Spin-1 gravitational waves

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Abstract

Gravitational fields invariant for a 2-dimensional Lie algebra of Killing fields \([X, Y] = Y\), with \(Y\) of light type, are analyzed. The conditions for them to represent gravitational waves are verified and the definition of energy and polarization is addressed; realistic generating sources are described.

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Introduction

Gravitational waves, that is a propagating warpage of space time generated from compact concentrations of energy, like neutron stars and black holes, have not yet been detected directly, although their indirect influence has been seen and measured with great accuracy. Presently there are, worldwide, many efforts to detect gravitational radiation, not only because a direct confirmation of their existence is interesting \textit{per se} but also because new insights on the nature of gravity and of the Universe itself could be gained. For these reasons exact solutions of the Einstein field equations deserve special attention when they are of propagative nature. The need of taking into full account the nonlinearity of Einstein’s equations when studying the generation of gravitational waves from strong sources is generally recognized \[26\]. Moreover, despite the great distance of the sources from Earth (where almost all the experimental devices, laser interferometers and resonant antennas, are located) there are situations where the non linear effects cannot be neglected. This is the case when the source is a binary coalescence: indeed it has been shown \[8\] that a secondary wave, called the Christodoulou memory is generated via the non linearity of Einstein’s field equations. The memory seems to be too weak to be detected from the present generation of interferometers \[26\] (even if its frequency is in the optimal band for the LIGO/VIRGO interferometers) but of the same order as the linear effects related to the same source, thus stressing the relevance of the nonlinearity of the Einstein’s equations also (soon) from an experimental point of view.

On the theoretical side, starting from the seventy’s new powerful mathematical methods have been developed to deal with nonlinear evolution equations. For instance, a suitable generalization of the \textit{Inverse Scattering Transform} allows to integrate \[4\] Einstein field equations for a metric of the form

\[
g = f(z, t) \left( dt^2 - dz^2 \right) + h_{11}(z, t) \, dx^2 + h_{22}(z, t) \, dy^2 + 2h_{12}(z, t) \, dx \, dy.\]

Indeed, the corresponding vacuum Einstein field equations reduce essentially\footnote{The function $f$ can be obtained by quadratures in terms of the matrix $H$.} to

\[
\left( \alpha H^{-1} H_{\xi} \right)_{\eta} + \left( \alpha H^{-1} H_{\eta} \right)_{\xi} = 0,
\]

where $H \equiv \| h_{ab} \|$, $\xi = (t + z) / \sqrt{2}$, $\eta = (t - z) / \sqrt{2}$, $\alpha = \sqrt{\text{det} H}$. This is a system of non-linear differential equations whose form is typical
for two-dimensional integrable systems. Its solution through the Inverse Scattering Transform, yields gravitational solitary waves solutions.

A geometric inspection of the metric above shows that it is invariant under translations along the $x, y$-axes, i.e. it admits two Killing fields, $\partial_x$ and $\partial_y$, closing on an Abelian\footnote{The study of metrics invariant for a Abelian 2-dimensional Killing Lie algebra goes back to Einstein and Rosen \cite{Einstein, Rosen}, Kompaneyets \cite{Kompaneyets}; recent analysis can be found in \cite{27, 15}.} two-dimensional Lie algebra $A_2$. Moreover, the distribution $\mathcal{D}$, generated by $\partial_x$ and $\partial_y$, is 2-dimensional and the distribution $\mathcal{D}^\perp$ orthogonal to $\mathcal{D}$ is integrable and transversal to $\mathcal{D}$.

Thus, it has been natural to consider \cite{25} the general problem of characterizing all gravitational fields $g$ admitting a Lie algebra $\mathcal{G}$ of Killing fields such that:

I. the distribution $\mathcal{D}$, generated by the vector fields of $\mathcal{G}$, is two-dimensional.
II. the distribution $\mathcal{D}^\perp$ orthogonal to $\mathcal{D}$ is integrable and transversal to $\mathcal{D}$.

The aim of this article is to study, among those solutions, the ones which show a propagative nature. A preliminary account of our results is given in \cite{7}.

The article is organized as follows. In section 1 gravitational fields invariant for a two dimensional Lie algebra are characterized by reducing the Einstein equations to the so called $\mu$-deformed Laplace equation. Harmonic coordinates are also introduced. Section 2 is devoted to the analysis of the wave-like character of the solutions through the Zel’manov and the Pirani criteria. In section 3 realistic sources are described. In section 4 Landau-Lifshitz’s and Bel’s energy-momentum pseudo-tensors are introduced and a comparison with the linearised theory is performed. Eventually, section 5 is devoted to the analysis of the polarization of the waves.

