GEOMETRY OF MULTISYMPLECTIC HAMILTONIAN FIRST-ORDER FIELD THEORIES

Arturo Echeverría-Enríquez,
Miguel C. Muñoz-Lecanda,*
Narciso Román-Roy†
Departamento de Matemática Aplicada y Telemática
Edificio C-3, Campus Norte UPC
C/ Jordi Girona 1. E-08034 BARCELONA. SPAIN

October 29, 2018

Abstract

In the jet bundle description of Field Theories (multisymplectic models, in particular), there are several choices for the multimomentum bundle where the covariant Hamiltonian formalism takes place. As a consequence, several proposals for this formalism can be stated, and, on each one of them, the differentiable structures needed for setting the formalism are obtained in different ways. In this work we make an accurate study of some of these Hamiltonian formalisms, showing their equivalence. In particular, the geometrical structures (canonical or not) needed for the Hamiltonian formalism, are introduced and compared, and the derivation of Hamiltonian field equations from the corresponding variational principle is shown in detail. Furthermore, the Hamiltonian formalism of systems described by Lagrangians is performed, both for the hyper-regular and almost-regular cases. Finally, the role of connections in the construction of Hamiltonian Field theories is clarified.

Key words: Jet bundles, Connections, First order Field Theories, Hamiltonian formalism.

AMS s. c. (2000): 53C05, 53C80, 55R10, 55R99, 58A20, 70S05.
PACS (1999): 02.40.Hw, 02.40.Vh, 11.10.Ef, 45.10.Na

*e-mail: MATMCML@MAT.UPC.ES
†e-mail: MATNRR@MAT.UPC.ES
Contents

1 Introduction 2

2 Geometrical background of the Lagrangian and Hamiltonian formalisms 4
  2.1 Lagrangian systems .................................................. 4
  2.2 Multimomentum bundles and Legendre maps ...................... 5
  2.3 Canonical forms ................................................... 10
  2.4 Regular and singular systems ...................................... 11

3 Hamiltonian formalism in the reduced multimomentum bundle 14
  3.1 Hamiltonian systems ................................................ 14
  3.2 Hamiltonian sections, Hamiltonian densities and connections .... 17
  3.3 Variational principle and field equations ......................... 19
  3.4 Hamiltonian system associated with a hyper-regular Lagrangian system .... 23
  3.5 Hamiltonian system associated with an almost-regular Lagrangian system ... 26
  3.6 Equivalence between the Lagrangian and Hamiltonian formalisms ............ 29

4 Hamiltonian formalism in the restricted multimomentum bundle $J^1\pi^*$. Relation with the formalism in $\Pi$ 32
  4.1 Hamiltonian systems ................................................ 32
  4.2 Hamiltonian system associated with a hyper-regular Lagrangian system .... 35
  4.3 Hamiltonian system associated with an almost-regular Lagrangian system ... 37

5 Examples 39
  5.1 Non-autonomous Mechanics ........................................ 39
  5.2 Electromagnetic field (with fixed background) .................... 40

6 Conclusions 43

A Geometrical structures in first-order jet bundles 45
1 Introduction

The application of techniques of differential geometry to the study of physical theories has been revealed as a very suitable method for better understanding many features of these theories. In particular, the geometric description of classical Field Theories is an area of increasing interest.

The standard geometrical techniques used for the covariant Lagrangian description of first-order Field Theories, involve first order jet bundles $J^1E \xrightarrow{\pi_1} E \xrightarrow{\pi} M$ and their canonical structures (see, for instance, [1], and references quoted therein). Nevertheless, for the covariant Hamiltonian formalism of these theories the situation is rather different, and there are different kinds of geometrical descriptions for this formalism. For instance, we can find models such as those described in [2], [10] and [37], which use $k$-symplectic forms, or in [3], [11], [12] and [13], where the essential geometric structure are the $k$-cosymplectic forms, or also as in [25], [29] and [30], where use is made of polysymplectic forms (in fact, $k$-symplectic, $k$-cosymplectic and polysymplectic structures are essentially equivalent objects). In this work, we consider only the polysymplectic models [1], [3], [24], [27], [28], [14], [15], and depending on the choice of the multimomentum phase space there are different ones. In fact:

1. There are some models where the multimomentum phase space is taken to be $\mathcal{M}\pi \equiv \Lambda^mT^*E$, the bundle of $m$-forms on $E$ ($m$ being the dimension of $M$) vanishing by the action of two $\pi$-vertical vector fields. This choice is made in works such as [22], [23] and [24], as a refinement of the techniques previously given in [31], [32] and [33] (see also [35] and [43]).

2. The multimomentum phase space $J^1\pi^* \equiv \Lambda^1_0T^*E/\Lambda^m_0T^*E$ (where $\Lambda^m_0T^*E$ is the bundle of $\pi$-semibasic $m$-forms in $E$) has been studied in [1] and used, later on, in [29], [36] and [37] for the analysis of different aspects of Hamiltonian Field Theories.

3. Finally, in [8], [16], [17], [19], [19], [30] and [31] the basic choice is the bundle $\Pi \equiv \pi^*TM \otimes V^*(\pi) \otimes \pi^*\Lambda^mT^*M$ (here $V^*(\pi)$ denotes the dual bundle of the $\pi$-vertical subbundle $V(\pi)$ of $TE$) which, in turn, is canonically related to $J^1E^* \equiv \pi^*TM \otimes T^*E \otimes \pi^*\Lambda^mT^*M$.

Although in [43] (and later papers by these authors), a covariant Hamiltonian formalism is constructed in $\mathcal{M}\pi$, in most of the works, this multimomentum bundle is not really used in order to establish a Hamiltonian formalism on $\mathcal{M}\pi$, but just for defining canonical differential structures which, translated to $J^1E$ and $J^1\pi^*$, are used for setting the Lagrangian and Hamiltonian formalisms, respectively. The choice of $J^1\pi^*$ or $\Pi$ as multimomentum phase space allows us to state covariant Hamiltonian formalisms for Field Theories. Nevertheless, none of them have canonical structures, so the Hamiltonian forms of the Hamiltonian formalism must be obtained from the canonical forms of the multicotangent bundle $\Lambda^mT^*E$. This is done by using sections of the projection $\mathcal{M}\pi \rightarrow J^1\pi^*$, (or $J^1E^* \rightarrow \Pi$) which are called Hamiltonian sections [3], or the so-called Hamiltonian densities [17], [49]. To our knowledge, a rigorous analysis comparing these formulations and their equivalence has not been done. The aim of this work is to carry out a comparative study of some of these Hamiltonian formulations, establishing the equivalence between them. In every case, the geometrical structures needed for setting the field equations in the Hamiltonian formalism are introduced, as well as the corresponding Legendre maps when the multimomentum bundles are related to a Lagrangian system.

The question of whether the use of connections in the bundle $\pi:E \rightarrow M$ is needed for the construction of the covariant formalisms in Field theories is studied. It was analyzed for the first time in [3], where a connection was used to define Hamiltonian densities in the Hamiltonian formalism, and in [11] for the case of the density of Lagrangian energy in the Lagrangian formalism. In this
work we make a deeper analysis on the role played by connections in the construction of Hamiltonian systems.

An obvious subject of interest is the statement of the Hamiltonian field equations. In all the multisymplectic models field equations are obtained by characterizing the critical sections which are solutions of the problem by means of the multisymplectic form \[1\], \[11\], \[15\], \[21\]. This characterization can be derived from a variational principle: the so-called Hamilton principle in the Lagrangian formalism and Hamilton-Jacobi principle in the Hamiltonian one. Nevertheless, this aspect of the theory is overlooked in many papers. We give an accurate derivation of the Hamiltonian equations starting from the Hamilton-Jacobi principle, and the role played by connections in the statement of covariant Hamiltonian equations is discussed.

An important kind of Hamiltonian systems are those which are the Hamiltonian counterpart of Lagrangian systems. The construction of such systems starting from the Lagrangian formalism is carried out by using a Legendre map associated with the Lagrangian density and the corresponding multimomentum bundle. This problem has been studied by different authors in the (hyper) regular case (see, for instance, \[3\], \[53\]), and in the singular (almost-regular) case \[17\], \[37\], \[49\]. In this work we review some of these constructions, developing new methods, and giving a unified perspective of all of them.

The structure of the work is as follows:

Section 2 is devoted to a review of the main features of the Lagrangian formalism of Field theories, and afterwards the definition of the different multimomentum bundles for the Hamiltonian formalism, as well as the construction and characterization of the canonical forms with which some of them are endowed. Furthermore, when these multimomentum bundles are related with a Lagrangian system, the corresponding Legendre maps are introduced for both the (hyper)-regular and the almost-regular cases.

In section 3, we undertake the construction of Hamiltonian systems in the multimomentum bundle \(\Pi\). As a first step, we will define the Hamiltonian forms which allow us to set the field equations in an intrinsic way. Since \(\Pi\) has no canonical geometric form, we must use the canonical forms with which \(M_\pi\) and \(J^1E^*\) are endowed. Ways of constructing Hamiltonian systems are studied and compared, and in this multimomentum bundle we make a careful deduction of the Hamiltonian equations from the variational principle. In addition, the Hamiltonian formalism associated to a Lagrangian system is developed, both for the hyper-regular and almost-regular cases. Finally, the equivalence between the Lagrangian and Hamiltonian formalisms is proved (for the hyper-regular case).

The construction of Hamiltonian systems in the multimomentum bundle \(J^1\pi^*\) is stated and analyzed in Section 4, following the same pattern as in the above section, and proving the equivalence between the formalisms developed for both multimomentum bundles.

As typical examples, time-dependent mechanics and the electromagnetic field are analyzed (in this context) in Section 5.

An appendix describing the basic geometrical structures in first-order jet bundles is included.
π₁, that is, V(π₁) = Ker Tπ₁, and by $X^{V(\pi^1)}(J^1E)$ the corresponding sections or vertical vector fields. Finally, $(x^\nu, y^A, v^\nu_\nu)$ (with $\nu = 1, \ldots, m$; $A = 1, \ldots, N$) will be natural local systems of coordinates in $J^1E$ adapted to the bundle $\pi: E \to M$, and such that $\omega = dx^1 \wedge \ldots \wedge dx^m \equiv d^m x$.

\section{Geometrical background of the Lagrangian and Hamiltonian formalisms}

\subsection{Lagrangian systems}

From the Lagrangian point of view, a first-order classical Field Theory is described by its configuration bundle $\pi: E \to M$, and a Lagrangian density which is a $\pi^1$-semibasic $m$-form on $J^1E$ (see the appendix for notation and terminology). A Lagrangian density is usually written as $\mathcal{L} = \mathcal{L}(\pi^1, \omega)$, where $\mathcal{L} \in C^\infty(J^1E)$ is the Lagrangian function associated with $\mathcal{L}$ and $\omega$. The Poincaré-Cartan $m$ and $(m + 1)$-forms associated with the Lagrangian density $\mathcal{L}$ are defined using the vertical endomorphism $\nu$ of the bundle $J^1E$:

$$\Theta_\mathcal{L} := i(\nu)\mathcal{L} + \mathcal{L} \equiv \theta_\mathcal{L} + \mathcal{L} \in \Omega^m(J^1E) \quad ; \quad \Omega_\mathcal{L} := -d\Theta_\mathcal{L} \in \Omega^{m+1}(J^1E)$$

In a natural chart in $J^1E$ we have

$$\Theta_\mathcal{L} = \frac{\partial \mathcal{L}}{\partial v^A_\nu} dy^A \wedge d^{m-1}x^\nu - \left( \frac{\partial \mathcal{L}}{\partial v^A_\nu} v^A_\nu - \mathcal{L} \right) d^m x$$

$$\Omega_\mathcal{L} = -\frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^2_\nu} d^{B} \wedge d^A \wedge d^{m-1}x^\nu - \frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^2_\nu} dy^B \wedge dy^A \wedge d^{m-1}x^\nu + \frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^2_\nu} v^A_\nu dy^B \wedge d^m x + \left( \frac{\partial^2 \mathcal{L}}{\partial y^B \partial v^2_\nu} v^A_\nu - \frac{\partial \mathcal{L}}{\partial y^B} + \frac{\partial \mathcal{L}}{\partial x^\nu \partial v^2_\nu} \right) dy^B \wedge d^m x$$

(See, for instance, [3], [11], [15], [21], [22] and [33], for details). Then a Lagrangian system is a couple $(J^1E, \Omega_\mathcal{L})$.

As we can see, the factor $E_\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial v^A_\nu} v^A_\nu - \mathcal{L}$ appears in the local expression of the Poincaré-Cartan $(m + 1)$-form, and it is recognized as the classical expression of the Lagrangian energy associated with the Lagrangian function $\mathcal{L}$. In fact, the existence of such a function as a global object, and by extension a density of Lagrangian energy, is closely related to the existence of a connection in the bundle $\pi: E \to M$, in the same way that happens in non-autonomous mechanics [10]. As shown in [11], we can define the density of Lagrangian energy using the vertical endomorphisms in $J^1E$. In fact, given a connection $\nabla$ in $\pi: E \to M$, we can identify $V^*(\pi)$ as a subbundle of $T^*E$. Then the operation $S^\nabla - \nabla$ makes sense, where $S$ and $\nabla$ are the vertical endomorphisms of the bundle $J^1E$, and $S^\nabla$ denotes the action of $S$ followed by the injection of $V^*(\pi)$ in $T^*E$ induced by $\nabla$ (see the appendix). Therefore:

**Definition 1** Let $(J^1E, \Omega_\mathcal{L})$ be a Lagrangian system and $\nabla$ a connection in the bundle $\pi: E \to M$. The density of Lagrangian energy associated with the Lagrangian density $\mathcal{L}$ and the connection $\nabla$ is given by

$$\mathcal{E}^\mathcal{L} = i(S^{\nabla} - \nabla) d\mathcal{L} - \mathcal{L} = i(S^{\nabla}) d\mathcal{L} - \Theta_\mathcal{L} \equiv \Theta^\nabla_\mathcal{L} - \Theta_\mathcal{L}$$

It is a $\pi^1$-vertical $m$-form in $J^1E$. Hence, we can write $\mathcal{E}^\mathcal{L} = E^\mathcal{L}(\pi^1, \omega)$, where $E^\mathcal{L} \in C^\infty(J^1E)$ is the Lagrangian energy function associated with $\mathcal{L}$, $\nabla$ and $\omega$. 
Remark:

• Note that every connection $\nabla$ in $\pi: E \rightarrow M$ allows us to split the Poincaré-Cartan forms as

$$\Theta_L = \Theta_L^\nabla - \mathcal{E}_L^\nabla, \quad \Omega_L = -d\Theta_L^\nabla + d\mathcal{E}_L^\nabla = \Omega_L^\nabla + d\mathcal{E}_L^\nabla$$

Using natural systems of coordinates, and $\Gamma^A_\nu$ being the component functions of the connection, we have the following local expressions

$$\mathcal{E}_L^\nabla = \left( \frac{\partial L}{\partial v^\nu} (v^A_\nu - \Gamma^A_\nu) \right) \text{d}m \quad ; \quad \mathcal{E}_L^\nabla = \frac{\partial L}{\partial v^\nu} (v^A_\nu - \Gamma^A_\nu) - \mathcal{L}$$

Observe also that if we take a local connection with $\Gamma^A_\nu = 0$, then the Lagrangian energy associated with this natural connection has the classical local expression given above.

A variational problem can be posed from the Lagrangian density $L$, which is called the Hamilton principle of the Lagrangian formalism: the states of the field are the sections of $\pi$ which are critical for the functional $L: \Gamma_c(M, E) \rightarrow \mathbb{R}$ defined by

$$L(\phi) := \int_M (j^1\phi)^* L \quad , \quad \text{for every } \phi \in \Gamma_c(M, E)$$

where $\Gamma_c(M, E)$ is the set of compact supported sections of $\pi$. These (compact-supported) critical sections can be characterized in several equivalent ways. In fact (see [11], [13], [36] and [53]):

**Theorem 1** The critical sections of the Hamilton’s principle are sections $\phi: M \rightarrow E$ whose canonical liftings $j^1\phi: M \rightarrow J^1E$ satisfy the following equivalent conditions:

1. $\frac{d}{dt} \big|_{t=0} \int_M (j^1\phi_t)^* L = 0 \ , \ \text{being } \phi_t = \sigma_t \circ \phi \ , \ \text{where } \{\sigma_t\} \ \text{denotes a local one-parameter group of any } \pi\text{-vertical vector field } Z \in \mathfrak{X}(E)$.

2. $\int_M (j^1\phi)^* L(j^1Z)L = 0 \ , \ \text{for every } Z \in \mathfrak{X}(\pi)(E)$.

3. $\int_M (j^1\phi)^* L(j^1Z)\Theta_L = 0 \ , \ \text{for every } Z \in \mathfrak{X}(\pi)(E)$.

4. $(j^1\phi)^* i^1(j^1Z)\Omega_L = 0 \ , \ \text{for every } Z \in \mathfrak{X}(\pi)(E)$.

5. $(j^1\phi)^* i(X)\Omega_L = 0 \ , \ \text{for every } X \in \mathfrak{X}(J^1E)$.

6. The coordinates of $\phi$ satisfy the Euler-Lagrange equations:

$$\frac{\partial L}{\partial y^A} \bigg|_{j^1\phi} - \frac{\partial L}{\partial x^\nu} \frac{\partial L}{\partial v^A_\nu} \bigg|_{j^1\phi} = 0 \ (\text{for } A = 1, \ldots, N)$$

2.2 Multimomentum bundles and Legendre maps

(See [13] for a more detailed study of all these constructions).

Let $\bar{y} \in J^1E$, with $\bar{y} \xrightarrow{\pi^1} y \xrightarrow{\pi} x$. We have that $T_{\bar{y}}J^1E = V_{\bar{y}}\pi^1$ is canonically isomorphic to $T_x^*M \otimes V_y\pi$, by means of the directional derivatives; therefore $V(\pi^1) = \tilde{\pi}^1*T^*M \otimes_{\mu_E} \pi^1*V(\pi)$. 
Moreover, if $\mathcal{D} \subset TJ^1E$ denotes the subbundle of total derivatives (which in a system of natural coordinates in $J^1E$, is generated by $\left\{ \frac{\partial}{\partial x^\nu} + v^A \frac{\partial}{\partial y^A} \right\}$), we have that $\pi^1*T\mathcal{E} = \pi^1*V(\pi) \oplus \mathcal{D}$ with $T_yE|_y = V_y \pi|_y \oplus D_y$ (see [53] for details). Hence there is a natural projection $\sigma: \pi^1*T\mathcal{E} \rightarrow \pi^1*V(\pi)$ and its dual injection $\sigma^*: \pi^1*V^*(\pi) \rightarrow \pi^1*T^*E$ and so we can consider the projection

$$\text{Id} \otimes \sigma: \pi^1*T^*M \otimes \pi^1*T\mathcal{E} \rightarrow \pi^1*T^*M \otimes \pi^1*V(\pi) = V(\pi)$$

In a natural chart $(x^\nu, y^A)$ adapted to the bundle $\pi: \mathcal{E} \rightarrow M$, the local expression of this mapping is

$$\sigma \left( \left( \frac{\partial}{\partial x^\nu} \bigg|_{\bar{y}} \right) \right) = -v^A(\bar{y}) \left( \frac{\partial}{\partial y^A} \bigg|_{\bar{y}} \right) \quad ; \quad \sigma \left( \left( \frac{\partial}{\partial y^A} \bigg|_{\bar{y}} \right) \right) = \left( \frac{\partial}{\partial y^A} \bigg|_{\bar{y}} \right)$$

and, if $\{\zeta^A\}$ is the dual basis of $\left\{ \frac{\partial}{\partial y^A} \right\}$ in $V^*(\pi)$, we have that $\sigma^*(\zeta^A) = dy^A - v^Adx^\nu$.

Now let $(J^1E, \Omega_{\mathcal{L}})$ be a Lagrangian system, and consider the restriction $\mathcal{L}_y: J^1E \rightarrow \Lambda^mT^*_xM$. Its differential map at $\bar{y} \in J^1yE$ is

$$T_{\bar{y}}J^1yE = V_{\bar{y}}\pi^1 \simeq (T^*_yM \otimes V_y\pi)_{\bar{y}} \xrightarrow{D_y\mathcal{L}_y} (\Lambda^mT^*_xM)_{\bar{y}}$$

Thus, using the defined projection $\sigma$, we have

$$T_{\bar{y}}J^1yE = V_{\bar{y}}\pi^1 \simeq (T^*_yM \otimes V_y\pi)_{\bar{y}} \xrightarrow{D_y\mathcal{L}_y} (\Lambda^mT^*_xM)_{\bar{y}}$$

$$\xrightarrow{\text{Id} \otimes \sigma_{\bar{y}}} (T^*_yM \otimes T_yE)_{\bar{y}}$$

**Definition 2**

1. The bundle (over $E$)

$$J^1E^* := \pi^*TM \otimes E \ T^*E \otimes E \pi^*\Lambda^mT^*M$$

is called the generalized multimomentum bundle associated with the bundle $\pi: \mathcal{E} \rightarrow M$. We denote the natural projections by $\tilde{\rho}^1: J^1E^* \rightarrow E$ and $\tilde{\rho}^1 := \pi \circ \tilde{\rho}^1: J^1E^* \rightarrow M$.

2. The generalized Legendre map associated with a Lagrangian density $\mathcal{L}$ is the $C^\infty$-map

$$\tilde{\mathcal{F}}\mathcal{L} : J^1E \rightarrow J^1E^* \quad \bar{y} \mapsto D_y\mathcal{L}_y \circ (\text{Id} \otimes \sigma)_{\bar{y}}$$

(We have departed a little from the notation in this definition, because $\sigma$ acts on $\pi^1*T\mathcal{E}$, and not on $T\mathcal{E}$. Then, given $\bar{y} \in J^1E$, the right way consists in taking $T_yE$ and lifting it to $\bar{y}$).

