NONLOCAL GAGLIARDO-NIRENBERG-SOBOLEV TYPE INEQUALITY

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ABSTRACT. We establish Gagliardo-Nirenberg-Sobolev type inequalities on nonlocal Sobolev spaces driven by \( p \)-Lévy integrable kernels, by imposing some appropriate growth conditions on the associated critical function. This naturally allows to devise Sobolev embeddings, as well as, compact embeddings of nonlocal Sobolev spaces into Orlicz type spaces. The Gagliardo-Nirenberg-Sobolev type inequalities, as in the classical context, turn out to have some reciprocity with Poincaré and Poincaré-Sobolev type inequalities. The classical fractional Sobolev inequality is also derived as a direct consequence.

1. INTRODUCTION

1.1. Motivation. Classical Sobolev inequalities are ubiquitous within the area of partial differential equations and calculus of variations, and have been investigated by a numerous number of mathematicians. They play crucial roles in existence theory and regularity theory. The Gagliardo-Nirenberg-Sobolev inequality, amongst many others, is certainly the most significant and influential inequalities of fractional Sobolev-Slobodeckij spaces. Our exposition aims to be as self-contained as possible. To begin with, let us introduce nonlocal Sobolev space with respect to a \( p \)-Lévy integrable kernel \( \nu \). Throughout this work the kernel \( \nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty) \), \( d \geq 1 \), is assumed to be the density of a symmetric \( p \)-Lévy measure with \( 1 \leq p < \infty \) that is \( \nu \) is symmetric, i.e., \( \nu(h) = \nu(-h) \) for \( h \in \mathbb{R}^d \setminus \{0\} \) and \( \nu \) is \( p \)-Lévy integrable, i.e.,

\[
\int_{\mathbb{R}^d} (1 \wedge |h|^p) \nu(h) \, dh < \infty. \tag{1.1}
\]

Hereafter, we write \(|h| = (h_1^2 + h_2^2 + \cdots + h_d^2)^{1/2}\) and \( a \wedge b = \min(a, b) \) for \( a, b \in \mathbb{R} \). The associated nonlocal Sobolev space is \( \mathcal{W}_p^\nu(\mathbb{R}^d) = \{ u \in L^p(\mathbb{R}^d) : |u|_{\mathcal{W}_p^\nu(\mathbb{R}^d)} < \infty \} \) where

\[
|u|_{\mathcal{W}_p^\nu(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx}{1 \wedge |x|^p} \right)^{1/p}. \tag{1.2}
\]

The space \( \mathcal{W}_p^\nu(\mathbb{R}^d) \) amounts to a Banach space under the norm

\[
\|u\|_{\mathcal{W}_p^\nu(\mathbb{R}^d)} = \left( \|u\|^p_{L^p(\mathbb{R}^d)} + |u|^{p}_p\mathcal{W}_p(\mathbb{R}^d) \right)^{1/p}.
\]

The terminology nonlocal Sobolev space to designate the space \( \mathcal{W}_p^\nu(\mathbb{R}^d) \) is justified since the latter naturally arises as the energy space associated with a nonlocal operator, which is a (non)linear \( p \)-Lévy integro-differential operator generated by \( \nu \), of the form

\[
Lu(x) := 2 \text{ p.v.} \int_{\mathbb{R}^d} |u(x) - u(y)|^{p-2} (u(x) - u(y)) \nu(x-y) \, dy, \quad (x \in \mathbb{R}^d).
\]

Indeed the \( p \)-Lévy operator \( L \) appears as the subdifferential of \( |u|_p^{\mathcal{W}_p^\nu(\mathbb{R}^d)} \) and, moreover, it is readily seen that \( \langle Lu, u \rangle = |u|_p^{\mathcal{W}_p^\nu(\mathbb{R}^d)} \) for a sufficiently smooth function \( u \). We refer interested reader

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As a result, see [Fog25], the conditions in (1.4) hold if and only if for all
\[\nu\]
\[A\text{ remarkable example of sequence } (\nu)\text{ is not fortuitous and is intrinsically related to the space } W^p_{\nu}(\mathbb{R}^d). \]
In fact, the p-Lévy integrability condition (1.1) can be self-generated from the seminorm \[\cdot \nu \in C^\infty(\mathbb{R}^d)\text{ and } p \in (0,1)\text{ if and only if } u \in C^\infty(\mathbb{R}^d)\text{ and } p \in (0,1)\text{ if and only if } W^p(\mathbb{R}^d) \subset W^p_{\nu}(\mathbb{R}^d)\text{ holds and is continuous; see Theorem 2.1. Here } W^p(\mathbb{R}^d)\text{ is the classical Sobolev space, i.e., the space of functions in } L^p(\mathbb{R}^d)\text{ whose first order distributional derivatives also lie in } L^p(\mathbb{R}^d). \text{ Recent studies regarding the function space } W^p_{\nu}(\mathbb{R}^d)\text{ and the analysis of related integro-differential equations can be found in } [Fog25,FK24].\]

The class of p-Lévy integrable kernels includes not only integrable functions, but also kernels with high singularities at the origin. A prototypical example is given by \[\nu(h) = s(1 - s)|h|^{d-1}p\text{, whereby it is readily seen that } \nu \in L^1(\mathbb{R}^d, 1 \wedge |h|p)\text{ if and only if } s \in (0,1).\]

Therefore \(W^p_{\nu}(\mathbb{R}^d)\) is the classical Sobolev space, i.e., the space of functions in \(L^p(\mathbb{R}^d)\) whose first order distributional derivatives also lie in \(L^p(\mathbb{R}^d)\). The resulting space is thus the well-known fractional Sobolev-Slobodeckij space \(W^{s,p}(\mathbb{R}^d)\) of order \(s\). Just like the latter, the nonlocal Sobolev space \(W^p_{\nu}(\mathbb{R}^d)\) also appears as a refinement space between \(L^p(\mathbb{R}^d)\) and the classical Sobolev space \(W^{1,p}(\mathbb{R}^d)\).

Furthermore, as \(s \to 1^-\), the space \(W^{s,p}(\mathbb{R}^d)\) reduces to the classical Sobolev space \(W^{1,p}(\mathbb{R}^d)\). Rigorously speaking, if \(|u|_{W^{1,p}(\mathbb{R}^d)}\) is given by (1.2) for \(\nu(h) = (1 - s)|h|^{d-1}p\), \(\nabla u\) denotes the distributional gradient of \(u\) and \(|u|_{W^{1,p}(\mathbb{R}^d)} = \|\nabla u\|_{L^p(\mathbb{R}^d)}\) is the \(L^p\)-norm of \(|\nabla u|\) then asymptotically we have

\[
\lim_{s \to 1^-} \int_{\mathbb{R}^d} (1 \wedge |h|^p) \nu_s(h) \, dh = 1 \quad \text{and for all } \delta > 0, \quad \lim_{\varepsilon \to 0^+} \int_{|h| > \delta} (1 \wedge |h|^p) \nu_s(h) \, dh = 0. \tag{1.4}
\]

As a result, see [Fog25], the conditions in (1.3) hold if and only if for all \(u \in W^{1,p}(\mathbb{R}^d)\),

\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu_s(x - y) \, dy \, dx = K_{d,p} |u|^p_{W^{1,p}(\mathbb{R}^d)}. \tag{1.5}
\]

We also refer to [Fog25,FG20,BBM01,FKV20] where the asymptotic formula (1.5) is established. A remarkable example of sequence \((\nu_s)\) generated after rescaling of a radial p-Lévy integrable function \(\nu : \mathbb{R}^d \setminus \{0\} \to (0, \infty)\) and satisfying the properties in (1.4) is defined as follows

\[
\nu_s(h) = \begin{cases} 
  \varepsilon^{-d}p \nu(h/\varepsilon) & \text{if } |h| \leq \varepsilon \\
  \varepsilon^{-d}|h|^{-p} \nu(h/\varepsilon) & \text{if } \varepsilon < |h| \leq 1 \quad \text{provided } \int_{\mathbb{R}^d} (1 \wedge |h|^p) \nu(h) \, dh = 1. \\
  \varepsilon^{-d} \nu(h) & \text{if } |h| > 1.
\end{cases}
\]

The simplest version of Gagliardo-Nirenberg-Sobolev inequality, reads as follows: for \(0 \leq s \leq 1\) and \(1 \leq p < \infty\), if the critical Sobolev exponent \(p^*_s\) also called the Sobolev conjugate of \(p\) satisfies

\[
\frac{1}{p^*_s} := \frac{1}{p} - \frac{s}{d} > 0,
\]

References to [FK24,FPS23,Fog25,DFK22] recent studies on Integro-Differential Equations (IDEs) associated with the p-Lévy operator \(L\). We emphasize that for \(p = 2\) the condition (1.1) is known in the theory of probability analysis as the Lévy condition and in this case the operator \(L\) naturally arises as the generator of a Lévy stochastic processes with pure jumps; see [Sat13] for more. Therefore terming the condition (1.1) naturally as the p-Lévy integrability condition is arguably much more congruent. It is worth noting that the p-Lévy integrability condition (1.1) is not fortuitous and is intrinsically related to the space \(W^p_{\nu}(\mathbb{R}^d)\).
then there exists a constant $C_s = C(d, s, p) > 0$ such that
\[
\left( \int_{\mathbb{R}^d} |u(x)|^{p^*_s} \, dx \right)^{1/p^*_s} \leq C_s |u|_{W^{s, p}(\mathbb{R}^d)} \quad \text{for all } u \in L^{p^*_s}(\mathbb{R}^d).
\] (1.6)

The inequality (1.6) is tautological for $s = 0$, with the convention $W^{0, p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$. The proof for $s = 1$ can be found in classical books, e.g., [AF03, Zie89, SC02]. Historically the inequality (1.6) is named as the Gagliardo-Nirenberg-Sobolev inequality because the case $s = 1$ and $1 < p < d$ is due to Sobolev [Sob33] whose original proof relies upon the boundedness of the Riesz potential from $L^p(\mathbb{R}^d)$ to $L^{p^*_s}(\mathbb{R}^d)$ whereas, Gagliardo [Gag59] and Nirenberg [Nir59] independently provided the complete picture of inequality (1.6) when $s = 1$ including the case $p = 1$ much later in a more general context of the well-known Gagliardo-Nirenberg interpolation inequality; see [FFRS21] for recent progress on this topic. For $s \in (0, 1)$, the direct proof that we present in Theorem 3.6, for the convenience of the reader, is apparently due to Haim Brezis and can be found in [Pon16, Proposition 15.5]. It is important to highlight that earlier proofs of the inequality (1.6) exist in the literature as well. For instance, a proof using basic analysis tools is well incorporated in [DNPV12, Section 6] which originally springs from [SVT11]. See also [BBM02, MS02] where the fractional inequality is established with a robust constant, i.e., with a constant $C$ such that $s \to 1$ in [SVT11] and in the case $p = 1$ in [FF60, Section 6]; see also [Zie89, Theorem 2.7.4] or [SC02] for more detailed proofs. The best constant of the fractional Gagliardo-Nirenberg-Sobolev inequality in the special case $p = 2$ and $s \in (0, 1)$ is established in [CT04]. Related Sobolev type inequalities with best constants such as log-Sobolev inequality, Hardy-Sobolev inequality or Gagliardo-Nirenberg inequality are referenced in [FS08, HT22].

### 1.2. Main goal

In view of the aforementioned classical (fractional) Gagliardo-Nirenberg-Sobolev inequality (1.6), it is reasonable to seek for an analogous inequality for the nonlocal Sobolev space $W^p_\nu(\mathbb{R}^d)$. Accordingly, we need to enforce adequate assumptions regarding the $p$-Lévy kernel $\nu$. First and foremost, for $r > 0$, consider the rearrangement radius function $\eta$ defined by
\[
\eta(r) = \left( \frac{r}{c_d} \right)^{1/d} \quad \text{with} \quad c_d = |B(0, 1)|.
\]

Indeed, for every measurable set $E \subset \mathbb{R}^d$, there holds that $|E| = |B(0, \eta(|E|))|$. Next we need introduce the potential $w : [0, \infty) \to [0, \infty)$ defined by
\[
w(r) = \left( |B(0, \eta(r))| \int_{B^c(0, \eta(r))} \nu(h) \, dh \right)^{1/p} \quad \text{equally} \quad \frac{w^p(r)}{r} = \int_{B^c(0, \eta(r))} \nu(h) \, dh. \tag{1.7}
\]

The potential $w$, which is essential in our analysis, avoids the eventual singularity of $\nu$ at the origin. For the sake of the reader’s convenience, we momentarily consider the following standing assumptions on $\nu$, that we improve later on. In what follows, the notation $\nu^{-1}$ stands for the reciprocal inverse of a bijective function $\nu$ and should not be confused with the fractional inverse $1/\nu$. In addition if $\nu$ is radial, we merely identify $\nu(|h|) = \nu(|h|)$, $h \in \mathbb{R}^d$.

