Eleven spherically symmetric constant density solutions with cosmological constant

Christian G. Böhmer

Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstrasse 8-10, A-1040 Wien, Österreich.

Abstract

Einstein’s field equations with cosmological constant are analysed for a static, spherically symmetric perfect fluid having constant density. Five new global solutions are described.

One of these solutions has the Nariai solution joined on as an exterior field. Another solution describes a decreasing pressure model with exterior Schwarzschild-de Sitter spacetime having decreasing group orbits at the boundary. Two further types generalise the Einstein static universe.

The other new solution is unphysical, it is an increasing pressure model with a geometric singularity.

Keywords: spherical symmetry, perfect fluid, cosmological constant, exact solutions
1 Introduction

Cosmological observations \cite{8,15} give strong indications for the presence of a positive cosmological constant, on the other hand the low energy limit of supersymmetry theories requires a negative sign \cite{21}.

Therefore it is interesting to analyse solutions to the field equations with cosmological constant representing for example relativistic stars. An overview of these solutions is given in table I, appendix A.

With vanishing cosmological constant the first static and spherically symmetric perfect fluid solution with constant density was already found by Schwarzschild in 1916 \cite{17}. In spherical symmetry Tolman \cite{20} and Oppenheimer and Volkoff \cite{14} reduced the field equations to the well known TOV equation. For simple equations of state Tolman integrated the TOV equation and discussed solutions. Although he included the cosmological constant in his calculations he stated that it is too small to produce effects. The geometrical properties of constant density were analysed by Stephani \cite{18}.

With non-vanishing cosmological constant $\Lambda \neq 0$ constant density solutions were first analysed by Weyl \cite{22}. The different possible spatial geometries were already pointed out and the possible coordinate singularity was mentioned. More than 80 years later Stuchlik \cite{19} analysed these solutions again. He integrated the TOV-$\Lambda$ equation for possible values of the cosmological constant up to the limit $\Lambda < 4\pi \rho_0$, where $\rho_0$ denotes this constant density. In these cases constant density solutions describe stellar models. In the present paper it is shown that a coordinate singularity occurs if $\Lambda \geq 4\pi \rho_0$, already mentioned by Weyl \cite{22}.

If the cosmological constant equals this upper bound the pressure vanishes at the mentioned coordinate singularity. In this case one has to use the Nariai solution \cite{10,11} to get the metric $C^1$ at the boundary. For larger cosmological constant the pressure will vanish after the coordinate singularity. The volume of group orbits is decreasing and there one has to join the Schwarzschild-de Sitter solution containing the $r = 0$ singularity. Increasing the cosmological constant further leads to generalisations of the Einstein static universe. These solutions have two regular centres with monotonically decreasing or increasing pressure from the first to the second centre. Certainly the Einstein cosmos itself is a solution.

Solutions with $\Lambda \geq 4\pi \rho_0$ have not been analysed in literature so far. Thus the described constant density solutions obeying this bound are considered to be new, except for the Einstein static universe.
Consider the following line element
\[ ds^2 = -\frac{1}{\lambda} e^{\lambda\nu(r)} dt^2 + e^{a(r)} dr^2 + r^2 d\Omega^2, \quad (2.1) \]
where \( G = 1 \) and \( \kappa = 8\pi\lambda^2 \) will be used. \( \lambda = 1 \) corresponds to Einstein’s theory of gravitation in geometrised units, roughly speaking \( \lambda \to 0 \) corresponds to the Newtonian theory [3, 4, 5].

