Supplementary to “A note on asymptotic distributions in directed exponential random graph models with bi-degree sequences”

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This is a supplementary material that contains proofs of the theorems 2 and 3.

1 Proof of Theorem 2

Note that both $d_i = \sum_{j \neq i} a_{i,j}$ and $b_i = \sum_{i \neq j} a_{i,j}$ are sum of $n - 1$ independent geometric random variables. Also note that $q_n \leq a_i^* + \beta_i^* \leq Q_n$ and $V = F'(\theta^*) \in \mathcal{L}_n(m, M)$, thus we have $(n - 1)Q_n^{-2} \leq v_{i,i} \leq (n - 1)q_n^{-2}$. Therefore, $v_{i,i}^{-1/2}(d_i - \mathbb{E}(d_i))$ is asymptotically standard normal if $Q_n/q_n = o(n^{1/2})$. Similarly, $v_{n+j,n+j}^{-1/2}(b_j - \mathbb{E}(b_j))$ is also asymptotically standard normal under the same condition. Similar to Proposition 2 in subsection 5.2 of the main texts, we have:

**Proposition 1.** Assume that $A \sim \mathbb{P}_{\theta^*}$ and $Q_n/q_n = o(n^{1/2})$, as $n \to \infty$, then $\bar{c}^\top S(g - \mathbb{E}(g))$ is asymptotically normally distributed with mean zero and variance

$$\sum_{i=1}^\infty \lambda_i^2 + \sum_{i,j=1}^\infty \lambda_i \lambda_j H_i H_j + \sum_{i=1}^\infty \kappa_i^2 + \sum_{i,j=1}^\infty \kappa_i \kappa_j H_{n+i} H_{n+j} - 2 \sum_{i,j=1}^\infty \lambda_i \kappa_j H_i H_{n+j},$$

where $\bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \ldots, \lambda_n v_{nn}^{1/2})^\top$, $\bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, \ldots, \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top$ and $\bar{c} = (\bar{\lambda}, \bar{\kappa})$.

Before proving Theorem 2, we present two lemmas. The proof of lemma 1 is similar to that of lemma 1 in subsection 5.2 of the main texts and we omit it.

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Lemma 1. Let \( \bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \ldots, \lambda_n v_{nn}^{1/2})^\top \), \( \bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, \ldots, \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top \) and \( \bar{c} = (\bar{\lambda}, \bar{\kappa}) \). Then

\[
\text{Var}[\bar{c}^\top W\{g - \mathbb{E}(g)\}] \leq \frac{c_1 M^3 + 3M^2m^2}{m^3(n - 1)} \times \left\{ \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 + \left( \sum_{i=1}^{n-1} |\kappa_i| \right)^2 + 2\left( \sum_{i=1}^{n} |\lambda_i| \right) \left( \sum_{j=1}^{n-1} |\kappa_j| \right) \right\}
\]

(1)

where \( m = 1/Q_n^2 \), \( M = 1/q_n^2 \) and \( c_1 \) given in proposition 1 in subsection 5.1 of the main texts.

The following lemma is due to Theorem 3 in Yan et al. (2016).

Lemma 2. Assume that \( \bar{\theta}^* \) satisfies \( q_n \leq \alpha_i^* + \beta_j^* \leq Q_n \) and \( A \sim \mathbb{P}_{\bar{\theta}^*} \), where \( \mathbb{P}_{\bar{\theta}^*} \) denote the probability distribution (1) of the main texts on \( A \) under the parameter \( \bar{\theta}^* \). If \( Q_n/q_n = o\{(n/\log n)^{1/18}\} \), then as \( n \) goes to infinity, with probability approaching one, the MLE \( \hat{\theta} \) exist and satisfies

\[
\| \hat{\theta} - \bar{\theta}^* \|_\infty = O_p\left( \frac{Q_n^9 (\log n)^{1/2}}{n^{1/2}q_n^9} \right) = o_p(1).
\]

Further, if the MLE exists, it is unique.