#### Assumption A

The function $\nu$ satisfies the $p$-Lévy integrability condition (1.1), is radial and is almost decreasing, i.e., there is $0 < \kappa \leq 1$ such that
\[
\kappa \nu(|x|) \leq \nu(|y|) \quad \text{for all } |x| \geq |y|. \tag{A}
\]

#### Assumption B

The mapping $t \mapsto 1/w(1/t)$ is invertible from $[0, \infty)$ to $[0, \infty)$, whose inverse $\phi$ is a Young function (see below for more details) and will be called the critical Young function associated with $\nu$. To be more precise, $\phi$ is defined by
\[
\phi(t) = \left( \frac{1}{w(1/t)} \right)^{-1} \quad \text{equivalently} \quad w(t) = \frac{1}{\phi^{-1}(1/t)}. \tag{B}
\]
**Assumption C:** The function $\phi_p : [0, \infty) \rightarrow [0, \infty)$ with $\phi_p(t) = \phi(t^{1/p})$ is convex, hence a Young function, and satisfies the growth condition: there is $\theta > 0$, such that

$$\phi_p\left(\theta^p s^\frac{1}{p} t\right) \leq \frac{\phi_p(s)}{\phi_p(t)} \quad \text{for all } s \leq t. \quad (C)$$

Clearly, this is equivalent to saying that for all $s \leq t$

$$\phi\left(\theta^p s^\frac{1}{p} t\right) \leq \frac{\phi(s)}{\phi(t)} \quad \text{or equivalently} \quad \theta \leq \phi^{-1}\left(s^\frac{1}{p} t\right) \frac{\phi^{-1}(t)}{\phi^{-1}(s)}$$

Putting $s = \frac{1}{\phi^{-1}(t')}$ and $t = \frac{1}{\phi^{-1}(s')}$ yields

$$\phi^{-1}\left(s^\frac{1}{p} t\right) \geq \theta \phi^{-1}(s') \quad \text{for all } s' \leq t'. \quad (C')$$

It is worth highlighting that $\phi$ only depends on $\nu, p$ and $d$ but, to alleviate the notations, we keep this implicit. Let us now provide basic notions on Young functions and associated Orlicz spaces. A thorough and extensive study of Orlicz spaces are carried out in the seminal textbooks [RR91], [RR02]. See also the traditional references [AF03], [HH19], [KR61], [KJF77], and the monographs [RGMP16], [DIHR11], where the latter offers a treatise on generalized Orlicz spaces, also known as Musielak-Orlicz spaces, including Lebesgue and Sobolev spaces with variable exponents. Recall that a function $\phi : [0, \infty) \rightarrow [0, \infty)$ is convex if, $\phi(s + t(t-s)) \leq \phi(s) + \tau(\phi(t) - \phi(s))$ for all $s, t \geq 0$ and $\tau \in [0, 1]$. **Young function:** A convex function $\phi : [0, \infty) \rightarrow [0, \infty]$ such that $\phi(0) = 0$ is termed a Young function. Consequently as a Young function $\phi$ is nondecreasing, the mapping $t \mapsto \phi(t)$ is nondecreasing on $(0, \infty]$, and, either $\phi \equiv 0$ (i.e. $\phi$ is identically zero) or $\phi(\infty) = \infty$. Moreover, it is well known that $\phi$ is continuous on its effective domain, i.e., on the set of elements in $t \in [0, \infty)$ where $\phi(t) < \infty$. A more advanced calculus, e.g., Jensen’s theorem [RR91] Theorem 1.3.1], yields the existence of a non-decreasing and right continuous function $b : [0, \infty) \rightarrow [0, \infty)$ called the density of $\phi$, such that $\phi(t) = \int_0^t b(s) \, ds$. This implies that $\phi$ has left and right derivatives that coincide except possibly on a countable set. To avoid unnecessary pathologies, it is customary to also assume that $\phi$ is neither identically zero nor identically infinite on $(0, \infty)$.

**Convex conjugate:** To a Young function $\phi$ one associates the convex complementary, also called the convex conjugate, $\hat{\phi} : [0, \infty) \rightarrow [0, \infty)$, which is simultaneously defined as follows

$$\hat{\phi}(t) = \sup \left\{ ts - \phi(s) : s > 0 \right\} = \int_0^t b(s) \, ds.$$ 

Here $\tilde{b}(t) = \sup\{s > 0 : b(s) < t\}$ is the right inverse of $b$. Clearly, $\tilde{\phi}$ is also a Young function, i.e., convex and $\tilde{\phi}(0) = 0$. Note that, by virtue of the Fenchel-Moreau theorem, the couple $(\phi, \tilde{\phi})$ is uniquely defined provided that $\phi$ is lower semi-continuous and additionally we have

$$\phi(t) = \sup \left\{ ts - \tilde{\phi}(s) : s > 0 \right\} = \int_0^t b(s) \, ds.$$ 

Analogously, the couple $(b, \tilde{b})$ is uniquely determined and $b$ is also the right inverse of $\tilde{b}$, i.e., $b(t) = \sup\{s > 0 : \tilde{b}(s) < t\}$. Furthermore, if $b$ is strictly increasing then $\tilde{b} = b^{-1}$, the inverse of $b$.

**N-function:** A Young function $\phi : [0, \infty) \rightarrow [0, \infty]$ is called a $N$–function (Nice Young function) if its density $b : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, right continuous and satisfies $0 < b(t) < \infty$ for $t > 0$, $\lim_{t \rightarrow 0^+} b(t) = 0$ and $\lim_{t \rightarrow \infty} b(t) = \infty$. This is equivalent to saying that $\phi$ is continuous, increasing, convex and in addition the mapping $t \mapsto \phi(t)$, $t > 0$ is increasing and satisfies

$$\lim_{t \rightarrow 0^+} \frac{\phi(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{\phi(t)} = 0. \quad (1.8)$$
Orlicz space: Next, we write \( K^\phi(\mathbb{R}^d) \) and \( L^\phi(\mathbb{R}^d) \) respectively to denote the Orlicz class and the Orlicz space with respect to the Young function \( \phi \) defined by

\[
K^\phi(\mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R} \text{ meas. : } \int_{\mathbb{R}^d} \phi(|u(x)|) \, dx < \infty \right\}, \\
L^\phi(\mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R} \text{ meas. : } \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) \, dx < \infty \text{ for some } \lambda > 0 \right\}.
\]

It is worthwhile noting that the Orlicz class \( K^\phi(\mathbb{R}^d) \) is a convex set of functions and that \( L^\phi(\mathbb{R}^d) \) is the linear hull of \( K^\phi(\mathbb{R}^d) \). In addition, \( u \in L^\phi(\mathbb{R}^d) \) if and only if \( u \in \lambda K^\phi(\mathbb{R}^d) \) for some \( \lambda > 0 \). The space \( L^\phi(\mathbb{R}^d) \) is a Banach space furnished with the Luxemburg norm \( \| \cdot \|_{L^\phi(\mathbb{R}^d)} \) defined as the Minkowski functional or gauge of \( K^\phi(\mathbb{R}^d) \) by

\[
\|u\|_{L^\phi(\mathbb{R}^d)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) \, dx \leq 1 \right\}.
\]

(1.9)

Obviously, by Fatou’s lemma we have

\[
\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\|u\|_{L^\phi(\mathbb{R}^d)}}\right) \, dx \leq 1.
\]

(1.10)

Beside the Luxemburg norm \( \| \cdot \|_{L^\phi(\mathbb{R}^d)} \), we have the Orlicz norm \( | \cdot |_{L^\phi(\mathbb{R}^d)} \), with

\[
|u|_{L^\phi(\mathbb{R}^d)} = \sup \left\{ \int_{\mathbb{R}^d} u(x)v(x) \, dx : \int_{\mathbb{R}^d} \tilde{\phi}(v(x)) \, dx \leq 1 \right\}.
\]

Moreover, the following comparison holds for all \( u \in L^\phi(\mathbb{R}^d) \),

\[
\|u\|_{L^\phi(\mathbb{R}^d)} \leq |u|_{L^\phi(\mathbb{R}^d)} \leq 2\|u\|_{L^\phi(\mathbb{R}^d)}.
\]

For the Young function \( \phi_L(t) = t^{p'}/p \) with \( 1 \leq p < \infty \), the Orlicz space \( L^{\phi_L}(\mathbb{R}^d) \) coincides with the well-known Lebesgue space \( L^p(\mathbb{R}^d) \). In addition for \( p \neq 1 \), we have \( \tilde{\phi}_L(t) = t^{p'}/p' \) satisfying the relation \( pp' = p + p' \). The computation of the Luxemburg norm is not often straightforward. To illustrate this, let \( E \subset \mathbb{R}^d \) be a measurable set with finite Lebesgue measure, i.e., \( |E| < \infty \) and consider \( \mathbb{1}_E \) to be its indicator function, i.e., \( \mathbb{1}_E(x) = 1 \) if \( x \in E \) and \( \mathbb{1}_E(x) = 0 \) elsewhere, then

\[
\|\mathbb{1}_E\|_{L^\phi(\mathbb{R}^d)} = \frac{1}{\phi^{-1}(1/|E|)}.
\]

(1.11)

Indeed, for \( \lambda > 0 \) we have \( \int_{\mathbb{R}^d} \phi\left(\frac{\mathbb{1}_E(x)}{\lambda}\right) \, dx = |E|\phi\left(\frac{1}{\lambda}\right) \) which is less than 1 if and only if

\[
\frac{1}{\phi^{-1}(1/|E|)} \leq \lambda \quad \text{and hence} \quad \frac{1}{\phi^{-1}(1/|E|)} \leq \|\mathbb{1}_E\|_{L^\phi(\mathbb{R}^d)}.
\]

Therefore, the formula (1.11) holds as it suffices to choose \( \lambda = \frac{1}{\phi^{-1}(1/|E|)} \) to have the reverse inequality. In the same spirit, using Jensen’s inequality one establishes that

\[
\|\mathbb{1}_E\|_{L^\phi(\mathbb{R}^d)} = |E|\tilde{\phi}^{-1}(1/|E|).
\]

Throughout this note, we only use the Luxemburg norm \( \| \cdot \|_{L^\phi(\mathbb{R}^d)} \) defined as in (1.9). From now, we assume the Orlicz space \( L^\phi(\mathbb{R}^d) \) associated with the critical function \( \phi \) is equipped with the norm \( \| \cdot \|_{L^\phi(\mathbb{R}^d)} \). Our main result (see Theorem 1.1) for a version involving non-radial kernel reads as follows.

**Theorem 1.1.** Let Assumption A, Assumption B and Assumption C be in force. Define \( \Theta_t = t[2k^2C_p(t)\phi(\frac{t}{k})]^{-1/p} \) with \( C_p(t) = \frac{t}{t^p} \), \( t \geq 2 \). The following inequality holds

\[
\|u\|_{L^\phi(\mathbb{R}^d)} \leq \Theta_t \left( \int_{\mathbb{R}^d} \left| u(x) - u(y) \right|^p \nu(x-y) \, dy \, dx \right)^{1/p} \quad \text{for all} \quad u \in L^\phi(\mathbb{R}^d).
\]

(1.12)

Accordingly, the embeddings \( W^p_p(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d) \) and \( W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d) \) are continuous.
Remark 1.2. (i) Note that $C_p(2) > 0$ only if $p > 1$. However the case \( p = 1 \) is easily included if we take $t > 2$. The constant $\Theta_t$ depends explicitly on $p, \phi, \theta, t, \kappa$ and implicitly on $d, \nu$ since $\phi$ depends on $d, p, \nu$. Moreover, we point out that the reliance on the free variable $t \geq 2$ is purely cosmetic and solely hinges from the approach used in proving Theorem 1.1.

(ii) The inequality (1.12) is easily obtained in the particular case $u(x) = \mathbb{1}_E(x)$, with $|E| < \infty$. In fact, the relations (1.3) and (1.11) imply $\|\mathbb{1}_E\|_{L^\phi(\mathbb{R}^d)} = w(|E|)$. Thus by Lemma 3.1 below we get

$$\|\mathbb{1}_E\|_{L^\phi(\mathbb{R}^d)} = w(|E|) \leq 2^{1/p} \kappa^{-2/p} |E|^{W_p(\mathbb{R}^d)},$$

which is an analogue of the inequality (1.12) as desired.

The embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d)$ reminisces the so-called Trudinger inequality [Mos71, Tru67] which implies that, for a smooth set $\Omega \subset \mathbb{R}^d$ and the Young function $\psi(t) = e^{c \ell/(d-1)} - 1$, $p = d > 1$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^\psi(\Omega)$ is continuous; $L^\psi(\Omega)$ is the Orlicz space on $\Omega$ associated with $\psi$. A substantial amount of results appeared in the wake of [Tru67], dealing with embedding of Sobolev spaces (eventually of Orlicz-Sobolev Spaces) into Orlicz spaces. A non-exhaustive list of references on this and related topics includes [Cia96, Cia04, Cia05, Iul17, PR18, CPS20].

Of course, in accordance to the fractional Sobolev inequality (1.6), we show later in Example 2.6 that taking the fractional kernel $\nu(h) = |h|^{-d-s_p}$ with $sp < d$, which fits the requirements of Theorem 1.1, gives $\phi(t) = ct^{p_s}$ whose corresponding Orlicz space is $L^{p_s}(\mathbb{R}^d)$. See also [ACPS21] for embeddings of fractional Orlicz-Sobolev spaces into Orlicz type spaces. Let us mention that, in this particular case, the critical exponent $p_s$ (or the critical Young function $\phi(t) = ct^{p_s}$) can be anticipated using an elementary scaling argument. Unfortunately it is not possible to forecast the critical Young function $\phi$ associated with a general kernel $\nu$, using a scaling argument. For instance, if $0 < s_1 < s_2 < 1$, the kernel $\nu(h) = \max(|h|^{-d-s_1p}, |h|^{-d-s_2p})$ or $\nu(h) = \min(|h|^{-d-s_1p}, |h|^{-d-s_2p})$, does not permit the use of a scaling argument. Observe however that we obtain the critical Young function $\phi$ in a more constructive, but still less explicit, manner. The abstract aspect of the kernel $\nu$ under consideration forces a more general setting. Therefore, we will see later in Theorem 3.10 that, Theorem 1.1 still holds under weakened assumptions on $\nu$. For instance, it appears that the radiality and the decaying condition A can be dispensed.

It appears as a natural question to know if it is possible to obtain the analog of the Gagliardo-Nirenberg-Sobolev inequality for functions restricted on an open set $\Omega$ different from $\mathbb{R}^d$. The answer to this important question turns out to be strongly related to the so-called Poincaré-Sobolev type inequality. Another aim of this note, is to formulate some interplay between Gagliardo-Nirenberg-Sobolev type inequalities, Poincaré-Sobolev type inequalities and Poincaré type inequalities; see [HK00] for classical case. We summarize the global idea as follows: if $\Omega \subset \mathbb{R}^d$ is sufficiently smooth then under the assumptions of Theorem 1.1 there is $C > 0$ also depending on $\Omega$ such that,

$$\|u - f_\Omega u\|_{L^\phi(\Omega)} \leq C \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all} \quad u \in L^\phi(\Omega).$$

1.3. Outline. The rest of the paper is structured as follows. In Section 2 we comment in details our standing assumptions by explaining their needfulness, and additionally providing some illustrative examples. Section 3 is dedicated to the proof of the main result Theorem 1.1 and its generalization in Theorem 3.10 with relaxed assumptions. This gives us an opportunity to revisit fractional Gagliardo-Nirenberg-Sobolev inequality with an alternative proof. Finally, in Section 4 we establish some reciprocity relations between Gagliardo-Nirenberg-Sobolev type inequalities, Poincaré-Sobolev type inequalities and Poincaré type inequalities.

