The generalised buoyancy/inertial forces and available energy of axisymmetric compressible stratified vortex motions

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Adiabatic and inviscid axisymmetric perturbations to a stable reference vortex in gradient wind balance gives rise to two kinds of restoring forces: a generalised buoyancy force aligned with the reference pressure gradient, proportional to the perturbation density, and a radial inertial/centrifugal force proportional to the squared angular momentum perturbation. In this paper, it is shown that the concept of available energy for finite amplitude axisymmetric vortex motions, previously constructed by Andrews (2006) and Codoban & Shepherd (2006), is best interpreted as a form of eddy energy that measures the work against such restoring forces required for moving a fluid parcel from its reference balanced vortex equilibrium position to its actual position; it is the sum of available acoustic energy, slantwise available potential energy, and available radial energy. If the reference entropy profile increases with height along surfaces of constant angular momentum and if the squared reference angular momentum increases with radius along isobaric surfaces, then the available energy is positive definite and the reference vortex is stable to all finite amplitude inviscid and adiabatic perturbations. In that case, either non-axisymmetric and/or diabatic/viscous effects are required to cause the available energy to grow with time. Thermodynamic and mechanical efficiencies can be constructed that determine the fraction of the sinks/sources of entropy and angular momentum that respectively modify the eddy available, mean available and background potential energies. When applied to the intensification of a cyclonic vortex by diabatic heating, the new framework vindicates the idea that the sustained creation of positive buoyancy anomalies (defined relative to the reference vortex state) near the eyewall is key for tropical cyclone intensification.

1. Introduction

The concepts of buoyancy and buoyancy forces are central to the theoretical description of most observed phenomena in stratified fluids (Turner 1973). The conventional buoyancy force is generally understood as a purely vertical force due to the restoring effect of gravity that a fluid parcel experiences when it works against the environment. Its magnitude is determined in large part by the background vertical gradient and near material invariance of specific entropy, of which the main relevant measure is the squared buoyancy frequency profile \( N^2(z) \). The conventional buoyancy force is typically introduced by splitting the total pressure and density fields as perturbations to background reference hydrostatic pressure and density fields \( p_r(z) \) and \( \rho_r(z) \) that are functions of height \( z \) alone. Unless the fluid is close to a state of rest, there is currently no agreement on how these should be defined; as a result, it is generally agreed that

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the conventional buoyancy is non-uniquely defined, as it is dependent on the particular choice of reference state. The energetics counterpart of the conventional buoyancy force is the so-called available potential energy (APE) density, which is a positive definite energy quantity (provided that $N^2_r > 0$) that measures the work against the buoyancy force needed to move a fluid parcel adiabatically from its equilibrium position in the notional reference state to its actual position (Andrews 1981; Holliday & McIntyre 1981; Tailleux 2013, 2018). The APE density, like the conventional buoyancy, can be defined for arbitrarily specified hydrostatic reference pressure and density fields $p_r(z)$ and $\rho_r(z)$ (which can also be assumed to be time dependent if needed). For a simple fluid whose equation of state depends only on temperature and pressure, the volume integral of APE density is equal to the globally defined APE of Lorenz (1955), provided that the reference state is chosen to coincide with the state of minimum potential energy that can be obtained from the actual state by means of an adiabatic re-arrangement of mass (Andrews 1981; Tailleux 2013). For a general reference state, the integral of APE density usually exceeds the Lorenz APE. In the case of a more complex fluid, such as a moist atmosphere or salty ocean, the issue is considerably more complicated, see Saenz et al. (2015); Hieronymus & Nycander (2015); Wong et al. (2016); Stansifer et al. (2017); Harris & Tailleux (2018) for discussions of some of the issues involved.

In many circumstances of practical importance, however, fluid parcels are so far away from their resting equilibrium position that their conventional buoyancy may differ considerably from their actual acceleration, thus losing all predictive value (Thorpe et al. 1989). For this reason, it has been found useful in many applications to define buoyancy relative to more locally defined reference pressure and density fields. This is the case, for example, for the concept of Convective Available Potential Energy (CAPE) (Moncrieff & Miller 1976), which refers to the total work done by the fluid parcel buoyancy — defined relative to a local atmospheric sounding — from its level of free convection (LFC) to an upper level of neutral buoyancy (LNB); in that case, the fluid parcel buoyancy also depends on another material invariant, namely total water content. In an oceanic context, McDougall (1987) proposed a generalisation of the concept of buoyancy valid for lateral displacements $\delta x$, for which he derived the expression $b = -N \cdot \delta x$, where $N = g(\alpha \nabla \theta - \beta \nabla S)$ is the so-called neutral vector, whose vertical component is equal to the locally defined squared buoyancy frequency. In the context of stratified turbulent mixing, Arthur et al. (2017) recently discussed some of the consequences for the estimation of turbulent mixing of using a globally- versus locally-defined reference state. In that case, the issue arises from the difficulty of connecting the study of turbulent mixing in small domains — for which the reference state is naturally defined in terms of a global horizontal average or adiabatically sorted state — with that in the field, which is usually based on the use of a locally-defined reference state, e.g. Thorpe (1977).

Motivated by confusion in the tropical cyclone literature about the role of buoyancy in tropical cyclones — due to the use of different definitions for the term — Smith et al. (2005) emphasised the importance of distinguishing between the local buoyancy, defined relative to a reference axisymmetric vortex in gradient wind balance, and the “system” buoyancy of the reference vortex defined relative to a notional state of rest. In terms of energetics, Smith et al. (2005)’s approach appears to be closely connected to that underlying the exact partitioning of the local APE density into mean and eddy components by Scotti & White (2014); Novak & Tailleux (2018); Tailleux (2018), in which the eddy and mean APE can be interpreted as related to the work done by the local and system buoyancies respectively, the system then being the mean flow. As shown by Smith et al. (2005), the generalised buoyancy force defined relative to a non-resting state is proportional to the local density anomaly times the pressure gradient of the reference state.
state, so that in addition to the unavoidable vertical component, it also possesses a horizontal component that is lacking in the conventional buoyancy force. In addition to the generalised buoyancy force, fluid parcels displaced adiabatically and inviscidly from a non-resting reference state will in general also experience inertial/centrifugal forces, thus complicating further the understanding of the energetics of displacements with both lateral and vertical components. These additional difficulties are not always recognised, however. For instance, they are neglected in the discussion of the work experienced by lateral displacements in McDougall (1987)’s theory of neutral surfaces. As a result, it is increasingly recognised that adiabatic and isohaline parcel exchanges taking place on neutral surfaces must experience the action of forces (Nycander 2011), and hence that neutral surfaces are in general not energetically optimal for lateral displacements (Tailleux 2016). To what extent such issues matter for our understanding of lateral stirring and mixing in the ocean is currently under active investigation.

As shown by Tailleux (2018), the general expression for the conventional (i.e. defined relative to a state of rest) local APE density for a compressible stratified binary fluid is \( \Pi = h(\eta, S, p) - h(\eta, S, p_r(z_r)) + g(z - z_r) + (p_r(z) - p)/\rho \), where \( h \) is the specific enthalpy, \( \eta \) the specific entropy, \( S \) chemical composition, \( p \) pressure, \( \rho \) density, and \( z_r \) the reference position of the fluid parcel in the reference state. It is easy to see that none of the key features of \( \Pi \) — its positive definite character and the possibility to further partition it as the sum of available acoustic energy (AAE) and the work of the conventional buoyancy force — require that \( p_r(z) \) and \( \rho_r(z) \) be functions of the vertical coordinate only; they could equally be functions of horizontal coordinates without altering its properties. As a result, it is possible to define an “eddy” form of APE density simply by using a more general reference pressure field \( p_m(x, y, z, t) \) in the definition of \( \Pi \) that also depends on horizontal position (and time if needed). In that case, however, the budget for the sum of the kinetic energy and of the eddy APE density thus defined is no longer closed, as a new term in its evolution equation equation appears that unsurprisingly involves the work against the horizontal pressure gradient \( \nabla_h p_m \). The main question addressed in this paper is whether it is possible to achieve a closed budget by incorporating a new form of available energy linked to the work done by the inertial/centrifugal forces. In this paper, we show that this is indeed possible for the particular case of axisymmetric compressible stratified vortex motions around a reference vortex in gradient wind balance. In that case, the construction of a closed energy budget is made possible by the fact that the inertial/centrifugal forces experienced by displaced fluid parcels have the nice property of being purely radial and proportional to the squared angular momentum anomaly, as previously shown and discussed by Rayleigh (1916) and Emanuel (1994). As a result, their energetics is very similar to that of the conventional buoyancy force, which makes it possible to introduce a new form of available radial energy (ARE) as the inertial/centrifugal counterpart of the APE density. The sum of the ARE and APE defines an eddy form of available energy that is formally similar to that previously constructed for a zonal flow by Codoban & Shepherd (2003) and for axisymmetric vortex motions by Codoban & Shepherd (2006) and Andrews (2006) (CSA06 thereafter). Whereas CSA06’s approach emphasises the Energy-Casimir method (Haynes 1988; Shepherd 1993), our approach emphasises a force-based viewpoint of available energetics, which we think is much simpler and more intuitive, as well as much less abstract. This paper is organised as follows. Section 2 describes the model formulation. Section 3 details the construction of the available energy, and provides illustrations of the new energetic concepts for an analytical axisymmetric vortex in a dry atmosphere. Section 4 demonstrates the potential usefulness of the framework by discussing the energetics of the growth and decay of an
2. Model formulation

