{1}{2}} \) norm has to be
applied to a term at large frequency in order to avoid an exponential loss. In the
present situation, the third term \( P_{k_3} \psi_3 \) is such a candidate. However, in
order to apply Lemma 3.3, we then need to place \( R_x \psi_1 P_{k_3} \psi_3 \) into \( S[k_2 + O(1)] \). This is easily feasible on account of the majorization \( \| P_k \phi \|_{S[k]} \leq C \| P_k \phi \|_{\tilde{X}_k^{\frac{1}{2}, -\frac{1}{2}}} \), which follows for example from (47):
\[
|| \sum_{k_2+100 \geq r \geq k_1+100} \sum_{k_2+100 \geq a \geq r-100} P_1[R_k, P_{k_1}, Q_r, \psi_1, \nabla^{-1}, P_{k_2} Q_{< r-100} \psi_2, \partial^\nu P_{k_1} Q_a \psi_3] ||_N[0] \\
\leq \sum_{k_2+100 \geq r \geq k_1+100} \sum_{k_2+100 \geq a \geq r-100} C_{2^\delta(a-k_2)} ||Q_r + O(1)\nabla[R_k, P_{k_1}, Q_r, \psi_1, \nabla^{-1}, P_{k_2} Q_{< r-100} \psi_2] ||_L^\infty ||P_{k_3} Q_a \partial^\nu \psi_3||_{X^{d-1/2}} \\
\leq \sum_{k_2+100 \geq r \geq k_1+100} \sum_{k_2+100 \geq a \geq r-100} C_{2^\delta(a-k_2)} ||Q_r + O(1)\nabla[R_k, P_{k_1}, Q_r, \psi_1, \nabla^{-1}, P_{k_2} Q_{< r-100} \psi_2] ||_{X^{d-1/2}} ||P_{k_3} Q_a \partial^\nu \psi_3||_{X^{d-1/2}} \\
\leq \sum_{k_2+100 \geq r \geq k_1+100} \sum_{k_2+100 \geq a \geq r-100} C_{2^\delta(a-k_2)} 2^{k_2-\frac{5}{2} - \frac{k_1}{2} - \frac{\nu}{2} - \frac{3k_1}{2} - \frac{3\nu}{2} - \frac{k_2}{2} - \frac{2}{2}} \prod_{l=1}^3 ||P_{k_l} \psi_l||_{L^\infty} \\
\leq \sum_{k_2+100 \geq r \geq k_1+100} \sum_{k_2+100 \geq a \geq r-100} C_{2^\delta(a-k_2)} 2^{k_2-\frac{5}{2} - \frac{k_1}{2} - \frac{\nu}{2} - \frac{3k_1}{2} - \frac{3\nu}{2} - \frac{k_2}{2} - \frac{2}{2}} \prod_{l=1}^3 ||P_{k_l} \psi_l||_{L^\infty} \\
\leq C_{2^\delta} 2^{\delta(k_1-k_2)} \prod_{l=1}^3 ||P_{k_l} \psi_l||_{L^\infty}
\]

for any \(0 < \delta' < \delta\).

1.1.2.b) \(P_{k_1} \psi_1\) at modulation \(2^\nu\) with \(k_1 + 100 \leq r \leq k_2 + 100\), \(P_{k_3} \psi_3\) at modulation \(\geq 2^{r-100}\) but \(\leq 2^{k_2+100}\), \(P_{k_2} \psi_2\) at modulation \(\geq 2^{r-100}\); this and the following 4 large modulation cases are routine and can be handled by the \(X^{d-1/2}\) and \(L^\infty L^2\) components of \(S[k]\). The only slight difficulty here is that one has to play the modulation of the term \(P_{k_3} Q_a \psi_2\) against its own frequency, i.e. one has to obtain an exponential gain in the difference \(a-k_2\). This requires the improvement of Bernstein’s inequality mentioned in the third section, i.e. (55).
1.1.2) \( P_k \psi_1 \) at modulation \( 2^r \) with \( k_1 + 100 \leq r \leq k_2 + 100 \), \( P_{k_3} \psi_3 \) at modulation \( > 2^{k_2 + 100} \). We have

\[
|| \sum_{k_2 + 100 \geq r \geq k_1 + 100} |P_0[R_{x} P_{k_1} Q_r \psi_1 \nabla^{-1} P_{k_2} \psi_2 \partial^{\nu} P_{k_3} Q_{k_2 + 100} \psi_3]| ||_{L^1_t L^2_x} \leq \sum_{k_2 + 100 \geq r \geq k_1 + 100} C ||R_{x} P_{k_1} Q_r \psi_1||_{L^2_t L^\infty_x} ||\nabla^{-1} P_{k_2} \psi_2||_{L^2_t L^\infty_x} ||\partial^{\nu} P_{k_3} Q_{k_2 + 100} \psi_3||_{L^2_t L^\infty_x} \leq C 2^{k_1 - \frac{5}{2} + \frac{3}{2} k_3 - 2} \prod_{l=1}^{3} ||P_{l} \psi_l||_{S[k_l]} (111)
\]

1.2.1) \( P_k \psi_1 \) at modulation \( > 2^{k_2 + 100} \), \( P_{k_3} \psi_3 \) at modulation \( \geq 2^{k_2 + 100} \):

\[
||P_0[R_{x} P_{k_1} Q_{k_2 + 100} \psi_1 \nabla^{-1} P_{k_2} Q_{k_2 + 100} \psi_2 \partial^{\nu} P_{k_3} \psi_3]| ||_{L^1_t L^2_x} \leq C ||R_{x} P_{k_1} Q_{k_2 + 100} \psi_1||_{L^2_t L^\infty_x} ||\nabla^{-1} P_{k_2} Q_{k_2 + 100} \psi_2||_{L^2_t L^\infty_x} ||\partial^{\nu} P_{k_3} \psi_3||_{L^\infty_t L^2_x} \leq C 2^{k_1 - k_2} \prod_{l=1}^{3} ||P_{l} \psi_l||_{S[k_l]} (112)
\]
1.2.2) $P_k \psi_1$ at modulation $> 2^{k_2+100}$, $P_k \psi_2$ at modulation $< 2^{k_2-100}$, $P_k \psi_3$ at modulation $\geq 2^{k_2-100}$.

$$\|P_0 [R_{r} P_k Q_{> k_2+100} \psi_1 \nabla^{-1} P_k Q_{< k_2-100} \psi_2 \partial^\nu P_k Q_{\geq k_2-100} \psi_3] \|_{L_t^1 L_x^2}$$

$$\leq C \|R_{r} P_k Q_{> k_2+100} \psi_1 \|_{L_t^\infty L_x^\infty} \|\nabla^{-1} P_k Q_{< k_2-100} \psi_2 \|_{L_t^\infty L_x^\infty} \|\partial^\nu P_k Q_{\geq k_2-100} \psi_3 \|_{L_t^1 L_x^2}$$

$$\leq C 2^{k_1-k_2} \prod_{l=1}^3 \|P_k \psi_l \|_{S[k_l]} \quad (113)$$

1.2.3) $P_k \psi_1$ at modulation $> 2^{k_2+100}$, $P_k \psi_2$ at modulation $< 2^{k_2-100}$, $P_k \psi_3$ at modulation $< 2^{k_2-100}$: note that provided $P_k \psi_1$ is at modulation $2^r$, $r > k_2 + 100$, then the output is at modulation $\sim 2^r$. Hence

$$\|P_0 \left[ \sum_{r > k_2+100} R_{r} P_k Q_{r \psi_1} \nabla^{-1} P_k Q_{< k_2-100} \psi_2 \partial^\nu P_k Q_{< k_2-100} \psi_3 \right] \|_{N[0]}$$

$$\leq \sum_{r > k_2+100} \|P_0 Q_{r \psi_1} \|_{L_t^\infty L_x^\infty} \|\nabla^{-1} P_k Q_{< k_2-100} \psi_2 \|_{L_t^\infty L_x^\infty} \|\partial^\nu P_k Q_{< k_2-100} \psi_3 \|_{L_t^1 L_x^2}$$

$$\leq \sum_{r > k_2+100} C 2^{-\frac{3}{2} + \frac{k_1}{2} + \frac{3k_1}{2}} \prod_{l=1}^3 \|P_k \psi_l \|_{S[k_l]} \quad (114)$$

$$\leq C 2^{k_1-k_2} \prod_{l=1}^3 \|P_k \psi_l \|_{S[k_l]}$$

2): $P_k \psi_1$ at modulation $< 2^{k_1+100}$.

First, reduce to the case that $P_k \psi_3$ has modulation $< 2^{k_1-100}$: note that

$$\|P_0 [R_{r} P_k Q_{< k_1+100} \psi_1 \nabla^{-1} P_k \partial^\nu P_k Q_{\geq k_1-100} \psi_3] \|_{L_t^1 L_x^2}$$

$$\leq C \|R_{r} P_k Q_{< k_1+100} \psi_1 \|_{L_t^\infty L_x^\infty} \|\nabla^{-1} P_k \partial^\nu P_k Q_{\geq k_1-100} \psi_3 \|_{L_t^1 L_x^2}$$

$$\leq C 2^{\frac{k_1}{2} + \frac{k_2}{2}} \prod_{l=1}^3 \|P_k \psi_l \|_{S[k_l]} \quad (115)$$

Finally, we have manoeuvred ourselves into a position to exploit the algebraic structure inherent in the trilinear form of the lemma, i.e. the '2nd half in (33)'. We thus have to estimate the following terms:
2.A): 

$$\Box (\nabla^{-1} P_{k_1} Q_{<k_1 + 100} \psi_1 P_{k_2} Q_{<k_1 - 100} \psi_2) \nabla^{-1} P_{k_2} \psi_2$$  \hspace{1cm} (116)$$

We first reduce $\nabla^{-1} P_{k_1} \psi$ to modulation $< 2^{k_1 - 100}$.

$$||\Box (\nabla^{-1} P_{k_1} Q_{k_1 - 100} \leq < k_1 + 100 \psi_1 P_{k_2} Q_{< k_1 - 100} \psi_2) \nabla^{-1} P_{k_2} \psi_2 ||_{(0)} \leq C 2^\delta (k_1 - k_2) ||Q_{< k_1 + O(1)} \Box (\nabla^{-1} P_{k_1} Q_{k_1 - 100} \leq < k_1 + 100 \psi_1 P_{k_2} Q_{< k_1 - 100} \psi_2) ||_{s_{k_2} + O(1)}$$

$$||P_k \psi_2 ||_{s[k_2]} \leq 2^\delta (k_1 - k_2) 2^\frac{\delta}{r} ||\nabla^{-1} P_{k_1} Q_{k_1 - 100} \leq < k_1 + 100 \psi_1 ||_{L^2_t L^2_x} ||P_{k_3} \psi_3 ||_{L^\infty_t L^2_x} ||P_{k_2} \psi_2 ||_{S[k_2]}$$  \hspace{1cm} (117)$$

$$\leq C 2^\delta (k_1 - k_2) \prod_{l=1}^3 ||P_k \psi_l ||_{s[k_1]}$$

We want to place $(\cdot)$ into $L^2_t L^2_x$, then apply lemma [13.3]. However, in order to avoid an exponential loss in $k_1$, we would have to place $\nabla^{-1} P_{k_1} \psi_1$ into $L^2_t L^\infty_x$, which is impossible. Hence we need to invoke null-frame spaces, and in particular the identity (14):

2.A.1): Both inputs of $(\cdot)$ have modulation $<<$ than the modulation of $(\cdot)$: assume that $(\cdot)$ is localized to modulation $2^r$. Hence by our assumptions $r \leq k_1 + O(1)$. Now assume that the inputs of $(\cdot)$ have modulation $\leq 2^{r - 100}$. Then using lemma (13.2) in [10], microlocalizing the inputs to the upper or lower half-space, we can restrict the projections to $S^2$ of the Fourier supports of the inputs to spherical caps $\pm \kappa, \pm \kappa'$ of size $2^{-\frac{\kappa}{2}} - 50$ at distance $C 2^{-\frac{\kappa}{2}}$, where the $\pm$-signs are assigned corresponding to whether the function is microlocalized in the upper or lower half-space $\tau > 0$. Utilizing [13] yields
\[ \|Q_r(\nabla^{-1} P_{k_1} Q_{<\tau-100} \psi_1 P_{k_3} Q_{<\tau-100} \psi_3)\|_{L^2_t L^2_x} \leq 4 \sup_{\pm, \pm} \sum_{\kappa, \kappa' \in \mathbb{K} \frac{r}{r_k} - 50} \|\nabla^{-1} P_{k_1, \kappa} Q^\pm_{<\tau-100} \psi_1 P_{k_3, \kappa'} Q^\pm_{<\tau-100} \psi_3\|_{L^2_t L^2_x} \]

\[ \leq C \sup_{\pm, \pm} \sum_{\kappa, \kappa' \in \mathbb{K} \frac{r}{r_k} - 50} \|\nabla^{-1} P_{k_1, \kappa} Q^\pm_{<\tau-100} \psi_1 P_{k_3, \kappa'} Q^\pm_{<\tau-100} \psi_3\|_{L^2_t L^2_x} \]

Now

\[ \sup_{\pm, \pm} \sum_{\kappa, \kappa' \in \mathbb{K} \frac{r}{r_k} - 50} \|\nabla^{-1} P_{k_1, \kappa} Q^\pm_{<\tau-100} \psi_1 P_{k_3, \kappa'} Q^\pm_{<\tau-100} \psi_3\|_{L^2_t L^2_x} \leq C \sup_{\pm, \pm} \sum_{\kappa, \kappa' \in \mathbb{K} \frac{r}{r_k} - 50} 2^{-\frac{2k_1}{2}} \|P_{k_1, \kappa} Q^\pm_{<\tau-100} \psi_1\|_{S[\kappa, \pm \kappa']} \]

\[ \leq C 2^{-\frac{2k_1}{2}} \sup_{\pm, \pm} \sum_{\kappa \in \mathbb{K} \frac{r}{r_k} - 50} \|P_{k_1, \kappa} Q^\pm_{<\tau-100} \psi_1\|_{S[\kappa_1, \pm \kappa]} \]

\[ \leq C 2^{-\frac{2k_1}{2}} \|P_{k_1 \psi_1}\|_{S[\kappa_1]} \|P_{k_3 \psi_3}\|_{S[k_3]} \]

Moreover
Now proceed as before to conclude 2.A.2): At least one input of (\ref{e:wave-equation}), has modulation \( \geq 2^{r-100} \).

\[
\|Q_r(\nabla^{-1} P_{k_1} Q_{<r-100} \psi_1 P_{k_3} Q_{<r-100} \psi_3) \nabla^{-1} P_{k_2} \psi_2 \|_{L^2_t L^\infty_x} \leq C \|\nabla^{-1} P_{k_1} Q_{<k_1-100} \psi_1 \|_{L^2_t L^\infty_x} \| P_{k_3} Q_{<k_1-100} \psi_3 \|_{L^\infty_t L^2_x} \leq C 2^{-\frac{t}{2}} \| P_{k_1} \psi_1 \|_{S[k_1]} \| P_{k_3} \psi_3 \|_{S[k_3]}
\]

Hence inserting the preceding two results into (\ref{e:wave-equation}) and utilizing lemma 3.3, we deduce

\[
\| Q_r(\nabla^{-1} P_{k_1} Q_{<r-100} \psi_1 P_{k_3} Q_{<r-100} \psi_3) \nabla^{-1} P_{k_2} \psi_2 \|_{L^2_t L^\infty_x} \leq C 2^{-\frac{t}{2}} \| P_{k_1} \psi_1 \|_{S[k_1]} \| P_{k_3} \psi_3 \|_{S[k_3]}
\]

