Modelling approach of a near-far-field model for bubble formation and transport

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Abstract. In this paper, we present a model based on a near-far-field bubble formation. We simulate the formation of a gas-bubble in a liquid, e.g., water and the transportation of such a gas-bubble in the liquid. The modelling approach is based on coupling the near-field model, which is done by the Young-Laplace equation, with the far-field model, which is done with a convection-diffusion equation. We decouple the small and large time- and space scales with respect to each adapted model. Such a decoupling allows to apply the optimal solvers for each near- or far-field model. We discuss the underlying solvers and present the numerical results for the near-far-field bubble formation and transport model.

Keywords: Near-far-field approach, Young-Laplace equation, Convection-diffusion equation, Level-set method, coupling analysis

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1 Introduction

We are motivated to model bubble formation and transport in liquid, which are applied in controlled production of gas bubbles in chemical-, petro-chemical-, plasma- or biomedical-processes, see [4], [8], [12] and [5].

We consider to decompose the formation process of bubbles, we call it near-field approach, and the transport process of bubbles, we call it far-field approach. Such a decomposition allows to separate the large scale-dependencies of the bubble formation, which has smaller time and space scales as the transport of the bubbles, which applies larger time and space scales, see [3]. For such a decomposition, we assume that the bubble is formed from an orifice in a solid surface and submerged in a liquid (viscous Newtonian liquid), see [10]. Therefore, the first process (formation) has to be finalized, when we start with the second process (transport). Such a decomposition allows to choose the optimal discretization and solver methods, i.e., we apply fast ODE-solvers for the near-field model and level-set methods for the far-field model.
The paper is organized as following: The modelling problems and their solvers are presented in Section 2. The coupling of the models are discussed in Section 3. The numerical experiments are presented in Section 4. In the contents, that are given in Section 5 we summarize our results.

2 Mathematical Model

The mathematical model is based on a real-life experiment, where gas-bubbles are formed in a liquid and are transported after the formation process, see the plasma-experiment in [5]. The experiment is given as a thin capillary, where the gas-bubbles are streamed in an homogeneous form and transported in a tube, which is filled with liquid, see the Figure 1.

![Sketch of the real-life experiment (capillary with gaseous outflow into a tube filled with water).](image)

We consider the profile of the tube and deal with the simplified approach of the experiment, which is given in Figure 2.

Based on the decoupling of formation and transport, while we assume, that the formation process is not influenced by the transport, see [1], we deal with two different decoupled models:

- **Near-field approach** based on a Young-Laplace equation, see [11], where we have a static shape after the formation of the bubbles.
- **Far-field approach** based on a convection-diffusion equation, see [3], where we have a rewriting into a level-set equation, such that we could transport the static bubble-shapes, see [9].

In the following, we discuss the different models.
Tube with water and periodical inflow–sources

Periodic sources
(stable bubble sources)

Fig. 2. Periodically inflow of the stable bubble sources.

2.1 Far-field approach

The first modelling approach is given with a convection-diffusion equation in cylindrical coordinate as:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -v \frac{\partial u}{\partial z} + D_L \frac{\partial^2 u}{\partial z^2} + \frac{D_t}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \quad (r, z, t) \in \Omega \times [0, T], \\
u(x, z, 0) &= u_{near}(x, z), \quad (r, z, t) \in \Omega,
\end{align*}
\]

where we assume \( u_{near} \) is the solution of the bubble-formation in the near-field and we assume to have Dirichlet-boundary conditions.

Here, we have the benefit and drawbacks of the modelling approach:

- Benefits:
  - The model is simple and fast to compute.
  - The model also allows to discuss a dynamical shape.
- Drawbacks:
  - The shape of the bubble is not preserved, while we assume a static shape.
  - The influence of the speed of motion in the outer normal direction is not possible.

2.2 Level-Set method

We apply an improved model, that allows to follow the shapes of the bubble, see [9].

The convection-diffusion equation is reformulated in the notation of a level-set equation, which is given as

\[
\frac{\partial u}{\partial t} = -\mathbf{v} \cdot \nabla u - F_0 |\nabla u|, \quad (x, t) \in \Omega \times [0, T],
\]

\[
u(x, 0) = u_0(x),
\]

\[
u(x, t) = 0.0, \quad (x, t) \in \partial \Omega \times [0, T],
\]

where \( \mathbf{v} \) is the convection vector and \( F_0 \) is the speed of motion in the outer normal direction. Further \( \Omega \) is the computational domain and \( T \) is the end time.

