FUNCTOR HOMOLOGY AND HOMOLOGY OF COMMUTATIVE MONOIDS

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The aim of this work is to clarify the relationship between homology theory of commutative monoids constructed à la Quillen [16], [17] and technology of Γ-modules as it was developed in [12],[13], [14], [15].

In Section 1 we recall the basics of Γ-modules and relation with commutative algebra (co)homology. In Section 2 we construct an analogue of Kähler differentials for commutative monoids. In Section 3 we construct homology theory for commutative monoids which and then we prove that commutative monoid homology are particular case of functor homology developed in [13].

It should be pointed out that the cohomology theory of commutative monoids first was constructed by P.-A. Grillet in the series of papers [2]-[7] (see also a recent work [1]). So our results shed light on Grillet theory. For instance we relate the commutative monoid (co)homology with André-Quillen (co)homology of corresponding monoid algebra. For another application we mention the Hodge decomposition for commutative monoid (co)homology which is an immediate consequence of our main result.

1. Γ-modules and commutative algebra (co)homology

1.1. Generalities on Γ-modules. Let $K$ be a commutative ring with unit which is fixed in the whole paper.

For any integer $n \geq 0$, we let $[n]$ be the set $\{0, 1, ..., n\}$ with basepoint 0. Let $\Gamma$ be the full subcategory of the category of pointed sets consisting of sets $[n]$. A left $\Gamma$-module is a covariant functor from $\Gamma$ to the category of $K$-modules, while a right $\Gamma$-module is a contravariant functor from $\Gamma$ to the category of $K$-modules. The category of all left $\Gamma$-modules is denoted by $\Gamma$-mod, while the category of all right modules is denoted by mod-$\Gamma$. It is well-known that the categories $\Gamma$-mod and mod-$\Gamma$ are abelian categories with sufficiently many projective and injective objects. For any $n \geq 0$ one defines the left $\Gamma$-module $\Gamma^n$ by

$$\Gamma^n(X) = K[X^n].$$

Here $K[S]$ denotes the free $K$-module generated by a set $S$. It is a consequence of the Yoneda lemma that for any left $\Gamma$-module $F$ one has natural isomorphisms

$$\text{Hom}_\Gamma(\Gamma^n, F) \cong F([n]).$$

Therefore $\Gamma^n$, $n \geq 0$, are projective generators of the category $\Gamma$-mod.

1.2. Hochschild and Harrison (co)homology of $\Gamma$-modules. The definition of these objects are based on the following pointed maps (see [9]). For any $0 \leq i \leq n + 1$, one defines a map

$$\epsilon^i : [n + 1] \rightarrow [n], \quad 0 \leq i \leq n + 1,$$
by

\[ \epsilon^i(j) = \begin{cases} 
  j & j \leq i, \\
  j - 1 & j > i \leq n, \\
  0 & j = i = n + 1.
\end{cases} \]

For a left $\Gamma$-module $F$ the Hochschild homology $HH_*(F)$ is defined as the homology of the chain complex

\[ F([0]) \leftarrow F([1]) \leftarrow F([2]) \leftarrow \cdots \leftarrow F([n]) \leftarrow \cdots \]

where the boundary map $\partial : F([n+1]) \to F([n])$ is given by $\sum_{i=0}^{n+1} (-1)^i F(\epsilon^i)$.

Quite similarly for a right $\Gamma$-module $T$ one defines the Hochschild cohomology $HH^*(T)$ as the cohomology of the following cochain complex

\[ T([0]) \to T([1]) \to \cdots \to T([n]) \to T([n+1]) \to \cdots \]

where the coboundary map $\delta : T([n]) \to T([n+1])$ is given by $\delta = \sum_{i=0}^{n+1} (-1)^i T(\epsilon^i)$.

We have the following obvious fact.

**Lemma 1.1.** Let $F$ be a left $\Gamma$-module, then $HH_0(F) = F([0])$ and $HH_1(F) = \text{Coker}(\partial : F([2]) \to F([1]))$.

Similarly, if $T$ is a right $\Gamma$-module, then $HH^0(T) = T([0])$ and $HH^1(T) = \ker(\delta : T([1]) \to T([2]))$.

Let $S_n$ be the symmetric group on $n$ letters, it acts as a group of automorphisms on $[n]$. For integers $p_1, \ldots, p_k$ with $n = p_1 + \cdots + p_k$, we set

\[ sh_{p_1, \ldots, p_k} = \sum \text{sgn}(\sigma) \sigma \in \mathbb{Z}[S_n] \]

where $\sigma \in S_n$ is running over all $(p_1, \ldots, p_k)$-shuffles. Each $sh_{p_1, \ldots, p_k}$ induces a map $T([n]) \to T([n])$, called the shuffle map. Let us denote by $\mathcal{T}_n$ the intersection of the kernels of all shuffle maps. These groups form a subcomplex of the Hochschild complex, called Harrison complex [9]. The groups $\text{Harr}^n(T)$, $n \geq 0$ are defined as the cohomology of the Harrison complex.

By duality we have also Harrison and Hochschild homology of left $\Gamma$-module.

The following is a theorem due to J.-L. Loday [9]. For alternative approach see [12].

