HPD-ININVARIANCE OF THE
TATE, BEILINSON AND PARSHIN CONJECTURES

GONÇALO TABUADA

Abstract. We prove that the Tate, Beilinson and Parshin conjectures are
invariant under Homological Projective Duality (=HPD). As an application,
we obtain a proof of these celebrated conjectures (as well as of the strong form
of the Tate conjecture) in the new cases of linear sections of determinantal
varieties and complete intersections of quadrics. Furthermore, we extend the
original conjectures of Tate, Beilinson and Parshin from schemes to stacks and
prove these extended conjectures for certain low-dimensional global orbifolds.

1. Introduction and statement of results

Let $k := F_q$ be a finite field of characteristic $p$ with $q = p^n$, $W(k)$ the associated
ring of $p$-typical Witt vectors, and $K := W(k)[1/p]$ the fraction field of $W(k)$.
Given a smooth projective $k$-scheme $X$, we will write $Z^*(X)\mathbb{Q}$ for the (graded)
$\mathbb{Q}$-vector space of algebraic cycles on $X$ up to rational equivalence, $Z^*(X)\mathbb{Q}/\sim_{num}$
for the quotient of $Z^*(X)\mathbb{Q}$ with respect to the numerical equivalence relation,
$H^*_\text{crys}(X) := H^*_\text{crys}(X/W(k)) \otimes_{W(k)} K$ for the crystalline cohomology groups of $X$,
and $K_* (X)\mathbb{Q}$ for the $\mathbb{Q}$-linearized algebraic $K$-theory groups of $X$.

Given a prime number $l \neq p$, consider the associated cycle class map
\begin{equation}
Z^*(X)\mathbb{Q}_l \longrightarrow H^2_{\text{ladic}}(X_k, \mathbb{Q}_l)(*)^{\text{Gal} (\overline{\mathbb{Q}}/k)}
\end{equation}
with values in $l$-adic cohomology. In the sixties, Tate [38] conjectured the following:

Conjecture $T^l(X)$: The cycle class map (1.1) is surjective.

In the same vein, consider the cycle class map
\begin{equation}
Z^*(X)\mathbb{Q}_p \longrightarrow H^2_{\text{crys}}(X)(*)^{\text{Fr}_p}
\end{equation}
with values in the $\mathbb{Q}_p$-vector subspace of those elements which are fixed by the
crystalline Frobenius $\text{Fr}_p$. Following [26], Tate’s conjecture admits the $p$-version:

Conjecture $T^p(X)$: The cycle class map (1.2) is surjective.

In the eighties, Beilinson (see [13, Conj. 50]) conjectured the following:

Conjecture $B(X)$: The equality $Z^*(X)\mathbb{Q} = Z^*(X)\mathbb{Q}/\sim_{num}$ holds.

Also in the eighties, Parshin (see [13, Conj. 51]) conjectured the following:

Conjecture $P(X)$: We have $K_i (X)\mathbb{Q} = 0$ for every $i \geq 1$.

Date: May 7, 2018.

Key words and phrases. Tate conjecture, Beilinson conjecture, Parshin conjecture, homologi-
projective duality, determinantal variety, quadric, Hasse-Weil zeta function, orbifold, algebraic
$K$-theory, topological periodic cyclic homology, noncommutative algebraic geometry.

The author was supported by a NSF CAREER Award.
All the above conjectures hold whenever \( \dim(X) \leq 1 \); see [8, 9, 13, 15, 37]. The conjectures \( T^i(X) \) and \( T^p(X) \) hold\(^1\) moreover for abelian varieties of dimension \( \leq 3 \) and for \( K3\)-surfaces; see [26, 41]. Besides these cases (and some other scattered cases), the aforementioned important conjectures remain wide open; consult Theorems 1.8 and 1.14 below for a proof of the Tate, Beilinson and Parshin conjectures in several new cases.

Recall from §2.1 that a differential graded (=dg) category \( \mathcal{A} \) is a category enriched over dg \( k \)-vector spaces. As explained in §3, given a smooth proper dg category \( \mathcal{A} \) in the sense of Kontsevich, the conjectures of Tate, Beilinson and Parshin admit noncommutative analogues \( T^i_{\text{nc}}(\mathcal{A}), T^p_{\text{nc}}(\mathcal{A}), B_{\text{nc}}(\mathcal{A}), \) and \( P_{\text{nc}}(\mathcal{A}) \), respectively. Examples of smooth proper dg categories include finite dimensional \( k \)-algebras of finite global dimension \( A \) as well as the canonical dg enhancement \( \text{perf}_{\text{dg}}(X) \) of the category of perfect complexes \( \text{perf}(X) \) of smooth proper \( k \)-schemes \( X \) (or, more generally, smooth proper algebraic stacks \( X \)); consult [16, 23].

**Theorem 1.3.** Given a smooth projective \( k \)-scheme \( X \), we have the equivalences:

\[
T^i(X) \iff T^i_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \quad T^p(X) \iff T^p_{\text{nc}}(\text{perf}_{\text{dg}}(X))
\]

\[
B(X) \iff B_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \quad P(X) \iff P_{\text{nc}}(\text{perf}_{\text{dg}}(X))
\]

Theorem 1.3 shows that the conjectures of Tate, Beilinson and Tate belong not only to the realm of algebraic geometry but also to the broad setting of smooth proper dg categories. Making use of this latter noncommutative viewpoint, we now prove that these celebrated conjectures are invariant under Homological Projective Duality (=HPD); for surveys on HPD we invite the reader to consult [22, 39].

Let \( X \) be a smooth projective \( k \)-scheme equipped with a line bundle \( \mathcal{L}_X(1) \); we write \( X \to \mathbb{P}(V) \) for the associated morphism, where \( V := H^0(X, \mathcal{L}_X(1))^* \). Assume that the triangulated category \( \text{perf}(X) \) admits a Lefschetz decomposition \( \langle A_0, A_1(1), \ldots, A_{i-1}(i-1) \rangle \) with respect to \( \mathcal{L}_X(1) \) in the sense of [21, Def. 4.1]. Following [21, Def. 6.1], let \( Y \) be the HP-dual of \( X \), \( \mathcal{L}_Y(1) \) the HP-dual line bundle, and \( Y \to \mathbb{P}(V^*) \) the morphism associated to \( \mathcal{L}_Y(1) \). Given a linear subspace \( L \subset V^* \), consider the linear sections \( X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp) \) and \( Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L) \).

**Theorem 1.4 (HPD-invariance).** Let \( X \) and \( Y \) be as above. Assume that \( X_L \) and \( Y_L \) are smooth\(^2\), that \( \dim(X_L) = \dim(X) - \dim(L) \) and \( \dim(Y_L) = \dim(Y) - \dim(L^\perp) \), and that the conjecture \( T^i_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}), \) resp. \( T^p_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}), \) resp. \( B_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}), \) resp. \( P_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}) \), holds, where \( A_{0\text{dg}}^{\text{nc}} \) stands for the dg enhancement of \( A_0 \) induced from \( \text{perf}_{\text{dg}}(X) \). Under these assumptions, we have the equivalence \( T^i(X_L) \iff T^i(Y_L) \), resp. \( T^p(X_L) \iff T^p(Y_L) \), resp. \( B(X_L) \iff B(Y_L) \), resp. \( P(X_L) \iff P(Y_L) \).

**Remark 1.5.** (i) Given a generic subspace \( L \subset V^* \), the sections \( X_L \) and \( Y_L \) are smooth, and \( \dim(X_L) = \dim(X) - \dim(L) \) and \( \dim(Y_L) = \dim(Y) - \dim(L^\perp) \).

(ii) The conjectures \( T^i_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}), T^p_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}), B_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}), \) and \( P_{\text{nc}}(A_{0\text{dg}}^{\text{nc}}) \), hold, in particular, whenever the triangulated category \( A_0 \) admits a full exceptional collection.

(iii) Theorem 1.4 holds more generally when \( Y \) is singular. In this case we need to replace \( Y \) by a noncommutative resolution of singularities \( \text{perf}_{\text{dg}}(Y; \mathcal{F}) \), where \( \mathcal{F} \) stands for a certain sheaf of noncommutative algebras over \( Y \) (consult [22, §2.4 for details), and conjecture \( T^i(Y) \), resp. \( T^p(Y) \), resp. \( B(Y) \), resp. \( P(Y) \),

---

\(^1\)As explained in §2.4 below, whenever \( \dim(X) \leq 3 \), we have \( T^i(X) \iff T^p(X) \) (for every \( l \neq p \)).

\(^2\)The linear section \( X_L \) is smooth if and only if the linear section \( Y_L \) is smooth; see [22, page 9].
by its noncommutative analogue \( T_{nc}(\text{perf}_{dg}(Y; \mathcal{F})) \), resp. \( T_{nc}(\text{perf}_{dg}(Y; \mathcal{F})) \), resp. \( B_{nc}(\text{perf}_{dg}(Y; \mathcal{F})) \), resp. \( P_{nc}(\text{perf}_{dg}(Y; \mathcal{F})) \).

To the best of the author's knowledge, Theorem 1.4 is new in the literature. In what follows, we illustrate its strength in the case of two important HP-dualities.

**Determinantal duality.** Let \( U_1 \) and \( U_2 \) be two \( k \)-vector spaces of dimensions \( d_1 \) and \( d_2 \), respectively, with \( d_1 \leq d_2 \), \( V := U_1 \otimes U_2 \), and \( 0 < r < d_1 \) an integer.

Consider the determinantal variety \( Z_{d_1, d_2}^r \subset \mathbb{P}(V) \) defined as the locus of those matrices \( U_2 \to U_1^* \) with rank \( r \). Recall that the determinantal varieties with \( r = 1 \) are the classical Segre varieties. For example, \( Z_{2, 2}^1 \subset \mathbb{P}^3 \) is the quadric surface defined as the zero locus of the \( 2 \times 2 \) minor \( v_0v_3 - v_1v_2 \). In contrast with the Segre varieties, the determinantal varieties \( Z_{d_1, d_2}^r \), with \( r \geq 2 \), are not smooth. The singular locus of \( Z_{d_1, d_2}^r \) consists of those matrices \( U_2 \to U_1^* \) with rank \( r \), i.e. it agrees with the closed subvariety \( Z_{d_1, d_2}^{r-1} \). Nevertheless, it is well-known that \( Z_{d_1, d_2}^r \) admits a canonical Springer resolution of singularities \( X_{d_1, d_2}^r \to Z_{d_1, d_2}^r \), which comes equipped with a projection \( q: X_{d_1, d_2}^r \to \text{Gr}(r, U_1) \) to the Grassmannian of \( r \)-dimensional subspaces in \( U_1 \).

Following [3, §3.3], the category \( \text{perf}(X) \), with \( X := X_{d_1, d_2}^r \), admits a Lefschetz decomposition \( \langle A_0, A_1, \ldots, A_{d_2 r-1}(d_2 r - 1) \rangle \), where \( A_0 = A_1 = \cdots = A_{d_2 r-1} = q^*(\text{perf}(\text{Gr}(r, U_1))) \cong \text{perf}(\text{Gr}(r, U_1)) \).

**Proposition 1.6.** The following conjectures hold:
\[
T_{nc}(A_{0}^{dg}) \quad T_{nc}(A_{q}^{dg}) \quad B_{nc}(A_{0}^{dg}) \quad P_{nc}(A_{q}^{dg}).
\]

Dually, consider the variety \( W_{d_1, d_2}^r \subset \mathbb{P}(V^*) \), defined as the loci of those matrices \( U_2 \to U_1 \) with corank \( r \), and the associated Springer resolutions of singularities \( Y := Y_{d_1, d_2}^r \to W_{d_1, d_2}^r \). As proved\(^3\) in [3, Prop. 3.4 and Thm. 3.5], \( X \) and \( Y \) are HP-dual to each other.

Given a generic linear subspace \( L \subseteq V^* \), consider the smooth linear sections \( X_L \) and \( Y_L \); note that whenever \( \mathbb{P}(L^\perp) \) does not intersects the singular locus of \( Z_{d_1, d_2}^r \), we have \( X_L = \mathbb{P}(L^\perp) \cap Z_{d_1, d_2}^r \). Theorem 1.4 yields the following result:

**Corollary 1.7.** We have the following equivalences:
\[
T(L)(X_L) \Leftrightarrow T(L)(Y_L) \quad T(\mathbb{P}(X_L)) \Leftrightarrow T(\mathbb{P}(Y_L)) \quad B(X_L) \Leftrightarrow B(Y_L) \quad P(X_L) \Leftrightarrow P(Y_L).
\]

By construction, \( \dim(X) = r(d_1 + d_2 - r) - 1 \) and \( \dim(Y) = r(d_1 - d_2 - r) + d_1 d_2 - 1 \). Consequently, we have \( \dim(X_L) = r(d_1 + d_2 - r) - 1 - \dim(L) \) and \( \dim(Y_L) = r(d_1 - d_2 - r) + d_1 d_2 + 1 + \dim(L) \). Since the Tate, Beilinson and Parshin conjectures hold in dimensions \( \leq 1 \), we hence obtain from Corollary 1.7 the following result:

**Theorem 1.8 (Linear sections of determinantal varieties).** Let \( X_L \) and \( Y_L \) be smooth linear sections of determinantal varieties as in Corollary 1.7.