The field equations for a perfect fluid are three independent equations, which imply energy-momentum conservation. One may either use the three independent field equations or one uses two field equations and the energy-momentum conservation equation. For the purpose of this paper it is more convenient to do the second. Hence consider the first two field equations and the conservation equation
\[ \frac{1}{\lambda r^2} e^{\lambda\nu(r)} \frac{d}{dr} \left( r - re^{-a(r)} \right) - \Lambda e^{\lambda\nu(r)} = 8\pi \rho(r) e^{\lambda\nu(r)}, \quad (2.2) \]
\[ \frac{1}{r^2} \left( 1 + r\lambda\nu'(r) - e^{a(r)} \right) + \lambda e^{a(r)} = 8\pi \lambda^2 P(r) e^{a(r)}, \quad (2.3) \]
\[ -\frac{\nu'(r)}{2} (\lambda P(r) + \rho) = P'(r). \quad (2.4) \]

There are four unknown functions. From a physical point of view there are two possibilities. Either a matter distribution \( \rho = \rho(r) \) or an equation of state \( \rho = \rho(P) \) is prescribed. The most physical case is to prescribe an equation of state. Instead, in the third section solutions with constant density are investigated.

Equation (2.2) can easily be integrated. Set the constant of integration equal to zero because of regularity at the centre. Introduce the definitions of mass up to \( r \) and mean density up to \( r \) by
\[ m(r) = 4\pi \int_0^r s^2 \rho(s) ds, \quad w(r) = \frac{m(r)}{r^3}, \quad (2.5) \]
respectively. Then the first field equation (2.2) yields to
\[ e^{-a(r)} = 1 - \lambda 2w(r)r^2 - \frac{\Lambda}{3} r^2. \quad (2.6) \]

Therefore one may write metric (2.1) as
\[ ds^2 = -\frac{1}{\lambda} e^{\lambda\nu(r)} dt^2 + \frac{dr^2}{1 - \lambda 2w(r)r^2 - \frac{\Lambda}{3} r^2} + r^2 d\Omega^2. \quad (2.7) \]

\[ 3 \]
The function $\nu'(r)$ can be eliminated from (2.3) and (2.4) and yields to the TOV-$\Lambda$ equation \[19, 23\]

\[
P'(r) = -r \frac{\left( \lambda 4\pi P(r) + w(r) - \frac{4}{3} \right) \left( \lambda P(r) + \rho(r) \right)}{1 - \lambda 2 w(r)r^2 - \lambda \frac{4}{3} r^2},
\]

(2.8)

which for $\Lambda = 0$ gives the Tolman-Oppenheimer-Volkoff equation, without cosmological term, short TOV equation \[14, 20\].

If an equation of state $\rho = \rho(P)$ is given \[16\], the conservation equation \[24\] can be integrated to give

\[
\nu(r) = -\int_{P_c}^{P(r)} \frac{2dP}{\lambda P + \rho(P)},
\]

(2.9)

where $P_c$ denotes the central pressure. Using the definition of $m(r)$ then \[24\] and \[28\] form an integro-differential system for $\rho(r)$ and $P(r)$. Differentiating \[28\] with respect to $r$ implies

\[
w'(r) = \frac{1}{r} \left( 4\pi \rho(P(r)) - 3w(r) \right).
\]

(2.10)

Therefore given $\rho = \rho(P)$ equations \[2.5\] and \[2.10\] are forming a system of differential equations in $P(r)$ and $w(r)$. In \[16\] existence and uniqueness for a given equation of state for this system was shown. The solution for $w(r)$ together with \[2.6\] gives the function $a(r)$, whereas the solution of $P(r)$ in \[2.9\] defines $\nu(r)$.

### 3 Solutions with constant density

For practical reasons the notation is changed to geometrised units where $c^2 = 1/\lambda = 1$. Assume a positive constant density distribution $\rho = \rho_0 = \text{const.}$ Then $w = \frac{4}{3} \rho_0$ gives \[2.5\] in the form

\[
P'(r) = -r \frac{\left( 4\pi P(r) + \frac{4}{3} \rho_0 - \frac{4}{3} \right) \left( P(r) + \rho_0 \right)}{1 - kr^2},
\]

(3.1)

where $k$ is given by

\[
k = \frac{8\pi}{3} \rho_0 + \frac{\Lambda}{3}.
\]

(3.2)

