Traces of creation-annihilation operators and Fredholm’s formulas

A. Chervov

Faculty of Mathematics and Mechanics
Moscow State University

and

Institute of Theoretical & Experimental Physics
117259 Moscow, Russia

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Abstract

We prove the formula for the traces of certain class of operators in bosonic and fermionic Fock spaces. Vertex operators belong to this class. Traces of vertex operators can be used for calculation of correlation functions and formfactors of integrable models (XXZ, Sine-Gordon, etc.), that is why we are interested in this problem. Also we show that Fredholm’s minor and determinant can be expressed by such traces. We obtain a short proof of the Fredholm’s formula for the solution of an integral equation.
1 Introduction.

In this paper we prove formulas for the traces of some class of operators in bosonic and fermionic Fock spaces. Such traces are used in mathematical physics, because correlators and formfactors of integrable models can be expressed in terms of such traces ([JM], [L] and [KLP2]). On the other hand Fredholm’s determinant and minor can be expressed in the terms of similar traces; and special case of our formula turns out to be Fredholm’s formula for the solution of an integral equation.

Let us discuss the formulas. Let $\mathbb{C}[x_1, x_2, ...]$ be the algebra of polynomials. Let $\mathbb{C}(x_1, x_2, ...)$ be the operator of multiplication on a polynomial $C(x_1, x_2, ...)$; let $\Lambda(\partial_{x_1}, \partial_{x_2}, ...) be a polynomial of operators $\partial_{x_1} = \frac{\partial}{\partial x_1}, \partial_{x_2} = \frac{\partial}{\partial x_2}, ...; let $\bar{\rho}$ be an arbitrary, degree preserving homomorphism of the algebra $\mathbb{C}[x_1, x_2, ...]$ onto itself. We prove the following formula for the trace over the space $\mathbb{C}[x_1, x_2, ...]$ of the product of these three operators:

$$Tr (\bar{\rho} A(\partial_{x_1}, \partial_{x_2}, ...) C(x_1, x_2, ...)) = (Tr \bar{\rho}) \left( A(\partial_{x_1}, \partial_{x_2}, ...) \right) \left( \frac{1}{1 - \rho} C(x_1, x_2, ...) \right)$$  \hspace{1cm} (1.1)

Here pairing $< ... | ... >$ is defined by the formula: $< (\partial_{x_1})^i \cdot (\partial_{x_2})^n | (x_1)^i \cdot (x_2)^n > = \delta_{i_1}^i \delta_{i_2}^n \delta_{i_3}^n! \delta_{i_4}^n!$. The trace of the operator $O$: $\mathbb{C}[x_1, x_2, ...] \to \mathbb{C}[x_1, x_2, ...]$ is the sum of diagonal elements in the natural basis $(x_1)^i (x_2)^j ... (x_k)^i$.

We shall also prove analogous formula for the anticommuting variables $\xi_i \xi_j = -\xi_j \xi_i$:

$$Tr_{\Lambda[\xi_1, \xi_2, ...]} (\bar{\rho} A(\partial_{\xi_1}, \partial_{\xi_2}, ...) C(\xi_1, \xi_2, ...)) = (Tr_{\Lambda[\xi_1, \xi_2, ...]} \bar{\rho}) \left( A(\partial_{\xi_1}, \partial_{\xi_2}, ...) \right) \left( \frac{1}{1 + \rho} C(\xi_1, \xi_2, ...) \right)$$ \hspace{1cm} (1.2)

The special cases of the first formula can be found in [JM], special cases of the second one in [SJM].

These formulas express matrix elements of the operator inverse to $(1 \pm \bar{\rho})$. If $\bar{\rho}$ is an integral operator then the second formula is equivalent to the Fredholm’s formula for the solution of integral equation. Thus we obtain short proof of the Fredholm’s formula, and interpretation of Fredholm’s minor as the trace over $\Lambda[\xi_1, \xi_2, ...]$ of some sort of operator, which seems to be unknown before. (The interpretation of Fredholm’s determinant as such trace is well known.) Also our first formula gives analogous formula of inversion not via the Fredholm’s determinants, but via Fredholm’s permanents.

Correlation functions and formfactors of different integrable models can be found, as traces of vertex operators. This was shown in [DFJMN], [JM] for XXZ-model and corresponding six-vertex model and in [L, KLP1, KLP2] and [KLP3] for SU(2)-invariant Thirring model and Sin− Gordon model. The so-called procedure of bosonization allows one to write down vertex operators, as operators in the space $\mathbb{C}[x_1, x_2, ...]$ of the type: $\bar{\rho} A(\partial_{x_1}, \partial_{x_2}, ...) C(x_1, x_2, ...)$. Our formula allows to compute the traces of such operators. Our initial purpose was to prove the formula $\dagger$ and its consequence $\dagger$ that are used in [KLP] to compute formfactors in SU(2)-invariant Thirring model. Subsequently we found general formula. Note that the traces of some concrete operators of such type were calculated previously in many papers (for example in [DFJMN], [L]) with the help of the Clavelli-Shapiro technique [CS].

Spaces $\mathbb{C}[x_1, x_2, ...]$, $\Lambda[\xi_1, \xi_2, ...]$ are called bosonic and fermionic Fock spaces, respectively. They are the spaces of states in quantum field theory. Observable values are given by the formula $Tr(\rho A)$, where $\rho$ is the density matrix, $A$ is the operator of some observable. So, we hope that our formulas, will be useful not only in solving the integrable models like SU(2)-invariant Thirring or Sin− Gordon, but also in the other problems of quantum field theory.

We shall give several proofs of the formulas $\dagger$ and $\dagger$. In the case when $A(\partial_{x_1}, \partial_{x_2}, ...) = C(x_1, x_2, ...)$ are polynomials of the first degree of $\partial_{x_1}, \partial_{x_2}, ...$ and $x_1, x_2, ...$ respectively, their commutator equals to the scalar and in this case it is possible to prove our formulas in few lines, and this proof is based only on commutation relations and the property $Tr AB = Tr BA$. The idea of another proof applicable in general case is the following: if operator $\rho$ is diagonalizable, then in the eigenvector basis the sum of diagonal elements factorizes to the product of simple sums. If the operator is not diagonalizable one can approximate it by diagonalizable ones, and taking the limit obtain the formula. The fact that trace factorizes to the product is based on the following idea: $\mathbb{C}[x_1, x_2, ...] = \mathbb{C}[x_1] \otimes \mathbb{C}[x_2] \otimes ...$ and $\Lambda[\xi_1, \xi_2, ...] = \Lambda[\xi_1] \otimes \Lambda[\xi_2] \otimes ...$, and if the operator $\rho$ is diagonal then $\rho = \rho_1 \otimes \rho_2 \otimes ...$. And it’s well-known that the trace of the tensor product of operators is product of traces. The third proof is the longest. In it we obtain formula for trace of our operator restricted on the space of polynomials of degree $N$. Since this space is not tensor product, the sum of diagonal elements is not factorizable, so the proof is rather intricate. But lemmas which we proved are rather interesting by themselves.

The algebras of polynomials $\mathbb{C}[x_1, x_2, ..., x_N], \Lambda[\xi_1, \xi_2, ..., \xi_N]$ are the synonyms for the symmetrical algebra $SV$ of linear space $V$ and the exterior algebra $AV$ of the space $V$ respectively. The space $V$ is such that $x_1, x_2, ..., x_N$ is a basis of $V$. In this paper it is more convenient for us to use the terminology of the symmetrical and exterior algebras instead of $\mathbb{C}[x_1, x_2, ..., x_N], \Lambda[\xi_1, \xi_2, ..., \xi_N]$, though, may be it makes the text not so transparent. For the case of infinite-dimensional $V$ the terminology of the symmetrical and exterior algebras is the only rigorous.

Our formula is valid for the case $\dim V = \infty$ as well as $\dim V < \infty$. One can easily reduce infinite-dimensional case to the finite-dimensional one, because any operator of the trace class can be approximated by the operators with the finite-dimensional image.

The paper is organized as follows. In sections 2 and 3 we prove our main formulas for the cases of fermionic and bosonic Fock spaces, respectively. Both cases are very similar, so in section 3 we omit all the details. In the fourth section we obtain

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1 permanent of square matrix is the sum of its elements standing in different columns and different lines.
Fredholm’s formula. In the fifth section we apply our formulas for calculation of the traces of vertex operators corresponding to $SU(2)$-invariant Thirring model. Further on follows several appendices. In the first we generalize the formulas to the case of the linear space with a countable basis, without any topology. In the second we discuss the regularization of the traces. In the third we mention the case, when there exist a continuous basis in $V$. The fourth appendix contains expressions for the traces of an operators in Fock spaces in terms of generating functions. In the fifth appendix we prove that infinite products obtained in calculations of the traces of vertex operators in section 5 are convergent.

After the work was completed we received a letter from V.E. Korepin who pointed out on his works [Kor], where the Fredholm’s determinants were used to express correlation functions of different integrable models. This expressions allowed Korepin and his coauthors to obtain a lot of important information about correlation functions such as long distance asymptotics, equations for correlation functions and so on.

2 Formula for fermionic Fock space.

In this section we prove the main trace formula (theorem 2.1) for the fermionic Fock space. From the mathematical point of view, fermionic Fock space is simply the space of polynomials of anticommuting variables or, more formally, the exterior algebra of some space $V$. We use the terminology of exterior algebra, because it’s more convenient and it’s rigorous in infinite-dimensional case.

Let $V$ be a linear space. Denote by $\Lambda^nV$ n-th antisymmetrical (exterior) tensor exponent of the space $V$, $\Lambda V = \bigoplus_{n=0}^\infty \Lambda^n V$ the exterior algebra of space $V$. If $a_1, a_2, ...$ is a basis of $V$, then $\Lambda V$ naturally is identified with $\Lambda[a_1, a_2, ...]$. Any operator $\rho$ on the space $V$ induces the action of operator $\bar{\rho}$ on the space $\Lambda V$ by the formula: $\bar{\rho} (\Lambda f) = \Lambda \rho(f)$, where $f$ is arbitrary elements from $V$. Note that correspondence $(V, \rho) \rightarrow (\Lambda V, \bar{\rho})$ is often called functor of secondary quantization [MM]. $\Lambda V$ is called fermionic Fock space. If $V$ is a Hilbert space, $\rho$ is a unitary operator, then on completion of $\Lambda V$ one can canonically introduce the structure of Hilbert space and $\bar{\rho}$ turns out to be unitary operator on $\Lambda V$. It’s natural to consider algebra $\Lambda V$ graded: if $w \in \Lambda^n V$ then $deg w = n$.

Let $V^*$ be a dual linear space. We shall extend the canonical pairing between $V^*$ and $V$ to the pairing between $\Lambda V$ and $\Lambda V^*$ according to Vick’s rule:

$$< \bigwedge_{i=1}^m f_i | \bigwedge_{j=1}^m u_j > = \sum_{\sigma \in S_m} (-1)^{sgn \sigma} \prod_{i=1}^m < f_i | u_{\sigma(i)} > = det < f_i | u_j > |_{i,j \leq m}$$  \hspace{1cm} (2.1)

If $e_i$ and $\bar{e}_i$ are dual basis in $V$ and $V^*$, respectively, then pairing may be defined in the following equivalent way:

$$< (\bar{e}_i)_1 \wedge ... \wedge (\bar{e}_i)_n | (e_1)_1 \wedge ... \wedge (e_1)_n > = \delta_{i_1}^{\sigma_1} ... \delta_{i_n}^{\sigma_n}$$  \hspace{0.5cm} $i_k = 0, 1$, $j_k = 0, 1$.

For the arbitrary $v \in LV$ one can define the operator of LEFTWARD multiplication on $v$, which we will call the creating operator and denote it by $C(v)$. By definition operator $C(v)$ acts from $\Lambda V$ to $\Lambda V$, as follows: $C(v)[\tilde{v}] = v \wedge \tilde{v}$. Analogically, for arbitrary $w \in LV^*$ one can consider operator $C(w) : \Lambda V^* \rightarrow \Lambda V^*$. Operator dual to the $C(w)$, which acts $\Lambda V \rightarrow \Lambda V^*$ we will call the annihilating operator and denote it by $A(w)$. For example, let us describe the action of operator $A(\tilde{e}_p)$ on $v = e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_l}$, where $e_i$ is a basis of $V$, $\bar{e}_i$ is dual basis of $V^*$ and $i_k \neq i_l$ at $k \neq l$. If $p \notin \{i_j\}$ then operator $A(\tilde{e}_p)$ kills $v$, if $p = i_1$ then operator erases $e_{i_1}$, if $p \in \{i_j\}$, but $p \neq i_1$, then one should put $e_p$ on first position, changing the sign appropriately, then erase it. If $w \in V^*$, then $A(w)$ is antideriving: $A(w)(v_1 \wedge v_2) = A(w)(v_1) \wedge v_2 + (-1)^{deg v_1} v_1 \wedge A(w)(v_2)$. Obviously, the following commutation relations holds:

$$A(w)C(v) + C(v)A(w) = < v | w > \hspace{0.5cm} v \in V, \hspace{0.5cm} w \in V^*$$  \hspace{1cm} (2.2)

$$A(w_1)A(w_2) = A(w_1w_2) \hspace{0.5cm} C(v_1)C(v_2) = C(v_1v_2)$$

$$\bar{\rho}C(v_1) = C(\rho(v_1))\bar{\rho} \hspace{0.5cm} A(w_1)\bar{\rho} = \bar{\rho}A(\rho(w_1))$$  \hspace{1cm} $v_1, v_2 \in \Lambda V$, $w_1, w_2 \in \Lambda V^*$

Here and further we will allow ourselves some sloppiness in notations: we will denote operator dual to $\bar{\rho}$ by the same symbol $\bar{\rho}$. So, in this notations: $< w | \tilde{v} > = < \bar{\rho} w | v >$.