(i) Whenever \( r(d_1 + d_2 - r) - 1 - \dim(L) \leq 1 \), the conjectures \( T(L)(Y_L), \quad T(\mathbb{P}(Y_L)), \quad B(Y_L), \quad \text{and} \quad P(Y_L), \) hold.

(ii) Whenever \( r(d_1 - d_2 - r) + d_1 d_2 + 1 + \dim(L) \leq 1 \), the conjectures \( T(L)(X_L), \quad T(\mathbb{P}(X_L)), \quad B(X_L), \quad \text{and} \quad P(X_L), \) hold.

\(^3\)In [3, Prop. 3.4 and Thm. 3.5] the authors worked over an algebraically closed field of characteristic zero. However, the same proof holds mutatis mutandis over \( k = \mathbb{F}_q \). Simply replace the reference [14] concerning the existence of a full strong exceptional collection on \( \text{perf}(\text{Gr}(r, U_1)) \) by the reference [5, Thm. 1.3] concerning the existence of a tilting bundle on \( \text{perf}(\text{Gr}(r, U_1)) \). The author is grateful to Marcello Bernardara for discussions concerning this issue.
To the best of the author’s knowledge, Theorem 1.8 is new in the literature. It proves the Tate, Beilinson and Parshin conjectures in several new cases. Here are two families of examples:

**Example 1.9 (Segre varieties).** Let \( r = 1 \). Thanks to Theorem 1.8(ii), whenever \( d_1 - d_2 - 2 + \dim(L) \leq 1 \), the conjectures \( T^i(X_L), T^p(X_L), B(X_L), \) and \( P(X_L) \), hold. In all these cases, \( X_L \) is a linear section of the Segre variety \( Z^1_{d_1,d_2} \) and the dimension of \( X_L \) is \( 2(d_2 - \dim(L)) \) or \( 2(d_2 - \dim(L)) + 1 \). Therefore, for example, by letting \( d_2 \to \infty \) and by keeping \( \dim(L) \) fixed, we obtain infinitely many new examples of smooth projective \( k \)-schemes \( X_L \), of arbitrary high dimension, satisfying the Tate, Beilinson and Parshin conjectures.

**Subexample 1.10.** Let \( r = 1, d_1 = 4, \) and \( d_2 = 2 \). In this particular case, the Segre variety \( Z^1_{4,2} \subset \mathbb{P}^7 \) agrees with the rational normal 4-fold scroll \( S_{1,1,1,1} \); see [10, Ex. 8.27]. Choose a generic linear subspace \( L \subset V^* \) of dimension 1 such that the hyperplane \( \mathbb{P}(L^\perp) \subset \mathbb{P}^7 \) does not contain any 3-plane of the ruling of \( S_{1,1,1,1} \). By combining Example 1.9 with [6, Prop. 2.5], we conclude that the rational normal 3-fold scroll \( X_L = S_{1,1,2} \) satisfies the Tate, Beilinson and Parshin conjectures.

**Example 1.11 (Square matrices).** Let \( d_1 = d_2 = d \). Thanks to Theorem 1.8(ii), whenever \( -r^2 - 1 + \dim(L) \leq 1 \), the conjectures \( T^i(X_L), T^p(X_L), B(X_L), \) and \( P(X_L) \) hold. In all these cases the dimension of \( X_L \) is \( 2(dr - \dim(L)) \) or \( 2(dr - \dim(L)) + 1 \). Therefore, for example, by letting \( d \to \infty \) and by keeping \( r \) and \( \dim(L) \) fixed, we obtain infinitely many new examples of smooth projective \( k \)-schemes \( X_L \), of arbitrary high dimension, satisfying the Tate, Beilinson and Parshin conjectures.

**Subexample 1.12.** Let \( d_1 = d_2 = 3 \) and \( r = 2 \). In this particular case, the determinantal variety \( Z^2_{3,3} \subset \mathbb{P}^8 \) has dimension 7 and its singular locus is the 4-dimensional Segre variety \( Z^3_{3,3} \subset Z^3_{3,3} \). Given a generic linear subspace \( L \subset V^* \) of dimension 5, the associated smooth linear section \( X_L \) is 2-dimensional and, thanks to Example 1.11, it satisfies the Tate, Beilinson and Parshin conjectures. Note that since \( \text{codim}(L^\perp) = 5 > 4 = \dim(Z^1_{3,3}) \), the subspace \( \mathbb{P}(L^\perp) \subset \mathbb{P}^8 \) does not intersects the singular locus \( Z^1_{3,3} \) of \( Z^3_{3,3} \). Therefore, for all the above choices of \( L \), the associated surface \( X_L \) is a linear section of the determinantal variety \( Z^3_{3,3} \).

**Veronese-Clifford duality.** Let \( W \) be a \( k \)-vector space of dimension \( d \) and \( X \) the associated projective space \( \mathbb{P}(W) \) equipped with the double Veronese embedding \( \mathbb{P}(W) \to \mathbb{P}(S^2W), [w] \mapsto [w \otimes w] \). Consider the Beilinson’s full exceptional collection \( \text{perf}(X) = \langle \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{O}_X(1), \ldots, \mathcal{O}_X(d-2) \rangle \) (see [2]) and set \( i := \lfloor d/2 \rfloor \) and

\[
\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_{i-2} := \langle \mathcal{O}_X(-1), \mathcal{O}_X \rangle, \quad \mathcal{A}_i := \begin{cases} \langle \mathcal{O}_X(-1), \mathcal{O}_X \rangle & \text{if } d = 2i \\ \langle \mathcal{O}_X(-1) \rangle & \text{if } d = 2i - 1. \end{cases}
\]

Under these notations, the category \( \text{perf}(X) \) admits the Lefschetz decomposition \( \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{i-1}(i-1) \rangle \) with respect to the line bundle \( \mathcal{L}_X(1) = \mathcal{O}_X(2) \). Remark 1.5(ii) hence implies the conjectures \( T^p_{\text{nc}}(A^4_0), T^p_{\text{nc}}(A^4_0), B_{\text{nc}}(A^4_0), \) and \( P_{\text{nc}}(A^4_0) \).

Let \( \mathcal{H} := X \times_{\mathbb{P}(S^2W)} Q \subset X \times \mathbb{P}(S^2W^*) \) be the universal hyperplane section, where \( Q \subset \mathbb{P}(S^2W) \times \mathbb{P}(S^2W^*) \) stands for the incidence quadric. By construction, the projection \( q: \mathcal{H} \to \mathbb{P}(S^2W^*) \) is a flat quadric fibration. As proved in [20, Thm. 5.4] (see also [1, Thm. 2.3.6]) the HP-dual \( Y \) of \( X \) is given by \( \text{perf}_{\text{dc}}(\mathbb{P}(S^2W^*); \mathcal{C}_0(q)) \) (see Remark 1.5(iii)), where \( \mathcal{C}_0(q) \) stands for the sheaf of even Clifford algebras associated to \( q \).
Let $L \subset S^2W^*$ be a generic linear subspace. On the one hand, $X_L$ corresponds to the smooth complete intersection of the $\dim(L)$ quadric hypersurfaces in $\mathbb{P}(W)$ parametrized by $L$. On the other hand, $Y_L$ is given by $\text{perf}_{dg}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)$. Theorem 1.14 yields the following result:

**Corollary 1.13.** We have the following equivalences:

\[ T^l(X_L) \leftrightarrow T^l_{\text{nc}}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)) \quad T^p(X_L) \leftrightarrow T^p_{\text{nc}}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)) \]

\[ B(X_L) \leftrightarrow B_{\text{nc}}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)) \quad P(X_L) \leftrightarrow P_{\text{nc}}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)) . \]

Recall that the space of quadrics $\mathbb{P}(S^2W^*)$ comes equipped with a canonical filtration $\Delta_d \subset \cdots \subset \Delta_2 \subset \Delta_1 \subset \mathbb{P}(S^2W^*)$, where $\Delta_i$ stands for the closed subscheme of those singular quadrics of corank $\geq i$.

**Theorem 1.14** (Intersection of two quadrics). Let $X_L$ be as in Corollary 1.13. Assume that $\dim(L) = 2$, that $\mathbb{P}(L) \cap \Delta_2 = \emptyset$, and that $p \neq 2$ when $d$ is odd. Under these assumptions, the conjectures $T^l(X_L)$, $T^p(X_L)$, $B(X_L)$, and $P(X_L)$, hold.

**Remark 1.15** (Intersection of even-dimensional quadrics). In the case of an intersection $X_L$ of (several) even-dimensional quadrics, we prove in Theorem 7.11 below that the conjectures $T^l(X_L)$ (for every $l \neq 2$), $T^p(X_L)$, $B(X_L)$, and $P(X_L)$, are equivalent to the corresponding conjectures for the discriminant 2-fold cover of the projective space $\mathbb{P}(L)$. To the best of the author’s knowledge, this (geometric) result is new in the literature.

The proof of Theorem 1.14 is based on the solution of the corresponding noncommutative conjectures of Corollary 1.13; consult §7 for details. In what concerns Tate’s conjecture, an alternative (geometric) proof, based on the notion of variety of maximal planes, was obtained by Reid\(^4\) in the early seventies; see [29, Thms. 3.14 and 4.14]. Therein, Reid proved the Hodge conjecture but, as Kahn kindly informed me, a similar proof works for the Tate conjecture. In what concerns the Beilinson and Parshin conjectures, Theorem 1.14 is new in the literature.

**Strong form of the Tate conjecture.** Given a smooth projective $k$-scheme $X$, let us write $\text{ord}_{z=i}\zeta(X, s)$ for the order of the pole of the Hasse-Weil zeta function $\zeta(X, s)$ of $X$ at $i \in \{0, 1, \ldots, \dim(X)\}$. In the sixties, Tate [38] also conjectured the following strong form of the Tate conjecture:\(^5\)

Conjecture $ST(X)$: The equality $\text{ord}_{z=i}\zeta(X, s) = \dim Z^l(X)_{Q}/_{\sim \text{num}}$ holds.

Let us write $Z^l(X)_{Q}/_{\sim \text{t-adiq}}$, resp. $Z^l(X)_{Q}/_{\sim \text{crys}}$, for the quotient of $Z^l(X)_{Q}$ with respect to the $l$-adic homological equivalence relation, resp. crystalline homological equivalence relation. Note that Beilinson’s conjecture $B(X)$ implies that $Z^l(X)_{Q}/_{\sim \text{t-adiq}} = Z^l(X)_{Q}/_{\sim \text{crys}} = Z^l(X)_{Q}/_{\sim \text{num}}$. Therefore, making use of [37, Thm. 2.9], resp. [26, Thm. 1.11], we conclude that $T^l(X) + B(X) \Rightarrow ST(X)$, resp. $T^p(X) + B(X) \Rightarrow ST(X)$. This implies that the strong form of the Tate conjecture also holds in the several new cases provided by Theorems 1.8 and 1.14.

\(^4\)Reid also assumed in loc. cit. that $\mathbb{P}(L) \cap \Delta_2 = \emptyset$; see [29, Def. 1.9].

\(^5\)As proved in [37, Thm. 2.9], resp. [26, Thm. 1.11], the strong form of the Tate conjecture $ST(X)$ implies the Tate conjecture $T^l(X)$ (for every $l \neq p$), resp. the $p$-version of the Tate conjecture $T^p(X)$.
Tate, Beilinson and Parshin conjectures for stacks. Theorem 1.3 allow us to easily extend the original conjectures of Tate, Beilinson and Parshin from smooth projective \( k \)-schemes \( X \) to smooth proper algebraic \( k \)-stacks \( \mathcal{X} \) by setting

\[
? \mathcal{X} := ?_{?} (\text{perf}_{dg}(\mathcal{X})) \quad \text{with} \quad ? \in \{ T^l, T^p, B, P \}.
\]

The next result proves these extended conjectures for “low-dimensional” orbifolds:

**Theorem 1.16.** Let \( G \) be a finite group of order \( s \), \( X \) a smooth projective \( k \)-scheme equipped with a \( G \)-action, and \( \mathcal{X} := [X/G] \) the associated global orbifold. If \( p \nmid s \), then we have the following implications

\[
\begin{align*}
(1.17) \quad \sum_{\sigma \leq G} T^l(X^\sigma \times \text{Spec}(k[\sigma])) & \Rightarrow T^l(\mathcal{X}) \quad \text{(for every } l \mid s) \\
(1.18) \quad \sum_{\sigma \leq G} T^p(X^\sigma \times \text{Spec}(k[\sigma])) & \Rightarrow T^p(\mathcal{X}) \\
(1.19) \quad \sum_{\sigma \leq G} B(X^\sigma \times \text{Spec}(k[\sigma])) & \Rightarrow B(\mathcal{X}) \\
(1.20) \quad \sum_{\sigma \leq G} P(X^\sigma \times \text{Spec}(k[\sigma])) & \Rightarrow P(\mathcal{X}),
\end{align*}
\]

where \( \sigma \) is an arbitrary cyclic subgroup of \( G \). Moreover, whenever \( s \mid (q - 1) \), resp. \( \dim(X) \leq 3 \), the \( k \)-schemes \( X^\sigma \times \text{Spec}(k[\sigma]) \) in (1.17)-(1.20), resp. in (1.17)-(1.18), can be replaced by the \( k \)-schemes \( X^\sigma \).

