Abstract

Most data sets comprise of measurements on continuous and categorical variables. In regression and classification Statistics literature, modeling high-dimensional mixed predictors has received limited attention. In this paper we study the general regression problem of inferring on a variable of interest based on high dimensional mixed continuous and binary predictors. The aim is to find a lower dimensional function of the mixed predictor vector that contains all the modeling information in the mixed predictors for the response, which can be either continuous or categorical. The approach we propose identifies sufficient reductions by reversing the regression and modeling the mixed predictors conditional on the response. We derive the maximum likelihood estimator of the sufficient reductions, asymptotic tests for dimension, and a regularized estimator, which simultaneously achieves variable (feature) selection and dimension reduction (feature extraction). We study the performance of the proposed method and compare it with other approaches through simulations and real data examples.
1. Introduction

Most data sets comprise of measurements on a mixture of categorical and continuous features. Examples abound in the biomedical and health sciences, neuro-imaging, genomics, finance, social media, and internet advertising. The first statistical approach to modeling the dependence structure of mixed data we found in the literature is the location model of Olkin and Tate (1961). The location model uses correlation as a measure of dependence and bypasses the mixed nature of the data by grouping the continuous variables using the categorical ones and requiring they be normally distributed with different means but same variance within the groups.

More recently, Markov Networks, or undirected graphical models, that encode pairwise conditional dependence relationships among random variables have been used to model multivariate mixed data. With few exceptions (Yang et al., 2014a, 2015; Chen et al., 2014), mixed continuous and categorical data are modeled with the Gaussian Graphical Model (GGM) in a manner similar to the location model. Binary variables are used to define the different categories and GGM requires the continuous variables be conditionally normal and pairwise conditionally independent within categories. References for GMMs for low-dimensional mixed data include Lauritzen and Wermuth (1989), Lauritzen (1996), Yuan and Lin (2007), Wainwright and Jordan (2008), and in the high-dimensional setting, Cheng et al. (2014, 2017) and Lee and Hastie (2015). In particular, Cheng et al. (2017) proposed a simplified version of the conditional Gaussian distribution that reduces the number of parameters significantly while maintaining flexibility.

Both GGMs and the location model are unsupervised approaches for mixed data that do not include an output of interest. In the case of a categorical output, approaches for the treatment of mixed, in particular, binary and continuous input variables, include methods based on nonparametric density estimation (Aitchison and Aitken, 1976), the use of logistic discrimination (Day and Kerridge, 1967), in which the probability of group membership is assumed to be a logistic function of the observed variates (Anderson, 1972, 1975), and a likelihood ratio classification rule (Krzanowski, 1975) based on the location model of Olkin and Tate (1961). Krzanowski (1993) surveys and summarizes the associated developments. More recently, the location model has been used in multiple imputation [see, e.g., Javaras and van Dyk (2003), Buuren (2018, Ch. 4, Sec. 4.4)].

In this paper we study the general regression and classification problem with high-dimensional mixed predictors. Specifically, we consider the conditional distribution of

$$Y \mid X, H,$$

where the response $Y$ is either continuous or categorical, $X = (X_1, X_2, \ldots, X_p)^T$ is a vector of $p$ continuous, and $H = (H_1, H_2, \ldots, H_q)^T$ is a vector of $q$ binary predictor variables. Our aim is to find a lower dimensional function of the mixed predictor vector $Z = (X^T, H^T)^T$ that encapsulates all information the mixed predictors contain for the response $Y$. Specifically, our target is the identification of a function, other than the identity, $R : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^d$ such that $F(Y \mid Z) = F(Y \mid R(Z))$, where $F(\cdot \mid \cdot)$ denotes the conditional cumulative distri-
bution function of the response given the predictors. Such a function $R$ is called a sufficient reduction of the regression of $Y$ on $Z$.

This seemingly ambitious goal turns out to be surprisingly simple using the inventive tool of inverse regression. When $Y$ and $Z$ are both random, inverse regression is based on the equivalence of the following two statements [see Cook (2007), Bura et al. (2016), Bura and Forzani (2015)],

(i) $Y \mid Z \overset{d}{=} Y \mid R(Z)$

(ii) $Z \mid (Y, R(Z)) \overset{d}{=} Z \mid R(Z)$

where $\overset{d}{=}$ signifies equal in distribution. Statement (i) is an alternative definition of a sufficient reduction for the forward regression in (1) and (ii) is the usual definition of a sufficient statistic for a parameter $Y$ indexing the distribution of the mixed $Z$. The equivalence of (i) and (ii) obtains that if one considers $Y$ as a parameter, the sufficient “statistic” for $Y$ is the sufficient reduction for the regression of $Y$ on $Z$. In consequence, in order to find a sufficient reduction for the forward regression of $Y$ on $Z$ in (1), we can equivalently solve the inverse problem of finding a sufficient statistic for the regression of $Z$ on $Y$.

Our approach exploits the factorization

$$F(X, H \mid Y) = F(X \mid Y, H)F(H \mid Y), \tag{2}$$

by allowing us to treat the continuous and binary predictors separately, while at the same time we account for their interdependence in their relationship with $Y$ in Section 2. An advantageous aspect of (2) is that it requires fewer parameters in order to characterize the distributional structure of the data.

In Section 3 we model $H \mid Y$ as multivariate Bernoulli, and $X \mid (Y, H)$ as multivariate normal, in analogy to the Gaussian graphical model and the location model in unsupervised multivariate analysis of mixed data. We show that the resulting distribution (2) belongs to the exponential family, and derive sufficient reductions for the regression $Y \mid (X, H)$ from the two separate regressions, $X \mid (Y, H)$ and $H \mid Y$ in Section 3. We compute the maximum likelihood estimator of the sufficient reduction in Section 4, its asymptotic distribution in Section 4.3, and an asymptotic test for the dimension of the sufficient reduction in Section 4.4. We complete our treatment with a method for simultaneous sufficient dimension reduction and variable selection in Section 5.

Section 6 contains an extensive simulation study that demonstrates the competitive performance of our approach. Furthermore, we show the superior performance of our methods as compared with generalized linear models and a version of principal component regression that allows for mixed predictors in the analysis of three data sets in Section 7.

Even though our focus in this paper is the regression of the usually univariate $Y$ on the mixed $Z$ vector, our development results in a new multivariate regression method for a mixed continuous and binary response, on which we comment further as we conclude in Section 8.

2. The Model

We start by specifying the notation we use throughout. The vec operator converts its matrix argument into a column vector. More precisely, if $G$ is an $m \times n$ matrix then vec($G$) is
an \( mn \times 1 \) vector obtained by stacking the columns of \( G \). The \text{unvec} operator is such that \text{unvec}(\text{vec}(G)) = G. We let \( k_q = q(q - 1)/2 \) and \( m_p = p(p + 1)/2 \). The \text{vec} operator converts the lower half of a matrix including the main diagonal to a vector. That is, if \( G \) is a square \( q \times q \) matrix then \text{vec}(G) is a \( m_q \times 1 \) vector obtained by stacking the columns of the lower triangular part of \( G \) including the diagonal. There is a unique \( D_q \in \mathbb{R}^{q(q+1)/2} \) and \( C_q \in \mathbb{R}^{q(q+1)/2 \times q^2} \) such that \text{vec}(G) = \( D_q \text{vec}(G) \) and \text{vec}(G) = \( C_q \text{vec}(G) \) for any \( G \) symmetric \( q \times q \) matrix.

The matrix \( L_q \in \mathbb{R}^{q(q+1)/2} \) has entries 1 and 0, so that \( L_q C_q \) is equal to \( C_q \) but replacing the values 1/2 by zeros. The matrix \( J_q \in \mathbb{R}^{q(q+1)/2} \) has entries 1 and 0, so that \( J_q C_q \) is equal to \( C_q \) but replacing the ones by zeros. A projection onto to the columns of \( b \) is denoted \( P_b \) and the projection onto the orthogonal complement of \( b \) will be denotes as \( Q_b \).

To regress \((X, H)\) on \( Y \), we model \( X \mid (Y, H) \) and \( H \mid Y \) separately and use the factorization in (2).

### 2.1 The distribution of \( X \mid (H, Y) \)

We let the \( p \)-dimensional vector of continuous random variables \( X \mid (H, Y) \) be multivariate normal with

\[
X \mid H, Y \sim N(\mu_X + A(f_Y - \bar{f}_Y) + \beta(H - \mu_H), \Delta),
\]

where \( \mu_X = E_X(X), \mu_H = E_H(H), f_Y : \mathbb{R} \rightarrow \mathbb{R}^r \) is a known function of \( Y \), \( \bar{f}_Y = E_Y(f_Y) \), \( A : p \times r \), and \( \beta : p \times q \), are unconstrained parameter matrices, and \( \Delta \) is a \( p \times p \) positive definite covariance matrix. For example, if the response is continuous, \( f_Y \) can be a vector of polynomials of order \( r \), or, in order to avoid multicollinearity, of a set of \( r \) orthonormal basis functions. If the response is categorical with values in one of \( h \) categories \( C_k, k = 1, \ldots, h \), we set \( r = h - 1 \) and let the \( k \)-th element of \( f_Y \) to be \( I(Y \in C_k) \), where \( I \) is the indicator function. To simplify notation, henceforth \( f_Y \) will signify the centered \( f_Y - \bar{f}_Y \).

The probability density function of \( X \mid (H, Y) \) in model (3) is

\[
f(X \mid H, Y = y) = \frac{1}{\sqrt{2\pi} \sqrt{\det \Delta}} \exp \left\{ -\frac{1}{2} \left( (X - \mu_X) - Af_y - \beta(H - \mu_H) \right)^T \Delta^{-1} \left( (X - \mu_X) - Af_y - \beta(H - \mu_H) \right) \right\}.
\]

### 2.2 The distribution of \( H \mid Y \)

The joint distribution of a random vector whose elements are binary random variables is modelled with the multivariate Bernoulli distribution [see Whittaker (2009); Dai (2012); Dai et al. (2013)]. Its probability mass function involves terms representing third and higher order moments of the random variables. The Ising model (Ising, 1925) is frequently used instead to alleviate the complexity of modeling as it includes up to second order interactions among the binary variables. For the multivariate binary regression \( H \mid Y \) we use the Ising with covariates model introduced in Cheng et al. (2014), where covariates are incorporated directly.

Let \( \mathcal{H} \) be all possible combinations of \( H \in \{0, 1\}^q \), \( H_{-j} = (H_1, \ldots, H_{j-1}, H_{j+1}, \ldots, H_q) \), \( H_{-i,-j} = (H_1, \ldots, H_{i-1}, H_{i+1}, \ldots, H_{j-1}, H_{j+1}, \ldots, H_q), i, j = 1, \ldots, q \). The joint probability
mass function of the \( q \)-dimensional vector of binary variables \( H \) conditional on \( Y \) is [see Cheng et al. (2014)]

\[
P(H \mid Y = y) = \frac{1}{G(\Gamma_y)} \exp \left\{ \vech^T(HH^T) \vech(\Gamma_y) \right\},
\]

where \( G(\Gamma_y) = \sum_{H \in \mathcal{H}} \exp \left( \vech^T(HH^T) \vech(\Gamma_y) \right) \), and \( \Gamma_y = (\gamma_{ij}^y) \) is a \( q \times q \) symmetric matrix with elements

\[
\gamma_{ij}^y = \log \left( \frac{\Pr(H_i = 1, H_j = 1 \mid H_{-i,j} = 0, y)}{1 - \Pr(H_i = 1 | H_{-i} = 0, y)} \right),
\]

\[
\gamma_{ij}^y = \log \left( \frac{\Pr(H_i = 1, H_j = 1 \mid y) \Pr(H_i = 0, H_j = 0 \mid y)}{\Pr(H_i = 1, H_j = 1 \mid y) \Pr(H_i = 0, H_j = 1 \mid y)} \right),
\]

for \( i \neq j \).

A linear model with independent variables \( f_Y \in \mathbb{R}^r \) is a natural choice for each \( \gamma_{ij}^y \),

\[
\gamma_{ij}^y = \tau_{ij,0}^y + \tau_{ij}^y f_Y, \quad i, j = 1, \ldots, q,
\]

where \( \tau_{ij}^y = (\tau_{ij,1}, \ldots, \tau_{ij,r}) \) is a vector of parameters independent of \( Y \), and \( \tau_{ij,0}^y \) is the intercept for each \((i, j)\). Here again, \( f_Y \) is also centered, and can be different from that in \( (4) \), even though, as will be seen later, choosing the same \( f_Y \) simplifies the formula for the joint distribution in \( (8) \) as well as the derivation of a sufficient reduction for the regression of \( Y \) on \( X, H \).

Next we define the \( q \times q \) matrices, \( \tau_0 \) and \( \tau_k, \ k = 1, \ldots, r \), as \( [\tau_0^k]_{ij} = \tau_{ij,0}^k \) and \( [\tau_k]_{ij} = \tau_{ij,k} \) with \( i, j = 1, \ldots, q \) and \( k = 1, \ldots, r \). We let \( \tau_0 = \vech(\tau_0^0) \), a \( q(q + 1)/2 \) vector, and \( \tau = (\vech(\tau_1), \ldots, \vech(\tau_r)) \), a \( q(q + 1)/2 \times r \) matrix, so that the \( q(q + 1)/2 \) vector \( \vech(\Gamma_y) \) is

\[
\vech(\Gamma_y) = \tau_0 + \tau f_y.
\]

Under \( (6) \) the probability mass function of \( H \mid Y \) in \( (5) \) is

\[
P(H \mid Y = y) = \frac{1}{G(\Gamma_y)} \exp \left\{ \vech^T(HH^T)(\tau_0 + \tau f_y) \right\},
\]

with \( G(\Gamma_y) = \sum_{H \in \mathcal{H}} \exp \left( \vech^T(HH^T)(\tau_0 + \tau f_y) \right) \).