Natural coordinates in $J^1E^*$ will be denoted by $(x^\nu, y^A, p^\nu_{\bar{y}}, p^A_{\bar{y}})$, and for every $y \in J^1E^*$, with $y \overset{\tilde{\rho}^1}{\rightarrow} \bar{y} \overset{\pi}{\rightarrow} x$, we have

$$y = \left. \frac{\partial}{\partial x^\nu} \right|_{\bar{y}} \otimes \left( p^\nu_{\bar{y}}(y)dx^\eta + p^A_{\bar{y}}(y)dy^A \right)_{\bar{y}} \otimes d^m|x_{\bar{y}}}$$
and the local expression of the generalized Legendre map is

\[
\begin{align*}
\mathcal{F}\mathcal{L}^* x^\nu &= x^\nu, & \mathcal{F}\mathcal{L}^* y^A &= y^A, & \mathcal{F}\mathcal{L}^* p_\nu^A &= -v_\nu^A \frac{\partial \mathcal{L}}{\partial v_\nu^A}, & \mathcal{F}\mathcal{L}^* p_A^\nu &= \frac{\partial \mathcal{L}}{\partial v_\nu^A}
\end{align*}
\]

(1)

Now, let \( \bar{y} \in J^1 E \) with \( \bar{y} \mapsto y \mapsto x \). We define the map \( \mathcal{L}_y: J^1_y E \rightarrow \Lambda^m T^*_x M \) as \( \mathcal{L}_y := \mathcal{L}|_{J^1_y E} \). It is a \( C^\infty \)-map of the affine space \( J^1_y E \), modeled on \( T^*_y M \otimes V_y(\pi) \), with values on \( \Lambda^m T^*_x M \). Then, the tangent map \( T_{\bar{y}} \mathcal{L}_y \) allows us to construct the following diagram (where the vertical arrows are canonical isomorphisms given by the directional derivatives)

\[
\begin{array}{ccc}
T_y J^1_y E & \longrightarrow & T_{\mathcal{L}_y(\bar{y})} \Lambda^m T^*_x M \\
\cong & & \cong \\
T^*_y M \otimes V_y \pi & \longrightarrow & \Lambda^m T^*_x M
\end{array}
\]

Hence, taking into account these identifications, we have that \( \tilde{T}_{\bar{y}} \mathcal{L}_y \) is an element of \( T_x M \otimes V_y(\pi) \otimes \Lambda^m T^*_x M \), and so, bearing in mind the analogy with classical mechanics, we define:

**Definition 3**

1. The bundle (over \( E \))

\[
\Pi := \pi^* T M \otimes_{E} V^*(\pi) \otimes_{E} \pi^* \Lambda^m T^* M
\]

is called the reduced multimomentum bundle associated with the bundle \( \pi: E \rightarrow M \). We denote the natural projections by \( \rho^1: \Pi \rightarrow E \) and \( \bar{\rho}^1 := \pi \circ \rho^1: \Pi \rightarrow M \).

2. The reduced Legendre map associated with a Lagrangian density \( \mathcal{L} \) is the \( C^\infty \)-map

\[
\mathcal{F}\mathcal{L} : J^1 E \rightarrow \Pi
\]

\[
\bar{y} \mapsto \tilde{T}_{\bar{y}} \mathcal{L}_y
\]

Natural coordinates in \( \Pi \) are denoted by \( (x^\nu, y^A, p_\nu^A) \), and for every \( \bar{y} \in \Pi \) with \( \bar{y} \mapsto y \mapsto x \),

\[
\bar{y} = p_A^\nu(\bar{y}) \frac{\partial}{\partial x^\nu} \otimes \zeta^A \otimes \mathrm{d}^m x|_y
\]

(We have departed from the notation by denoting the momentum coordinates in \( \Pi \) and \( J^1 E^* \) with the same symbol, \( p_A^\nu \). This departure will be repeated frequently throughout the work).

The local expression of the reduced Legendre map is

\[
\begin{align*}
\mathcal{F}\mathcal{L}^* x^\nu &= x^\nu, & \mathcal{F}\mathcal{L}^* y^A &= y^A, & \mathcal{F}\mathcal{L}^* p_\nu^A &= -v_\nu^A \frac{\partial \mathcal{L}}{\partial v_\nu^A}, & \mathcal{F}\mathcal{L}^* p_A^\nu &= \frac{\partial \mathcal{L}}{\partial v_\nu^A}
\end{align*}
\]

(2)

If we recall that \( J^1 E^* := \pi^* T M \otimes T^* E \otimes \pi^* \Lambda^m T^* M \), then the natural projection \( T^* E \rightarrow V^*(\pi) \) (which is the transpose of the natural injection \( V(\pi) \hookrightarrow TE \)) induces another one

\[
\delta: J^1 E^* \rightarrow \Pi
\]

**Proposition 1** The natural map \( \delta \) is onto, and \( \mathcal{F}\mathcal{L} = \delta \circ \mathcal{F}\mathcal{L} \).
Furthermore, we can introduce the following map:

**Definition 4** The canonical contraction in $J^1E^*$ is the map

\[ \iota: J^1E^* \equiv \pi^*TM \otimes T^*E \otimes \pi^*\Lambda^mT^*M \rightarrow \Lambda^mT^*E \]

defined as follows: \( \iota(y) := \alpha^k \wedge \pi^*i(u_k)\beta, \) for every \( y = u_k \odot \alpha^k \otimes \beta \in J^1E^* \).

In a chart of natural coordinates in $J^1E^*$, we have that

\[ \iota(y) = (p_y^\nu(y)dx^\nu + p_A^\nu(y)dy^A) \wedge i \left( \frac{\partial}{\partial x^\nu} \right) d^m x \bigg|_y = (p_y^\nu(y)d^m x + p_A^\nu(y)dy^A \wedge d^{m-1}x_\nu) \]

(3)

(\text{let us recall that } p_y^\nu \text{ denotes } \sum_{\nu=1}^m p_y^\nu).

For every \( y \in E \), we have that

\[ \iota(J^1E^*)_y = \{ \gamma \in \Lambda^mT^*_y E ; i(u_1)i(u_2)\gamma = 0, u_1, u_2 \in V_y(\pi) \} \equiv \Lambda^mT^*_y E \]

We will denote \( \iota_0: J^1E^* \rightarrow \iota(J^1E^*) = \Lambda^mT^*E = \bigcup_{y \in E} \{ (y, \alpha) ; \alpha \in \Lambda^mT^*_y E \} \) the restriction of \( \iota \) onto its image.

**Definition 5**

1. The bundle (over \( E \))

\[ \mathcal{M}_\pi := \Lambda^mT^*E \]

will be called the extended multimomentum bundle associated with the bundle \( \pi: E \rightarrow M \). We denote the natural projections by \( \tilde{\gamma}^1: \mathcal{M}_\pi \rightarrow E \) and \( \tilde{\gamma}^1: \mathcal{M}_\pi \rightarrow \mathcal{M} \).

2. The (first) extended Legendre map associated with a Lagrangian density \( \mathcal{L} \) is the $C^\infty$-map

\[ \mathcal{FL} := \iota_0 \circ \mathcal{F}\mathcal{L} \]

The (second) extended Legendre map is the $C^\infty$-map \( \mathcal{FL}: J^1E^* \rightarrow \mathcal{M}_\pi \) given by

\[ \mathcal{FL} = \mathcal{FL} + \pi^*\mathcal{L} \]

Natural coordinates in \( \mathcal{M}_\pi \) are denoted by \( (x^\nu, y^A, p_A^\nu, p_y^\nu) \), and for every \( y \in J^1E^* \) we have

\[ \nu: y \equiv (x^\nu, y^A, p_A^\nu, p_y^\nu) \mapsto \hat{y} \equiv (x^\nu, y^A, p_A^\nu, p = p_y^\nu) \]

The local expressions of the extended Legendre maps are

\[ \mathcal{FL}^* x^\nu = x^\nu, \quad \mathcal{FL}^* y^A = y^A, \quad \mathcal{FL}^* p_A^\nu = \frac{\partial \hat{E}}{\partial y^A_{\nu}}, \quad \mathcal{FL}^* p = -v_y^A \frac{\partial \hat{E}}{\partial y^A_{\nu}} \]

(4)

**Remarks:**

- It can be proved [6, 13] that \( \mathcal{M}_\pi \equiv \Lambda^mT^*E \) is canonically isomorphic to \( \text{Aff}(J^1E, \Lambda^mT^*M) \).
It is interesting to point out that, as $\Theta_L$ and $\theta_L$ can be thought of as $m$-forms on $J^1E$ along the projection $\pi^1: J^1E \to E$, the extended Legendre maps can be defined as

$$(\tilde{F}\mathcal{L}(\bar{y}))(Z_1, \ldots, Z_m) = (\theta_L)_y(\bar{Z}_1, \ldots, \bar{Z}_m)$$

$$(\tilde{F}\mathcal{L}(\bar{y}))(Z_1, \ldots, Z_m) = (\Theta_L)_y(\bar{Z}_1, \ldots, \bar{Z}_m)$$

where $\bar{y} \in J^1E$, $Z_1, \ldots, Z_m \in T_{\pi^1(\bar{y})}E$, and $\bar{Z}_1, \ldots, \bar{Z}_m \in T_{\bar{y}}J^1E$ are such that $T_{\bar{y}}\pi^1 \bar{Z}_\nu = Z_\nu$.

In addition, the (second) extended Legendre map can also be defined as the “first order vertical Taylor approximation to $£$” [6], [24].

For the construction of the last multimomentum bundle, observe that the sections of the bundle $\pi^*\Lambda^n T^*M \to E$ are the $\pi$-semibasic $m$-forms on $E$; therefore we introduce the notation $\Lambda^n_0 T^*E \equiv \pi^*\Lambda^n T^*M$, and then:

**Definition 6**

1. The bundle (over $E$)

$$J^1\pi^* := \Lambda^n_1 T^*E/\Lambda^n_0 T^*E \equiv M\pi/\Lambda^n_0 T^*E$$

will be called the restricted multimomentum bundle associated with the bundle $\pi: E \to M$. We denote the natural projections by $\tau^1: J^1\pi^* \to E$ and $\bar{\tau}^1 := \pi \circ \tau^1: J^1\pi^* \to M$.

2. The restricted Legendre map associated with a Lagrangian density $\mathcal{L}$ is the $C^\infty$-map

$$\mathcal{F}\mathcal{L} := \mu \circ \tilde{F}\mathcal{L} = \mu \circ \tilde{F}\mathcal{L}$$

where $\mu: \mathcal{M}\pi \to J^1\pi^*$ is the natural projection.

Natural coordinates in $J^1\pi^*$ will also be denoted as $(x^\nu, y^A, p_\nu^A)$. As is evident in this system, the local expression of the restricted Legendre map is

$$\mathcal{F}\mathcal{L}^* x^\nu = x^\nu \quad \mathcal{F}\mathcal{L}^* y^A = y^A \quad \mathcal{F}\mathcal{L}^* p_\nu^A = \frac{\partial \mathcal{L}}{\partial v_\nu^A}$$

(5)

**Theorem 2** The multimomentum bundles $J^1\pi^*$ and $\Pi$ are canonically diffeomorphic as vector bundles over $E$, and denoting this diffeomorphism by $\Psi: J^1\pi^* \to \Pi$, therefore $F\mathcal{L} = \mathcal{F}\mathcal{L} \circ \Psi$.

(Proof) Consider the diagram

We have that the maps $\mu \circ \iota_0$ and $\delta$ are surjective, linear on the fibers, and restrict to the identity on the base. On the other hand, for every $y \in E$, we have that $\ker \delta_y = \ker (\mu \circ \iota_0)_y$ (as can be shown from the corresponding expressions in coordinates). Hence we conclude that $J^1\pi^*$ and $\Pi$ are canonically isomorphic as vector bundles over $E$. 
(See [13] for another version of this proof, and an explicit construction of $\Psi$).

$\Pi$ and $J^1\pi^*$ are fiber bundles over $E$, then $\Psi$ is a fiber-diffeomorphism (it is the identity on the base), whose local expression in natural coordinates in $J^1\pi^*$ and $\Pi$ is

$$
\Psi^*x' = x' \quad \Psi^*y_A = y_A \quad \Psi^*p_A' = p_A' \quad (\forall \nu, A)
$$

### 2.3 Canonical forms

As is known [5], the multicotangent bundle $\Lambda^mT^*E$ is endowed with canonical forms: $\Theta \in \Omega^m(\Lambda^mT^*E)$ and the multisymplectic form $\Omega := -d\Theta \in \Omega^{m+1}(\Lambda^mT^*E)$. Then:

**Definition 7** The canonical $m$ and $(m + 1)$ forms of $J^1E^*$ are

$$
\hat{\Theta} = \iota^*\Theta \in \Omega^m(J^1E^*) \quad \hat{\Omega} := -d\hat{\Theta} = \iota^*\Omega \in \Omega^{m+1}(J^1E^*)
$$

On the other hand, observe that $\mathcal{M}\pi \equiv \Lambda^m_1T^*E$ is a subbundle of the multicotangent bundle $\Lambda^mT^*E$. Let

$$
\lambda: \Lambda^m_1T^*E \hookrightarrow \Lambda^mT^*E
$$

be the natural imbedding (hence $\lambda \circ \iota_0 = \iota$). Then:

**Definition 8** The canonical $m$ and $(m + 1)$ forms of $\mathcal{M}\pi$ (multimomentum Liouville $m$ and $(m+1)$ forms of $\mathcal{M}\pi$) are

$$
\Theta := \lambda^*\Theta \in \Omega^m(\mathcal{M}\pi) \quad \Omega = -d\Theta = \lambda^*\Omega \in \Omega^{m+1}(\mathcal{M}\pi)
$$

- Of course, $\hat{\Theta} = \iota_0^*\Theta$ and $\hat{\Omega} = \iota_0^*\Omega$
- $\Omega$ is 1-nondegenerate, and hence $(\mathcal{M}\pi, \Omega)$ is a multisymplectic manifold.

The canonical forms $\hat{\Theta}$ and $\Theta$ can also be characterized as follows (see [13]):

- $\hat{\Theta}$ is the only $m$-form in $J^1E^*$, such that if $y \in J^1E^*$, and $X_1, \ldots, X_m \in T_yJ^1E^*$, then

$$
\hat{\Theta}(y; X_1, \ldots, X_m) = \iota(y)[T_y\hat{\rho}^1(X_1), \ldots, T_y\hat{\rho}^1(X_m)]
$$

- In turn, considering the natural projection $\hat{\kappa}^1: \Lambda^m_1T^*E \to E$, then

$$
\Theta((y, \alpha); X_1, \ldots, X_m) := \alpha(y; T_{(y,\alpha)}\hat{\kappa}^1(X_1), \ldots, T_{(y,\alpha)}\hat{\kappa}^1(X_m))
$$

for every $(y, \alpha) \in \Lambda^m_1T^*E$ (where $y \in E$ and $\alpha \in \Lambda^m_1T^*_yE$), and $X_i \in \mathfrak{X}(\Lambda^m_1T^*E)$. 
Bearing in mind the following diagram

![Diagram](image)

we observe that the map \( \iota_0: J^1E^* \to M\pi \) is a form along the projection \( \hat{\rho}^1: J^1E^* \to E \). Then:

**Lemma 1** \( \hat{\Theta} = \iota_0^* \Theta = \hat{\rho}^{1*} \iota_0 \).

(Proof) Let \( y \in J^1E^* \), and \( X_1, \ldots, X_m \in T_y J^1E^* \). We have

\[
\iota_0^* \Theta(y; X_1, \ldots, X_m) = \Theta(\iota_0(y); T_y \iota_0(X_1), \ldots, T_y \iota_0(X_m))
\]

\[
= (\iota_0(y)(T_y \hat{\tau}^1 \circ \iota_0)(X_1), \ldots, T_y \hat{\tau}^1 \circ \iota_0)(X_m))
\]

\[
= (\iota_0(y)(T_y \hat{\rho}^1(X_1), \ldots, T_y \rho^1(X_m)) = (\hat{\rho}^{1*} \iota)(y; X_1, \ldots, X_m)
\]

In natural coordinates in \( J^1E^* \) and \( M\pi \), the local expressions of these forms are

\[
\hat{\Theta} = p^\eta \eta d^m x + p^\nu A d y^A \wedge d^{m-1} x_\nu, \quad \hat{\Omega} = -d p^\eta \eta \wedge d^m x - d p^\nu A \wedge d y^A \wedge d^{m-1} x_\nu
\]

\[
\Theta = p \eta d^m x + p^\nu A d y^A \wedge d^{m-1} x_\nu, \quad \Omega = -d p \wedge d^m x - d p^\nu A \wedge d y^A \wedge d^{m-1} x_\nu
\]

**Proposition 2** Let \( (J^1E, \Omega_L) \) be a Lagrangian system. Let \( \overline{FL} \) be the generalized Legendre map, and \( \overline{FL} \) and \( \overline{FL} \) the extended Legendre maps. Then

\[
\overline{FL}^* \Theta = \Theta_L - \mathcal{L} = \theta_L; \quad \overline{FL}^* \Omega = \Omega_L - d\mathcal{L} = -d\theta_L
\]

\[
\overline{FL}^* \Theta = \Theta_L - \mathcal{L} = \theta_L; \quad \overline{FL}^* \Omega = \Omega_L - d\mathcal{L} = -d\theta_L
\]

\[
\overline{FL}^* \Theta = \Theta_L; \quad \overline{FL}^* \Omega = \Omega_L
\]

### 2.4 Regular and singular systems

**Definition 9** Let \( (J^1E, \Omega_L) \) be a Lagrangian system.

1. \( (J^1E, \Omega_L) \) is said to be a regular or non-degenerate Lagrangian system if \( FL \), and hence, \( FL \) are local diffeomorphisms.

As a particular case, \( (J^1E, \Omega_L) \) is said to be a hyper-regular Lagrangian system if \( FL \), and hence \( FL \), are global diffeomorphisms.

2. Elsewhere \( (J^1E, \Omega_L) \) is said to be a singular or degenerate Lagrangian system.
The matrix of the tangent maps $\mathcal{F}L_*$ and $\mathcal{F}L_*$ in a natural coordinate system is

\[
\begin{pmatrix}
\text{Id} & 0 & 0 \\
0 & \text{Id} & 0 \\
\frac{\partial^2 L}{\partial x^i \partial v^p} & \frac{\partial^2 L}{\partial y^i \partial v^p} & \frac{\partial^2 L}{\partial v^i \partial v^p}
\end{pmatrix}
\]

where the sub-matrix \( \left( \frac{\partial^2 L}{\partial v^p \partial v^q} \right) \) is the partial Hessian matrix of $L$. Then, the regularity condition is equivalent to demanding that this matrix is regular everywhere in $J^1E$. This fact establishes the relation to the concept of regularity given in an equivalent way by saying that a Lagrangian system \((J^1E, \Omega_L)\) is regular if $\Omega_L$ is 1-nondegenerate. (See also [6] for a different definition of this concept).

**Proposition 3** (See [13] and [37]). Let \((J^1E, \Omega_L)\) be a hyper-regular Lagrangian system. Then

1. $\widetilde{FL}(J^1E)$ is a $m^2$-codimensional imbedded submanifold of $J^1E^*$, which is transverse to the projection $\delta$.
2. $\widetilde{FL}(J^1E)$ and $\widetilde{FL}(J^1E)$ are 1-codimensional imbedded submanifolds of $M\pi$, which are transverse to the projection $\mu$.
3. The manifolds $J^1\pi^*$, $\widetilde{FL}(J^1E)$, $\widetilde{FL}(J^1E)$, $\widetilde{FL}(J^1E)$ and $\Pi$ are diffeomorphic.

Hence, $\widetilde{FL}$, $\widetilde{FL}$ and $\widetilde{FL}$ are diffeomorphisms on their images; and the maps $\mu$, restricted to $\widetilde{FL}(J^1E)$ or to $\widetilde{FL}(J^1E)$, and $\iota_0$ and $\delta$, restricted to $\widetilde{FL}(J^1E)$, are also diffeomorphisms.

In this way we have the following diagram

```
\begin{array}{c}
\mathcal{F}L & \mathcal{F}L \\
\mathcal{F}L & \mathcal{F}L \\
\mathcal{F}L & \mathcal{F}L \\
\end{array}
```

\[
\begin{array}{c}
J^1E \\
J^1E \\
J^1E \\
\hline
\mu & \mu' & \lambda \\
\mu & \lambda & \Lambda^mT^*E \\
\mu & \lambda & \Lambda^mT^*E \\
\hline
\end{array}
\]

where the map $\mu': M\pi \rightarrow M\pi$ is defined by the relation

$$
\mu' := \widetilde{FL} \circ \mathcal{F}L^{-1} \circ \mu
$$

and it satisfies that $\mu \circ \mu' = \mu$. Observe also that the restriction $\mu': \widetilde{FL}(J^1E) \subset M\pi \rightarrow \widetilde{FL}(J^1E) \subset M\pi$, is a diffeomorphism, which is also defined by the relation $\mathcal{F}L = \mu' \circ \mathcal{F}L$.

For dealing with singular Lagrangians, we must assume minimal “regularity” conditions. Hence we introduce the following terminology:
Definition 10  A singular Lagrangian system \((J^1E, \Omega_L)\) is said to be almost-regular if:

1. \(\mathcal{P} := \mathcal{FL}(J^1E)\) and \(P := \mathcal{FL}(J^1E)\) are closed submanifolds of \(J^1\pi^*\) and \(\Pi\), respectively. (We will denote the corresponding imbeddings by \(j_0: \mathcal{P} \hookrightarrow J^1\pi^*\) and \(j_0: P \hookrightarrow \Pi\)).

2. \(\mathcal{FL}\), and hence \(\mathcal{FL}\), are submersions onto their images (with connected fibers).

3. For every \(\tilde{y} \in J^1E\), the fibers \(\mathcal{FL}^{-1}(\mathcal{FL}(\tilde{y}))\) and hence \(\mathcal{FL}^{-1}(\mathcal{FL}(\tilde{y}))\) are connected submanifolds of \(J^1E\).

(This definition is equivalent to that in reference \([37]\), but slightly different from that in references \([17], [49]\)).