Note that the assumption \( s \mid (q - 1) \) implies that \( p \nmid s \).

**Corollary 1.21.** (i) Assume that \( p \nmid s \). If \( \dim(X) \leq 1 \), then the conjectures

\[
T^l(\mathcal{X}) \quad \text{(for every } l \mid s), \quad T^p(\mathcal{X}), \quad B(\mathcal{X}), \quad \text{and } P(\mathcal{X}), \quad \text{hold}.
\]

(ii) Assume\(^6\) that \( s \mid (q - 1) \). If \( \dim(X) = 2 \), then \( T^l(\mathcal{X}) \Rightarrow T^l(\mathcal{X}) \quad \text{(for every } l \mid s), \quad T^p(\mathcal{X}) \Rightarrow B(\mathcal{X}), \quad B(\mathcal{X}) \Rightarrow B(\mathcal{X}), \quad \text{and } P(\mathcal{X}) \Rightarrow P(\mathcal{X}).
\]

**Example 1.22.** Let \( X \) be an abelian surface equipped with the \( \mathbb{Z}/2 \)-action \( a \mapsto -a \). Since the conjectures \( T^l(X) \) and \( T^p(X) \) hold, Corollary 1.21(ii) implies that the conjectures \( T^l(\mathcal{X}) \) (for every \( l \neq 2 \)) and \( T^p(\mathcal{X}) \) also hold.

We finish this section with the following “twisted” version of Corollary 1.21:

**Theorem 1.23.** Let \( G \) be a finite group of order \( s \), \( X \) a smooth projective \( k \)-scheme equipped with a \( G \)-action, \( \mathcal{X} := [X/G] \) the associated global orbifold, and \( \mathcal{F} \) a \( G \)-equivariant sheaf of Azumaya algebras over \( S \) of rank \( r \). Assume that \( s \mid (q - 1) \).

(i) If \( \dim(X) \leq 1 \), then the conjectures \( T^l(\mathcal{X}; \mathcal{F}) \quad \text{(for every } l \mid sr), \quad T^p(\mathcal{X}; \mathcal{F}), \quad B(\mathcal{X}; \mathcal{F}), \quad \text{and } P(\mathcal{X}; \mathcal{F}), \quad \text{hold}.
\]

(ii) If the \( G \)-action is faithful and \( \dim(X) = 2 \), then \( T^l(\mathcal{X}) \Rightarrow T^l(\mathcal{X}; \mathcal{F}) \quad \text{(for every } l \mid sr), \quad T^p(\mathcal{X}) \Rightarrow T^p(\mathcal{X}; \mathcal{F}), \quad B(\mathcal{X}) \Rightarrow B(\mathcal{X}; \mathcal{F}), \quad \text{and } P(\mathcal{X}) \Rightarrow P(\mathcal{X}; \mathcal{F}).
\]

2. Preliminaries

Throughout the article, \( k := \mathbb{F}_q \) is a finite field of characteristic \( p \) with \( q = p^n \).

2.1. Dg categories. For a survey on dg categories, we invite the reader to consult Keller’s ICM address [16]. Let \((\mathcal{C}(k), \otimes, k)\) be the category of dg \( k \)-vector spaces. A differential graded (=dg) category \( \mathcal{A} \) is a category enriched over \( \mathcal{C}(k) \) and a dg functor \( F : \mathcal{A} \to \mathcal{B} \) is a functor enriched over \( \mathcal{C}(k) \). In what follows, we will write \( \text{dgcat}(k) \) for the category of (essentially small) dg categories and dg functors.

Let \( \mathcal{A} \) be a dg category. The opposite dg category \( \mathcal{A}^{op} \) has the same objects and \( \mathcal{A}^{op}(x, y) := \mathcal{A}(y, x) \). A right dg \( \mathcal{A} \)-module is a dg functor \( M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(k) \) with

\(^6\)In the particular case of the (\( p \)-version of the) Tate conjecture, it suffices to assume that \( p \nmid s \).
values in the dg category $C_{dg}(k)$ of dg $k$-vector spaces. Following [16, §3.2], the derived category $\mathcal{D}(A)$ of $A$ is defined as the localization of the category of right dg $A$-modules $\mathcal{C}(A)$ with respect to the objectwise quasi-isomorphisms. In what follows, we will write $\mathcal{D}_c(A)$ for the triangulated subcategory of compact objects.

A dg functor $F: A \to B$ is called a Morita equivalence if it induces an equivalence on derived categories $\mathcal{D}(A) \simeq \mathcal{D}(B)$; see [16, §4.6]. As explained in [30, §1.6], the category $dgcat(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by $Hmo(k)$ the associated homotopy category.

The tensor product $A \otimes B$ of dg categories is defined as follows: the set of objects is $\text{obj}(A) \times \text{obj}(B)$ and $(A \otimes B)((x, w), (y, z)) := A(x, y) \otimes B(w, z)$. As explained in [16, §2.3], this construction gives rise to a symmetric monoidal structure on $dgcat(k)$ which descends to the homotopy category $Hmo(k)$.

A dg $A$-$B$-bimodule is a dg functor $B: A \otimes B^{op} \to C_{dg}(k)$ or, equivalently, a right dg $(A^{op} \otimes B)$-module. A standard example is the dg $A$-$B$-bimodule

$$ (2.1) \quad \rho B : A \otimes B^{op} \to C_{dg}(k) \quad (x, z) \mapsto B(z, F(x)) $$

associated to a dg functor $F: A \to B$. Following Kontsevich [17, 18, 19], a dg category $A$ is called smooth if the dg $A$-$A$-bimodule $id_A$ belongs to the category $\mathcal{D}_c(A^{op} \otimes A)$ and proper if $\sum_i \dim H^i(A, x, y) < \infty$ for any pair of objects $(x, y)$.

2.2. Additive invariants. Let $A$ and $B$ be two dg categories and $B$ a dg $A$-$B$-bimodule. Consider the following dg category $T(A, B; B)$: the set of objects is $\text{obj}(A) \amalg \text{obj}(B)$, the dg $k$-vector spaces of morphisms are given as follows

$$ T(A, B; B)(x, y) := \begin{cases} A(x, y) & \text{if } x, y \in A \\ B(x, y) & \text{if } x, y \in B \\ B(x, y) & \text{if } x \in A \text{ and } y \in B \\ 0 & \text{if } x \in B \text{ and } y \in A, \end{cases} $$

and the composition law is induced by the composition law of $A$ and $B$ and by the dg $A$-$B$-bimodule structure of $B$. Note that, by construction, we have canonical dg functors $\iota_A: A \to T(A, B; B)$ and $\iota_B: B \to T(A, B; B)$.

Recall from [30, Def. 2.1] that a functor $E: dgcat(k) \to D$, with values in an additive category, is called an additive invariant if it satisfies the following conditions:

(i) It sends the Morita equivalences to isomorphisms.

(ii) Given $A$, $B$, and $B$ as above, the dg functors $\iota_A$ and $\iota_B$ induce an isomorphism

$$ E(A) \oplus E(B) \xrightarrow{\sim} E(T(A, B; B)). $$

Let us write $\text{rep}(A, B)$ for the full triangulated subcategory of $\mathcal{D}(A^{op} \otimes B)$ consisting of those dg $A$-$B$-modules $B$ such that for every object $x \in A$ the associated right dg $B$-module $B(x, -)$ belongs to $\mathcal{D}_c(B)$. As explained in [30, §1.6.3], there is a natural bijection between $\text{Hom}_{Hmo(k)}(A, B)$ and the set of isomorphism classes of the category $\text{rep}(A, B)$. Under this bijection, the composition law of $Hmo(k)$ corresponds to the (derived) tensor product of bimodules. Therefore, since the dg $A$-$B$-bimodules (2.1) belong to $\text{rep}(A, B)$, we have the following symmetric monoidal functor:

$$ (2.2) \quad dgcat(k) \to Hmo(k) \quad A \mapsto A \quad (A \xrightarrow{\rho} B) \mapsto \rho B. $$

The additiveization of $Hmo(k)$ is the additive category $Hmo_0(k)$ with the same objects as $Hmo(k)$ and with abelian groups of morphisms $\text{Hom}_{Hmo_0(k)}(A, B)$ given by
the Grothendieck group $K_0 \operatorname{rep}(A,B)$ of the triangulated category $\operatorname{rep}(A,B)$. As explained in [30, §2.3], the following composition

$$
(2.3) \quad U : \text{dgcat}(k) \xrightarrow{(2.2)} \operatorname{Hmo}(k) \longrightarrow \operatorname{Hmo}_0(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad (\mathcal{A} \xrightarrow{f} \mathcal{B}) \mapsto [f_\mathcal{B}]
$$

is the universal additive invariant. Moreover, the symmetric monoidal structure of $\operatorname{Hmo}(k)$ extends to $\operatorname{Hmo}_0(k)$, making the above functor $(2.3)$ symmetric monoidal.

2.3. Noncommutative motives. For a book, resp. survey, on noncommutative motives, we invite the reader to consult [30], resp. [31]. Given a commutative ring $R$, recall from [30, §4.1] that the category of noncommutative Chow motives $\operatorname{NChow}(k)_R$ (with $R$-coefficients) is defined as the idempotent completion of the full subcategory of $\operatorname{Hmo}_0(k)_R$ consisting of those objects $U(A)_R$ with $A$ a smooth proper dg category. As explained in loc. cit., the category $\operatorname{NChow}(k)_R$ is $R$-linear, additive, and rigid symmetric monoidal. Moreover, we have natural isomorphisms:

$$
(2.4) \quad \operatorname{Hom}_{\operatorname{NChow}(k)_R}(U(A)_R, U(B)_R) := K_0(\operatorname{rep}(A^\text{opp} \otimes B)_R) \simeq K_0(A^\text{opp} \otimes B)_R .
$$

Given a $R$-linear, additive, rigid symmetric monoidal category $(C, \otimes, 1)$, its $\mathcal{N}$-ideal is defined as follows ($\text{tr}(g \circ f)$ stands for the categorical trace of $g \circ f$):

$$
\mathcal{N}(a,b) := \{ f \in \operatorname{Hom}_C(a,b) \mid \forall g \in \operatorname{Hom}_C(b,a) \text{ we have } \text{tr}(g \circ f) = 0 \} .
$$

Under these notations, recall from [30, §4.6] that the category of noncommutative numerical motives $\operatorname{NNum}(k)_R$ (with $R$-coefficients) is defined as the idempotent completion of the quotient category $\operatorname{NChow}(k)_R/\mathcal{N}$.

2.4. Tate conjecture for divisors. Let $X$ be a smooth projective $k$-scheme of dimension $d$. Given a prime number $l \neq p$, consider the Tate conjecture for divisors:

Conjecture $T^{1,1}(X)$: The cycle class map $(1.1)$ with $* = 1$ is surjective.

As proved in [37, Prop. 4.1], we have the implication $T^{1,1}(X) \Rightarrow T^{l,1}(X)$. Consequently, whenever $\dim(X) \leq 3$, we conclude that $T^{1,1}(X) \Leftrightarrow T^{l,1}(X)$.

Consider also the $p$-version of the Tate conjecture for divisors:

Conjecture $T^{p,1}(X)$: The cycle class map $(1.2)$ with $* = 1$ is surjective.

As proved in [27, Prop. 4.1], we have $T^{p,1}(X) \Leftrightarrow T^{1,1}(X)$ (for every $l \neq p$). Moreover, a proof similar to the one of [37, Prop. 5.1], with the commutative diagram (2.3) of [37] replaced by the commutative diagram of [27, page 25], shows that $T^{p,1}(X) \Rightarrow T^{p,d-1}(X)$. Consequently, whenever $\dim(X) \leq 3$, we conclude that $T^{p,1}(X) \Rightarrow T^{p,1}(X)$. This implies that whenever $\dim(X) \leq 3$, we have the equivalence $T^{1,1}(X) \Leftrightarrow T^{p,1}(X)$ (for every $l \neq p$).