The evolution of compressible vortex motions is most usefully described by writing the Navier-Stokes equations in cylindrical coordinates \((r, \phi, z)\):

\[
\frac{Du}{Dt} - \left( f + \frac{v}{r} \right) v + \nu \frac{\partial p}{\partial r} = D_u, \tag{2.1}
\]

\[
\frac{Dv}{Dt} + \left( f + \frac{v}{r} \right) u + \nu \frac{\partial p}{\partial \phi} = D_v, \tag{2.2}
\]

\[
\frac{Dw}{Dt} + \nu \frac{\partial p}{\partial z} = -g + D_w, \tag{2.3}
\]

\[
\frac{D\eta}{Dt} = \dot{q}_T, \tag{2.4}
\]

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( \rho r u \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left( \rho v \right) + \frac{\partial}{\partial z} \left( \rho w \right) = 0, \tag{2.5}
\]

\[
D = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial z}. \tag{2.6}
\]

where \((u, v, w)\) is the velocity field, \(p\) is pressure, \(\rho\) is the density, \(\nu = 1/\rho\) is the specific volume, \(\eta\) is the specific entropy, \(g\) is the acceleration of gravity, \(r\) is the radial coordinate increasing outward, \(z\) is height increasing upward. The terms \(D_i, i = u, v, w\) denote dissipative terms for momentum, while \(\dot{q}\) denotes diabatic heating. The thermodynamic equation of state is assumed in the form \(\rho = \rho(\eta, p)\) or \(\nu = \nu(\eta, p)\). For the developments that follow, it is useful to rewrite Eq. (2.2) for the azimuthal motion in terms of the specific angular momentum \(M = rv + fr^2/2\) as

\[
\frac{DM}{Dt} = rD_v - \frac{1}{\rho} \frac{\partial p}{\partial \phi}. \tag{2.7}
\]

As expected, \(M\) is materially conserved for purely axisymmetric motions \((\partial p/\partial \phi = 0)\) in the absence of the dissipative term \(D_v\). The following relations expressing various quantities in terms of \(M\) will prove useful:

\[
v = \frac{M}{r} - \frac{fr}{2}, \tag{2.8}
\]

\[
\frac{v^2}{2} = \frac{M^2}{2r^2} + \frac{f^2r^2}{8} - \frac{fM}{2} = \mu \chi + \frac{f^2}{16\chi} - \frac{f\sqrt{\mu}}{2}, \tag{2.9}
\]

\[
\left( f + \frac{v}{r} \right) v = \frac{M^2}{r^3} - \frac{f^2r}{4} = - \left( \mu - \frac{f^2}{16\chi^2} \right) \frac{\partial \chi}{\partial r}, \tag{2.10}
\]

where we have defined \(\chi = 1/(2r^2)\) and \(\mu = M^2\), similarly to Andrews (2006). Note that (2.9) assumes \(M > 0\) in order to write \(M = \sqrt{\mu}\). Other quantities of importance in the following discussions are the vorticity

\[
\xi = -\frac{\partial v}{\partial z} - \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) \hat{\phi} + \frac{1}{r} \frac{\partial (rv)}{\partial r} \hat{z}. \tag{2.11}
\]
and potential vorticity $Q = (\xi + f\hat{z}) \cdot \nabla \eta / \rho$. It is useful to remark that $M$ and $Q$ are linked through the relation:

$$Q = \frac{1}{\rho r} \frac{\partial (M, \eta)}{\partial (r, z)}.$$  \hspace{1cm} (2.12)

Potential vorticity is thus proportional to the Jacobian of the coordinate transformation allowing one to map the physical space $(r, z)$ to the space $(M, \eta)$ of the materially conserved quantities for axisymmetric motions. As discussed later on, the stability of axisymmetric compressible vortex motions depends crucially on $Q$ being single-signed over the domain considered.

3. Available energetics

3.1. Reference states

Following Andrews (2006), we define the reference state pertaining to the construction of momentum-constrained available energy as an axisymmetric solution of the inviscid form of Eqs. (2.1-2.5). For such a reference state, the azimuthal velocity $v_m(r, z)$, pressure $p_m(r, z)$ and density $\rho_m(r, z)$ are in gradient wind and hydrostatic balances:

$$\frac{1}{\rho_m} \frac{\partial p_m}{\partial r} = (f + \frac{v_m}{r}) v_m, \quad \frac{1}{\rho_m} \frac{\partial p_m}{\partial z} = -g.$$ \hspace{1cm} (3.1)

The corresponding reference profiles $\eta_m(r, z)$ and $M_m(r, z)$ for the specific entropy and angular momentum may then be inferred from the equation of state for density $\rho_m(r, z) = \rho(\eta_m(r, z), p_m(r, z))$, and via the definition of angular momentum $M_m(r, z) = rv_m(r, z) + fr^2/2$. For illustrative purposes, Fig. 1 shows a particular example of azimuthal wind speed associated with the analytic dry atmospheric vortex solution used by Smith et al. (2005), whose details can be found in Appendix A. This analytic solution serves as the basis for all subsequent illustrations.

The construction of a reference vortex solution in gradient wind balance from an actual unbalanced vortex is of interest in TC studies. From the viewpoint of available energy theory, the most logical approach is to regard the reference state as the state of minimum potential energy that can be obtained from the actual state by means of a re-arrangement of mass conserving both entropy and angular momentum, as argued by Cullen et al. (2015) (Scotti & Passagia (2019) argue for a similar construction but emphasising conservation of potential vorticity instead). In the TC literature, however, existing constructions such as that proposed by Nolan & Montgomery (2002) or Smith (2006) appear to conserve only angular momentum but not entropy. The precise details of how the reference state is constructed are important, because they ultimately determine the sign of the thermodynamic and mechanical efficiencies defined later on that control the fraction of the diabatic and viscous sources of entropy and angular momentum going into the production of mechanical energy. Any construction of reference state should aim to respect the sign of these efficiencies for fear of violating causality, as emphasised by Codoban & Shepherd (2003).

Regardless of how the reference state is constructed, we define the reference position $(r_\star, z_\star)$ of a fluid parcel as if the reference state were the state of minimum potential energy obtained by means of an adiabatic re-arrangement of mass conserving entropy and angular momentum, which imposes that it be a solution of:

$$M_m(r_\star, z_\star) = M, \quad \eta_m(r_\star, z_\star) = \eta.$$ \hspace{1cm} (3.2)

Eq. (3.2) generalises the level of neutral buoyancy (LNB) equation introduced in Tailleux
Figure 1: Azimuthal wind speed $v_m$ of the analytic dry vortex from Smith et al. (2005) used as a reference state to illustrate the momentum-constrained available energy. Contour labels indicate speed in m·s$^{-1}$.

In the case that the reference state is constructed such that Eq. (3.2) has no solution for some fluid parcels, the reference positions can be arbitrarily imposed to lie at the boundaries of the domain, as discussed by Tailleux (2013). To simplify subsequent derivations, one can rewrite (3.1) as

$$\frac{1}{\rho_m} \nabla p_m = \left( \frac{M_m^2}{r^3} - \frac{f^2 r}{4} \right) \nabla r - \nabla \Phi = - \left( \mu_m - \frac{f^2}{16\chi^2} \right) \nabla \chi - \nabla \Phi,$$

(3.3)

where $\Phi = g_0 z$ is the geopotential ($g_0$ is gravitational acceleration).

