Formalizing the Soundness of the Encoding Methods of SAT-based Model Checking

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Abstract—One of the effective model checking methods is to utilize the efficient decision procedure of SAT (or SMT) solvers. In a SAT-based model checking, a system and its property are encoded into a set of logic formulas and the safety is checked based on the satisfiability of the formulas. As the encoding methods are improved and crafted (e.g., k-induction and IC3/PDR), verifying their correctness becomes more important. This research aims at a formal verification of the SMC methods using the Coq proof assistant. Our contributions are twofold: (1) We specify the basic encoding methods, k-induction and (a simplified version of) IC3/PDR in Coq as a set of simple and modular encoding predicates. (2) We provide a formal proof of the soundness of the encoding methods based on our formalized lemmas on state sequences and paths. The specification of the SMC methods and the soundness proofs are available at https://github.com/dsksh/coq-smc/.

Keywords—verification, model checking, SAT, SMT, Coq.

I. INTRODUCTION

SAT-based model checking (SMC, Sect. II) is a symbolic model checking method that delegates the main reasoning/search process to efficient SAT (or SMT) solvers. An essential process in SMC is to encode a property of a state transition system into a propositional (or predicate) logic formula. Checking the satisfiability of the encoded formula entails the safety of the target system. Various SMC methods have been proposed, e.g., [1]–[6], each of which is carefully designed by e.g. unrolling of execution paths, inductions on paths, and over-approximation of states (Sect. II and III); each SMC method is typically formalized by a shallow embedding into Coq (Sect. IV-C). Also, we demonstrate that our Coq specification works as a prototype SMC tool (Sect. IV-C).

We formalize the SMC methods in a simplified manner that adapts the existing methods into a generic scheme described in Sect. II and III; each SMC method is presented as an implementation of the encoding methods, which are then formalized by a shallow embedding into Coq (Sect. IV-B). In the formalization, we explicitly specify the execution paths of a system. To facilitate this task, we develop a small yet useful theory of state sequences and paths, which contains lemmas on various properties and operations. Sect. VI discusses some characteristics of our verification results.

We consider that our result will help to provide a reliable SMC tool and serve as a formal scheme in the development of new encoding methods.

II. SAT-BASED MODEL CHECKING

This section introduces basics of SMC (Sect. II-A) and an encoding process for bounded safety (Sect. II-B).

We consider a set of states, each of which is typically encoded as a fixed-length bit vector or an integer; S denotes a set of states. A state transition system is specified by a pair \((I, T)\) of an initial condition \(I \subseteq S\) and a transition relation \(T \subseteq S \times S\). An initial path of a system is a sequence of states \(s_0 s_1 \cdots\) such that \(s_0 \in I\) and \(T(s_i, s_{i+1})\) for \(i \geq 0\). In the following, we denote the path by \(s_{[0..i]}\) and a path fragment \(s_i \cdots s_j\) by \(s_{[i..j]}\). We say a state \(s_i \in S\) is reachable iff there exists an initial path fragment \(s_{[0..i]}\). Given a property \(P \subseteq S\), we say a system \((I, T)\) is safe iff every reachable state satisfies \(P\); the safety is defined as

\[
\text{safety}(I, T, P) : \leftrightarrow \\
\forall i \in \mathbb{N}, \forall s_{[0..i]} , I(s_0) \rightarrow \text{path}_T(s_{[0..i]}) \rightarrow P(s_i),
\]

(1)

are considered explicitly in SMC. Our contributions are summarized as follows:

- We formally verify the soundness of the encoding methods; in particular, for the forward, backward, k-induction, and IC3/PDR methods (Sect. V). In the formalization, we explicitly specify the execution paths of a system. To facilitate this task, we develop a small yet useful theory of state sequences and paths, which contains lemmas on various properties and operations. Sect. VI discusses some characteristics of our verification results.

We consider that our result will help to provide a reliable SMC tool and serve as a formal scheme in the development of new encoding methods.
Algorithm 1: A generic SMC scheme.

Input : $I, T, P, k$
Output: true or false

1 $F := E_\ast(I, T, P, k)$;
2 for $(\forall s_{[0..m]}, f_i) \in F$ do
3 \hspace{1em} $F := F \land (\forall s_{[0..m]}, f_i) \mapsto \neg \text{CheckSat}(\neg f_i)$;
4 end
5 return Decide($F$);

Algorithm 2: Iterative process for unbounded MC.

