Stable Recovery from the Magnitude of Symmetrized Fourier Measurements

Philipp Walk  
Lehrstuhl für Theoretische Informationstechnik  
Technische Universität München  
Theresienstrasse 90, 80290 München  
Email: philipp.walk@tum.de

Peter Jung  
Lehrstuhl für Informationstheorie und Theoretische Informationstechnik  
Technische Universität Berlin  
Einsteinufer 25, 10587 Berlin  
Email: peter.jung@mk.tu-berlin.de

Abstract

In this note we show that stable recovery of complex-valued signals \(x \in \mathbb{C}^n\) up to global sign can be achieved from the magnitudes of \(4n-1\) Fourier measurements when a certain symmetrization and zero-padding is performed before measurement (\(4n-3\) is possible in certain cases). For real signals, symmetrization itself is linear and therefore our result is in this case a statement on uniform phase retrieval. Since complex conjugation is involved, such measurement procedure is not complex-linear but recovery is still possible from magnitudes of linear measurements on, for example, \((\text{Re}(x), \text{Im}(x))\).

I. INTRODUCTION

Recovering a signal from intensity (magnitude) measurements is known as the phase retrieval problem. This problem has a long history beginning in the 70’s by GERCHBERG and SAXTON [1] and later by Fienup [2], who gave explicit reconstruction algorithms for the phase from magnitude Fourier measurements. Since the magnitude of a linear measurement can not distinguish between numbers of unit modulus, stability and injectivity for such measurements can only hold up to a global phase resp. sign, i.e. up to a factor \(e^{i\omega}\) resp. \(\pm 1\). One of the challenging tasks in phase retrieval is to determine the necessary and sufficient number of linear measurements for stability or injectivity. For example, CANDES et.al. [3] have shown stable reconstruction of any \(n\)-dimensional complex-valued signal from the magnitude of \(O(n \log n)\) linear Gaussian-random measurements. A more principal result from BALAN et al. in [4] shows that a generic frame exists with injectivity at \(4n-2\) measurements. Moreover, they could give a fast reconstruction algorithm in [5]. In a recent result [6], BANDEIRA et al. conjecture that \(4n-4\) linear measurements are necessary for injectivity. However, a practical construction and implementation of measurements at this limiting number seem to be rather hard, but it serves as a theoretical bound.

More recently, non-linear or interference–based approaches are considered to provide unique phase reconstruction. For example, WANG [7] presented a method where interference with a known signal \(y \in \mathbb{C}^n\) helps to recover a signal \(x \in \mathbb{C}^n\) up to a global sign from only \(3n\) Fourier measurements \(|F(x + \omega y)|^2\) where \(\omega \in \mathbb{C}\) is a root of unity. For real \(k\)-sparse signals, ELDAR and MENDELSON [8] established stable recovery from \(O(k \log(en/k))\) subgaussian random measurements with high probability. A very recent result [9] from EHLER, FORNASIER and SIGL even extends this to the complex case and provides an explicit reconstruction algorithm. LU and VETTERLI also use sparsity for spectral factorization of real valued impulse responses [10]. Moreover, they also give a reconstruction algorithm. A recent result by WANG and XU [11] states injectivity for \(k\)-sparse complex-valued signals from \(4k-2\) generic measurements as long as \(k < n\). Unfortunately, so far (to the authors knowledge) there doesn’t exists a constructive or deterministic frame providing a recovery or even stable recovery.

