Optimal dividend and capital injection under spectrally positive Markov additive models

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Abstract

This paper studies De Finetti’s optimal dividend problem with capital injection under spectrally positive Markov additive models. Based on dynamic programming principle, we first study an auxiliary singular control problem with a final payoff at an exponential random time. The double barrier strategy is shown to be optimal and the optimal barriers are characterized in analytical form using fluctuation identities of spectrally positive Lévy processes. We then transform the original problem under spectrally positive Markov additive models into an equivalent series of local optimization problems with the final payoff at the regime-switching time. The optimality of the regime-modulated double barrier strategy can be confirmed for the original problem using results from the auxiliary problem and the fixed point argument for recursive iterations.

Keywords: Spectrally positive Lévy process, regime switching, De Finetti’s optimal dividend, capital injection, double barrier strategy, singular control

Mathematical Subject Classification (2020): 60G51, 93E20, 91G80

1 Introduction

De Finetti’s optimal dividend problem with capital injection has become a fast-growing research topic in insurance and corporate finance. The goal of the optimal control problem is to maximize the expected net present value (NPV) of dividends when the shareholders inject capital whenever necessary over an infinite horizon to bail out the company from ruin. In particular, abundant studies can be found when underlying risk processes follow general spectrally positive or spectrally negative Lévy processes; see among Avram et al. (2007), Avanzi et al. (2011), Bayraktar et al. (2013), Zhao et al. (2015), Zhao et al. (2017a), Zhao et al. (2017b), Pérez and Yamazaki (2017a), Pérez and Yamazaki (2017b), Pérez et al. (2018), Noba et al. (2018), Wang et al. (2022) and references therein.

On the other hand, the regime-switching model has been widely used thanks to its capability to capture the changes or transitions of market trends. A large amount of empirical studies on regime switching can be found in the literature; see, for example, Hamilton (1989), So et al. (1998), Ang and Bekaert (2002), Pelletier (2006), Ang and Timmermann (2012). In the context of optimal dividend with regime switching, we refer to a short list of Jiang and Pistorius (2012), Azcue and Muler (2015), Wei et al. (2016) and Yang and Zhu (2016). Recently, the optimality of a regime-modulated refraction-reflection strategy is verified in the optimal dividend problem with capital injection under

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spectrally negative Markov additive processes in Noba et al. (2020) when the dividend process is assumed to be absolutely continuous with a bounded rate. The spectrally negative Markov additive process can be understood as a family of spectrally negative Lévy processes switching via an independent Markov chain, in which a negative jump occurs when there is a regime change. The jump size random variable is independent of the family of Lévy processes and the Markov chain, which can be understood as the cost for the insurance company to get adapted to the new regime. To encode the Chapter 11 bankruptcy, the optimality of a barrier strategy in the optimal singular dividend control problem is addressed in Wang et al. (2021) under spectrally negative Lévy processes with endogenous regime switching and Parisian ruin with exponential delay. However, it remains an open problem whether the optimal dividend control fits the barrier type when the risk process follows the spectrally positive Markov additive process.

The present paper fills this gap, and we verify that the regime-modulated double barrier strategy attains the optimality among all singular dividend and capital injection controls under the spectrally positive Markov additive process. To be more precise, within each regime state $i$, we can find a positive barrier $b^*_i > 0$ such that: (i) when the risk process exceeds the barrier $b^*_i$, the company pays a dividend so that the surplus process reflects at the level $b^*_i$; (ii) when the risk process falls below 0, the capital is injected to bail out the risk process from ruin. Our methodology is based on fluctuation identities of spectrally positive Lévy processes and fixed point arguments for recursive iterations induced by dynamic programming. Comparing with Noba et al. (2020), we stress that our problem differs substantially as the dividend control process is not absolutely continuous and we work with spectrally positive Lévy processes. Distinct computations and proofs are required to handle the auxiliary optimal singular control problem with a final payoff. In addition, contrary to Noba et al. (2020), our value functions under the spectrally positive Markov additive process are unbounded due to upward jumps, causing some new difficulties in the fixed point arguments. We also note that, in the auxiliary optimal dividend problem with a final payoff, the value function and the optimal barrier can be expressed in a more concise way comparing with the results in Noba et al. (2020) thanks to different properties of spectrally positive Lévy processes.

The rest of the paper is organized as follows. In Section 2, we formulate the optimal dividend and capital injection problem under the spectrally positive Markov additive process and introduce some preliminaries of spectrally positive Lévy processes. In Section 3, we study an auxiliary optimal dividend and capital injection problem with a final payoff at an exponential terminal time. The optimality of a double-barrier strategy is verified using fluctuation identities of spectrally positive Lévy processes and smooth-fit principle. In Section 4, based on dynamic programming arguments, we prove the optimality of the regime-modulated double barrier strategy in the original control problem using fixed point arguments for recursive iterations and some results from the auxiliary control problem.

2 Problem Formulation under the Spectrally Positive Markov Additive Process

2.1 Problem Formulation

Let us consider the risk process modelled by the so-called spectrally positive Markov additive process $\{(X_t, Y_t); t \geq 0\}$; see, for example, the detailed introduction in Section XI of Asmussen (2003). Here,
\{Y_t; t \geq 0\} is a continuous time Markov chain with finite state space \(\mathcal{E}\) and the generator matrix \((q_{ij})_{i,j \in \mathcal{E}}\). Condition on that Markov chain \(Y\) is in the state \(i\), the process \(X\) evolves as a spectrally positive Lévy process \(X^t\) until the Markov chain \(Y\) switches to another state \(j \neq i\), at which instant there is a downward jump in \(X\) with a random amount \(J_{ij}\). We assume that \((X^t)_{i \in \mathcal{E}}, Y,\) and \((J_{ij})_{i,j \in \mathcal{E}}\) are mutually independent and are defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, \mathbb{P})\) satisfying the usual condition. Let us denote \(P_{x,i}\) the law of the process \({(X_t, Y_t); t \geq 0}\) conditioning on \(\{X_0 = x, Y_0 = i\}\).

We consider a bail-out dividend control problem in this Markov additive framework, where the beneficiaries of dividends are supposed to inject capitals into the surplus process so that the resulting surplus process are always non-negative, i.e., bankruptcy never occurs. To this end, let \(X\) denote the underlying risk process before dividends are deducted and capitals are injected into. We consider two non-decreasing, right-continuous, adapted processes \(\{D_t; t \geq 0\}\) and \(\{R_t; t \geq 0\}\) defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, \mathbb{P})\), which, respectively, represent the cumulative amount of dividends and injected capitals with \(D_{0-} = R_{0-} = 0\). The surplus process after taking into account the dividends and capital injection is defined by \(U_t := X_t - D_t + R_t, t \geq 0\). The value function of the de Finetti’s dividend control problem with capital injection is defined by

\[
V(x, i) = \sup_{D, R} E_{x,i} \left[ \int_0^\infty e^{-\int_0^t \delta_s \, ds} dD_t - \phi \int_0^\infty e^{-\int_0^t \delta_s \, ds} dR_t \right] \\
\text{subject to } U_t = X_t - D_t + R_t \geq 0 \text{ for all } t \geq 0, \\
\text{both } D_t \text{ and } R_t \text{ are non-decreasing, càdlàg and adapted processes,} \\
D_{0-} = R_{0-} = 0, \text{ and } \int_0^\infty e^{-\int_0^t \delta_s \, ds} dR_t < \infty, \mathbb{P}_{x,i}-\text{almost surely}, \tag{2.1}
\]

where \((\delta_i) \in (0, \infty)^\mathcal{E}\) is a discounting rate function that switches according to the economic environment depicted by the Markov chain \(Y\), and \(\phi > 1\) depicts the cost per unit capital injected. Our goal is to find the optimal strategy \((\bar{D}^*, \bar{R}^*)\) that attains the value functions \((V(x, i))_{i \in \mathcal{E}}\).

Following the similar proof of Proposition 3.4 in Noba et al. (2020), we can readily obtain the following dynamic programming result for the value function starting from the regime state \(i\), and its proof is hence omitted.

**Proposition 2.1** For \(x \in \mathbb{R}\) and \(i \in \mathcal{E}\), we have that

\[
V(x, i) = \sup_{D, R} E_{x,i} \left[ \int_0^{c_{\lambda_i}} e^{-\int_0^t \delta_s \, ds} dD_t - \phi \int_0^{c_{\lambda_i}} e^{-\int_0^t \delta_s \, ds} dR_t + e^{-\int_0^{c_{\lambda_i}} \delta_s \, ds} V(U_{c_{\lambda_i}}, Y_{c_{\lambda_i}}) \right], \tag{2.2}
\]

where \(c_{\lambda_i}\) is the first time \(Y\) switches the regime state under \(P_{x,i}\).

The next theorem is the main result of this paper, which confirms the optimality of a regime-modulated double barrier strategy for the stochastic control problem (2.1), whose proof is deferred to Section 4.