Let \((J^1E, \Omega_L)\) be an almost-regular Lagrangian system. Denote

\[
\hat{\mathcal{P}} := \mathcal{F}L(J^1E), \quad \hat{\mathcal{P}} := \mathcal{F}L(J^1E), \quad \hat{\mathcal{P}} := \mathcal{F}L(J^1E)
\]

Let \(j_0: \hat{\mathcal{P}} \hookrightarrow \mathcal{M}\pi, j_0: \hat{\mathcal{P}} \hookrightarrow \mathcal{M}\pi, j_0: \hat{\mathcal{P}} \hookrightarrow J^1E^*\) be the canonical inclusions, and

\[
\hat{\mu}: \hat{\mathcal{P}} \rightarrow \mathcal{P}, \quad \hat{\mu}: \hat{\mathcal{P}} \rightarrow \mathcal{P}, \quad \hat{i}_0: \hat{\mathcal{P}} \rightarrow \hat{\mathcal{P}}, \quad \hat{\delta}: \hat{\mathcal{P}} \rightarrow \hat{\mathcal{P}}, \quad \Psi_0: \hat{\mathcal{P}} \rightarrow \hat{\mathcal{P}}
\]

the restrictions of the maps \(\mu, i_0, \delta\) and the diffeomorphism \(\Psi\), respectively. Finally, define the restriction mappings

\[
\mathcal{FL}_0: J^1E \rightarrow \mathcal{P}, \quad \mathcal{FL}_0: J^1E \rightarrow \hat{\mathcal{P}}, \quad \mathcal{FL}_0: J^1E \rightarrow \hat{\mathcal{P}}, \quad \mathcal{FL}_0: J^1E \rightarrow \hat{\mathcal{P}}, \quad \mathcal{FL}_0: J^1E \rightarrow \hat{\mathcal{P}}
\]

Proposition 4  (See \([33], [37], [38]\)). Let \((J^1E, \Omega_L)\) be an almost-regular Lagrangian system. Then:

1. The maps \(\Psi_0\) and \(\hat{\mu}\) are diffeomorphisms.

2. For every \(\tilde{y} \in J^1E\),

\[
\mathcal{FL}_0^{-1}(\mathcal{FL}(\tilde{y})) = \mathcal{FL}_0^{-1}(\mathcal{FL}(\tilde{y})) = \mathcal{FL}_0^{-1}(\mathcal{FL}(\tilde{y})) \quad (8)
\]

3. \(\hat{\mathcal{P}}\) and \(\hat{\mathcal{P}}\) are submanifolds of \(\mathcal{M}\pi\), \(\hat{\mathcal{P}}\) is a submanifold of \(J^1E^*\), and \(j_0: \hat{\mathcal{P}} \hookrightarrow \mathcal{M}\pi, j_0: \hat{\mathcal{P}} \hookrightarrow \mathcal{M}\pi, j_0: \hat{\mathcal{P}} \hookrightarrow J^1E^*\) are imbeddings.

4. The restriction mappings \(\mathcal{FL}_0, \mathcal{FL}_0\) and \(\mathcal{FL}_0\) are submersions with connected fibers.

Thus we have the diagram

\[
\begin{array}{c}
J^1E \\
\mathcal{FL}_0 \quad \mathcal{FL}_0 \quad \mathcal{FL}_0 \\
\hat{\mu} \quad \hat{\mu} \quad \hat{\mu} \\
\mathcal{FL}_0 \quad \mathcal{FL}_0 \quad \mathcal{FL}_0 \\
\hat{\delta} \quad \hat{\delta} \quad \hat{\delta} \\
P \quad P \quad P \\
\hat{j}_0 \quad \hat{j}_0 \quad \hat{j}_0 \\
\mathcal{M}\pi \quad \mathcal{M}\pi \quad \mathcal{M}\pi \\
\mu \quad \mu \quad \mu \\
\mathcal{FL}_0^{-1}(\mathcal{FL}(\tilde{y})) \quad \mathcal{FL}_0^{-1}(\mathcal{FL}(\tilde{y})) \quad \mathcal{FL}_0^{-1}(\mathcal{FL}(\tilde{y})) \quad (8) \quad (9)
\end{array}
\]
where \( \hat{\mu}' : \hat{P} \to \hat{P} \) is defined by the relation \( \hat{\mu}' := \hat{\mu}^{-1} \circ \hat{\mu} \).

The maps \( \hat{\mu} \) and \( \hat{\mu}' \) are not diffeomorphisms in general, since \( \text{rank} \hat{\mathcal{L}} \geq \text{rank} \hat{\mathcal{L}} = \text{rank} \mathcal{L} \), as is evident from the analysis of the corresponding Jacobian matrices.

**Proposition 5** Let \((J^1E, \Omega_L)\) be an almost-regular Lagrangian system. Then:

\[
\ker F_* = \ker \mathcal{L}_* = \ker \hat{\mathcal{L}}_* = \ker \Omega_L \cap \mathfrak{X}^{(\pi)}(J^1E)
\]

(Proof) The first two equalities are immediate, since \( P := F\mathcal{L}(J^1E), P := \mathcal{L}(J^1E) \) and \( \hat{P} := \hat{\mathcal{L}}(J^1E) \) are diffeomorphic.

For the last equality, as \( F\mathcal{L}, \mathcal{L} \) and \( \hat{\mathcal{L}} \) are the identity on the basis \( E \) of the bundle \( \pi^1 : J^1E \to E \), first we have that

\[
\ker F_* = \ker \mathcal{L}_* = \ker \hat{\mathcal{L}} \subset \mathfrak{X}^{(\pi)}(J^1E)
\]

(and this relation holds also for the other Legendre maps). Then, for every \( X \in \ker \hat{\mathcal{L}}_* \) we have

\[
i(X)\Omega_L = i(X)(\hat{\mathcal{L}}^*\Omega) = \hat{\mathcal{L}}^*[[i(\hat{\mathcal{L}}_*X)\Omega] = 0
\]

and hence

\[
\ker \hat{\mathcal{L}}_* = \ker \mathcal{L}_* = \ker F_* \subset \ker \Omega_L \cap \mathfrak{X}^{(\pi)}(J^1E)
\]

Conversely, if \( X \in \mathfrak{X}^{(\pi)}(J^1E) \), in a natural system of coordinates in \( J^1E \) we have \( X = f^B_\nu \frac{\partial}{\partial v^B_\nu} \), and if, in addition, \( X \in \ker \Omega_L \), we obtain

\[
0 = i(X)\Omega_L = -\frac{\partial^2 \mathcal{L}}{\partial v^B_\eta \partial v^A_\nu} f^B_\eta dy^A \wedge d^{m-1}x_\nu + \frac{\partial^2 \mathcal{L}}{\partial v^B_\eta \partial v^A_\nu} v^A_\nu f^B_\eta \mathrm{d}m x
\]

this is equivalent to demanding that \( f^B_\eta \frac{\partial^2 \mathcal{L}}{\partial v^B_\eta \partial v^A_\nu} = 0 \), and this is the condition which characterizes locally the vector fields belonging to \( \ker F_* = \ker \mathcal{L}_* = \ker \hat{\mathcal{L}}_* \) (see (6)).

---

### 3 Hamiltonian formalism in the reduced multimomentum bundle

#### 3.1 Hamiltonian systems

(Compare this presentation with references [17], [19] and [49]).

The more standard way for constructing Hamiltonian systems in \( \Pi \) consists in using sections of the projection \( \delta \), and it is similar to that developed in [3] for the Hamiltonian formalism in \( J^1\pi^* \) (which we will review later). Thus:

**Definition 11** Consider the bundle \( \bar{\rho} : \Pi \to M \).

1. A section \( h_\delta : \Pi \to J^1E^* \) of the projection \( \delta \) is called a Hamiltonian section of \( \delta \).
2. The differentiable forms

\[ \Theta_{h_\delta} := h_\delta^\ast \hat{\Theta} = (\iota_0 \circ h_\delta)^\ast \Theta, \quad \Omega_{h_\delta} := -d\Theta_{h_\delta} = h_\delta^\ast \hat{\Omega} = (\iota_0 \circ h_\delta)^\ast \Omega \]

are called the Hamilton-Cartan m and \((m + 1)\) forms of \(\Pi\) associated with the Hamiltonian section \(h_\delta\).

3. The couple \((\Pi, \Omega_{h_\delta})\) is said to be a Hamiltonian system.

Using charts of natural coordinates in \(\Pi\) and \(J^1E^*\), a Hamiltonian section is specified by a set of local functions \(H_{\eta}^{\nu} \in C^\infty(U), \quad U \subset \Pi\), such that

\[ h_\delta(x^\nu, y^A, p_\nu^{A}) \equiv (x^\nu, y^A, p_\nu^{A} - H_{\eta}^{\nu}(x^\gamma, y^B, p_\gamma^{B}), p_\nu^{A}) \quad (10) \]

Then, the local expressions of these Hamilton-Cartan forms are

\[ \Theta_{h_\delta} = p_\nu^{A} dy^A \wedge dx^{m-1} - H_{h_\delta} dx^m \\
\Omega_{h_\delta} = -dp_\nu^{A} \wedge dy^A \wedge dx^{m-1} + dH_{h_\delta} \wedge dx^m \quad (11) \]

where \(H_{h_\delta} \equiv H_{\nu}^{\nu}\) is a local Hamiltonian function associated with the Hamiltonian section \(h_\delta\).

As \(\iota_0\) is a submersion, we can state the following:

**Definition 12** There is a natural equivalence relation in the set of Hamiltonian sections of \(\delta\), which is defined as follows: two Hamiltonian sections \(h_1^{\delta}, h_2^{\delta}\) are equivalent if \(\iota_0 \circ h_1^{\delta} = \iota_0 \circ h_2^{\delta}\). We denote by \(\{h_\delta\}\) the equivalence classes of this relation.

**Remarks:**

- Of course, Hamiltonian sections belonging to the same equivalence class give the same Hamilton-Cartan forms, and hence the same Hamiltonian system.
- Observe that all the Hamiltonian sections of the same equivalence class have the same local Hamiltonian function \(H_{h_\delta} \equiv H_{\nu}^{\nu}\) (in the same open set \(U \subset \Pi\)).

There is a relation between sections of \(\mu\) and of \(\delta\). In fact:

**Proposition 6** There is a bijective correspondence between the set of sections of the projection \(\mu: M\pi \to J^1\pi^*\) and the set of equivalence classes of sections of the projection \(\delta: J^1E^* \to \Pi\).

(Proof) In fact, this correspondence is established by the commutativity of the following diagram

\[ \begin{array}{ccc}
J^1\pi^* & \xymatrix{\Psi@{<-}^\Psi^\mu & \Pi \ar[l]
\ar[d]^\delta \ar[r]_{\iota_0} \ar[u]_{h_\delta} \ar[l]_{\iota_0} \ar[d]_{\Psi} \ar[r]_{h_\mu} & J^1\pi^*}
\end{array} \]
that is, a section $h_{\mu}: J^1 \pi^* \to \mathcal{M}\pi$ and a class $\{h_\delta\}: \Pi \to J^1 E^*$ are in correspondence if, and only if,

$$h_{\mu} = \iota_0 \circ h_\delta \circ \Psi$$

for every $h_\delta \in \{h_\delta\}$

and this correspondence is one-to-one.

Now we can study the structure of the set of Hamilton-Cartan forms, and hence of Hamiltonian systems. So, for every Hamiltonian section $h_\delta$ of $\delta$, consider the diagram

$$
\begin{array}{c}
\Pi \\
\overrightarrow{\delta} \\
J^1 E^* \\
\overrightarrow{\rho^1} \\
\hat{\rho}^1 \\
E \\
\overrightarrow{\hat{\rho}^1} \\
\Pi \\
\overrightarrow{\delta} \\
J^1 E^* \\
\overrightarrow{\rho^1} \\
\hat{\rho}^1 \\
\mathcal{M}\pi \\
\overrightarrow{\tau^1} \\
\Pi \\
\end{array}
$$

Lemma 2 Let $h^1_\delta, h^2_\delta: \Pi \to J^1 E^*$ be two Hamiltonian sections of $\delta$, then:

1. $h^1_\delta^* \hat{\Theta} - h^2_\delta^* \hat{\Theta} = (\iota_0 \circ h^1_\delta)^* \Theta - (\iota_0 \circ h^2_\delta)^* \Theta = \rho^1(\iota_0 \circ h^1_\delta - \iota_0 \circ h^2_\delta)$.
2. $\rho^1(\iota_0 \circ h^1_\delta - \iota_0 \circ h^2_\delta)$ is a $\hat{\rho}^1$-semibasic form in $\Pi$.

(Proof)

1. For every Hamiltonian section $h_\delta$, the map $\iota_0 \circ h_\delta$ is a form along the map $\hat{\rho}^1 \circ h_\delta = \rho^1$. Therefore, following the same pattern as in Lemma 1, we obtain that

$$(\iota_0 \circ h_\delta)^* \Theta = \rho^1(\iota_0 \circ h_\delta)$$

and hence the result is immediate.

2. As $\Psi \circ \mu \circ \iota_0 = \delta$, for every section $h_\delta$ we have that $\Psi \circ \mu \circ \iota_0 \circ h_\delta = \text{Id}_\Pi$, then $\mu \circ (\iota_0 \circ h^1_\delta - \iota_0 \circ h^2_\delta) = 0$, and therefore $\text{Im}(\iota_0 \circ h^1_\delta - \iota_0 \circ h^2_\delta) \in \ker \mu = \Lambda^0_m T^* E$ (that is, the $\hat{\rho}^1$-semibasic forms in $\Pi$.)

From the local expressions (3) and (10), for every $\tilde{y} \in U \subset \Pi$, we have

$$(\iota_0 \circ h^1_\delta - \iota_0 \circ h^2_\delta)(\tilde{y}) = (H_{h^1_\delta} - H_{h^2_\delta})(\tilde{y})d^m x|_{\tilde{y}}$$

which is the local expression (at $\tilde{y}$) of a $\hat{\rho}^1$-semibasic form in $\Pi$. In addition, this is also the local expression of the form

$$(\iota_0 \circ h^1_\delta)^* \Theta - (\iota_0 \circ h^2_\delta)^* \Theta = \Theta_{h^1_\delta} - \Theta_{h^2_\delta} \equiv \mathbb{H}$$
Definition 13 A \( \rho^1 \)-semibasic form \( \mathbb{H} \in \Omega^m(\Pi) \) is said to be a Hamiltonian density in \( \Pi \).

It can be written as \( \mathbb{H} = H(\rho^1 \omega) \), where \( H \in C^\infty(\Pi) \) is the global Hamiltonian function associated with \( \mathbb{H} \) and \( \omega \).

In this way, we have proved that two Hamiltonian systems generated by two Hamiltonian sections of \( \delta \) belonging to different equivalence classes, are related by means of a Hamiltonian density. We can state this result as follows:

Theorem 3 The set of Hamilton-Cartan \( m \)-forms associated with Hamiltonian sections of \( \delta \) is an affine space modelled on the set of Hamiltonian densities in \( \Pi \).

Remark:

• If \( (\Pi, \Omega_{h_\delta}) \) is a Hamiltonian system, taking into account (12) we have that every Hamiltonian section \( h'_{\delta} \) (such that \( h'_{\delta} \not\in \{h_\delta\} \)) allows us to split globally the Hamilton-Cartan forms as

\[
\Theta_{h_\delta} = \Theta_{h'_{\delta}} - \mathbb{H} ; \quad \Omega_{h_\delta} = \Omega_{h'_{\delta}} + d\mathbb{H} \quad (13)
\]

If \( (x^\nu, y^A, p_A^\nu) \) is a natural system of coordinates in \( \Pi \), such that \( \rho^1 \omega = d^m x \), and \( H_{h'_{\delta}}(x^\nu, y^A, p_A^\nu) \) is the local Hamiltonian function associated with the Hamiltonian section \( h'_{\delta} \), and \( \mathbb{H} = H(x^\nu, y^A, p_A^\nu)d^m x \), then

\[
\Theta_{h_\delta} = p_A^\nu dy^A \wedge d^{m-1}x_\nu - (H + H_{h'_{\delta}})d^m x
\]

\[
\Omega_{h_\delta} = -dp_A^\nu \wedge dy^A \wedge d^{m-1}x_\nu + d(H + H_{h'_{\delta}}) \wedge d^m x \quad (14)
\]

If \( H_{h_\delta} \) is the local Hamiltonian function associated with the Hamiltonian section \( h_\delta \), we have the relation \( H = H_{h_\delta} - H_{h'_{\delta}} \) (in an open set \( U \)). Hence, taking this into account, the local expressions (13) and (14) are really the same thing.

3.2 Hamiltonian sections, Hamiltonian densities and connections

In order to obtain a Hamiltonian density using two Hamiltonian sections, it is usual for one of them to be a linear section induced by a connection. This is a natural procedure for different reasons. For instance, when we construct the Hamiltonian formalism associated with a Lagrangian system, the Hamiltonian density must be related with the density of Lagrangian energy and, as this last is defined by using a connection, this same connection must be used for constructing the related Hamiltonian density (see sections 3.4 and 3.6).

Next, we are going to show how to define the linear Hamiltonian section induced by a connection. Hence, suppose that a connection \( \nabla \) has been chosen in \( \pi^*: E \rightarrow M \). It allows us to identify \( V^*(\pi) \) as a subbundle of \( T^*E \). So, if \( v_\nabla^*: V^*(\pi) \rightarrow T^*E \) is the dual injection of the vertical projection \( v_\nabla \) induced by \( \nabla \). Then:

Definition 14 The \( \rho^1 \) linear Hamiltonian section of \( \delta \) induced by the connection \( \nabla \) is the map

\[
h_\delta^\nabla : \Pi \rightarrow \pi^*TM \otimes T^*E \otimes \pi^*\Lambda^mT^*M := J^1E^*
\]

\[
\quad u_k \otimes \alpha^k \otimes \beta \quad \mapsto \quad u_k \otimes v_\nabla^*(\alpha^k) \otimes \beta
\]

that is, \( h_\delta^\nabla := \text{Id}_{\pi^*TM} \otimes v_\nabla^* \otimes \text{Id}_{\pi^*\Lambda^mT^*M} \).
Remark:

- Two linear sections $h^\infty_{\delta_1}$ and $h^\infty_{\delta_2}$ induced by two different connections $\nabla_1$ and $\nabla_2$ cannot belong to the same equivalence class of Hamiltonian sections, as can be proved comparing their coordinate expressions.

If $\hat{\Theta}$ is the canonical $m$-form in $J^1E^*$, the forms
\[
\Theta_{h^\infty_{\delta}} := (t_0 \circ h_{\delta}^\infty)^* \Theta = h_{\delta}^\infty * \Theta \in \Omega^m(\Pi) \quad , \quad \Omega_{h^\infty_{\delta}} := -d\Theta_{h^\infty_{\delta}} = h_{\delta}^\infty * \Omega \in \Omega^{m+1}(\Pi)
\]
(15)
are the Hamilton-Cartan $m$ and $m + 1$ forms of $\Pi$ associated with $\nabla$.

Remark:

- It can be proved [13] that the Hamilton-Cartan $m$-form associated with a connection $\nabla$ is the unique form $\Theta_{h^\infty_{\delta}} \in \Omega^m(\Pi)$ such that, if $\tilde{y} \in \Pi$ and $w_1, \ldots, w_m \in T_{\tilde{y}}\Pi$, then
\[
\Theta_{h^\infty_{\delta}}(\tilde{y}; w_1, \ldots, w_m) := (t_0 \circ h_{\delta}^\infty)(\tilde{y})(\rho^1(\tilde{y}); T_{\tilde{y}}\rho^1(w_1), \ldots, T_{\tilde{y}}\rho^1(w_m))
\]
\[
= [\rho^1(t_0 \circ h_{\delta}^\infty)](\tilde{y}; w_1, \ldots, w_m)
\]
(16)
that is, $\Theta_{h^\infty_{\delta}} = \rho^1*(t_0 \circ h_{\delta}^\infty)$

If $(x^\nu, y^A, p^\nu_A)$ is a system of natural coordinates in $\Pi$, and $\tilde{y} = p^\nu_A(\tilde{y})\frac{\partial}{\partial x^\nu} \otimes \zeta^A \otimes dx^m \in \Pi$, taking into account the local expression of $v^\nu_{\nabla}$ which, for a connection $\nabla = dx^\nu \otimes \left(\frac{\partial}{\partial x^\nu} + \Gamma^A_{\nu\mu} \frac{\partial}{\partial y^A}\right)$, is $v^\nu_{\nabla}(\zeta^A) = dy^A - \Gamma^A_{\nu\mu} dx^\nu$, we have that
\[
h_{\delta}(\tilde{y}) = h_{\delta}^\infty \left( p^\nu_A(\tilde{y}) \frac{\partial}{\partial x^\nu} \otimes \zeta^A \otimes dx^m \bigg|_{\rho^1(\tilde{y})}\right) = p^\nu_A(\tilde{y})\frac{\partial}{\partial x^\nu} \otimes (dy^A - \Gamma^A_{\nu\mu} dx^\nu) \otimes dx^m \bigg|_{\rho^1(\tilde{y})}
\]
\[
(t_0 \circ h_{\delta}^\infty)(\tilde{y}) = p^\nu_A(\tilde{y})(dy^A \wedge dx^m_{\nu} - \Gamma^A_{\nu\mu} dx^m_\mu)
\]
Observe that $t_0$ restricted to the image of $h_{\delta}^\infty$ is injective. Therefore
\[
\Theta_{h_{\delta}^\infty} = p^\nu_A(dy^A - \Gamma^A_{\nu\mu} dx^\mu) \wedge dx^m_{\nu} = p^\nu_A dy^A \wedge dx^m_{\nu} - p^\nu_A \Gamma^A_{\nu\mu} dx^m_\mu
\]
\[
\Omega_{h_{\delta}^\infty} = -dp_A \wedge dy^A \wedge dx^m_{\nu} + \Gamma^A_{\nu\mu} dp_A \wedge dx^m_{\mu} + p_A^\nu d\Gamma^A_{\nu\mu} \wedge dx^m
\]
(17)
Now, given a connection $\nabla$ and a Hamiltonian section $h_{\delta}$, from Lemma 2 we have that
\[
\rho^1*(t_0 \circ h_{\delta}^\infty - t_0 \circ h_{\delta}) = (t_0 \circ h_{\delta}^\infty)^* \Theta - (t_0 \circ h_{\delta})^* \Theta = \Theta_{h_{\delta}^\infty} - \Theta_{h_{\delta}} = h_{\delta}^\infty * \hat{\Theta} - h_{\delta}^\infty * \hat{\Theta} := \mathbb{H}_{h_{\delta}}^\infty
\]
is a $\rho^1$-semibasic $m$-form in $\Pi$. It is usually written as $\mathbb{H}_{h_{\delta}}^\infty = H_{h_{\delta}}^\infty(\rho^1*\omega)$, where $H_{h_{\delta}}^\infty \in C^\infty(\Pi)$ is the global Hamiltonian function associated with $\mathbb{H}_{h_{\delta}}^\infty$ and $\omega$.