**Definition 2.1:** operator $\rho$ on $V$ is called operator with trace, iff $\exists \lambda_i \in \mathbb{C}$, $y_i \in V$, $f_i \in V^*$ such that $\forall x \in V$ holds:

$$\rho(x) = \sum_{i>0} \lambda_i f_i(x) y_i$$  \hspace{1cm} (2.3)

Where $\sum_{i>0} |\lambda_i| < \infty$ \hspace{0.5cm} $f_i \in V^*$ \hspace{0.5cm} $|f_i| < C$ \hspace{0.5cm} $y_i \in V$ \hspace{0.5cm} $|y_i| < C$

(Small change generalizes this definition to the case of an arbitrary locally-convex space [SCH]. All results will be true in this case as well.)

**Definition 2.2:** $Tr \rho = \sum_{i>0} \lambda_i f_i(y_i)$.

Trace is well-defined i.e. trace is independent of the representation [2.3] if metric Grothendieck’s approximation property (AP) holds for the space $V$ ([Pietsch]). Let us recall the formulation of (AP): for any compact subset $K \subset V$ and $\forall \varepsilon > 0$ there exists operator $L : |L| \leq 1$ with finite-dimensional image, such that $\forall x \in K \hspace{1.5cm} |x - Lx| < \varepsilon$. 

2
As it is known, all typical spaces \( P^r, L^p[a,b], C^p[a,b] \) are spaces with AP, the construction of the space, where AP is not fulfilled, was long-time standing problem. Known separable examples of such spaces are rather artificial. Note that if there exists basis (in Schauder sense) in space \( V \), then \( V \) is space with AP. We shall also note that approximation property is closely connected to the following properties: any compact operator can be approximated in norm topology by operators with finite dimensional image; identical operator can be approximated by the compact operators uniformly on any precompact set.

It’s clear that if \( \rho \) is operator with trace, \( A \) is arbitrary operator, then \( \rho A \) and \( A \rho \) are operators with trace and \( TrA\rho = TrA\rho \).

**Proposition 2.1** If \( \rho \) is an operator with trace on space \( V \), then \( \rho \) is operator with trace on \( \Lambda V \) and \( Tr_{\Lambda V} \rho = \lim_{N \to \infty} {\hat{\rho}}_N \).

Here and further on we will always mean that \( V \) is a normed space with AP.

**Theorem 2.1** Let us formulate the following two simple facts, which are necessary for the proof of theorem 2.1.

- **Lemma 2.1**
- **Proposition 2.1**

\[ \text{Proposition 2.1} \]

The proof of proposition is based on straightforward expression for the \( Tr_{\Lambda V} \rho \):

\[
Tr_{\Lambda V} \rho = \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} \det |f_{i_1}(y_{i_1})|_{i_1 \leq k} \prod_{i=1}^k \lambda_i.
\]

One can obtain this expression straightforward or using the corollary 2.1 below. Due to Hadamard’s inequality \( \det |f_{i_1}(y_{i_1})|_{i_1 \leq k} \leq C^k \sqrt{k} \), hence \( |Tr_{\Lambda V} \rho| \leq C^k \sqrt{k} \), hence due to d’Alembert test the series \( \sum_k Tr_{\Lambda V} \rho \) is absolutely convergent.

**Proposition is proved.**

**Theorem 2.1**

Let \( \rho \) be operator with trace on \( V \); \( v, w \) arbitrary elements from \( \Lambda V \) and \( \Lambda V^* \) respectively. Then the following formulas hold:

\[
(a) \quad Tr_{\Lambda V} (\rho A(w)C(v)) = Tr_{\Lambda V} (\rho) \left\langle w \frac{1}{1+\rho} v \right\rangle
\]

\[
(b) \quad Tr_{\Lambda V} (\rho C(v)A(w)) = Tr_{\Lambda V} (\rho) \left\langle w \frac{\rho}{1+\rho} v \right\rangle
\]

**Remark 1:** In Proposition 3.2 we shall show that \( Tr_{\Lambda V} \rho = det_{\lambda} (1+\rho) \). In accordance with this, one must understand \( Tr_{\Lambda V} \rho \left\langle w \frac{1}{1+\rho} v \right\rangle \) as matrix element \( w \ldots v > \) of the augmented matrix, in the case if \( (1+\rho) \) is not invertable.

**Remark 2:** Note that the formula \( 2.5 \) (a) is equivalent to the formula \( 1.2 \).

Let us derive \( 2.5 \) (b) from \( 2.5 \) (a)

\[
\text{Let us derive} 2.5 (b) \text{from} 2.5 (a)
\]

\[
Tr_{\Lambda V} (\rho \rho (v)A(w)) = Tr_{\Lambda V} (A(w)\rho C(v)) = Tr_{\Lambda V} (\rho A(\rho(w))C(v)) =
\]

\[
= Tr_{\Lambda V} (\rho) < \rho(w) \frac{1}{1+\rho} v > = Tr_{\Lambda V} (\rho) < w \frac{\rho}{1+\rho} v >
\]

We shall give 3 proofs of the theorem 2.1. The first proof is very short, but, unfortunately, it is applicable only for the case \( w \in V^* \), \( v \in V \) i.e. \( \text{deg} w = \text{deg} v = 1 \). Another proof is simple too, and applicable in general case. The third proof is based on three propositions. The proof of the third is rather complicated, but it is rather interested by itself.

Let us formulate the following two simple facts, which are necessary for the proof of theorem 2.1.

**Lemma 2.1** If theorem 2.1 is true for any operator \( \rho \) with trace such that \( |\rho| < 1 \), then theorem 2.1 is true for any \( \rho \) with trace.

**Proof:**

Let \( \rho \) be an arbitrary operator with trace. Consider operator \( \lambda \rho \), where \( |\lambda| < \frac{1}{|\rho|} \). Since \( |\lambda| < 1 \), we see that theorem 2.1 is true for \( \lambda \rho \), hence:

\[
Tr_{\Lambda V} (\lambda \rho A(w)C(v)) = Tr_{\Lambda V} (\lambda \rho) \left\langle w \frac{1}{1+\lambda \rho} v \right\rangle
\]

Due to proposition 2.1 LHS and RHS of the equality above exist for all \( \lambda \notin \text{Spec} \rho \) and they are analytical functions of variable \( \lambda \). Hence, due to analicity, equality is true for all \( \lambda \notin \text{Spec} \rho \). Taking \( \lambda = 1 \) one obtains lemma for arbitrary \( \rho \).

**Lemma 2.2** If theorem 2.1 is true for arbitrary finite-dimensional space \( V \), then it’s true for any space \( V \).

**Proof:**

The idea of proof is easy - one can approximate any operator with trace by operators with finite-dimensional image.

Let \( \rho \) be an operator with trace on \( V \). Let us define operator \( \rho_N \) as follows: \( \rho_N(x) = \sum_{i_j > 0} \lambda_i f_i(x) y_i \), \( \lambda_i, y_i, f_i \) are defined according to definition 2.1. Let space \( V_N \) be linear span of vectors \( v, y_1, y_2, \ldots y_N \). Then \( \text{Im} \rho_N \subset V_N \) that is why \( Tr_{\Lambda V} \rho_N = Tr_{\Lambda V} \rho_N \). Since formula is true for any finite-dimensional space, hence \( Tr_{\Lambda V} (\hat{\rho}_N A(w)C(v)) = Tr_{\Lambda V} (\hat{\rho}_N) \left\langle w \frac{1}{1+\rho_N} v \right\rangle \).
Taking the limit $N \to \infty$ one obtains the necessary lemma. Passage to the limit is valid, because of proposition 2.1.

**Lemma 2.2 is proved.**

**The first Proof of theorem 2.1.** Only for case $\text{deg}w = \text{deg}v = 1$. (It is sufficient for Fredholm’s formulas.)

Recall formula 2.3 $A(w)C(v) + C(v)A(w) = \langle v | w \rangle$.

Due to lemma 2.1 it’s sufficient to prove only for such $\rho$ that $\rho^n \to 0, \ n \to \infty$.

The latest equality is obtained by sending $n \to \infty$, which is valid since $\rho^n \to 0$.

**Proof 1 is finished.**

Remark 1: The difficulty of applying this proof to the case, when $\text{deg}w > 1, \text{deg}v > 1$, is the following. In this case commutation relations between $A(w), C(v)$ are complicated. So the way to prove described above is possible, but very bulky, and we prefer to give another proof.

Remark 2: The proof that traces of vertex operators satisfy Knizhnik-Zamolodchikov equations is similar to proof 1 described above.

Remark 3: Let us note that we have not used that $\dim V < \infty$. We have only used commutation relations between $\bar{\rho}, A(w)$, $C(v)$ and the property that $Tr AB = Tr BA$. But this is not surprising, because trace maybe determined by the property that $Tr AB = Tr BA$.

**The second proof.**

Due to lemma 2.2 it’s sufficient to prove only for $\dim V = N < \infty$.

Proof is based on two simple ideas. The first idea is the following: it’s sufficient to consider the case when $\rho$ is diagonalizable operator, because any not diagonalizable operator can be approximated by diagonal ones. And easy to see that if $\rho_i \to \rho$ and theorem holds for $\rho_i$, then it holds for $\rho$. (Due to $\dim V < \infty$, hence $\dim \Lambda V < \infty$, we have no problems with approximation).

The second idea is the following: if one writes down the sum of diagonal elements in the basis of eigenvectors for $\rho$, then this sum factorizes. And each multiplier is the series similar to the geometric progression.

Let us consider the case $\dim V = 1$. Let $e \neq 0 \in V$; $\rho e = \lambda e$, let $\tilde{e}$ be the dual to $e$ element of $V^*$

$$Tr_{AV}(\tilde{e}A(e)C(e)) = <\tilde{e} | \rho e > = 1 + \lambda = det V (1 + \rho)$$

$$Tr_{AV}(\bar{\rho} \tilde{e} A(e) C(e)) = <\tilde{e} | \rho \tilde{e} > + <\tilde{e} | \rho \tilde{e} = 1 + 0 = 1$$

$$= 1 + \lambda <\tilde{e} | \frac{1}{1 + \rho} e > = Tr_{AV}(\tilde{e}A(e)C(e)) = <\tilde{e} | \frac{1}{1 + \rho} e > = 0$$

Since $< A^k V | A^k V >= 0$ for $i \neq k$.

In proving : we have obtained that $Tr_{AV} \bar{\rho} = 1 + \lambda$. Note that one can rewrite it as follows: $Tr_{AV} \bar{\rho} = det V (1 + \rho)$. Further we shall prove that it’s true not only for $\dim V = 1$, but also for any $V$ (even infinite-dimensional).

We see that transformations in formula 2.4 are possible only in the case $\lambda \neq -1$, i.e. $1 + \rho$ is invertible. Otherwise one must stop transformations on the step: $Tr_{AV}(\rho A(\tilde{e}) C(e)) = 1$, and understand it as matrix element $<\tilde{e} | ... e >$ of matrix augmented to $1 + \rho$, easily to see that this will be true in case $\dim V > 1$.

Thus theorem 2.1 is proved for the case $\dim V = 1$.

Let us come back to the case $\dim V = N$ where $N$ is arbitrary.

Recall that we want to prove:

$$Tr_{AV}(\bar{\rho} A(w) C(v)) = \langle (Tr_{AV} \bar{\rho}) | w \frac{1}{1 + \bar{\rho}} v \rangle$$

Note that LHS and RHS of the formula above are bilinear of $w, v$. Therefore it’s sufficient to prove only for basic vectors.
Let $c_1, c_2, ..., c_N$ be the basis of eigenvectors for operator $\rho$, (as we have already said it’s sufficient to prove for diagonal operators only). Let $\lambda_i$ be the eigenvalues of operator $\rho$ corresponding to vectors $c_i$. Let $\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_N$ be the dual basis in $V^*$.

