STRONGLY HIT AND FAR MISS HYPERTOPOLOGY &
HIT AND STRONGLY FAR MISS HYPERTOPOLOGY

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Dedicated to the Memory of Som Naimpally

Abstract. This article introduces the strongly hit and far-miss $\tau^{\Delta}$ as well as hit and strongly far miss $\tau_{w, \mathcal{R}}$ hypertopologies on $CL(X)$ associated with $\mathcal{R}$, a nonempty family of subsets on the topological space $X$. They result from the strong farness and strong nearness proximities. The main results in this paper stem from the Hausdorffness of $(CL(X), \tau^{\Delta})$ and $(RCL(X), \tau^{\Delta})$, where $RCL(X)$ is the space of regular closed subsets of $X$. To obtain the results, special local families are introduced.

1. Introduction

This article introduces some new hypertopologies on the space $CL(X)$ of non-empty closed subsets and on $RCL(X)$, the space of regular closed subsets of a topological space $X$. These new hypertopologies result from the introduction of strong farness [30] and strong nearness [31]. Such hypertopologies are located in the class of hit and miss ones. Significant examples of such topologies are Hausdorff, Fell and Attouch-Wets hypertopologies. Interest in this topic spans many years (see, e.g., [2, 3, 6, 8, 9, 5, 10, 11, 12, 23, 22, 13]) with a number of possible applications.

The strongly near proximity [31] and strongly far proximity [30] and [29], provide a foundation for the hypertopologies introduced, here. Strong nearness plays a role in the new hit sets, while strong farness leads to some new miss sets.

2. Preliminaries

In this work we focus our attention on two new kinds of hit and far-miss topologies. On the one hand, we use the concept of strong farness [30] and on the other hypertopology is based on strongly near proximity [31].

Strong proximities are associated with Lodato proximity and the Efremović property. Recall how a Lodato proximity is defined [19, 20, 21] (see, also, [26, 24, 17]).

Definition 2.1. Let $X$ be a nonempty set. A Lodato proximity $\delta$ is a relation on $\mathcal{P}(X)$, which satisfies the following properties for all subsets $A, B, C$ of $X$:

P0) $A \delta B \Rightarrow B \delta A$

P1) $A \delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$

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P2) \( A \cap B \neq \emptyset \Rightarrow A \delta B \)

P3) \( A \delta (B \cup C) \Leftrightarrow A \delta B \) or \( A \delta C \)

P4) \( A \delta B \) and \( \{ b \} \delta C \) for each \( b \in B \) \( \Rightarrow A \delta C \)

Further \( \delta \) is separated, if

P5) \( \{ x \} \delta \{ y \} \Rightarrow x = y \).

A \( \delta B \) reads ”\( A \) is near to \( B \)” and \( A \not\delta B \) reads ”\( A \) is far from \( B \)”.

A basic proximity is one that satisfies the \( \check{C}ech \) axioms \( P0) - P3) \) [4, §2.5, p. 439]. Lodato proximity or \( LO \)-proximity is one of the simplest proximities. We can associate a topology with the space \(( X, \delta )\) by considering as closed sets those sets that coincide with their own closure where, for a subset \( A \), we have

\[ \text{cl} A = \{ x \in X : x \delta A \} . \]

This is possible because of the correspondence of Lodato axioms with the well-known Kuratowski closure axioms.

By considering the gap between two sets in a metric space \(( d(A, B) = \inf \{ d(a, b) : a \in A, b \in B \} \) or \( \infty \) if \( A \) or \( B \) is empty \), Efremović introduced a stronger proximity called Efremović proximity or \( EF \)-proximity [15, 14].

**Definition 2.2.** An \( EF \)-proximity is a relation on \( \mathcal{P}(X) \) which satisfies \( P0) \) through \( P3) \) and in addition

\[ A \not\delta B \Rightarrow \exists E \subset X \text{ such that } A \not\delta E \text{ and } X \setminus E \not\delta B \text{ EF-property.} \]

A topological space has a compatible \( EF \)-proximity if and only if it is a Tychonoff space.

Any proximity \( \delta \) on \( X \) induces a binary relation over the powerset exp \( X \), usually denoted as \( \ll \delta \) and named the natural strong inclusion associated with \( \delta \), by declaring that \( A \) is strongly included in \( B \), \( A \ll \delta B \), when \( A \) is far from the complement of \( B \), \( A \not\delta X \setminus B \).

By strong inclusion the Efremović property for \( \delta \) can be written also as a between-ness property

\[ (EF) \quad \text{If } A \ll \delta B, \text{ then there exists some } C \text{ such that } A \ll \delta C \ll \delta \delta B. \]

We say that \( A \) and \( B \) are \( \delta \)-strongly far [31], where \( \delta \) is a Lodato proximity, and we write \( \not\delta \) if and only if \( A \not\delta B \) and there exists a subset \( C \) of \( X \) such that \( A \not\delta X \setminus C \) and \( C \not\delta B \), that is the Efremović property holds on \( A \) and \( B \).