In the following all solutions to the above differential equation are derived, see \[138, 119, 20\].
The central pressure \( P_c = P(r = 0) \) is always assumed to be positive and finite. Using that the density is constant in metric (2.7) reads

\[
\text{ds}^2 = - \left( \frac{P_c + \rho_0}{P(r) + \rho_0} \right)^2 \text{d}t^2 + \frac{\text{d}r^2}{1 - kr^2} + r^2 \text{d}\Omega^2. \tag{3.3}
\]

For non-vanishing \( k \) a new coordinate \( \alpha \) will be introduced

\[
k < 0, \quad r = \frac{1}{\sqrt{-k}} \sinh \alpha, \tag{3.4}
\]

\[
k > 0, \quad r = \frac{1}{\sqrt{k}} \sin \alpha. \tag{3.5}
\]

The metric is well defined for radial coordinates \( r \in [0, \hat{r}] \) if \( \Lambda > -8\pi \rho_0 \), where \( \hat{r} \) denotes the zero of \( g^{rr} \). If \( \Lambda \leq -8\pi \rho_0 \) the metric is well defined for all \( r \).

Solutions of differential equation (3.1) are uniquely determined by the three parameters \( \rho_0, P_c \) and \( \Lambda \). Therefore one has a 3-parameter family of solutions.

### 3.1 Stellar models with spatially hyperbolic geometry

\( \Lambda < -8\pi \rho_0 \)

If \( \Lambda < -8\pi \rho_0 \) then \( k < 0 \) and the spatial geometry of metric (3.3) is hyperbolic and the differential equation does not have a singularity. The volume of group orbits has no extrema, thus metric (3.3) has no coordinate singularities and is well defined for all \( r \).

Integration of (3.1) yields to

\[
P(\alpha) = \rho_0 \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right) \frac{1}{C \cosh \alpha - 3} - C \cosh \alpha. \tag{3.6}
\]

\( C \) is the constant of integration, evaluated by defining \( P(\alpha = 0) = P_c \) to be the central pressure. It is found that

\[
C = \frac{3P_c + \rho_0 - \frac{\Lambda}{4\pi}}{P_c + \rho_0}. \tag{3.7}
\]

The function \( P(\alpha) \) is well defined and monotonically decreasing for all \( \alpha \).

The pressure converges to \(-\rho_0\) as \( \alpha \to 0 \) or as the radius \( r \) tends to infinity. Thus there always exists an \( \alpha_b \) or \( R \) where the pressure vanishes. Vanishing of the numerator in (3.6) yields to

\[
\cosh \alpha_b = \frac{1}{C} \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right), \tag{3.8}
\]
the corresponding radius $R$ is given by

$$R^2 = \frac{3}{8\pi \rho_0 + \Lambda} \left( \frac{1}{C^2} \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right)^2 - 1 \right). \quad (3.9)$$

The new coordinate $\alpha$ simplifies expressions. Therefore the radial coordinate will only be used to give a physical picture or if it simplifies the calculations.

From equation (3.8) one can deduce the inverse function. Then one has the central pressure as a function of $\alpha_b$, $P_c = P_c(\alpha_b)$. To do this the explicit expression for $C$ from (3.7) is needed. One finds

$$P_c = \rho_0 \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right) \frac{(\cosh \alpha_b - 1)}{(1 - \frac{\Lambda}{4\pi \rho_0} - 3 \cosh \alpha_b)}. \quad (3.10)$$

The central pressure given by (3.10) should be finite. Therefore one obtains an analogue of the Buchdahl inequality. It reads

$$\cosh \alpha_b < \frac{1}{3} \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right). \quad (3.11)$$

Thus there exists an upper bound for $\alpha_b$ for given $\Lambda$ and $\rho_0$. Use that $\sinh(\text{arccosh}(\alpha)) = \sqrt{\alpha^2 - 1}$, then the corresponding radius $R$ is given by

$$R^2 < \frac{1}{3} \left( 4 - \frac{\Lambda}{4\pi \rho_0} \right). \quad (3.12)$$

Since the cosmological constant is negative the right-hand side of (3.12) is well defined. Using the definition of mass $M = (4\pi/3)\rho_0 R^3$ one can rewrite (3.12) in terms of $M$, $R$ and $\Lambda$. This leads to

$$3M < \frac{2}{3} R + R \sqrt{\frac{4}{9} - \frac{\Lambda}{3} R^2}. \quad (3.13)$$

This is the wanted analogue of the Buchdahl inequality.

