Remarks on Quantum Statistics

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Abstract

Some problems related to an algebraic approach to quantum statistics are discussed. Quantum statistics is described as a result of interactions. The Fock space representation is discussed. The problem of existence of well–defined scalar product is considered. An example of physical applications is also given.

Introduction

In the last years a few different approaches to quantum statistics which generalize the usual boson or fermion statistics has been intensively developed by several authors. The so–called $q$–statistics and corresponding $q$–relations have been studied by Greenberg [1, 2], Mohapatra [3], Fivel [4] and many others, see [5, 6, 7] for example. The deformation of commutation relations for bosons and fermions corresponding to quantum groups $SU_q(2)$ has been given by Pusz and Woronowicz [8, 9]. The $q$–relations corresponding to superparticles has been considered by Chaichian, Kulisch and Lukierski [10]. Quantum deformations have been also studied by Vokos [11], Fairle and Zachos [12] and many others.

Note that there is also an approach to particle systems with some non-standard statistics in low dimensional spaces based on the notion of the braid

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group $B_n$. In this approach the configuration space for the system of $n$–identical particles moving on a manifold $\mathcal{M}$ is given by the formula

$$Q_n(\mathcal{M}) = \left(\mathcal{M}^n - D\right)/S_n,$$

where $D$ is the subcomplex of the Cartesian product $\mathcal{M}^n$ on which two or more particles occupy the same position and $S_n$ is the symmetric group. The group $\pi_1(Q_n(\mathcal{M})) \equiv B_n(\mathcal{M})$ is known as the $n$–string braid group on $\mathcal{M}$. Note that the group $\Sigma_n(\mathcal{M})$ is a subgroup of $B_n(\mathcal{M})$ and is an extension of the symmetric group $S_n$ describing the interchange process of two arbitrary indistinguishable particles. It is obvious that the statistics of the given system of particles is determined by the group $\Sigma_n$. The mathematical formalism related to the braid group statistics has been developed intensively by Majid, see [15, 16, 17, 18, 19, 20, 21] for example. It is interesting that all commutation relations for particles equipped with arbitrary statistics can be described as representations of the so-called quantum Weyl or Wick algebra $\mathcal{W}$. Such algebraic formalism has been considered by Jörgensen, Schmith and Werner [31], and further developed by the author in the series of papers [22, 23, 24, 25, 26, 27, 28, 29, 30], and also by Ralowski [31, 32, 33, 34]. Similar approach has been also considered by others authors, see [35, 36, 37] and [38, 39]. An interesting approach to quantum statistics has been also given in [40, 41]. A proposal for the general algebraic formalism for description of particle systems equipped with an arbitrary generalized statistics based on the concept of monoidal categories with duality has been given by the author in [42]. The physical interpretation for this formalism was shortly indicated. A few examples of applications for this formalism are considered in [43, 44].

The generalization of the concept of quantum statistics is motivated by many different applications in quantum field theory and statistical physics. Some problems in condensed matter physics, magnetism or quantum optics lead to the study of particle systems obeying nonstandard statistics. It is interesting that in the last years new and highly organized structures of matter has been discovered. For example in fractional quantum Hall effect a system with well defined internal order has appears [13]. Another interesting structures appear in the so–called $\frac{1}{2}$ electronic magnetotransport anomaly [14, 17], high temperature superconductors or laser excitations of electrons. In these cases certain anomalous behaviour of electron have appear. An another example is given by the so–called Lutinger liquid [18]. The concept
of statistical–spin liquids has been studied by Byczuk and Spalek \[49\]. It is interesting that half of the available single–particle states are removed by the statistical interaction between the particles with opposite spins. The study of highly organized structures leads to the investigation of correlated systems of interacting particles. The essential problem in such study is to transform the system of interacting particles into an effective model convenient for the description of ordered structures. A system with generalized statistics seems to be one of the best candidate for such model. Hence there is an interest in the development of formalism related to the particle systems with unusual statistics and the possible physical applications. In this paper we would like to discuss some problems of possibility of application of systems with generalized statistics for the further description of ordered structures. All our considerations are based on the assumption that the quantum statistics of charged particles is determined by some specific interactions.