Proof of Theorem 2. By Lemma 2, with probability approaching one, we have

\[
\hat{\rho}_n := \max_{1 \leq i \leq 2n-1} |\hat{\rho}_i - \bar{\rho}_i^*| = O_p\left( \frac{Q_n^9 (\log n)^{1/2}}{n^{1/2}q_n^9} \right)
\]

(2)

The following calculations are based on the event (2). Let \( \hat{\gamma}_{i,j} = \hat{\alpha}_i + \hat{\beta}_j - \alpha_i^* - \beta_j^* \). For any \( i \neq j \), a direct calculation gives

\[
\frac{1}{\hat{\alpha}_i + \hat{\beta}_j} - \frac{1}{\alpha_i^* + \beta_j^*} = \frac{\alpha_i^* - \hat{\alpha}_i + \beta_j^* - \hat{\beta}_j}{(\hat{\alpha}_i + \hat{\beta}_j) + (\alpha_i^* + \beta_j^*)}
\]

\[
= \frac{\alpha_i^* - \hat{\alpha}_i + \beta_j^* - \hat{\beta}_j}{(\alpha_i^* + \beta_j^*)^2} \left( \frac{\alpha_i^* + \beta_j^*}{\hat{\alpha}_i + \hat{\beta}_j} - 1 \right) + \frac{\alpha_i^* - \hat{\alpha}_i + \beta_j^* - \hat{\beta}_j}{(\alpha_i^* + \beta_j^*)^2}
\]

\[
= h_{i,j} + \frac{\alpha_i^* - \hat{\alpha}_i + \beta_j^* - \hat{\beta}_j}{(\alpha_i^* + \beta_j^*)^2},
\]

where

\[
h_{i,j} = \frac{\hat{\gamma}_{i,j}^2}{(\hat{\alpha}_i + \hat{\beta}_j)(\alpha_i^* + \beta_j^*)^2},
\]

and \( 0 \leq \phi_{i,j} \leq 1 \). By the likelihood equations (4) in subsection 2.2 of the main texts, we
have
\[ g - \mathbb{E}(g) = V(\hat{\theta} - \theta^*) + h, \tag{3} \]
where \( h = (h_1, \ldots, h_{2n-1})^\top \) and
\[
\begin{align*}
    h_i &= \sum_{k=1,k \neq i}^n h_{i,k}, \ i = 1, \ldots, n, \\
    h_{n+i} &= \sum_{k=1,k \neq i}^n h_{k,i}, \ i = 1, \ldots, n - 1.
\end{align*}
\]
Equivalently,
\[ \hat{\theta} - \theta^* = V^{-1}(g - \mathbb{E}(g)) + V^{-1}h \]
By the definition of \( \hat{\rho}_n \), we have
\[
|h_{i,j}| \leq \frac{4\hat{\rho}_n^2}{q_n^2(q_n - \hat{\rho}_n)}, \quad |h_i| \leq \sum_{j \neq i} |h_{i,j}| \leq \frac{4(n-1)\hat{\rho}_n^2}{q_n^2(q_n - \hat{\rho}_n)}.
\]
Note that \((Sh)_i = h_i/v_{i,n} + (-1)^{1(i>n)}h_{2n}/v_{2n,n}, \ (n-1)Q_n^{-2} \leq v_{i,n} \leq (n-1)q_n^{-2}\) and \((V^{-1}h)_i = (Sh)_i + (Wh)_i\). By direct calculations, we have
\[
|(V^{-1}h)_i| = |(Sh)_i| + |(Wh)_i| \leq \frac{|h_i|}{v_{i,n}} + \frac{|h_{2n}|}{v_{2n,n}} + \{\|W\|_\infty \times [(2n-1) \max_i |h_i|] \}
\]
\[
\leq \frac{8Q_n^2\hat{\rho}_n^2}{q_n^3} + \frac{8c_1Q_n^6\hat{\rho}_n^2}{q_n^7},
\]
\[
= O\left(\frac{\log(n)Q_n^{24}}{nq_n^{21}}\right)
\]
where \(c_1\) given in proposition 1 in subsection 5.1 of the main texts.

