Optimal control of M/M/1 two-phase queueing system with state-dependent arrival rate, server breakdowns, delayed repair, and N-policy

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Abstract: In this paper, we examine a two-stage queueing system where the arrivals are Poisson with rate depends on the condition of the server to be specific: vacation, pre-service, operational or breakdown state. The service station is liable to breakdowns and deferral in repair because of non-accessibility of the repair facility. The service is in two basic stages, the first being bulk service to every one of the customers holding up on the line and the second stage is individual to each of them. The server works under N-policy. The server needs preliminary time (startup time) to begin batch service after a vacation period. Startup times, uninterrupted service times, the length of each vacation period, delay times and service times follows an exponential distribution. The closed form of expressions for the mean system size at different conditions of the server is determined. Numerical investigations are directed to concentrate the impact of the system parameters on the ideal limit N and the minimum base expected unit cost.

1. Introduction

In numerous genuine queueing systems for e.g. in assembling systems, computer communication networks systems and other communication networks the server is liable to unpredictable breakdowns and can be repaired. In this manner, it is important to perceive how the breakdown influences the level of execution of the queuing system. The functioning of such systems will be influenced by the breakdown of the service station and the postponement in repair because of no accessibility of the repair person or the device required for repair.

Gray et.al [1] carried out a multiple vacation queueing model, where the service station is liable to unpredictable breakdown while in operation. Vasanta Kumar and Chandan [3] presents the optimal operating policy for the two-phase M/μ/M/1 queueing system with server startup and N-policy with and without gating. Vasanta Kumar et al. [4, 5] provided the optimal policy of M/μ/M/1 and M/Ek/1 gated queueing systems with server startup and breakdowns. Vasantakumar et al. [6, 7] exhibited the optimal operating policy for a two-phase M'/M/1 and M'/Ek/1 queueing system under N-policy with server breakdowns and without gating. Vasantakumar and Srinivasarao [8] introduced the optimal operating policy for an N-policy two-phase queueing system with server startup, breakdowns and delayed repair. Hanumantharao et al. [2] imparted M/M/1 Two-Phase Gated Queueing System with Unreliable Server and State-Dependent Arrivals.
In the queuing models mentioned above it is assumed that the arrival rate of the customers into the system is independent of the state of the server. But in many realistic situations the customer’s arrival rate may depend on system state, or server’s status. Along these lines, the present review is gone for examining the M/M/1 two-stage queueing system with state-dependent arrival rate, server breakdowns, delayed repair, and N-policy.

The further discourse of the paper is sorted as follows: In section 2, the mathematical model is displayed. Section 3 manages the enduring steady state results. Explicit expressions for the mean number of customers in the system is determined and exhibited in section 4. Some other intriguing system performance measures are introduced in section 5. Optimal control policy and sensitivity analysis are exhibited in sections 6 and 7. Conclusions are introduced in section 8.

2. Mathematical Model

The following notations and assumptions are used to explore the steady state performance of the model under consideration.

A single server gives two stages of service, the first stage of service being batch service to all the waiting customers in the line and the second stage is individual service of each of them. The unremitting batch service and individual service times are assumed to follow an exponential distribution with means \(1/\beta\) and \(1/\mu\) respectively. On accomplishment of an individual phase of service, the server comes back to the batch service phase to serve the customers who have arrived for service. If customers are waiting, the server gives batch service followed by individual service. If no customer is waiting, the server receives a vacation and continues to take a vacation until \(N\) customers havetogether in the queue. After returning from vacation he begins startup. The startup times are expected to follow an exponential distribution with mean rate \(\theta\). Arrivals are assumed to follow Poisson distribution and they vary according to the state of the system: \(\lambda_1\) for idle or startup, \(\lambda_2\) for first phase or second phase service, and \(\lambda_3\) for breakdown and delay states. All the arrivals during startup and batch service phase are also accepted to join the batch which is in service. While serving in any phase of service, the server may breakdown at any point in time and repair won’t begin promptly because of non-accessibility of the repairman. The breakdown rate is assumed to follow Poisson distribution with mean \(\alpha_1\) during batch service and \(\alpha_2\) during individual service. Delay and repair times follow an exponential distribution with mean \(1/\xi\) and \(1/\eta\) respectively.