At $r = R$, where the pressure vanishes, the Schwarzschild-anti-de Sitter solution is joined.

**Joining interior and exterior solution**

At the $P = 0$ surface the Schwarzschild-anti-de Sitter solution is joined by using Gauss coordinates to the $r = \text{const.}$ hypersurfaces and defining the mass by $M = (4\pi/3)\rho_0 R^3$. If necessary one can rescale the time to get the $g_{tt}$-component continuous. With this rescaling and the use of Gauss coordinates one can get metric $C^1$ at the boundary.

Since the density is not continuous at the boundary the Ricci tensor is not, either. Therefore the metric is at most $C^1$. This cannot be improved.
3.2 Stellar models with spatially Euclidean geometry

\[ \Lambda = -8\pi \rho_0 \]

Assume that cosmological constant and constant density are chosen such that \(8\pi \rho_0 = -\Lambda\). Then \(k = 0\) and the denominator of (3.1) becomes one and the differential equation simplifies to

\[ \frac{dP}{dr} = -4\pi r (P + \rho_0)^2, \tag{3.14}\]

and the \(t = \text{const.}\) hypersurfaces of (3.3) are purely Euclidean. As in the former case metric (3.3) is well defined for all \(r\). Integration yields to

\[ P(r) = \frac{1}{2\pi r^2 + \frac{1}{P_c + \rho_0}} - \rho_0, \tag{3.15}\]

where the constant of integration is fixed by \(P(r = 0) = P_c\).

The denominator of the pressure distribution cannot vanish because central pressure and density are assumed to be positive. Therefore (3.15) has no singularities. As the radius tends to infinity the fraction tends to zero and thus the pressure converges to \(-\rho_0\).

This implies that there always exits a radius \(R\) where \(P(r = R) = 0\). Therefore all solutions to (3.14) in (3.3) describe stellar objects. Their radius is given by

\[ R^2 = \frac{1}{2\pi} \left( \frac{1}{\rho_0} - \frac{1}{P_c + \rho_0} \right). \tag{3.16}\]

One finds that the radius \(R\) is bounded by \(1/\sqrt{2\pi \rho_0}\). Inserting the definition of the density yields to

\[ R^2 < \frac{1}{2\pi \rho_0} = \frac{1}{2\pi} \frac{4\pi R^3}{3M}, \tag{3.17}\]

which implies

\[ 3M < 2R. \tag{3.18}\]

Thus (3.18) is the analogous Buchdahl inequality to equation (3.27) and equals (3.13) with \(\Lambda = -8\pi \rho_0 = -6M/R^3\).

At \(r = R\), where the pressure vanishes, the Schwarzschild-anti-de Sitter solution is joined uniquely as described. Since the density is not continuous at the boundary the Ricci tensor is not, either. Thus the metric is again at most \(C^1\).
3.3 Stellar models with spatially spherical geometry

\[ \Lambda > -8\pi \rho_0 \]

If \( \Lambda > -8\pi \rho_0 \) then \( k \) is positive. Equation (3.1) has a singularity at

\[ r = \hat{r} = \frac{1}{\sqrt{k}}. \tag{3.19} \]

In \( \alpha \) the singularity \( \hat{r} \) corresponds to \( \alpha(\hat{r}) = \pi/2 \), the equator of a 3-sphere. The spatial part of metric \( \text{(3.3)} \) now describes a part of the 3-sphere of radius \( 1/\sqrt{k} \). The metric is well defined for radii less than \( \hat{r} \).