To clarify the nature of the links with standard APE theory, we also consider a notional state of rest defined to be a function of height only and in hydrostatic balance only, viz.,

$$\frac{1}{\rho_0} \nabla p_0 = -\nabla \Phi$$

(3.4)

where $\rho_0(z) = \rho(\eta_0(z), p_0(z))$. In that case, the reference entropy profile is defined through $\rho_0(z) = \rho(\eta(z), p_0(z))$ and the reference angular momentum is a function of radius only: $M_0(r) = fr^2/2$. One may similarly define a reference position $(r_R, z_R)$ in such a reference state via the equations:

$$\eta_0(z_R) = \eta, \quad \frac{fr_R^2}{2} = M.$$

(3.5)

Eqs. (3.2) represent constraints similar to those underlying the Generalised Lagrangian Mean (GLM) theory of Andrews & McIntyre (1978) in which the zero mean assumption for the displacements is relaxed (see the recent paper by Gilbert & Vanneste (2018) for a recent revisiting of GLM theory).
3.2. Available energetics

Prior to discussing available energetics, one must start by identifying the appropriate form of total energy conservation pertaining to the system of equations considered. In the present case, the standard form of total energy is \( v^2/2 + e + \Phi = v^2/2 + h + \Phi - p/\rho \), where \( e \) is internal energy. Following the usual procedure, the latter can be shown to satisfy the following evolution equation:

\[
\rho \frac{D}{Dt} \left( \frac{v^2}{2} + h + \Phi - \frac{p}{\rho} \right) + \nabla \cdot (pv) = \rho v \cdot D + \rho \dot{q},
\]

(3.6)

For the system of equations considered to be energetically consistent, the viscous and diabatic terms \( D \) and \( \dot{q} \) must be such that the right-hand side of Eq. (3.6) is expressible as the divergence of some flux. Following Andrews (1981, 2006), we introduce the following identity:

\[
\rho \frac{D}{Dt} \left( \frac{p_m \rho}{\rho} \right) = \frac{\partial p_m}{\partial t} + \nabla \cdot (p_m v),
\]

(3.7)

which is valid for any arbitrary pressure field \( p_m = p_m(x, t) \), and add it to (3.6) to obtain the following alternative energy conservation equation:

\[
\rho \frac{D}{Dt} \left( \frac{v^2}{2} + \Pi_1 + h(\eta, p_m) + \Phi(z) \right) + \nabla \cdot [(p - p_m)v] = \rho v \cdot D + \rho \dot{q} + \frac{\partial p_m}{\partial t},
\]

(3.8)

where \( \Pi_1 = h(\eta, p) - h(\eta, p_m) + (p_m - p)/\rho \) is a positive definite energy quantity usually referred to as Available Acoustic Energy (AAE), e.g. Andrews (1981, 2006), Tailleux (2018). The positive definite character of \( \Pi_1 \) follows from the possibility to write it in the form

\[
\Pi_1 = \int_p^p \int_p^{p'} \nu_p(\eta, p'') \, dp'' \, dp' = \int_p^p \int_p^{p'} \frac{dp'' \, dp'}{\rho^2 c_s^2} \approx \frac{(p - p_m)^2}{2\rho_m c_s^2},
\]

(3.9)

where \( c_s^2 = (\partial \rho / \partial p)^{-1} \) is the squared speed of sound (Tailleux 2018). The advantage of using a reference pressure field that also depends on the horizontal coordinates is that it reduces the magnitude of the pressure perturbation \( p' \) as compared to standard APE theory, and hence reduces the contribution of AAE to the overall energy budget. Following Andrews (2006) and Codoban & Shepherd (2006), we decompose the total kinetic energy as the sum of the kinetic energies of the secondary (toroidal) and primary (azimuthal) circulations:

\[
\frac{v^2}{2} = \frac{u_s^2}{2} + \frac{v^2}{2},
\]

(3.10)

where \( u_s = (u, 0, w) \) is the velocity of the secondary circulation. Next, we define the vortex dynamic potential energy \( V \) as

\[
V = \frac{v^2}{2} + h(\eta, p_m) + \Phi(z),
\]

(3.11)

and define the vortex available energy \( A_e \) as the difference between the values of \( V \) in the actual and reference states:

\[
A_e = V - V_* = \frac{v^2}{2} - \frac{v^2}{2} + h(\eta, p_m) - h(\eta, p_*) + \Phi(z) - \Phi(z_*),
\]

(3.12)

where \( p_* = p_m(r_*, z_*, t) \) and \( \chi_* = \chi(r_*) \). Note that if it were not for the presence of the kinetic energy term \( v^2/2 - v^2_*/2 \), \( A_e \) would be nearly identical to the eddy APE term introduced by Tailleux (2018), which motivates us to regard \( A_e \) as a particular form of
eddy energy. To discuss the properties of $A_e$, it is more advantageous to express $v^2/2$ and $v^2_\star/2$ in terms of $\mu$ by using (2.9) as follows:

$$A_e = \mu(\chi - \chi_\star) + \frac{f^2}{16} \left( \frac{1}{\chi} - \frac{1}{\chi_\star} \right) + h(\eta, p_m) - h(\eta, p_\star) + \Phi(z) - \Phi(z_\star).$$  \hspace{1cm} (3.13)

Eq. (3.13) is the starting point for the subsequent derivations.

3.3. Interpretation of $A_e$ in terms of the work of a generalised buoyancy force

As can be seen from (3.13), a key property of $A_e$ is that it is a function of the actual and reference positions only at fixed $\eta$ and $\mu$ (or $M$). As a result, it is possible to write $A_e$ as the path integral

$$A_e = -\int_{x_*}^{x} b_e(\mu, \eta, x', t) \cdot dx', \hspace{1cm} (3.14)$$

thus allowing $A_e$ to be interpreted as the work against the generalised buoyancy force $b_e$ defined by

$$b_e = -\nabla A_e = \underbrace{(\nu_m - \nu_h)}_{b_e^T} \nabla p_m + \underbrace{(\mu_m - \mu)}_{b_e^M} \nabla \chi,$$

(3.15)

where the derivatives are taken by holding $\eta$ and $\mu$ constant, with $x = (r, z)$ and $x_* = (r_\star, z_\star)$. This result is interesting and important, because it generalises to $A_e$ the possibility first established by Andrews (1981) to interpret the APE density as the work needed to move a fluid parcel from its notional equilibrium position $x_\star$ in the reference state to its position $x$ in the actual state. Whether such a possibility also pertained to momentum-constrained available energy had remained unclear so far, as neither Codoban & Shepherd (2006) nor Andrews (2006) had discussed it.

In the following, we regard the part $b_e^T$ that is proportional to the specific volume anomaly as the thermodynamic component and the part $b_e^M$ that is proportional to the squared angular momentum anomaly as the mechanical component of the generalised buoyancy force $b_e$. The thermodynamic force $b_e^T$ has been discussed before by Smith et al. (2005), who sought to clarify the role played by buoyancy in tropical cyclones. In their paper, they write such a force in the form

$$b_e^T = \left( 1 - \frac{\rho_m}{\rho_h} \right) \frac{1}{\rho_m} \nabla p_m = \left( 1 - \frac{\rho_m}{\rho_h} \right) g_e,$$

(3.16)

where $g_e$ is a generalised acceleration defined by Smith et al. (2005) as

$$g_e = \left( \frac{v_m^2}{r} + f v_m, -g \right) = \frac{1}{\rho_m} \nabla p_m.$$  \hspace{1cm} (3.17)

The mechanical force $b_e^M$ has also been discussed before, for instance in relation to centrifugal waves, e.g. Markowski & Richardson (2010).

3.4. Partitioning of $A_e$ into mechanical and thermodynamic components

The available energy $A_e$ is a two-dimensional function of the radial and vertical displacements $\delta r = r - r_\star$ and $\delta z = z - z_\star$. For small enough displacements, $A_e$ reduces to a quadratic function $A_e \approx 1/2 \delta x^TH_A\delta x$, where $H_A$ is the Hessian matrix of $A_e$'s second derivatives. In that case, the most natural approach to establish the positive definite character of $A_e$ is to compute the (real) eigenvalues of $H_A$ and determine whether they are both positive. Since $b_e = -\nabla A_e \approx -H_A \delta x$, the eigenvectors of $H_A$ can be interpreted as the directions along which the restoring buoyancy force $b_e$ aligns perfectly with the
displacement $\delta x$. The eigenvalues thus represent the squared frequency of the natural oscillation taking place in the eigendirections. In the system of coordinates $(x_1, x_2)$ defined by the eigenvectors, the available energy $A_e$ can be written as the sum of two quadratic terms $A_e \approx \lambda_1 x_1^2/2 + \lambda_2 x_2^2/2$ that most clearly links its sign positive definite character to the sign of the eigenvalues $\lambda_1$ and $\lambda_2$. Such an approach is not available, however, for finite-amplitude displacements. In that case, previous authors have showed that $A_e$ can still be decomposed as the sum of two terms whose sign-definiteness can be linked to the sign of the gradients of $M_m$ and $\eta_m$. Such a decomposition is not unique in general, however, and different authors have proposed different ones.