Therefore, given a Hamiltonian system $(\Pi, \Omega_{h_{\delta}})$, taking into account (13), we have that every connection $\nabla$ in $\pi: E \to M$ allows us to split globally the Hamilton-Cartan forms as
\[
\Theta_{h_{\delta}} = \Theta_{h_{\delta}^\infty} - \mathbb{H}_{h_{\delta}}^\infty \quad , \quad \Omega_{h_{\delta}} = -d\Theta_{h_{\delta}} = \Omega_{h_{\delta}}^\infty + d\mathbb{H}_{h_{\delta}}^\infty
\]
(18)
In a natural system of coordinates in $\Pi$, such that $\rho^1*\omega = dx^m$, we write $\mathbb{H}_{h_{\delta}}^\infty = H_{h_{\delta}}^\infty(x^\nu, y^A, p^\nu_A) dx^m$, and
\[
\Theta_{h_{\delta}} = p_A^\nu dy^A \wedge dx^m_{\nu} - (H_{h_{\delta}}^\infty + p_A^\nu \Gamma^A_{\nu\mu}) dx^m
\]
\[
\Omega_{h_{\delta}} = -dp_A^\nu \wedge dy^A \wedge dx^m_{\nu} + d(H_{h_{\delta}}^\infty + p_A^\nu \Gamma^A_{\nu\mu}) \wedge dx^m
\]
(19)
Proposition 7 A couple \((\{h_\delta\}, \nabla)\) in \(\Pi\) is equivalent to a couple \((\mathbb{H}, \nabla)\) (that is, given a connection \(\nabla\), classes of Hamiltonian sections of \(\delta\) and Hamiltonian densities in \(\Pi\) are in one-to-one correspondence).

(Proof) Given a connection in \(\pi:E \to M\), we have just seen that all the Hamiltonian sections belonging to the same equivalence class \(\{h_\delta\}\) define a unique Hamiltonian density \(\mathbb{H}_{h_\delta}^\nabla\) and, hence, the same Hamilton-Cartan forms.

Conversely, given a Hamiltonian density \(\mathbb{H}\) and a connection \(\nabla\), we can construct an equivalence class of Hamiltonian sections \(\{h_\delta\}\) (which leads to the same Hamilton-Cartan forms), since, as \(\mathbb{H}:\Pi \to M\pi\) takes values in \(\bar{\rho}^1 \Lambda^m T^* M\), we have a map \(i_0 \circ h_\delta^\nabla - \mathbb{H}: \Pi \to M\pi\). From the local expression of this map, it is easy to prove that there exists a local section \(h_\delta^\nabla\), such that \(i_0 \circ h_\delta^\nabla = i_0 \circ h_\delta^\nabla - \mathbb{H}\). Then, using a partition of unity we can construct a global section fulfilling this condition, and hence a family of sections \(\{h_\delta\}\) defined by the relation \(i_0 \circ h_\delta^\nabla = i_0 \circ h_\delta^\nabla - \mathbb{H}\).

As a direct consequence of this proposition, we have another way of obtaining a Hamiltonian system, which consists in giving a couple \((\mathbb{H}, \nabla)\). In fact:

Proposition 8 Let \(\nabla\) be a connection in \(\pi:E \to M\), and \(\mathbb{H}\) a Hamiltonian density. There exist a unique class \(\{h_\delta\}\) of Hamiltonian sections of \(\delta\) such that

\[
\Theta_{h_\delta} = \Theta_{h_\delta^\nabla} - \mathbb{H}, \quad \Omega_{h_\delta} = -d\Theta_{h_\delta} = \Omega_{h_\delta^\nabla} + d\mathbb{H}
\]

Remark:

- If \(\pi:E\text{to}M\) is a trivial bundle, then there is a natural connection (the trivial one). So, in this case, there is a bijective correspondence between Hamiltonian systems and Hamiltonian densities.

This is the situation in classical non-autonomous mechanics [7], [9], [10].

3.3 Variational principle and field equations

Now we can establish the field equations for Hamiltonian systems. First we need to introduce the notion of prolongation of diffeomorphisms and vector fields from \(E\) to \(\Pi\).

Definition 15 Let \(\Phi:E \to E\) be a diffeomorphism of \(\pi\)-fiber bundles and \(\Phi_M:M \to M\) the induced diffeomorphism in \(M\). The prolongation of \(\Phi\) to \(\Pi\) is the diffeomorphism \(j^1\Phi:\Pi \to \Pi\) defined by

\[
j^1\Phi := (\Phi_M)_* \otimes \Phi^{-1} \otimes \Lambda^m \Phi_M^{-1}.
\]

Proposition 9 Let \(\Phi:E \to E\) be a diffeomorphism of fiber bundles, \(\Phi_M:M \to M\) its restriction to \(M\) and \(j^1\Phi\) its prolongation to \(\Pi\). Then

1. \(\rho^1 \circ j^1\Phi = \Phi \circ \rho^1\), \(\bar{\rho}^1 \circ j^1\Phi = \Phi_M \circ \bar{\rho}^1\).

2. If \(\Psi:E \to E\) is another fiber bundle diffeomorphism, then

\[
j^1(\Psi \circ \Phi) = j^1\Psi \circ j^1\Phi
\]
3. \( j^1 \Phi \) is a diffeomorphism of \( \rho^1 \)-bundles and \( \rho^1 \)-bundles, and \( (j^1 \Phi)^{-1} = j^1 \Phi^{-1} \).

**Definition 16** Let \( Z \in \mathfrak{X}(E) \) be a \( \pi \)-projectable vector field. The prolongation of \( Z \) to \( \Pi \) is the vector field \( j^1 Z \) whose local one-parameter group of diffeomorphisms are the extensions \( \{ j^1 \sigma_t \} \) of the local one-parameter group of diffeomorphisms \( \{ \sigma_t \} \) of \( Z \).

Now we can state:

**Definition 17** Let \( (\Pi, \Omega_{h^\delta}) \) be a Hamiltonian system. Let \( \Gamma_c(M, \Pi) \) be the set of compact-supported sections of \( \rho^1 \), and \( \psi \in \Gamma_c(M, \Pi) \). Consider the map
\[
H : \Gamma_c(M, \Pi) \rightarrow \mathbb{R}, \quad \psi \mapsto \int_M \psi^* \Theta_{h^\delta}
\]
The variational problem for this Hamiltonian system is the search of the critical (or stationary) sections of the functional \( H \), with respect to the variations of \( \psi \) given by \( \psi_t = j^1 \sigma_t \circ \psi \), where \( \{ \sigma_t \} \) is a local one-parameter group of every \( Z \in \mathfrak{X}^{V(\pi)}(E) \) (the module of \( \pi \)-vertical vector fields in \( E \)).

\[
\frac{d}{dt} \bigg|_{t=0} \int_M \psi_t^* \Theta_{h^\delta} = 0
\]

This is the so-called Hamilton-Jacobi principle of the Hamiltonian formalism.

**Theorem 4** Let \( (\Pi, \Omega_{h^\delta}) \) be a Hamiltonian system. The following assertions on a section \( \psi \in \Gamma_c(M, \Pi) \) are equivalent:

1. \( \psi \) is a critical section for the variational problem posed by \( \Theta_{h^\delta} \).
2. \( \int_M \psi^* L(j^1 Z) \Theta_{h^\delta} = 0 \), for every \( Z \in \mathfrak{X}^{V(\pi)}(E) \).
3. \( \psi^* i(j^1 Z) \Omega_{h^\delta} = 0 \), for every \( Z \in \mathfrak{X}^{V(\pi)}(E) \).
4. \( \psi^* i(X) \Omega_{h^\delta} = 0 \), for every \( X \in \mathfrak{X}(\Pi) \).
5. If \( (U; x^\nu, y^A, p_A^\nu) \) is a natural system of coordinates in \( \Pi \), then \( \psi = (x^\nu, y^A(x^\eta), p_A^\nu(x^\eta)) \) in \( U \) satisfies the system of equations
\[
\frac{\partial y^A}{\partial x^\nu} \bigg|_\psi = \frac{\partial H_{h^\delta}}{\partial p_A^\nu} \bigg|_\psi; \quad \frac{\partial p_A^\nu}{\partial x^\nu} \bigg|_\psi = -\frac{\partial H_{h^\delta}}{\partial y^A} \bigg|_\psi
\]
which are known as the Hamilton-De Donder-Weyl equations of the Hamiltonian formalism.

(Proof) \( 1 \Leftrightarrow 2 \) If \( Z \in \mathfrak{X}^{V(\pi)}(E) \) and \( \sigma_t \) is a one-parameter local group of \( Z \), we have
\[
\frac{d}{dt} \bigg|_{t=0} \int_M (j^1 \sigma_t \circ \psi)^* \Theta_{h^\delta} = \lim_{t \to 0} \frac{1}{t} \left( \int_M (j^1 \sigma_t \circ \psi)^* \Theta_{h^\delta} - \int_M \psi^* \Theta_{h^\delta} \right) = \lim_{t \to 0} \frac{1}{t} \left( \int_M \psi^*(j^1 \sigma_t)^* \Theta_{h^\delta} - \int_M \psi^* \Theta_{h^\delta} \right) = \lim_{t \to 0} \frac{1}{t} \left( \int_M \psi^*([j^1 \sigma_t]^* \Theta_{h^\delta} - \Theta_{h^\delta}) \right) = \int_M \psi^* L(j^1 Z) \Theta_{h^\delta}
\]
and the results follows immediately.

(2 ⇔ 3) Taking into account that

\[ L(j^1Z)\Theta_{h^5} = d\ i(j^1Z)\Theta_{h^5} + i(j^1Z)d\Theta_{h^5} = d\ i(j^1Z)\Theta_{h^5} - i(j^1Z)\Omega_{h^5} \]

we obtain that

\[ \int_M \psi^* L(j^1Z)\Theta_{h^5} = \int_M \psi^* d\ i(j^1Z)\Theta_{h^5} - \int_M \psi^* i(j^1Z)\Omega_{h^5} \]

and, as \( \psi \) has compact support, using Stoke’s theorem we have

\[ \int_M \psi^* d\ i(j^1Z)\Theta_{h^5} = \int_M d\psi^* i(j^1Z)\Theta_{h^5} = 0 \]

hence \( \int_M \psi^* L(j^1Z)\Theta_{h^5} = 0 \) (for every \( Z \in \mathfrak{X}^{\pi}(E) \)) if, and only if, \( \int_M \psi^* i(j^1Z)\Omega_{h^5} = 0 \), and, according to the fundamental theorem of variational calculus, this is equivalent to

\[ \psi^* i(j^1Z)\Omega_{h^5} = 0 \]

(3 ⇔ 5) Suppose that \( \psi \) is a section verifying that \( \psi^* i(j^1Z)\Omega_{h^5} = 0 \), for every \( Z \in \mathfrak{X}^{\pi}(E) \). In a natural chart in \( \Pi \), if \( Z = \beta^A \frac{\partial}{\partial y^A} \), then \( j^1Z = \beta^A \frac{\partial}{\partial y^A} - p^\nu \frac{\partial}{\partial y^A} \frac{\partial}{\partial y^\nu} \). Taking into account the local expression of \( \Omega_{h^5} \) given in [11], we have

\[ i(j^1Z)\Omega_{h^5} = \beta^A \left( dp^\nu \wedge d^{m-1}x_\nu + \frac{\partial H_{h^5}}{\partial y^A} dx^m \right) + p^\nu \frac{\partial}{\partial y^A} \frac{\partial}{\partial y^\nu} \left( dy^A \wedge d^{m-1}x_\nu - \frac{\partial H_{h^5}}{\partial p^A} dp^A \wedge dx^m \right) \]

As \( \psi = (x^\nu, f^A(x^\eta), g^\nu_\alpha(x^\eta)) \) is a section of \( \tilde{\rho} \), then on the points of the image of \( \psi \) we have \( y^A = f^A(x^\eta), p^\nu = g^\nu_\alpha(x^\eta) \), and we obtain

\[ 0 = \psi^* i(j^1Z)\Omega_{h^5} = \beta^A \left( \frac{\partial g^\nu_\alpha}{\partial x^\nu} + \frac{\partial H_{h^5}}{\partial y^A} \right) dx^m + g^\nu_\alpha \frac{\partial}{\partial y^A} \left( \frac{\partial f^A}{\partial x^\nu} - \frac{\partial H_{h^5}}{\partial p^A} \right) dx^m \]

and, as this holds for every \( Z \in \mathfrak{X}^{\pi}(E) \), this is equivalent to demanding that

\[ \beta^A \left( \frac{\partial g^\nu_\alpha}{\partial x^\nu} + \frac{\partial H_{h^5}}{\partial y^A} \right) + g^\nu_\alpha \frac{\partial}{\partial y^A} \left( \frac{\partial f^A}{\partial x^\nu} - \frac{\partial H_{h^5}}{\partial p^A} \right) = 0 \]

for every \( \beta^A(x^\nu, y^A) \). Therefore

\[ \beta^A \left( \frac{\partial g^\nu_\alpha}{\partial x^\nu} + \frac{\partial H_{h^5}}{\partial y^A} \right) = 0 ; \ g^\nu_\alpha \frac{\partial}{\partial y^A} \left( \frac{\partial f^A}{\partial x^\nu} - \frac{\partial H_{h^5}}{\partial p^A} \right) = 0 \]

From the first equalities we obtain the first group of the Hamiltonian equations. For the second ones, let \( (W; x^\nu, y^A, p^\nu) \) a natural chart, \( U = \tilde{\rho}(W) \), and \( \psi \) a critical section. Then, for every \( x \in U \) we have

\[ g^\nu_\alpha(x) \frac{\partial}{\partial y^A} \left( \frac{\partial f^A}{\partial x^\nu} - \frac{\partial H_{h^5}}{\partial p^A} \right) \bigg|_{\psi(x)} = 0 \]

A. Echeverría-Enríquez et al., Geometry of Multisymplectic Hamiltonian... 21
but, as there are critical sections passing through every point in \( W \), we obtain that
\[
\frac{\partial \beta^C}{\partial y^A}(x, f^A(x)) \left( \frac{\partial f^A}{\partial x^\nu} - \frac{\partial H_{h^*_k}}{\partial p^\nu_A} \right) \bigg|_{\psi(x)} = 0 \quad \text{(for every } B, \nu) \]

Now we can choose \( \beta^B \) such that \( \frac{\partial \beta^B}{\partial y^A} \) take arbitrary values, and then
\[
\frac{\partial f^A}{\partial x^\nu} - \frac{\partial H_{h^*_k}}{\partial p^\nu_A} = 0
\]

which is the second group of the Hamiltonian equations. The converse is trivial.

\( (4 \iff 5) \) Suppose that \( \psi \) is a section verifying that \( \psi^* i(X) \Omega_{h^*_k} = 0 \), for every \( X \in \mathfrak{X}(\Pi) \).

If \( X = \alpha^\nu \frac{\partial}{\partial x^\nu} + \beta^A \frac{\partial}{\partial y^A} + \gamma^\nu \frac{\partial}{\partial p^\nu_A} \), taking into account (3), we have
\[
i(X) \Omega_{h^*_k} = (-1)^n \alpha^\eta \left( dp^\nu_A \wedge dy^A \wedge d^{m-2}x_{\eta^\nu} - \frac{\partial H_{h^*_k}}{\partial p^\nu_A} dp^\nu_A \wedge d^{m-1}x_{\eta} \right)
\]
\[
+ \beta^A \left( dp^\nu_A \wedge d^{m-1}x_{\nu} + \frac{\partial H_{h^*_k}}{\partial y^A} d^m x \right) + \gamma^\nu \left( -dy^A \wedge d^{m-1}x_{\nu} + \frac{\partial H_{h^*_k}}{\partial p^\nu_A} dp^\nu_A \wedge d^m x \right)
\]

but as \( \psi = (x^\nu, f^A(x^\eta), g^\nu_A(x^\eta)) \) is a section of \( \tilde{\rho}^1 \), then on the points of the image of \( \psi \) we have \( y^A = f^A(x^\eta), p^\nu_A = g^\nu_A(x^\eta) \), and
\[
0 = \psi^* i(X) \Omega_{h^*_k} = (-1)^n \alpha^\eta \left( \frac{\partial f^A}{\partial x^\nu} - \frac{\partial H_{h^*_k}}{\partial p^\nu_A} \right) \frac{\partial g^\nu_A}{\partial x^\nu} d^m x +
\]
\[
\beta^A \left( \frac{\partial g^\nu_A}{\partial x^\nu} + \frac{\partial H_{h^*_k}}{\partial y^A} \right) d^m x + \gamma^\nu \left( -\frac{\partial f^A}{\partial x^\nu} + \frac{\partial H_{h^*_k}}{\partial p^\nu_A} \right) d^m x
\]

and, as this holds for every \( X \in \mathfrak{X}(\Pi) \), we obtain the Hamilton-De Donder-Weyl equations. The converse is trivial.

\[ \square \]

**Remark:**

- In relation to the equations (21), it is important to point out that they are not covariant, since the Hamiltonian function \( H_{h^*_k} \) is defined only locally, and hence it is not intrinsically defined.

In order to write a set of covariant Hamiltonian equations we must use a global Hamiltonian function, which can be obtained by introducing another Hamiltonian section \( h^*_\ell \), with \( h^*_\ell \not\in \{ h^*_k \} \) (as we have seen in section 3.1). It is usual to take the section induced by a connection \( \nabla \) in \( \pi: E \to M \), and hence we have the splitting given in (18) for the form \( \Omega_{h^*_k} \). Then, if \( \Gamma^B_\eta \) are the local component functions of \( \nabla \) in \( U \subset \Pi \), starting from the local expression (19), and following the same pattern as in the proof of the last item, we obtain for a critical section \( \psi = (x^\nu, y^A(x^\eta), p^\nu_A(x^\eta)) \) in \( U \) the covariant Hamiltonian equations:
\[
\frac{\partial y^A}{\partial x^\nu} \bigg|_{\psi} = \left( \frac{\partial H^\nabla_{h^*_k}}{\partial p^\nu_A} + \Gamma^A_\nu \right) \bigg|_{\psi} ; \quad \frac{\partial p^\nu_A}{\partial x^\nu} \bigg|_{\psi} = - \left( \frac{\partial H^\nabla_{h^*_k}}{\partial y^A} + p^\nu_\eta \frac{\partial \Gamma^B_\eta}{\partial y^A} \right) \bigg|_{\psi}
\]

Observe that, as \( H^\nabla_{h^*_k} = H_{h^*_k} - p^\nu_A \Gamma^A_\nu \) (on each open set \( U \subset \Pi \), where \( H_{h^*_k} \) is the corresponding local Hamiltonian function), then from these last equations we recover the Hamilton-De Donder-Weyl equations. (See [3] for comments on this subject).
3.4 Hamiltonian system associated with a hyper-regular Lagrangian system

It is evident that different choices of equivalence classes of Hamiltonian sections of $\delta$ lead to different Hamiltonian systems in $\Pi$. The question now is how to associate (if possible) a Hamiltonian system with a Lagrangian system. The answer to this question is closely related to the regularity of the Lagrangian system.

First, let $(J^1E, \Omega_L)$ be a hyper-regular Lagrangian system. Then:

**Lemma 3** For every section $h_\delta: \Pi \to J^1E^*$ of $\delta$, the relation

$$h_\mu := \mu' \circ \iota_0 \circ h_\delta \circ \Psi$$

defines a unique section of $\mu$, which is just $h_\mu = FL \circ FL^{-1}$.

**Proof** We have the diagram

Then, taking into account the commutativity of this diagram, we have

$$h_\mu = \mu' \circ \iota_0 \circ h_\delta \circ \Psi = FL \circ FL^{-1} \circ \mu \circ \iota_0 \circ h_\delta \circ \Psi$$

$$= FL \circ FL^{-1} \circ \Psi^{-1} \circ \Psi = FL \circ FL^{-1}$$

So $h_\mu$ is independent of $h_\delta$.

**Remarks:**

- This result is to be expected, since $FL(J^1E)$ is a 1-codimensional submanifold of $M\pi$, transverse to the projection $\mu$, and hence it defines a section $h_\mu$ of $\mu$. This is just the natural section used in [6] and [37] for associating a Hamiltonian system to a hyper-regular Lagrangian one (see section 4.2).

- Observe that a natural section $h_\delta$ of $\delta$ can be selected by making

$$h_\delta := FL \circ FL^{-1}$$

or, what is equivalent, its associated class can be defined by

$$\iota_0 \circ h_\delta := FL \circ FL^{-1} , \text{ for every } h_\delta \in \{h_\delta\}$$

Observe that this section $h_\delta$ is just the inverse of $\delta$ restricted to $FL(J^1E)$, and that $\iota_0 \circ h_\delta$ is a diffeomorphism.
Definition 18 Given a section $h_\delta: \Pi \to J^1 E^*$ of $\delta$, we define the Hamilton-Cartan forms

$$\Theta_{h_\delta} := (\mu' \circ \iota_0 \circ h_\delta)^* \Theta; \quad \Omega_{h_\delta} := (\mu' \circ \iota_0 \circ h_\delta)^* \Omega$$

Proposition 10 The Hamilton-Cartan forms are independent of the section $h_\delta$, and

$$FL^* \Theta_{h_\delta} = \Theta_L, \quad FL^* \Omega_{h_\delta} = \Omega_L$$

(22)

Then, $(\Pi, \Omega_{h_\delta})$ is the (unique) Hamiltonian system which is associated with the hyper-regular Lagrangian system $(J^1 E, \Omega_L)$.

