UNIVERSAL $\beta$-EXPANSIONS

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Abstract. Given $\beta \in (1,2)$, a $\beta$-expansion of a real $x$ is a power series in base $\beta$ with coefficients 0 and 1 whose sum equals $x$. The aim of this note is to study certain problems related to the universality and combinatorics of $\beta$-expansions. Our main result is that for any $\beta \in (1,2)$ and a.e. $x \in (0,1)$ there always exists a universal $\beta$-expansion of $x$ in the sense of Erdős and Komornik, i.e., a $\beta$-expansion whose complexity function is $2^n$. We also study some questions related to the points having less than a full branching continuum of $\beta$-expansions and also normal $\beta$-expansions.

1. Formulation of main results

Let $\beta \in (1,2)$ be our parameter, $\Sigma = \prod_{1}^{\infty} \{0,1\}$ and $x \geq 0$. We will call a sequence $\varepsilon \in \Sigma$ a $\beta$-expansion of $x$, if it satisfies

$$x = \pi_{\beta}(\varepsilon) := \sum_{n=1}^{\infty} \varepsilon_n \beta^{-n}. \tag{1.1}$$

Remark 1.1. Note that traditionally this term implies the greedy $\beta$-expansion of $x$ (see, e.g., [1]) but for our purposes it is better to use it in the above sense, because we will be interested in all $\beta$-expansions of a given $x$. We hope this will not cause any confusion.

It is clear that since $\varepsilon = (\varepsilon_n)_{1}^{\infty} \in \Sigma$, any $x$ representable in the form of the series (1.1), must belong to the interval $I_\beta := [0, 1/(\beta - 1)]$. On the other hand, each $x \in I_\beta$ does have at least one $\beta$-expansion, namely, the greedy $\beta$-expansion: if $x \in [0,1)$, let

$$T_\beta(x) = \beta x \mod 1,$$

and put

$$\varepsilon_n := \lfloor \beta T_\beta^{n-1} x \rfloor, \quad n \geq 1.$$
(here the power stands for the corresponding iteration, $\lfloor \cdot \rfloor$ denotes the integral part of a number and $\{ \cdot \}$ stands for its fractional part). If $x \in [1, (\beta - 1))$, then we put

$$\ell = \min \{ k \geq 1 : x - \beta^{-1} - \cdots - \beta^{-k} \in (0, 1) \}$$

and apply the greedy algorithm to $x - \beta^{-1} - \cdots - \beta^\ell$ to obtain the digits $\varepsilon_{\ell+1}, \varepsilon_{\ell+2}, \text{etc.}$ Finally, if $x = 1/(\beta - 1)$, then inevitably $\varepsilon_n \equiv 1$.

An important property of the greedy $\beta$-expansions consists in their monotonicity, i.e., if $x < y$, then the greedy $\beta$-expansion of $x$ is lexicographically less than the one of $y$. A detailed description of all possible greedy $\beta$-expansions for a given $\beta$ was given by Parry [10] and is briefly described in Section 2.

One of the intriguing questions regarding the $\beta$-expansions is as follows: does a given $x$ have $\beta$-expansions different from the greedy one, and if so, “how many”? (cardinality, dimension)

Recently the author proved the following metric result:

**Theorem 1.2.** [13, 14] For any $\beta \in (1, 2)$ a.e. $x \in I_\beta$ has a continuum of distinct $\beta$-expansions.

The proof given in [13] is deliberately elementary; however, in the survey paper [14] a more revealing (dynamical) proof of this result is given. In the present paper Theorem 1.2 comes in a slightly stronger form – see Theorem 3.6.

The main goal of this paper is to obtain a similar metric result about universal $\beta$-expansions which were introduced by Erdős and Komornik [5]. Recall their definition:

**Definition 1.3.** A $\beta$-expansion $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ is called universal if for any finite 0-1 word $w$ there exists $k \geq 1$ such that $\varepsilon_k \ldots \varepsilon_{k+N-1} = w$, where $N$ is the length of $w$ (notation: $N = |w|$). In other words, its complexity function must be $2^n$ (see, e.g., the survey paper [6] for the definition of complexity and dynamics-related results).