**Fundamental Assumptions**

The starting point for our discussion is a system of charged particles interacting with certain quantum field. The proper physical nature of the system is not essential for our considerations. The fundamental assumption is that the problem of interacting particles can be reduced to the study of a system consisting of charged particles and \(N\)-species of quanta of the field. In this way we can restrict our attention to study of such system. It is natural to expect that some new excited states of the system have appeared as a result of certain specific interaction. The existence of new ordered structures depends on the existence of such additional excitations. Hence we can restrict our attention to the study of possibility of appearance for these excitations. For the description of such possible excited states we use the concept of dressed particles. We assume that every charged particle is equipped with ability to absorb quanta of the external field. A system which contains a particle and certain number of quanta as a result of interaction with the external field is said to be dressed particle. A particle without quantum is called undressed or a quasihole. The particle dressed with two quanta of certain species is understood as a system of two new objects called quasiparticles. A quasiparticle is in fact the charged particle dressed with a single quantum. Two quasiparticles are said to be identical if they are dressed with quanta of the
same species. In the opposite case when the particle is equipped with two
different species of quanta then we have different quasiparticles. We describe
excited states as composition of quasiparticles and quasiholes. It is interest-
ing that quasiparticles and quasiholes have also their own statistics. We give
the following assumption for the algebraic description of excitation spectrum
of single dressed particle.

**Assumption 0: The ground state.** There is a state $|0\rangle \equiv 1$ called the
ground one. There is also the conjugate ground state $<0| \equiv 1^*$. This is the
state of the system before intersection.

**Assumption 1: Elementary states.** There is an ordered (finite) set of
single quasiparticle states

$$ S := \{x^i : i = 1, \ldots, N < \infty\}. \quad (1) $$

These states are said to be elementary (simple). They represent elementary
excitations of the system. We assume that the set $S$ of elementary states
forms a basis for a finite linear space $E$ over a field of complex numbers $\mathbb{C}$.

**Assumption 2: Elementary conjugate states.** There is also a corre-
sponding set of single quasihole states

$$ S^* := \{x^{*i} : i = N, N - 1, \ldots, 1\}. \quad (2) $$

These states are said to be conjugated. The set $S^*$ of conjugate states forms
a basis for the complex conjugate space $E^*$. The pairing $(\cdot, \cdot) : E^* \otimes E \rightarrow \mathbb{C}$
is given by

$$ (x^{*i}|x^j) := \delta^{ij}. \quad (3) $$

**Assumption 3: Composite states.** There is a set of projectors

$$ \Pi_n : E^\otimes n \rightarrow E^\otimes n \quad (4) $$
such that we have a $n$–multilinear mapping

$$ \odot_n : E^x n \rightarrow E^\otimes n. \quad (5) $$
defined by the following formula
\[ x^{i_1} \odot \cdots \odot x^{i_n} := \Pi_n(x^{i_1} \otimes \cdots \otimes x^{i_n}). \quad (6) \]

The set of \( n \)-multiquasiparticle states is denoted by \( P^n(S) \). All such states are result of composition (or clustering) of elementary ones. These states are also called composite states of order \( n \). They represent additional excitations charged particle under interaction. In this way for multiquasiparticle states we have the following set of states
\[ P^n(S) := \{ x^{i^{\sigma}} \equiv x^{i_1} \odot \cdots \odot x^{i_n} : \sigma = (i_1, \ldots, i_n) \in I \}, \quad (7) \]

Here \( I \) is a set of sequences of indices such that the above set of states forms a basis for a linear space \( A^n \). We have
\[ A^n = \text{Im}(\Pi_n). \quad (8) \]

Obviously we have where \( A^0 \equiv 1 \mathbb{C}, A^1 \equiv E \) and \( A^n \subset E^\otimes n \).

Assumption 4: Composite conjugated states. We also have a set of projectors
\[ \Pi_n^* : E^* \otimes n \rightarrow E^* \otimes n \quad (9) \]

and the corresponding set of composite conjugated states of length \( n \)
\[ P^n(S^*) := \{ x^{i^{\sigma}} \equiv x^{i_1^*} \odot \cdots \odot x^{i_n^*} : \sigma = (i_1, \ldots, i_n) \in I \}. \quad (10) \]

The set \( P^n(S^*) \) of composite conjugated states of length \( n \) forms a basis for a linear space \( A^*n \).