In the following, $\text{Kil} (g)$ will denote the Lie algebra of all Killing fields of a metric $g$ while $\text{Killing algebra}$ will denote a sub-algebra of $\text{Kil} (g)$.

Moreover, an integral (two-dimensional) submanifold of $\mathcal{D}$ will be called a Killing leaf, and an integral (two-dimensional) submanifold of $\mathcal{D}^\perp$ orthogonal leaf.
1 Invariant vacuum gravitational fields

Let \( g \) be a metric on the space-time \( M \) and \( G_2 \) one of its Killing algebras whose generators \( X, Y \) satisfy the commutation relation

\[
[X, Y] = \sigma Y, \quad \sigma = 0, 1.
\] (1)

The Frobenius distribution \( D \) is two-dimensional and a coordinate system \((x^\mu), \mu = 1, 2, 3, 4\) exists such that

\[
X = \partial / \partial x^3, \quad Y = \exp (\sigma x^3) \partial / \partial x^4.
\]

Such a coordinate system is called \[25\] semiadapted (to the Killing fields).

Condition II of the previous section allows to construct special semiaadapted charts such that the fields \( e_1 = \partial / \partial x^1, e_2 = \partial / \partial x^2 \), belong to \( D^\perp \). In such a chart, called from now on adapted, the most general \( G_2 \)-invariant metric has the form \[25\]

\[
g = g_{ij} dx^i dx^j + \left( \sigma^2 \lambda (x^4)^2 - 2 \sigma \mu x^4 + \nu \right) dx^3 dx^3 + 2 (\mu - \sigma \lambda x^4) dx^3 dx^4 + \lambda dx^4 dx^4, \quad i, j = 1, 2.
\] (2)

where \( g_{ij}, \lambda, \mu, \nu \) are arbitrary functions of \((x^1, x^2)\).

If the Killing field \( Y \) is not of light type, i.e. \( g(Y, Y) \neq 0 \), condition II follows automatically from I. The local structure of this class of Einstein metrics has been explicitly described; it turns out \[25\] that a third Killing field \( Z \) exists such that \((X, Y, Z)\) span the Killing algebra \( \text{so}(2, 1) \) and all invariant metrics are static and locally diffeomorphic to the pseudo-Schwarzschild metric. This metric was also found in the context of warped solutions \[11, 28\].

In the following the Killing field \( Y \) will be assumed to be of light type. In this case the general solution of vacuum Einstein equations, in the adapted coordinates \((x^1, x^2, x^3, x^4)\), is given by

\[
g = 2f(dx^1 dx^1 + dx^2 dx^2) + \mu [(w (x^1, x^2) - 2 \sigma x^4) dx^3 dx^3 + 2 dx^3 dx^4],
\] (2)

where \( \mu = D \Phi + B; \ D, B \in \mathcal{R}, \ \Phi \) is a non constant harmonic function of \( x^1 \) and \( x^2 \), \( f = (\nabla \Phi)^2 \sqrt{|\mu|/\mu}, \) and \( w (x^1, x^2) \) is a solution of the \( \mu \)-deformed Laplace equation:

\[
\Delta w + (\partial_{x^1} \ln |\mu|) \partial_{x^1} w + (\partial_{x^2} \ln |\mu|) \partial_{x^2} w = 0.
\] (3)
The solutions of the $\mu$-deformed Laplace equation will also be called $\mu$-harmonic functions.

In the particular case $\sigma = 1$, $f = 1/2$ and $\mu = 1$, the above metrics are locally diffeomorphic to a subclass of the vacuum Peres solutions [17], corresponding to a special choice of the harmonic function parameterising that metrics.

This is easily seen by introducing, for $\tilde{u} \neq 0$, new coordinates $(x, y, \tilde{u}, \tilde{v})$ defined by $x^1 = x$, $x^2 = y$, $x^3 = \ln |\tilde{u}|$, $x^4 = \tilde{w}$.

Of course, metrics (2) can also be obtained by imposing an additional isometry to metrics admitting a Killing vector field of light type and, indeed, they are an extension of the ones reported in [13].

When $w$ is constant the above family of Einstein metrics admits the time-like Killing vector field

$$T = \exp (-x^3) \frac{\partial}{\partial x^3} + \frac{1}{2} [(w - 2\sigma x^4) \exp (-x^3) + \exp (x^3)] \frac{\partial}{\partial x^4}$$

which is hypersurfaces orthogonal. This means that these Einstein metrics are just static gravitational fields.