Input : $I, T, P$
Output: true or false

1 for $k \geq 0$ do
2 \hspace{1em} if Alg1$_\ast(I, T, P, k)$ then
3 \hspace{2em} return true;
4 \hspace{1em} else if $\neg$Alg1$_\ast(I, T, P, k)$ then
5 \hspace{2em} return false;
6 end
7 end

where

$$\text{path}_T(s_{[0..i]}) := \left\{ \begin{array}{ll} \bigwedge_{0 \leq j < i} T(s_j, s_{j+1}) & \text{if } o < i, \\ \text{true} & \text{if } o = i. \end{array} \right.$$ (2)

Example 1 We can consider a flawed 3-bit shift register as a simple state transition system with $I(s) \iff s \geq 4$ and $T(s, s') := (2s + 1) \mod 8 = s'$, where bits are interpreted as decimal numbers. It is safe with the property $P(s) := s \neq 0$. An example initial path is $1002_2; 0012; 0112; 1112; 1112; \cdots$.

SAT-based model checking (SMC) is a formal verification method for transition systems, which exploits the efficient decision procedure of SAT (propositional satisfiability) or SMT (satisfiability modulo theories) solvers. In model checking processes, a bound $k \geq 0$ is given to restrict the depth of the search space. As a result, either (R1) a safety proof of the system, (R2) partial safety proof bounded by parameter $k$ or (R3) a counterexample of length $k$ or less will be obtained. Various SMC methods have been proposed, e.g., [2]–[3]. There exist a number of SMC tool implementations, e.g., nuXmv [8] and Kind 2 [9].

A. Generic Scheme of SMC

Given a target system $(I, T, P)$, a property $P$ and a bound $k$, an SMC procedure encodes the safety of the system into a logic formula $F$ and checks its satisfiability using a SAT/SMT solver. This procedure can be summarized as Alg. 1. Each SMC method provides their own encoding method; therefore, in the algorithm, it is parameterized as $E_\ast$ at Line 1. An encoding method generates a formula $F$ that describes a given system and a property by considering only a finite number (which depends on $k$) of states. As a result, multiple validity problems of the form $\forall s_{[0..m]}, f_i$ are generated as sub-formulas in $F$; then, they are discharged using a SAT/SMT solver, and those sub-formulas are substituted with the results, i.e., true or false (Line 3). Here, we denote the substitution of a sub-formula $G$ in $F$ with $H$ by $F'(G \mapsto H)$; the validity problem is solved by checking the unsatisfiability of the negation of $f_i$. An encoded formula $F$ can be an arbitrary logical combination of the sub-formulas, e.g., $\neg(\forall s_{[0..m]}, f_1) \land (\forall s_{[0..m]}, f_2)$ $\lor \cdots$. At Line 5, the validity of $F$ is determined, typically with a case analysis.

B. Encoding Bounded Safety

As a basic encoding method, we consider to encode a bounded safety property. For SMC with the bound $k$, we prepare $k + 1$ state variables $s_{[0..k]}$. Then, the following predicate formula states that every sequence $s_{[0..i]}$, which represents an initial path fragment of length $k$ or less, reaches a safe state.

$$E_\ast(I, T, P, k) := \bigwedge_{0 \leq i \leq k} (\forall s_{[0..i]} : I(s_0) \rightarrow \text{path}_T(s_{[0..i]}) \rightarrow P(s_i)).$$ (3)

In the above, $I, T$ and $P$ are predicates on states that specify the target system model and the property. By running Alg. 1 with $E_\ast$, which is supported by a SAT or SMT solver, we are able to verify whether the system is safe within the bound $k$ (R2) or there exists a counterexample (R3).

To use a SAT solver as the CheckSat procedure in Alg 1 the body of an encoded formula has to be (the negation of) a conjunction. Hence, the formula (3) should be modified as

$$\bigwedge_{0 \leq i \leq k} (\forall s_{[0..i]} : (I(s_0) \land \text{path}_T(s_{[0..i]}) \land \neg P(s_i))).$$ (4)

which is equivalent to (3) in classical logic.

III. UNBOUND Model CHECKING

For a SAT-based unbounded model checking, which concerns a safety for arbitrary lengths of path fragments (R1), we typically perform an iteration of encoding and satisfiability checking for $k \geq 0$. This process is illustrated as Alg. 2 which contains two calls for Alg. 1 (Lines 2 and 4) with different encoding methods for checking safety (R1) or unsafety (R3). Whenever the calls result in true or false, respectively, the process terminates. The termination depends on the completeness of the encoding methods.