Exterior algebra of $N$ variables is the tensor product of the exterior algebras of each variable, therefore $\Lambda V = \Lambda[c_1] \otimes \Lambda[c_2] \otimes ... \otimes \Lambda[c_N]$. Easy to see that $\overline{\rho} \prod_{i=1}^{N} A(\tilde{c}_i)^r_i \prod_{j=1}^{N} C(c_j)^{t_j} = \bigotimes_{i=1}^{N} \overline{\rho}_i A(\tilde{c}_i)^r_i C(c_i)^{t_i}$ where $\overline{\rho}_i$ acts in $\Lambda[c_i]$ as follows: $\overline{\rho}_i(e_i^j) = \tilde{\rho}_i(e_i^j) = \lambda_i^j(e_i^j)$ and $(r_i, t_j, k = 0, 1)$

That is why:

$$\text{Tr}_{\Lambda V} \left( \overline{\rho} \prod_{i=1}^{N} A(\tilde{c}_i)^r_i \prod_{j=1}^{N} C(c_j)^{t_j} \right) = \prod_{i=1}^{N} \text{Tr}_{\Lambda[c_i]} \left( \overline{\rho}_i A(\tilde{c}_i)^r_i C(c_i)^{t_i} \right) =$$

$$= \prod_{i=1}^{N} \left( \text{Tr}_{\Lambda[c_i]} \overline{\rho}_i \right) (\tilde{c}_i)^{r_i} > \sum_{i=1}^{N} \left( \tilde{c}_i \right)^t_i \left( \text{Tr}_{\Lambda V} \overline{\rho} \right) = \prod_{i=1}^{N} (\tilde{c}_i)^{r_i} > \prod_{i=1}^{N} C(c_i)^{t_i} >$$

The last equality is true due to: $\rho$ diagonal in basis $c_i$ and $<c_i|c_j>=0$ $i \neq j$

**Proof 2 is finished.**

Note that $\text{Tr}_{\Lambda N V} \overline{\rho} = \prod_{i=1}^{N} \text{Tr}_{\Lambda[c_i]} \overline{\rho} = \prod_{i=1}^{N} (1 + \lambda_i) = \text{det}_V (1 + \rho)$

Thus we have proved the following, well known proposition:

**Proposition 2.2** Let $V$ be a finite-dimensional space, let $\rho$ be an operator on $V$, then

$$\text{det}_V (1 + \rho) = \text{Tr}_{\Lambda V} \overline{\rho} \tag{2.7}$$

There is another more direct proof of this proposition:

Let $V = N$, then by definition of determinant $\text{det}_V O = \text{Tr}_{\Lambda N V} O$. Hence: $\text{det}_V (1 + \rho) = \text{Tr}_{\Lambda V} (1 + \rho)^{\Lambda N} = \text{Tr}_{\Lambda V} O \rho + \text{Tr}_{\Lambda V} \rho O + ... + \text{Tr}_{\Lambda V} \rho N \tag{2.9}$

Motivated by (2.7), one can introduce the following definition of determinant (applicable in infinite-dimensional case):

**Definition 2.3** Operator $O$ on space $V$ is called by operator with determinant, if operator $\rho = O - 1$ is operator with trace. Determinant of $O$ is defined by the formula: $\text{det}_V O = \text{Tr}_{\Lambda V} \overline{\rho}$.

Further we shall show that this definition is Fredholm’s definition, but written in more invariant way. Note that this is definition of usual, not somehow regularized, determinant. This determinant is equal to the product of the eigenvalues of operator $O$.

The third proof.

**Proposition 2.3** Let $e_i$ be an arbitrary basis of the finite-dimensional space $V$, $\tilde{e}_i$ be the dual basis, then for any operator $O$ on space $\Lambda^n V$, the following formula takes place:

$$\text{Tr}_{\Lambda^n V} O = \frac{1}{n!} \sum_{l_1, ..., l_n \geq 0} <\tilde{e}_{l_1}, \tilde{e}_{l_2}, ..., \tilde{e}_{l_n}, O e_{l_1} \wedge e_{l_2} \wedge ... \wedge e_{l_n}> \tag{2.8}$$

The proof is obvious, due to each basic vector is repeated in summation $n!$ times and it cancels with $n!$ in denominator.

Remark: If $V$ is Hilbert space, $e_i$ is it’s basis, $O$ is operator with trace on $\Lambda^n V$, then lemma is true also. The proof is the same, but instead of finite sums arises series, which are absolutely convergent, since $O$ is operator with trace.

**Corollary 2.1** Let $e_i$ be an arbitrary basis of the finite-dimensional space $V$, $\tilde{e}_i$ be the dual basis. Let $\rho$ be an operator on space $V$, $\overline{\rho}$ - induced operator on $\Lambda^n V$. Then:

$$\text{Tr}_{\Lambda^n V} \overline{\rho} = \frac{1}{n!} \sum_{l_1, ..., l_n \geq 0} \text{det} \begin{vmatrix} <\tilde{e}_{l_1}, |\rho e_{l_1}> & <\tilde{e}_{l_2}, |\rho e_{l_1}> & ... & <\tilde{e}_{l_n}, |\rho e_{l_1}> \\ ... & ... & ... & ... \\ <\tilde{e}_{l_1}, |\rho e_{l_n}> & <\tilde{e}_{l_2}, |\rho e_{l_n}> & ... & <\tilde{e}_{l_n}, |\rho e_{l_n}> \end{vmatrix} \tag{2.9}$$

To prove corollary one should apply Vick formula (2.1) to (2.9)

**Proposition 2.4** Let $\rho$ be an operator with trace on space $V$, $\overline{\rho}$ induced operator on $\Lambda^n V$, then $\text{Tr}_{\Lambda^n V} \overline{\rho}$ can be expressed through $\text{Tr}_{V} \rho$ as follows:

$$\text{Tr}_{\Lambda^n V} \overline{\rho} = \sum_{n_j \geq 0} (-1)^{\sum_{j=1}^{k} (n_j-1)} \prod_{j=1}^{k} (\text{Tr}_{V} \rho)^{n_j} \tag{2.10}$$

**Proposition 2.5** Let $\rho$ be an operator with trace on $V$; $v_i, w_i$ arbitrary elements from $V$ and $V^*$, respectively. Then the trace of $\rho \prod_{i=1}^{k} A(w_i) \prod_{j=1}^{k} C(v_j)$ over $\Lambda^n V$ can be expressed through $\text{Tr}_{V} \overline{\rho}$ and $<w_i|\overline{\rho} v_j>$ as follows:
The third proof of the theorem 2.1 is finished. 

Proof of theorem 2.1 easily follows from Proposition 2.5: LHS and RHS of equality in theorem 2.1 are bilinear of $w, v$, therefore it’s sufficient to prove only for $w = \bigwedge_{i=1}^k w_i$, $v = \bigwedge_{j=1}^n v_j$, where $\text{deg}w_i = \text{deg}v_1 = 1$. Easily to see that both LHS and RHS are equal to zero, if $n \neq k$. So it’s sufficient to consider the case $n = k$ only.

Due to lemma 2.1 it’s sufficient to prove for such $\rho$, that $|\rho| < 1$

$$Tr_{\Lambda^V} \left( \bar{\rho} \prod_{i=1}^k A(w_i) \prod_{j=1}^k C(v_j) \right) = \sum_{\rho, \bar{\rho}, q_j \geq 0 : \sum_j q_j + r = n} Tr_{\Lambda^V}(\bar{\rho}) \left( \prod_{j=1}^k w_j \prod_{j=1}^k (-\rho)^{q_j}(v_j) \right)$$

(2.11)

The third proof of the theorem 2.1 is finished.

The proof of Proposition 2.4 Similar to the proof of lemma 2.2, easy to show that it’s sufficient to consider finite-dimensional spaces $V$ only, because any operator with trace can be approximated by the operators with finite-dimensional image.

$$Tr_{\Lambda^V} \bar{\rho} = \frac{1}{n!} \sum_{\ell_1 \ldots \ell_n \geq 0} < \hat{\ell}_1 \wedge \hat{\ell}_2 \wedge \ldots \hat{\ell}_n | \bar{\rho} \epsilon \ell_1 \wedge \epsilon \ell_2 \wedge \ldots \epsilon \ell_n > =$$

$$= \frac{1}{n!} \sum_{\ell_1 \ldots \ell_n \geq 0} \sum_{\alpha \in \Lambda_n} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^n \epsilon \ell_i | \rho \epsilon \sigma(\ell_i) >$$

$$= \frac{1}{n!} \sum_{\alpha \in \Lambda_n} (-1)^{\sum_{i} (k_i - 1)} \prod_{i} (Tr V \rho^i)^{k_i}$$

where $k_i(\sigma)$ is the number of the cycles of length $i$ for $\sigma$

The amount of permutations that are product of $k_i$ cycles of length $i$, equals to $\frac{n!}{k_1! k_2! \ldots 1^{k_1} 2^{k_2} \ldots}$

hence, equality may be continued:

$$= \sum_{k_j \geq 0 : \sum_j j k_j = n} \frac{(-1)^{\sum_{i} (k_i - 1)} k_1! k_2! \ldots 1^{k_1} 2^{k_2} \ldots}{k_1! k_2! \ldots} \prod_{i} (Tr V \rho^i)^{k_i}$$

Proposition 2.4 is proved

The proof of Proposition 2.5 Similar to the proof of lemma 2.2, easy to show that it’s sufficient to consider only finite-dimensional spaces $V$ only, because any operator with trace can be approximated by the operators with finite-dimensional image.

The main instrument of proof is - is Vick’s formula (2.1). Note that Vick’s formula is conveniently rewritten rule of pairing between antisymmetrical tensors.

$$Tr_{\Lambda^V} \left( \bar{\rho} \prod_{i=1}^k A(w_i) \prod_{j=1}^k C(v_j) \right) =$$

$$= \frac{1}{n!} \sum_{\ell_1 \ldots \ell_n \geq 0} < \hat{\ell}_1 \wedge \hat{\ell}_2 \wedge \ldots \hat{\ell}_n | \bar{\rho} \prod_{i=1}^k A(w_i) \prod_{j=1}^k C(v_j) \epsilon \ell_1 \wedge \epsilon \ell_2 \wedge \ldots \epsilon \ell_n > =$$

$$= \frac{1}{n!} \sum_{\ell_1 \ldots \ell_n \geq 0} < w_1 \wedge \rho(\hat{\ell}_1) \wedge \rho(\hat{\ell}_2) \wedge \ldots \wedge \rho(\hat{\ell}_n) | \prod_{j=1}^k v_j \wedge \epsilon \ell_1 \wedge \epsilon \ell_2 \wedge \ldots \wedge \epsilon \ell_n > =$$

(2.12)
Let us introduce the following notation:

\[ \xi_i = v_i \quad \xi_i = w_i, \quad 1 \leq i \leq k; \quad \xi_i = e_{i+k}, \quad \xi_i = \rho(\tilde{e}_{i+k}), \quad k < i \leq n + k; \]

Let us apply Vick's formula to the 2.12:

\[ = \frac{1}{n!} \sum_{l_1, \ldots, l_n} \sum_{\sigma \in S_{n+k}} (-1)^{\sigma n_k} \prod_{i=1}^{n+k} \xi_{l_i} \xi_{l_{n+i}} > \]

Rewrite multipliers in following order:

\[ < w_1 | e_{(i)} > < \rho(\tilde{e}_{(i)}) | e_{(i)} > \ldots < \rho(\tilde{e}_{(n+1)}) | v_{(n+1)} > \]

\[ < w_2 | e_{(i)} > < \rho(\tilde{e}_{(i)}) | e_{(i)} > \ldots < \rho(\tilde{e}_{(n+2)}) | v_{(n+2)} > \]

\[ \ldots \]

\[ < w_k | e_{(i)} > < \rho(\tilde{e}_{(i)}) | e_{(i)} > \ldots < \rho(\tilde{e}_{(n+k)}) | v_{(n+k)} > \]

\[ \prod_{j \in J(\sigma)} < \rho(\tilde{e}_j) | e_{(j)} > \]

(2.13)

Where \( J(\sigma) \) is subset of \( \{1, \ldots, n\} \) that includes such elements that cannot be obtained from \( \{1, 2, \ldots, k\} \) by applying \( \sigma, \sigma^2, \sigma^3, \ldots \)

\[ r = Card(J(\sigma)). \quad (As \ usual, \ Card(M) \ is \ amount \ of \ elements \ in \ the \ set \ M ). \]

\( n_i \geq 0 \) - the least exponent such that \( \sigma^{n_i+1}(i) \) belongs to the set \( \{1, 2, 3, \ldots, k\} \). Denote \( I(\sigma) = \{1, 2, \ldots, n\} \setminus J(\sigma) \).

\[ Card(I(\sigma)) = \sum n_i. \]

Then \( r + \sum_i n_i = n. \)

Summating over \( l_i \) such that \( i \in I(\sigma) \) one obtains:

\[ = \frac{1}{n!} \sum_{\sigma \in S_{n+k}} (-1)^{\sigma n_k} \prod_{p=1}^{k} w_p | \bar{\rho}^n v_{\sigma^{n+1}(p)} > \sum_{l_j \in J(\sigma)} \prod_{j \in J(\sigma)} < \rho(\tilde{e}_j) | e_{(j)} > \]

Let \( \sigma = \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \), where \( \sigma_1 \) is the product of independent cycles which includes \( \{1, 2, \ldots, k\} \), \( \sigma_2 \) - the one which doesn’t includes \( \{1, 2, \ldots, k\} \). Obviously \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \) and \( \sigma(i) = \sigma_1(i) \quad i \in I(\sigma) \quad \sigma(i) = \sigma_2(i) \quad i \in J(\sigma) \).