As it will be argued in the next sections, when $w$ is not constant the above family of Einstein metrics may represent propagative gravitational fields.

Since the distribution $D^\perp$ is assumed to be transversal to $D$, the restriction of $g$ to any Killing leaf, say $S$, is non-degenerate. So, $(S, g|_S)$ is a homogeneous two-dimensional Riemannian manifold. In particular, the Gauss curvature $K(S)$ of the Killing leaves is constant. An explicit computation shows that $K(S)$ vanishes.

Thus, the space-time $\mathcal{M}$ has a fiber bundle structure

$$\pi : \mathcal{M} \longrightarrow \mathcal{W},$$

whose basis $\mathcal{W}$ is diffeomorphic to the orthogonal leaves and whose fibers are the Killing leaves and as such are flat two-dimensional Riemann manifolds.

Despite the non-linear nature of general relativity, gravitational fields (2) obey to two superposition laws. Indeed, with two harmonic functions $\Phi_1$ and $\Phi_2$ we can associate three gravitational fields (in facts a whole two-parameters family), that is, $g_{\Phi_1}$, $g_{\Phi_2}$ and $g_{a\Phi_1 + b\Phi_2}$; the last one, which is associated with the linear combination of $\Phi_1$ and $\Phi_2$, may be regarded as the superposition of the two associated solutions $g_{\Phi_1}$ and $g_{\Phi_2}$.

The second superposition law follows from the linearity of the $\mu$-deformed Laplace equation, so that with two $\mu$-harmonic functions $w_1$ and $w_2$ we
can associate three gravitational fields \( g_{w_1}, g_{w_2} \) and their sum \( g_{w_1+w_2} \equiv \frac{(g_{2w_1} + g_{2w_2})}{2} \.

1.1 The harmonic coordinates.

The coordinates \((x^3, x^4)\) on the Killing leaves \( S \) have a clear geometric meaning but are of difficult physical interpretation. Fortunately, being the Killing leaves flat manifolds, it is possible to introduce coordinates \((\tilde{z}, \tilde{t})\) diagonalizing the metric

\[ g|_S = \tilde{\mu}[(\tilde{w} - 2\sigma x^4)dx^3dx^3 + 2dx^3dx^4], \]

where \( \tilde{\mu} \) and \( \tilde{w} \), being the restriction of the functions \( \mu \) and \( w \) to the Killing leaves, are constant.

The coordinate system \((x, y, z, t)\), where for \( z > t \)

\[
\begin{align*}
x &= w_1(x^1, x^2) \\
y &= w_2(x^1, x^2) \\
z &= \frac{1}{2} \left[ (2x^4 - w(x^1, x^2)) \exp(x^3) + \exp(-x^3) \right] \\
t &= \frac{1}{2} \left[ (2x^4 - w(x^1, x^2)) \exp(x^3) - \exp(-x^3) \right],
\end{align*}
\]

is harmonic, \( w_1(x^1, x^2) \) and \( w_2(x^1, x^2) \) denoting any two independent \( \mu \)-harmonic functions. The generic Killing leaf \( S \) is mapped onto the half-plane \( z > t \), the line \( z = t \) representing the points with \( x^3 = +\infty \).

In these coordinates, metrics \((2)\) take the form

\[
g = 2\sqrt{\mu} \left( \frac{\nabla \Phi}{\mu} \right)^2 J^{-2} \left[ (\nabla y)^2dx^2 + (\nabla x)^2dy^2 - 2\nabla x \cdot \nabla y dxdy \right] + \mu [dz^2 - dt^2 + d(w) d(\ln|z-t|)] ,
\]

where \( J = \partial_{x^1}w_1 \partial_{x^2}w_2 - \partial_{x^1}w_1 \partial_{x^2}w_2 \) is the Jacobian determinant of the map \((x^1, x^2) \rightarrow (x, y)\).

In the case \( \mu = \text{const} \), the \( \mu \)-deformed Laplace equation reduces to the Laplace equation and \( w_1, w_2 \) reduce to be just harmonic functions. Thus, it is possible to choose \( x = x^1, y = x^2 \) so that in the harmonic coordinates \((x, y, z, t)\), and for \( \mu = 1 \), the above Einstein metrics take the particularly simple form

\[
g = 2f(dx^2 + dy^2) + dz^2 - dt^2 + d(w) d(\ln|z-t|) .
\]
This coordinates system explicitly shows that, when w is constant, the Einstein metrics given by Eq. (5) are static and, under the further assumption \( \Phi = x \sqrt{2} \), they reduce to the Minkowski one. Moreover, when w is not constant, gravitational fields (5) look like a disturbance moving, along the z direction on the Killing leaves, at light velocity. However, the last observation is neither rigorous nor covariant. Since the propagation direction is the most important ingredient in the study of the polarization, in the following sections a detailed analysis will be devoted to this question.