Instead, the concept of strongly near proximity arises from the need to introduce a relation yields information about the interiors of pairs of subsets that at least have non-empty intersection. We say that the relation \( \hat{\delta} \) on \( \mathcal{P}(X) \) is an strongly near proximity, provided it satisfies the following axioms. Let \( A, B, C \subset X \) and \( x \in X \).

\[ N0) \emptyset \not\hat{\delta} A, \forall A \subset X, \text{ and } X \hat{\delta} A, \forall A \subset X \]

\[ N1) A \hat{\delta} B \iff B \hat{\delta} A \]

\[ N2) A \hat{\delta} B \Rightarrow A \cap B \neq \emptyset \]
N3) If \( \text{int}(B) \) and \( \text{int}(C) \) are not equal to the empty set, \( A \delta B \) or \( A \delta C \) \( \Rightarrow \) \( A \delta (B \cup C) \)

N4) \( \text{int}A \cap \text{int}B \neq \emptyset \) \( \Rightarrow \) \( A \delta B \)

Example 2.3. Intersecting Interiors.

In Fig. 2.1, \( A \delta B \) (axiom N4) and \( E \delta (C \cup D) \) (axiom N3).

When we write \( A \delta B \), we read \( A \) is strongly near to \( B \) [29, 28] (see, also, [31]).

For each almost proximity we assume the following relations:

N5) \( x \in \text{int}(A) \) \( \Rightarrow \) \( \{x\} \delta A \)

N6) \( \{x\} \delta \{y\} \Leftrightarrow x = y \)

If we take the almost proximity related to non-empty intersection of interiors, we have that \( A \delta B \Leftrightarrow \text{int}A \cap \text{int}B \neq \emptyset \) provided \( A \) and \( B \) are not singletons; if \( A = \{x\} \), then \( x \in \text{int}(B) \), and if \( B \) too is a singleton, then \( x = y \). If \( A \subset X \) is an open set, then each \( x \in A \) is strongly near \( A \).

3. New Hypertopologies

Let \( \text{CL}(X) \) be the hyperspace of all non-empty closed subsets of a space \( X \).

Hit and miss and hit and far-miss topologies on \( \text{CL}(X) \) are obtained by the join of two halves. Well-known examples are Vietoris topology \([35, 36, 37, 38]\) (see, also, \([1, 2, 3, 6, 8, 9, 5, 10, 11, 12, 13, 17]\)) and Fell topology \([16]\). In \([30] \) and \([31]\), the following new hypertopologies are introduced.

\[ \tau_w \] is the hit and strongly far-miss topology having as subbase the sets of the form:

- \( V^- = \{E \in \text{CL}(X) : E \cap V \neq \emptyset\} \), where \( V \) is an open subset of \( X \),
- \( A_w = \{E \in \text{CL}(X) : E \delta X \setminus A\} \), where \( A \) is an open subset of \( X \)

\[ \tau^n \] is the strongly hit and far-miss topology having as subbase the sets of the form:

- \( V^n = \{E \in \text{CL}(X) : E \delta V\} \), where \( V \) is an open subset of \( X \),
- \( A^{++} = \{E \in \text{CL}(X) : E \delta X \setminus A\} \), where \( A \) is an open subset of \( X \)

where in both cases \( \delta \) is a Lodato proximity compatible with the topology on \( X \).

It is possible to consider several generalizations. For example, for the miss part we can look at subsets running in a family of closed sets \( \mathcal{B} \). So we define the strongly hit and far-miss topology on \( \text{CL}(X) \) associated with \( \mathcal{B} \) as the topology generated by the join of the hit sets \( A^w \), where \( A \) runs over all open subsets of \( X \), with the miss sets \( A^{++} \), where \( A \) is once again an open subset of \( X \), but more, whose complement runs in \( \mathcal{B} \). We can do the same with the hit and strongly far-miss topology.

4. Main Results

Next, consider strongly hit and far miss topologies and hit and strongly far miss topologies associated with families of subsets. We look at conditions that make these topologies \( T_2 \) topologies.
4.1. Hit and strongly far miss topologies.
Here, we consider results contained in [2], e.g., \( \tau_{w, B} \) is the topology having as subbase the sets of the form:

- \( V^- = \{ E \in \text{CL}(X) : E \cap V \neq \emptyset \} \), where \( V \) is an open subset of \( X \),
- \( A_w = \{ E \in \text{CL}(X) : E \notin X \setminus A \} \), where \( A \) is an open subset of \( X \) and \( X \setminus A \in B \)

**Definition 4.1.** Let \( X \) be a topological space and \( B \) a non-empty family of subsets of \( X \). We call \( B \) a strongly local family if and only if

\[
\forall x \in X \text{ and } \forall U \text{ nbhd of } x, \exists S \in B : x \in \text{int}(S) \subseteq S \ll_{w} U,
\]

where \( S \ll_{w} U \) means \( S \notin S \setminus U \).