Then

\[ = \frac{1}{n!} \sum_{\sigma_1} (-1)^{\sigma n_k} \prod_{p=1}^{k} w_p | \rho^n v_{\sigma^{n+1}(p)} > \sum_{l_j \in J(\sigma)} \prod_{j \in J(\sigma)} < \rho(\tilde{e}_j) | e_{(j)} > \]

\[ = \frac{1}{n!} \sum_{\sigma_1} (-1)^{\sigma n_k} \prod_{p=1}^{k} w_p | \rho^n v_{\sigma^{n+1}(p)} > r! T_{\Lambda V} \bar{\rho} \]

\[ = \frac{r!}{n!} \sum_{\sigma_1} (-1)^{\sigma n \delta} \sum_{r_1, n_1, \ldots, n_k \geq 0} \sum_{j, r = n} \prod_{p=1}^{k} w_p | (-\rho)^n v_{\sigma^{n+1}(p)} > T_{\Lambda V} \bar{\rho} \]

\[ = \frac{r!}{n!} \sum_{\sigma_1} (-1)^{\sigma n \delta} \sum_{r_1, n_1, \ldots, n_k \geq 0} \sum_{j, r = n} \frac{n!}{r!} \prod_{p=1}^{k} w_p | (-\rho)^n v_{\sigma^{n+1}(p)} > T_{\Lambda V} \bar{\rho} \]

\[ = \sum_{\sigma_1} (-1)^{\sigma n \delta} \sum_{r_1, n_1, \ldots, n_k \geq 0} \prod_{p=1}^{k} w_p | (-\rho)^n v_{\sigma^{n+1}(p)} > T_{\Lambda V} \bar{\rho} \]

\[ = \sum_{r_1, n_1, \ldots, n_k \geq 0} \prod_{p=1}^{k} w_p | (-\rho)^n v_{\sigma^{n+1}(p)} > T_{\Lambda V} \bar{\rho} \]

Where \( \Omega(r, n_1, \bar{\sigma}) \) is set of that and only that permutations \( \sigma_1 \) from \( S_k \), such that: \( \sigma_1^{n_1+1}(i) = \bar{\sigma}(i) \); \( \sigma_1(j) = j \quad j \in J(\sigma_1) \), and \( n_i \) - are the least exponents such that \( \sigma_1^{n_i+1}(i) = \bar{\sigma}(i) \) belongs to the set \( \{1, 2, \ldots, k\} \).

Easy to see that \( Card(\Omega(r, n_1, \bar{\sigma})) = \frac{n!}{r!} \)

Proposition 3 is proved.
Due to Corollary 2.1 one can reformulate definition 2.3 as follows:

**Proposition 2.6:** Let $\rho$ be an operator with trace on finite-dimensional space $V$, $e_i$ a basis of $V, \tilde{e}_i$ be the dual basis.

\[
\det_V(1 + \rho) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, i_2, \ldots, i_n} \det < \tilde{e}_{i_1}|\rho e_{i_1}| > |i_1, j \leq n
\] (2.14)

### 3 Formula for bosonic Fock space.

In this section we prove the main trace formula (theorem 3.1) for the bosonic Fock space. From the mathematical point of view, bosonic Fock space is simply the space of polynomials or, more formally, the symmetrical algebra of some space $V$. We use the terminology of the symmetrical algebra, because it’s more convenient and it’s rigorous in infinite-dimensional case.

Let $V$ be a linear space, $V^\otimes n$ - i-th tensor power of the space $V$, $SV$ - subspace of symmetric tensors, $SV = \oplus_{i=0}^{\infty} SV^n$ - symmetrical algebra of space $V$. If space $V$ is finitely-dimensional and $e_1, e_2, \ldots, e_N$ - its basis, then $SV = \mathbb{C}[e_1, e_2, \ldots, e_N]$. If $V$ is infinite-dimensional then one may naively treat $SV$ as $\mathbb{C}[e_1, e_2, \ldots, \ldots]$. Any operator $\rho$ on the space induce the action of operator $\bar{\rho}$ on the space $SV$ by the formula: $\bar{\rho}(\bigotimes_i v_i) = \rho(v_i)$ where $v_i$ - arbitrary elements from $V$. Note that correspondence $(V, \rho) \rightarrow (SV, \bar{\rho})$ is often called secondary quantization functor [MM]. $SV$ - bosonic Fock space and if $V$ - Hilbert space and $\rho$ unitary operator, then on completion of $SV$ one can canonically introduce the structure of Hilbert space and $\bar{\rho}$ turns out to be unitary operator on it. Naturally to consider algebra $SV$ graduated: if $w \in SV^0$ then $deg w = 0$.

Let $V^*$ be the dual space to $V$. Extend the pairing between $V$ and $V^*$ to the pairing between $SV$ and $SV^*$ according to Vick’s rule:

\[
< \prod_{i=1}^{m} f_i | \prod_{j=1}^{m} u_j >= \sum_{\sigma \in S_m} \prod_{i=1}^{m} < f_i | u_{\sigma(i)} >= p e r A | f_i | v_j > | i, j \leq m \quad f_i \in V^*, \quad u_j \in V
\] (3.1)

where $per A$ is permanent of matrix $A$. Recall that permanent of square matrix $A$ is sum of products of its elements, standing in different rows and columns.

Let us point out that $< w^k | v^b >= k! < w | v >^k$, at $w \in V^*$, v $\in V$. It’s convenient, because, if $x_i$ is basis in $V, \partial_i$ is dual basis in $V^*$ then differential operator $\frac{\partial}{\partial x_i}$ in space $SV$ is dual to operator of multiplication on $\partial_j$ in space $SV^*$, i.e.:

\[
< (\partial_1)^{i_1} \ldots (\partial_k)^{i_k+1} \ldots (\partial_n)^{i_n} | (x_1)^{j_1} \ldots (x_n)^{j_n} > = < (\partial_1)^{i_1} \ldots (\partial_k)^{i_k} \ldots (\partial_n)^{i_n} | \frac{\partial}{\partial x_i} [(x_1)^{j_1} \ldots (x_n)^{j_n}] >
\]

Analogously, operator of multiplication on $x_i$ in space $SV$ is dual to differential operator with respect to $\partial_j$ in space $SV^*$. Also, due to such choice of pairing proposition 3.3 holds.

Similar as it was done in previous section one can introduce creating-annihilating operators. For any $v \in SV$ creating operator $C(v) : SV \rightarrow SV$, for any $w \in SV^*$ annihilating operator $A(w) : SV \rightarrow SV$. In terms of space of polynomials creating operator is operator of multiplication on some polynomial, annihilating operator is polynomial of differential operators $\frac{\partial}{\partial x_i}$. Obviously, the following commutation relations holds:

\[
A(w)C(v) - C(v)A(w) = < v | w > \quad v \in V, \quad w \in V^*
\] (3.2)

\[
A(w_1)A(w_2) = A(w_2)A(w_1) = A(w_1w_2) \quad C(v_1)C(v_2) = C(v_2)C(v_1) = C(v_1v_2)
\]

\[
\bar{\rho}C(v_1) = C(\bar{\rho}(v_1))\bar{\rho} \quad A(w_1)\bar{\rho} = \bar{\rho}A(\bar{\rho}(w_1)) \quad v_1, v_2 \in SV, \quad w_1, w_2 \in SV^*
\]

\[
\rho : V \rightarrow V, \quad \bar{\rho} \text{ is induced operator : } SV \rightarrow SV
\]

**Proposition 3.1** If $\rho$ is a operator with trace on space $V$ and $|\rho| < 1$, then $\bar{\rho}$ is operator with trace on $SV$. For $v \in SV, w \in SV^*$ operator $\bar{\rho}A(w)C(v)$ is operator with trace.

Let us point out that analogous proposition 2.1 in previous section is true for all operators $\rho$ with trace, in contrast to the proposition 3.1, which is true only for $|\rho| < 1$. If $|\rho| > 1$, then $\bar{\rho}$ is unbounded operator on $SV$. Also note that norm of operator $A(w)$ on space $\oplus_{i<N} SV^i$ equals to $N|w|$, so it is unbounded on the whole space $SV$, but product $\bar{\rho}A(w)$ is operator with trace.

One can easy derive the proof of the proposition 3.1 from proposition 3.4.

**Theorem 3.1**

Let $\rho$ be an operator with trace on $V$ and $|\rho| < 1$; $v, w$ arbitrary elements from $SV$ and $SV^*$ respectively. Then the following formulas take place:

\[
(a) \quad \frac{Tr_{SV}(\bar{\rho}A(w)C(v))}{Tr_{SV}(\bar{\rho})} = \sum_{n=0}^{\infty} \langle w | \bar{\rho}^n v \rangle \quad (b) \quad \frac{Tr_{SV}(\bar{\rho}C(v)A(w))}{Tr_{SV}(\bar{\rho})} = \sum_{n=1}^{\infty} \langle w | \bar{\rho}^n v \rangle
\] (3.3)

Remark 1: Obviously, formula (3.3) (a) is equivalent to the formula (1.1) mentioned in introduction.

Remark 2: Actually if exist two of the expressions $Tr_{SV}(\bar{\rho}), Tr_{SV}(\bar{\rho}A(w)C(v))$, $\sum_{n=0}^{\infty} \langle w | \rho^n v \rangle$ then exists the third and formula (3.3) is valid. This can be simply shown using lemmas 2.2, 2.3.
Corollary 3.1
Let \( v, w \) - the elements of degree 1, from \( SV \) and \( SV^* \) respectively, i.e. \( v \in V, w \in V^* \) then:

\[
\frac{Tr_{SV}(\tilde{\rho}A(e^w)C(e^v))}{Tr_{SV}(\tilde{\rho})} = e^{\langle w, Tr_{SV}(\tilde{\rho})v \rangle}
\]  

(3.4)

One can prove corollary expanding to the series and applying theorem 3.1, or one can obtain straightforward proof of corollary 3.1, using commutation relations \( e^{A(v)}e^{C(v)} = e^{\langle w, v \rangle}e^{C(v)}e^{A(w)} \), and reasoning in the same way, as it was done in proof 1 of the theorem 2.1.

Proofs of theorem 3.1 are similar to the ones of theorem 2.1. One should only substitute sign plus for sign minus in proofs 1 and 3 of theorem 2.1 to obtain the ones of theorem 3.1. One can also change proof 2 of theorem 2.1 in order to obtain proof of theorem 3.1. But a little difference arises in this case. Let us consider it.

Let \( SV = \mathbb{C}[e_1, e_2, \ldots, e_N] = \mathbb{C}[e_1] \otimes \mathbb{C}[e_2] \otimes \ldots \otimes \mathbb{C}[e_N] \), where \( e_i \) any basis of \( V \). Hence, similar as it was done in previous section one can reduce the proof to the case \( dimV = 1 \). But \( dimSV = \infty \), in contrast to \( dimAV = 2 \). So the proofs differs for the case \( dimV = 1 \). Let us prove theorem 3.1 for the case \( dimV = 1 \).

Let \( x \neq 0 \in V; \rho x = \lambda x; \) hence: \( \tilde{\rho}(x^n) = \lambda^n x^n \). Let \( \partial_x \in V^* \) be dual to \( x \).

\[
Tr_{SV}(\tilde{\rho}A(x^k)C(x^k)) = Tr_{C[x]}(\tilde{\rho}) \frac{\partial^k}{\partial x^k} x^k = \sum_{n=0}^{\infty} \frac{(n+k)!}{k!} (\lambda)^n = \frac{\partial^k}{\partial \lambda^k} (1 - \lambda)^{-k} = (\tilde{\rho}A(\partial_x))^{k!} e^k = (\tilde{\rho}A(\partial_x))^{k!} e^k
\]

We have used obvious fact that \( Tr_{SV}(\tilde{\rho}) = \frac{1}{1 - \lambda} \). Note that one can rewrite it in the form: \( det_V \frac{1}{1 - \rho} = Tr_{SV}(\tilde{\rho}) \).

It is also evident that if \( k \neq l \), then \( Tr_{C[x]}(\tilde{\rho})^{k!} e^l = 0 \) as \( (\tilde{\rho})^{k!} e^l \).

So the case \( dimV = 1 \) is completed. Hence, proof 2 is finished.