Assumption 5: Algebra of states. The set of all composite states of arbitrary length is denoted by \( P(S) \). For this set of states we have the following linear space
\[ A := \bigoplus_n A^n. \quad (11) \]

If the formula
\[ m(s \otimes t)s \equiv s \circ t := \Pi_{m+n}(\tilde{s} \otimes \tilde{t}) \quad (12) \]

for \( s = \Pi_m(\tilde{s}), t = \Pi_n(\tilde{t}), \tilde{s} \in E^\otimes m, \tilde{t} \in E^\otimes n \), defines an associative multiplication in \( A \), then we say that we have an algebra of states. This algebra represents excitation spectrum for single dressed particle.
Assumption 6: Algebra of conjugated states. The set of composite conjugated states of arbitrary length is denoted by $P(S^*)$. We have here a linear space

$$\mathcal{A}^* := \bigoplus_n \mathcal{A}^n,$$  \hspace{1cm} (13)

If $m$ is the multiplication in $\mathcal{A}$, then the multiplication in $\mathcal{A}^*$ corresponds to the opposite multiplication in $\mathcal{A}$

$$m^\text{op}(t^* \otimes s^*) = (m(s \otimes t))^*.$$ \hspace{1cm} (14)

Creation and Annihilation Operators

We define creation operators for our model as multiplication in the algebra $\mathcal{A}$

$$a_s^+ t := s \otimes t, \text{ for } s, t \in \mathcal{A},$$ \hspace{1cm} (15)

where the multiplication is given by (12). For the ground state and annihilation operators we assume that

$$\langle 0 | 0 \rangle = 0, \hspace{0.5cm} a_s^* | 0 \rangle = 0 \text{ for } s^* \in \mathcal{A}^*.$$ \hspace{1cm} (16)

The proper definition of action of annihilation operators on the whole algebra $\mathcal{A}$ is a problem. For the pairing $\langle - | - \rangle^n : \mathcal{A}^n \otimes \mathcal{A}^n \rightarrow \mathbb{C}$ we assume in addition that we have the following formulae

$$\langle 0 | 0 \rangle^0 = 0, \hspace{0.5cm} \langle i | j \rangle^1 := (x^i | x^j) = \delta^{ij}, \hspace{0.5cm} \langle s | t \rangle^n := \langle \tilde{s} | P_n \tilde{t} \rangle^0 \text{ for } n \geq 2$$ \hspace{1cm} (17)

where $\tilde{s}, \tilde{t} \in E^\otimes n$, $P_n : E^\otimes n \rightarrow E^\otimes n$ is an additional linear operator and

$$\langle i_1 \cdots i_n | j_1 \cdots j_n \rangle^0 := \langle i_1 | j_1 \rangle \cdots \langle j_n | j_n \rangle.$$ \hspace{1cm} (18)

Observe that we need two sets $\Pi := \{ \Pi_n \}$ and $P := \{ P_n \}$ of operators and the action

$$a : s^* \otimes t \in \mathcal{A}^k \otimes \mathcal{A}^n \rightarrow a_{s^*}t \in \mathcal{A}^{n-k}.$$ \hspace{1cm} (19)

of annihilation operators for the algebraic description of our system. In this way the triple $\{ \Pi, P, a \}$, where $\Pi$ and $P$ are set of linear operators and $a$ is the action of annihilation operators, is the initial data for our model. The problem is to find and classify all triples of initial data which lead to the well-defined models. The general solution for this problem is not known for us. Hence we must restrict our attention for some examples.
**Definition:** If operators $P$ and $\Pi$ and the action $a$ of annihilation operators are given in such a way that there is unique, nondegenerate, positive definite scalar product, creation operators are adjoint to annihilation ones and vice versa, then we say that we have a well–defined system with generalized statistics.

