Characterization of classes of graphs with large general position number

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Abstract

Getting inspired by the famous no-three-in-line problem and by the general position subset selection problem from discrete geometry, the same is introduced into graph theory as follows. A set $S$ of vertices in a graph $G$ is a general position set if no element of $S$ lies on a geodesic between any two other elements of $S$. The cardinality of a largest general position set is the general position number $\text{gp}(G)$ of $G$. In [7] graphs $G$ of order $n$ with $\text{gp}(G) \in \{2, n, n-1\}$ were characterized. In this paper, we characterize the classes of all connected graphs of order $n \geq 4$ with the general position number $n-2$.

Key words: diameter; girth; general position set; general position number

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1 Introduction

The general position problem in graphs was introduced by P. Manuel and S. Klavžar [4] as a natural extension of the well known century old Dudeney’s no-three-in-line problem and the general position subset selection problem from discrete geometry [2, 3, 6]. The general position problem in graph theory was introduced in [4] as follows. A set $S$ of vertices in a graph $G$ is a general position set if no element of $S$ lies on a geodesic between any two other elements of $S$. A largest general position set
is called a gp-set and its size is the general position number (gp-number, in short), \( \text{gp}(G) \), of \( G \).

The same concept was in use two years earlier in [7] under the name geodetic irredundant sets. The concept was defined in a different method, see the preliminaries below. In [7] it is proved that for a connected graph of order \( n \), the complete graph of order \( n \) is the only graph with the largest general position number \( n \); and \( \text{gp}(G) = n - 1 \) if and only if \( G = K_1 + \bigcup_j m_j K_j \) with \( \sum m_j \geq 2 \) or \( G = K_n - \{e_1, e_2, \ldots, e_k\} \) with \( 1 \leq k \leq n - 2 \), where \( e_i \)'s all are edges in \( K_n \) which are incident to a common vertex \( v \). In [11], certain general upper and lower bounds on the gp-number are proved. In the same paper it is proved that the general position problem is NP-complete for arbitrary graphs. The gp-number for a large class of subgraphs of the infinite grid graph, for the infinite diagonal grid, and for Beneš networks were obtained in the subsequent paper [5]. Anand et al. [1] gives a characterization of general position sets in arbitrary graphs. As a consequence, the gp-number of graphs of diameter 2, cographs, graphs with at least one universal vertex, bipartite graphs and their complements were obtained. Subsequently, gp-number for the complements of trees, of grids, and of hypercubes were deduced in [1]. Recently, in [8] a sharp lower bound on the gp-number is proved for Cartesian products of graphs. In the same paper the gp-number for joins of graphs, coronas over graphs, and line graphs of complete graphs are determined. Recent developments on general position number can be seen in [9].

2 Preliminaries

Graphs used in this paper are finite, simple and undirected. The distance \( d_G(u, v) \) between \( u \) and \( v \) is the minimum length of an \( u, v \)-path. An \( u, v \)-path of minimum length is also called an \( u, v \)-geodesic. The maximum distance between all pairs of vertices of \( G \) is the diameter, \( \text{diam}(G) \), of \( G \). A subgraph \( H \) of a graph \( G \) is isometric subgraph if \( d_H(u, v) = d_G(u, v) \) for all \( u, v \in V(H) \). A The interval \( I_G[u, v] \) between vertices \( u \) and \( v \) of a graph \( G \) is the set of vertices that lie on some \( u, v \)-geodesic of \( G \). For \( S \subseteq V(G) \) we set \( I_G[S] = \bigcup_{u, v \in S} I_G[u, v] \). We may simplify the above notation by omitting the index \( G \) whenever \( G \) is clear from the context.