This section describes the dedicated encoding processes, $k$-induction (Sect. III-A) and IC3/PDR (Sect. III-B).
A. Encoding Unbounded Safety

Encoding methods for unbounded safety that assume a set of lasso-shaped paths and apply an induction have been proposed. Sheeran et al. [2] have proposed six algorithms for this purpose, which are intended for an efficient encoding and reasoning of lasso-shaped paths. Some of the algorithms also utilize the k-induction principle for an efficient SMC.

First, we prepare a shorthand to describe that \( s_{[0..k]} \) is a loop-free path.

\[
\text{loop}_T(s_{[0..k]}) \iff \text{path}_T(s_{[0..k]}) \land \bigwedge_{0 \leq i < j \leq k} s_i \neq s_j
\]  

Then, a sufficient condition for the safety (R1) can be encoded in two methods as follows (we refer to the whole formula as (11) and to each sub-formula as (a)–(e)).

\[
\begin{align*}
E_\square(I, T, P, k) &\iff E_\square(I, T, P, k) \land \\
& \quad \left( \forall s_{[0..k]}, I(s_0) \rightarrow \neg \text{loop}_T(s_{[0..k]}) \right) \tag{6} \\
E_\square(I, T, P, k) &\iff E_\square(I, T, P, k) \land \\
& \quad \left( \forall s_{[0..k]}, \text{loop}_T(s_{[0..k]}) \rightarrow P(s_k) \right) \tag{7}
\end{align*}
\]

Either of the encoding methods \( E_\square \) and \( E_\square \) considers a fixed point of the reachability analysis from the initial states or the unsafe states, respectively. The second quantified sub-formula of \( E_\square \) checks that all the initial paths of length \( k \) contain a loop. The implication sub-formula of \( E_\square \) is equivalent to \( \neg P(s_k) \rightarrow \neg \text{loop}_T(s_{[0..k]}) \) in classical logic; it checks that all paths of length \( k \) from an unsafe state contain a loop. Since the safety within the bound \( k \) is checked with \( E_\square \), the safety R1 is inferred as a result. As for \( E_\square \), we can modify the second parts of \( E_\square \) and \( E_\square \) as follows to have the sub-formulas of a conjunctive form.

\[
\begin{align*}
\forall s_{[0..k]}, \neg(I(s_0) \land \text{loop}_T(s_{[0..k]})) &\iff \bigwedge_{0 \leq i < j \leq k} s_i \neq s_j \quad (8) \\
\forall s_{[0..k]}, \neg(\text{loop}_T(s_{[0..k]}) \land \neg P(s_k)) &\iff \bigwedge_{0 \leq i < j \leq k} s_i \neq s_j \quad (9)
\end{align*}
\]

Example 2 For the system in Ex. 1, \( E_\square \) holds iff \( k \geq 4 \) since all initial paths are safe and they reach the invariant state \( 111_2 \) before the fourth step. \( E_\square \) holds iff \( k \geq 1 \) since the unsafe state \( 000_2 \) does not have a predecessor state.

Sheeran et al.’s first algorithm [2] is a hybrid method of the two, whose encoding method is equivalent to \( E_\square \lor E_\square \). Sheeran et al.’s subsequent algorithms apply a \( k \)-induction and encode the safety as follows.

\[
E_\square(I, T, P, k) \iff E_\square(I, T, P, k) \land \\
\left( \forall s_{[0..(k+1)]}, \text{path}_T(s_{[0..(k+1)]}) \rightarrow \bigwedge_{0 \leq i \leq k} P(s_i) \rightarrow P(s_{k+1}) \right) \tag{10}
\]

To have an unbounded model checker using our scheme of Alg. 2, \( E_\square \) in Alg1_true can be implemented as either \( E_\square \), \( E_\square \), \( E_\square \), or \( E_\square \lor E_\square \), and it can be implemented as \( E_\square \) in Alg1_false.

B. IC3/PDR Method

Bradley [4] has proposed an SMC method called IC3 or PDR (property directed reachability) [5] (we refer to the method as PDR in the sequel); it handles a sequence of over-approximations that encloses path fragments with sets of clauses; also, it uses the inductive relative relation between states to refine an over-approximation. In this paper, we do not go into the details of the method but focus on the post-conditions of the PDR method, given an over-approximation. We again adapt the method into our scheme of Alg. 1 and Alg. 2.