2.A.2): At least one input of (\ref{e:wave-equation}) has modulation \( \geq 2^{r-100} \).

\[
\|Q_r(\nabla^{-1} P_{k_1} Q_{<r-100} \psi_1 P_{k_3} Q_{<r-100} \psi_3) \nabla^{-1} P_{k_2} \psi_2 \|_{L^2_t L^\infty_x} \leq C \|\nabla^{-1} P_{k_1} Q_{<k_1-100} \psi_1 \|_{L^2_t L^\infty_x} \| P_{k_3} Q_{<k_1-100} \psi_3 \|_{L^\infty_t L^2_x} \leq C 2^{-\frac{t}{2}} \| P_{k_1} \psi_1 \|_{S[k_1]} \| P_{k_3} \psi_3 \|_{S[k_3]}
\]

Now proceed as before to conclude
\[
\begin{align*}
|| & \sum_{r<k_1+O(1)} \Box Q_r(\nabla^{-1} P_{k_1} Q_{<k_1-100\psi_1} P_{k_2} Q_{<k_1-100\psi_3})\nabla^{-1} P_{k_2} \psi_2||_{N[0]} \\
& \leq C 2^{\delta(k_1-k_2)} \prod_{l=1}^{3} ||P_{k_l} \psi_l||_{S[k_l]} 
\end{align*}
\tag{119}
\]

2.B):
\[
\Box(\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1}) P_{k_3} Q_{<k_1-100\psi_3} \nabla^{-1} P_{k_2} \psi_2 
\tag{120}
\]

We can estimate this directly:
\[
\begin{align*}
|| P_0 [\Box(\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1}) P_{k_3} Q_{<k_1-100\psi_3} \nabla^{-1} P_{k_2} \psi_2] ||_{N[0]} & \\
& \leq || P_0 [\Box(\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1}) P_{k_3} Q_{\geq k_1-100\psi_3} \nabla^{-1} P_{k_2} \psi_2] ||_{L^1_t L^2_x} \\
& \quad + || P_0 [\Box(\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1}) P_{k_3} \psi_3 \nabla^{-1} P_{k_2} \psi_2] ||_{L^1_t L^2_x} \\
& \leq C || \Box(\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1}) ||_{L^2_t L^\infty_x} || P_{k_3} Q_{\geq k_1-100\psi_3} ||_{L^1_t L^2_x} || \nabla^{-1} P_{k_2} \psi_2 ||_{L^\infty_t L^1_x} \\
& \quad + C || \Box(\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1}) ||_{L^2_t L^\infty_x} || P_{k_3} \psi_3 ||_{L^1_t L^1_x} || \nabla^{-1} P_{k_2} \psi_2 ||_{L^1_t L^1_x} 
\tag{121}
\end{align*}
\]

\[
\leq C 2^{\frac{k_1}{2}} \prod_{l=1}^{3} ||P_{k_l} \psi_l||_{S[k_l]} 
\]

which is of course acceptable because of \(k_1 < k_2 - 10 < O(1)\).

2.C):
\[
\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1} \Box(P_{k_3} Q_{<k_1-100\psi_3}) \nabla^{-1} P_{k_2} \psi_2 
\tag{122}
\]

This is again straightforward because
\[
\begin{align*}
|| P_0 [\nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1} \Box(P_{k_3} Q_{<k_1-100\psi_3}) \nabla^{-1} P_{k_2} \psi_2] ||_{L^1_t L^2_x} & \\
& \leq C || \nabla^{-1} P_{k_1} Q_{<k_1+100\psi_1} ||_{L^1_t L^\infty_x} || \nabla^{-1} P_{k_2} \psi_2 ||_{L^\infty_t L^1_x} || \Box(P_{k_3} Q_{<k_1-100\psi_3}) ||_{L^1_t L^2_x} \\
& \leq C 2^{\frac{k_1+k_2}{2}} \prod_{l=1}^{3} ||P_{k_l} \psi_l||_{S[k_l]} 
\tag{123}
\end{align*}
\]

This finishes the proof of the lemma, and thereby the proof of Case 1.A).
1.2): Output has modulation 2\(^l\) with \(k - 100 \leq l \leq k_1 + 100\), \(P_{k_3} \psi_3\) reduced to modulation \(2^{k-100}\): We use here the simple identity \(R_{ij} f R_{ij} g = \partial_j (\nabla^{-1} f R_{ij} g) - \partial_j (\nabla^{-1} f R_{ij} g)\) in order to pull out a derivative of \([.,.]\). This will allow us to play the modulation of the output against the larger frequencies of the inputs of \([.,.\).]

\[\sum_{k \leq \min \{k_1 + O(1), -10\}} P_0 \sum_{j=1}^3 \triangle^{-1} \partial_j P_k \{ R_{ij} P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_i P_{k_2} \psi_2 \} P_{k_3} \partial^\nu \psi_3 \]

We want to proceed in analogy to the case 1.A), by first reducing output and input \(P_{k_3} \psi_3\) to small modulation, in this case modulation \(< 2^{k-100}\), where \(k \leq \min \{k_1 + O(1), -10\}\) is held fixed. Since we are eventually summing over \(k\), we want to obtain an exponential gain in the difference \(k - k_1\). Keep in mind that \(k_1 = k_2 + O(1)\) for this case. We will use the "imbedded \(Q_{ij}\) null-form".

**Output has large modulation**

1.1): Output has modulation \(2^l\) with \(k - 100 \leq l \leq k_1 + 100\), \(P_{k_3} \psi_3\) reduced to modulation \(< 2^{k-100}\): We use here the simple identity \(R_{ij} f R_{ij} g = \partial_j (\nabla^{-1} f R_{ij} g) - \partial_j (\nabla^{-1} f R_{ij} g)\) in order to pull out a derivative of \([.,.]\). This will allow us to play the modulation of the output against the larger frequencies of the inputs of \([.,.]\):

\[\sum_{k - 100 \leq j \leq k_1 + 100} P_0 Q_l \left( \sum_{j=1}^3 P_k \triangle^{-1} \partial_j [R_{ij} P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_i P_{k_2} \psi_2] \right)\]

**1.C): Estimation of C.**

This is the case corresponding to high-high interactions in \([.,.]\), and can be rewritten as

\[\sum_{k \leq \min \{k_1 + O(1), -10\}} P_0 \left( \sum_{j=1}^3 \triangle^{-1} \partial_j P_k \left\{ R_{ij} P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_i P_{k_2} \psi_2 \right\} P_{k_3} \partial^\nu \psi_3 \right)\]

\[\sum_{k \leq \min \{k_1 + O(1), -10\}} P_0 \left( \sum_{j=1}^3 \triangle^{-1} \partial_j P_k \left\{ R_{ij} P_{k_1} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_i P_{k_2} \psi_2 \right\} P_{k_3} \partial^\nu \psi_3 \right)\]

1.2): Output has modulation \(2^l\) with \(k - 100 \leq l \leq k_1 + 100\), \(P_{k_3} \psi_3\) reduced to modulation \(\geq 2^{k-100}\):
1.3): Output has modulation $2^l$ with $l > k_1 + 100$, $P_{k_3} \psi_3$ reduced to modulation $< 2^{l-100}$: this is the case corresponding to very large modulation (by comparison with the occurring frequencies) of the output. This condition then entails that at least one input has at least comparable modulation. Thus we can write

$$\sum_{l > k_1 + 100} \sum_{j=1}^{3} P_0 Q_l (\sum_{j=1}^{3} P_k \vartriangle^{-1} \partial_j [R_\nu P_k \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} \psi_1 R_\nu P_{k_2} \psi_2] P_{k_3} Q_{< l-100} \vartheta^\nu \psi_3)$$

$$= \sum_{l > k_1 + 100} \sum_{j=1}^{3} P_0 Q_l (\sum_{j=1}^{3} P_k \vartriangle^{-1} \partial_j [R_\nu P_k Q_{\geq l-100} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_1} Q_{\geq l-100} \psi_1 R_\nu P_{k_2} \psi_2] P_{k_3} Q_{< l-100} \vartheta^\nu \psi_3)$$

$$+ \sum_{l > k_1 + 100} \sum_{j=1}^{3} P_0 Q_l (\sum_{j=1}^{3} P_k \vartriangle^{-1} \partial_j [R_\nu P_k Q_{\leq l-100} \psi_1 R_j P_{k_2} Q_{\geq l-100} \psi_2 - R_j P_{k_1} Q_{< l-100} \psi_1 R_\nu P_{k_2} Q_{\geq l-100} \psi_2] P_{k_3} Q_{< l-100} \vartheta^\nu \psi_3)$$

For example, we can estimate
\[
\sum_{l > k_1 + 100} \sum_{j=1}^{3} ||P_0 Q_l [\triangle^{-1} \partial_j P_k (R_{\nu} P_{k_1} Q_{\geq l-100} \psi_1 R_j P_{k_2} \psi_2) P_{k_3} Q_{\leq l-100}^\nu \psi_3]_N [0] ||
\leq \sum_{l > k_1 + 100} \sum_{j=1}^{3} C 2^{-\frac{j}{2}} ||\triangle^{-1} \partial_j P_k (R_{\nu} P_{k_1} Q_{\geq l-100} \psi_1 R_j P_{k_2} \psi_2)||_{L_t^1 L_x^\infty} ||P_{k_3} Q_{< l-100}^\nu \psi_3||_{L_t^1 L_x^2}
\]

\[
\sum_{l > k_1 + 100} \sum_{j=1}^{3} 2^{-\frac{j}{2}} 2^{2k} ||R_{\nu} P_{k_1} Q_{\geq l-100} \psi_1||_{L_t^1 L_x^2} ||R_j P_{k_2} \psi_2||_{L_t^1 L_x^2} ||P_{k_3} Q_{< l-100}^\nu \psi_3||_{L_t^1 L_x^2}
\]

(126)

\[
\sum_{l > k_1 + 100} \sum_{j=1}^{3} C 2^{k-k_1-1} \prod_{a=1}^{3} ||P_{k_a} \psi_a||_{S[k_a]}
\]

\[
\leq C 2^{2(k-k_1)} \prod_{a=1}^{3} ||P_{k_a} \psi_a||_{S[k_a]}
\]

The other terms in (125) are estimated similarly and therefore left out.

1.4): Output has modulation \(2^l\) with \(l > k_1 + 100\), \(P_{k_3} \psi_3\) reduced to modulation \(\geq 2^{l-100}\):

\[
\sum_{l > k_1 + 100} ||P_0 Q_l [P_k \triangle^{-1} \partial_j (R_{\nu} P_{k_1} \psi_1 R_j \psi_2 - R_j P_{k_1} \psi_1 R_{\nu} \psi_2) P_{k_3} Q_{\geq l-100}^\nu \psi_3]||_{L_t^1 L_x^2}
\]

\[
\leq \sum_{l > k_1 + 100} ||P_k \triangle^{-1} \partial_j (R_{\nu} P_{k_1} \psi_1 R_j \psi_2 - R_j P_{k_1} \psi_1 R_{\nu} \psi_2)||_{L_t^1 L_x^\infty} ||P_{k_3} Q_{\geq l-100}^\nu \psi_3||_{L_t^2 L_x^2}
\]

\[
\leq \sum_{l > k_1 + 100} C 2^{k-k_1} \prod_{a=1}^{3} ||P_{k_a} \psi_a||_{S[k_a]}
\]

(127)

\[
\leq C 2^{k-k_1} \prod_{a=1}^{3} ||P_{k_a} \psi_a||_{S[k_a]}
\]

Output has small modulation

2.1): Output has modulation \(< 2^{k-100}\), \(P_{k_3} \psi_3\) reduced to modulation \(\geq 2^{k+100}\); notice the identity
\[ P_{0}Q_{<k-100}[P_{k}\Delta^{-1}\partial_{j}(R_{u}P_{k_1}\psi_{1}R_{j}\psi_{2} - R_{j}P_{k_1}\psi_{1}R_{u}P_{k_2}\psi_{2})]P_{k_3}Q_{\geq k+100}\partial^\nu\psi_{3}] \]

\[ = \sum_{l \geq k+100} P_{0}Q_{<k-100}[P_{k}Q_{l+O(1)}\Delta^{-1}\partial_{j}(R_{u}P_{k_1}\psi_{1}R_{j}\psi_{2} - R_{j}P_{k_1}\psi_{1}R_{u}P_{k_2}\psi_{2})]P_{k_3}Q_{l}\partial^\nu\psi_{3}] \]

Next, observe that

\[ P_{k}Q_{l+O(1)}(P_{k_1}fP_{k_2}g) = \]

\[ P_{k}Q_{l+O(1)}(P_{k_1}P_{l+O(1)}Q_{<l-100}f)P_{k_2}Q_{<l-100}g) \]

\[ + P_{k}Q_{l+O(1)}(P_{k_1}Q_{\geq l-100}fP_{k_2}g) \]

\[ + P_{k}Q_{l+O(1)}(P_{k_1}Q_{<l-100}fP_{k_2}Q_{\geq l-100}g) \]

Thus we conclude that this case can be estimated by

\[ \sum_{l \geq k+100} C2^{2k}\|R_{u}P_{l+O(1)}P_{k_1}Q_{<l-100}\psi_{1}\|_{L^2_xL^2_t}\|R_{j}P_{k_2}Q_{<l-100}\psi_{2}\|_{L^2_xL^2_t}\|P_{k_3}Q_{l}\partial^\nu\psi_{3}\|_{L^2_xL^2_t} \]

\[ + \sum_{l \geq k+100} C2^{2k}\|R_{u}P_{k_1}Q_{\geq l-100}\psi_{1}\|_{L^2_xL^2_t}\|R_{j}P_{k_2}\psi_{2}\|_{L^2_xL^2_t}\|P_{k_3}Q_{l}\psi_{3}\|_{L^2_xL^2_t} \]

\[ + \sum_{l \geq k+100} 2^{2k}\|R_{u}P_{k_1}Q_{<l-100}\psi_{1}\|_{L^2_xL^2_t}\|R_{j}P_{k_2}Q_{\geq l-100}\psi_{2}\|_{L^2_xL^2_t}\|P_{k_3}Q_{l}\partial^\nu\psi_{3}\|_{L^2_xL^2_t} \]

\[ \leq \sum_{l \geq k+100, l = k_1 + O(1)} C2^{k-\frac{3}{l}} \prod_{a=1}^{3} \|P_{k_a}\psi_{a}\|_{S[k_a]} + \sum_{l \geq k+100} C2^{2k-\frac{3}{l}} \prod_{a=1}^{3} \|P_{k_a}\psi_{a}\|_{S[k_a]} \]

\[ \leq C2^{\frac{k+10}{k}} \prod_{a=1}^{3} \|P_{k_a}\psi_{a}\|_{S[k_a]} \]

Having reduced output and \( P_{k_3}\psi_{3} \) to small modulation, we can now invoke the identity \([105]\) and proceed in exact analogy with case 1.A. Since this does not entail any additional difficulties, it is left out. This then finishes the low-high case 1).

2): High-High Interactions: \([\_] \) at frequency \( 2^{k_3+O(1)} \geq 2^{-10}, k_3 \geq -10, \text{i.e.} \ k_3 \geq O(1) \).