Here we propose yet another decomposition $A_e = \Pi_e + \Pi_k$, which differs from previous ones in that it attempts to more cleanly separate the kinetic and potential energy parts of $A_e$. To that end, we exploit the fact that $A_e$ can be expressed as a path integral. Indeed, such a path can be broken into two components by introducing some yet-to-be-identified intermediate point $x_\mu = (r_\mu, z_\mu)$, thus allowing one to associate $\Pi_e$ and $\Pi_k$ with one of the integration sub-paths as follows:

$$A_e = \int_{x_*}^{x_\mu} b_e(M, \eta, x', t) \cdot dx' - \int_{x_\mu}^{x} b_e(M, \eta, x', t) \cdot dx'. \quad (3.18)$$

The same intermediate point $x_\mu$ can be similarly used to partition the exact expression (3.13) for $A_e$, thus yielding the following explicit expressions for $\Pi_e$ and $\Pi_k$:

$$\Pi_e = h(\eta, p_\mu) - h(\eta, p_*) + \Phi(z_\mu) - \Phi(z_*) + \mu(\chi_\mu - \chi_*) + \frac{f^2}{16} \left( \frac{1}{\chi_\mu} - \frac{1}{\chi_*} \right), \quad (3.19)$$

$$\Pi_k = h(\eta, p_m) - h(\eta, p_\mu) + \Phi(z) - \Phi(z_\mu) + \mu(\chi - \chi_\mu) + \frac{f^2}{16} \left( \frac{1}{\chi} - \frac{1}{\chi_\mu} \right), \quad (3.20)$$

where $p_\mu$ is shorthand for $p_m(r_\mu, z_\mu)$. In Eq. (3.20), the terms involving the specific enthalpy are clearly of a thermodynamic nature. In order for $\Pi_k$ to be purely mechanical in nature, these need to be removed. This is possible only if $(r_\mu, z_\mu)$ is chosen so that $p_\mu = p_m$. To further constrain $x_\mu$, we further impose that $\Pi_e$ and $\Pi_k$ are only contributed to by the thermodynamic and mechanical components of the generalised buoyancy force respectively; mathematically:

$$\Pi_e = -\int_{x_*}^{x_\mu} b_e^T \cdot dx' = \int_{x_*}^{x} (\nu_h - \nu_m) \nabla p_m \cdot dx' \quad (3.21)$$

$$\Pi_k = -\int_{x_\mu}^{x} b_e^M \cdot dx' = \int_{x_*}^{x} (\mu - \mu_m) \nabla \chi \cdot dx'. \quad (3.22)$$

For such expressions to hold, the work done by $b_e^M$ and $b_e^T$ must vanish on the first and second legs of the overall integration path respectively:

$$\int_{x_*}^{x_\mu} (\nu_h - \nu_m) \nabla p_m \cdot dx' = 0, \quad \int_{x_*}^{x} (\mu - \mu_m) \nabla \chi \cdot dx' = 0. \quad (3.23)$$

The most obvious way to fulfill (3.23) is by imposing $\mu_m = \mu$ on the first leg joining $x_*$ to $x_\mu$, while imposing to the second leg joining $x_\mu$ to $x$ that it follows an isobaric surface $\nabla p_m \cdot dx' = 0$. As a result, the intermediate point $x_\mu = (r_\mu, z_\mu)$ must lie at the intersection of the surface of constant angular momentum $\mu_m = \mu$ and isobaric surface $p_m = p_m(r, z)$. Its coordinates must therefore be solutions of

$$\mu_m(r_\mu, z_\mu) = \mu, \quad p_m(r_\mu, z_\mu) = p_m(r, z). \quad (3.24)$$
Such a construction and the two different integration paths are illustrated in Fig. 2 for the analytical vortex solution detailed in Appendix A. Moreover, Fig. 3 illustrates the superiority, at least visually, of the \((\mu_m, p_m)\) representation over the \((M_m, \eta_m)\) representation to achieve what looks like a near orthogonal finite-amplitude decomposition of available energy.

### 3.5. Sign definiteness of the eddy available potential energy \(\Pi_e\)

The key role played by isobaric and constant angular momentum surfaces for simplifying the partitioning of \(A_e\) motivates us to work in \((\mu_m, p_m)\) coordinates. Since

\[
\mu_m = (r^3/\rho_m)\partial p_m/\partial r + f^2 r^4/4,
\]

we have

\[
J_{\mu p} = \frac{\partial (\mu_m, p_m)}{\partial (r, z)} = -\rho_m gf^2 r^3 + \frac{\partial}{\partial r} \left( \frac{r^3}{\rho_m} \frac{\partial p_m}{\partial r} \right) \frac{\partial p_m}{\partial z} - \frac{\partial}{\partial z} \left( \frac{r^3}{\rho_m} \frac{\partial p_m}{\partial r} \right) \frac{\partial p_m}{\partial r}.
\] (3.25)

Close to a state of rest, \(J_{\mu p} \approx -\rho_m gf^2 r^3 < 0\), in which case the coordinate transformation is invertible and well defined, except at the origin \(r = 0\). In the following, we assume that this remains the case for a non-resting reference vortex state in gradient wind balance, as can be seen to be the case in Fig. 2 for our example analytical vortex. In the following, a tilde is used to denote functions of \((r, z)\) in their \((\mu_m, p_m)\) representation; for instance, \(\nu_m(r, z) = \tilde{\nu}_m(\mu_m, p_m)\) and \(\chi(r) = \tilde{\chi}(\mu_m, p_m)\).

Using the fact that \(\nabla p_m \cdot \mathbf{d}x' = dp'\), where \(p' = p_m(r', z')\), it is easily seen that Eq.
where \( \eta \) in Eq. (3.21) refers to \( \eta(p^\prime) \), that \( p_m(x_\star) = p_\star \), that \( p_m = p_m(x) \), and that \( \nu_m(r', z') = \nu_m(\mu, p') = \nu(\tilde{\eta}_m(\mu, p'), p') \) along the surface of constant angular momentum \( \mu_m = \mu \). Physically, Eq. (3.26) can be recognised as being similar to the conventional APE density (compare with Eq. (2.18) of Tailleux (2018)), for a definition of buoyancy defined relative to the horizontally-varying reference specific volume \( \tilde{\nu}_m(\mu, p) \) evaluated along a constant angular momentum surface. As a result, \( \Pi_e \) represents a “slantwise” APE density, by analogy with the concept of slantwise convective available potential energy (SCAPE) used in discussions of conditional symmetric instability (Bennetts & Hoskins 1979; Emanuel 1983b,a). To establish the positive definite character of \( \Pi_e \), note that (3.26) may be rewritten as

\[
\Pi_e = \int_{p_\star}^{p_m} \int_{\tilde{\eta}_m(\mu, p') \mu,p} \frac{\partial \nu}{\partial \eta} (\eta', p') \, d\eta' \, dp' = \frac{\partial \nu}{\partial \eta} (\eta, p_i) \int_{p_\star}^{p_m} \tilde{\eta}_m(\mu, p) \, dp' \,
\]

where we used the mean value theorem to take the adiabatic lapse rate \( \partial \nu/\partial \eta = \Gamma = \alpha T/(\rho c_p) \) out of the integral (\( \alpha \) is the isobaric thermal expansion and \( c_p \) is the isobaric specific heat capacity), where \( (\eta_i, p_i) \) represent some intermediate values of entropy and pressure, and used the fact that \( \eta = \tilde{\eta}_m(\mu, p) \) by definition. If the adiabatic lapse rate \( \Gamma \) is positive, as is normally the case, Eq. (3.27) shows that a sufficient condition for \( \Pi_e \) to be positive definite is

\[
\frac{\partial \tilde{\eta}_m}{\partial p} (\mu, p') < 0,
\]

regardless of \( p' \); this states that the specific entropy should increase with height (decrease with pressure) along surfaces of constant angular momentum, as expected. The special case where

\[
\frac{\partial \tilde{\eta}_m}{\partial z} (r, z) > 0, \quad \frac{\partial \tilde{\eta}_m}{\partial p} (\mu, p') > 0,
\]

would correspond to the so-called conditional symmetric instability (CSI), whereby the entropy profile is stable to upright vertical displacements but not to slantwise displacements. For small amplitude perturbations, a Taylor series expansion shows that (3.27) approximates to

\[
\Pi_e \approx -\Gamma_i \frac{\partial \tilde{\eta}_m}{\partial p}(\mu, p_i) \frac{(p_m - p_s)^2}{2}
\]

where \( \Gamma_i \) is shorthand for \( \partial \nu/\partial p(\eta_i, p_i) \). Note that this expression is essentially the same as the classical small-amplitude expression \( N^2 \delta z^2 / 2 \) for the conventional APE density in terms of an appropriate squared buoyancy frequency, where \( \delta z \) is the vertical displacement from the reference height. Note that in Eq. (3.27), we could equally have regarded pressure as a function of entropy to obtain a small amplitude approximation proportional to the squared entropy anomaly \( (\tilde{\eta}_m(\mu, p) - \tilde{\eta}_m(\mu, p_s))^2 / 2 \) instead if desired.
Figure 3: Available energy $A_e$ of a perturbed dry air parcel at $r = 40\,\text{km}$, $z = 5\,\text{km}$, in terms of (a) $M$ and $\eta$ perturbations and (b) $\mu$ and $p^*$ perturbations. The grey lines in (b) indicate the horizontal and vertical axes along which $\Pi_k$ and $\Pi_e$ change respectively, and the grey shading covers points in the space that are not sampled by the chosen perturbations of $M$ and $\eta$.