3. Steady state results

The different states of the queuing model are as follows

- (0, i, 0) is the state in which there are \(i(0 \leq i \leq N-1)\) units in the batch queue while the server is on vacation. Its probability is denoted by \(\Pi_{0,i,0}\).
- (1, i, 0) is the state in which there are \(i, (i \geq N)\) units in the batch queue while the server is doing startup work. Its probability is denoted by \(\Pi_{1,i,0}\).
- (2, i, 0) is the state in which there are \(i(i \geq 1)\) units in the batch service while the server is working in batch service. Its probability is denoted by \(\Pi_{2,i,0}\).
- (3, i, 0) is the state in which there are \(i(i \geq 1)\) units in the batch service, while the server is noticed to broken down and waiting for repair. Its probability is denoted by \(\Pi_{3,i,0}\).
- (4, i, 0) is the state in which there are \(i(i \geq 1)\) units in the first phase service while the server is under repair. Its probability is denoted by \(\Pi_{4,i,0}\).
- (5, i, j) is the state in which there are \(i(i \geq 0)\) units in the first phase service and \(j(j \geq 1)\) units in the second phase service while the server is working in individual service. Its probability is denoted by \(\Pi_{5,i,j}\).
- (6, i, j) is the state in which there are \(i(i \geq 0)\) units in the first phase service and \(j (j \geq 1)\) units in the individual service while the server is working in second phase service, but found to be broken-down and waiting for repair. Its probability is denoted by \(\Pi_{6,i,j}\).
- (7, i, j) is the state in which there are \(i(i \geq 0)\) units in the first phase service and \(j(j \geq 1)\) units in the second phase service while the server is working in individual service, but the server is under repair. Its probability is denoted by \(\Pi_{7,i,j}\).
The consistent steady state conditions fulfilled by the queuing system probabilities are as per the following

\[ \lambda_1 \Pi_{0,0,0} = \mu \Pi_{5,0,1}. \]  
(1)

\[ \lambda_1 \Pi_{0,i,0} = \lambda_1 \Pi_{0,i-1,0}, \quad i = 1, 2, 3, \ldots N - 1. \]
(2)

\[ (\lambda_1 + \theta) \Pi_{1,0,0} = \lambda_1 \Pi_{0,0,1} - \lambda_1 \Pi_{0,0,0}. \]
(3)

\[ (\lambda_1 + \theta) \Pi_{1,i,0} = \lambda_1 \Pi_{1,i-1,0}, \quad i = N + 1, N + 2, \ldots. \]
(4)

\[ (\lambda_2 + \beta + \alpha_1) \Pi_{2,0,0} = \lambda_2 \Pi_{2,1,0} + \mu \Pi_{5,0,1} + \gamma \Pi_{4,1,0}. \]
(5)

\[ (\lambda_2 + \beta + \alpha_2) \Pi_{2,i,0} = \lambda_2 \Pi_{2,i-1,0} + \mu \Pi_{5,i,1} + \gamma \Pi_{4,i,0} + \theta \Pi_{3,i,0}. \]
(6)

The following partial and joint probability generating functions are considered for the model under study:

\[ F_0(z) = \sum_{i=0}^{N-1} \Pi_{0,i,0} z^i, \quad F(z) = \sum_{i=0}^{\infty} \Pi_{i,0,0} z^i, \quad F_2(z) = \sum_{i=1}^{\infty} \Pi_{2,i,0} z^i, \quad F_3(z) = \sum_{i=1}^{\infty} \Pi_{3,i,0} z^i. \]

\[ F_4(z) = \sum_{i=1}^{\infty} \Pi_{4,i,0} z^i, \quad F_5(z, y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \Pi_{5,i,j} z^i y^j, \quad F_6(z, y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \Pi_{6,i,j} z^i y^j. \]

\[ F_7(z, y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \Pi_{7,i,j} z^i y^j, \quad T_j(z) = \sum_{i=0}^{\infty} \Pi_{7,i,j} z^i, \]

\[ R_j(z) = \sum_{i=0}^{\infty} \Pi_{5,i,j} z^i, \quad |z| \leq 1 \text{ and } |y| \leq 1. \]