Under \( (7) \) and \( (4) \), the joint distribution of the inverse regression \( (X, H \mid Y) \) has probability density function

\[
f(X, H \mid Y = y) = f(X \mid H, Y = y) f(H \mid Y = y)
\]

\[
= \frac{1}{\sqrt{2\pi \sqrt{\Delta}}} \exp \left\{ -\frac{1}{2} \left( (X - \mu_X) - Af_y - \beta(H - \mu_H) \right)^T \Delta^{-1} \left( (X - \mu_X) - Af_y - \beta(H - \mu_H) \right) \right\}
\]

\[
\times \frac{1}{G(\Gamma_y)} \exp \left\{ \vech^T(HH^T)(\tau_0 + \tau f_y) \right\}.
\]

Our regression model for the mixed vector \( Z \) is similar to the regression model of Fitzmaurice and Laird (1997) with the difference that we do not allow \( \mu_H \) to vary with \( Y \) in \( (4) \). This results in different maximum likelihood estimates for the parameters in \( (8) \) in Section 4.1.
3. Sufficient Reductions

We focus on the regression problem \((1)\), where we aim to identify a reduction \(R(Z)\) such that \(Y \mid Z \overset{d}{\sim} Y \mid R(Z)\). Since the latter is equivalent to \(Z \mid (Y, R(Z)) \overset{d}{\sim} Z \mid R(Z)\), as discussed in the introduction, we will derive the sufficient reduction that \(Y \mid \theta \sim Y \mid R(Z)\) using (2).

Of central importance to our development is showing that the density of \((X, H) \mid Y\) in (8) belongs to the exponential family of distributions. In Appendix 8, we express (8) as

\[
f(X, H \mid Y = y) = h(X, H) \exp \left( T^T (X, H) \eta_y - \psi(\eta_y) \right),
\]

which belongs to the natural exponential family of distributions [see, e.g., Morris (2006)]. In (9), \(h(X, H) = (2\pi)^{-1/2}\), the sufficient statistic is

\[
T(X, H) = \begin{pmatrix} X \\ H \\ -\frac{1}{2}D_p^T \eta_y \otimes \eta_y + \left( -\frac{1}{2}D_p^T \eta_y \otimes \eta_y + \left( \begin{array}{c} \eta_{g1} \\ \eta_{g2} \\ \eta_{g3} \\ \eta_{g4} \\ \eta_{g5} \end{array} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \end{pmatrix} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \\ \vartheta_4 \\ \vartheta_5 \end{pmatrix},
\]

(10)

the natural parameters are

\[
\eta_y = \begin{pmatrix} \eta_{g1} \\ \eta_{g2} \\ \eta_{g3} \\ \eta_{g4} \\ \eta_{g5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \\ \vartheta_4 \\ \vartheta_5 \end{pmatrix},
\]

(11)

with \(\vartheta^T = (\vartheta_1^T, \vartheta_2^T, \vartheta_3^T, \vartheta_4^T, \vartheta_5^T)^T\) where

\[
\vartheta_1 = (\vartheta_{1,0})^T = \begin{pmatrix} \Delta^{-1} \mu_x - \Delta^{-1} \beta \mu_H \\ \text{vec}(\Delta^{-1} A) \end{pmatrix} : (p + pr) \times 1,
\]

\[
\vartheta_2 = (\vartheta_{2,0}, \vartheta_{2,1})^T = \begin{pmatrix} -\beta^T \Delta^{-1} \mu_x + \beta^T \Delta^{-1} \beta \mu_H + L_q \tau_0 - \frac{1}{2} L_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta) \\ \text{vec}(L_q \tau - \beta^T \Delta^{-1} A) \end{pmatrix} : (q + qr) \times 1,
\]

\[
\vartheta_3 = \vartheta_{3,0} = \text{vec}(\Delta^{-1}) : k_p \times 1,
\]

\[
\vartheta_4 = \vartheta_{4,0} = \text{vec}(\Delta^{-1} \beta) : pq \times 1,
\]

\[
\vartheta_5 = (\vartheta_{5,0}, \vartheta_{5,1})^T = \begin{pmatrix} -\frac{1}{2} J_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta) + J_q \tau_0 \\ \text{vec}(J_q \tau) \end{pmatrix},
\]

and

\[
\psi(\eta_y) = -\frac{1}{2} \log |\text{unvec}(D_p \eta_{g3})| + \log(G(\Gamma_y)) + \frac{1}{2} \eta_{g1}^T (\text{unvec}(D_p \eta_{g3}))^{-1} \eta_{g1} \quad (13)
\]

\[
\psi_{1}(\eta_y) + \psi_{2}(\eta_y) + \psi_{3}(\eta_y),
\]

with

\[
G(\Gamma_y) = \sum_H \exp \left( \begin{pmatrix} J_q C_q \text{vec}(HH^T) \end{pmatrix}^T \begin{pmatrix} \eta_{g5} + J_q \frac{1}{2} D_q^T \text{vec}(\eta_{g4}^T (\text{unvec}(D_p \eta_{g3}))^{-1} \eta_{g4}) \end{pmatrix} \right)
\]

\[+ \text{H}^T \begin{pmatrix} \eta_{g2} + \eta_{g4}^T (\text{unvec}(D_p \eta_{g3}))^{-1} \eta_{g1} + \frac{1}{2} L_q D_q^T \text{vec}(\eta_{g4}^T (\text{unvec}(D_p \eta_{g3}))^{-1} \eta_{g4}) \end{pmatrix},
\]

(14)
where \( \bar{\eta}_y = \text{unvec}(\eta_y) \).

For any matrix \( V \), let \( S_V \) denote the span of the columns of \( V \); that is, \( S_V = \text{span}(V) \). Theorem 1 obtains the sufficient reduction for the regression of \( Y \) on \((X, H)\) using a result from Bura et al. (2016).

**Theorem 1** Suppose that \((X, H) \mid Y\) has density given by (9). The minimal sufficient reduction for the regression \( Y \mid (X, H) \) is

\[
R(X, H) = \alpha_a^T (T(X, H) - E(T(X, H))) \tag{15}
\]

where \( T(X, H) \) is given by (10) and \( \alpha_a \) is a basis for \( S_{\alpha_a} = \text{span}\{\eta_Y - E(\eta_Y), Y \in \mathcal{Y}\} \), with \( \eta_Y \) given in (11).

We provide the proof of Theorem 1 in Appendix 8, where we see that the reduction in (15) is characterized by the coefficients of the basis for \( \text{span}\{\eta_Y - E(\eta_Y), Y \in \mathcal{Y}\} = \text{span}(a) \) with

\[
a = \begin{pmatrix}
\Delta^{-1}A \\
L_q \tau - \beta^T \Delta^{-1}A \\
0 \\
0 \\
J_q \tau
\end{pmatrix} = \begin{pmatrix}
\text{unvec}(\vartheta_{1,1}) \\
\text{unvec}(\vartheta_{2,1}) \\
0 \\
0 \\
\text{unvec}(\vartheta_{5,1})
\end{pmatrix}.
\]

Since \( \eta_3 \approx \eta_{y3} \) and \( \eta_4 \approx \eta_{y4} \) do not depend on \( y \), Corollary 2 follows.

**Corollary 2** Suppose the density of \((X, H) \mid Y\) is given by (9). A minimal sufficient dimension reduction for the regression of \( Y \) on \((X, H)\) is given by

\[
R(X, H) = \alpha_b^T (t(X, H) - E(t(X, H))) \tag{16}
\]

where

\[
t(X, H) = \begin{pmatrix}
X^T, H^T, [J_q \text{vech}(HH^T)]^T
\end{pmatrix}^T,
\]

and \( \alpha_b \) is a basis for \( S_{\alpha_b} = \text{span}\{b\} \) with

\[
b = \begin{pmatrix}
\Delta^{-1}A \\
L_q \tau - \beta^T \Delta^{-1}A \\
0 \\
J_q \tau
\end{pmatrix} = \begin{pmatrix}
\text{unvec}(\vartheta_{1,1}) \\
\text{unvec}(\vartheta_{2,1}) \\
0 \\
\text{unvec}(\vartheta_{5,1})
\end{pmatrix} \tag{18}
\]

As the reduction in (16) is not only sufficient but also minimal, we call it optimal SDR in the sequel.

**Corollary 3** When the predictor vector contains only continuous variables; that is, \( q = 0 \) and \( Z = X \), the sufficient dimension reduction is

\[
R(X) = \alpha_1^T (X - E(X)) \tag{19}
\]

where \( S_{\alpha_1} = \text{span}(\alpha_1) = \text{span}(\Delta^{-1}A) \), and \( A : p \times r \) in (3).
The reduction (19) coincides with Principal Fitted Components (PFC) in Cook and Forzani (2008).

**Corollary 4** When the predictor vector contains only binary variables; that is, \( p = 0 \) and \( Z = H \), the sufficient dimension reduction is

\[
R(X) = \alpha^T (s(H) - E(s(H))),
\]

where

\[
s(H) = \left( H^T, \left[ J_q \text{vech}(HH^T) \right]^T \right)^T,
\]

and

\[
S_{\alpha_2} = \text{span}(\alpha_2) = \text{span} \left( \begin{pmatrix} L_q \tau \\ J_q \tau \end{pmatrix} \right).
\]

When the predictors are mixed, we derive a sufficient but not minimal reduction in Corollary (5), which we call sub-optimal SDR.

**Corollary 5** Suppose that \((X, H)|Y \) has density (9). A sufficient reduction for the regression of \( Y \) on \((X, H)\) is given by

\[
R(X, H) = \alpha^T (w(X, H) - E(w(X, H)) ),
\]

with

\[
w(X, H) = (X^T, H^T, \left[ \text{vech}(HH^T) \right]^T)^T,
\]

\[
S_{\alpha_c} = \text{span}(\alpha_c) = \text{span} \left( \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \right) = \text{span} \left( \begin{pmatrix} \Delta^{-1}A & 0 \\ -\beta^T \Delta^{-1}A & 0 \\ 0 & \tau \end{pmatrix} \right).
\]

If \( \text{rank}(c_1) = d_1 \leq \min\{r, p\} \) and \( \text{rank}(c_2) = d_2 \leq \min\{r, q(q+1)/2\} \), then

\[
c_1 = \left( \begin{pmatrix} \Delta^{-1}A \\ -\beta^T \Delta^{-1}A \end{pmatrix} \right) = \left( \begin{pmatrix} \alpha \xi \\ -\beta^T \alpha \xi \end{pmatrix} \right), \quad c_2 = \kappa \iota,
\]

where \( A = \alpha \xi, \quad \alpha \in \mathbb{R}^{p \times d_1}, \quad \xi \in \mathbb{R}^{d_1 \times r}, \quad \kappa \in \mathbb{R}^{q(q+1)/2 \times d_2} \) and \( \iota \in \mathbb{R}^{d_2 \times r} \) are full rank matrices. Therefore,

\[
\text{span}(c_1) = \text{span}\{ (\alpha^T, -\alpha^T \beta)^T \},
\]

\[
\text{span}(c_2) = \text{span}(\kappa).
\]

In Table 1, we summarize the results of this Section and tabulate the sufficient reductions for mixed normal and binary predictors.
Table 1: Sufficient Reductions in Regressions with Mixed Predictors.

| Predictor Distribution | Optimal SDR | Sub-optimal |
|------------------------|-------------|-------------|
| (X, H) | Y with density (9) | $\hat{\alpha}_o^T (t(X, H) - E(t(X, H)))$ | $\hat{\alpha}_e^T (w(X, H) - E(w(X, H)))$ |
| X | Y ~ $N(\mu_X + A(\delta_Y - \hat{\delta}_Y), \Delta)$ | $\hat{\alpha}_o^T (X - E(X))$ | $\hat{\alpha}_e^T (\delta_Y - E(\delta_Y))$ |
| H | Y with mass function (7) | $\hat{\alpha}_o^T (s(H) - E(s(H)))$ | $\hat{\alpha}_e^T (s(H))$ |

4. Reduction Estimators and their Asymptotic Distribution

In this section we derive maximum likelihood estimators for our optimal and sub-optimal sufficient reductions, the asymptotic normality of the projection matrix of the optimal SDR, with which we also obtain asymptotic tests of dimension of both optimal and sub-optimal reductions.

4.1 Parameter Estimation via Maximum Likelihood

We assume a random sample $(y_i, x_i, h_i), i = 1, \ldots, n$, is drawn from the joint distribution of $(Y, X, H)$ and that the conditional distribution models (5) and (3) hold. Finding the maximum likelihood estimators of the reductions derived in Section 3 requires first the estimation of the parameters $\Delta, \mu, \mu_H, A, \beta, \tau_0, \tau$, in the joint density (8) with log-likelihood

$$\sum_{i=1}^{n} \log f_{X,H}(x_i, h_i | y_i; \Delta, \mu, \mu_H, A, \beta, \tau_0, \tau).$$

We maximize (29) in two steps. First, we maximize $\sum_{i=1}^{n} \log f_{X}(x_i | y_i, h_i; \Omega)$ to estimate the parameters $\Omega = \{\Delta, \mu, \mu_H, A, \beta\}$. Since $X \mid (H, Y)$ follows a normal distribution, the maximum likelihood estimator (MLE) of $\Omega$ is obtained from fitting a multivariate normal linear model of $X$ on the centered $(H, Y)$ via MLE. The MLE of $A$ and $\beta$ are $(\hat{A}, \hat{\beta}) = \bar{X}^T L (L^T L)^{-1}$, where $\bar{X}$ denotes the $n \times p$ matrix with rows $(x_i - \bar{x})^T$, and $L$ the $n \times (r+q)$ matrix with rows $((f_y - \bar{f}_y)^T, (h_i - \bar{h})^T)$, $\bar{x} = \sum_{i=1}^{n} x_i/n, \bar{f}_y = \sum_{i=1}^{n} f_{y_i}/n$ and $\bar{h} = \sum_{i=1}^{n} h_i/n$.

The MLE of the covariance matrix is $\hat{\Delta} = \left(\bar{X}^T - (\hat{A}, \hat{\beta})L^T\right) \left(\bar{X}^T - (\hat{A}, \hat{\beta})L^T\right)^T/n$.