Here, we will show a concrete measurement procedure allowing stable recovery of any vector \(x \in \mathbb{C}^n\) with \(x_0 \in \mathbb{R}\) up to global sign from magnitudes of \(4n-3\) linear measurements. The measurements can implemented as linear mappings on, for example, \((\text{Re}(x), \text{Im}(x))\) or \((x, \bar{x})\). We want to stress the fact, that our measurements are not complex-linear, since we perform a non-linear symmetrization on the signal to obtain a symmetric auto-convolution, allowing magnitude measurements from \(4n-3\) linear Fourier measurements. However, this will have implications on certain (compressive) signal processing tasks since such type of measurements occur prior to I/Q–down conversion into a suitable complex baseband model. To prove stability for magnitude Fourier measurements, we will use our

\[ \text{Abstract} \]

In this note we show that stable recovery of complex-valued signals \(x \in \mathbb{C}^n\) up to global sign can be achieved from the magnitudes of \(4n-1\) Fourier measurements when a certain symmetrization and zero-padding is performed before measurement (\(4n-3\) is possible in certain cases). For real signals, symmetrization itself is linear and therefore our result is in this case a statement on uniform phase retrieval. Since complex conjugation is involved, such measurement procedure is not complex-linear but recovery is still possible from magnitudes of linear measurements on, for example, \((\text{Re}(x), \text{Im}(x))\).

I. INTRODUCTION

Recovering a signal from intensity (magnitude) measurements is known as the phase retrieval problem. This problem has a long history beginning in the 70’s by GERCHBERG and SAXTON [1] and later by Fienup [2], who gave explicit reconstruction algorithms for the phase from magnitude Fourier measurements. Since the magnitude of a linear measurement can not distinguish between numbers of unit modulus, stability and injectivity for such measurements can only hold up to a global phase resp. sign, i.e. up to a factor \(e^{i\omega}\) resp. \(\pm 1\). One of the challenging tasks in phase retrieval is to determine the necessary and sufficient number of linear measurements for stability or injectivity. For example, CANDES et.al. [3] have shown stable reconstruction of any \(n\)-dimensional complex-valued signal from the magnitude of \(O(n \log n)\) linear Gaussian-random measurements. A more principal result from BALAN et al. in [4] shows that a generic frame exists with injectivity at \(4n-2\) measurements. Moreover, they could give a fast reconstruction algorithm in [5]. In a recent result [6], BANDEIRA et al. conjecture that \(4n-4\) linear measurements are necessary for injectivity. However, a practical construction and implementation of measurements at this limiting number seem to be rather hard, but it serves as a theoretical bound.

More recently, non-linear or interference–based approaches are considered to provide unique phase reconstruction. For example, WANG [7] presented a method where interference with a known signal \(y \in \mathbb{C}^n\) helps to recover a signal \(x \in \mathbb{C}^n\) up to a global sign from only \(3n\) Fourier measurements \(|F(x + \omega y)|^2\) where \(\omega \in \mathbb{C}\) is a root of unity. For real \(k\)-sparse signals, ELDAR and MENDELSON [8] established stable recovery from \(O(k \log(en/k))\) subgaussian random measurements with high probability. A very recent result [9] from EHLER, FORNASIER and SIGL even extends this to the complex case and provides an explicit reconstruction algorithm. LU and VETTERLI also use sparsity for spectral factorization of real valued impulse responses [10]. Moreover, they also give a reconstruction algorithm. A recent result by WANG and XU [11] states injectivity for \(k\)-sparse complex-valued signals from \(4k-2\) generic measurements as long as \(k < n\). Unfortunately, so far (to the authors knowledge) there doesn’t exists a constructive or deterministic frame providing a recovery or even stable recovery.

Here, we will show a concrete measurement procedure allowing stable recovery of any vector \(x \in \mathbb{C}^n\) with \(x_0 \in \mathbb{R}\) up to global sign from magnitudes of \(4n-3\) linear measurements. The measurements can implemented as linear mappings on, for example, \((\text{Re}(x), \text{Im}(x))\) or \((x, \bar{x})\). We want to stress the fact, that our measurements are not complex-linear, since we perform a non-linear symmetrization on the signal to obtain a symmetric auto-convolution, allowing magnitude measurements from \(4n-3\) linear Fourier measurements. However, this will have implications on certain (compressive) signal processing tasks since such type of measurements occur prior to I/Q–down conversion into a suitable complex baseband model. To prove stability for magnitude Fourier measurements, we will use our
result in [12] for the \((s, f)\)–sparse zero-padded circular convolution. In view of sparsity, zero padding can also be seen as a particular structured sparse signal subclass in \(\mathbb{C}^{4n-3}\).