(Proof) The independence of the section $h_\delta$ is a consequence of lemma 3. Then, taking into account the commutativity of diagram (21), and proposition 3 for every section $h_\delta$, we have

$$FL^* \Theta_{h_\delta} = FL^* (\mu' \circ \iota_0 \circ h_\delta)^* \Theta = (\mu' \circ \iota_0 \circ h_\delta)^* FL^* \Theta = \tilde{FL}^* \Theta = \Theta_L$$

and the same result follows for $\Omega_{h_\delta}$. ■

Using charts of natural coordinates in $\Pi$ and $J^1 E^*$, and the expressions (1) and (2) of the Legendre maps, we have that the natural Hamiltonian section $h_\delta = FL \circ FL^{-1}$ has associated the local Hamiltonian function

$$H_{h_\delta}(x^\nu, y^A, p_A^\nu) = FL^{-1*} \left( p_A^\nu \frac{\partial L}{\partial v^\nu} - L \right) = p_A^\nu FL^{-1*} v^A - FL^{-1*} L$$

(23)

and for the Hamilton-Cartan forms:

$$\Theta_{h_\delta} = p_A^\nu dy^A \wedge d^{m-1} x^\nu - (p_A^\nu FL^{-1*} v^A - FL^{-1*} L) dx$$

$$\Omega_{h_\delta} = -dp_A^\nu \wedge dy^A \wedge d^{m-1} x^\nu + d(p_A^\nu FL^{-1*} v^A - FL^{-1*} L) \wedge dx$$

There is another way of obtaining this Hamiltonian system. In fact, suppose that a connection $\nabla$ is given in $\pi: E \to M$, and let $h_\delta^\nabla: \Pi \to J^1 E^*$ be the induced linear section of $\delta$. If we have used $\nabla$ for constructing the associated density of Lagrangian energy $E^\nabla_L \in \Omega^m(J^1 E)$ (see definition 1), the key is to define a Hamiltonian density $H^\nabla \in \Omega^m(\Pi)$ which is $FL$-related with $E^\nabla_L$. We can make this construction in two ways:

Proposition 11 1. The m-form $FL^* \Theta_{h_\delta^\nabla} - \Theta_L$ is $\bar{\pi}^1$-semibasic and

$$FL^* \Theta_{h_\delta^\nabla} - \Theta_L = E^\nabla_L$$

(24)

2. There exists a unique Hamiltonian density $H^\nabla \in \Omega^m(\Pi)$ such that

$$FL^* H^\nabla = FL^* \Theta_{h_\delta^\nabla} - \Theta_L = E^\nabla_L$$

(25)

Let $H^\nabla = H^\nabla(\bar{\beta}^1, \omega)$, with $H^\nabla \in C^\infty(\Pi)$. Then, $H^\nabla$ and $H^\nabla$ are called the Hamiltonian density and the Hamiltonian function associated with the Lagrangian system, the connection $\nabla$ and $\omega$. 
3. The Hamilton-Cartan forms of definition \( \Theta_{h_\delta} = \Theta_{h_\delta} - H^\nabla \), \( \Omega_{h_\delta} = -d\Theta_{h_\delta} = \Omega_{h_\delta} + dH^\nabla \) (26)

(Proof) 

1. Once again, it suffices to see it in a natural local system \((x^\nu, y^A, v^A_\nu)\). Then, if \( L = \mathcal{L}d^m x \), taking into account the corresponding local expressions we have that

\[
\mathcal{F}L^* \Theta_{h_\delta} - \Theta_L = \left( \frac{\partial L}{\partial v^A_\nu} (v^A_\nu - \Gamma^A_\nu) - \mathcal{L} \right) d^m x
\]

and the result holds. The last part follows, recalling the local expression of the density of Lagrangian energy. Thus this form is \( \bar{\pi}^1 \)-semibasic.

2. It is immediate, as \( \mathcal{F}L \) is a diffeomorphism.

3. From (24) and (25) we obtain that

\[
\mathcal{F}L^*(\Theta_{h_\delta} - H^\nabla) = \mathcal{F}L^* \Theta_{h_\delta} - \mathcal{F}L^* \Theta_{h_\delta} + \Theta_L = \Theta_L = \mathcal{F}L^* \Theta_{h_\delta}
\]

and therefore \( \mathcal{F}L^* \Omega_{h_\delta} = \Omega_L \) too. Then, the result follows because \( \mathcal{F}L \) is a diffeomorphism.

Remark:

- Notice that the item 1 holds even if \( \mathcal{F}L \) is not a diffeomorphism.

In a system of natural coordinates we have

\[
H^\nabla (x^\nu, y^A, p^A_\nu) = p^A_\nu (\mathcal{F}L^{-1} v^A_\nu - \Gamma^A_\nu) - \mathcal{F}L^{-1} \mathcal{L}
\]

and thus \( \mathcal{F}L^* H^\nabla = E^\nabla_L \).

An alternative way is to obtain this Hamiltonian density using only Hamiltonian sections.

**Proposition 12** Consider the Hamiltonian section \( h_\delta = \mathcal{F}L \circ \mathcal{F}L^{-1} \), and a connection \( \nabla \). Then we have that

\[
(\iota_0 \circ h_\delta)^* \Theta - (\mu' \circ \iota_0 \circ h_\delta)^* \Theta = H^\nabla
\]

and hence the splitting (26) holds

(Proof) We have the following diagram
Observe that \( \mu' \circ \iota_0 \circ h_\delta = \bar{F}\mathcal{L} \circ F\mathcal{L}^{-1} \). Therefore, taking into account definition \( \text{Proposition } 2 \), \( \text{(15)} \) and \( \text{(25)} \), we have

\[
(\iota_0 \circ h_\delta)\nabla \delta \star \Theta - (\mu' \circ \iota_0 \circ h_\delta)' \star \Theta = \Theta_{h_\delta} - (F\mathcal{L}^{-1})^* \mathcal{L} \star \Theta = \Theta_{h_\delta} - (F\mathcal{L}^{-1})^* \Theta = \mathbb{H}^\nabla
\]

Then the result for the splittings of the Hamilton-Cartan forms follows straightforwardly.

Remark:

- Note that the use of both extended Legendre maps is necessary for obtaining the Hamiltonian density in this way.

As a final remark, all the results stated in section 3.3 in relation to the variational principle and the characterization of critical sections are true. In particular, field equations are the Hamilton-De Donder-Weyl equations \( \text{(20)} \), where the local Hamiltonian function \( H_{h_\delta} \) is given by \( \text{(23)} \).

### 3.5 Hamiltonian system associated with an almost-regular Lagrangian system

Now, let \( (J^1E, \Omega_\mathcal{L}) \) be an almost-regular Lagrangian system. Bearing in mind diagram \( \text{Proposition } 2 \), first observe that the submanifold \( \jmath_0: P \rightarrow \Pi \), is a fiber bundle over \( E \) (and \( M \)), and the corresponding projections will be denoted \( \kappa_0^1: P \rightarrow E \) and \( \tilde{\kappa}_0^1: P \rightarrow M \), satisfying that \( \kappa_0^1 \circ \jmath_0 = \kappa_0^1 \) and \( \tilde{\kappa}_0^1 \circ \jmath_0 = \tilde{\kappa}_0^1 \).

**Proposition 13** The Lagrangian forms \( \Theta_\mathcal{L} \) and \( \Omega_\mathcal{L} \), are \( F\mathcal{L} \)-projectable.

(Proof) By Proposition \( 2 \), we have that \( \ker F\mathcal{L} = \ker \Omega_\mathcal{L} \cap X^{V(\pi)}(J^1E) \). Then, for every \( X \in \ker F\mathcal{L} \), we have that \( i(X)\Theta_\mathcal{L} = 0 \), since \( \Theta_\mathcal{L} \) is a \( \pi^1 \)-semibasic \( m \)-form, and in the same way \( L(X)\Theta_\mathcal{L} = 0 \). Therefore \( \Theta_\mathcal{L} \) is \( F\mathcal{L} \)-projectable.

As a trivial consequence of this fact, \( i(X)\Omega_\mathcal{L} = 0 \), and \( L(X)\Omega_\mathcal{L} = 0 \), and therefore \( \Omega_\mathcal{L} \) is also \( F\mathcal{L} \)-projectable.

**Definition 19** Given a section \( \hat{h}_\delta: P \rightarrow \hat{P} \) of \( \hat{\delta} \), we define the Hamilton-Cartan forms

\[
\Theta^0_\hat{h}_\delta := (\hat{\jmath}_0 \circ \hat{\mu}' \circ \iota_0 \circ \hat{h}_\delta)' \star \Theta \quad ; \quad \Omega^0_\hat{h}_\delta := (\hat{\jmath}_0 \circ \hat{\mu}' \circ \iota_0 \circ \hat{h}_\delta)' \star \Omega
\]

**Proposition 14** The Hamilton-Cartan forms \( \Theta^0_\hat{h}_\delta \) and \( \Omega^0_\hat{h}_\delta \) are independent of the section \( \hat{h}_\delta \) of \( \hat{\delta} \), and

\[
F\mathcal{L}^*_0 \Theta^0_\hat{h}_\delta = \Theta_\mathcal{L} \quad , \quad F\mathcal{L}^*_0 \Omega^0_\hat{h}_\delta = \Omega_\mathcal{L}
\]

Then \( (\Pi, P, \Omega^0_\hat{h}_\delta) \) is the unique Hamiltonian system associated with the almost-regular Lagrangian system \( (J^1E, \Omega_\mathcal{L}) \).
(Proof) We have the following diagram:

Then, taking into account the commutativity of this diagram, and proposition 2, for every section \( \hat{h}_\delta \) of \( \hat{\delta} \) we have that

\[
F L_0^* \Theta^0_{\hat{h}_\delta} = F L_0^*(\hat{j}_0 \circ \hat{\mu}' \circ \hat{i}_0 \circ \hat{h}_\delta) \ast \Theta = (\hat{j}_0 \circ \overline{F L}_0)^* \Theta = \overline{F L}^* \Theta = \Theta \]

and the same result follows for \( \Omega^0_{\hat{h}_\delta} \).

Remarks:

- Following the terminology of sections above, we have that all the sections \( \hat{h}_\delta \) belong to the same equivalence class.

- In the particular situation that \( \text{rank} \overline{F L}_0 = \text{rank} \overline{F L}_0 \), we have that \( \hat{i}_0 \) is a diffeomorphism and, as the fibers of \( \hat{\delta} \) are also the fibers of \( \hat{i}_0 \), then so is \( \hat{\delta} \). In this case there is only one map \( \hat{h}_\delta \), which is just \( \hat{\delta}^{-1} \).

As in the hyper-regular case, we can construct this Hamiltonian system using a connection. Thus, let \( \nabla \) be connection in \( \pi: E \to M \), and \( \hat{h}_\delta: \Pi \to J^1 E^* \) the induced linear section of \( \delta \). Let \( \mathcal{E}_L^\nabla \in \Omega^m(J^1 E) \) be the density of Lagrangian energy associated with \( \nabla \) (see definition [I]). Then:

**Proposition 15**

1. The density of Lagrangian energy \( \mathcal{E}_L^\nabla \) is \( F L \)-projectable.

2. The \( \pi^1 \)-semibasic \( m \)-form \( F L^* \Theta_{h^\nabla} \) is \( F L \)-projectable and denoting \( \Theta^0_{h^\nabla} = j^0 \Theta_{h^\nabla} \), we have

\[
\mathcal{E}_L^\nabla = F L^* \Theta_{h^\nabla} \Theta \Theta = F L^* \Theta^0_{h^\nabla} \Theta = \Theta_L
\]

3. There exists a unique \( \rho^1 \)-semibasic form \( H^\nabla_0 \in \Omega^m(P) \), such that

\[
F L_0^* H^\nabla_0 = \mathcal{E}_L^\nabla
\]

Let \( H^\nabla_0 = H^\nabla_0 (\rho^1_0 \omega) \), with \( H^\nabla_0 \in C^\infty(P) \). Then \( H^\nabla_0 \) and \( H^\nabla_0 \) are called the Hamiltonian density and the Hamiltonian function associated with the Lagrangian system, the connection \( \nabla \) and \( \omega \). Obviously we have that \( F L_0^* H^\nabla_0 = \mathcal{E}_L^\nabla \).

4. The Hamilton-Cartan forms of definition [I] split as

\[
\Theta^0_{h^\nabla} = j^0 \Theta_{h^\nabla} - H^\nabla_0 = \Theta^0_{h^\nabla} - H^\nabla_0
\]

\[
\Omega^0_{h^\nabla} = -d\Theta^0_{h^\nabla} = j^0 \Omega_{h^\nabla} + dH^\nabla_0 = \Omega^0_{h^\nabla} + dH^\nabla_0
\]
(Proof)

1. As $\mathcal{E}_L^\nabla = E_L^\nabla (\pi^1 \omega)$, it suffices to prove that the Lagrangian energy $E_L^\nabla$ is $\mathcal{F}L$-projectable. Then, for every $X \in \ker \mathcal{F}L$, using natural coordinates we have

$$L(X)E_L^\nabla = L \left( f_\eta \frac{\partial}{\partial v_\eta} \right) \left( \frac{\partial L}{\partial v^A_\nu} (v^A_\nu - \Gamma^A_\nu) - L \right)$$

$$= f_\eta \frac{\partial^2 L}{\partial v_\eta^2} (v^A_\nu - \Gamma^A_\nu) + f_\eta \frac{\partial L}{\partial v^A_\nu} \delta^A_\nu f_\eta \frac{\partial L}{\partial v_\eta} = 0$$

therefore $E_L^\nabla$ is $\mathcal{F}L$-projectable, and so is $\mathcal{E}_L^\nabla$.

2. It is immediate, taking into account that $\Theta_L$ is $\mathcal{F}L$-projectable, and the first item of Proposition 11.

3. The existence is assured, since $\mathcal{E}_L^\nabla$ is $\mathcal{F}L$-projectable and the uniqueness because $\mathcal{F}L_0$ is a submersión.

Next we prove that $\mathbb{H}_0^\nabla$ is $\hat{\rho}_0^1$-semibasic. As $\mathcal{F}L_0$ is a submersion, for every $y \in J^1 E$ and $\tilde{u} \in V_{\mathcal{F}L_0(y)}(\hat{\rho}_0^1)$, there exist $u \in T_y J^1 E$ such that $\tilde{u} = T_y \mathcal{F}L_0(u)$ and, in addition, $u \in V_y (\pi^1)$ because

$$T_y \pi^1 (u) = (T_{\mathcal{F}L_0(y)} \hat{\rho}_0^1 \circ T_y \mathcal{F}L_0)(u) = T_{\mathcal{F}L_0(y)} \hat{\Theta}_0^1 (\tilde{u}) = 0$$

Furthermore, $\mathcal{E}_L^\nabla$ is $\pi^1$-semibasic, and hence

$$0 = i(u)[\mathcal{E}_L^\nabla (\tilde{y})] = i(u) [(\mathcal{F}L_0^* \mathbb{H}_0^\nabla )(\tilde{y})] = (\mathcal{F}L_0)^* [i(\tilde{u}) (\mathbb{H}_0^\nabla (\mathcal{F}L_0(\tilde{y})))]
$$

then, for every $y \in J^1 E$ and $\tilde{u} \in V_{\mathcal{F}L_0(y)}(\hat{\rho}_0^1)$ we have $i(\tilde{u})(\mathbb{H}_0^\nabla (\mathcal{F}L_0(\tilde{y}))) \in \ker (\mathcal{F}L_0)^*_{\mathcal{F}L_0(\tilde{y})} = \{0\}$, since $\mathcal{F}L_0$ is a submersion. So $\mathbb{H}_0^\nabla$ is $\hat{\rho}_0^1$-semibasic.

4. Taking into account items 3 and 2, we obtain

$$\mathcal{F}L_0^* (\Theta^0_{h^\nabla} - \mathbb{H}_0^\nabla) = \mathcal{F}L_0^* \Theta^0_{h^\nabla} - \mathcal{F}L_0^* \mathbb{H}_0^\nabla = \mathcal{F}L_0^* \Theta^0_{h^\nabla} = \mathcal{E}_L^\nabla = \Theta_L = \mathcal{F}L_0^* \Theta^0_{h^\nabla}$$

and therefore $\Omega_L = \mathcal{F}L_0^* \Theta^0_{h^\nabla}$ too. Then the result follows because $\mathcal{F}L$ is a submersion.

We can construct the above Hamiltonian density in an alternative way, as follows:

**Proposition 16** Let $\hat{h}_\delta: P \to \hat{P}$ be a section of $\hat{\delta}$, and $\nabla$ a connection. Then, $h^\nabla_\delta$ induces a map $h^\nabla_\delta: P \to \hat{P}$ defined by the relation $j_0 \circ h^\nabla_\delta = h^\nabla_\delta \circ j_0$. Therefore

$$((j_0 \circ i_0 \circ h^\nabla_\delta)^* \Theta - (j_0 \circ \hat{\mu} \circ i_0 \circ h^\nabla_\delta)^* \Theta = \mathbb{H}_0^\nabla$$

and hence the splitting (28) holds.
Then, bearing in mind the second item of Proposition 15 and \((2 7)\), we have that \(\rho\) is stated in the same way, now using sections of \(\bar{\pi}\) that mechanics, this equivalence can be proved by using the (reduced) Legendre map.

One expects that both the Lagrangian and Hamiltonian formalism must be equivalent. As in 3.6 Equivalence between the Lagrangian and Hamiltonian formalisms

Taking into account the commutativity of this diagram, we have that every section \(\hat{\delta}\) of \(\hat{\delta}\) satisfies that

\[
\mu' \circ i_0 \circ \hat{\delta} \circ FL_0 = \hat{FL}_0
\]

Then, bearing in mind the second item of Proposition \(15\) and \((27)\), we have that

\[
FL_0^*[(\hat{j}_0 \circ i_0 \circ h_\delta^\nabla)^* \Theta - (\hat{j}_0 \circ \mu' \circ i_0 \circ \hat{\delta})^* \Theta] = (\hat{j}_0 \circ i_0 \circ \hat{\delta}^\nabla \circ FL_0)^* \Theta - (\hat{j}_0 \circ \mu' \circ i_0 \circ \hat{\delta} \circ FL_0)^* \Theta = (i_0 \circ \hat{\delta}^\nabla \circ \hat{j}_0 \circ FL_0)^* \Theta - (\hat{j}_0 \circ \hat{\delta} \circ FL_0)^* \Theta = (\hat{j}_0 \circ FL_0)^* \Theta h_\delta^\nabla - (\hat{j}_0 \circ \hat{\delta} \circ FL_0)^* \Theta = FL^* \Theta h_\delta^\nabla - FL^* \Theta = E_L^\nabla + \Theta_L = \Theta_L = E_L^\nabla
\]

and the result follows as a consequence of the third item of Proposition \(15\).

The statement for the splittings of \(\Theta^0_{h_\delta}\) and \(\Omega^0_{h_\delta}\) is immediate. \(\blacksquare\)

Note that, once again, the use of both extended Legendre maps is necessary to obtain the Hamiltonian density in this way.

Finally, in the almost-regular case, the Hamilton-Jacobi variational principle of definition \(14\) is stated in the same way, now using sections of \(\bar{\rho}_0^0: P \to M\), and the form \(\Theta^0_{h_\delta}\). So we look for sections \(\psi_0 \in \Gamma_c(M, P)\) which are stationary with respect to the variations given by \(\psi_0t = \sigma_t \circ \psi_0\), where \(\{\sigma_t\}\) is a local one-parameter group of any \(\bar{\rho}_0^0\)-vertical vector field \(Z \in \mathcal{X}(P)\), such that

\[
\frac{d}{dt}|_{t=0} \int_M \psi_0^* \Theta^0_{h_\delta} = 0
\]

Then these critical sections will be characterized by the condition (analogous to Theorem \(3\))

\[
\psi_0^* (\bar{X}^0) \Omega^0_{h_\delta} = 0 \quad \text{for every } \bar{X}^0 \in \mathcal{X}(P) \tag{29}
\]

### 3.6 Equivalence between the Lagrangian and Hamiltonian formalisms

One expects that both the Lagrangian and Hamiltonian formalism must be equivalent. As in mechanics, this equivalence can be proved by using the (reduced) Legendre map.

First, using the Legendre map, we can lift sections of \(\pi\) from \(E\) to \(\Pi\) as follows:
Definition 20 Let \((J^1 E, \Omega_L)\) be a hyper-regular Lagrangian system, \(FL\) the induced Legendre transformation, \(\phi: M \to E\) a section of \(\pi\) and \(j^1 \phi: M \to J^1 E\) its canonical prolongation to \(J^1 E\). The Lagrangian prolongation of \(\phi\) to \(\Pi\) is the section

\[
j^1 \phi := FL \circ j^1 \phi: M \to \Pi
\]

If \((J^1 E, \Omega_L)\) is an almost-regular Lagrangian system, the Lagrangian prolongation of a section \(\phi: M \to E\) to \(P\) is

\[
j^1 \phi := FL \circ j^1 \phi: M \to P
\]

Theorem 5 (Equivalence theorem for sections) Let \((J^1 E, \Omega_L)\) and \((\Pi, \Omega_{h_3})\) be the Lagrangian and Hamiltonian descriptions of a hyper-regular system.

If a section \(\phi \in \Gamma_c(M, E)\) is a solution of the Lagrangian variational problem (Hamilton principle) then the section \(\psi \equiv j^1 \phi := FL \circ j^1 \phi \in \Gamma_c(M, \Pi)\) is a solution of the Hamiltonian variational problem (Hamilton-Jacobi principle).

Conversely, if a section \(\psi \in \Gamma_c(M, \Pi)\) is a solution of the Hamiltonian variational problem, then the section \(\phi \equiv \rho^1 \circ \psi \in \Gamma_c(M, E)\) is a solution of the Lagrangian variational problem.

(Proof) Bearing in mind the diagram

\[
\begin{array}{ccc}
J^1 E & \xrightarrow{FL} & \Pi \\
\downarrow{j^1 \phi} & \uparrow{\rho^1} & \downarrow{\psi} \\
E & \uparrow{\pi^1} & P \\
\phi & \uparrow{\pi} & M
\end{array}
\]

(30)

If \(\phi\) is a solution of the Lagrangian variational problem then \((j^1 \phi)^* i(X)\Omega_L = 0\), for every \(X \in \mathfrak{X}(J^1 E)\) (Theorem \(\square\)); therefore, as \(FL\) is a local diffeomorphism,

\[
0 = (j^1 \phi)^* i(X)\Omega_L = (j^1 \phi)^* i(X)(FL^* \Omega_{h_3}) = (j^1 \phi)^* (i(F\pi^{-1} X)\Omega_{h_3}) = (FL \circ j^1 \phi)^* (i(X')\Omega_{h_3})
\]

which holds for every \(X' \in \mathfrak{X}(\Pi)\) and thus, by Theorem \(\square\), \(\psi \equiv FL \circ j^1 \phi\) is a solution of the Hamiltonian variational problem. (This proof holds also for the almost-regular case).

Conversely, let \(\psi \in \Gamma_c(M, \Pi)\) be a solution of the Hamiltonian variational problem. Reversing the above reasoning we obtain that \((FL^{-1} \circ \psi)^* i(X)\Omega_L = 0\), for every \(X \in \mathfrak{X}(J^1 E)\), and hence \(\sigma \equiv FL^{-1} \circ \psi \in \Gamma_c(M, J^1 E)\) is a critical section for the Lagrangian variational problem. Then, as we are in the hyper-regular case, \(\sigma\) must be a holonomic section \[12\], \[30\], \[53\], \(\sigma = j^1 \phi\), and since \(\square\) is commutative, \(\phi = \rho^1 \circ \psi \in \Gamma_c(M, E)\), necessarily.

Remarks:

- Observe that every section \(\psi: M \to \Pi\) which is solution of the Hamilton-Jacobi variational principle is necessarily a Lagrangian prolongation of a section \(\phi: M \to E\).
• In the almost-regular case, if \( \phi \) is a critical section of the Lagrangian problem, then \( \psi = F\mathcal{L} \circ j^1\phi \) is a critical section of the Hamiltonian problem. Furthermore, \( F\mathcal{L}: j^1\phi(M) \to F\mathcal{L}(j^1\phi(M)) \) is a diffeomorphism because sections of \( \pi^1 \) are transversal to the fibres of \( F\mathcal{L} \).