Integration of (3.1) gives

\[ P(\alpha) = \rho_0 \left( \frac{\Lambda}{4\pi \rho_0} - 1 \right) + C \cos \alpha \frac{3 - C \cos \alpha}{3 - C \cos \alpha}, \tag{3.20} \]

where the constant of integration \( C \) is given by (3.7). One finds

\[ P(\pi/2) = \rho_0 \left( \frac{\Lambda}{4\pi \rho_0} - 1 \right), \tag{3.21} \]

which is less than zero if \( \Lambda < 4\pi \rho_0 \), which is the restriction in [19] and the output of [12, 13]. Since this is the considered case the singularity of the pressure gradient is not important yet.

Since \( P(\pi/2) < 0 \) there exists an \( \alpha_b \) such that \( P(\alpha_b) = 0 \). \( \alpha_b \) can be derived from equation (3.20) and one obtains the analogue of (3.9)

\[ \cos \alpha_b = \frac{1}{C} \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right). \tag{3.22} \]

It remains to derive the analogue of the Buchdahl inequality for this case. One uses (3.22) to find

\[ P_c = P_c(\alpha_b) \]

\[ P_c = \rho_0 \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right) \frac{1 - \cos \alpha_b}{3 \cos \alpha_b - \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right)}, \tag{3.23} \]

which is similar to (3.19). Finiteness of the central pressure in (3.23) gives

\[ \cos \alpha_b > \frac{1}{3} \left( 1 - \frac{\Lambda}{4\pi \rho_0} \right). \tag{3.24} \]

With \( \sin(\arccos(\alpha)) = \sqrt{1 - \alpha^2} \), this yields to the wanted analogue

\[ R^2 < \frac{\frac{1}{3} \left( 4 - \frac{\Lambda}{4\pi \rho_0} \right)}{4\pi \rho_0}, \tag{3.25} \]
which equals \(3.12\). The greater sign reversed because \((\arccos \alpha)\) is decreasing. Thus it can also be rewritten to give \(3.13\). At \(\alpha = \alpha_b\) or \(r = R\) the Schwarzschild-anti-de Sitter solution is joined and the metric is \(C^1\). Without further assumptions this cannot be improved.

### 3.4 Solutions with vanishing cosmological constant

\(\Lambda = 0\)

Assume a vanishing cosmological constant. Then one can use all equations of the former case with \(\Lambda = 0\). Only one of these relations will be shown, namely the Buchdahl inequality \(\[2\]\). It is derived from \(3.25\)

\[
R^2 < \frac{1}{3\pi \rho_0},
\]

using \(M = (4\pi/3)\rho_0 R^3\) leads to

\[
M < \frac{4}{9} R, \quad 3M < \frac{4}{3} R.
\]  

(3.27)

### 3.5 Stellar models with spatially spherical geometry

\(0 < \Lambda < 4\pi \rho_0\)

Integration of the TOV-\(\Lambda\) equation gives \(3.20\). Because of \(3.21\) the boundary \(P(R) = 0\) exists. As in the former sections one finds \(3.12\) and written in terms of \(M, R\) and \(\Lambda\) again gives

\[
3M < \frac{2}{3} R + R \sqrt{\frac{4}{9} - \frac{\Lambda}{3} R^2}.
\]

(3.28)

Since the cosmological constant is positive the square root term is well defined if

\[
R \leq \sqrt{\frac{4}{3}} \frac{1}{\sqrt{\Lambda}}.
\]

(3.29)

In this section \(0 < \Lambda < 4\pi \rho_0\) is assumed. Using the definition of mass this can be rewritten to give

\[
R^2 \Lambda < 4\pi \rho_0 R^2 = \frac{3M}{R} < \frac{2}{3} + \sqrt{\frac{4}{9} - \frac{\Lambda}{3} R^2},
\]

(3.30)

where the right-hand side of \(3.28\) was used. By taking the \(2/3\) on the other side and squaring the equation this reduces to the simple inequality

\[
R^2 \Lambda < 1, \quad R < \frac{1}{\sqrt{\Lambda}}.
\]

(3.31)
which means that the boundary of the stellar object is located before the cosmological event horizon.

It is remarkable that bounds on mass and radius similar to (3.28) and (3.31) were derived in [12, 13] by considering the virial theorem in the presence of a positive cosmological constant. Furthermore the inequality \( \Lambda < 4\pi \rho \) is derived, which in the context of this paper may be seen as evidence that the exterior spacetime of stellar objects is the Schwarzschild-de Sitter spacetime with increasing group orbits at the boundary.