We will again utilize the algebraic structure of the null-form, but in a somewhat different fashion than before. The first step will involve reduction to comparably low modulation of output and third input. Having achieved this, we will try to pull out the \( \partial_{u} \) from the input of \( [\_] \) with larger frequency: Assume w.l.o.g. in the sequel that the 1st input of \( [\_] \) has larger frequency, i.e. \( k_2 \leq k_1 \).
2.1): $P_{k_3} \partial^\nu \psi_3$ has modulation $\geq 2^{k_3}$:

$$
||P_0 \sum_{j=1}^{3} \Delta^{-1} \partial_j [R_\nu \partial_{\nu} \psi_1 R_j P_{k_3} \psi_2 - R_j P_{k_3} \psi_1 R_\nu \partial_{\nu} P_{k_3} ||_{L^2_t L^2_x}^2
$$

$$
\leq C \sum_{j=1}^{3} 2^{-k_3} ||R_\nu \partial_{\nu} \psi_1 R_j P_{k_3} \psi_2 - R_j \partial_{\nu} P_{k_3} \psi_1 R_\nu ||_{L^2_t L^2_x}^2 \leq C 2^{-k_3} \sum_{\alpha=1}^{3} ||P_{k_3} \psi_\alpha||_{S[\kappa_3]}
$$

2.2): Output at modulation $\geq 2^0$, $P_{k_3} \psi_3$ at modulation $< 2^{k_3}$:

$$
||P_0 Q_{\geq 0} \sum_{j=1}^{3} \Delta^{-1} \partial_j [R_\nu \partial_{\nu} \psi_1 R_j P_{k_3} \psi_2 - R_j \partial_{\nu} P_{k_3} \psi_1 R_\nu ||_{L^2_t L^2_x}^2 \leq C 2^{-k_3} \sum_{\alpha=1}^{3} ||P_{k_3} \psi_\alpha||_{S[\kappa_3]}
$$

2.3): Output at modulation $< 2^0$, $P_{k_3} \psi_3$ at modulation $< 2^{k_3}$: A simple algebraic manipulation reduces this case to the sum

$$
||P_0 Q_{<0} \sum_{j=1}^{3} \Delta^{-1} \partial_j [R_\nu \partial_{\nu} \psi_1 R_j P_{k_3} \psi_2 \nabla^{-1} P_{k_3} \psi_1 ||_{L^2_t L^2_x}^2 \leq ||P_0 Q_{<0} \sum_{j=1}^{3} R_\nu \partial_{\nu} \psi_2 \nabla^{-1} P_{k_3} \psi_1 \nabla^{-1} P_{k_3} \psi_1 ||_{L^2_t L^2_x}^2
$$

For the first of these, we want to remove $\partial_{\nu} \psi$ from $[\cdot, \cdot]$, as this term might have very large modulation. We are in a favorable situation since letting $\partial_{\nu}$ fall on the output is harmless on account of our assumptions, while letting it fall on the third term $P_{k_3} \psi_3$ is quite useful, as it produces a $\Box$-operator. Thus we majorize the first term of (130) by

$$
\sum_{j=1}^{3} ||P_0 Q_{<0} \partial_{\nu} (\Delta^{-1} \partial_j [R_\nu \partial_{\nu} \psi_2 \nabla^{-1} P_{k_3} \psi_1 ||_{L^2_t L^2_x}^2
$$

$$
\sum_{j=1}^{3} ||P_0 Q_{<0} (\Delta^{-1} \partial_j [R_\nu \partial_{\nu} \psi_2 \nabla^{-1} P_{k_3} \psi_1 ||_{L^2_t L^2_x}^2
$$
For the first term in the immediately preceding, we have

\[
\sum_{j=1}^{3} ||P_0 Q_{<0} \partial_\nu (\Delta^{-1} \partial_j [R_j P_{k_2} \psi_2 \nabla^{-1} P_{k_1} \psi_1] P_{k_3} Q_{<k_3} \partial' \psi_3)||_{L^1_t L^2_x} \\
\leq C 2^{-k_1} ||P_{k_2} \psi_2||_{L^1_t L^2_x} ||P_{k_1} \psi_1||_{L^1_t L^2_x} ||P_{k_3} Q_{<k_3} \partial' \psi_3||_{L^1_t L^2_x} \\
\leq C 2^{-k_1} 2^{\frac{k_1+k_2+k_3}{3}} \prod_{\alpha=1}^{3} ||P_{k_\alpha} \psi_\alpha||_{S[k_\alpha]} \\
\leq C 2^{-k_1} 2^{\frac{k_1+k_2+k_3}{3}} \prod_{\alpha=1}^{3} ||P_{k_\alpha} \psi_\alpha||_{S[k_\alpha]} 
\]

(131)

We have used here that in the present situation \(k_1 \geq k_3 + O(1)\).

For the 2nd term, we have

\[
||P_0 Q_{<0} (\Delta^{-1} \partial_j [R_j P_{k_2} \psi_2 \nabla^{-1} P_{k_1} \psi_1] \Box P_{k_3} Q_{<k_3} \psi_3)||_{L^1_t L^2_x} \\
\leq C 2^{-k_1} ||R_j P_{k_2} \psi_2||_{L^1_t L^2_x} ||P_{k_1} \psi_1||_{L^1_t L^2_x} ||\Box P_{k_3} Q_{<k_3} \psi_3||_{L^1_t L^2_x} \\
\leq C 2^{-k_1} 2^{\frac{k_2-k_1}{3}} \prod_{\alpha=1}^{3} ||P_{k_\alpha} \psi_\alpha||_{S[k_\alpha]} \\
\leq C 2^{-k_1} 2^{\frac{k_2-k_1}{3}} \prod_{\alpha=1}^{3} ||P_{k_\alpha} \psi_\alpha||_{S[k_\alpha]} 
\]

(132)

Hence in order to finish case 2), we need to deal with the 2nd term in (131). But this is immediate, referring to lemma 4.5 in the preceding section, as well as the disposability of \(P_0 Q_{<0}, P_{k_3} Q_{<k_3}\).

3): High-Low Interactions: \([,] \) at frequency \(\geq 2^{-10}, k_3 < -10\).

This case is the most elementary on account of the fact that a low frequency term is hit by a derivative. Moreover, it can be dealt with by the same methods as in the immediately preceding case, so we shall discuss it only briefly. First, one reduces the output to modulation \(< 2^0\) and the third input \(P_{k_3} \psi_3\) to modulation \(< 2^{k_3}\). Then, as we only have to deal with the case when the inputs \(\psi_1, \psi_2\) are at frequencies \(2^{k_1}, 2^{k_2}\) with \(k_1 = k_2 + O(1) >> O(1)\) (otherwise, evaluate all inputs in \(L^3_t L^6_x\), we use as before the identity
\[
\sum_{j=1}^{3} P_{0}Q_{<0} (\Delta^{-1} \partial_j [R_{\nu}P_{k_3} \psi_1 R_j P_{k_2} \psi_2 - R_j P_{k_3} \psi_1 R_{\nu} P_{k_2} \psi_2] P_{k_3} Q_{<k_3} \partial^\nu \psi_3) \\
= \sum_{j=1}^{3} P_{0}Q_{<0} \partial_{\nu} (\Delta^{-1} \partial_j [\nabla^{-1} P_{k_1} \psi_1 R_j P_{k_2} \psi_2] \partial^\nu P_{k_3} Q_{<k_3} \psi_3) \\
- \sum_{j=1}^{3} P_{0}Q_{<0} (\Delta^{-1} \partial_j [\nabla^{-1} P_{k_1} \psi_1 R_j P_{k_2} \psi_2] \Box P_{k_3} Q_{<k_3} \psi_3) \\
- P_{0}Q_{<0} (\nabla^{-1} P_{k_1} \psi_1 R_{\nu} P_{k_2} \psi_2) \partial^\nu P_{k_3} Q_{<k_3} \psi_3
\]

(133)

Each of these can be easily estimated:

\[
\left| \sum_{j=1}^{3} P_{0}Q_{<0} \partial_{\nu} (\Delta^{-1} \partial_j [\nabla^{-1} P_{k_1} \psi_1 R_j P_{k_2} \psi_2] \partial^\nu P_{k_3} Q_{<k_3} \psi_3) \right|_{L^1 L^2} \\
\leq \sum_{j=1}^{3} \left| P_{0}Q_{<0} \partial_{\nu} (\Delta^{-1} \partial_j [\nabla^{-1} P_{k_1} \psi_1 R_j P_{k_2} \psi_2] \partial^\nu P_{k_3} Q_{<k_3} \psi_3) \right|_{L^1 L^2} \\
\leq \sum_{j=1}^{3} C \| \nabla^{-1} P_{k_1} \psi_1 \|_{L^2 L^2} \| R_j P_{k_2} \psi_2 \|_{L^2 L^2} \| \partial^\nu P_{k_3} Q_{<k_3} \psi_3 \|_{L^2 L^2} \\
\leq C 2^{-\frac{2k_3}{3}} 2^{-\frac{k_\psi}{6}} \prod_{a=1}^{3} \| P_{k_a} \psi_a \|_{S[k_a]}
\]

(134)

\[
\left| \sum_{j=1}^{3} P_{0}Q_{<0} (\Delta^{-1} \partial_j [\nabla^{-1} P_{k_1} \psi_1 R_j P_{k_2} \psi_2] \Box P_{k_3} Q_{<k_3} \psi_3) \right|_{L^1 L^2} \\
\leq \sum_{j=1}^{3} \left| P_{0}Q_{<0} (\Delta^{-1} \partial_j [\nabla^{-1} P_{k_1} \psi_1 R_j P_{k_2} \psi_2] \Box P_{k_3} Q_{<k_3} \psi_3) \right|_{L^1 L^2} \\
\leq \sum_{j=1}^{3} C \| \nabla^{-1} P_{k_1} \psi_1 R_j P_{k_2} \psi_2 \|_{L^2 L^2} \Box P_{k_3} Q_{<k_3} \psi_3 \|_{L^2 L^2} \\
\leq C 2^{-k_1} 2^{-\frac{2k_3}{3}} \prod_{a=1}^{3} \| P_{k_a} \psi_a \|_{S[k_a]}
\]

(135)

\[
\left| P_{0}Q_{<0} (\nabla^{-1} P_{k_1} \psi_1 R_{\nu} P_{k_2} \psi_2) \partial^\nu P_{k_3} Q_{<k_3} \psi_3 \right|_{L^1 L^2} \\
\leq C 2^{-\frac{2k_3}{3}} 2^{-\frac{k_\psi}{6}} \prod_{a=1}^{3} \| P_{k_a} \psi_a \|_{S[k_a]}
\]

(136)
Of course, in order to verify the statement of the Proposition, we can even discard the explicit gain in $k_1$. This finishes the proof of the Proposition.

5.4. The Gauge Change Estimate.

**Proposition 5.5.** Let $f(x)$ be a smooth function with all derivatives up to and including fourth order bounded. Then, provided $\tilde{\phi}_j, \psi$ are Schwartz functions with $\max\{||P_k\tilde{\phi}_j||_{S[k]}, ||P_k\psi||_{S[k]}\} \leq c_k$ for a frequency envelope $c_k$ as in the previous section, we have

$$||P_k(f(\Delta^{-1}\sum_{j=1}^{3} \partial_j \tilde{\phi}_j)\psi)||_{S[k]} \leq Cc_k$$  \hspace{1cm} (137)

**Proof** The proof of this assertion will consist in the careful analysis of many different cases. In particular, the following observation will be used many times: Let $u \in C^\infty(\mathbb{R}^n)$, and $F \in C^\infty(\mathbb{R})$.

**Lemma 5.6.** Let $X$ be a translation invariant norm defined on all measurable functions, and $F, u$ as before. Then

$$||P_0(F(u))||_X \leq C||P_0(\nabla uF'(u))||_X$$  \hspace{1cm} (138)

The proof of this is immediate: We have

$$P_0(F(u)) = P_0\tilde{P}_0(F(u)) = P_0(\sum_{j=1}^{n} \Delta^{-1}\partial_j \tilde{P}_0(F(u)))$$

$$= P_0 \sum_{j} \int_{\mathbb{R}^n} \Delta^{-1}\partial_j \hat{m}_0(x-y)\partial_j(F(u))(y)dy$$  \hspace{1cm} (139)

where $a_j(y) = \Delta^{-1}\partial_j(\hat{m}_0)(y)$, $m_0$ is the symbol of $\tilde{P}_0$, and $\tilde{P}_0$ is a Fourier multiplier like $P_0$ whose symbol equals 1 on the Fourier support of $P_0$. Since $a(y)$ has finite $L^1$-norm, the claim follows.

The reason why the previous lemma might be useful is that by hitting the function $f$ in the Proposition with a derivative, we gain an extra factor which is morally equivalent to $\phi$. The more such factors are present, the more freedom is gained in proving the necessary estimates. In particular, we must and can avoid to expand the function into a power series(which would require real analyticity anyways), for
then the crucial $L^\infty$ bound would be lost.

The following lemma will also be used many times in the sequel:

**Lemma 5.7.** Assume that $\|P_k \psi_j\|_{S[k]} \le c_k$, with $c_k$ a frequency envelope as before, $\psi_j \in \mathcal{S}(\mathbb{R}^{3+1})$, $j = 1, 2, 3$. Provided that $k \le j + O(1)$, we have

$$
\|P_k Q_j(f \sum_{j=1}^{3} \Delta^{-1} \partial_j \psi_j)\|_{L^2_t L^2_x} \le C 2^{-3j - \frac{3}{2}}
$$

(140)

**Proof** Replace $\sum_{j=1}^{3} \Delta^{-1} \partial_j \psi$ formally for simplicity’s sake by $\nabla^{-1} \psi$. For the sake of simplicity, assume that $|O(1)|$ in the formulation of the theorem and throughout the rest of the paper is $\le 50$. We split the expression as follows:

$$
P_k Q_j(f(\nabla^{-1} \psi)) = \square^{-1} P_k Q_j(\square \nabla^{-1} \psi f'(\nabla^{-1} \psi)) + \square^{-1} P_k Q_j(\partial_\nu \nabla^{-1} \psi \partial^\nu \nabla^{-1} \psi f''(\nabla^{-1} \psi)) + \square^{-1} P_k Q_j(P_{k+O(1)} \square \nabla^{-1} \psi P_{<k-100} f'(\nabla^{-1} \psi)) \quad (141)
$$

where $\square^{-1} P_k Q_j$ is a Fourier multiplier with symbol $\frac{m_k(|\xi|)m_\nu(|\tau| - |\xi|)}{|\tau^\nu|}$, This operator is actually disposable on account of $j \ge k + O(1)$, but its boundedness on $L^2_t L^2_x$ is entirely elementary. Indeed, its operator norm is dominated by $C 2^{-2j}$. The first 3 terms above represent the usual trichotomy into high-low, low-high, high-high interactions. We treat these first, the last term being most elementary.