From the steady-state equations (1) to (16), the following expressions for the generating functions are obtained

\[ F_0(z) = \frac{1-z^N}{1-z} \Pi_{0,0,0}. \]
(17)

\[ F_1(z) = \frac{\lambda_1 z \Pi_{0,0,0}}{(1-z)} \frac{1}{(1-z)^2}. \]
(18)

\[ [\lambda_2 (1 - z) + \beta + \alpha_2] F_2(z) = \mu R_1(z) + \theta F_1(z) - \lambda_1 \Pi_{0,0,0} + \gamma F_4(z). \]
(19)

\[ [\lambda_3 (1 - z) + \delta] F_3(z) = \alpha_1 F_2(z). \]
(20)

\[ [\lambda_3 (1 - z) + \gamma] F_4(z) = \delta F_3(z). \]
(21)

\[ [\lambda_3 (1 - z) + \gamma] F_5(z, y) = \alpha_2 F_5(z, y). \]
(22)

\[ [\lambda_3 (1 - z) + \gamma] F_7(z, y) = \delta F_5(z, y). \]
(23)

Substituting the value of \(F_6(z, y)\) in equation (24) we obtain

\[ [\lambda_3 (1 - z) + \gamma] F_7(z, y) = \frac{\delta \alpha_2 F_5(z, y)}{[\lambda_3 (1 - z) + \delta]}. \]
(24)

Substituting \(y = z\) in equation (22), we get

\[ [\lambda_3 (1 - z) + \alpha_2 z + \mu (z - 1)] F_5(z, y) = -\mu z R_1(z) + \gamma y F_7(z, y) + \beta F_2(z). \]
(25)

Substituting the values of \(F_2(z)\) and \(F_7(z, y)\) from equations (19) and (24) respectively in this equation, we get

\[ [\lambda_3 (1 - z) + \alpha_2 z + \mu (z - 1)] F_5(z, y) = \frac{\delta \alpha_2 F_5(z, y)}{[\lambda_3 (1 - z) + \delta]}. \]
\[
\frac{(\mu - \lambda_2 z)(\lambda_3 (1-z) + \delta)(\lambda_3 (1-z) + \gamma) + \alpha_2 z(\lambda_3^2 (z-1) - (\delta + \gamma))}{(\lambda_3 (1-z) + \delta)(\lambda_3 (1-z) + \gamma)} F_5(z, z)
\]

\[
= \frac{\lambda_2 z(\lambda_3 (1-z) + \delta)(\lambda_3 (1-z) + \gamma) - \alpha_2 z(\lambda_3^2 (z-1) - (\delta + \gamma))}{(\lambda_3 (1-z) + \delta)(\lambda_3 (1-z) + \gamma)} + \lambda_1 z\Pi_{0,0,0}(\theta(\frac{z^n-1}{z-1}) + \lambda_1),
\]

Simplifying this equation, we get
\[
Q(z) F_5(z, z) = \lambda_2 z(\lambda_3 (1-z) + \delta)(\lambda_3 (1-z) + \gamma) - \alpha_4 z(\lambda_3^2 (z-1) - (\lambda_3 - (\gamma + \delta)) + \lambda_1 z\Pi_{0,0,0}(\theta(\frac{z^n-1}{z-1}) + \lambda_1)(\lambda_3 (1-z) + \delta)(\lambda_3 (1-z) + \gamma)
\]

where,
\[
Q(z) =[-\lambda_2 \lambda_3^2 z^3 + (2\lambda_2 \lambda_3^2 + \lambda_2 \lambda_3 (\delta + \gamma) + \mu \lambda_3^2 + \lambda_3^3 \alpha_2) z^2 + (-\lambda_2 \lambda_3^2 - \lambda_2 \lambda_3 (\delta + \gamma) - \lambda_2 \delta \gamma - 2\mu \lambda_3^2 - \lambda_2 \mu (\delta + \gamma) - \lambda_2 \lambda_3 (\delta + \gamma)) z + (\mu \lambda_3^2 + \lambda_3 \mu (\delta + \gamma) + \mu \delta)].
\]