Next, we estimate $Y = (\tau_0, \tau)$ maximizing the conditional log-likelihood function

$$\sum_{i=1}^{n} \log f_{Y}(h_i ; y_i; Y).$$
Using parametrization (6), the joint probability mass function (5) can be written as
\[
P(H \mid Y = y) = \exp \left( \sum_{j=1}^{q} \tau_{j0}^{*} H_{j} + \sum_{j=1}^{q} \tau_{jj}^{*} f_{y} H_{j} \right)
+ \sum_{1 \leq j < j' \leq q} \tau_{jj'}^{*} H_{j} H_{j'} + \sum_{1 \leq j < j' \leq q} \tau_{jj'}^{T} f_{y} H_{j} H_{j'} \right) \frac{1}{G(f_{y})}.
\]

Following Cheng et al. (2014), we consider a single binary variable \( H_{j} \) and condition over the rest \( H_{-j} = (H_{1}, \ldots, H_{j-1}, H_{j+1}, \ldots, H_{q}) \) to obtain
\[
\log \frac{P(H_{j} = 1 \mid H_{-j}, Y)}{P(H_{j} = 0 \mid H_{-j}, Y)} = \tau_{j0}^{*} + \tau_{jj}^{*} f_{y} + \sum_{j \neq j'} \tau_{jj'}^{*} H_{j} H_{j'}.
\]

Thus, the conditional log-odds for a specific binary variable \( H_{j} \) is linear in the parameters. Moreover, the conditional maximum likelihood estimators for these parameters can be obtained by fitting a logistic regression of \( H_{j} \) on \( (f_{y}, H_{-j}, f_{y} H_{-j}) \), so that we obtain estimators for \( \tau_{0} \) and \( \tau \) by fitting \( q \) univariate logistic regressions. In particular, for the sample points \((h_{i}^{T}, y_{i}) = (h_{1}, \ldots, h_{q}, y_{i})\), for each binary variable \( j \) \((j = 1, \ldots, q)\), the conditional log-likelihood function is
\[
\ell_{j}(\tau_{0}, \tau; h_{i}, y_{i}) = \frac{1}{n} \sum_{i=1}^{n} \log P(h_{ij} \mid h_{i,-j}, y_{i}) = \frac{1}{n} \sum_{i=1}^{n} (h_{ij} \epsilon_{ij} - \log(1 + \exp(\epsilon_{ij}))),
\]
where \( h_{i,-j} = (h_{1}, \ldots, h_{i,j-1}, h_{i,j+1}, \ldots, h_{q}) \) and
\[
\epsilon_{ij} = \log \frac{P(h_{ij} = 1 \mid h_{i,-j}, y_{i})}{P(h_{ij} = 0 \mid h_{i,-j}, y_{i})} = \tau_{j0}^{*} + \tau_{jj}^{*} f_{y} + \sum_{j \neq j'} \tau_{jj'}^{*} h_{ij} + \sum_{j \neq j'} \tau_{jj'}^{T} f_{y} h_{ij}.
\]

To estimate \( \Gamma \) we use the joint estimation algorithm proposed by Cheng et al. (2014) that maximizes \( \sum_{j} \ell_{j}(\tau_{0}, \tau; h_{i}, y_{i}) \).

### 4.2 Maximum Likelihood Estimation of the Reductions

To estimate the **optimal SDR** \( \alpha_{b} \) in Corollary 2 and the **sub-optimal SDR** \( \alpha_{c} \) in Corollary 5, we need first to estimate \( b \) in (18) and \( c_{1} \) and \( c_{2} \) in (25). We use the ML estimators \((\hat{\Delta}, \hat{\mu}, \hat{\mu}_{H}, \hat{\Lambda}, \hat{\beta}, \hat{\tau}_{0}, \hat{\tau})\) of the corresponding parameters in (29) from Section 4.1.

#### 4.2.1 Optimal SDR

To estimate the minimal sufficient reduction in (16), or equivalently, derive a basis estimate of \( S_{\alpha_{b}} \), we need to first estimate \( b \) in (18). If \( d = \dim(S_{\alpha_{b}}) \), with \( d \leq \min\{r, p + q(q+1)/2\} \), the rank of \( b \) is also \( d \) with singular value decomposition
\[
b = U^{T} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} R,
\]
where \( k_{1} \geq \ldots \geq k_{d} > 0 \) are the singular values of \( b \), \( K = \text{diag}(k_{1}, \ldots, k_{d}) \), \( U^{T} = (U_{1}, U_{0}) \) is a \( m \times m \) orthogonal matrix with \( m = p + q(q+1)/2 \), \( U_{1} : m \times d \), \( U_{0} : m \times (m-d) \),
and $\mathbf{R}^T = (\mathbf{R}_1, \mathbf{R}_0)$ is an $r \times r$ orthogonal matrix with $\mathbf{R}_1 : r \times d$, $\mathbf{R}_0 : r \times (r - d)$. The submatrices satisfy $\mathbf{U}_1\mathbf{U}_1^T + \mathbf{U}_0\mathbf{U}_0^T = \mathbf{I}_m$, $\mathbf{U}_1^T\mathbf{U}_1 = \mathbf{I}_d$, $\mathbf{U}_0^T\mathbf{U}_0 = \mathbf{I}_{m - d}$, $\mathbf{U}_0^T\mathbf{U}_1 = \mathbf{0}$, $\mathbf{R}_1\mathbf{R}_1^T + \mathbf{R}_0\mathbf{R}_0^T = \mathbf{I}_r$, $\mathbf{R}_1^T\mathbf{R}_1 = \mathbf{I}_d$, $\mathbf{R}_0^T\mathbf{R}_0 = \mathbf{I}_{r - d}$, $\mathbf{R}_0^T\mathbf{R}_1 = \mathbf{0}$. Then,

$$\mathbf{b} = \mathbf{U}_1\mathbf{K}\mathbf{R}_1^T,$$

and, as a consequence, $\alpha_{\mathbf{b}}$ in Corollary 2 can be set to $\mathbf{U}_1$. Plugging in the ML estimators $(\hat{\Delta}, \hat{A}, \hat{\beta}, \hat{\tau})$ we obtain that the ML estimator of $\mathbf{b}$ is

$$\hat{\mathbf{b}} = \begin{pmatrix} \hat{\Delta}^{-1}\hat{A} \\ \mathbf{L}_q\hat{\tau} - \hat{\beta}^T\hat{\Delta}^{-1}\hat{A} \end{pmatrix} = \begin{pmatrix} \text{unvec}(\hat{\vartheta}_{1,1}) \\ \text{unvec}(\hat{\vartheta}_{2,1}) \\ \text{unvec}(\hat{\vartheta}_{5,1}) \end{pmatrix}.$$ (34)

The singular value decomposition of the MLE of $\mathbf{b}$ is

$$\tilde{\mathbf{b}} = \tilde{\mathbf{U}}^T \begin{pmatrix} \tilde{\mathbf{K}}_1 & 0 \\ 0 & \tilde{\mathbf{K}}_0 \end{pmatrix} \tilde{\mathbf{R}},$$

(35) where $\tilde{\mathbf{K}}_1 = \text{diag}(\tilde{k}_1, \ldots, \tilde{k}_d)$, $\tilde{\mathbf{K}}_0 = \text{diag}(\tilde{k}_{d+1}, \ldots, \tilde{k}_{\min(m,r)})$, $\tilde{k}_i$ are the singular values of $\tilde{\mathbf{b}}$ in decreasing order, $\tilde{\mathbf{U}}$ is an $m \times m$ orthogonal matrix whose columns are the left singular vectors of $\tilde{\mathbf{b}}$, and $\tilde{\mathbf{R}}$ is an $r \times r$ orthogonal matrix, whose columns are the right-singular vectors of $\tilde{\mathbf{b}}$. Let $\tilde{\mathbf{U}}_1$ be the first $d$ columns of $\tilde{\mathbf{U}}$, $\tilde{\mathbf{R}}_1$ the first $d$ columns of $\tilde{\mathbf{R}}^T$, and $\tilde{\mathbf{B}} = \tilde{\mathbf{K}}_1\tilde{\mathbf{R}}_1^T$. An estimator of $\mathbf{b}$ subject to $d = \text{dim}(\mathcal{S}_{\alpha_{\mathbf{b}}})$ is

$$\tilde{\mathbf{b}}^{(d)} = \tilde{\mathbf{U}}_1\tilde{\mathbf{K}}_1\tilde{\mathbf{R}}_1^T = \tilde{\mathbf{U}}_1\tilde{\mathbf{B}},$$

(36) and an estimator of the reduction $\alpha_{\mathbf{b}}$ in Corollary 2 is

$$\hat{\alpha}_{\mathbf{b}} = \tilde{\mathbf{U}}_1.$$ (37)

4.2.2 Sub-optimal SDR: $\hat{\mathcal{S}}_c$

To obtain an estimator for the space $\mathcal{S}_{\alpha_{\mathbf{c}}}$ in (25) that gives the sub-optimal sufficient reduction (23), we set $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2)$, where $\mathbf{c}_1$ and $\mathbf{c}_2$ are given in (26), with $\text{rank}(\mathbf{c}_1) = d_1$ and $\text{rank}(\mathbf{c}_2) = d_2$. Plugging in the MLE $(\hat{\Delta}, \hat{\mu}, \hat{\mu}_H, \hat{A}, \hat{\beta}, \hat{\tau}_0, \hat{\tau})$ of the corresponding parameters in (29) from Section 4.1, we obtain estimators of $\mathbf{c}_1$ and $\mathbf{c}_2$, 

$$\hat{\mathbf{c}}_1 = \begin{pmatrix} \hat{\Delta}^{-1}\hat{A} \\ -\hat{\beta}^T\hat{\Delta}^{-1}\hat{A} \end{pmatrix}, \quad \hat{\mathbf{c}}_2 = \hat{\tau}.$$ We then consider their respective SVD decompositions as in Section 4.2.1. Let $\tilde{\mathbf{U}}_{\mathbf{c}_1}$ denote the first $d_1$ left eigenvectors of $\hat{\mathbf{c}}_1$ and $\tilde{\mathbf{U}}_{\mathbf{c}_2}$ the first $d_2$ left eigenvectors of $\hat{\mathbf{c}}_2$. Then, an estimator for the sub-optimal sufficient reduction in (23) is defined as

$$\hat{\alpha}_{\mathbf{c}} = \begin{pmatrix} \hat{\mathbf{U}}_{\mathbf{c}_1} & 0 \\ 0 & \hat{\mathbf{U}}_{\mathbf{c}_2} \end{pmatrix}.$$
4.3 Asymptotic distribution of the optimal sufficient reduction estimator

In this section we derive the asymptotic distribution of the projection onto the column space of the estimated optimal sufficient reduction \( \hat{\alpha}_b \) in (37), \( P_{\hat{\alpha}_b} = \hat{\alpha}_b (\hat{\alpha}_b^T \hat{\alpha}_b)^{-1} \hat{\alpha}_b^T \). We use this result in the derivation of the asymptotic tests for dimension in Section 4.4 and for inference about the sufficient dimension reduction.

**Proposition 6** Suppose that \((X, H)|Y\) has probability mass function (9) with the natural parameters \( \eta_Y \) satisfying (11) and that \( b \) has rank \( d \). Then,

\[
\sqrt{n} \text{ vec} \left( P_{\hat{\alpha}_b} - P_{\alpha_b} \right) \xrightarrow{D} N \left( 0, V_{\hat{\alpha}_b} \right),
\]

with

\[
V_{\hat{\alpha}_b} = (I_{m^2} \otimes K_{mm}) (b^- \otimes Q_b)^T V_{rcl} (b^- \otimes Q_b) (I_{m^2} \otimes K_{mm}),
\]

where \( b^- \) is the Moore-Penrose generalized inverse of \( b \),

\[
V_{rcl} = WMVM^T W^T,
\]

with

\[
V^{-1} = E \left( F_y^T JF_y \right),
\]

\( F_y \) is defined in (11), \( J \) is the matrix of partial derivatives given by

\[
J = \frac{\partial^2 \psi(\eta_y)}{\partial \eta_y \partial \eta_y^T},
\]

\[
M = \begin{pmatrix}
0_{pr \times p} & I_{pr} & 0_{pr \times q} & 0_{pr \times qr} & 0_{pr \times m_p} & 0_{pr \times qp} & 0_{pr \times k_q} & 0_{pr \times r k_q} \\
0_{qr \times p} & 0_{qr \times pr} & 0_{qr \times q} & I_{qr} & 0_{qr \times m_q} & 0_{qr \times qp} & 0_{qr \times k_q} & 0_{qr \times r k_q} \\
0_{rk_q \times p} & 0_{rk_q \times pr} & 0_{rk_q \times q} & 0_{rk_q \times qr} & 0_{rk_q \times m_p} & 0_{rk_q \times qp} & 0_{rk_q \times k_q} & I_{rk_q}
\end{pmatrix},
\]

and

\[
W = \begin{pmatrix}
I_r \otimes \begin{pmatrix} I_p \\ 0_{q \times p} \end{pmatrix}, I_r \otimes \begin{pmatrix} 0_{p \times q} \\ I_q \end{pmatrix}, I_r \otimes \begin{pmatrix} 0_{p \times k_q} \\ 0_{q \times k_q} \end{pmatrix}, I_r \otimes \begin{pmatrix} 0_{k_q \times q} \\ I_{k_q} \end{pmatrix}
\end{pmatrix}.
\]

4.4 Tests for dimension

We propose two asymptotic tests for the dimension of the sufficient reduction in optimal SDR. We adapt these tests for the case of sub-optimal SDR, to estimate the dimension of the continuous predictors separately from the binary predictors.

The dimension of the sufficient reduction is the rank of \( b \) in (18). We estimate the rank \( d \) of \( b \) by sequentially testing the hypotheses

\[
H_0 : \text{rank}(b) = j \quad \text{vs.} \quad H_1 : \text{rank}(b) > j,
\]

for \( j = 0, 1, \ldots, \min(r, m) \), where \( m = p + q(q + 1)/2 \). For a fixed level \( \alpha \), the estimated rank is the smallest value of \( j \) for which the null is not rejected.
Bura and Yang (2011) proposed asymptotic tests for the rank of random matrices in sequential hypothesis testing. To construct the corresponding tests for dimension, we consider the singular value decomposition of \( \mathbf{b} \) in (33) and \( \hat{\mathbf{b}} \) in (36) with \( d = j \).