II. CIRCULAR CONVOLUTIONS, CORRELATIONS AND THE RNMP

Let \((F)_{kl} := n^{-2} \exp(i2\pi kl/n)\) be the \(k, l \in \{0, \ldots, n-1\}\) elements of the \(n \times n\) discrete Fourier transform (DFT) matrix. If dimension of a matrix is important it also will occur as a subscript, i.e. here \(F = F_n\). As well–known, \(F\) is unitary and \(\Gamma := F^2\) denotes time–reversal given by its action \(F^{-1}(x_0, \ldots, x_{n-1})^T := (x_0, x_{n-1}, \ldots, x_1)^T\). In particular \(\Gamma\) is an involution, i.e. \(\Gamma^2 = F^4 = 1\). The circular convolution \(\sum_{l=0}^{n-1} x_l y_{l+k} \) (\(\otimes\) and \(\otimes\) mean \(\pm\) modulo \(n\)) of two vectors \(x, y \in \mathbb{C}^n\) is a symmetric bilinear mapping given as:

\[
x \otimes y = \sqrt{n} F^*(Fx \otimes Fy) = y \otimes x
\]

and \(x \otimes x\) is called (circular) auto–convolution. Similarly, the circular correlation \(\sum_{l=0}^{n-1} x_l y_{l+k} \) is defined as \(x \otimes y := x \otimes \Gamma y\) and we have that Fourier transform of the auto–correlation:

\[
F(x \otimes x) = \sqrt{n} Fx \otimes F^\Gamma x = \sqrt{n} Fx \otimes \overline{Fx} = \sqrt{n}|Fx|^2
\]

is given as the squared magnitudes of the Fourier transform of \(x\). Furthermore, \((S^i)^l = \delta_{k \neq i,l}\) denotes the elements of \(l\)th power of the unit shift operator \(S\).

In [12] and [13] we have established a stability statement for zero-padded sparse circular convolutions. Let \(\text{supp}(x) := \{i : x_i \neq 0\}\) be the support of a vector in the canonical basis and \(\Sigma_k^s := \{x \in \mathbb{C}^n : |\text{supp}(x)| \leq k\}\) be the \(k\)–sparse vectors. We have the following result on the restricted norm multiplicativity property (RNMP) for the circular convolution of sparse zero–padded signals (see [12] for the general definition):

**Theorem 1** (RNMP for circular convolutions, [14], [13]). Let \(s, f, n \in \mathbb{N}\) with \(s \leq f \leq n\). Then there exist a constant \(\alpha_{n'} > 0\) with \(n' = n'(s, f, n) := \min\{\bar{n}(s, f, n), n\}\), such that for all \(x \in \Sigma_k^s, y \in \Sigma_f^y\) it holds

\[
\alpha_{n'} \cdot \|x\| \cdot \|y\| \leq \|x \otimes (y, 0)\| \leq \sqrt{s} \cdot \|x\| \cdot \|y\|,
\]

where \((x, 0), (y, 0) \in \mathbb{C}^{2n-1}\) denotes the vectors padded by \(n - 1\) zeros.

Note, that for sufficiently small \(s\) and \(f\) the constant \(\alpha_{n'}\) depends solely on the sparsity and not on the ambient dimension \(n\) [13]. Furthermore, without additional restrictions, zero padding is necessary to obtain a lower bound strictly greater than zero (see for example also [14] for an explicit example here). In fact, Theorem 1 is a statement on regular convolutions. However, it is natural to expect also a bound without zero padding in prime dimension. Moreover, from \(x \otimes y = Sx \otimes Sy\) follows that (3) holds whenever the zeros are contained in a cyclic block of size \(n - 1\).