**Example 1:** We assume here that $\Pi_n \equiv P_n \equiv id_{E \otimes n}$. This means that the algebra of states $\mathcal{A}$ is identical with the full tensor algebra $TE$ over the space $E$, and the second algebra $\mathcal{A}^*$ is identical with the tensor algebra $TE^*$. The action (19) of annihilation operators is given by the formula

$$a_{x^*_{i_1} \otimes \cdots \otimes x^*_{i_n}}(x^{j_1} \otimes \cdots \otimes x^{j_n}) := \delta_{i_1}^{j_1} \cdots \delta_{i_k}^{j_k} x^{j_{n-k+1}} \otimes \cdots \otimes x^{j_n}. \tag{20}$$

For the scalar product we have the equation

$$\langle i_n \cdots i_1 | j_1 \cdots j_n \rangle^n := \delta^{i_1 j_1} \cdots \delta^{i_n j_n} \tag{21}$$

It is easy to see that we have the relation and

$$a_{x^*_{i_1}} a_{x_{j}} := \delta_{j i_1}^1 \tag{22}$$

In this way we obtain the most simple example of well–defined system with generalized statistics. The corresponding statistics is the so–called infinite (Bolzman) statistics [1, 2].

**Example 2:** For this example we assume that $\Pi_n \equiv id_{E \otimes n}$. This means that $\mathcal{A} \equiv TE$ and $\mathcal{A}^* \equiv TE^*$. For the scalar product and for the action of annihilation operators we assume that there is a linear and invertible operator $T : E^* \otimes E \rightarrow E \otimes E^*$ defined by its matrix elements

$$T(x^{*i} \otimes x^j) = \sum_{k, s} T_{k^*l^i}^{j} x^k \otimes x^{*l}, \tag{23}$$

such that we have

$$(T_{k^*l})^{*ji} = T_{l^*k}^{j} = T_{l^*k}^{ji}, \text{ i.e. } T^* = T^t, \tag{24}$$

and $(T^t)_{k^*l}^{*ji} = T_{l^*k}^{*ji}$. Note that this operator not need to be linear, one can also consider the case of nonlinear one. We also assume that the operator $T^*$ act to the left, i.e. we have the relation

$$(x^{*j} \otimes x^l)T^* = \sum_{s, k} (x^l \otimes x^{*k}) T_{l^*s}^{*ji} \tag{25}$$
and
\[(T(x^i \otimes x^j))^* \equiv (x^{*j} \otimes x^i)T^*.
\] (26)
The operator \(T\) given by the formula (23) is said to be a twist or a cross operator. The operator \(T\) describes the cross statistics of quasiparticles and quasiholes. The set \(P\) of projectors is defined by induction
\[P_{n+1} := (id \otimes P_n) \circ R_{n+1},\]
(27)where \(P_1 \equiv id\) and the operator \(R_n\) is given by the formula
\[R_n := id + \tilde{T}(1) + \tilde{T}(1)\tilde{T}(2) + \cdots + \tilde{T}(1)\cdots\tilde{T}(n-1),\]
(28)where \(\tilde{T}(i) := id_E \otimes \cdots \otimes \tilde{T} \otimes \cdots \otimes id_E\), \(\tilde{T}\) on the \(i\)-th place, and
\[(\tilde{T})_{ki} = T_{i*j}^{*k}.
\] (29)
If the operator \(\tilde{T}\) is a bounded operator acting on some Hilbert space such that we have the following Yang-Baxter equation on \(E \otimes E \otimes E\)
\[(\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) = (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}),\]
(30)and \(||\tilde{T}|| \leq 1\), then according to Bożejko and Speicher [7] there is a positive definite scalar product
\[\langle s|t\rangle^n_T := \langle s|P_n t\rangle^n_0\]
(31)for \(s, t \in A^n \equiv E^{\otimes n}\). Note that the existence of nontrivial kernel of operator \(P_2 \equiv R_1 \equiv id_{E \otimes E} + \tilde{T}\) is essential for the nondegeneracy of the scalar product [31]. One can see that if this kernel is trivial, then we obtain well-defined system with generalized statistics [33, 34].

Example 3: If the kernel of \(P_2\) is nontrivial, then the scalar product (31) is degenerate. Hence we must remove this degeneracy by factoring the mentioned above scalar product by the kernel. We assume that there is an ideal \(I \subset TE\) generated by a subspace \(I_2 \subset kerP_2 \subset E \otimes E\) such that
\[a_{s^*}I \subset I\]
(32)for every \(s^* \in A^*\), and for the corresponding ideal \(I^* \subset E^* \otimes E^*\) we have
\[a_{s^*}t = 0\]
(33)
for every $t \in TE$ and $s^* \in I^*$. The above ideal $I$ is said to be Wick ideal \[31\]. We have here the following formulae

$$A := TE/I, \quad A^* := TE^*/I^*$$

for our algebras. The projection $\Pi$ is the quotient map

$$\Pi : \tilde{s} \in TE \longrightarrow s \in TE/I \equiv A$$

For the scalar product we have here the following relation

$$\langle s|t \rangle_{B,T} := \langle \tilde{s}|	ilde{t} \rangle_T$$

for $s = P_m(\tilde{s})$ and $t = P_n(\tilde{t})$. One can define here the action of annihilation operators in such a way that we obtain well–defined system with generalized statistics \[34\].