In [5] the authors concentrate their efforts mostly on the case $x = 1$; however, they present some results for general $x$ as well. They show in particular that there exists $\beta_0 > 1$ such that for each $\beta \in (1, \beta_0)$ every $x \in (0, 1/(\beta - 1))$ has a universal $\beta$-expansion. At the same time there exist larger $\beta$’s for which this is not the case (in particular, $\beta = \frac{1}{2}(1 + \sqrt{5})$ – see Counterexample below). The question about the maximal possible $\beta_0$ with this property remains open; one of the obstacles proves to be the fact that if $\beta$ is a Pisot number (an algebraic integer greater than 1 whose conjugates are all less than 1 in modulus), then, as was shown in [5], $x = 1$ cannot have a universal $\beta$-expansion.
It would be conceivable if for every $\beta \in (1, \frac{1}{2}(1 + \sqrt{5}))$ every $x$ had a universal $\beta$-expansion unless $\beta$ is a Pisot number.

The main result of the present paper is

**Theorem 1.4.** For every fixed $\beta \in (1, 2)$ a.e. $x \in I_\beta$ has a universal $\beta$-expansion.

It is natural to ask the question whether Theorems 1.2 and 1.4 are related. The answer is negative in one direction: namely, there exist $\beta$ and $x$ having a continuum of distinct $\beta$-expansions, none of which is universal:

**Counterexample.** Let $\beta = \frac{1}{2}(1 + \sqrt{5})$ and $x = 1/2\beta$. As was shown by Vershik and the author in [15], the space of all $\beta$-expansions for $x$ is $\prod_1^\infty \{100, 011\}$. It is easy to see that the word 1010 cannot occur in any of them, i.e., none of them is universal.

We believe (but have failed to show) that if $x$ has a universal $\beta$-expansion, then it has a continuum of distinct $\beta$-expansions. Since it is easy to show that any $x$ that has a finite number of $\beta$-expansions can be excluded (see Remark 3.7 below), the problem may be reformulated as follows:

**Open problem.** Prove or disprove that there exist $\beta \in (1, 2)$ and $x$ which has precisely $\aleph_0$ different $\beta$-expansions such that one of them is universal.

Section 3 contains a claim that improves Theorem 1.2; more precisely, we deal with the set of points $x$ whose branching compactum is not full (for example, such are points which have not more than countable set of $\beta$-expansions). We show that this set is in a way very close to the set of unique $\beta$-expansions studied in [8] (see Proposition 3.5 below) and in particular, has the Hausdorff dimension strictly less than 1.

Finally, in Section 4 we discuss more delicate questions related to normal $\beta$-expansions (see the definition at the beginning of Section 4).

## 2. Universality

This section is devoted completely to the proof of Theorem 1.4. Our method may be called anti-normalization; let us recall first some definitions and present a model example, for which our proof will be especially simple and revealing.

### 2.1. Necessary definitions.** Let the sequence $(a_n)_{n=1}^\infty$ be defined as follows: let $1 = \sum_{k=1}^\infty a_k \beta^{-k}$ be the greedy expansion of 1, i.e., $a_n = [\beta T_{T_n}^n - 1]$, $n \geq 1$. If the tail of the sequence $(a_n)$ differs from $0^\infty$, then we put $a_n \equiv a'_n$. Otherwise let $k = \max \{j : a'_j > 0\}$, and
\[(a_1, a_2, \ldots) := (a'_1, \ldots, a'_{k-1}, a'_k - 1)\infty.\]

In the seminal paper \[10\] it is shown that for each greedy expansion \(\varepsilon\) in base \(\beta\), \((\varepsilon_n, \varepsilon_{n+1}, \ldots)\) is lexicographically less (notation: \(<\)) than \((a_1, a_2, \ldots)\) for every \(n \in \mathbb{N}\). Moreover, it was shown that, conversely, every sequence with this property is actually the greedy expansion in base \(\beta\) for some \(x \in [0, 1)\).

Thus, the set of all greedy \(\beta\)-expansions for \(x \in [0, 1)\) is the \(\beta\)-compactum

\[(2.1) \quad X_\beta = \{\varepsilon \in \Sigma \mid (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec (a_1, a_2, \ldots), \ n \in \mathbb{N}\}\]

The sequences from the \(\beta\)-compactum will be called \(\beta\)-admissible (or simply \(\text{admissible}\) if it is clear which \(\beta\) is in question).