III. RECOVERY FROM THE MAGNITUDE OF SYMMETRIZED FOURIER MEASUREMENTS

Our contribution is motivated by the framework given in [12] on bilinear maps. Let \(B(x, y)\) be a symmetric bilinear map and denote its diagonal part by \(A(x) = B(x, x)\). Obviously there holds the binomial–type formula:

\[
A(x_1) - A(x_2) = B(x_1, x_1) - B(x_2, x_2) + B(x_1, x_2) - B(x_1, x_2) = B(x_1 - x_2, x_1 + x_2)
\]

establishing that such \(x_1\) and \(x_2\) can be (stable) distinguished modulo global sign on the basis of \(A(x_1)\) and \(A(x_2)\) whenever \(B(x_1 - x_2, x_1 + x_2)\) is well–separated from zero. More precisely, such a condition is given by the RNMP (given in [3] for the special case \(B(x, y) = x \otimes y\) to be considered here). Since \(B(x, y) = x \otimes y\) is symmetric it follows from (4), Theorem 1 and the results in [12] that each (zero padded) \(s\)–sparse \(x\) for sufficiently large \(n\) can be stable recovered modulo global sign from \(O(s \log n)\) compressive i.i.d. subgaussian (and suitable generalizations based on concentration properties) samples of its circular auto–convolution (which itself can have a sparsity up to \(s^2\)). However, more important is the estimation of \(x\) based on measurements on its auto–correlation \(x \otimes x\). In particular, for Fourier measurements this corresponds to the observation of intensity, see [2]. But, circular correlation \(x \otimes y\) is

\footnote{Our first approach on an explicit formula for \(\bar{n}(s, f)\) in [14] has been corrected in [13].}
only symmetric when \( x = \Gamma x \) (if and only if and the same also for \( y \)). In general, a symmetrization \( S : \mathbb{C}^n \to \mathbb{C}^{2n-1} \) is therefore necessary here:

\[
S(x) := \begin{pmatrix} x_0, x_1, \ldots, x_{n-1}, \bar{x}_{n-1}, \ldots, \bar{x}_1 \end{pmatrix}^T
\]

Let us stress the fact, that the symmetrization map is linear only for real vectors \( x \) since complex conjugation is involved. On the other hand, \( S \) can obviously be written as linear map on vectors like \((\text{Re}(x), \text{Im}(x))\) or \((x, \bar{x})\). Now, for \( x_0 = \bar{x}_0 \) the symmetry condition \( S(x) = \Gamma S(x) \) is fullfilled (note that here \( \Gamma = \Gamma_{2n-1} \)):

\[
S(x) = \begin{pmatrix} x \end{pmatrix} = \Gamma \begin{pmatrix} \bar{x} \end{pmatrix} = \Gamma \begin{pmatrix} x \end{pmatrix} = \Gamma S(x).
\]

Let us abbreviate therefore \( C_0^n := \{ x \in \mathbb{C}^n : x_0 \in \mathbb{R} \} \). Thus, for \( x, y \in C_0^n \), circular correlation of symmetrized vectors is symmetric and agrees with the circular convolution. To apply Theorem 1 we define the zero-padded symmetrization (first zero padding, then symmetrization) \( S_z : \mathbb{C}^n \to \mathbb{C}^{4n-3} \) by:

\[
S_z(x) := S\left(\begin{pmatrix} x \\ 0_{n-1} \end{pmatrix}\right).
\]

**Theorem 2.** Let \( n \in \mathbb{N} \), then \( \tilde{n} = 4n - 3 \) absolute-square Fourier measurements of zero padded symmetrized vectors in \( \mathbb{C}^{\tilde{n}} \), given by (7), are stable up to a global sign for \( x \in C_0^n \), i.e. for all \( x_1, x_2 \in C_0^n \) it holds

\[
\left\| \mathbf{F} S_z(x_1) \right\| ^2 - \left\| \mathbf{F} S_z(x_2) \right\| ^2 \geq c \left\| S_z(x_1 - x_2) \right\| \left\| S_z(x_1 + x_2) \right\| \tag{8}
\]

with \( c = c(\tilde{n}) = \alpha_{\tilde{n}} / \sqrt{\tilde{n}} > 0 \) and \( \mathbf{F} = \mathbf{F}_{\tilde{n}} \).