The first statistic we use to test (44) is \( \Lambda_1(j) = n \sum_{i=j+1}^{\min(m,r)} \hat{k}_i^2 \), where \( \hat{k}_i \)'s are the singular values of \( \hat{\mathbf{b}} \) in descending order. Proposition 6 obtains the asymptotic normality of \( \mathbf{b} \) with covariance \( \mathbf{V}_{rlc} \) in (39). When \( \text{rank}(\mathbf{b}) = j \),

\[
\Lambda_1(j) \xrightarrow{\mathcal{D}} \sum_{i=1}^{s} \omega_i X_i^2, \tag{45}
\]

where \( s = \min(\text{rank}(\mathbf{V}_{rlc})), \text{rank}(\mathbf{b}) \), \( (r-j)(m-j) \), \( X_i^2 \) are independent chi-squared random variables with 1 degree of freedom, and the weights are the descending eigenvalues of \( \mathbf{Q} = (\mathbf{R}_0^T \otimes \mathbf{U}_0^T) \mathbf{V}_{rlc}(\mathbf{R}_0 \otimes \mathbf{U}_0) \) [see Bura and Yang (2011)]. In practice, the weights \( \omega_i, i = 1, \ldots, s \) are replaced by \( \tilde{\omega}_1 \geq \tilde{\omega}_2 \geq \ldots \geq \tilde{\omega}_s \), the descending eigenvalues of

\[
\hat{\mathbf{Q}} = (\hat{\mathbf{R}}_0^T \otimes \hat{\mathbf{U}}_0^T) \hat{\mathbf{V}}_{rlc}(\mathbf{R}_0 \otimes \mathbf{U}_0), \tag{46}
\]

where \( \hat{\mathbf{V}}_{rlc} \) is a consistent estimate of \( \mathbf{V}_{rlc} \). This test rejects \( H_0 \) if \( \Lambda_1(j) > q_\alpha \), where \( q_\alpha \) is the \( (1 - \alpha) \) percentile of the distribution of \( \sum_{i=1}^{s} \hat{\omega}_i X_i^2 \). We estimate \( q_\alpha \) from the empirical distribution function of \( \Lambda_1 \), by generating 10000 realizations of \( \sum_{i=1}^{s} \hat{\omega}_i X_i^2 \) and computing the empirical quantile \( \hat{q}_\alpha \).

The second is a Wald test with test statistic, \( \Lambda_2(j) = n \text{vec}(\hat{\mathbf{K}}_0)^T \hat{\mathbf{Q}}^\dagger \text{vec}(\hat{\mathbf{K}}_0) \), where \( \hat{\mathbf{K}}_0 \) is defined in (35) and \( \hat{\mathbf{Q}}^\dagger \) is the Moore-Penrose inverse of \( \hat{\mathbf{Q}} \) in (46).

Following Bura and Yang (2011), since \( \hat{\mathbf{b}} \) is asymptotically normal, if \( j = \text{rank}(\mathbf{b}) \), then \( \Lambda_2(j) \xrightarrow{\mathcal{D}} \chi^2(s) \), where the degrees of freedom are \( s = \min(\text{rank}(\mathbf{V}_{rlc}), \text{rank}(\mathbf{b})) \), \( (r-j)(m-j) \). The rejection region is \( \Lambda_2(j) > \chi^2_\alpha(s) \), where \( \chi^2_\alpha(s) \) is the \( (1 - \alpha) \) percentile of the \( \chi^2(s) \) distribution.

5. Variable selection

Identifying variables that are not associated with the outcome is important for both interpretation and for improving the predictive power of a classifier or a regression model. We propose a method to simultaneously obtain the sufficient reduction and carry out variable selection by removing redundant variables from the reduction. This is obtained jointly with the estimate of the reduction by introducing structured regularization on a matrix factorization problem.

In particular, we exploit the factorization of the full rank maximum likelihood estimate \( \hat{\mathbf{b}} \) into a relevant full-rank factor \( \mathbf{C} \in \mathbb{R}^{p+q(q+1)/2 \times d} \), which determines the reduction, and a matrix \( \mathbf{B} \) that is immaterial.

The building block of the procedure proposed here is to note that the reduced rank estimator \( \hat{\mathbf{b}}(d) = \hat{\mathbf{U}}_1 \hat{\mathbf{B}} \) in (36) is also the solution to the least squares minimization problem

\[
\min_{\mathbf{C} \in \mathbb{R}^{p+q(q+1)/2 \times d}, \mathbf{C}^T \mathbf{C} = \mathbf{I}_{	ext{B}} \in \mathbb{R}^{d \times d}} (\text{vec}(\hat{\mathbf{b}}) - \text{vec}(\mathbf{CB}))^T (\text{vec}(\hat{\mathbf{b}}) - \text{vec}(\mathbf{CB})), \tag{47}
\]

where \( \hat{\mathbf{b}} \) is the maximum likelihood estimator of \( \mathbf{b} \). The solution can be expressed as \( \hat{\mathbf{C}} = \hat{\mathbf{U}}_1 \mathbf{V} \), for some orthogonal matrix \( \mathbf{V} \in \mathbb{R}^{d \times d} \), so that \( \text{span}(\hat{\mathbf{C}}) = \text{span}(\hat{\mathbf{U}}_1) \).
All sufficient reductions in Section 4.2.1 are of the form $R(X, H) = U_1^T (t(X, H) - E(t(X, H)))$. If $t_j$ is the $j$th component of $t(X, H)$, and $t_j$ is not associated with $Y$, the $j$th row of $U_1$ is zero. Therefore, identifying predictors that are conditionally independent of $Y$ corresponds to identifying the rows of $C$ that contain only 0. This can be achieved using mixed-norm regularizers that are known to induce structured sparsity in the estimates (Bach et al., 2012).

The proposed procedure is as follows. For a fixed $d$, once we obtain $\hat{B}^{(d)} = \hat{U}_1 \hat{B}$ in (36), we solve

$$\arg\min_{C \in \mathbb{R}^{(p+q(n+1)/2) \times d}, C \in \mathbb{I}} \left( \text{vec}(B) - \text{vec}(CB) \right)^T \left( \text{vec}(B) - \text{vec}(CB) \right) + \lambda \Omega(C),$$

(48)

where $\Omega(C)$ is a mixed-norm regularizer which penalizes the rows of $C$ in a similar manner to group-lasso. The specific form of $\Omega(C)$ depends on the type of predictor variables involved in the problem, as follows.

(a) When all predictors are continuous (normal), we use the penalty $\Omega(C) = \sum_{j=1}^{p} ||C_j||_2$, with $C_j$ the $j$th row of $C$. In this case the sufficient reduction contains no interaction terms and each row of $C$ affects a single element of $X$. Hence, by shrinking the $j$th row of $C$ to 0, the computed reduction becomes insensitive to the measured value of $X_j$. When all predictors are continuous, under the assumed model the optimization problem is indeed fairly similar to group lasso (Yuan and Lin, 2006) as can be seen after rewriting (48) as

$$\arg\min_{C \in \mathbb{R}^{(p+q(n+1)/2) \times d}, C \in \mathbb{I}} \| \text{vec}(B) - (\hat{B}^T \otimes I) \text{vec}(C) \|_2^2 + \lambda \sum_{j=1}^{p} ||C_j||_2.$$  

(b) When all predictors are binary, the sufficient reduction includes interaction effects $H_i H_j$. Thus, to discard the effect of a given binary variable, say $H_j$, we need to set all the entries in $C$ related to $H_j$ to zero. For a reduction of dimension $d$, there are $d$ such entries related to the main effects and $d(q-1)$ related to the interaction terms. The grouping of the entries of $C$ does not form a partition, since the entries affecting the interaction terms appear twice. For instance, assume for simplicity that $d = 1$. Parameter $\theta_{13}$ operates on variables $H_1$ and $H_3$ and then it enters the regularizer in groups $\{\eta_1, \theta_{12}, \theta_{13}, \ldots, \theta_1q\}$ and $\{\eta_3, \theta_{13}, \theta_{23}, \ldots, \theta_3q\}$. Both groups of parameters overlap at $\theta_{13}$. Thus, the regularizer inducing the desired sparsity structure is a mixed-norm regularizer with overlapping groups, $\Omega(C) = \sum_{g \in \mathcal{G}} ||C_g||_2$. Here, $g \subset \{1, \ldots, dq(q+1)/2\}$ indicates the subset of entries that affect the binary variable $H_i$ and $\mathcal{G}$ is the collection of such groups. Moreover, each binary variable is associated with two groups, one derived from the main effects and one from the interaction terms, since they typically have rather different scales. The obtained regularized problem can be solved using algorithms for overlapping group lasso, as proposed, for example, in Liu and Ye (2010).

(c) When the predictors are mixed normal and binary, we combine the regularizers described in (a) and (b) in a single penalty $\Omega(C) = \gamma \sum_{j=1}^{p} ||C_j||_2 + (1-\gamma) \sum_{g \in \mathcal{G}} ||C_g||_2$. 

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The value of $\gamma$ serves as a tuning weight for the amount of regularization in the continuous and binary parts, respectively. In sub-optimal SDR, we carry out variable selection separately for the continuous and binary variables as described in (a) and (b).

Selection of hyperparameters ($\lambda, \gamma$) is done using 10-fold cross validation, with prediction error as the optimization criterion. The procedure starts by estimating a maximum value $\lambda_m$ so that the whole estimate vanishes for any $\lambda > \lambda_m$. We then set a grid of $n_\lambda$ candidate values for $\lambda$, uniformly spaced on a logarithmic scale between 0 and $\lambda_m$. We typically use $n_\lambda = 100$. For $\gamma$ we test 11 values uniformly spaced in $[0,1]$. In each fold, an initial full-rank estimate of the reduction is computed using the training set and then factorized using truncated SVD to give $\hat{B}$ and an initial estimate for $C$. Problem (48) is solved for each pair of candidate values ($\lambda_k, \gamma_k$). The obtained reduction is applied to both the training and the test sample. With the reduced training set we fit a prediction model and then we evaluate the prediction error on the reduced test sample. The average prediction error over the ten cross-validation folds is then computed for each candidate pair ($\lambda_k, \gamma_k$). We pick the combination that attains the smallest mean prediction error.

6. Simulation Studies

We assess the performance of the proposed methods in estimating the sufficient reduction and its dimension, out-of-sample prediction, and variable selection in simulations.

In all our simulations the response is generated from the uniform distribution on the integers $\{1, \ldots, r + 1\}$, with $r = 5$, and set $f_y = I(y = j) - n_j/n$, where $I$ is the indicator function, $n$ denotes the total sample size and $n_j$ the number of observations in category $j$ for $j = 1, \ldots, r$. All reported results are based on sample sizes $n = 100, 200, 300, 500, 750, \text{ and } 1000$ repetitions.

6.1 Estimation, prediction and dimension tests

We assess the accuracy of estimating $\text{span}(\alpha)$ with $\text{span}(\hat{\alpha})$ using $||P_\alpha - P_{\hat{\alpha}}||_2$ [see Ye and Lim (2016)]. The prediction error is computed as $||P_{\alpha}^T(X_N, H_N) - P_{\hat{\alpha}}^T(X_N, H_N)||_2$, where $(X_N, H_N)$ is a new sample of size $N = 2000$ that is independent of the training sample. We estimate the sufficient reduction using the true $d$.

6.1.1 Continuous predictors

We generate $p$-variate continuous predictors as $X \mid Y = y \sim \mathcal{N}(\mu_y, \Delta)$ with $\mu_y = A f_y$ for $A = \Delta \alpha \xi$, where $\alpha \in \mathbb{R}^{d \times d}$ of rank($\alpha$) = $d$ and $\xi \in \mathbb{R}^{d \times r}$. We let $p = 20$ and $0_t, 1_t$ denote the $l$-vectors of zeros and ones, respectively.

(a) For $d = 1$, we set $\xi = 1_T^T \alpha = (0_{p/2}^T, 1_{p/2}^T)^T$, $\Delta = 5(I_p + \rho \alpha \alpha^T)$ with $\rho = 0.55$.

(b) For $d = 2$, we set

$$\xi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$
and $\alpha = (\alpha_1, \alpha_2)$ be an orthonormal basis of $\text{span}(\langle 0_{p/2}^T, 1_{p/2}^T \rangle^T, \langle 0_{p/4}^T, 1_{p/4}^T, -1_{p/4}^T \rangle^T)$, 
$\Delta = 5(I_p + \rho_1 \alpha_1 \alpha_1^T + \rho_2 \alpha_2 \alpha_2^T)$ for $\rho_1 = 0.55$ and $\rho_2 = 0.25$.

### 6.1.2 Binary predictors

We generate $q = 10$ binary predictors assuming that $H \mid Y$ follows an Ising model with parameters $\{\tau_0, \tau\}$, where $\tau = [\text{vech}(\tau_1), \ldots, \text{vech}(\tau_r)]$, $\tau_j$ are $q \times q$ matrices and set $\tau_0 = 0$.

(a) For $d = 1$, and $j = 1, \ldots, r$, $\tau_j = 3 \times K_1 / \sqrt{\sum_{ij}([K_1]_{ij})}$ with

$$
K_1 = \begin{pmatrix}
1 & 30 & 5 & 0 & 0 & 0 \\
30 & 1 & 10 & 0 & 0 & 0 \\
5 & 10 & 1 & 30 & 0 & 0 \\
0 & 0 & 30 & 1 & 30 & 0 \\
0 & 0 & 0 & 30 & 1 & 30 \\
0 & 0 & 0 & 0 & 30 & 1 \\
0 & 0 & 0 & 0 & 0 & 30 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0_{3 \times 1} & \cdots & \cdots & \cdots & \cdots & 0_{3 \times 1} \\
\end{pmatrix}.
$$

(b) For $d = 2$, $\tau_j = 3 \times K_1 / \sqrt{\sum_{ij}([K_1]_{ij})}$, for $j = 1, 3, 4, 5$, and

$$
\tau_2 = \frac{12}{\sqrt{6}} \times \begin{pmatrix}
I_6 & 0_{6 \times 4} \\
0_{4 \times 6} & 0_{4 \times 4} \\
\end{pmatrix}.
$$

### 6.1.3 Mixed predictors

(a) For $d = 1$, we use the same parameters as for continuous $X \mid (H, Y)$ and binary variables $H \mid Y$ in Sections 6.1.1, 6.1.2, respectively. Moreover, we set $\mu_H = 0$ and $\beta = (1_{p \times 6}/10, 0_{p \times 4}) \in \mathbb{R}^{p \times q}$, to induce sparsity in the binary predictors.