Notations: Through out, $B(x, r) := \{ y \in \mathbb{R}^d : |y - x| < r \}$ denotes the open ball with radius $r > 0$ and centered at $x \in \mathbb{R}^d$ and its closure is denoted by $\overline{B}(x, r)$. On many estimates, $C > 0$ is a generic constant depending on the local inputs.
2. Miscellaneous

In this section we discuss the aforementioned main assumptions and provide at the end, some examples of kernels. We also collect some useful basic results on Orlicz spaces needed in the sequel.

2.1. Discussion of the assumptions. Let us first comment on the aforementioned assumptions and explain their necessity.

**Assumption A:** Although, the class of radial and almost decreasing $p$-Lévy integrable kernels is fairly large, we will see later that this assumption can be dropped by the mean of the Schwartz symmetrization rearrangement, see Theorem 3.8. Nevertheless, having an almost decreasing $p$-Lévy integrable kernel, allows us to get a quicker constructive approach of the critical Young function $\phi$ as given in [11]. Beside this, the $p$-Lévy integrability condition, i.e., $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$, can neither be improved nor completely dropped. Indeed, this condition renders the space $W^p_\nu(\mathbb{R}^d)$ nontrivial, i.e., $W^p_\nu(\mathbb{R}^d) \neq \{0\}$ and more consistent, in a sense that it warrants the space $W^p_\nu(\mathbb{R}^d)$ to contain at least smooth functions of compact support. In fact, as mentioned earlier it can be shown that the $p$-Lévy integrability condition is very sharp and self-generated in the sense that $C^\infty_c(\mathbb{R}^d) \subset W^p_\nu(\mathbb{R}^d)$ if and only if $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$. This is the content of the following result whose proof can be found in [Fog25, Section 4]; see also the variant in [FK24].

**Theorem 2.1.** The following assertions are equivalent.

(i) The $p$-Lévy condition (111) holds, i.e. $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$.

(ii) The embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p_\nu(\mathbb{R}^d)$ is continuous.

(iii) $|u|_{W^p_\nu(\mathbb{R}^d)} < \infty$ for all $u \in W^{1,p}(\mathbb{R}^d)$.

(iv) $|u|_{W^p_\nu(\mathbb{R}^d)} < \infty$ for all $u \in C^\infty_c(\mathbb{R}^d)$.

(v) The space $W^p_\nu(\mathbb{R}^d)$ is nontrivial, i.e., $W^p_\nu(\mathbb{R}^d) \neq \{0\}$.

Moreover, this also remains true when $p = 1$ with $BV(\mathbb{R}^d)$ in place of $W^{1,1}(\mathbb{R}^d)$.

As illustrated by the next proposition, see details in [Fog23 Proposition 2.14] or [FG20], the $p$-Lévy integrability draws a borderline for which a space $W^p_\nu(\mathbb{R}^d)$ is trivial or not.

**Proposition 2.2.** Let $\nu : \mathbb{R}^d \to [0, \infty]$ be symmetric. The following assertions hold true.

(i) If $\nu \in L^1(\mathbb{R}^d)$ then $W^p_\nu(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ with equivalence in norm.

(ii) Define $C_\delta = \int_{B(0,\delta)} |h|^{p}(\nu(h)dh)$. If $\nu$ is radial and $C_\delta = \infty$ for all $\delta > 0$, thus $\nu \notin L^1(\mathbb{R}^d, 1 \wedge |h|^p)$, then all smooth functions contained in $W^p_\nu(\mathbb{R}^d)$ are constants.

(iii) If $\nu$ is radial then for $u \in W^{1,p}(\mathbb{R}^d)$ there is $\delta = \delta(u) > 0$ depending on $u$ such that

$$2^{-p}K_d,p C_\delta |u|_{W^{1,p}(\mathbb{R}^d)} \leq |u|_{W^p_\nu(\mathbb{R}^d)} \leq 2\|\nu\|_{L^1(\mathbb{R}^d, 1 \wedge |h|^p)}^1\|u\|_{W^{1,p}(\mathbb{R}^d)}. \quad (2.1)$$

**Warning!** The estimates in (2.1) do not imply that $W^{1,p}(\mathbb{R}^d) = W^p_\nu(\mathbb{R}^d)$.

**Assumption B:** The convexity part of the Assumption B turns out to be weaker than that of Assumption C. Indeed, the function $t \mapsto \phi_p(t) = \phi(t^{1/p})$ being a Young function and hence convex and nondecreasing implies that $\phi(t) = \phi_p(t^p)$ is also convex and hence a Young function. In fact the condition that $\phi_p$ is convex, can be viewed as a fair analog of the condition that $\frac{1}{p^d} > 0$ in the standard situation where $\nu(h) = s(1-s)|h|^{-d-sp}$. Furthermore, it is natural to require the function $\phi$ to be invertible as it rules out pathological functions. Note that, assuming $\phi$ is a Young function, one views from [13] that $\phi(0) = 0 = w(0)$ and $\phi(\infty) = \infty = w(\infty)$ and hence that $t \mapsto \phi(t)$ is invertible from $[0, \infty)$ to $[0, \infty)$ if and only if $r \mapsto w^p(r)$ is. Therefore, Assumption B is somewhat a great accessory to define the critical Young function $\phi$ and the associated Orlicz space $L^\phi(\mathbb{R}^d)$.

**Assumption C:** The Assumption C essentially constitutes the most fundamental and quite vital property needed on $\phi$ in order to establish our main result. Next, we explain how the
Assumption \( C \) globally infers certain growth conditions on \( \phi \). First of all, the growth condition \( (C) \) is clearly equivalent to saying that
\[
\phi(t^s/s) \leq \frac{\phi(s)}{\phi(t)} \quad \text{for all } s \leq t \quad \text{or equally} \quad \theta \leq \phi^{-1}(t^s/s) \quad \text{for all } s \leq t.
\]  
(2.2)

The latter suggests that the growth behavior of \( \phi \) is not far from that of a polynomial growth \( \text{[Mal85]} \); see for instance Proposition 2.3 below. This behavior can be expected, considering the inequality of interest \( (1.12) \) in Theorem 1.1. Since at the first glance, in comparison with the fractional Sobolev space \( W^{s,p}(\mathbb{R}^d) \), one can expect that the space \( W^{s,p}_v(\mathbb{R}^d) \) is embedded in another Lebesgue space. Nevertheless, it is worth noticing that fractional kernels of the form \( \nu(h) = |h|^{-d-np} \) with \( s \in (0,1) \) are the only radial functions satisfying Assumption A, Assumption B and Assumption C yielding polynomial critical Young functions of the form \( \phi(t) = ct^p \); see Example 2.6 and Theorem 3.7 below. Observe that letting \( s = rt \) with \( 0 \leq r \leq 1 \) then the condition \( (C) \) (see also (2.2)) is also to equivalent to saying that \( \phi \) satisfies sub-multiplicative condition
\[
\phi(t^s/s) \leq \phi(t^r/r) \quad \text{for all } t \geq 0 \quad \text{and} \quad \phi(r) \leq \phi(t) \quad \text{for all } t \geq 0 \quad \text{and} \quad r \in [0,1].
\]

or that \( \phi^{-1} \) satisfies the sub-multiplicative condition
\[
\theta \phi^{-1}(r) \leq \phi^{-1}(r) \quad \text{for all } t \geq 0 \quad \text{and} \quad r \in [0,1].
\]

Let us now highlight some facts about the Young function \( \phi(t) = \phi(t^{1/p}) \). Assume \( \phi \) is given as in \( [13] \) then taking \( t = \phi(t) \) for \( t > 0 \), in virtue of \( (1.7) \) and \( (1.8) \) we obtain
\[
\frac{\phi(t)}{t^p} = \frac{r}{(|\phi^{-1}(r)|)^p} = ru^p(1/r) = \int_{|h|^p \geq 1} \frac{\nu(h)}{r^p} dh.
\]  
(2.3)

In particular, if \( \lim_{t \to 0^+} \phi(t) = 0 \) and \( \lim_{t \to \infty} \phi(t) = \infty \) then
\[
\lim_{t \to \infty} \frac{\phi(t)}{t^p} = \int_{\mathbb{R}^d} \nu(h) dh \quad \text{and} \quad \lim_{t \to 0^+} \frac{\phi(t)}{t^p} = \lim_{r \to \infty} \int_{|h| \geq r} \nu(h) dh = 0.
\]  
(2.4)

The convexity of \( \phi_p(t) = \phi(t^{1/p}) \) induces certain geometry growths near 0 and near \( \infty \).

**Proposition 2.3.** Assume \( t \mapsto \phi_p(t) = \phi(t^{1/p}) \) is an invertible Young function.

(i) \( \phi \) is also a Young function.

(ii) The mappings \( t \mapsto \phi(t) \) and \( t \mapsto \frac{\phi(t)}{t^p} \) are increasing.

(iii) If \( 1 < p < \infty \) then \( \phi \) is an N-function.

(iv) Let \( \delta_0 = \frac{\phi(t_0)}{t_0^p} \) for fixed \( t_0 > 0 \) then we have
\[
\phi(t) \leq \delta_0 t^p \quad \text{if} \quad 0 \leq t \leq t_0 \quad \text{and} \quad \phi(t) \geq \delta_0 t^p \quad \text{if} \quad t \geq t_0.
\]

(v) Let \( \delta_0' = \frac{w^p(r_0)}{r_0^p} \) for fixed \( r_0 > 0 \) then we have
\[
\int_{B^c(0,r(\eta(r))]} \nu(h) dh = \frac{w^p(r)}{r} \leq \delta_0' \quad \text{if} \quad 0 \leq r \leq r_0 \quad \text{and} \quad \int_{B(0,r(\eta(r)))} \nu(h) dh = \frac{w^p(r) \eta(r)}{r} \leq \delta_0' \quad \text{if} \quad r \geq r_0.
\]

(vi) If \( \nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p) \) and \( \nu \notin L^1(\mathbb{R}^d) \) then \( \phi_p \) is an N-function, equally we have
\[
\lim_{t \to 0^+} \frac{\phi(t)}{t^p} = \lim_{t \to \infty} \frac{t^p}{\phi(t)} = 0.
\]

(vii) If \( \nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p) \) is radial, then \( r \mapsto \frac{d}{dr} \left( \frac{w^p(r)}{r} \right) \) satisfies the differential equation
\[
\nu(\eta(r)) = -\frac{d}{dr} \left( \frac{w^p(r)}{r} \right) \quad \text{and} \quad \lim_{r \to \infty} \frac{w^p(r)}{r} = 0.
\]  
(2.5)
Theorem 2.4. Let $D \subset \mathbb{R}^d$ be measurable and let $\phi_i$, $i = 1, 2$ be a pair of Young functions. If $|D| < \infty$ (resp. $|D| = \infty$) and $\phi_1(t) \leq \phi_2(ct)$ for all $t \geq t_0$ for some $c > 0$ and $t_0 > 0$ (resp. $t_0 = 0$) then the embedding $L^{\phi_2}(D) \hookrightarrow L^{\phi_1}(D)$ is continuous. The converse holds true as well.

Proof. The case $|D| = \infty$ and $t_0 = 0$ is straightforward and one has $\|u\|_{L^{\phi_1}(D)} \leq c \|u\|_{L^{\phi_2}(D)}$. Now, assume $|D| < \infty$, for $u \in L^{\phi_2}(D)$, consider $A = \{x \in D : \phi_2(u(x)) \leq c t_0 \|u\|_{L^{\phi_2}(D)}\}$ and put $T = \phi_1(t_0)|D| + 1$. Since $\phi_1(\frac{t}{T}) \leq \frac{1}{T} \phi_1(t)$ for $t > 0$, recalling (1.10), one gets

$$\int_D \frac{\phi_1(u(x))}{T \|u\|_{L^{\phi_2}(D)}} \, dx = \int_A \frac{\phi_1(u(x))}{T \|u\|_{L^{\phi_2}(D)}} \, dx + \int_{D\setminus A} \frac{\phi_1(u(x))}{T \|u\|_{L^{\phi_2}(D)}} \, dx \leq \frac{1}{T} \left( \phi_1(t_0)|A| + \int_{D\setminus A} \frac{\phi_2(u(x))}{\|u\|_{L^{\phi_2}(D)}} \, dx \right) \leq \frac{1}{T} \left( \phi_1(t_0)|D| + 1 \right) = 1.$$ 

Accordingly, this implies that $\|u\|_{L^{\phi_1}(D)} \leq c T \|u\|_{L^{\phi_2}(D)}$. Conversely, assume there is no constant $c > 0$ such that $\phi_1(t) \leq \phi_2(ct)$ for all $t > t_0 > 0$. Then one can construct an increasing sequence $0 < t_0 < \cdots < t_k < t_{k+1} \cdots$ such that $\phi_1(t_k) > \phi_2(2^k t_k)$. In particular, $\phi_2(t_k) > 2^k \phi_2(2^k t_k)$. Fix $D_0 \subset D$ such that $0 < |D_0| < \infty$ and let $D_k \subset D_0$ be disjoint measurable sets such that $|D_k| > 0$ and

$$|D_k| = \frac{\phi_2(t_k)|D_0|}{2^k \phi_2(2^k t_k)}, \quad \text{and hence} \quad \sum_{k=1}^{\infty} |D_k| < |D_0|.$$ 

We claim that the function $u = \sum_{k=1}^{\infty} k t_k \mathds{1}_{D_k}$ (which is supported in $D_0$) belongs in $L^{\phi_2}(\mathbb{R}^d)$ but not in $L^{\phi_1}(\mathbb{R}^d)$. Indeed, for any integer $n \geq 1$ we have

$$\int_D \phi_2(n u(x)) \, dx = \sum_{k=1}^{\infty} \phi_2(n k t_k) |D_k| \leq \sum_{k=1}^{n} \phi_2(n k t_k) |D_k| + \sum_{k \geq n+1} \phi_2(2^k t_k) |D_k| = \sum_{k=1}^{n} \phi_2(n k t_k) |D_k| + \phi_2(t_1) |D_0| \sum_{k \geq n+1} \frac{1}{2^k} < \infty.$$ 

Thus $u \in L^{\phi_2}(\mathbb{R}^d)$. However, for any $\varepsilon > 0$, recalling that $\phi_1(t_k) > 2^k \phi_2(2^k t_k)$ we have

$$\int_D \phi_1(\varepsilon u(x)) \, dx \geq \sum_{k \geq \frac{\varepsilon}{4^k}} \phi_1(\varepsilon k t_k) |D_k| \geq \sum_{k \geq \frac{\varepsilon}{4^k}} \phi_1(t_k) |D_k| \geq \sum_{k \geq \frac{\varepsilon}{4^k}} 2^k \phi_2(2^k t_k) |D_k| = \phi_2(t_1) |D_0| \sum_{k \geq \frac{\varepsilon}{4^k}} 1 = \infty.$$ 

This implies $u \notin L^{\phi_1}(\mathbb{R}^d)$. Hence $L^{\phi_2}(\mathbb{R}^d) \not\hookrightarrow L^{\phi_1}(\mathbb{R}^d)$. The proof is complete. \qed
Let $\phi \leq \delta_0 t^p$ for all $t \geq 1$. The claim follows from Theorem 2.4.