**Theorem 1.2.** If $K$ is a field of characteristic zero, then for any left $\Gamma$-module $F$ and right $\Gamma$-module $T$ there exist so called Hodge decompositions:

\[ HH_n(F) \cong \bigoplus_{i=1}^{n} HH_n^{(i)}(F), \quad n > 0, \]

\[ HH^n(F) \cong \bigoplus_{i=1}^{n} HH_n^{(i)}(T), \quad n > 0, \]

for suitable defined $HH_n^{(i)}(F)$ and $HH_n^{(i)}(T)$. Moreover, for $i = 1$ one has

\[ Harr_n(F) = HH_n^{(1)}(F), \quad Harr^n(T) = HH_n^{(1)}(T), \quad n > 0. \]
1.3. André-Quillen (co)homology of Γ-modules. We recall some material from [13].

A partition \( \lambda = (\lambda_1, \cdots, \lambda_k) \) is a sequence of natural numbers \( \lambda_1 \geq \cdots \geq \lambda_k \geq 1 \). The sum of partition is given by \( s(\lambda) := \lambda_1 + \cdots + \lambda_k \), while the group \( \Sigma(\lambda) \) is a product of the corresponding symmetric groups

\[
\Sigma(\lambda) := \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k},
\]

which is identified with the Young subgroup of \( \Sigma_{s(\lambda)} \). Let us observe that \( \Sigma_n = \text{Aut}_{T}([n]) \) and therefore \( \Sigma_n \) acts on \( F([n]) \) and \( T([n]) \) for any left \( \Gamma \)-module \( F \) and right \( \Gamma \)-module \( T \).

Let

\[
0 \to F_1 \to F \to F_2 \to 0
\]

be an exact sequence of left \( \Gamma \)-modules. It is called a \( \mathcal{Y} \)-exact sequence if for any partition \( \lambda \) with \( s(\lambda) = n \) the induced map

\[
F([n])^{\Sigma(\lambda)} \to F_2([n])^{\Sigma(\lambda)}
\]

is surjective. The class of \( \mathcal{Y} \)-exact sequences is proper in the sense of MacLane [11].

A left \( \Gamma \)-module \( F \) is \( \mathcal{Y} \)-projective, if the functor \( \text{Hom}_{\mathcal{Y}}(F, -) \) takes \( \mathcal{Y} \)-exact sequences to exact sequences. For example \( S^n\Gamma^1 \) is a \( \mathcal{Y} \)-projective [13]. Here \( S^n \) denotes the \( n \)-th symmetric power. According to [13] for any left \( \Gamma \)-module \( F \) there is a \( \mathcal{Y} \)-exact sequence

\[
0 \to F_1 \to F_0 \to F \to 0
\]

with \( \mathcal{Y} \)-projective \( F \). Hence one can take relative left derived functors of the functor \( HH_1 : \mathcal{Y}\text{-mod} \to \mathcal{K}\text{-mod} \). The values of these derived functors on a left \( \Gamma \)-module \( F \) is denoted by \( \pi^{\mathcal{Y}*}(F) \). So by the definition the functors \( \pi^{\mathcal{Y}*} \) are uniquely defined (up to isomorphism) by the following properties

**Lemma 1.3.** There exist unique family of functors \( \pi^{\mathcal{Y}*}_n : \mathcal{Y}\text{-mod} \to \mathcal{K}\text{-mod}, n \geq 0, \) such that

i) \( \pi^{\mathcal{Y}*}_0(F) = HH_1(F) \).

ii) For any \( \mathcal{Y} \)-exact sequence

\[
0 \to F_1 \to F \to F_2 \to 0
\]

there is a long exact sequence

\[
\cdots \to \pi^{\mathcal{Y}*}_{n+1}(F_2) \to \pi^{\mathcal{Y}*}_n(F_1) \to \cdots \to \pi^{\mathcal{Y}*}_0(F_1) \to \pi^{\mathcal{Y}*}_0(F) \to \pi^{\mathcal{Y}*}_0(F_2) \to 0.
\]

iii) The functor \( \pi^{\mathcal{Y}*}_n \) vanishes on \( \mathcal{Y} \)-projective objects, \( n \geq 1 \).

By a dual argument for any right \( \Gamma \)-module \( T \) one obtains \( \mathcal{K} \)-modules \( \pi^{\mathcal{Y}+*}(T) \).

1.4. \( \Gamma \)-modules and commutative algebras. The classical Hochschild cohomology (as well as the Harrison or André-Quillen (co)homology) of commutative algebras is a particular case of the cohomology of \( \Gamma \)-modules [9], [12], [13]. We recall the corresponding results. Let \( R \) be a commutative \( K \)-algebra and \( A \) be an \( R \)-module.

We have a left and right \( \Gamma \)-modules \( L_*(R, A) \) and \( L^*(R, A) \) defined on objects by

\[
L^*(R, A)([n]) := \text{Hom}(R \otimes^n A), \quad L_*(R, A)([n]) := R \otimes^n A.
\]

For a pointed map \( f : [n] \to [m] \), the action of \( L^*(R, A) \) on \( f \) is given by

\[
f^*(\psi)(a_1 \otimes \cdots \otimes a_n) = b_0 \psi(b_1 \otimes \cdots \otimes b_m)
\]

while for the functor \( L_*(R, A) \) one has:

\[
f_*(a_0 \otimes \cdots \otimes a_n) = b_0 \otimes \cdots \otimes b_m
\]

where \( b_j = \prod_{f(i) = j} a_i, \ j = 0, \cdots, n. \)

Then one has [9]:

\[
HH_*(L_*(R, A)) = HH_*(R, A), \quad HH^*(L^*(R, A)) = HH^*(R, A).
\]
\[ \text{Harr}_n(\mathcal{L}_c(R, A)) = \text{Harr}_m(R, A), \quad \text{Harr}_n(\mathcal{L}_c^*(R, A)) = \text{Harr}_m^*(R, A). \]

where \( HH_*(R, A) \) and \( \text{Harr}_*(R, A) \) (resp. \( HH^*(R, A), \text{Harr}^*(R, A) \)) are the Hochschild and Harrison (co)homology of \( R \) with coefficients in \( A \).