3.6. Sign definiteness of the mechanical eddy energy $\Pi_k$

We now turn to the problem of establishing the conditions for $\Pi_k$ to be positive definite. Eqs. (3.20), (3.22) and (3.23) show that $\Pi_k$ may be equivalently written as:

$$\Pi_k = \Phi(z) - \Phi(z_{\mu}) + \mu(\chi - \chi_{\mu}) + \frac{f^2}{16} \left( \frac{1}{\chi} - \frac{1}{\chi_{\mu}} \right) = \int_{\chi_{\mu}}^{\chi} (\mu - \mu_m) \nabla \chi \cdot \mathbf{d}x'. \quad (3.31)$$

As for $\Pi_e$, we find it useful to keep working in $(\mu_m, p_m)$ coordinates. To that end, we use the mathematical identity $(\mu - \mu_m) \nabla \chi = \nabla [(\mu - \mu_m) \chi] + \chi \nabla \mu_m$ (recall that $\mu$ is treated like a constant in such calculations), and the fact that by construction $p_{\mu} = p_m$, to rewrite $\Pi_k$ in the following equivalent ways:

$$\Pi_k = (\mu - \mu_m) \chi + \int_{\chi_{\mu}}^{\chi} \tilde{\chi}(\mu_m, p_m) \nabla \mu_m \cdot \mathbf{d}x' = \int_{\mu}^{\mu_m} [\tilde{\chi}(\mu', p_m) - \tilde{\chi}(\mu_m, p_m)] \, d\mu'. \quad (3.32)$$

Eq. (3.33) is very similar to the expression Eq. (3.21) for $\Pi_e$. It can be similarly expressed as a double integral,

$$\Pi_k = \int_{\mu}^{\mu_m} \int_{\mu_{\mu}}^{\mu'} \frac{\partial \tilde{\chi}}{\partial \mu}(\mu', p_m) \, d\mu'' \, d\mu', \quad (3.33)$$

which makes it clear that a sufficient condition for $\Pi_k$ to be positive definite is

$$\frac{\partial \tilde{\chi}}{\partial \mu}(\mu', p_m) < 0. \quad (3.34)$$

Physically, Eq. (3.34) corresponds to the condition that the reference squared angular momentum distribution increase with radius along isobaric surfaces. The violation of this condition corresponds to centrifugal instability, e.g. Drazin & Reid (1981). It is useful to remark that the partial derivative $\partial \tilde{\chi}/\partial \mu$ can be expressed in terms of the Jacobian $J_{\mu p}$ of the coordinate transformation from $(r, z)$ to $(\mu_m, p_m)$ coordinates as

$$\frac{\partial \tilde{\chi}}{\partial \mu} = \frac{\partial (\tilde{\chi}, p_m)}{\partial (r, z)} \left( \frac{\partial (\mu_m, p_m)}{\partial (r, z)} \right)^{-1} = \frac{\rho_m g}{r^3} \left( \frac{\partial (\mu_m, p_m)}{\partial (r, z)} \right)^{-1}, \quad (3.35)$$
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Figure 4: Estimate of the ratio $1/K_i$ as defined by Eq. (3.38) for the analytical vortex state described in Appendix A, assuming $M \approx M_m$.

which shows that the condition (3.34) is actually equivalent to the condition $J_{\mu p} < 0$, and hence to the requirement that the coordinates transformation from $(\mu_m, p_m)$ to $(r, z)$ be invertible.

By using the mean value theorem, it is possible to write

$$
\Pi_k = -\frac{\partial \tilde{\chi}}{\partial \mu}(\mu_i, p_m) \frac{(\mu - \mu_m)^2}{2}
$$

where $\mu_i$ is some intermediate value of $\mu$ within the interval $[\min(\mu, \mu_m), \max(\mu, \mu_m)]$. On the other hand, the Eulerian eddy kinetic energy $E_{vk} = (v - v_m)^2/2$ may be written as

$$
E_{vk} = \frac{(v - v_m)^2}{2} = \frac{1}{r^2(M_m + M)^2} \frac{(\mu_m - \mu)^2}{2}
$$

by using the relations $v = M/r - fr/2$, $v_m = M_m/r - fr/2$, $(v - v_m)^2 = (M - M_m)^2/r^2$ and $(\mu_m - \mu)^2 = (M_m + M)^2(M - M_m)^2$. The two quantities are therefore proportional to each other, i.e., $\Pi_k = K_i E_{vk}$, with the proportionality factor

$$
K_i = -r^2(M_m + M)^2 \frac{\partial \tilde{\chi}}{\partial \mu}(\mu_i, p_m).
$$

Near a state of rest, $M_m \approx M \approx fr^2/2$, then $\partial \tilde{\chi}/\partial \mu \approx -1/(f^2r^6)$, so that $K_i \approx 1$, and the two quantities are equivalent. In general, however, $K_i \neq 1$ but we were not able to develop a mathematical theory for its value. For illustrative purposes, Fig. 4 shows a particular estimate of the ratio $1/K_i$ for the particular example of the analytical reference vortex state described in Appendix A, where the approximation $\mu_i \approx \mu_m$ was used.

3.7. Mean “system” energies

The reference vortex state in gradient wind balance possesses conventional APE and KE relative to a notional state of rest characterised by the reference hydrostatic pressure and density fields $p_0(z)$ and $\rho_0(z)$. The conventional APE density — denoted $\Pi_m$ here
is naturally defined as
\[
\Pi_m = h(\eta, p_0) - h(\eta, p_R) + \Phi(z_0) - \Phi(z_R) = \int_{p_R}^{p_s} [\nu(\eta, p') - \nu_0(p')] \, dp',
\]  
(3.39)
and is equivalent to the APE density denoted by \( \Pi_2 \) in Tailleux (2018). The reference pressure \( p_R = p_0(z_R) \) at the reference depth \( z_R \) satisfies the level of neutral buoyancy (LNB) equation \( \nu(\eta, p_R) = \nu_0(p_R) \) originally introduced in Tailleux (2013), which is key to ensuring that \( \Pi_m \) is positive definite.

The mean kinetic energy of the reference vortex is simply equal to \( v_s^2/2 \). In contrast to the case \( f = 0 \) considered by Andrews (2006), the case of a finite rotation rate \( f \neq 0 \) considered in this paper gives rise to a background radial distribution of angular momentum \( M_R(r) = f r^2/2 \) that introduces a radial restoring inertial force, in the same way that a statically stable vertical gradient of entropy introduces a vertical restoring buoyancy force. As mentioned previously, a reference equilibrium radius \( r_R \) (for which \( v_R = 0 \)) can then be defined as the solution of \( M = f R^2/2 \). As a result, we may write \( v_* \) as
\[
v_* = \frac{M}{r_*} - \frac{f r_*}{2} = \frac{f r_R^2}{2 r_*} - \frac{f r_*}{2} = \frac{f (r_R + r_*)((r_R - r_*))}{2r_*},
\]  
(3.40)
which in turn implies for the kinetic energy \( v_*^2/2 \)
\[
\frac{v_*^2}{2} = \frac{f^2 (r_R + r_*)^2 (r_R - r_*)^2}{8r_*^2}.
\]  
(3.41)