Let us recall without proof the characterization of the intersection and the sum of Orlicz spaces, see [RGMP16]. Let $\phi_i, \ i = 1, 2$ be a pair of Young functions. The function $\phi_{\max}(t) = \max(\phi_1(t), \phi_2(t))$ is a Young function and $L^{\phi_1}(\mathbb{R}^d) \cap L^{\phi_2}(\mathbb{R}^d) = \phi_{\max}(\mathbb{R}^d)$. Moreover, if we equip $L^{\phi_1}(\mathbb{R}^d) \cap L^{\phi_2}(\mathbb{R}^d)$ with $\ell_1$ norm $\| \cdot \|_{L^{\phi_1}(\mathbb{R}^d) \cap L^{\phi_2}(\mathbb{R}^d)} = \max(\| \cdot \|_{L^{\phi_1}(\mathbb{R}^d)}, \| \cdot \|_{L^{\phi_2}(\mathbb{R}^d)})$, there holds that

$$
\frac{1}{2} \| u \|_{L^{\phi_{\max}}(\mathbb{R}^d)} \leq \| u \|_{L^{\phi_1}(\mathbb{R}^d) \cap L^{\phi_2}(\mathbb{R}^d)} \leq \| u \|_{L^{\phi_{\max}}(\mathbb{R}^d)}.
$$

(2.6)

In addition, for any other Young function $\psi$ such that $\psi(t) \leq \phi_{\max}(ct)$ for some $c > 0$, then by Theorem 2.4 the embedding $L^{\phi_1}(\mathbb{R}^d) \cap L^{\phi_2}(\mathbb{R}^d) \hookrightarrow L^{\psi}(\mathbb{R}^d)$ is continuous. Analogously, for the function $\phi_{\min}(t) = \min(\phi_1(t), \phi_2(t))$ we have $L^{\phi_1}(\mathbb{R}^d) + L^{\phi_2}(\mathbb{R}^d) = L^{\phi_{\min}}(\mathbb{R}^d)$. Note however, that $\phi_{\min}$ is not necessarily convex and thus, is identified with its greatest convex minorant $\phi_{\min}$.

$\phi_{\min}(t) = \int_0^t \frac{\phi_{\min}(s)}{s} \, ds = \int_0^t \min(\phi_1(s), \phi_2(s)) \, ds.$

So that, $\phi_{\min}(t) \leq \phi_{\min}(2t)$ and hence $L^{\phi_{\min}}(\mathbb{R}^d) = L^{\phi_{\min}}(\mathbb{R}^d)$. In what follows, we identify $\phi_{\min}$ with $\phi_{\min}$. Moreover, one can check that

$$
\frac{1}{2} \| u \|_{L^{\phi_{\min}}(\mathbb{R}^d)} \leq \| u \|_{L^{\phi_1}(\mathbb{R}^d) + L^{\phi_2}(\mathbb{R}^d)} \leq 2 \| u \|_{L^{\phi_{\min}}(\mathbb{R}^d)},
$$

where we recall that $\| \cdot \|_{L^{\phi_1}(\mathbb{R}^d) + L^{\phi_2}(\mathbb{R}^d)}$ is the natural norm on $L^{\phi_1}(\mathbb{R}^d) + L^{\phi_2}(\mathbb{R}^d)$

$$
\| u \|_{L^{\phi_1}(\mathbb{R}^d) + L^{\phi_2}(\mathbb{R}^d)} = \inf \left\{ \| u_1 \|_{L^{\phi_1}(\mathbb{R}^d)} + \| u_2 \|_{L^{\phi_2}(\mathbb{R}^d)} : u = u_1 + u_2, u_i \in L^{\phi_i}(\mathbb{R}^d) \right\}.
$$

In the particular case of Lebesgue spaces, $L^{p_1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d)$ resp. $L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d)$, $1 \leq p_1 \leq p_2$, is the Orlicz space associated with the function $t \mapsto \max(t^{p_1}, t^{p_2})$ resp. $t \mapsto \min(t^{p_1}, t^{p_2})$. Moreover, observe that we have $q \in [p_1, p_2]$ if and only if $\min(p_1, p_2) \leq t^q \leq \max(p_1, p_2)$ for all $t > 0$. Whence we deduce from Theorem 2.4 that $L^{p_1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \hookrightarrow L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d)$.

2.3. Examples of kernels. Let us provide examples for which the main inequality (1.12) holds. First of all, we deal with the standard fractional kernel $\nu(h) = |h|^{-d-sp}$, which allows us to recover the classical fraction Gagliardo-Nirenberg-Sobolev inequality.

Example 2.6. For $s \in (0, 1)$, consider the kernel $\nu(h) = |h|^{-d-sp}$, $h \neq 0$ so that $W^{s,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$. A painless computation through polar coordinates in [1, 7] and [3] yields that

$$
w(r) = \gamma_{s}^{1/p} \frac{r^{1/p_s}}{s} \frac{r^{1/p}}{p_s} \quad \text{and} \quad \phi(t) = \gamma_{s}^{p_t/p} \frac{r^{1/p_s}}{p_s},
$$

(2.7)

where, recalling $c_d = |B(0, 1)|$, we set

$$
\frac{1}{p_s} = \frac{1}{p} - \frac{s}{d} \quad \text{and} \quad \gamma_{s} = \frac{c_d^{1+p}}{sp}.
$$

Observe that $1/p_s > 0$ if and if $p_s > 1$ and, hence if and only if $\phi_p(t) = \phi(t^{1/p}) = \gamma_{s}^{p_t/p} \frac{r^{1/p_s}}{p_s}$ is convex. Moreover, for all $s, t > 0$ we have

$$
\phi_p \left( \frac{t^s}{s} \right) = \frac{\phi_p(s)}{\phi_p(t)} \quad \text{with} \quad \theta = \frac{1}{t^{1/p}}.
$$

Whence, Assumption A, Assumption B and Assumption C are fulfilled provided that $1/p_s > 0$. 

10
Example 2.7. Assume \( \nu \in L^1(\mathbb{R}^d) \) is radial and has full support so that \( \phi \) exists. The relation (2.3) implies \( \phi(t) \leq \|\nu\|_{L^1(\mathbb{R}^d)} t^p \) for all \( t > 0 \). Whence, according to Theorem 2.4 we get the continuous embedding \( L^p(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d) \) and we have
\[
\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq \|\nu\|_{L^1(\mathbb{R}^d)}^{1/p} \|u\|_{L^p(\mathbb{R}^d)} \quad \text{for all } u \in L^p(\mathbb{R}^d).
\]
Together with Corollary 2.5 we get the continuous embeddings \( L^p(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d) \hookrightarrow L^{p^*_0}(\mathbb{R}^d) \).

Example 2.8. For concrete examples, consider the Young function \( \phi^a(t) = \ln(a + e^{-t}) - \ln(a + 1) \) for all \( a > 0 \) is a fixed parameter. Note that \( \phi(t) \leq t^p \), for all \( t > 0 \). Clearly \( \phi^a_+(t) = \phi(t)^{1/p^*} = \ln(a + e^t) - \ln(a + 1) \) is also a Young function. Moreover, each \( \phi^a \) satisfies (C) with \( \theta = 1 \). Last one defines the kernel \( \nu^a \in L^1(\mathbb{R}^d) \) associated with \( \phi^a \) through the relation
\[
\nu^a(\eta(r)) = -\frac{d}{dr} \left( \frac{1}{r \xi^a(r)} \right) \text{ with } \xi^a(r) := [\phi^a_+]^{-1}(1/r) = \ln((a + 1)e^{1/r} - a).
\]
Each \( \nu^a \) fulfills Assumption A, Assumption B and Assumption C.

The next example exhibits a situation where the growth condition (C) fails but the inequality (1.12) still holds true with a possibly different constant.

Example 2.9. Fix \( 0 < s_1 < s_2 < 1 \), following the notations of Example 2.6 we define the Young function \( \phi(t) = \max(t^{p^*_1}, t^{p^*_2}) \), with \( 1/p^*_i > 0 \), \( i = 1, 2 \) so that \( L^p(\mathbb{R}^d) = L^{p^*_1}(\mathbb{R}^d) \cap L^{p^*_2}(\mathbb{R}^d) \). Clearly, \( \phi_p(t) = \phi(t^{1/p}) \) is convex since \( p^*_2 > p^*_1 > p \). Moreover the relation (13) gives
\[
u_p(r) = \frac{1}{\phi_p^{-1}(1/r)} = \max(r^{p/p^*_1}, r^{p/p^*_2}).
\]
Now, we differentiate the relation (17) and put \( \rho = \eta(r) \), equally \( r = c_d \rho^{d^*} \) to obtain that
\[
\nu(\rho) = \nu(\eta(r)) = -\frac{d}{d\rho} \left( \frac{\nu_p(r)}{r} \right) = \begin{cases} \frac{1}{\gamma_{s_2}} \rho^{-d-s_2p} & \text{if } \rho < \eta(1), \\ \frac{1}{\gamma_{s_1}} \rho^{-d-s_1p} & \text{if } \rho \geq \eta(1). \end{cases}
\]
Whence the kernel \( \nu \) is given by
\[
\nu(h) = \frac{1}{\gamma_{s_2}} \mathbb{1}_{B(0, \eta(1))}(h) |h|^{-d-s_2p} + \frac{1}{\gamma_{s_1}} \mathbb{1}_{B^c(0, \eta(1))}(h) |h|^{-d-s_1p}.
\]
One easily finds that \( c_1 \nu(h) \leq \max(|h|^{-d-s_1p}, |h|^{-d-s_2p}) \leq |h|^{-d-s_1p} + |h|^{-d-s_2p} \leq c_2 \nu(h) \) for some constants \( c_1, c_2 > 0 \). Note however, that the growth condition (C) cannot hold here, i.e., there is no constant \( \theta > 0 \) such that
\[
\phi(\theta s^t) \leq \phi(s) \phi(t) \quad \text{for all } s \leq t.
\]
It suffices to take \( s = 1 \) and tend \( t \to \infty \) to observe a contradiction. Nevertheless, according to Example 2.6 Theorem 1.1 applies on the kernels \( |h|^{-d-s_i p} \leq c_2 \nu(h) \), \( i = 1, 2 \) and since by (2.6), \( L^p(\mathbb{R}^d) = L^{p^*_1}(\mathbb{R}^d) \cap L^{p^*_2}(\mathbb{R}^d) \) and \( \|u\|_{L^p(\mathbb{R}^d)} \leq 2 \max(\|u\|_{L^{p^*_1}(\mathbb{R}^d)}, \|u\|_{L^{p^*_2}(\mathbb{R}^d)}) \), we get
\[
\|u\|_{L^p(\mathbb{R}^d)} \leq C(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x - y) dy dx)^{1/p} \quad \text{for all } u \in L^p(\mathbb{R}^d).
\]
In other words, inequality (1.12) still holds despite the failure of the growth condition (C).

The next example shows that the lack of convexity can sometimes be rectified.

Example 2.10. Fix \( 0 < s_1 < s_2 < 1 \), considering the notations of Example 2.6 define \( \phi(t) = \min(t^{p^*_1}, t^{p^*_2}) \), with \( p^*_i > 0 \), \( i = 1, 2 \). Note that \( \phi_p(t) = \phi(t^{1/p}) \) is not necessarily convex. However one rectifies this deficiency by defining
\[
\phi_{\min}(t) = \int_0^t \frac{\min(s^{p^*_1}, s^{p^*_2})}{s} ds \quad \text{so that } \phi_{\min}(t^{1/p}) = \frac{1}{p} \int_0^t \frac{\min(s^{p^*_1}/p, s^{p^*_2}/p)}{s} ds.
\]
Since, $p^*_s \geq p^*_1 \geq p$, one readily obtains that $t \mapsto \phi(t^{1/p})$ is convex. Furthermore,
\[
\phi(\frac{s}{t}) \leq \frac{\phi(s)}{\phi(t)} \quad \text{for all } s \leq t \quad \text{and} \quad \phi_{\min}(t) \leq \phi(t) \leq \phi_{\min}(2t) \quad \text{for all } t \geq 0.
\]

Combining altogether implies that
\[
\phi_{\min}(\frac{s}{2t}) \leq \phi(\frac{s}{2t}) \leq \frac{\phi(s/2)}{\phi(t)} \leq \frac{\phi_{\min}(s)}{\phi_{\min}(t)} \quad \text{for } s \leq t.
\]

It turns out that, $\phi_{\min}$ satisfies [C] with $\theta = \frac{1}{2}$ and thus the [Assumption C] Therefore, it is fair to identify $\phi$ with $\phi_{\min}$ so that $L^{\phi}(\mathbb{R}^d) = L^{\phi_{\min}}(\mathbb{R}^d) = L^{p^*_1}(\mathbb{R}^d) + L^{p^*_2}(\mathbb{R}^d)$. Next, we find the kernel associated with $\phi(t) = \min(t^{p^*_1}, t^{p^*_2})$. The relation [B] yields
\[
w^p(r) = \frac{1}{\phi^{-1}(1/r)} = \min(p/p^*_1, p/p^*_2).
\]

Using the relation [2.3] and put $\rho = \eta(r)$, equally $r = c_d\rho^d$ to obtain
\[
\nu(\rho) = \nu(\eta(r)) = -\frac{d}{\rho} \left(\frac{w^p(r)}{r}\right) = \begin{cases} 
\frac{1}{\gamma_1} \rho^{-d-s_2p} & \text{if } \rho \geq \eta(1), \\
\frac{1}{\gamma_2} \rho^{-d-s_1p} & \text{if } \rho < \eta(1).
\end{cases}
\]

Whence, the kernel $\nu$ is given by
\[
\nu(h) = \frac{1}{\gamma_1} \mathbb{I}_{B(0,\eta(1))}(h) |h|^{-d-s_1p} + \frac{1}{\gamma_2} \mathbb{I}_{B^c(0,\eta(1))}(h) |h|^{-d-s_2p}.
\]

One easily finds that $c_1 \nu(h) \leq \min(|h|^{-d-s_1p}, |h|^{-d-s_2p}) \leq c_2 \nu(h)$ for some constants $c_1, c_2 > 0$. Thus, identifying $\phi$ and $\phi_{\min}$ [Assumption A], [Assumption B] and [Assumption C] are satisfied.

