Existence results for nonexpansive multi-valued operators and nonlinear integral inclusions

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Abstract
In this paper, we establish some new variants of fixed point theorems for a large class of countably nonexpansive multi-valued mappings. Some fixed point theorems for the sum and the product of three multi-valued mappings defined on nonempty, closed convex set of Banach algebras are also presented. These results improve and complement a number of earlier works. As an application, we prove existence results for a broad class of nonlinear functional integral inclusions as well as nonlinear differential inclusions.

Keywords Countably condensing multi-valued · Countably nonexpansive multi-valued · Functional-differential inclusions · Banach algebras · Measure of weak noncompactness · Fixed point theorems

Mathematics Subject Classification 47H08 · 47H09 · 34K09 · 47B48 · 47H10

1 Introduction

Many problems arising in mathematical physics, chemistry, biology, medicine, etc., can be described, in a first formulation, using nonlinear differential inclusions as well as nonlinear integral inclusions, see [1, 5, 20, 26–31, 33–35, 37]. Thus, the pursuit of novel methodologies and techniques for tackling these problems has become an endeavor of utmost significance.

Fixed point theorems for nonexpansive mappings is one of the important generalizations of the well-known Banach fixed point theorem. That is why several authors have focused on the existence of fixed points of nonexpansive mappings in Banach spaces and have obtained many valuable results. We can cite for example Ben Amar et al. [11], Browder [12], Gohde [21], Ishikawa [25], Kirk [36]. These results have been extended by several authors to the case of multi-valued mappings. For examples, see [1, 2, 10, 22, 23]. In [10], Ben Amar
and O’Regan have established some fixed point theorems for $\beta$-nonexpansive multi-valued mappings, i.e.,

$$\beta(T(M)) \leq \beta(M),$$

(1.1)

for all bounded subset $M$ of $\Omega$ such that $T(M)$ is bounded.

Here $\beta$ is the measure of weak noncompactness of De Blasi, see [15].

In this paper, we investigate the existence of fixed point theorems for countably $\beta$-nonexpansive multi-valued mappings, i.e., the condition (1.1) holds only for countable bounded sets. Therefore, we prove new fixed point theorems of Krasnoselskii’s type for multi-valued mappings defined on Banach spaces under weak topology. Moreover, we establish the existence of solutions for the following hybrid fixed point inclusion on Banach algebras under the weak topology setting:

$$x \in Ax \cdot Bx + Cx.$$

(1.2)

The obtained results significantly extend and generalize some works in the literature. Our results are applied to discuss the existence of solutions to an abstract class of nonlinear integral inclusions of the form:

$$x(t) \in T_1(t, x(t)) \cdot \left( q(t) + \int_0^t k(t, s) H(s, x(s)) \, ds \right) + T_2(t, x(t)), \quad t \in J,$$

(1.3)

as well as the following class of nonlinear functional differential inclusions

$$\left\{ \left( \frac{x(t) - T_2(t, x(t))}{T_1(t, x(t))} \right) \right\}' \in k(t) H(t, x(t)), \quad t \in J;$$

(1.4)

where $k : J \rightarrow \mathbb{R}$, $X$ is a Banach algebra, $q : J \rightarrow X$, $T_i : J \times X \rightarrow X$ and $H : J \times X \rightarrow \mathcal{P}(X)$.

Some special cases of the nonlinear inclusion (1.4) have been studied in [16–18].

This work is organized as follows, Sect. 2 is devoted to some definitions and mainly to the basic tools which will be used in the sequel. The Sect. 3 deals with some fixed point results for countably $\mathcal{D}$-set-Lipschtzian and countably $\beta$-nonexpansive multi-valued mappings in Banach spaces under the weak topology setting. Moreover, we prove some fixed point theorems of Krasnoselskii’s type for multi-valued mappings defined on Banach spaces. In Sect. 4, we prove the existence of solutions for the inclusion (1.2).

In the last section, we investigate an existence theory of solutions for the nonlinear integral inclusion (1.3) and the differential inclusion (1.4).

## 2 Basics facts

Let $E$ be a Banach space endowed with the norm $\| \cdot \|$ and with the zero element $\theta$. For any $r > 0$, $B_E(x, r)$ denotes the closed ball of $E$ centered at $x$ with radius $r$, in particular $B_E(r) := B_E(\theta, r)$, and for any subset $S$ of $E$, we write $\text{co}(S)$ and $\overline{\text{co}}(S)$ to denote the convex hull and the closed convex hull of $S$, respectively. The sets $\mathcal{B}(E)$ and $\mathcal{W}(E)$ stand for the family of all nonempty bounded subsets of $E$ and all nonempty weakly compact subsets of $E$, respectively.
Let
\[
\mathcal{P}(E) = \{ S \subset E : S \neq \emptyset \},
\]
\[
\mathcal{P}_{cv}(E) = \{ S \in \mathcal{P}(E) : S \text{ is convex} \},
\]
\[
\mathcal{P}_{bd}(E) = \{ S \in \mathcal{P}(E) : S \text{ is bounded} \},
\]
\[
\mathcal{P}_c(E) = \{ S \in \mathcal{P}(E) : S \text{ is closed} \},
\]
\[
\mathcal{P}_{cl,cv}(E) = \{ S \in \mathcal{P}_{cl}(E) : S \text{ is convex} \}.
\]

Let \( S \) be a nonempty subset of a Banach space \( E \) and let \( F : S \rightarrow \mathcal{P}(E) \) be a multi-valued mapping. For every subset \( M \) of \( E \), we write
\[
F^{-1}(M) = \{ x \in S : F(x) \cap M \neq \emptyset \}
\]
and
\[
F(M) = \bigcup_{x \in M} F(x).
\]

**Definition 2.1** Let \( F : S \rightarrow \mathcal{P}(E) \) be a multi-valued operator. We say that:
(i) \( F \) has a weakly sequentially closed graph if for every sequence \( \{x_n\}_{n=1}^{\infty} \) of elements in \( S \) such that \( x_n \rightharpoonup x \) in \( S \) and for every sequence \( \{y_n\}_{n=1}^{\infty} \) of elements in \( E \) with \( y_n \in F(x_n) \) such that \( y_n \rightharpoonup y \) in \( E \), then \( y \in F(x) \). If \( F \) is a single-valued mapping, then \( F \) is called weakly sequentially continuous if for any sequence \( \{x_n\}_{n=1}^{\infty} \) in \( S \) such that \( x_n \rightharpoonup x \in S \), then \( F(x_n) \rightharpoonup F(x) \).
(ii) \( F \) is weakly compact, if \( F(A) \) is a relatively weakly compact, for all bounded subset \( A \) of \( S \).
(iii) \( F \) is sequentially weakly upper semi-compact in \( S \) (s.w.u.sco. for short) if for any weakly convergent sequence \( \{x_n\}_{n=1}^{\infty} \) of elements in \( S \) and for any arbitrary \( y_n \in F(x_n) \), the sequence \( \{y_n\}_{n=1}^{\infty} \) has a weakly convergent subsequence. \( \square \)

The theory of measures of weak noncompactness is an important tool used in this work. This measure was introduced by De Blasi in [15]. It is defined in the following way:
\[
\beta(W) := \inf \{ \varepsilon > 0 : \text{there exist } K \in \mathcal{W}(E) \text{ and } \varepsilon > 0 \text{ such that } W \subset K + B_E(\varepsilon) \}
\]
for all \( W \in \mathcal{B}(E) \).

This measure share several properties such as the subadditivity, the maximality, the positive homogeneity, the monotonicity, and the nonsingularity (see [15]).

Next, we recall some definitions. A detailed information can be found in [8, 24].

**Definition 2.2** Let \( F : \Omega \rightarrow \mathcal{P}(E) \) be a multi-valued mapping. We say that:
(i) \( F \) is \( \mathcal{D} \)-set-Lipschitzian (with respect to \( \beta \)), if there is a continuous nondecreasing function \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( \phi(0) = 0 \) such that for any bounded subset \( W \) of \( S \) with \( F(W) \in \mathcal{P}_{bd}(E) \), we have
\[
\beta(F(W)) \leq \phi(\beta(W)).
\]
If \( \phi(r) < r \) for \( r > 0 \), then \( F \) is called nonlinear \( \mathcal{D} \)-set-contraction. Moreover, if \( \phi(r) = kr \) with \( k < 1 \), we say that \( F \) is \( k \)-set-contraction.
(ii) \( F \) is \( \beta \)-condensing, if \( F \) is bounded and for any bounded subset \( W \) of \( \Omega \) with \( \beta(W) > 0 \), we have
\[
\beta(F(W)) < \beta(W).
\]
(iii) $F$ is $\beta$-nonexpansive, if $F$ is bounded and for any bounded subset $W$ of $S$, we have
\[ \beta(F(W)) \leq \beta(W). \]

\[ \square \]

**Definition 2.3** Let $F : S \rightarrow \mathcal{P}(E)$ be a multi-valued mapping. We say that:

(i) $F$ is countably $D$-set-Lipschitzian, if there is a continuous nondecreasing function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\Phi(0) = 0$ such that for all countably and bounded subset $W$ of $S$ with $F(W) \in \mathcal{P}_{bd}(E)$, we have
\[ \beta(F(W)) \leq \Phi(\beta(W)). \]

In addition, if $\Phi(r) < r$, $r > 0$, then $F$ is called countably $D$-set-contraction.