Assume a sequence \( R_0 R_1 \cdots \) (denoted by \( R_{[0..]} \)) of over-approximations of a set of reachable states in \( S \). Then, the post-condition of PDR for the true case is described as follows (we refer to the whole formula as (11) and to each sub-formula as (a)–(e)).

\[
E_\square(I, T, P, k) \iff \exists R_{[0..]} : \left( \forall i, \forall s, I(s) \rightarrow R_i(s) \right) \land \\
\quad \left( \forall i, \forall s, R_i(s) \rightarrow P(s) \right) \land \\
\quad \left( \forall i, \forall s, R_{i+1}(s) \rightarrow R_i(s') \right) \land \\
\quad \left( \forall i, \forall s, R_i(s) \rightarrow R_{i+1}(s') \right) \land \\
\quad \left( \forall s, R_{k+1}(s) \rightarrow R_k(s) \right) \tag{11}
\]

The original algorithm initializes over-approximations as \( R_0 := I \) and \( R_i := P \) for \( i \geq 1 \), and incrementally refines \( R_{[0..(k+1)]} \) (for \( k := 0, \ldots \)). The algorithm repeatedly (i) detects a bad state \( \hat{s} \) such that \( \exists s', R_k(\hat{s}) \land T(\hat{s}, s') \land \neg P(s') \), (ii) generalizes \( \neg \hat{s} \) to a clause \( C \) such that \( \forall s', R_i(s) \land C(s) \land T(s, s') \rightarrow C(s') \) (for some \( i \leq k \)), and (iii) refines \( R_{[0..(i+1)]} \) by adding \( C \) into the over-approximations. Finally, the algorithm terminates when the sub-formula (e) holds; the resulting over-approximation \( R_k \) is an inductive strengthening of \( P \), which is depicted in Fig. 1.

Note that the above post-condition is slightly modified from that of the original algorithm for the sake of simplicity of the verification process. First, although the original algorithm maintains a set of clauses for each \( R_i \), we do not distinguish the set and \( R_i \). Second, we weaken the sub-formula (e) from \( \exists i \leq k, \forall s, R_i(s) \rightarrow R_{i+1}(s) \) to \( \forall s \geq 1 \).

Example 3 For the system in Ex. 1 we can have a set of over-approximations specified as \( R_0 := I \) and \( R_i(s) := s \geq 1 \) for \( i \geq 1 \). It satisfies \( E_\square \) iff \( k \geq 1 \).
The post-condition for the false case, which is extracted from the PDR algorithm, is described below (the sub-formulas are referred to as (a)-(c)).

\[
E_{\Box}(I,T,P,k) \triangleq (\forall s, I(s) \rightarrow P(s)) \wedge (\forall s,s', I(s) \rightarrow T(s,s') \rightarrow P(s')) \wedge (\forall R_{[0..(k-1)]}, s[0..k], I(s_0) \wedge \bigwedge_{0 \leq i < k} (R_i(s_i) \wedge T(s_i, s_{i+1})) \rightarrow P(s_k))
\]

The sub-formulas (a) and (b) are for detecting length 0 and 1 counterexamples. The sub-formula (c) describes initial and bounded safe paths of length \(k\) or less, which is related with \(R_{[0..(k-1)]}\).

IV. SPECIFICATION OF SMC ENCODING LISTS

In this work, we have realized the SMC scheme in Fig. 1 in the Coq proof assistant (Sect. IVA). We first explain how we specify MC problems and encoding methods \(E_{\Box}\) in Coq (Sect. IVB). Next, we show that an SMC is performed as a theorem proving in Coq (Sect. IV-C).

A. Coq Proof Assistant and coq2smt

Coq (version 8.6.1) is a proof assistant based on the typed lambda calculus, which supports predicate logic formulas, algebraic types, higher-order functions, etc. to describe theorems and proofs. Using the tactic mechanism, a proof can be described/performed efficiently and simply. In a part of the verification, we rely on the excluded middle axiom of the Classical_Prop module (see Sect. IV-C).

A plug-in software for Coq, coq2smt (commit 604f72a), has developed to invoke SMT solvers within a Coq proof via the tactic smt solve. The tactic handles goals of a quantifier-free form involving boolean connectives, equalities, values/variables of type \(Z\), etc.; it discharges a goal using an SMT solver (e.g. CVC4 and Z3) and reflects the result in the proof context of Coq.