**Remark 2.11.** There are two key geometric observations emanating from Example 2.6 and Example 2.10. Firstly, modifying the $p$-Lévy integrable kernel $\nu$ at the origin or at the infinity may not change the topology of the nonlocal Sobolev space $W^p_0(\mathbb{R}^d)$.

Secondly, the geometric behavior of the kernel $\nu$ at the origin or at the infinity truly governs that of the associated critical function $\phi$ and hence influences the topology of the Orlicz space $L^\phi(\mathbb{R}^d)$. In other words a perturbation of the kernel $\nu$ at the origin or at the infinity can drastically change the resulting associated Orlicz space (or associated critical function). This geometric behavior also reads through the relation [2.3] which implies that
\[
\lim_{t \to \infty} \frac{\phi(t)}{t^p} = \int_{|h| \geq \eta(1/\gamma_1)} \nu(h) \, dh \quad \text{and} \quad \lim_{t \to 0^+} \frac{\phi(t)}{t^p} = \int_{|h| < \eta(1/\gamma_2)} \nu(h) \, dh.
\]

### 3. Main results

With a view to establish our main result, we need auxiliary results that are the milestones to prove Theorem 1.1. We begin with the following important lemma.

**Lemma 3.1.** Assume $\nu$ is almost decreasing, i.e., satisfies [A]. The following estimate holds
\[
\text{ess inf}_{x \in \mathbb{R}^d} \int_{E^c} \nu(x - y) \, dy \geq \kappa^2 \frac{w^p(|E|)}{|E|}, \quad \forall \, E \subset \mathbb{R}^d \text{ meas., } |E| < \infty.
\]

**Proof.** Let the ball $B(0, r_E)$ centered at the origin with radius $r_E$, be the symmetric rearrangement of $E$ that is $|E| = |B(0, r_E)|$, equally $r_E = (|E|/c_d)^{1/d} = \eta(|E|)$ where $c_d = |B(0, 1)|$. Noticing that $A \setminus B = A \setminus (A \cap B)$, one gets
\[
|B(x, r_E) \setminus E| = |B(x, r_E)| - |E \cap B(x, r_E)| = |E| - |E \cap B(x, r_E)| = |E \setminus B(x, r_E)|.
\]
Accordingly, using the fact that $\nu$ is almost decreasing, we get the sought estimate as follow
\[ \int_{E^c} \nu(x-y) \, dy = \int_{E^c \cap B(x,r_E)} \nu(x-y) \, dy + \int_{E^c \cap B^c(x,r_E)} \nu(x-y) \, dy \]
\[ \geq \kappa \nu(r_E) |E^c \cap B(0,r_E)| + \int_{E^c \cap B^c(x,r_E)} \nu(x-y) \, dy \]
\[ = \kappa \nu(r_E) |E^c \cap B^c(0,r_E)| + \int_{E^c \cap B^c(x,r_E)} \nu(x-y) \, dy \]
\[ \geq \kappa^2 \int_{E^c \cap B^c(x,r_E)} \nu(x-y) \, dy + \int_{E^c \cap B^c(x,r_E)} \nu(x-y) \, dy \]
\[ \geq \kappa^2 \int_{B^c(x,r_E)} \nu(x-y) \, dy = \kappa^2 \w^p |E| / |E|. \]

We also need the following lemma dealing with convexity of the critical function $\phi$.

**Lemma 3.2.** Assume that Assumption C is satisfied. Let $(a_k)_{k \in \mathbb{Z}}$ be a nonincreasing nonnegative sequence, i.e., $0 \leq a_{k+1} \leq a_k$, and $T > 0$. Then the following estimate holds true
\[ \phi_p \left( \frac{\theta_p}{T_k} \right) \sum_{k \in \mathbb{Z}} \left( \frac{1}{\phi^{-1}(1/a_k)} \right)^p T^k \leq \sum_{k \in \mathbb{Z}} a_{k+1} a_k \left( \frac{1}{\phi^{-1}(1/a_k)} \right)^p T^k. \] (3.1)

**Proof.** First, taking $s = \phi^{-1}(1/t')$ and $t = \phi^{-1}(1/s')$ the growth condition in C becomes
\[ \phi_p^{-1} \left( \frac{s'}{t'} \right) w^p(t') \geq \theta_p w^p(s') \quad \text{for all} \quad s' \leq t'. \] (3.2)

There is no loss of generality if we assume that the right-hand side of (3.1) is finite and that, for $n \geq 1$ sufficiently large, $\lambda_n > 0$ with
\[ \lambda_n = \sum_{|k| \leq n} w^p(a_k) T^k = \sum_{k \in \mathbb{Z}} w^p(a_k) T^k \]
where here, $a'_{k} = a_{k}$ if $|k| \leq n$ and $0$ if $|k| > n$. This makes sense as $w(0) = \phi_p(0) = 0$. In virtue of the Jensen inequality and the estimate (3.2) we obtain the following
\[ \sum_{k \in \mathbb{Z}} 2^k \lambda_n w^p(a_k) T^k \geq \sum_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} w^p(a_k) T^k \geq \lambda_n \sum_{k \in \mathbb{Z}} \phi_p^{-1} \left( \frac{a'_{k+1}}{a_k} \right) \frac{1}{\lambda_n} w^p(a_k) T^k \]
\[ \geq \lambda_n \phi_p \left( \frac{\theta_p}{\lambda_n} \sum_{k \in \mathbb{Z}} w^p(a_k) T^k \right) = \phi_p \left( \frac{\theta_p}{T} \right) \sum_{|k| \leq n} w^p(a_k) T^k. \]

Letting $n \to \infty$ gives the sought inequality since $w(t) = 1 / \phi^{-1}(1/t)$ as in (3). □

In connection with Lemma 3.2, we take $\phi(t) = t^q$, $q \geq 1$ to obtain the following particular result.

**Lemma 3.3.** For a nonnegative sequence $(a_k)_{k \in \mathbb{Z}}$, $T > 0$ and $q \geq 1$ we have
\[ \sum_{k \in \mathbb{Z}} a_k^{1/q} T^k \leq T^{q} \sum_{k \in \mathbb{Z}} a_k^{1/q} T^k. \] (3.3)

**Proof.** It suffices to assume that $0 < \sum_{k \in \mathbb{Z}} a_k^{1/q} T^k < \infty$. Let the counting measure $d\mu(k) = a_k^{1/q} T^k$ so that $d\mu(k+1) = T \left( \frac{a_{k+1}}{a_k} \right)^{1/q} d\mu(k)$. Jensen’s inequality yields the sought inequality since
\[ 1 = \left( \frac{1}{(\mu(\mathbb{Z}))} \int_{\mathbb{Z}} d\mu(k) \right)^q = \left( \frac{1}{\mu(\mathbb{Z})} \int_{\mathbb{Z}} T^{(a_{k+1}/a_k)^{1/q}} d\mu(k) \right)^q \leq \frac{1}{\mu(\mathbb{Z})} \int_{\mathbb{Z}} T^q a_{k+1} a_k d\mu(k). \]
Consequently, taking \( q = p_1^*/p \geq 1 \) in Lemma 3.3 results with the following inequality; compare with [DNPV12, Lemma 6.2] or [SV11, Lemma 5].

**Corollary 3.4.** Assume \( p_1^*/p = 1/p - \frac{s}{q} > 0 \), \( s \in (0,1) \). For every \( T > 0 \) and every nonnegative sequence \((a_k)_{k \in \mathbb{Z}}\) the following estimate holds

\[
\sum_{k \in \mathbb{Z}} a_k(d_{sp}/dT)^k \leq T^{(d_{sp}/dT)} \sum_{k \in \mathbb{Z}} a_{k+1}a_{k}^{-sp/dT}k.
\]

The next lemma is an immediate consequence of the relation in (3.5) and provides an interplay between a Luxemburg norm associated with a function \( \psi \) and that of the mapping \( t \mapsto \psi(t^q) \).

**Lemma 3.5.** Let \( \psi : [0, \infty) \to [0, \infty] \) be a Young function and \( q > 0 \). Assume \( \bar{\psi}_q(t) = \psi(t^q) \) is also a Young function then \( u \in L^{\bar{\psi}_q}(\mathbb{R}^d) \) if and only if \( u^q \in L^\psi(\mathbb{R}^d) \). Moreover, we have

\[
\|u\|_{L^\psi(\mathbb{R}^d)} = \|u^q\|_{L^\psi(\mathbb{R}^d)}^{1/q}.
\]

To prove Theorem 1.1, we use the measure theoretic decomposition of functions by level sets.

**Proof of Theorem 1.1** Without loss of the generality assume that \( u \geq 0 \) and \( |u|_{W^{s,p}_p(\mathbb{R}^d)} < \infty \). For each \( k \in \mathbb{Z} \) define

\[
A_k = \{ u > 2^k \} \quad \text{and} \quad D_k = A_k \setminus A_{k+1} = \{ 2^k < u \leq 2^{k+1} \},
\]

\[
a_k = \{|u > 2^k\}| \quad \text{and} \quad d_k = |D_k| = a_k - a_{k+1}.
\]

Note that \( A_{k+1} \subset A_k \) and hence \( a_{k+1} \leq a_k \). Moreover, \( D_k \)'s are disjoints, cover \( \mathbb{R}^d \) and we get

\[
A_{k+1}^c = \bigcup_{\ell \leq k} D_\ell \quad \text{and} \quad A_k = \bigcup_{\ell \geq k} D_\ell.
\]

(4.4)

Accordingly we find that

\[
a_k = \sum_{\ell \geq k} d_\ell \quad \text{and} \quad d_k = a_k - \sum_{\ell \geq k+1} d_\ell.
\]

(4.5)

Given \( x \in D_i \) and \( y \in D_j \) with \( j \leq i - 2 \), we have \(|u(x) - u(y)| \geq 2^i - 2^{i+1} \geq 2^{i-1} \). Therefore, according to (3.4) and Lemma 3.1 one deduces the following

\[
\int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \int_{D_i \cap D_j} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx
\]

\[
\geq 2 \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}: i < j-2} \int_{D_i \cap D_j} 2^{p(i-1)} \nu(x-y) \, dy \, dx
\]

\[
= 2 \sum_{i \in \mathbb{Z}} 2^{p(i-1)} \int_{D_i} \int_{A_{i-1}^c} \nu(x-y) \, dy \, dx
\]

\[
\geq 2 \kappa^2 \sum_{i \in \mathbb{Z}} 2^{p(i-1)} d_i \frac{w^p(a_{i-1})}{a_{i-1}}.
\]

(3.6)

In short, using the relation (B) we have

\[
|u|^p_{W^{s,p}_p(\mathbb{R}^d)} \geq 2 \kappa^2 \sum_{k \in \mathbb{Z}} 2^{pk} d_{k+1} \frac{1}{a_k^{p(1/q)}}.
\]

Recalling (3.5) and that \( d_i = a_{i} - a_{i+1} \), we get

\[
|u|^p_{W^{s,p}_p(\mathbb{R}^d)} \geq 2 \kappa^2 \sum_{i \in \mathbb{Z}} 2^{p(i-1)} a_i \frac{w^p(a_{i-1})}{a_{i-1}} - 2 \kappa^2 \sum_{i \in \mathbb{Z}} \sum_{a_{i-1} \neq 0} 2^{p(i-1)} d_i \frac{w^p(a_{i-1})}{a_{i-1}}.
\]

(3.7)
Since \( d_j \leq a_j \), using Fubini’s theorem and the formula
\[
\sum_{i \leq j} c^{i-1} = \frac{c^{j-1}}{c-1} \quad \text{for } c > 1,
\]
we find that
\[
\sum_{i \in \mathbb{Z}} \sum_{a_{i-1} \neq 0, i+1 \leq j} 2^p(i-1) d_j \frac{u^p(a_{i-1})}{a_{i-1}} = \sum_{i \in \mathbb{Z}} \sum_{a_{j-1} \neq 0, i \leq j-1} 2^p(i-1) d_j \frac{u^p(a_{i-1})}{a_{i-1}}
\leq \frac{1}{2^p-1} \sum_{a_{j-1} \neq 0} 2^p(j-1) d_j \frac{u^p(a_{j-1})}{a_{j-1}}.
\]
Combining this together with (3.6) and (3.7), recalling \( \nu_t \), we emphasize that \( d_{k+1} \leq a_{k+1} \leq a_k \). In virtue of Lemma 3.2 with \( T = 2^p \) and the fact that \( \phi^{-1} \) is nondecreasing, the following estimates hold true
\[
\sum_{k \in \mathbb{Z}} 2^{p_k} a_{k+1} \left( \frac{1}{\phi^{-1}(1/a_k)} \right)^p \geq \phi\left( \frac{\theta}{2} \right) \sum_{k \in \mathbb{Z}} 2^{p_k} \left( \frac{1}{\phi^{-1}(1/a_k)} \right)^p \geq \phi\left( \frac{\theta}{2} \right) \sum_{k \in \mathbb{Z}} 2^{p_k} \left( \frac{1}{\phi^{-1}(1/d_k)} \right)^p.
\]
Finally, since \( u^p = \sum_{k \in \mathbb{Z}} u^p \mathbb{1}_{D_k} \), in view of Lemma 3.5 and the relation (1.11) we find that
\[
\| u^p \|_{L^p(\mathbb{R}^d)} = \| u^p \|_{L^p(\mathbb{R}^d)} \leq \sum_{k \in \mathbb{Z}} \| u^p \mathbb{1}_{D_k} \|_{L^p(\mathbb{R}^d)} \leq 2^p \sum_{k \in \mathbb{Z}} 2^{p_k} \left( \frac{1}{\phi^{-1}(1/d_k)} \right)^p.
\]
Merging together (3.8), (3.9) and (3.10) gives the desired inequality for \( t = 2 \)
\[
| u^p \|_{W^p(\mathbb{R}^d)} \geq \frac{1}{\Theta^p_2} \| u^p \|_{L^p(\mathbb{R}^d)}, \quad \Theta^p_2 = 2^{p-1} [\kappa^2 C_p(2) \phi\left( \frac{\theta}{2} \right)]^{-1}.
\]
More generally, using the level sets decomposition \( A_k(t) = \{ u > t^k \} \) and \( D_k(t) = \{ t^k < u \leq t^{k+1} \} \), for \( t \geq 2 \), in place of \( A_k \) and \( D_k \) and repeating with a close look at the proof of the previous case, \( t=2 \), reveals the desired estimate (1.12). It immediately follows that the embedding \( W^p(\mathbb{R}^d) \hookrightarrow L^{\phi}(\mathbb{R}^d) \) is continuous while the continuity of the embedding \( W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\phi(\mathbb{R}^d) \) stems from the continuous embedding \( W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p(\mathbb{R}^d) \); see the relation (2.11). This ends the proof of Theorem 1.1. \( \square \)

We recall that by Example 2.6, the fractional Gagliardo-Nirenberg-Sobolev inequality (1.6) is a direct consequence of Theorem 1.1 where, \( \nu(h) = |h|^{-d-sp} \) is associated with the critical Young function \( \phi(t) = t^{s/p}/p^{s/p} \), wherein, the growth factor is given by \( \theta = 1/\gamma s/p \). However, for the convenience of the reader, we provide a simple proof credited to Haim Brezis and incorporated in [Pon16] Proposition 15.5].