(ii) $F$ is countably $k$-set-contraction, $0 \leq k < 1$, if for every countably and bounded subset $W$ of $S$ with $F(W) \in \mathcal{P}_{bd}(E)$, we have
\[ \beta(F(W)) \leq k \beta(W). \]

(iii) $F$ is countably $\beta$-condensing, if $F(S)$ is bounded and for every countably and bounded subset $W$ of $S$ with $\beta(W) > 0$, we have
\[ \beta(F(W)) < \beta(W). \]

(iv) $F$ is countably $\beta$-nonexpansive, if $F$ is bounded and for every countably and bounded subset $W$ of $S$, we have
\[ \beta(F(W)) \leq \beta(W). \]

\[ \square \]

Because it lacks the stability of convergence for the product sequences under the weak topology, A. Ben Amar, S. Chouayekh and A. Jeribi have introduced, in [7], a class of Banach algebras satisfying a certain sequential condition ($P$) :

\[ (P) \quad \text{for any sequences} \{\alpha_n\}_{n=1}^\infty \text{ and } \{\beta_n\}_{n=1}^\infty \text{ of } X \text{ such that } \alpha_n \to \alpha \text{ and } \beta_n \to \beta, \text{ then } \alpha_n \cdot \beta_n \to \alpha \cdot \beta. \]

This class includes the finite dimensional Banach algebra and the space $C(K, X)$, where $K$ is a compact Hausdorff space and $X$ satisfying $(P)$. In order to prove some fixed point theorems in Banach algebras satisfying $(P)$, the authors in [5] have introduced the concept of multi-valued mappings of the form $\left(\frac{I-C}{A}\right)$, where $A$ and $C$ define multi-valued mappings acting on Banach algebras.

**Definition 2.4** [5] Let $E$ be a Banach algebra and let $A, C : E \rightarrow \mathcal{P}(E)$ be multi-valued mappings. We say that the mapping $\frac{I-C}{A}$ is well defined on $x \in E$ and we write
\[ y \in \left(\frac{I-C}{A}\right)(x) \]
if $x \in yA(x) + C(x)$.

\[ \square \]

Following [13], we recall the next notion.
Definition 2.5 Let $F : S \rightarrow \mathcal{P}(E)$ be a multi-valued operator. We say that $F$ has weakly closed graph in $S \times E$, if for every sequence $\{x_n, n \in \mathbb{N}\} \subset S$ such that $x_n \rightarrow x$, $x \in S$ and $y_n \in Fx_n$ such that $y_n \rightarrow y$ then $Fx \cap L(x, y) \neq \emptyset$, where $L(x, y) := \{\alpha y + (1 - \alpha)x, \alpha \in [0, 1]\}$.

We say that $F$ has $w$-weakly closed graph in $S \times E$, if it has weakly closed graph in $S \times E$ with respect to the weak topology.

Using the concept of multi-valued mappings with $w$-weakly closed graphs, A. Ben Amar et al. have established the following useful result.

Theorem 2.1 [8] Let $K$ be a nonempty, closed, and convex subset of a Banach space $E$ and let $F : K \rightarrow \mathcal{P}_{cl,cv}(K)$ be a multi-valued operator such that:

(i) $F$ maps weakly compact sets into weakly relatively compact sets, and

(ii) $F$ is countably $\beta$-condensing with $w$-weakly closed graph.

Then $F$ has a fixed point.

3 Fixed point results for multi-valued mappings in Banach spaces

In this section, we establish some fixed point results for countably $\mathcal{D}$-set-Lipschitzian and countably $\beta$-nonexpansive multi-valued mappings in Banach spaces under the weak topology setting. Let $S$ be a nonempty, closed, and convex subset of a Banach space $E$.

To any multi-valued mapping $T : S \rightarrow \mathcal{P}(S)$ we associate a sequence $\{S_n^T, n \in \mathbb{N}\}$ of $\mathcal{P}_{cl,cv}(S)$ defined by

$$S_0^T = S \text{ and } S_{n+1}^T = \overline{co} \left\{ T \left( S_n^T \right) \right\}, \quad \text{for } n = 0, 1, \ldots$$

It is well known that $\{S_n^T, n \in \mathbb{N}\}$ is a decreasing sequence of nonempty subsets of $S$. In the following result we give a fixed point theorem on some class of multi-valued mappings, as special cases, includes the multi-valued mappings with weakly sequentially closed graphs, which are countably $\beta$-condensing, countably $\mathcal{D}$-set-contraction with respect to the De Blasi measure of weak noncompactness, and hemi-weakly compact, countably $\beta$-nonexpansive.

Theorem 3.1 Let $S$ be a nonempty, closed, and convex subset of $E$ and let $T : S \rightarrow \mathcal{P}(S)$ be a multi-valued mapping with a weakly sequentially closed graph such that:

(i) There exists a bounded subset $\Omega$ belongs to $\{S_n^T, n \in \mathbb{N}\}$,

(ii) $T$ is hemi-weakly compact and has closed, convex values on $\Omega$,

(iii) $T$ is countably $\mathcal{D}$-set-Lipschitzian on $\Omega$ with $\mathcal{D}$-function $\varphi$.

Then $T$ has at least one fixed point in $\Omega$ as soon as $\varphi(r) \leq r$ for $0 < r \leq \|\Omega\|$. □

Proof Let $n_0 \in \mathbb{N}$ such that $\Omega = S_{n_0}^T$. Let $\gamma \in (0, 1)$ and $y$ be fixed in $T^{n_0}(S)$. Let us define the multi-valued mapping $T_\gamma : S \rightarrow \mathcal{P}(S)$ by the formula

$$T_\gamma(x) = \gamma T(x) + (1 - \gamma)y.$$

Obviously $\{T^n(S), n \in \mathbb{N}\}$ is a decreasing sequence of nonempty subsets of $S$ since $S \neq \emptyset$ and $T(S) \subset S$. In view of the convexity of $S$, this leads to the conclusion that $T_\gamma(S) \subset S$. Consequently, $\{S_n^{T_\gamma}, n \in \mathbb{N}\}$ is, also, a decreasing sequence of nonempty subsets of $S$. In particular, we have

$$T_\gamma \left( S_{n_0}^{T_\gamma} \right) \subset \overline{co} \left\{ T_\gamma \left( S_{n_0}^{T_\gamma} \right) \right\} \subset S_{n_0}^{T_\gamma}.$$
Thus, $T_\gamma$ maps $S^T_{n_0}$ into $\mathcal{P}\left(S^T_{n_0}\right)$. Now, let us prove that $T_\gamma$ has convex values on $S^T_{n_0}$. To do this, we claim first that

$$S^T_{k} \subset S^T_{k} \text{ for all } k \in \{0, 1, \ldots, n_0\}. \quad (3.2)$$

We proceeding by induction argument. Evidently the inclusion (3.2) hold for $K = 0$. Assume that $S^T_{j} \subset S^T_{j}$ for some $J \in \{1, \ldots, n_0 - 1\}$. Then,

$$S^T_{j+1} = \overline{c_{\sigma}} \left\{ T_\gamma \left( S^T_{j} \right) \right\} = \overline{c_{\sigma}} \left\{ \gamma T \left( S^T_{j} \right) + (1 - \gamma) y \right\} \subset \overline{c_{\sigma}} \left\{ \gamma T \left( S^T_{j} \right) + (1 - \gamma) y \right\} \subset \overline{c_{\sigma}} \left\{ \gamma S^T_{j+1} + (1 - \gamma) y \right\}.$$

Since $y \in T^{j+1}(S) \subset S^T_{j+1}$ and the set $S^T_{j+1}$ is convex and closed, we reach to result that

$$\overline{c_{\sigma}} \left\{ \gamma S^T_{j+1} + (1 - \gamma) y \right\} \subset \overline{c_{\sigma}} \left\{ \gamma S^T_{j+1} + (1 - \gamma) S^T_{j+1} \right\} \subset S^T_{j+1}.$$

This proves the claim, and achieves that $S^T_{n_0}$ is bounded. Now, let $x$ be an arbitrary element of $S^T_{n_0}$ and let $t \in (0, 1)$ and $y_1, y_2 \in T_\gamma(x)$. Then, there exist $u_1, u_2 \in T(x)$ such that

$$y_1 = \gamma u_1 + (1 - \gamma) y \text{ and } y_2 = \gamma u_2 + (1 - \gamma) y.$$

Then, we have

$$t y_1 + (1 - t) y_2 = \gamma (t u_1 + (1 - t) u_2) + (1 - \gamma) y.$$

Since $T$ has convex values, we deduce that $t y_1 + (1 - t) y_2 \in T_\gamma(x)$. This prove the claim. Now let us show that $T_\gamma$ has closed values on $S^T_{n_0}$. Let $x \in S^T_{n_0}$ and $(z_n)_{n=0}^\infty \subset T_\gamma(x)$ be a convergent sequence to some $z \in S^T_{n_0}$. Then there exist a sequence $(w_n)_{n=0}^\infty \subset T(x)$ such that

$$z_n = \gamma w_n + (1 - \gamma) y \text{ for all } n \in \mathbb{N}.$$

This implies that $w_n \to \frac{1}{\gamma} (z - (1 - \gamma) y)$. Since $T$ has closed values, we get $\frac{1}{\gamma} (z - (1 - \gamma) y) \in T x$, that is $z \in T_\gamma(x)$. This achieves the proof of our claim. Now we show that $T_\gamma$ is countably $\beta$-condensing on $S^T_{n_0}$. Take an arbitrary countably subset $M$ of $S^T_{n_0}$. By using the boundedness of $T_\gamma(M)$ together with the properties of the De Blasi measure of weak noncompactness, we get

$$\beta(T_\gamma(M)) \leq \beta(\gamma T(M) + (1 - \gamma) y) \leq \gamma \varphi(\beta(M)). \quad (3.3)$$

If $\beta(M) > 0$ we get $\beta(T_\gamma(M)) < \beta(M)$, which means that $T_\gamma$ is countably $\beta$-condensing since $T_\gamma$ is bounded on $\Omega$.