As is shown in \[10\], the map \(\varphi_\beta : X_\beta \to [0, 1)\) defined by the formula

\[(2.2) \quad \varphi_\beta(\varepsilon) = \sum_{n=1}^{\infty} \varepsilon_n \beta^{-n},\]

is in fact one-to-one with the exception of a countable set of sequences.

Let \(\tau_\beta\) denote the shift on \(X_\beta\) (it is obvious from \(2.1\)) that the \(\beta\)-compactum is shift-invariant). Then \(T_\beta = \varphi_\beta \tau_\beta \varphi_\beta^{-1}\), and it is shown in \[12\] that there exists unique ergodic shift-invariant probability measure \(\nu_\beta\) on \(X_\beta\) such that \(\varphi_\beta(\nu_\beta)\) is equivalent to the Lebesgue measure. This measure is positive on all cylinders \([\varepsilon_1 = i_1, \ldots, \varepsilon_n = i_n]\) provided \((i_1, \ldots, i_n, 0\infty)\) is \(\beta\)-admissible.

Define the \(\beta\)-value of a 0-1 word \(w = (x_k)_1^N\) \((N \leq \infty)\) as follows:

\[
\text{val}_\beta(w) := \sum_{1}^{N} x_k \beta^{-k}.
\]

Let \(\Sigma_\beta\) denote the set of 0-1 sequences whose \(\beta\)-value is less than or equal to 1. The normalization (in base \(\beta\)) is the map from \(\Sigma_\beta\) to \(X_\beta\) defined by the formula

\[(2.3) \quad n_\beta := \varphi_\beta^{-1} \circ \pi_\beta,
\]

where \(\pi_\beta\) is given by \([1.1]\) and \(\varphi_\beta\) is given by \([2.2]\). We will identify sequences whose tail is \(0\infty\) with the corresponding finite words. So, the normalization of a finite word can be finite (see below). For more detail and “automatic” properties of normalization see \[7\].

Thus, the operation of normalization assigns to each 0-1 sequence the admissible sequence with the same \(\beta\)-value.

**Definition 2.1.** Two 0-1 words of the same length (finite or infinite) will be called equivalent, if they have the same \(\beta\)-value.
2.2. Special case of Theorem 1.4. Consider the case $\beta = G := \frac{1}{2}(1 + \sqrt{5})$. Let $W$ denote the set of all finite 0-1 words whose $\beta$-value is less than 1, and $V$ denote the set of all $\beta$-admissible 0-1 words (obviously, $V \subseteq W$). Note that in this case admissibility simply means that there are no two consecutive unities. As is well known, in this case the normalization of a finite $w \in W$ is a finite word of the same length (see, e.g., [13]).

Since the shift $(X_\beta, \nu_\beta, \tau_\beta)$ is ergodic, for any fixed $v \in V$, $\nu_G$-a.e. sequence $\varepsilon \in X_G$ contains $v$ infinitely many times. Let the equivalence class of $v$ be $[v] = \{w_1, \ldots, w_\ell\}$.

We perform the anti-normalization for $v$ as follows: take a generic sequence $\varepsilon$ in question; first time we hit $v$, we change it to $w_1$, next time we hit it – to $w_2$, etc., until we get to $w_\ell$. Note that this operation has not changed the $\beta$-value of $\varepsilon$. Now we fix some ordering of $V$, say, the lexicographic one: $V = \{v_1, v_2, \ldots\} = \{0, 1, 00, 01, 10, 000, 001, \ldots\}$ and perform this operation consecutively for $v_1, v_2, \ldots$. Since each word from $V$ occurs infinitely many times, we can avoid “overlaps”. The resulting “anti-normalized” sequence $\varepsilon'$ is a $\beta$-expansion of a Lebesgue-generic $x$, and by our construction, it contains all 0-1 words, i.e., is a universal sequence.

The actual reason why the case $\beta = G$ is so easy to deal with, is the fact that $G$ is a finitary Pisot number, i.e., the normalization of each finite word in base $G$ is finite as well. For an arbitrary (even Pisot) $\beta$ this is, generally speaking, not true, and we will need a more delicate argument.

2.3. General case. Note first that it suffices to prove Theorem 1.4 only for $x \in (0, 1)$. This is because for $x \in (1, 1/(\beta - 1))$ one can find $\ell \geq 1$ such that $y = x - \beta^{-1} - \cdots - \beta^{-\ell} \in [0, 1)$, then apply the theorem to $y$ (i.e., a generic $y$ has a universal $\beta$-expansion $(y_1, y_2, \ldots)$). Finally, $(1, \ldots, 1, y_1, y_2, \ldots)$ (with $\ell$ unities) is a universal $\beta$-expansion of $x$.