In the following we will assume that w is not constant.

2 Zelmanov’s and Pirani’s criteria

To check the wave character of gravitational fields (2), a Zakharov generalization of the Zel’manov criterion [31] will be applied which states that a vacuum solution of the Einstein equations is a gravitational wave if the components \( R_{\mu\nu\lambda\sigma} \) of the corresponding Riemann tensor field \( R \), satisfy a hyperbolic equation of the form

\[
g^{\alpha\beta} \nabla_\alpha \nabla_\beta R_{\mu\nu\lambda\sigma} = N_{\mu\nu\lambda\sigma}
\]

(6)

where \( \nabla_\beta \) denotes the Levi-Civita covariant derivative of the metric and \( N_{\mu\nu\lambda\sigma} \) denote the components of a tensor field \( N \) depending at most on first derivatives of the Riemann tensor itself.

For symmetric manifolds Eq. (6) with \( N = 0 \) is an identity because the Riemann tensor is covariantly constant, but it may become an identity also in the case of Einstein manifolds \( (R_{\alpha\beta} = \kappa g_{\alpha\beta}) \), for special choices of the tensor field \( N \). Hence, to exclude a priori these situations, the original Zel’manov criterion is formulated in the more restrictive assumptions:

- \( R_{\alpha\beta\gamma\delta} \) not covariantly constant\(^3\);
- \( g^{\alpha\beta} \nabla_\alpha \nabla_\beta R_{\mu\nu\lambda\sigma} = 0 \).

The metrics in Eq. (2) certainly do not define symmetric or Einstein manifolds, as can be checked from the components of the Ricci tensor given below. Hence, the first hypothesis is certainly satisfied while the second

\(^3\)That is the manifold is not symmetric.
one, i.e. \( N = 0 \), which ensures the applicability of the criterion to Einstein manifolds too, is not needed.

Concerning the physical meaning of this criterion, it can be shown that the characteristic hypersurface of the system of equations is identical with the characteristic hypersurface of the Einstein and Maxwell equations in curved space-time. Consequently, Eqs. describe the propagation of the discontinuities of the second derivatives of the Riemann tensor. This links the Zel’manov criterion to the intuitive concept of local wave of curvature. The criterion is independent on the explicit form of \( N_{\mu\nu\lambda\sigma} \); in fact, the characteristic hypersurface of a system of equations is determined only by the highest derivative term. Then we will not fix an explicit form of \( N_{\mu\nu\lambda\sigma} \) but just require that \( N_{\mu\nu\lambda\sigma} \) be a tensor containing at most first derivatives of the Riemann tensor. This clearly corresponds to a covariant criterion. Then a sufficient condition is

\[
g^{\alpha\beta} \partial_\alpha \partial_\beta R_{\mu\nu\lambda\sigma} = 0, \tag{7}\]

where \( \partial_\beta \) are the usual partial derivatives. In fact, if this is the case then \( N_{\mu\nu\lambda\sigma} \) is a tensor containing at most first derivatives of the Riemann tensor.

To start with, let us verify that gravitational fields do satisfy Eqs. (7).

In the harmonic coordinates system the only nonvanishing components of the Riemann and Ricci tensor fields corresponding to metrics are proportional to one of the following

\[
\begin{align*}
R_{txxx} &= \frac{(2f w_{,xx} + f_{,y} w_{,y} - f_{,x} w_{,x})}{4f(z-t)^2} \\
R_{txxy} &= \frac{(2f w_{,xy} - f_{,y} w_{,x} - f_{,x} w_{,y})}{4f(z-t)^2} \\
R_{tyzy} &= \frac{(2f w_{,yy} - f_{,y} w_{,y} + f_{,x} w_{,x})}{4f(z-t)^2} \\
R_{xyxy} &= \frac{f^2_{,y} + f^2_{,x} - f (f_{,xx} + f_{,yy})}{f}, \tag{8}
\end{align*}
\]

and

\[
R_{tt} = \frac{\Delta w}{2f(z-t)^2}, \quad R_{xx} = -\frac{(\nabla f)^2 - f \Delta f}{2f^2},
\]

respectively.
Moreover, the harmonicity condition for \( \Phi \) implies the last component vanishes. In fact, when \( \mu = \text{const} \), \( \Delta \Phi = 0 \) implies for \( f \) that
\[
f \Delta f - (\nabla f)^2 = 0. \tag{9}
\]