Substituting \(z=1\) and \(y=1\) in equations (17), (18), (19), (20), (21), (23), (25), and (26) we get
\[
F_0(1) = \Pi_{0,0,0},
\]
\[
F_1(1) = \frac{\lambda_1 \Pi_{0,0,0}}{\theta},
\]
\[
F_2(1) = \frac{\mu R_1(1)}{\theta},
\]
\[
F_3(1) = \frac{\alpha_1 F_2(1)}{\theta},
\]
\[
F_4(1) = \frac{\alpha_2 F_3(1)}{\theta},
\]
\[
F_5(1,1) = \frac{\gamma \delta}{\theta} \left[ \left(\frac{\lambda_2 + \lambda_1 (\alpha_1 + N\theta)}{\theta} \right) \Pi_{0,0,0} + \frac{\alpha_3 \lambda_2}{\gamma \delta} (\gamma + \delta) \right] \left( \frac{1}{Q(1)} \right),
\]
\[
F_0(1,1) = \frac{\alpha_2}{\theta} F_5(1,1),
\]
\[
F_7(1,1) = \frac{\alpha_2}{\gamma} F_5(1,1),
\]

where, \(R_1(1) = \sum_{i=0}^{\infty} \Pi_{5,i,1} = \frac{\lambda_2}{\mu} \).

The probability generating function of the queue size distribution is given by
\[
F(z, z) = F_0(z) + F_1(z) + F_2(z) + F_3(z) + F_4(z) + F_5(z, z) + F_0(z, z) + F_7(z, z)
\]

The Normalizing condition is
\[
F(1,1) = \frac{1}{1 - (1 + (\frac{\alpha_1}{\delta} + \frac{\alpha_1}{\gamma}) \frac{\lambda_2}{\theta} (1 + (\frac{\alpha_2}{\gamma} + \frac{\alpha_2}{\delta}) \frac{\gamma \delta}{\theta} (\lambda_2 + \frac{\alpha_1 \lambda_2 (\gamma + \delta)}{Q(1)})) \left( \frac{1}{\theta} + (\frac{\lambda_1 + N\theta}{\delta}) \frac{\alpha_3 \lambda_2}{\gamma \delta} \right) \left( \frac{Q(1)}{1 - Q(1)} \right)}{\lambda_1 + N\theta, \theta}
\]

Using the condition \(F(1,1) = 1\) in the forms \(\lim_{y \to 1} F(1,y) = 1\) and \(\lim_{z \to 1} F(z, 1) = 1\), we obtain
\[
R_1(1) = \left[ \frac{\alpha_2}{\lambda_2} \left( \frac{\lambda_2}{\theta} + \frac{\lambda_2}{\gamma} \frac{\alpha_1}{\delta} \right) - \frac{\lambda_1}{\mu} \frac{\lambda_1 (\alpha_1 + N\theta)}{\delta} \Pi_{0,0,0} \right] \left[ \frac{\gamma \delta}{\theta} \frac{Q(1)}{1 - Q(1)} \right].
\]

We now find the roots of \(Q(z) = 0\) for positive \(\lambda_3\). Examining to (26), \(Q(z)\) is cubic expression. \(Q(z)\) has three variations of sign and \(Q(-z)\) has no variation of sign. By Descarte’s rule of signs the equation \(Q(z)=0\) has three positive roots.

For the steady-state queue size distribution to exist, all the three zeros of the equation \(Q(z)=0\) should be greater than 1. Since in \(Q(z)\) the coefficient of \(z^3\) is negative the three zeros of \(Q(z) = 0\) will be greater than 1 if and only if \(Q(1) > 0\), \(Q'(1) < 0\) and \(Q''(1) > 0\).