We write \( \Sigma(B) \) for the collection of all finite unions of elements of \( B \).

![Fig. 3.1 A \ll_{\delta} U_x](image)

**Example 4.2. Strongly Local Family 1.**
A first simple example of strongly local family is the following. Take \( X \) as a locally compact topological space, \( B \) the family of compact subsets and \( \delta \) as the Alexandroff proximity defined by \( A \delta B \iff \text{cl}A \cap \text{cl}B \neq \emptyset \) or both \( \text{cl}A \) and \( \text{cl}B \) are non-compact. In fact, in this case, for each \( x \in X \) and each nbhd \( U \) of \( x \), it is always possible to find a compact subset containing \( x \) and strongly contained in \( U \). See, e.g., Fig. 3.1.

![Fig. 3.2 A \ll_{\delta} B_1 \ll_{\delta} C](image)

**Example 4.3. Strongly Local Family 2.**
A more general example is given by local proximity spaces \( (X, \delta, B) \) [15], where \( \delta \) is a proximity generally non-Efremović and \( B \) is a boundedness. In particular this family satisfies the following property: if \( A \in B \), \( C \subset X \) and \( A \ll_{\delta} C \) then there
exists some $B \in \mathcal{B}$ such that $A \ll_\delta B \ll_\delta C$, where $\ll_\delta$ is the natural strong inclusion associated with $\delta$. By this property follows that $\mathcal{B}$ is a strongly local family. See, e.g., Fig. 3.2.

**Theorem 4.4.** Let $(X, \tau)$ be a topological space, $\delta$ a compatible Lodato proximity on $X$ and $\mathcal{B}$ a non-empty subfamily of $\text{CL}(X)$. Suppose that any point that does not belong to a closed set is strongly far from the closed set. $(\text{CL}(X), \tau_{\omega, \mathcal{B}})$ is $T_2$ if and only if $\Sigma(\mathcal{B})$ is a strongly local family.

**Proof.** "$\Leftarrow\Rightarrow"$. We want to prove that $(\text{CL}(X), \tau_{\omega, \mathcal{B}})$ is $T_2$. Suppose that $A$ and $B$ are distinct closed subsets of $X$. So there exists a point $b \in B \cap (X \setminus A)$ and by the assumption there exists a subset $S \in \mathcal{B}$ such that $x \in \text{int}(S) \subset S \ll_\omega X \setminus A$. Then $B \in (\text{int}\,S)^-$ and $A \in (X \setminus S)_w$ with $(\text{int}\,S)^- \cap (X \setminus S)_w = \emptyset$.

"$\Rightarrow\Rightarrow"$. Consider now $x \in X$ and suppose $X \setminus A$ to be a nbhd of $x$. We have to prove that the special betweenness property holds. Take $A$ and $A \cup \{x\}$. By the hypothesis we know that there exist two open sets $\mathcal{A}_1, \mathcal{A}_2 \in \tau_{\omega, \mathcal{B}}$ such that $A \in \mathcal{A}_1$, $A \cup \{x\} \in \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Suppose

\[
\mathcal{A}_1 = (X \setminus S)_w \cap (\bigcap_{i=1}^n V_i^-) \\
\mathcal{A}_2 = (X \setminus T)_w \cap (\bigcap_{j=1}^m W_j^-)
\]

with $S, T \in \mathcal{B}$ and $V_1, \ldots, V_n, W_1, \ldots, W_m$ open subsets of $X$. Further we may assume that $S \cap V_i = \emptyset$, $\forall i = 1, \ldots, n$ and $T \cap W_j = \emptyset$, $\forall j = 1, \ldots, m$.

Consider now the sets $W_i \subseteq S_i$, $\forall i \in \{1, \ldots, n\}$. In fact if, by contradiction, $W_i \not\subseteq S$ $\forall i \in \{1, \ldots, n\}$, then there would exist $x_i \in W_i \cap (X \setminus S)$ $\forall i \in \{1, \ldots, n\}$. But now $A \cup \{x_1, \ldots, x_n\} \in \mathcal{A}_1 \cap \mathcal{A}_2$, and this is absurd. Hence there exists $\mathcal{W}_i \subseteq S$ and we have $x \in \mathcal{W}_i \subseteq \text{int}(S) \subset S \ll_\omega X \setminus A$.