The initialisation \( u(x, 0) \) is the results of the near-field computations. Such equations are well-known as level-set equations and can be solved like convection-diffusion equations, see [9].

In the following, we apply the explicit different discretization methods in space, while we apply the level-set equation with the explicit time-discretisation and apply upwind methods for the advection and outer normal direction term only in the \( x \)-direction, the same is also done with the \( y \)-direction.

We have the following terms:

\[
D_x^- u_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x},
\]

\[
D_x^+ u_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x},
\]

\[
|D_x^+ u_{i,j}| = \left( (\max(D_x^- u_{i,j}, 0))^2 + (\min(D_x^+ u_{i,j}, 0))^2 \right)^{1/2},
\]

\[
u_{i}^{n+1} = \nu_{i}^{n} - \Delta t \frac{\nu_{i}^{n} - \nu_{i-1}^{n}}{\Delta x} - \Delta t \frac{\nu_{i,j}^{n} - \nu_{i,j-1}^{n}}{\Delta x}
\]

+ \Delta t \frac{F_0 |D_x^+ u_{i,j}|}{
\]

where we assume \( v_x, v_y, F_0 \geq 0 \).

We discretize the Level-set equation with the explicit time-discretisation and apply upwind methods for the advection and outer normal direction term in the
\( x \)- and \( y \)-direction and have the following terms:

\[
\begin{align*}
D_x^- u_{i,j} &= \frac{u_{i,j} - u_{i-1,j}}{\Delta x}, \\
D_x^+ u_{i,j} &= \frac{u_{i+1,j} - u_{i,j}}{\Delta x}, \\
D_y^- u_{i,j} &= \frac{u_{i,j} - u_{i,j-1}}{\Delta y}, \\
D_y^+ u_{i,j} &= \frac{u_{i,j+1} - u_{i,j}}{\Delta y},
\end{align*}
\]

(10)

(11)

(12)

(13)

\[
|D_x^+ u_{i,j}| = \left( (\max(D_x^- u_{i,j}, 0))^2 + (\min(D_x^+ u_{i,j}, 0))^2 \right) \frac{1}{2}.
\]

(14)

(15)

\[
u_{i}^{n+1} = u_{i}^{n} - \Delta t \ v_x \frac{u_{i}^{n} - u_{i-1}^{n}}{\Delta x} - \Delta t \ v_y \frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta x} + \Delta t \ F_0 |D_x^+ u_{i,j}|,
\]

(16)

where we assume \( v_x, v_y, F_0 \geq 0 \).

Remark 1. An alternative approach of the shape transport can be done with the volume of fluid (VOF) method. Such a method is based on a free-surface modelling technique, while the method is tracking and locating the free surface, see also [6].

In the following, we discuss the near-field approach.

### 2.3 Near field model

The near-field model is discussed with respect to the formation of a drop or bubble, see [13] and [10].

The basic modelling idea is based on the so called Young-Laplace equation, see [2] and deals with the following simplified shape of the bubble, see Figure 3.

We deal with the following parameterisation, see [11]:

\[
\beta = -\rho g R_t^2 / \sigma,
\]

(17)

where \( \beta \) is the Bond number, \( \sigma \) is the surface tension, \( \rho \) is the liquid density, \( g \) is the gravity and \( R_t \) is the curvature of the drop.

The near-field equations are given as:

\[
\frac{dr}{ds} = \cos(\theta),
\]

(18)

\[
\frac{dz}{ds} = \sin(\theta),
\]

(19)

\[
\frac{d\theta}{ds} = 2 + \beta z - \frac{\sin(\theta)}{r},
\]

(20)
where $s$ is the arc length along the curve and $\theta$ the angle of elevation for its slope and $\alpha$ is the mono-layer surface tension.

We have the conditions:

$$r = a, \ z = 0, \ at \ s = 0, \quad (21)$$

$$\frac{dz}{ds} = \frac{dz}{dr} = 0, \ at \ s = L \ or \ (r = 0), \quad (22)$$

where $L$ is the arc length of the bubble which is a-priori unknown, so here we apply $r = a = 0$.