1) High-Low: Split this as follows:

$$
\square^{-1} P_k Q_j(P_{k+O(1)} \square \nabla^{-1} \psi P_{<k-100} f'(\nabla^{-1} \psi)) = \square^{-1} P_k Q_j(P_{k+O(1)}(Q_{<j+100} \square \nabla^{-1} \psi P_{<k-100} f'(\nabla^{-1} \psi))) + \square^{-1} P_k Q_j(P_{k+O(1)}(Q_{\ge j+100} \square \nabla^{-1} \psi P_{<k-100} f'(\nabla^{-1} \psi)))
$$

(142)

The $L^2_t L^2_x$-norm of the first term can be estimated by $C 2^{-\frac{j}{2} - 2^{-2j} c_k} \le C c_k 2^{-\frac{j}{2} - j^{-\frac{3}{2}}}$ by placing $P_{k+O(1)}(Q_{<j+100} \square \nabla^{-1} \psi$ into $L^2_t L^2_x$. As to the 2nd term, write it as

$$
\square^{-1} P_k Q_j \sum_{l \ge j+100} P_{k+O(1)} Q_l \square \nabla^{-1} \psi P_{<k-100} Q_{l+O(1)} f'(\nabla^{-1} \psi)
$$

(143)

where we have used the fact that if $(\tau_1, \xi_1)$ denotes a point in the Fourier support of $\square \nabla^{-1} \psi$ and $(\tau_2, \xi_2)$ denotes a point in the Fourier support of $P_{<k-100} f'(\nabla^{-1} \psi)$,
while \((\tau, \xi)\) denotes as point in the Fourier support of the output, we have the condition

\[
2^j \sim ||\tau| - |\xi|| = |\tau_1 \pm |\xi_1| + \tau_2 \pm |\xi_2| + \xi_1 \mp |\xi_2| \pm |\xi_1 + \xi_2| (144)
\]

where the signs have been chosen in such a way that \(|\tau_a \pm |\xi_a||, |\tau_1 + \tau_2 \pm |\xi_1 + \xi_2||, a \in 1, 2, \) This forces the factor \(P_{\leq k - 100} f'(\nabla^{-1} \psi)\) to be microlocalized at the indicated modulation. Now use the \(L^*_\ell L^*_2\)-boundedness of the operator \(\nabla^{-1}_t P_{t} Q_{t+O(1)}\) with symbol \(\frac{m_{\pm k}(\xi)^{m_{\pm 1}(||\tau| - |\xi||)}}{L^*_\ell L^*_2}\); indeed, this operator is given by convolution with a kernel whose \(L^1\)-norm is \(\leq C 2^{-\ell}\). This allows us to estimate this term by

\[
\begin{align*}
\| \sum_{l \geq j + 100} P_{k + O(1)} Q_l \nabla^{-1}_t \psi P_{k - 100} Q_{t + O(1)} (f'(\nabla^{-1} \psi)) \|_{L^*_\ell L^*_2} \\
\leq \| \sum_{l \geq j + 100} P_{k + O(1)} Q_l \nabla^{-1}_t \psi P_{k - 100} Q_{t + O(1)} (\partial_t \nabla^{-1}_t \psi f''(\nabla^{-1} \psi)) \|_{L^*_\ell L^*_2} \\
\leq \sum_{l \geq j + 100} \| P_{k + O(1)} Q_l \nabla^{-1}_t \psi \|_{L^*_\ell L^*_2} \| \nabla^{-1}_t P_{k - 100} Q_{t + O(1)} (\nabla^{-1}_t \partial_t \psi f''(\nabla^{-1} \psi)) \|_{L^*_\ell L^*_2} \\
\leq \sum_{l \geq j + 100} C 2 l c_k 2^{-\frac{\ell}{2} - k + \frac{j}{4}} \leq C c_k 2^{-\frac{\ell}{2} - k}
\end{align*}
\]

Hence the \(L^*_\ell L^*_2\)-norm of the 2nd term in (142) can be estimated by an expression of the desired form.

2): Low-High: Split this term as follows:

\[
\begin{align*}
\square^{-1} P_{k} Q_j (P_{k - 100} \square \nabla^{-1}_t \psi P_{k + O(1)} (f'(\nabla^{-1} \psi))) \\
= \square^{-1} P_{k} Q_j (P_{k - 100} Q_{j + 100} \square \nabla^{-1}_t \psi P_{k + O(1)} (f'(\nabla^{-1} \psi))) \\
+ \square^{-1} P_{k} Q_j (P_{k - 100} Q_{\geq j + 100} \square \nabla^{-1}_t \psi P_{k + O(1)} (f'(\nabla^{-1} \psi)))
\end{align*}
\]

We have

\[
\begin{align*}
\square^{-1} P_{k} Q_j (P_{k - 100} Q_{j + 100} \square \nabla^{-1}_t \psi P_{k + O(1)} (f'(\nabla^{-1} \psi))) \\
= \square^{-1} P_{k} Q_j (P_{k - 100} Q_{j + 100} \square \nabla^{-1}_t \psi \sum_{\alpha = 1}^{3} P_{k + O(1)} \triangle^{-1} \partial_{\alpha} \nabla^{-1}_t \psi f''(\nabla^{-1} \psi)))
\end{align*}
\]

whence

\[
\begin{align*}
\| \square^{-1} P_{k} Q_j (P_{k - 100} Q_{j + 100} \square \nabla^{-1}_t \psi P_{k + O(1)} (f'(\nabla^{-1} \psi))) \|_{L^*_\ell L^*_2} \\
\leq 2^{-2j} \| P_{k - 100} Q_{j + 100} \square \nabla^{-1}_t \psi \|_{L^*_\ell L^*_2} \sum_{\alpha = 1}^{3} P_{k + O(1)} \triangle^{-1} \partial_{\alpha} \nabla^{-1}_t \psi f''(\nabla^{-1} \psi)) \|_{L^*_\ell L^*_2} \\
\leq 2^{-2j} C 2^{\frac{\ell}{2} + \frac{\ell}{2} - k} \leq C 2^{-\frac{\ell}{2} - k}
\end{align*}
\]

Also,

\[
\begin{align*}
\sum_{l \geq j + 100} \square^{-1} P_{k} Q_j (P_{k - 100} Q_l \square \nabla^{-1}_t \psi P_{k + O(1)} Q_{t + O(1)} (f'(\nabla^{-1} \psi))) \\
= \sum_{l \geq j + 100} \square^{-1} P_{k} Q_j (P_{k - 100} Q_l \square \nabla^{-1}_t \psi \nabla^{-1}_t P_{k + O(1)} Q_{t + O(1)} (\partial_t \nabla^{-1}_t \psi f''(\nabla^{-1} \psi)))
\end{align*}
\]
Therefore
\[\sum_{l_j \geq j+100} ||\Box^{-1} P_k Q_{j} (P_{l_j-100} Q_{l_j} \Box \nabla^{-1} \psi P_{l_j+O(1)} Q_{l_j+O(1)} (f'(\nabla^{-1} \psi)))||_{L^2_t L^\infty_x} \]
\[\leq 2^{-2j} \sum_{l_j \geq j+100, r < k-100} ||P_{l_j} \Box \nabla^{-1} \psi||_{L^2_t L^\infty_x} 2^{-l_j} ||\partial_t \nabla^{-1} \psi f''(\nabla^{-1} \psi)||_{L^2_t L^3_x} \]
\[\leq 2^{-2j} \sum_{l_j \geq j+100, r < k-100} C 2^{\frac{-r-1}{2}} c_r \leq 2^{-2j} C \]

3): High-High interactions: We have
\[\sum_{l_1 = l_2 + O(1) \geq k+O(1)} \Box^{-1} P_{l_1} Q_{j} (P_{l_1} \Box \nabla^{-1} \psi P_{l_2} (f'(\nabla^{-1} \psi))) \]
\[= \sum_{l_1 = l_2 + O(1) \geq k+O(1)} \Box^{-1} P_{l_1} Q_{j} (P_{l_1} Q_{\leq l_1+10} (\Box \nabla^{-1} \psi) \sum_{a=1}^{3} P_{l_2} \triangle^{-1} \partial_a (\partial_a \nabla^{-1} \psi f''(\nabla^{-1} \psi))) \]
\[+ \sum_{l_1 = l_2 + O(1) \geq k+O(1)} \Box^{-1} P_{l_1} Q_{j} (P_{l_1} Q_{> l_1+10} (\Box \nabla^{-1} \psi) P_{l_2} (f'(\nabla^{-1} \psi))) \]
\[\leq 2^{-2j} \sum_{l_1 = l_2 + O(1) \geq k+O(1)} 2^k C c_{l_1} 2^{-l_2} \leq 2^{-2j} C \]

where in the last step we have used Bernstein’s inequality.

As to the 2nd term in (145), we have
\[\sum_{l_1 = l_2 + O(1) \geq k+O(1)} \Box^{-1} P_{l_1} Q_{j} (P_{l_1} Q_{> l_1+10} (\Box \nabla^{-1} \psi) P_{l_2} (f'(\nabla^{-1} \psi))) \]
\[= \sum_{l_1 = l_2 + O(1) \geq k+O(1)} \Box^{-1} P_{l_1} Q_{j} (P_{l_1} Q_{> l_1+10} (\Box \nabla^{-1} \psi) P_{l_2} (f'(\nabla^{-1} \psi))) \]
\[+ \sum_{l_1 = l_2 + O(1) \geq k+O(1)} \Box^{-1} P_{l_1} Q_{j} (P_{l_1} Q_{> l_1+10} (\Box \nabla^{-1} \psi) P_{l_2} (f'(\nabla^{-1} \psi))) \]

The 2nd term in the immediately preceding is estimated as follows:
\[\sum_{l_1 = l_2 + O(1) \geq k+O(1)} ||\Box^{-1} P_{l_1} Q_{j} (P_{l_1} Q_{> l_1+10} (\Box \nabla^{-1} \psi) P_{l_2} (f'(\nabla^{-1} \psi)))||_{L^2_t L^\infty_x} \]
\[\leq \sum_{j+100 \geq r > l_1+10} \sum_{l_1 = l_2 + O(1) \geq k+O(1)} ||\Box^{-1} P_{l_1} Q_{j} (P_{l_1} Q_{r} (\Box \nabla^{-1} \psi) P_{l_2} (f'(\nabla^{-1} \psi)))||_{L^2_t L^\infty_x} \]
\[\leq C 2^{-2j} ||P_{l_1} Q_{r} (\Box \nabla^{-1} \psi)||_{L^2_t L^\infty_x} ||P_{l_2} (f'(\nabla^{-1} \psi))||_{L^\infty_t L^\infty_x} \]
\[\leq C 2^{-2j} \sum_{j+100 \geq r > l_1+10} \sum_{l_1 = l_2 + O(1) \geq k+O(1)} 2^{\frac{-r-1}{2}} c_{l_1} \leq 2^{-\frac{3j}{2} - \frac{1}{2}} C \]
As to the 3rd term in (146), we have
\[
\sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (Q_r \Box \nabla^{-1} \psi P_2 \left( f''(\nabla^{-1} \psi) \right))
\]
\[
= \sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (Q_r \Box \nabla^{-1} \psi P_2 Q_{r+O(1)} \left( f''(\nabla^{-1} \psi) \right))
\]
Arguing as in the corresponding high-modulation-inputs case for the high-low or low-high interactions, we have
\[
\sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (P_1 Q_r \Box \nabla^{-1} \psi P_2 Q_{r+O(1)} \left( f''(\nabla^{-1} \psi) \right))
\]
\[
= \sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (P_1 Q_r \Box \nabla^{-1} \psi \nabla^{-1} P_2 Q_{r+O(1)} \partial_t \nabla^{-1} \psi f''(\nabla^{-1} \psi))
\]
\[
+ \sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (P_1 Q_r \Box \nabla^{-1} \psi \nabla^{-1} P_2 Q_{r+O(1)} \partial_t \nabla^{-1} \psi \sum_{a=1}^3 P_{g \leq l_2 - 100} \triangle^{-1} \partial_a \partial_a \nabla^{-1} \psi f''(\nabla^{-1} \psi))
\]
Now
\[
\sum_{l_1 = l_2 + O(1)} \sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (P_1 Q_r \Box \nabla^{-1} \psi \nabla^{-1} P_2 Q_{r+O(1)} \partial_t \nabla^{-1} \psi \sum_{a=1}^3 P_{g \leq l_2 - 100} \triangle^{-1} \partial_a \partial_a \nabla^{-1} \psi f''(\nabla^{-1} \psi))
\]
\[
P_{\leq l_2 - 100} f''(\nabla^{-1} \psi)) || L^1_t L^2_x \leq 2^{-2j} \sum_{l_1 = l_2 + O(1)} \sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (P_1 Q_r \Box \nabla^{-1} \psi \nabla^{-1} P_2 Q_{r+O(1)} \partial_t \nabla^{-1} \psi \sum_{a=1}^3 P_{g \leq l_2 - 100} \triangle^{-1} \partial_a \partial_a \nabla^{-1} \psi f''(\nabla^{-1} \psi))
\]
\[
P_{\leq l_2 - 100} f''(\nabla^{-1} \psi)) || L^1_t L^2_x \leq 2^{-2j} \sum_{l_1 = l_2 + O(1)} \sum_{r > \max\{j+100,l_1+10\}} 2^{r-2} \frac{1}{\triangle} c_1 2^{-r} c_2 \leq 2^{-2j} C
\]
Moreover
\[
\sum_{l_1 = l_2 + O(1)} \sum_{r > \max\{j+100,l_1+10\}} \Box^{-1} P_k Q_j (P_1 Q_r \Box \nabla^{-1} \psi \nabla^{-1} P_2 Q_{r+O(1)} \partial_t \nabla^{-1} \psi \sum_{a=1}^3 P_{g \leq l_2 - 100} \triangle^{-1} \partial_a \partial_a \nabla^{-1} \psi f''(\nabla^{-1} \psi)) || L^1_t L^2_x
\]
\[
\leq 2^{-2j} \sum_{l_1 = l_2 + O(1)} \sum_{r > \max\{j+100,l_1+10\}} 2^k \Box^{-1} P_k Q_j (P_1 Q_r \Box \nabla^{-1} \psi \nabla^{-1} P_2 Q_{r+O(1)} \partial_t \nabla^{-1} \psi || L^1_t L^2_x || L^1_t L^4_x) C 2^{-l_2}
\]
\[
|| \partial_a \nabla^{-1} \psi || L^1_t L^2_x \leq 2^{-2j} \sum_{l_1 = l_2 + O(1)} \sum_{r > \max\{j+100,l_1+10\}} C 2^k l_2 2^{\frac{1}{2} - r} \leq 2^{-2j} C
\]
where we have used Bernstein’s inequality for the last step.
The last term in (141) can be estimated easily by placing $\partial^\nu \nabla^{-1} \psi, \partial^\nu \nabla^{-1} \psi$ into $L^4_t L^4_x$.

Continuing with the proof of the Proposition, we split into High-Low, Low-High and High-High interactions:

$$P_0[f(\sum_{j=1}^{3} \triangle^{-1} \partial_j \tilde{\phi}_j) \psi] = P_{-10,10}f(\sum_{j=1}^{3} \triangle^{-1} \partial_j \tilde{\phi}_j)P_{<15} \psi + P_{<10}f(\sum_{j=1}^{3} \triangle^{-1} \partial_j \tilde{\phi}_j)P_{-10,10} \psi$$

$$+ \sum_{k_1>10,k_2=k_1+O(1)} P_{k_1} f(\sum_{j=1}^{3} \triangle^{-1} \partial_j \tilde{\phi}_j) P_{k_2} \psi$$

Introduce the notation $\sum_{j=1}^{3} \triangle^{-1} \partial_j \tilde{\phi}_j := \Phi$. 
High-Low Interactions.

This case is the most elementary, as one can immediately employ lemma \textit{5.6} in order to introduce a bilinear structure, without incurring any losses.