**Theorem 3.6** (Fractional Sobolev inequality). Let \( s \in (0,1) \) such that \( \frac{1}{p_s} = \frac{1}{p} - \frac{s}{d} > 0 \) then
\[
\| u \|_{L^{p_s}(\mathbb{R}^d)} \leq 2^{p_s/p} \| B(0,1) \|^{-1/p-s/d} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{1/p} \quad \text{for all } u \in L^{p_s}(\mathbb{R}^d).
\]
Proof. Fix $x \in \mathbb{R}^d$ and $r > 0$. Integrating the inequality $|u(x)| \leq |u(y)| + |u(x) - u(y)|$ over $y \in B(x, r)$ and using Jensen’s inequality implies

$$|u(x)| \leq \frac{1}{r^d} \int_{B(x, r)} |u(y)| dy + \frac{1}{r^d} \int_{B(x, r)} |u(x) - u(y)| dy$$

$$\leq \left( \frac{1}{r^d} \int_{B(x, r)} |u(y)|^p dy \right)^{1/p} + \left( \frac{1}{r^d} \int_{B(x, r)} |u(x) - u(y)|^p dy \right)^{1/p}$$

$$\leq \left( \frac{1}{r^d} \int_{B(x, r)} |u(y)|^p dy \right)^{1/p} + \left( \int_{B(x, r)} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy \right)^{1/p}$$

$$\leq r^{-d/p^*} |B(0, 1)|^{-1/p^*} \left( \int_{\mathbb{R}^d} |u(y)|^p dy \right)^{1/p^*} + r^s |B(0, 1)|^{-1/p} \left( \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy \right)^{1/p}.$$ 

Now we choose $r$ such that both summands of the last inequality are equal. To more be precise,

$$r(x) = r = \frac{|B(0, 1)|}{1/d-p/d} \left( \int_{\mathbb{R}^d} |u(y)|^p dy \right)^{1/p} \left( \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy \right)^{-1/d}.$$ 

Substituting this specific $r(x)$ in the preceding estimate leads to

$$|u(x)|^p \leq 2^{p^*} r^{-d}(x) |B(0, 1)|^{-1} \left( \int_{\mathbb{R}^d} |u(y)|^p dy \right)^{1/p^*} \left( \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy \right)^{-1/d}.$$ 

This implies an equivalent of the sought inequality, as integrating with respect to $x$ yields,

$$\int_{\mathbb{R}^d} |u(x)|^p dx \leq 2^{p^*} |B(0, 1)|^{-1-sp/d} \left( \int_{\mathbb{R}^d} |u(y)|^p dy \right)^{1-p/p^*} \left( \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy dx \right).$$

The next result is a sort of converse to Example 2.16. It implies that given a radial kernel $\nu$, then $\phi(t) = ct^q$ if and only if $\nu$ is a fractional kernel of the form $C|h|^{-d-sp}$, $s \in (0, 1)$.

**Theorem 3.7.** Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ be radial. Let [Assumption A], [Assumption B] and [Assumption C] be in force. Assume that $\nu$ is associated with the critical function $\phi(t) = ct^q$ for some $q, c > 0$. Then necessarily $\frac{d}{p} - \frac{1}{q} < \frac{1}{p}$ and there exists $s \in (0, 1)$, in fact, $s = \frac{d}{p} - \frac{d}{q}$, such that $\nu(h) = C_{p,q,d}|h|^{-d-sp}$ for some constant $C_{p,q,d}$ depending on $c, p, q$ and $d$.

Proof. First of all, in virtue of [Assumption C] observe that $\phi_p(t) = ct^{q/p}$ with $c > 0$ is convex if and only if $q \geq p \geq 1$. The relation (13) implies that $w(r) = c^{1/q} r^{1/q}$ and hence (1.7) amounts to

$$c^{p/q} q^{p/q - 1} = \int_{B^c(0, \eta(r))} \nu(h) |h|^d \nu(dh) = \int_{\eta(r)} \int_{\eta(r)} \nu(\tau) \tau^{-d-1} \tau' \nu(\eta(\tau')) d\tau' d\tau.$$ 

Differentiating both sides and letting $p = \eta(r) = \left(\frac{x}{r^d}\right)^{1/d}$ yields

$$\nu(\eta(r)) = c^{p/q} (1 - \frac{p}{q}) \int_{\eta(r)^d} \nu^{p/q - 2} - p^{-d+dp/q-d} C_{p,q,d} \rho^{-d-sp}$$ 

$$\text{with } s = \frac{d}{p} - \frac{d}{q} \in [0, d].$$

In short, $\nu(h) = C_{p,q,d}|h|^{-d-sp}$. Finally observe that, by [Assumption A] $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$ if and only if $s \in (0, 1)$ that is $\frac{1}{p} - \frac{1}{d} < \frac{1}{q}$ and $q > p$. This ends the proof.

**Symmetric rearrangement:** In view of generalizing Theorem 1.1 we need to bring into play the symmetric rearrangement. We want to weaken the assumptions on $\nu$, by possibly enlarging them to the class of non-radial kernels. Ultimately, we recall some essential notions of symmetric decreasing rearrangement; see for instance [Bae19, Gra14, LL01] for more details. Let $E \subset \mathbb{R}^d$ be a measurable set with $|E| < \infty$. The symmetric rearrangement of $E$ denoted $E^* = B(0, r_E)$ is the open ball
having the same volume with $E$, i.e., $r_E = \eta(|E|) = (\frac{|E|}{c_d})^{1/d}$. The symmetric rearrangement of a measurable function $u : \mathbb{R}^d \to \mathbb{R}$, is the function denoted $u^* : \mathbb{R}^d \to [0, \infty)$ and defined by

$$u^*(x) = u^*(|x|) = \int_0^\infty 1_{\{|u| > s\}}(x) \, ds = \inf\{s > 0 : |\{u| > s\}| \leq c_d|x|^d\}. \quad (3.11)$$

It is a routine to check that the identity in (3.11) holds true. Obviously the function $u^*$ is radial and radially nonincreasing, i.e., $u^*(x) \leq u^*(y)$ whenever $|x| \geq |y|$. Furthermore, $|\{u| > s\}|^* = |\{u^* > s\}|$, for all $s > 0$. This implies that $u^*$ and $u$ are equimeasurable, i.e., $|\{u| > s\}| = |\{u^* > s\}|$ for all $s > 0$ and that $u^*$ is lower semi-continuous. Next, assume $u^*(r) < \infty$, $r > 0$ and let $s_n = u^*(r) - \frac{1}{n}$, for $n \geq 1$ large. The inf characterization in (3.11) implies $s_n \notin \{s > 0 : |\{u| > s\}| \leq c_d r^d\}$. Thus, $\{|u| > u^*(r) - \frac{1}{n}\} > c_d r^d$ and hence letting $n \to \infty$ we get

$$|\{u| > u^*(r)\}| \geq c_d r^d = |B(0, r)|.$$

Since $u^*$ is radially nonincreasing, $\{|u^* > u^*(r)\} \subset B(0, r) \subset \{|u^* > u^*(r)\} \subset B(0, r) \supset \{|u| > u^*(r)\}$. Hence we get

$$|\{|u| > u^*(r)\}| = |\{u^* > u^*(r)\}| \leq |B(0, r)| = |\{u^* > u^*(r)\}|.$$

This combined with the previous inequality yields

$$|\{|u| > u^*(r)\}| \leq |B(0, r)| \leq |\{u| > u^*(r)\}|. \quad (3.12)$$

The equalities hold if $u^*$ is decreasing. The next result, compare with [JW19, Lemma 3.1], is a fair generalization of Lemma 3.1.

**Theorem 3.8.** Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ be measurable and note $\nu^*$ be its symmetric rearrangement. Let $E \subset \mathbb{R}^d$ be measurable such that $|E| < \infty$, define $r_E = \eta(|E|) = (\frac{|E|}{c_d})^{1/d}$, then

$$\text{ess inf}_{x \in \mathbb{R}^d} \int_{E^c} \nu(x - y) \, dy \geq \nu^\#(|E|), \quad \text{with} \quad \nu^\#(|E|) = \int_{\{\nu < \nu^*(r_E)\}} \nu(h) \, dh,$$

$$\text{ess sup}_{x \in \mathbb{R}^d} \int_{E} \nu(x - y) \, dy \leq \nu_{\#}(|E|), \quad \text{with} \quad \nu_{\#}(|E|) = \int_{\{\nu \geq \nu^*(r_E)\}} \nu(h) \, dh.$$ 

Moreover, $\nu^\#(|E|) \to \int_{\mathbb{R}^d} \nu(h) \, dh$ as $|E| \to 0$. If $\nu \in L^1(\mathbb{R}^d \setminus B(0, \delta))$ for some $\delta > 0$ then $\nu_{\#}(|E|) \to 0$ as $|E| \to \infty$.

**Proof.** We only prove the first inequality and the second follows analogously. Fixing $x \in \mathbb{R}^d$, it is sufficient to assume that $\int_{E^c} \nu(x - y) \, dy < \infty$. Let $E_x = x + E$ so that $|E_x| = |E|$ and hence we get

$$\int_{E^c} \nu(x - y) \, dy = \int_{E_x^c} \nu(h) \, dh + \int_{\nu^*(r_E) < \nu < \nu^*(r_E)} \nu(h) \, dh$$

$$\geq \int_{\nu < \nu^*(r_E)} \nu(h) \, dh - \nu^*(r_E)|E_x \cap \{\nu < \nu^*(r_E)\}| + \nu^*(r_E)|E_x^c \cap \{\nu \geq \nu^*(r_E)\}|$$

$$= \int_{\nu < \nu^*(r_E)} \nu(h) \, dh + \nu^*(r_E)|E_x^c \cap \{\nu \geq \nu^*(r_E)\}| - |E_x \cap \{\nu < \nu^*(r_E)\}|$$

$$\geq \int_{\nu < \nu^*(r_E)} \nu(h) \, dh.$$ 

The last inequality follows since, as inequality (3.12) implies $|\{\nu \geq \nu^*(r_E)\}| \geq |E|$, we have

$$|E_x^c \cap \{\nu \geq \nu^*(r_E)\}| - |E_x \cap \{\nu < \nu^*(r_E)\}|$$

$$= (|\{\nu \geq \nu^*(r_E)\}| - |E_x \cap \{\nu \geq \nu^*(r_E)\}|) - (|E| - |E_x \cap \{\nu \geq \nu^*(r_E)\}|)$$

$$= |\{\nu \geq \nu^*(r_E)\}| - |E| \geq |B(0, r_E)| - |E| = 0.$$

Meanwhile, $\nu^*(0) = \inf\{s > 0 : |\nu > s\} = 0 = \|\nu\|_{L^\infty(\mathbb{R}^d)}$, hence we get $\nu^\#(|E|) \to \int_{\mathbb{R}^d} \nu(h) \, dh$ as $|E| \to 0$. If $\nu \in L^1(\mathbb{R}^d \setminus B(0, \delta))$, then a convergence argument implies $\nu_{\#}(|E|) \to 0$ as $|E| \to \infty$. \hfill $\Box$
We mention in passing that Lemma 3.1 and Theorem 3.8 generalize [DNPV12, Lemma 6.1] focusing on the particular kernel, \( \nu(h) = |h|^{-d-sp} \) for \( s \in (0, 1) \). Indeed, the latter case follows by observing that \( \nu = \nu^* \). More generally, there holds the following.

**Corollary 3.9.** Let \( \nu = \frac{d+2}{d} \), \( \vartheta > 0 \). For \( \beta' > 0, 0 < \beta \leq d \) we have

\[
\text{ess inf}_{x \in \mathbb{R}^d} \int_{E^c} \frac{dy}{|x - y|^{d + \beta'}} \geq \gamma' E^{-\beta'/d} \quad \text{and} \quad \text{ess sup}_{x \in \mathbb{R}^d} \int_{E} \frac{dy}{|x - y|^{d + \beta}} \leq \gamma^d E^{\beta'/d}.
\]

We are now ready to state a refined version of Theorem 1.1 under weaker assumptions. For this, we need to redefine the potential \( w \) in (1.7) as follows

\[
w(r) = (rv^\#(r))^{1/p} = \left( |B(0, \eta(r))| \int_{\nu < \nu^*(\eta(r))} \nu(h) \, dh \right)^{1/p}, \quad \eta(r) = \left( \frac{r}{c_d} \right)^{1/d}.
\]

We emphasize that \( \nu^* \) is the symmetric rearrangement of \( \nu \).