Since \(Q(1) = \mu \gamma \delta - \lambda_2 \delta \gamma - \lambda_3 \alpha_2 (\delta + \gamma)\), we must assume that \(\mu \gamma \delta > \lambda_2 \delta \gamma + \lambda_3 \alpha_2 (\delta + \gamma)\)

or \(\frac{\lambda_2}{\mu} + \frac{\lambda_3 \alpha_2}{\gamma \delta} \frac{1}{\delta} < 1\).

Now (37) implies that \(\mu > \lambda_2 \text{ and } \lambda_3\), hence if (37) holds then
\[
Q'(1) = \lambda_3 (\alpha_3 - \mu) (\delta + \gamma) - \lambda_2 \delta \gamma - \lambda_3 \alpha_2 (\delta + \gamma) < 0,
\]
and
\[
Q''(1) = 2\lambda_2^2 (\mu - \lambda_2) + 2\lambda_2 \lambda_3 (\delta + \gamma) + 2\lambda_3^2 \alpha_2 > 0.
\]
Thus, if we assume that (37) holds, then the roots \( z_1, z_2 \) and \( z_3 \) of \( Q(z) = 0 \) will be greater than 1. From (37) and assume \( \lambda_2, \lambda_3, \) and \( \beta \) such that \( 0 < \Pi_{0,0,0} < 1. \)

Let \( k_1 = \frac{1}{z_1}, k_2 = \frac{1}{z_2} \) and \( k_3 = \frac{1}{z_3}. \)

Then \( Q(z) = \mu(z^3 + \lambda_3(\lambda + \gamma) + \gamma \delta)(1 - k_1 z)(1 - k_2 z)(1 - k_3 z). \)

Now from (26)

\[
\mu(\lambda_3^2 + \lambda_3(\lambda + \gamma) + \gamma \delta)(1 - k_1 z)(1 - k_2 z)(1 - k_3 z) = \lambda_2 z(\lambda_3(1 - z) + \delta)(\lambda_3(1 - z) + \gamma) - \alpha_1 z(\lambda_3^2(z - 1) - \lambda_3(\delta + \gamma))
\]

\[
\lambda_2 z \Pi_{0,0,0} \left(\frac{\theta(z^{-1})}{(\lambda_3(1 - z) + \theta)}\right)(\lambda_3(1 - z) + \delta)(\lambda_3(1 - z) + \gamma).
\]

4. Mean number of units in the system

Using the joint and marginal probability generating functions, the performance measures of the system viz. the mean number of units in the system while the server is on vacation \( (L_v) \), in startup \( (L_s) \), is working in first phase service \( (L_{bb}) \), is waiting for repair while in first phase service \( (L_{db}) \), is working in second phase service \( (L_{i}) \), is under repair when it is in first phase service \( (L_{di}) \), is working in second phase service \( (L_{db}) \), is waiting for repair while it is in second phase service \( (L_{d1}) \), and is under repair while it is in second phase service \( (L_{d2}) \) states are presented in this segment.

\[
L_v = \sum_{i=1}^{N-1} i \Pi_{0,1,0} = F'_2(1) = \frac{\mu}{\beta^2} \left(\lambda_2 + \lambda_3 \left(\frac{\alpha_1}{\gamma} + \frac{\alpha_1}{\delta}\right)\right)R_1(1) + \frac{\mu}{\beta} R'_1(1) + \frac{\theta}{\beta} F'_1(1).
\]

\[
L_s = \sum_{i=1}^{\infty} i \Pi_{0,1,0} = F'_1(1) = \lambda_3(\lambda + \gamma) + \gamma \delta - \lambda_3(\delta + \gamma) \left(\frac{F(1)}{\theta} + \lambda_3(1) + N\theta\right) + \frac{\lambda_3(1) + N\theta}{\theta} \gamma \delta.
\]

\[
L_{bb} = \sum_{i=1}^{N-1} i \Pi_{3,1,0} = F'_2(1) = \lambda_3^2(1) + \frac{\alpha_1}{\delta} \lambda_3 F'_2(1),
\]

\[
L_{db} = \sum_{i=1}^{N-1} i \Pi_{4,1,0} = F'_3(1) = \frac{\lambda_3}{\gamma^2} F'_3(1) + \frac{\lambda_3}{\gamma} F'_2(1),
\]

\[
L_d = \sum_{i=1}^{\infty} i \Pi_{5,1,1} = F'_5(1,1) = \lambda_3(1) + N\theta + \frac{\alpha_1}{\delta} \lambda_3 F'_3(1,1).
\]