Note that \( 2\|x\|^2 \geq \|S_z(x)\|^2 = \|x\|^2 + \|x^c\|^2 \geq \|x\|^2 \).

Thus, \( S_z(x) = 0 \) if and only if \( x = 0 \) and the stability in distinguishing \( x_1 \) and \( x_2 \) up to a global sign follows from the RHS of (8) and reads explicitly as:

\[
\left\| \mathbf{F} S_z(x_1) \right\| ^2 - \left\| \mathbf{F} S_z(x_2) \right\| ^2 \geq c \left\| x_1 - x_2 \right\| \left\| x_1 + x_2 \right\| .
\]

**Proof.** For symmetrized vectors \( S_z(x) \), auto–convolution agrees with auto–correlation and we get from (2):

\[
\mathbf{F} (S_z(x) \circledast S_z(x)) = \sqrt{n} \left\| \mathbf{F} S_z(x) \right\| ^2 .
\]

Putting things together we get for every \( x \in C_0^n \):

\[
\sqrt{n} \left\| \mathbf{F} S_z(x_1) \right\| ^2 - \left\| \mathbf{F} S_z(x_2) \right\| ^2 = \left\| \mathbf{F} (S_z(x_1, x_1) - S_z(x_2, x_2)) \right\| \tag{11}
\]

\( \mathbf{F} \) is unitary \( \implies \left\| S_z(x_1, x_1) - S_z(x_2, x_2) \right\| \tag{9} \left\| S_z(x_1 - x_2) \circledast S_z(x_1 + x_2) \right\| \)

Theorem 1 \( \implies \geq \alpha_{\tilde{n}} \left\| S_z(x_1 - x_2) \right\| \left\| S_z(x_1 + x_2) \right\| \tag{12}
\]

In the last step we use that Theorem 1 applies whenever the non–zero entries are contained in a cyclic block of length \( 2n-1 \).

In the real case (8) is equivalent to a stable linear embedding in \( \mathbb{R}^{4n-3} \) up to global sign (see here also [8] where Eldar and Mendelson used the \( \ell^1 \)–norm on the left side) and therefore this is an explicit phase retrieval statement for real signals. Recently, stable recovery also in the complex case up to global phase from the same number of subgaussian measurements has been achieved by Ehlert et al. in [9]. Such results hold with exponential high probability whereby our result is deterministic. But, since \( S_z \) is not complex–linear Theorem 2 can not directly be compared with the usual complex phase retrieval results. On the other hand, such an approach can now indeed distinguish the complex phase by the Fourier measurements and symmetrization provides injectivity for magnitude Fourier measurements up to global sign. To get rid of the odd definition \( C_0^n \) one could symmetrize (and zero padding) \( x \in \mathbb{C}^n \) also by:

\[
S_z(x) := (0_n, x_0 \ldots x_{n-1}, \bar{x}_{n-1} \ldots \bar{x}_0, 0_{n-1})^T \in C^{4n-1}
\tag{13}
again satisfying \( S_z(x) = \Gamma_{4n-1} \mathcal{S}_z(x) \) at the price of two further dimensions. Hence, we also have:

**Corollary 1.** Let \( n \in \mathbb{N} \), then \( \tilde{n} = 4n - 1 \) absolute-square Fourier measurements of zero padded and symmetrized vectors given by (13) are stable up to a global sign for \( x \in \mathbb{C}^n \), i.e. for all \( x_1, x_2 \in \mathbb{C}^n \) it holds

\[
\| \mathcal{F}_S(z)(x_1) \|^2 - \| \mathcal{F}_S(z)(x_2) \|^2 \geq 2c \| x_1 - x_2 \| \| x_1 + x_2 \| \quad (14)
\]

with \( c = c(\tilde{n}) = \alpha_{\tilde{n}} / \sqrt{\tilde{n}} > 0 \) and \( \mathcal{F} = \mathcal{F}_{\tilde{n}} \).