We have seen that \( det_V \frac{1}{1 - \rho} = Tr_{SV}(\tilde{\rho}) \) in case \( dimV = 1 \). Similar as it was done in previous section one can prove it for any \( V \). Hence the following proposition is true:

**Proposition 3.2:** Let \( \rho \) be an operator with trace on space \( V \) and \( |\rho| < 1 \), then

\[
\frac{1}{1 - \rho} = \sum_{n=0}^{\infty} Tr_{SV}(\tilde{\rho})^{n!} V
\]

Let us formulate analogs of propositions 2.3-6

**Proposition 3.3:** Let \( e_i \) be arbitrary basis of space \( V \), \( \tilde{e}_i \) be the dual basis, then for any operator \( O \) on space \( S^nV \) the following formula takes place:

\[
Tr_{SV}(O) = \frac{1}{n!} \sum_{l_1 \ldots l_n} \langle \tilde{e}_{l_1} \tilde{e}_{l_2} \ldots \tilde{e}_{l_n} | O e_{l_1} e_{l_2} \ldots e_{l_n} \rangle
\]

(3.5)

**Corollary 3.2:** Let \( \rho \) be an operator on finite-dimensional space \( V \), \( \tilde{e}_i \) be the dual basis, \( \tilde{\rho} \) is induced operator on \( S^nV \). \( e_i \) - basis of \( V \). Then the following formula takes place:

\[
Tr_{S^nV}(\tilde{\rho}) = \frac{1}{n!} \sum_{l_1 \ldots l_n} \langle \tilde{e}_{l_1} \tilde{e}_{l_2} \ldots \tilde{e}_{l_n} | \rho e_{l_1} e_{l_2} \ldots e_{l_n} \rangle
\]

(3.6)

**Proposition 3.4** Let \( \rho \) be an operator with trace on space \( V \), \( \rho \) is induced operator on \( S^nV \), then \( Tr_{S^nV}(\tilde{\rho}) \) can be expressed through \( Tr_{SV}(\tilde{\rho}) \) as follows:

\[
Tr_{S^nV}(\tilde{\rho}) = \sum_{n_1 \geq 0} \frac{1}{n_1! n_2! \ldots n_n! 2^{n_1} \ldots n_n! \prod_l (Tr_{SV}(\tilde{\rho}))^{n_l}}
\]

(3.7)

**Proposition 3.5** Let \( \rho \) be an operator with trace on space \( V \); \( v_j, w_j \) arbitrary elements from \( V \) and \( V^* \) respectively. Then trace of \( \tilde{\rho} \prod_{l=1}^{k} A(w_l) \prod_{j=1}^{k} C(v_j) \) over space \( S^nV \) can be expressed through \( Tr_{SV}(\tilde{\rho}) \) and \( \langle w_l, v_j \rangle \) as follows:
Lemma is proved.

We shall show that the trace formula for the bosonic Fock space (theorem 3.1) leads to the formula of

Proof: consider the case

Using lemma one can easily obtain that

Actually:

Recall the definitions of Fredholm’s determinant and Fredholm’s minor:

Let $V = C[a,b]$ (precisely speaking one must consider $C^*[a,b];$ look remark 4.1). Consider integral equation $\phi(x) + \int K(x,y)\phi(y) = f(x)$. Let $\rho$ be an integral operator with kernel $K(x,y)$.

Recall the definitions of Fredholm’s determinant and Fredholm’s minor:

Considering the set $\delta(x-s)$, at $s \in [a,b]$ as "continuous basis" in $C[a,b]$ one can obviously see that the definition of the Fredholm’s determinant is absolutely analogous to the formula \[ D^{fred}(1 + \rho) = 1 + \int K(\xi , \xi )d\xi + \frac{1}{2!} \int d\xi_1d\xi_2... \rho \begin{vmatrix} K(\xi_1 , \xi_1 ) & K(\xi_1 , \xi_2 ) \\ K(\xi_2 , \xi_1 ) & K(\xi_2 , \xi_2 ) \end{vmatrix} \]

$D^{fred}_{s,t}(1 + \rho) = K(s,t) + \int d\xi_1 \rho \begin{vmatrix} K(s,t) & K(s,\xi_1 ) \\ K(\xi_1 , t) & K(\xi_1 , \xi_1 ) \end{vmatrix} + \frac{1}{2!} \int d\xi_1 d\xi_2... \rho \begin{vmatrix} K(s,t) & K(s,\xi_1 ) & K(s,\xi_2 ) \\ K(\xi_1 , t) & K(\xi_1 , \xi_1 ) & K(\xi_1 , \xi_2 ) \\ K(\xi_2 , t) & K(\xi_2 , \xi_1 ) & K(\xi_2 , \xi_2 ) \end{vmatrix} + \ldots$ \hspace{1cm} (4.2)

$D^{fred}_{s,t}(1 + \rho) = K(s,t) + \int d\xi_1 \rho \begin{vmatrix} K(s,t) & K(s,\xi_1 ) \\ K(\xi_1 , t) & K(\xi_1 , \xi_1 ) \end{vmatrix} + \frac{1}{2!} \int d\xi_1 d\xi_2... \rho \begin{vmatrix} K(s,t) & K(s,\xi_1 ) & K(s,\xi_2 ) \\ K(\xi_1 , t) & K(\xi_1 , \xi_1 ) & K(\xi_1 , \xi_2 ) \\ K(\xi_2 , t) & K(\xi_2 , \xi_1 ) & K(\xi_2 , \xi_2 ) \end{vmatrix} + \ldots$ \hspace{1cm} (4.2)

The action of an integral operator with continuous kernel can be canonically extended to the action on delta-functions: $\int K(x,y)\delta(x-s)dx = K(s,y)$, obtained function is continuous.

Lemma 4.1 Let $\rho$ be an integral operator with continuous kernel $K(x,y)$, then

Proof: consider the case $n = 1$ hence we need to prove that: $Tr\rho = \int d\xi K(\xi,\xi)$. This fact is naively obvious. In order to prove it rigorously we note that any continuous function $K(x,y)$ can be represented in the form: $K(x,y) = \sum \lambda_i f_i(x) g_i(y)$, where $\sum |\lambda_i| < \infty$, $|f_i(x)| < C, |g_i(y)| < C$, $f_i, g_i$ are continuous functions. By definition 2.2 trace $\rho$ equals to $\sum \lambda_i \int d\xi f_i(\xi) g_i(\xi)$, hence it’s clear that $Tr\rho = \int d\xi K(\xi,\xi)$. In order to prove the formula for the case $n > 1$ one should use the formula \[ D^{fred}_{s,t}(1 + \rho) = D^{fred}_{s,t} + \rho C(\delta(s-t)) \]

Lemma is proved.

Using lemma one can easily obtain that $D^{fred}(1 + \rho) = Det(1 + \rho) = Tr_{AV} \rho$

Analogically Fredholm’s minor equals to:

$D^{fred}_{s,t}(1 + \rho) = Tr_{AV} A(\delta(x-s))C(\delta(x-t))$ \hspace{1cm} (4.4)

Actually:

$Tr_{AV} A(\delta(x-s))C(\delta(x-t)) = \frac{1}{n!} \int d\xi_1...d\xi_n < \delta(x-\xi_1)\ldots\delta(x-\xi_n)||A(\delta(x-s))> \hspace{1cm} (4.3)$
\[ \rho C(\delta(x-t)) \delta(x-\xi_1) \land \ldots \land \delta(x-\xi_n) \geq \frac{1}{n!} \int \ldots \int d\xi_1 \ldots d\xi_n < \delta(x-s) \land \delta(x-\xi_1) \land \ldots \land \delta(x-\xi_n) \]

\[ |\rho(\delta(x-t)) \land \rho(\delta(x-\xi_1)) \land \ldots \land \rho(\delta(x-\xi_n)) > \]

\[ = \frac{1}{n!} \int \ldots \int d\xi_1 \ldots d\xi_n \det \begin{vmatrix} K(s,t) & K(s,\xi_1) & \cdots & K(s,\xi_n) \\ K(\xi_1,t) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_n,t) & K(\xi_n,\xi_1) & \cdots & K(\xi_n,\xi_n) \end{vmatrix} \]

In order to make the reasonings above rigorous, let us note three facts. The first, \( \delta(x-s) \notin C[a,b] \), hence there is no problem in defining \( C(\delta(x-s)) \), but due to \( K(x, y) \) is continuous function the product \( \rho C(\delta(x-s)) \) is well-defined. The second, one can naturally extend the action of \( C(\delta(x-s)) \) and \( A(\delta(x-t)) \) on the delta-functions: \( A(\delta(x-t))|\delta(x-r)| = \delta(x-r) \). The third, operator \( A(\delta(x-s)) \per C(\delta(x-t)) \) on the space \( \Delta C[a,b] \) can be represented in the form: \( \sum_i \rho_i \leq L + \sum_i \rho_i T_i \), where \( \rho_i \) are operators with continuous kernels, \( T_i \) are operators of the form: \( T_i = t_i(x) \otimes \delta(x-t_i) \). Where \( t_i \) are continuous functions, \( \delta(x-t_i) \) are considered as functional on \( C[a,b] \). Hence that operator is operator with trace. And it’s easy to see that lemma 4.1 holds for this operator also, if action of this operator is defined on delta-functions as we have just described.

One can prove lemma 4.1 in another way. Really, expression

\[ \frac{1}{n!} \int \ldots \int d\xi_1 d\xi_2 \ldots d\xi_n \leq \delta(x-\xi_1) \land \ldots \land \delta(x-\xi_n) \bar{\rho}(\delta(x-\xi_1) \land \ldots \land \delta(x-\xi_n) > \]

can be transformed, in the same way, as we have done proving the proposition 2.4. But one should write integrals instead of sums. After the transformations one obtains:

\[ \sum_{j_1, \ldots, j_n = 0} \frac{1}{j_1! j_2! \ldots j_n!} \Pi_j (-1)^{j_1} \Pi_j K(\xi_1, \xi_2) K(\xi_1, \xi_3) \ldots K(\xi_1, \xi_n) n_i. \]

Applying Merser’s theorem \( (Tr \rho = \int dK(\xi, \xi)) \). One gets:

\[ \sum_{j_1, \ldots, j_n = 0} \frac{(-1)^n}{j_1! j_2! \ldots j_n!} \Pi_j (Tr \rho^j) \]

which is equal to \( Tr_{X^N} \bar{\rho} \) according to proposition 2.4. Proof is finished.

One can similar prove the formula \[ \square \] using the proposition 2.5 instead of 2.4.

Thus we obtained the following proposition:

**Proposition 4.1:**

\[ D_{fred}^f (1 + \rho) = Tr_{AV} \bar{\rho} = Det(1 + \rho) \] (4.5)

\[ D_{fred}^f (1 + \rho) = Tr_{AV} A(\delta(x-s)) \per C(\delta(x-t)) \] (4.6)

Hence, theorem 3.1 gives us expression for the matrix elements of the operator \( \frac{1}{1 + \rho} \):

\[ \langle \delta(x-s) \rangle \frac{1}{1 + \rho} \delta(x-t) = \frac{Tr_{AV} A(\delta(x-s)) \per C(\delta(x-t))}{Tr_{AV} \bar{\rho}} = \frac{D_{fred}^f (1 + \rho)}{D_{fred}^f (1 + \rho)} \]

\[ \langle \delta(x-s) \rangle \frac{1}{1 + \rho} \delta(x-t) = \delta(x-s) \delta(x-t) > \]

\[ \langle \delta(x-s) \rangle \frac{1}{1 + \rho} \delta(x-t) > \]

\[ \langle \delta(x-s) \rangle \frac{1}{1 + \rho} \delta(x-t) > \]

(4.7)

Thus, one obtains Fredholm’s formula for the solution of integral equation \( \phi(x) + \int K(x,y) \phi(y) = f(x) \):

\[ \phi(s) = f(s) - \int dt \frac{D_{fred}^f (1 + \rho)}{D_{fred}^f (1 + \rho)} f(t) \] (4.8)

Really, \( \phi(x) = \frac{1}{1 + \rho} f(x) \), hence, due to formula \[ \square \] \( \phi(s) = f(s) - \int dt \frac{D_{fred}^f (1 + \rho)}{D_{fred}^f (1 + \rho)} f(t) \)

Remark 4.1: Simpler to obtain formula \[ \square \] using not the \( Tr_{AV} \bar{\rho} A(\delta(x-s)) C(\delta(x-t)) \), but the trace: \( Tr_{AV} \bar{\rho} A(\delta(x-s)) C(f(x)) \). Handling this way, we shall obtain the same formula, but we would not leave the space \( C[a,b] \), as it happened when we consider \( Tr_{AV} \bar{\rho} A(\delta(x-s)) C(\delta(x-t)) \) (note that \( \delta(x-t) \notin C[a,b] \)). But we wanted to emphasize analogy between our formulas and Fredholm’s ones, so we have admitted ourselves some inaccurateness.