- When \( f \) is a constant function, Eqs. (8) reduce to
\[
R_{txzx} = w, \quad \frac{w}{2(z-t)^2} = 0, \quad R_{txzy} = w, \quad \frac{w}{2(z-t)^2}, \quad R_{tyzy} = w, \quad \frac{w}{2(z-t)^2} \tag{10}
\]
which, \( w(x, y) \) being a harmonic function, are all harmonic functions of \( x, y \). As a consequence, it is straightforward to check that the generalized Zel’manov criterion, in the form (7), is satisfied [7].

- When \( f \) is not a constant function, the generalized Zel’manov criterion is still satisfied in the form (9) thanks to a non-trivial combination of the harmonicity condition for \( w \) and Eq. (9).

- In the general case when \( f \) and \( \mu \) are not constant functions the Zel’manov criterion is satisfied in the form expressed by Eq. (6). This may be more conveniently checked in the adapted coordinates \( x^\mu \).

The Zel’manov criterion, even if it is covariant and allows a clear physical interpretation in terms of local waves of curvature, does not determine the propagation direction of the waves, that is the most important ingredient in the study of their polarization. In the next sections we will overcome this drawback of the Zel’manov criterion by using a suitable energy-momentum pseudo-tensor.

Besides the Zel’manov-Zakharov criterion, the Pirani algebraic criterion, which is based on the Petrov classification, is satisfied. First of all, let us recall that a vacuum solution of the Einstein equations is a gravitational wave according to Pirani if its Riemann tensor is of type \( \Pi \), \( N \) or \( \Pi \) in the Petrov classification [18]. Then, in light-cone coordinates \( u = (z - t)/\sqrt{2}, \quad v = (z + t)/\sqrt{2} \), where the metrics given by Eq. (5) read
\[
g = 2f(dx^2 + dy^2) + 2dudv + dw^2, \quad d\ln |u|, \tag{11}
\]
the vector fields \( \partial_u \) and \( \partial_v \) are both isotropic. Moreover, it is trivial to show that the only non-vanishing components of the Riemann tensor are
\[
R_{uinj} = \pm \frac{1}{2u^2} \partial_{ij}^2 w
\]
and this clearly corresponds to a type-$N$ Riemann tensor in the Petrov classification. Furthermore, it follows from the natural interpretation of the Pirani criterion \cite{20} that the gravitational wave propagates along the null vector field $\partial_{v}$, or, in other words, the gravitational wave \cite{5} propagates along the $z-$axis with velocity $c = 1$. Thus, the Pirani criterion, even if with a less clear physical interpretation, allows an easy and covariant determination of the propagation direction. It will be an important self-consistency check for our calculations to discover the same results by means of the energy-momentum pseudo-tensors.

3 The sources

In the theory of gravitational waves a crucial problem is to characterise realistic sources able to generate waves enough strong to be detected by the experimental devices.

The simplest source for metrics \cite{2} (with $\sigma = 1$) is dust with density $\rho$ and velocity $U^\mu$ and, then, characterized by an energy-momentum tensor $T_{\mu\nu} = \rho U_\mu U_\nu$. When $U^\mu$ is a light-like vector field, this kind of energy-momentum tensor can describe the energy and momentum of null electromagnetic waves, i.e., electromagnetic fields whose scalars, $F^{\mu\nu}F_{\mu\nu}$ and $\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$, both vanish. In this way it would be possible to describe the gravitational effects (in particular the emission of gravitational waves) of a very interesting astrophysical phenomenon, the $\gamma-$ray bursts (GRBs): emission of ultra-high energetic $\gamma$-rays ($\sim 10^{20}$ eV), whose origin is still to be fully understood.