A set of vertices \( S \subseteq V(G) \) is a general position set of \( G \) if no three vertices of \( S \) lie on a common geodesic in \( G \). A gp-set is thus a largest general position set. Call a vertex \( v \in T \subseteq V(G) \) to be an interior vertex of \( T \), if \( v \in I[T - \{v\}] \). Now, \( T \) is a general position set if and only if \( T \) contains no interior vertices. In this way general position sets were introduced in [7] under the name geodetic irredundant sets. The maximum order of a complete subgraph of a graph \( G \) is denoted by \( \omega(G) \). Let \( \eta(G) \) be the maximum order of an induced complete multipartite subgraph of
the complement of $G$. Finally, for $n \in \mathbb{N}$ we will use the notation $[n] = \{1, \ldots, n\}$.

In this paper, we make use of the following results.

**Theorem 2.1** [7] Let $G$ be a connected graph of order $n$ and diameter $d$. Then $\text{gp}(G) \leq n - d + 1$.

**Theorem 2.2** [7] For any cycle $C_n$ ($n \geq 5$), $\text{gp}(C_n) = 3$.

We recall the characterization of general position sets from [1], for which we need some additional information. Let $G$ be a connected graph, $S \subseteq V(G)$, and $P = \{S_1, \ldots, S_p\}$ a partition of $S$. Then $P$ is distance-constant if for any $i, j \in [p]$, $i \neq j$, the distance $d(u, v)$, where $u \in S_i$ and $v \in S_j$ is independent of the selection of $u$ and $v$. If $P$ is a distance-constant partition, and $i, j \in [p]$, $i \neq j$, then let $d(S_i, S_j)$ be the distance between a vertex from $S_i$ and a vertex from $S_j$. Finally, we say that a distance-constant partition $P$ is in-transitive if $d(S_i, S_k) \neq d(S_i, S_j) + d(S_j, S_k)$ holds for arbitrary pairwise different $i, j, k \in [p]$.

**Theorem 2.3** [1] Let $G$ be a connected graph. Then $S \subseteq V(G)$ is a general position set if and only if the components of $G[S]$ are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of $S$.

**Theorem 2.4** [1] If $\text{diam}(G) = 2$, then $\text{gp}(G) = \max\{\omega(G), \eta(G)\}$.

# 3 The characterization

In the following, we characterize all connected graphs $G$ of order $n \geq 4$ with the gp-number $n - 2$. Since the complete graph $K_n$ is the only connected graph of order $n$ with the gp-number $n$, by Theorem 2.1, we need to consider only graphs with diameter 2 or 3. First, we introduce four families of graphs with the diameter 3; and four families of graphs with the diameter 2.

Let $\mathcal{F}_1$ be the collection of all graphs obtained from the cycle $C : u_1, u_2, u_3, u_4, u_1$ by adding $k$ new vertices $v_1, v_2, \ldots, v_k (k \geq 1)$ and joining each $v_i, i \in [k]$ to the vertex $u_1$. Graphs from the family $\mathcal{F}_1$ are presented in Figure 1.

Let $\mathcal{F}_2$ be the collection of all graphs obtained from the path $P_2 : x, y$ and complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r} (r \geq 1)$, $K_{m_1}, K_{m_2}, \ldots, K_{m_s} (s \geq 1)$ and $K_{l_1}, K_{l_2}, \ldots, K_{l_t}$ (possibly complete graphs of this kind may be empty), by joining both $x$ and $y$ to all vertices of $K_{l_1}, K_{l_2}, \ldots, K_{l_t}$; joining $x$ to all vertices of $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$; and joining $y$ to all vertices of $K_{m_1}, K_{m_2}, \ldots, K_{m_s}$. Graphs from the family $\mathcal{F}_2$ are presented in Figure 2. Trees with diameter 3 are called double stars and they belong to the class $\mathcal{F}_2$. 

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Figure 1: Family $\mathcal{F}_1$

Figure 2: Family $\mathcal{F}_2$
Let $F_3$ be the collection of all graphs obtained from the path $P_4 : u, x, y, v$ and a complete graph $K_r (r \geq 1)$ by joining both $u$ and $x$ to all vertices of $K_r$ and joining $y$ to a subset $S$ of vertices of $V(K_r)$ (possibly $S$ may be empty or $S = V(K_r)$). Graphs from the family $F_3$ are presented in Figure 3.