As critical sections are integral manifolds of multivector fields \([36], [38], [53]\), then critical sections through different points in the same fiber of \( F\mathcal{L} \) have the same image by \( F\mathcal{L} \).

On the other hand, we can prove the equivalence between the Lagrangian and Hamiltonian formalisms from the variational point of view. First, we need the following lemma:

**Lemma 4** Let \( \beta \in \Omega^m(J^1E) \) and \( f \in C^\infty(J^1E) \). For every differentiable section \( \phi: U \subset M \to E \), the following conditions are equivalent:

1. \((j^1\phi)^*[f(\pi^1*\omega)] = (j^1\phi)^*\beta\).
2. \(\int_{j^1\phi} f(\pi^1*\omega) = \int_{j^1\phi} \beta\).

(Proof) Trivially 1 \(\Rightarrow\) 2.

Conversely, if we suppose 1 is not true, then there exists one section \( \phi: U \subset M \to E \) with \((j^1\phi)^*[f(\pi^1*\omega) - \beta] \neq 0 \) and hence there is \( x \in U \) and a closed neighbourhood \( V \) of \( x \) in \( U \) such that, taking \( \gamma: V \to E \) with \( \gamma = \phi|_V \), then \( \int_{j^1\gamma} [f(\pi^1*\omega) - \beta] \neq 0 \), so 2 is false.

Now, let \( \nabla \) be a connection in \( \pi: E \to M \), and let \( \mathcal{E}_\nabla = E_\nabla(\pi^1*\omega) \) be the density of Lagrangian energy associated with \( \nabla \) and \( \mathcal{L} \). Then:

**Theorem 6** The Lagrangian energy function is the unique function in \( J^1E \) verifying the following condition: for every section \( \phi: U \subset M \to E \),

\[
(j^1\phi)^*[E_\nabla(\pi^1*\omega)] = (j^1\phi)^*(F\mathcal{L}^*\Theta_{h^\nabla} - \mathcal{L})
\]

(Proof) (Uniqueness): Let \( f \) and \( g \) be two functions verifying this condition. Obviously \((j^1\phi)^*)((f - g)(\pi^1*\omega)) = 0\), but \(0 = (j^1\phi)^*[(f - g)(\pi^1*\omega)] = (f - g)(j^1\phi(x))(\pi^1*\omega)\), for every \( x \in U \). Hence, \((f - g)(j^1\phi(x)) = 0\), and this implies \( f - g = 0\), because every point in \( J^1E \) is in the image of some section \( j^1\phi \).

(Existence): From [24] we obtain

\[
(j^1\phi)^*(F\mathcal{L}^*\Theta_{h^\nabla} - \mathcal{L}) = (j^1\phi)^*[\Theta_{\mathcal{L}} + E_\nabla(\pi^1*\omega) - \mathcal{L}]
\]

\[
= (j^1\phi)^*[(i(V)d\mathcal{L} + \mathcal{L} + E_\nabla(\pi^1*\omega) - \mathcal{L})] = (j^1\phi)^*[E_\nabla(\pi^1*\omega)]
\]

since \((j^1\phi)^*(i(V)d\mathcal{L}) = 0\) (as can be proved by using expressions in coordinates). So, the energy function introduced in definition [32] satisfies this condition.

And this result leads to the following consequence:
Theorem 7 Let \((J^1 E, \Omega_L)\) and \((\Pi, \Omega_{h^\mu})\) be the Lagrangian and Hamiltonian descriptions of a hyper-regular system. Then, the Hamilton variational principle of the Lagrangian formalism and the Hamilton-Jacobi variational principle of the Hamiltonian formalism are equivalent. That is, for every section \(\phi \in \Gamma_c(M, E)\) we have that

\[
\int_\phi L = \int_\phi^* \Theta_{h^\mu}
\]

(Proof) The standpoint is the relation stated in Theorem 6 which, by Lemma 4, is equivalent to

\[
\int_\phi E^\pi_1(\pi^1, \omega) = \int_\phi (FL^* \Theta_{h^\mu} - L)
\]

therefore, from this equality, and using (24) and (22), we obtain

\[
\int_\phi L = \int_\phi [FL^* \Theta_{h^\mu} - E^\pi_1(\pi^1, \omega)] = \int_\phi \Theta_L = \int_\phi FL^* \Theta_{h^\mu} = \int_\phi L \circ j^1_\phi \Theta_{h^\mu} = \int_\phi^* \Theta_{h^\mu}
\]

It is important to remark the essential role played by the Lagrangian energy function in the proof of this equivalence.

(The above results are generalizations of others in non-autonomous mechanics \([10]\)).

4 Hamiltonian formalism in the restricted multimoment momentum bundle \(J^1 \pi^*\). Relation with the formalism in \(\Pi\)

4.1 Hamiltonian systems

The construction of the Hamiltonian formalism in \(J^1 \pi^*\) is posed, for the first time, in \([6]\), and the particular case of Hamiltonian systems associated with hyper-regular and almost-regular systems is stated in \([37]\). The procedure is essentially similar to that developed in section 3.1 for \(\Pi\). Next we sketch this construction, relating it with the above one in \(\Pi\). As we have proved the existence of the canonical diffeomorphism \(\Psi: \Pi \rightarrow J^1 \pi^*\), we can use it to prove the equivalence between the Hamiltonian formalisms in \(\Pi\) and \(J^1 \pi^*\).

Definition 21 Consider the bundle \(\tilde{\tau}^1: J^1 \pi^* \rightarrow M\).

1. A section \(h^\mu: J^1 \pi^* \rightarrow M\) of the projection \(\mu\) is called a Hamiltonian section of \(\mu\).

2. The differentiable forms

\[
\Theta_{h^\mu} := h^\ast_\mu \Theta, \quad \Omega_{h^\mu} := h^\ast_\mu \Omega
\]

are called the Hamilton-Cartan \(m\) and \((m+1)\) forms of \(J^1 \pi^*\) associated with the Hamiltonian section \(h^\mu\).

3. The couple \((J^1 \pi^*, \Omega_{h^\mu})\) is said to be a Hamiltonian system.
In a local chart of natural coordinates, a Hamiltonian section is specified by a local Hamiltonian function $H_{h_\mu} \in \mathcal{C}^\infty(U)$, $U \subset J^1\pi^*$, such that $H_{h_\mu}(x^\nu, y^A, p^\mu_B) \equiv (x^\nu, y^A, p = -H_{h_\mu}(x^\gamma, y^B, p^\mu_B), p^\nu_A)$. The local expressions of the Hamilton-Cartan forms associated with $h_\mu$ are similar to (11), but changing $H_{h_\delta}$ by $H_{h_\mu}$, and $p^\nu_A$ by $p^\nu_A$.

For Hamiltonian sections of $\mu$, we have a similar result to that in Lemma 2, and so, if $h^1_\mu, h^2_\mu$ are two sections of $\mu$, then
\[ \Theta_{h^1_\mu} - \Theta_{h^2_\mu} = h^1_\mu \Theta - h^2_\mu \Theta = \tau^1 \ast (h^1_\mu - h^2_\mu) \equiv \mathcal{H} \] (31)
is a $\tau^1$-semibasic $m$-form in $J^1\pi^*$.

**Definition 22** A $\tau^1$-semibasic form $\mathcal{H} \in \Omega^m(J^1\pi^*)$ is said to be a Hamiltonian density in $J^1\pi^*$.

It can be written as $\mathcal{H} = \Pi(\tau^1 \omega)$, where $\Pi \in \mathcal{C}^\infty(J^1\pi^*)$ is the global Hamiltonian function associated with $\mathcal{H}$ and $\omega$.

If (11) holds, then the relation between the global Hamiltonian function $H$ associated with $\mathcal{H}$, and the local Hamiltonian functions $H_{h_1^1}, H_{h_2^1}$ associated with $h^1_\mu$ and $h^2_\mu$ is $H = H_{h_1^1} - H_{h_2^1}$ (in an open set $U$).

In this way we have the analogous result as in Theorem 3.

**Theorem 8** The set of Hamilton-Cartan m-forms associated with Hamiltonian sections of $\mu$ is an affine space modelled on the set of Hamiltonian densities in $J^1\pi^*$.

Hence, if $(J^1\pi^*, \Omega_{h_\mu})$ is a Hamiltonian system, we have that every Hamiltonian section $h_\mu' \neq h_\mu$ allows us to split globally the Hamilton-Cartan forms as
\[ \Theta_{h_\mu'} = \Theta_{h_\mu} - \mathcal{H} \quad , \quad \Omega_{h_\mu'} = \Omega_{h_\mu} + d\mathcal{H} \]
The local expressions of these splittings are similar to (14), but changing $H_{h_\delta'}$ by $H_{h_\mu'}$, and $p_{A'}^\nu$ by $p_{A'}^\nu$.

Now, if we have a connection $\nabla$ in $\pi: E \to M$, it induces a linear section $h_{\nabla}^\delta: \Pi \to J^1E^*$ of $\delta$, and hence there exists another linear section $h_{\nabla}^\delta: J^1\pi^* \to M\pi$ of $\mu$ given by $h_{\nabla}^\delta \circ \psi^{-1} = \iota_0 \circ h_{\nabla}^\delta$ (see [10] for an alternative definition). Then, if $\Theta$ is the canonical $m$-form in $\Omega^m(M\pi)$, the forms
\[ \Theta_{h_{\nabla}^\delta} := h_{\nabla}^\delta \ast \Theta \in \Omega^m(J^1\pi^*) \quad , \quad \Omega_{h_{\nabla}^\delta} := -d\Theta_{h_{\nabla}^\delta} \in \Omega^{m+1}(J^1\pi^*) \]
are the Hamilton-Cartan $m$ and $(m+1)$ forms of $J^1\pi^*$ associated with the connection $\nabla$. Of course, a characterization of $\Theta_{h_{\nabla}^\delta}$ can be stated in the same way as in (10). The local expression of these Hamilton-Cartan forms associated with $\nabla$ is similar to (17).

Therefore, given a connection $\nabla$ and a Hamiltonian section $h_\mu$, from the above results we have that
\[ \tau^1 \ast (h_{\nabla}^\delta - h_\mu) = h_{\nabla}^\delta \ast \Theta - h_\mu \ast \Theta = \Theta_{h_{\nabla}^\delta} - \Theta_{h_\mu} := \mathcal{H}_{h_\mu} \]
is a Hamiltonian density in $J^1\pi^*$, which is written as $\mathcal{H}_{h_\mu} = \Pi_{h_\mu}(\tau^1 \omega)$, where $\Pi_{h_\mu} \in \mathcal{C}^\infty(J^1\pi^*)$ is the global Hamiltonian function associated with $\mathcal{H}_{h_\mu}$ and $\omega$. Then, the Hamilton-Cartan forms associated with $h_\mu$ split as
\[ \Theta_{h_\mu} = \Theta_{h_{\nabla}^\delta} - \mathcal{H}_{h_\mu} \quad , \quad \Omega_{h_\mu} = \Omega_{h_{\nabla}^\delta} + d\mathcal{H}_{h_\mu} \]
The local expressions of these splittings are similar to (19), but changing $H_{h^\mu}^\nabla$ by $H_{h^\mu}^\delta$, and $p^\nu_A$ by $p^\nu_A$.

If, conversely, we take a connection $\nabla$ and a Hamiltonian density $\mathcal{H}$, then making $h^\nabla \mu - H$ we obtain a Hamiltonian section $h^\mu$, since $\mathcal{H}:J^1\pi^* \to M\pi$ takes values in $\pi^*\Lambda^\text{m}T^*M$. Hence:

**Proposition 17** A couple $(h^\mu, \nabla)$ in $J^1\pi^*$ is equivalent to a couple $(\mathcal{H}, \nabla)$ (that is, given a connection $\nabla$, Hamiltonian sections of $\mu$ and Hamiltonian densities in $J^1\pi^*$ are in one-to-one correspondence).

Bearing in mind this last result, we have another way of obtaining a Hamiltonian system, which consists in giving a couple $(\mathcal{H}, \nabla)$. In fact:

**Proposition 18** Let $\nabla$ be a connection in $\pi:E \to M$, and $\mathcal{H}$ a Hamiltonian density. There exists a unique Hamiltonian section $h^\mu$ of $\mu$ such that

$$
\Theta_{h^\mu} = \Theta_{h^\mu}^\nabla - \mathcal{H}, \quad \Omega_{h^\mu} = -d\Theta_{h^\mu} = \Omega_{h^\delta} + d\mathcal{H} \quad (32)
$$

Concerning field equations, observe that diffeomorphisms in $E$ (and hence vector fields in $E$) can be lifted to $J^1\pi^*$, for instance, lifting them to $\Pi$ (see definitions 15 and 16), and translating them to $J^1\pi^*$ using the diffeomorphism $\Psi$. Hence, for a Hamiltonian system $(J^1\pi^*, \Omega_{h^\mu})$, we can set the Hamilton-Jacobi variational principle as in definition 17 (but with the form $\Omega_{h^\mu}$ instead of $\Omega_{h^\delta}$), and state the same results and comments as in Theorem 4.

Hamiltonian systems in $\Pi$ and $J^1\pi^*$ are equivalent. In fact; as a first result we have:

**Proposition 19** Let $\nabla$ be a connection in $\pi:E \to M$, and $\mathcal{H}$ and $\mathcal{H}$ Hamiltonian densities in $\Pi$ and $J^1\pi^*$, respectively, such that $\Psi^*\mathcal{H} = \mathcal{H}$. Then

$$
\Psi^*\Omega_{h^\mu}^\nabla = \Omega_{h^\delta}^\nabla, \quad \Psi^*\Omega_{h^\mu} = \Omega_{h^\delta}
$$

(Proof) The proof is based in the following fact:

$$
\Theta_{h^\mu} = h^\nabla^*\Theta = (\iota_0 \circ h^\delta \circ \Psi)^*\Theta = \Psi^*(\iota_0 \circ h^\delta)^*\Theta = \Psi^*\Theta_{h^\delta}
$$

and the result is immediate.

And therefore, as a direct consequence of Propositions 18 and 19, we can set the relation between the Hamiltonian systems in $\Pi$ and $J^1\pi^*$:

**Theorem 9** Every Hamiltonian system $(\Pi, \Omega_{h^\delta})$ is equivalent to a Hamiltonian system $(J^1\pi^*, \Omega_{h^\mu})$, and conversely.

At this point, we can study the relation between the set of connections $\nabla$ in the bundle $\pi:E \to M$, and the sets of linear (Hamiltonian) sections of the projections $\mu: M\pi \to J^1\pi^*$ and $\delta: J^1E^* \to \Pi$:
Theorem 10 The map $\nabla \mapsto h_\mu$ is a bijective affine map between the set of connections in the bundle $\pi: E \to M$ and the set of linear (Hamiltonian) sections of the projection $\mu: \mathcal{M}\pi \to J^1\pi^*$ or, what is equivalent, the set of linear (Hamiltonian) sections of the projection $\delta: J^1E^* \to \Pi$.

(Proof) Let the projection $\mu: \Lambda^m T^*E \to \Lambda^m_1 T^*E / \Lambda^m_0 T^*E$, and the set of linear sections of $\mu$

$$\Gamma(\mu) := \{ \ell \in \mathcal{L}(J^1\pi^*, \Lambda^m_1 T^*E) , \mu \circ \ell = \text{Id}_{J^1\pi^*}\}$$

which is an affine bundle modeled on the vector bundle

$$(J^1\pi^*)^* \otimes \Lambda^m_0 T^*E \simeq (\pi^* TM \otimes V^*(\pi) \otimes \pi^* \Lambda^m T^*M)^* \otimes \Lambda^m_0 T^*E$$

But this last bundle is just the vector bundle on which the affine bundle of the connection forms in $\pi: E \to M$ is modeled. Then the result follows.

Finally, the equivalence with the set of linear (Hamiltonian) sections of the projection $\delta: J^1E^* \to \Pi$ is proved by taking into account that every linear section of $\delta$ is associated with a connection $\nabla$, since this linear section defines a linear map from $V^*(\pi)$ to $T^*E$, and hence a projection $TE \to V(\pi)$ (that is, a connection).

4.2 Hamiltonian system associated with a hyper-regular Lagrangian system

The procedure is analogous to that in Section 3.4 (see also diagram (7)). Let $(J^1E, \Omega_\mathcal{L})$ be a hyper-regular Lagrangian system, then:

Definition 23 Let $h_\mu: J^1\pi^* \to \mathcal{M}\pi$ be the section of $\mu$ given by

$$h_\mu := \widetilde{FL} \circ FL^{-1}$$

which is a diffeomorphism connecting $J^1\pi^*$ and $\widetilde{FL}(J^1E)$ (observe that it is just the inverse of $\mu$ restricted to $\widetilde{FL}(J^1E)$). We define the Hamilton-Cartan forms

$$\Theta_{h_\mu} := h_\mu^* \Theta ; \quad \Omega_{h_\mu} := h_\mu^* \Omega$$

Proposition 20 The Hamilton-Cartan forms satisfy that

$$FL^* \Theta_{h_\mu} = \Theta_\mathcal{L} \quad , \quad FL^* \Omega_{h_\mu} = \Omega_\mathcal{L}$$

Then $(J^1\pi^*, \Omega_{h_\mu})$ is the (unique) Hamiltonian system associated with the hyper-regular Lagrangian system $(J^1E, \Omega_\mathcal{L})$.

(Proof) We have the diagram
Taking into account the commutativity of this diagram, and proposition 2, we have
\[ \mathcal{F}\mathcal{L}^*\Theta_{h_{\mu}} = \mathcal{F}\mathcal{L}^*h_{\mu}^*\Theta = \overline{\mathcal{F}\mathcal{L}}^*\Theta = \Theta_{\mathcal{L}} \]
and the same result follows for \( \Omega_{h_{\mu}} \).

Using charts of natural coordinates in \( J^1\pi^* \) and \( \mathcal{M}\pi \), and the expression (3) of the Legendre map, we obtain that the local Hamiltonian function \( H_{h_{\mu}} \) representing this Hamiltonian section is
\[ H_{h_{\mu}}(x^\nu, y^A, p_A^\nu) = \mathcal{F}\mathcal{L}^{-1*} \left( v_{\nu}^A \frac{\partial \mathcal{L}}{\partial v^\nu_A} - \mathcal{L} \right) = p_A^\nu \mathcal{F}\mathcal{L}^{-1*} v_{\nu}^A - \mathcal{F}\mathcal{L}^{-1*} \mathcal{L} \quad (33) \]
and the local expressions of the corresponding Hamilton-Cartan forms are
\[ \Theta_{h_{\mu}} = p_A^\nu dy^A \wedge d^{m-1}x_\nu - (p_A^\nu \mathcal{F}\mathcal{L}^{-1*} v_{\nu}^A - \mathcal{F}\mathcal{L}^{-1*} \mathcal{L})d^m x \]
\[ \Omega_{h_{\mu}} = -dp_A^\nu \wedge dy^A \wedge d^{m-1}x_\nu + d(p_A^\nu \mathcal{F}\mathcal{L}^{-1*} v_{\nu}^A - \mathcal{F}\mathcal{L}^{-1*} \mathcal{L}) \wedge d^m x \]

We can construct this Hamiltonian system using connections. Thus, if \( \nabla \) is a connection in \( \pi: E \to M \), and \( h_{\mu}^\nu \) is the linear Hamiltonian section of \( \mu \) associated with \( \nabla \), following the same pattern as in Proposition 1, we can prove:

**Proposition 21**

1. The \( m \)-form \( \mathcal{F}\mathcal{L}^*\Theta_{h_{\mu}} - \Theta_{\mathcal{L}} \) is \( \pi^1 \)-semibasic and
\[ \mathcal{F}\mathcal{L}^*\Theta_{h_{\mu}} - \Theta_{\mathcal{L}} = \mathcal{E}_{\mathcal{L}}^\nu \]

2. There exists a unique Hamiltonian density \( \mathcal{H}^\nu \in \Omega^m(J^1\pi^*) \) such that
\[ \mathcal{F}\mathcal{L}^*\mathcal{H}^\nu = \mathcal{F}\mathcal{L}^*\Theta_{h_{\mu}} - \Theta_{\mathcal{L}} = \mathcal{E}_{\mathcal{L}}^\nu \quad (34) \]
Then there exists a function \( H^\nu \in C^\infty(J^1\pi^*) \) such that \( \mathcal{H}^\nu = H^\nu(\pi^1\omega) \).
\( \mathcal{H}^\nu \) and \( H^\nu \) are called the Hamiltonian density and the Hamiltonian function associated with the Lagrangian system, the connection \( \nabla \) and \( \omega \).

3. The Hamilton-Cartan forms of definition (23) split as
\[ \Theta_{h_{\mu}} = \Theta_{h_{\mu}}^\nu - \mathcal{H}^\nu = \Theta_{h_{\mu}}^\nu - \mathcal{H}^\nu \]
\[ \Omega_{h_{\mu}} = -d\Theta_{h_{\mu}} = \Omega_{h_{\mu}}^\nu + d\mathcal{H}^\nu \quad (35) \]

We can obtain this Hamiltonian density using only Hamiltonian sections. In fact:

**Proposition 22**

1. Consider the Hamiltonian section \( h_{\mu} := \overline{\mathcal{F}\mathcal{L}} \circ \mathcal{F}\mathcal{L}^{-1} \), and a connection \( \nabla \).
Then we have
\[ h_{\mu}^\nu^* \Theta - h_{\mu}^* \Theta = \mathcal{H}^\nu \]
and hence the splitting (23) holds.

( Proof ) We have the diagram
The first item is a consequence of the third item of Proposition 3. For the second item, taking into account definition 8 and (32), Proposition 2, and (34), we have

\[ h^\nabla \ast \Theta - h^\ast \Theta = \Theta h^\nabla - (FL^{-1})^\ast \tilde{FL}^\ast \Theta = \Theta h^\nabla - (FL^{-1})^\ast \Theta = \mathcal{H}^\nabla \]

Then the result for the Hamilton-Cartan forms follows immediately.

Of course, all the results stated in section 3.3 concerning to the variational principle and the characterization of critical sections are true, and the local Hamiltonian function \( H_{h^\mu} \) appearing in the Hamilton-De Donder-Weyl equations is given by \( \mathcal{H}^\nabla \).