As before the Schwarzschild-de Sitter solution is joined \( C_1 \). Since the density is not continuous at the boundary this cannot be improved.

One can construct solutions without singularities. The group orbits are increasing up to \( R \). The boundary of the stellar object \( r = R \) can be put in region \( I \) of Penrose-Carter figure \( \mathbb{I} \) where the time-like killing vector is future directed. This leads to figure \( \mathbb{I} \) the spacetime still contains an infinite sequence of singularities \( r = 0 \) and space-like infinities \( r = \infty \). It is possible to put a second object with boundary \( r = R \) in region \( IV \) of Penrose-Carter figure \( \mathbb{I} \) where the time-like killing vector is past directed. This leads to figure \( \mathbb{I} \) and shows that there are no singularities in the spacetime. This spacetime is not globally static because of the remaining dynamical parts of the the Schwarzschild-de Sitter spacetime, regions \( II_C \) and \( IIC \). Penrose-Carter diagrams are in the appendix \( \mathbb{I} \).

A black hole alternative solution without singularities and without horizons was described by [9].

### 3.6 Solutions with exterior Nariai metric

\( \Lambda = 4\pi \rho_0 \)

In this special case integration of (3.31) gives

\[
P(\alpha) = \rho_0 \frac{C \cos \alpha}{3 - C \cos \alpha},
\]

The pressure (3.32) vanishes at \( \alpha = \alpha_b = \pi/2 \), the coordinate singularity in the radial coordinate \( r \), the interior metric reads

\[
ds^2 = -\left(1 - \frac{P_c}{P_c + \rho_0} \cos \alpha \right)^2 \left( \frac{P_c + \rho_0}{\rho_0} \right)^2 \, dt^2 + \frac{1}{\Lambda} \left[d\alpha^2 + d\Omega^2 \right].
\]

One would like to join the Schwarzschild-de Sitter metric, but this does not work, since the group orbits of the Schwarzschild-de Sitter metric are increasing whereas the group orbits of the interior metric have constant volume. But there is one other spherically symmetric vacuum solution to the Einstein field equations with cosmological constant, the Nariai solution.
Its metric is

$$ds^2 = \frac{1}{\Lambda} \left[ -(A \cos \log r + B \sin \log r)^2 dt^2 + \frac{1}{r^2} dr^2 + d\Omega^2 \right],$$  \hspace{1cm} (3.34)

where $A$ and $B$ are arbitrary constants. With $r = e^\alpha$ this becomes

$$ds^2 = \frac{1}{\Lambda} \left[ -(A \cos \alpha + B \sin \alpha)^2 dt^2 + d\alpha^2 + d\Omega^2 \right].$$  \hspace{1cm} (3.35)

Metrics (3.33) and (3.35) can be joined by fixing the constants $A$ and $B$. With

$$A = -\frac{P_c}{\rho_0} \sqrt{\Lambda}, \quad B = \frac{P_c + \rho_0}{\rho_0} \sqrt{\Lambda},$$  \hspace{1cm} (3.36)

the metric is $C^1$ at $\alpha = \pi/2$. As before, since the density is not continuous, the metric is at most $C^1$.

### 3.7 Solutions which have decreasing group orbits at the boundary

$4\pi\rho_0 < \Lambda < \Lambda_0$

Integration of the TOV-\(\Lambda\) equation again yields (3.20). Assume that the pressure vanishes before the second centre $\alpha = \pi$ of the 3-sphere is reached. The condition $P(\alpha = \pi) < 0$ leads to an upper bound for the cosmological constant. This bound is given by

$$\Lambda_0 := 4\pi\rho_0 \left( \frac{4 P_c/\rho_0 + 2}{P_c/\rho_0 + 2} \right).$$  \hspace{1cm} (3.37)

Then $4\pi\rho_0 < \Lambda < \Lambda_0$ implies the following:

The pressure is decreasing near the centre and vanishes for some $\alpha_b$, where $\pi/2 < \alpha_b < \pi$. Equation (3.3) together with the metric (3.3) implies that the volume of the group orbits is decreasing if $\alpha > \pi/2$.