(b) For $d = 2$, we generate $H \mid Y$ as in Section 6.1.2 with dimension 1 and $X \mid (H, Y)$ as in (a) with dimension 2.

In Figure 1, we plot the estimation error $\|P_\hat{\alpha} - P_\alpha\|_2$ and the prediction error $\|P_\hat{\alpha}^\top X_N - P_\alpha^\top X_N\|_2$ for optimal SDR on the $y$-axis versus the training sample size on the $x$-axis across all our simulation scenarios. For all types of predictors the prediction is smaller than the estimation error and both decrease as the sample size increases. Moreover, both increase as the dimension increases from 1 to 2 in the left and right panels, respectively, across types of predictors. When comparing types of predictors, continuous predictors exhibit higher estimation and prediction errors across sample sizes and mixed predictors result in the highest estimation and prediction errors.

In Figure 2 we plot the estimation and prediction error of sub-optimal SDR, where the continuous and binary variables are reduced separately. The pattern of behavior is consistent with that of optimal SDR in Figure 1, with the continuous variables inducing
Figure 1: Estimation error and out of sample prediction of optimal SDR with continuous, binary and mixed predictors for $d = 1$ (Left) and $d = 2$ (Right).
larger errors of both types across sample sizes and \( d = 1, 2 \). Again, the errors are smaller for dimension 1.

Under the same simulation settings, we also evaluate the performance of our simultaneous variable selection and dimension reduction method that is presented in Section 5. In Table 2 we report the proportion of variables correctly identified as non-relevant (true positives, TP) and the proportion of variables erroneously assessed as non-relevant (false negatives, FN). Between \( d = 1 \) and \( d = 2 \), TP is higher across sample sizes, whereas FN is lower. Both rates improve substantially as the sample size increases. When all predictors are continuous both rates are lower across sample sizes. This is expected since the inclusion of a binary variable results in second order interaction effects in the reduction. Therefore, to rule out a binary variable both its own-coefficient and all the coefficients of its interaction terms must be zero. Overall, our regularized SDR approach achieves high true positive and small false negative rates for reasonable sample sizes.

In Table 3 we report the proportion of times out of 100 replications that the dimension \( d \) was correctly estimated based on the sequential tests of dimension in Section 4.4 for all our simulation settings. The sample size has a noticeable effect in the accuracy of the estimation of dimension, as expected since both tests are asymptotic. The weighted \( \chi^2 \) test accuracy suffers more from increasing the dimension and all binary predictors as compared to that of the chi-squared test, across sample sizes. For mixed predictors, as well, the chi-squared test exhibits higher accuracy for both optimal and sub-optimal SDR across sample sizes.
Table 2: Accuracy of the regularized estimator in variable selection.

| Predictors | $d$ | Rates | 100   | 200   | 300   | 500   | 750   |
|------------|-----|-------|-------|-------|-------|-------|-------|
| Continuous | 1   | TP    | 0.653 | 0.751 | 0.796 | 0.851 | 0.889 |
|            |     | FN    | 0.314 | 0.17  | 0.095 | 0.044 | 0.012 |
|            | 2   | TP    | 0.521 | 0.591 | 0.629 | 0.748 | 0.843 |
|            |     | FN    | 0.165 | 0.048 | 0.014 | 0.004 | 0.002 |
| Binary     | 1   | TP    | 0.188 | 0.310 | 0.400 | 0.55  | 0.623 |
|            |     | FN    | 0.167 | 0.117 | 0.045 | 0.015 | 0.018 |
|            | 2   | TP    | 0.255 | 0.300 | 0.368 | 0.458 | 0.528 |
|            |     | FN    | 0.048 | 0.020 | 0.012 | 0.000 | 0.000 |
| Mixed      | 1   | TP    | 0.632 | 0.592 | 0.589 | 0.671 | 0.674 |
|            |     | FN    | 0.493 | 0.333 | 0.196 | 0.200 | 0.170 |
|            | 2   | TP    | 0.596 | 0.656 | 0.639 | 0.583 | 0.610 |
|            |     | FN    | 0.451 | 0.413 | 0.325 | 0.163 | 0.124 |

Table 3: Proportion of correct dimension estimation under the simulation settings in Section 6.

| Predictors | Dimension | Test Method | Sample Size | 100   | 200   | 300   | 500   | 750   |
|------------|-----------|-------------|-------------|-------|-------|-------|-------|-------|
| Continuous | $d = 1$   | Weighted $\chi^2$ |             | 0.60  | 0.82  | 0.83  | 0.89  | 0.95  |
|            |           |             |             | 0.00  | 0.00  | 0.00  | 1.00  | 0.98  |
|            | $d = 2$   | Weighted $\chi^2$ |             | 0.65  | 0.77  | 0.86  | 0.94  | 0.94  |
|            |           |             |             | 0.2   | 1.00  | 0.99  | 0.97  | 0.95  |
| Binary     | $d = 1$   | Weighted $\chi^2$ |             | 0     | 0.80  | 0.96  | 0.94  | 0.94  |
|            |           |             |             | 0     | 0.02  | 0.94  | 0.99  | 0.94  |
|            | $d = 2$   | Weighted $\chi^2$ |             | 0     | 0.02  | 0.30  | 0.66  | 0.96  |
|            |           |             |             | 0.02  | 0.20  | 0.92  | 0.96  | 0.94  |
| Mixed      | $d = 1$   | Weighted $\chi^2$ | Optimal     | 0     | 0.2   | 0.45  | 0.64  | 0.94  |
|            |           |             | Sub-optimal (cts) | 0.68  | 0.84  | 0.89  | 0.90  | 0.95  |
|            |           |             | Sub-optimal (bin) | 0.5   | 0.75  | 0.87  | 0.94  | 0.95  |
|            | $d = 2$   | Weighted $\chi^2$ | Optimal     | 0     | 0.18  | 1     | 0.98  | 0.98  |
|            |           |             | Sub-optimal (cts) | 0.95  | 0.98  | 0.98  | 0.98  | 0.95  |
|            |           |             | Sub-optimal (bin) | 0.18  | 0.94  | 0.98  | 0.98  | 0.96  |
|            | $d = 2$   | Weighted $\chi^2$ | Optimal     | 0     | 0.06  | 0.30  | 0.45  | 0.90  |
|            |           |             | Sub-optimal (cts) | 0.60  | 0.75  | 0.84  | 0.95  | 0.95  |
|            |           |             | Sub-optimal (bin) | 0     | 0.08  | 0.40  | 0.56  | 0.96  |
|            | $d = 2$   | Weighted $\chi^2$ | Optimal     | 0.08  | 0.36  | 0.64  | 0.92  | 0.93  |
|            |           |             | Sub-optimal (cts) | 0.12  | 0.98  | 0.99  | 0.96  | 0.95  |
|            |           |             | Sub-optimal (bin) | 0.22  | 0.30  | 0.96  | 0.96  | 0.95  |
7. Data Analyses

We compare our method with other approaches such as generalized linear models and principal component regression in two data applications. In particular, we compare our methods with PCA and PCAMIX in Sections 7.1 and 7.2. PCAMIX (Chavent et al., 2012, 2014) is a version of PCA that accommodates mixed variables and implements PCA with metrics; i.e., Generalized Singular Value Decomposition (GSVD) of pre-processed data [see Chavent et al. (2014) for details]. PCAMIX is ordinary standard PCA, when all variables are continuous, and standard multiple correspondence analysis (MCA), when all variables are categorical (Greenacre and Blasius (2006), Zhu et al. (2011), Camiz and Gomes (2013)).

7.1 Krzanowski Data Sets

Krzanowski (1975) studied the problem of discriminating between two groups in the presence of both binary and continuous explanatory variables. Krzanowski (1975) modeled the mixed predictors using the location model (Olkin and Tate, 1961) and proposed an allocation rule to two groups similar to Fisher’s discriminant function. The location model transforms the q binary variables $H_1, \ldots, H_q$ to the corresponding $2^q$-category multinomial vector and requires the continuous variables be conditionally normal in each of the $2^q$ categories with different means and same variance-covariance matrix. He showed that the simple linear discriminant function often gives satisfactory results, except when there is interaction between the mixed variables.

We analyze four of the five data sets in Krzanowski’s paper which contains continuous and binary predictors and a binary response.

1. **Data Set 1:** Ten variables recorded on 40 patients who were surgically treated for renal hypertension. Seven of the variables were continuous and three binary. After one year, 20 patients were classified as improved and 20 as unimproved.

2. **Data Set 2:** Seven variables recorded on 93 patients suffering from jaundice. Four of the variables were continuous and three binary. The two groups were patients requiring medical and surgical treatment.

3. **Data Set 3:** Twelve variables recorded on 62 patients suffering from jaundice. Eight of the variables were continuous and four binary. The two groups were patients requiring medical and surgical treatment.

4. **Data Set 4:** Eleven variables recorded on 186 patients who underwent ablative surgery for advanced breast cancer between 1958 and 1965 at Guy’s Hospital, London. Six of the variables were continuous and three binary. The two groups were patients for which the treatment was deemed to be successful and failure.

Some of the continuous variables were transformed to normality across all data sets. Since the response is binary, $f_q$ in (8) is a vector of frequencies with $r = 1$, so that the dimension either SDR method can detect cannot exceed 1. We reduced the mixed predictors using our two methods, SDR **OPTIMAL** and SDR **SUBOPTIMAL**, and also PCA and PCAMIX setting $d = 1$. In order to assess the classification accuracy of each method, the
Table 4: Leave-one-out misclassification rates and AUC values for four data sets in Krzanowski (1975).

| Set | Optimal | SubOpt. | Full | PCA | PCAmix | Location | Fisher | Logistic |
|-----|---------|---------|------|-----|--------|----------|--------|----------|
| 1   | MR 0.250 | 0.300   | 0.375| 0.325| 0.425  | 0.350    | 0.325  | 0.325    |
|     | AUC 0.918| 0.918   | 0.885| 0.675| 0.575  | -        | -      | -        |
| 2   | MR 0.280 | 0.204   | 0.258| 0.387| 0.290  | 0.290    | 0.280  | 0.301    |
|     | AUC 0.857| 0.858   | 0.837| 0.513| 0.469  | -        | -      | -        |
| 3   | MR 0.161 | 0.145   | 0.226| 0.484| 0.500  | 0.226    | 0.177  | 0.222    |
|     | AUC 0.949| 0.951   | 0.944| 0.623| 0.646  | -        | -      | -        |
| 4   | MR 0.296 | 0.290   | 0.392| 0.457| 0.430  | 0.328    | 0.382  | 0.371    |
|     | AUC 0.784| 0.785   | 0.738| 0.544| 0.572  | -        | -      | -        |

reduced predictors serve as independent variables in a logistic regression model. For comparison, we also fit an unreduced logistic regression model with all the original predictors, which we refer to as Full.

In Table 4 we report the leave-one-out misclassification rates and the area under the receiver operator characteristics curve, AUC (Pepe, 2003, p. 67), with the smallest and largest values, respectively, in boldface. Sub-optimal SDR emerges as the best method to summarize the mixed predictors with respect to misclassification error, followed by SDR Optimal that has better performance for data set 1. With respect to AUC, SDR Suboptimal is always the best.

In Table 4, we also provide the leave-one-out misclassification rates of Fisher’s LDA, logistic regression and Krzanowski’s allocation rule based on the location model, as reported in Krzanowski (1975, Tab. 3). Sub-optimal SDR exhibits better performance than Krzanowski’s location model across data sets. Optimal SDR performs the best in all data sets except for data set 2 where it is on par with Fisher’s linear discriminant analysis. Moreover, the Optimal and Sub-optimal SDR misclassification rates are smaller than all other methods in Krzanowski (1975), as well as mixed nonparametric kernel methods (Vlachonikolis and Marriott, 1982). Taken all together, our SDR methods for mixed predictors consistently produce targeted data reductions that provide better fit and prediction.

7.2 Governance index application

Considerable social science and economics research is devoted to the construction of indexes for descriptive and predictive purposes (Vyas and Kumaranayake, 2006; Kolenikov and Angeles, 2009; Filmer and Scott, 2012; Merola and Baulch, 2014; Forzani et al., 2018). An index is a statistical summary measure of change in a representative group of individual data points. It usually synthesizes the information contained in a set of \( p \) variables \( X \in \mathbb{R}^p \) via a linear combination, \( R(X) = \omega^T X \in \mathbb{R} \), where \( \omega \) is the vector of weights of the composite index.

In this example, we study the impact of governance on economic growth in the twelve South American countries as measured by per capita Gross Domestic Product (GDP) using the World Bank Governance Indicators.\(^1\) The World Bank considers the following six aggre-

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1. Governance Indicators and per capita GDP data can be downloaded from Worldwide Governance Indicators and The World Bank Data, respectively.
gate indicators of governance that combine the views of a large number of enterprise, citizen and expert survey respondents: control of corruption ($X_1$); rule of law ($X_2$); regulatory quality ($X_3$); government effectiveness ($X_4$); political stability ($X_5$); voice and accountability ($X_6$). They are standardized to have mean zero and standard deviation one, with values from -2.5 to 2.5, approximately, where higher values correspond to better governance. All six are highly positively correlated, and are all positively correlated with the per capita GDP; i.e., economic growth is positively associated with better governance indicators.

Our aim is to build a Composite Governance index (CG) to predict $Y$, the logarithm of per capita Gross Domestic Product (GDP), measured in 2010 US dollars, over the period 1996 to 2018. Using the set of governance indicator variables, we start by constructing the CG index via standard Principal Component Analysis (PCA) and Principal Fitted Components (PFC) [see Corollary 3] setting $d = 1$ and $f_y = \log(\text{GDP})$ in (8).