Moreover, with this source one can also preserve the symmetries \cite{25} of the vacuum solution. It can be shown \cite{6} that, in the adapted coordinates $(x^1, x^2, x^3, x^4)$, the corresponding solutions of Einstein equations represent the following gravitational fields

$$g = 2f(dx^1dx^1 + dx^2dx^2) + \mu[(\tilde{w}(x^1, x^2) - 2x^4)dx^3dx^3 + 2dx^3dx^4],$$

where $\mu = D\Phi + B$; $D, B \in \mathcal{R}$, $\Phi$ is a non constant harmonic function of $x^1$ and $x^2$, $f = (\nabla\Phi)^2\sqrt{\mu/\mu}$, while $\tilde{w}(x^1, x^2)$ is a solution of the $\mu$-deformed Poisson equation

$$\mu\Delta\tilde{w} + \nabla\mu \cdot \nabla\tilde{w} = 2f\mu^2\rho,$$

in which, to save writing, beyond $c = 1$, it has been taken $8\pi G = 1$, $G$ being the Newton gravitational constant.
It is worth to remark that spin-1 gravitational waves in vacuum necessarily are not square integrable. Thus, if they are requested to be asymptotically flat, then δ-like singularities appear in the plane \((x, y)\) in a neighborhood of the origin; in presence of matter sources the singularities are smoothed out.

4 The energy-momentum pseudo-tensors

The definition of momentun and energy associated with a gravitational field is an intrinsically controversial problem because these quantities are connected to the space-time translation invariance, whereas the group of invariance of general relativity is much bigger. With this cautionary remark in mind, various definitions are available which attain to different physical situations. When dealing with the solutions of the linearised Einstein equations in the vacuum (plane gravitational waves) a commonly accepted definition is based on the canonical energy-momentum pseudo-tensor (\([9, 30]\)):

\[
\tau_\mu^\nu = \frac{\partial L}{\partial (\partial_\nu g_{\alpha\beta})} \partial_\mu g_{\alpha\beta} - g_{\mu\nu} L,
\]

where

\[
L/\sqrt{|g|} = g^{\mu\nu} \left[ \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} - \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\sigma} \right]
\]

(13)
is the Ricci scalar deprived of terms containing the second derivative of the metric.

Then, for the wave solutions of the linearised Einstein equations the energy density \(\tau^0_0\) is expressed \([9]\) as the sum of squares of derivatives of some metric components which do represent the physical degrees of freedom of the metric.

Under a transformation preserving the propagation direction and the harmonic character of the coordinates system, in particular a rotation in the \((x, y)\) plane, the physical components of the metric transform like a spin-2 field. It is well known that in general \(\tau_\mu^\nu\) in Eq. (12) is not a tensor field but it does transform as a tensor field under those transformations which preserve the character of the field of consisting only of waves moving in the \(z\) direction, so that the \(g_{\mu\nu}\) remain functions of the single variable \(z - t\).

Thus, within the linearised theory, the canonical energy-momentum pseudo-tensor is a good tool to study the physical properties of the gravitational waves.
4.1 Comparison with the linearised theory

The exact gravitational wave

\[ g = dx^2 + dy^2 + dz^2 - dt^2 + d(w) d (\ln |z - t|), \]

(14)
given by Eq. (5) for \( \mu = 1, \ f = 1/2 \), has the physically interesting form of a perturbed Minkowski metric with \( h = dw d \ln |z - t| \).

Moreover, besides being an exact solution of the Einstein equations, it is a solution of the linearised Einstein equations on a flat background too:

\[
\begin{align*}
\eta^{\mu \nu} \partial_\mu \partial_\nu h &= 0 \\
\eta^{\mu \nu} (2h_{\mu \rho, \nu} - h_{\mu \nu, \rho}) &= 0
\end{align*}
\]

Then, to study its energy and polarization, the standard tools of the linearised theory and in particular the canonical energy-momentum pseudotensor, could be used. Nevertheless, with \( h = dw d (\ln |z - t|) \) the \( \tau^0_0 \) component of the canonical energy-momentum tensor vanishes. This is due to the fact that the components of the tensor \( h \) cannot be expressed in the transverse-traceless gauge since \( h \) has only one index in the plane transversal to the propagation direction.

Even if not explicitly declared, the standard textbooks analysis of the polarization is performed for the square integrable solutions of the wave-equation. Indeed, they can be always Fourier developed in terms of plane-wave functions with a light-like vector wave \( k_\mu \).