**Example 4:** If a linear and invertible operator $B : E \otimes E \longrightarrow E \otimes E$ defined by its matrix elements

$$B(x^i \otimes x^j) := B_{ij}^{kl}(x^k \otimes x^l)$$

is given such that we have the following conditions

$$B_{(1)}^{(1)} B_{(2)}^{(2)} B_{(1)}^{(1)} = B_{(2)}^{(2)} B_{(1)}^{(1)} B_{(2)}^{(2)},$$

$$B_{(1)}^{(1)} T_{(2)}^{(2)} T_{(1)}^{(1)} = T_{(2)}^{(2)} T_{(1)}^{(1)} B_{(2)}^{(2)},$$

$$(id_{E \otimes E} + \tilde{T})(id_{E \otimes E} - B) = 0,$$

then one can prove that there is well defined action of annihilation operators and scalar product. In this case we need two operators $T$ and $B$ satisfying the above consistency conditions for the model with generalized statistics \[32, 33, 34\].

**Example 5:** If $B = \frac{1}{\mu} \tilde{T}$, where $\mu$ is a parameter, then the third condition \[38\] is equivalent to the well known Hecke condition for $\tilde{T}$ and we obtain the well–known relations for Hecke symmetry and quantum groups \[8, 9, 50\].
Physical applications

Let us consider the system equipped with generalized statistics and described by two operators $T$ and $B$ like in Example 4. We assume here in addition that a linear and Hermitian operator $S : E \otimes E \rightarrow E \otimes E$ such that
\[ S^{(1)} S^{(2)} = S^{(2)} S^{(1)}, \quad \text{and} \quad S^2 = \text{id}_{E \otimes E}. \] (39)
is given. If we have the following relation
\[ \tilde{T} \equiv B \equiv S, \] (40)
then it is easy to see that the conditions (38) are satisfied and we have well-defined system with generalized statistics. Let us assume for simplicity that the operator $S$ is diagonal and is given by the following equation
\[ S(x^i \otimes x^j) = \epsilon^{ij} x^j \otimes x^i, \] (41)
for $i, j = 1, \ldots, N$, where $\epsilon^{ij} \in \mathbb{C}$, and $\epsilon^{ij} \epsilon^{ji} = 1$. In the general case we have
\[ \epsilon_{ij} = (-1)^{\Sigma_{ij}} q^{\Omega_{ij}}, \] (42)
where $\Sigma := (\Sigma_{ij})$ and $\Omega := (\Omega_{ij})$ are integer-valued matrices such that $\Sigma_{ij} = \Sigma_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$, $q \in \mathbb{C} \setminus \{0\}$ is a parameter \[51\]. The algebra $\mathcal{A}$ is here a quadratic algebra generated by relations
\[ x^i \circ x^j = \epsilon^{ij} x^j \circ x^i, \quad \text{and} \quad (x^i)^2 = 0 \quad \text{if} \quad \epsilon^{ii} = -1 \] (43)
We also assume that $\epsilon^{ii} = -1$ for every $i = 1, \ldots, N$. In this case the algebra $\mathcal{A}$ is denoted by $\Lambda_\epsilon(N)$. One can see that this is a $G$-graded $\epsilon$-commutative algebra \[52\]. Now let us study the algebra $\Lambda_\epsilon(2)$, where $\epsilon^{ii} = -1$ for $i = 1, 2$, and $\epsilon^{ij} = 1$ for $i \neq j$, in more details. In this case our algebra is generated by $x^1$ and $x^2$ such that we have
\[ x^1 \circ x^2 = x^2 \circ x^1, \quad (x^1)^2 = (x^2)^2 = 0 \] (44)
Note that the algebra $\Lambda_\epsilon(2)$ is an example of the so-called $Z_2 \oplus Z_2$-graded commutative colour Lie superalgebra \[53\]. Such algebra can be transformed into the usual Grassmann algebra $\Lambda_2$ generated by $\Theta^1$ and $\Theta^2$ such that we have the anticommutation relation
\[ \Theta^1 \Theta^2 = -\Theta^2 \Theta^1, \] (45)
and \((\Theta^1)^2 = (\Theta^2)^2 = 0\). In order to do such transformation we use the Clifford algebra \(C_2\) generated by \(e^1, e^2\) such that we have the relations

\[
e^i e^j + e^j e^i = 2\delta^{ij} \quad \text{for} \quad i, j = 1, 2.
\]