**Theorem 2.1** There exists a function \(b^* = (b^*_i)_{i \in \mathcal{E}} \in (0, \infty)^\mathcal{E}\) such that the double barrier dividend and capital injection strategy with the dynamic upper reflection barrier \(b^*_Y t\) and fixed lower reflection
barrier 0 is optimal that attains the value function in (2.1) that

\[ V_{0,b^*}(x,i) = V(x,i), \quad (x,i) \in \mathbb{R}_+ \times \mathcal{E}, \]

where \( V_{0,b^*}(x,i) \) represents the value function of the double barrier dividend and capital injection strategy with upper barrier \( b^*_Y \) and lower barrier 0.

### 2.2 Some preliminaries of spectrally positive Lévy processes

Let \( X = (X_t)_{t \geq 0} \) be a Lévy process defined on a probability space \( (\Omega, \mathcal{F}, P) \). For \( x \in \mathbb{R} \), we denote by \( P_x \) the law of \( X \) starting from \( x \) and write \( E_x \) the associated expectation. We also use \( P \) and \( E \) in place of \( P_0 \) and \( E_0 \). The Lévy process \( X \) is said to be spectrally positive if it has no negative jumps and it is not a subordinator. The Laplace exponent \( \psi : [0, \infty) \to \mathbb{R} \) satisfying

\[ E[e^{-\theta X_t}] =: \int_0^\infty e^{-sX_t} \psi(s) ds, \quad t, \theta \geq 0, \]

is given by the Lévy-Khintchine formula that

\[ \psi(\theta) := \gamma \theta + \frac{\sigma^2}{2} \theta^2 + \int_{(0, \infty)} (e^{-\theta z} - 1 + \theta z \mathbb{1}_{\{z < 1\}})v(dz), \quad \theta \geq 0, \]

where \( \gamma \in \mathbb{R}, \sigma \geq 0, \) and \( v \) is the Lévy measure of \( X \) on \( (0, \infty) \) that satisfies

\[ \int_{(0, \infty)} (1 \wedge z^2)v(dz) < \infty. \]

It is well-known that \( X \) has paths of bounded variation if and only if \( \sigma = 0 \) and \( \int_{(0, 1)} zv(dz) < \infty; \) in this case, we have

\[ X_t = -ct + S_t, \quad t \geq 0, \]

where

\[ c := \gamma + \int_{(0, 1)} zv(dz), \]

and \( (S_t)_{t \geq 0} \) is a driftless subordinator. As we have ruled out the case that \( X \) has monotone paths, it holds that \( c > 0 \). Its Laplace exponent is given by

\[ \psi(\theta) = c\theta + \int_{(0, \infty)} (e^{-\theta z} - 1)v(dz), \quad \theta \geq 0. \]

To exclude the trivial case, it is assumed throughout the paper that

\[ E[X_1] = -\psi'(0+) < \infty. \]

Let us also recall the \( q \)-scale function for the spectrally positive Lévy process \( X \). For \( q > 0 \), the \( q \)-scale function \( W_q : \mathbb{R} \to [0, \infty) \) is continuous and strictly increasing on \( (0, \infty) \) and takes value zero on \( (-\infty, 0) \) with its Laplace transform on \( [0, \infty) \) given by

\[ \int_0^\infty e^{-sx} W_q(x) dx = \frac{1}{\psi(s) - q}, \quad s > \Phi(q), \]
where \( \Phi(q) := \sup\{s \geq 0 : \psi(s) = q\} \). We also define \( Z_q(x) \) by

\[
Z_q(x) := 1 + q \int_0^x W_q(y) \, dy, \quad x \in \mathbb{R},
\]

and its anti-derivative

\[
Z_q(x) := \int_0^x Z_q(y) \, dy, \quad x \in \mathbb{R}.
\]

We recall that if \( X \) has paths of bounded variation, \( W_q(x) \in C^1((0, \infty)) \) if and only if the Lévy measure \( \nu \) has no atoms. If \( X \) has paths of unbounded variation, we have that \( W_q(x) \in C^1((0, \infty)) \). Moreover, if \( \sigma > 0 \), we have \( W_q(x) \in C^2((0, \infty)) \). Hence, we have that \( Z_q(x) \in C^1((0, \infty)) \), \( Z_q(x) \in C^1(\mathbb{R}) \) and \( Z_q(x) \in C^2((0, \infty)) \) for bounded variation case; and we have \( Z_q(x) \in C^1(\mathbb{R}) \), \( Z_q(x) \in C^2(\mathbb{R}) \) and \( Z_q(x) \in C^3((0, \infty)) \) for the unbounded variation case. We also know that

\[
W_q(0+) = \begin{cases} 0 & \text{if } X \text{ is of unbounded variation,} \\ 1/c & \text{if } X \text{ is of bounded variation.}
\end{cases}
\]

Let us define \( \tau_0^- := \inf\{t \geq 0; X_t < a\} \) and \( \tau_0^+ := \inf\{t \geq 0; X_t > b\} \). Then, for \( b \in (0, \infty) \) and \( x \in [0, b] \), we have

\[
E_x \left[ e^{-q_0^-} 1_{\{\tau_0^- < \tau_0^+\}} \right] = \frac{W_q(b - x)}{W_q(b)}, \tag{2.3}
\]

\[
E_x \left[ e^{-q_0^+} 1_{\{\tau_0^+ < \tau_0^-\}} \right] = Z_q(b - x) - \frac{Z_q(b)}{W_q(b)} W_q(b - x). \tag{2.4}
\]

### 3 Auxiliary Optimal Dividend Problem with A Final Payoff

In this section, we first consider an auxiliary optimal dividend and capital injection problem with a final payoff at an independent exponential terminal time in a single spectrally positive Lévy model. Let \( X_t \) be the underlying risk process that follows a single spectrally positive Lévy process and let \((D_t, R_t)_{t \geq 0}\) denote nondecreasing, right continuous and \( \mathcal{F}_t\)-adapted dividend and capital injection control processes starting from zero. The controlled surplus process \( U_t \) is defined by \( U_t := X_t - D_t + R_t \) and it is required that \( U_t \geq 0 \) a.s. for all \( t \geq 0 \). Let \( \omega(x) \) be a final payoff function and the terminal time exponential random variable is denoted by \( e_\lambda \) with parameter \( \lambda \).

Throughout this section, we assume that the payoff function \( \omega \) is continuous and concave over \([0, \infty)\) with \( \omega'_+(0+) \leq q \) and \( \omega'_+(\infty) \in [0, 1] \) where \( \omega'_+(x) \) denotes the right derivative of \( \omega \) at \( x \). For \( \delta > 0, \lambda > 0 \) and \( q = \delta + \lambda \), the expected net present value (NPV) of the dividends and capital injections with a final payoff at the random time \( e_\lambda \) is defined by

\[
V_{D,R}^\omega(x) := E_x \left[ \int_0^{e_\lambda} e^{-\delta t} \, dD_t - \phi \int_0^{e_\lambda} e^{-\delta t} \, dR_t + e^{-\delta e_\lambda} \omega(U_{e_\lambda}) \right]
\]

\[
= E_x \left[ \int_0^\infty \lambda e^{-\lambda s} \left[ \int_0^s e^{-\delta t} \, dD_t - \phi \int_0^s e^{-\delta t} \, dR_t + e^{-\delta s} \omega(U_s) \right] ds \right]
\]

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\[
V_{D,R}^0(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-qt} dD_t - \phi \int_0^\infty e^{-qt} dR_t + \lambda \int_0^\infty e^{-qt} \omega(U_t) dt \right].
\] (3.1)

The value function of the auxiliary stochastic control problem is then given by
\[
V_{D,R}^0(x) = \sup_{D,R} V_{D,R}^0(x) \quad \text{subject to} \quad U_t = X_t - D_t + R_t \geq 0 \text{ for all } t \geq 0,
\]
both \(D_t\) and \(R_t\) are non-decreasing, càdlàg and adapted processes,
\[
D_{0-} = R_{0-} = 0, \quad \text{and} \quad \int_0^\infty e^{-qt} dR_t < \infty, \quad \mathbb{P}_x-\text{almost surely.} \quad (3.2)
\]

We first follow some heuristic arguments to locate the optimal singular control in some smaller subset of admissible dividend and capital injection strategies. In fact, due to the time value of money (i.e., \(q > 0\)), it seems reasonable to inject capitals as late as possible. In addition, due to the transaction costs charged for each unit of capitals injected (\(\phi > 1\)), whenever capitals injection is required, the injected capital should be the amount to keep the surplus process non-negative, i.e., the surplus process will reflect from below at 0. The above intuitive arguments motivate the following Lemma 3.1. The proof is essentially similar to that of Lemma 4.2 in Wang et al. (2022) and is hence omitted.