Note that \( \bar{c} = (\bar{\lambda}, \bar{\kappa}), \ \bar{\lambda} = (\lambda_1v_{11}^{1/2}, \ldots, \lambda_nv_{nn}^{1/2})^\top \) and \( \bar{\kappa} = (\kappa_1v_{n+1,n+1}^{1/2}, \ldots, \kappa_{n-1}v_{2n-1,2n-1}^{1/2})^\top \).
Consequently, if \( \sum_{i=1}^{\infty} |\lambda_i| < \infty, \ \sum_{i=1}^{\infty} |\kappa_i| < \infty \) and \( Q_n/q_n = o(n^{1/50}/(\log n)^{1/25}) \), then as \( n \to \infty, \)
\[
|\sum_{i=1}^{2n-1} \tilde{c}_i(V^{-1}h)_i| = \sum_{i=1}^n \lambda_i\sqrt{v_{i,i}}(V^{-1}h)_i + \sum_{i=1}^{n-1} \kappa_i\sqrt{v_{n+i,n+i}}(V^{-1}h)_{n+i} \]
\[ = O\left(\frac{\log(n)Q_n^{24}}{n^{1/2}q_n^{23}} \left(\sum_{i=1}^n |\lambda_i| + \sum_{i=1}^{n-1} |\kappa_i|\right)\right) \to 0. \tag{4} \]
In view of (4) and Lemma 1, if \( Q_n/q_n = o(n^{1/50}/(\log n)^{1/25}) \) and condition (2) of the main
texts hold, then
\[
\bar{c}^\top (\hat{\theta} - \bar{\theta}^*) = \bar{c}^\top S(g - \mathbb{E}(g)) + \bar{c}^\top W\{g - \mathbb{E}(g)\} + \left| \sum_i \bar{c}_i (V^{-1} h)_i \right|
\]
\[
= \bar{c}^\top S(g - \mathbb{E}(g)) + a_p(1).
\]

Theorem 2 is immediately comes form Proposition 1.

2 Proof of theorem 3

Note that both \( d_i = \sum_{j \neq i} a_{i,j} \) and \( b_i = \sum_{i \neq j} a_{i,j} \) are sums of \( n - 1 \) independent geometric random variables. Also note that \( q_n \leq \bar{\alpha}_i^* + \bar{\beta}_i^* \leq Q_n \) and \( V = F'(\bar{\theta}^*) \in \mathcal{L}_n(m, M) \), here \( m = \frac{e^{Q_n}}{(e^n - 1)^2} \) and \( M = \frac{e^{Q_n}}{(e^n - 1)^2} \), thus we have
\[
\frac{e^{Q_n}}{(e^n - 1)^2} \leq v_{i,j} \leq \frac{e^{Q_n}}{(e^n - 1)^2}, \quad i = 1, ..., n, \quad j = n + 1, ..., 2n, \quad j \neq n + i,
\]
\[
\frac{(n - 1)e^{Q_n}}{(e^n - 1)^2} \leq v_{i,i} \leq \frac{(n - 1)e^{Q_n}}{(e^n - 1)^2}, \quad i = 1, ..., 2n.
\]

Note that if \( e^{Q_n/2}/q_n = o(n^{1/2}) \), \( v_{i,i}^{1/2}(d_i - \mathbb{E}(d_i)) \) \((i = 1, ..., n)\) is asymptotically standard normal and \( v_{n+j,n+j}^{-1/2}(b_j - \mathbb{E}(b_j)) \) \((j = 1, ..., n)\) is asymptotically standard normal. Similar to Proposition 2 in subsection 5.2 of the main texts, we have:

**Proposition 2.** Assume that \( e^{Q_n/2}/q_n = o(n^{1/2}) \), as \( n \to \infty \), then \( \bar{c}^\top S(g - \mathbb{E}(g)) \) is asymptotically normally distributed with mean zero and variance
\[
\sum_{i=1}^\infty \bar{\Lambda}_i^2 + \sum_{i,j=1}^\infty \lambda_i \lambda_j H_i H_j + \sum_{i=1}^\infty \kappa_i^2 + \sum_{i,j=1}^\infty \kappa_i \kappa_j H_{n+i} H_{n+j} - 2 \sum_{i,j=1}^\infty \lambda_i \kappa_j H_i H_{n+j},
\]
where \( \bar{\Lambda} = (\lambda_1 v_{11}^{1/2}, ..., \lambda_n v_{nn}^{1/2})^\top \), \( \bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, ..., \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top \) and \( \bar{c} = (\bar{\lambda}, \bar{\kappa}) \).