Our next task is to show that $T_\gamma$ maps weakly compact sets into relatively weakly compact sets. To do so, take a weakly compact set $M$ of $S^T_{n_0}$. Let $(y_n)_{n=1}^\infty$ be a sequence of $T_\gamma(M)$. Then there exists a sequence $(x_n)_{n=1}^\infty \subset M$ such that

$$y_n \in T_\gamma(x_n), \quad n = 0, 1, \ldots.$$
The use of (3.3) allows us to obtain that
\[
\beta \left( \{y_n\}_{n=1}^{\infty} \right) \leq \beta \left( T_{\gamma} \left( \{x_n\}_{n=1}^{\infty} \right) \right) \leq \gamma \psi \left( \beta \left( \{x_n\}_{n=1}^{\infty} \right) \right) = 0.
\]
Hence and in view of the Eberlein-Šmulian’s theorem, we infer that \(T_{\gamma}(M)\) is relatively weakly compact. Since \(T_{\gamma}\) has a weakly sequentially closed graph, then we may invoke Theorem 2.1 in order to conclude that there exists \(u_{\gamma} \in S_{n_0}^{T_{\gamma}}\) such that
\[
u_{\gamma} \in T_{\gamma}(u_{\gamma}).
\]
Now, let \(\{\gamma_m\}_{m=1}^{\infty}\) be a sequence in \((0, 1)\) such that \(\gamma_m \to 1\). From the above discussion and by using inclusion (3.2), there exists a sequence \(\{u_m\}_{m=1}^{\infty} \subset \Omega\) such that
\[
u_m \in T_{\gamma_m}(u_m).
\]
This leads to the conclusion that there exists \(w_m \in T(u_m)\) such that
\[
u_m = \gamma_m w_m + (1 - \gamma_m)y.
\]
On the other hand, it follows from inclusions
\[
T \left( \{u_m\}_{m=1}^{\infty} \right) \subset T(\Omega) \subset S_{n_0+1}^{T} \subset \Omega,
\]
that \(T(\{u_m\}_{m=1}^{\infty})\) is bounded in \(\Omega\). Then, we result that
\[
u_m - w_m = (\gamma_m - 1)w_m + (1 - \gamma_m)y \rightharpoonup \theta.
\]
Keeping in mind the hemi-weak compactness of \(T\), we deduce that \(\{u_m\}_{m=1}^{\infty}\) has a weakly convergent subsequence to some \(u\). Since \(T\) has a weakly sequentially closed graph, we reach the result that \(u \in T(u)\).

As an immediately consequence of Theorem 3.1 we reach the following fixed point theorem.

**Corollary 3.1** Let \(S\) be a nonempty, closed, and convex subset of \(E\) and let \(T : S \to \mathcal{P}(S)\) be a multi-valued mapping with a weakly sequentially closed graph such that:
(i) There exists a bounded subset \(\Omega\) belongs to \(\{S^T_n, n \in \mathbb{N}\}\),
(ii) \(T\) is hemi-weakly compact and has closed, convex values on \(\Omega\),
(iii) \(T\) is countably \(\beta\)-nonexpansive on \(\Omega\).
Then \(T\) has at least one fixed point in \(\Omega\).

The following results give a new fixed point theorem for countably \(D\)-set-contraction multi-valued mapping with \(D\)-contraction function \(\phi\) satisfying the estimates \(\phi(r) < r\) only on a bounded real interval.

**Corollary 3.2** Let \(S\) be a nonempty, closed, and convex subset of \(E\) and let \(T : S \to \mathcal{P}(S)\) be a multi-valued mapping with a weakly sequentially closed graph such that:
(i) There exists a bounded subset \(\Omega\) belongs to \(\{S^T_n, n \in \mathbb{N}\}\),
(ii) \(T\) has closed, convex values on \(\Omega\),
(iii) \(T\) is countably \(D\)-set-contraction on \(\Omega\), with \(D\)-function \(\phi\).
Then \(T\) has at least one fixed point in \(\Omega\).

**Proof** Let \(n_0 \in \mathbb{N}\) such that \(\Omega = S^{T}_{n_0}\). Thanks to Theorem 3.1, we have only to show that \(T : \Omega \to \mathcal{P}(\Omega)\) is hemi-weakly compact. To do this, let \(\{x_n\}_{n=1}^{\infty}\) be an arbitrary sequence
of $\Omega$ and let $y_n \in T(x_n)$ such that $\{x_n - y_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence. Furthermore, taking into account that

$$\{x_n\}_{n=1}^{\infty} \subseteq \{x_n - y_n\}_{n=1}^{\infty} + T(\{x_n\}_{n=1}^{\infty})$$

and bearing in mind the subadditivity of the measure of weak noncompactness of De Blasi, we arrive at the following conclusion:

$$\beta(\{x_n\}_{n=1}^{\infty}) \leq \beta(\{x_n - y_n\}_{n=1}^{\infty}) + \beta(T(\{x_n\}_{n=1}^{\infty})).$$

By using a contradiction argument we can prove that $\{x_n\}_{n=1}^{\infty}$ is relatively weakly compact.

Consequently, $\{x_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence, in view of the Eberlein-Šmulian’s theorem, which shows that $T$ is hemi-weakly compact.

Remark 3.1 Corollary 3.2 extends Theorem 3.1 in [5], and shows that the condition "$D$-set-contraction" can be relaxed by assuming that $T$ maps weakly compact sets into weakly relatively compact sets. Consequently, $\{x_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence, in view of the Eberlein-Šmulian’s theorem, which shows that $T$ is hemi-weakly compact.

The following result concerning the countably $\beta$-condensing multi-valued mappings and represents an extension of Theorem 2.1, since it does not require the boundness of $T(S)$.

Corollary 3.3 Let $S$ be a nonempty, closed, and convex subset of $E$ and let $T : S \to \mathcal{P}(S)$ be a multi-valued mapping with a weakly sequentially closed graph such that:

(i) There exists a bounded subset $\Omega$ belongs to $\{S_n^T, n \in \mathbb{N}\}$,
(ii) $T$ has closed, convex values on $\Omega$,
(iii) $T$ is countably $\beta$-condensing and maps weakly compact sets into weakly relatively compact sets.

Then $T$ has at least one fixed point in $\Omega$. □

Proof Let $n_0 \in \mathbb{N}$ such that $\Omega = S^T_{n_0}$. Since $T$ is countably $\beta$-condensing on $\Omega$, then

$$\beta(T(M)) \leq \beta(M)$$

for all $M \subset \Omega$ such that $\beta(M) > 0$. \hspace{2mm} (3.4)

On the other hand, taking into account that $T$ maps weakly compact sets into weakly relatively compact sets, we obtain that $\beta(T(M)) = 0$ for all weakly compact subset $M$ of $\Omega$. In particular, for every relatively weakly compact subset $M$ of $\Omega$, i.e $\beta(M) = 0$, we have

$$\beta(T(M)) \leq \beta(T(\overline{M}^{w} )) = 0.$$ \hspace{2mm} (3.5)

Using inequalities (3.4) and (3.5) we infer that $T$ is countably $D$-set-Lipschitzian with $D$-function $\phi$, given by $\phi(r) = r$ for all $0 \leq r \leq \|\Omega\|$. Now, by proceeding essentially as in the proof of Corollary 3.2 we can obtain that $T$ is hemi-weakly compact. Now, we may invoke Theorem 3.1 in order to conclude that $T$ has a fixed point $x$ in $\Omega$. □

Next, we establish some new hybrid fixed point theorems involving the sum of two multi-valued mappings defined on a nonempty, closed, and convex subsets of Banach spaces.

Theorem 3.2 Let $S$ be a nonempty, closed, and convex subset of a Banach space $E$, $A, B : S \to \mathcal{P}(cl, cv(E))$ be two multi-valued mappings have weakly sequentially closed graphs such that:

(i) $A$ is hemi-weakly compact and countably $D$-set-lipschitzian with $D$-function $\Phi$,
(ii) $B$ is weakly compact,

□ Springer
(iii) \((A + B)(S)\) is bounded in \(S\).

Then, the operator inclusion \((u \in Au + Bu)\) has at least one fixed point in \(S\) as soon as \(\Phi(r) \leq r\) for \(r > 0\).

\[\text{□}\]

**Proof** Since \(S\) is a nonempty convex subset of \(E\) and \((A + B)(S) \subset S\), then we may define a multi-valued mapping \(T\) by

\[
\begin{align*}
T : S^{A+B} &\to \mathcal{P}_{cl,cv} \left( S^{A+B} \right) \\
u &\mapsto Au + Bu
\end{align*}
\]

Recall that \(S^{A+B} = \text{co}(A + B)(S)\) is the second term of the sequence (3.1) associate to \(A + B\). Let \(M\) be a countably and bounded subset of \(S^{A+B}\). Keeping in mind the relatively weak compactness of \(B(M)\) and using the subadditivity of the De Blasi measure of weak noncompactness we get

\[
\beta(T(M)) \leq \beta(A(M)) + \beta(B(M)) \leq \Phi(\beta(M)).
\]

Then, \(T\) defines a countably \(\mathcal{D}\)-set-Lipschitzian multi-valued mapping on \(S^{A+B}\) with \(\mathcal{D}\)-function \(\Phi\). Now, we claim that \(T\) is hemi-weakly compact. To do this, let \(\{u_n\}_{n=1}^{\infty}\) be a sequence of elements in \(S^{A+B}\) and let \(v_n \in T(u_n)\) such that \(w_n := u_n - v_n \rightharpoonup w \in S^{A+B}\).

Since \(v_n \in T(u_n)\) then there exist \(a_n \in A(u_n)\) and \(b_n \in B(u_n)\) such that

\[
v_n = a_n + b_n,
\]

which implies that,

\[
u_n - a_n = w_n + b_n.
\]

It follows from the Eberlein-Šmulian’s theorem, that \(\{b_n, n \in \mathbb{N}\}\) has a weakly convergent subsequence. This shows from the hemi-weak compactness of \(A\), that \(\{u_n, n \in \mathbb{N}\}\) has a weakly convergent subsequence. It is easily seen that, \(T\) has a weakly sequentially closed graph. So, \(T\) has a fixed point in \(S^{A+B}\) due to Theorem 3.1. \[\text{□}\]

Now, we are in a position to apply Theorem 3.2 in order to deduce the following fixed point theorem.

**Corollary 3.4** Let \(S\) be a nonempty, closed, and convex subset of a Banach space \(E\), \(A, B : S \to \mathcal{P}_{cl,cv}(E)\) be two multi-valued mappings have weakly sequentially closed graphs such that:

(i) \(A\) is hemi-weakly compact and countably \(\beta\)-nonexpansive,

(ii) \(B\) is weakly compact,

(iii) \((A + B)(S)\) is a bounded subset of \(S\).