Since the proof is somewhat technical, we would like to present a sketch first and then fill up the details. Let $V, W$ be as in the previous subsection, $w \in W$; then there exists $v \in V$ whose $\beta$-value is slightly greater than the $\beta$-value of $w$ (just consider $v' = n_\beta(w)$, replace 0 by 1 at a sufficiently large coordinate of $v'$ and drop the rest of it).

A generic sequence $\varepsilon \in X_\beta$ is of the form $(\ldots, v, (tail))$ and we “anti-normalize” it into $(\ldots, w, (tail'))$, where the dots denote one and the same symbols, $tail' = (\varepsilon'_1, \varepsilon'_{n+1}, \ldots)$ is admissible, and $tail' = v + tail - w$ (we identify an admissible sequence with its $\beta$-value). Since we have chosen $v$ slightly greater than $w$, the $\beta$-value of $(\varepsilon'_1, \varepsilon'_{n+1}, \ldots)$
fills some interval \((a, b) \subset (0, 1)\), where \(a = a(w), b = b(w)\). Since \(\text{tail}\) is “random”, so is \(\text{tail'}\) (more precisely, its shift \((\varepsilon'_n, \varepsilon'_{n+1}, \ldots)\)). Hence we can repeat this procedure \emph{ad infinum}; the claim follows from the fact that for a.e. \(x \in (a, b)\), its greedy \(\beta\)-expansion contains every admissible word infinitely many times (this is a trivial consequence of Poincaré’s recurrence theorem for the shift \(\tau_\beta\)).

To turn this sketch into a real proof, we have to clarify the following points:

1. accurate choice of \(v\);
2. “randomness” of \(\text{tail'}\).

(1) Let \(w = x_1 \ldots x_k\) and \(v' = (\varepsilon_1, \ldots, \varepsilon_k, \ldots) = n_\beta(w)\). Put
\[
n := \min \{ j \geq 1 : \varepsilon_{k+j} = 0 \text{ and } \varepsilon_1 \ldots \varepsilon_{k+j-1} \text{ is } \beta\text{-admissible} \}.
\]
The number \(n\) is well defined, because the tail of \(v'\) does not coincide with the tail of \((a_i)_i^\infty\) (see the beginning of the section), whence one can always increase \(v'\) at a sufficiently large coordinate.

Put \(v := \varepsilon_1 \ldots \varepsilon_{k+n-1}1\). Now \(a := \text{val}_\beta(v) - \text{val}_\beta(w) \in (0, 1)\) (it is positive by the monotonicity of the greedy \(\beta\)-expansions – see Section [1]). To determine \(b\), we consider the sequence \(\tilde{v}\) which is defined as the largest possible \(\beta\)-admissible sequence beginning with \(v\). Then \(b := \text{val}_\beta(\tilde{v}) - \text{val}_\beta(w) < 1\).

(2) Put
\[
E_w^{(j)} := \{ x \in (0, 1) : \varepsilon = (\varepsilon_1, \ldots) \text{ is the greedy } \beta\text{-expansion of } x \text{ and } \varepsilon_{j+1} \ldots \varepsilon_{j+|v|} = v \},
\]
(here \(v = v(w)\) as above) and \(E_w := \bigcup_{j \geq 0} E_w^{(j)}\). Obviously, \(\mathcal{L}(E_w) = 1\), where \(\mathcal{L}\) denotes Lebesgue measure. The relation \(v + \text{tail} = w + \text{tail'}\) can be rewritten in the following way: we have for \(x \in E_w^{(j)}:\)
\[
\text{val}_\beta(v) + \beta^{-|v|} T_\beta^{j+|v|} x = \text{val}_\beta(w) + \beta^{-|w|} y,
\]
whence
\[
y = c_1(w) + c_2(w) T_\beta^{j+|v|} x \in (a(w), b(w)),
\]
where \(c_1(w), c_2(w)\) are some constants. Hence in view of \(T_\beta(\mathcal{L})\) being equivalent to \(\mathcal{L}\) ([12]), Lebesgue measure of all possible \(y\)’s in \((a(w), b(w))\) is full. Therefore, a generic \(x\) leads to a generic \(y\), and we may repeat this operation for all \(w\), thus constructing a universal \(\beta\)-expansion of a generic \(x\).