**Lemma 3.1** The optimal dividend and capital injection process \(\{(D_t, R_t); t \geq 0\}\) for the optimization problem (3.2) is such that \(0 \leq \Delta D_t \leq X_t - \) and
\[
R_t = -\inf_{s \leq t} (X_s - D_s) \land 0.
\] (3.3)

In particular, \(\{R_t; t \geq 0\}\) is continuous.

Thanks to Lemma 3.1, we are able to restrict ourselves to continuous capital injection process of the form (3.3) when characterizing the optimal dividend control to the problem (3.2). To continue, motivated by many existing studies in optimal dividend problems; see, for example, Azcue and Muler (2015), Avram et al. (2007), Pérez and Yamazaki (2017b), Pérez et al. (2018), Loeffen (2008), Loeffen (2009), Wang et al. (2022), Zhao et al. (2017a), we conjecture that the optimal singular dividend control in (3.2) also fits the barrier type. Recall that a barrier dividend strategy with the barrier \(b \in (0, \infty)\) refers to the singular dividend control process that any excessive surplus above \(b\) is deducted such that the resulting surplus process reflects from above at the level \(b\).

Based on our conjecture, let us first work with those dividend and capital injection strategies when dividends are paid according to an upper barrier \(b\) and capitals are injected according to the form (3.3), which we shall refer as the double barrier strategy with barriers \((0, b)\), denoted by \((D_t^{(0,b)}, R_t^{(0,b)})_{t \geq 0}\). That is, under a double barrier strategy \((D_t^{(0,b)}, R_t^{(0,b)})_{t \geq 0}\), the controlled surplus process will be reflected from above at \(b\) whenever it is to up-cross the level \(b\), and will be reflected from below at 0 whenever it is to down-cross the level 0. Let us also denote by \(U^{(0,b)} := X_t - D_t^{(0,b)} + R_t^{(0,b)}\) the resulting surplus process. The expected NPV with a final payoff under a double barrier strategy is defined by
\[
V_{0,b}^0(x) := \mathbb{E}_x \left[ \int_0^\infty e^{-qt} dD_t^{(0,b)} - \phi \int_0^\infty e^{-qt} dR_t^{(0,b)} + \lambda \int_0^\infty e^{-qt} \omega(U_t^{(0,b)}) dt \right].
\] (3.4)
In our previous conjecture, the value function (3.2) can be attained by a double barrier strategy with some particularly chosen 0 and \( b^* \), i.e., \( \mathcal{V}_{D^*, R^*}^0(x) = V_{0,b^*}^\omega(x) \). To verify this conjecture and characterize \( b^* \), let us first express the expected NPV in (3.4) with an arbitrary upper reflecting barrier \( b \in (0, \infty) \).

**Proposition 3.1** For a given \( b \in (0, \infty) \), the expected NPV in (3.4) can be written as

\[
V_{0,b}^\omega(x) = \begin{cases} 
-Z_q(b-x) - \frac{\psi'(0+)}{q} \lambda \int_0^b \omega'_+(y)Z_q(y-x)dy \\
+ \frac{Z_q(b-x)}{qW_q(b)} \left[ Z_q(b) - \phi - \lambda \int_0^b \omega'_+(y)W_q(y)dy \right], & x \in [0, b], \\
x-b + V_{0,b}^\omega(b), & x \in (b, \infty), \\
\phi x + V_{0,b}^\omega(0), & x \in (-\infty, 0).
\]

**Proof.** By Theorem 1 in Pistorius (2003), one has

\[
E_x \left[ \int_0^\infty e^{-qt} \omega(U_t^{0,b})dt \right] = \int_0^b \omega(y) \left[ \frac{Z_q(b-x)W_q(y)}{qW_q(b)} - W_q(y-x) \right] dy + \omega(0) \frac{Z_q(b-x)W_q(0+)}{qW_q(b)}, \quad x \in [0, b].
\]

Using (4.3) and (4.4) of Avram et al. (2007), we can get that

\[
E_x \left[ \int_0^\infty e^{-qt} dD_t^{0,b} \right] = -Z_q(b-x) - \frac{\psi'(0+)}{q} + \frac{Z_q(b)}{qW_q(b)} Z_q(b-x), \quad x \in [0, b],
\]

\[
E_x \left[ \int_0^\infty e^{-qt} dR_t^{0,b} \right] = \frac{Z_q(b-x)}{qW_q(b)}, \quad x \in [0, b].
\]

Considering (3.6)-(3.8) and then rearranging terms, we deduce that

\[
V_{0,b}^\omega(x) = -Z_q(b-x) - \frac{\psi'(0+)}{q} + \frac{Z_q(b)}{qW_q(b)} Z_q(b-x) - \frac{\phi Z_q(b-x)}{qW_q(b)} + \lambda \frac{\omega(0)}{q} - \int_0^b \omega'_+(y) \left[ \frac{Z_q(b-x)W_q(y)}{qW_q(b)} - \frac{Z_q(y-x)}{q} \right] dy,
\]

which is the desired result. \( \blacksquare \)

**Lemma 3.2** Let us define \( b^* \in (0, \infty) \) as the unique solution of the equation

\[
Z_q(x) - \phi - \lambda \int_0^x \omega'_+(y)W_q(y)dy = 0.
\]

We have \( b^* = \sup\{x \geq 0; q - \lambda \omega'_+(x) \leq 0\} \lor Z_q^{-1}(\phi) > 0 \). Then \( V_{0,b^*}^\omega(x) \) is continuously differentiable on \(( -\infty, \infty) \). Furthermore, if \( X \) has paths of unbounded variation, \( V_{0,b^*}^\omega(x) \) is twice continuously differentiable on \(( 0, \infty) \).
Proof. We first verify that (3.10) indeed admits a unique solution on \((0, \infty)\). By the concavity of \(\omega\) with \(\omega'_+(\infty) \in [0, 1]\), the function

\[
\ell(x) := Z_q(x) - \lambda \int_0^x \omega'_+(y) W_q(y)dy - \phi, \quad x \in [0, \infty),
\]

is first decreasing and then strictly increasing in \(x\) due to the fact that it’s right derivative

\[
\ell'_+(x) = W_q(x) (q - \lambda \omega'_+(x)) , \quad x \in (0, \infty),
\]

is first non-positive and then positive and tends to \(\infty\) as \(x\) goes to \(\infty\). As a consequence, one has \(\ell(\infty) = \infty\), which combined with the fact that \(\ell(0) = 1 - \phi < 0\) yields that there should be a unique zero of \(\ell(x)\), i.e., (3.10) has a unique solution \(b^\omega \in (0, \infty)\). In addition, it is easy to see from the above arguments and (3.10) that

\[
b^\omega > \sup\{x \geq 0; q - \lambda \omega'_+(x) \leq 0\} \lor Z^{-1}_q(\phi) > 0.
\]

By Proposition 3.1, one can derive that

\[
V^{\omega'}_{0,b}(x) = \frac{W_q(b-x)}{W_q(b)} \left[ \phi + \lambda \int_0^b \omega'_+(y) W_q(y)dy - Z_q(b) \right] + Z_q(b-x) - \lambda \int_0^b \omega'_+(y) W_q(y-x)dy, \quad x \in [0, b]. \tag{3.11}
\]

Combing (3.11) and the facts that \(b^\omega\) is the unique solution of (3.10) and \(\lim_{x \uparrow b} \int_0^b \omega'_+(y) W_q(y-x)dy = \lim_{x \uparrow b} x \omega'_+(y) W_q(y-x)dy = 0\), we have that

\[
V^{\omega'}_{0,b^\omega}(b^\omega -) = 1 + \frac{W_q(0^+)}{W_q(b^\omega)} \left[ - Z_q(b^\omega) + \phi + \lambda \int_0^{b^\omega} \omega'_+(y) W_q(y)dy \right] = 1 = V^{\omega'}_{0,b^\omega}(b^\omega +),
\]

and

\[
V^{\omega'}_{0,b^\omega}(0^+) = Z_q(b^\omega) - \lambda \int_0^{b^\omega} \omega'_+(y) W_q(y-x)dy = \phi,
\]

implying the continuous differentiability of \(V^{\omega'}_{0,b^\omega}(x)\) over \((-\infty, \infty)\). When \(X\) has paths of unbounded variation, the scale function \(W_q\) is continuously differentiable, and hence

\[
V^{\omega'}_{0,b^\omega}(b^\omega +) - V^{\omega'}_{0,b^\omega}(b^\omega -) = \frac{W_q(0^+)}{W_q(b^\omega)} \left[ Z_q(b^\omega) - \phi - \lambda \int_0^{b^\omega} \omega'_+(y) W_q(y)dy \right] + q W_q(0^+)
\]

\[
- \lambda \lim_{x \uparrow b} \int_0^{b^\omega} \omega'_+(y) W_q(y-x)dy
\]

\[
= 0,
\]

where, in the second equality, we have used the fact that \(\int_0^{b^\omega} \omega'_+(y) W_q(y-x)dy \leq \omega'_+(0^+) W_q(b^\omega - x) \to 0\) as \(x \uparrow b^\omega\). As a result, \(V^{\omega'}_{0,b^\omega}(x)\) is twice continuously differentiable over \((0, \infty)\) when \(X\) has paths of unbounded variation. ■
Recall that $b^\omega \in (0, \infty)$ is the unique solution of (3.10). The expected NPV $V_{0,b^\omega}^\omega(x)$ with the double barrier $(0, b^\omega)$ can be reduced to
\begin{equation}
V_{0,b^\omega}^\omega(x) = \begin{cases} 
-\frac{Z_q(b^\omega - x)}{q} + \frac{\lambda}{\omega} \left[ e(0) + \int_0^{b^\omega} \omega(y)Z_q(y-x)dy \right], & x \in [0, b^\omega], \\
x - b^\omega + V_{0,b^\omega}^\omega(b^\omega), & x \in (b^\omega, \infty), \\
\phi x + V_{0,b^\omega}^\omega(0), & x \in (-\infty, 0).
\end{cases}
\end{equation}