By [13] a similar result is also true for André-Quillen (co)homology of commutative rings. In order to state this result, let us first recall the definition of André-Quillen (co)homology [17].

Let \( \text{SCR} \) be the category of simplicial commutative rings and let \( \text{SS} \) be the category of simplicial sets and let \( U : \text{SCR} \to \text{SS} \) be a forgetful functor. According to [16] there is a unique closed model category structure on the category \( \text{SCR} \) such that a morphism \( f : X_\bullet \to Y_\bullet \) of \( \text{SCR} \) is weak equivalence (resp. fibration) if \( U(f) \) is a weak equivalence (resp. fibration) of simplicial sets. A simplicial commutative ring \( X_\bullet \) is called free if each \( X_n \) is a free commutative ring with a base \( S_n \), such that degeneracy operators \( s_i : X_n \to X_{n+1} \) maps \( S_n \) to \( S_{n+1} \), \( 0 \leq i \leq n \). Thanks to [16] any free simplicial commutative ring is cofibrant and any cofibrant object is a retract of a free simplicial commutative ring.

We let \( C^*(V^*) \) be the cochain complex associated to a cosimplicial abelian group \( V^* \). Let \( R \) be a commutative ring and let \( A \) be an \( R \)-module. Then the André-Quillen cohomology of \( R \) with coefficients in \( A \) is defined by (see [17]):

\[ D^*(R, A) := H^*(C^*(\text{Der}(P_\bullet, A))), \]

where \( P_\bullet \to R \) is a cofibrant replacement of the ring \( R \) considered as a constant simplicial ring and \( \text{Der} \) denotes the abelian group of all \( K \)-derivations.

The André-Quillen homology of \( R \) with coefficients in \( A \) is defined by

\[ D_*(R, A) := H_*(C_*(A \otimes_{P_\bullet} \Omega^1_{P_\bullet})), \]

where \( \Omega^1_{P_\bullet} \) is the Kähler differentials of a commutative \( K \)-algebra \( R \).

The main result of [13] claims that there are natural isomorphisms

\[ \pi^\mathcal{C}_*(\mathcal{L}_c(R, A)) = D_*(A, M), \quad \pi^\mathcal{C}_*(\mathcal{L}_c^*(R, A)) = D^*(A, M). \]

2. The category \( \mathcal{H}(C) \) associated to a commutative monoid \( C \)

2.1. Definition. Let \( C \) be a commutative monoid. Define the category \( \mathcal{H}(C) \) as follows. Objects of \( \mathcal{H}(C) \) are elements of \( C \). A morphism from an element \( a \in C \) to an element \( b \) is a pair \((c, a)\) of elements of \( C \) such that \( b = ca \). To simplify notations we write \( a \overrightarrow{\sim} ac \) for a morphism \((a, c) : a \to b = ac \). If \( a \overrightarrow{\sim} ac \) and \( ac \overrightarrow{\sim} acd \) are morphisms in \( \mathcal{H}(C) \), then the composite of these morphisms in \( \mathcal{H}(C) \) is \( a \overrightarrow{\sim} acd \).

It is clear that \( 1 \in C \) is an initial object of \( \mathcal{H}(C) \).

A left \( \mathcal{H}(C) \)-module is a covariant functor \( A : \mathcal{H}(C) \to \text{Ab} \), similarly a right \( \mathcal{H}(C) \)-module is a contravariant functor \( A : \mathcal{H}(C)^{\text{op}} \to \text{Ab} \).

We let \( \mathcal{H}(C)_{-\text{mod}} \) be the category of left \( \mathcal{H}(C) \)-modules, while \( \text{mod-} \mathcal{H}(C) \) denotes the category of right \( \mathcal{H}(C) \)-modules. If \( M \) is a left \( \mathcal{H}(C) \)-module. Then the value of \( M \) on the element \( a \in C \) (considered as object of \( \mathcal{H}(C) \)) is denoted by \( M(a) \). Moreover if \( a, b, c \in C \) and \( b = ca \), then we have an induced map \( c_* : M(a) \to M(b) \), with obvious properties \( 1_* = \text{id} \) and \( (c_1c_2)_* = c_1_*c_2_* \).

Quite similarly, if \( N \) is a right \( \mathcal{H}(C) \)-module, then the value of \( N \) on the element \( a \in C \) is denoted by \( N(a) \). Moreover if \( a, b, c \in C \) and \( b = ca \), then we have an induced map \( c^* : N(b) \to N(a) \), with obvious properties \( 1^* = \text{id} \) and \( (c_1c_2)^* = c_2^*c_1^* \).