Let $F_4$ be the collection of all graphs obtained from the path $P_3 : x, y, v$ and complete graphs $K_q, K_{n_1}, K_{n_2}, \ldots, K_{n_r} (r \geq 1), K_{m_1}, K_{m_2}, \ldots, K_{m_s} (s \geq 1)$ by joining $x$ to all vertices of $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$; joining $x$ and $v$ to all vertices of $K_{m_1}, K_{m_2}, \ldots, K_{m_s}$; joining $x$ and $y$ to all vertices of $K_q$. Graphs from the family $F_4$ are presented in Figure 4.

Next, we introduce four families of graphs with diameter 2.

Let $F_5$ be the collection of all graphs obtained from the complete graph $K_{n-2} (n \geq 5)$ by adding two new vertices $u$ and $v$, joining $u$ to all vertices of non-empty subset $S$ of $V(K_{n-2})$ of size at most $n - 3$; and joining $v$ to all vertices of non-empty subset $T$ of $V(K_{n-2})$ of size at most $n - 3$. The set $S$ must intersect with the set $T$ so that, the diameter of each graph from the family $F_5$ is 2. Graphs from the family $F_5$ are presented in Figure 5.

Let $F_6$ be the collection of all graphs obtained from the family $F_5$ by adding the edge $uv$. Moreover; in this case, the set $S$ may be disjoint with the set $T$. Graphs from the family $F_6$ are presented in Figure 6.

Let $F_7$ be the collection of all graphs obtained from the complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r} (r \geq 2)$ by adding two new vertices $x$ and $y$, joining $x$ to a non-empty subset $S_i$ of $V(K_{n_i})$ for all $i \in [r]$; and $y$ to a non-empty subset $T_i$ of $V(K_{n_i})$ for all $i \in [r]$ (the edges are in a way that for any $u \in V(K_{n_i})$ and $v \in V(K_{n_j})$ with $i \neq j$ must have a common neighbor). Moreover, for some $i \in [r]$; the set $S_i$ must intersect with the set $T_i$ so that, the diameter of each graph from the family $F_7$ is 2. Graphs from
Figure 4: Family $\mathcal{F}_4$

Figure 5: Family $\mathcal{F}_5$
the family $\mathcal{F}_7$ are presented in Figure 7. It is clear that both $C_4$ and $C_5$ belong to class $\mathcal{F}_7$.

Let $\mathcal{F}_8$ the collection of all graphs obtained from the family $\mathcal{F}_7$ by adding the edge $xy$. In this case, the set $S_i$ may be disjoint with the set $T_i$ for all $i \in [r]$. Graphs from the family $\mathcal{F}_8$ are presented in Figure 8.
Figure 7: Family $\mathcal{F}_7$

Figure 8: Family $\mathcal{F}_8$
Theorem 3.1 Let $G$ be a connected graph of order $n \geq 4$, then $gp(G) = n - 2$ if and only if $G$ belongs to the family $\cup_{i=1}^{8} F_i$.

Proof. First, suppose that $G$ is a connected graph of order $n$ with $gp(G) = n - 2$. Then it follows from Theorem 2.1 that $diam(G)$ is either 2 or 3. We consider the following two cases.

Case 1: $diam(G) = 3$. If $G$ is a tree, then $G$ is a double star and hence it belongs to $F_2$. So, assume that $G$ has cycles. Let $girth(G)$ denotes the length of a shortest cycle in $G$.

Let $C$ be any shortest cycle in $G$. Then it is clear that $C$ is an isometric subgraph of $G$. This shows that if $S$ is a general position set in $G$, then $S \cap V(C)$ is a general position set in $C$. Hence it follows from Theorem 2.2 that any general position set of $G$ contains at most three vertices from the cycle $C$. Now, since $gp(G) = n - 2$, we have that the length of $C$ is at most 5 and so $girth(G) \leq 5$.