Eq. (3.41) shows that \( v_*^2/2 \) is quadratic in the displacement amplitude \( \delta r = r_* - r_R \) from the equilibrium position \( r_R \) even for finite amplitude \( \delta r \). Moreover, Eq. (3.40) shows that creating a cyclonic circulation \( (v_* > 0 \text{ if } f > 0) \) requires \( r_R > r_* \), and hence the compression of the equilibrium constant angular momentum surfaces. In order to express \( v_*^2/2 \) as the work against the inertial restoring force, we may use (2.9) to write it in the form:
\[
\frac{v_*^2}{2} = \frac{v^2}{2} - \frac{v^2_R}{2} = \mu(\chi_* - \chi_R) + \frac{f^2}{16} \left( \frac{1}{\chi_*} - \frac{1}{\chi_R} \right).
\]  
(3.42)

making use of the fact that \( v^2_R = \mu \chi_R + f^2/(16 \chi_R) - f \sqrt{\mu}/2 = 0 \), which in turn may be written as the following integrals:
\[
\frac{v_*^2}{2} = \int_{r_R}^{r_*} (\mu - \mu_R(r')) \frac{\partial \chi}{\partial r}(r') \, dr' = \int_{\chi_R}^{\chi_*} (\mu - \mu_R(\chi')) \, d\chi'.
\]  
(3.43)

We propose to call the right-hand side of (3.41) and its integral expression (3.43) the available radial energy (ARE). Eq. (3.43) makes it clear that the restoring inertial force experienced by the fluid parcel as it experiences a radial displacement is \(-\left(\mu - \mu_R\right)\partial \chi / \partial r = \left(\mu - \mu_R\right)/r^3\), consistent with Emanuel (1994). For small displacements \( \delta r \), so that \( r_R \approx r_* \), the above expression approximates to \( v_*^2/2 \approx f^2(r_* - r_R)^2/2 \), which suggests that the natural frequency of the radial waves is \( f \), as would be the case for standard inertial waves. These results clearly establish that \( \Pi_k \) and \( \Pi_e \) are the natural generalisations of \( v_*^2/2 \) and \( \Pi_m \) for constructing the actual state from a non-resting reference state instead of a resting one.

4. Energetics of vortex growth and decay due to diabatic effects

For a stable reference vortex state, the above results show that the total eddy energy \( u^2_s/2 + \Pi_1 + \Pi_k + \Pi_e \) is globally conserved for purely adiabatic and inviscid axisymmetric
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disturbances. In that case, \( b_e \) acts a restoring force giving rise to a complex combination of internal and inertial/centrifugal waves, as discussed by Emanuel (1994) for instance. As long as the conditions for symmetric stability (3.28) and (3.34) are met, any transfer between the eddy and mean energies is forbidden, so that there cannot be any net growth or decay of the azimuthal circulation unless non-axisymmetric or diabatic/viscous effects are also considered. As an example of application of our framework, we show in the following how to use it to shed light on the issue of how diabatic/viscous effects may lead to the intensification of a cyclonic vortex, a central issue in the study of tropical cyclones. The discussion of non-axisymmetric effects, which is significantly more involved, is left to a future study.

4.1. Standard energetics viewpoint

Our aim in the following is to establish the conditions under which sinks and sources of specific entropy and angular momentum can lead to the intensification of an incipient cyclonic seed vortex. To that end, we place ourselves in a Northern Hemisphere-like situation \( (v > 0, f > 0) \). A standard viewpoint in the tropical cyclone literature, e.g., Smith et al. (2018), is to consider separate evolution equations for the azimuthal kinetic energy \( v^2/2 \) and the rest of the flow as follows:

\[
\rho \frac{D}{Dt} \left( \frac{u_v^2}{2} + \Phi + h - \frac{p}{\rho} \right) + \nabla \cdot (\rho u_s) \rho u_s + D_s + \rho \dot{q} + \left( f + \frac{v}{r} \right) \rho uv, \tag{4.1}
\]

\[
\rho \frac{D}{Dt} \frac{v^2}{2} = - \left( f + \frac{v}{r} \right) \rho uv + \rho v D_v, \tag{4.2}
\]

Since the dissipation term \( v D_v \) presumably acts as a brake on \( v \), Eq. (4.2) demonstrates that because \( v > 0 \) by design, the radial velocity must be negative \( (u < 0) \) in order for \( v \) to intensify. This is the only way that the energy conversion term \( -(f + v/r)uv \) can be positive and hence act as a source of energy for \( v \). This condition is of course well known and observed in numerical simulations of TCs. What the existing literature appears to be lacking, however, is a clear explanation for how the required radially inward motion is actually driven. Indeed, the emphasis tends to be on vertical motion associated with cumulus convection and the generation of strong updrafts by the release of latent heat. While in an axisymmetric model, mass conservation imposes that a low level radial inward motion should exist to replenish the mass lost by strong vertical motion near the TC centre, such an argument is not a physical explanation. While it suggests that radial forces of the correct sign must exist to do the job, it does not in itself explain what such forces are.

An alternative and purely Eulerian argument that calls for both \( u < 0 \) and \( w > 0 \) can be made from the angular momentum conservation equation (2.7) — which has essentially the same information content as (4.2) — written in the form:

\[
\frac{\partial M}{\partial t} = -u \frac{\partial M}{\partial r} - w \frac{\partial M}{\partial z} + r D_v. \tag{4.3}
\]

If the distribution of \( M \) is such that \( \partial M/\partial r > 0 \) and \( \partial M/\partial z < 0 \) as is seen to be the case for the analytical reference vortex case described in Appendix A and illustrated in Fig. 2, Eq. 4.3 makes it clear that both \( u < 0 \) and \( w > 0 \) will contribute to the local intensification of \( M \) and hence of \( v \). The understanding of axisymmetric TC intensification therefore boils down to understanding how viscous and diabatic effects cooperate to drive an upward and radially inward secondary circulation at low levels near the eyewall.
4.2. Generalised buoyancy/inertial force viewpoint

We now regard the azimuthal circulation as the sum of balanced and unbalanced parts \( v = v_\star + v'' \) (Lagrangian viewpoint) or \( v = v_m + v' \) (Eulerian viewpoint). In this view, the observed intensification of \( v \) may \textit{a priori} be due to the intensification of either \( v_m \) or \( v' \) (equivalently \( v_\star \) or \( v'' \)) or both, depending on how \( v_m \) is defined. Because there is some freedom in the specification of \( v_m \) in the present framework (for instance, it could be imposed to be time independent), we first discuss the case where the intensification of \( v \) may be primarily attributed to that of \( v' \) (restricting ourselves to the Eulerian viewpoint in the following). Evidence that such a case is relevant for the understanding of actual TC intensification is provided by the study of Bui et al. (2009), which suggests that the degree of unbalance of TCs is likely significant, especially in the boundary layer. Now, because \( v' = v - v_m = (M - M_m)/r = (\mu - \mu_m)/(r(M + M_m)) \), any increase in \( v' \) must result from the creation of a positive anomaly \( \mu' = \mu - \mu_m > 0 \) and hence from an increase in the mechanical energy reservoir \( H_k \), the only one that increases when \( |\mu - \mu_m| \) increases.

Prior to discussing energetics, it is useful to first discuss the forces at work in the system as this is what is most helpful to establish causal relationships. To that end, let us consider the form of the momentum equation for the secondary circulation \( u_\star \) that makes apparent the role of the generalised inertial/buoyancy force \( b_e \), viz.,

\[
\frac{D\mathbf{u}_e}{Dt} = \mathbf{b}_e - \frac{1}{\rho} \nabla p' - \nu \nabla p_m + \mathbf{D}_e, \tag{4.4}
\]

where from (3.15), the radial and vertical components of \( b_e \) may be written explicitly as follows:

\[
b_e^{(r)} = \frac{\nu_h - \nu_m}{r} \frac{\partial p_m}{\partial r} + \frac{1}{r^3} \left( \mu - \mu_m \right) = -\rho_m (\nu_h - \nu_m) \left( f + \frac{v_m}{r} \right) v_m + \frac{1}{r^3} \left( \mu - \mu_m \right), \tag{4.5}
\]

\[
b_e^{(z)} = -\nu_h \frac{\partial p_m}{\partial z} = \rho_m g (\nu_h - \nu_m). \tag{4.6}
\]