Then, the operator inclusion \((u \in Au + Bu)\) has at least one fixed point in \(S\). \[\text{□}\]

**Remark 3.2** Theorem 3.2 extends and improves Theorem 4.10 in [10]. Indeed, if there exist a bounded subset \(S_0\) in \(E\) and a sequence \(\{\lambda_n, n \in \mathbb{N}\} \subset (0, 1)\) with \(\lambda_n \to 1\) such that \((A + \lambda_n B)(S) \subset S_0\) for all \(n\), then \(S^{A+B}_1\) is bounded. \[\text{□}\]
4 Fixed Point results for multi-valued mappings in Banach algebras

In this section, we give sufficient conditions for the operator inclusion (1.2) which is acting on a Banach algebra satisfying (P) to have a fixed point. Firstly, we establish some hybrid fixed point theorems for (1.2) by looking at the multi-valued mapping

\[ T := A \cdot B + C. \]

**Theorem 4.1** Let \( S \) be a nonempty, closed, and convex subset of a Banach algebra \( E \) satisfying (P). Suppose that \( A, B, C : S \to \mathcal{P}(E) \) are three multi-valued mappings have weakly sequentially closed graphs such that:

(i) For each \( u \in S \), \( A(u) \cdot B(u) + C(u) \) is a closed, convex subset of \( S \).
(ii) There exists a bounded subset \( \Omega \) belongs to \( \{ S_{n}^{A+B+C} : n \in \mathbb{N} \} \).
(iii) \( A, B \) and \( C \) are countably \( D \)-set-lipschitzian on \( \Omega \) with \( D \)-functions \( \Phi_A, \Phi_B \) and \( \Phi_C \) respectively.
(iv) \( A(\Omega) \) and \( B(\Omega) \) are bounded.

Then, the operator inclusion (1.2) has at least one fixed point in \( \Omega \) provided that

\[ \| B(\Omega) \| \Phi_A(\beta(\Omega)) + \| A(\Omega) \| \Phi_B(\beta(\Omega)) + \Phi_A(\beta(\Omega)) \Phi_B(\beta(\Omega)). \]

**Proof** Let \( n_0 \in \mathbb{N} \) such that \( \Omega = S_{n_0}^{A+B+C} \). Proceeding as in the proof of Theorem 3.2, in view of assumption (i), we can define a multi-valued mapping \( T \) by

\[
\begin{align*}
T : \Omega &\to \mathcal{P}_{cl, cv}(\Omega) \\
u &\mapsto Au \cdot Bu + Cu.
\end{align*}
\]

First, we claim that \( T \) is countably \( D \)-set-lipschitzian. Indeed, let \( M \) be a countably and bounded subset of \( \Omega \). Combining Theorem 2.9 in [3] with Lemma 3.1 in [32] we get

\[ \beta(A(M)B(M)) \leq \| B(M) \| \Phi_A(\beta(M)) + \| A(M) \| \Phi_B(\beta(M)) + \Phi_A(\beta(M)) \Phi_B(\beta(M)). \]

Then, keeping in mind the subadditivity property of \( \beta \), we have that

\[
\beta(T(M)) \leq \beta(A(M)B(M)) + \beta(C(M)) \\
\leq \| B(M) \| \Phi_A(\beta(M)) + \| A(M) \| \Phi_B(\beta(M)) + \Phi_A(\beta(M)) \Phi_B(\beta(M)) + \Phi_C(\beta(M)).
\]

Thus, \( T \) defines a countably \( D \)-set-lipschitzian mapping on \( \Omega \) with \( D \)-function \( \Psi \) given by

\[ \Psi(r) = \| B(\Omega) \| \Phi_A(r) + \| A(\Omega) \| \Phi_B(r) + \Phi_A(r) \Phi_B(r) + \Phi_C(r) \]

for \( r \geq 0 \).

Now, we claim that \( T \) has a weakly sequentially closed graph. To this end, let \( \{ u_n \}_{n=1}^{\infty} \) be a sequence of elements in \( \Omega \) such that \( u_n \to u \in \Omega \) and let \( v_n \in T(u_n) \) such that \( v_n \to v \in \Omega \). Then there exist \( \sigma_n \in Au_n \) and \( \gamma_n \in Bu_n \) and \( \rho_n \in Cu_n \) such that

\[ v_n = \sigma_n \cdot \gamma_n + \rho_n. \]

Since \( A \) is countably \( D \)-set-lipschitzian, then

\[ \beta(\{ \sigma_n \}_{n=1}^{\infty}) \leq \Phi_A(\beta(\{ u_n \}_{n=1}^{\infty})) = 0. \]

From the Eberlein-Šmulian’s theorem, we can assume that \( \{ \sigma_n \}_{n=1}^{\infty} \) and \( \{ \gamma_n \}_{n=1}^{\infty} \) weakly converge to some points \( \sigma, \gamma \in E \), respectively. Further, taking into account that \( A \) and \( B \)
have weakly sequentially closed graphs we get \( \sigma \in Au \) and \( \gamma \in Bu \). From the condition \((P)\), it follows that
\[
\rho_n \rightarrow v - \sigma \cdot \gamma.
\]
Keeping in mind that \( C \) has a weakly sequentially closed graph, we derive that
\[
v - \sigma \cdot \gamma \in Cx.
\]
Consequently, we get
\[
v \in Au \cdot Bu + Cu,
\]
which achieves the proof of our claim.

Now, an application of Corollary 3.2 yields that there is \( u \in \Omega \) such that \( u \in Au \cdot Bu + Cu \).

\( \square \)

**Remark 4.1** Theorem 4.1 extends Theorem 4.2 in [5], and shows that the condition "\( \mathcal{D}\)-set-Lipschitzian" can be relaxed by assuming that \( F \) is countably \( \mathcal{D}\)-set-Lipschitzian, similarly the condition "\( A(S) \), \( B(S) \) and \( C(S) \) are bounded" can be relaxed by assuming that \( S_n^{A-B+C} \) is bounded for some \( n \in \mathbb{N} \), moreover it proves that the condition that \( A \), \( B \) and \( C \) are s.w.u.sco., i.e. Condition (3) in [5, Theorem 4.2], in the statement of that theorem is not needed. \( \square \)

**Remark 4.2** Notice that every s.w.u.sco., \( k \)-lipschitzian multi-valued mapping is \( \mathcal{D}\)-set-Lipschitzian, with \( \mathcal{D}\)-function \( \phi(r) = kr \), in particular is countably \( \mathcal{D}\)-set-Lipschitzian and \( A : B + C \) has closed values, by using the condition \((P)\) and the fact that \( A \), \( B \), and \( C \) have closed values. Thus, Theorem 4.1 generalizes Theorem 4.5 in [5], by taking \( n_0 = 0 \). \( \square \)

A special case of Theorem 4.1, which is useful in applications to differential and integral inclusions, is introduced in the following theorem.

**Corollary 4.1** Let \( S \) be a nonempty, closed, and convex subset of a Banach algebra \( E \) satisfying \((P)\). Suppose that \( A, B, C : S \rightarrow \mathcal{P}(E) \) are three multi-valued mappings have weakly sequentially closed graphs such that:

(i) For each \( u \in S \), \( A(u) \cdot B(u) + C(u) \) is a closed, convex subset of \( S \),

(ii) There exists a bounded subset \( \Omega \) belongs to \( \{ S_n^{A-B+C}, n \in \mathbb{N} \} \),

(iii) \( A \) and \( C \) are countably \( \mathcal{D}\)-set-lipschitzian with \( \mathcal{D}\)-functions \( \Phi_A \) and \( \Phi_C \) respectively,

(iv) \( B \) is weakly compact on \( \Omega \).

Then, the operator inclusion \((1.2)\) has at least one fixed point in \( S \) provided that
\[
\| B(\Omega) \| \Phi_A(r) + \Phi_C(r) < r \quad \text{for} \quad 0 < r \leq \| \Omega \|.
\]

\( \square \)

**Remark 4.3** Corollary 4.1 extends Theorem 4.3 in [5], and shows that the condition that \( A \) and \( C \) are s.w.u.sco., i.e. Condition (4) in [5, Theorem 4.3], in the statement of that theorem is not needed. \( \square \)

When \( A \) is countably \( \beta \)-nonexpansive we get the following result.

**Corollary 4.2** Let \( S \) be a nonempty, closed, and convex subset of a Banach algebra \( E \) satisfying \((P)\). Suppose that \( A, B, C : S \rightarrow \mathcal{P}(E) \) are three multi-valued mappings have weakly sequentially closed graphs such that:

(i) For each \( u \in S \), \( A(u) \cdot B(u) + C(u) \) is a closed, convex subset of \( S \),

(ii) There exists a bounded subset \( \Omega \) belongs to \( \{ S_n^{A-B+C}, n \in \mathbb{N} \} \),

(iii) \( A \) is countably \( \beta \)-nonexpansive and \( B \) is weakly compact on \( \Omega \),

(iv) \( C \) is countably \( \alpha \)-set-contraction on \( \Omega \).

Then, the operator inclusion \((1.2)\) has at least one fixed point in \( S \) provided that \( \| B(\Omega) \| + \alpha < 1 \). \( \square \)
The operator \( \left( \frac{I - C}{A} \right) \) was used in numerous works to establish some existence results for Hybrid fixed point theorems for several class of single-valued mappings in Banach algebras (see for example \([6, 7, 9]\)). In most of these works, the operator \( \left( \frac{I - C}{A} \right) \) play a fundamental role in their arguments. Recently, A. Ben Amar et al have introduced in \([5]\) an analogous definition of \( \left( \frac{I - C}{A} \right) \) for the multi-valued mappings, in order to investigate some fixed point theorems for (1.2). Next, we consider Hybrid fixed point theorems for (1.2) by looking at the multi-valued mapping

\[
T := \left( \frac{I - C}{A} \right)^{-1} B
\]

where \( A, B \) and \( C \) are countably \( \mathcal{D} \)-set-Lipschitzian or countably nonexpansive multi-valued mappings.