**Lemma 3.3** The expected NPV $V_{0,b^\omega}^\omega(x)$ is increasing and concave over $(-\infty, \infty)$. In addition, we have $(V_{0,b^\omega}^\omega)'(x) = \phi$ for $x \in (-\infty, 0]$, and $(V_{0,b^\omega}^\omega)'(x) = 1$ for $x \in [b^\omega, \infty)$. 

**Proof.** Define a process $\{V_t^{b^\omega}; t \geq 0\}$ that
\begin{equation}
V_t^{b^\omega} := b^\omega - X_t - \inf_{0 \leq s \leq t} (b^\omega - X_s) \land 0, \quad t \geq 0,
\end{equation}
which is the spectrally negative Lévy process $\{b^\omega - X_t; t \geq 0\}$ reflected at its infimum. In addition, denote
\begin{equation}
\sigma_t^+: = \inf\{t \geq 0; V_t^{b^\omega} \geq b^\omega\} = \inf\{t \geq 0; X_t - X_s \geq b^\omega \lor 0 \leq 0\}.
\end{equation}

Then, by Proposition 2 and Theorem 1 of **Pistorius (2004)**, we have that
\begin{equation}
E_x\left[e^{-q \sigma^+_t}\right] = \frac{Z_q(b^\omega - x)}{Z_q(b^\omega)}.
\end{equation}
\begin{equation}
\int_0^\infty e^{-qt}P_x\left(b^\omega - V_t^{b^\omega} \in dy, t < \sigma_t^+, \right)dt = \left[\frac{Z_q(b^\omega - x)}{Z_q(b^\omega)} W_q(y) - W_q(y - x)\right]1_{[0,b^\omega]}(y)dy.
\end{equation}

By (3.12), (3.15), (3.16) and the definition of $b^\omega$, we have that
\begin{align*}
(V_{0,b^\omega}^\omega)'(x) &= Z_q(b^\omega - x) - \lambda \int_0^{b^\omega} \omega'_+(y)W_q(y-x)dy \\
&= \phi - \lambda \int_0^{b^\omega} \omega'_+(y)\left[\frac{Z_q(b^\omega - x)}{Z_q(b^\omega)} - W_q(y - x)\right]dy \\
&= \phi - \lambda \int_0^{\infty} \omega'_+(y)\int_0^{\infty} e^{-qt}P_x\left(b^\omega - V_t^{b^\omega} \in dy, t < \sigma^+_t\right)dt \\
&= \phi - \lambda \int_0^{\infty} \omega'_+(y)\int_0^{\infty} e^{-qt}P_x\left(b^\omega - V_t^{b^\omega} \in dy, t < \sigma^+_t\right)dt
\end{align*}

By their definitions, we know that $V_t^{b^\omega}$ is non-increasing and $\sigma^+_t$ is non-decreasing with respect to the starting value $x$ of the process $X$, which combined with the concavity of $\omega$ results in the fact that the function $x \mapsto \phi - \lambda \int_0^{\infty} e^{-qt}P_x\left(b^\omega - V_t^{b^\omega} \in dy, t < \sigma^+_t\right)dt$ is non-increasing over $[0, b^\omega]$. Hence, the desired result is verified. ■
In our previous conjecture, the optimality of the control problem (3.2) can be attained by a double barrier dividend and capital injection strategy. To verify the optimality among all admissible dividend and capital injection singular controls, let us first characterize the optimal one among all admissible double barrier dividend and capital injection strategies. We then proceed to verify that the obtained optimal double barrier strategy is also the optimal control attaining the value function (3.2) among all admissible singular controls. With this two-step procedure in mind, we first establish the next result, which states that the double barrier strategy with the couple of barriers \((0, b^\omega)\) is the optimal one among all double barrier dividend and capital injection strategies with barriers \((0, b)\) such that \(b \in (0, \infty)\), i.e., the value function \(V_{0,b}^\omega(x)\) dominates the value function \(V_{0,b}^\omega(x)\) for all \(x \in (-\infty, \infty)\) and \(b \neq b^\omega\) with \(b \in (0, \infty)\).

**Lemma 3.4** Fix \(b \neq b^\omega\). Recall that the expected NPV \(V_{0,b}^\omega(x)\) is given by (3.5). Let us consider \(g(x) := V_{0,b}^\omega(x) - V_{0,b}^\omega(x)\). Then, \(g(x)\) is non-decreasing over \((-\infty, \infty)\) and satisfies that \(g(x) \geq 0\) for all \(x \in (-\infty, \infty)\).

**Proof.** We only provide the proof for the case \(b \in (b^\omega, \infty)\) because the proof for the case \(b \in (0, b^\omega)\) is very similar. By the fluctuation identity (2.4), we know that

\[
Z_q(b) - \frac{Z_q(b)}{W_q(b)}W_q(y) \geq 0, \quad y \in [0, b].
\]  

(3.18)

By (3.18) and the facts that \(\omega^+(y) < q/\lambda\) for \(y \geq b^\omega\), \(Z_q(b) - \phi - \lambda \int_0^b \omega^+(y)W_q(y)dy > 0\) for \(b > b^\omega\), and \(Z_q(b^\omega) - \phi - \lambda \int_0^{b^\omega} \omega^+(y)W_q(y)dy = 0\) (see Lemma 3.2), we have

\[
g(0) = -Z_q(b^\omega) \frac{\psi(0+)}{q} + \frac{\lambda}{q} \int_0^{b^\omega} \omega^+(y)Z_q(y)dy \tag{3.19}
\]

Using Theorem 8.7 in Chapter 8 of Kyprianou (2014), we arrive at

\[
\int_0^\infty e^{-qt}P_x(b - X_t \in dy, t < \tau_0^+ \wedge \tau_b^+)dt = \left[ \frac{W_q(b - x)}{W_q(b)}W_q(y) - W_q(y - x) \right]_{[0,b]}(y)dy,
\]

which implies that

\[
\frac{W_q(b - x)}{W_q(b)}W_q(y) - W_q(y - x) \geq 0, \quad y \in [0, b].
\]  

(3.20)
By Lemma 3.2, (3.20), and the definitions of $V_{0,b^\omega}$ and $V_{0,b^\omega}'$, one can check that

$$g'(x) = 1 - \left[ Z_q(b - x) - \lambda \int_0^b \omega'_+(y)W_q(y - x)dy \right]$$

$$+ \frac{W_q(b - x)}{W_q(b)} \left[ Z_q(b) - \phi - \lambda \int_0^b \omega'_+(y)W_q(y)dy \right]$$

$$- \frac{W_q(b - x)}{W_q(b)} \left[ Z_q(b^\omega) - \phi - \lambda \int_0^{b^\omega} \omega'_+(y)W_q(y)dy \right]$$

$$= \int_x^b (-q + \lambda \omega'_+(y)) W_q(y - x)dy$$

$$+ \frac{W_q(b - x)}{W_q(b)} \int_x^{b^\omega} (q - \lambda \omega'_+(y)) W_q(y)dy$$

$$= \int_x^{b^\omega} (q - \lambda \omega'_+(y)) \left[ W_q(b - x) \frac{W_q(b) - W_q(y) - W_q(y - x)}{W_q(b)} \right] dy$$

$$\geq 0, \quad x \in [b^\omega, b), \quad (3.21)$$

and

$$g'(x) = Z_q(b^\omega - x) - \lambda \int_0^{b^\omega} \omega'_+(y)W_q(y - x)dy - \left[ Z_q(b - x) - \lambda \int_0^b \omega'_+(y)W_q(y - x)dy \right]$$

$$+ \frac{W_q(b - x)}{W_q(b)} \left[ Z_q(b) - \phi - \lambda \int_0^b \omega'_+(y)W_q(y)dy \right]$$

$$- \frac{W_q(b - x)}{W_q(b)} \left[ Z_q(b^\omega) - \phi - \lambda \int_0^{b^\omega} \omega'_+(y)W_q(y)dy \right]$$

$$= \int_x^b (-q + \lambda \omega'_+(y)) W_q(y - x)dy + \frac{W_q(b - x)}{W_q(b)} \int_x^{b^\omega} (q - \lambda \omega'_+(y)) W_q(y)dy$$

$$= \int_x^{b^\omega} (q - \lambda \omega'_+(y)) \left[ W_q(b - x) \frac{W_q(b) - W_q(y) - W_q(y - x)}{W_q(b)} \right] dy$$

$$\geq 0, \quad x \in [0, b^\omega). \quad (3.22)$$

Putting (3.19), (3.21), (3.22), and the fact that $g'(x) \equiv 0$ over $(-\infty, 0) \cup (b, \infty)$ all together, we obtain the desired result. 