\textbf{Theorem 1.4 is proved.}

\footnote{Note that from the proof it follows that \(y\) being in \((a, b)\) (whereas \(x\) could assume any value in \((0, 1)\)) does not affect the choice of the next interval \((a, b)\).}
3. Combinatorics and branching

3.1. Unique expansions. We need first to recall some facts about unique \( \beta \)-expansions. Namely, \( x \in I_\beta \) will be said to have unique \( \beta \)-expansion if the greedy \( \beta \)-expansion is the only one \( \beta \)-expansion for \( x \). Let \( A_\beta \) denote the set of such \( x \in (0, 1/(\beta - 1)) \)'s (\( x = 0 \) and \( x = 1/(\beta - 1) \) obviously have a unique \( \beta \)-expansion). It is shown in [4] that if \( \beta < G \), then \( A_\beta = \emptyset \). A natural question to ask is about its properties when \( G \leq \beta < 2 \). The following theorem has been recently proved by P. Glendinning and the author:

**Theorem 3.1.** [8] The set \( A_\beta \) has measure zero for any \( \beta \in (1, 2) \).

The cardinality of the set \( A_\beta \) is

(i) \( \aleph_0 \) if \( \beta \in (G, \beta_*) \)

(ii) \( 2^{\aleph_0} \) if \( \beta \in [\beta_*, 2) \).

Moreover, if \( \beta = \beta_* \), then \( A_\beta \) is a Cantor set of zero Hausdorff dimension, and if \( \beta \in (\beta_*, 2) \), then \( 0 < \dim_H(A_\beta) < 1 \).

Here \( \beta_* = 1.787231650... \) is the Komornik-Loreti constant, i.e., the smallest \( \beta \) such that \( x = 1 \) has a unique \( \beta \)-expansion. In [9] it is shown that in fact \( \beta_* \) is the unique solution of the equation

\[
\sum_{n=1}^{\infty} m_n x^{-n+1} = 1,
\]

where \( m = (m_n)^\infty \) is the Thue-Morse sequence [3]:

\[
m = 0110 1001 1001 0110 1001 0110 0110 1001 ...
\]

In [3] we have given a symbolic description of unique \( \beta \)-expansions. Namely, let \( a = (a_1, a_2, ...) \) (see Section 2), and \( \sigma \) denote the shift on \( \Sigma \). Define

\[
U_\beta := \{ \varepsilon \in \Sigma : \mathbf{a} < \sigma^n \varepsilon < \mathbf{a}, \ n \geq 0 \},
\]

where bar denotes the inversion, i.e., \( \overline{0} = 1, \overline{1} = 0 \).

In [3] it is shown that any unique \( \beta \)-expansion which is neither \( 0^\infty \) nor \( 1^\infty \), is of the form \( 0^s \varepsilon \) or \( 1^s \varepsilon \), where \( s \geq 0 \), and \( \varepsilon \in U_\beta \).

3.2. “Less than continuum” of \( \beta \)-expansions. We will show that having “less than the full continuum” of possible \( \beta \)-expansions is almost the same cardinality-wise as having a unique one. Let \( \mathcal{R}_\beta(x) \) denote the set of all \( \beta \)-expansions of \( x \). We are going to construct by induction the branching compactum \( \Gamma_\beta(x) \subset \prod_1^\infty \{0, 1\} \).

Firstly, if \( x \in A_\beta \), then we define \( \Gamma_\beta(x) := \{0^\infty\} \); otherwise, there exists a branching, i.e., there exist \( (\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1}, \ldots) \in \mathcal{R}_\beta(x) \) and \( (\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1}', \varepsilon_{n+2}', \ldots) \in \mathcal{R}_\beta(x) \) for some \( n \geq 0 \), and \( \varepsilon_{n+1} \neq \varepsilon_{n+1}' \).
Thus, we can make a choice for the first symbol in $\Gamma_\beta(x)$: it is 1 if we choose the “lower branch” (i.e., zero at the n’th place) and 0 otherwise. Performing the same operation for $(\varepsilon_{n+1}, \varepsilon_{n+2}, \ldots)$ and $(\varepsilon'_{n+1}, \varepsilon'_{n+2}, \ldots)$ yields the second symbol in $\Gamma_\beta(x)$, etc. If one of the “tails” happens to be a unique $\beta$-expansion, we assume for simplicity that the remaining symbols are all 0’s.