**Theorem 4.2** Let \( S \) be a nonempty, closed, and convex subset of a Banach algebra \( E \) satisfying \((\mathcal{P})\). Assume that \( A, C : E \to \mathcal{P}_{cl,cv}(E) \) and \( B : S \to \mathcal{P}_{cl,cv}(E) \) are three multi-valued mappings have weakly sequentially closed graphs such that:

(i) \( A \) is weakly compact,
(ii) \( B \) is countably \( \mathcal{D} \)-set-contraction with \( \mathcal{D} \)-contraction \( \Phi \),
(iii) \( C \) is \( \mathcal{D} \)-set-contraction with \( \mathcal{D} \)-function \( \Psi \),
(iv) \( A(E), B(S), \) and \( C(E) \) are bounded,
(v) for each \( u \in S \), \( \left( \frac{I - C}{A} \right)^{-1} B(u) \) is a closed, convex subset of \( S \).

Then, the operator inclusion (1.2) has a solution in \( S \) whenever \( \|A(S)\|/\Phi(r) + \|\Psi(r)\| < r \) for \( r > 0 \).

**Proof** Let \( \gamma \in B(S) \) be fixed. From our assumptions we can define a multi-valued mapping \( Q \) by

\[
Q : E_{1}^{A \cdot \gamma + C} \to \mathcal{P}_{cl,cv} \left( E_{1}^{A \cdot \gamma + C} \right)
\]

\[
x \mapsto Ax \cdot \gamma + Cx.
\]

Recall that the bounded subset \( E_{1}^{A \cdot \gamma + C} = \overline{\sigma}(A \cdot \gamma + C)(E) \) is the second term of the sequence (3.1) associate to \( A \cdot \gamma + C \). Proceeding essentially as in the proof of Theorem 4.1, we can show that \( Q \) defines a countably \( \mathcal{D} \)-set-Lipschitzian, with \( \mathcal{D} \)-function \( \Psi \).

Now we claim that \( Q \) has a weakly sequentially closed graph. To do this, let \( \{u_n\}_{n=1}^{\infty} \) be a sequence of elements in \( E_{1}^{A \cdot \gamma + C} \) such that \( u_n \to u \) and let \( v_n \in Q(u_n) \) such that \( v_n \to v \). Then there exist two sequences \( \{\sigma_n\}_{n=1}^{\infty} \) and \( \{\rho_n\}_{n=1}^{\infty} \) with \( \sigma_n \in Au_n \) and \( \rho_n \in Cu_n \) such that

\[
v_n = \sigma_n \cdot \gamma + \rho_n.
\]

Since \( \beta(A(\{u_n\}_{n=1}^{\infty})) = 0 \), taking into account the Eberlein-Šmulian’s theorem we can suppose that \( \sigma_n \to \sigma \in E \). Keeping in mind that \( A \) and \( C \) have weakly sequentially closed graphs, we deduce that \( \sigma \in Au \), and \( v - \sigma \cdot \gamma \in Cu \). This means that \( v \in Tu \). Invoking Corollary 3.2, we infer that \( Q \) has a fixed point \( u \in E_{1}^{A \cdot \gamma + C} \), i.e.

\[
u \in Au \cdot \gamma + Cu.
\]

Thus,

\[
\gamma \in \left( \frac{I - C}{A} \right)(u),
\]
and so
\[(I - \frac{C}{A})(u) \cap B(S) \neq \emptyset.\]
This achieves that \((I - \frac{C}{A})^{-1}\) is well defined on \(B(S)\). Accordingly, we can define a nonempty, closed, and convex subset \(S_1^T := \overline{co}(\{(I - \frac{C}{A})^{-1} B(S)\})\) of \(S\), which is invariant by \(T := (I - \frac{C}{A})^{-1} B\) in view of assumption (v). Using assumption (iv) together with the inclusion
\[T(D) \subset AT(D) \cdot B(D) + CT(D) \text{ for all } D \subset S,\] in order to results that \(S_1^T\) is a bounded subset of \(S\). Now, let us consider the following multi-valued mapping:
\[
\begin{cases}
T : S_1^T \to \mathcal{P}_{cl, cv}(S_1^T) \\
u \mapsto \left(\left(I - \frac{C}{A}\right)^{-1} B(u)\right)
\end{cases}
\]
We will prove that \(T\) is countably condensing. To see this, let \(D\) be a countably subset of \(S_1^T\).
By combining Theorem 2.9 in [3] with Lemma 3.1 in [32], and using the subadditivity of \(\beta\) and the relatively weak compactness of \(A(S_1^T)\) it follows that
\[
\beta(T(D)) \leq \|A(S_1^T)\| \beta(B(D)) + \beta(C(T(D))) \\
\leq \|A(S)\| \Phi(\beta(D)) + \Psi(\beta(T(D))).
\] (4.2)
If \(\Phi(\beta(D)) = 0\), by using a contradiction argument we can show that \(T(D)\) is relatively weakly compact. So, we may assume that \(\Phi(\beta(D)) > 0\). Inequalities (4.2) imply that
\[
\beta(T(D)) - \Psi(\beta(T(D))) \leq \|A(S)\| \Phi(\beta(D)).
\]
Combining this inequality with our assumptions, we get
\[
\Phi(\beta(T(D))) < \Phi(\beta(D)).
\]
This means that \(T\) is countably condensing. In view of inequalities (4.2) and by using a contradiction argument, this leads to the conclusion that \(T\) maps weakly compact sets into relatively weakly compact sets. Now we claim that \(T\) has a weakly sequentially closed graph. To do this, let \(\{u_n\}_{n=1}^{\infty}\) be a sequence of elements in \(S_1^T\) which converges weakly to \(u\) and \(v_n \in T(u_n)\) which converges weakly to \(v\). Then, there exists a sequence \(\{w_n\}_{n=1}^{\infty}\) in \(E\) such that
\[
w_n \in \left(\frac{I - C}{A}\right)(v_n) \cap B(u_n), \quad \text{for all } n \in \mathbb{N}.
\] (4.3)
Using the fact that \(\{u_n\}_{n=1}^{\infty}\) is weakly convergent we get
\[
\beta\left(\{w_n\}_{n=1}^{\infty}\right) \leq \beta\left(B\left(\{u_n\}_{n=1}^{\infty}\right)\right) \leq \Phi(\beta\left(\{u_n\}_{n=1}^{\infty}\right)) = 0.
\]
Then, we can extracts a renamed subsequence of \(\{w_n\}_{n=1}^{\infty}\) such that \(w_n \to w\), which is a consequence of the Eberlein-Šmulian’s theorem. Keeping in mind that \(B\) has a weakly sequentially closed graph, we derive that
\[
w \in B(u).
\]
On the other hand, by using the Definition 2.4 together with inclusion (4.3), we deduce that
\[
v_n \in w_n \cdot A(v_n) + C(v_n), \quad n \in \mathbb{N}.
\]
Since $A$ and $C$ have weakly closed graphs and $E$ satisfying $(P)$, we infer that

$$v \in w \cdot A(v) + C(v),$$

and consequently

$$w \in \left( \frac{I - C}{A} \right)(v).$$

This means that $v \in T(u)$, and the claim is approved. The remained proof follows along the lines of Theorem 2.1.

**Theorem 4.3** Let $S$ be a nonempty, closed, and convex subset of a Banach algebra $E$ satisfying $(P)$. Assume that $A, C : E \rightarrow \mathcal{P}_{cl,cv}(E)$ and $B : S \rightarrow \mathcal{P}(E)$ are three multi-valued mappings have weakly sequentially closed graphs such that:

(i) $A$ is $D$-set-Lipschitzian with $D$-function $\Phi$,
(ii) $B$ is countably $\alpha$-set-contraction,
(iii) $C$ is $\gamma$-set-contraction,
(iv) $A(E), B(S)$ and $C(E)$ are bounded,
(v) for each $u \in S$, $(\frac{I - C}{A})^{-1} B(u)$ is a closed, convex subset of $S$.

Then, the operator inclusion (1.2) has a fixed point in $S$ whenever there exists a positif constant $\delta, \alpha < \delta < \frac{1 - \gamma}{\|A(S)\|}$, such that $\|B(S)\|\Phi(r) < (1 - 2\delta\|A(S)\| - \gamma)r$ for $r > 0$. □

**Proof** We shows, first, that $(\frac{I - C}{A})^{-1}$ exists on $B(S)$. To do this, let $v \in B(S)$ be fixed. Define a multi-valued mapping $Q$ by

$$Q : E \rightarrow \mathcal{P}_{cl,cv}(E)
\quad x \mapsto Ax \cdot v + Cx.$$  

An argument similar to that in the proof of Theorem 4.2 yields that $Q$ is countably $D$-set-contraction, with $D$-function $\varphi(r) = \|v\|\Phi(r) + \gamma r$, and has a weakly sequentially closed graph.

On the other hand, by using a contradiction argument, we may prove that $Q$ maps weakly compact sets into weakly relatively compact sets. Applying now Theorem 2.1 we obtain that there exists $u \in E$ such that

$$u \in Au \cdot v + Cu,$$

thus

$$v \in \left( \frac{I - C}{A} \right)(u).$$

As a result,

$$\left( \frac{I - C}{A} \right)(u) \cap B(S) \neq \emptyset.$$  

This achieves that $(\frac{I - C}{A})^{-1}$ is well defined on $B(S)$. Arguing as in the proof of Theorem 4.2, we can define a multi-valued mapping $T$ by

$$T : S^T_1 \rightarrow \mathcal{P}_{cl,cv}(S^T_1)
\quad u \mapsto \left( \frac{I - C}{A} \right)^{-1} B(u).$$
Proceeding essentially as in the proof of Theorem 4.2, we can prove that $T$ has a weakly sequentially closed graph. Our next task is to show that $T$ is countably $\frac{\alpha}{\beta}$-set-contraction. To achieve this, take an arbitrary countably and bounded subset $D \subset S_I^T$. Keeping in mind the inclusion (4.1) together with assumption (iv) we infer that $S_I^T$ is a bounded subset of $S$. Using again inclusion (4.1) combining with Theorem 2.9 in [3] with Lemma 3.1 in [32] we obtain

$$
\beta(T(D)) \leq \|B(S)\| \Phi(\beta(T(D))) + \alpha \|A(S)\| \beta(D) + \alpha \beta(D) \beta(A(T(D))) + \gamma \beta(T(D)) \\
\leq \|B(S)\| \Phi(\beta(T(D))) + 2\alpha \|A(S)\| \beta(D) + \gamma \beta(T(D)),
$$

(4.4)

which implies that

$$(1 - \gamma) \beta(T(D)) \leq \|B(S)\| \Phi(\beta(T(D))) + 2\alpha \|A(S)\| \beta(D).$$

Then, by using the inequality

$$\|B(S)\| \Phi(r) < (1 - (2 \delta \|A(S)\| + \gamma)) \frac{r}{r} \text{ for } r > 0,$$

we get

$$\beta(T(D)) \leq \frac{\alpha}{\delta} \beta(D).$$

This implies that $T$ is countably $\frac{\alpha}{\delta}$-set-contraction. By proceeding essentially as in the proof of Theorem 4.2, we can prove that $T$ has a weakly sequentially closed graph.