The proof of it is along the same steps as in Theorem 2. The direct extension to sparse signals as in [15] seems to be difficult since randomly chosen Fourier samples do not provide a sufficient measure of concentration property without further randomization.

**IV. Conclusion**

In this note we have shown stable recovery (up to global sign) of a signal \( x \) from magnitude measurements on the Fourier transform of its symmetrization \( S_z(x) \). For real signals this procedure is linear and establishes therefore a phase retrieval method. However, also in the complex case this has practical relevance and system design implications when considering linear measurements on \( (\text{Re}(x), \text{Im}(x)) \) or \( (x, \bar{x}) \). Our result is deterministic and uniform, i.e. it guarantees recovery up to global sign for any vector \( x \in \mathbb{C}^n \). Finally, the constant in the stability result depends only on the sparsity of \( x \) indicating a possible further reduction of the number of observations in the Fourier domain also in this case.

**References**

[1] R. W. Gerchberg and W. O. Saxton, “A practical algorithm for the determination of phase from image and diffraction plane pictures,” *Optik*, vol. 35, p. 237–246, 1972.
[2] J. R. Fienup, “Reconstruction of an object from the modulus of its fourier transform using a support constraint,” *JOSA A*, vol. 3, pp. 27–29, 1978.
[3] E. J. Candès, T. Strohmer, and V. Voroninski, “Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming,” *Communications on Pure and Applied Mathematics*, vol. 66, pp. 1241–1274, 2012.
[4] R. Balan, P. Casazza, and D. Edidin, “On signal reconstruction without phase,” *Applied and Computational Harmonic Analysis*, vol. 20, pp. 345–356, 2006.
[5] R. Balan, B. G. Bodmann, P. Casazza, and D. Edidin, “Fast algorithms for signal reconstruction without phase,” in *Proc. SPIE 6701*, 2007.
[6] A. S. Bandeira, J. Cahill, D. G. Mixon, and A. A. Nelson, “Saving phase: Injectivity and stability for phase retrieval,” *EURASIP Journal on Applied Signal Processing*, 2013.
[7] Y. Wang, “Minimal frames for phase retrieval,” in *Workshop of phaseless recovery*, 2013.
[8] Y. Eldar and S. Mendelson, “Phase retrieval: Stability and recovery guarantees,” *arxiv*, pp. 1–39, 2012.
[9] M. Ehler, M. Fornasier, and J. Siegl, “Quasi-linear compressed sensing,” *SIAM: Multiscale Modeling and Simulation*, vol. submitted, 2013.
[10] Y. Lu and M. Vetterli, “Sparse spectral factorization: Unicity and reconstruction algorithms,” 2011.
[11] Y. Wang and Z. Xu, “Phase retrieval for sparse signals,” *arxiv*, 2013.
[12] P. Walk and P. Jung, “Compressed sensing on the image of bilinear maps,” in *ISIT*, 2012, pp. 1291 – 1295.
[13] P. Walk, “Analysis of convolutions with finite support,” Ph.D. dissertation, TU München, 2013.
[14] P. Walk and P. Jung, “On a reverse \( \ell_2 \)-inequality for sparse circular convolutions,” in *ICASSP*, 2013.
[15] ——, “Approximation of Löwdin Orthogonalization to a Spectrally Efficient Orthogonal Overlapping PPM Design for UWB Impulse Radio,” *EURASIP Journal on Applied Signal Processing*, vol. 92, pp. 649–666, 2012.