4.2. **Strongly hit and far miss topologies.**

In this case, we want consider the family of regular closed subsets of $X$, $\text{RCL}(X)$. Recall that a set $F$ is regular closed if $F = \text{cl}(\text{int}F)$, that is $F$ coincides with the closure of its interior. Considering the nature of almost proximities, this family seems to be the most suitable to which refer. A well-known fact is that regular closed sets form a complete Boolean lattice [33]. Moreover there is a one-to-one correspondence between regular open ($\text{RO}(X)$) and regular closed sets. We have a regular open set $A$ when $A = \text{int}(\text{cl}A)$, that is $A$ is the interior of its closure. The correspondence between the two mentioned classes is given by $c : \text{RO}(X) \rightarrow \text{RCL}(X)$, where $c(A) = \text{cl}A$, and $o : \text{RCL}(X) \rightarrow \text{RO}(X)$, where $o(F) = \text{int}(F)$. By this correspondence it is possible to prove that also the family of regular open sets is a complete Boolean lattice. Furthermore it is shown that every complete Boolean lattice is isomorphic to the complete lattice of regular open sets in a suitable topology.

The importance of these families is also due to the possibility of using them for digital images processing, because they allow to satisfy certain common-sense physical
requirements.

Consider now $\tau^b_\mathcal{B}$, the strongly hit and far-miss topology associated to $\mathcal{B}$:

- $V^\approx = \{ E \in \text{RCL}(X) : E \not\approx V \}$, where $V$ is a regular open subset of $X$.
- $A^+ = \{ E \in \text{RCL}(X) : E \not\approx X \setminus A \}$, where $A$ is a regular open subset of $X$ and $X \setminus A \in \mathcal{B}$.

It is easy to prove that this family generates a topology on $\text{RCL}(X)$.

**Lemma 4.5.** Let $(X, \tau)$ be a topological space and $\delta^b$ strongly near proximity on $X$. For each regular open set $V$ on $\text{RCL}(X)$, we have that $V^\approx = \{ E \in \text{RCL}(X) : E \cap V \neq \emptyset \}$.

**Proof.** For each regular open set $V$ in $X$, it is straightforward that $V^\approx \subseteq \{ E \in \text{RCL}(X) : E \cap V \neq \emptyset \}$. Look now at the other inclusion and suppose that $A \not\approx \delta V$. Hence, by property (N4), we have that $\text{int}(A) \cap V = \emptyset$. But for $\text{RCL}(X)$, it follows that $A \cap V = \emptyset$. In fact, if this intersection is not empty, we can find an element $a \in A = \text{cl}(\text{int}(A))$ such that it is approximated by a net of elements in $\text{int}(A)$, $\{a_\lambda\}_{\lambda \in \Lambda}$. Now, since $V$ is a nhbd of $a$, $V$ must contain $\{a_\lambda\}_{\lambda \in \Lambda}$ residually. But this is absurd. □

**Definition 4.6.** Let $X$ be a topological space and $\mathcal{B}$ a non-empty family of subsets of $X$. We call $\mathcal{B}$ a regular local family if and only if

\[ \forall x \in X \text{ and } \forall U \text{regular open set containing } x, \exists S \in \mathcal{B} : x \in \text{int}(S) \subseteq S \ll \delta U, \]

where $S \ll \delta U$ means $S \not\approx X \setminus U$.

**Theorem 4.7.** Let $(X, \tau)$ be a topological space, $\delta^b$ an almost proximity on $X$, $\delta$ a compatible Lodato proximity on $X$ and $\mathcal{B}$ a non-empty subfamily of $\text{RCL}(X)$. If $\Sigma(\mathcal{B})$ is a regular local family, then $(\text{RCL}(X), \tau^b_\mathcal{B})$ is $T_2$.

**Proof.** The result simply follows by applying mostly the same procedure of thm. 4.4 and by lemma 4.5. □

**Remark 4.8.** Observe that in particular we could refer these results to a generalization of the Fell topology if we take as family $\mathcal{B}$ that one of compact subsets of a topological space.

**Example 4.9.** Regular local family.

An easy example of regular local family contained in $\text{RCL}(X)$ is obtained by considering $X = \mathbb{R}^N$ in which the family of closed convex subsets is a local family, i.e.,

\[ \forall x \in X \text{ and } \forall U \text{nbhd of } x, \exists C \text{ convex} : x \in \text{int}(C) \subseteq C \subset U. \]

See, e.g., Fig. 3.3 in the 3D spherical nbhd $C$ of $p$ is entirely in the interior of the nbhd $U$ of $p$. An important thing to notice is that in this case each open convex set is also regular open. ■

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