Hence, the desired result follows from Corollary 3.2. \qed

**Corollary 4.3** Let $S$ be a nonempty, closed, and convex subset of a Banach algebra $E$ satisfying the sequential condition $(\mathcal{P})$. Assume that $A, C : E \to \mathcal{P}_{cl,cv}(E), B : S \to \mathcal{P}(E)$ are three multi-valued mappings have weakly sequentially closed graphs such that:

(i) $A$ is countably $\beta$-nonexpansive,
(ii) $B$ is countably $\alpha$-set-contraction,
(iii) $C$ is $\gamma$-set-contraction,
(iv) $A(E), B(S)$ and $C(E)$ are bounded,
(v) for each $u \in S, \left(\frac{I - E}{A}\right)^{-1} B(u)$ is a closed, convex subset of $S$.

Then, the operator inclusion (1.2) has a fixed point in $S$ whenever there exists a positive constant $\delta, \alpha < \delta < \frac{1 - \gamma}{\|A(S)\|},$ such that $2\delta \|A(S)\| + \|B(S)\| + \gamma < 1.$ \qed

## 5 Nonlinear integral inclusions

Let $(X, \| \cdot \|)$ be a Banach algebra satisfying $(\mathcal{P})$, we denote by $E := C(J, X)$ the Banach algebra of all continuous functions from $J := [0, 1]$ to $X$, endowed with the supremum norm

$$\|f\|_\infty = \sup \{\|f(t)\| : t \in J\}, \text{ for each } f \in E.$$

For any multi-valued mapping $\Xi : J \times X \to \mathcal{P}(X)$ and for any $x \in E$ we denote

$$\int_0^t \Xi(s, x(s)) ds = \left\{ \int_0^t v(s) ds : v \text{ is Pettis integrable (P.I) and } v(t) \in \Xi(t, x(t)) \right\}$$

\[\Xi\] Springer
and
\[ \| \Xi(t, x) \| = \sup \{ \| v \| : v \in \Xi(t, x) \}. \]

In this section, we are mainly concerned with the existence results of solutions for the following problem of nonlinear integral inclusion on \( E \):
\[ x(t) \in T_1(t, x(t)) \cdot (q(t) + \int_0^t k(t, s) H(s, x(s)) \, ds) + T_2(t, x(t)), \quad t \in J, \quad (5.1) \]
where \( k : J \times J \to \mathbb{R}, q : J \to X, T_1 : J \times X \to X \) and \( H : J \times X \to \mathcal{P}_{cp,cv}(X) \).
Under the following assumptions, we could reach the solution of (5.1):

\begin{itemize}
  \item[(C_0)] The functions \( q : J \to X \) and \( k : J \times J \to \mathbb{R} \) are continuous.
  \item[(C_1)] For \( i = 1, 2, \) we have:
    \begin{itemize}
      \item[(i)] For all \( t \in J \), \( T_i(t, \cdot) \) are weakly sequentially continuous.
      \item[(ii)] There exist two continuous functions \( \vartheta_i : J \to [0, \infty) \) such that
        \[ \| T_1(t, x) - T_1(t', y) \| \leq \vartheta_1(t) \| x - y \| + |\vartheta_1(t) - \vartheta_1(t')| \| y \| \]
        for all \( x, y \in X \) and \( t, t' \in J \), and
        \[ \| T_2(t, x(t)) - T_2(t', x(t')) \| \leq |\vartheta_2(t) - \vartheta_2(t')| \| x \|_{\infty} \]
        for all \( x \in E \) such that \( \| x \|_{\infty} \leq R \) and \( t, t' \in J \).
      \item[(iii)] There exists a continuous, nondecreasing mapping \( \Psi : [0, +\infty) \to [0, +\infty) \) with \( \Psi(0) = 0 \) such that
        \[ \beta(T_2(J \times M)) \leq \Psi(\beta(M)) \]
        for all countably and bounded subset \( M \) of \( X \).
    \end{itemize}
  \item[(C_2)] The multi-valued mapping \( H : J \times X \to \mathcal{P}(X) \) is such that:
    \begin{itemize}
      \item[(i)] For each \( x \in E \), there are a scalarly measurable function \( w : J \to X \) with
        \( w(t) \in H(t, x(t)) \) a.e. \( t \in J \) such that \( w \) is Pettis integrable on \( J \).
      \item[(ii)] There exists \( R > 0 \) and \( h \in L^1(J, \mathbb{R}_+) \) such that
        \[ \| H(s, x) \| \leq h(s) \| x \|, \text{ for all } x \in B_X(R). \]
      \item[(iii)] There exists a continuous, nondecreasing mapping \( \Phi : [0, +\infty) \to [0, +\infty) \) with \( \Phi(0) = 0 \) such that
        \[ \beta(H(J \times M(J))) \leq \Phi(\beta(M(J))) \]
        for all countably and bounded subset \( M \) of \( B_E(R) \).
      \item[(iv)] \( H(t, \cdot) \) has weakly sequentially closed graph on \( B_X(R) \).
    \end{itemize}
\end{itemize}

Before reaching the main result in this section, the useful results for the sequel are stated.

**Theorem 5.1** [37] Let \( X \) be a Banach algebra and let \( H \subseteq C([0, T], X) \) be bounded and equi-continuous. Then the map \( t \mapsto \beta(H(t)) \) is continuous on \([0, T]\) and
\[ \beta(H) = \sup_{t \in [0, T]} \beta(H(t)) = \beta(H([0, T])), \]
where \( H(t) = \{ x(t) : x \in H \} \) and \( H([0, T]) = \bigcup_{t \in [0, T]} \{ x(t) : x \in H \} \). \( \square \)
Proposition 5.1 [19, 38] If \( x(\cdot) \) is Pettis integrable and \( h(\cdot) \) is a measurable and essentially bounded real-valued function, then \( h(\cdot)x(\cdot) \) is Pettis integrable. \( \square \)

Lemma 5.1 [14] Let \( X \) be a Banach space, and \( f : [a, b] \rightarrow X \) be a Pettis integrable function. Then

\[
\int_a^b f(s) \, ds \in (b - a)\text{co}\{f([a, b])\}.
\]

\( \square \)

Theorem 5.2 Suppose that the conditions \((C_0)-(C_2)\) are satisfied. Then, the problem \((5.1)\) has a solution \( x \) in \( B_E(R) \) whenever

\[
\left[ \| q \|_{\infty} + \| k(\cdot, \cdot) \|_{\infty} \| h \|_{L_1} R \right] \| \vartheta_1 \|_{\infty} R + 2 \| \vartheta_2 \|_{\infty} R + \| T_2(0, x(0)) \| \leq R \quad (5.2)
\]

and

\[
\left[ \| q \|_{\infty} + \| k(\cdot, \cdot) \|_{\infty} \| h \|_{L_1} R \right] \| \vartheta_1 \|_{\infty} r + 2 \| \vartheta_1 \|_{\infty} R \| k(\cdot, \cdot) \|_{\infty} \Phi(r) + \Psi(r) < r \quad (5.3)
\]

for \( 0 < r \leq R \). \( \square \)

Proof Let \( E = C(J, X) \) and

\[
S = \left\{ x \in E : \| x \|_{\infty} \leq R \text{ and } \| x(t) - x(\tau) \| \leq b(t, \tau), \text{ for } t, \tau \in J \right\},
\]

where

\[
b(t, \tau) = \frac{1}{(1 - \delta_1 \| \vartheta_1 \|_{\infty})} \left\{ R \delta_1 | \vartheta_1(t) - \vartheta_1(\tau) | + R | \vartheta_2(t) - \vartheta_2(\tau) | + \delta_2 \left[ \| q(t) - q(\tau) \| + R \sup_{s \in J} | k(t, s) - k(\tau, s) | \| h \|_{L_1} R + R \int_{\tau}^t | k(\tau, s) | h(s) \, ds \right] \right\},
\]

\[
\delta_1 = \| q \|_{\infty} + \| k \|_{\infty} \| h \|_{L_1} R,
\]

\[
\delta_2 = \| \vartheta_1 \|_{\infty} R + \| T_1(0, \vartheta) \|.
\]

Obviously, \( S \) is a closed, convex, bounded, and equi-continuous subset of \( E \). Let us consider three multi-valued mappings \( A, B \) and \( C \) defined on \( S \) by:

\[
(Ax)(t) = T_1(t, x(t)), \quad (Cx)(t) = T_2(t, x(t)),
\]

\[
Bx = \left\{ q(t) + \int_0^t k(t, s) \xi(s) \, ds ; \xi : J \rightarrow X \text{ is P.I., } \xi(s) \in H(s, x(s)), \text{ a.e. } s \in J \right\}.
\]

The integral inclusion \((5.9)\) can be written in the form:

\[
x(t) \in Ax(t) \cdot Bx(t) + Cx(t).
\]

In this section we will apply Theorem 4.1. So, the proof is composed by the following steps.