At $\alpha_b$ one uniquely joins the Schwarzschild-de Sitter solution by $M = (4\pi/3)\rho_0 R^3$. With Gauss coordinates relative to the $P(\alpha_b) = 0$ hypersurface the metric will be $C^1$. But there is a crucial difference to the former case with exterior Schwarzschild-de Sitter solution. Because of the decreasing group orbits at the boundary there is still the singularity $r = 0$ in the vacuum spacetime. Penrose-Carter diagram shows this interesting solution.

\[11\]
3.8 Decreasing solutions with two regular centres

\( \Lambda_0 \leq \Lambda < \Lambda_E \)

As before the pressure is given by (3.20). Assume that the pressure is decreasing near the first centre \( \alpha = 0 \). This gives an upper bound of the cosmological constant

\[
\Lambda_E := 4\pi \rho_0 \left( 3 \frac{P_c}{\rho_0} + 1 \right),
\]

(3.38)

where \( \Lambda_E \) is the cosmological constant of the Einstein static universe. These possible values \( \Lambda_0 \leq \Lambda < \Lambda_E \) imply:

The pressure is decreasing near the first centre \( \alpha = 0 \) but remains positive for all \( \alpha \) because \( \Lambda \geq \Lambda_0 \). Therefore there exists a second centre at \( \alpha = \pi \). At the second centre of the 3-sphere the pressure (3.20) becomes

\[
P(\alpha = \pi) = \rho_0 \left( \frac{4\pi \rho_0}{3P_c} - 1 \right) - C.
\]

(3.39)

It only vanishes if \( \Lambda = \Lambda_0 \). The solution is inextendible. The second centre is also regular. This is easily shown with Gauss coordinates.

Solutions of this kind are generalisations of the Einstein static universe. These 3-spheres have a homogeneous density but do not have constant pressure. They have a given central pressure \( P_c \) at the first regular centre which decreases monotonically towards the second regular centre. Generalisations of the Einstein static universe were also found earlier in Ref. [7] by Ibrahim and Nutku.\(^1\)

3.9 The Einstein static universe

\( \Lambda = \Lambda_E \)

Assume a constant pressure function. Then the pressure gradient vanishes and therefore the right-hand side of (3.1) has to vanish. This gives an unique relation between density, central pressure and cosmological constant. One obtains

\[
\Lambda = \Lambda_E = 4\pi \left( 3P_E + \rho_0 \right),
\]

(3.40)

where \( P_E = P_c \) was used to emphasise that the given central pressure corresponds to the Einstein static universe and is homogeneous. Its metric is of no further interest.

For a given density \( \rho_0 \) there exists for every choice of a central pressure \( P_c \) a unique cosmological constant given by (3.40) such that an Einstein static universe is the solution to (3.1).

\(^1\)I would like to thank Aysel Karafistan for bringing this reference to my attention.
3.10 Increasing solutions with two regular centres

$\Lambda_E < \Lambda < \Lambda_S$

Assume that the pressure (3.20) is finite at the second centre. This leads to an upper bound of the cosmological constant defined by

$$\Lambda_S := 4\pi \rho_0 \left(6 \frac{P_c}{\rho_0} + 4\right). \quad (3.41)$$

The possible values of the cosmological constant imply:

The pressure $P(\alpha)$ is increasing near the first regular centre. It increases monotonically up to $\alpha = \pi$, where one has a second regular centre. This situation is similar to the case where $\Lambda_0 \leq \Lambda < \Lambda_E$. These solutions are also describing generalisations of the Einstein static universe.