5. Some other system characteristics

In this segment, we obtain the expected system length when the server is in various states. Let the average length of vacation time \( (W_v) \), startup time \( (W_s) \), first phase service time \( (W_{bb}) \), delay time during first phase service \( (W_{db}) \), waiting time for repair during first phase service \( (W_{d1}) \), second phase service time \( (W_i) \), delay time during second phase service \( (W_{di}) \), and waiting time for repair during second phase service \( (W_{di}) \) be obtained. Then the expected length of complete cycle is given by

\[
L(N) = L_v + L_s + L_{bb} + L_{db} + L_{d1} + L_{d1} + L_{d2}.
\]

The long-run fractions of time the server is in different states are, respectively

\[
\frac{W_v}{W_c} = \frac{P_v}{P_s} = N\Pi_{0,0,0},
\]

\[
\frac{W_s}{W_c} = \frac{P_s}{P_s} = \frac{\lambda_3}{\theta},
\]

\[
\frac{W_{bb}}{W_c} = \frac{P_{bb}}{P_b} = \frac{\alpha_1 F_2(1)}{\delta},
\]

\[
\frac{W_{db}}{W_c} = \frac{P_{db}}{P_b} = \frac{\alpha_1 F_2(1)}{\delta},
\]

\[
\frac{W_{d1}}{W_c} = \frac{P_{d1}}{P_b} = \frac{\alpha_1 F_2(1)}{\delta},
\]

\[
\frac{W_{d2}}{W_c} = \frac{P_{d2}}{P_b} = \frac{\alpha_1 F_2(1)}{\delta}.
\]
The following system parameter and cost components are used

\[
\begin{align*}
\lambda_1, \lambda_2, \lambda_3, \mu, \theta, \alpha_1, \alpha_2, \text{ and cost components } C_h, C_c, C_m, C_b, C_r.
\end{align*}
\]

The expected length of vacation period, \( W_v = \frac{N}{\lambda_1} \). Substituting this in equation (48),

\[
W_c = \frac{1}{(\lambda_1 \Pi_{0,0,0})}.
\]

6. Optimal Operating Policy

In this section, we determine the ideal value of \( N \) that reduces the long-run average cost per unit time. To find the ideal limit value of \( N \) we consider the following linear cost composition.

Let \( TA(N) \) be the average cost per unit of time. Then

\[
TA(N) = C_h L(N) + C_o \left( \frac{W_d}{W_c} + \frac{W_i}{W_c} \right) + C_m \left( \frac{W_s}{W_c} \right) + C_h \left( \frac{W_{bb} + W_{db} + W_{bi} + W_{di}}{W_c} \right)
\]

\[
+ C_s \left( \frac{1}{W_c} - C_r \frac{W_v}{W_c} \right).
\]

where

\[
\begin{align*}
C_h & : \text{ Customer waiting cost per unit time,} \\
C_o & : \text{ Server busy time cost of the per unit time,} \\
C_m & : \text{ Startup cost per unit time,} \\
C_s & : \text{ Setup cost per cycle,} \\
C_r & : \text{ Server breakdown cost per unit time, } \text{ and} \\
C_t & : \text{ Reward per unit time.}
\end{align*}
\]

From (50) to (55), it is observed that \( \frac{W_b}{W_c}, \frac{W_{bb}}{W_c}, \frac{W_{db}}{W_c}, \frac{W_i}{W_c}, \frac{W_{bi}}{W_c}, \frac{W_{di}}{W_c} \) and \( \frac{W_{di}}{W_c} \) are independent of the decision variable \( N \). Hence for determination of the optimal operating \( N \)-policy, minimizing \( F(N) \) in (56) is equivalent to minimizing

\[
F_1(N) = C_h L(N) + C_o \left( \frac{W_s}{W_c} \right) + C_m \left( \frac{1}{W_c} - C_r \frac{W_v}{W_c} \right).
\]