Theorem 3.1 is analog of theorem 2.1 for bosonic Fock space. Using it one can obtain formulas similar to \[ \square \] But expressions involves not determinants, but permanents.

Let us consider an integral equation \( \phi(x) - \int K(x,y) \phi(y) = f(x) \), at condition \( |\rho|^n \to 0 \). (Where \( \rho \) is an integral operator with kernel \( K(x,y) \).)

Let us define \( P_{fred}^f (1 - \rho) \) and \( P_{fred}^f (1 - \rho) \) as follows:

\[ P_{fred}^f (1 - \rho) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \ldots \int d\xi_1 d\xi_2 \ldots d\xi_n \ per |K(\xi_1, \xi_2)|_{i,j \leq n} = \sum_{n=0}^{\infty} Tr_{AV} \bar{\rho} = det_{AV} \frac{1}{1 - \rho} \] (4.9)
\[ P_{s,t}^{fr\prime}(1 - \rho) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int d\xi_1 d\xi_2 \cdots d\xi_n \text{ per} \]
\[
\begin{pmatrix}
K(s) & K(s, \xi_1) & \ldots & K(s, \xi_n) \\
K(\xi_1, t) & K(\xi_1, \xi_1) & \ldots & K(\xi_1, \xi_n) \\
\vdots & \vdots & \ddots & \vdots \\
K(\xi_n, t) & K(\xi_n, \xi_1) & \ldots & K(\xi_n, \xi_n)
\end{pmatrix}
\]
\[
= \sum_{n=0}^{\infty} Tr_{SV} A(\delta(x - s)) \check{\rho} C(\delta(x - t))
\]

(4.10)

Hence theorem 3.1 gives us the formulas for matrix elements of the inverse operator:

\[
< \delta(x - s) | \frac{\rho}{1 - \rho} \delta(x - t) > = \frac{Tr_{SV} A(\delta(x - s)) \check{\rho} C(\delta(x - t))}{Tr_{SV} \check{\rho}} = \frac{P_{s,t}^{fr\prime}(1 - \rho)}{P_{fr\prime}(1 - \rho)}
\]

(5.10)

\[
< \delta(x - s) | \frac{1}{1 - \rho} \delta(x - t) > = < \delta(x - s) | \delta(x - t) > + \frac{P_{s,t}^{fr\prime}(1 - \rho)}{P_{fr\prime}(1 - \rho)}
\]

Hence, we obtain the analog of Fredholm’s formula for the solution of integral equation \( \phi(x) - \int K(x, y) \phi(y) = f(x) \):

\[
\phi(s) = f(s) + \int dt \frac{P_{s,t}^{fr\prime}(1 - \rho)}{P_{fr\prime}(1 - \rho)} f(t)
\]

(5.1)

Remark: maybe, the name "Fredholm’s permanents" is not acquitted, because \( P_{fr\prime}(1 - \rho) = Tr_{SV} \check{\rho} = det_V \frac{1}{1 - \rho} \). The last equality is Proposition 3.6.

5 Traces of intertwining operators.

In this section we apply general formulas obtained above for the the computing the traces of some concrete operators, which encounters during the solution of integrable models. Proposition 4 is used in paper [KLP2] for the computing formfactors of \( SU(2) \)-invariant Thirring model. It easily follows from the Proposition 2. Formula from the proposition 2 was suggested by the author’s scientific advisor S.M. Khoroshkin. And the proof of this formula was the initial purpose of this paper. Further the author came to the general formulas (theorems 2.1,3.1).

Actually, traces which we calculate in this section, turn out to be divergent that is why it is necessary to regularize them somehow. We carried out the discussion of regularization in appendix 2. Also on this moment it’s not known explicit topological description of the Fock space where the operators acts. But it’s clear that this space is subspace of the \( \mathbb{C}[[a_1, a_2, ...]] \) and monomials \( (a_1)^{i_1} (a_2)^{i_2} \cdots (a_k)^{i_k} \) are the basis of this space, in some sense of the word "basis". That is why one may understand these operators as infinite matrices, and the trace as the sum of diagonal elements, our theorem is valid and in this situation, this is discussed in appendix 1. In this section, we do not discuss all these problems but formally apply theorem 3.1 and its corollary 3.1.

Let \( V \) be a linear space.

Let \( \alpha(u), \beta(v) \) be functions with values in \( V^* \), \( V \) respectively, such that pairing between them is equal \( g(u - v) \).

Let operator \( e^{\gamma d} \) acts on \( V \) so that its pointwise action on functions can be written as follows: \( e^{\gamma d} \beta(z) = \beta(z + \gamma) \)

\[
(e^{\gamma d})^* \alpha(z) = \alpha(z - \gamma).
\]

Proposition 5.1.

\[
\frac{1}{Tr_{SV} e^{\gamma d}} Tr_{SV} e^{\gamma d} e^{\alpha(u)} e^{\beta(v)} = \sum_{n=0}^{\infty} e^{g(u - v - n \gamma)}
\]

(5.1)

As it have been already said, the traces of similar type are used for the solution of integrable models. This proposition is useful, because it provides the expression for the trace in terms of pairing only.

One can obtain the proof by simply applying corollary 3.1.

Let \( a_n \in V, a_{-n} \in V^* \) such elements that \( < a_{-m} | a_n > = na_{n,m} \).

Let \( \tilde{a}_+(u) = \sum_{i=1}^{\infty} \frac{a_{-i}}{i} u^{-i} \)

\( \tilde{a}_-(v) = \sum_{i=1}^{\infty} \frac{a_i}{i} v^i \).

Obviously \( < \tilde{a}_-(v), \tilde{a}_+(v) > = -log(1 - \frac{v}{u}) \).

Proposition 5.2.

\[
\frac{Tr_{SV}(e^{\gamma d} \prod_{i=1}^{l} \exp(k_i \tilde{a}_-(\alpha_i)) \prod_{j=1}^{j} \exp(l_j \tilde{a}_+(\beta_j)))}{Tr_{SV}(e^{\gamma d})} = \prod_{m=1}^{\infty} \prod_{i,j} (\beta_j - \alpha_i - m\gamma)^{-k_i l_j}
\]

(5.2)
where \( \sum k_i = 0 \sum l_j = 0 \).

Proof easily follows from corollary 3.1.

Note that obtained infinite product is convergent. Really, this follows from following proposition:

\[
\prod_{k_1, \ldots, k_n \geq 0} \prod_{p} \frac{a_m + \sum k_j \omega_j}{b_p + \sum k_j \omega_j}
\]

(5.3)

Is alighted when:

\[
\sum (a_m)^q = \sum (b_p)^q, \quad q = 0, 1, \ldots, n
\]

(5.4)

We prove this proposition in appendix 5.

And the conditions of this proposition are fulfilled, since \( \sum k_i = 0 \sum l_j = 0 \).

Actually, in physical applications one uses not the space of polynomials \( SV \subset \mathbb{C}[a_1, a_2, \ldots] \), but the sum of the infinite numbers of the copies of this space: \( F = \bigoplus_{M \in \mathbb{Z}/2} F_M \), where \( F_M^* = \mathbb{C}[a_{-1}, \ldots, a_{-n}, \ldots] \otimes (\mathbb{C}e^{M_0}) \).

Let us define the action of operator \( p \) as follows:

\[
p(a_{i_1}a_{i_2} \ldots a_{i_m} \otimes e^{u_0}) = 2n(a_{i_1}a_{i_2} \ldots a_{i_m} \otimes e^{u_0})
\]

(5.5)

Let us define the following functions: \( a_+(u) = \sum_{i=1}^{\infty} \frac{u}{i} + plog(u) \quad a_-(v) = \sum_{i=1}^{\infty} \frac{v}{i} + \frac{v_0}{2} \)

Let an operator \( e^{\gamma d} \) acts on the space \( F \) so that:

\[
e^{\gamma d} a_-(v) = a_-(v + \gamma)
\]

(5.6)

**Proposition 5.3:** Let us define operator \( O \) on space \( F \) as follows:

\[
O = \prod_{j=1}^{a} \exp(\gamma_j a_+(\beta_j)) \prod_{l=1}^{d} \exp(k_l a_-(\alpha_l)) \quad \text{where} \quad \sum k_i = 0 \sum l_j = 0
\]

Then takes place the following formula for his trace:

\[
\frac{Tr_F e^{\gamma d} O}{Tr_F e^{\gamma d}} = \prod_{m=0}^{\infty} \prod_{i,j} (\beta_j - \alpha_i - m\gamma)^{-k_i l_j}
\]

(5.7)

**Proof:**

It is obvious that due to condition \( \sum k_i = 0 \), each subspace \( F_M \subset F \) is invariant under the action of operator \( O \).

Due to proposition 2:

\[
\frac{Tr_{F_{\gamma d}} e^{\gamma d} O}{Tr_{F_{\gamma d}}} = \prod_{m=1}^{\infty} \prod_{i,j} (\beta_j - \alpha_i - m\gamma)^{-k_i l_j}
\]

Let us show that operator \( O e^{\gamma d} \) on the subspace \( F_M \) is equivalent to the operator \( \prod_j \beta_j^{2ML_j} O e^{2M \gamma d} \gamma d \) on subspace \( F^0 \), i.e. corresponding matrix elements are equal.

Thus, we have obtained, in particular, the following proposition:

**Lemma:** \( Tr_{F_M} e^{\gamma d} O = \prod_j \beta_j^{2ML_j} Tr_{F_{\gamma d}} e^{\gamma d} O e^{\gamma d} \)

By the direct computation, using Corollary 3.1, and making use of the conditions: \( \sum k_i = \sum l_j = 0 \) we obviously get

\[
\prod_j \beta_j^{2ML_j} Tr_{F_{\gamma d}} e^{\gamma d} O e^- = (Tr_{F_{\gamma d}})^{-k_i l_j}
\]

Hence, expression in LHS is not dependent on \( M \).
Thus, we see that traces of operators $e^{\gamma d} O_i e^{\gamma d}$ over all spaces $F^M$ are the same. Thus, their quotient, just like it was required in proposition 3 equals to:

$$\prod_{m=1}^{\infty} \prod_{i,j} (\beta_j - \alpha_i - m \gamma)^{-k_{i,j}}$$

(5.8)

Therefore, the quotient of the traces of these operators over the space $F = \bigoplus M F^M$ is equal to 5.8.

**Proof is finished.**

Let us introduce the following generating function:

$$\eta_+(z) = \lim_{K \to \infty} (2hK)^{-p/2} \sum_{k=0}^{K} e^{a_+(z-2kh) - a_+(z-h-2kh)}$$

(5.9)

It can easily be checked that it’s well-defined.

In paper [KLP1] it was shown that intertwining operators for basic representations of central extended Yangian double of $\mathfrak{sl}_2$ can be written as follows: $e^{\gamma_i a_-(\alpha)} \eta^\gamma_{\pm}(\alpha \pm h)$, where $\epsilon_i = \pm 1$

In paper [KLP2] the formfactors of $SU(2)$-invariant Thirring model were expressed through the traces of the products of intertwining operators. In that paper the following formula have been used:

**Proposition 5.4:**

$$\frac{\text{tr}_F \left( e^{\gamma d} \prod_j e^{k_j a_-(w_j)} \prod_k \rho A_k (z_k) \right)}{\text{tr}_F e^{\gamma d}} = \prod_{m=1}^{\infty} \prod_{i,j} \frac{(z_k - w_j - h - m \gamma - 2kh)^{k_j}_{p_k}}{(z_k - w_j - m \gamma - 2kh)^{k_j}_{p_k}}$$

where $\sum_j k_j = 0 \sum k p_k = 0$

Due to 5.3 and $\sum_j k_j = 0 \sum_k p_k = 0$ the product in RHS of the equality is convergent.

To prove the proposition one should substitute the definition of $\eta_+(z)$ (5.9) to the proposition 5.3.

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**Appendixes**

A.1 **Space with basis.**

In main text we have proved theorems for the normed spaces. In this appendix we shall show that if one consider infinite-size matrices then theorems are also valid, at some conditions on matrixes.

Note that one can consider infinite-size matrixes as operators acting: $C_0^\infty \to C^\infty$. Where $C_0^\infty$ is space of finite sequences, $C^\infty$ is the space of all sequences. Certainly, the product of two infinitely size matrixes is not always well-defined.

The trace of infinitesize matrix is sum of its diagonal elements.

Let $v, w \in C^\infty$. Define pairing as follows: $\langle v|w \rangle = \sum_i v_i w_i$

We shall say that trace, product, applying matrix on vector, pairing are WELL-DEFINED, iff the series arising in this operations are absolutely convergent.

One defines algebras $SC_0^\infty$, $SC^\infty$, $AC_0^\infty$, $AC^\infty$, in the same way as it was done above for normed $V$. For any $v \in C^\infty$ one defines creating-annihilating operators $A(v), C(v)$ acting:$C_0^\infty \to C^\infty$.