Before proving Theorem 3, we present two lemmas. The proof lemma 3 is similar to that of lemma 1 in subsection 5.2 of the main texts and we omit it. Lemma 4 is due to Theorem 5 in Yan et al. (2016).

**Lemma 3.** Let \( \bar{\Lambda} = (\lambda_1 v_{11}^{1/2}, ..., \lambda_n v_{nn}^{1/2})^\top \), \( \bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, ..., \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top \) and \( \bar{c} = (\bar{\Lambda}, \bar{\kappa}) \). Then
\[
Var[\bar{c}^\top W\{g - \mathbb{E}(g)\}] \leq \frac{c_1 M^3 + M^2 m}{m^3 n(n - 1)^2} \times \left\{ \left( \sum_{i=1}^n |\lambda_i| \right)^2 + \left( \sum_{i=1}^{n-1} |\kappa_i| \right)^2 + 2 \sum_{i=1}^n |\lambda_i| \sum_{j=1}^{n-1} |\kappa_j| \right\}
\]

where \( m = \frac{e^{Q_n}}{(e^n - 1)^2} \), \( M = \frac{e^{Q_n}}{(e^n - 1)^2} \) and \( c_1 \) given in proposition 1 in subsection 5.1 of the main texts.
Lemma 4. Assume that $\bar{\theta}^*$ satisfies $q_n \leq \bar{\alpha}_i^* + \bar{\beta}_j^* \leq Q_n$ for all $i \neq j$ and $A \sim P_{\bar{\theta}^*}$, where $P_{\bar{\theta}^*}$ denote the probability distribution (1) of the main texts on $A$ under the parameter $\bar{\theta}^*$. If $(1 + q_n^{-11})e^{6Q_n} = o(n^{1/2}/(\log n)^{1/2})$, then as $n$ goes to infinity, with probability approaching one, the MLE $\hat{\theta}$ exist and satisfies

$$\| \hat{\theta} - \bar{\theta}^* \| \propto = O_p(e^{3Q_n}(1 + 1/q_n^5)\sqrt{\log n/n}) = o_p(1).$$

Further, if the MLE exists, it is unique.

Proof of Theorem 3. By Lemma 4, with probability approaching one, we have

$$\hat{\rho}_n := \max_{1 \leq i \leq 2n-1} |\hat{\theta}_i - \bar{\theta}_i^*| = O(e^{3Q_n}(1 + 1/q_n^5)\sqrt{\log n/n}). \quad (6)$$

The following calculations are based on the event (6). Let $\hat{\gamma}_{i,j} = \hat{\alpha}_i + \hat{\beta}_j - \bar{\alpha}_i^* - \bar{\beta}_j^*$. By Taylor’s expansion, for any $i \neq j$, we have

$$\frac{1}{e^{\hat{\alpha}_i + \hat{\beta}_j} - 1} - \frac{1}{e^{\bar{\alpha}_i^* + \bar{\beta}_j^*} - 1} = -\frac{e^{\bar{\alpha}_i^* + \bar{\beta}_j^*}}{(e^{\bar{\alpha}_i^* + \bar{\beta}_j^*} - 1)^2} \hat{\gamma}_{i,j} + h_{i,j},$$

where

$$h_{i,j} = \frac{e^{\bar{\alpha}_i^* + \bar{\beta}_j^* + \eta_{i,j}\hat{\gamma}_{i,j}} (1 + e^{\bar{\alpha}_i^* + \bar{\beta}_j^* + \eta_{i,j}\hat{\gamma}_{i,j}})}{(e^{\bar{\alpha}_i^* + \bar{\beta}_j^* + \eta_{i,j}\hat{\gamma}_{i,j}} - 1)^3},$$