Differentiating \( F_1(N) \) with respect to \( N \) and assuming the result to 0, we get the optimal threshold \( N^* \) of \( N \). Hence

\[
N^* = \frac{-\lambda_1}{\theta} + \frac{\lambda_1 (\lambda_1 + 1)}{\theta (1 + \lambda_3 L)} + \frac{2 \lambda_1 K}{\theta (1 + \lambda_4 L)},
\]

where \( K = \frac{C_m + C_r + 8 C_s}{C_h} \) and \( L = \frac{\theta \delta}{Q(1)} \left( 1 + \frac{\alpha_2}{\gamma} + \frac{\alpha_3}{\delta} \right) \).

7. Sensitivity Analysis

We investigate the sensitivity of \( N^* \) in the light of changes in the values of the system parameters \( \lambda_1, \lambda_2, \lambda_3, \mu, \theta, \alpha_1, \alpha_2, \) and cost components \( C_h, C_c, C_m, C_b, C_r \).

The following system parameter and cost components are used

\[
\begin{align*}
\lambda_1 = 0.5, \lambda_2 = 3, \lambda_3 = 0.6, \mu = 15, \theta = 10, \alpha_1 = 0.2, \alpha_2 = 0.05, \gamma = 4, \delta = 2, C_h = 10, C_c = 25, C_b = 20, C_r = 0.1, C_m = 20, C_i = 20.
\end{align*}
\]

| Table 1: Effect of \( (\lambda_1, \lambda_2, \lambda_3) \) on \( N^*, L(N^*) \) and optimal average cost \( TA(N^*) \) |
| --- |
| \( \lambda_3 \) | \( N^* \) | \( L(N^*) \) | \( TA(N^*) \) | \( \lambda_2 \) | \( N^* \) | \( L(N^*) \) | \( TA(N^*) \) | \( \lambda_3 \) | \( N^* \) | \( L(N^*) \) | \( TA(N^*) \) |
| 0.5 | 3 | 0.95 | 17.12 | 3 | 0.84 | 17.12 | 0.6 | 3 | 0.85 | 16.28 |
| 1.0 | 4 | 1.52 | 28.00 | 4 | 0.79 | 24.68 | 0.9 | 3 | 0.76 | 15.42 |
One can notice from Table 1 that i) $N^*$ shows increasing trend with increase in $\lambda_1$ and insensitive with increase in $\lambda_2, \lambda_3$, and ii) $L(N^*)$ and $TA(N^*)$ increase with increase in the values of $\lambda_1, \lambda_2$, and decrease with increase in the values of $\lambda_3$.

**Table 2. Effect ($\mu, \beta, \theta$) on $N^*$, $L(N^*)$ and minimum average cost $TA(N^*)$**

| $\mu$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ | $\beta$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ | $\theta$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ |
|-------|-------|----------|------------|--------|-------|----------|------------|--------|-------|----------|------------|
| 18    | 3     | 0.95     | 16.48      | 11     | 3     | 0.96     | 16.83      | 3       | 3     | 0.90     | 16.27      |
| 21    | 3     | 0.96     | 16.05      | 12     | 3     | 0.97     | 16.59      | 4       | 3     | 0.88     | 15.84      |
| 24    | 3     | 0.96     | 15.73      | 13     | 3     | 0.98     | 16.39      | 5       | 3     | 0.87     | 15.58      |
| 27    | 3     | 0.97     | 15.48      | 14     | 3     | 0.99     | 16.23      | 6       | 3     | 0.86     | 15.41      |
| 30    | 3     | 0.97     | 15.29      | 15     | 3     | 1.00     | 16.09      | 7       | 3     | 0.85     | 15.28      |
| 33    | 3     | 0.97     | 15.14      | 16     | 3     | 1.01     | 15.96      | 8       | 3     | 0.85     | 15.19      |

From Table 2 we notice that i) with increase in the values of $\mu$ and $\beta$, $N^*$ is insensitive, whereas $N^*$ increases with increase in $\theta$, ii) $L(N^*)$ increases with increase in $\mu$ and $\beta$, and decrease with increase in $\theta$, and iii) $TA(N^*)$ decreases with increase in the value of $\mu$, $\beta$ and $\theta$.