3. Noncommutative conjectures

Throughout this section, $A$ denotes a smooth proper dg category.

Noncommutative Tate conjecture. Given a prime number $l \neq p$, consider the following abelian groups

$$
(3.1) \quad \operatorname{Hom}(\mathbb{Z}(l^\infty), \pi_{-1} L_{KU} K(A \otimes_{\mathbb{Z}_q} \mathbb{F}_{q^m})) \quad m \geq 1 ,
$$

where $\mathbb{Z}(l^\infty)$ stands for the Prüfer $l$-group, $K(A \otimes_{\mathbb{Z}_q} \mathbb{F}_{q^m})$ for the algebraic $K$-theory spectrum of the dg category $A \otimes_{\mathbb{Z}_q} \mathbb{F}_{q^m}$, and $L_{KU} K(A \otimes_{\mathbb{Z}_q} \mathbb{F}_{q^m})$ for the Bousfield localization of $K(A \otimes_{\mathbb{Z}_q} \mathbb{F}_{q^m})$ with respect to topological complex $K$-theory $KU$. Note that the abelian groups $(3.1)$ can, alternatively, be defined as the $l$-adic Tate
module of the abelian groups $\pi_{-1} L_{KU} K(\mathcal{A} \otimes_{F_p} F_{q^m})$, $m \geq 1$. Under these notations, Tate’s conjecture admits the following noncommutative analogue:

Conjecture $T^l_{nc}(\mathcal{A})$: The abelian groups (3.1) are trivial.

Remark 3.2. Note that the conjecture $T^l_{nc}(\mathcal{A})$ holds, for example, whenever the abelian groups $\pi_{-1} L_{KU} K(\mathcal{A} \otimes_{F_p} F_{q^m})$, $m \geq 1$, are finitely generated.

**Noncommutative $p$-version of the Tate conjecture.** By construction, the topological Hochschild homology $THH(\mathcal{A})$ of $\mathcal{A}$ carries a canonical $S^1$-action. This leads naturally to the spectrum of homotopy orbits $THH(\mathcal{A})_{hS^1}$, to the spectrum of homotopy fixed-points $TC^- (\mathcal{A}) := THH(\mathcal{A})^{hS^1}$, and also to the Tate construction $TP(\mathcal{A}) := THH(\mathcal{A})^{tS^1}$. As explained in [28, Cor. I.4.3], these spectra are related by the following cofiber sequence

$$
\Sigma THH(\mathcal{A})_{hS^1} \xrightarrow{N} THH(\mathcal{A})^{hS^1} \xrightarrow{\text{can}} THH(\mathcal{A})^{tS^1},
$$

where $N$ stands for the norm map. It is well-known that the abelian groups $THH_{i}(\mathcal{A})$ are $k$-linear. Hence, after inverting $p$, we have $\Sigma THH(\mathcal{A})_{hS^1} [1/p] \simeq \mathbb{S}$. Consequently, the above cofiber sequence (3.3) leads to a canonical isomorphism:

$$
\text{can}: TC^-_{0} (\mathcal{A})_{1/p} \xrightarrow{\sim} TP_{0}(\mathcal{A})_{1/p}.
$$

It is also well-known that the spectrum $THH(\mathcal{A})$ is bounded below, i.e. there exists an integer $m \gg 0$ such that $THH_{i}(\mathcal{A}) = 0$ for every $i < m$. This follows, for example, from Bökstedt’s celebrated computation $THH_{*}(k) \simeq k[u]$ (where the variable $u$ is of degree 2) and from the fact that $THH_{*}(\mathcal{A})$ is a dualizable $THH_{*}(k)$-module. Since the abelian groups $THH_{*}(\mathcal{A})$ are $k$-linear, the spectrum $THH(\mathcal{A})$ is moreover $p$-complete. Making use of [28, Lem. II 4.2], we hence obtain a “cyclotomic Frobenius” (which is defined before inverting $p$):

$$
\varphi_{p}: TC^-_{0} (\mathcal{A})_{1/p} \longrightarrow TP_{0}(\mathcal{A})_{1/p}.
$$

Let $\varphi := \varphi_{p} \circ \text{can}^{-1}$ be the associated endomorphism of $TP_{0}(\mathcal{A})_{1/p}$. It is also well-known that $TP_{0}(\mathcal{A})_{1/p}$ is a (finitely generated) module over $TP_{0}(k)_{1/p} \simeq K$, i.e. a (finite-dimensional) $K$-vector space. Moreover, the endomorphism $\varphi$ is $\varsigma$-semilinear with respect to the automorphism $\varsigma: K \rightarrow K$ that acts as $\lambda \mapsto \lambda^{p}$ on $k$. Hence, $\varphi^{n}$ becomes a $K$-linear endomorphism of $TP_{0}(\mathcal{A})_{1/p}$.

Recall from [32, Prop. 4.2] that the assignment $\mathcal{A} \mapsto TP_{0}(\mathcal{A})_{1/p}$ gives rise to a $K$-linear functor with values in the category of $K$-vector spaces:

$$
TP_{0}(-)_{1/p}: \text{NChow}(k) \longrightarrow \text{Vect}(K).
$$

This leads to the induced $K$-linear homomorphism:

$$
K_{0}(\mathcal{A})_{K} \simeq \text{Hom}(U(k)_{K}, U(\mathcal{A})_{K}) \xrightarrow{\theta} \text{Hom}(TP_{0}(k)_{1/p}, TP_{0}(\mathcal{A})_{1/p}) \simeq TP_{0}(\mathcal{A})_{1/p}.
$$

**Lemma 3.7.** The preceding homomorphism $\theta$ take values in the $K$-linear subspace $TP_{0}(\mathcal{A})_{1/p}^{\text{can}}$ of those elements which are fixed by the $K$-linear endomorphism $\varphi^{n}$.

**Proof.** On the one hand, the $K$-linear endomorphisms $\varphi^{n}: TP_{0}(\mathcal{A})_{1/p} \rightarrow TP_{0}(\mathcal{A})_{1/p}$ (parametrized by the smooth proper dg categories $\mathcal{A}$) give rise to a natural transformation of the above functor (3.6). On the other hand, thanks to the enriched Yoneda lemma, the $K$-linear natural transformations from the following functor

$$
K_{0}(\mathcal{A})_{K} \simeq \text{Hom}_{\text{NChow}(k)_{K}}(U(k)_{K}, -): \text{NChow}(k)_{K} \longrightarrow \text{Vect}(K)
$$


to the above functor (3.6) are in one-to-one correspondence with the elements of $TP_0(k)_{1/p} \simeq K$. Under this bijection, the identity element $1 \in K$ corresponds to the above homomorphisms $\theta$. Therefore, in order to prove Lemma 3.7, it suffices to show that the endomorphism $\varphi^n : TP_0(k)_{1/p} \to TP_0(k)_{1/p}$ sends $1$ to $1$. This follows from the following explicit descriptions
\[
\text{can: } W(k) [u, v] / (uv - p) \to W(k) [\delta, \delta^{-1}] \quad u \mapsto p\delta \quad v \mapsto \delta^{-1} \\
\varphi_p : W(k) [u, v] / (uv - p) \to W(k) [\delta, \delta^{-1}] \quad u \mapsto \delta \quad v \mapsto p\delta^{-1}
\]
of the homomorphisms can, $\varphi_p : TC_\sim (k) \to TP_0(k)$, where the variables $u$ and $\delta$ have degree $2$ and the variable $v$ has degree $-2$; see [4, Props. 6.2-6.3].

Thanks to Lemma 3.7, we have a $K$-linear homomorphism:

$$K_0(A)_K \to TP_0(A)_{1/p}^{\sim}.$$  

The $p$-version of Tate’s conjecture admits the following noncommutative analogue:

Conjecture $T_{nc}^p(A)$: The homomorphism (3.8) is surjective.

Noncommutative Beilinson conjecture. Recall from [30, §4.7] that the group $K_0(A) := K_0(D_c(A))$ comes equipped with the Euler bilinear pairing:

$$\chi : K_0(A) \times K_0(A) \to \mathbb{Z} \quad ([M], [N]) \mapsto \sum_i (-1)^i \text{dim} \text{Hom}_{D_c(A)}(M, N[-i]).$$

This bilinear pairing is, in general, not symmetric neither skew-symmetric. Nevertheless, as proved in [30, Prop. 4.24], the left and right kernels agree. Consequently, we obtain the numerical Grothendieck group $K_0(A)_{/\sim_{num}} := K_0(A) / \text{Ker}(\chi)$.

Notation 3.9. Let $K_0(A)_{/\sim_{num}} := K_0(A)_{/\sim_{num}} / \text{Ker}(\chi) \simeq (K_0(A)_{/\sim_{num}})_{\mathbb{Q}}$.

Beilinson’s conjecture admits the following noncommutative analogue:

Conjecture $B_{nc}(A)$: The equality $K_0(A)_\mathbb{Q} = K_0(A)_{/\sim_{num}}$ holds.

Noncommutative Parshin conjecture. Parshin’s conjecture admits the following noncommutative analogue:

Conjecture $P_{nc}(A)$: We have $K_i(A)_{\mathbb{Q}} = 0$ for every $i \geq 1$.

4. Proof of Theorem 1.3

As proved by Thomason in [40], the Tate conjecture $T^l(X)$ is equivalent to the vanishing of the abelian groups $\text{Hom}(\mathbb{Z}(l^\infty), \pi_{i-1}L_{KU} K(X \times \mathbb{F}_q \mathbb{F}_{q^m}), m \geq 1$. Therefore, the proof of the equivalence $T^l(X) \Leftrightarrow T^l_{nc}(\text{perf}_{dg}(X))$ follows from the canonical Morita equivalence between the dg categories $\text{perf}_{dg}(X \times \mathbb{F}_q \mathbb{F}_{q^m})$ and $\text{perf}_{dg}(X \times \mathbb{F}_q \mathbb{F}_{q^m})$; consult [34, Lem. 4.26].

Let us now prove the equivalence $T_p^l(X) \Leftrightarrow T^l_{nc}(\text{perf}_{dg}(X))$. Recall that the ring of $p$-typical Witt vectors $W(k)$ is the unramified extension of degree $m$ of the ring of $p$-adic integers $\mathbb{Z}_p$. Hence, we have an induced field extension $\mathbb{Q}_p \to K$. Note that the cycle class map (1.2) is surjective if and only if the $K$-linear homomorphism

$$\mathbb{Z}^*(X) \otimes_{\mathbb{Q}_p} K \to H_{crys}^{2*}(X)(*)^{Fr_p} \otimes_{\mathbb{Q}_p} K$$

is surjective. Therefore, making use of the following natural isomorphisms

$$H_{crys}^{2*}(X)(*)^{Fr_p} \otimes_{\mathbb{Q}_p} K \simeq H_{crys}^{2*}(X)^{Fr_p} \otimes_{\mathbb{Q}_p} K \simeq H_{crys}^{2*}(X)^{Fr_p} = H_{crys}^{2*}(X)^{Fr_p},$$
we conclude that the $p$-version of the Tate conjecture $T^p(X)$ is equivalent to the surjectivity of the induced $K$-linear cycle class map

\[(4.1) \quad Z^*(X)_K \rightarrow H^*_{\text{cris}}(X)_{\dagger p\text{Fr}_q}.
\]

On the one hand, since $\text{char}(K) = 0$, recall from [7, §18.3] that we have a natural isomorphism $K_0(\text{perf}_{\text{dg}}(X)) \simeq Z^*(X)_K$. On the other hand, recall from [33, Thm. 5.2] that we have a natural isomorphism $T^p_0(\text{perf}_{\text{dg}}(X))_{1/p} \simeq H^*_{\text{cris}}(X)$. Under these isomorphisms, the endomorphism $\varphi^n$ corresponds to the endomorphism $\frac{1}{q^n}\text{Fr}_q$ (see [11]) and the homomorphism (4.1) corresponds to the $K$-linear homomorphism (3.8). Consequently, (4.1) is surjective if and only if (3.8) is surjective.