For any function that is sufficiently differentiable, let us define an operator $\mathcal{A}$ on $f$ that

$$\mathcal{A}f(x) := \frac{1}{2} \sigma^2 f''(x) - \gamma f'(x) + \int_{(0,\infty)} (f(x + y) - f(x) - f'(x)y1_{(0,1)}(y)) \nu(dy),$$

where $x \in (-\infty, \infty)$. The next verification lemma establishes the connection between the associated HJB variational inequality and the value function of the auxiliary control problem (3.2). In particular, it enables us to compare the value function $V_{0,b^\omega}'(x)$ under the candidate optimal double barrier $(0, b^\omega)$ and the expected NPV $V_{(D,R)}'(x)$ under any admissible singular controls $(D, R)$. 

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Lemma 3.5 (Verification Lemma) Suppose that the function $f(x)$ is non-decreasing and continuously differentiable over $(-\infty, \infty)$. Furthermore, suppose that $f(x)$ is twice continuously differentiable over $(0, \infty)$ when $X$ has paths of unbounded variation, and that

$$\max\{(A - q)f(x) + \lambda \omega(x), 1 - f'(x), f'(x) - \phi\} \leq 0. \quad (3.23)$$

Then $f(x) \geq V_{(D, R)}^\alpha(x)$ for all $x \in \mathbb{R}$ and all admissible dividend and capital injection singular controls $(D, R)$.

Proof. Let $\mathcal{D}$ be the set of admissible dividend and capital injection strategy $(D_t, R_t)_{t \geq 0}$ with $R_t$ being continuous and of form (3.3). By Lemma 3.1, we only need to prove that $f(x)$ dominates the value function of any admissible dividend and capital injection strategies among $\mathcal{D}$. For a given strategy $(D, R) \in \mathcal{D}$, recall that $U_t = X_t - D_t + R_t$ for $t \geq 0$. We follow Theorem 2.1 in Kyprianou (2014) to denote $X_t$ as the sum of the independent processes $-\gamma t + \sigma B_t, \sum_{s \leq t} \Delta X_s 1_{\{\Delta X_s \geq 1\}}$, and $X_t + \gamma t - \sigma B_t - \sum_{s \leq t} \Delta X_s 1_{\{\Delta X_s \geq 1\}}$, with the latter one being a square integrable martingale. Denote by $\{U^c_t; t \geq 0\}$ and $\{D^c_t; t \geq 0\}$ as the continuous part of $\{U_t; t \geq 0\}$ and $\{D_t; t \geq 0\}$, respectively. By Theorem 4.57 (Itô’s formula) in Jacod and Shiryaev (2003), we have, for $x \in (0, \infty)$,

\[
e^{-qt} f(U_t) = f(x) - \int_0^t q e^{-qs} f(U_{s-}) ds + \int_0^t e^{-qs} f'(U_{s-}) dU_s
\]

\[
+ \frac{1}{2} \int_0^t e^{-qs} f''(U_{s-}) d(U^c(s), U^c(s))
\]

\[
+ \sum_{s \leq t} e^{-qs} (f(U_{s-} + \Delta U_s) - f(U_{s-}) - f'(U_{s-}) \Delta U_s)
\]

\[
e^{-qt} f(U_t) = f(x) - \int_0^t q e^{-qs} f(U_{s-}) ds + \int_0^t e^{-qs} f'(U_{s-}) d(-\gamma s + \sigma B_s)
\]

\[
+ \int_0^t e^{-qs} f'(U_{s-}) d(X_s + \gamma s - \sigma B_s - \sum_{r \leq s} \Delta X_r 1_{\{\Delta X_r \geq 1\}})
\]

\[
+ \int_0^t e^{-qs} f'(U_{s-}) d(R_s - D^c_s - \sum_{r \leq s} \Delta D_r)
\]

\[
+ \int_0^t e^{-qs} f'(U_{s-}) d(\sum_{r \leq s} \Delta X_r 1_{\{\Delta X_r \geq 1\}}) + \frac{\sigma^2}{2} \int_0^t e^{-qs} f''(U_{s-}) ds
\]

\[
+ \sum_{s \leq t} e^{-qs} [f(U_{s-} + \Delta X_s) - f(U_{s-}) - f'(U_{s-}) \Delta X_s]
\]

\[
+ \sum_{s \leq t} e^{-qs} [f(U_{s-} + \Delta U_s) - f(U_{s-}) + \Delta X_s] + f'(U_{s-}) \Delta D_s
\]

\[
e^{-qt} f(U_t) = f(x) - \int_0^t q e^{-qs} f(U_{s-}) ds + \int_0^t e^{-qs} f'(U_{s-}) d(-\gamma s + \sigma B_s)
\]

\[
+ \int_0^t e^{-qs} f'(U_{s-}) d(X_s + \gamma s - \sigma B_s - \sum_{r \leq s} \Delta X_r 1_{\{\Delta X_r \geq 1\}})
\]

\[
+ \int_0^t e^{-qs} f'(U_{s-}) d(R_s - D^c_s) + \frac{\sigma^2}{2} \int_0^t e^{-qs} f''(U_{s-}) ds
\]
\[ + \sum_{s \leq t} e^{-qs} [f(U_{s-} + \Delta X_s) - f(U_{s-}) - f'(U_{s-}) \Delta X_s 1_{\{\Delta X_s < 1\}}] + \sum_{s \leq t} e^{-qs} [f(U_{s-} + \Delta U_s) - f(U_{s-} + \Delta X_s)] , \] (3.24)

where \( \Delta D_s = D_s - D_{s-} \), \( \Delta X_s = X_s - X_{s-} \), and, \( \Delta U_s = U_s - U_{s-} = \Delta X_s - \Delta D_s \). By the fact that \( V'(x) \geq 1 \) for all \( x \in [0, \infty) \) (see; (3.23)), we have, for \( s \in [0, t) \),

\[ f(U_{s-} + \Delta U_s) - f(U_{s-} + \Delta X_s) + \Delta D_s \leq 0. \] (3.25)

Therefore, by (3.23), (3.24) and (3.25), we have

\[
e^{-qt} f(U_t) = f(x) + \int_0^t e^{-qs} (A - q)f(U_{s-})ds + \int_0^t e^{-qs} f'(U_{s-})dB_s \\
+ \int_0^t e^{-qs} f'(U_{s-})d\{X_s + \gamma s - \sigma B_s - \sum_{r \leq s} \Delta X_r 1_{\{\Delta X_r \geq 1\}}\} \\
+ \int_0^t e^{-qs} f'(U_{s-})d\{R_s - D_s\} + \int_0^t \int_0^\infty e^{-qs} [f(U_{s-} + y) - f(U_{s-})] \\
- f'(U_{s-})y 1_{\{0,1\}}(y) \mathbb{N}(ds, dy) + \sum_{s \leq t} e^{-qs} [f(U_{s-} + \Delta U_s) - f(U_{s-} + \Delta X_s)] \\
\leq f(x) - \lambda \int_0^t e^{-qs} \omega(U_{s-})ds + \phi \int_0^t e^{-qs} dR_s - \int_0^t e^{-qs} dD_s + \int_0^t e^{-qs} f'(U_{s-})dB_s \\
+ \int_0^t e^{-qs} f'(U_{s-})d\{X_s + \gamma s - \sigma B_s - \sum_{r \leq s} \Delta X_r 1_{\{\Delta X_r \geq 1\}}\} \\
- \sum_{s \leq t} e^{-qs} \Delta D_s + \int_0^t \int_0^\infty e^{-qs} (f(U_{s-} + y) - f(U_{s-})) \\
- f'(U_{s-})y 1_{\{0,1\}}(y) \mathbb{N}(ds, dy), \quad x \in (0, \infty). \] (3.26)

Define a sequence of stopping times \( T_m \) such that \( T_m := m \wedge \inf\{t \geq 0; U_t \geq m\} \), \( m \geq 1 \).