**Remark 2.** The ODE system can be solved with numerical methods, e.g., with the MATLAB function ode45. Based on the value of $R_t$, means the possible curvature of the bubble, we solve the half shape of the bubble and measure the different diameters.

### 3 Coupling Near-Field and Far-Field

The modelling assumes, that we could decouple the near and far-field, while we neglect the coalescence or ruptures of the bubbles, e.g., in the flow-field, see [7].

We assume that in terms of the bubble-density function:

$$f_b(r, z, x, y, t) = u(x, y, t)\delta((r - R(x, y, t)), (z - Z(x, y, t))), \quad (23)$$

where $u$ is the concentration of the bubble and $R$ and $Z$ are obtained with the bubble-formation equations, while $r$ and $z$ are the cylinder coordinates of the density function, that we do not have an influence means $r \approx R$ and $z \approx Z$ for the formation process.

We discuss the following different coupling ideas:

- Parameters of the ellipse are computed in the near-field and initialise the far-field bubble.
- The near-field computation is directly implemented into the far-field.
3.1 Decoupled computation of Near- and Far-Field

The near-field bubble is computed with the ODE’s given in (18)-(20).

We estimate the characteristic parameters of the ellipse in the Figure 4.

![Estimation of the bubble-parameters](image)

**Fig. 4.** Final bubble based on the near-field computation and estimation of the bubble-parameters (we assume an elliptic curve).

Based on the estimation of the elliptic-parameters, we obtain the curvature of the ellipse:

\[
\frac{(x - x_a)^2}{a^2} + \frac{(y - y_b)^2}{b^2} = 1. \tag{24}
\]

The ellipse is the curvature of the far-field, which is computed with the level-set method.

**Remark 3.** The transformation of the elliptic parameters of the near-field model allows to simplify the construction of the shape in the far-field. We only apply the ellipses in the far-field transport modell.

4 Numerical Experiments

In the following, we apply the bubble-formation based on the simplified model, i.e., Young-Laplace equation, and the bubble-transport model, based on the level-set equations.
4.1 Bubble-Formation: Experiment 1

The near-field equations are given as:

\[
\frac{dr}{ds} = \cos(\theta), \quad (25)
\]

\[
\frac{dz}{ds} = \sin(\theta), \quad (26)
\]

\[
\frac{d\theta}{ds} + \frac{\sin(\theta)}{r} = \frac{\Delta p}{\alpha}, \quad (27)
\]

where \( s \) is the arc length along the curve and \( \theta \) the angle of elevation for its slope and \( \alpha \) is the mono-layer surface tension.

We have the conditions:

\[
r = a, \quad z = 0, \text{ at } s = 0, \quad (28)
\]

\[
\frac{dz}{ds} = \frac{dr}{ds} = 0, \text{ at } s = L \text{ or } (r = 0), \quad (29)
\]

where \( L \) is the arc length of the bubble which is a-priori unknown, so here we apply \( r = 0 \).

We deal with the following domain: \((r, z) \in [0, b] \times [0, d] \), where \( a = 1, \ b = 3, \ d = 3 \). Further \( \alpha = 0.1, \ \rho = 0.1, \ g = 9.81, \ p_{\text{tube}} = 0.001 \).

The numerical results of bubble-formation is given in Figure 5.

![Figure 5](image)

**Fig. 5.** Bubble-formation, left figure with parameters \( a = 1, \ b = 3, \ d = 3, \ \alpha = 0.1, \ \rho = 0.1, \ g = 9.81, \ p_{\text{tube}} = 0.001 \) and right figure with parameters \( a = 10, \ b = 3, \ d = 3, \ \alpha = 0.2, \ \rho = 0.1, \ g = 9.81, p_{\text{tube}} = 0.4 \).

**Remark 4.** The Young-Laplace equation allows to formulate the bubble-formation such that we could obtain the radii of the different bubbles based on the various pressure parameters.
4.2 Bubble-Formation: Experiment 2

In the following, we couple the near-field and far-field computations. We have the following setting, see Figure 6.