One of the expected advantages of introducing a non-resting reference state is to minimise the role of \( \nabla p' \) in (4.4) and hence to maximise the ability of \( b_e \) to predict the actual acceleration \( D\mathbf{u}_e/Dt \). Assuming this to be the case, and recalling that \( (\mu - \mu_m)/r^3 > 0 \), Eq. (4.5) shows that a necessary condition for the radial component of \( b_e \) to point towards the centre of the cyclone is that the fluid parcels be positively buoyant,

\[
\nu_h - \nu_m > 0, \tag{4.7}
\]

in which case (4.6) shows that \( b_e^{(z)} \) will also point upward, as is expected physically. By definition, \( \nu_h = \nu(\eta, p_m(r, z)) \) and \( \nu_m = \nu(\eta_m(r, z), p_m(r, z)) \) so that

\[
\nu_h - \nu_m \approx \Gamma(\eta(r, z, t) - \eta_m(r, z)) \tag{4.8}
\]

is proportional to the local entropy anomaly \( \eta' = \eta - \eta_m \) (we have neglected the time variation of the reference variables, but these can be retained if desired). Since \( \Gamma > 0 \) in general, the creation of a positive specific volume anomaly requires a sustained diabatic source of entropy to increase \( \eta \). As discussed by Smith et al. (2005), whether (4.7) is satisfied depends critically on the choice of reference state used to define buoyancy. For instance, Brown (2002) found such a condition to be met for buoyancy defined relative to a relatively elaborate reference vortex state including even some degree of asymmetry. However, Zhang et al. (2000) found the parcels to be negatively buoyant for buoyancy defined relative to a rest state, the desired upward acceleration being then entirely provided by the pressure gradient term \( \nabla p' \).
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\[ \frac{u_s^2}{2} + \Pi_1 \]

\[ u(\mu - \mu) \]

\[ G_s \]

\[ \Pi_e \]

\[ \Pi_k \]

Figure 5: Hypothesised energy pathways associated with the intensification of a cyclonic vortex forced by sources of diabatic heating \( \dot{q} \) and angular momentum \( D\mu/Dt \).

Even if (4.7) holds, it is not sufficient to ensure that \( b_{e}^{(r)} \) be negative. Indeed, because \( (\mu - \mu_m)/r^3 > 0 \), (4.5) imposes a further constraint on the magnitude of positive buoyancy anomalies, namely:

\[ \nu_h - \nu_m > \left[ \rho_m \left( f + \frac{v_m}{r} \right) v_m \right]^{-1} \frac{\mu - \mu_m}{r^3}. \]  

(4.9)

If specific volume anomalies \( \nu_h - \nu_m \) are bounded, as is presumably the case in reality, (4.9) appears to impose an upper limit on the maximum angular momentum anomalies \( \mu - \mu_m \) and hence on the maximum intensity that the vortex can reach. This limit is \textit{a priori} different from the maximum potential intensity (MPI) predicted by Emanuel (1986) (see Emanuel (2018) for a recent review on this topic and wider TC research), which is reached when the production of available energy by surface enthalpy fluxes balances dissipation by surface friction in the region of maximum winds. Whether such a condition could account for why the intensity of many observed TCs remain significantly below their theoretical maximum intensity (Emanuel 2000) is left for future study.

4.3. Energy cycle

The generalised buoyancy/inertial force \( b_e \) and other forces that drive the secondary circulation do work and cause energy transfers between the different existing energy reservoirs, for which sources and sinks must exist in order for the system to achieve a steady state. In the following, we discuss the energy cycle associated with an intensifying cyclonic vortex whose intensification is dominated by the intensification of \( v' \). To that end, we find that the simplest and most economical description of the local energy cycle is one based on separate evolution equations for: the sum of the kinetic energy of the secondary circulation plus the AAE, \( u_s^2/2 + \Pi_1 \); the eddy slantwise APE \( \Pi_e \); and the eddy mechanical energy \( \Pi_k \). This leads to the following set of equations:

\[ \rho \frac{D}{Dt} \left( \frac{u_s^2}{2} + \Pi_1 \right) + \nabla \cdot (\rho' \mathbf{u}_s) = \rho (\mathbf{b}_e^T \cdot \mathbf{u}_s + \mathbf{b}_e^M \cdot \mathbf{u}_s) + \rho G_s, \]  

(4.10)
\[ G_s = \left( \frac{T - T_h}{T} \right) \dot{q} + \nu' \left( \frac{\partial p_m}{\partial t} + \mathbf{u}_s \cdot \nabla p_m \right), \]

\[ \frac{D \Pi_e}{Dt} = -\mathbf{b}_e^T \cdot \mathbf{u}_s + \left( \frac{T_h - T_s}{T} \right) \dot{q} + (\chi_\mu - \chi_*) D\mu \frac{\partial p_m}{\partial t} + \nu_h \frac{\partial p_m}{\partial t} - \nu_* \frac{\partial p_*}{\partial t}, \]

\[ D\Pi_k = -\mathbf{b}_e^M \cdot \mathbf{u}_s + (\chi - \chi_\mu) D\mu \frac{\partial p_m}{\partial t}. \]

For an intensifying vortex resulting from an increase in \( v' \), we established in the previous section that \( \nu_h - \nu_m > 0 \) and \( \mu - \mu_m > 0 \). The implications for the work against the generalised inertial and buoyancy forces \( \mathbf{b}_e^T \) and \( \mathbf{b}_e^M \) by the secondary circulation are:

\[ -\mathbf{b}_e^M \cdot \mathbf{u}_s = (\mu - \mu_m) \nabla \chi \cdot \mathbf{u}_s = -\frac{v(\mu - \mu_m)}{r^3} > 0, \]  

\[ -\mathbf{b}_e^T \cdot \mathbf{u}_s = (\nu_h - \nu_m) \nabla p_m \cdot \mathbf{u}_s = (\nu_h - \nu_m) \left[ u \frac{\partial p_m}{\partial r} - \rho_m g w \right] < 0. \]

The sign of such energy conversions suggest that the flow of energy follows the paths

\[ \Pi_e \rightarrow \frac{u^2_s}{2} + \Pi_1 \rightarrow \Pi_k, \]

as illustrated in Fig. 5. If we neglect the terms related to the time-dependence, the following term needs to be positive

\[ \left( \frac{T_h - T_s}{T} \right) \dot{q} + (\chi_\mu - \chi_*) D\mu \frac{\partial p_m}{\partial t} > 0. \]

If \( D\mu /Dt < 0 \) acts as a retarding effect, Fig. 2 shows that \( (r_* - r_\mu) > 0 \) and hence that \( (\chi_\mu - \chi_*) > 0 \), suggesting that the sink of angular momentum is of the wrong sign. Therefore, for (4.17) to act as a source term, the diabatic term must be positive and larger than the term proportional to the angular momentum sink term, viz.,

\[ \left( \frac{T_h - T_s}{T} \right) \dot{q} > \left| (\chi_\mu - \chi_*) D\mu \frac{\partial p_m}{\partial t} \right| > 0. \]

By definition, \( T_h = T(\eta, p_m) \) and \( T_* = T(\eta, p_*), \) so again from Fig. 2, \( p_m - p_* > 0 \) and therefore \( T_h - T_* > 0 \). Now, if we regard \( p_m = \tilde{p}_m(\eta_m, \mu_m) \) as a function of the reference entropy and squared angular momentum, we have

\[ \frac{(T_h - T_*)}{T} \approx \frac{1}{T} \frac{\partial T}{\partial p}(p_m - p_*) \approx \frac{1}{T} \left\{ \frac{\partial \tilde{p}_m}{\partial \eta_m}(\eta - \eta_m) + \frac{\partial \tilde{p}_m}{\partial \mu_m}(\mu - \mu_m) + \cdots \right\}. \]

Since in general pressure varies little with \( \mu_m \), it follows that the term is dominated by the entropy anomaly, which needs to be positive as \( \partial \tilde{p}_m / \partial \eta_m < 0 \). For the intensification of \( v' \) to proceed, a finite amplitude entropy anomaly \( \eta' \) needs to be produced in order for making the thermodynamic efficiency \( (T_h - T_*)/T \) large enough to satisfy the threshold relation (4.18).

\[ 4.4. \text{Evolution of the reference state and reference state variables} \]

As in all theories of available energy, there is some freedom in choosing the reference state. In this paper, we assumed it to be constant, thus allowing us to ascribe the intensification of a cyclonic vortex to the intensification of \( v' \). A theoretically important choice is to regard the reference state at any given time as the state of minimum energy that can be obtained by means of a re-arrangement of mass conserving entropy and angular momentum (alternatively potential vorticity) advocated by some authors.
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Cullen et al. (2015); Methven & Berrisford (2015); Scotti & Passagia (2019). Methven & Berrisford (2015) applied such an idea to the atmosphere using real data, and found the reference state to evolve only slowly. Whether this could be shown to be the case for tropical cyclones remains to be demonstrated, as the method to compute such a reference state is technically challenging. Existing methods for estimating a balanced reference state such as Smith (2006) or Nolan & Montgomery (2002) are consistent with angular momentum conservation, but inconsistent with an adiabatic re-arrangement.