Theorems 2.1 and 3.1 are true in this case:

**Theorem** Let $\rho$ be an infinitesize matrix, $w, v \in AC^\infty$. Assume that $\rho^n, Tr \rho^n, \langle w|\rho^n v \rangle$, $\langle w|1/\rho^n v \rangle$ are well-defined. Then $Tr_{AC_0^\infty \to AC^\infty} \hat{\rho} A(w) C(v)$ and $Tr_{AC_0^\infty \to AC^\infty} (\hat{\rho})$, are well-defined too and the following formula takes place:

$$\frac{Tr_{AC_0^\infty \to AC^\infty} (\hat{\rho} A(w) C(v))}{Tr_{AC_0^\infty \to AC^\infty} (\hat{\rho})} = \left\langle w | \frac{1}{1 + \rho} v \right\rangle$$

(A.1)

**Theorem** Let $\rho -$ be an infinitesize matrix, $w, v \in SC_0^\infty$. Assume that $\rho^n, Tr \rho^n, \langle w|\rho^n v \rangle$ are well-defined and series $\sum_i Tr \rho^i$. $\sum_i \langle w|\rho^n v \rangle$ are absolutely convergent. Then $Tr_{SC_0^\infty \to SC^\infty} \hat{\rho} A(w) C(v)$ and $Tr_{SC_0^\infty \to SC^\infty} (\hat{\rho})$ are well-defined and the following formula takes place:

$$\frac{Tr_{SC_0^\infty \to SC^\infty} (\hat{\rho} A(w) C(v))}{Tr_{SC_0^\infty \to SC^\infty} (\hat{\rho})} = \sum_{n=0}^{\infty} \langle w|\rho^n v \rangle$$

(A.2)
A.2 Regularization of the traces.

It is possible that in theorem 3.1 LHS of equality does not exist while RHS exists. That is why naturally to give the following definition.

Definition:

\[
\text{Reg} \frac{T_{SV}(\bar{\rho}A(w)C(v))}{T_{SV}(\bar{\rho})} = \begin{cases} \langle w|\frac{1}{1-\rho}v \rangle \\ \sum_{n=0}^{\infty} \langle w|\rho^n v \rangle \end{cases}
\]

We choose that value in RHS which exists, if both of them exists then they are obviously equal to each other.

According to proposition 3.2, \(T_{SV}\bar{\rho} = det_v \frac{1}{1-\rho}\) when \(|\rho| < 1\). That is why naturally to introduce the following definition.

Definition: \(\text{Reg}T_{SV}\bar{\rho} = det_v \frac{1}{1-\rho}\) if \(\frac{1}{1-\rho}\) exists.

The traces of such type are used for computing formfactors, in quantum field theory and statistical physics. In quantum field theory models traces usually turns out to be divergent, though we do not have rigorous proof that our regularization coincides with scaling limit from corresponding lattice model or satisfies Smirnov’s axioms ([31]) on formfactors or agreed with ultraviolet cut-off ([32]), but the concrete example of computing for \(SU(2)\)-invariant Thirring model shows that all approaches lead to the same results.

Further we shall show that in some cases such regularization coincides with certain limit of the quotient of traces, and in another cases with regularization by means of zeta function (if one exists, though, for \(SU(2)\)-invariant Thirring model or Sine-Gordon model such regularization through zeta function does not exist.)

regularization by passage to the limit.

Let us consider the operator \(\rho = e^{id}\) that was used in section 5. In natural basis \(a_1, a_2, \ldots\) it has the form:

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots \\
\gamma & 1 & 0 & \ldots \\
\frac{\gamma^2}{\pi} & \gamma & 1 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

and obviously \(T_{SV}\rho = \infty\), hence \(T_{SV}\bar{\rho}A(w)C(v) = \infty\), \(T_{SV}\bar{\rho} = \infty\). But nevertheless, for concrete \(w, v\) that were used in section 5, quotient of these traces was equal to \(\sum a_i < w|\rho^i(v)\) and was convergent. Further on we shall give yet another argument in favour that, in this case definition \(\text{Reg} \frac{T_{SV}(\bar{\rho}A(w)C(v))}{T_{SV}(\bar{\rho})} = \sum_{n=0}^{\infty} \langle w|\rho^n v \rangle\) is acquitted.

Let us introduce the operator \(T_i\) such that \(T_i(a_i) = (t)^i a_i\). Obviously, at \(0 < t < 1\) operators \(T_i, T_{SV}\) are operators with trace on the spaces \(V\) and \(SV\), respectively. \(T_{SV}T_i = \frac{1}{t^{n+1}}\), \(T_{SV}T_i = \frac{1}{t^n}\). Also, it is clear that \(T_i\rho\) is operator with trace and \((T_i\rho)^i \to 0\) as \(i \to \infty\), hence, \(\bar{T}_{SV}\rho\) is operator with trace on \(SV\). Diagonal elements of \(\bar{T}_{SV}\rho\) in basis \(a_1^k, a_2^k, a_3^k, \ldots\) tends to diagonal elements of \(\bar{\rho}\) as \(t \to 1\), hence, it’s natural to consider \(\text{Reg}(\bar{T}_{SV}\rho\bar{\rho}) = \lim_{l \to 1} \frac{T_{SV}(\bar{T}_{SV}\rho\bar{\rho}A(w)C(v))}{T_{SV}(\bar{\rho})} = \frac{\lim_{l \to 1} \sum a_i}{T_{SV}(\bar{\rho})}\).

Proposition: Assume that \(\lim_{l \to \infty} \langle \rho H_n \rangle^i = 0\) and series \(\sum_{n=0}^{\infty} \langle w|\rho^n v \rangle\) is convergent. Then \(T_{SV}H_n\bar{\rho} < \infty, T_{SV}H_n\bar{\rho}A(w)C(v) < \infty\) and

\[
\lim_{n \to \infty} \frac{T_{SV}(H_n\bar{\rho}C(w)C(v))}{T_{SV}(H_n\bar{\rho})} = \sum_{n=0}^{\infty} \langle w|\rho^n v \rangle \quad (A.3)
\]

Proof.

Proof is based on the observation that one can apply the theorem to to operators \(H_n\rho\), because \(\lim_{l \to \infty} \langle \rho H_n \rangle^i = 0\). Hence:

\[
\lim_{n \to \infty} \frac{T_{SV}(H_n\bar{\rho}C(w)C(v))}{T_{SV}(H_n\bar{\rho})} = \lim_{n \to \infty} \sum_{n=0}^{\infty} \langle w|\rho \bar{\rho}^n v \rangle = \sum_{n=0}^{\infty} \langle w|\rho^n v \rangle \quad (A.4)
\]
Last passage to the limit is acquitted, because $\tilde{\rho}^n \mathbf{v}$ is convergent, hence the set $\tilde{\rho}^n \mathbf{v} \quad n > 0 \in \mathbb{Z}$ is precompact set, consequently $H_n \to \text{id}$ is convergent uniformly on this set, so on can pass to the limit.

**Proposition is proved.**

Note that, apparently, such regularization is not the limit of quotient of traces over finite-dimensional subspaces exhausting every bit of $V$. If $\text{Tr}_{SV} \tilde{\rho} A(w)C(v) = \infty \quad \text{Tr}_{SV} \tilde{\rho} = \infty$ then, apparently, the limit of their quotient over finite-dimensional subspaces is is equal $\infty$ too.

For example consider the case: $\text{dim} V = 1, \rho = \text{id},$

\[
\lim_{N \to \infty} \sum_{n=0}^{N} \frac{\text{Tr}_{SV} \tilde{\rho}^n A(w)C(v)}{\text{Tr}_{SV} \tilde{\rho}^n} = \infty
\]

**Regularization by means of the analytical continuation.**

Regularization by means of passage to the limit was possible, when $|\rho| > 1$ then regularization through the limit is impossible.

It’s possible another variant of regularization by means of analytical continuation, in particular by means of zeta function, we shall show, that this variant coincides with our main regularization. Though note that in the previous example such method of regularization is not applicable.

Consider $\zeta(s) = \text{Tr}_{SV} \tilde{\rho}^{-s}$, assume that for $s$ large enough $\text{Tr}_{SV} \tilde{\rho}^{-s}$ is defined and assume that this function can be continued to meromorphic on the entire plane. Then we shall define $\text{Reg}_\tilde{\rho} \text{Tr}_{SV} \tilde{\rho} = \zeta(1).$ Note that typically $\text{Tr}_{SV} \rho^s$ has pole in unit. But $\text{Tr}_{SV} \tilde{\rho}^{-s} = \text{Det}_V \frac{1}{1-\rho}$ and this function is analytical in point $s = 1$, if $1 \notin \text{Spec } \rho$. Similar: we can define $\tilde{\zeta}(s) = \text{Tr}_{SV} \tilde{\rho}^{-s} A(w)C(v)$, and put $\text{Reg}_\tilde{\rho} \text{Tr}_{SV} \tilde{\rho} A(w)C(v) = \tilde{\zeta}(1)$.

Let $s$ be sufficiently large, then:

\[
\text{Reg}_\tilde{\rho} \frac{\text{Tr}_{SV} (\tilde{\rho}^s A(w)C(v))}{\text{Tr}_{SV} (\tilde{\rho}^s)} = \frac{\text{Tr}_{SV} (\tilde{\rho}^s A(w)C(v))}{\text{Tr}_{SV} (\tilde{\rho}^s)} = \left\langle w | \frac{1}{1-\rho^s} v \right\rangle
\]

(\text{A.5})

Function $\zeta(s)$ is analytical, hence, if equality is true at $s > N$ that it is true and always, i.e. $\text{Reg}_\tilde{\rho} \frac{\text{Tr}_{SV} (\tilde{\rho}^N A(w)C(v))}{\text{Tr}_{SV} (\tilde{\rho})} = \tilde{\zeta}(1) \frac{\text{Tr}_{SV} (\tilde{\rho}^N A(w)C(v))}{\text{Tr}_{SV} (\tilde{\rho})} = \left\langle w | \frac{1}{1-\rho^N} v \right\rangle$ = $\text{Reg}_\tilde{\rho} \text{Tr}_{SV} (\tilde{\rho}^N A(w)C(v))$ = \tilde{\zeta}(1)

The author thinks that the same situation will be with any other analytical regularization, because we always want to deform our operator somehow to the area, where traces are defined, continue the function from this area to the hole plain. But in the area, where traces are defined our formula is true, hence it will be true on the hole plain, because if two analytical functions are equal in some area , they are equal on the hole plain.

**A.3 Continuos basis.**

Considering the models of the quantum field theory, for example Sine-Gordon model \cite{KLP3, JKM}, it’s necessary to consider free bosons $a_{\lambda}$, where $t \in \mathbb{R}$. $a_{\lambda}$ plays role of ”continuous basis” in space $V$. Instead of giving rigorous definition of continuous basis, we prefer to consider two examples. The first is the following: any function belongs $L_2(\mathbb{R})$ one can expand to the Fourier’s integral: $f(x) = \int f(\lambda) e^{i\lambda x} d\lambda$ i.e. $e^{i\lambda x}$ at $\lambda \in \mathbb{R}$ is a continuous basis $L_2(\mathbb{R})$; the second: for any continuous function $f(x)$ holds that $f(x) = \int f(y) \delta(y-x) dy$ i.e. $\delta(y-x)$ at $x \in [a,b]$ is continuous basis in $C[a,b]$. Common in this two examples is the following: basic functions do not belong the space, but the elements of the space can be expressed as follows: $\int d\lambda \delta(\lambda) \epsilon_\lambda$, where $\epsilon_\lambda$ is continuous basis, $\delta(\lambda)$ is certain class of scalar functions determining the space.

In the case of continuously indexed bosons there must exist similar description of space $V$. Space $V$ is the set of the following elements: $\int d\lambda(\lambda) a_\lambda$, where $\delta(\lambda)$ belongs certain class of scalar functions. Usually $V$ is a Hilbert space, respectively, our theorem will be valid in this case.

If $a_{\lambda}$ is continuos basis of $V$, then trace of any operator $\rho$ on $V$ can be found by the formula: $\text{Tr}_{SV} \rho = \int_{\lambda > 0} < a_{\lambda} | \rho a_{\lambda} > d\lambda$, in case when integral is convergent.

Respectively, the trace over space $SV$, can be found by the formula similar to proposition 3.3:

\[
\text{Tr}_{SV} O = \sum_{n=0}^{\infty} \int_{\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \leq 0} \ldots \int_{\lambda_1 \leq \lambda_n \leq 0} < a_{\lambda_1} a_{\lambda_2} \ldots a_{\lambda_n} | O a_{\lambda_1} a_{\lambda_2} \ldots a_{\lambda_n} > \]

One can consider this equalities as definitions of traces. Theorem 2.1, 3.1 are valid, because one can apply proof 3, substituting sums for integrals.

**A.4 Traces and generating functions.**

In this appendix we shall introduce the ”by-product” of our work: initially the proof of our main results (theorems 2.1, 3.1) used the propositions below, but then we understood that they are not necessary. But may be it will be interesting for somebody.