![Figure 6](image)

**Fig. 6.** Left figure: Bubble-formation with the ODEs and right figure: Bubble-transport with the PDEs (level-set equations).

The numerical results of far-field bubble-transport is given in Figure 10. We apply the following parameters:

- Input-parameters of the near-field bubble computation: $r_0 = 10, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \Delta p = 0.9$.
- Output-parameters of the near-field bubble computation: $a_{\text{bubble}} = 0.1825, b_{\text{bubble}} = 0.2216$.
- Ellipse: $(x - 50)^2 + ((y - 50) \cdot a_{\text{bubble}}/b_{\text{bubble}})^2 - a_{\text{bubble}}^2$.

![Figure 7](image)

**Fig. 7.** Left figure with bubble formation (near-field) and right figure with the bubble transport (far-field).
The numerical results of the near-far-field coupled bubble-transport code, which is given in Figure 11.

**Fig. 8.** Upper figures: Transport of the first bubble with the level-set function, lower figures: Transport of the second bubble with the level-set function.

**Remark 5.** The coupling of the formation and transport of the bubbles are done with ordinary and partial differential equations. Based on decoupling such systems of mixed ordinary and partial differential equations, we could compute each separate part with the optimal numerical solvers.

### 4.3 Bubble-Formation: Multiple Bubble Experiment (2 Bubbles)

In the following, we couple the near-field and far-field computations with multiple bubbles. We have the following setting, see Figure 9.

The numerical results of far-field bubble-transport is given in Figure 10.

We apply the following parameters:

- Computation of the near-field bubbles (a representing bubble is computed)
  - Input-parameters of the near-field bubbles computation:
    - Bubble 1:
      \[ r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.05. \]
Fig. 9. Left figure: Bubble-formation with the ODEs and right figure: Bubble-transport with the PDEs (level-set equations).

* Bubble 2:
  \[ r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.2. \]

Output-parameters of the near-field bubble computation:

* Bubble 1:
  \[ a_{\text{bubble}1} = 0.5051, \ b_{\text{bubble}1} = 0.9909. \]

* Bubble 2:
  \[ a_{\text{bubble}2} = 0.4720, \ b_{\text{bubble}2} = 1.2147. \]

- Ellipse: \((x - x_{\text{bubble}i})^2 + \left((y - y_{\text{bubble}i})^2 a_{\text{bubble}i}/b_{\text{bubble}i}\right)^2 - a_{\text{bubble}i}^2, \)
  where \((x_{\text{bubble}i}, y_{\text{bubble}i})\) is the origin of the \(i\)-th bubble.

- Computation of the far-field bubbles (level-set initialisation):
  - Parameterisation of the level-set initial-function, e.g., two bubbles:

\[
\phi_0(x, y) = \begin{cases} 
(x - x_{\text{bubble}1})^2 + \left((y - y_{\text{bubble}1})^2 a_{\text{bubble}1}/b_{\text{bubble}1}\right)^2 - a_{\text{bubble}1}^2, & \text{if } a_x \leq x \leq 50, \ a_y \leq y \leq b_y, \\
(x - x_{\text{bubble}2})^2 + \left((y - y_{\text{bubble}2})^2 a_{\text{bubble}2}/b_{\text{bubble}2}\right)^2 - a_{\text{bubble}2}^2, & \text{if } 50 \leq x \leq b_x, \ a_y \leq y \leq b_y, 
\end{cases}
\]

where \((x_{\text{bubble}1}, y_{\text{bubble}1}) = (20, 50), (x_{\text{bubble}2}, y_{\text{bubble}2}) = (80, 50)\) with the coordinates of the grid \((a_x, a_y) = (0, 0)\) and \((b_x, b_y) = (100, 200)\).

The numerical results of the near-far-field coupled bubble-transport code, which is given in Figure 11.

Remark 6. The level-set method allows to deal with different level-set functions, such that we could transport multiple bubbles.

4.4 Bubble-Formation: Multiple Bubble Experiment (10 Bubbles)

In the following, we extend the near-field and far-field computations with 10 bubbles. We also apply the decomposition of near-field and far-field computations as given in Figure 9.

We apply the following parameters:
Fig. 10. Left figure with bubble formation (near-field) and right figure with the bubble transport (far-field).

Fig. 11. Multi-bubble transport with the level-set function.