(46)

For generators \(x^1, \) and \(x^2\) of the algebra \(\Lambda_\epsilon(2)\) the transformation is given by

\[
\Theta^1 := x^1 \otimes e^1, \quad \text{and} \quad \Theta^2 := x^2 \otimes e^2.
\]

(47)

It is interesting that the algebra \(\Lambda_\epsilon(2)\) can be represented by one grassmann variable \(\Theta, \Theta^2 = 0\)

\[
x^1 = (\Theta, 1), \quad x^2 = (1, \Theta).
\]

(48)

For the product \(x^1 \odot x^2\) we obtain

\[
x^1 \odot x^2 = (\Theta, \Theta).
\]

(49)

In physical interpretation generators \(\Theta^1\) and \(\Theta^2\) of the algebra \(\Lambda_2\) represents two fermions. They anticommute and according to the Pauli exclusion principle we can not put them into one energy level. Observe that the corresponding generators \(x^1\) and \(x^2\) of the algebra \(\Lambda_\epsilon(2)\) commute, their squares disappear and they describe two different quasiparticles. This means that these quasiparticles behave partially like bosons, we can put them simultaneously into one energy levels. This also means that single fermion can be transform under certain interactions into a system of two different quasiparticles.

References

[1] O. W. Greenberg, Phys. Rev. Lett. 64 705 (1990).
[2] O. W. Greenberg, Phys. Rev. D 43, 4111 (1991).
[3] R. N. Mohapatra, Phys. Lett. B 242, 407 (1990).
[4] D. I. Fivel, Phys. Rev. Lett. 65, 3361, (1990).
[5] D. Zagier, Commun. Math. Phys. 147, 199 (1992).
[6] S. Meljanac and A. Perica, Mod. Phys. Lett. A9 3293 (1994).
[7] M. Bożejko, R. Speicher, *Math. Ann.* **300**, 97, (1994).
[8] W. Pusz, *Rep. Math. Phys.* **27**, 394, (1989)
[9] W. Pusz and S.L. Woronowicz, *Rep. Math. Phys* **27**, 231, (1989)
[10] M. Chaichian, P. Kulisch, J. Lukierski, *Phys. Lett.* **B262**, 43, (1991).
[11] S.P. Vokos, *J. Math. Phys.* **32**, 2979, (1991).
[12] D.B. Fairle and C.K. Zachos, *Phys. Lett.* **B256**, 43, (1991)
[13] Y.S. Wu, *J.Math.Phys.* **52**, 2103, 1984
[14] T.D. Imbo and J. March–Russel, *Phys. Lett. B252*, 84, 1990
[15] S. Majid, *Int. J. Mod. Phys. A* **5**, 1 (1990).
[16] S. Majid, *J. Math. Phys.* **34**, 1176, (1993).
[17] S. Majid, *J. Math. Phys.* **34**, 4843, (1993).
[18] S. Majid, *J. Math. Phys.* **34**, 2045 (1993).
[19] S. Majid, Algebras and Hopf Algebras in Braided Categories, in Advanced in Hopf Algebras, Plenum 1993.
[20] S. Majid, *J. Geom. Phys.* **13**, 169 (1994).
[21] S. Majid, *AMS Cont. Math.* **134**, 219 (1992).
[22] W. Marcinek, *J. Math. Phys.* **33**, 1631 (1992).
[23] W. Marcinek, *Rep. Math. Phys.* **34**, 325 (1994).
[24] W. Marcinek, *Rep. Math. Phys. 33*, 117, (1993).
[25] W. Marcinek, *J. Math. Phys.* **35**, 2633, (1994).
[26] W. Marcinek, *Int. J. Mod. Phys.* **A10**, 1465-1481 (1995).
[27] W. Marcinek, On the deformation of commutation relations, in Proceedings of the XIII Workshop in Geometric Methods in Physics, July 1-7, 1994 Białowieza, Poland, ed. J. Antoine, Plenum Press 1995.
[28] W. Marcinek, On algebraic model of composite fermions and bosons, in Proceedings of the IXth Max Born Symposium, Karpacz, September 25 - September 28, 1996, Poland.