The harmonicity condition for the plane wave solutions \( h_{\mu \nu} = e_{\mu \nu} e^{i \rho} + e^*_{\mu \nu} e^{-i \rho} \) with \( \rho = k_\mu x^\mu \) and \( k_\mu k^\mu = 0 \), reduces to

\[
\frac{1}{2} k_\lambda \eta^{\mu \nu} e_{\mu \nu} = \eta^{\mu \nu} k_\nu e_{\mu \lambda}.
\]

(15)

It is trivial to see that the symmetry group of this equation, which encodes the harmonic nature of the coordinate system, reduces to linear transformations and more precisely to Poincaré transformations [30] (these are nothing but the usual "gauge transformations" of the linearised gravity). It can be easily shown that, for square integrable perturbations, one can always choose the transverse-traceless gauge. In other words, it is always possible to eliminate, with a suitable gauge transformation, the components of the perturbations with one index in the propagation direction [30], i.e. square integrable perturbations of spin-1 do not exist. For these reasons, the canonical
4.2 Landau–Lifshitz’s and Bel’s energy-momentum pseudotensors

Besides the canonical energy-momentum pseudo-tensor, a deep physical significance can be given to the Landau-Lifshitz energy-momentum pseudo-tensor $\tau_{\mu\kappa}$ [14] defined by

$$
\tau_{\mu\kappa} = \frac{1}{16\pi \kappa} \left\{ (2\Gamma^\nu_{\lambda\mu}\Gamma^\sigma_{\kappa\nu} - \Gamma^\nu_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\nu_{\lambda\nu}\Gamma^\sigma_{\mu\sigma}) (g^\rho\lambda g^{\kappa\mu} - g^\rho\kappa g^{\lambda\mu}) 
+ g^{\rho\lambda} g^{\mu\nu} (\Gamma^\kappa_{\lambda\mu} \Gamma^\sigma_{\kappa\nu} + \Gamma^\kappa_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} - \Gamma^\kappa_{\nu\sigma} \Gamma^\sigma_{\lambda\mu} - \Gamma^\kappa_{\lambda\mu} \Gamma^\sigma_{\nu\sigma}) 
+ g^{\rho\lambda} g^{\sigma\nu} (\Gamma^\rho_{\lambda\nu} \Gamma^\kappa_{\mu\sigma} - \Gamma^\rho_{\lambda\sigma} \Gamma^\kappa_{\mu\nu}) \right\}. \quad (16)
$$

There are strong evidences that, in some cases, it gives the correct definition of energy [19]. In fact, the energy flux radiated at infinity for an asymptotically flat space-time, evaluated with the Landau-Lifshitz energy-momentum pseudo-tensor, has been seen to agree with the Bondi flux [5] that is with the energy flux evaluated in the exact theory.

It is easy to check that the components $p^\mu \equiv \tau^\mu_0$ of the 4-momentum density are

$$
\begin{align*}
p^0 &= \frac{4}{(t-z)^2} C_1 (w_{,xx})^2 + C_2 (w_{,xy})^2 + \frac{4}{(t-z)^4} C_3 \nabla \cdot [|\nabla w|^2 \nabla w], \\
p^1 &= p^2 = 0, \quad p^3 = p^0,
\end{align*}
$$

where $C_i$ are some positive numerical constants, $\nabla = (\partial_x, \partial_y)$ and the harmonicity condition for $w$ has been used.

\footnote{Of course, it is possible to find [24] a coordinate system in which the perturbation $h$ has non vanishing components only in the transverse plane. However such a coordinate system is not harmonic.}
The use of the Bel’s superenergy tensor \( T^{\alpha\beta\lambda\mu} = \frac{1}{2} \left( R^{\alpha\rho\lambda\sigma} R_{\rho\sigma}^{\beta\mu} + * R^{\alpha\rho\lambda\sigma} * R_{\rho\sigma}^{\beta\mu} \right) \),

where the symbol * denotes the volume dual, leads to the same result. Indeed, in adapted coordinates the metric has the form

\[
g = dx^1 dx^1 + dx^2 dx^2 + (w(x^1, x^2) - 2x^4) dx^3 dx^3 + 2 dx^3 dx^4
\]

and the only non vanishing independent components of the covariant Riemann tensor \( R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R_{\beta\gamma\delta}^{\rho} \) are

\[
R_{1313} = -w_{,11}; \quad R_{1323} = -w_{,12}; \quad R_{2323} = -w_{,22}.
\]

It follows that the density energy represented by the Bel’s scalar

\[
W = T^{\alpha\beta\lambda\mu} U^\alpha U^\beta U^\lambda U^\mu,
\]

the \( U^\alpha \)'s denoting the components of a time-like unit vector field, depends on the squares of \( w_{,ij} \). Thus, both the Landau-Lifshitz pseudo tensor and the Bel superenergy tensor single out the same physical degrees of freedom. In particular, we can take the components \( h_{tx} \) and \( h_{ty} \) as fundamental degrees of freedom for the gravitational wave (14).