Finally, the equivalence between the Hamiltonian formalisms (in \( \Pi \) and \( J^1\pi^* \)) associated with a hyper-regular Lagrangian system, and between the Lagrangian formalism and the Hamiltonian formalism in \( J^1\pi^* \) is given by the following:

**Theorem 11** Let \((J^1E, \Omega_L)\) be a hyper-regular Lagrangian system. Then

\[ \Psi^\ast \Theta_{h^\delta} = \Theta_{h^\mu} \quad , \quad \Psi^\ast \Omega_{h^\delta} = \Omega_{h^\mu} \]

and hence its associated Hamiltonian systems \((\Pi, \Omega_{h^\delta})\) and \((J^1\pi^*, \Omega_{h^\mu})\) are equivalent.

*(Proof)* It is immediate. Observe also that the sections \( h^\mu \) and \( h^\delta \) are equivalent, by the commutativity of diagram \( \mathcal{D} \).

Observe that, as \( FL \) is a diffeomorphism, we also have that \( \Psi^\ast \Pi^\nabla = \mathcal{H}^\nabla \).

### 4.3 Hamiltonian system associated with an almost-regular Lagrangian system

The procedure is analogous to that in Section 3.5 (see also diagram \( \mathcal{D} \)). Now, \((J^1E, \Omega_L)\) is an almost-regular Lagrangian system, and the submanifold \( j_0: \mathcal{P} \hookrightarrow J^1\pi^* \), is a fiber bundle over \( E \) (and \( M \)). The corresponding projections will be denoted by \( \tau_0^1: \mathcal{P} \rightarrow E \) and \( \bar{\tau}_0^1: \mathcal{P} \rightarrow M \), satisfying that \( \tau_0 \circ j_0 = \tau_0^1 \) and \( \bar{\tau}_0 \circ j_0 = \bar{\tau}_0^1 \).

Taking into account Proposition 3, and following the same pattern as in Propositions 3, we can prove that the Lagrangian forms \( \Theta_L \) and \( \Omega_L \) are \( FL \)-projectable, and then:

**Definition 24** Given the diffeomorphism \( \tilde{h}_\mu = \tilde{\mu}^{-1} \), we define the Hamilton-Cartan forms

\[ \Theta^0_{h^\mu} = \tilde{h}^\ast \Theta \quad ; \quad \Omega^0_{h^\mu} = \tilde{h}^\ast \Omega \]

**Proposition 23** The Hamilton-Cartan forms satisfy that

\[ FL^0_\ast \Theta^0_{h^\mu} = \Theta_L \quad , \quad FL^0_\ast \Omega^0_{h^\mu} = \Omega_L \]

Then \((J^1\pi^*, \mathcal{P}, \Omega^0_{h^\mu})\) is the (unique) Hamiltonian system associated with the almost-regular Lagrangian system \((J^1E, \Omega_L)\).
and the same result follows for $\Omega^0_{\mu}$.  

We can construct this Hamiltonian system using a connection. Thus, let $\nabla$ be a connection in $\pi:E \to M$, and $h^\nabla_\mu:J^1\pi^* \to M\pi$ the induced linear section of $\mu$. Let $\mathcal{E}^\nabla_\mu \in \Omega^m(J^1E)$ be the density of Lagrangian energy associated with $\nabla$. Then, as in Proposition 15 we can prove:

**Proposition 24**

1. The $\tilde{\pi}^1$-semibasic m-form $\mathcal{F}\mathcal{L}^*\Theta_{h^\nabla_\mu} - \Theta_\mathcal{L}$ is $\mathcal{F}\mathcal{L}$-projectable and, if $\Theta^0_{h^\nabla_\mu} = j^0_\mu \Theta_{h^\nabla_\mu}$, then

$$\mathcal{E}^\nabla_\mu = \mathcal{F}\mathcal{L}^*\Theta_{h^\nabla_\mu} - \Theta_\mathcal{L} = \mathcal{F}\mathcal{L}^*\Theta^0_{h^\nabla_\mu} - \Theta_\mathcal{L}$$

2. There exists a unique $\tilde{\pi}^1_0$-semibasic form $\mathcal{H}^\nabla_0 \in \Omega^m(\mathcal{P})$, such that $\mathcal{F}\mathcal{L}^*\mathcal{H}^\nabla_0 = \mathcal{E}^\nabla_\mu$. Then, there is a function $\mathcal{H}^\nabla_0 \in C^\infty(\mathcal{P})$ such that $\mathcal{H}^\nabla_0 = \mathcal{H}^\nabla_0(\tilde{\pi}^1_0\omega)$. Obviously we have that $\mathcal{F}\mathcal{L}^*\mathcal{H}^\nabla_0 = \mathcal{E}^\nabla_\mu$.

$\mathcal{H}^\nabla_0$ and $\mathcal{H}^\nabla_0$ are called the Hamiltonian density and the Hamiltonian function associated with the Lagrangian system, the connection $\nabla$ and $\omega$.

3. The Hamilton-Cartan forms of definition 24 split as

$$\Theta^0_{h^\nabla_\mu} = j^0_\mu \Theta_{h^\nabla_\mu} - \mathcal{H}^\nabla_0 = \Theta^0_{h^\nabla_\mu} - \mathcal{H}^\nabla_0$$

$$\Omega^0_{h^\nabla_\mu} = -d\Theta^0_{h^\nabla_\mu} = j^0_\mu \Omega_{h^\nabla_\mu} + d\mathcal{H}^\nabla_0 = \Omega^0_{h^\nabla_\mu} + d\mathcal{H}^\nabla_0 \quad (36)$$

We can obtain this Hamiltonian system in the following equivalent way:

**Proposition 25** Consider the map $\tilde{h}^\nabla_\mu$, and a connection $\nabla$. Then $h^\nabla_\mu$ induces a map $\tilde{h}^\nabla_\mu: \mathcal{P} \to \tilde{\mathcal{P}}$, defined by the relation $\tilde{j}_0 \circ \tilde{h}^\nabla_\mu = h^\nabla_\mu \circ j_0$. Therefore

$$(\tilde{j}_0 \circ \tilde{h}^\nabla_\mu)^*\Theta - (j_0 \circ \tilde{h}^\nabla_\mu)^*\Theta = \mathcal{H}^\nabla_0$$

and hence the splitting $(36)$ holds.

(Proof) We have the diagram

The first part of the statement is a consequence of the fact that $\tilde{\mu}$ is a diffeomorphism. For the second part, taking into account the commutativity of this diagram, and bearing in mind the first item of Proposition 24 and Proposition 4, we have

$$\mathcal{F}\mathcal{L}^*\mathcal{H}^\nabla_0 = \mathcal{F}\mathcal{L}^*(\tilde{j}_0 \circ \tilde{h}^\nabla_\mu)^*\Theta - (j_0 \circ \tilde{h}^\nabla_\mu)^*\Theta = \mathcal{F}\mathcal{L}^*\Theta^0_{h^\nabla_\mu} - \mathcal{F}\mathcal{L}^*\tilde{h}^\nabla_\mu \Theta$$

$$= \mathcal{F}\mathcal{L}^*\Theta^0_{h^\nabla_\mu} - \mathcal{F}\mathcal{L}^*\Theta = \mathcal{E}^\nabla_\mu + \Theta_\mathcal{L} - \Theta_\mathcal{L} = \mathcal{E}^\nabla_\mu$$
and the result follows as a consequence of the above Proposition. The last statement is immediate.

Of course, the result stated in (29) concerning to the variational principle and the characterization of critical sections holds in the same way.

Finally, the equivalence between the Hamiltonian formalisms (in II and $J^1\pi^*$) associated with an almost-regular Lagrangian system, and between the Lagrangian formalism and the Hamiltonian formalism in $J^1\pi^*$ is given by the following:

**Theorem 12** Let $(J^1E, \Omega_L)$ be an almost-regular Lagrangian system. Then

$$\Psi^0_\ast \Theta^0_{h_\delta} = \Theta^0_{h_\mu}, \quad \Psi^0_\ast \Omega^0_{h_\delta} = \Omega^0_{h_\mu}$$

and hence its associated Hamiltonian systems $(\Pi, P, \Omega^0_{h_\mu})$ and $(J^1\pi^*, P, \Omega^0_{h_\mu})$ are equivalent.

**(Proof)** First, we have the following relation

$$\Theta^0_{h_\delta} = \theta^0_{h_\mu} = \theta^0_{h_\mu} = \psi^0 \ast \theta^0_{h_\delta} = (\psi \circ \theta^0) \ast \theta^0_{h_\delta} = \theta^0_{h_\mu} \ast \theta^0_{h_\delta} = \psi^0 \ast \theta^0_{h_\delta} = \theta^0_{h_\delta}$$

Furthermore, as $F_L = \psi_0 \circ F_L$, we have that

$$F^\ast_L = F \ast L_0 = F^\ast_L(\psi_0 \circ F_L) = (\psi_0 \ast F_L)^\ast \ast L_0 = \psi_0 \ast \ast F_L = \psi_0 \ast L_0 = L_0$$

since $F_L$ is a submersion. (In the same way $\psi_0 \ast L_0 = L_0$). Therefore, $\psi^0 \ast \theta^0_{h_\delta} = \theta^0_{h_\mu}$, and hence

$$\psi^0_\ast \Omega^0_{h_\delta} = \Omega^0_{h_\mu}.$$

Observe also that the section $\tilde{h}_\mu$ of $\mu$ and the family of sections $\tilde{h}_\delta$ of $\delta$ (given in Proposition 16) are equivalent, by the commutativity of diagram (9).

---

**5 Examples**

### 5.1 Non-autonomous Mechanics

The jet bundle description of time-dependent mechanical systems (see, for instance, [4] and [18]) takes $M = \mathbb{R}$, and $E = \mathbb{R} \times Q$, where $Q$ is a $N$-dimensional manifold (and thus $\pi: \mathbb{R} \times Q \to \mathbb{R}$ is a trivial bundle). Then $J^1E = \mathbb{R} \times TQ$. Natural adapted coordinates are denoted by $(t, q^i, v^i)$. Lagrangian densities are written $L = L dt$, where $L \in C^\infty(\mathbb{R} \times TQ)$ is a time-dependent Lagrangian function.

Next we will identify the different multimomentum bundles. First observe that, as $TM \simeq \mathbb{R} \times \mathbb{R}$, we obtain

$$\Lambda^m T^*M \equiv \Lambda^1 T^*M \simeq \mathbb{R} \times \mathbb{R}^*$$

Therefore we have:

**Generalized multimomentum bundle:** Observe that

$$\pi^* TM = \pi^* (\mathbb{R} \times \mathbb{R}) \simeq (\mathbb{R} \times Q) \times \mathbb{R} \simeq \mathbb{R} \times Q \times \mathbb{R}$$

$$T^* E = T^* (\mathbb{R} \times Q) = T^* \mathbb{R} \times T^* Q$$

$$\pi^* \Lambda^m T^*M \simeq \mathbb{R} \times Q \times \mathbb{R}^* \simeq Q \times T^* \mathbb{R}$$
and hence
\[ J^1E^* := \pi^*TM \times_E T^*E \times_E \pi^*\Lambda^m T^*M = (\mathbb{R} \times Q \times \mathbb{R}) \times_{(\mathbb{R} \times Q)} (T^*\mathbb{R} \times T^*Q) \times_{(\mathbb{R} \times Q)} (Q \times T^*\mathbb{R}) \]
\[ \simeq (\mathbb{R} \times Q \times \mathbb{R}) \times_{(\mathbb{R} \times Q)} (T^*\mathbb{R} \times T^*Q) \times_{(\mathbb{R} \times Q)} ((\mathbb{R} \times Q) \times \mathbb{R}^*) \simeq T^*\mathbb{R} \times T^*Q \simeq T^*(\mathbb{R} \times Q) \]

Notice that \( \dim J^1E^* = 2N + 2 \).

**Reduced multimomentum bundle:** As \( V^*(\pi) = \mathbb{R} \times T^*Q \), we have that
\[ \Pi := \pi^*TM \times_E V^*(\pi) \times_E \pi^*\Lambda^m T^*M = (\mathbb{R} \times Q \times \mathbb{R}) \times_{(\mathbb{R} \times Q)} (\mathbb{R} \times T^*Q) \times_{(\mathbb{R} \times Q)} (Q \times T^*\mathbb{R}) \]
\[ \simeq (\mathbb{R} \times Q \times \mathbb{R}) \times_{(\mathbb{R} \times Q)} (T^*\mathbb{R} \times T^*Q) \times_{(\mathbb{R} \times Q)} ((\mathbb{R} \times Q) \times \mathbb{R}^*) \simeq \mathbb{R} \times T^*Q \]
and \( \dim \Pi = 2N + 1 \).

**Extended multimomentum bundle:** Now we have
\[ \mathcal{M}\pi := \Lambda^m_0 T^*E \equiv \Lambda^1_0 T^*E = \Lambda^1_1 T^*E \simeq T^*E \simeq T^*(\mathbb{R} \times Q) \simeq T^*\mathbb{R} \times T^*Q \]
with \( \dim \mathcal{M}\pi = 2N + 2 \).

**Restricted multimomentum bundle:** Observe that \( \Lambda^m_0 T^*E \equiv \Lambda^1_0 T^*E = (\mathbb{R} \times Q) \times \mathbb{R}^* \) and then
\[ J^1\pi^* := \mathcal{M}\pi/\Lambda^m_0 T^*E \simeq (T^*\mathbb{R} \times T^*Q)/(\mathbb{R} \times Q \times \mathbb{R}^*) \simeq \mathbb{R} \times T^*Q \]
with \( \dim J^1\pi^* = 2N + 1 \).

**Comments:**

- It is interesting to point out that in the Hamiltonian formalism of non-autonomous mechanics, \( M \simeq \mathbb{R} \times T^*Q \) and \( J^1\pi^* \simeq \mathbb{R} \times T^*Q \) make the canonical diffeomorphism between the generalized and the extended multimomentum bundle evident. They correspond to the so-called extended momentum phase space of the symplectic formulation of time-dependent systems [3, 34, 48]. As a consequence, the generalized and the (first) extended Legendre maps are really the same.
- The case of singular (almost-regular) time-dependent mechanical systems has been extensively studied in this context in [3, 20].

### 5.2 Electromagnetic field (with fixed background)

In this case \( M \), is space-time endowed with a semi-Riemannian metric \( g \), \( E = T^*M \) is a vector bundle over \( M \) and \( \pi: T^*M \rightarrow M \) denotes the natural projection. Sections of \( \pi \) are the so-called electromagnetic potentials. Using the linear connection associated with the metric \( g \), one assures that \( J^1E \rightarrow T^*M \) is a vector bundle and, since \( VE = \pi^*T^*M \), we have \( J^1E = \pi^*T^*M \otimes_E \pi^*T^*M \).

Let \( \phi: M \rightarrow T^*M \) be a section of \( \pi \). Then \( J^1\phi: M \rightarrow \pi^*T^*M \times \pi^*T^*M \) is just \( J^1\phi = T\phi \) (observe that \( J^1\phi \) is a metric tensor on \( M \)). Now, considering \( \bar{y} \in J^1E \), and \( \phi: M \rightarrow T^*M \) being a representative of \( \bar{y} \); we have that the Lagrangian density is
\[ \mathcal{L} = \frac{1}{4}||d\phi||^2dV_g \]
where \( || \cdot || \) denotes the norm induced by the metric \( g \) on the 2-forms on \( M \), and \( dV_g \) is the volume element associated with the metric \( g \). Observe that \( d\phi \) is the skew-symmetric part of the matrix giving \( T\phi \) or, in other words, the skew-symmetric part of the metric \( T\phi \) on \( M \).
For simplifying calculations, we take $M = \mathbb{R}^3$ and the metric is $- + +$. Then $E = \mathbb{R}^3 \times \mathbb{R}^3$ and

$$J^1 E = (\mathbb{R}^3 \times \mathbb{R}^3) \times (\mathbb{R}^3 \times \mathbb{R}^3^*)$$

with dim $J^1 E = 15$. Coordinates in $J^1 E$ are usually denoted $(x^\nu, A^j, v_j^\nu)$, with $\nu, j = 0, 1, 2$. The coordinates $(A^1, A^2)$ constitute the vector potential and $A^0$ is the scalar potential. Then, locally $\phi = \phi_0 dx^\eta$, and therefore $J^1 \phi = \partial^2 \phi / \partial x^\nu \partial x^\nu \otimes dx^\eta$. It is usual to write $\phi$ in the form $A = \delta_{j\eta} A^j dx^\nu$ and $dA = \delta_{j\eta} v_j^\nu dx^\eta \wedge dx^\nu (\delta_{j\eta}$ is the Kronecker’s delta). Then, in natural coordinates we have the following expression for the Lagrangian function

$$\mathcal{L} = \frac{1}{4}[(v_2^j - v_2^0)^2 - (v_0^\nu - v_0^0)^2 - (v_1^j - v_1^0)^2]$$

Obviously, this is a singular Lagrangian, since its Hessian matrix

$$\frac{\partial^2 \mathcal{L}}{\partial v_0^j \partial v_0^\eta} = \frac{1}{2} \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

is singular (its rank is equal to 3).

Next we study the several Legendre maps associated with this system. As we know, all of them leave the coordinates $(x^\nu, A^j)$ invariant, thus we will write the relations only for the multimomentum coordinates.

**Generalized Legendre map:** The generalized multimomentum bundle is

$$J^1 E^* = (\mathbb{R}^3 \times \mathbb{R}^3^*) \times (\mathbb{R}^3 \otimes (\mathbb{R}^3 \otimes \mathbb{R}^3^*) \otimes \Lambda^3 \mathbb{R}^3^*)$$

From (1) we have

$$\begin{align*}
\overleftarrow{\mathcal{L}}^* p_0^0 &= 0 \\
\overleftarrow{\mathcal{L}}^* p_1^0 &= \frac{1}{2}(v_1^a - v_1^0) \\
\overleftarrow{\mathcal{L}}^* p_2^0 &= \frac{1}{2}(v_2^a - v_2^0)
\end{align*}$$

and for the additional multimomentum coordinates $(p_\eta^\nu$ in (1))

$$\begin{align*}
\overleftarrow{\mathcal{L}}^* p_0^0 &= \frac{1}{2}(v_0^a v_1^0 - v_0^0 v_1^a) + v_0^a (v_0^0 - v_0^a) \\
\overleftarrow{\mathcal{L}}^* p_1^0 &= \frac{1}{2}(v_1^a v_0^0 - v_1^0 v_0^a) + v_1^a (v_1^0 - v_1^a) \\
\overleftarrow{\mathcal{L}}^* p_2^0 &= \frac{1}{2}(v_2^a v_0^0 - v_2^0 v_0^a) + v_2^a (v_2^0 - v_2^a)
\end{align*}$$

and the Hamiltonian constraints

$$\begin{align*}
\dot{\xi}_1 &= p_0^0 = 0 \\
\dot{\xi}_2 &= p_0^0 + p_1^0 = 0 \\
\dot{\xi}_3 &= p_2^0 = 0 \\
\dot{\xi}_4 &= p_1^0 = 0 \\
\dot{\xi}_5 &= p_2^0 + p_1^0 = 0 \\
\dot{\xi}_6 &= p_1^0 + p_2^0 = 0
\end{align*}$$
and the additional ones
\[
\begin{align*}
\dot{\xi}^7 &\equiv p_0 p_7 (p_0^0 + p_1^1 - 2p_2^0) + (p_0^0)^2 p_0^0 + (p_1^1)^2 p_0^1 + p_1^1 (p_2^0 p_2^0 - p_0^0) = 0 \\
\dot{\xi}^8 &\equiv p_0 p_8 (p_0^0 + p_2^2 - 2p_0^0) + (p_0^0)^2 p_0^2 + (p_1^1)^2 p_2^0 + p_0^0 (p_2^0 p_2^0 - p_0^0) = 0 \\
\dot{\xi}^9 &\equiv p_0 p_9 (p_1^1 - p_2^2 - 2(p_2^0)^2) + (p_0^0)^2 p_2^1 + p_1^1 (p_2^0 p_2^0 - p_0^0) = 0
\end{align*}
\]
which define locally the submanifold \(\tilde{P}\) in \(J^1 E^*\).

Observe that \(\dim \tilde{P} = \dim J^1 E\), as rank \(\overline{F\mathcal{L}}_\ast\) is maximal. Then, taking into account the commutativity of diagram (9), we can conclude that, for this system, the degeneracy is on the fibers of the projection \(\delta\).

**Reduced Legendre map:** The reduced multimomentum bundle is

\[\Pi = (\mathbb{R}^3 \times \mathbb{R}^3^*) \times (\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \Lambda^3 \mathbb{R}^3^*)\]

For the reduced Legendre map the results are the same as for the restricted Legendre map, but changing the multimomentum coordinates \(p'_0\) by \(p_0\). So, from (2) we obtain

\[
\begin{align*}
F\mathcal{L}^* p_0^0 &= 0 & F\mathcal{L}^* p_2^0 &= 0 \\
F\mathcal{L}^* p_0^1 &= -\frac{1}{2}(v_1^1 - v_0^0) & F\mathcal{L}^* p_2^1 &= -\frac{1}{2}(v_1^2 - v_1^0) \\
F\mathcal{L}^* p_2^0 &= \frac{1}{2}(v_2^2 - v_1^2) & F\mathcal{L}^* p_2^2 &= 0
\end{align*}
\]

(\(F\mathcal{L}\) is a submersion onto its image, and hence the system is almost-regular).

Now we have the same Hamiltonian constraints

\[
\begin{align*}
\xi^1 &\equiv p_0^0 = 0 \\
\xi^2 &\equiv p_1^1 = 0 \\
\xi^3 &\equiv p_2^2 = 0
\end{align*}
\]

which define locally the submanifold \(P\) in \(\Pi\). Observe that, as these constraints are conditions of skew-symmetry, we have that

\[P = F\mathcal{L}(J^1 E) = \pi^* (\mathcal{A}(\mathbb{R}^3) \otimes \Lambda^3(\mathbb{R}^3^*))\]

where \(\mathcal{A}(\mathbb{R}^3)\) denotes the bundle whose sections are the 2-contravariant skew-symmetric tensor fields on \(\mathbb{R}^3\). Note also that \(\tilde{P}\) is not diffeomorphic to \(P\).