Next, we claim that there is no connected graph of order $n$ with $girth(G) = 5$ and $gp(G) = n - 2$. For, assume the contrary that there is a connected graph of order $n$ with $girth(G) = 5$ and $gp(G) = n - 2$. Let $C : u_1, u_2, u_3, u_4, u_5, u_1$ be a shortest cycle of length 5 in $G$. Since $girth(G) = 5$, it follows that the vertices from $N(u_i)$ are independent for all $i \in [5]$. Also, as above we have that any general position set of $G$ has at most three vertices from the cycle $C$. Let $S$ be a general position set in $G$. Since $gp(G) = n - 2$, we have that $S = V(G) \setminus \{u_i, u_j\}$. If $u_i$ and $u_j$ are successive vertices in $C$, then it follows that the induced subgraph of $S$ has a $P_3$, which is impossible. Hence without loss of generality, we may assume that $i = 1$ and $j = 3$. So $S = V(G) \setminus \{u_1, u_3\}$. Now, since $u_2, u_4, u_5 \in S$ and $N(u_i)$ is independent, by Theorem 2.3, it follows that $deg(u_i) \leq 3$ for $i = 2, 4, 5$. Now we claim that $deg(u_2) = deg(u_4) = deg(u_5) = 2$. Otherwise, we may assume that $deg(u_2) = 3$ and let $x$ be the neighbour of $u_2$ different from $u_1$ and $u_3$. Since $girth(G) = 5$, it follows that $x$ is not adjacent with the remaining vertices of $C$. Now, since $u_2, u_5, x \in S$, by Theorem 2.3, $d(u_5, x) = d(u_5, u_2) = 2$. Let $P : u_5, y, x$ be a $u_5, x$-geodesic of length 2. Then it is clear that $y \notin V(C)$ and so $y \in S$. This leads to the fact that induced subgraph of $S$ has a $P_3$, impossible in a general position set. Hence $deg(u_2) = 2$. Similarly $deg(u_4) = deg(u_5) = 2$.

Now, if $N(u_1) \neq \emptyset$, then $u_5 \in I[x, u_4]$ for all $x \in N(u_1)$ (otherwise $S$ contains an induced $P_3$), impossible. Hence $N(u_1) = \emptyset$. Similarly, $N(u_2) = \emptyset$. Hence $G \cong C_5$. But $gp(C_5) = 3 = n - 2$ and $diam(G) = diam(C_5) = 2$. Hence there is no connected graph of order $n$ with $diam(G) = 3$, $girth(G) = 5$ and $gp(G) = n - 2$. Hence $girth(G)$ is at most 4.

Now, assume that $girth(G) = 4$ and let $C : u_1, u_2, u_3, u_4, u_1$ be a shortest cycle of length 4 in $G$. Since $diam(G) = 3$, we have that $G \not\cong C_4$. Now, we may assume that $u_1 \in V(C)$ be a vertex such that $deg(u_1) \geq 3$ and let $x$ be a neighbour of $u_1$ such that
$x \notin V(C)$. Since $S$ is a general position set and $|S| = n - 2$, we have that $S$ contains exactly 2 vertices from $C$. We claim that $u_1 \notin S$. For otherwise assume that $u_1 \in S$. Since $|S| = n - 2$ and $x, u_1 \in S$, it follows from Theorem 2.3 that $u_2, u_4 \notin S$ and $u_3 \in S$. This shows that the path $x, u_1, u_2, u_3$ must be a $x, u_3$-geodesic (otherwise, since $|S| = n - 2$, $S$ contains an induced $P_3$. Hence $d(x, u_3) \neq d(u_1, u_3)$, which is impossible in a general position set. Hence $u_1 \notin S$.