**Step 1:** \( A \) and \( C \) maps \( S \) into \( E \) and \( B \) maps \( S \) into \( \mathcal{P}(E) \).

The claim regarding \( A \) and \( C \) is clear in view of Condition \((C_1)\). We corroborate now the claim for the multi-valued mapping \( B \). Firstly, it follows from Conditions \((C_2)-(i), (C_2)-(ii)\) and Proposition 5.1 in conjunction with Condition \((C_0)\) that \( B \) is well defined. Let \( x \in S \) be arbitrary and let \( y \in Bx \). Then, there exists a Pettis integrable mapping \( \xi : J \rightarrow X \) with \( \xi(s) \in H(s, x(s)) \) such that

\[
y(t) = q(t) + \int_0^t k(t, s) \xi(s) \, ds, \quad t \in J.
\]
Let \( \{t_n, \ n \in \mathbb{N}\} \) be any sequence of elements in \( J \) converging to \( t \). Then,
\[
\|y(t_n) - y(t)\| \leq \|q(t_n) - q(t)\| + \int_0^{t_n} |k(t_n, s) - k(t, s)| \|\xi(s)\| ds \\
+ \int_{t_n}^{t} |k(t, s)| \|\xi(s)\| ds.
\]
From the Hahn–Banach theorem there exists \( x^* \in X^* \) with \( \|x^*\| = 1 \) such that
\[
\left\| \int_0^{t_n} (k(t_n, s) - k(t, s)) \xi(s) ds \right\| = x^* \left( \int_0^{t_n} (k(t_n, s) - k(t, s)) \xi(s) ds \right)
\]
and
\[
\left\| \int_{t_n}^{t} k(t, s) \xi(s) ds \right\| = x^* \left( \int_{t_n}^{t} k(t, s) \xi(s) ds \right).
\]
Thus, in view of condition (C2)-(ii) we get
\[
\|y(t_n) - y(t)\| \leq \|q(t_n) - q(t)\| + \int_0^{t_n} |k(t_n, s) - k(t, s)| \|\xi(s)\| ds \\
+ \int_{t_n}^{t} |k(t, s)| \|\xi(s)\| ds \\
\leq \|q(t_n) - q(t)\| + R \left( \int_0^{t_n} |k(t_n, s) - k(t, s)| h(s) ds \right) \\
+ \int_{t_n}^{t} |k(t, s)| h(s) ds \\
:= \|q(t_n) - q(t)\| + \sum_{i=1}^{2} I_i(t_n, t),
\]
where
\[
I_1(t_n, t) = R \int_0^{t_n} |k(t_n, s) - k(t, s)| h(s) ds \quad \text{and} \quad I_2(t_n, t) = R \int_{t_n}^{t} |k(t, s)| h(s) ds.
\]
Using condition (C0) and the fact that \( h \in L^1(J, X) \) together with the dominated convergence theorem, we obtain \( I_1(t_n, t) \to 0 \) as \( t_n \to t \). Similarly, it is easy to see that \( I_2(t_n, t) \to 0 \) as \( t_n \to t \), since \( h \in L^1(J, X) \) and \( k \in L^\infty(J \times J) \). Hence, the continuity of \( q \) achieves that \( y(t_n) \to y(t) \), which implies that \( B(x) \in \mathcal{P}(E) \).

**Step 2:** \( A \) is countably \( \mathcal{D} \)-set-lipschitzian on \( S \).

The use of condition (C1)-(ii) allows us to deduce that \( A \) is lipschitzian with lipschitz constant \( \|\vartheta_1\|_\infty \). Since \( T_1(t, \cdot) \) is sequential weak continuous, it is well-known that \( A \) is \( \mathcal{D} \)-set-lipschitzian in light of Lemma 2.4 in [28] with \( \mathcal{D} \)-function \( \Phi_A(r) = \|\vartheta_1\|_\infty r \), in particular it is countably \( \|\vartheta_1\|_\infty \)-set-lipschitzian.

**Step 3:** \( B \) has a weakly sequentially closed graph and is countably \( \mathcal{D} \)-set-lipschitzian.

To see this, let \( (x_n)_{n} \) be a sequence of elements in \( S \) weakly converging to some element \( x \in S \) and let \( y_n \in B(x_n) \) such that \( y_n \) is weakly converging to some \( y \in E \). Thus, we can see that there exists a sequence of Pettis integrable mappings \( (\xi_n)_{n} \) with \( \xi_n(s) \in H(s, x_n(s)) \) such that
\[
y_n(t) = q(t) + \int_0^{t} k(t, s) \xi_n(s) ds.
\]
Fix \( t \in I \). Now, we may invoke Theorem 1.1 in [3] in order to conclude that \( x_n(s) \to x(s) \) and consequently the set \( \{x_n(s), n \in \mathbb{N}\} \) is relatively weakly compact for each \( s \in [0, t] \).

Using Condition \((\mathbf{C}_2)-(\mathbf{iii})\) we obtain

\[
\beta \left( H ([0, t] \times \{x_n(s), n \in \mathbb{N}\}) \right) \leq \beta \left( H (J \times \{x_n(s), n \in \mathbb{N}\}) \right) \leq \Phi \left( \{x_n(s), n \in \mathbb{N}\} \right).
\]

Then, we get

\[
\beta \left( H ([0, t] \times \{x_n(s), n \in \mathbb{N}\}) \right) = 0. \quad (5.4)
\]

This inequality in conjunction with inclusion

\[
\{\xi(s), n \in \mathbb{N}\} \subset H ([0, t] \times \{x_n(s), n \in \mathbb{N}\}) \quad \text{for a.e. } s \in [0, t],
\]

allows us to infer that \( \{\xi_n(s), n \in \mathbb{N}\} \) is relatively weakly compact for a.e. \( s \in [0, t] \). By the Eberlein-Šmulian’s Theorem, there exists a subsequence \( (\xi_{n_k}(s))_k \) of \( (\xi_n(s))_n \) which converges weakly to some element \( \xi(s) \). Taking into account the fact that \( H(t, \cdot) \) has a weakly sequentially closed graph, we get \( \xi(s) \in H(s, x(s)) \) a.e. \( s \in J \).

From the Condition \((\mathbf{C}_2)-(\mathbf{ii})\), it follows that

\[
\|\xi_n(s)\| \leq h(s) R.
\]

The dominated convergence theorem for the Pettis integral reach to result that \( \xi \) is Pettis integrable and

\[
y(t) = q(t) + \int_0^t k(t, s) \xi(s) \, ds, \quad t \in J. \quad (5.5)
\]

Next we prove that \( B \) is countably \( \mathcal{D} \)-set-lipschitzian on \( S \). Let \( M \) be a countably and bounded subset of \( S \). By the subadditivity property of \( \beta \) and Lemma 5.1 together with Condition \((\mathbf{C}_2)-(\mathbf{iii})\) we have for all \( t \in J \),

\[
\beta(B(M)(t)) \leq \beta \left( \bigcup \left\{ \int_0^t k(t, s) \, H(s, x(s)) \, ds, \, x \in M \right\} \right) \\
\leq \beta \left( \int_0^t k(t, s) \, H(s, M(s)) \, ds \right) \\
\leq \|k(\cdot, \cdot)\|_{\infty} \beta \left( \overline{\text{co}}(H(J, M(J))) \right) \\
\leq \|k(\cdot, \cdot)\|_{\infty} \Phi(B(M)).
\]

A similar reasoning as in the first steep, we can verify that \( B(S) \) is an equi-continuous subset of \( E \). Then invoking Theorem 5.1 and the facts that \( B(S) \) is bounded and \( S \) is equi-continuous, we get

\[
\beta(B(M)) \leq \|k(\cdot, \cdot)\|_{\infty} \Phi(B(M)),
\]

which implies that \( B \) is a countably \( \mathcal{D} \)-set-lipschitzian with \( \mathcal{D} \)-function \( \Phi_B(r) = \|k(\cdot, \cdot)\|_{\infty} \Phi(r) \).

**Step 4:** \( C \) is weakly sequentially continuous and is countably \( \mathcal{D} \)-set-contraction.

From condition \((\mathbf{C}_1)\) it follows that \( C \) is weakly sequentially continuous. Now, let \( M \) be a countably and bounded subset of \( S \). By using condition \((\mathbf{C}_1)-(\mathbf{iii})\) and the fact that \( \Psi \) is
nondecreasing, one sees that
\[
\beta(C(M)(t)) \leq \beta(T_2(t \times M(t))) \\
\leq \beta((T_2(J \times M(t)))) \\
\leq \Psi(\beta(M(t))).
\]

From Theorem 5.1 together with the boundedness of \( S \) and the fact that \( \Psi \) is nondecreasing, it follows that
\[
\beta(C(M)(t)) \leq \Psi(\beta(M)).
\]

According to condition \( C_1-i \), it follows that \( C(S) \) is an equi-continuous subset of \( E \). Taking into account the boundedness of \( C(S) \), and using Theorem 5.1 we obtain
\[
\beta(C(M)) \leq \Psi(\beta(M)).
\]

Hence, \( C \) define a countably \( D \)-set-lipschitzian mapping on \( S \), with \( D \)-function \( \Phi_C = \Psi \).

**Step 5:** \( (A \cdot B + C)(x) \) is a closed convex subset of \( S \) for each \( x \in S \).

Let \( x \in S \) and \( y \in (A \cdot B + C)(x) \) such that
\[
y(t) = T_1(t, x(t))(q(t) + \int_0^T k(t, s)\zeta(s)\,ds) + T_2(t, x(t)), \tag{5.6}
\]
where \( \zeta : J \rightarrow X \) is a Pettis integrable mapping with \( \zeta(s) \in H(s, x(s)) \). From the Hann-Banach’s theorem, it follows that
\[
\|y(t)\| \leq \|T_1(t, x(t))\| \|q(t) + \int_0^T k(t, s)\zeta(s)\,ds\| + \|T_2(t, x(t))\|
\]
\[
\leq \|T_1(t, x(t))\| \left(\|q(t)\| + \int_0^T |k(t, s)| \|\zeta(s)\|\,ds\right) + \|T_2(t, x(t))\|
\]
\[
\leq \|\vartheta_1\|_\infty \|R\| \left(\|q\|_\infty + \|k(t, \cdot)\|_\infty \|h\|_{L_1} R\right) + 2\|\vartheta_2\|_\infty R + \|T_2(0, x(0))\|
\]
Then, from our assumptions \( \|y\|_\infty \leq R \). Now, let \( x \in S \) and \( t, \tau \in J \). Then, from equality (5.6) it follows that
\[
\|y(t) - y(\tau)\| \leq \|A(x(t))(q(t) + \int_0^T k(t, s)\zeta(s)\,ds) - A(x(\tau))(q(\tau) + \int_0^T k(\tau, s)\zeta(s)\,ds)\|
\]
\[
+ \|T_2(t, x(t)) - T_2(\tau, x(\tau))\|
\]
\[
\leq \|A(x(t))(q(t) + \int_0^T k(t, s)\zeta(s)\,ds) - A(x(\tau))(q(\tau) + \int_0^T k(\tau, s)\zeta(s)\,ds)\|
\]
\[
+ \|T_2(t, x(t)) - T_2(\tau, x(\tau))\|
\]
\[
\leq \|A(x(\tau)) - A(x(t))\| \left(\|q(t) + \int_0^T k(t, s)\zeta(s)\,ds\| + \|q(\tau) + \int_0^T k(\tau, s)\zeta(s)\,ds\|\right)
\]
\[
+ \|C(x(t)) - C(x(\tau))\|
\]
\[
\leq \sum_{i=1}^3 J_i(t, \tau).
\]
We have,
\[
J_1(t, \tau) = \|A(x(t)) - A(x(\tau))\| \left\| (q(t) + \int_0^t k(t, s)\xi(s) \, ds) \right\|
\leq \delta_1 \left( |\vartheta_1(t) - x(t)| + |\vartheta_1(t) - \vartheta_1(\tau)| \|x(\tau)\| \right)
\leq \delta_1 \left[ \|\vartheta_1\|_{\text{sup}} b(t, \tau) + |\vartheta_1(t) - \vartheta_1(\tau)| R \right].
\]