B. State Transition Systems and Encoding Predicates

To specify a state transition system \((I, T)\), its property \(P\) and an over-approximation \(R\), we assume the following types based on the definitions in Sect. I

- **Definition state**: \(\text{Type := Z.}\)
- **Definition prop**: \(\text{Type := state \rightarrow Prop.}\)
- **Definition trans**: \(\text{Type := state \rightarrow state \rightarrow Prop.}\)

We specify the type \texttt{state} as \(Z\) to demonstrate a model checking using SMT solvers. To pursue the proof tasks in Sect. IV any type with equality = can be used as \texttt{state}. \(I, P\) and \(R\) are supposed to be typed as \texttt{prop}. For state sequences, we assume the following type.

- **Definition sseq**: \(\text{Type := nat \rightarrow Z.}\)

Here, we have not drawn on existing abstract types (e.g. list) because, in our proof scheme for SMC (Sect. IV-C), we intend to expand a proof goal into a flat formula, instead of an inductive reduction. For instance, the predicate \texttt{path} in Eq. (2), which requires a sequence \(SS[0..(len-o)]\) to be a path fragment, is specified recursively as follows. Indeed, this enables an expansion with an unfold application given specific argument values.

\[
\text{Fixpoint path}(T: \text{trans})(ss: \text{sseq})(o \text{ len}: \text{nat}) : \text{Prop} :=
\begin{align*}
\text{match} \text{len with} \\
\mid 0 & \Rightarrow \text{True} \\
\mid s \text{ len'} & \Rightarrow \text{path } T \text{ ss } o \text{ len'} \wedge T(s(s(o+\text{len}))(ss(s+c+len)) \\
\end{align*}
\]

Based on these types, for example, the method \(E_{\Box}\) is simply specified as the following predicate:

- **Definition forward_safety (I: prop)(T: trans)(P: prop)(k: nat): Prop := safety_k I T P k \wedge \text{lasso_fwd } I T k.\)

The predicates are defined separately for each sub-formula in Eq. (4): e.g., the predicate \texttt{lasso_fwd} is specified as

- **Definition lasso_fwd (I: prop)(T: trans)(k: nat): Prop := \forall ss: \text{sseq}. \neg(I(ss(0)) \wedge \text{loop_free } T ss 0 k).\)

In this way, these predicates are able to be shared between different encoding methods. For example, a hybrid method of \(E_{\Box}\) and \(E_{\triangledown}\) is specified as follows.

- **Definition sheeral1_safety (I: prop)(T: trans)(P: prop)(k: nat): Prop := safety_k I T P k \wedge (\text{lasso_fwd } I T k \lor \text{lasso_bwd } T P k).\)

A set of over-approximations (i.e. a sequence of \texttt{prop} values) handled by PDR is typed as the type \texttt{spsseq}.

- **Definition spseq**: \(\text{Type := nat \rightarrow Prop.}\)

Using this type, for instance the body of the sub-formula \(E_{\Box}\) is specified as

\[
\text{Fixpoint spseq_sseq(I: prop)(T: trans)(r: spseq)(ss: sseq)(i: nat): Prop :=}
\begin{align*}
\text{match } i \text{ with} \\
\mid 0 & \Rightarrow I ss[0] \\
\mid i & \Rightarrow \text{spseq_sseq } I T r ss i \wedge r+i ss[i] \wedge T ss[i] ss[i+1]
\end{align*}
\]

C. SMC as a Coq Proof Process

In this section, we demonstrate that our specification works as a simple SMC tool (except for PDR as it requires to prepare an over-approximation). To do so, a user first specifies a verification problem by defining \(I, T\) and \(P\) as objects of the types \texttt{trans} and \texttt{prop}, respectively. Then, an SMC is performed by describing a theorem and
its proof using a template script configured for each $E_*$. Type checking of this proof with Coq mimics an execution of the SMC procedure Alg. 1.

For example, a template theorem and proof for SMC with forward_safety (Eq. (3)) is described as

```
Theorem smc_example : forward_safety I T P k.
Proof.
  unfold I, T, P.
  unfold forward_safety; unfold ...
  repeat rewrite -> Nat.add_0_l; ...
  split.
  intros; smt solve; apply by_smt.
  repeat split; intros; smt solve; apply by_smt.
Qed.
```

Regarding Alg. 1 an $E_*$ method is implemented as a Coq predicate (e.g., forward_safety that implements $E_0$). $\neg$CheckSat($\neg f$) corresponds to discharging a sub-goal $f$ with smt solve, and Decide is represented as a proof template for each encoding method.