It follows that \( T_m \to \infty \) almost surely as \( m \to \infty \). In addition, \( U_{t-} \) is confined in the compact set \([0, m]\) for \( t \leq T_m \). By the Lévy-Itô decomposition theorem (see, Theorem 2.1 in Kyprianou (2014)) or Appendix A in Loeffen (2009), the stochastic integral

\[
\int_0^{t \wedge T_m} e^{-qs} f'(U_{s-})d\{X_s + \gamma s - \sigma B_s - \sum_{r \leq s} \Delta X_r 1_{\{\Delta X_r \geq 1\}}\}, \quad t \geq 0,
\]

is a martingale starting from zero. By Corollary 4.6 in Kyprianou (2014) and the facts that \( \int_0^1 y^2 \nu(dy) < \infty \) (because \( \nu \) is a Lévy measure) and \( \int_0^\infty y \nu(dy) < \infty \) (by the assumption that \( E[X_1] < \infty \)), the following stochastic integral with respect to the compensated Poisson random measure

\[
\int_0^{t \wedge T_m} \int_0^\infty e^{-qs} (f(U_{s-} + y) - f(U_{s-}) - f'(U_{s-})y 1_{\{0,1\}}(y)) \mathbb{N}(ds, dy), \quad t \geq 0,
\]

is a martingale starting from zero.
is a martingale starting from zero. Similarly, the following stochastic integral (see, Page 146 in Karatzas and Shreve (1991))
\[
\int_{0^-}^{t \wedge T_m} \sigma e^{-qs} f^t(U_{s^-}) dB_s, \quad t \geq 0,
\]
is a martingale starting from zero.

Taking expectations on both sides of (3.26) after localization by \( T_m \), we have
\[
f(x) \geq \text{E}_x \left[ e^{-q(t \wedge T_m)} f(U_{t \wedge T_m}) \right] - \phi \text{E}_x \left[ \int_{0^-}^{t \wedge T_m} e^{-qs} dR_s \right] \\
+ \text{E}_x \left[ \sum_{s \leq t \wedge T_m} e^{-qs} \Delta D_s + \int_{0}^{t \wedge T_m} e^{-qs} dD_s \right] + \lambda \text{E}_x \left[ \int_{0}^{t \wedge T_m} e^{-qs} \omega(U_{s^-}) ds \right] \\
\geq \text{E}_x \left[ e^{-q(t \wedge T_m)} f(0) \right] - \phi \text{E}_x \left[ \int_{0^-}^{t \wedge T_m} e^{-qs} dR_s \right] \\
+ \text{E}_x \left[ \int_{0^-}^{t \wedge T_m} e^{-qs} dD_s \right] + \lambda \text{E}_x \left[ \int_{0}^{t \wedge T_m} e^{-qs} \omega(U_s) ds \right], \quad x \in (0, \infty).
\] (3.27)

By setting \( n, t, m \to \infty \) in (3.27), and then taking use of the bounded convergence theorem (note that \( f(0) \) is bounded), we get
\[
f(x) \geq -\phi \text{E}_x \left[ \int_{0^-}^{\infty} e^{-qs} dR_s \right] + \text{E}_x \left[ \int_{0^-}^{\infty} e^{-qs} dD_s \right] + \lambda \text{E}_x \left[ \int_{0}^{\infty} e^{-qs} \omega(U_s) ds \right] \\
= V_{(D,R)}^{\omega}(x), \quad x \in (0, \infty).
\]
The arbitrariness of \((D,R)\) and the continuity of \( f \) imply that \( f(x) \geq V_{(D,R)}^{\omega}(x) \) for all \( x \in [0, \infty) \) and all admissible \((D,R) \in \mathcal{D}\). The reverse inequality is trivial, and the proof is completed. \( \blacksquare \)

**Lemma 3.6** It holds that \( 1 \leq V_{0,b^\omega}^{\omega'}(x) \leq \phi \) for \( x \in (0, \infty) \), and
\[
\begin{cases}
AV_{0,b^\omega}^{\omega'}(x) - qV_{0,b^\omega}^{\omega'}(x) + \lambda \omega(x) = 0, & x \in (0, b^\omega], \\
AV_{0,b^\omega}^{\omega'}(x) - qV_{0,b^\omega}^{\omega'}(x) + \lambda \omega(x) \leq 0, & x \in (b^\omega, \infty).
\end{cases}
\] (3.28)

**Proof.** That \( 1 \leq V_{0,b^\omega}^{\omega'}(x) \leq \phi \) for \( x \in (0, \infty) \) is a direct consequence of Lemma 3.3. Put \( \kappa := \tau_{0^-}^0 \wedge \tau_{b^\omega}^+ \).

Under the dividend and capital injection strategy \((D_t^{(0,b^\omega)}, R_t^{(0,b^\omega)})\), neither dividends will be paid out of the surplus process nor capitals will be injected into the surplus process prior to the time \( \kappa \), hence the controlled process \( U_t^{(0,b^\omega)} \) follows the same dynamics of \( X \) before \( \kappa \). By the strong Markov property of the process \( X \), we have that
\[
\text{E}_x \left[ \int_{0}^{\infty} e^{-qs} dD_t^{(0,b^\omega)} \right] = \text{E}_x \left[ \int_{0}^{\infty} e^{-qs} dR_t^{(0,b^\omega)} + \lambda \int_{0}^{\infty} e^{-qs} \omega(U_t^{(0,b^\omega)}) dt \right] \bigg| \mathcal{F}_{s \wedge \kappa} \\
+ \lambda \int_{0}^{\kappa} e^{-qs} \omega(X_t) dt
\]
\[ e^{-q(t\wedge\kappa)}V_{0,b^-}(X_{t\wedge\kappa}) + \lambda \int_{0}^{t\wedge\kappa} e^{-q(t\wedge\kappa)}\omega(X_t)dt \]

which implies that the right-hand side of the above equation is a martingale. By Itô's formula, it holds that

\[ e^{-q(t\wedge\kappa)}V_{0,b^-}(X_{t\wedge\kappa}) + \lambda \int_{0}^{t\wedge\kappa} e^{-q(t\wedge\kappa)}\omega(X_s)ds - V_{0,b^-}(x) \]

Following the same arguments in the proof of Lemma 3.5, we get that all the terms (except for the first one) on the right-hand side of the above equality are martingales starting from 0. Hence, by taking expectations on both sides of the above equation, we get that

\[ 0 = E_x \left[ \int_{0-}^{t\wedge\kappa} e^{-q(t\wedge\kappa)}((A - q)V_{0,b^-}(X_{t\wedge\kappa}) + \lambda\omega(X_{t\wedge\kappa}))ds \right], \quad t \geq 0, \quad x \in (0,b^\omega). \]

Dividing both sides of the above equation by \( t \) and then setting \( t \downarrow 0 \), we can obtain the equality in (3.28) for \( x \in (0,b^\omega) \) by the mean value theorem and the dominated convergence theorem. In addition, the equality in (3.28) for \( x = b^\omega \) follows from the continuity of \((A - q)V_{0,b^-}(x) + \lambda\omega(x)\) at \( b^\omega \). For a more detailed proof of (3.28), we refer to the proof of Lemma 4.2 in Kyprianou et al. (2010).

It remains to prove the inequality in (3.28). By the definition of the value function \( V_{0,b^-}(x) \), we get that

\[ (A - q)V_{0,b^-}(x) + \lambda\omega(x) = -\gamma + \int_{1}^{\infty} yu(dy) - q(x - b^\omega + V_{0,b^-}(b^\omega)) + \lambda\omega(x), \quad x \in (b^\omega, \infty), \]

which, combined with the concavity of \( \omega \) and the fact that \( b^\omega > \sup\{x \geq 0; q - \lambda\omega'_+(x) \leq 0\} \lor Z_q^{-1}(\phi) > 0 \) (see; Lemma 3.2), yields that

\[ [(A - q)V_{0,b^-}(x) + \lambda\omega(x)]' = -q + \lambda\omega'_+(x) \leq 0, \quad x \in (b^\omega, \infty). \quad (3.29) \]

In view of the fact that \((A - q)V_{0,b^-}(b^\omega) + \lambda\omega(b^\omega) = 0\) and (3.29), we conclude the desired inequality of (3.28).

**Theorem 3.1** The double barrier dividend and capital injection strategy with the upper barrier \( b^\omega \) and the lower barrier 0 dominates all admissible singular dividend and capital injection strategies that \( V_{0,b^-}(x) = \sup_{D,R} V_{D,R}(x) \).