1): $L_t^\infty L_x^2$-norm of the output. We have

\begin{equation}
\|P_0 \partial_t [P_{-10,10} f(\Phi) P_{<15} \psi] \|_{L_t^\infty L_x^2} \\
\leq \|P_0 [P_{-10,10} (\partial_t \Phi f'(\Phi)) P_{<15} \psi] \|_{L_t^\infty L_x^2} \\
+ \sum_{a=1}^{3} \|P_0 [\Delta^{-1} \partial_a P_{-10,10} (\partial_a \Phi f'(\Phi)) P_{<15} \psi] \|_{L_t^\infty L_x^2} \\
\leq C \sum_{k_1 \leq 0, k_2 < 15} \|P_{k_1} \nabla_{x,t} \Phi \|_{L_t^\infty L_x^4} \|P_{k_2} \psi \|_{L_t^\infty L_x^4} \\
+ C \sum_{k_1 > 0, k_2 < 15} \|P_{k_1} \nabla_{x,t} \Phi \|_{L_t^\infty L_x^2} \|P_{k_2} \psi \|_{L_t^\infty L_x^2} \\
\leq C \sum_{k_1 \leq 0, k_2 < 15} 2^{k_1+k_2} c_{k_1} c_{k_2} + \sum_{k_1 > 0, k_2 < 15} 2^{-k_1} 2^{k_2} c_{k_1} c_{k_2} \\
\leq C \tau_0^2
\end{equation}

2): $X_0^{1/2} \cdot \frac{3}{2}$-norm of the output.

2.1) Output at modulation $2^j$ with $j \leq 100$:

\begin{equation}
\|P_0 Q_j [P_{-10,10} f(\Phi) P_{<15} \psi] \|_{X_0^{1/2} \cdot \frac{3}{2}} \\
\leq \sum_{a=1}^{3} \|P_0 Q_j [\Delta^{-1} \partial_a P_{-10,10} (\partial_a \Phi f'(\Phi)) P_{<15} \psi] \|_{X_0^{1/2} \cdot \frac{3}{2}} \\
\leq C 2^{j} \sum_{a=1}^{3} \| \sum_{k_1 \leq -100} \Delta^{-1} \partial_a P_{-10,10} (\partial_a P_{k_1} \Phi f'(\Phi)) P_{<15} \psi \|_{L_t^2 L_x^2} \\
+ C 2^{j} \sum_{a=1}^{3} \| \sum_{k_1 > -100} \Delta^{-1} \partial_a P_{-10,10} (\partial_a P_{k_1} \Phi f'(\Phi)) P_{<15} \psi \|_{L_t^2 L_x^2} \\
\leq C 2^{j} \sum_{a,b=1}^{3} \| \sum_{k_1 \leq -100} \Delta^{-1} \partial_a P_{-10,10} (P_{k_1} \partial_a \Phi \Delta^{-1} \partial_b P_{-20,20} (\partial_b \Phi f''(\Phi)) P_{<15} \psi \|_{L_t^2 L_x^2} \\
+ C 2^{j} \sum_{a=1}^{3} \| \sum_{k_1 > -100} \Delta^{-1} \partial_a P_{-10,10} (\partial_a P_{k_1} \Phi f'(\Phi)) P_{<15} \psi \|_{L_t^2 L_x^2}
\end{equation}
The first of the two immediately preceding expressions can be estimated by

\[ C \sum_{k_1 \leq -10} \| P_{<-100} \nabla \Phi \|_{L^1_t L^1_x} \| \nabla \Phi \|_{L^1_t L^1_x} \| P_{<15} \psi \|_{L^\infty_t L^\infty_x} \leq C \sum_{k < 15} 2^k c_k \leq C c_0 \quad (149) \]

by definition of a frequency envelope.

The 2nd term can be estimated by

\[ C \sum_{k_1 > -100, k_2 < 15} \| P_{k_1} \nabla \Phi \|_{L^6_t L^3_x} \| P_{k_2} \psi \|_{L^6_t L^3_x} \leq C \sum_{k_1 > -100, k_2 < 15} 2^k c_{k_1} c_{k_2} \leq C c_0^2 \quad (150) \]

2.2) Output has modulation $2^j$ with $j > 100$:

A): $P_{<15} \psi$ has modulation $\geq 2^{j-100}$:

\[ \| P_0 Q_j [P_{-10,10} f(\Phi) P_{<15} Q_{\geq j-100} \psi] \|_{X^0_0, X^1_0} \leq C 2^{2j} \| P_{<15} Q_{\geq j-100} \psi \|_{L^2_t L^2_x} \leq C \sum_{k < 15} C 2^{2j} c_k \leq C c_0 \quad (151) \]

B): $P_{<15} \psi$ has modulation $< 2^{j-100}$; this implies that $P_{-10,10} f(\Phi)$ has to be at modulation $\sim 2^j$:

\[ 2^{2j} \| P_0 Q_j \partial_t [P_{-10,10} Q_j + O(1)] (f(\Phi)) P_{<15} Q_{<j-100} \psi \|_{L^2_t L^2_x} \leq 2^{2j} \| P_0 Q_j [P_{-10,10} Q_j + O(1)] (f(\Phi)) P_{<15} Q_{<j-100} \partial_t \psi \|_{L^2_t L^2_x} \]

\[ + 2^{2j} \| P_0 Q_j [P_{-10,10} Q_j + O(1)] (\partial_t f(\Phi)) P_{<15} Q_{<j-100} \psi \|_{L^2_t L^2_x} \quad (152) \]

The first summand is estimated by means of lemma 5.7:

\[ 2^{2j} \| P_0 Q_j [P_{-10,10} Q_j + O(1)] (f(\Phi)) P_{<15} Q_{<j-100} \partial_t \psi \|_{L^2_t L^2_x} \leq C 2^{2j} \| P_{-10,10} Q_j + O(1) \|_{L^2_t L^2_x} \| P_{<15} Q_{<j-100} \partial_t \psi \|_{L^\infty_t L^\infty_x} \leq C c_0 \quad (153) \]
The 2nd summand is estimated similarly.

Note that in the preceding we have established the boundedness of the $X^{\frac{3}{2}, \frac{3}{2}, 1}$-norm of the part of $P_0[\tilde{P}_{-10,10}(f(\Phi))P_{<15}\psi]$ with small modulation. In particular, we control the third component of the $S[0]$-norm, and are done with high-low interactions.
High-High Interactions.

This case is more difficult. In particular, we have to cope with the situation that
two high-frequency terms whose Fourier supports live very close to the light cone
result in an output of small frequency but very far away from the light cone, which
renders the \( X^{\frac{1}{2}, \frac{1}{2}} \)-component more difficult to control. This case would be im-
possible to handle in 2 space dimensions.

1): \( L^\infty_t L^2_x \)-norm of the output:

\[
||\partial_t \left( \sum_{k_1 > 10, k_1 = k_2 + O(1)} P_0[P_{k_1} (f(\Phi)) P_{k_2} \psi)] \right)||_{L^\infty_t L^2_x} \leq C \sum_{k_1 > 10, k_1 = k_2 + O(1)} ||P_{k_1} (f(\Phi)) \partial_t P_{k_2} \psi||_{L^\infty_t L^2_x} + C \sum_{k_1 > 10, k_1 = k_2 + O(1)} ||P_{k_1} (\partial_\phi f' (\Phi)) P_{k_2} \psi||_{L^\infty_t L^2_x} \leq C \sum_{k_1 > 10, k_1 = k_2 + O(1)} \triangle^{-1} P_{k_1} (\partial_{\alpha} \phi f' (\Phi)) \partial_t P_{k_2} \psi||_{L^\infty_t L^2_x} + C \sum_{k_1 > 10, k_1 = k_2 + O(1)} ||P_{k_1} (\partial_\phi f' (\Phi)) P_{k_2} \psi||_{L^\infty_t L^2_x} \leq \sum_{k_1 > 10, k_1 = k_2 + O(1)} C ||\nabla_{x,t} \phi||_{L^\infty_t L^2_x} (2^{-k_1} ||\partial_t P_{k_2} \psi||_{L^\infty_t L^2_x} + ||P_{k_2} \psi||_{L^\infty_t L^2_x}) \leq C \sum_{k_1 > 10} 2^{-k_1} c k_1 \leq C c_0
\]

2): \( \dot{X}^{\frac{1}{2}, \frac{1}{2}}_0 \)-norm of the output: We can easily control the \( \dot{X}^{\frac{1}{2}, \frac{1}{2}}_0 \)-norm of the output restricted to small modulations, which in particular controls the third com-
ponent of \( S[0] \) for the output. Hence consider now the case when the modulation
2\(^j\) of the output is very large, i.e. \( j >> 1 \). Split the output as follows:
The 2nd term in the immediately preceding is the most difficult, as we cannot employ the $X^{\frac{1}{2}, \infty}$-norm of the inputs. Instead, we will have to resort to angular localization of the inputs, and exploit the third component of $S[k]$:  

\[
\sum_{k_1 = k_2 + O(1) > 10}^\infty \sum_{j = k_1 + 10}^\infty \sum_{k_2} P_0 Q_j (P_{k_1} f(\Phi) P_{k_2} \psi)
\]

\[
= \sum_{k_1 = k_2 + O(1) > 10}^\infty \sum_{j = k_1 + 10}^\infty \sum_{k_2} P_0 Q_j (P_{k_1} Q_{<j} f(\Phi) P_{k_2} Q_{<j})
\]

\[
+ \sum_{k_1 = k_2 + O(1) > 10}^\infty \sum_{j = k_1 + 10}^\infty \sum_{k_2} P_0 Q_j (P_{k_1} Q_{>j} f(\Phi) P_{k_2} Q_{>j})
\]

\[
+ \sum_{k_1 = k_2 + O(1) > 10}^\infty \sum_{j = k_1 + 10}^\infty \sum_{k_2} P_0 Q_j (P_{k_1} f(\Phi) P_{k_2} Q_{<j})
\]

\[
+ \sum_{k_1 = k_2 + O(1) > 10}^\infty \sum_{j = k_1 + 10}^\infty \sum_{k_2} (P_{k_1} f(\Phi) P_{k_2} Q_{>j})
\]

(155)

The 2nd term in the immediately preceding is the most difficult, as we cannot employ the $X^{\frac{1}{2}, \infty}$-norm of the inputs. Instead, we will have to resort to angular localization of the inputs, and exploit the third component of $S[k]$:  

A): $\sum_{k_1 = k_2 + O(1) > 10}^\infty \sum_{j = k_1 + 10}^\infty \sum_{k_2} P_0 Q_j (P_{k_1} Q_{<j} f(\Phi) P_{k_2} Q_{<j})$: First, observe that if $(\tau_1, \xi_1), (\tau_2, \xi_2)$ are points in the Fourier supports of the inputs $P_{k_1} Q_{<j} f(\Phi), P_{k_2} Q_{<j}$ respectively that contribute toward the output, we need to have

\[
2^j \sim |\tau| - |\xi| = |\tau_1 \pm |\xi_1| + |\tau_2 \pm |\xi_2|| \mp |\xi_1| \mp |\xi_2| \pm |\xi_1 + \xi_2|
\]

where signs have been chosen in such fashion that $|\tau_i - |\xi_i| = |\tau_i \pm |\xi_i||, ||\tau_1 + \tau_2 - |\xi_1 + \xi_2| = |\tau_1 + |\xi_1 + |\xi_2||, i = 1, 2$. Now $|\xi_1 + \xi_2| \sim 2^j$, whence by our assumptions on the Fourier supports of the inputs we conclude that the signs of $\mp|\xi_1, \mp|\xi_2|$ must be identical; moreover $k_1 = k_2 + O(1) = j + O(1)$. Observe in the sequel that in this case $P_{k_1} Q_{<j}$ is a disposable operator, viz. the definition in section 3.

We need to estimate
the Fourier support of \( P_{k_1} \) and \( P_{k_2} \). We can microlocalize \( \Phi \) to have Fourier support in an approximately \( \tau \) cap, and then utilize the important bilinear inequality (156) respectively. First, note that \(|\xi_\alpha| \sim 2^{k_1}\) Next, \(|\tau_\alpha| \sim 2^{k_1-80}\), whence \(|\tau_\alpha| \sim 2^{k_1}\). Also, \( \tau_\alpha \) has the same sign as the \( \tau_1 \) of points \((\tau_1, \xi_1)\) in the Fourier support of \( P_{k_1} Q_{<j-10} (Q_{<j-10} \Phi P_{<j-10} f') \). Thus recalling the comments of the preceding paragraph, we can microlocalize \( Q_{<j-10} \Phi \), \( P_{k_2} Q_{<j-10} \psi \) to the same half-space \( \tau >> 0 \).

A.1.1): \( P_{<j-100} Q_{<j-10} f'(\Phi) \) at frequency \( \geq 2^0 \); fixing this frequency to be \( 2^l \), \( 0 \leq l \leq j-100 \) for now, we microlocalize the Fourier support of \( P_{k_1+O(1)} Q_{<j-10} \partial_\Phi \) to a cap of size \( 2^{3(l-k_1)} \) when projected onto \( S^2 \). This then implies that \( P_{k_2} Q_{<j-10} \psi \) can be restricted to have Fourier support in an approximately (up to \( O(1) \) choices) opposite cap. Now utilize the important bilinear inequality

\[
||\phi \psi||_{L^2_t L^2_x} \leq 2^{4j-k} \frac{|\kappa'|}{\text{dist}(\kappa, \kappa')} ||\phi||_{S[\kappa, \kappa']} ||\psi||_{S[\kappa', \kappa']}
\]

Hence

\[
2^j \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 0} P_0 Q_j \partial_\tau (P_{k_1} Q_{<j-10} f(\Phi) P_{k_2} Q_{<j-10} \psi) ||_{L^2_t L^2_x} \\
\leq 2^j \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 0} P_0 Q_j (P_{k_1} Q_{<j-10} f(\Phi) P_{k_2} Q_{<j-10} \psi) ||_{L^2_t L^2_x}
\]

\[
+ 2^j \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 0} P_0 Q_j (P_{k_1} Q_{<j-10} (\partial_\tau f'(\Phi)) P_{k_2} Q_{<j-10} \psi) ||_{L^2_t L^2_x}
\]

\[
\leq 2^j \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 0} P_0 Q_j (P_{k_1} Q_{<j-10} (\partial_\tau f'(\Phi)) P_{k_2} Q_{<j-10} \psi) ||_{L^2_t L^2_x}
\]

We estimate the last term here, the last term but one being dealt with in an identical manner.
\[
\|P_k Q_j [P_{k_1} Q_{j-10} \partial_t Q_{j-10} \Phi P_{j-10} \geq \partial_t Q_{j-10} (f'(\Phi)) P_{k_2} Q_{j-10} \psi]\|_{L_t^2 L_x^2}
\leq \sum_{\pm} \sum_{0 \leq l \leq j-100} \sum_{K, \kappa' \in K_{\frac{3}{4}(l-k_1)}, \text{dist}(\kappa, \kappa') \leq C 2^{\frac{3}{4}(l-k_1)}} \|P_k Q_j [P_{k_1} Q_{j-10} (P_{k_1} + O(1), \kappa \partial_t Q_{j-10} \Phi)\|_{L_t^2 L_x^2} (157)
\]

\[
P_k Q_{j-100} (f'(\Phi)) P_{k_2, \kappa'} \|_{L_t^\infty L_x^2}
\leq 4 \sup_{\pm} \sum_{0 \leq l \leq j-100} 2^{\frac{3}{4}(l-k_1)} \left( \sum_{\kappa \in K_{\frac{3}{4}(l-k_1)}} \|P_{k_1} Q_{j-10} (P_{k_1} + O(1), \kappa \partial_t Q_{j-10} \Phi)\|_{S[k_1 + O(1), \pm \kappa]} \right) ^\frac{3}{2}
\]

\[
\|P_{l} Q_{j-100} (f'(\Phi))\|_{L_t^\infty L_x^2}
\leq C \sup_{\pm} \sum_{0 \leq l \leq j-100} 2^{\frac{3}{4}(l-k_1)} \left[ \sum_{|a| \leq O(1)} \sum_{\kappa \in K_{\frac{3}{4}(l-k_1)}} \||P_{k_1 + a, \pm \kappa} \partial_t Q_{j-10} \phi|_{S[k_1 + O(1), \pm \kappa]} \right] ^\frac{3}{2}
\]

\[
\leq C \sum_{0 \leq l \leq j-100} 2^{\frac{3}{4}(l-k_1)} \|P_{k_1 + O(1)} \Phi\|_{S[k_1 + O(1)]} \|P_{k_2} \psi\|_{S[k_2]} \leq C 2^{-\frac{3}{4} c_{k_2}} \leq 2^{-\frac{3}{4} c_{j}}
\]

because of \(j = k_1 + O(1)\).

b): At least one of \(P_{k_1 + O(1)} \kappa \partial_t Q_{j-100} \Phi\), \(P_{k_2, \kappa'} Q_{j-10} \psi\) has modulation \(\geq 2^{\frac{3}{4}(l-k_1)}\): this is handled by placing the input with large modulation into \(L_t^2 L_x^2\) and the other input into \(L_t^\infty L_x^2\). One thereby obtains the upper bound

\[
\sum_{l \geq 0} C 2^{-\frac{3}{4} c_{k_1} c_{k_2}} \leq C 2^{-\frac{3}{4} c_{j}}
\]
Again, this suffices to establish the Proposition.