**Example 3.2.** For $\beta = G$ and $x = \beta^{-1}$, as is well known, $\mathfrak{R}_\beta(x) = \{10^\infty, 010110^\infty, \ldots\}$, whence $\Gamma_\beta(x) = \{0^\infty, 10^\infty, 010^\infty, 0010^\infty \ldots\}$. On the other hand, in Counterexample described in Section 1, $\Gamma_\beta(x) = \prod_1^\infty \{0, 1\}$.

By our construction, to every $\beta$-expansion of $x$ one can assign the (unique) sequence from $\Gamma_\beta(x)$, i.e., $\mathfrak{R}_\beta(x)$ and $\Gamma_\beta(x)$ are naturally isomorphic. Put

$$ C_\beta := \left\{ x \in (0, 1/(\beta - 1)) : \Gamma_\beta(x) \neq \prod_1^\infty \{0, 1\} \right\}. $$

Thus, the set of $x$’s, for which $\text{card } \mathfrak{R}_\beta(x) < 2^{\aleph_0}$, is a subset of $C_\beta$. Note that in [14, Theorem 2.18] we have in fact shown that for every $\beta$ the set $C_\beta$ has zero Lebesgue measure. Here we would like to make this result more precise.

The following auxiliary claim is straightforward:

**Lemma 3.3.** $x \in C_\beta$ if and only if there exists its $\beta$-expansion $\varepsilon$ and $n \geq 0$ such that $\sigma^n\varepsilon$ is a unique $\beta$-expansion.

This simple observation helps us to refine Theorem 1.2.

**Lemma 3.4.** There exists a map $\psi_\beta : C_\beta \to \mathcal{A}_\beta$ which is countable-to-one.

**Proof.** By the above, if $x \in C_\beta$, then $x = \sum_1^{n-1} \varepsilon_j \beta^{-j} + \beta^{-n}y$, where $\varepsilon_j \in \{0, 1\}$ and $y \in \mathcal{A}_\beta$. We define the map $\psi_\beta : x \mapsto y$. The choice of a specific $y$ is unimportant; for example, if there multiple $y$’s, choose the smallest $n$ first, and if there is still a choice, choose the lexicographically smallest $(\varepsilon_1, \ldots, \varepsilon_{n-1})$.

Now, if we have also $x' = \sum_1^{n'-1} \varepsilon'_j \beta^{-j} + \beta^{-n'}y'$, then $x - \beta^k x' \in \mathbb{Q}(\beta)$ for $k = n' - n$, whence for a given $x \in \psi_\beta^{-1}\{y\}$ there can be not more than a countable set of $x$’s from the same preimage. \(\square\)

Recall now that in [14, Theorem 3] quoted above, it was in fact shown that for any $\beta \in (1, G)$, $\text{card } \mathfrak{R}_\beta(x) = 2^{\aleph_0}$ for every $x \in (0, 1/(\beta - 3)$This in fact corresponds to the dynamical model described in detail in [14, §2].
This result is in a way best possible, because for $\beta = G$ there is already a countable set of points, each of which has $\aleph_0$ $\beta$-expansions (for instance, $x = 1$), and for $\beta > G$, as we know, there are points which even have a unique $\beta$-expansion.

Nevertheless, some improvement of Theorem 1.2 for $\beta > G$ is possible. Namely, we show that having a non-full branching is very close to having just a single $\beta$-expansion.

Let $w_n := m_2 \ldots m_{2^n+1}$, $n \geq 0$, where $m$ is the Thue-Morse sequence. That is, $w_0 = 1, w_1 = 11, w_2 = 1101, w_3 = 11010011$, etc.

**Proposition 3.5.** (1) For any $\beta \in (G, \beta_*)$ we have $C_\beta \subset Q(\beta)$. More precisely, every $x \in C_\beta$ has an eventually periodic $\beta$-expansion with the period $w_n\overline{w}_n$ for some $n \geq 0$.