The categories \( \mathcal{H}(C)_{-\text{mod}} \) and \( \text{mod-} \mathcal{H}(C) \) are abelian categories with enough projective and injective objects. For any element \( a \in C \) we let \( C^a \) and \( C_a \) be respectively the left and right \( \mathcal{H}(C) \)-modules defined by

\[ C^a(x) = \bigoplus_{c \in (x:a)} \mathbb{Z}, \quad C_a(x) = \bigoplus_{c \in (a:x)} \mathbb{Z}. \]
Here for elements $a, b \in C$ we let $(b : a)$ be the set of all elements $c \in C$ such that $b = ac$. By Yoneda lemma for any left $H(C)$-module $A$ and for any right $H(C)$-module $B$ one has isomorphisms
\[ \text{Hom}_{H(C)}(C^a, A) \cong A(a), \quad \text{Hom}_{H(C)}(C_a, B) \cong B(a). \]
It follows that $C^a$, $a \in C$ form a family of projective generators of the category $H(C)\text{-mod}$. Similarly $C_a$, $a \in C$ form a family of projective generators of the category $\text{mod-}H(C)$.

Let $N$ be a right $H(C)$-module and $M$ be a left $H(C)$-module. We let $N \otimes_{H(C)} M$ be the abelian group generated by elements of the form $x \otimes y$, where $x \in N(a)$, $y \in M(a)$, $a \in M$. These elements are subject to the following relations
\[ (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \]
\[ x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, \]
\[ c^*(z) \otimes y = z \otimes c_*(y). \]
Here $c \in M$, $x, x_1, x_2 \in N(a)$, $y, y_1, y_2 \in M(a)$, $z \in N_{ca}$. Then one has
\[ N \otimes_{H(C)} C^a \cong N(a), \quad C_a \otimes_{H(C)} M \cong M(a). \]

If $f : C \to C'$ is a homomorphism of monoids, then $f$ induces a functor $H(f) : H(C) \to H(C')$ in an obvious way. Thus for any left $H(C')$-module $M$ one has a left $H(C)$-module $f^*(M)$, which is given by
\[ f^*(M)(i) = M(f(i)). \]
In this way one obtains a functor $f^*$ from the category of (left or right) modules over $H(C')$ to the category of modules over $H(C)$.

2.2. $K[C]$-modules and $H(C)$-modules. We let $K[C]$ be the monoid algebra of the monoid $C$. Any $K[C]$-module $A$ gives rise to the left $H(C)$-module $j^*(A)$ which is defined by
\[ j^*(A)(a) = A \]
and for $b = ca$, the induced morphisms
\[ A = j_*(A)(a) \xrightarrow{c_*} j_*(A)(b) = A \]
is simply the multiplication by $c$.

If $M$ is a left $H(C)$-module, we let $j_*(M)$ be the following $K[C]$ module. As an abelian group one has
\[ j_*(M) = \bigoplus_{x \in C} M(x), \]
The module structure is defined as follows: for $x \in C$, $a \in M(x)$ and $c \in C$ one has
\[ ci_x(a) = i_{ca}(c_*(a)). \]
Here $i_x$ is the canonical inclusion $M(x) \to j_*(M)$, $x \in C$.

**Lemma 2.1.** The functor $j_*$ is a left adjoint functor to $j^*$.

**Proof.** For a left $H(C)$-module $M$ and a left $K[C]$-module $A$ an elements
\[ \xi \in \text{Hom}_{H(C)}(M, j^*(A)) \]
is given by the family of $K$-module homomorphisms $\xi_a : M(a) \to A$, $a \in A$ such that for any $c \in C$ the following
\[
\begin{array}{ccc}
M(a) & \xrightarrow{\xi_a} & A \\
\downarrow{c_*} & & \downarrow{c} \\
M(ac) & \xrightarrow{\xi_{ac}} & A
\end{array}
\]
The homomorphisms $\xi_a$, $a \in C$ defines a homomorphism of $K$-modules
\[ \hat{\xi} : j_*(M) = \bigoplus_{a \in C} M(a) \to A \]
which clearly is $K[M]$-homomorphism. So, $\xi \mapsto \hat{\xi}$ gives rise a homomorphism $\text{Hom}_{\mathcal{H}(C)}(M, j^*(A)) \to \text{Hom}_{K[C]}(j_*(M), A)$ which is obviously an isomorphism. \hfill \Box

2.3. Derivations, differentials and (co)homology in the theory of commutative algebras. Let $C$ be a commutative monoid and let $M$ be a left $\mathcal{H}(C)$-module. A derivation $\delta : C \to M$ with values in $M$ is a function which assigns to each element $a \in C$ an element $\delta(a) \in M(a)$, such that
\[ \delta(ab) = a_*(\delta(b)) + b_*(\delta(a)) \]
The abelian group of all derivations of $C$ with values in $M$ is denoted by $\text{Der}(C, M)$.
We claim that there exist a universal derivation. In fact we construct a left $\mathcal{H}(C)$-module $\Omega_C$, called differentials of a monoid $C$. It is a left $\mathcal{H}(C)$-module generated by symbols $da \in \Omega_C(a)$ one for every element $a \in C$, subject to relations
\[ d(ab) = a_*(d(b)) + b_*(d(a)) \]
for every $a$ and $b \in C$. It follows from the construction that $a \mapsto da$ is a derivation, which is clearly universal one, in the sense that for any derivation $\delta : C \to M$ there is a unique homomorphism of $\mathcal{H}(C)$-modules $\delta^* : \Omega_C \to M$ such that $\delta(a) = \delta^*(da)$. Thus for any left $\mathcal{H}(C)$-module $M$ one has a canonical isomorphism
\[ \text{Der}(C, M) \cong \text{Hom}_{\mathcal{H}(C)}(\Omega_C, M). \]

Lemma 2.2. One has an isomorphism of $K[C]$-modules
\[ j_*(\Omega_C) = \Omega^1_{K[C]} \]
Here $j_* : \mathcal{H}(C)\text{-mod} \to K[C]\text{-mod}$ is the functor constructed in Section 2.2 and $\Omega^1_{K[C]}$ is the Kähler differentials of the $K$-algebra $K[C]$.