A.1.2): $P_{<j-100}Q_{<j-100}f'(\Phi)$ at frequency $<2^0$: This is handled as in the preceding case, except that now $P_{k_1+O(1)}Q_{<j-100}\partial_t\Phi$, and $P_{k_2}Q_{<j-100}$ can be microlocalized in approximately opposite caps of size $2^{-\frac{3k_1}{4}}$. Also, $P_{<0}Q_{<j-100}(f'(\Phi))$ is estimated in the $L^\infty_tL^\infty_x$-norm. Otherwise, the argument is identical to the immediately preceding.

A.2): $f'(\Phi)$ at frequency $\geq 2^{-j-100}$.

\[
2^\frac{j}{2}\left|\sum_{k_1=k_2+O(1),k_1>10} P_0Q_j[P_{k_1}Q_{<j-100}(\partial_t\Phi P_{\geq j-100}(f'(\Phi)))P_{k_2}Q_{<j-100}]\right|_{L^2_tL^2_x}
\leq 2^\frac{j}{2}\left|\sum_{k_1=k_2+O(1),k_1>10} P_0Q_j[P_{k_1}Q_{<j-100}(\partial_t\Phi \sum_{a=1}^3 \partial_a \partial_t\Phi P_{\geq j-100}(\partial_a\Phi f''(\Phi)))]
\right|
\leq C \sum_{k_1=j+O(1)=k_2+O(1)} 2^{-\frac{j}{2}} c_{k_1} \leq Cc_0
\] (158)

A.3): $\partial_t\Phi$ at modulation $\geq 2^{j-100}$.

\[
2^\frac{j}{2}\left|\sum_{k_1=k_2+O(1),k_1>10} P_0Q_j[P_{k_1}Q_{\geq j-100}\partial_t\Phi P_{<k_1-100}Q_{<j-100}(f'(\Phi)))P_{k_2}Q_{<j-100}]\right|_{L^2_tL^2_x}
\leq C \sum_{k_1=j+O(1)} 2^\frac{j}{2}\left|\partial_t\Phi\right|_{L^\infty_tL^\infty_x} \left|\partial_t\Phi\right|_{L^4_tL^4_x} \left|P_{k_2}Q_{<j-100}\right|_{L^4_tL^4_x}
\leq C \sum_{k_1=j+O(1)} 2^\frac{j}{2} c_{k_1} \leq Cc_0
\] (159)

A.4): $f'(\Phi)$ at modulation $\geq 2^{j-100}$: use lemma 5.7 to conclude:
Note that \( B) \):

\[
2^{\frac{3 j}{2}} \left\| \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 10} P_0 Q_j [P_{k_1} Q_{\leq j-10}(\partial_\phi P_{\leq j} Q_{\geq j} \{ f(\Phi) \})] P_{k_2} Q_{\leq j-10} \psi \right\|_{L^2_t L^2_x} \leq 2^{\frac{j}{3}} C \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 10} \left\| P_{k_1} Q_{\leq j-10}(f(\Phi)) \right\|_{L^2_t L^\infty_x} \left\| P_{k_2} Q_{\leq j-10} \psi \right\|_{L^\infty_t L^2_x} \leq C \sum_{10 \leq k_1 = k_2 + O(1) \leq j + 10} C 2^{-k_1 c k_1} \leq C_0 \tag{160}
\]

Returning to the remaining terms of \( (159) \):

\[
2^{\frac{3 j}{2}} \left\| \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 10} P_0 Q_j (P_{k_1} Q_{\geq j} f(\Phi) P_{k_2} \psi) \right\|_{L^2_t L^2_x} \leq C 2^{\frac{j}{3}} \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 10} \left\| P_{k_1} Q_{\geq j} f(\Phi) \right\|_{L^2_t L^\infty_x} \left\| P_{k_2} \psi \right\|_{L^\infty_t L^2_x} \leq C \sum_{10 \leq k_1 = k_2 + O(1) \leq j + 10} C 2^{-k_1 c k_1} \leq C_0 \tag{161}
\]

\[
2^{\frac{3 j}{2}} \left\| \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 10} P_0 Q_j (P_{k_1} Q_{\leq j-10} f(\Phi) P_{k_2} Q_{\geq j-10} \psi) \right\|_{L^2_t L^2_x} \leq C \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 10} \sum_{10 \geq j-10} C 2^{\frac{j}{3}} \left\| P_{k_1} Q_{\leq j-10} \left( \sum_{a=1}^{3} \triangle^{-1} \partial_a (\partial_a f(\Phi)) \right) \right\|_{L^\infty_t L^2_x} \| P_{k_2} Q \psi \|_{L^2_t L^2_x} \leq C \sum_{10 \leq k_1 = k_2 + O(1) \leq j + 10} C 2^{-k_1 c k_1} \leq C_0 \tag{162}
\]

\[
2^{\frac{j}{3}} \sum_{k_1 = k_2 + O(1), j + 10 \geq k_1 > 10} P_0 Q_j (P_{k_1} f(\Phi) P_{k_2} Q_{\leq j-10} \psi) \]

The argument is a little more convoluted here, since \( P_{k_1} Q_j \) isn’t disposable anymore. Note that \( f(\Phi) \) can be restricted to modulation \( \geq 2^{\frac{j}{3} + O(1)} \).
Each of these terms is straightforward to estimate. We have

\[
2^{\frac{3j}{2}} \sum_{k_1 = k_2 + O(1) > j + 10} \|P_k Q_1 |P_1 f(\Phi) P_{\frac{3j}{2}} Q_{<j} - 10\psi| \|_{L_t^2 L_x^2} \\
\leq C 2^{\frac{3j}{2}} \sum_{k_1 = k_2 + O(1) > j + 10} \|P_k Q_{\geq j + O(1)} (f(\Phi)) \|_{L_t^2 L_x^2} \|P_{\frac{3j}{2}} Q_{<j} - 10\psi \|_{L_t^\infty L_x^2} \\
\leq C 2^{\frac{3j}{2}} \sum_{k_1 = k_2 + O(1) > j + 10} \| \sum_{a = 1}^{3} \Delta^{-1} \partial_a P_{k_1} Q_{\geq j + O(1)} \partial_a P_{k_1 + O(1)} Q_{\geq j + O(1)} \Phi \\
P_{<j-100} Q_{<j} - 100 (f'(\Phi)) \|_{L_t^2 L_x^2} \|P_{\frac{3j}{2}} Q_{<j} - 10\psi \|_{L_t^\infty L_x^2} \\
+ C 2^{\frac{3j}{2}} \sum_{k_1 = k_2 + O(1) > j + 10} \| \sum_{a = 1}^{3} \Delta^{-1} \partial_a P_{k_1} Q_{\geq j + O(1)} \partial_a P_{k_1 + O(1)} \Phi \\
P_{<j-100} Q_{\geq j} - 100 (f'(\Phi)) \|_{L_t^2 L_x^2} \|P_{\frac{3j}{2}} Q_{<j} - 10\psi \|_{L_t^\infty L_x^2} \\
+ C 2^{\frac{3j}{2}} \sum_{k_1 = k_2 + O(1) > j + 10} \| \sum_{a = 1}^{3} \Delta^{-1} \partial_a P_{k_1} Q_{\geq j + O(1)} \partial_a \Phi P_{<j-100} (f'(\Phi)) \|_{L_t^2 L_x^2} \\
\|P_{\frac{3j}{2}} Q_{<j} - 10\psi \|_{L_t^\infty L_x^2}
\]

Each of these terms is straightforward to estimate. We have
Here we have used lemma 5.7.

Finally

\[
2^{\frac{3j}{4}} \sum_{k_1=k_2+O(1)>j+10} \left\| \sum_{a=1}^3 \Delta^{-1} \partial_a P_{k_1} Q_{\geq j} + O(1) [\partial_a P_{k_1} + O(1) \Phi P_{j+100} \Phi] \right\|_{L_t^2 L_x^2}^2 \\
\leq C \sum_{k_1=k_2+O(1)>j+10} 2^{\frac{3j}{4}} 2^{-k_1} \left\| P_{k_1} \left\|_{L_t^\infty L_x^2} \right\| P_{j+100} \Phi \right\|_{L_t^2 L_x^2} \\
\leq C \sum_{k_1=k_2+O(1)>j+10} 2^{\frac{3j}{4}} 2^{-k_1} c_{k_1} 2^j 2^{-\frac{3j}{4}} 2^{-k_2} \\
\leq C \sum_{k_1=k_2+O(1)>j+10} 2^{-\frac{3j}{4}} c_{k_1} \leq C c_0
\]
Low-High Interactions

This case is hard as controlling the $X^{\frac{1}{2}, \infty}_2$-norm of the output for small modulations forces us to utilize lemma 5.6 when $f(\Phi)$ is at small frequency, hence incurring an exponential loss which we can only make up for by invoking another case of angular localization and employing the third component of $S[k]$.

1): $L^\infty_t L^2_x$-norm of the output:

\[
\|\partial_t [P_{<-10}(f(\Phi)) P_{-10,10} \psi]\|_{L^\infty_t L^2_x} \\
\leq C \|P_{-10,10} \partial_t \psi\|_{L^\infty_t L^2_x} + \|P_{<-10} (\partial_t \Phi f'(\Phi))\|_{L^\infty_t L^2} \|P_{-10,10} \psi\|_{L^\infty_t L^2_x} \\
\leq C C_0 \tag{163}
\]

2): $X^{\frac{1}{2}, \infty}_0$-norm of the output: For high modulations of the output, this can be done exactly as in the high-low case. Now assume that the modulation of the output is $2^j \leq 2^{-10}$.

\[
P_0 Q_j [P_{<j-100}(f(\Phi)) P_{-10,10} \psi] \\
= P_0 Q_j [P_{>j-100}(f(\Phi)) P_{-10,10} \psi] \\
+ P_0 Q_j [P_{<j-100} Q_{\geq j-100}(f(\Phi)) P_{-10,10} \psi] \\
+ P_0 Q_j [P_{<j-100} P_{\leq j-100} f(\Phi) P_{-10,10} Q_{>j-100} \psi] \tag{164}
\]

1): $P_0 Q_j [P_{>j-100}(f(\Phi)) P_{-10,10} \psi]$: reformulate this term as

\[
\sum_{j > j-100} P_0 Q_j (\sum_{a=1}^{3} \triangle^{-1} \partial_a P_j [\partial_a \Phi f'(\Phi)] P_{-10,10} \psi) \tag{165}
\]

1.1): $f'(\Phi)$ at frequency $< 2^{j-100}$ and modulation $< 2^{j-100}$, $\partial_a \Phi$ at modulation $< 2^{j-100}$; also, $\psi$ at modulation $< 2^{j-100}$. This is the worst possible scenario since we cannot introduce higher linearity by iterating lemma 5.6. We will have to resort to angular localization:
Using as usual the bilinear inequality (45), we now deduce:

\[
\sum_{j > j-100} P_0Q_j \left( \sum_{a=1}^{3} \Delta^{-1} \partial_a P_j^{(a)} \partial_a Q_{<j-100} \Phi \right) P_{<j-100} Q_{<j-100} f'(\Phi) [P_{-10,10} Q_{<j-100}] \\
= \sum_{j > j-100} P_0Q_j \left( \sum_{a=1}^{3} \Delta^{-1} \partial_a P_j^{(a)} \partial_a P_j^{(+O(1))} Q_{<j-100} \Phi \right) P_{<j-100} Q_{<j-100} f'(\Phi) \\
P_{-10,10} Q_{<j-100}] \\
= P_0Q_j \left( \int_{\mathbb{R}^3} a(y) P_{j+O(1)} Q_{<j-100} \partial_a \Phi(x-y) [P_{-10,10} Q_{<j-100}] f'(\Phi)(x-y)dy \\
P_{-10,10} Q_{<j-100} ] \\
= \int_{\mathbb{R}^3} a(y) P_0Q_j \left( P_{-15,15} Q_{j+O(1)} Q_{<j-100} \partial_a \Phi(x-y) [P_{-10,10} Q_{<j-100}] \psi(x) \\
P_{<j-100} Q_{<j-100} f'(\Phi)(x-y)dy \\
\right)
\]

Here \( a(x) \) denotes the kernel associated with the operator \( \Delta^{-1} \partial_a P_j \).