- Computation of the near-field bubbles (a representing bubble is computed)
  - Input-parameters of the near-field bubbles computation:
    * Bubble 1:
      \( r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.05. \)
    * Bubble 2:
      \( r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.06. \)
    * Bubble 3:
      \( r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.07. \)
    * Bubble 4:
      \( r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.08. \)
    * Bubble 5:
      \( r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.1. \)
    * Bubble 6:
      \( r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.12. \)
    * Bubble 7:
      \( r_0 = 0.1, z_0 = 0, \theta_0 = 0, \alpha = 0.2, \rho = 0.1, g = 9.81, \sigma = 0.2, r_t = 0.14. \)
• Output-parameters of the near-field bubble computation:

* Bubble 1:
  \( a_{\text{bubble}1} = 0.5051, \ b_{\text{bubble}1} = 0.9909. \)

* Bubble 2:
  \( a_{\text{bubble}2} = 0.5052, \ b_{\text{bubble}2} = 1.0127. \)

* Bubble 3:
  \( a_{\text{bubble}3} = 0.5034, \ b_{\text{bubble}3} = 1.0332. \)

* Bubble 4:
  \( a_{\text{bubble}4} = 0.5013, \ b_{\text{bubble}4} = 1.0529. \)

* Bubble 5:
  \( a_{\text{bubble}5} = 0.4959, \ b_{\text{bubble}5} = 1.0900. \)

* Bubble 6:
  \( a_{\text{bubble}6} = 0.4888, \ b_{\text{bubble}6} = 1.1208. \)

* Bubble 7:
  \( a_{\text{bubble}7} = 0.4797, \ b_{\text{bubble}7} = 1.1422. \)

* Bubble 8:
  \( a_{\text{bubble}8} = 0.4654, \ b_{\text{bubble}8} = 1.1639. \)

* Bubble 9:
  \( a_{\text{bubble}9} = 0.4685, \ b_{\text{bubble}9} = 1.1886. \)

* Bubble 10:
  \( a_{\text{bubble}10} = 0.4720, \ b_{\text{bubble}10} = 1.2147. \)

• Ellipse: \((x - x_{\text{bubble}i})^2 + \left((y - y_{\text{bubble}i}) \times a_{\text{bubble}i}/b_{\text{bubble}i}\right)^2 - a_{\text{bubble}}^2, \)
where \((x_{\text{bubble}i}, y_{\text{bubble}i})\) is the origin of the \(i\)-th bubble.

– Computation of the far-field bubbles (level-set initialisation):

• Parameterisation of the level-set initial-function, e.g., two bubbles:

\[
\phi_0(x, y) = \begin{cases} 
(x - x_{\text{bubble}1})^2 + \left((y - y_{\text{bubble}1}) \times a_{\text{bubble}1}/b_{\text{bubble}1}\right)^2 - a_{\text{bubble}1}^2, \\
(x - x_{\text{bubble}2})^2 + \left((y - y_{\text{bubble}2}) \times a_{\text{bubble}2}/b_{\text{bubble}2}\right)^2 - a_{\text{bubble}2}^2, \\
50 \leq x \leq b_x, \ a_y \leq y \leq b_y,
\end{cases}
\]

where \((x_{\text{bubble}1}, y_{\text{bubble}1}) = (20, 50), (x_{\text{bubble}2}, y_{\text{bubble}2}) = (80, 50)\) with the coordinates of the grid \((a_x, a_y) = (0, 0)\) and \((b_x, b_y) = (100, 200)\).

The numerical results of the near-far-field coupled bubble-transport code, which is given in Figure 12.

Remark 7. In the experiment, we deal with at least 10 bubbles, which are different formatted and transported via the level-set method. The numerical experiments allows to accelerate the formation and transport of such processes.
Fig. 12. Transport of 10 bubbles with the level-set function.

5 Conclusion

We present a bubble model, which is a coupled model based on a bubble formation and bubble transport model. The decoupling into near- and far-field models allows to apply optimal solver and discretization methods. We apply different numerical experiments, which shows the benefit of such a treatment. In future, we consider the fully coupled problem, while we deal with bubble density functions and the coupling between the formation and transport process. Such an extension allows to see the ruptures of the bubbles.

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