Regardless of whether the reference state depends on time or not, Eq. (3.2) shows that the diabatic and frictional effects $D\eta/Dt$ and $DM/Dt$ must in general cause the fluid parcel’s reference position $(r_*,z_*)$ to drift with time. Such a drift impacts some of the quantities that play a key role in the discussion of the energy cycle described above, such as the thermodynamic efficiency $(T_h-T_*/T)$, so that its importance needs to be assessed.

Evolution equations for $Dr_*/Dt = u_*$ and $Dz_*/Dt = w_*$ can be obtained by taking the material derivative of (3.2), which yields:

$$u_* \frac{\partial \eta_m}{\partial r} + w_* \frac{\partial \eta_m}{\partial z} = \frac{D\eta}{Dt} - \frac{\partial \eta_m}{\partial t},$$

$$u_* \frac{\partial M_m}{\partial r} + w_* \frac{\partial M_m}{\partial z} = \frac{DM}{Dt} - \frac{\partial M_m}{\partial t}.$$  \hspace{1cm} (4.20)

(4.21)

If the Jacobian $J_0 = \partial(M_m,\eta_m)/\partial(r,z)$ differs from zero, the above system can be inverted for $u_*$ and $w_*:

$$u_* = \frac{Dr_*}{Dt} = \frac{1}{J_0} \left\{ \frac{\partial \eta_m}{\partial z} \left( \frac{DM}{Dt} - \frac{\partial M_m}{\partial t} \right) - \frac{\partial M_m}{\partial z} \left( \frac{D\eta}{Dt} - \frac{\partial \eta_m}{\partial t} \right) \right\},$$

$$w_* = \frac{Dz_*}{Dt} = \frac{1}{J_0} \left\{ \frac{\partial \eta_m}{\partial r} \left( \frac{DM}{Dt} - \frac{\partial M_m}{\partial t} \right) + \frac{\partial M_m}{\partial r} \left( \frac{D\eta}{Dt} - \frac{\partial \eta_m}{\partial t} \right) \right\}.$$  \hspace{1cm} (4.22)

(4.23)

For the particular idealised reference vortex considered in this paper, $J_0 > 0$ everywhere, so that (4.20-4.21) is invertible. Moreover, the partial derivatives are such that $\partial \eta_m/\partial z > 0$, $\partial M_m/\partial z < 0$, $\partial \eta_m/\partial r < 0$ and $\partial M_m/\partial r > 0$. As a result, for a time-independent reference state, a positive diabatic entropy source $D\eta/Dt > 0$ and negative angular momentum sink $DM/Dt < 0$ have opposing effects on $u_*$ and $w_*$.

5. Discussion and conclusions

The conventional local APE density (Andrews 1981; Holliday & McIntyre 1981; Tailleux 2018) has long been known to be interpretable as the work against the conventional buoyancy force that is needed to move a fluid parcel from its notional resting position to its actual position. The arbitrariness of the reference state in the local APE theory appears therefore to reflect the well known non-unique character of the definition of the conventional buoyancy force that pervades the literature about stratified fluids (Turner 1973). The possibility of similarly interpreting the local available energy including momentum constraints for axisymmetric zonal and vortex motions derived by Codoban & Shepherd (2003, 2006); Andrews (2006) in terms of the work done against suitably defined forces had been lacking so far, however. The demonstration that such an interpretation is possible, and that the forces involved are the generalised buoyancy and inertial/centrifugal forces discussed by Smith et al. (2005) and Rayleigh (1916); Emanuel (1994) respectively, is one of the main achievements of the present paper. To formalise the fact that a radial gradient of angular momentum plays the same role as the vertical gradient of entropy in giving rise to an inertial/centrifugal restoring force that
is the direct counterpart of the conventional buoyancy force, we introduced a new form of available energy density associated with the radial distribution of squared angular momentum, baptized Available Radial Energy (ARE) density. When it is estimated relative to a reference angular momentum distribution of a resting rotating state, the ARE coincides with the vortex kinetic energy $v^2/2$.

Because it is defined relative to a non-resting reference state, the available energy including momentum constraints is best interpreted as a form of eddy energy related to the local buoyancy. In contrast, the available energy of the reference vortex state can be regarded as the system buoyancy, to use the terminology of Smith et al. (2005). In this paper, the eddy form of ARE is denoted by $\Pi_k$ and the eddy form of the conventional APE density is denoted by $\Pi_e$. Physically, $\Pi_e$ is naturally defined as a slantwise APE that has been extensively discussed in the literature, e.g., Bennetts & Hoskins (1979); Emanuel (1983). On the other hand, we are not aware that either ARE or its eddy form $\Pi_k$ has been previously discussed. For a symmetrically stable reference vortex, the sum of the available energy and kinetic energy is conserved for adiabatic and inviscid motions. In this paper, only two conditions are found to be sufficient to ensure symmetric stability: 1) that the squared angular momentum increases with radius along isobaric surfaces; 2) that the entropy increases vertically along surfaces of constant angular momentum. Although these cannot be regarded as new, they are given here in a significantly clearer and more explicit way than Andrews (2006) and Ilin (1991), for instance.

As in previous discussions of local available energetics and of the concept of buoyancy, we are unable to reach firm conclusions about the optimal strategy, provided there is one, for specifying the reference state in the present theory. Some key practical considerations seem worth noting, however. First, it is clear that one of the most desirable features of the reference state is that it minimises the role of the pressure gradient term $\nabla p'$ relative to that of the generalised buoyancy/inertial force $b_e$ in the momentum equation for the secondary circulation $u_s$. Second, it is important to note that, as emphasised by Codoban & Shepherd (2003), the choice of reference state impacts the sign and magnitude of the thermodynamic $(T_h - T_\star)/T$ and mechanical $(\chi - \chi_\star)$ efficiencies, which determine the relative fraction of the diabatic heating/cooling and friction contributing to the production/destruction of the available energy. This is of key importance for discussing causality in the considered system. There is some sense that the latter should be defined as the state of minimum energy obtainable by means of a re-arrangement of mass conserving both entropy and angular momentum, as suggested by Cullen et al. (2015) and Scotti & Passagia (2019) for instance.

We also derived evolution equations for the different energy reservoirs, accounting for the effects of sources/sinks of entropy and angular momentum, and applied the resulting framework to develop some understanding of how diabatic forcing may lead to the intensification of a cyclonic vortex and of the associated energy cycle. In the case where the vortex reference state can be assumed constant or slowly varying, our framework shows that intensification can only occur where positive buoyancy anomalies (defined relative to the non-resting reference vortex state) are created, in order for the generalised buoyancy force discussed by Smith et al. (2005) to be able to drive an inward and vertically directed secondary circulation. Creating such a positive buoyancy anomaly requires a source of diabatic heating. This makes it a priori possible for TC-like vortices to exist in a dry or semi-dry atmosphere provided that suitable sources can be shown to exist in such a case, as appears to be the case in Mrowiec et al. (2011); Cronin & Chavas (2019). In a moist atmosphere, however, there seems to be little choice but for the required source of diabatic heating to result from the release of latent heat associated with condensation, as is widely recognised. Our framework suggests an amplification mechanism that requires
the creation of a sufficiently large positive buoyancy anomaly that will in turn increase the thermodynamic efficiency of the system, thus initiating the creation of slantwise APE \( \Pi_e \), which is then converted into \( \Pi_k \). Demonstrating the relevance of such a mechanism for actual tropical cyclones requires generalising the present framework to account for humidity; this is beyond the scope of this paper and will be addressed in a forthcoming study.

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Appendix A. Analytical expression for vortex motions

Many of the illustrations of this paper are based on a dry idealised tropical cyclone axisymmetric vortex taken from Smith et al. (2005), the description of which is reproduced here. This idealised TC is defined by its pressure perturbation

\[
p(s, z) = (p_c - p_\infty(0)) \left[ 1 - \exp \left( \frac{-x}{s} \right) \right] \exp \left( \frac{-z}{z^*} \right) \cos \left( \frac{\pi z}{2 z_0} \right), \tag{A 1}\]

where \( p_c \) is the central pressure at the surface, \( p_\infty(0) \) is the surface pressure at large radial distance, \( s = r/r_m \) and \( x, r_m, z_0 \) and \( z^* \) are constants. We choose \( p_\infty(0) - p_c \) and \( x \) so that the maximum tangential wind speed is about \( 40 \text{ m} \cdot \text{s}^{-1} \) at a radius of about 40 km and declines to zero at an altitude of \( z_0 = 16 \text{ km} \): specifically \( p_\infty(0) - p_c = 50 \text{ mb} \), \( z^* = 8 \text{ km} \) and \( x = 1.048 \). The exponential decay with height approximately matches the decrease in the environmental density with height and is necessary to produce a reasonably realistic tangential wind distribution that decreases in strength with height.

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