The reason for reformulating the expression as above is that provided the inputs \( P_{j+O(1)} Q_{<j-100} \partial_a \Phi(x-y) \), \( P_{-10,10} Q_{<j-100} \psi(x) \) of \([\cdot] \) in the last term are microlocalized to a half-space \( \tau < 0 \), and such that the projection of their Fourier supports to \( S^2 \) are supported on caps \( \kappa_1, \kappa_2 \) respectively, of radius \( 2^{-\frac{\ell}{2^{j-10}}} \), then \( \pm \kappa_1 \), \( \pm \kappa_2 \) are at distance \( \sim 2^{-\frac{\ell}{2^{j-10}}} \). The signs are chosen to be \( +1 \) for the upper half-space and \( -1 \) for the lower half-space. For this simple geometric fact see lemma(13.2) in [13]. Hence we have the identity:

\[
P_{-15,15} Q_{j+O(1)} [P_{j+O(1)} Q_{<j-100} \partial_a \Phi P_{-10,10} Q_{<j-100} \psi(x)] \\
= \sum_{\pm} \sum_{\kappa, \kappa' \in K_{\ell^{-1}}} P_{-15,15} Q_{j+O(1)} [P_{j+O(1), \kappa} Q_{<j-100} \partial_a \Phi \\
P_{-10,10, \kappa} Q_{<j-100} \psi(x)] \\
= \sum_{\pm} \sum_{\kappa, \kappa' \in K_{\ell^{-1}}} P_{-15,15} Q_{j+O(1)} [P_{j+O(1), \kappa} Q_{<j-100} \partial_a \Phi \\
P_{-10,10, \kappa} Q_{<j-100} \psi(x)] \\
- \sum_{\pm} \sum_{\kappa, \kappa' \in K_{\ell^{-1}}} P_{-15,15} Q_{j+O(1)} [P_{j+O(1), \kappa} Q_{<j-100} \partial_a \Phi \\
P_{-10,10, \kappa} Q_{<j-100} \psi(x)]
\]

Using as usual the bilinear inequality (13), we now deduce:
Now as to the 2nd term in (166), note that

\[
\| \sum_{\kappa, \kappa' \in K} \sum_{\frac{1}{2} \leq | \kappa, \kappa'| \leq 10^{-10}, \text{dist}(\pm \kappa, \pm \kappa') \sim 2^{-\frac{1}{2}}} P_{-15,15} Q_{j+O(1)} [P_{j+O(1), \kappa} Q_{\leq j-100} \partial_a \Phi] \\
P_{-10,10, \kappa} Q_{\leq j-\psi(x)} \|_{L^2_t L^2_x} \\
\leq C \sum_{\kappa, \kappa' \in K} \sum_{\frac{1}{2} \leq | \kappa, \kappa'| \leq 10^{-10}, \text{dist}(\pm \kappa, \pm \kappa') \sim 2^{-\frac{1}{2}}} \| P_{j+O(1), \kappa} Q_{\leq j-100} \partial_a \Phi \|_{S[j, \kappa]} \quad (167)
\]

\[
\| P_{-10,10, \kappa} Q_{\leq j-\psi(x)} \|_{S[0, \kappa']} \leq C 2^\frac{1}{2} \left( \sum_{\kappa \in K} \| P_{j+O(1), \kappa} Q_{\leq j-100} \partial_a \Phi \|_{S[j, \kappa]} \right)^\frac{1}{2}
\]

\[
\leq C 2^\frac{1}{2} \sum_{\kappa \in K} \sum_{b \leq O(1)} \| P_{a, \kappa} Q_{\leq a+j-20} \partial_a \Phi \|_{S[a+b]} \left( \sum_{\kappa \in K} \| P_{a, \kappa} Q_{\leq a+j-20} \partial_a \Phi \|_{S[a+b]} \right)^\frac{1}{2}
\]

\[
\leq C 2^\frac{1}{2} \sum_{\kappa \in K} \sum_{b \leq O(1)} \sum_{a \leq O(1)} \| P_{a, \kappa} Q_{\leq a+j-20} \partial_a \Phi \|_{S[a+b]} \left( \sum_{\kappa \in K} \| P_{a, \kappa} Q_{\leq a+j-20} \partial_a \Phi \|_{S[a+b]} \right)^\frac{1}{2}
\]

\[
\leq C 2^\frac{1}{2} c_0 c_j
\]

Now as to the 2nd term in (166), note that

\[
\| \sum_{\kappa, \kappa' \in K} \sum_{\frac{1}{2} \leq | \kappa, \kappa'| \leq 10^{-10}, \text{dist}(\pm \kappa, \pm \kappa') \sim 2^{-\frac{1}{2}}} P_{-15,15} Q_{j+O(1)} [P_{j+O(1), \kappa} Q_{\leq j-100} \partial_a \Phi] \\
P_{-10,10, \kappa} Q_{j \geq j-100} \psi(x) \|_{L^2_t L^2_x} \\
\leq \left( \sum_{\kappa \in K} \| P_{j+O(1), \kappa} Q_{\leq j-100} \partial_a \Phi \|_{L^\infty_t L^\infty_x} \right)^\frac{1}{2}
\]

\[
\left( \sum_{\kappa \in K} \| P_{-10,10, \kappa} Q_{j \geq j-100} \psi(x) \|_{L^2_t L^2_x} \right)^\frac{1}{2}
\]

\[
\leq C 2^\frac{1}{2} \sum_{\kappa \in K} \sum_{b \leq O(1)} \| P_{a, \kappa} Q_{\leq a+j-20} \partial_a \Phi \|_{S[a+b]} \left( \sum_{\kappa \in K} \| P_{a, \kappa} Q_{\leq a+j-20} \partial_a \Phi \|_{S[a+b]} \right)^\frac{1}{2}
\]

\[
\leq C 2^\frac{1}{2} c_0 c_j
\]
By means of Bernstein’s and Strichartz’ inequality, we know that

\[
||P_k Q_j \phi||_{L^\infty_t L^2_x} \leq C 2^{\frac{3j}{2}} 2^{\frac{j^2}{4}} ||\phi||_{L^4_t L^2_x} \leq C 2^{\frac{3j}{2}} 2^{\frac{j^2}{4}} ||\phi||_{L^4_t L^2_x}
\]  

(169)

Therefore

\[
|| \sum_{\kappa, \kappa' \in K, dist(\kappa, \kappa') \sim 2^{-j}} P_{-15.15 Q_j + O(1)} [P_{j+O(1), \kappa} Q_j^{\pm}] \phi \partial_x \Phi - P_{-10.10 Q_j - j - 100} \psi(x)||_{L^2_t L^2_x} 
\]

\[
\leq C 2^{-\frac{j}{2}} 2^{\frac{j^2}{4}} ||P_{j+O(1)} Q_{< j - 100} \Phi||_{X^{\frac{1}{2}, \frac{1}{2}}_{j+O(1)}} ||P_{-10.10 Q_j - j - 100} \psi||_{X^{\frac{1}{2}, \frac{1}{2}}_{j+O(1)}}
\]

\[
\leq C 2^{-\frac{j}{2}} 2^{\frac{j^2}{4}} c_0 c_j
\]  

(170)

We can conclude that

\[
2^{\frac{j}{2}} || \sum_{j > j - 100} P_0 Q_j \left( \sum_{a=1}^{3} \triangle^{-1} \partial_a P_j [\partial_a \Phi f'(\Phi)] P_{-10.10} \psi \right) ||_{L^2_t L^2_x} 
\]

\[
\leq \sum_{j > j - 100} C (2^{\frac{j}{2}} + 2^{\frac{j^2}{4}}) c_j c_0 \leq C c_0
\]  

(171)

1.2): \( f'(\Phi) \) at frequency \( < 2^{j-100} \) and modulation \( < 2^{j-100} \), \( \partial_x \Phi \) at modulation \( \geq 2^{j-100} \), \( \psi \) at modulation \( < 2^{j-100} \):

\[
\sum_{j > j - 100} P_0 Q_j \left( \sum_{a=1}^{3} \triangle^{-1} \partial_a P_j [\partial_a P_j + O(1)] Q_{\geq j - 100} \Phi P_{< j - 100} Q_{< j - 100} f'(\Phi)] P_{-10.10} Q_{< j - 100} \psi \right)
\]

\[
= \sum_{j > j - 100} \sum_{l \geq j - 100} P_0 Q_j \left( \sum_{a=1}^{3} \triangle^{-1} \partial_a P_j [P_{j+O(1)} Q_l \Phi P_{< j - 100} Q_{< j - 100} f'(\Phi)] P_{-10.10} Q_{< j - 100} \psi \right)
\]

The \( X^{\frac{1}{2}, \frac{1}{2}}_{\infty} \)-norm of this can be majorized by

\[
C \sum_{j \geq j - 100} 2^{\frac{j}{2}} 2^{-j} c_j 2^{\frac{j^2}{4}} c_0 + \sum_{l > j} 2^{\frac{j}{2}} 2^{-j} c_j 2^{\frac{j^2}{4}} c_0 \leq C c_0
\]
1.3): \( f'(\Phi) \) has frequency \( \geq 2^{\tilde{j} - 10} \):

\[
2^{\tilde{j}} \| \Delta^{-1} \partial_a \Phi j^{\tilde{j}} \partial_a \Phi P \geq j - 10 (f'(\Phi)) \| P_{-10,10} Q_{< j - 100} \psi \| L^2 \| L^2_x
\]

\[
\leq \sum_{b=1}^{3} 2^{\tilde{j}} \| \Delta^{-1} \partial_a \Phi j^{\tilde{j}} \partial_a \Phi P \geq j - 10 \Delta^{-1} \partial_b (\partial_b \Phi f''(\Phi)) \| P_{-10,10} Q_{< j - 100} \psi \| L^2 \| L^2_x
\]

\[
\leq \sum_{b=1}^{3} 2^{\tilde{j}} \| \Delta^{-1} \partial_a \Phi j^{\tilde{j}} \partial_a \Phi P \geq j - 10 \Delta^{-1} \partial_b (\partial_b \Phi f''(\Phi)) \| \| P_{-10,10} Q_{< j - 100} \psi \| L^\infty \| L^2_x
\]

\[
\leq C 2^{\tilde{j} - j} c_0
\]

by placing \( \partial_a \Phi, \partial_b \Phi \) into \( L^4 \| L^4_x \), and \( P_{-10,10} Q_{< j - 100} \psi \) into \( L^\infty \| L^2_x \). This can be summed over \( \tilde{j} > j - 100 \).

1.4): \( f'(\Phi) \) has frequency between \( 2^{j - 100} \) and \( 2^{\tilde{j} - 10} \):

\[
2^{\tilde{j}} \| \Delta^{-1} \partial_a \Phi j^{\tilde{j}} \partial_a \Phi P_{j - 100 < \cdot, < j - 10} (f'(\Phi)) \| P_{-10,10} Q_{< j - 100} \psi \| L^2 \| L^2_x
\]

\[
\leq \sum_{b=1}^{3} 2^{\tilde{j}} \| \Delta^{-1} \partial_a \Phi j^{\tilde{j}} \partial_a \Phi P_{j - 100 < \cdot, < j - 10} \Delta^{-1} \partial_b (\partial_b \Phi f''(\Phi)) \| P_{-10,10} Q_{< j - 100} \psi \| L^2 \| L^2_x
\]

\[
\leq \sum_{b=1}^{3} 2^{\tilde{j}} \| \Delta^{-1} \partial_a \Phi j^{\tilde{j}} \partial_a \Phi P_{j - 100 < \cdot, < j - 10} \Delta^{-1} \partial_b (\partial_b \Phi f''(\Phi)) \| \| P_{-10,10} Q_{< j - 100} \psi \| L^\infty \| L^2_x
\]

\[
\leq \sum_{b=1}^{3} C 2^{\tilde{j} - j} j^{\tilde{j}} \| \partial_a \Phi \| L^4 \| L^4_x \| P_{j - 100 < \cdot, < j - 10} \Delta^{-1} \partial_b (\partial_b \Phi f''(\Phi)) \| \| P_{-10,10} Q_{< j - 100} \psi \| L^\infty \| L^2_x
\]

\[
\leq C 2^{\tilde{j} - j} c_0
\]

Again this can be summed over \( \tilde{j} > j - 100 \) to yield a bound of the required form.

1.5): \( f'(\Phi) \) has modulation \( \geq 2^{j - 100} \) and frequency \( < 2^{j - 100} \). Split into the cases \( P_{j + O(1)} Q_{\geq j - 100} \partial_b \Phi \) and \( P_{j + O(1)} Q_{< j - 100} \partial_b \Phi \). The first is dealt with as in 1.2), taking into account the disposability of the operator \( P_{< j - 100} Q_{\geq j - 100} \). Now
$$2^j \| \Delta^{-1} \partial_a P_j | P_{j+O(1)} Q_{<j-100} (\partial_a \Phi) P_{<j-100} Q_{\geq j-100} (f'(\Phi)) \|_{L_t^2 L_x^2} \leq C 2^j 2^{j-1} \| P_{j+O(1)} Q_{<j-100} \partial_a \Phi \|_{L_t^2 L_x^\infty} \| P_{<j-100} Q_{\geq j-100} (f'(\Phi)) \|_{L_t^2 L_x^\infty} \| P_{<j-100} Q_{<j-100} \psi \|_{L_t^\infty L_x^2} \leq C 2^j C_j c_0$$

where we have used (169). Summing over \( \tilde{j} \geq j - 100 \), this case is dealt with as well.

2): \( P_0 Q_j [P_{\leq j-100} Q_{\geq j-100} (f(\Phi)) P_{-10,10} \psi] \): This is handled using lemma 5.7:

$$2^j \| P_0 Q_j [P_{\leq j-100} Q_{\geq j-100} (f(\Phi)) P_{-10,10} \psi] \|_{L_t^2 L_x^2} \leq C 2^j \| P_{\leq j-100} Q_{\geq j-100} (f(\Phi)) \|_{L_t^2 L_x^\infty} \| P_{-10,10} \psi \|_{L_t^\infty L_x^2} \leq C c_0$$

(172)

3): \( P_0 Q_j [P_{\leq j-100} Q_{<j-100} f(\Phi) P_{-10,10} Q_{\geq j-100} \psi] \):

$$2^j \| P_0 Q_j [P_{\leq j-100} Q_{<j-100} f(\Phi) P_{-10,10} Q_{\geq j-100} \psi] \|_{L_t^2 L_x^2} \leq C 2^j \| P_{-10,10} Q_{\geq j-100} \psi \|_{L_t^2 L_x^2} \leq C c_0$$

(173)

We have simply placed \( P_{\leq j-100} Q_{<j-100} f(\Phi) ) \) into \( L_t^\infty L_x^\infty \).

3)

Finally, we have to deal with the 3rd component of \( S[0] \) in the low-high interaction case, i.e. we need to estimate

$$\sup_{\pm} \sup_{l \leq -10} \left( \sum_{\kappa \in K} \| P_{0,\pm,\kappa} Q_{\leq 2l} (f(\Phi) \psi) \|_{S[0,\kappa]}^2 \right)^{1/2}$$

(174)

We split into 4 cases:

1.a): \( \psi \) has modulation \( < 2^l \), \( f(\Phi) \) has frequency \( < 2^{l-10} \) and modulation \( < 2^{-10} \), where we have fixed \( l \leq -10 \):
\[
\sup_\pm \left( \sum_{\kappa \in K_l} \| P_{0,\pm \kappa} Q_{<2l}^{\pm} (P_{<l-10} Q_{<10} f(\Phi)) Q_{<2l}^{\pm} f(\Phi) \|_{S_{[0,\kappa]}}^2 \right) \frac{1}{2} \\
\leq \sup_\pm \left( \sum_{|a| < O(1)} \left( \sum_{\kappa' \in K_{l-10}, \kappa' \subset \kappa} P_{0,\pm \kappa} Q_{<2l}^{\pm} (P_{<l-10} Q_{<10} f(\Phi)) \right) \right) \frac{1}{2} \\
+ \sup_\pm \left( \sum_{|a| < 0(1)} \left( \sum_{\kappa' \in K_{l-10}, \kappa' \subset \kappa} P_{0,\pm \kappa} Q_{<2l}^{\pm} (P_{<l-10} Q_{<10} f(\Phi)) \right) \right) \frac{1}{2} \\
\leq C \sup_{|a| \leq 0(1)} \left( \sum_{\kappa' \in K_{l-10}} \| P_{0,\pm \kappa'} Q_{<a+2l-20}^{\pm} \|_{S_{[0,\kappa']}}^2 \right) \frac{1}{2} \\
+ C \sup_{|a| \leq 0(1)} \left( \sum_{\kappa' \in K_{l-10}} \| P_{0,\pm \kappa'} Q_{<a+2l-20}^{\pm} \|_{X^{1/2}_{0,1/2}} \right) \frac{1}{2} \\
\leq C \mathcal{C}_0
\]

We have used here the facts that \( \|fg\|_{S_{[k,\kappa]}} \leq C \|f\|_{L^\infty} \|g\|_{S_{[k,\kappa]}}, \|\phi\|_{S_{[k,\kappa]}} \leq C \|\phi\|_{S_{[k',\kappa']}} \) provided \( \kappa' \subset \kappa, k' = k + O(1) \), as well as the disposability of \( P_{0,\pm \kappa} Q_{<2l}^{\pm} \) and the inequality \( \|P_k \phi\|_{S_{[k,\kappa]}} \leq C \|\phi\|_{X^{1/2}_{0,1/2}} \). For these facts, see [10]. Also, we have used here that \( P_{k,\kappa} \) microlocalizes to a concentric cap inside \( \kappa \), of half its size.