**Table 3. Effect of ($\alpha_1, \alpha_2$) on $N^*$, $L(N^*)$ and optimal average cost $TA(N^*)$**

| $\alpha_1$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ | $\alpha_2$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ |
|------------|-------|----------|------------|------------|-------|----------|------------|
| 0.7        | 3     | 0.89     | 18.30      | 1.05       | 3     | 0.93     | 19.72      |
| 1.2        | 3     | 0.84     | 19.46      | 2.05       | 3     | 0.91     | 22.28      |
| 1.7        | 3     | 0.78     | 20.62      | 3.05       | 3     | 0.90     | 24.81      |
| 2.2        | 3     | 0.73     | 21.78      | 4.05       | 3     | 0.89     | 27.31      |
| 2.7        | 3     | 0.67     | 22.92      | 5.05       | 3     | 0.88     | 29.78      |
| 3.2        | 3     | 0.62     | 24.07      | 6.05       | 3     | 0.88     | 32.23      |

It is perceived from Table 3 that i) $N^*$ is insensitive with increase in the values of $\alpha_1$ and $\alpha_2$, and, $L(N^*)$ decreases with increase in the values of $\alpha_1$ and $\alpha_2$; $TA(N^*)$ decreases with increase in $\alpha_1$ and $\alpha_2$.

**Table 4. Effect of ($C_h, C_s, C_m$) on $N^*$, $L(N^*)$ and optimal average cost $TA(N^*)$**

| $C_h$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ | $C_s$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ | $C_m$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ |
|-------|-------|----------|------------|-------|-------|----------|------------|-------|-------|----------|------------|
| 30    | 3     | 0.95     | 17.49      | 250   | 5     | 1.56     | 27.42      | 225   | 4     | 1.25     | 24.46      |
It can be observed from Table 4 that i) $N^*$ and $L(N^*)$ are insensitive with an increase in the value of $C_b$, shows an increasing trend with $C_m$ and increases with $C_s$, and ii) $TA(N^*)$ increases with increase in the values of $C_b$, $C_s$, and $C_m$.

Table 5: Effect of $(C_h, C_b, C_r)$ on $N^*$, $L(N^*)$ and optimal average cost $TA(N^*)$

| $C_h$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ | $C_o$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ | $C_r$ | $N^*$ | $L(N^*)$ | $TA(N^*)$ |
|-------|-------|----------|-----------|------|-------|----------|-----------|------|-------|----------|-----------|
| 20    | 2     | 0.65     | 25.47     | 30   | 3     | 0.95     | 20.92     | 24   | 3     | 0.95     | 14.97     |
| 30    | 2     | 0.65     | 31.93     | 40   | 3     | 0.95     | 24.72     | 28   | 3     | 0.95     | 12.82     |
| 40    | 2     | 0.65     | 38.39     | 50   | 3     | 0.95     | 28.52     | 32   | 3     | 0.95     | 10.66     |
| 50    | 1     | 0.37     | 43.57     | 60   | 3     | 0.95     | 32.32     | 36   | 3     | 0.95     | 8.51      |
| 60    | 1     | 0.37     | 47.24     | 70   | 3     | 0.95     | 36.12     | 40   | 3     | 0.95     | 6.35      |
| 70    | 1     | 0.37     | 50.90     | 80   | 3     | 0.95     | 39.92     | 44   | 3     | 0.95     | 4.20      |

From Table 5, we observe that i) $N^*$ and $L(N^*)$ shows decreasing trend with an increase in the values of $C_h$ and insensitive with an increase in $C_o$ and $C_r$, and ii) $TA(N^*)$ increases with increase in $C_h$ and $C_o$, and decreases with $C_r$.

8. Conclusions

$M/M/1$ two-phase queueing system with state-dependent arrival rate, server breakdowns, delayed repair, and N-policy is analyzed. Furthermore, sensitivity analysis is performed for the optimal threshold value of $N$ with various system parameter values and the cost elements.

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