Let us now prove the equivalence $B(X) \Leftrightarrow B_{\text{nc}}(\text{perf}_{\text{dg}}(X))$. Note first that since $D_c(\text{perf}_{\text{dg}}(X)) \simeq \text{perf}(X)$, the Euler bilinear pairing is given as follows:

\[\chi: K_0(X) \times K_0(X) \rightarrow \mathbb{Z} \quad ([\mathcal{F}], [\mathcal{G}]) \rightarrow \sum_i (-1)^i \dim \text{Hom}_{\text{perf}(X)}(\mathcal{F}, \mathcal{G}[-i]).\]

Recall from [7, §19] that an algebraic cycle $\beta \in Z^*(X)_Q$ is numerically equivalent to zero if $\int_X \alpha \cdot \beta = 0$ for every $\alpha \in Z^*(X)_Q$. Recall also that we have the isomorphism

\[(4.2) \quad \tau: K_0(X)_Q \xrightarrow{\sim} Z^*(X)_Q \quad [\mathcal{F}] \mapsto \text{ch}(\mathcal{F}) \cdot \sqrt{\text{Td}_X},
\]

where $\text{ch}(\mathcal{F})$ stands for the Chern character of $\mathcal{F}$ and $\sqrt{\text{Td}_X}$ for the square root of the Todd class of $X$; see [7, §18.3]. Given any two perfect complexes $\mathcal{F}, \mathcal{G} \in \text{perf}(X)$, the Hirzebruch-Riemann-Roch theorem (see [7, Cor. 18.3.1]) yields the equality

\[(4.3) \quad \text{Eu}(\pi_*(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})) = \int_X \tau([\mathcal{F}^\vee]) \cdot \tau([\mathcal{G}]),
\]

where $\text{Eu}$ denotes the Euler characteristic and $\pi: X \rightarrow \text{Spec}(k)$ denotes the structural morphism of $X$. Since $\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G} \cong \text{Hom}(\mathcal{F}, \mathcal{G})$, where $\text{Hom}(-, -)$ stands for the internal Hom of the rigid symmetric monoidal category $\text{perf}(X)$, we hence conclude that $\text{Eu}(\pi_*(\text{Hom}(\mathcal{F}, \mathcal{G}))) \equiv (4.3)$ agrees with $\chi([\mathcal{F}], [\mathcal{G}])$. This implies that the above isomorphism (4.2) descends to the numerical quotients:

\[(4.4) \quad K_0(X)_Q \xrightarrow{\sim} Z^*(X)_Q \\
\quad \quad \xrightarrow{\tau} K_0(X)_Q/\sim_{\text{num}} \xrightarrow{\sim} Z^*(X)_Q/\sim_{\text{num}}.
\]

Consequently, the proof of the equivalence $B(X) \Leftrightarrow B_{\text{nc}}(\text{perf}_{\text{dg}}(X))$ follows now from the fact that $B(X)$, resp. $B_{\text{nc}}(\text{perf}_{\text{dg}}(X))$, is equivalent to the injectivity of the vertical homomorphism on the right-hand side, resp. left hand-side, of (4.4).

Finally, the proof of the equivalence $P(X) \Leftrightarrow P_{\text{nc}}(\text{perf}_{\text{dg}}(X))$ is clear.

5. PROOF OF THEOREM 1.4

By definition of the Lefschetz decomposition $\langle a_0, a_1(1), \ldots, a_{i-1}(i-1) \rangle$, we have a chain of admissible triangulated subcategories $A_{i-1} \subseteq \cdots \subseteq A_1 \subseteq A_0$ with $A_r(\mathcal{M}) := A_r \otimes \mathcal{L}_X(\mathcal{M})$. Note that $A_r(\mathcal{M}) \simeq A_r$. Let $a_r$ be the right orthogonal complement to $A_{r+1}$ in $A_r$; these are called the primitive subcategories in [21, §4]. By construction, we have the following semi-orthogonal decompositions:

\[(5.1) \quad A_r = \langle a_r, a_{r+1}, \ldots, a_{i-1} \rangle \quad 0 \leq r \leq i-1.
\]
As proved in [21, Thm. 6.3] (see also [1, Thm. 2.3.4]), the category $\text{perf}(Y)$ admits a HP-dual Lefschetz decomposition of the form $\langle B_{j-1}(1-j), B_{j-2}(2-j), \ldots, B_0 \rangle$ with respect to $L_Y(1)$; as above, we have a chain of admissible triangulated subcategories $B_{j-1} \subseteq B_{j-2} \subseteq \cdots \subseteq B_0$. Moreover, the primitive subcategories coincide (via a Fourier-Mukai type functor) with those of $\text{perf}(X)$ and we have the following semi-orthogonal decompositions:

\[
E_r = \langle a_0, a_1, \ldots, a_{\dim(V) - r - 2} \rangle \quad 0 \leq r \leq j - 1.
\]

Furthermore, the assumptions $\dim(X_L) \leq \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^+) \leq \dim(L)$ imply the existence of semi-orthogonal decompositions

\[
\text{perf}(X_L) = \langle C_L, \mathbb{A}_{\dim(V)}(1), \ldots, \mathbb{A}_{j-1}(i - \dim(V)) \rangle
\]

\[
\text{perf}(Y_L) = \langle B_{j-1}(\dim(L^+) - j), \ldots, B_{\dim(L^+)}(-1), C_L \rangle,
\]

where $C_L$ is a common (triangulated) category. Let us denote, respectively, by $C_L^{dg}$, $A_r^{dg}$, and $a_r^{dg}$, the dg enhancement of $C_L$, $A_r$, and $a_r$, induced from $\text{perf}^{dg}(X_L)$.

Similarly, let us denote by $C_L^{dg'}$ and $B^{dg'}$ the dg enhancement of $C_L$ and $B$, induced from $\text{perf}^{dg}(Y_L)$. Note that since by assumption the $k$-schemes $X_L$ and $Y_L$ are smooth (and proper), all the above dg categories are smooth (and proper).

Let us now prove the equivalence $T^f(X_L) \cong T^f(Y_L)$. Consider the functors

\[
E_m : \text{dgcat}(k) \longrightarrow \text{Mod}(\mathbb{Z}) \quad A \mapsto \pi_{-1}L_{KU}K(A \otimes_{F_q} \mathbb{F}_{q^m})
\]

with values in the additive category of abelian groups.

**Proposition 5.6.** The functors (5.5) are additive invariants.

**Proof.** Let $F : A \rightarrow B$ be a Morita equivalence and $m \geq 1$ an integer. As proved in [24, Prop. 7.1], the induced dg functor $F \otimes_{F_q} \mathbb{F}_{q^m} : A \otimes_{F_q} \mathbb{F}_{q^m} \rightarrow B \otimes_{F_q} \mathbb{F}_{q^m}$ is also a Morita equivalence. Therefore, since algebraic $K$-theory inverts Morita equivalences (see [30, §2.1]), the homomorphism $K(F \otimes_{F_q} \mathbb{F}_{q^m}) : K(A \otimes_{F_q} \mathbb{F}_{q^m}) \rightarrow K(B \otimes_{F_q} \mathbb{F}_{q^m})$ is invertible. By definition of the above functors (5.5), we hence conclude that the induced group homomorphism $E_m(A) \rightarrow E_m(B)$ is also invertible.

Now, let $A$ and $B$ be two dg categories and $B$ a dg $A$-$B$-bimodule. Following §2.2, we need to show that the dg functors $\iota_A$ and $\iota_B$ induce an isomorphism

\[
E_m(A) \oplus E_m(B) \longrightarrow E_m(T(A, B; B)).
\]

Consider the dg categories $A \otimes_{F_q} \mathbb{F}_{q^m}$ and $B \otimes_{F_q} \mathbb{F}_{q^m}$ and the dg bimodule $B \otimes_{F_q} \mathbb{F}_{q^m}$. Since algebraic $K$-theory is an additive invariant of dg categories, the dg functors $\iota_A \otimes_{F_q} \mathbb{F}_{q^m}$ and $\iota_B \otimes_{F_q} \mathbb{F}_{q^m}$ induce an isomorphism

\[
K(A \otimes_{F_q} \mathbb{F}_{q^m}) \oplus K(B \otimes_{F_q} \mathbb{F}_{q^m}) \cong K(T(A \otimes_{F_q} \mathbb{F}_{q^m}, B \otimes_{F_q} \mathbb{F}_{q^m}; B \otimes_{F_q} \mathbb{F}_{q^m})).
\]

Therefore, by definition of the above functors (5.5), we conclude from (5.8) that the homomorphism (5.7) is also invertible. This finishes the proof. \qed

Thanks to Proposition 5.6, the functors (5.5) are additive invariants. As explained in [30, Prop. 2.2], this implies that the above semi-orthogonal decompositions (5.3)-(5.4) give rise to direct sum decompositions of abelian groups:

\[
E_m(\text{perf}_{dg}(X_L)) \simeq E_m(C_L^{dg}) \oplus E_m(A_{\dim(V)}^{dg}) \oplus \cdots \oplus E_m(A_{j-1}^{dg})
\]

\[
E_m(\text{perf}_{dg}(Y_L)) \simeq E_m(B_{j-1}^{dg}) \oplus \cdots \oplus E_m(B_{\dim(L^+)}^{dg}) \oplus E_m(C_L^{dg}).
\]
Consequently, by applying the functor $\Hom(\mathbb{Z}(l^\infty), -)$ to the direct sum decompositions (5.9)-(5.10), we obtain the following equivalences of conjectures:

\begin{align}
(5.11) \quad & \quad T^l_{\text{nc}}(\text{perf}_{dg}(X_L)) \Leftrightarrow T^l_{\text{nc}}(C^dg_L) + T^l_{\text{nc}}(A^dg_{\dim(V)}) + \cdots + T^l_{\text{nc}}(A^dg_{l-1}) \\
(5.12) \quad & \quad T^l_{\text{nc}}(\text{perf}_{dg}(Y_L)) \Leftrightarrow T^l_{\text{nc}}(C^dg_{j-1}) + \cdots + T^l_{\text{nc}}(B^dg_{\dim(L^+)}) + T^l_{\text{nc}}(C^dg_L').
\end{align}

On the one hand, since by assumption the conjecture $T^l_{\text{nc}}(A^dg_0)$ holds, we conclude from the above semi-orthogonal decompositions (5.1)-(5.2) that the conjectures $T^l_{\text{nc}}(A^dg_r)$ and $T^l_{\text{nc}}(B^dg_r)$, with $0 \leq r \leq l - 1$, also hold. This implies that the right-hand side of (5.11), resp. (5.12), reduces to the conjecture $T^l_{\text{nc}}(C^dg_L)$, resp. $T^l_{\text{nc}}(C^dg_L')$. On the other hand, since the functor $\text{perf}(X_L) \to C_L \to \text{perf}(Y_L)$ is of Fourier-Mukai type, the dg categories $C^dg_L$ and $C^dg_L'$ are Morita equivalent. Using the fact that the functors (5.5) invert Morita equivalences, we hence conclude that $T^l_{\text{nc}}(C^dg_L) \Leftrightarrow T^l_{\text{nc}}(C^dg_L')$. Consequently, the proof follows now from the equivalences $T^l(X_L) \Leftrightarrow T^l_{\text{nc}}(\text{perf}_{dg}(X_L))$ and $T^l(Y_L) \Leftrightarrow T^l_{\text{nc}}(\text{perf}_{dg}(Y_L))$ of Theorem 1.4.

Let us now prove the equivalence $T^p(X_L) \Leftrightarrow T^p(Y_L)$. As explained in [30, Prop. 2.2], since the functor (2.3) is an additive invariant, the above semi-orthogonal decomposition (5.3) gives rise to the following direct sum decomposition

\begin{align}
(5.13) \quad & \quad U(\text{perf}_{dg}(X_L))_K \simeq U(C^dg_L)_K + U(A^dg_{\dim(V)})_K + \cdots + U(A^dg_{l-1})_K
\end{align}

in the $K$-linearized category $\text{Hmo}_K(k)$. Recall from the proof of Lemma 3.7 that the functor (3.6) comes equipped with the natural transformation $\varphi^n$. Therefore, by applying the $K$-linear functor (3.6) to the above direct sum decomposition (5.13), we conclude that the induced $K$-linear homomorphism

\begin{align}
K_0(\text{perf}_{dg}(X_L))_K \to T P_0(\text{perf}_{dg}(X_L))_{1/p}^{\varphi^n}
\end{align}

identifies with the induced (diagonal) $K$-linear homomorphism

\begin{align}
K_0(C^dg_L) \oplus \bigoplus_{r=\dim(V)} K_0(A^dg_{l-1})_K \to T P_0(C^dg_L)_{1/p}^{\varphi^n} \oplus \bigoplus_{r=\dim(V)} T P_0(A^dg_{l-1}).
\end{align}

This implies the following equivalence of conjectures:

\begin{align}
T^p_{\text{nc}}(\text{perf}_{dg}(X_L))_K \Leftrightarrow T^p_{\text{nc}}(C^dg_L) + T^p_{\text{nc}}(A^dg_{\dim(V)}) + \cdots + T^p_{\text{nc}}(A^dg_{l-1}).
\end{align}

All the above holds mutatis mutandis with $X_L$ replaced by $Y_L$. Consequently, the above semi-orthogonal decomposition (5.4) leads to the equivalence of conjectures

\begin{align}
T^p_{\text{nc}}(\text{perf}_{dg}(Y_L))_K \Leftrightarrow T^p_{\text{nc}}(B^dg_{j-1}) + \cdots + T^p_{\text{nc}}(B^dg_{\dim(L^+)}) + T^p_{\text{nc}}(C^dg_L').
\end{align}

The remainder of the proof is now similar to the proof of $T^l(X_L) \Leftrightarrow T^l(Y_L)$.