(2) For $\beta \in [\beta_*, 2)$,

$$\dim_H C_\beta = \dim_H A_\beta \in [0, 1).$$

**Proof.** (1) By [8, Proposition 13], $U_\beta$ contains only eventually periodic sequences with the period $w_n\overline{w}_n$ for some $n \geq 0$ if $\beta \in (G, \beta_*)$. Now the claim follows directly from Lemma 3.3.

(2) is a consequence of Lemma 3.4. □

As a corollary of [4, Theorem 3] and Proposition 3.5 we obtain

**Theorem 3.6.** The set

$$\{ x \in (0, 1/(\beta - 1)) : \text{card } \mathcal{A}_\beta(x) < 2^{\aleph_0} \}$$

is

- empty if $\beta \in (1, G)$;
- a proper subset of $Q(\beta)$ if $G \leq \beta < \beta_*$;
- a continuum of Hausdorff dimension 0 if $\beta = \beta_*$;
- a continuum of Hausdorff dimension strictly between 0 and 1 if $\beta \in (\beta_*, 2)$.

**Remark 3.7.** Note that if $\mathcal{A}_\beta(x)$ is finite, then for every $\varepsilon \in \mathcal{A}_\beta(x)$ there exists $n = n(\varepsilon) \geq 0$ such that $\sigma^n\varepsilon$ is unique. Hence such a sequence cannot be universal, because a unique $\beta$-expansion cannot contain, for instance, the word $0^s$ for $s$ large enough.

On the other hand, if $\mathcal{A}_\beta(x)$ is countable, there will be both sequences whose $n$’th shift is unique but also inevitably those not having this property. This is the main obstacle for an easy solution of the open problem mentioned in the end of Section 1.

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4Actually, from their proof it even follows that $C_\beta = \emptyset$ for every $\beta < G$. 

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4. NORMAL $\beta$-EXPANSIONS

We know from Theorem 1.4 that for a given $\beta$ a.e. $x$ has at least one universal $\beta$-expansion. Note that by the ergodicity of the shift $\tau_\beta$, each admissible block occurs in the greedy expansion of a generic $x$ with a positive limiting frequency. Thus, the proof given in Section 2 can be easily modified to yield

**Proposition 4.1.** Given $\beta \in (1, 2)$, a.e. $x \in I_\beta$ has a universal $\beta$-expansion with a positive limiting frequency of each 0-1 block.

It is thus natural to ask the following question: is it true that for every $\beta$ a.e. $x \in I_\beta$ has a normal $\beta$-expansion, i.e., the one for which the limiting frequency of each 0-1 block $B$ is exactly $2^{-|B|}$? A partial answer to this question is

**Theorem 4.2.** There exists a set $E \subset (1, 2)$ of full Lebesgue measure such that for each $\beta \in E$, Lebesgue-a.e. $x \in I_\beta$ has a normal $\beta$-expansion.

**Proof.** Let $p$ denote the product measure $\prod_1^{\infty} \{\frac{1}{2}, \frac{1}{2}\}$ on $\Sigma$, and $\mu_\beta = \pi_\beta(p)$, where $\pi_\beta$ is given by (1.1). This measure is called the Bernoulli convolution parameterized by $\beta$ (see, e.g., [11]).

Note first that the claim in question is valid for every $\beta$ and $\mu_\beta$-a.e. $x$ – it suffices to consider a set $\mathcal{N}$ of “normal” sequences in $\Sigma$ (which by the SLLN has $p$-measure 1) and take $\mathcal{N}_\beta := \pi_\beta(\mathcal{N})$. This set will have full $\mu_\beta$-measure, and clearly, every $x \in \mathcal{N}_\beta$ has a normal $\beta$-expansion.

To end the proof of the theorem, it suffices to recall that by the famous theorem due to B. Solomyak [16], for a.e. $\beta$ the Bernoulli convolution $\mu_\beta$ is absolutely continuous with respect to the Lebesgue measure on $I_\beta$, whence for a.e. $\beta$ the set $\mathcal{N}_\beta$ has Lebesgue measure 1 as well.

**Remark 4.3.** We believe $\mathcal{N}_\beta$ has Lebesgue measure 1 for all $\beta$, even if $\beta$ is a Pisot number (it is well known that $\mu_\beta$ in this case is singular [3]). We plan to return to this problem in the future.

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Yet again, we hope there will be no confusion with the notion of normal greedy $\beta$-expansions – see, e.g., [3].
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