In the left panel of Figure 3, we plot $\log(\text{GDP})$ versus the CG indexes based on PCA, which is the standard approach in such index construction (Mazziotta and Pareto, 2019). In the right panel of Figure 3, the response is plotted versus the index based on PFC. Both plots indicate dependence of the response on the indexes but the nature of relationship is the data pattern is hard to understand. A linear trend appears stronger in the right panel, which is reflected in the better fit of the linear regression model (black) with $R^2 = 0.27$ versus 0.17 for PCA. However, the PCA-based index in nonparametric kernel regression (blue) results in better fit. Using the \texttt{np R} package, the value of the nonparametric version of $R^2$ is 0.32 for the PFC-based CG index, which is much lower than 0.54, the value for the PCA-based index.

In Figure 4, we plot $\log(\text{GDP})$ versus the PCA and PFC composite governance indexes by country. The plots indicate that the PFC index gives a much better visualization of the relationship of $\log(\text{GDP})$ within each country, suggesting that adjusting the index by country could improve its predictive performance.

We add country effect by introducing eleven binary variables $H$. In Figure 5 we plot the log of GDP versus the CG index constructed by PCA for mixed variables (PCAmix) in the left panel and by our mixed optimal SDR approach in the right panel. Hardly any difference between the plots in the left panels of Figures 3 and 5 is noticeable. The PCAmix based CG index is very similar to the conventional PCA based CG that does not include
country effect, with $R^2$ equal to 0.17 and 0.61 for the linear and nonparametric models, respectively. Moreover, neither PCA based CG index exhibits an easy to understand or model relationship with the response.

In contrast, a very clear and simple pattern appears in the right panel of Figure 5, where the response is plotted versus our optimal SDR based index. The pattern suggests modeling $\log(GDP)$ as a linear function of the GC index. This is a distinct improvement over PCA and PCAMix (left panels of Figures 3 and 5) but also the SDR method PFC, which does not account for country effect (right panel of Figure 3). As a result, both the linear (black) and the kernel (blue) regression models for the regression of the log per capita GDP on the optimal SDR for mixed predictors based CG index have excellent fit with respective $R^2$ values of 0.91 and 0.93.

The average of the leave-one-out mean square prediction errors of the linear and kernel regression models in Table 5, provides an unbiased measure of predictive performance. The logarithm of the per capita GDP is regressed on the unsupervised CG indexes, constructed by PCA using only continuous predictors (PCA($X$)) and its extension for mixed variables (PCAMix($X, H$)), and the supervised CG Indexes, constructed by PFC only on continuous predictors (PFC($X$)) and our mixed predictor SDR methods, OPTIMAL($X, H$) and SUBOPTIMAL($X, H$).

The leave-one-out mean squared prediction errors of the supervised PFC based CG index are smaller than both PCA and PCAMix for the linear model, even though PFC
Table 5: Leave-one-out mean squared prediction errors for the per capita log GDP in South-American countries.

| Index Type   | Method          | Predictive Model | Linear | Non-Parametric |
|--------------|-----------------|------------------|--------|----------------|
| Unsupervised | PCA(X)          |                  | 0.319  | 0.189          |
|              | PCAmix(X, H)    |                  | 0.320  | 0.209          |
| Supervised   | PFC(X)          |                  | 0.292  | 0.282          |
|              | SDROptimal(X, H)|                  | 0.029  | 0.028          |
|              | SDRSubOptimal(X, H) |              | 0.028  | 0.022          |

The regularized estimation of the SDROptimal reduction selects all five continuous predictors except for rule of law. Political stability and voice and accountability have the highest weights in the CG index. Rule of law is the most correlated with four of the other variables, with correlation coefficient values over 0.80. We stipulate that our method drops it as its relationship with GDP is mostly absorbed by the other four. The binary variables are all selected. That is, our method finds a significant country effect on GDP.

8. Discussion

Our approach falls within model-based inverse regression for sufficient dimension reduction (SDR) (Cook (2007); Cook and Forzani (2008); Bura and Forzani (2015); Bura et al. (2016)). Model-based SDR requires knowledge of the family of distributions of the inverse predictors in contrast to moment-based SDR, such as SIR Li (1991), SAVE Cook and Weisberg (1991), or DR Li and Wang (2007), that impose conditions on the moments of the marginal distribution of the predictors. Because of this, our approach provides exhaustive identification and statistically efficient estimation of sufficient reductions for the conditional distribution of an output given mixed variables that contain all information in the mixed predictors for the output Y.

Furthermore, outside the context of dimension reduction for the forward regression problem of Y on mixed predictors Z, the modeling we use to accommodate the factorization in (2) in developing our SDR methods, is a new multivariate modeling approach for response vectors comprised of mixed variables. That is, if one were to only consider the multivariate regression of the mixed vector $Z = (X^T, H^T)^T$ on some other variables, say F, the models we use for the continuous and binary elements of Z in our development provides a new regression tool for mixed responses. Specifically, since the joint distribution of $Z \mid F$ belongs to the exponential family (9), our approach yields sufficient statistics for the unknown natural
parameters $\theta$ in (11), as well as optimal (efficient) maximum likelihood estimators, in a similar manner to generalized linear modeling for univariate responses.

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Appendix A. Proofs and Derivations for Section 3

Derivation of Eqn. (9)

From Eqn. (8), the density $f(X, H | Y = y)$, up to the constant $1/\sqrt{2\pi}$, equals

$$\exp \left\{ -\frac{1}{2} \left( (X - \mu_x) - Af_y - \beta (H - \mu_h) \right)^T \Delta^{-1} \left( (X - \mu_x) - Af_y - \beta (H - \mu_h) \right) ight\} + \text{vech}^T (HH^T) (\tau_0 + \tau f_y) + \frac{1}{2} \log(|\Delta|^{-1}) - \log(G(\Gamma_y)) \right\}.$$ 

After some algebra and rearrangement of terms we obtain

$$f(X, H | Y = y) = h(X, H) \exp \left( T^T(X, H) \eta_y - \psi(\eta_y) \right),$$

with $h(X, H) = (2\pi)^{-1/2}$,

$$T^T(X, H) \eta_y = X^T \Delta^{-1} \mu_x - X^T \Delta^{-1} \beta \mu_h + X^T \Delta^{-1} Af_y - H^T \beta^T \Delta^{-1} \mu_x + H^T \beta^T \Delta^{-1} \beta \mu_h - H^T \beta^T \Delta^{-1} Af_y - \frac{1}{2} X^T \Delta^{-1} X + X^T \Delta^{-1} \beta H - \frac{1}{2} H^T \beta^T \Delta^{-1} \beta H + \text{vech}^T (HH^T) \tau_0 + \text{vech}^T (HH^T) \tau f_y,$$  \hspace{1cm} (49)

and

$$\psi(\eta_y) = \frac{1}{2} T^T \Delta^{-1} \mu_x + \frac{1}{2} T^T \Delta^{-1} Af_y + \frac{1}{2} \mu_h \beta^T \Delta^{-1} \beta \mu_h + \frac{1}{2} \mu^T \Delta^{-1} \beta \mu_h - \frac{1}{2} \log(|\Delta|^{-1}) + \log(G(\Gamma_y)).$$  \hspace{1cm} (50)

Since $\text{tr}(A^T B) = \text{vec}(A)^T \text{vec}(B)$, $\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B)$ and $D_q$ in Section 2 is such that $\text{vec}(A) = D_q \text{vech}(A)$, (49) becomes

$$T^T(X, H) \eta_y = X^T (\Delta^{-1} \mu_x - \Delta^{-1} \beta \mu_h + (f_y^T \otimes I_p) \text{vec}(\Delta^{-1} A)) + H^T (-\beta^T \Delta^{-1} \mu_x + \beta^T \Delta^{-1} \beta \mu_h - (f_y^T \otimes I_q) \text{vec}(\beta^T \Delta^{-1} A)) - \frac{1}{2} (D_p D_p^T \text{vech}(XX^T)) T \text{vech}(\Delta^{-1}) + \text{vech}(XH^T) T \text{vech} \Delta^{-1} \beta) + \text{vech}(HH^T) T \left( -\frac{1}{2} D_q T \text{vec}(\beta^T \Delta^{-1} \beta) + \tau_0 + (f_y^T \otimes I_q (q+1)/2) \text{vec}(\tau) \right).$$
Finally, using the matrices $J_q$ and $L_q$ defined in Section 2, we obtain Eqns (10) and (11) from

$$
T^T(X, H)\eta_y = X^T(\Delta^{-1} \mu_X - \Delta^{-1} \beta \mu_H + (f^T_y \otimes I_p)\text{vec}((\Delta^{-1} A))
+ H^T(-\beta^T \Delta^{-1} \mu_X + \beta^T \Delta^{-1} \beta \mu_H - (f^T_y \otimes I_q)\text{vec}(\beta^T \Delta^{-1} A))
- \frac{1}{2}(D_p^T D_p^T \text{vech}(XX^T))^T \text{vech}(\Delta^{-1}) + \text{vec}(XH^T)^T \text{vec}(\Delta^{-1} \beta)
+ H^T\left(-\frac{1}{2}L_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta) + L_q \tau_0 + (f^T_y \otimes I_q)\text{vec}(L_q \tau)\right)
+ (J_q \text{vech}(HH^T))^T \left(-\frac{1}{2}J_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta) + J_q \tau_0 + (f^T_y \otimes I_{k_q})\text{vec}(J_q \tau)\right)
= X^T \eta_{y1} + H^T \eta_{y2} - \frac{1}{2}(D_p^T D_p \text{vec}(XX^T))^T \eta_3 + \text{vec}(XH^T)^T \eta_4
+ (J_q \text{vech}(HH^T))^T \eta_{y5},
$$

where $T(X, H)$ is defined in (10) and

$$
\eta_{y1} = \Delta^{-1} \mu_X - \Delta^{-1} \beta \mu_H + (f^T_y \otimes I_p)\text{vec}((\Delta^{-1} A) = F_{y1} \vartheta_1,
$$

with $F_{y1} = (I_p, f^T_y \otimes I_p)$, $\vartheta_1 = (\vartheta_{10}^T, \vartheta_{11}^T)^T$, $\vartheta_{10} = \Delta^{-1} \mu_X - \Delta^{-1} \beta \mu_H$, $\vartheta_{11} = \text{vec}(\Delta^{-1} A)$,

$$
\eta_{y2} = -\beta^T \Delta^{-1} \mu_X + \beta^T \Delta^{-1} \beta \mu_H - (f^T_y \otimes I_q)\text{vec}(\beta^T \Delta^{-1} A)
- \frac{1}{2}L_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta) + L_q \tau_0 + (f^T_y \otimes I_q)\text{vec}(L_q \tau)
= F_{y2} \vartheta_2,
$$

with $F_{y2} = (I_q, f^T_y \otimes I_q)$, $\vartheta_2 = (\vartheta_{20}^T, \vartheta_{21}^T)^T$, $\vartheta_{20} = -\beta^T \Delta^{-1} \mu_X + \beta^T \Delta^{-1} \beta \mu_H + L_q \tau_0 - \frac{1}{2}L_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta)$, $\vartheta_{21} = \text{vec}(L_q \tau - \beta^T \Delta^{-1} A)$,

$$
\eta_3 = \eta_{3y} = \text{vech}(\Delta^{-1}),
\eta_4 = \eta_{4y} = \text{vec}(\Delta^{-1} \beta),
$$

and

$$
\eta_{y5} = -\frac{1}{2}J_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta) + J_q \tau_0 + (f^T_y \otimes I_{k_q})\text{vec}(J_q \tau)
= F_{y5} \vartheta_5,
$$

with $F_{y5} = (I_{k_q}, f^T_y \otimes I_{k_q})$, $\vartheta_5 = (\vartheta_{50}^T, \vartheta_{51}^T)^T$, $\vartheta_{50} = -\frac{1}{2}J_q D_q^T \text{vec}(\beta^T \Delta^{-1} \beta) + J_q \tau_0$ and $\vartheta_{51} = \text{vec}(J_q \tau)$.

By Eqn. (7),

$$
G(\Gamma_y) = \sum_{H} \exp \left[ \text{vech}^T(HH^T) (\tau_0 + \tau f_y) \right].
$$

Plugging in matrices $J_q$, $L_q$, $D_q$ and $C_q$, defined in Section 2, and letting $\eta_4 = \text{unvec}(\eta_4)$, we obtain

$$
G(\Gamma_y) = \sum_{H} \exp \left[ (J_q \text{vec}(HH^T))^T \left( \eta_{y5} + \frac{1}{2}J_q D_q^T \text{vec}(\eta_4^T (\text{unvec}(D_p \eta_3))^{-1} \eta_4) \right) \right]
\left( \eta_{y2} + \frac{1}{2}L_q D_q^T \text{vec}(\eta_4^T (\text{unvec}(D_p \eta_3))^{-1} \eta_4) \right)
= (51).
$$
Finally, using the matrix $D_p$ defined in Section 2, Eqn. (50) yields

$$
\psi(\eta_y) = \frac{1}{2} \eta_y^T (\text{unvec}(D_p \eta_3))^{-1} \eta_y + \log |\text{unvec}(D_p \eta_3)| - \frac{1}{2} \log |\text{unvec}(D_p \eta_3)|
$$

with $G(\Gamma_y)$ given in (51).

Proof of Theorem 1

Since the density of $X, H \mid Y$ belongs to the full rank exponential family (Eqn. (9)), the minimal sufficient reduction for the regression $Y \mid (X, H)$ is given by

$$
R(X, H) = \alpha^T (T(X, H) - E(T(X, H))),
$$

where $\alpha$ is a basis for $S_{\alpha} = \text{span}\{\eta_Y - E(\eta_Y), Y \in \mathcal{Y}\}$, with $\eta_Y$ given in (11) [see Bura et al. (2016, Thm 1)]. Since $E(f_Y) = 0$, applying Eqns. (11) and (12) obtains

$$
\eta_y - E(\eta_y) = \begin{pmatrix}
(f_y^T \otimes I_p) \text{vec}(\Delta^{-1} A) \\
(f_y^T \otimes I_q) \text{vec}(L_q \tau - \beta^T \Delta^{-1} A) \\
0 \\
0 \\
(f_y^T \otimes I_{k_y}) \text{vec}(J_q \tau)
\end{pmatrix} = \begin{pmatrix}
\Delta^{-1} A f_y \\
(L_q \tau - \beta^T \Delta^{-1} A) f_y \\
0 \\
0 \\
J_q f_y
\end{pmatrix}.
$$

Then, $\text{span}(a) = \text{span}(\eta_y - E(\eta_Y), y \in \mathcal{Y})$ with

$$
a = \begin{pmatrix}
\Delta^{-1} A \\
L_q \tau - \beta^T \Delta^{-1} A \\
0 \\
0 \\
J_q \tau
\end{pmatrix}.
$$

Proof of Corollary 3

If follows from Corollary 2 since, in this case, $\theta_{2,1} = 0$ and $\theta_{5,1} = 0$.