Let us now prove the equivalence $B(X_L) \Leftrightarrow B(Y_L)$. As above, the semi-orthogonal decompositions (5.3)-(5.4) give rise to the following direct sum decompositions

\begin{align}
(5.14) \quad & \quad U(\text{perf}_{dg}(X_L))_Q \simeq U(C^dg_L)_Q + U(A^dg_{\dim(V)})_Q + \cdots + U(A^dg_{l-1})_Q \\
(5.15) \quad & \quad U(\text{perf}_{dg}(Y_L))_Q \simeq U(B^dg_{j-1})_Q + \cdots + U(B^dg_{\dim(L^+)})_Q + U(C^dg_L')._Q
\end{align}

in the $Q$-linearized category $\text{Hmo}_Q(k)$. As proved in [32, §6], given any smooth proper dg category $\mathcal{A}$, we have a natural isomorphism:

\begin{align}
(5.16) \quad & \quad \text{Hom}_{\text{Num}(k)}(U(k)_Q, U(\mathcal{A})_Q) \simeq K_0(\mathcal{A})_Q / -\text{num}.
\end{align}
Hence, by applying $\text{Hom}_{\text{NChow}(k)}(U(k)_{\mathbb{Q}}, -)$ and $\text{Hom}_{\text{NNum}(k)}(U(k)_{\mathbb{Q}}, -)$ to the direct sum decompositions (5.14)-(5.15) we obtain the equivalences of conjectures:

$$B_{nc}(\text{perf}_{dg}(X_L)) \Leftrightarrow B_{nc}(\mathbb{C}^{dg}_L) + B_{nc}(A^{dg}_{\dim(V)}) + \cdots + B_{nc}(A^{dg}_{i-1})$$

$$B_{nc}(\text{perf}_{dg}(Y_L)) \Leftrightarrow B_{nc}(\mathbb{E}^{dg}_{j-1}) + \cdots + B_{nc}(B^{dg}_{\dim(L)+1}) + B_{nc}(\mathbb{C}^{dg}_{i'}).$$

The remainder of the proof is now similar to the proof of $T^i(X_L) \Leftrightarrow T^i(Y_L)$.

Finally, let us prove the equivalence $P(X_L) \Leftrightarrow P(Y_L)$. Consider the functors

(5.17) $$K_i(-)_{\mathbb{Q}} : \text{dgcat}(k) \to \text{Vect}(\mathbb{Q}) \quad A \mapsto K_i(A)_{\mathbb{Q}}$$

with values in the category of $\mathbb{Q}$-vector spaces. As explained in [30, §2.2.1], these functors are additive invariants. Therefore, a proof similar to the one of the equivalence $T^i(X_L) \Leftrightarrow T^i(Y_L)$ allows us to conclude that $P(X_L) \Leftrightarrow P(Y_L)$.

6. PROOF OF PROPOSITION 1.6

As proved in [5, Thms. 1.3 and 1.7], the $\text{dg}$ category $\text{perf}_{dg}(\text{Gr}(r, U_1))$ is Morita equivalent to a finite dimensional $k$-algebra of finite global dimension $A$. Since $A^{dg} = \text{perf}_{dg}(\text{Gr}(r, U_1))$, we hence obtain the following equivalences of conjectures:

$$T^i_{nc}(A) \Leftrightarrow T^i_{nc}(A) \quad T^p_{nc}(A) \Leftrightarrow T^p_{nc}(A)$$

We start by proving the conjecture $T^i_{nc}(A)$. Recall that every finite field $k$ is perfect. Therefore, using the fact that the above functors ($\text{perf}_{dg}$, $\text{Gr}(r, U_1)$) are additive invariants, [35, Thm. 3.15] implies that $E_m(A) \simeq E_m(A/J(A)), m \geq 1$, where $J(A)$ stands for the Jacobson radical of $A$. By applying the functor $\text{Hom}(\mathbb{Z}(l^\infty), -)$ to these latter isomorphisms, we hence conclude that $T^i_{nc}(A) \Leftrightarrow T^i_{nc}(A/J(A))$.

Now, let us write $V_1, \ldots, V_s$ for the simple (right) $A/J(A)$-modules and $D_i := \text{End}_{A/J(A)}(V_i)$. Consequently, the center $Z_i$ of $D_i$ is a finite field extension of $k$ and $D_i$ is a central simple $Z_i$-algebra. Since the Brauer group of a finite field is trivial, this implies that $D_1 \times \cdots \times D_s$ is Morita equivalent to $Z_1 \times \cdots \times Z_s$. Consequently, we obtain the following equivalences:

$$T^i_{nc}(A) \Leftrightarrow T^i_{nc}(A/J(A)) \Leftrightarrow T^i_{nc}(D_1) + \cdots + T^i_{nc}(D_s) \Leftrightarrow T^i(Z_1) + \cdots + T^i(Z_s).$$

The proof of the conjecture $T^i_{nc}(A)$ follows now from the fact that the conjectures $T^i(Z_i), 1 \leq i \leq s$, hold because $\dim(Z_i) = 0$.

Let us now prove the conjecture $T^p_{nc}(A)$. Since the functor (2.3) is an additive invariant, [35, Thm. 3.15] implies that $U(A)_{K} \simeq U(A/J(A))_{K}$ in $\text{Hm}_0(k)_{K}$. Recall from the proof of Lemma 3.7 that the functor (3.6) comes equipped with the natural transformation $\varphi^n$. Therefore, by applying the $K$-linear functor (3.6) to the latter isomorphism, we obtain the following identification:

$$\left( K_0(A)_{K} \to TP_0(A)_{K} \right) \simeq \left( K_0(A/J(A))_{K} \to TP_0(A/J(A))_{K} \right).$$

This implies the equivalence of conjectures $T^p_{nc}(A) \Leftrightarrow T^p_{nc}(A/J(A))$. The proof of the conjecture $T^p_{nc}(A/J(A))$ is now similar to the proof of $T^i_{nc}(A/J(A))$.

Let us now prove the conjecture $B_{nc}(A)$. As above, we have an isomorphism $U(A)_{Q} \simeq U(A/J(A))_{Q}$ in $\text{Hm}_0(k)_{Q}$. Thanks to the natural isomorphisms (2.4) and (5.16), by applying $\text{Hom}_{\text{NChow}(k)}(U(k)_{Q}, -)$ and $\text{Hom}_{\text{NNum}(k)}(U(k)_{Q}, -)$ to
Corollary:

Moreover, the category $\text{perf}(\mathcal{O}_\mathcal{D})$ of noncommutative algebras over $\mathcal{D}$ is equivalent to $\text{perf}(\mathcal{P}(\mathcal{D}) \cap \Delta)$, where $\mathcal{D}$ is a sheaf of Azumaya algebras over $\mathcal{P}(\mathcal{D})$. This leads to a Morita equivalence between the dg categories $\text{perf}_\text{nc}(\mathcal{P}(\mathcal{D}); \mathcal{Cl}_0(q)_{|L})$ and $\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}); \mathcal{F})$. Consequently, making use of Corollary 1.13, we obtain the following equivalences of conjectures:

$$
\begin{align*}
(7.1) & \quad T^1(X_L) \Leftrightarrow T^1_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}); \mathcal{F})) & & T^p(X_L) \Leftrightarrow T^p_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}); \mathcal{F})) \\
(7.2) & \quad B(X_L) \Leftrightarrow B_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}); \mathcal{F})) & & P(X_L) \Leftrightarrow P_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}); \mathcal{F})).
\end{align*}
$$

Since by assumption $\dim(L) = 2$, the 2-fold cover $\mathcal{P}(\mathcal{D})$ is a smooth projective curve. Using the fact that the Brauer group of every smooth curve over a finite field $k$ is trivial (see [25, page 109]), we hence conclude that the right-hand side conjectures in (7.1)-(7.2) are equivalent to $T^1(\mathcal{P}(\mathcal{D})), T^p(\mathcal{P}(\mathcal{D})), B(\mathcal{P}(\mathcal{D})),$ and $P(\mathcal{P}(\mathcal{D}))$, respectively. The proof follows now from the fact that the Tate, Beilinson and Parshin conjectures hold for smooth projective curves.

We now assume that $d$ is odd and that $p \neq 2$. Following [20, §3.6] (see also [1, §1.7]), let $\mathcal{P}(\mathcal{D})$ be the discriminant stack associated to the pull-back $q_{|L}$ along $\mathcal{P}(\mathcal{D}) \subset \mathcal{P}(S^2W^*)$ of the flat quadric fibration $q: \mathcal{H} \to \mathcal{P}(S^2W^*)$. As explained in loc. cit., since by assumption $1/2 \in k$, $\mathcal{P}(\mathcal{D})$ is a smooth Deligne-Mumford stack. Moreover, using the fact that $\mathcal{P}(\mathcal{D})$ is a square root stack and that the critical locus of the flat quadric fibration $q_{|L}$ is the divisor $D$, we conclude from [12, Thm. 1.6] that $\text{perf}(\mathcal{P}(\mathcal{D})) = (\text{perf}(D), \text{perf}(\mathcal{P}(\mathcal{D})))$. Consequently, an argument similar to the one used in the proof of Theorem 1.4 yields the following equivalences of conjectures:

$$
\begin{align*}
(7.3) & \quad T^1_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}))) \Leftrightarrow T^1(D) + T^1(\mathcal{P}(\mathcal{D})) \\
(7.4) & \quad T^p_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}))) \Leftrightarrow T^p(D) + T^p(\mathcal{P}(\mathcal{D})) \\
(7.5) & \quad B_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}))) \Leftrightarrow B(D) + B(\mathcal{P}(\mathcal{D})) \\
(7.6) & \quad P_\text{nc}(\text{perf}_\text{dg}(\mathcal{P}(\mathcal{D}))) \Leftrightarrow P(D) + P(\mathcal{P}(\mathcal{D})).
\end{align*}
$$

Let us write $\mathcal{F}$ for the sheaf of noncommutative algebras $\mathcal{Cl}_0(q)_{|L}$ considered as a sheaf of noncommutative algebras over $\mathcal{P}(\mathcal{D})$. As proved in [20, §3.6] (see also [1, §1.7]), since by assumption $\mathcal{P}(\mathcal{D}) \cap \Delta_2 = \emptyset$, $\mathcal{F}$ is a sheaf of Azumaya algebras over $\mathcal{P}(\mathcal{D})$. Moreover, the category $\text{perf}(\mathcal{P}(\mathcal{D}); \mathcal{Cl}_0(q)_{|L})$ is equivalent (via a Fourier-Mukai
type functor) to $\text{perf}(\hat{P}(L); F)$. This leads to a Morita equivalence between the
dg categories $\text{perf}_{dg}(P(L); \text{Cl}_0(q)L)$ and $\text{perf}_{dg}(\hat{P}(L); F)$. Making use of Corollary
1.13, we hence obtain the following equivalences of conjectures:
\begin{align}
T^l(X_L) &\iff T^l_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) \\
T^p(X_L) &\iff T^p_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) \\
B(X_L) &\iff B_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) \\
P(X_L) &\iff P_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) .
\end{align}

Since by assumption $\dim(L) = 2$, we have $\dim(\hat{P}(L)) = 1$. Using the fact that the
Brauer group of every smooth curve over a finite field $k$ is trivial, we hence conclude
that in (7.7)-(7.8) we can replace $\text{perf}_{dg}(\hat{P}(L); F)$ by $\text{perf}_{dg}(\hat{P}(L))$. Consequently,
since $\dim(D) = 0$, the proof follows now from the combination of (7.3)-(7.6) with
the fact that the Tate, Beilinson and Parshin conjectures hold in dimensions $\leq 1$.

**Intersection of even-dimensional quadrics.** Given a smooth proper dg category $A$, a prime number $l \neq p$, and an integer $s \geq 1$, consider the $\mathbb{Z}[1/s]$-modules
\begin{equation}
\text{Hom}(\mathbb{Z}(l\infty), (\pi_{-1} L_{KU} K(A \otimes_{\mathbb{F}_q} \mathbb{F}_q^m))_{1/s}) \quad m \geq 1
\end{equation}
as well as the following variant of the noncommutative Tate conjecture:
Conjecture $T^l_{nc}(A; 1/s)$: The $\mathbb{Z}[1/s]$-modules (7.9) are trivial.