V. FORMAL VERIFICATION

We have verified the encoding methods in Sect. II and III. The correctness (including soundness, completeness and termination) is discussed rather informally in the original papers [2, 3]: our work aims at formalizing the soundness proofs with Coq, following the discussions in [2, 3]. In this paper, we do not formalize the completeness and leave it as a future work (see Sect. VIII). This section explains the process of formalization and proving.

Each encoding method represents a sufficient or necessary condition of the safety. Here, we describe the verified properties regarding Alg. 2 which returns either true or false using two encoding methods. In the following theorems, we relate the post-condition for true or false case with the safety (Eq. (1)). When the algorithm returns true using $E_*$, its soundness is described as follows.

**Theorem 1 (Soundness of True Case with $E_*$)**

\[ \forall I, T, P, k, E_*(I, T, P, k) \rightarrow safety(I, T, P). \]

Likewise, for the false case using $E_*$, its soundness is described as follows.

**Theorem 2 (Soundness of False Case with $E_*$)**

\[ \forall I, T, P, k, \neg E_*(I, T, P, k) \rightarrow \neg safety(I, T, P). \]

Note that Theorem 2 does not ensure the completeness of the true case with $E_*$, cf. $(\forall k, \neg q(k) \rightarrow \neg p)$ and $p \rightarrow (\exists k, q(k))$ are not logically equivalent.

Our verification is based on a shallow embedding of the transition systems and safety properties. The above theorems, which relate the safety and the encoded formulas $E_*$, are specified directly in the logic of Coq.

In the end, we have provided formal proofs for each of the following combinations. The proofs are explained in the following subsections.

- Theorem I with $E_0$ and with $E_0$: Sect. V-B
- Theorem I with $E_0$ (k-induction): Sect. V-C
- Theorem II with $E_0$: Sect. V-D
- Theorem II with $E_0$ (PDR): Sect. V-E
- Theorem II with $E_0$ (PDR): Sect. V-F

In the proof of Theorem I with $E_0$, we restate the consequent safety into the form that considers only loop-free paths. The following lemma is used for this deduction.

**Lemma 1** \( (\forall i, s_{[0..i]}, I(s_0) \rightarrow loopF_T(s_{[0..i]}) \rightarrow P(s_i)) \rightarrow safety(I, T, P). \)

Its proof is explained in Sect. V-G.

A. A Theory of State Sequences and Paths

For the ease of proofs, we have developed vocabularies and lemmas on state sequences (of the type $s_{seq}$) and paths ($s_{seq}$ values in which each pair of concatenated states satisfy $T$). Some of the lemmas utilized in the verification are shown in Fig. 2.

We introduce a suffix operation $\text{skip}_n$ for state sequences, which is defined as $\text{skip}_n(i, s) := s_{[(m+n..(n+i)]}$, and formalize the related properties. Using $\text{skip}_n$, we have the lemmas such as Fig. 2 (ss&sp 1–3).

Various split operations for paths, loop-free paths, etc. are useful in the verification. Therefore, we formalize those split relations as lemmas; for instance, we have lemmas Fig. 2 (ss&sp 4–6).

B. True Cases with $E_0$ and with $E_0$

Let $k$ be the parameter of the encoding method. To prove Theorem I for either $E_0$ or $E_0$, we split either of the proof tasks into two cases $i \leq k$ and $i > k$, where $i$ is that in Lemma I. We prove that the hypothesis in Lemma I holds for each case.

For the case $i \leq k$, we prove that the sub-formula $E_0(I, T, P, k)$ implies the safety within the bound $k$. We perform an induction on $k$ to show the correspondence between $E_0$, in which path is specified recursively, and the consequent specified with a universal quantifier.

For the case $i > k$, we split the path constraint in the proof context in either way of the following:

\[ I(s_0) \land \text{loopF}_T(s_{[0..k]}) \land \text{loopF}_T(s_{[k..i]}) \rightarrow P(s_i), \]
\[ I(s_0) \land \text{loopF}_T(s_{[0..(i-k)]}) \land \text{loopF}_T(s_{[i-(i-k)..i]}) \rightarrow P(s_i). \]

The split paths are illustrated in Fig. 3. The first $\text{loopF}_T$ of Eq. (13) (resp. the second $\text{loopF}_T$ of Eq. (14)) matches with the last sub-formula of Eq. (6) (resp. Eq. (7)). For Eq. (14) we utilize the $\text{skip}_n$ operator in the unification of formulas ($s_{[(i-k)..i]}$ is rewritten as $\text{skip}_n(i, s_{[0..k]})$).