Proceeding as in the steep 1, we obtain
\[
J_2(t, \tau) = \|A(x(\tau))\| \left\| (q(t) + \int_0^t k(t, s)\xi(s) \, ds) - (q(t) + \int_0^\tau k(t, s)\xi(s) \, ds) \right\|
\leq \delta_2 \left[ \|q(t) - q(\tau)\| + R \sup_{s \in J} |k(t, s) - k(\tau, s)| \|h\|_{L^1} + R \int_0^\tau |k(\tau, s)| h(s) \, ds \right].
\]
Moreover, we have
\[
J_3(t, \tau) = \|T_2(t, x(t)) - T_2(\tau, x(\tau))\|
\leq R |\vartheta_2(t) - \vartheta_2(\tau)|.
\]
Accordingly, we have
\[
\|y(t) - y(\tau)\| \leq \delta_1 \left[ \|\vartheta_1\|_{\text{sup}} b(t, \tau) + |\vartheta_1(t) - \vartheta_1(\tau)| R \right] + R |\vartheta_2(t) - \vartheta_2(\tau)|
+ \delta_2 \left[ \|q(t) - q(\tau)\| + R \sup_{s \in J} |k(t, s) - k(\tau, s)| \|h\|_{L^1} + R \int_0^\tau |k(\tau, s)| h(s) \, ds \right].
\]
By using the equality,
\[
(1 - \delta_1 \|\vartheta_1\|_{\text{sup}}) b(t, \tau) = \delta_1 \left( |\vartheta_1(t) - \vartheta_1(\tau)| R + R |\vartheta_2(\tau) - \vartheta_2(\tau)| + \delta_2 \left[ \|q(t) - q(\tau)\| + R \sup_{s \in J} |k(t, s) - k(\tau, s)| \|h\|_{L^1} + R \int_0^\tau |k(\tau, s)| h(s) \, ds \right],
\]
we obtain
\[
\|y(t) - y(\tau)\| \leq b(t, \tau).
\]
Consequently, we have
\[
(A \cdot B + C)(x) \in \mathcal{P}(S).
\]
Next we will prove that $A \cdot B + C$ has convex values. Let $x \in S$ and let $y_1, y_2 \in Ax \cdot Bx + Cx$. Then,
\[
y_1(t) = T_1(t, x(t))(q(t) + \int_0^t k(t, s)\xi_1(s) \, ds) + T_2(t, x(t)),
\]
and
\[
y_2(t) = T_1(t, x(t))(q(t) + \int_0^t k(t, s)\xi_2(s) \, ds) + T_2(t, x(t)).
\]
where $\zeta_1, \zeta_2 : J \rightarrow X$ are two Pettis integrable mappings with $\zeta_1(s), \zeta_2(s) \in H(s, x(s))$. So, for any $\alpha \in (0, 1)$, we have

\[
\alpha y_1(t) + (1 - \alpha) y_2(t) = \alpha T_1(t, x(t)) \left( q(t) + \int_0^t k(t, s)\zeta_1(s) \, ds \right) \\
+ (1 - \alpha) T_1(t, x(t)) \left( q(t) + \int_0^t k(t, s)\zeta_2(s) \, ds \right) + T_2(t, x(t)) \\
= T_1(t, x(t)) q(t) + T_1(t, x(t)) \int_0^t k(t, s)\alpha \zeta_1(s) + (1 - \alpha) \zeta_2(s) \, ds \\
+ T_2(t, x(t)).
\]

Since $H$ has convex values, we get

\[
\alpha \zeta_1(s) + (1 - \alpha) \zeta_2(s) \in H(s, x(s)) \text{ a.e. } s \in J.
\]

From the above discussion, it is clear that $\alpha \zeta_1(\cdot) + (1 - \alpha) \zeta_2(\cdot)$ is Pettis integrable and we have

\[
\alpha y_1 + (1 - \alpha) y_2 \in Ax \cdot Bx + Cx.
\]

It remains to prove that $Ax \cdot Bx + Cx$ is closed. Let $y_n$ be a sequence of elements in $Ax \cdot Bx + Cx$ which converging to some $y \in E$. Then,

\[
y_n(t) = T_1(t, x(t)) \left( q(t) + \int_0^t k(t, s) \xi_n(s) \, ds \right) + T_2(t, x(t)),
\]

where $\xi_n : J \rightarrow X$ is a sequence of Pettis integrable mapping with $\xi_n(s) \in H(s, x(s))$. The use of Condition (C2)-(iii), yields that $H$ has relatively weakly compact values. Thus, by using the inclusion

\[
(\xi_n(s))_n \subset H(s, x(s)),
\]

it follows that $(\xi_n(s))_n$ has a convergent subsequence $(\xi_{n_k}(s))_k$ to some $\xi \in H(s, x(s))$, in view of the Eberlein-Smulian’s Theorem. From our assumptions and by using the dominated convergence theorem of Pettis we can deduce that $\xi$ is Pettis integrable and

\[
\int_0^t k(t, s)\xi_n(s) \, ds \rightarrow \int_0^t k(t, s)\xi(s) \, ds.
\]

This implies that

\[
y_n(t) \rightarrow T_1(t, x(t)) \left( q(t) + \int_0^t k(t, s)\xi(s) \, ds \right) + T_2(t, x(t)),
\]

and consequently $y \in Ax \cdot Bx + Cx$.

**Step 6:** $\|B(S)\| \Phi_A(r) + \|A(S)\| \Phi_B(r) + \Phi_A(r) \Phi_B(r) + \Phi_C(r) < r$ for $0 < r < \|S\|$.

First let’s note that $A(S)$ and $B(S)$ are bounded subsets with bounds $\|v_1\| \|S\|$ and $\|q\| \|s\| + \|k(\cdot, \cdot)\| \|h\| \|L\| \|S\|$. From the above steeps, we have $\Phi_A(r) = \|v_1\| \|r\|$, $\Phi_B(r) = \|k(\cdot, \cdot)\| \|\Phi(r)\|$ and $\Phi_C(r) = \Psi(r)$, respectively. From the estimate (5.3) and since $S \subset B_E(R)$ we obtain

\[
\left[ \|q\| \|r\| + \|k(\cdot, \cdot)\| \|h\| \|L\| \|S\| \right] \|v_1\| \|r\| + \|v_1\| \|S\| \|k(\cdot, \cdot)\| \|\Phi(r)\| \\
+ \|v_1\| \|\Phi(r)\| \|\Psi(r)\| < r,
\]

for all $0 < r \leq \|S\|$.

\[\Box\]
As an application, we discuss existence results for the following nonlinear functional differential inclusion (FDI):

\[
\left( \frac{x(t) - T_2(t, x(t))}{T_1(t, x(t))} \right)' \in k(t) H(t, x(t)), \quad t \in J, \tag{5.7}
\]

satisfying the initial condition

\[
x(0) = x_0 \in X, \tag{5.8}
\]

where \( J = [0, 1] \), \( k : J \rightarrow \mathbb{R}, T_1 : J \times X \rightarrow X \) and \( H : J \times X \rightarrow \mathcal{P}_{cp,cv}(X) \). By a solution of the problem (5.7), (5.8) we mean a function \( x \in C(J, X) \) that satisfies the inclusions (5.7), (5.8) on \( J \). Our existence result for the (FDI) (5.7), (5.8) is

**Theorem 5.3** Suppose that the conditions (C1)–(C3) hold. Further if there exists a real number \( R > 0 \) such that the inequalities (5.2) and (5.3) hold with \( \|q\|_{\infty} = \frac{\|x_0 - T_2(0, x_0)\|}{\|T_1(0, x_0)\|} \), then the (FDI) (5.7), (5.8) has a solution \( x \) in \( \mathcal{B}_R \).

**Proof** Notice first that the (FDI) (5.7), (5.8) can be written as a fixed point problem

\[
\begin{cases}
  x(t) \in T_1(t, x(t)) \cdot \left[ \frac{x_0 - T_2(0, x_0)}{T_1(0, x_0)} + \int_0^t k(s) H(s, x(s)) \, ds \right] + T_2(t, x(t)), & t \in J, \\
  x(0) = x_0.
\end{cases}
\]

Now the desired conclusion of the theorem follows by a direct application of Theorem 5.3 with \( q(t) = \frac{x_0 - T_2(0, x_0)}{T_1(0, x_0)} \) and \( k(t, s) = k(s) \) for all \( t, s \in J \).

**Remark 5.1** Finally while concluding this work, we remark if \( T_1(t, x) = 1_X \) on \( J \times X \) and \( q(t) = \theta \) on \( J \), then Theorem 5.3 reduces to the existence results for the Perturbed Volterra Integral inclusion

\[
x(t) \in \int_0^t k(t, s) H(s, x(s)) \, ds + T(t, x(t)), \quad t \in J.
\]

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