**Lemma 7.10.** We have $T^l_{nc}(A) \iff T^l_{nc}(A; 1/s)$ for every $l \nmid s$.

**Proof.** Since by assumption $l \nmid s$, the localization homomorphisms
\[\pi_{-1} L_{KU} K(A \otimes_{\mathbb{F}_q} \mathbb{F}_q^m) \rightarrow (\pi_{-1} L_{KU} K(A \otimes_{\mathbb{F}_q} \mathbb{F}_q^m))_{1/s} \quad m \geq 1\]
induce an isomorphism between all the $l$-power torsion subgroups. Consequently,
by passing to the $l$-adic Tate modules, we conclude that the abelian groups (3.1)
are trivial if and only if the $\mathbb{Z}[1/s]$-modules (7.9) are trivial. $\square$

**Theorem 7.11.** Let $X_L$ be as in Corollary 1.13. Assume that $\hat{P}(L) \cap \Delta_2 = \emptyset$, that
the divisor $\hat{P}(L) \cap \Delta_1$ is smooth, and that $d$ is even. Under these assumptions, we
have the following equivalences:
\begin{align}
T^l(X_L) &\iff T^l(\hat{P}(L)) \text{ (for every } l \neq 2) \\
T^p(X_L) &\iff T^p(\hat{P}(L)) \\
B(X_L) &\iff B(\hat{P}(L)) \\
P(X_L) &\iff P(\hat{P}(L)) .
\end{align}

**Proof.** As explained in the proof of Theorem 1.14, we have the equivalences
\begin{align}
T^l(X_L) &\iff T^l_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) \\
T^p(X_L) &\iff T^p_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) \\
B(X_L) &\iff B_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) \\
P(X_L) &\iff P_{nc}(\text{perf}_{dg}(\hat{P}(L); F)) ,
\end{align}
where $F$ is a certain sheaf of Azumaya algebras over $\hat{P}(L)$ of rank $2^{(d/2)-1}$.

We start by proving that the first right-hand side conjecture in (7.12) is equivalent
to $T^l(\hat{P}(L))$ (for every $l \neq 2$). Consider the following functors
\[E_m(-)_{1/2}: \text{dgcat}(k) \rightarrow \text{Mod}(\mathbb{Z}[1/2]) \quad A \mapsto (\pi_{-1} L_{KU} K(A \otimes_{\mathbb{F}_q} \mathbb{F}_q^m))_{1/2}\]
with values in the additive category of $\mathbb{Z}[1/2]$-modules. Similarly to the proof
of Proposition 5.6, these functors are additive invariants. Consequently, using
the fact that the rank of the sheaf of Azumaya algebras $F$ is a power of 2 and
that the category $\text{Mod}(\mathbb{Z}[1/2])$ is $\mathbb{Z}[1/2]$-linear, we conclude from [35, Thm. 2.1]
that $E_m(\text{perf}_{dg}(\mathbb{P}(L)))_{1/2} \simeq E_m(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F}))_{1/2}$. By applying the functor $\text{Hom}(\mathbb{Z}(l^{\infty}), -)$ to these isomorphisms, we hence obtain the equivalences:

$$T^l_{nc}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F}); 1/2) \iff T^l_{nc}(\text{perf}_{dg}(\mathbb{P}(L)); 1/2) \iff T^l(\mathbb{P}(L); 1/2).$$

(7.14)

Thanks to Lemma 7.10, the preceding equivalence (7.14) yields the equivalence $T^l_{nc}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F})) \iff T^l(\mathbb{P}(L))$ (for every $l \neq 2$), and so the proof is finished.

Let us now prove that the second right-hand side conjecture in (7.12) is equivalent to $T^p(\mathbb{P}(L))$. Since the functor (2.3) is additive and $\text{char}(K) = 0$, [35, Thm. 2.1] implies that $U(\text{perf}_{dg}(\mathbb{P}(L)))_K \simeq U(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F}))_K$ in $\text{Hm}_0(k)_K$. Recall from the proof of Lemma 3.7 comes equipped with the natural transformation $\varphi^n$. Therefore, by applying the $K$-linear functor (3.6) to the latter isomorphism, we conclude that the induced $K$-linear homomorphism

$$K_0(\text{perf}_{dg}(\mathbb{P}(L)))_K \longrightarrow T_0(\text{perf}_{dg}(\mathbb{P}(L)))_{1/p}$$

identifies with the induced $K$-linear homomorphism

$$K_0(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F}))_K \longrightarrow T_0(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F}))_{1/p}.$$ 

This implies the following equivalences of conjectures:

$$T^p_{nc}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F})) \iff T^p_{nc}(\text{perf}_{dg}(\mathbb{P}(L))) \iff T^p(\mathbb{P}(L)).$$

Let us now prove that the first right-hand side conjecture in (7.13) is equivalent to $B(\mathbb{P}(L))$. As above, we have $U(\text{perf}_{dg}(\mathbb{P}(L)))_Q \simeq U(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F}))_Q$ in $\text{Hm}_0(k)_Q$. Thanks to the natural isomorphisms (2.4) and (5.16), by applying the functors $\text{Hom}_{\text{Num}}(U(k)_Q, -)$ and $\text{Hom}_{\text{Num}}(U(k)_Q, -)$ to the latter isomorphism, we hence obtain the following equivalences of conjectures:

$$B_{nc}(\text{perf}_{dg}(\mathbb{P}(L); \mathcal{F})) \iff B_{nc}(\text{perf}_{dg}(\mathbb{P}(L))) \iff B(\mathbb{P}(L)).$$

Finally, since the functors (5.17) are additive invariants (with values in the additive category of $\mathbb{Q}$-vector spaces), the proof that the second right-hand side conjecture in (7.13) is equivalent to $P(\mathbb{P}(L))$ is similar to the proof that the first right-hand side conjecture in (7.12) is equivalent to $T^l(\mathbb{P}(L))$ (for every $l \neq 2$).

**Remark 7.15** (Azumaya algebras). Let $X$ be a smooth projective $k$-scheme and $\mathcal{F}$ a sheaf of Azumaya algebras over $X$ of rank $r$. Note that an argument similar to the one used in the proof of Theorem 7.11 leads to the following equivalences:

$$T^l_{nc}(\text{perf}_{dg}(X; \mathcal{F})) \iff T^l(X) \quad \text{(for every } l \neq r)$$

$$T^p_{nc}(\text{perf}_{dg}(X; \mathcal{F})) \iff T^p(X)$$

$$B_{nc}(\text{perf}_{dg}(X; \mathcal{F})) \iff B(X)$$

$$P_{nc}(\text{perf}_{dg}(X; \mathcal{F})) \iff P(X).$$

**8. Proof of Theorem 1.16**

We start by proving Theorem 1.16 in what regards the Tate conjecture. Consider the following functors with values in the additive category of $\mathbb{Z}[1/s]$-modules:

$$(8.1) \quad E_m(-)_{1/s} : \text{dgcat}(k) \longrightarrow \text{Mod}(\mathbb{Z}[1/s]) \quad A \mapsto (\pi_{-1}L_{KU}K(A \otimes_{k} \mathbb{F}_q \mathcal{F}^n))_{1/s}.$$ 

Similarly to the proof of Proposition 5.6, these functors are additive invariants. Consequently, using the fact that $1/s \in k$ (since $p \nmid s$) and that category $\text{Mod}(\mathbb{Z}[1/s])$ is $\mathbb{Z}[1/s]$-linear, we conclude from [36, Thm. 1.1 and Rk. 1.4] that $E_m(\text{perf}_{dg}(\mathcal{X}))_{1/s}$
is a direct summand of $\bigoplus_{\sigma \subseteq G} E_m(\text{perf}_{dg}(X^\sigma \times \text{Spec}(k[\sigma])))_{1/s}$. By applying the functor $\text{Hom}(\mathbb{Z}(l^\infty), -)$, we hence obtain the following implication:

\begin{equation}
\sum_{\sigma \subseteq G} T^l(X^\sigma \times \text{Spec}(k[\sigma]); 1/s) \Rightarrow T^l(X; 1/s).
\end{equation}

The searched implication (1.17) follows then from the combination of (8.2) with Lemma 7.10. Assume now that $s \mid (q-1)$. Note that $s \mid (q-1)$ if and only if $k$ contains the $s^{\text{th}}$ roots of unity. Therefore, since $1/s \in k$, since the above functors (8.1) are additive invariants, and since the category $\text{Mod}(\mathbb{Z}[1/s])$ is $\mathbb{Z}[1/s]$-linear, we conclude from [36, Cor. 1.6(i)] that $E_m(\text{perf}_{dg}(X))_{1/s}$ is a direct summand of $\bigoplus_{\sigma \subseteq G} E_m(\text{perf}_{dg}(X^\sigma))_{1/s}^{\sigma r}$, where $r_{\sigma} \geq 1$ are certain integers. By applying the functor $\text{Hom}(\mathbb{Z}(l^\infty), -)$, we hence obtain the following implication:

\begin{equation}
\sum_{\sigma \subseteq G} T^l(X^\sigma; 1/s) \Rightarrow T^l(X; 1/s).
\end{equation}

As above, the searched implication $\sum_{\sigma \subseteq G} T^l(X^\sigma) \Rightarrow T^l(X)$ (for every $l \mid s$) follows then from the combination of (8.3) with Lemma 7.10. Assume finally that $\text{dim}(X) \leq 3$. In this case, the dimension of the smooth projective $k$-schemes $X^\sigma$ and $X^\sigma \times \text{Spec}(k[\sigma])$ is also $\leq 3$. Therefore, as explained in §2.4, it suffices to consider the Tate conjecture for divisors $T^{l,1}(-)$. As proved in [37, Thm. 5.2], we have the following equivalences:

$$T^{l,1}(X^\sigma \times \text{Spec}(k[\sigma])) \iff T^{l,1}(X^\sigma) + T^{l,1}(\text{Spec}(k[\sigma])) \iff T^{l,1}(X^\sigma).$$

This yields the equivalence of conjectures $T^l(X \times \text{Spec}(k[\sigma])) \iff T^l(X^\sigma)$ and hence the searched implication $\sum_{\sigma \subseteq G} T^l(X^\sigma) \Rightarrow T^l(X)$ (for every $l \mid s$).

Let us now prove Theorem 1.16 in what regards the $p$-version of the Tate conjecture. Since the functor (2.3) is additive, $1/s \in k$, and $\text{char}(K) = 0$, [36, Thm. 1.1 and Rk. 1.4] implies that $U(\text{perf}_{dg}(X))_K$ is a direct summand of the direct sum $\bigoplus_{\sigma \subseteq G} U(\text{perf}_{dg}(X^\sigma \times \text{Spec}(k[\sigma])))_K$. Recall from the proof of Lemma 3.7 that the functor (3.6) comes equipped with the natural transformation $\varphi^n$. Therefore, since the dg categories $\text{perf}_{dg}(X)$ and $\text{perf}_{dg}(X^\sigma \times \text{Spec}(k[\sigma]))$ are smooth proper, by applying the functor (3.6) we conclude that the induced $K$-linear homomorphism

$$K_0(\text{perf}_{dg}(X))_K \rightarrow TP_0(\text{perf}_{dg}(X))_1^{\sigma r}$$

is a direct summand of the induced (diagonal) $K$-linear homomorphism

$$\bigoplus_{\sigma \subseteq G} K_0(X^\sigma \times \text{Spec}(k[\sigma]))_K \rightarrow \bigoplus_{\sigma \subseteq G} TP_0(X^\sigma \times \text{Spec}(k[\sigma]))_1^{\sigma r}.$$ 

This yields the searched implication (1.18). Assume now that $s \mid (q-1)$. In this case, since $k$ contains the $s^{\text{th}}$ roots of unity, [36, Cor. 1.6(i)] implies that $U(\text{perf}_{dg}(X))_K$ is a direct summand of $\bigoplus_{\sigma \subseteq G} U(\text{perf}_{dg}(X^\sigma))_K^{\sigma r}$, where $r_{\sigma}$ are certain integers. Hence, an argument similar to the preceding one shows that $\sum_{\sigma \subseteq G} TP(X^\sigma) \Rightarrow TP(X)$.

Assume finally that $\text{dim}(X) \leq 3$. As explained in §2.4, since the dimension of $X^\sigma$ and $X^\sigma \times \text{Spec}(k[\sigma])$ is $\leq 3$, we have $TP(X^\sigma) \iff T^l(X^\sigma)$ and $TP(X^\sigma \times \text{Spec}(k[\sigma])) \iff T^l(X^\sigma \times \text{Spec}(k[\sigma]))$. Consequently, making use of the above equivalence of conjectures $T^l(X \times \text{Spec}(k[\sigma])) \iff T^l(X^\sigma)$, we conclude that $\sum_{\sigma \subseteq G} TP(X^\sigma) \Rightarrow TP(X)$.