Proof of Corollary 4

If follows from Corollary 2 since in this case $\theta_{1,1} = 0$ and $\theta_{2,1} = L_q \tau$.

Proof of Corollary 5

It suffices to show that $\text{span}(b) \subset \text{span}(\alpha_c)$. We can write $b$ as

$$
b = \begin{pmatrix}
\Delta A \\
-\beta^T \Delta^{-1} A \\
0 \\
L_1 \tau \\
L_r \\
0 \\
J_q \tau
\end{pmatrix} = \tilde{b} \begin{pmatrix}
I_r \\
I_r \\
I_r
\end{pmatrix},
$$

with $\text{span}(\tilde{b}) = \text{span}(\alpha_c)$. As a consequence, $\text{span}(b) \subset \text{span}(\alpha_c)$, and therefore $R(X, H)$ in Eqn. (23) is a sufficient dimension reduction, not necessary minimal. The rest of the corollary immediately follows.
Appendix B: Proof of Proposition 6

We first derive the asymptotic distribution of \( \hat{b} \) in (34) and prove auxiliary lemmas in Section in order to prove Proposition 6.

Asymptotic distribution of \( \hat{b} \)

**Proposition 7** \( \sqrt{n} \text{vec}(\hat{b} - b) \xrightarrow{D} N(0, V_{rel}) \) with

\[
V_{rel} = WMVM^T W^T,
\]

as in equation (39), where \( M, W \) and \( V \) are defined in Eqns. (42), (43) and (40), respectively.

**Proof** We rewrite

\[
b = \begin{pmatrix}
\Delta^{-1} A \\
L_q \tau - \beta^T \Delta^{-1} A
\end{pmatrix} = \begin{pmatrix}
\text{unvec}(\vartheta_{1,1}) \\
\text{unvec}(\vartheta_{2,1}) \\
\text{unvec}(\vartheta_{5,1})
\end{pmatrix},
\]

as follows. Let \( \tilde{b} = (\vartheta_{1,1}^T, \vartheta_{2,1}^T, \vartheta_{5,1}^T)^T \). Then, \( \tilde{b} = M\vartheta \), with \( M \) given in (42), so that

\[
\text{vec}(b) = W\tilde{b} = WM\vartheta,
\]

with \( W \) defined on (43). Then,

\[
\text{vec}(\hat{b}) = WM\hat{\vartheta}.
\]

The asymptotic normality of \( \hat{b} \) follows from the asymptotic normality of \( \hat{\vartheta} \), which is derived in Lemma 8, with

\[
\text{avar}(\sqrt{n}\hat{b}) = WM\text{avar}(\sqrt{n}\hat{\vartheta})M^T W^T = WMVM^T W^T.
\]

Lemma 8 If \( \text{avar}(\sqrt{n}\hat{\vartheta}) = V \), then

\[
V^{-1} = E \left[ F_y^T JF_y \right],
\]

as in (40), where \( F_y \) is defined in (11) and

\[
J = \frac{\partial^2 \psi(\eta_y)}{\partial \eta_y \partial \eta_y^T}
\]

in (41).
Proof Since \( \hat{\theta} \) is the maximum likelihood estimator,

\[
V = \text{avar} \left( \sqrt{n} \hat{\theta} \right) = - \left( E \left[ \frac{\partial^2 \log f(X, H \mid Y = y)}{\partial \theta \partial \theta^T} \right] \right)^{-1}
\]

We plug in \( \eta_y = F_y \theta \) (from Eqn. (11)) in Eqn. (9) to obtain

\[
\log f(X, H \mid Y = y) = \log h(X, H) + T^T(X, H) \eta_y - \psi(\eta_y) = \log h(X, H) + T^T(X, H) F_y \theta - \psi(F_y \theta).
\]

Then,

\[
\frac{\partial \log f(X, H \mid Y = y)}{\partial \text{vec}^T(\theta)} = T^T(X, H) F_y - \frac{\partial \psi(\eta_y)}{\partial \eta_y} F_y,
\]

\[
\frac{\partial^2 \log f(X, H \mid Y = y)}{\partial \text{vec}(\theta) \partial \text{vec}^T(\theta)} = - F_y^T \frac{\partial^2 \psi(\eta_y)}{\partial \eta_y \partial \eta_y} F_y = - F_y^T J F_y.
\]

Therefore \( V^{-1} = E \left[ F_y^T J F_y \right] \) from which Proposition 8 follows.

In order to compute \( J \), the first and second derivatives of \( \psi(\eta_y) \) with respect to \( \eta_y \) are required. The computation is carried out in Section ?? (Supplementary Material).

Auxiliary lemmas for Proposition 6

Lemma 9 Let \( \hat{H} = \hat{U}_1 \hat{K}_1 \hat{R}_1^T R_1 K^{-1} \). Then,

\[
\sqrt{n} \text{vec}(\hat{H} - U_1) \to N(0, (K^{-1} R_1^T \otimes I_m) V_{rlc}(R_1 K^{-1} \otimes I_m)).
\]

where \( V_{rlc} \) is defined in Eqn. (39), \( \hat{U}_1, \hat{K}_1 \) and \( \hat{R}_1 \) in Eqn. (18), and \( U_1, K \) and \( R_1 \) in Eqn. (32).

Proof By Eqn. (32), \( b = U_1 K R_1^T \) and by Eqn. (18), \( \hat{b} = \hat{U}_1 \hat{K}_1 \hat{R}_1^T + \hat{U}_0 \hat{K}_0 \hat{R}_0^T \). Then,

\[
\hat{H} - U_1 = \hat{U}_1 \hat{K}_1 \hat{R}_1^T R_1 K^{-1} - U_1 = \hat{b} R_1 K^{-1} - \hat{U}_0 \hat{K}_0 \hat{R}_0^T R_1 K^{-1} - U_1 = (\hat{b} - b) R_1 K^{-1} - \hat{U}_0 \hat{K}_0 \hat{R}_0^T R_1 K^{-1}.
\]

Thus,

\[
\sqrt{n} \text{vec}(\hat{H} - U_1) = \sqrt{n} \text{vec}(\hat{b} - b) R_1 K^{-1} - \hat{U}_0 \hat{K}_0 \hat{R}_0^T R_1 K^{-1} = \sqrt{n}(K^{-1} R_1^T \otimes I_m) \text{vec}(\hat{b} - b) - \text{vec}(\hat{U}_0 \hat{K}_0 \hat{R}_0^T R_1 K^{-1}).
\]

(55)

From Proposition 7 we have

\[
\sqrt{n} \text{vec}(\hat{b} - b) \xrightarrow{D} N(0, V_{rlc}),
\]

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so that
\[ \sqrt{n}(K^{-1}R_1^T \otimes I_m)\text{vec}(\hat{b} - b) \xrightarrow{D} N(0, \Sigma_U), \]  
with \( \Sigma_U = (K^{-1}R_1^T \otimes I_k)V_{rel}(K^{-1}R_1^T \otimes I_k)^T = (K^{-1}R_1^T \otimes I_k)V_{rel}(R_1K^{-1} \otimes I_k). \)

Since \( \sqrt{n}(\hat{U}_0\hat{K}_0\hat{R}_0^T) = O_p(1) \) and \( P_{R_i} = \left(P_{R_i} + O_p(n^{-1/2})\right) \), we get
\[
\sqrt{n}(\hat{U}_0\hat{K}_0\hat{R}_0^TR_1K^{-1}) = \sqrt{n}(\hat{U}_0\hat{K}_0\hat{R}_0^T)P_{R_i}R_1K^{-1} \\
= \sqrt{n}
\left(\hat{U}_0\hat{K}_0\hat{R}_0^T\right)\left(P_{R_i} + O_p(n^{-1/2})\right)R_1K^{-1} \\
= \sqrt{n}
\left(\hat{U}_0\hat{K}_0\hat{R}_0^T\right)O_p(n^{-1/2})R_1K^{-1} \\
= O_p(n^{-1/2}),
\]
where we use \( \hat{R}_0^TR_1 = 0 \). As a consequence, \( \sqrt{n}\text{vec}(\hat{U}_0\hat{K}_0\hat{R}_0^TR_1K^{-1}) \rightarrow 0 \) in probability, which, together with (56) in (55), obtain the result.

\[\blacksquare\]

**Lemma 10** Let \( \Gamma \) be a matrix of dimension \( p \times d \) of full rank \( d \) with \( d \leq p \), and let \( P_{\Gamma} \) be the orthogonal projection onto the column space of \( \Gamma \) and \( Q_{\Gamma} = I - P_{\Gamma} \). Also, let \( K_{pm} \in \mathbb{R}^{pm \times pm} \) be the unique matrix such that, for any symmetric \( p \times m \) matrix \( A \), \( \text{vec}(A^T) = K_{pm} \text{vec}(A) \). Then,
\[ \frac{\partial P_{\Gamma}}{\partial \text{vec}^T(\Gamma)} = (I_p^2 + K_{pp})(\Gamma^T\Gamma)^{-1} \otimes Q_{\Gamma}. \]  

**Proof** We will use the following two identities.

(i) Let \( X \) be a matrix and \( F(X) : m \times p \) and \( G(X) : p \times q \) differentiable matrix valued functions of \( X \). Then,
\[ \frac{\partial \text{vec}[F(X)G(X)]}{\partial \text{vec}^T(X)} = (G^T \otimes I_m)\frac{\partial \text{vec}[F(X)]}{\partial \text{vec}^T(X)} + (I_q \otimes F)\frac{\partial \text{vec}[G(X)]}{\partial \text{vec}^T(X)}. \]  

(ii) Let \( F(X) = X^T \) and \( G(X) = X \) with \( X : p \times q \). By (58),
\[ \frac{\partial \text{vec}(X^TX)}{\partial^2 \text{vec}(X)} = (I_q^2 + K_{qq})(I_q \otimes X^T). \]  
\[ \frac{\partial \text{vec}(X^TX)^{-1}}{\partial^2 \text{vec}(X)} = -((X^TX)^{-1} \otimes (X^TX)^{-1}))\frac{\partial \text{vec}(X^TX)}{\partial^2 \text{vec}(X)}. \]

Applying (58) yields
\[ \frac{\partial \text{vec}P_{\Gamma}}{\partial \text{vec}^T(\Gamma)} = \frac{\partial \text{vec}(\Gamma(\Gamma^T\Gamma)^{-1}\Gamma^T)}{\partial \text{vec}^T(\Gamma)} \\
= (\Gamma(\Gamma^T\Gamma)^{-1} \otimes I_p)\frac{\partial \text{vec}(\Gamma)}{\partial \text{vec}^T(\Gamma)} + (I_p \otimes \Gamma)(\Gamma \otimes \Gamma)^{-1}\Gamma^T)\frac{\partial \text{vec}((\Gamma^T\Gamma)^{-1}\Gamma^T)}{\partial \text{vec}^T(\Gamma)} \\
= (\Gamma(\Gamma^T\Gamma)^{-1} \otimes I_p) + (I_p \otimes \Gamma)\frac{\partial \text{vec}((\Gamma^T\Gamma)^{-1}\Gamma^T)}{\partial \text{vec}^T(\Gamma)}. \]
Let
\[ H = \frac{\partial \text{vec}((\Gamma^T \Gamma)^{-1} \Gamma^T)}{\partial \text{vec}^T(\Gamma)}. \]

Using (58), (59) and (60), we get
\[ H = (\Gamma \otimes I_d) \frac{\partial \text{vec}(\Gamma^T \Gamma)^{-1}}{\partial \text{vec}^T(\Gamma)} + (I_p \otimes (\Gamma^T \Gamma)^{-1}) K_{pd} \]
\[ = -(\Gamma \otimes I_d)((\Gamma^T \Gamma)^{-1} \otimes (\Gamma^T \Gamma)^{-1})) \right) (I_{d^2} + K_{dd})(I_d \otimes \Gamma^T) + (I_p \otimes (\Gamma^T \Gamma)^{-1}) K_{pd}. \]

Then,
\[ \frac{\partial \text{vec}P_{\Gamma}}{\partial \text{vec}^T(\Gamma)} = (\Gamma(\Gamma^T \Gamma)^{-1} \otimes I_p) + (I_p \otimes \Gamma) \times \]
\[ \left[-(\Gamma \otimes I_d)((\Gamma^T \Gamma)^{-1} \otimes (\Gamma^T \Gamma)^{-1}) (I_{d^2} + K_{dd})(I_d \otimes \Gamma^T) + (I_p \otimes (\Gamma^T \Gamma)^{-1}) K_{pd} \right] \]
\[ = (\Gamma(\Gamma^T \Gamma)^{-1} \otimes I_p) + (I_p \otimes \Gamma(\Gamma^T \Gamma)^{-1}) K_{pd} - (\Gamma(\Gamma^T \Gamma)^{-1} \otimes \Gamma(\Gamma^T \Gamma)^{-1}) \Gamma^T \right) \]
\[ = (I_{p^2} + K_{pp})(\Gamma(\Gamma^T \Gamma)^{-1} \otimes I_p) - (I_{p^2} + K_{pp})(\Gamma(\Gamma^T \Gamma)^{-1} \otimes P_{\Gamma}) \]
\[ = (I_{p^2} + K_{pp})(\Gamma(\Gamma^T \Gamma)^{-1} \otimes I_p - P_{\Gamma}) \]
\[ = (I_{p^2} + K_{pp})(\Gamma(\Gamma^T \Gamma)^{-1} \otimes Q_{\Gamma}). \]

\[ \textbf{Lemma 11} \text{ Suppose the two matrices } \hat{\Gamma} \text{ and } \Gamma \text{ are of order } p \times d \text{ with } d \leq p \text{ with } \Gamma \text{ of full rank } d. \text{ Assume that } \hat{\Gamma} \text{ is asymptotically normal with } \]
\[ \sqrt{n} \text{vec}(\hat{\Gamma} - \Gamma) \xrightarrow{D} \mathcal{N}(0, \Sigma). \]