The generalisations are symmetric with respect to the Einstein static universe. By symmetric one means the following. Instead of writing the pressure as a function of $\alpha$ depending on the given values $\rho_0$, $P_c$ and $\Lambda$ one can eliminate the cosmological constant with the pressure at the second centre $P(\alpha = \pi)$, given by (3.39). Then one easily finds that the pressure is symmetric to $\alpha = \pi/2$ if both central pressures are exchanged and therefore this is the converse situation to the case where $\Lambda_0 \leq \Lambda < \Lambda_E$. If the central pressures are equal the dependence on $\alpha$ vanishes and one is left with the Einstein static universe.

3.11 Solutions with geometric singularity

$\Lambda \geq \Lambda_S$

In this case it is assumed that $\Lambda$ exceeds the upper limit $\Lambda_S$. Then (3.20) implies that the pressure is increasing near the centre and diverges before $\alpha = \pi$ is reached. Thus these solutions have a geometric singularity with unphysical properties, as can be seen by considering the square of the Riemann tensor. Therefore they are of no further interest.

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## A Overview of constant density solutions

| cosmological constant | spatial geometry | short description of the solution |
|-----------------------|------------------|-----------------------------------|
| $\Lambda < -8\pi \rho_0$ | hyperboloid | stellar model with exterior Schwarzschild-anti-de Sitter solution |
| $\Lambda = -8\pi \rho_0$ | Euclidean | stellar model with exterior Schwarzschild-anti-de Sitter solution |
| $-8\pi \rho_0 < \Lambda < 0$ | 3–sphere | stellar model with exterior Schwarzschild-anti-de Sitter solution |
| $\Lambda = 0$ | 3–sphere | stellar model with exterior Schwarzschild solution |
| $0 < \Lambda < 4\pi \rho_0$ | 3–sphere | stellar model with exterior Schwarzschild-de Sitter solution |
| $\Lambda = 4\pi \rho_0$ | 3–sphere | stellar model with exterior Nariai solution |
| $4\pi \rho_0 < \Lambda < \Lambda_0$ | 3–sphere | decreasing pressure model with exterior Schwarzschild-de Sitter solution; the group orbits are decreasing at the boundary |
| $\Lambda_0 \leq \Lambda < \Lambda_E$ | 3–sphere | solution with two centres, pressure decreasing near the first; generalisation of the Einstein static universe |
| $\Lambda = \Lambda_E$ | 3–sphere | Einstein static universe |
| $\Lambda_E < \Lambda < \Lambda_S$ | 3–sphere | solution with two centres, solution increasing near the first; generalisation of the Einstein static universe |
| $\Lambda \geq \Lambda_S$ | 3–sphere | increasing pressure solution with geometric singularity |

Table I: Overview of constant density solutions
B  Penrose-Carter diagrams

Figure 1: Penrose-Carter diagram for Schwarzschild-de Sitter space. The Killing vector $\partial/\partial t$ is time-like and future-directed in regions $I$ and time-like and past-directed in regions $IV$. In the others regions it is space-like. The surfaces $r = r_+$ and $r = r_{++}$ are black-hole and cosmological event horizons, respectively. $\mathcal{J}^+$ and $\mathcal{J}^-$ are the space-like infinities.

Figure 2: Penrose-Carter diagram for Schwarzschild-anti-de Sitter space. The surface $r = r_+$ is the black-hole event horizons. $\mathcal{J}^+$ and $\mathcal{J}^-$ are the time-like infinities.
Figure 3: Penrose-Carter diagram with one stellar object having a radius $R$ which lies between the two horizons. The group orbits are increasing at the boundary. The shaded region is the matter solution with regular centre. There is still an infinite sequence of singularities $r = 0$ and space-like infinities $r = \infty$.

Figure 4: Penrose-Carter diagram with two stellar objects having radii $R$ which lie between the two horizons. Since the group orbits are increasing up to $R$ the vacuum part contains the cosmological event horizon $r_{++}$. This solution with two objects has no singularities. Because of regions $II_C$ and $III_C$ this spacetime is not globally static.
Figure 5: Penrose-Carter diagram with two stellar objects having radii $R$ which lie between the two horizons. The group orbits of the interior solutions are decreasing where the vacuum solution is joined. Thus the $r = 0$ singularity of the vacuum part is present.

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