Concerning the definition of the polarization, the above form for \( \tau_0^\mu \) is particularly appealing because, apart from a physically irrelevant total derivative that does not contribute to the total energy flux, the component \( \tau_0^0 \) representing the energy density is expressed as the sum of square amplitudes. The momentum \( p^i = \tau_0^i \) is non vanishing only in the \( z \)-direction and it is proportional to the energy with proportionality constant \( c = 1 \); that is these waves move with light velocity along the \( z \)-axis. Moreover, this result is perfectly consistent with the one obtained with the Pirani criterion.

5 Spin

The definition of spin or polarization for a theory, such as general relativity, which is non-linear and possesses a much bigger invariance than just the Poincaré one, deserves a careful analysis [21].

It is well known that the concept of particle together with its degrees of freedom like the spin may be only introduced for linear theories (for example
for the Yang-Mills theories, which are non linear, it is necessary to perform a perturbative expansion around the linearised theory). In these theories, when Poincaré invariant, the particles are classified in terms of the eigenvalues of two Casimir operators of the Poincaré group, $P^2$ and $W^2$ where $P^\mu$ are the translation generators and $W^\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^\rho^\sigma$ is the Pauli-Ljubanski polarization vector with $M^{\mu\nu}$ Lorentz generators.

As it has been shown, the gravitational fields of Eq. (14) represent gravitational waves moving at the velocity of light, that is, in the would be quantised theory, particles with zero rest mass. Thus, if a classification in terms of Poincaré group invariants could be performed, these waves would belong to the class of unitary (infinite-dimensional) representations of the Poincaré group characterized by $P^2 = 0$, $W^2 = 0$. But, in order for such a classification to be meaningful, $P^2$ and $W^2$ have to be invariants of the theory. This is not the case for general relativity, unless we restrict to a subset of transformations selected for example by some physical criterion or by experimental constraints. For the solutions of the linearised vacuum Einstein equations the choice of the harmonic gauge does the job \[30\]. There, the residual gauge freedom corresponds to the sole Lorentz transformations.

For these reasons, only gravitational fields represented by Eq. (14) will be considered, which, besides being exact solutions, solve the linearised vacuum Einstein equations as well. There exist several equivalent procedures to evaluate their polarization. For instance, one can look at the $\tau^0_0$ component of the Landau-Lifshitz pseudo-tensor and see how the metric components that appear in $\tau^0_0$ transform under an infinitesimal rotation $\mathcal{R}$ in the plane $(x, y)$ transverse\(^5\) to the propagation direction\(^6\).

The physical components of the metric are $h_{tx}$ and $h_{ty}$ and under the infinitesimal rotation $\mathcal{R}$ in the plane $(x, y)$ transform as a vector. Applied to any vector $(v_1, v_2)$ the infinitesimal rotation $\mathcal{R}$, has the effect

$$\mathcal{R}v_1 = v_2, \quad \mathcal{R}v_2 = -v_1,$$

from which

$$\mathcal{R}^2v_i = -v_i \quad i = 1, 2,$$

so that $i\mathcal{R}$ has the eigenvalues $\pm 1$.

\(^5\)With respect to the Minkowskian background metric, this plane is orthogonal to the propagation direction. With respect to the full metric this plane is transversal to the propagation direction and orthogonal only in the limit $|z-t| \rightarrow \infty$.

\(^6\)It has been said before, that this transformation preserves the harmonicity condition.
Thus, the components of $h_{\mu\nu}$ that contribute to the energy correspond to spin-1 fields\(^7\), provided that only Lorentz transformations are allowed.

Spin-0 and spin-1 gravitons have been considered, in a different context, in [2, 1, 16, 23].

### 6 Conclusions

It has been shown that gravitational fields (11) represent spin-1 gravitational waves and that the reason why it is commonly believed that spin-1 gravitational waves do not exist is that, in dealing with the linearised Einstein theory, all authors implicitly assume a *square integrable* perturbation. In other words, *square integrable* spin-1 gravitational waves are always *pure gauge*. However, it has been proven that there exist interesting *non square integrable* wave-like solutions of linearised Einstein equations that have spin-1. These solutions are very interesting at least for two reasons. Firstly, they are asymptotically flat (with at least a $\delta$-like singularity) in the plane orthogonal to the propagation direction. Secondly, they are solutions of the exact equations too, so that the spin-1 cannot be considered as an "artifact" of the linearised theory. Realistic sources able to smooth out the mentioned singularities have also been found.

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\(^7\)There is nothing to forbid the existence of two spin-1 fields, but one consequence is that particles with the same orientation repel and particles with opposite orientation attract.
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