**First extended Legendre map:** The extended multimomentum bundle is

\[\mathcal{M}\pi = (\mathbb{R}^3 \times \mathbb{R}^3^*) \times \Lambda^3(\mathbb{R}^3 \times \mathbb{R}^3^*)\]

From (3) we obtain that

\[
\begin{align*}
\overline{F\mathcal{L}}^* p_0^0 &= 0 & \overline{F\mathcal{L}}^* p_2^0 &= 0 \\
\overline{F\mathcal{L}}^* p_0^1 &= -\frac{1}{2}(v_1^0 - v_0^0) & \overline{F\mathcal{L}}^* p_2^1 &= -\frac{1}{2}(v_1^2 - v_1^0) \\
\overline{F\mathcal{L}}^* p_2^0 &= \frac{1}{2}(v_2^2 - v_1^2) & \overline{F\mathcal{L}}^* p_2^2 &= 0
\end{align*}
\]

and the additional relation

\[
\overline{F\mathcal{L}}^* p = -\frac{1}{2}((v_2^2 - v_1^2)^2 - (v_0^2 - v_2^0)^2 - (v_0^1 - v_1^0)^2) \equiv -2\mathcal{L}
\]

The corresponding Hamiltonian constraints are

\[
\begin{align*}
\dot{\chi}^1 &\equiv p_0^0 = 0 \\
\dot{\chi}^2 &\equiv p_1^1 = 0 \\
\dot{\chi}^3 &\equiv p_2^2 = 0 \\
\dot{\chi}^4 &\equiv p_0^1 + p_0^0 = 0 \\
\dot{\chi}^5 &\equiv p_2^0 + p_0^0 = 0 \\
\dot{\chi}^6 &\equiv p_2^1 + p_1^1 = 0
\end{align*}
\]
and the additional one
\[ \dot{\chi}^7 \equiv p + 2(p_1^2)^2 - 2(p_0^2)^2 - 2(p_0^1)^2 = 0 \]
All of them define locally the submanifold \( \tilde{\mathcal{P}} \) in \( \mathcal{M}_\pi \).

Second extended Legendre map: From (4) we obtain that
\[
\begin{align*}
\mathcal{F}_L^* p_0^0 & = 0 \\
\mathcal{F}_L^* p_0^1 & = \frac{1}{2} (v_0^2 - v_0^1) \\
\mathcal{F}_L^* p_0^2 & = \frac{1}{2} (v_0^2 - v_0^1) \\
\mathcal{F}_L^* p_1^0 & = 0 \\
\mathcal{F}_L^* p_1^1 & = -\frac{1}{2} (v_0^1 - v_1^0) \\
\mathcal{F}_L^* p_1^2 & = -\frac{1}{2} (v_0^1 - v_1^0) \\
\mathcal{F}_L^* p_2^0 & = 0 \\
\mathcal{F}_L^* p_2^1 & = \frac{1}{2} (v_2^1 - v_1^0) \\
\mathcal{F}_L^* p_2^2 & = \frac{1}{2} (v_2^1 - v_1^0)
\end{align*}
\]

and the additional relation
\[ \mathcal{F}_L^* p = -\frac{1}{4} [(v_2^1 - v_1^0)^2 - (v_0^1 - v_0^2)^2 - (v_0^1 - v_1^0)^2] \equiv -\mathcal{L} \]
and the Hamiltonian constraints are now
\[
\begin{align*}
\dot{\chi}^1 & \equiv p_0^0 = 0 \\
\dot{\chi}^2 & \equiv p_0^1 = 0 \\
\dot{\chi}^3 & \equiv p_0^2 + p_0^3 = 0 \\
\dot{\chi}^4 & \equiv p_0^0 + p_0^1 = 0 \\
\dot{\chi}^5 & \equiv p_0^2 + p_0^3 = 0 \\
\dot{\chi}^6 & \equiv p_2^0 + p_2^1 = 0
\end{align*}
\]
and the additional one
\[ \dot{\chi}^7 \equiv p + (p_1^2)^2 - (p_0^2)^2 - (p_0^1)^2 = 0 \]
All of them define locally the submanifold \( \tilde{\mathcal{P}} \) in \( \mathcal{M}_\pi \). Note that the last constraint identifies the extra coordinate \( p \) with the Hamiltonian function which, for this system, is \( H = -\frac{1}{4} ((p_1^2)^2 - (p_0^2)^2 - (p_0^1)^2) \).

Restricted Legendre map: The restricted multimomentum bundle is
\[ J^1 \pi^* = (\mathbb{R}^3 \times \mathbb{R}^3) \times [\Lambda_1^3(\mathbb{R}^3 \times \mathbb{R}^3) / \Lambda_0^3(\mathbb{R}^3 \times \mathbb{R}^3)] \]
From (3) we obtain
\[
\begin{align*}
\mathcal{F}_L^* p_0^0 & = 0 \\
\mathcal{F}_L^* p_0^1 & = -\frac{1}{2} (v_0^1 - v_1^0) \\
\mathcal{F}_L^* p_0^2 & = -\frac{1}{2} (v_0^1 - v_1^0) \\
\mathcal{F}_L^* p_1^0 & = \frac{1}{2} (v_0^1 - v_0^2) \\
\mathcal{F}_L^* p_1^1 & = 0 \\
\mathcal{F}_L^* p_1^2 & = -\frac{1}{2} (v_0^1 - v_1^0) \\
\mathcal{F}_L^* p_2^0 & = \frac{1}{2} (v_2^1 - v_1^0) \\
\mathcal{F}_L^* p_2^1 & = \frac{1}{2} (v_2^1 - v_1^0) \\
\mathcal{F}_L^* p_2^2 & = 0
\end{align*}
\]
Hence we obtain the following set of Hamiltonian constraints
\[
\begin{align*}
\chi^1 & \equiv p_0^0 = 0 \\
\chi^2 & \equiv p_0^1 = 0 \\
\chi^3 & \equiv p_0^2 = 0 \\
\chi^4 & \equiv p_0^0 + p_0^1 = 0 \\
\chi^5 & \equiv p_0^2 + p_0^3 = 0 \\
\chi^6 & \equiv p_2^0 + p_2^1 = 0
\end{align*}
\]
which define locally the submanifold \( \mathcal{P} \) in \( J^1 \pi^* \).

Observe that, for this example,
\[ \text{rank } \mathcal{F}_L = \text{rank } \mathcal{F}_L = \text{rank } \mathcal{F}_L = \text{rank } \mathcal{F}_L \]
and the submanifolds \( \mathcal{P}, \mathcal{P}, \tilde{\mathcal{P}} \) and \( \tilde{\mathcal{P}} \) are, in fact, diffeomorphic.

6 Conclusions

We have studied the Hamiltonian formalism for first-order Classical Field theories in the context of multisymplectic manifolds, taking different choices of multimomentum bundles as phase spaces, in particular the bundles \( \Pi \) and \( J^1 \pi^* \).
First we have reviewed the construction of these and other auxiliary multimomentum bundles \((J^1E^*\text{ and } M\pi)\), as well as the definition of suitable Legendre maps when all these bundles are thought of, in a certain sense, as the dual bundles of a Lagrangian system \((J^1E, \Omega_L)\). The key result is the existence of a canonical diffeomorphism between \(\Pi\) and \(J^1\pi^*\). (See section 2.2).

In order to state the Hamiltonian formalism on \(\Pi\) and \(J^1\pi^*\), some additional geometric element is needed for obtaining the Hamilton-Cartan forms from the canonical forms which \(J^1E^*\) and \(M\pi\) are endowed with. In particular, we can take sections of the projections \(\delta: J^1E^* \rightarrow \Pi\) and \(\mu: M\pi \rightarrow J^1\pi^*\), (which are called Hamiltonian sections, and are the elements carrying the “physical” information in this construction), which allows us to pull-back the canonical forms from \(J^1E^*\) and \(M\pi\) to \(\Pi\) and \(J^1\pi^*\) respectively. These are the Hamilton-Cartan forms which define the Hamiltonian system. Hamiltonian sections are associated with local Hamiltonian functions, which appear explicitly in the local expression of the corresponding Hamilton-Cartan forms. (See sections 3.1 and 4.1).

A relevant result is that different choices of Hamiltonian sections of \(\delta\) may lead to the same Hamilton-Cartan forms in \(\Pi\), and this allows us to establish an equivalence relation in the set of sections of the projection \(\delta\). Then, using the diffeomorphism \(\Psi\) between \(\Pi\) and \(J^1\pi^*\), it is proved that there is a one-to-one correspondence between sections of \(\mu\) and classes of equivalent sections of \(\delta\). Therefore, the Hamilton-Cartan forms in \(\Pi\) and \(J^1\pi^*\) are \(\Psi\)-related and hence, Hamiltonian systems in \(\Pi\) and \(J^1\pi^*\) are equivalent. (See sections 3.1 and 4.1). Furthermore, another one-to-one correspondence exists between the set of connections in the bundle \(E \rightarrow M\), and the set of linear sections of the respective projections \(\delta\) and \(\mu\). (See sections 3.2 and 4.1).

The difference between two Hamilton-Cartan \(m\)-forms defined by Hamiltonian sections is a semibasic \(m\)-form which is called a Hamiltonian density. Hence the set of Hamilton-Cartan forms can be thought of as an affine space modelled on the module of Hamiltonian densities. As a particular case, Given a connection, (classes of) Hamiltonian sections and Hamiltonian densities are in one-to-one correspondence. As a consequence of this fact, a Hamiltonian system can be also constructed starting from a Hamiltonian density and a connection. (See sections 3.2 and 4.1).

The field equations of the Hamiltonian formalism can be derived from the so-called Hamilton-Jacobi variational principle. Different but equivalent ways of characterizing the critical sections by means of the Hamilton-Cartan forms are shown. In particular, in natural coordinates of the multimomentum bundles \(\Pi\) or \(J^1\pi^*\), these sections are obtained as solutions of a local system of first-order partial differential equations, which are known as the Hamilton-De Donder-Weyl equations. Nevertheless, as the Hamiltonian function appearing in the local expression of the Hamilton-Cartan form is local, these equations are not covariant. Then, for obtaining a set of covariant equations, we must introduce a Hamiltonian density, and so a global Hamiltonian function. (See section 3.3).

The question of associating a Hamiltonian system to a Lagrangian one is also analyzed, both in the hyper-regular and the almost-regular cases. We can define this Hamiltonian system in three equivalent ways (see sections 3.4, 3.5, 4.2 and 4.3):

- Using a natural Hamiltonian section, which is defined using the Legendre maps, for obtaining the Hamilton-Cartan forms. These forms are related to the Poincaré-Cartan forms of the Lagrangian formalism, through the Legendre map.
- Using a connection, the density of Lagrangian energy of the Lagrangian formalism can be defined. Then, we construct a Hamiltonian density as the only semibasic \(m\)-form which is related to it by means of the suitable Legendre map.
This last Hamiltonian density can be obtained from a connection and the above natural Hamiltonian section. In this case, the extended Legendre maps must also be used, and in particular, for the construction in the reduced multimomentum bundle $\Pi$, both extended Legendre maps are needed (in the hyper-regular and in the almost-regular cases). This fact would justify the introduction of two extended Legendre maps.

- As an additional result, in the hyper-regular case, the equivalence between the Lagrangian and the Hamiltonian formalism is proved from a double point of view: showing the equivalence between the sections solution of the Lagrangian and Hamiltonian problems, and proving the equivalence of the Lagrangian and Hamiltonian variational principles. This equivalence is only partially proved in the almost-regular case. (See section [3.4]).

### A Geometrical structures in first-order jet bundles

(See [11], [13] and [33]).

Let $\pi: E \to M$ be a fiber bundle $(\dim M = m, \dim E = N + m)$, and $\pi^1: J^1E \to E$ the 1-jet bundle of local sections of $\pi$, which is also a differentiable bundle on $M$ with projection $\bar{\pi}^1 = \pi$. If $(x^\nu, y^A)$ (with $\nu = 1, \ldots, m; A = 1, \ldots, N$) is a local system of coordinates adapted to the bundle $\pi: E \to M$, then we denote by $(x^\nu, y^A, v^A_\nu)$ the local system of coordinates induced in $J^1E$.

Let $\phi: M \to E$ be a section of $\pi$, $x \in M$ and $y = \phi(x)$. The vertical differential of the section $\phi$ at the point $y \in E$ is the map

$$d_y^v \phi : T_y E \to V_y(\pi)$$

Then, considering $\bar{y} \in J^1E$ with $\bar{y} \xrightarrow{\pi^1} y \xrightarrow{\pi} x$ and $\bar{u} \in T_{\bar{y}} J^1E$. The structure canonical 1-form of $J^1E$, denoted by $\theta$, is defined by

$$\theta(\bar{y}; \bar{u}) := (d_y^v \phi)(T_{\bar{y}} \bar{\pi}^1(\bar{u}))$$

where $\phi$ is a representative of $\bar{y}$. Its expression in a natural local system is $\theta = (dy^A - v^A_\nu dx^\nu) \otimes \frac{\partial}{\partial y^A}$.

Consider the canonical isomorphism $S_{\bar{y}}: T_{\bar{y}}^*(\bar{y}) M \otimes V_{\bar{y}}(\pi) \to V_{\bar{y}}(\pi^1)$ which consists in associating to an element $\alpha \otimes v \in T_{\bar{y}}^*(\bar{y}) M \otimes V_{\bar{y}}(\pi)$ the directional derivative in $\bar{y}$ with respect to $\alpha \otimes v$. Taking into account that $\alpha \otimes v$ acts in $J^1E$ by translation, we have

$$S_{\bar{y}}(\alpha \otimes v) := D_{\alpha \otimes v}(\bar{y}) = \lim_{t \to 0} \frac{f(\bar{y} + t(\alpha \otimes v)) - f(\bar{y})}{t}$$

for $f \in C^\infty(J^1E)$. Then we have the following isomorphism of $C^\infty(J^1E)$-modules

$$S: \Gamma(J^1E, \bar{\pi}^1 \times M \otimes \pi^1 V(\pi)) \to \Gamma(J^1E, V(\pi^1))$$

which is called the vertical endomorphism $S$. (Here, $\Gamma(A, B)$ denotes the set of sections of the projection $A \to B$). Notice that $S \in \Gamma(J^1E, (\pi^1 V(\pi))^* \otimes V(\pi^1) \otimes \bar{\pi}^1 TM)$ (where all the tensor products are on $C^\infty(J^1E)$). Then, another vertical endomorphism $V$ arises from the natural contraction between the factor $\Gamma(J^1E, (\pi^1 V(\pi))^*)$ of $S$ and the factor $\Gamma(J^1E, \pi^1 V(\pi))$ of $\theta$:

$$V = \iota(S) \theta \in \Omega^1(J^1E) \otimes \Gamma(J^1E, V(\pi) \otimes \bar{\pi}^1 TM)$$
so it is a morphism
\[ V: \Gamma(J^1E, \pi^*(\pi^1) \otimes \pi^{1*}T^*M) \rightarrow \Omega^1(J^1E) \]

\( \mathcal{S} \) can also be thought of as a morphism
\[ \mathcal{S}: \Gamma(J^1E, \pi^*(\pi^1) \otimes \pi^{1*}T^*M) \rightarrow \Gamma(J^1E, (\pi^{1*}V(\pi))^*) \]

As every connection \( \nabla \) on \( \pi: E \rightarrow M \) gives an injection \( \nabla^v: \Gamma(J^1E, (\pi^{1*}V(\pi))^*) \hookrightarrow \Omega^1(J^1E) \), then it makes sense to define
\[ S^\nabla := \nabla^v \circ \mathcal{S}: \Gamma(J^1E, \pi^*(\pi^1) \otimes \pi^{1*}T^*M) \rightarrow \Omega^1(J^1E) \]

As a consequence of the foregoing, the operation \( S^\nabla - V \) is meaningful.

In a natural system of coordinates the local expressions of all these elements are
\[ S = \zeta^A \otimes \frac{\partial}{\partial v^A} \otimes \frac{\partial}{\partial x^\nu} \]
\[ V = (dy^A - v^A d\nu^A) \otimes \frac{\partial}{\partial v^A} \otimes \frac{\partial}{\partial x^\eta} \]
\[ S^\nabla = (dy^A - \Gamma^A_{\nu} d\nu^A) \otimes \frac{\partial}{\partial v^A} \otimes \frac{\partial}{\partial x^\eta} \]

where \( \{\zeta^A\} \) is the local basis of \( \Gamma(J^1E, \pi^{1*}V(\pi))^* \) which is dual of \( \left\{ \frac{\partial}{\partial y^A} \right\} \), and \( \Gamma^A_{\nu} \) are the component functions of the connection \( \nabla \).

Acknowledgments

We are grateful for the financial support of the CICYT TAP97-0969-C03-01 and the CICYT PB98-0821. We wish to thank Mr. Jeff Palmer for his assistance in preparing the English version of the manuscript.

References

[1] V. Aldaya, J.A. de Azcárraga, “Geometric formulation of classical mechanics and field theory”, Riv. Nuovo Cimento 3 (1980) 1-66.
[2] A. Awane, “k-symplectic structures”, J. Math. Phys. 32(12) (1992) 4046-4052.
[3] E. Binz, J. Sniatycki, H. Fisher, The Geometry of Classical fields, North Holland, Amsterdam, 1988.
[4] F. Cantrijn, L.A. Ibort, M. de León, “Hamiltonian Structures on Multisymplectic Manifolds”, Rnd. Sem. Math. Univ. Pol. Torino 54, (1996) 225-236.
[5] F. Cantrijn, L.A. Ibort, M. de León, “On the Geometry of Multisymplectic Manifolds”, J. Austral. Math. Soc. Ser. 66 (1999) 303-330.
[6] J.F. Cariñena, M. Crampin, L.A. Ibort, “On the multisymplectic formalism for first order Field Theories”, Diff. Geom. Appl. 1 (1991) 345-374.
[7] D. Chinea, M. de León, J.C. Marrero, “The constraint algorithm for time-dependent Lagrangians”, J. Math. Phys. 7 (1994) 3410-3447.
A. Echeverría-Enríquez et al, Geometry of Multisymplectic Hamiltonian...
[26] S.P. Hrabak. “On a Multisymplectic Formulation of the Classical BRST Symmetry for First-Order Field Theories (Part I): Algebraic Structures”, math-ph/9901012 (1999).

[27] S.P. Hrabak. “On a Multisymplectic Formulation of the Classical BRST Symmetry for First-Order Field Theories (Part II): Geometric Structures”. math-ph/9901013 (1999).

[28] L.A. Ibort, A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Invariant Forms and Automorphisms of Multisymplectic Manifolds”, math-dg/9805040 (1998).

[29] I. Kanatchikov, “Novel Algebraic Structures from the Polysymplectic Form in Field Theory”, GROUP21, Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras, Vol. 2, H.A. Doebner, W Scherer, C. Schulte Eds., World Scientific, Singapore (1997) 894.

[30] I.V. Kanatchikov, “Canonical structure of Classical Field Theory in the polymomentum phase space”, Rep. Math. Phys. 41(1) (1998) 49-90.

[31] J. Kijowski, “A Finite-dimensional Canonical Formalism in the Classical Field Theory”, Comm. Math. Phys. 30 (1973) 99-128.

[32] J. Kijowski, W. Szczyrba, “Multisymplectic Manifolds and the Geometrical Construction of the Poisson Brackets in the Classical Field Theory”, Géométrie Symplectique et Physique Mathématique Coll. Int. C.N.R.S. 237 (1975) 347-378.

[33] J. Kijowski, W.M. Tulczyjew, A Symplectic Framework for Field Theories, Lect. Notes Phys. 170, Springer-Verlag, Berlin (1979).

[34] R. Kuwabara, “Time-dependent mechanical symmetries and extended hamiltonian systems”, Rep. Math. Phys. 19 (1984) 27-38.

[35] J.K. Lawson, “A Frame Bundle Generalization of Multisymplectic Field Theories”, dg-ga/9706008 (1997).

[36] M. de León, J. Marín-Solano, J.C. Marrero, “Ehresmann Connections in Classical Field Theories”, Proc. III Fall Workshop: Differential Geometry and its Applications, Anales de Física, Monografías 2 (1995) 73-89.

[37] M. de León, J. Marín-Solano, J.C. Marrero, “A Geometrical approach to Classical Field Theories: A constraint algorithm for singular theories”, Proc. on New Developments in Differential geometry, L. Tamassii-J. Szenthe eds., Kluwer Acad. Press, (1996) 291-312.

[38] M. de León, J. Marín-Solano, J.C. Marrero, “The constraint algorithm in the jet formalism”, Diff. Geom. Appl. 6 (1996) 275-300.

[39] M. de León, I. Méndez, M. Salgado, “p-almost tangent structures”. Rend. Circolo Mat. Palermo ser. II, t XXXVII (1988) 282-294.

[40] M. de León, I. Méndez, M. Salgado, “Regular p-almost cotangent structures”. J. Korean Math. 25(2) (1988) 273-287.

[41] M. de León, E. Merino, J.A. Oubiña, P.R. Rodrigues, M. Salgado, “Hamiltonian Systems on k-cosymplectic Manifolds”. J. Math. Phys. 39(2) (1998) 876-893.

[42] M. de León, E. Merino, M. Salgado, “k-cosymplectic Manifolds and Lagrangian Formalism for Field Theories”. Preprint (1999).

[43] J.E. Marsden, S. Shkoller, “Multisymplectic Geometry, Covariant Hamiltonians and Water Waves”, Math. Proc. Camb. Phil. Soc. 125 (1999) 553-575.

[44] G. Martin, “Dynamical Structures for k-Vector Fields”, Int. J. Theor. Phys. 27(5) (1988) 571-585.
[45] G. Martin, “A Darboux Theorem for multisymplectic manifolds”, *Lett. Math. Phys.* **16** (1988) 133-138.

[46] L.K. Norris, “Generalized Symplectic Geometry in the Frame Bundle of a Manifold”, *Proc. Symposia in Pure Math.* **54**(2) (1993) 435-465.

[47] M. Puta, “Some remarks on the $k$-symplectic manifolds”, *Tensor N.S.* **47**(2) (1988) 109-115.

[48] M.F. Rañada, “Extended Legendre transformation approach to the time-dependent hamiltonian formalism”, *J. Phys. A: Math. Gen.* **25**(14) (1992) 4025-4035.

[49] G. Sardanashvily, *Generalized Hamiltonian formalism for Field Theory. Constraint Systems*, World Scientific, Singapore (1995).

[50] G. Sardanashvily, O. Zakharov, “The Multimomentum Hamiltonian formalism for Field Systems”, *Int. J. Theor. Phys.* **31** (1992) 1477-1504.

[51] G. Sardanashvily, O. Zakharov, “On application of the Hamilton formalism in fibred manifolds to Field Theory”, *Diff. Geom. Appl.* **3** (1993) 245-263.

[52] D.J. Saunders, “The Cartan form in Lagrangian field theories”, *J. Phys. A: Math. Gen.* **20** (1987) 333-349.

[53] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lect. Notes Ser. **142**, Cambridge, Univ. Press, 1989.