With the modifications of the proof contexts, the goals are discharged by matching them with the premises.
\[ \forall s \in \text{seq}, \forall j \in \mathbb{N}, i \geq j \rightarrow P(s_i) \rightarrow P(\text{skip}(i-j, s)). \]

\[ \forall T, \forall s \in \text{seq}, \forall j, k \in \mathbb{N}, \text{path}_T(s_{[j..(j+k)]}) \rightarrow \text{path}_T(\text{skip}(j, s)_{[0..k]}). \]

\[ \forall T, \forall s \in \text{seq}, \forall j, k \in \mathbb{N}, \exists \text{loop}_T(s_{[j..(j+k)]}) \rightarrow \text{no}\_\text{loop}_T(\text{skip}(j, s)_{[0..k]}). \]

Figure 2: Example lemmas on state sequences and paths.

Figure 3: Splitting of paths.

C. True Case with k-Induction

Theorem 1 with Eq. (10) is also proved with an application of Lemma 1 and a case split into \( i \leq k \) and \( i > k \).

The case \( i \leq k \) is proved as in Sect. V-B.

For the case \( i > k \), we apply a complete induction on the step number \( i \) in the predicate \textit{safety}; we are to prove that the \( i \)-th state satisfies the property \( P \) assuming \( \text{Eq. (10)} \) and the safety within the depth \( i-1 \). The proof is done in three steps.

First, we split the constraint \( \text{loop}_T(s_{[0..i]}) \) in the proof context into \( \text{loop}_T(s_{[0..(i-k)]}) \) and \( \text{loop}_T(s_{[(i-k)..i]}) \), as in Eq. (13). Second, we modify the indices in the sub-formula of \( \text{Eq. (10)} \) with the lemma

\[ \forall i, k, k < i \rightarrow (\forall s_{[0..i]}, \text{loop}_T(s_{[0..i]}) \rightarrow \bigwedge_{0 \leq k < i} P(s_i)) \rightarrow (\forall s_{[(i-k)..i]}, \text{loop}_T(s_{[(i-k)..i]}) \rightarrow \bigwedge_{i-k \leq j < i} P(s_j)). \]

Third, \( P \) is inferred by applying the last sub-formula of \( \text{Eq. (10)} \) and the hypothesis of the complete induction.

D. False Case with PDR

Theorem 2 with Eq. (16) is proved by showing that each of the three sub-formulas of \( \text{Eq. (16)} \) holds when assuming the safety of a system.

The sub-formulas (a) and (b) are easy to prove.

Before the proof of the sub-formula \( \text{Eq. (16)} \) (c), we prepare the following lemma that states a necessary condition.

\[ \forall I, T, P, R_{[0..(k-1)]}, s_{[0..k]}, \big(I(s_0) \land \bigwedge_{0 \leq i < k} (R_i(s_i) \land T(s_i, s_{i+1})) \land \neg P(s_{i+1})\big) \rightarrow I(s_0) \land \big(\forall i, i < k \rightarrow \text{path}_T(s_{[0..(i+1)]}) \land R_i(s_i)\big). \]

Note that the hypothesis is the negation of the body of \( \text{Eq. (16)} \) (c). It is proved by an induction on \( k \); in the induction step, we prove by a case split into the cases \( k = 0, 0 < i < k \) and \( 0 < k \leq i + 1 \).

The proof context of the sub-formula \( \text{Eq. (16)} \) (c) contains the hypothesis part of Eq. (15), so it is rewritten into the consequence part of Eq. (16). The proof goal \( P(s_{k+1}) \) is discharged by the assumption of the safety, but we need to show additionally that \( s_{[0..(k+1)]} \) is a path of the system. Using the rewritten premise above, we are able to prove it.
G. Reduction to Loop Free Paths

Lemma \(\text{II}1\) is proved by a complete induction on \(i\) in safety. In the induction step, we have to show the safety of the arbitrary paths of length \(i+1\), assuming the safety within the lengths less than \(i+1\) and the safety of the loop-free paths of length \(i+1\). It is proved based on the observation that the reachable states by looped paths can be reached by a shorter path.