Now, we claim that $u_1$ is the unique vertex in $C$ with degree at least 3. Assume the contrary that there exists $u_j \in C$ with $j \neq 1$ and $\text{deg}(u_j) \geq 3$. Then as above we have that $u_j \notin S$. Now, if $u_i$ and $u_j$ are adjacent vertices in $C$, then we can assume that $j = 2$. It follows from the fact that $S$ is a general position set of size $n - 2$, $d(u_3, x) = 3$ and $u_3, u_1, x$ is a geodesic in $G$, where $x$ is a neighbour of $u_1$ such that $x \notin V(C)$. This shows that the vertices $x, u_4, u_3, x$ lie on a common geodesic, a contradiction. Similarly if $u_1$ and $u_j$ are non-adjacent vertices in $C$ then $u_j = u_3$ and $u_2, u_4$ belong to $S$. Moreover, as above $S$ is a general position set of size $n - 2$, we have that $x, y \in S$ and $d(x, y) = 4$, where $x \in N(u_1) \setminus V(C)$ and $y \in N(u_4) \setminus V(C)$, which is impossible. Thus $u_1$ is the unique vertex in $C$ with $\text{deg}(u_1) \geq 3$. Also, since $\text{girth}(G) = 4$, we have that $N(u_1)$ induces an independent set. Hence the graph belongs to $\mathcal{F}_1$.

Now, consider $\text{girth}(G) = 3$ and $\text{diam}(G) = 3$. Let $P : u, x, y, v$ be a $u, v$-shortest path in $G$ of length 3. Then $S$ contains atmost 2 vertices from $V(P)$. Since $|S| = n - 2$, we have that $S$ contains exactly two vertices from $V(P)$. We consider the following four cases.

**Subcase 1.1:** $u, v \in S$. Then $x, y \notin S$. Moreover, $S = V(G) \setminus \{x, y\}$. Now, let $z$ be any neighbour of $u$. Since $S$ is a general position set of size $n - 2$, it follows that $I[z, v] \subseteq V(P)$. This shows that $d(z, v) \leq 3$. If $d(z, v) = 2$, then $z$ must be adjacent with $y$ and so $u, z, y, v$ is a $u - v$ geodesic, which contradicts the fact that $S$ is a general position set. Hence $d(z, v) = 3$ and since $I[z, v] \subseteq V(P)$, we have that $z$ is adjacent with $x$ but it is not adjacent with $y$. Similarly, we have that any neighbour of $v$ is adjacent with $y$ but non-adjacent with $x$. Now, assume that $z$ be any vertex in $G$ such that $z \notin V(P)$ and $z$ is non-adjacent with both $u$ and $v$. Then as in the previous case, we have that $I[z, v] \subseteq V(P)$. Also, we have $d(z, v) \in \{2, 3\}$ and $d(z, u) \in \{2, 3\}$. Hence it follows that $z$ is adjacent to $x$ or $y$ or both. Also, by Theorem 2.3, we have that the components of $S$ are in-transitive distance-constant cliques. Hence the graph reduces to the class $\mathcal{F}_2$.

**Subcase 1.2:** $u, x \in S$. Then $y, v \notin S$ and $S = V(G) \setminus \{y, v\}$. Now, let $z$ be any vertex in $G$ such that $z \notin V(P)$. Then, we have that $I[z, u] \subseteq V(P)$. Moreover, by Theorem 2.3, $d(z, u) = d(z, x)$. If $d(z, x) = 2$, then $I[z, x] \subseteq V(P)$, we have that $z$ is adjacent to $y$. But in this case $d(z, u)$ cannot be equal to 2. Similarly, if $d(z, x) = 3$ then $z$ is adjacent with $v$ but not $y$. Then it is clear that $d(z, u) \neq 3$. Hence it follows that $d(z, u) = d(z, x) = 1$. Again by Theorem 2.3, $V(G) \setminus \{y, v\}$ induces a clique.
Hence the graph reduces to the class $\mathcal{F}_3$.