[29] W. Marcinek, On quantum Weyl algebras and generalized quons, in Proceedings of the symposium: Quantum Groups and Quantum Spaces, Warsaw, November 20-29, 1995, Poland, ed. by R. Budzynski, W. Pusz and S. Zakrzewski, Banach Center Publications, Warsaw 1997.

[30] W. Marcinek, *Rep. Math. Phys.* **41**, 155 (1998).

[31] P.E.T. Jorgensen, L.M. Schmith, and R.F. Werner, Positive representation of general commutation relations allowing Wick ordering, *J. Funct. Anal.* **134**, 33 (1995).

[32] W. Marcinek and R. Rałowski, Particle operators from braided geometry, in "Quantum Groups, Formalism and Applications” XXX Karpacz Winter School in Theoretical Physics, 1994, Eds. J. Lukierski et al., 149-154 (1995).

[33] W. Marcinek and Robert Rałowski, On Wick Algebras with Braid Relations, *Preprint IFT UWr 876/9*, (1994) and *J. Math. Phys.* **36**, 2803, (1995).

[34] R. Rałowski, *J. Phys.* **A30**, 2633 (1997).

[35] R. Scipioni, *Phys. Lett.* **B327**, 56 (1994).

[36] Yu Ting and Wu Zhao-Yan, *Science in China* **A37**, 1472 (1994).

[37] S. Meljanac and A. Perica *Mod. Phys. Lett.* **A9**, 3293 (1994).

[38] M. Pillin, *Commun. Math. Phys.* **180**, 23 (1996).

[39] A. K. Mishra and G. Rajasekaran, *J. Math. Phys.* **38**, 466 (1997).

[40] G. Fiore and P. Schup, Statistics and Quantum Group Symmetries, in Proceedings of the symposium: Quantum Groups and Quantum Spaces, Warsaw, November 20-29, 1995, Poland, ed. by R. Budzynski, W. Pusz and S. Zakrzewski, Banach Center Publications, Warsaw 1997.
[41] S. Meljanac and M. Molekovic, *Int. J. Mod. Phys. Lett.* **A11**, 139 (1996).

[42] W. Marcinek, Categories and quantum statistics, in Proceedings of the symposium: Quantum Groups and their Applications in Physics, Poznan October 17-20, 1995, Poland, *Rep. Math. Phys.* **38**, 149-179 (1996).

[43] W. Marcinek, Topology and quantization, in Proceedings of the IVth International School on Theoretical Physics, Symmetry and Structural Properties, Zajaczkowo k. Poznania, August 29 - September 4 1996, Poland.

[44] W. Marcinek, *J. Math. Phys.* **39**, 818–830 (1998).

[45] A. Zee, Quantum Hall fluids in Field Theory, Topology and Condensed Matter Physics, ed. by H. D. Geyer, Lecture Notes in Physics, Springer 1995.

[46] J. K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989), *Phys. Rev. B* **40**, 8079 (1989); **41**, 7653 (1990).

[47] R. R. Du, H. L. Störmer, D. C. Tsui, A. S. Yeh, L. N. Pfeiffer and K. W. West, *Phys. Rev. Lett.* **73**, 3274 (1994).

[48] F. D. M. Haldane, *J. Phys.* **C14**, 2585 (1981).

[49] K. Byczuk and J. Spalek, *Phys. Rev. B* **51**, 7934 (1995).

[50] A. Kempf, *Let. Math. Phys.* **26**, 11, (1992).

[51] Z. Oziewicz, Lie algebras for arbitrary grading group, in Differential Geometry and Its Applications ed. by J. Janyska and D. Krupka, World Scientific, Singapore 1990

[52] M. Scheunert, *J. Math. Phys.* **20**, 712, (1979)

[53] J. Lukierski, V. Rittenberg, *Phys. Rev. D* **18**, 385, (1978).