Proof. Let $A$ be a $K[C]$-module. Then we have
\[ \text{Der}(C, j^*(A)) = \text{Hom}_{\mathcal{H}(C)}(\Omega_C, j^*(A)) = \text{Hom}_{K[C]}(j_*(\Omega), A). \]

On the other hand
\[ \text{Der}(C, j^*(A)) = \text{Der}(K[C], A) = \text{Hom}_{K[C]}(\Omega^1_{K[A]}, A) \]
and the result follows from the Yoneda lemma. \hfill \Box

2.4. The case $C = \mathbb{N}$. If $C$ is the free abelian monoid with a generator $t$, then a left $\mathcal{H}(C)$-module is nothing but a diagram of abelian groups
\[ M = (M_0 \xrightarrow{t} M_1 \xrightarrow{t} M_2 \xrightarrow{t} M_3 \xrightarrow{t} \cdots) \]
In particular the projective object $C^n$ corresponds to the diagram
\[ 0 \to 0 \to \cdots \to 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \cdots \]
where the first nontrivial group appears at the place $n$.

Quite similarly a right $\mathcal{H}(C)$-module is nothing but a diagram of abelian groups
\[ N = (\cdots \xrightarrow{t} N_3 \xrightarrow{t} N_2 \xrightarrow{t} N_1 \xrightarrow{t} N_0). \]

In particular the projective object $C_n$ corresponds to the diagram
\[ \cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \cdots \xrightarrow{1} \mathbb{Z} \]
where the first nontrivial group appears at the place $n$.

One easily observes that for any left $\mathcal{H}(C)$-module $M$ one has an isomorphism
\[ \text{Der}(C, M) \cong M_1 \]
which is given by $\delta \mapsto \delta(t)$. This follows from the fact that $\delta(t^n) = nt^{n-1}\delta(t)$. Thus
\[
\Omega_C = C^1 = (0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \cdots).
\]

2.5. **Product of two monoids.** Let $C$ be a product of two monoids: $C = C_1 \times C_2$. Then $H(C) = H(C_1) \times H(C_2)$. Assume $M_1$ and $M_2$ are (say left) $H(C_1)$ and $H(C_2)$-modules respectively. Then one can form a $H(C)$-module $M_1 \boxtimes M_2$ as follows:
\[
M_1 \boxtimes M_2(x_1, x_2) = M_1(x_1) \otimes M_2(x_2).
\]

**Lemma 2.3.** For any elements $c_1 \in C_1$ and $c_2 \in C_2$, one has
\[
C^{(c_1, c_2)} = C^{c_1} \boxtimes C^{c_2}
\]
and
\[
C_{(c_1, c_2)} = C_{c_1} \boxtimes C_{c_2}.
\]

**Proof.** By definition one has
\[
C^{c_1} \boxtimes C^{c_2}(x_1, x_2) = C^{c_1}(x_1) \otimes C^{c_2}(x_2)
\]
\[
= \left( \bigoplus_{a_1 \in C_1; a_1c_1 = x_1} \mathbb{Z} \right) \otimes \left( \bigoplus_{a_2 \in C_2; a_2c_2 = x_2} \mathbb{Z} \right)
\]
\[
= \bigoplus_{(a_1, a_2)(c_1, c_2) = (x_1, x_2)} \mathbb{Z}
\]
\[
= C^{(c_1, c_2)}(x_1, x_2).
\]

Similarly for the second isomorphism. \qed

We have homomorphisms
\[
\iota_1 : C_1 \to C, \quad \iota(c_1) = (c_1, 1), \quad \iota_2 : C_2 \to C, \quad \iota(c_2) = (1, c_2).
\]

For any left $H(C)$-module $M$ we set
\[
M^{(1)} = \iota_1^*(M), \quad M^{(2)} = \iota_2^*(M).
\]

**Lemma 2.4.** For any left $H(C)$-module $M$ one has
\[
\text{Der}(C, M) \cong \text{Der}(C_1, M^{(1)}) \oplus \text{Der}(C_2, M^{(2)}).
\]

**Proof.** This easily follows from the fact $(c_1, c_2) = (c_1, 1)(1, c_2)$. \qed

We also have projections $\pi_1 : C \to C_1$ and $\pi_2 : C \to C_2$, given respectively by $\pi_i(c_1, c_2) = c_i, \ i = 1, 2$.

**Lemma 2.5.** For any left $H(C_i)$-module $X_i$, $i = 1, 2$ and any left $H(C)$-module $M$, one has isomorphisms
\[
\text{Hom}_{H(C)}(\pi_1^* X_1, M) \cong \text{Hom}_{H(C_i)}(X_1, M^{(1)})
\]
and
\[
\text{Hom}_{H(C)}(\pi_2^* X_2, M) \cong \text{Hom}_{H(C_i)}(X_2, M^{(2)}).
\]