Then, \( \sqrt{n} \text{vec}(P_{\hat{\Gamma}} - P_{\Gamma}) \) is asymptotically normal with mean \( \mathbf{0} \) and variance
\[ (I_{p^2} + K_{pp})(\Gamma(\Gamma^T \Gamma)^{-1} \otimes Q_{\Gamma}) \Sigma (\Gamma(\Gamma^T \Gamma)^{-1} \otimes Q_{\Gamma})(I_{p^2} + K_{pp}). \]

\textbf{Proof} \ Let \( P_{\Gamma} = \Gamma(\Gamma^T \Gamma)^{-1} \Gamma^T \) be the orthogonal projection onto the column space of \( \Gamma \) and let \( g \) be a function defined in the subspace of the \( p \times d \) matrices of full rank \( d \) such that \( g(\Gamma) = \Gamma(\Gamma^T \Gamma)^{-1} \Gamma^T = P_{\Gamma}. \) From Lemma 10 we have that
\[ \nabla g(\Gamma) = \frac{\partial P_{\Gamma}}{\partial \text{vec}^T(\Gamma)} = (I_{p^2} + K_{pp})(\Gamma(\Gamma^T \Gamma)^{-1} \otimes Q_{\Gamma}). \]

By the Delta method,
\[ \sqrt{n} \left( \frac{g(\hat{\Gamma}) - g(\Gamma)}{\sqrt{\nabla g(\Gamma) \Sigma \nabla^T g(\Gamma)}} \right) \xrightarrow{D} \mathcal{N}(0, \Sigma), \]
which completes the proof.
Proof of Proposition 6

From (37), $\hat{\alpha}_b = \hat{U}_1$ and therefore $\alpha_b = U_1$ and $\text{span}(\hat{U}_1) = \text{span}(\hat{H})$ with $\hat{H} = \hat{U}_1 \hat{K}_1 \hat{R}_1^T \hat{R}_1 \hat{K}_1^{-1}$ defined in Lemma 9, which also provides the asymptotic distribution of $\hat{H}$. Applying Lemma 11 with $\hat{\Gamma} = \hat{H}$ and $\Gamma = H = U_1$ we obtain the asymptotic distribution with asymptotic variance

$$(I_p^2 + K_{pp})(U_1 K^{-1} R_1^T \otimes Q_{U_1}) V_{rict}(R_1 K^{-1} U_1^T \otimes Q_{U_1}) (I_p^2 + K_{pp})$$

since $U_1^T U_1 = I_d$. By (33), $b = U_1 K R_1^T$, therefore $b^- = R_1 K^{-1} U_1^T$ and the result follows.

References

J. Aitchison and C. G. G. Aitken. Multivariate binary discrimination by the kernel method. *Biometrika*, 63(3):413–420, 1976. ISSN 00063444. URL http://www.jstor.org/stable/2335719.

J. A. Anderson. Separate sample logistic discrimination. *Biometrika*, 59(1):19–35, 1972. ISSN 00063444. URL http://www.jstor.org/stable/2334611.

J. A. Anderson. Quadratic logistic discrimination. *Biometrika*, 62(1):149–154, 1975. ISSN 00063444. URL http://www.jstor.org/stable/2334497.

Francis Bach, Rodolphe Jenatton, Julien Mairal, and Guillaume Obozinski. Structured sparsity through convex optimization. *Statist. Sci.*, 27(4):450–468, 11 2012. doi: 10.1214/12-STS394. URL https://doi.org/10.1214/12-STS394.

E. Bura and J. Yang. Dimension estimation in sufficient dimension reduction: A unifying approach. *Journal of Multivariate Analysis*, 102(1):130 – 142, 2011. ISSN 0047-259X. doi: https://doi.org/10.1016/j.jmva.2010.08.007. URL http://www.sciencedirect.com/science/article/pii/S0047259X10001661.

E. Bura, S. Duarte, and L. Forzani. Sufficient reductions in regressions with exponential family inverse predictors. *Journal of the American Statistical Association*, 111(515):1313–1329, 2016.

Efstathia Bura and Liliana Forzani. Sufficient reductions in regressions with elliptically contoured inverse predictors. *Journal of the American Statistical Association*, 110(509):420–434, 2015. doi: 10.1080/01621459.2014.914440. URL https://doi.org/10.1080/01621459.2014.914440.

Stef van Buuren. *Flexible imputation of missing data*. CRC Press, 2nd edition, 2018.

S. Camiz and G.C. Gomes. Joint correspondence analysis versus multiple correspondence analysis: a solution to an undetected problem. In *Classification and data mining*. Stud. Classification Data Anal. Knowledge Organ., pages 11–18. Springer, Heidelberg, 2013.

M. Chavent, V. Kuentz-Simonet, B. Liquet, and J. Saracco. Orthogonal rotation in pcamix. *Advances in Data Analysis and Classification*, 6:131–146, 2012.
Marie Chavent, Vanessa Kuentz-Simonet, Amaury Labenne, and Jérôme Saracco. Multivariate analysis of mixed data: The r package pcamixdata, 2014.

Shizhe Chen, Daniela M. Witten, and Ali Shojaie. Selection and estimation for mixed graphical models. *Biometrika*, 102(1):47–64, 12 2014. ISSN 0006-3444. doi: 10.1093/biomet/asu051. URL https://doi.org/10.1093/biomet/asu051.

Jie Cheng, Elizaveta Levina, Pei Wang, and Ji Zhu. A sparse Ising model with covariates. *Biometrics*, 70(4):943–953, 2014. ISSN 0006-341X. doi: 10.1111/biom.12202. URL https://doi.org/10.1111/biom.12202.

Jie Cheng, Tianxi Li, Elizaveta Levina, and Ji Zhu. High-dimensional mixed graphical models. *Journal of Computational and Graphical Statistics*, 26(2):367–378, 2017. doi: 10.1080/10618600.2016.1237362. URL https://doi.org/10.1080/10618600.2016.1237362.

R.D. Cook. Fisher lecture: Dimension reduction in regression (with discussion). *Statistical Science*, 22:1–26, 2007.

R.D. Cook and L. Forzani. Principal fitted components for dimension reduction in regression. *Statistical Science*, 23:485–501, 2008.

R.D. Cook and S. Weisberg. Discussion of sliced inverse regression for dimension reduction. *Journal of the American Statistical Association*, 86:328–332, 1991.

Bin Dai. Multivariate bernoulli distribution models. Technical report, Dept. Statistics, Univ. Wisconsin, Madison, WI 53706, July 2012.

Bin Dai, Shilin Ding, and Grace Wahba. Multivariate bernoulli distribution. *Bernoulli*, 19(4):1465–1483, 09 2013. doi: 10.3150/12-BEJSP10. URL https://doi.org/10.3150/12-BEJSP10.

N. E. Day and D. F. Kerridge. A general maximum likelihood discriminant. *Biometrics*, 23(2):313–323, 1967. ISSN 0006341X, 15410420. URL http://www.jstor.org/stable/2528164.

D. Filmer and K. Scott. Assessing Asset Indices. *Demography*, 49:359–392, 2012.

Garrett M. Fitzmaurice and Nan M. Laird. Regression models for mixed discrete and continuous responses with potentially missing values. *Biometrics*, 53(1):110–122, 1997. ISSN 0006341X, 15410420. URL http://www.jstor.org/stable/2533101.

L. Forzani, R. García-Arancibia, P. Llop, and D. Tomassi. Supervised dimension reduction for ordinal predictors. *Computational Statistics and Data Analysis*, 125, 2018.

M. Greenacre and J. Blasius, editors. *Multiple correspondence analysis and related methods*. Statistics in the Social and Behavioral Sciences Series. Chapman & Hall/CRC, Boca Raton, FL, 2006. ISBN 978-1-58488-628-0; 1-58488-628-5. doi: 10.1201/9781420011319.

Ernst Ising. Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik*, 31 (1):253–258, Feb 1925. ISSN 0044-3328. doi: 10.1007/BF02980577. URL https://doi.org/10.1007/BF02980577.
Sufficient reductions for mixed predictors

Kristin N. Javaras and David A. van Dyk. Multiple imputation for incomplete data with semicontinuous variables. *Journal of the American Statistical Association*, 98(463):703–715, 2003. ISSN 01621459. URL http://www.jstor.org/stable/30045298.

S. Kolenikov and G. Angeles. Socioeconomic status measurement with discrete proxy variables: Is principal component analysis a reliable answer? *The Review of Income and Wealth*, 55(1):128–165, 2009.

W. J. Krzanowski. The location model for mixtures of categorical and continuous variables. *Journal of Classification*, 10(1):25–49, Jan 1993. ISSN 1432-1343. doi: 10.1007/BF02638452. URL https://doi.org/10.1007/BF02638452.

W.J. Krzanowski. Discrimination and classification using both binary and continuous variables. *Journal of the American Statistical Association*, 70(352):782–790, 1975.

S. L. Lauritzen. *Graphical Models*. Oxford University Press, Oxford, 1996.

S. L. Lauritzen and N. Wermuth. Graphical models for associations between variables, some of which are qualitative and some quantitative. *Ann. Statist.*, 17(1):31–57, 03 1989. doi: 10.1214/aos/1176347003. URL https://doi.org/10.1214/aos/1176347003.

Jason D. Lee and Trevor J. Hastie. Learning the structure of mixed graphical models. *Journal of Computational and Graphical Statistics*, 24(1):230–253, 2015. doi: 10.1080/10618600.2014.900500. URL https://doi.org/10.1080/10618600.2014.900500. PMID: 26085782.

B. Li and S. Wang. On directional regression for dimension reduction. *Journal of the American Statistical Association*, 102(479):997–1008, 2007.

K. C. Li. Sliced inverse regression for dimension reduction (with discussion). *Journal of the American Statistical Association*, 86:316–342, 1991.

Jun Liu and Jieping Ye. Fast overlapping group lasso. *arXiv:1009.0306v1*, 2010.

Matteo Mazziotta and Adriano Pareto. Use and misuse of pca for measuring well-being. *Social Indicators Research*, 142(2):451–476, Apr 2019. ISSN 1573-0921. doi: 10.1007/s11205-018-1933-0.

G. Merola and B. Baulch. Using sparse categorical principal components to estimate asset indices new methods with an application to rural south east asia. 2014.

Carl N. Morris. *Natural Exponential Families*. American Cancer Society, 2006. ISBN 9780471667193. doi: 10.1002/0471667196.ess1759.pub2. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/0471667196.ess1759.pub2.

I. Olkin and R. F. Tate. Multivariate correlation models with mixed discrete and continuous variables. *Ann. Math. Statist.*, 32(2):448–465, 06 1961. doi: 10.1214/aoms/1177705052. URL https://doi.org/10.1214/aoms/1177705052.

M.S. Pepe. *The Statistical Evaluation of Medical Tests for Classification and Prediction*. Oxford University Press, New York, 2003.
I. G. Vlachonikolis and F. H. C. Marriott. Discrimination with mixed binary and continuous data. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 31(1):23–31, 1982. ISSN 00359254, 14679876. URL [http://www.jstor.org/stable/2347071](http://www.jstor.org/stable/2347071).

S. Vyas and L. Kumaranayake. Constructing socio-economic status indices: How to use principal components analysis. *Health Policy and Planning*, 21(6):459–468, 2006.

Martin J. Wainwright and Michael I. Jordan. Graphical models, exponential families, and variational inference. *Foundations and Trends® in Machine Learning*, 1(1–2):1–305, 2008. ISSN 1935-8237. doi: 10.1561/2200000001. URL [http://dx.doi.org/10.1561/2200000001](http://dx.doi.org/10.1561/2200000001).

Joe Whittaker. *Graphical Models in Applied Multivariate Statistics*. Wiley Publishing, 2009. ISBN 0470743662, 9780470743669.

Eunho Yang, Yulia Baker, Pradeep Ravikumar, Genevera Allen, and Zhandong Liu. Mixed Graphical Models via Exponential Families. In Samuel Kaski and Jukka Corander, editors, *Proceedings of the Seventeenth International Conference on Artificial Intelligence and Statistics*, volume 33 of *Proceedings of Machine Learning Research*, pages 1042–1050, Reykjavik, Iceland, 22–25 Apr 2014a. PMLR. URL [http://proceedings.mlr.press/v33/yang14a.html](http://proceedings.mlr.press/v33/yang14a.html).

Eunho Yang, Pradeep Ravikumar, Genevera I. Allen, Yulia Baker, Ying-Wooi Wan, and Zhandong Liu. A general framework for mixed graphical models, 2014b.

Eunho Yang, Pradeep Ravikumar, Genevera I. Allen, Zh, and ong Liu. Graphical models via univariate exponential family distributions. *Journal of Machine Learning Research*, 16(115):3813–3847, 2015. URL [http://jmlr.org/papers/v16/yang15a.html](http://jmlr.org/papers/v16/yang15a.html).

Ke Ye and Lek-Heng Lim. Schubert varieties and distances between subspaces of different dimensions. *SIAM Journal on Matrix Analysis and Applications*, 37(3):1176–1197, 2016. doi: 10.1137/15M1054201. URL [https://doi.org/10.1137/15M1054201](https://doi.org/10.1137/15M1054201).

Ming Yuan and Yi Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society. Series B, statistical methodology*, 2006. ISSN 1369-7412.

Ming Yuan and Yi Lin. Model selection and estimation in the gaussian graphical model. *Biometrika*, 94(1):19–35, 2007. ISSN 00063444. URL [http://www.jstor.org/stable/20441351](http://www.jstor.org/stable/20441351).

Q Zhu, L. Lin, M.-L. Shyu, and S.-C. Chen. Effective supervised discretization for classification based on correlation maximization. *IEEE International Conference on Information Reuse & Integration*, pages 390–395, 2011.