**Proof.** The desired conclusion is a direct consequence of Lemmas 3.2, 3.3, 3.5 and 3.6. □
4 Optimalty of Regime-modulated Double Barrier Strategy

We continue to prove the main result Theorem 2.1 using results from the previous auxiliary control problem with a final payoff and the recursive iteration based on dynamic programming principle. As preparations, let us first consider the following space of functions

\[ \mathcal{B} := \{ f : \mathbb{R}_+ \times \mathcal{E} \to \mathbb{R} \mid \text{for each } i \in \mathcal{E}, \text{the function } x \mapsto f(x, i) - x \]

is continuous and bounded over \( [0, \infty) \),

endowed with the metric

\[ \rho(f, g) := \max_{i \in \mathcal{E}} \sup_{x \geq 0} |f(x, i) - g(x, i)| = \max_{i \in \mathcal{E}} \sup_{x \geq 0} |(f(x, i) - x) - (g(x, i) - x)|. \]

It is straightforward to check that the metric space \((\mathcal{B}, \rho)\) is complete.

The following Lemma 4.1 states that the value function \( V \) is an element of \( \mathcal{B} \).

**Lemma 4.1** Denote \( \delta := \min_{i \in \mathcal{E}} \delta_i \), \( \overline{X}_t := \sup_{s \leq t} X_s \), and \( X_t := \inf_{s \leq t} X_s \). We have that

\[ V(x, i) := x + \phi E_{0, i} \left[ \int_0^\infty e^{-\delta t} d(X_t \wedge 0) \right] \leq V(x, i) \leq x + E_{0, i} \left[ \int_0^\infty e^{-\delta t} d(\overline{X}_t \vee 0) \right] =: \nabla(x, i), \]

for all \((x, i) \in \mathbb{R}_+ \times \mathcal{E} \).

**Proof.** The lower bound of \( V(x, i) \) can be derived if we consider the extreme admissible dividend and capital injection strategy where the manager of the company pays whatever she has as dividends at time 0, and pays no dividends afterwards and bail out all deficits by injecting capitals. The upper bound is derived by considering an admissible dividend and capital injection strategy \((\hat{D}, \hat{R})\) where the manager of the company pays every dollar accumulated by \( X \) as dividends as early as possible all the way (i.e., \( \hat{D}_t := \overline{X}_t \vee 0 \) ), and cover all deficits by capital injection (i.e., \( \hat{R}_t := -\inf_{s \leq t}(X_s - (\overline{X}_t \vee 0)) \)). Furthermore, for any an admissible dividend and capital injection strategy \((D, R)\), by integrating by parts, we have

\[ E_{x, i} \left[ \int_0^\infty e^{-\delta \int_0^t \delta_s ds} d\hat{D}_t \right] = E_{x, i} \left[ \int_0^\infty e^{-\int_0^t \delta_s ds} d\hat{D}_t - \phi \int_0^\infty e^{-\int_0^t \delta_s ds} d\hat{R}_t \right] \]

\[ = E_{x, i} \left[ \int_0^\infty \delta_t e^{-\int_0^t \delta_s ds} \left( \hat{D}_t - D_t + \phi R_t \right) dt \right] \]

\[ \geq E_{x, i} \left[ \int_0^\infty \delta_t e^{-\int_0^t \delta_s ds} (X_t - D_t + R_t) dt \right] \geq 0, \tag{4.1} \]

where the first inequality follows from the facts that \( \hat{D}_t = \overline{X}_t \vee 0 \geq X_t, R_t \geq 0 \) and \( \phi > 1 \), and the second inequality is because the strategy \((D, R)\) is admissible. Then, using (4.1) and the arbitrariness of the admissible strategy \((D, R)\) implies that

\[ V(x, i) \leq E_{x, i} \left[ \int_0^\infty e^{-\int_0^t \delta_s ds} d\hat{D}_t \right] = E_{x, i} \left[ \int_0^\infty e^{-\int_0^t \delta_s ds} d(\overline{X}_t \vee 0) \right] \]

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\[= x + E_{0,i} \left[ \int_0^\infty e^{-\int_0^t \delta_{Y_i} ds} d(\overline{X}_t \lor 0) \right] \leq \overline{V}(x,i), \quad (4.2)\]

where the second equality has used the spatial homogeneity of Lévy processes. This completes the proof. \(\blacksquare\)

For any function \(f : [0,\infty) \times \mathcal{E} \to \mathbb{R}\), we define a function \(\tilde{f} : [0,\infty) \times \mathcal{E} \to \mathbb{R}\) that

\[
\tilde{f}(x,i) := \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \int_{-\infty}^0 \left[ (f(x+y,j) - x - y) 1_{\{-y \leq x\}} + (f(0,j) - 0) 1_{\{-y > x\}} + \phi(x+y) 1_{\{-y > x\}} + y 1_{\{-y \leq x\}} - x 1_{\{-y > x\}} \right] dF_{ij}(y), \quad (4.3)
\]

where \(\lambda_i = \sum_{j \neq i} \lambda_{ij}\), and \(F_{ij}\) is the distribution function of \(J_{ij}\) for \(i, j \in \mathcal{E}\). Note that

\[
\left| \tilde{f}(x,i) - x \right| = \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \int_{-\infty}^0 \left[ (f(x+y,j) - x - y) 1_{\{-y \leq x\}} + (f(0,j) - 0) 1_{\{-y > x\}} + \phi(x+y) 1_{\{-y > x\}} + y 1_{\{-y \leq x\}} - x 1_{\{-y > x\}} \right] dF_{ij}(y) \leq \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \rho(f(x) + (\phi + 1)E[J_{ij}]), \quad (x,i) \in [0,\infty) \times \mathcal{E}, \quad (4.4)
\]

where we have used the fact that \(0 < |x| \lor |x + y| < |y|\) on \(\{-y > x\}\). By (4.4) and \(\max_{i,j \in \mathcal{E}} E[J_{ij}] < \infty\), one gets that \(\tilde{f} \in \mathcal{B}\) when \(f \in \mathcal{B}\).

For any function \(b = (b_i) \in [0,\infty)^\mathcal{E}\), denote by \(V_{0,b}(x,i)\) the value function (i.e., the NPV of the accumulated differences between dividends and the costs of capital injections) of the double barrier dividend and capital injection strategy with dynamic upper barrier \(b\) and constant lower barrier 0. In addition, let us define a mapping \(T_b\) acting on \(f \in \mathcal{B}\) such that

\[
T_b f(x,i) := E_x^i \left[ \int_0^\infty e^{-q_i t} dB_t^{b_i} - \phi \int_0^\infty e^{-q_i t} dR_t^{b_i} + \lambda_i \int_0^\infty e^{-q_i t} \tilde{f}(U_t^{b_i},i) dt \right], \quad (4.5)
\]

where \(q_i = \delta_i + \lambda_i\) and \(E_x^i\) denotes the expectation operator with respect to the law of the process \(X^i\) conditioned on the event \(\{X_0^i = x\}\). The process \(U_t^{b_i}\) is the double-reflected process with upper reflecting barrier \(b_i \geq 0\), lower reflecting barrier 0, and the underlying risk process \(X^i\); and \(D_t^{b_i}\), \(R_t^{b_i}\) are the cumulative dividends paid and capitals injected, respectively. In what follows, the scale functions of \(X^i\) will be denoted by \(W_{q,i}, Z_{q,i}\) and \(\overline{Z}_{q,i}\), whose definitions are given in Section 2.2 where the subscript \(i\) is absent.

**Lemma 4.2** For \(b \in [0,\infty)^\mathcal{E}\) and \((x,i) \in R \times \mathcal{E}\), we have \(V_{0,b}(x,i) = T_b V_{0,b}(x,i)\).

**Proof.** When \(Y_0 = i\), let \(e_{\lambda_i}\) be the first time \(Y\) switches the regime state. By the Markov property, the proof of Proposition 3.1, Theorem 1 in Pistorius (2003), as well as the independence between \((X^i)_{i \in \mathcal{E}}, Y\) and \((J_{ij})_{i,j \in \mathcal{E}}\), we can derive

\[
V_{0,b}(x,i) = E_{x,i} \left[ \int_0^{e_{\lambda_i}} e^{-\delta t} dB_t^{b_i} - \phi \int_0^{e_{\lambda_i}} e^{-\delta t} dR_t^{b_i} + e^{-\delta t} e_{\lambda_i} V_{0,b}(U_{e_{\lambda_i}}^{b_i} + J_{e_{\lambda_i}} Y_{e_{\lambda_i}}) \right]. \]

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\[
V_0,b(x,i) = (x - b_i + V_0,b(b_i, i)) 1_{[b_i, \infty)}(x) + (\phi x + V_0,b(0, i)) 1_{(-\infty, 0)}(x).
\]
4.2 \quad \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \int_{-x}^{0} \left[ f(x + y, j) - (\phi(x + y) + f(0, j)) \right] dF_{ij}(y) \right]
\leq \rho(f, g) \sup_{i \in \mathcal{E}} E_i \left[ e^{-\delta_i \epsilon} \right]
:= \beta \rho(f, g), \quad \beta \in (0, 1).  \quad (4.11)

By (4.11), for \( f \in \mathcal{B} \), \( (T^n_b f)_{n \geq 1} \) is a Cauchy sequence. Hence, we have that
\[ T^\infty_b f := \lim_{n \to \infty} T^n_b f = T_b (\lim_{n \to \infty} T^n_b f) = T_b (T^\infty_b f), \quad f \in \mathcal{B}, \]
which implies that \( T^\infty_b f \) is a fixed point of the mapping \( T_b \). By lemma 4.2, we obtain (4.8) as desired.