**Theorem 3.10.** Let \( \nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty) \) be measurable. Let \( w : [0, \infty) \to [0, \infty) \) with \( w(r) = (rv^\#(r))^{1/p} \) as in (3.13). Assume that \( r \mapsto w(r) \) is invertible, \( t \mapsto \phi_p(t) = \phi(t^{1/p}) \) is a Young function where \( \phi(t) = 1/w^{-1}(1/t) \) and that there is \( \theta > 0 \) such that

\[
\phi\left( t \frac{s}{t} \right) \leq \frac{\phi(s)}{\phi(t)} \quad \text{for all} \quad 0 \leq s \leq t.
\]

Define \( \Theta_t = t[2C_p(t)\phi(\frac{t}{2})]^{-1/p} \) with \( C_p(t) = \frac{t}{2} - 2, t \geq 2 \). Then the following inequality holds

\[
\|u\|_{L^p(\mathbb{R}^d)} \leq \Theta_t \left( \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p} \quad \text{for all} \quad u \in L^\vartheta(\mathbb{R}^d).
\]

**Proof.** The proof follows exactly the lines of the proof of Theorem 1.1 as the only major change is the analog of the estimate (3.6) which easily derives from Theorem 3.8. \( \square \)

**Remark 3.11.** The assertions (i) – (vi) of Proposition 2.3 remain true for a general kernel \( \nu \) such that \( w(r) = (rv^\#(r))^{1/p} \) and \( \phi(t) = 1/w^{-1}(1/t) \) exist.

We now present two observations for which, it is possible to eschew the lack of certain assumptions of Theorem 3.10 and retrieve a similar conclusion. The first observation implies that lack of growth condition \( (\text{C}) \) may sometime not be a direct obstacle. For instance we saw in Example 2.9 our inequality remains true although \( \phi : t \mapsto \min(t^{p_{i_1}}, t^{p_{i_2}}), p_{i_2} > p_{i_1} > p \) does not satisfy the growth condition \( (\text{C}) \), but this is compensated by the fact that each \( t \mapsto t^{p_{i_1}}, i = 1, 2 \) verifies the growth condition \( (\text{C}) \). The second observation implies that one can compensate the lack of convexity. For instance, as we mentioned in Example 2.10 our inequality remains true although \( \phi : t \mapsto \min(t^{p_{i_1}}, t^{p_{i_2}}), p_{i_2} > p_{i_1} > p \) is not convex but the inequality \( (\text{1.12}) \) remains true. The lack of convexity is compensated by the minorant convex. This is the goal of the next result.

**Theorem 3.12.** Let \( \nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty) \) be measurable and \( \phi \) be the corresponding critical function. Assume there exist \( \nu_i, i = 1, 2 \) such that \( t \mapsto \phi_i(t^{1/p}) \) is convex and satisfies the growth condition: \( \phi_i(t \phi_i(\theta t^2)) \leq \phi_i(s) \) for all \( s \leq t \). Consider the following two scenarios.

(i) We have \( \phi(t) = \max(\phi_1(t), \phi_2(t)) \) and there exist \( \nu_i : \mathbb{R}^d \setminus \{0\} \to [0, \infty), i = 1, 2 \) associated with \( \phi_i \) such that \( c^{-1} \nu(h) \leq \max(\nu_1(h), \nu_2(h)) \leq c \nu(h) \) for some \( c \geq 1 \).

(ii) We have \( \phi(t) = \min(\phi_1(t), \phi_2(t)) \) which is identified with \( \phi_{\min} \),

\[
\phi_{\min}(t) = \int_0^t \frac{\min(\phi_1(s), \phi_2(s))}{s} \, ds.
\]

Then in either scenario (i) and (ii) the following inequality remains valid

\[
\|u\|_{L^p(\mathbb{R}^d)} \leq C \left( \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p} \quad \text{for all} \quad u \in L^\vartheta(\mathbb{R}^d).
\]
Proof. (i) Assume \( \phi(t) = \max(\phi_1(t), \phi_2(t)) \) so that \( \|u\|_{L^\theta(\mathbb{R}^d)} \leq 2 \max(\|u\|_{L^{\phi_1}(\mathbb{R}^d)}, \|u\|_{L^{\phi_2}(\mathbb{R}^d)}) \) by (2.6). Since Theorem 3.10 holds for each couple \((\phi_i, \nu_i), i = 1, 2\), the desired inequality follows. Indeed, for \( u \in L^{\phi}(\mathbb{R}^d) = L^{\phi_1}(\mathbb{R}^d) \cap L^{\phi_2}(\mathbb{R}^d) \),
\[
\|u\|_{L^{\phi}(\mathbb{R}^d)} \leq C \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu_1(x-y) \, dy \, dx \right)^{1/p} \\
\leq C \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p}, \quad C = \max(C_1, C_2)c_1^{1/p}.
\]

(ii) Assume \( \phi(t) = \min(\phi_1(t), \phi_2(t)) \). In virtue of Proposition 2.3 \( t \mapsto \frac{\phi(t)}{t} \), \( i = 1, 2 \) is increasing, it follows that \( t \mapsto \phi_{(t)}(t^{1/p}) \) is convex since
\[
\phi_{(t)}(t^{1/p}) = \frac{1}{p} \int_0^t \min(\phi_1(s), \phi_2(s)) \, ds.
\]
Moreover, putting \( \theta' = \min(\theta_1, \theta_2) \) one easily checks that
\[
\phi(t) \leq \frac{\phi(t)}{t} \quad \text{for all } s \leq t \quad \text{and} \quad \phi_{(t)}(t^{1/p}) \leq \phi_{(t)}(t) \leq \phi_{(t)}(2t) \quad \text{for all } t > 0.
\]
Altogether, this implies that \( \phi_{(t)} \) satisfies the growth condition (C) with \( \theta = \frac{1}{2} \min(\theta_1, \theta_2) \). Indeed,
\[
\phi_{(t)} \left( \frac{t}{2} \right) \leq \phi(s/2) \leq \phi(t) \leq \phi_{(t)}(t)
\]
for \( s \leq t \).

It turns out that Lemma 3.2 applies to \( \phi_{(t)} \) and hence, since \( \phi_{(t)}(1/r) \leq \phi_{(t)}(1/r) \leq \phi_{(t)}(2t) \), as a substitute for the inequality 3.1 one readily obtains that
\[
\phi_{(t)} \left( \frac{t}{2} \right) \sum_{k \in \mathbb{Z}} \left( \frac{1}{\phi_{(t)}(1/d_k)} \right) p T^k \leq 2^p \sum_{k \in \mathbb{Z}} \frac{d_{k+1}}{a_k} \left( \frac{1}{\phi_{(t)}(1/a_k)} \right) p T^k.
\]
Whence, a mere adaptation of the proof of Theorem 1.1 provides the desired inequality. \( \square \)

An immediate consequence of Theorem 3.10 is given by the following embeddings.

**Corollary 3.13.** Assume the assumptions of Theorem 3.10 are in force, with \( \nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p) \). Let \( \psi \) be a Young function such that \( \psi(ct) \leq \max(tp, \phi(t)) \) for all \( t > 0 \) and for some constant \( c > 0 \). The embeddings \( W^p_\psi(\mathbb{R}^d) \hookrightarrow L^\psi(\mathbb{R}^d) \) and \( W^{1,p}_\psi(\mathbb{R}^d) \hookrightarrow L^{\psi}(\mathbb{R}^d) \) are continuous.

**Proof.** Clearly, Theorem 3.10 implies \( W^p_\psi(\mathbb{R}^d) \hookrightarrow L^\psi(\mathbb{R}^d) \) and we naturally have \( W^p_\psi(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \). Hence \( W^p_\psi(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \cap L^\psi(\mathbb{R}^d) = L^{\max(p, \phi(t))}(\mathbb{R}^d) \), by Theorem 2.4. Therefore, since \( \psi(ct) \leq \max(tp, \phi(t)) \), Theorem 2.4 implies that \( W^p_\psi(\mathbb{R}^d) \hookrightarrow L^\psi(\mathbb{R}^d) \) and hence \( W^{1,p}_\psi(\mathbb{R}^d) \hookrightarrow W^p_\psi(\mathbb{R}^d) \) see the relation 2.1. \( \square \)

Since \( q \in [p, p_\ast] \) if and only if \( t^q \leq \max(tp, t^{p_\ast}) \) we deduce the following.

**Corollary 3.14.** If \( p_\ast > 0 \), \( s \in (0, 1) \), then for every \( q \in [p, p_\ast] \) the embedding \( W^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \) is continuous.

More generally, in order to capture the above embeddings in Corollary 3.13 on arbitrarily open sets, we need to introduce extension domain with respect to the kernel \( \nu \).

**Definition 3.15.** An open set \( \Omega \subset \mathbb{R}^d \) will be called an \( W^p_\nu \)-extension domain if there exist a linear operator \( E : W^p_\nu(\Omega) \to W^p_\nu(\mathbb{R}^d) \) and a constant \( C := C(\Omega, \nu, d, p) \) such that
\[Eu|_{\Omega} = u \quad \text{and} \quad \|Eu\|_{W^p_\nu(\mathbb{R}^d)} \leq C\|u\|_{W^p_\nu(\Omega)}, \quad \text{for all} \quad u \in W^p_\nu(\Omega).
\]
According to [Zho15], an open set \( \Omega \subset \mathbb{R}^d \) is an \( W^{s,p}_\nu \)-extension domain, \( s \in (0, 1) \) if and only if \( \Omega \) satisfies the measure density condition, i.e., there is \( c > 0 \) such that for every \( x \in \Omega \) and \( r > 0 \) we have \( |\Omega \cap B(x, r)| > cr^d \). For some authors this condition also means that \( \Omega \) is a \( d \)-set. Note that every Lipschitz domain is an \( W^{s,p}_\nu \)-extension domain even for \( s = 1 \); see [HKT08].
If \( \nu \) is radially almost decreasing then a bounded bi-Lipschitz domain \( \Omega \subset \mathbb{R}^d \) is an \( \mathcal{W}_p^s \)-extension domain; see [FG20] Theorem 3.78. For instance, in this context, \( \Omega = \mathbb{R}^d \), the half space \( \Omega = \mathbb{R}^d_+ \) and Balls \( \Omega = B(a,r), r > 0, a \in \mathbb{R}^d \) are \( \mathcal{W}_p^s \)-extension domains.

**Corollary 3.16.** Let the assumptions of Theorem 3.10 be in force. Assume \( \Omega \subset \mathbb{R}^d \) is an \( \mathcal{W}_p^s \)-extension domain. Let \( \psi \) be a Young function such that \( \psi(ct) \leq \max(t^p, \phi(t)) \) for all \( t > 0 \) and for some constant \( c > 0 \). Then the embedding \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^q(\Omega) \) is continuous. Moreover, if \( |\Omega| < \infty \) then \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^q(\Omega) \) is continuous when \( \psi(ct) \leq \max(t, \phi(t)) \) for all \( t \geq 1 \).

**Proof.** Clearly, Theorem 3.10 and the extension property of \( \Omega \) imply \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^{\phi}(\Omega) \) and we naturally have \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^p(\Omega) \). Hence \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^p(\Omega) \cap L^{\phi}(\Omega) = L^{\max(t, \phi(t))}(\Omega) \), where the equality is a consequence of the equivalence (2.6). This, combined with the fact that \( \psi(ct) \leq \max(t^p, \phi(t)) \) and Theorem 2.4 imply that \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^\psi(\Omega) \).

Analogously, if \( |\Omega| < \infty \) then \( L^p(\Omega) \hookrightarrow L^1(\Omega) \) so that we have \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^1(\Omega) \cap L^{\phi}(\Omega) = L^{\max(t, \phi(t))}(\Omega) \) and since \( \psi(ct) \leq \max(t^p, \phi(t)) \), \( t \geq 1 \) by Theorem 2.4 we get \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^\psi(\Omega) \). \( \square \)

Note that \( \psi(ct) \leq \max(t, \phi(t)) \) for all \( t \geq 1 \) implies that \( \psi(ct) \leq \max(t^p, \phi(t)) \) for all \( t \geq 1 \). Specializing Corollary 3.16 to the particular case \( \nu(h) = |h|^{-d-sp} \) yields the following well-known embedding result.

**Corollary 3.17.** Assume \( \Omega \subset \mathbb{R}^d \) is an \( \mathcal{W}_s^p \)-extension domain, \( s \in (0,1) \). If \( p_s^* > 0 \) then \( \mathcal{W}_s^p(\Omega) \hookrightarrow L^q(\Omega) \) is continuous for every \( q \in [p, p_s^*] \). Moreover, \( |\Omega| < \infty \) then \( \mathcal{W}_s^p(\Omega) \hookrightarrow L^q(\Omega) \) is continuous for every \( q \in [1, p_s^*] \).

**Proof.** It suffices to observe that \( q \in [p, p_s^*] \) if and only if \( t^q \leq \max(t^p, t^{p_s^*}) \) for all \( t > 0 \). The case \( |\Omega| < \infty \) follows analogously since \( q \in [1, p_s^*] \) implies \( t^q \leq \max(t^p, t^{p_s^*}) \) for all \( t \geq 1 \). \( \square \)

A natural question arises concerning the compactness of the embeddings. We begin with the following important remark.

**Remark 3.18.** It is not reasonable to expect that the embedding \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^{\phi}(\Omega) \) is compact. This assertion aligns with the well-known fact that in the classical case where \( \nu(h) = |h|^{-d-sp} \), the embedding \( \mathcal{W}_s^p(\Omega) \hookrightarrow L^p(\Omega) \) is never compact even for sufficiently smooth \( \Omega \). However, the next result devises the compact embedding into Orlicz space \( L^\psi(\Omega) \). To this end, we point out that some conditions on \( \nu \) and \( \Omega \) under which the embedding \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^p(\Omega) \) is compact are referenced for instance [FG20], Section 3.7, and [Fog25, FK24, DMT23].

**Theorem 3.19** (Compact embedding). Let the assumptions of Theorem 3.10 be in force. Let \( \Omega \subset \mathbb{R}^d \) be open and bounded. Assume the embedding \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^p(\Omega) \) is compact. Let \( \psi \) be a Young function such that, for all \( t \geq 1 \), \( \psi(ct) \leq \max(t, \phi(t)) \) (in particular \( \psi(ct) \leq \max(t^p, \phi(t)) \)) for some \( c > 0 \). Then the embedding \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^\psi(\Omega) \) is compact provided that \( \psi \) grows essentially more slowly than \( \phi \) at infinity, that is,

\[
\lim_{t \to \infty} \frac{\psi(t)}{\phi(t)} = 0.
\]

In particular if \( \Omega \) is bounded Lipschitz and \( p_s^* > 0 \), then the embedding \( \mathcal{W}_s^p(\Omega) \hookrightarrow L^q(\Omega) \) with \( q \in [1, p_s^*] \), is compact.