To compare paths with or without a loop reaching a same state, we introduce the predicate 
\[
\text{shorter}_ss(o,k,d,s,s')
\]
defined as
\[
\forall i \in \mathbb{N}, \ (i \leq k - o \rightarrow s_{k-i} = s'_{k-i}) \land s_{k+d+1} = s'_{k+i}).
\]

Fig. 4 (left) illustrates the definition. It represents the fact that a sequence \(s'\) is a shortened sequence of \(s\) where the fragment \(s[(k+1), (k+d-1)]\) is removed from \(s\); thus, two fragments of each of the two sequences coincide as 
\[
s_{[o,k]} = s'_{[o,k]} \quad \text{and} \quad s_{[(k+d)-]} = s'_{[(k+d)-]}.
\]

With this predicate, we formalize the lemma
\[
\forall i \in \mathbb{N}, \forall s \in \text{ssseq}, \left( \bigvee_{0 \leq m < n \leq i} s_m = s_n \right) \rightarrow \\
\neg \left( \forall s' \in \text{ssseq}, \forall k, d \in \mathbb{N}, \ k + d > i \lor d \leq 0 \lor \\
\text{shorter}_ss(0, k, d, s, s') \land s_i \neq s'_{i-d} \right).
\]

Note that the consequent part can be transformed into the formula below in classical logic.
\[
\exists s' \in \text{ssseq}, \exists k, d \in \mathbb{N}, \ k + d \leq i \land d > 0 \land \\
\text{shorter}_ss(0, k, d, s, s') \land s_i = s'_{i-d}
\]

Fig. 4 (right) illustrates this lemma. It states that, when a sequence \(s\) contains a loop, there exists a shortened sequence \(s'\) and the same states are reachable with the two paths. The consequent holds by matching the variables as \(k = m\) and \(d = n - m\).

To prove Lemma \(\text{II}1\) we perform a proof by contradiction: we assume that an unsafe path of length \(k\) exists when the safety is checked for loop-free paths of length \(k\). Such path should be looped; thus, we can show that there are unsafe shorter paths, which contradicts the induction hypothesis.

VI. REVIEW ON THE VERIFICATION RESULT

This section describes the statistical data (Sect. VI-A) and discussions (Sect. VI-B and VI-C) regarding the overall results of the verification with Coq described in Sect. V.
contradiction, to obtain the contraposition of Eq. (17), and to decompose the negation of the loop_free clause.

VII. RELATED WORK

As far as we know, there has been no formally verified SMC tool. Issues in the SMC tools include the possibility of flaws in encoding methods and their implementations. Our tool demonstrates that a reliable SMC is possible by directly using a verified implementation. On the other hand, lack of certification for (especially “safe”) verification results becomes another issue [9], [10].

For generic model checking, there exist several verified tools. Sprenger [11] has formalized the modal μ-calculus and a dedicated model checker in Coq. Amjad [12] has proposed to embed a symbolic model checker with its underlying BDD within the HOL tool. More recently, Esparza et al. [13] have developed more practical and verified LTL model checker, which has been specified and verified in Isabelle/HOL and then extracted as an ML implementation. Wimmer and Lammich [14] have proposed a verified model checker based on timed automata, in which model checking with abstraction of continuous states has been formalized and verified. In those verification tasks, the correctness of model checking algorithms was verified by relating the outputs of algorithms and the semantics of properties. When compared to our work, formalization with state sequences and paths that considers lengths/shifting/splitting seems specific in our work. In the previous work except [13], the formalizations were based on the set of states and paths were not considered explicitly. In [13], paths were considered but operations seemed simpler than ours.

As a slightly different line of work, there have been methods that generate proof certificates from provers e.g. SAT/SMT solvers [9], [15] and model checkers [10]. They propose to verify a result computed by the provers by verifying the generated certificates on a theorem prover. Our purpose is different from those works but integration of our work with certification might enable constructive combined SMC and deduction.

VIII. CONCLUSIONS

We specified the SMC methods and formalized their soundness proofs on Coq. We consider that our result provides an example formal proof, which is not trivial to perform on a proof assistant like Coq. The specification of the SMC methods and the soundness proofs are available at https://github.com/dsksh/coq-smc.

There are several future work directions. First, we can continue the verification task to obtain the formal correctness proof. A difficulty in the proof of completeness will be the formalization of properties such as “an infinite sequence of finite states should contain a loop.” We consider that it requires another effort in the development of the theory of paths. Second direction is to consider other SMC methods (including newly improved methods), target systems and properties, e.g. liveness properties and the k-Liveness method (e.g. [5]). Otherwise, formalization of the encoding methods based on the bounded semantics [11] will be interesting. Third, a verification and extraction of a practical SMC tool will be valuable.

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