**Proof.** Let $\eta \in \text{Hom}_{H(C)}(\pi_1^* X_1, M)$. Thus $\eta$ is a collection of homomorphisms of $K$-modules
\[
\eta_{(a_1, a_2)} : X_{a_1} \to M_{(a_1, a_2)}
\]
such that for any elements $c_1 \in C_1, c_2 \in C_2$ the following diagram commutes
\[
\begin{array}{ccc}
X_{a_1} & \xrightarrow{\eta_{(a_1, a_2)}} & M_{(a_1, a_2)} \\
\downarrow c_1 & & \downarrow (c_1, c_2) \\
X_{a_1c_1} & \xrightarrow{\eta_{(a_1c_1, a_2c_2)}} & M_{(a_1c_1, a_2c_2)}
\end{array}
\]
it follows that $\eta(a_1, a_1) = (1, a_2)_* \circ \eta(a_1, 1)$. It is clear that the family of homomorphisms $\eta_{a_1, 1}$, $a_1 \in C_1$ defines the morphism $\tilde{\eta} \in \text{Hom}_{H(C)}(X_1, M^{(1)})$ and the previous equality shows that $\eta \mapsto \tilde{\eta}$ is really a bijection. \qed

Corollary 2.6. If $C = C_1 \times C_2$, then

$$\Omega_C = \pi_1^* \Omega_{C_1} \oplus \pi_2^* \Omega_{C_1},$$

where $\pi_i : C \to C_i$, $i = 1, 2$ is the canonical projection.

Proof. For any $H(C)$-module $M$ one has

$$\text{Hom}_{H(C)}(\Omega_C, M) = \text{Der}(C, M) = \text{Der}(C_1, M^{(1)}) \oplus \text{Der}(C_2, M^{(2)})$$

$$= \text{Hom}_{H(C)}(\Omega_{C_1}, M^{(1)}) \oplus \text{Hom}_{H(C)}(\Omega_{C_2}, M^{(2)})$$

$$= \text{Hom}_{H(C)}(\pi_1^* \Omega_{C_1}, M) \oplus \text{Hom}_{H(C)}(\pi_2^* \Omega_{C_2}, M)$$

and the result follows from the Yoneda lemma. \qed

3. Commutative monoid (co)homology and $\Gamma$-modules

3.1. $\Gamma$-modules related to monoids. Let $C$ be a commutative monoid and let $M$ be a left $H(C)$-module. Define a right $\Gamma$-module $G^*(C, M)$ as follows. On objects it is given by

$$G^*(C, M)([n]) = \prod_{(a_1, \ldots, a_n) \in C^n} M_{a_1 \cdots a_n}.$$ 

Thus $\eta \in G(C, M)([n])$ is a function which assigns to any $n$-tuple of elements $(a_1, \ldots, a_n)$ of $C$ an element $\eta(a_1, \ldots, a_n) \in M_{a_1 \cdots a_n}$. Let $f : [n] \to [m]$ be a pointed map and $\xi \in G(C, M)([m])$. Then the function $f^*(\xi) \in G(C, M)([n])$ is given by

$$f^*(\xi)(a_1, \ldots, a_n) = b_{0*}(\xi(b_1, \ldots, b_m)).$$

Quite similarly, let $N$ be a right $H(C)$-module. Define left $\Gamma$-module $G_*(C, N)$ as follows. On objects it is given by

$$G_*(C, N)([n]) = \bigoplus_{(a_1, \ldots, a_n) \in C^n} N(a_1 \cdots a_n).$$

In order, to extend the definition on morphism, we let

$$\iota_{(a_1, \ldots, a_n)} : N(a_1 \cdots a_n) \to G_*(C, N)([n])$$

be the canonical inclusion. Let $f : [n] \to [m]$ be a pointed map. Then the homomorphism

$$f_* : G_*(C, N)([n]) \to G_*(C, N)([m])$$

is defined by

$$f_*[a_1, \ldots, a_n](x) = \iota_{(b_1, \ldots, b_m)}((b_0)_*[x]),$$

where $x \in N(a_1 \cdots a_n)$ and

$$b_j = \prod_{f(i) = j} a_i, \quad j = 0, \ldots, n.$$ 

Here we used the convention that $b_0 = 1$ provided $f^{-1}([0]) = \{0\}$.

Lemma 3.1. Let $C = \mathbb{N}$ be a free commutative monoid with a generator $t$, and let $C_n$ be the standard projective right $H(C)$-module, $n \geq 0$, see Section 2.4. Then one has an isomorphism of left $\Gamma$-modules

$$G_*(C, C_n) = \bigoplus_{k=0}^n S^k \circ \Gamma^1$$
In particular, $G_*(C, C_n)$ is $\mathcal{Y}$-projective.

Proof. Since $\Gamma^1([m])$ is a free $K$-module spanned on $x_1, \ldots, x_m$, it follows that $S^k \circ \Gamma^1([m])$ is a free $K$-module spanned by all monomials of degree $k$ on the variables $x_1, \ldots, x_m$. On the other hand we have

$$G_*(C, C_n) ([m]) = \bigoplus_{k=0}^{n} \bigoplus_{n_1 + \ldots + n_m = k} \mathbb{Z}.$$  

To see the expected isomorphism, it is enough to assign to a basis element of $\bigoplus_{n_1 + \ldots + n_m = m} \mathbb{Z}$ the monomial $x_1^{n_1} \ldots x_m^{n_m}$.

Lemma 3.2. Let $C = C_1 \times C_2$ be product of two monoids and $N_i$ be right $\mathcal{H}(C_i)$ modules, $i = 1, 2$. Then one has

$$G_*(C, N_1 \boxtimes N_2) = G_*(C, N_1) \otimes G_*(C, N_2).$$

The proof is straightforward.