**Proof.** The compactness of the embedding \( \mathcal{W}_s^p(\Omega) \hookrightarrow L^q(\Omega) \) with \( q \in [1, p_s^*] \), is a consequence of the first statement since in this case \( \phi(t) = t^{p_s^*/p} p_s^* \) and \( t^q/t^{p_s^*} \xrightarrow{t \to \infty} 0 \). To prove the first statement, let \( (u_n)_n \subset \mathcal{W}_p^s(\Omega) \) be bounded sequence in \( \mathcal{W}_p^s(\Omega) \). By Corollary 3.16 the Sobolev embedding \( \mathcal{W}_p^s(\Omega) \hookrightarrow L^\psi(\Omega) \) is continuous. Moreover, taking \( \psi = \phi \), one finds that \( (u_n)_n \) is also bounded \( L^\phi(\Omega) \). By assumption, we can assume without lost of generality that there is \( u \in \mathcal{W}_p^s(\Omega) \) such that \( \|u_n - u\|_{L^\phi(\Omega)} \xrightarrow{n \to \infty} 0 \). For \( \varepsilon > 0 \), consider \( t_\varepsilon > 0 \) such that \( \psi(t) \leq \varepsilon \phi(t) \) for all \( t > t_\varepsilon \). Moreover, we know from Proposition 2.3 that \( t \mapsto \frac{\phi(t)}{t} = t^{-1} \phi(t) \) is increasing. Therefore, for all
t ∈ (0, tε) we find that
\[ \frac{\psi(t)}{t} \leq \frac{1}{c} \max \left(1, \frac{\phi(t/c)}{t/c}\right) \leq \frac{1}{c} \max \left(1, \frac{\phi(tε/c)}{(tε/c)}\right) =: C(ε). \]

Whence, we have just proved that for every ε > 0 there is C(ε) > 0 such that
\[ \psi(t) \leq εφ(t) + C(ε)t \quad \text{for all } t > 0. \]

From this one readily deduces that, for every ε > 0 there is C(ε) > 0
\[ \|u_n - u\|_{L^p(\Omega)} \leq ε\|u_n - u\|_{L^p(\Omega)} + C(ε)\|u_n - u\|_{L^1(\Omega)} \leq εM + C(ε)|\Omega|^{1/p} \|u_n - u\|_{L^p(\Omega)} \]

where we put \( M = \|u\|_{L^p(\Omega)} + \sup_{n \geq 1} \|u_n\|_{L^p(\Omega)}. \) Since \( \|u_n - u\|_{L^p(\Omega)} \xrightarrow{n \to \infty} 0, \) we get
\[ \limsup_{n \to \infty} \|u_n - u\|_{L^p(\Omega)} \leq εM. \]

Whence since ε > 0 is arbitrarily chosen, we deduce that \( \|u_n - u\|_{L^p(\Omega)} \xrightarrow{n \to \infty} 0. \)

\[ \square \]

4. Poincaré-Sobolev inequality

In this section, we wish to establish Gagliardo-Nirenberg-Sobolev inequality for functions restricted on a domain \( \Omega \subset \mathbb{R}^d. \) First and foremost, observe that the inequality
\[ \|u\|_{L^p(\Omega)} \leq C\left( \int_{\Omega} |u(x) - u(y)|^p ν(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\Omega), \]
cannot hold for an arbitrary bounded \( \Omega \subset \mathbb{R}^d. \) In fact, if u is a nonzero constant, then the right-hand side is zero but the left-hand side is not. Accordingly, we need to rule out constant functions in this context. For instance, if we replace the integrand on the left-hand side by \( u - f_\Omega u \) then it fully makes sense to think of an inequality of the form
\[ \|u - f_\Omega u\|_{L^p(\Omega)} \leq C\left( \int_{\Omega} |u(x) - u(y)|^p ν(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\Omega), \]
where \( f_\Omega u = \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx \) denotes the mean value of u over \( \Omega. \) This particular type of inequality is customarily well known as a Sobolev-Poincaré type inequality and turns out to have a strong reciprocity with the Poincaré type inequalities. To be more precise, see Theorem 4.3, the validity of Sobolev-Poincaré type inequality implies that of Poincaré type inequality and vice versa. Let us recall some Poincaré type inequalities of interest here. Another consequence of Theorem 4.3 is given by the Poincaré-Friedrichs type inequality.

**Theorem 4.1** (Poincaré type inequalities). Let \( \Omega \subset \mathbb{R}^d \) be measurable with \( |\Omega| < \infty \) and let \( ν : \mathbb{R}^d \setminus \{0\} \to [0, \infty) \) be measurable with full support.

**Poincaré-Friedrichs inequality:** Let \( L^p_\Omega(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d) : u = 0, \ a.e \ on \ \Omega^c\} \) and let \( ν^\# \) be defined as in Theorem 4.3. Letting \( C = [2ν^\#(|\Omega|)]^{-1/p}, \) there holds that
\[ \|u\|_{L^p(\Omega)} \leq C\left( \int_{\mathbb{R}^d \setminus \mathbb{R}^d} |u(x) - u(y)|^p ν(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p_\Omega(\mathbb{R}^d). \]

**Poincaré inequality:** Assume ν is radially almost decreasing, i.e. \( ν(|x|) \leq ν(|y|) \) if \( |x| \geq |y|, \) then letting \( C = [κ|\Omega| ν(R)]^{-1/p} \) where \( R = \text{diam}(\Omega) \) is the diameter of \( \Omega, \) we have
\[ \|u - f_\Omega u\|_{L^p(\Omega)} \leq C\left( \int_{\Omega} |u(x) - u(y)|^p ν(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\Omega). \]
Theorem 4.3 yields the Poincaré-Friedrichs inequality as follows
\[
|u|_{W^p_c(\mathbb{R}^d)}^p \geq 2 \int_{\Omega} |u(x)|^p \, dx \int_{\Omega} \nu(x-y) \, dy \geq 2 \nu^* (|\Omega|) \|u\|_{L^p(\Omega)}^p.
\]
Next let \( R = \text{diam}(\Omega) \) and assume \( \nu \) is almost decreasing, so that we get \( \nu(x-y) \geq \kappa \nu(R) \) for all \( x, y \in \Omega \). For \( u \in L^p(\Omega) \), Jensen’s inequality yields,
\[
\int_{\Omega} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \geq C^p \int_{\Omega} \int_{\Omega} |u(x) - u(y)|^p \, dy \, dx \geq C^p \|u - f_\Omega u\|_{L^p(\Omega)}^p.
\]

More recent and improved versions of Poincaré type inequalities including the situation where \( \nu \) is not fully supported and/or \( \Omega \) is bounded only in one direction have been established in [Fog25]. In particular, one is able to obtain the Poincaré inequality even if \( \nu \) is not almost decreasing, by requiring that the embedding \( W^p_0(\Omega) \hookrightarrow L^p(\Omega) \) is compact; see [FG20,Fog25]. The next corollary follows from Theorem 3.10.

**Corollary 4.2** (Poincaré-Friedrichs-Sobolev type inequality). Let \( \Omega \subset \mathbb{R}^d \) be open and define \( L^p_0(\mathbb{R}^d) = \{ u \in L^p(\mathbb{R}^d) : u = 0, \text{ a.e. on } \Omega \} \). Under the assumptions of Theorem 3.10 we get
\[
\|u\|_{L^p_0(\mathbb{R}^d)} \leq \Theta(t) \left( \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p_0(\mathbb{R}^d).
\]

The next result is somewhat a side consequence of the convexity assumption on \( \phi_p \) and shows the equivalence between the Sobolev inequality and the Poincaré-Sobolev inequality.

**Theorem 4.3** (Poincaré-Sobolev inequalities). Assume assumptions of Theorem 3.10 hold. Let \( \Omega \subset \mathbb{R}^d \) be measurable such that \( 0 < |\Omega| < \infty \). If the Poincaré-Sobolev inequality holds true \( \Omega \), i.e.,
\[
\|u - f_\Omega u\|_{L^p(\Omega)} \leq C \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\Omega),
\]
then the Poincaré inequality also holds true, i.e.,
\[
\|u - f_\Omega u\|_{L^p(\Omega)} \leq C \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\Omega).
\]
The converse holds true if in addition, \( \Omega \) is an \( W^p_0 \) extension domain.

**Proof.** By Proposition 2.1 we get \( \phi(t) \geq \delta_0 t^p \) for all \( t \geq t_0 \) with fixed \( t_0 > 0 \). Thus, the embedding \( L^p(\Omega) \hookrightarrow L^p(\Omega) \) is continuous, by Theorem 2.1. Together with the Poincaré inequality yields
\[
\|u - f_\Omega u\|_{L^p(\Omega)} \leq C \|u - f_\Omega u\|_{L^p(\Omega)} \leq C \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p}.
\]
Conversely assume, the Poincaré inequality holds and \( \Omega \) is an \( W^p_0 \) extension domain. Let \( \pi \in W^p_0(\mathbb{R}^d) \) be an extension of \( u_0 = u - f_\Omega u \) with \( u \in W^p_0(\Omega) \). Applying Theorem 3.10 reveals that \( \pi \in L^p(\mathbb{R}^d) \) and we deduce the Poincaré-Sobolev inequality as follows
\[
\|u - f_\Omega u\|_{L^p(\Omega)} \leq \|\pi\|_{L^p(\mathbb{R}^d)} \leq C \left( \int_{\mathbb{R}^d} |\pi(x) - \pi(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p}
\leq C \left( \int_{\Omega} |u(x) - f_\Omega u|^p \, dx + \int_{\Omega} |u_0(x) - u_0(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p}
\leq C \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x-y) \, dy \, dx \right)^{1/p}.
\]
As a direct consequence of Theorem 3.10 and Theorem 4.3 combined with Theorem 4.1 we get.

**Corollary 4.4.** Let the assumptions of Theorem 3.10 be in force. If \( \nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty) \) is almost decreasing and \( \Omega \subseteq \mathbb{R}^d \) is an \( W^p_2 \)-extension domain then Poincaré-Sobolev inequality holds, that is, there is a constant \( C = C(\Omega, \nu, p, d) > 0 \) only depends on \( \Omega, \nu, d \) and \( p \) such that

\[
\|u - f_\Omega u\|_{L^p(\Omega)} \leq C \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\Omega). \tag{4.1}
\]

In particular, if \( s \in (0, 1) \) and \( p_s^* > 0 \) then

\[
\|u - f_\Omega u\|_{L^{p_s^*}(\Omega)} \leq C \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^{p_s^*}(\Omega).
\]

**Remark 4.5.** Let \( \psi \) be a Young function such that, for all \( t \geq 1 \), \( \psi(ct) \leq \max(t, \phi(t)) \) for some \( c > 0 \). Assume that the Sobolev-Poincaré inequality \( 4.1 \) holds then there is \( C' > 0 \) such that

\[
\|u - f_\Omega u\|_{L^p(\Omega)} \leq C' \left( \int_{\Omega} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\Omega).
\]

**Remark 4.6.** By a straightforward scaling argument one finds \( C = C(d, p, s) \) such that,

\[
\|u - f_{B_r} u\|_{L^{p_s^*}(B_r)} \leq C \left( \int_{B_r} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^{p_s^*}(B_r), \tag{4.2}
\]

where \( B_r = B(0, r), r > 0 \) is any ball and \( C > 0 \) in \( 4.2 \) is independent on \( r > 0 \). Letting \( r \to \infty \), one verifies that the uniform estimate in \( 4.2 \) implies the Gagliardo-Nirenberg-Sobolev inequality given in Theorem 3.10. More generally the next result infers that the uniform Poincaré-Sobolev inequality on balls with respect to \( \phi \) implies the nonlocal Gagliardo-Nirenberg-Sobolev inequality.

**Theorem 4.7.** Let \( \nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p) \) be nonnegative and let \( \phi(t) = 1/w^{-1}(1/t) \) be defined as in Theorem 3.10 with \( w(r) = (r\nu^*(r))^{1/p} \). Assume there is a universal constant \( \Theta > 0 \) such that for all balls \( B \subseteq \mathbb{R}^d \), the following Poincaré-Sobolev inequality holds true,

\[
\|u - f_B u\|_{L^\phi(B)} \leq \Theta \left( \int_{BB} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^\phi(B).
\]

Then following inequality holds true as well

\[
\|u\|_{L^\phi(\mathbb{R}^d)} \leq \Theta \left( \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} \quad \text{for all } u \in L^\phi(\mathbb{R}^d).
\]

**Proof.** Let \( B_r = B(0, r), r > 0 \). Using the assumption and the formula assumption 1.11 gives

\[
\|u\|_{L^\phi(B_r)} \leq \|u - f_{B_r} u\|_{L^\phi(B_r)} + \|1_{B_r}\|_{L^\phi(\mathbb{R}^d)} \left| \int_{B_r} u \right| \\
\leq \Theta \left( \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x - y) \, dy \, dx \right)^{1/p} + \frac{|B_r|^{-1/p}}{\phi^{-1}(1/|B_r|)} \left( \int_{\mathbb{R}^d} |u(x)|^p \, dx \right)^{1/p}.
\]

In virtue of Theorem 3.3, we know that \( \nu^*(r) \to 0 \) as \( r \to \infty \) so, by the definition of \( \phi \) we obtain

\[
\frac{|B_r|^{-1/p}}{\phi^{-1}(1/|B_r|)} = |B_r|^{-1/p} \psi(|B_r|) = (\nu^*(r))^{1/p} \to 0 \quad \text{as } r \to \infty.
\]

Since \( \Theta > 0 \) is independent of \( r \), tending \( r \to \infty \) in the foregoing yields the desired inequality. \( \square \)

**Open Questions:** (i) Regarding Theorem 3.12, can the growth condition \( (C) \) be improved? (ii) Is the critical function \( \phi \) optimal? Indeed if \( \psi \) is another Young function satisfying the Gagliardo-Nirenberg-Sobolev inequality \( 1.12 \) then it also holds true for the Young function \( t \to \max(\phi(t), \psi(t)) \). Note that we call \( \phi \) optimal if there exists \( c > 0 \) such that \( \psi(t) \leq \phi(ct) \) for all
\( t > 0 \), this is equivalent (see Theorem 2.4) to saying that \( L^p(\mathbb{R}^d) \hookrightarrow L^s(\mathbb{R}^d) \); in other words \( L^p(\mathbb{R}^d) \) is the smallest Orlicz space satisfying the inequality. (1.12).

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