As it is known, considering affine algebras it’s convenient to work with generating functions i.e. with functions with values in algebra or the space of its representation. For example, it’s easier to write down the commutation relations in terms of generating functions, but not the concrete generators.
Initially, our purpose was to prove the concrete proposition \([5.2]\). In this proposition we considered \(v = a_-(\alpha), w = a_+(\beta)\). \(a_\pm(z)\) are the generating functions, the action of operator \(\rho = e^{\gamma d}\) can be easily written in terms of generating functions, but rather complicated in terms of concrete generators. Therefore it was natural to try to express the trace in terms of generating functions.

So the main purpose of this section to find expressions for the traces in terms of generating functions. Turns out to be, that it’s possible to each operator on space \(V\) assign some integral operator acting on generating functions. Generating functions plays role of the "continuous basis" and the trace can be expressed as some integral.

We shall only formulate the propositions, we shall not prove them, because the proofs are trivial.

Let \(V\) be a linear space, \(\dim V = N \leq \infty\). \(V^*\) - its dual. Let \(v_i\) \(0 \leq i \leq N - 1\) be a basis in \(V\); \(w_j\) \(1 \leq j \leq N\) - basis in \(V^*\).

Let us denote by: \(\beta(x) = \sum_{j=1}^{N} w_j(x)^{-j}\alpha(y) = \sum_{i=0}^{N-1} v_i(y)^i\).

Let us introduce "identical pairing tensor":

\[
d(x, y) = \frac{1}{x} \sum_{i=0}^{N-1} \frac{y^i}{x^i} \tag{A.6}
\]

**Proposition A. 4.1:** Vectors \(v_i, w_j\) are the dual basis in \(V, V^*\) respectively, iff \(<\beta(x)\alpha(y)> = d(x, y)\).

**Definition:** We shall define the formal integral from power series \(F(x)\) as follows: \(\int F(x)dx\) equals to coefficient at \(\frac{1}{x}\).

**Proposition A. 4.2** Let \(A\) be an operator on space \(V\). The function: \(\tilde{A}(x, y) = \sum_{0 \leq i,j \leq N-1} a^i_j \frac{x^y}{x^i}\) corresponds to \(A\), where \(a^i_j\) are matrix elements of the operator \(A\) in basis \(v_i\). Then the following formulas holds:

\[
(\text{id})(x, y) = d(x, y) \tag{A.7}
\]

\[
A(\alpha(z)) = \int \tilde{A}(x, z)\alpha(x)dx \tag{A.8}
\]

\[
\overline{AB}(x, y) = \int \tilde{B}(x, z)\tilde{A}(z, y)dz \tag{A.9}
\]

\[
TrA = \int A(x, x)dx \tag{A.10}
\]

\[
\sum_i w_i \otimes v_i(x, y) = <\beta(x)|\alpha(y)> \tag{A.11}
\]

**Remark 1:** Formal integral defined above, coincides with ordinary integral, if we choose the appropriate contour of integration. In the case \(\dim V < \infty\) one can choose arbitrary contour which includes the zero and not includes \(\infty\). In case \(\dim V = \infty\) the functions \(\alpha(z), \beta(z), A(x, y)\) may have some poles, therefore one must choose contour with respect to them. General principle is the following: typically we consider integrals from the following expressions: for the integral: \(\int F_1(z)F_2(z)dz\), where \(F_1(z)\) is the formal power series of \(\frac{1}{z}\), \(F_2(z)\) is formal power series of \(z\), contour must contain singular points of \(F_1(z)\) and does not contain singular points of \(F_2(z)\). In examples under consideration only integrals of such type arise. For example in formula \(\frac{1}{x}\) contour must be selected as follows: at each \(x\) contour must contain all singular points for \(A(x, z)\) and does not contain all singular points for \(\alpha(z)\). In proposition A.4.3 contour must contain singular points for \(\beta(z)\) and does not contain singular points for \(A(\alpha(z))\). Note that that singular points for functions \(A(\alpha(z))\) and \(\alpha(z)\) may be different, since, operator \(A\) is not obliged to be bounded. Also we want to note that broadly speaking, it’s not necessarily to require the analytical continuation of functions \(A(\alpha(z))\), \(\beta(z)\) to the hole plain, it’s sufficient to require that areas where they not analiticaly one can separate by a contour, and integrate over it.

**Remark 2:** In this section we mean that \(TrA\) is sum of diagonal elements in basis \(v_i\). In the case \(\dim V = \infty\) we shall consider only such operators \(A\) that sum of their diagonal elements is absolutely convergent.

**Proposition A. 4.3** Let us assume that \(<\beta(x)|\alpha(y)> = d(x, y)\), then for the trace of any operator \(A\) over space \(V\) takes place the formula:

\[
Tr_{V^*}A = \int <\beta(x)|\alpha(x)> dx \tag{A.12}
\]

**Remark:** if \(e_i, e^i\) are dual basis in \(V\) and \(V^*\) respectively, then for any operator \(A\) holds that: \(TrA = \sum_i <e^i|Ae_i>\). It’s clear that formula \(\frac{1}{A}\) is the analogous to above formula of linear algebra, if one considers \(\alpha(z), \beta(z)\) as dual basis indexed by \(z\). One may be surprised why we call by dual basis, such basis that pairing between them is equal \(d(u, v) = \sum_{v} \frac{N-1}{v-n}\), but not the delta-function \(\delta(u, v)\). But \(d(u, v)\) may be considered as \(\delta(u, v)\), since \(\int d(u, v)P(u)du = P(v)\) for any polinomial \(P(u)\) of degree not exceeding \(N - 1\). This property is similar to the one of delta-function. One can see that this is the only property, which is necessary.
Proposition A.4.4. Let us denote by \( g(x, y) = \langle \beta(x)|\alpha(y) \rangle \). Due to the property \[ g(x, y) \] is the "kernel" of operator \( \sum_i w_i \otimes v_i \). Let \( g^{-1}(x, y) \) be the "kernel" of the inverse operator. Note, that integral operators with "kernels" \( g(x, y) \) and \( g^{-1}(x, y) \) mutually inverse.

\[
Tr V A = \int \int g^{-1}(x, y) < \beta(x)|A\alpha(x) > dx dy
\] (A.13)

Remark: this Proposition is the analogue of following simple fact of linear algebra: \( Tr A = \sum_i g_i^j \langle w^j|Av_i \rangle \), where \( g_i^j = < w^j|v_j > \).

In some cases \( g^{-1}(x, y) \) can be found by the following simple method.

Proposition A.4.5 Let us that \( g(x, y) = A_y d(x, y) \). Where \( A_y \) is self-adjoint operator acting on functions from variable \( y \). Then \( g^{-1}(x, y) = A_y^{-1}d(x, y) \)

Proof:

\[
\int g^{-1}(z, y)g(x, y)dy = \int A_y^{-1}d(z, y)A_y d(x, y)dy =
\]

\[
= \int d(z, y)A_y^{-1}A_y d(x, y)dy = \int d(z, y)d(x, y)dy = d(z, y)
\]

Similar one can obtain:

Proposition A.4.6: Let us assume that \( g(x, y) = A_x d(x, y) \). Where \( A_x \) is self-adjoint operator acting on functions from variable \( x \). Then \( g^{-1}(x, y) = A_x^{-1}d(x, y) \)

Similar as it have been done above, one can obtain the following propositions, analogous to lemmas 2.1, 3.1.

Proposition A.4.7 Let us assume that \( < \beta(x)|\alpha(y) >= d(x, y) \), then for the trace of any operator \( A \) over the space \( S^n V, \Lambda V \) respectively, takes place the formulas:

\[
Tr_{S^n V} A = \frac{1}{n!} \int ... \int < \beta(x_1)\beta(x_2)\ldots\beta(x_n)|A\alpha(x_1)\alpha(x_2)\ldots\alpha(x_n) > \prod dx_i
\]

(A.14)

\[
Tr_{\Lambda^n V} A = \frac{1}{n!} \int ... \int < \beta(x_1) \wedge \beta(x_2) \wedge \ldots \wedge \beta(x_n)|A\alpha(x_1) \wedge \alpha(x_2) \wedge \ldots \wedge \alpha(x_n) > \prod dx_i
\]

(A.15)

Proposition A.4.8 Let us assume that \( < \beta(x)|\alpha(y) >= g(x, y) \), then for the trace of any operator \( A \) over the space \( S^n V, \Lambda V \) respectively, takes place the formulas:

\[
Tr_{S^n V} A = \frac{1}{n!} \int ... \int \prod_{i=1}^{n} g^{-1}(x_i, y_i) \prod_{i=1}^{n} \beta(x_i)|A \prod_{i=1}^{n} \alpha(y_i) > \prod_{i=1}^{n} dx_idy_i
\]

(A.16)

\[
Tr_{\Lambda^n V} A = \frac{1}{n!} \int ... \int \prod_{i=1}^{n} g^{-1}(x_i, y_i) \prod_{i=1}^{n} \beta(x_i)|A \prod_{i=1}^{n} \alpha(y_i) > \prod_{i=1}^{n} dx_idy_i
\]

(A.17)

A.5 Convergence of infinite product.

Proposition A.5.1 If

\[
\sum_{m=0}^{\infty} (a_m)^q = \sum_{p=0}^{\infty} (b_p)^q, \quad q = 0, 1, \ldots, n
\]

then the following infinite-product is convergent:

\[
\prod_{k=0}^{\infty} \frac{\prod_{m=1}^{M} (a_m + \sum_{j=0}^{k} k_j \omega_j)}{\prod_{p=1}^{P} (b_p + \sum_{j=0}^{k} k_j \omega_j)} \quad \text{Where } \text{Re} \omega_i < 0
\]

(A.19)

Apparentely, Barns [B] was the first who studied similar products in his theory of generalized gamma-functions. The proof described below was communicated to the author by his scientific adviser S.M. Khoroshkin.

Proof:

Note, that M=P (otherwise n-th the member of product will not tends to 1).

Proof is based on integral representation for \[ f(x) \]

Let us recall Frullani’s formula: if integral \( \int_A^{\infty} \frac{f(ax) - f(bx)}{x} \) exists \( \forall A > 0 \), then \( \int_0^\infty \frac{f(ax) - f(bx)}{x} = \ln(b/a)f(0) \).
Hence:

\[
\ln\left(\prod_{m=1}^{N}(a_m + k\omega)\prod_{p=1}^{M}(b_p + k\omega)\right) = \int_{0}^{\infty} \frac{\sum_{m=1}^{N} e^{(a_m + k\omega)x} - \sum_{p=1}^{M} e^{(b_p + k\omega)x}}{x} \mathrm{d}x
\]  \hspace{1cm} (A.20)

We can consider \(a_i, b_i < 0\). Since for the large enough number \(k_j\) holds: \((a_m + \sum k_j \omega_j)\) and \((b_p + \sum k_j \omega_j) < 0\). Let us prove at first that

\[
\prod_{k \geq 0} \prod_{m=1}^{N}(a_m + k\omega) = \exp\left(\int_{0}^{\infty} \frac{\sum_{m=1}^{N} e^{a_m x} - \sum_{p=1}^{M} e^{b_p x}}{x} \mathrm{d}x(1 - \exp(\omega x))\right)
\]  \hspace{1cm} (A.21)

where \(\text{Re}\omega < 0\)

Really:

\[
\ln\left(\prod_{k \geq 0} \prod_{m=1}^{N}(a_m + k\omega)\right) = \sum_{k \geq 0} \ln\left(\prod_{m=1}^{N}(a_m + k\omega)\prod_{p=1}^{M}(b_p + k\omega)\right) = \\
\sum_{k \geq 0} \int_{0}^{\infty} \frac{\sum_{m=1}^{N} e^{(a_m + k\omega)x} - \sum_{p=1}^{M} e^{(b_p + k\omega)x}}{x} \mathrm{d}x
\]

\[
\int_{0}^{\infty} \sum_{m=1}^{N} \left(e^{a_m x} - e^{b_m x}\right) \sum_{k \geq 0} e^{k\omega x} = \int_{0}^{\infty} \sum_{m=1}^{N} \frac{e^{a_m x} - e^{b_m x}}{1 - \exp(\omega x)x} \mathrm{d}x
\]

Similar

\[
\prod_{k_1,\ldots,k_n \geq 0} \prod_{m=1}^{M}(a_m + \sum k_j \omega_j) = \exp\left(\int_{0}^{\infty} \sum_{m=1}^{M} \frac{e^{a_m x} - e^{b_m x}}{x} \mathrm{d}x(1 - \exp(\omega x))\right)
\]  \hspace{1cm} (A.22)

Hence, the convergence of the product is equivalent to the convergence of the integral. The integral is obviously convergent in the point \(\infty\). Conditions of convergence in point zero give us

\[
\sum_{m} (a_m)^q = \sum_{p} (b_p)^q, \hspace{1cm} q = 0, 1, \ldots, n
\]  \hspace{1cm} (A.23)

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