Corollary 3.3. Let $C$ be a finitely generated free commutative monoid and let $N$ be a projective object in the category of right $\mathcal{H}(C)$-modules. Then $G_*(C, N)$ is a $\mathcal{Y}$-projective left $\Gamma$-module.

Proof. Since, any projective object is a retract of a direct sum of standard projective modules $C^c$, it is enough to restrict ourself with the case when $C = C^c$. Assume $C = \mathbb{N}^k$. We will work by induction on $k$. If $k = 1$, then the result was already established, see Lemma 3.1. Rest follows from Lemma 2.4 and Lemma 3.2 and the fact that tensor product of two $\mathcal{Y}$-projective objects is $\mathcal{Y}$-projective see [13].

3.2. Homology and cohomology of commutative monoids. Let $CM$ be the category of all commutative monoids and let $SCM$ be the category of all simplicial commutative monoids. There is a forgetful functor $U' : SCM \to SS$. By [16] there is unique closed model category structure on the category $SCM$ such that a morphism $f : X_\cdot \to Y_\cdot$ of $SCM$ is a weak equivalence (resp. fibration) if $U'(f)$ is a weak equivalence (resp. fibration) of simplicial sets. A simplicial commutative monoid $X_\cdot$ is called free if each $X_n$ is a free commutative monoid with a base $Y_n$, such that degeneracy operators $s_i : X_n \to X_{n+1}$ maps $Y_n$ to $Y_{n+1}$, $0 \leq i \leq n$. According to [16] any free simplicial commutative monoid is cofibrant and any cofibrant object is a retract of a free simplicial commutative monoid.

If $C' \to C$ is a morphism of commutative monoids then it gives rise to a functor $\mathcal{H}(C') \to \mathcal{H}(C)$, which allows us to consider any left or right $\mathcal{H}(C)$-module as a module over $\mathcal{H}(C')$. In particular if $P_\cdot \to C$ is an augmented simplicial monoid and $M$ is a left $\mathcal{H}(C)$-module, one can considered $M$ as a left $\mathcal{H}(P_k)$-module, for all $k \geq 0$. The same holds for right $\mathcal{H}(C)$-modules.

Let $M$ be a left $\mathcal{H}(C)$-module. Then the Grillet cohomology of $C$ with coefficients in $M$ is defined by

$$D^*(C, M) := H^*(\mathcal{D}(P_\cdot, M)),$$

where $P_\cdot \to C$ is a cofibrant replacement of the monoid $C$ considered as a constant simplicial monoid.

Let $N$ be a right $\mathcal{H}(C)$-module. Then the Grillet homology of $C$ with coefficients in $N$ is defined by

$$D_*(C, N) := H_*(C_\cdot(\Omega P_\cdot \otimes_{\mathcal{H}(P_\cdot)} N)),$$

where $P_\cdot \to C$ is a cofibrant replacement of the monoid $C$ considered as a constant simplicial monoid.

The definition of the cohomology essentially goes back to Grillet (see [2]-[5]), but the definition of the Grillet homology is new.

By comparing the definition we obtain the following basic fact, which is missing in (see [2]-[5]).
Lemma 3.4. Let C be a commutative monoid and A be a \( K[C] \)-module. Then one has the isomorphisms:

\[
\begin{align*}
\mathcal{D}^*(C,j^*(A)) & \cong \mathcal{D}^*(K[C],A), \\
\mathcal{D}_*(C,j^*(A)) & \cong \mathcal{D}_*(K[C],A).
\end{align*}
\]

Proof. The isomorphism in the dimension zero is obvious, compare with Lemma 2.2. Rest follows from the fact that if \( P_\ast \rightarrow C \) is a cofibrant replacement of \( C \) in the category \( \text{SCR} \), then \( K[P_\ast] \rightarrow K[C] \) is a cofibration replacement of \( K[C] \) in the category \( \text{SCR} \).

3.3. The main Theorem. Now we are in the situation to state our main theorem, which relates Grillet (co)homology of the monoid \( M \) with the Andre-Quillen (co)homology of the \( \Gamma \)-modules \( \mathcal{G}_*(C,N) \) and \( \mathcal{G}^*(C,M) \).

Theorem 3.5. Let \( C \) be a commutative monoid, \( M \) be a left and \( N \) be a right \( \mathcal{H}(C) \)-modules. Then one has the following isomorphisms

\[
\begin{align*}
\mathcal{D}^*(C,M) & = \pi \mathcal{H}^*(\mathcal{G}^*(C,M)), \\
\mathcal{D}_*(C,M) & = \pi \mathcal{H}_*(\mathcal{G}_*(C,N)).
\end{align*}
\]

The proof is based on several steps. The idea is to reduce the theorem to the case when \( M \) is a free commutative monoid with one generator. In this case, the theorem is proved using direct computation. We give proof only for homology, a dual argument works for cohomology.

We need some lemmata.

Lemma 3.6. Let \( C \) be a commutative monoid, \( N \) be a right \( \mathcal{H}(C) \)-module. Then one has natural isomorphisms

\[
\mathcal{H}H_1(\mathcal{G}_*(C,N)) \cong N \otimes_{\mathcal{H}(C)} \Omega_C.
\]

Proof. Thanks to Lemma 1.1 one has \( \mathcal{H}H_1(\mathcal{G}_*(C,N)) \) is isomorphic to the cokernel of the map

\[
\partial : \bigoplus_{a,b \in C} N(ab) \rightarrow \bigoplus_{a \in C} N(a)
\]

As usual, we let \( i_a : N(a) \rightarrow \bigoplus_{a \in C} N(a) \) be the canonical inclusion. For an element \( x \in N(a) \), the class of \( i_a(x) \) in \( \mathcal{H}H_1(\mathcal{G}_*(C,N)) \) is denoted by \( \text{cl}(a;x) \). Then

\[
\text{cl}(a;x) \mapsto x \otimes da
\]

defines the isomorphism \( \mathcal{H}H_1(\mathcal{G}_*(C,N)) \cong N \otimes_{\mathcal{H}(C)} \Omega_C \).