Let us define another space of functions that
\[ \mathcal{C} := \{ f \in \mathcal{B} \mid \hat{f}(x, i) \text{ is concave and } 1 \leq \hat{f}(x, i) \leq \phi \text{ for all } x \in [0, \infty) \text{ and } i \in \mathcal{E} \}. \quad (4.12) \]

**Lemma 4.4** Suppose that \( f \in \mathcal{B} \cap C^1(\mathbb{R}_+) \) is concave, non-decreasing, and satisfies \( 1 \leq f'(x, i) \leq \phi \) over \( \mathbb{R}_+ \times \mathcal{E} \), we have that \( f \in \mathcal{C} \).

**Proof.** By definition, \( \hat{f} \) can be rewritten as
\[ \hat{f}(x, i) = \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \left[ \int_{-x}^{0} \left[ f(x + y, j) - (\phi(x + y) + f(0, j)) \right] dF_{ij}(y) + \phi(x + E[J_{ij}]) + f(0, j) \right], \]
which implies that
\[ \hat{f}'(x, i) = \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \left[ \phi + \int_{-x}^{0} \left[ f'(x + y, j) - \phi \right] dF_{ij}(y) \right]. \quad (4.13) \]
This result, together with the concavity of \( f \), yields the concavity of \( \hat{f}(x, i) \). In addition, by (4.13) and the fact that \( 1 - \phi \leq f'(x + y, j) - \phi \leq 0 \), one can deduce that
\[ 1 = \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \leq \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \left[ \phi + \int_{-x}^{0} (1 - \phi) dF_{ij}(y) \right] \leq \hat{f}'(x, i) \leq \phi \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} = \phi. \]
The proof is completed. \[ \blacksquare \]

For \( f \in \mathcal{C} \) and \((x, i) \in \mathbb{R}_+ \times \mathcal{E} \), let us define another operator \( T_{\sup} \) that
\[ T_{\sup} f(x, i) := \sup_{D, R} E_{x,i} \left[ \int_{0}^{\epsilon_{\lambda_i}} e^{-\delta_{t} i} dD_{t} - \phi \int_{0}^{\epsilon_{\lambda_i}} e^{-\delta_{t} i} dR_{t} + e^{-\delta_{t} i} \hat{f}(U_{\epsilon_{\lambda_i}, i}) \right]
= \sup_{D, R} E_{x,i} \left[ \int_{0}^{\infty} e^{-q_{t} i} dD_{t} - \phi \int_{0}^{\infty} e^{-q_{t} i} dR_{t} + \lambda_i \int_{0}^{\infty} e^{-q_{t} i} \hat{f}(U_{t, i}) dt \right], \quad (4.14) \]
where \( U_t^i = X_t^i - D_t^i + R_t^i \) represents the controlled surplus process with control \((D^i, R^i) \) and driving process \( X^i \).

We denote \( V_0 := V \) and \( \overline{V}_0 := \overline{V} \) as well as \( V_n := T_{\sup}(V_{n-1}) \) and \( \overline{V}_n := T_{\sup} (\overline{V}_{n-1}) \), for \( n \geq 1 \).
**Lemma 4.5** We have $V_n \leq V \leq \overline{V}_n$ on $\mathbb{R}_+ \times \mathcal{E}$ for all $n \geq 1$, and

$$V(x, i) = \lim_{n \uparrow \infty} V_n(x, i) = \lim_{n \uparrow \infty} \overline{V}_n(x, i), \quad (x, i) \in \mathbb{R}_+ \times \mathcal{E},$$

(4.15)

where the convergence is under the metric $\rho(\cdot, \cdot)$. Moreover, we have $V \in \mathcal{C}$.

**Proof.** The first claim of Lemma 4.5 can be verified by the method of induction. In fact, by Lemma 4.1, we have $V_0 \leq V \leq \overline{V}_0$. Suppose that $V_{n-1} \leq V \leq \overline{V}_{n-1}$, then

$$V_n = T_{\sup} V_{n-1} \leq T_{\sup} V \leq T_{\sup} \overline{V}_{n-1} = \overline{V}_n,$$

which, together with the fact that $V$ is a fixed point of the mapping $T_{\sup}$, implies the desired claim that $V_n \leq V \leq \overline{V}_n$ for all $n \geq 1$.

To prove the second claim of Lemma 4.5, for any $f \in \mathcal{C}$ and $i \in \mathcal{E}$, Theorem 3.1 guarantees the existence of $b_i^V \in (0, \infty)$ such that the second equality of (4.14) is achieved by the expected NPV under a double barrier strategy with upper barrier $b_i^V$ and lower barrier 0. Denote $b^V = (b_i^V)_{i \in \mathcal{E}}$, it follows that $T_{\sup} f = T_{b^V} f$ over $\mathbb{R}_+ \times \mathcal{E}$, which, together with Lemmas 3.2-3.3, implies that $T_{\sup} f(\cdot, i) \in C^1(0, \infty)$ and it is concave as well as $1 \leq (T_{\sup} f)'(\cdot, i) \leq \phi$ over $[0, \infty)$ for all $i \in \mathcal{E}$. Hence, $T_{\sup} f \in \mathcal{C}$ (see; Lemma 4.4), and

$$\rho(T_{\sup} f, T_{\sup} g) = \rho(T_{b^V} f, T_{b^V} g) = \rho(\sup_{b} T_{b} f, \sup_{b} T_{b} g) \leq \sup_{b} \rho(T_{b} f, T_{b} g) \leq \beta \rho(f, g), \quad \beta \in (0, 1),$$

i.e., $T_{\sup}$ is a contraction mapping from $\mathcal{C}$ to itself. Hence, the Cauchy sequences $(V_n)_{n \geq 1}$ and $(\overline{V}_n)_{n \geq 1}$ converge to the unique fixed point $V$ of $T_{\sup}$. In addition, by (4.15) and the dominated convergence theorem, we have that

$$\hat{V}(x, i) = \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \int_{-\infty}^{0} \left[ V(x + y, j) 1_{\{-y \leq x\}} + (\phi(x + y) + V(0, j)) 1_{\{-y > x\}} \right] dF_{ij}(y)$$

$$= \lim_{n \to \infty} \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \int_{-\infty}^{0} \left[ V_n(x + y, j) 1_{\{-y \leq x\}} + (\phi(x + y) + V_n(0, j)) 1_{\{-y > x\}} \right] dF_{ij}(y)$$

$$= \lim_{n \to \infty} \sum_{j \in \mathcal{E}, j \neq i} \frac{\lambda_{ij}}{\lambda_i} \int_{-\infty}^{0} \left[ \overline{V}_n(x + y, j) 1_{\{-y \leq x\}} + (\phi(x + y) + \overline{V}_n(0, j)) 1_{\{-y > x\}} \right] dF_{ij}(y)$$

$$= \lim_{n \to \infty} \hat{V}_n(x, i) = \lim_{n \to \infty} \overline{V}_n(x, i), \quad (x, i) \in \mathbb{R}_+ \times \mathcal{E}. \quad (4.16)$$

By (4.16) and the facts that $(V_n)_{n \geq 1} \subseteq \mathcal{C}$ and $(\overline{V}_n)_{n \geq 1} \subseteq \mathcal{C}$, we obtain that $V \in \mathcal{C}$. ■

Finally, we can give the proof of Theorem 2.1 using the previous preparations.

**Proof of Theorem 2.1.** From Lemma 4.5, it follows that $V \in \mathcal{C}$. This result and Theorem 3.1 imply that there exists a function $b^V = (b_i^V)_{i \in \mathcal{E}} \in (0, \infty)^{\mathcal{E}}$ such that $V(x, i) = T_{\sup} V(x, i) = T_{b^V} V(x, i)$ for all $(x, i) \in \mathbb{R}_+ \times \mathcal{E}$. Therefore, by (4.8) and $V \in \mathcal{B}$ (as $V \in \mathcal{C}$), we have that

$$V(x, i) = \lim_{n \uparrow \infty} T_{b^V}^n V(x, i) = V_{0, b^V}(x, i).$$
i.e., \( b^* := b^V = (b_i^V)_{i \in E} \) is the desired barrier function such that the conclusion of Theorem 2.1 holds.

Acknowledgements: W. Wang is supported by the National Natural Science Foundation of China under no. 12171405 and no. 11661074. X. Yu is supported by the Hong Kong Polytechnic University research grant under no. P0031417.

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