1.b): \( \psi \) has modulation \( < 2^l \), \( f(\Phi) \) has frequency \( < 2^{l-10} \) and modulation \( \geq 2^{-10} \), where we have fixed \( l \leq -10 \): This is much more elementary using lemma [5,7] hence left out.

2): \( \psi \) has modulation \( \geq 2^{2l} \) and \( f(\Phi) \) has frequency \( < 2^{l-10} \): we have

\[
\left( \sum_{\kappa \in K_l} \| P_{0,\pm \kappa} Q_{<2l}^{\pm} (P_{<l-10} f(\Phi)) Q_{\geq 2l}^{\pm} f(\Phi) \|_{S_{[0,\kappa]}}^2 \right)^{\frac{1}{2}} \\
\leq C \left( \sum_{r < 2l} \left( \sum_{\kappa \in K_l} \| P_{0,\pm \kappa} Q_{<2l}^{\pm} (P_{<l-10} f(\Phi)) Q_{\geq 2l}^{\pm} f(\Phi) \|_{S_{[0,\kappa]}}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
\leq 2^{\frac{r}{2}} C \left( \sum_{r < 2l} \| P_{<l-10} f(\Phi) \|_{L_{1,2}^{l-2}} \right)^{\frac{1}{2}} \\
\leq C \sum_{r < 2l} 2^{\frac{r}{2} - l} C_0 \leq C \mathcal{C}_0
\]

3): \( f(\Phi) \) has frequency \( \geq 2^{2l-10} \):
\[
(\sum_{\kappa \in K} ||P_{0,\pm \kappa} Q_{<2l}^\pm (P_{\geq l-10}[f(\Phi)]P_{-10,10}\psi)||_S^2)^{\frac{1}{2}} \\
\leq C \sum_{r < 2l} ||P_0 Q_r^\pm (P_{\geq l-10}[f(\Phi)]P_{-10,10}\psi)||_{X_{\alpha}^{\pm,1}} \\
\leq C \sum_{r < 2l} 2^{r}\sum_{\alpha = 1}^3 \Delta^{-1} \partial_\alpha P_{\geq l-10}(\partial_\alpha \Phi f'(\Phi))P_{-10,10}\psi||_{L_t^2 L_x^2} \\
\leq C \sum_{r < 2l} C2^{r-1}\sum_{\alpha = 1}^3 \partial_\alpha \Phi f'(\Phi)||_{L_t^4 L_x^4} ||P_{-10,10}\psi||_{L_t^2 L_x^2} \\
\leq C \sum_{r < 2l} 2^{r-1}c_0 \leq Cc_0 
\]

This finishes the low-high case, and the Proposition is established.

**Remark**: Note that we have proved more than the Proposition states: provided that \(||P_k \psi||_{S[k]} \leq \mu c_k, ||P_k \hat{\phi}||_{S[k]} \leq M c_k\), and \(\sqrt{\sum_k c_k^2} \leq \epsilon\), we have that

\[
||P_k[f(\Delta^{-1} \partial_\alpha \hat{\phi}) \psi]||_{S[k]} \leq C(Q(M\epsilon) + 1)\mu c_k 
\]

for some polynomial \(Q(x)\). This is the statement needed to close the bootstrapping argument.
Lemma 5.8. (T. Tao): Let \( j \leq \min(k_1, k_2) + O(1) \). Then

\[
\| P_k(F\psi) \|_{N[k]} \leq C 2^{-\delta_1(k - \max(k_1, k_2))} 2^{j - \delta_2(j - \min(k_1, k_2))} \| F \|_{X^{\frac{1}{4}, -\frac{1}{4}}_{\psi}} \| \psi \|_{S[k_2]}
\]

for all Schwartz functions \( F \) with Fourier support at frequency \( 2^{k_1} \) and modulation \( 2^j \) while \( \psi \) is at frequency \( 2^{k_2} \), \( \delta_1, \delta_2 > 0 \).

Proof

We need only prove the high-high interaction case. Proceeding as in Tao’s paper, we split into the following cases:

Rescale to \( k_1 = k_2 + O(1) = 0 \), whence \( k \leq O(1) \). Also, let \( C >> 1 \):

1): The estimate for \( P_k(FQ_{\geq j-C}\psi) \):

1a): \( j < 100k \):

\[
2^{-\frac{j}{2}} \| P_k(FQ_{\geq j-C}\psi) \|_{L_t^\infty L_x^2} \leq 2^{-\frac{j}{2}} \| F \|_{L_t^\infty L_x^\infty} \| Q_{\geq j-C}\psi \|_{L_t^\infty L_x^2} \\
\leq 2^{-\frac{j}{2}} 2^j 2^{-\frac{j}{2}} \| F \|_{L_t^\infty L_x^2} \| \psi \|_{S[k_2]}
\]

where \( \delta \) can be chosen to be \( \frac{1}{4} \), by the improved Bernstein’s inequality (55). Clearly the gain \( 2^\frac{j}{2} \) makes more than up for the \( 2^{-\frac{j}{2}} \)-loss.

1b): \( j \geq 100k \): use Bernstein’s inequality to get

\[
2^{-\frac{j}{2}} \| P_k(Fk_1FPk_2Q_{\geq j-C}\psi) \|_{L_t^\infty L_x^2} \leq C 2^{-\frac{j}{2}} 2^{\frac{k}{2}} \| P_k(Fk_1FPk_2Q_{\geq j-C}\psi) \|_{L_t^\infty L_x^2} \\
\leq 2^k \| F \|_{L_t^\infty L_x^2} \| Q_{\geq j-C}\psi \|_{L_t^\infty L_x^2} \leq 2^k 2^{-\frac{j}{2}} \| F \|_{L_t^\infty L_x^2} \| \psi \|_{S[k_2]}
\]

This is acceptable.

2): The estimate for \( P_kQ_{\geq j-C}(FQ_{<j-C}\psi) \):

2a): \( j < 100k \):

\[
2^{-\frac{j}{2}} 2^{-\frac{k}{2}} \| P_kQ_{\geq j-C}(FQ_{<j-C}\psi) \|_{L_t^\infty L_x^2} \leq 2^{-\frac{j}{2}} 2^{-\frac{k}{2}} \| F \|_{L_t^\infty L_x^\infty} \| Q_{<j-C}\psi \|_{L_t^\infty L_x^2} \\
\leq C 2^{-\frac{k}{2}} 2^{-\frac{j}{2}} 2^{\frac{k}{2}} \| F \|_{L_t^\infty L_x^2} \| \psi \|_{S[k_2]}
\]

Once again, we use that \( \delta = \frac{1}{4} \), in order to conclude this case.
2b): 100k ≤ j < O(1): Use Bernstein’s inequality:

\[
2^{-\frac{3}{2}k} 2^{-\frac{j}{2}} \|P_k Q_{j>C}(FQ_{j>C}\psi)\|_{L^2_t L^2_x} \leq C2^{-\frac{1}{2}k} 2^{-\frac{j}{2}} \|P_k Q_{j>C}(FQ_{j>C}\psi)\|_{L^2_t L^2_x} \\
\leq 2^{-\frac{j}{2}} 2^{k} \|F\|_{L^2_t L^2_x} \|\psi\|_{S[k]}
\]

This is again acceptable.

3): The estimate for \(P_k Q_{j>C}(FQ_{j>C}\psi)\): this is the only case where we have to work a little more carefully, and deviate from Tao’s proof: note that \(j \leq k + O(1)\) here. The idea in this case is to use the angular separation of the inputs \(F, Q_{j>C}\psi\), which is obtained from Tao’s lemma (13.2) for the ”imbalanced case”: this allows us to conclude that upon localizing \(F, Q_{j>C}\psi\) in caps of size \(2^{-10}2^{-\frac{j}{2}k}\), their angular separation in Tao’s signed sense has to be \(\sim 2^{-\frac{j}{2}k}\). In particular, fixing such a cap for \(F\), there can be at most \(O(1)\)-possible caps for \(Q_{j>C}\psi\). Moreover, we can also conclude from that lemma that the angular separation between the output and \(F\) is \(\sim 2^{-\frac{j}{2}k}\). Now utilize the fundamental inequality

\[
\|\phi\psi\|_{NFA[k]} \leq C \frac{|k'|2^{-\frac{j}{2}}}{2^{\frac{1}{2}\text{dist}(\kappa, \kappa')}} \|\phi\|_{L^2_t L^2_x} \|\psi\|_{S[k', \kappa']}
\]

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to estimate the previous by (letting \(l_1 = \frac{k + j}{2} - 10, l_2 = \frac{j - k}{2} - 10\))

\[
\|P_k Q_{j>C}(FQ_{j>C}\psi)\|_{N[k]} \\
\leq 2^{-\frac{j}{2}} \sum_{\pm, \pm, \pm} \sum_{\kappa' \in K_{l_2}} \sum_{\kappa \in K_{l_1}, \text{dist}(\pm \kappa', \kappa) \sim 2^l} P_{k, \kappa'} Q_{j-C}(P_{k_1, \kappa'} Q_{j}^\pm F P_{k, \kappa'} Q_{j-C}\psi)_{NFA[\kappa']}
\]

\[
\leq C2^{-\frac{j}{2}} 2^{\frac{1}{2}} \sum_{\pm, \pm} \sum_{\kappa' \in K_{l_2}} \sum_{\kappa \in K_{l_1}, \text{dist}(\pm \kappa', \kappa) \sim 2^l} \|P_{k_1, \kappa'} \|_{L^2_t L^2_x}^2
\]

\[
\leq C2^{-\frac{j}{2}} 2^{\frac{1}{2}} [2^{-\frac{j}{2}} \|Q_j F\|_{L^2_t L^2_x}] \sum_{\kappa \in K_{l_1}} \|P_{k_2, \pm \kappa'} Q_{j+20}^\pm F_{j+20} Q_{j+20}\psi\|_{S[k_2, \kappa']} \]

\[
+ 2^{\frac{j}{2}} |k|2^{-\frac{j}{2}} \|Q_j F\|_{L^2_t L^2_x} \|Q_{j-C(j+k+k_2-20)} P_{k_2} \psi\|_{X^{-\frac{1}{2}, \infty}_{q_2}}
\]

where we have discarded the summation over \(\kappa''\) on account of the distance condition \(\text{dist}(\kappa', \kappa'') \sim 2^{l_2}\). Of course, the above is acceptable.

Lemma 5.9. (T. Tao) Let \(\phi, \psi\) be Schwarz functions on \(\mathbb{R}^{3+1}\). Then letting \(k_1 = k_2 + O(1) \geq O(1)\), we have

\[
\|P_0[R_{k_1} P_{k_1} \phi \partial^\nu P_{k_2} \psi]\|_{N[0]} \leq C2^{-\delta k_1} \|P_{k_1} \phi\|_{S[k_1]} \|P_{k_2} \psi\|_{S[k_2]}
\]

for some \(\delta > 0\).
Proof We only have to prove this high-high interaction case. As usual, we split into the cases corresponding to large modulations of the inputs, when the $\dot{X}^{\pm}_{H,1,\infty}$-component becomes effective, and small modulations of the inputs, when the $Q_0$-structure and lemma 3.3 kick in.

1): $P_0[R_\nu P_{k_1} Q_{\geq k_1+100} \phi \partial^\nu P_{k_2} Q_{< k_2-100}]$

\[
\|P_0[R_\nu P_{k_1} Q_{\geq k_1+100} \phi \partial^\nu P_{k_2} Q_{< k_2-100}]\|_{X^0_{H_\nu} - \frac{\nu}{4}} \\
\leq \sum_{t \geq k_1+100} \|P_0 Q_{t+O(1)}[R_\nu P_{k_1} Q_{t} \phi \partial^\nu P_{k_2} Q_{< k_2-100}]\|_{X^0_{H_\nu} - \frac{\nu}{4}}
\]
\[
\leq \sum_{t \geq k_1+100} C 2^{-t} \| R_\nu P_{k_1} Q_{t} \phi \|_{L^2_x L^2_t} \| \partial^\nu P_{k_2} Q_{< k_2-100} \|_{L^\infty_x L^2_t} \\
\leq \sum_{t \geq k_1+100} C 2^{-t} 2^{-\nu - k_1} \| P_{k_1} \phi \|_{S[k_1]} 2^\nu \| P_{k_2} \psi \|_{S[k_2]}
\]
\[
\leq C 2^{-k_1} \| P_{k_1} \phi \|_{S[k_1]} \| P_{k_2} \psi \|_{S[k_2]}
\]

2): $P_0[R_\nu P_{k_1} Q_{\geq k_1+100} \phi \partial^\nu P_{k_2} Q_{\geq k_2-100}]

\[
\|P_0[R_\nu P_{k_1} Q_{\geq k_1+100} \phi \partial^\nu P_{k_2} Q_{\geq k_2-100}]\|_{L^1_x L^2_t}
\]
\[
\leq C \| R_\nu P_{k_1} Q_{\geq k_1+100} \phi \|_{L^2_x L^2_t} \| \partial^\nu P_{k_2} Q_{\geq k_2-100} \|_{L^\infty_x L^2_t}
\]
\[
\leq C 2^{-k_1} \| P_{k_1} \phi \|_{S[k_1]} \| P_{k_2} \psi \|_{S[k_2]}
\]

3): $P_0[R_\nu P_{k_1} Q_{< k_1+100} \phi \partial^\nu P_{k_2} Q_{\geq k_2+200}]$: This is like case 1).

4): $P_0[R_\nu P_{k_1} Q_{< k_1+100} \phi \partial^\nu P_{k_2} Q_{< k_2+200}]$: Use the $Q_0$-structure to reduce this to the estimation of the following terms:

4.1): $\Box P_0[P_{k_1} Q_{< k_1+100} \phi \partial^\nu P_{k_2} Q_{< k_2+200}]

\[
\|\Box P_0[P_{k_1} Q_{< k_1+100} \phi \partial^\nu P_{k_2} Q_{< k_2+200}]\|_{X^0[0]}
\]
\[
\leq \|\Box P_0 Q_{< k_1+200}[P_{k_1} Q_{< k_1+100} \phi \partial^\nu P_{k_2} Q_{< k_2+200}]\|_{X^0_{H_\nu} - \frac{\nu}{4}}
\]
\[
\leq C 2^{-k_1} \| P_{k_1} \phi \|_{L^2_x L^2_t} \| P_{k_2} \psi \|_{L^2_x L^2_t}
\]
\[
\leq C 2^{-k_1} \| P_{k_1} \phi \|_{S[k_1]} \| P_{k_2} \psi \|_{S[k_2]}
\]

4.2): $P_0[P_{k_1} Q_{< k_1+100} \Box \phi \partial^\nu P_{k_2} Q_{< k_2+200}]$: use lemma 5.3 to conclude that
\[ P_0 [P_{k_1} Q_{<k_1} + 100 \Box P_{k_2} Q_{<k_2} + 200 \psi] \leq C 2^{-5k_1} \left\| \left( P_{k_1} \phi \right) \right\|_{X_{k_1}^{1, \frac{3}{2}, \infty}} \left\| \left( P_{k_2} Q_{<k_2} + 200 \psi \right) \right\|_{S[k_2]} \]

4.3): \[ P_0 [P_{k_1} Q_{<k_1} + 100 \Box P_{k_2} Q_{<k_2} + 200 \Box \phi] : \text{this is similar to the preceding case.} \]

\[ \left( \frac{1}{2} \right) \]

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