and $0 \leq \eta_{i,j} \leq 1$. It is not difficult to verify that

$$g - \mathbb{E}(g) = V(\hat{\theta} - \bar{\theta}^*) + h,$$

where $h = (h_1, \ldots, h_{2n-1})^\top$ and

$$h_i = \sum_{k=1, k \neq i}^{n} h_{i,k}, \quad i = 1, \ldots, n,$$

$$h_{n+i} = \sum_{k=1, k \neq i}^{n} h_{k,i}, \quad i = 1, \ldots, n-1.$$

Equivalently,

$$\hat{\theta} - \bar{\theta}^* = V^{-1}(g - \mathbb{E}(g)) + V^{-1}h.$$

Since $\bar{\alpha}_i^* + \bar{\beta}_j^* > 0$, $q_n - \hat{\rho}_n < \alpha_i + \beta_j + \eta_{i,j}\hat{\gamma}_{i,j} \leq Q_n + q_n$ and $\hat{\rho}_n$ is sufficiently small, by
the definition of $\hat{\rho}_n$, we have
\[ |h_{i,j}| \leq \frac{e^{(q_n - \hat{\rho}_n)}(1 + e^{(q_n - \hat{\rho}_n)})}{(e^{(q_n - \hat{\rho}_n)} - 1)^3} \hat{\rho}_n^2, \quad |h_i| \leq \sum_{j \neq i} |h_{i,j}| \leq (n - 1) \frac{e^{(q_n - \hat{\rho}_n)}(1 + e^{(q_n - \hat{\rho}_n)})}{(e^{(q_n - \hat{\rho}_n)} - 1)^3} \hat{\rho}_n. \]

Note that $(Sh)_i = h_i/v_{i,i} + (-1)^{i > n}h_{2n}/v_{2n,2n}$. By direct calculations, we have
\[ |(V^{-1}h)_i| = |(Sh)_i| + |(Wh)_i| \leq \frac{|h_i|}{v_{i,i}} + \frac{|h_{2n}|}{v_{2n,2n}} + \left\{ ||W||_{\infty} \times [(2n - 1) \max_i |h_i|] \right\} \leq O\left(\frac{\log(n)}{n^2} \times e^{9Q_n(1 + q_n^{-15})}\right). \]

Let $\bar{c} = (\bar{\lambda}, \bar{\kappa})$, $\bar{\lambda} = (\lambda_1v_{1,1}^{1/2}, \ldots, \lambda_nv_{n,n}^{1/2})^\top$ and $\bar{\kappa} = (\kappa_1v_{n+1,n+1}^{1/2}, \ldots, \kappa_{n-1}v_{2n-1,2n-1}^{1/2})^\top$. Consequently, if $\sum_{i=1}^n |\lambda_i| < \infty$, $\sum_{i=1}^\infty |\kappa_i| < \infty$ and $e^{9Q_n(1 + q_n^{-15})} = o(n^{1/2}/\log n)$, then as $n \to \infty$,
\[ \left| \sum_{i=1}^{2n-1} \bar{c}_i (V^{-1}h)_i \right| = \left| \sum_{i=1}^n \lambda_i \sqrt{v_{i,i}} (V^{-1}h)_i + \sum_{i=1}^{n-1} \kappa_i \sqrt{v_{n+i,n+i}} (V^{-1}h)_{n+i} \right| \leq O\left(\frac{\log(n)}{n^{1/2}} \times e^{9Q_n(1 + q_n^{-15})}(\sum_{i=1}^n |\lambda_i| + \sum_{i=1}^{n-1} |\kappa_i|) \right) \to 0 \] (7)

In view of (7) and Lemmas 3 and 4, if $e^{9Q_n(1 + q_n^{-15})} = o(n^{1/2}/\log n)$ and condition(2) of the main texts hold, then
\[ \bar{c}^\top (\hat{\theta} - \tilde{\theta}^*) = \bar{c}^\top S(g - E(g)) + \bar{c}^\top W\{g - E(g)\} + \left| \sum_i \bar{c}_i (V^{-1}h)_i \right| \]
\[ = \bar{c}^\top S(g - E(g)) + o_p(1) \]

Theorem 3 is immediately comes from Proposition 2. \qed

References

Yan, T., Leng, C., and Zhu, J. (2016). Asymptotics in directed exponential random graph models with an increasing bi-degree sequence. The Annals of Statistics, 44(1):31–57.