Let us now prove Theorem 1.16 in what regards the Beilinson conjecture. As above, $U(\text{perf}_{dg}(X))_0$ is a direct summand of $\bigoplus_{\sigma \subseteq G} U(\text{perf}_{dg}(X^\sigma \times \text{Spec}(k[\sigma])))_0$. Therefore, thanks to the natural isomorphisms (2.4) and (5.16), by applying the
functions \( \text{Hom}_{\text{NChow}(k)}(U(k) \otimes k, -) \) and \( \text{Hom}_{\text{Num}(k)}(U(k) \otimes k, -) \), we obtain the implication (1.19). Once again as above, whenever \( m \mid (q - 1) \), \( U(\text{perf}_{dg}(\mathcal{X})) \mathbb{Q} \) is a direct summand of \( \bigoplus_{\sigma \subseteq G} U(\text{perf}_{dg}(X^\sigma)) \mathbb{Q} \). Hence, an argument similar to the preceding one shows that \( \sum_{\sigma \subseteq G} B(X^\sigma) \Rightarrow B(\mathcal{X}) \).

Finally, since the functors (5.17) are additive invariants (with values in the additive category of \( \mathbb{Q} \)-vector spaces), the proof of Theorem 1.16 in what regards the Parshin conjecture is similar to the above proof of Theorem 1.16 in what regards the Tate conjecture.

**Proof of Theorem 1.23**

We start by proving Theorem 1.23 in what regards the Tate conjecture. Since by assumption \( s \mid (q - 1) \), \( k \) contains the \( s \)th roots of unity and \( 1/s \in k \). Therefore, using the fact that the above functors (8.1) (with \( s \) replaced by \( sr \)) are additive invariants and that the category \( \text{Mod}(\mathbb{Z}[1/sr]) \) is \( \mathbb{Z}[1/sr] \)-linear, we conclude from [36, Cor. 1.29(ii)] that \( E_m(\text{perf}_{dg}(\mathcal{X}; \mathcal{F}))_{1/sr} \) is a direct summand of \( \bigoplus_{\sigma \subseteq G} E_m(\text{perf}_{dg}(Y_\sigma))_{1/sr} \), where \( Y_\sigma \) is a certain \( \sigma^G \)-Galois cover of \( X^\sigma \). By applying the functor \( \text{Hom}(\mathbb{Z}(l^\infty), -) \), we hence obtain the following implication:

\[
\sum_{\sigma \subseteq G} T^l(Y_\sigma; 1/sr) \Rightarrow T^l(\mathcal{X}; \mathcal{F}; 1/sr).
\]

Lemma 7.10 yields then the following implication:

\[
(8.4) \sum_{\sigma \subseteq G} T^l(Y_\sigma) \Rightarrow T^l(\mathcal{X}; \mathcal{F}) \quad (\text{for every } l \nmid sr).
\]

On the one hand, if \( \dim(X) \leq 1 \), then the dimension of \( Y_\sigma \) is also \( \leq 1 \) for every \( \sigma \subseteq G \). Consequently, the conjecture \( T^l(\mathcal{X}; \mathcal{F}) \) follows from the implication (8.4).

On the other hand, if the \( G \)-action is faithful and \( \dim(X) = 2 \), then \( \dim(Y_\sigma) = \dim(X^\sigma) \leq 1 \) for every non-trivial cyclic subgroup \( \sigma \subseteq G \). Consequently, in the above implication (8.4) the \( k \)-schemes \( Y_\sigma \) can be replaced by the single \( k \)-scheme \( X \).

Let us now prove Theorem 1.23 in what regards the \( p \)-version of the Tate conjecture. Since the functor (2.3) is additive, \( k \) contains the \( s \)th roots of unity, and \( \text{char}(K) = 0 \), [36, Cor. 1.29(ii)] implies that \( U(\text{perf}_{dg}(\mathcal{X}; \mathcal{F}))_K \) is a direct summand of \( \bigoplus_{\sigma \subseteq G} U(\text{perf}_{dg}(Y_\sigma))_K \). Recall from the proof of Lemma 3.7 that the functor (3.6) comes equipped with the natural transformation \( \varphi^n \). Therefore, since the \( dg \) categories \( \text{perf}_{dg}(\mathcal{X}; \mathcal{F}) \) and \( \text{perf}_{dg}(Y_\sigma) \) are smooth proper, by applying the functor (3.6) we conclude that the induced \( K \)-linear homomorphism

\[
K_0(\text{perf}_{dg}(\mathcal{X}; \mathcal{F}))_K \rightarrow T^p_0(\text{perf}_{dg}(\mathcal{X}; \mathcal{F}))(1/p)^{\varphi^n}
\]

is a direct summand of the induced (diagonal) \( K \)-linear homomorphism

\[
\bigoplus_{\sigma \subseteq G} K_0(\text{perf}_{dg}(Y_\sigma))_K \rightarrow \bigoplus_{\sigma \subseteq G} T^p_0(\text{perf}_{dg}(Y_\sigma))(1/p)^{\varphi^n}.
\]

This yields the implication \( \sum_{\sigma \subseteq G} T^p(Y_\sigma) \Rightarrow T^p(\mathcal{X}; \mathcal{F}) \). The remainder of the proof is now similar to the above proof, concerning the Tate conjecture, with the implication (8.4) replaced by the latter implication.

Let us now prove Theorem 1.23 in what regards the Beilinson conjecture. As above, \( U(\text{perf}_{dg}(\mathcal{X}; \mathcal{F}))_Q \) is a direct summand of \( \bigoplus_{\sigma \subseteq G} U(\text{perf}_{dg}(Y_\sigma))_Q \). Therefore, thanks to the natural isomorphisms (2.4) and (5.16), by applying the functors
\[ \text{Hom}_{\text{NChow}(k)}(U(k) \mathbb{Q}, -) \] and \[ \text{Hom}_{\text{Num}(k)} U(k) \mathbb{Q}, -) \), we obtain the implication
\[ \sum_{\sigma \subseteq G} B(Y_{\sigma}) \Rightarrow B(X; F). \] The remainder of the proof is now similar to the above proof, concerning the Tate conjecture, with (8.4) replaced by the latter implication.

Finally, since the functors (5.17) are additive invariants (with values in the additive category of \( \mathbb{Q} \)-vector spaces), the proof of Theorem 1.23 in what regards the Parshin conjecture is similar to the above proof of Theorem 1.23 in what regards the Tate conjecture.

Acknowledgments: I am grateful to Bruno Kahn for his interest in the arguments used in the proof of Theorem 1.14 and for the reference [29], to Kiran Kedlaya for asking me if the \( p \)-version of the Tate conjecture would admit a noncommutative analogue, and to Lars Hesselholt and Peter Scholze for useful discussions concerning topological periodic cyclic homology. I also would like to thank the Hausdorff Research Institute for Mathematics in Bonn for its hospitality.

References
[1] A. Auel, M. Bernardara and M. Bolognesi, Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems. J. Math. Pures Appl. (9) 102 (2014), no. 1, 249–291.
[2] A. Beilinson, Coherent sheaves on \( \mathbb{P}^n \) and problems in linear algebra. Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 68–69.
[3] M. Bernardara, M. Bolognesi and D. Faenzi, Homological projective duality for determinantal varieties. Adv. Math. 296 (2016), 181–209.
[4] B. Bhatt, M. Morrow and P. Scholze, Topological Hochschild homology and integral \( p \)-adic Hodge theory. Available at arXiv:1802.03261.
[5] R. Buchweitz, G. Leuschke and M. Van den Bergh, On the derived category of Grassmannians in arbitrary characteristic. Compos. Math. 151 (2015), no. 7, 1242–1264.
[6] A. Conca and D. Faenzi, A remark on hyperplane sections of rational normal scrolls. Available at arXiv:1709.08332.
[7] W. Fulton, Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 2. Springer-Verlag, Berlin, 1984.
[8] D. Grayson, Finite generation of \( K \)-groups of a curve over a finite field (after Daniel Quillen). Algebraic \( K \)-theory, Part I (Oberwolfach, 1980), pp. 69–90, LNM 966, 1982.
[9] G. Harder, Die Kohomologie \( S \)-arithmetischer Gruppen über Funktionenkörpern. Invent. Math. 42 (1977), 135–175.
[10] J. Harris, Algebraic geometry. A first course. Graduate Texts in Mathematics, 133. Springer-Verlag, New York, 1992.
[11] L. Hesselholt, Topological periodic cyclic homology and the Hasse-Weil zeta function. Available at arXiv:1602.01980.
[12] A. Ishii and K. Ueda, The special McKay correspondence and exceptional collections. Tohoku Math. J. (2) 67 (2015), no. 4, 555–609.
[13] B. Kahn, Algebraic \( K \)-theory, algebraic cycles and arithmetic geometry. Handbook of Algebraic \( K \)-theory, Berlin, New York. Springer-Verlag, pp. 351–428, 2005.
[14] M. Kapranov, On the derived categories of coherent sheaves on some homogeneous spaces. Invent. Math. 92 (1988), no. 3, 479–508.
[15] K. Kato and S. Saito, Unramified class field theory of arithmetical surfaces. Ann. of Math. 118 (1983), 241–275.
[16] B. Keller, On differential graded categories. International Congress of Mathematicians (Madrid), Vol. II, 151–190. Eur. Math. Soc., Zurich (2006).
[17] M. Kontsevich, Mixed noncommutative motives. Talk at the Workshop on Homological Mirror Symmetry, Miami, 2010. Notes available at www.math.mit.edu/~auroux/frg/miami10-notes.
[18] , Notes on motives in finite characteristic. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 213–247, Progr. Math., 270, Birkhäuser Boston, MA, 2009.
[19] , Noncommutative motives. Talk at the IAS on the occasion of the 61st birthday of Pierre Deligne (2005). Available at http://video.ias.edu/Geometry-and-Arithmetic.
[20] A. Kuznetsov, Derived categories of quadric fibrations and intersections of quadrics. Adv. Math. 218 (2008), no. 5, 1340–1369.

[21] A. Kuznetsov, Homological projective duality. Publ. Math. IHÉS, no. 105 (2007), 157–220.

[22] A. Kuznetsov, Semiorthogonal decompositions in algebraic geometry. Available at arXiv:1404.3143.

[23] V. Lunts and D. Orlov, Uniqueness of enhancement for triangulated categories. J. Amer. Math. Soc. 23 (2010), no. 3, 853–908.

[24] M. Marcolli and G. Tabuada, Noncommutative Artin motives. Selecta Math. (N.S.) 20 (2014), no. 1, 315–358.

[25] J. Milne, Étale cohomology. Princeton Mathematical Series, 33 (1980).

[26] J. Milne, The Tate conjecture over finite fields. AIM talk. Available at Milne’s personal webpage http://www.jmilne.org/math/articles/2007e.pdf.

[27] M. Morrow, A variational Tate conjecture in crystalline cohomology. Available at arXiv:1408.6783.

[28] T. Nikolaus and P. Scholze, On topological cyclic homology. Available at arXiv:1707.01799.

[29] M. Reid, The complete intersection of two or more quadrics. Ph. D. thesis. Available at https://homepages.warwick.ac.uk/~masda/3folds/qu.pdf.

[30] G. Tabuada, Noncommutative Motives. With a preface by Yuri I. Manin. University Lecture Series, 63. American Mathematical Society, Providence, RI, 2015.

[31] G. Tabuada, Recent developments on noncommutative motives. Available at arXiv:1611.05439. To appear in Contemporary Mathematics, AMS.

[32] G. Tabuada, Noncommutative motives in positive characteristic and their applications. Available at arXiv:1707.04248.

[33] G. Tabuada, On Grothendieck’s standard conjectures of type C and D in positive characteristic. Available at arXiv:1710.04644.

[34] G. Tabuada and M. Van den Bergh, The Gysin triangle via localization and A^1-homotopy invariance. Transactions of the American Mathematical Society 370 (2018), no. 1, 421–446.

[35] G. Tabuada, Noncommutative motives of Azumaya algebras. J. Inst. Math. Jussieu 14 (2015), no. 2, 379–403.

[36] G. Tabuada, Additive invariants of orbifolds. Available at arXiv:1612.03162. To appear in Geometry and Topology.

[37] J. Tate, Conjectures on algebraic cycles in ℓ-adic cohomology. Motives (Seattle, WA, 1991), 71–83, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[38] J. Tate, Algebraic cycles and poles of zeta functions. Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963) pp. 93–110. Harper & Row, New York 1965.

[39] R. Thomason, Notes on HPD. Available at arXiv:1512.08985. To appear in Proceedings of the 2015 AMS Summer Institute, Salt Lake City.

[40] R. Thomason, A finiteness condition equivalent to the Tate conjecture over ℤ_q. Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), 385–392, Contemp. Math., 83, Amer. Math. Soc., Providence, RI, 1989.

[41] B. Totaro, Recent progress on the Tate conjecture. Bull. Amer. Math. Soc. 54 (2017), 575–590.

GONÇALO TABUADA, DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139, USA
E-mail address: tabuada@math.mit.edu
URL: http://math.mit.edu/~tabuada