**Subcase 1.3:** $u, y \in S$. Then $x, v \not\in S$ and $S = V(G) \setminus \{x, v\}$. Now, for any $z \not\in V(P)$, we have that $I[z, y] \subseteq V(P)$ and $I[z, u] \subseteq V(P)$. Thus $d(z, y) \leq 3$ for all $z \not\in V(P)$. If $d(z, y) = 3$, then $z$ must be adjacent to $u$ and so by Theorem 2.3, $d(u, y) = 3$, a contradiction. Thus $d(z, y) \in \{1, 2\}$. If $d(z, y) = 1$, then again by Theorem 2.3, we have that $d(u, z) = 2$ and so $z$ must be adjacent to $x$. Moreover, $\{z \not\in V(P) : d(z, y) = 1\}$ induces a clique. Now, if $d(z, y) = 2$, then by using the same argument, we have that $z$ is either adjacent to $x$ or $z$ is adjacent to both $x$ and $v$. Hence the graph reduces to class $\mathcal{F}_4$.

**Subcase 1.4:** $x, y \in S$. Then $u, v \not\in S$ and $S = V(G) \setminus \{u, v\}$. Now, for any $z \not\in V(P)$, as in the previous case we have that $I[z, x] \subseteq V(P)$ and $I[z, y] \subseteq V(P)$. Moreover, by Theorem 2.3, $d(z, x) = d(z, y)$. Now, if $d(z, x) \neq 1$, then $d(z, y) \neq 1$. This shows that $z$ must be adjacent to both $u$ and $v$, which is impossible. Hence $d(z, x) = d(z, y) = 1$. Hence it follows from Theorem 2.3, $V(G) \setminus \{u, v\}$ induces a clique. Moreover, since both $x$ and $y$ belong to $S$, it is clear that $d(u, z) = d(v, z) = 2$ for all $z \not\in V(P)$. Hence in this case the graph reduces to the family $\mathcal{F}_2$.

**Case 2:** $\text{diam}(G) = 2$. Then by Theorem 2.4 we have $\text{gp}(G) = \max\{\omega(G), \eta(G)\} = n - 2$. We consider the following two subcases.

**Subcase 2.1:** $\omega(G) \geq \eta(G)$. Then $\text{gp}(G) = \omega(G) = n - 2$. Let $K$ be a clique of order $n - 2$ and let $u, v \in V(G)$ be such that $u, v \not\in V(K)$. Then it is clear that $1 \leq \deg(u) \leq n - 3$ and $1 \leq \deg(v) \leq n - 3$. Now, if $u$ and $v$ are adjacent in $G$, then $G$ belongs to the family $\mathcal{F}_6$. Otherwise, $G$ belongs to the family $\mathcal{F}_5$.

**Subcase 2.2:** $\eta(G) > \omega(G)$. Then $\text{gp}(G) = \eta(G) = n - 2$. This shows that the complement of $G$ has complete multipartite subgraph $H$ of order $n - 2$. Thus the components of the induced subgraphs of $H$ in $G$ are cliques, say $K_{n_1}, K_{n_2}, \ldots, K_{n_s}$. Moreover $d(u, v) = 2$ for all $u \in V(K_{n_i})$ and $v \in V(K_{n_j})$. Now, let $x$ and $y$ be the vertices in $G$ such that $x, y \not\in V(H)$. Then it is clear that the graph reduces to the family $\mathcal{F}_9$, when $x$ and $y$ are adjacent in $G$. Otherwise it belongs to the family $\mathcal{F}_7$.

On the other hand, if $G$ belongs to the family $\bigcup_{i=1}^{9} \mathcal{F}_i$, by Theorems 2.1 and 2.3, one can easily verify that $\text{gp}(G) = n - 2$. This completes the proof.

\[\square\]

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