Lemma 3.7. Let \( C \) be a commutative monoid and let

\[
0 \rightarrow N_1 \rightarrow N \rightarrow N_2 \rightarrow 0
\]

be a short exact sequence of right \( \mathcal{H}(C) \)-modules, then

\[
0 \rightarrow \mathcal{G}_*(C,N_1) \rightarrow \mathcal{G}_*(C,N) \rightarrow \mathcal{G}_*(C,N_2) \rightarrow 0
\]

is a \( \mathcal{Y} \)-exact sequence of left \( \Gamma \)-modules.

Proof. For a partition \( \lambda \) of \( n \) and a set \( P \) we denote by \( P^\lambda \) the set of orbits of the cartesian product \( P^n \) under the action of the group \( \Sigma(\lambda) \subset \Sigma_n \). In particular we have a set \( C^\lambda \). For any element \( \mu \in C^\lambda \) we put \( N_\mu := N(a_1 \cdots a_n) \), where \( (a_1, \cdots, a_n) \in \mu \). Since

\[
\mathcal{G}_*(C,N)([n])^{\Sigma(\lambda)} = \bigoplus_{\mu \in C^\lambda} N_\mu
\]

the result follows. \( \square \)

By the same argument we have also the following.
Lemma 3.8. Let \( f : D \to C \) be a surjective homomorphism of commutative monoids, then for any right \( \mathcal{H}(C) \)-module \( N \) the induced morphism of left \( \Gamma \)-modules
\[
G_* (D, M) \to G_* (C, M)
\]
is a \( \mathcal{Y} \)-epimorphism.

Proof. In the notation of the proof Lemma 3.7 the map \( D^\lambda \to C^\lambda \) is surjective and the result follows. \( \square \)

Lemma 3.9. Let \( \epsilon : X_* \to C \) be a simplicial resolution in the category of commutative monoids and \( N \) be a right \( \mathcal{H}(C) \)-module. Then the associated chain complexes of the simplicial left \( \Gamma \)-module \( G_* (X_* , N) \to G_* (C, N) \) is a \( \mathcal{Y} \)-resolution.

Proof. Since \( X_*^\lambda \to C^\lambda \) is a weak equivalence the result follows. \( \square \)

3.4. Proof of Theorem 3.5. Thanks to Lemma 3.6 Theorem is true for \( i = 0 \). First we consider the case, when \( C = \mathbb{N} \) is the free commutative monoid with a generator \( t \). In this case \( D_i (C, -) = 0 \), if \( i > 0 \). On the other hand \( G_* (C, C_n) \) is \( \mathcal{Y} \)-projective thanks to Lemma 3.1. Therefore the result is true in this case. It follows from Lemma 2.3, Lemma 3.2 and Lemma 4.2 of [13] that the result is true if \( C \) is a free commutative monoid and \( N \) is projective. By Lemma 3.7 the functor \( \pi^\lambda_0 (G_* (C, -)) \) assigns the long exact sequence to a short exact sequence of right \( \mathcal{H}(C) \)-modules. Therefore we can consider such an exact sequence associated to a short exact sequence of right \( \mathcal{H}(C) \)-modules
\[
0 \to N_1 \to F \to N \to 0
\]
with projective \( F \). Since the result is true if \( i = 0 \) one obtains by induction on \( i \), that \( AQ_i (G_* (C, -)) = 0 \) provided \( i > 0 \) and \( C \) is free commutative monoid.

Now consider the general case. Let \( P_* \to C \) be a free simplicial resolution in the category of commutative monoids. Then we have
\[
N \otimes_{\mathcal{H}(P)} \Omega \cong \pi^\lambda_0 (G_* (C, N))
\]
Thanks to Lemma 3.9 \( C_* (G_* (P_* , N)) \to G_* (C, N) \) is a \( \mathcal{Y} \)-resolution consisting with \( AQ_* \)-acyclic objects and the result follows.

3.5. Applications. Let \( C \) be a commutative monoid, \( M \) be a left \( \mathcal{H}(C) \)-module and \( N \) be a right \( \mathcal{H}(C) \)-module. For the \( \Gamma \)-modules \( G_* (C, N) \) and \( G^* (C, M) \) one can apply the reach theory of functor homology developped in [9], [12], [13]. For example if one applies the Hochschild and Harrison theories one gets groups \( HH_*(C, N) \), \( Harr_*(C, N) \) and \( HH^*(C, M) \), \( Harr^*(C, M) \). Comparing with definitions one sees that \( HH^*(C, M) \) is nothing but Leech cohomology [8]. If \( K \) is a field of characteristic zero, then we have
\[
D_* (C, N) = Harr_{*+1} (C, N), \quad D^* (C, M) = Harr^{*+1} (C, M)
\]
this follows from general results valid for arbitrary \( \Gamma \)-modules [12], [13]. In particular this solves the cocycle problem for Grillet cohomology [6]. By theorem 1.2 we also obtain that the Grillet cohomology is a direct summand of Leech cohomology.

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